Fibonacci konferenciakötet külső borító

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Sierpinski-like triangle-patterns in Bi- and Fibo-nomial triangles

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Abstract

In this paper we introduce the notion of generalized (p-order) Sierpinski-like triangle-pattern, and we define the Bi- and Fibo-nomial triangles (P_Δ, F_Δ) and their divisibility patterns ($P_{\Delta(p)}, F_{\Delta(p)}$), respect to p. We proof that if p is an odd prime then these divisibility patterns actually are generalized Sierpinski-like triangle-patterns.

Keywords: Fibonacci sequence, Binomial triangle, Fibonomial triangle, Sierpinski pattern

MSC: 11B39, 11B65

1. Introduction

Several authors investigated the divisibility patterns of Bi- and Fibo-nomial triangles. Long (see [1]) showed that, modulo p (where p denotes a prime), Binomial triangles (also called Pascal's triangle) have self-similar structures (upon scaling by the factor p). Holte proofed similar results for Fibonomial triangles (see [2, 3]). Wells investigated (see [4]) the parallelisms between modulo 2 patterns of Bi- and Fibo-nomial triangles. In this paper we introduce the notion of generalized (p-order) Sierpinski-like triangle-pattern, and we proof that if p is an odd prime then the divisibility patterns, respect to p, of the Bi- and Fibo-nomial triangles are generalized Sierpinski-like triangle-patterns.

2. Sierpinsky like binary triangle patterns

Definition 2.1. We define $S(a, p, k)$ as generalized Sierpinsky-like binary triangle pattern, where: a denotes the side-length of the starting triangle, p denotes the order of the pattern, and k denotes the level of the pattern. The first level pattern is an equilateral number triangle with side-lengths equal to a, and all elements equal to 1 (row i, $1 \leq i \leq a$, contains i elements equal to 1). We construct the p-th order ($p > 1$), $(k+1)$-th level pattern from the p-th order, k-th level pattern ($k \geq 1$) as follows:

- We multiply the k-th level triangle $1+2+\ldots+p$ times and we arrange them in p rows (row i, $1 \leq i \leq p$, will contain i k-th level triangle) in such a way that each triangle touches its neighbour triangles at a corner.
- The remaining free positions are filled by zeros.

Figure 1 shows the 3rd order, 1st, 2nd and 3rd level patterns, if the starting side-length is 3. If we choose as starting side-length 4, then we have the patterns from Figure 2.

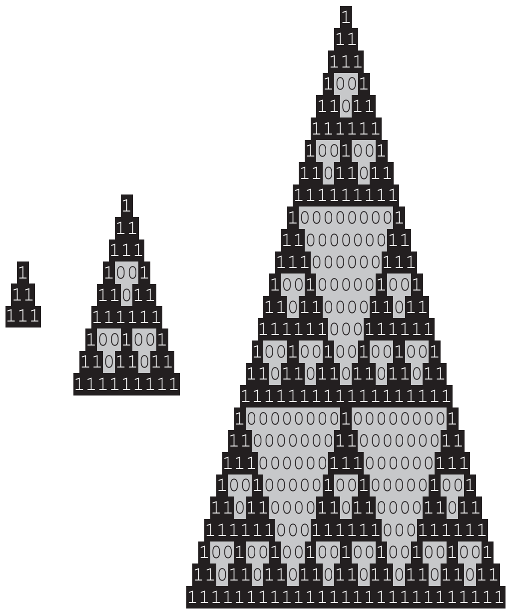

Figure 1: The 3$^{\text{rd}}$ order, 1$^{\text{st}}$ (a), 2$^{\text{nd}}$ (b) and 3$^{\text{rd}}$ (c) level patterns, if the starting side-length is 3.

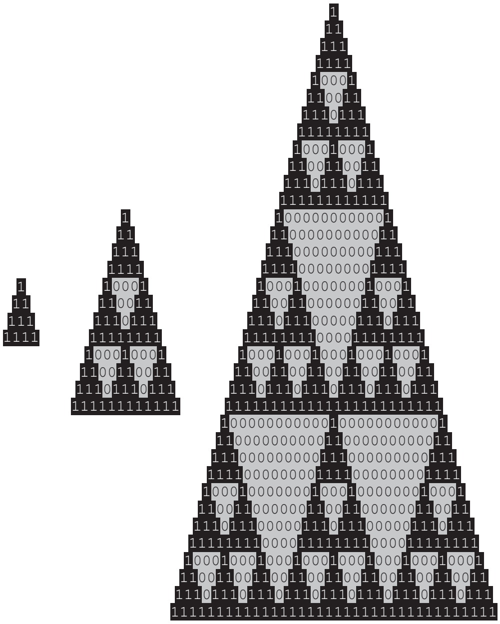

Figure 2: The 3rd order, 1st (a), 2nd (b) and 3rd (c) level patterns, if the starting side-length is 4.

3. Patterns in the prime-factorization of n and F_n

Definition 3.1. For any prime $p \geq 2$, we define sequence $x(r,p)_{r\geq 1}$ as the sequence of the powers of p in the prime-factorization of n.

Let a_p denote the subscript of the first natural number which is divisible by p. Evidently, $a_p = p$.

It is trivial that sequence $x(r,p)_{r\geq 1}$ can be constructed as follows:

- Step 0: We start with $x(r,p)_{r\geq 1} = 0$
- Step 1: All a_p-th elements 0 are increased with 1.
- Step 2: All p-th elements 1 are increased with 1.
- Step k: ... All p-th elements equal to $(k-1)$ are increased with 1 ...

Let n_k denote the subscript of the first term of sequence $x(r,p)_{r\geq 1}$ that is equal to a given $k \geq 1$. Evidently, $n_k = p^k$.

Definition 3.2. The well-known Fibonacci sequence is defined as follows:

$$F_0 = 0, F_1 = 1$$
$$F_r = F_{r-1} + F_{r-2}, r > 1$$

Definition 3.3. For any prime $p \geq 2$, we define sequence $y(r,p)_{r\geq 1}$ as the sequence of the powers of p in the prime-factorization of F_r.

Let b_p denote the subscript of the first Fibonacci number which is divisible by p (restricted period of $F \pmod p$). Two well-known results (for proofs see [5, 6]):

Lemma 3.4. *For any $i \geq 1$, $b_i \mid r$ if and only if $i \mid F_r$.*

Lemma 3.5. *Let p be an odd prime and suppose p^t divides F_r but p^{t+1} does not divide F_r for some $t \geq 1$. If p does not divide v then p^{t+1} divides $F_{r\cdot v\cdot p}$ but p^{t+2} does not divide $F_{r\cdot v\cdot p}$.*

A well-known conjecture in this subject:

Conjecture. For any prime p, F_{b_p} is divisible by p exactly once.

Assuming the validity of the above conjecture an immediate consequence of lemmas 3.4 and 3.5 is that sequence $y(r,p)_{r\geq 1}$ can be constructed as follows:

- Step 0: We start with $y(r,p)_{r\geq 1} = 0$
- Step 1: All b_p-th elements 0 are increased with 1.
- Step 2: All p-th elements 1 are increased with 1. (for $p = 2$ all p-th elements 1 are increased with 2)
- Step k: ... All p-th elements appeared in step $(k-1)$ are increased with 1 ...

Let m_k denote the subscript of the first term of sequence $y(r,p)_{r\geq 1}$ that is equal to a given $k \geq 1$. Evidently, $m_1 = b_p$.

Two immediate properties of sequence y are:

Property 1. Sequence y is characterized by several symmetry points: terms from symmetric positions are identical.

$$y_r = y_{j\cdot m_k - r} = y_{j\cdot m_k + r} = y_{p\cdot m_k - r}, \text{ for any } 0 < r < m_k,\ j = 1\ldots(p-1).$$
$$y_r = y_{j\cdot(m_k/p) - r} = y_{j\cdot(m_k/p)+r} = y_{m_k - r}, \text{ for any } 0 < r < \frac{m_k}{p},\ j = 1\ldots(p-1).$$

Proof. Trivially results from Lemmas 3.4 and 3.5. □

Property 2. For a fixed d the sum of the terms of a subsequence of length d is minimal for the leftmost (starting with index 1) subsequence and maximal for the rightmost (ending with index m_k) one. We define

$$v(i,d) = y_{i+1-d} + \ldots + y_i,\ d = 1\ldots m_k,\ i = d\ldots m_k,$$
$$u(i,d) = y_i + \ldots + y_{i+d-1},\ d = 1\ldots m_k,\ i = 1\ldots(m_k+1-d).$$

We have for a fixed d

a) $v(i,d) < v(m_k,d)$, for any $i = d \dots m_k - 1$

b) $u(1,d) \leq u(i,d)$, for any $i = 2 \dots (m_k + 1 - d)$

Proof. (a): According to the way sequence y was built we have:

- Step 0: All terms are 0 and consequently $v(i,d) = v(m_k,d)$, for any $i = d \dots m_k - 1$.
- Steps $1 \dots (k-1)$: Since the increasing operations take place in equidistant positions, and the term from position m_k is increased in each step, we have $v(i,d) \leq v(m_k,d)$, for any $i = d \dots m_k - 1$.
- Step k: Since in this step only the term from position m_k is increased, we have $v(i,d) < v(m_k,d)$, for any $i = d \dots m_k - 1$. □

Proof. (b): According to the way sequence y was built we have:

- Step 0: All terms are 0 and consequently $u(1,d) = u(i,d)$, for any $i = 2 \dots (m_k + 1 - d)$.
- Steps $1 \dots k$: The number of equidistant increases along a fixed length sequence decreases as the position of the first increase increases. Since in each step the position of the first increase (if it exists) of the leftmost subsequence of length d is maximal (relative to subsequences that start in positions $i > 1$), we have $u(1,d) \leq u(i,d)$, for any $i = 2 \dots (m_k + 1 - d)$. □

Note that properties 1 and 2 hold even we do not assume the validity of the above conjecture. Since sequences x and y were constructed in a similar way, Lemmas 3.4 and 3.5 hold for sequence x too (m_k has to be replaced by n_k).

4. Bi- and Fibo-nomial triangles

Definition 4.1. We define the r rows height Binomial triangle (also called Pascal triangle) ($P_\Delta(r)$) as an equilateral number triangle with rows indexed by $i = 0 \dots (r-1)$, the elements of rows indexed by $j = 0 \dots i$, and term (i,j) equal to:

$$P_\Delta[i,j] = \frac{\prod_{i+1-j}^{i} t}{\prod_{1}^{j} t}$$

Changing t by F_t in the definition of Binomial triangle we receive the corresponding Fibonomial triangle.

Definition 4.2. We define the r rows height Fibonomial triangle ($F_\Delta(r)$) as an equilateral number triangle with rows indexed by $i = 0 \dots (r-1)$, the elements of rows indexed by $j = 0 \dots i$, and term (i,j) equal to

$$F_\Delta[i,j] = \frac{\prod\limits_{i+1-j}^{i} F_t}{\prod\limits_{1}^{j} F_t}$$

Definition 4.3. We also define the mod p binary Bi- and Fibo-nomial triangles $(P_{\Delta(p)}, F_{\Delta(p)})$ as follows: term (i,j) in the binary triangle is 0, if p divide term (i,j) in the corresponding Bi- or Fibo-nomial triangle, otherwise it is 1.

$$P_{\Delta(p)}[i,j] = \begin{cases} 0 & \text{if } p \mid P_{\Delta(p)}[i,j] \\ 1 & \text{otherwise} \end{cases}$$

$$F_{\Delta(p)}[i,j] = \begin{cases} 0 & \text{if } p \mid F_{\Delta(p)}[i,j] \\ 1 & \text{otherwise} \end{cases}$$

Figures 1 and 2 (triangles c) shows the $n_3 = 27$ and $m_3 = 36$ row height mod 3 binary Bi- and Fibo-nomial triangles, respectively.

5. Main result

Lemma 5.1. *Considering triangle $F_{\Delta(p)}$ (p an odd prime), for any i $(0 \le i < m_k)$ segments $F_{\Delta(p)}[i, 0\dots i]$, $F_{\Delta(p)}[m_k+i, 0\dots i]$ and $F_{\Delta(p)}[m_k+i, m_k\dots(m_k+i)]$ are identical.*

Proof. For $i = 0$ the validity of this lemma results trivially from the definition of $F_{\Delta(p)}$. In the case of $0 < i < m_k$ terms $F_{\Delta(p)}[i,j]$ and $F_{\Delta(p)}[m_k+i,j]$ $(j = 1\dots i)$ are identical since the denominators of terms $F_{\Delta(p)}[i,j]$ and $F_{\Delta(p)}[m_k+i,j]$ are identical, and the exponents of p in the factorizations of the numerators of these terms are also identical. These exponents, $\sum_{m_k+r+1-i}^{m_k+r} x_t$ and $\sum_{r+1-i}^{r} x_t$, are equals since $y_{m_k+j} = y_j$ for any $j = 1\dots r$. Since both row i and row m_k+i are symmetrical, it results that the segments of the first $i+1$ and last $i+1$ elements of row m_k+i are identical. □

Lemma 5.2. *Considering triangle $F_{\Delta(p)}$ (p an odd prime), for any i and j, where $0 \le i < m_k$ and $i+1 \le j < m_k$, term $F_{\Delta(p)}[m_k+i,j]$ equals zero.*

Proof. With respect to the exponent of p in the factorizations of term $F_\Delta[m_k+i,j]$ we have

$$\sum_{m_k+r+1-i}^{m_k+r} x_t - \sum_{1}^{i} x_t = \sum_{m_k+r+1-i}^{m_k} x_t - \sum_{r+1}^{i} x_t > 0.$$

The equality results from Property 1 and the inequality results from Property 2.b. Consequently, $F_\Delta[m_k+i,j]$ is dividable by p. □

Lemma 5.3. *Considering triangle $F_{\Delta(p)}$ (p an odd prime), segments $F_{\Delta(p)}[m_k + i, 0\dots m_k+i]$ and $F_{\Delta(p)}[f\cdot m_k+i, g\dots(g+m_k+i)]$, where $0 \le i < m_k, 1 < f < p$ and $0 \le g < f$, are identical.*

Proof. With respect to the exponent of p in the factorizations of term $F_\Delta[f \cdot m_k + i, g + j]$, where $0 \le j \le m_k + i$, we have

$$\begin{aligned}
&\sum_{f\cdot m_k+r+1-(g\cdot m_k+i)}^{f\cdot m_k+r} x_t - \sum_{1}^{g\cdot m_k+i} x_t \\
&= \sum_{f\cdot m_k+r+1-(g\cdot m_k+i)}^{f\cdot m_k+r-g\cdot m_k} x_t + \sum_{f\cdot m_k+r-g\cdot m_k+1}^{f\cdot mk} x_t + \sum_{f\cdot m_k+1}^{f\cdot m_k+r} x_t - \sum_{1}^{g\cdot m_k} x_t - \sum_{g\cdot m_k+1}^{g\cdot m_k+i} x_t \\
&= \sum_{f\cdot m_k+r+1-(g\cdot m_k+i)}^{f\cdot m_k+r-g\cdot m_k} x_t + \sum_{f\cdot m_k+r-g\cdot m_k+1}^{f\cdot mk} x_t + \sum_{(f-g)\cdot m_k+1}^{(f-g)\cdot m_k+r} x_t - \sum_{1}^{g\cdot m_k} x_t - \sum_{g\cdot m_k+1}^{g\cdot m_k+i} x_t \\
&= \sum_{f\cdot m_k+r+1-(g\cdot m_k+i)}^{f\cdot m_k+r-g\cdot m_k} x_t + \sum_{(f-g)\cdot m_k+1}^{f\cdot mk} x_t - \sum_{1}^{g\cdot m_k} x_t - \sum_{g\cdot m_k+1}^{g\cdot m_k+i} x_t \\
&= \sum_{f\cdot m_k+r+1-(g\cdot m_k+i)}^{f\cdot m_k+r-g\cdot m_k} x_t - \sum_{g\cdot m_k+1}^{g\cdot m_k+i} x_t \\
&= \sum_{(f-g)\cdot m_k+r+1-i}^{(f-g)\cdot m_k+r} x_t - \sum_{g\cdot m_k+1}^{g\cdot m_k+i} x_t = \sum_{m_k+r+1-i}^{m_k+r} x_t - \sum_{1}^{i} x_t.
\end{aligned}$$

Which equals to the exponent of p in the factorizations of term $F_\Delta[m_k + i, j]$. □

Theorem 5.4. *For odd prime $p, P_{\Delta(p)}(n_k)$ is identical with $S(n_1, p, k)$.*

The proof of this theorem follows the same train of thought as the next one.

Theorem 5.5. *For odd prime $p, F_{\Delta(p)}(m_k)$ is identical with $S(m_1, p, k)$.*

Proof. We use mathematical induction. For $k = 1$ it is trivial that $F_{\Delta(p)}(1)$ is identical with $S(m_1, p, 1)$. Assuming that $F_{\Delta(p)}(k)$ is identical with $S(m_1, p, k)$, we prove that $F_{\Delta(p)}\ (k + 1)$ is identical with a $S(m_1, p, k + 1)$. Lemmas 5.1 and 5.2 show that rows $[m_k \dots 2 \cdot m_k)$ follow the Sierpinski pattern. Lemma 5.3 shows: since segments $[j \cdot m_k \dots (j+1)m_k)$, $(j = 2 \dots (p-1))$ can be viewed as translations of segment $[m_k \dots 2 \cdot m_k)$, these also follow the Sierpinski pattern. □

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Tiling approach to obtain identities for generalized Fibonacci and Lucas numbers

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Abstract

In Proofs that Really Count [2], Benjamin and Quinn have used "square and domino tiling" interpretation to provide tiling proofs of many Fibonacci and Lucas formulas. We explore this approach in order to provide tiling proofs of some generalized Fibonacci and Lucas identities.

Keywords: Generalized Fibonacci and Lucas numbers; Tiling proofs.

MSC: 05A19, 11B39, 11B37.

1. Introduction

Let U_n and V_n denote the generalized Fibonacci and Lucas numbers defined, respectively, by

$$U_n = aU_{n-1} + bU_{n-2} \quad (n \geq 2)\,, \tag{1.1}$$

with the initial conditions $U_0 = 1,\ U_1 = a$, and by

$$V_n = aV_{n-1} + bV_{n-2} \quad (n \geq 2)\,, \tag{1.2}$$

with the initial conditions $V_0 = 2,\ V_1 = a$, where a and b are non-negative integers.

In [1], the generalized Fibonacci number U_n is interpreted as the number of ways to tile a $1 \times n$ board with cells labeled $1, 2, \ldots, n$ using colored squares (1×1 tiles) and dominoes (1×2 tiles), where there are a different colors for squares and b different colors for dominoes. In fact, there is one way to tile a empty board ($U_0 = 1$), since a board of length one can be covered by one colored square ($U_1 = a$), so this satisfy the initial Fibonacci conditions. Now for $n \geq 2$, if the first tile is a square, then there are a possibilities to color the square and U_{n-1} ways to tile $1 \times (n-1)$ board. If the first tile is a domino, then there are b choices for the domino and U_{n-2} ways to tile $1 \times (n-2)$ board. This gives the relation (1.1).

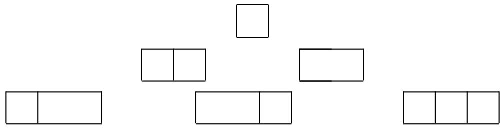

Figure 1: Tilings of length 1, 2 and 3 using squares and dominoes

Similarly, the generalized Lucas numbers count the number of ways to tile a circular $1 \times n$ board with squares and dominoes (termed $1 \times n$ bracelet). We call a $1 \times n$ bracelet in-phase if there is no domino occupying cells n and 1, and out-of phase if there is a domino occupying cells n and 1. The empty bracelet can be either in-phase or out-of phase, then $V_0 = 2$. Since a 1×1 bracelet can be tiled only by a square $V_1 = a$. For $n \geq 2$, a $1 \times n$ bracelet can be obtained from a $1 \times (n-1)$ bracelet by adding a square to the left of the first tile or from a $1 \times (n-2)$ bracelet by adding a domino to the left of the first tile. Then for $n \geq 2$ we have the relation (1.2).

Benjamin and Quinn, have used this approach to provide tiling proofs of many Fibonacci relations. Our goal is to use this interpretation to provide tiling proofs for the following two identities:

$$U_n - \sum_{k=0}^{m-1} \binom{n-k}{k} b^k a^{n-2k} = b^m \sum_{0 \leq j \leq k \leq n-2m} U_{n-k-2m} \frac{a^k}{k!} \begin{bmatrix} k \\ j \end{bmatrix} m^j, \tag{1.3}$$

where $\begin{bmatrix} k \\ j \end{bmatrix}$ are the Stirling numbers of the first kind.

$$2U_{n+m-1} = V_m U_{n-1} + V_n U_{m-1}. \tag{1.4}$$

To prove these identities we need the following Lemma.

Lemma 1.1 ([2]). *The number of* $1 \times n$ *tilings using exactly* k *colored dominoes is*

$$\binom{n-k}{k} b^k a^{n-2k}, \quad (k = 0, 1, \ldots, \lfloor n/2 \rfloor). \tag{1.5}$$

2. Combinatorial identities

Our first identity generalizes identity (1) given in [3]. It counts the number of ways to tile a $1 \times (n+2)$ board with at least one colored domino

$$U_{n+2} - a^{n+2} = b \sum_{k=0}^{n} U_k a^{n-k} \quad (n \geq 0). \tag{2.1}$$

Note that for $a = b = 1$, relation (2.1) gives the well known Lucas identity

$$f_{n+2} - 1 = \sum_{k=0}^{n} f_k,$$

where f_n is the shifted Fibonacci number defined recurrently by

$$f_n = f_{n-1} + f_{n-2} \quad (n \geq 2)\,, \tag{2.2}$$

with the initials $f_0 = f_1 = 1$.

The following identity counts the number of $1 \times n$ tilings with at least m colored dominoes.

Identity 1. *For $m \geq 1$ and $n \geq 2m$, we have*

$$U_n - \sum_{k=0}^{m-1} \binom{n-k}{k} b^k a^{n-2k} = b^m \sum_{0 \leq j \leq k \leq n-2m} U_{n-k-2m} \frac{a^k}{k!} \begin{bmatrix} k \\ j \end{bmatrix} m^j.$$

Proof. The left hand side counts the number of tilings of length n excluding the tilings with exactly $0, 1, \ldots, m-1$ dominoes. Now, let $k+1$, $k+2$ $(0 \leq k \leq n-2m)$ be the position of the m-*th* (from the right to the left) domino (see figure 2), then there are U_k ways to tile the first k cells, b ways to color the domino at position $k+1$, $k+2$, and there are $\binom{n-m-k-1}{m-1} b^{m-1} a^{n-2m-k}$ ways to tiles cells from $k+3$ to n with exactly $m-1$ dominoes. Hence there are $\binom{n-m-k-1}{m-1} U_k b^m a^{n-2m-k}$ possible ways to tile an $1 \times n$ board with the m-*th* domino at the positions $k+1$, $k+2$. Summing over all $0 \leq k \leq n-2m$, we obtain

$$b^m \sum_{k=0}^{n-2m} U_k a^{n-k-2m} \binom{n-k-m-1}{m-1} = b^m \sum_{k=0}^{n-2m} U_{n-k-2m} a^k \binom{k+m-1}{m-1}. \tag{2.3}$$

Now, we express the binomial coefficient in terms of Stirling numbers of the first kind: $\binom{k+m-1}{m-1} = \frac{(m+k-1)\cdots(m+1)m}{k!} = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{m^j}{k!}$, this gives the right hand side of the identity. $\square$

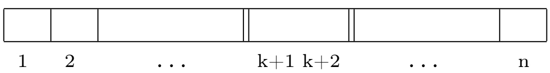

Figure 2: A $1 \times n$ tiling with the m-th domino at cells $k+1$, $k+2$

Remark 2.1. We can consider the intermediate identity (2.3), as given in the proof without using Stirling numbers.

Corollary 2.2. *Let $a = b = 1$, using relation* (2.3) *we have for $m = 1, 2, 3$ respectively*

$$\sum_{k=0}^{n} f_k = f_{n+2} - 1 \quad \textit{(E. Lucas, 1878)}$$
$$\sum_{k=0}^{n} k f_k = n f_{n+2} - f_{n+3} + 3 \quad \textit{(Brother. U. Alfred, 1965)}$$
$$\sum_{k=0}^{n} k^2 f_k = (n^2 + 2) f_{n+2} - (2n - 3) f_{n+3} - 13 \quad \textit{(Brother. U. Alfred, 1965)}$$

Now, we give tiling proof for the relation (1.4), for an algebraic proof, see for instance (V16a, pp 26, [5]).

Identity 2. *For $m \geq 1$ and $n \geq 1$, we have*

$$2U_{n+m-1} = V_m U_{n-1} + V_n U_{m-1}.$$

Proof. The left hand side counts the number of ways to tile a $1 \times (n+m-1)$ board. For the right hand side we suppose that we have a $1 \times (n+m-1)$ tiling. There is two cases:

Case 1. The $1 \times (n+m-1)$ tiling is breakable at m-th cell (there is not a domino covering positions m and $m+1$), then the $1 \times (n+m-1)$ tiling can be split into a $1 \times m$ tiling and a $1 \times (n-1)$ tiling. Now we attach the right side of the m-th cell to the left side of the first cell of the $1 \times m$ tiling, thus we form a in-phase $1 \times m$ bracelet. We denote the number of ways to tile an in-phase m-bracelet by V'_m.

Case 2. The $1 \times (n+m-1)$ tiling is not breakable at the m-th cell (there is a domino covering positions m and $m+1$), then it is breakable at $(m-1)$-th cell. In this case, we create a $1 \times (m-1)$ tiling and an out-of phase $1 \times n$ bracelet. We denote the number of ways to tile an out-phase $1 \times n$ bracelet by V''_n.

Now, we apply the same approach for the n-th cell, by considering either $1 \times (n+m-1)$ tiling is breakable at n-th cell or not. So, we obtain

$$\begin{aligned} 2U_{n+m-1} &= V'_m U_{n-1} + U_{m-1} V''_n + V'_n U_{m-1} + U_{n-1} V''_m \\ &= U_{n-1}(V'_m + V''_m) + U_{m-1}(V'_n + V''_n). \end{aligned}$$

We conclude by the fact that $V'_m + V''_m = V_m$ and $V'_n + V''_n = V_n$. □

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On factors of sums of consecutive Fibonacci and Lucas numbers

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Abstract

The Problem B-1 in the first issue of the Fibonacci Quarterly is the starting point of an extensive exploration of conditions for factorizations of several types of sums involving Fibonacci and Lucas numbers.

Keywords: Fibonacci number, Lucas number, factor, sum

MSC: Primary 11B39, 11Y55, 05A19

1. Introduction

Recall the Problem B-1 proposed by I. D. Ruggles of San Jose State College on the page 73 in the initial issue of the journal Fibonacci Quarterly in February 1963.

Problem B-1. *Show that the sum of twenty consecutive Fibonacci numbers is divisible by F_{10}.*

In the third issue of this first volume on pages 76 and 77 there is a solution using induction by Marjorie R. Bicknell also of San Jose State College.

With a little help from computers one can easily solve the above problem (using Maple V or Mathematica) and discover many other similar results. It is the purpose of this paper to present some of these discoveries. The proofs of all our claims could be done by induction. We shall leave them as the challenge to the readers.

There are many nice summation formulas for Fibonacci and Lucas numbers in the literature (see, for example, [1], [2], [3], [4] and [5]). We hope that the readers will find the ones that follow also interesting.

2. Sums of $4\,i+4$ consecutive Fibonacci numbers

In the special case (for $i = 4$) the following theorem provides another solution of the Problem B-1. It shows that the sums $\sum_{j=0}^{4i+3} F_{k+j}$ have the Fibonacci number F_{2i+2} as a common factor.

Theorem 2.1. *For integers $i \geq 0$ and $k \geq 0$, the following identities hold:*

$$\sum_{j=0}^{4i+3} F_{k+j} = F_{2i+2}\,L_{k+2i+3} = F_{k+4i+5} - F_{k+1} = F_{2i}\,L_{k+2i+5} + L_{k+3} =$$
$$L_{2i+1}\,F_{k+2i+4} + F_{k+2} = L_{2i}\,F_{k+2i+5} - 3\,F_{k+3} = F_{2i+1}\,L_{k+2i+4} - L_{k+2}.$$

The other identities in Theorem 1 have some importance in computations because they show that in order to get the big sum we need to know initial terms and two terms in the middle. The second representation is not suitable as the number F_{k+4i+5} is rather large.

3. The alternating sums

It is somewhat surprising that the (opposites of the) alternating sums of $4i+4$ consecutive Fibonacci numbers also have $F_{2\,i+2}$ as a common factor. Hence, the alternating sums of twenty consecutive Fibonacci numbers are all divisible by F_{10}.

Theorem 3.1. *For integers $i \geq 0$ and $k \geq 0$, the following identities hold:*

$$-\sum_{j=0}^{4i+3} (-1)^j\,F_{k+j} = F_{2i+2}\,L_{k+2i} = F_{k+4i+2} - F_{k-2} = L_{2i}\,F_{k+2i+2} - 3\,F_k$$
$$= F_{2i+1}\,L_{k+2i+1} - L_{k-1} = F_{2i-1}\,L_{k+2i+3} - 2\,L_{k+1} = L_{2i-1}\,F_{k+2i+3} + 4\,F_{k+1}.$$

4. Sums of $4\,i+2$ consecutive Fibonacci numbers

Similar results hold also for the (alternating) sums of $4\,i+2$ consecutive Fibonacci numbers. The common factor is the Lucas number L_{2i+1}. Hence, all (alternating) sums of twenty-two consecutive Fibonacci numbers are divisible by L_{11}.

Theorem 4.1. *For integers $i \geq 0$ and $k \geq 0$, the following identities hold:*

$$\sum_{j=0}^{4i+1} F_{k+j} = L_{2i+1}\,F_{k+2i+2} = F_{k+4i+3} - F_{k+1} = L_{2i-1}\,F_{k+2i+4} + L_{k+3}$$
$$= L_{2i+2}\,F_{k+2i+1} - L_k = F_{2i+3}\,L_{k+2i} - 3\,F_{k-1} = F_{2i+5}\,L_{k+2i-1} - 7\,F_{k-3}.$$

$$-\sum_{j=0}^{4i+1} (-1)^j F_{k+j} = L_{2i+1} F_{k+2i-1} = F_{k+4i} - F_{k-2} = F_{2i-1} L_{k+2i+1} - 3 F_k$$
$$= L_{2i} F_{k+2i} - L_{k-1} = L_{2i-2} F_{k+2i+2} - 2 L_{k+1} = F_{2i-2} L_{k+2i+2} + 4 F_{k+1}.$$

5. Sums with $4i+1$ and $4i+3$ terms

One can ask about the formulas for the (alternating) sums of $4i+1$ and $4i+3$ consecutive Fibonacci numbers. The answer provides the following theorem. These sums do not have common factors. However, they are sums of two familiar type of products (like $F_{2i} F_{k+2i+3}$ and $F_{2i+1} F_{k+2i}$).

Theorem 5.1. *For integers $i \geq 0$ and $k \geq 0$, the following identities hold:*

$$\sum_{j=0}^{4i} F_{k+j} = F_{2i} F_{k+2i+3} + F_{2i+1} F_{k+2i} =$$
$$F_{2i} L_{k+2i} + L_{2i+1} F_{k+2i} = F_{k+4i+2} - F_{k+1} = L_{2i+2} F_{k+2i} - 2 F_k$$
$$= F_{2i-1} L_{k+2i+3} - 2 F_{k+3} = L_{2i+1} F_{k+2i+1} - F_{k-1}.$$

$$\sum_{j=0}^{4i} (-1)^j F_{k+j} = F_{2i-1} F_{k+2i-1} + F_{2i+2} F_{k+2i-2} =$$
$$L_{2i+1} F_{k+2i} - F_{2i} L_{k+2i} = F_{k+4i-1} + F_{k-2} = L_{2i-1} F_{k+2i} + 2 F_k.$$

$$\sum_{j=0}^{4i+2} F_{k+j} = F_{2i+2} F_{k+2i+4} - F_{2i+1} F_{k+2i+1} = F_{k+4i+4} - F_{k+1} =$$
$$L_{2i+1} F_{k+2i+3} + F_k = L_{2i+2} F_{k+2i+2} - F_{k+2} = F_{2i+2} L_{k+2i+2} - F_{k-1}.$$

$$\sum_{j=0}^{4i+2} (-1)^j F_{k+j} = F_{2i+1} F_{k+2i+1} + F_{2i+2} F_{k+2i-2} =$$
$$L_{2i+3} F_{k+2i+1} - 2 F_{2i+2} L_{k+2i} = F_{k+4i+1} + F_{k-2} = F_{2i} L_{k+2i+1} + 2 F_k.$$

6. Sums of consecutive Lucas numbers

The above results suggests to consider many other sums especially when they are products or when they have very simple values.

The first that come to mind are the same sums of consecutive Lucas numbers. A completely analogous study could be done in this case. Here we only give a sample of two such identities.

$$\sum_{j=0}^{4i+3} L_{k+j} = 5\, F_{2i+2} F_{k+2i+3}, \quad \sum_{i=0}^{4i+1} L_{k+j} = L_{2i+1} L_{k+2i+2}.$$

7. Sums of consecutive products

Let us now consider sums of consecutive products of consecutive Fibonacci numbers. For an even number of summands the Fibonacci number F_{2i+2} is a common factor. Let $A = (-1)^k$.

$$\sum_{j=0}^{2i} F_{k+j}\, F_{k+j+1} = \tfrac{L_{2i+1} L_{2k+2i+1} - A}{5},$$

$$\sum_{j=0}^{2i+1} F_{k+j}\, F_{k+j+1} = F_{2i+2} F_{2k+2i+2}.$$

The same for the Lucas numbers gives the following identities:

$$\sum_{j=0}^{2i} L_{k+j}\, L_{k+j+1} = L_{2i+1} L_{2k+2i+1} + A,$$

$$\sum_{j=0}^{2i+1} L_{k+j}\, L_{k+j+1} = 5\, F_{2i+2} F_{2k+2i+2}.$$

We shall get similar identities in the two cases when Fibonacci and Lucas numbers both appear in each summand on the left hand side.

$$\left(\sum_{j=0}^{2i} F_{k+j}\, L_{k+j+1} \right) + A = \left(\sum_{j=0}^{2i} L_{k+j}\, F_{k+j+1} \right) - A = L_{2i+1} F_{2k+2i+1},$$

$$\sum_{j=0}^{2i+1} F_{k+j}\, L_{k+j+1} = \sum_{j=0}^{2i+1} L_{k+j}\, F_{k+j+1} = F_{2i+2} L_{2k+2i+2}.$$

8. Sums of squares of consecutive numbers

Our next step is to consider sums of squares of consecutive Fibonacci and Lucas numbers. Note that once again the summation of even and odd number of terms each lead to a separate formula. In fact, we consider a more general situation when multiples of a fixed number are used as indices of the terms in the sum. Only the parity of this number determines the form of the formula for the sum.

Theorem 8.1. *For all integers i, $k \geq 0$ and $v \geq 1$, we have*

$$\sum_{j=0}^{2i} F_{k+2vj}^2 = \frac{F_{2v(2i+1)} L_{2k+4vi}}{5\, F_{2v}} - \frac{2\, A}{5},$$

$$\sum_{j=0}^{2i+1} F_{k+2vj}^2 = \frac{F_{4v(i+1)} L_{2k+2v(2i+1)}}{5\, F_{2v}} - \frac{4\, A}{5},$$

$$\sum_{j=0}^{2i} L_{k+2vj}^2 = \frac{F_{2v(2i+1)} L_{2k+4vi}}{F_{2v}} + 2\, A,$$

$$\sum_{j=0}^{2i+1} L_{k+2vj}^2 = \frac{F_{4v(i+1)} L_{2k+2v(2i+1)}}{F_{2v}} + 4\, A,$$

Theorem 8.2. *For all integers i, $k \geq 0$ and $v \geq 0$, we have*

$$\sum_{j=0}^{2i} F_{k+(2v+1)j}^2 = \frac{L_{(2i+1)(2v+1)} L_{2k+2i(2v+1)}}{5\, L_{2v+1}} - \frac{2\, A}{5},$$

$$\sum_{j=0}^{2i+1} F_{k+(2v+1)j}^2 = \frac{F_{2(i+1)(2v+1)} F_{2k+(2i+1)(2v+1)}}{L_{2v+1}}.$$

$$\sum_{j=0}^{2i} L_{k+(2v+1)j}^2 = \frac{L_{(2i+1)(2v+1)} L_{2k+2i(2v+1)}}{L_{2v+1}} + 2\, A,$$

$$\sum_{j=0}^{2i+1} L_{k+(2v+1)j}^2 = \frac{5\, F_{2(i+1)(2v+1)} F_{2k+(2i+1)(2v+1)}}{L_{2v+1}}.$$

In particular, for $v = 0$ and $i = 9$, we conclude that the sums of squares of twenty consecutive Fibonacci numbers are divisible by F_{20} and the same sums of Lucas numbers by $5\, F_{20}$.

9. More sums of products

Here are some additional sums that are products or very close to the products.

$$\sum_{j=1}^{2i} F_j\, F_{k+j} = F_{2i-2}\, F_{k+2i+3} + F_{k+3} = F_{2i}\, F_{k+2i+1},$$

$$\sum_{j=1}^{2i+1} F_j\, F_{k+j} = F_{2i}\, F_{k+2i+3} + F_{k+1} = F_{2i+2}\, F_{k+2i+1}.$$

$$\sum_{j=0}^{2i} L_j\, L_{k+j} = L_{k+4i+1} + L_{k-2} = F_{2i+1}\, L_{k+2i+1} + F_{2i+2}\, L_{k+2i-2},$$

$$\sum_{j=0}^{2i+1} L_j\, L_{k+j} = L_{k+4i+3} - L_{k-1} = 5\, F_{2i+2}\, F_{k+2i+1}.$$

$$\sum_{j=0}^{2i} L_j\, F_{k+j} = F_{2i+2}\, L_{k+2i-1} + F_{k-1} = F_{2i}\, L_{k+2i+1} + 2\, F_k =$$

$$F_{2i+1}\, L_{k+2i-2} + L_{2i+2}\, F_{k+2i-2} = F_{2i}\, L_{k+2i-1} + L_{2i}\, F_{k+2i},$$

$$\sum_{j=0}^{2i+1} L_j\, F_{k+j} = \sum_{j=1}^{2i+1} F_j\, L_{k+j} = F_{2i}\, L_{k+2i+3} + L_{k+1} =$$

$$L_{2i+1}\, F_{k+2i+2} + F_k = F_{2i+2}\, L_{k+2i+1}.$$

$$\sum_{j=1}^{2i} F_j\, L_{k+j} = F_{2i+2}\, L_{k+2i-1} - L_{k-1} = L_{2i+1}\, F_{k+2i} - F_k =$$

$$F_{2i+1}\, L_{k+2i} - L_k = L_{2i}\, F_{k+2i+1} - 2\, F_{k+1} = F_{2i}\, L_{k+2i+1},$$

10. Sums of products of three numbers

In this final section we shall consider two sums of three consecutive Fibonacci and Lucas numbers when once again the common factor appears.

Theorem 10.1. *Let u be either $4i+1$ or $4i+3$. For all integers $i \geq 0$ and $k \geq 0$, we have*

$$\sum_{j=1}^{u} F_{k+j}\, F_{k+2j}\, F_{k+3j} = F_{u+1} \left[\tfrac{P}{4} - \tfrac{Q-A\,S}{10} - \tfrac{A\,R}{6} \right],$$

$$\sum_{j=1}^{u} L_{k+j}\, L_{k+2j}\, L_{k+3j} = 5\, F_{u+1} \left[\tfrac{Q-2\,P}{4} - \tfrac{A\,R}{2} + \tfrac{A\,S}{6} \right],$$

with

$$P = F_{3k+20i+10} + F_{3k+12i+6} + F_{3k+4i+2}, \qquad R = F_{k+12i+12} + 4\,F_{k+4i+4},$$
$$S = L_{k+12i+12} + 2\,L_{k+4i+4}, \qquad Q = L_{3k+20i+10} + L_{3k+12i+6} + L_{3k+4i+2},$$

if $u = 4i + 1$ *and*

$$P = F_{3k+20i+20} + F_{3k+12i+12} + F_{3k+4i+4}, \qquad R = F_{k+12i+12} + 4\,F_{k+4i+4},$$
$$S = L_{k+12i+12} + 2\,L_{k+4i+4}, \qquad Q = L_{3k+20i+20} + L_{3k+12i+12} + L_{3k+4i+4},$$

if $u = 4i + 3$.

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Some identities for Jacobsthal and Jacobsthal-Lucas numbers satisfying higher order recurrence relations

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Abstract

The Jacobsthal recurrence relation is extended to higher order recurrence relations and the basic list of identities provided by A. F. Horadam [10] is expanded and extended to several identities for some of the higher order cases.

Keywords: sequences, recurrence relations

MSC: 11B37 11B83 11A67 11Z05

1. Introduction

Horadam, in [10], exhibited a plethora of identities for the second order Jacobsthal and Jacobsthal-Lucas numbers. He then went on to explore their relationships and those of a variety of associated and representative sequences. The aim here is to present some additional identities and analogous relationships for numbers arising from some higher order Jacobsthal recurrence relations.

Obtaining properties by extending the Jacobsthal sequence to the third and higher orders depends on the choice of initial conditions. For example, this was done in [3] by taking all of the conditions to be zero, except the last, which was assigned the value 1. The procedure here will be to extend by using other initial values.

2. The second order Jacobsthal case

The second-order recurrence relations for the Jacobsthal numbers, J_n, and for the Jacobsthal-Lucas numbers, j_n, and a few of their relationships are given here for reference. Namely,

Recurrence relations

$$J_{n+2} = J_{n+1} + 2J_n,\ J_0 = 0, J_1 = 1,\ n \geq 0$$

$$j_{n+2} = j_{n+1} + 2j_n,\ j_0 = 2, j_1 = 1,\ n \geq 0$$

Table of values

n	0	1	2	3	4	5	6	7	8	9	10	...
J_n	0	1	1	3	5	11	21	43	85	171	341	...
j_n	2	1	5	7	17	31	65	127	257	511	1025	...

Binet forms

$$J_n = \frac{2^n - (-1)^n}{3} \text{ and } j_n = 2^n + (-1)^n$$

Simson/Cassini/Catalan identities

$$\begin{vmatrix} J_{n+1} & J_n \\ J_n & J_{n-1} \end{vmatrix} = (-1)^n 2^{n-1}, \quad \begin{vmatrix} j_{n+1} & j_n \\ j_n & j_{n-1} \end{vmatrix} = 9(-1)^{n-1} 2^{n-1}$$

Ordinary generating functions

$$\sum_{k=0}^{\infty} J_k x^k = \frac{x}{1 - x - 2x^2}$$

$$\sum_{k=0}^{\infty} j_k x^k = \frac{2 - x}{1 - x - 2x^2}$$

Exponential generating functions

$$\sum_{k=0}^{\infty} J_k \frac{x^k}{k!} = \frac{e^{2x} - e^{-x}}{3}$$

$$\sum_{k=0}^{\infty} j_k \frac{x^k}{k!} = e^{2x} + e^{-x}$$

Although these are not given in [10] the exponential generating functions are easily obtained using the Maclaurin series for the exponential function and can be useful in establishing identities. For example, using the method provided in [2, 12,

p. 232ff] the following can be obtained. Let $A = e^x$ and $B = \dfrac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta}$ where $\alpha = 2$ and $\beta = -1$. Then

$$B = \frac{1}{\alpha - \beta}\left[\frac{(\alpha - \beta)x}{1!} + \frac{(\alpha^2 - \beta^2)x^2}{2!} + \cdots\right] = \sum_{k=0}^{\infty} J_k \frac{x^k}{k!}.$$

Using the well known double sum identity

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} F(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} F(k,n-k)$$

found in [2, 15, p. 56] AB can be written as

$$AB = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{\infty} J_k \frac{x^k}{k!} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} J_k \frac{x^{n+k}}{n!k!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} J_k \frac{x^{(n-k)+k}}{(n-k)!k!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} J_k\right)\frac{x^n}{n!}.$$

In addition AB can also be written as

$$AB = \frac{e^{(\alpha+1)x} - e^{(\beta+1)x}}{\alpha - \beta} = \frac{e^{(2+1)x} - e^{(-1+1)x}}{2-(-1)} = \frac{e^{3x} - 1}{3} = \frac{1}{3}\cdot 0 + \sum_{n=1}^{\infty} 3^{n-1}\frac{x^n}{n!}$$

and so it follows that

$$\sum_{k=0}^{n}\binom{n}{k} J_k = 3^{n-1}.$$

Similarly with $B = \dfrac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta}$ and $A = e^{-3x}$ it follows that

$$\sum_{k=0}^{n}\binom{n}{k}(-2)^{n-1} J_k = (-3)^{n-1},$$

and if $B = e^{\alpha x + \beta x}$ then

$$\sum_{k=0}^{n}\binom{n}{k} J_k j_{n-k} = 2^n J_n.$$

Other summation identities can be obtained in a similar fashion.

3. The third order Jacobsthal case

First we consider extending the Jacobsthal and Jacobsthal-Lucas numbers to the third order, denoted as $J_n^{(3)}$ and $j_n^{(3)}$ respectively, with the following initial conditions:

Recurrence relations

$$J^{(3)}_{n+3} = J^{(3)}_{n+2} + J^{(3)}_{n+1} + 2J^{(3)}_{n},\ J^{(3)}_0 = 0, J^{(3)}_1 = 1, J^{(3)}_2 = 1\ n \geq 0.$$

$$j^{(3)}_{n+3} = j^{(3)}_{n+2} + j^{(3)}_{n+1} + 2j^{(3)}_{n},\ j^{(3)}_0 = 2, j^{(3)}_1 = 1, j^{(3)}_2 = 5\ n \geq 0.$$

Table of values

n	0	1	2	3	4	5	6	7	8	9	10	...
$J_n^{(3)}$	0	1	1	2	5	9	18	37	73	146	293	...
$j_n^{(3)}$	2	1	5	10	17	37	74	145	293	586	1169	...

Note that we extend to 3^{rd} order using initial conditions $\{0, 1, 1\}$ in the spirit of extending the Fibonacci initial conditions $\{0, 1\}$ to Tribonacci $\{0, 1, 1\}$ and those initial conditions for the Jacobsthal-Lucas numbers in a natural way from the second order case.

Binet forms

Using standard techniques for solving recurrence relations, the auxiliary equation, and its roots are given by

$$x^3 - x^2 - x - 2 = 0;\ x = 2, \text{and } x = \frac{-1 \pm i\sqrt{3}}{2}.$$

Note that the latter two are the complex conjugate cube roots of unity. Call them ω_1 and ω_2, respectively. Thus the Binet formulas can be written as

$$J^{(3)}_n = \frac{2}{7}2^n - \frac{3 + 2i\sqrt{3}}{21}\omega_1^n - \frac{3 - 2i\sqrt{3}}{21}\omega_2^n,$$

and

$$j^{(3)}_n = \frac{8}{7}2^n + \frac{3 + 2i\sqrt{3}}{7}\omega_1^n + \frac{3 - 2i\sqrt{3}}{7}\omega_2^n. \tag{3.1}$$

Simson's identities

$$\begin{vmatrix} J^{(3)}_{n+2} & J^{(3)}_{n+1} & J^{(3)}_{n} \\ J^{(3)}_{n+1} & J^{(3)}_{n} & J^{(3)}_{n-1} \\ J^{(3)}_{n} & J^{(3)}_{n-1} & J^{(3)}_{n-2} \end{vmatrix} = -2^{n-1}, \quad \begin{vmatrix} j^{(3)}_{n+2} & j^{(3)}_{n+1} & j^{(3)}_{n} \\ j^{(3)}_{n+1} & j^{(3)}_{n} & j^{(3)}_{n-1} \\ j^{(3)}_{n} & j^{(3)}_{n-1} & j^{(3)}_{n-2} \end{vmatrix} = -9 \cdot 2^{n+1}. \tag{3.2}$$

The identities above can be proved using mathematical induction. As an example an inductive proof for the J_n case is provided: For $n = 2, 3, 4$ and 5, the determinants are routinely computed to be $-2, -4, -8, -16$, respectively. So we surmise the general case to be as given in (3.2). Assuming the n^{th} case is true and expanding that determinant by the 3^{rd} column and expanding the $(n+1)^{th}$ determinant by the 1^{st} column yields the following:

$$\begin{vmatrix} J^{(3)}_{n+3} & J^{(3)}_{n+2} & J^{(3)}_{n+1} \\ J^{(3)}_{n+2} & J^{(3)}_{n+1} & J^{(3)}_{n} \\ J^{(3)}_{n+1} & J^{(3)}_{n} & J^{(3)}_{n-1} \end{vmatrix} = 2\begin{vmatrix} J^{(3)}_{n+2} & J^{(3)}_{n+1} & J^{(3)}_{n} \\ J^{(3)}_{n+1} & J^{(3)}_{n} & J^{(3)}_{n-1} \\ J^{(3)}_{n} & J^{(3)}_{n-1} & J^{(3)}_{n-2} \end{vmatrix} + C,$$

where

$$C = (J^{(3)}_{n+2} + J^{(3)}_{n+1}) \begin{vmatrix} J^{(3)}_{n+1} & J^{(3)}_{n} \\ J^{(3)}_{n} & J^{(3)}_{n-1} \end{vmatrix} - (J^{(3)}_{n+1} + J^{(3)}_{n}) \begin{vmatrix} J^{(3)}_{n+2} & J^{(3)}_{n+1} \\ J^{(3)}_{n} & J^{(3)}_{n-1} \end{vmatrix}$$
$$+ (J^{(3)}_{n} + J^{(3)}_{n-1}) \begin{vmatrix} J^{(3)}_{n+2} & J^{(3)}_{n+1} \\ J^{(3)}_{n+1} & J^{(3)}_{n} \end{vmatrix}.$$

By expanding C it is easy to see that the expression is 0 and so the conjecture is valid.

Ordinary generating functions

The ordinary generating functions are obtained by standard methods [12, p 237ff] as briefly illustrated here.

Let $g(x) = \sum_{k=0}^{\infty} J_k x^k$ and $h(x) = \sum_{k=0}^{\infty} j_k x^k$. Compute $(1-x-x^2-2x^3)g(x)$ and $(1-x-x^2-2x^3)h(x)$ and apply the initial conditions for the third order Jacobsthal and Jacobsthal-Lucas numbers, respectively, to obtain the following generating functions.

$$\sum_{k=0}^{\infty} J^{(3)}_k x^k = \frac{x}{1-x-x^2-2x^3}.$$

$$\sum_{k=0}^{\infty} j_k x^k = \frac{2-x+2x^2}{1-x-x^2-2x^3}.$$

Exponential generating functions

The exponential generating functions can be obtained from the Maclaurin series for the exponential function as follows. Note that

$$\frac{1}{21}\left(6e^{2x} - (3+2i\sqrt{3})e^{\omega_1 x} - (3+2i\sqrt{3})e^{\omega_2 x}\right) =$$
$$\sum_{k=0}^{\infty} \frac{1}{21}\left(6(2^k) - (3+2i\sqrt{3})\omega_1^k - (3+2i\sqrt{3})\omega_2^k\right)\frac{x^k}{k!} = \sum_{k=0}^{\infty} J_k \frac{x^k}{k!}.$$

Also, since

$$(3+2i\sqrt{3})e^{\omega_1 x} + (3+2i\sqrt{3})e^{\omega_2 x} = e^{-\frac{1}{2}x}\left((3+2i\sqrt{3})e^{\frac{\sqrt{3}}{2}ix} + (3+2i\sqrt{3})e^{\frac{\sqrt{3}}{2}ix}\right)$$
$$= e^{-\frac{1}{2}x}\left(6\cos\frac{\sqrt{3}x}{2} + 4\sqrt{3}\sin\frac{\sqrt{3}x}{2}\right),$$

the exponential generating function for the 3^{rd} order Jacobsthal numbers becomes

$$\sum_{k=0}^{\infty} J^{(3)}_k \frac{x^k}{k!} = \frac{1}{21}\left(6e^{2x} + e^{-\frac{1}{2}x}\left(6\cos\frac{\sqrt{3}x}{2} + 4\sqrt{3}\sin\frac{\sqrt{3}x}{2}\right)\right).$$

Similarly the exponential generating function for the 3^{rd} order Jacobsthal-Lucas numbers can be written as

$$\sum_{k=0}^{\infty} j_k^{(3)} \frac{x^k}{k!} = \frac{1}{7}\left(8e^{2x} + e^{-\frac{1}{2}x}\left(6\cos\frac{\sqrt{3}x}{2} + 4\sqrt{3}\sin\frac{\sqrt{3}x}{2}\right)\right).$$

4. Additional identities for third order Jacobsthal numbers

Summation formulas

$$\sum_{k=0}^{n} J_k^{(3)} = \begin{cases} J_{n+1}^{(3)} & \text{if } n \not\equiv 0 \bmod 3 \\ J_{n+1}^{(3)} - 1 & \text{if } n \equiv 0 \bmod 3 \end{cases}, \quad \sum_{k=0}^{n} j_k^{(3)} = \begin{cases} j_{n+1}^{(3)} - 2 & \text{if } n \not\equiv 0 \bmod 3 \\ j_{n+1}^{(3)} + 1 & \text{if } n \equiv 0 \bmod 3 \end{cases}.$$

Miscellaneous identities

$$3J_n^{(3)} + j_n^{(3)} = 2^{n+1}. \tag{4.1}$$

$$j_n^{(3)} - 3J_n^{(3)} = 2j_{n-3}^{(3)}. \tag{4.2}$$

$$j_{n+1}^{(3)} + j_n^{(3)} = 3J_{n+1}^{(3)}.$$

$$\left(j_n^{(3)}\right)^2 - 9\left(J_n^{(3)}\right)^2 = 2^{n+1} j_{n-3}^{(3)}.$$

$$\begin{cases} j_{3n-1}^{(3)} &= J_{3n+1}^{(3)} \\ j_{3n}^{(3)} &= J_{3n+2}^{(3)} + 1 \\ j_{3n+1}^{(3)} &= J_{3n+3}^{(3)} - 1 \end{cases}.$$

$$\begin{cases} j_{3n-1}^{(3)} - 4J_{3n-1}^{(3)} &= 1 \\ j_{3n}^{(3)} - 4J_{3n}^{(3)} &= 2 \\ j_{3n+1}^{(3)} - 4J_{3n+1}^{(3)} &= -3 \end{cases}.$$

$$j_n^{(3)} - 4j_{n-2}^{(3)} = \begin{cases} -3 & \text{if } n \text{ is even} \\ 6 & \text{if } n \text{ is odd} \end{cases}.$$

Squaring both sides of (4.1) and (4.2) and subtracting the results, it follows that

$$J_n^{(3)} j_n^{(3)} = \frac{1}{3}\left(4^n - \left(j_{n-3}^{(3)}\right)^2\right).$$

Note that some observations on generating functions for the Jacobsthal polynomials can be found in [7, 8]. Papers on generating functions for a variety of sequential numbers are abundant. See, for example [1, 4, 5, 6, 9, 13, 14, 16].

As an illustration of how ordinary generating functions can be used to derive identities, we use the technique of Gould, see [4] and used for Fibonacci identities in [2]. Making use of the properties of α and β for Fibonacci numbers as needed, it follows that

$$\begin{aligned}\sum_{k=0}^{\infty} J_k^{(3)} F_k x^k &= \sum_{k=0}^{\infty} J_k^{(3)} \frac{\alpha^k - \beta^k}{\alpha - \beta} x^k \\ &= \frac{\alpha x}{1 - \alpha x - \alpha^2 x^2 - 2\alpha^3 x^3} + \frac{\beta x}{1 - \beta x - \beta^2 x^2 - 2\beta^3 x^3} \\ &= \frac{x - x^3 - 2x^4}{1 - x - 4x^2 - 5x^3 + 4x^5 - 4x^6}.\end{aligned}$$

Similarly if we write (3.1) as $j_n^{(3)} = \frac{8}{7} 2^n + \frac{A}{7}\omega_1^n + \frac{B}{7}\omega_2^n$ and make use of the fact that $A\omega_1 = \frac{-9 + i\sqrt{3}}{2}, B\omega_2 = \frac{-9 - i\sqrt{3}}{2}, \omega_1^2 = \omega_2$, and $\omega_2^2 = \omega_1, \omega_1\omega_2 = \omega_1^3 = \omega_2^3 = 1$ then the following generating function is obtained:

$$\begin{aligned}\sum_{k=0}^{\infty} J_k^{(3)} j_k^{(3)} &= \frac{1}{7} \sum_{k=0}^{\infty} J_k^{(3)} \left(8(2x)^k + A(\omega_1 x)^k + B(\omega_2 x)^k\right) \\ &= \frac{13x + 20x^2 + 47x^3 - 16x^4 + 8x^5 - 40x^6 - 32x^7}{7(1 - 2x - 4x^2 - 16x^3)(1 + x + 2x^2 - 5x^3 - x^4 - 2x^5 + 4x^6)}.\end{aligned}$$

5. Higher order Jacobsthal numbers

As seen in [3] one way to generalize the Jacobsthal recursion is as follows.

$$J_{n+k}^{(k)} = \sum_{j=1}^{k-1} J_{n+k-j}^{(k)} + 2J_n^{(k)}$$

with $n \geq 0$ and initial conditions $J_i^{(k)} = 0$, for $i = 0, 1, \ldots k-2$ and $J_{k-1}^{(k)} = 1$, has characteristic equation $(x-2)(x^{k-1} + x^{k-2} + \cdots + x^2 + x + 1) = 0$ with eigenvalues 2 and $\omega_j = e^{\frac{2\pi i m}{k}}$ for $j = 1, 2, \ldots, k-1$, which yields the Binet form:

$$J_n^{(k)} = \frac{1}{\prod_{j=1}^{k-1}(2 - \omega_j)} \left(2^n - \sum_{j=1}^{k-1} \prod_{m \neq j}^{k-1} \frac{2 - \omega_m}{\omega_j - \omega_m} \omega_j^n \right).$$

In this paper we generalize the Jacobsthal recursion as

$$J_{n+k}^{(k)} = \sum_{j=1}^{k-1} J_{n+k-j}^{(k)} + 2J_n^{(k)},$$

with $n \geq 0$ and initial conditions $J_0^{(k)} = 0$ and $J_i^{(k)} = 1$ for $i = 1, \ldots k-1$. For the k^{th} order Jacobsthal -Lucas numbers $j_n^{(k)}$ we use the same recursion with initial conditions $j_i^{(k)} = j_i^{(k-1)}$ for $i = 0 \ldots k-1$. With the change of initial conditions a similar compact form for k^{th} order Binet formulae appears to be unobtainable as indicated in the examples below.

Ordinary generating function

A formula for the ordinary generating function for all generalized Fibonacci numbers has been addressed in other papers. For example, that given in [11] for the recurrence

$$a_n = b_{k-1}a_{n-1} + b_{k-2}a_{n-2} + \cdots + b_0 a_{n-k}$$

with arbitrary constant coefficients, b_j, and with arbitrary initial conditions is

$$g(x) = \frac{a_0 + \sum_{i=1}^{k-1}\left(a_i - \sum_{j=0}^{i} b_{k-i+j}a_j\right)x^i}{1 - \sum_{i=1}^{k} b_{k-i}x^i}. \tag{5.1}$$

Here we exhibit (5.1) for the k^{th} order Jacobsthal case (which could also be obtained by using the same procedure used in deriving the generating function for the 3^{rd} order case) namely

$$\sum_{i=0}^{\infty} J_i^{(k)} x^i = \frac{J_0^{(k)} + (J_1^{(k)} - J_0^{(k)})x + \cdots + (J_{k-1}^{(k)} - J_{k-2}^{(k)} - \cdots 2J_0^{(k)})x^{k-1}}{1 - x - x^x - \cdots - 2x^k}.$$

Examples

(1) The Fourth Order Jacobsthal and Jacobsthal–Lucas numbers

Recurrence relations

$$J_{n+4}^{(4)} = J_{n+3}^{(4)} + J_{n+2}^{(4)} + J_{n+1}^{(4)} + 2J_n^{(4)},$$

where $n \geq 0$ and $J_0^{(4)} = 0, J_1^{(4)} = J_2^{(4)} = J_3^{(4)} = 1$.

$$j_{n+4}^{(4)} = j_{n+3}^{(4)} + j_{n+2}^{(4)} + j_{n+1}^{(4)} + 2j_n^{(4)},$$

where $n \geq 0$ and $j_0^{(4)} = 2, j_1^{(4)} = 1, j_2^{(4)} = 5, j_3^{(4)} = 10$.

Table of values

n	0	1	2	3	4	5	6	7	8	9	10	...
$J_n^{(4)}$	0	1	1	1	3	7	13	25	51	103	205	...
$j_n^{(4)}$	2	1	5	10	20	37	77	154	308	613	1229	...

Binet form

The auxiliary equation, and its roots are given by

$$x^4 - x^3 - x^2 - x - 2 = 0, \; x_1 = 2, x_2 = -1, x_3 = i, x_4 = -i,$$

and the Binet formulas can be written as

$$J_n^{(4)} = \frac{1}{8+i}\left(2^n - \frac{1}{2}(1+8i)i^n + \frac{1}{2}(3+i)(-1)^n - \frac{1}{2}(4-7i)(-i)^n\right)$$

and

$$j_n^{(4)} = \frac{104(1-3i)2^n - 15(11+3i)i^n - 6(6+17i)(-1)^n - 15(7+9i)(-i)^n}{4(16-63i)}.$$

Rewriting these in terms of the roots of unity, ω_j does not suggest a pattern when compared with the 2^{nd} and 3^{rd} order cases.

Simson's identity

$$\begin{vmatrix} J_{n+3}^{(4)} & J_{n+2}^{(4)} & J_{n+1}^{(4)} & J_n^{(4)} \\ J_{n+2}^{(4)} & J_{n+1}^{(4)} & J_n^{(4)} & J_{n-1}^{(4)} \\ J_{n+1}^{(4)} & J_n^{(4)} & J_{n-1}^{(4)} & J_{n-2}^{(4)} \\ J_n^{(4)} & J_{n-1}^{(4)} & J_{n-2}^{(4)} & J_{n-3}^{(4)} \end{vmatrix} = 0, \begin{vmatrix} j_{n+3}^{(4)} & j_{n+2}^{(4)} & j_{n+1}^{(4)} & j_n^{(4)} \\ j_{n+2}^{(4)} & j_{n+1}^{(4)} & j_n^{(4)} & j_{n-1}^{(4)} \\ j_{n+1}^{(4)} & j_n^{(4)} & j_{n-1}^{(4)} & j_{n-2}^{(4)} \\ j_n^{(4)} & j_{n-1}^{(4)} & j_{n-2}^{(4)} & j_{n-3}^{(4)} \end{vmatrix} = 2^{n-2}\cdot 3^5.$$

Summation formulas

$$\sum_{k=0}^{n} J_k^{(4)} = \begin{cases} J_{n+1}^{(4)} & \text{if } n \equiv \pm 1 \bmod 4 \\ J_{n+1}^{(4)} - 1 & \text{if } n \equiv 0 \bmod 4 \\ J_{n+1}^{(4)} + 1 & \text{if } n \equiv 2 \bmod 4 \end{cases}, \sum_{k=0}^{n} j_k^{(4)} = \begin{cases} j_{n+1}^{(4)} - 2 & \text{if } n \not\equiv 0 \bmod 4 \\ j_{n+1}^{(4)} + 1 & \text{if } n \equiv 0 \bmod 4 \end{cases}.$$

Miscellaneous fourth order identities

$$6J_n^{(4)} + j_n^{(4)} = \begin{cases} j_{n+1}^{(4)} + 1 & \text{if } n \equiv 0 \bmod 4 \\ j_{n+1}^{(4)} + 2 & \text{if } n \equiv 1 \bmod 4 \\ j_{n+1}^{(4)} + 1 & \text{if } n \equiv 2 \bmod 4 \\ j_{n+1}^{(4)} - 4 & \text{if } n \equiv 3 \bmod 4 \end{cases}.$$

$$j_n^{(4)} - 6J_n^{(4)} = \begin{cases} 2 & \text{if } n \equiv 0 \bmod 4 \\ -5 & \text{if } n \equiv 1 \bmod 4 \\ -1 & \text{if } n \equiv 2 \bmod 4 \\ 4 & \text{if } n \equiv 3 \bmod 4 \end{cases}.$$

$$J_n^{(4)} + j_n^{(4)} = \begin{cases} J_{n+2}^{(4)} & \text{if } n \equiv 0 \bmod 4 \\ J_{n+2}^{(4)} + 2 & \text{if } n \equiv 1 \bmod 4 \\ J_{n+2}^{(4)} - 1 & \text{if } n \equiv 2 \bmod 4 \\ J_{n+2}^{(4)} - 1 & \text{if } n \equiv 3 \bmod 4 \end{cases}.$$

In this case the product of the Jacobsthal and Jacobsthal–Lucas functions is somewhat less appealing than in previous cases:

$$24J_n^{(4)}j_n^{(4)} = \begin{cases} (j_{n+1}^{(4)}+1)^2-4 & \text{if } n\equiv 0 \bmod 4 \\ (j_{n+1}^{(4)}+2)^2-25 & \text{if } n\equiv 1 \bmod 4 \\ (j_{n+2}^{(4)}+1)^2-1 & \text{if } n\equiv 2 \bmod 4 \\ (j_{n+2}^{(4)}-4)^2-16 & \text{if } n\equiv 3 \bmod 4 \end{cases}.$$

(2) The Fifth Order Jacobsthal and Jacobsthal–Lucas numbers
Recurrence relations

$$J_{n+5}^{(5)} = J_{n+4}^{(4)} + J_{n+3}^{(5)} + J_{n+2}^{(5)} + J_{n+1}^{(5)} + 2J_n^{(5)},$$

where $n \geq 0$ and $J_0^{(5)} = 0, J_1^{(5)} = J_2^{(5)} = J_3^{(5)} = J_4^{(5)} = 1$.

$$j_{n+5}^{(5)} = j_{n+4}^{(5)} + j_{n+3}^{(5)} + j_{n+2}^{(5)} + j_{n+1}^{(5)} + 2j_n^{(5)},$$

where $n \geq 0$ and $j_0^{(5)} = 2, j_1^{(5)} = 1, j_2^{(5)} = 5, j_3^{(5)} = 10, j_4^{(5)} = 20$.
Table of values

n	0	1	2	3	4	5	6	7	8	9	10	...
$J_n^{(5)}$	0	1	1	1	1	4	9	17	33	65	132	...
$j_n^{(5)}$	2	1	5	10	20	40	77	157	314	628	1256	...

Binet form

The auxiliary equation, and its roots are given by

$$x^5 - x^4 - x^3 - x^2 - x - 2 = 0, x_1 = 2, x_2 = \omega_1, x_3 = \omega_2, x_4 = \omega_3, x_5 = \omega_4,$$

where for $m = 1,2,3,4$, $\omega_m = \exp\left(\dfrac{2\pi i m}{5}\right)$. The Binet formulas can be written as

$$\begin{aligned} J_n^{(5)} &= \frac{-4}{33}2^n - \frac{24+43\omega_1+37\omega_2-59\omega_3-45\omega_4}{155}\omega_1^n \\ &+ \frac{24-59\omega_1+43\omega_2-45\omega_3+37\omega_4}{155}\omega_2^n + \frac{24+37\omega_1-45\omega_2+43\omega_3-59\omega_4}{155}\omega_3^n \\ &- \frac{24-45\omega_1-59\omega_2+37\omega_3+43\omega_4}{155}\omega_4^n, \end{aligned}$$

and similarly

$$\begin{aligned} j_n^{(5)} &= \frac{42}{33}2^n + \frac{3(14-24\omega_1-12\omega_2+25\omega_3-3\omega_4)}{155(\omega_1-\omega_2-\omega_3-\omega_4)}\omega_1^n \\ &+ \frac{3(14+25\omega_1-24\omega_2-3\omega_3+12\omega_4)}{155(\omega_1-\omega_2-\omega_3-\omega_4)}\omega_2^n + \frac{3(14-12\omega_1-3\omega_2-24\omega_3+25\omega_4)}{155(\omega_1-\omega_2-\omega_3-\omega_4)}\omega_3^n \\ &- \frac{3(14-3\omega_1+25\omega_2-12\omega_3-24\omega_4)}{155(\omega_1-\omega_2-\omega_3-\omega_4)}\omega_4^n, \end{aligned}$$

Simson's identity

$$\begin{vmatrix} J_{n+4}^{(5)} & J_{n+3}^{(5)} & J_{n+2}^{(5)} & J_{n+1}^{(5)} & J_{n}^{(5)} \\ J_{n+3}^{(5)} & J_{n+2}^{(5)} & J_{n+1}^{(5)} & J_{n}^{(5)} & J_{n-1}^{(5)} \\ J_{n+2}^{(5)} & J_{n+1}^{(5)} & J_{n}^{(5)} & J_{n-1}^{(5)} & J_{n-2}^{(5)} \\ J_{n+1}^{(5)} & J_{n}^{(5)} & J_{n-1}^{(5)} & J_{n-2}^{(5)} & J_{n-3}^{(5)} \\ J_{n}^{(5)} & J_{n-1}^{(5)} & J_{n-2}^{(5)} & J_{n-3}^{(5)} & J_{n-4}^{(5)} \end{vmatrix} = 2^{n-2} \cdot 11.$$

$$\begin{vmatrix} j_{n+4}^{(5)} & j_{n+3}^{(5)} & j_{n+2}^{(5)} & j_{n+1}^{(5)} & j_{n}^{(5)} \\ j_{n+3}^{(5)} & j_{n+2}^{(5)} & j_{n+1}^{(5)} & j_{n}^{(5)} & j_{n-1}^{(5)} \\ j_{n+2}^{(5)} & j_{n+1}^{(5)} & j_{n}^{(5)} & j_{n-1}^{(5)} & j_{n-2}^{(5)} \\ j_{n+1}^{(5)} & j_{n}^{(5)} & j_{n-1}^{(5)} & j_{n-2}^{(5)} & j_{n-3}^{(5)} \\ j_{n}^{(5)} & j_{n-1}^{(5)} & j_{n-2}^{(5)} & j_{n-3}^{(5)} & j_{n-4}^{(5)} \end{vmatrix} = 2^{n-3} \cdot 3^4 \cdot 19.$$

Summation formulas

$$\sum_{k=0}^{n} J_k^{(5)} = \begin{cases} J_{n+1}^{(5)} & \text{if } n \equiv \pm 1 \bmod 5 \\ J_{n+1}^{(5)} - 1 & \text{if } n \equiv 0 \bmod 5 \\ J_{n+1}^{(5)} + 1 & \text{if } n \equiv 2 \bmod 5 \\ J_{n+1}^{(5)} + 2 & \text{if } n \equiv 3 \bmod 5 \end{cases}, \sum_{k=0}^{n} j_k^{(5)} = \begin{cases} j_{n+1}^{(5)} - 2 & \text{if } n \not\equiv 0 \bmod 5 \\ j_{n+1}^{(5)} + 1 & \text{if } n \equiv 0 \bmod 5 \end{cases}.$$

Miscellaneous fifth order identities

$$j_n^{(5)} + 6J_n^{(5)} = \begin{cases} 2^{n+1} & \text{if } n \equiv 0 \bmod 5 \\ 2^{n+1} + 3 & \text{if } n \equiv 1 \bmod 5 \\ 2^{n+1} + 3 & \text{if } n \equiv 2 \bmod 5 \\ 2^{n+1} & \text{if } n \equiv 3 \bmod 5 \\ 2^{n+1} - 6 & \text{if } n \equiv 4 \bmod 5 \end{cases}. \qquad (5.2)$$

$$j_n^{(5)} - 6J_n^{(5)} = \begin{cases} 2^{n-1} - 3(J_{n-3}^{(5)} - 1) & \text{if } n \equiv 0 \bmod 5 \\ 2^{n-1} - 3(J_{n-3}^{(5)} + 2) & \text{if } n \equiv 1 \bmod 5 \\ 2^{n-1} - 3(J_{n-3}^{(5)} + 2) & \text{if } n \equiv 2 \bmod 5 \\ 2^{n-1} - 3J_{n-3}^{(5)} & \text{if } n \equiv 3 \bmod 5 \\ 2^{n-1} - 3(J_{n-3}^{(5)} - 3) & \text{if } n \equiv 4 \bmod 5 \end{cases}. \qquad (5.3)$$

If we let the right hand side of (5.2) be M and that of (5.3) N, then the following are noted

$$j_n^{(5)} = \frac{M+N}{2}, J_n^{(5)} = \frac{M-N}{12}.$$

$$j_n^{(5)} + J_n^{(5)} = \frac{7M+5N}{12}, j_n^{(5)} - J_n^{(5)} = \frac{5M+7N}{12}, J_n^{(5)} j_n^{(5)} = \frac{M^2-N^2}{24}.$$

and finally

$$(j_n^{(5)})^2 - 36(J_n^{(5)})^2 = MN \text{ and } (j_n^{(5)})^2 + 36(J_n^{(5)})^2 = \frac{M^2+N^2}{2}.$$

6. Concluding comments

The authors believe that most of these results are new but unfortunately, many of them do not seem to fall into a convenient pattern for generalization to an n^{th} order case. While investigating the Simson (Cassini/Catalan) identity for higher order Jacobsthal numbers a general Simson identity for an arbitrary n^{th} order recursive relation was discovered and proved. This generalized Simson identity has resulted in a short paper that will be submitted to the Fibonacci Quarterly. Certainly many more identities could be generated from those obtained here and by investigating Jacobsthal and Jacobsthal-Lucas polynomials. For example, using the methods presented in [1, 2, 6, 13, 16] a plethora of identities generated from ordinary generating functions should be possible; and similarly using [2, 5, 12, 14], identities obtained from the exponential generating functions should arise. Further investigations for these and other methods useful in discovering identities for the higher order Jacobsthal and Jacobsthal-Lucas numbers will be addressed in a future paper.

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Identities in the spirit of Ramanujan's amazing identity

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Abstract

Motivated by an amazing identity by Ramanujan in his "lost notebook", a proof of Ramanujan's identity suggested by Hirschhorn using an algebraic identity, and an algorithm by Chen to find such an algebraic identity, we will establish several identities similar to Ramanujan's amazing identity. For example, if

$$\sum_{n\geq 0} a_n x^n = \frac{9+3609x-135x^2}{1-6888x+6888x^2-x^3},$$

$$\sum_{n\geq 0} b_n x^n = \frac{10-1478x+172x^2}{1-6888x+6888x^2-x^3},$$

$$\sum_{n\geq 0} c_n x^n = \frac{12+1146x+138x^2}{1-6888x+6888x^2-x^3},$$

then

$$a_n^3+b_n^3=c_n^3+1.$$

Keywords: Ramanujan, identity

MSC: 11A55

1. Introduction

In his "lost notebook", Ramanujan [4] stated the following amazing identity. If

$$\sum_{n\geq 0} a_n x^n = \frac{1+53x+9x^2}{1-82x-82x^2+x^3},$$

$$\sum_{n\geq 0} b_n x^n = \frac{2-26x-12x^2}{1-82x-82x^2+x^3},$$
$$\sum_{n\geq 0} c_n x^n = \frac{2+8x-10x^2}{1-82x-82x^2+x^3},$$

then

$$a_n^3 + b_n^3 = c_n^3 + (-1)^n.$$

Hirschhorn [2] demonstrated that using the algebraic identity from the "lost notebook",

$$(x^2+7xy-9y^2)^3 + (2x^2-4xy+12y^2)^3 = (2x^2+10y^2)^3 + (x^2-9xy-y^2)^3, \quad (1.1)$$

Ramanujan could have proved his identity. Chen [1] gave an algorithm to produce similar algebraic identities and Ramanujan-like identities. Our goal is to use this procedure to find explicit algebraic identities and Ramanujan-like identities.

2. Third power algebraic identity to Ramanujan-like identity

The following algebraic identity was suggested by Chen [1] and the theorem and proof were suggested by Hirschhorn [2].

Theorem 2.1. *Let*

$$\begin{aligned}(r_1x^2+s_1xy+t_1y^2)^3 &+ (r_2x^2+s_2xy+t_2y^2)^3 \\ =(r_3x^2+s_3xy+t_3y^2)^3 &+ (x^2-s_4xy-t_4y^2)^3,\end{aligned} \quad (2.1)$$

be an algebraic identity in variables x and y and integer constants r_1, r_2, r_3, s_1, s_2, s_3, s_4, t_1, t_2, t_3, and t_4. Then if

$$\sum_{n\geq 0} a_n x^n = \frac{r_1+(s_1s_4+t_1-r_1t_4)x-t_1t_4x^2}{1-(s_4^2+t_4)x-(s_4^2t_4+t_4^2)x^2+t_4^3x^3},$$
$$\sum_{n\geq 0} b_n x^n = \frac{r_2+(s_2s_4+t_2-r_2t_4)x-t_2t_4x^2}{1-(s_4^2+t_4)x-(s_4^2t_4+t_4^2)x^2+t_4^3x^3},$$
$$\sum_{n\geq 0} c_n x^n = \frac{r_3+(s_3s_4+t_3-r_3t_4)x-t_3t_4x^2}{1-(s_4^2+t_4)x-(s_4^2t_4+t_4^2)x^2+t_4^3x^3}$$

then

$$a_n^3 + b_n^3 = c_n^3 + (-t_4)^{3n}.$$

Proof. Let $w_0 = 0$, $w_1 = 1$, and

$$w_{n+2} = s_4 w_{n+1} + t_4 w_n.$$

The generating function for the sequence $\{w_n\}$ is given by

$$w(x) = \sum_{n\geq 0} w_n x^n = \frac{x}{1 - s_4 x - t_4 x^2}.$$

Now, if $x = w_{n+1}$ and $y = w_n$, then

$$\begin{aligned} x^2 - s_4 xy - t_4 y^2 &= w_{n+1}^2 - s_4 w_{n+1} w_n - t_4 w_n^2 \\ &= w_{n+1}^2 - w_n(s_4 w_{n+1} + t_4 w_n) \\ &= w_{n+1}^2 - w_n w_{n+2} = (-t_4)^n. \end{aligned}$$

The last equality can be proved by induction on n.

Now, let

$$\begin{aligned} a_n &= r_1 x^2 + s_1 xy + t_1 y^2 = r_1 w_{n+1}^2 + s_1 w_{n+1} w_n + t_1 w_n^2, \\ b_n &= r_2 x^2 + s_2 xy + t_2 y^2 = r_2 w_{n+1}^2 + s_2 w_{n+1} w_n + t_2 w_n^2, \\ c_n &= r_3 x^2 + s_3 xy + t_3 y^2 = r_3 w_{n+1}^2 + s_3 w_{n+1} w_n + t_3 w_n^2. \end{aligned}$$

We can show that

$$a_n^3 + b_n^3 = c_n^3 + (-t_4)^{3n}.$$

But, using generating function techniques, we can show that

$$\begin{aligned} \sum_{n\geq 0} w_n^2 x^n &= \frac{x - t_4 x^2}{1 - (s_4^2 + t_4)x - (s_4^2 t_4 + t_4^2)x^2 + t_4^3 x^3}, \\ \sum_{n\geq 0} w_{n+1}^2 x^n &= \frac{1 - t_4 x}{1 - (s_4^2 + t_4)x - (s_4^2 t_4 + t_4^2)x^2 + t_4^3 x^3}, \\ \sum_{n\geq 0} w_n w_{n+1} x^n &= \frac{s_4 x}{1 - (s_4^2 + t_4)x - (s_4^2 t_4 + t_4^2)x^2 + t_4^3 x^3}. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{n\geq 0} a_n x^n &= \frac{r_1 + (s_1 s_4 + t_1 - r_1 t_4)x - t_1 t_4 x^2}{1 - (s_4^2 + t_4)x - (s_4^2 t_4 + t_4^2)x^2 + t_4 x^3}, \\ \sum_{n\geq 0} b_n x^n &= \frac{r_2 + (s_2 s_4 + t_2 - r_2 t_4)x - t_2 t_4 x^2}{1 - (s_4^2 + t_4)x - (s_4^2 t_4 + t_4^2)x^2 + t_4^3 x^3}, \\ \sum_{n\geq 0} c_n x^n &= \frac{r_3 + (s_3 s_4 + t_3 - r_3 t_4)x - t_3 t_4 x^2}{1 - (s_4^2 + t_4)x - (s_4^2 t_4 + t_4^2)x^2 + t_4^3 x^3}, \end{aligned}$$

and the proof is complete. □

3. Search for third power algebraic identities

We will attempt to find particular integer constants involving all the r's, s's, and t's which satisfy equation (2.1) with the following procedure.

