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CONTROLLABILITY AND HYERS-ULAM STABILITY RESULTS OF INITIAL VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS VIA GENERALIZED PROPORTIONAL-CAPUTO FRACTIONAL DERIVATIVE

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Abstract. This paper concerns the investigation of controllability and Hyers-Ulam stability of initial value problems for fractional differential equations via generalized proportional-Caputo fractional derivatives. The main results are obtained by means of the Krasnoselskii's fixed point theorem.

2010 *Mathematics Subject Classification:* 26A33; 93B05; 34K35

Keywords: Hyers-Ulam stability, controllability, proportional-Caputo derivatives, Krasnoselskii's fixed point theorem

1. INTRODUCTION

Fractional differential equations have been studied extensively in the literature because of their applications in various fields of engineering and science; see the books [16,18,22]. In the few past years, the variety of definitions of fractional operators has become visible to those interested in fractional calculus.Here, we focus on the most ranking kinds including Liouville, Caputo, Hadamard, Atangana-Baleanu, Caputo-Fabrizio derivatives and etc. For instance, see the books [15, 23] and the papers [3, 8, 19–21] and the references quoted therein.

Recently, Khalil et al. [14] introduce a new definition of fractional derivative, called the conformable fractional derivative, with an obstacle that it does not tend to the original function as the order α tends to zero. The new definition has attracted good efforts of many researchers to establish some useful results; see, for example, [5–7, 17, 26].

In control theory, a proportional derivative controller for controller output u at time t with two tuning parameters has the algorithm

$$u(t) = \kappa_P \mathcal{E}(t) + \kappa_d \frac{d}{dt}\mathcal{E}(t),$$

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where κ_P and κ_d are the proportional control parameter and the derivative control parameter, respectively. The function $\mathcal{E}$ is the error between the state variable and the process variable. This control law enables Dawei et al. [11] to present the control of complex networks models.

Inspired by the above concept of the proportional derivative controller, Anderson et al. [2] were able to define the proportional (conformable) derivative of order α by

$$ {}_0^P D_t^\alpha g(t) = k_1(\alpha,t)g(t) + k_0(\alpha,t)g'(t), $$

where g is differentiable function and $k_0, k_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ are continuous functions of the variable t and the parameter $\alpha \in [0,1]$ which satisfy the following conditions for all $t \in \mathbb{R}$:

$$ \lim_{\alpha\to 0^+} k_0(\alpha,t) = 0, \quad \lim_{\alpha\to 1^-} k_0(\alpha,t) = 1, \quad k_0(\alpha,t) \neq 0, \ \alpha \in (0,1], \tag{1.1} $$

$$ \lim_{\alpha\to 0^+} k_1(\alpha,t) = 1, \quad \lim_{\alpha\to 1^-} k_1(\alpha,t) = 0, \quad k_1(\alpha,t) \neq 0, \ \alpha \in [0,1). \tag{1.2} $$

This newly defined local derivative tends to the original function as the order α tends to zero and hence improved the conformable derivatives. In [13], Jarad et al. discussed a special case of the proportional derivatives when $k_1(\alpha,t) = 1-\alpha$ and $k_0(\alpha,t) = \alpha$.

In [4], Baleanu et al. proposed a new more general proportional fractional derivative as a linear combination of a Riemann-Liouville integral and a Caputo derivative, also they obtained some amazing results relevant to the newly hybrid fractional operator such as the Laplace transform and its inversion. Further, they solved some differential equations involved that new hybrid derivative and got the solution in terms of a new bivariate Mittag-Leffler function.

Inspired by the new results in [4], we investigate initial value problems for fractional differential equations via generalized proportional-Caputo fractional derivatives. Precisely, we consider the following IVP:

$$ \begin{cases} {}_0^{PC} D_t^\alpha x(t) = f(t,x(t)) + Bu(t), \ \ t \in J = [0,b], \ b < \infty, \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{1.3} $$

where ${}_0^{PC} D_t^\alpha$ denotes the proportional-Caputo fractional derivative of order $\alpha \in (0,1]$, the function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous, the control function $u(\cdot)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space, and B is a bounded linear operator from U to $\mathbb{R}$.

Controllability is one of the fundamental notions of modern control theory, which enables one to steer the control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls where initial and final state may vary over the entire space. The problem of controllability of nonlinear systems represented by fractional differential equations has been extensively studied by several authors; see, for example, [1, 9, 10, 25] and the references therein.

To our knowledge there are no similar contributions to the controllability and Hyers-Ulam stability of fractional differential equations via generalized proportional-Caputo fractional derivatives.

2. PRELIMINARIES

In this section we collect some definitions, properties and propositions of the new generalized proportional-Caputo hybrid fractional derivative.

Definition 1 ([4])**.** The proportional-Caputo hybrid fractional derivative of order $\alpha \in (0,1)$ of a differentiable function $g(t)$ is given by

$$ {}_0^{PC}D_t^{\alpha} g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \big(k_1(\alpha,\tau)g(t) + k_0(\alpha,\tau)g'(t)\big)(t-\tau)^{-\alpha}\, d\tau, \tag{2.1} $$

where the function space domain is given by requiring that g is differentiable and both g and g' are locally L^1 functions on the positive reals.

Definition 2 ([4])**.** The inverse operator of the proportional-Caputo hybrid fractional derivative of order $\alpha \in (0,1)$ is given by

$$ {}_0^{PC}I_t^{\alpha} g(t) = \int_0^t \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds\right) \frac{{}_0^{RL}D_u^{1-\alpha} g(u)}{k_0(\alpha,u)}\, du, \tag{2.2} $$

where ${}_0^{RL}D_u^{1-\alpha}$ denotes the Riemann-Liouville fractional derivative of order $1-\alpha$ and is given by

$$ {}_0^{RL}D_u^{1-\alpha} g(u) = \frac{1}{\Gamma(\alpha)} \frac{d}{du} \int_0^u (u-s)^{\alpha-1} g(s)\, ds. \tag{2.3} $$

For more details, we refer the reader to the book of Kilbas et al. [15].

Proposition 1 ([4])**.** *The following inversion relations*

$$ {}_0^{PC}D_t^{\alpha}\, {}_0^{PC}I_t^{\alpha} g(t) = g(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \lim_{t\to 0} {}_0^{RL}I_t^{\alpha} g(t), \tag{2.4} $$

$$ {}_0^{PC}I_t^{\alpha}\, {}_0^{PC}D_t^{\alpha} g(t) = g(t) - \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds\right) g(0) \tag{2.5} $$

are satisfied.

Proposition 2 ([4])**.** *The proportional-Caputo hybrid fractional derivative operator* ${}_0^{PC}D_t^{\alpha}$ *is non-local and singular.*

Remark 1 ([4]). In the limiting cases $\alpha \to 0$ and $\alpha \to 1$, we recover the following special cases:

$$ \lim_{\alpha\to 0} {}_0^{PC}D_t^{\alpha} g(t) = \int_0^t g(\tau)\, d\tau, $$
$$ \lim_{\alpha\to 1} {}_0^{PC}D_t^{\alpha} g(t) = g(t). $$

Theorem 1 (Krasnoselskii's fixed point theorem, [24]). *Let Ω be a closed convex and non-empty subset of a Banach space $\mathbb{X}$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$, be two operators such that*

(i) $\mathcal{P}_1 x + \mathcal{P}_2 y \in \Omega$, *for all* $x, y \in \Omega$,
(ii) $\mathcal{P}_1$ *is compact and continuous,*
(iii) $\mathcal{P}_2$ *is a contraction mapping.*

Then there exists $z \in \Omega$ *such that* $z = \mathcal{P}_1 z + \mathcal{P}_2 z$.

3. Controllability Results

In this section, we employ the generalized proportional Caputo fractional derivative operator to discuss the controllability of the IVP (1.3).

Let $C(J,\mathbb{R})$ be the Banach space of all real-valued continuous functions from J into $\mathbb{R}$ equipped by the norm $\|x\| = \sup_{t\in[0,T]} |x(t)|$.

Firstly, we consider the following auxiliary lemma.

Lemma 1. *Let $0 < \alpha \leq 1$ and $h \in C(J,\mathbb{R})$. Then the solution of the following linear fractional differential equation*

$$\begin{cases} {}^{PC}_{0}D^{\alpha}_{t}x(t) = h(t), \;\; t \in J, \\ x(0) = x_0, \end{cases} \tag{3.1}$$

is equivalent to the Volterra integral equation

$$\begin{aligned} x(t) = \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)x_0 \\ + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}h(\tau)\,d\tau\,du. \end{aligned} \tag{3.2}$$

Proof. Applying the operator ${}^{PC}_{0}I^{\alpha}_{t}(\cdot)$ on both sides of (3.1), we get

$${}^{PC}_{0}I^{\alpha}_{t}\,{}^{PC}_{0}D^{\alpha}_{t}x(t) = {}^{PC}_{0}I^{\alpha}_{t}h(t).$$

Using (2.2) and (2.3) together with Proposition 1, we get

$$x(t) - \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)x(0) = \int_0^t \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{{}^{RL}_{0}D^{1-\alpha}_{u}h(u)}{k_0(\alpha,u)}\,du.$$

In view of the following elementary fact:

$${}^{RL}_{0}D^{1-\alpha}_{u}h(u) = {}^{RL}_{0}I^{-(1-\alpha)}_{u}h(u) = {}^{RL}_{0}I^{\alpha-1}_{u}h(u) = \frac{1}{\Gamma(\alpha-1)}\int_0^u (u-\tau)^{\alpha-2}h(\tau)\,d\tau,$$

one can easily obtain the desired integral equation (3.3). The converse follows by direct computation. This completes the proof. □

By virtue of Lemma 1, the solution of the IVP (1.3) is given by

$$\begin{aligned} x(t) &= \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)x_0 \\ &\quad + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}f(\tau,x(\tau))\,d\tau\,du \\ &\quad + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}Bu_x(\tau)\,d\tau\,du. \end{aligned} \tag{3.3}$$

The following definition is helpful in the discussion of the controllability of the IVP (1.3).

Definition 3. The IVP (1.3) is said to be controllable on the interval J if, for every $x_0, x_1 \in \mathbb{R}$, there exists a control $u \in L^2(J,U)$ such that a solution x of equation (1.3) satisfies $x(b) = x_1$.

The following assumptions will be imposed.

(**A1**) The function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous.

(**A2**) There exists a constant $L > 0$ such that

$$|f(t,x) - f(t,y)| \le L|x-y|, \quad \text{for all } t \in J,\ x,y \in \mathbb{R}.$$

(**A3**) The linear operator $W : L^2(J,U) \to \mathbb{R}$, defined by

$$Wu = \frac{1}{\Gamma(\alpha-1)}\int_0^b\int_0^u \exp\left(-\int_u^b \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}Bu(\tau)\,d\tau\,du$$

has an induced inverse operator W^{-1} which takes values in $L^2(J,U)/\ker W$, where the kernel space of W is defined by $\ker W = \{x \in L^2(J,U) : Wx = 0\}$ and there exist constants $M_1, M_2 > 0$ such that $\|B\| \le M_1$ and $\|W^{-1}\| \le M_2$.

Now we formulate the first main theorem of the paper.

Theorem 2. *If the assumptions* (**A1**) – (**A3**) *are satisfied. Then the IVP (1.3) is controllable on J, provided that*

$$\frac{M_1 M_2 b^{2\alpha} L}{M_k^2 \Gamma^2(\alpha+1)} < 1, \tag{3.4}$$

where $\inf_{t\in J}|k_0(\alpha,t)| = M_k \neq 0$.

Proof. Set $\sup_{t\in J}|f(t,0)| = M_f < \infty$.

We consider the set $\mathcal{B}_r = \{x \in C(J,\mathbb{R}) : \|x\| \le r)\}$ with $r \ge \dfrac{\Lambda_1}{1-\Lambda_2}$.

For the purpose of expediency, we define the two constants $\Lambda_1 > 0$ and $0 < \Lambda_2 < 1$ as

$$\Lambda_1 = |x_0| + \frac{M_f b^\alpha}{M_k\Gamma(\alpha+1)} + \frac{M_1 M_2 b^\alpha}{M_k\Gamma(\alpha+1)}\left[|x_1| + |x_0| + \frac{M_f b^\alpha}{M_k\Gamma(\alpha+1)}\right],$$

$$\Lambda_2 = \frac{Lb^{\alpha}}{M_k\Gamma(\alpha+1)}\Big(1+\frac{M_1M_2b^{\alpha}}{M_k\Gamma(\alpha+1)}\Big).$$

Define the control $u_x(t)$ by

$$u_x(t) = W^{-1}\Big[x_1 - \exp\Big(-\int_0^b \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\Big)x_0 + \frac{1}{\Gamma(\alpha-1)}\int_0^b\int_0^u \exp\Big(-\int_u^b \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\Big)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}f(\tau,x(\tau))\,d\tau\,du\Big](t),\ t\in J.$$

Later, we shall use the following estimates:

$$\begin{aligned}
\|u_x\| &= \sup_{t\in J}|u_x(t)|\\
&\le M_2\sup_{t\in J}\Big\{|x_1|+|x_0|+\frac{1}{\Gamma(\alpha-1)}\int_0^b\int_0^u\frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|}|(f(\tau,x(\tau))-f(\tau,0))+f(\tau,0)|\,d\tau\,du\Big\}\\
&\le M_2\sup_{t\in J}\Big\{|x_1|+|x_0|+\frac{1}{\Gamma(\alpha-1)}\int_0^b\int_0^u\frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|}\big(|f(\tau,x(\tau))-f(\tau,0)|+|f(\tau,0)|\big)\,d\tau\,du\Big\}\\
&\le M_2\sup_{t\in J}\Big\{|x_1|+|x_0|+\frac{1}{\Gamma(\alpha-1)}\int_0^b\int_0^u\frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|}\big(L|x(\tau)|+|f(\tau,0)|\big)\,d\tau\,du\Big\}\\
&\le M_2\Big[|x_1|+|x_0|+\frac{b^{\alpha}}{M_k\Gamma(\alpha+1)}\big(L\|x\|+M_f\big)\Big],
\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}
\|u_x-u_y\| &= \sup_{t\in J}|u_x(t)-u_y(t)|\\
&\le M_2\sup_{t\in J}\left\{\frac{1}{\Gamma(\alpha-1)}\int_0^b\int_0^u\frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|}|f(\tau,x(\tau))-f(\tau,y(\tau))|\,d\tau\,du\right\}\\
&\le \frac{M_2L}{M_k\Gamma(\alpha-1)}\sup_{t\in J}\left\{\int_0^b\int_0^u(u-\tau)^{\alpha-2}|x(\tau)-y(\tau)|\,d\tau\,du\right\}\\
&\le \frac{M_2Lb^{\alpha}}{M_k\Gamma(\alpha+1)}\|x-y\|.
\end{aligned}\tag{3.6}$$

Using the control $u_x(t)$, we define the operators $\mathcal{P}_1,\mathcal{P}_2$ on $\mathcal{B}_r$ as:

$$\begin{aligned}
(\mathcal{P}_1x)(t) &= \exp\Big(-\int_0^t\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\Big)x_0\\
&\quad+\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u\exp\Big(-\int_u^t\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\Big)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}f(\tau,x(\tau))\,d\tau\,du,\\
(\mathcal{P}_2x)(t) &= \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u\exp\Big(-\int_u^t\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\Big)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}Bu_x(\tau)\,d\tau\,du.
\end{aligned}$$

Clearly, one can notice that $(\mathcal{P}_1 x + \mathcal{P}_2 x)(b) = x_1$. This means that u_x steers the IVP (1.3) from x_0 to x_1 in finite time b, which implies that the IVP (1.3) is controllable on J.

The proof is divided into three main steps.

Step 1. $\mathcal{P}_1 x + \mathcal{P}_2 y \in \mathcal{B}_r, \quad \forall x, y \in \mathcal{B}_r$.
For each $t \in J$ and $x, y \in \mathcal{B}_r$, using (3.5), one has

$$
\begin{aligned}
\|\mathcal{P}_1 x + \mathcal{P}_2 y\| &= \sup_{t\in J} |(\mathcal{P}_1 x)(t) + (\mathcal{P}_2 y)(t)| \\
&\leq \sup_{t\in J} \Bigg\{ \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds\right) |x_0| \\
&\quad + \frac{1}{\Gamma(\alpha-1)} \int_0^b \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds\right) \\
&\quad \times \frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|} |(f(\tau,x(\tau)) - f(\tau,0)) + f(\tau,0)| \, d\tau \, du \\
&\quad + \frac{1}{\Gamma(\alpha-1)} \int_0^b \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds\right) \frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|} |Bu_y(\tau)| \, d\tau \, du \Bigg\} \\
&\leq |x_0| + \frac{1}{M_k \Gamma(\alpha-1)} \int_0^b \int_0^u (u-\tau)^{\alpha-2} \big(L\|x\| + M_f\big) \, d\tau \, du \\
&\quad + \frac{1}{M_k \Gamma(\alpha-1)} \int_0^b \int_0^u (u-\tau)^{\alpha-2} \|B\| \|u_y\| \, d\tau \, du \\
&\leq |x_0| + \frac{b^\alpha}{M_k \Gamma(\alpha+1)} \big(Lr + M_f\big) \\
&\quad + \frac{M_1 M_2 b^\alpha}{M_k \Gamma(\alpha+1)} \left[|x_1| + |x_0| + \frac{b^\alpha}{M_k \Gamma(\alpha+1)} \big(Lr + M_f\big)\right] \\
&\leq |x_0| + \frac{M_f b^\alpha}{M_k \Gamma(\alpha+1)} + \frac{M_1 M_2 b^\alpha}{M_k \Gamma(\alpha+1)} \left[|x_1| + |x_0| + \frac{M_f b^\alpha}{M_k \Gamma(\alpha+1)}\right] \\
&\quad + \frac{L b^\alpha}{M_k \Gamma(\alpha+1)} \Big(1 + \frac{M_1 M_2 b^\alpha}{M_k \Gamma(\alpha+1)}\Big) r \\
&= \Lambda_1 + \Lambda_2 r \leq r.
\end{aligned}
$$

Thus, we conclude that $\mathcal{P}_1 x + \mathcal{P}_2 y \in \mathcal{B}_r$.

Step 2. $\mathcal{P}_1$ is compact and continuous. Firstly, we show that $\mathcal{P}_1$ is continuous. Let $\{x_n\}$ be a sequence such that $x_n \to x$ as $n \to \infty$ in $\mathcal{B}_r$. Thus, for each $t \in J$, we have

$$
\begin{aligned}
\|\mathcal{P}_1 x_n - \mathcal{P}_1 x\| &= \sup_{t\in J} |(\mathcal{P}_1 x_n)(t) - (\mathcal{P}_1 x)(t)| \\
&\leq \frac{1}{M_k \Gamma(\alpha-1)} \int_0^t \int_0^u (u-\tau)^{\alpha-2} \|(f(\cdot, x_n(\cdot)) - f(\cdot, x(\cdot))\| \, d\tau \, du.
\end{aligned}
$$

Therefore, the continuity of f implies that $\mathcal{P}_1$ is continuous.

Next, we show that $\mathcal{P}_1$ is uniformly bounded on $\mathcal{B}_r$. For each $t \in J$ and $x \in \mathcal{B}_r$, one has

$$\begin{aligned}
\|\mathcal{P}_1 x\| &= \sup_{t\in J} |(\mathcal{P}_1 x)(t)| \\
&\leq \sup_{t\in J} \left\{ \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) |x_0| + \frac{1}{\Gamma(\alpha-1)} \int_0^b \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) \right. \\
&\quad \left. \times \frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|} |(f(\tau,x(\tau)) - f(\tau,0)) + f(\tau,0)| \, d\tau \, du \right\} \\
&\leq |x_0| + \frac{b^\alpha}{M_k \Gamma(\alpha+1)} \big(Lr + M_f \big),
\end{aligned}$$

which implies that $\mathcal{P}_1$ is uniformly bounded on $\mathcal{B}_r$.

It remains to show that $\mathcal{P}_1$ is equicontinuous. For each $t_1, t_2 \in J$, $t_1 < t_2$ and $x \in \mathcal{B}_r$, one obtain that:

$$\begin{aligned}
&\|(\mathcal{P}_1 x)(t_2) - (\mathcal{P}_1 x)(t_1)\| \\
&\leq \left\| \exp\left(-\int_0^{t_2} \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) x_0 - \exp\left(-\int_0^{t_1} \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) x_0 \right\| \\
&\quad + \frac{1}{\Gamma(\alpha-1)} \left\| \int_0^{t_2} \int_0^u \left[\exp\left(-\int_0^{t_2} \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) - \exp\left(-\int_0^{t_1} \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) \right] \right. \\
&\quad \left. \times \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} (f(\tau,x(\tau)) \, d\tau \, du \right\| \\
&\quad + \frac{1}{\Gamma(\alpha-1)} \left\| \int_{t_1}^{t_2} \int_0^u \exp\left(-\int_0^{t_1} \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} (f(\tau,x(\tau)) \, d\tau \, du \right\| \\
&= \left\| \frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)} x_0 \exp\left(-\int_0^{\xi} \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) (t_2 - t_1) \right\| \\
&\quad + \frac{1}{\Gamma(\alpha-1)} \left\| \int_0^{t_2} \int_0^u \frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)} \exp\left(-\int_0^{\xi} \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) \right. \\
&\quad \left. (t_2 - t_1) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} (f(\tau,x(\tau)) \, d\tau \, du \right\| \\
&\quad + \frac{1}{\Gamma(\alpha-1)} \left\| \int_{t_1}^{t_2} \int_0^u \exp\left(-\int_0^{t_1} \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} (f(\tau,x(\tau)) \, d\tau \, du \right\| \\
&\leq \left| \frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)} x_0 \right| (t_2 - t_1) + \frac{\bar{f}}{\Gamma(\alpha+1)} \left| \frac{k_1(\alpha,\xi)}{k_0^2(\alpha,\xi)} \right| t_2 (t_2 - t_1) + \frac{\bar{f}}{\Gamma(\alpha+1)} \left| \frac{1}{k_0(\alpha,\xi)} \right| (t_2^\alpha - t_1^\alpha),
\end{aligned}$$

where $\bar{f} = \sup_{t\in J\times\mathcal{B}_r}|f(t,x(t)|$ and $\xi \in (t_1,t_2)$. As $t_1 \to t_2$, the right hand side of the above inequality tends to zero independently of $x \in \mathcal{B}_r$. As a consequence of the Arzelà-Ascoli theorem, we deduce that $\mathcal{P}_1$ is compact on $\mathcal{B}_r$.

Step 3. $\mathcal{P}_2$ is a contraction on $\mathcal{B}_r$.
For each $t \in J$ and $x, y \in \mathcal{B}_r$, using (3.6), one has

$$\begin{aligned}
\|\mathcal{P}_2x - \mathcal{P}_2y\| &= \sup_{t\in J}|(\mathcal{P}_2x)(t) - (\mathcal{P}_2y)(t)| \\
&= \sup_{t\in J}\left\{\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}B(u_x(\tau)-u_y(\tau))\,d\tau\,du\right\} \\
&\leq \frac{M_1 b^\alpha}{M_k\Gamma(\alpha+1)}\|u_x - u_y\| \\
&\leq \frac{M_1M_2b^{2\alpha}L}{M_k^2\Gamma^2(\alpha+1)}\|x-y\|.
\end{aligned}$$

In view of the condition (3.4), we conclude that $\mathcal{P}_2$ is a contraction mapping.

Therefore, all the assumptions of Krasnoselskii's fixed point theorem (Theorem 1) are satisfied. Hence, the IVP (1.3) is controllable on J. This completes the proof. □

4. Hyers-Ulam Stability

Here, we elucidate Hyers-Ulam stability of the IVP (1.3). We begin with the following essential definition.

Definition 4 ([12])**.** The integral equation (3.3) is said to be Hyers-Ulam stable, if there exists a constant $\mu > 0$ satisfying: for every $\varepsilon > 0$, if

$$\begin{aligned}
&\Bigg|x(t) - \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)x_0 \\
&-\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}f(\tau,x(\tau))\,d\tau\,du \\
&-\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}Bu_x(\tau)\,d\tau\,du\Bigg| \leq \varepsilon,
\end{aligned}$$

there exists a continuous function $x^*(t)$ satisfying

$$\begin{aligned}
x^*(t) =& \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)x_0 \\
&+\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}f(\tau,x^*(\tau))d\tau du \\
&+\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}Bu_{x^*}(\tau)d\tau du, \qquad (4.1)
\end{aligned}$$

such that

$$|x(t)-x^*(t)| \leq \mu\,\varepsilon, \quad \forall\, t \in J.$$

Theorem 3. *Assume that the assumptions* $(\mathbf{A1})-(\mathbf{A3})$ *are satisfied. Then the IVP (1.3) is Hyers-Ulam stable.*

Proof. With the help of Theorem 2, let $x(t)$ be unique solution of (3.3) and $x^*(t)$ be any other solution satisfying (4.1). Then, by a similar way in the proof of Theorem 2 and by virtue of (3.6), one has

$$\begin{aligned}
|x(t)-x^*(t)| &\leq \frac{1}{M_k\Gamma(\alpha-1)}\int_0^t\int_0^u (u-\tau)^{\alpha-2}\,|f(\tau,x(\tau))-f(\tau,x^*(\tau))|\;d\tau\,du \\
&\quad+\frac{M_1}{M_k\Gamma(\alpha-1)}\int_0^t\int_0^u (u-\tau)^{\alpha-2}\,|u_x(\tau)-u_{x^*}(\tau)|\;d\tau\,du \\
&\leq \frac{L}{M_k\Gamma(\alpha-1)}\int_0^t\int_0^u (u-\tau)^{\alpha-2}\,|x(\tau)-x^*(\tau)|\;d\tau\,du \\
&\quad+\frac{M_1}{M_k\Gamma(\alpha-1)}\int_0^t\int_0^u (u-\tau)^{\alpha-2}\,|u_x(\tau)-u_{x^*}(\tau)|\;d\tau\,du \\
&\leq \left(\frac{b^\alpha L}{M_k\Gamma(\alpha+1)}+\frac{M_1M_2b^{2\alpha}L}{M_k^2\Gamma^2(\alpha+1)}\right)\|x-x^*\| \\
&= \mu\,\|x-x^*\|,
\end{aligned}$$

where

$$\mu := \frac{b^\alpha L}{M_k\Gamma(\alpha+1)}+\frac{M_1M_2b^{2\alpha}L}{M_k^2\Gamma^2(\alpha+1)}.$$

Therefore, the integral equation (3.3) is Hyers-Ulam stable. Consequently, the IVP (1.3) is Hyers-Ulam stable. The proof is finished. □

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ASYMPTOTIC STABILITY ANALYSIS OF RIEMANN-LIOUVILLE FRACTIONAL STOCHASTIC NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. The novelty of our paper is to establish results on asymptotic stability of mild solutions in pth moment to Riemann-Liouville fractional stochastic neutral differential equations (for short Riemann-Liouville FSNDEs) of order $\alpha \in (\frac{1}{2}, 1)$ using a Banach's contraction mapping principle. The core point of this paper is to derive the mild solution of FSNDEs involving Riemann-Liouville fractional time-derivative by applying the stochastic version of variation of constants formula. The results are obtained with the help of the theory of fractional differential equations, some properties of Mittag-Leffler functions and asymptotic analysis under the assumption that the corresponding fractional stochastic neutral dynamical system is asymptotically stable.

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Keywords: Riemann-Liouville fractional derivative, fractional stochastic neutral dynamical systems, existence and uniqueness, asymptotic stability, continuity of mild solutions in pth moment

1. INTRODUCTION

Over the years, many results have been investigated on the theory and applications of stochastic differential equations (SDEs) [14, 27, 29]. The deterministic models often oscillate due to noise. Certainly, the extension of these models is essential to consider stochastic models, in which the connected parameters are considered as appropriate Brownian motion and stochastic processes. The modeling of most problems in real-world problems is described by stochastic differential equations rather than deterministic equations. Thus, it is of great importance to design stochastic effects in the study of fractional-order dynamical systems. In particular, fractional stochastic differential equations (FSDEs) which are a generalization of differential equations by the use of fractional and stochastic calculus are more popular due to their applications in modeling and mathematical finance. Recently, FSDEs are intensively applied to model mathematical problems in finance [11, 42], dynamics of complex systems in engineering [45] and other areas [15, 22]. Most of the results on fractional stochastic

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dynamical systems are limited to prove existence and uniqueness of mild solutions using fixed point theorem [3, 5, 16, 19, 31, 36, 41].

Several outcomes have been investigated on the qualitative theory and applications of fractional stochastic functional differential equations (FSDEs) [2, 13, 38, 40]. For instance, Mahmudov and McKibben [21] studied the approximate controllability of fractional evolution equations involving generalized Riemann-Liouville fractional derivative in Hilbert spaces. Moreover, the stability theory for stochastic differential equations has been considered by means of exponential functions which lead to attain more efficient consequences in [23]. Especially, Cao and Zhu [6] considered pth exponential stability of impulsive stochastic functional differential equations based on vector Lyapunov function. Based on average dwell-time method, Razumikhin-type technique and vector Lyapunov function, some novel stability method are obtained for impulsive stochastic functional differential systems. On a study of different types of stability studies for FSDEs can be found in [1, 4, 10, 17, 24, 30, 44].

Results on the asymptotic behavior of solutions of fractional differential equations with Caputo and Riemann-Liouville fractional time-derivative are relatively scarce in the literature. Mahmudov [20] derived an explicit solution formula to linear inhomogeneous delayed Langevin equation involving two Riemann-Liouville fractional derivatives and studied existence and uniqueness, and Ulam-Hyers stability of solutions. In [8], Cong et al. investigate the asymptotic behavior of solutions of the perturbed linear fractional differential system. Cong et al. [7] proved the theorem of linearized asymptotic stability for fractional differential equations. More precisely, they showed that an equilibrium of a nonlinear Caputo fractional differential equation is asymptotically stable if its linearization at the equilibrium is asymptotically stable. There are only a few papers related to asymptotic stability of solutions of fractional stochastic differential equations which can be found in [34, 39]. Sakthivel et al. [35] studied existence and asymptotic stability in pth moment of a mild solution to a class of nonlinear fractional neutral stochastic differential equations with infinite delays in Hilbert spaces. The same asymptotic stability in pth moment of a mild solutions of nonlinear impulsive stochastic differential equations and impulsive stochastic partial differential equations with infinite delays was discussed in [32] and [33], respectively.

To the best of our knowledge, the asymptotic stability of mild solutions for fractional stochastic neutral differential equations with Riemann-Liouville fractional derivative are an untreated topic in the present literature. Due to lack of asymptotic stability of mild solutions to Riemann-Liouville FSNDEs, this motivates us to establish new results on the asymptotic analysis of fractional stochastic differential equations with Riemann-Liouville fractional time-derivative involving matrix coefficients.

Therefore, the plan of this paper is systematized as below: Section 2 is a preparatory section where we recall some basic notions and results from fractional calculus and fractional differential equations. Then we resort the setting for main results of the theory and we impose certain assumptions, definitions of stability and

asymptotic stability in pth moment of mild solutions to Riemann-Liouville FSNDEs stochastic analysis. In Section 3, first we verify the continuity of operator in pth moment on $[0,\infty)$. Then we show global existence and uniqueness of mild solution under various assumptions by a Banach's contraction mapping principle. Section 4 is devoted to proving asymptotic stability of mild solutions to Riemann-Liouville FSNDEs. With the help of properties of Mittag-Leffler functions, we show that Ψ is well-defined. Finally, we study asymptotic stability in pth of Riemann-Liouville FSNDEs of fractional-order $\alpha \in (\frac{1}{2},1)$. Section 5 is for the conclusion and future work by providing several open problems.

To conclude this introductory section, we introduce the norm of the matrix which are used throughout the paper. For any constant matrix $A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, the norm of the matrix A, according to the maximum norm on $\mathbb{R}^n$ is

$$\|A\| = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|.$$

Throughout this paper, we assume that $p \ge 2$ is an integer.

2. Mathematical preliminaries and statement of the main results

We embark on this section by briefly presenting some necessary facts and definitions from fractional calculus and special functions which are used throughout the paper.

2.1. *Fractional Calculus*

Definition 1 ([25, 28])**.** The Riemann–Liouville integral operator of fractional order $\alpha > 0$ is defined by

$$I_{0+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} g(r)\, \mathrm{d}r, \quad \text{for} \quad t > 0.$$

The Riemann–Liouville derivative operator of fractional order $\alpha > 0$ is defined by

$$D_{0+}^{\alpha} g(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left(I_{0+}^{n-\alpha} g(t)\right), \quad \text{where} \quad n-1 < \alpha \le n.$$

Definition 2 ([9, 28, 37])**.** Suppose that $\alpha > 0$, $t > 0$. The Caputo derivative operator of fractional order α is defined by:

$${}^{C}D_{0+}^{\alpha} g(t) = I_{0+}^{n-\alpha} \left(\frac{\mathrm{d}^n}{\mathrm{d}t^n} g(t)\right), \quad \text{where} \quad n-1 < \alpha \le n.$$

In particular, for $\alpha \in (0,1)$

$$I_{0^+}^{\alpha}\, {}^{C}D_{0^+}^{\alpha} f(t) = f(t) - f(0).$$

The Riemann–Liouville fractional integral operator and the Caputo fractional derivative have the following property for $\alpha \geq 0$ [18, 28]:

$$I_{0+}^{\alpha}({}^{C}D_{0+}^{\alpha}g(t)) = g(t) - \sum_{k=0}^{n-1} \frac{t^k g^{(k)}(0)}{\Gamma(k+1)}.$$

The relationship between the Riemann–Liouville and Caputo fractional derivatives is as follows:

$${}^{C}D_{0+}^{\alpha}g(t) = D_{0+}^{\alpha}g(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha} g^{(k)}(0)}{\Gamma(k-\alpha+1)}, \quad \alpha \geq 0.$$

Lemma 1. *The relationship between Riemann-Liouville fractional derivative and integral as below:*

(i) *If $f \in C(0,T]$, then for any point $t \in (0,T]$*

$${}^{RL}D_{0+}^{\alpha}\left(I_{0+}^{\alpha}f(t)\right) = f(t).$$

(ii) *If $f \in C(0,T]$ and $I_{0+}^{1-\alpha}f(t) \in C(0,T]$, then for any point $t \in (0,T]$*

$$I_{0+}^{\alpha}\left({}^{RL}D_{0+}^{\alpha}f(t)\right) = f(t) - \frac{I_{0+}^{1-\alpha}f(t)|_{t=0}}{\Gamma(\alpha)}t^{\alpha-1}.$$

The Mittag-Leffler function is a generalization of the exponential function, first proposed as a single parameter function of one variable by using an infinite series [26]. Extensions to two or three parameters are well known and thoroughly studied in textbooks such as [12].

Definition 3. We consider the matrix Mittag-Leffler functions with one and two parameters which are defined by

$$E_{\alpha}(t^{\alpha}A) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)},$$
$$E_{\alpha,\beta}(t^{\alpha}A) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+\beta)}.$$

Throughout this paper, we define

$$\Lambda_{\alpha}^{s} := \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg(\lambda)| > \frac{\alpha\pi}{2}\}.$$

For any matrix $A \in \mathbb{R}^{n\times n}$, the set $spec(A)$ is the spectrum of A, i.e

$$spec(A) := \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of the matrix } A\}.$$

It is well-known that the trivial solution of linear time-invariant fractional differential equation is asymptotically stable if and only if the spectrum $spec(A)$ of the matrix $A \in \mathbb{R}^{n\times n}$ satisfies the condition

$$spec(A) \subset \Lambda_{\alpha}^{s}.$$

In our further consideration, we will use the following result.

Lemma 2 ([43]). *Let $A \in \mathbb{R}^{n\times n}$ and suppose that $spec(A) \subset \Lambda^s_\alpha$. Then the following statements are valid.*

- *There exists $t_0 > 0$ and a positive constant $\tilde{M}$ which depends on parameters t_0, α, A such that*

$$t^{\alpha-1}\|E_{\alpha,\alpha}(t^\alpha A)\| \leq \frac{\tilde{M}}{t^{\alpha+1}}, \quad \forall t \geq t_0.$$

- *The quantity*

$$t^{1-\alpha}\int_0^t (t-\tau)^{\alpha-1}\|E_{\alpha,\alpha}((t-\tau)^\alpha A)\|\tau^{\alpha-1}\mathrm{d}\tau$$

is bounded on $[0,\infty)$, i.e.

$$\sup_{t\geq 0} t^{1-\alpha}\int_0^t (t-\tau)^{\alpha-1}\|E_{\alpha,\alpha}((t-\tau)^\alpha A)\|\tau^{\alpha-1}\mathrm{d}\tau < \infty.$$

Variation of constants method. The first step in our approach is to use the fractionally modified version of the variation of constants method which is established in [9, Theorem 7.2 and Remark 7.1] in order to obtain the solution of (2.5) by applying stochastic version of variation of constants formula in Section 3. Namely, the solution of a single inhomogeneous Riemann-Liouville fractional differential equation of the form

$$\begin{cases} {}^{RL}D^\alpha_{0+}y(t) = Ay(t) + h(t), \quad t > 0; \\ I^{k-\alpha}_{0+}y(t)|_{t=0} = b_k, \quad k = 1,2,\ldots,n, \end{cases} \tag{2.1}$$

where $n-1 < \alpha \leq n$ and $A \in \mathbb{R}^{n\times n}$ constant matrix. The problem (2.1) was analytically solved in [37] by successive iteration method. However, with the help of Laplace transform, one can obtain the same solution directly and more quickly. Indeed, taking into account initial conditions and using the inverse Laplace transform, problem (2.1) can be expressed in the form:

$$y(t) = \sum_{k=1}^{n} b_k t^{\alpha-k}E_{\alpha,\alpha-k+1}(At^\alpha) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t-\tau)^\alpha)h(\tau)\mathrm{d}\tau.$$

The following results play a necessary role on proofs of main results in Section 3.

Suppose that $\alpha_1,\alpha_2 > 1$ and $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = 1$. If $|f(t)|^{\alpha_1}$, $|g(t)|^{\alpha_2} \in L^1(\Omega)$, then $|f(t)g(t)| \in L^1(\Omega)$ and

$$\int_\Omega |f(t)g(t)|\mathrm{d}t \leq \left(\int_\Omega |f(t)|^{\alpha_1}\mathrm{d}t\right)^{\frac{1}{\alpha_1}}\left(\int_\Omega |g(t)|^{\alpha_2}\mathrm{d}t\right)^{\frac{1}{\alpha_2}}, \tag{2.2}$$

where $L^1(\Omega)$ represents the Banach space of all Lebesgue measurable functions $f : \Omega \to \mathbb{R}$ with $\int_\Omega |f(t)|\mathrm{d}t < \infty$. Especially, when $\alpha_1 = \alpha_2 = 2$, the inequality (2.2)

reduces to the Cauchy-Schwartz inequality

$$\left(\int_\Omega |f(t)g(t)|\mathrm{d}t\right)^2 \le \int_\Omega |f(t)|^2\mathrm{d}t \int_\Omega |g(t)|^2\mathrm{d}t. \tag{2.3}$$

Let $n \in \mathbb{N}$, $q > 1$ and $x_i \in \mathbb{R}_+$, $i = 1,2,\dots,n$. Then, the following inequality holds true

$$\|\sum_{i=1}^n x_i\|^q \le n^{q-1}\sum_{i=1}^n \|x_i\|^q.$$

In particular, when $q = 2$, we use the following inequality within the estimations in this paper:

$$\|\sum_{i=1}^n x_i\|^2 \le n\sum_{i=1}^n \|x_i\|^2. \tag{2.4}$$

2.2. *Statement of Main Results*

Let $\mathbb{R}^n$ be endowed with the standard Euclidean norm $\|\cdot\|$. For each $t \ge [0,T]$ let $\Xi_t := \mathbb{L}^p(\Omega, \mathscr{F}_T, \mathbb{P})$ denote the space of all $\mathscr{F}_t$ measurable integrable random variables with values in $\mathbb{R}^n$.

We consider Riemann-Liouville fractional stochastic neutral differential equations of order $\alpha \in (\frac{1}{2}, 1)$ has the following form on $[0,T]$:

$$\begin{cases} D^{\alpha}_{0+}\left(X(t) + g(t,X(t))\right) = AX(t) + b(t,X(t)) + \sigma(t,X(t))\frac{\mathrm{d}W_t}{\mathrm{d}t}, \\ I^{1-\alpha}_{0+}\left(X(t) + g(t,X(t))\right)|_{t=0} = \rho. \end{cases} \tag{2.5}$$

The coefficients $g, b, \sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ are measurable and bounded functions, $A \in \mathbb{R}^{n\times n}$ and the initial condition ρ in integral form is an $\mathscr{F}_0$-measurable. Let $(W_t)_{t\ge 0}$ denote a standard scalar Brownian motion on a complete probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ with filtration $\mathbb{F} := \{\mathscr{F}_t\}_{t\ge 0}$.

Let H be the Banach space of all $\mathbb{R}^n$-valued $\mathbb{F}_t$-adapted process ξ satisfying the following norm defined by

$$\|\xi\|^p_H := \sup_{t\ge 0} \mathbf{E}\|t^{1-\alpha}\xi(t)\|^p < \infty.$$

Let us recall the definition of mild solution to Riemann-Liouville FSNDEs (2.5).

Definition 4. A stochastic process $\{X(t), t \in [0,T]\}$ is called a mild solution of (2.5) if

- $X(t)$ is adapted to $\{\mathscr{F}_t\}_{t\ge 0}$ with $\int_0^t \|X(t)\|^p_H \mathrm{d}t < \infty$ almost everywhere;
- $X(t) \in H$ has continuous path on $t \in [0,T]$ a.s. and satisfies the following Volterra integral equation of second kind for each $t \in [0,T]$:

$$X(t) = \frac{t^{\alpha-1}\rho}{\Gamma(\alpha)} - g(t,X(t)) + \frac{1}{\Gamma(\alpha)}\int_0^t A(t-\tau)^{\alpha-1} g(\tau, X(\tau))\mathrm{d}\tau$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} b(\tau,X(\tau))\mathrm{d}\tau + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\sigma(\tau,X(\tau))\mathrm{d}W_\tau. \tag{2.6}$$

Applying variation of constants formula for deterministic Riemann-Liouville fractional differential equations and then adapting this formula for stochastic case, for each $\rho \in \mathbb{R}^n$, we obtain

$$\begin{aligned} X(t) = {} & t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha A)\rho - g(t,X(t)) - \int_0^t A(t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^\alpha A)g(\tau,X(\tau))\mathrm{d}\tau \\ & + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^\alpha A)b(\tau,X(\tau))\mathrm{d}\tau \\ & + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^\alpha A)\sigma(\tau,X(\tau))\mathrm{d}W(\tau). \end{aligned} \tag{2.7}$$

Theorem 1 (Coincidence of between the notion of integral equation and mild solution)**.** *The integral equation* (2.6) *and the mild solution* (2.1) *of Riemann-Liouville FSNDEs coincide each other.*

Proof. Using martingale representation and Itô's isometry, this theorem can be proved in a similar way shown in [3] by assuming $p \geq 2$. □

We impose the following assumptions on (2.5):

Assumption 2.1. *Let $A \in \mathbb{R}^{n\times n}$ be constant matrix and suppose that $spec(A) \subset \Lambda_\alpha^s$.*

Assumption 2.2. *The coefficients g,b,σ satisfy global Lipschitz continuity: there exists $L_g, L_b, L_\sigma > 0$ such that for all $x,y \in \mathbb{R}^n$, $t \geq 0$,*

$$\|b(t,x)-b(t,y)\| \leq L_b\|x-y\|, \qquad \|g(t,x)-g(t,y)\| \leq L_g\|x-y\|,$$

$$\|\sigma(t,x)-\sigma(t,y)\| \leq L_\sigma\|x-y\|.$$

Assumption 2.3. *$b(\cdot,0)$ is $\mathbb{L}^p$ integrable i.e. $\int_0^T \|b(\tau,0)\|^p\mathrm{d}\tau < \infty$, and $\sigma(\cdot,0)$ is essentially bounded i.e.*

$$ess \sup_{\tau\in[0,T]} \|\sigma(\tau,0)\| < \infty.$$

Assumption 2.4. *We impose the following condition on measurable functions g,b,σ which are used on certain estimations in Section 3.*

$$g(t,0) = 0, \quad b(t,0) = 0, \quad \sigma(t,0) = 0.$$

Assumption 2.5. *The following condition will be used in Section 3 to derive asymptotic stability results.*

$$\Theta := 4^{p-1}\left(L_g^p\|A\|^p M^p \left(B\left(\frac{p\alpha-1}{p-1},\frac{p\alpha-1}{p-1}\right)\right)^{p-1} T^{p\alpha-1}\right.$$

$$+L_b^p M^p \left(B\left(\frac{p\alpha-1}{p-1}, \frac{p\alpha-1}{p-1} \right) \right)^{p-1} T^{p\alpha-1}$$
$$+C_p L_\sigma^p M^p T^{p(\alpha-1)+\frac{p}{2}} \left(B(2\alpha-1,2\alpha-1) \right)^{p/2} \Big) < 1,$$

where $C_p = \left(\frac{p(p-1)}{2} \right)^{p/2}$, $M := \sup_{t\in[0,T]} \|E_{\alpha,\alpha}(t^\alpha A)\|$ *and* B *is the beta function.*

Definition 5. Let $p \geq 2$ be an integer. Equation 2.5 is said to be stable in pth moment if for arbitrarily given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\sup_{t\geq 0} \mathbf{E}\left(\|X(t)\|^p \right) < \varepsilon, \quad \text{when} \quad \|\rho\| < \delta. \tag{2.8}$$

Definition 6. Let $p \geq 2$ be an integer. Equation 2.5 is said to be asymptotically stable in pth moment if it is stable in pth moment and for any $\rho \in \mathbb{R}^n$ it holds

$$\lim_{t\to\infty} \sup_{t\geq T} \mathbf{E}\left(\|X(t)\|^p \right) = 0. \tag{2.9}$$

Now let us state the following well-known lemma (Da Prato and Zabczyk, 1992) which will be used in the sequel of the proofs of the main results.

Lemma 3. *Let* $p \geq 2, t > 0$ *and let* Φ *be an* $\mathbb{R}^n$*-valued predictable process such that* $\boldsymbol{E} \int_0^t \|\Phi(\tau)\|^p d\tau < \infty$. *Then*

$$\sup_{\tau\in[0,T]} \boldsymbol{E} \left\| \int_0^\tau \Phi(u) dW(u) \right\|^p \leq \left(\frac{p(p-1)}{2} \right)^{p/2} \left(\int_0^t \left(\boldsymbol{E}\|\Phi(\tau)\|^p \right)^{2/p} d\tau \right)^{p/2}.$$

3. Existence and uniqueness of Riemann-Liouville fractional stochastic neutral differential equations

In this section, to show global existence and uniqueness results for Riemann-Liouville FSNDEs (2.5), we state and prove the following theorem.

Theorem 2. *Suppose that Assumptions 2.2-2.4 hold. Then for any* $\rho \in \mathbb{R}^n$, *there exists a unique mild solution of* (2.5).

Proof. To prove the asymptotic stability it is sufficient to show that the operator Ψ has a fixed point in H. To show this result, we use the contraction mapping principle.

Associated with the fixed point problem, we define an operator $\Psi : H \to H$ for $t \in [0,T]$ by

$$(\Psi X)(t) = t^{\alpha-1} E_{\alpha,\alpha}(t^\alpha A)\rho - g(t,X(t)) - \int_0^t A(t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha A) g(\tau,X(\tau)) d\tau$$
$$+ \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha A) b(\tau,X(\tau)) d\tau$$

$$+\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^{\alpha}A)\sigma(\tau,X(\tau))\mathrm{d}W(\tau). \tag{3.1}$$

First, we verify mean square continuity of Ψ on $[0,T]$. Let $X\in H$, $t_1\geq 0$ and r be sufficiently small and show that

$$\begin{aligned}\mathbf{E}\|(t_1+r)^{1-\alpha}(\Psi X)(t_1+r)-t_1^{1-\alpha}(\Psi X)(t_1)\|^p\\ \leq 5^{p-1}\sum_{i=1}^{5}\mathbf{E}\|(t_1+r)^{1-\alpha}I_i(t_1+r)-t_1^{\alpha}I_i(t_1)\|^p.\end{aligned} \tag{3.2}$$

Note that

$$\begin{aligned}&\mathbf{E}\|(t_1+r)^{1-\alpha}I_1(t_1+r)-t_1^{1-\alpha}I_1(t_1)\|^p\\ &=\mathbf{E}\|\left[(t_1+r)^{1-\alpha}(t_1+r)^{\alpha-1}E_{\alpha,\alpha}((t_1+r)^{\alpha}A)-t_1^{1-\alpha}t_1^{\alpha-1}E_{\alpha,\alpha}(t_1^{\alpha}A)\right]\rho\|^p\\ &=\mathbf{E}\|\left[E_{\alpha,\alpha}((t_1+r)^{\alpha}A)-E_{\alpha,\alpha}(t_1^{\alpha}A)\right]\rho\|^p.\end{aligned} \tag{3.3}$$

The strong continuity of $E_{\alpha,\alpha}(t^{\alpha}A)$ implies that the right-hand-side of (3.3) goes to zero as $r\to 0$.

By Assumptions 2.2 and 2.4, we get

$$\begin{aligned}&\mathbf{E}\|(t_1+r)^{1-\alpha}I_2(t_1+r)-t_1^{1-\alpha}I_2(t_1)\|^p\\ &=\mathbf{E}\|(t_1+r)^{1-\alpha}g(t_1+r,X(t_1+r))-t_1^{1-\alpha}g(t_1,X(t_1))\|^p\\ &\leq 2^{p-1}\mathbf{E}\|(t_1+r)^{1-\alpha}g(t_1+r,X(t_1+r))-(t_1+r)^{1-\alpha}g(t_1+r,X(t_1))\|^p\\ &\quad+2^{p-1}\mathbf{E}\|(t_1+r)^{1-\alpha}g(t_1+r,X(t_1))-t_1^{1-\alpha}g(t_1,X(t_1))\|^p\\ &\leq 2^{p-1}L_g^p\mathbf{E}\|(t_1+r)^{1-\alpha}\left(X(t_1+r)-X(t_1)\right)\|^p\\ &\quad+2^{p-1}\mathbf{E}\|(t_1+r)^{1-\alpha}g(t_1+r,X(t_1))-t_1^{1-\alpha}g(t_1,X(t_1))\|^p\to 0,\quad \text{as}\quad r\to 0.\end{aligned} \tag{3.4}$$

In view of the Hölder's inequality and Assumptions 2.2 and 2.4, we have

$$\begin{aligned}&\mathbf{E}\|(t_1+r)^{1-\alpha}I_3(t_1+r)-t_1^{1-\alpha}I_3(t_1)\|^p\\ &=\mathbf{E}\|\int_0^{t_1+r}A(t_1+r)^{1-\alpha}(t_1+r-\tau)^{\alpha-1}E_{\alpha,\alpha}((t_1+r-\tau)^{\alpha}A)g(\tau,X(\tau))\mathrm{d}\tau\\ &\quad-\int_0^{t_1}At_1^{1-\alpha}(t_1-\tau)^{\alpha-1}E_{\alpha,\alpha}((t_1-\tau)^{\alpha}A)g(\tau,X(\tau))\mathrm{d}\tau\|^p\\ &\leq 2^{p-1}\mathbf{E}\|\int_0^{t_1+r}A\Big[(t_1+r)^{1-\alpha}(t_1+r-\tau)^{\alpha-1}E_{\alpha,\alpha}((t_1+r-\tau)^{\alpha}A)\\ &\quad-t_1^{1-\alpha}(t_1-\tau)^{\alpha-1}E_{\alpha,\alpha}((t_1-\tau)^{\alpha}A)\Big]g(\tau,X(\tau))\mathrm{d}\tau\|^p\\ &\quad+2^{p-1}\mathbf{E}\left\|\int_{t_1}^{t_1+r}At_1^{1-\alpha}(t_1-\tau)^{\alpha-1}E_{\alpha,\alpha}((t_1-\tau)^{\alpha}A)g(\tau,X(\tau))\mathrm{d}\tau\right\|^p\end{aligned}$$

$$
\begin{aligned}
&\le 2^{p-1}\mathbf{E}\Big(\int_0^{t_1+r}\|A\|\Big[\tau^{\alpha-1}(t_1+r)^{1-\alpha}(t_1+r-\tau)^{\alpha-1}E_{\alpha,\alpha}((t_1+r-\tau)^{\alpha}A)\\
&\quad-\tau^{\alpha-1}t_1^{1-\alpha}(t_1-\tau)^{\alpha-1}E_{\alpha,\alpha}((t_1-\tau)^{\alpha}A)\Big]\|\tau^{1-\alpha}g(\tau,X(\tau))\|\mathrm{d}\tau\Big)^p\\
&\quad+2^{p-1}\mathbf{E}\Big(\int_{t_1}^{t_1+r}\|A\|t_1^{1-\alpha}\tau^{\alpha-1}(t_1-\tau)^{\alpha-1}\|E_{\alpha,\alpha}((t_1-\tau)^{\alpha}A)\|\|\tau^{1-\alpha}g(\tau,X(\tau))\|\mathrm{d}\tau\Big)^p\\
&\le 2^{p-1}L_g^p\|A\|^p\Big(\int_0^{t_1+r}\|\tau^{\alpha-1}(t_1+r)^{1-\alpha}(t_1+r-\tau)^{\alpha-1}E_{\alpha,\alpha}((t_1+r-\tau)^{\alpha}A)\\
&\quad-\tau^{\alpha-1}t_1^{1-\alpha}(t_1-\tau)^{\alpha-1}E_{\alpha,\alpha}((t_1-\tau)^{\alpha}A)\|^{\frac{p}{p-1}}\mathrm{d}\tau\Big)^{p-1}\int_0^{t_1+r}\mathbf{E}\|\tau^{1-\alpha}X(\tau)\|^p\mathrm{d}\tau\\
&\quad+2^{p-1}L_g^p\|A\|^p\Big(\int_{t_1}^{t_1+r}t_1^{\frac{p(1-\alpha)}{p-1}}((t_1-\tau)\tau)^{\frac{p(\alpha-1)}{p-1}}\|E_{\alpha,\alpha}((t_1-\tau)^{\alpha}A)\|^{\frac{p}{p-1}}\mathrm{d}\tau\Big)^{p-1}\\
&\quad\int_{t_1}^{t_1+r}\mathbf{E}\|\tau^{1-\alpha}X(\tau)\|^p\mathrm{d}\tau. \qquad (3.5)
\end{aligned}
$$

Thus, the right-hand-side of above inequality tends to zero as $r\to 0$.
Next, we consider

$$
\begin{aligned}
&\mathbf{E}\|(t_1+r)^{1-\alpha}I_4(t_1+r)-t_1^{1-\alpha}I_4(t_1)\|^p\\
&=\mathbf{E}\|\int_0^{t_1+r}(t_1+r)^{1-\alpha}(t_1+r-\tau)^{\alpha-1}E_{\alpha,\alpha}((t_1+r-\tau)^{\alpha}A)b(\tau,X(\tau))\mathrm{d}\tau\\
&\quad-\int_0^{t_1}t_1^{\alpha}(t_1-\tau)^{\alpha-1}E_{\alpha,\alpha}((t_1-\tau)^{\alpha}A)b(\tau,X(\tau))\mathrm{d}\tau\|^p\\
&\le 2^{p-1}\mathbf{E}\|\int_0^{t_1+r}\Big[(t_1+r)^{1-\alpha}(t_1+r-\tau)^{\alpha-1}E_{\alpha,\alpha}((t_1+r-\tau)^{\alpha}A)\\
&\quad-t_1^{1-\alpha}(t_1-\tau)^{\alpha-1}E_{\alpha,\alpha}((t_1-\tau)^{\alpha}A)\Big]b(\tau,X(\tau))\mathrm{d}\tau\|^p\\
&\quad+2^{p-1}\mathbf{E}\Big\|\int_{t_1}^{t_1+r}t_1^{1-\alpha}(t_1-\tau)^{\alpha-1}E_{\alpha,\alpha}((t_1-\tau)^{\alpha}A)b(\tau,X(\tau))\mathrm{d}\tau\Big\|^p\\
&\le 2^{p-1}\mathbf{E}\Big(\int_0^{t_1+r}\|\tau^{\alpha-1}(t_1+r)^{1-\alpha}(t_1+r-\tau)^{\alpha-1}E_{\alpha,\alpha}((t_1+r-\tau)^{\alpha}A)\\
&\quad-\tau^{\alpha-1}t_1^{1-\alpha}(t_1-\tau)^{\alpha-1}E_{\alpha,\alpha}((t_1-\tau)^{\alpha}A)\|\|\tau^{1-\alpha}b(\tau,X(\tau))\|\mathrm{d}\tau\Big)^p\\
&\quad+2^{p-1}\mathbf{E}\left(\int_{t_1}^{t_1+r}\tau^{\alpha-1}(t_1-\tau)^{\alpha-1}\|E_{\alpha,\alpha}((t_1-\tau)^{\alpha}A)\|\|\tau^{1-\alpha}b(\tau,X(\tau))\|\mathrm{d}\tau\right)^p\\
&\le 2^{p-1}L_g^p\Big(\int_0^{t_1+r}\|\tau^{\alpha-1}(t_1+r)^{1-\alpha}(t_1+r-\tau)^{\alpha-1}E_{\alpha,\alpha}((t_1+r-\tau)^{\alpha}A)\\
&\quad-\tau^{\alpha-1}t_1^{1-\alpha}(t_1-\tau)^{\alpha-1}E_{\alpha,\alpha}((t_1-\tau)^{\alpha}A)\|^{\frac{p}{p-1}}\mathrm{d}\tau\Big)^{p-1}\int_0^{t_1+r}\mathbf{E}\|\tau^{1-\alpha}X(\tau)\|^p\mathrm{d}\tau\\
&\quad+2^{p-1}L_g^p\left(\int_{t_1}^{t_1+r}((t_1-\tau)\tau)^{\frac{p(\alpha-1)}{p-1}}\|E_{\alpha,\alpha}((t_1-\tau)^{\alpha}A)\|^{\frac{p}{p-1}}\mathrm{d}\tau\right)^{p-1}\int_{t_1}^{t_1+r}\mathbf{E}\|\tau^{1-\alpha}X(\tau)\|^p\mathrm{d}\tau. \qquad (3.6)
\end{aligned}
$$

Therefore, the right-hand-side of the above inequality tends to zero as $r \to 0$ and ε is sufficiently small.

Further, we have

$$
\begin{aligned}
&\mathbf{E}\|(t_1+r)^{1-\alpha} I_5(t_1+r) - t_1^{1-\alpha} I_5(t_1)\|^p \\
&= \mathbf{E}\| \int_0^{t_1+r} (t_1+r)^{1-\alpha}(t_1+r-\tau)^{\alpha-1} E_{\alpha,\alpha}((t_1+r-\tau)^{\alpha} A)\sigma(\tau, X(\tau))\mathrm{d}W(\tau) \\
&\quad - \int_0^{t_1} t_1^{1-\alpha}(t_1-\tau)^{\alpha-1} E_{\alpha,\alpha}((t_1-\tau)^{\alpha} A)\sigma(\tau, X(\tau))\mathrm{d}W(\tau)\|^p \\
&\le 2^{p-1}\mathbf{E}\| \int_0^{t_1+r} \Big[(t_1+r)^{1-\alpha}(t_1+r-\tau)^{\alpha-1} E_{\alpha,\alpha}((t_1+r-\tau)^{\alpha} A) \\
&\quad - t_1^{1-\alpha}(t_1-\tau)^{\alpha-1} E_{\alpha,\alpha}((t_1-\tau)^{\alpha} A)\Big]\sigma(\tau, X(\tau))\mathrm{d}W(\tau)\|^p \\
&\quad + 2^{p-1}\mathbf{E}\left\| \int_{t_1}^{t_1+r} t_1^{1-\alpha}(t_1-\tau)^{\alpha-1} E_{\alpha,\alpha}((t_1-\tau)^{\alpha} A)\sigma(\tau, X(\tau))\mathrm{d}W(\tau)\right\|^p \\
&\le 2^{p-1} C_p \mathbf{E}\Big(\int_0^{t_1+r} \Big[\tau^{\alpha-1}(t_1+r)^{1-\alpha}(t_1+r-\tau)^{\alpha-1} E_{\alpha,\alpha}((t_1+r-\tau)^{\alpha} A) \\
&\quad - \tau^{\alpha-1} t_1^{1-\alpha}(t_1-\tau)^{\alpha-1} E_{\alpha,\alpha}((t_1-\tau)^{\alpha} A)\Big]^2 \|\tau^{1-\alpha}\sigma(\tau, X(\tau))\|^2 \mathrm{d}\tau\Big)^{p/2} \\
&\quad + 2^{p-1} C_p \mathbf{E}\left(\int_{t_1}^{t_1+r} \tau^{2\alpha-2}(t_1-\tau)^{2\alpha-2} \|E_{\alpha,\alpha}((t_1-\tau)^{\alpha} A)\|^2 \|\tau^{1-\alpha}\sigma(\tau, X(\tau))\|^2 \mathrm{d}\tau\right)^{p/2} \\
&\le 2^{p-1} C_p L_\sigma^p \Big(\int_0^{t_1+r} \Big[\tau^{\alpha-1}(t_1+r)^{1-\alpha}(t_1+r-\tau)^{\alpha-1} E_{\alpha,\alpha}((t_1+r-\tau)^{\alpha} A) \\
&\quad - \tau^{\alpha-1} t_1^{1-\alpha}(t_1-\tau)^{\alpha-1} E_{\alpha,\alpha}((t_1-\tau)^{\alpha} A)\Big]^2 \mathbf{E}\|\tau^{1-\alpha} X(\tau)\|^2 \mathrm{d}\tau\Big)^{p/2} \\
&\quad + 2^{p-1} C_p L_\sigma^p \left(\int_{t_1}^{t_1+r} \tau^{2\alpha-2}(t_1-\tau)^{2\alpha-2} \|E_{\alpha,\alpha}((t_1-\tau)^{\alpha} A)\|^2 \mathbf{E}\|\tau^{1-\alpha} X(\tau)\|^2 \mathrm{d}\tau\right)^{p/2} \\
&\le 2^{p-1} C_p L_\sigma^p \Big(\int_0^{t_1+r} \Big[\tau^{\alpha-1}(t_1+r)^{1-\alpha}(t_1+r-\tau)^{\alpha-1} E_{\alpha,\alpha}((t_1+r-\tau)^{\alpha} A) \\
&\quad - \tau^{\alpha-1} t_1^{1-\alpha}(t_1-\tau)^{\alpha-1} E_{\alpha,\alpha}((t_1-\tau)^{\alpha} A)\Big]^{\frac{2p}{p-2}} \mathrm{d}\tau\Big)^{\frac{p-2}{2}} \int_0^{t_1+r} \mathbf{E}\|\tau^{1-\alpha} X(\tau)\|^p \mathrm{d}\tau \\
&\quad + 2^{p-1} C_p L_\sigma^p \Big(\int_{t_1}^{t_1+r} \big((t_1-\tau)^{2\alpha-2} \|E_{\alpha,\alpha}((t_1-\tau)^{\alpha} A)\|^2\big)^{\frac{p}{p-2}} \mathrm{d}\tau\Big)^{\frac{p-2}{2}} \\
&\quad \int_{t_1}^{t_1+r} \mathbf{E}\|\tau^{1-\alpha} X(\tau)\|^p \mathrm{d}\tau. \qquad (3.7)
\end{aligned}
$$

As $r \to 0$ the right-hand-side of the above inequality tends to zero. Thus, by taking into (3.3)-(3.7) account Ψ is continuous in pth moment on $[0,T]$.

Next, to prove the global existence and uniqueness of solutions, we will show that the operator Ψ has a unique fixed point. Indeed for any $X, Y \in H$, we have

$$
\sup_{t\in[0,T]} \mathbf{E}\|t^{1-\alpha}(\Psi X)(t) - t^{1-\alpha}(\Psi Y)(t)\|^p
$$

$$
\begin{aligned}
&\le 4^{p-1}\sup_{t\in[0,T]}\mathbf{E}\|t^{1-\alpha}g(t,(\Psi X)(t))-t^{1-\alpha}g(t,(\Psi Y)(t))\|^p\\
&\quad+4^{p-1}\sup_{t\in[0,T]}\mathbf{E}\|\int_0^t At^{1-\alpha}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^{\alpha}A)[g(\tau,X(\tau))-g(\tau,Y(\tau))]\mathrm{d}\tau\|^p\\
&\quad+4^{p-1}\sup_{t\in[0,T]}\mathbf{E}\|\int_0^t t^{1-\alpha}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^{\alpha}A)[b(\tau,X(\tau))-b(\tau,Y(\tau))]\mathrm{d}\tau\|^p\\
&\quad+4^{p-1}\sup_{t\in[0,T]}\mathbf{E}\|\int_0^t t^{1-\alpha}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^{\alpha}A)[\sigma(\tau,X(\tau))-\sigma(\tau,Y(\tau))]\mathrm{d}W(\tau)\|^p\\
&\le 4^{p-1}L_g^p\sup_{t\in[0,T]}\mathbf{E}\|t^{1-\alpha}(\Psi X)(t)-t^{1-\alpha}(\Psi Y)(t)\|^p\\
&\quad+4^{p-1}L_g^p\|A\|^p\Big(\sup_{t\in[0,T]}\int_0^t\left(t^{1-\alpha}(t-\tau)^{\alpha-1}\tau^{\alpha-1}\|E_{\alpha,\alpha}((t-\tau)^{\alpha}A)\|\right)^{\frac{p}{p-1}}\mathrm{d}\tau\Big)^{p-1}\\
&\quad\times\sup_{t\in[0,T]}\mathbf{E}\|t^{1-\alpha}X(t)-t^{1-\alpha}Y(t)\|^p\\
&\quad+4^{p-1}L_b^p\Big(\sup_{t\in[0,T]}\int_0^t\left(t^{1-\alpha}(t-\tau)^{\alpha-1}\tau^{\alpha-1}\|E_{\alpha,\alpha}((t-\tau)^{\alpha}A)\|\right)^{\frac{p}{p-1}}\mathrm{d}\tau\Big)^{p-1}\\
&\quad\times\sup_{t\in[0,T]}\mathbf{E}\|t^{1-\alpha}X(t)-t^{1-\alpha}Y(t)\|^p\\
&\quad+4^{p-1}C_p\sup_{t\in[0,T]}\mathbf{E}\Big[\int_0^t\|t^{1-\alpha}(t-\tau)^{\alpha-1}\tau^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^{\alpha}A)[\tau^{1-\alpha}\sigma(\tau,X(\tau))\\
&\quad-\tau^{1-\alpha}\sigma(\tau,Y(\tau))]\|_X^2\mathrm{d}\tau\Big]^{\frac{p}{2}}\\
&\le 4^{p-1}L_g^p\|\Psi X-\Psi Y\|_H^p\\
&\quad+4^{p-1}L_g^p\|A\|^pM^p\Big(\sup_{t\in[0,T]}\int_0^t\left(t^{1-\alpha}(t-\tau)^{\alpha-1}\tau^{\alpha-1}\|\right)^{\frac{p}{p-1}}\mathrm{d}\tau\Big)^{p-1}\|X-Y\|_H^p\\
&\quad+4^{p-1}L_b^pM^p\Big(\sup_{t\in[0,T]}\int_0^t\left(t^{1-\alpha}(t-\tau)^{\alpha-1}\tau^{\alpha-1}\right)^{\frac{p}{p-1}}\mathrm{d}\tau\Big)^{p-1}\|X-Y\|_H^p\\
&\quad+4^{p-1}C_pL_\sigma^pM^p\Big(\sup_{t\in[0,T]}\int_0^t t^{2-2\alpha}(t-\tau)^{2\alpha-2}\tau^{2\alpha-2}\mathrm{d}\tau\Big)^{p/2}\|X-Y\|_H^p\\
&\le 4^{p-1}L_g^p\|\Psi X-\Psi Y\|_H^p+4^{p-1}L_g^p\|A\|^pM^p\left(B\left(\frac{p\alpha-1}{p-1},\frac{p\alpha-1}{p-1}\right)\right)^{p-1}T^{p\alpha-1}\|X-Y\|_H^p\\
&\quad+4^{p-1}L_b^pM^p\left(B\left(\frac{p\alpha-1}{p-1},\frac{p\alpha-1}{p-1}\right)\right)^{p-1}T^{p\alpha-1}\|X-Y\|_H^p\\
&\quad+4^{p-1}C_pL_\sigma^pM^pT^{p(\alpha-1)+\frac{p}{2}}\left(B(2\alpha-1,2\alpha-1)\right)^{p/2}\|X-Y\|_H^p.
\end{aligned}
$$

Thus,

$$\|\Psi X-\Psi Y\|_H^p\le\frac{\Theta}{1-4^{p-1}L_g^p}\|X-Y\|_H^p,$$

where

$$\Theta := 4^{p-1}\Big[L_g^p\|A\|^p M^p\left(B\left(\frac{p\alpha-1}{p-1},\frac{p\alpha-1}{p-1}\right)\right)^{p-1}T^{p\alpha-1}$$
$$+L_b^p M^p\left(B\left(\frac{p\alpha-1}{p-1},\frac{p\alpha-1}{p-1}\right)\right)^{p-1}T^{p\alpha-1}$$
$$+C_p L_\sigma^p M^p T^{p(\alpha-1)+\frac{p}{2}}\left(B(2\alpha-1,2\alpha-1)\right)^{p/2}\Big].$$

Therefore, by Assumption 2.4, Ψ is a contraction mapping and hence there exists a unique fixed point, which is a mild solution of (2.5) on $[0,T]$. □

4. Asymptotic stability analysis of Riemann-Liouville fractional neutral differential equations

We assume that $\mathscr{D}$ is a space of all $\mathbb{R}^n$-valued $\mathbb{F}_t$-adapted process φ satisfying the following norm defined by

$$\|\varphi\|_{\mathscr{D}}^p = \sup_{t\geq 0}\mathbf{E}\|\varphi(t)\|^p.$$

where φ is continuous in t such that $\mathbf{E}\|\varphi(t)\|^p \to 0$ as $t\to 0$. It is clear that $\mathscr{D}$ is a Banach space with respect to the norm $\|\cdot\|_{\mathscr{D}}$.

Theorem 3. *Let $p\geq 2$ be an integer and suppose that Assumptions 2.1- 2.4 hold. Then, the mild solution of (2.5) is asymptotically stable in the pth moment.*

Proof. First, we show that $\Psi(\mathscr{D})\subset\mathscr{D}$. Let $X\in\mathscr{D}$. We have

$$\mathbf{E}\|(\Psi X)(t)\|^p \leq 6^{p-1}\mathbf{E}\|t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha A)\rho\|^p + 6^{p-1}\mathbf{E}\|g(t,X(t))\|^p$$
$$+6^{p-1}\mathbf{E}\left\|\int_0^t A(t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^\alpha A)g(\tau,X(\tau))\mathrm{d}\tau\right\|^p$$
$$+6^{p-1}\mathbf{E}\left\|\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^\alpha A)b(\tau,X(\tau))\mathrm{d}\tau\right\|^p$$
$$+6^{p-1}\mathbf{E}\left\|\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^\alpha A)\sigma(\tau,X(\tau))\mathrm{d}W(\tau)\right\|^p. \quad (4.1)$$

Now we estimate the terms on the right-hand-side of (4.1). By Assumption 2.1 and according to Lemma 2 (i), first we have

$$6^{p-1}\mathbf{E}\|t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha A)\rho\|^p \to 0, \quad \text{as} \quad t\to\infty.$$

For $X(t)\in\mathscr{D}$ and for any $\varepsilon>0$, there exists a $t_1>0$ such that $\mathbf{E}\|X(t)\|^p\leq\varepsilon$ for $t\geq t_1$. Therefore,

$$6^{p-1}\mathbf{E}\|g(t,X(t))\|^p = 6^{p-1}\mathbf{E}\|g(t,X(t))-g(t,0)+g(t,0)\|^p$$
$$\leq 6^{p-1}\mathbf{E}\|g(t,X(t))-g(t,0)\|^p$$

$$\leq 6^{p-1}L_g^p\mathbf{E}\|X(t)\|^p \to 0, \quad \text{as} \quad t\to\infty.$$

For the fourth term, by Assumptions 2.2 and 2.4:

$$\begin{aligned}
&6^{p-1}\mathbf{E}\left\|\int_0^t A(t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^{\alpha}A)g(\tau,X(\tau))\mathrm{d}\tau\right\|^p\\
&\leq 6^{p-1}\mathbf{E}\left(\int_0^t \|A\|(t-\tau)^{\alpha-1}\|E_{\alpha,\alpha}((t-\tau)^{\alpha}A)\|\|g(\tau,X(\tau))\|\mathrm{d}\tau\right)^p\\
&\leq 6^{p-1}L_g^p\|A\|^p\left\|\int_0^t\left((t-\tau)^{\alpha-1}\|E_{\alpha,\alpha}((t-\tau)^{\alpha}A)\|\right)^{\frac{p}{p-1}}\mathrm{d}\tau\right\|^{p-1}\int_0^t\mathbf{E}\|X(\tau)\|^p\mathrm{d}\tau\\
&\leq 6^{p-1}L_g^p\|A\|^p\left(\int_0^t\left(\tau^{\alpha-1}\|E_{\alpha,\alpha}(\tau^{\alpha}A)\|\right)^{\frac{p}{p-1}}\mathrm{d}\tau\right)^{p-1}\int_0^t\mathbf{E}\|X(\tau)\|^p\mathrm{d}\tau\to 0, \text{ as } t\to\infty.
\end{aligned}$$

Similarly, applying Hölder inequality, Assumptions 2.2 and 2.4, we also have

$$\begin{aligned}
&6^{p-1}\mathbf{E}\left\|\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^{\alpha}A)b(\tau,X(\tau))\mathrm{d}\tau\right\|^p\\
&\leq 6^{p-1}\mathbf{E}\left(\int_0^t (t-\tau)^{\alpha-1}\|E_{\alpha,\alpha}((t-\tau)^{\alpha}A)\|\|b(\tau,X(\tau))\|\mathrm{d}\tau\right)^p\\
&\leq 6^{p-1}L_b^p\left(\int_0^t\left(\tau^{\alpha-1}\|E_{\alpha,\alpha}(\tau^{\alpha}A)\|\right)^{\frac{p}{p-1}}\mathrm{d}\tau\right)^{p-1}\int_0^t\mathbf{E}\|X(\tau)\|^p\mathrm{d}\tau\to 0, \text{ as } t\to\infty.
\end{aligned}$$

For the sixth term of (4.1), we arrive at

$$\begin{aligned}
&6^{p-1}\mathbf{E}\left\|\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^{\alpha}A)\sigma(\tau,X(\tau))\mathrm{d}W(\tau)\right\|^p\\
&\leq 6^{p-1}C_p\mathbf{E}\left(\int_0^t (t-\tau)^{2\alpha-2}\|E_{\alpha,\alpha}((t-\tau)^{\alpha}A)\|^2\|\sigma(\tau,X(\tau))\|^2\mathrm{d}\tau\right)^{p/2}\\
&\leq 6^{p-1}C_pL_\sigma^p\left(\int_0^t\left[(t-\tau)^{2\alpha-2}\|E_{\alpha,\alpha}((t-\tau)^{\alpha}A)\|^2\right]^{\frac{p}{p-2}}\mathrm{d}\tau\right)^{\frac{p-2}{2}}\int_0^t\mathbf{E}\|X(\tau)\|^p\mathrm{d}\tau\\
&\leq 6^{p-1}C_pL_\sigma^p\left(\int_0^t\left[\tau^{2\alpha-2}\|E_{\alpha,\alpha}(\tau^{\alpha}A)\|^2\right]^{\frac{p}{p-2}}\mathrm{d}\tau\right)^{\frac{p-2}{2}}\int_0^t\mathbf{E}\|X(\tau)\|^p\mathrm{d}\tau\to 0, \text{ as } t\to\infty.
\end{aligned}$$

Now we are in a position to prove asymptotic stability of the mild solution of (2.5). To do so, as the first step, we have to prove the stability in pth moment. Let $\varepsilon>0$ be given and choose $\delta>0$ such that $\delta<\varepsilon$ satisfies

$$\begin{aligned}
&6^{p-1}M^pT^{p(\alpha-1)}\delta+6^{p-1}\Big[L_g^p+L_g^p\|A\|^pM^p\left(\frac{p-1}{p\alpha-1}\right)^{p-1}T^{p\alpha-1}\\
&\qquad+L_b^pM^p\left(\frac{p-1}{p\alpha-1}\right)^{p-1}T^{p\alpha-1}+C_pL_\sigma^pM^p\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{p/2}\Big]\varepsilon<\varepsilon.
\end{aligned}$$

If $X(t) = X(t,\rho)$ is a mild solution of (2.5) with $\|\rho\|^p < \delta$, then $(\Psi X)(t) = X(t)$ satisfies $\mathbf{E}\|X(t)\|^p < \varepsilon$ for every $t \geq 0$. If there exists $\hat{t}$ such that $\mathbf{E}\|X(\hat{t})\|^p = \varepsilon$ and $\mathbf{E}\|X(\tau)\|^p < \varepsilon$ for $\tau \in [0,\hat{t}]$. Then,

$$\begin{aligned}\mathbf{E}\|X(\hat{t})\|^p &\leq 6^{p-1}M^pT^{p(\alpha-1)}\delta \\ &\quad + 6^{p-1}\Big[L_g^p + L_g^p\|A\|^pM^p\left(\frac{p-1}{p\alpha-1}\right)^{p-1}T^{p\alpha-1} \\ &\quad + L_b^pM^p\left(\frac{p-1}{p\alpha-1}\right)^{p-1}T^{p\alpha-1} + C_pL_\sigma^pM^p\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{p/2}\Big]\varepsilon < \varepsilon,\end{aligned}$$

which contradicts to the definition of $\hat{t}$. Therefore, the mild solution of (2.5) is asymptotically stable in pth moment. □

Remark 1. In particular, if we replace Riemann-Liouville fractional derivative with Caputo one when $p = 2$, we have already proved the existence and uniqueness of mild solution to Caputo FSNDEs in [3]. By the help of [3], Theorem 3 can be adapted with regard to the appropriate conditions as follows:

Theorem 4. *Suppose that Assumptions 2.2-2.5 hold. Then, Caputo FSNDEs are mean square asymptotically stable if*

$$4\left(L_g^2\|A\|^2M^2\frac{T^{2\alpha-1}}{2\alpha-1} + L_b^2M^2\frac{T^{2\alpha-1}}{2\alpha-1} + L_\sigma^2M^2\frac{T^{2\alpha-1}}{2\alpha-1}\right) < 1.$$

5. Conclusion

In this paper, we have studied asymptotic stability of solutions in pth moment to Riemann-Liouville FSNDEs with fractional order $\alpha \in (\frac{1}{2},1)$. For this class of systems, we can easily derive mean square asymptotic stability under appropriate conditions by assuming $p = 2$. Also, we can continue the research in the direction by extending this study to the case that $p = p(x)$ is a variable continuous function which greater or equals than 2. To do so, stochastic integration in $L_{p(x)}$ space should be investigated and Lemma 3 should be adapted to the case $p = p(x)$ for further research on asymptotic stability, so we leave this as an open problem for future work.

The main contribution of our results is that we have opened the possibility for a cooperative investigation to solve several issues, for instance, combining the methods of this paper to study the control theory, one may solve controllability of nonlinear case of our problem in finite and infinite dimensional spaces and one can also discuss finite-time stability of semiliniear fractional stochastic differential equations.

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THE FIRST THREE LARGEST NUMBERS OF SUBUNIVERSES OF SEMILATTICES

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Abstract. Let $(L,\vee)$ be a finite n-element semilattice where $n \geq 5$. We prove that the first largest number of subuniverses of an n-element semilattice is 2^n, while the second largest number is $28\cdot 2^{n-5}$ and the third one is $26\cdot 2^{n-5}$. Also, we describe the n-element semilattices with exactly 2^n, $28\cdot 2^{n-5}$ or $26\cdot 2^{n-5}$ subuniverses.

2010 *Mathematics Subject Classification:* 06A12; 06B99

Keywords: finite lattice, sublattice, subuniverse, finite semilattice, number of subuniverses

1. Introduction and our result

For a semilattice $(L,\vee)$, $\mathrm{Sub}(L,\vee)$ will denote its *subuniverse-lattice*. By a subuniverse, we mean a subsemilattice or the emptyset. All semilattices occurring in this paper will be assumed to be finite. On a semilattice $(L,\leq)$, we have a natural partial ordering defined by

$$x \leq y \iff x\vee y = y.$$

Conversely, if $(L,\leq)$ is partial order in which any two elements x,y have a least upper bound $x\vee y$, then $(L,\vee)$ is a semilattice. For any x,y in a join-semilattice, $x\wedge y$ is defined by their infimum provided it exists; if this infimum does not exist, then $x\wedge y$ is undefined. Let P and Q be posets with disjoint underlying sets. Then the *ordinal sum* $P+_{ord}Q$ is the poset on $P\cup Q$ with $s\leq t$ if either $s,t\in P$ and $s\leq t$; or $s,t\in Q$ and $s\leq t$; or $s\in P$ and $t\in Q$. To draw the Hasse diagram of $P+_{ord}Q$, we place the Hasse diagram of Q above that of P and then connect any minimal element of Q with any maximal element of P; see Figure 1. If K with 1 and L with 0 are finite posets, then their glued sum $K+_{\mathrm{glu}}L$ is the ordinal sum of the posets $K\setminus\{1_K\}$, the singleton poset, and $L\setminus\{0_L\}$, in this order; see Figure 2. Note that $+_{\mathrm{glu}}$ is an associative but not a commutative operation.

The semilattices H_3 and H_4 will be used later, see Figure 3.

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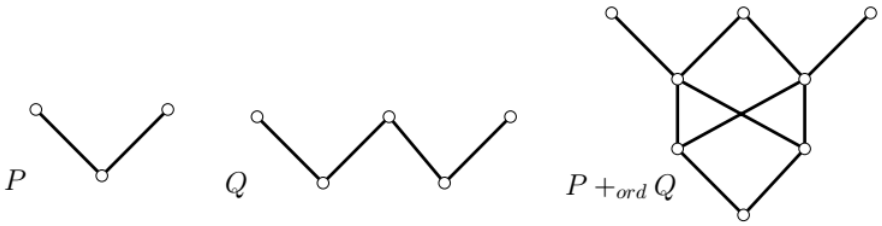

FIGURE 1. The ordinal sum $P +_{ord} Q$ of P and Q

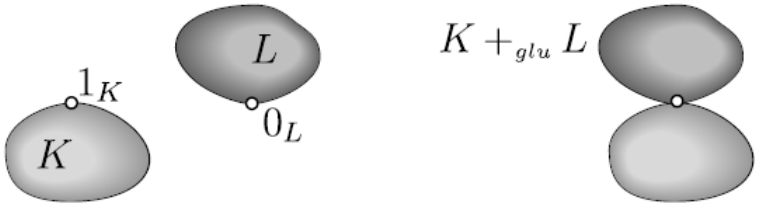

FIGURE 2. The glued sum $K +_{\mathrm{glu}} L$ of K and L

Our result is motivated by similar results, see Ahmed and Horváth [1], Czédli [3–7] and Czédli and Horváth [8]. To obtain more information about lattice theory and semilattices we direct the reader to the bibliography indicated in [9, 10] and [2], respectively.

For a natural number $n \in \mathbb{N}^+ := \{1, 2, 3, \dots\}$, let

$$\mathrm{NS}(n) := \{|\operatorname{Sub}(L)| : L \text{ is a semilattice of size } |L| = n\}.$$

Theorem 1. *If $5 \leq n \in \mathbb{N}^+$, then the following assertions hold.*

(i) The largest number in $\mathrm{NS}(n)$ is $2^n = 32 \cdot 2^{n-5}$. Furthermore, an n-element semilattice $(L, \vee)$ has exactly 2^n subuniverses if and only if $(L, \vee)$ is a chain.

(ii) The second largest number in $\mathrm{NS}(n)$ is $28 \cdot 2^{n-5}$. Furthermore, an n-element semilattice $(L, \vee)$ has exactly $28 \cdot 2^{n-5}$ subuniverses if and only if $(L, \vee) \cong H_3 +_{\mathrm{glu}} C_1$ or $(L, \vee) \cong C_0 +_{ord} H_3 +_{\mathrm{glu}} C_1$, where C_0 and C_1 are finite chains.

(iii) The third largest number in $\mathrm{NS}(n)$ is $26 \cdot 2^{n-5}$. Furthermore, an n-element semilattice $(L, \vee)$ has exactly $26 \cdot 2^{n-5}$ subuniverses if and only if $(L, \vee) \cong H_4 +_{\mathrm{glu}} C_1$ or $(L, \vee) \cong C_0 +_{ord} H_4 +_{\mathrm{glu}} C_1$, where C_0 and C_1 are finite chains.

2. TWO PREPARATORY LEMMAS

An element u of a semilattice L is called a narrow element, or a *narrows* for short, if $u \neq 1_L$ and $L = {\uparrow}u \cup {\downarrow}u$. That is, if $u \neq 1_L$ and $x \| u$ holds for no $x \in L$.

The notion of a binary partial algebra is well known, but the reader can refresh his/her knowledge from [4]. Let $\mathcal{A}$ be a finite n-element binary partial algebra. A *subuniverse* of $\mathcal{A}$ is a subset X of $\mathcal{A}$ such that X is closed with respect to all partial operations. The set of subuniverses of $\mathcal{A}$ will be denoted by $\mathrm{Sub}(\mathcal{A})$. The *relative number of subuniverses of* $\mathcal{A}$ denoted by $\sigma_k(\mathcal{A})$ is defined as follows:

$$\sigma_k(\mathcal{A}) = |\mathrm{Sub}(\mathcal{A})| \cdot 2^{k-n}.$$

In order to comply with [8], will use $k = 5$. The original definition of σ_k is given in the paper of Czédli [4], there he used $k = 8$.

Lemma 1. *If $(K,\vee)$ is a subsemilattice and H is a subset of a finite semilattice $(L,\vee)$, then the following three assertions hold.*

(i) With the notation $t := |\{H \cap S : S \in Sub(L,\vee)\}|$, we have that

$$\sigma_k(L,\vee) \leq t \cdot 2^{k-|H|}.$$

(ii) $\sigma_k(L,\vee) \leq \sigma_k(K,\vee)$.

(iii) Assume, in addition, that $(K,\vee)$ has no narrows. Then $\sigma_k(L,\vee) = \sigma_k(K,\vee)$ if and only if $(L,\vee)$ is (isomorphic to) $C_0 +_{ord} (K,\vee) +_{\mathrm{glu}} C_1$, where C_1 is a chain, and C_0 is a chain or the emptyset.

Proof. Parts (i) and (ii) can be extracted from the proof of in Lemma 2.3 of [4]. The argument there yields a bit more than stated in (i) and (ii); namely, for later reference, note the following.

$$\text{If } \sigma_k(L,\vee) = \sigma_k(K,\vee), \text{ then for every } S \in \mathrm{Sub}(K,\vee) \text{ and every subset } X \text{ of } L\backslash K \text{ we have that } S \cup X \in \mathrm{Sub}(L,\vee). \tag{2.1}$$

Next, to prove part (iii), let $n := |(L,\vee)|$ and $m := |(K,\vee)|$. Let $k := 5$, the case of another k is analogous. Assume that $(K,\vee)$ has no narrows. First, let $(L,\vee) = C_0 +_{ord} (K,\vee) +_{\mathrm{glu}} C_1$. It is obvious that whenever $X \subseteq L\backslash K$ and $S \in \mathrm{Sub}(K,\vee)$, then $S \cup X \in \mathrm{Sub}(L,\vee)$. Since $L\backslash K$ has $2^{|L|-|K|}$ subsets, $|\mathrm{Sub}(L,\vee)| \geq |\mathrm{Sub}(K,\vee)| \cdot 2^{|L|-|K|}$. Dividing this inequality by $2^{n-5} = 2^{m-5} \cdot 2^{|L|-|K|}$ we obtain the required equality, as the converse inequality given in part (ii).

Conversely, assume the equality in (iii). We claim that

$$\text{for all } y \in K \text{ and for all } x \in L \setminus K,\ y \nparallel x. \tag{2.2}$$

Suppose the contrary. If $y \in K$, then $\{y\} \in \mathrm{Sub}(K)$. If $x \in L \setminus K$ and $y||x$, then $\{y,x\}$ is not a subuniverse of L, contradicting (2.1).

We claim that

$$\text{for all } x \in L \setminus K,\ x \not> 1_K \text{ implies that for all } y \in K,\ x < y. \tag{2.3}$$

Suppose the contrary and pick x in $L \setminus K$ and $y \in K$ such that $x \not> 1_K$ and $x \not< y$. Using (2.2) and $x \neq y$, we have that $y < x < 1_K$. Let $p := \bigvee\{s \in K : s < x\}$; this exists by finiteness and $y \leq p \leq x$. In fact, $p \in K$ as K is a subsemilattice of L but $x \notin K$, so $y \leq p < x$. Now let $u \in K$ such that $u \not\leq p$. We know from (2.2) that $u \nparallel x$.

If we had $u \leq x$ (in fact, $u < x$ since $x \notin K$), then u would be one of the joinands defining p and so $u \leq p$ would be a contradiction. Hence $x < u$, and so $p < x < u$ implies $p < u$. We have seen that, for any $u \in K$, $u \not\leq p$ implies $p < u$. In other words, $K = \uparrow_K p \cup \downarrow_K p$, which means p is a narrows, contradicting our assumption on K. Thus, (2.3) holds. Finally, we show that $L \setminus K$ is a chain. Indeed, if $L \setminus K$ is not a chain, say $a||b$, $a \in L \setminus K$ and $b \in L \setminus K$, then $\varnothing \in \mathrm{Sub}(K)$ extended by $\{a,b\} \notin \mathrm{Sub}(L)$ would contradict (2.1). Define $C_1 = \{x \in L \setminus K : x \geq 1_K\}$; it is a chain (a subchain of $L \setminus K$). Let $C_0 = (L \setminus K) \setminus C_1$; it is either a chain or empty. If C_0 is empty, then L is $K +_{glue} C_1$, as required. If C_0 is nonempty, then its elements are less than any element of K by (2.3), and so $L = C_0 +_{ord} K +_{glue} C_1$, as required. □

The following lemma can be proved by a computer program, but for the reader's convenience, we give its proof.

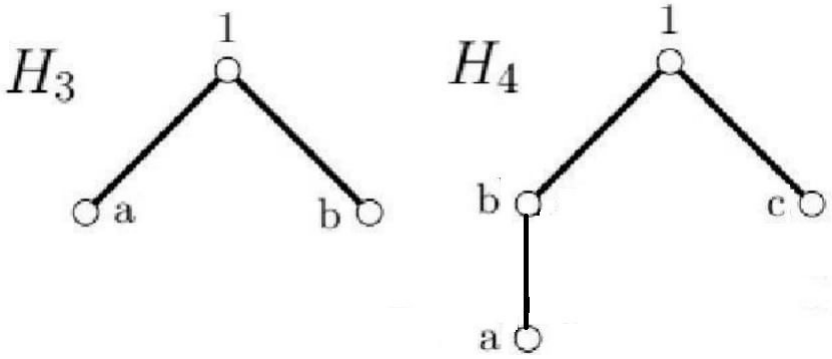

FIGURE 3. H_3 and H_4

Lemma 2. *For the join-semilattices given in Figure 3 the following assertions hold.*

(i) $\sigma_5(H_3) = 28$,
(ii) $\sigma_5(H_4) = 26$,

Proof. The notations given by Figure 3 will be used. For later reference, note that if $(L, \vee)$ is a chain then $|Sub(L, \vee)| = 2^{|(L,\vee)|}$.

For (i), observe that

$$|\{S \in \mathrm{Sub}(H_3, \vee) : a \notin S\}| = 4, \qquad \text{(S is chain)},$$
$$|\{S \in \mathrm{Sub}(H_3, \vee) : a \in S, \{b\} \cap S = \varnothing\}| = 2, \text{ and}$$
$$|\{S \in \mathrm{Sub}(H_3, \vee) : a \in S, \{b\} \cap S \neq \varnothing\}| = 1,$$

whereby $|\mathrm{Sub}(H_3, \vee)| = 4 + 2 + 1 = 7 = 28 \cdot 2^{3-5}$, note that $\sigma_5(H_3) = 28$ proves (i).

For (ii), let us compute

$$|\{S \in \mathrm{Sub}(H_4,\vee) : c \notin S\}| = 7, \qquad \text{by (i)},$$
$$|\{S \in \mathrm{Sub}(H_4,\vee) : c \in S, \{a,b\} \cap S = \varnothing\}| = 2, \text{ and}$$
$$|\{S \in \mathrm{Sub}(H_4,\vee) : c \in S, \{a,b\} \cap S \neq \varnothing\}| = 4.$$

Hence, $|\mathrm{Sub}(H_4,\vee)| = 7+2+4 = 13 = 26 \cdot 2^{4-5}$, while $\sigma_5(H_4) = 26$ proves (ii).

Remark 1. For counting subsemilattices, a computer program is available on G. Czédli's webpage: http://www.math.u-szeged.hu/~czedli/

□

3. The rest of the proof

Proof of Theorem 1. Part (i) is trivial. For part (ii) let $(L,\vee)$ be an n-element semilattice. We know from Lemma 1 (iii) that if

$$(L,\vee) \cong H_3 +_{\mathrm{glu}} C_1 \text{ or } (L,\vee) \cong C_0 +_{ord} H_3 +_{\mathrm{glu}} C_1, \text{ where } C_0 \text{ and } C_1 \text{ are chains,} \tag{3.1}$$

then $\sigma_5(L) = \sigma_5(H_3) = 28$, indeed. Conversely, assume that $\sigma_5(L) = 28$. Then it follows from part (i) that L is not a chain. So L has two incomparable elements, a and b. Clearly, $\{a,b,a\vee b\}$ is a join-subsemilattice isomorphic to H_3. But $\sigma_5(H_3)$ is also 28 by Lemma 2 (i). Thus, Lemma 1 (iii) immediately yields that L is of the desired form. By this we completed the proof of part (ii) of Theorem 1.

We prove part (iii).

Assume that $(L,\vee)$ is of the given form, then $\sigma_5(L,\vee) = 26$ is clear from Lemma 2 (ii) and Lemma 1 (iii). In order to prove the converse, that is, the nontrivial implication, assume that $\sigma_5(L,\vee) = 26$. By Theorem 1 (i), $(L,\vee)$ has two incomparable elements, a and b. By part Theorem 1 (ii), $\{a,b\}$ is not the only 2-element antichain in L since otherwise $\sigma_5(L,\vee)$ would be 28. To complete the proof, consider the following cases.

Case 1: There is an antichain $\{c,d\}$ disjoint from $\{a,b\}$, where the elements a, b, c, d are distict. Let $x := a\vee b$ and $y := c\vee d$. There are cases depending on $t := |\{a,b,c,d,x,y\}|$, which is 4, 5, or 6. The number of possible cases can be reduced by symmetry: a and b play a symmetric role, so do c and d, and so do $\{a,b\}$ and $\{c,d\}$ and thus x and y. We consider three sub-cases as we will see below:

Sub-case 1a: Now $t = 6$. Take the partial algebra $U_1 = \{a,b,c,d,x,y\}$ with $a\vee b = x$ and $c\vee d = y$. This six-element partial algebra has $\sigma_5(U_1) = 24.5$, this can be checked by the mentioned computer program. By Lemma 2.3 from [4] we obtain that $\sigma_5(L) \leq \sigma_5(U_1) \leq 24.5$, contradicting $\sigma_5(L) = 26$. Thus, this case is excluded.

Sub-case 1b: Now $t = 5$. By symmetry, $y = c\vee d$ is not a new element, so $y = c\vee d$ is either x or a. The case $y = b$ need not be considered because a is symmetric to b. Therefore, this sub-case 1b is split in two cases as follows:

First, where $y = x$, this case is captured by taking the partial algebra $U_2 = \{a,b,c,d,x\}$ with $a \vee b = x$, $c \vee d = x$. It has $\sigma_5(U_2) = 25$ by the mentioned computer program. Like above, this implies $\sigma_5(L) \leq \sigma_5(U_2) \leq 25$, contradicting $\sigma_5(L) = 26$.

Second, where $y = a$, this case is captured by taking the partial algebra $U_3 = \{a,b,c,d,x\}$ with $a \vee b = x$, $c \vee d = a$. This five-element partial algebra has $\sigma_5(U_3) = 24 < 26$, and we get a contradiction as above.

By the above we can note that the sub-case 1b is excluded since so are both of its subcases.

Sub-case 1c: Here $t = 4$. Then $x = a \vee b$ is one of c and d. By symmetry, we can assume that $a \vee b = c$. However, then $a < c < c \vee d$, $b < c < c \vee d$, whereby $y = c \vee d$ is none of the elements a,b,c,d, contradicting $t = 4$. So this case is excluded.

After having all of its sub-cases excluded, we obtain that Case 1 is excluded. That is, no two-element antichain is disjoint from $\{a,b\}$. But remember that there is another two-element antichain, whereby, by a-b symmetry, we consider **Case 2:** there is an element c such that a and c are incomparable. Again there are two sub-cases according to the position of b and c.

Sub-case 2a: Here b and c are also incomparable. Here, we have to investigate, how many of $a \vee b$, $a \vee c$, and $b \vee c$ are equal to $a \vee b \vee c$. Since the answer could be $0,1,2$ or 3 (and using symmetry), it suffices to consider only the following four join-semilattices. The first join-semilattice is $K_0 = \{a,b,c,z,x,y,1\}$ with edges ax, bx, by, cy, az, cz, $x1$, $y1$, $z1$ and equalities $a \vee b = x$, $b \vee c = y$, $a \vee c = z$; this gives $\sigma_5(K_0) = 15.25$. The second join-semilattice is $K_1 = \{a,b,c,x,y,1\}$ with edges ax, bx, by, cy, $x1$, $y1$ and equalities $a \vee b = x$, $b \vee c = y$, $a \vee c = 1$; this gives $\sigma_5(K_1) = 18.5$. The third is $K_2 = \{a,b,c,x,1\}$ with edges $ax, bx, x1, c1$ and constraints $a \vee b = x$, $a \vee c = 1$, $b \vee c = 1$; this gives $\sigma_5(K_2) = 22$. The fourth is $K_3 = \{a,b,c,1\}$ with edges $a1, b1, c1$, and equalities $a \vee b = 1$, $a \vee c = 1$, $b \vee c = 1$; this gives $\sigma_5 = 24$. Since one of K_0, K_1, K_2, and K_3 is a subsemilattice of L and all the four σ_5 values of these join-semilattices are smaller than 26, therefore sub-case 2a is excluded.

Sub-case 2b: Here b and c are comparable in addition to that a and c are incomparable and so do a and b. At present, b and c play a symmetric role. So we can assume that $b < c$. Suppose, for contradiction, that $x := a \vee b < a \vee c =: 1$. By the incomparabilities assumed, $|\{a,b,c,x,1\}| = 5$; for example if $x = a \vee b = c$ is impossible since it would yield $a < c$. The mentioned constraints are defining K. Now $\sigma_5(K) = 23 < 26$ gives a contradiction. So $a \vee b < a \vee c$ fails but $a \vee b \leq a \vee c$ since $b < c$. Therefore, with $1 = a \vee b = a \vee c$, $\{a,b,c,1\}$ is a subsemilattice (isomorphic to) H_4.

Now that all other possibilities have been excluded, we know that H_4 is a join-subsemilattice of $(L, \vee)$. Observe that H_4 has no narrows. Therefore, by Lemma 1 (iii), $(L, \vee)$ is of the desired form. By this, the proof of Theorem 1 is completed.

□

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SOLUTIONS FORMULAS FOR SOME GENERAL SYSTEMS OF DIFFERENCE EQUATIONS

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Abstract. In this paper, we give explicit formulas of the solutions of the two general systems of difference equations

$$x_{n+1} = f^{-1}\left(ag(y_n) + bf(x_{n-1}) + cg(y_{n-2}) + df(x_{n-3})\right),$$
$$y_{n+1} = g^{-1}\left(af(x_n) + bg(y_{n-1}) + cf(x_{n-2}) + dg(y_{n-3})\right),$$

and

$$x_{n+1} = f^{-1}\left(a + \frac{b}{g(y_n)} + \frac{c}{g(y_n)f(x_{n-1})} + \frac{d}{g(y_n)f(x_{n-1})g(y_{n-2)}}\right),$$
$$y_{n+1} = g^{-1}\left(a + \frac{b}{f(x_n)} + \frac{c}{f(x_n)g(y_{n-1})} + \frac{d}{f(x_n)g(y_{n-1})f(x_{n-2)}}\right),$$

where $n \in \mathbb{N}_0$, $f, g : D \longrightarrow \mathbb{R}$ are "$1-1$" continuous functions on $D \subseteq \mathbb{R}$, the initial values x_{-i}, y_{-i}, $i = 0,1,2,3$ are arbitrary real numbers in D and the parameters a, b, c and d are arbitrary real numbers. Our results considerably extend some existing results in the literature.

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1. Introduction and Preliminaries

Difference equations are used to describes real discrete models in various branches of modern sciences such as, biology, economy, control theory. This explain why a big number of papers is devoted to this subject, see for example ([1] - [20]). It is clear that if we want to understand our models, we need to know the behavior of the solutions of the equations of the models, and this fact will be possible if we can solve in closed form these equations. One can find in the literature a lot of works on difference equations where explicit formulas of the solutions are given, see for instance [1], [2], [5], [8], [9], [10], [7], [12], [11], [13], [16], [15], [14], [17], [18], [21], [20], [22]. Such type of difference equations and systems is called solvable difference equations. In the present work, we continue our interest in solvable difference equations, more

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precisely, we will solve the following two general systems of difference equations

$$x_{n+1} = f^{-1}\left(ag(y_n) + bf(x_{n-1}) + cg(y_{n-2}) + df(x_{n-3})\right),$$
$$y_{n+1} = g^{-1}\left(af(x_n) + bg(y_{n-1}) + cf(x_{n-2}) + dg(y_{n-3})\right),$$

and

$$x_{n+1} = f^{-1}\left(a + \frac{b}{g(y_n)} + \frac{c}{g(y_n)f(x_{n-1})} + \frac{d}{g(y_n)f(x_{n-1})g(y_{n-2)}}\right),$$
$$y_{n+1} = g^{-1}\left(a + \frac{b}{f(x_n)} + \frac{c}{f(x_n)g(y_{n-1})} + \frac{d}{f(x_n)g(y_{n-1})f(x_{n-2)}}\right),$$

where $n \in \mathbb{N}_0$, $f, g : D \longrightarrow \mathbb{R}$ are one to one ("$1-1$") continuous functions on $D \subseteq \mathbb{R}$, the initial values x_{-i}, y_{-i}, $i = 0,1,2,3$ are arbitrary real numbers in D and the parameters a, b, c and d are arbitrary real numbers.

In our study, we are inspired and motivated by the ideas, the equations and the systems of some recent published papers. The papers, [1], [2] and especially [15] are our main motivation in the present work. The obtained results considerably generalize some existing results in the literature, see [1], [2], [3], [4], [5], [12], [11], [13], [16], [15], [14], [17], [18], [21].

Now, we will recall some known results that will be useful, in solving in closed form our systems. To this end, let consider the two following homogeneous fourth order linear difference equations

$$R_{n+1} = aR_n + bR_{n-1} + cR_{n-2} + dR_{n-3}, n \in \mathbb{N}_0, \tag{1.1}$$
$$S_{n+1} = -aS_n + bS_{n-1} - cS_{n-2} + dS_{n-3}, n \in \mathbb{N}_0, \tag{1.2}$$

where the initial values R_0, R_{-1}, R_{-2}, R_{-3}, S_0, S_{-1}, S_{-2} and S_{-3} and the parameters a, b, c and d, $d \neq 0$ are real numbers. Noting that the formulas of the solutions of equations (1.1) and (1.2), see [2], are expressed in terms of the the sequence $(J_n)_{n=0}^{+\infty}$ defined by the recurrent relation

$$J_{n+4} = aJ_{n+3} + bJ_{n+2} + cJ_{n+1} + dJ_n, \quad n \in \mathbb{N}_0, \tag{1.3}$$

and the special initial values

$$J_0 = 0, \quad J_1 = 0, \quad J_2 = 1 \text{ and } J_3 = a. \tag{1.4}$$

The closed form of the solutions of $(J_n)_{n=0}^{+\infty}$ and many properties of them are well known in the literature see for example [19], [6].

The following result from [2], gave the general terms of (1.1) and (1.2) in terms of the sequence $(J_n)_{n=0}^{+\infty}$. The obtained formulas will be very useful to obtain the formulas of the solutions of our systems.

Lemma 1. *We have for all $n \in \mathbb{N}_0$,*

$$R_n = dJ_{n+1}R_{-3} + (cJ_{n+1} + dJ_n)R_{-2} + (J_{n+3} - aJ_{n+2})R_{-1} + J_{n+2}R_0, \tag{1.5}$$
$$S_n = (-1)^{n+1}\left[dJ_{n+1}S_{-3} - (cJ_{n+1} + dJ_n)S_{-2} + (J_{n+3} - aJ_{n+2})S_{-1} - J_{n+2}S_0\right]. \tag{1.6}$$

Consider the following fourth order linear system of difference equations

$$u_{n+1} = av_n + bu_{n-1} + cv_{n-2} + du_{n-3},\ v_{n+1} = au_n + bv_{n-1} + cu_{n-2} + dv_{n-3},\ n \in \mathbb{N}_0, \tag{1.7}$$

where the initial values u_{-i}, v_{-i}, $i = 0,1,2,3$ are nonzero real numbers.
In [2], the authors got the solutions of (1.7), in terms of the sequence $(J_n)_{n=0}^{+\infty}$ and the initial values u_{-i}, v_{-i}, $i = 0,1,2,3$.
For the interest of the readers we show how we can solve system (1.7).

Lemma 2. *The solutions of the system* (1.7) *are given by the following formulas*

$$u_{2n-1} = dJ_{2n}u_{-3} + (cJ_{2n} + dJ_{2n-1})\,v_{-2} + (J_{2n+2} - aJ_{2n+1})\,u_{-1} + J_{2n+1}v_0,\ n \in \mathbb{N}, \tag{1.8}$$

$$u_{2n} = dJ_{2n+1}v_{-3} + (cJ_{2n+1} + dJ_{2n})\,u_{-2} + (J_{2n+3} - aJ_{2n+2})\,v_{-1} + J_{2n+2}u_0,\ n \in \mathbb{N}_0, \tag{1.9}$$

$$v_{2n-1} = dJ_{2n}v_{-3} + (cJ_{2n} + dJ_{2n-1})\,u_{-2} + (J_{2n+2} - aJ_{2n+1})\,v_{-1} + J_{2n+1}u_0,\ n \in \mathbb{N}, \tag{1.10}$$

$$v_{2n} = dJ_{2n+1}u_{-3} + (cJ_{2n+1} + dJ_{2n})\,v_{-2} + (J_{2n+3} - aJ_{2n+2})\,u_{-1} + J_{2n+2}v_0,\ n \in \mathbb{N}_0. \tag{1.11}$$

Proof. From (1.7), we get

$$u_{n+1} + v_{n+1} = a(v_n + u_n) + b(u_{n-1} + v_{n-1}) + c(v_{n-2} + u_{n-2}) + d(u_{n-3} + v_{n-3}),\ n \in \mathbb{N}_0$$

and

$$u_{n+1} - v_{n+1} = a(v_n - u_n) + b(u_{n-1} - v_{n-1}) + c(v_{n-2} - u_{n-2}) + d(u_{n-3} - v_{n-3}),\ n \in \mathbb{N}_0.$$

Putting

$$R_n = u_n + v_n, \quad S_n = u_n - v_n,\ n \geq -3, \tag{1.12}$$

we obtain two homogeneous linear difference equations of fourth order:

$$R_{n+1} = aR_n + bR_{n-1} + cR_{n-2} + dR_{n-3},\ n \in \mathbb{N}_0,$$

and

$$S_{n+1} = -aS_n + bS_{n-1} - cS_{n-2} + dS_{n-3},\ n \in \mathbb{N}_0. \tag{1.13}$$

Using (1.12), we get

$$u_n = \frac{1}{2}(R_n + S_n),\ v_n = \frac{1}{2}(R_n - S_n),\ n \geq -3.$$

From Lemma 1, we obtain

$$\begin{aligned}
u_{2n-1} &= dJ_{2n}u_{-3} + (cJ_{2n} + dJ_{2n-1})\,v_{-2} + (J_{2n+2} - aJ_{2n+1})\,u_{-1} + J_{2n+1}v_0,\ n \in \mathbb{N},\\
u_{2n} &= dJ_{2n+1}v_{-3} + (cJ_{2n+1} + dJ_{2n})\,u_{-2} + (J_{2n+3} - aJ_{2n+2})\,v_{-1} + J_{2n+2}u_0,\ n \in \mathbb{N}_0,\\
v_{2n-1} &= dJ_{2n}v_{-3} + (cJ_{2n} + dJ_{2n-1})\,u_{-2} + (J_{2n+2} - aJ_{2n+1})\,v_{-1} + J_{2n+1}u_0,\ n \in \mathbb{N},\\
v_{2n} &= dJ_{2n+1}u_{-3} + (cJ_{2n+1} + dJ_{2n})\,v_{-2} + (J_{2n+3} - aJ_{2n+2})\,u_{-1} + J_{2n+2}v_0, n \in \mathbb{N}_0.
\end{aligned}$$

□

2. Solutions of a first general system of difference equations

In this part, we will focus our interest on our first general system of difference equations, that is the system

$$\begin{cases} x_{n+1} = f^{-1}\left(ag(y_n) + bf(x_{n-1}) + cg(y_{n-2}) + df(x_{n-3})\right), \\ y_{n+1} = g^{-1}\left(af(x_n) + bg(y_{n-1}) + cf(x_{n-2}) + dg(y_{n-3})\right), \end{cases} \tag{2.1}$$

where $n \in \mathbb{N}_0$, $f, g : D \longrightarrow \mathbb{R}$ are continuous functions, with $D = D_f = D_g$, that is f and g have the same domain, and it is also assumed that f, g are "$1-1$" on $D \subseteq \mathbb{R}$, the initial values x_{-i}, y_{-i}, $i = 0,1,2,3$ are arbitrary real numbers in D and the parameters a, b, c and d are arbitrary real numbers.

2.1. *Explicit formulas of solutions of system (2.1) with $d \neq 0$*

In the following result, we solve in closed form (2.1).

