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ANALYSIS OF NONLINEAR TWO DIMENSIONAL LAMINAR NATURAL FLOW AND MIXED CONVECTION OVER VARIABLE SURFACE WITH FREE STREAM CONDITIONS

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Abstract. Chemical reaction, heat and mass transfer over an accelerating surface with heat source and thermal stratification in the presence of suction and injection are studied. The governing partial differential equations of this problem, subjected to their boundary conditions are transformed and solved numerically by applying R.K. Gill method. It has been observed that in the presence of mass diffusion (1) Due to the suction of the accelerating surface the increase of the thermal stratification effect decelerates the fluid motion and increases the temperature distribution and concentration of the fluid along the surface and for injection, it accelerates the fluid motion and decreases the temperature distribution and concentration of the fluid along the accelerating surface (2) Due to the increase of the values of the thermal stratification parameter with constant suction and injection the skin friction and rate of mass transfer decrease and the rate of heat transfer of the fluid increases with an increase of the strength of the chemical reaction.

Mathematical Subject Classification: 76V05
Keywords: chemical reaction, thermal stratification, suction/injection, heat and mass transfer, R. K. Gill method, Boussinesq's approximation

Nomenclature

a	Stretching rate
C	Species concentration in the fluid
C_∞	Species concentration with fluid away from the surface
C_w	Species concentration near the surface
c_p	Specific heat at constant pressure
D	Chemical molecular diffusivity
g	Acceleration due to gravity

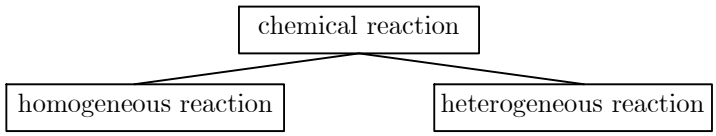
G_{r_x}	Grashof number
G_{c_x}	Modified Grashof number
k	Thermal conductivity
m	Pressure gradient parameter
n	Parameter associated with thermal stratification
P_r	Prandtl number
R_{e_x}	Reynolds number
S_c	Schmidt number
S	Suction or Injection parameter
T	Temperature of the fluid near the wall
T_w	Temperature of the wall
T_∞	Temperature of the fluid far away from the wall
u	Axial velocity
α	Thermal diffusivity
β	Coefficient of volume expansion
β^*	Coefficient of expansion with concentration
ν	Kinematic viscosity
ρ	Density of the fluid
γ	Dimensionless chemical reaction parameter
δ	Non-dimensional longitudinal coordinate

1. **Introduction**

Mixed convection flow occurs frequently in nature. The temperature distribution varies from layer to layer and these types of flows have wide applications in industry, agriculture and oceanography. Further they are especially used in dyeing-industries. One of the most significant types of flow which has many practical applications in industrial manufacturing processes is the boundary layer behavior over a moving continuous solid surface. For example, heat treated materials travelling on a conveyor belt possess the characteristics of a moving continuous surface.

The effect of power law surface temperature and power law surface heat flux in the heat transfer characteristics of a continuous linear stretching surface was investigated by Chen and Char [1]. Processes involving the mass transfer effect have long been recognized as important, principally in chemical processing equipment. Crane [2], Vlegger [3] and Gupta and Gupta [4] analyzed the problem of a stretching surface temperature. Georgantopoulos et. al [5] have studied the effects of free convective and mass transfer in a conducting liquid, when the fluid is subjected to a transverse magnetic field. Heat and mass transfer on hydromagnetic flow over a stretching surface with chemical reaction and thermal stratification effects was analyzed by Kandasamy and Anjali Devi [6]. Recently, Acharya et. al [7] have studied heat and mass transfer on an accelerating surface subjected to both power law surface temperature and power law heat flux variations with a temperature dependent heat source in the presence of suction and blowing. These investigations have a bearing on the tacit assumption that the moving strip is inextensible.

Many practical diffusional operations involve the molecular diffusion of a species in the presence of chemical reaction within or at the boundary. There are two types of reactions.



A homogeneous reaction is one that occurs uniformly throughout a given phase. The species generation in a homogeneous reaction is analogous to internal source of heat generation. The study of heat and mass transfer with chemical reaction is of great practical importance to engineers because of its almost universal occurrence in many branches of science and engineering. In the present work, chemical reaction, heat and mass transfer over an accelerating surface with heat source and thermal stratification in the presence of suction and injection are analysed. The fluid is assumed to be viscous and Boussinesq. The governing partial differential equations of the problem subject to their boundary conditions are solved using efficient R. K. Gill numerical technique. In the absence of chemical reaction, the results are in excellent agreement with that of [7] which elucidates the efficiency of the numerical technique that has been used. Numerical calculations for different values of dimensionless parameters entering the problem under consideration are obtained for the purpose of illustrating the results graphically. Examination of such flow models reveals the influence of chemical reaction field on skin friction, rate of heat and mass transfer profiles. The analysis of the results obtained shows that the flow field is influenced by the presence of chemical reaction.

2. **Mathematical Analysis**

Consider a steady viscous and Boussinesq fluid flowing over an accelerating surface in the presence of a temperature dependent heat source. The problem is considered to be of boundary layer type and two-dimensional. Due to the coordinate system, the x-axis is parallel to the vertical surface and the y-axis is chosen normal to it. The fluid properties are also assumed to be constant in a limited temperature range. The concentration of diffusing species is very small in comparison with other chemical species and hence the species thermal diffusion and diffusion thermal energy effects are neglected and viscous dissipation in the energy equation is negligible. The chemical reactions are taking place in the flow and the physical properties ρ, μ, D and the rate of chemical reaction, k_1 are constant throughout the fluid. Under these conditions, the governing boundary layer equations of momentum, energy and species concentration for mixed convective flow with Boussinesq's approximation are as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + U + g\beta\left(T - T_\infty\right) + g\beta^*\left(C - C_\infty\right) \tag{2.2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}+\left(\beta_1\frac{u}{\rho_{c_p}}\right)(T_\infty-T) \tag{2.3}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D\frac{\partial^2 C}{\partial y^2}-k_1 C \tag{2.4}$$

The boundary conditions are

$$\begin{aligned} u &= ax,\ v=v_0,\ C=C_\infty+A_0x^r,\ T=T_\infty+A_1x^r \text{ at } y=0 \\ u &= 0,\ C=C_\infty,\ T=T_\infty(x)=(1-n)\,T_o+nT_w(x) \text{ as y} \longrightarrow \infty \end{aligned} \tag{2.5}$$

where r is the temperature parameter. For $r=0$, thermal boundary conditions become isothermal and n is constant, such that $0\le n<1$. The n defined as above is called thermal stratification parameter and it is equal to $\frac{m_1}{(1+m_1)}$ of [8, 9] where m_1 is a constant. T_o is constant reference temperature, say $T_\infty(0)$. The suffixes w and α denote surface and ambient conditions.

As in [7] we introduce the following change of variables

$$\Psi(x,y)=(\nu a)^{1/2}\,xf(\eta) \tag{2.6}$$

$$\eta(x,y)=y\left(\frac{a}{\nu}\right)^{1/2}. \tag{2.7}$$

The velocity components are given by

$$u=\frac{\partial\Psi}{\partial y},\ v=-\frac{\partial\Psi}{\partial x}. \tag{2.8}$$

It can be easily verified that the continuity equation (2.1) is identically satisfied and we set $C_\infty=0$ and introduce the non-dimensional form of temperature and the concentration as

$$\theta=\frac{T-T_\infty}{T_w-T_\infty} \tag{2.9}$$

$$\phi=\frac{C-C_\infty}{C_w-C_\infty},\qquad \delta=\frac{\beta_1 x}{\rho_{c_p}} \tag{2.10}$$

$$R_{e_x}=\frac{Ux}{\nu}\qquad \text{(Reynolds number)} \tag{2.11}$$

$$G_{r_x}=\frac{\nu g\beta\,(T_w-T_\infty)}{U^3}\qquad \text{(Grashof number)} \tag{2.12}$$

$$G_{c_x}=\frac{\nu g\beta^*\,(C_w-C_\infty)}{U^3}\qquad \text{(Modified Grashof number)} \tag{2.13}$$

$$P_r=\frac{\mu c_p}{k}\qquad \text{(Prandtl number)} \tag{2.14}$$

$$S_c=\frac{\nu}{D}\qquad \text{(Schmidt number)} \tag{2.15}$$

$$\gamma=\frac{\nu k_1}{U^2}\qquad \text{(Chemical reaction parameter)} \tag{2.16}$$

$$S=-v_0\,(\nu a)^{-1/2}\qquad \text{(suction/injection parameter)}$$

where m is a positive integer and U is the reference velocity.

Equations (2.2), (2.3) and (2.4) become

$$f''' + G_{c_x}\phi + G_{r_x}R_{e_x}\theta + ff'' - (f')^2 = 0 \tag{2.17}$$

$$\theta'' - P_r\left(\frac{n}{1-n}\right)f'r - P_r\theta\,(r+\delta)\,f' + P_r f\theta' = 0 \tag{2.18}$$

$$\phi'' - S_c f'\phi\,(r_h + \gamma R_{e_x}) + S_c f\phi' = 0 \tag{2.19}$$

with boundary conditions

$$\begin{aligned} f(0) &= -v_o\,(\nu a)^{-1/2} = S,\ f'(0) = 1,\ \phi(0) = 1,\ \theta(0) = 1 \\ f'(\alpha) &= 0,\ \phi(\alpha) = 0,\ \theta(\alpha) = 0 \end{aligned} \tag{2.20}$$

where v_0 is the velocity of suction if $v_0 < 0$ and injection if $v_0 > 0$.

Equations (2.17) to (2.19) with boundary conditions (2.20) are integrated using Runge-Kutta Gill method. Heat and mass transfer of the fluid are studied for different values of chemical reaction and thermal stratification effects. In the following section, we discuss the results in detail.

3. **Results and discussions**

In order to get a clear insight into the physical problem, numerical results are displayed with the help of graphical illustrations.

In the absence of chemical reaction, the results have been compared with those of a previous work [7] and it is found that they are in good agreement. The obtained numerical results are illustrated by means of Figures 1-6.

The effect of thermal stratification over the velocity, temperature distribution and concentration of the fluid are elucidated with the help of Figures 1, 2 and 3.

Figure 1 depicts the dimensionless velocity profiles $f'(\eta)$ for different values of thermal stratification effect with suction ($S > 0$) and injection ($S < 0$) respectively. Due to the suction of the accelerating surface, it is observed that the component of the velocity of the fluid along the surface decreases with increase of the thermal stratification effect and for injection, the velocity of the fluid along the surface increases with increase of the thermal stratification effect. On the other hand, due to the suction of the accelerating surface, the dimensionless temperature $\theta(\eta)$ and concentration $\phi(\eta)$ of the fluid increase and for injection, the temperature and concentration of the fluid decrease with increase of the thermal stratification effect and these are shown in Figures 2 and 3, respectively. So, for suction, the increase of the thermal stratification effect decelerates the fluid motion and increases the temperature distribution and concentration of the fluid along the surface. On the other hand, in the case of injection, the increase of the thermal stratification effect accelerates the fluid motion and decreases the temperature distribution and concentration of the fluid along the accelerating surface.

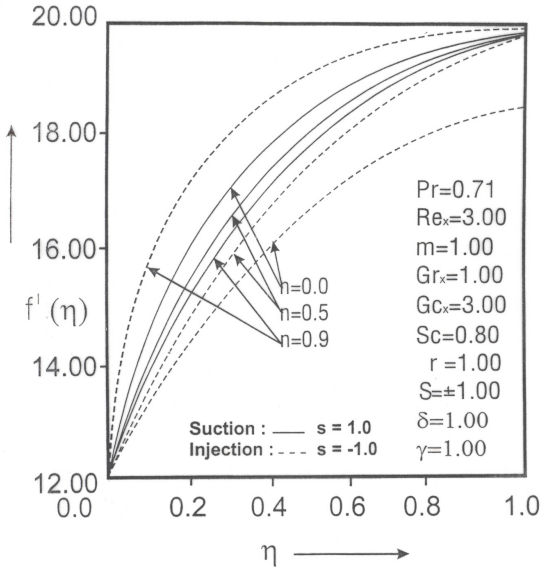


Figure 1. Effect of thermal stratification over the velocity profiles

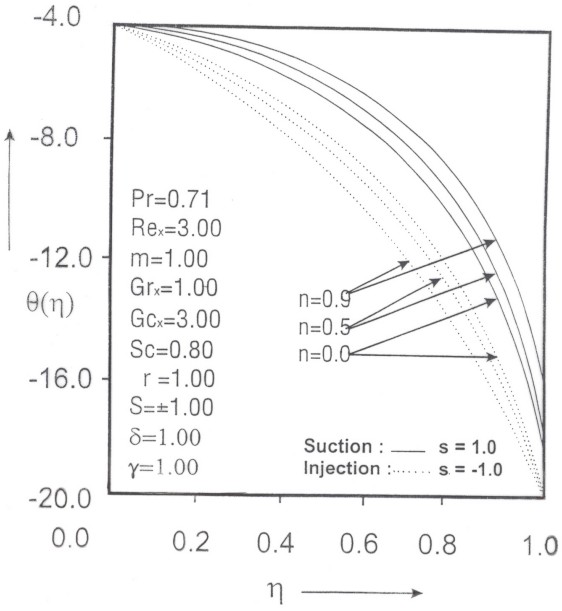


Figure 2. Influence of thermal stratification over the temperature profiles

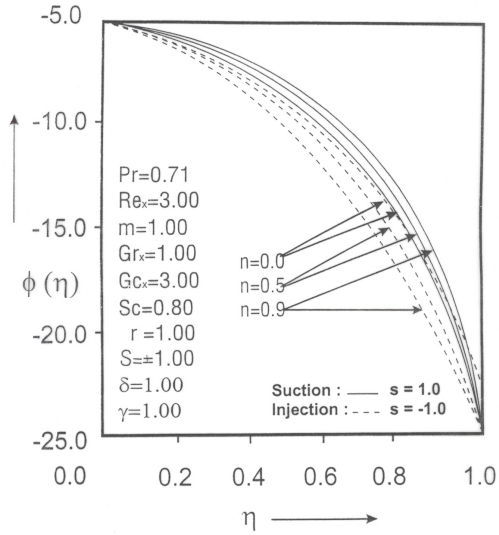


Figure 3. Effect of thermal stratification over the concentration profiles

The influence of chemical reaction over the skin friction, rate of heat and mass transfer are shown in Figures 4, 5 and 7.

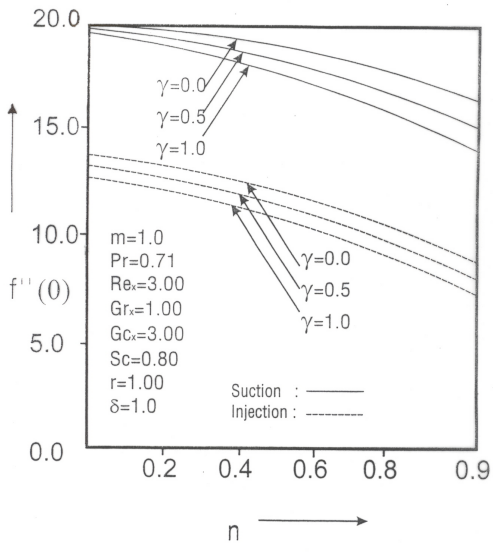


Figure 4. Influence of chemical reaction over the skin friction profiles

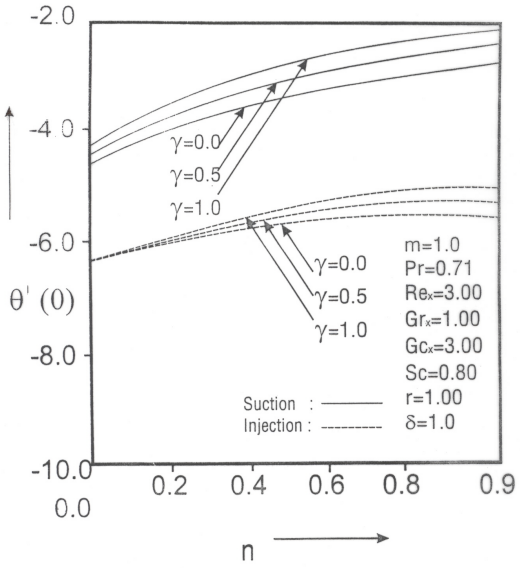


Figure 5. Effect of chemical reaction over the rate of heat transfer profiles

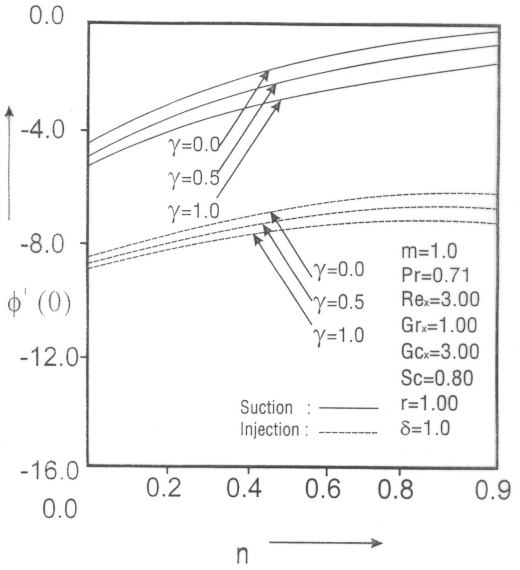


Figure 6. Influence of chemical reaction over the rate of mass transfer profiles

Figure 4 depicts the dimensionless skin friction profiles $f''(0)$ for different values of chemical reaction parameter $\gamma = 0.00$, 1.00 and 2.00. Due to the increase of thermal stratification parameter $0 \leq n < 1$ for both the cases of suction and injection, it is observed that the skin friction of the fluid decreases with increase of chemical reaction parameter γ.

Figure 5 represents the dimensionless rate of heat transfer profiles $\theta(0)$ for different values of chemical reaction parameter $\gamma = 0.00$, 1.00 and 2.00. Due to the increase of thermal stratification parameter $0 \leq n < 1$ for both the cases of suction and injection, it is clear that the rate of heat transfer of the fluid increases with increase of chemical reaction parameter γ.

Figure 6 stands for the dimensionless rate of mass transfer profiles $\phi'(0)$ for different values of chemical reaction parameter $\gamma = 0.00$, 1.00 and 2.00. Due to the increase of thermal stratification parameter $0 \leq n < 1$ for both the cases of suction and injection, it is observed that the rate of mass transfer of the fluid decreases with increase of chemical reaction parameter γ.

4. **Conclusion**

In the absence of chemical reaction, in general the results are identical to those of Acharya et.al [7].

We conclude the following from the previous results and discussions:

Due to the suction of the accelerating surface, the increase of the thermal stratification effect decelerates the fluid motion and increases the temperature distribution and concentration of the fluid along the surface and for injection, it accelerates the fluid motion and decreases the temperature distribution and concentration of the fluid along the accelerating surface.

Due to the increase of the strength of thermal stratification with constant suction and injection, the increase of the effect of chemical reaction decelerates the skin friction and rate of mass transfer of the fluid and accelerates the rate of heat transfer of the fluid and accelerates the rate of heat transfer of the fluid along the surface.

It is hoped that the present results can be used for understanding more complex two dimensional problems involving the flow of electrically conducting fluids.

Acknowledgement. One of the authors (R. K) wishes to thank Dr. R. Vadivel, Principal and Dr. K. Periyasamy, Assistant Professor of Chemistry, IRTT, Erode, for their encouragement.

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OUTLINES OF THE THEORY OF LARGE PERTURBATIONS OF LINEAR SYSTEMS WITH FINITE DEGREES OF FREEDOM

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Abstract. The aim of this paper is to give closed, exact and general formulae for the eigenvalues and eigenvectors of a (in an arbitrarily large measure) perturbated conservative linear oscillator, which are also valid for the case of invariant (persistent) and degenerated (or having both properties) eigenvalues. Such formulae are established and their properties are investigated. Some special inverse eigenvalue problems are presented and solved using the formulae established; in this manner a so-called deflation can also be solved. We show that the Lancaster formula [11] (not systematically derived there) is a special case of our results. We embed these results in some relevant problems of the literature.

Mathematical Subject Classification: 70J15, 34A30, 34A55
Keywords: linear oscillator, finite degrees of freedom, large perturbation, inverse eigenvalue problem

1. Introduction. Formulation of the problem

The most simple form of the problem raised in the title can be formulated as follows. Let us consider the differential equation

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}_1\mathbf{x} = 0$$

where $\mathbf{M}$ and $\mathbf{C}_1$ are symmetric, constant and real matrices of order n, $\mathbf{M}$ is positive definite, $\mathbf{C}_1$ is positive definite or semidefinite, $\mathbf{x}$ is an unknown column matrix of order n, differentiation with respect to time is denoted by dotted letters, accordingly $\ddot{\mathbf{x}}$ is the second time derivative of $\mathbf{x}$. From physical point of view this equation can describe the oscillations of linear mechanical, electrical or electromechanical systems. In the first case $\mathbf{m}$ is the mass (inertia-) matrix and $\mathbf{C}_1$ is the matrix of restoring forces (moments); the name spring matrix is also usual. Looking for a particular solution of the form

$$\mathbf{x} = \mathbf{x}_o \sin \alpha t$$

where $\mathbf{x}_o$ and α are unknown constants, we obtain the so-called original (or starting) eigenvalue problem

$$\left(-\alpha^2\mathbf{M} + \mathbf{C}_1\right)\mathbf{x}_o = 0\,. \tag{1.1}$$

Designating its solutions by

$$\alpha_1^2 \leq \alpha_2^2 \leq \ldots \alpha_n^2 \leq ; \qquad \mathbf{x}_{o1}, \mathbf{x}_{o2} \ldots \mathbf{x}_{on} \tag{1.2a}$$

and considering them as known quantities the most simple form of the problem raised in the title reads: writing

$$\mathbf{M} + \Delta\mathbf{M} \quad \text{for} \quad \mathbf{M} \quad \text{and} \quad \mathbf{C}_1 + \Delta\mathbf{C}_1 \quad \text{for} \quad \mathbf{C}_1$$

in equation (1.1) – here $\Delta\mathbf{M}$ and $\Delta\mathbf{C}_1$ are real symmetric and constant matrices such that $\mathbf{M} + \Delta\mathbf{M}$ is positive definite and $\mathbf{C}_1 + \Delta\mathbf{C}_1$ positive definite is or semidefinite – how the solutions of the eigenvalue problem perturbated according to the above, i.e., the eigenvalues and eigenvectors

$$\alpha_1'^2 \leq \alpha_2'^2 \leq \ldots \alpha_n'^2 ; \qquad \mathbf{x}'_{o1}, \mathbf{x}'_{o2} \ldots \mathbf{x}'_{on} \tag{1.2b}$$

can be found by making use of the solutions (1.2a) of the original one.

In the following this most simple case of our problem will be examined.

The relevant literature gave in the beginning such approximate formulas for

$$\alpha_i'^2 \qquad \text{and} \qquad \mathbf{x}'_{oi}\,, \quad i = 1, 2, ..., n$$

which were the more exact, the smaller was some measure of $\Delta\mathbf{M}$ and $\Delta\mathbf{C}_1$. Up to the time of its publication, Patkó [1] summarizes the most important relevant results from the point of view of the practice, it generalizes them also; for the generalization the same assertion is valid, which was mentioned in our preceding sentence.

Probably the paper [2] by Fraeijs de Veubeke is the first in which the idea taking $\Delta\mathbf{M}$ and $\Delta\mathbf{C}_1$ in a special diadic form is published and exact equalities are found for the quantities sought, therefore it is not necessary to limit some measure of $\Delta\mathbf{M}_1$ and $\Delta\mathbf{C}_1$ in the interest of a prescribed accuracy.

The significance of dyads are emphasized more and more, e.g. especially by Egerváry [3], Rózsa [4], Tevan [5], Zurmühl and Falk [6], Housholder [7]. Additional material can be found from Bazley's thoughts [8] in connection with the theory of perturbation of linear operators and from Fichera's works e.g. [9].

Hereinafter starting from the works cited above – generalizing them to a certain extent - we try to give first a solution for the above quoted most simple form, then we also intend to show some other applications and certain generalizations.

As regards the perturbated problem, a practical approach is to produce its solution numerically with a computer. However we would like to emphasize: in this paper we intend to express explicitly the eigenelements (eigenvalues and eigenvectors) of the modified (perturbated) problem by those of the original (starting) one. The method followed in this way can facilitate the solution of certain inverse eigenvalue problems and can give a comprehensive insight into the theoretical background of these problems, stimulating the raising perhaps of new thoughts in this way.

2. **Solution to the most simple problem**

Without violating generality it is sufficient to investigate instead of equation (1.1) the problem

$$\left(-\alpha^2\mathbf{I}+\mathbf{C}\right)\mathbf{y}_o=\mathbf{0}, \tag{2.1}$$

where $\mathbf{I}$ is the unit matrix of order n, nevertheless $\mathbf{C}=\left(\mathbf{R}^T\right)^{-1}\mathbf{C}_1\mathbf{R}^{-1}$, where $\mathbf{R}$ is the upper triangle matrix in the Cholesky-decomposition $\mathbf{M}=\mathbf{R}^T\mathbf{R}$ of the positive definite matrix $\mathbf{M}$. If the positive definite matrix $\mathbf{M}$ is diagonal, $\mathbf{C}=\mathbf{M}^{-1/2}\mathbf{C}_1\mathbf{M}^{-1/2}$, then the properties of definiteness of $\mathbf{C}$ are the same as those of $\mathbf{C}_1$.

In the following we regard equation (2.1) as our point of departure: equation (2.1) will be considered to be the original problem and its solutions will be designated by

$$\alpha_1^2\le\alpha_2^2\le\ldots\le\alpha_n^2 \qquad \text{and} \qquad \mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_n\,. \tag{2.2}$$

It is well known that the eigenvectors $\mathbf{y}_i$ $(i=1,2,...,n)$ are orthogonal to each other and can be normed in such a way that

$$\mathbf{y}_i^T\mathbf{y}_j=\delta_{ij}; \qquad i,j=1,2,...,n. \tag{2.3a}$$

(If we work instead of $\mathbf{I}$ with the general inertia matrix $\mathbf{M}$, the norming

$$\mathbf{y}_i^T\mathbf{M}\mathbf{y}_i=\delta_{ij}; \qquad i,j=1,2,...,n \tag{2.3b}$$

must be applied to maintain the validity of the subsequent formulae.) As is known, the eigenvalue problem (2.1) can also have degenerated (multiple) eigenvalues. If α_j^2 is an eigenvalue with the multiplicity s_j, then s_j linearly independent eigenvector(s) $\mathbf{y}_j$, $j=1,2,...,s_j$ belong to α_j^2. These can be chosen such that the complete system of eigenvectors is orthonormal.

Let $\mathbf{v}_i$ $(i=1,2,...,n)$ be real, linearly independent column matrices of order n. Later on it will turn out that the most suitable forms for $\Delta\mathbf{I}$ (written in place of $\Delta\mathbf{M}$) and $\Delta\mathbf{C}$ are sums of dyads:

$$\Delta\mathbf{I}=\sum_{i=1}^{n}a_i\mathbf{v}_i\mathbf{v}_i^T \qquad \text{and} \qquad \Delta\mathbf{C}=\sum_{i=1}^{n}b_i\mathbf{v}_i\mathbf{v}_i^T\,. \tag{2.4}$$

Here a_i and b_i $(i=1,2,...,n)$ are real constants. (By this assumption the symmetry of the modification is ensured.)

We shall make some comments on the physical background of equation (1.1) (more precisely of equation (2.1) or (2.4)):

(i) If $\mathbf{C}$ is diagonal, furthermore $\Delta\mathbf{C}$ is non-diagonal, but $\Delta\mathbf{I}$ is diagonal, then perturbation (2.4) can be interpreted in such a way that the original system composed of isolated oscillators (each with one degree of freedom) was rendered through the perturbation elastically coupled. If $\Delta\mathbf{C}$ is tridiagonal (keeping in mind that now $\Delta\mathbf{I}$ is diagonal), the result of the "coupling" is a longitudinal chainlike oscillator which consists of pointlike masses of which only the immediate neighboring mass points are connected by (massless) springs.

It is possible that the massless spring connects (springs connect) might involve springs attached to a fixed point.

(ii) If $\mathbf{C}$ and $\Delta\mathbf{C}$ are diagonal and $\Delta\mathbf{I}$ is non-diagonal, then according to equation (2.4) the perturbation can be interpreted in such a way that the original oscillator consisting of a number of n isolated oscillators with one degree of freedom is made by the perturbation to be inertia-coupled.

(iii) Let $\mathbf{C}$ and $\Delta\mathbf{C}$ be tridiagonal. Further let $\Delta\mathbf{I}$ be diagonal. Then the original chainlike structure remains of the same structure. Also the second sentence of item (i) is valid here.

3. Solution of problem (2.1) perturbated according to (2.4)

Consequently, we seek the solution of equation

$$\left[-\alpha^2\left(\mathbf{I}+\sum_{i=1}^{n}a_i\mathbf{v}_i\mathbf{v}_i^T\right)+\left(\mathbf{C}+\sum_{i=1}^{n}b_i\mathbf{v}_i\mathbf{v}_i^T\right)\right]\mathbf{y}_o=\mathbf{0} \tag{3.1}$$

for α^2 and $\mathbf{y}_o$ if the original eigenelements are known. (For the sake of simplicity we have applied the notation $\mathbf{y}_o$ in the same way here as in equation (2.1) though their meanings are different.)

The homogeneous problem (3.1) can be written in an inhomogeneous form, and both sides can be multiplied by $\sin\alpha t$; referring to the fact, that after multiplication the problem can be considered to be one in connection with steady forced vibrations. This inhomogeneous form is

$$\left(-\alpha^2\mathbf{I}+\mathbf{C}\right)\mathbf{y}_o\sin\alpha t=\mathbf{f}_o\sin\alpha t \tag{3.2}$$

where

$$\mathbf{f}_o=\left(\alpha^2\sum_{i=1}^{n}a_i\mathbf{v}_i\mathbf{v}_i^T-\sum_{i=1}^{n}b_i\mathbf{v}_i\mathbf{v}_i^T\right)\mathbf{y}_o. \tag{3.3}$$

Introducing diagonal matrices $\langle a_1,a_2,...,a_n\rangle$ $\langle b_1,b_2,...,b_n\rangle$ and using the identity valid for any vectors $\mathbf{u}_i,\mathbf{v}_i$ and scalars c_i:

$$[\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_n]\,\langle c_1,c_2,...,c_n\rangle\begin{bmatrix}\mathbf{v}_1^T\\ \mathbf{v}_2^T\\ \vdots\\ \mathbf{v}_n^T\end{bmatrix}\equiv\sum_{k=1}^{n}c_k\mathbf{v}_k\mathbf{v}_k^T \tag{3.4}$$

we get from (3.3) that

$$\mathbf{f}_o=\alpha^2\,[\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_n]\,\langle a_1,a_2,...,a_n\rangle\begin{bmatrix}\mathbf{v}_1^T\\ \mathbf{v}_2^T\\ \vdots\\ \mathbf{v}_n^T\end{bmatrix}\mathbf{y}_o-$$

$$- [\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n] \langle b_1, b_2, ..., b_n \rangle \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} \mathbf{y}_o =$$

$$= [\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n] \left(\alpha^2 \langle a_1, a_2, ..., a_n \rangle - \langle b_1, b_2, ..., b_n \rangle \right) \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} \mathbf{y}_o \quad (3.5)$$

As is well known when solving (3.2) for $\mathbf{y}_o$ one has to distinguish two cases.

Characteristic for the case denoted by (a) is that $\alpha^2 \neq \alpha_i^2$, $i = 1, 2, ..., n$, i.e., none of the wanted perturbated eigenvalues coincides with an eigenvalue of the original system, that is perturbation does not permit any of the original system's eigenvalues to be invariant. Usually this case is characterized also by the designation: there is (are) no persistent eigenvalue(s).

Characteristic for the case denoted by (b) is that, after the perturbation at least one - let us say the j^{th}- original eigenvalue remains invariant, i.e. $\alpha^2 = \alpha_j^2$. Usually in this case we speak about persistent eigenvalue(s). For the sake of generality we shall assume that α_j^2 is degenerated, its multiplicity is s_j and the linearly independent eigenvectors $\mathbf{y}_j$, $(l = 1, 2, ..., s_j)$ belong to it.

In case (a) solution of equation (3.5) assumes the form (omitting the common factor $\sin \alpha t$):

$$\mathbf{y}_o = \sum_{k=1}^{n} \frac{\mathbf{y}_k^T \mathbf{f}_o}{\alpha_k^2 - \alpha^2} \mathbf{y}_k \ . \quad (3.6a)$$

In case (b) however, the solution is (substituting, in accordance with our assumption, α_j instead of α):

$$\mathbf{y}_o = \sum_{k=1}^{n}{}' \frac{\mathbf{y}_k^T \mathbf{f}_o}{\alpha_k^2 - \alpha_j^2} \mathbf{y}_k + \sum_{l=1}^{s_j} d_l \mathbf{y}_{j_l} \ , \quad (3.6b)$$

where the constants d_l $(l = 1, 2, ..., s_j)$ can be calculated from additional reasonings; the comma on the first $\sum$ means that from this summing all quantities (in a number s_j) which belong to the invariant (and degenerated) α_j^2 must be left out.

For case (b) solution exists if and only if

$$\mathbf{y}_{j_l}^T \cdot \mathbf{f}_o = 0, \qquad l = 1, 2, ..., s_j \ . \quad (3.6c)$$

First case (a) will be examined. For this reason we substitute expression (3.5) for $\mathbf{f}_o$ in equation (3.6a) and get:

$$\mathbf{y}_o = \sum_{k=1}^{n} \frac{\mathbf{y}_k^T}{\alpha_k^2 - \alpha^2} \left\{ [\mathbf{v}_1, ..., \mathbf{v}_n] \left(\alpha^2 \langle a_1, ..., a_n \rangle - \langle b_1, ..., b_n \rangle \right) \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} \right\} \mathbf{y}_o \mathbf{y}_k$$

in which the coefficient of the last $\mathbf{y}_k$ on the right side is a scalar. Consequently, we can write

$$\mathbf{y}_o = \sum_{k=1}^{n} \frac{\mathbf{y}_k \mathbf{y}_k^T}{\alpha_k^2 - \alpha^2} \left\{ [\mathbf{v}_1, ..., \mathbf{v}_n] \left(\alpha^2 \langle a_1, ..., a_n \rangle - \langle b_1, ..., b_n \rangle \right) \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} \right\} \mathbf{y}_o \,. \tag{3.7}$$

Equation (3.7) is a homogeneous linear system of equations for the perturbated eigenelements. Since we are interested only in solutions different from zero, the necessary and sufficient condition for this is the disappearance of the determinant:

$$\left| \mathbf{I} - \sum_{k=1}^{n} \frac{\mathbf{y}_k \mathbf{y}_k^T}{\alpha_k^2 - \alpha^2} \left\{ [\mathbf{v}_1, ..., \mathbf{v}_n] \left(\alpha^2 \langle a_1, ..., a_n \rangle - \langle b_1, ..., b_n \rangle \right) \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} \right\} \right| = 0 \,. \tag{3.8}$$

With the knowledge of $\mathbf{y}_k$, α_k, $\mathbf{v}_k$, and b_k $(k = 1, 2, ..., n)$ only α^2 is unknown. The values of α^2, i.e., the perturbated eigenvalues can be computed by making use of a numerical method. Determination of the corresponding eigenvectors $\mathbf{y}_o$ requires the solution of equations (3.7).

We can see that no matter how "large" perturbations we have, in principle we can work with accurate equations.

To apply those general reasonings which are the main subject of this paper, determinant (3.8) should be transformed into a more auspicious form. This transformation is based on the following statement: if the system of equation $\mathbf{A}(\alpha)\,\mathbf{x} = \mathbf{0}$ with quadratical coefficient matrix is multiplied from the left by a matrix $\mathbf{V}$, independent of α, and with non-vanishing determinant, then from equation $\mathbf{VA}(\alpha)\,\mathbf{x} = \mathbf{0}$ regarded as a condition for $\mathbf{x} \neq \mathbf{0}$ we find the same values of α, when working with the original equation.