Procedure to search for third power algebraic identities

1. Pick one particular set of integers r_1, r_2, and r_3 such that

$$r_1^3 + r_2^3 = r_3^3 + 1. \tag{3.1}$$

2. Select a collection of sets of integers t_1, t_2, t_3, and t_4 such that

$$t_1^3 + t_2^3 = t_3^3 - t_4^3. \tag{3.2}$$

Also, select a range of integer values for s_1 and s_2 to search.

a. For each t_1, t_2, t_3, t_4, s_1, and s_2, compute s_3 and s_4 using the equations

$$\begin{aligned} s_3 &= \frac{s_1 t_1^2 + s_2 t_2^2 + r_1^2 s_1 t_4^2 + r_2^2 s_2 t_4^2}{r_3^2 t_4^2 + t_3^2}, \\ s_4 &= r_3^2 s_3 - r_1^2 s_1 - r_2^2 s_2. \end{aligned}$$

Make sure these constants can be computed and that they are integers.

b. Check the following conditions.

$$\begin{aligned} &3r_1 t_1^2 + 3s_1^2 t_1 + 3r_2 t_2^2 + 3s_2^2 t_2 = 3r_3 t_3^2 + 3s_3^2 t_3 + 3t_4^2 - 3s_4^2 t_4, \\ &6r_1 s_1 t_1 + s_1^3 + 6r_2 s_2 t_2 + s_2^3 = 6r_3 s_3 t_3 + s_3^3 + 6s_4 t_4 - s_4^3, \\ &3r_1^2 t_1 + 3r_1 s_1^2 + 3r_2^2 t_2 + 3r_2 s_2^2 = 3r_3^2 t_3 + 3r_3 s_3^2 - 3t_4 + 3s_4^2. \end{aligned}$$

c. If all the above conditions are satisfied (every equation is true), the resulting collection of r's, s's, and t's form an algebraic identity satisfying equation (2.1).

To prove that the procedure above will produce an algebraic identity, cube the trinomials in (2.1) to obtain

$$\begin{aligned} &t_1^3 y^6 + 3s_1 t_1^2 x y^5 + (3r_1 t_1^2 + 3s_1^2 t_1) x^2 y^4 + (6r_1 s_1 t_1 + s_1^3) x^3 y^3 \\ &\qquad + (3r_1^2 t_1 + 3r_1 s_1^2) x^4 y^2 + 3r_1^2 s_1 x^5 y + r_1^3 x^6 \\ &+ t_2^3 y^6 + 3s_2 t_2^2 x y^5 + (3r_2 t_2^2 + 3s_2^2 t_2) x^2 y^4 + (6r_2 s_2 t_2 + s_2^3) x^3 y^3 \\ &\qquad + (3r_2^2 t_2 + 3r_2 s_2^2) x^4 y^2 + 3r_2^2 s_2 x^5 y + r_2^3 x^6 \\ &= t_3^3 y^6 + 3s_3 t_3^2 x y^5 + (3r_3 t_3^2 + 3s_3^2 t_3) x^2 y^4 + (6r_3 s_3 t_3 + s_3^3) x^3 y^3 \\ &\qquad + (3r_3^2 t_3 + 3r_3 s_3^2) x^4 y^2 + 3r_3^2 s_3 x^5 y + r_3^3 x^6 \\ &- t_4^3 y^6 - 3s_4 t_4^2 x y^5 + (3t_4^2 - 3s_4^2 t_4) x^2 y^4 + (6s_4 t_4 - s_4^3) x^3 y^3 \end{aligned} \tag{3.3}$$

$$+ (-3t_4 + 3s_4^2)x^4y^2 - 3s_4x^5y + x^6.$$

Collecting like terms in (3.3), we obtain the following equation.

$$\begin{aligned}
&(t_1^3 + t_2^3)y^6 + (3s_1t_1^2 + 3s_2t_2^2)xy^5 + (3r_1t_1^2 + 3s_1^2t_1 + 3r_2t_2^2 + 3s_2^2t_2)x^2y^4 \\
&\quad + (6r_1s_1t_1 + s_1^3 + 6r_2s_2t_2 + s_2^3)x^3y^3 + (3r_1^2t_1 + 3r_1s_1^2 + 3r_2^2t_2 + 3r_2s_2^2)x^4y^2 \\
&\quad + (3r_1^2s_1 + 3r_2^2s_2)x^5y + (r_1^3 + r_2^3)x^6 \\
&= (t_3^3 - t_4^3)y^6 + (3s_3t_3^2 - 3s_4t_4^2)xy^5 + (3r_3t_3^2 + 3s_3^2t_3 + 3t_4^2 - 3s_4^2t_4)x^2y^4 \\
&\quad + (6r_3s_3t_3 + s_3^3 + 6s_4t_4 - s_4^3)x^3y^3 + (3r_3^2t_3 + 3r_3s_3^2 - 3t_4 + 3s_4^2)x^4y^2 \\
&\quad + (3r_3^2s_3 - 3s_4)x^5y + (r_3^3 + 1)x^6.
\end{aligned} \tag{3.4}$$

Step 1 in the procedure insures that the coefficients of x^6 in the algebraic identity are equal. In addition, we would like r_1, r_2, and r_3 to be positive integers. For Ramanujan's algebraic identity this condition is trivially true since

$$1^3 + 2^3 = 2^3 + 1.$$

Other trivial values of r_1, r_2, and r_3 which satisfy (3.1) are $r_1 = 1$ and $r_2 = r_3 = r$, where r is a positive integer.

Appendix I gives positive integer values of r_1, r_2, and r_3 ($r_1 < r_2$ and $r_2 \neq r_3$) which satisfy (3.1). These values were determined by a C++ program.

In step 2, we select a collection of t's satisfying (3.4) to try. This guarantees that the coefficients of y^6 in the algebraic identity are equal. In the spirit of Ramanujan, we assume $t_4 = \pm 1$. To obtain nontrivial results, we also require that $t_1 \neq t_2$, $t_1 \neq -t_2$, $t_1 \neq -1$, and $t_2 \neq -1$. Otherwise, some of the t's could be positive or negative integers and cancel each other. Appendix II contains some of the t's which satisfy (3.2). Again, this appendix was constructed with the help of a C++ program.

Also, in step 2 we search a range of integers s_1 and s_2 (via a C++ program). Some typical ranges for s_1 and s_2 were from -1500 to 1500. With the r's, t's, s_1, and s_2 fixed, the constants left are s_3 and s_4. For step 2a, we compute integers s_3 and s_4. The formulas in step 2a are equivalent to the equations equating the coefficients in the xy^5 and x^5y terms in (3.4). These equations are

$$\begin{aligned}
3s_1t_1^2 + 3s_2t_2^2 &= 3s_3t_3^2 - 3s_4t_4^2, \\
3r_1^2s_1 + 3r_2^2s_2 &= 3r_3^2s_3 - 3s_4.
\end{aligned}$$

Step 2a merely solves them for s_3 and s_4 since they are linear equations in those two variables. We also require that $s_4 > 0$.

For step 2b, the conditions we check are the equations resulting from equating the coefficients of the terms x^2y^4, x^3y^3 and x^4y^2 on each side of equation (3.4). In step 2c, if all of these conditions are satisfied, the constants determine an algebraic identity.

4. Third power results

We found the following results. The constants in each row of the following table satisfy (2.1). We include the leading coefficient of 1 in the last trinomial. Recall that the form of the last trinomial is $x^2 - s_4xy - t_4y^2$.

r_1, s_1, t_1	r_2, s_2, t_2	r_3, s_3, t_3	1, s_4, t_4
1,556,-65601	2,-364,83802	2,-36,67402	1,756,1
1,61,-791	2,-40,1010	2,-4,812	1,83,-1
1,7,-9	2,-4,12	2,0,10	1,9,1
1,-25,135	2,-32,138	2,-36,172	1,9,1
1,-227,11161	2,-292,11468	2,-328,14258	1,83,-1
9,412,-11161	10,-180,14258	12,112,11468	1,756,1
9,-126,3753	10,236,-3230	12,96,2676	1,430,-1
9,45,-135	10,-20,172	12,12,138	1,83,-1
9,-169,791	10,-180,812	12,-220,1010	1,9,1
9,-1539,65601	10,-1640,67402	12,-2004,83802	1,83,-1
3753,-126,9	4528,200,-8	5262,84,6	1,430,-1
11161,3481,-791	11468,-1300,1010	14258,1292,812	1,6887,-1
11161,412,-9	11468,-112,12	14258,180,10	1,756,1

The bounds on s_1 and s_2 varied depending on the speed of the search. Note that the third row is the algebraic identity discovered by Ramanujan. This gives Ramanujan's amazing identity. The eighth row gives the algebraic identity

$$\begin{aligned}&(9x^2+45xy-135y^2)^3+(10x^2-20xy+172y^2)^3\\ =&(12x^2+12xy+138y^2)^3+(x^2-83xy+y^2)^3.\end{aligned}$$

This produces the Ramanujan-like identity result found in the abstract. The seventh row gives the algebraic identity

$$\begin{aligned}&(9x^2-126xy+3753y^2)^3+(10x^2+236xy-3230y^2)^3\\ =&(12x^2+96xy+2676y^2)^3+(x^2-430y+y^2)^3.\end{aligned}$$

This produces the following Ramanujan-like identity. If

$$\sum_{n\geq 0} a_nx^n = \frac{9-54172x+3753x^2}{1-184899x+184899x^2-x^3},$$

$$\sum_{n\geq 0} b_nx^n = \frac{10+98260x-3230x^2}{1-184899x+184899x^2-x^3},$$

$$\sum_{n\geq 0} c_nx^n = \frac{12+43968x+2676x^2}{1-184899x+184899x^2-x^3},$$

then

$$a_n^3+b_n^3=c_n^3+1.$$

5. Fourth power algebraic identities to Ramanujan-like identities

McLaughlin [3] found ten sequences whose sums of their first through fifth powers are equal. We will not be so ambitious. The following identity was suggested by Chen [1] and the theorem and proof were suggested by Hirschhorn [2].

Theorem 5.1. *Let*

$$\begin{aligned}&(x^2+s_1xy+t_1y^2)^4+(mx^2+s_2xy+t_2y^2)^4+(nx^2+s_3xy+t_3y^2)^4 \\ =&(mx^2+s_4xy+t_4y^2)^4+(nx^2+s_5xy+t_5y^2)^4+(x^2-s_6xy-t_6y^2)^4,\end{aligned} \tag{5.1}$$

be an algebraic identity in variables x and y and integer constants m, n, s_1, s_2, s_3, s_4, s_5, s_6, t_1, t_2, t_3, t_4, t_5, and t_6. Then if

$$\sum_{n\geq 0} a_nx^n = \frac{1+(s_1s_6+t_1-t_6)x-t_1t_6x^2}{1-(s_6^2+t_6)x-(s_6^2t_6+t_6^2)x^2+t_6^3x^3},$$

$$\sum_{n\geq 0} b_nx^n = \frac{m+(s_2s_6+t_2-mt_6)x-t_2t_6x^2}{1-(s_6^2+t_6)x-(s_6^2t_6+t_6^2)x^2+t_6^3x^3},$$

$$\sum_{n\geq 0} c_nx^n = \frac{n+(s_3s_6+t_3-nt_6)x-t_3t_6x^2}{1-(s_6^2+t_6)x-(s_6^2t_6+t_6^2)x^2+t_6^3x^3},$$

$$\sum_{n\geq 0} d_nx^n = \frac{m+(s_4s_6+t_4-mt_6)x-t_4t_6x^2}{1-(s_6^2+t_6)x-(s_6^2t_6+t_6^2)x^2+t_6^3x^3},$$

$$\sum_{n\geq 0} e_nx^n = \frac{n+(s_5s_6+t_5-nt_6)x-t_5t_6x^2}{1-(s_6^2+t_6)x-(s_6^2t_6+t_6^2)x^2+t_6^3x^3}$$

then

$$a_n^4+b_n^4+c_n^4=d_n^4+e_n^4+(-t_6)^{4n}.$$

Proof. The proof of this theorem is similar to the proof of Theorem 2.1. □

6. Search for fourth power algebraic identities

We will attempt to find particular integer constants involving m, n, and all the s's and t's which satisfy equation (5.1) with the following procedure.

Procedure to search for fourth power algebraic identities

1. Pick one particular set of integers m and n.

2. Select a collection of sets of integers t_1, t_2, t_3, t_4, t_5, and $t_6=\pm 1$ such that $t_1^4+t_2^4+t_3^4=t_4^4+t_5^4+1$. Also, select a range of integer values for s_1, s_2, s_3, and s_4 to search.

a. For each t_1, t_2, t_3, t_4, t_5, t_6, s_1, s_2, s_3, and s_4, compute s_5 and s_6 using the equations

$$s_5 = \frac{s_1 t_1^3 + s_2 t_2^3 + s_3 t_3^3 - s_4 t_4^3 - s_1 t_6^3 - m^3 s_2 t_6^3 - n^3 s_3 t_6^3 + m^3 s_4 t_6^3}{n^3 t_6^3 + t_5^3},$$
$$s_6 = -s_1 - m^3 s_2 - n^3 s_3 + m^3 s_4 + n^3 s_5.$$

Make sure these constants can be computed and that they are integers.

b. Check the following conditions.

$$\begin{aligned}
&4t_1^3 + 6s_1^2 t_1^2 + 4mt_2^3 + 6s_2^2 t_2^2 + 4nt_3^3 + 6s_3^2 t_3^2 \\
&\quad = 4mt_4^3 + 6s_4^2 t_4^2 + 4nt_5^3 + 6s_5^2 t_5^2 - 4t_6^3 + 6s_6^2 t_6^2, \\
&12 s_1 t_1^2 + 4s_1^3 t_1 + 12 m s_2 t_2^2 + 4s_2^3 t_2 + 12 n s_3 t_3^2 + 4s_3^3 t_3 \\
&\quad = 12 m s_4 t_4^2 + 4s_4^3 t_4 + 12 n s_5 t_5^2 + 4s_5^3 t_5 - 12 s_6 t_6^2 + 4s_6^3 t_6, \\
&6t_1^2 + 12 s_1^2 t_1 + s_1^4 + 6m^2 t_2^2 + 12 m s_2^2 t_2 + s_2^4 + 6n^2 t_3^2 + 12 n s_3^2 t_3 + s_3^4 \\
&\quad = 6m^2 t_4^2 + 12 m s_4^2 t_4 + s_4^4 + 6n^2 t_5^2 + 12 n s_5^2 t_5 + s_5^4 + 6t_6^2 - 12 s_6^2 t_6 + s_6^4, \\
&12 s_1 t_1 + 4s_1^3 + 12 m^2 s_2 t_2 + 4ms_2^3 + 12 n^2 s_3 t_3 + 4ns_3^3 \\
&\quad = 12 m^2 s_4 t_4 + 4ms_4^3 + 12 n^2 s_5 t_5 + 4ns_5^3 + 12 s_6 t_6 - 4s_6^3, \\
&4t_1 + 6s_1^2 + 4m^3 t_2 + 6m^2 s_2^2 + 4n^3 t_3 + 6n^2 s_3^2 \\
&\quad = 4m^3 t_4 + 6m^2 s_4^2 + 4n^3 t_5 + 6n^2 s_5^2 - 4t_6 + 6s_6^2.
\end{aligned}$$

c. If all the above conditions are satisfied (every equation is true), the resulting collection of m, n, s's, and t's form an algebraic identity satisfying equation (5.1).

The proof that this procedure yields an algebraic identity is similar to the previous procedure.

We need to make a couple of remarks. First of all, we pick positive integers m and n with $m < n$. Again, in the spirit of Ramanujan, we assume $t_6 = \pm 1$. We first note that once a solution is found, we have many other similar solutions since every one of the t's could be positive or negative. We list out the nontrivial values of the t's ($1 < t_1 < t_2 < t_3$ and $t_1 \leq t_4$) in Appendix III. This appendix was constructed with the help of a C++ program. Some typical ranges for s_1, s_2, s_3, and s_4 were from -20 to 20. Finally, we require that $s_6 > 0$.

7. Fourth power results

We found the following results. The constants in each row of the following table satisfy (5.1). Again, we include the leading coefficient of 1 in the last trinomial. Recall that the form of the last trinomial is $x^2 - s_6 xy - t_6 y^2$.

$m = 1$ and $n = 2$

1, s_1, t_1	1, s_2, t_2	2, s_3, t_3	1, s_4, t_4	2, s_5, t_5	1, s_6, t_6
1,-4,4	1,-6,9	2,-10,13	1,-7,11	2,-10,12	1,3,-1
1,-3,4	1.-8,9	2,-11,13	1,-9,12	2,-11,11	1,2,1
1,-1,4	1,-2,9	2,-3,13	1,7,-12	2,-3,-11	1,10,-1
1,-4,5	1,-6,6	2,-10,11	1,-7,9	2,-10,10	1,3,-1
1,0,5	1,-2,6	2,-2,11	1,7,-9	2,-2,-10	1,9,1
1,-4,5	1,-5,6	2,-9,11	1,-7,10	2,-9,9	1,2,1
1,-5,6	1,-10,23	2,-15,29	1,-11,26	2,-15,27	1,4,-1
1,-4,6	1,-12,23	2,-16,29	1,-13,27	2,-16,26	1,3,1
1,0,6	1,-4,23	2,-4,29	1,11,-27	2,-4,-26	1,15,-1
1,-6,7	1,-7,14	2,-13,21	1,-9,18	2,-13,19	1,4,-1
1,-4,7	1,-12,14	2,-16,21	1,-13,19	2,-16,18	1,3,1
1,-4,7	1,0,14	2,-4,21	1,9,-19	2,-4,-18	1,13,-1
1,-7,8	1,-6,11	2,-13,19	1,-9,16	2,-13,17	1,4,-1
1,-5,8	1,-10,11	2,-15,19	1,-11,16	2,-15,17	1,4,-1
1,-3,8	1,0,11	2,-3,19	1,9,-16	2,-3,-17	1,12,1
1,-6,8	1,-6,11	2,-12,19	1,-9,17	2,-12,16	1,3,1

$m = 2$ and $n = 3$

1, s_1, t_1	2, s_2, t_2	3, s_3, t_3	2, s_4, t_4	3, s_5, t_5	1, s_6, t_6
1,-1,7	2,-2,14	3,-3,21	2,10,-19	3,-6,-18	1,16,-1
1,-8,8	2,-10,11	3,-18,19	2,-14,16	3,-17,17	1,3,-1
1,0,8	2,-2,11	3,-2,19	2,10,-16	3,-5,-17	1,15,1
1,-7,8	2,-9,11	3,-16,19	2,-13,17	3,-15,16	1,2,1
1,-8,10	2,-12,19	3,-20,29	2,-16,26	3,-19,25	1,3,1
1,-4,10	2,0,19	3,-4,29	2,12,-26	3,-7,-25	1,19,-1
1,-3,11	2,0,16	3,-3,27	2,12,-23	3,-6,-24	1,18,1
1,0,11	2,-4,39	3,-4,50	2,16,-46	3,-9,-45	1,25,-1
1,-8,13	2,-10,13	3,-18,26	2,-14,22	3,-17,23	1,3,-1
1,4,-13	2,0,-13	3,4,-26	2,4,-22	3,3,-23	1,1,1
1,-1,14	2,-3,41	3,-4,55	2,17,-49	3,-9,-50	1,26,1
1,-8,15	2,-12,19	3,-20,34	2,-16,30	3,-19,29	1,3,1
1,-5,16	2,-1,55	3,-6,71	2,19,-65	3,-11,-64	1,30,-1
1,-6,19	2,0,57	3,-6,76	2,20,-68	3,-11,-69	1,31,1
1,-2,21	2,-6,64	3,10,-113	2,10,-112	3,6,-69	1,22,-1
1,14,-116	2,0,-155	3,14,-271	2,12,-236	3,11,-235	1,1,-1

$m = 3$ and $n = 5$

1, s_1, t_1	3, s_2, t_2	5, s_3, t_3	3, s_4, t_4	5, s_5, t_5	1, s_6, t_6
1,-2,21	3,-4,41	5,6,-71	3,6,-69	5,4,-49	1,22,-1

The bounds on s_1, s_2, s_3, and s_4 varied depending on the speed of the search. The first row of the table for $m = 1$ and $n = 2$ gives the algebraic identity

$$(x^2 - 4xy + 4y^2)^4 + (x^2 - 6xy + 9y^2)^4 + (2x^2 - 10xy + 13y^2)^4$$
$$= (x^2 - 7xy + 11y^2)^4 + (2x^2 - 10xy + 12y^2)^4 + (x^2 - 3xy + y^2)^4.$$

This produces the following Ramanujan-like identity. If

$$\sum_{n\geq 0} a_n x^n = \frac{1 - 7x + 4x^2}{1 - 8x + 8x^2 - x^3},$$
$$\sum_{n\geq 0} b_n x^n = \frac{1 - 8x + 9x^2}{1 - 8x + 8x^2 - x^3},$$
$$\sum_{n\geq 0} c_n x^n = \frac{2 - 15x + 13x^2}{1 - 8x + 8x^2 - x^3},$$
$$\sum_{n\geq 0} d_n x^n = \frac{1 - 9x + 11x^2}{1 - 8x + 8x^2 - x^3},$$
$$\sum_{n\geq 0} e_n x^n = \frac{2 - 16x + 12x^2}{1 - 8x + 8x^2 - x^3},$$

then

$$a_n^4 + b_n^4 + c_n^4 = d_n^4 + e_n^4 + 1.$$

The row in the table for $m = 3$ and $n = 5$ gives the algebraic identity

$$(x^2 - 2xy + 21y^2)^4 + (3x^2 - 4xy + 41y^2)^4 + (5x^2 + 6xy - 71y^2)^4$$
$$= (3x^2 + 6xy - 69y^2)^4 + (5x^2 + 4xy - 49y^2)^4 + (x^2 - 22xy + y^2)^4.$$

This produces the following Ramanujan-like identity. If

$$\sum_{n\geq 0} a_n x^n = \frac{1 - 22x + 21x^2}{1 - 483x + 483x^2 - x^3},$$
$$\sum_{n\geq 0} b_n x^n = \frac{3 - 44x + 41x^2}{1 - 483x + 483x^2 - x^3},$$
$$\sum_{n\geq 0} c_n x^n = \frac{5 + 66x + 71x^2}{1 - 483x + 483x^2 - x^3},$$
$$\sum_{n\geq 0} d_n x^n = \frac{3 + 66x + 69x^2}{1 - 483x + 483x^2 - x^3},$$
$$\sum_{n\geq 0} e_n x^n = \frac{5 + 44x + 49x^2}{1 - 483x + 483x^2 - x^3},$$

then

$$a_n^4 + b_n^4 + c_n^4 = d_n^4 + e_n^4 + 1.$$

8. Questions

The previous data suggests several questions.

1. In the third power case, we were unable to find any nontrivial algebraic identities like (2.1) with $r_1 = 1$ and $r_2 = r_3 = r$ where $r \geq 3$. We would like to know if any exist and if so, what are they?

2. We were unable to find any fourth power algebraic identities of the form

$$\begin{aligned}(r_1x^2 + s_1xy + t_1y^2)^4 + (r_2x^2 + s_2xy + t_2y^2)^4 + (r_3x^2 + s_3xy + t_3y^2)^4 \\ = (r_4x^2 + s_4xy + t_4y^2)^4 + (x^2 - s_5xy - t_5y^2)^4,\end{aligned}$$

   where the r's are positive integers and the s's and t's are nontrivial. Do such identities exist?

3. In the fourth power case, we found algebraic identities for every pair we tried where m if a positive integer and $n = m+1$. Is this always true? In addition, is there any other algebraic identity where $n \neq m + 1$ other than the one we found where $m = 3$ and $n = 5$?

Appendix I: $r_1^3 + r_2^3 = r_3^3 + 1$

r_1	r_2	r_3
9	10	12
64	94	103
73	144	150
135	235	249
244	729	738
334	438	495
368	1537	1544
577	2304	2316
1010	1897	1988
1033	1738	1852
1126	5625	5640
1945	11664	11682
3088	21609	21630
3097	3518	4184
3753	4528	5262
3987	9735	9953
4083	8343	8657
4609	36864	36888
5700	38782	38823
5856	9036	9791
6562	59049	59076
7364	83692	83711
9001	90000	90030
10876	31180	31615
11161	11468	14258
11767	41167	41485
11980	131769	131802
13294	19386	21279
15553	186624	186660
16617	35442	36620

r_1	r_2	r_3
19774	257049	257088
20848	152953	153082
24697	345744	345786
26914	44521	47584
27238	33412	38599
27784	35385	40362
27835	72629	73967
30376	455625	455670
35131	76903	79273
36865	589824	589872
38305	51762	57978
39892	151118	152039
44218	751689	751740
49193	50920	63086
50313	80020	86166
59728	182458	184567
65601	67402	83802
99457	222574	229006
107258	278722	283919
135097	439312	443530
158967	312915	326033
190243	219589	259495
191709	579621	586529
198550	713337	718428
243876	547705	563370
294121	325842	391572
336820	583918	619111
372106	444297	518292
434905	780232	822898
590896	734217	844422

Appendix II: $t_1^3 + t_2^3 = t_3^3 - t_4^3$

t_1	t_2	t_3	t_4
-9	6	-8	1
6	-9	-8	1
-9	8	-6	1
8	-9	-6	1
-8	-6	-9	-1
-6	-8	-9	-1
-8	9	6	-1
9	-8	6	-1
-6	9	8	-1
9	-6	8	-1
6	8	9	1
8	6	9	1
-12	9	-10	-1
9	-12	-10	-1
-12	10	-9	-1
10	-12	-9	-1
-10	-9	-12	1
-9	-10	-12	1
-10	12	9	1
12	-10	9	1
-9	12	10	1
12	-9	10	1
9	10	12	-1
10	9	12	-1
-103	64	-94	-1
64	-103	-94	-1
-103	94	-64	-1
94	-103	-64	-1
-94	-64	-103	1
-64	-94	-103	1
-94	103	64	1
103	-94	64	1
-64	103	94	1
103	-64	94	1
64	94	103	-1
94	64	103	-1

t_1	t_2	t_3	t_4
-144	71	-138	1
71	-144	-138	1
-144	138	-71	1
138	-144	-71	1
-138	-71	-144	-1
-71	-138	-144	-1
-138	144	71	-1
144	-138	71	-1
-71	144	138	-1
144	-71	138	-1
71	138	144	1
138	71	144	1
-150	73	-144	-1
73	-150	-144	-1
-150	144	-73	-1
144	-150	-73	-1
-144	-73	-150	1
-73	-144	-150	1
-144	150	73	1
150	-144	73	1
-73	150	144	1
150	-73	144	1
73	144	150	-1
144	73	150	-1
-172	135	-138	1
135	-172	-138	1
-172	138	-135	1
138	-172	-135	1
-138	-135	-172	-1
-135	-138	-172	-1
-138	172	135	-1
172	-138	135	-1
-135	172	138	-1
172	-135	138	-1
135	138	172	1
138	135	172	1

Appendix III: $t_1^4 + t_2^4 + t_3^4 = t_4^4 + t_5^4 + 1$

t_1	t_2	t_3	t_4	t_5
2	31	47	14	49
2	31	47	49	14
2	35	47	19	50
2	35	47	50	19
2	47	173	71	172
2	47	173	172	71
2	148	191	56	206
2	148	191	206	56
3	6	21	16	19
3	6	21	19	16
3	7	8	2	9
3	7	8	9	2
3	7	44	24	43
3	7	44	43	24
3	21	36	2	37
3	21	36	37	2
3	24	111	77	104
3	24	111	104	77
4	9	13	11	12
4	9	13	12	11
4	18	19	6	22
4	18	19	22	6
4	41	103	58	101
4	41	103	101	58
4	49	75	25	78
4	49	75	78	25
4	76	105	54	110
4	76	105	110	54
4	83	100	32	110
4	83	100	110	32
5	6	11	9	10
5	6	11	10	9
6	14	37	22	36
6	14	37	36	22
6	19	31	9	32
6	19	31	32	9
6	23	29	26	27
6	23	29	27	26

t_1	t_2	t_3	t_4	t_5
6	25	29	15	32
6	25	29	32	15
6	29	47	23	48
6	29	47	48	23
6	31	41	24	43
6	31	41	43	24
6	47	71	43	72
6	47	71	72	43
6	138	165	100	178
6	138	165	178	100
7	14	21	18	19
7	14	21	19	18
7	27	157	109	147
7	27	157	147	109
7	57	73	9	79
7	57	73	79	9
7	76	107	83	104
7	76	107	104	83
7	109	148	121	142
7	109	148	142	121
8	11	19	16	17
8	11	19	17	16
8	43	51	47	48
8	43	51	48	47
8	109	132	62	144
8	109	132	144	62
9	25	34	30	31
9	25	34	31	30
9	34	193	152	171
9	34	193	171	152
9	197	200	45	236
9	197	200	236	45
10	14	103	80	92
10	14	103	92	80
10	19	29	25	26
10	19	29	26	25
10	39	41	32	45
10	39	41	45	32

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On h-perfect numbers

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Abstract

Let $\sigma(x)$ denote the sum of the divisors of x. The diophantine equation $\sigma(x)+\sigma(y)=2(x+y)$ equalizes the abundance and deficiency of x and y. For $x=n$ and $y=hn$ the solutions n are called h-perfect since the classical perfect numbers occur as solutions for $h=1$. Some results on h-perfect numbers are determined.

Keywords: perfect numbers, amicable numbers

MSC: 11A25

1. Introduction

Let $\sigma(n)$ denote the sum of the divisors of n, that is,

$$\sigma(n)=\prod_{i=1}^{r}\frac{p_i^{\alpha_i+1}-1}{p_i-1}\quad\text{for}\quad n=\prod_{i=1}^{r}p_i^{\alpha_i}.$$

Since the classical antiquity there exist two famous problems for $\sigma(n)$.

At first it is asked for perfect numbers n fulfilling

$$\sigma(n)=2n.$$

All even perfect numbers are of the form $n=(2^p-1)2^{p-1}$ where p is a prime number and where 2^p-1 is a so-called Mersenne prime number, too. Nearly 50 such prime numbers are known. The existence of odd perfect numbers is still unknown.

Secondly, it is asked for amicable number pairs x,y such that

$$\sigma(x)-x=y\quad\text{and}\quad\sigma(y)-y=x.$$

Several thousand pairs are known. It remains unknown whether there are infinitely many pairs.

Nonperfect numbers n are called abundant if $\sigma(n) > 2n$ and called deficient if $\sigma(n) < 2n$. Then it may be asked for perfect number pairs x, y fulfilling the diophantine equation

$$\sigma(x) + \sigma(y) = 2(x + y), \tag{1.1}$$

that is, x and y equalize abundance and deficiency.

There exist many solutions x, y of (1.1). For fixed d let X and Y be the sets of solutions x and y of $\sigma(x) = 2x + d$ and $\sigma(y) = 2y - d$, respectively. The sets X and Y are finite (see [1], p. 169). Then all pairs x, y with $x \in X$ and $y \in Y$ are solutions of (1.1).

It may be remarked that perfect and amicable numbers are special cases of (1.1): Perfect numbers for $x = y$ and amicable numbers for $\sigma(x) = \sigma(y)$.

Here it is proposed to consider the special class of solutions of (1.1) when y is a multiple of x, that is,

$$\sigma(n) + \sigma(hn) = 2(n + hn) = 2n(h + 1). \tag{1.2}$$

If $h = 1$ then n is a perfect number. Therefore solutions n of (1.2) may be called h-perfect numbers. Some results on h-perfect numbers are determined in the following.

2. Powers of two

For $h = 2^t$ all h-perfect numbers are dependent on a sequence of certain prime numbers being similar to Mersenne prime numbers.

Theorem 2.1. *A number n is 2^t-perfect, $t \geq 1$, if and only if it holds $n = 2^\alpha((2^t + 1)2^\alpha - 1)$ where $(2^t + 1)2^\alpha - 1$ is a prime number.*

Proof. Suppose that n is 2^t-perfect, $t \geq 1$.

If $(n, 2) = 1$ then equation (1.2) implies

$$\sigma(n) + \sigma(n2^t) = \sigma(n)(1 + 2^{t+1} - 1) = \sigma(n)2^{t+1} = 2n(1 + 2^t).$$

Since the left term of (1.2) is divisible by 2^{t+1} whereas the right term of (1.2) is divisible by 2 only, odd 2^t-perfect numbers do not exist.

If $n = s2^\alpha$, $\alpha \geq 1$, $(s, 2) = 1$ then equation (1.2) yields

$$\sigma(s2^\alpha) + \sigma(s2^{t+\alpha}) = 2(s2^\alpha + s2^{t+\alpha}).$$

This is equivalent to

$$\sigma(s)((2^t + 1)2^\alpha - 1) = (2^t + 1)2^\alpha s \quad \text{with} \quad s = v((2^t + 1)2^\alpha - 1), \ v \geq 1, \tag{2.1}$$

since $((2^t + 1)2^\alpha - 1, (2^t + 1)2^\alpha) = 1$.

If $v > 1$ then equation (2.1) determines

$$v((2^t + 1)2^\alpha - 1) + v + 1 \leq \sigma(v((2^t + 1)2^\alpha - 1)) = v(2^t + 1)2^\alpha,$$

a contradiction.

If $v = 1$ and if $s = (2^t + 1)2^\alpha - 1$ is a composite number then equation (2.1) yields

$$(2^t + 1)2^\alpha < \sigma((2^t + 1)2^\alpha - 1) = (2^t + 1)2^\alpha,$$

again a contradiction.

If $v = 1$ and if $s = (2^t + 1)2^\alpha - 1$ is a prime number then equations (2.1) and (1.2) are fulfilled and $n = s2^\alpha$ is 2^t-perfect. □

In [2] the first 16 and 12 prime numbers $p = (2^t+1)2^\alpha-1$ are listed for $t = 1$ and $t = 2$, respectively. Thus 10, 44, 184, 752, 12224, 49024,... are the first 2-perfect numbers. The question for odd 2^t-perfect numbers, $t \geq 1$, is completely answered by nonexistence whereas it is still open in the classical case of perfect numbers.

3. Nonexistence

For some classes of values of h it can be proved that h-perfect numbers do not exist.

Theorem 3.1. *For $h = c2^t$, $(c, 2) = 1$, $c \geq 3$, there are no even h-perfect numbers if $c + 2 < 2^{t+2}$ and there are no h-perfect numbers if $c + 2 < 2^{t+1}$.*

Proof. For even n let $n = r2^\alpha$, $\alpha \geq 1$, $(r, 2) = 1$. Now suppose that n is $c2^t$-perfect for $c + 2 < 2^{t+2}$. Equation (1.2) implies

$$(2^{\alpha+1} - 1)\sigma(r) + (2^{\alpha+t+1} - 1)\sigma(cr) = r2^{\alpha+1}(c2^t + 1).$$

Using $\sigma(cr) \geq cr + \sigma(r)$ it follows

$$\sigma(r)(2^{\alpha+1} - 1 + 2^{\alpha+t+1} - 1) \leq (2^{\alpha+1} + c)r.$$

Then $\sigma(r) \geq r$ together with $\alpha \geq 1$ determines

$$2^{t+1} \leq 2^{\alpha+t+1} \leq c + 2,$$

a contradiction.

For odd n suppose that n is $c2^t$-perfect for $c+2 < 2^{t+1}$. Equation (1.2) implies

$$\sigma(n) + (2^{t+1} - 1)\sigma(cn) = 2n(1 + c2^t).$$

With $\sigma(cn) \geq cn + \sigma(n)$ it follows

$$2^{t+1}\sigma(n) \leq (c + 2)n$$

and with $\sigma(n) \geq n$ the contradiction

$$2^{t+1} \leq c + 2$$

is obtained. □

For $h < 100$ by Theorem 3.1 no h-perfect numbers occur if $h = 12$, 20, 24, 40, 48, 56, 72, 80, 88, or 92.

The following theorem presents another example of partial nonexistence.

Theorem 3.2. *There is no even 3^t-perfect number, $t \geq 1$.*

Proof. Suppose that $n = r2^\alpha$ is an h-perfect number for $h = 3^t$, $t \geq 1$, $\alpha \geq 1$, $(r, 2) = 1$. Equation (1.2) yields

$$\sigma(r)(2^{\alpha+1} - 1) + \sigma(r3^t)(2^{\alpha+1} - 1) = r2^{\alpha+1}(1 + 3^t). \tag{3.1}$$

Case I: $(r, 3) = 1$. It follows

$$\sigma(r)(2^{\alpha+1} - 1)(1 + (3^{t+1} - 1)/2) = r2^{\alpha+1}(1 + 3^t)$$

and equivalently

$$\sigma(r)(2^{\alpha+1} - 1)(1 + 3^{t+1}) = r2^{\alpha+2}(1 + 3^t).$$

With $\sigma(r) \geq r$ the inequality

$$(2^{\alpha+1} - 1)(1 + 3^{t+1}) \leq 2^{\alpha+2}(1 + 3^t)$$

is obtained being equivalent to

$$(3^t - 1)2^{\alpha+1} \leq 1 + 3^{t+1}.$$

This is a contradiction for $\alpha, t \geq 1$ excluded $\alpha = t = 1$. Then, however, the left term of (3.1) is divisible by 3 and, in the contrary, 3 does not divide the right term of (3.1) due to $(r, 3) = 1$.

Case II: $r = s3^\beta$, $\beta \geq 1$, $(s, 3) = 1$, and $(s, 2) = 1$ since $(r, 2) = 1$. By equation (3.1) it follows

$$\sigma(s)(2^{\alpha+1} - 1)(3^{\beta+1} + 3^{\beta+t+1} - 2) = s2^{\alpha+2}3^\beta(1 + 3^t)$$

and with $\sigma(s) \geq s$

$$2^{\alpha+1}3^{\beta+1} + 2^{\alpha+1}3^{t+\beta+1} - 2^{\alpha+2} - 3^{\beta+1} - 3^{t+\beta+1} + 2 \leq 2^{\alpha+2}3^{t+\beta} + 2^{\alpha+2}3^\beta.$$

This inequality is equivalent to

$$(3^\beta(1 + 3^t) - 2)(2^{\alpha+1} - 3) \leq 4$$

yielding a contradiction for $\alpha, \beta, t \geq 1$. $\square$

4. Even perfect-perfect numbers

For some values of h there exist only a small number of h-perfect numbers.

Theorem 4.1. *For $h = 6$ only 13 is h-perfect and for any other even perfect number h there are no h-perfect numbers.*

Proof. Let $h = (2^p - 1)2^{p-1}$ be an even perfect number, that is, p and $2^p - 1$ both are prime numbers. Suppose that n is an h-perfect number.

For even n, that is, $n = r2^\alpha$, $\alpha \geq 1$, $(r, 2) = 1$, Theorem 3.1 implies the condition $2^p + 1 \geq 2^{p+1}$ being impossible.

For odd n two cases are distinguished.

Case I: $n = r(2^p - 1)^\alpha = rq^\alpha$, $\alpha \geq 1$, $(r, 2^p - 1) = (r, q) = 1$. By equation (1.2),

$$\sigma(rq^\alpha) + \sigma(r2^{p-1}q^{\alpha+1}) = 2rq^\alpha(1 + q2^{p-1})$$

and hence

$$\sigma(r)(q^{\alpha+1} - 1 + (2^p - 1)(q^{\alpha+2} - 1)) = r(q - 1)(2q^\alpha + 2^p q^{\alpha+1}).$$

With $\sigma(r) \geq r$ and $2^p - 1 = q$ this yields

$$q^{\alpha+1} - 1 + q^{\alpha+3} - q \leq 2q^{\alpha+1} + q^{\alpha+3} + q^{\alpha+2} - 2q^\alpha - q^{\alpha+2} - q^{\alpha+1}$$

and thus the contradiction

$$2q^\alpha \leq q + 1.$$

Case II: $(n, 2^p - 1) = (n, q) = 1$. Equation (1.2) yields

$$\sigma(n) + \sigma(nq2^{p-1}) = 2n(1 + q2^{p-1}),$$

$$\sigma(n) + \sigma(n)(2^p - 1)(q + 1) = n(2 + q2^p),$$

and thus

$$\sigma(n)(1 + q(q + 1)) = n(2 + q(q + 1)).$$

Since $(1 + q(q + 1), 2 + q(q + 1)) = 1$ it is necessary that

$$\sigma(n) = v(2 + q(q + 1)) \quad \text{with} \quad n = v(1 + q(q + 1)),\ v \geq 1. \tag{4.1}$$

If $v > 1$ in equation (4.1) then

$$v(1 + q(q + 1)) + v + 1 \leq \sigma(n) = v(2 + q(q + 1))$$

is a contradiction.

If $v = 1$ in equation (4.1) and if $1 + q(q + 1)$ is a composite number then

$$2 + q(q + 1) < \sigma(n) = 2 + q(q + 1)$$

is a contradiction.

It remains that $v = 1$ in equation (4.1) and $1 + q(q + 1)$ is a prime number. This, however, is impossible for odd prime numbers p since 3 divides $1 + q(q + 1) = 1 + (2^p - 1)2^p$ due to $2^p \equiv -1 \pmod 3$. Thus $p = 2$ determines $1 + q(q + 1) = 13$ as the unique solution of equations (4.1) and (1.2) for $h = (2^2 - 1)2^{2-1} = 6$. □

5. Small values of h

For $h \leq 16$ the discussion is completed for $h = 2$, 4, 6, 8, 12, and 16. For $h = 3$, 9, and 10 even h-perfect numbers do not exist. So far no h-perfect numbers are known for $h = 3$, 9, 10, and 13. The numbers $n = 14$ and $n = 7030$ are 5-perfect, $n = 135$ and $n = 1365$ are 7-perfect, $n = 182$ is 11-perfect, $n = 5$ and $n = 118$ are 14-perfect, and $n = 455$ is 15-perfect.

Finally, there are two corollaries for the Fibonacci number $F_7 = 13$ as consequences of Theorems 3.1 and 4.1.

Corollary 5.1. *Only 13 is an h-perfect number for any even perfect number h.*

Corollary 5.2. *Only 13 is a* $3 \cdot 2^t$*-perfect number for any* $t \geq 1$.

References

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Proof of the Tojaaldi sequence conjectures

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Abstract

Heuristically, the base b, size a Tojaaldi sequence of size k, $\mathcal{T}_k^{(a,b)}$, is the sequence of initial digits of the $(k+1)-$digit Generalized Fibonaaci numbers, defined by $F_0^{(a)} = 0, F_1^{(a)} = 1, F_n^{(a)} = aF_{n-1}^{(a)} + F_{n-2}^{(a)}, n \geq 2$. For example, $\mathcal{T}_2^{(1,10)} = \langle 1, 2, 3, 6, 9 \rangle$ corresponding to the initial digits of the three-digit Fibonacci numbers, 144, 233, 377, 610, 987. In [1] we showed that (eventually) there are at most b Tojaaldi sequences and conjectured that there are exactly b Tojaaldi sequences. Based on computer studies we also conjectued that the Tojaaldi sequences are Benford distributed. We prove these two conjectures

Keywords: Tojaaldi, Fibonacci, initial digits, Benford

MSC: 11B37 11B39

1. Introduction and goals

The goal of this paper is to prove the two conjectures presented in [1]. For purposes of completeness we will repeat the necessary definitions, conventions and theorems from [1]. For pedagogic purposes we will also repeat key illustrative examples. However, the reader should consult [1] for details on proofs and the well-definedness of definitions.

An outline of this paper is as follows: In this section we present all necessary definitions and propositions. In the next section we state the main Theorems of [1] as well as the two conjectures. In the final section we prove the conjectures.

Notational Conventions. Throughout this paper if $\{n \in \mathbb{N} : P(n)\}$ is the set of integers with property P then we notationally indicate the sequence of such integers (with the natural order inherited from the integers) by $\langle n \in \mathbb{N} : P(n)\rangle$. Throughout this paper discrete sequences and sets will be notationally indicated with angle brackets and braces respectively.

Definition 1.1. For integers $a \geq 1, n \geq 0$, the *generalized Fibonacci numbers* are defined by

$$F_0^{(a)} = 0, F_1^{(a)} = 1, F_n^{(a)} = aF_{n-1}^{(a)} + F_{n-2}^{(a)}, \qquad n \geq 2.$$

The generalized Fibonacci numbers can equivalently be defined by their Binet form

$$F_n^{(a)} = \frac{\alpha_a^n - \beta_a^n}{D}, D = \alpha_a - \beta_a = \sqrt{a^2+4}, \alpha_a = \frac{a+D}{2}, \beta_a = \frac{a-D}{2}. \qquad (1.1)$$

When speaking about the generalized Fibonacci numbers, if we wish to explicitly note the dependence on a, we will use the phrase *the a-Fibonacci numbers.*

The following identity is useful when making estimates.

Lemma 1.2. *For integers $k \geq 1, m \geq 1$,*

$$F_{m+k}^{(a)} = \alpha_a^k F_m^{(a)} + F_k^{(a)} \beta_a^m. \qquad (1.2)$$

Definition 1.3. The base b, *a-Tojaaldi sequence* of size k is defined and notationally indicated by

$$\mathcal{T}_k^{(a,b)} = \left\langle \left\lfloor \frac{F_n^{(a)}}{b^k} \right\rfloor : n \geq 1, b^k \leq F_n^{(a)} < b^{k+1} \right\rangle, \qquad k \geq 0. \qquad (1.3)$$

The base b, *a-Tojaaldi set* (of the a-Fibonacci numbers) is defined and notationally indicated by

$$\mathcal{T}^{(a,b)} = \{\mathcal{T}_k^{(a,b)} : 0 \leq k < \infty\}.$$

Example 1.4. Heuristically, a Tojaaldi sequence is the sequence of initial digits of all base b size a Fibonacci numbers, with a fixed number of digits. So, for example, $\mathcal{T}_2^{(1,10)} = \langle 1, 2, 3, 6, 9\rangle$, corresponding to the initial digits of the 3-digit Fibonacci numbers: 144,233,377,610,987.

Remark 1.5. The theorems of this paper carry over to the generalized Lucas numbers with extremely minor modifications.

The Tojaaldi sequences were initially studied by Tom Barrale who manually compiled tables of them from 1997-2007. Michael Sluys then contributed computing resources enabling computation of Tojaaldi sequences for the first (approximately) half million Fibonacci numbers. This computer study was replicated by Hendel using alternate algorithms. This computer study contains important information about the distribution of Tojaaldi sequences which is the basis of the conjecture that the Tojaaldi sequences are Benford distributed.

The name Tojaaldi is an acronym formed from the initial two letters of Barrale's family: Thomas, Jared, Allison, and Dianne, his eldest, second eldest son, daughter and wife respectively. (The third letter of "Thomas" was used rather than the second because it is a vowel.)

Definition 1.6. For integers $b \geq 2, a \geq 1$, $n_0(a,b)$ is the smallest positive integer such that

$$F_n^{(a)} = i \cdot b^j, \text{ is not solvable for integers } 1 \leq i \leq b-1, n \geq n_0(a,b). \tag{1.4}$$

Example 1.7. Clearly, $n_0(1,10) = 1$, $n_0(2,10) = 1$ and $n_0(1,12) = 13$.

Definition 1.8. For integer k, *n(k) = n(k,a,b)* is the unique integer defined by the equation

$$F_{n(k)}^{(a)} < b^k \leq F_{n(k)+1}^{(a)}, \qquad k \geq 1. \tag{1.5}$$

Definition 1.9. For fixed integers $a \geq 1$ and $b \geq 2$, *j(a,b)* is the unique non-negative integer satisfying the inequality,

$$\alpha_a^{j(a,b)} < b < \alpha_a^{j(a,b)+1}. \tag{1.6}$$

Definition 1.10. Let $k_1(a,b)$ be the smallest positive integer such that for all $k \geq k_1(a,b)$, (i) $n(k) \geq n_0(a,b)$, and (ii) $n(k) \geq j(a,b)$. An integer $k \geq k_1(a,b)$ will be called *non-trivial* while other positive integers will be called *trivial.* Similarly, a Tojaaldi sequence $\mathcal{T}_k^{(a,b)}$ will be called *non-trivial* if k is non-trivial. We notationally indicate the set of all non-trivial, base b, a-Tojaaldi sequences, by $\overline{\mathcal{T}}^{(a,b)}$.

Lemma 1.11. *For non-trivial k,*

$$\#\mathcal{T}_k^{(a,b)} \in \{j(a,b), j(a,b)+1\}. \tag{1.7}$$

Proof. [1, Proposition 2.5]. □

Example 1.12. $j(1,10) = 4, n_0(1,10) = 1,$ and $n(1,1,10) = 6$. Hence, by (1.7), $\mathcal{T}_0^{(1,10)}$ is the only base 10, 1-Tojaaldi sequence with 6 elements.

Lemma 1.13. *If k is non-trivial then (i) $F_{n(k)}^{(a)} \leq i \cdot b^k, 1 \leq i \leq b-1 \Rightarrow F_{n(k)}^{(a)} < i \cdot b^k$ (ii) $\#\mathcal{T}_k^{(a,b)} \in \{j(a,b), j(a,b)+1\}$, (iii) $F_{n(k)+p}^{(a)} > b^k \Leftrightarrow \alpha_a^p F_{n(k)}^{(a)} > b^k, 1 \leq p \leq j(a,b)+1$.*

Proof. [1, Proposition 2.8]. □

Remark 1.14. Non-triviality was introduced to avoid only a few aberrent Tojaaldi sequences such as $\mathcal{T}_0^{(1,10)}$. In general, restricting ourselves to non-trivial sequences is not that restrictive. For example, $k_1(1,10) = 1$ and $k_1(1,12) = 3$.

Definition 1.15. For fixed a, b, and $x \in [\alpha_a^{-1}, 1)$, the *base b, real, a-Tojaaldi sequence of x* is defined by

$$T_x^{(a,b)} = \langle \lfloor \alpha_a^k x \rfloor : 1 \leq k \leq m, \text{ with } m \text{ defined by } \alpha_a^m x < b \leq \alpha_a^{m+1} x \rangle.$$

Remark 1.16. $\mathcal{T}_z^{(a)}$ has different definitions depending on whether z is an integer or non-integer. This should cause no confusion in the sequel since the meaning will always be clear from the context.

Definition 1.17. For integer k, $a \geq 1$, and $b \geq 2$,

$$x = x(k) = x(k, a, b) = \frac{F_{n(k)}^{(a)}}{b^k}, \qquad k \geq 1. \tag{1.8}$$

Lemma 1.18. *For integer k, $a \geq 1$, and $b \geq 2$,*

$$\mathcal{T}_{x(k)}^{(a,b)} = \mathcal{T}_k^{(a,b)}, \tag{1.9}$$

and

$$x(k) \in (\alpha_a^{-1}, 1). \tag{1.10}$$

Proof. [1, Proposition 2.14] □

Definition 1.19. For each integer, $1 \leq i \leq b$, *e(i) = e(i,a)* is the unique integer satisfying. $\alpha_a^{e(i)-1} \leq i < \alpha_a^{e(i)}$.

Definition 1.20. The *(a, b)-partition* refers to

$$\langle B_i : 1 \leq i \leq b+1 \rangle = \langle 1, \frac{i}{\alpha_a^{e(i)}} : 1 \leq i \leq b \rangle. \tag{1.11}$$

Remark 1.21. By our notational convention on the use of angle brackets, the B_i simply sequentially order the $\{\frac{j}{\alpha_a^{e(j)}}\}_{1 \leq j \leq b}$. Consequently, the $B_i, 1 \leq i \leq b+1$, partition the interval $[\frac{1}{\alpha_a}, 1)$, into b semi-open intervals with $B_1 = \alpha_a^{-1}$ and $B_{b+1} = 1$.

Example 1.22. Table 1 presents the (1,10)-parition and other useful information.

Lemma 1.23. *For a fixed $a \geq 1, b \geq 2, (a, b) - partition, \langle B_i : 1 \leq i \leq b+1 \rangle$, and a real $y \in [B_m, B_{m+1}), 1 \leq m \leq b$,*

$$\mathcal{T}_{B_m}^{(a,b)} = \mathcal{T}_y^{(a,b)}. \tag{1.12}$$

Proof. [1, Proposition 2.15]. □

Example 1.24. $x = x(1, 1, 10) = 0.8$. Inspecting Table 1,

$$x \in [B_6, B_7) = [0.76, 0.81).$$

It is then straightforward to verify, as shown in Table 1, that

$$\mathcal{T}_{0.8}^{(1,10)} = \langle 1, 2, 3, 5, 8 \rangle = \mathcal{T}_1^{(1,10)}. \tag{1.13}$$

$\frac{1}{\alpha}$	$\frac{7}{\alpha^5}$	$\frac{3}{\alpha^3}$	$\frac{8}{\alpha^5}$	$\frac{5}{\alpha^4}$	$\frac{2}{\alpha^2}$	$\frac{9}{\alpha^5}$	$\frac{6}{\alpha^4}$	$\frac{10}{\alpha^5}$	$\frac{4}{\alpha^3}$	1
0.62	0.63	0.71	0.72	0.73	0.76	0.81	0.88	0.90	0.94	1.00
11246	11247	11347	11348	11358	12358	12359	12369	1236	1246	
3,888	21,250*	3,396*	2,068*	8,515	11,158**	13,980*	5,465*	8,515	10,583*	88,818

Table 1: Row 3 of this table contains the ten base 10, 1-Tojaaldi sequences of size at least 1. Row 4 presents the numerical frequencies of Tojaaldi sequences. Row 1 contains the (1,10)-partition of $[\alpha^{-1}, 1)$ by $B_i, 1 \leq i \leq b$, defined in Definition 1.17. Row 2 contains two digit numerical approximations of the B_i. In row 4, the number of asterisks indicate the difference between (actual) observed and Benford (predicted) frequencies, $88818 \cdot \frac{\log(B_{i+1}) - \log(B_i)}{\log(1) - \log(\alpha^{-1})}$. To illustrate our notation, there are 11158 occurrences of the Tojaaldi sequence $\langle 1, 2, 3, 5, 8 \rangle$ among the Tojaaldi sequences of sizes 1 to 88818. The Benford densities described in Definition 1.28 and Proposition 1.29, predict there should be $88818 \cdot \frac{\left(\log(9) - \log(\alpha^5)\right) - \left(\log(2) - \log(\alpha^2)\right)}{\log(\alpha)} \approx 11156$ occurrences, and hence we have placed two asterisks on the 11158 entry to indicate a difference of two between the observed and predicted frequencies.

In the sequel we will assume integers a, b are fixed. This will allow us to ease notation and drop the functional dependency on a, b. So for example we will speak about k_1 instead of $k_1(a, b)$.

In the sequel we will speak about an integer $K \geq k_1(a, b)$. In several proofs we will speak about the effect of K growing arbitrarily large.

Definition 1.25. The sequence $\{y(k)\}_{k \geq K}$, is recursively defined by

$$y(K) = x(K) = \frac{F^{(a)}_{n(K)}}{b^K},$$
$$y(k) = y(k-1) \begin{cases} \frac{\alpha_a^{j+1}}{b}, & \text{if } y(k-1)\frac{\alpha_a^{j+1}}{b} < 1, \\ \frac{\alpha_a^{j}}{b}, & \text{if } y(k-1)\frac{\alpha_a^{j+1}}{b} > 1, \end{cases} \qquad \text{for } k > K. \tag{1.14}$$

Definition 1.26. The sequence $\{n_y(k)\}_{k \geq K}$, is defined by $n_y(k) = 0$, for $k < K$, and

$$n_y(k) = n_y = \#\{K \leq i \leq k : y(i)\frac{\alpha_a^{j+1}}{b} > 1\}, \qquad k \geq K. \tag{1.15}$$

Lemma 1.27.

$$y(k) = \frac{F^{(a)}_{n(K)}}{b^K} \left(\frac{\alpha_a^j}{b}\right)^{k-K} \alpha_a^{n_y(k-1)}, \qquad \textit{for } k \geq K. \tag{1.16}$$

Proof. A straightforward induction. □

Definition 1.28. The sequence $\{n_x(k)\}_{k\geq K}$, is defind by $n_x(k)=0$, for $k<K$, and

$$n_x(k)=\#\{K\leq i\leq k: x(i)\frac{\alpha_a^{j+1}}{b}>1\}, \qquad k\geq K. \tag{1.17}$$

Remark 1.29. The definitions and propositions we have just presented are almost identical to those in [1, Section 3]. The sole difference is that [1] restricts these definitions and propositions to the case $K=k_1$ while here, we have allowed $K>k_1$. It is this small subtlety which will allow us to prove that most $x(k)$ are arbitrarily close to $y(k)$ for large enough $k>K$.

Example 1.30. Let $a=1, b=10$. Then $k_1(a,b)=1$. By (1.14) and (1.8),

$$\{y(1),\ldots,y(4)\}=\{\frac{F_6}{10}=0.8, 0.8872, 0.9839, 0.6744\}\approx$$
$$\{x(1),\ldots,x(4)\}=\{\frac{8}{10},\frac{89}{100},\frac{987}{1000},\frac{6765}{10000}\}.$$

Note that $x(i)-y(i)\approx 0.003$.

Definition 1.31. An integer $k\geq K$ will be called *exceptional* relative to (a,b) if $n_x(k-1)\neq n_y(k-1)$. Otherwise, k will be called *non-exceptional.*

Example 1.32. let $a=1, b=10$. Then $j(a,b)=4$ and $n(1,a,b)=6$.

By Definition 1.14, $x(44)=\frac{F_{212}}{10^{44}}=0.9034$, to four decimal places. By Definition 1.21, $y(44)=0.9006$. But $y(44)\frac{\alpha_a^5}{10}=0.9988<1$, while $x(44)\frac{\alpha_a^5}{10}=1.0019>1$, and consequently $x(44)\neq y(44)$, implying by Definition 1.26 that 45 is exceptional.

Note, that by Definition 1.21, $y(45)=0.9988$. while by Definition 1.14, $x(45)=0.6192$.

Hence, for the exceptional value of 45, $x(45)$ and $y(45)$ are not close. In fact, $y(45)-x(45)>0.37$. The "spikes" in Figures 1 and 2 correspond to the exceptional integers and show that they are rare.

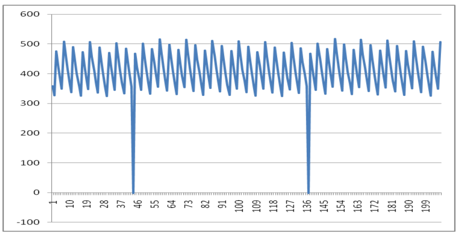

Figure 1: Distribution of $\lfloor\frac{1}{x(n)-y(n)}+0.5\rfloor$ for $2\leq n\leq 200$, for the 1-Fibonacci numbers and base 10. The $x(n)$ and $y(n)$ are defined in Definitions 1.14 and 1.21 respectively.

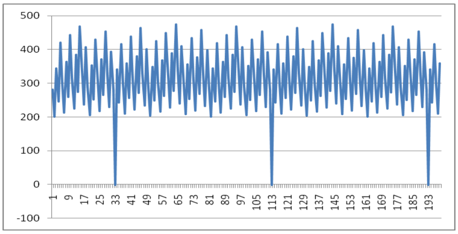

Figure 2: Distribution of $\lfloor \frac{1}{x(n)-y(n)} + 0.5 \rfloor$ for $2 \leq n \leq 200$, for the 2-Fibonacci numbers and base 10. The $x(n)$ and $y(n)$ are defined in Definitions 1.14 and 1.21 respectively.

Definition 1.33. Let $[a, b)$ be an interval on the real line and let $\mathbf{X} \sim Uniform([a, b))$ be a random variable uniformly distributed over this space. If for some constant $c > 1$, the random variable $\mathbf{Y}$ satisfies $\mathbf{Y} = c^{\mathbf{X}}, c > 1$, over the space $[c^a, c^b)$, then we say that $\mathbf{Y}$ is *Benford* distributed over $[c^a, c^b)$, and we notationally indicate this by $\mathbf{Y} \sim Benford([c^a, c^b)$.

Lemma 1.34. *If* $\mathbf{Y} \sim Benford([c^a, c^b)$, *then for* $c^a \leq c_1 \leq c_2 \leq c^b$,

$$Prob(c_1 < \mathbf{Y} < c_2) = \frac{\log_c(\frac{c_2}{c_1})}{b - a}.$$

Remark 1.35. For a proof see [1, Proposition 4.3]. For general references on the Benford distribution see the bibliography in [1]. Notice that the restriction of the spaces and random variables $\mathbf{Y}$ and $\mathbf{X}$ to spaces of countable dense subsets of $[a, b)$ does not change the proposition conclusion.

Example 1.36. Table 1, which presents 88,818 Tojaaldi sequences, allows illustration of the Benford sequence (and Conjecture 2).

Each of these 88,818 Tojaaldi sequences involve 4 or 5 Fibonacci numbers. Thus the 88,818 Tojaaldi sequences involve $3888 \times 5 + 21250 \times 5 + \ldots + 10583 \times 4 = 424992$ Fibonacci numbers. Since the Fibonacci numbers are Benford distributed, we *expect* $\log_{10}(\frac{10}{9}) \times 88818 = 19446.6$ Fibonacci numbers beginning with 9. But $\langle 1, 2, 3, 5, 9 \rangle$ and $\langle 1, 2, 3, 6, 9 \rangle$ are the only Tojaaldi sequences having Fibonacci numbers beginning with 9; so we *observe* $13980 + 5465 = 19445$ Fibonacci numbers beginning with 9.

We can repeat this numerical exercise for each digit (besides 9). We can then compute the $\chi-$square statistic, $\chi^2 = \sum_{i=1}^{9} \frac{(O_i - P_i)^2}{P_i} = 0.0004$ showing a very strong agreement between theory and observed frequency for the Fibonacci-number frequencies.

Similarly, as outlined in the caption to Table 1, we may compute *observed* and *expected* Tojaaldi-sequence frequencies; the associated $\chi-$square statistic is 0.0013, suggesting that the Tojaaldi sequences are Benford distributed. This numerical study motivates Conjecture 2 which will be formally stated in the next section and proven in the final section of this paper.

Definition 1.37. The uniform discrete measure used when making statements about frequency of Tojaaldi sequences on initial segments of integers, is given by the following discrete probablity measure.

$$P_{\mathbf{L}}(\mathcal{T}_{k_0}^{(a,b)}) = \frac{\#\{k : \mathcal{T}_k^{(a,b)} = \mathcal{T}_{k_0}^{(a,b)}, 1 \le k \le L\}}{\#\{\mathcal{T}_k^{(a,b)} : 1 \le k \le L\}}, \qquad L \ge 1, \tag{1.18}$$

with # indicating cardinality and k_0, k, L are integers.

2. Main theorems and conjectures

Conjecture 1. For all $b \ge 2, a \ge 1$, $\#\overline{\mathcal{T}}^{(a,b)} = b$.

Theorem 2.1. *For $b > 1$, and arbitrary $a \ge 1$,*

$$\#\overline{\mathcal{T}}^{(a,b)} \le b.$$

Proof. [1, Theorem 2.9] □

Lemma 2.2. *For given (a, b) let $\langle B_i : 1 \le i \le b+1\rangle$ be an (a, b)-partition, and let z_0 be an arbitrary point in the real space $[\alpha_a^{-1}, 1)$ with the continuous uniform measure. Then*

$$Prob(\mathcal{T}_z = \mathcal{T}_{z_0}) = \frac{\mu([B_i, B_{i+1}))}{\mu([\alpha_a^{-1}, 1))},$$

where i is picked so that $[B_i, B_{i+1})$ contains z_0.

Proof. [1, Theorem 4.1] □

Theorem 2.3. *For any integer $K \ge k_1$, $\{y(i) : i \ge K\}$ is Benford distributed over the space $[\alpha^{-1}, 1)$.*

Proof. [1, Theorem 4.5] (with K replacing k_1 throughout the proof.) □

Conjecture 2. The $\{x(i)\}_{i \ge k_1}$ are Benford distributed.

3. Proof of the two conjectures

In this section we prove the two main conjectures which we restate as theorems. Prior to doing so we will need some preliminary propositions.

Lemma 3.1. *$\{n_y(k)\}_{k \ge K}$ is non-decreasing and unbounded as k goes to infinity.*

Proof. By Definition 1.22, $n_y(k)$ is non-decreasing. Suppose contrary to the proposition there is a k_0 such that for all $k \geq k_0$, $n_y(k) = n_y(k_0)$. We proceed to derive a contradiction, proving that $n_y(k)$ is unbounded as k goes to infinity.

First, we show, using an inductive argument, that $y(k) \in (\alpha_a^{-1}, 1)$, for $k > K$. The base case, when $k = K$ is established by Definition 1.21 and equation (1.10). The induction step is established by Definitions 1.21 and 1.8.

Returning to the proof of Proposition 3.1, note that according to Definition 1.21, there are two cases to consider, according to whether $y(k_0)\frac{\alpha_a^{j+1}}{b} < 1$, or $y(k_0)\frac{\alpha_a^{j+1}}{b} > 1$. We assume $y(k_0)\frac{\alpha_a^{j+1}}{b} < 1$, the treatment of the other case being almost identical. Then since we assumed $n_y(k) = n_y(k_0), k \geq k_0$, we have $y(k_0)\left(\frac{\alpha_a^{j+1}}{b}\right)^n < 1$, for all integer $n \geq 0$, a contradiction, since by Definition 1.8, $\left(\frac{\alpha_a^{j+1}}{b}\right)^n$ goes to infinity as n gets arbitrary large. This contradiction shows that our original assumption that $y(k)$ is bounded is false. This completes the proof. □

Lemma 3.2. *For non-exceptional $k > K$*

$$|x(k) - y(k)| \in \left(\alpha_a^{-2n(K)-1}, \alpha_a^{-2n(K)}\right). \tag{3.1}$$

Proof. [1, Proposition 3.6] with K replacing k_1 in both the proposition statement and throughout the proof. □

Remark 3.3. As noted in the previous section, because we replaced k_1 by K, the lower bound estimate of the difference in (3.1) is going to 0. Consequently $\{x(i)\}_{i \geq K}$ is asymptotically approaching $\{y(i)\}_{i \geq K}$. Formally, we have the following Corollary.

Corollary 3.4. *As k varies over non-exceptional k,*

$$\lim_{k \to \infty} |x(k) - y(k)| = 0.$$

Proof. Immediate, by combining Propositions 3.1 and 3.2. □

Lemma 3.5. *Using Definition 1.17, let $\langle B_i : 1 \leq i \leq b+1 \rangle = \langle 1, \frac{i}{\alpha_a^{e(i)}} : 1 \leq i \leq b \rangle$ be an (a,b)-partition. Then the $\#\{T_{B_i}, 1 \leq i \leq b\} = b$, that is, the T_{B_i} are distinct.*

Proof. Following [1, Proposition 2.15], define a $b \times j(a,b) + 1$ matrix, $A(k,l) = B_k \alpha_a^l, 1 \leq k \leq b, 1 \leq l \leq j(a,b) + 1$, so that by Definition 1.13
$T_{B_k} = \langle \lfloor A(k,1) \rfloor, \ldots, \lfloor A(k,m) \rfloor \rangle$, and by Definitions 1.16, 1.17 and 1.8, m equals $j(a,b)$ or $j(a,b)+1$. Recall the following facts about the matrix A :
(I) $A(k, e(i(k))) = i(k)$; (II) no other cell entries (besides $(k, e(i(k)))$) can have exact integer values; (III) A is strictly increasing as one goes from top to bottom and left to right, that is, $A(k,l) < A(k',l')$ if either (i)$l < l'$ or (ii) $l = l', k < k'$.

Using these three facts we see that $\lfloor A(k', e(i(k)) \rfloor < A(k, e(i(k))$, for $k' < k$, $1 \leq k \leq b, i(k) \neq b$. Hence, $T_{B_{k'}} \neq T_{B_k}$, for $k' < k$. An almost identical argument applies when $i(k) = b$. Hence the T_{B_i} are distinct as was to be shown. □

Example 3.6. We can illustrate the proof using Table 1. By Table 1, $B_4 = \frac{8}{\alpha^5}$, implying that the 5th member of the sequence T_{B_4} equals 8 and the 5th member of the previous sequences, $T_{B_k}, 1 \leq k < 4$, are strictly less than 8 as confirmed by Table 1.

Note also the special case $B_9 = \frac{10}{\alpha^5}$, implying that the 5th member of the sequence T_{B_9} is empty while the 5th member of the previous sequences, $T_{B_k}, 1 \leq k \leq 8$, are non-empty, as confirmed by Table 1.

The next three propositions show that exceptional k (as defined in Definition 1.26) are rare. First we prove the following proposition, which provides an alternate recursive definition to $x(k)$, defined in Definition 1.14.

Lemma 3.7. *The sequence* $\{x(k)\}_{k \geq K}$, *is recursively defined by*

$$x(K) = \frac{F^{(a)}_{n(K)}}{b^K},$$

$$x(k) = \begin{cases} x(k-1)\frac{\alpha_a^{j+1}}{b} + F^{(a)}_{j+1}\frac{\beta_a^{n(k-1)}}{b^k}, & \text{if } x(k-1)\frac{\alpha_a^{j+1}}{b} < 1, \\ x(k-1)\frac{\alpha_a^{j}}{b} + F^{(a)}_{j}\frac{\beta_a^{n(k-1)}}{b^k}, & \text{if } x(k-1)\frac{\alpha_a^{j+1}}{b} > 1, \qquad \text{for } k > K. \end{cases} \quad (3.2)$$

Proof. If $k = K$ the proposition is true by Definition 1.14. If $k > K$, then by Definitions 1.3, 1.7 and Proposition 1.10

$$n(k) - n(k-1) = \#T^{(a,b)}_{k-1} \in \{j, j+1\}.$$

Consequently, there are two cases to consider. We treat the case $n(k) = n(k-1) + j$, the treatment of the other case, $n(k) = n(k-1) + j + 1$, being similar.

But then, by Proposition 1.12,

$$F^{(a)}_{n(k)} = \alpha_a^j F^{(a)}_{n(k-1)} + F^{(a)}_j \beta_a^{n(k-1)}.$$

Equation (3.2), follows by dividing both sides of this last equation by b^k and applying Defintion 1.14. □

Prior to stating the next two propositions, it may be useful to numerically illustrate the proof method. The following example continues Example 1.27.

Example 3.8. Let $a = 1, b = 10$. Then by Definition 1.8, $j(a,b) = 4$. By Definitions 1.22 and 1.24,

$$n_y(43) = n_x(43),$$

implying by Definition 1.26, that 44 is not exceptional. By Definition 1.21, $y(44) = 0.9006$; by Definition 1.14, $x(44) = 0.9034$. Application of Defintions 1.22 and 1.24 require use of $\frac{\alpha_a^5}{10} = 0.9017$. Observe that

$$y(44) = 0.9006 < 0.9017 < 0.9034 = x(44).$$

Consequently,

$$y(44)\frac{\alpha_a^5}{10} < 1; x(44)\frac{\alpha_a^5}{10} > 1.$$

Therefore, by Definitions 1.22 and 1.24

$$n_y(44) = n_y(43) + 1; n_x(44) = n_x(43).$$

Hence, by Definition 1.26, $k = 45$ is an exceptional value. Notice that $y(44)$ and $x(44)$ are close in value as predicted by Proposition 3.2. The values of $x(45)$ and $y(45)$ may now be computed using Definition 1.22 and Proposition 3.6,

$$y(45) = 0.9988; x(45) = 0.6192.$$

Here, $y(45)$ and $x(45)$ are not close. More precisely, $y(45)$ is close to 1 while $x(45)$ is close to α_a^{-1}.

But by applying Definitions 1.22 and 1.24 we see that

$$n_y(45) = n_y(44); n_x(45) = n_x(44) + 1,$$

implying that

$$n_y(45) = n_x(45),$$

in other words, 46 is not exceptional. We in fact confirm that $y(46)$ and $x(46)$ are indeed close as required.

$$y(46) = 0.6846 < 0.6867 = x(46).$$

We may summarize this numerical example as follows: (I) Most k are non-exceptional. (II) For an exceptional k to occur, one of $x(k-1), y(k-1)$ must be greater than $\frac{\alpha_a^{j+1}}{b}$ while the other is less. (III) This occurs rarely because most k are non-exceptional and hence, by Proposition 3.2, $x(k)$ and $y(k)$ are usually numerically close. (IV) If k is exceptional then $x(k)$ will be close to α_a^{-1} while $y(k)$ will be close to 1. (V) Consequently $k+1$ will not be exceptional and in fact $x(k+1)$ and $y(k+1)$ will again be close to each other.

The next proposition formalizes this example.

Lemma 3.9. *If k is exceptional then $k-1$ and $k+1$ are non-exceptional.*

Proof. Assume that k is exceptional and $k-1$ is not exceptional. This assumption is allowable, since by Definitions 1.21, 1.22 and 1.24, K and $K+1$ are not exceptional and therefore the "first" exceptional k must be preceded by a non-exceptional value. We proceed to show that $k+1$ is not exceptional. Therefore, the "2nd" exceptional k is preceded by a non-exceptional k. Proceeding in this manner we will always be justified if we assume the predecessor of an exceptional k is not exceptional. Consequently, we have left to prove that $k+1$ is not exceptional.

By Definitions 1.26, 1.22 and 1.24, for k to be exceptional we must have one of $x(k-1)\frac{\alpha_a^{j+1}}{b}$ and $y(k-1)\frac{\alpha_a^{j+1}}{b}$ greater than one while the other is less than one. We treat one of these cases, the treatment of the other case being similar.

Accordingly, we assume

$$n_y(k-2) = n_x(k-2) \longrightarrow k-1 \text{ is not exceptional,} \tag{3.3}$$

and we further assume

$$y(k-1) < \Big(\frac{\alpha_a^{j+1}}{b}\Big)^{-1} < x(k-1) \longrightarrow y(k-1)\frac{\alpha_a^{j+1}}{b} < 1, x(k-1)\frac{\alpha_a^{j+1}}{b} > 1. \quad (3.4)$$

Combining Proposition 3.2 with (3.4) we obtain

$$y(k-1) > \Big(\frac{\alpha_a^{j+1}}{b}\Big)^{-1} - \frac{1}{\alpha_a^{2n(K)}}, x(k-1) < \Big(\frac{\alpha_a^{j+1}}{b}\Big)^{-1} + \frac{1}{\alpha_a^{2n(K)}}. \quad (3.5)$$

Hence, by Definition 1.21 and Proposition 3.6,

$$y(k) = y(k-1)\frac{\alpha_a^{j+1}}{b}, x(k) = x(k-1)\frac{\alpha_a^{j}}{b} + F_j^{(a)}\frac{\beta_a^{n(k-1)}}{b^k}. \quad (3.6)$$

Using equation (3.4), Definitions 1.26, 1.22 and 1.24, we confirm that

$$n_y(k-1) = n_y(k-2), n_x(k-1) = n_x(k-2)+1 \longrightarrow k \text{ is exceptional.} \quad (3.7)$$

Again, by Definition 1.26, to decide whether $k+1$ is exceptional we need to compute $n_y(k)$ and $n_x(k)$. We first compute $n_y(k)$.

Appplying equations (3.6) and (3.5) to Definition 1.22, we have

$$y(k)\frac{\alpha_a^{j+1}}{b} = y(k-1)\Big(\frac{\alpha_a^{j+1}}{b}\Big)^2 > \frac{\alpha_a^{j+1}}{b} - \frac{\alpha_a^{2j+2}}{b^2\alpha_a^{2n(K)}}. \quad (3.8)$$

j and b are $O(1)$ (relative to the choice of K) while we may chose K arbitrarily large. It follows that as K goes to infinity,

$$y(k)\frac{\alpha_a^{j+1}}{b} > \frac{\alpha_a^{j+1}}{b} - \frac{\alpha_a^{2j+2}}{b^2\alpha^{2n(K)}} \approx \frac{\alpha_a^{j+1}}{b} > 1. \quad (3.9)$$

Consequently by (3.9), Definition 1.22, and (3.7)

$$n_y(k) = n_y(k-1)+1 = n_y(k-2)+1. \quad (3.10)$$

.

We now carry out a similar analysis on $x(k)$. By Proposition 3.6 we have

$$x(k)\frac{\alpha_a^{j+1}}{b} = \Big(x(k-1)\frac{\alpha_a^{j}}{b} + F_j^{(a)}\frac{\beta_a^{n(k-1)}}{b^k}\Big)\frac{\alpha_a^{j+1}}{b} \quad (3.11)$$

Applying the upper bound for $x(k-1)$ presented in (3.5) we obtain after some straightforward manipulations

$$x(k)\frac{\alpha_a^{j+1}}{b} < \frac{\alpha_a^{j}}{b} + \alpha_a^{2j+1-2n(K)}\frac{1}{b^2} + F_j^{(a)}\alpha_a^{(j+1)}\frac{\beta_a^{n(k-1)}}{b^{k+1}} \approx \frac{\alpha_a^{j}}{b} < 1. \quad (3.12)$$

Hence, by Definition 1.24 and equation (3.7),

$$n_x(k) = n_x(k-1) = n_x(k-2)+1. \quad (3.13)$$

Equations (3.10) and (3.13) together imply that $n_x(k) = n_y(k)$, and hence, by Definition 1.26, $k+1$ is not exceptinoal as was to be shown.

This completes the proof. □

Lemma 3.10. *$Prob(\{k : k$ is exceptional$\}) = 0$.*

Proof. By Proposition 3.8, exceptional k occur as singletons (that is, two consecutive integers cannot be exceptional). Furthermore, by Proposition 3.2, if k is exceptional $k-1$ is non-exceptional and

$$x(k-1), y(k-1) \in \Big(\big(\frac{\alpha_a^{j+1}}{b}\big)^{-1} - \big(\alpha_a^{2n(K)}\big)^{-1}, \big(\frac{\alpha_a^{j+1}}{b}\big)^{-1} + \big(\alpha_a^{2n(K)}\big)^{-1} \Big).$$

By Theorem 2.3 the $\{y(i)\}_{i \geq K}$ are Benford distributed and hence the probability of $y(k-1)$ being in an open interval whose width is going to 0, may be made as small as we please.

But by Proposition 3.8 every exceptional k is uniquely associated with a non-exceptional k.

This completes the proof. □

We can now prove the two conjectures.

Theorem 3.11. *The $\{x(n)\}_{n \geq 1}$ are Benford distributed.*

Proof. Consider an arbitrary set (of reals), $B \subset (\alpha_a^{-1}, 1)$. To prove the theorem, we must show that $Prob(B \cap \{x(n)\}_{n \geq 1})$ equals the desired Benford-distribution probability.

By Definition 1.28 and Proposition 1.29 we know that $Prob(B \cap \{y(n)\}_{n \geq 1}) = \frac{\log(M_y) - \log(m_y)}{\log(1) - \log(\alpha_a^{-1})}$, with $M_y = \sup(B \cap \{y(n)\}_{n \geq 1})$ and $m_y = \inf(B \cap \{y(n)\}_{n \geq 1})$. Define $M_x = \sup(B \cap \{x(n)\}_{n \geq 1})$ and $m_x = \inf(B \cap \{x(n)\}_{n \geq 1})$. By Corollary 3.3, $|M_y - M_x|$ and $|m_y - m_x|$ can be made arbitrarily small. The result immediately follows. □

Theorem 3.12. *For all $b \geq 2, a \geq 1$, $\#\overline{\mathcal{T}}^{(a,b)} = b$.*

Proof. By Theorem 2.1, $\#\overline{\mathcal{T}}^{(a,b)} \leq b$. It therefore suffices to prove $\#\overline{\mathcal{T}}^{(a,b)} \geq b$. The proof is constructive.