Definition 1. A solution $\{x_n, y_n\}_{n \geq -3}$ of system (2.1) is said to be well-defined if for all $n \in \mathbb{N}_0$, we have

$$ag(y_n) + bf(x_{n-1}) + cg(y_{n-2}) + df(x_{n-3}) \in D_{f^{-1}},$$

and

$$af(x_n) + bg(y_{n-1}) + cf(x_{n-2}) + dg(y_{n-3}) \in D_{g^{-1}}.$$

Theorem 1. *Let $\{x_n, y_n\}_{n \geq -3}$ be a well-defined solution of the system (2.1), then we have the following representation*

$$\begin{cases} x_{2n-1} & = f^{-1}\left(dJ_{2n} f(x_{-3}) + (cJ_{2n} + dJ_{2n-1})\, g(y_{-2}) + (J_{2n+2} - aJ_{2n+1})\, f(x_{-1})\right. \\ & \left. + J_{2n+1} g(y_0)\right), n \in \mathbb{N}, \\ x_{2n} & = f^{-1}\left(dJ_{2n+1} g(y_{-3}) + (cJ_{2n+1} + dJ_{2n})\, f(x_{-2}) + (J_{2n+3} - aJ_{2n+2})\, g(y_{-1})\right. \\ & \left. + J_{2n+2} f(x_0)\right), n \in \mathbb{N}_0, \end{cases} \tag{2.2}$$

$$\begin{cases} y_{2n-1} & = g^{-1}\left(dJ_{2n} g(y_{-3}) + (cJ_{2n} + dJ_{2n-1})\, f(x_{-2}) + (J_{2n+2} - aJ_{2n+1})\, g(y_{-1})\right. \\ & \left. + J_{2n+1} f(x_0)\right), n \in \mathbb{N}, \\ y_{2n} & = g^{-1}\left(dJ_{2n+1} f(x_{-3}) + (cJ_{2n+1} + dJ_{2n})\, g(y_{-2}) + (J_{2n+3} - aJ_{2n+2})\, f(x_{-1})\right. \\ & \left. + J_{2n+2} g(y_0)\right), n \in \mathbb{N}_0. \end{cases} \tag{2.3}$$

Proof. Since the functions f, g are "$1-1$", then from (2.1) we get

$$\begin{cases} f(x_{n+1}) = ag(y_n) + bf(x_{n-1}) + cg(y_{n-2}) + df(x_{n-3}), \\ g(y_{n+1}) = af(x_n) + bg(y_{n-1}) + cf(x_{n-2}) + dg(y_{n-3}), \ \ n \in \mathbb{N}_0. \end{cases} \tag{2.4}$$

By the change of variables

$$X_n = f(x_n), \qquad Y_n = g(y_n), n \geq -3, \tag{2.5}$$

system (2.4) is transformed to the following one

$$\begin{cases} X_{n+1} = aY_n + bX_{n-1} + cY_{n-2} + dX_{n-3}, \\ Y_{n+1} = aX_n + bY_{n-1} + cX_{n-2} + dY_{n-3}, \ \ n \in \mathbb{N}_0. \end{cases} \tag{2.6}$$

Clearly (2.6) is in the form of system (1.7), by Lemma (2), we obtain the following representation of solutions

$$\begin{cases} X_{2n-1} = dJ_{2n}X_{-3} + (cJ_{2n} + dJ_{2n-1})Y_{-2} + (J_{2n+2} - aJ_{2n+1})X_{-1} + J_{2n+1}Y_0, n \in \mathbb{N}, \\ X_{2n} = dJ_{2n+1}Y_{-3} + (cJ_{2n+1} + dJ_{2n})X_{-2} + (J_{2n+3} - aJ_{2n+2})Y_{-1} + J_{2n+2}X_0, n \in \mathbb{N}_0, \end{cases} \tag{2.7}$$

$$\begin{cases} Y_{2n-1} = dJ_{2n}Y_{-3} + (cJ_{2n} + dJ_{2n-1})X_{-2} + (J_{2n+2} - aJ_{2n+1})Y_{-1} + J_{2n+1}X_0, n \in \mathbb{N}, \\ Y_{2n} = dJ_{2n+1}X_{-3} + (cJ_{2n+1} + dJ_{2n})Y_{-2} + (J_{2n+3} - aJ_{2n+2})X_{-1} + J_{2n+2}Y_0, n \in \mathbb{N}_0. \end{cases} \tag{2.8}$$

Now, by (2.5) we get that

$$\begin{cases} x_{2n-1} & = f^{-1}\left[dJ_{2n}f(x_{-3}) + (cJ_{2n} + dJ_{2n-1})g(y_{-2}) + (J_{2n+2} - aJ_{2n+1})f(x_{-1}) \right. \\ & \left. + J_{2n+1}g(y_0)\right], n \in \mathbb{N}, \\ x_{2n} & = f^{-1}\left[dJ_{2n+1}g(y_{-3}) + (cJ_{2n+1} + dJ_{2n})f(x_{-2}) + (J_{2n+3} - aJ_{2n+2})g(y_{-1}) \right. \\ & \left. + J_{2n+2}f(x_0)\right], n \in \mathbb{N}_0, \end{cases} \tag{2.9}$$

$$\begin{cases} y_{2n-1} & = g^{-1}\left[dJ_{2n}g(y_{-3}) + (cJ_{2n} + dJ_{2n-1})f(x_{-2}) + (J_{2n+2} - aJ_{2n+1})g(y_{-1}) \right. \\ & \left. + J_{2n+1}f(x_0)\right], n \in \mathbb{N}, \\ y_{2n} & = g^{-1}\left[dJ_{2n+1}f(x_{-3}) + (cJ_{2n+1} + dJ_{2n})g(y_{-2}) + (J_{2n+3} - aJ_{2n+2})f(x_{-1}) \right. \\ & \left. + J_{2n+2}g(y_0)\right], n \in \mathbb{N}_0. \end{cases} \tag{2.10}$$

□

Remark 1. Moreover, if $g \equiv f$ and $y_{-i} = x_{-i}$, $i = \overline{0,3}$ then, the system (2.1) will be the equation

$$x_{n+1} = f^{-1}\left(af(x_n) + bf(x_{n-1}) + cf(x_{n-2}) + df(x_{n-3})\right) \tag{2.11}$$

and then the representation of the well-defined solutions are given by

$$\begin{cases} x_{2n-1} & = f^{-1}\left[dJ_{2n}f(x_{-3}) + (cJ_{2n} + dJ_{2n-1})f(x_{-2}) + (J_{2n+2} - aJ_{2n+1})f(x_{-1}) \right. \\ & \left. + J_{2n+1}f(x_0)\right], n \in \mathbb{N}, \\ x_{2n} & = f^{-1}\left[dJ_{2n+1}f(x_{-3}) + (cJ_{2n+1} + dJ_{2n})f(x_{-2}) + (J_{2n+3} - aJ_{2n+2})f(x_{-1}) \right. \\ & \left. + J_{2n+2}f(x_0)\right], n \in \mathbb{N}_0. \end{cases} \tag{2.12}$$

In [15], Stevic studied the equation (2.11).

Now as applications of theorem 1, we give the following examples.

Example 1. Let

$$f(t)=t^{2j+1}, g(t)=t^{2k+1}, \quad j,k\in\mathbb{N}_0. \tag{2.13}$$

Then, $D_f=D_g=\mathbb{R}$, clearly the functions f and g are "$1-1$" continuous functions on $\mathbb{R}$ and the system (2.1) becomes

$$\begin{cases} x_{n+1}=\left[ay_n^{2k+1}+bx_{n-1}^{2j+1}+cy_{n-2}^{2k+1}+dx_{n-3}^{2j+1}\right]^{\frac{1}{2j+1}}, \\ y_{n+1}=\left[ax_n^{2j+1}+by_{n-1}^{2k+1}+cx_{n-2}^{2j+1}+dy_{n-3}^{2k+1}\right]^{\frac{1}{2k+1}}, \ n\in\mathbb{N}_0. \end{cases} \tag{2.14}$$

Then from (2.2) and (2.3), we obtain that general solution of the equation (2.14) is

$$\begin{cases} x_{2n-1} &= \Big[dJ_{2n}x_{-3}^{2j+1}+(cJ_{2n}+dJ_{2n-1})y_{-2}^{2k+1}+(J_{2n+2}-aJ_{2n+1})x_{-1}^{2j+1} \\ &\quad +J_{2n+1}y_0^{2k+1}\Big]^{\frac{1}{2j+1}}, n\in\mathbb{N}, \\ x_{2n} &= \Big[dJ_{2n+1}y_{-3}^{2k+1}+(cJ_{2n+1}+dJ_{2n})x_{-2}^{2j+1}+(J_{2n+3}-aJ_{2n+2})y_{-1}^{2k+1} \\ &\quad +J_{2n+2}x_0^{2j+1}\Big]^{\frac{1}{2j+1}}, n\in\mathbb{N}_0, \end{cases} \tag{2.15}$$

$$\begin{cases} y_{2n-1} &= \Big[dJ_{2n}y_{-3}^{2k+1}+(cJ_{2n}+dJ_{2n-1})x_{-2}^{2j+1}+(J_{2n+2}-aJ_{2n+1})y_{-1}^{2k+1} \\ &\quad +J_{2n+1}x_0^{2j+1}\Big]^{\frac{1}{2k+1}}, n\in\mathbb{N}, \\ y_{2n} &= \Big[dJ_{2n+1}x_{-3}^{2j+1}+(cJ_{2n+1}+dJ_{2n})y_{-2}^{2k+1}+(J_{2n+3}-aJ_{2n+2})x_{-1}^{2j+1} \\ &\quad +J_{2n+2}y_0^{2k+1}\Big]^{\frac{1}{2k+1}}, n\in\mathbb{N}_0. \end{cases} \tag{2.16}$$

Example 2. Let

$$f(t)=\frac{1}{t^{2j+1}}, g(t)=\frac{1}{t^{2k+1}}, \quad j,k\in\mathbb{N}_0. \tag{2.17}$$

Then, $D_f=D_g=\mathbb{R}-\{0\}$, clearly the functions f and g are "$1-1$" continuous functions on $\mathbb{R}-\{0\}$ and the system (2.1) becomes

$$\begin{cases} x_{n+1}=\left[\frac{a}{y_n^{2k+1}}+\frac{b}{x_{n-1}^{2j+1}}+\frac{c}{y_{n-2}^{2k+1}}+\frac{d}{x_{n-3}^{2j+1}}\right]^{\frac{-1}{2j+1}}, \\ y_{n+1}=\left[\frac{a}{x_n^{2j+1}}+\frac{b}{y_{n-1}^{2k+1}}+\frac{c}{x_{n-2}^{2j+1}}+\frac{d}{y_{n-3}^{2k+1}}\right]^{\frac{-1}{2k+1}}, \ n\in\mathbb{N}_0, \end{cases} \tag{2.18}$$

or equivalently

$$\begin{cases} x_{n+1} = \left[\dfrac{y_n^{2k+1} x_{n-1}^{2j+1} y_{n-2}^{2k+1} x_{n-3}^{2j+1}}{a x_{n-1}^{2j+1} y_{n-2}^{2k+1} x_{n-3}^{2j+1} + b y_n^{2k+1} y_{n-2}^{2k+1} x_{n-3}^{2j+1} + c y_n^{2k+1} x_{n-1}^{2j+1} x_{n-3}^{2j+1} + d y_n^{2k+1} x_{n-1}^{2j+1} y_{n-2}^{2k+1}} \right]^{\frac{1}{2j+1}}, \\ y_{n+1} = \left[\dfrac{x_n^{2j+1} y_{n-1}^{2k+1} x_{n-2}^{2j+1} y_{n-3}^{2k+1}}{a y_{n-1}^{2k+1} x_{n-2}^{2j+1} y_{n-3}^{2k+1} + b x_n^{2j+1} x_{n-2}^{2j+1} y_{n-3}^{2k+1} + c x_n^{2j+1} y_{n-1}^{2k+1} y_{n-3}^{2k+1} + d x_n^{2j+1} y_{n-1}^{2k+1} x_{n-2}^{2j+1}} \right]^{\frac{1}{2k+1}}, \quad n \in \mathbb{N}_0. \end{cases} \tag{2.19}$$

Then from (2.2) and (2.3), we obtain that general solution of system (2.19) is

$$\begin{cases} x_{2n-1} &= \Big[d J_{2n} x_{-3}^{-(2j+1)} + (c J_{2n} + d J_{2n-1}) y_{-2}^{-(2k+1)} + (J_{2n+2} - a J_{2n+1}) x_{-1}^{-(2j+1)} \\ &\quad + J_{2n+1} y_0^{-(2k+1)} \Big]^{-\frac{1}{2j+1}}, n \in \mathbb{N}, \\ x_{2n} &= \Big[d J_{2n+1} y_{-3}^{-(2k+1)} + (c J_{2n+1} + d J_{2n}) x_{-2}^{-(2j+1)} + (J_{2n+3} - a J_{2n+2}) y_{-1}^{-(2k+1)} \\ &\quad + J_{2n+2} x_0^{-(2j+1)} \Big]^{-\frac{1}{2j+1}}, n \in \mathbb{N}_0, \end{cases} \tag{2.20}$$

$$\begin{cases} y_{2n-1} &= \Big[d J_{2n} y_{-3}^{-(2k+1)} + (c J_{2n} + d J_{2n-1}) x_{-2}^{-(2j+1)} + (J_{2n+2} - a J_{2n+1}) y_{-1}^{-(2k+1)} \\ &\quad + J_{2n+1} x_0^{-(2j+1)} \Big]^{-\frac{1}{2k+1}}, n \in \mathbb{N}, \\ y_{2n} &= \Big[d J_{2n+1} x_{-3}^{-(2j+1)} + (c J_{2n+1} + d J_{2n}) y_{-2}^{-(2k+1)} + (J_{2n+3} - a J_{2n+2}) x_{-1}^{-(2j+1)} \\ &\quad + J_{2n+2} y_0^{-(2k+1)} \Big]^{-\frac{1}{2k+1}}, n \in \mathbb{N}_0. \end{cases} \tag{2.21}$$

If $j = k = 0$, then system (2.18) becomes

$$\begin{cases} x_{n+1} = \dfrac{y_n x_{n-1} y_{n-2} x_{n-3}}{a x_{n-1} y_{n-2} x_{n-3} + b y_n y_{n-2} x_{n-3} + c y_n x_{n-1} x_{n-3} + d y_n x_{n-1} y_{n-2}}, \\ y_{n+1} = \dfrac{x_n y_{n-1} x_{n-2} y_{n-3}}{a y_{n-1} x_{n-2} y_{n-3} + b x_n x_{n-2} y_{n-3} + c x_n y_{n-1} y_{n-3} + d x_n y_{n-1} x_{n-2}}, \quad n \in \mathbb{N}_0. \end{cases} \tag{2.22}$$

The form of the well-defined solutions of (2.22), can be obtained by putting $j = k = 0$, in the formulas of the solutions of (2.18). The solutions of the equation, see [15]

$$x_{n+1} = \frac{x_n x_{n-1} x_{n-2} x_{n-3}}{a x_{n-1} x_{n-2} x_{n-3} + b x_n x_{n-2} x_{n-3} + c x_n x_{n-1} x_{n-3} + d x_n x_{n-1} x_{n-2}}, \quad n \in \mathbb{N}_0, \tag{2.23}$$

can be obtained from the solutions of (2.22) by taking $y_{-i} = x_{-i}$, $i = 0, 1, 2, 3$.

2.2. *Particular cases of system* (2.1)

2.2.1. *The case* $d = 0$ *and* $c \neq 0$

In this case the system (2.1) takes the form:

$$\begin{cases} x_{n+1} = f^{-1}\left(ag(y_n) + bf(x_{n-1}) + cg(y_{n-2})\right), \\ y_{n+1} = g^{-1}\left(af(x_n) + bg(y_{n-1}) + cf(x_{n-2})\right), \quad n \in \mathbb{N}_0. \end{cases} \tag{2.24}$$

Using the change of variables (2.5), with $n \geq -2$, we get the third order linear system

$$X_{n+1} = aY_n + bX_{n-1} + cY_{n-2}, Y_{n+1} = aX_n + bY_{n-1} + cX_{n-2}, n \geq -2. \tag{2.25}$$

Consider the sequence $\left(\widetilde{J}_n\right)_{n\geq 0}$ defined by

$$\widetilde{J}_{n+3} = a\widetilde{J}_{n+2} + b\widetilde{J}_{n+1} + c\widetilde{J}_n, \quad n \in \mathbb{N}_0, \tag{2.26}$$

and the special initial values

$$\widetilde{J}_0 = 0, \widetilde{J}_1 = 1, \widetilde{J}_2 = a.$$

The sequence $\left(\widetilde{J}_n\right)_{n\geq 0}$ is obtained from the sequence $(J_n)_{n\geq 0}$ defined by (1.3):

$$J_{n+4} = aJ_{n+3} + bJ_{n+2} + cJ_{n+1} + dJ_n, J_0 = 0, \quad J_1 = 0, \quad J_2 = 1 \text{ and } J_3 = a, n \in \mathbb{N}_0.$$

For $d = 0$, we obtain

$$J_{n+4} = aJ_{n+3} + bJ_{n+2} + cJ_{n+1}.$$

Putting

$$\widetilde{J}_n = J_{n+1}, n \in \mathbb{N}_0,$$

we get the sequence (2.26). Noting that in this case, the corresponding sequences $(R_n)_{n\geq 0}$, $(S_n)_{n\geq 0}$ will be

$$R_{n+1} = aR_n + bR_{n-1} + cR_{n-2}, S_{n+1} = -aS_n + bS_{n-1} - cS_{n-2}, n \in \mathbb{N}_0,$$

with the initial values R_0, R_{-1}, R_{-2}, S_0, S_{-1}, S_{-2}. The formulas of the solutions of these equations are expressed using the sequence $\left(\widetilde{J}_n\right)_{n\geq 0}$, see [2].

The formulas of the solutions of (2.25) and (2.24), can be obtaining from those of solutions of (1.7) and solutions of (2.1) by changing J_n by $\widetilde{J}_{n-1}$.

In summary we have the following result.

Corollary 1. *Let* $\{x_n, y_n\}_{n\geq -2}$ *be a well-defined solution of system* (2.24)*, then*

$$x_{2n-1} = f^{-1}\left[c\widetilde{J}_{2n-1}g(y_{-2}) + \left(\widetilde{J}_{2n+1} - a\widetilde{J}_{2n}\right) f(x_{-1}) + \widetilde{J}_{2n}g(y_0)\right], n \in \mathbb{N},$$

$$x_{2n} = f^{-1}\left[c\widetilde{J}_{2n}f(x_{-2}) + \left(\widetilde{J}_{2n+2} - a\widetilde{J}_{2n+1}\right) g(y_{-1}) + \widetilde{J}_{2n+1}f(x_0)\right], n \in \mathbb{N}_0,$$

$$y_{2n-1} = g^{-1}\left[c\widetilde{J}_{2n-1}f(x_{-2}) + \left(\widetilde{J}_{2n+1} - a\widetilde{J}_{2n}\right) g(y_{-1}) + \widetilde{J}_{2n}f(x_0)\right], n \in \mathbb{N},$$

$$y_{2n}=g^{-1}\left[c\widetilde{J}_{2n}g(y_{-2})+\left(\widetilde{J}_{2n+2}-a\widetilde{J}_{2n+1}\right)f(x_{-1})+\widetilde{J}_{2n+1}g(y_0)\right],n\in\mathbb{N}_0.$$

Remark 2. If $g\equiv f$ and $y_{-i}=x_{-i}$, $i=0,1,2$ then, system (2.24) becomes

$$x_{n+1}=f^{-1}\left[af(x_n)+bf(x_{n-1})+cf(x_{n-2})\right] \tag{2.27}$$

and by Corollary 1, the every well defined solution is given by

$$\begin{cases} x_{2n-1}=f^{-1}\left[c\widetilde{J}_{2n-1}f(x_{-2})+\left(\widetilde{J}_{2n+1}-a\widetilde{J}_{2n}\right)f(x_{-1})+\widetilde{J}_{2n}f(x_0)\right],n\in\mathbb{N}, \\ x_{2n}=f^{-1}\left[c\widetilde{J}_{2n}f(x_{-2})+\left(\widetilde{J}_{2n+2}-a\widetilde{J}_{2n+1}\right)f(x_{-1})+\widetilde{J}_{2n+1}f(x_0)\right],n\in\mathbb{N}_0, \end{cases} \tag{2.28}$$

which can written in a unified form as

$$x_n=f^{-1}\left[c\widetilde{J}_nf(x_{-2})+\left(\widetilde{J}_{n+2}-a\widetilde{J}_{n+1}\right)f(x_{-1})+\widetilde{J}_{n+1}f(x_0)\right],n\in\mathbb{N}_0. \tag{2.29}$$

Noting again that this equation, was studied by Stevic in [15].

2.2.2. *Case* $d=0,c\neq 0$ *and* $a=0$

In this case we get the system

$$\begin{cases} x_{n+1}=f^{-1}\left[bf(x_{n-1})+cg(y_{n-2})\right], \\ y_{n+1}=g^{-1}\left[bg(y_{n-1})+cf(x_{n-2})\right], \ n\in\mathbb{N}_0. \end{cases} \tag{2.30}$$

Here, $\left(\widetilde{J}_n\right)_{n\geq 0}$ will be the sequence defined by

$$\mathcal{P}_{n+3}=b\mathcal{P}_{n+1}+c\mathcal{P}_n,\quad n\in\mathbb{N}_0, \tag{2.31}$$

and the special initial values

$$\mathcal{P}_0=0,\qquad \mathcal{P}_1=1 \text{ and } \mathcal{P}_2=0, \tag{2.32}$$

so, the solutions are expressed in terms of $(\mathcal{P})_{n\geq 0}$ and are given by

$$\begin{aligned} x_{2n-1}&=f^{-1}\left[c\mathcal{P}_{2n-1}g(y_{-2})+\mathcal{P}_{2n+1}f(x_{-1})+\mathcal{P}_{2n}g(y_0)\right],n\in\mathbb{N}, \\ x_{2n}&=f^{-1}\left[c\mathcal{P}_{2n}f(x_{-2})+\mathcal{P}_{2n+2}g(y_{-1})+\mathcal{P}_{2n+1}f(x_0)\right],n\in\mathbb{N}_0, \\ y_{2n-1}&=g^{-1}\left[c\mathcal{P}_{2n-1}f(x_{-2})+\mathcal{P}_{2n+1}g(y_{-1})+\mathcal{P}_{2n}f(x_0)\right],n\in\mathbb{N}, \\ y_{2n}&=g^{-1}\left[c\mathcal{P}_{2n}g(y_{-2})+\mathcal{P}_{2n+2}f(x_{-1})+\mathcal{P}_{2n+1}g(y_0)\right],n\in\mathbb{N}_0, \end{aligned}$$

for the system (2.30) and by

$$x_n=f^{-1}\left[c\mathcal{P}_nf(x_{-2})+\mathcal{P}_{n+2}f(x_{-1})+\mathcal{P}_{n+1}f(x_0)\right],n\in\mathbb{N}_0,$$

for its one dimensional version, that is the equation

$$x_{n+1}=f^{-1}\left(bf(x_{n-1})+cf(x_{n-2})\right).$$

If $b\neq 0$, $(\mathcal{P})_{n\geq 0}$ will be a generalized Padovan sequence and if $b=c=1$, then $(\mathcal{P}_n)_{n\geq 0}$ will be the famous Padovan sequence.
Noting that system (2.30) generalize for example the works of [5] and [21].

2.2.3. *Case $c = d = 0$ and $b \neq 0$*

In this case, we get the system

$$\begin{cases} x_{n+1} = f^{-1}\left[ag(y_n) + bf(x_{n-1})\right], \\ y_{n+1} = g^{-1}\left[af(x_n) + bg(y_{n-1})\right], \ n \in \mathbb{N}_0. \end{cases} \tag{2.33}$$

By the same philosophy, we obtain the sequence $\left(\widetilde{F}_n\right)_{n\geq 0} = \left(\widetilde{J}_{n+1}\right)_{n\geq 0}$, defined by

$$\widetilde{F}_{n+2} = a\widetilde{F}_{n+1} + b\widetilde{F}_n, \widetilde{F}_0 = 1, \widetilde{F}_1 = a, n \in \mathbb{N}_0,$$

and the solutions of (2.33) and its one dimensional version, are obtained from the solutions of (2.24) and (2.27), by writing $\widetilde{F}_{n-1}$ instead of $\widetilde{J}_n$. If $a \neq 0$, $\left(\widetilde{F}_n\right)_{n\geq 0}$ is a generalized Fibonacci sequence and if $a = b = 1$, $\left(\widetilde{F}_n\right)_{n\geq 0}$ will be the famous Fibonacci sequence.
System (2.33) and its one dimensional versions, generalize for example the works of [17], [18].

2.2.4. *Case $b = c = d = 0$ and $a \neq 0$*

In this case, we get the system

$$x_{n+1} = f^{-1}\left(ag(y_n)\right), y_{n+1} = g^{-1}\left(af(x_n)\right), \ n \in \mathbb{N}_0. \tag{2.34}$$

Using the fact that f, g are one to one functions, and the change of variables

$$X_n = f(x_n), Y_n = g(y_n), n \geq 0$$

the system (2.34), will be

$$X_{n+1} = aY_n, Y_{n+1} = aX_n, \ n \in \mathbb{N}_0.$$

So,

$$X_{2n} = a^{2n}X_0, Y_{2n} = a^{2n}Y_0, X_{2n+1} = a^{2n+1}Y_0, Y_{2n+1} = a^{2n+1}X_0, n \in \mathbb{N}_0.$$

Hence

$$x_{2n} = f^{-1}(a^{2n}f(x_0)), y_{2n} = g^{-1}(a^{2n}g(y_0)), n \in \mathbb{N}_0$$

and

$$x_{2n+1} = f^{-1}(a^{2n+1}g(y_0)), y_{2n+1} = g^{-1}(a^{2n+1}f(x_0)), n \in \mathbb{N}_0.$$

3. Solutions of a second general system of difference equations

In this part, we are interested in the following system of difference equations given by

$$\begin{cases} x_{n+1} = f^{-1}\left(a + \frac{b}{g(y_n)} + \frac{c}{g(y_n)f(x_{n-1})} + \frac{d}{g(y_n)f(x_{n-1})g(y_{n-2})}\right), \\ y_{n+1} = g^{-1}\left(a + \frac{b}{f(x_n)} + \frac{c}{f(x_n)g(y_{n-1})} + \frac{d}{f(x_n)g(y_{n-1})f(x_{n-2})}\right), \end{cases} \tag{3.1}$$

where $n \in \mathbb{N}_0$, $f, g : D \longrightarrow \mathbb{R}$ are continuous functions, with $D = D_f = D_g$, that is f and g have the same domain, in addition we assume that f, g are "$1-1$" on $D \subseteq \mathbb{R}$, the initial values x_{-i}, y_{-i}, $i = 0,1,2$, are arbitrary real numbers in D and the parameters a, b, c and d are arbitrary real numbers.

Definition 2. A solution $\{x_n, y_n\}_{n\geq -2}$ of system (3.1) is said to be well-defined if for all $n \in \mathbb{N}_0$, we have

$$a + \frac{b}{g(y_n)} + \frac{c}{g(y_n)f(x_{n-1})} + \frac{d}{g(y_n)f(x_{n-1})g(y_{n-2})} \in D_{f^{-1}},$$

and

$$a + \frac{b}{f(x_n)} + \frac{c}{f(x_n)g(y_{n-1})} + \frac{d}{f(x_n)g(y_{n-1})f(x_{n-2})} \in D_{g^{-1}}.$$

We solve in closed form (3.1) and we investigated particular cases of it. The philosophy is the same as in the previous section (2), so we will brief in presenting our formulas of the solutions.

3.1. *Explicit formulas of solutions of system* (3.1) *with* $d \neq 0$

The following result is devoted to the formulas of well-defined solutions of (3.1).

Theorem 2. *Let* $\{x_n, y_n\}_{n\geq -2}$ *be a well-defined solution of system* (3.1). *Then, for all* $n \in \mathbb{N}_0$ *we have*

$$\begin{aligned}
x_{2n+1} &= f^{-1}[(dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\, g(y_{-2}) + (J_{2n+4} - aJ_{2n+3})\, f(x_{-1})g(y_{-2}) \\
&\quad + J_{2n+3}g(y_0)f(x_{-1})g(y_{-2}))(dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n})\, g(y_{-2}) \\
&\quad + (J_{2n+3} - aJ_{2n+2})\, f(x_{-1})g(y_{-2}) + J_{2n+2}g(y_0)f(x_{-1})g(y_{-2}))^{-1}], \\
x_{2n+2} &= f^{-1}[(dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2})\, f(x_{-2}) + (J_{2n+5} - aJ_{2n+4})\, g(y_{-1})f(x_{-2}) \\
&\quad + J_{2n+4}f(x_0)g(y_{-1})f(x_{-2}))(dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\, f(x_{-2}) \\
&\quad + (J_{2n+4} - aJ_{2n+3})\, g(y_{-1})f(x_{-2}) + J_{2n+3}f(x_0)g(y_{-1})f(x_{-2}))^{-1}], \\
y_{2n+1} &= g^{-1}[(dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\, f(x_{-2}) + (J_{2n+4} - aJ_{2n+3})\, g(y_{-1})f(x_{-2}) \\
&\quad + J_{2n+3}f(x_0)g(y_{-1})f(x_{-2}))(dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n})\, f(x_{-2}) \\
&\quad + (J_{2n+3} - aJ_{2n+2})\, g(y_{-1})f(x_{-2}) + J_{2n+2}f(x_0)g(y_{-1})f(x_{-2}))^{-1}], \\
y_{2n+2} &= g^{-1}[(dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2})\, g(y_{-2}) + (J_{2n+5} - aJ_{2n+4})\, f(x_{-1})g(y_{-2}) \\
&\quad + J_{2n+4}g(y_0)f(x_{-1})g(y_{-2}))(dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\, g(y_{-2}) \\
&\quad + (J_{2n+4} - aJ_{2n+3})\, f(x_{-1})g(y_{-2}) + J_{2n+3}g(y_0)f(x_{-1})g(y_{-2}))^{-1}].
\end{aligned}$$

Proof. Using the fact that the functions f, g are one to one and using the change of variables (2.5), with $n \geq -2$, the system (3.1) becomes

$$\begin{cases} X_{n+1} = a + \dfrac{b}{Y_n} + \dfrac{c}{Y_n X_{n-1}} + \dfrac{d}{Y_n X_{n-1} Y_{n-2}}, \\ Y_{n+1} = a + \dfrac{b}{X_n} + \dfrac{c}{X_n Y_{n-1}} + \dfrac{d}{X_n Y_{n-1} X_{n-2}}, \quad n \in \mathbb{N}_0, \end{cases} \tag{3.2}$$

or equivalently,

$$\begin{cases} X_{n+1} = \dfrac{aY_nX_{n-1}Y_{n-2} + bX_{n-1}Y_{n-2} + cY_{n-2} + d}{Y_nX_{n-1}Y_{n-2}}, \\ Y_{n+1} = \dfrac{aX_nY_{n-1}X_{n-2} + bY_{n-1}X_{n-2} + cX_{n-2} + d}{X_nY_{n-1}X_{n-2}}, \quad n \in \mathbb{N}_0. \end{cases} \tag{3.3}$$

This system was solved in [2], and for $n \in \mathbb{N}_0$, the solutions of (3.3) takes the form

$$X_{2n+1} = \frac{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})Y_{-2} + (J_{2n+4} - aJ_{2n+3})X_{-1}Y_{-2} + J_{2n+3}Y_0X_{-1}Y_{-2}}{dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n})Y_{-2} + (J_{2n+3} - aJ_{2n+2})X_{-1}Y_{-2} + J_{2n+2}Y_0X_{-1}Y_{-2}}, \tag{3.4}$$

$$X_{2n+2} = \frac{dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2})X_{-2} + (J_{2n+5} - aJ_{2n+4})Y_{-1}X_{-2} + J_{2n+4}X_0Y_{-1}X_{-2}}{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})X_{-2} + (J_{2n+4} - aJ_{2n+3})Y_{-1}X_{-2} + J_{2n+3}X_0Y_{-1}X_{-2}}, \tag{3.5}$$

$$Y_{2n+1} = \frac{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})X_{-2} + (J_{2n+4} - aJ_{2n+3})Y_{-1}X_{-2} + J_{2n+3}X_0Y_{-1}X_{-2}}{dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n})X_{-2} + (J_{2n+3} - aJ_{2n+2})Y_{-1}X_{-2} + J_{2n+2}X_0Y_{-1}X_{-2}}, \tag{3.6}$$

$$Y_{2n+2} = \frac{dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2})Y_{-2} + (J_{2n+5} - aJ_{2n+4})X_{-1}Y_{-2} + J_{2n+4}Y_0X_{-1}Y_{-2}}{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})Y_{-2} + (J_{2n+4} - aJ_{2n+3})X_{-1}Y_{-2} + J_{2n+3}Y_0X_{-1}Y_{-2}}, \tag{3.7}$$

where $(J_n)_{n\in\mathbb{N}_0}$ is the sequence defined by (1.3).

Using (2.5), (3.4), (3.5), (3.6) and (3.7), we get that for $n \in \mathbb{N}_0$, every well-defined solution of system (3.1) has the following representation

$$\begin{aligned}
x_{2n+1} &= f^{-1}[(dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\,g(y_{-2}) + (J_{2n+4} - aJ_{2n+3})\,f(x_{-1})g(y_{-2}) \\
&\quad + J_{2n+3}g(y_0)f(x_{-1})g(y_{-2}))(dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n})\,g(y_{-2}) \\
&\quad + (J_{2n+3} - aJ_{2n+2})\,f(x_{-1})g(y_{-2}) + J_{2n+2}g(y_0)f(x_{-1})g(y_{-2}))^{-1}], \\
x_{2n+2} &= f^{-1}[(dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2})\,f(x_{-2}) + (J_{2n+5} - aJ_{2n+4})\,g(y_{-1})f(x_{-2}) \\
&\quad + J_{2n+4}f(x_0)g(y_{-1})f(x_{-2}))(dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\,f(x_{-2}) \\
&\quad + (J_{2n+4} - aJ_{2n+3})\,g(y_{-1})f(x_{-2}) + J_{2n+3}f(x_0)g(y_{-1})f(x_{-2}))^{-1}], \\
y_{2n+1} &= g^{-1}[(dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\,f(x_{-2}) + (J_{2n+4} - aJ_{2n+3})\,g(y_{-1})f(x_{-2}) \\
&\quad + J_{2n+3}f(x_0)g(y_{-1})f(x_{-2}))(dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n})\,f(x_{-2}) \\
&\quad + (J_{2n+3} - aJ_{2n+2})\,g(y_{-1})f(x_{-2}) + J_{2n+2}f(x_0)g(y_{-1})f(x_{-2}))^{-1}], \\
y_{2n+2} &= g^{-1}[(dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2})\,g(y_{-2}) + (J_{2n+5} - aJ_{2n+4})\,f(x_{-1})g(y_{-2}) \\
&\quad + J_{2n+4}g(y_0)f(x_{-1})g(y_{-2}))(dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\,g(y_{-2}) \\
&\quad + (J_{2n+4} - aJ_{2n+3})\,f(x_{-1})g(y_{-2}) + J_{2n+3}g(y_0)f(x_{-1})g(y_{-2}))^{-1}].
\end{aligned}$$

□

Remark 3. (1) In [2], to solve the system (3.3), the authors used the change of variables

$$X_n = \frac{u_n}{v_{n-1}}, Y_n = \frac{v_n}{u_{n-1}}, n \geq -2,$$

to obtain the fourth linear system (1.7).

(2) When $g \equiv f$ and $y_{-i} = x_{-i}$, $i = \overline{0,2}$ then system (3.1) becomes the equation

$$x_{n+1} = f^{-1}\left[a + \frac{b}{f(x_n)} + \frac{c}{f(x_n)f(x_{n-1})} + \frac{d}{f(x_n)f(x_{n-1})f(x_{n-2})}\right], n \in \mathbb{N}_0, \quad (3.8)$$

and the form of every well-defined solution of (3.8) is given by

$$\begin{aligned}
x_{2n+1} &= f^{-1}[(dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})f(x_{-2}) + (J_{2n+4} - aJ_{2n+3})f(x_{-1})f(x_{-2}) \\
&\quad + J_{2n+3}f(x_0)f(x_{-1})f(x_{-2}))(dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n})f(x_{-2}) \\
&\quad + (J_{2n+3} - aJ_{2n+2})f(x_{-1})f(x_{-2}) + J_{2n+2}f(x_0)f(x_{-1})f(x_{-2}))^{-1}], \\
x_{2n+2} &= f^{-1}[(dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2})f(x_{-2}) + (J_{2n+5} - aJ_{2n+4})f(x_{-1})f(x_{-2}) \\
&\quad + J_{2n+4}f(x_0)f(x_{-1})f(x_{-2}))(dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})f(x_{-2}) \\
&\quad + (J_{2n+4} - aJ_{2n+3})f(x_{-1})f(x_{-2}) + J_{2n+3}f(x_0)f(x_{-1})f(x_{-2}))^{-1}],
\end{aligned}$$

which can be represented in the unified form

$$\begin{aligned}
x_{n+1} &= f^{-1}[(dJ_{n+2} + (cJ_{n+2} + dJ_{n+1})f(x_{-2}) + (J_{n+4} - aJ_{n+3})f(x_{-1})f(x_{-2}) \\
&\quad + J_{n+3}f(x_0)f(x_{-1})f(x_{-2}))(dJ_{n+1} + (cJ_{n+1} + dJ_n)f(x_{-2}) \\
&\quad + (J_{n+3} - aJ_{n+2})f(x_{-1})f(x_{-2}) + J_{n+2}f(x_0)f(x_{-1})f(x_{-2}))^{-1}].
\end{aligned}$$

Now we give some applications of Theorem 2.

Example 3. Let

$$f(t) = t^{2j+1}, g(t) = t^{2k+1}, j, k \in \mathbb{N}_0.$$

We have the functions f and g are one to one continuous functions on $\mathbb{R} = D_f = D_g$. In this case, system (3.1) becomes

$$\begin{cases}
x_{n+1} = \left[a + \frac{b}{y_n^{2k+1}} + \frac{c}{y_n^{2k+1}x_{n-1}^{2j+1}} + \frac{d}{y_n^{2k+1}x_{n-1}^{2j+1}y_{n-2}^{2k+1}}\right]^{\frac{1}{2j+1}}, \\
y_{n+1} = \left[a + \frac{b}{x_n^{2j+1}} + \frac{c}{x_n^{2j+1}y_{n-1}^{2k+1}} + \frac{d}{x_n^{2j+1}y_{n-1}^{2k+1}x_{n-2}^{2j+1}}\right]^{\frac{1}{2k+1}}, \quad n \in \mathbb{N}_0.
\end{cases} \quad (3.9)$$

Then by Theorem 2, we obtain that the solutions of system (3.9) have the following form

$$x_{2n+1} = \left[\frac{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})y_{-2}^{2k+1} + (J_{2n+4} - aJ_{2n+3})x_{-1}^{2j+1}y_{-2}^{2k+1} + J_{2n+3}y_0^{2k+1}x_{-1}^{2j+1}y_{-2}^{2k+1}}{dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n})y_{-2}^{2k+1} + (J_{2n+3} - aJ_{2n+2})x_{-1}^{2j+1}y_{-2}^{2k+1} + J_{2n+2}y_0^{2k+1}x_{-1}^{2j+1}y_{-2}^{2k+1}}\right]^{\frac{1}{2j+1}}, \quad (3.10)$$

$$x_{2n+2} = \left[\frac{dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2})x_{-2}^{2j+1} + (J_{2n+5} - aJ_{2n+4})y_{-1}^{2k+1}x_{-2}^{2j+1} + J_{2n+4}x_0^{2j+1}y_{-1}^{2k+1}x_{-2}^{2j+1}}{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})x_{-2}^{2j+1} + (J_{2n+4} - aJ_{2n+3})y_{-1}^{2k+1}x_{-2}^{2j+1} + J_{2n+3}x_0^{2j+1}y_{-1}^{2k+1}x_{-2}^{2j+1}}\right]^{\frac{1}{2j+1}}, \quad (3.11)$$

$$y_{2n+1} = \left[\frac{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})x_{-2}^{2j+1} + (J_{2n+4} - aJ_{2n+3})y_{-1}^{2k+1}x_{-2}^{2j+1} + J_{2n+3}x_0^{2j+1}y_{-1}^{2k+1}x_{-2}^{2j+1}}{dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n})x_{-2}^{2j+1} + (J_{2n+3} - aJ_{2n+2})y_{-1}^{2k+1}x_{-2}^{2j+1} + J_{2n+2}x_0^{2j+1})y_{-1}^{2k+1}x_{-2}^{2j+1}}\right]^{\frac{1}{2k+1}}, \quad (3.12)$$

$$y_{2n+2}=\left[\frac{dJ_{2n+3}+(cJ_{2n+3}+dJ_{2n+2})y_{-2}^{2k+1}+(J_{2n+5}-aJ_{2n+4})x_{-1}^{2j+1}y_{-2}^{2k+1}+J_{2n+4}y_0^{2k+1}x_{-1}^{2j+1}y_{-2}^{2k+1}}{dJ_{2n+2}+(cJ_{2n+2}+dJ_{2n+1})y_{-2}^{2k+1}+(J_{2n+4}-aJ_{2n+3})x_{-1}^{2j+1}y_{-2}^{2k+1}+J_{2n+3}y_0^{2k+1}x_{-1}^{2j+1}y_{-2}^{2k+1}}\right]^{\frac{1}{2k+1}}. \tag{3.13}$$

Example 4. Let

$$f(t)=\frac{1}{t^{2j+1}},\ g(t)=\frac{1}{t^{2k+1}},\ j,k\in\mathbb{N}_0.$$

We have the functions f and g are one to one continuous functions on $\mathbb{R}-\{0\}=D_f=D_g$. Then, system (3.1) becomes

$$\begin{cases} x_{n+1}=\left[\dfrac{1}{a+by_n^{2k+1}+cy_n^{2k+1}x_{n-1}^{2j+1}+dy_n^{2k+1}x_{n-1}^{2j+1}y_{n-2}^{2k+1}}\right]^{\frac{1}{2j+1}}, \\ y_{n+1}=\left[\dfrac{1}{a+bx_n^{2j+1}+cx_n^{2j+1}y_{n-1}^{2k+1}+dx_n^{2j+1}y_{n-1}^{2k+1}x_{n-2}^{2j+1}}\right]^{\frac{1}{2k+1}}, \quad n\in\mathbb{N}_0. \end{cases} \tag{3.14}$$

Then by Theorem 2, the solutions of system (3.14) have the following representation

$$x_{2n+1}=\left[\frac{dJ_{2n+1}y_0^{2k+1}x_{-1}^{2j+1}y_{-2}^{2k+1}+(cJ_{2n+1}+dJ_{2n})y_0^{2k+1}x_{-1}^{2j+1}+(J_{2n+3}-aJ_{2n+2})y_0^{2k+1}+J_{2n+2}}{dJ_{2n+2}y_0^{2k+1}x_{-1}^{2j+1}y_{-2}^{2k+1}+(cJ_{2n+2}+dJ_{2n+1})y_0^{2k+1}x_{-1}^{2j+1}+(J_{2n+4}-aJ_{2n+3})y_0^{2k+1}+J_{2n+3}}\right]^{\frac{1}{2j+1}}, \tag{3.15}$$

$$x_{2n+2}=\left[\frac{dJ_{2n+2}x_0^{2j+1}y_{-1}^{2k+1}x_{-2}^{2j+1}+(cJ_{2n+2}+dJ_{2n+1})x_0^{2j+1}y_{-1}^{2k+1}+(J_{2n+4}-aJ_{2n+3})x_0^{2j+1}+J_{2n+3}}{dJ_{2n+3}x_0^{2j+1}y_{-1}^{2k+1}x_{-2}^{2j+1}+(cJ_{2n+3}+dJ_{2n+2})x_0^{2j+1}y_{-1}^{2k+1}+(J_{2n+5}-aJ_{2n+4})x_0^{2j+1}+J_{2n+4}}\right]^{\frac{1}{2j+1}}, \tag{3.16}$$

$$y_{2n+1}=\left[\frac{dJ_{2n+1}x_0^{2j+1}y_{-1}^{2k+1}x_{-2}^{2j+1}+(cJ_{2n+1}+dJ_{2n})x_0^{2j+1}y_{-1}^{2k+1}+(J_{2n+3}-aJ_{2n+2})x_0^{2j+1}+J_{2n+2}}{dJ_{2n+2}x_0^{2j+1}y_{-1}^{2k+1}x_{-2}^{2j+1}+(cJ_{2n+2}+dJ_{2n+1})x_0^{2j+1}y_{-1}^{2k+1}+(J_{2n+4}-aJ_{2n+3})x_0^{2j+1}+J_{2n+3}}\right]^{\frac{1}{2k+1}}, \tag{3.17}$$

$$y_{2n+2}=\left[\frac{dJ_{2n+2}y_0^{2k+1}x_{-1}^{2j+1}y_{-2}^{2k+1}+(cJ_{2n+2}+dJ_{2n+1})y_0^{2k+1}x_{-1}^{2j+1}+(J_{2n+4}-aJ_{2n+3})y_0^{2k+1}+J_{2n+3}}{dJ_{2n+3}y_0^{2k+1}x_{-1}^{2j+1}y_{-2}^{2k+1}+(cJ_{2n+3}+dJ_{2n+2})y_0^{2k+1}x_{-1}^{2j+1}+(J_{2n+5}-aJ_{2n+4})y_0^{2k+1}+J_{2n+4}}\right]^{\frac{1}{2k+1}}. \tag{3.18}$$

3.1.1. *The system* $x_{n+1}=\frac{1}{a+by_n+cy_nx_{n-1}+dy_nx_{n-1}y_{n-2}}$,

$$y_{n+1}=\frac{1}{a+bx_n+cx_ny_{n-1}+dx_ny_{n-1}x_{n-2}}$$

Here we will focus our study on the system of difference equations

$$\begin{cases} x_{n+1}=\dfrac{1}{a+by_n+cy_nx_{n-1}+dy_nx_{n-1}y_{n-2}}, \\ y_{n+1}=\dfrac{1}{a+bx_n+cx_ny_{n-1}+dx_ny_{n-1}x_{n-2}}, \end{cases} \quad n\in\mathbb{N}_0, \tag{3.19}$$

which is a particular case of system (3.14) with $j=k=0$. Noting that system (3.19), generalize the studies in [12], [11], [13]. Then, putting $j=k=0$ in the formulas of well-defined solutions of system (3.14), we obtain the following result.

Corollary 2. *Let* $\{x_n,y_,\}_{n\geq-2}$ *be a well-defined solution of* (3.19)*, then for* $n\in\mathbb{N}_0$*, we have*

$$x_{2n+1}=\frac{dJ_{2n+1}y_0x_{-1}y_{-2}+(cJ_{2n+1}+dJ_{2n})y_0x_{-1}+(J_{2n+3}-aJ_{2n+2})y_0+J_{2n+2}}{dJ_{2n+2}y_0x_{-1}y_{-2}+(cJ_{2n+2}+dJ_{2n+1})y_0x_{-1}+(J_{2n+4}-aJ_{2n+3})y_0+J_{2n+3}},$$

$$x_{2n+2}=\frac{dJ_{2n+2}x_0y_{-1}x_{-2}+(cJ_{2n+2}+dJ_{2n+1})x_0y_{-1}+(J_{2n+4}-aJ_{2n+3})x_0+J_{2n+3}}{dJ_{2n+3}x_0y_{-1}x_{-2}+(cJ_{2n+3}+dJ_{2n+2})x_0y_{-1}+(J_{2n+5}-aJ_{2n+4})x_0+J_{2n+4}},$$
$$y_{2n+1}=\frac{dJ_{2n+1}x_0y_{-1}x_{-2}+(cJ_{2n+1}+dJ_{2n})x_0y_{-1}+(J_{2n+3}-aJ_{2n+2})x_0+J_{2n+2}}{dJ_{2n+2}x_0y_{-1}x_{-2}+(cJ_{2n+2}+dJ_{2n+1})x_0y_{-1}+(J_{2n+4}-aJ_{2n+3})x_0+J_{2n+3}},$$
$$y_{2n+2}=\frac{dJ_{2n+2}y_0x_{-1}y_{-2}+(cJ_{2n+2}+dJ_{2n+1})y_0x_{-1}+(J_{2n+4}-aJ_{2n+3})y_0+J_{2n+3}}{dJ_{2n+3}y_0x_{-1}y_{-2}+(cJ_{2n+3}+dJ_{2n+2})y_0x_{-1}+(J_{2n+5}-aJ_{2n+4})y_0+J_{2n+4}}.$$

Moreover, if we choose $a=b=c=d=1$, the sequence $(J_n)_{n=0}^{+\infty}$ will be nothing other than the famous Tetranacci sequence defined for $n\in\mathbb{N}_0$ by

$$T_{n+4}=T_{n+3}+T_{n+2}+T_{n+1}+T_n,\ T_0=T_1=0,\quad T_2=T_3=1, \tag{3.20}$$

and for this choice of the parameters, the solutions takes the form

$$x_{2n+1}=\frac{T_{2n+1}y_0x_{-1}y_{-2}+(T_{2n+1}+T_{2n})y_0x_{-1}+(T_{2n+3}-T_{2n+2})y_0+T_{2n+2}}{T_{2n+2}y_0x_{-1}y_{-2}+(T_{2n+2}+T_{2n+1})y_0x_{-1}+(T_{2n+4}-T_{2n+3})y_0+T_{2n+3}},$$
$$x_{2n+2}=\frac{T_{2n+2}x_0y_{-1}x_{-2}+(T_{2n+2}+T_{2n+1})x_0y_{-1}+(T_{2n+4}-T_{2n+3})x_0+T_{2n+3}}{T_{2n+3}x_0y_{-1}x_{-2}+(T_{2n+3}+T_{2n+2})x_0y_{-1}+(T_{2n+5}-T_{2n+4})x_0+T_{2n+4}},$$
$$y_{2n+1}=\frac{T_{2n+1}x_0y_{-1}x_{-2}+(T_{2n+1}+T_{2n})x_0y_{-1}+(T_{2n+3}-T_{2n+2})x_0+T_{2n+2}}{T_{2n+2}x_0y_{-1}x_{-2}+(T_{2n+2}+T_{2n+1})x_0y_{-1}+(T_{2n+4}-T_{2n+3})x_0+T_{2n+3}},$$
$$y_{2n+2}=\frac{T_{2n+2}y_0x_{-1}y_{-2}+(T_{2n+2}+T_{2n+1})y_0x_{-1}+(T_{2n+4}-T_{2n+3})y_0+T_{2n+3}}{T_{2n+3}y_0x_{-1}y_{-2}+(T_{2n+3}+T_{2n+2})y_0x_{-1}+(T_{2n+5}-T_{2n+4})y_0+T_{2n+4}}.$$

Now, we will study the stability of the equilibrium points of system (3.19) with $a=b=c=d=1$, that is the system

$$\begin{cases} x_{n+1}=f_1(x_n,x_{n-1},x_{n-2},y_n,y_{n-1},y_{n-2})=\frac{1}{1+y_n+x_{n-1}y_n+y_{n-2}x_{n-1}y_n},\\ y_{n+1}=f_2(x_n,x_{n-1},x_{n-2},y_n,y_{n-1},y_{n-2})=\frac{1}{1+x_n+y_{n-1}x_n+x_{n-2}y_{n-1}x_n}. \end{cases} \tag{3.21}$$

For the stability of the equilibrium points, we assume that the initial values are positive real numbers.

The points $\left(\frac{1}{\alpha},\frac{1}{\alpha}\right)$, $\left(\frac{1}{\beta},\frac{1}{\beta}\right)$, $\left(\frac{1}{\gamma},\frac{1}{\gamma}\right)$ and $\left(\frac{1}{\delta},\frac{1}{\delta}\right)$ are solutions of the system

$$\begin{cases} x=\frac{1}{1+y+xy+xy^2},\\ y=\frac{1}{1+x+yx+x^2y}, \end{cases}$$

where α, β, γ and δ are roots of polynomial characteristic associated to the equation (3.20), see [2]. It follows that $(\bar{x},\bar{y})=\left(\frac{1}{\alpha},\frac{1}{\alpha}\right)$ is the only equilibrium point for system (3.21) in $(0,+\infty)^2$.

For the equilibrium point $(\bar{x},\bar{y})=\left(\frac{1}{\alpha},\frac{1}{\alpha}\right)$, we have the following result.

Theorem 3. *The equilibrium point* $(\bar{x},\bar{y}) = \left(\frac{1}{\alpha},\frac{1}{\alpha}\right)$ *is globally asymptotically stable.*

Proof. The Jacobean matrix associated to the system (3.21) around the equilibrium point $(\bar{x},\bar{y}) = \left(\frac{1}{\alpha},\frac{1}{\alpha}\right)$, is given by

$$A = \begin{pmatrix} 0 & -\frac{\alpha+1}{\alpha^4} & 0 & -\frac{\alpha-1}{\alpha} & 0 & -\frac{1}{\alpha^4} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{\alpha-1}{\alpha} & 0 & -\frac{1}{\alpha^4} & 0 & -\frac{\alpha+1}{\alpha^4} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of A is

$$P(\lambda) = -h(\lambda)h(-\lambda), \text{ where } h(\lambda) = \lambda^3 - \frac{\alpha-1}{\alpha}\lambda^2 + \frac{\alpha+1}{\alpha^4}\lambda - \frac{1}{\alpha^4}. \tag{3.22}$$

Consider the two functions

$$h_1(\lambda) = \lambda^3,\ h_2(\lambda) = \frac{\alpha-1}{\alpha}\lambda^2 - \frac{\alpha+1}{\alpha^4}\lambda + \frac{1}{\alpha^4}.$$

We have

$$|h_2(\lambda)| \leq \left|\frac{\alpha-1}{\alpha}\right| + \left|\frac{\alpha+1}{\alpha^4}\right| + \left|\frac{1}{\alpha^4}\right| < 1 = |h_1(\lambda)|, \forall \lambda \in \mathbb{C} : |\lambda| = 1.$$

It follows by Rouché's Theorem, that all the roots of $h(\lambda)$ lie in the open unit disk, and so it is for the roots of $P(\lambda)$. Thus the equilibrium point $\left(\frac{1}{\alpha},\frac{1}{\alpha}\right)$ is locally asymptotically stable.
It remains to prove that

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} y_n = \frac{1}{\alpha}.$$

To this end we will, use the fact

$$\lim_{n\to\infty} \frac{T_{n+k}}{T_n} = \alpha^k, \quad \forall k \in \mathbb{N}. \tag{3.23}$$

Using the formula of the solutions, we have

$$\begin{aligned} \lim_{n\to\infty} x_{2n+1} &= \lim_{n\to\infty} \frac{T_{2n+1}y_0x_{-1}y_{-2} + (T_{2n+1}+T_{2n})y_0x_{-1} + (T_{2n+3}-T_{2n+2})y_0 + T_{2n+2}}{T_{2n+2}y_0x_{-1}y_{-2} + (T_{2n+2}+T_{2n+1})y_0x_{-1} + (T_{2n+4}-T_{2n+3})y_0 + T_{2n+3}} \\ &= \lim_{n\to\infty} \frac{\frac{T_{2n+1}}{T_{2n}}y_0x_{-1}y_{-2} + \left(\frac{T_{2n+1}}{T_{2n}}+\frac{T_{2n}}{T_{2n}}\right)y_0x_{-1} + \left(\frac{T_{2n+3}}{T_{2n}}-\frac{T_{2n+2}}{T_{2n}}\right)y_0 + \frac{T_{2n+2}}{T_{2n}}}{\frac{T_{2n+2}}{T_{2n}}y_0x_{-1}y_{-2} + \left(\frac{T_{2n+2}}{T_{2n}}+\frac{T_{2n+1}}{T_{2n}}\right)y_0x_{-1} + \left(\frac{T_{2n+4}}{T_{2n}}-\frac{T_{2n+3}}{T_{2n}}\right)y_0 + \frac{T_{2n+3}}{T_{2n}}} \end{aligned}$$

$$= \frac{\alpha y_0 x_{-1} y_{-2} + (\alpha+1) y_0 x_{-1} + (\alpha^3 - \alpha^2) y_0 + \alpha^2}{\alpha^2 y_0 x_{-1} y_{-2} + (\alpha^2 + \alpha^1) y_0 x_{-1} + (\alpha^4 - \alpha^3) y_0 + \alpha^3}$$
$$= \frac{1}{\alpha}.$$

Similarly, we show that

$$\lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} y_{2n+2} = \frac{1}{\alpha},$$

that is

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \frac{1}{\alpha}. \tag{3.24}$$

This complete the proof. □

As a consequence of Corollary (2), when choosing the initial values to satisfies $y_{-i} = x_{-i}$, $i = 0,1,2$, we get the formulas of the well-defined solutions of the equation

$$x_{n+1} = \frac{1}{a + bx_n + cx_{n-1}x_n + dx_{n-2}x_{n-1}x_n}, \; n \in \mathbb{N}_0. \tag{3.25}$$

These formulas are given in the next result.

Corollary 3. *For all $n \in \mathbb{N}_0$, the form of every well-defined solution of equation (3.25) is given by*

$$x_{2n+1} = \frac{dJ_{2n+1}x_0x_{-1}x_{-2} + (cJ_{2n+1} + dJ_{2n})x_0x_{-1} + (J_{2n+3} - aJ_{2n+2})x_0 + J_{2n+2}}{dJ_{2n+2}x_0x_{-1}x_{-2} + (cJ_{2n+2} + dJ_{2n+1})x_0x_{-1} + (J_{2n+4} - aJ_{2n+3})x_0 + J_{2n+3}},$$
$$x_{2n+2} = \frac{dJ_{2n+2}x_0x_{-1}x_{-2} + (cJ_{2n+2} + dJ_{2n+1})x_0x_{-1} + (J_{2n+4} - aJ_{2n+3})x_0 + J_{2n+3}}{dJ_{2n+3}x_0x_{-1}x_{-2} + (cJ_{2n+3} + dJ_{2n+2})x_0x_{-1} + (J_{2n+5} - aJ_{2n+4})x_0 + J_{2n+4}},$$

which can represented in a unified form as

$$x_{n+1} = \frac{dJ_{n+1}x_0x_{-1}x_{-2} + (cJ_{n+1} + dJ_n)x_0x_{-1} + (J_{n+3} - aJ_{n+2})x_0 + J_{n+2}}{dJ_{n+2}x_0x_{-1}x_{-2} + (cJ_{n+2} + dJ_{n+1})x_0x_{-1} + (J_{n+4} - aJ_{n+3})x_0 + J_{n+3}}. \tag{3.26}$$

Moreover, if $a = b = c = d = 1$, then (3.25) becomes

$$x_{n+1} = \frac{1}{1 + x_n + x_{n-1}x_n + x_{n-2}x_{n-1}x_n} \tag{3.27}$$

and the solutions are expressed in terms of Tetranacci numbers as follows

$$x_{2n+1} = \frac{T_{2n+1}x_0x_{-1}x_{-2} + (T_{2n+1} + T_{2n})x_0x_{-1} + (T_{2n+3} - T_{2n+2})x_0 + T_{2n+2}}{T_{2n+2}x_0x_{-1}x_{-2} + (T_{2n+2} + T_{2n+1})x_0x_{-1} + (T_{2n+4} - T_{2n+3})x_0 + T_{2n+3}}, \tag{3.28}$$
$$x_{2n+2} = \frac{T_{2n+2}x_0x_{-1}x_{-2} + (T_{2n+2} + T_{2n+1})x_0x_{-1} + (T_{2n+4} - T_{2n+3})x_0 + T_{2n+3}}{T_{2n+3}x_0x_{-1}x_{-2} + (T_{2n+3} + T_{2n+2})x_0x_{-1} + (T_{2n+5} - T_{2n+4})x_0 + T_{2n+4}}, \tag{3.29}$$

or equivalently in the unified form,

$$x_{n+1}=\frac{T_{n+1}x_0x_{-1}x_{-2}+(T_{n+1}+T_n)x_0x_{-1}+(T_{n+3}-T_{n+2})x_0+T_{n+2}}{T_{n+2}x_0x_{-1}x_{-2}+(T_{n+2}+T_{n+1})x_0x_{-1}+(T_{n+4}-T_{n+3})x_0+T_{n+3}}. \tag{3.30}$$

Remark 4. It is not hard to see that $\bar{x}=\dfrac{1}{\alpha}$ is the unique equilibrium point for (3.27) in $(0,+\infty)$ when taking the initial values positive real numbers.

The linearized equation of (3.27) about the equilibrium point $\bar{x}=\dfrac{1}{\alpha}$ is

$$w_{n+1}=-\frac{\alpha-1}{\alpha}w_n-\frac{\alpha+1}{\alpha^4}w_{n-1}-\frac{1}{\alpha^4}w_{n-2}. \tag{3.31}$$

We have $\frac{\alpha-1}{\alpha}$, $\frac{\alpha+1}{\alpha^4}$, $\frac{1}{\alpha^4}$ are real numbers and

$$\left|\frac{\alpha-1}{\alpha}\right|+\left|\frac{\alpha+1}{\alpha^4}\right|+\left|\frac{1}{\alpha^4}\right|<1,$$

thus, $\bar{x}=\frac{1}{\alpha}$ is locally asymptotically stable.
Now, using (3.23) and (3.30), we get

$$\begin{aligned}
\lim_{n\to\infty}x_{n+1}&=\lim_{n\to\infty}\frac{T_{n+1}x_0x_{-1}x_{-2}+(T_{n+1}+T_n)x_0x_{-1}+(T_{n+3}-T_{n+2})x_0+T_{n+2}}{T_{n+2}x_0x_{-1}x_{-2}+(T_{n+2}+T_{n+1})x_0x_{-1}+(T_{n+4}-T_{n+3})x_0+T_{n+3}}\\
&=\lim_{n\to\infty}\frac{\frac{T_{n+1}}{T_n}x_0x_{-1}x_{-2}+\left(\frac{T_{n+1}}{T_n}+\frac{T_n}{T_n}\right)x_0x_{-1}+\left(\frac{T_{n+3}}{T_n}-\frac{T_{n+2}}{T_n}\right)x_0+\frac{T_{n+2}}{T_n}}{\frac{T_{n+2}}{T_n}x_0x_{-1}x_{-2}+\left(\frac{T_{n+2}}{T_n}+\frac{T_{n+1}}{T_n}\right)x_0x_{-1}+\left(\frac{T_{n+4}}{T_n}-\frac{T_{n+3}}{T_n}\right)x_0+\frac{T_{n+3}}{T_n}}\\
&=\frac{\alpha x_0x_{-1}x_{-2}+(\alpha+1)x_0x_{-1}+(\alpha^3-\alpha^2)x_0+\alpha^2}{\alpha^2x_0x_{-1}x_{-2}+(\alpha^2+\alpha)x_0x_{-1}+(\alpha^4-\alpha^3)x_0+\alpha^3}\\
&=\frac{1}{\alpha}.
\end{aligned}$$

In summary, the equilibrium point $\bar{x}=\frac{1}{\alpha}$ is globally asymptotically stable.

3.2. *Particular cases of system* (3.1)

3.2.1. *The case* $d=0$ *and* $c\neq 0$

In this case the system (3.1) take the form:

$$\begin{cases}x_{n+1}=f^{-1}\left[a+\frac{b}{g(y_n)}+\frac{c}{g(y_n)f(x_{n-1})}\right],\\ y_{n+1}=g^{-1}\left[a+\frac{b}{f(x_n)}+\frac{c}{f(x_n)g(y_{n-1})}\right],\quad n\in\mathbb{N}_0.\end{cases} \tag{3.32}$$

Corollary 4. *The formulas of well-defined solutions of system* (3.32)*, are given, for all* $n \in \mathbb{N}_0$*, by*

$$x_{2n+1} = f^{-1}\left[\frac{c\widetilde{J}_{2n+1} + \left(\widetilde{J}_{2n+3} - a\widetilde{J}_{2n+2}\right) f(x_{-1}) + \widetilde{J}_{2n+2} g(y_0) f(x_{-1})}{c\widetilde{J}_{2n} + \left(\widetilde{J}_{2n+2} - a\widetilde{J}_{2n+1}\right) f(x_{-1}) + \widetilde{J}_{2n+1} g(y_0) f(x_{-1})}\right],$$

$$x_{2n+2} = f^{-1}\left[\frac{c\widetilde{J}_{2n+2} + \left(\widetilde{J}_{2n+4} - a\widetilde{J}_{2n+3}\right) g(y_{-1}) + \widetilde{J}_{2n+3} f(x_0) g(y_{-1})}{c\widetilde{J}_{2n+1} + \left(\widetilde{J}_{2n+3} - a\widetilde{J}_{2n+2}\right) g(y_{-1}) + \widetilde{J}_{2n+2} f(x_0) g(y_{-1})}\right],$$

$$y_{2n+1} = g^{-1}\left[\frac{c\widetilde{J}_{2n+1} + \left(\widetilde{J}_{2n+3} - a\widetilde{J}_{2n+2}\right) g(y_{-1}) + \widetilde{J}_{2n+2} f(x_0) g(y_{-1})}{c\widetilde{J}_{2n} + \left(\widetilde{J}_{2n+2} - a\widetilde{J}_{2n+1}\right) g(y_{-1}) + \widetilde{J}_{2n+1} f(x_0) g(y_{-1})}\right],$$

$$y_{2n+2} = g^{-1}\left[\frac{c\widetilde{J}_{2n+2} + \left(\widetilde{J}_{2n+4} - a\widetilde{J}_{2n+3}\right) f(x_{-1}) + \widetilde{J}_{2n+3} g(y_0) f(x_{-1})}{c\widetilde{J}_{2n+1} + \left(\widetilde{J}_{2n+3} - a\widetilde{J}_{2n+2}\right) f(x_{-1}) + \widetilde{J}_{2n+2} g(y_0) f(x_{-1})}\right].$$

If $g = f$ *and* $y_{-i} = x_{-i}$*,* $i = 0,1,2$ *then, system* (3.32) *becomes*

$$x_{n+1} = f^{-1}\left[a + \frac{b}{f(x_n)} + \frac{c}{f(x_n) f(x_{n-1})}\right], \quad n \in \mathbb{N}_0, \tag{3.33}$$

hence, every well-defined solution of equation (3.33) *is given by*

$$x_{2n+1} = f^{-1}\left[\frac{c\widetilde{J}_{2n+1} + \left(\widetilde{J}_{2n+3} - a\widetilde{J}_{2n+2}\right) f(x_{-1}) + \widetilde{J}_{2n+2} f(x_0) f(x_{-1})}{c\widetilde{J}_{2n} + \left(\widetilde{J}_{2n+2} - a\widetilde{J}_{2n+1}\right) f(x_{-1}) + \widetilde{J}_{2n+1} f(x_0) f(x_{-1})}\right],$$

$$x_{2n+2} = f^{-1}\left[\frac{c\widetilde{J}_{2n+2} + \left(\widetilde{J}_{2n+4} - a\widetilde{J}_{2n+3}\right) f(x_{-1}) + \widetilde{J}_{2n+3} f(x_0) f(x_{-1})}{c\widetilde{J}_{2n+1} + \left(\widetilde{J}_{2n+3} - a\widetilde{J}_{2n+2}\right) f(x_{-1}) + \widetilde{J}_{2n+2} f(x_0) f(x_{-1})}\right],$$

or in the unified form

$$x_{n+1} = f^{-1}\left[\frac{c\widetilde{J}_{n+1} + \left(\widetilde{J}_{n+3} - a\widetilde{J}_{n+2}\right) f(x_{-1}) + \widetilde{J}_{n+2} f(x_0) f(x_{-1})}{c\widetilde{J}_{n} + \left(\widetilde{J}_{n+2} - a\widetilde{J}_{n+1}\right) f(x_{-1}) + \widetilde{J}_{n+1} f(x_0) f(x_{-1})}\right], \quad n \in \mathbb{N}_0,$$

where $\left(\widetilde{J}_n\right)_{n\geq 0}$ *is the sequence defined by*

$$\widetilde{J}_{n+3} = a\widetilde{J}_{n+2} + b\widetilde{J}_{n+1} + c\widetilde{J}_n, \widetilde{J}_0 = 0, \widetilde{J}_1 = 1, \widetilde{J}_2 = a, n \in \mathbb{N}_0.$$

3.2.2. *The system* $x_{n+1} = \frac{1}{a+by_n+cx_{n-1}y_n}, y_{n+1} = \frac{1}{a+bx_n+cy_{n-1}x_n}$

In the following, we are interested in a particular system of (3.32) and some particular equations of its one dimensional version.

Let $f(t)=\frac{1}{t}$ and $g(t)=\frac{1}{t}$, then, system (3.32) becomes

$$x_{n+1}=\frac{1}{a+by_n+cx_{n-1}y_n},\ y_{n+1}=\frac{1}{a+bx_n+cy_{n-1}x_n}. \tag{3.34}$$

So, by Corollary 4, we get the following result.

Corollary 5. *For all $n\in\mathbb{N}_0$, the representation of well-defined solution of system (3.34) is*

$$x_{2n+1}=\frac{c\widetilde{J}_{2n}y_0x_{-1}+\left(\widetilde{J}_{2n+2}-a\widetilde{J}_{2n+1}\right)y_0+\widetilde{J}_{2n+1}}{c\widetilde{J}_{2n+1}y_0x_{-1}+\left(\widetilde{J}_{2n+3}-a\widetilde{J}_{2n+2}\right)y_0+\widetilde{J}_{2n+2}},$$

$$x_{2n+2}=\frac{c\widetilde{J}_{2n+1}x_0y_{-1}+\left(\widetilde{J}_{2n+3}-a\widetilde{J}_{2n+2}\right)x_0+\widetilde{J}_{2n+2}}{c\widetilde{J}_{2n+2}x_0y_{-1}+\left(\widetilde{J}_{2n+4}-a\widetilde{J}_{2n+3}\right)x_0+\widetilde{J}_{2n+3}},$$

$$y_{2n+1}=\frac{c\widetilde{J}_{2n}x_0y_{-1}+\left(\widetilde{J}_{2n+2}-a\widetilde{J}_{2n+1}\right)x_0+\widetilde{J}_{2n+1}}{c\widetilde{J}_{2n+1}x_0y_{-1}+\left(\widetilde{J}_{2n+3}-a\widetilde{J}_{2n+2}\right)x_0+\widetilde{J}_{2n+2}},$$

$$y_{2n+2}=\frac{c\widetilde{J}_{2n+1}y_0x_{-1}+\left(\widetilde{J}_{2n+3}-a\widetilde{J}_{2n+2}\right)y_0+\widetilde{J}_{2n+2}}{c\widetilde{J}_{2n+2}y_0x_{-1}+\left(\widetilde{J}_{2n+4}-a\widetilde{J}_{2n+3}\right)y_0+\widetilde{J}_{2n+3}}.$$

Moreover, the solutions of the following equation

$$x_{n+1}=\frac{1}{a+bx_n+cx_{n-1}x_n}, \tag{3.35}$$

follows directly from those of the system (3.34) by taking $y_{-i}=x_{-i}$, $i=0,1$. Then, representation of well-defined solution is

$$x_{2n+1}=\frac{c\widetilde{J}_{2n}x_0x_{-1}+\left(\widetilde{J}_{2n+2}-a\widetilde{J}_{2n+1}\right)x_0+\widetilde{J}_{2n+1}}{c\widetilde{J}_{2n+1}x_0x_{-1}+\left(\widetilde{J}_{2n+3}-a\widetilde{J}_{2n+2}\right)x_0+\widetilde{J}_{2n+2}}, \tag{3.36}$$

$$x_{2n+2}=\frac{c\widetilde{J}_{2n+1}x_0x_{-1}+\left(\widetilde{J}_{2n+3}-a\widetilde{J}_{2n+2}\right)x_0+\widetilde{J}_{2n+2}}{c\widetilde{J}_{2n+2}x_0x_{-1}+\left(\widetilde{J}_{2n+4}-a\widetilde{J}_{2n+3}\right)x_0+\widetilde{J}_{2n+3}}. \tag{3.37}$$

Recently in [12], the authors studied the particular difference equations

$$x_{n+1}=\frac{1}{-1+x_n+x_{n-1}x_n}, \tag{3.38}$$

$$x_{n+1}=\frac{1}{1+x_n-x_{n-1}x_n}, \tag{3.39}$$

$$x_{n+1}=\frac{1}{1-x_n+x_{n-1}x_n}, \tag{3.40}$$

$$x_{n+1}=\frac{1}{-1-x_n-x_{n-1}x_n}, \tag{3.41}$$

and as a generalization of (3.38), (3.39), (3.40), (3.41) the authors studied again in [11], the equation

$$x_{n+1}=\frac{1}{\frac{\beta}{\gamma}+\frac{\alpha}{\gamma}x_n+\frac{1}{\gamma}x_{n-1}x_n}, \gamma\neq 0. \tag{3.42}$$

Clearly these equations are particular cases of equation (3.35).
In fact to obtain equation (3.38) it suffices to take $a=-1$, $b=1$ and $c=1$ in (3.35), when choosing $a=1$, $b=1$ and $c=-1$ in (3.35) we get equation (3.39). Equation (3.40) follows from (3.35) for the choice $a=1$, $b=-1$ and $c=1$. Again if we take $a=-1$, $b=-1$ and $c=-1$ in (3.35) we get the equation (3.41), finally for the choice $a=\frac{\beta}{\gamma}$, $b=\frac{\alpha}{\gamma}$ and $c=\frac{1}{\gamma}$ in (3.35) we find the equation (3.42).
Noting, that the authors studied in [13] the following systems

$$\begin{cases} x_{n+1}=\dfrac{1}{1+y_n+x_{n-1}y_n}, \\ y_{n+1}=\dfrac{1}{1+x_n+y_{n-1}x_n}, \end{cases} \tag{3.43}$$

$$\begin{cases} x_{n+1}=\dfrac{1}{-1+y_n-x_{n-1}y_n}, \\ y_{n+1}=\dfrac{1}{-1+x_n-y_{n-1}x_n}. \end{cases} \tag{3.44}$$

Systems (3.43) and (3.44) are particular cases of system (3.34) for the choices $a=b=c=1$ and $a=-1, b=1, c=-1$ respectively.
Now, we will investigate each of these cases separately.

(1) Consider $a=-1, b=1, c=1$, equation (3.35) becomes (3.38). For this choice of the parameters, we get the sequence $\left(\widetilde{J}_n\right)_{n=0}^{\infty}$ defined by:

$$\widetilde{J}_{n+3}=-\widetilde{J}_{n+2}+\widetilde{J}_{n+1}+\widetilde{J}_n, \widetilde{J}_0=0, \widetilde{J}_1=1, \widetilde{J}_2=-1.$$

It is easy to see that

$$\widetilde{J}_n=-\left(\frac{1}{4}+\frac{1}{2}n\right)(-1)^n+\frac{1}{4}, \quad n\in\mathbb{N}_0.$$

From this it follows that

$$\widetilde{J}_{2n}=-n, \widetilde{J}_{2n+1}=n+1.$$

Replacing in (3.36) and (3.37), we obtain

$$x_{2n+1}=\frac{-nx_{-1}x_0+n+1}{(n+1)x_{-1}x_0+x_0-(n+1)},$$

$$x_{2n+2}=\frac{(n+1)x_0x_{-1}+x_0-(n+1)}{-(n+1)x_0x_{-1}+n+2},$$

and these are the formulas given in [12] for the solutions of (3.38).

(2) Consider $a=1, b=1, c=-1$, equation (3.35) becomes (3.39). For this choice of the parameters, we get the sequence $\left(\widetilde{J}_n\right)_{n=0}^{\infty}$ defined by:

$$\widetilde{J}_{n+3}=\widetilde{J}_{n+2}+\widetilde{J}_{n+1}-\widetilde{J}_n, \widetilde{J}_0=0, \widetilde{J}_1=1, \widetilde{J}_2=1.$$

It is easy to see that

$$\widetilde{J}_n=\frac{1}{4}+\frac{1}{2}n-\frac{1}{4}(-1)^n, \quad n\in\mathbb{N}_0.$$

From this it follows that

$$\widetilde{J}_{2n}=n, \widetilde{J}_{2n+1}=n+1.$$

Replacing in (3.36) and (3.37), we get

$$x_{2n+1}=\frac{nx_{-1}x_0-(n+1)}{(n+1)x_{-1}x_0-x_0-(n+1)},$$
$$x_{2n+2}=\frac{(n+1)x_0x_{-1}-x_0-(n+1)}{(n+1)x_0x_{-1}-(n+2)},$$

and these are the formulas given in [12] for the solutions of (3.39).

(3) Consider $a=1, b=-1, c=1$, equation (3.35) becomes (3.40). For this choice of the parameters, we get the sequence $\left(\widetilde{J}_n\right)_{n=0}^{\infty}$ defined by:

$$\widetilde{J}_{n+3}=\widetilde{J}_{n+2}-\widetilde{J}_{n+1}+\widetilde{J}_n, \widetilde{J}_0=0, \widetilde{J}_1=1, \widetilde{J}_2=1.$$

We have

$$\begin{aligned}\widetilde{J}_n&=\frac{1}{2}+\frac{1}{2(i-1)}(i)^n+\frac{-1}{2(i+1)}(-i)^n\\&=\frac{1}{2}+\left(\frac{-1}{4}-\frac{1}{4}i\right)(i)^n+\left(\frac{-1}{4}+\frac{1}{4}i\right)(-i)^n, \quad n\in\mathbb{N}_0.\end{aligned}$$

From this it follows that

$$\widetilde{J}_{2n}=\frac{1}{2}-\frac{1}{2}(-1)^n, \widetilde{J}_{2n+1}=\frac{1}{2}+\frac{1}{2}(-1)^n,$$

depending on n is even or odd, we get

$$\widetilde{J}_{4n}=0, \widetilde{J}_{4n+1}=1, \widetilde{J}_{4n+2}=1, \widetilde{J}_{4n+3}=0.$$

Replacing in (3.36) and (3.37), we obtain

$$x_{4n+1}=\frac{1}{x_{-1}x_0-x_0+1},$$

$$x_{4n+2} = \frac{x_{-1}x_0 - x_0 + 1}{x_{-1}x_0},$$
$$x_{4n+3} = x_{-1},$$
$$x_{4n+4} = x_0,$$

and these are the formulas given in [12] for the solutions of (3.39).

(4) Consider $a = b = c = -1$, equation (3.35) becomes (3.41). For this choice of the parameters, we get the sequence $\left(\widetilde{J}_n\right)_{n=0}^{\infty}$ defined by:

$$\widetilde{J}_{n+3} = -\widetilde{J}_{n+2} - \widetilde{J}_{n+1} - \widetilde{J}_n, \widetilde{J}_0 = 0, \widetilde{J}_1 = 1, \widetilde{J}_2 = -1.$$

We have

$$\begin{aligned}\widetilde{J}_n &= \frac{-1}{2}(-1)^n + \frac{1}{2(1+i)}(i)^n + \frac{1}{2(1-i)}(-i)^n \\ &= \frac{-1}{2}(-1)^n + \left(\frac{1}{4} - \frac{1}{4}i\right)(i)^n + \left(\frac{1}{4} + \frac{1}{4}i\right)(-i)^n, \quad n \in \mathbb{N}_0.\end{aligned}$$

From this it follows that

$$\widetilde{J}_{2n} = \frac{-1}{2} + \frac{1}{2}(-1)^n, \widetilde{J}_{2n+1} = \frac{1}{2} + \frac{1}{2}(-1)^n,$$

depending on n is even or odd, we get

$$\widetilde{J}_{4n} = 0, \widetilde{J}_{4n+1} = 1, \widetilde{J}_{4n+2} = -1, \widetilde{J}_{4n+3} = 0.$$

Replacing in (3.36) and (3.37), we get

$$x_{4n+1} = \frac{1}{-x_{-1}x_0 - x_0 - 1},$$
$$x_{4n+2} = \frac{-x_{-1}x_0 - x_0 - 1}{x_{-1}x_0},$$
$$x_{4n+3} = x_{-1},$$
$$x_{4n+4} = x_0,$$

and these are the formulas given in [12] for the solutions of (3.41).

(5) Consider $a = b = c = 1$, system (3.34) becomes (3.43). For this choice of the parameters, the sequence $\left(\widetilde{J}_n\right)_{n=0}^{+\infty}$ will be nothing other than the famous Tribonacci sequence defined for $n \in \mathbb{N}_0$ by

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n, T_0 = 0, \quad T_1 = T_2 = 1. \tag{3.45}$$

The solutions are given by

$$\begin{cases} x_{2n+1} = \dfrac{T_{2n}y_0x_{-1} + (T_{2n+2} - T_{2n+1})y_0 + T_{2n+1}}{T_{2n+1}y_0x_{-1} + (T_{2n+3} - T_{2n+2})y_0 + T_{2n+2}}, \\ x_{2n+2} = \dfrac{T_{2n+1}x_0y_{-1} + (T_{2n+3} - T_{2n+2})x_0 + T_{2n+2}}{T_{2n+2}x_0y_{-1} + (T_{2n+4} - T_{2n+3})x_0 + T_{2n+3}}, \\ y_{2n+1} = \dfrac{T_{2n}x_0y_{-1} + (T_{2n+2} - T_{2n+1})x_0 + T_{2n+1}}{T_{2n+1}x_0y_{-1} + (T_{2n+3} - T_{2n+2})x_0 + T_{2n+2}}, \\ y_{2n+2} = \dfrac{T_{2n+1}y_0x_{-1} + (T_{2n+3} - T_{2n+2})y_0 + T_{2n+2}}{T_{2n+2}y_0x_{-1} + (T_{2n+4} - T_{2n+3})y_0 + T_{2n+3}}. \end{cases} \tag{3.46}$$

(6) Consider $a = -1, b = 1, c = -1$, system (3.34) becomes (3.44). For this choice of the parameters, we get the sequence $\left(\widetilde{J}_n\right)_{n=0}^{\infty}$ defined by:

$$\widetilde{J}_{n+3} = -\widetilde{J}_{n+2} + \widetilde{J}_{n+1} - \widetilde{J}_n, \widetilde{J}_0 = 0, \widetilde{J}_1 = 1, \widetilde{J}_2 = -1.$$

Using the fact that $\widetilde{J}_n = (-1)^{n+1}T_n$ ([1]), we get from Corollary 5 that the representation of well-defined solutions of system (3.44) is

$$\begin{cases} x_{2n+1} = \dfrac{-(T_{2n}y_0x_{-1} + (T_{2n+1} - T_{2n+2})y_0 + T_{2n+1})}{T_{2n+1}y_0x_{-1} - (T_{2n+1} + T_{2n})y_0 + T_{2n+2}}, \\ x_{2n+2} = \dfrac{-(T_{2n+1}x_0y_{-1} + (T_{2n+2} - T_{2n+3})x_0 + T_{2n+2})}{T_{2n+2}x_0y_{-1} - (T_{2n+2} + T_{2n+1})x_0 + T_{2n+3}}, \\ y_{2n+1} = \dfrac{-(T_{2n}x_0y_{-1} + (T_{2n+1} - T_{2n+2})x_0 + T_{2n+1})}{T_{2n+1}x_0y_{-1} - (T_{2n+1} + T_{2n})x_0 + T_{2n+2}}, \\ y_{2n+2} = \dfrac{-(T_{2n+1}y_0x_{-1} + (T_{2n+2} - T_{2n+3})y_0 + T_{2n+2})}{T_{2n+2}y_0x_{-1} - (T_{2n+2} + T_{2n+1})y_0 + T_{2n+3}}, \end{cases} \tag{3.47}$$

and these are the formulas given in [13] for the solutions of (3.44).

3.2.3. *Case $d = 0, c \neq 0$ and $a = 0$*

In this case, we obtain the following system

$$\begin{cases} x_{n+1} = f^{-1}\left[\frac{b}{g(y_n)} + \frac{c}{g(y_n)f(x_{n-1})}\right], \\ y_{n+1} = g^{-1}\left[\frac{b}{f(x_n)} + \frac{c}{f(x_n)g(y_{n-1})}\right], \quad n \in \mathbb{N}_0. \end{cases} \tag{3.48}$$

Here, $\left(\widetilde{J}_n\right)_{n\geq 0}$ will be the sequence $(\mathcal{P}_n)_{n\geq 0}$ defined by

$$\mathcal{P}_{n+3} = b\mathcal{P}_{n+1} + c\mathcal{P}_n, \mathcal{P}_0 = 0, \mathcal{P}_1 = 1, \mathcal{P}_2 = 0, n \in \mathbb{N}_0. \tag{3.49}$$

So, well-defined solutions are expressed in terms of $(\mathcal{P}_n)_{n\geq 0}$ and takes the form

$$x_{2n+1} = f^{-1}\left[\frac{c\mathcal{P}_{2n+1} + \mathcal{P}_{2n+3}f(x_{-1}) + \mathcal{P}_{2n+2}g(y_0)f(x_{-1})}{c\mathcal{P}_{2n} + \mathcal{P}_{2n+2}f(x_{-1}) + \mathcal{P}_{2n+1}g(y_0)f(x_{-1})}\right], n \in \mathbb{N}_0,$$

$$x_{2n+2} = f^{-1}\left[\frac{c\mathcal{P}_{2n+2} + \mathcal{P}_{2n+4}g(y_{-1}) + \mathcal{P}_{2n+3}f(x_0)g(y_{-1})}{c\mathcal{P}_{2n+1} + \mathcal{P}_{2n+3}g(y_{-1}) + \mathcal{P}_{2n+2}f(x_0)g(y_{-1})}\right], n \in \mathbb{N}_0,$$

$$y_{2n+1} = g^{-1}\left[\frac{c\mathcal{P}_{2n+1} + \mathcal{P}_{2n+3}g(y_{-1}) + \mathcal{P}_{2n+2}f(x_0)g(y_{-1})}{c\mathcal{P}_{2n} + \mathcal{P}_{2n+2}g(y_{-1}) + \mathcal{P}_{2n+1}f(x_0)g(y_{-1})}\right], n \in \mathbb{N}_0,$$

$$y_{2n+2} = g^{-1}\left[\frac{c\mathcal{P}_{2n+2} + \mathcal{P}_{2n+4}f(x_{-1}) + \mathcal{P}_{2n+3}g(y_0)f(x_{-1})}{c\mathcal{P}_{2n+1} + \mathcal{P}_{2n+3}f(x_{-1}) + \mathcal{P}_{2n+2}g(y_0)f(x_{-1})}\right], n \in \mathbb{N}_0.$$

for system (3.48) and by

$$x_{n+1} = f^{-1}\left[\frac{c\mathcal{P}_{n+1} + \mathcal{P}_{n+3}f(x_{-1}) + \mathcal{P}_{n+2}f(x_0)f(x_{-1})}{c\mathcal{P}_n + \mathcal{P}_{n+2}f(x_{-1}) + \mathcal{P}_{n+1}f(x_0)f(x_{-1})}\right], n \in \mathbb{N}_0,$$

for its one dimensional version, that is the equation

$$x_{n+1} = f^{-1}\left[\frac{b}{f(x_n)} + \frac{c}{f(x_n)f(x_{n-1})}\right], \ n \in \mathbb{N}_0.$$

Also, (3.48) generalize some works in the literature, see, for example [5] and [21].

3.2.4. *Case* $c = d = 0, b \neq 0$

In this case, we get the system

$$x_{n+1} = f^{-1}\left[a + \frac{b}{g(y_n)}\right], y_{n+1} = g^{-1}\left[a + \frac{b}{f(x_n)}\right], \ n \in \mathbb{N}_0. \tag{3.50}$$

In this case, the well-defined solutions will be expressed using terms of the sequence $\left(\widetilde{F}_n\right)_{n\geq 0} = \left(\widetilde{J}_{n+1}\right)_{n\geq 0}$, defined by

$$\widetilde{F}_{n+2} = a\widetilde{F}_{n+1} + b\widetilde{F}_n, \widetilde{F}_0 = 1, \widetilde{F}_1 = a, n \in \mathbb{N}_0$$

and the solutions for (3.50) and its one dimensional version, are obtained from Corollary (4), by writing $\widetilde{F}_{n-1}$ instead of $\widetilde{J}_n$. System (3.48) and its one dimensional version, generalize some existing works, for example [17], [18].

Remark 5. If $b = c = d = 0, a \neq 0$, we get

$$x_n = f^{-1}(a), y_n = g^{-1}(a), n = 1, 2, \cdots.$$

4. Conclusion

In this work, we have presented formulas of well-defined solutions of some general systems of difference equations defined by one to one functions on a set D of real numbers. Noting that the obtained formulas of the solutions of our systems are expressed using some remarkable sequences, like Fibonacci, Tribonaci, Padovan, Tetranacci and their generalizations. Our work generalizes a lot of existing works in the literature on solvable difference equations and systems. Although, the condition "one to one functions", reduces the set of possibilities surveyed by this work. But our results, can be used to obtain the formulas of well-defined solutions of other systems, that their solvability, seems for the first sight impossible. So, under appropriate

choice of the the set D, we can solve complicated difference equations and systems involving for example functions like "tan", "ln" and others.

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ON NEW CHEBYSHEV INEQUALITIES VIA FRACTIONAL OPERATORS

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Abstract. The aim of this paper is to establish several fractional integral inequalities related to the weighted and the extended Chebyshev functional. We use generalized fractional integral operators, new conformable fractional integral operators and Saigo fractional integral operators to prove our results. This study states that our findings are more convenient and efficient than other available results.

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1. Introduction and Preliminaries

The integral inequality theory is very important in applied sciences. For more details, we refer the reader to [1,2] and the references therein. Moreover, the study of the integral inequalities using fractional integration theory is also of great importance, we refer to [8,9,14,15] for some applications. In this paper, we will be concerned with the extended and the weighted Chebyshev functional [2,9].

So, in this section, we introduce some definitions and some published results that have motivated the present work; we begin by recalling the following definition.

Definition 1. If f and g are two integrable functions on $[a,b]$ and p is a positive and integrable function on [a,b], Chebyshev functional is given by the following quantity

$$\mathrm{T}(f,g,p) := \int_a^b p(x)dx \int_a^b f(x)g(x)p(x)dx - \int_a^b f(x)p(x)dx \int_a^b g(x)p(x)dx.$$

Related to the above quantity, in [6], N. Elezovic *et al.* proved the following integral result:

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Theorem 1. [6, Theorem B.] *The inequality*

$$|\mathrm{T}(f,g,p)| \leq \frac{1}{2}\left(\int_a^b\int_a^b p(x)p(y)|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}\left|\int_y^x |f'(t)|^{\alpha}dt\right|^{\frac{\gamma}{\alpha}}dxdy\right)^{\frac{1}{\gamma}}$$
$$\times\left(\int_a^b\int_a^b p(x)p(y)|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}\left|\int_y^x |g'(t)|^{\beta}dt\right|^{\frac{\gamma'}{\beta}}dxdy\right)^{\frac{1}{\gamma'}}$$
$$\leq \frac{1}{2}||f'||_{\alpha}||g'||_{\beta}\left(\int_a^b\int_a^b p(x)p(y)|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}dxdy\right)$$

is valid provided that $f' \in \mathrm{L}^{\alpha}([a,b])$ *and* $g' \in \mathrm{L}^{\beta}([a,b])$*;* $\alpha > 1$, $\beta > 1$, $\gamma > 1$, $\alpha^{-1}+\alpha'^{-1}=1$, $\beta^{-1}+\beta'^{-1}=1$, $\gamma^{-1}+\gamma'^{-1}=1$.

Then, by means of the following fractional integral definition:

Definition 2. Let $f \in L_1[a,b]$. The Riemannn-Liouville integral of order $a \geq 0$ is defined by

$$J_a^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \qquad a \leq x \leq b.$$

The authors of the paper [5] established the following theorem:

Theorem 2. [5, Theorem 3.1]

$$2|\mathrm{J}^{\sigma}p(t)\mathrm{J}^{\sigma}pfg(t) - \mathrm{J}^{\sigma}pf(t)\mathrm{J}^{\sigma}pg(t)|$$
$$\leq \frac{||f'||_{\alpha}||g'||_{\beta}}{\Gamma^2(\sigma)}\int_0^t\int_0^t (t-x)^{\sigma-1}(t-y)^{\sigma-1}|x-y|p(x)p(y)dxdy,$$

where $f' \in \mathrm{L}^{\alpha}([0,\infty])$ *and* $g' \in \mathrm{L}^{\beta}([0,\infty])$*;* $\alpha > 1$, $\beta > 1$, $\alpha^{-1}+\beta^{-1}=1$.

Furthermore, by considering a second weighted function q (which is supposed to be positive and integrable) on $[a,b]$, we consider the extended Chebyshev's functional [3, 9]:

$$\tilde{\mathrm{T}}(f,g,p,q) := \int_a^b q(x)dx\int_a^b p(x)f(x)g(x)dx + \int_a^b p(x)dx\int_a^b q(x)f(x)g(x)dx$$
$$-\int_a^b p(x)f(x)dx\int_a^b q(x)g(x)dx - \int_a^b q(x)f(x)dx\int_a^b p(x)g(x)dx.$$

Then, we recall the following two theorems [4].

Theorem 3. [4, Theorem 3.1] *Let* f *and* g *be two differentiable functions on* $[0,\infty)$ *and* p *be positive and integrable function on* $[0,\infty)$*. If* $f' \in L^{\alpha}([0,\infty))$, $g' \in L^{\beta}([0,\infty))$,

$\alpha,\beta,\gamma>1$ with $\frac{1}{\alpha}+\frac{1}{\alpha'}=1$, $\frac{1}{\beta}+\frac{1}{\beta'}=1$ and $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$, then for all $t>0$, $\delta>0$, we have the inequality

$$\begin{aligned}
&2\left|J^{\delta}p(t)J^{\delta}pfg(t)-J^{\delta}pg(t)J^{\delta}pf(t)\right| \\
&\leq\left(\frac{||f'||_{\alpha}^{\gamma}}{\Gamma(\delta)}\int_0^t\int_0^t(t-x)^{\delta-1}(t-y)^{\delta-1}p(x)p(y)|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}dxdy\right)^{\frac{1}{\gamma}} \\
&\quad\times\left(\frac{||g'||_{\beta}^{\gamma}}{\Gamma(\alpha)}\int_0^t\int_0^t(t-x)^{\delta-1}(t-y)^{\delta-1}p(x)p(y)|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}dxdy\right)^{\frac{1}{\gamma}} \\
&\leq\frac{||f'||_{\alpha}||g'||_{\beta}}{\Gamma(\delta)^2}\left(\int_0^t\int_0^t(t-x)^{\delta-1}(t-y)^{\delta-1}|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}p(x)p(y)dxdy\right).
\end{aligned}\tag{1.1}$$

Theorem 4. [4, Theorem 3.3] *Let f and g be two differentiable functions on $[0,\infty)$ and p,q be two positive and integrable function on $[0,\infty)$. If $f'\in L^{\alpha}([0,\infty))$, $g'\in L^{\beta}([0,\infty))$, α, β, $\gamma>1$ with $\frac{1}{\alpha}+\frac{1}{\alpha'}=1$, $\frac{1}{\beta}+\frac{1}{\beta'}=1$ and $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$, then for all $t>0$, $\delta>0$, we have:*

$$\begin{aligned}
&\left|J^{\delta}q(t)J^{\delta}pfg(t)+J^{\delta}p(t)J^{\delta}qfg(t)-J^{\delta}pf(t)J^{\delta}qg(t)-J^{\delta}qf(t)J^{\delta}pg(t)\right| \\
&\leq\frac{||f'||_{\alpha}||g'||_{\beta}}{\Gamma(\delta)^2}\left(\int_0^t\int_0^t(t-x)^{\delta-1}(t-y)^{\delta-1}|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}p(x)q(y)dxdy\right).
\end{aligned}\tag{1.2}$$

The main aim of this paper is to establish new fractional integral inequalities related to the weighted and the extended Chebyshev functional. We use the generalized fractional integral operator to establish some of our main results. Then, by considering the conformable fractional integral approach, we establish other integral results. Finally, by Saigo fractional integral operator, we present to the reader other results on integral inequalities. Some published results related to integral inequalities in the sense or Riemann-Liouville [4, 5] are deduced as some special cases.

2. Main Results

Let's first recall the generalized fractional integral operators definition we used in the proof of Theorem 5 and Theorem 6.

In [10], Raina introduced a generalized class of Mittag-Leffler functions defined formally by

$$\mathcal{F}_{\rho,\lambda}^{\sigma}(x)=\mathcal{F}_{\rho,\lambda}^{\sigma(0),\sigma(1),\cdots}(x)=\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}x^k\qquad(\rho,\lambda>0;\ |x|<\mathbf{R}),\tag{2.1}$$

where the coefficients $\sigma(k)$, $(k\in\mathbb{N}=\mathbb{N}\cup\{0\})$ is a bounded sequence of positive real numbers and $\mathbf{R}$ is the set of real numbers. With the help of (2.1), Raina [10] and

Agarwal *et al.* [1] defined the following left-sided and right-sided fractional integral operators respectively, as follows:

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi\right)(x)=\int_a^x (x-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(x-t)^{\rho}]\varphi(t)dt \qquad (x>a), \tag{2.2}$$

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi\right)(x)=\int_x^b (t-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-x)^{\rho}]\varphi(t)dt \qquad (x<b), \tag{2.3}$$

where λ, $\rho>0$, $w\in\mathbb{R}$ and $\varphi(t)$ is such that the integral on the right side exits. In recently some integral inequalities this operator involving have appeared in the literature (see, e.g., [13–17]).

It is easy to verify that $\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi(t)$ and $\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi(t)$ are bounded integral operators on $L(a,b)$, if

$$\mathfrak{M}:=\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]<\infty.$$

In fact, for $\varphi\in L(a,b)$, we have

$$||\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi(x)||_1\le\mathfrak{M}(b-a)^{\lambda}||\varphi||_1$$

and

$$||\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi(x)||_1\le\mathfrak{M}(b-a)^{\lambda}||\varphi||_1,$$

where

$$||\varphi||_p:=\left(\int_a^b|\varphi(x)|^p dx\right)^{\frac{1}{p}}.$$

Here, many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. For instance the classical Riemann–Liouville fractional integrals J^{α}_{a+} and J^{α}_{b-} of order α follow easily by setting $\lambda=\alpha$, $\sigma(0)=1$ and $w=0$ in (2.2) and (2.3).

We begin our main results by proving the following theorem.

Theorem 5. *Let f and g be two differentiable functions on $[0,\infty)$ and p be positive and integrable function on $[0,\infty)$. If $f'\in L_r[a,b]$, $g'\in L_s[a,b]$, r, s, $\gamma>1$ with $\frac{1}{r}+\frac{1}{r'}=1$, $\frac{1}{s}+\frac{1}{s'}=1$ and $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$, then for all $t>0$, α, ρ, $w\ge 0$, we have;*

$$\begin{aligned}
&2\left|\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}p(t)\,\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}pfg(t)-\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}pg(t)\,\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}pf(t)\right|\\
&\le\left[||f'||_r^{\gamma}\int_a^t\int_a^t(t-x)^{\lambda-1}(t-y)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-x)^{\rho}]\mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-y)^{\rho}]\right.\\
&\qquad\left.\times|x-y|^{\frac{1}{r'}+\frac{1}{s'}}p(x)p(y)dxdy\right]^{\frac{1}{\gamma}}
\end{aligned}\tag{2.4}$$

$$\times \left[||g'||_s^{\gamma} \int_a^t \int_a^t (t-x)^{\lambda-1}(t-y)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-x)^{\rho}] \mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-y)^{\rho}] \right.$$
$$\left. \times |x-y|^{\frac{1}{r}+\frac{1}{s}} p(x)p(y)dxdy \right]^{\frac{1}{\gamma}}$$
$$\leq ||f'||_r ||g'||_s \int_a^t \int_a^t (t-x)^{\lambda-1}(t-y)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-x)^{\rho}] \mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-y)^{\rho}]$$
$$\times |x-y|^{\frac{1}{r}+\frac{1}{s}} p(x)p(y)dxdy.$$

Proof. We consider the function H defined by:

$$H(x,y) := (f(x)-f(y)(g(x)-g(y))); \qquad x,y \in (0,t), t>0.$$

Some easy techniques allow us to write

$$\int_a^t (t-x)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-x)^{\rho}] p(x) H(x,y)dx = \mathcal{J}^{\sigma}_{\rho,\lambda,a+;w} pfg(t)$$
$$- g(y)\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w} pf(t) - f(y)\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w} pg(t) + f(y)g(y)\,\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w} p(t).$$

By the same arguments as above, it yields that

$$\int_a^t \int_a^t (t-x)^{\lambda-1}(t-y)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-x)^{\rho}] \mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-y)^{\rho}] p(x)p(y)H(x,y)dxdy$$
$$= 2\left[\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w} p(t)\, \mathcal{J}^{\sigma}_{\rho,\lambda,a+;w} pfg(t) - \mathcal{J}^{\sigma}_{\rho,\lambda,a+;w} pg(t)\, \mathcal{J}^{\sigma}_{\rho,\lambda,a+;w} pf(t) \right].$$

Also, we can see that

$$H(x,y) := \int_y^x \int_y^x f'(u)g'(w)dudw.$$

By using Hölder inequality, we have

$$|f(x)-f(y)| \leq |x-y|^{\frac{1}{\gamma}} \left| \int_y^x |f'(u)|^r du \right|^{\frac{1}{r}}$$

and

$$|g(x)-g(y)| \leq |x-y|^{\frac{1}{\gamma}} \left| \int_y^x |g'(w)|^s dw \right|^{\frac{1}{s}}.$$

Therefore,

$$|H(x,y)| \leq |x-y|^{\frac{1}{r}+\frac{1}{s}} \left| \int_y^x |f'(u)|^r du \right|^{\frac{1}{r}} \left| \int_y^x |g'(w)|^s du \right|^{\frac{1}{s}}.$$

Hence,

$$
\begin{aligned}
&2\left|\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}p(t)\,\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}pfg(t)-\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}pg(t)\,\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}pf(t)\right|\\
&\leq\int_a^t\int_a^t(t-x)^{\lambda-1}(t-y)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-x)^{\rho}]\mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-y)^{\rho}]p(x)p(y)\\
&\quad\times|x-y|^{\frac{1}{r}+\frac{1}{s}}\left|\int_y^x|f'(u)|^r du\right|^{\frac{1}{r}}\left|\int_y^x|g'(w)|^s dw\right|^{\frac{1}{s}}dxdy.
\end{aligned}
$$

Again, by using Hölder's inequality, we obtain

$$
\begin{aligned}
&2\left|\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}p(t)\,\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}pfg(t)-\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}pg(t)\,\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}pf(t)\right|\\
&\leq\left[\int_a^t\int_a^t(t-x)^{\lambda-1}(t-y)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-x)^{\rho}]\mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-y)^{\rho}]\right.\\
&\qquad\left.\times|x-y|^{\frac{1}{r}+\frac{1}{s}}\left|\int_y^x|f'(u)|^r du\right|^{\frac{\gamma}{r}}p(x)p(y)dxdy\right]^{\frac{1}{\gamma}}\\
&\quad\times\left[\int_a^t\int_a^t(t-x)^{\lambda-1}(t-y)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-x)^{\rho}]\mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-y)^{\rho}]\right.\\
&\qquad\left.\times|x-y|^{\frac{1}{r}+\frac{1}{s}}\left|\int_y^x|g'(w)|^s dw\right|^{\frac{\gamma'}{s}}p(x)p(y)dxdy\right]^{\frac{1}{\gamma'}}.
\end{aligned}
\tag{2.5}
$$

Since, we know that

$$
\left|\int_x^y|f'(u)|^r du\right|\leq||f'||_r^r,\qquad\left|\int_x^y|g'(w)|^s dw\right|\leq||g'||_s^s,
$$

then (2.5) can be written as follows:

$$
\begin{aligned}
&2\left|\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}p(t)\,\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}pfg(t)-\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}pg(t)\,\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}pf(t)\right|\\
&\leq\left[||f'||_r^{\gamma}\int_a^t\int_a^t(t-x)^{\lambda-1}(t-y)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-x)^{\rho}]\mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-y)^{\rho}]\right.\\
&\qquad\left.\times|x-y|^{\frac{1}{r}+\frac{1}{s}}p(x)p(y)dxdy\right]^{\frac{1}{\gamma}}\\
&\quad\times\left[||g'||_s^{\gamma'}\int_a^t\int_a^t(t-x)^{\lambda-1}(t-y)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-x)^{\rho}]\mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-y)^{\rho}]\right.\\
&\qquad\left.\times|x-y|^{\frac{1}{r}+\frac{1}{s}}p(x)p(y)dxdy\right]^{\frac{1}{\gamma'}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&2\left|\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}p(t)\,\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}pfg(t)-\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}pg(t)\,\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}pf(t)\right|\\
&\leq ||f'||_r||g'||_s\int_a^t\int_a^t(t-x)^{\lambda-1}(t-y)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-x)^{\rho}]\mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-y)^{\rho}]\\
&\quad\times|x-y|^{\frac{1}{r'}+\frac{1}{s'}}p(x)p(y)dxdy.
\end{aligned}
$$

This completes the proof. □

Remark 1. If we choose $\lambda=\alpha$, $\sigma(0)=1$ *and* $w=0$ in Theorem 5, then the inequality (2.4) reduces to the inequality (1.1).

We prove also the following result.

Theorem 6. *Let f and g be two differentiable functions on $[0,\infty)$ and p, q be positive and integrable function on $[0,\infty)$. If $f'\in L_r[a,b]$, $g'\in L_s[a,b]$, r, s, $\gamma>1$ with $\frac{1}{r}+\frac{1}{r'}=1$, $\frac{1}{s}+\frac{1}{s'}=1$ and $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$, then for all $t>0$, α, ρ, $w\geq 0$, we have;*

$$
\begin{aligned}
&\Big|\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}q(t)\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}pfg(t)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}p(t)\,\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}qfg(t)\\
&\quad-\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}pf(t)\,\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}qg(t)-\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}qf(t)\,\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}pg(t)\Big| \qquad (2.6)\\
&\leq ||f'||_r||g'||_s\int_a^t\int_a^t(t-x)^{\lambda-1}(t-y)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-x)^{\rho}]\mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-y)^{\rho}]\\
&\quad\times|x-y|^{\frac{1}{r'}+\frac{1}{s'}}p(x)q(y)dxdy.
\end{aligned}
$$

Proof. We use the same arguments as in the proof of Theorem 5. The proof of this theorem is thus omitted. □

Remark 2. If we take $\lambda=\alpha$, $\sigma(0)=1$ *and* $w=0$ in Theorem 6, then the inequality (2.6) reduces to the inequality (1.2).

For the conformable integral approach, we recall the following definition.

Definition 3. [7, Definition 2.1 & Definition 2.2] The left and right-fractional conformable integrals of order $\beta\in\mathbb{C}$, $Re(\beta)>0$, are defined by

$$
\begin{aligned}
{}^{\beta}_{a}I^{\alpha}f(x)&=\frac{1}{\Gamma(\beta)}\int_a^x\left(\frac{(x-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)dt}{(t-a)^{1-\alpha}},\\
{}^{\beta}I^{\alpha}_{b}f(x)&=\frac{1}{\Gamma(\beta)}\int_x^b\left(\frac{(b-x)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)dt}{(b-t)^{1-\alpha}}.
\end{aligned}
$$

Notice that, if $(Qf)(t)=f(a+b-t)$, then we have $({}^{\beta}_{a}I^{\alpha}Qf)(x)=Q({}^{\beta}I^{\alpha}_{b}f)(x)$. Based on this definition, we present to the reader the following two results.

Theorem 7. *Let f and g be two differentiable functions on $[0,\infty)$ and p be positive and integrable function on $[0,\infty)$. If $f' \in L_r[a,b]$, $g' \in L_s[a,b]$, r, s, $\gamma > 1$ with $\frac{1}{r}+\frac{1}{r'}=1$, $\frac{1}{s}+\frac{1}{s'}=1$ and $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$, then for all $t>0$, $\beta \in \mathbb{C}$ and $Re(\beta)>0$ we have;*

$$
\begin{aligned}
&2\left|{}_a^{\beta}I^{\alpha}p(t)\,{}_a^{\beta}I^{\alpha}pfg(t)-{}_a^{\beta}I^{\alpha}pg(t)\,{}_a^{\beta}I^{\alpha}pf(t)\right| \\
&\leq \left[\frac{||f'||_r^{\gamma}}{\Gamma(\beta)}\int_a^t\int_a^t\left(\frac{(t-a)^{\alpha}-(x-a)^{\alpha}}{\alpha}\right)^{\beta-1}\left(\frac{(t-a)^{\alpha}-(y-a)^{\alpha}}{\alpha}\right)^{\beta-1}\right. \\
&\qquad\left.\times\frac{1}{(x-a)^{1-\alpha}}\frac{1}{(y-a)^{1-\alpha}}|x-y|^{\frac{1}{r'}+\frac{1}{s'}}p(x)p(y)dxdy\right]^{\frac{1}{\gamma}} \\
&\quad\times\left[\frac{||g'||_s^{\gamma'}}{\Gamma(\beta)}\int_a^t\int_a^t\left(\frac{(t-a)^{\alpha}-(x-a)^{\alpha}}{\alpha}\right)^{\beta-1}\left(\frac{(t-a)^{\alpha}-(y-a)^{\alpha}}{\alpha}\right)^{\beta-1}\right. \\
&\qquad\left.\times\frac{1}{(x-a)^{1-\alpha}}\frac{1}{(y-a)^{1-\alpha}}|x-y|^{\frac{1}{r'}+\frac{1}{s'}}p(x)p(y)dxdy\right]^{\frac{1}{\gamma'}} \qquad (2.7)\\
&\leq \frac{||f'||_r||g'||_s}{\Gamma^2(\beta)}\int_a^t\int_a^t\left(\frac{(t-a)^{\alpha}-(x-a)^{\alpha}}{\alpha}\right)^{\beta-1}\left(\frac{(t-a)^{\alpha}-(y-a)^{\alpha}}{\alpha}\right)^{\beta-1} \\
&\quad\times\frac{1}{(x-a)^{\alpha}}\frac{1}{(y-a)^{\alpha}}|x-y|^{\frac{1}{r'}+\frac{1}{s'}}p(x)p(y)dxdy.
\end{aligned}
$$

Proof. Let us reconsider the function

$$H(x,y) := (f(x)-f(y)(g(x)-g(y))); \qquad x,y\in(0,t), t>0. \qquad (2.8)$$

Multiplying (2.8) by $\frac{1}{\Gamma(\beta)}\left(\frac{(t-a)^{\alpha}-(x-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{1}{(x-a)^{1-\alpha}}p(x)$ and integrating the resulting identity with respect to x from a to t, we can write

$$
\begin{aligned}
&\frac{1}{\Gamma(\beta)}\int_a^t\left(\frac{(t-a)^{\alpha}-(x-a)^{\alpha}}{\alpha}\right)^{\beta-1}\times\frac{1}{(x-a)^{1-\alpha}}p(x)H(x,y)dx \\
&\qquad={}_a^{\beta}I^{\alpha}pfg(t)-g(y)\,{}_a^{\beta}I^{\alpha}pf(t)-f(y)\,{}_a^{\beta}I^{\alpha}pg(t)+f(y)g(y)\,{}_a^{\beta}I^{\alpha}p(t).
\end{aligned}
$$

In the same manner, we get:

$$
\begin{aligned}
&\frac{1}{\Gamma^2(\beta)}\int_a^t\int_a^t\left(\frac{(t-a)^{\alpha}-(x-a)^{\alpha}}{\alpha}\right)^{\beta-1}\left(\frac{(t-a)^{\alpha}-(y-a)^{\alpha}}{\alpha}\right)^{\beta-1} \\
&\qquad\times\frac{1}{(x-a)^{1-\alpha}}\frac{1}{(y-a)^{1-\alpha}}p(x)p(y)H(x,y)dxdy \\
&\qquad=2\left[{}_a^{\beta}I^{\alpha}p(t)\,{}_a^{\beta}I^{\alpha}pfg(t)-{}_a^{\beta}I^{\alpha}pg(t)\,{}_a^{\beta}I^{\alpha}pf(t)\right].
\end{aligned}
$$

To achieve the proof, we use the same arguments as in the proof of Theorem 5. □

Remark 3. If we choose $a = 0$ and $\alpha = 1$ in Theorem 7, then the inequality (2.7) reduces to the inequality (1.1).

Theorem 8. *Let f and g be two differentiable functions on $[0,\infty)$ and p, q be positive and integrable function on $[0,\infty)$. If $f' \in L_r[a,b]$, $g' \in L_s[a,b]$, r, s, $\gamma > 1$ with $\frac{1}{r} + \frac{1}{r'} = 1$, $\frac{1}{s} + \frac{1}{s'} = 1$ and $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$, then for all $t > 0$, $\beta \in \mathbb{C}$ and $Re(\beta) > 0$, we have*

$$\begin{aligned}
&\Big| {}_a^{\beta}I^{\alpha} q(t) {}_a^{\beta}I^{\alpha} pfg(t) + {}_a^{\beta}I^{\alpha} p(t) {}_a^{\beta}I^{\alpha} qfg(t) \\
&- {}_a^{\beta}I^{\alpha} pf(t) {}_a^{\beta}I^{\alpha} qg(t) - {}_a^{\beta}I^{\alpha} qf(t) {}_a^{\beta}I^{\alpha} pg(t) \Big| \\
&\leq \frac{||f'||_r ||g'||_s}{\Gamma^2(\beta)} \int_a^t \int_a^t \left(\frac{(t-a)^{\alpha} - (x-a)^{\alpha}}{\alpha} \right)^{\beta-1} \left(\frac{(t-a)^{\alpha} - (y-a)^{\alpha}}{\alpha} \right)^{\beta-1} \\
&\quad \times \frac{1}{(x-a)^{\alpha}} \frac{1}{(y-a)^{\alpha}} |x-y|^{\frac{1}{r'}+\frac{1}{s'}} p(x) q(y) dx dy.
\end{aligned} \tag{2.9}$$

Proof. The proof is evident and hence it is omitted. □

Remark 4. If we choose $a = 0$ and $\alpha = 1$ in Theorem 8, then the inequality (2.9) reduces to the inequality (1.2).

Let us now recall the Saigo integral definition [8, 11, 12]:

Definition 4. Let $\alpha > 0$, $\beta, \eta \in \mathbb{R}$, then the Saigo fractional integral $\mathrm{J}_{0,t}^{\alpha,\beta,\eta}$ of order α for real-valued continuous function $f(t)$ is defined by

$$\mathrm{I}_{0,t}^{\alpha,\beta,\eta}\{f(t)\} = \frac{t^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} {}_2\mathrm{F}_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}\right) f(x) dx, \tag{2.10}$$

here, the function ${}_2\mathrm{F}_1(-)$ in the right-hand side of (2.10) is the Gaussian hypergeometric function defined by

$${}_2\mathrm{F}_1(a,b;c;t) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!},$$

and $(a)_n$ is the Pochhammer symbol

$$(a)_0 = 1, \qquad (a)_n = a(a+1)\dots(a+n-1).$$

Using this definition and with some arguments used in Theorem 5, we can establish the following two results.