Consequently let us multiply equation (3.7) reduced to zero from the left by the matrix

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} ,$$

the determinant of which does not depend on α, and does not vanish, because - according to our starting point - $\mathbf{v}_1, ..., \mathbf{v}_n$ are linearly independent. Hence the product

in question is:

$$\left[\begin{array}{c}\mathbf{v}_1^T\\ \vdots\\ \mathbf{v}_n^T\end{array}\right]\left[\mathbf{I}-\sum_{k=1}^{n}\frac{\mathbf{y}_k\mathbf{y}_k^T}{\alpha_k^2-\alpha^2}\left\{[\mathbf{v}_1,...,\mathbf{v}_n]\left(\alpha^2\langle a_1,...,a_n\rangle-\langle b_1,...,b_n\rangle\right)\left[\begin{array}{c}\mathbf{v}_1^T\\ \vdots\\ \mathbf{v}_n^T\end{array}\right]\right\}\right]\mathbf{y}_o=$$

$$=\left[\mathbf{I}-\sum_{k=1}^{n}\frac{1}{\alpha_k^2-\alpha^2}\left\{\left[\mathbf{v}_1^T\mathbf{v}_1,...,\mathbf{v}_1^T\mathbf{v}_n\right]\cdot\left\langle\alpha^2a_1-b_1,...,\alpha^2a_n-b_n\right\rangle\left[\begin{array}{c}\mathbf{v}_1^T\\ \vdots\\ \mathbf{v}_n^T\end{array}\right]\right\}\right]\mathbf{y}_o=\mathbf{0}\,. \tag{3.9}$$

Stating the above equation we used the fact that the unit matrix is commutable with an arbitrary matrix; furthermore the rule of associativity. Owing to the transformations we can switch over instead of determinant (3.8) to the determinant following from equation (3.9)

$$\left|\mathbf{I}-\sum_{k=1}^{n}\frac{1}{\alpha_k^2-\alpha^2}\left[\begin{array}{c}\mathbf{v}_1^T\mathbf{y}_k\\ \vdots\\ \mathbf{v}_n^T\mathbf{y}_k\end{array}\right]\left\{\left[\mathbf{y}_k^T\mathbf{v}_1,...,\mathbf{y}_k^T\mathbf{v}_n\right]\cdot\left\langle\alpha^2a_1-b_1,...,\alpha^2a_n-b_n\right\rangle\right\}\right|\,. \tag{3.10}$$

Knowing $\mathbf{y}_k$, α_k, $\mathbf{v}_k$, a_k, b_k $(k=1,2,...,n)$ this latter determinant - which must be equal to zero - also makes possible the calculation of the perturbated eigenvalues no matter how "large" the perturbations are. Equation (3.8) has yet the favourable property that by choosing

$$\mathbf{v}_k=\mathbf{y}_k,\qquad k=1,2,...,n^1 \tag{3.11}$$

we get due to equation (2.3a) the following form:

$$\left|\mathbf{I}-\sum_{k=1}^{n}\frac{1}{\alpha_k^2-\alpha^2}\left[\begin{array}{c}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{array}\right]\begin{array}{l}(1\\ \\ \\ (k\\ \\ \\ (n\end{array}\left[\overset{1}{\underset{\smile}{0}}\ \ldots\ 0\ \overset{k}{\underset{\smile}{1}}\ 0\ \ldots\ \overset{n}{\underset{\smile}{0}}\right]\left\langle\alpha^2a_1-b_1,...,\alpha^2a_n-b_n\right\rangle\right|=$$

$$=\left|\mathbf{I}-\sum_{k=1}^{n}\frac{1}{\alpha_k^2-\alpha^2}\left\langle 0\ \ldots\ 0\ \overset{k}{\underset{\smile}{1}}\ 0\ \ldots\ 0\right\rangle\left\langle\alpha^2a_1-b_1,...,\alpha^2a_n-b_n\right\rangle\right|=$$

[1] In place of equation (3.11) - if we work not with $\mathbf{I}$, but with the "general" inertia matrix $\mathbf{M}$ - it follows that $\mathbf{v}_k=\mathbf{M}\mathbf{y}_k$, and equation (2.3a) should be replaced by (2.3b); the further reasonings are valid in unaltered form.

$$= \left| \mathbf{I} - \sum_{k=1}^{n} \frac{1}{\alpha_k^2 - \alpha^2} \left\langle 0, \ldots, 0, \overset{k}{\breve{\alpha^2}} a_k, 0, \ldots, 0 \right\rangle \right| . \tag{3.12}$$

In what follows when seeking the solution of equation (3.2) we shall investigate case (a) only.

Let us change over to case (b), in which to sum up - let us say that α_j^2 with the multiplicity s_j remains invariant (persistent). Our following formulas apply to this α_j.

Let us substitute back the expression (3.5) for $\mathbf{f}_o$ into (3.6b) and (3.6c) and according to the assumption of invariance α_j^2 in place of α^2; the result is:

$$\mathbf{y}_o = \sum_{k=1}^{n}{}' \left\{ \frac{\mathbf{y}_k \mathbf{y}_k^T}{\alpha_k^2 - \alpha_j^2} \left[\mathbf{v}_1, ..., \mathbf{v}_n \right] \left\langle \alpha_j^2 a_1, -b_1, ..., \alpha_j^2 a_n - b_n \right\rangle \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} \right\} \mathbf{y}_o + \sum_{l=1}^{s_j} d_l \, \mathbf{y}_{j_l} , \tag{3.13a}$$

$$\mathbf{y}_{j_l}^T \left\{ \left[\mathbf{v}_1, ..., \mathbf{v}_n \right] \left\langle \alpha_j^2 a_1, -b_1, ..., \alpha_j^2 a_n - b_n \right\rangle \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} \right\} \mathbf{y}_o = 0, \quad l = 1, 2, ..., s_j. \tag{3.13b}$$

The homogeneous linear system of equations (3.13a) and (3.13b) can be recast in the partitioned system of equation introducing the new unknown vector $\mathbf{u}^T = [\mathbf{y}_o^T, d_1, ..., d_{s_j}] = [\mathbf{y}_o^T, \mathbf{d}^T]$ (here $\mathbf{d}^T = [d_1, ..., d_{s_j}]$):

$$\begin{bmatrix} \underset{(n \times n)}{\mathbf{U}}{}^{11} & \underset{(n \times s_j)}{\mathbf{U}}{}^{12} \\ \\ \underset{(s_j \times n)}{\mathbf{U}}{}^{21} & \mathbf{0} \end{bmatrix} \mathbf{u} = \mathbf{0} \tag{3.14}$$

in which

$$\mathbf{U}_{11} = \mathbf{I} - \sum_{k=1}^{n}{}' \frac{\mathbf{y}_k \mathbf{y}_k^T}{\alpha_k^2 - \alpha_j^2} \left\{ \left[\mathbf{v}_1, ..., \mathbf{v}_n \right] \left\langle \alpha_j^2 a_1 - b_1, ..., \alpha_j^2 a_n - b_n \right\rangle \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} \right\} , \tag{3.15a}$$

$$\mathbf{U}_{12} = \left[\mathbf{y}_{j_1}, ..., \mathbf{y}_{j_{s_j}} \right] , \tag{3.15b}$$

$$\mathbf{U}_{21} = \left\{ \begin{bmatrix} \mathbf{y}_{j_1}^T \\ \vdots \\ \mathbf{y}_{j_{s_j}}^T \end{bmatrix} \left[\mathbf{v}_1, ..., \mathbf{v}_n \right] \left\langle \alpha_j^2 a_1 - b_1, ..., \alpha_j^2 a_n - b_n \right\rangle \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} \right\} \tag{3.15c}$$

Naturally only those solutions of equation (3.14) have to be taken into consideration for which $\mathbf{y}_o^T \neq \mathbf{0}$. This is only possible if and only if $\varrho\left(\mathbf{U}_{21}\right) < n$, where ϱ is the rank of the matrix in the parenthesis. In the following we inquire if this inequality is valid.

As can be seen from the definitions (3.15b) and (3.15c)

$$\mathbf{U}_{21} = \mathbf{U}_{12}^T \left[\mathbf{v}_1, ..., \mathbf{v}_n\right] \left\langle \alpha_j^2 a_1 - b_1, ..., \alpha_j^2 a_n - b_n \right\rangle \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} . \tag{3.16}$$

Since the vectors $\mathbf{y}_{j_l}$ $(l = 1, 2, ..., s_j)$ are linearly independent: $\varrho\left(\mathbf{U}_{21}\right) = s_j$, and because multiplication cannot increase the rank, it follows from equation (3.16)

$$\varrho\left(\mathbf{U}_{21}\right) \leq s_j \ . \tag{3.17}$$

Let us investigate if s_j and n should satisfy some general inequality. First we would like to prove that $s_j \neq n$. (It is clear from the definitions of s_j and n that $s_j > n$ is impossible. Int the following we show that $s_j < n$. As a matter of fact in the extreme case for which the original system has only one n-fold (n-fold degenerated) eigenvalue invariant in spite of the perturbation, it holds that $s_j = n$. This extreme case nevertheless cannot occur, because in this case the n-dimensional vector $\mathbf{f}_o$ (carrying the perturbation) according to (3.6b) had to be orthogonal to the linearly independent vectors $\mathbf{y}_{j_l}$ $(l = 1, 2, ..., s_j = n)$, therefore $\mathbf{f}_o$ should be the vector zero, thus there is no perturbation. Consequently

$$s_j < n \ . \tag{3.18}$$

If we compare the above inequality and (3.17), we obtain

$$\varrho\left(\mathbf{U}_{21}\right) < n, \tag{3.19}$$

from which it follows for the case we are interested in that the fulfillment of the inequality

$$\mathbf{y}_o^T \neq \mathbf{0}$$

is possible. According to an investigation [10] (here not detailed), conditions for the existence of a solution for $\mathbf{y}_o^T \neq \mathbf{0}$ are as follows:

(b_1) either (if $\det \mathbf{U}_{11} \neq 0$)

$$\det \mathbf{U}_{21}\mathbf{U}_{11}^{-1}\mathbf{U}_{12} = 0, \qquad \varrho\left(\mathbf{U}_{21}\right) = s_j, \qquad s_j < n; \qquad \text{then } \mathbf{d} \neq \mathbf{0}, \tag{3.20a}$$

(b_2) or (if $\det \mathbf{U}_{11} = \underline{0}$); here two subcases are possible:

(b_{21}) if the condition $\mathbf{d} = \mathbf{0}$ is to be satisfied

$$\varrho\left(\mathbf{U}_{12}\right) < n$$

is necessary

(b_{22}) if we prescribe $\mathbf{d} \neq \mathbf{0}$ either

$$\varrho\left(\mathbf{U}_{12}\right) = s_j, \quad s_j < n \tag{3.21}$$

or

$$\varrho\left(\mathbf{U}_{12}\right) < \min\left(s_j, n\right) \tag{3.22}$$

is necessary.

The general considerations of this paper in turn can get a more auspicious form, if we multiply equation (3.14) from the left by the hypermatrix

$$\mathbf{U} = \begin{bmatrix} \underset{(n\times n)}{\mathbf{V}} & \underset{(n\times s_j)}{\mathbf{0}} \\ \\ \underset{(s_j\times n)}{\mathbf{0}} & \mathbf{I}_{s_j} \end{bmatrix} \tag{3.23}$$

which is nonsingular and independent of α where the meaning of $\mathbf{V}$ is the same as it was defined before, $\mathbf{I}_{s_j}$, however, is the unit matrix of order s_j.

After performing the multiplication we obtain

$$\mathbf{U}\begin{bmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{U}_{21} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{y}_o \\ \mathbf{d} \end{bmatrix} = \begin{bmatrix} \mathbf{V}\mathbf{U}_{11} & \mathbf{y}_o + \mathbf{V}\mathbf{U}_{12} & \mathbf{d} \\ \mathbf{U}_{21}\mathbf{y}_o & \mathbf{0} & \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{3.24}$$

Let the first member of the first block-equation on the left side of equation (3.24) be transmuted as we did it in connection with equation (3.9).

In this way we obtain

$$\mathbf{V}\mathbf{U}_{11}\,\mathbf{y}_o = \left[\mathbf{I} - \sum_{k=1}^{n}{}' \frac{1}{\alpha_k^2 - \alpha^2}\begin{bmatrix} \mathbf{v}_1^T\mathbf{y}_k \\ \vdots \\ \mathbf{v}_n^T\mathbf{y}_k \end{bmatrix}\left\{\left[\mathbf{y}_k^T\mathbf{v}_1, ..., \mathbf{y}_k^T\mathbf{v}_n\right]\left\langle \alpha_j^2 a_1 - b_1, ..., \alpha_j^2 a_n - b_n\right\rangle\right\}\right]\begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix}\mathbf{y}_o = \mathbf{0} \tag{3.25}$$

Therefore equation (3.24) has the form

$$\begin{bmatrix} \mathbf{V}_{11} & \mathbf{V}\mathbf{U}_{12} \\ \mathbf{V}_{21} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{V}\mathbf{y}_o \\ \mathbf{d} \end{bmatrix} = \mathbf{0} \tag{3.26}$$

where

$$\mathbf{V}_{11} = \mathbf{I} - \sum_{k=1}^{n}{}' \frac{1}{\alpha_k^2 - \alpha_j^2}\begin{bmatrix} \mathbf{v}_1^T\mathbf{y}_k \\ \vdots \\ \mathbf{v}_n^T\mathbf{y}_k \end{bmatrix}\left[\mathbf{y}_k^T\mathbf{v}_1, ..., \mathbf{y}_k^T\mathbf{v}_n\right]\left\langle \alpha_j^2 a_1 - b_1, ..., \alpha_j^2 a_n - b_n\right\rangle \tag{3.27}$$

$$\mathbf{V}_{21} = \begin{bmatrix} \mathbf{y}_{j_1}^T\mathbf{v}_1, ..., \mathbf{y}_{j_1}^T\mathbf{v}_n \\ \mathbf{y}_{j_{s_j}}^T\mathbf{v}_1, ..., \mathbf{y}_{j_{s_j}}^T\mathbf{v}_n \end{bmatrix}\left\langle \alpha_j^2 a_1 - b_1, ..., \alpha_j^2 a_n - b_n\right\rangle. \tag{3.28}$$

This transformation makes it possible to express conditions (3.20a,b,c) substituting $\mathbf{U}_{11}, \mathbf{U}_{12}, \mathbf{U}_{21}$ succesively for $\mathbf{V}_{11}, \mathbf{V}\mathbf{U}_{12}$, and $\mathbf{V}_{21}$. Our formulas will be again more simple using the choice (3.11).

If there are more than one invariant and distinct eigenvalues, then the problem can be described by as many determinant equations – according to equations (3.14) or (3.15a) – as the number of the invariant eigenvalues prescribed.

One can raise the question, if in the case of single eigenvalues such a non zero perturbation exists which leaves every single eigenvalue invariant. This problem is described by a number of n, from equation (3.14) following determinant equations (individually of order $n+1$). If we consider the vectors $\mathbf{v}_1$ to be fixed and consider for example the coefficients a_i, b_i $(i = 1, ..., n)$ to be unknowns, then we can generally assume arbitrarily n from them. For instance, the coordination of the (persistent) eigenvalues with other eigenvectors is feasible, as before.

In what follows, based on our previous results, we shall consider solutions of various problems of common interest.

4. **Perturbation with one term for single original eigenvalues**

We shall call the perturbation "having one term referring to the index r", if the coefficients a_i, b_i are non-zero only for the index $i = r$, i.e., if

$$\mathbf{f}_o = [\mathbf{v}_1, ..., \mathbf{v}_r, ...\mathbf{v}_n] \left\langle 0, ..., \alpha^2 a_r - b_r, , ..., 0 \right\rangle \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_r^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix}, \tag{4.1}$$

and in accordance with the method presented above $a_r \neq 0$, $b_r \neq 0$.

The determinant (3.11) that provides the perturbated eigenvalues assumes the form

$$\left| \mathbf{I} - \sum_{k=1}^{n} \frac{1}{\alpha_k^2 - \alpha^2} \begin{bmatrix} \mathbf{v}_1^T \mathbf{y}_k \\ \vdots \\ \mathbf{v}_r^T \mathbf{y}_k \\ \vdots \\ \mathbf{v}_n^T \mathbf{y}_k \end{bmatrix} \left[\mathbf{y}_k^T \mathbf{y}_1, ..., \mathbf{y}_k^T \mathbf{v}_r, ..., \mathbf{y}_k^T \mathbf{v}_n \right] \left\langle 0, ..., \alpha^2 a_r - b_r, ..., 0 \right\rangle \right|.$$

Taking the opportunity of the choice $\mathbf{v}_k = \mathbf{y}_k$ this determinant is very similar to that defined by equation (3.12):

$$\left| \mathbf{I} - \sum_{k=1}^{n} \frac{1}{\alpha_k^2 - \alpha^2} \left\langle 0, 0, ...\alpha^2 a_r - b_r, 0, ..., 0 \right\rangle \right|$$
$$= \det \left\langle 1, 1, ..., 1 - \frac{\alpha^2 a_r - b_r}{\alpha_r^2 - \alpha^2}, 1, ..., 1 \right\rangle = 1 - \frac{\alpha^2 a_r - b_r}{\alpha_r^2 - \alpha^2}. \tag{4.2}$$

The determinant will be equal to zero if

$$\alpha^2 = \frac{\alpha_r^2 + b_r}{1 + a_r}, \tag{4.3}$$

i.e., if one single eigenvalue is perturbated. This formula does not say yet what index the perturbated eigenvalue has.

Inserting the eigenvalues (4.3) in (3.9) we get the components $\mathbf{v}_k^T\mathbf{y}_o$ $(k = 1, 2, ..., n)$ of the perturbated eigenvector projected on $\mathbf{v}_k = \mathbf{y}_k$; this equation reads disregarding the details of the calculations (observing that on the left side of equation (3.9) only the member with index $k = r$ is not zero):

$$\left\langle 1, 1, ..., \overset{r}{\breve{0}}, 1, ..., 1 \right\rangle \begin{bmatrix} \mathbf{y}_1^T\mathbf{y}_o \\ \vdots \\ \mathbf{y}_r^T\mathbf{y}_o \\ \vdots \\ \mathbf{y}_n^T\mathbf{y}_o \end{bmatrix} = 0\,, \tag{4.4}$$

from which

$$\mathbf{y}_1^T\mathbf{y}_o = 0\,, \qquad \text{etc.} \qquad \mathbf{y}_{r-1}^T\mathbf{y}_o = 0\,, \qquad \mathbf{y}_{r+1}^T\mathbf{y}_o = 0\,, \qquad \text{etc.} \qquad \mathbf{y}_n^T\mathbf{y}_o = 0\,,$$

but in turn it must be $\mathbf{y}_r^T\mathbf{y}_o \neq 0$, otherwise the unknown vector $\mathbf{y}_o$ was orthogonal to the linearly independent n-dimensional vectors $\mathbf{y}_1, ..., \mathbf{y}_n$, and therefore it would be zero. The product $\mathbf{y}_r^T\mathbf{y}_o$ can be assumed as an arbitrary real number and we can agree to choose it to 1:

$$\mathbf{y}_r^T\mathbf{y}_o = 1\,. \tag{4.5}$$

Equations (4.4) and (4.5) will be interpreted in such a way that to the single perturbated eigenvalue belongs as eigenvector the original eigenvector with index r.

According to equation (4.3) - as we saw it - only one single eigenvalue is perturbated. In the following we shall check if the circumstance that the rest of the eigenvalues remain invariant fits in - as needed - the formerly elucidated thoughts in connection with the invariant (persistent) eigenvalues.

The check consists in investigating whether the single eigenvalues

$$\alpha_1^2,\ \alpha_2^2, ..., \alpha_{r-1}^2,\ \alpha_{r+1}^2, ..., \alpha_n^2$$

one after the other satisfy the conditions (3.20a) or (3.21) formulated by now in terms of $\mathbf{V}_{11}$, $\mathbf{V}\mathbf{U}_{12}$ and $\mathbf{V}_{21}$. First we assume case (b_1), i.e., (3.20a) is valid, which now means that we substitute $\mathbf{V}_{11}$ for $\mathbf{U}_{11}$ $(\det\mathbf{V}_{11} \neq 0)$. In our case with the choice $\mathbf{v}_k = \mathbf{y}_k$

$$\mathbf{V}_{11} = \mathbf{I} - \sum_{k-1}^{n\ '} \frac{1}{\alpha_k^2 - \alpha^2} \begin{bmatrix} \mathbf{y}_1^T\mathbf{y}^k \\ \vdots \\ \mathbf{y}_n^T\mathbf{y}_k \end{bmatrix} \left[\mathbf{y}_k^T\mathbf{y}_1, ..., \underset{-k}{y^T\mathbf{y}_n}\right] \left\langle 0, 0, ..., \alpha^2 a_r - b_r, ..., 0, 0 \right\rangle .$$

If we want to check the values of α_1^2, then we must substitute α_1^2 for α^2, and we have to omit the index $k = 1$ from the above sum. In this way we get

$$\mathbf{V}_{11} = \mathbf{I} - \left\{ \frac{1}{\alpha_2^2 - \alpha_1^2} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} [0, 1, 0, ..., 0] + \frac{1}{\alpha_3^2 - \alpha_1^2} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} [0, 0, 1, ..., 0] + ... + \right.$$

$$+\frac{1}{\alpha_r^2-\alpha_1^2}\begin{bmatrix}0\\ \vdots\\ 1\\ 0\\ 0\end{bmatrix}\,{}_{(r}\quad\left[0,0,...,\overset{r}{\smile}1,0,0\right]+,...,$$

$$+\frac{1}{\alpha_n^2-\alpha_1^2}\begin{bmatrix}0\\ 0\\ \vdots\\ 0\\ 1\end{bmatrix}[0,...,0,1]\left\langle 0,0,...,\alpha_1^2a_r-b_r,0,...,0\right\rangle\Bigg\}\left\langle 0,0,...,\alpha_1^2a_r-b_r,0,...,0\right\rangle=$$

$$=\mathbf{I}-\frac{1}{\alpha_r^2-\alpha_1^2}\left\langle 0,0,...,\alpha_1^2a_r-b_r,0,...,0\right\rangle=\left\langle 1,1,...,1-\frac{\alpha_1^2a_r-b_r}{\alpha_r^2-\alpha_1^2},1,...,1\right\rangle.$$

If we intend to check generally the values α_i^2 $(i=1,2,...,r-1,r+1,...,n)$, then we should write

$$\mathbf{V}_{11}=\mathbf{I}-\Bigg\{\frac{1}{\alpha_1^2-\alpha_i^2}\begin{bmatrix}1\\ 0\\ \vdots\\ 0\\ 0\end{bmatrix}[1,...,0]+\frac{1}{\alpha_2^2-\alpha_i^2}\begin{bmatrix}0\\ 1\\ \vdots\\ 0\\ 0\end{bmatrix}[0,1,0,...,0]+...+$$

$$+\frac{1}{\alpha_{i-1}^2-\alpha_i^2}\begin{bmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{bmatrix}\,{}_{(i-1}\quad\left[0,...,0,\overset{i-1}{\smile}1,0,...,0\right]+$$

$$+\frac{1}{\alpha_{i+1}^2-\alpha_i^2}\begin{bmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{bmatrix}\,{}_{(i+1}\quad\left[0,...,0,\overset{i+1}{\smile}1,0,...,0\right]+$$

$$+\frac{1}{\alpha_r^2-\alpha_i^2}\begin{bmatrix}0\\ \vdots\\ 1\\ 0\\ 0\end{bmatrix}\,{}_{(r}\quad\left[0,...,0,\overset{r}{\smile}1,0,0\right]+$$

$$+...+\frac{1}{\alpha_n^2-\alpha_i^2}\left.\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}[0,...,1]\right\}\left\langle 0,0,...,\alpha_i^2a_r-b_r,...,0\right\rangle=$$

$$=\mathbf{I}-\frac{1}{\alpha_r^2-\alpha_i^2}\begin{bmatrix}0\\0\\\vdots\\1\\\vdots\\0\\0\end{bmatrix}\begin{matrix}(r\\\\\\\\\\\\\end{matrix}\quad\left[0,0,...,\overset{r}{\underset{\smile}{1}},...,0,0\right]\;\left\langle 0,0,...,\alpha_i^2a_r-b_r,...,0\right\rangle$$

$$=\left\langle 1,1,...,1-\frac{\alpha_i^2a_r-b_r}{\alpha_r^2-\alpha_i^2},1,...,1\right\rangle.$$

Summing up: if we substitute α_i^2 for any eigenvalue different from α_r^2, i.e., α_i^2 is an eigenvalue, but $\alpha_i^2\neq\alpha_r^2$, then

$$|\mathbf{V}_{11}|=1-\frac{\alpha_i^2a_r-b_r}{\alpha_r^2-\alpha_i^2},\quad\text{and thus }\;|\mathbf{V}_{11}|\neq 0\,,$$

which is in accordance with our expectations.

We shall check making use of the relation (3.20a) whether the equality

$$\det\left(\mathbf{U}_{21}\mathbf{U}_{11}^{-1}\mathbf{U}_{12}\right)=\det\left(\mathbf{V}_{21}\mathbf{V}_{11}^{-1}\mathbf{V}\mathbf{V}_{12}\right)$$

holds. It can be seen for $i=1,2,...,r-1,r+1,...,n$ that

$$\mathbf{V}_{21}\mathbf{V}_{11}^{-1}\mathbf{V}\mathbf{U}_{12}=$$

$$=\left[\mathbf{y}_i^T\mathbf{v}_1,...,\mathbf{y}_i^T\mathbf{v}_r,...,\mathbf{y}_i^T\mathbf{v}_n\right]\left\langle 0,0,...,\alpha_i^2a_r-b_r,0,...,0\right\rangle\times$$

$$\times\left\langle 1,...,1,\frac{\alpha_r^2-\alpha_i^2}{\alpha_r^2-\alpha_i^2-\alpha_i^2a_r+b_r},...,1\right\rangle\begin{bmatrix}\mathbf{v}_1^T\\\vdots\\\mathbf{v}_r^T\\\vdots\\\mathbf{v}_n^T\end{bmatrix}\mathbf{y}_i=$$

$$=[0,0,0,...,0]\left\langle 1,...,1,\frac{\overset{r}{\underset{\smile}{\alpha_r^2-\alpha_i^2}}}{\alpha_r^2-\alpha_i^2-\alpha_i^2a_r+b_r},...,1\right\rangle\begin{bmatrix}0\\\vdots\\1\\\vdots\\0\end{bmatrix}\begin{matrix}(i\neq r\\\\\\\\\end{matrix}\quad=\mathbf{0}$$

therefore $\det\left(\mathbf{V}_{21}\mathbf{V}_{11}^{-1}\mathbf{V}\mathbf{U}_{12}\right)=0$ as it should be.

Investigation of condition (b_2) and its subcases (b_{21}) and (b_{22}) will be not presented.

It may be of interest, however, to examine the perturbated eigenvectors belonging to these invariant eigenvalues. Based on equation (3.26) and the foregoing equation for $\mathbf{V}\mathbf{y}_o$ containing the components (projected on $\mathbf{v}_1, ..., \mathbf{v}_n$) of the eigenvectors searched we write

$$\begin{bmatrix} 1 & & & & & & & & 0 \\ & 1 & & & & & & & 0 \\ & & \ddots & & & & & & \vdots \\ & & & 1-\frac{\alpha_i^2 a_r - b_r}{\alpha_r^2 - \alpha_i^2} & & & & & 1 \\ & & & & 1 & & & & 0 \\ & & & & & \ddots & & & \vdots \\ & & & & & & 1 & & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} (i \begin{bmatrix} \mathbf{v}_1^T \mathbf{y}_o \\ \vdots \\ \mathbf{v}_r^T \mathbf{y}_o \\ \vdots \\ d_1 \end{bmatrix} = \mathbf{0}.$$

The rank of its coefficient matrix is n, because $1 - \frac{\alpha_i^2 a_r - b_r}{\alpha_r^2 - \alpha_i^2} \neq 0$ for $i \neq r$ thus the defect is 1, which is the number of the free unknowns. From the first n equations then follows

$$\mathbf{v}_1^T \mathbf{y}_o = 0, ..., \mathbf{v}_r^T \mathbf{y}_o = 0, ..., \mathbf{v}_i^T \mathbf{y}_o + d_1 = 0, ..., \mathbf{v}_n^T \mathbf{y}_o = 0 \, .$$

The $n+1^{th}$ equation is automatically satisfied. d_1 can be taken as the free unknown; choosing $d_1 = -1$, $\mathbf{v}_i^T \mathbf{y}_o = 1$, – because now $\mathbf{v}_i^T = \mathbf{y}_i$ - the perturbated eigenvector is the same as the original eigenvector that belongs to α_i^2.

Summing up our results: in the case of single original eigenvalues, and of a one-term perturbation referring only to the index r and using the special choice (3.11), nearly all eigenvalues remain invariant; among the eigenvalues only the one having the index r changes to $\frac{\alpha_r^2 + b_r}{1 + a_r}$. (However, it is not certain if this will be the r^{th} eigenvalue of the perturbated system.)

5. **Two subcases of the investigation of the preceding item**

In the subcase characterized by the equation

$$\frac{\alpha_r^2 + b_r}{1 + a_r} = \alpha_i^2, \qquad i \neq r \tag{5.1}$$

the perturbation makes the originally i^{th} eigenvalue α_i^2 a double one. If

$$\alpha_{r-1}^2 < \frac{\alpha_r^2 + b_r}{1 + a_r} < \alpha_{r+1}^2,$$

then the perturbated r^{th} eigenvalue remains the r^{th} eigenvalue of the perturbated system. If this inequality chain is invalid, the perturbation alters the original order of the eigenvalues.

Let us investigate how the perturbated eigenvectors can be formed if for fixed i and r equation (5.1) holds. This equation can be interpreted in such a way that the i^{th} original eigenvalue remains invariant. The corresponding blocks in the partitioned hypermatrix of equation (3.26), which gives the perturbated eigenvectors, are as follows ($\alpha^2 = \alpha_i^2 = \frac{\alpha_r^2 + b_r}{1+a_r}$, and we have omitted the term with index $k = i$ from the sum and have chosen $\mathbf{v}_j = \mathbf{y}_j$):

$$\mathbf{V}_{11} = \mathbf{I} - \left\{ \frac{1}{\alpha_1^2 - \alpha_i^2} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} [1,0,...,0] \left\langle 0,0,...,\alpha_i^2 a_r - b_r,...,0,...,0 \right\rangle + ...+ \right.$$

$$+ \frac{1}{\alpha_{i-1}^2 - \alpha_i^2} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} (i-1 \quad \begin{bmatrix} \overset{i-1}{\smile} \\ 0,...,\ 1\ ,...,0 \end{bmatrix} \left\langle 0,0,...,\alpha_i^2 a_r - b_r,0,...,0 \right\rangle \ +$$

$$+ \frac{1}{\alpha_{i+1}^2 - \alpha_i^2} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} (i+1 \quad \begin{bmatrix} \overset{i+1}{\smile} \\ 0,...,\ 1\ ,...,0 \end{bmatrix} \left\langle 0,0,...,\alpha_i^2 a_r - b_r,...,0,...,0 \right\rangle \ +...+$$

$$\left. + \frac{1}{\alpha_n^2 - \alpha_i^2} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{bmatrix} [0,...,0,...,1] \left\langle 0,0,...,\alpha_i^2 a_r - b_r,0,...,0 \right\rangle \right\}$$

$$= \mathbf{I} - \frac{1}{\alpha_r^2 - \alpha_i^2} \left\langle 0,...,0,\alpha_i^2 a_r - b_r,0,...,0 \right\rangle = \left\langle 1,1,...,\overset{r}{\smile}{1} - \frac{\alpha_i^2 a_r - b_r}{\alpha_r^2 - \alpha_i^2},1,1,...,1 \right\rangle ;$$

$$\mathbf{V}_{21} = \begin{bmatrix} \overset{i}{\smile} \\ 0,0,...,\ 1\ ,0,...,0 \end{bmatrix} \left\langle 0,0,...,\alpha_i^2 a_r - b_r,0,...,0 \right\rangle = [0,...,0] ;$$

$$\mathbf{VU}_{12} = \begin{bmatrix} \mathbf{y}_1^T \\ \vdots \\ \mathbf{y}_n^T \end{bmatrix} [\mathbf{y}_i] = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad (i$$

Algebraical details helping calculation of $\mathbf{V}_{11}$:

$$1 - \frac{\alpha_i^2 a_r - b_r}{\alpha_r^2 - \alpha_i^2} = \frac{\alpha_r^2 - \alpha_i^2 - \alpha_i^2 a_r + b_r}{\alpha_r^2 - \alpha_i^2} =$$

$$= \frac{\alpha_r^2 + b_r - \alpha_i^2(1 + a_r)}{\alpha_r^2 - \alpha_i^2} = \frac{\alpha_r^2 + b_r - \frac{\alpha_r^2 + b_r}{1+a_r}(1 + a_r)}{\alpha_r^2 - \frac{\alpha_r^2 + b_r}{1+a_r}} = 0.$$

Equation (3.26) reads:

$$\begin{bmatrix} 1 & & & & & & & & & 0 \\ & 1 & & & & & & & & 0 \\ & & \ddots & & & & & & & \vdots \\ & & & 1 & & & & & & 1 \\ & & & & 0 & & & & & 0 \\ & & & & & 1 & & & & 0 \\ & & & & & & \ddots & & & \vdots \\ & & & & & & & 1 & & 0 \\ & & & & & & & & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ (i \\ \\ \\ \\ \\ \\ \end{matrix} \begin{bmatrix} \mathbf{v}_1^T \mathbf{y}_o \\ \vdots \\ \mathbf{v}_i^T \mathbf{y}_o \\ \vdots \\ \mathbf{v}_n^T \mathbf{y}_o \\ d_1 \end{bmatrix} = \mathbf{0}$$

$$(r$$

from which it follows for the non-trivial solutions

$$\mathbf{v}_1^T \mathbf{y}_o = 0, \qquad \mathbf{v}_r^T \mathbf{y}_o \neq 0, \qquad \text{etc.} \qquad , \mathbf{v}_i^T \mathbf{y}_o + d_1 = 0, \qquad \text{etc.} \qquad \mathbf{v}_n^T \mathbf{y}_o = 0.$$

Thus $\mathbf{y}_o$ can have non-zero components only in the directions $\mathbf{v}_r = \mathbf{y}_r$ and $\mathbf{v}_i = \mathbf{y}_i$. It is possible to state the assumption $\mathbf{y}_o = c_r \mathbf{y}_r + c_i \mathbf{v}_i$. We can norm it with the prescription $\mathbf{y}_o^T \mathbf{y}_o = 1$, which yields $c_r^2 + c_i^2 = 1$ and either c_r or c_i can be assumed.

6. **Solution of an inverse eigenvalue problem**

The above reasonings make it possible to design a system with n degrees of freedom with prescribed eigenvalues. We start with a known system and modify it using a perturbation having n terms, and we are looking for the coefficients

$$a_i, \; b_i, \qquad (i = 1, 2, ..., n).$$

Let us suppose that the prescribed eigenvalues $\alpha_i^{'2}$ $(i = 1, 2, ..., n)$ satisfy the conditions

$$0 < \alpha_1^{'2} < \alpha_2^{'2} < ... < \alpha_n^{'2}$$

and

$$\alpha_i^{\prime 2} \neq \alpha_j^{'2}, \qquad i = 1, 2, ..., n; \quad j = 1, 2, ..., n,$$

consequently the prescribed eigenvalue are differing from each other, and there are no invariant eigenvalues. Further let $0 < \alpha_1^2 < \alpha_2^2 < ... < \alpha_n^2$.

This task can indeed be solved by a perturbation having n terms. A more detailed expanding of determinant (3.12) is:

$$\left| i - \sum_{k=1}^{n} \frac{1}{\alpha_k^2 - \alpha^2} \left\langle 0, 0, ..., 0, \alpha^2 a_k - b_k, 0, ...0 \right\rangle \right| = \left| \left\langle 1 - \frac{\alpha^2 a_1 - b_1}{\alpha_1^2 - \alpha^2}, 1 - \frac{\alpha^2 a_2 - b_2}{\alpha_2^2 - \alpha^2}, ..., 1 - \frac{\alpha^2 a_n - b_n}{\alpha_n^2 - \alpha^2} \right\rangle \right|.$$

This diagonal determinant can be set to zero by setting to zero at least one of its elements. In this way - wishing the perturbated eigenvalues to be just the prescribed ones - we get the equations below with the agreement that the first element of the diagonal is set to zero by making use of the first eigenvalue and so on:

$$1 - \frac{\alpha_1'^2 a_1 - b_1}{\alpha_1^2 - \alpha_1'^2} = 0, \qquad 1 - \frac{\alpha_2'^2 a_2 - b_2}{\alpha_2^2 - \alpha_2'^2} = 0, \qquad 1 - \frac{\alpha_n'^2 a_n - b_n}{\alpha_n^2 - \alpha_n'^2} = 0.$$

It follows from these equations that the coefficients a_i, b_i should satisfy the equations

$$-\alpha_i^{'2} a_i + b_i = -\alpha_i^2 + \alpha_i^{'2}, \quad i = 1, 2, ..., n.$$

Having altogether $2n$ coefficients we can posit arbitrarily a number of n, or a number of n surplus stipulations can be taken.

Equation for the components $\mathbf{y}_i^T \mathbf{y}_{oi}$ of the perturbated eigenvectors $\mathbf{y}_{oi}$ projected on the vectors $\mathbf{y}_i$ is of the form

$$\left\langle 1 - \frac{\alpha_i'^2 a_1 - b_1}{\alpha_1^2 - \alpha_i'^2}, 1 - \frac{\alpha_i'^2 a_2 - b_2}{\alpha_2^2 - \alpha_i'^2}, ..., 1 - \frac{\alpha'^2 a_i - b_i}{\alpha_i^2 - \alpha_i'^2}, ..., 1 - \frac{\alpha_i'^2 a_n - b_n}{\alpha_n^2 - \alpha_i'^2} \right\rangle \begin{bmatrix} \mathbf{y}_1^T \mathbf{y}_{oi} \\ \vdots \\ \mathbf{y}_i^T \mathbf{y}_o \\ \vdots \\ \mathbf{y}_n \mathbf{y}_o \end{bmatrix} = \mathbf{0}\,.$$

Only the i^{th} element of the above determinant will be zero, i.e., only the i^{th} element of the unknown column vector can differ from zero, the other elements must be equal to zero. In other words this vector (with an appropriate agreement for norming) can be equal to the i^{th} eigenvector of the original system.