Using Definition 1.17, let $\langle B_i : 1 \leq i \leq b+1 \rangle = \langle 1, \frac{i}{\alpha_a^{e(i)}} : 1 \leq i \leq b \rangle$ be an (a,b)-partition. For $1 \leq i \leq b$, pick a non-exceptional $x(n_i) \in (B_i, B_{i+1})$, for some integer n_i. $x(n_i)$ exists since by Theorem 3.7, $\{x(n)\}_{n \geq 1}\}$ is Benford distributed and hence dense in $(\alpha_a^{-1}, 1)$.

But then by Proposition 1.19, $T_{x(n_i)} = T_{B_i}$; by Proposition 3.4, the T_{B_i} are distinct; and by Proposition 1.15, $T_{x(n_i)} = T_{n_i}$. Hence, we have produced at least b distinct Tojaaldi sequences as was to be shown. □

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Sums of powers of Fibonacci and Lucas polynomials in terms of Fibopolynomials

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Abstract

We consider sums of powers of Fibonacci and Lucas polynomials of the form $\sum_{n=0}^{q} F_{tsn}^{k}(x)$ and $\sum_{n=0}^{q} L_{tsn}^{k}(x)$, where s,t,k are given natural numbers, together with the corresponding alternating sums $\sum_{n=0}^{q}(-1)^n F_{tsn}^{k}(x)$ and $\sum_{n=0}^{q}(-1)^n L_{tsn}^{k}(x)$. We give conditions on s,t,k for express these sums as some proposed linear combinations of the s-Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m=1,2,\ldots,tk$.

Keywords: Sums of powers; Fibonacci and Lucas polynomials; Z Transform.

MSC: 11B39

1. Introduction

We use $\mathbb{N}$ for the natural numbers and $\mathbb{N}'$ for $\mathbb{N}\cup\{0\}$. We follow the standard notation $F_n(x)$ for Fibonacci polynomials and $L_n(x)$ for Lucas polynomials. Binet's formulas

$$F_n(x)=\frac{1}{\sqrt{x^2+4}}\left(\alpha^n(x)-\beta^n(x)\right)\quad\text{and}\quad L_n(x)=\alpha^n(x)+\beta^n(x),\tag{1.1}$$

where

$$\alpha(x)=\frac{1}{2}\left(x+\sqrt{x^2+4}\right)\quad\text{and}\quad\beta(x)=\frac{1}{2}\left(x-\sqrt{x^2+4}\right),\tag{1.2}$$

will be used extensively (without further comments). We will use also the identities

$$\frac{F_{(2p-1)s}(x)}{F_s(x)}=\sum_{k=0}^{p-1}(-1)^{sk}L_{2(p-k-1)s}(x)-(-1)^{s(p-1)},\tag{1.3}$$

$$\frac{F_{2ps}(x)}{F_s(x)} = \sum_{k=0}^{p-1} (-1)^{sk} L_{(2p-2k-1)s}(x), \tag{1.4}$$

$$F_M(x) F_N(x) - F_{M+K}(x) F_{N-K}(x) = (-1)^{N-K} F_{M+K-N}(x) F_K(x), \tag{1.5}$$

where $p \in \mathbb{N}$ in (1.3) and (1.4), and $M, N, K \in \mathbb{Z}$ in (1.5). (Identity (1.5) is a version of the so-called "index-reduction formula"; see [7] for the case $x = 1$.) Two variants of (1.5) we will use in section 5 are

$$F_M(x) L_N(x) - F_{M+K}(x) L_{N-K}(x) = (-1)^{N-K+1} L_{M+K-N}(x) F_K(x), \tag{1.6}$$

$$\left(x^2+4\right) F_M(x) F_N(x) - L_{M+K}(x) L_{N-K}(x) = (-1)^{N-K+1} L_{M+K-N}(x) L_K(x). \tag{1.7}$$

Given $n \in \mathbb{N}'$ and $k \in \{0, 1, \ldots, n\}$, the *s-Fibopolynomial* $\binom{n}{k}_{F_s(x)}$ is defined by $\binom{n}{0}_{F_s(x)} = \binom{n}{n}_{F_s(x)} = 1$, and

$$\binom{n}{k}_{F_s(x)} = \frac{F_{sn}(x) F_{s(n-1)}(x) \cdots F_{s(n-k+1)}(x)}{F_s(x) F_{2s}(x) \cdots F_{ks}(x)}. \tag{1.8}$$

(These mathematical objects were used before in [19], where we called them "s-polyfibonomials". However, we think now that "s-Fibopolynomials" is a better name to describe them.)

Plainly we have symmetry for s-Fibopolynomials: $\binom{n}{k}_{F_s(x)} = \binom{n}{n-k}_{F_s(x)}$. We can use the identity

$$F_{s(n-k)+1}(x) F_{sk}(x) + F_{sk-1}(x) F_{s(n-k)}(x) = F_{sn}(x),$$

(which comes from (1.5) with $M = sn$, $N = 1$ and $K = -sk+1$), to conclude that

$$\binom{n}{k}_{F_s(x)} = F_{s(n-k)+1}(x) \binom{n-1}{k-1}_{F_s(x)} + F_{sk-1}(x) \binom{n-1}{k}_{F_s(x)}. \tag{1.9}$$

Formula (1.9) and a simple induction argument, show that s-Fibopolynomials are indeed polynomials (with $\deg \binom{n}{k}_{F_s(x)} = sk(n-k)$). The case $s = x = 1$ corresponds to Fibonomials $\binom{n}{k}_F$, introduced by V. E. Hoggatt, Jr. [5] in 1967 (see also [23]), and the case $x = 1$ corresponds to s-Fibonomials $\binom{n}{k}_{F_s}$, first mentioned also in [5], and studied recently in [18]. We comment in passing that Fibonomials are important mathematical objects involved in many interesting research works during the last few decades (see [4, 8, 9, 11, 24]).

The well-known identity

$$\sum_{n=0}^{q} F_n^2 = F_q F_{q+1} = \binom{q+1}{2}_F, \tag{1.10}$$

was the initial motivation for this work. We will see that (1.10) is just a particular case of the following polynomial identities ((4.20) in section 4)

$$
\begin{aligned}
&(-1)^{(s+1)q} L_s(x) \sum_{n=0}^{q} (-1)^{(s+1)n} F_{sn}^2(x) \\
&= (-1)^{sq} F_s(x) \sum_{n=0}^{q} (-1)^{sn} F_{2sn}(x) = F_{s(q+1)}(x) F_{sq}(x).
\end{aligned}
$$

To find closed formulas for sums of powers of Fibonacci and Lucas numbers $\sum_{n=0}^{q} F_n^k$ and $\sum_{n=0}^{q} L_n^k$, and for the corresponding alternating sums of powers $\sum_{n=0}^{q} (-1)^n F_n^k$ and $\sum_{n=0}^{q} (-1)^n L_n^k$, is a challenging problem that has been in the interest of many mathematicians along the years (see [1, 3, 12, 13, 15, 16, 21], to mention some). There are also some works considering variants of these sums and/or generalizations (in some sense) of them (see [2, 10, 14, 22], among many others).

This work presents, on one hand, a generalization of the problem mentioned above, considering Fibonacci and Lucas polynomials (instead of numbers) and involving more parameters in the sums. On the other hand, we are not interested in any closed formulas for these sums, but only in sums that can be written as certain linear combinations of certain s-Fibopolynomials (as in (1.10)). More precisely, in this work we obtain sufficient conditions (on the positive integer parameters t, k, s), for the polynomial sums of powers $\sum_{n=0}^{q} F_{tsn}^k(x)$, $\sum_{n=0}^{q} L_{tsn}^k(x)$, $\sum_{n=0}^{q} (-1)^n F_{tsn}^k(x)$ and $\sum_{n=0}^{q} (-1)^n L_{tsn}^k(x)$, can be expressed as linear combinations of the s-Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m = 1, 2, \ldots, tk$, according to some proposed expressions ((3.3), (3.15), (4.5) and (4.6), respectively). (We conjecture that these sufficient conditions are also necessary, see remark 3.2.)

In Section 2 we recall some facts about Z transform, since some results related to the Z transform of the sequences $\left\{F_{tsn}^k(x)\right\}_{n=0}^{\infty}$ and $\left\{L_{tsn}^k(x)\right\}_{n=0}^{\infty}$ (obtained in a previous work) are the starting point of the results in this work.

The main results are presented in Section 3 and 4. Propositions 3.1 and 3.3 in section 3 contain, respectively, sufficient conditions on the positive integers t, k, s for the sums of powers $\sum_{n=0}^{q} F_{tsn}^k(x)$ and $\sum_{n=0}^{q} L_{tsn}^k(x)$ can be written as linear combinations of the mentioned s-Fibopolynomials, and propositions 4.1 and 4.3 in section 4 contain, respectively, sufficient conditions on the positive integers t, k, s for the alternating sums of powers $\sum_{n=0}^{q} (-1)^n F_{tsn}^k(x)$ and $\sum_{n=0}^{q} (-1)^n L_{tsn}^k(x)$ can be written as linear combinations of those s-Fibopolynomials. Surprisingly, there are some intersections on the conditions on t and k in Proposition 3.1 and 4.1 (and also in Proposition 3.3 and 4.3), allowing us to write results for sums of powers of the form $\sum_{n=0}^{q} (-1)^{sn} F_{tsn}^k(x)$ or $\sum_{n=0}^{q} (-1)^{(s+1)n} F_{tsn}^k(x)$ (and similar sums for Lucas polynomials), that work at the same time for sums $\sum_{n=0}^{q} F_{tsn}^k(x)$ and alternating sums $\sum_{n=0}^{q} (-1)^n F_{tsn}^k(x)$ as well, depending on the parity of s. These results are presented in section 4: Corollary 4.2 (for the Fibonacci case) and 4.4 (for the Lucas case).

Finally, in Section 5 we show some examples of identities obtained as derivatives of some of the results obtained in previous sections.

2. Preliminaries

The Z transform maps complex sequences $\{a_n\}_{n=0}^{\infty}$ into holomorphic functions $A : U \subset \mathbb{C} \to \mathbb{C}$, defined by the Laurent series $A(z) = \sum_{n=0}^{\infty} a_n z^{-n}$ (also written as $\mathcal{Z}(a_n)$, defined outside the closure of the disk of convergence of the Taylor series $\sum_{n=0}^{\infty} a_n z^n$). We also write $a_n = \mathcal{Z}^{-1}(A(z))$ and we say the the sequence $\{a_n\}_{n=0}^{\infty}$ is the inverse Z transform of $A(z)$. Some basic facts we will need are the following:

(a) $\mathcal{Z}$ is linear and injective (same for $\mathcal{Z}^{-1}$).

(b) If $\{a_n\}_{n=0}^{\infty}$ is a sequence with Z transform $A(z)$, then the Z transform of the sequence $\{(-1)^n a_n\}_{n=0}^{\infty}$ is

$$\mathcal{Z}((-1)^n a_n) = A(-z). \tag{2.1}$$

(c) If $\{a_n\}_{n=0}^{\infty}$ is a sequence with Z transform $A(z)$, then the Z transform of the sequence $\{na_n\}_{n=0}^{\infty}$ is

$$\mathcal{Z}(na_n) = -z\frac{d}{dz}A(z). \tag{2.2}$$

Plainly we have (for given $\lambda \in \mathbb{C}$, $\lambda \neq 0$)

$$\mathcal{Z}(\lambda^n) = \frac{z}{z-\lambda}. \tag{2.3}$$

For example, if $t, k \in \mathbb{N}'$ are given, we can write the generic term of the sequence $\left\{F_{tsn}^k(x)\right\}_{n=0}^{\infty}$ as

$$\begin{aligned} F_{tsn}^k(x) &= \left(\frac{1}{\sqrt{x^2+4}}\left(\alpha^{tsn}(x) - \beta^{tsn}(x)\right)\right)^k \\ &= \frac{1}{(x^2+4)^{\frac{k}{2}}}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\left(\alpha^{tsl}(x)\,\beta^{ts(k-l)}(x)\right)^n. \end{aligned} \tag{2.4}$$

The linearity of $\mathcal{Z}$ and (2.3) give us

$$\mathcal{Z}\left(F_{tsn}^k(x)\right) = \frac{1}{(x^2+4)^{\frac{k}{2}}}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\frac{z}{z-\alpha^{tsl}(x)\,\beta^{ts(k-l)}(x)}. \tag{2.5}$$

Similarly, since the generic term of the sequence $\left\{L_{tsn}^k(x)\right\}_{n=0}^{\infty}$ can be expressed as

$$L_{tsn}^k(x) = \left(\alpha^{tsn}(x) + \beta^{tsn}(x)\right)^k = \sum_{l=0}^{k}\binom{k}{l}\left(\alpha^{tsl}(x)\,\beta^{ts(k-l)}(x)\right)^n, \tag{2.6}$$

we have that

$$\mathcal{Z}\left(L_{tsn}^k(x)\right) = \sum_{l=0}^{k}\binom{k}{l}\frac{z}{z-\alpha^{tsl}(x)\,\beta^{ts(k-l)}(x)}. \tag{2.7}$$

Observe that formulas

$$\mathcal{Z}\left(F_n\left(x\right)\right)=\frac{z}{z^2-xz-1}\qquad\text{and}\qquad\mathcal{Z}\left(L_n\left(x\right)\right)=\frac{z\left(2z-x\right)}{z^2-xz-1},\tag{2.8}$$

are the simplest cases ($k=t=s=1$) of (2.5) and (2.7), respectively.

In a recent work [20] (inspired by [6], among others), we proved that expressions (2.5) and (2.7) can be written in a special form. The result is that (2.5) can be written as

$$\begin{aligned}&\mathcal{Z}\left(F_{tsn}^k\left(x\right)\right)\\&=z\frac{\sum_{i=0}^{tk}\sum_{j=0}^{i}\left(-1\right)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}F_{ts(i-j)}^k\left(x\right)z^{tk-i}}{\sum_{i=0}^{tk+1}\left(-1\right)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{tk+1}{i}_{F_s(x)}z^{tk+1-i}},\end{aligned}\tag{2.9}$$

and (2.7) can be written as

$$\begin{aligned}&\mathcal{Z}\left(L_{tsn}^k\left(x\right)\right)\\&=z\frac{\sum_{i=0}^{tk}\sum_{j=0}^{i}\left(-1\right)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}L_{ts(i-j)}^k\left(x\right)z^{tk-i}}{\sum_{i=0}^{tk+1}\left(-1\right)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{tk+1}{i}_{F_s(x)}z^{tk+1-i}}.\end{aligned}\tag{2.10}$$

From (2.9) and (2.10) we obtained that $F_{tsn}^k\left(x\right)$ and $L_{tsn}^k\left(x\right)$ can be expressed as linear combinations of the s-Fibopolynomials $\binom{n+tk-i}{tk}_{F_s(x)}$, $i=0,1,\ldots,tk$, according to

$$\begin{aligned}&F_{tsn}^k\left(x\right)\\&=\left(-1\right)^{s+1}\sum_{i=0}^{tk}\sum_{j=0}^{i}\left(-1\right)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}F_{ts(i-j)}^k(x)\binom{n+tk-i}{tk}_{F_s(x)},\end{aligned}\tag{2.11}$$

and

$$\begin{aligned}&L_{tsn}^k\left(x\right)\\&=\left(-1\right)^{s+1}\sum_{i=0}^{tk}\sum_{j=0}^{i}\left(-1\right)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}L_{ts(i-j)}^k(x)\binom{n+tk-i}{tk}_{F_s(x)}.\end{aligned}\tag{2.12}$$

The denominator in (2.9) (or (2.10)) is a $(tk+1)$-th degree z-polynomial, which we denote as $D_{s,tk+1}\left(x,z\right)$, that can be factored as

$$\begin{aligned}&\sum_{i=0}^{tk+1}\left(-1\right)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{tk+1}{i}_{F_s(x)}z^{tk+1-i}\\&=\left(-1\right)^{s+1}\prod_{j=0}^{tk}\left(z-\alpha^{sj}\left(x\right)\beta^{s(tk-j)}\left(x\right)\right).\end{aligned}\tag{2.13}$$

(See proposition 1 in [20].) Moreover, if tk is even, $tk = 2p$ say, then (2.13) can be written as

$$D_{s,2p+1}(x;z) = (-1)^{s+1}(z-(-1)^{sp})\prod_{j=0}^{p-1}\left(z^2-(-1)^{sj}L_{2s(p-j)}(x)\,z+1\right), \quad (2.14)$$

and if tk is odd, $tk = 2p-1$ say, we have

$$D_{s,2p}(x;z) = (-1)^{s+1}\prod_{j=0}^{p-1}\left(z^2-(-1)^{sj}L_{s(2p-1-2j)}(x)\,z+(-1)^{(2p-1)s}\right). \quad (2.15)$$

(See (40) and (41) in [20].)

3. The main results (I)

Let us consider first the Fibonacci case. From (2.11) we can write the sum $\sum_{n=0}^{q} F_{tsn}^{k}(x)$ in terms of a sum of s-Fibopolynomials in a trivial way, namely

$$\sum_{n=0}^{q} F_{tsn}^{k}(x) \quad (3.1)$$
$$= (-1)^{s+1}\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}F_{ts(i-j)}^{k}(x)\sum_{n=0}^{q}\binom{n+tk-i}{tk}_{F_s(x)}.$$

The point is that we can write (3.1) as

$$\sum_{n=0}^{q} F_{tsn}^{k}(x) \quad (3.2)$$
$$= (-1)^{s+1}\sum_{m=1}^{tk}\sum_{i=0}^{tk-m}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}F_{ts(i-j)}^{k}(x)\binom{q+m}{tk}_{F_s(x)}$$
$$+ (-1)^{s+1}\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}F_{ts(i-j)}^{k}(x)\sum_{n=0}^{q}\binom{n}{tk}_{F_s(x)}.$$

Expression (3.2) tells us that the sum $\sum_{n=0}^{q} F_{tsn}^{k}(x)$ can be written as a linear combination of the s-Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m = 1, 2, \ldots, tk$, according to

$$\sum_{n=0}^{q} F_{tsn}^{k}(x) \quad (3.3)$$
$$= (-1)^{s+1}\sum_{m=1}^{tk}\sum_{i=0}^{tk-m}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}F_{ts(i-j)}^{k}(x)\binom{q+m}{tk}_{F_s(x)},$$

if and only if

$$\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}F^k_{ts(i-j)}(x)=0. \tag{3.4}$$

Observe that from (2.5) and (2.9) we can write

$$\begin{aligned}&\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}F^k_{ts(i-j)}(x)\,z^{tk-i} \qquad (3.5)\\ &=\frac{1}{(x^2+4)^{\frac{k}{2}}}\left(\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\frac{1}{z-\alpha^{lts}(x)\,\beta^{(k-l)ts}(x)}\right)\\ &\quad\times\left(\sum_{i=0}^{tk+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{tk+1}{i}_{F_s(x)}z^{tk+1-i}\right).\end{aligned}$$

Let us consider the factors in parentheses of the right-hand side of (3.5), namely

$$\Pi_1(x,z)=\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\frac{1}{z-\alpha^{lts}(x)\,\beta^{(k-l)ts}(x)}, \tag{3.6}$$

and

$$\Pi_2(x,z)=\sum_{i=0}^{tk+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{tk+1}{i}_{F_s(x)}z^{tk+1-i}. \tag{3.7}$$

Clearly any of the conditions

$$\Pi_1(x,1)=0, \tag{3.8}$$

or

$$\Pi_1(x,1)<\infty \text{ and } \Pi_2(x,1)=0. \tag{3.9}$$

imply (3.4).

Proposition 3.1. *The sum $\sum_{n=1}^{q}F^k_{tsn}(x)$ can be written as a linear combination of the s-Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m=1,2,\ldots,tk$, according to (3.3), in the following cases*

	t	k	s
(a)	*even*	*odd*	*even*
(b)	*odd*	$\equiv 2 \bmod 4$	*odd*
(c)	$\equiv 0 \bmod 4$	*odd*	*any*

Proof. Observe that in each of the three cases the product tk is even. Then, according to (2.14) we can write

$$\Pi_2(x,z)=(-1)^{s+1}\left(z-(-1)^{\frac{kts}{2}}\right)\prod_{j=0}^{\frac{tk}{2}-1}\left(z^2-(-1)^{sj}L_{2s\left(\frac{tk}{2}-j\right)}(x)\,z+1\right). \tag{3.10}$$

(a) Let us suppose that t is even, k is odd and s is even. In this case the factor $\left(z-(-1)^{\frac{kts}{2}}\right)$ of the right-hand side of (3.10) is $(z-1)$, so we have $\Pi_2(x,1)=0$. It remains to check that $\Pi_1(x,1)$ is finite. In fact, by writing k as $2k-1$, and using that t and s are even, one can check that

$$\Pi_1(x,1)=\sqrt{x^2+4}\sum_{l=0}^{k-1}\binom{2k-1}{l}(-1)^{l+1}\frac{F_{(2k-1-2l)ts}(x)}{2-L_{(2k-1-2l)ts}(x)}, \tag{3.11}$$

so we have that $\Pi_1(x,1)$ is finite, and then the right-hand side of (3.5) is equal to zero when $z=1$, as wanted.

(b) Suppose now that t is odd, $k\equiv 2 \bmod 4$ and that s is odd. In this case the factor $\left(z-(-1)^{\frac{kts}{2}}\right)$ of the right-hand side of (3.10) is $(z+1)$, so $\Pi_2(x,1)\neq 0$. However, by writing k as $2(2k-1)$ and using that t and s are odd, we can see that

$$\Pi_1(x,1)=\sum_{l=0}^{2k-2}\binom{2(2k-1)}{l}(-1)^l-\frac{1}{2}\binom{2(2k-1)}{2k-1}=0.$$

Thus, the right-hand side of (3.5) is equal to 0 when $z=1$, as wanted.

(c) Let us suppose that $t\equiv 0 \bmod 4$, k is odd, and s is any positive integer. In this case the factor $\left(z-(-1)^{\frac{kts}{2}}\right)$ of the right-hand side of (3.10) is $(z-1)$, so we have $\Pi_2(x,1)=0$. By writing k as $2k-1$, and using that t is multiple of 4, we can see that formula (3.11) is valid for any $s\in\mathbb{N}$, so we conclude that $\Pi_1(x,1)$ is finite. Thus the right-hand side of (3.5) is 0 when $z=1$, as wanted. □

An example from the case (c) of proposition 3.1 is the following identity (corresponding to $t=4$ and $k=1$), valid for any $s\in\mathbb{N}$

$$\begin{aligned}&\sum_{n=0}^{q}F_{4sn}(x) \qquad (3.12)\\ &=F_{4s}(x)\left(\binom{q+1}{4}_{F_s(x)}+(-1)^{s+1}L_{2s}(x)\binom{q+2}{4}_{F_s(x)}+\binom{q+3}{4}_{F_s(x)}\right).\end{aligned}$$

Remark 3.2. A natural question about proposition 3.1 is if the given conditions on t, k and s are also necessary (for expressing the sum $\sum_{n=1}^{q}F_{tsn}^{k}(x)$ as a linear combination of the s-Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m=1,2,\ldots,tk$, according to (3.3)). We believe that the answer is yes, and we think that this conjecture (together with similar conjectures in propositions 3.3, 4.1 and 4.3) can be a good topic for a future work. Nevertheless, we would like to make some comments about this point in the case of proposition 3.1. The cases where we do not have the conditions on t,k,s stated in proposition 3.1 are the following (e = even, o = odd)

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)
t	e	e	o	$\equiv 2 \bmod 4$	o	o	o
k	e	e	e	o	$\equiv 0 \bmod 4$	o	o
s	e	o	e	o	o	e	o

Thus, in order to prove the necessity of the mentioned conditions we need to show that (3.4) *does not hold* in each of these 7 cases. For example, the case $t = 1$ and $k = 3$ is included in (vi) and (vii). In this case the left-hand side of (3.4) is (for any $s \in \mathbb{N}$)

$$\sum_{i=0}^{3}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{4}{j}_{F_s(x)}F^3_{s(i-j)}(x) = -F^3_s(x)\left(2L_s(x)+1+(-1)^s\right).$$

That is, (3.4) does not hold, which means that the sum of cubes $\sum_{n=0}^{q}F^3_{sn}(x)$ can not be written as the linear combination of the s-Fibopolynomials $\binom{q+1}{3}_{F_s(x)}$, $\binom{q+2}{3}_{F_s(x)}$ and $\binom{q+3}{3}_{F_s(x)}$ proposed in (3.3). However, it is known that $\sum_{n=0}^{q}F^3_n = \frac{1}{10}\left(F_{3q+2}-6(-1)^q F_{q-1}+5\right)$ (see [1]). In fact, the case of sums of odd powers of Fibonacci and Lucas numbers has been considered for several authors (see [3], [16], [21]). It turns out that some of their nice results belong to some of the cases (i) to (vii) above, so they can not be written as in (3.3).

Let us consider now the case of sums of powers of Lucas polynomials. From (2.12) we see that

$$(-1)^{s+1}\sum_{n=0}^{q}L^k_{tsn}(x) \tag{3.13}$$
$$= \sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}L^k_{ts(i-j)}(x)\sum_{n=0}^{q}\binom{n+tk-i}{tk}_{F_s(x)},$$

which can be written as

$$\sum_{n=0}^{q}L^k_{tsn}(x) \tag{3.14}$$
$$= (-1)^{s+1}\sum_{m=1}^{tk}\sum_{i=0}^{tk-m}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}L^k_{ts(i-j)}(x)\binom{q+m}{tk}_{F_s(x)}$$
$$+ (-1)^{s+1}\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}L^k_{ts(i-j)}(x)\sum_{n=0}^{q}\binom{n}{tk}_{F_s(x)}.$$

Expression (3.14) tells us that the sum $\sum_{n=0}^{q}L^k_{tsn}(x)$ can be written as a linear combination of the s-Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m = 1, 2, \ldots, tk$, according to

$$\sum_{n=0}^{q}L^k_{tsn}(x) \tag{3.15}$$
$$= (-1)^{s+1}\sum_{m=1}^{tk}\sum_{i=0}^{tk-m}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}L^k_{ts(i-j)}(x)\binom{q+m}{tk}_{F_s(x)},$$

if and only if

$$\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}L_{ts(i-j)}^{k}(x)=0. \tag{3.16}$$

From (2.7) and (2.10) we can write

$$\begin{aligned}&\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{tk+1}{j}_{F_s(x)}L_{ts(i-j)}^{k}(x)\,z^{tk-i} \\ &=\left(\sum_{i=0}^{tk+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{tk+1}{i}_{F_s(x)}z^{tk+1-i}\right) \\ &\quad\times\sum_{l=0}^{k}\binom{k}{l}\frac{1}{z-\alpha^{lts}(x)\,\beta^{(k-l)ts}(x)}.\end{aligned} \tag{3.17}$$

We have again the factor $\Pi_2(x,z)$ considered in the Fibonacci case (see (3.7)), and the factor

$$\widetilde{\Pi}_1(x,z)=\sum_{l=0}^{k}\binom{k}{l}\frac{1}{z-\alpha^{lts}(x)\,\beta^{(k-l)ts}(x)}. \tag{3.18}$$

Plainly any of the conditions: (a) $\widetilde{\Pi}_1(x,1)=0$, or, (b) $\widetilde{\Pi}_1(x,1)<\infty$ and $\Pi_2(x,1)=0$, imply (3.16).

Proposition 3.3. *The sum $\sum_{n=0}^{q}L_{tsn}^{k}(x)$ can be written as a linear combination of the s-Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m=1,2,\ldots,tk$, according to (3.15), in the following cases*

	t	k	s
(a)	*even*	*odd*	*even*
(b)	$\equiv 0 \bmod 4$	*odd*	*any*

Proof. In both cases we have tk even, so it is valid the factorization (3.10) for $\Pi_2(x,z)$.

(a) Suppose that t and s are even and that k is odd. In this case the factor $\left(z-(-1)^{\frac{kts}{2}}\right)$ in $\Pi_2(x,z)$ is $(z-1)$, so we have $\Pi_2(x,1)=0$. It remains to check that $\widetilde{\Pi}_1(x,1)$ is finite. In fact, by writing k as $2k-1$ and using that t and s are even, one can see that $\widetilde{\Pi}_1(x,1)=4^{k-1}$.

(b) Suppose now that $t\equiv 0 \bmod 4$, k is odd and s is any positive integer. In this case the factor $\left(z-(-1)^{\frac{kts}{2}}\right)$ in $\Pi_2(x,z)$ is again $(z-1)$, so we have $\Pi_2(x,1)=0$. With a similar calculation to the case (a), we can see that in this case we have also $\widetilde{\Pi}_1(x,1)=4^{k-1}$. □

An example from the case (b) of proposition 3.3 is the following identity (corresponding to $t = 4$ and $k = 1$), valid for any $s \in \mathbb{N}$

$$\begin{aligned}&\sum_{n=0}^{q} L_{4sn}(x) \qquad (3.19)\\&= 2\binom{q+4}{4}_{F_s(x)} + \left(-L_{4s}(x) + 2(-1)^{s+1} L_{2s}(x)\right)\binom{q+3}{4}_{F_s(x)}\\&\quad + (-1)^s \left(L_{6s}(x) + L_{2s}(x) + 2(-1)^s\right)\binom{q+2}{4}_{F_s(x)} - L_{4s}(x)\binom{q+1}{4}_{F_s(x)}.\end{aligned}$$

Examples from the cases (a) and (b) of proposition 3.1, and from the case (a) of proposition 3.3, will be given in section 4, since some variants of them work also as examples of alternating sums of powers of Fibonacci or Lucas polynomials, to be discussed in section 4 (see corollaries 4.2 and 4.4).

4. The main results (II): alternating sums

According to (2.1), (2.5), (2.7), (2.9) and (2.10), the Z transform of the alternating sequences $\left\{(-1)^n F_{tsn}^k(x)\right\}_{n=0}^{\infty}$ and $\left\{(-1)^n L_{tsn}^k(x)\right\}_{n=0}^{\infty}$ are

$$\begin{aligned}\mathcal{Z}\left((-1)^n F_{tsn}^k(x)\right) &= \frac{1}{(x^2+4)^{\frac{k}{2}}} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \frac{z}{z + \alpha^{lts}(x)\beta^{(k-l)ts}(x)} \qquad (4.1)\\&= -z \frac{\sum_{i=0}^{tk} \sum_{j=0}^{i} (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x)(-z)^{tk-i}}{\sum_{i=0}^{tk+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{tk+1}{i}_{F_s(x)} (-z)^{tk+1-i}},\end{aligned}$$

and

$$\begin{aligned}\mathcal{Z}\left((-1)^n L_{tsn}^k(x)\right) &= \sum_{l=0}^{k} \binom{k}{l} \frac{z}{z + \alpha^{lts}(x)\beta^{(k-l)ts}(x)} \qquad (4.2)\\&= -z \frac{\sum_{i=0}^{tk} \sum_{j=0}^{i} (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{tk+1}{j}_{F_s(x)} L_{ts(i-j)}^k(x)(-z)^{tk-i}}{\sum_{i=0}^{tk+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{tk+1}{i}_{F_s(x)} (-z)^{tk+1-i}}.\end{aligned}$$

By using (2.11) and (2.12) it is possible to establish expressions, for the case of alternating sums, similar to expressions (3.2) and (3.14), namely

$$\sum_{n=0}^{q}(-1)^n F_{tsn}^{k}(x) = (-1)^{s+1+tk+q} \tag{4.3}$$
$$\times \sum_{m=1}^{tk}\sum_{i=0}^{tk-m}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i+m}\binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^{k}(x)\binom{q+m}{tk}_{F_s(x)}$$
$$+(-1)^{s+1+tk}\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i}\binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^{k}(x)\sum_{n=0}^{q}(-1)^n\binom{n}{tk}_{F_s(x)},$$

and

$$\sum_{n=0}^{q}(-1)^n L_{tsn}^{k}(x) = (-1)^{s+1+tk+q} \tag{4.4}$$
$$\times \sum_{m=1}^{tk}\sum_{i=0}^{tk-m}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i+m}\binom{tk+1}{j}_{F_s(x)} L_{ts(i-j)}^{k}(x)\binom{q+m}{tk}_{F_s(x)}$$
$$+(-1)^{s+1+tk}\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i}\binom{tk+1}{j}_{F_s(x)} L_{ts(i-j)}^{k}(x)\times$$
$$\times\sum_{n=0}^{q}(-1)^n\tbinom{n}{tk}_{F_s(x)},$$

respectively. From (4.3) and (4.4) we see that the alternating sums of powers $\sum_{n=0}^{q}(-1)^n F_{tsn}^{k}(x)$ and $\sum_{n=0}^{q}(-1)^n L_{tsn}^{k}(x)$ can be written as linear combinations of the s-Fibopolynomials $\binom{q+m}{tk}_{F_s(x)}$, $m=1,2,\ldots,tk$, according to

$$\sum_{n=0}^{q}(-1)^n F_{tsn}^{k}(x) = (-1)^{s+1+tk+q} \tag{4.5}$$
$$\times \sum_{m=1}^{tk}\sum_{i=0}^{tk-m}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i+m}\binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^{k}(x)\binom{q+m}{tk}_{F_s(x)},$$

and

$$\sum_{n=0}^{q}(-1)^n L_{tsn}^{k}(x) = (-1)^{s+1+tk+q} \tag{4.6}$$
$$\times \sum_{m=1}^{tk}\sum_{i=0}^{tk-m}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i+m}\binom{tk+1}{j}_{F_s(x)} L_{ts(i-j)}^{k}(x)\binom{q+m}{tk}_{F_s(x)},$$

if and only if we have that

$$\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i}\binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^{k}(x) = 0, \tag{4.7}$$

and

$$\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i}\binom{tk+1}{j}_{F_s(x)}L_{ts(i-j)}^{k}(x)=0, \tag{4.8}$$

respectively. Observe that, according to (4.1) and (4.2), we have

$$\begin{aligned}&\left(x^2+4\right)^{\frac{k}{2}}\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i}\binom{tk+1}{j}_{F_s(x)}F_{ts(i-j)}^{k}(x)\,z^{tk-i}\\&=\left(\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\frac{1}{z+\alpha^{lts}(x)\,\beta^{(k-l)ts}(x)}\right)\\&\quad\times\left(\sum_{i=0}^{tk+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}+i}\binom{tk+1}{i}_{F_s(x)}z^{tk+1-i}\right),\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}&\sum_{i=0}^{tk}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}+i}\binom{tk+1}{j}_{F_s(x)}L_{ts(i-j)}^{k}(x)\,z^{tk-i}\\&=\left(\sum_{l=0}^{k}\binom{k}{l}\frac{1}{z+\alpha^{lts}(x)\,\beta^{(k-l)ts}(x)}\right)\\&\quad\times\left(\sum_{i=0}^{tk+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}+i}\binom{tk+1}{i}_{F_s(x)}z^{tk+1-i}\right),\end{aligned} \tag{4.10}$$

respectively. Then we need to consider now the following factors

$$\Omega_1(x,z)=\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\frac{1}{z+\alpha^{lts}(x)\,\beta^{(k-l)ts}(x)}, \tag{4.11}$$

$$\widetilde{\Omega}_1(x,z)=\sum_{l=0}^{k}\binom{k}{l}\frac{1}{z+\alpha^{lts}(x)\,\beta^{(k-l)ts}(x)}, \tag{4.12}$$

and

$$\Omega_2(x,z)=\sum_{i=0}^{tk+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}+i}\binom{tk+1}{i}_{F_s(x)}z^{tk+1-i}. \tag{4.13}$$

Plainly (4.7) is concluded from any of the conditions: (a) $\Omega_1(x,1)=0$, or, (b) $\Omega_1(x,1)<\infty$ and $\Omega_2(x,1)=0$, and (4.8) is concluded from any of the conditions: (a) $\widetilde{\Omega}_1(x,1)=0$, or, (b) $\widetilde{\Omega}_1(x,1)<\infty$ and $\Omega_2(x,1)=0$. In this section we give conditions on the parameters t, k and s, that imply (4.7) (for the Fibonacci case: proposition 4.1), and that imply (4.8) (for the Lucas case: proposition 4.3).

In the Fibonacci case we have the following result.

Proposition 4.1. *The alternating sum* $\sum_{n=0}^{q}(-1)^n F_{tsn}^k(x)$ *can be written as a linear combination of the s-Fibopolynomials* $\binom{q+m}{tk}_{F_s(x)}$, $m=1,2,\ldots,tk$, *according to (4.5), in the following cases*

	t	k	s
(a)	any	$\equiv 0 \bmod 4$	any
(b)	any	even	even
(c)	$\equiv 2 \bmod 4$	any	odd
(d)	even	even	any

Proof. Observe that in all the four cases we have tk even. Thus, according to (2.14) (with z replaced by $-z$) we can factor $\Omega_2(x,z)$ as

$$\Omega_2(x,z)=(-1)^s\left(z+(-1)^{\frac{tsk}{2}}\right)\prod_{j=0}^{\frac{tk}{2}-1}\left(z^2+(-1)^{sj}L_{2s\left(\frac{tk}{2}-j\right)}(x)\,z+1\right). \quad (4.14)$$

(a) Suppose that $k\equiv 0 \bmod 4$ and that t and s are any positive integers. In this case the factor $\left(z+(-1)^{\frac{tsk}{2}}\right)$ of (4.14) is $z+1$, so we have $\Omega_2(x,1)\neq 0$. However, by setting $z=1$ in (4.11), with k replaced by $4k$, we get

$$\Omega_1(x,1)=\sum_{l=0}^{2k-1}\binom{4k}{l}(-1)^l+\frac{1}{2}\binom{4k}{2k}=0.$$

Thus (4.7) holds, as wanted.

(b) Suppose that k and s are even, and that t is any positive integer. In this case we have $z+(-1)^{\frac{tsk}{2}}=z+1$, and then $\Omega_2(x,1)\neq 0$. By setting $z=1$ in (4.11), with k and s substituted by $2k$ and $2s$, respectively, we get

$$\begin{aligned}\Omega_1(x,1)&=\sum_{l=0}^{2k}\binom{2k}{l}(-1)^l\frac{1}{1+\alpha^{2lts}(x)\,\beta^{(2k-l)2ts}(x)}\\&=\sum_{l=0}^{k-1}\binom{2k}{l}(-1)^l+\frac{1}{2}\binom{2k}{k}(-1)^k=0.\end{aligned}$$

Thus (4.7) holds, as wanted.

(c) Suppose that s is odd, $t\equiv 2 \bmod 4$, and k is any positive integer. If in (4.14) we set $z=1$ and replace t by $2(2t-1)$, we obtain that

$$\Omega_2(x,1)=\left(1+(-1)^k\right)\prod_{j=0}^{(2t-1)k-1}\left((-1)^jL_{2s((2t-1)k-j)}(x)+2\right). \quad (4.15)$$

We consider two sub-cases:

(c1) Suppose that k is even. In this case we have $\Omega_2(x,1) \neq 0$. But if we set $z = 1$ in (4.11), replace k by $2k$, and use that $t \equiv 2 \bmod 4$ and that s is odd, we obtain that

$$\Omega_1(x,1) = \sum_{l=0}^{k-1} \binom{2k}{l} (-1)^l + \frac{1}{2}\binom{2k}{k}(-1)^k = 0.$$

Thus (4.7) holds when k is even.

(c2) Suppose that k is odd. In this case we have clearly that $\Omega_2(x,1) = 0$. We check that $\Omega_1(x,1)$ is finite. If we set $z = 1$ in (4.11), substitute k by $2k-1$, and use that $t \equiv 2 \bmod 4$ and that s is odd, we obtain that

$$\Omega_1(x,1) = \sqrt{x^2+4}\sum_{l=0}^{k-1} \binom{2k-1}{l} (-1)^{l+1} \frac{F_{(2k-1-2l)ts}(x)}{2 + L_{(2k-1-2l)ts}(x)}.$$

Then we have $\Omega_1(x,1) < \infty$, as wanted. That is, expression (4.7) holds when k is odd.

(d) Suppose that k and t are even and s is any positive integer. In this case the factor $\left(z + (-1)^{\frac{tsk}{2}}\right)$ of (4.14) is $(z+1)$, so we have $\Omega_2(x,1) \neq 0$. Observe that, replacing k and t by $2k$ and $2t$, respectively (and letting s be any natural number) we obtain the same expression for $\Omega_1(x,1)$ of the case (b), namely

$$\Omega_1(x,1) = \sum_{l=0}^{2k} \binom{2k}{l} (-1)^l \frac{1}{1 + \alpha^{2lts}(x)\,\beta^{(2k-l)2ts}(x)},$$

which is equal to 0. That is, in this case we have also that $\Omega_1(x,1) = 0$, and we conclude that (4.7) holds. □

Corollary 4.2. *(a) If t is odd and $k \equiv 2 \bmod 4$, we have the following identity valid for any $s \in \mathbb{N}$*

$$\sum_{n=0}^{q} (-1)^{(s+1)n} F_{tsn}^k(x) = (-1)^{(s+1)(tk+q)} \tag{4.16}$$
$$\times \sum_{m=1}^{tk}\sum_{i=0}^{tk-m}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}+(s+1)(i+m)} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x) \binom{q+m}{tk}_{F_s(x)}.$$

(b) If $t \equiv 2 \bmod 4$ and k is odd, we have the following identity valid for any $s \in \mathbb{N}$

$$\sum_{n=0}^{q} (-1)^{sn} F_{tsn}^k(x) = (-1)^{s(1+tk+q)+1} \tag{4.17}$$
$$\times \sum_{m=1}^{tk}\sum_{i=0}^{tk-m}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}+s(i+m)} \binom{tk+1}{j}_{F_s(x)} F_{ts(i-j)}^k(x) \binom{q+m}{tk}_{F_s(x)}.$$

Proof. (a) When t is odd and $k \equiv 2 \bmod 4$, formula (4.16) with s odd gives the result (3.3) of case (b) of proposition 3.1 (which is valid for t odd, $k \equiv 2 \bmod 4$ and s odd). Similarly, for t odd and $k \equiv 2 \bmod 4$, formula (4.16) with s even, gives the result (4.5) of case (b) of proposition 4.1 (which is valid for k and s even and any t).

(b) When $t \equiv 2 \bmod 4$ and k is odd, formula (4.17) with s even, gives the result (3.3) of case (a) of proposition 3.1 (which is valid for t even, k odd and s even). Similarly, for $t \equiv 2 \bmod 4$ and k odd, formula (4.17) with s odd, gives the result (4.5) of case (c) of proposition 4.1 (which is valid for $t \equiv 2 \bmod 4$, s odd and any k). □

We give examples from the cases considered in corollary 4.2. Beginning with the case (a), by setting $t = 1$ and $k = 2$ in (4.16), we have the following identity, valid for $s \in \mathbb{N}$

$$\sum_{n=0}^{q} (-1)^{(s+1)(n+q)} F_{sn}^{2}(x) = F_{s}^{2}(x) \binom{q+1}{2}_{F_s(x)}. \tag{4.18}$$

The case $t = s = x = 1$ and $k = 6$ of (4.16) is

$$\sum_{n=0}^{q} F_n^6 = \binom{q+1}{6}_F + \binom{q+5}{6}_F - 11\left(\binom{q+2}{6}_F + \binom{q+4}{6}_F\right) - 64\binom{q+3}{6}_F.$$

This identity is mentioned in [17] (p. 259), and previously was obtained in [15] with the much more simple right-hand side $\frac{1}{4}\left(F_q^5 F_{q+3} + F_{2q}\right)$.

An example from the case (b) of corollary 4.2 is the following identity, valid for $s \in \mathbb{N}$ (obtained by setting $t = 2$ and $k = 1$ in (4.17))

$$\sum_{n=0}^{q} (-1)^{s(n+q)} F_{2sn}(x) = F_{2s}(x) \binom{q+1}{2}_{F_s(x)}. \tag{4.19}$$

From (4.18) and (4.19) we see that

$$\begin{aligned} &(-1)^{(s+1)q} L_s(x) \sum_{n=0}^{q} (-1)^{(s+1)n} F_{sn}^2(x) \\ &= (-1)^{sq} F_s(x) \sum_{n=0}^{q} (-1)^{sn} F_{2sn}(x) = F_{s(q+1)}(x) F_{sq}(x). \end{aligned} \tag{4.20}$$

The simplest example from the case (a) of proposition 4.1, corresponding to $k = 4$ and $t = 1$, is the following identity valid for any $s \in \mathbb{N}$

$$\begin{aligned} &\sum_{n=0}^{q} (-1)^{n+q} F_{sn}^4(x) \\ &= F_s^4(x)\left(\binom{q+1}{4}_{F_s(x)} + \binom{q+3}{4}_{F_s(x)} + \left(3(-1)^s L_{2s}(x) + 4\right)\binom{q+2}{4}_{F_s(x)}\right). \end{aligned} \tag{4.21}$$

With some patience one can see that the case $x = 1$ of (4.21) is

$$\sum_{n=0}^{q} (-1)^n F_{sn}^4 = \frac{(-1)^q F_{sq} F_{s(q+1)} \left(L_s L_{sq} L_{s(q+1)} - 4L_{2s}\right)}{5L_s L_{2s}},$$

demonstrated by Melham [13].

An example from the case (d) of proposition 4.1, corresponding to $t = k = 2$, is the following identity valid for any $s \in \mathbb{N}$

$$\sum_{n=0}^{q} (-1)^{n+q} F_{2sn}^2 (x) \tag{4.22}$$
$$= F_{2s}^2 (x) \left(\binom{q+1}{4}_{F_s(x)} + (-1)^{s+1} L_{2s} (x) \binom{q+2}{4}_{F_s(x)} + \binom{q+3}{4}_{F_s(x)} \right).$$

Now we consider alternating sums of powers of Lucas polynomials.

Proposition 4.3. *The alternating sum* $\sum_{n=0}^{q} (-1)^n L_{tsn}^k (x)$ *can be written as a linear combination of the s-Fibopolynomials* $\binom{q+m}{tk}_{F_s(x)}$, $m = 1, 2, \ldots, tk$, *according to (4.6), if* s *and* k *are odd positive integers and* $t \equiv 2 \bmod 4$.

Proof. We will show that in the case stated in the proposition we have $\widetilde{\Omega}_1 (x, 1) < \infty$ and $\Omega_2 (x, 1) = 0$, which implies (4.8). Since s and k are odd, and $t \equiv 2 \bmod 4$, the factor $\left(z + (-1)^{\frac{tsk}{2}}\right)$ of (4.14) is $(z - 1)$, so we have $\Omega_2 (x, 1) = 0$. Let us see that $\widetilde{\Omega}_1 (x, 1) < \infty$. If in (4.12) we set $z = 1$ and replace k by $2k - 1$, we get for $t \equiv 2 \bmod 4$ and s odd that $\widetilde{\Omega}_1 (x, 1) = 4^{k-1}$, as wanted. □

Corollary 4.4. *If* $t \equiv 2 \bmod 4$ *and* k *is odd, we have the following identity valid for any* $s \in \mathbb{N}$

$$\sum_{n=0}^{q} (-1)^{sn} L_{tsn}^k (x) = (-1)^{s(1+tk+q)+1} \tag{4.23}$$
$$\times \sum_{m=1}^{tk} \sum_{i=0}^{tk-m} \sum_{j=0}^{i} (-1)^{\frac{(sj+2(s+1))(j+1)}{2} + s(i+m)} \binom{tk+1}{j}_{F_s(x)} L_{ts(i-j)}^k (x) \binom{q+m}{tk}_{F_s(x)}.$$

Proof. When $t \equiv 2 \bmod 4$ and k is odd, formula (4.23) with s even, gives the result (3.15) of case (a) of Proposition 3.3 (which is valid for t even, k odd and s even). Similarly, if $t \equiv 2 \bmod 4$ and k is odd, formula (4.23) with s odd, gives the result (4.6) of Proposition 4.3 (which is valid for $t \equiv 2 \bmod 4$, k odd and s odd). □

An example of (4.23) is the following identity (corresponding to $t = 2$ and $k = 1$), valid for any $s \in \mathbb{N}$

$$\sum_{n=0}^{q} (-1)^{s(n+q)} L_{2sn} (x) = 2\binom{q+2}{2}_{F_s(x)} - L_{2s} (x) \binom{q+1}{2}_{F_s(x)}, \tag{4.24}$$

which can be written as

$$\sum_{n=0}^{q}(-1)^{s(n+q)}L_{2sn}(x)=\frac{1}{F_s(x)}L_{sq}(x)F_{s(q+1)}(x). \tag{4.25}$$

5. Further results

To end this work we want to present (in two propositions) some examples of identities obtained as derivatives of some of our previous results. We will use the identities

$$\frac{d}{dx}F_n(x)=\frac{1}{x^2+4}\left(nL_n(x)-xF_n(x)\right). \tag{5.1}$$

$$\frac{d}{dx}L_n(x)=nF_n(x). \tag{5.2}$$

One can see that these formulas are true by checking that both sides of each one have the same Z transform. By using (2.2) and (2.8) we see that the Z transform of both sides of (5.1) is $z^2\left(z^2-xz-1\right)^{-2}$, and that the Z transform of both sides of (5.2) is $z\left(z^2+1\right)\left(z^2-xz-1\right)^{-2}$.

Proposition 5.1. *The following identities hold*

$$\begin{aligned}&(-1)^{(s+1)q}2L_s(x)\sum_{n=0}^{q}(-1)^{(s+1)n}nF_{2sn}(x)\\&=2qF_{s(2q+1)}(x)+F_{sq}(x)L_{s(q+1)}(x)-\frac{\left(x^2+4\right)F_s(x)}{L_s(x)}F_{s(q+1)}(x)F_{sq}(x).\end{aligned} \tag{5.3}$$

$$\begin{aligned}&(-1)^{sq}2F_s(x)\sum_{n=0}^{q}(-1)^{sn}nL_{2sn}(x)\\&=2qF_{s(2q+1)}(x)+F_{sq}(x)L_{s(q+1)}(x)-\frac{L_s(x)}{F_s(x)}F_{s(q+1)}(x)F_{sq}(x).\end{aligned} \tag{5.4}$$

Proof. We will use (4.20), which contains two identities, namely

$$(-1)^{(s+1)q}L_s(x)\sum_{n=0}^{q}(-1)^{(s+1)n}F_{sn}^2(x)=F_{s(q+1)}(x)F_{sq}(x), \tag{5.5}$$

and

$$(-1)^{sq}F_s(x)\sum_{n=0}^{q}(-1)^{sn}F_{2sn}(x)=F_{s(q+1)}(x)F_{sq}(x). \tag{5.6}$$

By using that $L_{sq}(x) F_{s(q+1)}(x) + F_{sq}(x) L_{s(q+1)}(x) = 2F_{s(2q+1)}(x)$ (see (1.6)), we can see that

$$\begin{aligned} &\frac{d}{dx}\left(F_{s(q+1)}(x) F_{sq}(x)\right) \\ &= \frac{1}{x^2+4}\left(2sqF_{s(2q+1)}(x) + sF_{sq}(x) L_{s(q+1)}(x) - 2xF_{s(q+1)}(x) F_{sq}(x)\right). \end{aligned}$$

The derivative of the left-hand side of (5.5) is

$$\begin{aligned} &\frac{d}{dx}\left((-1)^{(s+1)q} L_s(x) \sum_{n=0}^{q} (-1)^{(s+1)n} F_{sn}^2(x)\right) \\ &= (-1)^{(s+1)q} L_s(x) \sum_{n=0}^{q} \frac{(-1)^{(s+1)n}}{x^2+4} 2snF_{2sn}(x) \\ &\quad + \left(sF_s(x) - \frac{2xL_s(x)}{x^2+4}\right) \frac{1}{L_s(x)} F_{s(q+1)}(x) F_{sq}(x). \end{aligned}$$

Then, the derivative of (5.5) is

$$\begin{aligned} &(-1)^{(s+1)q} L_s(x) \sum_{n=0}^{q} \frac{(-1)^{(s+1)n}}{x^2+4} 2snF_{2sn}(x) \\ &\quad + \left(sF_s(x) - \frac{2xL_s(x)}{x^2+4}\right) \frac{1}{L_s(x)} F_{s(q+1)}(x) F_{sq}(x) \\ &= \frac{1}{x^2+4}\left(2sqF_{s(2q+1)}(x) + sF_{sq}(x) L_{s(q+1)}(x) - 2xF_{s(q+1)}(x) F_{sq}(x)\right) \end{aligned}$$

from where (5.3) follows.

The derivative of the left-hand side of (5.6) is

$$\begin{aligned} &\frac{d}{dx}\left((-1)^{sq} F_s(x) \sum_{n=0}^{q} (-1)^{sn} F_{2sn}(x)\right) \\ &= \frac{(-1)^{sq}}{x^2+4} F_s(x) \sum_{n=0}^{q} (-1)^{sn} 2snL_{2sn} + \frac{sL_s(x) - 2xF_s(x)}{F_s(x)(x^2+4)} F_{s(q+1)}(x) F_{sq}(x). \end{aligned}$$

Thus, the derivative of (5.6) is

$$\begin{aligned} &\frac{(-1)^{sq}}{x^2+4} F_s(x) \sum_{n=0}^{q} (-1)^{sn} 2snL_{2sn} + \frac{sL_s(x) - 2xF_s(x)}{F_s(x)(x^2+4)} F_{s(q+1)}(x) F_{sq}(x) \\ &= \frac{1}{x^2+4}\left(2sqF_{s(2q+1)}(x) + sF_{sq}(x) L_{s(q+1)}(x) - 2xF_{s(q+1)}(x) F_{sq}(x)\right), \end{aligned}$$

from where (5.4) follows. □

Proposition 5.2. *The following identity holds*

$$\begin{aligned}&(-1)^{sq} F_s(x) \sum_{n=0}^{q} (-1)^{sn} n F_{2sn}(x) \\ &= \frac{1}{x^2+4}\left(\frac{(-1)^{s+1}}{F_s(x)} F_{2sq}(x) + q L_{s(2q+1)}(x)\right).\end{aligned} \tag{5.7}$$

Proof. Identity (5.7) is the derivative of (4.25), together with

$$F_s(x) L_{s(q+1)}(x) - L_s(x) F_{s(q+1)}(x) = 2(-1)^{s+1} F_{sq}(x),$$

and

$$\left(x^2+4\right) F_{sq}(x) F_{s(q+1)}(x) + L_{sq}(x) L_{s(q+1)}(x) = 2L_{s(2q+1)}(x).$$

(See (1.6) and (1.7).) We leave the details of the calculations to the reader. □

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Some relationships between poly-Cauchy numbers and poly-Bernoulli numbers*

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Abstract

In this paper, we show some relationships between poly-Cauchy numbers introduced by T. Komatsu and poly-Bernoulli numbers introduced by M. Kaneko.

Keywords: Bernoulli numbers; Cauchy numbers; poly-Bernoulli numbers; poly-Cauchy numbers

MSC: 05A15, 11B75

1. Introduction

Let n and k be positive integers. Poly-Cauchy numbers of the first kind $c_n^{(k)}$ are defined by

$$c_n^{(k)} = \underbrace{\int_0^1 \dots \int_0^1}_{k} (x_1 x_2 \dots x_k)(x_1 x_2 \dots x_k - 1) \dots (x_1 x_2 \dots x_k - n + 1) dx_1 dx_2 \dots dx_k$$

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(see in [7]). The concept of poly-Cauchy numbers is a generalization of that of the classical Cauchy numbers $c_n = c_n^{(1)}$ defined by

$$c_n = \int_0^1 x(x-1)\dots(x-n+1)dx$$

(see e.g. [2, 8]). The generating function of poly-Cauchy numbers ([7, Theorem 2]) is given by

$$\mathrm{Lif}_k(\ln(1+x)) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!}\,,$$

where

$$\mathrm{Lif}_k(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+1)^k}$$

is the k-th polylogarithm factorial function. An explicit formula for $c_n^{(k)}$ ([7, Theorem 1]) is given by

$$c_n^{(k)} = (-1)^n \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-1)^m}{(m+1)^k} \quad (n \ge 0, k \ge 1)\,, \tag{1.1}$$

where $\left[{n \atop m} \right]$ are the (unsigned) Stirling numbers of the first kind, arising as coefficients of the rising factorial

$$x(x+1)\dots(x+n-1) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} x^m$$

(see e.g. [4]).

On the other hand, M. Kaneko ([6]) introduced the poly-Bernoulli numbers $B_n^{(k)}$ by

$$\frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!}\,,$$

where

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}$$

is the k-th polylogarithm function. When $k = 1$, $B_n = B_n^{(1)}$ is the classical Bernoulli number with $B_1^{(1)} = 1/2$, defined by the generating function

$$\frac{xe^x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}\,.$$

An explicit formula for $B_n^{(k)}$ ([6, Theorem 1]) is given by

$$B_n^{(k)} = (-1)^n \sum_{m=0}^{n} \left\{ {n \atop m} \right\} \frac{(-1)^m m!}{(m+1)^k} \quad (n \ge 0, k \ge 1)\,, \tag{1.2}$$

where $\left\{ {n \atop m} \right\}$ are the Stirling numbers of the second kind, determined by

$$\left\{ {n \atop m} \right\} = \frac{1}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (m-j)^n$$

(see e.g. [4]).

In this paper, we show some relationships between poly-Cauchy numbers and poly-Bernoulli numbers.

2. Main result

Poly-Bernoulli numbers can be expressed by poly-Cauchy numbers ([7, Theorem 8]).

Theorem 2.1. *For $n \ge 1$ we have*

$$B_n^{(k)} = \sum_{l=1}^{n} \sum_{m=1}^{n} m! \left\{ {n \atop m} \right\} \left\{ {m-1 \atop l-1} \right\} c_l^{(k)}\,.$$

On the other hand,

$$\begin{aligned}
c_2^{(k)} &= \frac{1}{2!} B_2^{(k)} + \frac{3}{2} B_1^{(k)} \\
&= \frac{1}{2!} (B_2^{(k)} + 3B_1^{(k)})\,, \\
c_3^{(k)} &= \frac{1}{3!} B_3^{(k)} + 2B_2^{(k)} + \frac{23}{6} B_1^{(k)} \\
&= \frac{1}{3!} (B_3^{(k)} + 12B_2^{(k)} + 23B_1^{(k)})\,, \\
c_4^{(k)} &= \frac{1}{4!} B_4^{(k)} + \frac{5}{4} B_3^{(k)} + \frac{215}{24} B_2^{(k)} + \frac{55}{4} B_1^{(k)} \\
&= \frac{1}{4!} (B_4^{(k)} + 30B_3^{(k)} + 215B_2^{(k)} + 330B_1^{(k)})\,, \\
c_5^{(k)} &= \frac{1}{5!} B_5^{(k)} + \frac{1}{2} B_4^{(k)} + \frac{207}{24} B_3^{(k)} + \frac{95}{2} B_2^{(k)} + \frac{1901}{30} B_1^{(k)} \\
&= \frac{1}{5!} (B_5^{(k)} + 60B_4^{(k)} + 1035B_3^{(k)} + 5700B_2^{(k)} + 7604B_1^{(k)})\,, \\
c_6^{(k)} &= \frac{1}{6!} B_6^{(k)} + \frac{7}{48} B_5^{(k)} + \frac{707}{144} B_4^{(k)} + \frac{1015}{16} B_3^{(k)} + \frac{13279}{45} B_2^{(k)} + \frac{4277}{12} B_1^{(k)}
\end{aligned}$$

$$= \frac{1}{6!}\left(B_6^{(k)} + 105B_5^{(k)} + 3535B_4^{(k)} + 45675B_3^{(k)} + 212464B_2^{(k)} + 256620B_1^{(k)}\right).$$

In general, we have the following identity, expressing poly-Cauchy numbers $c_n^{(k)}$ by using poly-Bernoulli numbers $B_n^{(k)}$.

Theorem 2.2. *For $n \geq 1$ we have*

$$c_n^{(k)} = (-1)^n \sum_{l=1}^{n} \sum_{m=1}^{n} \frac{(-1)^m}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ l \end{bmatrix} B_l^{(k)}.$$

Proof. By (1.1) and (1.2), we have

$$\begin{aligned}
\text{RHS} &= (-1)^n \sum_{l=1}^{n} \sum_{m=1}^{n} \frac{(-1)^m}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ l \end{bmatrix} (-1)^l \sum_{i=0}^{l} \begin{Bmatrix} l \\ i \end{Bmatrix} \frac{(-1)^i i!}{(i+1)^k} \\
&= (-1)^n \sum_{m=1}^{n} \frac{(-1)^m}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \sum_{l=0}^{n} \begin{bmatrix} m \\ l \end{bmatrix} (-1)^l \sum_{i=0}^{l} \begin{Bmatrix} l \\ i \end{Bmatrix} \frac{(-1)^i i!}{(i+1)^k} \\
&= (-1)^n \sum_{m=1}^{n} \frac{(-1)^m}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \sum_{i=0}^{n} \frac{(-1)^i i!}{(i+1)^k} \sum_{l=i}^{n} (-1)^l \begin{bmatrix} m \\ l \end{bmatrix} \begin{Bmatrix} l \\ i \end{Bmatrix} \\
&= (-1)^n \sum_{m=0}^{n} \frac{(-1)^m}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-1)^m m!}{(m+1)^k} (-1)^m \\
&= (-1)^n \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-1)^m}{(m+1)^k} = \text{LHS}.
\end{aligned}$$

Note that $\left[\begin{smallmatrix} m \\ 0 \end{smallmatrix}\right] = 0$ $(m \geq 1)$ and $\left[\begin{smallmatrix} m \\ l \end{smallmatrix}\right] = 0$ $(l > m)$, and

$$\sum_{l=i}^{m} (-1)^{m-l} \begin{bmatrix} m \\ l \end{bmatrix} \begin{Bmatrix} l \\ i \end{Bmatrix} = \begin{cases} 1 & (i = m); \\ 0 & (i \neq m). \end{cases}$$

□

3. Poly-Cauchy numbers of the second kind

Poly-Cauchy numbers of the second kind $\hat{c}_n^{(k)}$ are defined by

$$\hat{c}_n^{(k)} = \underbrace{\int_0^1 \dots \int_0^1}_{k} (-x_1 x_2 \dots x_k)(-x_1 x_2 \dots x_k - 1)$$

$$\dots (-x_1 x_2 \dots x_k - n + 1) dx_1 dx_2 \dots dx_k$$

(see in [7]). If $k=1$, then $\hat{c}_n^{(1)}=\hat{c}_n$ is the classical Cauchy numbers of the second kind defined by

$$\hat{c}_n=\int_0^k(-x)(-x-1)\dots(-x-n+1)dx$$

(see e.g. [2, 8]). The generating function of poly-Cauchy numbers of the second kind ([7, Theorem 5]) is given by

$$\mathrm{Lif}_k(-\ln(1+x))=\sum_{n=0}^\infty \hat{c}_n^{(k)}\frac{x^n}{n!}\,.$$

An explicit formula for $\hat{c}_n^{(k)}$ ([7, Theorem 4]) is given by

$$\hat{c}_n^{(k)}=(-1)^n\sum_{m=0}^n\begin{bmatrix}n\\m\end{bmatrix}\frac{1}{(m+1)^k}\quad(n\ge 0,k\ge 1)\,. \tag{3.1}$$

In a similar way, we have a relationship, expressing poly-Cauchy numbers of the second kind $\hat{c}_n^{(k)}$ by using poly-Bernoulli numbers $B_n^{(k)}$. The proof is similar and omitted.

Theorem 3.1. *For $n\ge 1$ we have*

$$\hat{c}_n^{(k)}=(-1)^n\sum_{l=1}^n\sum_{m=1}^n\frac{1}{m!}\begin{bmatrix}n\\m\end{bmatrix}\begin{bmatrix}m\\l\end{bmatrix}B_l^{(k)}\,.$$

In addition, we also obtain the corresponding relationship to Theorem 2.1.

Theorem 3.2. *For $n\ge 1$ we have*

$$B_n^{(k)}=(-1)^n\sum_{l=1}^n\sum_{m=1}^n m!\left\{\begin{matrix}n\\m\end{matrix}\right\}\left\{\begin{matrix}m\\l\end{matrix}\right\}\hat{c}_l^{(k)}\,.$$

Proof. By (1.2) and (3.1), we have

$$\begin{aligned}\mathrm{RHS}&=(-1)^n\sum_{l=1}^n\sum_{m=1}^n m!\left\{\begin{matrix}n\\m\end{matrix}\right\}\left\{\begin{matrix}m\\l\end{matrix}\right\}(-1)^l\sum_{i=0}^l\begin{bmatrix}l\\i\end{bmatrix}\frac{1}{(i+1)^k}\\&=(-1)^n\sum_{m=1}^n m!\left\{\begin{matrix}n\\m\end{matrix}\right\}\sum_{l=0}^n\left\{\begin{matrix}m\\l\end{matrix}\right\}(-1)^l\sum_{i=0}^l\begin{bmatrix}l\\i\end{bmatrix}\frac{1}{(i+1)^k}\\&=(-1)^n\sum_{m=1}^n m!\left\{\begin{matrix}n\\m\end{matrix}\right\}\sum_{i=0}^n\frac{1}{(i+1)^k}\sum_{l=i}^n(-1)^l\left\{\begin{matrix}m\\l\end{matrix}\right\}\begin{bmatrix}l\\i\end{bmatrix}\end{aligned}$$

$$= (-1)^n \sum_{m=0}^{n} m! \left\{ {n \atop m} \right\} \frac{1}{(m+1)^k} (-1)^m$$
$$= (-1)^n \sum_{m=0}^{n} \left\{ {n \atop m} \right\} \frac{(-1)^m m!}{(m+1)^k} = \text{LHS}\,.$$

Note that

$$\sum_{l=i}^{m} (-1)^{m-l} \left\{ {m \atop l} \right\} \left[{l \atop i} \right] = \begin{cases} 1 & (i = m); \\ 0 & (i \neq m). \end{cases}$$ □

4. Poly-Cauchy polynomials and poly-Bernoulli polynomials

Poly-Cauchy polynomials of the first kind $c_n^{(k)}(z)$ are defined by

$$c_n^{(k)}(z) = n! \underbrace{\int_0^1 \dots \int_0^1}_{k} (x_1 x_2 \dots x_k - z)(x_1 x_2 \dots x_k - 1 - z) \cdots (x_1 x_2 \dots x_k - (n-1) - z) dx_1 dx_2 \dots dx_k,$$

and are expressed explicitly in terms of Stirling numbers of the first kind ([5, Theorem 1])

$$c_n^{(k)}(z) = \sum_{m=0}^{n} \left[{n \atop m} \right] (-1)^{n-m} \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k}\,.$$

Poly-Cauchy polynomials of the second kind $\hat{c}_n^{(k)}(z)$ are defined by

$$\hat{c}_n^{(k)}(z) = n! \underbrace{\int_0^1 \dots \int_0^1}_{k} (-x_1 x_2 \dots x_k + z)(-x_1 x_2 \dots x_k - 1 + z) \cdots (-x_1 x_2 \dots x_k - (n-1) + z) dx_1 dx_2 \dots dx_k,$$

and are expressed explicitly in terms of Stirling numbers of the first kind ([5, Theorem 4].

$$\hat{c}_n^{(k)}(z) = \sum_{m=0}^{n} \left[{n \atop m} \right] (-1)^{n} \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k}\,.$$

In 2010, Coppo and Candelpergher [3], 2011 Bayad and Hamahata [1, (1.5)] introduced the poly-Bernoulli polynomials $B_n^{(k)}(z)$ given by

$$\frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}} e^{-xz} = \sum_{n=0}^{\infty} B_n^{(k)}(z) \frac{x^n}{n!},$$

and

$$\frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}}e^{xz} = \sum_{n=0}^{\infty} B_n^{(k)}(z)\frac{x^n}{n!}\,,$$

respectively, satisfying $B_n^{(k)}(0) = B_n^{(k)}$.

If we define still different poly-Bernoulli polynomials $B_n^{(k)}$ by

$$B_n^{(k)}(z) = (-1)^n \sum_{m=0}^{n} \left\{ {n \atop m} \right\} (-1)^m m! \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k},$$

satisfying $B_n^{(k)}(0) = B_n^{(k)}$ $(n \geq 0,\ k \geq 1)$, then we have relationships between the poly-Bernoulli polynomials and poly-Cauchy polynomials similar to those between the poly-Bernoulli numbers and the poly-Cauchy numbers.

Theorem 4.1. *For $n \geq 1$ we have*

$$\begin{aligned} B_n^{(k)}(z) &= \sum_{l=1}^{n}\sum_{m=1}^{n} m! \left\{ {n \atop m} \right\} \left\{ {m-1 \atop l-1} \right\} c_l^{(k)}(z)\,, \\ &= (-1)^n \sum_{l=1}^{n}\sum_{m=1}^{n} m! \left\{ {n \atop m} \right\} \left\{ {m \atop l} \right\} \hat{c}_l^{(k)}(z)\,, \\ c_n^{(k)}(z) &= (-1)^n \sum_{l=1}^{n}\sum_{m=1}^{n} \frac{(-1)^m}{m!} \left[{n \atop m} \right] \left[{m \atop l} \right] B_l^{(k)}(z) \\ \hat{c}_n^{(k)}(z) &= (-1)^n \sum_{l=1}^{n}\sum_{m=1}^{n} \frac{1}{m!} \left[{n \atop m} \right] \left[{m \atop l} \right] B_l^{(k)}(z)\,. \end{aligned}$$

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Algebraic relations with the infinite products generated by Fibonacci numbers

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Abstract

In this paper, we establish explicit algebraic relations among infinite products including Fibonacci and Lucas numbers with subscripts in geometric progressions. The algebraic relations given in this paper are obtained by using general criteria for the algebraic dependency of such infinite products.

Keywords: Algebraic independence, Infinite products, Fibonacci numbers, Mahler functions.

MSC: 11J81, 11J85.

1. Introduction

Let α and β be real algebraic numbers with $|\alpha| > 1$ and $\alpha\beta = -1$. We define

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n \quad (n \geq 0). \tag{1.1}$$

If $\alpha = (1+\sqrt{5})/2$, we have $U_n = F_n$ and $V_n = L_n$ $(n \geq 0)$, where the sequences $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ are the Fibonacci numbers and the Lucas numbers defined, respectively, by $F_{n+2} = F_{n+1} + F_n$ $(n \geq 0)$, $F_0 = 0$, $F_1 = 1$ and by $L_{n+2} = L_{n+1} + L_n$ $(n \geq 0)$, $L_0 = 2$, $L_1 = 1$.

Throughout this paper, we adopt the following notation. Let $d \geq 2$ be a fixed integer and $\zeta_m = e^{2\pi i/m}$ a primitive m-th root of unity. For $\tau \in \mathbb{C}$ with $|\tau| = 1$, we define the set $\Omega_j(\tau) := \left\{ z \in \mathbb{C} \mid z^{d^j} = \tau \text{ or } z^{d^j} = \overline{\tau} \right\}$ for $j = 0, 1, \ldots$. Let $S_k(\tau)$ be a subset of $\Omega_k(\tau)$ such that for any $\gamma \in S_k(\tau)$ the numbers $\zeta_d \gamma$ and $\overline{\gamma}$ belong to $S_k(\tau)$, where $\overline{\gamma}$ indicates the complex conjugate of γ. Namely, $S_k(\tau)$ satisfies $S_k(\tau) = \zeta_d S_k(\tau)$ and $S_k(\tau) = \overline{S_k(\tau)}$. For example, if $d = 2$, $\tau = 1$, and $k = 3$ we have $\Omega_3(1) = \{e^{k\pi i/4} \mid 0 \leq k \leq 7\}$ and so we can choose $S_3(1) = \{\pm e^{\pi i/4}, \pm e^{3\pi i/4}\}$. We define the following sets that are determined depending only on $S_k(\tau)$:

$$\Lambda_i(\tau) = \left\{ \gamma^{d^{k-i}} \mid \gamma \in S_k(\tau) \right\} \quad (0 \leq i \leq k-1),$$

$$\Gamma_i(\tau) = \{\gamma \in \Omega_i(\tau) \mid \gamma^d \in \Lambda_{i-1}(\tau)\} \setminus \Lambda_i(\tau) \quad (1 \leq i \leq k-1).$$

Then we put

$$\mathcal{E}_k(\tau) = \left(\bigcup_{i=1}^{k-1} \Gamma_i(\tau) \right) \bigcup S_k(\tau) \tag{1.2}$$

and

$$\mathcal{F}_k(\tau) = \begin{cases} \mathcal{E}_k(\tau) \bigcup \{\tau, \overline{\tau}\} & \text{if } \tau \notin \mathcal{E}_k(\tau), \\ \mathcal{E}_k(\tau) \setminus \{\tau, \overline{\tau}\} & \text{otherwise.} \end{cases}$$

In [1] we established necessary and sufficient conditions for the infinite products generated by each of the sequences in (1.1) to be algebraically dependent over $\mathbb{Q}$ and obtained the following:

Theorem 1.1. *Let $\{U_n\}_{n \geq 0}$ be the sequence defined by* (1.1) *and d be an integer greater than* 1. *Let $a_1, \ldots, a_m$ be nonzero distinct real algebraic numbers. Then the numbers*

$$\prod_{\substack{k=0 \\ U_{d^k} \neq -a_i}}^{\infty} \left(1 + \frac{a_i}{U_{d^k}} \right) \quad (i = 1, \ldots, m)$$

are algebraically dependent if and only if d is odd and there exist distinct τ_1, $\tau_2 \in \mathbb{C}$ with $|\tau_1| = |\tau_2| = 1$ and $\mathcal{F}_{k_1}(\tau_1), \mathcal{F}_{k_2}(\tau_2)$ for some $k_1, k_2 \geq 1$ such that $\mathcal{F}_{k_1}(\tau_1) \cap \mathcal{F}_{k_2}(\tau_2) \subset \{\tau_1, \overline{\tau_1}, \tau_2, \overline{\tau_2}\}$ and $\{a_1, \ldots, a_m\}$ contains

$$-\frac{1}{\alpha - \beta}(\gamma + \overline{\gamma})$$

for all $\gamma \in (\mathcal{F}_{k_1}(\tau_1) \bigcup \mathcal{F}_{k_2}(\tau_2)) \setminus \{\pm\sqrt{-1}\}$.

Theorem 1.2. *Let $\{V_n\}_{n \geq 0}$ be the sequence defined by* (1.1) *and d be an integer greater than* 1. *Let $a_1, \ldots, a_m$ be nonzero distinct real algebraic numbers. Then the numbers*

$$\prod_{\substack{k=0 \\ V_{d^k} \neq -a_i}}^{\infty} \left(1 + \frac{a_i}{V_{d^k}} \right) \quad (i = 1, \ldots, m)$$

are algebraically dependent if and only if at least one of the following properties is satisfied:

1. $d = 2$ *and the set* $\{a_1, \dots, a_m\}$ *contains* $b_1, \dots, b_l$ $(l \geq 3)$ *with* $b_1 < -2$ *satisfying*

$$b_2 = -b_1, \quad b_j = b_{j-1}^2 - 2 \quad (j = 3, \dots, l-1), \quad b_l = -b_{l-1}^2 + 2.$$

2. $d = 2$ *and there exist* $\tau \in \mathbb{C}$ *with* $|\tau| = 1$ *and* $\mathcal{F}_k(\tau)$ *for some* $k \geq 1$ *such that* $\{a_1, \dots, a_m\}$ *contains*

$$-(\gamma + \overline{\gamma})$$

for all $\gamma \in \mathcal{F}_k(\tau) \setminus \{\pm\sqrt{-1}\}$.

3. $d \geq 4$ *is even and there exist distinct* τ_1, $\tau_2 \in \mathbb{C}$ *with* $|\tau_1| = |\tau_2| = 1$ *and* $\mathcal{F}_{k_1}(\tau_1), \mathcal{F}_{k_2}(\tau_2)$ *for some* $k_1, k_2 \geq 1$ *such that* $\mathcal{F}_{k_1}(\tau_1) \cap \mathcal{F}_{k_2}(\tau_2) \subset \{\tau_1, \overline{\tau_1}, \tau_2, \overline{\tau_2}\}$ *and* $\{a_1, \dots, a_m\}$ *contains*

$$-(\gamma + \overline{\gamma})$$

for all $\gamma \in (\mathcal{F}_{k_1}(\tau_1) \bigcup \mathcal{F}_{k_2}(\tau_2)) \setminus \{\pm\sqrt{-1}\}$.

Note that Theorems 1.1 and 1.2 above are generalizations of [2, Theorems 1 and 2], respectively.

Corollary 1.3 (cf. [3])**.** *Let* $d \geq 2$ *be a fixed integer and* $a \neq 0$ *a real algebraic number. Then the numbers*

$$\prod_{\substack{k=1 \\ U_{d^k} \neq -a}}^{\infty} \left(1 + \frac{a}{U_{d^k}}\right) \quad \text{and} \quad \prod_{\substack{k=1 \\ V_{d^k} \neq -a}}^{\infty} \left(1 + \frac{a}{V_{d^k}}\right)$$

are transcendental, except for only two algebraic numbers

$$\prod_{k=1}^{\infty} \left(1 - \frac{1}{V_{2^k}}\right) = \frac{\alpha^4 - 1}{\alpha^4 + \alpha^2 + 1}, \quad \prod_{k=1}^{\infty} \left(1 + \frac{2}{V_{2^k}}\right) = \frac{\alpha^2 + 1}{\alpha^2 - 1}. \tag{1.3}$$

Corollary 1.4. *Let* a *be a nonzero real algebraic number with* $a \neq -V_{2^k} - 2$ $(k \geq 1)$. *Then the number*

$$\prod_{k=1}^{\infty} \left(1 + \frac{a}{V_{2^k} + 2}\right)$$

is transcendental, except when $a = -3, -2$*; indeed*

$$\prod_{k=1}^{\infty} \left(1 - \frac{2}{V_{2^k} + 2}\right) = \frac{\alpha^2 - 1}{\alpha^2 + 1}, \quad \prod_{k=1}^{\infty} \left(1 - \frac{3}{V_{2^k} + 2}\right) = \frac{(\alpha^2 - 1)^2}{\alpha^4 + \alpha^2 + 1}. \tag{1.4}$$

Proof. Using the equality

$$1+\frac{a}{V_{2^k}+2}=\left(1+\frac{a+2}{V_{2^k}}\right)\left(1+\frac{2}{V_{2^k}}\right)^{-1}$$

and the second equality in (1.3), we have

$$\prod_{k=1}^{\infty}\left(1+\frac{a}{V_{2^k}+2}\right)=\frac{\alpha^2-1}{\alpha^2+1}\prod_{k=1}^{\infty}\left(1+\frac{a+2}{V_{2^k}}\right). \tag{1.5}$$

By Corollary 1.3 we see that the infinite product in the right-hand side of (1.5) is algebraic only if $a=-3,-2$. The equalities (1.4) follow immediately from (1.5) with (1.3). □

Applying Corollary 1.4 with $\alpha=(1+\sqrt{5})/2$, we obtain the transcendence of

$$\prod_{k=1}^{\infty}\left(1+\frac{a}{L_{2^k}+2}\right)$$

for any nonzero algebraic number $a\neq-3,-2,-L_{2^k}-2$ $(k\geq 1)$, and the equalities

$$\prod_{k=1}^{\infty}\left(1-\frac{2}{L_{2^k}+2}\right)=\frac{1}{\sqrt{5}},\quad \prod_{k=1}^{\infty}\left(1-\frac{3}{L_{2^k}+2}\right)=\frac{1}{4}. \tag{1.6}$$

It should be noted that Corollaries 1.3 and 1.4 hold even if the number a is a nonzero complex algebraic number (see [3]).

2. Algebraic dependence relations

Theorems 1.1 and 1.2 in the introduction are useful to obtain the explicit algebraic dependence relations among the infinite products generated by the Fibonacci and Lucas numbers as well as their transcendence degrees. We exhibit such examples in this section and their proofs in the next section.

Example 2.1. Let a be a nonzero real algebraic number. The transcendental numbers

$$s_1=\prod_{\substack{k=0\\ F_{3^k}\neq-a}}^{\infty}\left(1+\frac{a}{F_{3^k}}\right),\quad s_2=\prod_{\substack{k=0\\ F_{3^k}\neq a}}^{\infty}\left(1-\frac{a}{F_{3^k}}\right)$$

are algebraically dependent if and only if $a=\pm1/\sqrt{5}$. If $a=1/\sqrt{5}$, then

$$s_1s_2^{-1}=2+\sqrt{5}.$$

Example 2.2. The transcendental numbers

$$s_1 = \prod_{k=0}^{\infty}\left(1+\frac{a_1}{F_{5^k}}\right),\ s_2 = \prod_{k=0}^{\infty}\left(1+\frac{a_2}{F_{5^k}}\right),$$
$$s_3 = \prod_{k=0}^{\infty}\left(1-\frac{a_1}{F_{5^k}}\right),\ s_4 = \prod_{k=0}^{\infty}\left(1-\frac{a_2}{F_{5^k}}\right)$$

with $a_1 = (-5+\sqrt{5})/10,\ a_2 = (5+\sqrt{5})/10$ satisfy

$$s_1 s_2 s_3^{-1} s_4^{-1} = 2+\sqrt{5},$$

while $\text{trans.}\deg_{\mathbb{Q}} \mathbb{Q}(s_1,s_2,s_3,s_4) = 3$.