Theorem 9. *Let f and g be two differentiable functions on $[0,\infty)$ and p be positive and integrable function on $[0,\infty)$. If $f' \in L_r[a,b]$, $g' \in L_s[a,b]$, r, s, $\gamma > 1$ with*

$\frac{1}{r}+\frac{1}{r'}=1$, $\frac{1}{s}+\frac{1}{s'}=1$ *and* $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$, *then for all* $t>0$, $\alpha>0$ *and* β, $\eta\in\mathbb{R}$ *with* $\alpha+\beta\geq 0$ *and* $\eta\leq 0$ *we have*

$$
\begin{aligned}
&2\left|\mathrm{I}_{0,t}^{\alpha,\beta,\eta}p(t)\,\mathrm{I}_{0,t}^{\alpha,\beta,\eta}pfg(t)-\mathrm{I}_{0,t}^{\alpha,\beta,\eta}pg(t)\,\mathrm{I}_{0,t}^{\alpha,\beta,\eta}pf(t)\right|\\
&\leq\left[||f'||_r^{\gamma}\frac{t^{-\alpha-\beta}}{\Gamma(\beta)}\int_0^t\int_0^t(t-x)^{\alpha-1}(t-y)^{\alpha-1}\,{}_2\mathrm{F}_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right)\right.\\
&\qquad\left.\times|x-y|^{\frac{1}{r'}+\frac{1}{s'}}p(x)p(y)dxdy\right]^{\frac{1}{\gamma}}\\
&\quad\times\left[||g'||_s^{\gamma}\frac{t^{-\alpha-\beta}}{\Gamma(\beta)}\int_0^t\int_0^t(t-x)^{\alpha-1}(t-y)^{\alpha-1}\,{}_2\mathrm{F}_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{y}{t}\right)\right.\\
&\qquad\left.\times|x-y|^{\frac{1}{r'}+\frac{1}{s'}}p(x)p(y)dxdy\right]^{\frac{1}{\gamma'}}\\
&\leq||f'||_r||g'||_s\frac{t^{-2\alpha-2\beta}}{\Gamma^2(\beta)}\int_0^t\int_0^t(t-x)^{\alpha-1}(t-y)^{\alpha-1}\,{}_2\mathrm{F}_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right)\\
&\quad\times{}_2\mathrm{F}_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{y}{t}\right)|x-y|^{\frac{1}{r'}+\frac{1}{s'}}p(x)p(y)dxdy.
\end{aligned}
$$

Proof. Let us define

$$H(x,y):=(f(x)-f(y)(g(x)-g(y)));\qquad x,y\in(0,t),t>0. \tag{2.11}$$

Multiplying (2.11) by $\frac{t^{-\alpha-\beta}}{\Gamma(\beta)}(t-x)_2^{\alpha-1}\mathrm{F}_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right)p(x)$ and integrating the resulting identity with respect to x from 0 to t, we can write

$$
\begin{aligned}
&\frac{t^{-\alpha-\beta}}{\Gamma(\beta)}\int_0^t(t-x)_2^{\alpha-1}\mathrm{F}_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right)p(x)H(x,y)dx\\
&=\mathrm{I}_{0,t}^{\alpha,\beta,\eta}pfg(t)-g(y)\,\mathrm{I}_{0,t}^{\alpha,\beta,\eta}pf(t)-f(y)\,\mathrm{I}_{0,t}^{\alpha,\beta,\eta}pg(t)+f(y)g(y)\,\mathrm{I}_{0,t}^{\alpha,\beta,\eta}p(t).
\end{aligned}\tag{2.12}
$$

Again, multiplying (2.12) by $\frac{t^{-\alpha-\beta}}{\Gamma(\beta)}(t-y)_2^{\alpha-1}\mathrm{F}_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{y}{t}\right)p(y)$ and integrating the resulting identity with respect to y from 0 to t, we can write

$$
\begin{aligned}
&\frac{t^{-2\alpha-2\beta}}{\Gamma^2(\beta)}\int_0^t\int_0^t(t-x)^{\alpha-1}(t-y)_2^{\alpha-1}\mathrm{F}_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right)\\
&\qquad\times{}_2\mathrm{F}_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{y}{t}\right)p(x)p(y)H(x,y)dxdy\\
&\qquad=2\left[\mathrm{I}_{0,t}^{\alpha,\beta,\eta}p(t)\,\mathrm{I}_{0,t}^{\alpha,\beta,\eta}pfg(t)-\mathrm{I}_{0,t}^{\alpha,\beta,\eta}pg(t)\,\mathrm{I}_{0,t}^{\alpha,\beta,\eta}pf(t)\right].
\end{aligned}
$$

Using the same arguments as in the proof of Theorem 5, we obtain the desired result. □

We end this paper by presenting to the reader the following result.

Theorem 10. *Let f and g be two differentiable functions on $[0,\infty)$ and p, q be positive and integrable function on $[0,\infty)$. If $f' \in L_r[a,b]$, $g' \in L_s[a,b]$, r, s, $\gamma > 1$ with $\frac{1}{r}+\frac{1}{r'}=1$, $\frac{1}{s}+\frac{1}{s'}=1$ and $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$, then for all $t>0$, $\alpha>0$ and β, $\eta \in \mathbb{R}$ with $\alpha+\beta \geq 0$ and $\eta \leq 0$ we have*

$$\begin{aligned}
&\Big| \mathrm{I}_{0,t}^{\alpha,\beta,\eta} q(t) \mathrm{I}_{0,t}^{\alpha,\beta,\eta} pfg(t) + \mathrm{I}_{0,t}^{\alpha,\beta,\eta} \mathrm{I}_{0,t}^{\alpha,\beta,\eta} qfg(t) \\
&- \mathrm{I}_{0,t}^{\alpha,\beta,\eta} pf(t) \mathrm{I}_{0,t}^{\alpha,\beta,\eta} qg(t) - \mathrm{I}_{0,t}^{\alpha,\beta,\eta} qf(t) \mathrm{I}_{0,t}^{\alpha,\beta,\eta} pg(t) \Big| \\
&\leq ||f'||_r ||g'||_s \frac{t^{-2\alpha-2\beta}}{\Gamma^2(\beta)} \int_0^t \int_0^t (t-x)^{\alpha-1} (t-y)^{\alpha-1} {}_2\mathrm{F}_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right) \\
&\quad \times {}_2\mathrm{F}_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{y}{t}\right) |x-y|^{\frac{1}{r'}+\frac{1}{s'}} p(x)q(y)dxdy.
\end{aligned}$$

Proof. Multiplying (2.12) by $\frac{t^{-\alpha-\beta}}{\Gamma(\beta)}(t-y)^{\alpha-1} {}_2\mathrm{F}_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{y}{t}\right) q(y)$ and integrating the resulting identity with respect to y from a to t, we can write

$$\begin{aligned}
&\Big| \mathrm{I}_{0,t}^{\alpha,\beta,\eta} q(t) \mathrm{I}_{0,t}^{\alpha,\beta,\eta} pfg(t) + \mathrm{I}_{0,t}^{\alpha,\beta,\eta} p(t) \mathrm{I}_{0,t}^{\alpha,\beta,\eta} qfg(t) \\
&- \mathrm{I}_{0,t}^{\alpha,\beta,\eta} pf(t) \mathrm{I}_{0,t}^{\alpha,\beta,\eta} qg(t) - \mathrm{I}_{0,t}^{\alpha,\beta,\eta} qf(t) \mathrm{I}_{0,t}^{\alpha,\beta,\eta} pg(t) \Big| \\
&\leq \frac{t^{-2\alpha-2\beta}}{\Gamma^2(\beta)} \int_0^t \int_0^t (t-x)^{\alpha-1} (t-y)^{\alpha-1} {}_2\mathrm{F}_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right) \\
&\qquad {}_2\mathrm{F}_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{y}{t}\right) \\
&\times |x-y|^{\frac{1}{r'}+\frac{1}{s'}} \left| \int_y^x |f'(u)|^r du \right|^{\frac{1}{r}} \left| \int_y^x |g'(w)|^s dw \right|^{\frac{1}{s}} p(x)q(y)dxdy.
\end{aligned}$$

Some easy calculations allow us to achieve the proof of this theorem. □

3. Conclusion

In this paper, we presented several fractional integral inequalities related to the weighted and the extended Chebyshev functional using generalized fractional integral operators, new conformable fractional integral operators and Saigo fractional integral operator, respectively. The results exhibited in Section 2 generalized the work earlier done by Dahmani [4] for Riemann–Liouville fractional integral operator.

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CONGRUENCES RELATED TO BELL POLYNOMIALS VIA A DIFFERENTIAL OPERATOR

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Abstract. In this paper, by the differential operator we give some congruences and new proof for some known congruence concerning the Bell polynomials, derangement polynomials and r-Lah polynomials.

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Keywords: Congruence, differential operator, Bell polynomials, Derangement polynomials, r-Lah polynomials

1. INTRODUCTION

Recall that the n-th Bell polynomial (named also Touchard polynomial) $\mathcal{B}_n(x)$ and the n-th r-Bell polynomial $\mathcal{B}_{n,r}(x)$ are defined by

$$\mathcal{B}_0(x)=1 \quad \text{and} \quad \mathcal{B}_n(x)=\sum_{k=0}^{n}\left\{ n \atop k \right\} x^k,$$

$$\mathcal{B}_{0,r}(x)=1 \quad \text{and} \quad \mathcal{B}_{n,r}(x)=\sum_{k=0}^{n}\left\{ n+r \atop k+r \right\}_r x^k,$$

where $\left\{ n \atop k \right\}$ is (n,k)-th Stirling number of the second kind which counts the number of partition of the set $[n]$ into k non-empty subsets, and $\left\{ n \atop k \right\}_r$ is the r-Stirling numbers of the second kind which counts the number of partition of the set $[n]$ into k non-empty subsets such that the numbers $1,\ldots,r$ are in distinct subsets see [2].
The numbers $\mathcal{B}_n := \mathcal{B}_n(1)$ is the n-th Bell number which counts the number of all partitions of the set $[n] := \{1,\ldots,n\}$ and $\mathcal{B}_{n,r} := \mathcal{B}_{n,r}(1)$ is the n-th r-Bell number which counts the number of all partitions of set $[n+r]$ into $k+r$ non-empty subsets such that the first r elements are in distinct subsets, see [6]. These polynomials are also given by the Dobinski formula

$$\mathcal{B}_n(x)=\exp(-x)\sum_{j\geq 0} j^n \frac{x^j}{j!} \quad \text{and} \quad \mathcal{B}_{n,r}(x)=\exp(-x)\sum_{j\geq 0}(j+r)^n \frac{x^j}{j!}. \tag{1.1}$$

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In 1933 Touchard [11] established for any prime number p the following congruence

$$\mathcal{B}_{n+p} \equiv \mathcal{B}_{n+1} + \mathcal{B}_n \pmod{p}, \qquad n \in \mathbb{N}.$$

Later Gertsch and Robert [3], by the umbral calculus method proved that

$$\mathcal{B}_{n+p}(x) \equiv \mathcal{B}_{n+1}(x) + x^p \mathcal{B}_n(x) \pmod{p\mathbb{Z}_p[x]}, \qquad n \in \mathbb{N}.$$

Benyattou and Mihoubi [1] proved that

$$\mathcal{B}_{n+p,r}(x) \equiv \mathcal{B}_{n+1,r}(x) + x^p \mathcal{B}_{n,r}(x) \pmod{p\mathbb{Z}_p[x]}, \qquad n,r \in \mathbb{N},$$

where $\mathbb{Z}_p$ denotes the ring of p-adic integers.

Several congruences involved the Bell polynomials are given in [1,3,9,10] and are linked to other polynomials such the Lah and derangement polynomials. Motivated by the congruences studied in the above references we use in this paper the differential operator and its properties to establish some congruences and to give a new proof for some known congruence concerning the Bell polynomials, r-Bell polynomials, derangement polynomials and r Lah polynomials. This paper is organized as follow: In the next section by the differential operator we define the Bell polynomials also we give a new proofs of some known congruences. In the third section we present some congruences on the r-Lah polynomials and the derangement polynomials.
In the remainder of this paper we use the notations

$$(x)_0 := 1, \qquad (x)_n := x(x-1)\cdots(x-n+1) \quad \text{if} \quad n \geq 1,$$
$$\langle x \rangle_0 := 1 \qquad \langle x \rangle_n := x(x+1)\cdots(x+n-1) \quad \text{if} \quad n \geq 1,$$

and

$$(x)_n = \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} x^k, \qquad x^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} (x)_k,$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ is the absolute Stirling number of the first kind which counts the number of permutations of the set $[n] := \{1,\ldots,n\}$ into k cycles.

2. Congruences on Bell and r-Bell polynomials

In this section we give some properties of the differential operator and a new proof of some known congruences on the Bell and r-Bell polynomials.

Lemma 1. *Let $\mathbf{D} = \frac{d}{dx}$ be the differential operator and $\mathbf{P} = x + x\mathbf{D}$ and let f be a polynomial. Then for any non-negative integers n, r there hold*

$$(\mathbf{P})_r \mathbf{P}^n 1 = \mathbf{P}^n x^r = x^r \mathcal{B}_{n,r}(x), \tag{2.1}$$
$$(\mathbf{P})_r f(\mathbf{P}) 1 = x^r f(\mathbf{P} + r) 1. \tag{2.2}$$

In particular, we get

$$\mathcal{B}_n(x) = \mathbf{P}^{n-m} \mathcal{B}_m(x), \quad 0 \leq m \leq n, \tag{2.3}$$
$$x^r = (\mathbf{P})_r 1, \tag{2.4}$$

where $(\mathbf{x})_r$ *is the Pochhammer symbol defined above.*

Proof. To prove the identity $\mathbf{P}^n x^r = x^r \mathcal{B}_{n,r}(x)$ we proceed by induction $n \geq 0$. For $n = 0$ or $n = 1$, we have

$$\mathbf{P}^0 x^r = x^r \quad \text{and} \quad \mathbf{P}^1 x^r = (x + x\mathbf{D}) x^r = x^{r+1} + r x^r = x^r (x + r) = x^r \mathcal{B}_{1,r}(x).$$

Assume that $\mathbf{P}^k x^r = x^r \mathcal{B}_{k,r}(x)$ for all $k \in \{0, 1, \ldots, n\}$. Then

$$\mathbf{P}^{n+1} x^r = \mathbf{P}(\mathbf{P}^n x^r) = (x + x\mathbf{D}) x^r \mathcal{B}_{n,r}(x) = (r + x) x^r \mathcal{B}_{n,r}(x) + x^{r+1} \mathbf{D} \mathcal{B}_{n,r}(x),$$

and from (1.1) we obtain

$$x\mathbf{D}\mathcal{B}_{n,r}(x) = \mathcal{B}_{n+1,r}(x) - (r + x)\mathcal{B}_{n,r}(x),$$

hence

$$\mathbf{P}^{n+1} x^r = (r + x) x^r \mathcal{B}_{n,r}(x) + x^r (\mathcal{B}_{n+1,r}(x) - (r + x)\mathcal{B}_{n,r}(x)) = x^r \mathcal{B}_{n+1,r}(x),$$

which completes the step induction. We also have

$$(\mathbf{P})_r \mathbf{P}^n 1 = \sum_{k=0}^{r} (-1)^{r-k} \begin{bmatrix} r \\ k \end{bmatrix} \mathbf{P}^{n+k} 1 = \mathbf{P}^n \sum_{k=0}^{r} (-1)^{r-k} \begin{bmatrix} r \\ k \end{bmatrix} \mathcal{B}_k(x) = \mathbf{P}^n x^r = x^r \mathcal{B}_{n,r}(x).$$

The identity (2.2) follows because

$$(\mathbf{P})_r \mathbf{P}^n 1 = x^r \mathcal{B}_{n,r}(x) = x^r \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \mathcal{B}_k(x) = x^r \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \mathbf{P}^k 1 = x^r (\mathbf{P} + r)^n 1.$$

For the particular cases, by taking $r = 0$ in (2.1) we get $\mathcal{B}_n(x) = \mathbf{P}^n 1$. This means that

$$\mathcal{B}_n(x) = \mathbf{P}(\mathbf{P}^{n-1} 1) = \mathbf{P}\mathcal{B}_{n-1}(x) = \cdots = \mathbf{P}^m \mathcal{B}_{n-m}(x),$$

by taking $n = 0$ in (2.1) we get $(\mathbf{P})_r 1 = x^r$. □

Remark 1. Since $\mathcal{B}_{n+m}(x) = \mathbf{P}^n \mathcal{B}_m(x) = \mathbf{P}^m \mathcal{B}_n(x)$, the following symmetric identity follows

$$\mathcal{B}_{n+m}(x) = \sum_{r=0}^{m} \left\{ \begin{matrix} m \\ r \end{matrix} \right\} x^r \mathcal{B}_{n,r}(x) = \sum_{r=0}^{n} \left\{ \begin{matrix} n \\ r \end{matrix} \right\} x^r \mathcal{B}_{m,r}(x).$$

Let $r_1, \ldots, r_q, n \geq 0$ be integers such that $r_1 \leq \cdots \leq r_q$ and let

$$\mathbf{r}_q = (r_1, \ldots, r_q), \qquad |\mathbf{r}_q| = r_1 + \cdots + r_q.$$

Recall the $\mathbf{r}_q$-Stirling numbers and the $\mathbf{r}_q$ -Bell polynomials introduced and studied by Maamra and Mihoubi [4, 5, 7], can be defined by

$$\sum_{j=0}^{n+|\mathbf{r}_{q-1}|} \left\{ \begin{matrix} n + |\mathbf{r}_q| \\ j + r_q \end{matrix} \right\}_{\mathbf{r}_q} (x)_j = (x + r_q)_{r_1} \cdots (x + r_q)_{r_{q-1}} (x + r_q)^n,$$

$$\mathcal{B}_n(x; \mathbf{r}_q) = \sum_{j=0}^{n+|\mathbf{r}_{q-1}|} \left\{ \begin{matrix} n + |\mathbf{r}_q| \\ j + r_q \end{matrix} \right\}_{\mathbf{r}_q} x^j.$$

Lemma 2. *There holds*

$$(\mathbf{P})_{r_1}\cdots(\mathbf{P})_{r_{q-1}}\mathbf{P}^n x^{r_q} = (\mathbf{P})_{r_1}\cdots(\mathbf{P})_{r_q}\mathbf{P}^n 1 = x^{r_q}\mathcal{B}_n(x;\mathbf{r}_q).$$

Proof. Setting $(u)_{r_1}\cdots(u)_{r_q} = \sum_{k=0}^{|\mathbf{r}_q|} a_k(\mathbf{r}_q)u^k$.
From Theorem 7 given in [4] and Lemma 1 we have

$$\begin{aligned} x^{r_q}\mathcal{B}_n(x;\mathbf{r}_q) &= \sum_{k=0}^{|\mathbf{r}_q|} a_k(\mathbf{r}_q)\mathcal{B}_{n+k}(x) = \sum_{k=0}^{|\mathbf{r}_q|} a_k(\mathbf{r}_q)\mathbf{P}^{n+k}1 = \mathbf{P}^n\sum_{k=0}^{|\mathbf{r}_q|} a_k(\mathbf{r}_q)\mathbf{P}^k 1 \\ &= (\mathbf{P})_{r_1}\cdots(\mathbf{P})_{r_{q-1}}(\mathbf{P})_{r_q}\mathbf{P}^n 1 = (\mathbf{P})_{r_1}\cdots(\mathbf{P})_{r_{q-1}}\mathbf{P}^n x^{r_q}. \end{aligned}$$

□

Lemma 3. *Let f be a polynomial in $\mathbb{Z}[x]$. For any non negative integers $n, s \geq 1$, and for any prime p there holds*

$$f(\mathbf{P})(\mathbf{P}^{p^s} - \mathbf{P})1 \equiv (x^p + x^{p^2} + \cdots + x^{p^s})f(\mathbf{P})1 \pmod{p\mathbb{Z}_p[x]}.$$

Proof. It suffices to take $f(x) = x^n$ and proceed by induction on s. Indeed, for $s = 1$, use the Touchard's congruence for polynomials

$$B_{n+p}(x) \equiv B_{n+1}(x) + x^p B_n(x) \pmod{p\mathbb{Z}_p[x]}$$

to get

$$\begin{aligned} \mathbf{P}^n(\mathbf{P}^p - \mathbf{P})1 &= \mathbf{P}^{n+p}1 - \mathbf{P}^{n+1}1 = B_{n+p}(x) - B_{n+1}(x) \\ &\equiv x^p B_n(x) = x^p\mathbf{P}^n 1 \pmod{p\mathbb{Z}_p[x]}. \end{aligned}$$

Assume it is true for s. Then

$$\begin{aligned} \mathbf{P}^n(\mathbf{P}^{p^{s+1}} - \mathbf{P})1 &= ((\mathbf{P}^{p^s} - \mathbf{P} + \mathbf{P})^p - \mathbf{P})\mathbf{P}^n 1 \\ &\equiv ((\mathbf{P}^{p^s} - \mathbf{P})^p + \mathbf{P}^p - \mathbf{P})\mathbf{P}^n 1 \\ &= (\mathbf{P}^{p^s} - \mathbf{P})^p\mathbf{P}^n 1 + (\mathbf{P}^p - \mathbf{P})\mathbf{P}^n 1 \\ &\equiv (x^p + x^{p^2} + \cdots + x^{p^s})(\mathbf{P}^{p^s} - \mathbf{P})^{p-1}\mathbf{P}^n 1 + x^p\mathbf{P}^n 1 \\ &\vdots \\ &\equiv (x^p + x^{p^2} + \cdots + x^{p^s})^p\mathbf{P}^n 1 + x^p\mathbf{P}^n 1 \\ &\equiv (x^{p^2} + x^{p^3} + \cdots + x^{p^{s+1}})\mathbf{P}^n 1 + x^p\mathbf{P}^n 1 \\ &= (x^p + x^{p^2} + \cdots + x^{p^{s+1}})\mathbf{P}^n 1 \pmod{p\mathbb{Z}_p[x]} \end{aligned}$$

hence, the proof is completed. □

Now we give new proof for a congruence concerning $\mathcal{B}_n(x;\mathbf{r}_q)$.

Proposition 1. *Let $n,r,r_q \geq \cdots \geq r_1 \geq 0$ and $s \geq 1$ be non-negative integers. For any prime p there holds*

$$\mathcal{B}_{n+p^s}(x;\mathbf{r}_q) \equiv (x^p+\cdots+x^{p^s})\mathcal{B}_n(x;\mathbf{r}_q)+\mathcal{B}_{n+1}(x;\mathbf{r}_q) \pmod{p\mathbb{Z}_p[x]}. \tag{2.5}$$

In particular, for $s=q=1$, $r_q=r$ and or $r=0$ we get

$$\mathcal{B}_{n+p,r}(x) \equiv x^p\mathcal{B}_{n,r}(x)+\mathcal{B}_{n+1,r}(x) \pmod{p\mathbb{Z}_p[x]},$$
$$\mathcal{B}_{n+p}(x) \equiv x^p\mathcal{B}_n(x)+\mathcal{B}_{n+1}(x) \pmod{p\mathbb{Z}_p[x]}.$$

Proof. For (2.5), by Lemma 2 and Lemma 3 we have

$$\begin{aligned}
x^{r_q}\mathcal{B}_{n+p^s}(x;\mathbf{r}_q) &= x^{r_q}(\mathcal{B}_{n+p^s}(x;\mathbf{r}_q)-\mathcal{B}_{n+1}(x;\mathbf{r}_q))+x^{r_q}\mathcal{B}_{n+1}(x;\mathbf{r}_q)\\
&= (\mathbf{P})_{r_1}\cdots(\mathbf{P})_{r_q}\mathbf{P}^n(\mathbf{P}^{p^s}-\mathbf{P})1+x^{r_q}\mathcal{B}_{n+1}(x;\mathbf{r}_q)\\
&\equiv (x^p+x^{p^2}+\cdots+x^{p^s})(\mathbf{P})_{r_1}\cdots(\mathbf{P})_{r_q}\mathbf{P}^n1+x^{r_q}\mathcal{B}_{n+1}(x;\mathbf{r}_q)\\
&= x^{r_q}(x^p+\cdots+x^{p^s})\mathcal{B}_n(x;\mathbf{r}_q)+x^{r_q}\mathcal{B}_{n+1}(x;\mathbf{r}_q) \pmod{p\mathbb{Z}_p[x]}.
\end{aligned}$$

This completes the proof. □

3. Congruences on the r-Lah polynomials and the derangement polynomials

In this section, we give some general congruences on the r-Lah polynomials and the derangement polynomials.To start, let us give a short introduction to these polynomials. Recall that the (n,k)-*th* r-Lah number $L_r(n,k)$ counts the number of partitions of the set $[n+r]$ into $k+r$ ordered list, such that the numbers of set $[r]$ are in distinct lists, see also [8], and the r-Lah polynomials associated to r-Lah number are defined by

$$\mathcal{L}_{n,r}(x)=\sum_{k=0}^{n}L_r(n,k)x^k,$$

with exponential generating function

$$\sum_{i=0}^{n}\mathcal{L}_{n,r}(x)\frac{t^n}{n!}=\frac{1}{(1-t)^{2r}}\exp\left(\frac{t}{1-t}x\right).$$

The derangement polynomials are defined by

$$\mathcal{D}_n(x)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{D}_{n-k}x^k=\sum_{k=0}^{n}\binom{n}{k}k!\,(x-1)^{n-k},$$

where $\mathcal{D}_n(0):=\mathcal{D}_n$ is the number of derangements of n elements, see [10].

Lemma 4. *For any non-negative integer n,r there hold*

$$\mathcal{L}_{n,r}(x)=\langle\mathbf{P}+2r\rangle_n 1, \tag{3.1}$$
$$\mathcal{D}_n(1-x)=(-1)^n(\mathbf{P}-1)_n 1. \tag{3.2}$$

Proof. Since $\langle x\rangle_n := (x+n-1)_n$ and the property

$$\langle x\rangle_{n+m} = \langle x\rangle_n \langle x+n\rangle_m,$$

then, from the known identity [8] $\langle x+2r\rangle_n = \sum_{k=0}^{n} L_r(n,k)\, x^k$, and by the relation (2.4) the r-Lah polynomials can be written as

$$\mathcal{L}_{n,r}(x) = \sum_{k=0}^{n} L_r(n,k)\, x^k = \sum_{k=0}^{n} L_r(n,k)\, (\mathbf{P})_k\, 1 = \langle \mathbf{P}+2r\rangle_n\, 1. \tag{3.3}$$

We also have by the relation (2.4)

$$(\mathbf{P}-1)_n\, 1 = \sum_{k=0}^{n} \binom{n}{k} (-1)_k\, (\mathbf{P})_{n-k}\, 1 = \sum_{k=0}^{n} \binom{n}{k} (-1)_k\, x^{n-k} = (-1)^n\, \mathcal{D}_n(1-x).$$

□

Proposition 2. *For any prime number p and any integers $n, m, r \geq 0$, there holds*

$$\mathcal{L}_{n+mp,r}(x) \equiv x^{mp}\, \mathcal{L}_{n,r}(x) \pmod{p\mathbb{Z}_p[x]}.$$

Proof. From (3.1) we have

$$\begin{aligned}
\mathcal{L}_{n+mp,r}(x) &= \langle \mathbf{P}+2r\rangle_{n+mp} 1 = \langle \mathbf{P}+2r\rangle_p \langle \mathbf{P}+2r+p\rangle_{n+(m-1)p} 1 \\
&\equiv (\mathbf{P}+2r+p-1)_p \langle \mathbf{P}+2r\rangle_{n+(m-1)p} 1 \\
&= \langle \mathbf{P}+2r\rangle_{n+(m-1)p} \sum_{k=0}^{p} \binom{p}{k} (2r+p-1)_{p-k}\, (\mathbf{P})_k\, 1 \\
&\equiv \langle \mathbf{P}+2r\rangle_{n+(m-1)P} \Big((2r+p-1)_p + (\mathbf{P})_p \Big) 1 \\
&= p! \binom{2r+p-1}{p} \langle \mathbf{P}+2r\rangle_{n+(m-1)p} 1 + (\mathbf{P})_p \langle \mathbf{P}+2r\rangle_{n+(m-1)p} 1 \\
&\equiv x^p \langle \mathbf{P}+2r+p\rangle_{n+(m-1)p} 1 \equiv x^p\, \mathcal{L}_{n+(m-1)p,r}(x) \pmod{p\mathbb{Z}_p[x]}.
\end{aligned}$$

So, we get successively

$$\begin{aligned}
\mathcal{L}_{n+mp,r}(x) &\equiv x^p\, \mathcal{L}_{n+(m-1)p,r}(x) \equiv \dots \\
&\equiv x^{(m-1)p}\, \mathcal{L}_{n+p,r}(x) \equiv x^{mp}\, \mathcal{L}_{n,r}(x) \pmod{p\mathbb{Z}_p[x]}.
\end{aligned}$$

□

Proposition 3. *Let p be a prime number and m, n be non-negative numbers. There holds*

$$\mathcal{D}_{n+mp}(1-x) \equiv (-x)^{mp}\, \mathcal{D}_n(1-x) \pmod{p\mathbb{Z}_p[x]}.$$

For $x = 1$, we obtain

$$\mathcal{D}_{n+mp} \equiv (-1)^{mp}\, \mathcal{D}_n \pmod{p}.$$

Proof. By (3.2), the identity $(x)_{n+m} = (x)_n (x-n)_m$ and the congruence

$$\binom{p}{j} \equiv 0 \pmod{p}, \quad 1 \le j \le p-1,$$

we obtain

$$\begin{aligned}
\mathcal{D}_{n+p}(1-x) &= (-1)^{n+p} (\mathbf{P}-1)_{n+p} 1 = (-1)^{n+p} (\mathbf{P}-1)_p (\mathbf{P}-1-p)_n 1 \\
&= (-1)^{n+p} \sum_{j=0}^{p} \binom{p}{j} (-1)_{p-j} (\mathbf{P})_j (\mathbf{P}-1-p)_n 1 \\
&\equiv (-1)^{n+p} (-1)_p (\mathbf{P}-1)_n + (-1)^{n+p} (\mathbf{P})_p (\mathbf{P}-1)_n 1 \\
&\equiv (-1)^{n+p} x^p (\mathbf{P}+p-1)_n 1 \equiv (-1)^{n+p} x^p (\mathbf{P}-1)_n 1 \\
&= (-x)^p \mathcal{D}_n (1-x) \pmod{p\mathbb{Z}_p[x]}
\end{aligned}$$

and one can proceed easily by induction on $m \ge 0$ to complete the proof.

□

Proposition 4. *For any non-negative integers n, r and for any prime number p, there holds*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k (2r+n)_{n-k} \mathcal{D}_k (1-x) = \mathcal{L}_{n,r}(x). \tag{3.4}$$

In particular for $n = p-1$ we get

$$\sum_{k=0}^{p-1} (2r+p-1)_{p-1-k} \mathcal{D}_k (1-x) \equiv \mathcal{L}_{p-1,r}(x) \pmod{p\mathbb{Z}_p[x]}, \tag{3.5}$$

for $n = p-1$, $r = 0$ we get

$$\sum_{k=0}^{p-1} \frac{\mathcal{D}_k (1-x)}{k!} \equiv -\mathcal{L}_{p-1}(x) \pmod{p\mathbb{Z}_p[x]}. \tag{3.6}$$

Proof. For (3.4) we have

$$\begin{aligned}
\mathcal{L}_{n,r}(x) &= \langle \mathbf{P}+2r \rangle_n 1 = (\mathbf{P}+2r+n-1)_n 1 \\
&= \sum_{k=0}^{n} \binom{n}{k} (2r+n)_{n-k} (\mathbf{P}-1)_k 1 \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^k (2r+n)_{n-k} \mathcal{D}_k (1-x).
\end{aligned}$$

For (3.5) and (3.6) take $n = p-1$ or $n = p-1$, $r = 0$ and use the congruences

$$\binom{p-1}{k} \equiv (-1)^k \pmod{p}, \quad k! (p-1)_{p-1-k} \equiv -1 \pmod{p} \quad 0 \le k \le p-1.$$

□

Proposition 5. *For any non-negative integers n and for any prime number p, there holds*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\mathcal{B}_{n-k}(x)=\sum_{k=0}^{n}\left\{\begin{matrix}n\\k\end{matrix}\right\}(-1)^k\mathcal{D}_k(1-x). \tag{3.7}$$

In particular for $n=p-1$ or $n=p$ we obtain

$$\sum_{k=0}^{p-1}\mathcal{B}_{p-1-k}(x)\equiv\sum_{k=0}^{p-1}\left\{\begin{matrix}p-1\\k\end{matrix}\right\}(-1)^k\mathcal{D}_k(1-x)\pmod{p\mathbb{Z}_p[x]}, \tag{3.8}$$

$$\mathcal{D}_p(1-x)\equiv -x^p\pmod{p\mathbb{Z}_p[x]}. \tag{3.9}$$

Proof. For (3.7) we have

$$(\mathbf{P}-1)^n 1=\sum_{k=0}^{n}\left\{\begin{matrix}n\\k\end{matrix}\right\}(\mathbf{P}-1)_k 1.$$

Then

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\mathbf{P}^{n-k}1=\sum_{k=0}^{n}\binom{n}{k}(-1)^k\mathcal{B}_{n-k}(x)=\sum_{k=0}^{n}\left\{\begin{matrix}n\\k\end{matrix}\right\}(-1)^k\mathcal{D}_k(1-x).$$

For (3.8) and (3.9) take $n=p-1$ or $n=p$ and use the congruences

$$\binom{p-1}{k}\equiv(-1)^k\pmod{p},\quad \left\{\begin{matrix}p\\k\end{matrix}\right\}\equiv 0\pmod{p}\quad 1<k<p.$$

□

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THEORETICAL JUSTIFICATION OF VENTCEL'S BOUNDARY CONDITIONS FOR A THIN LAYER THREE-DIMENSIONAL THERMOELASTICITY PROBLEM

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Abstract. In this paper, we consider a model of linear thermoelasticity for an elastic three-dimensional body with high thermal conductivity covered by a thin layer on one of its faces. We show that Ventcel's boundary conditions may be obtained, as the thickness of the rigid body goes to zero.

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1. Introduction and statement of the problem

It is well known that there are two main approaches for the study of problems involving thin layers. We can consider directly the thin layers problems and use adapted numerical methods (see, for example [2, 10, 15]), or we incorporate the thin layer effect through an approximate boundary condition in an approximate way, see for example [5, 8, 13] and the references therein. The second approach will be illustrated in this work by the study of a three-dimensional thermoelasticity coupled system for an elastic body covered on one of its faces by a thin layer or a thin shell of thickness ε.

More precisely, let us consider a three-dimensional elastic body with high thermal conductivity and occupying the set $\overline{\Omega_+} = [0,1]\times[0,1]\times[0,1]$. The boundary of Ω_+ is denoted by $\partial\Omega_+ = \overline{\Sigma}\cup\Gamma_+$, where $\Sigma =]0,1[\times]0,1[\times\{0\}$. The body is clamped on the portion Γ_+ of its boundary and reinforced by a thin shell $\overline{\Omega_-^\varepsilon} = [0,1]\times[0,1]\times[-\varepsilon,0]$, with $\varepsilon > 0$ sufficiently small, on the other part Σ. The boundary of Ω_-^ε is denoted $\partial\Omega_-^\varepsilon = \overline{\Sigma^\varepsilon}\cup\overline{\Sigma}\cup\overline{\Gamma_-^\varepsilon}$, where $\Sigma^\varepsilon =]0,1[\times]0,1[\times\{-\varepsilon\}$ and the whole domain is $\Omega^\varepsilon = \Omega_+\cup\Omega_-^\varepsilon\cup\Sigma$.

In what follows, the variables $u = (u_1,u_2,u_3)$ and θ represent respectively, a displacement vector field and the temperature. The functions $f = (f_1,f_2,f_3)$ and g are the volume forces exerted on the body Ω^ε. We also denote by w', w'' for the time

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derivatives of a function w. We then consider the following transmission system:

$$\begin{cases} u'' - \frac{1}{p}\mathrm{div}(p\nabla u) + \alpha\nabla\theta = f & \text{in } \Omega^\varepsilon \times (0,T) \\ \theta' - \frac{1}{p}\mathrm{div}(p\nabla\theta) + \alpha\mathrm{div}(u') = g & \text{in } \Omega^\varepsilon \times (0,T) \end{cases} \tag{1.1}$$

with Dirichlet conditions on the portion of the boundary $\Gamma_+ \cup \Gamma_-^\varepsilon$

$$u = 0, \quad \theta = 0 \quad \text{on } (\Gamma_+ \cup \Gamma_-^\varepsilon) \times (0,T). \tag{1.2}$$

The boundary conditions on the face Σ^ε are given by

$$-p\partial_\nu u = 0, \quad -p\partial_\nu\theta + \alpha p\gamma_n(u') = 0 \quad \text{on } \Sigma^\varepsilon \times (0,T) \tag{1.3}$$

where ∂_ν (respectively, γ_n) stands for the normal derivative (respectively, normal trace) operator. Here, $\alpha > 0$ is a coupling parameter and p is a discontinuous function on Σ, defined by

$$p = \begin{cases} 1 & \text{on } \Omega_+, \\ 1/\varepsilon & \text{on } \Omega_-^\varepsilon. \end{cases}$$

We define also the transmission conditions through Σ by

$$[u] = 0, \quad [\theta] = 0, \quad [-p\partial_\nu u] = 0, \quad \left[-p\partial_\nu\theta + \alpha p\gamma_n(u')\right] = 0, \text{ on } \Sigma \times (0,T) \tag{1.4}$$

where $[.]$ denotes the skip function through the transmission surface Σ. We associate with the system (1.1) the initial conditions

$$u(0) = u^0, \quad u'(0) = u^1, \quad \theta(0) = \theta^0 \quad \text{in } \Omega^\varepsilon \tag{1.5}$$

where u^0, u^1 and θ^0 are respectively the initial states of the displacement vector, the movement speed and the function of heat dissipation.

Recently, numerical modeling of thermoelasticity problems in a three dimensional region have been studied in [27] and [28]. However, in our case and due to the small thickness of the layer, these numerical computations can become cumbersome, especially for three-dimensional problems. So, our aim is to derive and justify approximate boundary conditions which replace the effect of the thin layer at the junction interface.

The analysis of such kind of problems in the two-dimensional case can be found in [4,6,11,12,16,22–25]. Let us mention the works [1,3,7,14] in the three-dimensional case.

The paper is organized as follows. In Section 2, we show that Problem (1.1)–(1.5) admits a weak solution (u,θ) in a sense that we will specify later on. In Section 3, we construct an approximate problem depending on the thickness ε of the thin layer by scaling argument. In Section 4, we establish a priori estimates which allow us to extract a weakly convergent subsequence. Finally, in Section 5 we show that the weak limit of the obtained approximate problem is a solution to a problem with Ventcel's boundary conditions.

2. Resolution of Problem (1.1)–(1.5)

In this section, we will show that under some assumptions, Problem (1.1)–(1.5) admits a solution in a weak sense that will be specified later on. So, we will need some Sobolev spaces, which we recall in the following definitions:

$$\mathcal{W}(\Omega^\varepsilon) = \left\{ v \in \mathcal{H}^1(\Omega^\varepsilon) \mid v = 0 \text{ on } \Gamma_+ \cup \Gamma_-^\varepsilon \right\}$$

with

$$\mathcal{H}^1(\Omega^\varepsilon) = \left\{ v \in \mathcal{L}^2(\Omega^\varepsilon) \mid \partial_x v \in \mathcal{L}^2(\Omega^\varepsilon),\ \partial_y v \in \mathcal{L}^2(\Omega^\varepsilon),\ \partial_z v \in \mathcal{L}^2(\Omega^\varepsilon) \right\}$$

where

$$\mathcal{L}^2(\Omega^\varepsilon) = \left\{ v = (v_1, v_2, v_3)/v_i \in L^2(\Omega^\varepsilon) \mid i = 1,2,3 \right\}$$

and

$$W(\Omega^\varepsilon) = \left\{ \varphi \in H^1(\Omega^\varepsilon) \mid \varphi = 0 \text{ on } \Gamma_+ \cup \Gamma_-^\varepsilon \right\}$$

with

$$H^1(\Omega^\varepsilon) = \left\{ \varphi \in L^2(\Omega^\varepsilon) \mid \varphi' \in L^2(\Omega^\varepsilon) \right\}.$$

Lemma 1. *$\mathcal{W}(\Omega^\varepsilon)$ and $W(\Omega^\varepsilon)$ are Hilbert spaces.*

Proof. $\Gamma_+ \cup \Gamma_-^\varepsilon$ is a piece of the boundary $\partial\Omega^\varepsilon$ of nonzero measure, the two spaces $\mathcal{W}(\Omega^\varepsilon)$ and $W(\Omega^\varepsilon)$ respectively with norms $\|\nabla v\|_{\mathcal{L}^2(\Omega^\varepsilon)}$ and $\|\nabla \varphi\|_{L^2(\Omega^\varepsilon)}$ are Hilbert spaces, see ([20], Chapter 1, paragraph 1.8). □

2.1. *Weak formulation*

By a formal calculation and assuming that the Green formula

$$\langle \operatorname{div}(u), \varphi \rangle_{\Omega^\varepsilon} = -\langle u, \nabla \varphi \rangle_{\Omega^\varepsilon} + \langle \gamma_n(u), \gamma_0(\varphi) \rangle_{\partial\Omega^\varepsilon}$$

holds, we obtain a weak formulation (P^*) of the problem in the form

$$\begin{aligned} &\frac{d}{dt}\left(\langle p\partial_t u, v\rangle_{\Omega^\varepsilon}\right) + \langle p\nabla u, \nabla v\rangle_{\Omega^\varepsilon} + \frac{d}{dt}\langle p\theta, \varphi\rangle_{\Omega^\varepsilon} \\ &+ \langle p\nabla\theta, \nabla\varphi\rangle_{\Omega^\varepsilon} + \alpha\langle p\nabla\theta, v\rangle_{\Omega^\varepsilon} - \alpha\langle p\partial_t u, \nabla\varphi\rangle_{\Omega^\varepsilon} = \langle pf, v\rangle_{\Omega^\varepsilon} + \langle pg, \varphi\rangle_{\Omega^\varepsilon} \end{aligned} \tag{2.1}$$

with

$$\langle pu(0,.), v\rangle_{\Omega^\varepsilon} = \left\langle pu^0, v\right\rangle_{\Omega^\varepsilon} \tag{2.2}$$

$$\frac{d}{dt}\langle pu, v\rangle_{\Omega^\varepsilon}(0,.) = \left\langle pu^1, v\right\rangle_{\Omega^\varepsilon} \tag{2.3}$$

$$\langle p\theta, \varphi\rangle_{\Omega^\varepsilon}(0,.) = \left\langle p\theta^0, \varphi\right\rangle_{\Omega^\varepsilon} \tag{2.4}$$

for all $(v, \varphi) \in \mathcal{W}(\Omega^\varepsilon) \times W(\Omega^\varepsilon)$.

2.2. *Existence and uniqueness of a weak solution*

Proposition 1. *Let us assume that*

$$f \in L^2(0,T;\mathcal{L}^2(\Omega^\varepsilon)), \qquad g \in L^2(0,T;L^2(\Omega^\varepsilon),$$

$$u^0 \in \mathcal{W}(\Omega^\varepsilon), \qquad u^1 \in \mathcal{L}^2(\Omega^\varepsilon), \qquad \theta^0 \in W(\Omega^\varepsilon),$$

then the weak problem (2.1)–(2.4) admits a unique solution (u,θ) *satisfying*

$$u \in L^\infty(0,T;\mathcal{W}(\Omega^\varepsilon)), \qquad \partial_t u \in L^\infty(0,T;\mathcal{L}^2(\Omega^\varepsilon)),$$

$$\theta \in L^\infty(0,T;L^2(\Omega^\varepsilon)) \cap L^2(0,T;W(\Omega^\varepsilon)).$$

Proof. We will demonstrate the above proposition in three steps. First, we establish *a priori* estimates which will allow us at second time through the Faedo–Galerkin method to construct convergent sequences to a solution of our problem in the weak sense and finally we establish uniqueness.

Step 1: *A priori* estimate

Let us now consider the energy function, see the definition e.g. in [17, 19, 21] and references therein. Then, we denote by $E(t)$ the energy of the system for $t \neq 0$

$$E(t) = \frac{1}{2}\left(\|\partial_t u\|^2_{\Omega_+} + \frac{1}{\varepsilon}\|\partial_t u\|^2_{\Omega^\varepsilon_-} + \|\nabla u\|^2_{\Omega_+} + \frac{1}{\varepsilon}\|\nabla u\|^2_{\Omega^\varepsilon_-} + \|\theta\|^2_{\Omega_+} + \frac{1}{\varepsilon}\|\theta\|^2_{\Omega^\varepsilon_-} \right)$$

then we set

$$E(0) = \frac{1}{2}\left(\|u^1\|^2_{\Omega_+} + \frac{1}{\varepsilon}\|u^1\|^2_{\Omega^\varepsilon_-} + \|\nabla u^0\|^2_{\Omega_+} + \frac{1}{\varepsilon}\|\nabla u^0\|^2_{\Omega^\varepsilon_-} + \|\theta^0\|^2_{\Omega_+} + \frac{1}{\varepsilon}\|\theta^0\|^2_{\Omega^\varepsilon_-} \right).$$

It suffices then to take in the weak problem $(2.1)-(2.4)$ $v = \partial_t u$ and $\varphi = \theta$ to obtain the so-called energy equality

$$E(t) + \int_0^t \left(\|\nabla\theta\|^2_{\Omega_+} + \frac{1}{\varepsilon}\|\nabla\theta\|^2_{\Omega^\varepsilon_-} \right) ds = E(0) + \int_0^t \left(\langle f, \partial_t u\rangle_{\Omega_+} + \langle f, \partial_t u\rangle_{\Omega^\varepsilon_-} + \langle g,\theta\rangle_{\Omega_+} + \langle g,\theta\rangle_{\Omega^\varepsilon_-} \right) ds.$$

In virtue of Cauchy–Schwarz inequality, we establish an estimate of the energy in function of the initial conditions and data of the problem

$$E(t) \leq c\left(E(0) + \int_0^t \left(\|f\|^2_{\Omega^\varepsilon} + \|g\|^2_{\Omega^\varepsilon} \right) ds \right) \qquad \forall t \in (0,T) \tag{2.5}$$

where c is a positive constant. Therefore, thanks to the assumptions of the proposition, the energy of the system is bounded in time

$$E(t) \leq c \qquad \forall t \in (0,T) \tag{2.6}$$

where c is another positive constant which depends only on the data. This also gives an *a priori* estimate of θ in $L^2(0,T;W(\Omega^\varepsilon))$. Indeed, we have

$$\int_0^T \left(\|\nabla\theta\|^2_{\Omega_+} + \frac{1}{\varepsilon}\|\nabla\theta\|^2_{\Omega^\varepsilon_-} \right) ds \le c \tag{2.7}$$

where c is another positive constant which depends only on the data.

Step 2: Existence - Faedo–Galerkin method.

$\mathcal{W}(\Omega^\varepsilon)$ and $W(\Omega^\varepsilon)$ are separable Hilbert spaces. Let $(v_i)_i$ (respectively, $(\varphi_i)_i$) a basis of $\mathcal{W}(\Omega^\varepsilon)$ (respectively, $W(\Omega^\varepsilon)$). We introduce then the approximate solution $(u_m(t),\theta_m(t))$ defined by

$$u_m(t) = \sum_{i=1}^m h_{im}(t)v_i \quad \text{and} \quad \theta_m(t) = \sum_{i=1}^m k_{im}(t)\varphi_i$$

which has to satisfy the approximate problem

$$\begin{aligned} &\frac{d}{dt}\left(\langle p\partial_t u_m, v_j\rangle_{\Omega^\varepsilon}\right) + \langle p\nabla u_m, \nabla v_j\rangle_{\Omega^\varepsilon} + \frac{d}{dt}\langle p\theta_m, \varphi_j\rangle_{\Omega^\varepsilon} + \langle p\nabla\theta_m, \nabla\varphi_j\rangle_{\Omega^\varepsilon} \\ &+ \alpha\langle p\nabla\theta_m, v_j\rangle_{\Omega} - \alpha\langle p\partial_t u_m, \nabla\varphi_j\rangle_{\Omega^\varepsilon} = \langle pf, v_j\rangle_{\Omega^\varepsilon} + \langle pg, \varphi_j\rangle_{\Omega^\varepsilon} \end{aligned} \tag{2.8}$$

with

$$\begin{cases} u_m(0) = u^{0m} = \sum\limits_{j=1}^m h_{jm}(0)v_j, \quad \partial_t u_m(0) = u^{1m} = \sum\limits_{j=1}^m h'_{jm}(0)v_j, \\ \theta_m(0) = \theta^{0m} = \sum\limits_{j=1}^m k_{jm}(0)\varphi_j, \end{cases} \tag{2.9}$$

for all $j = 1,\dots,m$. The coefficients $h_{jm}(0)$ and $k_{jm}(0)$ will be chosen such that

$$u^{0m} \to u^0 \text{ in } \mathcal{W}(\Omega^\varepsilon), \qquad u^{1m} \to u^1 \text{ in } \mathcal{L}^2(\Omega^\varepsilon) \qquad \text{and } \theta^{0m} \to \theta^0 \text{ in } W(\Omega^\varepsilon).$$

For each $j = 1,\dots,m$, the system (2.8) is a system of ordinary differential equations with initial Cauchy conditions (2.9). So, the system (2.8)–(2.9) of the unknowns $h_{jm}(t)$ and $k_{jm}(t)$ has a unique solution, see for example [26]. Note that Estimates (2.6) and (2.7) remain valid for the approximate solutions, then

$$\|\partial_t u_m\|^2_{\Omega_+} + \frac{1}{\varepsilon}\|\partial_t u_m\|^2_{\Omega^\varepsilon_-} \le c, \qquad \|\nabla u_m\|^2_{\Omega_+} + \frac{1}{\varepsilon}\|\nabla u_m\|^2_{\Omega^\varepsilon_-} \le c$$

and

$$\|\theta_m\|^2_{\Omega_+} + \frac{1}{\varepsilon}\|\theta_m\|^2_{\Omega^\varepsilon_-} \le c, \qquad \int_0^T \left(\|\nabla\theta_m\|^2_{\Omega_+} + \frac{1}{\varepsilon}\|\nabla\theta_m\|^2_{\Omega^\varepsilon_-} \right) ds \le c$$

where c denotes various constants independent of m. This means that

$$(u_m,\theta_m) \text{ is bounded in } L^2(0,T;\mathcal{W}(\Omega^\varepsilon)) \times L^2(0,T;W(\Omega^\varepsilon)).$$

So, we can extract a subsequence from (u_m,θ_m) (that we continue to denote (u_m,θ_m)) such that,

$$u_m \to u \text{ weakly in } L^2(0,T;\mathcal{W}(\Omega^\varepsilon)) \tag{2.10}$$

and

$$\theta_m \to \theta \ \text{ weakly in } \ L^2(0,T;W(\Omega^\varepsilon)). \tag{2.11}$$

As inequality (2.5) is still valid for the new subsequence u_m, then $\partial_t u_m$ is bounded in $L^2(0,T;\mathcal{L}^2(\Omega^\varepsilon))$. So, it is possible to extract a subsequence denoted again u_m such that $\partial_t u_m$ converges weakly to some χ in $L^2(0,T;\mathcal{L}^2(\Omega^\varepsilon))$. We deduce that $\chi = \partial_t u$ almost everywhere because the equality holds in the distributional sense. Then,

$$\partial_t u_m \to \partial_t u \ \text{ weakly in } L^2\ (0,T;\mathcal{L}^2(\Omega^\varepsilon)). \tag{2.12}$$

It remains to show that (u,θ) thus constructed is the unique solution of Problem (2.1)–(2.4). For this purpose, we multiply the equation (2.10) by a test function $\psi \in \mathcal{D}(0,T)$, the space of infinitely differentiable functions with compact support in $(0,T)$, and then we integrate with respect to the time variable and we obtain

$$\begin{aligned}
&\int_0^T \frac{d}{dt}\left(\langle p\partial_t u_m, v_j\rangle_{\Omega^\varepsilon}\right)\psi ds + \int_0^T \langle p\nabla u_m, \nabla v_j\rangle_{\Omega^\varepsilon}\psi ds + \int_0^T \frac{d}{dt}\langle p\theta_m, \varphi_j\rangle_{\Omega^\varepsilon}\psi ds \\
&+ \int_0^T \langle p\nabla\theta_m, \nabla\varphi_j\rangle_{\Omega^\varepsilon}\psi ds + \alpha\int_0^T \langle p\nabla\theta_m, v_j\rangle_{\Omega^\varepsilon}\psi ds - \alpha\int_0^T \langle p\partial_t u_m, \nabla\varphi_j\rangle_{\Omega^\varepsilon}\psi ds \\
&= \int_0^T \langle pf, v_j\rangle_{\Omega^\varepsilon}\psi ds + \int_0^T \langle pg, \varphi_j\rangle_{\Omega^\varepsilon}\psi ds
\end{aligned}$$

for all $j = 1,\dots,m$. Integrating by parts with respect to the time variable, we obtain

$$\begin{aligned}
&\int_0^T \langle pu_m, v_j\rangle_{\Omega^\varepsilon}\psi'' ds + \int_0^T \langle p\nabla u_m, \nabla v_j\rangle_{\Omega^\varepsilon}\psi ds - \int_0^T \langle p\theta_m, \varphi_j\rangle_{\Omega^\varepsilon}\psi' ds \\
&+ \int_0^T \langle p\nabla\theta_m, \nabla\varphi_j\rangle_{\Omega^\varepsilon}\psi ds + \alpha\int_0^T \langle p\nabla\theta_m, v_j\rangle_{\Omega^\varepsilon}\psi ds + \alpha\int_0^T \langle pu_m, \nabla\varphi_j\rangle_{\Omega^\varepsilon}\psi' ds \\
&= \int_0^T \langle pf, v_j\rangle_{\Omega^\varepsilon}\psi ds + \int_0^T \langle pg, \varphi_j\rangle_{\Omega^\varepsilon}\psi ds
\end{aligned}$$

for all $j = 1,\dots,m$. Summing over j, and making the passage to the limit, the previous equality remains true for all $v \in \mathcal{W}(\Omega^\varepsilon)$ and for all $\varphi \in W(\Omega^\varepsilon)$. By density arguments, the previous equality remains true for all $v \in L^2(0,T;\mathcal{W}(\Omega^\varepsilon))$ and for all $\varphi \in L^2(0,T;W(\Omega^\varepsilon))$, see for example[18]. Passing to the limit and using (2.10), (2.11) and (2.12), (u,θ) is a weak solution of the Problem (2.1)–(2.4).

Step 3: Uniqueness

For uniqueness, we set

$$f = g = u^0 = u^1 = 0, \quad \text{and } \ \theta^0 = 0$$

in Problem (2.1)–(2.4). Thanks to inequality (2.5), we obtain

$$E(t) = 0 \qquad \forall t \in (0,T).$$

This implies that

$$\theta = 0 \ \text{ on } \Omega_+ \quad \text{and } \ \theta = 0 \ \text{ on } \ \Omega_-^\varepsilon$$

and in virtue of the transmission condition (1.4), it follows that

$$\theta = 0 \ \text{ on } \Omega^\varepsilon.$$

Similarly, we can show that

$$\nabla u = 0 \ \text{ on } \Omega_+ \quad \text{ and } \ \nabla u = 0 \ \text{ on } \ \Omega^\varepsilon_-.$$

As $\|\nabla u\|$ is an equivalent norm to the norm of $\mathcal{W}(\Omega^\varepsilon)$, it follows that

$$u = 0 \ \text{ on } \Omega_+ \quad \text{ and } \ u = 0 \ \text{ on } \ \Omega^\varepsilon_-.$$

Finally, thanks to the transmission condition (1.4), we have

$$u = 0 \ \text{ on } \Omega^\varepsilon.$$

□

3. SCALED PROBLEM

3.1. *Scaling*

In order to study the asymptotic behavior of the solution of Problem (2.1)–(2.4) as $\varepsilon \to 0$, and since Ω^ε vary with ε, we first transform the body Ω^ε into a fixed body (independent of ε) by means of a change of scaling. Then, the displacement field u and the heat propagation θ can be expressed in function of ε. So, let us consider the following change of variable $T_\varepsilon\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T_\varepsilon(x,y,z) = \begin{cases} (x,y,z) & \text{if } z \geq 0 \\ (x,y,\varepsilon z) & \text{if } z < 0 \end{cases} \tag{3.1}$$

which is a similarity operating on the z variable and it makes stiff (independent of ε) the open domain Ω^ε. We note then

$$\begin{aligned} \Omega &= T_\varepsilon^{-1}(\Omega^\varepsilon) =]0.1[^2 \times]-1,1[, & & \\ \Omega_+ &= T_\varepsilon^{-1}(\Omega_+), & \Omega_- &= T_\varepsilon^{-1}(\Omega^\varepsilon_-) =]0.1[^2 \times]-1,0[, \\ \Sigma &= T_\varepsilon^{-1}(\Sigma), & \Sigma_- &= T_\varepsilon^{-1}(\Sigma^\varepsilon) =]0.1[^2 \times \{-1\}, \\ \Gamma_+ &= T_\varepsilon^{-1}(\Gamma_+), & \Gamma_- &= T_\varepsilon^{-1}(\Gamma^\varepsilon_-). \end{aligned}$$

The transformation T_ε permits also to define the scaling of the state variables as follows: to the scalar function φ, we associate the function

$$\varphi^\varepsilon = \varphi \circ T_\varepsilon \tag{3.2}$$

and to the vector field v, we associate the field

$$v^\varepsilon = T_\varepsilon \circ v \circ T_\varepsilon. \tag{3.3}$$

This allows us to express the Sobolev spaces on the open set Ω by defining $\mathcal{W}^{\varepsilon}(\Omega)$ and $W^{\varepsilon}(\Omega)$ as follows:

$$\mathcal{W}^{\varepsilon}(\Omega)=\left\{v_{\pm}^{\varepsilon}\in\mathcal{H}^{1}(\Omega_{\pm})\mid [v]=0 \text{ on } \Sigma \text{ and } v=0 \text{ on } \Gamma_{+}\cup\Gamma_{-}\right\}$$
$$W^{\varepsilon}(\Omega)=\left\{\varphi\in H^{1}(\Omega)\mid [\varphi]=0 \text{ on } \Sigma \text{ and } \varphi=0 \text{ on } \Gamma_{+}\cup\Gamma_{-}\right\}.$$

We naturally endow $\mathcal{W}^{\varepsilon}(\Omega)$ and $W^{\varepsilon}(\Omega)$ by the standard norms $\|\nabla v\|_{\mathcal{L}^2(\Omega^{\varepsilon})}$ and $\|\nabla\varphi\|_{L^2(\Omega^{\varepsilon})}$, respectively. It is obvious that $\mathcal{W}^{\varepsilon}(\Omega)$ and $W^{\varepsilon}(\Omega)$ are Hilbert spaces for the simple reason that $v^{\varepsilon}\in\mathcal{W}^{\varepsilon}(\Omega)$ is equivalent to $v\in\mathcal{W}(\Omega^{\varepsilon})$ and $\varphi^{\varepsilon}\in W^{\varepsilon}(\Omega)$ is equivalent to $\varphi\in W(\Omega^{\varepsilon})$.

3.2. *Weak formulation in the fixed open set* Ω

We will now set the variational problem (2.1)–(2.4) in the open set Ω and we associate to the data f, g, u^0, u^1 and θ^0 of Problem (2.1)–(2.4) new data defined on the set Ω and obtained by changing of variables and scale, which we denote f^{ε}, g^{ε}, $u^{\varepsilon 0}$, $u^{\varepsilon 1}$ and $\theta^{\varepsilon 0}$, respectively.

Proposition 2. *Under the assumptions of Proposition 1,* $(u^{\varepsilon},\theta^{\varepsilon})$ *satisfies*

$$u^{\varepsilon}\in L^{\infty}(0,T;\mathcal{W}(\Omega)),\qquad \partial_t u^{\varepsilon}\in L^{\infty}(0,T;\mathcal{L}^2(\Omega)),$$

$$\text{and } \theta^{\varepsilon}\in L^{\infty}(0,T;L^2(\Omega))\cap L^2(0,T;W(\Omega))$$

and it is the unique solution of Problem (P^{ε}) *below*

$$\begin{aligned}
&\frac{d}{dt}\left(\langle\partial_t u^{\varepsilon},v^{\varepsilon}\rangle_{+}+\langle\partial_t u_1^{\varepsilon},v_1^{\varepsilon}\rangle_{-}+\langle\partial_t u_2^{\varepsilon},v_2^{\varepsilon}\rangle_{-}+\frac{1}{\varepsilon^2}\langle\partial_t u_3^{\varepsilon},v_3^{\varepsilon}\rangle_{-}\right)\\
&+\langle\nabla u^{\varepsilon},\nabla v^{\varepsilon}\rangle_{+}+\langle\partial_x u_1^{\varepsilon},\partial_x v_1^{\varepsilon}\rangle_{-}+\langle\partial_y u_1^{\varepsilon},\partial_y v_1^{\varepsilon}\rangle_{-}+\frac{1}{\varepsilon^2}\langle\partial_z u_1^{\varepsilon},\partial_z v_1^{\varepsilon}\rangle_{-}\\
&+\langle\partial_x u_2^{\varepsilon},\partial_x v_2^{\varepsilon}\rangle_{-}+\langle\partial_y u_2^{\varepsilon},\partial_y v_2^{\varepsilon}\rangle_{-}\frac{1}{\varepsilon^2}\langle\partial_z u_2^{\varepsilon},\partial_z v_2^{\varepsilon}\rangle_{-}+\frac{1}{\varepsilon^2}\langle\partial_x u_3^{\varepsilon},\partial_x v_3^{\varepsilon}\rangle_{-}\\
&+\frac{1}{\varepsilon^2}\langle\partial_y u_3^{\varepsilon},\partial_y v_3^{\varepsilon}\rangle_{-}+\frac{1}{\varepsilon^4}\langle\partial_z u_3^{\varepsilon},\partial_z v_3^{\varepsilon}\rangle_{-}+\frac{d}{dt}\left(\langle\theta^{\varepsilon},\varphi^{\varepsilon}\rangle_{+}+\langle\theta^{\varepsilon},\varphi^{\varepsilon}\rangle_{-}\right)\\
&+\langle\nabla\theta^{\varepsilon},\nabla\varphi^{\varepsilon}\rangle_{+}+\langle\partial_x\theta^{\varepsilon},\partial_x\varphi^{\varepsilon}\rangle_{-}+\langle\partial_y\theta^{\varepsilon},\partial_y\varphi^{\varepsilon}\rangle_{-}+\frac{1}{\varepsilon^2}\langle\partial_z\theta^{\varepsilon},\partial_z\varphi^{\varepsilon}\rangle_{-}\\
&+\alpha\langle\nabla\theta^{\varepsilon},\overrightarrow{v}^{\varepsilon}\rangle_{+}+\alpha\langle\partial_x\theta^{\varepsilon},v_1^{\varepsilon}\rangle_{-}+\alpha\langle\partial_y\theta^{\varepsilon},v_2^{\varepsilon}\rangle_{-}+\frac{1}{\varepsilon^2}\alpha\langle\partial_z\theta^{\varepsilon},v_3^{\varepsilon}\rangle_{-}\\
&-\alpha\langle\partial_t\overrightarrow{u}^{\varepsilon},\nabla\varphi^{\varepsilon}\rangle_{+}-\alpha\langle\partial_t u_1^{\varepsilon},\partial_x\varphi^{\varepsilon}\rangle_{-}-\alpha\langle\partial_t u_2^{\varepsilon},\partial_y\varphi^{\varepsilon}\rangle_{-}-\frac{1}{\varepsilon^2}\alpha\langle\partial_t u_3^{\varepsilon},\partial_z\varphi^{\varepsilon}\rangle_{-}\\
&=\langle f^{\varepsilon},v^{\varepsilon}\rangle_{+}+\langle f_1^{\varepsilon},v_1^{\varepsilon}\rangle_{-}+\langle f_2^{\varepsilon},v_2^{\varepsilon}\rangle_{-}+\frac{1}{\varepsilon^2}\langle f_3^{\varepsilon},v_3^{\varepsilon}\rangle_{-}+\langle g^{\varepsilon},\varphi^{\varepsilon}\rangle_{+}+\langle g^{\varepsilon},\varphi^{\varepsilon}\rangle_{-}
\end{aligned}$$

with initial conditions

$$\langle u^\varepsilon(0), v^\varepsilon\rangle_+ + \langle u_1^\varepsilon(0), v_1^\varepsilon\rangle_- + \langle u_2^\varepsilon(0), v_2^\varepsilon\rangle_- + \frac{1}{\varepsilon^2}\langle u_3^\varepsilon(0), v_3^\varepsilon\rangle_- = \langle u^{\varepsilon 0}, v^\varepsilon\rangle_+ + \langle u_1^{\varepsilon 0}, v_1^\varepsilon\rangle_- + \langle u_2^{\varepsilon 0}, v_2^\varepsilon\rangle_- + \frac{1}{\varepsilon^2}\langle u_3^{\varepsilon 0}, v_3^\varepsilon\rangle_-$$

$$\langle \partial_t u^\varepsilon(0), v^\varepsilon\rangle_+ + \langle \partial_t u_1^\varepsilon(0), v_1^\varepsilon\rangle_- + \langle \partial_t u_2^\varepsilon(0), v_2^\varepsilon\rangle_- + \frac{1}{\varepsilon^2}\langle \partial_t u_3^\varepsilon(0), v_3^\varepsilon\rangle_- = \langle u^{\varepsilon 1}, v^\varepsilon\rangle_+ + \langle u_1^{\varepsilon 1}, v_1^\varepsilon\rangle_- + \langle u_2^{\varepsilon 1}, v_2^\varepsilon\rangle_- + \frac{1}{\varepsilon^2}\langle u_3^{\varepsilon 1}, v_3^\varepsilon\rangle_-$$

$$\langle \theta^\varepsilon(0), \varphi^\varepsilon\rangle_+ + \langle \theta^\varepsilon(0), \varphi^\varepsilon\rangle_- = \langle \theta^{\varepsilon 0}, \varphi^\varepsilon\rangle_+ + \langle \theta^{\varepsilon 0}, \varphi^\varepsilon\rangle_-$$

for all $(v^\varepsilon, \varphi^\varepsilon) \in \mathcal{W}^\varepsilon(\Omega) \times W^\varepsilon(\Omega)$, *where* $\langle .,. \rangle_\pm$ *denotes* $\langle .,. \rangle_{\Omega_\pm}$.

Proof. This is a direct consequence of Proposition 1 by changing variable and scale. □

4. A PRIORI ESTIMATES AND THEIR CONSEQUENCES

Proposition 3. *Under the following assumptions:*

(i) $E^\varepsilon(0)$ *is bounded independently of* ε,

(ii) f_+^ε *and* g_+^ε *are bounded independently of* ε *in* $\mathcal{L}^2(\Omega_+)$ *and* $L^2(\Omega_+)$, *respectively,*

(iii) f_{1-}^ε, f_{2-}^ε, $\frac{1}{\varepsilon} f_{3-}^\varepsilon$ *and* g_-^ε *are bounded independently of* ε *in* $L^2(\Omega_-)$,

we have the following a priori estimates:

$$\|\partial_t u^\varepsilon\|_+^2 \le c, \qquad \|\partial_t u_1^\varepsilon\|_-^2 \le c, \qquad \|\partial_t u_2^\varepsilon\|_-^2 \le c, \qquad \frac{1}{\varepsilon^2}\|\partial_t u_3^\varepsilon\|_-^2 \le c, \tag{4.1}$$

$$\|\nabla u^\varepsilon\|_+^2 \le c, \qquad \|\partial_x u_1^\varepsilon\|_-^2 \le c, \qquad \|\partial_y u_1^\varepsilon\|_-^2 \le c, \qquad \frac{1}{\varepsilon^2}\|\partial_z u_1^\varepsilon\|_-^2 \le c, \tag{4.2}$$

$$\|\partial_x u_2^\varepsilon\|_-^2 \le c, \qquad \|\partial_y u_2^\varepsilon\|_-^2 \le c, \qquad \frac{1}{\varepsilon^2}\|\partial_z u_2^\varepsilon\|_-^2 \le c \tag{4.3}$$

$$\frac{1}{\varepsilon^2}\|\partial_x u_3^\varepsilon\|_-^2 \le c, \qquad \frac{1}{\varepsilon^2}\|\partial_y u_3^\varepsilon\|_-^2 \le c, \qquad \frac{1}{\varepsilon 4}\|\partial_z u_3^\varepsilon\|_-^2 \le c,$$

$$\|\theta^\varepsilon\|_+^2 \le c, \qquad \|\theta^\varepsilon\|_-^2 \le c$$

$$\int_0^T \|\nabla\theta^\varepsilon\|_+^2\, ds \le c \qquad \int_0^T \|\partial_x\theta^\varepsilon\|_-^2\, ds \le c,$$

$$\int_0^T \|\partial_y\theta^\varepsilon\|_-^2\, ds \le c, \qquad \int_0^T \frac{1}{\varepsilon^2}\|\partial_z\theta^\varepsilon\|_-^2\, ds \le c$$

where c *denotes various constants independent of* ε.

Proof. This is a direct consequence of the energy equality and the following inequality:

$$E^\varepsilon(t) \le c\left(E^\varepsilon(0) + \int_0^T \left(\|f^\varepsilon\|_+^2 + \|f_1^\varepsilon\|_-^2 + \|f_2^\varepsilon\|_-^2 + \frac{1}{\varepsilon^2}\|f_3^\varepsilon\|_-^2 + \|g^\varepsilon\|_+^2 + \|g^\varepsilon\|_-^2\right) ds\right)$$

established in the same way as in Proposition 1. □

From the *a priori* estimates of Proposition 3, there exists a subsequence of $(u^\varepsilon, \theta^\varepsilon)$ which we denote again $(u^\varepsilon, \theta^\varepsilon)$, such that

$$u^\varepsilon \text{ converges weakly-* in } L^\infty(0,T;\mathcal{W}(\Omega))$$

$$\partial_t u^\varepsilon \text{ converges weakly-* in } L^\infty(0,T;\mathcal{L}^2(\Omega))$$

θ^ε converges weakly-* in $L^\infty(0,T;L^2(\Omega))$ and converges weakly in $L^2(0,T;W(\Omega))$. We define the following Hilbert spaces:

$$\mathcal{V}(\Omega_+) = \left\{v \in \mathcal{H}^1(\Omega_+) \mid v = 0 \text{ on } \Gamma_+,\ v_1 \in H_0^1(\Sigma), v_2 \in H_0^1(\Sigma) \text{ and } v_3 = 0 \text{ on } \Sigma\ \right\}$$
$$\mathcal{H}^1(\Omega_+) = \left\{v \in \mathcal{L}^2(\Omega_+) \mid\ v|_\Sigma \in \mathcal{L}^2(\Sigma)\right\}$$
$$V(\Omega_+) = \left\{\varphi \in H^1(\Omega_+) \mid \varphi = 0 \text{ on } \Gamma_+\ ,\ \varphi|_\Sigma \in H_0^1(\Sigma)\ \right\}$$
$$H^1(\Omega_+) = \left\{\varphi \in L^2(\Omega_+) \mid\ \varphi|_\Sigma \in L^2(\Sigma)\right\}.$$

We will need the following lemma, see [9].

Lemma 2. *Let I be a bounded interval of $\mathbb{R}$, $p \in [1,+\infty]$ and $u \in W^{1,p}(I)$. Then, there exists $\widetilde{u} \in C^0(I)$ such that*

$$u = \widetilde{u}|_I \quad \text{and} \quad \widetilde{u}(b) - \widetilde{u}(a) = \int_a^b u'(t)dt$$

for all $a, b \in I$.

Proposition 4. *If (u,θ) is the weak limit of $(u^\varepsilon, \theta^\varepsilon)$, then*

(i) $u_+ \in L^\infty(0,T;\mathcal{V}(\Omega_+))$, $\partial_t u_+ \in L^\infty(0,T;\mathcal{H}^1(\Omega_+))$ *and*

$$\begin{cases} u_{1-} = u_{1+}|_\Sigma & \text{on } \Omega_-; \\ u_{2-} = u_{2+}|_\Sigma & \text{on } \Omega_-; \\ u_{3-} = 0 & \text{on } \Omega_-. \end{cases}$$

(ii) $\theta_+ \in L^\infty(0,T;H^1(\Omega_+)) \cap L^2(0,T;V(\Omega_+))$ *and* $\theta_- = \theta_+|_\Sigma$ *on* Ω_-.

Proof.

(i) Thanks to (4.1), (4.2) and (4.3), we deduce by passing to the limit that u_{1-} and u_{2-} depend only of x and y on Ω_- and u_{3-} is independent of x , y and z on Ω_-.

Since $u(.,.,z) \in W^{1,2}(]-1,0[)$ and thanks to Lemma 2, there exists a continuous extension $\widetilde{u}$ of u to $[-1,0]$. In addition,

$$\forall Z \in [-1,0]: \qquad \widetilde{u}(.,.,0) - \widetilde{u}(.,.,Z) = \int_Z^0 \partial_z u(.,.,t)dt.$$

As $\int_Z^0 \partial_z u(.,.,t)dt$ vanishes, it follows that

$$\widetilde{u}(.,.,Z) = \widetilde{u}(.,.,0)$$

which means that $\widetilde{u}$ on $[-1,0]$ is determined by its value $\widetilde{u}(.,.,0)$ and then u is also determined by its value at the boundary.

(ii) Similarly, we also establish the results on θ_+ and θ_-.

□

5. PASSAGE TO THE LIMIT

In this section we will show that the asymptotic action (when $\varepsilon \to 0$) of the thin shell Ω_-^ε on the solid Ω_+ is modelled by evolutionary boundary conditions on the face Σ of the boundary of Ω_+. Theses boundary conditions are of Ventcel type.

5.1. *Weak formulation of the Ventcel's problem*

Under some conditions, the solution $(u^\varepsilon, \theta^\varepsilon)$ of the problem (P^ε) converges in the weak sense to a limit (u,θ), itself solution of a limit problem (P).

Proposition 5. *Suppose that*

(i) $(f_+^\varepsilon, \int_{-1}^0 f_1^\varepsilon(.,.,z)dz, \int_{-1}^0 f_2^\varepsilon(.,.,z)dz)$ *converges weakly to* (f_+, f^*, g^*) *in* $\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega_+)) \times L^2(\Sigma) \times L^2(\Sigma)$,

(ii) $(u_+^{\varepsilon 0}, \int_{-1}^0 u_1^{\varepsilon 0}(.,.,z)dz, \int_{-1}^0 u_2^{\varepsilon 0}(.,.,z)dz)$ *converges weakly to* (u_+^0, u_1^0, u_2^0) *in* $\mathcal{H}^1(\Omega_+) \times H^1(\Sigma) \times H^1(\Sigma)$,

(iii) $(u_+^{\varepsilon 1}, \int_{-1}^0 u_1^{\varepsilon 1}(.,.,z)dz, \int_{-1}^0 u_2^{\varepsilon 1}(.,.,z)dz)$ *converges weakly to* (u_+^1, u_1^1, u_2^1) *in* $\mathcal{L}^2(\Omega_+) \times L^2(\Sigma) \times L^2(\Sigma)$,

(iv) $(g_+^\varepsilon, \int_{-1}^0 g^\varepsilon(.,.,z)dz)$ *converges weakly to* (g_+, g) *in* $L^2(0,T;L^2(\Omega_+)) \times L^2(\Sigma)$,

(v) $(\theta_+^{\varepsilon 0}, \int_{-1}^0 \theta^{\varepsilon 0}(.,.,z)dz)$ *converges weakly to* (θ_+^0, θ^1) *in* $L^2(\Omega_+) \times L^2(\Sigma)$,

then,

(a) $(u_+^\varepsilon, \int_{-1}^0 u_1^\varepsilon(.,.,z)dz, \int_{-1}^0 u_2^\varepsilon(.,.,z)dz)$ *converges weakly-* to* (u_+, u_1, u_2) *in* $\mathcal{L}^\infty(0,T;\mathcal{H}^1(\Omega_+)) \times H^1(\Sigma) \times H^1(\Sigma)$,

(b) $(\partial_t u_+^\varepsilon, \int_{-1}^0 \partial_t u_1^\varepsilon(.,.,z)dz, \int_{-1}^0 \partial_t u_2^\varepsilon(.,.,z)dz)$ *converges weakly-* to* $(\partial_t u_+, \partial_t u_1, \partial_t u_2)$ *in* $\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega_+)) \times L^2(\Sigma) \times L^2(\Sigma)$,

(c) $(\theta_+^\varepsilon, \int_{-1}^0 \theta^\varepsilon(.,.,z)dz)$ *converges weakly-* to* (θ_+, θ) *in* $L^\infty(0,T;H^1(\Omega_+)) \times H^1(\Sigma)$.

In addition, (u,θ) *is a weak solution of the following problem (P): to find*

$$u \in L^\infty(0,T;\mathcal{V}(\Omega_+)) :\ \partial_t u \in L^\infty(0,T;\mathcal{H}^1(\Omega_+))$$

$$\text{and}\ \ \theta \in L^\infty(0,T;L^2(\Omega_+)) \cap L^2(0,T;V(\Omega_+))$$

such that

$$\begin{aligned}
&\frac{d}{dt}\left(\langle \partial_t u, v\rangle_+ + \langle \partial_t u_1, v_1\rangle_\Sigma + \langle \partial_t u_2, v_2\rangle_\Sigma\right) + \langle \nabla u, \nabla v\rangle_+ + \langle \partial_x u_1, \partial_x v_1\rangle_\Sigma \\
&+ \langle \partial_y u_1, \partial_y v_1\rangle_\Sigma + \langle \partial_x u_2, \partial_x v_2\rangle_\Sigma + \langle \partial_y u_2, \partial_y v_2\rangle_\Sigma + \frac{d}{dt}\left(\langle \theta, \varphi\rangle_+ + \langle \theta, \varphi\rangle_\Sigma\right) \\
&+ \langle \nabla\theta, \nabla\varphi\rangle_+ + \langle \partial_x\theta, \partial_x\varphi\rangle_\Sigma + \langle \partial_y\theta, \partial_y\varphi\rangle_\Sigma + \alpha\langle \nabla\theta, v\rangle_+ + \alpha\langle \partial_x\theta, v_1\rangle_\Sigma \\
&+ \alpha\langle \partial_y\theta, v_2\rangle_\Sigma - \alpha\langle \partial_t u, \nabla\varphi\rangle_+ - \alpha\langle \partial_t u_1, \partial_x\varphi\rangle_\Sigma - \alpha\langle \partial_t u_2, \partial_y\varphi\rangle_\Sigma \\
&= \langle f, v\rangle_+ + \langle f_1, v_1\rangle_\Sigma + \langle f_2, v_2\rangle_\Sigma + \langle g, \varphi\rangle_+ + \langle g, \varphi\rangle_\Sigma
\end{aligned}$$

with

$$\begin{aligned}
\langle u(0), v\rangle_+ + \langle u_1(0), v_1\rangle_\Sigma + \langle u_2(0), v_2\rangle_\Sigma &= \left\langle u^0, v\right\rangle_+ + \left\langle u_1^0, v_1\right\rangle_\Sigma + \left\langle u_2^0, v_2\right\rangle_\Sigma \\
\langle \partial_t u(0), v\rangle_+ + \langle \partial_t u_1(0), v_1\rangle_\Sigma + \langle \partial_t u_2(0), v_2\rangle_\Sigma &= \left\langle u^1, \overrightarrow{v}\right\rangle_+ + \left\langle u_1^1, v_1\right\rangle_\Sigma + \left\langle u_2^1, v_2\right\rangle_\Sigma \\
\langle \theta(0), \varphi\rangle_+ + \langle \theta(0), \varphi\rangle_\Sigma &= \left\langle \theta^0, \varphi\right\rangle_+ + \left\langle \theta^0, \varphi\right\rangle_\Sigma
\end{aligned}$$

for all $(v,\varphi) \in \mathcal{V}(\Omega) \times V(\Omega)$.

Proof. It suffices to note that the weak formulation is as follows:

$$\begin{aligned}
&\frac{d}{dt}\left(\langle \partial_t u^\varepsilon, \overrightarrow{v}^\varepsilon\rangle_+ + \langle \partial_t u_1^\varepsilon, v_1^\varepsilon\rangle_- + \langle \partial_t u_2^\varepsilon, v_2^\varepsilon\rangle_- + \frac{1}{\varepsilon^2}\langle \partial_t u_3^\varepsilon, v_3^\varepsilon\rangle_-\right) \\
&+ \langle \nabla u^\varepsilon, \nabla v^\varepsilon\rangle_+ + \langle \partial_x u_1^\varepsilon, \partial_x v_1^\varepsilon\rangle_- + \langle \partial_y u_1^\varepsilon, \partial_y v_1^\varepsilon\rangle_- + \frac{1}{\varepsilon^2}\langle \partial_z u_1^\varepsilon, \partial_z v_1^\varepsilon\rangle_- \\
&+ \langle \partial_x u_2^\varepsilon, \partial_x v_2^\varepsilon\rangle_- + \langle \partial_y u_2^\varepsilon, \partial_y v_2^\varepsilon\rangle_- + \frac{1}{\varepsilon^2}\langle \partial_z u_2^\varepsilon, \partial_z v_2^\varepsilon\rangle_- + \frac{1}{\varepsilon^2}\langle \partial_x u_3^\varepsilon, \partial_x v_3^\varepsilon\rangle_- \\
&+ \frac{1}{\varepsilon^2}\langle \partial_y u_3^\varepsilon, \partial_y v_3^\varepsilon\rangle_- + \frac{1}{\varepsilon^4}\langle \partial_z u_3^\varepsilon, \partial_z v_3^\varepsilon\rangle_- + \frac{d}{dt}\left(\langle \theta^\varepsilon, \varphi^\varepsilon\rangle_+ + \langle \theta^\varepsilon, \varphi^\varepsilon\rangle_-\right) \\
&+ \langle \nabla\theta^\varepsilon, \nabla\varphi^\varepsilon\rangle_+ + \langle \partial_x\theta^\varepsilon, \partial_x\varphi^\varepsilon\rangle_- + \langle \partial_y\theta^\varepsilon, \partial_y\varphi^\varepsilon\rangle_- + \frac{1}{\varepsilon^2}\langle \partial_z\theta^\varepsilon, \partial_z\varphi^\varepsilon\rangle_- \\
&+ \alpha\langle \nabla\theta^\varepsilon, v^\varepsilon\rangle_+ + \alpha\langle \partial_x\theta^\varepsilon, v_1^\varepsilon\rangle_- + \alpha\langle \partial_y\theta^\varepsilon, v_2^\varepsilon\rangle_- + \frac{1}{\varepsilon^2}\alpha\langle \partial_z\theta^\varepsilon, v_3^\varepsilon\rangle_- \\
&- \alpha\langle \partial_t u^\varepsilon, \nabla\varphi^\varepsilon\rangle_+ - \alpha\langle \partial_t u_1^\varepsilon, \partial_x\varphi^\varepsilon\rangle_- - \alpha\langle \partial_t u_2^\varepsilon, \partial_y\varphi^\varepsilon\rangle_- - \frac{1}{\varepsilon^2}\alpha\langle \partial_t u_3^\varepsilon, \partial_z\varphi^\varepsilon\rangle_- \\
&= \langle f^\varepsilon, v^\varepsilon\rangle_+ + \langle f_1^\varepsilon, v_1^\varepsilon\rangle_- + \langle f_2^\varepsilon, v_2^\varepsilon\rangle_- + \frac{1}{\varepsilon^2}\langle f_3^\varepsilon, v_3^\varepsilon\rangle_- + \langle g^\varepsilon, \varphi^\varepsilon\rangle_+ + \langle g^\varepsilon, \varphi^\varepsilon\rangle_- .
\end{aligned}$$

As we have (see, [9])

$$\int\int\int_{\Omega_-} \dots dxdydz = \int_{-1}^{0} \left(\int\int_{\Sigma} \dots dxdy \right) dz = \int\int_{\Sigma} \left(\int_{-1}^{0} \dots dz \right) dxdy,$$

then the weak formulation becomes:

$$\begin{aligned}
&\frac{d}{dt}\left(\langle \partial_t u^\varepsilon, v^\varepsilon \rangle_+ + \int_{-1}^{0} \left(\int\int_{\Sigma} \partial_t u_1^\varepsilon v_{1-}^\varepsilon dxdy \right) dz + \int_{-1}^{0} \left(\int\int_{\Sigma} \partial_t u_2^\varepsilon v_{2-}^\varepsilon dxdy \right) dz \right. \\
&\left. + \frac{1}{\varepsilon^2} \int_{-1}^{0} \left(\int\int_{\Sigma} \partial_t u_3^\varepsilon v_{3-}^\varepsilon dxdy \right) dz \right) + \langle \nabla u^\varepsilon, \nabla v^\varepsilon \rangle_+ + \int_{-1}^{0} \left(\int\int_{\Sigma} \partial_x u_1^\varepsilon \partial_x v_1^\varepsilon dxdy \right) dz \\
&+ \int_{-1}^{0} \left(\int\int_{\Sigma} \partial_y u_1^\varepsilon \partial_y v_1^\varepsilon dxdy \right) dz + \frac{1}{\varepsilon^2} \int_{-1}^{0} \left(\int\int_{\Sigma} \partial_z u_1^\varepsilon \partial_z v_1^\varepsilon dxdy \right) dz \\
&+ \int_{-1}^{0} \left(\int\int_{\Sigma} \partial_x u_2^\varepsilon \partial_x v_2^\varepsilon dxdy \right) dz + \int_{-1}^{0} \left(\int\int_{\Sigma} \partial_y u_2^\varepsilon \partial_y v_2^\varepsilon dxdy \right) dz \\
&+ \frac{1}{\varepsilon^2} \int_{-1}^{0} \left(\int\int_{\Sigma} \partial_z u_2^\varepsilon \partial_z v_2^\varepsilon dxdy \right) dz + \frac{1}{\varepsilon^2} \int_{-1}^{0} \left(\int\int_{\Sigma} \partial_x u_3^\varepsilon \partial_x v_3^\varepsilon dxdy \right) dz \\
&+ \frac{1}{\varepsilon^2} \int_{-1}^{0} \left(\int\int_{\Sigma} \partial_y u_3^\varepsilon \partial_y v_3^\varepsilon dxdy \right) dz + \frac{1}{\varepsilon^4} \int_{-1}^{0} \left(\int\int_{\Sigma} \partial_z u_3^\varepsilon \partial_z v_3^\varepsilon dxdy \right) dz \\
&+ \frac{d}{dt}\left(\langle \theta^\varepsilon, \varphi^\varepsilon \rangle_+ + \int_{-1}^{0} \left(\int\int_{\Sigma} \theta^\varepsilon \varphi^\varepsilon dxdy \right) dz \right) + \langle \nabla \theta^\varepsilon, \nabla \varphi^\varepsilon \rangle_+ \\
&+ \int_{-1}^{0} \left(\int\int_{\Sigma} \partial_x \theta^\varepsilon \partial_x \varphi^\varepsilon dxdy \right) dz + \int_{-1}^{0} \left(\int\int_{\Sigma} \partial_y \theta^\varepsilon \partial_y \varphi^\varepsilon dxdy \right) dz \\
&+ \frac{1}{\varepsilon^2} \int_{-1}^{0} \left(\int\int_{\Sigma} \partial_z \theta^\varepsilon \partial_z \varphi^\varepsilon dxdy \right) dz + \alpha \langle \nabla \theta^\varepsilon, v^\varepsilon \rangle_+ + \alpha \int_{-1}^{0} \left(\int\int_{\Sigma} \partial_x \theta^\varepsilon v_1^\varepsilon dxdy \right) dz \\
&+ \alpha \int_{-1}^{0} \left(\int\int_{\Sigma} \partial_y \theta^\varepsilon v_2^\varepsilon dxdy \right) dz + \frac{\alpha}{\varepsilon^2} \int_{-1}^{0} \left(\int\int_{\Sigma} \partial_z \theta^\varepsilon v_3^\varepsilon dxdy \right) dz - \alpha \langle \partial_t u^\varepsilon, \nabla \varphi^\varepsilon \rangle_+
\end{aligned}$$

$$
\begin{aligned}
&-\alpha\int_{-1}^{0}\left(\int\int_{\Sigma}\partial_t u_1^\varepsilon\partial_x\varphi^\varepsilon dxdy\right)dz-\alpha\int_{-1}^{0}\left(\int\int_{\Sigma}\partial_t u_2^\varepsilon\partial_y\varphi^\varepsilon dxdy\right)dz\\
&-\frac{\alpha}{\varepsilon^2}\int_{-1}^{0}\left(\int\int_{\Sigma}\partial_t u_3^\varepsilon\partial_z\varphi^\varepsilon dxdy\right)dz\\
&=\langle f^\varepsilon,v^\varepsilon\rangle_+ +\int_{-1}^{0}\left(\int\int_{\Sigma}f_1^\varepsilon v_1^\varepsilon dxdy\right)dz+\int_{-1}^{0}\left(\int\int_{\Sigma}f_2^\varepsilon v_2^\varepsilon dxdy\right)dz\\
&+\frac{1}{\varepsilon^2}\int_{-1}^{0}\left(\int\int_{\Sigma}f_3^\varepsilon v_3^\varepsilon dxdy\right)dz+\langle g^\varepsilon,\varphi^\varepsilon\rangle_+ +\int_{-1}^{0}\left(\int\int_{\Sigma}g^\varepsilon\varphi^\varepsilon dxdy\right)dz.
\end{aligned}
$$

We take then v^ε and φ^ε such that

$$v_+^\varepsilon\in\mathcal{V}(\Omega_+)=\left\{v\in\mathcal{H}^1(\Omega_+)\mid v=0\text{ on }\Gamma_+\,,\,v_1\in H_0^1(\Sigma),v_2\in H_0^1(\Sigma),v_3=0\text{ on }\Sigma\right\}$$

v_-^ε as follows:

$$\begin{cases} v_{1-}^\varepsilon=v_{1+}|_\Sigma & \text{on }\Omega_-;\\ v_{2-}^\varepsilon=v_{2+}|_\Sigma & \text{on }\Omega_-;\\ v_{3-}^\varepsilon=0 & \text{on }\Omega_-;\end{cases}$$

$$\varphi_+^\varepsilon\in V(\Omega_+)=\left\{\varphi\in H^1(\Omega_+)\mid\varphi=0\text{ on }\Gamma_+\,,\ \varphi|_\Sigma\in H_0^1(\Sigma)\ \right\}$$

and

$$\varphi_-^\varepsilon=\varphi_+^\varepsilon\big|_\Sigma\ \text{ on }\ \Omega_-.$$

Then, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\langle\partial_t u^\varepsilon,v^\varepsilon\rangle_+ +\int\int_{\Sigma}\left(\int_{-1}^{0}\partial_t u_1^\varepsilon dz\right)v_{1-}^\varepsilon dxdy+\int\int_{\Sigma}\left(\int_{-1}^{0}\partial_t u_2^\varepsilon dz\right)v_{2-}^\varepsilon dxdy\right)\\
&+\langle\nabla u^\varepsilon,\nabla v^\varepsilon\rangle_+ +\int\int_{\Sigma}\left(\int_{-1}^{0}\partial_x u_1^\varepsilon dz\right)\partial_x v_1^\varepsilon dxdy+\int\int_{\Sigma}\left(\int_{-1}^{0}\partial_y u_1^\varepsilon dz\right)\partial_y v_1^\varepsilon dxdy\\
&+\int\int_{\Sigma}\left(\int_{-1}^{0}\partial_x u_2^\varepsilon dz\right)\partial_x v_2^\varepsilon dx+\int\int_{\Sigma}\left(\int_{-1}^{0}\partial_x u_1^\varepsilon dz\right)\partial_x v_1^\varepsilon dxdy\\
&+\int\int_{\Sigma}\left(\int_{-1}^{0}\partial_y u_1^\varepsilon dz\right)\partial_y v_1^\varepsilon dxdy+\int\int_{\Sigma}\left(\int_{-1}^{0}\partial_x u_2^\varepsilon dz\right)\partial_x v_2^\varepsilon dxdy\\
&+\int\int_{\Sigma}\left(\int_{-1}^{0}\partial_y u_2^\varepsilon dz\right)\partial_y v_2^\varepsilon dxdy+\frac{d}{dt}\left(\langle\theta^\varepsilon,\varphi^\varepsilon\rangle_+ +\int\int_{\Sigma}\left(\int_{-1}^{0}\theta^\varepsilon dz\right)\varphi^\varepsilon dxdy\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\langle\nabla\theta^\varepsilon,\nabla\varphi^\varepsilon\rangle_+ + \int\int_\Sigma \left(\int_{-1}^{0}\partial_x\theta^\varepsilon dz\right)\partial_x\varphi^\varepsilon dxdy + \int\int_\Sigma \left(\int_{-1}^{0}\partial_y\theta^\varepsilon dz\right)\partial_y\varphi^\varepsilon dxdy\\
&+\alpha\langle\nabla\theta^\varepsilon,v^\varepsilon\rangle_+ + \alpha\int\int_\Sigma \left(\int_{-1}^{0}\partial_x\theta^\varepsilon dz\right)v_1^\varepsilon dxdy + \alpha\int\int_\Sigma \left(\int_{-1}^{0}\partial_y\theta^\varepsilon dz\right)v_2^\varepsilon dxdy\\
&-\alpha\langle\partial_t u^\varepsilon,\nabla\varphi^\varepsilon\rangle_+ - \alpha\int\int_\Sigma \left(\int_{-1}^{0}\partial_t u_1^\varepsilon dz\right)\partial_x\varphi^\varepsilon dxdy\\
&-\alpha\int\int_\Sigma \left(\int_{-1}^{0}\partial_t u_2^\varepsilon dz\right)\partial_y\varphi^\varepsilon dxdy\\
=\ &\langle f^\varepsilon,v^\varepsilon\rangle_+ + \int\int_\Sigma \left(\int_{-1}^{0} f_1^\varepsilon dz\right)v_1^\varepsilon dxdy + \int\int_\Sigma \left(\int_{-1}^{0} f_2^\varepsilon dz\right)v_2^\varepsilon dxdy +\\
&\langle g^\varepsilon,\varphi^\varepsilon\rangle_+ + \int\int_\Sigma \left(\int_{-1}^{0} g^\varepsilon dz\right)\varphi^\varepsilon dxdy
\end{aligned}
$$

and we deduce the result by passing to the limit. □

5.2. *Strong formulation of the Ventcel's problem*

Under the conditions of Proposition 5 and by using an integration by parts on the boundary, the obtained weak problem can be expressed as follows:

$$
\begin{cases}
u_+'' - \triangle u_+ + \alpha\nabla\theta_+ = f_+ & \text{on } \Omega_+\times(0,T);\\
\theta_+' - \triangle\theta_+ + \alpha\,\mathrm{div}(u_+') = g_+ & \text{on } \Omega_+\times(0,T);
\end{cases}
$$

with Dirichlet conditions on the fixed portion of the boundary Γ_+

$$u_+ = 0, \quad \theta_+ = 0 \quad \text{on } \Gamma_+\times(0,T),$$

Ventcel type conditions on the side of junction Σ

$$
\begin{cases}
u_1'' - \partial_{x^2}^2 u_1 - \partial_{y^2}^2 u_1 - \partial_z u_{1+} + \alpha\partial_x\theta = f_1 & \text{on } \Sigma\times(0,T);\\
u_2'' - \partial_{x^2}^2 u_2 - \partial_{y^2}^2 u_2 - \partial_z u_{2+} + \alpha\partial_y\theta = f_2 & \text{on } \Sigma\times(0,T);\\
u_3 = 0 & \text{on } \Sigma\times(0,T);\\
\theta' - \partial_{x^2}^2\theta - \partial_{y^2}^2\theta - \partial_z\theta_+ + \alpha\partial_x u_1' + \alpha\partial_y u_2' = g & \text{on } \Sigma\times(0,T);
\end{cases}
$$

and Cauchy initial conditions

$$u_+(0) = u_+^0, \quad u_+'(0) = u_+^1, \quad \theta_+(0) = \theta_+^0 \text{ on } \Omega_+$$

and

$$u_1(0) = u_1^0, \quad u_2(0) = u_2^0, \quad u_1'(0) = u_1^1, \quad u_2'(0) = u_2^1, \quad \theta(0) = \theta_1 \text{ on } \Sigma.$$

CONCLUSION

The model thus constructed has, on the face covered with a thin shell, boundary conditions involving tangential and time derivatives of the same order as that of the interior differential operator. These conditions are called of Ventcel type.

This work allows also to have a reference in three-dimensional case for approximate boundary conditions as effect of a thin layer.

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AN APPROACH TO PILLAI'S PROBLEM WITH THE PELL SEQUENCE AND THE POWERS OF 3

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Abstract. In this paper, we consider the Diophantine equation $P_n - 3^a = \nu$ and find all ν having at least two representations. In the proof of the main theorem, we use a version of the Baker-Davenport reduction method.

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Keywords: Pillai's problem, Pell numbers, Diophantine equation, Baker's theory

1. INTRODUCTION

Let us consider the Diophantine equation

$$a^x - b^y = \nu \tag{1.1}$$

where a, b are given fixed positive integers, $\nu \geq 1$ and $\min\{x, y\} \geq 2$. This equation is also known as Pillai's equation. The $\nu = 1$ case, known also as the Catalan's conjecture, was proved by Mihăilescu [24]. In 1936 and 1937 (see [25, 26]), Pillai conjectured that only integers ν with at least two representations of the form $2^x - 3^y$ are as follows:

$$2^3 - 3^2 = -1 = 2^1 - 3^1, \quad 2^5 - 3^3 = 5 = 2^3 - 3^1, \quad 2^8 - 3^5 = 13 = 2^4 - 3^1.$$

This conjecture is a continuation of the work of Herschfeld [20, 21] in 1935 and verified by Stroeker and Tijdeman in 1982 [28]. The problem that Pillai deals with in the general case has recently begun to be studied in integer sequences thanks to Baker's theory. For example in [13], authors shows that the set of integers admitting at least two representations of the form $F_n - 2^a$ is $\{-30, -11, -3, -1, 0, 1, 5, 85\}$. Since then, many studies including integer sequences were handled by the authors (see [3, 6–12, 15–19]).

We consider Pell sequence which is defined by $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$. Some of the terms of the Pell sequence are given by

$$0, 1, 2, 5, 12, 29, 70, \ldots$$

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Its characteristic polynomial is of the form $x^2-2x-1=0$ whose roots are $\alpha=1+\sqrt{2}$ and $\beta=1-\sqrt{2}$. Binet's formula enables us to rewrite the Pell sequence by using the roots α and β as

$$P_n=\frac{\alpha^n-\beta^n}{2\sqrt{2}}. \tag{1.2}$$

Also, it is known that

$$\alpha^{n-2}\leq P_n\leq \alpha^{n-1}. \tag{1.3}$$

More detailed information on Pell sequences can be found in [22].

In this article, we aimed to prove the following result.

Theorem 1. *The only integers ν having at least two representations of the form*

$$P_n-3^a=\nu \tag{1.4}$$

are $\{-1,2\}$. Furthermore, all the representations of these integers as P_n-3^a with integers $n\geq 0$ and $a\geq 0$ are given by

$$P_2-3^1=-1=P_0-3^0$$
$$P_5-3^3=2=P_3-3^1.$$

2. Preliminaries

2.1. *Linear forms in logarithms.*

Let us give the concept of logarithmic height of an algebraic number and its some properties, which is a component of an important theorem that we will use for proof.

Definition 1. Let ξ be an algebraic number of degree d with minimal polynomial

$$a_0x^d+a_1x^{d-1}+\cdots+a_d=a_0\cdot\prod_{i=1}^{d}(x-\xi_i)$$

where a_i's are relatively prime integers with $a_0>0$ and ξ_i's are conjugates of ξ. Then

$$h(\xi)=\frac{1}{d}\left(\log a_0+\sum_{i=1}^{d}\log\left(\max\{|\xi_i|,1\}\right)\right)$$

is called *the logarithmic height of* ξ. The following proposition gives some properties of logarithmic height that can be found in [27].

Proposition 1. *Let $\xi,\xi_1,\xi_2,\ldots,\xi_t$ be elements of an algebraic closure of $\mathbb{Q}$ and $m\in\mathbb{Z}$. Then*

(1) $h(\xi_1\cdots\xi_t)\leq\sum_{i=1}^{t}h(\xi_i)$
(2) $h(\xi_1+\cdots+\xi_t)\leq\log t+\sum_{i=1}^{t}h(\xi_i)$
(3) $h(\xi^m)=|m|\,h(\xi)$.

We use a Baker-type lower bound (see in [2]) for a nonzero linear form in logarithms of algebraic numbers several times to prove our main result Theorem 1. In the literature, there are many such bounds but we will use the result of Matveev stated in the following theorem (see [23] or Theorem 9.4 in [5]).

Theorem 2. *Let $\gamma_1,\gamma_2,\ldots,\gamma_s$ be nonzero elements of a real algebraic number field $\mathbb{F}$ of degree D, $b_1,b_2,\ldots,b_s$ rational integers. Set*

$$\Lambda := \gamma_1^{b_1}\ldots\gamma_s^{b_s} - 1$$

and

$$B := \max\{|b_1|,\ldots,|b_s|\}.$$

If Λ is nonzero, then

$$\log|\Lambda| > -3\cdot 30^{s+4}\cdot(s+1)^{5.5}\cdot D^2\cdot(1+\log D)\cdot(1+\log(sB))\cdot A_1\cdots A_s$$

where

$$A_i \geq \max\{D\cdot h(\gamma_i), |\log\gamma_i|, 0.16\}$$

for all $1\leq i\leq s$. If $\mathbb{F}=\mathbb{R}$, then

$$\log|\Lambda| > -1.4\cdot 30^{s+3}\cdot t^{4.5}\cdot D^2\cdot(1+\log D)\cdot(1+\log B)\cdot A_1\cdots A_s.$$

2.2. *Baker-Davenport reduction lemma.*

We use some of the results of the continued fractions theory to reduce the upper bounds on variables, which are quite large, that we have obtained throughout the calculations. The following lemma we specifically use, and taken from [4], is a variation of the result due to [14] which is a generalization of a result of [1].

Lemma 1. *Let A, B, μ be some real numbers with $A>0$ and $B>1$ and let γ be an irrational number and M be a positive integer. Take p/q as a convergent of the continued fraction of γ such that $q>6M$. Set $\varepsilon := \|\mu q\| - M\|\gamma q\| > 0$ where $\|\cdot\|$ denotes the distance from the nearest integer. Then there is no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w}$$

in positive integers u, v and w with

$$u\leq M \qquad \text{and} \qquad w\geq \frac{\log\frac{Aq}{\varepsilon}}{\log B}.$$

3. Proof of Theorem 1

Let us assume that there exist nonnegative integers n,a,n_1,a_1 such that $(n,a)\neq(n_1,a_1)$ and

$$P_n - 3^a = P_{n_1} - 3^{a_1}. \tag{3.1}$$

If we take $a = a_1$ then we obtain $P_n = P_{n_1}$ and so $(n,a) = (n_1,a_1)$ which contadicts our assumption. Thus, without loss of the generality, we can assume that $a > a_1$. By rewriting the equation (3.1) as

$$P_n - P_{n_1} = 3^a - 3^{a_1}$$

we get $n > n_1$ from the positivity of left hand side of the equality. Using the equality above and the inequality (1.3), we obtain

$$\alpha^{n-4} \leq P_{n-2} \leq P_n - P_{n_1} = 3^a - 3^{a_1} < 3^a, \tag{3.2}$$

$$\alpha^{n-1} \geq P_n > P_n - P_{n_1} = 3^a - 3^{a_1} \geq 3^{a-1}. \tag{3.3}$$

These two inequalities yield immediately that

$$1 + \left(\frac{\log 3}{\log \alpha}\right)(a-1) < n < \left(\frac{\log 3}{\log \alpha}\right) a + 4. \tag{3.4}$$

If $n \leq 200$ then we have $a \leq 158$ from the above inequality. When we examine the solutions of the equation (1.4) in the Mathematica program for these bounds, we see that there is no other solution than the solutions stated in the Theorem 1. Therefore, we assume that $n > 200$ from now on and from (3.4) we have that $a > 158$. So, to solve the equation (1.4), it suffices to find an upper bound for n.

3.1. *Upper bound for n*

If we substitute the formula (1.2) in equation (3.1), we get

$$\left|\frac{\alpha^n}{2\sqrt{2}} - 3^a\right| = \left|\frac{\beta^n}{2\sqrt{2}} + \frac{\alpha^{n_1} - \beta^{n_1}}{2\sqrt{2}} - 3^{a_1}\right| = \frac{1}{2}\left(\frac{\alpha^{n_1}}{\sqrt{2}} + \sqrt{2}\right) + 3^{a_1} < 3\max\{\alpha^{n_1}, 3^{a_1}\}.$$

Multiplying both sides by 3^{-a}, using the relation (3.2) and the fact that $\alpha < 3$, we obtain

$$\left|\left(2\sqrt{2}\right)^{-1} \alpha^n 3^{-a} - 1\right| < 3\max\left\{\frac{\alpha^{n_1}}{3^a}, 3^{a_1-a}\right\} ¡ \max\left\{\alpha^{n_1-n+6}, 3^{a_1-a+1}\right\}. \tag{3.5}$$

Let us apply the Theorem 2 to the above inequality. Set $\Delta_1 := \left(2\sqrt{2}\right)^{-1} \alpha^n 3^{-a} - 1$. Δ_1 is not zero. If it were zero, it would be $\alpha^{2n} \in \mathbb{Z}$, which is impossible. Let us take $s := 3$, $(\gamma_1,\gamma_2,\gamma_3) := \left(2\sqrt{2}, \alpha, 3\right)$ and $(b_1,b_2,b_3) := (-1,n,-a)$. We have $D := 2$ since each γ_i belongs to $\mathbb{Q}\left(\sqrt{2}\right)$ and we can choose A_1, A_2, A_3 and B as follows:

$$\begin{aligned}
A_1 &:= 2.1 > 2.079 \simeq 2 \cdot \log\left(2\sqrt{2}\right) = 2 \cdot h(\gamma_1) \\
A_2 &:= 0.9 > 0.8813 \simeq \log \alpha = 2 \cdot h(\gamma_2) \\
A_3 &:= 2.2 > 2.1972 \simeq 2 \cdot \log 3 = 2 \cdot h(\gamma_3) \\
B &:= 2n
\end{aligned}$$

since $\max\{1,n,a\} \leq 2n$. Then from Theorem 2, we get

$$\log|\Delta_1| > -1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2\cdot(1+\log 2)\cdot(1+\log(2n))\cdot 2.1\cdot 0.9\cdot 2.2.$$

By comparing with (3.5), we obtain

$$\min\{(n-n_1-6)\log\alpha,(a-a_1-1)\log 3\} < 4.04\cdot 10^{12}\cdot(1+\log(2n))$$

which gives

$$\min\{(n-n_1)\log\alpha,(a-a_1)\log 3\} < 4.1\cdot 10^{12}\cdot(1+\log(2n)). \tag{3.6}$$

Now, let us consider the above result as two cases.

Case 1. $\min\{(n-n_1)\log\alpha,(a-a_1)\log 3\} = (n-n_1)\log\alpha$.
In this case, we rewrite the equation (3.1) as

$$\left|\left(\frac{\alpha^{n-n_1}-1}{2\sqrt{2}}\right)\alpha^{n_1}-3^a\right| = \left|\frac{\beta^n-\beta^{n_1}}{2\sqrt{2}}-3^{a_1}\right| < 1+3^{a_1} < 3^{1+a_1},$$

which implies

$$\left|\left(\frac{\alpha^{n-n_1}-1}{2\sqrt{2}}\right)\alpha^{n_1}3^{-a}-1\right| < 3^{a_1-a+1}. \tag{3.7}$$

Set $\Delta_2 := \left(\frac{\alpha^{n-n_1}-1}{2\sqrt{2}}\right)\alpha^{n_1}3^{-a}-1$. If $\Delta_2 = 0$, then we get

$$\alpha^n-\alpha^{n_1} = 2\sqrt{2}\cdot 3^a.$$

Conjugating this equation in $\mathbb{Q}\left(\sqrt{2}\right)$ yields that

$$\beta^n-\beta^{n_1} = -2\sqrt{2}\cdot 3^a. \tag{3.8}$$

From the triangle inequality, absolute value of the left hand side of the equation (3.8) is less than 2, but that of the right hand side is greater than 2 for $a > 158$. This is a contradiciton and hence $\Delta_2 \neq 0$. So, we can apply Theorem 2 to the equation (3.7) with the parameters

$$s = 3, \quad (\gamma_1,\gamma_2,\gamma_3) := \left(\frac{\alpha^{n-n_1}-1}{2\sqrt{2}},\alpha,3\right) \quad \text{and} \quad (b_1,b_2,b_3) := (1,n_1,-a).$$

Since the minimal polynomial of γ_1 divides

$$8x^2-8P_{n-n_1}x-\left((-1)^{n-n_1}+1-Q_{n-n_1}\right)$$

where $\{Q_t\}_{t\geq 0}$ is the Pell-Lucas sequence, we get

$$\begin{aligned} h(\gamma_1) &\leq \frac{1}{2}\left(\log 8+\log\left(\frac{\alpha^{n-n_1}+1}{2\sqrt{2}}\right)\right) < \frac{1}{2}\log\left(4\sqrt{2}\alpha^{n-n_1}\right) \\ &< \frac{1}{2}(n-n_1+2)\log\alpha < 4.2\cdot 10^{11}\cdot(1+\log 2n). \end{aligned}$$

So, we can choose A_1, A_2, A_3 and B as follows:

$$A_1 := 8.4\cdot 10^{11}\cdot(1+\log 2n) > 2\cdot h(\gamma_1)$$

$$A_2 := 0.9 > 2 \cdot h(\gamma_2)$$
$$A_3 := 2.2 > 2 \cdot h(\gamma_3)$$
$$B := 2n$$

since $\max\{1, n_1, a\} \leq 2n$. Thus, we obtain that

$$\log|\Delta_2| > -C \cdot 8.4 \cdot 10^{11} \cdot (1 + \log 2n) \cdot 0.9 \cdot 2.2$$

where

$$C := 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1 + \log 2) \cdot (1 + \log(2n))$$

and this inequality implies that

$$\log|\Delta_2| > -1.61 \cdot 10^{24} \cdot (1 + \log 2n)^2 .$$

Comparing the last inequality with the ineqıality (3.7), we get that

$$(a - a_1)\log 3 < 1.62 \cdot 10^{24} \cdot (1 + \log 2n)^2 .$$

Case 2. $\min\{(n - n_1)\log\alpha, (a - a_1)\log 3\} = (a - a_1)\log 3$.
In this case, we rewrite the equation (3.1) as

$$\left| \frac{\alpha^n}{2\sqrt{2}} - 3^{a_1}\left(3^{a-a_1} - 1\right) \right| = \left| \frac{\beta^n + \alpha^{n_1} - \beta^{n_1}}{2\sqrt{2}} \right| < \frac{\alpha^{n_1} + 2}{2\sqrt{2}} < \alpha^{n_1}$$

which implies

$$\begin{aligned} \left| \left(2\sqrt{2}\left(3^{a-a_1} - 1\right)\right)^{-1} \alpha^n 3^{-a_1} - 1 \right| &< \frac{\alpha^{n_1}}{3^a - 3^{a_1}} \\ &< \frac{3\alpha^{n_1}}{3^a} < 3\alpha^{n_1 - n + 4} < \alpha^{n_1 - n + 6}. \end{aligned} \tag{3.9}$$

from (3.2). Let

$$\Delta_3 := \left(2\sqrt{2}\left(3^{a-a_1} - 1\right)\right)^{-1} \alpha^n 3^{-a_1} - 1.$$

Clearly, $\Delta_3 \neq 0$. If the opposite were true, that is, if $\Delta_3 = 0$, then $\alpha^{2n} \in \mathbb{Z}$, which is a contradiction. In this aplication of Theorem 2, we take

$$s = 3, \qquad (\gamma_1, \gamma_2, \gamma_3) := \left(2\sqrt{2}\left(3^{a-a_1} - 1\right), \alpha, 3\right)$$
$$\text{and} \qquad (b_1, b_2, b_3) := (-1, n, -a_1).$$

Since the minimal polynomial of γ_1 is $x^2 - 8\left(3^{a-a_1} - 1\right)^2$ we get,

$$h(\gamma_1) = \log\left(2\sqrt{2}\left(3^{a-a_1} - 1\right)\right) < (a - a_1 + 1)\log 3 \quad < 4.2 \cdot 10^{12} \cdot (1 + \log(2n))$$

from (3.6). So, we can take

$$A_1 := 8.4 \cdot 10^{12} \cdot (1 + \log 2n) \qquad A_3 := 2.2$$
$$A_2 := 0.9 \qquad B := 2n$$

since $\max\{1,n,a_1\} \leq 2n$. Then, we obtain

$$\log|\Delta_3| > -C \cdot 8.4 \cdot 10^{12} \cdot (1+\log(2n)) \cdot 0.9 \cdot 2.2,$$

where

$$C := 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1+\log 2) \cdot (1+\log(2n)).$$

Hence,

$$\log|\Delta_3| > -1.61 \cdot 10^{25} \cdot (1+\log(2n))^2.$$

Comparing the above result with the inequality (3.9), we get

$$(n-n_1)\log\alpha < 1.7 \cdot 10^{25} \cdot (1+\log(2n))^2.$$

So, from Case 1 and Case 2, we have

$$\begin{aligned} \min\{(n-n_1)\log\alpha, (a-a_1)\log 3\} &< 4.1 \cdot 10^{12} \cdot (1+\log(2n)), \\ \max\{(n-n_1)\log\alpha, (a-a_1)\log 3\} &< 1.7 \cdot 10^{25} \cdot (1+\log(2n))^2. \end{aligned} \tag{3.10}$$

Finally, we rewrite the equation (3.1) as

$$\left|\left(\frac{\alpha^{n-n_1}-1}{2\sqrt{2}}\right)\alpha^{n_1} - (3^a - 3^{a_1})\right| = \left|\frac{\beta^n - \beta^{n_1}}{2\sqrt{2}}\right| < |\beta^{n_1}| < 1.$$

Dividing the leftmost and rightmost sides by $(3^a - 3^{a_1})$ in the above inequality, we get

$$\left|\left(\frac{\alpha^{n-n_1}-1}{2\sqrt{2}\,(3^{a-a_1}-1)}\alpha^{n_1}3^{-a_1} - 1\right)\right| < \left|\frac{1}{(3^a - 3^{a_1})}\right| \leq \frac{3}{3^a} \leq 3\alpha^{-(n-4)} \leq \alpha^{6-n} \tag{3.11}$$

from (3.2). Now, we can apply Theorem 2 to the above inequality with

$$\begin{aligned} s &= 3, & (\gamma_1,\gamma_2,\gamma_3) &:= \left(\frac{\alpha^{n-n_1}-1}{2\sqrt{2}\,(3^{a-a_1}-1)}, \alpha, 3\right) \\ \text{and} \quad & & (b_1,b_2,b_3) &:= (1, n_1, -a_1). \end{aligned}$$

Let us compute $h(\gamma_1)$ for A_1.

$$\begin{aligned} h(\gamma_1) = h\left(\frac{\alpha^{n-n_1}-1}{2\sqrt{2}\,(3^{a-a_1}-1)}\right) &\leq h\left(\frac{\alpha^{n-n_1}-1}{2\sqrt{2}}\right) + h\left(3^{a-a_1}-1\right) \\ &< \frac{1}{2}(n-n_1+2)\log\alpha + (a-a_1)\log 3 < 2.1 \cdot 10^{25} \cdot (1+\log(2n))^2. \end{aligned}$$

So, we can take

$$\begin{aligned} A_1 &:= 4.2 \cdot 10^{25} \cdot (1+\log(2n))^2 & A_3 &:= 2.2 \\ A_2 &:= 0.9 & B &:= 2n \end{aligned}$$

since $\max\{1,n_1,a_1\} \leq 2n$. If we take

$$\Delta_4 := \frac{\alpha^{n-n_1}-1}{2\sqrt{2}\,(3^{a-a_1}-1)}\alpha^{n_1}3^{-a_1} - 1$$

one can easily see that $\Delta_4 \neq 0$ as in the previous ones. So, from the Theorem 2, we obtain

$$\log|\Delta_4| > -C \cdot 4.2 \cdot 10^{25} \cdot (1+\log(2n))^2 \cdot 0.9 \cdot 2.2$$

where $C := 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1+\log 2) \cdot (1+\log(2n))$. This bound and (3.11) gives

$$n - 6 < 8.06 \cdot 10^{37} \cdot 4.2 \cdot 10^{25} \cdot (1+\log(2n))^3$$

which can be simplified as

$$n < 8.52 \cdot 10^{43}. \tag{3.12}$$

Now, let us try to reduce the bound on n.