In what has gone before we adhered to the convention that the first element of the diagonal matrix in question will be set to zero by the prescribed eigenvalue with the index 1, and so on. Let us depart now from this agreement for instance in such a way that, the prescribed eigenvalue with the index 1 should set to zero the n^{th} element, the prescribed eigenvalue with the index 2, however, the $(n-1)^{th}$ element, and so on. If n is even, we can agree that the prescribed eigenvalue with the index j sets the $(n-j)^{\text{th}}$ element to zero. If n is odd, this agreement does not refer to the $\frac{n-1}{2}{}^{th}$ element; this will be made zero also now by the $\frac{n-1}{2}{}^{th}$ prescribed eigenvalue.

Making use of this agreement if n is even, only the $n-j^{th}$ element of the eigenvector belonging to the prescribed eigenvector with the index j can differ from zero, i.e., the

perturbated eigenvector can be considered equal to the original eigenvector with the index $n-j$.

7. Designing an oscillator with n degrees of freedom and one degenerated eigenvalue α_d^2 with multiplicity n

We shall assume that the eigenvalues of the original system fulfill the relations

$$0 < \alpha_1^2 < \alpha_2^2 < ... < \alpha_n^2$$

and

$$\alpha_d^2 \neq \alpha_i^2, \quad i = 1, 2, ..., n.$$

Making use of the determinant (3.12) again, we obtain

$$1 - \frac{\alpha_d^2 a_1 - b_1}{\alpha_1^2 - \alpha_d^2} = 0, \qquad 1 - \frac{\alpha_d^2 a_2 - b_2}{\alpha_2^2 - \alpha_d^2} = 0, \qquad 1 - \frac{\alpha_d^2 a_n - b_n}{\alpha_n^2 - \alpha_d^2} = 0.$$

It is possible anew to prescribe yet n conditions, or n coefficients can be chosen arbitrarily from the coefficients a_i, b_i $(i = 1, 2, ..., n)$.

8. About the inverse eigenvalue task of Lancaster

Lancaster in his paper [11] supplies - inter alia - it seems that without a detailed demonstration – a perturbation (falling in the topic of the present paper), which leaves the eigenvectors and zero eigenvalues (if they exist) invariant. We would like to show the connection between this perturbation and that in the present paper. In order to make Lancaster's formulas close to ours, we shall set forth them with a designation differing a little from Lancaster's original ones.

Lancaster starts - with our denotations - with the eigenvalue task (1.1):

$$\left(-\alpha^2 \mathbf{M} + \mathbf{C}_1\right) \mathbf{x} = \mathbf{0},$$

then divides the perturbation task in two parts, considering first the problem

$$\left(-\alpha^2 \hat{\mathbf{M}} + \mathbf{C}_1\right) \mathbf{y_o} = \mathbf{0}, \tag{8.1}$$

then the problem

$$\left(-\alpha^2 \mathbf{M} + \hat{\mathbf{C}}_1\right) \mathbf{y_o} = \mathbf{0}, \tag{8.2}$$

where $\hat{\mathbf{M}}$ and $\hat{\mathbf{C}}_1$ are appropriately perturbated forms of $\mathbf{M}$ and $\mathbf{C}_1$ respectively. In the sequel - combining the two problems - we shall deal with the problem

$$\left(-\alpha^2 \hat{\mathbf{M}} + \hat{\mathbf{C}}_1\right) \mathbf{y_o} = \mathbf{0}. \tag{8.3}$$

We introduce the modal matrix $\mathbf{X}$ formed by the eigenvectors $\mathbf{x}_{oi}$ $(i = 1, ..., n)$ of problem (1.1) as

$$[\mathbf{x}_{o1}, \mathbf{x}_{o2}, ..., \mathbf{x}_{on}] = \mathbf{X},$$

which is assumed to satisfy the orthonormality condition $\mathbf{X}^T\mathbf{M}\mathbf{X} = \mathbf{I}$. From the latter we get

$$\left(\mathbf{X}^T\right)^{-1} = \mathbf{M}\mathbf{X} \quad \text{and} \quad \mathbf{X}^{-1} = \mathbf{X}^{\mathbf{T}}\mathbf{M}\,. \tag{8.4}$$

Lancaster's special perturbation is

$$\overset{\wedge}{\mathbf{M}} = \mathbf{M} - \left(\mathbf{X}^T\right)^{-1} \langle \delta_1, ..., \delta_n \rangle \left(\mathbf{I} + \langle \delta_1, ..., \delta_n \rangle\right)^{-1} \mathbf{X}^{-1}, \tag{8.5}$$

and

$$\overset{\wedge}{\mathbf{C}}_{\mathbf{1}} = \mathbf{C}_{\mathbf{1}} + \left(\mathbf{X}^T\right)^{-1} \left\langle \alpha_1^2, ..., \alpha_n^2 \right\rangle \langle \gamma_1, ..., \gamma_n \rangle \mathbf{X}^{-1}, \tag{8.6}$$

where δ_i and γ_i are the so-called "perturbational factors" – we remark that Lancaster took also δ_i instead of γ_i $(i = 1, ..., n)$; and α_i^2 $(i = 1, ..., n)$ are the eigenvalues of (1.1).

Equation (8.3) can be written in the inhomogeneous form corresponding to (3.2), where taking (8.5) and (8.6) into consideration:

$$\mathbf{f}_o = \Big\{ -\alpha^2 \mathbf{M}\mathbf{X} \langle \delta_1, ..., \delta_n \rangle \left(\mathbf{I} + \langle \delta_1, ..., \delta_n \rangle\right)^{-1} \mathbf{X}^T\mathbf{M} - \mathbf{M}\mathbf{X} \left\langle \alpha_1^2, ..., \alpha_n^2 \right\rangle \langle \gamma_1, ..., \gamma_n \rangle \mathbf{X}^T\mathbf{M} \Big\} \mathbf{y}_o. \tag{8.7}$$

Comparing equation (8.7) and our formula (3.5) with the special assumption $\mathbf{v}_k = \mathbf{M}\mathbf{x}_{ok}$ in (3.5) we find that Lancaster's formulae can be derived from ours, if we make use of the choices

$$\langle a_1, ..., a_n \rangle = -\langle \delta_1, ..., \delta_n \rangle \left(\mathbf{I} - \langle \delta_1, ..., \delta_n \rangle\right)^{-1} \tag{8.8a}$$

and

$$\langle b_1, ..., b_n \rangle = \left\langle \alpha_1^2, ..., \alpha_n^2 \right\rangle \langle \gamma_1, ..., \gamma_n \rangle\,. \tag{8.8b}$$

It follows from (8.8a,b) that

$$a_i = \frac{\delta_i}{1 + \delta_i} \qquad \text{and} \qquad b_i = \alpha_i^2 \gamma_i \quad (i = 1, ..., n) \tag{8.9}$$

consequently the relation corresponding to (4.3) expressed in terms of δ_i and γ_i is:

$$\alpha^2 = \alpha_i^2 (1 + \gamma_i)(1 + \delta_i), \quad i = 1, ..., n. \tag{8.10}$$

This means Lancaster's formula is a subcase of ours and at the same time we have given a proof for his formulas.

9. **Perturbation of a simply connected chain-like oscillator leaving these properties invariant**

The system mentioned in the title can be described by equation (2.1), where $\mathbf{C}$ is tridiagonal and excepting for one row (which can be either the first one or the last one) the sum of the elements should be zero. The perturbation must be such that after perturbation $\mathbf{I}$ remains diagonal and $\mathbf{C}$ also remains tridiagonal.

The perturbation in the form (2.4) can leave $\mathbf{I}$ diagonal by $\mathbf{v}_i \neq \mathbf{0}$ $(i = 1, 2, ..., n)$, if for

$$a_i \neq 0, \;\; i = 1, 2, ..., n \qquad \text{the matrix} \qquad \sum^{n} a_i \mathbf{v}_i \mathbf{v}_i^T$$

is diagonal, and $\mathbf{C}$ will be perturbated to be tridiagonal, if $\sum_{i=1}^{n} b_i \mathbf{v}_i \mathbf{v}_i^T$ is tridiagonal.

Assume that $\mathbf{v}_i \mathbf{v}_i^T$ is not diagonal for all i. Then the constants a_i can only ensure that the matrix $\sum_{i=1}^{n} a_i \mathbf{v}_i \mathbf{v}_i^T$ is diagonal if they satisfy more than n conditions, which is generally not possible. Accordingly, $\mathbf{v}_i \mathbf{v}_i^T$ $(i = 1, 2, ..., n)$ must be diagonal, which is only possible, if each $\mathbf{v}_i$ contains only one element different from zero.

In accordance with all that was said above $a_i \neq 0$, $(i = 1, 2, ..., n)$ and $\mathbf{v}_i \mathbf{v}_i^T$ $(i = 1, 2, ..., n)$ should contain in the main diagonal at most one non-zero element. Therefore $\sum_{i=1}^{n} b_i \mathbf{v}_i \mathbf{v}_i^T$ can perturbate at most elements of $\mathbf{C}$ in the main diagonal. This means that if $\mathbf{C}$ belongs to a single connected chain-like system, then the perturbated spring matrix cannot be interpreted unconditionally belonging to a simply connected chain-like system. By that namely - as has been said above - (excepting for example the first or the last row), the sum of the elements of each row must be zero. In general the perturbated spring matrix will characterize such a system, to which not only the immediately neighbouring masses are connected by a spring but in general there belong springs to each mass coupling it to a fixed point.

However, if we assume that $a_i = 0$, $(i = 1, 2, ..., n)$, then we can proceed in the same way as in case (iiii) – see page 120. Since then $\Delta \mathbf{I} = \mathbf{0}$, the perturbated mass matrix is again the unit matrix. Let us survey in this case some possibilities for inserting the vectors $\mathbf{v}_i$.

If we take only the first element of $\mathbf{v}_1$, then only the first and second one of $\mathbf{v}_2, ...$, afterwards only the $(n-2)^{th}$ and $(n-1)^{th}$ element of $\mathbf{v}_{n-1}$, thereafter only the $(n-1)^{st}$ and n^{st} element of $\mathbf{v}_n$ as differing from zero, then the perturbation adheres to the tridiagonal form of $\mathbf{C}$. For the case of fixed vectors b_i, however, in general the property mentioned above does not remain in connection with the sum of the elements in the rows of the perturbated $\mathbf{C}$.

10. **Remarks helping to find relationship with some relevant results**

According to Falk [12] the differential equation (1.1) - if there are no degenerated eigenvalues - can be transformed by the (diagonal) $\mathbf{S}$ and real quadratical $\mathbf{T}$ matrices - which can be calculated without the knowledge of the eigenvalues leaving these invariant - to the form

$$\mathbf{S}^T \mathbf{T}^T \mathbf{M} \mathbf{T} \mathbf{S} \ddot{\mathbf{x}} + \mathbf{S}^T \mathbf{T}^T \mathbf{C}_1 \mathbf{T} \mathbf{S} \mathbf{x} = \mathbf{0},$$

which corresponds to a chain-like longitudinal oscillator without springs connecting not immediately neighboring masses.

Rutishauser [13], however, gives an orthogonal Hessenberg matrix (constructing it with the help of a known eigenvector) which transforms (knowing also the eigenvalues belonging to this known eigenvector) the original problem with a similarity transformation without changing the band width of the original matrices to an eigenvalue

problem which has a zero eigenvalue corresponding to the known eigenvector used to the construction of the transformation matrix mentioned, and the other original eigenvalues remain invariant. Usually one speaks in this case about "deflation". By this procedure it is not warranted that the transformed problem corresponds to chain-like oscillator without springs connected not immediately to the neighboring masses, even if the starting problem was characterized with this property.

To Rutishauser's transformation always can be ordered such a series of dyad-separations, which results in the same effect as Rutishauser's similarity transformation.

In the preceding paragraph we mentioned shortly the notion of deflation. We give first a very simple definition of this idea suiting to our main subject. Let us suppose, we have an oscillator with the eigenfrequencies

$$\alpha_1'^2, \alpha_2'^2, ..., \alpha_{k-1}'^2, \alpha_k'^2, \alpha_{k+1}'^2, ..., \alpha_n'^2.$$

How can we very simply describe another oscillator by making use of the eigenfrequencies

$$\alpha_1'^2, \alpha_2'^2, ..., \alpha_{k-1}'^2, \alpha_{k+1}'^2, ..., \alpha_n'^2,$$

i.e., a series consisting of the original eigenfrequencies with $\alpha_k'^2$ as a missing eigenfrequency. At the same time - if we wish - (naturally the following is only possible by an oscillator with finite degrees of freedom) we can diminish by one the number of its degrees of freedom. The above thoughts about the solution of the inverse eigenvalue problems give an immediate possibility to solve the problem of deflation with or without diminishing the number of its degrees of freedom.

11. **Concluding remarks**

We truly hope that we could show using the dyads the outlines promised in the title and perhaps more generally and comprehensively as before. It was not necessary to limit some measure of the perturbations in the interest of a prescribed measure of the perturbations in the interest of a prescribed measure of exactness; nonpersistent and persistent eigenvalues can be examined as well. Applicability of the results was explained to simpler and more complex problems. Multiplying the original equation by two kinds of partitioned matrices allowed an expressive interpretation of results. Lancaster's known formulas showed themselves to be a subcase of our results. We investigated possible forms of the dyadic perturbation which leave the main properties of a simply connected chain-like oscillator unchanged.

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TIME RATES OF TENSORS IN CONTINUUM MECHANICS UNDER ARBITRARY TIME DEPENDENT TRANSFORMATIONS PART II.

SYSTEMS OF MATERIALLY OBJECTIVE TIME RATES OF TENSORS

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Abstract. Relations have been deduced for the material time rates of deformation tensors of a body in coordinate systems moving arbitrarily with respect to each other. The concept of materially objective time rates of tensors is associated with the concept of co-ordinates systems moving arbitrarily (the transformations are time dependent and arbitrary) which respect to each other, rather than with rigid body motion (the transformations are orthogonal) as generally accepted in the literature. It has been shown that one part of the materially objective time rates of tensors found in literature and also those proposed in this paper are materially objective for arbitrary time dependent transformations, while their other part for orthogonal transformations only.

Mathematical Subject Classification: 73A05
Keywords: moving co-ordinate systems, deformation of a body, material time rates, materially objective time rates, comparisons

1. Introduction

1.1. The first part of this paper (Kozák [14]) gives the kinematic quantities for coordinate systems moving arbitrarily (capable of deformation) with respect to each other. It has also been shown how the material time rates defined in these coordinate systems are related to each other.

Making use of the results of the first part, the present second part gives a new and more general definition for the materially (physically) objective time rates of continuum mechanics. In addition, they are arranged into a system. An outline is provided about the issue how the physically objective time rates of continuum mechanics are related to each other both for those being defined in the paper and for those taken from the literature.

1.2. Section 2 is devoted to the material time rates of the tensors describing the deformations of continuum both in the fixed coordinate system and in another one moving arbitrarily with respect to the fixed one. The spin tensors of the principal directions of strains and the rotation tensor are also determined in these coordinate systems.

In Section 3 materially objective tensors determined by the velocity field and the deformation of continuum are sought and then the basic system of materially objective time rates of a tensor is defined by making use of the relations established between the material time rates defined in coordinate systems moving arbitrarily with respect to each other in Section 4 of Part I. It is also shown how the objective time rates found in the literature follow from the basic system and the objective tensors defined in the first part of the present section. In addition to this, new materially objective time rates are defined.

1.3. Notations and notational conventions are the same as in the first part. When citing equations of the first part, the equation number is followed by a point and the roman numeral "I".

2. **Material time rates of deformation tensors of a body**

2.1. For our later considerations the material time rates of the scalar line elements $\mathrm{d}s$, the stretch λ_e, the deformation gradient $\boldsymbol{F}$, the rotation tensor $\boldsymbol{FR}$, the left stretch tensor $\boldsymbol{V}$ and the eigenvectors $\mathbf{n}_p$ as well as the conclusions obtained from the investigation of these rates deserve special attention.

Consider first the material time rates defined in the coordinate system $\{x^p\}$.

For the vectorial line element $\mathrm{d}\mathbf{r}$, the scalar line element $\mathrm{d}s$, the stretch λ_e and the deformation gradient $\boldsymbol{F}$ we may write the known formulae

$$ {}^{(\mathrm{x})}(\mathrm{d}\mathbf{r})^{\cdot} = \mathrm{d}\,{}^{(\mathrm{x})}\mathbf{v} = {}^{(\mathrm{x})}\boldsymbol{L}\cdot\mathrm{d}\mathbf{r}\,, \tag{2.1} $$

$$ \frac{{}^{(\mathrm{x})}(\mathrm{d}s)^{\cdot}}{\mathrm{d}s} = \mathbf{e}\cdot{}^{(\mathrm{x})}\boldsymbol{D}\cdot\mathbf{e} = e^p\,{}^{(\mathrm{x})}d_{pq}e^q\,, \tag{2.2} $$

$$ \frac{{}^{(\mathrm{x})}\dot{\lambda}_e}{\lambda_e} = {}^{(\mathrm{x})}(\ln\lambda_e)^{\cdot} = \mathbf{e}\cdot{}^{(\mathrm{x})}\boldsymbol{D}\cdot\mathbf{e}\,, \tag{2.3} $$

$$ {}^{(\mathrm{x})}\boldsymbol{F}^{\cdot} = {}^{(\mathrm{x})}\boldsymbol{L}\cdot\boldsymbol{F}\,. \tag{2.4} $$

2.2. Let

$$ {}^{(\bar{\mathrm{n}}\mathrm{x})}\overline{\boldsymbol{W}} = {}^{(\bar{\mathrm{n}}\mathrm{x})}\overline{W}^K{}_L\,\overline{\mathbf{G}}_K\overline{\mathbf{G}}^L \tag{2.5} $$

be the spin tensor referring to the coordinate system $\{\overline{\nu}^p\}$ in the reference configuration (see Subsection 3.4. of Part I.). Further let

$$ {}^{(\mathrm{nx})}\boldsymbol{W} = {}^{(\mathrm{nx})}W^p_q\mathbf{g}_p\mathbf{g}^q \tag{2.6} $$

be the spin tensor referring to the co-ordinate system $\{\nu^p\}$ in the present configuration. ${}^{(\bar{\mathrm{n}}\mathrm{x})}\overline{\boldsymbol{W}}$ and ${}^{(\mathrm{nx})}\boldsymbol{W}$ are both skew tensors.

The eigenvectors $\overline{\mathbf{n}}_p$ and $\mathbf{n}_q$ of the principal axes of strains $\overline{\boldsymbol{U}}$ and $\boldsymbol{V}$ remain unit vectors during the deformation of the body and are always orthonormal to each other. Consequently, the following relations hold in the coordinate systems $\{\overline{\nu}^p\}$ and $\{\nu^p\}$ rotating together with the principal axes

$$ {}^{(\overline{\mathrm{n}})}\dot{\overline{\mathbf{n}}}_p = \mathbf{0} \quad \text{and} \quad {}^{(\mathrm{n})}\dot{\mathbf{n}}_q = \mathbf{0}\,. \tag{2.7} $$

Apply formulae (4.13.I) and determine the material time rates of the eigenvectors $\overline{\mathbf{n}}_p$ and $\mathbf{n}_q$ i.e., the corotate rates, taking into account that the motion of coordinate systems $\{\overline{\nu}^p\}$ and $\{\nu^p\}$ with respect to the coordinate system $\{x^p\}$ is a rigid body motion, i.e., ${}^{(\overline{\mathrm{n}}\mathrm{x})}\boldsymbol{L} = {}^{(\overline{\mathrm{n}}\mathrm{x})}\overline{\boldsymbol{W}}$ and ${}^{(nx)}\boldsymbol{L} = {}^{(nx)}\boldsymbol{W}$. In this way we have

$$ {}^{(\mathrm{x})}\dot{\overline{\mathbf{n}}}_p = {}^{(\overline{\mathrm{n}})}\dot{\overline{\mathbf{n}}}_p - \overline{\mathbf{n}}_p \cdot {}^{(\overline{\mathrm{n}}\mathrm{x})}\overline{\boldsymbol{W}} = {}^{(\overline{\mathrm{n}}\mathrm{x})}\overline{\boldsymbol{W}} \cdot \overline{\mathbf{n}}_p, \tag{2.8} $$

$$ {}^{(\mathrm{x})}\dot{\mathbf{n}}_q = {}^{(\mathrm{n})}\dot{\mathbf{n}}_q - \mathbf{n}_q \cdot {}^{(\mathrm{nx})}\boldsymbol{W} = {}^{(\mathrm{nx})}\boldsymbol{W} \cdot \mathbf{n}_q. \tag{2.9} $$

Following this we proceed to determine the material time rate of the expression $\mathbf{n}_p = \boldsymbol{R} \cdot \overline{\mathbf{n}}_p$ (see 3.15.I):

$$ {}^{(\mathrm{x})}\dot{\mathbf{n}}_p = {}^{(\mathrm{x})}\dot{\boldsymbol{R}} \cdot \overline{\mathbf{n}}_p + \boldsymbol{R} \cdot {}^{(\mathrm{x})}\dot{\overline{\mathbf{n}}}_p. $$

After substitutions and taking into account that $\mathbf{n}_p$ can be any of the eigenvectors, we obtain:

$$ {}^{(nx)}\boldsymbol{W} = {}^{(\mathrm{x})}\dot{\boldsymbol{R}} \cdot \boldsymbol{R}^{\mathrm{T}} + \boldsymbol{R} \cdot {}^{(\overline{\mathrm{n}}\mathrm{x})}\overline{\boldsymbol{W}} \cdot \boldsymbol{R}^{\mathrm{T}}. \tag{2.10} $$

Since both ${}^{(\mathrm{nx})}\boldsymbol{W}$ and ${}^{(\overline{\mathrm{n}}\mathrm{x})}\overline{\boldsymbol{W}}$ are skew and the product $\boldsymbol{R} \cdot {}^{(\overline{\mathrm{n}}\mathrm{x})}\overline{\boldsymbol{W}} \cdot \boldsymbol{R}^{\mathrm{T}}$ is also skew, it follows that so is ${}^{(\mathrm{x})}\dot{\boldsymbol{R}} \cdot \boldsymbol{R}^{\mathrm{T}}$. Let the corresponding spin tensor (the rate of rotation tensor) be

$$ {}^{(\mathrm{x})}\dot{\boldsymbol{R}} \cdot \boldsymbol{R}^{\mathrm{T}} = {}^{(\mathrm{Rx})}\boldsymbol{W}\,, \tag{2.11} $$

from which we obtain the material time rate of the rotation $\boldsymbol{R}$:

$$ {}^{(\mathrm{x})}\dot{\boldsymbol{R}} = {}^{(\mathrm{Rx})}\boldsymbol{W} \cdot \boldsymbol{R}. \tag{2.12} $$

As can be seen from (2.10) the spin tensors ${}^{(\overline{\mathrm{n}}\mathrm{x})}\overline{\boldsymbol{W}}$, ${}^{(\mathrm{nx})}\boldsymbol{W}$ and ${}^{(\mathrm{Rx})}\boldsymbol{W}$ we have introduced should meet the relation

$$ {}^{(\mathrm{nx})}\boldsymbol{W} = {}^{(\mathrm{Rx})}\boldsymbol{W} + \boldsymbol{R} \cdot {}^{(\overline{\mathrm{n}}\mathrm{x})}\overline{\boldsymbol{W}} \cdot \boldsymbol{R}^{\mathrm{T}}. \tag{2.13} $$

2.3. According to (3.16.I) it holds for the left stretch tensor in the coordinate system $\{\nu^p\}$ rotating together with the eigenvectors that

$$ {}^{(\mathrm{n})}\dot{\boldsymbol{V}} = {}^{(\mathrm{n})}\dot{\lambda}_{\underline{p}}\delta^p_q \mathbf{n}_p \mathbf{n}^q\,. \tag{2.14} $$

Therefore it follows from (4.14.I) that

$$ \begin{aligned} {}^{(\mathrm{x})}\dot{\boldsymbol{V}} &= {}^{(\mathrm{n})}\dot{\boldsymbol{V}} + {}^{(\mathrm{nx})}\boldsymbol{W} \cdot \boldsymbol{V} - \boldsymbol{V} \cdot {}^{(\mathrm{nx})}\boldsymbol{W} = \\ &= {}^{(\mathrm{x})}\dot{\lambda}_{\underline{p}}\delta^p_q \mathbf{n}_p \mathbf{n}^q + {}^{(\mathrm{nx})}\boldsymbol{W} \cdot \boldsymbol{V} - \boldsymbol{V} \cdot {}^{(\mathrm{nx})}\boldsymbol{W}. \end{aligned} \tag{2.15} $$

Here we have utilized the relation ${}^{(\mathrm{n})}\dot{\lambda}_{\underline{p}} = {}^{(\mathrm{x})}\dot{\lambda}_{\underline{p}}$ which holds in the coordinate systems $\{x^p\}$ and $\{\nu^p\}$ which perform a rigid body motion with respect to each other (For the sake of comparison it is worth recalling formula (2.21), which we present later).

2.4. Recalling the polar decomposition (3.14.I) for the deformation gradient $\boldsymbol{F}$, it follows from (2.4)

$$^{(\mathrm{x})}\boldsymbol{L} = {}^{(\mathrm{x})}\boldsymbol{F}^{\cdot} \cdot \boldsymbol{F}^{-1} = \left({}^{(\mathrm{x})}\boldsymbol{V}^{\cdot} \cdot \boldsymbol{R} + \boldsymbol{V} \cdot {}^{(\mathrm{x})}\boldsymbol{R}^{\cdot}\right) \cdot \boldsymbol{R}^{\mathrm{T}} \cdot \boldsymbol{V}^{-1} .$$

Making use of equations (2.15) and (2.12) we have

$$^{(\mathrm{x})}\boldsymbol{L} = {}^{(\mathrm{x})}(\ln \boldsymbol{V})^{\circ} + {}^{(\mathrm{nx})}\boldsymbol{W} + \boldsymbol{V}\left({}^{(\mathrm{Rx})}\boldsymbol{W} - {}^{(\mathrm{nx})}\boldsymbol{W}\right) \cdot \boldsymbol{V}^{-1} , \tag{2.16}$$

where

$${}^{(\mathrm{x})}(\ln \boldsymbol{V})^{\circ} = \left({}^{(\mathrm{x})}\dot{\lambda}_{\underline{p}} \delta^p_q \mathbf{n}_p \mathbf{n}^q\right) \cdot \boldsymbol{V}^{-1} = {}^{(\mathrm{x})}\left(\ln_{\underline{p}}\right)^{\cdot} \delta^p_q \mathbf{n}_p \mathbf{n}^q . \tag{2.17}$$

2.5. When formulating relative material time rates, we shall utilize the relations being set up between material time rates defined in the coordinate systems $\{x^p\}$ and $\{\widehat{x}^k\}$ in Section 4 of Part I.

In the case of the vectorial line element we obtain from (4.30.I), (2.1) and (3.27.I) that

$${}^{(\widehat{\mathrm{x}})}(\mathrm{d}\mathbf{r}^*)^{\cdot} = \mathrm{d}^{(\widehat{\mathrm{x}})}\mathbf{v} = {}^{(\mathrm{x})}(\mathrm{d}\mathbf{r})^{\cdot} - {}^{(\mathrm{Gx})}\boldsymbol{L} \cdot \mathrm{d}\mathbf{r} = {}^{(\widehat{\mathrm{x}})}\boldsymbol{L} \cdot \mathrm{d}\mathbf{r} . \tag{2.18}$$

For the scalar line element the result follows from the expression

$${}^{(\widehat{\mathrm{x}})}(\mathrm{d}s \cdot \mathrm{d}s)^{\cdot} = 2^{(\widehat{\mathrm{x}})}(\mathrm{d}s)^{\cdot}\, \mathrm{d}s = {}^{(\widehat{\mathrm{x}})}(\mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{r})^{\cdot} .$$

If we substitute equation (2.18) and take (3.28.I) also into consideration we obtain:

$$\frac{{}^{(\widehat{\mathrm{x}})}(\mathrm{d}s)^{\cdot}}{\mathrm{d}s} = \mathbf{e} \cdot {}^{(\widehat{\mathrm{x}})}\boldsymbol{D} \cdot \mathbf{e} = \mathbf{e} \cdot \left({}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{Gx})}\boldsymbol{D}\right) \cdot \mathbf{e} = \frac{{}^{(\mathrm{x})}(\mathrm{d}s)^{\cdot}}{\mathrm{d}s} - \mathbf{e} \cdot {}^{(\mathrm{Gx})}\boldsymbol{D} \cdot \mathbf{e} , \tag{2.19}$$

$$\frac{{}^{(\widehat{\mathrm{x}})}\dot{\lambda}_e}{\lambda_e} = {}^{(\widehat{\mathrm{x}})}(\ln \lambda_e)^{\cdot} = {}^{(\mathrm{x})}(\ln \lambda_e)^{\cdot} - \mathbf{e} \cdot {}^{(\mathrm{Gx})}\boldsymbol{D} \cdot \mathbf{e} . \tag{2.20}$$

Here $\mathbf{e}$ is the unit vector in the direction of the line element.

It follows from (2.17) and (2.20) that

$${}^{(\mathrm{x})}(\ln \boldsymbol{V})^{\circ} = {}^{(\mathrm{x})}\left(\ln \lambda_{\underline{p}}\right)^{\cdot} \delta^p_q \mathbf{n}_p \mathbf{n}^q = \left[{}^{(\widehat{\mathrm{x}})}\left(\ln \lambda_{\underline{p}}\right)^{\cdot} + \mathbf{n}_{\underline{p}} \cdot {}^{(\mathrm{Gx})}\boldsymbol{D} \cdot \mathbf{n}^{\underline{q}}\right] \delta^p_q \mathbf{n}_p \mathbf{n}^q . \tag{2.21}$$

2.6. In the case of the deformation gradient equation (4.30.I) should be applied since $\boldsymbol{F}$ has only one index relating to the present configuration. Substituting (2.4) and (3.27.I) equation

$${}^{(\widehat{\mathrm{x}})}(\boldsymbol{F}^*)^{\cdot} = {}^{(\mathrm{x})}\boldsymbol{F}^{\cdot} - {}^{(\mathrm{Gx})}\boldsymbol{D} \cdot \boldsymbol{F} = {}^{(\widehat{\mathrm{x}})}\boldsymbol{L} \cdot \boldsymbol{F} \tag{2.22}$$

is obtained.

As regards the left stretch tensor $\boldsymbol{V}$, equation (4.14.I) should be followed:

$${}^{(\widehat{\mathrm{x}})}(\boldsymbol{V}^*_*)^{\cdot} = {}^{(\mathrm{x})}\boldsymbol{V}^{\cdot} - {}^{(\mathrm{Gx})}\boldsymbol{D} \cdot \boldsymbol{V} + \boldsymbol{V} \cdot {}^{(\mathrm{Gx})}\boldsymbol{D} .$$

In view of equation (2.15) we find

$$
{}^{(\hat{\mathrm{x}})}\left(\boldsymbol{V}^{*}_{*}\right)^{\cdot} = {}^{(\mathrm{x})}\dot{\lambda}_{\underline{p}}\delta^{p}_{q}\mathbf{n}_{p}\mathbf{n}^{q} - {}^{(\mathrm{Gx})}\boldsymbol{D}\cdot\boldsymbol{V} + \boldsymbol{V}\cdot{}^{(\mathrm{Gx})}\boldsymbol{D} + \\
+ \left({}^{(\mathrm{nx})}\boldsymbol{W} - {}^{(\mathrm{Gx})}\boldsymbol{W}\right)\cdot\boldsymbol{V} - \boldsymbol{V}\cdot\left({}^{(\mathrm{nx})}\boldsymbol{W} - {}^{(\mathrm{Gx})}\boldsymbol{W}\right). \tag{2.23}
$$

The relative velocity gradient is obtained with the aid of equation (3.27.I). Substituting equation (2.16) we can write

$$
{}^{(\hat{\mathrm{x}})}\boldsymbol{L} = {}^{(\mathrm{x})}\left(\ln\boldsymbol{V}\right)^{\circ} + \left({}^{(\mathrm{nx})}\boldsymbol{W} - {}^{(\mathrm{Gx})}\boldsymbol{W}\right) - \\
- \boldsymbol{V}\cdot\left[\left({}^{(\mathrm{nx})}\boldsymbol{W} - {}^{(\mathrm{Gx})}\boldsymbol{W}\right) - \left({}^{(\mathrm{Rx})}\boldsymbol{W} - {}^{(\mathrm{Gx})}\boldsymbol{W}\right)\right] - {}^{(\mathrm{Gx})}\boldsymbol{D}, \tag{2.24}
$$

where

$$
{}^{(\mathrm{x})}\left(\ln\boldsymbol{V}\right)^{\circ} = \left[{}^{(\hat{\mathrm{x}})}\left(\ln\lambda_{\underline{p}}\right)^{\cdot} + \mathbf{n}_{\underline{p}}\cdot{}^{(\mathrm{Gx})}\boldsymbol{D}\cdot\mathbf{n}^{\underline{q}}\right]\delta^{p}_{q}\mathbf{n}_{p}\mathbf{n}^{q}\,.
$$

Remark: The result (2.24) could also be deduced from the equation for the relative velocity gradient ${}^{(\hat{\mathrm{x}})}\boldsymbol{L} = {}^{(\hat{\mathrm{x}})}\left(\boldsymbol{F}^{*}\right)^{\cdot}\cdot\boldsymbol{F}$, which follows from (2.22).

2.7. The well-known material time rate of the Eulerian strain tensor, i.e., the relation

$$
\boldsymbol{E} = \frac{1}{2}\left[\boldsymbol{I} - \left(\boldsymbol{F}^{-1}\right)^{\mathrm{T}}\cdot\boldsymbol{F}^{-1}\right] \tag{2.25}
$$

valid in the coordinate system $\{x^{p}\}$ is obtained by utilizing equation (2.4)

$$
{}^{(\mathrm{x})}\boldsymbol{E}^{\cdot} = {}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{x})}\boldsymbol{L}^{\mathrm{T}}\cdot\boldsymbol{E} - \boldsymbol{E}\cdot{}^{(\mathrm{x})}\boldsymbol{L}\,. \tag{2.26}
$$

Making use of equation (4.13.I) we can write in the coordinate system $\left\{\widehat{x}^{k}\right\}$

$$
{}^{(\hat{\mathrm{x}})}\left(\boldsymbol{E}_{**}\right)^{\cdot} = {}^{(\hat{\mathrm{x}})}\boldsymbol{D} - {}^{(\hat{\mathrm{x}})}\boldsymbol{L}^{\mathrm{T}}\cdot\boldsymbol{E} - \boldsymbol{E}\cdot{}^{(\hat{\mathrm{x}})}\boldsymbol{L} + {}^{(\mathrm{Gx})}\boldsymbol{D}\,. \tag{2.27}
$$

2.8. Now we introduce the quantities

$$
{}^{(\mathrm{n}\hat{\mathrm{x}})}\boldsymbol{W} = {}^{(\mathrm{nx})}\boldsymbol{W} - {}^{(\mathrm{Gx})}\boldsymbol{W}, \quad {}^{(\mathrm{n}\hat{\mathrm{x}})}w^{p}_{\ q} = {}^{(\mathrm{nx})}w^{p}_{\ q} - {}^{(\mathrm{Gx})}w^{p}_{\ q}\,, \tag{2.28}
$$

and

$$
{}^{(\mathrm{R}\hat{\mathrm{x}})}\boldsymbol{W} = {}^{(\mathrm{Rx})}\boldsymbol{W} - {}^{(\mathrm{Gx})}\boldsymbol{W}, \quad {}^{(\mathrm{R}\hat{\mathrm{x}})}w^{p}_{\ q} = {}^{(\mathrm{Rx})}w^{p}_{\ q} - {}^{(\mathrm{Gx})}w^{p}_{\ q}. \tag{2.29}
$$

where ${}^{(\mathrm{n}\hat{\mathrm{x}})}\boldsymbol{W}$ and ${}^{(\mathrm{nx})}\boldsymbol{W}$ are both the spin tensors of the coordinate system $\{\nu^{p}\}$. Moreover ${}^{(\mathrm{R}\hat{\mathrm{x}})}\boldsymbol{W}$ and ${}^{(\mathrm{Rx})}\boldsymbol{W}$ are both rates of the rotation tensor.

One can come to further conclusions from formulae (2.16) and (2.24) giving velocity gradients in the coordinate systems $\{x^{p}\}$ and $\left\{\widehat{x}^{k}\right\}$. For this purpose we shall write the formulae mentioned in the coordinate system $\{\nu^{p}\}$ of the principal axes by taking the additive decomposition of ${}^{(\mathrm{x})}\boldsymbol{L}$ and ${}^{(\hat{\mathrm{x}})}\boldsymbol{L}$ also into consideration:

$$
{}^{(\mathrm{x})}\left(\ln\lambda_{\underline{p}}\right)^{\cdot}\delta^{p}_{q} + {}^{(\mathrm{nx})}w^{p}_{\ q}\left(1 - \frac{\lambda_{\underline{p}}}{\lambda_{\underline{q}}}\right) + {}^{(\mathrm{Rx})}w^{p}_{\ q}\frac{\lambda_{\underline{p}}}{\lambda_{\underline{q}}} = {}^{(\mathrm{x})}d^{p}_{q} + {}^{(\mathrm{x})}w^{p}_{\ q}\,, \tag{2.30}
$$

$$
{}^{(\hat{\mathrm{x}})}\left(\ln\lambda_{\underline{p}}\right)^{\cdot}\delta^{p}_{q} + {}^{(\mathrm{n}\hat{\mathrm{x}})}w^{p}_{\ q}\left(1 - \frac{\lambda_{\underline{p}}}{\lambda_{\underline{q}}}\right) + {}^{(\mathrm{R}\hat{\mathrm{x}})}w^{p}_{\ q}\frac{\lambda_{\underline{p}}}{\lambda_{\underline{q}}} = \\
= -\left(\mathbf{n}_{\underline{p}}\cdot{}^{(\mathrm{Gx})}\boldsymbol{D}\cdot\mathbf{n}^{\underline{q}}\right)\delta^{p}_{q} + {}^{(\mathrm{x})}d^{p}_{q} + {}^{(\hat{\mathrm{x}})}w^{p}_{\ q}\,. \tag{2.31}
$$

It should be emphasized that equations (2.30), (2.31) and formulae (2.32)-(2.39) presented later are all regarded in the coordinate system $\{\nu^p\}$ of the principal strains without drawing the reader's attention to this fact by a separate notation.