Remark 2.3. The infinite products $\prod_{k=0}^{\infty}(1+a_i/F_{d^k})$ for odd d and $\prod_{k=1}^{\infty}(1+a_i/L_{d^k})$ for even d are easily expressed as the values at an algebraic number of $\Phi_i(z)$ defined by (3.2) with $b=1$, which will be shown in (3.3) of Section 3. Hence, for simplicity, we take $k \geq 1$ in the following examples.

Example 2.4. Let $a \neq 2,-1$ be a real algebraic number. The transcendental numbers

$$s_1 = \prod_{\substack{k=1 \\ L_{2^k}\neq -a}}^{\infty}\left(1+\frac{a}{L_{2^k}}\right),\quad s_2 = \prod_{\substack{k=1 \\ L_{2^k}\neq a}}^{\infty}\left(1-\frac{a}{L_{2^k}}\right)$$

are algebraically dependent if and only if $a = \pm\sqrt{2}$. If $a=\pm\sqrt{2}$, using the relation $L_{2^k}^2 = L_{2^{k+1}}+2$ $(k\geq 1)$ and the first equality in (1.6), we have

$$s_1 s_2 = \prod_{k=2}^{\infty}\left(1-\frac{2}{L_{2^k}+2}\right) = \frac{5}{3}\cdot\frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{3}.$$

Example 2.5. The transcendental numbers

$$s_1 = \prod_{k=1}^{\infty}\left(1-\frac{\sqrt{3}}{L_{4^k}}\right), s_2 = \prod_{k=1}^{\infty}\left(1+\frac{\sqrt{3}}{L_{4^k}}\right),$$
$$s_3 = \prod_{k=1}^{\infty}\left(1-\frac{1}{L_{4^k}}\right), s_4 = \prod_{k=1}^{\infty}\left(1+\frac{2}{L_{4^k}}\right)$$

satisfy

$$s_1 s_2 s_3 s_4^{-1} = \frac{5}{8},$$

while $\text{trans.}\deg_{\mathbb{Q}} \mathbb{Q}(s_1,s_2,s_3,s_4) = 3$.

Example 2.6. The transcendental numbers

$$s_1 = \prod_{k=1}^{\infty}\left(1-\frac{1}{L_{6^k}}\right),\quad s_2 = \prod_{k=1}^{\infty}\left(1+\frac{1}{L_{6^k}}\right),\quad s_3 = \prod_{k=1}^{\infty}\left(1+\frac{2}{L_{6^k}}\right),$$

$$s_4 = \prod_{k=1}^{\infty}\left(1+\frac{\sqrt{3}}{L_{6^k}}\right), \quad s_5 = \prod_{k=1}^{\infty}\left(1-\frac{\sqrt{3}}{L_{6^k}}\right)$$

satisfy

$$s_1 s_2 s_3 s_4^{-1} s_5^{-1} = \frac{\sqrt{5}}{2},$$

while $\operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}(s_1, s_2, s_3, s_4, s_5) = 4$.

Example 2.7. The transcendental numbers

$$s_i = \prod_{k=1}^{\infty}\left(1+\frac{a_i}{L_{4^k}}\right) \quad (i = 1, \dots, 8),$$

where

$$a_1 = -(\zeta_{16}^{1} + \zeta_{16}^{15}), \quad a_2 = -(\zeta_{16}^{5} + \zeta_{16}^{11}), \quad a_3 = -(\zeta_{16}^{7} + \zeta_{16}^{9}), \quad a_4 = -(\zeta_{64}^{3} + \zeta_{64}^{61}),$$
$$a_5 = -(\zeta_{64}^{13} + \zeta_{64}^{51}), \quad a_6 = -(\zeta_{64}^{19} + \zeta_{64}^{45}), \quad a_7 = -(\zeta_{64}^{29} + \zeta_{64}^{35}), \quad a_8 = 2,$$

satisfy

$$s_1 s_2 \cdots s_7 s_8^{-2} = \frac{25}{7(7-\sqrt{2-\sqrt{2}})}.$$

Example 2.8. The transcendental numbers

$$s_i = \prod_{k=1}^{\infty}\left(1+\frac{a_i}{L_{4^k}}\right) \quad (i = 1, \dots, 10),$$

where

$$a_1 = -\frac{3}{2}, \quad a_2 = \frac{\sqrt{7}}{2}, \quad a_3 = \frac{3}{2}, \quad a_4 = -\frac{\sqrt{7}}{2}, \quad a_5 = \frac{31}{16},$$
$$a_6 = -\frac{4}{\sqrt{5}}, \quad a_7 = \frac{2}{\sqrt{5}}, \quad a_8 = \frac{4}{\sqrt{5}}, \quad a_9 = -\frac{2}{\sqrt{5}}, \quad a_{10} = \frac{14}{25},$$

satisfy

$$s_1 s_2 s_3 s_4 s_5^{-1} s_6^{-1} s_7^{-1} s_8^{-1} s_9^{-1} s_{10} = \frac{3024}{3575},$$

while $\operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}(s_1, s_2, \dots, s_{10}) = 9$.

3. Proofs of the examples

Let $\{R_n\}_{n\geq 0}$ be the sequence $\{U_n\}_{n\geq 0}$ or $\{V_n\}_{n\geq 0}$ defined by (1.1). Let $d \geq 2$ be a fixed integer and $a_1, \dots, a_m$ nonzero real algebraic numbers. Define

$$(p_i, b) := \begin{cases} ((\alpha-\beta)a_i, -(-1)^d) & \textit{if } R_n = U_n, \\ (a_i, (-1)^d) & \textit{if } R_n = V_n, \end{cases} \tag{3.1}$$

and

$$\Phi_i(z) := \prod_{k=0}^{\infty}\left(1+\frac{p_i z^{d^k}}{1+bz^{2d^k}}\right) \quad (i=1,\dots,m). \tag{3.2}$$

Taking an integer $N \geq 1$ such that $|R_{d^k}| > \max\{|a_1|,\dots,|a_m|\}$ for all $k \geq N$, we have

$$\begin{aligned}\Phi_i(\alpha^{-d^N}) &= \prod_{k=N}^{\infty}\left(1+\frac{p_i\alpha^{-d^k}}{1+b\alpha^{-2d^k}}\right)\\ &= \prod_{k=N}^{\infty}\left(1+\frac{p_i}{\alpha^{d^k}+b(-1)^{d^k}\beta^{d^k}}\right) = \prod_{k=N}^{\infty}\left(1+\frac{a_i}{R_{d^k}}\right) \quad (i=1,\dots,m),\end{aligned}$$

so that

$$\prod_{\substack{k=0\\ R_{d^k}\neq -a_i}}^{\infty}\left(1+\frac{a_i}{R_{d^k}}\right) = \Phi_i(\alpha^{-d^N})\prod_{\substack{k=0\\ R_{d^k}\neq -a_i}}^{N-1}\left(1+\frac{a_i}{R_{d^k}}\right) \quad (i=1,\dots,m). \tag{3.3}$$

We note that (3.3) is valid also for $N=0$ only if d is odd and $R_{d^k} \neq -a_i$ $(k \geq 0)$.

Proof of Example 2.1. First we show that s_1 and s_2 are algebraically dependent only if $a = \pm 1/\sqrt{5}$, using the case of $m=2$ in Theorem 1.1. If s_1 and s_2 are algebraically dependent, then $\{\tau_1,\tau_2\} = \{1,-1\}$, since $\mathcal{F}_k(\tau)$ consists of at least four elements if $\tau \neq \pm 1$. If $d=3$, $m=2$, and $\{\tau_1,\tau_2\}=\{1,-1\}$, it is easily seen that $\mathcal{F}_1(\tau_1)\bigcup\mathcal{F}_1(\tau_2) = \{\zeta_3,\overline{\zeta_3},-\zeta_3,-\overline{\zeta_3}\}$ and so $\{a_1,a_2\} = \{1/\sqrt{5},-1/\sqrt{5}\}$.

Next we show the equality $s_1 s_2^{-1} = 2+\sqrt{5}$ by proving a general relation which holds for the functions $\Phi_i(z)$ $(1 \leq i \leq d-1)$ defined by (3.2), where $d \geq 3$ is an odd integer. Put

$$p_1 = -(\zeta_d+\overline{\zeta_d}),\ p_2 = -(\zeta_d^2+\overline{\zeta_d}^2),\dots,\ p_{\frac{d-1}{2}} = -(\zeta_d^{\frac{d-1}{2}}+\overline{\zeta_d}^{\frac{d-1}{2}})$$

in the equation (3.2) with $b=1$. Then we have

$$\begin{aligned}&\Phi_1(z)\cdots\Phi_{\frac{d-1}{2}}(z)\\ &= \prod_{k=0}^{\infty}\left(\frac{1}{(1+z^{2d^k})^{\frac{d-1}{2}}}\frac{1-z^{d^{k+1}}}{1-z^{d^k}}\right) = \frac{1}{1-z}\prod_{k=0}^{\infty}\frac{1}{(1+z^{2d^k})^{\frac{d-1}{2}}}.\end{aligned}$$

Moreover, putting

$$p_{\frac{d-1}{2}+1} = \zeta_d+\overline{\zeta_d},\ p_{\frac{d-1}{2}+2} = \zeta_d^2+\overline{\zeta_d}^2,\dots,\ p_{d-1} = \zeta_d^{\frac{d-1}{2}}+\overline{\zeta_d}^{\frac{d-1}{2}},$$

we get

$$\Phi_{\frac{d-1}{2}+1}(z)\cdots\Phi_{d-1}(z)$$

$$= \prod_{k=0}^{\infty} \left(\frac{1}{(1+z^{2d^k})^{\frac{d-1}{2}}} \frac{1+z^{d^{k+1}}}{1+z^{d^k}} \right) = \frac{1}{1+z} \prod_{k=0}^{\infty} \frac{1}{(1+z^{2d^k})^{\frac{d-1}{2}}}.$$

Hence, we have

$$\Phi(z) := \frac{\Phi_1(z) \cdots \Phi_{\frac{d-1}{2}}(z)}{\Phi_{\frac{d-1}{2}+1}(z) \cdots \Phi_{d-1}(z)} = \frac{1+z}{1-z}. \tag{3.4}$$

If $d = 3$, then $p_1 = -(\zeta_3 + \overline{\zeta_3}) = 1$, $p_2 = \zeta_3 + \overline{\zeta_3} = -1$, and so

$$a_1 = \frac{1}{\alpha - \beta} p_1 = \frac{1}{\sqrt{5}}, \qquad a_2 = \frac{1}{\alpha - \beta} p_2 = -\frac{1}{\sqrt{5}}$$

by (3.1). Then, by the equation (3.3) with $N = 0$, we have

$$\Phi(\alpha^{-1}) = s_1 {s_2}^{-1} = \frac{\alpha + 1}{\alpha - 1} = 2\alpha + 1 = 2 + \sqrt{5}. \qquad \square$$

Proof of Example 2.2. We consider the case of $d = 5$ in (3.4). Then

$$p_1 = -(\zeta_5 + \overline{\zeta_5}) = \frac{1 - \sqrt{5}}{2}, \quad p_2 = -(\zeta_5^2 + \overline{\zeta_5^2}) = \frac{1 + \sqrt{5}}{2},$$

$$p_3 = \zeta_5 + \overline{\zeta_5} = \frac{-1 + \sqrt{5}}{2}, \quad p_4 = \zeta_5^2 + \overline{\zeta_5^2} = -\frac{1 + \sqrt{5}}{2}.$$

By (3.1) we have

$$a_1 = \frac{-5 + \sqrt{5}}{10}, \qquad a_2 = \frac{5 + \sqrt{5}}{10}, \qquad a_3 = \frac{5 - \sqrt{5}}{10}, \qquad a_4 = -\frac{5 + \sqrt{5}}{10}.$$

Then, by the equation (3.3) with $N = 0$ and (3.4), we have

$$\Phi(\alpha^{-1}) = \frac{s_1 s_2}{s_3 s_4} = \frac{\alpha + 1}{\alpha - 1} = 2 + \sqrt{5}.$$

Finally, we prove that $\operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}(s_1, s_2, s_3, s_4) = 3$, using Theorem 1.1. Let $\tau_1 = 1$, $\tau_2 = -1$, $S_1(\tau_1) = \mathcal{E}_1(\tau_1) = \{\zeta_5, \overline{\zeta_5}, \zeta_5^2, \overline{\zeta_5^2}, 1\}$, and $S_1(\tau_2) = \mathcal{E}_1(\tau_2) = \{-\zeta_5, -\overline{\zeta_5}, -\zeta_5^2, -\overline{\zeta_5^2}, -1\}$. Then $\mathcal{F}_1(\tau_1) = \{\zeta_5, \overline{\zeta_5}, \zeta_5^2, \overline{\zeta_5^2}\}$ and $\mathcal{F}_1(\tau_2) = \{-\zeta_5, -\overline{\zeta_5}, -\zeta_5^2, -\overline{\zeta_5^2}\}$. It is enough to show that s_1, s_2, and s_3 are algebraically independent, which is equivalent to the fact that a_1, a_2, and a_3 do not satisfy Theorem 1.1 with $m = 3$. By (1.2) with $S_1(\tau_i) = \mathcal{E}_1(\tau_i)$ $(i = 1, 2)$, considering the number of the elements of $S_k(\tau_i)$ with $k \geq 2$ satisfying $S_k(\tau_i) = \zeta_5 S_k(\tau_i)$ and $S_k(\tau_i) = \overline{S_k(\tau_i)}$, we see that $\{a_1, a_2, a_3, a_4\}$ is the minimal set of $-(\gamma + \overline{\gamma})/\sqrt{5}$ with $\gamma \in \mathcal{F}_{k_1}(\tau_1) \cup \mathcal{F}_{k_2}(\tau_2) \setminus \{\pm\sqrt{-1}\}$ satisfying Theorem 1.1 with $m = 4$. $\square$

Proof of Example 2.4. First we prove directly that $s_1 s_2 = \sqrt{5}/3$ if $a = \pm\sqrt{2}$. Let $\tau = \sqrt{-1}$ and $S_1(\tau) = \mathcal{E}_1(\tau) = \{\zeta_8, \overline{\zeta_8}, -\zeta_8, -\overline{\zeta_8}\}$ in the property 2 of Theorem 1.2. Then $\mathcal{F}_1(\tau) = \{\zeta_8, \overline{\zeta_8}, -\zeta_8, -\overline{\zeta_8}, \sqrt{-1}, -\sqrt{-1}\}$. Putting

$$p_1 = -(\zeta_8 + \overline{\zeta_8}) = -\sqrt{2}, \quad p_2 = \zeta_8 + \overline{\zeta_8} = \sqrt{2}$$

in the equation (3.2) with $b = 1$, we have

$$\Phi_1(z)\Phi_2(z) = \prod_{k=0}^{\infty}(z^{2^k} - \zeta_8)(z^{2^k} - \overline{\zeta_8})(z^{2^k} + \zeta_8)(z^{2^k} + \overline{\zeta_8})\frac{1}{(1+z^{2\cdot 2^k})^2}$$

$$= \prod_{k=0}^{\infty}\frac{1+z^{2^{k+2}}}{(1+z^{2^{k+1}})^2} = \prod_{k=0}^{\infty}\frac{1+z^{2^{k+2}}}{1+z^{2^{k+1}}}\frac{1-z^{2^{k+1}}}{1-z^{2^{k+2}}} = \frac{1-z^2}{1+z^2}.$$

By the equation (3.3) with $N = 1$ and $\alpha = (1+\sqrt{5})/2$, we get

$$s_1 s_2 = \Phi_1(\alpha^{-2})\Phi_2(\alpha^{-2}) = \frac{\alpha^4 - 1}{\alpha^4 + 1}.$$

Hence, noting that $\alpha^4 = (\alpha+1)^2 = 3\alpha + 2$, we have

$$s_1 s_2 = \frac{1}{3}\cdot\frac{3\alpha+1}{\alpha+1} = \frac{1}{3}(2\alpha - 1) = \frac{\sqrt{5}}{3}.$$

Conversely, if s_1 and s_2 are algebraically dependent for some algebraic number a, then by the property 2 of Theorem 1.2 with $m = 2$ the set $\mathcal{F}_k(\tau) \setminus \{\pm\sqrt{-1}\}$ must consist of four elements, which is achieved only if $\tau = \pm\sqrt{-1}$ and $k = 1$. □

Proof of Example 2.5. We use the property 3 of Theorem 1.2. Let $\tau_1 = \zeta_3$, $\tau_2 = 1$, $S_1(\tau_1) = \mathcal{E}_1(\tau_1) = \{\zeta_{12}, \overline{\zeta_{12}}, \zeta_{12}^4, \overline{\zeta_{12}^4}, \zeta_{12}^5, \overline{\zeta_{12}^5}, \zeta_{12}^2, \overline{\zeta_{12}^2}\}$, and $S_1(\tau_2) = \mathcal{E}_1(\tau_2) = \{1, -1, \sqrt{-1}, -\sqrt{-1}\}$. Then $\mathcal{F}_1(\tau_1) = \{\zeta_{12}, \overline{\zeta_{12}}, \zeta_{12}^5, \overline{\zeta_{12}^5}, \zeta_{12}^2, \overline{\zeta_{12}^2}\}$ and $\mathcal{F}_1(\tau_2) = \{-1, \sqrt{-1}, -\sqrt{-1}\}$. Putting

$$p_1 = -(\zeta_{12} + \overline{\zeta_{12}}) = -\sqrt{3}, \quad p_2 = -(\zeta_{12}^5 + \overline{\zeta_{12}^5}) = \sqrt{3}, \quad p_3 = -(\zeta_{12}^2 + \overline{\zeta_{12}^2}) = -1,$$

and $p_4 = 2$ in the equation (3.2) with $b = 1$, we have

$$\Phi_1(z)\Phi_2(z)\Phi_3(z)$$

$$= \prod_{k=0}^{\infty}(z^{4^k} - \zeta_{12})(z^{4^k} - \overline{\zeta_{12}})(z^{4^k} - \zeta_{12}^5)(z^{4^k} - \overline{\zeta_{12}^5})(z^{4^k} - \zeta_{12}^2)(z^{4^k} - \overline{\zeta_{12}^2})\frac{1}{(1+z^{2\cdot 4^k})^3}$$

$$= \prod_{k=0}^{\infty}\frac{(z^{4^{k+1}} - \zeta_{12}^4)(z^{4^{k+1}} - \overline{\zeta_{12}^4})}{(z^{4^k} - \zeta_{12}^4)(z^{4^k} - \overline{\zeta_{12}^4})}\frac{1}{(1+z^{2\cdot 4^k})^3}$$

$$= \frac{1}{(z - \zeta_{12}^4)(z - \overline{\zeta_{12}^4})}\prod_{k=0}^{\infty}\frac{1}{(1+z^{2\cdot 4^k})^3},$$

and

$$\Phi_4(z) = \prod_{k=0}^{\infty}\frac{1 + 2z^{4^k} + z^{2\cdot 4^k}}{1 + z^{2\cdot 4^k}} = \prod_{k=0}^{\infty}\frac{(1+z^{4^k})^2(1+z^{2\cdot 4^k})^2}{(1+z^{2\cdot 4^k})^3}$$

$$= \prod_{k=0}^{\infty}\frac{1}{(1+z^{2\cdot 4^k})^3}\left(\frac{1-z^{4^{k+1}}}{1-z^{4^k}}\right)^2 = \frac{1}{(1-z)^2}\prod_{k=0}^{\infty}\frac{1}{(1+z^{2\cdot 4^k})^3}. \quad (3.5)$$

Hence, we get

$$\Phi(z) := \Phi_1(z)\Phi_2(z)\Phi_3(z)\Phi_4^{-1}(z) = \frac{(1-z)^2}{1+z+z^2}.$$

By the equation (3.3) with $N = 1$ and $\alpha = (1+\sqrt{5})/2$, we have

$$s_1 s_2 s_3 s_4^{-1} = \Phi(\alpha^{-4}) = \frac{\alpha^8 - 2\alpha^4 + 1}{\alpha^8 + \alpha^4 + 1} = \frac{7\alpha^4 - 2\alpha^4}{7\alpha^4 + \alpha^4} = \frac{5}{8},$$

since

$$\alpha^8 + 1 = (3\alpha+2)^2 + 1 = 21\alpha + 14 = 7\alpha^4. \tag{3.6}$$

To prove that $\operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}(s_1, s_2, s_3, s_4) = 3$, it is enough to show that s_2, s_3, and s_4 are algebraically independent, which is equivalent to the fact that p_2, p_3, and p_4 do not satisfy the property 3 of Theorem 1.2 with $m = 3$. By (1.2) with $S_1(\tau_i) = \mathcal{E}_1(\tau_i)$ $(i = 1, 2)$, considering the number of the elements of $S_k(\tau_i)$ with $k \geq 2$ satisfying $S_k(\tau_i) = \zeta_4 S_k(\tau_i)$ and $S_k(\tau_i) = \overline{S_k(\tau_i)}$, we see that $\{-\sqrt{3}, \sqrt{3}, -1, 2\}$ is the minimal set of $-(\gamma + \overline{\gamma})$ with $\gamma \in \mathcal{F}_{k_1}(\tau_1) \cup \mathcal{F}_{k_2}(\tau_2) \setminus \{\pm\sqrt{-1}\}$ satisfying the property 3 of Theorem 1.2 with $m = 4$. □

Proof of Example 2.6. We use the property 3 of Theorem 1.2. Let $\tau_1 = 1$, $\tau_2 = -1$, $S_1(\tau_1) = \mathcal{E}_1(\tau_1) = \{\zeta_6, \zeta_6^2, -1, \zeta_6^4, \zeta_6^5, 1\}$, and $S_1(\tau_2) = \mathcal{E}_1(\tau_2) = \{\zeta_{12}, \sqrt{-1}, \zeta_{12}^5, \zeta_{12}^7, -\sqrt{-1}, \zeta_{12}^{11}\}$. Then $\mathcal{F}_1(\tau_1) = \{\zeta_6, \zeta_6^2, -1, \zeta_6^4, \zeta_6^5\}$ and $\mathcal{F}_1(\tau_2) = \{\zeta_{12}, \sqrt{-1}, \zeta_{12}^5, \zeta_{12}^7, -\sqrt{-1}, \zeta_{12}^{11}, -1\}$.

We show the equality $s_1 s_2 s_3 s_4^{-1} s_5^{-1} = \sqrt{5}/2$ by proving a general relation among the functions $\Phi_i(z)$ defined by (3.2). Let $d \geq 6$ be an even integer. Putting

$$p_0 = -2,\ p_1 = -(\zeta_d + \overline{\zeta_d}),\ p_2 = -(\zeta_d^2 + \overline{\zeta_d}^2), \ldots,\ p_{\frac{d}{2}} = -(\zeta_d^{\frac{d}{2}} + \overline{\zeta_d}^{\frac{d}{2}}) = 2$$

in the equation (3.2) with $b = 1$, we have

$$\begin{aligned}&\left(\Phi_0(z) \cdot \Phi_1^2(z)\Phi_2^2(z) \cdots \Phi_{\frac{d}{2}-1}^2(z) \cdot \Phi_{\frac{d}{2}}(z)\right) \cdot \Phi_0^{-1}(z) \\ &= \prod_{k=0}^{\infty} \left(\frac{1}{(1+z^{2d^k})^{d-1}} \frac{(z^{d^{k+1}}-1)^2}{(z^{d^k}-1)^2} \right) = \frac{1}{(z-1)^2} \prod_{k=0}^{\infty} \frac{1}{(1+z^{2d^k})^{d-1}}.\end{aligned}$$

In the same way, putting

$$p_{\frac{d}{2}+1} = -(\zeta_{2d} + \overline{\zeta_{2d}}),\ p_{\frac{d}{2}+2} = -(\zeta_{2d}^3 + \overline{\zeta_{2d}}^3), \ldots,\ p_d = -(\zeta_{2d}^{d-1} + \overline{\zeta_{2d}}^{d-1}),$$

we get

$$\begin{aligned}&\Phi_{\frac{d}{2}+1}^2(z)\Phi_{\frac{d}{2}+2}^2(z) \cdots \Phi_d^2(z) \cdot \Phi_{\frac{d}{2}}^{-1}(z) \\ &= \prod_{k=0}^{\infty} \left(\frac{1}{(1+z^{2d^k})^{d-1}} \frac{(z^{d^{k+1}}+1)^2}{(z^{d^k}+1)^2} \right) = \frac{1}{(z+1)^2} \prod_{k=0}^{\infty} \frac{1}{(1+z^{2d^k})^{d-1}}.\end{aligned}$$

Hence, noting that $\Phi_i(0) = 1$ $(1 \le i \le d)$, we have

$$\Phi(z) := \frac{\Phi_1(z)\Phi_2(z)\cdots\Phi_{\frac{d}{2}}(z)}{\Phi_{\frac{d}{2}+1}(z)\Phi_{\frac{d}{2}+2}(z)\cdots\Phi_d(z)} = \frac{1+z}{1-z}. \tag{3.7}$$

Now assume that $d = 6$ in (3.7). Noting that $p_5 = 0$ and putting

$$a_1 = p_1 = -1, \quad a_2 = p_2 = 1, \quad a_3 = p_3 = 2, \quad a_4 = p_4 = -\sqrt{3}, \quad a_5 = p_6 = \sqrt{3}$$

in the equation (3.3) with $N = 1$ and $\alpha = (1+\sqrt{5})/2$, we have

$$\Phi(\alpha^{-6}) = \frac{s_1 s_2 s_3}{s_4 s_5} = \frac{\alpha^6+1}{\alpha^6-1}.$$

Since $\alpha^6 = 8\alpha + 5$, we get

$$\frac{s_1 s_2 s_3}{s_4 s_5} = \frac{\alpha^6+1}{\alpha^6-1} = \frac{1}{2}(2\alpha - 1) = \frac{\sqrt{5}}{2}.$$

The transcendence degree is obtained in the same way as in the proof of Example 2.5. □

Proof of Example 2.7. We use the property 3 of Theorem 1.2. Let $\tau_1 = \sqrt{-1}$, $\tau_2 = 1$,

$$S_2(\tau_1) = \{\zeta_{64}^3, \zeta_{64}^{13}, \zeta_{64}^{19}, \zeta_{64}^{29}, \zeta_{64}^{35}, \zeta_{64}^{45}, \zeta_{64}^{51}, \zeta_{64}^{61}\},$$

and

$$S_1(\tau_2) = \mathcal{E}_1(\tau_2) = \{1, -1, \sqrt{-1}, -\sqrt{-1}\}.$$

Then

$$\Lambda_1(\tau_1) = \{\zeta_{16}^3, \zeta_{16}^{13}\}, \ \Gamma_1(\tau_1) = \{\zeta_{16}^1, \zeta_{16}^5, \zeta_{16}^7, \zeta_{16}^9, \zeta_{16}^{11}, \zeta_{16}^{15}\}, \ \Lambda_0(\tau_1) = \{\sqrt{-1}, -\sqrt{-1}\},$$

and so

$$\mathcal{F}_2(\tau_1) = \{\zeta_{64}^3, \zeta_{64}^{13}, \zeta_{64}^{19}, \zeta_{64}^{29}, \zeta_{64}^{35}, \zeta_{64}^{45}, \zeta_{64}^{51}, \zeta_{64}^{61}, \zeta_{16}^1, \zeta_{16}^5, \zeta_{16}^7, \zeta_{16}^9, \zeta_{16}^{11}, \zeta_{16}^{15}, \sqrt{-1}, -\sqrt{-1}\},$$
$$\mathcal{F}_1(\tau_2) = \{-1, \sqrt{-1}, -\sqrt{-1}\}.$$

Putting

$$p_1 = -(\zeta_{16}^1 + \zeta_{16}^{15}), \quad p_2 = -(\zeta_{16}^5 + \zeta_{16}^{11}), \quad p_3 = -(\zeta_{16}^7 + \zeta_{16}^9),$$

$$p_4 = -(\zeta_{64}^3 + \zeta_{64}^{61}), \quad p_5 = -(\zeta_{64}^{13} + \zeta_{64}^{51}), \quad p_6 = -(\zeta_{64}^{19} + \zeta_{64}^{45}), \quad p_7 = -(\zeta_{64}^{29} + \zeta_{64}^{35})$$

in the equation (3.2) with $b = 1$, we get

$$\Phi_1(z)\Phi_2(z)\cdots\Phi_7(z)$$
$$= \prod_{k=0}^{\infty}\left(\frac{1}{(1+z^{2\cdot 4^k})^6}\frac{(z^{4^{k+1}} - \zeta_{16}^3)(z^{4^{k+1}} - \zeta_{16}^{13})}{(z^{4^k} - \zeta_{16}^3)(z^{4^k} - \zeta_{16}^{13})}\frac{z^{2\cdot 4^{k+1}}+1}{z^{2\cdot 4^k}+1}\right)$$

$$= \frac{1}{(z^2+1)(z-\zeta_{16}^3)(z-\zeta_{16}^{13})}\prod_{k=0}^{\infty}\frac{1}{(1+z^{2\cdot 4^k})^6}.$$

Letting $p_8 = 2$ and using (3.5) in the proof of Example 2.5, we have

$$\Phi(z) := \frac{\Phi_1(z)\Phi_2(z)\cdots\Phi_7(z)}{\Phi_8^2(z)} = \frac{(z-1)^4}{(z^2+1)(z-\zeta_{16}^3)(z-\zeta_{16}^{13})}.$$

Putting $a_i = p_i$ $(1 \le i \le 8)$ in the equation (3.3) with $N = 1$ and $\alpha = (1+\sqrt{5})/2$ and using (3.6), we obtain

$$\begin{aligned}\frac{s_1\cdots s_7}{s_8^2} &= \Phi(\alpha^{-4}) = \frac{(\alpha^4-1)^4}{(\alpha^8+1)(\alpha^8+1-(\zeta_{16}^3+\zeta_{16}^{13})\alpha^4)}\\ &= \frac{(7\alpha^4-2\alpha^4)^2}{7\alpha^4(7\alpha^4-(\zeta_{16}^3+\zeta_{16}^{13})\alpha^4)}\\ &= \frac{25}{7(7-\sqrt{2-\sqrt{2}})},\end{aligned}$$

since $\zeta_{16}^3+\zeta_{16}^{13} = 2\cos(3\pi/8) = \sqrt{2-\sqrt{2}}$. □

Proof of Example 2.8. Let $d \ge 2$ be an integer. Let γ and η be complex numbers with $|\gamma| = |\eta| = 1$. We show a general relation which holds for the functions $\Phi_i(z)$ $(1 \le i \le 2d+2)$ defined by (3.2). Putting

$$p_1 = -(\gamma+\overline{\gamma}),\ p_2 = -(\gamma\zeta_d+\overline{\gamma\zeta_d}),\ldots,\ p_d = -(\gamma\zeta_d^{d-1}+\overline{\gamma\zeta_d^{d-1}}),$$

and $p_{d+1} = -(\gamma^d+\overline{\gamma}^d)$ in the equation (3.2) with $b = 1$, we have

$$\begin{aligned}&\Phi_1(z)\cdots\Phi_d(z)\Phi_{d+1}^{-1}(z)\\ &= \prod_{k=0}^{\infty}\left(\frac{1}{(1+z^{2d^k})^{d-1}}\frac{1}{\left(1+p_{d+1}z^{d^k}+z^{2d^k}\right)}\prod_{i=1}^{d}(1+p_iz^{d^k}+z^{2d^k})\right)\\ &= \prod_{k=0}^{\infty}\left(\frac{1}{(1+z^{2d^k})^{d-1}(z^{d^k}-\gamma^d)(z^{d^k}-\overline{\gamma}^d)}\prod_{i=0}^{d-1}(z^{d^k}-\gamma\zeta_d^i)(z^{d^k}-\overline{\gamma\zeta_d^i})\right)\\ &= \frac{1}{(z-\gamma^d)(z-\overline{\gamma}^d)}\prod_{k=0}^{\infty}\frac{1}{(1+z^{2d^k})^{d-1}}.\end{aligned}$$

Moreover, putting

$$p_{d+2} = -(\eta+\overline{\eta}),\ p_{d+3} = -(\eta\zeta_d+\overline{\eta\zeta_d}),\ldots,\ p_{2d+1} = -(\eta\zeta_d^{d-1}+\overline{\eta\zeta_d^{d-1}}),$$

and $p_{2d+2} = -(\eta^d+\overline{\eta}^d)$, we get

$$\Phi(z) := \frac{\Phi_1(z)\cdots\Phi_d(z)}{\Phi_{d+1}(z)}\cdot\frac{\Phi_{2d+2}(z)}{\Phi_{d+2}(z)\cdots\Phi_{2d+1}(z)} = \frac{(z-\eta^d)(z-\overline{\eta}^d)}{(z-\gamma^d)(z-\overline{\gamma}^d)}. \tag{3.8}$$

Substituting $z = \alpha^{-4}$ into (3.8) and using (3.6), we get

$$\Phi(\alpha^{-4}) = \frac{\alpha^8 + p_{2d+2}\alpha^4 + 1}{\alpha^8 + p_{d+1}\alpha^4 + 1} = \frac{7 + p_{2d+2}}{7 + p_{d+1}}. \tag{3.9}$$

For Example 2.8, we take $d = 4$ and

$$\gamma = \frac{3 + \sqrt{-7}}{4}, \qquad \eta = \frac{2 + \sqrt{-1}}{\sqrt{5}}.$$

Noting that $\gamma^4 \neq \eta^4$ and taking $\tau_1 = \gamma^4$ and $\tau_2 = \eta^4$ in the property 3 of Theorem 1.2, we have

$$\begin{aligned} S_1(\tau_1) = \mathcal{E}_1(\tau_1) &= \{\gamma, \sqrt{-1}\gamma, -\gamma, -\sqrt{-1}\gamma, \overline{\gamma}, \sqrt{-1}\overline{\gamma}, -\overline{\gamma}, -\sqrt{-1}\overline{\gamma}\}, \\ S_1(\tau_2) = \mathcal{E}_1(\tau_2) &= \{\eta, \sqrt{-1}\eta, -\eta, -\sqrt{-1}\eta, \overline{\eta}, \sqrt{-1}\overline{\eta}, -\overline{\eta}, -\sqrt{-1}\overline{\eta}\}, \end{aligned}$$

and so

$$\begin{aligned} \mathcal{F}_1(\tau_1) &= \{\gamma, \sqrt{-1}\gamma, -\gamma, -\sqrt{-1}\gamma, \overline{\gamma}, \sqrt{-1}\overline{\gamma}, -\overline{\gamma}, -\sqrt{-1}\overline{\gamma}, \gamma^4, \overline{\gamma}^4\}, \\ \mathcal{F}_1(\tau_2) &= \{\eta, \sqrt{-1}\eta, -\eta, -\sqrt{-1}\eta, \overline{\eta}, \sqrt{-1}\overline{\eta}, -\overline{\eta}, -\sqrt{-1}\overline{\eta}, \eta^4, \overline{\eta}^4\}, \end{aligned}$$

since γ and η are not roots of unity. Then we have

$$p_1 = -\frac{3}{2},\ p_2 = \frac{\sqrt{7}}{2},\ p_3 = \frac{3}{2},\ p_4 = -\frac{\sqrt{7}}{2},\ p_5 = \frac{31}{16},$$
$$p_6 = -\frac{4}{\sqrt{5}},\ p_7 = \frac{2}{\sqrt{5}},\ p_8 = \frac{4}{\sqrt{5}},\ p_9 = -\frac{2}{\sqrt{5}},\ p_{10} = \frac{14}{25},$$

since

$$\gamma^4 = -\frac{31 - 3\sqrt{-7}}{32}, \qquad \eta^4 = -\frac{7 - 24\sqrt{-1}}{25}.$$

Using (3.9), we get

$$\frac{s_1 \cdots s_4}{s_5} \cdot \frac{s_{10}}{s_6 \cdots s_9} = \frac{7 + 14/25}{7 + 31/16} = \frac{3024}{3575}$$

by the equation (3.3) with $N = 1$. The transcendence degree is obtained in the same way as in the proof of Example 2.5. $\square$

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On divisibility properties of some differences of Motzkin numbers

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Abstract

We discuss divisibility properties of some differences of Motzkin numbers M_n. The main tool is the application of various congruences of high prime power moduli for binomial coefficients and Catalan numbers combined with some recurrence relevant to these combinatorial quantities and the use of infinite disjoint covering systems.

We find proofs of the fact that, for different settings of a and b, more and more p-ary digits of $M_{ap^{n+1}+b}$ and M_{ap^n+b} agree as n grows.

Keywords: Catalan number, Motzkin number, harmonic number, divisibility

MSC: 11B83, 11B50, 11A07, 05A19

1. Introduction

The differences of certain combinatorial quantities, e.g., Motzkin numbers, exhibit interesting divisibility properties. Motzkin numbers are defined as the number of certain random walks or equivalently (cf. [2]) as

$$M_n = \sum_{k=0}^{n} \binom{n}{2k} C_k, n \geq 0, \tag{1.1}$$

where C_k is the kth Catalan number

$$C_k = \frac{1}{k+1}\binom{2k}{k}, k \geq 0.$$

We need some basic notation. Let n and k be positive integers, p be a prime, $d_p(k)$ and $\nu_p(k)$ denote the sum of digits in the base p representation of k and the highest power of p dividing k, respectively. The latter one is often referred to as the p-adic order of k. For the rational n/k we set $\nu_p(n/k) = \nu_p(n) - \nu_p(k)$.

We rely on the p-adic order of the differences of Catalan numbers $C_{ap^{n+1}+b} - C_{ap^n+b}$ (cf. Theorems 3.8 and 3.9) with a prime p, $(a,p) = 1$, and $n \geq n_0$ for some integer $n_0 \geq 0$.

As n grows, eventually more and more binary digits of M_{a2^n+b} and $M_{a2^{n+1}+b}$ agree, starting with the least significant bit, for every fixed $a \geq 1$ and $b \geq 0$, as stated by Theorem 2.3. We also determine lower bounds on the rate of growth in the number of matching digits in Corollary 2.4, Theorems 2.1 and 2.5. Conjecture 5.1 suggests finer details for $p = 2$. Conjectures 5.3 and 5.5 propose the exact value of $\nu_2(M_{a2^{n+1}+b} - M_{a2^n+b})$ if $p = 2$, $a = 1$, and $b = 0, 1, 2$, as well as $\nu_p(M_{ap^{n+1}} - M_{ap^n})$ if $p = 3$ and $(a, 3) = 1$, or $p \geq 5$ prime and $a = 1$, in addition to half of the odd a values if $p = 2$ and n is odd. We present Conjectures 5.1-5.3 that concern upper and lower bounds on $\nu_2(M_{a2^{n+1}+b} - M_{a2^n+b})$ and its exact value, respectively, with special interest in the cases with $a = 1$, $b = 0, 1, 2, 3$, and more generally, $b = 2^q - 1$, 2^q, and $2^q + 1, q \geq 1$. Further extensions and improvements are given in Theorems 5.6 and 5.7 (cf. [8]). All results involving the exact orders of differences or lower bounds on them can be easily rephrased in terms of super congruences for the underlying quantities.

Section 2 collects some of the main results (cf. Theorems 2.1 and 2.5) while Section 3 is devoted to known results and their direct consequences regarding congruential and p-adic properties of the binomial coefficients and Catalan numbers. We provide the proofs of Theorems 2.1 and 2.5 in Sections 4 and 5, respectively. We also prove Theorems 2.2-2.3 and state four conjectures (cf. Conjectures 5.1-5.3 and 5.5) related to Motzkin numbers in Section 5, including lower bounds on the order of differences for all primes.

2. Main results

In this section we list our main results regarding the differences of certain Motzkin numbers. Except for Theorem 2.3, they all determine lower bounds on the rate of growth in the number of matching p-ary digits in the differences.

Theorem 2.1. *For $p = 2$, $n \geq 2$, $a \geq 1$ odd, and $b = 0$ or 1, we have*

$$\nu_2(M_{a2^{n+1}+b} - M_{a2^n+b}) = n - 1, \textit{if } n \textit{ is even}$$

and

$$\nu_2(M_{a2^{n+1}+b} - M_{a2^n} + b) \geq n, \textit{if } n \textit{ is odd.}$$

Theorem 2.2 provides us with a lower bound on $\nu_2(M_{a2^{n+1}+b} - M_{a2^n+b})$ on a recursive fashion in b and potentially, it can give the exact order if $a = 1$.

Theorem 2.2. *For $a \geq 1$ odd and $n \geq n_0$ with some $n_0 = n_0(a, b) \geq 1$, we get that for $b \geq 2$ even*

$$\nu_2(M_{a2^{n+1}+b} - M_{a2^n+b}) =$$

$$= \min\{\nu_2(M_{a2^{n+1}+b-1} - M_{a2^n+b-1}),\\ \nu_2(M_{a2^{n+1}+b-2} - M_{a2^n+b-2})\} - \nu_2(b+2)$$

if the two expressions under the minimum operation are not equal. However, it they are, then it is at least

$$\nu_2(M_{a2^{n+1}+b-1} - M_{a2^n+b-1}) + 1 - \nu_2(b+2).$$

On the other hand, if $b \geq 3$ is odd then we have

$$\nu_2(M_{a2^{n+1}+b} - M_{a2^n+b}) = \nu_2(M_{a2^{n+1}+b-1} - M_{a2^n+b-1}) \tag{2.1}$$

if $\nu_2(M_{a2^{n+1}+b-2} - M_{a2^n+b-2}) + \nu_2(b-1) > \nu_2(M_{a2^{n+1}+b-1} - M_{a2^n+b-1})$, and

$$\nu_2(M_{a2^{n+1}+b} - M_{a2^n+b}) = \nu_2(M_{a2^{n+1}+b-2} - M_{a2^n+b-2})\\ + \nu_2(b-1)$$

if $\nu_2(M_{a2^{n+1}+b-2} - M_{a2^n+b-2}) + \nu_2(b-1) < \nu_2(M_{a2^{n+1}+b-1} - M_{a2^n+b-1})$, and otherwise, it is at least $\nu_2(M_{a2^{n+1}+b-1} - M_{a2^n+b-1})$. Note however the stipulation that in all equalities above, if the right hand side value is at least $n - 2\nu_2(b+2)$ then the equality turns into the inequality $\nu_2(M_{a2^{n+1}+b} - M_{a2^n+b}) \geq n - 2\nu_2(b+2)$.

Theorem 2.2 guarantees that as n grows, eventually more and more binary digits of M_{a2^n+b} and $M_{a2^{n+1}+b}$ agree, starting with the least significant bit for every fixed $a \geq 1$ and $b \geq 0$.

Theorem 2.3. *For every $a \geq 1$, $b \geq 0$, and $K \geq 0$ integers, there exists an $n_0 = n_0(a, b, K)$ so that $\nu_2(M_{a2^{n+1}+b} - M_{a2^n+b}) \geq K$ for all $n \geq n_0$.*

For the asymptotic growth of the Motzkin numbers we have $M_n \sim c3^n/n^{3/2}$ with some integer $c > 0$. Unfortunately, neither this fact nor Theorem 2.3 helps in assessing the rate of growth of matching digits, i.e., $\nu_2(M_{a2^{n+1}+b} - M_{a2^n+b})$. However, Theorem 2.2 and $\sum'_{i \leq b+2} \nu_2(i) = \nu_2((b+2)!) = (b+2) - d_2(b+2)$, with the summation running through even values of i only, imply the following, although rather coarse, lower bound.

Corollary 2.4. *For $a \geq 1$ odd, $b \geq 0$, and $n \geq n_0$ with some $n_0 = n_0(a, b) \geq 1$, we have*

$$\nu_2(M_{a2^{n+1}+b} - M_{a2^n+b}) \geq n - (b+2) + d_2(b+2).$$

Theorem 2.5 gives a lower bound on $\nu_3(M_{3^{n+1}+b} - M_{3^n+b})$ with $b = 0$ or 1, and $\nu_p(M_{p^{n+1}+b} - M_{p^n+b})$ for $p \geq 5$ and $0 \leq b \leq p-3$.

Theorem 2.5. *For $p \geq 3$ prime and $n \geq n_0$ with some integer $n_0 = n_0(p) \geq 0$, we have*

$$\nu_p(M_{p^{n+1}} - M_{p^n}) \geq n. \tag{2.2}$$

Assuming that $n \geq n_0$, for $p = 3$, we have

$$\nu_3(M_{3^{n+1}+1} - M_{3^n+1}) \geq n - 1,$$

and for $p \geq 5$, we have

$$\nu_p(M_{p^{n+1}+b} - M_{p^n+b}) \geq n$$

with $0 \leq b \leq p-3$.

3. Preparation

We note that there are many places in the literature where relevant divisibility and congruential properties of the binomial coefficients are discussed. Excellent surveys can be found in [5] and [11]. The following three theorems comprise the most basic facts regarding divisibility and congruence properties of the binomial coefficients. We assume that $0 \le k \le n$.

Theorem 3.1 (Kummer, 1852). *The power of a prime p that divides the binomial coefficient $\binom{n}{k}$ is given by the number of carries when we add k and $n-k$ in base p.*

Theorem 3.2 (Legendre, 1830). *We have*

$$\nu_p\big(\tbinom{n}{k}\big) = \tfrac{n-d_p(n)}{p-1} - \tfrac{k-d_p(k)}{p-1} - \tfrac{n-k-d_p(n-k)}{p-1} = \tfrac{d_p(k)+d_p(n-k)-d_p(n)}{p-1}.$$

In particular, $\nu_2(\binom{n}{k}) = d_2(k) + d_2(n-k) - d_2(n)$ represents the carry count in the addition of k and $n-k$ in base 2.

From now on M and N will denote integers such that $0 \le M \le N$.

Theorem 3.3 (Lucas, 1877). *Let $N = (n_d, \ldots, n_1, n_0)_p = n_0 + n_1 p + \cdots + n_d p^d$ and $M = m_0 + m_1 p + \cdots + m_d p^d$ with $0 \le n_i, m_i \le p-1$ for each i, be the base p representations of N and M, respectively.*

$$\binom{N}{M} \equiv \binom{n_0}{m_0}\binom{n_1}{m_1}\cdots\binom{n_d}{m_d} \bmod p.$$

Lucas' theorem has some remarkable extensions.

Theorem 3.4 (Anton, 1869, Stickelberger, 1890, Hensel, 1902). *Let $N = (n_d, \ldots, n_1, n_0)_p = n_0 + n_1 p + \cdots + n_d p^d, M = m_0 + m_1 p + \cdots + m_d p^d$ and $R = N - M = r_0 + r_1 p + \cdots + r_d p^d$ with $0 \le n_i, m_i, r_i \le p-1$ for each i, be the base p representations of $N, M,$ and $R = N - M$, respectively. Then with $q = \nu_p\big(\binom{N}{M}\big)$,*

$$(-1)^q \frac{1}{p^q}\binom{N}{M} \equiv \left(\frac{n_0!}{m_0! r_0!}\right)\left(\frac{n_1!}{m_1! r_1!}\right)\cdots\left(\frac{n_d!}{m_d! r_d!}\right) \bmod p.$$

Davis and Webb (1990) and Granville (1995) have independently generalized Lucas' theorem and its extension Theorem 3.4. For a given integer n and prime p, we define $(n!)_p = n!/(p^{\lfloor n/p \rfloor}\lfloor n/p \rfloor!)$ to be the product of positive integers not exceeding n and not divisible by p, and which is closely related to the p-adic Morita gamma function.

Theorem 3.5 (Granville, 1995 in [5]). *Let $N = (n_d, \ldots, n_1, n_0)_p = n_0 + n_1 p + \cdots + n_d p^d, M = m_0 + m_1 p + \cdots + m_d p^d$ and $R = N - M = r_0 + r_1 p + \cdots + r_d p^d$ with $0 \le n_i, m_i, r_i \le p-1$ for each i, be the base p representations of $N, M,$ and $R = N - M$, respectively. Let $N_j = n_j + n_{j+1} p + \cdots + n_{j+k-1} p^{k-1}$ for each $j \ge 0$,* *i.e., the least positive residue of* $\lfloor N/p^j \rfloor \bmod p^k$ *with some integer* $k \geq 1$*; also make the corresponding definitions for* M_j *and* R_j*. Let* ϵ_j *be the number of carries when adding* M *and* R *on and beyond the* j*th digit. Then with* $q = \epsilon_0 = \nu_p\left(\binom{N}{M}\right)$*,*

$$\frac{1}{p^q}\binom{N}{M} \equiv (\pm 1)^{\epsilon_{k-1}}\left(\frac{(N_0!)_p}{(M_0!)_p(R_0!)_p}\right)\left(\frac{(N_1!)_p}{(M_1!)_p(R_1!)_p}\right)\cdots\left(\frac{(N_d!)_p}{(M_d!)_p(R_d!)_p}\right) \bmod p^k$$

where ± 1 *is* -1 *except if* $p = 2$ *and* $k \geq 3$*.*

We also use the following generalization of the Jacobstahl–Kazandzidis [1] congruences.

Theorem 3.6 (Corollary 11.6.22 [1])**.** *Let* M *and* N *such that* $0 \leq M \leq N$ *and* p *prime. We have*

$$\binom{pN}{pM} \equiv \begin{cases} \left(1 - \frac{B_{p-3}}{3}p^3 NM(N-M)\right)\binom{N}{M} & \bmod p^4 NM(N-M)\binom{N}{M}, \text{ if } p \geq 5, \\ (1 + 45NM(N-M))\binom{N}{M} & \bmod p^4 NM(N-M)\binom{N}{M}, \text{ if } p = 3, \\ (-1)^{M(N-M)}P(N,M)\binom{N}{M} & \bmod p^4 NM(N-M)\binom{N}{M}, \text{ if } p = 2, \end{cases}$$

where $P(N,M) = 1 + 6NM(N-M) - 4NM(N-M)(N^2 - NM + M^2) + 2(NM(N-M))^2$.

Remark 3.7. It is well known that $\nu_p(B_n) \geq -1$ by the von Staudt–Clausen theorem. If the prime p divides the numerator of B_{p-3}, i.e., $\nu_p(B_{p-3}) \geq 1$, or equivalently $\binom{2p}{p} \equiv 2 \bmod p^4$, then it is sometimes called a Wolstenholme prime [1]. The only known Wolstenholme primes up to 10^9 are $p = 16843$ and 2124679.

Based on the above theorems, we state some of the main tools regarding the differences of Catalan numbers (cf. [7] for details and proofs). For the p-adic orders we obtain the following theorem.

Theorem 3.8. *For any prime* $p \geq 2$ *and* $(a,p) = 1$*, we have*

$$\nu_p(C_{ap^{n+1}} - C_{ap^n}) = n + \nu_p\left(\binom{2a}{a}\right), \quad n \geq 1.$$

We can introduce an extra additive term $b \geq 1$ into Theorem 3.8.

Theorem 3.9. *For* $p = 2$*,* a *odd, and* $n \geq n_0 = 2$ *we have*

$$\nu_2(C_{a2^{n+1}+1} - C_{a2^n+1}) = n + \nu_2\left(\binom{2a}{a}\right) - 1,$$

and in general, for $b \geq 1$ *and* $n \geq n_0 = \lfloor \log_2 2b \rfloor + 1$

$$\nu_2(C_{a2^{n+1}+b} - C_{a2^n+b}) = n + \nu_2\left(\binom{2a}{a}\right) + \nu_2(g(b))$$

$$= n + d_2(a) + d_2(b) - \lceil \log_2(b+2) \rceil - \nu_2(b+1) + 1$$

where $g(b) = 2\binom{2b}{b}(b+1)^{-1}(H_{2b} - H_b - 1/(2(b+1))) = 2C_b(H_{2b} - H_b - 1/(2(b+1)))$ *with* $H_n = \sum_{j=1}^{n} 1/j$ *being the nth harmonic number.*

For any prime $p \geq 3$, $(a,p) = 1$, *and* $b \geq 1$ *we have that*

$$\nu_p(C_{ap^{n+1}+b} - C_{ap^n+b}) = n + \nu_p\left(\binom{2a}{a}\right) + \nu_p(g(b)),$$

with $n \geq n_0 = \max\{\nu_p(g(b)) + 2r - \nu_p(C_b) + 1, r+1\} = \max\{\nu_p(2(H_{2b} - H_b - 1/(2(b+1)))) + 2r + 1, r+1\}$ *and* $r = \lfloor \log_p 2b \rfloor$.

In general, for any prime $p \geq 2$, $(a,p) = 1, b \geq 1$, *and* $n > \lfloor \log_p 2b \rfloor$, *we have*

$$\nu_p(C_{ap^{n+1}+b} - C_{ap^n+b}) \geq n + \nu_p\left(\binom{2a}{a}\right) + \nu_p\left(\binom{2b}{b}\right) - \lfloor \log_p 2b \rfloor - \nu_p(b+1).$$

Note. Clearly, $\nu_p(g(b)) \geq 0$ for $1 \leq b \leq (p-1)/2$. We note that in general, for $b \geq 1$ we have $\nu_p(g(b)) \geq \nu_p(\binom{2b}{b}) - \lfloor \log_p 2b \rfloor - \nu_p(b+1)$ if $p \geq 2$ while $\nu_2(g(b)) = d_2(b) - \lceil \log_2(b+2) \rceil - \nu_2(b+1) + 1 = d_2(b+1) - \lceil \log_2(b+2) \rceil$ if $p = 2$.

We note that as a byproduct, we proved some generalization of the observation from [10] that for any $n \geq 2$ the remainders $C_{2^{n+m-1}-1} \bmod 2^n$ are equal for each $m \geq 0$ (see [9], [12], and [13],too) in [7]:

Theorem 3.10. *For any prime* $p \geq 2$, $(a,p) = 1, b \geq 0$, *we have that* $C_{ap^m+b} \bmod p^n$ *is constant for* $m \geq n + \nu_p(b+1) + \max\{0, \lfloor \log_p 2b \rfloor\}, n \geq 1$.

We also note that

$$\nu_2(C_k) = d_2(k) - \nu_2(k+1) = d_2(k+1) - 1 \tag{3.1}$$

holds, i.e., $\nu_2(C_{2^{n+1}}) = \nu_2(C_{2^n}) = 1$. It follows that C_k is odd if and only if $k = 2^q - 1$ for some integer $q \geq 0$.

4. The proof of Theorem 2.1

In this section we present

The proof of Theorem 2.1. We prove the case with $b = 0$ and then we note that the case with $b = 1$ is practically identical. Thus, we assume that $b = 0$.

First we deal with the case with $a = 1$. We use the identity (1.1)

$$M_{2^n} = \sum_{k=0}^{2^n} \binom{2^n}{2k} C_k,$$

rely on identity (3.1) and select an infinite incongruent disjoint covering system (IIDCS). The difference of the appropriate Motzkin numbers can be rewritten as

$$M_{2^{n+1}} - M_{2^n} = \sum_{k=1}^{2^{n-1}} \left(\binom{2^{n+1}}{2(2k)} C_{2k} - \binom{2^n}{2k} C_k \right) + \sum_{k \equiv 1 \bmod 2}^{2^n} \binom{2^{n+1}}{2k} C_k \tag{4.1}$$

after removing the superfluous term with $k = 0$ in the first sum. We break the first summation in (4.1) into parts according to the IIDCS $\{2^q \pmod{2^{q+1}}\}_{q \geq 0}$, which allows us to write every positive integer uniquely in the form of $2^q + 2^{q+1}K$ for some q and $K \geq 0$.

$$\begin{aligned}
&\sum_{k=1}^{2^{n-1}} \left(\binom{2^{n+1}}{2(2k)} C_{2k} - \binom{2^n}{2k} C_k \right) \\
&= \sum_{q=0}^{n-1} \sum_{\substack{k=2^q+2^{q+1}K \\ 0 \leq K \leq \frac{2^{n-q-1}-1}{2}}} \left(\binom{2^{n+1}}{2(2k)} C_{2k} - \binom{2^n}{2k} C_k \right) \\
&= \sum_{q=0}^{n-2} \sum_{\substack{k=2^q+2^{q+1}K \\ 0 \leq K \leq 2^{n-q-2}-1}} \left(\binom{2^{n+1}}{2(2k)} C_{2k} - \binom{2^n}{2k} C_k \right) \\
&+ (C_{2^n} - C_{2^{n-1}}).
\end{aligned} \tag{4.2}$$

We introduce the following quantities

$$M'_r(n, m) = \sum_{\substack{k \equiv r \bmod m \\ 1 \leq k \leq 2^n}} \binom{2^{n+1}}{2k} C_k$$

and focus on cases when m is a power of two.

The second summation in (4.1) is $M'_1(n, 2)$. Its 2-adic order is at least n according to

Theorem 4.1. *For integers $n \geq q \geq 1$, we have*

$$\nu_2(M'_{2^q}(n, 2^{q+1})) = n + 1 - q.$$

If $q = 0$ then $\nu_2(M'_1(n, 2)) = n$ if n is odd, otherwise the 2-adic order is at least $n + 1$.

We can gain more insight into the 2-adic structure of the terms of the sum (4.2) by checking how the 2-adic orders of the terms $\binom{2^n}{2k} C_k$ and $\binom{2^{n+1}}{2(2k)} C_{2k}$ with $k = 2^q + 2^{q+1}K$ behave in $M'_{2^q}(n-1, 2^{q+1})$ and $M'_{2^{q+1}}(n, 2^{q+2})$, respectively.

If $1 \leq q \leq n-1$ then both 2-adic orders are equal to $n - q + d_2(K)$. Indeed, the range for K is $0 \leq K \leq 2^{n-q-2} - 1$ if $q \leq n-2$ and $K = 0$ if $q = n-1$ in both cases, and more importantly, the difference $A_{q,K} = \binom{2^{n+1}}{2(2k)} C_{2k} - \binom{2^n}{2k} C_k =$

$\binom{2^{n+1}}{2(2k)}(C_{2k}-C_k)+\left(\binom{2^{n+1}}{2(2k)}-\binom{2^n}{2k}\right)C_k$ has 2-adic order $n+d_2(K)$ by Theorems 3.1, 3.8, and 3.6. Note that $\nu_2(A_{q,K})$ is determined by the 2-adic order of the first term in the last sum, and it is given by combining $\nu_2\left(\binom{2^{n+1}}{2(2k)}\right)=n-1-q$ and $\nu_2(C_{2k}-C_k)=q+d_2(1+2K)=q+d_2(K)+1$. Therefore, $\nu_2(\sum_K A_{q,K})=n$ for each $q\ge 1$ and it is due to the term with $K=0$.

If $q=0$, i.e., $k=1+2K$, then $A_{0,K}=M_2'(n,4)-M_1'(n-1,2)$ and $\nu_2(A_{0,K})\ge n-1$ since $\nu_2\left(\binom{2^{n+1}}{4(1+2K)}C_{2+4K}\right)=n-1+d_2(3+4K)-1=n+d_2(K)$ and

$$\nu_2\left(\binom{2^n}{2(1+2K)}C_{1+2K}\right)=n-1+d_2(2+2K)-1=n-2+d_2(1+K)\ge n-1. \quad (4.3)$$

The latter minimum value is taken exactly for $n-1$ values of K since in the range $0\le K\le 2^{n-2}-1$ there are exactly $n-1$ terms with $K=2^r-1, r=0,1,\dots,n-2$, leading to $d_2(K+1)=1$. Thus, the 2-adic order of the corresponding sum $\sum_K A_{0,K}$ is $n-1$ if n is even and at least n if n is odd.

The proof is now complete for the case $a=1$. The proof with an arbitrary $a\ge 1$ odd is very similar except it requires a more detailed analysis of the terms in (4.4) than we had in (4.1). In any case, the first term with $q=0$ in the right hand side of (4.2) and (4.5), i.e., $A_{0,K}=M_2'(n,4)-M_1'(n-1,2)$ and $A_{0,K,a}=M_{2,a}'(n,4)-M_{1,a}'(n-1,2)$ (cf. notation below), respectively, determines the 2-adic order.

We use the binary representation of $a=\sum_{i=0}^{\infty}a_i2^i=\sum_{i\in S}2^i$ with $0\in S=\{i|a_i=1\}$ since a is odd.

We rewrite the difference

$$M_{a2^{n+1}}-M_{a2^n}=\sum_{k=1}^{a2^{n-1}}\left(\binom{a2^{n+1}}{2(2k)}C_{2k}-\binom{a2^n}{2k}C_k\right)+\sum_{k\equiv 1 \bmod 2}^{a2^n}\binom{a2^{n+1}}{2k}C_k. \quad (4.4)$$

We break the first summation in (4.4) into parts according to the covering system used in (4.2)

$$\sum_{k=1}^{a2^{n-1}}\left(\binom{a2^{n+1}}{2(2k)}C_{2k}-\binom{a2^n}{2k}C_k\right)= \quad (4.5)$$

$$=\sum_{q=0}^{\lfloor\log_2 a2^{n-1}\rfloor}\sum_{\substack{k=2^q+2^{q+1}K\\ 0\le K\le\frac{a2^{n-q-1}-1}{2}}}\left(\binom{a2^{n+1}}{2(2k)}C_{2k}-\binom{a2^n}{2k}C_k\right).$$

Now we introduce

$$M_{r,a}'(n,m)=\sum_{\substack{k\equiv r \bmod m\\ 1\le k\le a2^n}}\binom{a2^{n+1}}{2k}C_k$$

and note that the second term in (4.4) is $M'_{1,a}(n,2)$. Its 2-adic order is at least n. In fact, for a general term in the sum $M'_{1,a}(n,2)$, we get that

$$\nu_2\left(\binom{a2^{n+1}}{2k}C_k\right) \geq (n+1-1)+(d_2(2+2K)-1) = n-1+d_2(1+K) \geq n \quad (4.6)$$

with $0 \leq k = 1+2K \leq a2^n$, i.e., $0 \leq K \leq a2^{n-1}-1$. We want equalities in (4.6) in order to determine $\nu_2(M'_{1,a}(n,2))$. While in the case of $a=1$ it trivially follows that $\nu_2(\binom{a2^{n+1}}{2k}) = n$, now we have to deal with the possibility that $2k > 2^{n+1}$. By Theorem 3.1, the first inequality turns into equality exactly if

$$K = j + \sum_{i\in S'\subseteq S\setminus\{0\}} 2^{i+n-1}$$

with $0 \leq j \leq 2^{n-1}-1$, while the second one becomes an equality when $d_2(K+1) = |S'| + d_2(j+1) = 1$, i.e., $S' = \emptyset$ and $j = 2^r - 1$ and thus, $K = 2^r - 1$ with $r = 0, 1, \ldots, n-1$. Therefore, this case turns out to be identical to that of $a=1$ and hence, $\nu_2(M'_{1,a}(n,2)) \geq n$ with equality if and only if n is odd. (By the way, this argument is also used at the end of the proof of Theorem 4.1 below. Note that Theorem 4.1 remains valid even after introducing the parameter a, i.e., if we replace $M'_{2^q}(n,2^{q+1})$ with $M'_{2^q,a}(n,2^{q+1})$, cf. Theorem 4.2.)

Now we turn to the analysis of (4.5). We have three cases: either $1 \leq q \leq n-1$, or $q \geq n$, or $q=0$. We consider the difference with $k = 2^q + 2^{q+1}K$

$$\begin{aligned} A_{q,K,a} &= \binom{a2^{n+1}}{2(2k)}C_{2k} - \binom{a2^n}{2k}C_k \\ &= \binom{a2^{n+1}}{2(2k)}(C_{2k}-C_k) + \left(\binom{a2^{n+1}}{2(2k)} - \binom{a2^n}{2k}\right)C_k. \end{aligned} \quad (4.7)$$

If $1 \leq q \leq n-1$ then it has 2-adic order $n + d_2(K)$ by Theorems 3.1, 3.8, and 3.6. Note that $\nu_2(A_{q,K,a})$ is determined by the 2-adic order of the first term in the last sum and it is given by combining $\nu_2(\binom{a2^{n+1}}{2(2k)}) = n-1-q$ and $\nu_2(C_{2k}-C_k) = q + d_2(1+2K) = q + d_2(K) + 1$. Therefore, $\nu_2(\sum_K A_{q,K,a}) = n$ for each $q \geq 1$ and it is due to the term with $K=0$.

If $q \geq n$ then both terms of the last sum in (4.7) have a 2-adic order of at least $n+1$ by Theorems 3.1, 3.8, and 3.6. For example, for the first term we see that $\nu_2(C_{2k}-C_k) = q + d_2(1+2K) \geq n+1+d_2(K) \geq n+1$.

If $q=0$, i.e., $k = 1+2K$, then $\nu_2(A_{0,K,a}) = n-1$ since $\nu_2(\binom{a2^{n+1}}{4(1+2K)}C_{2+4K}) = n-1+d_2(3+4K)-1 = n+d_2(K)$ and

$$\nu_2\left(\binom{a2^n}{2(1+2K)}C_{1+2K}\right) \geq n-1+d_2(2+2K)-1 = n-2+d_2(1+K) \geq n-1. \quad (4.8)$$

In a similar fashion to (4.6), the latter minimum value is taken exactly for $n-1$ values of K since in the range $0 \le K \le a2^{n-2}-1$ there are exactly $n-1$ terms with $K = 2^r-1, r = 0, 1, \dots, n-2$, leading to $d_2(K+1) = 1$ so that $\nu_2\big(\binom{a2^n}{2(1+2K)}\big) = n-1$. Thus, the 2-adic order of the corresponding sum $\sum_K A_{0,K,a}$ is $n-1$ if n is even and at least n if n is odd.

If $b = 1$ then we observe that the 2-adic orders of $\binom{a2^n+1}{2k}$ and $\binom{a2^n}{2k}$ are equal. By switching from $a2^n$ and $a2^{n+1}$ to $a2^n+1$ and $a2^{n+1}+1$, respectively, the proof is almost identical to that of the case with $b = 0$. Note that the only term that requires some extra work is the second term $\big(\binom{a2^{n+1}+1}{2(2k)} - \binom{a2^n+1}{2k}\big)C_k$ in the revised version of (4.7). In fact, its 2-adic order is at least n (more precisely, after making b_1 more specific below, it is $\nu_2\big(2k\binom{a2^n}{2k}\big)$), as it follows by Theorem 3.6:

$$\begin{aligned}
\binom{a2^{n+1}+1}{4k} - \binom{a2^n+1}{2k} = \\
&= \frac{a2^{n+1}+1}{a2^{n+1}+1-4k}\binom{a2^{n+1}}{4k} - \frac{a2^n+1}{a2^n+1-2k}\binom{a2^n}{2k} \\
&= \frac{a2^{n+1}+1}{a2^{n+1}+1-4k}\binom{a2^n}{2k}(1+b_1 2^{n+1}) - \frac{a2^n+1}{a2^n+1-2k}\binom{a2^n}{2k} \\
&= \left(\frac{a2^{n+1}+1}{a2^{n+1}+1-4k} - \frac{a2^n+1}{a2^n+1-2k} + b_2 2^{n+1}\right)\binom{a2^n}{2k} \\
&\equiv \frac{2k}{(a2^{n+1}+1-4k)(a2^n+1-2k)}\binom{a2^n}{2k} \equiv a2^n\binom{a2^n-1}{2k-1} \\
&\equiv 0 \pmod{2^n}
\end{aligned}$$

where $b_i, i = 1$ and 2 are some numbers with $\nu_2(b_i) \ge 0$. □

Apparently, cases with $b \ge 2$ call for more refined methods. It also appears that proving Conjecture 5.5 for $p = 2$ might require congruences modulo 2^{n+1} for both $\binom{a2^{n+1}}{2(2k)}(C_{2k} - C_k)$ in (4.7) and $\binom{a2^n}{2(1+2K)}C_{1+2K}$ in (4.8). In fact, it helped proving Theorem 5.6 (cf. Section 5 below).

Now we prove Theorem 4.1.

The proof of Theorem 4.1. For the 2-adic orders of the terms of $M'_{2^q}(n, 2^{q+1})$ with $1 \le q \le n$, we get that

$$\begin{aligned}
\nu_2\left(\binom{2^{n+1}}{2k}C_k\right) &= n - \nu_2(k) + \nu_2(C_k) = n - q + d_2(1 + 2^q + 2^{q+1}K) - 1 \\
&= n - q + 1 + d_2(K) \ge n - q + 1,
\end{aligned}$$

and the lower bound is met exactly if $K = 0$.

If $q = 0$ then we have $\nu_2(M'_1(n, 2)) \ge n$ by (4.3). In fact, as it was explained above in the proof of Theorem 2.1 but now using $n+1$ rather than n and $0 \le k =$

$1 + 2K \leq 2^n$, i.e., $0 \leq K \leq 2^{n-1} - 1$ in the summation resulting in $M_1'(n, 2)$, the minimum 2-adic value n is taken by n terms with $K = 2^r - 1, r = 0, 1, \ldots, n-1$. Therefore, for the 2-adic order of the sum, we get n exactly if n is odd. □

We also have the following

Theorem 4.2. *For integers $n \geq q \geq 1$, we have*

$$\nu_2(M'_{2^q,a}(n, 2^{q+1})) = n + 1 - q.$$

If $q = 0$ then $\nu_2(M'_{1,a}(n, 2)) = n$ if n is odd, otherwise the 2-adic order is at least $n + 1$.

We omit the proof but mention that the case with $q = 0$ has already been proven in the proof of Theorem 2.1 by using (4.6) while the case with $1 \leq q \leq n$ can be taken care of similarly to the proof of Theorem 4.1.

5. More proofs, facts, and conjectures for Motzkin numbers

Here we present the proofs of Theorems 2.2, 2.3, and 2.5, and four conjectures on the order of the difference of certain Motzkin numbers including cases with any prime $p \geq 3$.

Proof of Theorem 2.2. We use a recurrence for the Motzkin numbers:

$$M_m = \frac{3(m-1)M_{m-2} + (2m+1)M_{m-1}}{m+2}, m \geq 0, \tag{5.1}$$

with $m = a2^{n+1} + b$ and $a2^n + b$. We take the difference and simplify it. It turns out that the common denominator on the right hand side is odd when b is odd and has 2-adic order $2\nu_2(b+2)$ when b is even. In the numerator only the two terms $3(b-1)(b+2)\big(M_{a2^{n+1}+b-2} - M_{a2^n+b-2}\big)$ and $(2b+1)(b+2)\big(M_{a2^{n+1}+b-1} - M_{a2^n+b-1}\big)$, and possibly two additive terms with 2-adic order at least n matter (due to the possibility that either $\nu_2(2^n 3M_{a2^{n+1}+b-1}) = n$ or $\nu_2(2^n 9M_{a2^{n+1}+b-2}) = n$ or both). The details are straightforward. □

Proof of Theorem 2.3. We prove by induction on b for any fixed $a \geq 1$ odd since it suffices to consider only such values of a. The cases with $b = 0$ and 1 are covered by Theorem 2.1. Assume that the statement is true for all values $0, 1, \ldots, b-2, b-1$. We set $K' = K + 2\nu_2(b+2)$ and $n_0 = n_0(a, b, K) = \max\{n_0(a, b-2, K'), n_0(a, b-1, K')\}$ and apply Theorem 2.2 which yields that $\nu_2(M_{a2^{n+1}+b} - M_{a2^n+b}) \geq K' - 2\nu_2(b+2) = K$ for $n \geq n_0(a, b, K)$. □

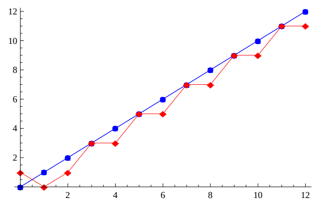
Further numerical evidence suggests a refinement of Corollary 2.4 on the rate of growth (cf. Figure 1 for illustration).

Conjecture 5.1. *For all integers $a \geq 1$ odd, $b \geq 0$ and n sufficiently large, there exist two constants $c_1(a,b)$ and $c_2(a,b)$ so that $n - c_1(a,b) \leq \nu_2(M_{a2^{n+1}+b} - M_{a2^n+b}) \leq n + c_2(a,b)$. In particular, we have $c_1(1,b) \leq c\log_2 b$ with some constant $c > 0$, $c_2(1,b) \leq 1$, and $c_1(1,2^q-1) \leq q$ and $c_2(1,2^q) \leq -1$ for $q \geq 2$.*

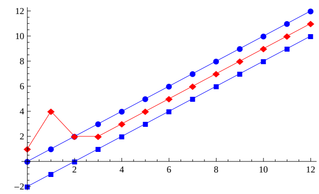
We also believe that following conjecture is true.

Conjecture 5.2. *The sequences $\{\nu_2(M_{2^{n+1}+b} - M_{2^n+b})\}_{n \geq n_0}$ with $b = 2^q$ and $b = 2^q + 1, q \geq 1$, become identical for some sufficiently large $n_0 = n_0(q)$.*

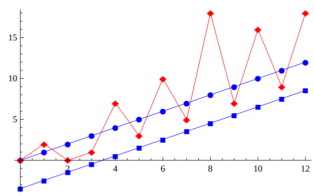
This means that, in this special case, equality (2.1) holds with a value which is less than n in Theorem 2.2. By the way, this seems to happen in many cases when we compare $M_{2^{n+1}+b} - M_{2^n+b}$ with $M_{2^{n+1}+b+1} - M_{2^n+b+1}$ with b even.



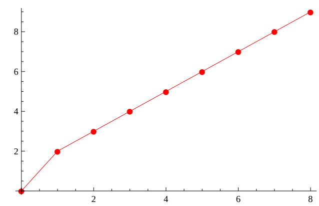
(a) $p = 2, a = 1, b = 1$ (which agrees with $a = 1, 5, 9$, or 13, and $b = 0$ for $n \geq 1$, cf. Theorem 2.1, Conjectures 5.3 and 5.5)



(b) $p = 2, a = 1, b = 4$ (which agrees with $a = 1, b = 5$ for $n \geq 3$, cf. Conjecture 5.2)



(c) $p = 2, a = 3, b = 11$, cf. Conjecture 5.1



(d) $p = 3, a = 2, b = 0$, cf. Conjecture 5.5

Figure 1: The function $\nu_p(M_{ap^{n+1}+b} - M_{ap^n+b}), 0 \leq n \leq 12$ (with $y = n$ and $n - \log_2 b$ included for $p = 2$)

We have a "conditional proof" of Conjecture 5.2 under assumptions on $c_1(1, 2^q - 1)$ and $c_2(1, 2^q)$. The inequalities of Conjecture 5.1 combined with equality (2.1) would already prove Conjecture 5.2 for $q \geq 2$. Indeed, in this case we have $\nu_2(M_{2^{n+1}+2^q+1} - M_{2^n+2^q+1}) = \nu_2(M_{2^{n+1}+2^q} - M_{2^n+2^q})$ since $\nu_2(M_{2^{n+1}+2^q-1} - M_{2^n+2^q-1}) + \nu_2(2^q + 1 - 1) \geq n - c_1(1, 2^q - 1) + q \geq n > n - 1 \geq n + c_2(1, 2^q) \geq \nu_2(M_{2^{n+1}+2^q} - M_{2^n+2^q})$.

This argument would not work for $q = 1$, i.e., for $b = 2$ and 3. However, by assuming the "right" patterns for $b = 1$ and 2, we can prove the case with $b = 3$.

Indeed, Conjecture 5.3 and equality (2.1) immediately imply the statement of Conjecture 5.2 for n odd and $b = 3$. If n is even and $b = 3$ then a slight fine tuning in the proof of Theorem 2.2 will suffice since $\nu_2(M_{2^{n+1}+2}) = 1$ for $n \geq 3$ and $\nu_2(M_{2^{n+1}+1}) = 0$ for $n \geq 1$ (by Conjecture 5.3 and the facts that $\nu_2(M_{18}) = 1$ and $\nu_2(M_5) = 0$).

We add that Theorem 2.1 states a similar fact about identical sequences with $b = 0$ and 1 for a odd and n even.

Conjecture 5.3. *If $n \geq 2$, and $b = 0$ or 1 then*

$$\nu_2(M_{2^{n+1}+b} - M_{2^n+b}) = \begin{cases} n-1, & \text{if } n \text{ is even,} \\ n, & \text{if } n \text{ is odd.} \end{cases}$$

If $n \geq 3$, and $b = 2$ then

$$\nu_2(M_{2^{n+1}+b} - M_{2^n+b}) = \begin{cases} n, & \text{if } n \text{ is even,} \\ n-2, & \text{if } n \text{ is odd.} \end{cases}$$

Remark 5.4. The case with $b = 0$ or 1, and $n \geq 2$ even has already been proven as part of Theorem 2.1 (with $a = 1$). On the other hand, we obtained only a lower bound if n is odd and otherwise, this case remains open. Therefore, the former case can be left out from the conjecture and was included only for the sake of uniformity.

The case with $a = 1$ and $b = 0$ is further extended in

Conjecture 5.5. *For $p = 2$, $a \equiv 1 \pmod 4$, and $n \geq 2$, we have*

$$\nu_2(M_{a2^{n+1}} - M_{a2^n}) = n, \text{ if } n \text{ is odd.}$$

For $p = 3$, $(a, 3) = 1$, and $n \geq n_0 = n_0(a)$ with some integer $n_0(a) \geq 0$, we have

$$\nu_3(M_{a3^{n+1}} - M_{a3^n}) = n + \nu_3\left(\binom{2a}{a}\right).$$

For $p \geq 5$ prime and $n \geq n_0 = n_0(p)$ with some integer $n_0(p) \geq 0$, we have

$$\nu_p(M_{p^{n+1}} - M_{p^n}) = n.$$

The panels (a) and (d) of Figure 1 demonstrate this conjecture in some cases with $0 \leq n \leq 12$. If $p = 2$, $a \geq 1$ any odd, and $n \geq 2$ even then the 2-adic order is $n - 1$ as it has already been proven in Theorem 2.1.

The proof of Theorem 2.5. We give only a sketch of the proof.

We prove the case with $b = 0$ first and use the IIDCS

$$\{ip^q \pmod{p^{q+1}}\}_{i=1,2,\dots,p-1;q\geq 0}$$

which allows us to write every positive integer uniquely in the form of $ip^q + Kp^{q+1}$ with some integers $K \geq 0$, i, and q. In a similar fashion to the proof of Theorem 2.1, the difference of the appropriate Motzkin numbers can be rewritten as

$$\begin{aligned} M_{p^{n+1}} - M_{p^n} &= \sum_{k=1}^{p^n/2} \left(\binom{p^{n+1}}{p(2k)} C_{pk} - \binom{p^n}{2k} C_k \right) + \sum_{i=1}^{p-1} \Big(\sum_{k \equiv i \bmod p}^{p^{n+1}/2} \binom{p^{n+1}}{2k} C_k \Big) \\ &= \sum_{q=0}^{n-1} \sum_{i=1}^{p-1} \Bigg(\sum_{\substack{k=ip^q+Kp^{q+1} \\ 0 \leq K \leq \frac{p^{n-q}-2i}{2p}}} \left(\binom{p^{n+1}}{p(2k)} C_{pk} - \binom{p^n}{2k} C_k \right) \Bigg) \qquad (5.2) \\ &+ \sum_{i=1}^{p-1} \Big(\sum_{k \equiv i \bmod p}^{p^{n+1}/2} \binom{p^{n+1}}{2k} C_k \Big) \qquad (5.3) \end{aligned}$$

after removing the superfluous term with $k = 0$ in the first sum. The first term (5.2) can be rewritten as

$$\begin{aligned} &\sum_{q=0}^{n-1} \sum_{i=1}^{p-1} \sum_{\substack{k=ip^q+Kp^{q+1} \\ 0 \leq K \leq \frac{p^{n-q}-2i}{2p}}} \left(\binom{p^{n+1}}{p(2k)} C_{pk} - \binom{p^n}{2k} C_k \right) \\ &= \sum_{q=0}^{n-1} \sum_{i=1}^{p-1} \sum_{\substack{k=ip^q+Kp^{q+1} \\ 0 \leq K \leq \frac{p^{n-q}-2i}{2p}}} \left(\binom{p^{n+1}}{p(2k)} (C_{pk} - C_k) + \left(\binom{p^{n+1}}{p(2k)} - \binom{p^n}{2k} \right) C_k \right). \end{aligned}$$

For the p-adic order of every term in the summation, we obtain that $\nu_p(\binom{p^{n+1}}{p(2k)}(C_{pk} - C_k)) \geq n - q + q = n$ by Theorem 3.8, and $\nu_p((\binom{p^{n+1}}{p(2k)} - \binom{p^n}{2k})C_k) \geq n + 2$ by Theorem 3.6 and Remark 3.7.