3.2. *Reducing the bound on n*

We want to improve the bound on n we find in (3.12) until it is small enough. We will use Lemma 1 several times for this. Let us go back to the inequality (3.5). Set

$$\Gamma := n\log\alpha - a\log 3 - \log\left(2\sqrt{2}\right).$$

Assume that $\min\{n-n_1, a-a_1\} \geq 20$. If we consider Δ_1, Δ_2 and Δ_3 again then since we assume that $\min\{n-n_1, a-a_1\} \geq 20$ we obtain

$$\left|e^{\Gamma} - 1\right| = |\Delta_1| < \frac{1}{4}$$

which implies

$$|\Gamma| < \frac{1}{2}.$$

Since the inequality $|\xi| < 2\left|e^{\xi} - 1\right|$ holds for all $\xi \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, we get

$$|\Gamma| < 2\max\left\{\alpha^{n_1-n+6}, 3^{a_1-a+1}\right\} \leq \max\left\{\alpha^{n_1-n+7}, 3^{a_1-a+2}\right\}.$$

Let us assume that $\Gamma > 0$. Then we have

$$\begin{aligned} 0 &< n\left(\frac{\log\alpha}{\log 3}\right) - a - \frac{\log\left(2\sqrt{2}\right)}{\log 3} < \max\left\{\alpha^{n_1-n+7}, 3^{a_1-a+2}\right\} \\ &< \max\left\{\frac{\alpha^7}{(\log 3)\cdot\alpha^{n-n_1}}, \frac{9}{(\log 3)\cdot 3^{a-a_1}}\right\} < \max\left\{436\cdot\alpha^{-(n-n_1)}, 9\cdot 3^{-(a-a_1)}\right\}. \end{aligned}$$

Now, we apply Lemma 1 with

$$\gamma := \frac{\log\alpha}{\log 3}, \quad \mu := \frac{\log\left(1/\left(2\sqrt{2}\right)\right)}{\log 3}, \quad (A,B) := (436,\alpha) \quad \text{or} \quad (A,B) := (9,3).$$

We can choose $M := 8.52 \cdot 10^{43} > n$ from the inequality (3.12). Computing 86th convergent of γ with the help of the Mathematica gives

$$q := 525422721605994540683823357818903486740995467$$

where $q > 6M$. For this value of q, it can be seen that $\varepsilon > 0.35$. So, we have either

$$n - n_1 < 125 \qquad \text{or} \qquad a - a_1 < 97. \tag{3.13}$$

In case of $\Gamma < 0$, variables change as

$$\gamma := \frac{\log 3}{\log \alpha}, \quad \mu := \frac{\log\left(2\sqrt{2}\right)}{\log \alpha}, \quad (A,B) := (543,\alpha) \quad \text{or} \quad (A,B) := (11,3).$$

So, we get $\varepsilon > 0.42$ but this does not affect the result stated in (3.13).
Let us consider separately the results obtained in the inequality (3.13).
Firstly, let $n - n_1 < 125$. Assumed that $a - a_1 \geq 20$ and consider the inequality (3.7). Set

$$\Gamma_1 := n_1 \log \alpha - a \log 3 + \log\left(\frac{\alpha^{n-n_1} - 1}{2\sqrt{2}}\right).$$

Then, we get

$$|\Gamma_1| < \frac{6}{3^{a-a_1}}$$

from the inequality (3.7). Let us assume that $\Gamma_1 > 0$. In this case, we have

$$0 < n_1\left(\frac{\log \alpha}{\log 3}\right) - a + \frac{\log\left(\left(\alpha^{n-n_1} - 1\right)/\left(2\sqrt{2}\right)\right)}{\log 3} < \frac{6}{(\log 3)\cdot 3^{a-a_1}} < \frac{6}{3^{a-a_1}}$$

Let we again apply the Lemma 1 with the parameters

$$\gamma := \frac{\log \alpha}{\log 3}, \qquad \mu := \frac{\log\left(\left(\alpha^{n-n_1} - 1\right)/\left(2\sqrt{2}\right)\right)}{\log 3}, \qquad (A,B) := (6,3)$$

for $n - n_1 \in \{1,2,\cdots 124\}$. All values of ε are greater than 0.00046 for the 88th convergent of γ which also satisfies the condition $q > 6M$. So, we get

$$a - a_1 < \frac{\log\left(6q/0.00046\right)}{\log 3} < 106.$$

Thus, $n - n_1 < 125$ implies that $a - a_1 < 106$. In case $\Gamma_1 < 0$, the same results are obtained.
Secondly, now, let us assumed that $a - a_1 < 97$. Set

$$\Gamma_2 := n \log \alpha - a_1 \log 3 + \log\left(1/\left(2\sqrt{2}\left(3^{a-a_1} - 1\right)\right)\right)$$

from the inequality (3.9) for Δ_3 and assume that $n - n_1 \geq 20$. Hence, we get

$$|\Gamma_2| < \frac{3\alpha^6}{\alpha^{n-n_1}}.$$

Assume that $\Gamma_2 > 0$. Then, we obtain

$$0 < n\left(\frac{\log \alpha}{\log 3}\right) - a_1 + \frac{\log\left(1/\left(2\sqrt{2}\left(3^{a-a_1} - 1\right)\right)\right)}{\log 3} < \frac{3\alpha^6}{\log 3 \cdot \alpha^{n-n_1}} < \frac{541}{\alpha^{n-n_1}}.$$

If we again apply Lemma 1 with the parameters

$$\gamma := \frac{\log\alpha}{\log 3}, \qquad \mu := \frac{\log\left(1/\left(2\sqrt{2}\,(3^{a-a_1}-1)\right)\right)}{\log 3}, \qquad (A,B) := (541,\alpha)$$

for $a-a_1 \in \{1,2,\cdots,96\}$ we get $\varepsilon > 0.005$ and

$$n - n_1 < 134.$$

The same result is obtained for $\Gamma_2 < 0$. Hence, if $n-n_1 < 125$, then $a-a_1 < 106$, and if $a-a_1 < 97$, then $n-n_1 < 134$. Consequently, the results $n-n_1 \leq 134$ and $a-a_1 \leq 105$ are always valid.
Finally, set

$$\Gamma_3 := n_1\log\alpha - a_1\log 3 + \log\left(\frac{\alpha^{n-n_1}-1}{2\sqrt{2}\,(3^{a-a_1}-1)}\right)$$

using the inequality (3.11). Since $n > 200$, inequality (3.11) implies that

$$|\Gamma_3| < \frac{3}{\alpha^{n-6}} = \frac{3\alpha^6}{\alpha^n}.$$

Assume that $\Gamma_3 > 0$. Then, we have

$$0 < n_1\left(\frac{\log\alpha}{\log 3}\right) - a_1 + \frac{\log\left((\alpha^{n-n_1}-1)/\left(2\sqrt{2}\,(3^{a-a_1}-1)\right)\right)}{\log 3} < \frac{3\alpha^6}{\log 3\cdot\alpha^n} < \frac{541}{\alpha^n}.$$

If we again apply Lemma 1 to the above inequality for $n-n_1 \in \{1,\cdots,134\}$ and $a-a_1 \in \{1,\cdots,105\}$ except for $(n-n_1, a-a_1) = (4,1)$we get $\varepsilon > 6.70754\times 10^{-6}$ and

$$n < 146. \tag{3.14}$$

A similar result is obtained for $\Gamma_3 < 0$. Now, let $(n-n_1, a-a_1) = (4,1)$. Since

$$\frac{\alpha^4-1}{4\sqrt{2}} = \alpha^2$$

we have

$$\Gamma_3 := (n_1+2)\log\alpha - a_1\log 3.$$

Hence, we get

$$\left|\gamma - \frac{a_1}{n_1+2}\right| < \frac{541}{(n_1+2)\,\alpha^n}.$$

Since $n > 200$, $\alpha^n > 2\times 541\times\left(9\times 10^{43}\right) > 2\times 541\times(n_1+2)$which implies that right hand side of above inequality is at most $2/(n_1+2)^2$. Let $[s_0,s_1,s_2,\cdots] = [0,1,4,\cdots]$ be the continued fraction expression of the above γ and let p_t/q_t be its tth convergent. By Legendre's criterion, we obtain that $a_1/(n_1+2) = p_t/q_t$ for

some $t \in \{1,2,\cdots,85\}$. Here we use that $n_1+2 < 9\times 10^{43} < q_{85}$. Then putting $s(M) := \max\{s_i : i = 1,\cdots,85\} = 181$ we get

$$\left|\gamma - \frac{p_t}{q_t}\right| > \frac{1}{(s(M)+2)\,q_t^2} = \frac{1}{183q_t^2}.$$

Therefore, we obtain

$$\frac{1}{183q_t^2} < \frac{541}{\alpha^n q_t}$$

which implies that

$$\alpha^n < 541\times 183\times 9\times 10^{43}.$$

So, we get

$$n < 128. \tag{3.15}$$

Bounds for n in (3.14) and (3.15) are all contradicts our assumption that $n > 200$. This completes the proof of Theorem 1.

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VARIETIES CORRESPONDING TO CLASSES OF COMPLEMENTED POSETS

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Abstract. As algebraic semantics of the logic of quantum mechanics there are usually used orthomodular posets, i.e. bounded posets with a complementation which is an antitone involution and where the join of orthogonal elements exists and the orthomodular law is satisfied. When we omit the condition that the complementation is an antitone involution, then we obtain skew-orthomodular posets. To each such poset we can assign a bounded λ-lattice in a non-unique way. Bounded λ-lattices are lattice-like algebras whose operations are not necessarily associative. We prove that any of the following properties for bounded posets with a unary operation can be characterized by certain identities of an arbitrary assigned λ-lattice: complementarity, orthogonality, almost skew-orthomodularity and skew-orthomodularity. Moreover, we prove corresponding independence results. Finally, we show that the variety of skew-orthomodular λ-lattices is congruence permutable as well as congruence regular.

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It is well-known that an algebraic semantics of the logic of quantum mechanics is provided by means of orthomodular lattices as shown by G. Birkhoff and J. von Neumann [2] or, independently, by K. Husimi [9]. The details of this construction can be found e.g. in the monograph by L. Beran [1]. However, it was shown later that in the logic of quantum mechanics the connective disjunction represented by the lattice operation $\vee$ need not exist for elements that are not orthogonal. Hence the concept of an orthomodular poset was introduced as follows:

A bounded *poset* $\mathbf{P} = (P, \leq, ', 0, 1)$ with a unary operation is called *orthomodular* (see e.g. [4]) if $'$ is an antitone involution on $(P, \leq)$ which is a complementation, i.e. $x \leq y$ implies $y' \leq x'$, $x'' = x$, $\sup(x, x')$ exists for all $x \in P$ and it is equal to 1, and $\inf(x, x')$ exists for all $x \in P$ and it is equal to 0; moreover, $\sup(x, y)$ must exist in case

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$x \leq y'$; finally, for all $x, y \in P$ with $x \leq y$ there exists $x \vee (x' \wedge y)$ and it is equal to y. Due to De Morgan's laws, also dually, $y \wedge (y' \vee x)$ exists for all $x, y \in P$ with $x \leq y$ and it is equal to x. The last property is called the *orthomodular law* and can be expressed in the case of lattices alternatively in the form of the equivalent identities

$$x \vee (x' \wedge (x \vee y)) \approx x \vee y,$$
$$y \wedge (y' \vee (x \wedge y)) \approx x \wedge y.$$

It was shown by V. Snášel [10] that every bounded poset can be organized into a so-called bounded λ-lattice. Bounded λ-lattices can be considered as bounded lattices whose binary operations are not necessarily associative. More precisely, a bounded λ-lattice is a bounded lattice if and only if its binary operations are associative, see [6] for details. The notion of a λ-lattice was successfully used by the first two authors for constructing a variety of λ-lattices which corresponds to the class of orthomodular posets. This variety turns out to be congruence permutable and congruence regular. Of course, it is of advantage to work with varieties of algebras instead of classes of posets since for varieties the well-known methods of Universal Algebra can be applied.

Back to the logic of quantum mechanics, we take as an appropriate structure for the algebraic semantics complemented posets in which the join of two orthogonal elements exists and which satisfy the orthomodular law. We do not ask this complementation to be an antitone involution. The concept of complemented lattices which satisfy the orthomodular law, but whose complementation need not be an antitone involution was introduced in [3] and studied in [7]. This concept was generalized by the first and third author to posets in [8]. In the present paper we show that similarly to the case of orthomodular posets, for the posets described above the method of considering assigned λ-lattices can be successfully applied. In fact, it turns out that important properties of certain posets can be characterized by identities of assigned λ-lattices.

Let $\mathbf{P} = (P, \leq, ', 0, 1)$ be a bounded poset with a unary operation and $a, b \in P$. We define

$$L(a,b) := \{x \in P \mid x \leq a, b\},$$
$$U(a,b) := \{x \in P \mid a, b \leq x\}.$$

If there exists $\sup(a,b)$ or $\inf(a,b)$ then we will denote these elements by $a \vee b$ or $a \wedge b$, respectively.

We call $\mathbf{P}$ *complemented* if it satisfies the identities $x \vee x' \approx 1$ and $x \wedge x' \approx 0$. In this case the operation $'$ is called a *complementation*. We say that a, b are *orthogonal elements* of P, shortly $a \perp b$, if $a \leq b'$. We call a *complemented poset* $\mathbf{P}$ *orthogonal* if $x \vee y$ exists for arbitrary orthogonal elements x, y of P. We call an orthogonal poset $\mathbf{P}$ *almost skew-orthomodular* if $x \vee (x' \wedge y)$ exists for all $x, y \in P$ with $x \leq y$. We call

an almost skew-orthomodular poset $\mathbf{P}$ *skew-orthomodular* if $x \vee (x' \wedge y) = y$ for all $x, y \in P$ with $x \leq y$.

Example 1. The poset shown in Fig. 1

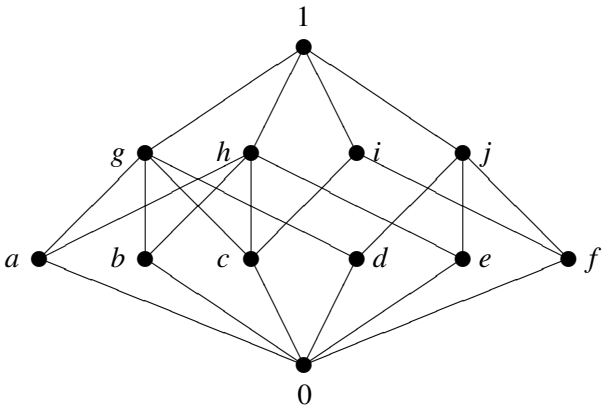

Fig. 1

with

x	0	a	b	c	d	e	f	g	h	i	j	1
x'	1	j	j	j	i	i	h	f	f	e	c	0

is skew-orthomodular, but is not a lattice. Moreover, $'$ is antitone, but not an involution.

In the following, let $\mathcal{P}c$, $\mathcal{P}o$, $\mathcal{P}a$ and $\mathcal{P}s$ denote the class of all complemented, orthogonal, almost skew-orthomodular and skew-orthomodular posets, respectively. We are going to show that these classes do not coincide, i.e. the inclusions are proper.

Theorem 1. *We have*

$$\mathcal{P}s \subsetneqq \mathcal{P}a \subsetneqq \mathcal{P}o \subsetneqq \mathcal{P}c.$$

Proof. The poset shown in Fig. 2

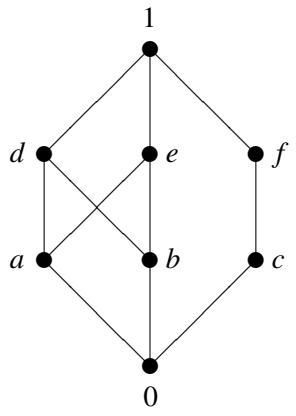

Fig. 2

with

x	0	a	b	c	d	e	f	1
x'	1	c	c	d	f	f	a	0

is not a lattice and belongs to $\mathcal{P}a \setminus \mathcal{P}s$ since $a \leq d$, but $a \vee (a' \wedge d) = a \vee (c \wedge d) = a \vee 0 = a \neq d$. Moreover, $'$ is neither antitone nor an involution. The poset shown in Fig. 3

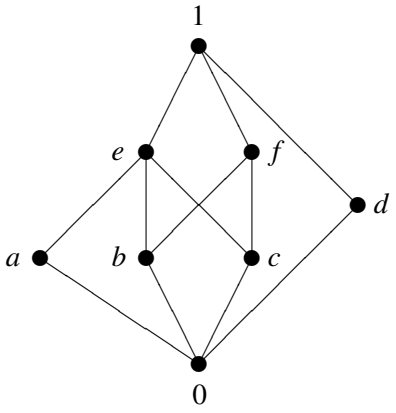

Fig. 3

with

x	0	a	b	c	d	e	f	1
x'	1	f	d	d	a	d	d	0

belongs to $\mathcal{P}o \setminus \mathcal{P}a$ since $a \leq e$, but $a' \wedge e = f \wedge e$ does not exist. Moreover, $'$ is neither antitone nor an involution. The poset shown in Fig. 4

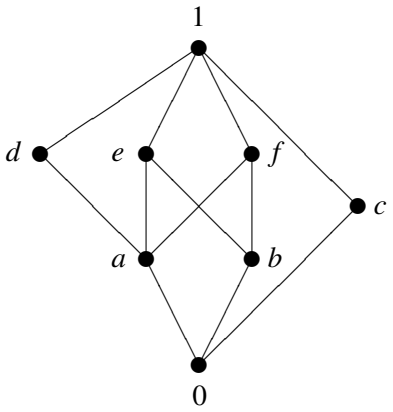

Fig. 4

with

x	0	a	b	c	d	e	f	1
x'	1	c	d	a	c	c	c	0

belongs to $\mathcal{P}c \setminus \mathcal{P}o$ since $a \leq d = b'$, but $a \vee b$ does not exist. Moreover, $'$ is neither antitone nor an involution. □

Now we introduce the concept of a bounded λ-lattice taken from [10].

A *bounded λ-lattice* is an algebra $(L, \sqcup, \sqcap, 0, 1)$ of type $(2,2,0,0)$ satisfying the identities

$$x \sqcup y \approx y \sqcup x,\ x \sqcap y \approx y \sqcap x,$$
$$x \sqcup ((x \sqcup y) \sqcup z) \approx (x \sqcup y) \sqcup z,\ x \sqcap ((x \sqcap y) \sqcap z) \approx (x \sqcap y) \sqcap z,$$
$$x \sqcup (x \sqcap y) \approx x,\ x \sqcap (x \sqcup y) \approx x,$$
$$x \sqcup 0 \approx x,\ x \sqcup 1 \approx 1.$$

Hence the class of bounded λ-lattices forms a variety. Notice that every bounded λ-lattice satisfies the identities

$$x \sqcap 0 \approx 0 \text{ and } x \sqcap 1 \approx x.$$

Recall from [6] that every variety of bounded λ-lattices is congruence distributive. It is well-known that in every bounded λ-lattice $x \sqcup y = y$ is equivalent to $x \sqcap y = x$.

Let $\mathbf{P} = (P, \leq, ', 0, 1)$ be a bounded poset with a unary operation. We introduce binary operations $\sqcup$ and $\sqcap$ on P as follows ($x, y \in P$): If $x \vee y$ exists then $x \sqcup y := x \vee y$. Otherwise $x \sqcup y = y \sqcup x$ is an arbitrary element of $U(x,y)$. If $x \wedge y$ exists then $x \sqcap y := x \wedge y$. Otherwise $x \sqcap y = y \sqcap x$ is an arbitrary element of $L(x,y)$. Then $(P, \sqcup, \sqcap, ', 0, 1)$ is a bounded λ-lattice with a unary operation which we call a λ-*lattice assigned* to the bounded poset $\mathbf{P}$. Let $\mathbb{A}(\mathbf{P})$ denote the set of all λ-lattices assigned to $\mathbf{P}$.

To every bounded λ-lattice $\mathbf{L} = (L, \sqcup, \sqcap, ', 0, 1)$ with a unary operation we assign a bounded poset $\mathbb{P}(\mathbf{L}) = (L, \leq, ', 0, 1)$ as follows:

$$x \leq y \text{ if and only if } x \sqcup y = y$$

($x, y \in L$). It was shown in [10] that $(L, \leq, 0, 1)$ is a bounded poset and

$$x \leq y \text{ if and only if } x \sqcap y = x.$$

Moreover, using the absorption laws, we easily derive the identities

$$x \sqcup x \approx x \text{ and } x \sqcap x \approx x.$$

For $i \in \{c, o, a, s\}$ let $\mathcal{L}i$ denote the class of all bounded λ-lattices $\mathbf{L}$ with a unary operation satisfying $\mathbb{P}(\mathbf{L}) \in \mathcal{P}i$. Hence $\mathcal{L}i$ can be considered as a representation of $\mathcal{P}i$. This means that the properties of $\mathcal{L}i$ may be considered as properties of $\mathcal{P}i$.

In the following we will characterize the above mentioned properties of bounded posets with a unary operation by means of identities of assigned λ-lattices. Surprisingly, this works despite the fact that this assignment is not unique. Hence, classes of complemented, orthogonal, almost skew-orthomodular and skew-orthomodular posets will be characterized by means of varieties of bounded λ-lattices. We start with complemented posets.

Theorem 2. *Let* $\mathbf{P} = (P, \leq, ', 0, 1)$ *be a bounded poset with a unary operation and* $\mathbf{L} = (P, \sqcup, \sqcap, ', 0, 1) \in \mathbb{A}(\mathbf{P})$. *Then* $\mathbf{P}$ *is complemented if and only if* $\mathbf{L}$ *satisfies the identities*

$$(x \sqcup y) \sqcup (x' \sqcup y) \approx 1, \tag{1}$$

$$(x \sqcap y) \sqcap (x' \sqcap y) \approx 0. \tag{2}$$

Hence $\mathcal{L}c$ *is a variety.*

Proof. Let $a, b \in P$. First assume $\mathbf{P} \in \mathcal{P}c$. Then

$$a \leq a \sqcup b \leq (a \sqcup b) \sqcup (a' \sqcup b),$$

$$a' \leq a' \sqcup b \leq (a \sqcup b) \sqcup (a' \sqcup b)$$

and hence $(a \sqcup b) \sqcup (a' \sqcup b) \in U(a, a') = \{1\}$, i.e., $(a \sqcup b) \sqcup (a' \sqcup b) = 1$. Dually,

$$(a \sqcap b) \sqcap (a' \sqcap b) \leq a \sqcap b \leq a,$$

$$(a \sqcap b) \sqcap (a' \sqcap b) \leq a' \sqcap b \leq a'$$

and hence $(a \sqcap b) \sqcap (a' \sqcap b) \in L(a, a') = \{0\}$, i.e., $(a \sqcap b) \sqcap (a' \sqcap b) = 0$. Conversely, suppose $\mathbf{L}$ to satisfy identities (1) and (2). If $a, a' \leq b$ then $a \sqcup b = a' \sqcup b = b$ and hence

$$b = b \sqcup b = (a \sqcup b) \sqcup (a' \sqcup b) = 1$$

showing $a \vee a' = 1$. Similarly, $b \leq a, a'$ implies $a \sqcap b = a' \sqcap b = b$ and therefore

$$b = b \sqcap b = (a \sqcap b) \sqcap (a' \sqcap b) = 0$$

showing $a \wedge a' = 0$. Hence $\mathbf{P} \in \mathcal{P}c$. □

The identities

$$x \sqcup x' \approx 1,$$
$$x \sqcap x' \approx 0.$$

are necessary, but not sufficient for a bounded λ-lattice to be complemented.

Orthogonal posets can be characterized by an identity that is a bit more complicated than the previous ones (1) and (2).

Theorem 3. *Let* $\mathbf{P} = (P, \leq, ', 0, 1) \in \mathcal{P}c$ *and* $\mathbf{L} = (P, \sqcup, \sqcap, ', 0, 1) \in \mathbb{A}(\mathbf{P})$. *Then* $\mathbf{P}$ *is orthogonal if and only if* $\mathbf{L}$ *satisfies the identity*

$$(((x \sqcap y') \sqcup z) \sqcup (y \sqcup z)) \sqcap ((x \sqcap y') \sqcup y) \approx (x \sqcap y') \sqcup y. \tag{3}$$

Hence $\mathcal{L}o$ *is a variety.*

Proof. Let $a, b, c \in P$. First assume $\mathbf{P} \in \mathcal{P}o$. Then $(a \sqcap b') \vee b$ exists. Now

$$a \sqcap b' \leq (a \sqcap b') \sqcup c \leq ((a \sqcap b') \sqcup c) \sqcup (b \sqcup c),$$
$$b \leq b \sqcup c \leq ((a \sqcap b') \sqcup c) \sqcup (b \sqcup c)$$

and hence $((a \sqcap b') \sqcup c) \sqcup (b \sqcup c) \in U(a \sqcap b', b)$ which yields

$$(a \sqcap b') \sqcup b \leq ((a \sqcap b') \sqcup c) \sqcup (b \sqcup c)$$

which is equivalent to identity (3). Conversely, suppose $\mathbf{L}$ to satisfy identity (3). Assume $a \perp b$. Then $a \sqcap b' = a$. If $a, b \leq c$ then $a \sqcup c = b \sqcup c = c$ and hence

$$a \sqcup b = (a \sqcap b') \sqcup b = (((a \sqcap b') \sqcup c) \sqcup (b \sqcup c)) \sqcap ((a \sqcap b') \sqcup b) \leq$$
$$\leq ((a \sqcap b') \sqcup c) \sqcup (b \sqcup c) = (a \sqcup c) \sqcup c = c \sqcup c = c$$

showing $a \sqcup b = a \vee b$, i.e. $a \vee b$ exists. Hence $\mathbf{P} \in \mathcal{P}o$. □

Similarly as above we can characterize almost skew-orthomodular posets.

Theorem 4. *Let* $\mathbf{P} = (P, \leq, ', 0, 1) \in \mathcal{P}o$ *and* $\mathbf{L} = (P, \sqcup, \sqcap, ', 0, 1) \in \mathbb{A}(\mathbf{P})$. *Then* $\mathbf{P}$ *is almost skew-orthomodular if and only if* $\mathbf{L}$ *satisfies the identity*

$$(((x \sqcap y)' \sqcap z) \sqcap (y \sqcap z)) \sqcup ((x \sqcap y)' \sqcap y) \approx (x \sqcap y)' \sqcap y. \tag{4}$$

Hence $\mathcal{L}a$ *is a variety.*

Proof. Let $a, b, c \in P$. First assume $\mathbf{P} \in \mathcal{P}a$. Then $(a \sqcap b)' \wedge b$ exists. Now

$$((a \sqcap b)' \sqcap c) \sqcap (b \sqcap c) \leq (a \sqcap b)' \sqcap c \leq (a \sqcap b)',$$
$$((a \sqcap b)' \sqcap c) \sqcap (b \sqcap c) \leq b \sqcap c \leq b$$

and hence $((a \sqcap b)' \sqcap c) \sqcap (b \sqcap c) \in L((a \sqcap b)', b)$ which yields

$$((a \sqcap b)' \sqcap c) \sqcap (b \sqcap c) \leq (a \sqcap b)' \sqcap b$$

which is equivalent to identity (4). Conversely, suppose $\mathbf{L}$ to satisfy identity (4). Assume $a \leq b$. Then $a \sqcap b = a$. If $c \leq a', b$ then $a' \sqcap c = b \sqcap c = c$ and hence

$$\begin{aligned} c = c \sqcap c = (a' \sqcap c) \sqcap c = ((a \sqcap b)' \sqcap c) \sqcap (b \sqcap c) \leq \\ \leq (((a \sqcap b)' \sqcap c) \sqcap (b \sqcap c)) \sqcup ((a \sqcap b)' \sqcap b) = (a \sqcap b)' \sqcap b = a' \sqcap b \end{aligned}$$

showing $a' \sqcap b = a' \wedge b$, i.e. $a' \wedge b$ exists. Moreover, $a \sqcup b = b$. If $a, a' \sqcap b \leq c$ then $a \sqcup c = (a' \sqcap b) \sqcup c = c$ and hence

$$\begin{aligned} a \sqcup (a' \sqcap b) = (b \sqcap a') \sqcup a = (((b \sqcap a') \sqcup c) \sqcup (a \sqcup c)) \sqcap ((b \sqcap a') \sqcup a) \leq \\ \leq ((b \sqcap a') \sqcup c) \sqcup (a \sqcup c) = ((a' \sqcap b) \sqcup c) \sqcup c = c \sqcup c = c \end{aligned}$$

showing $a \sqcup (a' \sqcap b) = a \vee (a' \wedge b)$, i.e. $a \vee (a' \wedge b)$ exists. Hence $\mathbf{P} \in \mathcal{P}a$. □

Example 2. The poset shown in Fig. 5

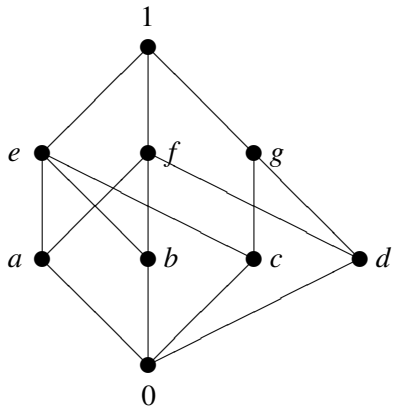

Fig. 5

with

x	0	a	b	c	d	e	f	g	1
x'	1	g	g	1	1	g	g	1	g

satisfies identity (4), but does not belong to $\mathcal{P}c$ since $c \wedge c' = c \wedge 1 = c \neq 0$, and it is not a lattice. Moreover, $'$ is antitone, but not an involution.

Next we characterize skew-orthomodular posets by identities of assigned λ-lattices. Since in almost skew-orthomodular λ-lattices we have

$$x \sqcup (x' \sqcap (x \sqcup y)) = x \vee (x' \wedge (x \vee y)),$$

we only need to add a single identity. Let us note that the poset shown in Fig. 1 is almost skew-orthomodular, but not skew-orthomodular. For skew-orthomodularity we have the following result. The proof is evident.

Corollary 1. *Let* $\mathbf{P} = (P, \leq, ', 0, 1)$ *be a bounded poset with a unary operation and* $\mathbf{L} = (P, \sqcup, \sqcap, ', 0, 1) \in \mathbb{A}(\mathbf{P})$. *Then* $\mathbf{P}$ *is skew-orthomodular if and only if* $\mathbf{L}$ *satisfies the identities* (1) – (5) *where*

$$x \sqcup (x' \sqcap (x \sqcup y)) \approx x \sqcup y. \tag{5}$$

Hence $\mathcal{L}s$ *is a variety.*

In the following we show some important congruence properties of the variety $\mathcal{L}s$.

Let $\mathcal{V}$ be a variety. The variety $\mathcal{V}$ is called *congruence permutable* if $\Theta \circ \Phi = \Phi \circ \Theta$ for all $\mathbf{A} \in \mathcal{V}$ and all $\Theta, \Phi \in \operatorname{Con}\mathbf{A}$. The variety $\mathcal{V}$ is called *congruence regular* if for each $\mathbf{A} = (A, F) \in \mathcal{V}$, $a \in A$ and $\Theta, \Phi \in \operatorname{Con}\mathbf{A}$ with $[a]\Theta = [a]\Phi$ we have $\Theta = \Phi$. It is well-known (cf. [5], Theorems 3.1.8 and 6.1.3) that $\mathcal{V}$ is congruence permutable if and only if there exists a so-called *Malcev term*, i.e. a ternary term p satisfying

$$p(x,x,y) \approx p(y,x,x) \approx y$$

and it is regular if and only if there exists a positive integer n and ternary terms $t_1, \ldots, t_n$ such that

$$t_1(x,y,z) = \cdots = t_n(x,y,z) = z \text{ if and only if } x = y.$$

Theorem 5. *Let* $\mathcal{V}$ *be a variety of bounded* λ*-lattices* $(L, \sqcup, \sqcap, ', 0, 1)$ *with a unary operation satisfying the identities* $x \sqcap x' \approx 0$ *and* (5). *Then* $\mathcal{V}$ *is congruence permutable. In particular,* $\mathcal{L}s$ *is congruence permutable.*

Proof. The term

$$p(x,y,z) := (x \sqcup (y' \sqcap (y \sqcup z))) \sqcap (z \sqcup (y' \sqcap (y \sqcup x)))$$

is a Malcev term since

$$\begin{aligned} p(x,x,z) &\approx (x \sqcup (x' \sqcap (x \sqcup z))) \sqcap (z \sqcup (x' \sqcap (x \sqcup x))) \approx (x \sqcup z) \sqcap (z \sqcup (x' \sqcap x)) \approx \\ &\approx (x \sqcup z) \sqcap (z \sqcup 0) \approx (x \sqcup z) \sqcap z \approx z, \\ p(x,z,z) &\approx (x \sqcup (z' \sqcap (z \sqcup z))) \sqcap (z \sqcup (z' \sqcap (z \sqcup x))) \approx (x \sqcup (z' \sqcap z)) \sqcap (z \sqcup x) \approx \\ &\approx (x \sqcup 0) \sqcap (z \sqcup x) \approx x \sqcap (z \sqcup x) \approx x. \end{aligned}$$

□

We are going to show also congruence regularity of the variety $\mathcal{L}s$.

Theorem 6. *Let* $\mathcal{V}$ *be a variety of bounded* λ*-lattices* $(L, \sqcup, \sqcap, ', 0, 1)$ *with a unary operation satisfying the identities* $0' \approx 1$, $x \sqcap x' \approx 0$ *and* (5). *Then* $\mathcal{V}$ *is congruence regular. In particular, the variety* $\mathcal{L}s$ *is congruence regular.*

Proof. Put

$$t(x,y) := (x' \sqcap (x \sqcup y)) \sqcup (y' \sqcap (x \sqcup y)).$$

Then

$$t(x,x) \approx (x' \sqcap (x \sqcup x)) \sqcup (x' \sqcap (x \sqcup x)) \approx (x' \sqcap x) \sqcup (x' \sqcap x) \approx 0 \sqcup 0 \approx 0,$$

and if $t(x,y)=0$ then $x' \sqcap (x \sqcup y) = y' \sqcap (x \sqcup y) = 0$ and hence

$$x = x \sqcup 0 = x \sqcup (x' \sqcap (x \sqcup y)) = x \sqcup y = y \sqcup x = y \sqcup (y' \sqcap (y \sqcup x)) = y \sqcup 0 = y.$$

If we put

$$t_1(x,y,z) := t(x,y) \sqcup z,$$
$$t_2(x,y,z) := (t(x,y))' \sqcap z$$

then

$$t_1(x,x,z) \approx t(x,x) \sqcup z \approx 0 \sqcup z \approx z,$$
$$t_2(x,x,z) \approx (t(x,x))' \sqcap z \approx 0' \sqcap z \approx 1 \sqcap z \approx z,$$

and if $t_1(x,y,z) = t_2(x,y,z) = z$ then $t(x,y) \leq z \leq (t(x,y))'$ and hence $t(x,y) = t(x,y) \sqcap (t(x,y))' = 0$ whence $x = y$. (Observe that in $\mathcal{L}c$ we have $0' \approx 0 \sqcup 0' \approx 1$.) □

Finally, we are going to show the independence of identities (1) – (4).

Theorem 7. *Within the variety of bounded λ-lattices with a unary operation the following hold:*

(i) *Identities* (1) – (4) *are independent,*
(ii) *identities* (1) – (4) *do not imply identity* (5).

Proof.

(i) The λ-lattice $(\{0,1\}, \sqcup, \sqcap, 0, 1)$ with

x	0	1
x'	0	0

satisfies (2), (3) and (4), but not (1) since

$$(0 \sqcup 0) \sqcup (0' \sqcup 0) = 0 \sqcup (0 \sqcup 0) = 0 \neq 1.$$

The λ-lattice $(\{0,1\}, \sqcup, \sqcap, 0, 1)$ with

x	0	1
x'	1	1

satisfies (1), (3) and (4), but not (2) since

$$(1 \sqcap 1) \sqcap (1' \sqcap 1) = 1 \sqcap (1 \sqcap 1) = 1 \neq 0.$$

The λ-lattice shown in Fig. 6

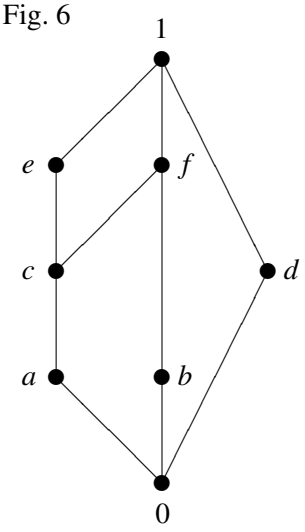

Fig. 6

with $a \sqcup b = 1$, $b \sqcup c = f$, $e \sqcap f = c$ and

x	0	a	b	c	d	e	f	1
x'	1	d	e	d	a	b	d	0

satisfies (1), (2) and (4), but not (3) since

$$\begin{aligned}(((a \sqcap b') \sqcup f) \sqcup (b \sqcup f)) \sqcap ((a \sqcap b') \sqcup b) &= (((a \sqcap e) \sqcup f) \sqcup f) \sqcap ((a \sqcap e) \sqcup b) = \\ &= ((a \sqcup f) \sqcup f) \sqcap (a \sqcup b) = (f \sqcup f) \sqcap 1 = \\ &= f \neq 1 = a \sqcup b = (a \sqcap e) \sqcup b = \\ &= (a \sqcap b') \sqcup b.\end{aligned}$$

The λ-lattice shown in Fig. 7

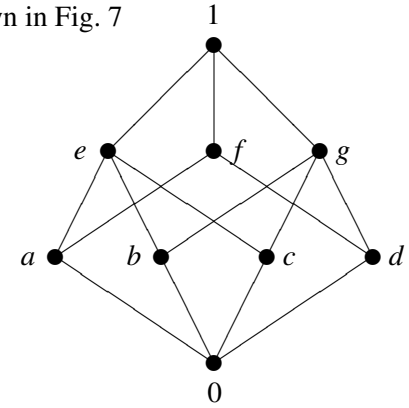

Fig. 7

with $a \sqcup b = e$, $a \sqcup c = e$, $a \sqcup d = f$, $b \sqcup c = 1$, $b \sqcup d = g$, $c \sqcup d = g$, $e \sqcap f = a$, $e \sqcap g = c$, $f \sqcap g = d$ and

x	0	a	b	c	d	e	f	g	1
x'	1	g	f	f	e	d	b	a	0

satisfies (1), (2) and (3), but not (4) since

$$\begin{aligned}(((d \sqcap g)' \sqcap b) \sqcap (g \sqcap b)) \sqcup ((d \sqcap g)' \sqcap g) &= ((d' \sqcap b) \sqcap b) \sqcup (d' \sqcap g) = \\ &= ((e \sqcap b) \sqcap b) \sqcup (e \sqcap g) = (b \sqcap b) \sqcup c = \\ &= b \sqcup c = 1 \neq c = e \sqcap g = d' \sqcap g = \\ &= (d \sqcap g)' \sqcap g.\end{aligned}$$

(ii) The λ-lattice shown in Fig. 8

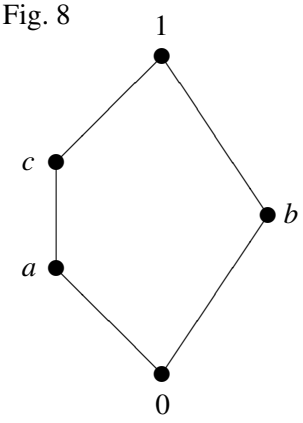

Fig. 8

with

x	0	a	b	c	1
x'	1	b	a	b	0

satisfies (1) – (4), but not (5) since

$$a \sqcup (a' \sqcap (a \sqcup c)) = a \sqcup (b \sqcap c) = a \sqcup 0 = a \neq c = a \sqcup c.$$

□

It should be mentioned that the λ-lattices in Fig. 6 (with $a \vee b = f$) and in Fig. 8 are in fact lattices.

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AN EFFICIENT METHOD FOR SOLVING SECOND-ORDER DELAY DIFFERENTIAL EQUATION

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Abstract. In this paper, the initial-value problem for a linear second order delay differential equation is considered. To solve this problem numerically, an appropriate difference scheme is constructed by using the method of integral identities which contains basis functions and interpolating quadrature rules with weight and remainder term in integral form. Besides, the method is proved to be first-order convergent in discrete maximum norm. The numerical illustration provided support the theoretical results. Finally, the proposed method is compared with the implicit Euler method.

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Keywords: delay differential equation, initial value problem, finite difference method, error estimate, convergence

1. INTRODUCTION

Delay differential equations (DDEs) are very important and useful in many areas of science and engineering, for instance, physics, biomathematics, medicine, economics, chemistry, etc [7, 8, 13, 16]. Firstly, DDEs occur in modeling effects and interactions between cancer cells, namely tumor population [17]. In engineering, time delays often arise in many life systems like sensors and dynamical processes [14]. Additionally, DDEs can be used in modelling in bridge constructions, traffic control, robot technology, physiological processes and diseases, climate systems, signal transmission [2, 12, 13].

Besides, metal cutting processes [9], bistable device [22], the controlled flexible structures [23] are modelled by second order DDEs. For example, the damped linear oscillator with delayed velocity feedback described by the second order delay differential equation

$$x''(t)+cx'(t)+kx(t)=vx(t-\tau)+wx'(t-\tau)+f\cos(\lambda t),$$

where $c>0$, $k>0$, v and w are constants, representing the damping coefficient, the stiffness coefficient, the feedback gains of displacement path and velocity path, respectively and τ is constant time delay, f is the external force (for details, see [10]).

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Motivated by the above works, we consider the following initial value problem with delay in the interval $\bar{I} = [0,T]$:

$$Lu := u''(t) + a(t)u'(t) + b(t)u'(t-r) + c(t)u(t-r) = f(t), \qquad t \in I, \tag{1.1}$$

$$u(t) = \varphi(t), \qquad t \in I_0; \qquad u'(0) = \gamma \tag{1.2}$$

where $I = (0,T] = \cup_{p=1}^{m} I_p$, $I_p = \{t : r_{p-1} < t \leq r_p\}$, $1 \leq p \leq m$ and $r_s = sr$, $0 \leq s \leq m$, and $I_0 = [-r,0]$ (for simplicity we suppose that T/r is integer; i.e., $T = mr$). $a(t) \geq \alpha > 0$, $b(t)$, $c(t)$, $f(t)$ and $\varphi(t)$ are sufficiently smooth functions on $\bar{I}$ and I_0, γ is a real number, r is a constant large delay. The existence and uniqueness of a solution to second order DDEs is discussed in [3, 5, 6, 11, 13, 24].

In this study, the delay term r is chosen as large. Because if r is taken as small, it can be easily reduced to ordinary differential equation with the help of Taylor series, which such equations take place a lot in the literature. However, the interest in large delay equations has been remarkable in recent years.

Even if this problem is linear, it may not always be possible to find the exact solution. Therefore, it is important to develop effective numerical methods to solve DDEs. In the past two decades, several numerical methods have been proposed for the second order DDEs problem including finite difference method [4, 20], two-point block method [18], Adams–Moulton method [19], variational iteration method [15], Legendre-Gauss spectral collocation method [25], initial value method [21].

The aim of this study is to present a more effective numerical method than classical methods such as Euler and Runge–Kutta. This method to be used for the numerical solution of (1.1)–(1.2) consists of a finite difference scheme given on a uniform mesh. The scheme is constructed by the method based on using appropriate quadrature rules with the weight and remainder terms in integral form. These results in a local truncation error containing only first derivative of the exact solution and hence facilitates examination of the convergence. This method of approximation has the advantage that the schemes can also be more effective than classical schemes such as Euler in the case when the continuous problem is considered under certain restrictions.

The remainder of this paper is as follows. We put forward some important properties of the exact solution in Section 2 and introduce the finite difference discretization in Section 3. In Section 4, we analyze the error estimates for the approximate solution and prove the convergence in the discrete maximum norm. In Section 5, we present numerical results which confirm the theoretical analysis. The paper ends with a summary of the main conclusions.

Throughout the paper, C denotes a generic positive constant. Some specific, fixed constants of this kind are indicated by subscripting C and $\overline{C}$.

2. Properties of Exact Solution

Here, we give some properties of the exact solution of (1.1)–(1.2), which are needed in the analysis of appropriate numerical solution. For any continuous function

$g(t)$, we use

$$\|g\|_\infty \equiv \|g\|_{\infty,\bar{I}} = \max_{0\le t\le T} |g(t)|, \qquad \|g\|_1 \equiv \|g\|_{1,I} = \int_0^T |g(t)|\,dt,$$
$$\|g\|_{\infty,p} \equiv \|g\|_{\infty,I_p}, \qquad \|g\|_{1,p} \equiv \|g\|_{1,I_p}, \qquad 0 \le p \le m.$$

Lemma 1. *If $a, b, c, f \in C(\bar{I})$ and $\varphi \in C^1(I_0)$, then for the solution u of the problem (1.1)–(1.2) we have*

$$\|u\|_{\infty,p} \le C_p, \qquad 1 \le p \le m, \tag{2.1}$$
$$\|u'\|_{\infty,p} \le \overline{C}_p, \qquad 1 \le p \le m, \tag{2.2}$$

where

$$C_1 = \alpha^{-1}(|\gamma| + [\|f\|_{1,1} + \|b\|_{1,1}\|\varphi'\|_{\infty,0} + \|c\|_{1,1}\|\varphi\|_{\infty,0}]),$$
$$\overline{C}_1 = |\gamma| + \alpha^{-1}[\|f\|_{\infty,1} + \|b\|_{\infty,1}\|\varphi'\|_{\infty,0} + \|c\|_{\infty,1}\|\varphi\|_{\infty,0}],$$
$$C_p = \alpha^{-1}(\overline{C}_{p-1} + \|f\|_{1,p} + \|b\|_{1,p}\overline{C}_{p-1} + \|c\|_{1,p}C_{p-1}), \qquad 2 \le p \le m,$$
$$\overline{C}_p = \overline{C}_{p-1} + \alpha^{-1}[\|f\|_{\infty,p} + \|b\|_{\infty,p}\overline{C}_{p-1} + \|c\|_{\infty,p}C_{k-1}], \qquad 2 \le p \le m.$$

Proof. The proof is by induction in p. First, for $t \in I_p$, from (1.1) we have

$$u'(t) = u'(r_{p-1})e^{-\int_{r_{p-1}}^{t} a(s)ds} + \int_{r_{p-1}}^{t} F(\xi)e^{-\int_{\xi}^{t} a(s)ds}\,d\xi, \tag{2.3}$$

$$u(t) = u'(r_{p-1})\int_{r_{p-1}}^{t} e^{-\int_{r_{p-1}}^{\xi} a(s)ds}\,d\xi + \int_{r_{p-1}}^{t} ds \int_{r_{p-1}}^{s} F(\xi)e^{-\int_{\xi}^{t} a(\eta)d\eta}\,d\xi$$
$$= u'(r_{p-1})\int_{r_{p-1}}^{t} e^{-\int_{r_{p-1}}^{\xi} a(s)ds}\,d\xi + \int_{r_{p-1}}^{t} d\xi F(\xi)\int_{\xi}^{t} e^{-\int_{\xi}^{s} a(\eta)d\eta}\,ds \tag{2.4}$$

with

$$F(t) = f(t) - b(t)u'(t-r) - c(t)u(t-r).$$

Now, for $p = 1$ ($t \in I_1$), from (2.4) we have

$$|u(t)| \le u'(0)\int_0^t e^{-\int_0^\xi \alpha ds}\,d\xi + \int_0^t d\xi\,|F(\xi)|\int_\xi^t e^{-\int_\xi^s a(\eta)d\eta}\,ds$$
$$\le |\gamma|\int_0^t e^{-\alpha\xi}d\xi$$

$$+\int_0^t [|f(\xi)|+|b(\xi)|\,|u'(\xi-r)|+|c(\xi)|\,|u(\xi-r)|]\int_\xi^t e^{-\alpha(s-\xi)}ds$$
$$\le \alpha^{-1}|\gamma|(1-e^{-\alpha\xi})$$
$$+\alpha^{-1}\int_0^t [|f(\xi)|+|b(\xi)|\,|u'(\xi-r)|+|c(\xi)\,|u(\xi-r)|\,|](1-e^{-\alpha(t-\xi)})d\xi.$$

So, we get

$$|u(t)| \le \alpha^{-1}[|\gamma|+\|f\|_{1,1}+\|b\|_{1,1}\|\varphi'\|_{\infty,0}+\|c\|_{1,1}\|\varphi\|_{\infty,0}] \equiv C_1.$$

Next, from (2.3), we have

$$|u'(t)| \le |u'(0)|e^{-\alpha t}+\int_0^t [|f(\xi)|+|b(\xi)|\,|u'(\xi-r)|+|c(\xi)\,|u(\xi-r)|\,|]e^{-\alpha(t-\xi)}d\xi$$
$$\le |\gamma|+\alpha^{-1}[\|f\|_{\infty,1}+\|b\|_{\infty,1}\|\varphi'\|_{\infty,0}+\|c\|_{\infty,1}\|\varphi\|_{\infty,0}](1-e^{-\alpha t}) \equiv \overline{C}_1,$$

thus the inequalities (2.1) and (2.2) are valid for $p=1$. And now, let the inequalities (2.1) and (2.2) be true for $p=k$. That is,

$$C_k = \alpha^{-1}\left[\overline{C}_{k-1}+\|f\|_{1,k}+\|b\|_{1,k}\overline{C}_{k-1}+\|c\|_{1,k}C_{k-1}\right],$$
$$\overline{C}_k = \overline{C}_{k-1}+\alpha^{-1}[\|f\|_{\infty,k}+\|b\|_{\infty,k}\overline{C}_{k-1}+\|c\|_{\infty,k}C_{k-1}].$$

For $t\in I_{k+1}$, because of (2.4) we get

$$|u(t)| \le |u'(r_k)|\int_{r_k}^t e^{-\alpha(\xi-r_k)}d\xi+\int_{r_k}^t d\xi\,|F(\xi)|\int_\xi^t e^{-\alpha(s-\xi)}ds$$
$$\le \alpha^{-1}|u'(r_k)|(1-e^{-\alpha(t-r_k)})$$
$$+\alpha^{-1}\int_{r_k}^t (|f(\xi)|+|b(\xi)|\,|u'(\xi-r)|+|c(\xi)|\,|u(\xi-r)|)(1-e^{-\alpha(t-\xi)})d\xi$$
$$\le \alpha^{-1}\overline{C}_k+\alpha^{-1}[\|f\|_{1,k+1}+\|b\|_{1,k+1}\overline{C}_k+\|c\|_{1,k+1}C_k]$$

and from (2.3) we have

$$|u'(t)| \le |u'(r_k)|e^{-\alpha(t-r_k)}$$
$$+\int_{r_k}^t (|f(\xi)|+|b(\xi)|\,|u'(\xi-r)|+|c(\xi)|\,|u(\xi-r)|)e^{-\alpha(t-\xi)}d\xi$$
$$\le \overline{C}_k+\alpha^{-1}[\|f\|_{\infty,k+1}+\|b\|_{\infty,k+1}\overline{C}_k+\|c\|_{\infty,k+1}C_k](1-e^{-\alpha(t-r_k)}) \equiv \overline{C}_{k+1}.$$

Therefore the inequalities (2.1) and (2.2) hold for $p = k+1$. □

3. THE DIFFERENCE SCHEME AND MESH

We further denote by ω_{N_0} the uniform mesh on $\bar{I}$:

$$\omega_{N_0} = \{t_i = ih,\ i = 1,2,...,N_0,\ h = T/N_0 = r/N\}$$

which contains N mesh points at each subinterval I_p $(1 \le p \le m)$:

$$\omega_{N_p} = \{t_i : (p-1)N+1 \le i \le pN\}, \qquad 1 \le p \le m,$$

and consequently

$$\omega_{N_0} = \bigcup_{p=1}^{m} \omega_{N_p}.$$

For any mesh function $g(t)$, we set $g_i = g(t_i)$ and y_i denote by an approximation of $u(t)$ at t_i. We use the notation

$$g_{\bar{t},i} = \frac{g_i - g_{i-1}}{h}, \qquad g_{t,i} = \frac{g_{i+1} - g_i}{h},$$

$$g_{\overset{0}{t},i} = \frac{g_{i+1} - g_{i-1}}{2h}, \qquad g_{\bar{t}t,i} = \frac{g_{i+1} - 2g_i + g_{i-1}}{h^2},$$

$$\|g\|_{\infty,p} \equiv \|g\|_{\infty,\omega_{N_p}} := \max_{1\le i\le N} |g_i|, \qquad 1 \le p \le m.$$

To construct the numerical method, we begin with the identity

$$h^{-1}\int_{t_{i-1}}^{t_{i+1}} Lu(t)\psi_i(t)dt = h^{-1}\int_{t_{i-1}}^{t_{i+1}} f(t)\psi_i(t)dt,\ 1 \le i \le N_0, \tag{3.1}$$

with basis functions

$$\psi_i(t) = \begin{cases} \psi_i^{(1)}(t), & t_{i-1} < t < t_i, \\ \psi_i^{(2)}(t), & t_i < t < t_{i+1}, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\psi_i^{(1)} = \frac{\int_{t_{i-1}}^{t} e^{\int_s^t a(\eta)d\eta}ds}{\int_{t_{i-1}}^{t_i} e^{\int_s^{t_i} a(\eta)d\eta}ds},$$

$$\psi_i^{(2)}(t) = \left(1 - \frac{\int_{t_i}^{t} e^{\int_s^{t_{i+1}} a(\eta)d\eta}ds}{\int_{t_i}^{t_{i+1}} e^{\int_s^{t_{i+1}} a(\eta)d\eta}ds}\right) e^{\int_{t_i}^{t} a(\tau)d\tau},$$

are the solutions of the following problems, respectively:

$$\begin{cases} \psi_i''(t) - (a(t)\psi_i(t))' = 0, & t_{i-1} < t < t_i, \\ \psi_i(t_{i-1}) = 0,\ \psi_i(t_i) = 1, \end{cases}$$

$$\begin{cases} \psi_i''(t) - (a(t)\psi_i(t))' = 0, & t_i < t < t_{i+1}, \\ \psi_i(t_i) = 1,\ \psi_i(t_{i+1}) = 0. \end{cases}$$

The relation (3.1) is rewritten as

$$\begin{aligned} -h^{-1}\int_{t_{i-1}}^{t_{i+1}} u'(t)\psi_i'(t)dt + h^{-1}\int_{t_{i-1}}^{t_{i+1}} a(t)u'(t)\psi_i(t)dt \\ + h^{-1}\int_{t_{i-1}}^{t_{i+1}} b(t)u'(t-r)\psi_i(t)dt + h^{-1}\int_{t_{i-1}}^{t_{i+1}} c(t)u(t-r)\psi_i(t)dt \\ = h^{-1}\int_{t_{i-1}}^{t_{i+1}} f(t)\psi_i(t)dt \end{aligned} \tag{3.2}$$

for $t \in (t_{i-1}, t_{i+1})$. Using the formulas (2.1) and (2.2) from [1] on each interval (t_{i-1}, t_i) and (t_i, t_{i+1}) taking into account (3.2) we have the following precise relation

$$\ell u_i \equiv A_i u_{\bar{t}t,i} + B_i u_{\bar{t}t,i-N} + C_i u^0_{t,i} + D_i u^0_{t,i-N} + E_i u_{i-N} = F_i + R_i,\ 1 \le i \le N_0,$$

with

$$\begin{aligned} A_i &= 1 - \frac{1}{2}\int_{t_{i-1}}^{t_i} a(t)\psi_i^{(1)}(t)dt + \frac{1}{2}\int_{t_i}^{t_{i+1}} a(t)\psi_i^{(2)}(t)dt, \\ B_i &= -\frac{1}{2}\int_{t_{i-1}}^{t_i} [b(t) + (t-t_i)c(t)]\,\psi_i^{(1)}(t)dt \\ &\quad + \frac{1}{2}\int_{t_i}^{t_{i+1}} [b(t) + (t-t_i)c(t)]\,\psi_i^{(2)}(t)dt, \\ C_i &= h^{-1}\int_{t_{i-1}}^{t_i} a(t)\psi_i^{(1)}(t)dt + h^{-1}\int_{t_i}^{t_{i+1}} a(t)\psi_i^{(2)}(t)dt, \\ D_i &= h^{-1}\int_{t_{i-1}}^{t_i} [b(t) + (t-t_i)c(t)]\,\psi_i^{(1)}(t)dt \\ &\quad + h^{-1}\int_{t_i}^{t_{i+1}} [b(t) + (t-t_i)c(t)]\,\psi_i^{(2)}(t)dt, \\ E_i &= h^{-1}\int_{t_{i-1}}^{t_i} c(t)\psi_i^{(1)}(t)dt + h^{-1}\int_{t_i}^{t_{i+1}} c(t)\psi_i^{(2)}(t)dt, \\ F_i &= h^{-1}\int_{t_{i-1}}^{t_i} f(t)\psi_i^{(1)}(t)dt + h^{-1}\int_{t_i}^{t_{i+1}} f(t)\psi_i^{(2)}(t)dt, \\ R_i &= h^{-1}\int_{t_{i-1}}^{t_{i+1}} dt \left[c(t)\psi_i(t) - \frac{d}{dt}(b(t)\psi_i(t))\right] \int_{t_{i-1}}^{t_{i+1}} u'(\xi - r)K_0(t,\xi)d\xi, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} K_0(t,\xi) &= T_0(t-\xi) - h^{-1}(t - t_{i-1}), \\ T_0(\lambda) &= 1, \qquad \lambda \ge 0; \qquad\qquad T_0(\lambda) = 0, \qquad \lambda < 0. \end{aligned}$$

On the other hand, in order to define an approximation for the initial condition (1.2), we begin with the identity

$$h^{-1}\int_{t_0}^{t_1} Lu(t)\psi^{(0)}(t)dt = h^{-1}\int_{t_0}^{t_1} f(t)\psi^{(0)}(t)dt,$$

with

$$\psi^{(0)}(t) = \left(1 - \frac{\int_{t_0}^{t} e^{\int_s^{t_1} a(\eta)d\eta}ds}{\int_{t_0}^{t_1} e^{\int_s^{t_1} a(\eta)d\eta}ds}\right) e^{\int_{t_0}^{t} a(\tau)d\tau},$$

which is the solution of the problem

$$\psi^{(0)}(t) = \begin{cases} \psi^{(0)\prime\prime}(t) - (a(t)\psi^{(0)}(t))' = 0, & t_0 < t < t_1, \\ \psi^{(0)}(t_0) = 1, & \psi^{(0)}(t_1) = 0. \end{cases}$$

In a similar way as above, we have

$$A_0 u_{t,0} = B_0 + r^{(0)},$$

where

$$\begin{aligned} A_0 &= h^{-1}\left(1 + \int_{t_0}^{t_1} a(t)\psi^{(0)}(t)dt\right), \\ B_0 &= h^{-1}\gamma + h^{-1}\int_{t_0}^{t_1} f(t)\psi^{(0)}(t)dt - h^{-1}u_{-N}\int_{t_0}^{t_1} c(t)\psi^{(0)}(t)dt \\ &\quad - h^{-1}u_{t,-N}\int_{t_0}^{t_1}[b(t) + tc(t)]\psi^{(0)}(t)dt, \\ r^{(0)} &= -h^{-1}\int_{t_0}^{t_1} dt\,(b(t)\psi^{(0)}(t))'\int_{t_0}^{t_1} u'(\xi - r)K_0(t,\xi)d\xi \\ &\quad + h^{-1}\int_{t_0}^{t_1} dt\; c(t)\psi^{(0)}(t)\int_{t_0}^{t_1} u'(\xi - r)K_0(t,\xi)d\xi. \end{aligned} \tag{3.4}$$

Consequently, we propose the following difference scheme for approximating the problem (1.1)–(1.2):

$$\ell y_i \equiv A_i y_{\bar{t}t,i} + B_i y_{\bar{t}t,i-N} + C_i y^0_{t,i} + D_i y^0_{t,i-N} + E_i y_{i-N} = F_i, \qquad 1 \le i \le N_0, \tag{3.5}$$

$$y_i = \varphi_i, \qquad -N \le i \le 0; \qquad \ell y_0 \equiv A_0 y_{t,0} = B_0. \tag{3.6}$$

Moreover, we can easily propose the implicit Euler method as an alternative to the approximate solution of the problem (1.1)–(1.2):

$$\ell y_i \equiv y_{\bar{t}t,i} + a_i y^0_{t,i} + b_i y^0_{t,i-N} + c_i y_{i-N} = f_i, \qquad 1 \le i \le N_0, \tag{3.7}$$

$$y_i = \varphi_i, \qquad -N \le i \le 0; \qquad \ell y_0 \equiv y_{t,0} = \gamma. \tag{3.8}$$

4. Convergence analysis of the method

To analyze the convergence of the method, we define the error function $z_i = y_i - u_i$, $0 \le i \le N_0$, which is the solution of the following discrete problem

$$\ell z_i = R_i, \qquad 1 \le i \le N_0, \tag{4.1}$$

$$z_i = 0, \qquad -N_0 \le i \le 0; \qquad \ell z_0 = -r^{(0)}, \tag{4.2}$$

where the truncation errors R_i and $r^{(0)}$ are given by (3.3) and (3.4), respectively.

Lemma 2. *If a, c, $f \in C(\bar{I})$, $b \in C^1(\bar{I})$, and $\varphi \in C^1(I_0)$, then for the truncation errors R_i and $r^{(0)}$, we have*

$$\|R\|_{\infty,N_0} \le CN^{-1}, \tag{4.3}$$

$$\left|r^{(0)}\right| \le CN^{-1}. \tag{4.4}$$

Proof. From (3.3), if we rewrite R_i,

$$|R_i| \le Ch^{-1} \int_{t_{i-1}}^{t_{i+1}} dt[(|c(t)| + |b'(t)| + |a(t)|\,|b(t)|)\psi_i(t)] \int_{t_{i-1}}^{t_{i+1}} |u'(\xi - r)|\, d\xi$$

$$\le Ch^{-1} \int_{t_{i-1}}^{t_{i+1}} dt\psi_i(t) \int_{t_{i-1}}^{t_{i+1}} |u'(\xi - r)|\, d\xi$$

and, by virtue of Lemma 1 and $0 < \psi_i(t) \le C$,

$$|R_i| \le Ch,$$

which implies (4.3). Now, we have to prove (4.4). From (3.4), we get

$$\left|r^{(0)}\right| \le Ch^{-1} \int_{t_0}^{t_1} dt[|c(t)| + |b'(t)| + |a(t)|\,|b(t)|]\psi^{(0)}(t) \int_{t_0}^{t_1} |u'(\xi - r)|\, d\xi$$

and also, due to Lemma 1 and $0 < \psi_0(t) \le C$, we have

$$\left|r^{(0)}\right| \le Ch.$$

□

Lemma 3. *For z_i, which is the solution of the problem (4.1)–(4.2), the following estimate holds:*

$$|z_k| \le \lambda_1 \left|r^{(0)}\right| + \lambda_2 \sum_{j=1}^{k-1} \|R\|_{\infty,j}, \quad 1 \le k \le N_0, \tag{4.5}$$

where

$$\lambda_1 = TA_0^{-1} e^{(\|b\|_\infty + \|c\|_\infty)T}, \qquad \lambda_2 = e^{(\|b\|_\infty + \|c\|_\infty)T}.$$

Proof. If $v_i = z_{t,i}$, then we can write the equation (4.1) as follows,

$$A_i z_{\bar{t}t,i} + B_i z_{\bar{t}t,i-N} + C_i z^0_{t,i} + D_i z^0_{t,i-N} + E_i z_{i-N} = R_i$$

$$A_i v_{\bar{t},i} + \frac{C_i}{2}(v_i + v_{i-1}) = G_i$$

where

$$G_i = R_i - B_i z_{\bar{t}t,i-N} - D_i z_{0\atop t,i-N} - E_i z_{i-N}.$$

Solving this difference equation with respect to v_i, we obtain

$$v_i = v_0 Q_i + \sum_{k=1}^{i} \phi_k Q_{i,k} \tag{4.6}$$

where

$$Q_{i,k} = \begin{cases} 1, & k = i, \\ \prod_{j=k+1}^{i} \frac{2A_i - hC_i}{2A_i + hC_i}, & 1 \le k \le i-1, \end{cases} \qquad \phi_k = \frac{2hG_i}{2A_i + hC_i}.$$

Taking into account $v_i = z_{t,i}$ in (4.6), we write

$$z_{i+1} = z_i + hv_0 Q_i + h\sum_{k=1}^{i} \phi_k Q_{i,k}. \tag{4.7}$$

Solving the first-order difference equation (4.7), we obtain

$$z_i = z_1 + h\sum_{k=1}^{i-1} z_{t,0} Q_k + h\sum_{k=1}^{i-1}\sum_{j=1}^{k} \phi_j Q_{k,j} = -\frac{hr^{(0)}}{A_0} - \frac{hr^{(0)}}{A_0}\sum_{k=1}^{i-1} Q_k + h\sum_{k=1}^{i-1}\sum_{j=1}^{k} \phi_j Q_{k,j}.$$

Next, since $2A_i + hC_i > 0$ and $0 < (2A_i - hC_i)/(2A_i + hC_i) < 1$ $(1 \le i \le N_0)$,

$$\begin{aligned} |z_i| &\le \frac{h\left|r^{(0)}\right|}{A_0} + \frac{h\left|r^{(0)}\right|(i-1)}{A_0} \\ &\quad + h\sum_{k=1}^{i-1}\left[\|R\|_{\infty,k} + (\|b\|_\infty + \|c\|_\infty)(|z_{k-N+1}| + |z_{k-N}| + |z_{k-N-1}|)\right] \\ &\le \frac{hN_0\left|r^{(0)}\right|}{A_0} + h\sum_{k=1}^{i-1}\left[\|R\|_{\infty,k} + (\|b\|_\infty + \|c\|_\infty)|z_k|\right]. \end{aligned}$$

Thus, we have

$$|z_i| \le \frac{T\left|r^{(0)}\right|}{A_0} + h\sum_{k=1}^{i-1}\|R\|_{\infty,k} + h(\|b\|_\infty + \|c\|_\infty)\sum_{k=1}^{i-1}|z_k|.$$

Using difference analogue of Gronwall's inequality, we get

$$|z_i| \le \left[TA_0^{-1}\left|r^{(0)}\right| + \sum_{k=1}^{i-1}\|R\|_{\infty,k}\right] e^{(\|b\|_\infty + \|c\|_\infty)t_i}$$

which gives (4.5). □

Now, we formulate the theorem, which expresses the main result of the paper.

Theorem 1. *For a, c, $f \in C(\bar{I})$, $b \in C^1(\bar{I})$, and $\varphi \in C^1(I_0)$, the solution of the difference problem (3.1)–(3.2) is first order convergent to the solution of (1.1)–(1.2):*

$$\|y-u\|_{\infty,\omega_{N_0}} \leq CN^{-1}.$$

Proof. It immediately follows from (4.5) by taking into consideration (4.3)–(4.4). □

5. Numerical results

In this section, we present the numerical results obtained not only by the proposed method (3.5)–(3.6) but also by the other method (3.7)–(3.8), confirming the theoretical results.

Example 1. We consider the particular problem:

$$u''(t)+2u'(t)+u'(t-1)-u(t-1)+e^{-t}=0, \qquad 0<t\leq 2 \tag{5.1}$$

$$u(t)=e^t, \qquad -1\leq t\leq 0, \qquad u'(0)=1. \tag{5.2}$$

whose the exact solution is

$$u(t)=\begin{cases} 1+e^{-t}-e^{-2t}, & t\in(0,1], \\ \frac{1}{2}(1+t)+(1-2e)e^{-t}+\frac{1}{2}[(1+3t)e^2-2]e^{-2t}, & t\in(1,2]. \end{cases}$$

Also, we define the exact error e_i^N and the computed maximum pointwise error e^N as follows:

$$e_i^N=|y_i-u_i|, \qquad e^N=\max_{1\leq i\leq N} e_i^N.$$

The computational results for solving the problem (5.1)–(5.2) obtained by using both the present method (PM) and the implicit Euler method (EM) are shown in the Tables 1–3.

6. Conclusion

In this paper, we have developed the finite difference method for solving the linear second order DDE. We have shown that the method has first order convergence in the discrete maximum norm. The numerical results for the test problem are computed for different values N in Tables 1–3. The graph comparing the numerical results obtained from both methods is shown in Fig. 1. The computational results with both methods in Table 3 and Fig. 1 show that the proposed method is more effective than the other method. The ideas presented method here can be used for the study of initial or boundary value problems for linear differential equations with delay as well as neutral type.

TABLE 1. The numerical results on $(0,2]$ for Example 1(PM).

t_i	u_i	$y_i(N=64)$	e_i^{64}	$y_i(N=128)$	e_i^{128}
0.125	1.1036961	1.1036953	$8.277E-7$	1.1036959	$2.069E-7$
0.250	1.1722701	1.1722687	$1.472E-6$	1.1722698	$3.681E-7$
0.375	1.2149227	1.2149208	$1.974E-6$	1.2149222	$4.936E-7$
0.500	1.2386512	1.2386489	$2.365E-6$	1.2386506	$5.914E-7$
0.625	1.2487566	1.2487540	$2.670E-6$	1.2487560	$6.675E-7$
0.750	1.2492364	1.2492335	$2.907E-6$	1.2492357	$7.268E-7$
0.875	1.2430881	1.2430850	$3.092E-6$	1.2430873	$7.730E-7$
1.000	1.2325442	1.2325409	$3.235E-6$	1.2325433	$8.089E-7$
1.125	1.2203862	1.2203905	$4.365E-6$	1.2203872	$1.091E-6$
1.250	1.2123285	1.2123368	$8.219E-6$	1.2123306	$2.055E-6$
1.375	1.2122725	1.2122821	$9.654E-6$	1.2122749	$2.413E-6$
1.500	1.2219502	1.2219598	$9.585E-6$	1.2219526	$2.396E-6$
1.625	1.2417225	1.2417311	$8.636E-6$	1.2417246	$2.159E-6$
1.750	1.2711252	1.2711324	$7.223E-6$	1.2711270	$1.806E-6$
1.875	1.3092394	1.3092450	$5.617E-6$	1.3092408	$1.404E-6$
2.000	1.3549343	1.3549382	$3.990E-6$	1.3549353	$9.974E-7$

TABLE 2. The numerical results on $(0,2]$ for Example 1 (EM).

t_i	u_i	$y_i(N=64)$	e_i^{64}	$y_i(N=128)$	e_i^{128}
0.125	1.1036961	1.1062957	$2.600E-3$	1.1049940	$1.298E-3$
0.250	1.1722701	1.1768911	$4.621E-3$	1.1745781	$2.308E-3$
0.375	1.2149227	1.2211157	$6.193E-3$	1.2180167	$3.094E-3$
0.500	1.2386512	1.2460666	$7.415E-3$	1.2423569	$3.706E-3$
0.625	1.2487566	1.2571224	$8.366E-3$	1.2529384	$4.182E-3$
0.750	1.2492364	1.2583412	$9.105E-3$	1.2537885	$4.552E-3$
0.875	1.2430881	1.2527673	$9.679E-3$	1.2479284	$4.840E-3$
1.000	1.2325442	1.2426699	$1.013E-2$	1.2376087	$5.065E-3$
1.125	1.2203862	1.2307176	$1.033E-2$	1.2255527	$5.167E-3$
1.250	1.2123285	1.2226009	$1.027E-2$	1.2174656	$5.137E-3$
1.375	1.2122725	1.2223646	$1.009E-2$	1.2173198	$5.047E-3$
1.500	1.2219502	1.2318340	$9.884E-3$	1.2268938	$4.944E-3$
1.625	1.2417225	1.2514273	$9.705E-3$	1.2465769	$4.854E-3$
1.750	1.2711252	1.2807138	$9.589E-3$	1.2759216	$4.796E-3$
1.875	1.3092394	1.3187915	$9.552E-3$	1.3140176	$4.778E-3$
2.000	1.3549343	1.3645361	$9.602E-3$	1.3597371	$4.803E-3$

TABLE 3. Comparison of e^N for both methods on $(0,2]$ for Example 1.

N	e^N(EM)	e^N(PM)	N	e^N(EM)	e^N(PM)
32	$2.066E-2$	$3.906E-5$	256	$2.584E-3$	$6.104E-7$
64	$1.034E-2$	$9.766E-6$	512	$1.292E-3$	$1.526E-7$
128	$5.169E-3$	$2.442E-6$	1024	$6.462E-4$	$3.817E-8$

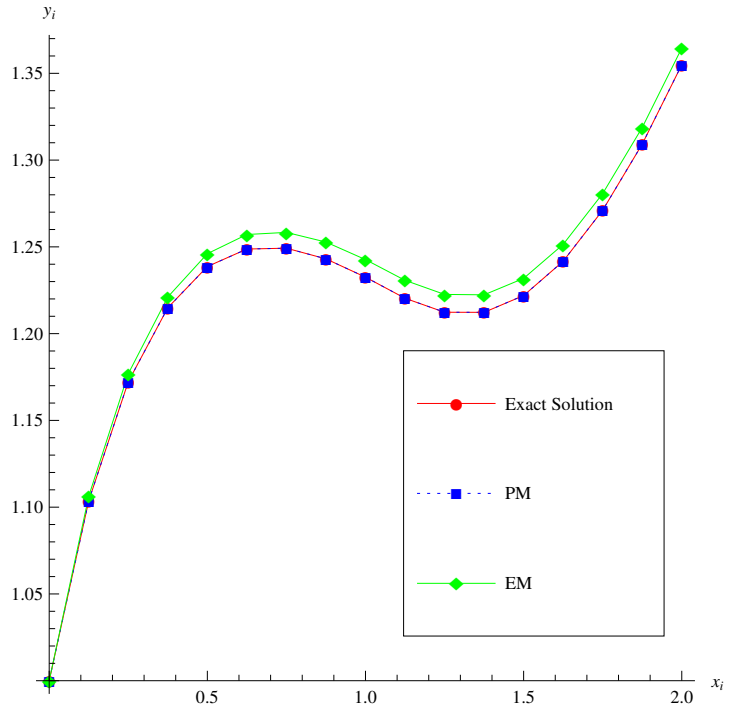

FIGURE 1. Computational results of Example 1 for $N = 64$.

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A NUMERICAL SOLUTION OF VOLTERRA INTEGRAL-ALGEBRAIC EQUATIONS USING BERNSTEIN POLYNOMIALS

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Abstract. The principal aim of this paper is to serve the numerical solution for a linear integral-algebraic equation (IAE) by using Bernstein polynomials. Employing of Bernstein polynomials, the system of integral equations is approximated by the Gaussian quadrature formula with respect to the Legendre weight function. The proposed method reduces the system of integral equations to a system of algebraic equations that can be easily solved by any usual numerical method. Moreover, the convergence analysis of this algorithm will be shown by preparing some theorems. Several examples are included to illustrate the efficiency and accuracy of the proposed technique and also the results are compared with the different methods.

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Keywords: Bernstein polynomials, integral algebraic equations, Gauss-Legendre collocation method, approximate solutions, convergence analysis

1. INTRODUCTION

Integral equation is encountered in a variety of applications in many fields including optimal control systems [9, 31], potential theory [13], geophysics [12], renewal theory [1], radiation [23], optimization [20], mathematical economics [30], mathematical problems of radioactive equilibrium [34], particle transport problems of astrophysics and reactor theory [22], acoustics [33], fluid mechanics [19] and etc.

Systems of Volterra integral equations with identically singular matrices in the principal part are called integral algebraic equations. Such equations and systems frequently arise in theoretical [7, 9, 14, 25, 29, 35] and many applied problems especially in the fields of dynamic processes in chemical reactors [17], identification of memory kernels in heat conduction [32] viscoelastic materials [15], evolution of a chemical reaction within a small cell [16] and Kirchhoff's laws. The theory of IAEs appeared from early attempts by Gear in the 1990 that determined the difficulties of these equations. He introduced the "index reduction procedure" for IAEs system in [11] similar to that in [10] for differential algebraic equations in which if the process

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is terminated, then the index is determined. This means that under suitable conditions, there is a solution for the resulting regular system of integral equations.

Several authors have investigated the existence, uniqueness and numerical analysis of IAEs systems. Bulatov et al. [6], gave the existence and uniqueness results of solution for IAE systems with convolution kernels and defined the index in analogy to Gear's approach. Kauthen [18] applied the polynomial spline collocation method for a semi-explicit IAEs with index-1 and established global convergence as well as local super convergence. Furthermore, Brunner [4] defined the index-1 tractable for a semi-explicit form of IAEs and investigated the existence of a unique solution for these types of systems.

In this article, we study numerical methods for solving the system of Volterra integral equations of the first and second kind. More precisely, we consider the integral algebraic equation

$$\mathbf{A}(t)\mathbf{X}(t) = \mathbf{F}(t) + \int_0^t \mathbf{K}(t,s)\mathbf{X}(s)ds, \qquad 0 \le t \le 1, \tag{1.1}$$

where

$$\mathbf{A}(t) = [a_{ij}(t)], \;\; i,j = 1,2,\cdots,p, \quad \mathbf{F}(t) = [y_1(t)\; y_2(t)\; \cdots\; y_m(t)]^T,$$
$$\mathbf{K}(t,s) = [k_{ij}(t,s)], \qquad i,j = 1,2,\cdots,p$$

are given functions and $\mathbf{X}(t) = [x_1(t)\; x_2(t)\; \cdots\; x_m(t)]^T$ is the solution to be determined. If $\det \mathbf{A}(t) = 0$, this system is called as Volterra Integral-Algebraic Equations (IAEs). Under the condition $\det \mathbf{A}(t) = 0$, the system can have several solutions or no solution at all. We apply the following theorem for the existence of the solution of (1.1).

Theorem 1 ([2]). *Assume that system* (1.1) *with* $\det \boldsymbol{A}(s) = 0$ *satisfies the following conditions:*

1. $\mathbf{A}(t) \in C^1_{[0,1]}, \mathbf{X}(t) \in C^1_{[0,1]}$ *and* $\mathbf{K}(t,s) \in C^1_{\triangle}$, *where* $\triangle = \{0 \le s \le t \le 1\}$.
2. $Rank\mathbf{A}(t) = deg(\det[\lambda\mathbf{A}(t) + \mathbf{K}(t,t)]) = k$, *for all* $t \in [0,1]$, *where* k *is constant and* λ *is a scalar.*
3. $Rank\mathbf{A}(0) = Rank[\mathbf{A}(0)|\mathbf{F}(0)]$.

Then the system has a unique continuous solution.

Several researchers have adopted different techniques for solving system of integral algebraic equations [2, 5, 8, 21, 24, 28]. Our suggestion is by employing of Bernstein polynomials and combining with some other concepts to obtain a numerical solution for (1.1).

The paper is organized in the following way. In Section 2, we present the definition of the Bernstein polynomials and some fundamental properties are studied. In Section 3, we apply the Bernstein polynomials for solving system (1.1). A convergence analysis is presented in Section 4. We offer some numerical examples to illustrate the efficiency of these algorithms in Section 5.

2. PRELIMINARIES

Some necessary preliminaries and notations of Bernstein polynomials are given below which are usable later.

Definition 1 ([27]). The Bernstein polynomials of degree n defined on the interval $[0,1]$ are given by

$$B_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}, \quad k = 0,1,2,\cdots,n, \tag{2.1}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad n \in \mathbb{N}.$$

It should be note that each of these $n+1$ polynomials having degree n and satisfies the following properties:

(i) $B_{n,k}(t) = 0$, if $k < 0$ or $k > n$.
(ii) $B_{n,k}(0) = B_{n,k}(1) = 0$, for $1 \le k \le n-1$.
(iii) $\sum_{k=0}^{n} B_{n,k}(t) = 1$.

The Bernstein's approximation, $B_n(f)$ to a function $f : [0,1] \to \mathbb{R}$ is the polynomial

$$B_n(f) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{n,k}(t), \tag{2.2}$$

where $B_{n,k}$ is a polynomial of degree n, which is defined by (2.1).

Bernstein in [3] used this approximation to give the first constructive proof of the Weierstrass theorem.

Theorem 2 ([26]). *For all functions f in $C[0,1]$, the sequence $B_n(f)$, $n = 1,2,3,\cdots$ converges uniformly to f, where $B_n(f)$ is defined by (2.2).*

Suppose that $H = L^2[0,1]$ is a Hilbert space with the inner product defined by $\langle f,h\rangle = \int_0^1 f(t)h(t)^T dt$ and $V = \text{Span}\{B_{n,0}(t), B_{n,1}(t), \cdots, B_{n,n}(t)\}$. Since V is a finite-dimensional subspace of H, it is closed and convex. Thus, for $f \in H$, there exists a unique best approximation out of V such as g, such that [27]

$$\|f-g\| \le \| f-h \|, \quad \forall h \in V.$$

Since $g \in V$, there exists a unique coefficients $c_0, c_1, \cdots, c_n$ such that

$$f \simeq g = \sum_{k=0}^{n} c_k B_{n,k}(t) = \mathbf{C}^T \psi(t),$$

where

$$\psi^T = [B_{n,0}, B_{n,1}, \cdots, B_{n,n}] \quad \text{and} \quad \mathbf{C}^T = [c_0, c_1, \cdots, c_n].$$

Then, $\mathbf{C}^T$ can be obtained by

$$\mathbf{C}^T = \mathbf{D}^{-1} \langle f, \psi \rangle, \tag{2.3}$$

where

$$\langle f,\psi\rangle = \int_0^1 f(t)\psi(t)^T dt,$$

and $\mathbf{D}$ is an $(n+1)\times(n+1)$ matrix which is said the dual matrix of $\psi(t)$ and will be introduced in the form

$$\mathbf{D} = \langle \psi,\psi\rangle = \int_0^1 \psi(t)\psi(t)^T dt.$$

3. Employing of Bernstein polynomials

Any unknown vector-valued function $\mathbf{X}$ can be expand in the Bernstein basis as

$$B_n(\mathbf{X}(\mathbf{t})) = [B_n(x_1(t)), B_n(x_2(t)), \cdots, B_n(x_m(t))]^T, \tag{3.1}$$

$$B_n(x_j(t)) = \sum_{k=0}^{n} c_{j,k} B_{n,k}(t), \quad t\in[0,1],\ j = 1,2,\cdots,p, \tag{3.2}$$

where n is chosen as any positive integer and $B_{n,k}(t), k = 0,1,\cdots,n$ are the Bernstein polynomials defined in the previous section.

In order to approximate the solution of (1.1), we consider the system of linear Volterra integral algebraic equations with variable coefficients in the following form:

$$\sum_{j=1}^{p} a_{ij}(t)x_j(t) = f_i(t) + \sum_{j=1}^{p}\int_0^t k_{ij}(t,s)x_j(s)ds, \quad i = 1,2,\cdots,p, \quad 0\le t\le 1. \tag{3.3}$$

Substituting (3.2) in (3.3), we obtain

$$\sum_{j=1}^{p} a_{ij}(t)B_n(x_j(t)) = f_i(t) + \sum_{j=1}^{p}\int_0^t k_{ij}(t,s)B_n(x_j(s))ds, \quad i = 1,2,\cdots,p. \tag{3.4}$$

Putting (3.2) in (3.4) we have

$$\sum_{j=1}^{p} a_{ij}(t)\sum_{k=0}^{n} c_{j,k}B_{n,k}(t) = f_i(t) + \sum_{j=1}^{p}\int_0^t k_{ij}(t,s)\sum_{k=0}^{n} c_{j,k}B_{n,k}(s)ds \quad i = 1,2,\cdots,p. \tag{3.5}$$

Then we get

$$\sum_{j=1}^{p}\sum_{k=0}^{n}\left(a_{ij}(t)B_{n,k}(t) - \int_0^t k_{ij}(t,s)B_{n,k}(s)ds\right)c_{j,k} = f_i(t), \quad i = 1,2,\cdots,p. \tag{3.6}$$

In order to find $c_{j,k}, j = 1,2,\cdots,p$ and $k = 0,1,2,\cdots,n$, we convert the above equation to a linear system of equations by collocating (3.6) in $(n+1)$ Newton-Cotes collocation points $t_l = \frac{2l-1}{2(n+1)}$, $l = 0,1,\cdots,n$ on the interval $[0,1]$.

Thus we have

$$\sum_{j=1}^{p}\sum_{k=0}^{n}\left(a_{ij}(t_l)B_{n,k}(t_l) - \int_0^{t_l} k_{ij}(t_l,s)B_{n,k}(s)ds\right)c_{j,k} = f_i(t_l). \tag{3.7}$$

In our approach, we use the quadrature rule to approximate the integrals involved in this system, which achieves zero error integration for polynomial integrands of degree less than or equal to $2n+2$ with $n+1$ Legendre-Gauss nodes. For this purpose, a simple linear transformation must be made. So, the intervals $[0,t_l]$, $l=0,1,\cdots,n$ are transferred to a fixed interval $[-1,1]$ by means of the transformation

$$s=\frac{t_l}{2}+\frac{t_l}{2}\tau \quad \text{and} \quad ds=\frac{t_l}{2}d\tau.$$

Hence, employing this transformation on the second term in the left hand side of (3.7) the results is

$$\sum_{j=1}^{p}\sum_{k=0}^{n}\left(a_{ij}(t_l)B_{n,k}(t_l)-\frac{t_l}{2}\int_{-1}^{1}k_{ij}(t_l,\frac{t_l}{2}+\frac{t_l}{2}\tau)B_{n,k}(\frac{t_l}{2}+\frac{t_l}{2}\tau)d\tau\right)c_{j,k}=f_i(t_l). \quad (3.8)$$

Applying the Gaussian quadrature, system (3.8) is converted to

$$\sum_{j=1}^{p}\sum_{k=0}^{n}\left(a_{ij}(t_l)B_{n,k}(t_l)-\frac{t_l}{2}\sum_{r=0}^{n}\omega_r k_{ij}(t_l,\frac{t_l}{2}+\frac{t_l}{2}\tau_r)B_{n,k}(\frac{t_l}{2}+\frac{t_l}{2}\tau_r)\right)c_{j,k}=f_i(t_l), \quad (3.9)$$

where τ_r, $r=0,1,\cdots,n$ are the $n+1$ Legendre-Gauss nodes and ω_r, $r=0,1,\cdots,n$ are the weights on $[-1,1]$.

For brevity, we define below symbols as:

$$T_{l,k}^{(i,j)}=a_{ij}(t_l)B_{n,k}(t_l), \qquad R_{l,k}^{(i,j)}=\frac{t_l}{2}\sum_{r=0}^{n}\omega_r k_{ij}(t_l,\frac{t_l}{2}+\frac{t_l}{2}\tau_r)B_{n,k}(\frac{t_l}{2}+\frac{t_l}{2}\tau_r).$$

Now we can write system (3.9) in the form

$$\sum_{j=1}^{p}\sum_{k=0}^{n}\left(T_{l,k}^{(i,j)}-R_{l,k}^{(i,j)}\right)c_{j,k}=f_i(t_l), \quad i=1,2,\cdots,p, \quad l=0,1,\cdots,n. \quad (3.10)$$

Consequently, expression (3.10) can be summarized in a matrix form as follows:

$$\mathcal{A}\mathcal{X}=\mathcal{Y}, \quad (3.11)$$

where

$$\begin{aligned}\mathcal{X}&=[c_{1,0},\cdots,c_{1,n},\cdots,c_{j,0},\cdots,c_{j,n},\cdots,c_{p,0},\cdots,c_{p,n}]^T, \quad (3.12)\\ \mathcal{Y}&=[f_1(t_0),\cdots,f_1(t_n),\cdots,f_i(t_0),\cdots,f_i(t_n),\cdots,f_p(t_0),\cdots,f_p(t_n)]^T,\end{aligned}$$

and

$$\mathcal{A}=\begin{bmatrix}\mathcal{A}^{(1,1)} & \cdots & \mathcal{A}^{(1,p)}\\ \vdots & \ddots & \vdots\\ \mathcal{A}^{(p,1)} & \cdots & \mathcal{A}^{(p,p)}\end{bmatrix}.$$

Parochial matrix $\mathcal{A}^{(i,j)}$ for $(i,j)=(1,2,\cdots,p)$ is defined by

$$\mathcal{A}^{(i,j)}=\begin{bmatrix}\mathcal{A}_{0,0}^{(i,j)} & \cdots & \mathcal{A}_{0,n}^{(i,j)}\\ \vdots & \ddots & \vdots\\ \mathcal{A}_{n,0}^{(i,j)} & \cdots & \mathcal{A}_{n,n}^{(i,j)}\end{bmatrix},$$

where

$$\mathcal{A}_{l,k}^{(i,j)}=T_{l,k}^{(i,j)}-R_{l,k}^{(i,j)}.$$

The resulting generalized linear system can be solved for $c_{j,k}$ for $j=1,2,\cdots,p$ and $k=0,1,\cdots,n$ by a standard method, and hence $x_j(t)$ is obtained.

4. Convergence analysis

In this section we proceed to prove that the present numerical method converges to the exact solutions of system (1.1). For this purpose, let us consider the Banach space $C=C[0,1],\times\cdots\times C[0,1]$ (p times) endowed with the norm

$$\|\mathbf{X}(t)\|=\max_{1\le j\le p}\max_{0\le t\le 1}|x_j(t)|.$$

Theorem 3. *Consider the system of Volterra integral-algebraic equations* (1.1). *Assume that* $\bar{B}_n(\mathbf{X}(t))$ *is the best approximation of* $V=Span\{B_{n,0}(t),B_{n,1}(t),\cdots,B_{n,n}(t)\}$ *and* $\hat{B}_n(\mathbf{X}(t))$ *is the approximation solution of* (1.1). *Then*

$$\|\bar{B}_n(\mathbf{X}(t))-\hat{B}_n(\mathbf{X}(t))\|\le\max_{1\le j\le p}\max_{0\le k\le n}|\bar{c}_{j,k}-\hat{c}_{j,k}|, \tag{4.1}$$

where

$$\bar{B}_n(\mathbf{X}(t))=\left[\sum_{k=0}^{n}\bar{c}_{1,k}B_{n,k}(t),\sum_{k=0}^{n}\bar{c}_{2,k}B_{n,k}(t),\cdots,\sum_{k=0}^{n}\bar{c}_{p,k}B_{n,k}(t)\right]^T$$

and

$$\hat{B}_n(\mathbf{X}(t))=\left[\sum_{k=0}^{n}\hat{c}_{1,k}B_{n,k}(t),\sum_{k=0}^{n}\hat{c}_{2,k}B_{n,k}(t),\cdots,\sum_{k=0}^{n}\hat{c}_{p,k}B_{n,k}(t)\right]^T,$$

where $\bar{c}_{j,k}$, $j=1,2,\cdots,p$, $k=0,1,2,\cdots,n$ *is defined by* (2.3) *and* $\hat{c}_{j,k}$, $j=1,2,\cdots,p$, $k=0,1,2,\cdots,n$ *is the solution of* (3.11) *computed through our presented method.*

Proof. We can write

$$\begin{aligned}\|\bar{B}_n(\mathbf{X}(t))-\hat{B}_n(\mathbf{X}(t))\|&=\max_{1\le j\le p}\max_{0\le t\le 1}\Big|\sum_{k=0}^{n}(\bar{c}_{j,k}-\hat{c}_{j,k})B_{n,k}(t)\Big|\\&\le\max_{1\le j\le p}\max_{0\le t\le 1}\max_{0\le k\le n}|\bar{c}_{j,k}-\hat{c}_{j,k}|\sum_{k=0}^{n}B_{n,k}(t),\end{aligned} \tag{4.2}$$

where

$$\sum_{k=0}^{n} B_{n,k}(t) = 1, \quad 0 \le t \le 1.$$

This implies that

$$\|\bar{B}_n(\mathbf{X}(t)) - \hat{B}_n(\mathbf{X}(t))\| \le \max_{1\le j\le p} \max_{0\le k\le n} |\bar{c}_{j,k} - \hat{c}_{j,k}|, \tag{4.3}$$

and the proof is completed. □

Now we will study the convergence of our considered method for the system of (1.1).

Theorem 4. *Considering the assumptions in Theorem (3) for (1.1), let* $\mathbf{F} \in C([0,1], \mathbb{R}^p)$ *and* $\mathbf{K} \in C([0,1]\times[0,1], \mathbb{R}^p)$. *Moreover, assume that there are positive numbers* M_1 *and* M_2 *where* $\|\mathbf{A}(t)\| \le M_1$ *and* $\|\mathbf{K}(t,s)\| \le M_2$ *such that* $M_1 - M_2 > 0$. *If* $\hat{B}_n(\mathbf{X}(t))$ *is the approximation solution of (1.1), then for sufficiently large n we have*

$$\|\mathbf{X}(t) - \hat{B}_n(\mathbf{X}(t))\| \le \frac{M_1 + M_2}{M_1 - M_2} \max_{1\le j\le p} \max_{0\le k\le n} |\bar{c}_{j,k} - \hat{c}_{j,k}|, \tag{4.4}$$

Proof. The system of Volterra integral-algebraic equation (1.1) can be written as

$$\mathbf{A}(t)\mathbf{X}(t) = \mathbf{F}(t) + \int_0^t \mathbf{K}(t,s)\mathbf{X}(s)ds. \tag{4.5}$$

Substituting $\hat{B}_n(\mathbf{X}(t))$ in this system yields

$$\mathbf{A}(t)\hat{B}_n(\mathbf{X}(t)) = \hat{\mathbf{F}}(t) + \int_0^t \mathbf{K}(t,s)\hat{B}_n(\mathbf{X}(s))ds. \tag{4.6}$$

Since $\bar{B}_n(\mathbf{X}(t))$ is the best approximation, we have

$$\mathbf{A}(t)\bar{B}_n(\mathbf{X}(t)) = \mathbf{F}(t) + \int_0^t \mathbf{K}(t,s)\bar{B}_n(\mathbf{X}(s))ds. \tag{4.7}$$

Subtracting (4.5) from (4.6), we get

$$\mathbf{A}(t)\left(\mathbf{X}(t) - \hat{B}_n(\mathbf{X}(t))\right) = \mathbf{F}(t) - \hat{\mathbf{F}}(t) + \int_0^t \mathbf{K}(t,s)\left(\mathbf{X}(s) - \hat{B}_n(\mathbf{X}(s))\right), \tag{4.8}$$

hence

$$\|\mathbf{A}(t)\|\|\mathbf{X}(t) - \hat{B}_n(\mathbf{X}(t))\| \le \|\mathbf{F}(t) - \hat{\mathbf{F}}(t)\| + \|\mathbf{K}(t,s)\|\|\mathbf{X}(s) - \hat{B}_n(\mathbf{X}(s))\|, \tag{4.9}$$

or

$$\left(\|\mathbf{A}(t)\| - \|\mathbf{K}(t,s)\|\right)\|\mathbf{X}(t) - \hat{B}_n(\mathbf{X}(t))\| \le \|\mathbf{F}(t) - \hat{\mathbf{F}}(t)\|, \tag{4.10}$$

therefore, we can write

$$\|\mathbf{X}(t) - \hat{B}_n(\mathbf{X}(t))\| \le \frac{1}{M_1 - M_2}\|\mathbf{F}(t) - \hat{\mathbf{F}}(t)\|. \tag{4.11}$$

In order to find a bound on $\|\mathbf{F}(t)-\hat{\mathbf{F}}(t)\|$, subtracting (4.6) from (4.7), one obtains

$$\mathbf{F}(t)-\hat{\mathbf{F}}(t)=\mathbf{A}(t)\left(\bar{B}_n(\mathbf{X}(t))-\hat{B}_n(\mathbf{X}(t))\right)-\int_0^t \mathbf{K}(t,s)\left(\bar{B}_n(\mathbf{X}(s))-\hat{B}_n(\mathbf{X}(s))\right)ds. \tag{4.12}$$

Then, we get

$$\begin{aligned}\|\mathbf{F}(t)-\hat{\mathbf{F}}(t)\| &\le \|\mathbf{A}(t)\|\|\bar{B}_n(\mathbf{X}(t))-\hat{B}_n(\mathbf{X}(t))\|+\|\mathbf{K}(t,s)\|\|\bar{B}_n(\mathbf{X}(s))-\hat{B}_n(\mathbf{X}(s))\|,\\ &\le (\|\mathbf{A}(t)\|+\|\mathbf{K}(t,s)\|)\,\|\bar{B}_n(\mathbf{X}(t))-\hat{B}_n(\mathbf{X}(t))\|\\ &\le (M_1+M_2)\|\bar{B}_n(\mathbf{X}(t))-\hat{B}_n(\mathbf{X}(t))\|. \end{aligned}\tag{4.13}$$

According to Theorem 3, one can write

$$\|\mathbf{F}(t)-\hat{\mathbf{F}}(t)\| \le (M_1+M_2)\max_{1\le j\le m}\max_{0\le k\le n}|\bar{c}_{j,k}-\hat{c}_{j,k}|. \tag{4.14}$$

Substituting this bound in (4.11), the purpose will be achieved. □

5. Numerical Examples

In this section some examples using our considered numerical solution are given for (1.1) to show accuracy and efficiency of this method. Exact solution of these examples are available. Comparing exact solution with approximate solution shows that this method is very accurate. A comparison of our proposed method with some other known methods has also performed. We use maximum absolute errors for selected points, defined as

$$\|E_j(t)\|_\infty=\max|x_j(t)-\hat{x}_j(t)|, \qquad j=1,2,$$

where $x_j(t)$ and $\hat{x}_j(t)$ denotes the exact and approximate solution, respectively.
It should be noted that all of the numerical computations have been performed on computer using a program written in MATLAB.

Example 1. Consider the linear Volterra integral-algebraic equations

$$\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}y(t)\\z(t)\end{bmatrix}=\begin{bmatrix}f_1(t)\\f_2(t)\end{bmatrix}+\int_0^t\begin{bmatrix}1+t-s & e^{t-s}\\1 & e^{2t-s}\end{bmatrix}\begin{bmatrix}y(s)\\z(s)\end{bmatrix}ds.$$

Here

$$\begin{aligned}f_1(t)&=3te-t-2e^{-t}-t+\frac{1}{2}\cos t-\frac{1}{2}\sin t-\frac{1}{2}e^t+1,\\ f_2(t)&=e^{-t}(t+1)+\frac{1}{2}e^t\cos t-\frac{1}{2}e^t\sin t-\frac{1}{2}e^{2t}-1,\end{aligned}$$

with the exact solution $y(t)=te^{-t}$, $z(t)=\cos t$.

Let us check that this system satisfies the conditions required in Theorem 1.
It is clear that

$$A(t)=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \text{ and } \mathbf{K}(t,s)=\begin{bmatrix}1+t-s & e^{t-s}\\ 1 & e^{2t-s}\end{bmatrix}$$

satisfy the first condition required in Theorem 1.
On other hand Rank$\mathbf{A}(t)=1$ and

$$\lambda\mathbf{A}(t)+\mathbf{K}(t,t)=\begin{bmatrix}\lambda & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}1 & 1\\ 1 & e^{t}\end{bmatrix}=\begin{bmatrix}\lambda+1 & 1\\ 1 & e^{t}\end{bmatrix}.$$

Hence

$$\begin{vmatrix}\lambda+1 & 1\\ 1 & e^{t}\end{vmatrix}=(\lambda+1)e^{t}-1\Rightarrow \deg[\det(\lambda\mathbf{A}(t)+\mathbf{K}(t,t))]=1$$

and

$$\mathbf{A}(0)\mid\mathbf{F}(0)=\left[\begin{array}{cc|c}1 & 0 & 1\\ 0 & 0 & 0\end{array}\right].$$

Therefore

$$\text{Rank}\mathbf{A}(0)=\text{Rank}[\mathbf{A}(0)\mid\mathbf{F}(0)]=1.$$

Thus, this example satisfies the conditions required in Theorem 1 and therefore it has a unique continuous solution on $[0,1]$.

TABLE 1. Absolute errors of $y(t)$ and $z(t)$ at the selected points for Example 1

t	$n=8$		$n=11$	
	$e_1(t)$	$e_2(t)$	$e_1(t)$	$e_2(t)$
0	2.6411e-09	2.2546e-09	1.6636e-13	7.2164e-14
0.1	1.7224e-10	2.4399e-10	3.6499e-15	3.8858e-14
0.2	4.8195e-11	3.8055e-11	5.5511e-16	1.1435e-14
0.3	1.6081e-11	4.5866e-11	1.2490e-15	1.2879e-14
0.4	3.5780e-11	4.8837e-11	1.5543e-15	5.3291e-15
0.5	4.2929e-11	2.7642e-11	1.6653e-15	7.3275e-15
0.6	5.0793e-11	4.4997e-11	2.2204e-15	4.9960e-15
0.7	7.3545e-11	1.0888e-11	2.6645e-15	1.0436e-14
0.8	4.8906e-11	5.5758e-11	3.6637e-15	2.6090e-14
0.9	2.5692e-10	6.0195e-11	1.4988e-15	2.2415e-13
1	2.1918e-09	4.7665e-09	1.1147e-13	9.2568e-12

In this example Table 1 shows absolute errors for $n=8$ and $n=11$ by using the present method at selected points. In Table 2, we compare the maximum absolute error for the present method at $n=11$ with method [21] and method [8]. Figure 1

TABLE 2. Comparison of the maximum absolute error for Example 1

Error	Present method	Method of [21]	Method of [8]
	$n = 11$	$n = 32$	$n = 32$
$E_1(t)$	1.6636e-13	3/0473e-05	4.51e-03
$E_2(t)$	9.2568e-12	9/9757e-03	4.20e-03

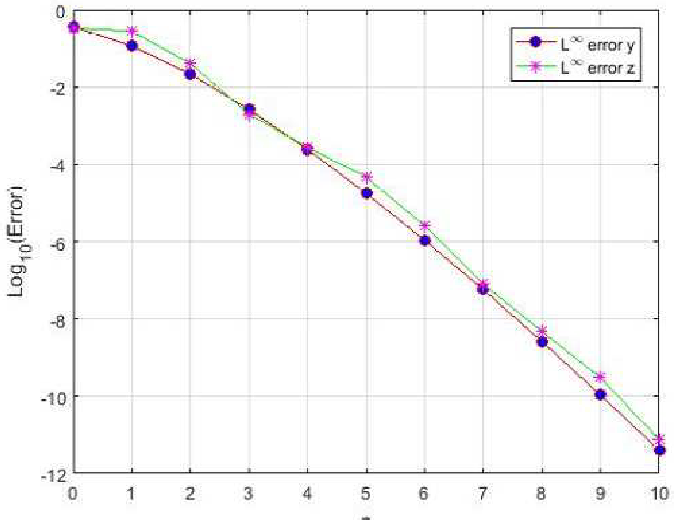

FIGURE 1. Maximum absolute error for Example 1

contains the numerical comparison of errors of our solutions using the present method for different values of n.

Example 2. Consider the linear Volterra integral-algebraic equations

$$\begin{bmatrix} 1 & t \\ t & t^2 \end{bmatrix} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} e^t(t+1)+te^{-t} \\ 2te^t+1+e^{-t}(t^2-1) \end{bmatrix} + \int_0^t \begin{bmatrix} -e^{t-s} & 0 \\ -e^{t+s} & -e^{-2s} \end{bmatrix} \begin{bmatrix} y(s) \\ z(s) \end{bmatrix} ds,$$

with the exact solution $y(t) = e^t$ and $z(t) = e^{-t}$.

In Table 3, we show the absolute errors for $n = 8$ and $n = 11$ by using the present method. In Table 4, we compare the maximum absolute error for the present method at $n = 11$ with method [2], method [5] and method [24]. Figure 2 contains the numerical comparison of errors of our solutions using the present method for different values of n.

TABLE 3. Absolute errors of $y(t)$ and $z(t)$ at the selected points for Example2

t	$n=8$		$n=11$	
	$e_1(t)$	$e_2(t)$	$e_1(t)$	$e_2(t)$
0	7.7345e-10	5.9966e-10	2.7700e-13	2.9332e-13
0.1	7.2120e-11	6.5572e-11	9.5479e-15	6.3283e-15
0.2	5.6464e-12	1.7860e-11	4.2188e-15	9.1038e-15
0.3	1.7329e-11	1.1736e-11	3.1086e-15	1.9984e-15
0.4	1.4032e-11	9.9728e-12	3.7748e-15	2.1094e-15
0.5	1.3195e-11	5.2208e-12	3.9968e-15	1.7764e-15
0.6	1.3600e-11	4.5389e-12	4.6629e-15	3.1086e-15
0.7	9.7931e-12	2.7638e-12	1.3323e-15	2.2204e-15
0.8	1.8867e-11	1.2439e-12	4.4409e-15	2.8866e-15
0.9	3.5453e-11	8.9101e-13	8.4377e-15	2.5424e-14
1	9.7940e-10	3.3398e-10	9.9210e-13	1.4697e-12

TABLE 4. Comparison of the maximum absolute error for Example 2

Error	Present method	method [2]	method [5]	method [24]
	$n=11$	$m=32$	$n=10$	$n=5\ m=10$
$e_1(t)$	9.9210e-13	2.83e-02	1.10e-04	1.68e-12
$e_2(t)$	1.4697e-12	2.34e-02	1.10e-04	1.48e-12

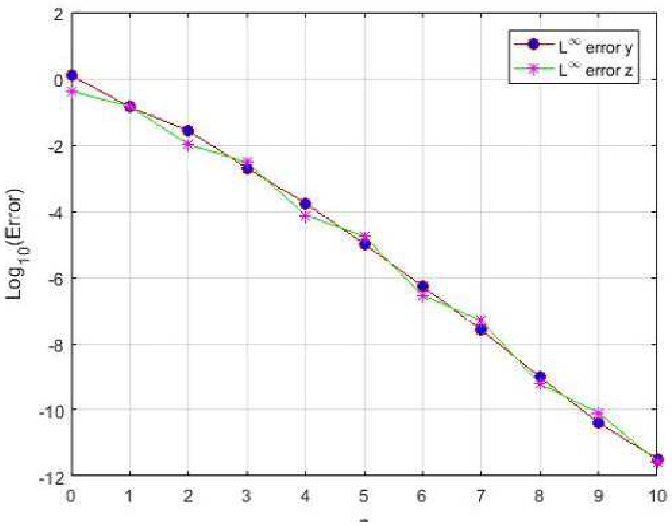

FIGURE 2. Maximum absolute error for Example 2

Example 3. As a final example, consider the linear Volterra integral-algebraic equations

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} + \int_0^t \begin{bmatrix} s+t & s^2+t^2 \\ s-t^2 & s+t+1 \end{bmatrix} \begin{bmatrix} y(s) \\ z(s) \end{bmatrix} ds,$$

where

$$f_1(t) = -t - 2(\sin t)t^2 + 2\sin t,$$
$$f_2(t) = t^2 - 2\sin t + t\cos t - t^2\cos t + 1 - \cos t - 2t\sin t,$$

with the exact solution $y(t) = \sin t$ and $z(t) = \cos t$.

In Table 5, we show the absolute errors for $n = 8$ and $n = 11$ by using the present method. In Table 6, we compare the maximum absolute error for the present method with $n = 11$ and method [28]. Figure 3 contains the numerical comparison of errors of our solutions using the present method for different values of n.

TABLE 5. Absolute errors of $y(t)$ and $z(t)$ at the selected points for Example 3

t	$n = 8$		$n = 11$	
	$e_1(t)$	$e_2(t)$	$e_1(t)$	$e_2(t)$
0	4.3709e-10	3.0667e-11	3.0552e-14	0.0000e+00
0.1	2.9435e-11	3.2249e-12	7.7716e-16	6.3283e-15
0.2	6.7472e-12	3.8103e-13	1.1102e-16	2.8866e-15
0.3	8.3311e-13	2.9913e-12	1.1102e-16	2.2204e-16
0.4	3.6615e-12	1.8513e-12	1.6653e-16	1.1102e-15
0.5	4.5935e-12	3.5471e-12	1.6653e-16	0.0000e+00
0.6	5.7888e-12	2.5377e-12	2.2204e-16	9.9920e-16
0.7	1.0052e-11	4.9851e-12	5.5511e-16	2.9976e-15
0.8	6.6127e-12	3.8752e-12	8.8818e-16	4.1078e-15
0.9	4.4303e-11	8.9651e-12	1.4433e-15	2.2982e-14
1	3.6263e-10	7.9842e-10	1.3434e-14	8.5454e-13

TABLE 6. Comparison of the maximum absolute error for Example 3

Error	Present method	method [28]
	$n = 11$	$m = 32$
$e_1(t)$	3.0552e-14	7.6e-13
$e_2(t)$	8.5454e-13	1.4e-11

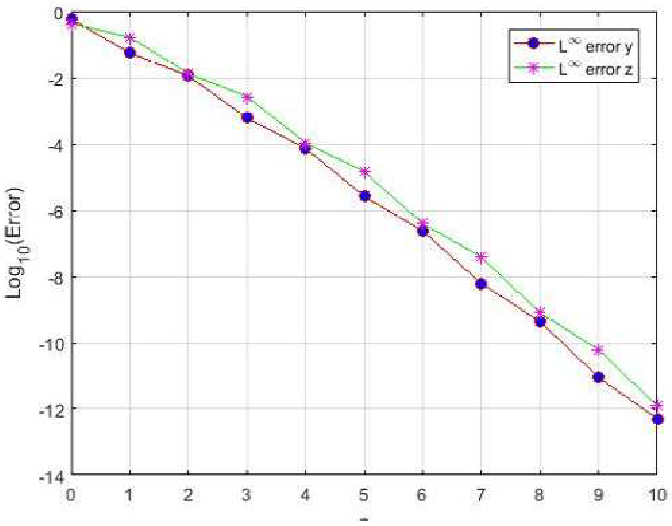

FIGURE 3. Maximum absolute error for Example 3

CONCLUSION

The aim of this manuscript was to suggest the employing of Bernstein polynomials to obtain a numerical solution for a Volterra integral-algebraic equations. The process of our considered method is that the system of integral equations has been reduced to a system of algebraic equations. In this methods, the system of integral equations was approximated by the Gauss quadrature formula with respect to the Legendre weight function and then the convergence of the proposed method was discussed. Several numerical example were presented to show that the superiority and efficiency of our numerical solution respect to some other well known methods.

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SS-LIFTING MODULES AND RINGS

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Abstract. A module M is called *ss-lifting* if for every submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $A \cap M_2 \subseteq Soc_s(M)$, where $Soc_s(M) = Soc(M) \cap Rad(M)$. In this paper, we provide the basic properties of *ss*-lifting modules. It is shown that: (1) a module M is *ss*-lifting iff it is amply *ss*-supplemented and its *ss*-supplement submodules are direct summand; (2) for a ring R, ${}_RR$ is *ss*-lifting if and only if it is *ss*-supplemented iff it is semiperfect and its radical is semisimple; (3) a ring R is a left and right artinian serial ring and $Rad(R) \subseteq Soc({}_RR)$ iff every left R-module is *ss*-lifting. We also study on factor modules of *ss*-lifting modules.

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1. INTRODUCTION

In this study R is used to show a ring which is associative and has an identity. All mentioned modules will be unital left R-module. Let M be an R-module. The notation $A \leq M$ means that A is a submodule of M. A proper submodule A of M is called *small* in M and showed by $A \ll M$ whenever $A + C \neq M$ for all proper submodule C of M. A module M is called *hollow* if every submodule of M is small in M. By $Rad(M)$, namely *radical*, we will denote the sum of all small submodules of M. Equivalently, $Rad(M)$ is the intersection of all maximal submodules of M. A hollow module M with maximal radical is *local*. As a dual notion of a small submodule, a submodule $E \subseteq M$ is called *essential* in M, denoted by $E \trianglelefteq M$, if $E \cap K \neq 0$ for every nonzero submodule K of M. The socle of M which is the sum of all simple submodules of M is denoted by $Soc(M)$. It is well known that $Soc(M)$ is the intersection of all essential submodules of M. The relation between radical and socle of a module M is not determined. In [8], the sum of all simple submodules of the module M that is small is denoted by $Soc_s(M)$. It is shown in [4, Lemma 2] that $Soc_s(M) = Soc(M) \cap Rad(M)$.