Assuming the velocity gradient ${}^{(\mathrm{x})}\boldsymbol{L}$ to be known, formulae (2.30) can be regarded as a linear system of equations with nine unknowns by which are meant the three values ${}^{(\mathrm{x})}(\ln\lambda_p)^{\cdot}$ and the three independent components of the skew tensors ${}^{(\mathrm{nx})}w^p{}_q$ and ${}^{(\mathrm{Rx})}w^p{}_q$ each. To find the unknowns from (2.30) we shall set up two sets of linear equations:

$$ {}^{(\mathrm{x})}(\ln\lambda_p)^{\cdot} = \frac{{}^{(\mathrm{x})}\dot{\lambda}_p}{\lambda_{\underline{p}}} = {}^{(\mathrm{x})}d^{\underline{p}}_{\underline{p}}\,, \qquad p = q\,, \tag{2.32} $$

$$ {}^{(\mathrm{nx})}w^p{}_q\left(1-\frac{\lambda_{\underline{p}}}{\lambda_{\underline{q}}}\right) + {}^{(\mathrm{Rx})}w^p{}_q\frac{\lambda_{\underline{p}}}{\lambda_{\underline{q}}} = {}^{(\mathrm{x})}d^p_q + {}^{(\mathrm{x})}w^p{}_q\,, \qquad p\neq q\,, \tag{2.33} $$

$$ -{}^{(\mathrm{nx})}w^p{}_q\left(1-\frac{\lambda_{\underline{q}}}{\lambda_{\underline{p}}}\right) - {}^{(\mathrm{Rx})}w^p{}_q\frac{\lambda_{\underline{q}}}{\lambda_{\underline{p}}} = {}^{(\mathrm{x})}d^p_q - {}^{(\mathrm{x})}w^p{}_q\,, \qquad p\neq q\,. \tag{2.34} $$

Assuming different eigenvalues, i.e., $\lambda_1\neq\lambda_2\neq\lambda_3\neq\lambda_1$, the solution of equation systems (2.33), (2.34) takes the form

$$ {}^{(\mathrm{nx})}w^p{}_q = {}^{(\mathrm{x})}w^p{}_q - \frac{\left(\lambda_{\underline{p}}\right)^2+\left(\lambda_{\underline{q}}\right)^2}{\left(\lambda_{\underline{p}}\right)^2-\left(\lambda_{\underline{q}}\right)^2}\,{}^{(\mathrm{x})}d^p_q\,, \qquad p\neq q\,, \tag{2.35} $$

$$ {}^{(\mathrm{Rx})}w^p{}_q = {}^{(\mathrm{x})}w^p{}_q - \frac{\lambda_{\underline{p}}-\lambda_{\underline{q}}}{\lambda_{\underline{p}}+\lambda_{\underline{q}}}\,{}^{(\mathrm{x})}d^p_q\,, \qquad p\neq q\,. \tag{2.36} $$

The spin tensor of the coordinate system $\{\overline{\nu}^p\}$ in the reference configuration is obtained from (2.13):

$$ {}^{(\overline{\mathrm{n}}\mathrm{x})}w^K{}_L = -R_p{}^K\frac{2\lambda_{\underline{p}}\lambda_{\underline{q}}}{\left(\lambda_{\underline{p}}\right)^2-\left(\lambda_{\underline{q}}\right)^2}\,{}^{(\mathrm{x})}d^p_q R^q{}_L, \qquad K\neq L. \tag{2.37} $$

For coinciding eigenvalues the above results change to a certain extent. If, for example, $\lambda_1=\lambda_2\neq\lambda_3$, then

- for $p=1$ and $p=2$

$$ {}^{(\mathrm{Rx})}w^1{}_2 = {}^{(\mathrm{x})}w^1{}_2 \quad \text{while} \quad {}^{(\mathrm{nx})}w^1{}_2 \quad \text{is undetermined,} $$

- for $p\neq q$ and $p,q=2,3$ formulae (2.35), (2.36) remain valid.

If $\lambda_1=\lambda_2=\lambda_3$ and $p\neq q$, then

$$ {}^{(\mathrm{Rx})}w^p{}_q = {}^{(\mathrm{x})}w^p{}_q \quad \text{while} \quad {}^{(\mathrm{nx})}w^p{}_q \quad \text{is undetermined.} $$

In view of the formal similarity of (2.30) to (2.31) it follows at once by repeating the preceding line of thought that on the one hand:

$$ {}^{(\widehat{\mathrm{x}})}\left(\ln \lambda_p\right)^{\cdot} = {}^{(\widehat{\mathrm{x}})}d^p_{\underline{p}} = {}^{(\mathrm{x})}d^p_{\underline{p}} - {}^{(\mathrm{Gx})}d^p_{\underline{p}} $$

[which has already been known - see (2.20)], and on the other hand:

$$ {}^{(\mathrm{n}\widehat{\mathrm{x}})}w^p{}_q = {}^{(\widehat{\mathrm{x}})}w^p{}_q - \frac{\left(\lambda_{\underline{p}}\right)^2 + \left(\lambda_{\underline{q}}\right)^2}{\left(\lambda_{\underline{p}}\right)^2 - \left(\lambda_{\underline{q}}\right)^2}\,{}^{(\mathrm{x})}d^p_q\,, \qquad p \neq q\,, \tag{2.38} $$

$$ {}^{(\mathrm{R}\widehat{\mathrm{x}})}w^p{}_q = {}^{(\widehat{\mathrm{x}})}w^p{}_q - \frac{\lambda_{\underline{p}} - \lambda_{\underline{q}}}{\lambda_{\underline{p}} + \lambda_{\underline{q}}}\,{}^{(\mathrm{x})}d^p_q\,, \qquad p \neq q\,. \tag{2.39} $$

2.9. Comparing (2.35) and (2.38) as well as (2.36) and (2.39), further results are obtained for the spin tensors:

$$ {}^{(\widehat{\mathrm{x}})}\boldsymbol{W} - {}^{(\mathrm{n}\widehat{\mathrm{x}})}\boldsymbol{W} = {}^{(\mathrm{x})}\boldsymbol{W} - {}^{(\mathrm{nx})}\boldsymbol{W}, \tag{2.40} $$

$$ {}^{(\widehat{\mathrm{x}})}\boldsymbol{W} - {}^{(\mathrm{R}\widehat{\mathrm{x}})}\boldsymbol{W} = {}^{(\mathrm{x})}\boldsymbol{W} - {}^{(\mathrm{Rx})}\boldsymbol{W}. \tag{2.41} $$

By subtracting (2.40) from (2.41) we have

$$ {}^{(\mathrm{n}\widehat{\mathrm{x}})}\boldsymbol{W} - {}^{(\mathrm{R}\widehat{\mathrm{x}})}\boldsymbol{W} = {}^{(\mathrm{nx})}\boldsymbol{W} - {}^{(\mathrm{Rx})}\boldsymbol{W}. \tag{2.42} $$

2.10. In the case when ${}^{(\mathrm{x})}d^p_q = 0$ for $p \neq q$, i.e., when the angles formed by the principal axes of the stretch tensors do not change at the instant under consideration:

$$ {}^{(nx)}\boldsymbol{W} = {}^{(Rx)}\boldsymbol{W} = {}^{(x)}\boldsymbol{W}, \quad {}^{(\overline{n}x)}\boldsymbol{W} = 0 \quad \text{and} \quad {}^{(n\widehat{x})}\boldsymbol{W} = {}^{(R\widehat{x})}\boldsymbol{W} = {}^{(\widehat{x})}\boldsymbol{W}. \tag{2.43} $$

3. **Materially objective time rates of tensors and their system**

3.1. In the present paper tensors are understood as materially (or physically) objective (or objective for short) quantities if their components follow the general transformation laws of tensors in coordinate systems moving arbitrarily (being deformable) with respect to each other.

Summarizing briefly: an objective tensor and its objective time rates are invariant under arbitrary time dependent transformations, more briefly: if they are objective, they can be defined independently of the choice of coordinate systems.

In literature invariance under orthogonal time dependent transformations is understood as the criterion for objectivity. For this reason we distinguish, in the sequel, tensors being objective under arbitrary time dependent transformation from those being objective for orthogonal time dependent transformation only.

3.2. On the basis of the results presented in Sections 2, 3 and 4 of Part I we can conclude that the tensors describing the time rate of change of tensors and being, therefore, defined in a certain co-ordinate system do not satisfy, in general, the above criterion of objectivity. As examples we could mention, among others, the velocity

vector fields, the velocity gradient, the strain rate tensor, the spin tensors (3.36.I)-(3.39.I), the material time rates (4.13.I)-(4.16.I) of a tensor, and the material time rates of line elements and those of stretches (2.19), (2.20).

At the same time we can also conclude on the basis of formulae (2.40)-(2.42) that some tensors, which are defined as a difference between various spin tensors, are independent of the choice of a coordinate system even if the coordinate systems under consideration move arbitrarily with respect to each other. They are, therefore, objective.

If we restrict the criterion for objectivity to orthogonal transformations only, additional objective tensors can be obtained. In this case, ${}^{(\mathrm{Gx})}\boldsymbol{D} = 0$ and for example according to (2.19), (2.20), the material time rates of the scalar line element and the stretch are objective.

In the present Section the objective tensors are considered first and then their objective time rates are investigated.

Since the sum, difference and product of objective tensors are also objective additional objective tensors can be established with the aid of the known ones.

In what follows, marking the co-ordinate system $\{x^p\}$ as such in which the tensors are defined, we collect the objective tensors introduced in Section 2 and, in addition, we also deduce further objective tensors by means of the rule mentioned in the preceding paragraph.

The tensors and the time rates of tensors that are objective under arbitrary time dependent transformation will be separated from those being objective under orthogonal transformation only.

3.3. *Objective tensors under arbitrary time dependent transformations*:

- according to (2.40)-(2.42):

$$ {}^{(\mathrm{x})}\boldsymbol{W} - {}^{(\mathrm{nx})}\boldsymbol{W}, \quad {}^{(\mathrm{x})}\boldsymbol{W} - {}^{(\mathrm{Rx})}\boldsymbol{W}, \quad {}^{(\mathrm{nx})}\boldsymbol{W} - {}^{(\mathrm{Rx})}\boldsymbol{W}, \tag{3.1} $$

- according to (3.1):

$$ {}^{(\mathrm{x})}\boldsymbol{L} - {}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{nx})}\boldsymbol{W}, \quad {}^{(\mathrm{x})}\boldsymbol{L} - {}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{Rx})}\boldsymbol{W}, \quad \boldsymbol{V}\cdot\Big({}^{(\mathrm{nx})}\boldsymbol{W} - {}^{(\mathrm{Rx})}\boldsymbol{W}\Big)\cdot\boldsymbol{V}^{-1}, \tag{3.2} $$

- according to (3.2):

$$ {}^{(\mathrm{x})}\boldsymbol{L} - {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}} \quad \text{where} \quad {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}} = {}^{(\mathrm{x})}\boldsymbol{D} + {}^{(\mathrm{nx})}\boldsymbol{W} - \boldsymbol{V}\cdot\Big({}^{(\mathrm{nx})}\boldsymbol{W} - {}^{(\mathrm{Rx})}\boldsymbol{W}\Big)\cdot\boldsymbol{V}^{-1}. \tag{3.3} $$

3.4. *Objective tensors under orthogonal time dependent transformations only.* In the case of an orthogonal transformation motion of the grid (i.e. of the coordinate system $\{\widehat{x}^k\}$ relative to the coordinate system $\{x^p\}$) is that of a rigid body. Consequently, in this case the tensors listed below are all objective:

- according to (3.28.I) the strain rate tensor:

$$ {}^{(\mathrm{x})}\boldsymbol{D}, \tag{3.4} $$

- according to (3.2) and (3.4):

$$ {}^{(\mathrm{x})}\boldsymbol{L} - {}^{(\mathrm{nx})}\boldsymbol{W}, \quad {}^{(\mathrm{x})}\boldsymbol{L} - {}^{(\mathrm{Rx})}\boldsymbol{W}, \tag{3.5} $$

- according to (2.19) and (2.20):

$$ {}^{(\mathrm{x})}(\mathrm{d}s)^{\cdot}, \quad {}^{(\mathrm{x})}(\lambda_e)^{\cdot}, \quad {}^{(\mathrm{x})}(\ln\lambda_e)^{\cdot}, \tag{3.6} $$

- according to (2.17) and (3.6):

$$ {}^{(\mathrm{x})}(\ln\boldsymbol{V})^{\circ} = {}^{(\mathrm{x})}\left(\ln_{\underline{p}}\right)^{\cdot}\delta^p_q\mathbf{n}_p\mathbf{n}^q, \tag{3.7} $$

- according to (3.2), (3.4) and (3.7)

$$ {}^{(\mathrm{x})}\boldsymbol{L} - {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{II}}, \tag{3.8a} $$

where

$$ {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{II}} = {}^{(\mathrm{nx})}\boldsymbol{W} + {}^{(\mathrm{x})}(\ln\boldsymbol{V})^{\circ} = {}^{(\mathrm{x})}\boldsymbol{V}^{\cdot}\cdot\boldsymbol{V}^{-1} + \boldsymbol{V}\cdot{}^{(\mathrm{nx})}\boldsymbol{W}\cdot\boldsymbol{V}^{-1}, $$

and

$$ {}^{(\mathrm{x})}\boldsymbol{L} - {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{III}}, \tag{3.8b} $$

where

$$ {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{III}} = {}^{(\mathrm{Rx})}\boldsymbol{W} + {}^{(\mathrm{x})}(\ln\boldsymbol{V})^{\cdot}, $$

- according to (3.5) and (3.6):

$$ {}^{(\mathrm{x})}\boldsymbol{L} - {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{IV}}, \tag{3.9} $$

where $ {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{IV}} = {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}} - {}^{(\mathrm{x})}\boldsymbol{D} = {}^{(\mathrm{nx})}\boldsymbol{W} - \boldsymbol{V}\cdot\left({}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{Rx})}\boldsymbol{W}\right)\cdot\boldsymbol{V}^{-1}$.

3.5. Making use of the equations (4.13.I)-(4.16.I) one can define the objective time rate of an arbitrary tensor (the term objective refers to the objectivity of the materially objective time rate as a tensor).

3.6. *Objective time rates for arbitrary time dependent transformations.* Substitution of ${}^{(Gx)}\boldsymbol{L}$ from (3.27.I) into (4.13.I) yields:

$$ {}^{(\hat{\mathrm{x}})}(\boldsymbol{A}_{**})^{\cdot} + {}^{(\hat{\mathrm{x}})}\boldsymbol{L}^{\mathrm{T}}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot{}^{(\hat{\mathrm{x}})}\boldsymbol{L} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + {}^{(\mathrm{x})}\boldsymbol{L}^{\mathrm{T}}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot{}^{(\mathrm{x})}\boldsymbol{L}. $$

Similarly, other three relations, which are not detailed here, can be obtained from (4.14.I)-(4.16.I).

Now we can define the so called *basic system of the time rates* [written here in the coordinate system $\{x^p\}$] *being objective under any time dependent transformations* (or *the basic system of objective rates* for brevity's sake). Denoting them by a small triangle used as superscript we may write:

$$ \text{I.} \quad (\boldsymbol{A}_{**})^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + {}^{(\mathrm{x})}\boldsymbol{L}^{\mathrm{T}}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot{}^{(\mathrm{x})}\boldsymbol{L}, \tag{3.10} $$

$$ \text{II.} \quad (\boldsymbol{A}^{*}_{\ *})^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{L}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot{}^{(\mathrm{x})}\boldsymbol{L}, \tag{3.11} $$

$$ \text{III.} \quad (\boldsymbol{A}_{*}^{\ *})^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + {}^{(\mathrm{x})}\boldsymbol{L}^{\mathrm{T}}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot{}^{(\mathrm{x})}\boldsymbol{L}^{\mathrm{T}}, \tag{3.12} $$

$$ \text{IV.} \quad (\boldsymbol{A}^{**})^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{L}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot{}^{(\mathrm{x})}\boldsymbol{L}^{\mathrm{T}}. \tag{3.13} $$

We write only equation (3.11) in indicial notation:

$$\left(a^{p}{}_{q}\right)^{\nabla} = {}^{(\mathrm{x})}\left(a^{p}{}_{q}\right)^{\cdot} - {}^{(\mathrm{x})}l^{p}{}_{s}a^{s}{}_{q} + a^{p}{}_{s}\,{}^{(\mathrm{x})}l^{s}{}_{q}. \tag{3.14}$$

Equation (3.10) can be identified as the *Cotter-Rivlin rate* (1955, [8]); (3.13) as the *Oldroyd* (1950, [6]) or *Trusdell* (1955, [7]) *rate*. All the four objective rates have been given by *Atluri* (1984, [2]).

The objective time rates obey the general rules valid for addition and multiplication of tensors.

Decomposing the tensor ${}^{(\mathrm{x})}\boldsymbol{L}$ and introducing the quantity

$${}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{w}} = {}^{(\mathrm{x})}\boldsymbol{W}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot{}^{(\mathrm{x})}\boldsymbol{W}, \tag{3.15}$$

the basic system (3.10)-(3.13) can be transformed into other form:

$$\text{I.}\quad (\boldsymbol{A}_{**})^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + {}^{(\mathrm{x})}\boldsymbol{D}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot{}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{w}}, \tag{3.16}$$

$$\text{II.}\quad (\boldsymbol{A}^{*}_{*})^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{D}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot{}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{w}}, \tag{3.17}$$

$$\text{III.}\quad (\boldsymbol{A}_{*}^{*})^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + {}^{(\mathrm{x})}\boldsymbol{D}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot{}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{w}}, \tag{3.18}$$

$$\text{IV.}\quad (\boldsymbol{A}^{**})^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{D}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot{}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{w}}. \tag{3.19}$$

From the basic system of objective rates additional objective rates can be deduced with the help of objective tensors (see Subsection 3.3. on page 148). Consider (3.1) as an example. Since the tensor

$$\left({}^{(\mathrm{x})}\boldsymbol{W} - {}^{(\mathrm{nx})}\boldsymbol{W}\right)\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\left({}^{(\mathrm{x})}\boldsymbol{W} - {}^{(\mathrm{nx})}\boldsymbol{W}\right)$$

is also physically objective, new objective rates are obtained by adding it to the right hand side of equations (3.16)-(3.19). Following the same procedure in respect of equation (3.1), let us introduce the quantities

$$\begin{aligned}{}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{wn}} &= {}^{(\mathrm{x})}\boldsymbol{W}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot{}^{(\mathrm{x})}\boldsymbol{W} - \left({}^{(\mathrm{x})}\boldsymbol{W} - {}^{(\mathrm{nx})}\boldsymbol{W}\right)\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\left({}^{(\mathrm{x})}\boldsymbol{W} - {}^{(\mathrm{nx})}\boldsymbol{W}\right)\\ &= {}^{(\mathrm{nx})}\boldsymbol{W}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot{}^{(\mathrm{nx})}\boldsymbol{W},\end{aligned} \tag{3.20}$$

$$\begin{aligned}{}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{wR}} &= {}^{(\mathrm{x})}\boldsymbol{W}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot{}^{(\mathrm{x})}\boldsymbol{W} - \left({}^{(\mathrm{x})}\boldsymbol{W} - {}^{(\mathrm{Rx})}\boldsymbol{W}\right)\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\left({}^{(\mathrm{x})}\boldsymbol{W} - {}^{(\mathrm{Rx})}\boldsymbol{W}\right)\\ &= {}^{(\mathrm{Rx})}\boldsymbol{W}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot{}^{(\mathrm{Rx})}\boldsymbol{W}\end{aligned} \tag{3.21}$$

by the use of which we obtain the objective rates:

$$\text{I.}\quad (\boldsymbol{A}_{**})^{\nabla}_{\mathrm{J}} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + {}^{(\mathrm{x})}\boldsymbol{D}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot{}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{J}}, \tag{3.22}$$

$$\text{II.}\quad (\boldsymbol{A}^{*}_{*})^{\nabla}_{\mathrm{J}} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{D}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot{}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{J}}, \tag{3.23}$$

$$\text{III.}\quad (\boldsymbol{A}_{*}^{*})^{\nabla}_{\mathrm{J}} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + {}^{(\mathrm{x})}\boldsymbol{D}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot{}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{J}}, \tag{3.24}$$

$$\text{IV.}\quad (\boldsymbol{A}^{**})^{\nabla}_{\mathrm{J}} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{D}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot{}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{J}}. \tag{3.25}$$

where "J" is either "wn" or "wR".

New objective rates are obtained with the help of the objective tensor (3.3). According to (3.10) and (3.3) the rate

$$(\boldsymbol{A}_{**})^{\nabla}_{\mathrm{I}} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + {}^{(\mathrm{x})}\boldsymbol{L}^{\mathrm{T}} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot {}^{(\mathrm{x})}\boldsymbol{L} -$$

$$- \left[\left({}^{(\mathrm{x})}\boldsymbol{L} - {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}} \right)^{\mathrm{T}} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot \left({}^{(\mathrm{x})}\boldsymbol{L} - {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}} \right) \right] = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + \left({}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}} \right)^{\mathrm{T}} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}}$$

is, for example, objective. In this way we find the following objective rates:

$$\text{I.} \quad (\boldsymbol{A}_{**})^{\nabla}_{\mathrm{I}} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + \left({}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}} \right)^{\mathrm{T}} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}}, \tag{3.26}$$

$$\text{II.} \quad (\boldsymbol{A}^{*}{}_{*})^{\nabla}_{\mathrm{I}} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}}, \tag{3.27}$$

$$\text{III.} \quad (\boldsymbol{A}_{*}{}^{*})^{\nabla}_{\mathrm{I}} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + \left({}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}} \right)^{\mathrm{T}} \cdot \boldsymbol{A} - \boldsymbol{A} \cdot \left({}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}} \right)^{\mathrm{T}}, \tag{3.28}$$

$$\text{IV.} \quad (\boldsymbol{A}^{**})^{\nabla}_{\mathrm{I}} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}} \cdot \boldsymbol{A} - \boldsymbol{A} \cdot \left({}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}} \right)^{\mathrm{T}}. \tag{3.29}$$

Since ${}^{(\mathrm{x})}\mathbf{v} = \mathbf{0}$ and ${}^{(\mathrm{x})}\boldsymbol{L} = \mathbf{0}$ in the convected coordinate system, making use of the convected material time rate deduced in the same manner as (4.11.I), we obtain

$$\text{I.} \quad (\boldsymbol{A}_{**})^{\nabla} = {}^{(\mathrm{X})}(\boldsymbol{A}_{**})^{\cdot} = \left. \frac{\partial a_{KL}}{\partial t} \right|_{(\mathrm{X})} \mathbf{G}^{K}\mathbf{G}^{L}, \tag{3.30}$$

$$\text{II.} \quad (\boldsymbol{A}^{*}{}_{*})^{\nabla} = {}^{(\mathrm{X})}(\boldsymbol{A}^{*}{}_{*})^{\cdot} = \left. \frac{\partial a^{K}{}_{L}}{\partial t} \right|_{(\mathrm{X})} \mathbf{G}_{K}\mathbf{G}^{L}, \tag{3.31}$$

$$\text{III.} \quad (\boldsymbol{A}_{*}{}^{*})^{\nabla} = {}^{(\mathrm{X})}(\boldsymbol{A}_{*}{}^{*})^{\cdot} = \left. \frac{\partial a_{K}{}^{L}}{\partial t} \right|_{(\mathrm{X})} \mathbf{G}^{K}\mathbf{G}_{L}, \tag{3.32}$$

$$\text{IV.} \quad (\boldsymbol{A}^{**})^{\nabla} = {}^{(\mathrm{X})}(\boldsymbol{A}^{**})^{\cdot} = \left. \frac{\partial a^{KL}}{\partial t} \right|_{(\mathrm{X})} \mathbf{G}_{K}\mathbf{G}_{L} \tag{3.33}$$

for the basic system of objective rates in convected coordinates. Equations (3.30)-(3.33) have already been given by *Sedov* (1960, [3]) and *Atluri* (1984, [2]).

3.7. *Objective time rates for orthogonal time dependent transformations.* According to (3.4) the strain rate tensor is objective for orthogonal time dependent transformations. Consequently, from equations (3.16)-(3.19) and (3.22)-(3.25) objective rates independent of the index positions are at once obtained:

$$\boldsymbol{A}^{\nabla}_{\mathrm{w}} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{W} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot {}^{(\mathrm{x})}\boldsymbol{W}, \tag{3.34}$$

$$\boldsymbol{A}^{\nabla}_{\mathrm{wn}} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{nx})}\boldsymbol{W} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot {}^{(\mathrm{nx})}\boldsymbol{W}, \tag{3.35}$$

$$\boldsymbol{A}^{\nabla}_{\mathrm{wR}} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{Rx})}\boldsymbol{W} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot {}^{(\mathrm{Rx})}\boldsymbol{W}. \tag{3.36}$$

Equation (3.34) can be identified as the *Jaumann rate* (1911, [1]); (3.35) as the *Soverby-Chu rate* (1984, [12]); (3.36) as the *Green-Naghdi* (1965, [9]) or *Green-McInnis* (1967, [10]) or *Dienes* (1984, [11]) or *Atluri* (1984, [2]) *rate*. These equations have also been given by *Dubey* (1987, [4]), who employed a method that he called principal axis technique.

Since ${}^{(\mathrm{x})}\boldsymbol{D}$ is objective, so is the rate

$$\boldsymbol{A}^{\nabla}_{\mathrm{S}} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{W}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot{}^{(\mathrm{x})}\boldsymbol{W} + {}^{(\mathrm{x})}D_{\mathrm{I}}\boldsymbol{A} \tag{3.37}$$

independently of the position of indices; ${}^{(\mathrm{x})}D_{\mathrm{I}}$ is the first scalar invariant of ${}^{(\mathrm{x})}\boldsymbol{D}$.

Objective rates depending on the position of indices can be obtained from the basic system (3.10)-(3.13) if we utilize tensors (3.8a)-(3.9) which are objective for orthogonal transformation only:

$$\text{I.}\quad (\boldsymbol{A}_{**})^{\nabla}_{\mathrm{K}} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + \left({}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{K}}\right)^{\mathrm{T}}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot{}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{K}}, \tag{3.38}$$

$$\text{II.}\quad (\boldsymbol{A}^{*}_{*})^{\nabla}_{\mathrm{K}} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{K}}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot{}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{K}}, \tag{3.39}$$

$$\text{III.}\quad (\boldsymbol{A}_{*}^{*})^{\nabla}_{\mathrm{K}} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + \left({}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{K}}\right)^{\mathrm{T}}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot\left({}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{K}}\right)^{\mathrm{T}}, \tag{3.40}$$

$$\text{IV.}\quad (\boldsymbol{A}^{**})^{\nabla}_{\mathrm{K}} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{K}}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot\left({}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{K}}\right)^{\mathrm{T}}. \tag{3.41}$$

Here "K = II, III, IV" and the tensors ${}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{K}}$ are identified by equations (3.8a)-(3.9).

For "K = IV" equations (3.39)-(3.41) coincide with the *Balla-Szabó rate* (1988, [5]).

Using ${}^{(\mathrm{x})}D_{\mathrm{I}}$ from (3.13) we get an objective rate

$$(\boldsymbol{A}^{**})^{\nabla}_{\mathrm{T}} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{L}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot{}^{(\mathrm{x})}\boldsymbol{L}^{\mathrm{T}} + {}^{(\mathrm{x})}D_{\mathrm{I}}\boldsymbol{A} \tag{3.42}$$

which, when written for the stress tensor, can be identified as the *Trusdell rate* (1955, [7]).

The arithmetic mean of equations (3.42) and (3.37), i.e., the tensor

$$(\boldsymbol{A}^{**})^{\nabla}_{\mathrm{D}} = \frac{1}{2}\left((\boldsymbol{A}^{**})^{\nabla}_{\mathrm{T}} + \boldsymbol{A}^{\nabla}_{\mathrm{S}}\right) =$$

$$= {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - \left(\frac{1}{2}\,{}^{(\mathrm{x})}\boldsymbol{D} + {}^{(\mathrm{x})}\boldsymbol{W}\right)\cdot\boldsymbol{A} - \boldsymbol{A}\cdot\left(\frac{1}{2}\,{}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{x})}\boldsymbol{W}\right) + {}^{(\mathrm{x})}D_{\mathrm{I}}\boldsymbol{A} \tag{3.43}$$

is also an objective rate which can be identified as the *Durban-Baruch rate* (1977, [13]).

3.8. In the remainder of the present Section we shall consider some particular cases.

For the metric tensor g_{pq} it follows on the basis of equations (3.10) and (3.13) that

$$(g_{pq})^{\nabla} = 2^{(\mathrm{x})}d_{pq}, \qquad (g^{pq})^{\nabla} = -\,2^{(\mathrm{x})}d^{pq} \tag{3.44}$$

which hold also for any arbitrary transformation.

For orthogonal transformation $(g_{pq})^{\nabla}_{\mathrm{w}} = (g_{pq})^{\nabla}_{\mathrm{wn}} = (g_{pq})^{\nabla}_{\mathrm{wR}} = 0$.

3.9. Assuming an orthogonal transformation, objective rates of the left stretch tensor $\boldsymbol{V}$ are provided by equations (2.15) and (3.34)-(3.36). If, in addition, we

utilize equation (2.32) we get

$$\boldsymbol{V}^{\nabla}_{\mathrm{w}} = \lambda_{\underline{p}}\ {}^{(\mathrm{x})}d^{\underline{q}}_{\underline{p}}\delta^{p}_{q}\mathbf{n}_{p}\mathbf{n}^{q} - \left({}^{(\mathrm{x})}\boldsymbol{W} - \ {}^{(\mathrm{nx})}\boldsymbol{W}\right)\cdot\boldsymbol{V} + \boldsymbol{V}\cdot\left({}^{(\mathrm{x})}\boldsymbol{W} - \ {}^{(\mathrm{nx})}\boldsymbol{W}\right), \tag{3.45}$$

$$\boldsymbol{V}^{\nabla}_{\mathrm{wn}} = \lambda_{\underline{p}}\ {}^{(\mathrm{x})}d^{\underline{q}}_{\underline{p}}\delta^{p}_{q}\mathbf{n}_{p}\mathbf{n}^{q}, \tag{3.46}$$

$$\boldsymbol{V}^{\nabla}_{\mathrm{wR}} = \lambda_{\underline{p}}\ {}^{(\mathrm{x})}d^{\underline{q}}_{\underline{p}}\delta^{p}_{q}\mathbf{n}_{p}\mathbf{n}^{q} - \left({}^{(\mathrm{Rx})}\boldsymbol{W} - \ {}^{(\mathrm{nx})}\boldsymbol{W}\right)\cdot\boldsymbol{V} + \boldsymbol{V}\cdot\left({}^{(\mathrm{Rx})}\boldsymbol{W} - \ {}^{(\mathrm{nx})}\boldsymbol{W}\right). \tag{3.47}$$

Making use of relations (2.35) and (2.36) we write formulae (3.45) and (3.47) entirely in the coordinate system $\{\nu^{p}\}$ of the principal axes:

$$\boldsymbol{V}^{\nabla}_{\mathrm{w}} = \lambda_{\underline{p}}\ {}^{(\mathrm{x})}d^{\underline{q}}_{\underline{p}}\delta^{p}_{q}\mathbf{n}_{p}\mathbf{n}^{q} + \frac{\left(\lambda_{\underline{p}}\right)^{2} + \left(\lambda_{\underline{q}}\right)^{2}}{\lambda_{\underline{p}} + \lambda_{\underline{q}}}\left({}^{(\mathrm{x})}d^{p}_{q} - {}^{(\mathrm{x})}\ d^{\underline{q}}_{\underline{p}}\delta^{p}_{q}\right)\mathbf{n}_{p}\mathbf{n}^{q}, \tag{3.48}$$

$$\boldsymbol{V}^{\nabla}_{\mathrm{wR}} = \lambda_{\underline{p}}\ {}^{(\mathrm{x})}d^{\underline{q}}_{\underline{p}}\delta^{p}_{q}\mathbf{n}_{p}\mathbf{n}^{q} + \frac{2\lambda_{\underline{p}}\lambda_{\underline{q}}}{\lambda_{\underline{p}} + \lambda_{\underline{q}}}\left({}^{(\mathrm{x})}d^{p}_{q} - {}^{(\mathrm{x})}\ d^{\underline{q}}_{\underline{p}}\delta^{p}_{q}\right)\mathbf{n}_{p}\mathbf{n}^{q}. \tag{3.49}$$

3.10. As regards the Eulerian strain tensor, from equations (3.10) and (2.26) we get for arbitrary time dependent transformations that

$$\left(\boldsymbol{E}_{**}\right)^{\nabla} = \ {}^{(\mathrm{x})}\boldsymbol{D}\ . \tag{3.50}$$

3.11. For orthogonal time dependent transformations the Jaumann rate is obtained from equations (3.34) and (2.26):

$$\begin{aligned}\boldsymbol{E}^{\nabla}_{\mathrm{w}} &= \ {}^{(\mathrm{x})}\boldsymbol{E}^{\cdot} - \ {}^{(\mathrm{x})}\boldsymbol{W}\cdot\boldsymbol{E} + \boldsymbol{E}\cdot\ {}^{(\mathrm{x})}\boldsymbol{W} = \\ &= \ {}^{(\mathrm{x})}\boldsymbol{D} - \ {}^{(\mathrm{x})}\boldsymbol{D}\cdot\boldsymbol{E} + \boldsymbol{E}\cdot\ {}^{(\mathrm{x})}\boldsymbol{D}.\end{aligned} \tag{3.51}$$

Let us generalize formulae (3.34)-(3.36) for the Eulerian Hill's strain tensor:

$$\widetilde{\boldsymbol{E}} = f\left(\lambda_{\underline{p}}\right)\delta^{p}_{q}\mathbf{n}_{p}\mathbf{n}^{q}\,, \tag{3.52}$$

and for arbitrary spin tensors ${}^{(\mathrm{x})}\widetilde{\boldsymbol{W}}$ (${}^{(\mathrm{x})}\widetilde{\boldsymbol{W}}$ is a skew tensor):

$$\widetilde{\boldsymbol{E}}^{\nabla} = \ \widetilde{\boldsymbol{E}}^{\cdot} + \widetilde{\boldsymbol{E}}\cdot\ {}^{(\mathrm{x})}\widetilde{\boldsymbol{W}} - \ {}^{(\mathrm{x})}\widetilde{\boldsymbol{W}}\cdot\widetilde{\boldsymbol{E}}. \tag{3.53}$$

One can raise the following question: what tensors $\widetilde{\boldsymbol{E}}$ and ${}^{(\mathrm{x})}\widetilde{\boldsymbol{W}}$ satisfy equation

$$\widetilde{\boldsymbol{E}}^{\nabla} = \ \widetilde{\boldsymbol{E}}^{\cdot} + \widetilde{\boldsymbol{E}}\cdot\ {}^{(\mathrm{x})}\widetilde{\boldsymbol{W}} - \ {}^{(\mathrm{x})}\widetilde{\boldsymbol{W}}\cdot\widetilde{\boldsymbol{E}} = \ {}^{(\mathrm{x})}\boldsymbol{D} \tag{3.54}$$

which is formally objective.