Clearly, the p-adic order of every term in (5.3) is at least $n + 1$.

Unfortunately, the above treatment cannot be easily extended to higher values of b, however, recurrence (5.1) comes to the rescue. Indeed, if $p = 3$ and $b = 1$, or $p \geq 5$ and $1 \leq b \leq p - 3$ then we use (5.1) with $m = p^{n+1} + b$ and $p^n + b$, and by easily adapting the proof of Theorem 2.2, we prove the statement step by step for $b = 1$, then for $b = 2$, ..., and finally for $b = p - 3$. In the initial case of $b = 1$, the multiplying factor $m - 1$ of M_{m-2} in (5.1) is divisible by p^n in both settings of m while the terms with M_{m-1} are covered by the case of $b = 0$. Starting with $b = 2$, we can use the already proven statement with $b - 1$ and $b - 2$. This proof cannot be directly extended beyond $b = p - 3$ since the common denominator in the recurrence has p-adic order $2\nu_p(b + 2)$, and this is the reason for the potential drop in the 3-adic order when $b = 1$. □

Note that we have recently succeeded in proving the following extensions and improvements to Conjecture 5.5 and Theorem 2.5 in [8], by applying congruential recurrences and refining the techniques used in this paper. The last part of the first theorem confirms Conjecture 5.5 for $p = 2$ and $a = 1$ given that n is odd. The case with n even has been settled by Theorem 2.1.

Theorem 5.6. *For $p = 2$, we have that*

$$M(2^{n+1}) - M(2^n) = \begin{cases} 3 \cdot 2^{n-1} \bmod 2^{n+1}, & \text{if } n \geq 4 \text{ and even,} \\ 2^n \bmod 2^{n+1}, & \text{if } n \geq 3 \text{ and odd.} \end{cases}$$

For $n \geq 2$, we have

$$\nu_2(M(2^{n+1}) - M(2^n)) = \begin{cases} n-1, & \text{if } n \text{ is even,} \\ n, & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 5.7. *For any prime $p \geq 3$ and integer $n \geq 2$, we have that $\nu_p(M(p^{n+1}) - M(p^n)) = n$. In particular, with the Legendre symbol $(\frac{p}{3})$, we have*

$$M(p^{n+1}) - M(p^n) \equiv \begin{cases} \frac{p-1}{2} p^n \bmod p^{n+1}, & \text{if } (\frac{p}{3}) \equiv 0 \text{ or } 1 \bmod p, \\ (\frac{p+1}{4} + (-1)^n \frac{p-3}{4}) p^n \bmod p^{n+1}, & \text{if } (\frac{p}{3}) \equiv -1 \bmod p. \end{cases}$$

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On the Fibonacci distances of ab, ac and bc*

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Abstract

For a positive real number x let the Fibonacci distance $\|x\|_F$ be the distance from x to the closest Fibonacci number. Here, we show that for integers $a > b > c \geq 1$, we have the inequality

$$\max\{\|ab\|_F, \|ac\|_F, \|bc\|_F\} > \exp(0.034\sqrt{\log a}).$$

Keywords: Fibonacci distance, Fibonacci diophantine triples

MSC: 11D72

1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. For a positive real number x we put

$$\|x\|_F = \min\{|x - F_n| : n \geq 0\}.$$

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In [4], it was shown that there are no positive integers $a > b > c$ such that $ab + 1 = F_\ell$, $ac + 1 = F_m$ and $bc + 1 = F_n$ for some positive integers ℓ, m, n. Note that if such a triple would exist, then $\max\{\|ab\|_F, \|ac\|_F, \|bc\|_F\} \le 1$. This suggests investigating the more general problem of the triples of positive integers $a > b > c$ in which all three distances $\|ab\|_F$, $\|ac\|_F$ and $\|bc\|_F$ are small. We have the following result.

Theorem 1.1. *If $a > b > c \ge 1$ are integers then*

$$\max\{\|ab\|_F, \|ac\|_F, \|bc\|_F\} > \exp(0.034\sqrt{\log a}).$$

We have the following numerical corollary.

Corollary 1.2. *If $a > b > c \ge 1$ are positive integers such that*

$$\max\{\|ab\|_F, \|ac\|_F, \|bc\|_F\} \le 2,$$

then $a \le \exp(415.62)$. In fact, the solution with maximal a of the above inequality is the following:

$$(a, b, c) = (235, 11, 1).$$

2. The proof of Theorem 1.1

2.1. Preliminary results

We put $(\alpha, \beta) = ((1 + \sqrt{5})/2, (1 - \sqrt{5})/2)$ and recall the Binet formula

$$F_k = \frac{\alpha^k - \beta^k}{\sqrt{5}} \qquad \text{valid for all} \qquad k \ge 0. \tag{2.1}$$

We write $(L_k)_{k\ge0}$ for the Lucas companion of the Fibonacci sequence $(F_k)_{k\ge0}$ given by $L_0 = 2$, $L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \ge 0$. Its Binet formula is $L_k = \alpha^k + \beta^k$ for all $k \ge 0$. Furthermore, the inequalities

$$\alpha^{k-2} \le F_k \le \alpha^{k-1} \qquad \text{and} \qquad \alpha^{k-1} \le L_k \le \alpha^{k+1} \qquad \text{hold for all} \qquad k \ge 1. \tag{2.2}$$

We put

$$M = \max\{\|ab\|_F, \|ac\|_F, \|bc\|_F\}. \tag{2.3}$$

Lemma 2.1. *We have $M \ge 1$.*

Proof. Assume that $M = 0$. Then

$$6 \le ab = F_n, \quad 3 \le ac = F_m, \quad 2 \le bc = F_\ell$$

for some positive integers $n > m > \ell \ge 3$. If $n > 12$, then, by Carmichael's Primitive Divisor Theorem (see [2]), there exists a prime $p \mid F_n$ which does not divide F_k for any $1 \le k < n$. In particular, p cannot divide $F_m F_\ell = F_n c^2$, which is impossible. Thus, $n \le 12$. A case by case analysis shows that there is no solution. □

We put

$$ab + u = F_n, \qquad ac + v = F_m, \qquad bc + w = F_\ell, \tag{2.4}$$

where $|u| = \|ab\|_F$, $|v| = \|ac\|_F$ and $|w| = \|bc\|_F$. In the above, ℓ, m, n are positive integers and since $F_1 = F_2$, we may assume that $\min\{\ell, m, n\} \geq 2$. Furthermore,

$$\max\{|u|, |v|, |w|\} = M.$$

We treat first the case when $a \leq 4M$.

Lemma 2.2. *If $a \leq 4M$, then*

$$\max\{\ell, m, n\} \leq 5 \log(3M).$$

Proof. If $a \leq 4M$, then

$$\alpha^{n-2} \leq F_n = ab + u \leq 4M(4M - 1) + M < 16M^2,$$

so

$$\begin{aligned} n &\leq 2 + \frac{2\log(4M)}{\log \alpha} < 2 + 2.1\log(4M) \\ &= 2 + 2.1\log(4/3) + 2.1\log(3M) \\ &< 2.7 + 2.1\log(3M) < 5\log(3M). \end{aligned}$$

A similar argument works for ℓ and m. □

From now on, we assume that $a > 4M$.

Lemma 2.3. *Assume that $a > 4M$. Then*

(i) $n > \max\{\ell, m\}$;

(ii) $a > \sqrt{F_n}$;

(iii) $n \geq 3$.

Proof. (i) Note that

$$F_n = ab + u \geq ab - M > ac + M \geq ac + v = F_m,$$

where the middle inequality $ab - M > ac + M$ holds because it is equivalent to $a(b-c) > 2M$, which holds because $a > 4M$ and $b > c$, so $b - c \geq 1$. Hence, $n > m$. In the same way,

$$F_n = ab + u \geq ab - M > bc + M \geq bc + w = F_\ell.$$

The middle inequality is $ab - M > bc + M$, which is equivalent to $b(a - c) > 2M$. If $a - c \geq 2M$, then indeed $b(a - c) > 2M$ because $b > 1$. If $a - c < 2M$, it follows that $b > c > a - 2M > 2M$ (because $a > 4M$), and $a - c > 1$, so again the inequality $b(a-c) > 2M$ holds. This implies (i).

(ii) Here, by the previous argument, we have

$$a^2 > ab + M \geq ab + u = F_n.$$

This implies (ii).

(iii) is a consequence of (i) and of the fact that $\min\{\ell, m\} \geq 2$. □

Lemma 2.4. *When $a > 4M$, it is not possible to have $u = v = 0$.*

Proof. If $u = v = 0$, then, since $n > m$ by (i) of Lemma 2.3, we have

$$a \leq \gcd(ab, ac) = \gcd(F_n, F_m) = F_{\gcd(n,m)} = F_{n/d} \leq \alpha^{n/d-1},$$

where $d > 1$ is some divisor of n and where in the above we used the second inequality in (2.2). Hence, by (ii) of Lemma 2.3 and inequality (2.2), we get

$$\alpha^{n/2-1} \leq \sqrt{F_n} < a \leq \alpha^{n/d-1} \leq \alpha^{n/2-1},$$

a contradiction. □

The following lemma follows immediately by the Pigeon–Hole Principle and is well–known (see Lemma 1 in [3], for example).

Lemma 2.5. *Let $X \geq 3$ be a real number. Let a and b be nonnegative integers with* $\max\{a, b\} \leq X$*. Then there exist integers λ, ν not both zero with* $\max\{|\lambda|, |\nu|\} \leq \sqrt{X}$ *such that $|a\lambda + b\nu| \leq 3\sqrt{X}$.*

2.2. Some biquadratic numbers

We write

$$\begin{aligned} F_n - u &= \frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right) - u = \frac{1}{\sqrt{5}}\left(\alpha^n - (-\alpha^{-1})^n\right) - u \\ &= \frac{\alpha^{-n}}{\sqrt{5}}\left(\alpha^{2n} - \sqrt{5}u\alpha^n - (-1)^n\right) \\ &= \frac{\alpha^{-n}}{\sqrt{5}}\left(\alpha^n - u_{1,n}\right)\left(\alpha^n - u_{2,n}\right). \end{aligned} \tag{2.5}$$

In the above,

$$u_{i,n} = \frac{\sqrt{5}u + (-1)^i\sqrt{5u^2 + 4(-1)^n}}{2}, \qquad i \in \{1, 2\}. \tag{2.6}$$

In the same way,

$$F_m - v = \frac{\alpha^{-m}}{\sqrt{5}}\left(\alpha^m - v_{1,m}\right)\left(\alpha^m - v_{2,m}\right), \tag{2.7}$$

where

$$v_{j,m} = \frac{\sqrt{5}v + (-1)^j\sqrt{5v^2 + 4(-1)^m}}{2}, \qquad j \in \{1,2\}. \tag{2.8}$$

Observe that $u_{2,n} = (-1)^{n+1}u_{1,n}^{-1}$ and $v_{2,m} = (-1)^{m+1}v_{1,m}^{-1}$. Furthermore, both $u_{1,n}$, $u_{2,n}$ are roots of the polynomial

$$f_{u,n}(X) = (X^2 - (-1)^n)^2 - 5u^2X^2 = X^4 - (5u^2 + 2(-1)^n)X^2 + 1.$$

Similarly, both $v_{1,m}$ and $v_{2,m}$ are roots of the polynomial

$$f_{v,m}(X) = (X^2 - (-1)^m)^2 - 5v^2X^2 = X^4 - (5v^2 + 2(-1)^m)X^2 + 1.$$

Put $\mathbb{K} = \mathbb{Q}(\sqrt{5}, u_{1,n}, v_{1,m})$. Then the degree $d = [\mathbb{K} : \mathbb{Q}]$ of $\mathbb{K}$ over $\mathbb{Q}$ is a divisor of 32. Further, $\mathbb{K}$ contains α, $u_{1,n}$, $u_{2,n}$, $v_{1,m}$, $v_{2,m}$ and all their conjugates. It follows easily that all conjugates $u_{i,n}^{(s)}$ for $s = 1, \ldots, d$ satisfy

$$u_{i,n}^{(s)} = \frac{1}{2}\left(\pm\sqrt{5}u \pm \sqrt{5u^2 + 4(-1)^n}\right), \qquad i = 1,2, \qquad s = 1, \ldots, d,$$

therefore the inequality

$$|u_{i,n}^{(s)}| \le \frac{1}{2}\left(\sqrt{5}|u| + \sqrt{5u^2 + 4}\right) \le \frac{1}{2}\left(\sqrt{5}M + \sqrt{5M^2 + 4}\right) < 3M \tag{2.9}$$

holds for $i = 1,2$ and $s = 1, \ldots, d$. Similarly the inequality

$$|v_{j,m}^{(s)}| < 3M \tag{2.10}$$

holds for $j = 1,2$ and $s = 1, \ldots, d$.

2.3. The first upper bound on n

The key step of the proof is writing

$$a \mid \gcd(ab, ac) = \gcd(F_n - u, F_m - v),$$

and passing in the above relation at the level of principal ideals in $\mathcal{O}_{\mathbb{K}}$. Using relations (2.5) and (2.7), we can write in $\mathcal{O}_{\mathbb{K}}$:

$$\begin{aligned} a\mathcal{O}_{\mathbb{K}} &\mid \gcd\left((\alpha^n - u_{1,n})(\alpha^n - u_{2,n})\mathcal{O}_{\mathbb{K}}, (\alpha^m - v_{1,m})(\alpha^m - v_{2,m})\mathcal{O}_{\mathbb{K}}\right) \\ &\mid \prod_{\substack{1\le i\le 2\\ 1\le j\le 2}} \gcd\left((\alpha^n - u_{i,n})\mathcal{O}_{\mathbb{K}}, (\alpha^m - v_{j,m})\mathcal{O}_{\mathbb{K}}\right). \end{aligned} \tag{2.11}$$

Passing to the norms in $\mathbb{K}$, we get

$$a^d = N_{\mathbb{K}/\mathbb{Q}}(a\mathcal{O}_{\mathbb{K}}) \le \prod_{\substack{1\le i\le 2\\ 1\le j\le 2}} N_{\mathbb{K}/\mathbb{Q}}\left(\gcd\left((\alpha^n - u_{i,n})\mathcal{O}_{\mathbb{K}}, (\alpha^m - v_{j,m})\mathcal{O}_{\mathbb{K}}\right)\right). \tag{2.12}$$

For $i, j \in \{1, 2\}$ put

$$I_{i,n,j,m} = \gcd\left((\alpha^n - u_{i,n})\,\mathcal{O}_{\mathbb{K}}, (\alpha^m - v_{j,m})\,\mathcal{O}_{\mathbb{K}}\right). \tag{2.13}$$

In order to bound the norm of $I_{i,n,j,m}$ in $\mathbb{K}$, we use the following lemma.

Lemma 2.6. *When $a > 4M$, there exist coprime integers λ, ν satisfying $\max\{|\lambda|, |\nu|\} \le \sqrt{n}$ such that $|n\lambda + m\nu| \le 3\sqrt{n}$ and*

$$\alpha^{n\lambda+m\nu} - u_{i,n}^{\lambda} v_{j,m}^{\nu} \in I_{i,n,j,m}. \tag{2.14}$$

Proof. The existence of a pair of integers λ, ν not both zero such that the inequalities $\max\{|\lambda|, |\nu|\} \le \sqrt{n}$ and $|n\lambda + m\nu| \le 3\sqrt{n}$ hold follows from Lemma 2.6 for $(a, b, X) = (n, m, X)$. The condition $X \ge 3$ is fulfilled for our case by (iii) of Lemma 2.3. The fact that λ and ν can be chosen to be in fact coprime follows by replacing the pair (λ, ν) by $(\lambda/\gcd(\lambda, \nu), \nu/\gcd(\lambda, \nu))$. Finally, observing that

$$\alpha^n \equiv u_{i,n} \pmod{I_{i,n,j,m}} \qquad \text{and} \qquad \alpha^m \equiv v_{j,m} \pmod{I_{i,n,j,m}},$$

exponentiating the first of the above congruences to power λ, the second to power ν, and multiplying the resulting congruences, we get containment (2.14). □

In what follows, in this section we make the following assumption:

Assumption 2.7. *Assume that that pair (λ, ν) from the conclusion of Lemma 2.6 satisfies*

$$\alpha^{n\lambda+m\nu} - u_{i,n}^{\lambda} v_{j,m}^{\nu} \ne 0 \qquad \textit{for all} \qquad i, j \in \{1, 2\}. \tag{2.15}$$

The main result of this section is the following.

Lemma 2.8. *Under the Assumption 2.7, when $a > 4M$, we have*

$$a \le 2^4 (3M)^{8\sqrt{n}}. \tag{2.16}$$

Proof. By congruence (2.14), we have

$$I_{i,n,j,m} \mid \left(\alpha^{n\lambda+m\nu} - u_{i,n}^{\lambda} v_{j,m}^{\nu}\right) \mathcal{O}_{\mathbb{K}},$$

and taking norms in $\mathbb{K}$ we get

$$N_{\mathbb{K}/\mathbb{Q}}(I_{i,n,j,m}) \mid N_{\mathbb{K}/\mathbb{Q}}\left((\alpha^{n\lambda+m\nu} - u_{i,n}^{\lambda} v_{j,m}^{\nu})\mathcal{O}_{\mathbb{K}}\right) = N_{\mathbb{K}/\mathbb{Q}}(\alpha^{n\lambda+m\nu} - u_{i,n}^{\lambda} v_{j,m}^{\nu}).$$

Since the number appearing on the right above is not zero by Assumption 2.7, we get

$$N_{\mathbb{K}/\mathbb{Q}}(I_{i,n,j,m}) \le N_{\mathbb{K}/\mathbb{Q}}\left(\alpha^{n\lambda+m\nu} - u_{i,n}^{\lambda} v_{j,m}^{\nu}\right),$$

therefore

$$N_{\mathbb{K}/\mathbb{Q}}(I_{i,n,j,m}) \le \prod_{s=1}^{d} \left|(\alpha^{(s)})^{n\lambda+m\nu} - (u_{i,n}^{(s)})^{\lambda}(v_{j,m}^{(s)})^{\nu}\right|.$$

Inequalities (2.9) and (2.10) together with the inequalities for λ and ν from the statement of Lemma 2.6 and the fact that $\alpha^{(s)} \in \{\alpha, \beta\}$ imply that

$$\left|(\alpha^{(s)})^{n\lambda+m\nu} - (u_{i,n}^{(s)})^{\lambda}(v_{j,m}^{(s)})^{\nu}\right| \leq |\alpha|^{3\sqrt{n}} + (3M)^{2\sqrt{n}} < 2(3M)^{2\sqrt{n}},$$

for $s = 1, \ldots, d$, where for the last inequality we used $(3M)^2 \geq 3^2 > \alpha^3$. Hence,

$$N_{\mathbb{K}/\mathbb{Q}}(I_{i,n,j,m}) \leq 2^d(3M)^{2d\sqrt{n}},$$

Thus, by inequality (2.12), we get

$$a^d \leq \prod_{\substack{1\leq i\leq 2\\ 1\leq j\leq 2}} N_{\mathbb{K}/\mathbb{Q}}(I_{i,n,j,m}) \leq 2^{4d}(3M)^{8d\sqrt{n}},$$

giving

$$a \leq 2^4(3M)^{8\sqrt{n}},$$

which is what we wanted to prove. □

Lemma 2.8 has the following consequence.

Lemma 2.9. *Under the Assumption 2.7, when $a > 4M$, we have*

$$n < (41\log(3M))^2. \tag{2.17}$$

Proof. Combining the inequality (2.16) of Lemma 2.8 for a with (ii) of Lemma 2.3 and inequality (2.2), we get

$$\alpha^{n/2-1} \leq \sqrt{F_n} < a \leq 2^4(3M)^{8\sqrt{n}}.$$

It gives

$$\frac{n}{2} - 1 < \frac{4\log 2}{\log\alpha} + \left(\frac{8\log(3M)}{\log\alpha}\right)\sqrt{n} < 5.8 + 16.7\log(3M)\sqrt{n},$$

or

$$n < \left(\frac{13.6}{\log(3M)\sqrt{n}} + 33.4\right)\log(3M)\sqrt{n} < (41\log M)\sqrt{n},$$

because $n \geq 3$. So

$$n < (41\log(3M))^2,$$

which is what we wanted to prove. □

From now on, we assume that

$$n \geq (41\log(3M))^2. \tag{2.18}$$

Lemma 2.2 tells us that if this the case, then also the inequality $a > 4M$ holds. In particular, for such values of n Assumption 2.7 cannot hold. This is the case we study next.

2.4. General remarks when Assumption 2.7 does not hold

From now on, we study the cases when Assumption 2.7 does not hold. In this case, there exist $i_0, j_0 \in \{1,2\}$ such that

$$\alpha^{n\lambda+m\nu} = u_{i_0,n}^{\lambda} v_{j_0,m}^{\nu}. \tag{2.19}$$

In particular

$$(\alpha^4)^{n\lambda+m\nu} = (u_{i_0,n}^4)^{\lambda}(v_{j_0,m}^4)^{\nu}. \tag{2.20}$$

Observe that if $u = 0$, then

$$u_{i,n} = (-1)^i\sqrt{(-1)^n}, \qquad i \in \{1,2\},$$

therefore $u_{i_0,4}^4 = 1$. Similarly, if $v = 0$, then $v_{j_0,m}^4 = 1$. If $u \neq 0$, then write

$$5u^2 + 4(-1)^n = d_{u,n}y_{u,n}^2,$$

where $d_{u,n}$ is a positive square free integer and $y_{u,n}$ is some positive integer. Observe that $d_{u,n}$ is coprime to 5 so $5d_{u,n}$ is square free. Observe further that $5u^2$ and $d_{u,n}y_{u,n}^2$ have the same parity and

$$u_{i,n}^2 = \frac{1}{2}\left(\frac{5u^2 + d_{u,n}y_{u,n}^2}{2} + (-1)^i\sqrt{5d_{u,n}}\,uy_{u,n}\right) \in \mathbb{Q}(\sqrt{5d_{u,n}}) = \mathbb{K}_{u,n}$$

for $i = 1,2$. Moreover, $u_{1,n}^2$ is an algebraic integer and a unit in the quadratic field $\mathbb{K}_{u,n}$ the inverse of which is $u_{2,n}^2$. Similarly, if $v \neq 0$, we write

$$5v^2 + 4(-1)^m = d_{v,m}y_{v,m}^2,$$

where $d_{v,m}$ is some positive square free integer and $y_{v,m}$ is some positive integer. As in the case of $u_{i,n}^2$, we have

$$v_{j,m}^2 \in \mathbb{Q}(\sqrt{5d_{v,m}}) = \mathbb{K}_{v,m}$$

is a unit in the quadratic field $\mathbb{K}_{v,m}$. We continue with the following result.

Lemma 2.10. *In case when $uv \neq 0$, and inequality* (2.18) *holds, it is not possible that $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{5d_{u,n}})$ and $\mathbb{Q}(\sqrt{5d_{v,m}})$ are three distinct quadratic fields.*

Proof. Assume that the three quadratic fields $\mathbb{Q}(\sqrt{5})$, $\mathbb{K}_{u,n}$ and $\mathbb{K}_{v,m}$ were distinct. Then $d_{u,n}$ and $d_{v,m}$ are distinct square free integers larger than 1 which are coprime to 5. By Galois theory, there is an automorphism of $\mathbb{Q}(\sqrt{5}, \sqrt{5d_{u,n}}, \sqrt{5d_{v,m}})$, let's call it σ, such that $\sigma(\sqrt{5}) = -\sqrt{5}$, $\sigma(\sqrt{d_{u,n}}) = -\sqrt{d_{u,n}}$ and $\sigma(\sqrt{d_{v,m}}) = -\sqrt{d_{v,m}}$. Observe that σ leaves both $\sqrt{5d_{u,n}}$ and $\sqrt{5d_{v,m}}$ invariant, therefore $\sigma(u_{i,n}^2) = u_{i,n}^2$ and $\sigma(v_{j,m}^2) = v_{j,m}^2$ for $i,j \in \{1,2\}$, while $\sigma(\alpha) = \beta$. Applying σ to the equation (2.20), we get

$$(\beta^4)^{\lambda m+\nu n} = (u_{i_0,n}^4)^{\lambda}(v_{j_0,m}^4)^{\nu}. \tag{2.21}$$

Multiplying relations (2.20) and (2.21), we get

$$1 = (u_{i_0,n}^2)^{4\lambda}(v_{j_0,m}^2)^{4\nu} \qquad \text{or} \qquad (u_{i_0,n}^2)^{4\lambda} = (v_{j_0,m}^2)^{-4\nu}.$$

Thus, $u_{i_0,n}^{4\lambda}$ is in $\mathbb{Q}(\sqrt{5d_{u,n}}) \cap \mathbb{Q}(\sqrt{5d_{v,m}}) = \mathbb{Q}$. Since $u_{i_0,n}^2$ is in fact a positive unit ditinct from 1 in $\mathbb{K}_{u,n}$, we get that $\lambda = 0$, and then also $\nu = 0$, which is not allowed. □

We now put

$$\mathbb{U} = \mathbb{Q}(\sqrt{5}, u_{1,n}^4, v_{1,m}^4).$$

If $u = 0$, then $u_{1,n}^4 = 1$, so that $\mathbb{U}$ has degree 2 or 4 over $\mathbb{Q}$. The same holds when $v = 0$. Finally, when $uv \neq 0$, then $u_{1,n}^4 \in \mathbb{Q}(\sqrt{5d_{u,n}})$ and $v_{1,m}^4 \in \mathbb{Q}(\sqrt{5d_{v,m}})$, so

$$\mathbb{U} \subseteq \mathbb{Q}(\sqrt{5}, \sqrt{5d_{u,n}}, \sqrt{5d_{v,m}}).$$

Lemma 2.9 implies that the field appearing in the right hand side of the above containment cannot have degree 8 over $\mathbb{Q}$. Hence, $\mathbb{U}$ must have degree 2 or 4 over $\mathbb{Q}$ in case $uv \neq 0$ as well.

We shall refer to the case when $[\mathbb{U} : \mathbb{Q}] = 4$ as the *rank two case*, and to the case when $[\mathbb{U} : \mathbb{Q}] = 2$ as the *rank one case*.

2.5. The rank two case

We start with the following result.

Lemma 2.11. *Assume that inequality* (2.18) *holds. Then in the rank two case, we have* $uv \neq 0$.

Proof. Assume, for example, that $u = 0$. Then, since we are in the rank two case, it follows that $d_{v,m} > 1$. Now equation (2.20) implies that

$$(\alpha^4)^{n\lambda n + m\nu} = (u_{i_0,n}^4)^{\lambda}(v_{j_0,m}^4)^{\nu} = (v_{j_0,m}^4)^{\nu}.$$

This shows that $(v_{j_0,m}^4)^{\nu} \in \mathbb{Q}(\sqrt{5}) \cap \mathbb{Q}(\sqrt{5d_{v,m}}) = \mathbb{Q}$. Since $v_{j_0,m}^2$ is in fact a unit of infinite order in $\mathbb{K}_{v,m}$, we get that $\nu = 0$, which implies that also $n\lambda + m\nu = 0$, therefore $n\lambda = 0$. Thus, $\lambda = \nu = 0$, which is not allowed. The same contradiction is obtained when $v = 0$. □

Lemma 2.12. *Assume that inequality* (2.18) *holds. Then in the rank two case, we have* $d_{u,n} = d_{v,m} > 1$.

Proof. If this were not so, then we would either have $d_{u,n} = 1$ and $d_{v,m} > 1$ or $d_{u,n} > 1$ and $d_{v,m} = 1$. Assume say that $d_{u,n} = 1$ and $d_{v,m} > 1$. Then $u_{i_0,n}^4 \in \mathbb{Q}(\sqrt{5})$. Relation (2.20) now shows that

$$(\alpha^4)^{n\lambda + m\nu}(u_{i_0,n}^{-4})^{\lambda} = (v_{j_0,m}^4)^{\nu}.$$

The above relation shows that $(v_{j_0,m}^4)^\nu \in \mathbb{Q}(\sqrt{5}) \cap \mathbb{Q}(\sqrt{5d_{v,m}}) = \mathbb{Q}$. This implies easily that $\nu = 0$. Now relation (2.20) shows that $(\alpha^4)^{n\lambda} = (u_{i_0,n}^4)^{n\lambda}$. Since λ and $\nu = 0$ are coprime, we get that $\lambda = 1$, and so $\alpha^{4n} = u_{i_0,n}^4$. This shows that $\alpha^n = \pm u_{i_0,n}$. In particular,

$$\alpha^n = |u_{i_0,n}| < 3M$$

(see inequality (2.9)), so that

$$n \leq \frac{\log(3M)}{\log \alpha} < 3\log(3M),$$

which contradicts inequality (2.18). □

Lemma 2.13. *Assume that inequality* (2.18) *holds. Then we cannot be in the rank two case.*

Proof. Assume that we are in the rank two case. By Lemma 2.12, we have $d_{u,n} = d_{v,m} > 1$. Put $D = d_{u,n}$. We then have the following relations

$$5u^2 - Dy_{u,n}^2 = 4(-1)^{n+1};$$
$$5v^2 - Dy_{v,m}^2 = 4(-1)^{m+1}.$$

By a result of Nagell (see Theorem 3 in [5]), we have $n \equiv m \pmod 2$. Further, put $\varepsilon = (-1)^{n+1}$ and let $(X, Y) = (a, b)$ be the minimal solution in positive integers of the Diophantine equation

$$5X^2 - DY^2 = 4\varepsilon. \tag{2.22}$$

Then all other positive integer solutions (X, Y) of the above equation (2.22) are of the form

$$\frac{\sqrt{5}X + \sqrt{D}Y}{2} = \left(\frac{\sqrt{5}a + \sqrt{D}b}{2}\right)^k$$

for some odd positive integer k. In particular, putting $\zeta = (\sqrt{5}a + \sqrt{D}b)/2$, we then have

$$\frac{\sqrt{5}|u| + \sqrt{D}y_{u,n}}{2} = \zeta^{k_u} \qquad \text{and} \qquad \frac{\sqrt{5}|v| + \sqrt{D}y_{v,m}}{2} = \zeta^{k_v}$$

for some odd positive integers k_u and k_v. We now see invoking (2.6) that

$$u_{i,n} = \operatorname{sign}(u)\left(\frac{\sqrt{5}|u| + (-1)^i \operatorname{sign}(u)\sqrt{D}y_{u,n}}{2}\right) = \operatorname{sign}(u)\zeta^{\eta_{i,u}k_u},$$

where $\eta_{i,u} = 1$ if $\operatorname{sign}(u) = (-1)^i$ and $\eta_{i,u} = -1$ if $\operatorname{sign}(u) = (-1)^{i+1}$. Similarly,

$$v_{j,m} = \operatorname{sign}(v)\zeta^{\eta_{j,v}k_v}$$

where $\eta_{j,v} \in \{\pm 1\}$. Going back to relation (2.19), we get

$$\alpha^{n\lambda+m\nu} = \operatorname{sign}(u)^{\lambda}\operatorname{sign}(v)^{\nu}\zeta^{\eta_{i_0,u}\lambda k_u+\eta_{j_0,v}\nu k_v}.$$

Since α and ζ are multiplicatively independent, we get that

$$n\lambda + m\nu = 0, \qquad \operatorname{sign}(u)^{\lambda}\operatorname{sign}(v)^{\nu} = 1, \qquad \eta_{i_0,u}\lambda k_u + \eta_{j_0,v}\nu k_v = 0.$$

From the left relation above we get that λ and ν have opposite signs. From the right relation above, we get that $\lambda/\nu = -\eta_{j_0,v}\eta_{i_0,u}k_v/k_u$, and since λ and ν are coprime, we get that they are both odd and that $\eta_{i_0,u} = \eta_{j_0,v}$. Finally, since λ and ν are both odd, from the middle relation above we get that $\operatorname{sign}(u) = \operatorname{sign}(v)$. Put $e = \gcd(k_u, k_v)$. Writing $k_u = e\ell_u$, $k_v = e\ell_v$, and putting $\delta = \operatorname{sign}(u)$ and $\eta = \eta_{i_0,u}$, we get that

$$u_{i_0,n} = \delta(\zeta^{\eta e})^{\ell_u} = (\delta\zeta^{\eta e})^{\ell_u} \qquad \text{and} \qquad v_{j_0,m} = \delta(\zeta^{\eta e})^{\ell_v} = (\delta\zeta^{\eta e})^{\ell_v}.$$

Writing $\zeta_1 = \delta\zeta^{\eta e}$, we get that

$$u_{i_0,n} = \zeta_1^{\ell_u} \qquad \text{and} \qquad v_{j_0,m} = \zeta_1^{\ell_v}.$$

Further, $\ell_u/\ell_v = k_u/k_v = -\nu/\lambda = n/m$, so that if we put $k = \gcd(m, n)$, then $n = \ell_u k$ and $m = \ell_v k$. Since $u_{1,n}u_{2,n} = \varepsilon = v_{1,m}v_{2,m}$, it follows that if i_1 and j_1 are such that $\{i_0, i_1\} = \{j_0, j_1\} = \{1, 2\}$, then

$$u_{i_1,n} = \varepsilon\zeta_1^{-\ell_u} = \zeta_2^{\ell_u} \qquad \text{and} \qquad v_{j_1,m} = \varepsilon\zeta_1^{-\ell_v} = \zeta_2^{\ell_v},$$

where $\zeta_2 = \varepsilon\zeta_1^{-1}$. Thus,

$$\begin{aligned}
\alpha^n - u_{i_0,n} &= (\alpha^k)^{\ell_u} - \zeta_1^{\ell_u};\\
\alpha^n - u_{i_1,n} &= (\alpha^k)^{\ell_u} - \zeta_2^{\ell_u};\\
\alpha^m - v_{j_0,m} &= (\alpha^k)^{\ell_v} - \zeta_1^{\ell_v};\\
\alpha^m - v_{j_1,m} &= (\alpha^k)^{\ell_v} - \zeta_2^{\ell_v}.
\end{aligned}$$

Since ℓ_u and ℓ_v are coprime, it follows that

$$I_{i_0,n,j_0,m} = \gcd\left(\left((\alpha^k)^{\ell_u} - \zeta_1^{\ell_u}\right)\mathcal{O}_{\mathbb{K}}, \left((\alpha^k)^{\ell_v} - \zeta_1^{\ell_v}\right)\mathcal{O}_{\mathbb{K}}\right) = (\alpha^k - \zeta_1)\mathcal{O}_{\mathbb{K}}. \quad (2.23)$$

Similarly,

$$I_{i_1,n,j_1,m} = \gcd\left(\left((\alpha^k)^{\ell_u} - \zeta_2^{\ell_u}\right)\mathcal{O}_{\mathbb{K}}, \left((\alpha^k)^{\ell_v} - \zeta_2^{\ell_v}\right)\mathcal{O}_{\mathbb{K}}\right) = (\alpha^k - \zeta_2)\mathcal{O}_{\mathbb{K}}. \quad (2.24)$$

As for $I_{i_0,n,j_1,m}$, we have

$$(\alpha^k)^{\ell_u} \equiv \zeta_1^{\ell_u} \pmod{I_{i_0,n,j_1,m}} \qquad \text{and} \qquad (\alpha^k)^{\ell_v} \equiv \zeta_2^{\ell_v} \pmod{I_{i_0,n,j_1,m}}.$$

Exponentiating the first congruence above to ℓ_v and the second to ℓ_u, and comparing the resulting congruences, we get

$$\zeta_1^{\ell_u \ell_v} \equiv \zeta_2^{\ell_u \ell_v} \pmod{I_{i_0,n,j_1,m}}$$

so that

$$I_{i_0,n,j_1,m} \mid (\zeta_1^{2\ell_u \ell_v} - \varepsilon)\mathcal{O}_{\mathbb{K}}, \tag{2.25}$$

and the principal ideal on the right above is not zero. Similarly,

$$I_{i_1,n,j_0,m} \mid (\zeta_2^{2\ell_u \ell_v} - \varepsilon)\mathcal{O}_{\mathbb{K}}. \tag{2.26}$$

Hence, divisibility relation (2.11) together with relations (2.23)–(2.26) now implies

$$a \mid (\alpha^k - \zeta_1)(\alpha^k - \zeta_2)(\zeta_1^{2\ell_u \ell_v} - \varepsilon)(\zeta_2^{2\ell_u \ell_v} - \varepsilon).$$

Taking norms in $\mathbb{K}$, we get that

$$a^d \le |N_{\mathbb{K}/\mathbb{Q}}(\alpha^k - \zeta_1)||N_{\mathbb{K}/\mathbb{Q}}(\alpha^k - \zeta_2)||N_{\mathbb{K}/\mathbb{Q}}(\zeta_1^{2\ell_u \ell_v} - \varepsilon)||N_{\mathbb{K}/\mathbb{Q}}(\zeta_2^{2\ell_u \ell_v} - \varepsilon)|. \tag{2.27}$$

Since

$$u_{i_0,n}^{(s)} = (\zeta_1^{(s)})^{\ell_u}$$

and $\ell_u \ge 1$, it follows, by (2.9), that

$$|\zeta_1^{(s)}| < 3M.$$

Similarly, $|\zeta_2^{(s)}| < 3M$. Furthermore,

$$\zeta \ge \frac{\sqrt{5} + \sqrt{3}}{2} > \alpha.$$

Since

$$\zeta^{e\ell_u} = |u_{i,n}| \qquad \text{for some} \qquad i \in \{1, 2\},$$

we get that

$$\ell_u \le e\ell_u \le \frac{\log(3M)}{\log \alpha} < 2.1 \log(3M).$$

Similarly, $\ell_v \le 2.1 \log(3M)$. It now follows that

$$|(\alpha^{(s)})^k - \zeta_1^{(s)}| \le \alpha^k + 3M \le 6M\alpha^k \qquad \text{for all} \qquad s = 1, \ldots, d.$$

Similarly,

$$|(\alpha^{(s)})^k - \zeta_2^{(s)}| \le \alpha^k + 3M \le 6M\alpha^k \qquad \text{for all} \qquad s = 1, \ldots, d.$$

Finally,

$$|(\zeta_1^{(s)})^{2\ell_u \ell_v} - \varepsilon| \le (|(\zeta_1^{(s)})^{\ell_u}|)^{2\ell_v} + 1 = |u_{i_0,n}^{(s)}|^{2\ell_v} + 1 < 2(3M)^{4.2 \log(3M)},$$

for all $s = 1, \ldots, d$ and a similar inequality holds with ζ_1 replaced by ζ_2. We thus get that

$$|N_{\mathbb{K}/\mathbb{Q}}(\alpha^k - \zeta_i)| < (6M)^d \alpha^{dk}, \qquad |N_{\mathbb{K}/\mathbb{Q}}(\zeta_i^{2\ell_u \ell_v} - \varepsilon)| < 2^d (3M)^{4.2d \log(3M)}$$

for $i = 1, 2$, which together with (2.27) gives

$$a^d < (6M)^{2d} \alpha^{2dk} 2^{2d} (3M)^{8.4d \log(3M)},$$

or

$$a < 16(3M)^{2+8.4 \log(3M)} \alpha^{2k}. \tag{2.28}$$

Observe that $k = n/\ell_u = m/\ell_v$, and $n > m$ (by (i) of Lemma 2.3) and $\ell_u > \ell_v$ are odd and coprime. Thus, $\ell_u \geq 3$. If $\ell_u = 3$, then $\ell_v = 1$, so $m = n/3$. If this is the case, then

$$a \leq ac = F_m - v \leq F_m + M < F_m + a/2$$

(because $a > 4M$), therefore $a < 2F_m = 2F_{n/3}$. With (ii) of Lemma 2.3 and inequality (2.2), we get

$$\alpha^{n/2-1} < \sqrt{F_n} < a < 2F_{n/3} < 2\alpha^{n/3-1},$$

therefore

$$n < \frac{6 \log 2}{\log \alpha}, \qquad \text{so} \qquad n \leq 4,$$

a contradiction. Thus, we conclude that it is not possible that $\ell_u = 3$. Thus, $\ell_u \geq 5$. Hence, $k \leq n/5$. Inequality (2.28) together with (ii) of Lemma 2.3 and (2.2) give

$$\alpha^{n/2-1} < \sqrt{F_n} < a < 16(3M)^{2+8.4 \log(3M)} \alpha^{2n/5}.$$

Then

$$\begin{aligned} \frac{n}{10} &< 1 + \frac{\log 16}{\log \alpha} + \left(\frac{2 + 8.4 \log(3M)}{\log \alpha} \right) \log(3M) \\ &< 7.8 + 2.1(2 + 8.4 \log(3M)) \log(3M) \\ &< 7.8 + 22(\log(3M))^2, \end{aligned}$$

so

$$n < 78 + 220(\log(3M))^2 < 300(\log(3M))^2,$$

which contradicts inequality (2.18). □

In particular, if inequality (2.18) holds, then we are in the rank one case.

2.6. The rank one case

Lemma 2.14. *Assume that* (2.18) *holds. We have* $u = \pm F_t$ *and* $v = \pm F_s$ *for some nonnegative integers* t, s *which are either zero or satisfy* $n \equiv t \pmod 2$ *and* $m \equiv s \pmod 2$.

Proof. Since we are in the rank one case, it follows that $u^2_{i_0,n} \in \mathbb{Q}(\sqrt{5})$. So, if $u \neq 0$, it follows that $d_{u,n} = 1$, so that $5u^2 + 4(-1)^n = y^2_{u,n}$. In particular, $y^2_{u,n} - 5u^2 = 4(-1)^n$. It is well–known that if (X, Y) are positive integers such that $Y^2 - 5X^2 = 4(-1)^k$ for some integer k, then $X = F_t$ for some nonnegative integer $t \equiv k \pmod 2$ (and the value of Y is L_k). In particular, $|u| = F_t$ for some integer t which is congruent to n modulo 2. The statement about v can be proved in the same way. □

We now have

$$ab = F_n - u = F_n - \operatorname{sign}(u)F_t = F_{(n-t_1)/2}L_{(n+t_1)/2},$$

where $t_1 = \varepsilon_{u,t,n}t$ and $\varepsilon_{u,t,n} \in \{\pm 1\}$ depends on the sign of u as well as on the residue classes of n and t modulo 4. Similarly, we have

$$ac = F_m - v = F_m - \operatorname{sign}(v)F_s = F_{(m-s_1)/2}L_{(m+s_1)/2},$$

and $s_1 = \varepsilon_{v,m,s}s$ for some $\varepsilon_{v,m,s} \in \{\pm 1\}$. Observe also that either $t = 0$, or $t \geq 1$ and

$$\alpha^{t-2} \leq F_t \leq M,$$

so that

$$t \leq 2 + \frac{\log M}{\log \alpha} < 2 + 2.1 \log M < 2.1 \log(3M). \tag{2.29}$$

The same inequality holds with t replaced by $|t_1|$, s, $|s_1|$. Note also that

$$n \pm t_1 \geq n - t > (41 \log(3M))^2 - 2.1 \log(3M) > 0.$$

Lemma 2.15. *One of the following holds:*

(i) $n - t_1 = m - s_1$;

(ii) $n + t_1 = m + s_1$;

(iii) $s = 0$, $m = (n - t_1)/2$ *and* $b = L_{(n+t_1)/2}c$.

Proof. As a warm up, we start with the case when $t = 0$. Then

$$\begin{aligned} a &\leq \gcd(ab, ac) = \gcd(F_n, F_{(m-s_1)/2}L_{(m+s_1)/2}) \\ &\leq \gcd(F_n, F_{(m-s_1)/2}) \gcd(F_n, L_{(m+s_1)/2}) \\ &\leq F_{\gcd(n,(m-s_1)/2)}L_{\gcd(n,(m+s_1)/2)}. \end{aligned}$$

In the above argument, we used the fact that $\gcd(F_p, F_q) = F_{\gcd(p,q)}$ and that $\gcd(F_p, L_q) \le L_{\gcd(p,q)}$ for positive integers p and q. Put

$$\gcd(n, (m-t_1)/2) = n/d_1 \qquad \text{and} \qquad \gcd(n, (m+t_1)/2) = n/d_2.$$

If $d_1 = 1$, then $n \mid (m-t_1)/2$, therefore $n - t_1 > m - t_1 \ge 2n$, or

$$n \le -t_1 \le t < 2.1 \log(3M),$$

contradicting inequality (2.18). A similar inequality holds if $d_2 = 1$. So, from now on, we assume that $\min\{d_1, d_2\} \ge 2$. If $\min\{d_1, d_2\} \ge 10$, we then have

$$\alpha^{n/2-1} < \sqrt{F_n} < a \le F_{n/d_1} L_{n/d_2} \le \alpha^{n/d_1 + n/d_2} \le \alpha^{n/5},$$

giving $n/2 - 1 < n/5$, so $n \le 3$, a contradiction.

So, we may assume that $\min\{d_1, d_2\} \le 9$. Assume that $\max\{d_1, d_2\} \le 9$. Write $n/d_1 = (m - s_1)/d_3$ and $n/d_2 = (m+s_1)/d_4$. If $d_3 \ge d_1 + 1$, we then get

$$m - s_1 = \frac{d_3 n}{d_1} \ge n + \frac{n}{d_1} > m + \frac{n}{d_1},$$

so

$$n < -d_1 s_1 \le d_1 s \le 9 \times 2.1 \log(3M) < 20 \log(3M),$$

contradicting inequality (2.18). Thus, $\max\{d_1, d_2\} \ge 10$. If $\min\{d_1, d_2\} \ge 3$, we then get that

$$\alpha^{n/2-1} < \sqrt{F_n} < a \le F_{n/d_1} L_{n/d_2} \le \alpha^{n/d_1 + n/d_2} \le \alpha^{n/3 + n/10},$$

giving $n < 15$, which is impossible. Thus, $\min\{d_1, d_2\} = 2$ giving

$$\text{either} \qquad n/2 = \gcd(n, (m-s_1)/2), \qquad \text{or} \qquad n/s = \gcd(n, (m+s_1)/2).$$

Thus, either $n/2 = (m - s_1)/2d_3$, or $n/2 = (m+s_1)/2d_4$ for some divisors d_3 or d_4 of $(m - s_1)/2$ and $(m + s_1)/2$, respectively. If we are in the first case and $d_3 > 1$, then

$$m - s_1 = d_3 n \ge 2n > m + n$$

giving $n < -s_1 \le s < 2.1 \log(3M)$, a contradiction. The same inequality is obtained if $n/2 = (m + s_1)/2d_4$ for some divisor $d_4 > 1$ of $(m + s_1)/2$. The last case is $n/2 = (m - s_1)/2$ (or $n = m - s_1$), or $n/2 = (m + s_1)/2$ (or $n = m + s_1$), which is (ii) for the particular case when $t = 0$.

Assume next that $st \ne 0$. In this case,

$$\begin{aligned} a &\le \gcd(ab, ac) = \gcd(F_{(n-t_1)/2} L_{(n+t_1)/2}, F_{(m-s_1)/2} L_{(m+s_1)/2}) \\ &\le \gcd(F_{(n-t_1)/2}, F_{(m-s_1)/2}) \gcd(F_{(n-t_1)/2}, L_{(m+s_1)/2}) \\ &\times \gcd(L_{(n+t_1)/2}, F_{(m-s_1)/2}) \gcd(L_{(n+t_1)/2}, L_{(m+s_1)/2}) \\ &\le F_{\gcd((n-t_1)/2,(m-s_1)/2)} L_{\gcd((n-t_1)/2,(m+s_1)/2)} \end{aligned}$$

$$\times L_{\gcd((n+t_1)/2,(m-s_1)/2)} L_{\gcd((n+t_1)/2,(m+s_1)/2)}. \tag{2.30}$$

Write

$$\gcd\left(\frac{n-t_1}{2}, \frac{m-s_1}{2}\right) = \frac{n-t_1}{2d_1};$$
$$\gcd\left(\frac{n-t_1}{2}, \frac{m+s_1}{2}\right) = \frac{n-t_1}{2d_2};$$
$$\gcd\left(\frac{n+t_1}{2}, \frac{m-s_1}{2}\right) = \frac{n+t_1}{2d_3};$$
$$\gcd\left(\frac{n+t_1}{2}, \frac{m+s_1}{2}\right) = \frac{n+t_1}{2d_4}$$

for some positive integers d_1, d_2, d_3, d_4. Assume that $\min\{d_1, d_2, d_3, d_4\} \geq 10$. Then

$$\alpha^{n/2-1} < \sqrt{F_n} < a \leq F_{(n-t_1)/2d_1} L_{(n-t_1)/2d_2} L_{(n+t_1)/2d_3} L_{(n+t_1)/2d_4}$$
$$< \alpha^{(n-t_1)/2d_1+(n-t_1)/2d_2+(n+t_1)/2d_3+(n+t_1)/2d_4+2} \leq \alpha^{(n+t)/5+2},$$

giving

$$n < \frac{10}{3}\left(3 + \frac{t}{5}\right) < 10 + \frac{4.2}{3}\log(3M) < 12\log(3M),$$

contradicting inequality (2.18). Suppose $\min\{d_1, d_2, d_3, d_4\} \leq 9$. Assume that there exist $i \neq j$ such that both $d_i \leq 9$ and $d_j \leq 9$. Just to fix ideas, we assume that $i = 1$, $j = 3$. Put

$$\frac{n-t_1}{2d_1} = \frac{m-s_1}{2d_5}, \qquad \text{and} \qquad \frac{n+t_1}{2d_3} = \frac{m-s_1}{2d_7}. \tag{2.31}$$

Assume say that $d_5 \geq d_1 + 1$. Then

$$m - s_1 = \frac{d_5(n-t_1)}{d_1} \geq n - t_1 + \frac{n-t_1}{d_1} > m - t_1 + \frac{n-t_1}{d_1},$$

so

$$n \leq t_1 + d_1(t_1 - s_1) \leq t + 9(s+t) < 20\max\{s,t\} < 42\log(3M),$$

contradicting inequality (2.18). A similar contradiction is obtained if one supposes that $d_7 \geq d_3 + 1$. Thus, we may assume that $d_5 \leq d_1 \leq 9$ and $d_7 \leq d_3 \leq 9$. Equations (2.31) give

$$d_5 n - d_1 m = d_5 t_1 - d_1 s_1;$$
$$d_7 n - d_3 m = -d_7 t_1 - d_3 s_1.$$

One checks that the above system has a unique solution (m, n), and the same is true for the other values of $i \neq j$ in $\{1, 2, 3, 4\}$, not only for $(i, j) = (1, 3)$. We solve the system by Cramer's rule getting

$$\begin{vmatrix} d_5 & -d_1 \\ d_7 & -d_3 \end{vmatrix} n = \begin{vmatrix} d_5 t_1 - d_1 s_1 & -d_1 \\ -d_7 t_1 - d_3 s_1 & -d_3 \end{vmatrix}.$$

Thus, using Hadamard's inequality,

$$\begin{aligned} n &\le \begin{vmatrix} d_5t_1 - d_1s_1 & -d_1 \\ -d_7t_1 - d_3s_1 & -d_3 \end{vmatrix} \\ &\le \sqrt{d_1^2 + d_3^2} \times \sqrt{(d_5t_1 - d_1s_1)^2 + (d_7t_1 + d_3s_1)^2} \\ &\le 9\sqrt{2} \times 9 \times 2 \times \sqrt{2}\max\{s,t\} < 700\log(3M), \end{aligned}$$

which contradicts inequality (2.18). So, we may assume that there exists at most one $i \in \{1,2,3,4\}$ such that $d_i \le 9$. If $d_i \ge 2$, then

$$\begin{aligned} \alpha^{n/2-1} < \sqrt{F_n} < a &\le F_{(n-t_1)/2d_1}L_{(n-t_1)/2d_2}L_{(n+t_1)/2d_3}L_{(n+t_1)/2d_4} \\ &\le \alpha^{(n-t_1)/2d_1+(n-t_1)/2d_2+(n+t_1)/2d_3+(n+t_1)/2d_4+2} \\ &\le \alpha^{(n+t)/4+3(n+t)/20+2}, \end{aligned}$$

which gives

$$\frac{n}{10} < 3 + \frac{2}{5}t, \quad \text{therefore} \quad n < 30 + 4t < 30 + 8.4\log(3M) < 40\log(3M),$$

which contradicts inequality (2.18). Thus, it remains to consider the case $d_i = 1$. Say $i = 1$. We then get $(n-t_1)/2 \mid (m-s_1)/2$. If $(m-s_1)/2$ is a proper multiple of $(n-t_1)/2$, we then get that

$$(m-s_1)/2 \ge 2 \times (n-t_1)/2 = n - t_1 > m/2 + n/2 - t_1,$$

giving

$$n \le 2t_1 - s_1 \le 2t + s \le 6.3\log(3M),$$

which contradicts inequality (2.18). Thus, it remains the consider $n - t_1 = m - s_1$. This was when $d_i = 1$ and $i = 1$. For $i = 2,3,4$, we get that $n - t_1 = m + s_1$, $n + t_1 = m - s_1$, $n + t_1 = m + s_1$, respectively. Let us see that not all four possibilities occur.

Suppose say that $n - t_1 = m + s_1$. Then, as we have seen,

$$\gcd((n-t_1)/2, (m-s_1)/2) = \gcd((n-t_1)/2, (n-t_1)/2 - s_1) \mid s_1 \mid s,$$

$$\gcd((n+t_1)/2, (m+s_1)/2) = \gcd((n+t_1)/2, (n-t_1)/2) \mid t_1 \mid t,$$

and

$$\gcd((n+t_1)/2, (m-s_1)/2) = \gcd((n+t_1)/2, (n-t_1)/2 - s_1) \mid t_1 + s_1.$$

Observe that $s_1 + t_1 \ne 0$, for if $s_1 + t_1 = 0$, then since also $n - t_1 = m + s_1$, or $n = m + (s_1 + t_1) = m + 0$, we would get that $n = m$, a contradiction. Divisibilities (2.30) show that

$$a \le F_{\gcd((n-t_1)/2,(m-s_1)/2)} \gcd(F_{(n-t_1)/2}, L_{(m+s_1)/2}) L_{\gcd((n+t_1)/2,(m-s_1)/2)}$$

$$\times L_{\gcd((n+t_1)/2,(m+s_1)/2)} \le F_s \times 2 \times L_{t+s} \times L_t,$$

where we used the fact that $\gcd(F_k, L_k) \mid 2$ for all positive integers k with $k = (n-t_1)/2 = (m+s_1)/2$. Thus,

$$a \le 2\alpha^{2s+2t+1} < \alpha^{3+8.4\log(3M)}.$$

Since also $a > \sqrt{F_n} > \alpha^{n/2-1}$, we get

$$\frac{n}{2} - 1 < 3 + 8.4\log(3M), \qquad \text{therefore} \qquad n < 25\log(3M),$$

contradicting inequality (2.18). A similar argument applies when $n + t_1 = m - s_1$. Hence, we either have $n - t_1 = m - s_1$, or $m + t_1 = n + s_1$, which is (i).

Finally, let's us discuss the case $s = 0$. We follow the previous program. We have

$$\begin{aligned} a &\le \gcd(ab, ac) = \gcd(F_{(n-t_1)/2}L_{(n+t_1)/2}, F_m) \\ &\le \gcd(F_{(n-t_1)/2}, F_m)\gcd(L_{(n+t_1)/2}, F_m) \\ &\le F_{\gcd((n-t_1)/2,m)}L_{\gcd((n+t_1)/2,m)}. \end{aligned}$$

As in previous arguments, put

$$\gcd((n-t_1)/2, m) = (n-t_1)/2d_1, \quad \text{and} \quad \gcd((n+t_1)/2, m) = (n+t_1)/2d_2.$$

If $\min\{d_1, d_2\} \ge 5$, we have

$$\alpha^{n/2-1} < a \le F_{(n-t_1)/2d_1}L_{(n+t_1)/2d_2} \le \alpha^{(n-t_1)/2d_1 + (n+t_1)/2d_2} \le \alpha^{(n+t)/5},$$

so that

$$n < \frac{10}{3}\left(1 + \frac{t}{5}\right) < 4 + \frac{4.2}{3}\log(3M) < 6\log(3M),$$

contradicting inequality (2.18). Assume now that both $d_1 \le 4$ and $d_2 \le 4$. Put d_3 and d_4 such that $m/d_3 = (n-t_1)/2d_1$ and $m/d_4 = (n+t_1)/2d_2$. If $d_3 \ge 2d_1 + 1$, we then have

$$m = \frac{d_3}{2d_1}(n - t_1) \ge n - t_1 + \frac{n-t_1}{2d_1} > m - t_1 + \frac{n-t_1}{2d_1},$$

so

$$n \le (2d_1+1)t_1 \le (2d_1+1)t \le 9 \times 2.1\log(3M) < 20\log(3M),$$

contradicting inequality (2.18). A similar contradiction is obtained if we assume that $d_4 \ge 2d_2 + 1$. Thus, $d_3 \le 2d_1 \le 8$ and $d_4 \le 2d_2 \le 8$. We then get

$$\frac{n+t_1}{n-t_1} = \frac{d_2d_3}{d_1d_4},$$

so that

$$n(d_1d_4 - d_2d_3) = -t_1(d_1d_4 + d_2d_3).$$

Therefore

$$n \leq t(d_1 d_4 + d_2 d_3) \leq 64 \times 2.1 \log(3M) < 400 \log(3M),$$

contradicting inequality (2.18). Assume $\min\{d_1, d_2\} \leq 4$ and $\max\{d_1, d_2\} \geq 5$. If $\min\{d_1, d_2\} \geq 2$, we then get

$$\alpha^{n/2-1} < a < \alpha^{(n-t_1)/2d_1+(n+t_1)/2d_2} \leq \alpha^{(n+t)(1/4+1/10)},$$

giving

$$n < \frac{20}{3}\left(1 + \frac{7}{20}t\right) < 7 + \frac{7}{3} \times 2.1 \log(3M) < 12 \log(3M),$$

which contradicts inequality (2.18). So, the last possibility is $\min\{d_1, d_2\} = 1$. Hence, we either have $\gcd((n - t_1)/2, m) = (n - t_1)/2$, or $\gcd((n + t_1)/2, m) = (n + t_1)/2$. In particular, $m = \delta(n - t_1)/2$, or $m = \delta(n + t_1)/2$ for some positive integer δ. If $\delta \geq 3$, we get

$$n > m \geq \frac{3(n \pm t_1)}{2} \geq \frac{3(n-t)}{2},$$

giving $n < 3t < 10 \log(3M)$, a contradiction. If $\delta = 2$, we get that $m = n - t_1$ or $m = n + t_1$, which is (i) because $s = 0$. Suppose now that $\delta = 1$. Then either $m = (n - t_1)/2$, or $m = (n + t_1)/2$. Assume that $m = (n + t_1)/2$. Then

$$\begin{aligned} a &\leq \gcd(ab, ac) = \gcd(F_{(n-t_1)/2} L_{(n+t_1)/2}, F_{(n+t_1)/2}) \\ &\leq \gcd(F_{(n-t_1)/2}, F_{(n+t_1)/2}) \gcd(L_{(n+t_1)/2}, F_{(n+t_1)/2}) \leq 2F_t, \end{aligned}$$

so we get that

$$\alpha^{n/2-1} \leq 2F_t < \alpha^{t+1}, \qquad \text{therefore} \qquad n < 4 + 2t < 10 \log(3M),$$

a contradiction. Finally, in case $m = (n - t_1)/2$, we then have

$$ab = F_{(n-t_1)/2} L_{(n+t_1)/2}, \qquad ac = F_m = F_{(n-t_1)/2},$$

therefore

$$ab = (ac) L_{(n+t_1)/2}, \qquad \text{so} \qquad b = L_{(n+t_1)/2} c,$$

which is (iii). □

We can now give a lower bound for b.

Lemma 2.16. *Assume that inequality* (2.18) *holds. Then*

$$b > \alpha^{n/2-14 \log(3M)}. \tag{2.32}$$

Proof. If we are in case (iii) of Lemma 2.15, then

$$b \geq L_{(n+t_1)/2} \geq \alpha^{n/2-t/2-1} \geq \alpha^{n/2-1-1.05\log(3M)} \geq \alpha^{n/2-3\log(3M)}.$$

Assume next that $n - t_1 = m - s_1$ and $st \neq 0$. Then

$$\gcd((n-t_1)/2, (m+s_1)/2) = \gcd((n-t_1)/2, (n-t_1)/2 + s_1) \mid s_1 \mid s,$$

$$\gcd((n+t_1)/2, (m-s_1)/2) = \gcd((n+t_1)/2, (n-t_1)/2) \mid t_1 \mid t,$$

and

$$\gcd((n+t_1)/2, (m+s_1)/2) = \gcd((n+t_1)/2, (n-t_1)/2 + s_1) \mid t_1 - s_1.$$

Observe that $t_1 - s_1 \neq 0$ since if $t_1 - s_1 = 0$, then $n - m = t_1 - s_1 = 0$, so $n = m$, which is impossible. Now relation (2.30) shows that

$$\begin{aligned} a &\leq F_{(n-t_1)/2} L_s L_t L_{t+s} \leq \alpha^{(n+t)/2+2s+t+2} \\ &\leq \alpha^{n/2+2+3.5\max\{s,t\}} < \alpha^{n/2+10\log(3M)}. \end{aligned} \tag{2.33}$$

Since $|u| \leq M < a$, it follows that

$$\alpha^{n-2} < F_n = ab + u \leq ab + |u| \leq ab + M < 2ab < 2b\alpha^{n/2+10\log(3M)},$$

giving

$$b > 2^{-1}\alpha^{n/2-2-10\log(3M)} > \alpha^{n/2-4-10\log(3M)} > \alpha^{n/2-14\log(3M)},$$

which is the desired inequality. A similar argument applies when $n + t_1 = m + s_1$ and $st \neq 0$.

Assume next that $t = 0$. Then $n = m - s_1$ or $n = m + s_1$. Assume say that $n = m - s_1$. Then

$$\begin{aligned} a &\leq \gcd(F_n, F_{(m-s_1)/2} L_{(m+s_1)/2}) \leq F_{\gcd(n,(m-s_1)/2)} L_{\gcd(n,(m+s_1)/2} \\ &= F_{n/2} L_{\gcd(n,n/2+s_1)} \leq F_{n/2} L_s, \end{aligned}$$

so

$$a \leq \alpha^{n/2+s} \leq \alpha^{n/2+2.1\log(3M)},$$

which is an inequality better than (2.33). In turn, we get that inequality (2.32) holds. A similar argument applies when $t = 0$ and $n = m + s_1$, and also when $s = 0$ and either $m = n - t_1$ or $m = n + t_1$. We give no further details here. □

We now write

$$b \leq \gcd(ab, bc) = \gcd(F_n - u, F_\ell - w).$$

Write, as we did in Section 2.2,

$$F_\ell - w = \frac{\alpha^{-\ell}}{\sqrt{5}} \left(\alpha^\ell - w_{1,\ell}\right) \left(\alpha^\ell - w_{2,\ell}\right), \tag{2.34}$$

where

$$w_{k,\ell} = \frac{\sqrt{5}w + (-1)^k\sqrt{5w^2 + 4(-1)^\ell}}{2}, \qquad k \in \{1, 2\}. \tag{2.35}$$

As for the numbers $u_{i,n}$ and $v_{j,m}$ (see inequalities (2.9) and (2.10)), we also have that $w_{k,\ell}$ and all its conjugates $w_{k,\ell}^{(s)}$ satisfy

$$|w_{k,\ell}^{(s)}| < 3M.$$

We put $= \mathbb{Q}(\sqrt{5}, u_{1,n}, w_{1,\ell})$, and use the argument from the beginning of Section 2.3, in particular an analog of inequality (2.11) to say that

$$\begin{aligned} b\mathcal{O} &\mid \gcd\left(\left(\alpha^n - u_{1,n}\right)\left(\alpha^n - u_{2,n}\right)\mathcal{O}, \left(\alpha^\ell - w_{1,\ell}\right)\left(\alpha^\ell - w_{2,\ell}\right)\mathcal{O}\right) \\ &\mid \prod_{\substack{1\le i\le 2\\ 1\le k\le 2}} \gcd\left(\left(\alpha^n - u_{i,n}\right)\mathcal{O}, \left(\alpha^\ell - w_{k,\ell}\right)\mathcal{O}\right). \end{aligned} \tag{2.36}$$

Put

$$I_{i,n,k,\ell} = \gcd\left((\alpha^n - u_{i,n})\mathcal{O}, (\alpha^\ell - w_{k,\ell})\mathcal{O}\right), \qquad i, k \in \{1, 2\}.$$

Using Lemma 2.6, we construct coprime integers λ', ν' satisfying the inequalities $\max\{|\lambda'|, |\nu'|\} \le \sqrt{n}$, $|n\lambda' + \ell\nu'| \le 3\sqrt{n}$ and furthermore

$$\alpha^{n\lambda'+\ell\nu'} - u_{i,n}^{\lambda'} w_{k,\ell}^{\nu'} \in I_{i,n,k,\ell}.$$

As in Section 2.3, we make the following assumption.

Assumption 2.17. *Assume that the pair (λ', ν') satisfies*

$$\alpha^{n\lambda'+\ell\nu'} - u_{i,n}^{\lambda'} w_{k,\ell}^{\nu'} \ne 0 \qquad \textit{for all} \qquad i, k \in \{1, 2\}.$$

Then the argument of Lemma 2.8 shows that

$$b \le 2^4(3M)^{8\sqrt{n}}.$$

Combined with Lemma 2.16, we get that

$$\alpha^{n/2-14\log(3M)} < 2^4(3M)^{8\sqrt{n}},$$

therefore

$$n/2 - 14\log(3M) < \frac{\log(16)}{\log\alpha} + \left(\frac{8\log(3M)}{\log\alpha}\right)\sqrt{n} < 5.8 + 16.7\log(3M)\sqrt{n},$$

so

$$n < \left(\frac{11.6}{\sqrt{n}} + \frac{28\log(3M)}{\sqrt{n}} + 16.7\log(3M)\right)\sqrt{n}.$$

Since n satisfies inequality (2.18), we have that $\sqrt{n} > 41\log(3M)$, therefore

$$\frac{11.6}{\sqrt{n}} < 2 \qquad \text{and} \qquad \frac{28\log(3M)}{\sqrt{n}} < 1.$$

Hence, we get that

$$\sqrt{n} < 3 + 16.7\log(3M) < 20\log(3M),$$

contradicting inequality (2.18). The conclusion is:

Lemma 2.18. *If inequality* (2.18) *holds, then Assumption 2.17 cannot hold.*

Thus, there exist $i_1, k_1 \in \{1,2\}$ such that

$$\alpha^{n\lambda'+\ell\nu'} = u_{i_1,n}^{\lambda'} w_{k_1,\ell}^{\nu'}.$$

Since we already know that $u_{i_1,n}^2 \in \mathbb{Q}(\sqrt{5})$ (because we are in the rank one case), it follows that $w_{k_1,\ell}^{2\nu'} \in \mathbb{Q}(\sqrt{5})$. In particular, either $w = 0$, or $w \neq 0$ but $5w^2 + 4(-1)^\ell = y_{w,\ell}^2$ holds for some positive integer ℓ. In particular, $w = \pm F_r$ for some nonnegative integer r which is either 0 or is congruent to ℓ modulo 2. Thus

$$bc = F_\ell - w = F_{(\ell-r_1)/2}L_{(\ell+r_1)/2}$$

where $r_1 = \pm r$. Since $|w| \leq M$, we also have $r < 2.1\log(3M)$.

We now show that both m and ℓ are large.

Lemma 2.19. *Assume that inequality* (2.18) *holds. Then*

$$\min\{\ell, m\} > n/2 - 17\log(3M). \tag{2.37}$$

Proof. Since $b > \alpha^{n/2-14\log(3M)}$ by Lemma 2.16, and since n satisfies inequality (2.18), it follows that $b > 2M$. Indeed, this last inequality is implied by

$$\alpha^{n/2-14\log(3M)} > 2M,$$

or

$$n/2 - 14\log(3M) > \frac{\log 2M}{\log\alpha},$$

which in turn is implied by

$$n/2 - 14\log(3M) > 2.1\log(3M),$$

which in turn is implied by $n > 33\log(3M)$, which holds when n satisfies inequality (2.18). Hence,

$$\begin{aligned}\alpha^{\ell-1} &> F_\ell = bc + w \geq bc - M \geq b - M > b/2 \\ &\geq 2^{-1}\alpha^{n/2-14\log(3M)} > \alpha^{n/2-2-14\log(3M)},\end{aligned}$$

giving

$$\ell - 1 > n/2 - 2 - 14\log(3M), \qquad \text{or} \qquad \ell > n/2 - 17\log(3M).$$

The same argument works for m. □

We now return to Lemma 2.15 and get the following result.

Lemma 2.20. *If inequality* (2.18) *holds, then part (iii) of Lemma 2.15 cannot hold.*

Proof. Assume that (iii) of Lemma 2.15 holds. Then

$$bc = L_{(n+t_1)/2}c^2 = F_{(\ell-r_1)/2}L_{(\ell+r_1)/2}.$$

Since n satisfies inequality (2.18), we have that

$$(n+t_1)/2 > (n-t)/2 > ((41\log(3M))^2 - 2.1\log(3M))/2 > 12,$$

therefore $L_{(n+t_1)/2}$ has a primitive prime factor p. Its order of appearance in the Fibonacci sequence is $n+t_1$. Since $p \mid F_{(\ell-r_1)/2}L_{(\ell+r_1)/2}$, it follows that either $(\ell-r_1)/2$ is a multiple of $n+t_1$, or $\ell+r_1$ is a multiple of $n+t_1$. But obviously

$$(\ell+r_1)/2 < (n+r)/2 < n-t \le n+t_1,$$

where the middle inequality holds because it is equivalent to $n > 2r+t$, which is implied by (2.18) since then

$$n > (41\log(3M))^2 > 6.3\log(3M) > r+2t.$$

Thus, the only possibility is that $\ell+r_1$ is a multiple of $n+t_1$. Since

$$2(n+t_1) \ge 2n-2t > n+r > \ell+r \ge \ell+r_1,$$

it follows that the only possibility is that $\ell+r_1 = n+t_1$. Hence,

$$L_{(n+t_1)/2}c^2 = F_{(\ell-r_1)/2}L_{(\ell+r_1)/2} = F_{(\ell-r_1)/2}L_{(n+t_1)/2},$$

giving $F_{(\ell-r_1)/2} = c^2$. Since the largest square in the Fibonacci sequence is $F_{12} = 12^2$ (see [1] for a more general result), we get that $(\ell-r_1)/2 \le 12$, so

$$\ell \le 24 + r_1 \le 24 + r < 30\log(3M). \tag{2.38}$$

However, this last inequality contradicts the inequality (2.37) because n satisfies inequality (2.18). This shows that indeed part (iii) of Lemma 2.15 cannot happen. □

We now revisit the argument of Lemma 2.15 and prove in exactly the same way the following result.

Lemma 2.21. *Assume that inequality* (2.18) *holds. Then one of the following holds:*

(i) $n - t_1 = \ell - r_1$;

(ii) $n + t_1 = \ell + r_1$.

Proof. We follow the proof of Lemma 2.15. The relevant inequality here is, instead of (2.30),

$$b \le \gcd(ab, bc) = \gcd(F_{(n-t_1)/2}L_{(n+t_1)/2}, F_{(\ell-r_1)/2}L_{(\ell+r_1)/2}). \tag{2.39}$$

In the proof of Lemma 2.15 we used the lower bound $a > \alpha^{n/2-1}$, whereas here we use the lower bound $b > \alpha^{n/2-14\log(3M)}$ given by Lemma 2.16. We only go through a couple scenarios which have not been contemplated in the proof of Lemma 2.15.

One of them is when $u = w = 0$. Then

$$\alpha^{n/2-14\log(3M)} < b = \gcd(F_n, F_\ell) = F_{\gcd(n,\ell)}.$$

Clearly, $\gcd(n, \ell) = n/d_1$ for some divisor $d_1 > 1$ of n because $\ell < n$. If $d_1 \ge 3$, we get

$$\alpha^{n/2-14\log(3M)} < F_{n/d_1} < \alpha^{n/d_1} \le \alpha^{n/3},$$

or $n < 84\log(3M)$, contradicting inequality (2.18). Hence, $\gcd(n, \ell) = n/2$, and the only possibility is $\ell = n/2$. But then

$$bc = F_{n/2}, \qquad ab = F_n = F_{n/2}L_{n/2}, \qquad \text{giving} \qquad a = L_{n/2}c.$$

Hence,

$$F_{(m-s_1)/2}L_{(m+s_1)/2} = ac = L_{n/2}c^2.$$

Since n is large, $L_{n/2}$ has primitive divisors whose order of appearance in the Fibonacci sequence is exactly n. We deduce that n divides either $(m - s_1)/2$ or $m + s_1$. Since we have $(m - s_1)/2 \le (m + s)/2 < (n + s)/2 < n$ and $m + s_1 \le m + s < n + s < 2n$ whenever n satisfies inequality (2.18), we conclude that the only possibility is that $m + s_1 = n$. Thus, we get the equations $L_{n/2}c^2 = F_{(m-s_1)/2}L_{(m+s_1)/2} = F_{(m-s_1)/2}L_{n/2}$, so $F_{(m+s_1)/2} = c^2$, giving $(m + s_1)/2 \le 12$. This gives

$$m \le 24 - s_1 \le 24 + s < 24 + 2.1\log(3M),$$

which contradicts inequality (2.37) of Lemma 2.19 when n satisfies inequality (2.18).

This shows that we cannot have u and w be simultaneously zero.

Next we follow along the proof of Lemma 2.15 replacing (m, s, s_1) by (ℓ, r, r_1). Everything works out until we arrive at the analogue of (iii) of Lemma 2.15, which for us is $w = r = 0$, $\ell = (n - t_1)/2$ and $a = L_{(n+t_1)/2}c$. But in this case

$$L_{(n+t_1)/2}c^2 = ac = F_{(m-s_1)/2}L_{(m+s_1)/2}.$$

Using again the information that $(n + t_1)/2$ is large and $L_{(n+t_1)/2}$ has primitive prime divisors, we conclude that the only possible scenario is $m + s_1 = n + t_1$, leading to $F_{(m-s_1)/2} = c^2$, which gives that $(m - s_1)/2$ is small, contradicting inequality (2.37). We give no further details. □

We can now give a lower bound for c.

Lemma 2.22. *Assume that inequality* (2.18) *holds. Then*

$$c > \alpha^{n/2-31\log(3M)}. \tag{2.40}$$

Proof. This is very similar to the proof of Lemma 2.16. Assume, for example, that $n - t_1 = \ell - r_1$ and $tr \neq 0$. Then

$$\gcd((n-t_1)/2, (\ell+r_1)/2) = \gcd((n-t_1)/2, (n-t_1)/2 + r_1) \mid r_1 \mid r,$$

$$\gcd((n+t_1)/2, (\ell-r_1)/2) = \gcd((n+t_1)/2, (n-t_1)/2) \mid t_1 \mid t,$$

and

$$\gcd((n+t_1)/2, (\ell+r_1)/2) = \gcd((n+t_1)/2, (n-t_1)/2 + r_1) \mid t_1 - r_1.$$

Observe that $t_1 - r_1 \neq 0$ since if $t_1 - r_1 = 0$, then $n - \ell = t_1 - r_1 = 0$, so $n = \ell$, which is impossible. Now relation (2.39) implies that

$$\begin{aligned} b &\leq F_{(n-t_1)/2} L_r L_t L_{t+r} \leq \alpha^{(n+t)/2+2r+t+2} \\ &\leq \alpha^{n/2+2+3.5\max\{r,t\}} < \alpha^{n/2+10\log(3M)}. \end{aligned} \tag{2.41}$$

Since $|w| \leq M < b$, it follows, by inequality (2.37), that

$$\alpha^{n/2-17\log(3M)-2} \leq \alpha^{\ell-2} \leq F_\ell = bc + w \leq bc + M < 2bc < 2c\alpha^{n/2+10\log(3M)},$$

giving

$$c > 2^{-1}\alpha^{n/2-2-27\log(3M)} > \alpha^{n/2-4-27\log(3M)} > \alpha^{n/2-31\log(3M)},$$

which is the desired inequality. A similar argument applies when $n + t_1 = \ell + r_1$ and $tr \neq 0$.

A similar proof works when either $t = 0$ or $r = 0$ providing better lower bounds for c. We give no further details here. □

We now revisit the argument of Lemma 2.15 and prove in exactly the same way the following result.

Lemma 2.23. *Assume that inequality* (2.18) *holds. Then one of the following holds:*

(i) $m - s_1 = \ell - r_1$*;*

(ii) $m + s_1 = \ell + r_1$*.*

Proof. This is entirely similar with the proof of Lemma 2.15, except that we use the relation

$$c \leq \gcd(ac, bc) = \gcd(F_{(m-s_1)/2} L_{(m+s_1)/2}, F_{(\ell-r_1)/2} L_{(\ell+r_1)/2})$$

and the lower bound (2.40) on c. We give no further details. □

Finally, we prove the following result.

Lemma 2.24. *Inequality* (2.18) *does not hold.*

Proof. From Lemmas 2.15, 2.21 and 2.23, one gets easily that either $n-t_1 = m-s_1 = \ell-r_1$ or $n+t_1 = m+s_1 = \ell+r_1$. Assume say that $N = n-t_1 = m-s_1 = \ell+r_1$. Then

$$ab = F_N L_{N+2t_1}, \qquad ac = F_N L_{N+2s_1}, \qquad bc = F_N L_{N+2r_1}.$$

If U and V denote any two of the numbers $N, N+2r_1, N+2s_1, N+2t_1$, then $U/2 < V < 2U$ because n satisfies inequality (2.18). Also, all the above four numbers exceed 12. Using again the primitive divisor theorem, we conclude that $N+2r_1$ is one of the numbers $\{N, N+2s_1, N+2t_1\}$, so $r_1 \in \{0, s_1, t_1\}$. But if $r_1 = s_1$, then since also $\ell-r_1 = m-s_1$, we get $m = \ell$, so $ac = F_{(m-s_1)/2}L_{(m+s_1)/2} = F_{(\ell-r_1)/2}L_{(\ell+r_1)/2} = bc$, contradicting the fact that $a > b > c \geq 1$. Thus, $r_1 = 0$. Similarly, we get $s_1 = t_1 = 0$, therefore $n = m = \ell$, which is not allowed. A similar argument works when $n+t_1 = m+s_1 = \ell+r_1$. □

Proof of Theorem 1.1. We are now ready to finish the proof of Theorem 1.1. Indeed,

$$2a \leq ab = F_n + u \leq F_n + M.$$

So, either $a \leq M$, or $a > M$ in which case $a \leq 2a - M \leq F_n < \alpha^n$ giving

$$\frac{\log a}{\log \alpha} < n < (41\log(3M))^2.$$

The above inequality implies that

$$\log M > 41^{-1}\sqrt{2}\sqrt{\log a} > 0.034\sqrt{\log a}. \tag{2.42}$$

In case $a \leq M$, we get $\log M \geq \log a > 0.034\sqrt{\log a}$ because $a \geq 3$ so $\log a > 1$. Hence, inequality (2.42) always holds, showing that $M > \exp(0.034\sqrt{\log a})$, which is what we wanted to prove. □

3. The proof of Corollary 1.2

The condition $a < \exp(415.62)$ (coming directly from Theorem 1.1) implies $n \leq 1730$ via the inequalities $\alpha^{n-2} < F_n < a^2$. It is easy to see that $n \geq 8$ entails $n > m$, moreover from $n \geq 8$ and $m \geq 7$ we conclude $m \geq \ell$. These make it possible to apply a computer search for checking all the candidates (n, m, ℓ). Obviously $n \geq 5$ must be fulfilled, therefore we can verify individually the cases $5 \leq n \leq 7$. Totally 222 solutions to the system (2.4) have been found in $(a, b, c, u, v, w, n, m, \ell)$, the largest a is occurring in

$$(a, b, c, u, v, w, n, m, \ell) = (235, 11, 1, -1, -2, 2, 18, 13, 8).$$

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Algebraic independence results for the infinite products generated by Fibonacci numbers

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Abstract

The aim of this paper is to investigate the algebraic independence between two infinite products generated by the Fibonacci numbers $\{F_n\}_{n\geq 0}$ whose indices run in certain geometric progressions or binary recurrent sequences. As an application, we determine all the integers $m \geq 1$ such that the infinite products

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{F_{2^k}}\right) \quad \text{and} \quad \prod_{k=1}^{\infty}\left(1+\frac{1}{F_{2^k+m}}\right)$$

are algebraically independent over $\mathbb{Q}$.