A module M is called *extending* if every submodule of M is essential in a direct summand of M [3]. Dually, a module M is *lifting* if for every submodule A of M lies over a direct summand, that is, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$, $A \cap M_2 \ll M_2$. A characterization of lifting modules is given with help of

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supplemented modules in [3]. Here a module M is *supplemented* if every submodule A of M has a supplement B in M, that is, $M = A + B$ and $A \cap \ll B$. M is called *amply supplemented* if whenever $M = A + B$, B contains a supplement of A in M. Clearly, direct summands are supplements. By [7], M is lifting if and only if M is amply supplemented and every supplement submodule of M is a direct summand of it.

Since $Soc_s(X) = Soc(X) \cap Rad(X) \ll X$ for any module X, the authors call a submodule V of a module M *ss-supplement* of a submodule U in M if $M = U + V$ and $U \cap V \subseteq Soc_s(V)$ (see [4]). It is shown in [4, Lemma 3] that a submodule V of M is *ss*-supplement of some submodule U in M if and only if V is a supplement of U in M and $U \cap V$ is semisimple. Following [4], a module M is said to be *ss-supplemented* if every submodule A of M has an *ss*-supplement B in M, and it is called *amply ss-supplemented* if whenever $M = A + B$, B contains an *ss*-supplement of A in M. Clearly, the class of *ss*-supplemented modules is between the class of semisimple modules and the class of supplemented modules. The basic properties and characterizations of *ss*-supplemented modules are given in the same paper.

Considering all of these definitions, we can define *ss*-lifting modules. A module M is called *ss-lifting* if for every submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$, $A \cap M_2 \ll M$ and $A \cap M_2$ is semisimple. In this paper, some fundamental properties of *ss*-lifting modules will be examined. It is proved that a module M is *ss*-lifting module if and only if it is amply *ss*-supplemented and every *ss*-supplement submodule of M is direct summand. It is shown that every π-projective and *ss*-supplemented module is *ss*-lifting. It is proved that for a ring R, ${}_RR$ is *ss*-lifting if and only if R is semiperfect and its radical is semisimple. Moreover, it is shown that R is a left and right artinian serial ring and $Rad(R) \subseteq Soc({}_RR)$ if and only if every left R-module is *ss*-lifting. Nevertheless, it is proved that any factor module generated by submodule of a weakly distributive module is *ss*-lifting.

2. SS-Lifting Modules

In this section, we examine the basic properties of *ss*-lifting modules. In particular, we give characterizations of some ring classes via *ss*-lifting modules. Let us begin with the following definition.

Definition 1. Let M be a module. M is called *ss-lifting* if, for every submodule U of M, M has a decomposition $M = U' \oplus V$ such that $U' \subseteq U$ and $U \cap V \subseteq Soc_s(V)$.

It can be seen that a module M is *ss*-lifting if and only if, for every submodule U of M, M has a decomposition $M = U' \oplus V$ such that $U' \subseteq U$ and $U \cap V \subseteq Soc_s(M)$. Note that we shall freely use this fact in this paper. It is clear that every *ss*-lifting module is lifting. The following example shows that in general a lifting module need not be *ss*-lifting.

Example 1. Let R be a local Dedekind domain and K be the quotient field of R. Put $M =_R K$. Then M is hollow and so it is lifting. Since R is a commutative domain and

M is an injective module, it follows that $M = Rad(M)$ and $Soc(M) = 0$. Therefore $Soc_s(M) = Soc(M) \cap Rad(M) = 0$. So M is not *ss*-lifting.

Lemma 1. *Let M be an ss-lifting module. Then M is amply ss-supplemented.*

Proof. Let U be any submodule of M. By the hypothesis, there are a submodule V of M and $U' \leq U$ such that $M = U' \oplus V$ and $U \cap V \subseteq Soc_s(V)$. Therefore $M = U + V$. It means that V is an *ss*-supplement of U in M and so M is *ss*-supplemented. It follows from [4, Proposition 26] that U' is *ss*-supplemented as a direct summand of M. Now, by modularity law, we can write $U = U \cap M = U \cap (U' \oplus V) = = U' \oplus (U \cap V)$ and then U is *ss*-supplemented by [4, Corollary 24] since $U \cap V$ is semisimple. Hence M is amply *ss*-supplemented according to [4, Proposition 33]. □

In [4], a module M is said to be *strongly local* if M is local and its radical is semisimple. Using Lemma 1, we have the next result.

Corollary 1. *For the non-zero hollow module M, the following are equivalent:*

(1) *M is strongly local.*
(2) *M is ss-lifting.*
(3) *M is amply ss-supplemented.*

Proof. (1) ⇒ (2) Let U be a proper submodule of M. Since M is strongly local, we have $U \subseteq Rad(M) \subseteq Soc(M)$ and then U is semisimple. Now, if $U' = 0$ and $V = M$ are taken, we obtain that $M = U' \oplus M$, $U' \leq U$ and $U \cap M = U$ is semisimple. Thus M is *ss*-lifting.

(2) ⇒ (3) It is clear that by Lemma 1.

(3) ⇒ (1) By [4, Proposition 15]. □

Observe from Corollary 1 that the local $\mathbb{Z}$-module $\mathbb{Z}_8$ is lifting which is not *ss*-lifting.

Lemma 2. *Let M be a module and $A \leq M$. The following conditions are equivalent:*

(1) *There is a direct summand X of M such that $X \leq A$ and $\frac{A}{X} \subseteq Soc_s\left(\frac{M}{X}\right)$.*
(2) *There are a direct summand X of M and a submodule Y of M such that $X \leq A$, $A = X + Y$ and $Y \subseteq Soc_s(M)$.*
(3) *There is a decomposition $M = X \oplus X'$ with $X \subseteq A$ and $X' \cap A \subseteq Soc_s(M)$.*
(4) *A has an ss-supplement X' in M such that $X' \cap A$ is a direct summand in A.*
(5) *There is a homomorphism $e: M \longrightarrow M$ with $e^2 = e$ such that $e(M) \leq A$ and $(1-e)(A) \subseteq Soc_s(1-e)(M)$.*

Proof. (1)⇒(2) Since X is a direct summand of M, there exists a submodule X' of M with $M = X \oplus X'$. If both sides of the equality are taken with A, we get that $A = X + (X' \cap A)$. Since $\frac{A}{X} \ll \frac{M}{X}$ and $\frac{A}{X}$ is semisimple, $\Psi\left(\frac{A}{X}\right) = X' \cap A \ll X$ and $X' \cap A$ is semisimple where $\Psi: M \to X$ is the canonical projection.

(2)⇒(3) By the hypothesis, we can write $M = X \oplus X'$ for some submodule X' of M. Then X' is a *ss*-supplement of X in M and so a *ss*-supplement of $A = X + Y$ in M by [4, Lemma 22]. Therefore $X' \cap A \subseteq Soc_s(X')$.

(3)⇒(4) If we take an intersection the equality $M = X \oplus X'$ with A, we can write $A = X \oplus (X' \cap A)$. Hence X' is a *ss*-supplement of A in M.

(4)⇒(5) From the hypothesis, we have $M = A + X'$, $X' \cap A \subseteq Soc_s(X')$ and $A = (X' \cap A) \oplus X$ for some $X \subseteq A$. Then $M = A + X' = (X' \cap A) + X + X' = X + X'$ and $(X' \cap A) \cap X = 0$ and $M = X + X'$. Let $e: M \to M$ be the projection such that $e(m) = x$, $m = x + x'$, $x \in X$, $x' \in X'$. Then $e(M) \subseteq X \subseteq U$. Since $(1-e)(M) = X'$, we get that $(1-e)(A) = X' \cap A \ll X' = (1-e)(M)$ and $X' \cap A$ is semisimple.

(5)⇒(1) Let $X = e(M)$. Since e is an idempotent, we have $M = e(M) \oplus (1-e)(M)$. Then $M = X \oplus (1-e)(M)$ with $X \subseteq A$. We will consider the isomorphism $\Phi: \frac{M}{X} \to (1-e)(M)$. From here, $\Phi\left(\frac{A}{X}\right) = (1-e)(A) \ll (1-e)(M) = \Phi\left(\frac{M}{X}\right)$. Since Φ^{-1} is an isomorphism, we can get $\frac{A}{X} \ll \frac{M}{X}$ and $\Phi^{-1}((1-e)(A)) = \frac{A}{X}$ is semisimple. Therefore $\frac{A}{X} \subseteq Soc_s\left(\frac{M}{X}\right)$. □

Note that every direct summand of a module is an *ss*-supplement submodule of the module and *ss*-supplement submodules are supplement.

Theorem 1. *For a module M, the following conditions are equivalent:*

(1) *M is ss-lifting.*
(2) *Every submodule A of M can be written as $A = N \oplus S$ with N is a direct summand of M and $S \subseteq Soc_s(M)$.*
(3) *M is an amply ss-supplemented module and every ss-supplement submodule of M is a direct summand.*

Proof. (1) ⇔ (2) By Lemma 2.

(1) ⇒ (3) It follows from Lemma 1 that M is an amply *ss*-supplemented module. Since every supplement submodule of a lifting module is a direct summand of the module, it follows from (1) that every every *ss*-supplement in M is a direct summand.

(3) ⇒ (1) Let A be a submodule of M. By the hypothesis, A has an *ss*-supplement X and X has an *ss*-supplement Y such that $Y \leq A$ and Y is a direct summand of M. Then there exists a submodule T of M with $M = Y \oplus T$. Hence we get that $A = Y \oplus (A \cap T)$ and $A = Y + (A \cap X)$. If we consider the projection $\pi: Y \oplus T \to T$, we can obtain that $\pi(A) = \pi(Y + (A \cap X)) = A \cap T$. In this way, we say that there is a decomposition $M = Y \oplus T$ such that $Y \leq A$, $A \cap T \ll M$ and $A \cap T \subseteq Soc_s(M)$ and so M is *ss*-lifting. □

Theorem 2. *Let M be a π-projective and ss-supplemented module. Then M is ss-lifting.*

Proof. By Proposition 37 of [4], M is amply *ss*-supplemented. Since M is *ss*-supplemented, there exists a submodule V of M such that $M = U + V$ and $U \cap V \subseteq Soc_s(V)$. On the other side, there exists a submodule U of M such that

$M = U' + V$, $U' \subseteq U$ and $U' \cap V \subseteq Soc_s(U')$ because M is amply *ss-supplemented. Hence* U' and V are mutual *ss-supplements*. By 41.14 (2) in [7], we can write $U' \cap V = 0$. It means that $M = U' \oplus V$. Thus M is *ss-lifting.* □

Theorem 3. *Let M be an ss-lifting module. Then every direct summand of M is ss-lifting.*

Proof. Let X be a direct summand of M with $M = X \oplus X'$ for some submodule X' of M and $U \leq X$. Since M is *ss*-lifting, there there exists a submodule V of M such that $M = U + V$, $U \cap V \subseteq Soc_s(V)$. Then we can write $X = U' \oplus (X \cap V)$ and $U \cap (X \cap V) = (U \cap X) \cap V = U \cap V$ is semisimple. It follows from $U \cap V \ll M$ and $U \cap V \ll X$ because X is a direct summand of M. Hence $X = U' \oplus (X \cap V)$ implies that $U \cap V \ll X \cap V$. Thus X is *ss*-lifting. □

Now, we will give necessary conditions for any lifting module to be *ss*-lifting.

Theorem 4. *Let M be a module with small radical. Then the following statements are equivalent:*

(1) *M is ss-lifting.*
(2) *M is lifting and $Rad(M) \subseteq Soc(M)$.*

Proof. (1) $\Rightarrow$ (2) Since $Rad(M)$ is a small submodule of M and M is *ss*-lifting, M is an *ss*-supplement of $Rad(M)$ in M and so $Rad(M) \cap M = Rad(M)$ is semisimple.
(2) $\Rightarrow$ (1) Let $U \leq M$. Since M is lifting, there is a decomposition for a submodule V of M, $M = U' \oplus V$, $U' \leq U$ and $U \cap V \ll V$. It follows that $U \cap V \subseteq Rad(V) \subseteq$ $\subseteq Rad(M)$ is semisimple. Thus M is *ss*-lifting. □

Since a projective supplemented module has small radical, we have the following fact as a result of Theorem 4.

Corollary 2. *Let M be a projective module. Then M is ss-lifting if and only if it is lifting and its radical is semisimple.*

Recall from [7, 43.9] that a ring whose all left modules are supplemented is *left perfect*. It follows from [7, 43.9] that a ring R is left perfect if and only if R is semilocal and $Rad(R)$ is right t-nilpotent if and only if every left R-module has a projective cover, that is, for any left R-module M, there exist a projective module P and an epimorphism $f: P \longrightarrow M$ with small kernel. R is called *semiperfect* if every finitely generated left (or right) R-module is supplemented. Now we give a characterization of semiperfect (left perfect) rings.

Lemma 3. *Let R be an arbitrary ring. Then ${}_RR$ is ss-lifting if and only if R is semiperfect and $Rad(R) \subseteq Soc({}_RR)$.*

Proof. By Theorem 4. □

Theorem 5. *The following statements are equivalent for a ring R.*

(1) ${}_RR$ *is ss-lifting.*
(2) ${}_RR$ *is ss-supplemented.*
(3) *Every left R-module is ss-supplemented.*
(4) *R is semiperfect and* $Rad(R) \subseteq Soc({}_RR)$.

Proof. (1) $\Rightarrow$ (2) It is clear.
(2) $\Rightarrow$ (3) $\Rightarrow$ (4) It follows from [4, Theorem 41].
(4) $\Rightarrow$ (1) By Theorem 4. □

Now we characterize the rings with the property that every left module is *ss*-lifting. Firstly, we need following lemma.

Lemma 4. *Let M be a lifting module and* $Rad(M) \subseteq Soc(M)$ *.Then M is ss-lifting.*

Proof. The proof is clear. □

Theorem 6. *The following statements are equivalent:*

(1) *R is a left and right artinian serial ring and* $Rad(R) \subseteq Soc({}_RR)$.
(2) *Every left R-module is ss-lifting.*

Proof. (1)⇒(2) By the hypothesis and Lemma 3, it is clear that $Rad(R) \subseteq Soc({}_RR)$. On the other side, if every left R-module is semisimple lifting, then every left R-module is lifting by [2, 29.10].

(2)⇒(1) Since $Rad(R) \subseteq Soc({}_RR)$, we have $Rad(R)^2 = 0$ by [7, 21.12 (4)]. Moreover, we say that every left R-module is lifting by [2, 29.10]. We can write $Rad(M) = Rad(R)M \subseteq Soc({}_RR)M \subseteq Soc(M)$ because R is an artinian ring. Therefore M is *ss*-lifting by previous Lemma. □

Example 2. Consider the local ring $R = \mathbb{Z}_4$ is left and right artinian serial ring and $Rad(R) = \{0,2\} = Soc({}_RR)$ and so every left R-module is *ss*-lifting by Theorem 6.

Theorem 7. *Let M be a ss-lifting module. If* $\frac{K+X}{X}$ *is a direct summand of* $\frac{M}{X}$ *for every direct summand K of M, then* $\frac{M}{X}$ *is ss-lifting.*

Proof. Let $\frac{A}{X} \leq \frac{M}{X}$. Since M is *ss*-lifting, there exists a direct summand K of M with $K \leq A$ and $\frac{A}{K} \subseteq Soc_s\left(\frac{M}{K}\right)$ by Lemma 2. It is clear that $\frac{K+X}{X} \leq \frac{A}{X}$. If we say $\frac{A}{K+X} \subseteq Soc_s\left(\frac{M}{K+X}\right)$, the proof is completed. Since $\frac{\left(\frac{M}{K}\right)}{\left(\frac{K+X}{K}\right)} \cong \frac{M}{K+X}$, we get that $\frac{A}{K+X} \subseteq Soc_s\left(\frac{M}{K+X}\right)$. Therefore, $\frac{M}{X}$ is a *ss*-lifting module by Lemma 2. □

Recall from [2] that a submodule U of M is called *fully invariant* if $f(U)$ is contained in U for every R-endomorphism f of M. Recall from [2] that a module M is called *duo* if every submodule of M is fully invariant in M.

Theorem 8. *Let M be a ss-lifting module and X be a fully invariant submodule of M. Then* $\frac{M}{X}$ *is ss-lifting.*

Proof. Suppose that $M = K \oplus L$. Then $e(M) = K$ and $(1-e)(M) = L$ for some $e \in End(M)$. Since X is fully invariant, $e(X) = X \cap K$ and $(1-e)(X) = X \cap L$. From here, $X = e(X) \oplus (1-e)(X) = (X \cap K) \oplus (X \cap L)$ and we can write $\frac{K+X}{X} = \frac{K+[(X\cap K)\oplus(X\cap L)]}{X} = \frac{K\oplus(X\cap L)}{X}$ and $\frac{L+X}{X} = \frac{L+[(X\cap K)\oplus(X\cap L)]}{X} = \frac{L\oplus(X\cap K)}{X}$. Hence $M = K + X + L + X = [K \oplus (X \cap L)] + L + X$ implies that $[K \oplus (X \cap L)] \cap [L+X] = [K \oplus (X \cap L)] \cap [L + (X \cap K)] = (X \cap K) \oplus (X \cap L) = X$ and $\frac{M}{X} = \left(\frac{K\oplus(X\cap L)}{X}\right) \oplus \left(\frac{L+X}{X}\right)$. Thus $\frac{M}{X}$ is *ss*-lifting by the previous theorem. □

Recall from [1] that a submodule U is called a *weak distributive* of M if $U = (U \cap X) + (U \cap Y)$ for all submodules $X, Y \leq M$ such that $M = X + Y$. A module M is said to be *weakly distributive* if every submodule of M is a weak distributive submodule of M.

Theorem 9. *Let M be a weakly distributive module and $X \leq M$. Then $\frac{M}{X}$ is ss-lifting.*

Proof. Let $M = K \oplus L$. Then we have $\frac{M}{X} = \left(\frac{K+X}{X}\right) + \left(\frac{L+X}{X}\right)$ and $X = X + K \cap L = (X+K) \cap (X+L)$. Thus $\frac{M}{X} = \left(\frac{K+X}{X}\right) \oplus \left(\frac{L+X}{X}\right)$ and so $\frac{M}{X}$ is *ss*-lifting by Theorem 7. □

Theorem 10. *Let $M = M_1 \oplus M_2$ be a duo module. If M_1 and M_2 are ss-lifting modules, then M is ss-lifting.*

Proof. Suppose that L be a submodule of M. We can write $L = \bigoplus_{i=1}^{2} (L \cap M_i)$ by Lemma 2.1 of [2]. For each $i \in \{1,2\}$, there exists a direct summand D_i of M_i such that $M_i = D_i \oplus D_i'$ with $D_i \leq L \cap M_i$ and $L \cap D_i' \subseteq Soc_s(D_i')$. From here

$$M = M_1 \oplus M_2 = \left(D_1 \oplus D_1'\right) \oplus \left(D_2 \oplus D_2'\right) = (D_1 \oplus D_2) \oplus \left(D_1' \oplus D_2'\right).$$

It is clear that $D_1 \oplus D_2 \leq L$. Since $L \cap D_i' \subseteq Soc_s(D_i')$, $L \cap (D_1' \oplus D_2') \subseteq Soc_s(D_1' \oplus D_2')$. Therefore M is *ss*-lifting. □

Lemma 5 (see [5, Lemma 5])**.** *The following statements are equivalent for a module $M = M_1 \oplus M_2$.*
(i) M_2 is M_1-projective.
(ii) For each submodule N of M with $M = M_1 + N$ there exists a submodule N' of N such that $M = M_1 \oplus N'$.

Theorem 11. *Let the module $M = M_1 \oplus M_2$ with M_1 and M_2 are relatively projective modules. If M_1 is semisimple and M_2 is ss-lifting, then M is ss-lifting.*

Proof. Suppose that K be a non-zero submodule of M.

Case 1: Assume that $T = M_1 \cap (K + M_2) \neq 0$. Since M_1 is semisimple, we can write $M_1 = T \oplus T_1$ for some submodule T_1 of M_1 and so $M = T \oplus T_1 \oplus M_2 = = [(M_1 \cap (K + M_2))] \oplus T_1 \oplus M_2 = K \oplus (M_2 \oplus T_1)$. Using Prop. 4.31, Prop. 4.32 and

Prop. 4.33 in [6], we can say that T is $M_2 \oplus T_1$-projective. By 41.14 in [7], there exists a submodule K_1 of K such that $M = K_1 \oplus (M_2 \oplus T_1)$. Let A be any submodule of M_2 and $K \cap (M_2 \oplus T_1) \neq 0$. Since $K \cap (A+T_1) \leq A \cap (K+T_1) + T_1 \cap (K+A)$ and $T_1 \cap (K+A) = 0$, then $K \cap (A+T_1) \leq A \cap (K+T_1)$. Similarly, $A \cap (K+T_1) \leq K \cap (A+T_1)$. Hence $A \cap (K+T_1) = K \cap (A+T_1)$. Moreover, if we consider M_2 is *ss*-lifting, then there exists a submodule X_1 of $M_2 \cap (K+T_1) = K \cap (M_2+T_1)$ such that $M_2 = X_1 \oplus X_2$ and $X_2 \cap (K+T_1) \subseteq Soc_s(X_2)$ for some submodule X_2 of M_2. Therefore $M = (K_1 \oplus X_1) \oplus (X_2 \oplus T_1)$, $K_1 \oplus T_1 \leq K$ and $K \cap (X_2 \oplus T_1) = = X_2 \cap (K+T_1) \subseteq Soc_s(X_2 \oplus T_1)$.

Case 2: Assume that $T = M_1 \cap (K+M_2) = 0$. From here T is a submodule of M_2. Since M_2 is *ss*-lifting, there exists a submodule Y_1 of K such that $M_2 = Y_1 \oplus Y_2$, $K \cap Y_2 \subseteq Soc_s(Y_2)$ for some submodule Y_2 of M_2. Thus $M = M_1 \oplus M_2 = = M_1 \oplus (Y_1 \oplus Y_2) = Y_1 \oplus (M_1 \oplus Y_2)$ and $K \cap (M_1 \oplus Y_2) = K \cap Y_2 \subseteq Soc_s(M_1 \oplus Y_2)$. As a result M is *ss*-lifting. □

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STABILITY, NEIMARK-SACKER BIFURCATION AND CHAOS CONTROL FOR A PREY-PREDATOR SYSTEM WITH HARVESTING EFFECT ON PREDATOR

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Abstract. This paper deals with the dynamic behavior of a prey predator model obtained by the forward Euler method. We investigate the complex dynamics of discrete-time prey-predator system related to predator population which is subject to the effects of harvesting. The stability of equilibrium point of the model and also the existence and the direction of Neimark-Sacker bifurcation are analyzed. We show that the system undergoes Neimark-Sacker bifurcations by using center manifold theorem and bifurcation theory. A state feedback method is applied in order to control the Neimark-Sacker bifurcation. Moreover, numerical simulations are carried out to demonstrate the theoretical results obtained for stability, bifurcation and chaos control strategy.

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1. INTRODUCTION

Changes in population size result from interactions between species such as predatory, cooperative, mutualistic, commensal interactions. One of the fundamental structure in population dynamics is prey-predator interactions which has been described firstly by two pioneers Lotka-Volterra [23, 33]. In recent times, many researchers have considered effects of harvesting on various types of prey-predator models [2, 10, 16, 24, 25, 34–36]. Prey-predator harvesting which is applied on predator population as well as prey population have a strong impact on the dynamic evaluation [3, 4]. Discrete-time and continuous-time models are used for predicting the size of a population. Discrete-time models are suitable for obtaining more accurate numerical simulations for non-overlapping generations [9, 11, 21, 22, 26]. Therefore, the studies on discretization of prey-predator models governed by difference

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equations have received remarkable attention [15, 28, 29, 37, 38]. Bifurcation theory is widely used in the mathematical research of dynamic systems [12, 20]. For the some type of bifurcation analysis of discrete-time population models, we refer to [1, 5, 7, 13–15, 17, 21, 22, 28, 29, 37, 38]. Dynamic models are also often used in chaos theory studies. A chaotic behavior occurs suddenly and includes unpredictable behavior. Chaos control theory offers some strategies avoid to the chaotic situation that occurs with these unpredictable behaviors [5–8, 19, 27, 30–32]. So, the chaotic behavior can be delayed or can be prevented completely. With this aspect, this theory is one of the important study subjects for dynamic systems. Lotka-Volterra model subject to harvesting effect on predator population is presented the following form:

$$\begin{aligned} \frac{dx_n}{dt} &= ax_n(1-x_n) - cx_ny_n \\ \frac{dy_n}{dt} &= bx_ny_n - ey_n - Hy_n. \end{aligned} \tag{1.1}$$

where x_n and y_n denote the numbers of prey and predator, respectively. Moreover the parameters a, b,c,e,H are positive and the initial conditions x_0, y_0 are positive real numbers.

In this paper, we will focus discrete-time version of the system (1.1) by applying the forward Euler scheme as follows:

$$\begin{aligned} x_{n+1} &= x_n + h[ax_n(1-x_n) - cx_ny_n] \\ y_{n+1} &= y_n + h[bx_ny_n - ey_n - Hy_n] \end{aligned} \tag{1.2}$$

Here H denotes harvesting effect. The important ıssues in discrete-time models to investigate the existence and the direction of the bifurcation and chaos phenomenon which are performed either by using numerical simulations or by using the center manifold theorem and bifurcation theory. Also, control of chaotic behavior in dynamic model using control strategies are remarkable .

Our purpose in this study is to obtain the equilibrium points of the system (1.2); and to investigate the stability conditions of these equilibrium points. Additionally, the direction and the existence of the Neimark-Sacker bifurcation of the system (1.2) is studied by choosing b as the bifurcation parameter; and controlled the chaos behavior of the model via control strategy. Moreover, we show the dynamic properties of the system (1.2), by means of trajectories, bifurcation diagrams and phase portraits.

This study consists of four sections. In Section II, we investigate the existence and local asymptotic stability of the equilibrium points of the system (1.2) in $\mathbb{R}^2_+$. In Section III, we discuss the undergoing of the system (1.2) into a Neimark-Sacker bifurcation. In Section IV, we apply chaos control method to provide control of the system (1.2). Section V includes numerical simulations to support the theoretical results. In the last section, the results are briefly presented.

2. The Existence and Stability of Equilibrium Points

In this section, we discuss the existence and stability of equilibrium points of the system (1.2) in the close first quadrant $\mathbb{R}_2^+$. A equilibrium point of (1.2) is a point $(\bar{x},\bar{y})$ that satisfies

$$\begin{aligned} \bar{x} &= f(\bar{x},\bar{y}), \\ \bar{y} &= g(\bar{x},\bar{y}). \end{aligned} \tag{2.1}$$

We easy obtain the equilibrium points of system (1.2) by using (2.1)

$$\begin{aligned} \bar{x} &= \bar{x} + h[a\bar{x}(1-\bar{x}) - c\bar{x}\bar{y}] \\ \bar{y} &= \bar{y} + h[b\bar{x}\bar{y} - e\bar{y} - H\bar{y}]. \end{aligned} \tag{2.2}$$

It is clear that the system (1.2) has the equilibrium points $(0,0)$ and $(1,0)$ for all positive parameters. Additionaly, in the case of $b > e+H$, the system (1.2) has a unique positive coexistence equilibria $E_2 = (\frac{e+H}{b}, \frac{a(b-e-H)}{bc})$. So, we reach the results for existence of the equilibrium points of the system (1.2) as follows:

Lemma 1. *For the system (1.2), the following cases hold:*

(i) *The system (1.2) has two equilibrium points. These are an trivial (extinction) equilibrium point $E_0 = (0,0)$ and an exclusion (axial) equilibrium point $E_1 = (1,0)$ for all positive parameters;*

(ii) *If $b > e+H$, then the system (1.2) has a uniqe positive coexistence equilibrium point $E_2 = (\frac{e+H}{b}, \frac{a(b-e-H)}{bc})$.*

The Jacobian matrix of a system (1.2) evaluated at any equilibrium point $(\bar{x},\bar{y})$ is determined by

$$J_{(\bar{x},\bar{y})} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{2.3}$$

and the characteristic equation of matrix $J_{(\bar{x},\bar{y})}$ can be given with

$$\lambda^2 - trJ_{(\bar{x},\bar{y})}\lambda + \det J_{(\bar{x},\bar{y})} = 0. \tag{2.4}$$

Suppose that λ_1 and λ_2 be two roots of $F(\lambda) = 0$. Then the equilibrium point $(\bar{x},\bar{y})$ is local asymptotic stable, if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, and it is called as a sink point. Otherwise, $(\bar{x},\bar{y})$ is always unstable, and it is known source (repeller). And, the equilibrium point $(\bar{x},\bar{y})$ is called a saddle point if $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$). If $|\lambda_1| = 1$ or $|\lambda_2| = 1$, then $(\bar{x},\bar{y})$ is known as non-hyperboic point.

Now, let us study the local dynamics of the equilibrium points by obtaining the Jacobian matrix evaluated at E_0, E_1 and E_2. Firstly, by considering (1.2), we can get the Jacobian matrix as

$$J_{E_0} = \begin{pmatrix} 1+ah & 0 \\ 0 & 1-eh-hH \end{pmatrix} \tag{2.5}$$

evaluated at E_0, and the eigenvalues of J_{E_0} are

$$\lambda_1 = 1 + ah, \quad \lambda_2 = 1 - eh - hH.$$

Thus, we can express the topological classification of the equilibrium point E_0.

Lemma 2. *For the extinction equilibrium point E_0, the following cases hold:*

(i) *The equilibrium point E_0 cannot be a sink point. It is a source point for $eh + hH > 2$.*
(ii) *If $eh + hH < 2$, then the equilibrium point E_0 is a saddle point.*

Secondly, we can get the Jacobian matrix evaluated at E_1 as

$$J_{E_1} = \begin{pmatrix} 1 - ah & -ch \\ 0 & 1 + bh - eh - hH \end{pmatrix} \tag{2.6}$$

and the eigenvalues of J_{E_1} are

$$\lambda_1 = 1 - ah, \quad \lambda_2 = 1 + bh - eh - hH.$$

From there, we can get the topological classification of the equilibrium point E_1.

Lemma 3. *For the extinction equilibrium point E_1, the following cases hold:*

(i) *If $ah < 2$ and $(b < e + H$ and $(e + H - b)h < 2)$, then the equilibrium point E_1 is a sink point.*
(ii) *If $ah < 2$ and $(b > e + H$ or $(H + e - b)h > 2)$ or $ah > 2$ and $(b < e + H$ and $(e + H - b)h < 2)$, then the equilibrium point E_1 is a saddle point.*
(iii) *If $ah > 2$ and $(b > e + H$ or $(H + e - b)h > 2)$, then the equilibrium point E_1 is a source point.*

In order to investigate the dynamics of a unique positive equilibrium point E_2, we need the following lemma.

Lemma 4. *[22] Let $F(\lambda) = \lambda^2 - trJ\lambda + \det J$. Suppose that $F(1) > 0$, and λ_1 and λ_2 are two roots of $F(\lambda) = 0$. Then,*

(i) *$|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $\det J < 1$;*
(ii) *$|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$;*
(iii) *$|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $\det J > 1$;*
(iv) *$\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if $F(-1) = 0$ and $-trJ \neq 0, 2$;*
(v) *λ_1 and λ_2 are complex and $|\lambda_1| = |\lambda_2| = 1$ if and only if $(-trJ)^2 - 4\det J < 0$ and $\det J = 1$.*

At last, we can give the Jacobian matrix as

$$J_{E_2} = \begin{pmatrix} \frac{b - ah(e+H)}{b} & -\frac{ch(e+H)}{b} \\ \frac{ah(b-e-H)}{c} & 1 \end{pmatrix} \tag{2.7}$$

evaluated at E_2, and the characteristic polynomial of J_{E_2} is

$$F(\lambda)=\lambda^2-\left[1+\frac{b-ah(e+H)}{b}\right]\lambda+\left[\frac{b-ah(e+H)}{b}+\frac{ach^2(b-e-H)(e+H)}{bc}\right].$$

By using Lemma 4, we can get the topological classification of the coexistence positive equilibrium E_2.

Lemma 5. *For the coexistence positive equilibrium point E_2, the following cases hold:*

(i) *If*

$$\frac{2aeh+ae^2h^2+2ahH+2aeh^2+ah^2H^2}{4+aeh^2+ah^2H}<b<\frac{1+eh+hH}{h},$$

then the equilibrium point E_2 is a sink point.

(ii) *If*

$$b>\frac{1+eh+hH}{h},$$

then the equilibrium point E_2 is a source point.

(iii) *If*

$$b<\frac{2aeh+ae^2h^2+2ahH+2aeh^2+ah^2H^2}{4+aeh^2+ah^2H},$$

then the equilibrium point E_2 is a saddle point.

(iv) *If*

$$b=\frac{2aeh+ae^2h^2+2ahH+2aeh^2+ah^2H^2}{4+aeh^2+ah^2H}$$

or

$$b=\frac{1+eh+hH}{h},$$

then the equilibrium point E_2 is non-hyperbolic point.

Example 1. For the parameter values $a=2.5$, $b=2.25$, $e=0.2$, $c=1$, $H=1$ and $h=0.9$ the initial conditons $(x_0,y_0)=(0.3,0.4)$, the equilibrium point E_2 of the model (1.2) is locally asymptotically stable. Moreover, for $b=2.31$, the equilibrium point E_2 of the model (1.2) is non-hyperbolic point. To confirm the theoretical result, the trajectories of the model (1.2) are shown in Figure 1.

3. Neimark-Sacker Bifurcations

In this section, the direction and existence of Neimark–Sacker bifurcation are obtained for the system (1.2). Additionally, if the system (1.2) provides eigenvalue assignment, transversality and non resonance conditions, then a Neimark–Sacker bifurcation occurs at a bifurcation point [12,30]. Considering Lemma 4-(v), the bifurcation occurs in non-hyperbolic point E_2 where the roots of J_{E_2} are complex conjugate with module one. In order to work Neimark–Sacker bifurcation in the system (1.2),

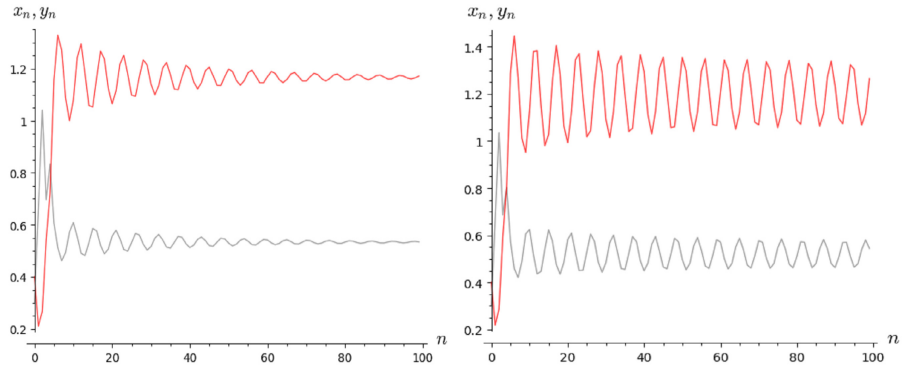

FIGURE 1. (a) Trajectories of the prey-predator system (1.2) with the parameter values $a = 2.5, b = 2.25, e = 0.2, c = 1, H = 1$ and $h = 0.9$ the initial conditions $(x_0, y_0) = (0.3, 0.4)$. (b) Trajectories of the prey-predator system (1.2) with the parameter values $a = 2.5, b = 2.31, e = 0.2, c = 1, H = 1$ and $h = 0.9$ the initial conditions $(x_0, y_0) = (0.3, 0.4)$.

we define the parameters providing non-hyperbolic conditions by taking the bifurcation parameter b as follows:

$$NSB_{E_2} = \left\{ \begin{array}{c} a, b, c, eh,\ H \in \mathbb{R}^+ : b > \frac{e+H}{2} + \frac{1}{2}\sqrt{ae + e^2 + (a+2e)H + H^2} \\ \text{and } b = b_1 = \frac{1+eh+hH}{h} \end{array} \right\}. \tag{3.1}$$

The eigenvalues of J_{E_2} have a pair of conjugate complex numbers which are of module one for $b = b_1$. These eigenvalues are found as

$$\lambda, \bar{\lambda} = \frac{trJ_{E_2}}{2} \mp \frac{\sqrt{4 \det J_{E_2} - (trJ_{E_2})^2}}{2} \mid_{b=b_1} \tag{3.2}$$

where

$$\lambda, \bar{\lambda} = \frac{1}{2b}\left[2b - aeh - ahH \mp ih\sqrt{a(e+H)[4b^2 - ae - 4be - aH - 4bH]}\right] \mid_{b=b_1} \tag{3.3}$$

such that

$$|\lambda| = \left|\bar{\lambda}\right| = 1.$$

For $b \in NSB_{E_2}$, we get

$$\frac{\partial |\lambda_i(b)|}{\partial b} \mid_{b=b_1} \neq 0\,, \qquad i = 1, 2. \tag{3.4}$$

Additionally, if

$$trJ_{E_2} \mid_{b=b_1} \neq 0, -1, \tag{3.5}$$

then, we reach the following:

$$\lambda^k(b_1) \neq 1, \qquad k = 1,2,3,4. \tag{3.6}$$

Here, the value trJ_{E_2} at the point $b = b_1$ is

$$trJ_{E_2} = \frac{2eH - ah^2(e+H) + 2(1+hH)}{1+eh+hH}.$$

Moreover, the eigenvalues of J_{E_2} evaluated at the point $b = b_1$ are obtained as

$$\lambda, \overline{\lambda} = \frac{-ah^2(e+H)}{2(eH+hH+1)} \mp \frac{ih\sqrt{a(e+H)[(\frac{4(eH+hH+1)(-h(e+H)+eH+hH+1)}{h^2} - a(e+H))]}}{2(eH+hH+1)} + 1.$$

and

$$\frac{\partial |\lambda_i(b)|}{\partial b}|_{b=b_1} = \frac{ah^2(e+H)h\sqrt{a(e+H)\frac{4(eH+hH+1)(-h(e+H)+eH+hH+1)}{h^2}}}{(2(eH+hH+1)^2\sqrt{a(e+H)\frac{4(eH+hH+1)(-h(e+H)+eH+hH+1)}{h^2}}) - a(e+H)}$$
$$\frac{-a(e+H) - iah(e+H) - 2i(e+H)(eH+hH+1)}{2(eH+hH+1)^2\sqrt{a(e+H)\frac{4(eH+hH+1)(-h(e+H)+eH+hH+1)}{h^2}} - a(e+H)}.$$

Let $q, p \in \mathbb{C}^2$ be two eigenvectors which correspond to the eigenvalues λ of the matrix $J(\Omega_{NSB_{E2}})$ and the eigenvalues $\overline{\lambda}$ of the matrix $J(\Omega_{NSB_{E2}})^T$, respectively. These eigenvectors are calculated as

$$q \sim \left(\frac{-1 - \frac{1}{2}[-1 - M - i\sqrt{2M - 1 - M^2 - 4NS}]}{S}, 1 \right) \tag{3.7}$$

and

$$p \sim \left(\frac{-1 - \frac{1}{2}[-1 + M + i\sqrt{2M - 1 - M^2 - 4NS}]}{N}, 1 \right) \tag{3.8}$$

such that

$$M = \frac{1 + eH - ah^2(e+H) + hH}{1+eH+hH},$$
$$N = -\frac{ch^2(e+H)}{1+eH+hH}$$
and
$$S = \frac{a(1+eH-eh)}{c}.$$

By using the scalar product in $\mathbb{C}^2 :< p,q >= \overline{p_1}q_1 + \overline{p_2}q_2$, we define the following vector in order to normalize p according to q

$$p \sim \left(\frac{(M-1) - i\sqrt{2M-1-M^2-4NS}}{R^*}, \frac{1}{R^*} \right) \tag{3.9}$$

where $R^* = \frac{(M+1)+i\sqrt{2M-1-M^2-4NS}}{4NS} + 1$ such that $< p,q >= 1$.

In order to translate the equilibrium point E_2 of the system (1.2) into the origin $(0,0)$, we take

$$u_n = x_n - \frac{e+H}{b},\ v_n = y_n - \frac{a(b-e-H)}{bc}. \tag{3.10}$$

Then, we get the following map:

$$\begin{pmatrix} u \\ v \end{pmatrix} \to J_{E_2} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} F_1(u,v) \\ F_2(u,v) \end{pmatrix}, \tag{3.11}$$

where

$$F_1(u,v) = -chuv - ahu^2 + O(\|U\|^3)$$
$$F_2(u,v) = bhuv + O(\|U\|^3).$$

such that $U_t = (u,v)^T$. Additionally, the system (1.2) can be written as

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} \to J_{E_2} \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \frac{1}{2}B(u_n,v_n) + \frac{1}{6}C(u_n,v_n,w_n) + O(\|U_n\|^4), \tag{3.12}$$

with the multi-linear vector functions of $u;v;w \in \mathbb{R}^2$:

$$B(u,v) = \begin{pmatrix} B_1(u,v) \\ B_2(u,v) \end{pmatrix}$$

and

$$C(u,v) = \begin{pmatrix} B_1(u,v,w) \\ B_2(u,v,w) \end{pmatrix}.$$

These vectors are expressed by

$$B_1(u,v) = \sum_{j,k=1}^{2} \frac{\partial^2 F_1}{\partial\xi_j \partial\xi_k} |_{\xi=0}\, u_j v_k = -2ahu_1v_1 - ch(u_2v_1 + u_1v_2),$$

$$B_2(u,v) = \sum_{j,k=1}^{2} \frac{\partial^2 F_2}{\partial\xi_j \partial\xi_k} |_{\xi=0}\, u_j v_k = bh(u_2v_1 + u_1v_2),$$

$$C_1(u,v,w) = \sum_{j,k=1}^{2} \frac{\partial^3 F_1}{\partial\xi_j \partial\xi_k \xi_l} |_{\xi=0}\, u_j v_k w_l = 0,$$

$$C_2(u,v,w) = \sum_{j,k=1}^{2} \frac{\partial^3 F_2}{\partial\xi_j \partial\xi_k \xi_l} |_{\xi=0}\, u_j v_k w_l = 0.$$

$\forall U \in \mathbb{R}^2$ can be uniquely shown as

$$U = zq + \overline{z}\overline{q} \tag{3.13}$$

for some $z \in \mathbb{C}$. Here, $\bar{z}$ is the conjugate of the complex number z, and $z =< p, U >$. For all sufficiently small $|b|$, we can transform the system (1.2) into the form,

$$z \to \lambda(b)z + g(z, \bar{z}, b), \tag{3.14}$$

where $\lambda(r) = (1 + \omega(b))e^{i\arctan(b)}$ with $\omega(b_1) = 0$ and $g(z, \bar{z}, b)$ is a complex valued smooth function of z and $\bar{z}$. The Taylor expression of g with respect to $g(z, \bar{z})$ is as follows:

$$g(z, \bar{z}, b) = \sum_{k+l\geq 2} \frac{1}{k!l!} g_{kl}(b) z^k \bar{z}^l, \tag{3.15}$$

with

$$\begin{aligned} g_{20}(b_1) &=< p, B(q, q) > \\ g_{11}(b_1) &=< p, B(q, \overline{q}) > \\ g_{02}(b_1) &=< p, B(\overline{q}, \overline{q}) > \\ g_{21}(b_1) &=< p, C(q, q, \overline{q}) > . \end{aligned}$$

In order to come out as the Neimark-Sacker bifurcation for the system (3.11), we need that the coefficient $\varphi(b_1)$ must not be zero. This coefficient is

$$\begin{aligned} \varphi(b_1) = Re\left(\frac{e^{-i\arctan(b_1)}}{2} g_{21}\right) - Re\left(\frac{(1 - 2e^{i\arctan(b_1)})e^{-2i\arctan(b_1)}}{2(1 - e^{i\arctan(b_1)})} g_{20} g_{11}\right) \\ - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2 \end{aligned} \tag{3.16}$$

where $e^{i\arctan(b_1)} = \lambda(b_1)$. Consequently, we have the following theorem on Neimark-Sacker bifurcation:

Theorem 1. *If (3.5) holds, $\varphi(b_1) \neq 0$ and the parameter b changes its value in a small vicinity of NSB_{E2}, then the system (1.2) passes through a Neimark-Sacker bifurcation at the only equilibrium point E_2. Moreover, if $\varphi(b_1) < 0$ ($\varphi(b_1) > 0$), then there exists a unique attracting (repelling) invariant closed curve which bifurcates from E_2.*

4. Chaos Control

Stabilizing unstable periodic orbits in chaotic dynamics has been an interesting research topic in recent years [5–7]. Chaos can be controlled by various methods (see [8, 19, 27, 31, 37]). The first feedback control strategy known as the OGY method was proposed by Ott et al. [27]. We study to control the chaos via chaos controlling strategy based a state feedback control method [8, 19]. Now, we suppose that system

(1.2) undergoes Neimark-Sacker bifurcation at equilibrium point $(\bar{x},\bar{y})$, then corresponding controlled system can be taken as follows:

$$\begin{aligned} x_{n+1} &= x_n + h(ax_n(1-x_n) - cx_ny_n) - k_1(x_n-\bar{x}) - k_2(y_n-\bar{y}) \\ y_{n+1} &= y_n + h[(bx_ny_n - ey_n - Hy_n] \end{aligned} \tag{4.1}$$

such that $-k_1(x_n-\bar{x}) - k_2(y_n-\bar{y})$ is the control force; and k_1 and k_2 define as the feedback gains. The Jacobian matrix of $J(\bar{x},\bar{y})$ of controlled system (4.1) can be written as

$$J(\bar{x},\bar{y}) = \begin{bmatrix} \frac{b-ah(e+H)-bk_1}{b} & \frac{-ch(e+H)+bk_2}{b} \\ \frac{ah(b-e-H)}{c} & 1 \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix $J(\bar{x},\bar{y})$ is given by

$$\mu^2 - \frac{2b-ah(e+H)-bk_1}{b}\mu \tag{4.2}$$
$$+\frac{c(-ah(e+H)(1+eh+hH)+b(1+ah^2(e+H))-bck_1+abh(b-e-h)k_2}{bc} = 0.$$

Let μ_1 and μ_2 be the roots of characteristic equation system (4.2), then we have

$$\mu_1+\mu_2 = \frac{2b-ah(e+H)-bk_1}{b} \tag{4.3}$$

$$\begin{aligned} \mu_1\mu_2 &= \frac{c(-ah(e+H)(1+eh+hH)+b(1+ah^2(e+H))-bck_1}{bc} \\ &\quad + \frac{abh(b-e-h)k_2}{bc}. \end{aligned} \tag{4.4}$$

To determine the marginal stability lines, we must find the solution of the equations $\mu_1 = \pm 1$ and $\mu_1\mu_2 = 1$. These conditions confirm that μ_1 and μ_2 have absolute value less than 1. Let us take $\mu_1\mu_2 = 1$, then from Eq. (4.4), we get

$$l_1 := \frac{bck_1 + ah(c(e+H)(1+h(-b+e+H)+b(-b+e+H)k_2)}{bc} = 0. \tag{4.5}$$

Now, we suppose that $\mu_1 = 1$, then Eq. (4.3) and Eq. (4.4) imply

$$\begin{aligned} l_2 &:= 1 + \frac{c(-ah(e+H)(2+h(e+H))+b(4+ah^2(e+H))-2bk_1)}{bc} \\ &\quad + \frac{abh(b-e-H)k_2}{bc} - \frac{-2b+ah(e+H)+bk_1}{h} = 0. \end{aligned} \tag{4.6}$$

Moreover, from $\mu_1 = -1$, by using Eq. (4.3) and Eq. (4.4), we obtain

$$\begin{aligned} l_3 &:= 1 + \frac{c(-ah(e+H)(2+h(e+H))+b(4+ah^2(e+H))-2bk_1)}{bc} \\ &\quad + \frac{abh(b-e-H)k_2}{bc} + \frac{-2b+ah(e+H)+bk_1}{h} = 0. \end{aligned} \tag{4.7}$$

The triangular region determined by the lines l_1, l_2 and l_3 in k_1k_2 plane is the region of the values that make the eigenvalues less than 1.

5. Numerical Simulations

In this section, theoretical results are supported with graphics by using Maple [32] and SageMath [18] programming. Some numerical simulations are given to demonstrate the existence of Neimark-Sacker bifurcation for the system (1.2). Here, bifurcation diagrams and phase portraits are illustrated by taking b as the bifurcation parameter.

Example 2. Let us consider the following system for the parameter values $e = 0.2$, $a = 2.5$, $c = 1$, $H = 1$, $h = 0.9$ and $b = 2.3111$,

$$\begin{aligned} x_{n+1} &= x_n + 2.25x_n(1-x_n) - 0.9x_ny_n \\ y_{n+1} &= y_n + 2.07999x_ny_n - 0.18y_n - 0.9y_n. \end{aligned} \tag{5.1}$$

The coexistence positive equilibrium point is $(\bar{x}, \bar{y}) = (0.519233, 1.20192)$, and the Jacobian matrix evaluated $(\bar{x}, \bar{y})$ is as follows:

$$J_{(\bar{x},\bar{y})} = \begin{pmatrix} -0.168276 & -0.46731 \\ 2.49998 & 1 \end{pmatrix}.$$

If the eigenvalues are calculated, then we obtain

$$\lambda_{1,2} = 0.415862 \mp 0.909422i$$

such that $|\lambda_{1,2}| = 1$. Let the complex eigenvectors $q, p \in \mathbb{C}^2$ be correspond to $\lambda_{1,2}$, respectively.

$$q \sim (-0.3637718555 - 0.233656893i, i)^T$$

and

$$p \sim (-0.3637718555 + 0.233656893i, -i)^T$$

If the normalized vector is taken as

$$q \sim (-0.3637718555 - 0.233656983i, i)^T$$

to obtain the normalization $< p, q >= 1$, the vector is found as

$$p \sim (0.336308 - 0.315341i, 0.193293 + 1.04866i)^T.$$

By transformation of variables

$$u_n = x_n - 0.519233, \; v_n = y_n - 1.20192$$

the system (5.1) can be written as follows:

$$\begin{aligned} u_{n+1} &= -0.168277u_n - 0.46731v_n - 0.9u_nv_n - 2.25u_n^2 \\ y_{n+1} &= 2.49998u_n + v_n + 2.07999u_nv_n. \end{aligned}$$

When the coefficients of the form (3.15) are calculated, we get

$$g_{20}(b_1) = -1.62334 - 1.59177i$$
$$g_{11}(b_1) = -0.313611 + 0.901413i$$
$$g_{02}(b_1) = 1.48092 - 0.932656i$$
$$g_{21}(b_1) = 0.$$

Here, $\theta = 1.141903497$; and from (3.16), we get $\varphi(b_{NS}) = 0.606259 > 0$. Consequently, a sub critical Neimark-Sacker bifurcation emerges at $b_{NS} = 2.3111$.

To confirm the theoretical result, bifurcation diagrams and phase portraits of the prey-predator system (1.2) are shown in Figures 1-3 where the initial point is $(x_0, y_0) = (0.3, 0.4)$. From Figure 2(a) and Figure 2(b), we see that the equilibrium point $(\bar{x}, \bar{y})$

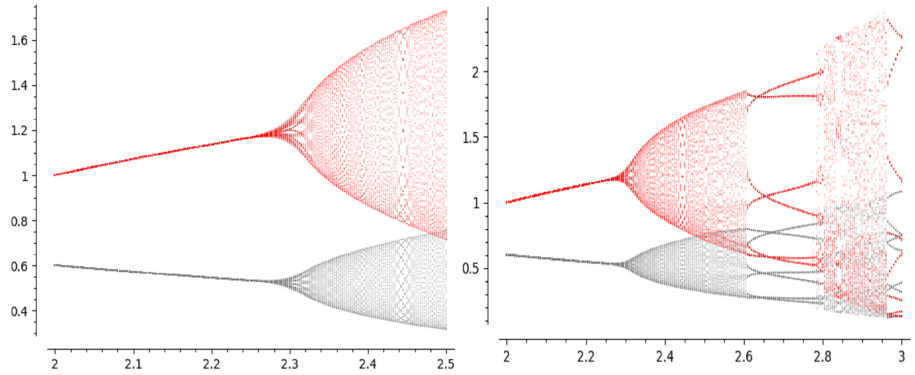

FIGURE 2. (a) Bifurcations Diagrams of the prey-predator system (1.2) for different values $b \in (2, 2.5)$ with the parameter values $a = 2.5, e = 0.2, c = 1, H = 1$ and $h = 0.9$ the initial conditions $(x_0, y_0) = (0.3, 0.4)$. (b) Bifurcations Diagrams of the prey-predator system (1.2) for different values $b \in (2, 3)$ with the parameter values $a = 2.5, e = 0.2, c = 1, H = 1$ and $h = 0.9$ the initial conditions $(x_0, y_0) = (0.3, 0.4)$.

of the system (1.2) is stable for $b < 2.3111$ and loses its stability at $b = 2.3111$, and close invariant curves appear around the positive equilibrium point if $b > 2.3111$. More clearly, when the parameter b exceeds 2.3111, the equilibrium point is unstable and a repelling invariant closed curve bifurcates from the positive equilibrium point. The phase portaits of bifurcation diagrams in Figure 2(a) and 2(b) for different values of b are shown in Figure 3, which clearly demonstrates the process how smooth invariant curve bifurcates from the stable equilibrium point and increases its radius.

Example 3. We consider system (1.2) with the parameter values $a = 2.5$, $e = 0.2$, $c = 1$, $H = 1$, $h = 0.9$, and initial condition $(x_0, y_0) = (0.3, 0.4)$. When $2 \leq b <$

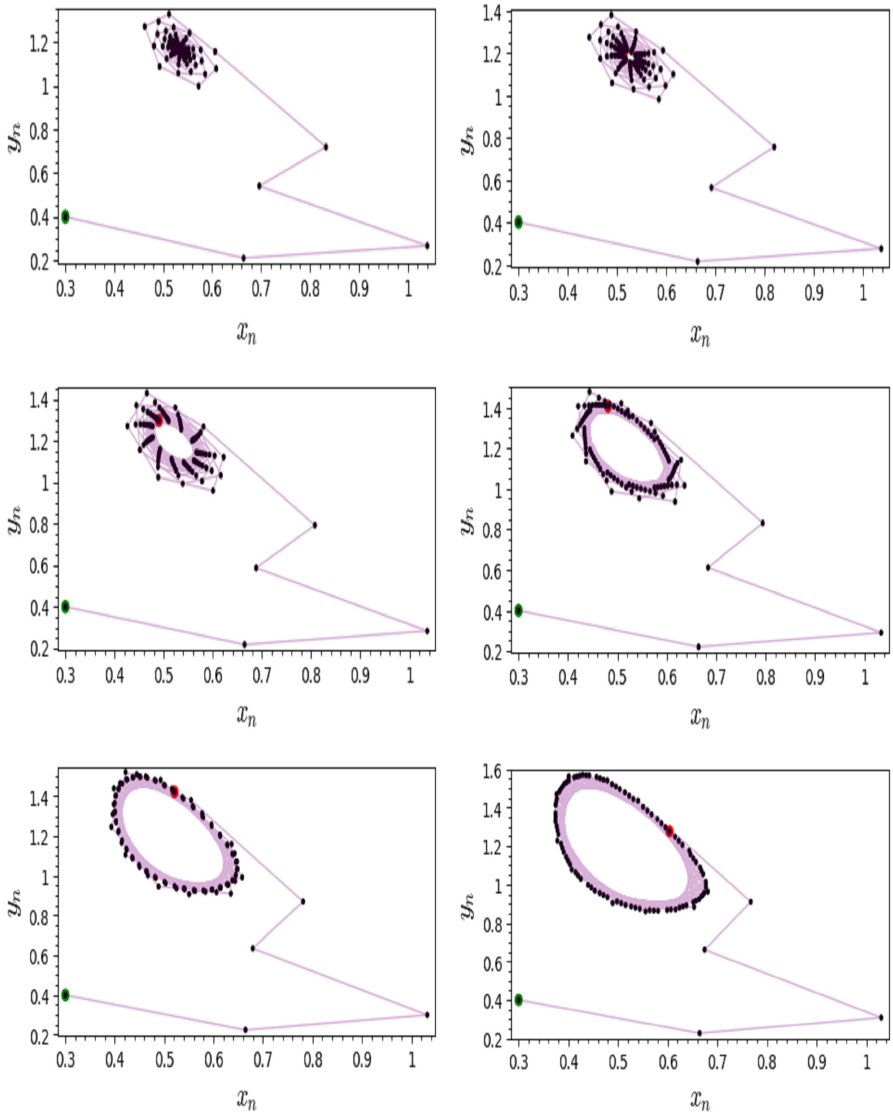

FIGURE 3. The phase portraits of the prey-predator system (1.2) for different values of b with the parameter values $a = 2.5$, $e = 0.2$, $c = 1$, $H = 1$, $h = 0.9$ and the initial conditions $(x_0, y_0) = (0.3, 0.4)$.

2.3111, the positive equilibrium points $(\bar{x}, \bar{y})$ are locally asymptotic stable and this equilibrium point $(\bar{x}, \bar{y})$ are unstable for $2.3111 \leq b \leq 3$. Furthermore, at the value

$b = 2.3111$, the positive equilibrium point $(\bar{x},\bar{y})$ becomes unstable and also is produced closed invariant curve enclosing the unique positive unstable equilibrium point $(\bar{x},\bar{y})$. So, Neimark Sacker bifurcation at $b = 2.3111$ can be emerge. To clarify this situation, we obtain, $\lambda^2 - 0.837124\lambda + 1.00541 = 0$, the characteristic equation of Jacobian matrix evaluated at $(\bar{x},\bar{y}) = (0.519233, 1.20192)$ with the parametric values $(e,a,c,H,h,b) = (0.2,2.5,1,1,0.9,2.3111)$. The eigenvalues of characteristic equation are $\lambda_{1,2} = 0.415862 \mp 0.909422i$ with $|\lambda_{1,2}| = 1$. Consequently; we say that $(e,a,c,H,h,b) = (0.2,2.5,1,1,0.9,2.3111) \in NSB_{E_2}$

Example 4. In this example, we apply a state feedback control method by taking parameter values $(e,a,c,H,h,b) = (0.2,2.5,1,1,0.9,2.5)$ and initial condition $(x_0,y_0) = (0.3,0.4)$. Then corresponding controlled form is given by

$$\begin{aligned} x_{n+1} &= x_n + 0.9(2.5x_n(1 - x_n - x_n y_n) - k_1(x_n - 0.48) - k_2(y_n - 1.3) \\ y_{n+1} &= y_n + 0.9[(2.5x_n y_n - 0.2y_n - y_n]. \end{aligned} \tag{5.2}$$

If marginal lines l_1, l_2 and l_3 in Eq.(4.5)-(4.7) are considered, we get

$$\begin{aligned} l_1 &= -0.1836 + k_1 - 2.925k_2 = 0 \\ l_2 &= 3.1036 - 2k_1 + 2.925k_2 = 0 \end{aligned}$$

and

$$l_3 = 1.2636 + 2.925k_2 = 0.$$

The marginal lines l_1, l_2 and l_3 determine the stable triangular region in the k_1k_2 plane. The region bounded by this lines of the controlled system (5.2) is plotted in Figure 4.

6. Conclusions

This paper contains the complex dynamic behavior of a discrete-time the prey-predator system (1.2). We investigate stability conditions of the equilibrium points of the system (1.2), and show that the system (1.2) displays a Neimark-Sacker bifurcations at coexistence equilibrium point. So, we apply the stabilizing feedback control methods to avoid chaos.

We find that the system (1.2) has a trivial (extinction) equilibrium E_0, an exclusion equilibrium E_1 and a coexistence equilibrium E_2. We get the asymptotic stability conditions of these equilibria by using the linearization method. It is clear that there is a unique positive coexistence equilibrium E_2 of the system (1.2) with $b > e + H$. Moreover, it is proved that the system (1.2) undergoes Neimark-Sacker bifurcation under the condition $b = \frac{1+eh+hH}{h}$ by using mathematical techniques of bifurcation theory. It is seen that the Neimark–Saker bifurcation appears when the parameters vary on the neighborhood

$$NSB_{E_2} = \left\{ a,b,e \in \mathbb{R}^+ : b > \frac{e+H}{2} + \frac{1}{2}\sqrt{ae + e^2 + aH + 2eH + H^2} \text{ and } b = b_1 \right\}.$$

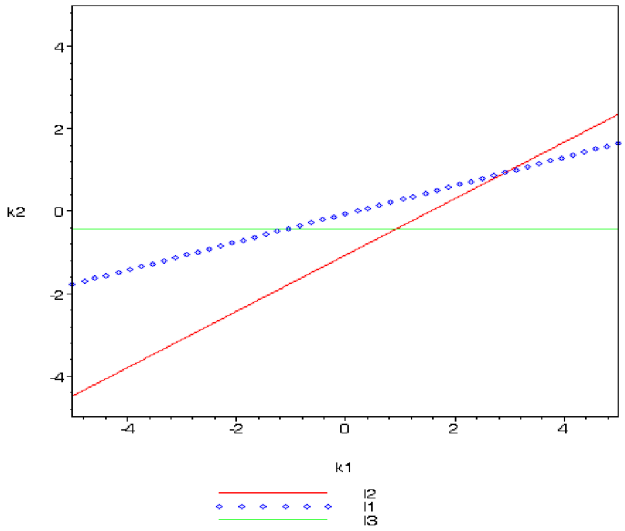

FIGURE 4. Stability region of the controlled system (5.2) in k_1k_2 plane

Some Figures present the dynamic properties of system (1.2) which has an chaos for $b > 2.3111$, $e = 0.2$, $a = 2.5$, $c = 1$, $H = 1$, $h = 0.9$ and the initial conditions $(x_0, y_0) = (0.3, 0.4)$. We conclude that the parameter b has a different effects on the dynamics of the system (1.2). Later, we obtain the conditions to control the chaos of the model via a state feedback method with the parameter $e = 0.2$, $a = 2.5$, $b = 2.5$, $c = 1$, $H = 1$, $h = 0.9$. Chaos is successfully controlled by using control strategy.

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THE CONVERGENCE OF EXPONENTIAL OPERATORS CONNECTED WITH x^3 ON FUNCTIONS OF BOUNDED VARIATION

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Abstract. In the present paper, we estimate the rate of convergence of exponential type operators connected with x^3 on functions of bounded variation.

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Keywords: exponential operators, bounded variation, rate of convergence

1. Exponential operators

For $x \in (0,\infty)$ one of the exponential operators introduced in [11, (3.11)] is defined as

$$(Q_n f)(x) = \int_0^\infty \phi_n(x,t) f(t) dt, \tag{1.1}$$

where the kernel is given by

$$\phi_n(x,t) = \sqrt{\frac{n}{2\pi}} \frac{1}{t\sqrt{t}} \exp\left(\frac{n}{x} - \frac{nt}{2x^2} - \frac{n}{2t}\right).$$

These operators satisfy the partial differential equation

$$x^3 \frac{\partial}{\partial x}[(Q_n f)(x)] = n(Q_n \psi_x(t) f)(x),$$

where $\psi_x(t) = t - x$. As per our knowledge these operators have not been studied by researchers in the last four decades due to their complicated behaviour. Very recently in [6] the author estimated moments using the concept of moment generating function and established some of the approximation properties of these operators. In the past several years the convergence estimation of many well-know operators is an active area of research amongst researchers. Many known operators have been appropriately modified and their approximation behaviours have been discussed. In this direction some summation-integral operators were introduced and studied in [4, 10, 15]. For the statistical convergence, we refer the readers to [13], concerning difference between two operators [7], simultaneous approximation [9], Bézier

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bases with shape parameter [16] and moment estimations of a generalized class of Szász–Mirakyan–Durrmeyer operators [8].

The convergence rate on functions of bounded variation is also an important area of research in the recent past decades, we mention here some of the work done earlier on different operators viz. Bézier variant of the Baskakov-Kantorovich operators [1], exponential operators connected with $p(x) = 2x^{3/2}$ [2], MKZ operators [3], Baskakov-Durrmeyer type operators [4], Baskakov Bézier operators [5], nonlinear integral operators [12], Bézier variant of the Bleimann–Butzer–Hahn operators [14], Kantorovich variant of the Bleimann, Butzer and Hahn operators [18], Szász–Bézier integral operators [17], general family of operators of Durrmeyer type [15] etc. Also some better bounds to have different basis were established in [19].

We estimate in the present article the rate of convergence for the operators Q_n on functions of bounded variation, by using the optimum upper bound of the basis function.

2. Auxiliary Lemmas

Lemma 1. *If we denote* $\nu_{n,m}(x) = (Q_n(\psi_x^m(t))(x), m = 0,1,2,\cdots$, *then following [[6, Remark 1]], we have*

$$\nu_{n,m}(x) = \left[\frac{\partial^m}{\partial F^m}\left\{\exp\left(\frac{n}{x}\left(1-\sqrt{\frac{n-2x^2F}{n}}\right)-xF\right)\right\}\right]_{F=0}.$$

Some of the central moments of the operators are given by

$$\nu_{n,0}(x) = 1, \quad \nu_{n,1}(x) = 0, \quad \nu_{n,2}(x) = \frac{x^3}{n}.$$

Also, for each $x \in (0,\infty)$, *we get* $\nu_{n,m}(x) = O_x(n^{-[(m+1)/2]})$, *where* $[\alpha]$ *represents the integral part of* α.

Lemma 2. *Let* $x \in (0,\infty)$ *and the kernel* $\phi_n(x,t)$ *is defined in (1.1), then we have*

$$\eta_n(x,y) = \int_0^y \phi_n(x,t)dt \le \frac{x^3}{n(x-y)^2}, \quad 0 < y < x$$

$$1-\eta_n(x,z) = \int_z^\infty \phi_n(x,t)dt \le \frac{x^3}{n(z-x)^2}, \quad x < z < \infty.$$

Lemma 3. *For each* $x \in (0,\infty)$, *we have*

$$\int_0^x \phi_n(x,t)dt \le \frac{1}{2} + \frac{\sqrt{x}}{2\sqrt{2\pi n}}.$$

Proof. Simple analysis leads us to

$$\int_0^x \phi_n(x,t)dt = \sqrt{\frac{n}{2\pi}}\int_0^x \frac{1}{t\sqrt{t}}\exp\left(\frac{n}{x}-\frac{nt}{2x^2}-\frac{n}{2t}\right)dt$$

$$
\begin{aligned}
&= \sqrt{\frac{n}{2\pi x}} \int_0^1 \frac{2}{t^2} \exp\left[-\frac{n}{2x}\left(t-\frac{1}{t}\right)^2\right] dt \\
&= \sqrt{\frac{n}{2\pi x}} \left[\int_0^1 \left(1+\frac{1}{t^2}\right) \exp\left[-\frac{n}{2x}\left(t-\frac{1}{t}\right)^2\right] dt \right. \\
&\quad \left. - \exp\left(\frac{2n}{x}\right) \int_0^1 \left(1-\frac{1}{t^2}\right) \exp\left[-\frac{n}{2x}\left(t+\frac{1}{t}\right)^2\right] dt\right] \\
&= \sqrt{\frac{n}{2\pi x}} \left[\int_{-\infty}^0 \exp\left(-\frac{n}{2x}z^2\right) dz + \exp\left(\frac{2n}{x}\right) \int_2^\infty \exp\left(-\frac{n}{2x}z^2\right) dz\right] \\
&\le \sqrt{\frac{n}{2\pi x}} \left[\int_{-\infty}^0 \exp\left(-\frac{n}{2x}z^2\right) dz + \exp\left(\frac{2n}{x}\right) \int_2^\infty \frac{z}{2} \exp\left(-\frac{n}{2x}z^2\right) dz\right] \\
&= \sqrt{\frac{n}{2\pi x}} \left[\frac{1}{2}\left(\frac{2x\pi}{n}\right)^{1/2} + \frac{x}{2n}\right] = \frac{1}{2} + \frac{\sqrt{x}}{2\sqrt{2\pi n}}.
\end{aligned}
$$

Thus the desired result follows. □

3. Rate of Convergence

Theorem 1. *Let f be a function of bounded variation on each finite subinterval of $(0,\infty)$ satisfying the growth condition $f(t) = O(e^{\gamma t})$ as $t \to +\infty$. Then, for n large, there holds*

$$
\begin{aligned}
&\left|(Q_n f)(x) - \frac{1}{2}(f(x+)+f(x-))\right| \\
&\le \frac{\sqrt{x}}{2\sqrt{2\pi n}} |f(x+)-f(x-)| + \frac{2x+1}{n} \sum_{k=1}^{n} v_{x_k^-}^{x_k^+}(f_x) \\
&\quad + \frac{xe^{\gamma x}}{n} + \frac{\sqrt{x}}{\sqrt{n}} \left(\exp\left(2\gamma x + \frac{x^3(2\gamma)^2}{2n} + \frac{x^5(2\gamma)^3}{2n^2} + \cdots\right)\right)^{1/2},
\end{aligned}
$$

where $x_k^- = x - x/\sqrt{k}$, $x_k^+ = x + x/\sqrt{k}$ and

$$
f_x(t) = \begin{cases} f(t)-f(x-), & 0 < t < x \\ f(t)-f(x+), & x < t < \infty \\ 0, & t = x. \end{cases}
$$

Proof. Let $x \in (0,\infty)$, starting with

$$
\begin{aligned}
&\left|(Q_n f)(x) - \frac{1}{2}(f(x+)+f(x-))\right| \\
&\qquad \le \frac{1}{2}|f(x+)-f(x-)|.|(Q_n sign(t-x))(x)| + |(Q_n f_x)(x)|.
\end{aligned}
$$

Using the constant preservation of the operators i.e $(Q_n e_0)(1) = 1$, we have

$$\begin{aligned}(Q_n sign(t-x))(x) &= \left(\int_x^\infty - \int_0^x\right) \phi_n(x,t)dt \\ &= 2\left[\frac{1}{2} - \int_0^x \phi_n(x,t)dt\right].\end{aligned}$$

Thus by Lemma 3, we have

$$|(Q_n sign(t-x))(x)| \leq \frac{\sqrt{x}}{\sqrt{2\pi n}}.$$

Next we estimate $(Q_n f_x)(x)$ as follows:

$$\begin{aligned}(Q_n f_x)(x) &= \int_0^\infty \phi_n(x,t) f_x(t)dt \\ &= \left(\int_0^{x_n^-} + \int_{x_n^-}^{x_n^+} + \int_{x_n^+}^\infty\right) \phi_n(x,t) f_x(t)dt \\ &=: e_1 + e_2 + e_3.\end{aligned}$$

First, integrating by parts

$$\begin{aligned}e_1 &= \int_0^{x_n^-} f_x(t) d_t(\eta_n(x,t)) \\ &= f_x(x_n^-)\eta_n(x,x_n^-) - \int_0^{x_n^-} \eta_n(x,t) d_t(f_x(t)).\end{aligned}$$

Since $|f_x(y)| \leq v_y^x(f_x)$, we have

$$|e_1| \leq v_{x_n^-}^x(f_x)\eta_n(x,x_n^-) + \int_0^{x_n^-} \eta_n(x,t) d_t(-v_t^x(f_x)).$$

Applying Lemma 2, and in the next step integrating by parts, we get

$$\begin{aligned}|e_1| &\leq 2v_{x_n^-}^x(f_x) + \frac{x^3}{n}\int_0^{x_n^-} \frac{1}{(x-t)^2} d_t(-v_t^x(f_x)) \\ &= \frac{x^3}{n}\left[\frac{1}{x^2}v_0^x(f_x) + 2\int_0^{x_n^-} \frac{1}{(x-t)^3} v_t^x(f_x)dt\right] \\ &= \frac{x^3}{n}\left[\frac{1}{x^2}v_0^x(f_x) + \frac{1}{x^2}\sum_{k=1}^n v_{x_k^-}^x(f_x)\right] \\ &\leq \frac{2x}{n}\sum_{k=1}^n v_{x_k^-}^x(f_x).\end{aligned}$$

Next for $t \in [x_n^-, x_n^+]$ and by fact $\int_{x_n^-}^{x_n^+} d_t(\eta_n(x,t)) \leq 1$, we conclude that

$$|e_2| \leq \frac{1}{n} \sum_{k=1}^{n} v_{x_k^-}^{x_k^+}(f_x).$$

Finally

$$\begin{aligned} e_3 &= \left(\int_{x_n^+}^{2x} + \int_{2x}^{\infty} \right) \phi_n(x,t) dt \\ &=: e_{31} + e_{32}. \end{aligned}$$

Arguing analogously as in estimate of e_1, we have

$$|e_{31}| \leq \frac{2x}{n} \sum_{k=1}^{n} v_x^{x_k^+}(f_x).$$

Using the growth $f_x(t) = O(e^{\gamma t}), t \to \infty$, applying Lemma 1 and [6, Lemma 1], we have

$$\begin{aligned} |e_{32}| &\leq \int_{2x}^{\infty} \phi_n(x,t)(e^{\gamma x} + e^{\gamma t}) dt \\ &\leq \frac{e^{\gamma x}}{x^2} \int_0^{\infty} \phi_n(x,t)(t-x)^2 dt + \frac{1}{x} \int_0^{\infty} \phi_n(x,t) e^{\gamma t} |t-x| dt \\ &= \frac{e^{\gamma x}}{x^2} . \nu_{n,2}(x) + \frac{1}{x} . \left(\nu_{n,2}(x) . Q_n(e^{2\gamma t}, x) \right)^{1/2} \\ &= \frac{x e^{\gamma x}}{n} + \frac{\sqrt{x}}{\sqrt{n}} \left(\exp\left(2\gamma x + \frac{x^3 (2\gamma)^2}{2n} + \frac{x^5 (2\gamma)^3}{2n^2} + \cdots \right) \right)^{1/2}. \end{aligned}$$

Collecting the estimates of e_1, e_2, e_3, we get the desired result. □

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ON G-RADICAL SUPPLEMENT SUBMODULES

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Abstract. In this work, we give some new properties of Rad-supplement and g-radical supplement submodules. Let V be a g-radical supplement of U in M and U or V be essential submodule of M. Then $Rad_gV = V \cap Rad_gM$. Let V be a g-radical supplement of U in M, U or V be essential submodule of M and $x \in V$. Then $Rx \ll_g V$ if and only if $Rx \ll_g M$. In this work, some relations between Rad-supplement, g-radical supplement, β^* and β_g^* relations are also studied. Let $X\beta_g^*Y$ in M. If V is a g-radical supplement of X in M and $V \trianglelefteq M$, then V is also a g-radical supplement of Y in M. Let M be an R-module. It is proved that M is semilocal (g-semilocal) if every submodule of M β^* equivalent to a Rad-supplement (g-radical supplement) submodule in M.

2010 *Mathematics Subject Classification:* 16D10; 16D70

Keywords: small submodules, radical, g-supplemented modules, g-radical supplemented modules

1. INTRODUCTION

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an R-module. We will denote a submodule N of M by $N \leq M$. Let M be an R-module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small submodule* of M and denoted by $N \ll M$. Let M be an R-module and $N \leq M$. If there exists a submodule K of M such that $M = N + K$ and $N \cap K = 0$, then N is called a *direct summand* of M and it is denoted by $M = N \oplus K$. A submodule N of an R-module M is called an *essential submodule* of M, denoted by $N \trianglelefteq M$, in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equivalently, $N \cap K = 0$ implies that $K = 0$. Let M be an R-module and K be a submodule of M. K is called a *generalized small* (briefly, *g-small*) *submodule* of M if for every $T \trianglelefteq M$ with $M = K + T$ implies that $T = M$, this is written by $K \ll_g M$ (in [13], it is called an *e-small submodule* of M and denoted by $K \ll_e M$). It is clear that every small submodule is a generalized small submodule but the converse is not true generally. Let U and V be submodules of M. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V << V$, then V is called a *supplement* of U in M. M is called a *supplemented module* if every

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submodule of M has a supplement in M. Let M be an R-module and $U,V \leq M$. If $M = U+V$ and $M = U+T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U+V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. The intersection of all maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$. M is said to be *semilocal* if $M/RadM$ is semisimple, i. e. every submodule of $M/RadM$ is a direct summand of $M/RadM$. Let M be an R-module and $U,V \leq M$. If $M = U+V$ and $U \cap V \leq RadV$, then V is called a *generalized* (*radical*) *supplement* (briefly *Rad-supplement*) of U in M. M is said to be *generalized* (*radical*) *supplemented* (briefly *Rad-supplemented*) if every submodule of M has a Rad-supplement in M. The intersection of all essential maximal submodules of an R-module M is called the *generalized radical* (briefly *g-radical*) of M and denoted by Rad_gM (in [13], it is denoted by Rad_eM). If M have no essential maximal submodules, then we denote $Rad_gM = M$. If M/Rad_gM is semisimple, i. e. every submodule of M/Rad_gM is a direct summand of M/Rad_gM, then M is called a *g-semilocal* module. Let M be an R-module. We say submodules X and Y of M are β^* equivalent, $X\beta^*Y$, if and only if $Y+K = M$ for every $K \leq M$ such that $X+K = M$ and $X+T = M$ for every $T \leq M$ such that $Y+T = M$. We say submodules X and Y of M are β_g^* equivalent, $X\beta_g^*Y$, if and only if $Y+K = M$ for every $K \trianglelefteq M$ such that $X+K = M$ and $X+T = M$ for every $T \trianglelefteq M$ such that $Y+T = M$. Let M be an R-module and $X \leq Y \leq M$. If $Y/X \ll M/X$, then we say Y *lies above* X in M.

More informations about supplemented modules are in [2, 8, 12]. More informations about g-small submodules and g-supplemented modules are in [3, 7, 10]. The definition of generalized supplemented modules and some properties of them are in [11]. The definition of g-semilocal modules and some properties of them are in [5]. The definition of β^* relation and some results of this relation are in [1]. The definition of β_g^* relation and some results of this relation are in [9].

Lemma 1. *Let M be an R-module. The following assertions hold.*

(1) *For every $m \in Rad_gM$, $Rm \ll_g M$.*
(2) *If $N \leq M$, then $Rad_gN \leq Rad_gM$.*
(3) $Rad_gM = \sum_{L \ll_g M} L$.

Proof. See [4, Lemma 2 and Lemma 3]. □

2. G-RADICAL SUPPLEMENT SUBMODULES

Definition 1. Let M be an R-module and $U,V \leq M$. If $M = U+V$ and $U \cap V \leq Rad_gV$, then V is called a g-radical supplement of U in M. If every submodule of M has a g-radical supplement in M, then M is called a g-radical supplemented module. (See [4, 6].)

Clearly we can see that every g-supplemented module is g-radical supplemented. But the converse is not true in general. Every Rad-supplemented module is g-radical supplemented.

Lemma 2. *Let V be a g-radical supplement of U in M and $U \trianglelefteq M$. Then $Rad_gV = V \cap Rad_gM$.*

Proof. By Lemma 1, $Rad_gV \leq V \cap Rad_gM$. Let T be an essential maximal submodule of V. Then $U \cap V \leq Rad_gV \leq T$ holds. By $\frac{M}{U+T} = \frac{U+T+V}{U+T} \cong \frac{V}{V\cap(U+T)} =$ $= \frac{V}{U\cap V+T} = \frac{V}{T}$ and $U+T \trianglelefteq M$, $U+T$ is an essential maximal submodule of M and $Rad_gM \leq U+T$. Hence $V \cap Rad_gM \leq V \cap (U+T) = U \cap V + T = T$. Thus $Rad_gV = V \cap Rad_gM$, as desired. □

Theorem 1. *Let V be a g-radical supplement of U in M, $U \trianglelefteq M$ and $x \in V$. Then $Rx \ll_g V$ if and only if $Rx \ll_g M$.*

Proof.

$\Longrightarrow$ Clear.

$\Longleftarrow$ Since $Rx \ll_g M$, by Lemma 1, $Rx \leq Rad_gM$ and $x \in Rad_gM$. Then $x \in V \cap Rad_gM$. By Lemma 2, $Rad_gV = V \cap Rad_gM$. Hence $x \in Rad_gV$ and by Lemma 1, $Rx \ll_g V$. We can also prove this part as follows:

Let T be an essential maximal submodule of V. Here $U \cap V \leq Rad_gV \leq T$. Assume that $Rx \nleq T$. Then $Rx+T = V$ and $M = U+V = U+Rx+T$. Since $Rx \ll_g M$ and $U+T \trianglelefteq M$, $U+T = M$. Then $V = V \cap M = V \cap (U+T) =$ $= U \cap V + T = T$, a contradiction. Hence $Rx \leq T$ for every essential maximal submodule T of V and $Rx \leq Rad_gV$. Thus $x \in Rad_gV$ and by Lemma 1, $Rx \ll_g V$.

□

Corollary 1. *Let V be a Rad-supplement of U in M and $U \trianglelefteq M$. Then $Rad_gV = V \cap Rad_gM$.*

Proof. Clear from Lemma 2. □

Corollary 2. *Let V be a Rad-supplement of U in M, $U \trianglelefteq M$ and $x \in V$. Then $Rx \ll_g V$ if and only if $Rx \ll_g M$.*

Proof. Clear from Theorem 1. □

Theorem 2. *Let V be a g-radical supplement of U in M, $V \trianglelefteq M$ and $x \in V$. The following assertions hold.*

(1) $Rad_gV = V \cap Rad_gM$.
(2) $Rx \ll_g V$ *if and only if* $Rx \ll_g M$.

Proof.

(1) By Lemma 1, $Rad_gV \leq V \cap Rad_gM$. Let T be an essential maximal submodule of V. Then $U \cap V \leq Rad_gV \leq T$ holds. Since $T \trianglelefteq V$ and $V \trianglelefteq M$, then $T \trianglelefteq M$ and $U+T \trianglelefteq M$. Then by $\frac{M}{U+T} = \frac{U+T+V}{U+T} \cong \frac{V}{V\cap(U+T)} = \frac{V}{U\cap V+T} = \frac{V}{T}$, $U+T$ is an essential maximal submodule of M and $Rad_gM \leq U+T$. Hence $V \cap Rad_gM \leq V \cap (U+T) = U \cap V + T = T$. Thus $Rad_gV = V \cap Rad_gM$, as desired.

(2) $\Longrightarrow$ Clear.

$\Longleftarrow$ Since $Rx \ll_g M$, by Lemma 1, $Rx \leq Rad_gM$ and $x \in Rad_gM$. Then $x \in V \cap Rad_gM$. By Theorem 2 (1), $Rad_gV = V \cap Rad_gM$. Hence $x \in Rad_gV$ and by Lemma 1, $Rx \ll_g V$. We can also prove this part as follows:

Let T be an essential maximal submodule of V. Here $U \cap V \leq Rad_gV \leq \leq T$. Assume that $Rx \nleq T$. Then $Rx + T = V$ and $M = U + V = = U + Rx + T$. Since $T \trianglelefteq V$ and $V \trianglelefteq M$, then $T \trianglelefteq M$ and $U+T \trianglelefteq M$. Since $Rx \ll_g M$, $U+T = M$. Then $V = V \cap M = V \cap (U+T) = = U \cap V + T = T$, a contradiction. Hence $Rx \leq T$ for every essential maximal submodule T of V and $Rx \leq Rad_gV$. Thus $x \in Rad_gV$ and by Lemma 1, $Rx \ll_g V$.

□

Corollary 3. *Let V be a Rad-supplement of U in M and $V \trianglelefteq M$. Then $Rad_gV = V \cap Rad_gM$.*

Proof. Clear from Theorem 2 (1). □

Corollary 4. *Let V be a Rad-supplement of U in M, $V \trianglelefteq M$ and $x \in V$. Then $Rx \ll_g V$ if and only if $Rx \ll_g M$.*

Proof. Clear from Theorem 2 (2). □

Example 1. Consider the $\mathbb{Z}$-module $\mathbb{Q}$. For ${}_{\mathbb{Z}}\mathbb{Z} \leq_{\mathbb{Z}} \mathbb{Q}$, $Rad_g\mathbb{Z} = Rad\mathbb{Z} = 0$. Since $Rad_g\mathbb{Q} = Rad\mathbb{Q} = \mathbb{Q}$, $\mathbb{Z} \cap Rad_g\mathbb{Q} = \mathbb{Z} \cap \mathbb{Q} = \mathbb{Z}$. Hence $Rad_g\mathbb{Z} \neq \mathbb{Z} \cap Rad_g\mathbb{Q}$.

Proposition 1. *Let $X\beta_g^*Y$ in M. If V is a g-radical supplement of X in M and $V \trianglelefteq M$, then V is also a g-radical supplement of Y in M.*

Proof. By hypothesis, $M = X + V$ and $X \cap V \leq Rad_gV$. Since $X\beta_g^*Y$ and $V \trianglelefteq M$, $Y + V = M$. Let T be any essential maximal submodule of V. Since $T \trianglelefteq V$ and $V \trianglelefteq M$, then $T \trianglelefteq M$. Assume that $Y \cap V \nleq T$. Then $Y \cap V + T = V$. Here $M = = Y + V = Y + Y \cap V + T = Y + T$ and since $X\beta_g^*Y$ and $T \trianglelefteq M$, $X + T = M$. Then $V = V \cap M = V \cap (X+T) = V \cap X + T$ and since $X \cap V \leq Rad_gV \leq T$, $V = V \cap X + T = T$. This is a contradiction. Hence $Y \cap V \leq T$ for every essential maximal submodule of V and $Y \cap V \leq Rad_gV$. Thus V is a g-radical supplement of Y in M. □

Lemma 3. *Let $X\beta^*Y$ in M. If X and Y have Rad-supplements in M, then they have the same Rad-supplements in M.*

Proof. Let C be a Rad-supplement of X in M. Then $M = X + C$ and $X \cap C \leq RadC$. Since $X\beta^* Y$, $Y + C = M$. Let T be any maximal submodule of C. Assume that $Y \cap C \not\leq T$. Then $Y \cap C + T = C$. Here $M = Y + C = Y + Y \cap C + T = Y + T$ and since $X\beta^* Y$, $X + T = M$. Then $C = C \cap M = C \cap (X + T) = X \cap C + T$ and since $X \cap C \leq RadC \leq T$, $C = X \cap C + T = T$. This is a contradiction. Hence $Y \cap C \leq T$ for every maximal submodule of C and $Y \cap C \leq RadC$. Thus C is a Rad-supplement of Y in M. Similarly, the interchanging the roles of X and Y, we can prove that every Rad-supplement of Y in M is also a Rad-supplement of X in M. □

Corollary 5. *Let X lies above Y in M. If X and Y have Rad-supplements in M, then they have the same Rad-supplements in M.*

Proof. Clear from Lemma 3. □

Lemma 4. *Let $X\beta^* Y$ in M. If X has a g-radical supplement V in M, then V is also a g-radical supplement of Y in M.*

Proof. By hypothesis, $M = X + V$ and $X \cap V \leq Rad_g V$. Since $X\beta^* Y$, $Y + V = M$. Let T be any essential maximal submodule of V. Assume that $Y \cap V \not\leq T$. Then $Y \cap V + T = V$. Here $M = Y + V = Y + Y \cap V + T = Y + T$ and since $X\beta^* Y$, $X + T = M$. Then $V = V \cap M = V \cap (X + T) = X \cap V + T$ and since $X \cap V \leq$ $\leq Rad_g V \leq T$, $V = V \cap X + T = T$. This is a contradiction. Hence $Y \cap V \leq T$ for every essential maximal submodule of V and $Y \cap V \leq Rad_g V$. Thus V is a g-radical supplement of Y in M. □

Corollary 6. *Let X lies above Y in M. If X and Y have g-radical supplements in M, then they have the same g-radical supplements in M.*

Proof. Clear from Lemma 4. □

Lemma 5. *Let $X\beta^* Y$ and Y be a Rad-supplement of U in M. Then $U \cap X \leq RadM$.*

Proof. Since Y is a Rad-supplement of U in M, $M = U + Y$ and $U \cap Y \leq RadY \leq$ $\leq RadM$. Since $M = U + Y$ and $X\beta^* Y$, $M = U + X$. Let T be any maximal submodule of M. Here $U \cap Y \leq RadM \leq T$. Assume that $U \cap X \not\leq T$. Then $U \cap X + T = M$ and since $M = U + X$, by [2, Lemma 1.24], $X + U \cap T = M$. Since $X\beta^* Y$, $Y + U \cap T = M$ and since $U + T = M$, by [2, Lemma 1.24] again, $U \cap Y + T = M$. Then by $U \cap Y \leq T$, $M = U \cap Y + T = T$. This is a contradiction. Hence $U \cap X \leq T$ for every maximal submodule T of M and $U \cap X \leq RadM$. □

Corollary 7. *Let X lies above Y and Y be a Rad-supplement of U in M. Then $U \cap X \leq RadM$.*

Proof. Clear from Lemma 5. □

Lemma 6. *Let M be an R-module. If every submodule of M is β^* equivalent to a Rad-supplement submodule in M, then M is semilocal.*

Proof. Let $X/RadM \leq M/RadM$. Since $X \leq M$, by hypothesis, there exists a Rad-supplement submodule Y in M such that $X\beta^*Y$. Let Y be a Rad-supplement of U in M. By Lemma 5, $U \cap X \leq RadM$. Since $X\beta^*Y$ and $Y+U=M$, $X+U=M$. Then $\frac{M}{RadM} = \frac{X+U}{RadM} = \frac{X}{RadM} + \frac{U+RadM}{RadM}$ and $\frac{X}{RadM} \cap \frac{U+RadM}{RadM} = \frac{X\cap(U+RadM)}{RadM} = \frac{U\cap X+RadM}{RadM} = = \frac{RadM}{RadM} = 0$. Hence $\frac{M}{RadM} = \frac{X}{RadM} \oplus \frac{U+RadM}{RadM}$ and $M/RadM$ is semisimple. Thus M is semilocal. □

Corollary 8. *Let M be an R-module. If every submodule of M lies above a Rad-supplement submodule in M, then M is semilocal.*

Proof. Clear from Lemma 6. □

Theorem 3. *Let $X\beta^*Y$ and Y be a g-radical supplement of U in M. Then $U \cap X \leq Rad_gM$.*

Proof. Since Y is a g-radical supplement of U in M, $M = U+Y$ and $U \cap Y \leq \leq Rad_gY \leq Rad_gM$. Since $M = U+Y$ and $X\beta^*Y$, $M = U+X$. Let T be any essential maximal submodule of M. Here $U \cap Y \leq Rad_gM \leq T$. Assume that $U \cap X \nleq T$. Then $U \cap X + T = M$ and since $M = U+X$, by [2, Lemma 1.24], $X+U \cap T = M$. Since $X\beta^*Y$, $Y+U \cap T = M$ and since $U+T=M$, by [2, Lemma 1.24] again, $U \cap Y + T = M$. Then by $U \cap Y \leq T$, $M = U \cap Y + T = T$. This is a contradiction. Hence $U \cap X \leq T$ for every essential maximal submodule T of M and $U \cap X \leq Rad_gM$. □

Corollary 9. *Let X lies above Y and Y be a g-radical supplement of U in M. Then $U \cap X \leq Rad_gM$.*

Proof. Clear from Theorem 3. □

Theorem 4. *Let M be an R-module. If every submodule of M is β^* equivalent to a g-radical supplement submodule in M, then M is g-semilocal.*

Proof. Let $X/Rad_gM \leq M/Rad_gM$. Since $X \leq M$, by hypothesis, there exists a g-radical supplement submodule Y in M such that $X\beta^*Y$. Let Y be a g-radical supplement of U in M. By Theorem 3, $U \cap X \leq Rad_gM$. Since $X\beta^*Y$ and $Y+U=M$, $X+U=M$. Then $\frac{M}{Rad_gM} = \frac{X+U}{Rad_gM} = \frac{X}{Rad_gM} + \frac{U+Rad_gM}{Rad_gM}$ and $\frac{X}{Rad_gM} \cap \frac{U+Rad_gM}{Rad_gM} = = \frac{X\cap(U+Rad_gM)}{Rad_gM} = \frac{U\cap X+Rad_gM}{Rad_gM} = \frac{Rad_gM}{Rad_gM} = 0$. Hence $\frac{M}{Rad_gM} = \frac{X}{Rad_gM} \oplus \frac{U+Rad_gM}{Rad_gM}$ and M/Rad_gM is semisimple. Thus M is g-semilocal. □

Corollary 10. *Let M be an R-module. If every submodule of M lies above a g-radical supplement submodule in M, then M is g-semilocal.*

Proof. Clear from Theorem 4. □

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ON EIGHT SOLVABLE SYSTEMS OF DIFFERENCE EQUATIONS IN TERMS OF GENERALIZED PADOVAN SEQUENCES

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Abstract. In this study we show that the systems of difference equations

$$x_{n+1}=f^{-1}\left(af\left(p_{n-1}\right)+bf\left(q_{n-2}\right)\right),\ \ y_{n+1}=f^{-1}\left(af\left(r_{n-1}\right)+bf\left(s_{n-2}\right)\right),$$

for $n\in\mathbb{N}_0$, where the sequences p_n, q_n, r_n and s_n are some of the sequences x_n and y_n, $f:D_f\longrightarrow\mathbb{R}$ is a "$1-1$" continuous function on its domain $D_f\subseteq\mathbb{R}$, initial values x_{-j}, y_{-j}, $j\in\{0,1,2\}$, are arbitrary real numbers in D_f and the parameters a,b are arbitrary complex numbers, with $b\neq 0$, can be explicitly solved in terms of generalized Padovan sequences. Some analytical examples are given to demonstrate the theoretical results.

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Keywords: system of difference equations, solution of explicit form, Padovan number

1. INTRODUCTION

Firstly, recall that $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, stand for natural, non-negative integer, integer, real and complex numbers, respectively. If $m,n\in\mathbb{Z}$, $m\leq n$ the notation $i=\overline{m,n}$ stands for $\{i\in\mathbb{Z}:m\leq i\leq n\}$.

Difference equations for which the solutions can be constructed explicitly are useful due to numerous applications. As particular, difference equations related to Fibonacci, Lucas, Padovan, Tetranacci, Horadam, Pell, Jacobsthal, and Jacobsthal-Lucas sequences and their generalizations are of much interest. Many related references can be found, for example, in [5–7, 10, 11, 17–19, 21, 23].

The equation

$$x_{n+1}=\frac{ax_{n-l}x_{n-k}}{bx_{n-p}\pm cx_{n-q}},\ \ n\in\mathbb{N}_0, \tag{1.1}$$

where the initial conditions are arbitrary positive real numbers, k, l, p, q are non-negative integers and a, b, c are positive constants, is one of the difference equations whose solutions are associated with number sequences. Positive solutions of concrete special cases of equation (1.1) have been studied by several authors. For the first time, Elabbasy et al., in [6, 7], obtained positive solutions of some special cases of equation (1.1) by using induction principle. In addition, they didn't give theoretical

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explanation of how solutions were obtained. One of the special cases is

$$x_{n+1}=\frac{x_{n-1}x_{n-2}}{x_{n-1}+x_{n-2}},\ \ n\in\mathbb{N}_0, \tag{1.2}$$

whose solutions are associated with the well known Padovan numbers in literature. Moreover, the multi-dimensional expansion of the concrete some special cases of equation (1.1) can be seen in the literature (see, for example, [2–4, 8, 9, 13, 14, 24]). Another equation

$$x_{n+1}=a+\frac{b}{x_n}+\frac{c}{x_nx_{n-1}},\ \ n\in\mathbb{N}_0, \tag{1.3}$$

where the parameters a,b,c and initial values x_{-1} and x_0 are complex numbers and $c\neq 0$, which is one of these equations. The solutions of equation (1.3) are associated with number sequences, has been studied in [16]. Unlike the method used to obtain solutions of some special cases of (1.1), by using convenient transformation the equation in (1.3) reduce to the next third-order linear difference equation with constant coefficients

$$x_{n+1}=ax_n+bx_{n-1}+cx_{n-2},\ \ n\in\mathbb{N}_0, \tag{1.4}$$

which has actually the general solution

$$x_n=x_0S_n+x_{-1}\left(S_{n+1}-aS_n\right)+cx_{-2}S_{n-1},\ \ n\in\mathbb{N}_0, \tag{1.5}$$

where $(S_n)_{n\geq -2}$ of equation (1.4) satisfying the initial values $S_{-2}=S_{-1}=0$, $S_0=1$. Quite recently in [15], among other things, a generalization of (1.4) is treated as

$$x_n=f^{-1}\left(af\left(x_{n-1}\right)+bf\left(x_{n-2}\right)+cf\left(x_{n-3}\right)\right),\ n\in\mathbb{N}_0, \tag{1.6}$$

where $f:D_f\longrightarrow\mathbb{R}$ is a "$1-1$" continuous function on its domain $D_f\subseteq\mathbb{R}$, parameters a,b,c and the initial values x_{-3},x_{-2} and x_{-1} are real numbers. In addition, the authors obtained the solution of the equation (1.6) in relation to the solution given in (1.5).

On the other hand, one of the popular topics for system of difference equations is also symmetric and close-to-symmetric systems such as

$$x_{n+1}=g\left(p_{n-k},q_{n-l}\right),\ \ y_{n+1}=g\left(r_{n-k},s_{n-l}\right),\ \ n\in\mathbb{N}_0, \tag{1.7}$$

where the sequences p_n, q_n, r_n, s_n are some of the sequences x_n and y_n and k, l are fixed natural numbers. There are studies related to some special cases of the system (1.7) (see, for example, [1, 12, 20, 22]).

Motivated by this line of investigations, here we show that the systems of difference equations

$$x_{n+1}=f^{-1}\left(af\left(p_{n-1}\right)+bf\left(q_{n-2}\right)\right),\ \ y_{n+1}=f^{-1}\left(af\left(r_{n-1}\right)+bf\left(s_{n-2}\right)\right), \tag{1.8}$$

for $n\in\mathbb{N}_0$, where the sequences p_n, q_n, r_n and s_n are some of the sequences x_n and y_n, $f:D_f\longrightarrow\mathbb{R}$ is a "$1-1$" continuous function on its domain $D_f\subseteq\mathbb{R}$, the initial values x_{-j}, y_{-j}, $j\in\{0,1,2\}$ are arbitrary real numbers and the parameters and a,b

are arbitrary complex numbers, can be solved. To do this, we will use the solutions given in (1.5) and the solutions obtained by rearranging these solutions. In this way we also give analytical examples for the general solution of special cases of system (1.8).

2. Main Results

In this section, we consider the eight special cases of systems (1.8), where the sequences p_n, q_n, r_n, s_n are some of the sequences x_n and y_n, for $n \geq -2$, and initial values x_{-j}, y_{-j}, $j \in \{0,1,2\}$, are arbitrary real numbers.

2.1. *Case 1:* $p_n = x_n$, $q_n = x_n$, $r_n = y_n$, $s_n = y_n$

In this case, system (1.8) is expressed as

$$x_{n+1} = f^{-1}\left(af(x_{n-1}) + bf(x_{n-2})\right), \quad y_{n+1} = f^{-1}\left(af(y_{n-1}) + bf(y_{n-2})\right), \tag{2.1}$$

for $n \in \mathbb{N}_0$. Since f is "$1-1$", from (2.1)

$$f(x_{n+1}) = af(x_{n-1}) + bf(x_{n-2}), \quad f(y_{n+1}) = af(y_{n-1}) + bf(y_{n-2}), \tag{2.2}$$

for $n \in \mathbb{N}_0$. By using the change of variables

$$f(x_n) = u_n, \text{ and } f(y_n) = v_n, \quad n \geq -2, \tag{2.3}$$

system (2.2) is transformed to the following one

$$u_{n+1} = au_{n-1} + bu_{n-2}, \quad v_{n+1} = av_{n-1} + bv_{n-2}, \tag{2.4}$$

for $n \in \mathbb{N}_0$. By taking $a = 0$, $b = a$, $c = b$ in (1.4) and $S_n = J_{n+1}$, for all $n \geq -2$, which is called generalized Padovan sequence, in (1.5), the solutions to equations in (2.4) are given by

$$u_n = u_0 J_{n+1} + u_{-1} J_{n+2} + bu_{-2} J_n, \tag{2.5}$$

$$v_n = v_0 J_{n+1} + v_{-1} J_{n+2} + bv_{-2} J_n, \tag{2.6}$$

for $n \in \mathbb{N}_0$. From (2.3), (2.5) and (2.6), it follows that the general solution to system (2.2) is given by

$$x_n = f^{-1}\left(f(x_0) J_{n+1} + f(x_{-1}) J_{n+2} + bf(x_{-2}) J_n\right), \; n \geq -2, \tag{2.7}$$

$$y_n = f^{-1}\left(f(y_0) J_{n+1} + f(y_{-1}) J_{n+2} + bf(y_{-2}) J_n\right), \; n \geq -2. \tag{2.8}$$

2.2. *Case 2:* $p_n = x_n$, $q_n = x_n$, $r_n = x_n$, $s_n = x_n$

In this case, system (1.8) becomes

$$x_{n+1} = f^{-1}\left(af(x_{n-1}) + bf(x_{n-2})\right), \quad y_{n+1} = f^{-1}\left(af(x_{n-1}) + bf(x_{n-2})\right), \tag{2.9}$$

for $n \in \mathbb{N}_0$. It should be first note that from the equations in (2.9) it immediately follows that $x_n = y_n$, for all $n \in \mathbb{N}$. From (2.7), the general solution to system (2.9) is

$$x_n = y_n = f^{-1}\left(f(x_0) J_{n+1} + f(x_{-1}) J_{n+2} + bf(x_{-2}) J_n\right), \; n \in \mathbb{N}. \tag{2.10}$$

2.3. *Case 3:* $p_n = y_n$, $q_n = y_n$, $r_n = y_n$, $s_n = y_n$

In this case, we obtain the system

$$x_{n+1} = f^{-1}\left(af(y_{n-1}) + bf(y_{n-2})\right), \quad y_{n+1} = f^{-1}\left(af(y_{n-1}) + bf(y_{n-2})\right), \tag{2.11}$$

for $n \in \mathbb{N}_0$, which is an analogue of the system (2.9). By interchanging the variables x_n and y_n, the system in (2.9) is transformed into (2.11). So, by interchanging x_j and y_j for $j \in \{0,1,2\}$, the formula in (2.10) is transformed into the formula

$$x_n = y_n = f^{-1}\left(f(y_0)J_{n+1} + f(y_{-1})J_{n+2} + bf(y_{-2})J_n\right), \; n \in \mathbb{N}. \tag{2.12}$$

2.4. *Case 4:* $p_n = x_n$, $q_n = x_n$, $r_n = y_n$, $s_n = x_n$

In this case, system (1.8) is written as in the form

$$x_{n+1} = f^{-1}\left(af(x_{n-1}) + bf(x_{n-2})\right), \quad y_{n+1} = f^{-1}\left[af(y_{n-1}) + bf(x_{n-2})\right], \tag{2.13}$$

for $n \in \mathbb{N}_0$. Since f is "$1-1$", from (2.13)

$$f(x_{n+1}) = af(x_{n-1}) + bf(x_{n-2}), \quad f(y_{n+1}) = af(y_{n-1}) + bf(x_{n-2}), \tag{2.14}$$

for $n \in \mathbb{N}_0$. By using the change of variables

$$f(x_n) = u_n, \;\; n \geq -2, \;\; \text{and} \;\; f(y_n) = v_n, \;\; n \geq -1, \tag{2.15}$$

system (2.14) is transformed to the following one

$$u_{n+1} = au_{n-1} + bu_{n-2}, \;\; v_{n+1} = av_{n-1} + bu_{n-2}, \tag{2.16}$$

for $n \in \mathbb{N}_0$. From (2.5), we can write the solution of the first equation in (2.16) as

$$u_n = u_0 J_{n+1} + u_{-1} J_{n+2} + bu_{-2} J_n, \; n \in \mathbb{N}_0. \tag{2.17}$$

By subtracting the second one from the first equations in (2.16), we have

$$u_{n+1} - v_{n+1} = a(u_{n-1} - v_{n-1}), \; n \in \mathbb{N}_0. \tag{2.18}$$

From (2.18) we see that the sequence $(u_n - v_n)_{n \geq -1}$ satisfies the difference equation

$$w_n = aw_{n-2}, \; n \geq 1, \tag{2.19}$$

from which it follows that

$$u_{2n+i} - v_{2n+i} = a^{n+1}(u_{i-2} - v_{i-2}), \tag{2.20}$$

for $n \in \mathbb{N}_0$, $i \in \{1,2\}$.

From (2.17) and (2.20), we get

$$\begin{aligned} v_{2n+i} &= u_{2n+i} - a^{n+1}u_{i-2} + a^{n+1}v_{i-2}, \\ &= u_0 J_{2n+i+1} + u_{-1} J_{2n+i+2} + bu_{-2} J_{2n+i} - a^{n+1}u_{i-2} + a^{n+1}v_{i-2}, \end{aligned} \tag{2.21}$$

for $n \in \mathbb{N}_0$, $i \in \{1,2\}$.

Employing (2.17) and (2.21) in (2.15) and after some calculation, we obtain

$$x_n = f^{-1}\left(f(x_0)J_{n+1} + f(x_{-1})J_{n+2} + bf(x_{-2})J_n\right), \; n \geq -2, \tag{2.22}$$

$$y_{2n+1} = f^{-1}\big(f(x_0)J_{2n+2} + f(x_{-1})\left(J_{2n+3} - a^{n+1}\right) + bf(x_{-2})J_{2n+1} + a^{n+1}f(y_{-1})\big),\ n \geq -1, \tag{2.23}$$

$$y_{2n+2} = f^{-1}\big(f(x_0)\left(J_{2n+3} - a^{n+1}\right) + f(x_{-1})J_{2n+4} + bf(x_{-2})J_{2n+2} + a^{n+1}f(y_0)\big),\ n \geq -1. \tag{2.24}$$

2.5. *Case 5:* $p_n = x_n$, $q_n = y_n$, $r_n = y_n$, $s_n = y_n$

In this case, system (1.8) is expressed as

$$x_{n+1} = f^{-1}\big(af(x_{n-1}) + bf(y_{n-2})\big),\ \ y_{n+1} = f^{-1}\big(af(y_{n-1}) + bf(y_{n-2})\big), \tag{2.25}$$

for $n \in \mathbb{N}_0$. Note that system (2.25) is obtained from equations (2.13) by interchanging the letters x and y, and hence all the statements concerning solutions to the equations follow from the corresponding statements in Case 4.

The general solution to the system (2.25) is given

$$x_{2n+1} = f^{-1}\big(f(y_0)J_{2n+2} + f(y_{-1})\left(J_{2n+3} - a^{n+1}\right) + bf(y_{-2})J_{2n+1} + a^{n+1}f(x_{-1})\big),\ n \geq -1, \tag{2.26}$$

$$x_{2n+2} = f^{-1}\big(f(y_0)\left(J_{2n+3} - a^{n+1}\right) + f(y_{-1})J_{2n+4} + bf(y_{-2})J_{2n+2} + a^{n+1}f(x_0)\big),\ n \geq -1, \tag{2.27}$$

$$y_n = f^{-1}\big(f(y_0)J_{n+1} + f(y_{-1})J_{n+2} + bf(y_{-2})J_n\big),\ n \geq -2. \tag{2.28}$$

2.6. *Case 6:* $p_n = y_n$, $q_n = y_n$, $r_n = x_n$, $s_n = x_n$

In this case, we obtain the system

$$x_{n+1} = f^{-1}\big(af(y_{n-1}) + bf(y_{n-2})\big),\ \ y_{n+1} = f^{-1}\big(af(x_{n-1}) + bf(x_{n-2})\big), \tag{2.29}$$

for $n \in \mathbb{N}_0$. Since f is "$1-1$", from (2.29)

$$f(x_{n+1}) = af(y_{n-1}) + bf(y_{n-2}),\ \ f(y_{n+1}) = af(x_{n-1}) + bf(x_{n-2}), \tag{2.30}$$

for $n \in \mathbb{N}_0$. By using the change of variables

$$f(x_n) = u_n,\ \text{ and }\ f(y_n) = v_n,\ \ n \geq -2, \tag{2.31}$$

system (2.30) is transformed to the following one

$$u_{n+1} = av_{n-1} + bv_{n-2},\ \ v_{n+1} = au_{n-1} + bu_{n-2},\ n \in \mathbb{N}_0. \tag{2.32}$$

By summing the equations in (2.32) we get

$$u_{n+1} + v_{n+1} = a\left(u_{n-1} + v_{n-1}\right) + b\left(u_{n-2} + v_{n-2}\right),\ n \in \mathbb{N}_0, \tag{2.33}$$

whereas by subtracting the second one from the first, we have

$$u_{n+1} - v_{n+1} = -a\left(u_{n-1} - v_{n-1}\right) - b\left(u_{n-2} - v_{n-2}\right),\ n \in \mathbb{N}_0. \tag{2.34}$$

From (2.5), we can write the solution of equation (2.33) as

$$u_n + v_n = (u_0 + v_0) J_{n+1} + (u_{-1} + v_{-1}) J_{n+2} + b (u_{-2} + v_{-2}) J_n, \tag{2.35}$$

for $n \geq -2$. On the other hand, by taking $a = 0$, $b = -a$, $c = -b$ in (1.4) and $S_n = (-1)^n J'_{n+1}$, for all $n \geq -2$, which is called generalized Padovan sequence, in (1.5), from (2.34), we also have

$$u_n - v_n = (-1)^n \left((u_0 - v_0) J'_{n+1} - (u_{-1} - v_{-1}) J'_{n+2} + b (u_{-2} - v_{-2}) J'_n \right), \tag{2.36}$$

for $n \geq -2$. From (2.36) we obtain

$$u_{2n} - v_{2n} = (u_0 - v_0) J'_{2n+1} - (u_{-1} - v_{-1}) J'_{2n+2} + b (u_{-2} - v_{-2}) J'_{2n}, \tag{2.37}$$

for $n \geq -1$. From (2.35)

$$u_{2n} + v_{2n} = (u_0 + v_0) J_{2n+1} + (u_{-1} + v_{-1}) J_{2n+2} + b (u_{-2} + v_{-2}) J_{2n}, \tag{2.38}$$

for $n \geq -1$. By summing the equations (2.37) and (2.38) we get

$$\begin{aligned} u_{2n} = & \frac{\left(J_{2n+1} + J'_{2n+1}\right) u_0 + \left(J_{2n+1} - J'_{2n+1}\right) v_0 + \left(J_{2n+2} - J'_{2n+2}\right) u_{-1}}{2} \\ & + \frac{\left(J_{2n+2} + J'_{2n+2}\right) v_{-1} + b \left(J_{2n} + J'_{2n}\right) u_{-2} + b \left(J_{2n} - J'_{2n}\right) v_{-2}}{2}, \end{aligned} \tag{2.39}$$

for $n \geq -1$. By subtracting equation (2.37) from equation (2.38), we have

$$\begin{aligned} v_{2n} = & \frac{\left(J_{2n+1} - J'_{2n+1}\right) u_0 + \left(J_{2n+1} + J'_{2n+1}\right) v_0 + \left(J_{2n+2} + J'_{2n+2}\right) u_{-1}}{2} \\ & + \frac{\left(J_{2n+2} - J'_{2n+2}\right) v_{-1} + b \left(J_{2n} - J'_{2n}\right) u_{-2} + b \left(J_{2n} + J'_{2n}\right) v_{-2}}{2}, \end{aligned} \tag{2.40}$$

for $n \geq -1$. From (2.36) we have

$$u_{2n+1} - v_{2n+1} = -(u_0 - v_0) J'_{2n+2} + (u_{-1} - v_{-1}) J'_{2n+3} - b (u_{-2} - v_{-2}) J'_{2n+1}, \tag{2.41}$$

for $n \geq -1$. From (2.35)

$$u_{2n+1} + v_{2n+1} = (u_0 + v_0) J_{2n+2} + (u_{-1} + v_{-1}) J_{2n+3} + b (u_{-2} + v_{-2}) J_{2n+1}, \tag{2.42}$$

for $n \geq -1$. By summing the equations (2.41) and (2.42) we get

$$\begin{aligned} u_{2n+1} = & \frac{\left(J_{2n+2} - J'_{2n+2}\right) u_0 + \left(J_{2n+2} + J'_{2n+2}\right) v_0 + \left(J_{2n+3} + J'_{2n+3}\right) u_{-1}}{2} \\ & + \frac{\left(J_{2n+3} - J'_{2n+3}\right) v_{-1} + b \left(J_{2n+1} - J'_{2n+1}\right) u_{-2}}{2} + \frac{b \left(J_{2n+1} + J'_{2n+1}\right) v_{-2}}{2}, \end{aligned} \tag{2.43}$$

for $n \geq -1$. By subtracting equation (2.41) from equation (2.42), we have

$$v_{2n+1} = \frac{\left(J_{2n+2} + J'_{2n+2}\right) u_0 + \left(J_{2n+2} - J'_{2n+2}\right) v_0 + \left(J_{2n+3} - J'_{2n+3}\right) u_{-1}}{2}$$

$$+\frac{\left(J_{2n+3}+J'_{2n+3}\right)v_{-1}+b\left(J_{2n+1}+J'_{2n+1}\right)u_{-2}}{2}+\frac{b\left(J_{2n+1}-J'_{2n+1}\right)v_{-2}}{2}, \tag{2.44}$$

for $n\geq -1$. Employing (2.39), (2.40), (2.43), (2.44) in (2.31) and after some calculation, we obtain

$$\begin{aligned}x_{2n}=f^{-1}\Bigg(&\frac{\left(J_{2n+1}+J'_{2n+1}\right)f(x_0)+\left(J_{2n+1}-J'_{2n+1}\right)f(y_0)}{2}\\&+\frac{\left(J_{2n+2}-J'_{2n+2}\right)f(x_{-1})+\left(J_{2n+2}+J'_{2n+2}\right)f(y_{-1})+b\left(J_{2n}+J'_{2n}\right)f(x_{-2})}{2}\\&+\frac{b\left(J_{2n}-J'_{2n}\right)f(y_{-2})}{2}\Bigg),\end{aligned} \tag{2.45}$$

$$\begin{aligned}y_{2n}=f^{-1}\Bigg(&\frac{\left(J_{2n+1}-J'_{2n+1}\right)f(x_0)+\left(J_{2n+1}+J'_{2n+1}\right)f(y_0)}{2}\\&+\frac{\left(J_{2n+2}+J'_{2n+2}\right)f(x_{-1})+\left(J_{2n+2}-J'_{2n+2}\right)f(y_{-1})+b\left(J_{2n}-J'_{2n}\right)f(x_{-2})}{2}\\&+\frac{b\left(J_{2n}+J'_{2n}\right)f(y_{-2})}{2}\Bigg),\end{aligned} \tag{2.46}$$

$$\begin{aligned}x_{2n+1}=f^{-1}\Bigg(&\frac{\left(J_{2n+2}-J'_{2n+2}\right)f(x_0)+\left(J_{2n+2}+J'_{2n+2}\right)f(y_0)}{2}\\&+\frac{\left(J_{2n+3}+J'_{2n+3}\right)f(x_{-1})+\left(J_{2n+3}-J'_{2n+3}\right)f(y_{-1})}{2}\\&+\frac{b\left(J_{2n+1}-J'_{2n+1}\right)f(x_{-2})+b\left(J_{2n+1}+J'_{2n+1}\right)f(y_{-2})}{2}\Bigg),\end{aligned} \tag{2.47}$$

and

$$\begin{aligned}y_{2n+1}=f^{-1}\Bigg(&\frac{\left(J_{2n+2}+J'_{2n+2}\right)f(x_0)+\left(J_{2n+2}-J'_{2n+2}\right)f(y_0)}{2}\\&+\frac{\left(J_{2n+3}-J'_{2n+3}\right)f(x_{-1})+\left(J_{2n+3}+J'_{2n+3}\right)f(y_{-1})}{2}\\&+\frac{b\left(J_{2n+1}+J'_{2n+1}\right)f(x_{-2})+b\left(J_{2n+1}-J'_{2n+1}\right)f(y_{-2})}{2}\Bigg),\end{aligned} \tag{2.48}$$

for $n\geq -1$.