As Reinhardt and Dubey [16] have shown if

$$\widetilde{\boldsymbol{E}} = \ln\boldsymbol{V} = \ln\lambda_{\underline{p}}\delta^{p}_{q}\mathbf{n}_{p}\mathbf{n}^{q}\,,$$

then

$${}^{(\mathrm{x})}\widetilde{w}^{p}{}_{q} = \ {}^{(\mathrm{nx})}w^{p}{}_{q} + \frac{1}{\ln\lambda_{\underline{p}} - \ln\lambda_{\underline{q}}}\ {}^{(\mathrm{x})}d^{p}_{q}, \quad \lambda_{\underline{p}} \neq \lambda_{\underline{q}}, \quad p \neq q, \tag{3.55}$$

in the coordinate system $\{\nu^p\}$, where according to equation (2.35)

$$ {}^{(\mathrm{nx})}w^p{}_q = {}^{(\mathrm{x})}w^p{}_q - \frac{\left(\lambda_{\underline{p}}\right)^2 + \left(\lambda_{\underline{q}}\right)^2}{\left(\lambda_{\underline{p}}\right)^2 - \left(\lambda_{\underline{q}}\right)^2}\, {}^{(\mathrm{x})}d^p_q, \quad \lambda_{\underline{p}} \neq \lambda_{\underline{q}}, \quad p \neq q\,, $$

and ${}^{(\mathrm{x})}\widetilde{\boldsymbol{W}}$ is the so-called logarithmic spin tensor. The second term on the right side can be manipulated further:

$$ \frac{\left(\lambda_{\underline{p}}\right)^2 - \left(\lambda_{\underline{q}}\right)^2}{\left(\lambda_{\underline{p}}\right)^2 + \left(\lambda_{\underline{q}}\right)^2} = \frac{\frac{\lambda_p}{\lambda_{\underline{q}}} - \frac{\lambda_q}{\lambda_{\underline{p}}}}{\frac{\lambda_p}{\lambda_{\underline{q}}} + \frac{\lambda_q}{\lambda_{\underline{p}}}} = \frac{e^{\ln \frac{\lambda_p}{\lambda_{\underline{q}}}} - e^{-\ln \frac{\lambda_p}{\lambda_{\underline{q}}}}}{e^{\ln \frac{\lambda_p}{\lambda_{\underline{q}}}} + e^{-\ln \frac{\lambda_p}{\lambda_{\underline{q}}}}} = \tanh \ln \frac{\lambda_{\underline{p}}}{\lambda_{\underline{q}}} = \tanh\left(\ln \lambda_{\underline{p}} - \ln \lambda_{\underline{q}}\right)\,. $$

If we take the foregoing into account, equation (3.55) assumes the form:

$$ {}^{(\mathrm{x})}\widetilde{w}^p{}_q = {}^{(\mathrm{x})}w^p{}_q + \left(\frac{1}{\ln \lambda_{\underline{p}} - \ln \lambda_{\underline{q}}} - \frac{1}{\tanh\left(\ln \lambda_{\underline{p}} - \ln \lambda_{\underline{q}}\right)}\right) {}^{(\mathrm{x})}d^p_q, \quad \lambda_{\underline{p}} \neq \lambda_{\underline{q}}, \quad p \neq q. \tag{3.56} $$

Xiao at all. [15] have shown that equation (3.54) has a solution if and only if $\widetilde{\boldsymbol{E}} = \ln \boldsymbol{V}$. Summarizing what has been said above, we can conclude that equation (3.54) has the following solution

$$ (\ln \boldsymbol{V})^{\nabla} = (\ln \boldsymbol{V})^{\cdot} + \ln \boldsymbol{V} \cdot {}^{(\mathrm{x})}\widetilde{\boldsymbol{W}} - {}^{(\mathrm{x})}\widetilde{\boldsymbol{W}} \cdot \ln \boldsymbol{V} = {}^{(\mathrm{x})}\boldsymbol{D}, \tag{3.57} $$

where ${}^{(\mathrm{x})}\widetilde{\boldsymbol{W}}$ is given by equation (3.56) and with regard to (2.9)

$$ \begin{aligned} (\ln \boldsymbol{V})^{\cdot} &= \left(\ln \lambda_{\underline{p}}\right)^{\cdot} \delta^p_q \mathbf{n}_p \mathbf{n}^q + \ln \lambda_{\underline{p}} \left(\dot{\mathbf{n}}_p \mathbf{n}^q + \mathbf{n}_p \dot{\mathbf{n}}^q\right) = \\ &= \left(\ln \lambda_{\underline{p}}\right)^{\cdot} \delta^p_q \mathbf{n}_p \mathbf{n}^q + {}^{(\mathrm{nx})}\boldsymbol{W} \cdot \ln \boldsymbol{V} - \ln \boldsymbol{V} \cdot {}^{(\mathrm{nx})}\boldsymbol{W}. \end{aligned} \tag{3.58} $$

The objective time rate tensor $(\ln \boldsymbol{V})^{\nabla}$ is the so-called logarithmic strain rate tensor which can be obtained by comparing formulae (3.55), (3.57) and (3.58). It is remarkable that $(\ln \boldsymbol{V})^{\cdot} \neq (\ln \boldsymbol{V})^{\circ}$.

4. **Concluding remarks**

In the present paper invariance under arbitrary time dependent transformations (valid for co-ordinate systems moving arbitrarily with respect to each other) is regarded as a criterion for material objectivity of tensors. This means that the components of materially objective tensors and those of their objective time rates follow the general transformation rules valid for tensors in coordinate systems moving arbitrarily with respect to each other. In these cases the matrix of transformation is a function of time. The investigations that have been carried out are based on the relations we established in Part I (Kozák [14]) between material time rates defined in various co-ordinate systems.

Section 2 of the present Part II treats the material time rates of tensors describing the deformation of the continuum both in the fixed co-ordinate system and in the co-ordinate system moving arbitrarily with respect to the fixed one. The spin tensors of the principal directions of strains and of the rotation tensor are also determined in both co-ordinate systems. In this way we introduce some objective tensors (being objective partly for arbitrary partly for orthogonal time dependent transformations) each of which is determined directly or indirectly by the left stretch tensor and the velocity gradient.

Section 3 is devoted to materially objective time rates of tensors. By applying the relations, which have been deduced for the material time rates taken in coordinate systems moving arbitrarily with respect to each other, to an arbitrary tensor we have found a basic system of the objective time rates which is valid for arbitrary time dependent transformations and can be given in any coordinate system. Regarding the basic system as a point of departure and making use of the objective tensors which describe the state of velocity and deformation of a continuum, we have established objective time rates being partly new and partly published in literature.

It has been shown that one part of the objective time rates found in literature is objective under arbitrary time dependent transformation, while the other part under orthogonal transformation only.

Although the chain of thought is detailed for second-order tensors only, the non-particular results are valid for a tensor of any order.

The paper restricts its attention to the issue of objectivity of time rates and disregards the part these rates play in constitutive equations.

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ON THE MULTIPLE SOLUTION OF AXISYMMETRIC MINIMUM SURFACES

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Abstract. The optimum condition for the unloaded shape of prestressed tents is met by minimum surfaces. The simplest type of such structures is a rotationally symmetric catenary surface. In this paper, the authors present an analysis which shows that catenary surfaces cannot fit arbitrarily chosen boundary circles of the tent, and also that if a solution of the problem exists, also a dual solution can be found.

Mathematical Subject Classification: 35J65, 49Q05
Keywords: optimum shape, minimum surface, envelope curve

1. Introduction

The problem about the existence and uniqueness of minimum surfaces under certain geometric constraints arose in connection with the optimum shapes of prestressed tents. The shape of a tent is considered optimum, if the fabric can be brought into a uniform stress state by prestressing.

In a previous paper [5] the mathematical background for the analysis of optimum shapes of tents with fixed boundaries has been published. It was shown that surfaces which meet the statical optimum condition are minimum surfaces as well. Also an iterative solution of the non-linear partial differential equation for the shape-function of membranes developing a self-stress state of uniform membrane forces has been presented. In certain cases the iteration proved divergent, which led to the analysis of conditions for the existence of the solution.

For the sake of simplicity, axisymmetric membranes will be dealt with. However, the nature of conditions for the existence of optimum shapes shows that they may also apply to other surfaces.

2. The shape of the meridian

In the case of axisymmetric membranes the problem of optimum shapes can be mathematically reduced to the analysis of the meridian. This analysis can be made analytically.

For an unloaded membrane of axisymmetric shape the following connection of the membrane forces and principal radii (principal curvatures) holds [1]:

$$\frac{N_\alpha}{R_\alpha} + \frac{N_\vartheta}{R_\vartheta} = 0, \tag{2.1}$$

where N_α, N_ϑ are the membrane forces in meridian and annular directions, and R_α, R_ϑ are the radii of curvatures in the same directions. These curvatures are the principal curvatures.

As the membrane of optimum shape develops a uniform tension self-stress state

$$N = N_\alpha = N_\vartheta \,.$$

Equation (2.1) leads to the geometric condition

$$\frac{1}{R_\alpha} = -\frac{1}{R_\vartheta}. \tag{2.2}$$

If the meridian of the surface is given as a function of z, i.e., in the form $r = r(z)$ then the principal curvatures can be expressed as

$$\frac{1}{R_\alpha} = \frac{r''}{\left[1 + (r')^2\right]^{3/2}}, \qquad \frac{1}{R_\vartheta} = \frac{-1}{r\left[1 + (r')^2\right]^{1/2}}, \tag{2.3}$$

where the differentiation with respect to z is denoted by $()'$.

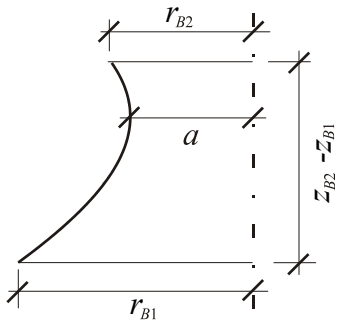

Figure 1. The meridian of the surface

Substituting equations (2.3) into equation (2.2), a non-linear second-order differential equation can be formed for r:

$$r''r - (r')^2 - 1 = 0\,. \tag{2.4}$$

The general solution of this differential equation is given in [3] as

$$r\,(z) = a\cosh\left(\frac{z-b}{a}\right), \tag{2.5}$$

where a and b are parameters of the solution. It follows from equation (2.5) that the meridians of rotationally symmetric membranes are *chain-curves*. The parameter 'a' can be interpreted as the radius of the throat circle of the catenoid surface (Figure 1) (or the height of the deep point of the chain-curve) while 'b' is its distance from the $z = 0$ co-ordinate plane.

If the levels z_{B1}, z_{B2} and the radii r_{B1}, r_{B2} of the boundary circles are given values (Figure 1), the real parameters a and b have to be calculated by solving the conditional equation system:

$$r_{B1} = a\cosh\left(\frac{z_{B1}-b}{a}\right), \qquad r_{B2} = a\cosh\left(\frac{z_{B2}-b}{a}\right). \tag{2.6}$$

We have found, on the one hand, that the above equation system does not always have a real solution for the parameters a and b, and, on the other hand, if there exist a pair of real numbers for a and b which fulfill the equation system, then another pair of a and b can be found. These facts give rise to two questions: one, which conditions decide that the mechanical problem has or does not have any solution, and another, which of the multiple mathematical solutions is the real solution for the mechanical problem.

In the subsequent sections, first, conditions for the existence of the solution will be dealt with, then the minimum property of the surfaces obtained by the multiple mathematical solutions will be checked.

3. The envelope of chain-curves intersecting at a common point

On the basis of the general solution of equation (2.5) a transformed form can be derived which makes analyzing our problem easier. First, let r/a be replaced by ρ, z/a by ζ, b/a by β, then let a common multiplier $\cosh\beta$ of ρ and ζ be introduced into the general solution. In this way we can arrive at the equation

$$\rho(\zeta,\beta) = \frac{1}{\cosh\beta}\cosh(\zeta\cosh\beta - \beta)\,. \tag{3.1}$$

This form makes analyzing our problem easier because one of the boundary points of the meridian always gets to the point $\zeta = 0$, $\rho = 1$. Setting parameter β to different values, equation (3.1) generates a set of chain-curves passing through that common point – see Figure 2. The problem of fitting a chain-curve to the points with co-ordinates (z_{B1}, r_{B1}) and (z_{B2}, r_{B2}) can be transformed to that of selecting chain-curves from the set generated by equation (3.1), which passes through the point of the co-ordinates

$$\zeta_B = \frac{z_{B2}-z_{B1}}{r_{B1}}, \quad \rho_B = \frac{r_{B2}}{r_{B1}}\,. \tag{3.2}$$

A glance at the diagram of the series of chain-curves generated by equation (3.1) (see Figure 2) makes it obvious that there is an envelope which divides the co-ordinate plane (ζ,ρ) into a part where points cannot be reached by chain-curves passing through the point (0,1), and another part, where it is possible. Figure 2 also shows the shape of the envelope resembles a shifted chain-curve $\rho = \cosh\zeta - 1$,

however, it must differ from that curve because the envelope has to get closer to the diagram of $\rho = \cosh\zeta$ if ζ has a larger absolute value.

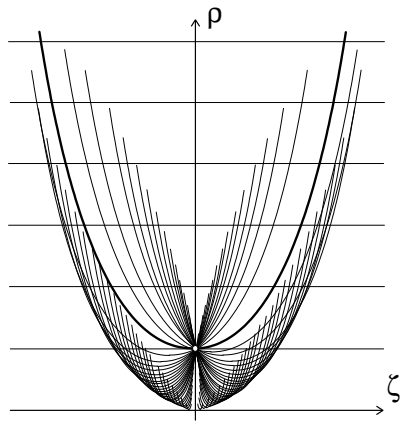

Figure 2. Chain-curves passing through the point $(0, 1)$

The equation of the envelope of the series of curves generated by a parametric equation is determined by the conditions that points of the envelope are also points of those curves touching the envelope, and the first order variation of the parametric equation with respect to its parameter must vanish at the touching points. These two conditions can be expressed as

$$F(\rho, \zeta, \beta) = 0\,, \tag{3.3}$$

$$\frac{\partial F(\rho, \zeta, \beta)}{\partial \beta} = 0 \tag{3.4}$$

where equation (3.3) is the equation of the curves with parameter β. The equation of the envelope can be obtained by eliminating β from equations (3.3) and (3.4) [4].

In our case $F(\rho, \zeta, \beta)$ is the same as equation (3.1) arranged to zero, conditions (3.3) and (3.4) yield a system of transcendent equations as follows:

$$\rho\cosh\beta - \cosh(\zeta\cosh\beta - \beta) = 0\,, \tag{3.5}$$

$$\rho\sinh\beta - \sinh(\zeta\cosh\beta - \beta)(\zeta\sinh\beta - 1) = 0\,. \tag{3.6}$$

Though β cannot be analytically eliminated from the above equation system, the envelope can be plotted point by point using numerical solutions of equations (3.5) and (3.6) for different values of β.

The plot of the envelope is the continuous line in Figure 3. For small values of ζ its shape seems to osculate the curve of the function $\rho = \cosh\zeta - 1$, which is shown with dashed line in Figure 3.

However, for large values of β, the numerical solutions of equations (3.5) and (3.6) are getting more and more inaccurate and they do not permit us even to settle the question whether the envelope starts with a zero or nonzero slope at the origin.

This question can be answered by analyzing the curve of the deep points of the chain-curves in Figure 2. Co-ordinates of the deep points of chain-curves passing

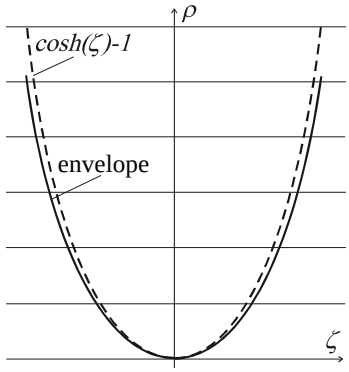

Figure 3. The envelope curve

through the point $(0, 1)$ are

$$\rho_0 = \frac{1}{\cosh\beta} \qquad \zeta_0 = \frac{\beta}{\cosh\beta}\,.$$

These expressions permit us to eliminate parameter β and to express the equation of the curve of the deep points as

$$\zeta_0 = \rho_0 \cdot \operatorname{arccosh}\frac{1}{\rho_0}\,. \tag{3.7}$$

Though equation (3.7) cannot be made explicit for ρ_0, it enables us to answer whether the curve of the deep points starts with a zero or with nonzero slope from the origin. Producing the first derivative of equation (3.7) as

$$\frac{\mathrm{d}\,\zeta_0}{\mathrm{d}\,\rho_0} = \operatorname{arccosh}\frac{1}{\rho_0} - \frac{1}{\rho_0 \cdot \sqrt{1-\rho_0^2}},$$

then performing the limit transition for its reciprocal at $\rho_0 = 0$, we find

$$\lim_{\rho_0 \to 0} \left(\frac{\mathrm{d}\,\zeta_0}{\mathrm{d}\,\rho_0}\right)^{-1} = 0,$$

which means that the curve starts with a zero slope.

Since the deep points of the chain-curves are above the envelope curve, if the curve of the deep points starts with a zero slope at the origin, the envelope does the same.

Again the diagram of the chain-curves generated by equation (3.1) shows that each point (ζ_B, ρ_B) of the domain of possible solutions is a point of intersection of two curves of the series, that is, if we have a solution of our problem, we always have a dual solution as well. If the point (ζ_B, ρ_B) is exactly on the envelope curve, then the two meridians coincide.

The domain of the possible solutions can be divided into a part (A), where the deep points of the intersecting chain-curves lie on different sides of the co-ordinate plane split by axis ρ, and another part (B), where the deep points lie on the same side

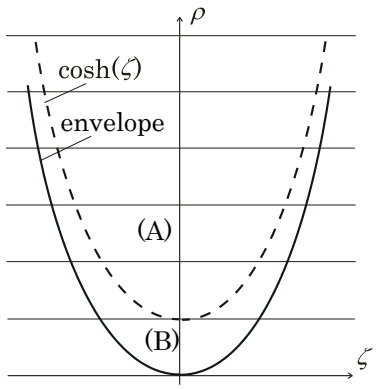

Figure 4. Two regions of the domain of solutions

The two regions are separated by the chain-curve the deep point of which lies on axis ρ (Figure 4). Since we could not derive a closed formula for the envelope, its shape can be analyzed only numerically. The approximate formula

$$\mathrm{Env}\zeta \approx \cosh\zeta - 1$$

yields fairly good first estimates both for small values of ρ and for larger values as well. For values of ζ larger than about 0.7 we find a more accurate envelope by assuming that the condition

$$\frac{\mathrm{d}\,\mathrm{Env}(\zeta_e)}{\mathrm{d}\zeta_e} = \frac{\mathrm{d}\,F(\rho_e, \zeta_e, \beta)}{\mathrm{d}\zeta_e}$$

holds for the first estimate, where the index e refers to the place of the envelope. In this way we can write

$$\sinh\zeta_e \approx \sinh(\zeta_e \cosh\beta - \beta)\,,$$

which permits us to approximate ζ_e in the form

$$\zeta_e \approx \frac{\beta}{\cosh\beta - 1}, \tag{3.8}$$

and then the corresponding ρ_e as

$$\rho_e = \frac{1}{\cosh\beta}\cosh\left(\frac{1}{\cosh\beta - 1}\right). \tag{3.9}$$

In Figure 5 the plots of the first estimate (dashed line) and the refined envelope calculated using equations (3.8) and (3.9) are shown

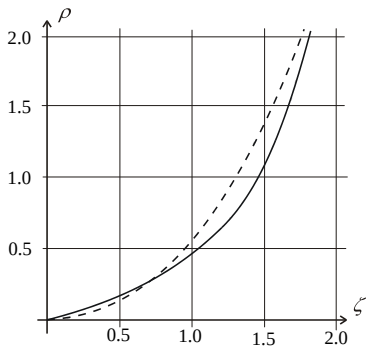

Figure 5. Approximate envelopes

The envelope gives an answer to our first question: the data of the boundaries determine that in the given geometrical case a membrane with optimum shape does or does not exist. For any pairs of the radii of the boundary rings there exists a maximum value of height of the surface that permits us to connect the rings with a surface of optimal shape. If the height is less than this value, we can find two surfaces which meet the optimum condition, if the height is chosen bigger than this value, we cannot find solutions.

4. **The minimum surface**

It was shown in [5] that the surface of the optimum shape of unloaded membranes is minimum. However, in the last section two solutions were found for the optimum shape. This result raises the problem: which curve specifies the minimum-surface and what extreme property the other curve exhibits.

To answer this question two types of numerical investigation were performed.

First we shall consider a combination of the two solutions.

Let the functions of the curves passing through an arbitrarily chosen point within the envelope be called $\rho_1(\zeta)$ and $\rho_2(\zeta)$. Let $\rho_1(\zeta)$ belong to the curve which runs higher between the points of intersection than the other curve.

The curves representing the function

$$\rho_\alpha(\alpha, \zeta) = \alpha\ \rho_1 + (1 - \alpha\ )\ \rho_2$$

also pass through the end points of the curves belonging to $\rho_1(\zeta)$ and $\rho_2(\zeta)$. If $\alpha = 1$, then $\rho_\alpha(\zeta) = \rho_1(\zeta)$, if $\alpha = 0$, then $\rho_\alpha(\zeta) = \rho_2(\zeta)$, if $0 < \alpha < 1$, then the curve of $\rho_\alpha(\zeta)$ lies between that of $\rho_1(\zeta)$ and $\rho_2(\zeta)$.

The area $A(\alpha)$ of the surface assigned by $\rho_\alpha(\alpha, \zeta)$ can be analytically expressed as a fairly complicated definite integral. Instead of using this integral, we numerically calculated the surface area for some values of α, and then plotted the results.

Figure 6 shows a typical plot of $A(\alpha)$. In this case co-ordinates ζ of the deep points of the curves have the same sign, however, plots belonging meridians with deep points

that lie at values of ζ with different signs show the same characteristics. The plots clearly show that meridians at $\alpha = 1$ result in local minima of surface areas, and at $\alpha = 0$ in their local maxima. Hence, meridians $\rho_1(\zeta)$ can be stable solutions, but $\rho_2(\zeta)$ are always unstable ones.

It may cause difficulties to explain that at certain negative values of α the computed surface area gets lower than the minimum at $\alpha = 1$. The reason may be that the combined meridians do not always characterize surfaces with a physically realistic surface. If the meridian intersects the axis of rotation, between the points of intersection the integral takes into account a negative surface area. However, the plots of the meridians show that in some cases the area of the combined surface can get smaller

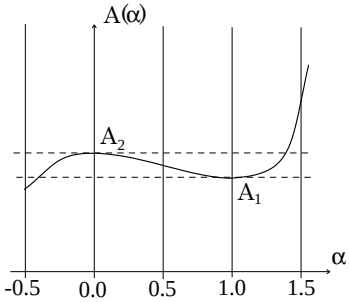

Figure 6. The plot of $A(\alpha)$

than the local minimum before this intersection happens. To clarify this unexpected result, an individual analysis of the extreme properties of the surfaces was performed.

Secondly, we consider a variation of different functions.

The extreme property of the meridians $\rho_1(\zeta)$ and $\rho_2(\zeta)$ can be individually investigated by a numerical variational analysis. For that purpose we have to choose functions which take zero values at the boundaries like

$$\delta\rho = \zeta\,(\zeta_B - \zeta)\,,$$
$$\delta\rho = \sin\left(\frac{\pi\,\zeta}{\zeta_B}\right),$$
$$\delta\rho = \cosh\left(\zeta - \frac{\zeta_B}{2}\right) - \cosh\left(\frac{\zeta_B}{2}\right),$$

and to use them to vary the meridians and the surfaces. By adding $\varepsilon\,\delta\rho$ to functions $\rho_1(\zeta)$ and $\rho_2(\zeta)$ we can numerically calculate the surface area

$$A(\rho_1) + \delta A(\rho_1) = A(\rho_1 + \varepsilon\,\delta\rho),$$
$$A(\rho_2) + \delta A(\rho_2) = A(\rho_2 + \varepsilon\,\delta\rho)$$

for different values of ε, then plot the results in a common co-ordinate system.

Figure 7 shows some characteristic results of the analysis made in this way. On the left hand-side of Figure 7 the variation of the surface area is shown for a pair of meridians fitting the same boundary points. Plots A_1 and A_2 clearly show that both

$A(\rho_1)$ and $A(\rho_2)$ have local extreme values at $\varepsilon = 0$. That is, both $A(\rho_1)$ and $A(\rho_2)$ are extremals of the numeric variational problem, because their first order variation is zero. On the other hand, $A(\rho_1)$ has a local minimum and $A(\rho_2)$ also has a local maximum at $\varepsilon = 0$, which means, their second order variations are differently signed. This difference indicates different extreme properties: the surface which belongs to ρ_1 is a stable minimum surface, while the other is unstable.

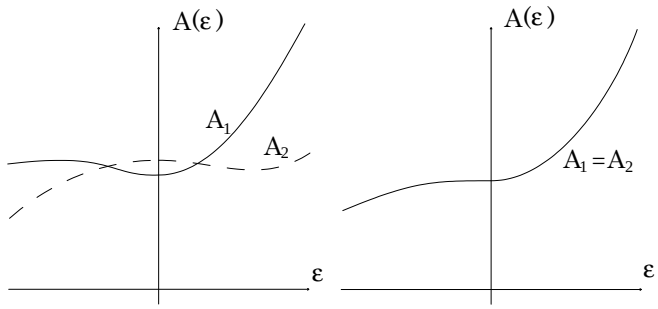

Figure 7. Variation of surface areas

On the right-hand side of Figure 7 the variation of the surface area is shown which belongs to coinciding meridians $\rho_1 = \rho_2$. In this case both the first and second order variations are zero and the surface assigned by the meridians is at the limit of the stability.

The plots again show that the condition stated by equation (2.4) for axisymmetric minimum surfaces is not a global minimum condition. It can be met by surfaces which are locally minimum surfaces and also by surfaces which are not minimum surfaces at all. Moreover, it may happen that the area of the varied surface is smaller than that of the minimum surface.

5. **Connection with the areas of the boundary rings**

If we neglect the small effect of gravity, the shape of soap films stretched between concentric circular boundary rings is an annular plate, which turns into a catenoid shaped minimum surface if the planes of the boundary rings get separated. Experiments with such soap films can be used to check the results of the above analysis. Sometimes they show an interesting phenomenon: by increasing the distance of the rings, the catenoid gets more and more laced, and at a certain distance it snaps into two separate circular plates stretched on the two boundary rings.

This snapping is usually explained by the reasoning that the catenoid snaps into separate circles when its area gets equal to the sum of the area of the two circles. However, the existence of axisymmetric surfaces with a smaller area than the stable minimum surface and also the local nature of the minimum property of minimum surfaces give rise to serious doubts about the tenability of that reasoning.

Our results can also be used to settle this problem. The area of the catenoid surfaces generated by equation (3.1) can be analytically expressed as

$$A\left(\zeta_B,\beta\right)=\frac{2\pi}{\cosh^2\beta}\int\limits_0^{\zeta_B}\cosh^2\left(\zeta\cdot\cosh\beta-\beta\right)\mathrm{d}\zeta\,\cosh\beta=$$

$$=\frac{\pi}{\cosh^2\beta}\left[\frac{\sinh 2\left(\zeta_B\cdot\cosh\beta-\beta\right)}{2}+\left(\zeta_B\cdot\cosh\beta-\beta\right)\right]$$

and $A(\zeta_B,\beta)$ can be matched with the sum of the area of the boundary rings

$$A_R(\zeta_B,\beta)=\pi\left[\frac{\cosh^2\left(\zeta_B\cdot\cosh\beta-\beta\right)}{\cosh^2\beta}+1\right]$$

Parameters, when $A(\zeta_B,\beta)=A_R(\zeta_B,\beta)$, specify pairs of coordinates ζ_B and ρ_B of boundary circles of our interest

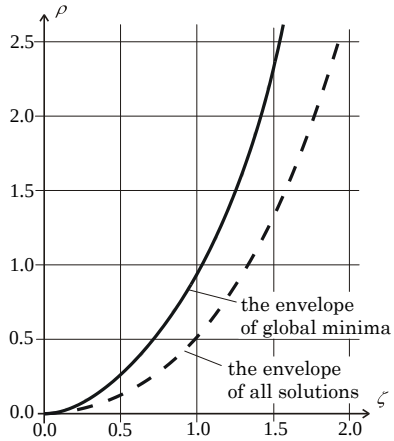

Figure 8. The envelope of global minima

In Figure 8 a continuous line represents the boundary points of those meridians for which the area of the stable minimum surface is equal to the sum of the areas of their boundary circles. The first (and everywhere conservative) estimate of the envelope is also plotted (dashed line). The figure shows that the two curves permit solutions of the rotationally symmetric minimum surface problem, where the area of the minimum surface is larger than the sum of area of the boundary circles.

The continuous line can also be considered as an envelope. If the points within the continuous line are connected to the point (0,1) by chain-curves we obtain minimum surfaces with globally minimum surface area.

6. **Conclusions**

The analysis has shown that the differential equation for the meridian of axisymmetric minimum surfaces cannot be solved for arbitrary values of the boundary radii and

if a solution can be found, also a dual solution exists, unless the two solutions coincide. It means that in typical cases, besides the minimum surface, another surface of revolution exists, which also develops uniform self-stress state. Both surfaces are catenary surfaces, but the area of the minimum surface behaves as a minimum, while that of the dual solution behaves as a maximum against small variations. In case of coinciding solutions only stationarity can be found.

Both minimum and maximum properties are of local nature. It is quite obvious for the surface maximum, but more or less unexpected for the minimum surface, because in case of appropriate values of radii and distance of boundary circles, the area of the minimum surface is larger than that of properly chosen surfaces fitting the same boundaries. The analysis has also clarified under what conditions the area of the minimum surface is the global minimum of the problem.

These results make it clear why does the stability of our iterative method depend both on the geometrical data, and on the mesh of the discretizing net and why does it also exhibit the same sensibility in any minimum surface problems characterized by two or more boundary curves. Similarly to the catenary surface problem, the existence of these surfaces is also conditioned by the data of the boundaries and the solutions are multiple solutions as well. The closer the multiple solutions are to each other the poorer the stability of the iterative solution is. Refinement of the mesh does not only improves the solution but also makes the iteration more stable because it decreases the chance to drop from a convergent path into a closely divergent one.

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A MAPPING TECHNIQUE FOR A HEAT CONDUCTION PROBLEM ON MOVING MESH USING THE *HP*-VERSION OF THE FINITE ELEMENT METHOD

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Abstract. The numerical solution of the thermal part of a coupled thermo-mechanical contact problem is presented. The *hp*-version finite element method is used for the discretization of the temperature field in space, and finite difference method in time. The mesh is modified during the computation, therefore the temperature field has to be transferred between different meshes. A mapping technique is developed for the *p*-version of the finite element method. Numerical examples are presented.

Mathematical Subject Classification: 74S05
Keywords: heat conduction, *hp*-version of the finite element method, mapping technique

1. Introduction

This paper is concerned with the numerical solution of the thermal part of the coupled thermo-mechanical contact problem. Many papers deal with contact problems but without thermo-dynamical effects [1, 2, 3]. When friction is considered and the bodies slide on each other, heat generation and heat conduction have to be taken into account.

The contact problems are usually treated with the *h*-version of the finite element method. In recent years some papers tried to treat the contact problem with the *p*-version of the finite element method [1, 2, 4, 5]. The advantages of the *p*-version are the higher accuracy, faster convergence and coarser mesh [6]. The numerical solution of the heat conduction problem, (both the *h*- and *p*-versions), is well known [6, 7], but the coupling with the mechanical contact problem causes some difficulties. During the computation of the mechanical problem, a minor modification of the mesh is needed to avoid oscillations in the numerical solution [1, 2]. Heat conduction is a temporal process, and the temporal part of the discretization is usually treated with the finite difference method. The temperature field of the previous time step is needed

to determine the current temperature field. The temperature field of the previous and current time steps appears in the same system of equations, that is why they have to be given in the same nodal points, i.e. in the same mesh. When the mesh is modified, the temperature parameters have to be transferred from the old mesh to the new mesh. There are many mapping techniques in the literature, but mostly for the h-version. Some papers dealing with plastic deformation apply mapping techniques for data transfer between different meshes [8, 9, 10, 11]. In our investigation a mapping technique is developed for the p-version.

In this paper three dimensional axisymmetrical bodies are investigated with a two dimensional mathematical model. Numerical examples are presented.

2. The heat conduction problem

Let us consider two disjoint regions $\Omega^e\,(e=1,2)$ with piecewise smooth boundaries $\partial\Omega^e\,(e=1,2)$ occupied by two continuous deformable bodies. We intend to investigate the heat conduction during the sliding contact of the bodies. Let the initial temperature be equal to the ambient temperature everywhere in Ω^e. The surfaces of the bodies can be split into two parts: the contact area of the bodies, which is denoted by $\Gamma_c^e\,(\Gamma_c^e \in \partial\Omega^e)$, and the rest of the surfaces, which is denoted by $\Gamma_s^e\,(\Gamma_s^e \in \partial\Omega^e)$, where $\Gamma_c^e \cup \Gamma_s^e = \partial\Omega^e$ and $\Gamma_c^e \cap \Gamma_s^e = \emptyset$. When the bodies slide on each other, heat is generated on the surfaces Γ_c^e due to friction. The generated heat partially flows into the bodies and partially flows across the surfaces Γ_s^e. During this heat conduction process the temperature distribution of the bodies changes. To determine the temperature distribution $T^e(\mathbf{r},t)$ the equation of heat conduction can be applied

$$\rho^e c^e \dot{T}^e(\mathbf{r},t) = \nabla\cdot(k^e \nabla T^e(\mathbf{r},t)) + Q^e \qquad \mathbf{r}\in\Omega^e \tag{2.1}$$

where $\mathbf{r}$ is the position vector, t is time, ρ^e is density, c^e is the specific heat, k^e is thermal conductivity, Q^e is the generated heat within Ω^e and $e=1,2$. The temperature distribution $T^e(\mathbf{r},t)$ has to satisfy the following initial and boundary conditions:

$$T^e(\mathbf{r},0) = T_0^e(\mathbf{r}) \qquad \mathbf{r}\in\Omega^e \tag{2.2}$$

$$q_c^e = -k\nabla T^e(\mathbf{r},t)\cdot\mathbf{n}^e = q_{fr}^e(\mathbf{r},t) + q_{ex}^e(\mathbf{r},t) \quad \mathbf{r}\in\Gamma_c^e \tag{2.3}$$

$$q_s^e = -k\nabla T^e(\mathbf{r},t)\cdot\mathbf{n}^e = q_{co}^e(\mathbf{r},t) \qquad \mathbf{r}\in\Gamma_s^e \tag{2.4}$$

where $T_0^e(\mathbf{r})$ is the initial temperature, $\mathbf{n}^e$ is the outward normal unit vector of $\partial\Omega^e$, q_c^e is the heat flux flowing out of the e^{th} body through Γ_c^e and q_s^e is the heat flux flowing out of the e^{th} body through Γ_s^e. The heat flux q_c^e consists of two parts. The first part arises from the frictional dissipation:

$$q_{fr}^1(\mathbf{r},t) = -c_D\beta v(\mathbf{r},t)p(\mathbf{r},t) \quad \text{and} \quad q_{fr}^2(\mathbf{r},t) = -c_D(1-\beta)v(\mathbf{r},t)p(\mathbf{r},t), \tag{2.5}$$

where c_D a constant from experiment ($c_D \le 1$), $\beta = \dfrac{k^1}{k^1+k^2}$ and $0\le\beta\le 1$, $v(\mathbf{r},t)$ and $p(\mathbf{r},t)$ are the relative velocity of the bodies and the contact pressure at the point

r and at time t, respectively. The second part is the heat exchange between the bodies arising from the different temperatures of the bodies at the contact interface:

$$q^e_{ex}(\mathbf{r},t) = \hat{\alpha}(T^e(\mathbf{r},t) - T^{e^*}(\mathbf{r},t)), \tag{2.6}$$

where $\hat{\alpha}$ is the coefficient of surface heat transfer between the bodies, e and e^* are the number of the bodies. If $e = 1$, then $e^* = 2$, and if $e = 2$, then $e^* = 1$. The heat flux q^e_{co} denotes convective heat flux, defined as

$$q^e_{co}(\mathbf{r},t) = \alpha^e(T^e(\mathbf{r},t) - T_\infty), \tag{2.7}$$

where α^e is a coefficient of surface heat transfer and T_∞ is the specified ambient temperature of the surrounding medium. The area of the contact surface Γ^e_c and the contact pressure can be obtained by solving the contact problem with the appropriate boundary conditions.