Keywords: Algebraic independence, Infinite products, Fibonacci numbers, Mahler-type functional equation

MSC: 11J85

1. Introduction and the results

Let $\{R_n\}_{n\geq 0}$ be the binary recurrence defined by

$$R_{n+2} = A_1 R_{n+1} + A_2 R_n, \quad n \geq 0, \tag{1.1}$$

where A_1 and A_2 are nonzero integers and the initial values R_0 and R_1 are integers, not both zero. Suppose that $|A_2| = 1$ and $A_1^2 + 4A_2 > 0$. If $A_1 = A_2 = 1$ and $R_0 = 0, R_1 = 1$, then we have $R_n = F_n$ $(n \geq 0)$, where F_n is the nth Fibonacci number.

Let $d \geq 2$ be a fixed integer. The second author [6] investigated necessary and sufficient conditions for the infinite product generated by the sequence (1.1) to be algebraic. As an application, the transcendence of the infinite product $\prod_{k=1}^{\infty}(1 + \frac{1}{F_{d^k}})$ was deduced. In [3], the algebraic independence over $\mathbb{Q}$ of the sets of infinite products

$$\prod_{\substack{k=1 \\ F_{d^k} \neq -b_i}}^{\infty} \left(1 + \frac{b_i}{F_{d^k}}\right) \quad (i = 1, \dots, m)$$

was proved for any nonzero distinct integers $b_1, \dots, b_m$. In particular, the numbers

$$\prod_{k=1}^{\infty} \left(1 + \frac{1}{F_{2^k}}\right) \quad \text{and} \quad \prod_{k=2}^{\infty} \left(1 - \frac{1}{F_{2^k}}\right)$$

are algebraically independent over $\mathbb{Q}$. Recently, the authors [4] proved algebraic independence results for the infinite products generated by two distinct binary recurrences; for example, the two numbers

$$\prod_{k=1}^{\infty} \left(1 + \frac{1}{F_{2^k}}\right) \quad \text{and} \quad \prod_{k=1}^{\infty} \left(1 + \frac{1}{L_{2^k}}\right)$$

are algebraically independent over $\mathbb{Q}$, where the sequence $\{L_n\}_{n\geq 0}$ is the Lucas companion of the Fibonacci sequence defined by

$$L_{n+2} = L_{n+1} + L_n \quad (n \geq 0), \quad L_0 = 2, \quad L_1 = 1.$$

In what follows, let $\{R_n\}_{n\geq 0}$ be the binary recurrence given by (1.1) with $A_1 = A_2 = 1$. Then the sequence $\{R_n\}_{n\geq 0}$ is expressed as

$$R_n = g_1\alpha^n + g_2\beta^n, \quad n \geq 0, \tag{1.2}$$

where $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, and

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -\beta & 1 \\ \alpha & -1 \end{pmatrix} \begin{pmatrix} R_0 \\ R_1 \end{pmatrix}.$$

In this paper, we prove some algebraic independence results for the infinite products generated by Fibonacci numbers and the sequence (1.2). We state our results.

Theorem 1.1. *Let $d \geq 2$ be a fixed integer and $\{R_n\}_{n\geq 0}$ be the sequence defined by* (1.2) *with* $(R_0, R_1) \neq (0, 1)$*. Let*

$$\eta := \prod_{k=1}^{\infty}\left(1+\frac{1}{F_{d^k}}\right) \quad \text{and} \quad \nu := \prod_{\substack{k=1\\ R_{d^k}\neq 0,-1}}^{\infty}\left(1+\frac{1}{R_{d^k}}\right).$$

Then the following conditions are equivalent:
(i) *The numbers η and ν are algebraically dependent over $\mathbb{Q}$.*
(ii) *The number ν is algebraic.*
(iii) *$d = 2$ and either the condition $g_1 + g_2 = 1$ or the condition $g_1 = g_2 = -1$ is satisfied.*

Corollary 1.2. *Let $d \geq 2$ and $\{R_n\}_{n\geq 0}$ the sequence defined by* (1.2)*. If $d \geq 3$, then the numbers*

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{F_{d^k}}\right) \quad \text{and} \quad \prod_{\substack{k=1\\ R_{d^k}\neq 0,-1}}^{\infty}\left(1+\frac{1}{R_{d^k}}\right)$$

are algebraically independent over $\mathbb{Q}$. The same holds for the case of $d = 2$ and $R_0 \notin \{-2, 0, 1\}$.

Corollary 1.3. *Let $d \geq 2$ be an integer and let $\gamma \neq 1$ be a nonzero rational number. Then the infinite products*

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{F_{d^k}}\right) \quad \text{and} \quad \prod_{\substack{k=1\\ F_{d^k}\neq -\gamma}}^{\infty}\left(1+\frac{\gamma}{F_{d^k}}\right)$$

are algebraically independent over $\mathbb{Q}$.

It should be noted that Corollary 1.3 holds even if γ is a nonzero algebraic number (cf. [1]).

Corollary 1.4. *Let $d \geq 2$ and $m \geq 1$ be integers. Then the infinite products*

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{F_{d^k}}\right) \quad \text{and} \quad \prod_{k=1}^{\infty}\left(1+\frac{1}{F_{d^k+m}}\right) \tag{1.3}$$

are algebraically dependent over $\mathbb{Q}$ if and only if $(d, m) = (2, 1), (2, 2)$. In the two exceptional cases above, we have

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{F_{2^k+1}}\right) = \frac{3(\sqrt{5}-1)}{2}, \quad \prod_{k=1}^{\infty}\left(1+\frac{1}{F_{2^k+2}}\right) = 6 - 2\sqrt{5}.$$

The proofs of Theorem 1.1 and the corollaries will be given in Section 3.

2. Lemmas

Let $d \geq 2$ be a fixed integer and let $\{R_n\}_{n\geq 0}$ be the sequence defined by (1.2). Define

$$\Phi(x) := \prod_{k=0}^{\infty} \left(1 + \frac{g_1^{-1} x^{d^k}}{1 + (-1)^d g_1^{-1} g_2 x^{2d^k}}\right). \tag{2.1}$$

The function $\Phi(x)$ converges in $|x| < 1$ and satisfies the functional equation

$$\Phi(x^d) = c(x)\Phi(x), \tag{2.2}$$

with

$$c(x) = \frac{1 + (-1)^d g_1^{-1} g_2 x^2}{1 + g_1^{-1} x + (-1)^d g_1^{-1} g_2 x^2}.$$

To prove Theorem 1.1, we use the following lemma.

Lemma 2.1 (Special case of [6, Theorem 7])**.** *Let $d \geq 2$ be an integer. Let a and b be nonzero algebraic numbers and*

$$G(x) = \prod_{k=0}^{\infty} \left(1 + \frac{a x^{d^k}}{1 - b x^{2d^k}}\right), \quad |x| < 1.$$

Then the function $G(x)$ is a rational function with the algebraic coefficients if and only if $d = 2$ and either the condition $a + b = 1$ or the condition $a = b = -1$ is satisfied.

Lemma 2.2. *Let $\Phi(x)$ be the function given in* (2.1)*. Then the following conditions are equivalent*:
(i) *The function $\Phi(x)$ is algebraic over $\mathbb{Q}(\alpha, x)$.*
(ii) *The function $\Phi(x)$ is a rational function with algebraic coefficients.*
(iii) *$d = 2$ and either the condition $g_1 + g_2 = 1$ or the condition $g_1 = g_2 = -1$ is satisfied.*

Proof. First we prove (i)⇒(ii). Suppose that $\Phi(x)$ is algebraic over $\mathbb{Q}(\alpha, x)$. Then, by the functional equation (2.2) and [5, Theorem 1.3] with $C = \overline{\mathbb{Q}}$, we see that $\Phi(x)$ is a rational function over some algebraic number field $\mathbb{L} \supseteq \mathbb{Q}(\alpha)$. The assertions (ii)⇒(iii) and (iii)⇒(i) follow immediately from Lemma 2.1. □

Remark 2.3. If the property (iii) in Lemma 2.2 is satisfied, then the corresponding infinite products $\Phi(x)$ are expressed as rational functions explicitly. Indeed, in the case of $d = 2$ and $g_1 + g_2 = 1$, we have

$$\Phi(x) = \prod_{k=0}^{\infty} \left(1 + \frac{(1-b)x^{2^k}}{1 - bx^{2^{k+1}}}\right) = \prod_{k=0}^{\infty} \frac{(1 + x^{2^k})(1 - bx^{2^k})}{1 - bx^{2^{k+1}}} = \frac{1 - bx}{1 - x} \tag{2.3}$$

with $b = -g_1^{-1}g_2$. If $d = 2$ and $g_1 = g_2 = -1$, then

$$\Phi(x) = \prod_{k=0}^{\infty}\left(1 + \frac{-x^{2^k}}{1+x^{2^{k+1}}}\right) = \prod_{k=0}^{\infty}\frac{(1+\omega^{2^k}x^{2^k})(1+\omega^{-2^k}x^{2^k})}{1+x^{2^{k+1}}}$$
$$= \frac{1-x^2}{(1-\omega x)(1-\omega^{-1}x)} = \frac{1-x^2}{1+x+x^2}, \quad (2.4)$$

where ω is a primitive cubic root of unity.

Let $\mathbb{K}$ be an algebraic number field. For an integer $d \geq 2$, we define the subgroup H_d of the group $\mathbb{K}(x)^\times$ of nonzero elements of $\mathbb{K}(x)$ by

$$H_d = \left\{ \frac{g(x^d)}{g(x)} \;\middle|\; g(x) \in \mathbb{K}(x)^\times \right\}.$$

Let $\mathbb{K}[[x]]$ be the ring of formal power series with coefficients in $\mathbb{K}$.

Lemma 2.4 (Kubota [2, Corollary 8]). *Let $f_1(x), \dots, f_m(x) \in \mathbb{K}[[x]] \setminus \{0\}$ satisfy the functional equations*

$$f_i(x^d) = c_i(x)f_i(x), \quad c_i(x) \in \mathbb{K}(x)^\times \quad (i = 1, \dots, m). \quad (2.5)$$

Then $f_1(x), \dots, f_m(x)$ are algebraically independent over $\mathbb{K}(x)$ if and only if the rational functions $c_1(x), \dots, c_m(x)$ are multiplicatively independent modulo H_d.

Lemma 2.5 (Kubota [2], see also Nishioka [5, Theorem 3.6.4]). *Suppose that the functions $f_1(x), \dots, f_m(x) \in \mathbb{K}[[x]]$ converge in $|x| < 1$ and satisfy the functional equations* (2.5) *with $c_i(x)$ defined and nonzero at $x = 0$. Let γ be an algebraic number with $0 < |\gamma| < 1$ such that $c_i(\gamma^{d^k})$ are defined and nonzero for all $k \geq 0$. If $f_1(x), \dots. f_m(x)$ are algebraically independent over $\mathbb{K}(x)$, then the values $f_1(\gamma), \dots, f_m(\gamma)$ are algebraically independent over $\mathbb{Q}$.*

3. Proofs of Theorem 1.1 and the corollaries

Putting $g_1 = -g_2 = 1/\sqrt{5}$ in (2.1), we have

$$\Psi(x) := \prod_{k=0}^{\infty}\left(1 + \frac{\sqrt{5}x^{d^k}}{1-(-1)^d x^{2d^k}}\right).$$

By Lemma 2.2, the function $\Psi(x)$ is transcendental over $\mathbb{K}(x)$. Let η and ν be as in Theorem 1.1. Take an integer N such that $|R_{d^k}| > 1$ for all $k \geq N > 1$. Then, using (2.2), we get

$$\eta = p_N\Psi(\alpha^{-d^N}) = p_N\Psi(\alpha^{-1})\prod_{i=0}^{N-1} b(\alpha^{-d^i}), \quad (3.1)$$

$$\nu \quad = q_N \Phi(\alpha^{-d^N}) = q_N \Phi(\alpha^{-1}) \prod_{i=0}^{N-1} c(\alpha^{-d^i}), \tag{3.2}$$

where

$$b(x) = \frac{1-(-1)^d x^2}{1+\sqrt{5}x-(-1)^d x^2}, \quad c(x) = \frac{1+(-1)^d g_1^{-1} g_2 x^2}{1+g_1^{-1}x+(-1)^d g_1^{-1} g_2 x^2}$$

and p_N and q_N are nonzero rational numbers given by

$$p_N = \prod_{k=1}^{N-1}\left(1+\frac{1}{F_{d^k}}\right), \quad q_N = \prod_{\substack{k=1 \\ R_{d^k}\neq 0,-1}}^{N-1}\left(1+\frac{1}{R_{d^k}}\right).$$

Proof of Theorem 1.1. The assertion (ii)⇒(i) is trivial. If the condition (iii) holds, then, by Remark 2.3, the function $\Phi(x)$ is a rational function as in (2.3) or (2.4). Hence, by (3.2), we see that the number ν is algebraic and so the property (ii) is satisfied. Thus, we have only to prove (i)⇒(iii).

Suppose that η and ν are algebraically dependent over $\mathbb{Q}$. Then so are the values $\Phi(\alpha^{-1})$ and $\Psi(\alpha^{-1})$ by (3.1) and (3.2). Since $\Psi(x)$ and $\Phi(x)$ satisfy the functional equation (2.2), they are algebraically dependent over $\mathbb{K}(x)$ by Lemma 2.5. Thus, we see by Lemma 2.4, that the rational functions $b(x)$ and $c(x)$ are multiplicatively dependent modulo H_d, namely, there exist integers e_1, e_2, not both zero, and $g(x) \in \mathbb{K}(x)^\times$ such that

$$b(x)^{e_1} c(x)^{e_2} = g(x^d)/g(x), \tag{3.3}$$

where 0 is neither a pole nor a root of $g(x)$ because $b(0)c(0) = 1$. To simplify notations, we rewrite the equation (3.3), as

$$F(x) := \left(\frac{1-(-1)^d x^2}{1+\sqrt{5}x-(-1)^d x^2}\right)^{e_1} \left(\frac{1+g_1^{-1} g_2 (-1)^d x^2}{1+g_1^{-1}x+g_1^{-1} g_2 (-1)^d x^2}\right)^{e_2}, \tag{3.4}$$

where e_1 and e_2 are nonzero integers and

$$F(x) = \frac{A(x^d)B(x)}{A(x)B(x^d)} \tag{3.5}$$

with $A(x)$ and $B(x)$ being the polynomials without common roots with algebraic coefficients such that $g(x) = A(x)/B(x)$. We also assume that $e_1 > 0$, otherwise we replace the pair of exponents (e_1, e_2) by the pair $(-e_1, -e_2)$ and interchange $A(x)$ and $B(x)$. We distinguish four cases.

Case I). $e_1 e_2 > 0$. By (3.4) and (3.5), we have

$$\begin{aligned} A(x)B(x^d)(1-(-1)^d x^2)^{e_1}(1+g_1^{-1} g_2 (-1)^d x^2)^{e_2} \\ = A(x^d)B(x)P(x)^{e_1} Q(x)^{e_2}, \end{aligned} \tag{3.6}$$

where $e_1, e_2 \geq 1$ and

$$P(x) = 1 + \sqrt{5}x - (-1)^d x^2, \quad Q(x) = 1 + g_1^{-1}x + g_1 g_2^{-1}(-1)^d x^2.$$

Let γ_1 and γ_2 be the real roots of $P(x)$. Noting that

$$\gamma_1, \gamma_2 = \frac{(-1)^d\sqrt{5} \pm \sqrt{5 + 4(-1)^d}}{2} = \begin{cases} (\pm 3 + \sqrt{5})/2, & d : \text{even}, \\ (\pm 1 - \sqrt{5})/2, & d : \text{odd}, \end{cases} \tag{3.7}$$

we may put $|\gamma_1| > 1 > |\gamma_2|$.

First we suppose $|g_1^{-1}g_2| > 1$. Then the absolute values of the roots of the polynomial

$$(1 - (-1)^d x^2)^{e_1}(1 + g_1^{-1}g_2(-1)^d x^2)^{e_2}$$

appearing in the left hand side in (3.6) are not greater than 1. Let γ ($|\gamma| \geq |\gamma_1| > 1$) be the root of the polynomial appearing in the right hand side in (3.6) with the largest absolute value. Substituting $x = \gamma$ into (3.6), we have $A(\gamma)B(\gamma^d) = 0$, so that $A(\gamma) = 0$ or $B(\gamma^d) = 0$. If $A(\gamma) = 0$, substituting $x = \gamma^{1/d}$ into (3.6) again and noting that $|\gamma^{1/d}| > 1$, we have $A(\gamma^{1/d}) = 0$. Repeating this process, we obtain $A(\gamma^{1/d^k}) = 0$ for all $k \geq 0$, a contradiction. Thus we have $B(\gamma^d) = 0$. Substituting $x = \gamma^d$ into (3.6) and noting that $|\gamma^d| > 1$, we get $B(\gamma^{d^k}) = 0$ for all $k \geq 0$, a contradiction.

A similar contradiction is deduced in the case of $|g_1^{-1}g_2| \leq 1$.

Case II). $e_1 e_2 < 0$. In this case, we have

$$\begin{aligned} &A(x)B(x^d)(1 - (-1)^d x^2)^{h_1} Q(x)^{h_2} \\ &= A(x^d)B(x)(1 + g_1^{-1}g_2(-1)^d x^2)^{h_2} P(x)^{h_1}, \end{aligned} \tag{3.8}$$

where $h_1, h_2 \geq 1$.

First we prove that d is even. Suppose on the contrary that $d \geq 3$ is odd. The assumption $(R_0, R_1) \neq (0, 1)$ in Theorem 1.1 implies that $(g_1, g_2) \neq (1/\sqrt{5}, -1/\sqrt{5})$. Hence, at least one of the roots of $P(x)$ is not a root of $Q(x)$. Let γ ($|\gamma| \neq 1$) be as in (3.7) with $Q(\gamma) \neq 0$. Then, substituting $x = \gamma$ into (3.8), we have $A(\gamma)B(\gamma^d) = 0$, so that $A(\gamma) = 0$ or $B(\gamma^d) = 0$. Assume that $A(\gamma) = 0$. Since $d \geq 3$ and $\deg Q(x) = 2$, there exists a determination of $\gamma^{1/d}$ such that $Q(\gamma^{1/d}) \neq 0$. Hence, substituting $x = \gamma^{1/d}$ into (3.8) again and noting that $|\gamma^{1/d}| \neq 1$, we have $A(\gamma^{1/d}) = 0$. Repeating this process, we find a sequence $\{\gamma^{1/d^k}\}_{k \geq 0}$ of roots of γ such that $A(\gamma^{1/d^k}) = 0$ $(k \geq 0)$. This is a contradiction. Thus, we have $B(\gamma^d) = 0$. Let $\zeta_d = e^{2\pi i/d}$ be primitive d-th root of unity. Then the number $\zeta_d\gamma$ is neither real nor purely imaginary because d is odd. Hence, substituting $x = \zeta_d\gamma$ into (3.8), we have $B(\zeta_d\gamma) = 0$, since

$$A(\gamma^d)(1 + g_1^{-1}g_2(-1)^d(\zeta_d\gamma)^2)P(\zeta_d\gamma) \neq 0.$$

Furthermore, noting that $d \geq 3$ and $\deg Q(x) = 2$, we see that there exists a complex nonreal number $\zeta_{d^2}\gamma^{1/d}$ such that

$$A(\zeta_d\gamma)(1 + g_1^{-1}g_2(-1)^d(\zeta_{d^2}\gamma^{1/d})^2)P(\zeta_{d^2}\gamma^{1/d}) \neq 0.$$

Hence, substituting $x = \zeta_{d^2}\gamma^{1/d}$ into (3.8), we get $B(\zeta_{d^2}\gamma^{1/d}) = 0$. Repeating this process, we obtain $B(\zeta_{d^{k+1}}\gamma^{1/d^k}) = 0$ for all $k \geq 0$, a contradiction.

Thus, we see that d is even and so the equation (3.8) becomes

$$A(x)B(x^d)(1 - x^2)^{h_1}Q(x)^{h_2} = A(x^d)B(x)(1 + g_1^{-1}g_2x^2)^{h_2}P(x)^{h_1}. \tag{3.9}$$

Comparing the orders at $x = 1$ of both sides of (3.9), we obtain $g_1^{-1}g_2 = -1$ and $h_1 = h_2$. Dividing the both sides of (3.9) by $(1 - x^2)^{h_1}$, we have

$$A(x)B(x^d)(1 + g_1^{-1}x - x^2)^{h_1} = A(x^d)B(x)(1 + \sqrt{5}x - x^2)^{h_1}. \tag{3.10}$$

Note that the polynomial $Q(x) = 1 + g_1^{-1}x - x^2$ has the real roots ξ_1, ξ_2 with $|\xi_1| > 1 > |\xi_2|$. Let γ_1 and γ_2 be the roots of $1 + \sqrt{5}x - x^2$ given by (3.7). Then $\gamma_i \neq \xi_j$ $(1 \leq i, j \leq 2)$ because $g_1^{-1} \neq \sqrt{5}$. Hence, substituting $x = \gamma_1$ into (3.10), we have $A(\gamma_1)B(\gamma_1^d) = 0$, so that either $A(\gamma_1) = 0$ or $B(\gamma_1^d) = 0$. Assume that $A(\gamma_1) = 0$. Since $|\xi_1| > 1 > |\xi_2|$, we can choose $\gamma_1^{1/d}$ $(|\gamma_1^{1/d}| > 1)$ such that $Q(\gamma_1^{1/d}) \neq 0$. Thus, substituting $x = \gamma_1^{1/d}$ into (3.10), we have $A(\gamma_1^{1/d}) = 0$. Continuing in this way, we create a sequence of complex numbers $\{\gamma^{1/d^k}\}_{k\geq 0}$ which are all roots of $A(x)$, a contradiction. In the case of $B(\gamma_1^d) = 0$, substituting $x = \zeta_d\gamma_1$ $(\neq \gamma_1)$ into (3.10), we get $B(\zeta_d\gamma_1) = 0$. Similarly, we obtain $B(\zeta_{d^{k+1}}\gamma_1^{1/d^k}) = 0$ for all $k \geq 0$, a contradiction.

Case III). $e_1 = 0$. By (2.2) and (3.3)

$$g(x)\Phi(x^{d^k})^{e_2} = \Phi(x)^{e_2}g(x^{d^k}) \qquad (k \geq 0).$$

Taking the limit as $k \to \infty$, we obtain $g(x) = \Phi(x)^{e_2}g(0)$ $(|x| < 1)$, so that $\Phi(x)$ is algebraic over $\mathbb{K}(x)$. Hence, by Lemma 2.2, we see that that $d = 2$ and one of the conditions $g_1 + g_2 = 1$ or $g_1 = g_2 = -1$ is satisfied, which is the property (iii) in Theorem 1.1.

Case IV). $e_2 = 0$. Similarly to the proof in Case III, we see that the function $\Psi(x)$ is algebraic over $\mathbb{K}(x)$. This contradicts Lemma 2.2.

Therefore the proof of Theorem 1.1 is completed. □

Next we prove the corollaries. Corollaries 1.2 and 1.3 follow immediately from Theorem 1.1. We prove Corollary 1.4.

Proof of Corollary 1.4. Let $R_n := F_{n+m}$ $(n \geq 0)$. Then the sequence $\{R_n\}_{n\geq 0}$ is expressed as $R_n = g_1\alpha^n + g_2\beta^n$, where

$$g_1 = \alpha^m/(\alpha - \beta), \quad g_2 = -\beta^m(\alpha - \beta).$$

Note that $g_1, g_2 \neq -1$ for any integer $m \geq 1$. If the infinite products (1.3) are algebraically dependent over $\mathbb{Q}$, then the condition (iii) in Theorem 1.1 is satisfied, namely, $d = 2$ and

$$1 = g_1 + g_2 = \frac{\alpha^m - \beta^m}{\alpha - \beta} = F_m.$$

Thus, we have $m = 1, 2$. Conversely, if $(d, m) = (2, 1)$ or $(2, 2)$, then we have by (2.3) and (3.1)

$$\prod_{k=1}^{\infty}\left(1 + \frac{1}{F_{2^k+1}}\right) = \frac{3(\sqrt{5}-1)}{2}, \quad \prod_{k=1}^{\infty}\left(1 + \frac{1}{F_{2^k+2}}\right) = 6 - 2\sqrt{5}.$$

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Sonification of multiple Fibonacci-related sequences*

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Abstract

Expanding on previous musical exploration involving the sonification of Fibonacci-related number sequences, five contrasting stereo electro-acoustic compositions utilizing multiple integer sequences simultaneously are presented and analyzed.

Keywords: music, sonification, musical composition

MSC: 00A65

1. Introduction

This article expands on previous research into the direct sonification of Fibonacci-related number sequences. Direct sonification, or pure sonification, is a technique in which the composer attempts to create a musical or sonic graph that represents as accurate a likeness of the mathematical object sonified as is possible. In two previous papers, a system of tunings based on the Fibonacci sequence and the golden ratio was introduced, and various sonifications of Fibonacci-related integer sequences and Zeckendorf representations were presented [1] [2]. In this paper, we focus for the first time on the sonification of multiple integer sequences simultaneously.

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1.1. Terminology and definitions

This article makes use of the terms *dynamic parameters* and *static parameters*. Dynamic parameters are those musical parameters that the integer sequence sonified controls – those musical parameters that change according to the values of the sequence. This can include parameters such as frequency, level of loudness, spatial location, etc. Static parameters are, in contrast, those musical parameters that do not change according to the sequence and remain the same throughout the sonification. This can include many things, such as the number of audio channels used, the timbre used or the duration attributed to each integer, etc. If a sonification utilizes more than one sequence simultaneously, then dynamic and static parameters can be either *individual* or *global*. Individual parameters are those that pertain only to a specific sequence, whereas global pertain to all sequences that are being simultaneously sonified.

In the following, let $\varphi = \frac{\sqrt{5}-1}{2}$ and $\Phi = \frac{\sqrt{5}+1}{2}$. Let F_n represent the Fibonacci sequence, where $F_0 = 0$ and $F_1 = 1$. Let G_n represent a generalized Fibonacci sequence. Sequences are enclosed in $<>$ brackets.

2. Five compositions analyzed

In these analyses, we will utilize slightly modified and abridged versions of the scores of the works for practical purposes, as we have in previous papers. The original scores are Internet-based and in color, which can be very useful when graphing multiple sequences and in differentiating between individual and global parameters. It is highly recommended that the reader view and listen to the Quicktime files and see the original scores for each composition as well, to which links are provided. These pieces were all composed in 2011 and 2012 using the author's Objective-C++ program *Virahanka*, created for composing with various types of number sequences utilizing Csound [3] as an integrated sound synthesis engine.

2.1. φ and Φ signature sequences no. 3

If R is a positive irrational number and we arrange the set of all numbers $i + jR$ in order, where i and j are positive integers, $i_1 + j_1R$, $i_2 + j_2R$, $i_3 + j_3R$, $\ldots$, then $\langle i_1, i_2, i_3, \ldots \rangle$ is the signature sequence of R [4]. In the case of this composition, the two signature sequences used are those based on the smaller and larger golden ratio values, where R $= \varphi$ and R $= \Phi$, respectively (A084532 and A084531 in Sloane's OEIS [5]).

In this audio-visual work from *Collection XI* (B1237 in [6]), two signature sequences are sonified simultaneously with a synchronized point graph. The original score, Quicktime file (1920×1080 pixels) and other information can be found at `http://caseymongoven.com/b1237`.

The synthesis technique utilized in this work is based on the short-time Fourier transform (STFT), in which a monaural signal – in this case a wine glass from

Daniel Gehrs Winery being struck – is broken up into many smaller overlapping pieces of equal length and analyzed in the frequency domain as it changes over time in order to enable frequency shifting without time-stretching (among other possible manipulations). This means that each note articulated in this piece is derived from spectral data of the same single original sample of a wineglass being struck.

One of the challenges with signature sequences in particular, due to the relatively even distribution of the integers in such sequences, is finding a point of termination. Generally, a point of termination at either the highest point the sequence has reached, or at an occurrence of the integer 1, sounds best. Regardless of where one stops, the compositional result with signature sequences – and many other sequences with such uniform distribution of the integers – ends up sounding a bit as if one had suddenly torn it off at the end, reminiscent in a way of György Ligeti's instructions at the end of his *Continuum* for harpsichord: "Stop suddenly, as though torn off." In the case of this composition, 351 members of both sequences were utilized, beginning with the first members of the sequences. This resulted in a range of integers of 1-21 for the signature sequence where $R = \varphi$, and a range of 1-33 for the signature sequence where $R = \Phi$.

Each integer is attributed a static duration of .05 seconds, resulting in a piece duration of $351 * .05$ seconds $= 17.55$ seconds. Two dynamic parameters are used: frequency and simulated location. The latter is an individual dynamic parameter (each sequence has been given its own location),[1] the former is global. The unit interval of the tuning used in this piece is $\varphi^7 + 1$, and the pitch orientation is descending, meaning that higher integers are represented by lower pitches. Dynamic level (loudness) is a static parameter: each piece is attributed the dynamic level mezzo forte (medium loud). Attack and release values – also static parameters in this composition – of .0055 and .0089 seconds were used for each integer.[2]

Collection XI

φ and Φ Signature Sequences no. 3
Casey Mongoven
October 29, 2011

classification of work: `audio-visual`
synthesis engine: `Csound` `show Csound orchestra`
synthesis technique: `STFT-based phase vocoder`

description of sequences sonified

A084532 `Arranging the numbers s + j`φ `in increasing order, where s and j are positive integers, the sequence of s's is the signature sequence of` φ.

[1]The simulated location is given in degrees. 0 degrees represents straight ahead, while positive numbers represent sound sources emanating from the right and negative values represent those emanating from the left.

[2]Attack and release values are part of the so-called *envelope* of a note. Attack is the time in the very beginning of the note (often on a micro-sound time scale) where it becomes louder, and release is the part at the very end when it becomes quieter. Without attack and release values, it is possible that a click can result using certain synthesis techniques.

φ is equal to (-1 + sqrt(5))/2. 351 members used

A084531 Arranging the numbers s + jΦ in increasing order, where s and j are positive integers, the sequence of s's is the signature sequence of Φ. Φ is equal to (1 + sqrt(5))/2. 351 members used

global static parameters:

offset: 1
pitch orientation: descending
temperament: phi^7 + 1
number of channels: 2
note value: 0.05 seconds
piece length: 17.55 seconds
spectral data: gehrs glass seven
dynamic: mf
attack: .0055 seconds
release: .0089 seconds

global and individual dynamic parameters:

A084532

integer	approximate frequency Hz	simulated location degrees
1	873.0	-1.635
2	843.9	-2.394
3	815.8	-3.152
4	788.7	-3.910
5	762.4	-4.669
6	737.0	-5.427
7	712.5	-6.185
8	688.8	-6.944
⋮	⋮	⋮
19	474.6	-15.286
20	458.8	-16.044
21	443.5	-16.802

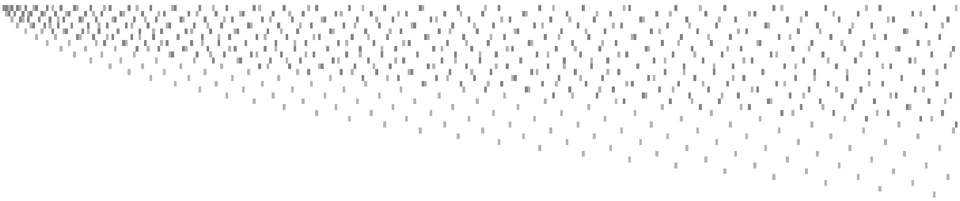
A084531

integer	approximate frequency Hz	simulated location degrees
1	873.0	25.903
2	843.9	25.144
3	815.8	24.386
4	788.7	23.628
5	762.4	22.869
6	737.0	22.111
7	712.5	21.353
8	688.8	20.594
⋮	⋮	⋮
31	316.1	3.152
32	305.6	2.394
33	295.4	1.635

351 values used of Sloane's A084532: 1, 1, 2, 1, 2, 1, 3, 2, 1, 3, 2, 4, 1, 3, 2, 4, 1, 3, 5, 2, 4, ..., 14, 6, 19, 11, 3, 16, 8, 21

351 values used of Sloane's A084531: 1, 2, 1, 3, 2, 4, 1, 3, 5, 2, 4, 1, 6, 3, 5, 2, 7, 4, 1, 6, 3, ..., 12, 25, 4, 17, 30, 9, 22, 1

graph of sequences: 1) A084532, 351 values; 2) A084531, 351 values.



Traditional European music theory has often expressed the aesthetic appeal of contrary motion (voices moving in opposite directions) in composition. If we zoom in and take a closer look at the beginning of the graph, we can see that the voices exhibit consistent contrary motion; after the first two members of the sequence, when one voice ascends, the other descends and vice versa:

In the original online version of the score, it is possible to hover over the individual integers in the graph with the cursor and view the dynamic parameters attributed to each integer.

2.2. Min and Max Fibbit Running no. 4

I learned about the sequences used in this work (B1359 in [6]) from Ron Knott. The first is based on the number of runs of equal bits in the Zeckendorf binary representations [7]. For example, 100010100 (the Zeckendorf representation of the integer $66 = 55 + 8 + 3$) has six runs of equal bits, namely 1, 000, 1, 0, 1, and 00. The following diagram shows the first 13 binary Zeckendorf representations with the number of runs of equal bits (A104324 in [5]):

integer	Zeckendorf representation	number of runs
1	1	1
2	10	2
3	100	2
4	101	3
5	1000	2
6	1001	3
7	1010	4
8	10000	2
9	10001	3
10	10010	4
11	10100	4
12	10101	5
13	10000	2
↓	↓	↓

Ron Knott named the first composition written with this sequence *Min Fibbit Running* (B101 in [6]), as the Zeckendorf representation is sometimes called the *minimum* representation (in that it requires the smallest number of Fibonacci numbers), *bit* because of the binary representation, and *running* because the sequence counts *runs*.

The second sequence (A104325 in [5]) follows the same exact principle of counting runs, only using the dual Zeckendorf binary representations, described in [8]. Whereas the standard Zeckendorf binary representations contain no consecutive 1s, the dual Zeckendorf representations contain no consecutive 0s. The dual Zeckendorf binary representation can be created by starting with the standard Zeckendorf representation and applying a left-to-right algorithm recursively in which each occurrence of the bit string 100 is replaced with 011 until no consecutive zeros are found in the representation. The result is as follows:

integer	dual Zeckendorf representation	number of runs
1	1	1
2	10	2
3	11	1
4	101	3
5	110	2
6	111	1
7	1010	4
8	1011	3
9	1101	3
10	1110	2
11	1111	1
12	10101	5
13	10110	4
↓	↓	↓

In this composition, wavetable synthesis was used, a technique based on the periodic reproduction of a single cycle of a waveform. $F_{15}-1 = 609$ members of each sequence were used; this number was chosen to preserve the symmetry of these self-similar sequences. In contrast to signature sequences, which are relatively uniform throughout, these sequences have clear "seams" at $F_n - 1$, which are convenient points to end a sonification. In addition, both sequences will always end with the same integer if the point of termination is $F_n - 1$, because there is only one possible representation of an integer $F_n - 1$ as a sum of Fibonacci numbers [10].

In this piece, the duration of each integer is slightly slower: .065 seconds. Location is a static parameter – one sequence is placed on each speaker. The wavetables were each derived from a single cycle of a wave from a viola built by Anne Cole; the viola was named "Bluebonnet" by its maker. The tuning used is the harmonic series with a fundamental frequency of 86 Hz, i.e. the integers of the sequence $a(n)$ were mapped directly to partials $a(n) \rightarrow a(n) * 86$ Hz, which makes the highest integer $13 \rightarrow 1118$ Hz. Each sequence is attributed its own wavetable and therefore its own timbre. Frequency, loudness, attack and release are the dynamic parameters utilized in this composition – all of them are global, applying to both sequences.

In the original online scores of Collection XIII, almost any element in the score can be hovered over with the cursor in order to gain more information and clarify meaning.

Collection XIII

Min and Max Fibbit Running no. 4
Casey Mongoven
March 13, 2012

classification of work: `audio-visual`
synthesis engine: `Csound` `show Csound orchestra`
synthesis technique: `wavetable with FFT resynthesis`

description of sequences sonified

A104324 `number of runs in the minimal Fibonacci (binary) representation of n 609 members used`
A104325 `number of runs in the maximal Fibonacci (binary) representation of n 609 members used`

global static parameters:

offset: `1`
temperament: `series of harmonic partials`
pitch orientation: `ascending`
number of channels: `2`
note duration: `0.065 seconds`
piece duration: `39.585 seconds`

individual static parameters:

location:
-30°
30°

wavetables:
cole bluebonnet 4
cole bluebonnet 5

global dynamic parameters:

A104324

integer	frequency Hz	loudness	attack s	release s
1	86.000000	pp	0.012300	0.014400
2	172.000000	pp	0.011908	0.013942
3	258.000000	p	0.011517	0.013483
4	344.000000	p	0.011125	0.013025
5	430.000000	p	0.010733	0.012567
6	516.000000	mp	0.010342	0.012108
7	602.000000	mp	0.009950	0.011650
8	688.000000	mp	0.009558	0.011192
9	774.000000	mf	0.009167	0.010733
10	860.000000	mf	0.008775	0.010275
11	946.000000	mf	0.008383	0.009817
12	1032.000000	f	0.007992	0.009358
13	1118.000000	f	0.007600	0.008900

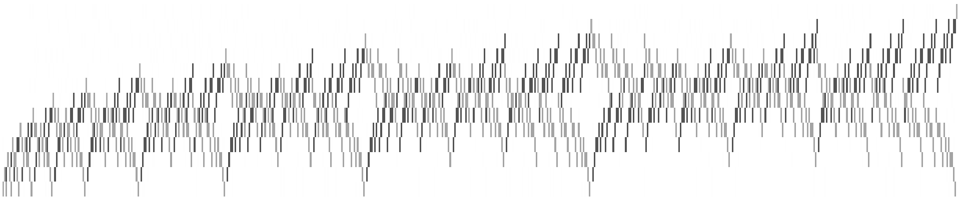
A104325

integer	frequency Hz	loudness	attack s	release s
1	86.000000	pp	0.012300	0.014400
2	172.000000	pp	0.011908	0.013942
3	258.000000	p	0.011517	0.013483
4	344.000000	p	0.011125	0.013025
5	430.000000	p	0.010733	0.012567
6	516.000000	mp	0.010342	0.012108
7	602.000000	mp	0.009950	0.011650
8	688.000000	mp	0.009558	0.011192
9	774.000000	mf	0.009167	0.010733
10	860.000000	mf	0.008775	0.010275
11	946.000000	mf	0.008383	0.009817
12	1032.000000	f	0.007992	0.009358
13	1118.000000	f	0.007600	0.008900

609 values used of Sloane's A104324: 1, 2, 2, 3, 2, 3, 4, 2, 3, 4, 4, 5, 2, 3, 4, 4, 5, 4, 5, 6, 2, ..., 10, 11, 12, 10, 11, 12, 12, 13

609 values used of Sloane's A104325: 1, 2, 1, 3, 2, 1, 4, 3, 3, 2, 1, 5, 4, 3, 4, 3, 3, 2, 1, 6, 5, ..., 4, 3, 4, 3, 3, 2, 1, 13

graph of sequences: 1) A104324, 609 values; 2) A104325, 609 values.



These two sequences combined exhibit particularly beautiful contrary motion.

2.3. Q(n) and U(n) rep sequences no. 1

The first sequence used in this work (B1361 in [6]) is based on the number of possible representations of an integer n as a sum of distinct elements of the Lucas sequence beginning $\langle 1, 3, 4, 7, 11, \dots \rangle$ (A003263 in [5]).[3] For example, the integer 8 can be represented in two ways: $7+1$ or $4+3+1$, so $a_8 = 2$. Integers that are not representable (e.g. 2 and 6) are sonified as silence. The second sequence sonified follows the exact same principle, but uses elements of the generalized Fibonacci sequence $\langle 1, 4, 5, 9, 14, \dots \rangle$ instead (A103344 in [5] – called U(n) here). 1363 members of the first sequence were used, and 1740 of the second. This means that in this case, the first sequence has an earlier point of termination than the second. This was done in order to highlight a certain relationship between these sequences: if one removes all of the 0s from both sequences, which are represented as silence in this sonification, then the sequences are identical. The musical form resulting from the combination of these sequences is therefore a unique type of mensural canon by augmentation.

The synthesis technique used in this composition was granular synthesis with a resonance filter. Granular synthesis is a general sound synthesis technique that operates on the microsound time scale in which small fragments of sound called *grains* (generally lasting between 1 to 50 milliseconds) are utilized. In the sonification of integer sequences, a grain can be used to represent an integer. This technique can be highly useful because, compared to other synthesis techniques, a much larger number of integers can be heard in a short timespan. If a resonance filter is used to filter a grain, as it is here, then each integer can be attributed its own resonance center frequency. Similarly, each grain can be attributed its own location according to the sequence or its own level of loudness. The grain used was from a different Anne Cole viola, named “1980” after the year it was made. The note duration here is a speedy .025 seconds. The unit interval of the tuning is again $\varphi^7 + 1$, this time with ascending frequency orientation (higher integers are represented by higher frequencies). The natural filtered attack of the grain was left unaltered, but a static release value of .0089 was used at the end of each note.

As in the last composition, location is a static parameter and each sequence is placed on an individual loudspeaker. Three global dynamic parameters are utilized: resonance filter center frequency, resonance filter bandwidth and loudness.

[3]This sequence was referred to as Q(n) in *Fibonacci and Related Number Theoretical Tables* [9].

Collection XIII

Q(n) and U(n) Rep Sequences no. 1
Casey Mongoven
March 18, 2012

classification of work: audio-visual
synthesis engine: Csound show Csound orchestra
synthesis technique: granular synthesis with resonance filter

description of sequences sonified

A003263 number of possible representations of n as a sum using distinct elements of the Lucas sequence beginning 1,3,4,7,11,... 1363 members used

A103344 number of possible representations of n as a sum using distinct elements of the Fibonacci-type sequence beginning 1,4,5,9,14,... 1740 members used

global static parameters:

offset: 1
temperament: $\text{phi}^7 + 1$
pitch orientation: ascending
number of channels: 2
note duration: 0.025 seconds
piece duration: 43.5 seconds
grain: cole 1980 24
release: 0.0089 seconds

individual static parameters:

location:
-30°
30°

global dynamic parameters:

A003263

integer	resonance center frequency Hz	resonance bandwidth Q factor	loudness
1	850.000000	8.333333	pp
2	879.275576	8.474576	pp
3	909.559456	8.620690	pp
4	940.886370	8.771930	p
5	973.292241	8.928571	p
⋮	⋮	⋮	⋮
24	1852.076217	13.513514	ff
25	1915.865155	13.888889	ff
26	1981.851103	14.285714	ff

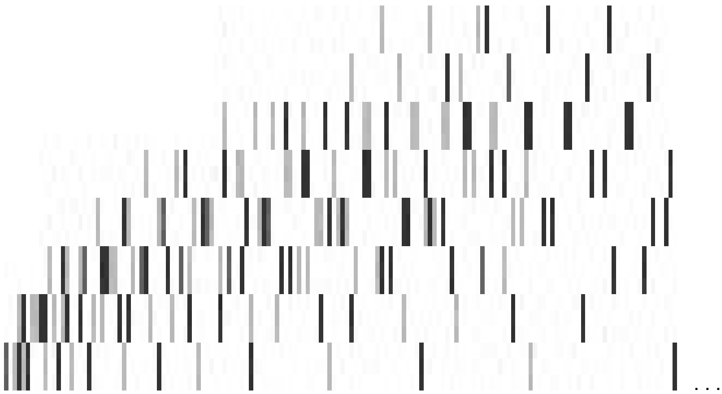
A103344

integer	resonance center frequency Hz	resonance bandwidth Q factor	loudness
1	850.000000	8.333333	pp
2	879.275576	8.474576	pp
3	909.559456	8.620690	pp
4	940.886370	8.771930	p
5	973.292241	8.928571	p
⋮	⋮	⋮	⋮
24	1852.076217	13.513514	ff
25	1915.865155	13.888889	ff
26	1981.851103	14.285714	ff

1363 values used of Sloane's A003263: 1, 0, 1, 2, 1, 0, 2, 2, 0, 1, 3, 2, 0, 2, 3, 1, 0, 3, 3, 0, 2, ..., 6, 12, 6, 0, 7, 7, 0, 1

1740 values used of Sloane's A103344: 1, 0, 0, 1, 2, 1, 0, 0, 2, 2, 0, 0, 1, 3, 2, 0, 0, 2, 3, 1, 0, ..., 6, 0, 0, 7, 7, 0, 0, 1

graph of sequences: 1) A003263, 1363 values; 2) A103344, 1740 values.



The above graph of this work had to be truncated for practical reasons. The canonic principle inherent in this work was taken to an even higher level in the eight-channel work *Rep Sequences no. 1* (B1117 in [6]), in which eight such sequences are utilized at once, one per speaker.

2.4. Absent and unique residues no. 3

The two sequences sonified in this work (B1411 in [6]) are based on the Fibonacci sequence under a modulus. One sequence counts the number of unique residues absent in F_n modulo m, while the other counts the number of unique residues present. In the diagram below, F_n modulo m has been reduced to the length of its period.

m	F_n modulo m	**unique**	**absent**
1	$\langle 0\rangle$	1	0
2	$\langle 0,1,1\rangle$	2	0
3	$\langle 0,1,1,2,0,2,2,1\rangle$	3	0
4	$\langle 0,1,1,2,3,1\rangle$	4	0
5	$\langle 0,1,1,2,3,0,3,3,1,4,0,4,4,3,2,0,2,2,4,1\rangle$	5	0
6	$\langle 0,1,1,2,3,5,2,1,3,4,1,5,0,5,5,4,3,1,4,5,3,2,5,1\rangle$	6	0
7	$\langle 0,1,1,2,3,5,1,6,0,6,6,5,4,2,6,1\rangle$	7	0
8	$\langle 0,1,1,2,3,5,0,5,5,2,7,1\rangle$	6	2
9	$\langle 0,1,1,2,3,5,8,4,3,7,1,8,0,8,8,7,6,4,1,5,6,2,8,1\rangle$	9	0
10	$\langle 0,1,1,2,3,5,8,3,1,4,5,9,\ldots,9,5,4,9,3,2,5,7,2,9,1\rangle$	10	0
11	$\langle 0,1,1,2,3,5,8,2,10,1\rangle$	7	4
12	$\langle 0,1,1,2,3,5,8,1,9,10,7,5,0,5,5,10,3,1,4,5,9,2,11,1\rangle$	11	1
13	$\langle 0,1,1,2,3,5,8,0,8,8,3,11,\ldots,8,5,0,5,5,10,2,12,1\rangle$	9	4
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$

The sequence of unique residues is placed on the left speaker, and the sequence of absent residues on the right. A single grain created from the sound of pieces of raw lump charcoal colliding together was used as a sound source – a light percussive sound that is rich mostly in higher frequency components. The only dynamic parameter in this work is loudness, ranging from extremely quiet to loud (*pppp* to *f*). In both sequences, each member lasts .074 seconds, and $F_{16} = 987$ members are used of each sequence, resulting in a piece duration of 73.038 seconds. 0s in the absent residues sequence were sonified as silence, resulting in periodic gaps on the right speaker.

Collection XIV

Absent and Unique Residues no. 3
Casey Mongoven
May 18, 2012

classification of work: `audio-visual`
synthesis engine: `Csound` `show Csound orchestra`
synthesis technique: `granular synthesis`

description of sequences sonified

A066853 `number of unique residues in Fibonacci sequence mod n 987 members used`
A118965 `number of residues absent in Fibonacci sequence mod n 987 members used`

global static parameters:

sequence offset: `1`
number of channels: `2`
duration of single member of sequence: `0.074 seconds`
piece duration: `73.038 seconds`
grain: `raw charcoal 8`

individual static parameters:

location:

-30°
30°

global dynamic parameters:

A066853

integer	loudness
1	pppp
2	pppp
3	pppp
4	pppp
5	pppp
⋮	⋮
745	mf
750	mf
875	f

A118965

integer	loudness
1	pppp
2	pppp
4	pppp
5	pppp
7	pppp
⋮	⋮
928	ff
937	ff
966	ff

987 values used of Sloane's A066853: 1, 2, 3, 4, 5, 6, 7, 6, 9, 10, 7, 11, 9, 14, 15, 11, 13, 11, 12, 20, 9, ..., 555, 149, 614, 739, 61, 745, 94, 21

987 values used of Sloane's A118965: 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 4, 1, 4, 0, 0, 5, 4, 7, 7, 0, 12, ..., 425, 832, 368, 244, 923, 240, 892, 966

graph of sequences: 1) A003263, 987 values; 2) A103344, 987 values.

The graph at the end of the score had to be truncated for practical reasons. As can be seen, these sequences also display natural contrary motion when sonified together; however, the sequences' relatively erratic behavior obscures this to some degree, as does the fact that only a single dynamic parameter is used in the sonification, which is not frequency.

2.5. Stolarsky and Wythoff arrays no. 4

Unlike the preceding works, all of which utilized two integer sequences, four integer sequences based on the Wythoff and Stolarsky arrays are sonified simultaneously in *Stolarsky and Wythoff Arrays no. 4* (B1412 in [6]). The Wythoff array can be created by taking the non-negative integers $\mathbb{Z}_{\geq 0}$ and the Beatty sequence $\lfloor \Phi(\mathbb{Z}_{\geq 0} + 1) \rfloor$ as starting points for rows of G_n [11], as follows:

$\mathbb{Z}_{\geq 0}$	$\lfloor \Phi(\mathbb{Z}_{\geq 0} + 1) \rfloor$											
0	1	1	2	3	5	8	13	21	34	55	89	→
1	3	4	7	11	18	29	47	76	123	199	322	→
2	4	6	10	16	26	42	68	110	178	288	466	→
3	6	9	15	24	39	63	102	165	267	432	699	→
4	8	12	20	32	52	84	136	220	356	576	932	→
5	9	14	23	37	60	97	157	254	411	665	1076	→
6	11	17	28	45	73	118	191	309	500	809	1309	→
7	12	19	31	50	81	131	212	343	555	898	1453	→
8	14	22	36	58	94	152	246	398	644	1042	1686	→
9	16	25	41	66	107	173	280	453	733	1186	1919	→
10	17	27	44	71	115	186	301	487	788	1275	2063	→
11	19	30	49	79	128	207	335	542	877	1419	2296	→
12	21	33	54	87	141	228	369	597	966	1563	2529	→
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↘

The portion in black is the Wythoff array.

In the Stolarsky array, the first integer in each row k is the lowest that has not yet occurred in any row above. The integer that follows k is given by $[\Phi * k]$ [12]:

k	$[\Phi * k]$									
1	2	3	5	8	13	21	34	55	89	→
4	6	10	16	26	42	68	110	178	288	→
7	11	18	29	47	76	123	199	322	521	→
9	15	24	39	63	102	165	267	432	699	→
12	19	31	50	81	131	212	343	555	898	→
14	23	37	60	97	157	254	411	665	1076	→
17	28	45	73	118	191	309	500	809	1309	→
20	32	52	84	136	220	356	576	932	1508	→
22	36	58	94	152	246	398	644	1042	1686	→
25	40	65	105	170	275	445	720	1165	1885	→
27	44	71	115	186	301	487	788	1275	2063	→
30	49	79	128	207	335	542	877	1419	2296	→
33	53	86	139	225	364	589	953	1542	2495	→
35	57	92	149	241	390	631	1021	1652	2673	→
38	61	99	160	259	419	678	1097	1775	2872	→
41	66	107	173	280	453	733	1186	1919	3105	→
43	70	113	183	296	479	775	1254	2029	3283	→
46	74	120	194	314	508	822	1330	2152	3482	→
48	78	126	204	330	534	864	1398	2262	3660	→
51	83	134	217	351	568	919	1487	2406	3893	→
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↘

The sequence that gives the column in which an integer n occurs in the Wythoff array is called the horizontal para-Fibonacci sequence; the sequence that gives the row is called the vertical para-Fibonacci sequence (A035614 and A019586 in [5]). One can create two more similar sequences by applying the same principle to the Stolarsky array (A098861 and A098862 in [5]). All four of these sequences are sonified simultaneously in this composition.

In this work, wavetable synthesis is utilized. The purity of tone using this technique can increase clarity of representation when sonifying a significant number of sequences simultaneously. The wavetable used for all sequences was derived from the sound of an opening door. The dynamic parameters used here are frequency, attack and release, and simulated location. Of these parameters, all are global except for simulated location, which is individual for each sequence. $\varphi^7 + 1$ is the unit interval in the temperament used, and the frequencies in the work span from a soaring 2355 Hz to about 119.6 Hz (the frequency orientation is descending). A somewhat slower tempo of .12 seconds and 233 members of each sequence are utilized, resulting in a piece duration of 27.96 seconds. A static level of loudness of mezzo-forte (medium-loud) was chosen in order not to obscure either the lower or higher members of the sequence.

Collection XIV

Stolarsky and Wythoff Arrays no. 4
Casey Mongoven
May 20, 2012

classification of work: `audio-visual`
synthesis engine: `Csound` `show Csound orchestra`
synthesis technique: `wavetable with FFT resynthesis`

description of sequences sonified

A035614 gives number of column in Wythoff array that contains n 233 members used
A019586 gives number of row in Wythoff array that contains n 233 members used
A098862 gives number of column in Stolarsky array that contains n 233 members used
A098861 gives number of row in Stolarsky array that contains n 233 members used

global static parameters:

sequence offset: 1
temperament: phi^7 + 1
frequency orientation: descending
number of channels: 2
duration of single member of sequence: 0.12 seconds
piece duration: 27.96 seconds
wavetable: door sound 1
loudness: mf

global and individual dynamic parameters:

A035614

integer	frequency Hz	attack s	release s	simulated location degrees
0	2355.000000	0.012300	0.014400	-27.665
1	2276.590020	0.012247	0.014338	-27.565
2	2200.790708	0.012193	0.014275	-27.465
3	2127.515142	0.012140	0.014213	-27.365
4	2056.679295	0.012086	0.014150	-27.265
⋮	⋮	⋮	⋮	⋮
9	1736.345543	0.011819	0.013838	-26.765
10	1678.533730	0.011766	0.013775	-26.665
11	1622.646767	0.011713	0.013713	-26.565

A019586

integer	frequency Hz	attack s	release s	simulated location degrees
0	2355.000000	0.012300	0.014400	-12.155
1	2276.590020	0.012247	0.014338	-12.055
2	2200.790708	0.012193	0.014275	-11.955
3	2127.515142	0.012140	0.014213	-11.855
4	2056.679295	0.012086	0.014150	-11.755
⋮	⋮	⋮	⋮	⋮
86	128.017015	0.007707	0.009025	-3.554
87	123.754675	0.007653	0.008962	-3.454
88	119.634250	0.007600	0.008900	-3.354

A098862

integer	frequency Hz	attack s	release s	simulated location degrees
0	2355.000000	0.012300	0.014400	3.354
1	2276.590020	0.012247	0.014338	3.454
2	2200.790708	0.012193	0.014275	3.554
3	2127.515142	0.012140	0.014213	3.654
4	2056.679295	0.012086	0.014150	3.754
⋮	⋮	⋮	⋮	⋮
9	1736.345543	0.011819	0.013838	4.254
10	1678.533730	0.011766	0.013775	4.354
11	1622.646767	0.011713	0.013713	4.454

A098861

integer	frequency Hz	attack s	release s	simulated location degrees
0	2355.000000	0.012300	0.014400	18.864
1	2276.590020	0.012247	0.014338	18.964
2	2200.790708	0.012193	0.014275	19.064
3	2127.515142	0.012140	0.014213	19.164
4	2056.679295	0.012086	0.014150	19.264
⋮	⋮	⋮	⋮	⋮
86	128.017015	0.007707	0.009025	27.465
87	123.754675	0.007653	0.008962	27.565
88	119.634250	0.007600	0.008900	27.665

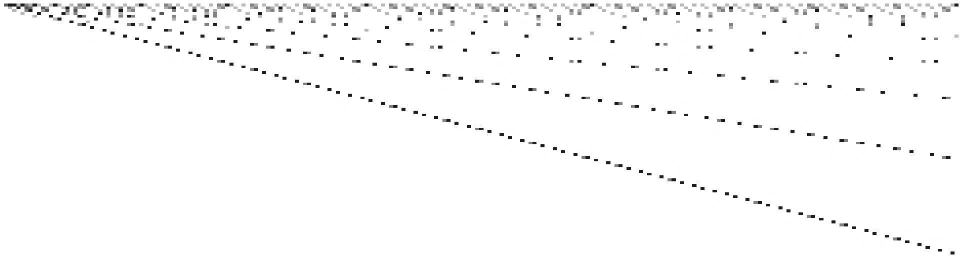
233 values used of Sloane's A035614: 0, 1, 2, 0, 3, 0, 1, 4, 0, 1, 2, 0, 5, 0, 1, 2, 0, 3, 0, 1, 6, 0, ..., 0, 1, 4, 0, 1, 2, 0, 11

233 values used of Sloane's A019586: 0, 0, 0, 1, 0, 2, 1, 0, 3, 2, 1, 4, 0, 5, 3, 2, 6, 1, 7, 4, 0, 8, ..., 86, 53, 12, 87, 54, 33, 88, 0

233 values used of Sloane's A098862: 0, 1, 2, 0, 3, 1, 0, 4, 0, 2, 1, 0, 5, 0, 1, 3, 0, 2, 1, 0, 6, 0, ..., 0, 1, 3, 0, 2, 1, 0, 11

233 values used of Sloane's A098861: 0, 0, 0, 1, 0, 1, 2, 0, 3, 1, 2, 4, 0, 5, 3, 1, 6, 2, 4, 7, 0, 8, ..., 86, 53, 20, 87, 33, 54, 88, 0

graph of sequences: 1) A035614, 233 values; 2) A019586, 233 values; 3) A098862, 233 values; 4) A098861, 233 values.



3. Ongoing research

New challenges are presented when composing sonifications of multiple integer sequences simultaneously. Which sequences can be sonified together and how? Which parameters should we allow to be individual and which global? When should such a sonification terminate? Currently, the author is working on theoretical criteria for the sonification of integer sequences that attempt to posit potential answers to such questions. In addition, an experiment involving more than 150 participants is being carried out that compares various sonified Fibonacci-related mathematical objects to analog mathematical objects more or less unrelated to F_n, in an attempt to gain some insight into the aesthetic value of F_n and the golden ratio in musical composition. This work is part of the author's doctoral dissertation in the Media Arts and Technology Department at UCSB.

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Primary classes of compositions of numbers*

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Dedicated to the memory of P. A. MacMahon on the occasion of the 158th anniversary of his birth

Abstract

The compositions, or ordered partitions, of integers, fall under certain natural classes. In this expository paper we highlight the most important classes by means of bijective proofs. Some of the proofs rely on the properties of zig-zag graphs - the graphical representations of compositions introduced by Percy A. MacMahon in his classic book *Combinatory Analysis*.

Keywords: composition, conjugate, zig-zag graph, line graph, bit-encoding, direct detection.

MSC: 05A19.

1. Introduction

A composition of a positive integer n is a representation of n as a sequence of positive integers which sum to n. The terms are called parts of the composition.

Denote the number of compositions of n by $c(n)$. The formula for $c(n)$ may be obtained from the classical recurrence relation:

$$c(n+1) = 2c(n), \quad c(1) = 1. \tag{1.1}$$

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Indeed a composition of $n+1$ may be obtained from a composition of n either by adding 1 to the first part, or by inserting 1 to the left of the previous first part. The recurrence gives the well-known formula: $c(n) = 2^{n-1}$.

For example, the following are the compositions of $n = 1, 2, 3, 4$:

$$(1)$$
$$(2), (1,1)$$
$$(3), (1,2), (2,1), (1,1,1)$$
$$(4), (1,3), (2,2), (3,1), (1,1,2), (1,2,1), (2,1,1), (1,1,1,1)$$

When the order of parts is fixed we obtain the partitions of n. For example, 4 has just 5 partitions – $(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$.

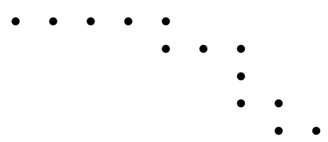
This is an expository paper devoted to a classification of compositions according to certain natural criteria afforded by their rich symmetry. We will mostly employ the extensive beautiful machinery developed by P. A. MacMahon in his classic text [3]. His original analysis of the properties of compositions seems to have received scarce attention in the literature during the last half-century.

Percy Alexander MacMahon was born in Malta on 26 September 1854, the son of brigadier general. He attended a military academy and later became an artillery officer, attaining the rank of Major, all the while doing top-class mathematics research.

According to his posthumous contemporary biographer, Paul Garcia [2],

"*MacMahon did pioneering work in invariant theory, symmetric function theory, and partition theory. He brought all these strands together to bring coherence to the discipline we now call combinatorial analysis. . . .*"

MacMahon's study of compositions was influenced by his pioneering work in partitions. For instance, he devised a graphical representation of a composition, called a *zig-zag graph*, which resembles the partition Ferrers graph except that the first dot of each part is aligned with the last part of its predecessor. Thus the zig-zag graph of the composition $(5,3,1,2,2)$ is



(1.2)

The conjugate of a composition is obtained by reading its graph by columns, from left to right: the graph (1.2) gives the conjugate of the composition $(5,3,1,2,2)$ as $(1,1,1,1,2,1,3,2,1)$.

The zigzag graph possesses a rich combinatorial structure providing several equivalent paths to the conjugate composition. The latter are outlined in Section 2.

We will sometimes write $C \models n$ to indicate that C is a composition of n, and the integer n will be referred to as the *weight* of C. A k-composition is a composition

with k parts. The conjugate of C will be denoted by C'.
Now following MacMahon, we define, relative to a composition $C = (c_1, c_2, \ldots, c_k)$:

The *inverse* of C is the reversal composition $\overline{C} = (c_k, c_{k-1}, \ldots, c_2, c_1)$.

C is called *self-inverse* if $C = \overline{C}$.

C is *inverse-conjugate* if it's inverse coincides with its conjugate: $C' = \overline{C}$.

The zigzag graph of a composition C can be read in four ways to give generally different compositions namely $C, C', \overline{C}, \overline{C}'$. Exceptions occur when C is self-inverse, or when C is inverse-conjugate, in which case only two readings are obtained.

We deliberately refrain from applying generating function techniques in this paper for the simple reason that the apparent efficacy of their use has largely been responsible for obscuring the methods discussed.

2. The conjugate composition

In this section we outline five different paths to the conjugate composition.

ZG: The Zig-zag Graph, already defined above.

LG: The Line graph (also introduced by MacMahon [3, Sec. IV, Ch. 1, p. 151])

The number n is depicted as a line divided into n equal segments and separated by $n-1$ spaces. A composition $C = (c_1, \ldots, c_k)$ then corresponds to a choice of $k-1$ from the $n-1$ spaces, indicated with nodes. The conjugate C' is obtained by placing nodes on the other $n-k$ spaces. Thus the line graph of the composition $(5, 3, 1, 2, 2)$ is

 ,

from which we deduce that $C' = (1, 1, 1, 1, 2, 1, 3, 2, 1)$. It follows that C' has $n - k + 1$ parts.

SubSum: Subset Partial Sums:

There is a bijection between compositions of n into k parts and $(k-1)$-subsets of $\{1, \ldots, n-1\}$ via partial sums (see also [6]) given by

$$C = (c_1, \ldots, c_k) \mapsto \{c_1, c_1 + c_2, \ldots, c_1 + c_2 + \cdots + c_{k-1}\} = L. \tag{2.1}$$

Hence C' is the composition corresponding to the set $\{1, \ldots, n-1\} \setminus L$.

BitS: Encoding by Binary Strings

It is sometimes necessary to express compositions as bit strings. The procedure for such *bit-encoding* consists of converting the set L into a unique bit string $B = (b_1, \ldots, b_{n-1}) \in \{0, 1\}^{n-1}$ such that

$$b_i = \begin{cases} 1 & \text{if } i \in L \\ 0 & \text{if } i \notin L. \end{cases}$$

The complementary bit string B', obtained from B by swapping the roles of 1 and 0, is then the bit encoding of C'.

DD: Direct Detection of Conjugates

There is an easily-mastered rule for writing down the conjugate of a composition by inspection. A sequence of x consecutive equal parts $c, \ldots, c$ will be abbreviated as c^x. First, the general composition has two forms, subject to inversion:

(1) $C = (1^{a_1}, b_1, 1^{a_2}, b_2, 1^{a_3}, b_3, \ldots),\ a_i \geq 0, b_i \geq 2;$

(2) $E = (b_1, 1^{a_1}, b_2, 1^{a_2}, b_3, 1^{a_3}, \ldots),\ a_i \geq 0, b_i \geq 2.$

The conjugate, in either case, is given by the rule:

(1c) $C' = (a_1 + 1, 1^{-1+b_1-1}, 1 + a_2 + 1, 1^{-1+b_2-1}, 1 + a_3 + 1, \ldots)$
$= (a_1 + 1, 1^{b_1-2}, a_2 + 2, 1^{b_2-2}, a_3 + 2, \ldots).$

Similarly,

(2c) $E' = (1^{b_1-1}, a_1 + 2, 1^{b_2-2}, a_2 + 2, \ldots).$

For example, $(1, 3, 4, 1^3, 2, 1^2, 6)'$ is given by

$$(1 + 1, 1^{3-2}, 1 + 1, 1^{4-2}, 1 + 1^3 + 1, 1 + 1^2 + 1, 1^{6-1}) = (2, 1, 2, 1^2, 5, 4, 1^5).$$

The various approaches to the conjugate composition obviously have their merits and demerits. The strength of the **DD** method is that it often provides a general form of the conjugate composition explicitly.

3. Special classes of compositions

We will need the following algebraic operations:

If $A = (a_1, \ldots, a_i)$ and $B = (b_1, \ldots, b_j)$ are compositions, we define the concatenation of the parts of A and B by

$A|B = (a_1, \ldots, a_i, b_1, \ldots, b_j).$

In particular for a nonnegative integer c, we have $A|(c) = (A, c)$ and $(c)|A = (c, A)$. Define the *join* of A and B as

$A \uplus B = (a_1, \ldots, a_{i-1}, a_i + b_1, b_2, \ldots, b_j).$

The following rules are easily verified:

1. $\overline{A|B} = \overline{B}|\overline{A}$.
2. $(A|B)' = A' \uplus B'$.

Note that $(A, 0) \uplus B = A \uplus (0, B) = A|B$.

3.1. Equitable decomposition by conjugation

The conjugation operation immediately implies the following identity:

Proposition 3.1. *The number of compositions of n with k parts equals the number of compositions of n with $n - k + 1$ parts.*

The two classes consist of different compositions except when n is odd and $k = (n+1)/2 = n-k+1$. In the latter case the two classes are coincident. Indeed since there are $c(n,k) = \binom{n-1}{k-1}$ compositions of n with k parts, we see at once that $c(n,k) = c(n, n-k+1)$.

Thus the set $W(n)$ of compositions of n may be economically stored by keeping only the sets $W(n,k)$ of k-compositions, $k = 1, \ldots, \lfloor \frac{n+1}{2} \rfloor$, whereby the remaining compositions are accessible via conjugation.

Looking closely at this idea, assume that the elements of each set $W(n,k)$ are arranged in lexicographic order, and list the sets in increasing order of lengths of members as follows:

$$\underbrace{W(n,1), W(n,2), \ldots, W(n, \lfloor \tfrac{n+1}{2} \rfloor)}_{\text{generates } W(n) \text{ via conjugation}}, W(n, \lfloor \tfrac{n+1}{2} \rfloor + 1), \ldots, W(n, n-1), W(n,n). \quad (3.1)$$

This arrangement implies one of the beautiful symmetries exhibited by many sets of compositions:

If the set divisions are removed to reveal a single list of all compositions of n, then the j-th composition from the left and the j-th composition from the right are mutual conjugates. In other words, the j-th composition is the conjugate of the $(n-j+1)$-th composition, from either end.

This arrangement is illustrated for compositions of $n = 1, 2, 3, 4$ (see Section 1).

3.2. Equitable four-way decomposition

Define a $1c2$-*composition* as a composition with the first part equal to 1 and last part > 1. The following are analogously defined: $2c1$-*composition,* $1c1$-*composition*, and $2c2$-*composition.*

Then observe that the $2c1$-compositions are inverses of $1c2$-compositions, and that the set of $2c2$-compositions form the set of conjugates of the $1c1$-compositions. It turns out that the set of compositions of n splits naturally into four subsets of equal cardinality corresponding to the four types of compositions.

Theorem 3.2. *Let n be a natural number > 1. Then the following classes of compositions are equinumerous:*

(i) $1c1$-compositions of n.

(ii) $1c2$-compositions of n.

(iii) $2c1$-compositions of n.

(iv) $2c2$-compositions of n.

Each class is enumerated by $c(n-2)$.

Proof. By the remark immediately preceding the theorem, it suffices to establish a bijection: (i) $\Longleftrightarrow$ (ii). An object in (ii) has the form $C = (1, c_2, \ldots, c_k), c_k > 1$. Deleting the initial 1 and subtracting 1 from c_k gives $(c_2, \ldots, c_k - 1) = T$, a composition of $n-2$. Now pre-pend and append 1 to obtain $(1, c_2, \ldots, c_k - 1, 1)$, which is a unique composition in (i). Lastly, also note that the passage from C to

T is a bijection from (i) to the class of compositions of $n-2$. In other words the common number of compositions in each of the classes is $c(n-2)$. □

Example. When $n = 5$, the four classes are given by:
(i) $(1,3,1),(1,2,1,1),(1,1,2,1),(1,1,1,1,1)$;
(ii) $(1,4),(1,2,2),(1,1,3),(1,1,1,2)$;
(iii) $(4,1),(2,2,1),(3,1,1),(2,1,1,1)$;
(iv) $(2,1,2),(2,3),(3,2),(5)$.

Remark 3.3. An Application: Since Theorem 3.2 implies $c(n) = 4c(n-2)$, it can be applied to the generation of compositions of n from those of $n-2$ in an obvious way. Such algorithm is clearly more efficient than the classical recursive procedure via the compositions of $n-1$ (see (1.1)). Thus to compute the compositions of 5, for example, it suffices to use the set $W(3) = \{(3),(2,1),(1,2),(1,1,1)\}$, together with the quick generation procedures corresponding to the bijections in the proof of Theorem 3.2.

A further saving of storage space can be attained by combining this four-way decomposition with the conjugation operation. Then to store the set $W(n)$ of compositions of n it would suffice to hold only one half of $W(n-2)$, arranged as previously described.

As a mixed refinement of Theorem 3.2 we have the following identity, which is a consequence of conjugation.

Proposition 3.4. *The number of compositions of n with one or two 1's which can appear only as a first and/or last part equals the number of compositions of n into 1's and 2's whose first and/or last part is 2.*

For example, when $n = 5$, the two classes of compositions mentioned in the proposition are:

$$(1,4),(4,1),(1,2,2),(1,3,1),(2,2,1);$$

$$(2,1,1,1),(1,1,1,2),(2,2,1),(2,1,2),(1,2,2).$$

3.3. Self-inverse compositions

Self-inverse compositions constitute the next easily distinguishable class of compositions. Their enumeration is usually straightforward. The number of parts of a composition C will also be referred to as its *length*, denoted by $\ell(C)$.

We remark that MacMahon [3] proved most of the results in this sub-section, in the case of k-compositions, using the **LG** method.

Proposition 3.5.

(*i*) *The number of self-inverse compositions of $2n$ is $c(n+1)$.*

(*ii*) *The number of self-inverse compositions of $2n-1$ is $c(n)$.*

Proof. We prove only part (i) (the proof of part (ii) is similar). Firstly, if C is a self-inverse composition with $\ell(C)$ odd, then C has the form:
$C = (c_1, \dots, c_{k-1}, c_k, c_{k-1}, \dots, c_1)$, where c_k is even. Thus
$C = (c_1, \dots, c_{k-1}, c_k/2) \uplus (c_k/2, c_{k-1}, \dots, c_1) \equiv A \uplus \overline{A}$,
where $A = (c_1, \dots, c_{k-1}, c_k/2)$ runs over all compositions of n.

If $\ell(C)$ is even, then C has the form $C = (c_1, \dots, c_{k-1}, c_k, c_k, c_{k-1} \dots, c_1) \equiv B|\overline{B}$, where $B = (c_1, \dots, c_{k-1}, c_k)$ runs over all compositions of n.

It follows that there are as many self-inverse compositions of $2n$ into an odd number of parts as into an even number of parts. Using the above notations, a simple bijection is $C \equiv A \uplus \overline{A} \mapsto A|\overline{A}$, and conversely, $C \equiv B|\overline{B} \mapsto B \uplus \overline{B}$. □

The essential results on self-inverse compositions are summarized below.

Theorem 3.6. *The following sets of compositions have the same number of elements:*
(i) self-inverse compositions of $2n-1$.
(ii) self-inverse compositions of $2n$ *of odd lengths.*
(iii) self-inverse compositions of $2n$ *of even lengths.*
(iv) self-inverse compositions of $2n-2$.
(v) compositions of n.

Proof. (i) $\iff$ (ii): if $(c_1, \dots, c_{k-1}, c_k, c_{k-1}, \dots, c_1)$ is in (i), then

$$(c_1, \dots, c_{k-1}, c_k + 1, c_{k-1}, \dots, c_1)$$

is in (ii), and conversely.
(i) $\iff$ (iv): if $(c_1, \dots, c_{k-1}, c_k, c_{k-1}, \dots, c_1)$ and $(c_1, \dots, c_{k-1}, c_k, c_k, c_{k-1}, \dots, c_1)$ belong to (iv), then (i) contains $(c_1, \dots, c_{k-1}, c_k + 1, c_{k-1}, \dots, c_1)$ and

$$(c_1, \dots, c_{k-1}, c_k, 1, c_k, c_{k-1}, \dots, c_1),$$

respectively.
Lastly, since the cases (ii) $\iff$ (iii) $\iff$ (v) have been demonstrated with the proof of Proposition 3.5, the theorem follows. □

4. Inverse-conjugate compositions

Let C be a k-composition. If C is inverse-conjugate, then $k = |C| - k + 1$ or $|C| = 2k-1$. Thus inverse-conjugate compositions are defined only for odd weights. In fact, every odd integer > 1 has a nontrivial inverse-conjugate composition. For instance, $(1, 2^{k-1})$ and $(1^{k-1}, k)$ are both inverse-conjugate compositions of $2k-1$.

Consider a general composition,

$$C = (1^{a_1}, b_1, 1^{a_2}, b_2, \dots, 1^{a_r}, b_r),\ a_i \geq 0, b_i \geq 2.$$

Then, using the **DD** conjugation rule in Section 2, we obtain

$$C' = (a_1 + 1, 1^{b_1 - 2}, a_2 + 2, 1^{b_2 - 2}, \dots, 1^{b_{r-1} - 2}, a_r + 2, 1^{b_r - 1}).$$

Thus the conditions for C to be inverse-conjugate are

$$b_r = a_1 + 1,\ b_{r-1} = a_2 + 2, \dots, b_1 = a_r + 2.$$

Hence we have proved:

Lemma 4.1. *An inverse-conjugate composition C (or its inverse) has the form:*

$$C = (1^{b_r-1}, b_1, 1^{b_{r-1}-2}, b_2, 1^{b_{r-2}-2}, b_3, \dots, b_{r-1}, 1^{b_1-2}, b_r),\ b_i \geq 2. \tag{4.1}$$

Note that the sum of the parts is $2(b_1 + \cdots + b_r) - (r-1)(2) - 1 \equiv 1 \pmod 2$, as expected.

Let $(c_1, \dots, c_k)$ be an inverse-conjugate composition of $n > 1$. For any index $j < k$ with $c_{j+1} \neq 1$, consider the sub-composition $(c_1, \dots, c_j)$. First, notice the following relation between the two "halves" of (4.1):

$$\overline{(1^{b_r-1}, b_1, \dots, b_j, 1^{b_{r-j}-2})} = (b_{r-j} - 1, 1^{b_j-2}, b_{r-j+1}, \dots, 1^{b_1-2}, b_r)'. \tag{4.2}$$

Therefore, if $|C| = 2k - 1$, it is possible for the weight of either side of (4.2) to be exactly $k - 1$. The latter case implies an instructive dissection of C:

$$\begin{aligned} C &= (1^{b_r-1}, b_1, \dots, b_j, 1^{b_{r-j}-2})|(1) \uplus (b_{r-j} - 1, 1^{b_j-2}, b_{r-j+1}, \dots, 1^{b_1-2}, b_r) \\ &= (1^{b_r-1}, b_1, \dots, b_j, 1^{b_{r-j}-2})|(1) \uplus \overline{(1^{b_r-1}, b_1, \dots, b_j, 1^{b_{r-j}-2})'}. \end{aligned}$$

where the last equality follows by conjugating both sides of (4.2).

The gist of the foregoing discussion is summarized in the next theorem.

Theorem 4.2. *If $C = (c_1, \dots, c_k)$ is an inverse-conjugate composition of $n = 2k - 1 > 1$, or its inverse, then there is an index j such that $c_1 + \cdots + c_j = k - 1$ and $c_{j+1} + \cdots + c_k = k$ with $c_{j+1} > 1$. Moreover,*

$$\overline{(c_1, \dots, c_j)} = (c_{j+1} - 1, c_{j+2}, \dots, c_k)' \tag{4.3}$$

Thus C can be written in the form

$$C = A|(1) \uplus B \quad \textit{such that} \quad B' = \overline{A}, \tag{4.4}$$

where A and B are generally different compositions of $k - 1$.

It follows that an inverse-conjugate composition C of $n > 1$ cannot be self-inverse, even though $\overline{C}$ is also inverse-conjugate (in contrast with the so-called *self-conjugate* partitions of $n > 2$ [1, 4]).

The theorem implies the following result of MacMahon which he demonstrated using the **LG** method.

Theorem 4.3 (MacMahon)**.** *The number of inverse-conjugate compositions of an odd integer $n > 0$ equals the number of compositions of n which are self-inverse.*

Proof. We describe a bijection α between the two classes of compositions by invoking Theorem 4.2. If $C \models 2k-1$ is inverse-conjugate, then C can be written in the form $C = A|(1) \uplus B$ or $C = A \uplus (1)|B$ for certain compositions A, B, of $k-1$ satisfying $B' = \overline{A}$.

In the first case we use (4.3) to get $\alpha(C) = A|[(1) \uplus B]'$, which is a self-inverse composition of the type $A|(1)|\overline{A}$.

The second case, $C = A \uplus (1)|B$, implies that there is a part $m > 1$ such that $C = X|(m)|B$, with $X \models M < k-1$. Now split m between the two compositions as follows: $X|(m-1) \uplus (1)|B = (X, m-1) \uplus (1, B)$, which is in the first-case form. Hence $\alpha(C) = (X, m-1) \uplus (1, B)'$, giving a self-inverse composition of the type $Y|(d)|\overline{Y}$, with d an odd integer > 1.

Conversely given a self-inverse composition, $T = (b_1, \ldots, b_r) \equiv B|(d)|\overline{B}$ of $2k-1$, we first write T as the join of two compositions of $k-1$ and k, by splitting the middle part. The middle part, by weight, is b_{j+1} such that $s_j = b_1 + \cdots + b_j \leq k-1$ and $s_j + b_{j+1} \geq k$. Thus

$T \mapsto (b_1, \ldots, b_j)|(k-1-s_j) \uplus (k-t_j)|(b_{j+2}, \ldots, b_r) \equiv X|(k-1-s_j) \uplus (k-t_j)|\overline{X}$, where $t_j = b_{j+2} + \cdots + b_k$.