2.7. *Case 7:* $p_n=y_n$, $q_n=x_n$, $r_n=x_n$, $s_n=y_n$

In this case, system (1.8) is expressed as

$$x_{n+1}=f^{-1}\left(af(y_{n-1})+bf(x_{n-2})\right),\quad y_{n+1}=f^{-1}\left(af(x_{n-1})+bf(y_{n-2})\right), \tag{2.49}$$

for $n \in \mathbb{N}_0$. Since f is "$1-1$", from (2.49)

$$f(x_{n+1}) = af(y_{n-1}) + bf(x_{n-2}), \quad f(y_{n+1}) = af(x_{n-1}) + bf(y_{n-2}), \tag{2.50}$$

for $n \in \mathbb{N}_0$. By using the change of variables

$$f(x_n) = u_n, \text{ and } f(y_n) = v_n, \quad n \geq -2, \tag{2.51}$$

system (2.50) is transformed to the following one

$$u_{n+1} = av_{n-1} + bu_{n-2}, \quad v_{n+1} = au_{n-1} + bv_{n-2}, \; n \in \mathbb{N}_0. \tag{2.52}$$

By summing the equations in (2.52) we get

$$u_{n+1} + v_{n+1} = a(u_{n-1} + v_{n-1}) + b(u_{n-2} + v_{n-2}), \; n \in \mathbb{N}_0, \tag{2.53}$$

whereas by subtracting the second one from the first, we have

$$u_{n+1} - v_{n+1} = -a(u_{n-1} - v_{n-1}) + b(u_{n-2} - v_{n-2}), \; n \in \mathbb{N}_0. \tag{2.54}$$

From (2.5), we can write the solution of equation (2.53) as

$$u_n + v_n = (u_0 + v_0)J_{n+1} + (u_{-1} + v_{-1})J_{n+2} + b(u_{-2} + v_{-2})J_n, \tag{2.55}$$

for $n \geq -2$. On the other hand, by taking $a = 0$, $b = -a$, $c = b$ in (1.4) and $S_n = J'_{n+1}$, for all $n \geq -2$, which is called generalized Padovan sequence, in (1.5), from (2.54), we also have that

$$u_n - v_n = (u_0 - v_0)J'_{n+1} + (u_{-1} - v_{-1})J'_{n+2} + b(u_{-2} - v_{-2})J'_n, \tag{2.56}$$

for $n \geq -2$. By summing the equations (2.55) and (2.56) we get

$$\begin{aligned} u_n = {} & \frac{J_{n+1} + J'_{n+1}}{2}u_0 + \frac{J_{n+1} - J'_{n+1}}{2}v_0 + \frac{J_{n+2} + J'_{n+2}}{2}u_{-1} \\ & + \frac{J_{n+2} - J'_{n+2}}{2}v_{-1} + b\frac{J_n + J'_n}{2}u_{-2} + b\frac{J_n - J'_n}{2}v_{-2}, \; n \geq -2. \end{aligned} \tag{2.57}$$

By subtracting equation (2.56) from equation (2.55), we have

$$\begin{aligned} v_n = {} & \frac{J_{n+1} - J'_{n+1}}{2}u_0 + \frac{J_{n+1} + J'_{n+1}}{2}v_0 + \frac{J_{n+2} - J'_{n+2}}{2}u_{-1} \\ & + \frac{J_{n+2} + J'_{n+2}}{2}v_{-1} + b\frac{J_n - J'_n}{2}u_{-2} + b\frac{J_n + J'_n}{2}v_{-2}, \; n \geq -2. \end{aligned} \tag{2.58}$$

From (2.51), (2.57) and (2.58) and after some calculation, we obtain

$$\begin{aligned} x_n = f^{-1}\Bigg(& \frac{J_{n+1} + J'_{n+1}}{2}f(x_0) + \frac{J_{n+1} - J'_{n+1}}{2}f(y_0) \\ & + \frac{J_{n+2} + J'_{n+2}}{2}f(x_{-1}) + \frac{J_{n+2} - J'_{n+2}}{2}f(y_{-1}) \\ & + b\frac{J_n + J'_n}{2}f(x_{-2}) + b\frac{J_n - J'_n}{2}f(y_{-2})\Bigg), \; n \geq -2, \end{aligned} \tag{2.59}$$

and

$$y_n = f^{-1}\left(\frac{J_{n+1}-J'_{n+1}}{2}f(x_0)+\frac{J_{n+1}+J'_{n+1}}{2}f(y_0)\right.$$
$$+\frac{J_{n+2}-J'_{n+2}}{2}f(x_{-1})+\frac{J_{n+2}+J'_{n+2}}{2}f(y_{-1})$$
$$\left.+b\frac{J_n-J'_n}{2}f(x_{-2})+b\frac{J_n+J'_n}{2}f(y_{-2})\right),\ n\geq -2. \tag{2.60}$$

2.8. *Case 8:* $p_n = x_n$, $q_n = y_n$, $r_n = y_n$, $s_n = x_n$

In this case, system (1.8) is written as in the form

$$x_{n+1} = f^{-1}\left(af(x_{n-1})+bf(y_{n-2})\right),\ \ y_{n+1} = f^{-1}\left(af(y_{n-1})+bf(x_{n-2})\right), \tag{2.61}$$

for $n\in\mathbb{N}_0$. Since f is "$1-1$", from (2.61)

$$f(x_{n+1}) = af(x_{n-1})+bf(y_{n-2}),\ \ f(y_{n+1}) = af(y_{n-1})+bf(x_{n-2}), \tag{2.62}$$

for $n\in\mathbb{N}_0$. By using the change of variables

$$f(x_n) = u_n,\ \text{ and }\ f(y_n) = v_n,\ \ n\geq -2, \tag{2.63}$$

system (2.62) is transformed to the following one

$$u_{n+1} = au_{n-1}+bv_{n-2},\ \ v_{n+1} = av_{n-1}+bu_{n-2},\ n\in\mathbb{N}_0. \tag{2.64}$$

By summing the equations in (2.64) we get

$$u_{n+1}+v_{n+1} = a(u_{n-1}+v_{n-1})+b(u_{n-2}+v_{n-2}),\ n\in\mathbb{N}_0, \tag{2.65}$$

whereas by subtracting the second one from the first, we have

$$u_{n+1}-v_{n+1} = a(u_{n-1}-v_{n-1})-b(u_{n-2}-v_{n-2}),\ n\in\mathbb{N}_0. \tag{2.66}$$

From (2.5), we can write the solution of equation (2.65) as

$$u_n+v_n = (u_0+v_0)J_{n+1}+(u_{-1}+v_{-1})J_{n+2}+b(u_{-2}+v_{-2})J_n, \tag{2.67}$$

for $n\geq -2$. On the other hand, by taking $a=0$, $b=a$, $c=-b$ in (1.4) and $S_n = (-1)^n J_{n+1}$, for all $n\geq -2$, which is called generalized Padovan sequence, in (1.5), from (2.66), we also have that

$$u_n-v_n = (-1)^n\left((u_0-v_0)J_{n+1}-(u_{-1}-v_{-1})J_{n+2}+b(u_{-2}-v_{-2})J_n\right), \tag{2.68}$$

for $n\geq -2$. From (2.68) we have

$$u_{2n}-v_{2n} = (u_0-v_0)J_{2n+1}-(u_{-1}-v_{-1})J_{2n+2}+b(u_{-2}-v_{-2})J_{2n}, \tag{2.69}$$

for $n\geq -1$ and

$$u_{2n+1}-v_{2n+1} = -(u_0-v_0)J_{2n+2}+(u_{-1}-v_{-1})J_{2n+3}-b(u_{-2}-v_{-2})J_{2n+1}, \tag{2.70}$$

for $n\geq -1$. From (2.67)

$$u_{2n}+v_{2n} = (u_0+v_0)J_{2n+1}+(u_{-1}+v_{-1})J_{2n+2}+b(u_{-2}+v_{-2})J_{2n}, \tag{2.71}$$

for $n \geq -1$. By summing the equations (2.69) and (2.71)

$$u_{2n} = u_0 J_{2n+1} + v_{-1} J_{2n+2} + b u_{-2} J_{2n},\ n \geq -1. \tag{2.72}$$

By subtracting equation (2.69) from equation (2.71), we have

$$v_{2n} = v_0 J_{2n+1} + u_{-1} J_{2n+2} + b v_{-2} J_{2n},\ n \geq -1. \tag{2.73}$$

From (2.67)

$$u_{2n+1} + v_{2n+1} = (u_0 + v_0) J_{2n+2} + (u_{-1} + v_{-1}) J_{2n+3} + b (u_{-2} + v_{-2}) J_{2n+1}, \tag{2.74}$$

for $n \geq -1$. By summing the equations (2.70) and (2.74) we get

$$u_{2n+1} = v_0 J_{2n+2} + u_{-1} J_{2n+3} + b v_{-2} J_{2n+1},\ n \geq -1. \tag{2.75}$$

By subtracting equation (2.70) from equation (2.74), we have

$$v_{2n+1} = u_0 J_{2n+2} + v_{-1} J_{2n+3} + b u_{-2} J_{2n+1},\ n \geq -1. \tag{2.76}$$

From (2.63), (2.72), (2.73), (2.75), (2.76) and after some calculation, we obtain

$$x_{2n} = f^{-1}\left(f(x_0) J_{2n+1} + f(y_{-1}) J_{2n+2} + b f(x_{-2}) J_{2n}\right),\ n \geq -1, \tag{2.77}$$

$$y_{2n} = f^{-1}\left(f(y_0) J_{2n+1} + f(x_{-1}) J_{2n+2} + b f(y_{-2}) J_{2n}\right),\ n \geq -1, \tag{2.78}$$

$$x_{2n+1} = f^{-1}\left(f(y_0) J_{2n+2} + f(x_{-1}) J_{2n+3} + b f(y_{-2}) J_{2n+1}\right),\ n \geq -1, \tag{2.79}$$

and

$$y_{2n+1} = f^{-1}\left(f(x_0) J_{2n+2} + f(y_{-1}) J_{2n+3} + b f(x_{-2}) J_{2n+1}\right), n \geq -1. \tag{2.80}$$

3. Analytical Examples

In this section, we give examples for Case 1, 4 and 7. Examples for the other cases can be constructed similarly.

Example 1. Let

$$f(t) = t. \tag{3.1}$$

Then, $D_f = \mathbb{R}$ and system (2.1) becomes

$$x_{n+1} = a x_{n-1} + b x_{n-2},\ \ y_{n+1} = a y_{n-1} + b y_{n-2},\ \ n \in \mathbb{N}_0. \tag{3.2}$$

Here we can also assume that parameters a, b and initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}$, and y_0 are complex numbers, since function (3.1) is "$1-1$" on $D_f = \mathbb{C}$. Function (3.1) is obviously an involution:

$$f^{-1}(t) = f(t),\quad t \in D_f.$$

We see that formulas (2.7) and (2.8) hold. Using (3.1) in (2.7) and (2.8), we obtain that the general solution to system (3.2) is

$$\begin{aligned} x_n &= f^{-1}\left(f(x_0) J_{n+1} + f(x_{-1}) J_{n+2} + b f(x_{-2}) J_n\right) \\ &= x_0 J_{n+1} + x_{-1} J_{n+2} + b x_{-2} J_n,\ n \geq -2, \end{aligned} \tag{3.3}$$

$$\begin{aligned} y_n &= f^{-1}\left(f(y_0)J_{n+1} + f(y_{-1})J_{n+2} + bf(y_{-2})J_n\right) \\ &= y_0 J_{n+1} + y_{-1}J_{n+2} + by_{-2}J_n,\ n \geq -2. \end{aligned} \tag{3.4}$$

Example 2. Let

$$f(t) = \frac{1}{t}. \tag{3.5}$$

Then $D_f = \mathbb{R} \setminus \{0\}$ and system (2.13) becomes

$$x_{n+1} = \left(\frac{a}{x_{n-1}} + \frac{b}{x_{n-2}}\right)^{-1},\ \ y_{n+1} = \left(\frac{a}{y_{n-1}} + \frac{b}{x_{n-2}}\right)^{-1},\ \ n \in \mathbb{N}_0. \tag{3.6}$$

Here we can also assume that parameters a, b and initial values $x_{-2}, x_{-1}, x_0, y_{-1}$ and y_0 are complex numbers, since function (3.5) is "$1-1$" on $D_f = \mathbb{C} \setminus \{0\}$.

Clearly, function (3.5) is an involution. We see that (2.22)–(2.24) hold. Using (3.5) in (2.22)–(2.24), we obtain the general solution to system (3.6):

$$\begin{aligned} x_n &= f^{-1}\left(f(x_0)J_{n+1} + f(x_{-1})J_{n+2} + bf(x_{-2})J_n\right) \\ &= \left(\frac{1}{x_0}J_{n+1} + \frac{1}{x_{-1}}J_{n+2} + \frac{b}{x_{-2}}J_n\right)^{-1} \\ &= \frac{x_0 x_{-1} x_{-2}}{x_{-1}x_{-2}J_{n+1} + x_0 x_{-2}J_{n+2} + bx_0 x_{-1}J_n},\ n \geq -2, \end{aligned} \tag{3.7}$$

$$\begin{aligned} y_{2n+1} &= f^{-1}\left(f(x_0)J_{2n+2} + f(x_{-1})\left(J_{2n+3} - a^{n+1}\right) + bf(x_{-2})J_{2n+1} + a^{n+1}f(y_{-1})\right) \\ &= \left(\frac{1}{x_0}J_{2n+2} + \frac{1}{x_{-1}}\left(J_{2n+3} - a^{n+1}\right) + \frac{b}{x_{-2}}J_{2n+1} + \frac{a^{n+1}}{y_{-1}}\right)^{-1} \\ &= \left(\frac{x_{-1}x_{-2}y_{-1}J_{2n+2} + x_0 x_{-2}y_{-1}\left(J_{2n+3} - a^{n+1}\right)}{x_0 x_{-1} x_{-2} y_{-1}}\right. \\ &\quad \left. + \frac{bx_0 x_{-1} y_{-1} J_{2n+1} + a^{n+1} x_0 x_{-1} x_{-2}}{x_0 x_{-1} x_{-2} y_{-1}}\right)^{-1},\ n \geq -1, \end{aligned} \tag{3.8}$$

$$\begin{aligned} y_{2n+2} &= f^{-1}\left(f(x_0)\left(J_{2n+3} - a^{n+1}\right) + f(x_{-1})J_{2n+4} + bf(x_{-2})J_{2n+2} + a^{n+1}f(y_0)\right) \\ &= \left(\frac{1}{x_0}\left(J_{2n+3} - a^{n+1}\right) + \frac{1}{x_{-1}}J_{2n+4} + \frac{b}{x_{-2}}J_{2n+2} + \frac{a^{n+1}}{y_0}\right)^{-1} \\ &= \left(\frac{x_{-1}x_{-2}y_0\left(J_{2n+3} - a^{n+1}\right) + x_0 x_{-2} y_0 J_{2n+4}}{x_0 x_{-1} x_{-2} y_0}\right. \\ &\quad \left. + \frac{bx_0 x_{-1} y_0 J_{2n+2} + a^{n+1} x_0 x_{-1} x_{-2}}{x_0 x_{-1} x_{-2} y_0}\right)^{-1},\ n \geq -1. \end{aligned} \tag{3.9}$$

Example 3. Let

$$f_k(t) = t^{2k+1},\ \ k \in \mathbb{N}_0. \tag{3.10}$$

Then $D_{f_k}=\mathbb{R}$ and system (2.49) becomes

$$x_{n+1}=\left(ay_{n-1}^{2k+1}+bx_{n-2}^{2k+1}\right)^{\frac{1}{2k+1}},\quad y_{n+1}=\left(ax_{n-1}^{2k+1}+by_{n-2}^{2k+1}\right)^{\frac{1}{2k+1}},\quad n\in\mathbb{N}_0. \tag{3.11}$$

Here we can also assume that parameters a, b and initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ are complex numbers, since function (3.10) is "$1-1$" on $D_{f_k}=\mathbb{C}$.

Function (3.10) is an involution:

$$f_k^{-1}(t)=t^{\frac{1}{2k+1}},\quad t\in D_{f_k}.$$

We see that (2.59) and (2.60) hold. Using (3.10) in (2.59) and (2.60), we obtain the general solution to system (3.11):

$$\begin{aligned}x_n&=f_k^{-1}\left(\frac{J_{n+1}+J'_{n+1}}{2}f_k(x_0)+\frac{J_{n+1}-J'_{n+1}}{2}f_k(y_0)+\frac{J_{n+2}+J'_{n+2}}{2}f_k(x_{-1})\right.\\&\quad\left.+\frac{J_{n+2}-J'_{n+2}}{2}f_k(y_{-1})+b\frac{J_n+J'_n}{2}f_k(x_{-2})+b\frac{J_n-J'_n}{2}f_k(y_{-2})\right)\\&=\left(\frac{J_{n+1}+J'_{n+1}}{2}x_0^{2k+1}+\frac{J_{n+1}-J'_{n+1}}{2}y_0^{2k+1}+\frac{J_{n+2}+J'_{n+2}}{2}x_{-1}^{2k+1}\right.\\&\quad\left.+\frac{J_{n+2}-J'_{n+2}}{2}y_{-1}^{2k+1}+b\frac{J_n+J'_n}{2}x_{-2}^{2k+1}+b\frac{J_n-J'_n}{2}y_{-2}^{2k+1}\right)^{\frac{1}{2k+1}},\end{aligned} \tag{3.12}$$

for $n\geq -2$,

$$\begin{aligned}y_n&=f_k^{-1}\left(\frac{J_{n+1}-J'_{n+1}}{2}f_k(x_0)+\frac{J_{n+1}+J'_{n+1}}{2}f_k(y_0)+\frac{J_{n+2}-J'_{n+2}}{2}f_k(x_{-1})\right.\\&\quad\left.+\frac{J_{n+2}+J'_{n+2}}{2}f_k(y_{-1})+b\frac{J_n-J'_n}{2}f_k(x_{-2})+b\frac{J_n+J'_n}{2}f_k(y_{-2})\right)\\&=\left(\frac{J_{n+1}-J'_{n+1}}{2}x_0^{2k+1}+\frac{J_{n+1}+J'_{n+1}}{2}y_0^{2k+1}+\frac{J_{n+2}-J'_{n+2}}{2}x_{-1}^{2k+1}\right.\\&\quad\left.+\frac{J_{n+2}+J'_{n+2}}{2}y_{-1}^{2k+1}+b\frac{J_n-J'_n}{2}x_{-2}^{2k+1}+b\frac{J_n+J'_n}{2}y_{-2}^{2k+1}\right)^{\frac{1}{2k+1}},\end{aligned} \tag{3.13}$$

for $n\geq -2$.

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ON SPECIAL DEVELOPABLE RULED SURFACES WITH POINTWISE 1-TYPE GAUSS MAP

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Abstract. In this paper, some special developable ruled surfaces with point-wise Gauss map are studied. Three different special developable ruled surfaces called rectifying ruled surface, generalized normal ruled surface and osculating type ruled surface are considered. The conditions for such surfaces to have 1-type Gauss map and also to be minimal are introduced. Finally, some examples are given for the obtained results.

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Keywords: ruled surface, Laplacian, Gauss map, minimal surface

1. INTRODUCTION

Studies on finite type submanifolds in the Euclidean and semi-Euclidean spaces have started in 1970's by Chen and then this subject has been used for characterizing and investigating other important submanifolds [4, 5]. After this subject has been introduced, it has gained interest among geometers and developed rapidly. Especially, the application of the notion of finite type to Gauss map of submanifolds has been an important development [1–3, 6, 7, 13, 14, 16]. The notion of 1-type Gauss map is described by the equality of $\Delta G = \lambda(G+C)$ where Δ is the Laplacian of a surface M, G is the Gauss map of M, λ is a real number and C is a constant vector [13]. Some of the well-known surfaces with 1-type Gauss map in the Euclidean 3-space E^3 are catenoids, helicoids and right cones [6].

In the current paper, our purpose is to study some special ruled surfaces that have pointwise 1-type Gauss map whose details are given in Section 2. These surfaces are called rectifying, generalized normal and osculating type ruled surfaces defined in [11, 12, 15]. We obtain characterizations for such surfaces to have pointwise 1-type Gauss map. We also give results when such surfaces are minimal.

2. PRELIMINARIES

An isometric immersion (a smooth map, respectively) $x : M \to E^3$ of a submanifold (a 2-dimensional Riemannian manifold, respectively) M in the Euclidean 3-space

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E^3 is of finite type if the position vector field x of the submanifold M can be given as a finite sum of the eigenvectors of Laplacian Δ of M which is $x = x_0 + \sum_{i=1}^{j} x_i$ where x_0 is a constant map and $x_1, x_2, ..., x_j$ are non-constant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $1 \leq i \leq j$. If the eigenvalues $\lambda_1, \lambda_2, ...\lambda_j$ are different from each other, then M is of j-type. Especially, if one of the eigenvalues λ_i vanishes, then M is of null j-type [4, 5].

Let M be a connected (not necessarily compact) surface in the Euclidean 3-space E^3. Then the position vector field x and the mean curvature vector H of the surface M in the Euclidean 3-space E^3 satisfy the equation $\Delta x = -2\mathrm{H}$ where $\mathrm{H} = HG$, H is the mean curvature and G is the Gauss map of the surface. This equality leads to a well-recognized result which is: A surface M in the Euclidean 3-space E^3 is a minimal surface if and only if all coordinate functions of E^3 that are restricted to the surface M are harmonic functions, i.e.,

$$\Delta x = 0 \tag{2.1}$$

[4, 16].

In a Euclidean space, a submanifold M is called to be of 1-type, i.e., the position vector field x of M satisfies the equation $\Delta x = \lambda x$, $\lambda \in \mathbb{R}$ if and only if either the submanifold minimal ($\lambda = 0$) or M is a minimal submanifold of a hypersphere centered at the origin ($\lambda \neq 0$) of the Euclidean space which M is in [16].

Let M be a surface in E^3 with the parametric form $x = x(u, v)$. The Gauss map (or the unit surface normal) of M is defined by

$$G : M \to S^2 \subset E^3, \quad G(u, v) = \left(\frac{x_u \times x_v}{\|x_u \times x_v\|} \right)(u, v)$$

where S^2 denotes the unit sphere and $x_u = \partial x / \partial u$, $x_v = \partial x / \partial v$ are partial derivatives [8].

Let the matrix (g_{ij}) consist of the components of the Riemannian metric on M. Then,

$$(g_{ij}) = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} \langle x_u, x_u \rangle & \langle x_u, x_v \rangle \\ \langle x_v, x_u \rangle & \langle x_v, x_v \rangle \end{bmatrix}.$$

If the matrix $\left(g^{ij}\right)$ is the inverse of (g_{ij}) and g is the determinant of (g_{ij}), then, the Laplacian Δ on M is given by

$$\Delta = -\frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x_j} \right). \tag{2.2}$$

The surface M has pointwise 1-type Gauss map if the Gauss map G satisfy the condition

$$\Delta G = f(G + C) \tag{2.3}$$

where $f = f(u, v)$ is a real valued function and C is a constant vector. A pointwise 1-type Gauss map is called proper if the function f given in (2.3) is non-constant.

If the vector C vanishes, then the surface is called a surface with pointwise 1-type Gauss map of the first kind. If not, it is called of the second kind [6].

Here, we give some examples of surfaces with pointwise 1-type Gauss map.

Example 1. Let the surface M_1 be given by the parametrization $x(u,v) = (u,0,0) + v(0,\cos u,\sin u)$, whose Gauss map is $G(u,v) = \frac{1}{\sqrt{v^2+1}}(-v,-\sin u,\cos u)$. Then, we get $\Delta G = \frac{2}{(v^2+1)^2}G$ where $f = \frac{2}{(v^2+1)^2}$ and $C = 0$. Therefore, the surface M_1 has pointwise 1-type Gauss map of the first kind. On the other hand, the Laplacian of the position vector x of the surface M_1 vanishes, i.e., $\Delta x = 0$. Then, M_1 is minimal. The graph of the surface M_1 is given in Figure 1.

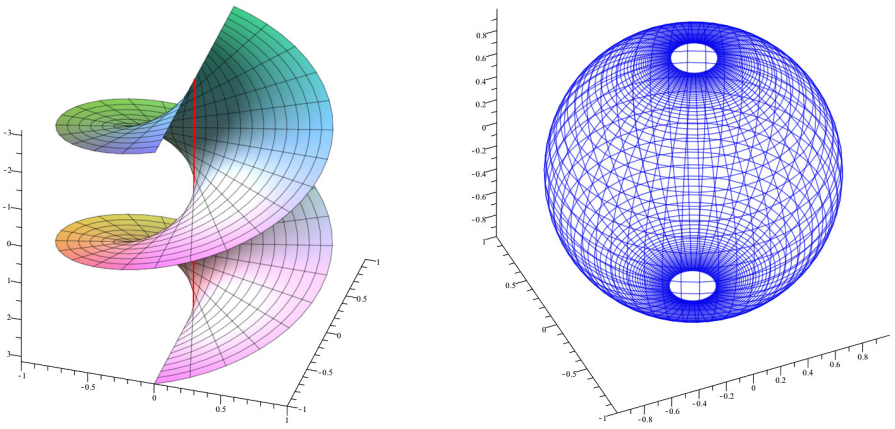

FIGURE 1. M_1 with its base curve α (red) and Gauss map (blue).

Example 2. Let the surface M_2 be given by the parametrization $x(u,v) = (v\cos u, v\sin u, v)$, whose Gauss map is $G(u,v) = \pm\frac{\sqrt{2}}{2}(\cos u,\sin u,-1)$. Then, we obtain $\Delta G = v^2\left[G \pm \left(0,0,\frac{\sqrt{2}}{2}\right)\right]$ where $f = v^2$ and $C = \pm\left(0,0,\frac{\sqrt{2}}{2}\right)$. Therefore, the surface M_2 has pointwise 1-type Gauss map of second kind. The surface M_2 does not have harmonic type position vector, thus it is not minimal. The graph of the surface M_2 is given in Figure 2.

Let $\alpha : I \subset \mathbb{R} \to E^3$ be a regular curve in the Euclidean 3-space E^3 given by the arc length parameter u and with Frenet frame $\{T,N,B\}$, curvature κ and torsion τ. The vector fields T,N,B defined by $T = \alpha'$, $N = \alpha''/\|\alpha''\|$, $B = T \times N$ are called unit tangent, principal normal and binormal vector, respectively, where the prime denotes the derivative with respect to u. Then, $\{T,N,B\}$ is an orthogonal moving frame along α and called Frenet frame of it. The derivative formulae (or Frenet formulae) of that frame are given by $T' = \kappa N$, $N' = -\kappa T + \tau B$, $B' = -\tau N$. The

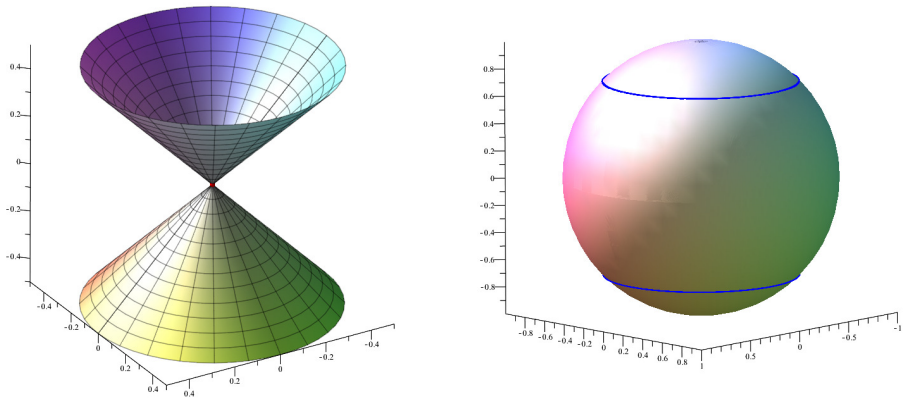

FIGURE 2. M_2 with its base curve α (red) and Gauss map (blue).

planes $Sp\{T,N\}$, $Sp\{T,B\}$ and $Sp\{N,B\}$ are called the osculating, the rectifying and the normal planes of α, respectively [8].

A surface $M : I \times J \to \mathbb{R}^3$, $I,J \subset \mathbb{R}$ given by the parametrization $x(u,v) = \alpha(u) + vq(u)$ is called a ruled surface where $u \in I$ and $v \in J$. The curve α is called base curve and q is called director curve. The straight lines $v \to \alpha(u) + vq(u)$ are called rulings [10]. Clearly, a ruled surface is constructed by the continuous motion of a straight line and that makes this surface important in some areas such as computer aided geometric design and architecture [9]. A ruled surface is called developable if and only if $\det(\alpha', q, q') = 0$ [10].

3. DEVELOPABLE GENERAL RECTIFYING RULED SURFACES WITH POINTWISE 1-TYPE GAUSS MAP

The parametric form of a general rectifying ruled surface M is given by $x(u,v) = \alpha(u) + vq_r(u)$, $q_r(u) = a_1(u)T(u) + a_2(u)B(u)$ where u is the arc-length parameter of α, $a_1 = a_1(u)$, $a_2 = a_2(u)$ are smooth functions of u such that $a_1^2 + a_2^2 = 1$ and the index "r" is used to remark that the ruling always lies on the general rectifying plane of the base curve α [15]. The general rectifying ruled surface M is developable if and only if $a_1\kappa - a_2\tau = 0$ [15]. Hereafter, we will assume that the general rectifying ruled surface M is developable.

The Gauss map G of the developable general rectifying surface (dgr-surface) M is $G(u,v) = -N(u)$. Then, we obtain

$$G_u = \kappa T - \tau B, \quad G_{uu} = \kappa' T + \left(\kappa^2 + \tau^2\right) N - \tau' B, \quad G_v = G_{uv} = G_{vu} = G_{vv} = 0.$$

Also, the components and the determinant g of the matrix (g_{ij}) are obtained as

$$g_{11} = \left(1 + va'_1\right)^2 + v^2(a'_2)^2,\ g_{12} = g_{21} = a_1,\ g_{22} = 1 \text{ and } g = \left(1 + va'_1\right)^2 + v^2(a'_2)^2 - a_1^2$$

respectively. From (2.2), the Laplacian operator on the *dgr*-surface M is

$$\Delta = -\xi\left[\frac{\partial}{\partial u}\left(\xi\frac{\partial}{\partial u} - a_1\xi\frac{\partial}{\partial v}\right) + \frac{\partial}{\partial v}\left(-a_1\xi\frac{\partial}{\partial u} + \xi\left[\left(1+va_1'\right)^2 + v^2(a_2')^2\right]\frac{\partial}{\partial v}\right)\right]$$

where ξ is a smooth function given by $\xi(u,v) = \frac{1}{\sqrt{\left(1+va_1'\right)^2+v^2(a_2')^2-a_1^2}}$. Then, the Laplacian of the Gauss map of M is computed as

$$\Delta G = -\xi\left(\kappa(\xi_u - a_1\xi_v) + \kappa'\xi\right)T - \left(\kappa^2+\tau^2\right)\xi^2 N + \xi\left(\tau(\xi_u - a_1\xi_v) + \tau'\xi\right)B. \quad (3.1)$$

From (3.1) we have that $\Delta G = 0$ if and only if $\kappa = 0$, $\tau = 0$. Then, the following theorem is obtained:

Theorem 1. *A dgr-surface M has harmonic type Gauss map G if and only if its base curve α is a straight line.*

Moreover, from (3.1), it yields that $\Delta G = fG$, $f = \left(\kappa^2+\tau^2\right)\xi^2$ if and only if the equations $\kappa(\xi_u - a_1\xi_v) + \kappa'\xi = 0$, $\tau(\xi_u - a_1\xi_v) + \tau'\xi = 0$ hold and we have the following theorem:

Theorem 2. *A dgr-surface M has point-wise 1-type Gauss map of the first kind if and only if the curvatures do not vanish at the same time and the system*

$$\kappa(\xi_u - a_1\xi_v) + \kappa'\xi = 0, \quad \tau(\xi_u - a_1\xi_v) + \tau'\xi = 0 \quad (3.2)$$

holds.

Theorem 2 gives us the following special cases:

Case 1: Let the base curve α be a plane curve different from a straight line. Then, $\kappa \neq 0$, $\tau = 0$. From the developable condition we have $a_1 = 0$, $a_2 = 1$, $\xi = 1$ and (3.1) becomes $\Delta G = -\left(\kappa' T + \kappa^2 N\right)$. We have that $\Delta G = fG$ with $f = \kappa^2$ if and only if κ is constant. Then, we obtain the following theorem:

Theorem 3. *Let the base curve α be a plane curve. Then the dgr-surface M has the pointwise 1-type Gauss map of the first kind if and only if α is a circle.*

Case 2: Let the base curve α be a circular helix with non-zero curvatures κ, τ. Then, from the system (3.2) we have that $\xi_u - a_1\xi_v = 0$ which gives the following theorem:

Theorem 4. *Let the base curve α be a circular helix. Then the dgr-surface M has the pointwise 1-type Gauss map of the first kind if and only if $\xi_u - a_1\xi_v = 0$ holds.*

Case 3: Assume that one of the curvatures κ, τ is non-zero constant. Then, from the system (3.1), we have that the other one is also non-zero constant and following theorem is obtained:

Theorem 5. *If one of the curvature functions κ, τ of base curve α is a non-zero constant, then the dgr-surface M has the pointwise 1-type Gauss map of the first kind if and only if α is a circular helix.*

Now let us examine the Laplacian of the position vector x of dgr-surface M. For any surface M we have that $\Delta x = 0$ holds if and only if M is minimal [16]. Also, we know that the dgr-surface M is minimal if and only if $(1+va'_1)\kappa - va'_2\tau = 0$ holds [15]. Then, by taking the partial derivatives of x, from (2.2) it follows

$$\begin{aligned}\Delta x = &-\xi\Big\{\left(1+va'_1\right)\xi_u + (va''_1 - 2a_1a'_1)\xi - a_1(a_1\xi)_u \\ &\qquad -\left(1+va'_1\right)a_1\xi_v + a_1\left[\left(\left(1+va'_1\right)^2 + v^2(a'_2)^2\right)\xi\right]_v\Big\}T \\ &-\xi^2\left\{\left(1+va'_1\right)\kappa - va'_2\tau\right\}N \\ &-\xi\Big\{va'_2\xi_u + (va''_2 - 2a_1a'_2)\xi - a_2(a_1\xi)_u \\ &\qquad -va_1a'_2\xi_v + a_2\left[\left(\left(1+va'_1\right)^2 + v^2(a'_2)^2\right)\xi\right]_v\Big\}B.\end{aligned} \tag{3.3}$$

We have that $\Delta x = 0$ if and only if the system

$$\begin{cases}\left(1+va'_1\right)(\xi_u - a_1\xi_v) + (va''_1 - 2a_1a'_1)\xi \\ \quad -a_1(a_1\xi)_u + a_1\left[\left(\left(1+va'_1\right)^2 + v^2(a'_2)^2\right)\xi\right]_v = 0 \\ \left(1+va'_1\right)\kappa - va'_2\tau = 0 \\ va'_2(\xi_u - a_1\xi_v) + (va''_2 - 2a_1a'_2)\xi \\ \quad -a_2(a_1\xi)_u + a_2\left[\left(\left(1+va'_1\right)^2 + v^2(a'_2)^2\right)\xi\right]_v = 0\end{cases} \tag{3.4}$$

holds. Then, we get the followings:

Theorem 6. *Let M be a dgr-surface. Then, the followings are equivalent:*

(i) M has harmonic type position vector.
(ii) M is minimal.
(iii) $(1+va'_1)\kappa - va'_2\tau = 0$ holds.
(iv) The system in (3.4) holds.

Using the equality $(1+va'_1)\kappa - va'_2\tau = 0$, the first and third equalities in (3.4) become

$$va'_2\frac{\tau}{\kappa}(\xi_u - a_1\xi_v) + (va''_1 - 2a_1a'_1)\xi - a_1(a_1\xi)_u + a_1\left[\left(v^2(a'_2)^2\left(1+\frac{\tau^2}{\kappa^2}\right)\right)\xi\right]_v = 0,$$

$$va'_2(\xi_u - a_1\xi_v) + (va''_2 - 2a_1a'_2)\xi - a_2(a_1\xi)_u + a_2\left[\left(v^2(a'_2)^2\left(1+\frac{\tau^2}{\kappa^2}\right)\right)\xi\right]_v = 0,$$

respectively. Then, we get

$$va'_2(\xi_u - a_1\xi_v)\left(a_2\frac{\tau}{\kappa} - a_1\right) + a_2(va''_1 - 2a_1a'_1)\xi - a_1(va''_2 - 2a_1a'_2)\xi = 0.$$

Since the surface is developable, we have $a_1\kappa - a_2\tau = 0$ which gives $a_2(va_1'' - 2a_1a_1') - a_1(va_2'' - 2a_1a_2') = 0$ and following theorem is given:

Theorem 7. *If the dgr-surface M has the harmonic type position vector (or, equivalently minimal), then*

$$\frac{\tau}{\kappa} = \frac{va''_1 - 2a_1a'_1}{va''_2 - 2a_1a'_2}$$

holds.

Example 3. Let the space curve α be given by the parametrization $\alpha(u) = \left(\cos\frac{u}{\sqrt{2}}, \sin\frac{u}{\sqrt{2}}, \frac{u}{\sqrt{2}}\right)$. Then, by calculating its Frenet elements and choosing $a_1(u) = a_2(u) = 1/\sqrt{2}$, we obtain the *dgr*-surface M_3 with the parametrization $x(u,v) = \left(\cos\frac{u}{\sqrt{2}}, \sin\frac{u}{\sqrt{2}}, \frac{u}{\sqrt{2}} + v\right)$. The Gauss map of *dgr*-surface M_3 and the Laplacian of Gauss map are computed as $G(u,v) = \left(\cos\frac{u}{\sqrt{2}}, \sin\frac{u}{\sqrt{2}}, 0\right)$ and $\Delta G = \left(\cos\frac{u}{\sqrt{2}}, \sin\frac{u}{\sqrt{2}}, 0\right)$, respectively. Finally, we obtain that $\Delta G = G$ which indicates that M_3 has 1-type Gauss map of first kind. This result satisfies Theorem 2. The second equation given in Theorem 6 is not satisfied since $(1 + va'_1)\kappa - va'_2\tau = \frac{1}{2} \neq 0$. Then, the *dgr*-surface M_3 is not minimal. So, it does not have harmonic type position vector. The graph of the *dgr*-surface M_3 is given in Figure 3.

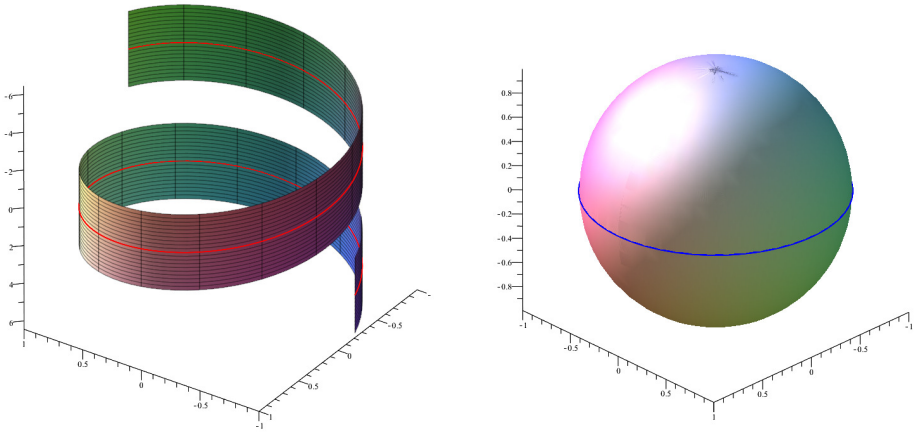

FIGURE 3. M_3 with its base curve α (red) and Gauss map (blue).

4. DEVELOPABLE GENERALIZED NORMAL RULED SURFACES WITH POINTWISE 1-TYPE GAUSS MAP

The parametric form of a generalized normal surface M is given by $x(u,v) = \alpha(u) + vq_n(u)$, $q_n(u) = b_1(u)T(u) + b_2(u)B(u)$ where u is the arc length parameter

of base curve α, $b_1 = b_1(u)$, $b_2 = b_2(u)$ are smooth functions of u such that $b_1^2 + b_2^2 = 1$ and the index "n" is used to remark that the ruling always lies on the normal plane of the base curve α [12]. Such a surface is developable if and only if $b_1' b_2 - b_1 b_2' - \tau = 0$ holds. For the developable generalized normal surface (*dgn*-surface) M, we also have $b_1\tau + b_2' = 0$, $b_1' - b_2\tau = 0$ [12].

The Gauss map G of the dgn-surface M is obtained as $G(u,v) = -b_2(u)N(u) + b_1(u)B(u)$. The matrix (g_{ij}) has the form

$$(g_{ij}) = \begin{pmatrix} (1 - vb_1\kappa)^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

The determinant g is computed as $g = (1 - vb_1\kappa)^2$. Finally, from (2.2) the Laplacian operator on *dgn*-surface M is

$$\Delta = -\frac{1}{\psi}\left[\frac{\partial}{\partial u}\left(\frac{1}{\psi}\frac{\partial}{\partial u}\right) + \frac{\partial^2}{\partial v^2}\right] \tag{4.1}$$

where ψ is a differentiable function defined by $\psi(u,v) = 1 - vb_1\kappa$. Then, from (4.1) the Laplacian of the Gauss map is

$$\Delta G = \frac{[b_2\kappa\psi]_u}{\psi^3}T - \frac{b_2\kappa^2}{\psi^2}N \tag{4.2}$$

and we have the followings:

Theorem 8. *A dgn-surface M has pointwise 1-type Gauss map of the first kind, that is $\Delta G = fG$ where $f = \kappa^2$ if and only if $\kappa =$ constant and $b_1 = 0$.*

Proof. From (4.2), we obtain that $\Delta G = fG$ holds if and only if $[b_2\kappa\psi]_u = 0$ and $b_1 = 0$ which lead us to the desired result. □

Now, let us examine the Laplacian of the position vector x of the dgn-surface M. From (4.1), we obtain

$$\Delta x = \frac{b_2\kappa}{\psi}(-b_2N + b_1B). \tag{4.3}$$

Then, from the equality $\Delta x = 0$, we get the following:

Theorem 9. *A dgn-surface M has harmonic type position vector (or, equivalently minimal) if and only if $b_2 = 0$ or $\kappa = 0$, i.e., α is a straight line.*

Moreover, from (4.3), we also have the following theorem:

Theorem 10. *The position vector x and Gauss map G of a dgn-surface M satisfy the equation $\Delta x = \frac{b_2\kappa}{\psi}G$.*

Using the equality $\Delta x = -2\mathrm{H}$ given in [4], from Theorem 10 we have that the mean curvature H of a *dgn*-surface M is $H = -\frac{b_2\kappa}{2\psi}$, where $\mathrm{H} = HG$ is the mean curvature vector and H is the mean curvature of M. This result also coincides with the calculated value of mean curvature H of a *dgn*-surface M in [12].

Example 4. Let choose the unit circle $\alpha(u) = (\cos u, \sin u, 0)$ as the base curve. By taking, $b_1 = 0$, $b_2 = 1$, the parametric form of a dgn-surface M_4 is obtained as $x(u,v) = (\cos u, \sin u, v)$. The Gauss map of M_4 is $G = (\cos u, \sin u, 0)$. Then, by a simple calculation, we obtain $\Delta G = G = (\cos u, \sin u, 0)$ which shows that the *dgn*-surface M_4 has 1-type Gauss map. Since $b_2 \neq 0$ or $\kappa \neq 0$, by Theorem 9, the *dgn*-surface M_4 is not minimal. Then, it does not have harmonic type position vector. The graph of the *dgn*-surface M_4 is given in Figure 4.

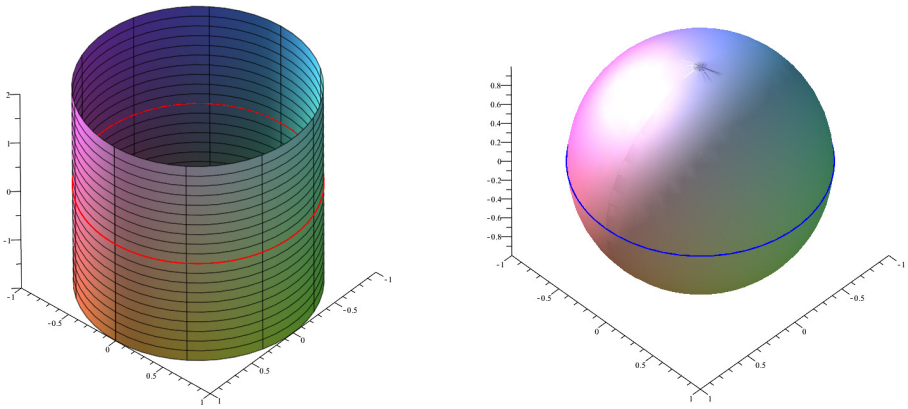

FIGURE 4. M_4 with its base curve α (red) and Gauss map (blue).

5. DEVELOPABLE OSCULATING TYPE RULED SURFACES WITH 1-TYPE GAUSS MAP

An osculating type ruled surface M has the parametric form $x(u,v) = \alpha(u) + vq_o(u)$, $q_o(u) = \cos\theta T(u) + \sin\theta N(u)$ where u is the arclength parameter of base curve α, $\theta = \theta(u)$ is a smooth function of u and the index "o" is used to remark that the ruling always lies on the osculating plane of the base curve α [11]. A new orthonormal frame $\{q_o, B, r\}$ has been defined for osculating type ruled surfaces where $r = q_o \times B$. The derivative formulae for this frame are

$$\begin{bmatrix} q_o' \\ B' \\ r' \end{bmatrix} = \begin{bmatrix} 0 & \mu & -\eta \\ -\mu & 0 & \omega \\ \eta & -\omega & 0 \end{bmatrix} \begin{bmatrix} q_o \\ B \\ r \end{bmatrix}$$

where $\mu = \tau\sin\theta$, $\omega = \tau\cos\theta$, $\eta = \theta' + \kappa$ [11]. The osculating type ruled surface M is developable if and only if $\tau = 0$ or $\sin\theta = 0$. In both cases, we get $\mu = 0$.

From (2.2), the Laplacian operator on developable osculating type ruled surface (*dot*-surface) M is obtained as

$$\Delta = -\zeta\left[\frac{\partial}{\partial u}\left(\zeta^3\frac{\partial}{\partial u}-\zeta^3\cos\theta\frac{\partial}{\partial v}\right)+\frac{\partial}{\partial v}\left(-\zeta^3\cos\theta\frac{\partial}{\partial u}+\zeta^3\left[\zeta^{-2}+\cos^2\theta\right]\frac{\partial}{\partial v}\right)\right]$$

where $\zeta(u,v)=\frac{1}{\sin\theta-v\eta}$ is a smooth function. The Gauss map of the *dot*-surface M is obtained as $G(u,v)=B(u)$. In order to compute the Laplacian of the Gauss map, the partial derivatives of G, components and determinant of the matrix (g_{ij}) are obtained as $G_u=-\tau N, G_{uu}=\kappa\tau T-\tau' N-\tau^2 B, G_v=G_{uv}=G_{vv}=0, \langle x_u,x_u\rangle=(\sin\theta-v\eta)^2+\cos^2\theta, \langle x_u,x_v\rangle=\langle x_v,x_u\rangle=\cos\theta, \langle x_v,x_v\rangle=1, g=(\sin\theta-v\eta)^2$ respectively. Then, the Laplacian of Gauss map is

$$\Delta G=-\kappa\tau\zeta T+\left[-\tau\zeta^3\left(\zeta^{-1}\right)_u+\tau'\zeta^2-\tau\cos\theta\zeta\zeta_v\right]N+\tau^2\zeta^2 B. \tag{5.1}$$

If $\tau=0$, then the surface is a plane and $\Delta G=0$ holds. So, we consider the second case $\sin\theta=0$ for the Laplacian of Gauss map. For this case, we have $\cos\theta=1$ and from (5.1) it follows $\Delta G=-\frac{1}{v^2\eta^2}\left(\kappa\tau T-\tau' N-\tau^2 B\right)$ which gives that $\Delta G\neq fG$ since we assume $\tau\neq 0$. In this case, the *dot*-surface M does not have 1-type Gauss map. Therefore, we have the following theorem:

Theorem 11. *A dot-surface M has pointwise 1-type Gauss map of first kind if and only if M is a plane.*

For the Laplacian of the position vector x of M, we calculate the partial derivatives as follows $x_{uu}=\kappa N+v\left(-\eta' r-\eta^2 q_o\right)$, $x_{uv}=-\eta r$, $x_{vv}=0$ and investigate the following cases:

Case 1: Let $\tau=0$ and $\sin\theta\neq 0$. Then, the Laplacian of the position vector x of the surface M is obtained as

$$\begin{aligned}\Delta x=&-\zeta\left\{-\zeta^2\left(\zeta^{-1}\right)_u-\zeta_v\cos\theta\right.\\&+\left[-v\eta^2\zeta-\left(\zeta\cos\theta\right)_u+\left(\zeta(\zeta^{-2}+\cos^2\theta)\right)_v\right]\cos\theta\\&\left.+\left[v\eta\zeta^2\left(\zeta^{-1}\right)_u-v\eta'\zeta+2\eta\zeta\cos\theta-v\eta\zeta_v\cos\theta\right]\sin\theta\right\}T\\&-\zeta\left\{\kappa\zeta+\left[-v\eta^2\zeta-\left(\zeta\cos\theta\right)_u+\left(\zeta(\zeta^{-2}+\cos^2\theta)\right)_v\right]\sin\theta\right.\\&\left.-\left[v\eta\zeta^2\left(\zeta^{-1}\right)_u-v\eta'\zeta+2\eta\zeta\cos\theta-v\eta\zeta_v\cos\theta\right]\cos\theta\right\}N.\end{aligned} \tag{5.2}$$

When $\tau=0$, i.e., the base curve is planar, then we have that the dot-surface is a plane. Then, the equality given in (5.2) is always equal to zero, $\Delta x=0$, in Case 1. Thus, we have the following corollary:

Corollary 1. *A dot-surface M with a planar base curve satisfy the equations*

$$\begin{cases}-\zeta^2\left(\zeta^{-1}\right)_u-\zeta_v\cos\theta+y_1\cos\theta+y_2\sin\theta=0\\ \kappa\zeta+y_1\sin\theta-y_2\cos\theta=0\end{cases}$$

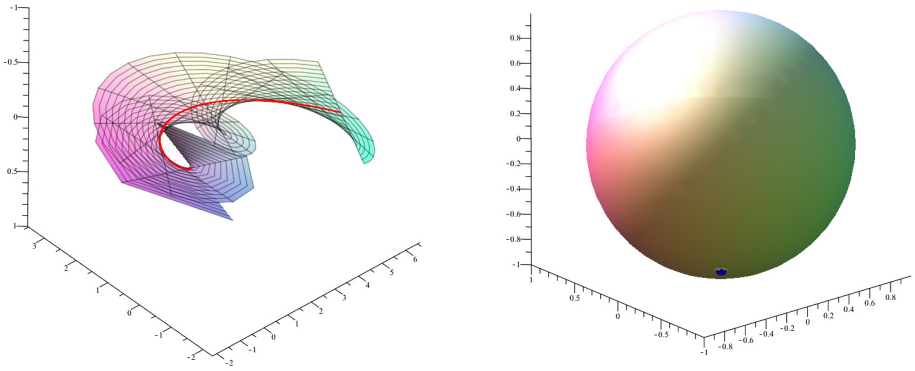

FIGURE 5. M_5 with base curve α (red) and Gauss map (blue).

where

$$y_1 = -v\eta^2\zeta - (\cos\theta\zeta)_u + \left(\zeta(\zeta^{-2} + \cos^2\theta)\right)_v,$$
$$y_2 = v\eta\zeta^2\left(\zeta^{-1}\right)_u - v\eta'\zeta + 2\eta\cos\theta\zeta - v\eta\cos\theta\zeta_v.$$

Case 2: Let $\tau \neq 0$ and $\sin\theta = 0$. Then, the Laplacian of the position vector x of M is given by $\Delta x = \frac{2}{v^3\eta^2}T + \frac{\theta'}{v^2\eta^2}N$. Therefore, in this case, we get $\Delta x \neq 0$.

From the above cases, the following theorem is obtained:

Theorem 12. *A dot-surface M has harmonic type position vector (or, equivalently minimal) if and only if its base curve is planar.*

Example 5. Let consider the plane curve α given by the parametrization

$$\alpha(u) = \left(\left(\frac{u}{\sqrt{2}} + 1\right)\cos\left(\ln\left(\frac{u}{\sqrt{2}} + 1\right)\right), \left(\frac{u}{\sqrt{2}} + 1\right)\sin\left(\ln\left(\frac{u}{\sqrt{2}} + 1\right)\right), 0\right).$$

Then, by choosing $\theta(u) = u/\sqrt{2}$, a *dot*-surface M_5 can be given by the parametrization

$$x(u,v) = \alpha(u) + v\left(\cos\left(\frac{u}{\sqrt{2}}\right)T(u) + \sin\left(\frac{u}{\sqrt{2}}\right)N(u)\right)$$

where $T(u)$ and $N(u)$ are tangent vector and principal vector, respectively. The Gauss map of *dot*-surface M_5 is computed as $G(u,v) = (0,0,-1)$ and finally, we obtain that $\Delta G = 0$ which indicates that the dot-surface M_5 has pointwise 1-type Gauss map of the first kind. Furthermore, since the base curve α is planar, the surface M_5 is a plane. Then, it is minimal and has harmonic type position vector. The graph of the *dot*-surface M_5 is given in Figure 5.

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GREEN'S FUNCTIONS OF THE WAVE EQUATION IN DIFFERENT DIMENSIONS

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Abstract. Relations of different Green's functions of the wave equation in different dimensions, with each other and with the Green's function of the Laplace equation, are investigated. These are used to obtain the Green's functions of the wave equation in different dimensions.

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Keywords: Green's function, wave equation, Laplace equation, different dimensions

1. INTRODUCTION

The simple wave equation is a hyperbolic partial differential equation (see [12] for example). This equation arises in many problems. Among these are the vibrations of a string, the vibrations of a membrane, the propagation of sound in fluids ([5–7] respectively, for example). Another important example is the propagation of light, or electromagnetic field [4]. All of these phenomena have one thing in common, and that thing is propagation with finite speed. In more technical terms, the effect of a point source is not propagated outside the wave cone (light cone in the case of light [10]).

As the wave equation is linear, one way to investigate the effect of a point source on the wave is to study the Green's function [1]. One important question is whether the effect of point source is propagated only on the wave cone or on the wave cone and inside the wave cone as well. One can also ask whether this behaviour depends on the dimension of the space or not.

The wave equation is a hyperbolic equation, as mentioned before. But if one performs an analytic continuation on the time variable, it is seen that for imaginary time the wave equation is transformed to the Laplace equation, [13]. The Laplace equation arises in many cases as well, including the problem of finding the electrostatic potential [2, 8, 9]. While the wave equation is hyperbolic, the Laplace equation is elliptic and the properties of the solutions of the wave equation and the Laplace equation are totally different. Yet there arises a question on the possible relation of a solution of the Laplace equation with a solution of the wave with imaginary time coordinate. To

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be more specific, is it possible to obtain the Green's function of the wave equation by performing an analytic continuation on the Green's function of the Laplace equation?

Finally, the qualitative behaviour of the Green's functions of the Laplace equation in different dimensions are almost similar, specially if the dimension of the space is larger than two. Is this the case for the Green's function of the wave equation as well?

The aim of this paper is to address these questions. It is shown that the Green's function of the wave equation is related to that of the Laplace equation through an analytic continuation. But this is the case only for a certain kind of the Green's functions of the wave equation, which is the Feynman Green's function. This kind of the Green's function does not arise very often in the context of classical physics, but it plays a crucial role in quantum physics [3]. Yet it is possible to express the retarded and advanced Green's function (which arise more often in classical physics) in terms of the Feynman Green's function. Using these, various kinds of the Green's function of the wave equation in different dimensions are obtained. As a result, it is seen that for odd space dimensions (except for the one dimensional space) the effect of a point source is propagated only on the wave cone, while for even dimensional spaces propagation takes place inside the wave cone as well.

The scheme of the paper is the following. In Section 2 the wave equation and the equation governing its Green's function are introduced. In Section 3 various kinds of the Green's function of the wave equation are discussed. In Section 4 the relations between these Green's functions are studied. In Section 5 a recursive relation in terms of the dimension of the space is presented for the Green's function of the wave equation. In Section 6 the Feynman Green's function is obtained as an analytic continuation of the Green's function of the Laplace equation. Using that, other Green's functions of the wave equation are obtained as well. Section 7 is devoted to concluding remarks.

2. The wave equation and its Green's function

The (homogeneous) simple wave equation is

$$\left(-\frac{\partial^2}{\partial t^2}+\nabla^2\right)\psi(t,\boldsymbol{r})=0, \tag{2.1}$$

where $(t,\boldsymbol{r})$ is a point in the space-time, and the units are chosen so that the speed of the wave is unity. The general solution of this equation is an arbitrary linear combination of plain waves $\exp(-\mathrm{i}\,\omega t+\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{r})$, where ω and $\boldsymbol{k}$ satisfy the dispersion relation

$$\omega^2-\boldsymbol{k}\cdot\boldsymbol{k}=0. \tag{2.2}$$

The wave equation with source is

$$\left(-\frac{\partial^2}{\partial t^2}+\nabla^2\right)\psi(t,\boldsymbol{r})=j(t,\boldsymbol{r}), \tag{2.3}$$

where j (the source) is an arbitrary function of the space-time. As this equation is linear, its solution is

$$\psi(t,\boldsymbol{r}) = \int \mathrm{d}t\,\mathrm{d}V\; G(t,\boldsymbol{r};t',\boldsymbol{r}')\, j(t',\boldsymbol{r}') + \psi_{\mathrm{h}}(t,\boldsymbol{r}). \tag{2.4}$$

ψ_{h} is an arbitrary function satisfying the homogeneous wave equation (2.1), and G is the Green's function of the wave equation, satisfying

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right) G(t,\boldsymbol{r};t',\boldsymbol{r}') = \delta(t-t')\,\delta^D(\boldsymbol{r}-\boldsymbol{r}'). \tag{2.5}$$

D is the dimension of the space. The wave equation is space-time-independent, in the sense that in the differential operator in the left-hand side, there are no explicit functions of space-time. This results in the fact that one can choose G to be a function of the difference of the primed and unprimed variables:

$$G(t,\boldsymbol{r};t',\boldsymbol{r}') = G(t-t',\boldsymbol{r}-\boldsymbol{r}'). \tag{2.6}$$

Using this, equation (2.5) is rewritten as

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right) G(t,\boldsymbol{r}) = \delta(t)\,\delta^D(\boldsymbol{r}). \tag{2.7}$$

3. Different Green's functions of the wave equation

The solution to (2.5) or (2.7) is not unique. In fact, adding any solution of the homogenous equation (2.1) to an arbitrary solution of (2.7), another solution of (2.7) is obtained. Another way to see this, is using the Fourier transformation. The Fourier transform of (2.7) is

$$(\omega^2 - \boldsymbol{k}\cdot\boldsymbol{k})\,\tilde{G}(\omega,\boldsymbol{k}) = 1. \tag{3.1}$$

The solution to this is not unique: Any function of the form

$$\tilde{G}_{\mathrm{h}} = f(\omega,\boldsymbol{k})\,\delta(\omega^2 - \boldsymbol{k}\cdot\boldsymbol{k}) \tag{3.2}$$

satisfies the homogenous equation

$$(\omega^2 - \boldsymbol{k}\cdot\boldsymbol{k})\,\tilde{G}_{\mathrm{h}}(\omega,\boldsymbol{k}) = 0, \tag{3.3}$$

where f is an arbitrary function. The meaning of (3.2) is that G_{h} is a superposition of plane waves, satisfying the dispersion relation (2.2). So adding such a function $\tilde{G}_{\mathrm{h}}$ to a solution of (3.1), gives another solution of (3.1). Another point (of course related to this) is that to obtain the Green's function from its Fourier transform, this Fourier transform should be rigorously defined. The solution to (3.1) is

$$\tilde{G}(\omega,\boldsymbol{k}) = \mathrm{pf}\,\frac{1}{\omega^2 - \boldsymbol{k}\cdot\boldsymbol{k}} + \tilde{G}_{\mathrm{h}}(\omega,\boldsymbol{k}), \tag{3.4}$$

where $\mathrm{pf}\,(1/x)$ is a distribution the effect of which on the test function ϕ is [11]

$$\langle \mathrm{pf}\,(1/x), \phi(x)\rangle = \lim_{\varepsilon\to 0^+}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right)\frac{\mathrm{d}x\,\phi(x)}{x}. \tag{3.5}$$

Three special choices for $\tilde{G}$ are the retarded Green's function:

$$\tilde{G}_{\mathrm{R}} = \frac{1}{(\omega + \mathrm{i}\varepsilon)^2 - \boldsymbol{k}\cdot\boldsymbol{k}}, \tag{3.6}$$

the advanced Green's function:

$$\tilde{G}_{\mathrm{A}} = \frac{1}{(\omega - \mathrm{i}\varepsilon)^2 - \boldsymbol{k}\cdot\boldsymbol{k}}, \tag{3.7}$$

and the Feynman Green's function:

$$\tilde{G}_{\mathrm{F}} = \frac{1}{\omega^2 - \boldsymbol{k}\cdot\boldsymbol{k} + \mathrm{i}\varepsilon}. \tag{3.8}$$

In all cases, the limit $\varepsilon \to 0^+$ is meant. The relation

$$G(t,\boldsymbol{r}) = \frac{1}{(2\pi)^{D+1}} \int \mathrm{d}\omega \mathrm{d}^D k\, \tilde{G}(\omega,\boldsymbol{k}) \exp(-\mathrm{i}\omega t + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}) \tag{3.9}$$

relates each of these Green's functions to its Fourier transform. The integral over ω can be transformed (for $t > 0$ and $t < 0$) to an integral over a closed path on the complex ω plane:

$$G(t,\boldsymbol{r}) = \frac{1}{(2\pi)^{D+1}} \int \mathrm{d}^D k \int_{S_\pm} \mathrm{d}\omega\, \tilde{G}(\omega,\boldsymbol{k}) \exp(-\mathrm{i}\omega t + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}) \quad \pm t > 0. \tag{3.10}$$

S_+ is a path beginning at $\omega \to -\infty$, going on the real line to $\omega \to +\infty$, and returning on an infinite semicircle in the half-plane $\mathrm{Im}(\omega) < 0$ to $\omega \to -\infty$. S_- is a path beginning at $\omega \to -\infty$, going on the real line to $\omega \to +\infty$, and returning on an infinite semicircle in the half-plane $\mathrm{Im}(\omega) > 0$ to $\omega \to -\infty$. This shows that

$$G_{\mathrm{R}} = 0, \qquad t < 0, \tag{3.11}$$

and

$$G_{\mathrm{A}} = 0, \qquad t > 0, \tag{3.12}$$

as in each case the integrand as a function of ω is analytic inside the integration contour. The reason for calling these Green's functions retarded and advanced, is that the retarded Green's function is nonzero only for $t > 0$, while the advanced Green's function is nonzero only for $t < 0$. In the case of the Feynman Green's function, it is seen that in $t > 0$ only terms with positive frequency (ω) contribute, while for $t < 0$ only terms with negative frequency (ω) contribute.

Consider two auxiliary functions G_+ and G_-. Their definition is similar to (3.9), except that the integration path in the complex ω plane is not the real line:

$$G_\pm(t,\boldsymbol{r}) = \frac{1}{(2\pi)^{D+1}} \int \mathrm{d}^D k \oint_{C_\pm} \mathrm{d}\omega\, \tilde{G}(\omega,\boldsymbol{k}) \exp(-\mathrm{i}\omega t + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}). \tag{3.13}$$

C_+ is a contour going counter-clockwise so that the point $\omega = (\boldsymbol{k}\cdot\boldsymbol{k})^{1/2}$ is inside it and the point $\omega = -(\boldsymbol{k}\cdot\boldsymbol{k})^{1/2}$ is outside it. C_- is a contour going counter-clockwise so that the point $\omega = -(\boldsymbol{k}\cdot\boldsymbol{k})^{1/2}$ is inside it and the point $\omega = (\boldsymbol{k}\cdot\boldsymbol{k})^{1/2}$ is outside it.

These functions don't satisfy (2.7) but, as will be seen, the retarded, advanced, and Feynman Green's functions can be written in terms of them.

4. Relation of different Green's functions with each other

From the definitions of the previous section, it is seen that

$$\begin{aligned} G_{\mathrm{R}}(t,\boldsymbol{r}) &= -\theta(t)\,[G_+(t,\boldsymbol{r}) + G_-(t,\boldsymbol{r})],\\ G_{\mathrm{A}}(t,\boldsymbol{r}) &= \theta(-t)\,[G_+(t,\boldsymbol{r}) + G_-(t,\boldsymbol{r})],\\ G_{\mathrm{F}}(t,\boldsymbol{r}) &= -\theta(t)\,G_+(t,\boldsymbol{r}) + \theta(-t)\,G_-(t,\boldsymbol{r}). \end{aligned} \tag{4.1}$$

Using (3.13), one can relate G_+ and G_- to each other and their complex conjugates. One has

$$G_+(-t,\boldsymbol{r}) = -G_-(t,\boldsymbol{r}). \tag{4.2}$$

To obtain this, one uses the change of variable $z = -\omega$. The integration contour for the variable z is C_-, every point of which is the negative of a point on C_+. One also has

$$G_+(-t,\boldsymbol{r}) = -G_+^*(t,\boldsymbol{r}). \tag{4.3}$$

To obtain this, the change of variable $s = \omega^*$ has been used. The integration contour for the variable s is a contour every point of which is the complex conjugate of a point on C_+. This is the same as C_+, but going clockwise. So the integral on this contour is the negative of the integral on C_+. The final result is

$$\begin{aligned} G_+(-t,\boldsymbol{r}) &= -G_-(t,\boldsymbol{r}) = -G_+^*(t,\boldsymbol{r}),\\ G_-(-t,\boldsymbol{r}) &= -G_+(t,\boldsymbol{r}) = -G_-^*(t,\boldsymbol{r}). \end{aligned} \tag{4.4}$$

Using these and the relations (4.1), all of these functions can be written in terms of the Feynman Green's function:

$$\begin{aligned} G_+(t,\boldsymbol{r}) &= -\theta(t)\,G_{\mathrm{F}}(t,\boldsymbol{r}) + \theta(-t)\,G_{\mathrm{F}}^*(t,\boldsymbol{r}),\\ G_-(t,\boldsymbol{r}) &= -\theta(t)\,G_{\mathrm{F}}^*(t,\boldsymbol{r}) + \theta(-t)\,G_{\mathrm{F}}(t,\boldsymbol{r}),\\ G_{\mathrm{R}}(t,\boldsymbol{r}) &= \theta(t)\,[G_{\mathrm{F}}(t,\boldsymbol{r}) + G_{\mathrm{F}}^*(t,\boldsymbol{r})],\\ G_{\mathrm{A}}(t,\boldsymbol{r}) &= \theta(-t)\,[G_{\mathrm{F}}(t,\boldsymbol{r}) + G_{\mathrm{F}}^*(t,\boldsymbol{r})]. \end{aligned} \tag{4.5}$$

So, knowing the Feynman Green's function one can obtain the other four functions.

5. Recursion relation for the Green's function in terms of the dimension

The general form of the five functions introduced is

$$G(t,\boldsymbol{r}) = \frac{1}{(2\pi)^{D+1}} \int \mathrm{d}^D k \int_C \mathrm{d}\omega\, \frac{1}{\omega^2 - \boldsymbol{k}\cdot\boldsymbol{k}} \exp(-\mathrm{i}\,\omega t + \mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{r}). \tag{5.1}$$

The integral on the right-hand side depends on the integration contour C. The five functions introduced in the previous section correspond to five different contours. The above expression can be written as

$$G^{(D)}(t,\boldsymbol{r}) = \frac{S_{D-2}}{(2\pi)^{D+1}} \int_0^\infty k^{D-1}\,\mathrm{d}k \int_0^\pi \sin^{D-2}\theta\,\mathrm{d}\theta \times \int_C \mathrm{d}\omega\, \frac{1}{\omega^2-k^2}\, \exp(-\mathrm{i}\omega t + \mathrm{i}kr\cos\theta). \tag{5.2}$$

S_D is the area of a D-dimensional sphere of radius one. To arrive at this result, use has been made of the fact that the contour C depends on only the length of k. Differentiating the above expression with respect to r, one obtains

$$\frac{\partial}{\partial r} G^{(D)}(t,\boldsymbol{r}) = \frac{S_{D-2}}{(2\pi)^{D+1}} \int_0^\infty k^{D-1}\,\mathrm{d}k \int_0^\pi \sin^{D-2}\theta\,\mathrm{d}\theta \times \int_C \mathrm{d}\omega\, \frac{1}{\omega^2-k^2}\, \exp(-\mathrm{i}\omega t + \mathrm{i}kr\cos\theta)\,\mathrm{i}k\cos\theta. \tag{5.3}$$

Integrating by parts on θ results in

$$\frac{\partial}{\partial r} G^{(D)}(t,\boldsymbol{r}) = \frac{S_{D-2}}{(2\pi)^{D+1}} \int_0^\infty k^{D-1}\,\mathrm{d}k \int_0^\pi \sin^{D-2}\theta\,\mathrm{d}\theta \times \int_C \mathrm{d}\omega\, \frac{1}{\omega^2-k^2}\, \exp(-\mathrm{i}\omega t + \mathrm{i}kr\cos\theta)\, \frac{-k^2 r\sin^2\theta}{D-1}. \tag{5.4}$$

From this,

$$\frac{\partial}{r\partial r} G^{(D)}(t,\boldsymbol{r}) = \frac{-4\pi^2 S_{D-2}}{(D-1)S_D}\, G^{(D+2)}(t,\boldsymbol{r}). \tag{5.5}$$

Substituting

$$S_D = \frac{2\pi^{(D+1)/2}}{\Gamma[(D+1)/2]}, \tag{5.6}$$

one obtains

$$G^{(D+2)}(t,\boldsymbol{r}) = -\frac{1}{\pi}\frac{\partial}{\partial r^2} G^{(D)}(t,\boldsymbol{r}). \tag{5.7}$$

This relation is true for all of the five functions introduced. It is, however, assumed that $D \geq 1$. Otherwise there is no integration on k. In fact for $D = 0$, there is no space for the wave to propagate in.

6. Determination of the Feynman Green's function from the Green's function of the Laplace equation

The Green's function of the Laplace equation in $D+1$ dimensions satisfy

$$\left(\frac{\partial}{\partial t^2} + \nabla^2\right) G_{\mathrm{E}}(t,\boldsymbol{r}) = \delta(t)\,\delta^D(\boldsymbol{r}). \tag{6.1}$$

This is the analogue of (2.7) for the Green's function of the wave equation. One obtains

$$G_{\mathrm{E}}(t,\boldsymbol{r}) = \frac{1}{(2\pi)^{D+1}} \int \mathrm{d}^D k \int \mathrm{d}\omega \, \frac{1}{-\omega^2 - \boldsymbol{k}\cdot\boldsymbol{k}} \exp(-\mathrm{i}\,\omega t + \mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{r}). \tag{6.2}$$

Integrating over ω,

$$G_{\mathrm{E}}(t,\boldsymbol{r}) = \frac{-1}{(2\pi)^{D}} \int \mathrm{d}^D k \, \frac{\theta(t)\, e^{-kt} + \theta(-t)\, e^{kt}}{2k} \exp(\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{r}). \tag{6.3}$$

Using (3.8) and (3.9) for the Feynman Green's function,

$$G_{\mathrm{F}}(t,\boldsymbol{r}) = \frac{-i}{(2\pi)^{D}} \int \mathrm{d}^D k \, \frac{\theta(t)\, e^{-\mathrm{i}kt} + \theta(-t)\, e^{\mathrm{i}kt}}{2k} \exp(\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{r}). \tag{6.4}$$

Compare it with (6.3). (6.3) is for real t. But one can analytically continuate (6.3) to complex t with nonzero real part:

$$G_{\mathrm{E}}(t,\boldsymbol{r}) = \frac{-1}{(2\pi)^{D}} \int \mathrm{d}^D k \, \frac{\theta[\mathrm{Re}(t)]\, e^{-kt} + \theta[-\mathrm{Re}(t)]\, e^{kt}}{2k} \exp(\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{r}). \tag{6.5}$$

It is seen that this function is analytic for $\mathrm{Re}(t) \neq 0$. Now compare (6.4) and (6.5). It is seen that for real t,

$$G_{\mathrm{F}}(t,\boldsymbol{r}) = \mathrm{i} \lim_{\varepsilon\to 0^+} G_{\mathrm{E}}(\mathrm{i}t\, e^{-\mathrm{i}\varepsilon}, \boldsymbol{r}). \tag{6.6}$$

Note that the real part of $\xi := \mathrm{i}\, e^{-\mathrm{i}\varepsilon}$ is positive (for positive ε). So for real t,

$$\theta[\mathrm{Re}(\xi\, t)] = \theta(t). \tag{6.7}$$

To arrive at (6.6), (6.7) has been used.

So, to obtain the Feynman Green's function in a $D+1$ dimensional spacetime, it is sufficient to know the Green's function of the Laplace equation in $D+1$ dimensions. For $D > 1$, it is

$$G_{\mathrm{E}}(t,\boldsymbol{r}) = \frac{R^{1-D}}{(1-D)\, S_D}, \tag{6.8}$$

where

$$R := \sqrt{t^2 + \boldsymbol{r}\cdot\boldsymbol{r}}. \tag{6.9}$$

To arrive at (6.8), one uses a spherically symmetric ansatz for G_{E} and applies the Gauss' theorem. (6.6) and (6.8) result in

$$G_{\mathrm{F}}(t,\boldsymbol{r}) = \mathrm{i}\, \frac{(r^2 - t^2 + \mathrm{i}\varepsilon)^{(1-D)/2}}{(1-D)\, S_D}. \tag{6.10}$$

($2\varepsilon t^2$ has been substituted with ε.) It is seen that this expression satisfies (5.7).

Using this function, one can obtain the other four functions. To do so, one way is to determine the real and imaginary parts of the Feynman Green's function. From

now on, one should treat even and odd D's separately. For even D's, $(D-1)/2$ is half integer and

$$\mathrm{Re}[G_{\mathrm{F}}(t,\boldsymbol{r})] = -(-1)^{D/2}\,\mathrm{pf}\left[\frac{(t^2-r^2)^{(1-D)/2}}{(1-D)\,S_D}\,\theta(t^2-r^2)\right], \qquad (D/2)\in\mathbb{Z}, \quad (6.11)$$

and

$$\mathrm{Im}[G_{\mathrm{F}}(t,\boldsymbol{r})] = \mathrm{pf}\left[\frac{(r^2-t^2)^{(1-D)/2}}{(1-D)\,S_D}\,\theta(r^2-t^2)\right], \qquad (D/2)\in\mathbb{Z}. \quad (6.12)$$

In these relations, the action of $\mathrm{pf}[x^{-n-(1/2)}\,\theta(x)]$ on the test function ϕ has been defined through the recursion relation

$$\int \mathrm{d}x\,\mathrm{pf}[x^{-n-1-(1/2)}\,\theta(x)]\,\phi(x) := \frac{1}{n+(1/2)}\int \mathrm{d}x\,\mathrm{pf}[x^{-n-(1/2)}\,\theta(x)]\,\phi'(x). \quad (6.13)$$

$x^{-(1/2)}\,\theta(x)$ is locally integrable and the sign pf in front of it has no effect.

For odd D's, $(D-1)/2$ is integer and

$$\mathrm{Re}[G_{\mathrm{F}}(t,\boldsymbol{r})] = -\frac{1}{4\pi^{(D-1)/2}}\,\delta^{[(D-3)/2]}(t^2-r^2), \qquad [(D-1)/2]\in\mathbb{Z}, \quad (6.14)$$

and

$$\mathrm{Im}[G_{\mathrm{F}}(t,\boldsymbol{r})] = \mathrm{pf}\left[\frac{(r^2-t^2)^{(1-D)/2}}{(1-D)\,S_D}\right], \qquad [(D-1)/2]\in\mathbb{Z}. \quad (6.15)$$

To obtain these relations, use has been made of

$$\frac{1}{x+\mathrm{i}\varepsilon} = \mathrm{pf}\,\frac{1}{x} - \mathrm{i}\pi\,\delta(x) \quad (6.16)$$

and its derivatives [11]. Note that the behaviour of the Feynman Green's function is completely different in even and odd dimensions. Specially, the support of its real part is the wave cone for odd space dimensions, while for even space dimensions inside the wave cone is also part of the support.

The case $D=1$ should be considered separately. for $D=1$, the relations (6.8) and (6.9) are modified to

$$G_{\mathrm{E}}(t,x) = \frac{1}{4\pi}\,\ln\frac{x^2+t^2}{a^2}, \qquad D=1, \quad (6.17)$$

and

$$G_{\mathrm{F}}(t,x) = \frac{i}{4\pi}\,\ln\frac{x^2-t^2+\mathrm{i}\varepsilon}{a^2}, \qquad D=1, \quad (6.18)$$

where a is an arbitrary constant. So,

$$\mathrm{Re}[G_{\mathrm{F}}(t,x)] = -\frac{1}{4}\,\theta(t^2-x^2), \qquad D=1, \quad (6.19)$$

and

$$\mathrm{Im}[G_{\mathrm{F}}(t,x)] = \frac{1}{4\pi}\ln\left|\frac{x^2-t^2}{a^2}\right|, \qquad D=1. \tag{6.20}$$

One then finds the retarded and advanced Green's functions in various dimensions:

$$G_{\mathrm{R}}(t,\boldsymbol{r}) = -2(-1)^{D/2}\,\mathrm{pf}\left[\frac{(t^2-r^2)^{(1-D)/2}}{(1-D)S_D}\,\theta(t^2-r^2)\right]\theta(t), \qquad (D/2)\in\mathbb{Z}, \tag{6.21}$$

and

$$G_{\mathrm{R}}(t,\boldsymbol{r}) = -\frac{1}{2\pi^{(D-1)/2}}\,\theta^{[(D-1)/2]}(t^2-r^2)\,\theta(t), \qquad [(D-1)/2]\in\mathbb{Z}. \tag{6.22}$$

The advanced Green's function is the same as the retarded Green's function, with the substitution of $\theta(-t)$ instead of $\theta(t)$.

7. Concluding remarks

The Green's functions of the wave equation in various dimensions were discussed. It was shown that there are various kinds of Green's functions. Among these Green's functions, the relation of the Feynman Green's with the Green's function of the Laplace equation was found to be particularly simple: these two are simply related to each other through an analytic continuation of the time coordinate to imaginary values. Using this, the retarded and advanced Green's functions were also obtained. An interesting result is that while for odd space dimensions greater than one the support of these Green's functions is on the wave cone, for even space dimensions the support of these Green's functions contains inside the wave cone as well. As a result, for odd space dimensions greater than one the effect of a point source in space time is felt at any point of the space only at a single instant, while for even space dimensions the effect of a point source in space time is felt at any point of the space from a specific time and is continued afterwards.

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RELATIONS OF COMBINATORIC CONVOLUTION SUMS FOR RESTRICTED DIVISOR FUNCTIONS AND BERNOULLI POLYNOMIALS

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Abstract. In this paper, we study combinatoric convolution sums involving divisor functions. First, we establish some explicit formulas for certain combinatoric convolution sums of divisor functions derived from Bernoulli polynomials. Second, we show a formula of the fourth order of convolution sums of divisor functions expressed by their divisor functions and linear combination of Bernoulli polynomials.

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1. Introduction and statement of main results

The symbols $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the set of natural numbers, the ring of integers and the field of real numbers, respectively. Over the years Bernoulli polynomials have been used to prove important mathematical theorems. Since the 17th century, many mathematicians in different fields have been interested in Bernoulli polynomials. The Bernoulli polynomials $B_k(x)$, which are usually defined by the exponential generating function

$$\frac{te^{xt}}{e^t-1}=\sum_{k=0}^{\infty}\frac{B_k(x)}{k!}t^k,$$

play an important and quite mysterious role in mathematics and various fields like analysis, number theory and differential topology. The Bernoulli numbers B_k are defined to be $B_k := B_k(0)$.

The Bernoulli polynomial [7] is expressed through the respective numbers

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k},$$

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$$B_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}B_k(x)y^{n-k},$$
$$B_n'(x)=nB_{n-1}(x),$$
$$B_n(1-x)=(-1)^nB_n(x) \text{ and}$$
$$B_n(1+x)-B_n(x)=nx^{n-1}.$$

For $n\in\mathbb{N}$ and $k\in\mathbb{Z}$, we define two divisor functions

$$\sigma_k(n):=\sum_{d|n}d^k, \qquad \hat{\sigma}_k(n):=\sum_{d|n}(-1)^{\frac{n}{d}-1}d^k, \quad \sigma_k^*(n):=\sum_{\substack{d|n\\ \frac{n}{d}\text{odd}}}d^k.$$

The identity

$$\sum_{k=1}^{n-1}\sigma(k)\sigma(n-k)=\frac{5}{12}\sigma_3(n)+\left(\frac{1}{12}-\frac{1}{2}n\right)\sigma(n)$$

for the basic convolution sum first appeared in a letter from Besge to Liouville in 1862 ([3]). Hahn [6, (4.8)] considered

$$36\sum_{m<n}\hat{\sigma}(m)\hat{\sigma}(n-m)=\begin{cases}-3\hat{\sigma}(n)+3\widetilde{\sigma}_3(n), & \text{if } n \text{ is odd},\\ -3\hat{\sigma}(n)-5\widetilde{\sigma}_3(n)+4\widetilde{\sigma}_3(\frac{n}{2}), & \text{if } n \text{ is even}.\end{cases} \tag{1.1}$$

It is well-known that $\hat{\sigma}_k(n)=\sigma_k(n)-2\sigma_k(\frac{n}{2})$ ([6, (1.13)]).

In this article, we are trying to focus on the combinatorial convolution sums. For positive integers k, l, n, p and q the combinatorial convolution sums

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\hat{\sigma}_{2k-2s-1}(m)\hat{\sigma}_{2s+1}(n-m)$$

and

$$\sum_{\substack{1\le m\le p-1\\ 1\le m'\le q-1\\ a,b,c,d \text{ odd}\\ a+b+c+d=2l}}\binom{2l}{a,b,c,d}\hat{\sigma}_a(m)\hat{\sigma}_b(p-m)\hat{\sigma}_c(m')\hat{\sigma}_d(q-m')$$

can be evaluated explicitly in terms of divisor functions and a sum involving Bernoulli polynomials.

We are motivated by Ramanujan's recursion formula for sums of the product of two Eisenstein series [2] and its proof, and also the following identities ([1],[4],[9]):

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1}(N-m)$$
$$=\frac{2k+3}{4k+2}\sigma_{2k+1}(N)+\left(\frac{k}{6}-N\right)\sigma_{2k-1}(N)$$

$$+\frac{1}{2k+1}\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(N).$$

Using these new formulas and addition theorem of Bernoulli polynomials, we derive the explicit formulas for quadnomial convolution sums of divisor functions. We define $\hat{B}_{2k+1}(n)$ that if n is even, then $\sum_{d|n}B_{2k+1}(d)-2\sum_{d|\frac{n}{2}}B_{2k+1}(d)$ and if n is odd, then $\sum_{d|n}B_{2k+1}(d)$. Similarly, $\hat{\hat{B}}_{2k+1}(n)$ that if n is even, then $\sum_{d|n}\hat{B}_{2k+1}(d)-2\sum_{d|\frac{n}{2}}\hat{B}_{2k+1}(d)$ and if n is odd, then $\sum_{d|n}\hat{B}_{2k+1}(d)$. More precisely, we prove the following results.

Theorem 1. *If $k\geq 1$ and $n\geq 2$, then*

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\hat{\sigma}_{2k-2s-1}(m)\hat{\sigma}_{2s+1}(n-m)$$
$$=\frac{1}{2}\sigma_{2k+1}(n)-\frac{1}{2}\hat{\sigma}_{2k}(n)-\frac{1}{2k+1}\hat{B}_{2k+1}(n).$$

Theorem 2. *If $k\geq 2$ and $N\geq 1$, then*

$$\sum_{s=1}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{2N-1}(-1)^m\hat{\sigma}_{2k-2s-1}(m)\hat{\sigma}_{2s+1}(2N-m)$$
$$=\frac{1}{2}\sigma_{2k+1}(N)-\frac{1}{2}\hat{\sigma}_{2k}(2N)-\frac{1}{2k+1}\hat{B}_{2k+1}(2N).$$

Corollary 1. *For $N\geq 1$, we have*

$$\sum_{\substack{m=1\\ m\ \text{odd}\\ l\ \text{even}}}^{2N-1}\hat{\sigma}_{l-1}(m)\hat{\sigma}(2N-m)$$
$$=\frac{1}{2l}\left[\sigma_{l+1}(N)+\sigma_{l+1}(2N)\right]-\frac{1}{l}\hat{\sigma}_l(2N)-\frac{2}{l(l+1)}\hat{B}_{l+1}(2N).$$

Corollary 2. *For $k\geq 2$ and $N\geq 1$, we have*

$$\sum_{s=1}^{k-1}\binom{2k}{2s+1}\sum_{\substack{m=1\\ m\ odd}}^{2N-1}\hat{\sigma}_{2k-2s-1}(m)\hat{\sigma}_{2s+1}(2N-m)=2^{2k-1}\sigma^*_{2k+1}(N)$$
$$-\frac{1}{4}\left[\sigma_{2k+1}(N)+\sigma_{2k+1}(2N)\right]-\frac{1}{2}\hat{\sigma}_{2k}(2N)+\frac{1}{2k+1}\hat{B}_{2k+1}(2N).$$

Theorem 3. *Let $m\in\mathbb{N}$, $m\geq 1$ and $x,y\in\mathbb{R}$. Then we have*

$$\sum_{k=1}^{m-1}\binom{2m}{2k}\frac{B_{2k+1}(x)B_{2m-2k+1}(y)}{(2k+1)(2m-2k+1)}=\frac{1}{4m+4}\left\{B_{2m+2}(y-x)\right.$$
$$\left.-B_{2m+2}(x+y)\right\}-\frac{1}{2m+1}\left\{xB_{2m+1}(y)+yB_{2m+1}(x)\right\}$$

$$+\frac{1}{4m+2}\{B_{2m+1}(x)+B_{2m+1}(y)+(x+y-1)B_{2m+1}(x+y)$$
$$+(x-y)B_{2m+1}(y-x)\}.$$

The following theorem gives a formula of the fourth order of convolution sums of divisor functions expressed by their divisor functions and linear combination of Bernoulli polynomials.

Theorem 4. *Let $l,p,q\in\mathbb{N}$ with greater than* 1. *Then*

$$\sum_{\substack{1\le m\le p-1\\ 1\le m'\le q-1\\ a,b,c,d\ odd\\ a+b+c+d=2l}}\binom{2l}{a,b,c,d}\hat{\sigma}_a(m)\hat{\sigma}_b(p-m)\hat{\sigma}_c(m')\hat{\sigma}_d(q-m')$$
$$=\frac{1}{8(2l+1)}\Big\{\hat{\hat{B}}_{2l+1}(q+p+1)+\hat{\hat{B}}_{2l+1}(q-p+1)-\hat{\hat{B}}_{2l+1}(q+p)-\hat{\hat{B}}_{2l+1}(q-p)$$
$$+\sum_{\substack{d|p\\ d'|q}}d'd\left[B_{2l+1}(d'+d+1)+B_{2l+1}(d'-d+1)-B_{2l+1}(d'+d)-B_{2l+1}(d'-d)\right]$$
$$-\sum_{d|p}d\left[\hat{B}_{2l+1}(q+d+1)+\hat{B}_{2l+1}(q-d+1)-\hat{B}_{2l+1}(q+d)-\hat{B}_{2l+1}(q-d)\right]$$
$$-\sum_{d'|q}d'\left[\hat{B}_{2l+1}(d'+p+1)+\hat{B}_{2l+1}(d'-p+1)-\hat{B}_{2l+1}(d'+p)-\hat{B}_{2l+1}(d'-p)\right]\Big\}$$
$$+\frac{1}{4(2l+1)}\Big\{2\hat{\hat{B}}_{2l+1}(p+q)+\hat{\hat{B}}_{2l+1}(p-q)+\hat{\hat{B}}_{2l+1}(q-p)$$
$$-\sum_{d|p}d\left[\hat{B}_{2l+1}(q+d)+\hat{B}_{2l+1}(q-d)\right]-\sum_{d'|q}d'\left[\hat{B}_{2l+1}(p+d')+\hat{B}_{2l+1}(p-d')\right]\Big\}$$
$$+\frac{1}{2(2l+1)}\Big\{(p+q-1)\hat{\hat{B}}_{2l+1}(p+q)+(p-q)\hat{\hat{B}}_{2l+1}(q-p)$$
$$-(2p-1)\hat{\hat{B}}_{2l+1}(q)-(2q-1)\hat{\hat{B}}_{2l+1}(p)$$
$$+\hat{B}_{2l+1}(p)\left[\sigma_1(q)-\sigma_0(q)+2\sigma_0\left(\frac{q}{2}\right)\right]$$
$$+\hat{B}_{2l+1}(q)\left[\sigma_1(p)-\sigma_0(p)+2\sigma_0\left(\frac{p}{2}\right)\right]\Big\}$$
$$+\left[\frac{1}{4}\sigma_1(p)-\sigma_1\left(\frac{p}{2}\right)\right]\left[\sigma_{2l+1}(q)-\sigma_{2l}(q)+2\sigma_{2l}\left(\frac{q}{2}\right)\right]$$
$$+\left[\sigma_{2l+1}(p)-\sigma_{2l}(p)+2\sigma_{2l}\left(\frac{p}{2}\right)\right]\left[\frac{1}{4}\sigma_1(q)-\sigma_1\left(\frac{q}{2}\right)\right]$$

$$-\frac{1}{4(l+1)}\left[\hat{\hat{B}}_{2l+2}(p+q)-\hat{\hat{B}}_{2l+2}(q-p)\right].$$

2. Properties of combinatoric convolution sums of divisor functions derived from Bernoulli polynomials

The purpose of this section is to give proofs of the main results. Also, we would like to enrich this section with examples. To proof of the theorems, we need the auxiliary results and lemmas.

Lemma 1. *([5]) Let k,n be positive integers. Then*

$$\begin{aligned}\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\hat{\sigma}_{2k+1-2j}(n) &= (2k+1)\sigma^{*}_{2k+1}(n)\\ &\quad-\left(\frac{2k+3}{2}\right)\hat{\sigma}_{2k+1}(n)-\left(\frac{k(2k+1)}{6}\right)\hat{\sigma}_{2k-1}(n)\\ &\quad-(2k+1)\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\hat{\sigma}_{2k-2s-1}(m)\hat{\sigma}_{2s+1}(n-m).\end{aligned}$$

Lemma 2. *([8]) Let $m\in\mathbb{N}$ and $x,y\in\mathbb{R}$. Then we have*

(a)
$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}\frac{B_{k+1}(x)}{k+1}\frac{B_{m-k+1}(y)}{m-k+1}\\ &=(x+y-1)\frac{B_{m+1}(x+y)}{m+1}-\frac{B_{m+2}(x+y)}{m+2}-\frac{B_{m+2}(x)}{(m+1)(m+2)}-\frac{B_{m+2}(y)}{(m+1)(m+2)}.\end{aligned}$$

(b)
$$\begin{aligned}&\sum_{k=0}^{m}(-1)^{k}\binom{m}{k}\frac{B_{k+1}(x)}{k+1}\frac{B_{m-k+1}(y)}{m-k+1}\\ &=(x-y)\frac{B_{m+1}(y-x)}{m+1}+\frac{B_{m+2}(y-x)}{m+2}+\frac{B_{m+2}(1-x)}{(m+1)(m+2)}+\frac{B_{m+2}(y)}{(m+1)(m+2)}.\end{aligned}$$

Now we are ready to prove the main results of the article.

Proof of the Theorem 1. Consider

$$\begin{aligned}T&:=\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\hat{\sigma}_{2k+1-2j}(n)\\ &=\sum_{j=0}^{2k+1}\binom{2k+1}{j}B_{j}\hat{\sigma}_{2k+1-j}(n)-\sum_{j=0}^{k}\binom{2k+1}{2j+1}B_{2j+1}\hat{\sigma}_{2k+1-(2j+1)}(n)\end{aligned}$$

$$-\binom{2k+1}{0}B_0\hat{\sigma}_{2k+1}(n)-\binom{2k+1}{2}B_2\hat{\sigma}_{2k+1-2}(n).$$

It is well-known that $B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$ and $B_{2j+1}=0$ $(j\geq 1)$. Thus

$$\begin{aligned}T&=\sum_{j=0}^{2k+1}\binom{2k+1}{j}B_j\hat{\sigma}_{2k+1-j}(n)-\binom{2k+1}{0}B_0\hat{\sigma}_{2k+1}(n)\\&\quad-\binom{2k+1}{1}B_1\hat{\sigma}_{2k}(n)-\binom{2k+1}{2}B_2\hat{\sigma}_{2k-1}(n)\\&=\sum_{j=0}^{2k+1}\binom{2k+1}{j}B_j\hat{\sigma}_{2k+1-j}(n)-\hat{\sigma}_{2k+1}(n)+\left(k+\frac{1}{2}\right)\hat{\sigma}_{2k}(n)-\frac{k(2k+1)}{6}\hat{\sigma}_{2k-1}(n).\end{aligned}\tag{2.1}$$

Using $B_n(x)=\sum_{l=0}^{n}\binom{n}{l}B_l x^{n-l}$, we obtain

$$\begin{aligned}&\sum_{j=0}^{2k+1}\binom{2k+1}{j}B_j\left(\sigma_{2k+1-j}(n)-2\sigma_{2k+1-j}\left(\frac{n}{2}\right)\right)\\&\qquad=\sum_{j=0}^{2k+1}\binom{2k+1}{j}B_j\left(\sum_{d|n}d^{2k+1-j}-2\sum_{d|\frac{n}{2}}d^{2k+1-j}\right)\\&\qquad=\sum_{d|n}B_{2k+1}(d)-2\sum_{d|\frac{n}{2}}B_{2k+1}(d)\\&\qquad=\hat{B}_{2k+1}(n).\end{aligned}\tag{2.2}$$

Combining (2.1), Lemma 1 and (2.2) we deduce that

$$\begin{aligned}&\sum_{j=0}^{2k+1}\binom{2k+1}{j}B_j\hat{\sigma}_{2k+1-j}(n)-\hat{\sigma}_{2k+1}(n)+(k+\frac{1}{2})\hat{\sigma}_{2k}(n)-\frac{k(2k+1)}{6}\hat{\sigma}_{2k-1}(n)\\&\quad=\hat{B}_{2k+1}(n)-\hat{\sigma}_{2k+1}(n)+(k+\frac{1}{2})\hat{\sigma}_{2k}(n)-\frac{k(2k+1)}{6}\hat{\sigma}_{2k-1}(n)\\&\quad-(2k+1)\sigma^{*}_{2k+1}(n)+\left(\frac{2k+3}{2}\right)\hat{\sigma}_{2k+1}(n)+\left(\frac{k(2k+1)}{6}\right)\hat{\sigma}_{2k-1}(n)\\&\quad=-(2k+1)\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\hat{\sigma}_{2k-2s-1}(m)\hat{\sigma}_{2s+1}(n-m),\end{aligned}$$

and we obtain

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\hat{\sigma}_{2k-2s-1}(m)\hat{\sigma}_{2s+1}(n-m)$$
$$=\frac{1}{2}\sigma_{2k+1}(n)-\frac{1}{2}\hat{\sigma}_{2k}(n)-\frac{1}{2k+1}\hat{B}_{2k+1}(n).$$

□

Remark 1. With $k=1$ in Theorem 1, we recover (1.1).

Example 1. With $k=1,2,3$ in Theorem 1 look as

$$\text{(a)}\qquad \hat{B}_3(n)=\frac{3}{4}\sigma_3(n)-\frac{3}{2}\hat{\sigma}_2(n)-3\sum_{m=1}^{n-1}\hat{\sigma}_1(m)\hat{\sigma}_1(n-m),$$

$$\text{(b)}\qquad \hat{B}_5(n)=\frac{5}{2}\sigma_5(n)-\frac{5}{2}\hat{\sigma}_4(n)-40\sum_{m=1}^{n-1}\hat{\sigma}_1(m)\hat{\sigma}_3(n-m),$$

$$\text{(c)}\qquad \hat{B}_7(n)=\frac{7}{2}\sigma_7(n)-\frac{7}{2}\hat{\sigma}_6(n)-84\sum_{m=1}^{n-1}\hat{\sigma}_1(m)\hat{\sigma}_5(n-m)$$
$$-140\sum_{m=1}^{n-1}\hat{\sigma}_3(m)\hat{\sigma}_3(n-m).$$

Lemma 3. *([4]) If $k\geq 1$ and $N\in N$, then*

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N}\hat{\sigma}_{2k-2s-1}(2m-1)\hat{\sigma}_{2s+1}(2N-2m+1)=\frac{1}{4}\sigma^{*}_{2k+1}(2N).$$

Proof of the Theorem 2. Using Theorem 1, we obtain

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{2N-1}\hat{\sigma}_{2k-2s-1}(m)\hat{\sigma}_{2s+1}(2N-m) \tag{2.3}$$
$$=\frac{1}{2}\left[\sigma_{2k+1}(2N)-\hat{\sigma}_{2k}(2N)\right]-\frac{1}{2k+1}\hat{B}_{2k+1}(2N).$$

Using (2.3) and Lemma 3, we get

$$\sum_{s=1}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\hat{\sigma}_{2k-2s-1}(2m)\hat{\sigma}_{2s+1}(2N-2m) \tag{2.4}$$
$$=\frac{1}{4}\left[\sigma_{2k+1}(2N)+\sigma_{2k+1}(N)\right]-\frac{1}{2}\hat{\sigma}_{2k}(2N)-\frac{1}{2k+1}\hat{B}_{2k+1}(2N).$$

With (2.4) and Lemma 3, we get

$$\sum_{s=1}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{2N-1}(-1)^m\hat{\sigma}_{2k-2s-1}(m)\hat{\sigma}_{2s+1}(2N-m)$$

$$= \frac{1}{2}\sigma_{2k+1}(N) - \frac{1}{2}\hat{\sigma}_{2k}(2N) - \frac{1}{2k+1}\hat{B}_{2k+1}(2N).$$

□

Example 2. With $k = 2,3,4$ in Theorem 2, we have

(a)

$$40B_3(x)B_3(y) - 6\left[B_3(x) + B_3(y)\right] = -12\left[xB_5(y) + yB_5(x)\right] + 6\left[(x+y-1)B_5(x+y) + (x-y)B_5(y-x)\right] + 5\left[B_6(y-x) + B_6(x+y)\right],$$

(b)

$$B_3(x)B_5(y) + B_5(x)B_3(y) = \frac{1}{16}\left[B_8(y-x) - B_8(x+y)\right] - \frac{1}{7}\left[xB_7(y) + yB_7(x)\right] + \frac{1}{14}\left[B_7(x) + B_7(y) + (x+y-1)B_7(x+y) + (x-y)B_7(y-x)\right],$$

(c)

$$B_3(x)B_7(y) + B_7(x)B_3(y) = \frac{3}{120}\left[B_10(y-x) - B_10(x+y)\right] - \frac{1}{12}\left[xB_9(y) + yB_9(x)\right] + \frac{1}{24}\left[B_9(x) + B_9(y) + (x+y-1)B_9(x+y) + (x-y)B_9(y-x)\right] - \frac{21}{10}B_5(x)B_5(y).$$

Proof of the Corollary 1. Using Theorem 1 and Theorem 2, we obtain

$$(2k)\sum_{\substack{m=1 \\ m \text{ odd}}}^{2N-1} \hat{\sigma}_{2k-1}(m)\hat{\sigma}(2N-m) = \frac{1}{2}\sigma_{2k+1}(N) + \frac{1}{2}\sigma_{2k+1}(2N) - \hat{\sigma}_{2k}(2N) - \frac{2}{2k+1}\hat{B}_{2k+1}(2N). \tag{2.5}$$

Replacing $2k$ by l in (2.5), we get proof of the corollary. □

Proof of the Corollary 2. Let m is odd. Using Theorem 1 and Theorem 2, we obtain

$$(2k)\sum_{\substack{m=1 \\ m \text{ odd}}}^{2N-1} \hat{\sigma}_{2k-1}(m)\hat{\sigma}(2N-m) + 2\sum_{s=1}^{k-1}\binom{2k}{2s+1}\sum_{\substack{m=1 \\ m \text{ odd}}}^{2N-1} \hat{\sigma}_{2k-2s-1}(m)\hat{\sigma}_{2s+1}(2N-m) = \frac{1}{2}\left[\sigma_{2k+1}(2N) - \sigma_{2k+1}(N)\right]. \tag{2.6}$$

It is clearly known that $\sigma_k^*(2N)=\sigma_k(2N)-\sigma_k(N)$ and $\sigma_k^*(2N)=2^k\sigma_k^*(N)$. After we used it in (2.6), we get proof of the corollary. □

Proof of the Theorem 3. Replacing $2m$ by m in Lemma 2, we get

$$\begin{aligned}\sum_{k=0}^{m}\binom{2m}{2k}&\frac{B_{2k+1}(x)B_{2m-2k+1}(y)}{(2k+1)(2m-2k+1)}\\&=\frac{1}{2(2m+1)}\left\{(x+y-1)B_{2m+1}(x+y)+(x-y)B_{2m+1}(y-x)\right\}\\&+\frac{1}{2(2m+2)}\left\{B_{2m+2}(y-x)-B_{2m+2}(x+y)\right\}.\end{aligned}$$

Using $B_k(1-x)=(-1)^kB_k(x)$ with $k\geq 0$, we derive

$$\begin{aligned}\sum_{k=1}^{m-1}\binom{2m}{2k}\frac{B_{2k+1}(x)B_{2m-2k+1}(y)}{(2k+1)(2m-2k+1)}&=\frac{1}{4m+4}\left\{B_{2m+2}(y-x)-B_{2m+2}(x+y)\right\}\\&-\frac{1}{2m+1}\left\{xB_{2m+1}(y)+yB_{2m+1}(x)\right\}+\frac{1}{4m+2}\left\{B_{2m+1}(x)+B_{2m+1}(y)\right.\\&\left.++(x+y-1)B_{2m+1}(x+y)+(x-y)B_{2m+1}(y-x)\right\}.\end{aligned}$$

□

Example 3. With $m=2,3,4$ in Theorem 3, we have

(a)

$$\begin{aligned}40B_3(x)B_3(y)-6\left[B_3(x)+B_3(y)\right]&=-12\left[xB_5(y)+yB_5(x)\right]\\&+6\left[(x+y-1)B_5(x+y)+(x-y)B_5(y-x)\right]+5\left[B_6(y-x)+B_6(x+y)\right],\end{aligned}$$

(b)

$$\begin{aligned}B_3(x)B_5(y)+B_5(x)B_3(y)&=\frac{1}{16}\left[B_8(y-x)-B_8(x+y)\right]-\frac{1}{7}\left[xB_7(y)+yB_7(x)\right]\\&+\frac{1}{14}\left[B_7(x)+B_7(y)+(x+y-1)B_7(x+y)+(x-y)B_7(y-x)\right],\end{aligned}$$

(c)

$$\begin{aligned}B_3(x)B_7(y)+B_7(x)B_3(y)&=\frac{3}{120}\left[B_{10}(y-x)-B_{10}(x+y)\right]\\&-\frac{1}{12}\left[xB_9(y)+yB_9(x)\right]+\frac{1}{24}\left[B_9(x)+B_9(y)\right.\\&\left.+(x+y-1)B_9(x+y)+(x-y)B_9(y-x)\right]-\frac{21}{10}B_5(x)B_5(y).\end{aligned}$$

3. Divisor functions and linear combination of Bernoulli polynomials

We consider the polynomial

$$\sum_{\substack{1\le m\le p-1\\ 1\le m'\le q-1\\ a,b,c,d \text{ odd}\\ a+b+c+d=2l}} \binom{2l}{a,b,c,d}\hat{\sigma}_a(m)\hat{\sigma}_b(p-m)\hat{\sigma}_c(m')\hat{\sigma}_d(q-m')$$

once more. Let's call this polynomial $\mathfrak{C}$. Actually, the polynomials $\mathfrak{C}$ are combinations of Bernoulli polynomials and divisor functions.