3. The weak formulation of the heat conduction problem

The weak formulation of the above mentioned heat conduction problem can be obtained by applying Galerkin's method. Let us multiply equation (2.1) by virtual temperature Θ^e and integrate it on Ω^e:

$$\sum_{e=1}^{2} \int_{\Omega^e} (\rho^e c^e \dot{T}^e - \nabla\cdot(k^e \nabla T^e) - Q^e)\Theta^e d\Omega = 0\,.$$

After integrating by parts and employing the Gauss theorem, the weak form will be the following:

$$\sum_{e=1}^{2} \left[\int_{\Omega^e} \rho^e c^e \dot{T}^e \Theta^e d\Omega + \int_{\partial\Omega^e} q^e_n \Theta^e d\Gamma + \int_{\Omega^e} k^e \nabla T^e \cdot \nabla\Theta^e d\Omega - \int_{\Omega^e} Q^e \Theta^e d\Omega \right] \tag{3.1}$$

where $q^e_n = -k^e \nabla T \cdot \mathbf{n}^e$ is the heat flux orthogonal to the surfaces of the bodies, and positive if the heat flows out of the bodies. The bodies can exchange heat with each other through the surfaces Γ^e_c. To take into account this effect and the other boundary conditions, equation (3.1) must be specialized to bodies 1 and 2. The superscript indices denote the appropriate bodies:

$$\begin{aligned}
&\int_{\Omega^1} \rho^1 c^1 \dot{T}^1 \Theta^1 d\Omega + \int_{\Omega^1} k^1 \nabla T^1 \cdot \nabla\Theta^1 d\Omega + \int_{\Gamma^1_s} \alpha^e (T^1 - T_\infty)\Theta^1 d\Gamma - \int_{\Gamma^1_c} c_D \beta v p \Theta^1 d\Gamma + \\
&\quad + \int_{\Gamma^1_c} \hat{\alpha}(T^1 - T^2)\Theta^1 d\Gamma + \int_{\Omega^2} \rho^2 c^2 \dot{T}^2 \Theta^2 d\Omega + \int_{\Omega^2} k^2 (\nabla T^2 \cdot \nabla\Theta^2) d\Omega + \\
&\quad + \int_{\Gamma^2_s} \alpha^e (T^2 - T_\infty)\Theta^2 d\Gamma - \int_{\Gamma^2_c} c_D (1-\beta) v p \Theta^2 d\Gamma + \int_{\Gamma^2_c} \hat{\alpha}(T^2 - T^1)\Theta^2 d\Gamma = 0\,.
\end{aligned} \tag{3.2}$$

4. **Finite element discretization**

The temperature field T^e $(e = 1, 2)$ is a function of the position vector and the time. We cannot give the exact solution of (3.2) in a closed form, that is why T^e can be determined only approximately. To find an approximation to T^e, we create a set of functions by subdividing Ω^e into a number of domains, called finite elements. We define a set of basis functions on Ω in such a way that each of the basis functions is nonzero over individual elements. Applying orthogonal basis functions the round off error can be minimized [6]. In this way the temperature distribution can be written as:

$$T^e(\mathbf{r}, t) = \sum_{i=1}^{n^e} \sum_{j=1}^{N} T_{ij}^e(t) N_j(\mathbf{r}) \,, \tag{4.1}$$

where n^e is the number of finite elements of the body e, N is the number of shape functions, $N_j(\mathbf{r})$ are the shape functions and T_{ij}^e are the nodal temperatures and parameters of the i^{th} element. In a similar way, the virtual temperature also can be approximated:

$$\Theta^e(\mathbf{r}, t) = \sum_{i=1}^{n^e} \sum_{j=1}^{N} \Theta_{ij}^e(t) N_j(\mathbf{r}) \tag{4.2}$$

We restrict our investigation to axially symmetric problems, where a three dimensional problem can be treated with a two dimensional mathematical model. Using the summation convention

$$\sum_{j=1}^{N} T_{ij}^e N_j(\mathbf{r}) \equiv T_{ij}^e N_j(\mathbf{r}),$$

the functional (3.2) can be written with cylindrical coordinates and with the approximated temperature $T^e(\mathbf{r}, t)$ and virtual temperature $\Theta^e(\mathbf{r}, t)$ in the following form:

$$\begin{aligned}
0 = 2\pi \sum_{q=1}^{n^1} \Bigg[& \int_{\Omega^1} \rho^1 c^1 N_i \dot{T}_{qi}^1 N_k \Theta_{qk}^1 r dr dz + \int_{\Omega^1} k^1 \left(\frac{\partial N_i}{\partial r} T_{qi}^1 \frac{\partial N_k}{\partial r} \Theta_{qk}^1 + \frac{\partial N_i}{\partial z} T_{qi}^1 \frac{\partial N_k}{\partial z} \Theta_{qk}^1 \right) r dr dz + \\
& + \int_{\Gamma_s^1} \alpha^1 (N_i T_{qi}^1 - T_\infty) N_k \Theta_{qk}^1 r ds - \int_{\Gamma_c^1} c_D \beta v p N_i \Theta_{qi}^1 r ds + \int_{\Gamma_c^1} \hat{\alpha} (N_i T_{qi}^1 - N_k T_{q^* k}^2) N_l \Theta_{ql}^1 r ds \Bigg] + \\
+ 2\pi \sum_{q=1}^{n^2} \Bigg[& \int_{\Omega^2} \rho^2 c^2 N_i \dot{T}_{qi}^2 N_k \Theta_{qk}^2 r dr dz + \int_{\Omega^2} k^2 \left(\frac{\partial N_i}{\partial r} T_{qi}^2 \frac{\partial N_k}{\partial r} \Theta_{qk}^2 + \frac{\partial N_i}{\partial z} T_{qi}^2 \frac{\partial N_k}{\partial z} \Theta_{qk}^2 \right) r dr dz + \\
& + \int_{\Gamma_s^2} \alpha^2 (N_i T_{qi}^2 - T_\infty) N_k \Theta_{qk}^2 r ds - \int_{\Gamma_c^2} c_D (1 - \beta) v p N_i \Theta_{qi}^2 r ds - \int_{\Gamma_c^2} \hat{\alpha} (N_i T_{qi}^2 - N_k T_{q^* k}^1) N_l \Theta_{ql}^2 r ds \Bigg]
\end{aligned} \tag{4.3}$$

where $ds = \sqrt{dr^2 + dz^2}$. The finite elements at the contact zone are in front of each other, so that the nodes of this elements are in coincidence. If the q^{th} element is in

the contact zone, q^* concerns the element which is in the other body in front of the q^{th} element. Let us use the following notation:

$$M_{ik}^{(1)} = \rho^1 c^1 \int\limits_{\Omega^1} N_i N_k r dr dz \qquad M_{ik}^{(2)} = \rho^2 c^2 \int\limits_{\Omega^2} N_i N_k r dr dz$$

$$\tilde{K}_{ik}^{(1)} = k^1 \int\limits_{\Omega^1} \left(\frac{\partial N_i}{\partial r}\frac{\partial N_k}{\partial r} + \frac{\partial N_i}{\partial z}\frac{\partial N_k}{\partial z} \right) r dr dz \qquad \tilde{K}_{ik}^{(2)} = k^2 \int\limits_{\Omega^2} \left(\frac{\partial N_i}{\partial r}\frac{\partial N_k}{\partial r} + \frac{\partial N_i}{\partial z}\frac{\partial N_k}{\partial z} \right) r dr dz$$

$$\tilde{C}_{ik}^{(1)} = \alpha^1 \int\limits_{\Gamma_s^1} N_i N_k r ds \qquad \tilde{C}_{ik}^{(2)} = \alpha^2 \int\limits_{\Gamma_s^2} N_i N_k r ds$$

$$\tilde{\tilde{C}}_{ik}^{(1)} = \hat{\alpha} \int\limits_{\Gamma_c^1} N_i N_k r ds \qquad \tilde{\tilde{C}}_{ik}^{(2)} = \hat{\alpha} \int\limits_{\Gamma_c^2} N_i N_k r ds$$

$$\tilde{F}_i^{(1)} = c_D \beta \int\limits_{\Gamma_c^1} v p N_i r ds \qquad \tilde{F}_i^{(2)} = c_D (1-\beta) \int\limits_{\Gamma_c^2} v p N_i r ds$$

For the sake of simplicity let T_∞ be equal to zero. Obviously, for the element whose side is not on Γ_c^e the integral in $\tilde{\tilde{C}}_{ik}^{(e)}$ and $\tilde{F}_i^{(e)}$ vanish. With the help of this notation and the fact that equation (4.3) can be split into two parts associated with coefficients Θ_{ik}^1, Θ_{ik}^2 we have

$$\begin{aligned} &\sum_{q=1}^{n^1} \left(M_{ik}^{(1)} \dot{T}_{qk}^1 + \tilde{K}_{ik}^{(1)} T_{qk}^1 + \tilde{C}_{ik}^{(1)} T_{qk}^1 + \tilde{\tilde{C}}_{ik}^{(1)} T_{qk}^1 - \tilde{\tilde{C}}_{ik}^{(1)} T_{q^*k}^2 - \tilde{F}_i^{(1)} \right) = 0 \\ &\sum_{q=1}^{n^2} \left(M_{ik}^{(2)} \dot{T}_{qk}^2 + \tilde{K}_{ik}^{(2)} T_{qk}^2 + \tilde{C}_{ik}^{(2)} T_{qk}^2 + \tilde{\tilde{C}}_{ik}^{(2)} T_{qk}^2 - \tilde{\tilde{C}}_{ik}^{(2)} T_{q^*k}^1 - \tilde{F}_i^{(2)} \right) = 0 \end{aligned} \tag{4.4}$$

We have to distinguish two cases. In the first one the elements are not situated on Γ_c^e. In this case the integrals $\tilde{\tilde{C}}_{ik}^{(e)}$ and $\tilde{F}_i^{(e)}$ vanish, and equation (4.4) can be written in a simpler form.

$$\sum_{q=1}^{n_n^e} (\underbrace{M_{ik}^{(e)}}_{{}^{(q)}M_{ik}} \underbrace{\dot{T}_{qk}^e}_{\dot{T}_{qk}} + \underbrace{(\tilde{K}_{ik}^{(e)} + \tilde{C}_{ik}^{(e)})}_{{}^{(q)}\tilde{K}_{ik}} \underbrace{T_{qk}^e}_{T_{qk}}) = 0 \tag{4.5}$$

where n_n^e is the number of the elements, which are not situated in the contact zone. In the second case the elements situated on Γ_c^e are considered. Now the integrals in $\tilde{\tilde{C}}_{ik}^{(e)}$ and $\tilde{F}_i^{(e)}$ do not vanish. Because of the coupling in equation (4.4) two systems of linear equations have to be computed simultaneously. These equations concern the elements situated in front of each other in Γ_c^e.

$$\sum_{q=1}^{n_c^e} \left\{ \underbrace{\begin{bmatrix} M_{ik}^{(1)} & 0 \\ 0 & M_{ik}^{(2)} \end{bmatrix}}_{{}^{(q)}M_{ik}} \underbrace{\begin{bmatrix} \dot{T}_{qk}^1 \\ \dot{T}_{q^*k}^2 \end{bmatrix}}_{\dot{T}_{qk}} + \underbrace{\begin{bmatrix} \tilde{K}_{ik}^{(1)} + \tilde{C}_{ik}^{(1)} + \tilde{\tilde{C}}_{ik}^{(1)} & -\tilde{\tilde{C}}_{ik}^{(1)} \\ -\tilde{\tilde{C}}_{ik}^{(2)} & \tilde{K}_{ik}^{(2)} + \tilde{C}_{ik}^{(2)} + \tilde{\tilde{C}}_{ik}^{(2)} \end{bmatrix}}_{{}^{(q)}\tilde{K}_{ik}} \underbrace{\begin{bmatrix} T_{qk}^1 \\ T_{q^*k}^2 \end{bmatrix}}_{T_{qk}} - \underbrace{\begin{bmatrix} \tilde{F}_i^{(1)} \\ \tilde{F}_i^{(2)} \end{bmatrix}}_{{}^{(q)}\tilde{F}_i} \right\} = 0 \tag{4.6}$$

where q runs from 1 to the number of contacting elements of one body (e.g. n_c^1), and q^* means the element in front of the actual element (q), and it is situated in the other body. The superscript (q) on the left concerns the numbering of the elements. ${}^{(q)}M_{ik}$, ${}^{(q)}\tilde{K}_{ik}$ and ${}^{(q)}\tilde{F}_i$ must be computed for every element and be added to the global convection matrix M_{ik}, conductivity matrix $\tilde{K}_{ik}$ and load vector $\tilde{F}_i$, respectively. Obviously, the element level temperature parameters are added to a global vector of temperature parameters (T_k). Equations (4.5) and (4.6) are first order differential equations in time. The usual way of discretizing such equations is the so-called ϑ-*method*

$$M_{ik}\frac{T_k^{\{n+1\}}-T_k^{\{n\}}}{\Delta t}+\tilde{K}_{ik}(\vartheta T_k^{\{n+1\}}+(1-\vartheta)T_k^{\{n\}})-(\vartheta\tilde{F}_i^{\{n+1\}}+(1-\vartheta)\tilde{F}_i^{\{n\}})=0$$

where the superscript indices in brace denote the number of time steps, Δt means the length of the time step and ϑ is a real number between 0 and 1. Since $T_k^{\{n\}}$ is known from the previous time step, a system of linear equations is obtained:

$$\underbrace{(M_{ik}+\Delta t\tilde{K}_{ik}\vartheta)}_{K_{ik}}\underbrace{T_k^{\{n+1\}}}_{T_k}-\underbrace{((M_{ik}-\Delta t\tilde{K}_{ik}(1-\vartheta))T_k^{\{n\}}+\Delta t(\vartheta\tilde{F}_i^{\{n+1\}}+(1-\vartheta)\tilde{F}_i^{\{n\}}))}_{F_i}=0\,, \tag{4.7}$$

The final system of linear equations has the form

$$K_{ik}T_k=F_i \tag{4.8}$$

where the indices i and k run from 1 to m, i.e., the number of degrees of freedom of the problem.

5. **A mapping technique**

Let us consider an axisymmetrical contact problem. The two contacting bodies slide over each other, which causes heat generation and wear on the contact surface. The generated heat flows into the bodies, therefore the temperature distribution of the bodies changes. The heat conduction equation has to be solved because the temperature field is needed for computing the thermal expansion, which is part of the displacement field. Due to the wear and the thermal expansion, the contact area changes continuously. When the contact problem is solved, the boundary of the contact zone is unknown *a priori*. In order to treat such contact problems, an adaptive *hp*-version method is required [1, 2].

One of the advantages of the p-version is that for smooth problems only coarse meshes are needed, since the error in energy norm decreases exponentially when the polynomial degree of elements is increased [6]. When the p-version is used, then the accuracy is typically high enough for the singularities to induce oscillations in the numerical solution. In the boundary of the contact region the normal stress may have jumps in its derivatives, which are regarded as singularities. Concerning the finite element discretization, we have a problem of category C [6], i.e., the mesh in 2D cannot be constructed so that the points, where the solutions are not analytic, are at nodal points. Our aim is to convert the problem of category C into one of category B

[6]. The oscillations are minimized when nodes (or nodes and edges) in two dimensions (three dimensional problems) are located at the boundary of the contact zone [1].

The coupled thermo-mechanical contact problem can be solved via the operator split technique [12]. The problem is split into separately solved thermal and mechanical parts. A possible way to solve the mechanical part is detailed in [1, 12]. In this paper we focus our attention on the thermal part.

The heat conduction problem is both a spatial and a temporal process, which are discretized by the finite element method and the finite difference method, respectively [7]. Computation of a time step consists of the computation of the contact problem, and after that the computation of the heat conduction problem, iteratively. The heat conduction problem has to be solved in the modified, new mesh, where the nodal points are located on the new boundary of the contact region. For the determination of the new temperature field, the temperature field given in the old mesh is needed. There are several mapping techniques in the literature (e.g. in connection with plastic deformation), but mostly for the h-version. For a high order approximation in FEM one can find two approaches in the literature. The first is based on the Lagrangian polynomials, the shape functions are associated with nodal point parameters. The nodal point parameter is the actual value of the field in question at the location of the node. (We can speak of e.g. nodal displacement, nodal temperature, and so on.) For the rest of the nodes with the same shape function the field value is zero. The state variables (e.g. temperature) can be easily interpolated from the nodal points of the old mesh to the nodal points of the new mesh [8, 9, 10, 11].

In the second approach, which is applied here, Legendre polynomials are used for the approximation. It should be emphasized that the shape functions constructed by the Legendre polynomials are not associated with nodal parameters. We have so called side modes, internal modes (bubble modes), edge modes and so on. That is, the parameters of the shape functions do not correspond to a field value in a specific location. Therefore the parameters have no direct physical meaning contrary to the nodal parameters. We can speak only of parameters associated with displacements and temperatures [6]. The least-squares method is more suitable for determining these parameters. The least-squares method has already been applied successfully for the h-version, where the mapped fields have to respect the local and the global equilibriums [13], or when the mesh is coarsed [14]. The data transfer can be applied similarly in the p-version via the least-squares method as has been done in the above mentioned h-version.

When the mesh modification is performed, the new temperature field can be solved by equation (4.7). In this equation F_i depends on the temperature field of the previous time step, which is given in the old mesh, hence the temperature field has to be transferred to the new mesh. The nodal values and parameters of the temperature field have to be known in the new mesh to perform the matrix multiplication in F_i.

These values can be determined by the least-squares method:

$$\sum_{i=1}^{n} \iint\limits_{S^i} \left[T(r,z) - \tilde{T}(r,z)\right]^2 dr dz = \min.\,, \tag{5.1}$$

where $n = n^1 + n^2$, S^i is the surface of one finite element, $T(r,z)$ is the temperature in the new mesh, $\tilde{T}(r,z)$ is the temperature given in the old mesh and n is the number of elements in the new mesh. Let us substitute (4.1) into equation (5.1), and transform it into the $[-1,1]$ interval, so that the numerical integration can be computed via Gauss quadrature.

$$\sum_{i=1}^{n} \int_{-1}^{1}\int_{-1}^{1} \left[N_j(\xi,\eta)T_{ij} - N_j(\tilde{\xi},\tilde{\eta})\tilde{T}_{\tilde{i}j}\right]^2 \left| \begin{array}{cc} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{array} \right| d\xi d\eta = \min.\,, \tag{5.2}$$

where the subscripts i and $\tilde{i}$ denote the elements in the new mesh and those elements in the old mesh, which coincide with the point with coordinates (ξ,η) in the new mesh, respectively. The subscript index j refers to the shape functions. Let T_l be a vector. The size of this vector equals the number of degrees of freedom of the problem. After the summation in (5.2) the components of T_{ij} are added to the appropriate element of T_l. This element of T_l can be determined by the connection between the local and global numbering [6]. After deriving equation (5.2) with respect to T_{ij} and making the summation for every element, the following linear system of equations is obtained:

$$\sum_{l=1}^{m} A_{kl} T_l = b_k\,, \tag{5.3}$$

where m is the number of degrees of freedom of the temperature field,

$$A_{kl} = \sum_{i=1}^{n} \int_{-1}^{1}\int_{-1}^{1} N_k(\xi,\eta)N_l(\xi,\eta) \left| \begin{array}{cc} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{array} \right| d\xi d\eta\,,$$

and

$$b_k = \sum_{i=1}^{n} \int_{-1}^{1}\int_{-1}^{1} N_k(\xi,\eta)N_j(\tilde{\xi},\tilde{\eta})\tilde{T}_{\tilde{i}j} \left| \begin{array}{cc} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{array} \right| d\xi d\eta\,.$$

From this equation the vector of temperature parameters T_l can be computed. The values of temperature have to be known in Gauss or Lobatto points (ξ,η) because of the numerical integration. However, the Gauss or Lobatto integration points in the new mesh and in the old mesh do not coincide. So the problem is to find the point P in the appropriate element $(\tilde{i})$ with local coordinates $(\tilde{\xi},\tilde{\eta})$ in the other mesh with practically the same contour – see Figure 1.

Instead of checking each element on whether it contains the point P or not (as in [10]), the searching process is realized by a new recursive algorithm. It is practical to create a list of the surrounding elements of each element, and a list of already checked elements [15]. Within a step one element is examined. If the element has already been checked, the step is finished. The local coordinates $(\tilde{\xi},\ \tilde{\eta})$ of the point P have to

be computed by the Newton-Raphson method. If the local coordinates are between -1 and 1, the algorithm is finished. If $\tilde{\xi}$ and $\tilde{\eta}$ are outside the interval $[-1, 1]$, the element is marked as "checked". After that, the neighbors of the element have to be checked. The neighbors can be determined with the help of the list of the surrounding elements. An order have to be defined among the neighbors. On ground of this order the search goes on as a new step. The order can be arbitrary, but the process is faster if a suitable order is chosen. A possible way is the following. Let R be a ray that starts from the point Q (see Figure 2), and passes through the elements to the point P. The ray crosses one of the edges of the starting element. The element next to

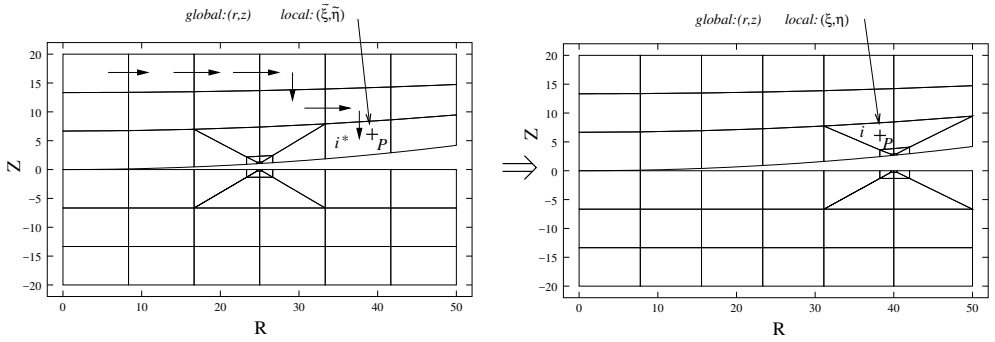

Figure 1. If element i and point (r, z) are known, the appropriate element i^* and local coordinate $(\tilde{\xi}, \tilde{\eta})$ are sought on the old mesh.

this edge will be the first in the order. If the ray passes through a node, an arbitrary neighboring edge of this node can be considered. With this algorithm in hand the element containing the point P can be rapidly found.

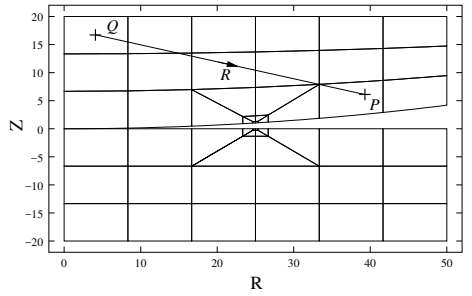

Figure 2. Let R be a ray that starts from the point Q, and passes through the elements to the point P. R crosses a side of the element containing Q, and enters the neighboring element. This neighboring element is the first in the further search process.

6. **Numerical examples**

Let us consider an axisymmetrical body with inner radius $R_i = 80$ mm and outer radius $R_o = 120$ mm. The height of the body can be considered infinite. In other words this is an infinitely long tube. The heat flux on the inner and outer surface equals zero, i.e., the body is thermally isolated. The initial temperature $f(r)$ is given. Because of its symmetry this problem can be treated with a one dimensional mathematical model. The following differential equation has to be solved:

$$c\rho\frac{\partial T(r,t)}{\partial t} = k\left(\frac{\partial^2 T(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial T(r,t)}{\partial r}\right) \qquad R_i \le r \le R_o\,, \tag{6.1}$$

where the initial and boundary conditions are written as:

$$T(r,0) = f(r) \qquad R_i \le r \le R_o \tag{6.2}$$

$$-k\frac{\partial T(r,t)}{\partial r}\bigg|_{r=R_i} = 0 \qquad t \ge 0$$

$$-k\frac{\partial T(r,t)}{\partial r}\bigg|_{r=R_o} = 0 \qquad t \ge 0\,.$$

The specific heat is $c = 460$ J/kg°C, the material density is $\rho = 7850\text{kg/m}^3$, and the coefficient of heat conduction is $k = 55$ W/m°C. The exact solution of the problem is the following (see Appendix):

$$T_{ex}(r,t) = T_s + 2\sum_{n=1}^{\infty} C_n \frac{U_0(\omega_n r)}{R_o^2(U_0(\omega_n R_o))^2 - R_i^2(U_0(\omega_n R_i))^2} e^{-\kappa\omega_n t}\,, \tag{6.3}$$

where

$$U_0(\omega_n r) = J_0(\omega_n r)Y_1(\omega_n R_i) - Y_0(\omega_n r)J_1(\omega_n R_i),$$

$$C_n = \int_{R_i}^{R_o} (f(\xi) - T_s)\xi U_0(\omega_n \xi)d\xi,$$

$$T_s = \frac{2}{R_o^2 - R_i^2}\int_{R_i}^{R_o} \xi f(\xi)d\xi,$$

$\kappa = \dfrac{k}{c\rho}$, ω_n are the roots of the equation

$$U_1(\omega R_o) = J_1(\omega R_o)Y_1(\omega R_i) - Y_1(\omega R_o)N_1(\omega R_i) = 0$$

and $J_0(\xi)$, $J_1(\xi)$, $Y_0(\xi)$, $Y_1(\xi)$ are the first and second order Bessel functions of the first kind and the first and second order Bessel functions of the second kind, respectively [16]. One can see that the temperature field will be equalized if the bodies are thermally isolated. When the temperature gradient vanishes everywhere in the bodies, the temperature field will change no more. Accordingly, when the time becomes infinitely high in (6.3), the solution of the heat conduction problem will be the constant temperature.

Due to the symmetry of the problem, it is enough to solve the heat conduction problem between two parallel planes perpendicular to axis z. Let the two planes be

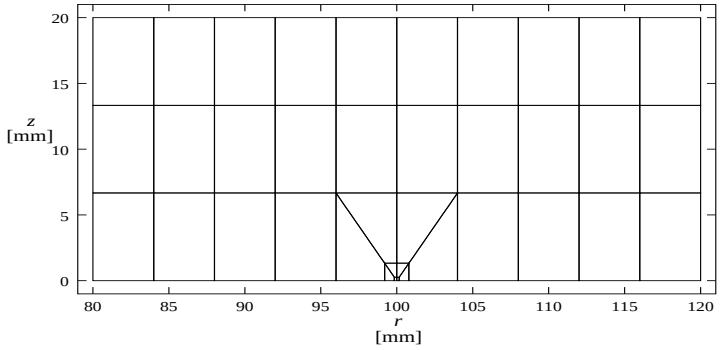

Figure 3. The cross-section of the body between $z_1 = 0$ mm and $z_2 = 20$ mm

situated at $z_1 = 0$ mm and at $z_2 = 20$ mm. The cross-section of the body bounded by the two planes is divided into finite elements – see Figure 3. The polynomial degree of the shape functions is 8, the number of degrees of freedom is 2522. The initial

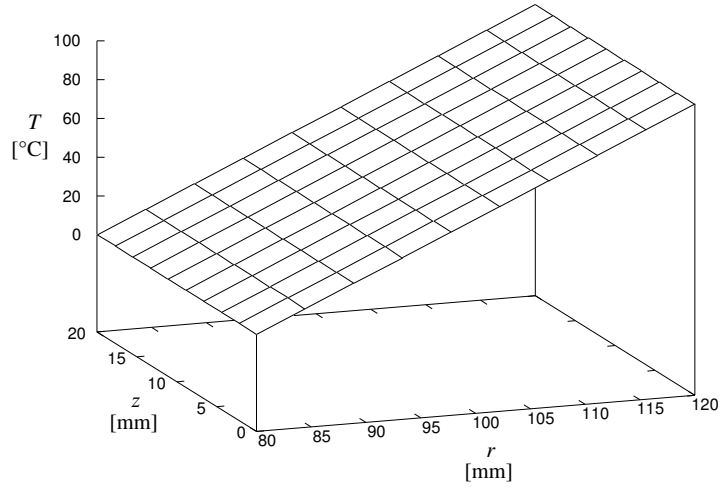

Figure 4. The initial temperature distribution for the first and second example

temperature is $T(r, 0) = f(r) = (2.5r - 200)°\mathrm{C}$ (see Figure 4) and the length of a time step is $\Delta t = 0.01$ s. To ensure the accuracy of six digits in the exact solution, the first 100 terms of the series are considered in (6.3).

The mesh is modified in every time step, so that the node, whereto the mesh is graded, moves from $r = 90$ mm to $r = 110$ mm. We will investigate the influence

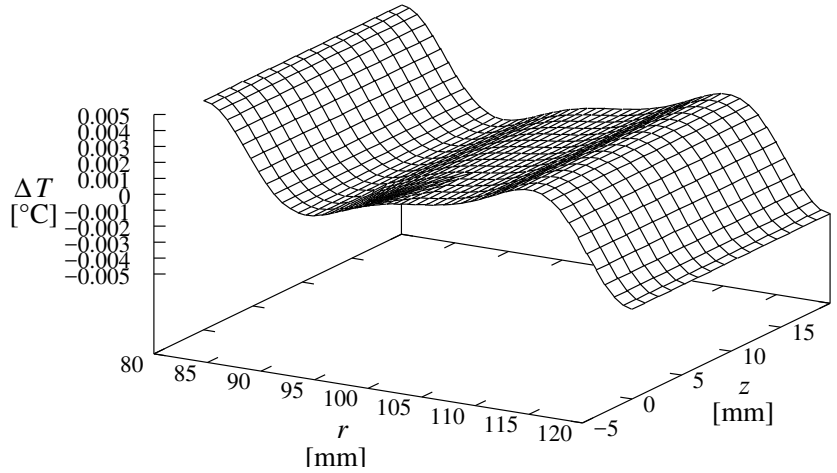

Figure 5. The difference between the exact solution and the finite element approximation

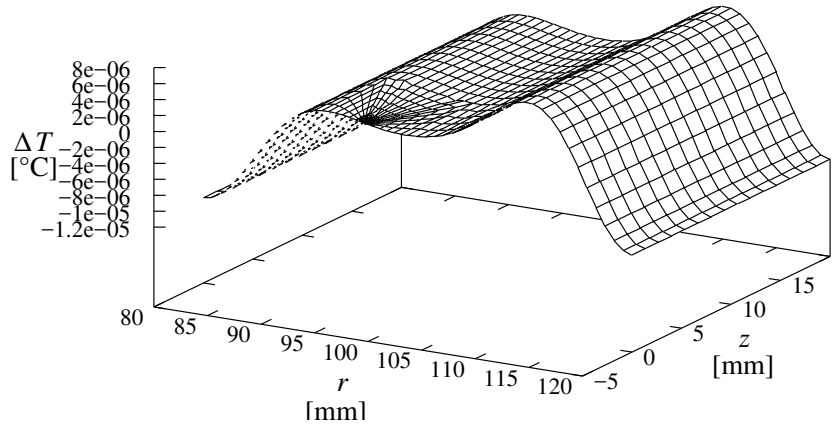

Figure 6. The difference between the finite element approximation on fixed mesh and finite element approximation on continuously changing mesh

of this modification on the finite element solution. The difference between the exact solution and the finite element approximation at $t = 1$ s can be seen in Figure 5. The relative error is solved by $e_1 = \Delta T_1/|T_{ex}|$, where $\Delta T_1 = |T_{ap} - T_{ex}|$, T_{ap} is the approximated temperature and T_{ex} is the exact temperature. The relative error is less than 0.01% everywhere in the bodies. The modification of the mesh has no significant influence on the accuracy of the finite element approximation. Now let us compare two finite element approximations. In the first case the mesh is fixed, and in the second case the mesh is modified in every time step, so that the node,

whereto the mesh is graded, moves from $r = 90$ mm to $r = 110$ mm. The boundary conditions are the same as in the first example. The time considered is $t = 1$ s, and the length of a time step is $\Delta t = 0.01$ s. One can see in Figure 6 that the difference between the two fields is small. The relative error is defined as $e_2 = \Delta T_2 / T_{ap}$, where $\Delta T_2 = |T_{am} - T_{ap}|$, T_{ap} is the approximated temperature in the fixed mesh and T_{am} is the approximated temperature in the modified mesh. The relative error is less than $1.2 \cdot 10^{-7}\%$, there is no significant difference between the two approximated temperature fields. After these two examples we can suppose that this technique

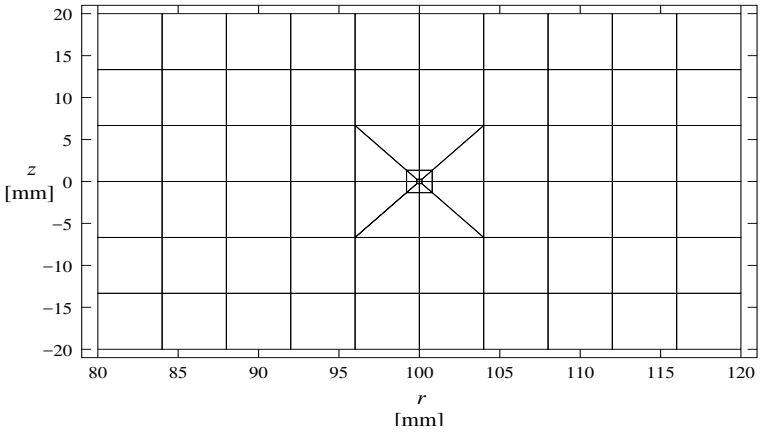

Figure 7. The finite element mesh of the cross-section of the bodies. The contact surface is the $z = 0$ plane. The boundaries of the contact region are the $r = 80$ mm edge and the point where the smallest element is situated, i.e., $r = 100$ mm

provides a reliable solution of the heat conduction problem in a continuously changing mesh. In the third example a possible application of this technique will be introduced. Let us consider a coupled thermo-mechanical contact problem. The system consists of two bodies, which slide on each other. The lower body is fixed, and the upper one is rotated with angular velocity $\omega = 1/\text{s}$. The inner and outer radii of the bodies are $R_i = 80$ mm and $R_o = 120$ mm, respectively, and their height is $h = 20$ mm. The contact surface is the $r\varphi$ plane – see Figure 7. The problem can be split into separately solved mechanical and thermo-dynamical problems. In this paper we disregard the computation of the contact problem. Instead a parabolic contact pressure distribution is assumed. The thermal boundary conditions are the same as equation (2.2). Let $\alpha^1 = \alpha^2 = 44$ W/m^2°C, $c^1 = c^2 = 460$ J/kg°C, $\rho^1 = \rho^2 = 7850$ kg/m^3, $k^1 = k^2 = 55$ W/m°C, $\vartheta = 2/3$, $\Delta t = 0.01$ s, $c_D = 0.8$. The applied polynomial degree is 8. When the mesh is modified, the pressure distribution changes. The pressure can be written as: $p(r) = p_{max}\left[1 - (r - R_i)^2/(r_b - R_i)^2\right]$, where p_{max} is the maximum pressure, and r_b is the r coordinate of the node whereto the mesh is graded – see Figure 8. The initial temperature distribution is zero. It is supposed, that the boundary of the contact region, its radius is r_b, moves continuously from $r = 90$ mm

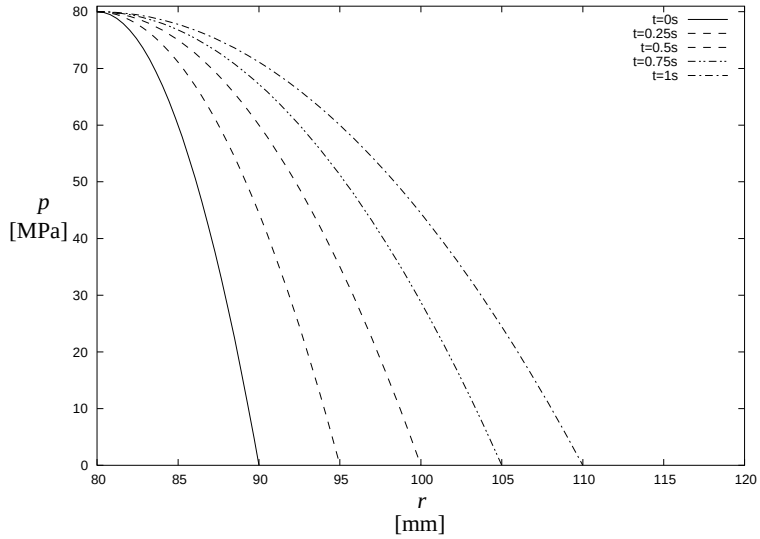

Figure 8. The contact pressure distribution at $t = 0$s, 0.25s, 0.5s, 0.75s, 1s

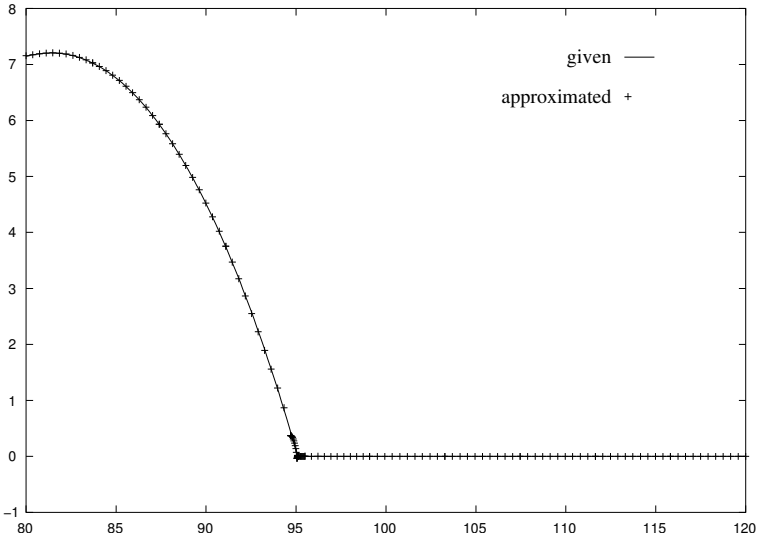

Figure 9. The given heat flux (see equation (2.2)) and the heat flux obtained from the approximated temperature field

to $r = 110$ mm during the process. One of the ways to verify the numerical solution is to check whether the boundary conditions meet the heat flux obtained from the

temperature field approximated: $q_{ap}^{e} = -k\dfrac{\partial T^{e}(r,t)}{\partial r}\Big|_{r\in\Gamma_c^e}$. Figure 9 shows that the derivative of the temperature field with respect to r gives accurately the boundary condition. The temperature distributions for the upper body at $t = 0$ s, $t = 0.25$ s, $t = 0.5$ s, $t = 0.75$ s and $t = 1$ s can be seen in Figures 10-14. Due to the fact, that

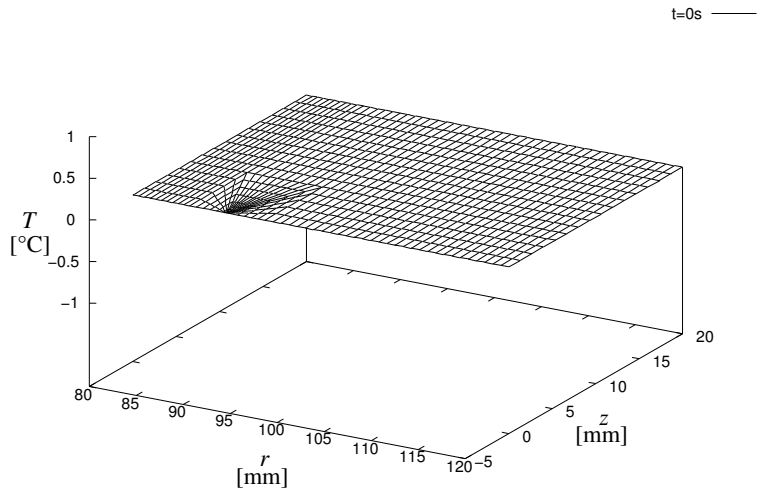

Figure 10. The approximated temperature distribution of the upper body at $t = 0$ s

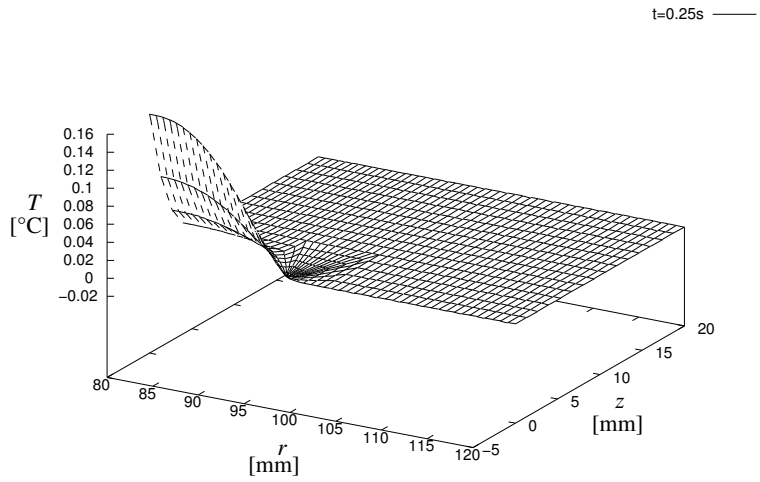

Figure 11. The approximated temperature distribution of the upper body at $t = 0.25$ s

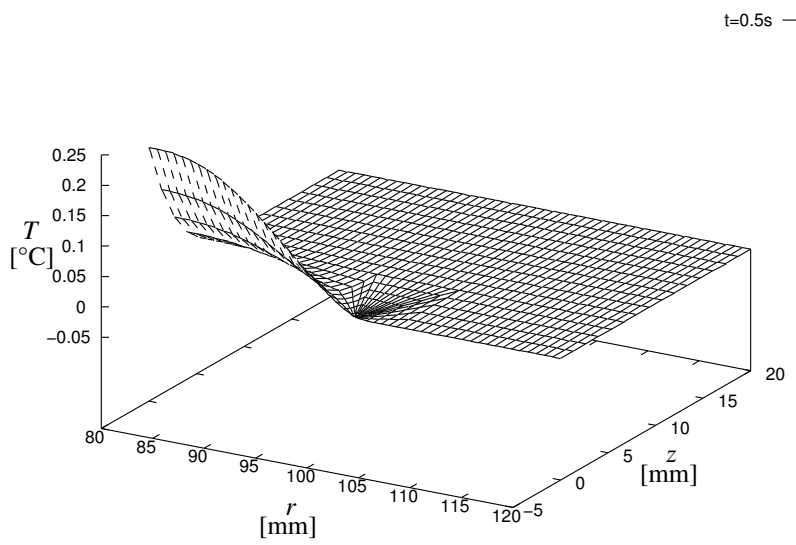

Figure 12. The approximated temperature distribution of the upper body at $t = 0.5$ s

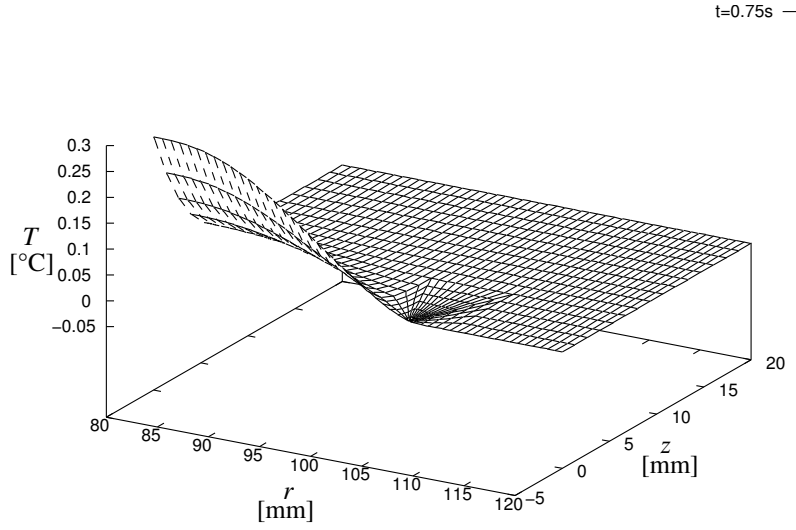

Figure 13. The approximated temperature distribution of the upper body at $t = 0.75$ s

the boundary conditions on the lower and upper bodies are the same, the temperature distributions of the lower and upper bodies are mirror images of each other.