Hence $\alpha^{-1}(T) = X|(k-1-s_j) \uplus (k-t_j, \overline{X})'$, which is inverse-conjugate. □

Example. Consider the inverse-conjugate composition of 15 given by

$$C = (1, 1, 1, 2, 3, 1, 2, 4).$$

Then since $1 + 1 + 1 + 2 < 7$ and $1 + 1 + 1 + 2 + 3 > 7$, we have

$$C = (1, 1, 1, 2)|(3)|(1, 2, 4) \to (1, 1, 1, 2, 2) \uplus (1, 1, 2, 4)' = (1, 1, 1, 2, 2) \uplus (3, 2, 1, 1, 1),$$

which gives $T = (1, 1, 1, 2, 5, 2, 1, 1, 1)$, a self-inverse composition of 15. Conversely,

$$(1, 1, 1, 2, 5, 2, 1, 1, 1) \to (1, 1, 1, 2, 2) \uplus (3, 2, 1, 1, 1)' = (1, 1, 1, 2, 2) \uplus (1, 1, 2, 4),$$

which gives back $(1, 1, 1, 2, 3, 1, 2, 4)$.

It can also be verified that $C' = (4, 2, 1, 3, 2, 1, 1, 1)$ corresponds to the self-inverse composition $(4, 2, 1, 1, 1, 2, 4) = T'$ under the bijection.

Corollary 4.4. *There are as many inverse-conjugate compositions of* $2n-1$ *as there are compositions of* n.

Proof. The proof can be deduced from Theorem 3.6 and Theorem 4.3, but we give a direct proof. If $n = 1$, the composition (1) belongs trivially to the two classes of compositions. So assume $n > 1$.

Let $(c_1, \ldots, c_n)$ be any inverse-conjugate composition of $2n-1$. Then by (4.4) there is an index j such that $c_1 + \cdots + c_j = n-1$ or $c_{k-j+1} + \cdots + c_k = n-1$.

There are $c(n-1)$ inverse-conjugate compositions $(c_, \ldots, c_n)$ in which $c_1 + \cdots + c_j = n-1$, $n > 1$, and there distinct conjugates (i.e., inverses). Since there are no self-inverse inverse-conjugate compositions, the total number of inverse-conjugate compositions of $2n-1$ is $2c(n-1) = c(n)$, as required.

We can also give a bijection. According to Theorem 4.2 every inverse-conjugate composition $(c_1, \dots, c_n)$ satisfies $c_1 + \cdots + c_j = n - 1$ and $c_{j+1} + \cdots + c_n = n$ with $c_{j+1} > 1$, or $c_1 + \cdots + c_j = n$ and $c_{j+1} + \cdots + c_n = n - 1$ with $c_j > 1$, for a certain index j. Now with each inverse-conjugate composition of the first type associate the composition of n given by $(c_1, \dots, c_j, 1)$, and with each of the second type associate, $(c_1, \dots, c_j)$, which is already a composition of n.

This gives the required bijection. □

Example. We illustrate the second part of the proof of Corollary 4.4. There are 8 inverse-conjugate compositions of 7:

$$(1,1,1,4), (1,1,2,3), (1,2,2,2), (1,3,1,2),$$
$$(2,1,3,1), (2,2,2,1), (3,2,1,1), (4,1,1,1).$$

The corresponding list of compositions of 4, under the bijection, is:

$$(1,1,1,1), (1,1,2), (1,2,1), (1,3), (2,1,1), (2,2), (3,1), (4).$$

5. Further consequences

The machinery developed here can be used to relate compositions directly with bit strings, that is, finite sequences of 0's and 1's.

Theorem 5.1.
(i) The number of compositions of $n+1$ without the part m equals the number of n-bit strings that avoid a run of $m-1$ ones.

(ii) The number of compositions of $n+1$ in which m may appear only as a first or last part equals the number of n-bit strings that avoid $01^{m-1}0$.

Proof. To prove part (ii) we give a bijection between the two sets, using the **Sub-Sum** and **BitS** conjugation methods. If $C = (m, c_1, c_2, \dots) \models n+1, c_i \neq m > 1$, then the image of C under the bijection (2.1) is $L = (m, m + c_1, m + c_1 + c_2, \dots)$. Since $c_i \neq m$ for all i, no pair of consecutive terms in L are separated by $m-1$ elements. So the bit encoding of C avoids $10^{m-1}1$. The same conclusion obviously holds if we start with a composition that does not contain m as a part. Thus the desired bijection is the map that takes a composition C of n with no intermediate m's to the bit encoding of the conjugate C'.

The proof of part (i) is similar. □

It turns out that the two classes of compositions in Theorem 5.1 are equinumerous, for $m = 2$, provided the weights differ by unity.

Theorem 5.2. *The number of compositions of n in which 2 may appear only as a first or last part equals the number of compositions of $n+1$ without 2's.*

Proof. We provide a recursive proof. Let d_n be the number of compositions of n in which 2 may appear only as a first or last part, and let c_n be the number of compositions of n without 2's.

Then, we first observe that

$$d_n = c_n + 2c_{n-2} + c_{n-4}, \tag{5.1}$$

since d_n enumerates the set consisting of compositions without 2's, those with exactly one 2 at either end, and those with two 2's at both ends.

The enumerator c_n fulfills the following recurrence relations.

$$c_n = 2c_{n-1} - c_{n-2} + c_{n-3}; \tag{5.2}$$

$$c_n = c_{n-1} + c_{n-2} + c_{n-4}; \tag{5.3}$$

with the initial values $c_1 = c_2 = 1$.
For (5.2), we note that a composition counted by c_n can be found in three ways:

(i) by adding 1 to the last part of a composition counted by c_{n-1}, provided we exclude compositions of $n-1$ with last part 1;

(ii) by appending 1 to a composition counted by c_{n-1}; and

(iii) by appending 3 to a composition counted by c_{n-3}, since the previous two types exclude the latter.

The numbers of compositions of n generated are, respectively, $c_{n-1} - c_{n-2}$, c_{n-1} and c_{n-3}. Hence altogether we obtain (5.2).

For (5.3), note that compositions counted by c_n with first part 1 are also counted by c_{n-1}; those with first part > 1, that is, first part ≥ 3, are counted by c_{n-2}, with the exception of those with first part equal to 4. The latter are obtained by appending 4 to compositions of $n-4$ with no 2's. Hence the result.

Now using (5.3) and (5.2), we obtain
$d_n = c_n+2c_{n-2}+c_{n-4} = c_n+2c_{n-2}+c_n-c_{n-1}-c_{n-2} = 2c_n-c_{n-1}+c_{n-2} = c_{n+1}$, as required. □

We are presently unable to give a direct bijection between the two sets of compositions in Theorem 5.2. The theorem can, of course, be formulated in terms of bit strings using the **BitS** conjugation method (cf. Theorem 5.1):

Corollary 5.3. *The number of n-bit strings avoiding* 010 *is equal to the number of* $(n+1)$*-bit strings avoiding isolated* 1*'s.*

However, even in this new form, the difficulty of finding a bijective proof seems to persist. It is possible to give a recursive proof of Corollary 5.3 that is similar to the proof of Theorem 5.2.

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On second order non-homogeneous recurrence relation

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Abstract

We consider here the sequence g_n defined by the non-homogeneous recurrence relation $g_{n+2} = g_{n+1} + g_n + At^n$, $n \geq 0$, $A \neq 0$ and $t \neq 0$, α, β where α and β are the roots of $x^2 - x - 1 = 0$ and $g_0 = 0$, $g_1 = 1$.

We give some basic properties of g_n.Then using Elmore's technique and exponential generating function of g_n we generalize g_n by defining a new sequence G_n. We prove that G_n satisfies the recurrence relation $G_{n+2} = G_{n+1} + G_n + At^n e^{xt}$.

Using Generalized circular functions we extend the sequence G_n further by defining a new sequence $Q_n(x)$. We then state and prove its recurrence relation. Finally we make a note that sequences $G_n(x)$ and $Q_n(x)$ reduce to the standard Fibonacci Sequence for particular values.

1. Introduction

The Fibonacci Sequence $\{F_n\}$ is defined by the recurrence relation

$$F_{n+2} = F_{n+1} + F_n, n \geq 0 \tag{1.1}$$

with

$$F_0 = 0, \quad \text{and} \quad F_1 = 1.$$

We consider here a slightly more general non-homogeneous recurrence relation which gives rise to a generalized Fibonacci Sequence which we call *The Pseudo Fibonacci Sequence.* But before defining this sequence let us state some identities for the Fibonacci Sequence.

2. Some Identities for $\{F_n\}$

Let α and β be the distinct roots of $x^2 - x - 1=0$, with

$$\alpha = \frac{(1+\sqrt{5})}{2} \quad \text{and} \quad \beta = \frac{(1-\sqrt{5})}{2}. \tag{2.1}$$

Note that

$$\alpha + \beta = 1, \quad \alpha\beta = -1 \quad \text{and} \quad \alpha - \beta = \sqrt{5}. \tag{2.2}$$

Binets formula for $\{F_n\}$ is given by

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}. \tag{2.3}$$

Generating function for $\{F_n\}$ is

$$F(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{(1-x-x^2)}. \tag{2.4}$$

Exponential Generating Function for $\{F_n\}$ is given by

$$E(x) = \sum_{n=0}^{\infty} \frac{F_n x^n}{n!} = \frac{e^{\alpha x} - e^{\beta x}}{\sqrt{5}}. \tag{2.5}$$

3. Elmores Generalisation of $\{F_n\}$

Elmore [1] generalized the Fibonacci Sequence $\{F_n\}$ as follows. He takes $E_0(x) = E(x)$ as in (2.5) and then defines $E_n(x)$ of the generalized sequence $\{E_n(x)\}$ as the n^{th} derivatives with respect to x of $E_0(x)$. Thus we see from (2.5) that

$$E_n(x) = \frac{\alpha^n e^{\alpha x} - \beta^n e^{\beta x}}{\sqrt{5}}.$$

Note that

$$E_n(0) = \frac{\alpha^n - \beta^n}{\sqrt{5}} = F_n.$$

The Recurrence relation for $\{E_n\}$ is given by

$$E_{n+2}(x) = E_{n+1}(x) + E_n(x).$$

4. Definiton of Pseudo Fibonacci Sequence

Let $t \neq \alpha, \beta$ where α, β are as in (2.1). We define the Pseudo Fibonacci Sequence $\{g_n\}$ as the sequence satisfying the following non-homogeneous recurrence relation.

$$g_{n+2} = g_{n+1} + g_n + At^n, n \geq 0, A \neq 0 \quad \text{and} \quad t \neq 0, \alpha, \beta \tag{4.1}$$

with $g_0 = 0$ and $g_1 = 1$. The few initial terms of $\{g_n\}$ are

$$g_2 = 1 + A,$$
$$g_3 = 2 + A + At.$$

Note that for A $= 0$ the above terms reduce to those for $\{F_n\}$.

5. Some Identities for $\{g_n\}$

Binet's formula: Let

$$p = p(t) = \frac{A}{t^2 - t - 1}. \tag{5.1}$$

Then g_n is given by

$$g_n = c_1\alpha^n + c_2\beta^n + \frac{At^n}{t^2 - t - 1} \tag{5.2}$$
$$= c_1\alpha^n + c_2\beta^n + pt^n, \tag{5.3}$$

where

$$c_1 = \frac{1 - p(t)(t - \beta)}{\alpha - \beta}, \tag{5.4}$$

$$c_2 = \frac{p(t)(t - \alpha) - 1}{\alpha - \beta}. \tag{5.5}$$

The Generating Function $G(x) = \sum_{n=0}^{\infty} g_n x^n$ is given by

$$G(x) = \frac{x + x^2(A - t)}{(1 - xt)(1 - x - x^2)}, \quad 1 - xt \neq 0. \tag{5.6}$$

Note from (5.6) that if A $= 0$

$$G(x) = \frac{x}{1 - x - x^2},$$

which, as in section (2.4), is the generating function for $\{F_n\}$.

The Exponential Generating Function $E^*(x) = \sum_{n=0}^{\infty} \frac{g_n x^n}{n!}$ is given by

$$E^*(x) = c_1 e^{\alpha x} + c_2 e^{\beta x} + pe^{xt}, \tag{5.7}$$

where c_1 and c_2 are as in (5.4) and (5.5) respectively. Note that if A$=0$ we see from (5.3), (5.4) and (5.5) that

$$p = 0, \quad c_1 = \frac{1}{\sqrt{5}}, \quad c_2 = \frac{-1}{\sqrt{5}},$$

so that $E^*(x)$ reduces to $\frac{e^{\alpha x} - e^{\beta x}}{\sqrt{5}}$ which, as in (2.5), is the Exponential generating function for $\{F_n\}$.

6. Generalization of $\{g_n\}$ by applying Elmore's Method

Let

$$E_0^*(x) = E^*(x) = c_1 e^{\alpha x} + c_2 e^{\beta x} + p e^{xt}$$

be the Exponential Generating Function of $\{g_n\}$ as in (5.7). Further, let $G_n(x)$ of the sequence $\{G_n(x)\}$ be defined as the n^{th} derivative with respect to x of $E_0^*(x)$, then

$$G_n(x) = c_1 \alpha^n e^{\alpha x} + c_2 \beta^n e^{\beta x} + p t^n e^{xt}. \tag{6.1}$$

Note that

$$G_n(0) = c_1 \alpha^n + c_2 \beta^n + p t^n = g_n, \tag{6.2}$$

which, in turn, reduces to F_n if $A = 0$.

Theorem 6.1. *The sequence $\{G_n(x)\}$ satisfies the non-homogeneous recurrence relation*

$$G_{n+2}(x) = G_{n+1}(x) + G_n(x) + A t^n e^{xt}. \tag{6.3}$$

Proof.

$$\begin{aligned} \text{R.H.S.} &= c_1 \alpha^{n+1} e^{\alpha x} + c_2 \beta^{n+1} e^{\beta x} + p t^{n+1} e^{xt} \\ &\quad + c_1 \alpha^n e^{\alpha x} + c_2 \beta^n e^{\beta x} + p t^n e^{xt} + A t^n e^{xt} \\ &= c_1 \alpha^n e^{\alpha x}(\alpha + 1) + c_2 \beta^n e^{\beta x}(\beta + 1) \\ &\quad + p t^n e^{xt}(t+1) + p(t^2 - t - 1) t^n e^{xt}. \end{aligned} \tag{6.4}$$

Since α and β are the roots of $x^2 - x - 1 = 0$, $\alpha + 1 = \alpha^2$ and $\beta + 1 = \beta^2$ so that (6.4) reduces to

$$\text{R.H.S} = c_1 \alpha^{n+2} e^{\alpha x} + c_2 \beta^{n+2} e^{\beta x} + p t^{n+2} e^{xt} = G_{n+2}(x). \qquad \square$$

7. Generalization of Circular Functions

The Generalized Circular Functions are defined by Mikusinsky [2] as follows: Let

$$N_{r,j} = \sum_{n=0}^{\infty} \frac{t^{nr+j}}{(nr+j)!}, \quad j = 0, 1, \ldots, r-1; \quad r \geq 1, \tag{7.1}$$

$$M_{r,j} = \sum_{n=0}^{\infty} (-1)^r \frac{t^{nr+j}}{(nr+j)!}, \quad j = 0, 1, \ldots, r-1; \quad r \geq 1. \tag{7.2}$$

Observe that

$$\begin{aligned} N_{1,0}(t) &= e^t, & N_{2,0}(t) &= \cosh t, & N_{2,1}(t) &= \sinh t, \\ M_{1,0}(t) &= e^{-t}, & M_{2,0}(t) &= \cos t, & M_{2,1}(t) &= \sin t. \end{aligned}$$

Differentiating (7.1) term by term it is easily established that

$$N_{r,0}^{(p)}(t) = \begin{cases} N_{r,j-p}(t), & 0 \leq p \leq j \\ N_{r,r+j-p}(t), & 0 \leq j < j < p \leq r \end{cases} \tag{7.3}$$

In particular, note from (7.3) that

$$N_{r,0}^{(r)}(t) = N_{r,0}(t),$$

so that in general

$$N_{r,0}^{(nr)}(t) = N_{r,0}(t), r \geq 1. \tag{7.4}$$

Further note that

$$N_{r,0}(0) = N_{r,0}^{(nr)}(0) = 1.$$

8. Application of Circular functions to generalize $\{g_n\}$

Using Generalized Circular Functions and Pethe-Phadte technique [3] we define the sequence $Q_n(x)$ as follows. Let

$$Q_0(x) = c_1 N_{r,0}(\alpha^* x) + c_2 N_{r,0}(\beta^* x) + p N_{r,0}(t^* x), \tag{8.1}$$

where $\alpha^* = \alpha^{1/r}$, $\beta^* = \beta^{1/r}$ and $t^* = t^{1/r}$, r being the positive integer. Now define the sequence $\{Q_n(x)\}$ successively as follows:

$$Q_1(x) = Q_0^{(r)}(x), \quad Q_2(x) = Q_0^{(2r)}(x),$$

and in general

$$Q_n(x) = Q_0^{(nr)}(x),$$

where derivatives are with respect to x. Then from (8.1) and using (7.4) we get

$$\begin{aligned} Q_1(x) &= c_1 \alpha N_{r,0}(\alpha^* x) + c_2 \beta N_{r,0}(\beta^* x) + pt N_{r,0}(t^* x), \\ Q_2(x) &= c_1 \alpha^2 N_{r,0}(\alpha^* x) + c_2 \beta^2 N_{r,0}(\beta^* x) + pt^2 N_{r,0}(t^* x), \\ Q_n(x) &= c_1 \alpha^n N_{r,0}(\alpha^* x) + c_2 \beta^n N_{r,0}(\beta^* x) + pt^n N_{r,0}(t^* x). \end{aligned} \tag{8.2}$$

Observe that if $r = 1$, $x = 0$, $A = 0$, $\{Q_n(x)\}$ reduces to $\{F_n\}$.

Theorem 8.1. *The sequence $\{G_n(x)\}$ satisfies the non-homogeneous recurrence relation*

$$Q_{n+2}(x) = Q_{n+1}(x) + Q_n(x) + At^n N_{r,0}(t^* x). \tag{8.3}$$

Proof.

$$\begin{aligned}\text{R.H.S.} &= c_1\alpha^{n+1}N_{r,0}(\alpha^*x) + c_2\beta^{n+1}N_{r,0}(\beta^*x) + pt^{n+1}N_{r,0}(t^*x) \\ &\quad + c_1\alpha^n N_{r,0}(\alpha^*x) + c_2\beta^n N_{r,0}(\beta^*x) + pt^n N_{r,0}(t^*x) + At^n N_{r,0}(t^*x) \\ &= c_1\alpha^n N_{r,0}(\alpha^*x)(\alpha+1) + c_2\beta^n N_{r,0}(\beta^*x)(\beta+1) + t^n N_{r,0}(t^*x)(pt+p+A). \quad (8.4)\end{aligned}$$

Using the fact that α and β are the roots of $x^2 - x - 1 = 0$ and (5.1) in (8.4) we get

$$\text{R.H.S.} = c_1\alpha^{n+2}N_{r,0}(\alpha^*x) + c_2\beta^{n+2}N_{r,0}(\beta^*x) + pt^{n+2}N_{r,o}(t^*x) = Q_{n+2}(x). \quad \square$$

It would be an interesting exercise to prove 7 identities for $Q_n(x)$ similar to those proved in Pethe-Phadte with respect to $P_n(x)$ [3].

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Some aspects of Fibonacci polynomial congruences

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Abstract

This paper formulates a definition of Fibonacci polynomials which is slightly different from the traditional definitions, but which is related to the classical polynomials of Bernoulli, Euler and Hermite. Some related congruence properties are developed and some unanswered questions are outlined.

Keywords: Congruences, recurrence relations, Fibonacci sequence, Lucas sequences, umbral calculus.

MSC: 11B39;11B50;11B68

1. Introduction

The purpose of this paper is to consider some congruences associated with a generalized Fibonacci polynomial which is defined in the next section in relation to two generalized arbitrary order ($r \geq 2$) Fibonacci sequences, $\{u_n\}$ and $\{v_n\}$:

$$\left.\begin{array}{ll} u_n = \sum_{j=1}^{r}(-1)^{j+1}P_j u_{n-j} & n>0 \\ u_n = 1 & n=0 \\ u_n = 0 & n<0 \end{array}\right| \tag{1.1}$$

and

$$\left.\begin{array}{ll} v_n = \sum_{j=1}^{r}(-1)^{j+1}P_j v_{n-j} & n \geq r \\ v_n = \sum_{j=1}^{r}\alpha_j^n & 0 \leq n < r \\ v_n = 0 & n < 0 \end{array}\right| \tag{1.2}$$

where the P_j are arbitrary integers and the α_j are the roots, assumed distinct, of the auxiliary equation for the recurrence relations above, namely,

$$0 = x^r - \sum_{j=1}^{r}(-1)^{j+1}P_j x^{r-j}.$$

For example, when $r = 2$ we have $u_n = P_1 u_{n-1} - P_2 u_{n-2}$ with $u_0 = 1$, $u_1 = P_1$, $u_2 = P_1^2 - P_2$, and so on. These are referred to as the Lucas *fundamental* numbers (see [8]). When $r = 2$ the $\{v_n$ correspond to the Lucas *primordial* numbers with $v_0 = 2$, $v_1 = \alpha_1 + \alpha_2 = P_1$, $v_2 = \alpha_1^2 + \alpha_2^2 = P_1^2 - 2P_2$ and so on (see [5], Table 1).

n	0	1	2	3	$\cdots$
u_n	1	P_1	$P_1^2 - P_2$	$P_1^3 - 2P_1P_2 + P_3$	$\cdots$
v_n	r	P_1	$P_1^2 - 2P_2$	$P_1^3 - 3P_1P_2 + rP_3$	$\cdots$

Table 1: First four terms of $\{u_n\}$ and $\{v_n\}$

In [11] the ordinary generating function

$$\sum_{n=0}^{\infty} u_n x^n = \prod_{j=1}^{r}(1 - \alpha_j x)^{-1} \tag{1.3}$$

is used to show that

$$\sum_{n=0}^{\infty} u_n x^n = \exp\left(\sum_{m=1}^{\infty} v_m \frac{x^m}{m}\right) \tag{1.4}$$

thus suggesting a generalized Fibonacci polynomial $u_n(x)$ defined formally as

$$\sum_{n=0}^{\infty} u_n(x)\frac{t^n}{n!} = \exp\left(xt + \sum_{m=1}^{\infty} v_m \frac{t^m}{m}\right). \tag{1.5}$$

Then from (1.4) and (1.5) we get (1.6) and (1.7)

$$u_n(0) = u_n n! \tag{1.6}$$

and thus

$$\sum_{n=0}^{\infty} u_n(x)\frac{t^n}{n!} = e^{xt}\sum_{n=0}^{\infty} u_n(0)\frac{t^n}{n!} \tag{1.7}$$

by analogy with the polynomials of Bernoulli, Euler and Hermite (see [2, 9]). Other analogies with these polynomials can also be obtained in [12].

We also note that there are many other ways of defining Fibonacci polynomials and their generalizations in literature, (see [1, 3, 6]). The aim in this paper is to extend some of the results associated with (1.5) to congruences (see [7]). Some of these properties for Fibonacci numbers were explored in [13]. Daykin, Dresel and Hilton also obtained some similar results by combining the roots of the auxiliary equation to aid their study of the structure of a second order recursive sequence in a finite field (see [4]).

2. Fibonacci polynomials

We emphasize that the concern here is with the formal aspects of the theory and in the term-by-term differentiation of series we assume that conditions of continuity and uniform convergence are satisfied in the appropriate closed intervals. Thus a result we shall find useful is a recurrence relation for these Fibonacci polynomials

$$u_{n+1}(x) = xu_n(x) + \sum_{j=0}^{n} n^{\underline{j}} v_{j+1} u_{n-j}(x) \tag{2.1}$$

in which $n^{\underline{j}}$ is the falling factorial coefficient.

Proof of (2.1). Since

$$\sum_{n=0}^{\infty} u_n(x)\frac{t^n}{n!} = \exp\left(xt + \sum_{m=1}^{\infty} v_m \frac{t^m}{m}\right)$$

and

$$\frac{\partial}{\partial t}\sum_{n=0}^{\infty} u_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} u_{n+1}(x)\frac{t^n}{n!}$$

and

$$\frac{\partial}{\partial t}\left(\exp\left(xt + \sum_{m=1}^{\infty} v_m \frac{t^m}{m}\right)\right) = \left(x + \sum_{m=0}^{\infty} v_{m+1}t^m\right)\exp\left(xt + \sum_{m=1}^{\infty} v_m \frac{t^m}{m}\right),$$

we have that

$$\begin{aligned}\sum_{n=0}^{\infty} u_{n+1}(x)\frac{t^n}{n!} &= \left(x + \sum_{m=0}^{\infty} v_{m+1}t^m\right)\sum_{n=0}^{\infty} u_n(x)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty} xu_n(x)\frac{t^n}{n!} + \left(\sum_{m=0}^{\infty} v_{m+1}t^m\right)\left(\sum_{n=0}^{\infty} u_n(x)\frac{t^n}{n!}\right)\\ &= \sum_{n=0}^{\infty} xu_n(x)\frac{t^n}{n!} + \sum_{n=0}^{\infty}\sum_{j=0}^{n} n^{\underline{j}} v_{j+1}u_{n-j}(x)\frac{t^n}{n!}\end{aligned}$$

which yields the required result on equating coefficients of t. $\square$

When $x = 0$ this becomes

$$(n+1)\,u_{n+1} = \sum_{j=0}^{n} v_{j+1} u_{n-j} \tag{2.2}$$

since $n! = n^{\underline{j}}(n-j)!$. When $r = 2$ and $P_1 = -P_2 = 1$, equation (2.2) becomes the known (see [5])

$$nF_{n+1} = \sum_{j=0}^{n-1} L_{j+1} F_{n-j}.$$

Now from (1.5) it follows that

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x)\frac{t^n}{n!} &= \exp(xt)\exp\left(\sum_{m=1}^{\infty} v_m \frac{t^m}{m}\right)\\ &= \sum_{k=0}^{\infty} x^k \frac{t^k}{k!} \sum_{j=0}^{\infty} u_j t^j\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{n!}{k!} u_{n-k} x^k \frac{t^n}{n!}.\end{aligned}$$

So that on equating coefficients of t we get

$$u_n(x) = \sum_{k=0}^{n} \frac{n!}{k!} u_{n-k} x^k \tag{2.3}$$

and with (1.6)

$$u_n(x) = \sum_{k=0}^{n} \frac{n!}{k!} \frac{u_{n-k}(0)}{(n-k)!} x^k$$

so that

$$u_n(x) = \sum_{k=0}^{n} \binom{n}{k} u_{n-k}(0) x^k. \tag{2.4}$$

Then

$$u_0(x) = u_0 = 1.$$

It is of interest to note another connection between these Fibonacci polynomials and the classical polynomials. We can write equation (2.4) in the suggestive form

$$u_n(x) = (x + u_n(0))^n \tag{2.5}$$

which is analogous to the well-known

$$B_n(x) = (x + B_n(0))^n \tag{2.6}$$

for the Bernoulli polynomials, and in which it is understood that after the expansion of the right hand sides of (2.1)and (2.2), terms of the form a^k are replaced by a_k as in the umbral calculus (see [10]).

3. Fibonacci polynomial congruences

We now use induction on t and n to prove that

$$u_{n+tm}(x) \equiv u_n(x)\left(u_m(x)\right)^t \pmod{m} \tag{3.1}$$

Proof of (3.1). When $t = 0$, the result is obvious for all n. When $t = 1$ and $n = 1$, we note from (2.1) that $u_1(x) = x + v_1$, and

$$\begin{aligned} u_{m+1}(x) &= (x+v_1)u_m(x) + \sum_{j=1}^{m} m^{\underline{j}} v_{j+1} u_{m-j}(x) \\ &\equiv (x+v_1)u_m(x) \pmod{m} \\ &\equiv u_1(x)u_m(x) \pmod{m}. \end{aligned}$$

Assume the result is true for $t = 1$, and $n = 1, 2, \cdots, s$; that is,

$$u_{m+n}(x) \equiv u_m(x)u_n(x) \pmod{m}, \qquad n = 1, 2, \cdots, s.$$

Then

$$\begin{aligned} u_{m+s+1}(x) &= (x+v_1)u_{m+s}(x) + \sum_{j=1}^{m+s} (m+s)^{\underline{j}} v_{j+1} u_{m+s-j}(x) \\ &\equiv (x+v_1)u_{m+s}(x) + \sum_{j=1}^{s} s^{\underline{j}} v_{j+1} u_{m+s-j}(x) \pmod{m} \end{aligned}$$

since

$$\begin{aligned} (m+s)^{\underline{j}} &= (m+s)(m+s-1)\cdots(m+s-j+1) \\ &\equiv s(s-1)\cdots(s-j+1) \pmod{m}. \end{aligned}$$

Thus

$$\begin{aligned} u_{m+s+1}(x) &\equiv (x+v_1)u_m(x)u_s(x) + \sum_{j=1}^{s} s^{\underline{j}} v_{j+1} u_{s-j}(x)u_m(x) \pmod{m} \\ &= u_m(x)\left((x+v_1)u_s(x) + \sum_{j=1}^{s} s^{\underline{j}} v_{j+1} u_{s-j}(x)\right) \pmod{m} \\ &= u_m(x)u_{s+1}(x) \pmod{m}. \end{aligned}$$

So when $t = 1$, for all n,

$$u_{n+m}(x) \equiv u_n(x)\left(u_m(x)\right)^1 \pmod{m},$$

when $t = 2$, for all n,

$$u_{n+2m}(x) \equiv u_n(x)\left(u_m(x)\right)^2 \pmod{m}.$$

Assume the result holds for $t = 3, 4, \cdots, k$:

$$\begin{aligned} u_{n+(k+1)m}(x) &\equiv u_{n+km}(x)u_m(x) \pmod m \\ &\equiv \left(u_n(x)\left(u_m(x)\right)^k\right) u_m(x) \pmod m \\ &\equiv u_n(x)\left(u_m(x)\right)^{k+1} \pmod m \end{aligned}$$

and this completes the proof of (3.1). □

As a simple illustration of (3.1), if $r = 2$, $m = 2$, $n = 3$, and $t = 1$, then from (2.3)

$$\begin{aligned} u_5(x) &= \sum_{k=0}^{5} \frac{5!}{k!} u_{5-k} x^k \\ &\equiv \frac{5!}{4!} u_1 x^4 + \frac{5!}{5!} u_0 x^5 \pmod 2 \\ &\equiv 5x^4 + x^5 \pmod 2 \\ &\equiv x^4 + x^5 \pmod 2 \end{aligned}$$

and similarly,

$$\begin{aligned} u_3(x) &\equiv 3x^2 + x^3 \pmod 2 \\ &\equiv x^2 + x^3 \pmod 2 \\ u_2(x) &\equiv x^2 \pmod 2 \end{aligned}$$

or

$$u_5(x) \equiv u_3(x)\left(u_2(x)\right) \pmod 2.$$

It follows that for $n = 2, 3, \cdots$,

$$u_n(x)\left(u_m(x)\right)^t - u_{n+tm}(x) = \sum_{j=-ntm}^{tn} B_j(n) u_{n+j}(x) \tag{3.2}$$

in which the $B_j(n) = B_j(n; t, m)$ are also polynomials in n with integer coefficients modulo m. We may also assume that in the summation $B_j(n) = 0$ $(-ntm \leq j < -n)$.

4. Conclusion

The $\{u_s(0)\}$ satisfy recurrence relations with variable coefficients:

$$\begin{aligned} u_n(0) &= n! u_n \\ &= n! \sum_{j=1}^{r} (-1)^{j+1} P_j u_{n-j} \end{aligned}$$

$$= \sum_{j=1}^{r} (-1)^{j+1} P_j \frac{n!}{(n-j)!} u_{n-j}(0).$$

This may be worthy of further separate investigation, as may two-dimensional polynomials of the form $\{u_{m,n}(x)\}$ to correspond with horizontal and vertical tilings of Fibonacci numbers.

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The On-Line Encyclopedia of Integer Sequences

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Abstract

We all recognize $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ but what about $1, 2, 4, 6, 3, 9, 12, 8, 10, 5, 15, \ldots$? If you come across a number sequence and want to know if it has been studied before, there is only one place to look, the *On-Line Encyclopedia of Integer Sequences* (or *OEIS*). Now in its 49th year, the OEIS contains over 220,000 sequences and 20,000 new entries are added each year. This article will briefly describe the OEIS and its history. It will also discuss some sequences generated by recurrences that are less familiar than Fibonacci's, due to Greg Back and Mihai Caragiu, Reed Kelly, Jonathan Ayres, Dion Gijswijt, and Jan Ritsema van Eck.

Keywords: Fibonacci, sequences, recurrences.

MSC: Primary 11B

1. The Fibonacci numbers

The Fibonacci numbers have been in the On-Line Encyclopedia of Integer Sequences® (or OEIS®) right from the beginning. When I started collecting sequences as a graduate student in 1964, the Fibonacci numbers became sequence A000045 (incidentally, 49 years later, sequences being added have A-numbers around A222000[1]). Over 3000 sequences in the OEIS mention Fibonacci's name in their definition.

Some especially noteworthy variations on the Fibonacci numbers were recently defined by Back and Caragiu [2] in the *Fibonacci Quarterly*. The simplest of their

[1] As of February 2013. Throughout this article, six-digit numbers prefixed by A refer to entries in the OEIS [15]. As in the OEIS, we adopt the convention that $a(n)$ denotes the nth term of the sequence being discussed.

examples replaces the Fibonacci recurrence by

$$a(n) = \text{gpf}(a(n-1) + a(n-2)), \tag{1.1}$$

where gpf stands for *greatest prime factor* (A006530). If we start with $1, 1$ we get

$$1, 1, 2, 3, 5, 2, 7, 3, 5, 2, 7, \ldots \tag{1.2}$$

(A175723), and the cycle $3, 5, 2, 7$ repeats for ever. Back and Caragiu show that no matter what the initial values are, (1.1) always becomes periodic and that $3, 5, 2, 7$ is the only nontrivial cycle. On the other hand, consider

$$a(n) = \text{gpf}(a(n-1) + a(n-2) + a(n-3)). \tag{1.3}$$

If we start with $1, 1, 1$ we get

$$1, 1, 1, 3, 5, 3, 11, 19, 11, 41, 71, 41, 17, 43, 101, 23, \ldots \tag{1.4}$$

(A177904), which after 86 steps enters a cycle of length 212. Now it is only a conjecture that (1.3) always becomes periodic, for any initial values.

Another interesting variant of the Fibonacci sequence[2] was very recently introduced into the OEIS by Reed Kelly [12]. Kelly's recurrence is

$$a(n) = \frac{a(n-1) + a(n-3)}{\gcd\{a(n-1), a(n-3)\}}, \tag{1.5}$$

with initial values $1, 1, 1$:

$$1, 1, 1, 2, 3, 4, 3, 2, 3, 2, 2, 5, 7, 9, 14, 3, 4, 9, 4, 2, \ldots \tag{1.6}$$

(A214551). This sequence appears to grow exponentially ($a(n) \approx \text{const.} \cdot e^{0.123\ldots n}$?), but essentially nothing has been proved about it.

The OEIS is an endless source of lovely problems!

2. How the OEIS is used

However, the main use for the OEIS is as a reference work for identifying sequences and telling you what is known about them. If you come across a sequence of numbers, and you want to know if it has been studied before, there is only one place to look, the OEIS [15] (`http://oeis.org`).

You enter the first few terms[3], and click "Submit". If you are lucky, the OEIS will return one or more sequences that match what you entered, and, for each one, it will tell you such things as:

[2] Or, more precisely, of another medieval sequence, the Narayana cows sequence, A000930.

[3] When looking up a sequence, it is recommended that you omit the first term or two, since different people may start a sequence in different ways.

- The definition of the sequence
- The first 10, or 10,000, or sometimes 500,000 terms
- Comments explaining further properties of the sequence
- Formulas for generating the sequence
- Computer programs for producing the sequence
- References to books and articles where the sequence is mentioned
- Links to web pages on the Internet where the sequence has appeared
- The name of the person who submitted the sequence to the OEIS
- Examples illustrating some of the terms of the sequence (for example, sequence A000124, which gives the maximal number of pieces that can be obtained when cutting a circular pancake with n cuts, is illustrated with pictures showing the pieces obtained with 1, 2, 3, 4 and 5 cuts)
- The history of each entry in the OEIS as it has evolved over time

You can also view graphs or plots of the sequence, or listen to it when it is converted to sounds.

If your sequence is *not* found, you will be encouraged to submit it. This will establish your priority over the sequence, and will help the next person who comes across it. Only sequences of general interest should be submitted. The sequence of primes whose decimal expansion begins with 2012 is an example of a sequence that would not be of general interest. Published sequences are almost always acceptable.

If your sequence was not in the OEIS, you should also try sending it to our email server *Superseeker* (see `http://oeis.org/ol.html`), which will try hard to find an explanation for your sequence. For example, Superseeker might suggest a recurrence or generating function for your sequence, or tell you that it can be obtained by applying one of over a hundred different transformations to one of the over 200,000 sequences in the OEIS. Superseeker is a very powerful tool for analyzing sequences.

Accuracy has always been one of the top priorities in the OEIS. Its standards are those of a mathematics reference work. Ideally, every number, formula, computer program, etc., should be absolutely correct. Formulas that are stated unconditionally should be capable of being proved, and otherwise should be labeled as conjectures. Of course, as the database has grown, these goals have become harder and harder to achieve. Many non-mathematicians have difficulty in understanding the difference between a theorem and a conjecture. ("My formula fits the first 30 terms, so obviously it *must* be correct.")

The OEIS has often been called one of the most useful mathematical sites on the Internet. There is a web page (`http://oeis.org/wiki/Works\_Citing\_OEIS`) that lists over 3000 articles and books that reference it.

3. History of the OEIS

I started collecting sequences in 1964, entering them on punched cards (the original motivation was to find an explanation for various sequences that had arisen in my dissertation, the simplest of which was the sequence that became A000435). Eventually two books were published ([16] in 1973, with 2372 entries, and [17], written with Simon Plouffe, in 1995, with 5847 sequences).

In 1996, when the number of entries had risen to 10,000, I put the database on the Internet, calling it the *The On-Line Encyclopedia of Integer Sequences* or *OEIS*. By 2009, the database had grown to over 150,000 entries, and was becoming too big for one person to manage, so I set up a foundation, *The OEIS Foundation Inc* (`http://oeisf.org`), whose goals are to own the intellectual property of the OEIS, to maintain it, and to raise funds to support it.

With major help from Russ Cox (of Google) and my colleague David Applegate (at AT&T), I moved the OEIS off my home page at AT&T to a commercial hosting service, and attempted to set it up as a "wiki." However, this proved to be extremely difficult, and it required a tremendous amount of work by Russ Cox before it started working properly. It was not until November 11, 2010 that the OEIS was officially launched in its new home at `http://oeis.org`. This would not have been possible without the help that Russ Cox and David Applegate provided.

The fact that the OEIS is now a wiki means that I no longer have to process all the updates myself. Once a user has registered[4], he or she can propose new sequences or updates to existing sequences. All submissions are reviewed by a panel of about 80 editors. Nearly two years after it was launched, the wiki system is working well. Since November 2010 the database has grown from 180,000 sequences to its current number of around 220,000. From 1996 to the present, the database has grown at between 10,000 and 20,000 new sequences per year, with about an equal number of entries that are updated.

More about the history of the OEIS can be found on the OEIS Foundation web site, `http://oeisf.org`.[5]

4. The poster and the OEIS movie

To celebrate the creation of the the OEIS Foundation, David Applegate and I made a poster that shows 25 especially interesting sequences (several of which will be mentioned in this article). It can be downloaded (along with a key) from the Foundation web site.

Also, Tony Noe made a movie that shows graphs of the first thousand terms of a thousand sequences from the OEIS: it is quite spectacular. It runs for 8.5 minutes,

[4] All readers are encouraged to register: go to `http://oeis.org/wiki` and click "Register."

[5] As President, it would be remiss of me not to mention that the OEIS Foundation is a charitable organization and donations are tax-deductible in the USA. The web site is free, and none of the trustees receive a salary. To make a donation, please go to `http://oeisf.org`.

and it too can be found on the Foundation web site. It is also on YouTube (search for "OEIS movie").

5. Puzzles

One of the goals of the OEIS has always been to help people get higher scores on IQ tests, and the database includes many sequences that have appeared as puzzles. The following are a few examples. If you can't solve them, you know where to find the answers!

- 61, 21, 82, 43, ...
- 2, 4, 6, 30, 32, 34, 36, 40, 42, 44, 46, 50, 52, 54, 56, 60, 62, 64, 66, 2000, ...
- 0, 0, 0, 0, 4, 9, 5, 1, 1, 0, 55, 55, ...
- 5, 8, 12, 18, 24, 30, 36, 42, 52, 60, ...
- 1, 2, 6, 21, 85, 430, 2586, 18109, 144880, ...

The last one is a bit tricky, but it did appear on a quiz.

6. Two sequences that agree for a long time

People often ask if it is possible for two sequences to agree for many terms yet not be the same. Here is an extreme example. The sequences

$$\left\lfloor \frac{2n}{\log 2} \right\rfloor \quad \text{and} \quad \left\lceil \frac{2}{2^{1/n}-1} \right\rceil$$

both begin

$$2, 5, 8, 11, 14, 17, 20, 23, 25, 28, 31, 34, 37, \ldots$$

(A078608). In fact they agree for the first 777451915729367 terms! There are infinitely many disagreements, the positions of which form sequence A129935:

$$777451915729368, 140894092055857794, 1526223088619171207, 3052446177238342414, \ldots$$

7. Theorems resulting from the OEIS

Another question that is often asked is if there are any theorems that have resulted from the OEIS. The answer is that there are many such examples. In the list of papers that cite the OEIS (`http://oeis.org/wiki/Works\_Citing\_OEIS`) one will find numerous acknowledgments that say things like "This result was discovered with the help of the OEIS."

I will give three concrete examples of theorems that were discovered with the help of the OEIS. The first concerns the remainder term in Gregory's series for $\pi/2$,

$$\frac{\pi}{2} = 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k+1} = 2\Big(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\Big), \tag{7.1}$$

which is famous for converging very slowly. In 1987, Joseph North observed that if one truncates the series after 50,000 terms, the answer is of course wrong. There is an error in the fifth decimal place. Surprisingly, he noticed that the next nine digits are correct, then there is an error, then there are nine more correct digits, another error, and so on. Here is the decimal expansion of the truncated sum followed by the true value of $\pi/2$ (the sequences of digits form A013706 and A019669). The digits that differ are in bold-face.

1.5707**9**63267948**9**661923132**1**6916397**5144**20985**84699**687... (truncated)
1.5707**8**63267948**9**761923132**1**1916397**5205**20985**83314**687... (true value)

The differences between the corresponding bold-faced terms are

$$1, -1, 5, -61, 1385, \ldots.$$

Jonathan Borwein looked up this sequence in [16], and found that (apart from signs) it appeared to be the Euler numbers, A000364. The end result of this investigation was a new theorem.

Theorem 7.1 (Borwein, Borwein and Dilcher [4]; see also [3, pp. 28–29], [5])**.**

$$\frac{\pi}{2} - 2\sum_{k=1}^{N/2} \frac{(-1)^{k+1}}{2k+1} \sim \sum_{m=0}^{\infty} \frac{E_m}{N^{2m+1}}, \tag{7.2}$$

where the E_m *are the Euler numbers (A000364):*

$$1, 1, 5, 61, 1385, 50521, 2702765, 199360981, 19391512145, \ldots.$$

The second example is one that I was involved with personally. It began when Eric W. Weisstein (at Wolfram Research, and creator of *MathWorld*) wrote to me about a discovery he had made. He had been classifying real matrices of 0's and 1's according to various properties, and he found that the numbers of such matrices all of whose eigenvalues were positive were $1, 3, 25, 543, 29281$ for matrices of orders $1, 2, \ldots, 5$. He observed that these numbers coincided with the beginning of sequence A003024 (whose definition on the surface seemed to have nothing to do with eigenvalues), and he conjectured that the sequences should in fact be identical. He was right, and this led to the following theorem.

Theorem 7.2 ([14])**.** *The number of acyclic directed graphs with* n *labeled vertices is equal to the number of* $n \times n$ *matrices of* 0*'s and* 1*'s all of whose eigenvalues are real and positive.*

The third example is a result of Deutsch and Sagan [8]. It is well-known that the famous Catalan numbers

$$C_n := \frac{1}{n+1}\binom{2n}{n}$$

(A000108) are odd if and only if $n = 2^k - 1$ for some k. Deutsch and Sagan proved (among other things) an analogous result for the almost equally-famous Motzkin numbers (A001006),

$$M_n := \sum_{k=1}^{n} \binom{n}{2k} C_k.$$

Theorem 7.3 ([8]). *M_n is even if and only if $n \in 4S-2$ or $4S-1$, where*

$$S := (1, 3, 4, 5, 7, 9, 11, 12, 13, 15, \ldots)$$

lists the numbers whose binary expansion ends with an even number of 0's (A003159).

8. Three unusual recurrences

The Fibonacci recurrence is very nice, but it is 800 years old. In the last section of this article I will discuss some *modern* recurrences that I find very appealing.

8.1. The EKG sequence

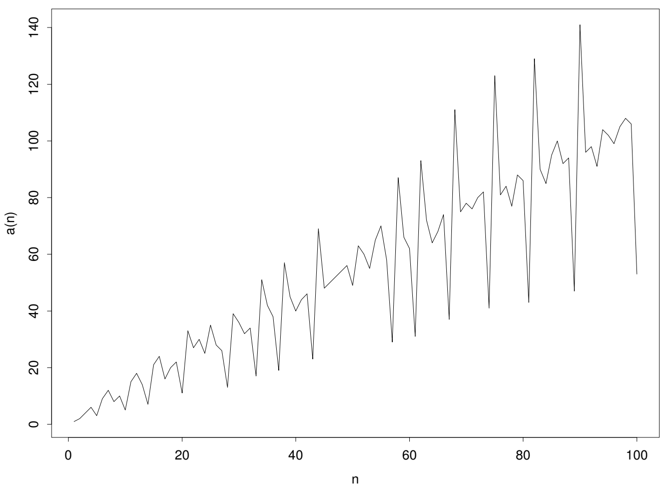

Figure 1: The first 100 terms of the EKG sequence, with successive points joined by lines

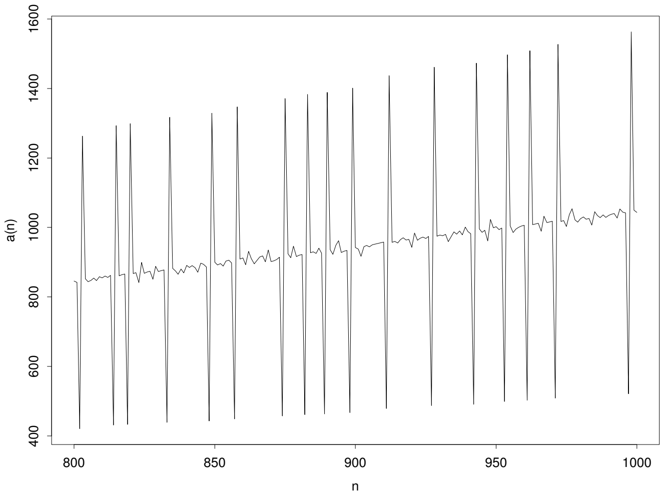

Figure 2: Terms 800 to 1000 of the EKG sequence

Jonathan Ayres contributed this to the OEIS in 2001 [1]. The first 18 terms are

$$1, 2, 4, 6, 3, 9, 12, 8, 10, 5, 15, 18, 14, 7, 21, 24, 16, 20, \ldots$$

(A064413), and the defining recurrence is $a(1) = 1$, $a(2) = 2$, and, for $n \geq 3$,

$a(n)$ is the smallest natural number not yet in the sequence which has a common factor $>$ 1 with the previous term.

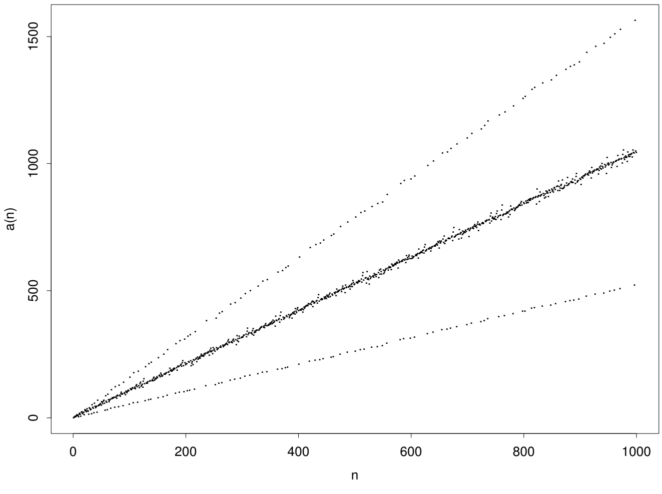

Figure 3: Scatter-plot of the first 1000 terms of the EKG sequence. They lie roughly on three almost-straight lines.

Thus $a(3)$ must have a common factor with 2, i.e. it must be even, and 4 is the smallest candidate, so $a(3) = 4$. The next term must also be even, so $a(4) = 6$. The smallest number not yet in the sequence which has a common factor with 6 is 3, so $a(5) = 3$. Similarly, $a(6) = 9$, $a(7) = 12$, $a(8) = 8$, $a(9) = 10$, $a(11) = 5$, $a(12) = 15$, and so on. Jeffrey Lagarias, Eric Rains and I studied this sequence in [13]. We called it the EKG sequence, since it looks like an electrocardiogram when plotted (Figs. 1, 2).

It is not difficult to show that the primes appear in increasing order, and that each odd prime p is either preceded by $2p$ and followed by $3p$, or is preceded by $3p$ and followed by $2p$ (as we just saw, 3 was preceded by 6 and followed by 9, 5 is preceded by 10 and followed by 15).[6]

By definition, no number can can be repeated. But does every number appear? The answer is Yes.

Theorem 8.1. *The EKG sequence is a permutation of the natural numbers.*

Sketch of Proof. (i) If infinitely many multiples of some prime p occur in the sequence, then every multiple of p must occur. (For if not, let kp be the smallest missing multiple of p. Every number below kp either appears or it doesn't, but once we get to a multiple of p beyond all those terms, the next term must be kp, which is a contradiction.) (ii) If every multiple of a prime p appears, then every number appears. (The proof is similar.) (iii) Every number appears. (For if there are only finitely many different primes among the prime factors of all the terms, then some prime must divide infinitely many terms, and the result follows from (i) and (ii). On the other hand, if infinitely many different primes p appear, then there are infinitely many terms $2p$, as noted above, so 2 appears infinitely often, and again the result follows from (i) and (ii).) □

Although the initial terms of the sequence jump around, when we look at the big picture we find that the points lie very close to three almost-straight lines (Fig. 3). This is somewhat similar to the behavior of the prime numbers, which are initially erratic, but lie close to a smooth curve (since the nth prime is roughly $n \log n$) when we look at the big picture – see Don Zagier's lecture on "The first 50 million prime numbers" [18].

In fact, we have a precise conjecture about the three lines on which the points lie. We believe – but are unable to prove – that almost all $a(n)$ satisfy the asymptotic formula $a(n) \sim n(1 + 1/(3 \log n))$ (the central line in Fig. 3), and that the exceptional values $a(n) = p$ and $a(n) = 3p$, for p a prime, produce the points on the lower and upper lines. We were able to show that the sequence has essentially linear growth (there are constants c_1 and c_2 such that $c_1 n < a(n) < c_2 n$ for all n), but the proof of even this relatively weak result was quite difficult. It would be nice to have better bounds.

[6] We conjectured that p was always preceded by $2p$ rather than $3p$. This was later proved by Hofman and Pilipczuk [11].

8.2. Gijswijt's sequence and the Curling Number Conjecture

1	1	2																	
1	1	2	2	2	3														
1	1	2																	
1	1	2	2	2	3	2													
1	1	2																	
1	1	2	2	2	3														
1	1	2																	
1	1	2	2	2	3	2	2	2	3	2	2	2	3	3	2				
1	1	2																	
1	1	2	2	2	3														
1	1	2																	
1	1	2	2	2	3	2													
1	1	2																	
1	1	2	2	2	3														
1	1	2																	
1	1	2	2	2	3	2	2	2	3	2	2	2	3	3	2	2	2	3	2

Table 1: The first 98 terms of Gijswijt's sequence (A090822)

We start by defining the curling number of a sequence. Let S be a finite nonempty sequence of integers. By grouping consecutive terms, it is always possible to write it as $S = X\,Y\,Y \cdots Y = X\,Y^k$, where X and Y are sequences of integers and Y is nonempty. There may be several ways to do this: choose the one that maximizes the value of k: this k is the *curling number* of S.

For example, if $S = 0\,1\,2\,2\,1\,2\,2\,1\,2\,2$, we could write it as $X\,Y^2$, where $X = 0\,1\,2\,2\,1\,2\,2\,1$ and $Y = 2$, or as $X\,Y^3$, where $X = 0$ and $Y = 1\,2\,2$. The latter representation is to be preferred, since it has $k = 3$, and as $k = 4$ is impossible, the curling number of this S is 3.

In 2004, Dion Gijswijt, then a graduate student at the University of Amsterdam and also the puzzle editor for the Dutch magazine *Pythagoras*, contributed the following sequence to the OEIS. Start with $a(1) = 1$, and, for $n \geq 2$, use the recurrence

$$\boxed{a(n) = \text{curling number of } a(1), \ldots, a(n-1).}$$

The beginning of the sequence is shown in Table 1 (it has been broken up into sections to show where the curling number drops back to 1):

This sequence was analyzed by Gijswijt, Fokko van de Bult, John Linderman, Allan Wilks and myself [6]. The first time a 4 appears is at $a(220)$. We computed several million terms without finding a 5, and for a while we wondered if perhaps no term greater than 4 was ever going to appear. However, we were able to show that a 5 does eventually appear, although the universe would grow cold before a direct search would find it. The first 5 appears at about term

$$10^{10^{23}}.$$

We also showed that the sequence is actually unbounded, and we conjecture that the first time that a number m $(= 5, 6, 7, \ldots)$ appears is at about term number

$$2^{2^{3^{4^{\cdot^{\cdot^{\cdot^{m-1}}}}}}},$$

a tower of height $m - 1$.

Our arguments could be considerably simplified if the *Curling Number Conjecture* were known to be true. This states that:

> If one starts with any initial sequence of integers, and extends it by repeatedly calculating the curling number and appending it to the sequence, the sequence will eventually reach 1.

The conjecture is still open. One way to tackle it is to consider starting sequences S_0 that contain only 2's and 3's, and to see how far such a sequence will extend (by repeatedly appending the curling number) before reaching a 1.

Let $\mu(n)$ denote the maximal length that can be achieved before a 1 appears, for any starting sequence S_0 consisting of n 2's and 3's. For $n = 4$, for example, $S_0 = 2\,3\,2\,3$ produces the sequence

$$2\,3\,2\,3\,2\,2\,2\,3\,1\,\ldots,$$

and no other starting string does better, so $\mu(4) = 8$. The Curling Number Conjecture would imply that $\mu(n) < \infty$ for all n. Reference [6] gave $\mu(n)$ for $1 \leq n \leq 30$, and Benjamin Chaffin and I have determined $\mu(n)$ for all $n \leq 48$ [7]. By making certain plausible assumptions about S_0, we have also computed lower bounds on $\mu(n)$ (which we conjecture to be the true values) for all $n \leq 80$. The results are shown in Table 2 and Figure 4. The values of $\mu(n)$ also form sequence A094004 in [15].

As can be seen from Fig. 4, up to $n = 80$, it appears that $\mu(n)$ increases in a piecewise linear manner. At the values $n = 1, 2, 4, 6, 8, 9, 10, 11, 14, 19, 22, 48, 68, 76, 77$ (A160766), assuming that the values in Table 2 are correct, there is a jump, but at the other values of n, $\mu(n)$ is simply $\mu(n-1) + 1$. Table 3 gives the starting sequences where $\mu(n) > \mu(n-1) + 1$ for $n \leq 48$.

For example, Table 2 shows that

$$\mu(n) = n + 120 \quad \text{for } 22 \leq n \leq 47. \tag{8.1}$$

In this range one cannot do any better than taking the starting sequence for $n = 22$ and prefixing it by an irrelevant sequence of $47 - n$ 2's and 3's. However, at $n = 48$ a new record-holder appears, and it seems that

$$\mu(n) = n + 131 \text{ for } 48 \leq n \leq 67. \tag{8.2}$$

n	1	2	3	4	5	6	7	8	9	10	11	12
$\mu(n)$	1	4	5	8	9	14	15	66	68	70	123	124
n	13	14	15	16	17	18	19	20	21	22	23	24
$\mu(n)$	125	132	133	134	135	136	138	139	140	142	143	144
n	25	26	27	28	29	30	31	32	33	34	35	36
$\mu(n)$	145	146	147	148	149	150	151	152	153	154	155	156
n	37	38	39	40	41	42	43	44	45	46	47	48
$\mu(n)$	157	158	159	160	161	162	163	164	165	166	167	179
n	49	50	51	52	53	54	55	56	57	58	59	60
$\mu(n)$	180	181	182	183	184	185	186	187	188	189	190	191
n	61	62	63	64	65	66	67	68	69	70	71	72
$\mu(n)$	192	193	194	195	196	197	198	200	201	202	203	204
n	73	74	75	76	77	78	79	80	81	82	83	84
$\mu(n)$	205	206	207	209	250	251	252	253	?	?	?	?

Table 2: Lower bounds on $\mu(n)$, the record for a starting sequence of n 2's and 3's. Entries for $n \leq 48$ are known to be exact (and we conjecture the other entries are exact).

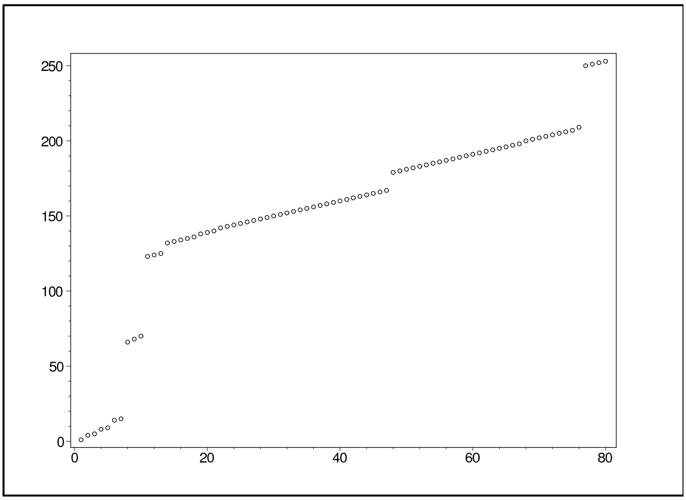

Figure 4: Scatter-plot of lower bounds on $\mu(n)$, the record for a starting sequence of n 2's and 3's. Entries for $n \leq 48$ are known to be exact (and we conjecture the other entries are exact).

We have not succeeded in finding any algebraic constructions for good starting sequences. For more about the Curling Number Conjecture see [7].

n	Starting sequence
1	2
2	2 2
4	2 3 2 3
6	2 2 2 3 2 2
8	2 3 2 2 2 3 2 3
9	2 2 3 2 2 2 3 2 3
10	2 3 2 3 2 2 2 3 2 2
11	2 2 3 2 3 2 2 2 3 2 2
14	2 2 3 2 3 2 2 2 3 2 2 3 2 3
19	2 2 3 2 2 3 2 3 2 2 2 3 2 2 3 2 2 3 2
22	2 3 2 2 3 2 2 3 2 3 2 2 2 3 2 3 2 2 3 2 2 3
48	2 2 3 2 2 3 2 3 2 2 2 3 2 2 2 3 2 2 3 2 2 2 3 2 2 3 2 3 2 2 2 3 2 2 2 3 2 2 3 2 2 2 3 2 2 3 2 3

Table 3: Starting sequences of n 2's and 3's for which $\mu(n) > \mu(n-1)+1$. This is complete for $n \leq 48$ and is believed to be complete for $n \leq 67$.

8.3. Van Eck's sequence

In 2010, Jan Ritsema van Eck [9] contributed a sequence to the OEIS that is defined by yet another unusual recurrence. Again we start with $a(1) = 0$, and then for $n \geq 2$,

> $a(n)$ is the number of steps backwards before the previous appearance of $a(n-1)$, or $a(n) = 0$ if $a(n-1)$ has never appeared before.

Since $a(1) = 0$ has never appeared before, $a(2) = 0$. Now 0 has appeared one step before, at $a(1)$, so $a(3) = 1$. We have not seen a 1 before, so $a(4) = 0$. We had an earlier 0 at $a(2)$, so $a(5) = 4 - 2 = 2$. This is the first 2 we have seen, so $a(6) = 0$. And so on. The first 36 terms are shown in Table 4.

0,	0,	1,	0,	2,	0,	2,	2,	1,	6,	0,	5,	
0,	2,	6,	5,	4,	0,	5,	3,	0,	3,	2,	9,	
0,	4,	9,	3,	6,	14,	0,	6,	3,	5,	15,	0,	
5,	3,	5,	2,	17,	0,	6,	11,	0,	3,	8,	0,	...

Table 4: The first 48 terms of Van Eck's sequence (A181391)

Figure 5 shows a scatter-plot of the first 800 terms. The plot suggests that after n terms, there are occasionally terms around n, or in other words that $\limsup a(n)/n \approx 1$. This is confirmed by looking at the first million terms, and the data also strongly suggests that every number appears in the sequence. However, at present these are merely conjectures.

Van Eck was able to show that there are infinitely many 0's in the sequence, or, equivalently, that the sequence is unbounded.

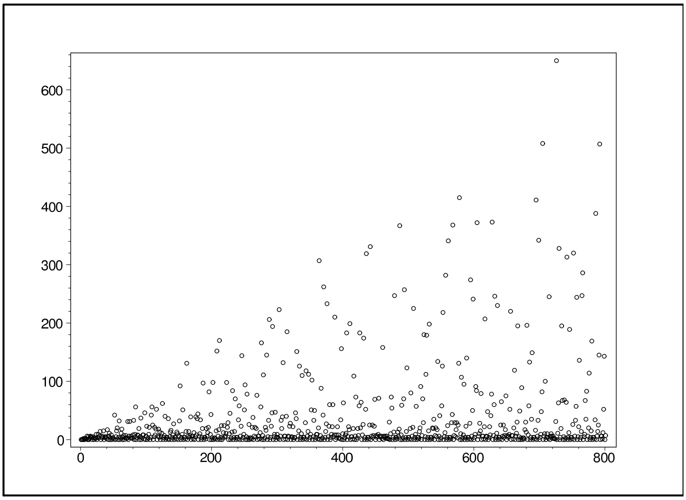

Figure 5: Scatter-plot of the first 800 terms of Van Eck's sequence A181391

Theorem 8.2 (Van Eck, personal communication)**.** *The sequence contains infinitely many* 0*'s.*

Proof. Suppose, seeking a contradiction, that there are only finite number of 0's in the sequence. Then after a certain point no new terms can appear, so the sequence is bounded. Let M be the largest term. This means that any block of M successive terms determines the sequence. But there are only M^M different possible blocks. So a block must repeat and the sequence is eventually periodic. Furthermore, the period cannot contain a 0.

Suppose the period has length p, and starts at term r, with $a(r) = x, \ldots, a(r + p - 1) = z, a(r + p) = x, \ldots$. There is another z after $q \le p$ steps, which is immediately followed by q. But this q implies that $a(r - 1) = z$. Therefore the periodic part really began at step $r - 1$.

Repeating this argument shows that the periodic part starts at $a(1)$. But $a(1) = 0$, and the periodic part cannot contain a 0. Contradiction. □

It would be nice to know more about this fascinating sequence!

9. Conclusion

I will end with a few general remarks.

- The OEIS needs more editors. If you are interested in helping, please write to me or one of the other Editors-in-Chief. There are no formal duties, everything is done on a volunteer basis, and you will get to see a lot of interesting new problems.
- Everyone should register with the OEIS – see Sect. 3.
- If you write a paper that mentions a sequence in the OEIS, please do two things. Add it to the list of papers that cite the OEIS – see Sec. 2, and add a reference pointing to your paper to any entries in the OEIS that it mentions.
- The same thing if you come across a sequence in your work, in the library, or on a web site: send it in to the OEIS if it is missing (it need not be your own sequence – just mention the source) or add a reference to the source if it is already in the OEIS. It is these cross-connections that make the database so valuable.

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A representation of the natural numbers by means of cycle-numbers, with consequences in number theory

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Abstract

In this paper we give rules for creating a number triangle **T** in a manner analogous to that for producing Pascal's arithmetic triangle; but all of its elements belong to $\{0, 1\}$, and cycling of its rows is involved in the creation. The method of construction of any one row of **T** from its preceding rows will be defined, and that, together with starting and boundary conditions, will suffice to define the whole triangle, by sequential continuation.

We shall use this triangle in order to define the so-called *cycle-numbers*, which can be mapped to the natural numbers. **T** will be called the '*cycle-number triangle*'.

First we shall give some theorems about relationships between the cycle-numbers and the natural numbers, and discuss the cycling of patterns within the triangle's rows and diagonals. We then begin a study of figures (i.e. (0,1)-patterns, found on lines, triangles and squares, etc.) within **T**. In particular, we shall seek relationships which tell us something about the prime numbers. For our later studies, we turn the triangle onto its side and work with a doubly-infinite matrix **C**.

We shall find that a great deal of cycling of figures occurs within **T** and **C**, and we exploit this fact whenever we can. The phenomenon of cycling patterns leads us to muse upon a 'music of the integers', indeed a 'symphony of the integers', being played out on the cycle-number triangle or on **C**. Like Pythagoras and his 'music of the spheres', we may well be the only persons capable of hearing it!

Keywords: cycle-number triangle, cycle-number, prime cycle-numbers

MSC: 11R99

Music is the pleasure the human mind experiences from counting without being aware that it is counting.
G. W. Leibnitz (1646–1714)

1. Introduction

In his scholarly book on Pascal's '*Arithmetical Triangle*', the author AWF Edwards [2] says of the *Triangle*: "It reveals patterns which delight the eye, raises questions which tax the number-theorists, and ..." (he adds, quoting D. Knuth) "amongst the coefficients there are so many relations present that, when someone finds a new identity, there aren't many people who get excited about it any more, except the discoverer!"

Pascal, in his own publication on the famous triangle, in 1654, said that he was fascinated by the mathematical richness of the patterns that he had discovered in it, and that: "He had had to leave out more than he could put in!"

In the triangle that we are about to define, none of its elements rise above 1 (its alphabet is $\{0, 1\}$), and yet (echoing Pascal) we have found a great richness in the geometric patterns of 0s and 1s that arise, many of which carry with them secrets about the prime numbers. Further (again echoing Pascal), we have had to leave out much more than we could put in.

We have called our triangle '*the cycle-number triangle*' because of the many cyclic phenomena involved in the $(0, 1)$-patterns of most interest, and because we derive from it a new representation of the natural numbers, each one exhibiting cyclic behaviour. We have given our triangle the general label **T**.

We declare that we have not seen this triangle defined before in the literature, but, of course, it may well have been described several times in the past and produced nothing of sufficient interest to keep mathematicians using and mentioning it. If we may quote Pascal yet again, he wrote in his autobiography, apropos his triangle: "Let no one say I have said nothing new. The arrangement of the subject is new. When we play tennis, we both play with the same ball, but one of us places it better!"

Whatever is the case, we hope that our methods and studies of the cycle-number triangle contain something new and worthy of their presentation.

2. Example, definition and construction of T

We begin by showing the cycle-number triangle **T**, down to row 6, in Figure 1 below. The apex triangle and directions for the central axis and i,j reference axes are shown.

Note in Figure 1 the left- and right-boundaries of **T**, two sloping lines, each containing the sequence **0**, **1**, **0**, **0**, **0**. These are the given elements, to start construction of the triangle.

To reference an element in **T** we may use two coordinates $(\mathbf{i}, \mathbf{j})$, with the i, j$^{\text{th}}$ element occurring at the intersection of the sloping lines $\mathbf{R_i}$ (the ith diagonal parallel to the right-boundary) and $\mathbf{L_j}$ (the j$^{\text{th}}$ diagonal parallel to the left-boundary). The two directed reference lines are indicated in Figure 1 on either side of the triangle $\mathbf{T(6)}$.

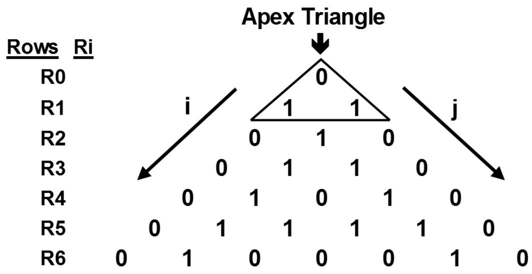

Figure 1: The Cycle-number Triangle $\mathbf{T(6)}$ (with apex triangle defined)

The general rules for generating **T** now follow.

2.1. Constructing the Triangle T

(1) The apex and triangle boundaries

The cycle-number triangle **T** is constructed according to the following rules:

(i) The elements have alphabet $\{\mathbf{0}, \mathbf{1}\}$;

(ii) The apex element is **0**;

(iii) The left-boundary **L** is (from the apex downwards to the left) $\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \ldots$;

(iv) The right-boundary **R** runs from the apex downwards to the right, with the same (**0**,**1**)-sequence as that of **L**.

The rows following the apex triangle are then constructed, row by row, by making a sequence of ‘neck-tie’ applications, as explained next.

(2) The ‘neck-tie’ figure and its uses

The figure which we use repeatedly to generate the elements of **T**, row-by-row, is called a *neck-tie* in view of its shape. It consists of an equi-sided triangle, ∇, supported by two long, sloping legs which are potentially infinite.

When applied to row **Ri** of **T**, the top side of the triangle is marked with the $(0, 1)$ elements of **Ri**, and the other two sides take the same markings, in order, cycling around the triangle (see Figure 2 for an example applied to **R3**).

The side-length, defined to be the number of elements on each side, increases by one with each row application.

The neck-tie (designated **ni**) is completed by adding the two spreading legs from the lowest vertex of the neck triangle. The left leg slopes down to the left, and has the (0,1)-pattern from the right side of ∇ appearing in it, cycling ever downwards.

Similarly, the right leg slopes down to the right, with the (0,1)-pattern from the left side of ∇, cycling ever downwards, appearing in it.

Figure 2 below shows how the Cycle-Number Triangle **T** is constructed, row by row, down to row **R6**. It also includes an expanded view of the neck-tie which is applied to row **R3**.

Notation. We designate by **T(i)** or **Ti** the sub-triangle of **T** which extends from the apex down to row **Ri**. And **ni** will designate the neck-tie which is applied to row **Ri** of **T**. It must be noted that the constructed neck-tie lines are notional. They do not appear normally in diagrams of **T**.

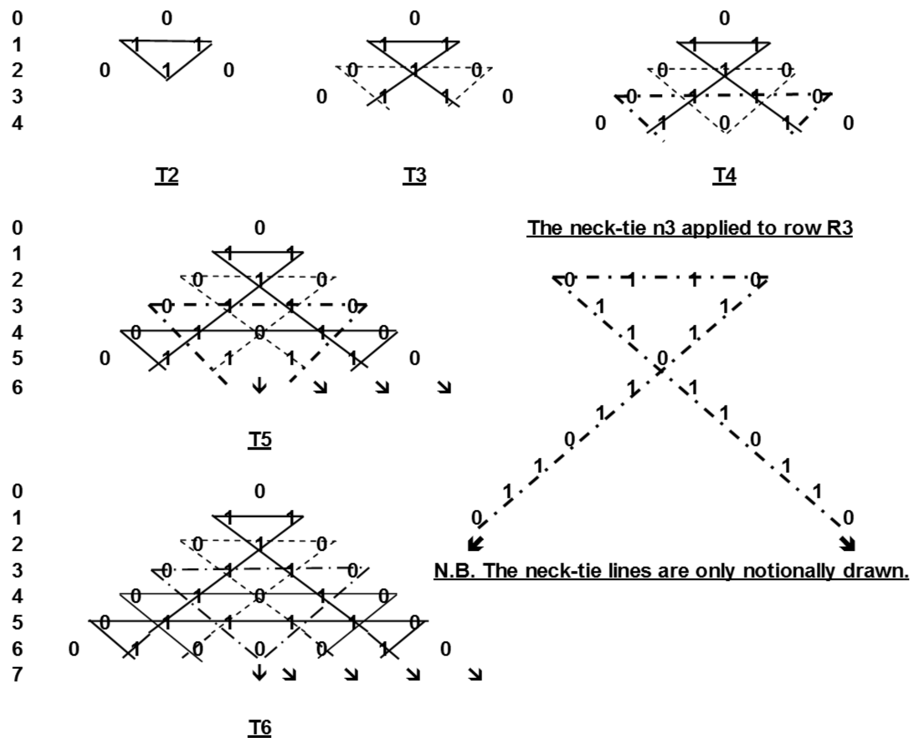

Figure 2: Constructing triangle **T** row-by-row $\mathbf{T(2)} \to \mathbf{T(6)}$

(3) The cycle-numbers and their fundamental cycles (f.c.s)

Observe how the string (110) from the top of the neck-tie **n3** cycles around the neck (in both directions). It also cycles down the left and right legs. This string is defined to be *the fundamental cycle (f.c.)* of the cycle-number $\underline{\mathbf{3}}$. Some general properties of cycle-numbers will now be developed.

The n^{th} cycle-number is designated by $\underline{\mathbf{n}}$, and its fundamental cycle by $\underline{\mathbf{n}}$**'**.

Definition 2.1. The cycle-number $\underline{\mathbf{n}}$ is the infinite string obtained by cycling its fundamental cycle $\underline{\mathbf{n}}$**'** indefinitely: e.g. $\underline{\mathbf{3}} = \overline{110}$.

Evidently, for each value of n > 0, two pictorial representations of $\underline{\mathbf{n}}$ occur in triangle **T**, cycling down the left and right legs respectively of the corresponding neck-tie.

Definition 2.2. The infinite sequence of cycle-numbers with n > 0 will be denoted by $\underline{\mathbf{N}}$. (It corresponds one-to-one with the natural number sequence **N**.)

Later we shall display the cycle-numbers as rows of a doubly infinite matrix **C**.

Since we cannot apply a neck-tie to $\underline{\mathbf{0}}$ in row **R0**, we have to define its cycle-number specially.

Definition 2.3. The zero cycle-number is $\underline{\mathbf{0}} \equiv 01000\cdots = (01)\overline{0}$. (This is the string on the R and L diagonals of **T**. It is special in that its cycling does not begin until after (01) occurs. With all the other cycle-numbers ($n > 0$), the cycling begins with the first digit of the string for $\underline{\mathbf{n}}$.)

With Figures 1 and 2 to guide us, we can make and prove the following general observations, as our first theorems about the cycle-number triangle, and the cycle-numbers derived from it.

Theorem 2.4. *The first six fundamental cycles (f.c.s), taken from the rows **R1** to **R6** of **T6**, are (1), (10), (110), (1010), (11110) and (100010). Generally:*

*(i) Every f.c. after **R0** is of length n (it has n letters);*

*(ii) Every f.c. after **R1** begins with a 1 and ends with a 10.*

Proof. The proofs of each item follow immediately from the neck-tie construction and applications to the sequence of rows, or from each other, so they will not be spelled out. □

Note that, like $\underline{\mathbf{0}}$, the cycle-number $\underline{\mathbf{1}} = \overline{1}$ is 'special', arising from the second row of the apex triangle. It cycles from its f.c. $\underline{\mathbf{1}}$**'** indefinitely, never acquiring a 0 (c.f. Peano's first two axioms, which are needed to establish 0, and its successor S(0) which is later labelled 1: both 'special numbers').

Theorem 2.5 (Palindrome Principles)**.**

(i) The $(0,1)$*-string on the upper side of a neck-tie* ∇ *is a palindrome; In general, row* $\boldsymbol{Ri}$ *of* $\boldsymbol{T}$ *is a* $(0,1)$*-palindrome of length* $i+1$*, for* $\boldsymbol{i} = 1, 2, 3, \ldots$*.*

(ii) The first $n-1$ *elements of a fundamental cycle (after* $\boldsymbol{R0}$*) form a palindrome.*

Proof. The proofs of each item follow immediately from the neck-tie construction and applications to the sequence of rows of **T**, or from each other, so they will not be spelled out. □

As an example, the complete (0,1)-string from the upper side of neck-tie **n6** is (0100010), which is a palindrome. And the f.c. of cycle-number $\underline{6}$ is $\underline{\mathbf{6}}$**'** = (100010), whose first five digits form a palindrome.

N.B. The Palindrome Principles, simple though they are, turn out to be powerful tools in enabling us to look ahead in **T** to discern patterns in number sequences.

3. Some theorems on lines in T

We have shown how to define the cycle-numbers, and given a few results about their $(0,1)$-patterns, and their fundamental cycles. We now begin a study of the $(0,1)$-patterns which occur on lines in **T**.

Theorem 3.1. *We already defined above (see Figure 1) how to reference elements* $\boldsymbol{(i, j)}$ *in* $\boldsymbol{T}$ *using the sloping reference diagonals for* $\boldsymbol{i}$ *and* $\boldsymbol{j}$ *coordinates. As explained above,* $\boldsymbol{L_i}$ *is the* $\boldsymbol{i}^{th}$ *sloping line in* $\boldsymbol{T}$ *parallel to the* $\boldsymbol{L}$*-boundary, and* $\boldsymbol{R_j}$ *is the* $\boldsymbol{j}^{th}$ *sloping line in* $\boldsymbol{T}$ *parallel to the* $\boldsymbol{R}$*-boundary. (Thus* $\boldsymbol{R_i}//\boldsymbol{R}$ *and* $\boldsymbol{L_i}//\boldsymbol{L}$*.) Then:*

(i) $\boldsymbol{L_1}$ *and* $\boldsymbol{R_1}$ *are both sequences of cycled 1s, which we designate as unit-cycle lines;*

(ii) $\boldsymbol{L_2}$ *and* $\boldsymbol{R_2}$ *are both sequences of cycles of* $(1,0)$ *(starting after the boundary element* ***0****), which we designate as* ***2****-cycle lines, having pattern* $\overline{10}$*;*

(iii) $\boldsymbol{L_3}$ *and* $\boldsymbol{R_3}$ *are both sequences of cycles of* $(1,1,0)$ *(starting after the boundary element* ***0****), which we designate as* ***3****-cycle lines, having pattern* $\overline{110}$*;*

(iv) $\boldsymbol{L_4}$ *and* $\boldsymbol{R_4}$ *are both sequences of cycles of* $(1,0,1,0)$ *(starting after the boundary element* ***0****), which we designate as* ***4****-cycle lines* $\overline{1010}$*;*

(v) This sequence of pairs $(\boldsymbol{L_i}, \boldsymbol{R_i})$ *of i-cycle lines, with* $\boldsymbol{L_i} = \boldsymbol{R_i}$*, continues indefinitely as* $\boldsymbol{i}$ *increases by 1 at each row-step in* $\boldsymbol{T}$*.*

Proof. The proofs of each item follow immediately from the neck-tie construction and applications to the sequence of rows of **T**, or from each other, so they will not be spelled out. □

The above theorem has shown that there is an ordered sequence of cycling $(0,1)$-strings down the left- and right- diagonals, equal in pairs. The next Theorem 3.2 will prove that the same sequences occur in vertical columns, the pairs being equidistant from the central axis of **T**. Before presenting this theorem, let us define the Cartesian axis frame, to which we can refer elements in horizontal and vertical directions.

3(2) The $(\mathbf{x}, \mathbf{y})$ Cartesian axes

The origin of the frame is at the apex of **T** so the apex is at point $(\mathbf{0}, \mathbf{0})$. The Cartesian y-axis is oriented vertically, with direction downwards. The Cartesian x-axis is the horizontal through the apex, with positive direction to the right. Its scale unit is that distance which separates the columns of **T** (equally spaced).