Proof of the Theorem 4. We note that

$$\binom{2m}{2k}\binom{2k}{2s+1}\binom{2m-2k}{2s'+1} = \frac{(2m)!}{(2s+1)!(2k-2s-1)!(2s'+1)!(2m-2k-2s'-1)!}$$
$$\binom{2m}{2s+1,2k-2s-1,2s'+1,2m-2k-2s'-1} = \binom{2m}{a,b,c,d}. \tag{3.1}$$

By Theorem 1 and (3.1), we obtain

$$\begin{aligned}
\mathfrak{C} &= \sum_{\substack{1\le m\le p-1\\ 1\le m'\le q-1\\ a,b,c,d \text{ odd}\\ a+b+c+d=2l}} \binom{2l}{a,b,c,d}\hat{\sigma}_a(m)\hat{\sigma}_b(p-m)\hat{\sigma}_c(m')\hat{\sigma}_d(q-m')\\
&= \sum_{w=1}^{l-1}\binom{2l}{2w}\left[\sum_{s=0}^{w-1}\binom{2w}{2s+1}\sum_{m=1}^{p-1}\hat{\sigma}_{2w-2s-1}(m)\hat{\sigma}_{2s+1}(p-m)\right]\\
&\quad\times\left[\sum_{s'=0}^{l-w-1}\binom{2l-2w}{2s'+1}\sum_{m'=1}^{q-1}\hat{\sigma}_{2l-2w-2s'-1}(m')\hat{\sigma}_{2s'+1}(q-m')\right]\\
&= \sum_{w=1}^{l-1}\binom{2l}{2w}\left\{\frac{1}{2}\sigma_{2w+1}(p)-\frac{1}{2}\sigma_{2w}(p)+\sigma_{2w}\left(\frac{p}{2}\right)-\sum_{d|p}\frac{B_{2w+1}(d)}{2w+1}\right.\\
&\quad\left.+2\sum_{d|\frac{p}{2}}\frac{B_{2w+1}(d)}{2w+1}\right\}\times\left\{\frac{1}{2}\sigma_{2l-2w+1}(q)-\frac{1}{2}\sigma_{2l-2w}(q)+\sigma_{2l-2w}\left(\frac{q}{2}\right)\right.\\
&\quad\left.-\sum_{d'|q}\frac{B_{2l-2w+1}(d')}{2l-2w+1}+2\sum_{d'|\frac{q}{2}}\frac{B_{2l-2w+1}(d')}{2l-2w+1}\right\}.
\end{aligned}$$

Consider the terms of $\mathfrak{C}$, we obtain

$$\mathfrak{C}_{(1)} := \frac{1}{4}\sum_{w=1}^{l-1}\binom{2l}{2w}\sigma_{2w+1}(p)\sigma_{2l-2w+1}(q),$$

$$\mathfrak{C}_{(2)} := -\frac{1}{4}\sum_{w=1}^{l-1}\binom{2l}{2w}\sigma_{2w+1}(p)\sigma_{2l-2w}(q),$$

$$\mathfrak{C}_{(3)} := \frac{1}{2}\sum_{w=1}^{l-1}\binom{2l}{2w}\sigma_{2w+1}(p)\sigma_{2l-2w}\left(\frac{q}{2}\right),$$

$$\mathfrak{C}_{(4)} := -\frac{1}{4}\sum_{w=1}^{l-1}\binom{2l}{2w}\sigma_{2w}(p)\sigma_{2l-2w+1}(q),$$

$$\mathfrak{C}_{(5)} := \frac{1}{4}\sum_{w=1}^{l-1}\binom{2l}{2w}\sigma_{2w}(p)\sigma_{2l-2w}(q),$$

$$\mathfrak{C}_{(6)} := -\frac{1}{2}\sum_{w=1}^{l-1}\binom{2l}{2w}\sigma_{2w}(p)\sigma_{2l-2w}\left(\frac{q}{2}\right),$$

$$\mathfrak{C}_{(7)} := \frac{1}{2}\sum_{w=1}^{l-1}\binom{2l}{2w}\sigma_{2w}\left(\frac{p}{2}\right)\sigma_{2l-2w+1}(q),$$

$$\mathfrak{C}_{(8)} := -\frac{1}{2}\sum_{w=1}^{l-1}\binom{2l}{2w}\sigma_{2w}\left(\frac{p}{2}\right)\sigma_{2l-2w}(q),$$

$$\mathfrak{C}_{(9)} := \sum_{w=1}^{l-1}\binom{2l}{2w}\sigma_{2w}\left(\frac{p}{2}\right)\sigma_{2l-2w}\left(\frac{q}{2}\right),$$

$$\mathfrak{C}_{(10)} := -\frac{1}{2}\sum_{d|p}\sum_{w=1}^{l-1}\binom{2l}{2w}\frac{B_{2w+1}(d)}{2w+1}\sigma_{2l-2w+1}(q),$$

$$\mathfrak{C}_{(11)} := \frac{1}{2}\sum_{d|p}\sum_{w=1}^{l-1}\binom{2l}{2w}\frac{B_{2w+1}(d)}{2w+1}\sigma_{2l-2w}(q),$$

$$\mathfrak{C}_{(12)} := -\sum_{d|p}\sum_{w=1}^{l-1}\binom{2l}{2w}\frac{B_{2w+1}(d)}{2w+1}\sigma_{2l-2w}\left(\frac{q}{2}\right),$$

$$\mathfrak{C}_{(13)} := -\frac{1}{2}\sum_{d'|q}\sum_{w=1}^{l-1}\binom{2l}{2w}\frac{B_{2l-2w+1}(d')}{2l-2w+1}\sigma_{2w+1}(p),$$

$$\mathfrak{C}_{(14)} := \frac{1}{2}\sum_{d'|q}\sum_{w=1}^{l-1}\binom{2l}{2w}\frac{B_{2l-2w+1}(d')}{2l-2w+1}\sigma_{2w}(p),$$

$$\mathfrak{C}_{(15)} := -\sum_{d'|q}\sum_{w=1}^{l-1}\binom{2l}{2w}\frac{B_{2l-2w+1}(d')}{2l-2w+1}\sigma_{2w}\left(\frac{p}{2}\right),$$

$$\mathfrak{C}_{(16)} := \sum_{d|\frac{p}{2}}\sum_{w=1}^{l-1}\binom{2l}{2w}\frac{B_{2w+1}(d)}{2w+1}\sigma_{2l-2w+1}(q),$$

$$\mathfrak{C}_{(17)} := -\sum_{d|\frac{p}{2}}\sum_{w=1}^{l-1}\binom{2l}{2w}\frac{B_{2w+1}(d)}{2w+1}\sigma_{2l-2w}(q),$$

$$\mathfrak{C}_{(18)} := 2\sum_{d|\frac{p}{2}}\sum_{w=1}^{l-1}\binom{2l}{2w}\frac{B_{2w+1}(d)}{2w+1}\sigma_{2l-2w}\left(\frac{q}{2}\right),$$

$$\mathfrak{C}_{(19)} := \sum_{d'|\frac{q}{2}}\sum_{w=1}^{l-1}\binom{2l}{2w}\frac{B_{2l-2w+1}(d')}{2l-2w+1}\sigma_{2w+1}(p),$$

$$\mathfrak{C}_{(20)} := -\sum_{d'|\frac{q}{2}}\sum_{w=1}^{l-1}\binom{2l}{2w}\frac{B_{2l-2w+1}(d')}{2l-2w+1}\sigma_{2w}(p),$$

$$\mathfrak{C}_{(21)} := 2\sum_{d'|\frac{q}{2}}\sum_{w=1}^{l-1}\binom{2l}{2w}\frac{B_{2l-2w+1}(d')}{2l-2w+1}\sigma_{2w}\left(\frac{p}{2}\right),$$

$$\mathfrak{C}_{(22)} := \sum_{\substack{d|p\\ d'|q}}\sum_{w=1}^{l-1}\binom{2l}{2w}\frac{B_{2w+1}(d)}{2w+1}\frac{B_{2l-2w+1}(d')}{2l-2w+1},$$

$$\mathfrak{C}_{(23)} := 4\sum_{\substack{d|\frac{p}{2}\\ d'|\frac{q}{2}}}\sum_{w=1}^{l-1}\binom{2l}{2w}\frac{B_{2w+1}(d)}{2w+1}\frac{B_{2l-2w+1}(d')}{2l-2w+1},$$

$$\mathfrak{C}_{(24)} := -2\sum_{\substack{d|\frac{p}{2}\\ d'|q}}\sum_{w=1}^{l-1}\binom{2l}{2w}\frac{B_{2w+1}(d)}{2w+1}\frac{B_{2l-2w+1}(d')}{2l-2w+1},$$

and

$$\mathfrak{C}_{(25)} := -2\sum_{\substack{d|p\\ d'|\frac{q}{2}}}\sum_{w=1}^{l-1}\binom{2l}{2w}\frac{B_{2w+1}(d)}{2w+1}\frac{B_{2l-2w+1}(d')}{2l-2w+1}.$$

From the binomial theorem,

$$\sum_{w=1}^{l-1}\binom{2l}{2w}d'^{2l-2w}d^{2w} = \frac{1}{2}\left[(d'+d)^{2l}+(d'-d)^{2l}\right]-d'^{2l}-d^{2l},$$

we obtain

$$\mathfrak{C}_{(1)} = \frac{1}{8}\sum_{\substack{d|p\\ d'|q}}d'd[(d'+d)^{2l}+(d'-d)^{2l}]-\frac{1}{4}\sum_{\substack{d|p\\ d'|q}}dd'^{2l+1}-\frac{1}{4}\sum_{\substack{d|p\\ d'|q}}d^{2l+1}d'.$$

By the property of Bernoulli polynomial,

$$B_n(x+1)-B_n(x) = nx^{n-1},$$

we get

$$\mathfrak{C}_{(1)} = \frac{1}{8(2l+1)} \sum_{\substack{d|p \\ d'|q}} d'd \left[B_{2l+1}(d'+d+1) + B_{2l+1}(d'-d+1) \right.$$
$$\left. -B_{2l+1}(d'+d) - B_{2l+1}(d'-d) \right] - \frac{1}{4} \left[\sigma_1(p)\sigma_{2l+1}(q) + \sigma_{2l+1}(p)\sigma_1(q) \right].$$

Similarly, we derive that

$$\mathfrak{C}_{(2)} = -\frac{1}{8(2l+1)} \sum_{\substack{d|p \\ d'|q}} d \left[B_{2l+1}(d'+d+1) + B_{2l+1}(d'-d+1) - B_{2l+1}(d'+d) \right.$$
$$\left. -B_{2l+1}(d'-d) \right] + \frac{1}{4} \left[\sigma_1(p)\sigma_{2l}(q) + \sigma_{2l+1}(p)\sigma_0(q) \right],$$
$$\mathfrak{C}_{(3)} = \frac{1}{4(2l+1)} \sum_{\substack{d|p \\ d'|\frac{q}{2}}} d \left[B_{2l+1}(d'+d+1) + B_{2l+1}(d'-d+1) - B_{2l+1}(d'+d) \right.$$
$$\left. -B_{2l+1}(d'-d) \right] - \frac{1}{2} \left[\sigma_1(p)\sigma_{2l}\left(\frac{q}{2}\right) + \sigma_{2l+1}(p)\sigma_0\left(\frac{q}{2}\right) \right],$$
$$\mathfrak{C}_{(4)} = -\frac{1}{8(2l+1)} \sum_{\substack{d|p \\ d'|q}} d' \left[B_{2l+1}(d'+d+1) + B_{2l+1}(d'-d+1) - B_{2l+1}(d'+d) \right.$$
$$\left. -B_{2l+1}(d'-d) \right] + \frac{1}{4} \left[\sigma_0(p)\sigma_{2l+1}(q) + \sigma_{2l}(p)\sigma_1(q) \right],$$
$$\mathfrak{C}_{(5)} = \frac{1}{8(2l+1)} \sum_{\substack{d|p \\ d'|q}} \left[B_{2l+1}(d'+d+1) + B_{2l+1}(d'-d+1) \right.$$
$$\left. -B_{2l+1}(d'+d) - B_{2l+1}(d'-d) \right] - \frac{1}{4} \left[\sigma_0(p)\sigma_{2l}(q) + \sigma_{2l}(p)\sigma_0(q) \right],$$
$$\mathfrak{C}_{(6)} = -\frac{1}{4(2l+1)} \sum_{\substack{d|p \\ d'|\frac{q}{2}}} \left[B_{2l+1}(d'+d+1) + B_{2l+1}(d'-d+1) - B_{2l+1}(d'+d) \right.$$
$$\left. -B_{2l+1}(d'-d) \right] + \frac{1}{2} \left[\sigma_0(p)\sigma_{2l}\left(\frac{q}{2}\right) + \sigma_{2l}(p)\sigma_0\left(\frac{q}{2}\right) \right],$$
$$\mathfrak{C}_{(7)} = \frac{1}{4(2l+1)} \sum_{\substack{d|\frac{p}{2} \\ d'|q}} d' \left[B_{2l+1}(d'+d+1) + B_{2l+1}(d'-d+1) - B_{2l+1}(d'+d) \right.$$
$$\left. -B_{2l+1}(d'-d) \right] - \frac{1}{2} \left[\sigma_0\left(\frac{p}{2}\right)\sigma_{2l+1}(q) + \sigma_{2l}\left(\frac{p}{2}\right)\sigma_1(q) \right],$$

$$\mathfrak{C}_{(8)} = -\frac{1}{4(2l+1)} \sum_{\substack{d|\frac{p}{2} \\ d'|q}} \left[B_{2l+1}(d'+d+1) + B_{2l+1}(d'-d+1) - B_{2l+1}(d'+d) - B_{2l+1}(d'-d)\right] + \frac{1}{2}\left[\sigma_0\left(\frac{p}{2}\right)\sigma_{2l}(q) + \sigma_{2l}\left(\frac{p}{2}\right)\sigma_0(q)\right],$$

$$\mathfrak{C}_{(9)} = \frac{1}{2(2l+1)} \sum_{\substack{d|\frac{p}{2} \\ d'|\frac{q}{2}}} \left[B_{2l+1}(d'+d+1) + B_{2l+1}(d'-d+1) - B_{2l+1}(d'+d) - B_{2l+1}(d'-d)\right] - \left[\sigma_0\left(\frac{p}{2}\right)\sigma_{2l}\left(\frac{q}{2}\right) + \sigma_{2l}\left(\frac{p}{2}\right)\sigma_0\left(\frac{q}{2}\right)\right].$$

We obtain

$$\begin{aligned}
\mathfrak{C}_{(10)} &= -\frac{1}{2}\sum_{w=1}^{l-1}\binom{2l}{2w}\sum_{d|p}\frac{B_{2w+1}(d)}{2w+1}\sigma_{2l-2w+1}(q) \\
&= -\frac{1}{2(2l+1)}\sum_{w=1}^{l-1}\frac{(2l+1)(2l)!}{(2w+1)!(2l-2w)!}\sum_{\substack{d|p \\ d'|q}} B_{2w+1}(d)\sigma_{2l+1-2w}(q) \\
&= -\frac{1}{4(2l+1)}\sum_{\substack{d|p \\ d'|q}} d'\left[B_{2l+1}(d+d') + B_{2l+1}(d-d')\right] \\
&\quad + \frac{1}{2}\sigma_1(p)\sigma_{2l+1}(q) - \frac{1}{4}\sigma_0(p)\sigma_{2l+1}(q) + \frac{1}{2(2l+1)}\sigma_1(q)\sum_{d|p} B_{2l+1}(d).
\end{aligned}$$

Similarly, like $\mathfrak{C}_{(10)}$, we get

$$\begin{aligned}
\mathfrak{C}_{(11)} &= \frac{1}{4(2l+1)}\sum_{\substack{d|p \\ d'|q}}\left[B_{2l+1}(d+d') + B_{2l+1}(d-d')\right] \\
&\quad - \frac{1}{2}\sigma_1(p)\sigma_{2l}(q) + \frac{1}{4}\sigma_0(p)\sigma_{2l}(q) - \frac{1}{2(2l+1)}\sigma_0(q)\sum_{d|p} B_{2l+1}(d), \\
\mathfrak{C}_{(12)} &= -\frac{1}{2(2l+1)}\sum_{\substack{d|p \\ d'|fracq2}}\left[B_{2l+1}(d+d') + B_{2l+1}(d-d')\right] + \sigma_1(p)\sigma_{2l}\left(\frac{q}{2}\right) \\
&\quad - \frac{1}{2}\sigma_0(p)\sigma_{2l}\left(\frac{q}{2}\right) + \frac{1}{2l+1}\sigma_0\left(\frac{q}{2}\right)\sum_{d|p} B_{2l+1}(d), \\
\mathfrak{C}_{(13)} &= -\frac{1}{4(2l+1)}\sum_{\substack{d|p \\ d'|q}} d\left[B_{2l+1}(d'+d) + B_{2l+1}(d'-d)\right] + \frac{1}{2}\sigma_{2l+1}(p)\sigma_1(q)
\end{aligned}$$

$$
\begin{aligned}
&\quad -\frac{1}{4}\sigma_{2l+1}(p)\sigma_0(q)+\frac{1}{2(2l+1)}\sigma_1(p)\sum_{d'|q}B_{2l+1}(d'),\\
\mathfrak{C}_{(14)} &= \frac{1}{4(2l+1)}\sum_{\substack{d|p\\ d'|q}}\left[B_{2l+1}(d'+d)+B_{2l+1}(d'-d)\right]-\frac{1}{2}\sigma_{2l}(p)\sigma_1(q)\\
&\quad +\frac{1}{4}\sigma_{2l}(p)\sigma_0(q)-\frac{1}{2(2l+1)}\sigma_0(p)\sum_{d'|q}B_{2l+1}(d'),\\
\mathfrak{C}_{(15)} &= -\frac{1}{2(2l+1)}\sum_{\substack{d|\frac{p}{2}\\ d'|q}}\left[B_{2l+1}(d'+d)+B_{2l+1}(d'-d)\right]+\sigma_{2l}\left(\frac{p}{2}\right)\sigma_1(q)\\
&\quad -\frac{1}{2}\sigma_{2l}\left(\frac{p}{2}\right)\sigma_0(q)+\frac{1}{2l+1}\sigma_0\left(\frac{p}{2}\right)\sum_{d'|q}B_{2l+1}(d'),\\
\mathfrak{C}_{(16)} &= \frac{1}{2(2l+1)}\sum_{\substack{d|\frac{p}{2}\\ d'|q}}d'\left[B_{2l+1}(d'+d)+B_{2l+1}(d-d')\right]-\sigma_1\left(\frac{p}{2}\right)\sigma_{2l+1}(q)\\
&\quad +\frac{1}{2}\sigma_0\left(\frac{p}{2}\right)\sigma_{2l+1}(q)-\frac{1}{2l+1}\sigma_1(q)\sum_{d|\frac{p}{2}}B_{2l+1}(d),\\
\mathfrak{C}_{(17)} &= -\frac{1}{2(2l+1)}\sum_{\substack{d|\frac{p}{2}\\ d'|q}}\left[B_{2l+1}(d+d')+B_{2l+1}(d-d')\right]+\sigma_1\left(\frac{p}{2}\right)\sigma_{2l}(q)\\
&\quad -\frac{1}{2}\sigma_0\left(\frac{p}{2}\right)\sigma_{2l}(q)+\frac{1}{2l+1}\sigma_0(q)\sum_{d|\frac{p}{2}}B_{2l+1}(d),\\
\mathfrak{C}_{(18)} &= \frac{1}{2l+1}\sum_{\substack{d|\frac{p}{2}\\ d'|\frac{q}{2}}}\left[B_{2l+1}(d+d')+B_{2l+1}(d-d')\right]-2\sigma_1\left(\frac{p}{2}\right)\sigma_{2l}\left(\frac{q}{2}\right)\\
&\quad +\sigma_0\left(\frac{p}{2}\right)\sigma_{2l}\left(\frac{q}{2}\right)-\frac{2}{2l+1}\sigma_0\left(\frac{q}{2}\right)\sum_{d|\frac{p}{2}}B_{2l+1}(d),\\
\mathfrak{C}_{(19)} &= \frac{1}{2(2l+1)}\sum_{\substack{d|p\\ d'|\frac{q}{2}}}d\left[B_{2l+1}(d'+d)+B_{2l+1}(d'-d)\right]-\sigma_{2l+1}(p)\sigma_1\left(\frac{q}{2}\right)\\
&\quad +\frac{1}{2}\sigma_{2l+1}(p)\sigma_0\left(\frac{q}{2}\right)-\frac{1}{2l+1}\sigma_1(p)\sum_{d'|\frac{q}{2}}B_{2l+1}(d'),
\end{aligned}
$$

$$\mathfrak{C}_{(20)} = -\frac{1}{2(2l+1)} \sum_{\substack{d|p \\ d'|\frac{q}{2}}} \left[B_{2l+1}(d'+d) + B_{2l+1}(d'-d)\right] + \sigma_{2l}(p)\sigma_1\left(\frac{q}{2}\right)$$
$$-\frac{1}{2}\sigma_{2l}(p)\sigma_0\left(\frac{q}{2}\right) + \frac{1}{2l+1}\sigma_0(p) \sum_{d'|\frac{q}{2}} B_{2l+1}(d')$$

and

$$\mathfrak{C}_{(21)} = \frac{1}{2l+1} \sum_{\substack{d|\frac{p}{2} \\ d'|\frac{q}{2}}} \left[B_{2l+1}(d'+d) + B_{2l+1}(d'-d)\right] - 2\sigma_{2l}\left(\frac{p}{2}\right)\sigma_1\left(\frac{q}{2}\right)$$
$$+\sigma_{2l}\left(\frac{p}{2}\right)\sigma_0\left(\frac{q}{2}\right) - \frac{2}{2l+1}\sigma_0\left(\frac{p}{2}\right) \sum_{d'|\frac{q}{2}} B_{2l+1}(d').$$

From Lemma 2, then we get

$$\mathfrak{C}_{(22)} = \sum_{\substack{d|p \\ d'|q}} \sum_{w=1}^{l-1} \binom{2l}{2w} \frac{B_{2w+1}(d)}{2w+1} \frac{B_{2l-2w+1}(d')}{2l-2w+1}$$
$$= \frac{1}{2(2l+1)} \sum_{\substack{d|p \\ d'|q}} \left[(d+d'-1)B_{2l+1}(d+d') + (d-d')B_{2l+1}(d'-d)\right.$$
$$\left.-(2d-1)B_{2l+1}(d') - (2d'-1)B_{2l+1}(d)\right]$$
$$-\frac{1}{4(l+1)} \sum_{\substack{d|p \\ d'|q}} \left[B_{2l+2}(d+d') - B_{2l+2}(d'-d)\right],$$
$$\mathfrak{C}_{(23)} = \frac{4}{2(2l+1)} \sum_{\substack{d|\frac{p}{2} \\ d'|\frac{q}{2}}} \left[(d+d'-1)B_{2l+1}(d+d') + (d-d')B_{2l+1}(d'-d)\right.$$
$$\left.-(2d-1)B_{2l+1}(d') - (2d'-1)B_{2l+1}(d)\right]$$
$$-\frac{4}{4(l+1)} \sum_{\substack{d|\frac{p}{2} \\ d'|\frac{q}{2}}} \left[B_{2l+2}(d+d') - B_{2l+2}(d'-d)\right],$$
$$\mathfrak{C}_{(24)} = -\frac{2}{2(2l+1)} \sum_{\substack{d|\frac{p}{2} \\ d'|q}} \left[(d+d'-1)B_{2l+1}(d+d') + (d-d')B_{2l+1}(d'-d)\right.$$
$$\left.-(2d-1)B_{2l+1}(d') - (2d'-1)B_{2l+1}(d)\right]$$

$$+\frac{2}{4(l+1)}\sum_{\substack{d|\frac{p}{2}\\ d'|q}}\left[B_{2l+2}(d+d')-B_{2l+2}(d'-d)\right],$$

$$\mathfrak{C}_{(25)}=-\frac{2}{2(2l+1)}\sum_{\substack{d|p\\ d'|\frac{q}{2}}}\left[(d+d'-1)B_{2l+1}(d+d')+(d-d')B_{2l+1}(d'-d)\right.$$

$$\left.-(2d-1)B_{2l+1}(d')-(2d'-1)B_{2l+1}(d)\right]$$

$$+\frac{2}{4(l+1)}\sum_{\substack{d|p\\ d'|\frac{q}{2}}}\left[B_{2l+2}(d+d')-B_{2l+2}(d'-d)\right].$$

Summing $\mathfrak{C}_{(i)}(i=1,...,25)$, we drive the Theorem. □

Example 4. In Theorem 4, for $l=a=b=c=d=1$ and $p=q=2$, we get

$$\sum_{\substack{1\le m\le p-1\\ 1\le m'\le q-1\\ a,b,c,d \text{ odd}\\ a+b+c+d=2l}}\binom{2l}{a,b,c,d}\hat{\sigma}_a(m)\hat{\sigma}_b(p-m)\hat{\sigma}_c(m')\hat{\sigma}_d(q-m')=4!.$$

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SOME PROPERTIES OF COFINITELY WEAK ESSENTIAL SUPPLEMENTED MODULES

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Abstract. Let M be an $R-$module. If every cofinite essential submodule of M has a weak supplement in M, then M is called a cofinitely weak essential supplemented (or briefly cwe-supplemented) module. In this work, some properties of these modules are investigated. It is proved that any sum of cwe-supplemented modules is cwe-supplemented. It is also proved that every factor module and every homomorphic image of a cwe-supplemented module are cwe-supplemented.

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1. INTRODUCTION

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an $R-$module. We will denote a submodule N of M by $N \leq M$. Let M be an $R-$module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* (or *superfluous*) submodule of M and denoted by $N \ll M$. A submodule N of an R -module M is called an *essential* submodule and denoted by $N \trianglelefteq M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equvalently, $N \cap L = 0$ for $L \leq M$ implies that $L = 0$. A submodule K of M is called a *cofinite* submodule of M if M/K is finitely generated. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is called a *supplemented* module if every submodule of M has a supplement in M. M is called an *essential supplemented* module if every essential submodule of M has a supplement in M. M is called a *cofinitely supplemented* module if every cofinite submodule of M has a supplement in M. M is called a *cofinitely essential supplemented* module if every cofinite essential submodule of M has a supplement in M. Let M be an $R-$module and $U \leq M$. If for every $V \leq M$ such that $M = U + V$, U has a supplement V' with $V' \leq V$, we say U has *ample supplements* in M. If every

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submodule of M has ample supplements in M, then M is called an *amply supplemented* module. If every essential submodule of M has ample supplements in M, then M is called an *amply essential supplemented* module. If every cofinite submodule of M has ample supplements in M, then M is called an *amply cofinitely supplemented* module. If every cofinite essential submodule of M has ample supplements in M, then M is called an *amply cofinitely essential supplemented* module. Let M be an $R-$module and $U,V \leq M$. If $M = U+V$ and $U \cap V \ll M$, then V is called a *weak supplement* of U in M. M is said to be *weakly supplemented* if every submodule of M has a weak supplement in M. M is said to be *cofinitely weak supplemented* if every cofinite submodule of M has a weak supplement in M. M is called a *weakly essential supplemented* module if every essential submodule of M has a weak supplement in M. The intersection of all maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$. Let M be an $R-$module and $U,K \leq M$. We say U *lies above* K in M if $K \leq U$ and $U/K \ll M/K$.

More informations about (amply) supplemented modules are in [4, 12–14]. The definitions of (amply) essential supplemented modules and some properties of them are in [8, 10, 11]. The definitions of (amply) cofinitely supplemented modules and some properties of them are in [1]. The definitions of (amply) cofinitely essential supplemented modules and some details of them are in [6, 7]. Some details about weakly supplemented and cofinitely weak supplemented modules are in [2, 4]. The definition of weakly essential supplemented modules and some properties of these modules are in [9].

2. Cofinitely weak essential supplemented modules

Definition 1. Let M be an $R-$module. If every cofinite essential submodule of M has a weak supplement in M, then M is called a cofinitely weak essential supplemented (or briefly cwe-supplemented) module. (See also [5])

Lemma 1. *Let M be a finitely generated $R-$module. Then M is weakly essential supplemented if and only if M is cwe-supplemented.*

Proof. Clear from definitions. □

Proposition 1. *Let M be a cwe-supplemented module. Then $M/RadM$ have no proper cofinite essential submodules.*

Proof. Let $\frac{K}{RadM}$ be any cofinite essential submodule of $\frac{M}{RadM}$. By $\frac{M}{K} \cong \frac{M/RadM}{K/RadM}$, K is a cofinite submodule of M. Since $\frac{K}{RadM} \trianglelefteq \frac{M}{RadM}$, then $K \trianglelefteq M$ and since M is cwe-supplemented, K has a weak supplement V in M. Here $M = K+V$ and $K \cap V \ll M$. Since $M = K+V$, $\frac{M}{RadM} = \frac{K}{RadM} + \frac{V+RadM}{RadM}$. Since $K \cap V \ll M$, by [13, 21.5], $K \cap V \leq RadM$. Then $\frac{K}{RadM} \cap \frac{V+RadM}{RadM} = \frac{K \cap V+RadM}{RadM} = 0$ and $\frac{M}{RadM} = \frac{K}{RadM} \oplus \frac{V+RadM}{RadM}$.

Since $\frac{M}{RadM} = \frac{K}{RadM} \oplus \frac{V+RadM}{RadM}$ and $\frac{K}{RadM} \trianglelefteq \frac{M}{RadM}$, $\frac{K}{RadM} = \frac{M}{RadM}$. Hence $\frac{M}{RadM}$ have no proper cofinite essential submodules. □

Proposition 2. *Let M be a cwe-supplemented module. If K is a proper cofinite essential submodule of M and $RadM \leq K$, then $K/RadM$ is not essential in $M/RadM$.*

Proof. Since $RadM \leq K$ and $K \neq M$, $K/RadM \neq M/RadM$. Since M is cwe-supplemented, K has a weak supplement V in M. Here $M = K+V$ and $K \cap V \ll M$. Since $M = K+V$, $\frac{M}{RadM} = \frac{K}{RadM} + \frac{V+RadM}{RadM}$. By $K \cap V \leq RadM$, $\frac{K}{RadM} \cap \frac{V+RadM}{RadM} = \frac{K \cap V+RadM}{RadM} = 0$ and $\frac{M}{RadM} = \frac{K}{RadM} \oplus \frac{V+RadM}{RadM}$. Following these we have $\frac{V+RadM}{RadM} \neq 0$ and since $\frac{K}{RadM} \cap \frac{V+RadM}{RadM} = 0$, $K/RadM$ is not essential in $M/RadM$. □

Lemma 2. *Let M be an $R-$module, U be a cofinite essential submodule of M and $M_1 \leq M$. If M_1 is cwe-supplemented and $U+M_1$ has a weak supplement in M, then U has a weak supplement in M.*

Proof. Let X be a weak supplement of $U+M_1$ in M. Then $M = U+M_1+X$ and $X \cap (U+M_1) \ll M$. Since U is a cofinite submodule of M and $\frac{M/U}{(U+X)/U} \cong \frac{M}{U+X} = \frac{M_1+U+X}{U+X} \cong \frac{M_1}{M_1 \cap (U+X)}$, $M_1 \cap (U+X)$ is a cofinite submodule of M_1. Since $U \trianglelefteq M$, $(U+X) \trianglelefteq M$ and $(U+X) \cap M_1 \trianglelefteq M_1$. Then by M_1 being cwe-supplemented, $(U+X) \cap M_1$ has a weak supplement Y in M_1. Here $M_1 = (U+X) \cap M_1 + Y$ and $(U+X) \cap Y = (U+X) \cap M_1 \cap Y \ll M_1 \leq M$. Then $M = U+M_1+X = U+X+(U+X) \cap M_1 + Y = U+X+Y$ and $U \cap (X+Y) \leq (U+X) \cap Y + (U+Y) \cap X \leq (U+M_1) \cap X + (U+X) \cap Y \ll M$. Hence $X+Y$ is a weak supplement of U in M. □

Corollary 1. *Let U be a cofinite essential submodule of M and $M_i \leq M$ for $i = 1,2,...,n$. If M_i is cwe-supplemented for every $i = 1,2,...,n$ and $U+M_1+M_2+...+M_n$ has a weak supplement in M, then U has a weak supplement in M.*

Proof. Clear from Lemma 2. □

Lemma 3. *Any sum of cwe-supplemented modules is cwe-supplemented.*

Proof. Let U be a cofinite essential submodule of M and $M = \sum_{\lambda \in \Lambda} M_\lambda$ for $M_\lambda \leq M$ and M_λ be cwe-supplemented for every $\lambda \in \Lambda$. Since U is a cofinite submodule of M, then there exist $\lambda_1, \lambda_2, ..., \lambda_n \in \Lambda$ such that $M = U+M_{\lambda_1}+M_{\lambda_2}+...+M_{\lambda_n}$. Then 0 is a weak supplement of $U+M_{\lambda_1}+M_{\lambda_2}+...+M_{\lambda_n}$ in M. Since M_{λ_i} is cwe-supplemented for every $i = 1,2,...,n$, by Corollary 1, U has a weak supplement in M. Hence M is cwe-supplemented. □

Corollary 2. *Let M be a cwe-supplemented $R-$module. Then $M^{(\Lambda)}$ is cwe-supplemented for every index set Λ.*

Proof. Clear from Lemma 3. □

Lemma 4. *Every factor module of a cwe-supplemented module is cwe-supplemented.*

Proof. Let M be a cwe-supplemented $R-$module and $\frac{M}{K}$ be any factor module of M. Let $\frac{U}{K}$ be a cofinite essential submodule of $\frac{M}{K}$. Then U is a cofinite essential submodule of M and since M is cwe-supplemented, U has a weak supplement V in M. Here $M = U + V$ and $U \cap V \ll M$. Following we have $\frac{M}{K} = \frac{U}{K} + \frac{V+K}{K}$ and $\frac{U}{K} \cap \frac{V+K}{K} = \frac{U\cap V+K}{K} \ll \frac{V+K}{K}$. Hence $\frac{V+K}{K}$ is a weak supplement of $\frac{U}{K}$ in $\frac{M}{K}$ and $\frac{M}{K}$ is cwe-supplemented. □

Corollary 3. *Every homomorphic image of a cwe-supplemented module is cwe-supplemented.*

Proof. Clear from Lemma 4. □

Lemma 5. *Let M be a cwe-supplemented module. Then every $M-$generated $R-$module is cwe-supplemented.*

Proof. Let N be a $M-$generated $R-$module. Then there exist an index set Λ and an $R-$module epimorphism $f : M^{(\Lambda)} \longrightarrow N$. Since M is cwe-supplemented, by Corollary 2, $M^{(\Lambda)}$ is cwe-supplemented. Then by Corollary 3, N is cwe-supplemented. □

Lemma 6. *Let M be an $R-$module, $K \ll M$ and $\frac{U+K}{K} \trianglelefteq \frac{M}{K}$ for every $U \trianglelefteq M$. If M/K is cwe-supplemented, then M is also cwe-supplemented.*

Proof. Let U be any cofinite essential submodule of M. Since U is a cofinite submodule of M, we clearly see that $U + K$ is a cofinite submodule of M. By $\frac{M/K}{(U+K)/K} \cong \frac{M}{U+K}$, $(U + K)/K$ is a cofinite submodule of M/K. By hypothesis, $\frac{U+K}{K} \trianglelefteq \frac{M}{K}$ and since M/K is cwe-supplemented, $\frac{U+K}{K}$ has a weak supplement V/K in M/K. Here $\frac{M}{K} = \frac{U+K}{K} + \frac{V}{K} = \frac{U+V}{K}$ and $\frac{U\cap V+K}{K} = \frac{U+K}{K} \cap \frac{V}{K} \ll \frac{M}{K}$. Since $\frac{M}{K} = \frac{U+V}{K}$, then $M = U + V$. Let $U \cap V + T = M$ with $T \leq M$. Then $\frac{U\cap V+K}{K} + \frac{T+K}{K} = \frac{M}{K}$ and since $\frac{U\cap V+K}{K} \ll \frac{M}{K}$, $\frac{T+K}{K} = \frac{M}{K}$ and we have $T + K = M$. Since $K \ll M$, we have $T = M$. Hence $U \cap V \ll M$ and V is a weak supplement of U in M. Therefore, M is cwe-supplemented. □

Corollary 4. *Let $f : M \longrightarrow N$ be an $R-$module epimorphism, $Ker(f) \ll M$ and $f(U) \trianglelefteq N$ for every $U \trianglelefteq M$. If N is cwe-supplemented, then M is also cwe-supplemented.*

Proof. Clear from Lemma 6. □

Proposition 3. *Let R be a ring. The following assertions are equivalent.*
(i) ${}_RR$ is weakly essential supplemented
(ii) ${}_RR$ is cwe-supplemented.
(iii) $R^{(\Lambda)}$ is cwe-supplemented for every index set Λ.
(iv) Every $R-$module is cwe-supplemented.

Proof. $(i) \Longleftrightarrow (ii)$ Clear from Lemma 1, since ${}_RR$ is finitely generated.
$(ii) \Longleftrightarrow (iii)$ Clear from Corollary 2.

$(iii) \Longrightarrow (iv)$ Let M be an $R-$module. Then there exist an index set Λ and an $R-$module epimorphism $f : R^{(\Lambda)} \longrightarrow M$. Since $R^{(\Lambda)}$ is cwe-supplemented, by Corollary 3, M is also cwe-supplemented.

$(iv) \Longrightarrow (ii)$ Clear. □

Let M be an $R-$module. We say submodules X and Y of M are β^* equivalent, $X\beta^*Y$, if and only if $(X+Y)/X \ll M/X$ and $(X+Y)/Y \ll M/Y$. More details about β^* relation are in [3].

Lemma 7. *Let M be an $R-$module. If every cofinite essential submodule of M is β^* equivalent to a weak supplement submodule in M, then M is cwe-supplemented.*

Proof. Let U be any cofinite essential submodule of M. By hypothesis, there exists a weak supplement submodule X such that $U\beta^*X$ in M. Since X is a weak supplement submodule in M, there exists $V \leq M$ such that X is a weak supplement of V in M. Then V is a weak supplement of X in M. Since $U\beta^*X$, by [3, Theorem 2.6 (ii)], V is a weak supplement of U in M. Hence M is cwe-supplemented. □

Corollary 5. *Let M be an $R-$module. If every cofinite essential submodule lies above a weak supplement submodule in M, then M is cwe-supplemented.*

Proof. Let U be any cofinite essential submodule of M. By hypothesis, there exists a weak supplement submodule X such that U lies above X in M. Since U lies above X in M, we clearly see that $U\beta^*X$ in M. Hence every cofinite essential submodule of M is β^* equivalent to a weak supplement submodule in M and by Lemma 7, M is cwe-supplemented. □

Corollary 6. *Let M be an $R-$module. If every cofinite essential submodule of M is β^* equivalent to a supplement submodule in M, then M is cwe-supplemented.*

Proof. Clear from Lemma 7, since every supplement submodule is a weak supplement submodule in M. □

Corollary 7. *Let M be an $R-$module. If every cofinite essential submodule lies above a supplement submodule in M, then M is cwe-supplemented.*

Proof. Clear from Corollary 5, since every supplement submodule is a weak supplement submodule in M. □

Lemma 8. *Let M be a cwe-supplemented $R-$module. If every weak supplement of any cofinite essential submodule of M is a supplement in M, then M is cofinitely essential supplemented.*

Proof. Let U be any cofinite essential submodule of M. Since M is cwe-supplemented, U has a weak supplement V in M. Here $M = U+V$ and $U \cap V \ll M$. By hypothesis, V is a supplement in M. Since $U \cap V \ll M$, by [12, Lemma 2.5], $U \cap V \ll V$. Hence V is a supplement of U in M. Therefore, M is cofinitely essential supplemented. □

Corollary 8. *Let M be a finitely generated cwe-supplemented $R-$module. If every weak supplement submodule in M is a supplement in M, then M is essential supplemented.*

Proof. Clear from Lemma 8, since every submodule of M is cofinite. □

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SOME PROPERTIES OF (AMPLY) G-RADICAL SUPPLEMENTED MODULES

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Abstract. In this work, some new properties of (amply) g-radical supplemented modules are investigated. It is proved that every factor module and every homomorphic image of an amply g-radical supplemented module are amply g-radical supplemented. Let M be a π-projective and g-radical supplemented module. Then M is amply g-radical supplemented. Let M be a projective and g-radical supplemented module. Then every finitely M-generated module is amply g-radical supplemented. Let R be any ring. Then ${}_RR$ is g-radical supplemented if and only if every finitely generated R-module is amply g-radical supplemented.

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1. INTRODUCTION

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an R-module. We will denote a submodule N of M by $N \leq M$. Let M be an R-module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small submodule* of M and denoted by $N \ll M$. Let M be an R-module and $N \leq M$. If there exists a submodule K of M such that $M = N + K$ and $N \cap K = 0$, then N is called a *direct summand* of M and it is denoted by $M = N \oplus K$. A submodule N of an R-module M is called an *essential submodule* of M, denoted by $N \trianglelefteq M$, if $K = 0$ for every $K \leq M$ with $K \cap N = 0$. Let M be an R-module and K be a submodule of M. K is called a *generalized small* (briefly, *g-small*) *submodule* of M if for every $T \trianglelefteq M$ with $M = K + T$ implies that $T = M$, this is written by $K \ll_g M$ (in [14], it is called an *e-small submodule* of M and denoted by $K \ll_e M$). Let M be an R-module. M is called a *hollow module* if every proper submodule of M is small in M. M is called a *local module* if M has the largest submodule, i. e. a proper submodule which contains all other proper submodules. Let U and V be submodules of M. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement*

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of U in M. M is called a *supplemented module* if every submodule of M has a supplement in M. Let M be an R-module and $U,V \leq M$. If $M = U+V$ and $M = U+T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U+V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. Let M be an R-module and $U \leq M$. If for every $V \leq M$ such that $M = U+V$, U has a supplement (g-supplement) V' in M with $V' \leq V$, then we say U has *ample supplements* (*g-supplements*) in M. If every submodule of M has ample supplements (g-supplements) in M, then M is called an *amply supplemented* (*g-supplemented*) *module*. The intersection of all maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$. M is said to be *semilocal* if $M/RadM$ is semisimple, i. e. every submodule of $M/RadM$ is a direct summand of $M/RadM$. Let M be an R-module and $U,V \leq M$. If $M = U+V$ and $U \cap V \leq RadV$, then V is called a *generalized* (*radical*) *supplement* (briefly, *Rad-supplement*) of U in M. M is said to be *generalized (radical) supplemented* (briefly, *Rad-supplemented*) if every submodule of M has a Rad-supplement in M. Let M be an R-module and $U \leq M$. If for every $V \leq M$ such that $M = U+V$, U has a Rad-supplement V' in M with $V' \leq V$, then we say U has *ample generalized* (*radical*) *supplements* (briefly, *ample Rad-supplements*) in M. If every submodule of M has ample Rad-supplements in M, then M is called an *amply generalized* (*radical*) *supplemented* (briefly, *amply Rad-supplemented*) *module*. The intersection of all essential maximal submodules of an R-module M is called the *generalized radical* (briefly, *g-radical*) of M and denoted by Rad_gM (in [14], it is denoted by Rad_eM). If M have no essential maximal submodules, then we denote $Rad_gM = M$. M is said to be *g-semilocal* if M/Rad_gM is semisimple, i. e. every submodule of M/Rad_gM is a direct summand of M/Rad_gM. Let M be an R-module. We say submodules X and Y of M are β^* equivalent, $X\beta^*Y$, if and only if $Y+K = M$ for every $K \leq M$ such that $X+K = M$ and $X+T = M$ for every $T \leq M$ such that $Y+T = M$. We say submodules X and Y of M are β_g^* equivalent, $X\beta_g^*Y$, if and only if $Y+K = M$ for every $K \trianglelefteq M$ such that $X+K = M$ and $X+T = M$ for every $T \trianglelefteq M$ such that $Y+T = M$. Let M be R-module and $X \leq Y \leq M$. If $Y/X \ll M/X$, then we say Y *lies above* X in M.

More information about (amply) supplemented modules are in [2, 6, 13]. More informations about g-small submodules and (amply) g-supplemented modules are in [4, 8, 11]. The definition of (amply) generalized supplemented modules and some properties of them are in [12]. The definition of g-semilocal modules and some properties of them are in [5]. The definition of β^* relation and some results of this relation are in [1]. The definition of β_g^* relation and some results of this relation are in [10].

Lemma 1. *Let M be an R-module. The following assertions hold.*

(1) For every $m \in Rad_gM$, $Rm \ll_g M$.
(2) If $N \leq M$, then $Rad_gN \leq Rad_gM$.
(3) $Rad_gM = \sum_{L \ll_g M} L$.

Proof. See [3, Lemma 2 and Lemma 3]. □

2. G-RADICAL SUPPLEMENTED MODULE

Definition 1. Let M be an R-module and $U,V \leq M$. If $M = U+V$ and $U \cap V \leq Rad_gV$, then V is called a g-radical supplement of U in M. If every submodule of M has a g-radical supplement in M, then M is called a g-radical supplemented module. (See [3, 7].)

Clearly we can see that every g-supplemented module is g-radical supplemented. But the converse is not true in general. Every Rad-supplemented module is g-radical supplemented.

Proposition 1. *Let M be an R-module and $Rad_gV = V \cap Rad_gM$ for every $V \leq M$. Then M is g-radical supplemented if and only if M is g-semilocal.*

Proof. ($\Longrightarrow$) Clear from [3, Theorem 1].

($\Longleftarrow$) Let $U \leq M$. Since M is g-semilocal, $(U+Rad_gM)/Rad_gM$ is a direct summand of M/Rad_gM. By this, there exists $V/Rad_gM \leq M/Rad_gM$ such that $\frac{U+Rad_gM}{Rad_gM} \oplus \frac{V}{Rad_gM} = \frac{M}{Rad_gM}$. Then $\frac{U+Rad_gM}{Rad_gM} + \frac{V}{Rad_gM} = \frac{M}{Rad_gM}$ and $\frac{U+Rad_gM}{Rad_gM} \cap \frac{V}{Rad_gM} = 0$. Here $\frac{M}{Rad_gM} = \frac{U+Rad_gM}{Rad_gM} + \frac{V}{Rad_gM} = \frac{U+Rad_gM+V}{Rad_gM} = \frac{U+V}{Rad_gM}$ and $0 = \frac{U+Rad_gM}{Rad_gM} \cap \frac{V}{Rad_gM} = \frac{(U+Rad_gM)\cap V}{Rad_gM} = \frac{U\cap V+Rad_gM}{Rad_gM}$. Since $\frac{U+V}{Rad_gM} = \frac{M}{Rad_gM}$, $U+V=M$. Since $\frac{U\cap V+Rad_gM}{Rad_gM} = 0$, $U \cap V + Rad_gM = Rad_gM$ and $U \cap V \leq Rad_gM$. By $U \cap V \leq V$ and $Rad_gV = = V \cap Rad_gM$, $U \cap V \leq V \cap Rad_gM = Rad_gV$. Hence V is a g-radical supplement of U in M and M is g-radical supplemented. □

Proposition 2. *Let M be an R-module and $RadV = V \cap RadM$ for every $V \leq M$. Then M is Rad-supplemented if and only if M is semilocal.*

Proof. ($\Longrightarrow$) Clear from [12, Proposition 2.6 (2)].

($\Longleftarrow$) Let $U \leq M$. Since M is semilocal, $(U+RadM)/RadM$ is a direct summand of $M/RadM$. By this, there exists $V/RadM \leq M/RadM$ such that $\frac{U+RadM}{RadM} \oplus \frac{V}{RadM} = \frac{M}{RadM}$. Then $\frac{U+RadM}{RadM} + \frac{V}{RadM} = \frac{M}{RadM}$ and $\frac{U+RadM}{RadM} \cap \frac{V}{RadM} = 0$. Here $\frac{M}{RadM} = \frac{U+RadM}{RadM} + \frac{V}{RadM} = \frac{U+RadM+V}{RadM} = \frac{U+V}{RadM}$ and $0 = \frac{U+RadM}{RadM} \cap \frac{V}{RadM} = \frac{(U+RadM)\cap V}{RadM} = \frac{U\cap V+RadM}{RadM}$. Since $\frac{U+V}{RadM} = \frac{M}{RadM}$, $U+V=M$. Since $\frac{U\cap V+RadM}{RadM} = 0$, $U \cap V + RadM = RadM$ and $U \cap V \leq RadM$. By $U \cap V \leq V$ and $RadV = V \cap RadM$, $U \cap V \leq V \cap RadM = RadV$. Hence V is a Rad-supplement of U in M and M is Rad-supplemented. □

Lemma 2. *Let $X\beta_g^*Y$ in M, Y be a g-radical supplement of U in M and $U \trianglelefteq M$. Then $U \cap X \leq Rad_gM$.*

Proof. Since Y is a g-radical supplement of U in M, $M = U+Y$ and $U \cap Y \leq Rad_gY \leq Rad_gM$. Since $M = U+Y$ and $U \trianglelefteq M$ and $X\beta_g^*Y$, $M = U+X$. Let T be any essential maximal submodule of M. Here $U \cap Y \leq Rad_gM \leq T$. Assume

that $U \cap X \not\leq T$. Then $U \cap X + T = M$ and since $M = U + X$, by [2, Lemma 1.24], $X + U \cap T = M$. Since $U \trianglelefteq M$ and $T \trianglelefteq M$, $U \cap T \trianglelefteq M$. Since $X \beta_g^* Y$, $Y + U \cap T = M$ and since $U + T = M$, by [2, Lemma 1.24] again, $U \cap Y + T = M$. Then by $U \cap Y \leq T$, $M = U \cap Y + T = T$. This is a contradiction. Hence $U \cap X \leq T$ for every essential maximal submodule T of M and $U \cap X \leq Rad_g M$. □

Corollary 1. *Let $X \beta^* Y$ in M, Y be a g-radical supplement of U in M and $U \trianglelefteq M$. Then $U \cap X \leq Rad_g M$.*

Proof. Clear from Lemma 2. □

Corollary 2. *Let X lies above Y, Y be a g-radical supplement of U in M and $U \trianglelefteq M$. Then $U \cap X \leq Rad_g M$.*

Proof. Clear from Lemma 2. □

Definition 2. Let M be an R-module and $V \leq M$. If V is a g-radical supplement of an essential submodule in M, then V is called an eg-radical supplement submodule (briefly, eg-radical supplement) in M.

Lemma 3. *Let M be an R-module. If every submodule of M is β_g^* equivalent to an eg-radical supplement in M, then M is g-semilocal.*

Proof. Let $X/Rad_g M \leq M/Rad_g M$. Since $X \leq M$, by hypothesis, there exists an eg-radical supplement Y in M such that $X \beta^* Y$. Since Y is an eg-radical supplement in M, there exists $U \trianglelefteq M$ such that Y is a g-radical supplement of U in M. By Lemma 2, $U \cap X \leq Rad_g M$. Since $X \beta_g^* Y$ and $U \trianglelefteq M$ and $Y + U = M$, $X + U = M$. Then $\frac{M}{Rad_g M} = \frac{X+U}{Rad_g M} = \frac{X}{Rad_g M} + \frac{U+Rad_g M}{Rad_g M}$ and $\frac{X}{Rad_g M} \cap \frac{U+Rad_g M}{Rad_g M} = \frac{X \cap (U+Rad_g M)}{Rad_g M} = \frac{U \cap X + Rad_g M}{Rad_g M} = \frac{Rad_g M}{Rad_g M} = 0$. Hence $\frac{M}{Rad_g M} = \frac{X}{Rad_g M} \oplus \frac{U+Rad_g M}{Rad_g M}$ and $M/Rad_g M$ is semi-simple. Thus M is g-semilocal. □

Corollary 3. *Let M be an R-module. If every submodule of M lies above an eg-radical supplement in M, then M is g-semilocal.*

Proof. Clear from Lemma 3. □

3. Amply g-radical supplemented modules

Definition 3. Let M be an R-module and $U \leq M$. If for every $V \leq M$ such that $M = U + V$, U has a g-radical supplement V' in M with $V' \leq V$, then we say U has ample g-radical supplements in M. If every submodule of M has ample g-radical supplements in M, then M is called an amply g-radical supplemented module. (See also [9].)

Clearly we can see that every amply g-radical supplemented module is g-radical supplemented. Since $RadM \leq Rad_g M$ for every R-module M, every amply Rad-supplemented module is amply g-radical supplemented. Since every amply supplemented

module is amply Rad-supplemented, every amply supplemented module is amply g-radical supplemented. Hollow and local modules are amply g-radical supplemented.

Proposition 3. *Let $M = M_1 + M_2 + \cdots + M_n$. If M_i is amply g-radical supplemented for every $i = 1,2,\ldots,n$, then M is g-radical supplemented.*

Proof. Since M_i is amply g-radical supplemented for every $i = 1,2,\ldots,n$, M_i is g-radical supplemented. Then by [3, Corollary 4], M is g-radical supplemented. □

Proposition 4. *Let M be an amply g-supplemented R-module. Then M is amply g-radical supplemented.*

Proof. Let $M = U + V$. Since M is amply g-supplemented, U has a g-supplement V' with $V' \leq V$. Here $M = U + V'$ and $U \cap V' \ll_g V'$. Since $U \cap V' \ll_g V'$, by Lemma 1, $U \cap V' \leq Rad_g V'$. Hence V' is a g-radical supplement of U in M. Moreover, $V' \leq V$. Hence M is amply g-radical supplemented. □

Proposition 5. *Let M be an R-module, $U_1, U_2 \leq M$ and $M = U_1 + U_2$. If U_1 and U_2 have ample g-radical supplements in M, then $U_1 \cap U_2$ has also ample g-radical supplements in M.*

Proof. Let $U_1 \cap U_2 + T = M$. Then by [2, Lemma 1.24], $M = U_1 + U_2 \cap T = U_2 + U_1 \cap T$. Since U_1 and U_2 have ample g-radical supplements in M, then U_1 has a g-radical supplement V_1 with $V_1 \leq U_2 \cap T$ and U_2 has a g-radical supplement V_2 with $V_2 \leq U_1 \cap T$. Since $M = U_1 + V_1$ and $V_1 \leq U_2$, by Modular Law, $U_2 = U_2 \cap (U_1 + V_1) = U_1 \cap U_2 + V_1$. Similarly we have $U_1 = U_1 \cap U_2 + V_2$. Then $M = U_1 + U_2 = U_1 \cap U_2 + V_2 + U_1 \cap U_2 + V_1 = U_1 \cap U_2 + V_1 + V_2$ and by Lemma 1, $U_1 \cap U_2 \cap (V_1 + V_2) = U_1 \cap (V_1 + U_2 \cap V_2) = U_1 \cap V_1 + U_2 \cap V_2 \leq Rad_g V_1 + Rad_g V_2 \leq Rad_g (V_1 + V_2)$. Hence $V_1 + V_2$ is a g-radical supplement of $U_1 \cap U_2$ and since $V_1 + V_2 \leq T$, $U_1 \cap U_2$ has ample g-radical supplements in M. □

Lemma 4. *Every factor module of an amply g-radical supplemented module is amply g-radical supplemented.*

Proof. Let M be any amply g-radical supplemented module and $K \leq M$. Let $U/K \leq M/K$ and $M/K = U/K + V/K$ with $V/K \leq M/K$. Then $M = U + V$ and since M is amply g-radical supplemented, there exists a g-radical supplement T of U with $T \leq V$. Then by [3, Lemma 8], $(T + K)/K$ is a g-radical supplement of U/K in M/K. Moreover, $(T + K)/K \leq V/K$. Hence U/K has ample g-radical supplements in M/K and M/K is amply g-radical supplemented. □

Corollary 4. *The homomorphic image of an amply g-radical supplemented module is amply g-radical supplemented.*

Proof. Clear from Lemma 4. □

Lemma 5. *Let M be an R-module. If every submodule of M is g-radical supplemented, then M is amply g-radical supplemented.*

Proof. Let $U \leq M$ and $M = U + V$ with $V \leq M$. By hypothesis, V is g-radical supplemented and $U \cap V$ has a g-radical supplement T in V. Here $V = U \cap V + T$ and $U \cap V \cap T \leq Rad_g T$. Then $M = U + V = U + U \cap V + T = U + T$ and $U \cap T = U \cap V \cap T \leq Rad_g T$. Hence U has ample g-radical supplements in M and M is amply g-radical supplemented. □

Proposition 6. *Let R be any ring. Then every R-module is g-radical supplemented if and only if every R-module is amply g-radical supplemented.*

Proof. ($\Longrightarrow$) Let M be an R-module. Since every R-module is g-radical supplemented, every submodule of M is g-radical supplemented. Then by Lemma 5, M is amply g-radical supplemented, as desired.
($\Longleftarrow$) Clear. □

Lemma 6. *Let M be a π-projective and g-radical supplemented module. Then M is amply g-radical supplemented.*

Proof. Let $M = U + V$ and X be a g-radical supplement of U in M. Since M is π-projective and $M = U + V$, there exists an R-module homomorphism $f : M \to M$ such that $Imf \subset V$ and $Im(1-f) \subset U$. So, we have $M = f(M) + (1-f)(M) = f(U) + f(X) + U = U + f(X)$. Suppose that $a \in U \cap f(X)$. Since $a \in f(X)$, there exists $x \in X$ such that $a = f(x)$. Since $a = f(x) = f(x) - x + x = x - (1-f)(x)$ and $(1-f)(x) \in U$, we have $x = a + (1-f)(x) \in U$. Thus $x \in U \cap X$ and so $a = f(x) \in f(U \cap X)$. Therefore we have $U \cap f(X) \leq f(U \cap X) \leq f(Rad_g X) \leq Rad_g f(X)$. This means that $f(X)$ is a g-radical supplement of U in M. Moreover, $f(X) \subset V$. Therefore M is amply g-radical supplemented. □

Corollary 5. *If M is a projective and g-radical supplemented module, then M is an amply g-radical supplemented module.*

Proof. Clear from Lemma 6. □

Lemma 7. *Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$. If M_i is projective and g-radical supplemented for every $i = 1, 2, \ldots, n$, then M is amply g-radical supplemented.*

Proof. Since M_i projective for every $i = 1, 2, \ldots, n$, by [2, 4.3], M is projective. Since M_i is g-radical supplemented for every $i = 1, 2, \ldots, n$, by [3, Corollary 4], M is g-radical supplemented. Then by Corollary 5, M is amply g-radical supplemented, as desired. □

Lemma 8. *Let M be a projective and g-radical supplemented module. Then every finitely M-generated module is amply g-radical supplemented.*

Proof. Let N be a finitely M-generated R-module. Then there exist a finite index set Λ and an R-module epimorphism $f : M^{(\Lambda)} \longrightarrow N$. Since M is projective and

g-radical supplemented, by Lemma 7, $M^{(\Lambda)}$ is amply g-radical supplemented. Then by Corollary 4, N is amply g-radical supplemented. □

Proposition 7. *Let R be any ring. The following assertions are equivalent.*

(i) ${}_RR$ *is g-radical supplemented.*
(ii) ${}_RR$ *is amply g-radical supplemented.*
(iii) ${}_RR^{(\Lambda)}$ *is g-radical supplemented for every finite index set* Λ.
(iv) *Every finitely generated $R-$module is g-radical supplemented.*
(v) *Every finitely generated $R-$module is amply g-radical supplemented.*

Proof. (i) $\Longleftrightarrow$ (ii) Clear from Corollary 5, since ${}_RR$ is projective.
(i) $\Longleftrightarrow$ (iii) Clear from [3, Corollary 4].
(i) $\Longleftrightarrow$ (iv) Clear from [3, Corollary 6].
(i) $\Longleftrightarrow$ (v) Clear from Lemma 8. □

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A GENERALIZATION OF MENON-RAO-SURY'S IDENTITIES TO ADDITIVE CHARACTERS BY TÓTH'S METHOD

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Abstract. In this paper, we obtain a new Menon-type identity:

$$\sum_{\substack{a_1,\ldots,a_s,b_1,\ldots,b_r=1\\ \gcd(a_1,\ldots,a_s,n)=1}}^{n} \gcd(a_1-c_1,\ldots,a_s-c_s,b_1,\ldots,b_r,n)^s \lambda_1(b_1)\cdots\lambda_r(b_r)$$
$$= \varphi_s(n)\sigma_r(\gcd(w_1,\ldots,w_r,n)),$$

where $\lambda_j(b) := \exp(2\pi i w_j b/n)$ is an additive character of $\mathbb{Z}/n\mathbb{Z}$ for $1 \leqslant j \leqslant r$, $(c_1,\ldots,c_s) \in \mathbb{Z}^s$ is a fixed vector such that $\gcd(c_1,\ldots,c_s,n) = 1$, $\varphi_s(n) = n^s \prod_{p|n}(1-p^{-s})$ is the Jordan's totient function and $\sigma_r(n) = \sum_{d|n} d^r$ is the rth divisor function. This extends Rao's identity ([9]) and Sury's identity ([11]) to additive characters. Following the method of Tóth [14], we also generalize the above identity to arbitrary arithmetic function.

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1. INTRODUCTION

Classical Menon's identity states that, for every $n \in \mathbb{N} := \{1,2,\ldots\}$,

$$\sum_{\substack{a=1\\ \gcd(a,n)=1}}^{n} \gcd(a-1,n) = \varphi(n)\tau(n), \tag{1.1}$$

where $\gcd(\ ,\ )$ represents the greatest common divisor, φ denotes Euler's totient function and $\tau(n)$ is the number of divisors of n. This interesting arithmetic identity was proved by P. K. Menon [6] in 1965 and has lots of generalizations.

Using finite Fourier representations and Cauchy composite of totally even functions, K. Nageswara Rao [9] generalized Menon's identity as follows: for $s \in \mathbb{N}$,

$$\sum_{\substack{a_1,\ldots,a_s=1\\ \gcd(a_1,\ldots,a_s,n)=1}}^{n} \gcd(a_1-c_1,\ldots,a_s-c_s,n)^s = \varphi_s(n)\tau(n), \tag{1.2}$$

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where $(c_1,\dots,c_s)\in\mathbb{Z}^s$ is a fixed vector such that $\gcd(c_1,\dots,c_s,n)=1$ and

$$\varphi_s(n):=\#\{(a_1,\dots,a_s)\in\mathbb{Z}^s \mid 1\leqslant a_1,\dots,a_s\leqslant n,\ \gcd(a_1,\dots,a_s,n)=1\} \tag{1.3}$$

is the well known arithmetic function defined by Jordan, usually called Jordan's totient function. Explicitly, Jordan's totient can be computed by

$$\varphi_s(n)=n^s\prod_{p|n}(1-p^{-s}), \tag{1.4}$$

where p runs over all prime divisors of n ([1, p.147–155], [5, p.1–2]).

In 2009, B. Sury [11] proved that for $r\geqslant 0$,

$$\sum_{\substack{a,b_1,\dots,b_r=1\\ \gcd(a,n)=1}}^{n}\gcd(a-1,b_1,\dots,b_r,n)=\varphi(n)\sigma_r(n), \tag{1.5}$$

where $\sigma_r(n)=\sum_{d|n}d^r$ by using the Cauchy-Frobenius-Burnside lemma.

Recently, Zhao and Cao [19] derived the following elegant Menon-type identity with Dirichlet character:

$$\sum_{\substack{a=1\\ \gcd(a,n)=1}}^{n}\gcd(a-1,n)\chi(a)=\varphi(n)\tau\left(\frac{n}{d}\right), \tag{1.6}$$

where χ is a Dirichlet character modulo n with conductor d.

In [4], Li, Hu and Kim further extended identities (1.5) and (1.6) to multiplicative and additive characters:

$$\begin{aligned}&\sum_{\substack{a,b_1,\dots,b_r=1\\ \gcd(a,n)=1}}^{n}\gcd(a-1,b_1,\dots,b_r,n)\chi(a)\lambda_1(b_1)\cdots\lambda_r(b_r)\\ =&\varphi(n)\sigma_r\left(\gcd\left(n/d,w_1,\dots,w_r\right)\right),\end{aligned} \tag{1.7}$$

where for $1\leqslant j\leqslant r$, $b\mapsto\lambda_j(b):=\exp(2\pi i w_j b/n)$ with $w_j\in\mathbb{Z}$ is an additive character and d is the conductor of a Dirichlet character χ.

In [14], Tóth presented a generalization of (1.7) to arbitrary arithmetic function F, whose proof is short and elegant,

$$\begin{aligned}&\sum_{a_1,\dots,a_s,b_1,\dots,b_r=1}^{n}F(\gcd(a_1-c_1,\dots,a_s-c_s,b_1,\dots,b_r,n))\chi_1(a_1)\cdots\chi_s(a_s)\\ &\lambda_1(b_1)\cdots\lambda_r(b_r)=\varphi(n)^s\chi_1^*(c_1)\cdots\chi_s^*(c_s)\sum_{\substack{m|\gcd(n/d_1,\dots,n/d_s,w_1,\dots,w_r)\\ \gcd(n/m,c_1\cdots c_s)=1}}\frac{m^r(\mu*F)(n/m)}{\varphi(n/m)^s},\end{aligned} \tag{1.8}$$

where $(c_1,c_2,\dots,c_s)\in\mathbb{Z}^s$ is an arbitrary vector; χ_j are Dirichlet characters modulo n with conductors d_j and associated primitive characters $\chi_j^*(1\leqslant j\leqslant s)$; $\mu*F$ is the Dirichlet convolution of F and Möbius function μ.

Note that when $s=1$, $c_1=1$ and $F(a)=a$, (1.8) reduces to (1.7) by $\mu * F=\varphi$.

For other related works on Menon's identity, see [2, 3, 7, 8, 10, 12, 13, 15–18] and references therein.

The initial aim of this note is to generalize Rao's identity (1.2) and Sury's identity (1.5) to additive characters, which is the following theorem.

Theorem 1. *Let $n\in\mathbb{N}$. For $s\in\mathbb{N}$, let $(c_1,\dots,c_s)\in\mathbb{Z}^s$ be a fixed vector such that $\gcd(c_1,\dots,c_s,n)=1$. For $r\geqslant 0$, denote $b\mapsto\lambda_j(b)=\exp(2\pi i w_j b/n)$ to be an additive character of $\mathbb{Z}/n\mathbb{Z}$ where $1\leqslant j\leqslant r$. Then, the following identity holds.*

$$\begin{aligned}&\sum_{\substack{a_1,\dots,a_s,b_1,\dots,b_r=1\\ \gcd(a_1,\dots,a_s,n)=1}}^{n}\gcd(a_1-c_1,\dots,a_s-c_s,b_1,\dots,b_r,n)^s\lambda_1(b_1)\cdots\lambda_r(b_r)\\ =&\varphi_s(n)\sigma_r(\gcd(w_1,\dots,w_r,n)),\end{aligned}\tag{1.9}$$

where $\varphi_s(n)$ is the Jordan's totient function of order s determined by (1.3) or (1.4), and $\sigma_r(n)=\sum_{d|n}d^r$ is the r-th divisor function.

In fact, following the method of [14], we will get a more general result:

Theorem 2. *Let F be an arbitrary arithmetic function. For $s\in\mathbb{N}$, let $(c_1,\dots,c_s)\in\mathbb{Z}^s$ be an arbitrary vector with $\gcd(c_1,\dots,c_s,n)=g$, and λ_j be additive characters as defined above, with $w_j\in\mathbb{Z}(1\leqslant j\leqslant r)$. Then*

$$\begin{aligned}&\sum_{\substack{a_1,\dots,a_s,b_1,\dots,b_r=1\\ \gcd(a_1,\dots,a_s,n)=1}}^{n}F(\gcd(a_1-c_1,\dots,a_s-c_s,b_1,\dots,b_r,n))\lambda_1(b_1)\cdots\lambda_r(b_r)\\ =&\varphi_s(n)\sum_{\substack{m\mid\gcd(w_1,\dots,w_r,n)\\ \gcd(n/m,g)=1}}\frac{m^r(\mu*F)(n/m)}{\varphi_s(n/m)}.\end{aligned}\tag{1.10}$$

Note that for $F(n)=n^s$, $\mu*F=\varphi_s$. Therefore, (1.10) reduces to (1.9) whenever $F(n)=n^s$ and $\gcd(c_1,\dots,c_s,n)=1$. So we will only prove Theorem 2.

2. Proof of the main result

Denote $\mathbb{Z}_n:=\mathbb{Z}/n\mathbb{Z}$ to be the quotient ring of $\mathbb{Z}$ modulo n and let

$$\mathcal{M}_s(n):=\{(a_1,\dots,a_s)\in\mathbb{Z}_n^s\mid\gcd(a_1,\dots,a_s,n)=1\}\tag{2.1}$$

i.e., $\mathcal{M}_s(n)$ consists of vectors of $\mathbb{Z}_n^s$ with maximal order. Clearly the cardinality of $\mathcal{M}_s(n)$ is $\varphi_s(n)$.

To prove Theorem 2, we need the following lemma.

Lemma 1. *Assume $m\mid n$. Under the natural homomorphism*

$$\mathbb{Z}_n^s\xrightarrow{\pi}\mathbb{Z}_m^s:(a_1,\dots,a_s)\longmapsto(a_1,\dots,a_s)\mod m,$$

$\mathcal{M}_s(n)$ maps surjectively to $\mathcal{M}_s(m)$ and each vector of $\mathcal{M}_s(m)$ has $\varphi_s(n)/\varphi_s(m)$ pre-images in $\mathcal{M}_s(n)$.

Proof. Clearly, $\pi(\mathcal{M}_s(n)) \subset \mathcal{M}_s(m)$. For the left part, first consider the prime power case: $n = p^u$. Assume $m = p^v$ with $0 \leqslant v \leqslant u$. The case $v = 0$ is trivial since $\mathcal{M}_s(m) = \mathbb{Z}_m^s$ has only one element. For $v > 0$, a vector β of $\mathbb{Z}_m^s$ belongs to $\mathcal{M}_s(m)$ if and only if some component of β does not divide by p. This implies that each pre-image of $\beta \in \mathcal{M}_s(m)$ in $\mathbb{Z}_n^s$ lies in $\mathcal{M}_s(n)$. Therefore, each vector of $\mathcal{M}_s(m)$ has $(n/m)^s = \varphi_s(n)/\varphi_s(m)$ pre-images in $\mathcal{M}_s(n)$.

The general case can be deduced by the Chinese remainder theorem.

Let $n = \prod_{j=1}^{t} p_j^{u_j}$ be the prime factorization of n. Denote $n_j = p_j^{u_j}$ for $1 \leqslant j \leqslant t$. We have the natural isomorphism

$$\mathbb{Z}_n^s \cong \mathbb{Z}_{n_1}^s \times \mathbb{Z}_{n_2}^s \times \cdots \times \mathbb{Z}_{n_t}^s.$$

Under this isomorphism,

$$\mathcal{M}_s(n) \overset{1:1}{\longleftrightarrow} \mathcal{M}_s(n_1) \times \mathcal{M}_s(n_2) \times \cdots \times \mathcal{M}_s(n_t). \tag{2.2}$$

Similarly, let $m = \prod_{j=1}^{t} p_j^{v_j}$ and $m_j = p_j^{v_j}$ for $1 \leqslant j \leqslant t$. Then

$$\mathcal{M}_s(m) \overset{1:1}{\longleftrightarrow} \mathcal{M}_s(m_1) \times \mathcal{M}_s(m_2) \times \cdots \times \mathcal{M}_s(m_t). \tag{2.3}$$

Clearly, the following diagram commutes.

$$\begin{array}{ccc} \mathcal{M}_s(n) & \overset{1:1}{\longrightarrow} & \mathcal{M}_s(n_1) \times \mathcal{M}_s(n_2) \times \cdots \times \mathcal{M}_s(n_t) \\ \downarrow \pi & & \downarrow \pi_1 \times \cdots \times \pi_t \\ \mathcal{M}_s(m) & \overset{1:1}{\longrightarrow} & \mathcal{M}_s(m_1) \times \mathcal{M}_s(m_2) \times \cdots \times \mathcal{M}_s(m_t) \end{array} \tag{2.4}$$

We have proved that for each j, $\mathcal{M}_s(m_j)$ has $\varphi_s(n_j)/\varphi_s(m_j)$ pre-images in $\mathcal{M}_s(n_j)$. By (2.4) and the multiplicative property of φ_s, we get the desired result. □

Proof of Theorem 2. Let S be the left hand side of (1.10). Using the identity $F(n) = \sum_{m|n} (\mu * F)(m)$, we get

$$\begin{aligned} S &= \sum_{\substack{a_1,\ldots,a_s,b_1,\ldots,b_r=1 \\ \gcd(a_1,\ldots,a_s,n)=1}}^{n} \lambda_1(b_1)\cdots\lambda_r(b_r) \sum_{m|\gcd(a_1-c_1,\ldots,a_s-c_s,b_1,\ldots b_r,n)} (\mu * F)(m) \\ &= \sum_{m|n} (\mu * F)(m) \sum_{\substack{a_1,\ldots,a_s=1 \\ \gcd(a_1,\ldots,a_s,n)=1 \\ (a_1,\ldots,a_s) \\ \equiv (c_1,\ldots,c_s) \pmod m}}^{n} 1 \sum_{\substack{b_1,\ldots,b_r=1 \\ (b_1,\ldots,b_r) \\ \equiv (0,\ldots,0) \pmod m}}^{n} \lambda_1(b_1)\cdots\lambda_r(b_r). \end{aligned} \tag{2.5}$$

The condition $(a_1,\dots,a_s) \equiv (c_1,\dots,c_s) \pmod m$ requires that

$$\gcd(g,m) = \gcd(c_1,\dots,c_s,m) = \gcd(a_1,\dots,a_s,m) = 1$$

since $\gcd(a_1,\dots,a_s,n) = 1$. Then using Lemma 1, we get

$$\sum_{\substack{a_1,\dots,a_s=1 \\ \gcd(a_1,\dots,a_s,n)=1 \\ (a_1,\dots,a_s) \\ \equiv(c_1,\dots,c_s) \pmod m}}^{n} 1 = \frac{\varphi_s(n)}{\varphi_s(m)} \tag{2.6}$$

if $\gcd(g,m) = 1$; and otherwise it equals to 0.

Substituting (2.6) to (2.5) and using the orthogonality of characters, we get

$$\begin{aligned} S &= \sum_{\substack{m\mid n \\ \gcd(m,g)=1 \\ \frac{n}{m}\mid\gcd(w_1,\dots,w_r)}} (\mu * F)(m) \frac{\varphi_s(n)}{\varphi_s(m)} \left(\frac{n}{m}\right)^r \\ &= \varphi_s(n) \sum_{\substack{m\mid\gcd(w_1,\dots,w_r,n) \\ \gcd(n/m,g)=1}} m^r \frac{(\mu * F)(n/m)}{\varphi_s(n/m)} \end{aligned} \tag{2.7}$$

which concludes the proof. □

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RQWDFB: ANALYSIS OF RETRIAL QUEUES WITH DELAYED FEEDBACK

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Abstract. The models of retrial queues M/M/K/K with delayed feedback are considered. If upon arrival of primary call (*p*-call) there are free channels then it occupies one of them for a some exponentially distributed time; otherwise, it in accordance to Bernoulli scheme either leaves the system forever or go to orbit for retrial calls (*r*-calls). After the customer is served completely, it will decide based on Bernoulli scheme either to join the orbit for re-service after some delay (delayed feedback) or to leave the system forever. If upon arrival *r*-call there is a free server, then the arriving call begins to be served immediately; otherwise, it in accordance to Bernoulli scheme either leaves the orbit or return to orbit. An approximate method to calculation of the steady-state probabilities of the appropriate two-dimensional Markov chain as well as performance measures of the system is developed. Results of numerical experiments are demonstrated.

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Keywords: multi-channel queuing system, retrial calls, delayed feedback, finite and infinite orbits, balking from orbit, calculation method

1. INTRODUCTION

Retrial of calls are a common phenomenon in queuing systems. In retrial queuing (RQ) systems arriving new calls who find all servers and waiting room occupied is decide to leave the system forever or to join the retrial group to try again for their service after a random period of time. Performance modeling of RQ systems has attracted many researchers since they adequately describe many real life problems, e.g. in switching systems, telecommunication networks, computer networks, inventory systems etc. [1, 10]. For comprehensive survey and list of references in this field see [11, 23].

Retrial phenomenon is observed in queuing systems with feedback (FB) as well. In such systems serviced calls require repeated service depending on some factors, e.g. quality of service, response time in the system, etc. For the state of the art of the theoretical and practical researches in queuing systems with FB see [19].

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Note that almost all papers on RQ have considered the systems without feedback. However, retrial queuing systems with delayed feedback (RQwDFB) are suitable models for many practical situations, e.g. multiple access telecommunication systems, where data that is transmitted erroneously are sent again can be modeled as retrial queue with feedback.

In the literature there are few works devoted to study RQwDFB. It seems that RQwDFB for the first time has been investigated in paper [5]. In this paper authors investigated an RQwDFB $M/M/c/c$ with infinite orbit capacity when $c = 1, 2$. An arriving call is served immediately if one of the servers is idle, and if all servers are busy, it will either join the retrial group to try again after a random period of time or leave the system forever without service. On retrials, each customer is treated the same as a primary customer. If retrial call finds free servers, it receives service immediately, and if all the servers are busy, it will either leave the system forever or join the retrial group. All calls in the retrial group act independently of each other. The retrial time is exponentially distributed with finite mean and is independent of all previous retrial times and all other stochastic processes in the system. After the call is served completely, it will decide either to join the retrial group again for another service or to leave the system forever. They found the joint probability generating function (PGF) of the number of busy servers and the number of customers in the retrial group for $c = 1, 2$.

In [21] considered an RQwDFB $M/G/1/1$ with single server vacation and constant retrial rate where the server is subjected to starting failures. It is assumed that the retrial time is governed by an arbitrary distribution. The server leaves for a vacation as soon as the system becomes empty. When the server returns from the vacation and finds no calls, he waits free for the first customer to arrive from outside the system. If the server is started successfully (with a certain probability), the call gets service immediately; otherwise, the repair for the server starts immediately and the call must leave for the orbit and make a retrial later. The system size distribution at random points and various performance measures are derived.

In [2, 3, 8] considered Markovian models of single server RQwDFB. Models with two types of calls under non-pre-emptive and pre-emptive priority service disciplines are investigated in [3] and [2], respectively. In both papers [3] and [2] the retrial, loss and feedback allowed for low priority calls only. If the server is free at the time of the arrival of low priority call, then an incoming call immediately begins to be serviced; after completion of service, low priority call either may re-join the orbit of infinite size or leaves the system. If the server is busy then the arriving low priority call goes to orbit and becomes a source of repeated calls. If the server is free at the time of the arrival of high priority customer, then the arriving call begins to be served immediately and high priority customer leaves the system after service completion. In [3] the non-pre-emptive priority service discipline is used, i.e. if the server is engaging with low priority call and the higher priority call comes then the

high priority call will get service only after completion of the service of low priority call who is in service. In [2] the pre-emptive priority service is used, i.e. if the server is engaging with low priority call and the higher priority call enters then the high priority call will get service immediately and the low priority call who is in service goes to orbit without completion of his service. The model of single server RQwDFB with batch service under multiple server vacations is considered in [8]. It is assumed that upon completion the service batch of calls in accordance to Bernoulli scheme either may join the queue or leaves the system. If after completion of the service number of calls in the queue less than a given threshold, then server leaves for a multiple vacation period with exponential distribution. The arriving call in accordance to Bernoulli scheme either may join the queue or leaves the system when server is busy or in vacation.

Model of RQwDFB $M/M/1/N$, $N<\infty$, with two orbits is considered in [4]. If upon arrival, the calls find the queue full, then they decide to stay for an attempt to get service or leave the system. Arriving customers who decide to stay for an attempt have to choose one of the orbits in accordance with the probabilities that depend on thresholds in orbits. After the call is served completely, it may decide to either attempt for re-service or leave the system forever. In this case, the selection of one of the orbits occurs according to the Bernoulli scheme.

Recently in [25] considered an RQwDFB $M/G/1$ with negative calls under working vacation and working breakdowns. The PGFs for the numbers of calls in the orbit under different server status are derived. Various system performance measures and some special cases were discussed.

To study models of RQwDFB in [4, 5, 8, 21, 25] PGF approach is used while in [2, 3] matrix-geometric method [22] is applied.

Note that indicated above works devoted to single-server RQwDFBs. Moreover, in these works it is assumed that the retrial and feedback probabilities are constant quantities, i.e. they do not depend on the state of the system. These assumptions significantly limit the application areas of the proposed models, since in practice RQwDFB contains many servers and in these systems decisions about going into orbit or leaving orbit without getting re-service often depend on the current state of the system.

Models of multi-server RQwDFBs, in contrast its single-server counterpart, are not studied (exception is [5] in which two-server system is investigated as well). This paper is motivated by the work [5]. The novelty of our contribution is threefold. Firstly, we consider models of multi-server RQwDFB with both finite and infinite capacity of orbit. Secondly, we take into account that retrial, feedback probabilities as well as balking probabilities from the orbit are state-dependent; in other words, we assume that both arrived and departure calls as well as retrial calls have information about number of calls in orbit, i.e. appropriate decisions depend on number of calls in orbit. Thirdly, we propose an approximate method to study the proposed models

of RQwDFB. The developed method allows calculate the steady-state probabilities as well as performance measures of the models under study for arbitrary number of channels and capacity of orbits for retrial and feedback calls. Moreover, calculations are realized by explicit formulas that contains tabulated quantities.

Proposed here approach to study models of RQwDFB based on the idea of theory of space merging of Markov chains [12]. The space-merging algorithm (SMA) for the first time was used to calculation of the steady-state probabilities of the two-dimensional Markov chain (2D MC) in [6, 17, 18]. Recently hierarchical SMA to study 3D MC is developed in [13]. In last decade many researchers develop similar SMAs to study 2D MC in various areas of applications [7,9,14–16,24]. In all papers, authors note high efficiency of SMA (in both sense of accuracy and complexity) in comparison with other numerical methods.

This paper is structured as follows. Basic mathematical model of the investigated RQwDFB is described in Section 2. Generator matrix of the constructed 2D MC is created in Section 3. Here exact method to calculate its steady-state probabilities is indicated and explicit formulas for performance measures of the investigated system is developed as well. In Section 4, approximate method for solving indicated above problems is proposed. Results of numerical experiments that show high accuracy of the approximate method as well as behavior of performance measures versus system parameters are demonstrated in Section 5. Conclusions are given in Section 6.

2. MODEL DESCRIPTION

First consider the model with finite capacity of orbit, i.e. let maximum capacity of orbit is R, $R < \infty$.The system contains $K > 1$ identical channels and a new arriving calls (or primary calls, p-calls) form Poisson flow with rate λ. An incoming p-call is accepted for service if at this moment there is a free channel; otherwise, i.e. if upon arrival of p-call all channels are busy, then it either go to orbit with a probability (w.p.) α_r or leave the system w.p. $1-\alpha_r$ when there are r calls in the orbit, $\alpha_R = 0$.

Calls from orbit (r-calls) independently each other generates request to service. It is assumed that times between arrivals of r-calls have common exponential distribution with mean η^{-1} , i.e. if there are r calls in orbit then arrival intensity is $r\eta$ (linear retrial rate). If upon arrival of r-call all channels are busy, then it either leave the orbit w.p. σ_r or return to orbit w.p. $1-\sigma_r$ when there are r calls in the orbit.

After completion of the service process, the call, according to the Bernoulli scheme, either w.p. β_r go to orbit in order to requires repeated service after some positive delay, or with a complementary probability $1-\beta_r$ leaves the system forever when there are r calls in the orbit, $\beta_R = 0$.

Service times for both types of p-calls and r-calls are independent and identically distributed (i.i.d.) random variables (r.v.) and suppose that their distribution function is exponential one with a common average $1/\mu$.

It is assumed that the arriving of p-calls from outside, the arriving of r-calls from orbit and service of calls, are independent of each other.

The problem is to determine the steady-state probabilities of a given system. The solution of this problem allows us to find the following performance measures of the system: loss probabilities of x-calls, P_x, $x \in \{p,r\}$; the average number of busy channels $(K_{av})$and the average number of r-calls in orbit (L_r).

3. Generation of Q-matrix and formulas for performance measures

First consider model with finite capacity of orbit. The state of the system at arbitrary moment of the time is defined by two-dimensional vector (b, r), where b is total number of busy channels and r is number of r-calls in orbit. Taking into account the type of distribution functions of all r.v. that are involved in the formation of the model we conclude that the state space of the appropriate two-dimensional Markov chain (2D MC) is determined as follows:

$$S = \{1, 2, ..., K\} \times \{0, 1, ..., R\} . \tag{3.1}$$

Transition intensity from state $(b,\ r)$ to state $(b',\ r')$ is denoted by $q((b,r),(b',r'))$. These quantities are determined as follows:

For case $0 \le b \le K-1$:

$$q\left((b,r),\left(b',r'\right)\right) = \begin{cases} \lambda, & \text{if } (b',r') = (b+1,r) , \\ b\mu\beta_r, & \text{if } r < R,\ (b',r') = (b-1,r+1) , \\ b\mu(1-\beta_r) , & \text{if } r < R,\ (b',r') = (b-1,r) , \\ b\mu, & \text{if } r = R,\ (b',r') = (b-1,R) , \\ r\eta , & \text{if } (b',r') = (b+1,r-1) . \end{cases} \tag{3.2}$$

For case $b = K$:

$$q\left((K,r),\left(b',r'\right)\right) = \begin{cases} \lambda\alpha_r, & \text{if } r < R,\ (b',r') = (K,r+1) , \\ K\mu\beta_r, & \text{if } r < R,\ (b',r') = (K-1,r+1) , \\ K\mu(1-\beta_r), & \text{if } r < R,\ (b',r') = (K-1,r) , \\ K\mu, & \text{if } r = R,\ (b',r') = (K-1,R) , \\ r\eta\sigma_r, & \text{if } (b',r') = (b+1,r-1) . \end{cases} \tag{3.3}$$

From relations (3.2) and (3.3) we conclude that all states of space (3.1) are communicated each other, i.e. steady-state probabilities are existing. Let $p(b,r)$ denote stationary probability of state $(b,r) \in S$.

The performance measures of the system are determined as follows:

$$K_{av} = \sum_{b=1}^{K} b \sum_{r=0}^{R} p\,(b,r) ; \tag{3.4}$$

$$L_r = \sum_{r=1}^{R} r \sum_{b=0}^{K} p\,(b,r) ; \tag{3.5}$$

$$P_p = p(K,R) + \sum_{r=0}^{R-1} (1-\alpha_r)\, p(K,r); \tag{3.6}$$

$$P_r = \sum_{r=1}^{R} \sigma_r p(K,r). \tag{3.7}$$

For the model with moderate dimension of state space (3.1) the steady-state probabilities might be calculated from system of equilibrium equations (SEE). Due to the complex structure of the obtained GM (see formulas (3.2) and (3.3)), it is not possible to find analytical solution of SEE. To solve the indicated problem Neuts' matrix-geometric method and its modifications as well as spectral expansion method [20] might be used. However, these methods often suffer from computational instability for large values of number of channels and orbit capacity. Therefore, an alternative (approximate) method for solving this problem is given below. It is based on the space merging approach for multidimensional MCs and it allows calculate the steady-state probabilities and performance metrics via explicit formulas. Moreover, the obtained formulas contain quantities that are tabulated even, i.e. Erlang's loss formulas etc.

4. Approximate Method Based on Space Merging Approach

For the correct application of the developed method, it is required satisfying the following condition: the state space (3.1) is decomposed into classes in a such a way that the transitions between states within classes are much larger than transitions between states of different classes. This condition is fulfilled under certain ratios of the initial parameters of the system. Let us assume that arrival rate of p-calls is much larger than arrival rate of r-calls, i.e. below we assume that the following conditions are fulfilled: $\eta << \lambda$.

When fulfilling this assumption, we consider the following splitting of the state space (3.1):

$$S = \bigcup_{r=0}^{R} S_r, \qquad S_r \bigcap S_{r'} = \varnothing, \qquad r \neq r', \tag{4.1}$$

where $S_r = \{(b,r) \in S:\ b = 0, 1, ..., K\}$, $r = 0, 1, ..., R$.

Under accepted above assumptions the transitions between states within classes S_r are much larger than transitions classes between states of different classes. Based on splitting (4.1) in the state space (3.1), the following merge function is defined:

$$U(b,r) = < r >, \qquad (b,r) \in S_r, \tag{4.2}$$

where $< r >$ is the merged state which includes all states from the class S_r, $r = 0, 1, ..., R$. We denote $\Omega = \{< r >:\ r = 0, 1, ..., R\}$.

Approximate values of the steady-state probabilities of the initial 2D MC are denoted by $\tilde{p}(b,r)$. They are determined as follows:

$$\tilde{p}(b,r) = \rho_r(b)\,\pi(<r>), \tag{4.3}$$

where $\rho_r(b)$ is the probability of state (b,r) within splitting model with state space S_r, $\pi(<r>)$ is the probability of merged state $<r>\in\Omega$.

Remark 1. Because of in splitting model with state space S_r states (b,r) are uniquely defined by component b, hereinafter these states are denoted as scalar b, $b=0,1,\ldots,K$.

From (3.2) and (3.3) we conclude that the probabilities of the states of splitting models with spaceS_r, $0\le r\le R-1$, coincide with the state probabilities of the one-dimensional birth-death process (1D BDP) in which the birth intensity is equal to λ and the death intensity in the state b is equal to $b\mu(1-\beta_r)$. So, steady-state probabilities $\rho_r(b)$, $0\le r\le R-1$, coincide with steady-state probabilities of Erlang's model $M/M/K/K$ with load $\nu_r=\lambda/\mu(1-\beta_r)$, i.e.

$$\rho_r(b) = \frac{\nu_r^b}{b!}\Bigg/\sum_{i=0}^{K}\frac{\nu_r^i}{i!},\ b=0,1,\ldots,K. \tag{4.4}$$

By same way we conclude that the probabilities of the states of splitting model with space S_R coincide with the state probabilities of the 1D BDP in which the birth intensity is equal to λ and the death intensity in the state b is equal to $b\mu$. In other words, steady-state probabilities$\rho_R(b)$, $0\le b\le K$, coincide with steady-state probabilities of Erlang's model $M/M/K/K$ with load $\nu=\lambda/\mu$.

Merged model with state space Ω represents 1D BDP as well. Transition rates between its states denote by $q(<r>,<r'>)$. It is clear that $q(<r>,<r'>)=0$ if $|r-r'|>1$. Positive transition rates are calculated as follows:

For $0\le r\le R-1$:

$$\begin{aligned} q(<r>,<r+1>) &= \lambda\alpha_r\rho_r(K)+\mu\beta_r\sum_{b=1}^{K}b\rho_r(b) \\ &= \lambda\alpha_r E_B(\nu_r,K)+\mu\beta_r B_{av}(\nu_r,K). \end{aligned} \tag{4.5}$$

For $1\le r\le R$:

$$\begin{aligned} q(<r>,<r-1>) &= r\eta\sum_{b=1}^{K-1}\rho_r(b)+r\eta\sigma_r\rho_r(K) \\ &= r\eta\left(1-(1-\sigma_r)E_B(\nu_r,K)\right). \end{aligned} \tag{4.6}$$

Hereinafter, $E_B(x,\ K)$ and $B_{av}(x,K)$ denote the Erlang B-formula and the average number of busy channels for the $M/M/K/K$ system with a load x, respectively, i.e.

$$E_B(x,K) = \frac{x^K}{K!}\Bigg/\sum_{k=0}^{K}\frac{x^k}{k!};\ B_{av}(x,K) = x\left(1-E_B(x,K)\right).$$

Therefore, from relations (4.5) and (4.6) we conclude that the probabilities of merged states are calculated by classical formulae for 1D BDP:

$$\pi(<r>)=\prod_{i=0}^{r-1}\frac{\Lambda_i}{\mathrm{M}_{i+1}}\pi(<0>),\ r=1,...,R, \tag{4.7}$$

where $\Lambda_i=q(<i>,<i+1>)$, $\mathrm{M}_i=q(<i>,<i-1>)$ and $\pi(<0>)$ is found from normalizing condition, i.e. $\sum_{r=0}^{R}\pi(<r>)=1$.

Finally, taking into account relations (4.3), (4.4) and (4.7), approximate values of the steady-state of the initial 2D MC are found. Further, after certain mathematical transformations, we obtain the following approximate formulas for calculating the performance measures (3.4)-(3.7) of the system under study:

$$K_{av}\approx\sum_{b=1}^{K}b\sum_{r=0}^{R}\rho_r(b)\,\pi(<r>); \tag{4.8}$$

$$L_r\approx\sum_{r=1}^{R}r\pi(<r>); \tag{4.9}$$

$$P_p\approx E_B(\nu_R,K)\,\pi(<R>)+\sum_{r=0}^{R-1}(1-\alpha_r)\,E_B(\nu_r,K)\,\pi(<r>); \tag{4.10}$$

$$P_r\approx\sum_{r=1}^{R}\sigma_r E_B(\nu_r,K)\,\pi(<r>). \tag{4.11}$$

Case of infinite orbit capacity. Note that in the case of constants probabilities α_r, β_r and σ_r for all the possible r, the formulas (4.8)-(4.11) are even more simplified. Moreover, in this case it is possible to obtain explicit solutions for models with an infinite orbit capacity.

So, for a model with an infinite orbit capacity $(R=\infty)$, the state probabilities inside all splitting models with space S_r, $r=0,1,2,\ldots$ coincide with the state probabilities of the Erlang model *M/M/K/K* with load $\nu/(1-\beta)$ (see formula (4.4)). From relations (4.5) and (4.6) we conclude that for this model the probabilities of merged states coincide with the probabilities of the states of the model *M/M/*∞ with the load $\theta=\Lambda/\Psi$, i.e.

$$\pi(<r>)=\frac{\theta^r}{r!}\exp(-\theta), \qquad r=0,1,2,\ldots \tag{4.12}$$

where $\Lambda=\lambda\alpha E_B\left(\frac{\nu}{1-\beta},K\right)+\mu\beta B_{av}\left(\frac{\nu}{1-\beta},K\right)$, $\mathrm{M}=\eta\left(1-(1-\sigma)E_B\left(\frac{\nu}{1-\beta},K\right)\right)$.

Then by using (4.12) from formulas (4.8)-(4.11) we obtain that the performance measures of the model with an infinite orbit capacity are approximately determined as follows:

$$K_{av}\approx B_{av}\left(\frac{\nu}{1-\beta},K\right); \tag{4.13}$$

$$L_r \approx \sum_{r=1}^{R} r\pi(< r >) = \theta; \tag{4.14}$$

$$P_p \approx (1-\alpha) E_B\left(\frac{\nu}{1-\beta}, K\right); \tag{4.15}$$

$$P_r \approx \sigma(1-\exp(-\theta)) E_B\left(\frac{\nu}{1-\beta}, K\right). \tag{4.16}$$

5. Numerical Results

Here results of numerical experiments are demonstrated. The goal of these experiments is investigation of the accuracy of the developed approximate formulas. For concreteness, we investigate the behavior of the accuracy versus the rate of r-calls (η).

As a measure of proximity of two distributions of steady-state probabilities (exact and approximate) the following norms are used:

Cosine similarity:

$$\|\mathrm{N}\|_1 = \frac{\sum_{(b,r)\in S} p(b,r)\,\tilde{p}(b,r)}{\left(\sum_{(b,r)\in S}(p(b,r))^2\right)^{\frac{1}{2}}\left(\sum_{(b,r)\in S}(\tilde{p}(b,r))^2\right)^{\frac{1}{2}}}; \tag{5.1}$$

Euclidean distance:

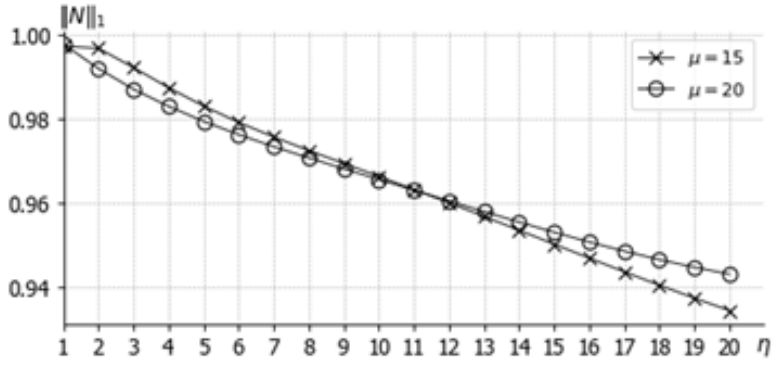
$$\|\mathrm{N}\|_2 = \left(\sum_{(b,r)\in S}(p(b,r)-\tilde{p}(b,r))^2\right)^{\frac{1}{2}}. \tag{5.2}$$

Remark 2. In general, cosine similarity norm (5.1) is used to estimate the orientation of two vectors. However, in our experiments it estimates not only the orientation of the steady-state probability vectors in various (exact and approximate) approaches, as well as their Euclidean proximity. This is because, according to the normalization condition, the ends of these two vectors are in the same hyper plane of the Euclidean space of dimension $(K+1)(R+1)$, i.e. $\sum_{(b,r)\in S} p(b,r) = \sum_{(b,r)\in S} \tilde{p}(b,r) = 1$. Nonetheless, here values of Euclidean distance is presented as well.

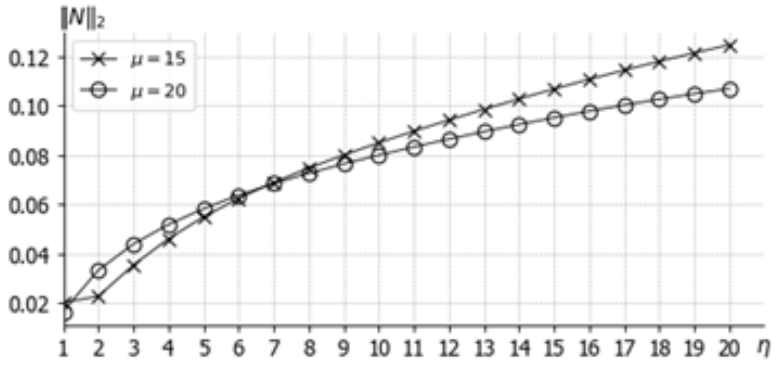
Firstly, consider results of numerical experiments for the model with finite capacity of orbit. In these experiments it is assumed that retrial and feedback probabilities as well as balking probabilities from the orbit are state-dependent and these parameters satisfies the following conditions: $0 < \alpha_{r+1} \le \alpha_r < 1$; $0 < \beta_{r+1} \le \beta_r < 1$; $0 < \sigma_r \le \sigma_{r+1} < 1$. For concreteness, we assume that $\alpha_r = 1/(r+1)$; $\beta_r = 1/(r^2+1)$; $\sigma_r = r/(r+1)$.

Below in all experiments (for both models with finite and infinite capacity of orbit) we assume that $K = 3$, $\lambda = 40$. And in experiments for model with finite capacity of orbit it is assumed that $R = 20$.

In Figure 1 the dependence of similarity norms (5.1) and (5.2) on the rate of r-calls is demonstrated. Exact values of the steady-state probabilities are calculated from the corresponding SEE. From these graphs we conclude that the steady-state probabilities are calculated with high accuracy, i.e. in wide range of changing rate of r-calls the value of similarity norm (5.1) is close to one (see Figure 1A) while Euclidean distance (5.2) is small enough (see Figure 1B).



(A)



(B)

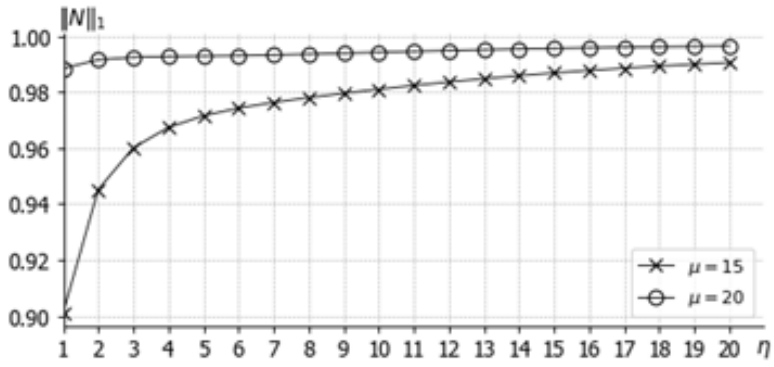
FIGURE 1. Dependence norms (5.1) (A) and (5.2) (B) on parameter η for the model with finite capacity of orbit

It is important to note that when the rate of r-calls increases accuracy of calculation of the steady-state probabilities decreases, i.e. the norm (5.1) is decreasing function versus to the rate of r-calls while norm (5.2) increases relative to the indicated parameter. These facts were expected since when the rate of r-calls is small then the transitions between states of different classes S_r, $r = 0, 1, ..., R$, in splitting (4.1)

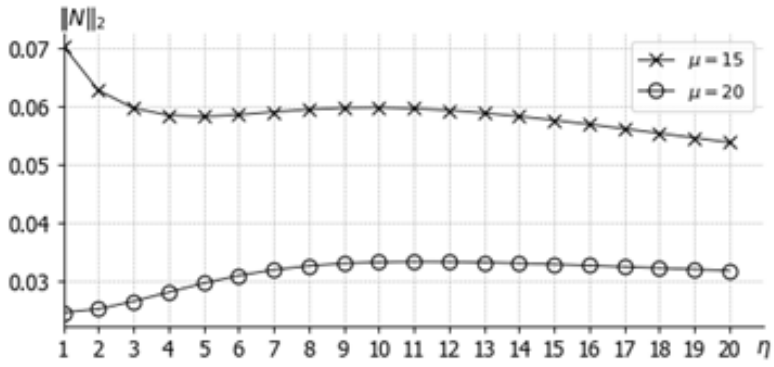
is rare, i.e. accuracy of proposed formulas becomes very high. Note that values of norms (5.1) and (5.2) are almost independent on service rate.

Now consider the results of numerical experiments for the model with infinite capacity of orbit, i.e. when $R = \infty$. In these experiments we assume that $\alpha = 0.3$, $\beta = 0.1$, $\sigma = 0.4$.

The corresponding results for the dependence of similarity norms (5.1) and (5.2) on the rate of r-calls are shown in Figure 2.



(A)



(B)

FIGURE 2. Dependence norms (5.1) (A) and (5.2) (B) on parameter η for the model with infinite capacity of orbit

It can be seen from the graphs that in this model accuracy of calculation of approximate values of steady-state probabilities is much better than in model with finite capacity of orbit (see Figure 1). Here exact values of the steady-state probabilities are calculated from the corresponding SEE by using the Python SciPy software package.

We choose R to be so large that state probabilities $p(b,R)$, $b=\overline{0,3}$, are equal to the machine zero (in our experiments $R=100$). In this model values of norms (5.1) and (5.2) are essentially dependent on service rate, i.e. the higher the service intensity, the high value of norm (5.1) and lower value of norm (5.2).

6. Conclusion

The paper proposes mathematical models of RQwDFB in which the probabilities of retrial, feedback and balking from the orbit depend on the current number of calls in the orbit. We studied both models with common finite and infinite orbit capacity for retrial and feedback calls and developed exact and approximate methods for calculating their performance measures.

As directions for further research, it is proposed to study similar models in which retrial and feedback calls are distinguished. Of particular interest are also the problems of optimizing RQwDFB regarding the selected criteria for the quality of their functioning. An important area of further research is also the study of the possibility of using special functions [26] to solve the problems investigated here. These problems are the subject of special studies.

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ON PROXIMAL CONTRACTIONS VIA IMPLICIT RELATIONS AND BEST PROXIMITY POINTS

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Abstract. In this paper, we employ two types of implicit relations to define some new kind of proximal contractions and study about their best proximity points. More precisely, we use two class of functions $\mathcal{A}$ and $\mathcal{A}'$ to explore proximal $\mathcal{A}$, $\mathcal{A}'$-contractions of first and second kind and strong proximal $\mathcal{A}$, $\mathcal{A}'$-contractions. We investigate the existence of best proximity points results of the same. It is worth mentioning that the well-known results of Sadiq Basha [J. Approx. Theory, 2011] on proximal contractions are the special cases of our obtained results. We authenticate our results by suitable examples. Finally, we point out some areas where our obtained results can be applied.

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Keywords: Best proximity point, proximal contraction, strong proximal contraction, approximative compactness.

1. INTRODUCTION

Best proximity point theory deals with a natural generalization of fixed point theory by routing the method of computing an optimal approximate solution to the equation $Sx = x$, where $S : G \to H$ is a non-self mapping, G, H being two disjoint subsets of a metric space (M,d). Since for $x \in G$, we always have $d(x,Sx) \geq \text{dist}(G,H)$, where $\text{dist}(G,H) = \inf\{d(x,y) : x \in G, y \in H\}$, it follows that an element $x \in G$ will be an approximate optimal solution of $Sx = x$ if $d(x,Sx) = \text{dist}(G,H)$. Such a point '$x$' is known as a best proximity point of S, and the branch of mathematics dealing with best proximity points is known as the best proximity point theory. There are numerous articles that analyze several kinds of contractions for the existence of best proximity point(s) for single-valued as well as multivalued mappings. Interested readers may consult with the papers [3,4,9,13,15,18,19] for single-valued mappings and [2,11,12,20,22] for multivalued mappings.

The study of the best proximity point theory by using different contractions had been enriched in 2011 with a new kind of contraction by Sadiq Basha [17]. In [17],

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he came with some new kind of contractions such as proximal contractions of the first kind, proximal contractions of the second kind, strong proximal contractions of the first kind.

Definition 1. ([17, p. 1774, Definitions 2.2-2.4]). Let (M,d) be a metric space and G,H two non-empty subsets of M. A mapping $S : G \to H$ is said to be a

(i) proximal contraction of the first kind if there exists $\alpha \in [0,1)$ satisfying

$$\left.\begin{array}{l} d(u_1,Sx_1) = \text{dist}(G,H) \\ d(u_2,Sx_2) = \text{dist}(G,H) \end{array}\right\} \Longrightarrow d(u_1,u_2) \leq \alpha\, d(x_1,x_2)$$

for all $u_1,u_2,x_1,x_2 \in G$,

(ii) proximal contraction of the second kind if there exists $\alpha \in [0,1)$ satisfying

$$\left.\begin{array}{l} d(u_1,Sx_1) = \text{dist}(G,H) \\ d(u_2,Sx_2) = \text{dist}(G,H) \end{array}\right\} \Longrightarrow d(Su_1,Su_2) \leq \alpha\, d(Sx_1,Sx_2)$$

for all $u_1,u_2,x_1,x_2 \in G$,

(iii) strong proximal contraction of the first kind if there exists $\alpha \in [0,1)$ such that for all $u_1,u_2,x_1,x_2 \in G$ and for all $\gamma \in [1,2)$

$$\left.\begin{array}{l} d(u_1,Sx_1) \leq \gamma\, \text{dist}(G,H) \\ d(u_2,Sx_2) \leq \gamma\, \text{dist}(G,H) \end{array}\right\} \Longrightarrow d(u_1,u_2) \leq \alpha\, d(x_1,x_2) + (\gamma-1)dist(G,H).$$

In the above definitions of proximal contractions, we see that the definitions involves the displacement $d(x_1,x_2)$ only. It is known that for two points x_1,x_2, the other displacements are $d(Sx_1,x_1), d(Sx_2,x_2), d(Sx_1,x_2)$ and $d(Sx_2,x_1)$, and there are a plenty number of contractions which involves these displacements, and these contractions play a crucial role in the theory of fixed point and best proximity point. If we compare Definition 2 with some usual well-known contractions, then one can notice that u_1,u_2 play the roles of Sx_1,Sx_2 in Definition 2. So if someone requires to extend the proximal contractions by using the displacements $d(Sx_1,x_1), d(Sx_2,x_2), d(Sx_1,x_2)$, $d(Sx_2,x_1)$, then one has to work with $d(u_1,x_1)$, (u_2,x_2), $d(u_1,x_2)$, $d(u_2,x_1)$ respectively. So it will be impressive works if the concepts of proximal contractions can be enlarged by involving the displacements $d(u_1,x_1)$, (u_2,x_2), $d(u_1,x_2)$, $d(u_2,x_1)$.

Motivated by this fact, in the current paper, we broaden the proximal contractions by associating all the five displacements $d(x_1,x_2)$, $d(u_1,x_1)$, $d(u_2,x_2)$, $d(u_1,x_2)$ and $d(u_2,x_1)$. To continue this, we introduce proximal $\mathcal{A}$-contractions which involve $d(x_1,x_2)$, $d(u_1,x_1)$ and $d(u_2,x_2)$; and proximal $\mathcal{A}'$-contractions which involve $d(x_1,x_2)$, $d(u_1,x_2)$ and $d(u_2,x_1)$. More specifically, we define *proximal $\mathcal{A}$-contractions of first and second kind; proximal $\mathcal{A}'$-contractions of first and second kind; strong proximal $\mathcal{A}$-contractions* and *strong proximal $\mathcal{A}'$-contractions*. After this, we study on adequate sufficient conditions to ensure the existence of best proximity point(s) of the above-mentioned contractions, and access the required adequate sufficient conditions which will be presented in next section. Along with this, we give a number of examples to support the validity of our proven results.

Throughout this paper, $\mathcal{A}$ and $\mathcal{A}'$ will contain all functions $f:\mathbb{R}_+^3\to\mathbb{R}$ having the properties $(\mathcal{A}_1)$-$(\mathcal{A}_2)$ and $(\mathcal{A}_1')$-$(\mathcal{A}_3')$ respectively, where

- $(\mathcal{A}_1)$ there exists $k\in[0,1)$ such that if $r\le f(s,s,r)$ or $r\le f(r,s,s)$, then $r\le ks$ for all $r,s\in\mathbb{R}_+$;
- $(\mathcal{A}_2)$ there exists $\alpha\in[0,1)$ such that $f(r,0,0)\le\alpha r$;

and

- $(\mathcal{A}_1')$ there exists $k\in[0,1)$ such that if $r\le f(s,0,r+s)$, then $r\le ks$ for all $r,s\in\mathbb{R}_+$;
- $(\mathcal{A}_2')$ if $t\le t_1$, then $f(r,s,t)\le f(r,s,t_1)$ for all $r,s,t,t_1\in\mathbb{R}_+$;
- $(\mathcal{A}_3')$ if $r\le f(r,r,r)$, then $r=0$.

For examples and properties of such collections of mappings, we refer the readers to [1, 7, 14].

2. Main Results

Throughout this section, (M,d) will denote a metric space and G, H will denote two non-empty subsets of M, and G_0, H_0 will denote the following:

$$G_0=\{x\in G: d(x,y)=\operatorname{dist}(G,H) \text{ for some } y\in H\}$$
$$H_0=\{y\in H: d(x,y)=\operatorname{dist}(G,H) \text{ for some } x\in G\}.$$

First, we define proximal $\mathcal{A}$, $\mathcal{A}'$ -contractions of the first kind in the following way:

Definition 2. A mapping $S:G\to H$ is said to be a

(i) proximal $\mathcal{A}$-contraction of the first kind if there exists an $f\in\mathcal{A}$ satisfying

$$\left.\begin{array}{l} d(u_1,Sx_1)=\operatorname{dist}(G,H)\\ d(u_2,Sx_2)=\operatorname{dist}(G,H)\end{array}\right\}\Longrightarrow d(u_1,u_2)\le f(d(x_1,x_2),d(u_1,x_1),d(u_2,x_2))$$

for all $u_1,u_2,x_1,x_2\in G$,

(ii) proximal $\mathcal{A}'$-contraction of the first kind if there exists an $f\in\mathcal{A}'$ satisfying

$$\left.\begin{array}{l} d(u_1,Sx_1)=\operatorname{dist}(G,H)\\ d(u_2,Sx_2)=\operatorname{dist}(G,H)\end{array}\right\}\Longrightarrow d(u_1,u_2)\le f(d(x_1,x_2),d(u_1,x_2),d(u_2,x_1))$$

for all $u_1,u_2,x_1,x_2\in G$.

Our first two results regarding the existence of best proximity point(s) of the above two proximal contractions are as follows:

Theorem 1. *Suppose that (M,d) is complete, G,H are closed and $G_0\neq\varnothing$. Let $S:G\to H$ be a continuous proximal $\mathcal{A}$-contraction of the first kind such that $S(G_0)$ resides in H_0. Then S has a unique best proximity point.*

Proof. Since G_0 is non-empty, we choose an element $u_0\in G$. Then $Su_0\in S(G_0)\subset H_0$. Then we find an element $u_1\in G_0$ such that $d(u_1,Su_0)=\operatorname{dist}(G,H)$. Similarly, $Su_1\in H_0$ and in the same way we find an element $u_2\in G_0$ such that $d(u_2,Su_1)=$

dist(G,H). Continuing this process, we arrive at a sequence $\{u_n\}$ of elements of G_0 such that

$$d(u_{n+1},Su_n) = \text{dist}(G,H) \text{ for all } n \in \mathbb{N}.$$

Now note that

$$d(u_n,Su_{n-1}) = \text{dist}(G,H)$$

and

$$d(u_{n+1},Su_n) = \text{dist}(G,H)$$

for all $n \in \mathbb{N}$. Since S is a proximal $\mathcal{A}$-contraction of the first kind, there exists an $f \in \mathcal{A}$ such that

$$d(u_n,u_{n+1}) \leq f(d(u_{n-1},u_n),d(u_n,u_{n-1}),d(u_{n+1},u_n)).$$

So there exists a $k \in [0,1)$ such that

$$d(u_n,u_{n+1}) \leq kd(u_{n-1},u_n)$$

for all $n \in \mathbb{N}$ which, in fact, implies that

$$d(u_n,u_{n+1}) \leq k^n d(u_1,u_0).$$

Now for any $m,n \in \mathbb{N}$, we have

$$\begin{aligned} d(u_{m+n},u_n) &\leq d(u_{m+n},u_{m+n-1}) + d(u_{m+n-1},u_{m+n-2}) + \cdots + d(u_{n+1},u_n) \\ &\leq (k^{m+n-1} + k^{m+n-2} + \cdots + k^n) d(u_1,u_0) \\ &= k^n \frac{1-k^m}{1-k} d(u_1,u_0) \longrightarrow 0 \text{ as } m,n \to \infty. \end{aligned}$$

Therefore, $\{u_n\}$ is a Cauchy sequence in G. Being a closed subset of a complete metric space (M,d), G supplies an element u such that $u_n \longrightarrow u$ as $n \to \infty$. Then, by continuity of S, we get $Su_n \to Su$ as $n \to \infty$ and consequently $d(u_{n+1},Su_n) \to d(u,Su)$. Now $d(u_{n+1},Su_n) = \text{dist}(G,H)$ for all $n \in \mathbb{N}$, confirms that $d(u,Su) = \text{dist}(G,H)$ which shows that u is a best proximity point of S.

Let $u^* \in G$ be such that $d(u^*,Su^*) = \text{dist}(G,H)$. Then we have

$$d(u,u^*) \leq f((d(u,u^*),d(u,u),d(u^*,u^*)) = f(d(u,u^*),0,0)$$

which implies that

$$d(u,u^*) \leq k \cdot 0 = 0.$$

Hence $u = u^*$ and the theorem is proved. □

Theorem 2. *Suppose that (M,d) is complete, G,H are closed and $G_0 \neq \varnothing$. Let $S : G \to H$ be a continuous proximal $\mathcal{A}'$-contraction of the first kind such that $S(G_0)$ resides in H_0. Then S has a unique best proximity point in G.*

Proof. We consider a sequence $\{u_n\}$ of elements of G_0, defined as in Theorem 1, such that

$$d(u_{n+1}, Su_n) = \text{dist}(G,H) \text{ for all } n \in \mathbb{N}.$$

Now note that

$$d(u_n, Su_{n-1}) = \text{dist}(G,H)$$

and

$$d(u_{n+1}, Su_n) = \text{dist}(G,H)$$

for all $n \in \mathbb{N}$.

Since S is a proximal $\mathcal{A}'$-contraction of the first kind, there exists an $f \in \mathcal{A}'$ such that

$$\begin{aligned} d(u_n, u_{n+1}) &\leq f(d(u_{n-1}, u_n), d(u_n, u_n), d(u_{n+1}, u_{n-1})) \\ &\leq f(d(u_{n-1}, u_n), 0, d(u_{n+1}, u_n) + d(u_n, u_{n-1})). \end{aligned}$$

So there exists a $k \in [0,1)$ such that

$$d(u_n, u_{n+1}) \leq k d(u_{n-1}, u_n)$$

for all $n \in \mathbb{N}$. Proceeding as in Theorem 1, we can show that $\{u_n\}$ is a Cauchy sequence in G. Since G is a closed subset of the complete metric space (M,d), $u_n \longrightarrow u$ as $n \to \infty$ for some $u \in G$. That u is a best proximity point of S follows by the similar arguments as in Theorem 1.

Let $u^* \in G$ be such that $d(u^*, Su^*) = \text{dist}(G,H)$. Since S is a proximal $\mathcal{A}'$-contraction of the first kind, we have

$$d(u, u^*) \leq f(d(u,u^*), d(u^*,u), d(u,u^*))$$

which implies that

$$d(u,u^*) = 0.$$

Hence $u = u^*$ and the proof is complete. □

Next, we give the following supporting examples:

Example 1. We take $M = \mathbb{R}$, d as the usual metric and choose $G = [2,\infty)$, $H = (-\infty,-1]$. Also we take $f \in \mathcal{A}$ defined by $f(r,s,t) = \frac{3}{4}\max\{r,s,t\}$ and define $S : G \to H$ by $Sx = \frac{2-3x}{4}$ for all $x \in G$.

Let $u_1, u_2, x_1, x_2 \in G$ be such that $d(u_1, Sx_1) = \text{dist}(G,H)$ and $d(u_2, Sx_2) = \text{dist}(G,H)$. Then

$$4u_1 + 3x_1 = 14 \text{ and } 4u_2 + 3x_2 = 14.$$

Now,

$$\begin{aligned} d(u_1,u_2) = |u_1 - u_2| &= \left| \frac{14-3x_1}{4} - \frac{14-3x_2}{4} \right| \\ &= \frac{3}{4}|x_1 - x_2| = \frac{3}{4} d(x_1,x_2), \end{aligned}$$

which yields that

$$d(u_1,u_2) \leq \frac{3}{4} f(d(x_1,x_2),d(u_1,x_1),d(u_2,x_2)).$$

Therefore, S is a proximal $\mathcal{A}$-contraction of first kind. So by Theorem 1, S has a unique best proximity point, viz., $u = 2$.

Example 2. We choose $M = \mathbb{R}$, d as the usual metric; $G = [6,7]$, $H = [2,3]$; $f(r,s,t) = \frac{49}{50}\max\{s,t\}$ and define $S : G \to H$ be defined by $Sx = 9 - x$ for all $x \in G$. Let $u_1,u_2,x_1,x_2 \in G$ be such that $d(u_1,Sx_1) = \text{dist}(G,H)$ and $d(u_2,Sx_2) = \text{dist}(G,H)$. Then

$$u_1 + x_1 = 12 \text{ and } u_2 + x_2 = 12.$$

Without loss of generality, let us suppose that $x_1 \geq x_2$. Then

$$d(u_1,u_2) = |u_1 - u_2| = |12 - x_1 - 12 + x_2| = x_1 - x_2.$$

Also,

$$d(u_1,x_1) = 12 - 2x_1 \text{ and } d(u_2,x_2) = 12 - 2x_2.$$

Now,

$$\begin{aligned} &f(d(x_1,x_2),d(u_1,x_1),d(u_2,x_2)) \\ &\quad = \frac{49}{50}\max\{x_1 - x_2, 2x_1 - 12, 2x_2 - 12\} \\ &\quad = \frac{49}{50}(2x_1 - 12)\Big[\because x_1 \geq x_2, \text{ so, } 2x_1 - 12 \geq 2x_2 - 12\Big]. \end{aligned}$$

Therefore,

$$d(u_1,u_2) \leq f(d(x_1,x_2),d(u_1,x_1),d(u_2,x_2))$$

which shows that S is a proximal $\mathcal{A}$-contraction of first kind. So by Theorem 1, S possesses a unique best proximity point, viz., $u = 6$.