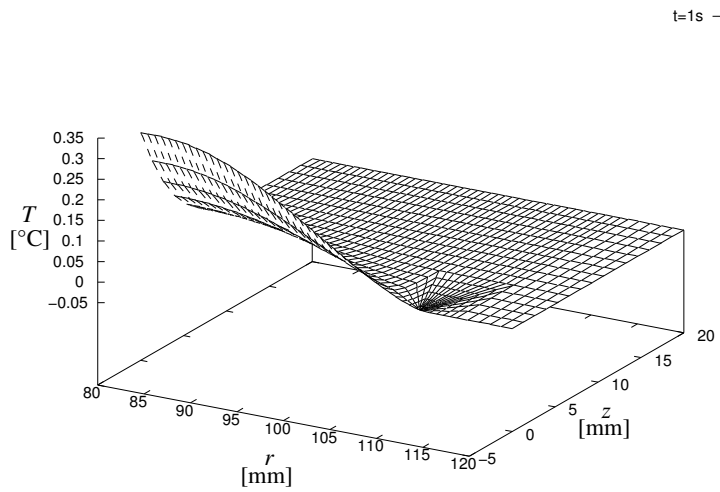

Figure 14. The approximated temperature distribution of the upper body at $t = 1$ s

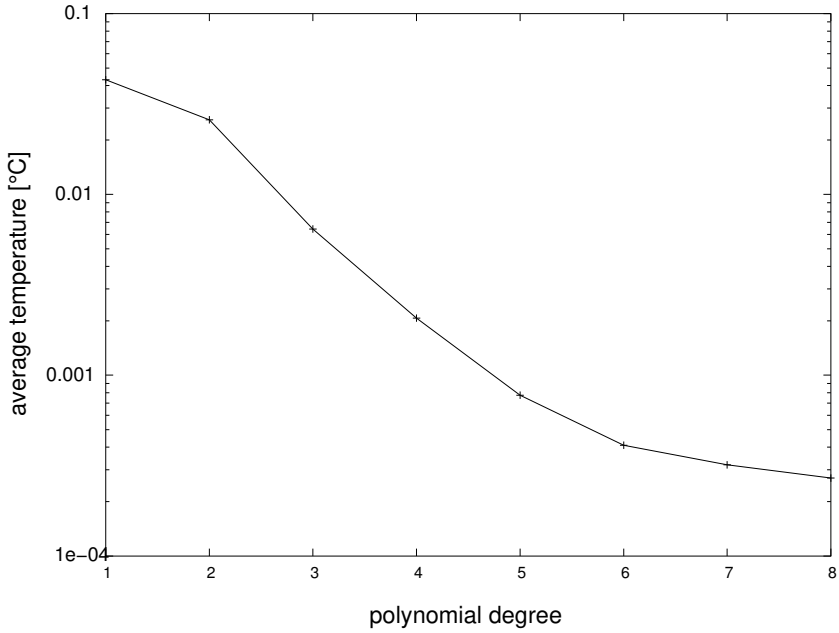

Figure 15. When the polynomial degree is increased, the error in average temperature is decreased

The convergence of the solution was also investigated with different polynomial degrees. At the end of the time interval the average temperature $\bar{T}$, which should be

proportional to the thermal energy, was computed

$$\bar{T} = \frac{1}{\Omega} \int_{\Omega} T(r,z) d\Omega \,. \tag{6.4}$$

This average temperature as a function of p is represented in Figure 15. When the polynomial degree is increased, the error in average temperature is decreased and the average temperature converges to a given value.

7. Conclusions

A special form of the *hp*-version of the finite element method has been presented for the solution of the heat conduction problem for axially symmetric bodies. The mesh was adjusted in every time step. The purpose was to simulate the thermal part of a thermo-mechanical contact problem. When the contact problem is solved, the mesh has to be adjusted so that the boundary of the contact zone is a nodal point. The high polynomial degree and the employment of small elements around the border of the contact region ensure high accuracy in the numerical solution. When the contact region changes the mesh has to be modified. Using the mapping technique, the new temperature field can be computed in the modified mesh with a great accuracy, and the heat conduction problem can be solved accurately. Exact solutions and finite element approximations were compared with each other to prove the reliability of the computation.

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APPENDIX

Consider the following partial differential equation:

$$\frac{\partial \vartheta(r,t)}{\partial t} = \kappa \left(\frac{\partial^2 \vartheta(r,t)}{\partial r^2} + \frac{1}{r} \frac{\partial \vartheta(r,t)}{\partial r} \right), \qquad \begin{matrix} r_1 \le r \le r_2 \\ 0 \le t \le t_{\max} \end{matrix} \tag{A.1}$$

$$\vartheta(r,0) = f(r) \tag{A.2}$$

$$k \frac{\partial \vartheta(r,t)}{\partial r}\bigg|_{r=r_1} = 0 \tag{A.3}$$

$$k \frac{\partial \vartheta(r,t)}{\partial r}\bigg|_{r=r_2} = 0 \tag{A.4}$$

where $\kappa = \dfrac{k}{\rho c}$, $f(r)$ is a given function, r_1 and r_2 are the inner and outer radii, respectively, and $t_{\max}$ is the length of the time interval considered. The solution is searched for as the sum of the solution of a steady-state problem and a solution of a transient problem: $\vartheta(r,t) = u(r) + v(r,t)$.

1. The steady-state problem:

$$\kappa \left(\frac{d^2 u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr} \right) = 0, \quad r_1 \le r \le r_2 \tag{A.5}$$

$$k \frac{du(r)}{dr}\bigg|_{r=r_1} = 0 \tag{A.6}$$

$$k \frac{du(r)}{dr}\bigg|_{r=r_2} = 0 \tag{A.7}$$

As is well known, the general solution of equation (A.5) is given by the formula:

$$u(r) = c_1 \ln r + c_2 \,, \tag{A.8}$$

where c_1 can be determined from the boundary conditions:

$$\left.\begin{aligned} \left.\frac{du(r)}{dr}\right|_{r=r_1} &= \frac{c_1}{r_1} = 0 \\ \left.\frac{du(r)}{dr}\right|_{r=r_2} &= \frac{c_1}{r_2} = 0 \end{aligned}\right\} \quad \Rightarrow \quad c_1 = 0 \tag{A.9}$$

c_2 is determined later with the help of the solution of the transient problem. At this moment the solution of equation (A.5) is

$$u(r) = c_2 (= \text{const.}) \,. \tag{A.10}$$

2. The transient problem:

$$\frac{\partial v(r,t)}{\partial r} = \kappa \left(\frac{\partial^2 v(r,t)}{\partial r^2} + \frac{1}{r} \frac{\partial v(r,t)}{\partial r} \right), \qquad \begin{matrix} r_1 \le r \le r_2 \\ 0 \le t \le t_{\max} \end{matrix} \tag{A.11}$$

$$v(r,t)|_{t=0} = f(r) - u(r) \,, \tag{A.12}$$

$$k \left. \frac{\partial v(r,t)}{\partial r} \right|_{r=r_1} = 0 \,, \tag{A.13}$$

$$k \left. \frac{\partial v(r,t)}{\partial r} \right|_{r=r_2} = 0 \,. \tag{A.14}$$

Let us try to find the solution of equations (A.11-A.14) as the product of two functions:

$$v(r,t) = \varphi(r)\psi(t) \tag{A.15}$$

Substituting equation (A.15) into equation (A.11) and separating the variables r and t we obtain

$$\frac{1}{\kappa} \frac{1}{\psi(t)} \frac{d\psi(t)}{dt} = \frac{1}{\varphi(r)} \frac{d^2\varphi(r)}{dr^2} + \frac{1}{r\varphi(r)} \frac{d\varphi(r)}{dr} = \pm\omega^2. \tag{A.16}$$

Because the left and right hand sides are independent, they must equal to the same constant. It is worth giving this constant in the form: $\pm\omega^2$. With this in hand two ordinary differential equations have to be solved instead of a partially differential equation.

a.

$$\frac{d\psi(t)}{dt} = \pm\kappa\omega^2\psi(t) \,. \tag{A.17}$$

The solution of equation (A.17) has the form

$$\psi(t) = c_3 e^{\pm\kappa\omega^2 t} \,. \tag{A.18}$$

If the exponent is greater than zero for $t \to \infty$ then the function $\psi(t)$ is divergent. That is reason why only the negative sign is acceptable:

$$\psi(t) = c_3 e^{-\kappa\omega^2 t} \tag{A.19}$$

b.

$$r\frac{d^2\varphi(r)}{dr^2}+\frac{d\varphi(r)}{dr}+\omega^2 r\varphi(r)=0\,. \tag{A.20}$$

The solution of equation (A.20) assumes the form

$$\varphi(r)=c_4\mathrm{J}_0(\omega r)+c_5\mathrm{Y}_0(\omega r)\,, \tag{A.21}$$

where $\mathrm{J}_0(r)$ and $\mathrm{Y}_0(r)$ are the first order Bessel functions.

Let us substitute (A.19) and (A.21) into (A.15)

$$v(r,t)=e^{-\kappa\omega^2 t}(\underbrace{c_3c_4}_{c_6}\mathrm{J}_0(\omega r)+\underbrace{c_3c_5}_{c_7}\mathrm{Y}_0(\omega r)) \tag{A.22}$$

We can now substitute the solution (A.22) into the boundary condition (A.13)

$$-\,k\,\frac{\partial v(r,t)}{\partial r}\bigg|_{r=r_1}=ke^{-\kappa\omega^2 t}(c_6\omega\mathrm{J}_1(\omega r_1)+c_7\omega\mathrm{Y}_1(\omega r))=0\,, \tag{A.23}$$

where $\mathrm{J}_1(r)$ and $\mathrm{Y}_1(r)$ are the second order Bessel functions. The boundary conditions can be satisfied only if $c_6=C\mathrm{Y}_1(\omega r)$ and $c_7=C\mathrm{J}_1(\omega r)$, where C is a constant. The following notations will be employed:

$$\mathrm{U}_0(\omega r):=\mathrm{Y}_1(\omega r_1)\mathrm{J}_0(\omega r)-\mathrm{J}_1(\omega r_1)\mathrm{Y}_0(\omega r) \tag{A.24}$$

and

$$\mathrm{U}_1(\omega r):=\mathrm{Y}_1(\omega r_1)\mathrm{J}_1(\omega r)-\mathrm{J}_1(\omega r_1)\mathrm{Y}_1(\omega r)\,. \tag{A.25}$$

It can be proved that

$$\frac{d\mathrm{U}_0(\omega r)}{dr}=-\omega\mathrm{U}_1(\omega r)\,. \tag{A.26}$$

With the help of equation (A.24) the solution of equation (A.11) can be written in the following form:

$$v(r,t)=Ce^{-\kappa\omega^2 t}\mathrm{U}_0(\omega r)\,. \tag{A.27}$$

Let us substitute equation (A.27) into the boundary condition (A.14)

$$-\,k\,\frac{\partial v(r,t)}{\partial r}\bigg|_{r=r_2}=ke^{-\kappa\omega^2 t}\omega\mathrm{U}_1(\omega r_2)=0\,. \tag{A.28}$$

This means that the roots of $\mathrm{U}_1(\omega r)=0$ have to be determined to satisfy equation (A.14). Let us denote the roots by ω_n $(n=1,2,\dots)$. The functions $\mathrm{U}_0(\omega_n r)$ are independent, hence the solution of (A.11) can be written as a series

$$v(r,t)=\sum_{n=1}^{\infty}C_n e^{-\kappa\omega_n^2 t}\mathrm{U}_0(\omega_n r)\,. \tag{A.29}$$

Making use of the orthogonality of the Bessel functions one can prove the orthogonality condition

$$\int_{r_1}^{r_2} r\mathrm{U}_0(\omega_n r)\mathrm{U}_0(\omega_m r)dr=\begin{cases}\dfrac{1}{2}\left[r_1^2\mathrm{U}_0^2(\omega_n r_1)-r_2^2\mathrm{U}_0^2(\omega_n r_2)\right] & \text{if } n=m\\ 0 & \text{if } n\neq m\,.\end{cases} \tag{A.30}$$

Let us substitute series (A.29) into the boundary condition (A.12):

$$v(r,t)|_{t=0} = \sum_{n=1}^{\infty} C_n \mathrm{U}_0(\omega_n r) = f(r) - u(r)\,. \tag{A.31}$$

Let us multiply equation (A.31) by $r\mathrm{U}_0(\omega_m r)$ and integrate it on the $[r_1, r_2]$ interval

$$\int_{r_1}^{r_2} (f(r) - u(r))\mathrm{U}_0(\omega_m r) r dr = \sum_{n=1}^{\infty} C_n \int_{r_1}^{r_2} \mathrm{U}_0(\omega_n r)\mathrm{U}_0(\omega_m r) r\, dr\,. \tag{A.32}$$

The constant C_n can be determined with the aid of equation (A.30):

$$C_n = \frac{\displaystyle\int_{r_1}^{r_2} (f(r) - u(r))\mathrm{U}_0(\omega_n r) r dr}{\dfrac{1}{2}\left[r_1^2 \mathrm{U}_0^2(\omega_n r_1) - r_2^2 \mathrm{U}_0^2(\omega_n r_2)\right]}\,. \tag{A.33}$$

The solution of equation (A.11) is of the form

$$v(r,t) = \sum_{n=1}^{\infty} \frac{\displaystyle\int_{r_1}^{r_2} (f(r) - u(r))\mathrm{U}_0(\omega_n r) r dr}{\dfrac{1}{2}\left[r_1^2 \mathrm{U}_0^2(\omega_n r_1) - r_2^2 \mathrm{U}_0^2(\omega_n r_2)\right]} e^{-\kappa\omega_n^2 t}\mathrm{U}_0(\omega_n r) \tag{A.34}$$

Finally the constant c_2 in equation (A.10) has to be determined. Let us multiply (A.1) by r and integrate it over the interval $[r_1, r_2]$

$$\int_{r_1}^{r_2} \frac{\partial \vartheta(r,t)}{\partial t} r dr = \int_{r_1}^{r_2} \kappa \left[\frac{\partial^2 \vartheta(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial \vartheta(r,t)}{\partial r}\right] r dr\,. \tag{A.35}$$

It can be easily proved that

$$\left[\frac{\partial^2 \vartheta(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial \vartheta(r,t)}{\partial r}\right] r = \frac{\partial}{\partial r}\left(r \frac{\partial \vartheta(r,t)}{\partial r}\right)\,. \tag{A.36}$$

With the aid of (A.36), the boundary conditions (A.3) and (A.4) and the equation $\kappa = k/\rho c$ we can calculate the right side of equation (A.35)

$$\int_{r_1}^{r_2} \frac{\partial \vartheta(r,t)}{\partial t} r dr = r \underbrace{\kappa \frac{\partial \vartheta(r,t)}{\partial r}\Big|_{r=r_2}}_{=0} - r \underbrace{\kappa \frac{\partial \vartheta(r,t)}{\partial r}\Big|_{r=r_1}}_{=0} = 0 \tag{A.37}$$

or

$$\frac{\partial}{\partial t}\int_{r_1}^{r_2} \vartheta(r,t) r dr = 0 \quad \Rightarrow \quad \int_{r_1}^{r_2} \vartheta(r,t) r dr = \text{constant}\,. \tag{A.38}$$

The temperature distribution $\vartheta(r,t)$ is decomposed into a steady-state and a transient temperature distributions

$$\int_{r_1}^{r_2} \underbrace{\vartheta(r,0)}_{f(r)} r dr = \int_{r_1}^{r_2} \vartheta(r,\infty) r dr = \int_{r_1}^{r_2} \underbrace{u(r)}_{c_2} r dr + \underbrace{\int_{r_1}^{r_2} v(r,\infty) r dr}_{=0} = c_2(r_2^2 - r_1^2)\,. \quad \text{(A.39)}$$

The constant c_2 follows from equation (A.39)

$$c_2 = \frac{1}{r_2^2 - r_1^2} \int_{r_1}^{r_2} f(r) r dr \quad \text{(A.40)}$$

Making use of the foregoing we can write

$$\vartheta(r,t) = \frac{1}{r_2^2 - r_1^2} \int_{r_1}^{r_2} f(\zeta)\zeta d\zeta + \\ + \sum_{n=1}^{\infty} \frac{\displaystyle\int_{r_1}^{r_2} \left(f(\eta) - \frac{1}{r_2^2 - r_1^2} \int_{r_1}^{r_2} f(\xi)\xi d\xi \right) \mathrm{U}_0(\omega_n \eta)\eta d\eta}{\dfrac{1}{2}\left[r_1^2 \mathrm{U}_0^2(\omega_n r_1) - r_2^2 \mathrm{U}_0^2(\omega_n r_2) \right]} e^{-\kappa\omega_n^2 t} \mathrm{U}_0(\omega_n r)$$

which gives the temperature distribution we wanted to determine.

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EXACT SOLUTION OF THE PLANE PROBLEM OF ELASTICITY IN A RECTANGULAR REGION

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Abstract. The exact solution of the quasi-static boundary value problem of plane elasticity in a rectangular region is constructed. This solution consisting of two parts satisfies both the conditions imposed on the boundary of the rectangle and the original relationships. The first part depends on the resultant vector and moment of the tractions and the second one depends on the self-equilibrated part of the tractions. Necessary existence conditions are deduced for the solution.

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1. Introduction

In our opinion, the problem of constructing solutions to the quasi-static boundary value problems of elasticity and thermoelasticity in bounded regions with corner points is well elucidated by Grinchenko [1] and Grinchenko and Ulitko [2]. Also a method for solving such problems in a rectangular region has been proposed in [1, 2]. The method consists in superposing some solutions each of which is an exact one for the corresponding infinite regions without corner points. It is obvious that seeking solutions in such a form is complicated with regard to the requirement that the boundary conditions should be satisfied at the corner points of the region.

In general, while constructing exact solutions to elasticity or thermoelasticity problems in regions having corner points, e.g. for a rectangle, a parallelepiped etc., one encounters great difficulties without a method for separating variables in the governing equations of elasticity – the Láme equations in terms of displacements or the Beltrami equations in terms of stresses. However, Saint-Venant's principle makes it possible to represent solutions to the equations of elasticity in terms of stresses in the form of a superposition of two parts: the first one depending on the resultant vector and moment of the tractions, and the second one being the self-equilibrated part. This gives the idea that eigenfunctions, which should be found for the stress tensor components, could be helpful here. Unfortunately, separation of variables, e.g. in the

biharmonic equation for the plane elasticity problem in a rectangle, is complicated since there are two boundary conditions to be satisfied on each side of the rectangle. That is the reason why the two boundary conditions are equivalently replaced by an integral one in this paper when we integrate the equilibrium equations. This makes it possible to separate variables in the governing fourth-order integro-differential equation and to construct a complete set of functions for the appropriate component of the stress tensor, which consists of the eigenfunctions and the associated ones thus corresponding to Saint-Venant's principle, and, finally, to construct the exact solution to the problem raised.

2. **Formulation of the problem**

Let us consider the quasi-static boundary value problem of plane elasticity in terms of stresses for the rectangular region $D = \{(x, y) \in [-a, a] \times [-b, b]\}$ provided that the material is isotropic and homogeneous and there are no body forces. This problem is governed – see [3, 4] for details – by

the equilibrium equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \qquad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0, \qquad (x, y) \in D \tag{2.1}$$

and the compatibility equation:

$$\Delta(\sigma_x + \sigma_y) = 0, \qquad \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2. \tag{2.2}$$

We shall assume that the normal stresses

$$\sigma_x|_{x=a} = -p_1(y), \quad \sigma_x|_{x=-a} = -p_2(y), \quad \sigma_y|_{y=b} = -p_3(x), \quad \sigma_y|_{y=-b} = -p_4(x) \tag{2.3}$$

and the shear stresses

$$\sigma_{xy}|_{x=a} = q_1(y), \quad \sigma_{xy}|_{x=-a} = q_2(y), \quad \sigma_{xy}|_{y=b} = q_3(x), \quad \sigma_{xy}|_{y=-b} = q_4(x) \tag{2.4}$$

are imposed on the boundary of the rectangle; the prescribed values are denoted by $p_1, \ldots, p_4$ and $q_1, \ldots, q_4$.

To determine the stress tensor components from the set of equations (2.1) and (2.2) under the boundary conditions (2.3) and (2.4), it is useful to replace boundary conditions (2.4) for shear stress by the corresponding conditions for derivatives of normal stresses

$$\left.\frac{\partial \sigma_x}{\partial x}\right|_{x=a} = -\frac{dq_1}{dy}, \quad \left.\frac{\partial \sigma_x}{\partial x}\right|_{x=-a} = -\frac{dq_2}{dy}, \quad \left.\frac{\partial \sigma_y}{\partial y}\right|_{y=b} = -\frac{dq_3}{dx}, \quad \left.\frac{\partial \sigma_y}{\partial y}\right|_{y=-b} = -\frac{dq_4}{dy}. \tag{2.5}$$

Such a replacement can be carried out with the help of equilibrium equations (2.1), which should be fulfilled on the boundary.

Further on, both the boundary conditions for normal stresses (2.3) and (2.5) and those for the shear stress (2.4) will be used to construct the solution of equations (2.1)–(2.2).

By means of direct integration of the equilibrium equations (2.1) and separating the variables in the governing integro-differential equations [5, 6, 7, 8] formed separately for the stress components σ_y and σ_x on the basis of equation (2.2), we can construct complete sets of the orthogonal eigenfunctions and associated functions $\{1, x, \cos(\gamma_n x/a), \sin(\lambda_n x/a)\}$ and $\{1, y, \cos(\gamma_n y/b), \sin(\lambda_n y/b)\}$ $(n = 1, 2, ...)$ in order to decompose the normal stresses

$$\begin{aligned} \sigma_x &= X_0^1(x) + yX_0^2(x) + \sum_{n=1}^{\infty} \left(X_n^1(x) \cos \gamma_n \tfrac{y}{b} + X_n^2(x) \sin \lambda_n \tfrac{y}{b} \right) , \\ \sigma_y &= Y_0^1(y) + xY_0^2(y) + \sum_{n=1}^{\infty} \left(Y_n^1(y) \cos \gamma_n \tfrac{x}{a} + Y_n^2(y) \sin \lambda_n \tfrac{x}{a} \right) \end{aligned} \tag{2.6}$$

and the tractions

$$\begin{aligned} p_i &= a_0^i + yb_0^i + \sum_{n=1}^{\infty} \left(a_n^i \cos \gamma_n \tfrac{y}{b} + b_n^i \sin \lambda_n \tfrac{y}{b} \right) , \\ \frac{dq_i}{dy} &= c_0^i + yd_0^i + \sum_{n=1}^{\infty} \left(c_n^i \cos \gamma_n \tfrac{y}{b} + d_n^i \sin \lambda_n \tfrac{y}{b} \right) , \quad i = 1, 2, \\ p_j &= a_0^j + xb_0^j + \sum_{n=1}^{\infty} \left(a_n^j \cos \gamma_n \tfrac{x}{a} + b_n^j \sin \lambda_n \tfrac{x}{a} \right) , \\ \frac{dq_j}{dx} &= c_0^j + xd_0^j + \sum_{n=1}^{\infty} \left(c_n^j \cos \gamma_n \tfrac{x}{a} + d_n^j \sin \lambda_n \tfrac{x}{a} \right) , \quad j = 3, 4 . \end{aligned} \tag{2.7}$$

Here $\gamma_n = n\pi$, $\lambda_n > 0$ are the roots of equation $\tan \lambda = \lambda$ $(n = 1, 2, ...)$.

The associated functions $1, y$ and $1, x$ define the so-called "elementary" parts in decompositions (2.6). They depend on the parts of the tractions in (2.7) which are not self-equilibrated. The eigenfunctions $\cos(\gamma_n y/b)$, $\sin(\lambda_n y/b)$ and $\cos(\gamma_n x/a)$, $\sin(\lambda_n x/a)$ $(n = 1, 2, ...)$ satisfying the homogeneous integral equilibrium conditions

$$\int_{-b}^{b} \sigma_x^s dy = \int_{-b}^{b} y\sigma_x^s dy = 0, \qquad \int_{-a}^{a} \sigma_y^s dx = \int_{-a}^{a} x\sigma_y^s dx = 0 \tag{2.8}$$

respectively, determine in (2.7) the parts under the summation signs which correspond to self-equilibrated parts of tractions (2.7). Therefore we can present the normal stresses in the form

$$\sigma_x = \sigma_x^0 + \sigma_x^s , \qquad \sigma_y = \sigma_y^0 + \sigma_y^s . \tag{2.9}$$

Here and in the sequel the superscripts "0", "s" denote the elementary (not self-equilibrated) and self-equilibrated parts of solutions (2.6) or tractions, respectively. The latter should also be presented in the form

$$p_i = p_i^0 + p_i^s , \qquad q_i = q_i^0 + q_i^s , \qquad i = 1, 2, 3, 4. \tag{2.10}$$

In what follows, when no confusion can arise, the indices introduced will be omitted for the sake of brevity in writing.

Now we shall consider how to derive the elementary and self-equilibrated parts of the normal stresses. Also the same parts of the shear stress will be constructed by using the method of direct integration of the equilibrium equations [7].

3. Elementary solutions

The elementary parts of the solution of the problem (2.1)–(2.3), (2.5) for the normal stresses

$$\sigma_x^0 = X_0^1(x) + yX_0^2(x), \qquad \sigma_y^0 = Y_0^1(y) + xY_0^2(y) \tag{3.1}$$

should be solutions of equations (2.1), (2.2), and should satisfy (as non-self-equilibrated tractions (2.7)) both the boundary conditions (2.3), (2.5), i.e., equations

$$\begin{aligned}
\left(X_0^1 + yX_0^2\right)\Big|_{x=(-1)^{i+1}a} &= -\frac{1}{2b}\int\limits_{-b}^{b} p_i dy - \frac{3y}{2b^3}\int\limits_{-b}^{b} p_i y dy\,,\\
\left(\frac{dX_0^1}{dx} + y\frac{dX_0^2}{dx}\right)\Big|_{x=(-1)^{i+1}a} &= -\frac{1}{2b}\int\limits_{-b}^{b} \frac{dq_i}{dy} dy - \frac{3y}{2b^3}\int\limits_{-b}^{b} \frac{dq_i}{dy} y dy\,, \quad i = 1, 2,\\
\left(Y_0^1 + xY_0^2\right)\Big|_{y=(-1)^{j+1}b} &= -\frac{1}{2a}\int\limits_{-a}^{a} p_j dx - \frac{3x}{2a^3}\int\limits_{-a}^{a} p_j x dx\,,\\
\left(\frac{dY_0^1}{dy} + x\frac{dY_0^2}{dy}\right)\Big|_{y=(-1)^{j+1}b} &= -\frac{1}{2a}\int\limits_{-a}^{a} \frac{dq_j}{dx} dx - \frac{3x}{2a^3}\int\limits_{-a}^{a} \frac{dq_j}{dx} x dx, \quad j = 3, 4\,,
\end{aligned} \tag{3.2}$$

and the non-homogeneous integral equilibrium conditions [7]

$$\begin{aligned}
2\int\limits_{-a}^{a} \sigma_y dx &= -\int\limits_{-a}^{a} (p_3 + p_4) dx + \int\limits_{-b}^{b} (q_2 - q_1)\,\mathrm{sign}(y - \xi) d\xi,\\
2\int\limits_{-a}^{a} x\sigma_y dx &= -\int\limits_{-a}^{a} (p_3 + p_4) x dx + \int\limits_{-b}^{b} (p_1 - p_2)|y - \xi| d\xi +\\
&\quad + \int\limits_{-a}^{a} ((y - b)q_3 + (y + b)q_4) dx - a\int\limits_{-b}^{b} (q_1 + q_2)\mathrm{sign}(y - \xi) d\xi,\\
2\int\limits_{-b}^{b} \sigma_x dy &= -\int\limits_{-b}^{b} (p_1 + p_2) dy + \int\limits_{-a}^{a} (q_4 - q_3)\mathrm{sign}(x - \eta) d\eta,\\
2\int\limits_{-b}^{b} y\sigma_x dy &= -\int\limits_{-b}^{b} (p_1 + p_2) y dy + \int\limits_{-a}^{a} (p_3 - p_4)|x - \eta| d\eta +\\
&\quad + \int\limits_{-b}^{b} ((x - a)q_1 + (x + a)q_2) dy - b\int\limits_{-a}^{a} (q_3 + q_4)\mathrm{sign}(x - \eta) d\eta\,.
\end{aligned} \tag{3.3}$$

The latter have been obtained by integrating the equilibrium equations (2.1) and taking the boundary conditions (2.3) and (2.4) into account.

Also the constituents X_0^i, Y_0^i $(i = 1, 2)$ of the solution (3.1) should satisfy the coordination conditions

$$X_0^1 = \frac{1}{2b}\int_{-b}^{b}\sigma_x dy, \quad X_0^2 = \frac{3}{2b^3}\int_{-b}^{b} y\sigma_x dy, \quad Y_0^1 = \frac{1}{2a}\int_{-a}^{a}\sigma_y dx, \quad Y_0^2 = \frac{3}{2a^3}\int_{-a}^{a} x\sigma_y dx \tag{3.4}$$

which follow directly from equations (3.1). It is obvious that the static equilibrium conditions

$$\begin{aligned} &\int_{-b}^{b}(p_2 - p_1)dy + \int_{-a}^{a}(q_3 - q_4)dx = \int_{-a}^{a}(p_4 - p_3)dx + \int_{-b}^{b}(q_1 - q_2)dy = 0, \\ &\int_{-b}^{b}(p_2 - p_1)ydy + b\int_{-a}^{a}(q_3 + q_4)dx = \int_{-a}^{a}(p_4 - p_3)xdx + a\int_{-b}^{b}(q_1 + q_2)dy \end{aligned} \tag{3.5}$$

should also be satisfied by the tractions imposed on the boundary of the rectangle.

Since the governing equations (3.2)–(3.5) are overdetermined, one should expect that for the existence of solutions (3.1) some necessary conditions should be laid upon the tractions.

On the basis of equations (2.1), (2.2) the relations

$$\frac{\partial^2\sigma_x^0}{\partial x^2} - \frac{\partial^2\sigma_y^0}{\partial y^2} = 0 \qquad \frac{\partial^2\sigma_x^0}{\partial x^2} + \frac{\partial^2\sigma_y^0}{\partial y^2} = 0$$

follow for the solutions (3.1). Consequently

$$\frac{\partial^2\sigma_x^0}{\partial x^2} = \frac{\partial^2\sigma_y^0}{\partial y^2} = 0\,.$$

According to these equations, by means of (3.1), (3.3), (3.4) and (2.10), we arrive at the following equations for the non-self-equilibrated part of the tractions:

$$\begin{aligned} &\frac{d}{dx}(q_4^0 - q_3^0) + \frac{3y}{b^2}\left(p_3^0 - p_4^0 - b\tfrac{d}{dx}(q_3^0 + q_4^0)\right) = 0, \\ &\frac{d}{dy}(q_2^0 - q_1^0) + \frac{3x}{a^2}\left(p_1^0 - p_2^0 - a\tfrac{d}{dy}(q_1^0 + q_2^0)\right) = 0\,. \end{aligned}$$

They obviously lead to the evident conditions

$$\begin{aligned} \frac{d}{dx}(q_4^0 - q_3^0) = \frac{d}{dy}(q_2^0 - q_1^0) = p_3^0 - p_4^0 - b\frac{d}{dx}(q_3^0 + q_4^0) = \\ = p_1^0 - p_2^0 - a\frac{d}{dy}(q_1^0 + q_2^0) = 0. \end{aligned} \tag{3.6}$$

In accordance with decompositions (2.7), the necessary conditions

$$\begin{gathered} aq_{43}^{-}=\int\limits_{-a}^{a}(q_4-q_3)dx=\int\limits_{-b}^{b}(p_2-p_1)dy, \quad bq_{21}^{-}=\int\limits_{-b}^{b}(q_2-q_1)dy=\int\limits_{-a}^{a}(p_4-p_3)dx, \\ abq_{34}^{+}=b\int\limits_{-a}^{a}(q_3+q_4)dx+\int\limits_{-a}^{a}(p_3-p_4)xdx, \quad abq_{12}^{+}=a\int\limits_{-b}^{b}(q_1+q_2)dy+\int\limits_{-b}^{b}(p_1-p_2)ydy, \\ q_3(a)-q_3(-a)=q_4(a)-q_4(-a), \quad q_1(b)-q_1(-b)=q_2(b)-q_2(-b), \end{gathered} \tag{3.7}$$

to be satisfied by the tractions follow from equations (3.6). Here

$$\begin{gathered} q_{21}^{-}=(q_2-q_1)|_{y=b}+(q_2-q_1)|_{y=-b}=2(q_3(-a)-q_3(a)), \\ q_{43}^{-}=(q_4-q_3)|_{x=a}+(q_4-q_3)|_{x=-a}=2(q_1(-b)-q_1(b)), \\ q_{12}^{+}=(q_1+q_2)|_{y=b}+(q_1+q_2)|_{y=-b}, \quad q_{34}^{+}=(q_3+q_4)|_{x=a}+(q_3+q_4)|_{x=-a}. \end{gathered}$$

The equality $q_{12}^{+}=q_{34}^{+}$, which reflects the duality of the shear stress at the corner points of the rectangle, should also be fulfilled.