We shall use **Cn** to denote the n^{th} column, which contains the (**0**,**1**)-string which appears down the vertical line $x = n$, for $n = 1, 2, 3, \ldots$.

3(3) The axis line and its (0,1) pattern

The axis line is $x = 0$. Thus the column **C0** is the **T-**triangle axis, and each digit appearing on it below the apex triangle is the lower vertex of a neck-triangle, which in turn is a cycling of a boundary digit **0** (after **n1**). Hence the axis bears the same $(0,1)$ pattern as do the boundaries, viz. 01000. . . .

To the left of the axis, x will take corresponding negative values.

It follows from Theorem 2.5(i) (Palindrome Principle), that we need only deal with the columns **Cn** when **n** is positive. The corresponding columns in the negative direction will carry the same cycle-number sequence. On the few occasions which we refer to columns to the left of the axis we shall write **C(-n).**

We have shown that the sets of **L**-diagonals and **R**-diagonals are equal in pairs, w.r.t. their (**0**,**1**)-patterns. The next Figure and theorem shows that this same phenomenon occurs in the vertical columns, when taken in pairs equidistant from the axis of **T**.

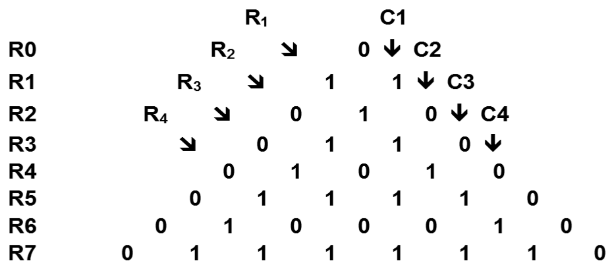

Figure 3: Triangle $\mathbf{T(7)}$, indicating the rows **Rj** and cols. **Cj** for $\mathbf{j} = 1, 2, 3, 4$

Theorem 3.2 (The vertical column patterns).
Diagram. *Refer to Figure 3. Recall that* $\boldsymbol{R_i}//\boldsymbol{R}$*, where* $\boldsymbol{R}$ *is the right-boundary of* $\boldsymbol{T}$*.*

Let us abbreviate the phrase 'the (0,1)-pattern in column ***Cn****' to 'the* ***Cn****-pattern'.*

Subsection 3(3) above proved the case $\boldsymbol{C0} = \boldsymbol{R_0}$*.*

Now we assert that (refer to Figure 3):

(i) The ***Cj****-pattern is equal to the* $\boldsymbol{R_j}$*-pattern, for* $\boldsymbol{j} = 1, 2, 3, \ldots$
And by axis-symmetry of ***T*** *the* ***C(-j)****-pattern is also equal to the* $\boldsymbol{R_j}$*-pattern.*

(ii) The ***Cj****-pattern lies along both the lines* $x = j$ *and* $x = -j$*.*

Proof. (i) Referring to the diagram of **T7** above, we construct an inductive proof, using properties of the neck-tie triangles (see Figure 2 and 3): we begin by showing the theorem to be true for j = 1.

We note that the (0,1) elements of **C1** occur in rows **R1, R3, R5**, etc. (i.e. in the odd rows). The first element of **C1** is 1, by definition of the diagonal pattern in $\mathbf{R_0}$.

Then the following statements are evidently true:

The second element of **C1** is in row **R3**, which is the third element of diagonal $\mathbf{R_1}$, which is equal to 1 (by Theorem 3.1(i); $\mathbf{R_1}$ is a unit-line).

The third element of **C1** is in row **R5**, which is the fifth element of $\mathbf{R_1}$, which equals 1 since $\mathbf{R_1}$ is a unit-line.

In general, the i^{th} element of **C1** is in row $\mathbf{R(i{+}2)}$, which is the $(\mathbf{i{+}2})^{\mathrm{th}}$ element of $\mathbf{R_1}$ and hence is equal to 1.

The proof that the **C1**-pattern is a string of 1s is now easily completed by induction, using the neck-tie construction rules. Then, by the Palindrome Principle, we can assert that the **C(-1)** pattern is also a string of 1s.

We can apply the same arguments to determine that the **C2**-pattern is the same as the $\mathbf{R_2}$-pattern, and equals the **C(-2)**-pattern.

Induction can now be used to generalize this overall argument, to prove the statement that for all j > 0 the **Cj**-pattern is the same as the $\mathbf{R_j}$-pattern, and equals the **C(-j)**-pattern. This will complete the proof of Theorem 3.2, for the (0,1)-patterns in the columns of **T**. □

Before going on to study (0,1)-patterns in **T** other than those occurring in straight lines, as treated above, we shall now present an alternative method for generating the elements of **T**. The starting triangle for this method bears comparison with the Pascal triangle in 'binomial coefficient form'. Moreover, it immediately shows how the rows of **T** relate to the natural numbers in **N** (in two directions) and their 'coprimeness properties'.

4. A second method for generating the cycle-number triangle T

4.1. The enteger triangle E

The cycle-number triangle **T(6)** was shown in Figure 1, Section 2, and then it was shown how to generate **T** generally by the neck-tie algorithm. Now we shall obtain the triangle **T(6)** by writing down a triangle **E** of ordered pairs of integers, called *entegers*, and then operating on each enteger by the so-called "coprime-function" named *kappa* ($\boldsymbol{\kappa}$). (N.B. We introduced the notion of 'enteger' in [4] and [5]. We write $\mathrm{n_m}$ for an enteger. Two entegers are added as with vectors.) Before defining kappa, below we give the enteger triangle **E** (on the left) of entegers down to **R6**, and (on the right) the triangle after the transformations by kappa have taken place.

R0 $\mathbf{0_0}$
R1 $\mathbf{1_0 \quad 1_1}$
R2 $\mathbf{2_0 \quad 2_1 \quad 2_2}$
R3 $\mathbf{3_0 \quad 3_1 \quad 3_2 \quad 3_3}$
R4 $\mathbf{4_0 \quad 4_1 \quad 4_2 \quad 4_3 \quad 4_4}$
R5 $\mathbf{5_0 \quad 5_1 \quad 5_2 \quad 5_3 \quad 5_4 \quad 5_5}$
R6 $\mathbf{6_0 \quad 6_1 \quad 6_2 \quad 6_3 \quad 6_4 \quad 6_5 \quad 6_6}$
The enteger triangle E(6)

T = κE ⟶

0
1 1
0 1 0
0 1 1 0
0 1 0 1 0
0 1 1 1 1 0
0 1 0 0 0 1 0
The cycle-number triangle T(6)

Figure 4: The triangle ***E*** of entegers (ordered pairs), transformed by $\boldsymbol{\kappa}$ to **T**

We shall define **E** by giving its $\mathrm{n^{th}}$ line, then define the function kappa, and then establish the validity of the general transformation $\boldsymbol{\kappa}\mathbf{E} = \mathbf{T}$

Definition 4.1. The general row **Rn** for the enteger triangle **E** is:

$$n_0,\ n_1,\ n_2,\ \dots\ ,\ n_{n-1},\ n_n.$$

(For comparison, in Pascal's triangle the row is $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n-1}, \binom{n}{n}$.)

Lemma 4.2. *Given the triangle* ***T(n)****, we can extend it to* ***T(n+1)*** *thus: add* 1_0 *('vectorial' enteger addition) to the last enteger in each left diagonal* $\boldsymbol{L_i}$ *for* $i = 0, 1, 2, \dots, n$*, and add* 1_1 *to the last enteger* n_n *of* ***Rn****.*

Definition 4.3. Let $\mathrm{e = s_t}$, with $\mathrm{s,t} \in \mathbf{N}$. Then the '*coprime operator (kappa)*' is defined as follows:

$$\kappa(e) \equiv \begin{cases} 1, & \text{if } s \text{ and } t \text{ are coprime;} \\ 0, & \text{otherwise (see also special cases below).} \end{cases}$$

Special cases: $\kappa(0_0) \equiv 0$; $\kappa(1_0) \equiv 1$; $\kappa(t_0) \equiv 0$ for all $t > 1$, where $e = s_t$ is the general notation for an enteger, with s and t written diagonally.

Without using the notion of divisibility, we can determine when a number pair is coprime by means of the 'repeated-pair-subtraction method' used in the simplest version of Euclid's Algorithm (call it EA) (some examples of this are given in Figure 5).

Lemma 4.4. *Let kappa be applied (element-wise) to the n^{th} row of **E**. Then the result is a palindromic* $(0, 1)$*-string.*

Proof. Consider the elements of **Rn** of **E**, taken in pairs symmetrically placed relative to the axis of **E**. The i^{th} pair, after applying kappa to each, is $\kappa(n_i)$ and $\kappa(n_{n\text{-}i})$. Applying only the first subtraction in Euclid's Algorithm, we find from the second of our pairs that after this one subtraction $\kappa(n_{n\text{-}i}) = \kappa(i_n) = \kappa(n_i)$. This is true for all pairs in the row (if there is a single central enteger, as in the even-numbered rows, then it is immaterial what kappa-value it takes) so a palindromic (0,1)-string results from the row **Rn**. □

4.2. Relationships between E and T

We claimed in Figure 4 that $\boldsymbol{\kappa}(\mathbf{E}) = \mathbf{T}$, the cycle-number triangle. This sub-section is concerned with proving this claim.

We have already shown that the n^{th} rows of both triangles $\boldsymbol{\kappa}(\mathbf{E})$ and **T** have the same lengths n+1, and that these rows each consist of a (0,1)-string which is palindromic. So both triangles are symmetric w.r.t. their axis-lines.

It is immediate that they both have the same **R** and **L** diagonals, and that in both, the $\mathbf{R_1}$ and $\mathbf{L_1}$ diagonals are unit-lines. The two both have the same axis lines, since in **E** the axis $x = 0$ carries the entegers 0_0, 2_1, 4_2, ..., which is an A.P. of entegers having common difference 2_1. (We can extend Lemma 4.2 to show that this sequence extends indefinitely.) Applying kappa to the sequence, we get 0, 1, 0, 0, ..., which is the axis pattern in **T**.

Recall that **T** was constructed by applying a neck-tie construction, say **nn,** to each row **Rn**, We shall show how the same type of construction can be used to build **E**, and moreover that the same type of cycling around and down the neck-ties, as in **T**, then occurs in $\boldsymbol{\kappa}\mathbf{E}$. The construction of neck-tie $\mathbf{nn} = \nabla_\mathbf{n}$ for **E** requires the following steps:

(i) the top side of ∇_n is n_0, n_1, n_2, ..., $n_{n\text{-}1}$, n_n (see Definition 4.1)

(ii) the left side of ∇_n is n_0, $(n{+}1)_1$, $(n{+}2)_2$, ..., $(2n\text{-}1)_{n\text{-}1}$, $(2n)_n$

(iii) the right side of ∇_n is n_n, $(n{+}1)_n$, $(n{+}2)_n$, ..., $(2n\text{-}1)_n$, $(2n)_n$

(iv) the left leg is a continuation of the sequence in (iii)

(v) the right leg is a continuation of the sequence in (ii)

To exemplify this definition of ∇_n we will show the neck-tie ∇_3, and on its right, show what happens to it when certain of its elements are reduced by applying the 'pair subtraction' method of Euclid's Algorithm to them (recall that such applications do not change the kappa values from those of the original pairs). We show only the neck, and one cycle of the left leg and the right leg. It will be evident how the leg cycles must continue.

$\mathbf{3_0 \quad 3_1 \quad 3_2 \quad 3_3}$	$\mathbf{3_0 \quad 3_1 \quad 3_2 \quad 0_3}$	$\mathbf{0 \quad 1 \quad 1 \quad 0}$
$\mathbf{4_1 \qquad 4_3}$	$\mathbf{3_1 \qquad 1_3}$	$\mathbf{1 \qquad 1}$
$\mathbf{5_2 \quad 5_3}$	$\mathbf{3_2 \quad 2_3}$	$\mathbf{1 \quad 1}$
$\mathbf{6_3}$	$\mathbf{3_0}$	$\mathbf{0}$
$\mathbf{7_3 \quad 7_4}$	$\mathbf{1_3 \quad 3_1}$	$\mathbf{1 \quad 1}$
$\mathbf{8_3 \qquad 8_5}$	$\mathbf{2_3 \qquad 3_2}$	$\mathbf{1 \qquad 1}$
$\mathbf{9_3 \qquad\quad 9_6}$	$\mathbf{0_3 \qquad\quad 3_0}$	$\mathbf{0 \qquad\quad 0}$

Figure 5: The neck-tie $\nabla_{\mathbf{3}}$; then same with appropriate EA changes; then $\kappa(\nabla_{\mathbf{3}})$

Remarks. The left diagram shows the neck-tie as defined for row **R3** of **E.** The centre diagram is attractive, found by applying EA (Euclid's Algorithm) subtractions appropriately, to elements of $\nabla_{\mathbf{3}}$. It shows how the kappa values of elements of **n3** must cycle in the desired way and become equal to corresponding values in **T**.

Its generalization to the neck-tie $\nabla_{\mathbf{n}}$ and to **T** is automatic.

The claim that $\boldsymbol{\kappa}(\mathbf{E}) = \mathbf{T}$ now follows by induction on $\nabla_{\mathbf{n}}$. Thus, if we verify it for $n = 1$ and 2, we can extend those to verify it for $n = 3$, and so on.

4.3. On 'Coprimeness' and 'Primeness' in the Cycle-Numbers

This is an appropriate moment for us to link the cycle-numbers to the natural numbers with regard to the concepts of coprimeness and primeness, in such a way that the two number systems can be said to represent one another exactly in those regards.

Consider the two triangles **T** and **E** in Figure 4, where $\mathbf{T} = \boldsymbol{\kappa}\mathbf{E}$. It is seen that each row of **T** (e.g. **Rn**) carries a complete record of what we shall call the coprimeness relation of **n** with each of the integers $i = 1, 2, 3, \ldots, n$. This gives rise to the following lemma, which directly relates fundamental cycles and the totient function of Euler:

Lemma 4.5. *Let $\omega(\underline{\mathrm{n}})$, the weight of cycle-number $\underline{\mathrm{n}}$, be the sum of the elements of **n'** in row **Rn** of **T**. Then $\omega(\underline{n}) = \varphi(n)$, where φ (i.e. phi) is Euler's totient function.*

Proof. This follows immediately from Definition 4.1, $\mathbf{T} = \boldsymbol{\kappa}\mathbf{E}$ and the definition of φ. □

We can with reason speak of the *coprimeness* of a cycle-number $\underline{\mathrm{n}}$ and assign a measure of it by the ratio (or index) $\omega(\underline{\mathrm{n}})/\mathrm{n} = \varphi(\mathrm{n})/\mathrm{n}$.

We continue these notions by coining the slogan that "coprimeness begets primeness", and presenting the following definitions to add precision to it.

Definition 4.6 (Coprimeness index, and primeness of $\underline{\mathbf{n}}$)**.** We define a *cycle-number* $\underline{\mathrm{n}}$ to be *prime* if its coprimeness index has value $(n-1)/n = 1 - 1/n$. The value of the index can never be 1, since $\kappa(\mathrm{n_n}) = 0$.

Note that with this definition, it can be shown that $\underline{\mathrm{n}}$ is a prime cycle-number iff n is a prime integer. For if n is not prime, there exists some integer $m < n$ such that $\kappa(\mathrm{n_m}) = 0$, and the coprimeness index of $\underline{\mathrm{n}}$ is less than $1 - 1/n$, so $\underline{\mathrm{n}}$ is not prime.

Conversely, if n is prime, then $\varphi(n) = n - 1 = \omega(\underline{\mathrm{n}})$, the coprimeness index is (n-1)/n , and so $\underline{\mathrm{n}}$ is prime.

Examples may be seen in rows 2, 3, 5, and 7 of **T** in Figure 3.

5. Three operations on cycle-numbers

5.1. Definitions of the operators

The following three operations and their symbols are defined on the cycle-numbers, which allow us to discover and develop various algebraic relationships between the cycle-numbers. We shall not report on the subsequent algebra further than we need to, in order to study (0,1)-patterns in the triangle **T** and a later-derived matrix C.

The three operators and their symbols are:

(1) 'star' ($*$) , **(2)** 'add' ('$+$'), and **(3)** 'multiply' ($\wedge$).

Definition 5.1. The 'star' (or 'conjoin') operation is one of conjoinment of two given (0,1)-vectors or strings. Thus if $\underline{\mathbf{m}}$ and $\underline{\mathbf{n}}$ are two (0,1)-strings, then $\underline{\mathbf{m}} * \underline{\mathbf{n}}$ is the string obtained by writing first the $\underline{\mathbf{m}}$-string, and then continuing with the $\underline{\mathbf{n}}$-string, thus creating a string of length m+n. Clearly this operation does not generally commute.
It can be extended in the obvious way to deal with three or more strings.

Care must be taken when interpreting the conjoin of two f.c.s of cycle-numbers. The result is *not necessarily* another cycle-number f.c.; in fact, *it usually isn't*.

Definition 5.2. Two cycle-numbers $\underline{\mathrm{m}}$ and $\underline{\mathrm{n}}$ are added in a natural way as follows (letting their 'sum' be $\underline{\mathrm{m}}$ '$+$' $\underline{\mathrm{n}} \equiv \underline{\mathrm{s}}$).
Let s = m+n (sum of the two cycle-number f.c. lengths), and find from row **R**(m+n) of the enteger triangle **E** the enteger string which, on applying κ to its elements, yields **s'**. Using Def. 4.1 we find the required string to be $(\mathrm{m+n})_1$, $(\mathrm{m+n})_2$, ... , $(\mathrm{m+n})_{\mathrm{s-1}}$, $(\mathrm{m+n})_{\mathrm{s}}$. Applying κ to this string yields $\underline{\mathrm{s}}$', and hence $\underline{\mathrm{s}}$.

Note: To make full sense of this operation, we must think of it as taking place between rows **Rm** and **Rn** of the enteger triangle **E**. We are extending the triangle in the 'natural way', by 'adding' (strictly 'appending') *n* further rows to it (down from **Rm**) according to theorems previously given for patterns on the diagonal lines.

Finally, having reached $\mathbf{R_{m+n}} = \mathbf{R_s}$, we drop the first element s_0 and we are left with $\underline{s}$' as required. (As an example, see Figure 3 and add $\underline{2}$ to $\underline{3}$ to get $\underline{5}$.)

Definition 5.3 (The cap product)**.** Two cycle-numbers $\underline{m}$ and $\underline{n}$ in **N** (not including $\underline{0}$) are 'multiplied' (in a not-so-natural way), as defined below. Again we carry out the initial operations upon the two respective f.c.s, $\underline{m}$' and $\underline{n}$' and arrive at the fundamental cycle of a new cycle-number which we shall call the Boolean Product (B.P.), or the 'cap product', of the two cycle-numbers. This 'product' is a powerful tool for us in our study of cycle-number patterns. Its definition is as follows:
Let mn = k. Then the Boolean Product of $\underline{m}$ and $\underline{n}$ is a cycle-number whose f.c. is of length k, and is found from the following formula: $\underline{k}' = (n * \underline{m}') \wedge (m * \underline{n}')$.
The left-hand bracket contains the (0,1)-string of n conjoined cycles of the f.c. of $\underline{m}$, and the right-hand bracket contains the (0,1)-string of m conjoined cycles of the f.c. of $\underline{n}$. The cap symbol between the two bracketed terms indicates that an element-wise product has to be computed, according to the following binary multiplication table (Boolean): $0\wedge0 = 0\wedge1 = 1\wedge0 = 0$, and $1\wedge1 = 1$.

It is easy to see that the two multiplication sets from **N(x)** and $\underline{\mathbf{N}}(\wedge)$ are isomorphic.

A simple example will illustrate the use of the operation $\wedge$**.**

Example. Let $\underline{m} = \underline{2}$ and $\underline{n} = \underline{3}$. Then

$$(n * \underline{m}')\wedge(m * \underline{n}') = (\mathbf{101010})\wedge(\mathbf{110110}).$$

Note that each string is of length 2x3 =6. Applying $\wedge$ element-wise gives the result (**100010**), which is the f.c. of $\underline{6}$.

The reader should check this result in Fig. 3, and observe how the $\underline{2}$-cycles and $\underline{3}$-cycles arrive at **R6** in their respective neck-ties, with their end 0s filling three places in $\underline{6}$'. If one lays either $\underline{2}$' or $\underline{3}$' along the length of $\underline{6}$', as with two moving rulers, one finds that each ruler cycles $\underline{6}$' exactly, with regard to their end 0s. We say (using Euclid's language) that both $\underline{2}$' and $\underline{3}$' *measure* $\underline{6}$' because of this.

It is helpful to place $n * \underline{m}$' above $m * \underline{n}$', and apply the cap products vertically, in the k resulting 2x1 columns. Thus, with the example:

$$3 * \underline{2}' = 101010$$
$$2 * \underline{3}' = 101110$$
$$\wedge \quad = 100010$$

Note that for a 1 to occur in the result, there must be two 1s above it. We shall exploit this fact later.

Before leaving our discussion of the cycle-number triangle **T**, we remark that all manner of patterns can be discovered in **T**, based on the arrangements of the 0s and 1s, and how they are related to the cycle-numbers.

We have already discussed many of the most obvious patterns, and built definitions and theorems about them. We shall end this Section by presenting an interesting theorem that demonstrates a fractal property, namely that **T** can properly include a copy of itself ... indeed an infinite sequence of such copies. Our proof will be 'pictorial', extending to two inclusions only.

Theorem 5.4. $\mathbf{T} \supset \mathbf{T} \supset \mathbf{T} \supset \dots$ *(proper inclusions).*

Proof. Pictorial Proof (See Figure 6). □

The Figure 6 first shows **T** to row 10 (i.e. **T(10)**), with its first two neckties, coloured black and blue respectively. The second and third **T**s of the theorem are shown below the first one.

Clearly the second two triangles have elements which map directly to themselves and to those of the original **T**. They are mapped from the original neck-ties, but with changed Euclidean shapes. They remain similar in congruent triples. In terms of cycle-numbers their necks are still equi-sided triangles, and their legs still carry the same (0,1)-patterns. The neck-ties undergo anti-clockwise, Euclidean rotations.

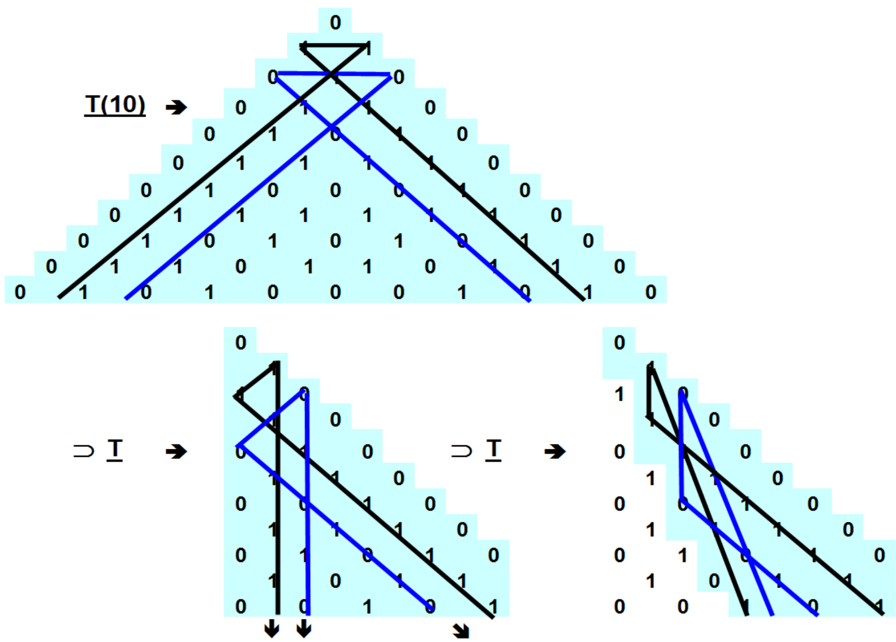

Figure 6: A fractal property of **T**

This sequence of included **T**s and their corresponding triangles and neck-ties can be extended, and other Ts and sequences of Ts can be found elsewhere in the original triangle.

6. Definition of the cycle-number matrix C

Our construction of the cycle-number triangle **T** enabled us to introduce the notion of cycle-numbers and define various of their properties and operations on them. We now wish to display all the cycle-numbers in a doubly-infinite matrix called **C,** which provides a more convenient view-point of their domain for us to proceed with their study. It is easy to produce **C**, for it is just a matter of turning triangle **T** 'on its side' and 'dropping' the boundary diagonals **R** and **L**. Sub-section 6.1 clarifies this.

6.1. Producing C by using the f.c.s from T

To be more precise, we place the fundamental cycles of the cycle-numbers in the rows of **C**, with n' (from 1' onwards) occupying the first n elements of row **Rn**. Then we allow each number to cycle indefinitely in its row, from the leading diagonal (l.d.) towards the right, potentially filling all the rows of **C**.

To reference elements in the matrix, we shall envisage perpendicular Cartesian axes y (vertically down) and x (horizontally across) both taking all values in positive **N**. The following diagram exemplifies all these arrangements up to $n = 13$.

Row	**y/x**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
R1	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
R2	**2**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**
R3	**3**	**1**	**1**	**0**	**1**	**1**	**0**	**1**	**1**	**0**	**1**	**1**	**0**	**1**
R4	**4**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**
R5	**5**	**1**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**0**	**1**	**1**	**1**
R6	**6**	**1**	**0**	**0**	**0**	**1**	**0**	**1**	**0**	**0**	**0**	**1**	**0**	**1**
R7	**7**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**1**
R8	**8**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**
R9	**9**	**1**	**1**	**0**	**1**	**1**	**0**	**1**	**1**	**0**	**1**	**1**	**0**	**1**
R10	**10**	**1**	**0**	**1**	**0**	**0**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**
R11	**11**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**1**
R12	**12**	**1**	**0**	**0**	**0**	**1**	**0**	**1**	**0**	**0**	**0**	**1**	**0**	**1**
R13	**13**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**

(l. d.)

Figure 7: The Cycle-Number matrix **C(13)**

Observe that the triangle beneath (and including) the l.d., is the cycle-number triangle 'left-justified' and with the zero-lines removed. Its rows are the fundamental cycles (f.c.s) of the cycle-numbers.

Remarks. (On the column elements in C(13).) We are now going to observe what happens in columns C1, C2, ... as the rows are introduced sequentially, R1 to R2 to R3 etc. In order to describe what we are doing we need to add to our vocabulary several graphic terms and notations, such as 'potential prime (pP)', 'potential twin prime (pT or pTP)', 'stalactite in Cj (j-stal)' and 'n-sieve' or 'p-sieve'. Each of these will be defined when introduced.

Observations. (Many have already been noted earlier, from **T**.)

(i) **C** is symmetric about the leading diagonal, so **Rn** = **Cn**.

(ii) The leading diagonal (l.d.) is 1, 0, 0, 0, ...

(iii) The f.c. n' of cycle-number n, in **Rn**, runs across from **C1** to the l.d. Its transpose, in **Cn**, is equal to it and runs from **R1** down to the l.d.

(iv) All elements in **R1** are 1, being placed there by 1' as it cycles along to the right.

(v) We say that all elements in the columns of **R1** are *potentially prime* (pP), and that each begins '*growing a stalactite of* 1s' in its column (c.f. a real stalactite, starting to grow down from the roof of a cave).

(vi) All elements in **R2** are produced by 2' cycling to the right; thus $1, 0, 1, 0, \ldots$ are the elements placed in **R2** of columns **C1, C2**, etc.

(vii) We now think of the process in (vi) as being a 'sieving' action, thus: the 0s are placed in the even cols., and each one 'stops' the stalactite above it from growing its column of 1s any further. Thus all stalactites in the even columns are now 'stopped' at **R2**.

(viii) The stalactite in **C2** 'has reached' the l.d. of **C**, and the f.c. composition (10) satisfies our definition of primeness. So we say that the 2-stal is prime; sometimes we say that the stalactite in col. 2 is prime, and even that **C2** is prime (if n is prime, then **Cn** contains a prime stalactite). Thus '2 is P'.

(ix) In all even cols. after **C2**, the pP stalactites are stopped when the **2**-sieve cycles by; and their stalactites become nonP (or nP, or not-prime). When this happens, we say that the stalactite has reached its final length in its column. This length is the number of 1s acquired, plus a 1 for the final 0.

(x) Stalactites which reach the l.d. are 'prime stals'. Their columns are 'prime cols'.

(xi) In all odd columns the stalactites are not 'stopped' by the **2**-sieve. Their lengths all 'grow' by 1, and they remain as potential primes (pPs). We say they have passed through the **2**-sieve. Now we imagine the **3**-sieve beginning its cycling, and we ask how many of the remaining pP stalactites will survive its passage.

(xii) We can carry on this process for ever, letting the row sieves pass along to the right, here and there stopping a stalactite from growing further.

(xiii) Observe that some pairs of rows have identical (0,1)-patterns. Examples are: **R2, R4, R8**; and **R3, R9**; and **R6, R12**. Conditions for this are given by:

Theorem 6.1. *Two cycle-numbers* $\underline{m}$ *and* $\underline{n}$ *have the same* $(0,1)$*-pattern in their rows if and only if m and n have the same radical; that is, iff* $r(m) = r(n)$*. In case* $m < n$*, we have m measures n, and* $\underline{m}$*' cycles in* $\underline{n}$*'.*

Proof. The proof is left to the reader. □

Example.

(i) $\underline{2}$, $\underline{4}$, and $\underline{8}$ have the same $(0,1)$ pattern, since $r(2) = 2 = r(4) = r(8)$. The f.c.s are respectively 10, 1010, and 10101010; $\underline{2}$' cycles in $\underline{4}$' and $\underline{8}$'; and $\underline{4}$' cycles in $\underline{8}$'.

(ii) $\underline{6}$ and $\underline{12}$ have the same $(0,1)$ pattern, since $r(6) = 6 = 2 \cdot 3$ and $r(12) = r(2^2 \cdot 3) = 2 \cdot 3$. We have $\underline{6}$'= 100010, which cycles in $\underline{12}$'= 100010100010.

Definition 6.2. The relation 'has the same (0,1)-pattern' is denoted by ρ(rho). It is easy to show that ρ is an equivalence relation on the rows of **C**. Hence the set of rows of **C** is partitioned by ρ.

We now introduce a matrix derived from **C**, denoted by **PBPS(C),** and obtained by sequentially computing its rows.

6.2. The PBPS matrix

A useful pictorial device is obtained by transforming the matrix **C** as we go along, row by row, and placing the modified rows in a new matrix, say $\mathbf{S} \equiv$ **PBPS(C).** The acronym stands for **P**artial **B**oolean **P**roduct (row)-**S**equence. The rows of **C** form the sequence $\underline{\mathbf{N}}$, of the cycle-numbers $\underline{n}$. The i^{th} partial BP of this sequence, denoted by $\mathbf{s_i}$, is the i^{th} row of **S**. Thus the rules of the computations are as follows:

Row computation Rules:
Let $\underline{\mathbf{n}}$ denote the n^{th} row of **C**, and $\underline{\mathbf{s}}_\mathbf{n}$ denote the corresponding row in **S**. Then

(i) $\mathbf{s_1} = \underline{1} = \overline{1}$; and

(ii) $\mathbf{s_n}$ = factorial $\underline{n}$ (using BP multiplication) = primorial $\underline{i}$ (using BP multiplication, and lemma 6.3 below).

Example.

$$\begin{aligned}\mathbf{s}_6 &= \underline{1}\wedge\underline{2}\wedge\underline{3}\wedge\underline{4}\wedge\underline{5}\wedge\underline{6} \text{ which reduces to}\\ &= \underline{2}\wedge\underline{3}\wedge\underline{5},\\ &= \text{primorial } \underline{5}.\end{aligned}$$

(The notation for this (see [6]) is **5#** where cap or BP multiplication is understood. We occasionally use a personal notation $\mathbf{X_i}$ for primorial $\mathrm{p_i}$, where **X** is capital 'chi'. Thus for example, $\mathbf{5\#} = \mathbf{X_3}$.)

The sequence of primorials rises rapidly in lengths, since $\mathbf{p_{n+1}\#} = \mathbf{p_{n+1}} \wedge \mathbf{p_n\#}$.

Lemma 6.3. *factorial* $\underline{n}$ = *primorial* $\boldsymbol{p_n}$, *where* $\boldsymbol{p_n}$ *is the greatest prime cycle-number less than or equal to* $\underline{n}$.

Proof. Any **n**-sieve which is not a prime sieve cannot supply a 0 to a column which has an unstopped stalactite in it, and hence can be ignored. For if **n** were not prime, it would be measurable by one or more primes, **p** say, with **p**<**n**, and one of the **p**-sieves arising from them would already have stopped the stalactite. □

Example. In $\mathbf{s_6}$, the $\underline{4}$ and $\underline{6}$ cycle-numbers are not prime.

(i) Now $\underline{2} \wedge \underline{4} \equiv \underline{2}$ (in its whole (0,1)-string) so a stalactite which has passed through the **2**-sieve must also pass through the **4**-sieve. Thus the $\underline{4}$ may be ignored.

(ii) For $\underline{6}$, the other non-prime, we have $\underline{6} = \underline{2} \wedge \underline{3}$. Therefore any 0 presented to a column in $\underline{6}$! by the 4-sieve or the 6-sieve will find that the stalactite has already been stopped by either the 2-sieve or the 3-sieve. The computation of $\underline{6}$' shows how this must happen:

$$\begin{array}{rr} & 101010\\ \wedge & \underline{110110}\\ & \underline{100010}\end{array}$$

The resulting **PBPS** matrix need show only the **C1** column of 1s, and the completed stalactites (with their final 0s) in the other columns. In the leading diagonal we place a P in each prime column, for ease of locating prime rows. All other entries in the matrix are 0s, and these are not shown (i.e. their cells are left blank).

Below is the reduced matrix **PBPS(C13)**. (See Lemma 6.3 for explanation of why we can write factorial $\underline{n}$ for obtaining each row **Rn** . For each prime row, we could write $\underline{\mathbf{p}}\#$, and for each non-prime row we would have $\underline{\mathbf{n}}\mathbf{!} \equiv \underline{\mathbf{p}}\#$ where p is the greatest prime < n; and all multiplication is ∧.)

	y/x	1	2	3	4	5	6	7	8	9	10	11	12	13
1!	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2!	2	1	P	1	0	1	0	1	0	1	0	1	0	1
3!	3	1		P		1		1		0		1		1
4!	4	1			0	1		1				1		1
5!	5	1				P		1				1		1
6!	6	1					0	1				1		1
7!	7	1						P				1		1
8!	8	1							0			1		1
9!	9	1								0		1		1
10!	10	1									0	1		1
11!	11	1										P		1
12!	12	1											0	1
13!	13	1												P

Figure 8: The matrix **PBPS(C13)** (a prime rib diagram)

The reader will appreciate why we have added the bracketed phrase to the figure's caption. The prime and twin prime stalactites stand out like ribs in a rib-cage.

Note in particular that when a growing stalactite acquires a 0 from a passing sieve, it 'stops growing'. More precisely, its pattern now ends in (1,0), and the next $\wedge$ operation in its column is $1 \wedge 0 = 0$. This happens to all stalactites eventually (except the stal. in **C1**). Those which reach the l.d. become primes; whilst those pPs which are not destined to become primes are stopped by a 0 above the l.d.

Much more can be said, and deduced from, the **C** and **PBPS** matrices. This must all be left for a segue paper. To end this one, we shall include a Section 7 which muses upon the 'music' made by cycling (0,1)-patterns formed within **T** and **C**.

7. Musings on the 'music of the cycle-numbers'

At the head of this paper (p. 2), we gave a quote by Leibnitz which expresses very beautifully a relationship he claims between music, man and mathematics, one with which we whole-heartedly empathize. Here we take up his theme with some of our own feelings (well, Turner's anyway!) about musical images arising from studies of the cycle-numbers in **T**. The author du Sautoy, in his book *Music of the Primes* [1], traces many connections between mathematics and music, stemming from work due to Pythagoras, Euler, and so on up to the present day, where his focus of attention is on the distribution of the primes and their relations to the zeros of the zeta function and Riemann's Hypothesis, and on related musical ideas.

With our cycle-numbers, we have shown that each number **n** has an interior pattern, or structure, with a fundamental cycle (a (0,1)-string of length n) which cycles around its **n**-necktie and down the two legs indefinitely, within the cycle-number triangle **T**. Similarly, in matrix **C**, the cycling takes place linearly in two (and more) directions. It is easy to compare these 'movements' with the vibrations of tuned strings on a musical instrument, or on bars of a xylophone. One can even turn the cycled patterns into music, by clapping or drumming the (0,1)-strings using Morse-code rhythms, and accenting beats on the starts of each cycle. For example, the number $\underline{\mathbf{2}}$ has f.c. (10), which can be clapped in 2/4 time as it cycles, thus: **da-di, da-di, da-di,** ... with stresses on each **da** . Similarly $\underline{\mathbf{3}}$, with f.c. (110), can be clapped in 3/4 or 3/8 time as **da-da-di, da-da-di,** etc. with stresses on each first **da**. The notion of polyphony is easily introduced via the cap product. For example, $\underline{\mathbf{2}}$ and $\underline{\mathbf{3}}$ can oscillate together, as the joint vector $\underline{\mathbf{2}} \wedge \underline{\mathbf{3}} = \underline{\mathbf{6}}$. This has f.c. (100010) (clapped as **da-di-di-di-da-di**),which can be stressed in various ways to produce differing rhythms and 'sounds'.

In this manner, one can think of each twin prime having its own distinctive rhythms and sounds; e.g. ($\underline{\mathbf{3}}$, $\underline{\mathbf{5}}$) resonates with $\underline{\mathbf{15}}$, and so on. These patterns, or pieces of linear patterns, occur and recur in different ways and places throughout the matrix **C**, causing 'overtones' or 'harmonics' in the 'music'.

One interesting comment, about the entrance of each successive prime, will suffice to end this musing. When a new prime arises in **T**, it breaks various previous symmetries, and introduces its own distinctive rhythm into the music which is sounding within and about its new linear 'melodies', on its own grid in **T** or **C**.

Perhaps, like Pythagoras and his 'music of the spheres', we (i.e. Turner) may well be the only person capable of hearing the 'music of the cycle-numbers'.

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Bridges between different known integer sequences

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Abstract

In this paper a new method of generating identities for Fibonacci and Lucas numbers is presented. This method is based on some fundamental identities for powers of the golden ratio and its conjugate. These identities give interesting connections between Fibonacci and Lucas numbers and Bernoulli numbers, Catalan numbers, binomial coefficients, δ-Fibonacci numbers, etc.

Keywords: Fibonacci and Lucas numbers, Bernoulli numbers, Bell numbers, Dobinski's formula

MSC: 11B83, 11A07, 39A10

1. Introduction

The authors' fascination with Fibonacci, Lucas and complex numbers has been reflected in the following two nice identities (discovered independently by Rabinowitz [10] and Wituła [7] and, probably, many other, former and future admirers of the Fibonacci and Lucas numbers):

$$(1+\xi+\xi^4)^n = F_{n+1}+F_n(\xi+\xi^4) \quad \text{and} \quad (1+\xi^2+\xi^3)^n = F_{n+1}+F_n(\xi^2+\xi^3), \tag{1.1}$$

where $\xi^5 = 1$, $\xi \in \mathbb{C}$ and $\xi \neq 1$, and F_n denotes the nth Fibonacci number.

2. Basic identities

Let

$$\alpha := 2\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \beta := -2\cos\left(\frac{2}{5}\pi\right) = \frac{1-\sqrt{5}}{2}.$$

Then we have

$$\alpha + \beta = 1, \qquad \alpha\beta = -1 \tag{2.1}$$

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad n \in \mathbb{Z}, \tag{2.2}$$

$$L_n = \alpha^n + \beta^n, \qquad n = 0, 1, 2, \ldots, \tag{2.3}$$

where L_n denotes the nth Lucas number [3, 9].

Then, identities (1.1) can be written in the form

$$F_{n+1} + x^{-1} F_n = x^n, \tag{2.4}$$

for every $x \in \{\alpha, \beta\}$. In other words, we get the divisibility relation of polynomials

$$(x^2 - x - 1) \mid (x^{n+1} - F_{n+1}x - F_n).$$

Similarly (by induction) we can generate the identity

$$L_{n+1} + x^{-1} L_n = (2x - 1)\, x^n, \tag{2.5}$$

for every $x \in \{\alpha, \beta\}$. This implies the following divisibility relation of polynomials

$$(x^2 - x - 1) \mid ((2x - 1)x^{n+1} - L_{n+1}x - L_n).$$

Remark 2.1. If the values F_n and L_n were defined for real subscripts $n \in [0, 1)$ (see [15]), then from formulae (2.4) and (2.5) we could easily extend these definitions for any other real subscripts.

In particular, if functions $[0, 1] \ni n \mapsto F_n$ and $[0, 1] \ni n \mapsto L_n$ are continuous, then from formulae (2.4) and (2.5) we could obtain the continuous extensions of these functions. With this problem also some special problem is connected (see Corollary 2.6 – Dobinski's formula problem).

Immediately from identities (2.4) and (2.5) the next result follows.

Theorem 2.2 (Golden ratio power factorization theorem). *Let $\{k_n\}_{n=1}^{\infty}$ be a sequence of positive integers. Then the following identities hold true*

$$\prod_{n=1}^{N} \Big(F_{k_n+1} + \frac{\sqrt{5}-1}{2} F_{k_n} \Big) = \Big(\frac{1+\sqrt{5}}{2} \Big)^{\sum_{n=1}^{N} k_n},$$

$$\prod_{n=1}^{N} \Big(F_{k_n+1} - \frac{\sqrt{5}+1}{2} F_{k_n} \Big) = \Big(\frac{1-\sqrt{5}}{2} \Big)^{\sum_{n=1}^{N} k_n},$$

or in equivalent compact form

$$\prod_{n=1}^{N} \Big(F_{k_n+1} + x^{-1} F_{k_n} \Big) = x^{\sum_{n=1}^{N} k_n},$$

$$\prod_{n=1}^{N}\Big(L_{k_n}+(2x-1)\,F_{k_n}\Big)=2^N\,x^{\sum\limits_{n=1}^{N}k_n},$$

for every $x\in\{\alpha,\beta\}$, and

$$\prod_{n=1}^{N}\Big(L_{k_n+1}+\frac{\sqrt{5}-1}{2}\,L_{k_n}\Big)=(\sqrt{5})^N\,\Big(\frac{1+\sqrt{5}}{2}\Big)^{\sum\limits_{n=1}^{N}k_n},$$

$$\prod_{n=1}^{N}\Big(L_{k_n+1}-\frac{\sqrt{5}+1}{2}\,L_{k_n}\Big)=(-\sqrt{5})^N\,\Big(\frac{1-\sqrt{5}}{2}\Big)^{\sum\limits_{n=1}^{N}k_n},$$

or in equivalent compact form

$$\prod_{n=1}^{N}\Big(L_{k_n+1}+x^{-1}L_{k_n}\Big)=(2x-1)^N x^{\sum\limits_{n=1}^{N}k_n},$$

for every $x\in\{\alpha,\beta\}$. The above identities are called "Golden Gate" relations.

We note that these identities act as links between Fibonacci and Lucas sequences and many other special sequences of numbers, especially many known linear recurrence sequences. Now we will present the collection of such relations.

First let us consider the Bernoulli numbers B_r defined by the following recursion formula [6, 11]:

$$B_0=1,\quad \binom{n}{n-1}B_{n-1}+\binom{n}{n-2}B_{n-2}+\ldots+\binom{n}{0}B_0=0,\qquad n=2,3,\ldots$$

(we note that $B_{2k+1}=0$, $k=1,2,\ldots$). Moreover, $B_k(y)$ denotes here the k-th Bernoulli polynomial defined by

$$B_k(y)=\sum_{l=0}^{k}\binom{k}{l}B_l\,y^{k-l}.$$

Corollary 2.3 (A bridge between Fibonacci, Lucas and Bernoulli numbers)**.** *We have*

$$\prod_{n=1}^{N-1}\Big(F_{n^k+1}+x^{-1}F_{n^k}\Big)=x^{\int_0^N B_k(y)\,dy},$$

$$\prod_{n=1}^{N-1}\Big(L_{n^k}+(2x-1)\,F_{n^k}\Big)=2^{N-1}\,x^{\int_0^N B_k(y)\,dy}$$

and

$$\prod_{n=1}^{N-1}\Big(L_{n^k+1}+x^{-1}L_{n^k}\Big)=(2x-1)^{N-1}x^{\int_0^N B_k(y)\,dy},$$

for every $x\in\{\alpha,\beta\}$.

Proof. The identities result from the following known relation [6, 11]:

$$\sum_{n=1}^{N-1} n^k = \int_0^N B_k(y)\,dy = \sum_{r=0}^{k} \binom{k}{r} B_r \,\frac{N^{k-r+1}}{k-r+1}. \qquad \square$$

Corollary 2.4 (A bridge between Fibonacci numbers, Lucas numbers and binomial coefficients)**.** *We have*

$$\prod_{k=1}^{\lfloor (n+1)/2\rfloor} \left(F_{\binom{n-k}{k-1}+1} + x^{-1} F_{\binom{n-k}{k-1}}\right) = x^{F_n},$$

$$\prod_{k=1}^{\lfloor (n+1)/2\rfloor} \left(L_{\binom{n-k}{k-1}} \pm \sqrt{5}\, F_{\binom{n-k}{k-1}}\right) = 2^{\lfloor (n+1)/2\rfloor} \left(\frac{1\pm\sqrt{5}}{2}\right)^{F_n},$$

$$\prod_{k=1}^{\lfloor (n+1)/2\rfloor} \left(L_{\binom{n-k}{k-1}+1} + x^{-1} L_{\binom{n-k}{k-1}}\right) = (2x-1)^{\lfloor (n+1)/2\rfloor}\, x^{F_n},$$

for every $x \in \{\alpha, \beta\}$.

Proof. All the above identities follow from relation (see [9]):

$$F_n = \sum_{k=1}^{\lfloor (n+1)/2\rfloor} \binom{n-k}{k-1}. \qquad \square$$

Note that similar and simultaneously more general relations could be obtained for the incomplete Fibonacci and Lucas $p-$numbers (see [12, 13]).

Next corollary concerns the Catalan numbers defined in the following way

$$C_n := \frac{1}{n+1}\binom{2n}{n}, \quad n = 0, 1, \ldots$$

Corollary 2.5 (A bridge between Fibonacci numbers, Lucas numbers and Catalan numbers)**.** *We have*

$$\prod_{n=0}^{N} \left(F_{1+C_{N-n}C_n} + x^{-1} F_{C_{N-n}C_n}\right) = x^{C_{N+1}}, \tag{2.6}$$

$$\prod_{n=0}^{N} \left(L_{C_{N-n}C_n} + (2x-1)\, F_{C_{N-n}C_n}\right) = 2^{N+1} x^{C_{N+1}} \tag{2.7}$$

and

$$\prod_{n=0}^{N} \left(L_{1+C_{N-n}C_n} + x^{-1} L_{C_{N-n}C_n}\right) = (2x-1)^{N+1} x^{C_{N+1}}, \tag{2.8}$$

for every $x \in \{\alpha, \beta\}$.

Moreover, if p is prime and $p \equiv 3\,(mod\,4)$, then we have

$$\sqrt[p]{x^2 F_{1+C_{\frac{p-1}{2}}} + x\, F_{C_{\frac{p-1}{2}}}} = x^{\frac{2+C_{(p-1)/2}}{p}}, \tag{2.9}$$

$$\sqrt[p^2]{\left(F_{1+\frac{1}{2}C_{\frac{p^2-1}{2}}} + x^{-1}F_{\frac{1}{2}C_{\frac{p^2-1}{2}}}\right)\left(F_{1+\binom{p-1}{\frac{p-1}{2}}} + x^{-1}F_{\binom{p-1}{\frac{p-1}{2}}}\right)} = x^{\frac{\frac{1}{2}C_{(p^2-1)/2}+\binom{p-1}{\frac{p-1}{2}}}{p^2}}, \tag{2.10}$$

for every $x \in \{\alpha, \beta\}$.

Proof. Identities (2.6)-(2.8) can be obtained from the recursive relation for C_n

$$C_{N+1} = \sum_{n=0}^{N} C_{N-n}C_n, \quad N = 0, 1, \ldots$$

Whereas relations (2.9) and (2.10) result from the fact that if p is prime and $p \equiv 3\,(mod\,4)$, then $p|(2 + C_{\frac{p-1}{2}})$ and $p^2|\left(\frac{1}{2}C_{\frac{p^2-1}{2}} + \binom{p-1}{\frac{p-1}{2}}\right)$ (see [1]). □

Next conclusion is connected with the Bell numbers B_n, $n = 0, 1, \ldots$ [6].

Corollary 2.6 (A bridge between Fibonacci numbers, Lucas numbers and Bell numbers)**.** *We have*

$$\prod_{n=0}^{N}\left(F_{\binom{N}{n}\mathbf{B}_n+1} + x^{-1}F_{\binom{N}{n}\mathbf{B}_n}\right) = x^{\mathbf{B}_{N+1}},$$

$$\prod_{n=0}^{N}\left(L_{\binom{N}{n}\mathbf{B}_n} + (2x-1)\,F_{\binom{N}{n}\mathbf{B}_n}\right) = 2^{N+1}x^{\mathbf{B}_{N+1}},$$

$$\prod_{n=0}^{N}\left(L_{\binom{N}{n}\mathbf{B}_n+1} + x^{-1}L_{\binom{N}{n}\mathbf{B}_n}\right) = (2x-1)^{N+1}x^{\mathbf{B}_{N+1}},$$

for every $x \in \{\alpha, \beta\}$.

Proof. All the above identities follow from the well known recursive relation

$$\mathbf{B}_0 := 1,$$

$$\mathbf{B}_{N+1} = \sum_{n=0}^{N}\binom{N}{n}\mathbf{B}_n, \quad N = 0, 1, \ldots$$ □

We note that for the Bell numbers the following interesting relation, called Dobinski's formula [6], holds:

$$\mathbf{B}_N = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^N}{k!}, \quad N = 0, 1, 2, \ldots$$

In connection with the above formula we formulate a certain problem which can be expressed in the following way. Is it possible to generalize the definition of Fibonacci numbers F_n onto real indices (of Lucas numbers L_n, respectively) such that the following equality will be fulfilled:

$$\prod_{k=0}^{\infty}\left(F_{1+\frac{e^{-1}k^N}{k!}}+x^{-1}F_{\frac{e^{-1}k^N}{k!}}\right)=x^{B_N},$$

for every $x\in\{\alpha,\beta\}$ and $N\in\mathbb{N}$, or

$$\prod_{k=0}^{\infty}\frac{L_{1+\frac{e^{-1}k^N}{k!}}+x^{-1}L_{\frac{e^{-1}k^N}{k!}}}{2x-1}=x^{B_N},$$

for every $x\in\{\alpha,\beta\}$ and $N\in\mathbb{N}$, respectively?

Next corollary concerns the connection with the δ-Fibonacci numbers defined by relations (see [14]):

$$a_n(\delta)=\sum_{k=0}^{n}\binom{n}{k}F_{k-1}\,(-\delta)^k \tag{2.11}$$

and

$$b_n(\delta)=\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\,F_k\,\delta^k, \tag{2.12}$$

for $\delta\in\mathbb{C}$.

Corollary 2.7 (A bridge between Fibonacci, Lucas and δ-Fibonacci numbers)**.** *For positive integers δ and n we get*

$$\prod_{k=0}^{n}\left(F_{1+\binom{n}{k}F_{k-1}\delta^k}+x^{-1}F_{\binom{n}{k}F_{k-1}\delta^k}\right)=x^{a_n(-\delta)},$$

$$\prod_{k=1}^{n}\left(F_{1+\binom{n}{k}F_k\delta^k}+x^{-1}F_{\binom{n}{k}F_k\delta^k}\right)=x^{-b_n(-\delta)},$$

$$\prod_{k=0}^{n}\left(L_{\binom{n}{k}F_{k-1}\delta^k}+(2x-1)\,F_{\binom{n}{k}F_{k-1}\delta^k}\right)=2^{n+1}\,x^{a_n(-\delta)},$$

$$\prod_{k=1}^{n}\left(L_{\binom{n}{k}F_k\delta^k}\pm\sqrt{5}\,F_{\binom{n}{k}F_k\delta^k}\right)=2^n\,x^{-b_n(-\delta)},$$

$$\prod_{k=0}^{n}\left(L_{1+\binom{n}{k}F_{k-1}\delta^k}+x^{-1}L_{\binom{n}{k}F_{k-1}\delta^k}\right)=(2x-1)^{n+1}x^{a_n(-\delta)},$$

$$\prod_{k=1}^{n}\left(L_{1+\binom{n}{k}F_k\delta^k}+x^{-1}L_{\binom{n}{k}F_k\delta^k}\right)=(2x-1)^{n}x^{-b_n(-\delta)},$$

etc., for every $x\in\{\alpha,\beta\}$. Moreover, we define here $F_{n+1}=F_n+F_{n-1}$, $n\in\mathbb{Z}$.

Let us note that similar relations we have for the incomplete $\delta-$Fibonacci numbers $a_{n,r}(\delta)$ and $b_{n,s}(\delta)$ where

$$a_{n,r}(\delta) := \sum_{k=0}^{r} \binom{n}{k} F_{k-1}(-\delta)^k, \qquad 0 \le r \le n,$$
$$b_{n,s}(\delta) := \sum_{k=1}^{s} \binom{n}{k} (-1)^{k-1} F_k \delta^k, \quad 1 \le s \le n.$$

Now we consider the $r-$generalized Fibonacci sequence $\{G_n\}$ defined as follows

$$G_n = \begin{cases} 0, & \text{if} \quad 0 \le n < r-1, \\ 1, & \text{if} \quad n = r-1, \\ G_{n-1} + G_{n-2} + \ldots + G_{n-r}, & \text{if} \quad n \ge r. \end{cases}$$

Corollary 2.8 (A bridge between Fibonacci, Lucas and classic r-Fibonacci numbers)**.** *Let $r \in \mathbb{N}$, $r \ge 2$. Then the following identities hold true [8]:*

$$\Big(F_{1+2^{r-1}G_{n-r}} + x^{-1} F_{2^{r-1}G_{n-r}}\Big) \prod_{k=1}^{r-1} \Big(F_{1+(\sum_{i=k}^{r-1} 2^{i-1})G_{n-r-k}} + x^{-1} F_{(\sum_{i=k}^{r-1} 2^{i-1})G_{n-r-k}}\Big) = x^{G_n},$$

for every $n \ge 2\,r-1$, and

$$\Big[\prod_{k=0}^{n} (F_{1+G_k^2} + x^{-1} F_{G_k^2})\Big] \times \Big[\prod_{i=2}^{r-1} \prod_{k=0}^{n-i} (F_{1+G_k G_{k+i}} + x^{-1} F_{G_k G_{k+i}})\Big] = x^{G_n G_{n+1}},$$

the special case of which is the following Lucas identity

$$\prod_{k=1}^{n} \Big(F_{1+F_k^2} + x^{-1} F_{F_k^2}\Big) = x^{F_n F_{n+1}},$$

for every $x \in \{\alpha, \beta\}$.

Corollary 2.9. *We have also ($x \in \{\alpha, \beta\}$):*

$$\Big(F_{F_{n+1}+1} + x^{-1} F_{F_{n+1}}\Big) \Big(F_{F_{n-1}+1} + x^{-1} F_{F_{n-1}}\Big) = x^{L_n},$$
$$\Big(L_{F_{n+1}} \pm \sqrt{5} F_{F_{n+1}}\Big)\Big(L_{F_{n-1}} \pm \sqrt{5}\, F_{F_{n-1}}\Big) = 4\Big(\frac{1 \pm \sqrt{5}}{2}\Big)^{L_n},$$
$$\Big(L_{F_{n+1}+1} + x^{-1} L_{F_{n+1}}\Big) \Big(L_{F_{n-1}+1} + x^{-1} L_{F_{n-1}}\Big) = 5\, x^{L_n},$$

since $F_{n+1} + F_{n-1} = L_n$, $n \in \mathbb{N}$. Furthermore, we have

$$\Big(F_{L_{n+1}+1} + x^{-1} F_{L_{n+1}}\Big) \Big(F_{L_{n-1}+1} + x^{-1} F_{L_{n-1}}\Big) = x^{5\,F_n},$$

$$\Big(L_{L_{n+1}} \pm \sqrt{5}\, F_{L_{n+1}}\Big)\Big(L_{L_{n-1}} \pm \sqrt{5}\, F_{L_{n-1}}\Big) = 4\left(\frac{1 \pm \sqrt{5}}{2}\right)^{5\,F_n},$$
$$\Big(L_{L_{n+1}+1} + x^{-1} L_{L_{n+1}}\Big)\Big(L_{L_{n-1}+1} + x^{-1} L_{L_{n-1}+1}\Big) = 5\, x^{5\,F_n},$$

since $L_{n+1} + L_{n-1} = 5\,F_n$, $n \in \mathbb{N}$.

Remark 2.10. Note that Theorem 2.2 is connected, in some way, with the following very important Zeckendorf's theorem [6]:

For every number $n \in \mathbb{N}$ *there exists exactly one increasing sequence* $2 \le k_1 < \ldots < k_r$, *where* $r = r(n) \in \mathbb{N}$, *such that* $k_{i+1} - k_i \ge 2$ *for* $i = 1, 2, \ldots, r-1$, *and*

$$n = F_{k_1} + F_{k_2} + \ldots + F_{k_r}.$$

For example, we have

$$1000 = 987 + 13 = F_{16} + F_7,$$

that is

$$\big(\sqrt{5}\, F_{987} \pm L_{987}\big)\big(\sqrt{5}\, F_{13} \pm L_{13}\big) = 2\, L_{1000} \pm 2\sqrt{5}\, F_{1000} =$$
$$= \big(L_{987} \pm \sqrt{5}\, F_{987}\big)\big(L_{13} \pm \sqrt{5}\, F_{13}\big) = 4\left(\frac{1 \pm \sqrt{5}}{2}\right)^{1000}.$$

3. Final remark

Finally, we note that identities (2.4), considered at the beginning of this paper, were discussed by many authors. For example, S. Alikhani and Y. Peng [2] basing on (2.4) have proven that α^n, for every $n \in \mathbb{N}$, cannot be a root of any chromatic polynomial. Furthermore, D. Gerdemann [5] has used the first of identities (2.4) for analyzing the, so called, Golden Ratio Division Algorithm. Consequently, he has discovered a semi-combinatorial proof of the following beautiful theorem.

Theorem 3.1. *For nonconsecutive integers* $a_1, \ldots, a_k$, *the following two statements are equivalent (for every* $m \in \mathbb{N}$*):*

$$m\, F_n = F_{n+a_1} + F_{n+a_2} + \ldots + F_{n+a_k},$$
$$m = \alpha^{a_1} + \alpha^{a_2} + \ldots + \alpha^{a_k}.$$

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When do the Fibonacci invertible classes modulo M form a subgroup?

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Abstract

In this paper, we look at the invertible classes modulo M representable as Fibonacci numbers and we ask when these classes, say $\mathcal{F}_M$, form a multiplicative group. We show that if M itself is a Fibonacci number, then $M \leq 8$; if M is a Lucas number, then $M \leq 7$. We also show that if $x \geq 3$, the number of $M \leq x$ such that $\mathcal{F}_M$ is a multiplicative subgroup is $O(x/(\log x)^{1/8})$.

Keywords: Fibonacci and Lucas numbers, congruences, multiplicative group

MSC: 11B39

1. Introduction

Let $\{F_k\}_{k\geq 0}$ be the Fibonacci sequence given by $F_0 = 0,\ F_1 = 1$ and

$$F_{k+2} = F_{k+1} + F_k \quad \text{for all} \quad k \geq 0,$$

with the corresponding Lucas companion sequence $\{L_k\}_{k\geq 0}$ satisfying the same recurrence with initial conditions $L_0 = 2$, $L_1{=}1$. The distribution of the Fibonacci numbers modulo some positive integer M has been extensively studied. Here, we put

$$\mathcal{F}_M = \{F_n \pmod{M} : \gcd(F_n, M) = 1\}$$

and ask when is $\mathcal{F}_M$ a multiplicative group. We present the following conjecture.

Conjecture 1.1. *There are only finitely many M such that $\mathcal{F}_M$ is a multiplicative group.*

Shah [5] and Bruckner [1] proved that if p is prime and $\mathcal{F}_p$ is the entire multiplicative group modulo p, then $p \in \{2, 3, 5, 7\}$. We do not know of many results in the literature addressing the multiplicative order of a Fibonacci number with respect to another Fibonacci number, although in [3] it was shown that if $F_n F_{n+1}$ is coprime to F_m and F_{n+1}/F_n has order $s \notin \{1, 2, 4\}$ modulo F_m, then $m < 500s^2$. Moreover, Burr [2] showed that $F_n \pmod m$ contains a complete set of residues modulo m if and only if m is of the forms: $\{1, 2, 4, 6, 7, 14, 3^j\} \cdot 5^k$, where $k \geq 0, j \geq 1$.

In this paper, we prove that if $M = F_m$ is a Fibonacci number itself, or $M = L_m$, then Conjecture 1.1 holds in the following strong form.

Theorem 1.2. *If $M = F_m$ and $\mathcal{F}_M$ is a multiplicative group, then $m \leq 6$. If $M = L_m$ and $\mathcal{F}_M$ is a multiplicative group, then $m \leq 4$.*

We also show that for most positive integers M, $\mathcal{F}_M$ is not a multiplicative group.

Theorem 1.3. *For $x \geq 3$, the number of $M \leq x$ such that $\mathcal{F}_M$ is a multiplicative subgroup is $O(x/(\log x)^{1/8})$. In particular, the set of M such that $\mathcal{F}_M$ is a multiplicative subgroup is of asymptotic density 0.*

2. Proof of Theorem 1.2

We first deal with the case of the Fibonacci numbers. It is well-known that the Fibonacci sequence is purely periodic modulo every positive integer M. When $M = F_m$, then the period is at most $4m$. Thus, $\#\mathcal{F}_M \leq 4m$, Let $\omega(m)$ be the number of distinct prime factors of m. Assume that X is some positive integer such that

$$\pi(X) \geq \omega(m) + 4. \tag{2.1}$$

Here, $\pi(X)$ is the number of primes $p \leq X$. Then there exist three odd primes $p < q < r \leq X$ none of them dividing m. For a triple $(a, b, c) \in \{0, 1, \ldots, \lfloor (4m)^{1/3} \rfloor\}$, we look at the congruence class $F_p^a F_q^b F_r^c \pmod M$. There are $(\lfloor (4m)^{1/3} \rfloor + 1)^3 > 4m \geq \#\mathcal{F}_M$ such elements modulo M, so they cannot be all distinct. Thus, there are $(a_1, b_1, c_1) \neq (a_2, b_2, c_2)$ such that

$$F_p^{a_1} F_q^{b_1} F_r^{c_1} \equiv F_p^{a_2} F_q^{b_2} F_r^{c_2} \pmod M.$$

Hence, $F_p^{a_1 - a_2} F_q^{b_1 - b_2} F_r^{c_1 - c_2} \equiv 1 \pmod M$. Observe that the rational number $x = F_p^{a_1 - a_2} F_q^{b_1 - b_2} F_r^{c_1 - c_2} - 1$ cannot be zero because $F_p,\ F_q,\ F_r$ are all larger than 1 and coprime any two. Thus, M divides the numerator of the nonzero rational number x, and so we get

$$F_m = M \leq F_p^{|a_1 - a_2|} F_q^{|b_1 - b_2|} F_r^{|c_1 - c_2|}. \tag{2.2}$$

We now use the fact that

$$\alpha^{k-2} \le F_k \le \alpha^{k-1} \quad \text{for all} \quad k = 1, 2 \ldots,$$

where $\alpha = (1+\sqrt{5})/2$, to deduce from (2.2) that

$$\alpha^{m-2} \le F_m \le (F_p F_q F_r)^{(4m)^{1/3}} < (\alpha^{X-1})^{3(4m)^{1/3}},$$

so that

$$m < 3(4m)^{1/3}X + 2 - 3(4m)^{1/3} < 3(4m)^{1/3}X,$$

therefore

$$m < 6\sqrt{3}X^{3/2}. \tag{2.3}$$

Let us now get some bounds on m. We take $X = m^{1/2}$. Assuming $X > 17$ (so, $m > 17^2$), we have, by Theorem 2 in [4], that

$$\pi(X) > \frac{X}{\log X} = \frac{2m^{1/2}}{\log m}.$$

Since $2^{\omega(m)} \le m$, we have that

$$\omega(m) \le \frac{\log m}{\log 2}.$$

Thus, inequality (2.1) holds for our instance provided that

$$\frac{2m^{1/2}}{\log m} > \frac{\log m}{\log 2} + 4,$$

which holds for all $m > 5000$. Now inequality (2.3) tells us that

$$m < 6\sqrt{3}m^{3/4}, \quad \text{therefore} \quad m < (6\sqrt{3})^4 < 12000. \tag{2.4}$$

Let us reduce the above bound on m. Since

$$2 \times 3 \times 5 \times 7 \times 11 \times 13 = 30030 > m,$$

it then follows that $\omega(m) \le 5$, therefore it is enough to choose $X = 23$ to be the 9th prime and then inequality (2.1) holds. Thus, (2.3) tells us that $m \le 6\sqrt{3} \times 23^{3/2} < 1200$. We covered the rest of the range with Mathematica. That is, for each $m \in [10, 1200]$, we took the first two odd primes p and q which do not divide m and checked whether for some positive integer $n \le 4m$ both congruences $F_p^n \equiv 1 \pmod{F_m}$ and $F_q^n \equiv 1 \pmod{F_m}$. The only m's that passed this test were $m = 10, 11$. We covered the rest by hand. The only values m that satisfy the hypothesis of the theorem are $m = 1, 2, 3, 4, 5, 6$.

If $M = L_m$, then, the argument is similar to the one above up and we point out the differences only. The period of the Fibonacci numbers modulo a Lucas number

L_m is at most $8m$, and so $\#\mathcal{F}_M \le 8m$. As before, one takes X as in (2.1), and the triple $(a,b,c) \in \{0,1,\ldots,\lfloor 2m^{1/3}\rfloor\}$, implying an inequality as in (2.2), namely

$$L_m = M \le F_p^{|a_1-a_2|} F_q^{|b_1-b_2|} F_r^{|c_1-c_2|}. \tag{2.5}$$

Since for all $k \ge 1$, $\alpha^{k-1} \le L_k \le \alpha^{k+1}$, then

$$\alpha^{m-1} \le L_m \le (F_p F_q F_r)^{2m^{1/3}} \le \alpha^{6(X+1)m^{1/3}},$$

and so, $m < 6m^{1/3}X + 1 + 6m^{1/3} < 13m^{1/3}X$, which implies

$$m < 13^{3/2} X^{3/2}. \tag{2.6}$$

The argument we used before with $X = m^{1/2}$ works here, as well, rendering the bound $m < 13^6 = 4{,}826{,}809$. We can decrease the bound by using the fact that the product of all primes up to 19 is 9,699,690 $>$ 4,826,809, and so, $\omega(m) \le 7$, therefore, it is enough to choose $X = 31$ (the 11th prime) for the inequality (2.1) to hold. We use $X = 31$ in the formula before (2.6) to get $m - 192 \cdot m^{1/3} - 1 < 0$, which implies $m < 14^3 = 2744$ (to see that, label $y := m^{1/3}$ and look at the sign of the polynomial $y^3 - 192y - 1$).

To cover the range from 10 to 2744, we used the same trick as before (which works, since by $F_{2m} = L_m F_m$, then $\gcd(F_p, L_m) = \gcd(F_p, F_{2m}/F_m) | \gcd(F_p, F_{2m}) = F_{\gcd(p,2m)}$). To speed up the computation we used the fact that one can choose one of the primes p, q to be 5, since a Lucas number is never divisible by 5. The only m's that passed the test were $10, 12, 15, 21$, which are easily shown (by displaying the corresponding residues) not to generate a multiplicative group structure. The only values of m, for which we do have a multiplicative groups structure for $\mathcal{F}_M$ when $M = L_m$ are $m \in \{1,2,3,4\}$.

3. Proof of Theorem 1.3

Consider the following set of primes

$$\mathcal{P} = \left\{ p > 5 : \left(\frac{5}{p}\right) = 1,\ \left(\frac{11}{p}\right) = \left(\frac{46}{p}\right) = -1 \right\}.$$

Here, for an integer a and an odd prime p, we use $\left(\frac{a}{p}\right)$ for the Legendre symbol of a with respect to p. Let $\mathcal{M}$ be the set of M such that $\mathcal{F}_M$ is a multiplicative subgroup. We show that M is free of primes from $\mathcal{P}$. Since $\mathcal{P}$ is a set of primes of relative density 1/8 (as a subset of all primes), the conclusion will follow from the Brun sieve (see [6, Chapter I.4, Theorem 3]). To see that M is free of primes from p, observe that since $F_3 = 2$, $F_4 = 3$, and $\mathcal{F}_M$ is a multiplicative subgroup, it follows that there exists n such that $F_n \equiv 6 \pmod{M}$. If $p \mid M$ for some $p \in \mathcal{P}$, it follows that

$$F_n - 6 \equiv 0 \pmod{p}. \tag{3.1}$$

Since $\left(\frac{5}{p}\right) = 1$, it follows that both $\sqrt{5}$ and α are elements of $\mathbb{F}_p$. With the Binet formula, we have

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}.$$

Put $t_n = \alpha^n$, $\varepsilon_n = (-1)^n$. Thus, $\beta^n = (-\alpha^{-1})^n = \varepsilon_n t_n^{-1}$, so congruence (3.1) becomes

$$\frac{t_n - \varepsilon_n t_n^{-1}}{\sqrt{5}} - 6 \equiv 0 \pmod{p}$$

giving

$$t_n^2 - 6\sqrt{5}\, t_n - \varepsilon_n \equiv 0 \pmod{p}.$$

Thus, one of the quadratic equations $t^2 - 6\sqrt{5}\, t \pm 1 = 0$ must have a solution t modulo p. Since the discriminants of the above quadratic equations are $176 = 16 \times 11$ and $184 = 4 \times 46$, respectively, and since neither 11 nor 46 is a quadratic residue modulo p, we get the desired conclusion.

4. Comments

The bound $O(x/(\log x)^{1/8})$ of Theorem 1.3 is too weak to allow one to decide via the Abel summation formula whether

$$\sum_{M \in \mathcal{M}} \frac{1}{M}$$

is finite or not. Of course Conjecture 1.1 would imply that the above sum is finite. We leave it as a problem to the reader to improve the bound on the counting function of $\mathcal{M} \cap [1, x]$ from Theorem 1.3 enough to decide that indeed the sum of the above series is convergent.

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Problem proposals

compiled by Clark Kimberling

These problems were posed by participants of the Fifteenth International Conference on Fibonacci Numbers and Their Applications, Institute of Mathematics and Informatics, Eszterházy Károly College, Eger, Hungary, June 27, 2012.

Problem 1 (posed by Heiko Harborth)**.**
For $F_{13} = 233$ and $F_{18} = 2584$, this holds:

$$\sigma(F_{13}) + \sigma(F_{18}) = 2(F_{13} + F_{18}).$$

Are there further pairs of Fibonacci numbers equalizing their abundance and deficiency?

Problem 2 (posed by Heiko Harborth)**.**
For 5 and 14, this holds: 5 is 14-perfect and 14 is 5-perfect, where n is h-perfect if

$$\sigma(n) + \sigma(nh) = 2(n + hn).$$

Are there further pairs a, b such that a is b-perfect and b is a-perfect?

Problem 3 (posed by Heiko Harborth)**.**
Find numbers n that are h-perfect for more than one value of h, where n is h-perfect if

$$\sigma(n) + \sigma(nh) = 2(n + hn).$$

Examples: 135 is 7-perfect and 55-perfect, and 5 is h-perfect for $h \in \{14, 806, 1166\}$.

Problem 4 (posed by Clark Kimberling)**.**
Let r_n be the greatest eigenvalue of the n^{th} principal submatrix of the Fibonacci self-fusion matrix, M. Let s_n be the greatest eigenvalue of the n^{th} principal submatrix of the Fibonacci self-fission matrix, $\widetilde{M}$. Prove or disprove:

$$\lim_{n\to\infty} \frac{r_{n+1}}{r_n} = \lim_{n\to\infty} \frac{s_{n+1}}{s_n} = \frac{3+\sqrt{5}}{2}$$

(The matrices M and $\widetilde{M}$ are presented in the *Online Encyclopedia of Integer Sequences* at A202453 and A202503.)

Problem 5 (posed by Bill Webb).
A monic polynomial, all of whose coefficients are negative, will be called a negative polynomial. Characterize polynomials that divide some negative polynomial. (For example, every linear polynomial divides a negative polynomial.)

Problem 6 (posed by Joseph Lahr).
Evaluate these sums:

$$\sum_{n=1}^{k} F_{n^2} \quad \text{and} \quad \sum_{n=1}^{k} L_{n^2}.$$

These sums are comparable to $\sum_{n=1}^{k} e^{n^2}$, which occurs in the Fourier transform of chirp-signals, as typifed by the equation $S_n = A\cos(an^2)$.

Problem 7 (posed by Larry Ericksen).
Let $p(n)$ denote the n^{th} prime, and let n_k denote the k^{th} value of n for which $p(n)+2$ is prime. Find all k such that $k(k+1)$ divides $p(n_k)+1$. Example: $k = 8$, $n_8 = 20$, $p(20) = 71$, $p(20)+1 = 8 \cdot 9$. In other words, $k(k+1)$ divides the average of the twin primes $p(n_k)$ and $p(n_k)+2$.

Problem 8 (posed by Larry Ericksen).
Let $p(m)$ denote the m^{th} prime. Find all pairs (m,n) such that reversing the digits of m yields n and reversing the digits of $p(m)$ yields $p(n)$. Example: $m = 12$, $n = 21$, $p(m) = 37$, $p(n) = 73$.

Problem 9 (posed by Lawrence Somer).
Let $ax^2+bxy+cy^2$ be a binary quadratic form with a,b,c integers and discriminant $D = b^2 - 4ac \neq 0$. Suppose that p is a prime such that $p \nmid D$.

(a) Do there exist integers x_0, y_0 such that

$$\left(\frac{ax_0^2 + bx_0y_0 + cy_0^2}{p}\right) = -1,$$

where $\left(\frac{n}{p}\right)$ denotes the Legendre symbol?

(b) Answer (a) with $a = 1$.

(c) Answer (a) with $a = 1$ and $c = \pm 1$.

(d) Answer (a) with $a = 1$ and p such that $\left(\frac{-D}{p}\right) = 1$.

Problem 10 (posed by Neville Robbins).
A *Wilf partition* of n is a partition such that all distinct parts have distinct multiplicities, as in $6 = 4+1+1$. Let $f(n)$ be the number of Wilf partitions of n, as typified by

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$f(n)$	1	1	2	2	4	5	7	10	13	15	21	28	31

and sequence A098859 in the *Online Encyclopedia of Integer Sequences*.

(a) Prove that $f(n)$ is strictly increasing for $n \geq 3$.

(b) Obtain an explicit formula or recurrence for $f(n)$.

Problem 11 (posed by Gabriele Gelatti).
Examples gleaned from visual art suggest that if N is a positive integer, then the product

$$F_{n-4}F_{n-3}F_{n-2}F_{n-1}F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}$$

is equal to a polynomial function of $F_n, F_n^2, \dots, F_n^9$. Following the presentation of this problem, Kristóf Huszár sketched a proof that $F_{n-k}F_{n+k} = F_n^2 + (-1)^{n-k+1}F_k^2$, which implies that

$$F_n \prod_{i=1}^{k} F_{n-i}F_{n+i} = F_n \prod_{i=1}^{k} (F_n^2 + (-1)^{n-i+1}F_k^2),$$

a polynomial in F_n of degree $2k+1$. Subsequently, Bill Webb described a general form of identity, as follows. Let $k = 4t+1$, where $t > 0$ (or, one may also start with $k = 4t$ or $k = 4t+2$ or $k = 4t+3$.) For any given $j_1, j_2, \dots, j_k$, the product

$$F_{n+j_1}F_{n+j_2}\cdots F_{n+j_k}$$

can be written in the form

$$\begin{aligned} &a_1F_{kn} + a_2(-1)^nF_{(k-2)n} + a_3F_{(k-3)n} + \cdots + a_{2t+1}F_n \\ &+ b_1F_{k(n+1)} + b_2(-1)^nF_{(k-2)(n+1)} + b_3F_{(k-3)(n+1)} + \cdots + b_{2t}F_{n+1}. \end{aligned} \tag{1}$$

The values of a_i and b_i are easily calculated as solutions of $k+1$ linear equations. The terms $F_{r(n+1)}$ can be replaced by F_{rn+s_i} or L_{rn+s_i}, and similarly for the terms F_{rn}. It appears likely that the correspondence between $(j_1, j_2, \dots, j_k)$ and the coefficients a_i and b_i includes interesting cases; for example, when is (1) "short"?

Problem 12 (posed by Clark Kimberling, Heiko Harborth, and Peter Moses).
Discuss the triangular arrangements (as indicated by the example below, or of other sorts) of the numbers $1, 2, \dots, n(n+1)/2$ that have interlacing rows; i.e., each term in the first $n-1$ rows is between the two numbers just below it. For $n = 3$:

$$\begin{array}{ccccc} & & 3 & & \\ & 2 & & 5 & \\ 1 & & 4 & & 6 \end{array} \qquad \begin{array}{ccccc} & & 3 & & \\ & 2 & & 5 & \\ 1 & & 4 & & 6 \end{array} \qquad \begin{array}{ccccc} & & 3 & & \\ & 2 & & 5 & \\ 4 & & 1 & & 6 \end{array}$$

$$\begin{array}{ccccc} & & 3 & & \\ & 2 & & 4 & \\ 5 & & 1 & & 6 \end{array} \qquad \begin{array}{ccccc} & & 3 & & \\ & 2 & & 4 & \\ 6 & & 1 & & 5 \end{array}$$

$$\begin{array}{ccccc} & & 4 & & \\ & 2 & & 5 & \\ 1 & & 3 & & 6 \end{array} \qquad \begin{array}{ccccc} & & 4 & & \\ & 2 & & 5 & \\ 1 & & 6 & & 3 \end{array} \qquad \begin{array}{ccccc} & & 4 & & \\ & 2 & & 5 & \\ 3 & & 1 & & 6 \end{array}$$

$$\begin{array}{ccccc} & & 4 & & \\ & 3 & & 5 & \\ 1 & & 6 & & 2 \end{array} \qquad \begin{array}{ccccc} & & 4 & & \\ & 3 & & 5 & \\ 2 & & 6 & & 1 \end{array}$$

Problem 13 (posed by Curtis Cooper).
Find, or prove the nonexistence of, an algebraic identity of the form

$$(r_1x^2 + s_1xy + t_1y^2)^4 + (r_2x^2 + s_2xy + t_2y^2)^4$$
$$= (r_3x^2 + s_3xy + t_3y^2)^4 + (r_4x^2 + s_4xy + t_4y^2)^4 + (r_5x^2 - s_5xy - t_5y^2)^4,$$

where x and y are variables, r_i are positive integers, s_i and t_i are nontrivial integers, $s_5 > 0$, and $t_5 = \pm 1$.

Problem 14 (posed by Augustine Munagi).
Give an explicit bijective proof of the following proposition. The number of compositions of n in which 2 may appear only as a first or last part equals the number of compositions of $n+1$ in which 2 is not a part.

Example: A005251$(n+2)$ is the number of compositions of n having at most two 2s, which may occur only at endpoints; e.g., for $n = 4$, the compositions are (4), $(1,3)$, $(3,1)$, $(1,1,1,1)$, $(2,2)$, $(1,1,2)$, $(2,1,1)$. For the other kind, A005251$(n+1)$ is the number of compositions of n having no 2; e.g., for $n = 5$, the compositions are (5), $(1,4)$, $(4,1)$, $(1,1,3)$, $(1,3,1)$, $(3,1,1)$, $(1,1,1,1)$.