Consequently the elementary solutions (3.1), which satisfy equations (2.1), (2.2) and conditions (3.2)–(3.5), take the form

$$\begin{gathered} \sigma_x^0=\frac{1}{4b}\int\limits_{-b}^{b}\left((p_2-p_1)\frac{x}{a}-p_1-p_2\right)dy+\frac{3y}{4b^3}\int\limits_{-b}^{b}\left((p_2-p_1)\frac{x}{a}-p_1-p_2\right)ydy, \\ \sigma_y^0=\frac{1}{4a}\int\limits_{-a}^{a}\left((p_4-p_3)\frac{y}{b}-p_3-p_4\right)dx+\frac{3x}{4a^3}\int\limits_{-a}^{a}\left((p_4-p_3)\frac{y}{b}-p_3-p_4\right)xdx \end{gathered} \tag{3.8}$$

if the necessary conditions (3.7) are fulfilled. The last formulae show that in the presence of conditions (3.7) the elementary solutions to the normal stresses are linear in their coordinates and depend only on p_i $(i=1,2)$ and p_j $(j=3,4)$, respectively.

Now we can apply the principle of superposition. By direct integration of the equilibrium equations (2.1) and taking the boundary conditions (2.3) and (2.4) into account, one can determine the elementary part of shear stresses – for the normal stresses see (3.8) – as the sum of two constituents

$$\begin{aligned} \sigma_{xy}^0=\frac{1}{4}\left[q_{12}^{+}-q_{21}^{-}\frac{x}{a}-q_{43}^{-}\frac{y}{b}\right]+\frac{3(x^2-a^2)}{8a^3}\left[aq_{34}^{+}-\int\limits_{-a}^{a}(q_3+q_4)dx\right]+ \\ +\frac{3(y^2-b^2)}{8b^3}\left[bq_{12}^{+}-\int\limits_{-b}^{b}(q_1+q_2)dy\right]. \end{aligned} \tag{3.9}$$

These constituents depend on the tractions q_i $(i=1,2)$ and q_j $(j=3,4)$, respectively and are polynomial functions of the coordinates with a degree not higher than two.

It should be noted that $\{1,\ x,\ x^2,\ \sin(\gamma_n x/a),\ \cos\lambda_n-\cos(\lambda_n x/a)\}$ $(n=1,2,...)$ is the complete set of functions for separating variables in the problem (2.1)–(2.4) for

σ_{xy} in respect of the variable x. Similarly, $\{1,\ y,\ y^2,\ \sin(\gamma_n y/b),\ \cos\lambda_n - \cos(\lambda_n y/b)\}$ $(n = 1, 2, ...)$ is the complete set in the variable y. This means that the solution to σ_{xy} can be presented in the form

$$\sigma_{xy} = Y^1_{xo}(y) + Y^2_{xo}(y)x + Y^3_{xo}(y)x^2 + \\ + \sum_{n=1}^{\infty}[Y^1_{xn}(y)\sin\gamma_n\frac{x}{a} + Y^2_{xn}(y)(\cos\lambda_n - \cos\lambda_n\frac{x}{a})] \quad (3.10)$$

or

$$\sigma_{xy} = X^1_{yo}(x) + X^2_{yo}(x)y + X^3_{yo}(x)y^2 + \\ + \sum_{n=1}^{\infty}[X^1_{yn}(x)\sin\gamma_n\frac{y}{b} + X^2_{yn}(x)(\cos\lambda_n - \cos\lambda_n\frac{y}{b})]\,,$$

where the functions $\sin(\gamma_n x/a)$, $\cos\lambda_n - \cos(\lambda_n x/a)$ and $\sin(\gamma_n y/b)$, $\cos\lambda_n - \cos(\lambda_n y/b)$ $(n = 1, 2, ...)$ are the eigenfunctions satisfying the homogeneous conditions

$$\sigma^s_{xy}(a) = \sigma^s_{xy}(-a) = \int_{-a}^{a}\sigma^s_{xy}dx = 0, \quad \sigma^s_{xy}(b) = \sigma^s_{xy}(-b) = \int_{-b}^{b}\sigma^s_{xy}dy = 0,$$

respectively.

It can be checked with ease that the necessary conditions (3.7) are also sufficient for the boundary conditions (3.2) to be satisfied by the stresses (3.8).

The integral equilibrium conditions (3.7) set up for the tractions are special if they are compared with the general equilibrium conditions (3.5) since the latter are satisfied identically if conditions (3.7) are fulfilled. However, conditions (3.7) ensure the fulfillment of the principle of superposition for problem (2.1)–(2.5) in the sense that the solution can be considered as a sum of two other solutions – one for $p_i, q_i \neq 0$ $(i = 1, 2)$ the other for $p_j = q_j = 0$ $(j = 3, 4)$, and vice versa: $p_i = q_i = 0$ and $p_j, q_j \neq 0$.

The necessary equilibrium conditions (3.7) mean that in the absence of shear stresses at the corner points of the rectangle, i.e., when $q^-_{21} = q^-_{43} = q^+_{12} = q^+_{34} = 0$, for the existence of the solution of problem (2.1)–(2.5) in the form (2.6) satisfying the superposition and Saint-Venant's principles, the tractions imposed on the boundary of the region should be equilibrated separately for p_i, q_i $(i = 1, 2)$ and for p_j, q_j $(j = 3, 4)$. This is so because within the general equilibrium conditions (3.5) only, superposition of the solution would be impossible.

It should be finally emphasized that within the framework of the theory of elasticity, i.e., when $\sigma_{xy} = Ge_{xy}$ (G is the shear modulus) due to the duality of the shear stresses at the corner points of the region D, fulfillment of conditions $(3.7)_{5,6}$ is necessary for the total shear strains to be equal to zero at the corner points of the rectangle.

4. Solutions to self-equilibrated tractions

The self-equilibrated normal stresses (2.6)

$$\begin{aligned} \sigma_x^s &= \sum_{n=1}^{\infty} \left(X_n^1(x) \cos \gamma_n \tfrac{y}{b} + X_n^2(x) \sin \lambda_n \tfrac{y}{b} \right) , \\ \sigma_y^s &= \sum_{n=1}^{\infty} \left(Y_n^1(y) \cos \gamma_n \tfrac{x}{a} + Y_n^2(y) \sin \lambda_n \tfrac{x}{a} \right) \end{aligned} \tag{4.1}$$

should satisfy equations (2.1), (2.5) and the homogeneous integral equilibrium conditions (2.8). We shall construct both the normal stresses and the self-equilibrated shear stresses (3.10), the latter being presented in the form of the sum of two solutions

$$\begin{aligned} \sigma_{xy}^s = \sum_{n=1}^{\infty} \left[Y_{xn}^1(y) \sin \gamma_n \frac{x}{a} + Y_{xn}^2(y)(\cos \lambda_n - \cos \lambda_n \frac{x}{a}) \right] + \\ + \sum_{n=1}^{\infty} \left[X_{yn}^1(x) \sin \gamma_n \frac{y}{b} + X_{yn}^2(x)(\cos \lambda_n - \cos \lambda_n \frac{y}{b}) \right] , \end{aligned} \tag{4.2}$$

if we make use of the method proposed in [8] for constructing similar functions for the plane boundary value problem of thermoelasticity in a rectangle. Therefore

$$\sigma_x^s = \sum_{i=1}^{\infty} \sigma_x^{(i)}, \quad \sigma_y^s = \sum_{i=1}^{\infty} \sigma_y^{(i)}, \quad \sigma_{xy}^s = \sum_{i=1}^{\infty} \sigma_{xy}^{(i)}, \tag{4.3}$$

where

$$\begin{aligned} \sigma_x^{(2i-1)} = \frac{1}{2} \frac{\partial^2}{\partial y^2} \int_{-a}^{a} \sigma_y^{(2i-1)} |x-\eta| d\eta, \quad \sigma_{xy}^{(2i-1)} = -\frac{1}{2} \frac{\partial}{\partial y} \int_{-a}^{a} \sigma_y^{(2i-1)} \operatorname{sign}(x-\eta) d\eta, \\ \sigma_y^{(2i)} = \frac{1}{2} \frac{\partial^2}{\partial x^2} \int_{-b}^{b} \sigma_x^{(2i)} |y-\xi| d\xi, \quad \sigma_{xy}^{(2i)} = -\frac{1}{2} \frac{\partial}{\partial x} \int_{-b}^{b} \sigma_x^{(2i)} \operatorname{sign}(y-\xi) d\xi. \end{aligned} \tag{4.4}$$

Here and further on $i = 1, 2, ...$

The constituents of normal stresses

$$\begin{aligned} \sigma_x^{(2i)} &= \sum_{m=1}^{\infty} \left(X_{1m}^{(2i)}(x) \cos \gamma_m \tfrac{y}{b} + X_{2m}^{(2i)}(x) \sin \lambda_m \tfrac{y}{b} \right) , \\ \sigma_y^{(2i-1)} &= \sum_{n=1}^{\infty} \left(Y_{1n}^{(2i-1)}(y) \cos \gamma_n \tfrac{x}{a} + Y_{2n}^{(2i-1)}(y) \sin \lambda_n \tfrac{x}{a} \right) \end{aligned} \tag{4.5}$$

will be sought as the solutions of the integro-differential equations

$$\begin{aligned} \frac{\partial^2 \sigma_y^{(2i-1)}}{\partial x^2} + 2 \frac{\partial^2 \sigma_y^{(2i-1)}}{\partial y^2} + \frac{1}{2} \frac{\partial^4}{\partial y^4} \int_{-a}^{a} \sigma_y^{(2i-1)} |x-\eta| d\eta = \frac{\partial^2 l^{(2i-1)}}{\partial y^2} - \frac{\partial^2 l^{(2i-2)}}{\partial x^2} \\ \frac{\partial^2 \sigma_x^{(2i)}}{\partial y^2} + 2 \frac{\partial^2 \sigma_x^{(2i)}}{\partial x^2} + \frac{1}{2} \frac{\partial^4}{\partial x^4} \int_{-b}^{b} \sigma_x^{(2i)} |y-\xi| d\xi = \frac{\partial^2 l^{(2i)}}{\partial x^2} - \frac{\partial^2 l^{(2i-1)}}{\partial y^2}, \end{aligned} \tag{4.6}$$

when the boundary conditions

$$\begin{gathered}\sigma_x^{(2i)}\Big|_{x=\pm a}=\begin{cases}-p_1^s\delta_{i,1},\\-p_2^s\delta_{i,1},\end{cases}\qquad \sigma_y^{(2i-1)}\Big|_{y=\pm b}=\begin{cases}-p_3^s\delta_{i,1},\\-p_4^s\delta_{i,1},\end{cases}\\ \frac{\partial\sigma_x^{(2i)}}{\partial x}\Big|_{x=\pm a}=\begin{cases}-dq_1^s/dy\delta_{i,1},\\-dq_2^s/dy\delta_{i,1},\end{cases}\qquad \frac{\partial\sigma_y^{(2i-1)}}{\partial y}\Big|_{y=\pm b}=\begin{cases}-dq_3^s/dx\delta_{i,1},\\-dq_4^s/dx\delta_{i,1},\end{cases}\end{gathered}\tag{4.7}$$

which follow from (2.3) and (2.5) due to equations (2.7), are satisfied. Here $\delta_{i,j}$ is the Kronecker delta; $l^{(0)}\equiv 0$;

$$\begin{gathered}l^{(2i-1)}=a\sum_{n=1}^{\infty}\left[\frac{d^2Y_{1n}^{(2i-1)}}{dy^2}\frac{a\cos\gamma_n}{\gamma_n^2}+x\frac{d^2Y_{2n}^{(2i-1)}}{dy^2}\frac{\cos\lambda_n}{\lambda_n}\right]=\\ =\sum_{m=1}^{\infty}\left\{(\tilde\alpha_{1m}^{(2i-1)}+x\tilde\beta_{1m}^{(2i-1)})\cos\gamma_m\frac{y}{b}+(\tilde\alpha_{2m}^{(2i-1)}+x\tilde\beta_{2m}^{(2i-1)})\sin\lambda_m\frac{y}{b}\right\};\\ l^{(2i)}=b\sum_{m=1}^{\infty}\left[\frac{d^2X_{1m}^{(2i)}}{dx^2}\frac{b\cos\gamma_m}{\gamma_m^2}++y\frac{d^2X_{2m}^{(2i)}}{dx^2}\frac{\cos\lambda_m}{\lambda_m}\right]=\\ =\sum_{n=1}^{\infty}\left\{\alpha_{1n}^{(2i)}+y\beta_{1n}^{(2i)}\cos\gamma_n\frac{x}{a}+(\alpha_{2n}^{(2i)}+y\beta_{2n}^{(2i)})\sin\lambda_n\frac{x}{a}\right\}.\end{gathered}\tag{4.8}$$

Thus, the possibility of separating variables in equations (4.6) has been established for representations (4.5) taking expressions (4.8) into account. For the functions in question, $Y_{jn}^{(2i-1)}$, $X_{jm}^{(2i)}$ $(j=1,2)$, we arrive at the problem of solving the ordinary fourth-order differential equations

$$\begin{gathered}\left[\frac{d^4}{dy^4}-2\left(\frac{z_{jn}}{a}\right)^2\frac{d^2}{dy^2}+\left(\frac{z_{jn}}{a}\right)^4\right]Y_{jn}^{(2i-1)}=-\left(\frac{z_{jn}}{a}\right)^4\left(\alpha_{jn}^{(2i-2)}+y\beta_{jn}^{(2i-2)}\right),\\ \left[\frac{d^4}{dx^4}-2\left(\frac{z_{jm}}{b}\right)^2\frac{d^2}{dx^2}+\left(\frac{z_{jm}}{b}\right)^4\right]X_{jm}^{(2i)}=-\left(\frac{z_{jm}}{b}\right)^4\left(\tilde\alpha_{jm}^{(2i-1)}+x\tilde\beta_{jm}^{(2i-1)}\right)\end{gathered}\tag{4.9}$$

with boundary conditions

$$\begin{gathered}X_{jm}^{(2i)}\Big|_{x=\pm a}=\begin{cases}P_{jm}^{1(i)},\\P_{jm}^{2(i)},\end{cases}\qquad Y_{jn}^{(2i-1)}\Big|_{y=\pm b}=\begin{cases}S_{jn}^{1(i)},\\S_{jn}^{2(i)},\end{cases}\\ \frac{dX_{jm}^{(2i)}}{dx}\Big|_{x=\pm a}=\begin{cases}P_{jm}^{3(i)},\\P_{jm}^{4(i)},\end{cases}\qquad \frac{dY_{jn}^{(2i-1)}}{dy}\Big|_{y=\pm b}=\begin{cases}S_{jn}^{3(i)},\\S_{jn}^{4(i)},\end{cases}\end{gathered}\tag{4.10}$$

where

$$P_{1m}^{j(i)}=-\frac{\delta_{i,1}}{b}\int_{-b}^{b}p_j\cos\gamma_m\frac{y}{b}dy,\quad P_{2m}^{j(i)}=-\frac{\delta_{i,1}}{b\sin^2\lambda_m}\int_{-b}^{b}p_j\sin\lambda_m\frac{y}{b}dy,$$

$$S_{1n}^{j(i)}=-\frac{\delta_{i,1}}{a}\int_{-a}^{a}p_{j+2}\cos\gamma_n\frac{x}{a}dx,\quad S_{2n}^{j(i)}=-\frac{\delta_{i,1}}{a\sin^2\lambda_n}\int_{-a}^{a}p_{j+2}\sin\lambda_n\frac{x}{a}dx,$$

$$P_{1m}^{j+2(i)} = -\frac{\delta_{i,1}}{b}\int_{-b}^{b}\frac{dq_j}{dy}\cos\gamma_m\frac{y}{b}dy\,, \quad P_{2m}^{j+2(i)} = -\frac{\delta_{i,1}}{b\sin^2\lambda_m}\int_{-b}^{b}\frac{dq_j}{dy}\sin\lambda_m\frac{y}{b}dy\,,$$
$$S_{1n}^{j+2(i)} = \frac{\delta_{i,1}}{a}\int_{-a}^{a}\frac{dq_{j+2}}{dx}\cos\gamma_n\frac{x}{a}dx\,, \quad S_{2n}^{j+2(i)} = -\frac{\delta_{i,1}}{a\sin^2\lambda_n}\int_{-a}^{a}\frac{dq_{j+2}}{dx}\sin\lambda_n\frac{x}{a}dx,$$
$$z_{jk} = (2-j)\gamma_k + (j-1)\lambda_k, \quad j = 1,2; \quad k = \{n,m\}.$$

The solution of problem (4.9), (4.10) can be given in the following form

$$\begin{aligned} X_{jm}^{(2i)} &= \sum_{k=1}^{4}\Delta_k(\tfrac{z_{jm}}{b},a,x)P_{jm}^{k(i)} + \tilde{\alpha}_{jm}^{(2i-1)}f_1(\tfrac{z_{jm}}{b},a,x) + \tilde{\beta}_{jm}^{(2i-1)}f_2(\tfrac{z_{jm}}{b},a,x),\\ Y_{jn}^{(2i-1)} &= \sum_{k=1}^{4}\Delta_k(\tfrac{z_{jn}}{a},b,y)S_{jn}^{k(i)} + \alpha_{jn}^{(2i-2)}f_1(\tfrac{z_{jn}}{a},b,y) + \beta_{jn}^{(2i-2)}f_2(\tfrac{z_{jn}}{a},b,y), \end{aligned} \tag{4.11}$$

where

$$\tilde{\alpha}_{1m}^{(2i-1)} = \frac{(-1)^m}{b}\sum_{n=1}^{\infty}K(\tfrac{\gamma_n}{a},\tfrac{\gamma_m}{b},a)\Big[K_1(\tfrac{\gamma_n}{a},\tfrac{\gamma_m}{b},b)(S_{1n}^{3(i)} - S_{1n}^{4(i)}) + K_2(\tfrac{\gamma_n}{a},\tfrac{\gamma_m}{b},b)(S_{1n}^{1(i)} + S_{1n}^{2(i)} + 2\alpha_{1n}^{(2i-2)})\Big]\,,$$

$$\tilde{\beta}_{1m}^{(2i-1)} = \frac{(-1)^m}{ab}\sum_{n=1}^{\infty}\lambda_n K(\tfrac{\lambda_n}{a},\tfrac{\gamma_m}{b},a)\Big[K_1(\tfrac{\lambda_n}{a},\tfrac{\gamma_m}{b},b)(S_{2n}^{3(i)} - S_{2n}^{4(i)}) + K_2(\tfrac{\lambda_n}{a},\tfrac{\gamma_m}{b},b)(S_{2n}^{1(i)} + S_{2n}^{2(i)} + 2\alpha_{2n}^{(2i-2)})\Big]\,,$$

$$\tilde{\alpha}_{2m}^{(2i-1)} = \frac{1}{b^2\sin\lambda_m}\sum_{n=1}^{\infty}K(\tfrac{\gamma_n}{a},\tfrac{\lambda_m}{b},a)\Big[K_3(\tfrac{\gamma_n}{a},\tfrac{\lambda_m}{b},b)(S_{1n}^{1(i)} - S_{1n}^{2(i)}) + K_4(\tfrac{\gamma_n}{a},\tfrac{\lambda_m}{b},b)(S_{1n}^{3(i)} + S_{1n}^{4(i)}) + K_5(\tfrac{\gamma_n}{a},\tfrac{\lambda_m}{b},b)\beta_{1n}^{(2i-2)})\Big]\,,$$

$$\tilde{\beta}_{2m}^{(2i-1)} = \frac{1}{b^2a\sin\lambda_m}\sum_{n=1}^{\infty}\lambda_n K(\tfrac{\lambda_n}{a},\tfrac{\lambda_m}{b},a)\Big[K_3(\tfrac{\lambda_n}{a},\tfrac{\lambda_m}{b},b)(S_{2n}^{1(i)} - S_{2n}^{2(i)}) + K_4(\tfrac{\lambda_n}{a},\tfrac{\lambda_m}{b},b)(S_{2n}^{3(i)} + S_{2n}^{4(i)}) + K_5(\tfrac{\lambda_n}{a},\tfrac{\lambda_m}{b},b)\beta_{2n}^{(2i-2)})\Big]\,,$$

$$\alpha_{1n}^{(2i)} = \frac{(-1)^n}{a}\sum_{m=1}^{\infty}K(\tfrac{\gamma_m}{b},\tfrac{\gamma_n}{a},b)\Big[K_1(\tfrac{\gamma_m}{b},\tfrac{\gamma_n}{a},a)(P_{1m}^{3(i)} - P_{1m}^{4(i)}) + K_2(\tfrac{\gamma_m}{b},\tfrac{\gamma_n}{a},a)(P_{1m}^{1(i)} + P_{1m}^{2(i)} + 2\tilde{\alpha}_{1m}^{(2i-1)})\Big]\,,$$

$$\beta_{1n}^{(2i)} = \frac{(-1)^n}{ab}\sum_{m=1}^{\infty}\lambda_m K(\tfrac{\lambda_m}{b},\tfrac{\gamma_n}{a},b)\Big[K_1(\tfrac{\lambda_m}{b},\tfrac{\gamma_n}{a},a)(P_{2m}^{3(i)} - P_{2m}^{4(i)}) + K_2(\tfrac{\lambda_m}{b},\tfrac{\gamma_n}{a},a)(P_{2m}^{1(i)} + P_{2m}^{2(i)} + 2\tilde{\alpha}_{2m}^{(2i-1)})\Big]\,,$$

$$\alpha_{2n}^{(2i)} = \frac{1}{a^2 \sin\lambda_n} \sum_{m=1}^{\infty} K(\frac{\gamma_m}{b}, \frac{\lambda_n}{a}, b) \Big[K_3(\frac{\gamma_m}{b}, \frac{\lambda_n}{a}, a)(P_{1m}^{1(i)} - P_{1m}^{2(i)}) + \\ + K_4(\frac{\gamma_m}{b}, \frac{\lambda_n}{a}, a)(P_{1m}^{3(i)} + P_{1m}^{4(i)}) + K_5(\frac{\gamma_m}{b}, \frac{\lambda_n}{a}, a)\tilde{\beta}_{1m}^{(2i-1)}) \Big] ,$$

$$\beta_{2n}^{(2i)} = \frac{1}{a^2 b \sin\lambda_n} \sum_{m=1}^{\infty} \lambda_m K(\frac{\lambda_m}{b}, \frac{\lambda_n}{a}, b) \Big[K_3(\frac{\lambda_m}{b}, \frac{\lambda_n}{a}, a)(P_{2m}^{1(i)} - P_{2m}^{2(i)}) + \\ + K_4(\frac{\lambda_m}{b}, \frac{\lambda_n}{a}, a)(P_{2m}^{3(i)} + P_{2m}^{4(i)}) + K_5(\frac{\lambda_m}{b}, \frac{\lambda_n}{a}, a)\tilde{\beta}_{2m}^{(2i-1)}) \Big] ;$$

$$f_1(p,h,t) = 2\frac{(\sinh(ph) + ph\cosh(ph))\cosh(pt) - pt\sinh(ph)\sinh(pt)}{\sinh(2ph) + 2ph} - 1 ,$$

$$f_2(p,h,t) = 2\frac{h^2 p \sinh(ph)\sinh(pt) + (\sinh(ph) - ph\cosh(ph))t\cosh(pt)}{\sinh(2ph) - 2ph} - t ;$$

$$\Delta_j(p,h,t) = [(\sinh(ph) + ph\cosh(ph))\cosh(pt) - pt\sinh(ph)\sinh(pt)] \,/\, [\sinh(2ph) + 2ph] - \\ - (-1)^j \,[(\cosh(ph) + ph\sinh(ph))\sinh(pt) - pt\cosh(ph)\cosh(pt)] \,/\, [\sinh(2ph) - 2ph] ,$$

$$\Delta_{j+2}(p,h,t) = [t\sinh(ph)\cosh(pt) - h\cosh(ph)\sinh(pt)] \,/\, [\sinh(2ph) - 2ph] - \\ - (-1)^j \,[t\cosh(ph)\sinh(pt) - h\sinh(ph)\cosh(pt)] \,/\, [\sinh(2ph) + 2ph] , \quad j = 1, 2;$$

$$K(p,q,h) = \frac{\cos(ph)}{(p^2+q^2)^2}, \qquad K_1(p,q,h) = \frac{(p^2+3q^2)\sinh(2ph) + 2ph(p^2+q^2)}{\sinh(2ph) + 2ph},$$

$$K_2(p,q,h) = -\frac{4pq^2\sinh^2(ph)}{\sinh(2ph) + 2ph}, \quad K_5(p,q,h) = -\frac{8q^2}{p}\frac{(\sinh(ph) - ph\cosh(ph))^2}{\sinh(2ph) - 2ph},$$

$$K_3(p,q,h) = \frac{(q^2-p^2)\sinh(2ph) + 2(p^2+q^2)ph - 4pq^2h\cosh^2(ph)}{\sinh(2ph) - 2ph},$$

$$K_4(p,q,h) = \frac{h(p^2+3q^2)\sinh(2ph) - 2(p^2+q^2)ph^2 - 4q^2\sinh^2(ph)/p}{\sinh(2ph) - 2ph} .$$

Having determined the constituents (4.11) of the decompositions (4.5), in accordance with formulae (4.3), (4.4) we get the self-equilibrated parts of the solution (4.1), (4.2):

$$\begin{aligned}
\sigma_x^s &= \sum_{m=1}^{\infty} \big(X_{1m}(x)\cos\gamma_m\tfrac{y}{b} + X_{2m}(x)\sin\lambda_m\tfrac{y}{b}\big) + \\
&+ a^2 \sum_{n=1}^{\infty} \Big[\tfrac{d^2Y_{1n}}{dy^2}\tfrac{\cos\gamma_n - \cos(\gamma_n x/a)}{\gamma_n^2} + \tfrac{d^2Y_{2n}}{dy^2}\tfrac{\lambda_n(x/a)\cos\lambda_n - \sin(\lambda_n x/a)}{\lambda_n^2} \Big], \\
\sigma_y^s &= \sum_{n=1}^{\infty} \big(Y_{1n}(y)\cos\gamma_n\tfrac{x}{a} + Y_{2n}(y)\sin\lambda_n\tfrac{x}{a}\big) + \\
&+ b^2 \sum_{m=1}^{\infty} \Big[\tfrac{d^2X_{1m}}{dx^2}\tfrac{\cos\gamma_m - \cos(\gamma_m y/b)}{\gamma_m^2} + \tfrac{d^2X_{2m}}{dx^2}\tfrac{\lambda_m(y/b)\cos\lambda_m - \sin(\lambda_m y/b)}{\lambda_m^2} \Big], \\
\sigma_{xy}^s &= -a \sum_{n=1}^{\infty} \Big[\tfrac{dY_{1n}}{dy}\tfrac{\sin(\gamma_n x/a)}{\gamma_n} + \tfrac{dY_{2n}}{dy}\tfrac{\cos\lambda_n - \cos(\lambda_n x/a)}{\lambda_n} \Big] - \\
&- b \sum_{m=1}^{\infty} \Big[\tfrac{dX_{1m}}{dx}\tfrac{\sin(\gamma_m y/b)}{\gamma_m} + \tfrac{dX_{2m}}{dx}\tfrac{\cos\lambda_m - \cos(\lambda_m y/b)}{\lambda_m} \Big].
\end{aligned} \tag{4.12}$$

Here $Y_{jn} = \sum\limits_{i=1}^{\infty} Y_{jn}^{(2i-1)}$, $X_{jm} = \sum\limits_{i=1}^{\infty} X_{jm}^{(2i)}$ $(j = 1, 2)$.

Equivalence of representations (4.1), (4.2) to representations (4.12) is rigorously ensured by the completeness of the corresponding sets of functions.

Finally solution to the problem (2.1)–(2.5) is written for normal stresses in the form (2.9), where the elementary and self-equilibrated parts are given by formulae (3.8), $(4.12)_{1,2}$. The solution to the shear stress is given by formulae (3.9), (3.10), and $(4.12)_3$.

The solutions constructed correspond to Saint-Venant's principle. Moreover, they correspond to it strictly, as it was Saint-Venant who pointed out the existence of the associated functions 1, y and 1, x for normal stresses σ_x, σ_y, which correspond to the tension and bending of a body with a rectangular cross-section. The eigenfunctions, which determine the parts of the solutions which depend on self-equilibrated tractions imposed on the boundary, have essential influence only not far from the ends $x = \pm a$ and $y = \pm b$ of the rectangular region.

5. **Numerical results**

Let us consider some computational results providing a plane stress state if the rectangle is subjected to the loads

$$p_1 = p_2 = p(b^2 - y^2), \qquad p_j = q_k = 0, \quad j = 3, 4; \quad k = 1, 2, 3, 4; \qquad p = \text{const}.$$

Figure 1 shows how the normal stress σ_x divided by the traction p depends on the x-coordinate for the regions having constant width $2b = 2$ while the lengths are

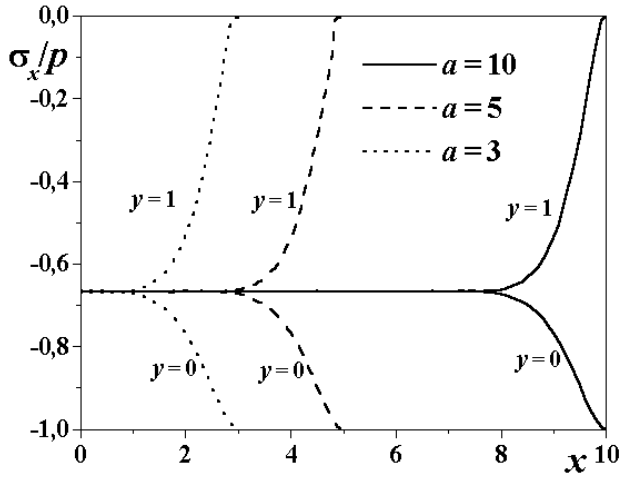

Figure 1. x–distribution of the stress σ_x/p for $a = 10; 5; 3$ and $b = 1$

$2a = 6, 10, 20$. It is clear from the graph that the stresses satisfy the boundary conditions even at the approximate distance $(2 - 2,5)b$ from the boundary, and tend to

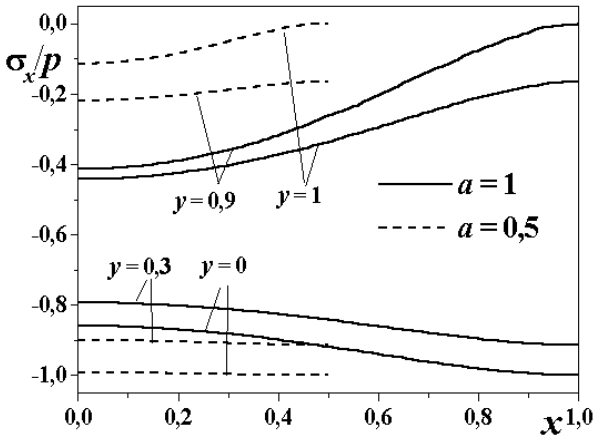

Figure 2. Influence of the self-equilibrated tractions on distribution of the stress σ_x/p under x–shortening of the rectangle

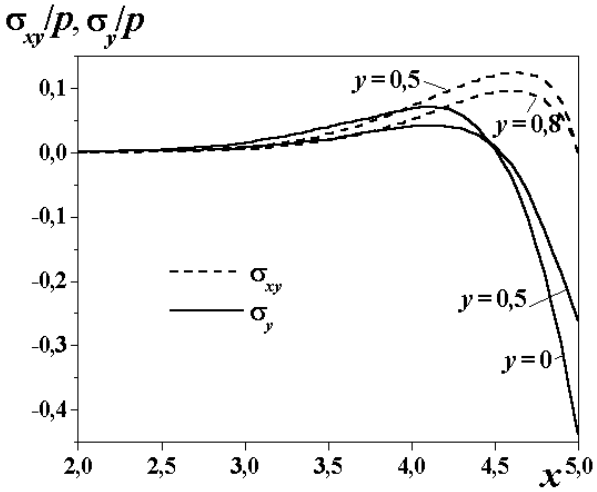

Figure 3. Behavior of the stresses σ_y/p and σ_{xy}/p for $a = 5$ and $b = 1$

a constant compressive stress along the region's width, which is equal to $\sigma_x^0/p = -2/3$ according to the elementary solution.

Figure 2 presents the behavior of σ_x/p both in a square ($a = b = 1$) and in a rectangle with sides $a = 0.5, b = 1$ and for four different cross-sections $y = 0; 0.3; 0.9; 1$. So, if the regions have a short x-dimension compared to the width, the area of the disturbed stress caused by the boundary conditions captures the whole region. If

$a = 0.5$ and y tends to zero, the normal stress has a constant distribution in accordance with the boundary conditions, i.e., it is independent of x.

The x-distribution of the stresses depicted in Figure 3 shows that the stresses are self-equilibrated. Self-equilibration of shear stress is caused by the fact that σ_{xy} is an odd function of x and y.

6. Conclusions

By making use of the proposed method of direct integration, the exact solution of the boundary value problem of plane elasticity in a rectangle is constructed. This solution is the sum of the self-equilibrated and elementary parts. The last ones (3.8) determined by the associated functions 1, y and 1, x in the equations for the normal stresses σ_x, σ_y, respectively, correspond to the tension and bending. Naturally, they depend on non self-equilibrated tractions imposed on the boundary. The self-equilibrated parts (4.12) determined by the eigenfunctions, expressing the stresses caused by the self-equilibrated tractions, have an essential influence only not far from the boundary, tending to zero when moving away from it. Therefore, the decompositions (2.6) correspond to Saint-Venant's principle.

The elementary solutions (3.8), (3.9) of the problem (2.1)–(2.4) are either linear functions of the coordinates for normal stresses or parabolic ones for shear stresses.

For the solutions to exist, the non self-equilibrated parts of the tractions should satisfy the equilibrium conditions (3.7).

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Notes for Contributors

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Aims and scope. The aim of the journal is to publish research papers on theoretical and applied mechanics. Special emphasis is given to articles on computational mechanics, continuum mechanics (mechanics of solid bodies, fluid mechanics, heat and mass transfer) and dynamics. Review papers on a research field and materials effective for teaching can also be accepted and are published as review papers or classroom notes. Papers devoted to mathematical problems relevant to mechanics will also be considered.

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A Short History of the Publications of the University of Miskolc

The University of Miskolc (Hungary) is an important center of research in Central Europe. Its parent university was founded by the Empress Maria Teresia in Selmecbánya (today Banska Štiavnica, Slovakia) in 1735. After the first World War the legal predecessor of the University of Miskolc moved to Sopron (Hungary) where, in 1929, it started the series of university publications with the title *Publications of the Mining and Metallurgical Division of the Hungarian Academy of Mining and Forestry Engineering* (Volumes I.-VI.). From 1934 to 1947 the Institution had the name Faculty of Mining, Metallurgical and Forestry Engineering of the József Nádor University of Technology and Economic Sciences at Sopron. Accordingly, the publications were given the title *Publications of the Mining and Metallurgical Engineering Division* (Volumes VII.-XVI.). For the last volume before 1950 – due to a further change in the name of the Institution – *Technical University, Faculties of Mining, Metallurgical and Forestry Engineering, Publications of the Mining and Metallurgical Divisions* was the title.

For some years after 1950 the Publications were temporarily suspended.

After the foundation of the Mechanical Engineering Faculty in Miskolc in 1949 and the movement of the Sopron Mining and Metallurgical Faculties to Miskolc, the Publications restarted with the general title *Publications of the Technical University of Heavy Industry* in 1955. Four new series - Series A (Mining), Series B (Metallurgy), Series C (Machinery) and Series D (Natural Sciences) - were founded in 1976. These came out both in foreign languages (English, German and Russian) and in Hungarian.

In 1990, right after the foundation of some new faculties, the university was renamed to University of Miskolc. At the same time the structure of the Publications was reorganized so that it could follow the faculty structure. Accordingly three new series were established: Series E (Legal Sciences), Series F (Economic Sciences) and Series G (Humanities and Social Sciences). The latest series, i.e., the series H (European Integration Studies) was founded in 2001. The eight series are formed by some periodicals and such publications which come out with various frequencies.

Papers on computational and applied mechanics were published in the

Publications of the University of Miskolc, Series D, Natural Sciences.

This series was given the name Natural Sciences, Mathematics in 1995. The name change reflects the fact that most of the papers published in the journal are of mathematical nature though papers on mechanics also come out.

The series

Publications of the University of Miskolc, Series C, Fundamental Engineering Sciences

founded in 1995 also published papers on mechanical issues. The present journal, which is published with the support of the Faculty of Mechanical Engineering as a member of the Series C (Machinery), is the legal successor of the above journal.

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