Example 3. We choose (M,d) as the usual metric space $(\mathbb{R},d)$ and $G = [3,5]$, $H = [0,1]$. We take $f \in \mathcal{A}'$ as $f(r,s,t) = \frac{1}{3}(s+t)$ and consider the mapping $S : G \to H$ defined by

$$Sx = \begin{cases} 1 & \text{if } x \in [3,4]; \\ 5 - x & \text{if } x \in [4,5]. \end{cases}$$

Let $u_1,u_2,x_1,x_2 \in G$ be such that $d(u_1,Sx_1) = \text{dist}(G,H) = d(u_2,Sx_2)$. We now consider the following cases:

Case 1: Let $x_1,x_2 \in [3,4]$. Then

$$|u_1 - 1| = 2 \Longrightarrow u_1 = 3.$$

Similarly, $u_2 = 3$. So, it is obvious that

$$d(u_1,u_2) \leq f(d(x_1,x_2),d(u_1,x_2),d(u_2,x_1)).$$

Case 2: Let $x_1, x_2 \in [4,5]$. Then

$$\begin{aligned} |u_1 - (5 - x_1)| = 2 &\Longrightarrow |u_1 + x_1 - 5| = 2 \\ &\Longrightarrow u_1 + x_1 = 7. \end{aligned}$$

Similarly, $u_2 + x_2 = 7$. Therefore, $d(u_1,u_2) = |u_1 - u_2| = |x_1 - x_2|$. Without loss of generality, we assume that $x_1 \geq x_2$. Again,

$$d(u_1,x_2) = |u_1 - x_2| = |7 - x_1 - x_2| = x_1 + x_2 - 7.$$

Similarly, $d(u_2,x_1) = x_1 + x_2 - 7$. Therefore,

$$\begin{aligned} 3d(u_1,u_2) - \{d(u_1,x_2) + d(u_2,x_1)\} & \\ = 3(x_1 - x_2) - \{x_1 + x_2 - 7 + x_1 + x_2 - 7\} & \\ = 3x_1 - 3x_2 - 2x_1 - 2x_2 + 14 = x_1 - 5x_2 + 14 & \\ \leq 5 - 20 + 14 = -1 < 0 & \end{aligned}$$

which gives

$$d(u_1,u_2) \leq \frac{1}{3}\{d(u_1,x_2) + d(u_2,x_1)\}$$

that is

$$d(u_1,u_2) \leq f(d(x_1,x_2), d(u_1,x_2), d(u_2,x_1)).$$

Case 3: Let $x_1 \in [3,4]$ and $x_2 \in [4,5]$. Then as in the above cases, we have $u_1 = 3$ and $u_2 + x_2 = 7$. Therefore,

$$d(u_1,u_2) = |u_1 - u_2| = |3 - u_2| = u_2 - 3 = 4 - x_2.$$

Now,

$$\begin{aligned} d(u_1,x_2) + d(u_2,x_1) &= |3 - x_2| + |u_2 - x_1| \\ &= x_2 - 3 + x_1 + x_2 - 7 = x_1 + 2x_2 - 10. \end{aligned}$$

Therefore, as in case-2, it can be shown that

$$d(u_1,u_2) \leq f(d(x_1,x_2), d(u_1,x_2), d(u_2,x_1)).$$

Hence combining all the cases, we see that S is a proximal $\mathcal{A}'$-contraction of first kind. Hence Theorem 2 ensures that S admits a unique best proximity point. Note that the best proximity point is 3.

Next, we give the definitions of proximal $\mathcal{A}$, $\mathcal{A}'$- contractions of the second kind.

Definition 3. A mapping $S : G \to H$ is said to be a

(i) proximal $\mathcal{A}$-contraction of the second kind if there exists an $f \in \mathcal{A}$ satisfying

$$\left.\begin{aligned} d(u_1,Sx_1) = \operatorname{dist}(G,H) \\ d(u_2,Sx_2) = \operatorname{dist}(G,H) \end{aligned}\right\}$$

$$\Longrightarrow d(Su_1,Su_2) \leq f(d(Sx_1,Sx_2), d(Su_1,Sx_1), d(Su_2,Sx_2))$$

for all $u_1, u_2, x_1, x_2 \in G$,

(ii) proximal $\mathcal{A}'$-contraction of the second kind if there exists an $f \in \mathcal{A}'$ satisfying

$$\left.\begin{array}{l} d(u_1, Sx_1) = \text{dist}(G,H) \\ d(u_2, Sx_2) = \text{dist}(G,H) \end{array}\right\}$$
$$\Longrightarrow d(Su_1, Su_2) \leq f(d(Sx_1, Sx_2), d(Su_1, Sx_2), d(Su_2, Sx_1))$$

for all $u_1, u_2, x_1, x_2 \in G$.

Our upcoming two results deal with the existence of best proximity point(s) of the aforementioned contractions. Before presenting these results, we first recall the following definition:

Definition 4. ([17, p. 1774, Definition 2.1]). G is said to be approximatively compact with respect to H if every sequence $\{x_n\}$ in G with $d(y, x_n) \to d(y, G)$ for some y in H, has a convergent subsequence in G.

Theorem 3. *Suppose that (M,d) is complete, G, H are closed, G is approximately compact with respect to H and $G_0 \neq \varnothing$. Let $S : G \to H$ be a continuous proximal $\mathcal{A}$-contraction of the second kind such that $S(G_0)$ resides in H_0. Then S has a best proximity point in G. Moreover, if S is injective, then the best proximity point is unique.*

Proof. Since G_0 is non-empty, we choose an element $v_0 \in G$. Then $Sv_0 \in S(G_0) \subset H_0$. Then there is an element $v_1 \in G_0$ such that $d(v_1, Sv_0) = \text{dist}(G,H)$. Similarly, $Sv_1 \in H_0$ and in the same way we find an element $v_2 \in G_0$ such that $d(v_2, Sv_1) = \text{dist}(G,H)$. Therefore, continuing this process we arrive at a sequence $\{v_n\}$ of elements of G_0 such that

$$d(v_{n+1}, Sv_n) = \text{dist}(G,H) \text{ for all } n \in \mathbb{N}.$$

Now note that

$$d(v_n, Sv_{n-1}) = \text{dist}(G,H)$$

and

$$d(v_{n+1}, Sv_n) = \text{dist}(G,H)$$

for all $n \in \mathbb{N}$. Since S is a proximal $\mathcal{A}$-contraction of the second kind, there exists an $f \in \mathcal{A}$ such that

$$d(Sv_n, Sv_{n+1}) \leq f(d(Sv_{n-1}, Sv_n), d(Sv_n, Sv_{n-1}), d(Sv_{n+1}, Sv_n)).$$

So there exists a $k \in [0,1)$ such that

$$d(Sv_n, Sv_{n+1}) \leq k d(Sv_{n-1}, Sv_n)$$

for all $n \in \mathbb{N}$ which, in fact, implies that

$$d(Sv_n, Sv_{n+1}) \leq k^n d(Sv_1, Sv_0).$$

Now for any $m, n \in \mathbb{N}$, we have

$$d(Sv_{m+n}, Sv_n) \leq d(Sv_{m+n}, Sv_{m+n-1}) + d(Sv_{m+n-1}, Sv_{m+n-2}) + \cdots + d(Sv_{n+1}, Sv_n)$$

$$\leq (k^{m+n-1}+k^{m+n-2}+\cdots+k^n)d(Sv_1,Sv_0)$$
$$= k^n\frac{1-k^m}{1-k}d(Sv_1,Sv_0) \longrightarrow 0 \text{ as } m,n\to\infty.$$

This shows that $\{Sv_n\}$ is a Cauchy sequence in H. Now closedness of H in the complete metric space (M,d) ensures the existence of an element $v\in H$ such that $Sv_n \longrightarrow v$ as $n\to\infty$. Now,

$$\begin{aligned}\operatorname{dist}(v,G) &\leq d(v,v_n)\leq d(v,Sv_{n-1})+d(Sv_{n-1},v_n)\\ &= d(v,Sv_{n-1})+\operatorname{dist}(G,H)\leq d(v,Sv_{n-1})+\operatorname{dist}(v,G)\end{aligned}$$

which implies that $d(v,v_n)\to\operatorname{dist}(v,G)$ as $n\to\infty$. Since G is proximally compact with respect to H, $\{v_n\}$ has a convergent subsequence $\{v_{n_k}\}$ in G. Let $v_{n_k}\to u$ for some $u\in G$. Then

$$d(u,v)=\lim_{k\to\infty}d(v_{n_k},Sv_{n_k-1})=\operatorname{dist}(G,H).$$

Therefore, $u\in G_0$. Since S is continuous, $Sv_{n_k}\to Su$ as $k\to\infty$. Again we have, $Sv_{n_k}\to v$ as $k\to\infty$. Hence $v=Su$. Thus, $d(u,Su)=\operatorname{dist}(G,H)$.

Finally, let S be injective. Let u^* be another element in G such that $d(u^*,Su^*)=\operatorname{dist}(G,H)$. Then,

$$d(Su,Su^*)\leq f(d(Su,Su^*),d(Su,Su),d(Su^*,Su^*))=f(d(Su,Su^*),0,0)$$

which implies that

$$d(Su,Su^*)\leq k\cdot 0=0.$$

Hence $Su=Su^*$. Since S is injective, we have $u=u^*$ and the proof is complete. □

Theorem 4. *Suppose that (M,d) is complete, G,H are closed, G is approximately compact with respect to H and $G_0\neq\varnothing$. Let $S:G\to H$ be a continuous proximal $\mathcal{A}'$-contraction of the second kind such that $S(G_0)$ resides in H_0. Then S has a best proximity point in G. Moreover, if S is injective, then the best proximity point is unique.*

Proof. Proceeding as in Theorem 3, we can construct a sequence $\{v_n\}$ of elements of G_0 such that

$$d(v_{n+1},Sv_n)=\operatorname{dist}(G,H) \text{ for all } n\in\mathbb{N}.$$

Now note that

$$d(v_n,Sv_{n-1})=\operatorname{dist}(G,H)$$

and

$$d(v_{n+1},Sv_n)=\operatorname{dist}(G,H)$$

for all $n\in\mathbb{N}$. Since S is a proximal $\mathcal{A}'$-contraction of the second kind, there exists an $f\in\mathcal{A}'$ such that

$$\begin{aligned}d(Sv_n,Sv_{n+1}) &\leq f(d(Sv_{n-1},Sv_n),d(Sv_n,Sv_n),d(Sv_{n+1},Sv_{n-1}))\\ &\leq f(d(Sv_{n-1},Sv_n),0,d(Sv_{n+1},Sv_n)+d(Sv_n,Sv_{n-1})).\end{aligned}$$

So there exists a $k \in [0,1)$ such that

$$d(Sv_n, Sv_{n+1}) \leq kd(Sv_{n-1}, Sv_n)$$

for all $n \in \mathbb{N}$. Following similar arguments, as in Theorem 3, we can prove that $\{Sv_n\}$ is a Cauchy sequence in H. Since H is a closed subset of the complete metric space (M,d), we get an element $v \in H$ such that $Sv_n \longrightarrow v$ as $n \to \infty$. Now,

$$\begin{aligned} \text{dist}(v,G) &\leq d(v,v_n) \leq d(v,Sv_{n-1}) + d(Sv_{n-1},v_n) \\ &= d(v,Sv_{n-1}) + dist(G,H) \leq d(v,Sv_{n-1}) + \text{dist}(v,G) \end{aligned}$$

which implies that $d(v,v_n) \to \text{dist}(v,G)$ as $n \to \infty$. Since G is proximally compact with respect to H, $\{v_n\}$ has a convergent subsequence $\{v_{n_k}\}$ in G. Let $v_{n_k} \to u$ for some $u \in G$. Now

$$d(u,v) = \lim_{k\to\infty} d(v_{n_k}, Sv_{n_k-1}) = \text{dist}(G,H).$$

This implies that $u \in G_0$. Applying continuity of S, we get $Sv_{n_k} \to Su$ as $k \to \infty$. Again we have, $Sv_{n_k} \to v$ as $k \to \infty$. Hence $v = Su$. Thus, $d(u,Su) = \text{dist}(G,H)$.

We now take S to be injective. Let u^* be another element in G such that $d(u^*,Su^*) = \text{dist}(G,H)$. Then,

$$d(Su,Su^*) \leq f(d(Su,Su^*), d(Su,Su^*), d(Su^*,Su))$$

which implies that

$$d(Su,Su^*) = 0.$$

Hence $Su = Su^*$. Since S is injective, we have $u = u^*$ and the proof is complete. □

Remark 1. In the above two theorems, to ensure the uniqueness of best proximity point, injectiveness of S is not necessary, which follows from the following examples.

Example 4. We take $(M,d) = (\mathbb{R}^2, d)$ where

$$d\Big((x_1,y_1),(x_2,y_2)\Big) = |x_1 - x_2| + |y_1 - y_2|$$

for all $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$;

$$\begin{aligned} G &= \{(x,y) \in \mathbb{R}^2 : 4 \leq x \leq 5, 0 \leq y \leq 1\}, \\ H &= \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}; \\ f(r,s,t) &= \frac{1}{2}r + \frac{1}{5}(s+t), \end{aligned}$$

and define $S : G \to H$ by

$$S(x,y) = \left(1, \frac{y}{2}\right)$$

for all $(x,y) \in G$. Let $u_1 = (u_1', u_1''), u_2 = (u_2', u_2''), x_1 = (x_1', x_1''), x_2 = (x_2', x_2'') \in G$ be such that

$$d(u_1,Sx_1) = \text{dist}(G,H) = 3 \text{ and } d(u_2,Sx_2) = \text{dist}(G,H) = 3.$$

Then, $Sx_1 = \left(1, \frac{x_1''}{2}\right)$ and $Sx_2 = \left(1, \frac{x_2''}{2}\right)$. Now,

$$\begin{aligned} & d(u_1, Sx_1) = 3 \\ & \Longrightarrow d\left((u_1', u_1''), \left(1, \frac{x_1''}{2}\right)\right) = 3 \\ & \Longrightarrow |u_1' - 1| + \left|u_1'' - \frac{x_1''}{2}\right| = 3 \\ & \Longrightarrow u_1' - 1 + \left|u_1'' - \frac{x_1''}{2}\right| = 3 \\ & \Longrightarrow u_1' + \left|u_1'' - \frac{x_1''}{2}\right| = 4 \end{aligned}$$

which implies that $u_1' = 4$ and $u_1'' = \frac{x_1''}{2}$. Similarly, $d(u_2, Sx_2) = 3$ gives $u_2' = 4$ and $u_2'' = \frac{x_2''}{2}$. Therefore,

$$\begin{aligned} d(Su_1, Su_2) &= d\left(\left(1, \frac{u_1''}{2}\right), \left(1, \frac{u_2''}{2}\right)\right) \\ &= \left|\frac{u_1''}{2} - \frac{u_2''}{2}\right| = \left|\frac{x_1''}{4} - \frac{x_2''}{4}\right| = \frac{1}{4}|x_1'' - x_2''|. \end{aligned}$$

Also,

$$d(Sx_1, Sx_2) = d\left(\left(1, \frac{x_1''}{2}\right), \left(1, \frac{x_2'')}{2}\right)\right) = \frac{1}{2}|x_1'' - x_2''|.$$

Now,

$$\begin{aligned} & d(Su_1, Su_2) - f(d(Sx_1, Sx_2), d(Su_1, Sx_1), d(Su_2, Sx_2)) \\ & = d(Su_1, Su_2) - \left\{\frac{1}{2}d(Sx_1, Sx_2) + \frac{1}{5}\Big(d(Su_1, Sx_1) + d(Su_2, Sx_2)\Big)\right\} \\ & = \frac{1}{4}|x_1'' - x_2''| - \frac{1}{4}|x_1'' - x_2''| - \frac{1}{5}\Big(d(Su_1, Sx_1) + d(Su_2, Sx_2)\Big) \leq 0 \end{aligned}$$

which yields that

$$d(Su_1, Su_2) \leq f(d(Sx_1, Sx_2), d(Su_1, Sx_1), d(Su_2, Sx_2))$$

which, in turn, implies that S is a proximal $\mathcal{A}$-contraction of the second kind. It is easy to check that $(4,0)$ is the unique best proximity point of S and S is not injective.

Example 5. In this example, we take the metric space (M,d) as above and choose

$$G = \Big\{(x,y) \in \mathbb{R}^2 : x = 2, 0 \leq y \leq 3\Big\} \bigcup \Big\{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 2, y = 2\Big\}$$

and

$$H = \Big\{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\Big\}.$$

Then $\text{dist}(G,H)=1$. We define $S:G\to H$ by

$$S(x,y)=\left(\frac{x}{2},0\right)$$

for all $(x,y)\in G$. Also we choose $f\in\mathcal{A}'$ which is defined by $f(r,s,t)=\frac{1}{4}(s+t)$. Let $u_1,u_2,x_1,x_2\in G$ be such that $d(u_1,Sx_1)=\text{dist}(G,H)=d(u_2,Sx_2)$. Then we have

$$d\left((u_1',u_1''),\left(\frac{x_1'}{2},0\right)\right)=1$$
$$\Longrightarrow\left|u_1'-\frac{x_1'}{2}\right|+|u_1''|=1$$

which implies that $u_1''\leq 1$ and $u_1'=2$. Similarly, we get $u_2''\leq 1$ and $u_2'=2$. Now,

$$d(Su_1,Su_2)=d\left(\left(\frac{u_1'}{2},0\right),\left(\frac{u_2'}{2},0\right)\right)=\left|\frac{u_1'}{2}-\frac{u_2'}{2}\right|=0.$$

Therefore,

$$d(Su_1,Su_2)\leq f(d(Sx_1,Sx_2),d(Sx_1,u_2),d(Sx_2,u_1))$$

whence S is a proximal $\mathcal{A}'$-contraction of second kind. One can easily verify that $(4,0)$ is the unique best proximity point of S and S is not injective.

Remark 2. In Theorem 3 and Theorem 4, the injectiveness of S can't be dropped, which follows from the following example.

Example 6. Let us take $M=\mathbb{R}$, d as the usual metric and $G=\left[-1,-\frac{1}{2}\right]\cup\left[\frac{1}{2},-1\right]$, $H=\{0\}$. We define $S:G\to H$ by $Sx=0$ for all $x\in G$. Then one can check that S is proximal $\mathcal{A}$, $\mathcal{A}'$-contractions of the second kind and S has two best proximity points viz., $-\frac{1}{2},\frac{1}{2}$. It may be noted that S is not an injection.

Next, we come up with the notions of strong proximal contractions, and present two results exhibiting the sufficient conditions in order to get best proximity points of strong proximal contractions.

Definition 5. A mapping $S:G\to H$ is said to be

(i) a strong proximal $\mathcal{A}$-contraction if there exists an $f\in\mathcal{A}$ such that for all $u_1,u_2,x_1,x_2\in G$ and for all $\gamma\in[1,2)$

$$\left.\begin{array}{l} d(u_1,Sx_1)\leq\gamma\,\text{dist}(G,H)\\ d(u_2,Sx_2)\leq\gamma\,\text{dist}(G,H)\end{array}\right\}$$
$$\Longrightarrow d(u_1,u_2)\leq f(d(x_1,x_2),d(u_1,x_1),d(u_2,x_2))+(\gamma-1)dist(G,H),$$

(ii) a strong proximal $\mathcal{A}'$-contraction if there exists an $f\in\mathcal{A}'$ such that for all $u_1,u_2,x_1,x_2\in G$ and for all $\gamma\in[1,2)$

$$\left.\begin{array}{l} d(u_1,Sx_1)\leq\gamma\,\text{dist}(G,H)\\ d(u_2,Sx_2)\leq\gamma\,\text{dist}(G,H)\end{array}\right\}$$
$$\Longrightarrow d(u_1,u_2)\leq f(d(x_1,x_2),d(u_1,x_2),d(u_2,x_1))+(\gamma-1)dist(G,H).$$

Theorem 5. *Suppose that (M,d) is complete, G,H are closed and $dist(G,H)>0$. Let $S:G\to H$ be a continuous strong proximal $\mathcal{A}$-contraction such that there exists a sequence $\{x_n\}$ in G with $d(x_n,Sx_n)\to dist(G,H)$ as $n\to\infty$. Then S has a unique best proximity point and $\{x_n\}$ has a subsequence converging to that best proximity point.*

Proof. For each $p\in\mathbb{N}$, we define

$$F_p=\left\{x\in G: d(x,Sx)\leq\left(1+\frac{1}{p}\right)\text{dist}(G,H)\right\}.$$

Since $d(x_n,Sx_n)\to\text{dist}(G,H)$, there exists an $n_p\in\mathbb{N}$ such that

$$d(x_{n_p},Sx_{n_p})\leq\left(1+\frac{1}{p}\right)\text{dist}(G,H)$$

which implies that F_p is non-empty for each $p\in\mathbb{N}$. Since S is continuous, each F_p is closed. It is also evident that $F_{p+1}\subset F_p$ for each $p\in\mathbb{N}$. If x and x^* are two elements of F_p, then we have

$$d(x,Sx)\leq\left(1+\frac{1}{p}\right)\text{dist}(G,H)$$

and

$$d(x^*,Sx^*)\leq\left(1+\frac{1}{p}\right)\text{dist}(G,H).$$

Since S is a strong proximal $\mathcal{A}$-contraction, there exists $f\in\mathcal{A}$ such that

$$\begin{aligned}d(x,x^*)&\leq f(d(x,x^*),d(x,x),d(x^*,x^*))+\frac{1}{p}\text{dist}(G,H)\\&=f(d(x,x^*),0,0)+\frac{1}{p}\text{dist}(G,H)\\&\leq\alpha d(x,x^*)+\frac{1}{p}\text{dist}(G,H)\text{ for some }\alpha\in[0,1).\end{aligned}$$

Therefore, we get

$$d(x,x^*)\leq\frac{1}{(1-\alpha)p}\text{dist}(G,H).$$

Hence $diam(A_p)\to 0$ as $p\to\infty$. Therefore by Cantor's intersection theorem, we have

$$\bigcap_p F_p=\{u\}$$

for some $u\in G$. From this we see that,

$$dist(G,H)\leq d(u,Su)\leq\left(1+\frac{1}{p}\right)\text{dist}(G,H)$$

for each p. Hence we have

$$d(u,Su)=\text{dist}(G,H).$$

For the last part, it is to be noted that

$$d(x_{n_p},u) \leq \frac{1}{(1-\alpha)p}\text{dist}(G,H).$$

Hence the subsequence $\{x_{n_p}\}$ converges to u and the theorem follows. □

Remark 3. The conclusions of the above theorem also hold if S is a strong proximal $\mathcal{A}'$-contraction instead of strong proximal $\mathcal{A}$-contraction. The proof being similar to the above theorem, we omit it.

We conclude this paper by presenting an example in support of Theorem 5 followed by a couple of remarks.

Example 7. Let us take $(M,d) = (\mathbb{R},d)$, d being the usual metric; $G = [0,1]$ and $H = [5,6]$ and take $f \in \mathcal{A}$, where $f(r,s,t) = \frac{1}{4}(r+s+t)$. We define $S : G \to H$ by $Sx = 6-x$ for all $x \in G$.

Let $u_1,u_2,x_1,x_2 \in G$ be such that $d(u_1,Sx_1) \leq \gamma\,\text{dist}(G,H)$ and $d(u_2,Sx_2) \leq \gamma\,\text{dist}(G,H)$ for all $\gamma \in [1,2]$. Then

$$\begin{aligned} |u_1 - Sx_1| \leq 4\gamma \quad &\Longrightarrow |u_1 - 6 + x_1| \leq 4\gamma \\ &\Longrightarrow 6 - u_1 - x_1 \leq 4\gamma \\ &\Longrightarrow u_1 \geq 6 - 4\gamma - x_1. \end{aligned}$$

Similarly,

$$u_2 \geq 6 - 4\gamma - x_2.$$

Without loss of generality, we assume that $u_1 \geq u_2$. Therefore,

$$\begin{aligned} d(u_1,u_2) = |u_1 - u_2| = u_1 - u_2 &\leq u_1 - (6 - 4\gamma - x_2) \\ &= u_1 - 6 + 4\gamma + x_2 \leq 1 + 1 - 6 + 4\gamma = 4(\gamma - 1) = (\gamma - 1)d(A,B). \end{aligned}$$

So, we get

$$d(u_1,u_2) \leq f(d(x_1,x_2),d(u_1,x_1),d(u_2,x_2)) + (\gamma - 1)\text{dist}(G,H)$$

for any $f \in \mathcal{A}$ which implies that S is a strong proximal $\mathcal{A}$-contraction. Consequently by Theorem 5, S has a unique best proximity point. Also, the unique best proximity point is 1.

Remark 4. The best proximity point results of different kinds of proximal contractions due to Sadiq Basha [17] can be obtained from our results by choosing $f(r,s,t) = \alpha r$, where $\alpha \in [0,1)$.

Remark 5. By selecting different f in Theorem 1, Theorem 3 and Theorem 5, we can obtain the best proximity point results of the proximal versions of the contractions of Kannan [8] $\left(f(r,s,t) = \alpha(s+t),\ \text{where}\ 0 \leq \alpha < \frac{1}{2}\right)$, Reich [16] $\left(f(r,s,t) = \alpha_1 r + \alpha_2 s + \alpha_3 t,\ \text{where}\ 0 \leq \alpha_1,\alpha_2,\alpha_3 < 1; \alpha_1+\alpha_2+\alpha_3 < 1\right)$, Bianchini [5] $\left(f(r,s,t) = \alpha\max\{s,t\},\ \text{where}\ 0 \leq \alpha < 1\right)$ and Khan [10] $\left(f(r,s,t) = \alpha\sqrt{st},\ \text{where}\ 0 \leq \alpha < 1\right)$.

Remark 6. In this remark, we point out some suitable areas where our obtained results can be utilized. In many cases of sciences and engineering, instead of solutions of certain kinds of integro-differential equations, one need optimal solutions of the same. On the other hand, if an integral or differential equation does not possess any solution, then we often show interest on the best proximity solutions of the equation (see [21], [6]). In the aforementioned two cases, the desired optimal and best proximity solutions can be dealt by best proximity point results only. Thus in such cases, our obtained best proximity point results can be applied.

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ESSENTIAL G-RADICAL SUPPLEMENTED MODULES

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Abstract. Let M be an R-module. If every essential submodule of M has a g-radical supplement in M, then M is called an essential g-radical supplemented (or briefly eg-radical supplemented) module. In this work, some properties of these modules are investigated. It is proved that every factor module and every homomorphic image of an eg-radical supplemented module are eg-radical supplemented. Let M be an eg-radical supplemented module. Then every finitely M-generated R-module is eg-radical supplemented.

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1. INTRODUCTION

Throughout this paper all rings are associative with identity and all modules are unital left modules.

Let R be a ring and M be an R-module. We denote a submodule N of M by $N \leq M$. A module M is said to be *simple* if M have no submodules with distinct from 0 and M. The sum of all simple submodules of a module M is called the *socle* of M and denoted by $\mathrm{Soc}(M)$. M is called a *semisimple* module, if M is a direct sum of simple modules (it is equivalent to $\mathrm{Soc}(M) = M$). Let M be an R-module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* (or *superfluous*) submodule of M and denoted by $N \ll M$. A submodule N of an R-module M is called an *essential* submodule, denoted by $N \trianglelefteq M$, in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equvalently, $N \cap L = 0$ for $L \leq M$ implies that $L = 0$. Let M be an R-module and K be a submodule of M. K is called a *generalized small* (briefly *g-small*) submodule of M if for every essential submodule T of M with the property $M = K + T$ implies that $T = M$, we denote this by $K \ll_g M$ (in [13], it is called an *e-small submodule* of M and denoted by $K \ll_e M$). Let M be an R-module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. M is said to be *essential supplemented* (briefly *e-supplemented*)

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if every essential submodule of M has a supplement in M. Let M be an R-module and $U,V \leq M$. If $M = U+V$ and $M = U+T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U+V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. M is said to be *essential g-supplemented* if every essential submodule of M has a g-supplement in M. The intersection of all maximal submodules of an R-module M is called the *radical* of M and denoted by $\mathrm{Rad}(M)$. If M have no maximal submodules, then we denote $\mathrm{Rad}(M) = M$. Let M be an R-module and $U,V \leq M$. If $M = U+V$ and $U \cap V \leq \mathrm{Rad}(V)$, then V is called a *generalized* (*Radical*) *supplement* (briefly *Rad-supplement*) of U in M. M is said to be *generalized* (*Radical*) *supplemented* (briefly *Rad-supplemented*) if every submodule of M has a Rad-supplement in M. The intersection of all essential maximal submodules of an R-module M is called the *generalized radical* (briefly *g-radical*) of M and denoted by $\mathrm{Rad}_g M$ (in [13] it is denoted by $\mathrm{Rad}_e M$). If M have no essential maximal submodules, then we denote $\mathrm{Rad}_g M = M$. Let M be an R-module and $U,V \leq M$. If $M = U+V$ and $U \cap V \leq \mathrm{Rad}_g V$, then V is called a *g-radical supplement* of U in M. M is said to be *g-radical supplemented* if every submodule of M has a g-radical supplement in M. Let M be an R-module and $K \leq V \leq M$. We say V *lies above* K in M if $V/K \ll M/K$.

More details about supplemented modules are in [2,11]. More details about essential supplemented modules are in [7,8]. More informations about g-small submodules and g-supplemented modules are in [3,4]. The definition of essential g-supplemented modules and some properties of them are in [5]. More details about generalized (Radical) supplemented modules are in [10,12]. The definition of g-radical supplemented modules and some properties of them are in [4].

Lemma 1 ([4, Lemma 3])**.** *The following assertions are hold.*

(1) *If* $N \leq M$, *then* $\mathrm{Rad}_g N \leq \mathrm{Rad}_g M$.
(2) *If* $K,L \leq M$, *then* $\mathrm{Rad}_g K + \mathrm{Rad}_g L \leq \mathrm{Rad}_g(K+L)$.
(3) *If* $f: M \longrightarrow N$ *is an* R*-module homomorphism, then* $f(\mathrm{Rad}_g M) \leq \mathrm{Rad}_g N$.

2. Essential g-radical supplemented modules

Definition 1 ([6, Definition 1])**.** Let M be an R-module. If every essential submodule of M has a g-radical supplement in M, then M is called an essential g-radical supplemented (or briefly eg-radical supplemented) module.

Clearly we can see that every essential g-supplemented module is eg-radical supplemented. But the converse is not true in general (see Example 1 and Example 2). Every g-radical supplemented module is eg-radical supplemented.

Proposition 1. *Let M be an eg-radical supplemented R-module. If every nonzero submodule of M is essential in M, then M is g-radical supplemented.*

Proof. Clear from definitions. □

Lemma 2. *Let M be an eg-radical supplemented module. Then $M/\mathrm{Rad}_g M$ have no proper essential submodules.*

Proof. Let $U/\mathrm{Rad}_g M \trianglelefteq M/\mathrm{Rad}_g M$. Then $U \trianglelefteq M$ and since M is eg-radical supplemented, U has a g-radical supplement V in M. Here $M = U+V$ and $U \cap V \leq \mathrm{Rad}_g V \leq \mathrm{Rad}_g M$. Then $M/\mathrm{Rad}_g M = (U+V)/\mathrm{Rad}_g M = U/\mathrm{Rad}_g M + (V+\mathrm{Rad}_g M)/\mathrm{Rad}_g M$ and $U/\mathrm{Rad}_g M \cap (V+\mathrm{Rad}_g M)/\mathrm{Rad}_g M = (U \cap V+\mathrm{Rad}_g M)/\mathrm{Rad}_g M = \mathrm{Rad}_g M/\mathrm{Rad}_g M = 0$. Hence $M/\mathrm{Rad}_g M = U/\mathrm{Rad}_g M \oplus (V+\mathrm{Rad}_g M)/\mathrm{Rad}_g M$ and since $U/\mathrm{Rad}_g M \trianglelefteq M/\mathrm{Rad}_g M$, $U/\mathrm{Rad}_g M = M/\mathrm{Rad}_g M$. Thus $M/\mathrm{Rad}_g M$ have no proper essential submodules. □

Corollary 1. *Let M be an eg-radical supplemented module. Then $M/\mathrm{Rad}_g M$ is semisimple.*

Proof. Since M is eg-radical supplemented, by Lemma 2, $M/\mathrm{Rad}_g M$ have no proper essential submodules. Then by [11, Section 21.1], $\mathrm{Soc}(M/\mathrm{Rad}_g M) = M/\mathrm{Rad}_g M$ and $M/\mathrm{Rad}_g M$ is semisimple. □

Corollary 2. *Let M be an essential g-supplemented module. Then $M/\mathrm{Rad}_g M$ is semisimple.*

Proof. Clear from Corollary 1. □

Lemma 3. *Let M be an R-module, $U \trianglelefteq M$ and $N \leq M$. If $U+N$ has a g-radical supplement in M and N is eg-radical supplemented, then U has a g-radical supplement in M.*

Proof. Let X be a g-radical supplement of $U+N$ in M. Since $U \trianglelefteq M$, $U+X \trianglelefteq M$ and $(U+X)\cap N \trianglelefteq N$. Since N is eg-radical supplemented, $(U+X)\cap N$ has a g-radical supplement Y in N. Since X is a g-radical supplement of $U+N$ in M, $M = U+N+X$ and $(U+N)\cap X \leq \mathrm{Rad}_g X$. Since Y is a g-radical supplement of $(U+X)\cap N$ in N, $N = (U+X)\cap N + Y$ and $(U+X)\cap Y = (U+X)\cap N \cap Y \leq \mathrm{Rad}_g Y$. Then $M = U+N+X = U+(U+X)\cap N+Y+X = U+X+Y$ and, by Lemma 1, $U\cap(X+Y) \leq (U+X)\cap Y + (U+Y)\cap X \leq \mathrm{Rad}_g Y + (U+N)\cap X \leq \mathrm{Rad}_g Y + \mathrm{Rad}_g X \leq \mathrm{Rad}_g(X+Y)$. Hence $X+Y$ is a g-radical supplement of U in M. □

Lemma 4. *Let $M = M_1 + M_2$. If M_1 and M_2 are eg-radical supplemented, then M is also eg-radical supplemented.*

Proof. Let $U \trianglelefteq M$. Then $U+M_1 \trianglelefteq M$ and since $U+M_1+M_2$ has a trivial g-radical supplement 0 in M and M_2 is eg-radical supplemented, by Lemma 3, $U+M_1$ has a g-radical supplement in M. Since M_1 is eg-radical supplemented and $U \trianglelefteq M$, by Lemma 3 again, U has a g-radical supplement in M. Hence M is eg-radical supplemented. □

Corollary 3. *Let $M = M_1 + M_2 + \cdots + M_n$. If M_i is eg-radical supplemented for every $i = 1, 2, \ldots, n$, then M is also eg-radical supplemented.*

Proof. Clear from Lemma 4. □

Lemma 5. *Let $f: M \to N$ be an R-module epimorphism, $U, V \leq M$ and $\mathrm{Ker}(f) \leq U$. If V is a g-radical supplement of U in M, then $f(V)$ is a g-radical supplement of $f(U)$ in N.*

Proof. Since V is a g-radical supplement of U in M, $M = U + V$ and $U \cap V \leq \mathrm{Rad}_g V$. Then $N = f(M) = f(U+V) = f(U) + f(V)$. Let $x \in f(U) \cap f(V)$. Then there exist $u \in U$ and $v \in V$ with $x = f(u) = f(v)$. Here $f(v-u) = f(v) - f(u) = 0$ and $v - u \in \mathrm{Ker}(f) \leq U$. Then $v = v - u + u \in U$ and since $v \in V$, $v \in U \cap V$. Hence $x = f(v) \in f(U \cap V)$ and $f(U) \cap f(V) \leq f(U \cap V)$. Here clearly we can see that $f(U \cap V) \leq f(U) \cap f(V)$ and $f(U) \cap f(V) = f(U \cap V)$. Since $U \cap V \leq \mathrm{Rad}_g V$, by Lemma 1, $f(U) \cap f(V) = f(U \cap V) \leq f(\mathrm{Rad}_g V) \leq \mathrm{Rad}_g f(V)$. Hence $f(V)$ is a g-radical supplement of $f(U)$ in N, as desired. □

Lemma 6. *Every homomorphic image of an eg-radical supplemented module is eg-radical supplemented.*

Proof. Let M be an eg-radical supplemented R-module and $f: M \to N$ be an R-module epimorphism. Let $U \trianglelefteq N$. By [11, Section 17.3 (3)], $f^{-1}(U) \trianglelefteq M$ and since M is eg-radical supplemented, $f^{-1}(U)$ has a g-radical supplement V in M. Since $\ker(f) \leq f^{-1}(U)$, by Lemma 5, $f(V)$ is a g-radical supplement of $f\left(f^{-1}(U)\right) = U$ in N. Hence N is eg-radical supplemented, as desired. □

Corollary 4. *Every factor module of an eg-radical supplemented module is eg-radical supplemented.*

Proof. Clear from Lemma 6. □

Lemma 7. *Let M be an eg-radical supplemented R-module. Then every finitely M-generated R-module is eg-radical supplemented.*

Proof. Let N be a finitely M-generated R-module. Then there exist a finite index set Λ and an R-module epimorphism $f: M^{(\Lambda)} \to N$. Since M is eg-radical supplemented, by Corollary 3, $M^{(\Lambda)}$ is eg-radical supplemented. Then by Lemma 6, N is eg-radical supplemented, as desired. □

Proposition 2. *Let R be a ring. Then the R-module ${}_RR$ is eg-radical supplemented if and only if every finitely generated R-module is eg-radical supplemented.*

Proof. ($\Longrightarrow$) Clear from Lemma 7.
($\Longleftarrow$) Clear, since ${}_RR$ is finitely generated. □

Definition 2. Let M be an R-module and $X \leq M$. If X is a g-radical supplement of an essential submodule of M, then X is called an eg-radical supplement submodule in M.

Let M be an R-module. It is defined the relation β^* on the set of submodules of an R-module M by $X\beta^*Y$ if and only if $Y+K=M$ for every $K\leq M$ such that $X+K=M$ and $X+T=M$ for every $T\leq M$ such that $Y+T=M$ (see [1]). It is defined the relation β_g^* on the set of submodules of an R-module M by $X\beta_g^*Y$ if and only if $Y+K=M$ for every $K\trianglelefteq M$ such that $X+K=M$ and $X+T=M$ for every $T\trianglelefteq M$ such that $Y+T=M$ (see [9]).

Lemma 8. *Let M be an R-module. If every essential submodule of M is β_g^* equivalent to an eg-radical supplement submodule in M, then M is eg-radical supplemented.*

Proof. Let $X\trianglelefteq M$. By hypothesis, there exists an eg-radical supplement submodule V in M with $X\beta_g^* V$. Let V be a g-radical supplement of an essential submodule U in M. Then $M=U+V$ and $U\cap V\leq \mathrm{Rad}_g V$. Since $U\trianglelefteq M$, by hypothesis, there exists an eg-radical supplement submodule Y in M with $U\beta_g^*Y$. Let Y be a g-radical supplement of S in M and $S\trianglelefteq M$. Then $M=S+Y$ and $S\cap Y\leq \mathrm{Rad}_g Y$. Since $X\beta_g^* V$ and $M=U+V$, $M=X+U$ and since $U\beta_g^*Y$ and $X\trianglelefteq M$, $M=X+Y$. Assume $X\cap Y\not\leq \mathrm{Rad}_g Y$. Then there exists an essential maximal submodule T of Y such that $X\cap Y+T=Y$. By using [2, Lemma 1.24], we can see that $M=S+Y=X\cap Y+S+T=Y+X\cap(S+T)=U+X\cap(S+T)=X+U\cap(S+T)=V+U\cap(S+T)=U\cap V+S+T$. Since T is an essential maximal submodule of Y, by $\frac{M}{S+T}=\frac{Y+S+T}{S+T}\cong\frac{Y}{Y\cap(S+T)}=\frac{Y}{S\cap Y+T}=\frac{Y}{T}$ and $S+T\trianglelefteq M$, $S+T$ is an essential maximal submodule of M and hence $U\cap V\leq \mathrm{Rad}_g V\leq S+T$. Then $M=U\cap V+S+T=S+T$. This is a contradiction. Hence $X\cap Y\leq \mathrm{Rad}_g Y$ and Y is a g-radical supplement of X in M. Thus M is eg-radical supplemented. □

Corollary 5. *Let M be an R-module. If every essential submodule of M is β^* equivalent to an eg-radical supplement submodule in M, then M is eg-radical supplemented.*

Proof. Clear from Lemma 8. □

Corollary 6. *Let M be an R-module. If every essential submodule of M lies above an eg-radical supplement submodule in M, then M is eg-radical supplemented.*

Proof. Clear from Corollary 5. □

Corollary 7. *Let M be an R-module. If every essential submodule of M is an eg-radical supplement submodule in M, then M is eg-radical supplemented.*

Proof. Clear from Corollary 6. □

Lemma 9. *Let M be an R-module. If every submodule of M is β^* equivalent to an eg-radical supplement submodule in M, then M is g-radical supplemented.*

Proof. Let $X\leq M$. By hypothesis, there exists an eg-radical supplement submodule V in M with $X\beta^*V$. Let V be a g-radical supplement of an essential submodule U

in M. Then $M = U + V$ and $U \cap V \leq \mathrm{Rad}_g V$. By hypothesis, there exists an eg-radical supplement submodule Y in M with $U\beta^* Y$. Let Y be a g-radical supplement of an essential submodule S in M. Then $M = S + Y$ and $S \cap Y \leq \mathrm{Rad}_g Y$. Since $X\beta^* V$ and $M = U + V$, $M = X + U$ and since $U\beta^* Y$, $M = X + Y$. Assume $X \cap Y \not\leq \mathrm{Rad}_g Y$. Then there exists an essential maximal submodule T of Y such that $X \cap Y + T = Y$. By using [2, Lemma 1.24], we can see that $M = S+Y = X \cap Y + S + T = Y + X \cap (S+T) = U + X \cap (S+T) = X + U \cap (S+T) = V + U \cap (S+T) = U \cap V + S + T$. Since T is an essential maximal submodule of Y, by $\frac{M}{S+T} = \frac{Y+S+T}{S+T} \cong \frac{Y}{Y \cap (S+T)} = \frac{Y}{S \cap Y + T} = \frac{Y}{T}$ and $S + T \trianglelefteq M$, $S + T$ is an essential maximal submodule of M and hence $U \cap V \leq \mathrm{Rad}_g V \leq S + T$. Then $M = U \cap V + S + T = S + T$. This is a contradiction. Hence $X \cap Y \leq \mathrm{Rad}_g Y$ and Y is a g-radical supplement of X in M. Thus M is g-radical supplemented. □

Corollary 8. *Let M be an R-module. If every submodule of M lies above an eg-radical supplement submodule in M, then M is g-radical supplemented.*

Proof. Clear from Lemma 9. □

Corollary 9. *Let M be an R-module. If every submodule of M is an eg-radical supplement submodule in M, then M is g-radical supplemented.*

Proof. Clear from Lemma 9. □

Example 1. Consider the $\mathbb{Z}$-module $\mathbb{Q}$. Since $\mathrm{Rad}_g \mathbb{Q} = \mathrm{Rad}(\mathbb{Q}) = \mathbb{Q}$, ${}_{\mathbb{Z}}\mathbb{Q}$ is eg-radical supplemented. But, since ${}_{\mathbb{Z}}\mathbb{Q}$ is not supplemented and every nonzero submodule of ${}_{\mathbb{Z}}\mathbb{Q}$ is essential in ${}_{\mathbb{Z}}\mathbb{Q}$, ${}_{\mathbb{Z}}\mathbb{Q}$ is not essential g-supplemented.

Example 2. Consider the $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ for a prime p. It is easy to check that $\mathrm{Rad}_g \mathbb{Z}_{p^2} \neq \mathbb{Z}_{p^2}$. By [4, Lemma 4], $\mathrm{Rad}_g\left(\mathbb{Q} \oplus \mathbb{Z}_{p^2}\right) = \mathrm{Rad}_g \mathbb{Q} \oplus \mathrm{Rad}_g \mathbb{Z}_{p^2} \neq \mathbb{Q} \oplus \mathbb{Z}_{p^2}$. Since $\mathbb{Q}$ and $\mathbb{Z}_{p^2}$ are eg-radical supplemented, by Lemma 4, $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ is eg-radical supplemented. But $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ is not essential g-supplemented.

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STABILITY AND HYPERSTABILITY OF MULTI-ADDITIVE-CUBIC MAPPINGS

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Abstract. In this article, we introduce the multi-additive-cubic mappings and then unify the system of functional equations defining a multi-additive-cubic mapping to a single equation. Using a fixed point theorem, we study the generalized Hyers-Ulam stability of such equation. As a result, we show that the multi-additive-cubic functional equation can be hyperstable.

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1. INTRODUCTION

A classical question in the theory of functional equation is the following: "when is it true that a function which approximately satisfies a functional equation $\mathcal{F}$ must be close to an exact solution of $\mathcal{F}$?" If the problem accepts a solution, we say that the equation $\mathcal{F}$ is stable.

A stimulating and famous talk presented by Ulam [22] in 1940, motivated the study of stability problems for various functional equations. He gave a wide range talk before a Mathematical Colloquium at the University of Wisconsin in which he presented a list of unsolved problems. In 1941, Hyers [14] was the first Mathematician to present the result concerning the stability of functional equations. He brilliantly answered the question of Ulam, the problem for the case of approximately additive mappings on Banach spaces. In course of time, the theorem formulated by Hyers was generalized by Aoki [1], Th. M. Rassias [20] and J. M. Rassias [19] for additive mappings.

Let V be a commutative group, W be a linear space, and $n \geq 2$ be an integer. Recall from [11] that a mapping $f : V^n \longrightarrow W$ is said to be *multi-additive* if it is additive (satisfies Cauchy's functional equation $A(x+y) = A(x) + A(y)$) in each variable. Some facts on such mappings can be found in [17] and many other sources. Ciepliński

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in [11] showed that f is multi-additive if and only if it satisfying the equation

$$f(x_1+x_2)=\sum_{j_1,j_2,\ldots,j_n\in\{1,2\}} f(x_{1j_1},x_{2j_2},\ldots,x_{nj_n}), \tag{1.1}$$

where $x_j=(x_{1j},x_{2j},\ldots,x_{nj})\in V^n$ with $j\in\{1,2\}$. Moreover, f is called *multi-quadratic* if it is quadratic (satisfies the quadratic functional equation $Q(x+y)+Q(x-y)=2Q(x)+2Q(y)$) in each variable [12]. In [23], Zhao et al. proved that the mapping $f:V^n\longrightarrow W$ is multi-quadratic if and only if the following relation holds

$$\sum_{t\in\{-1,1\}^n} f(x_1+tx_2)=2^n\sum_{j_1,j_2,\ldots,j_n\in\{1,2\}} f(x_{1j_1},x_{2j_2},\ldots,x_{nj_n}). \tag{1.2}$$

For the generalized form and Jensen type of multi-quadratic mappings refer to [6] and [21], respectively. In [11] and [12], Ciepliński studied the generalized Hyers-Ulam stability of multi-additive and multi-quadratic mappings in Banach spaces, respectively (see also [23]). Furthermore, the mentioned mapping f is also called a *multi-cubic* if it is cubic in each variable, i.e., satisfies the equation

$$C(2x+y)+C(2x-y)=2C(x+y)+2C(x-y)+12C(x) \tag{1.3}$$

in each variable [15]. In [7], the second author and Shojaee, introduced the multi-cubic mappings and proved the multi-cubic functional equations can be hyperstable, that is, every approximately multi-cubic mapping under some conditions is multi-cubic; for other forms of cubic functional equations and their stabilities refer to [4, 5, 16, 19]. Various versions of multi-cubic mappings and functional equations which are recently studied can be found in [13] and [18].

In this paper, we define the multi-additive-cubic mappings and present a characterization of such mappings. In other words, we reduce the system of n equations defining the multi-additive-cubic mappings to obtain a single functional equation. We also prove the generalized Hyers-Ulam stability for multi-additive-cubic functional equations by using the fixed point method which was introduced and used for the first time by Brzdęk [8]; for more applications of this approach and alternative version for the stability of multi-Cauchy-Jensen mappings in Banach spaces and 2-Banach spaces see [2, 3] and [10], respectively.

2. Characterization of multi-additive-cubic mappings

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers, $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, $\mathbb{R}_+:=[0,\infty)$. For any $l\in\mathbb{N}_0$, $m\in\mathbb{N}$, $t=(t_1,\ldots,t_m)\in\{-1,1\}^m$ and $x=(x_1,\ldots,x_m)\in V^m$ we write $lx:=(lx_1,\ldots,lx_m)$ and $tx:=(t_1x_1,\ldots,t_mx_m)$, where rx stands, as usual, for the rth power of an element x of the linear space V.

Let V and W be linear spaces, $n\in\mathbb{N}$ and $k\in\{0,\ldots,n\}$. A mapping $f:V^n\longrightarrow W$ is called k-additive and $n-k$-cubic (briefly, multi-additive-cubic) if f is additive in each of some k variables and is cubic in each of the other variables (see equation (1.3)). In this note, we suppose for simplicity that f is additive in each of the first k variables,

but one can obtain analogous results without this assumption. Let us note that for $k=n$ ($k=0$), the above definition leads to the so-called multi-additive (multi-cubic) mappings.

In what follows, we assume that V and W are vector spaces over the rationals. Moreover, we identify $x=(x_1,\dots,x_n)\in V^n$ with $(x^k,x^{n-k})\in V^k\times V^{n-k}$, where $x^k:=(x_1,\dots,x_k)$ and $x^{n-k}:=(x_{k+1},\dots,x_n)$, and we adopt the convention that $(x^n,x^0):=x^n:=(x^0,x^n)$. Put $x_i^k=(x_{i1},\dots,x_{ik})\in V^k$ and $x_i^{n-k}=(x_{i,k+1}\dots,x_{in})\in V^{n-k}$ where $i\in\{1,2\}$. In addition, we put

$$\mathcal{M}=\left\{(N_{k+1},\dots,N_n)|\ N_j\in\{x_{1j}\pm x_{2j},x_{1j}\},\ j\in\{k+1,\dots,n\}\right\}.$$

Consider

$$\mathcal{M}_m^{n-k}:=\left\{\mathfrak{N}_{n-k}=(N_{k+1},\dots,N_n)\in\mathcal{M}|\ \mathrm{Card}\{N_j:N_j=x_{1j}\}=m\right\},$$

for any $m\in\{0,\dots,n-k\}$. From now on, we use the following notation:

$$f\left(x_i^k,\mathcal{M}_m^{n-k}\right):=\sum_{\mathfrak{N}_{n-k}\in\mathcal{M}_m^{n-k}}f\left(x_i^k,\mathfrak{N}_{n-k}\right)\qquad(i\in\{1,2\}).$$

Note that in the above notations, if $k=0$ then we obtain the same notation for multi-cubic mappings which are used in [7]. Here, we reduce the system of n equations defining the k-additive and $n-k$-cubic mapping to obtain a single functional equation.

Proposition 1. *Let $n\in\mathbb{N}$ and $k\in\{0,\dots,n\}$. If the mapping $f:V^n\longrightarrow W$ is k-additive and $n-k$-cubic mapping, then f satisfies the equation*

$$\sum_{t\in\{-1,1\}^{n-k}}f\left(x_1^k+x_2^k,2x_1^{n-k}+tx_2^{n-k}\right)=\sum_{m=0}^{n-k}2^{n-k-m}12^m\sum_{i\in\{1,2\}}f\left(x_i^k,\mathcal{M}_m^{n-k}\right)\quad(2.1)$$

for all $x_i^k=(x_{i1},\dots,x_{ik})\in V^k$ and $x_i^{n-k}=(x_{i,k+1}\dots,x_{in})\in V^{n-k}$ where $i\in\{1,2\}$.

Proof. Without loss of generality, we assume that $k\in\{0,\dots,n-1\}$. For any $x^{n-k}\in V^{n-k}$, define the mapping $g_{x^{n-k}}:V^k\longrightarrow W$ by $g_{x^{n-k}}\left(x^k\right):=f\left(x^k,x^{n-k}\right)$ for $x^k\in V^k$. By assumption, $g_{x^{n-k}}$ is k-additive, and hence Theorem 2 from [11] implies that

$$g_{x^{n-k}}\left(x_1^k+x_2^k\right)=\sum_{j_1,j_2,\dots,j_k\in\{1,2\}}g_{x^{n-k}}\left(x_{j_11},x_{j_22},\dots,x_{j_kk}\right),\qquad\left(x_1^k,x_2^k\in V^k\right).$$

It now follows from the above equality that

$$f\left(x_1^k+x_2^k,x^{n-k}\right)=\sum_{j_1,j_2,\dots,j_k\in\{1,2\}}f\left(x_{j_11},x_{j_22},\cdots,x_{j_kk},x^{n-k}\right)\quad(2.2)$$

for all x_1^k, $x_2^k\in V^k$ and $x^{n-k}\in V^{n-k}$. Similarly to the above, for any $x^k\in V^k$ consider the mapping $h_{x^k}:V^{n-k}\longrightarrow W$ defined through $h_{x^k}\left(x^{n-k}\right):=f(x^k,x^{n-k}),\ x^{n-k}\in V^{n-k}$

which is $n-k$-cubic and so we conclude from Proposition 2.2 of [7] that

$$\sum_{t\in\{-1,1\}^{n-k}} h_{x^k}\left(2x_1^{n-k}+tx_2^{n-k}\right)=\sum_{m=0}^{n-k}2^{n-k-m}12^m h_{x^k}\left(\mathcal{M}_k^{n-k}\right). \tag{2.3}$$

for all x_1^{n-k}, $x_2^{n-k}\in V^{n-k}$, where

$$h_{x^k}\left(\mathcal{M}_k^{n-k}\right):=\sum_{\mathfrak{N}_{n-k}\in\mathcal{M}_k^{n-k}} h_{x^k}\left(\mathfrak{N}_{n-k}\right).$$

By the definition of h_{x^k}, relation (2.3) is equivalent to

$$\sum_{t\in\{-1,1\}^{n-k}} f\left(x^k,2x_1^{n-k}+tx_2^{n-k}\right)=\sum_{m=0}^{n-k}2^{n-k-m}12^m f\left(x^k,\mathcal{M}_k^{n-k}\right) \tag{2.4}$$

for all $x_1^{n-k},x_2^{n-k}\in V^{n-k}$ and $x^k\in V^k$. Plugging equality (2.2) into (2.4), we get

$$\begin{aligned}\sum_{t\in\{-1,1\}^{n-k}} f\left(x_1^k+x_2^k,2x_1^{n-k}+tx_2^{n-k}\right)&=\sum_{m=0}^{n-k}2^{n-k-m}12^m f\left(x_1^k+x_2^k,\mathcal{M}_m^{n-k}\right)\\&=\sum_{m=0}^{n-k}2^{n-k-m}12^m\sum_{j_1,j_2,\ldots,j_n\in\{1,2\}} f\left(x_{j_1}1,x_{j_2}2,\ldots,x_{j_k}k,\mathcal{M}_m^{n-k}\right)\\&=\sum_{m=0}^{n-k}2^{n-k-m}12^m\sum_{i\in\{1,2\}} f\left(x_i^k,\mathcal{M}_m^{n-k}\right)\end{aligned}$$

for all $x_i^k=(x_{i1},\ldots,x_{ik})\in V^k$ and $x_i^{n-k}=(x_{i,k+1}\ldots,x_{in})\in V^{n-k}$, which proves that f satisfies equation (2.1). □

We remember that in Proposition 1, if $k=0$, we arrive to the upcoming equation. In other words, it is proved in [7, Proposition 2.2] that every multi-cubic mapping $f:V^n\longrightarrow W$ satisfying

$$\sum_{t\in\{-1,1\}^n} f(2x_1+tx_2)=\sum_{m=0}^{n}2^{n-m}12^m f(\mathcal{M}_m^n). \tag{2.5}$$

In the sequel, $\binom{n}{k}$ is the binomial coefficient defined for all $n,k\in\mathbb{N}$ with $n\geq k$ by $\frac{n!}{k!(n-k)!}$. We say the mapping $f:V^n\longrightarrow W$ satisfies *the 3-power condition* in the jth variable if

$$f(z_1,\ldots,z_{j-1},2z_j,z_{j+1},\ldots,z_n)=8f(z_1,\ldots,z_{j-1},z_j,z_{j+1},\ldots,z_n),\ ((z_1,\ldots,z_n)\in V^n).$$

Remark 1. It is easily verified that if the mapping C satisfying equation (1.3), then

$$C(2x)=8C(x). \tag{2.6}$$

But the converse is not true. Let $(\mathcal{A},\|\cdot\|)$ be a Banach algebra. Fix the vector a_0 in $\mathcal{A}$ (not necessarily unit). Define the mapping $h:\mathcal{A}\longrightarrow\mathcal{A}$ by $h(a)=\|a\|^3a_0$ for

any $a \in \mathcal{A}$. Obviously, for each $x \in \mathcal{A}$, (2.6) is true while (1.3) does not hold for h. Therefore, condition (2.6) does not imply that f is a cubic mapping.

Lemma 1. *Suppose that the mapping $f : V^n \longrightarrow W$ satisfies equation (2.1). Then, $f(2x) = 2^{3n-2k} f(x)$. In particular,*

(i) $f(0) = 0$;

(ii) *if f satisfies equation (1.1) or equivalently is multi-additive (the case $k = n$), then $f(2x) = 2^n f(x)$;*

(iii) *if f satisfies equation (2.5) (the case $k = 0$), then $f(2x) = 2^{3n} f(x)$.*

Proof. We firstly rewrite (2.1) as follows:

$$\sum_{t\in\{-1,1\}^{n-k}} f\left(x_1^k + x_2^k, 2x_1^{n-k} + tx_2^{n-k}\right) = \sum_{m=0}^{n-k} 2^{n-k-m} 12^m$$

$$\sum_{j_1,j_2,\ldots,j_n\in\{1,2\}} f(x_{j_1 1},\ldots,x_{j_k k},\overbrace{x_{1,k+1} \pm x_{2,k+1},\ldots,x_{1n} \pm x_{2n}}^{n-k-m-\text{times}},\overbrace{x_{1,k+1},\ldots,x_{1n}}^{m-\text{times}}). \quad (2.7)$$

Note that by the definition of $\mathcal{M}_m^{n-k}$, the elements of set $\{x_{1,k+1},\ldots,x_{1n}\}$ have m choice in the last $n-k$ components. Putting $x_1^k = x_2^k = x^k$ and $x_1^{n-k} = x^{n-k}$, $x_2^{n-k} = 0$ in (2.7), we have

$$\begin{aligned} 2^{n-k} f(2x) &= \sum_{m=0}^{n-k} \binom{n-k}{m} 2^{n-k-m} 12^m 2^k 2^{n-k-m} f(x) \\ &= 2^{2n-k} \sum_{m=0}^{n-k} \binom{n-k}{m} 3^m 1^{n-k-m} f(x) \\ &= 2^{2n-k}(3+1)^{n-k} f(x) \\ &= 2^{4n-3k} f(x). \end{aligned} \quad (2.8)$$

Therefore, $f(2x) = 2^{3n-2k} f(x)$. □

Let $0 \le p \le k$ and $0 \le q \le n-k$. Put

$$\mathcal{K}_{(p,q)} = \left\{ {}_{(p,q)}x := (\overbrace{0,\ldots,0,x_{i_1},0,\ldots,0,x_{i_p},0,\ldots,0}^{k-\text{times}},\overbrace{0,\ldots,0,x_{j_1},0,\ldots,0,x_{j_q},0,\ldots,0}^{n-k-\text{times}}) \in V^n \right\}$$

where $1 \le i_1 < \cdots < i_p \le k$ and $1 \le j_1 < \cdots < j_q \le n-k$. In other words, $\mathcal{K}_{(p,q)}$ is the set of all vectors in V^n that exactly their $p+q$-components are non-zero such that p components of them are coordinates of x^k and q components of them are just coordinates of x^{n-k}.

We wish to show that if the mapping $f : V^n \longrightarrow W$ satisfies equation (2.1), then it is multi-additive-cubic. In order to do this, we bring the next lemma.

Lemma 2. *If the mapping $f : V^n \longrightarrow W$ fulfilling equation (2.1) and the 3-power condition in the last $n-k$ variables, then $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero.*

Proof. We argue by induction on $p+q$ that for each ${}_{(p,q)}x \in \mathcal{K}_{(p,q)}$, $f\left({}_{(p,q)}x\right) = 0$ for $0 \leq p \leq k$ and $0 \leq q \leq n-k$. For $p+q = 0$, it follows from Lemma 1 that $f(0,\ldots,0) = 0$. Assume that for each ${}_{(p,q)}x \in \mathcal{K}_{(p,q)}$, $f\left({}_{(p,q)}x\right) = 0$ with $p+q = s-1$. We show that if ${}_{(p,q)}x \in \mathcal{K}_{(p,q)}$, then $f\left({}_{(p,q)}x\right) = 0$ for $p+q = s$. By a suitable replacement in (2.1), that is p coordinates of x^k and q coordinates of x^{n-k} are non-zero and using the assumption, we get

$$\begin{aligned} 2^{n-k}2^{3q}f\left({}_{(p,q)}x\right) &= \sum_{m=0}^{n-k-q}\binom{n-k-q}{m}2^{n-k-m}12^m2^{k-p}2^{n-k-m}f\left({}_{(p,q)}x\right) \\ &= 2^{2n-k-p}\sum_{m=0}^{n-k-q}\binom{n-k-q}{m}3^m1^{n-k-q-m}f\left({}_{(p,q)}x\right) \\ &= 2^{2n-k-p}(3+1)^{n-k-q}f\left({}_{(p,q)}x\right) \\ &= 2^{4n-3k-p-2q}f\left({}_{(p,q)}x\right). \end{aligned}$$

Hence, $f\left({}_{(p,q)}x\right) = 0$. Note that we have used the same computations of (2.8) of Lemma 1 in the above relations. This shows that $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero. □

It follows from Remark 1 that the 3-power condition does not imply f is cubic in the jth variable. Adding this condition for f, we show that if f satisfies equation (2.1), then it is k-additive and $n-k$-cubic (multi-additive-cubic) mapping as follows.

Proposition 2. *If the mapping $f : V^n \longrightarrow W$ satisfies equation (2.1) and the 3-condition in the last $n-k$ variables, then it is multi-additive-cubic mapping.*

Proof. Putting $x_2^{n-k} = (0,\cdots,0)$ in the left side of (2.1) and applying the hypothesis, we obtain

$$2^{n-k}f\left(x_1^k+x_2^k,2x_1^{n-k}\right) = 2^{n-k}\times 2^{3(n-k)}f\left(x_1^k+x_2^k,x_1^{n-k}\right). \tag{2.9}$$

On the other hand, by using Lemma 2, the right side of (2.1) will be

$$\begin{aligned} &\sum_{m=0}^{n-k}\binom{n-k}{m}2^{n-k-m}12^m2^{n-k-m}\sum_{j_1,j_2,\cdots,j_n\in\{1,2\}}f\left(x_{j_11},x_{j_22},\ldots,x_{j_kk},x_1^{n-k}\right) \\ &= \sum_{m=0}^{n-k}\binom{n-k}{m}4^{n-k-m}12^m\sum_{j_1,j_2,\cdots,j_k\in\{1,2\}}f\left(x_{j_11},x_{j_22},\ldots,x_{j_kk},x_1^{n-k}\right) \\ &= 2^{4(n-k)}\sum_{j_1,j_2,\ldots,j_k\in\{1,2\}}f\left(x_{j_11},x_{j_22},\ldots,x_{j_kk},x_1^{n-k}\right). \end{aligned} \tag{2.10}$$

Comparing relations (2.9) and (2.10), we find

$$f\left(x_1^k+x_2^k,x_1^{n-k}\right)=\sum_{j_1,j_2,\cdots,j_n\in\{1,2\}} f\left(x_{j_1}1,x_{j_2}2,\ldots,x_{j_n}n,x_1^{n-k}\right)$$

for all $x_1^k,x_2^k\in V^n$ and $x_1^{n-k}\in V^{n-k}$. In light of [11, Theorem 2], we see that f is additive in each of the k first variables. Furthermore, by putting $x_2^k=(0,\ldots,0)$ in (2.1) and using Lemma 2, we have

$$\sum_{t\in\{-1,1\}^{n-k}} f\left(x_1^k,2x_1^{n-k}+tx_2^{n-k}\right)=\sum_{m=0}^{n-k}2^{n-k-m}12^m f\left(x_1^k,\mathcal{M}_m^{n-k}\right)$$

for all $x_1^k\in V^k$ and $x_1^{n-k},x_2^{n-k}\in V^{n-k}$, and thus [7, Proposition 2.3] now completes the proof. □

3. Stability of (2.1)

In this section, we prove the generalized Hyers-Ulam stability of equation (2.1) by a fixed point result (Theorem 1) in Banach spaces. Throughout, for two sets X and Y, the set of all mappings from X to Y is denoted by Y^X. Here, we introduce the oncoming three hypotheses:

(A1) Y is a Banach space, $\mathcal{S}$ is a nonempty set, $j\in\mathbb{N}$, $g_1,\ldots,g_j:\mathcal{S}\longrightarrow\mathcal{S}$ and $L_1,\ldots,L_j:\mathcal{S}\longrightarrow\mathbb{R}_+$,

(A2) $\mathcal{T}:Y^{\mathcal{S}}\longrightarrow Y^{\mathcal{S}}$ is an operator satisfying the inequality

$$\|\mathcal{T}\lambda(x)-\mathcal{T}\mu(x)\|\leq\sum_{i=1}^{j}L_i(x)\,\|\lambda(g_i(x))-\mu(g_i(x))\|,\quad \lambda,\mu\in Y^{\mathcal{S}},x\in\mathcal{S},$$

(A3) $\Lambda:\mathbb{R}_+^{\mathcal{S}}\longrightarrow\mathbb{R}_+^{\mathcal{S}}$ is an operator defined through

$$\Lambda\delta(x):=\sum_{i=1}^{j}L_i(x)\delta(g_i(x))\qquad \delta\in\mathbb{R}_+^{\mathcal{S}},x\in\mathcal{S}.$$

In the next theorem, we present a fundamental result in fixed point theory [9, Theorem 1]. This result plays a key tool to obtain our aim in this paper.

Theorem 1. *Let hypotheses* (A1)-(A3) *hold and the function* $\theta:\mathcal{S}\longrightarrow\mathbb{R}_+$ *and the mapping* $\phi:\mathcal{S}\longrightarrow Y$ *fulfils the following two conditions:*

$$\|\mathcal{T}\phi(x)-\phi(x)\|\leq\theta(x),\quad \theta^*(x):=\sum_{l=0}^{\infty}\Lambda^l\theta(x)<\infty\qquad (x\in\mathcal{S}).$$

Then, there exists a unique fixed point ψ *of* $\mathcal{T}$ *such that*

$$\|\phi(x)-\psi(x)\|\leq\theta^*(x)\qquad (x\in\mathcal{S}).$$

Moreover, $\psi(x)=\lim_{l\to\infty}\mathcal{T}^l\phi(x)$ *for all* $x\in\mathcal{S}$.

Here and subsequently, for the mapping $f : V^n \longrightarrow W$, we consider the difference operator $\mathcal{D}f : V^n \times V^n \longrightarrow W$ by

$$\mathcal{D}f(x_1,x_2) := \sum_{q\in\{-1,1\}^{n-k}} f\left(x_1^k + x_2^k, 2x_1^{n-k} + qx_2^{n-k}\right) - \sum_{m=0}^{n-k} 2^{n-k-m}12^m \sum_{i\in\{1,2\}} f\left(x_i^k, \mathcal{M}_m^{n-k}\right)$$

for all $x_i^k = (x_{i1},\dots,x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{i,k+1}\dots,x_{in}) \in V^{n-k}$. We have the following stability result for the functional equation (2.1).

Theorem 2. *Let $\beta \in \{-1,1\}$, V be a linear space and W be a Banach space. Suppose that $\phi : V^n \times V^n \longrightarrow \mathbb{R}_+$ is a mapping satisfying the inequality*

$$\sum_{l=0}^{\infty} \left(\frac{1}{2^{(3n-2k)\beta}}\right)^l \phi\left(2^{\beta l - \frac{|\beta-1|}{2}} x_1, 2^{\beta l - \frac{|\beta-1|}{2}} x_2\right) < \infty \tag{3.1}$$

for all $x_1,x_2 \in V^n$ and

$$\Phi(x) := \frac{1}{2^{(3n-2k)\frac{|\beta+1|}{2}+n-k}} \sum_{l=0}^{\infty} \left(\frac{1}{2^{(3n-2k)\beta}}\right)^l \phi\left(2^{\beta l - \frac{|\beta-1|}{2}} x, \left(2^{\beta l - \frac{|\beta-1|}{2}} x^k, 0\right)\right) < \infty$$

for all $x \in V^n$. Assume also $f : V^n \longrightarrow W$ is a mapping fulfilling the inequality

$$\|\mathcal{D}f(x_1,x_2)\| \leqslant \phi(x_1,x_2) \tag{3.2}$$

for all $x_1,x_2 \in V^n$. Then, there exists a unique solution $\mathcal{F} : V^n \longrightarrow W$ of (2.1) such that

$$\|f(x) - \mathcal{F}(x)\| \leq \Phi(x) \tag{3.3}$$

for all $x = (x^k, x^{n-k}) \in V^n$.

Proof. Putting $x_1^k = x_2^k = x^k$, $x_1^{n-k} = x^{n-k}$, $x_2^{n-k} = 0$ in (3.2) and by relation (2.8) of Lemma 1, we have

$$\left\|2^{n-k} f(2x) - 2^{4n-3k} f(x)\right\| \leq \phi\left(x, \left(x^k, 0\right)\right) \tag{3.4}$$

and so

$$\left\|f(2x) - 2^{3n-2k} f(x)\right\| \leq \frac{1}{2^{n-k}} \phi\left(x, \left(x^k, 0\right)\right). \tag{3.5}$$

for all $x = x_1 \in V^n$. Set

$$\xi(x) := \frac{1}{2^{(3n-2k)\frac{|\beta+1|}{2}+n-k}} \phi\left(\frac{x}{2^{\frac{|\beta-1|}{2}}}, \left(\frac{x^k}{2^{\frac{|\beta-1|}{2}}}, 0\right)\right)$$

and $\mathcal{T}\xi(x) := \frac{1}{2^{(3n-2k)\beta}} \xi(2^\beta x)$ where $\xi \in W^{V^n}$. Then, relation (3.5) can be modified as

$$\|f(x) - \mathcal{T}f(x)\| \leq \xi(x) \qquad (x \in V^n).$$

Define $\Lambda\eta(x) := \frac{1}{2^{(3n-2k)\beta}}\eta(2^\beta x)$ for all $\eta \in \mathbb{R}_+^{V^n}, x = (x^k, x^{n-k}) \in V^n$. We now see that Λ has the form described in (A3) with $\mathcal{S} = V^n$, $g_1(x) = 2^\beta x$ and $L_1(x) = \frac{1}{2^{(3n-2k)\beta}}$ for all $x \in V^n$. On the other hand, for each $\lambda, \mu \in W^{V^n}$ and $x \in V^n$, we get

$$\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| = \left\| \frac{1}{2^{(3n-2k)\beta}} \left[\lambda\left(2^\beta x\right) - \mu\left(2^\beta x\right)\right] \right\| \leq L_1(x)\, \|\lambda(g_1(x)) - \mu(g_1(x))\|.$$

The above relation shows that the hypothesis (A2) holds. By induction on l, one can check for any $l \in \mathbb{N}_0$ and $x \in V^n$ that

$$\begin{aligned} \Lambda^l \xi(x) :&= \left(\frac{1}{2^{(3n-2k)\beta}}\right)^l \xi\left(2^{\beta l} x\right) \\ &= \frac{1}{2^{(3n-2k)\frac{|\beta+1|}{2}+n-k}} \left(\frac{1}{2^{(3n-2k)\beta}}\right)^l \phi\left(2^{\beta l - \frac{|\beta-1|}{2}} x, \left(2^{\beta l - \frac{|\beta-1|}{2}} x^k, 0\right)\right) \end{aligned} \tag{3.6}$$

for all $x \in V^n$. The relations (3.1) and (3.6) necessitate that all assumptions of Theorem 1 are satisfied. Hence, there exists a unique mapping $\mathcal{F} : V^n \longrightarrow W$ such that

$$\mathcal{F}(x) = \lim_{l\to\infty} \left(\mathcal{T}^l f\right)(x) = \frac{1}{2^{(3n-2k)\beta}} \mathcal{F}\left(2^\beta x\right) \qquad (x \in V^n),$$

and (3.3) holds as well. We shall to prove that

$$\|\mathcal{D}(\mathcal{T}^l f)(x_1, x_2)\| \leq \left(\frac{1}{2^{(3n-2k)\beta}}\right)^l \phi\left(2^{\beta l} x_1, 2^{\beta l} x_2\right) \tag{3.7}$$

for all $x_1, x_2 \in V^n$ and $l \in \mathbb{N}_0$. We argue by induction on l. The validity of (3.7) for $l = 0$ obtains by (3.2). Assume that (3.7) is true for an $l \in \mathbb{N}_0$. Then

$$\begin{aligned} \left\|\mathcal{D}\left(\mathcal{T}^{l+1} f\right)(x_1, x_2)\right\| &= \Bigg\| \sum_{q\in\{-1,1\}^{n-k}} \left(\mathcal{T}^{l+1} f\right)\left(x_1^k + x_2^k, 2x_1^{n-k} + q x_2^{n-k}\right) \\ &\quad - \sum_{m=0}^{n-k} 2^{n-k-m} 12^m \sum_{i\in\{1,2\}} \left(\mathcal{T}^{l+1} f\right)\left(x_i^k, \mathcal{M}_m^{n-k}\right) \Bigg\| \\ &= \frac{1}{2^{(3n-2k)\beta}} \Bigg\| \sum_{q\in\{-1,1\}^{n-k}} \left(\mathcal{T}^l f\right)\left(2^\beta\left(x_1^k + x_2^k\right), 2^\beta\left(2x_1^{n-k} + q x_2^{n-k}\right)\right) \\ &\quad - \sum_{m=0}^{n-k} 2^{n-k-m} 12^m \sum_{i\in\{1,2\}} \left(\mathcal{T}^l f\right)\left(2^\beta x_i^k, 2^\beta \mathcal{M}_m^{n-k}\right) \Bigg\| \\ &= \frac{1}{2^{(3n-2k)\beta}} \left\|\mathcal{D}\left(\mathcal{T}^l f\right)\left(2^\beta x_1, 2^\beta x_2\right)\right\| \\ &\leq \left(\frac{1}{2^{(3n-2k)\beta}}\right)^{l+1} \phi\left(2^{\beta(l+1)} x_1, 2^{\beta(l+1)} x_2\right) \end{aligned}$$

for all $x_1, x_2 \in V^n$. Letting $l \to \infty$ in (3.7) and applying (3.1) we arrive at $\mathcal{D}\mathcal{F}(x_1, x_2) = 0$ for all $x_1, x_2 \in V^n$. This means that the mapping $\mathcal{F}$ satisfies (2.1) which finishes the proof. □

In the next corollary, we show that the functional equation (2.1) is stable when the norm $\mathcal{D}f(x_1,x_2)$ is controlled by a small positive real number.

Corollary 1. *Given $\delta > 0$. Let also V be a normed space and W be a Banach space. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\mathcal{D}f(x_1,x_2)\| \leq \delta$$

for all $x_1,x_2 \in V^n$, then there exists a unique solution $\mathcal{F} : V^n \longrightarrow W$ of (2.1) such that

$$\|f(x) - \mathcal{F}(x)\| \leq \frac{\delta}{2^{n-k}(2^{3n-2k}-1)}$$

for all $x \in V^n$.

Proof. Setting the constant function $\phi(x_1,x_2) = \delta$ for all $x_1,x_2 \in V^n$, and applying Theorem 2 in the case $\beta = 1$, one can obtain the desired result. □

Corollary 2. *Let $\alpha \in \mathbb{R}$ with $\alpha \neq 3n-2k$. Let also V be a normed space and W be a Banach space. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\mathcal{D}f(x_1,x_2)\| \leq \sum_{i=1}^{2}\sum_{j=1}^{n}\|x_{ij}\|^{\alpha}$$

for all $x_1,x_2 \in V^n$, then there exists a unique solution $\mathcal{F} : V^n \longrightarrow W$ of (2.1) such that

$$\|f(x) - \mathcal{F}(x)\| \leq \frac{1}{2^{n-k}(|2^{3n-2k}-2^{\alpha}|)}\left(2\sum_{j=1}^{k}\|x_{1j}\|^{\alpha} + \sum_{j=k+1}^{n}\|x_{1j}\|^{\alpha}\right)$$

for all $x = x_1 \in V^n$.

Proof. The result can be obtained by choosing the function $\phi(x_1,x_2) = \sum_{i=1}^{2}\sum_{j=1}^{n}\|x_{ij}\|^{\alpha}$ for all $x_1,x_2 \in V^n$ and using Theorem 2. □

Recall that a functional equation Γ is *hyperstable* if any mapping f satisfying the equation Γ approximately is a true solution of Γ. Under some conditions the functional equation (2.1) can be hyperstable as follows.

Corollary 3. *Let $\delta > 0$. Suppose that $\alpha_{ij} > 0$ for $i \in \{1,2\}$ and $j \in \{1,\cdots,n\}$ fulfill $\sum_{i=1}^{2}\sum_{j=1}^{n}\alpha_{ij} \neq 3n-2k$. Let V be a normed space and W be a Banach space. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\mathcal{D}f(x_1,x_2)\| \leq \delta\prod_{i=1}^{2}\prod_{j=1}^{n}\|x_{ij}\|^{\alpha_{ij}}$$

for all $x_1,x_2 \in V^n$, then f satisfies equation (2.1). In particular, if f satisfies the 3-power condition in the last $n-k$ variables, then it is multi-additive-cubic.

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MAIA TYPE FIXED POINT RESULTS FOR MULTIVALUED F-CONTRACTIONS

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Abstract. In this paper, by considering the Wardowski's technique, we present fixed point results for multivalued mapping on a space with two metrics. Also, taking into account α-admissibility of a multivalued mapping, we provide some more general results.

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1. Introduction and preliminaries

In 2012, Wardowski [18] introduced a new concept for contraction mappings called F-contraction by considering a class of real valued functions. Let $\mathcal{F}$ be the set of all functions $F : (0,\infty) \longrightarrow \mathbb{R}$ satisfying the following conditions:

(F1) F is strictly increasing, i.e., for all $\alpha,\beta \in (0,\infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$,

(F2) For each sequence $\{a_n\}$ of positive numbers

$$\lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{n\to\infty} F(a_n) = -\infty,$$

(F3) There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Then a self mapping T of a metric space (X,d) is said to be F-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\forall x,y \in X,\ d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y)). \tag{1.1}$$

Taking different functions $F \in \mathcal{F}$ in (1.1) one gets a variety of F-contractions, some of them are of a type known in the literature. For example, let $F_1 : (0,\infty) \to \mathbb{R}$ be given by the formula $F_1(\alpha) = \ln\alpha$. It is clear that $F_1 \in \mathcal{F}$. Then each mapping $T : X \to X$ is an F-contraction such that

$$d(Tx,Ty) \leq e^{-\tau} d(x,y),\ \text{for all } x,y \in X \text{ with } Tx \neq Ty. \tag{1.2}$$

Therefore every Banach contraction mapping with contractive constant $0 < L < 1$ is an F-contraction with $F_1(\alpha) = \ln\alpha$ and $\tau = -\ln L > 0$. Also by condition (F1),

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every F-contraction is a contractive mapping and hence it is continuous. From the Banach and Edelstein fixed point theorems, we know that every Banach contraction mapping on a complete metric space has a unique fixed point and every contractive mapping on a compact metric space has a unique fixed point. That is, passing from Banach to Edelstein fixed point theorem, when the class of mapping is expending by contractive condition, the structure of the space is restricted. Now, it may come to mind, is there any change of structure of the space when investigating the existence of the fixed points of F-contractions. Therefore, Wardowski [18] proved the following result without restricting the structure of the space:

Theorem 1. *Let (X,d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point in X.*

In the literature, there are many generalization of Theorem 1 (see [4,7,11,12,17]), one of them as follows:

Theorem 2 ([11])**.** *Let (X,d) be a complete metric space and let $T : X \to X$ be a mapping. If there exist $F \in \mathcal{F}$ and $\tau > 0$ such that*

$$\forall x,y \in X,\ d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(m(x,y)), \tag{1.3}$$

where

$$m(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(y,Tx)]\right\},$$

then T has a unique fixed point in X provided that T or F is continuous.

It is our main aim in this work to give a fixed point theory for generalized multivalued F-contraction on a space with two metrics. First we recall some useful properties of multivalued mappings.

In 1969, using the concept of the Hausdorff metric, Nadler [13] introduced the notion of multivalued contraction mapping and proved a multivalued version of the well known Banach contraction principle. Let (X,d) be a metric space. Denote by

- $P_d(X)$ the family of all nonempty subsets of X,
- $C_d(X)$ the family of all nonempty, closed subsets of X,
- $CB_d(X)$ the family of all nonempty, closed and bounded subsets of X,
- $K_d(X)$ the family of all nonempty, compact subsets of X.

It is well known that the function $H_d : CB_d(X) \times CB_d(X) \to \mathbb{R}$ defined by

$$H_d(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}$$

for every $A,B \in CB_d(X)$, is a metric on $CB_d(X)$, which is called Hausdorff metric induced by d, where $d(x,B) = \inf\{d(x,y) : y \in B\}$. Let $T : X \to CB_d(X)$ be a map,

then T is called a multivalued contraction if for all $x,y \in X$ there exists $L \in [0,1)$ such that

$$H_d(Tx,Ty) \leq Ld(x,y).$$

Nadler proved that every multivalued contraction mapping on a complete metric space has a fixed point [13].

Furthermore, let X and Y be two metric spaces. Then, a multivalued mapping $T : X \to P(Y)$ is said to be upper semicontinuous (lower semicontinuous) if the inverse image of closed sets (open sets) is closed (open). A multivalued mapping is continuous if it is upper as well as lower semicontinuous. T is a closed multivalued mapping if the graph $G(T) = \{(x,y) : x \in X, y \in Tx\}$ is a closed subset of $X \times Y$. If T is a closed multivalued mapping, then it is closed values. Conversely, if T is both upper semicontinuous and closed values, then it is a closed multivalued mapping (see Proposition 2.17 of [9]).

Taking into account the ideas of Wardowski and Nadler, Altun et al [4] introduced the concept of multivalued F-contractions and obtained a fixed point result for these type mappings on complete metric space. After Acar et al in [1] presented the following definition and proved the following theorem:

Let (X,d) be a metric space and $T : X \to CB_d(X)$ be a mapping. Then T is said to be a generalized multivalued F-contraction with respect to d if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that

$$x,y \in X, H_d(Tx,Ty) > 0 \Rightarrow \tau + F(H_d(Tx,Ty)) \leq F(M(x,y)),$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Tx)]\right\}.$$

Theorem 3 ([1]). *Let (X,d) be complete metric space and $T : X \to K_d(X)$ be a generalized multivalued F-contraction with respect to d. If T or F is continuous, then T has a fixed point in X.*

In 2012, Samet et al [16] introduced the concept of α-ψ-contractive and α-admissible mapping and established various fixed point theorems. Also, Asl et al [5] defined the notion of α-admissible and α_*-admissible multivalued mappings as follows:

Let (X,d) be a metric space, $T : X \to P_d(X)$ and $\alpha : X \times X \to [0,\infty)$ be a function. Then T is said to be an α-admissible mapping if for each $x \in X$ and $y \in Tx$ with $\alpha(x,y) \geq 1$ implies $\alpha(y,z) \geq 1$ for all $z \in Ty$ and T is an α_*-admissible mapping if for each $x \in X$ and $y \in Tx$ with $\alpha(x,y) \geq 1$ implies $\alpha_*(Tx,Ty) \geq 1$, where $\alpha_*(Tx,Ty) = \inf\{\alpha(a,b) : a \in Tx, b \in Ty\}$. It is clear that an α_*-admissible mapping is also α-admissible.

Subsequently, in 2016, Durmaz and Altun [8] presented a fixed point results for α-admissible multivalued mappings.

Let (X,d) be a metric space, $T : X \to CB_d(X)$ and $\alpha : X \times X \to [0,\infty)$ be two mappings. Define a set

$$T_\alpha = \{(x,y) : \alpha(x,y) \geq 1 \text{ and } H_d(Tx,Ty) > 0\} \subset X \times X.$$

Given $F \in \mathcal{F}$, then T is said to be a multivalued (α,F)-contraction with respect to d if there exists a function $\tau : (0,\infty) \to (0,\infty)$ such that

$$\tau(d(x,y)) + F(H_d(Tx,Ty)) \leq F(d(x,y)) \tag{1.4}$$

for all $(x,y) \in T_\alpha$. In this case, the function τ is called the contractive factor of T.

Theorem 4 ([8]). *Let (X,d) be a complete metric space and $T : X \to K_d(X)$ be an α-admissible and multivalued (α,F)-contraction with respect to d with contractive factor τ. Suppose that*

$$\liminf_{t \to s^+} \tau(t) > 0, \text{ for all } s \geq 0 \tag{1.5}$$

and there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$. If T is a closed multivalued mapping, then T has a fixed point.

On the other hand, fixed point theory studies can be established both for single valued and multivalued contraction type mappings on space with two metrics. Unlike the conventional, here it is accepted that the mapping is contraction or contraction type according to the one metric when the space is complete for the other metric. It can be find the fundamental version of these type fixed point results in [2,6,10,14,15].

In this paper, we will present fixed point results for multivalued F-contraction mappings on a space with two metrics. Then, using α-admissibility of a multivalued mapping, we will give some more general results.

2. Main Results

Theorem 5. *Let (X,ρ) be a complete metric space, d another metric on X and $T : X \to K_d(X)$ be a generalized multivalued F-contraction with respect to d. Suppose that there exists $c > 0$ such that*

$$\rho(x,y) \leq cd(x,y) \text{ for each } x,y \in X. \tag{2.1}$$

If T is a closed multivalued mapping (with respect to ρ), then T has a fixed point in X.

Proof. Let $x_0 \in X$. Since Tx is nonempty for all $x \in X$, we can choose $x_1 \in Tx_0$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T and so the proof is completed. Let $x_1 \notin Tx_1$. Then $d(x_1,Tx_1) > 0$ since Tx_1 is closed. On the other hand, from (F1) and $d(x_1,Tx_1) \leq H_d(Tx_0,Tx_1)$, we get

$$F(d(x_1,Tx_1)) \leq F(H_d(Tx_0,Tx_1)).$$

Since T be a generalized multivalued F-contraction with respect to d, we obtain that

$$F(d(x_1,Tx_1)) \leq F(H_d(Tx_0,Tx_1)) \leq F(M(x_0,x_1)) - \tau \tag{2.2}$$

$$
\begin{aligned}
&= F\left(\max\left\{\begin{array}{c} d(x_0,x_1), d(x_0,Tx_0), d(x_1,Tx_1), \\ \frac{1}{2}\left[d(x_0,Tx_1)+d(x_1,Tx_0)\right] \end{array}\right\}\right) - \tau \\
&\leq F\left(\max\left\{d(x_0,x_1), d(x_1,Tx_1), \frac{1}{2}d(x_0,Tx_1)\right\}\right) - \tau \\
&\leq F\left(\max\left\{\begin{array}{c} d(x_0,x_1), d(x_1,Tx_1), \\ \frac{1}{2}\left[d(x_0,x_1)+d(x_1,Tx_1)\right] \end{array}\right\}\right) - \tau \\
&= F\left(\max\left\{d(x_0,x_1), d(x_1,Tx_1)\right\}\right) - \tau.
\end{aligned}
$$

If $d(x_0,x_1) \leq d(x_1,Tx_1)$, then we have $F(d(x_1,Tx_1)) \leq F(d(x_1,Tx_1)) - \tau$, which is a contradiction since $\tau > 0$. Thus we get

$$F(d(x_1,Tx_1)) \leq F(d(x_0,x_1)) - \tau.$$

Since Tx_1 is compact with respect to d, we obtain that $x_2 \in Tx_1$ such that $d(x_1,x_2) = d(x_1,Tx_1)$. Therefore, from (2.2)

$$F(d(x_1,x_2)) \leq F(H_d(Tx_0,Tx_1)) \leq F(d(x_0,x_1)) - \tau.$$

If we continue recursively, then we obtain a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ and

$$F(d(x_n,x_{n+1})) \leq F(d(x_n,x_{n-1}) - \tau \tag{2.3}$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0} \in Tx_{n_0}$, then x_{n_0} is a fixed point of T. Therefore, suppose that for every $n \in \mathbb{N}$, $x_n \notin Tx_n$. Denote $d_n = d(x_n,x_{n+1})$, for $n = 0,1,2,\ldots$. Then, $d_n > 0$ for all n and, using (2.3), the following holds:

$$F(d_n) \leq F(d_{n-1}) - \tau \leq F(d_{n-2}) - 2\tau \leq \ldots \leq F(d_0) - n\tau. \tag{2.4}$$

From (2.4), we obtain $\lim_{n\to\infty} F(d_n) = -\infty$. Hence, from (F2), we have

$$\lim_{n\to\infty} d_n = 0.$$

By (F3), there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} d_n^k F(d_n) = 0.$$

From (2.4), the following holds for all $n \in \mathbb{N}$

$$d_n^k F(d_n) - d_n^k F(d_0) \leq -d_n^k n\tau \leq 0. \tag{2.5}$$

Letting $n \to \infty$ in (2.5), we obtain that

$$\lim_{n\to\infty} n d_n^k = 0.$$

Hence, there exists $n_1 \in \mathbb{N}$ such that $nd_n^k \leq 1$ for all $n \geq n_1$. Therefore, we have

$$d_n \leq \frac{1}{n^{1/k}} \tag{2.6}$$

for all $n \geq n_1$. In order to show that $\{x_n\}$ is a Cauchy sequence, consider $m,n \in \mathbb{N}$ such that $m > n \geq n_1$. By (2.6) and using the triangular inequality for the metric, we have

$$\begin{aligned} d(x_n,x_m) &\leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \ldots + d(x_{m-1},x_m) \\ &= d_n + d_{n+1} + \ldots + d_{m-1} \\ &= \sum_{i=n}^{m-1} d_i \leq \sum_{i=n}^{\infty} d_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

From the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$, we obtain $\lim_{n\to\infty} d(x_n,x_m) = 0$. Thus $\{x_n\}$ is a Cauchy sequence in (X,d). From (2.1) the sequence $\{x_n\}$ is a Cauchy in (X,ρ) too. Since (X,ρ) is a complete metric space, there exists $x \in X$ with $\rho(x_n,x) \to 0$ as $n \to \infty$.

Since T is a closed multivalued mapping (with respect to ρ), we have $x \in Tx$. Thus, T has a fixed point in X. □

Remark 1. If we take $d = \rho$ in Theorem 5, then Theorem 3 holds.

Corollary 1. *Let (X,ρ) be a complete metric space, d another metric on X which satisfies (2.1) and $T : X \to K_d(X)$ be a multivalued operator. Suppose that there exist $F \in \mathcal{F}$ and $\tau > 0$ such that*

$$x,y \in X, H_d(Tx,Ty) > 0 \Longrightarrow \tau + F(H_d(Tx,Ty)) \leq F(d(x,y)). \tag{2.7}$$

If T is a closed multivalued mapping (with respect to ρ), then T has a fixed point in X.

Remark 2. Note that in Theorem 5, Tx is compact for all $x \in X$. Thus, one can ask: Does T has a fixed point if $T : X \to CB_d(X)$ is a generalized multivalued F-contraction? The following example, which is modified from Example 3.2 of [3], shows the answer is negative.

Example 1. Let $X = [0,1]$, ρ is the usual metric and

$$d(x,y) = \begin{cases} 0, & x = y \\ 1 + |x-y|, & x \neq y \end{cases}.$$

Then it is clear that (X,ρ) is a complete metric space and $\rho(x,y) \leq d(x,y)$ for all $x,y \in X$. Since τ_d is discrete topology, then all subsets of X are closed and also they are bounded. Therefore all subsets of X are closed and bounded with respect to d. Define a map $T : X \to CB_d(X)$,

$$Tx = \begin{cases} Q, & x \in X \backslash Q \\ X \backslash Q, & x \in Q \end{cases},$$

where Q is the set of all rational numbers in X. Therefore T has no fixed point. Now, define $F:(0,\infty)\to\mathbb{R}$ by

$$F(\alpha)=\begin{cases}\ln\alpha, & \alpha\leq 1\\ \\ \alpha, & \alpha>1\end{cases},$$

then we can see that $F\in\mathcal{F}$. Now we show that T is a generalized multivalued F-contraction with respect to d, that is

$$\forall x,y\in X\ [H_d(Tx,Ty)>0\Rightarrow 1+F(H_d(Tx,Ty))\leq F(M(x,y))].$$

Note that $H_d(Tx,Ty)>0\Rightarrow\{x,y\}\cap Q$ is a singleton. Therefore for $x,y\in X$ with $H(Tx,Ty)>0$, we have

$$H_d(Tx,Ty)=H_d(Q,X\backslash Q)=1<1+|x-y|=d(x,y)\leq M(x,y)$$

and so

$$1+F(H_d(Tx,Ty))\leq F(M(x,y)).$$

However, by adding the following condition on F, we can consider $CB_d(X)$ instead of $K_d(X)$ in Theorem 5:

(F4) $F(\inf A)=\inf F(A)$ for all $A\subset(0,\infty)$ with $\inf A>0$.

Remark 3. Note that if F is right-continuous and satisfies (F1), then it satisfies (F4).

Denote by $\mathcal{F}_*$, the set of all functions F satisfying (F1)-(F4). It is clear that $\mathcal{F}_*\subset\mathcal{F}$.

Theorem 6. *Let (X,ρ) be a complete metric space, d another metric on X which satisfies (2.1) and $T:X\to CB_d(X)$ be a generalized multivalued F-contraction with $F\in\mathcal{F}_*$. If T is a closed multivalued mapping (with respect to ρ), then T has a fixed point in X.*

Proof. Let $x_0\in X$. Since Tx is nonempty for all $x\in X$, we can choose $x_1\in Tx_0$. If $x_1\in Tx_1$, then x_1 is a fixed point of T and so the proof is completed. Let $x_1\notin Tx_1$. Then $d(x_1,Tx_1)>0$ since Tx_1 is closed. On the other hand, from (F1) and $d(x_1,Tx_1)\leq H_d(Tx_0,Tx_1)$, we get

$$F(d(x_1,Tx_1))\leq F(H_d(Tx_0,Tx_1)).$$

Since T be a generalized multivalued F-contraction, we can write that

$$F(d(x_1,Tx_1))\leq F(H_d(Tx_0,Tx_1))\leq F(M(x_1,x_0))-\tau. \tag{2.8}$$

By (F4), we get

$$F(d(x_1,Tx_1))=\inf_{y\in Tx_1}F(d(x_1,y))$$

since $d(x_1,Tx_1)>0$. Hence, from (2.8) we have

$$\inf_{y\in Tx_1}F(d(x_1,y))\leq F(M(x_1,x_0))-\tau<F(M(x_1,x_0))-\frac{\tau}{2}. \tag{2.9}$$

Therefore, from (2.9) there exists $x_2 \in Tx_1$ such that

$$F(d(x_1,x_2)) \leq F(M(x_1,x_0)) - \frac{\tau}{2}.$$

Otherwise, by the same way we obtain $x_3 \in Tx_2$ such that

$$F(d(x_2,x_3)) \leq F(M(x_2,x_1)) - \frac{\tau}{2}.$$

We continue iterative, so we get a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ and for all $n = 1,2,3,...$

$$F(d(x_n,x_{n+1})) \leq F(M(x_n,x_{n-1})) - \frac{\tau}{2}.$$

The rest of the proof can be completed as in the proof of Theorem 5. □

Now, we present a new fixed point result for multivalued F-contraction by α-admissibility of a multivalued mappings on space with two metrics.

Theorem 7. *Let (X,ρ) be a complete metric space, d another metric on X which satisfies (2.1) and $T : X \to K_d(X)$ be an α-admissible and multivalued (α,F)-contraction with respect to d with contractive factor τ satisfying (1.5). Suppose that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$. If T is a closed multivalued mapping (with respect to ρ), then T has a fixed point.*

Proof. Suppose that T has no fixed point. Then $d(x,Tx) > 0$, for all $x \in X$. Let x_0 and x_1 be as mentioned in the hypothesis, then $H_d(Tx_0,Tx_1) > 0$. So $(x_0,x_1) \in T_\alpha$, thus we can use (1.4) for x_0 and x_1. By (F1) we have

$$F(d(x_1,Tx_1)) \leq F(H_d(Tx_0,Tx_1)) \leq F(d(x_1,x_0)) - \tau(d(x_1,x_0)). \tag{2.10}$$

Since Tx_1 is compact, there exists $x_2 \in Tx_1$ such that $d(x_1,x_2) = d(x_1,Tx_1)$. Therefore, from (2.10)

$$F(d(x_1,x_2)) \leq F(H_d(Tx_0,Tx_1)) \leq F(d(x_1,x_0)) - \tau(d(x_1,x_0)).$$

Now, since T is an α-admissible mapping we have $\alpha(x_1,x_2) \geq 1$ and also $H_d(Tx_1,Tx_2) > 0$. Therefore, $(x_1,x_2) \in T_\alpha$, so we can use (1.4) for x_1 and x_2. Thus

$$F(d(x_2,Tx_2)) \leq F(H_d(Tx_1,Tx_2)) \leq F(d(x_2,x_1)) - \tau(d(x_2,x_1)).$$

Since Tx_2 is compact, there exists $x_3 \in Tx_2$ such that $d(x_2,x_3) = d(x_2,Tx_2)$. Therefore, we have

$$F(d(x_2,x_3)) \leq F(H_d(Tx_1,Tx_2)) \leq F(d(x_2,x_1)) - \tau(d(x_2,x_1)).$$

By induction, we obtain a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$, $(x_n,x_{n+1}) \in T_\alpha$ and for all $n \in \mathbb{N}$

$$F(d(x_n,x_{n+1})) \leq F(d(x_n,x_{n-1})) - \tau(d(x_n,x_{n-1})). \tag{2.11}$$

Denote $d_n = d(x_n, x_{n+1})$, for $n = 0, 1, 2, \ldots$. Then, $d_n > 0$ for all n and, using (2.11), $\{d_n\}$ is decreasing and hence convergent. By (1.5), there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $\tau(d_n) > \delta$ for all $n > n_0$. Thus, we get

$$\begin{aligned} F(d_n) &\leq F(d_{n-1}) - \tau(d_{n-1}) \\ &\leq F(d_{n-2}) - \tau(d_{n-1}) - \tau(d_{n-2}) \\ &\vdots \\ &\leq F(d_0) - \tau(d_{n-1}) - \tau(d_{n-2}) - \cdots - \tau(d_0) \\ &\leq F(d_0) - \tau(d_{n-1}) - \tau(d_{n-2}) - \cdots - \tau(d_{n_0}) \\ &\leq F(d_0) - (n - n_0)\delta \end{aligned} \tag{2.12}$$

for all $n > n_0$. Letting $n \to \infty$ in the above inequality, we obtain $\lim_{n\to\infty} F(d_n) = -\infty$. Hence, from (F2), we have

$$\lim_{n\to\infty} d_n = 0.$$

By (F3), there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} d_n^k F(d_n) = 0. \tag{2.13}$$

From (2.12), the following holds for all $n > n_0$

$$d_n^k F(d_n) - d_n^k F(d_0) \leq d_n^k [F(d_0) - (n - n_0)\delta] - d_n^k F(d_0) = -d_n^k (n - n_0)\delta \leq 0.$$

Taking into account (2.13), we obtain that from the above inequality

$$\lim_{n\to\infty} n d_n^k = 0.$$

Hence, there exists $n_1 \in \mathbb{N}$ such that $n d_n^k \leq 1$ for all $n \geq n_1$. Therefore, we have

$$d_n \leq \frac{1}{n^{1/k}}.$$

for all $n \geq n_1$. In order to show that $\{x_n\}$ is a Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Then, we get

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \\ &< \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

From the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$, we obtain $\lim_{n\to\infty} d(x_n, x_m) = 0$. Thus $\{x_n\}$ is a Cauchy sequence in (X, d). From (2.1) the sequence $\{x_n\}$ is a Cauchy in (X, ρ) too. Since (X, ρ) is a complete metric space, there exists $x \in X$ with $\rho(x_n, x) \to 0$ as $n \to \infty$.

Since T is closed multivalued mapping (with respect to ρ), then we have $x \in Tx$, which is a contradiction.

Therefore, T has a fixed point in X. □

Remark 4. We can take $CB_d(X)$ instead of $K_d(X)$ in Theorem 7 with $F \in \mathcal{F}_*$.

Theorem 8. *Let (X,ρ) be a complete metric space, d another metric on X which satisfies (2.1) and $T : X \to CB_d(X)$ be an α-admissible and multivalued (α,F)-contraction with $F \in \mathcal{F}_*$ and contractive factor τ satisfying (1.5). Suppose that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$. If T is a closed multivalued mapping (with respect to ρ), then T has a fixed point.*

Proof. We start as in the proof of Theorem 7. Considering the condition (F4), we get

$$F(d(x_1,Tx_1)) = \inf_{y\in Tx_1} F(d(x_1,y)).$$

Therefore

$$F(d(x_1,Tx_1)) \leq F(H_d(Tx_0,Tx_1)) \leq F(d(x_1,x_0)) - \tau(d(x_1,x_0)),$$

and we can write

$$\inf_{y\in Tx_1} F(d(x_1,y)) \leq F(d(x_1,x_0)) - \tau(d(x_1,x_0)) < F(d(x_1,x_0)) - \frac{\tau(d(x_1,x_0))}{2}.$$

Thus, there exists $x_2 \in Tx_1$ such that

$$F(d(x_1,x_2)) \leq F(d(x_1,x_0)) - \frac{\tau(d(x_1,x_0))}{2}.$$

The rest of the proof can be completed as in the proof of Theorem 7. □

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UNITARINESS OF OPERATORS

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Abstract. In this paper, we explain some sufficient conditions for unitariness of Toeplitz operators and little Hankel operators on the Bergman space.

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1. INTRODUCTION

The Bergman space is the Hilbert space of all holomorphic functions f on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, denoted as $A^2(\mathbb{D})$ for which

$$\|f\|_{A^2(\mathbb{D})} = \left(\int \left(|f(z)|^2 dA(z) \right) \right)^{\frac{1}{2}} < \infty,$$

where $dA(z)$ is the normalized Lebesgue area measure on the open unit disk $\mathbb{D}$. If $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and $k(z) = \sum_{n=0}^{\infty} b_n z^n$ are two functions in $A^2(\mathbb{D})$, then the inner product of h and k is given by

$$\langle h, k \rangle = \int_{\mathbb{D}} h(z)\overline{k(z)} dA(z) = \sum_{n=0}^{\infty} \frac{a_n \overline{b_n}}{n+1}.$$

The Bergman reproducing kernel is the function $K_z \in A^2(\mathbb{D})$ for $z \in \mathbb{D}$ such that $f(z) = \langle f, K_z \rangle$ for all $f \in A^2(\mathbb{D})$ and normalized reproducing kernel k_z is the function $\frac{K_z}{\|K_z\|_2}$. Here the norm $\|.\|_2$ and the inner product $\langle .,. \rangle$ are taken in the space $L^2(\mathbb{D}, dA)$. For any integer, $n \geq 0$, let $e_n(z) = \sqrt{n+1} z^n$. Then, $\{e_n\}_{n=0}^{\infty}$ forms an orthonormal basis for $A^2(\mathbb{D})$. The Toeplitz operator T_ϕ with symbol $\phi \in L^\infty(\mathbb{D})$ on $A^2(\mathbb{D})$ is defined by $T_\phi f = P(\phi f)$; here P is an orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $A^2(\mathbb{D})$.

Let $\overline{A^2(\mathbb{D})}$ be the space of conjugate analytic functions in $L^2(\mathbb{D}, dA)$. Then $\overline{A^2(\mathbb{D})} = \{\overline{g} : g \in A^2(\mathbb{D})\}$ is closed in $L^2(\mathbb{D}, dA)$. Let $\phi \in L^\infty(\mathbb{D})$, the little Hankel operator $h_\phi : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ be defined by $h_\phi f = \overline{P}(\phi f), f \in A^2(\mathbb{D})$ where $\overline{P}$ is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $\overline{A^2(\mathbb{D})}$. There are also numerous equivalent

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ways of defining little Hankel operators on $A^2(\mathbb{D})$. For illustration, define the map $S_\phi : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ by $S_\phi f = PJ(\phi f)$, where J is selfadjoint, unitary mapping from $L^2(\mathbb{D}, dA)$ into itself given by $Jh(z) = h(\overline{z})$. Observe that, $JS_\phi = h_\phi$. So S_ϕ is unitarily equivalent to h_ϕ.

Let $Aut(\mathbb{D})$ be the Lie group of all automorphisms (biholomorphic mappings) of $\mathbb{D}$. We can define for each $a \in \mathbb{D}$, an automorphism ϕ_a in $Aut(\mathbb{D})$ such that

(i) $(\phi_a \, o \, \phi_a)(z) \equiv z$;
(ii) $\phi_a(0) = a, \phi_a(a) = 0$;
(iii) ϕ_a has a unique fixed point in $\mathbb{D}$.

In fact, $\phi_a(z) = \frac{a-z}{1-\overline{a}z}$ for all a and z in $\mathbb{D}$. It is easy to verify that the derivative of ϕ_a at z is equal to $-k_a(z)$. It implies the real Jacobian determinant of ϕ_a at z is

$$J_{\phi_a(z)} = |k_a(z)|^2 = \frac{\left(1-|a|^2\right)^2}{|1-\overline{a}z|^4}.$$

Given $a \in \mathbb{D}$ and f (any measurable function on $\mathbb{D}$), let us define a function $U_a f$ on $\mathbb{D}$ by $U_a f(w) = k_a(w) f(\phi_a(w))$. Notice that U_a is a bounded linear operator on $L^2(\mathbb{D}, dA)$ and $A^2(\mathbb{D})$ for all $a \in \mathbb{D}$. Further, it can be verified that $U_a^2 = I$, the identity operator, $U_a^* = U_a, U_a(A^2(\mathbb{D})) \subset A^2(\mathbb{D})$ and $U_a((A^2(\mathbb{D}))^\perp) \subset (A^2(\mathbb{D}))^\perp$ for all $a \in \mathbb{D}$. Thus $U_a P = P U_a$ for all $a \in \mathbb{D}$ (see [11]).

Let $H^\infty(\mathbb{D})$ denote the space of bounded analytic functions on $\mathbb{D}$. Let $\mathcal{L}(H)$ denote the algebra of bounded, linear operators from a Hilbert space H into itself and let $T \in \mathcal{L}(H)$. Then the Berezin transform of T is denoted by $\widetilde{T}$, a complex valued function on $\mathbb{D}$ defined by $\widetilde{T(z)} = \langle Tk_z, k_z \rangle$. Let $ker(T)$ denotes kernel of T and an operator T is normaloid if, $\|T\| = sup\{|\langle Tx, x\rangle|; \|x\| = 1\}$. An operator T is paranormal if, $\|Tx\|^2 \leq \|T^2 x\| \|x\|$ and p-hyponormal if $\|T\|^{2p} \geq \|T^*\|^{2p}$ for $0 < p \leq 1$.

In this article, we establish some sufficient conditions for Toeplitz operators and little Hankel operators on the Bergman space $A^2(\mathbb{D})$ to be unitary and average of unitaries.

2. Results

2.1. *Preliminary considerations*

In this section we introduce the basic concepts and known results used in the course of our investigation. Our presentation is on the Bergman space. For more details, refer [1, 6, 8, 12].

Corollary 1 ([6]). *Let $T, S \in \mathbb{B}(H)$ with T invertible. If $T^{-1}S = ST^*$ with $0 \notin cl(W(S))$ and if T is hyponormal, then T is unitary.*

Proposition 1 ([12]). *For an invertible operator $T \in \mathbb{B}(H)$, the following are equivalent: (1) $T\Theta T^{-1}$ and $\|T\| \leq 1$ (2) T is unitary.*

Corollary 2 ([1]). *Let $T_1 \in \mathcal{L}(H)$ be injective $w-$ hyponormal operator and $T_2 \in \mathcal{L}(H)$ be an isometry. Assume that there exists quasiaffinities X and Y such that $T_1X = XT_2$ and $YT_1 = T_2Y$, if either X or Y is compact, then T_1 and T_2 are unitary operators and unitarily equivalent.*

Theorem 1 ([8]). *If a normal operator N_1 is a quasi-affine transform of a normal operator N_2, then N_1 is unitarily equivalent to N_2.*

Moreover, the present study has certain advanced applications which are related to the outcomes of the recent results of Padhy *et al.* [3] and Srivastava *et al.* [14].

3. Main Result

3.1. *Toeplitz operators*

In this section, we present some sufficient conditions for Toeplitz operators on $A^2(\mathbb{D})$ to be unitary.

Theorem 2. *Let ϕ be an essentially bounded Lebesgue measurable function on $\mathbb{D}$ such that $\|\phi\|_\infty \leq 1$. Then T_ψ can be represented as an average of n unitary operators, where $\psi = (1-\frac{1}{n})\phi + \frac{1}{n}$, $n > 2$ and $n \in \mathbb{N}$.*

Proof. Since $\|\phi\|_\infty \leq 1$,

$$\|I - T_{\frac{1+\phi}{2}}\| = \|I - \frac{I}{2} - \frac{T_\phi}{2}\| = \|\frac{I}{2} - \frac{T_\phi}{2}\| \leq \frac{1}{2} + \frac{1}{2}\|T_\phi\| \leq \frac{1+\|\phi_\infty\|}{2} < 1.$$

Therefore, $T_{\frac{1+\phi}{2}}$ is invertible. Let $T_{\frac{1+\phi}{2}}$ has a polar decomposition VP, where P is a positive operator and V is a unitary operator. Therefore, $T_{\frac{1+\phi}{2}} = \frac{V(P_1+P_2)}{2}$, where $P_1 = P + i\sqrt{I-P^2}$ and $P_2 = P - i\sqrt{I-P^2}$ are unitary operators. So, $T_{1+\phi} = VP_1 + VP_2$. Similarly there are unitary operators $V_1, V_2, ..., V_{n-1}$ and $W_1, W_2, ..., W_{n-1}(= V_n)$ such that

$$\begin{aligned} T_{1+\phi} &= V_1 + W_1 \\ T_{1+2\phi} &= V_1 + W_1 + T_\phi. \end{aligned}$$

Therefore,

$$\begin{aligned} T_{1+(n-1)\phi} &= V_1 + W_1 + T_{(n-2)\phi} \\ &= V_1 + V_2 + W_2 + T_{(n-3)\phi} \\ &= ... \\ &= V_1 + V_2 + ... + (V_n = W_{n-1}). \end{aligned}$$

Then,

$$T_\psi = T_{(1-\frac{1}{n})\phi + \frac{1}{n}} = \frac{1}{n}T_{1+(n-1)\phi} = \frac{1}{n}(V_1 + V_2 + ... + V_n).$$

Hence the result holds. □

Theorem 3. *Let $\phi \in L^\infty(\mathbb{D})$ and $\phi \geq 0$. If $T_\phi T_{\phi-1} = 0$, then T_ψ is unitary where $\psi = 2(\phi \circ \phi_a) - 1$.*

Proof. Since $\phi \geq 0, T_\phi \geq 0$. Then T_ϕ is self-adjoint. Further, Since $T_\phi T_{\phi-1} = 0$, that implies $T_\phi(T_\phi - I) = 0$ or $T_\phi^2 = T_\phi$. It is well known that $U_a T_\phi U_a = T_{\phi\circ\phi_a}$ for all $a \in \mathbb{D}$. Now

$$\begin{aligned}
T_\psi^* T_\psi &= T^*_{2(\phi\circ\phi_a)-1} T_{2(\phi\circ\phi_a)-1} \\
&= (2T^*_{\phi\circ\phi_a} - I)(2T_{\phi\circ\phi_a} - I) \\
&= (2(U_a^* T_\phi^* U_a^*) - I)(2(U_a T_\phi U_a) - I) \\
&= 4(U_a^* T_\phi^* U_a^* U_a T_\phi U_a) - 2U_a^* T_\phi^* U_a^* - 2U_a T_\phi U_a + I \\
&= 4U_a T_\phi^2 U_a - 4U_a T_\phi U_a + I \qquad (\because U_a^* = U_a \ , \ T_\phi^* = T_\phi \text{ and } U_a^2 = I) \\
&= 4U_a T_\phi U_a - 4U_a T_\phi U_a + I \qquad (\because T_\phi^2 = T_\phi) \\
&= I.
\end{aligned}$$

Similarly, one can verify that $T_\psi T_\psi^* = I$. Hence T_ψ is unitary. □

Theorem 4. *Let ϕ be an essentially bounded Lebesgue measurable function on $\mathbb{D}$. If T_ϕ is an isometry and $T_{\overline{\phi}\circ\phi_a}$ is hyponormal for $a \in \mathbb{D}$. Then T_ϕ is unitary.*

Proof. For all $a \in \mathbb{D}$, it is known that

$$T_\phi U_a = U_a T_{\phi\circ\phi_a}. \tag{3.1}$$

Now,

$$U_a T^*_{\phi\circ\phi_a} = U_a (U_a T_\phi U_a)^* = U_a U_a^* T_\phi^* U_a^* = T_\phi^* U_a.$$

Multiplying $T_{\overline{\phi}}$ on the left of equation (3.1), we obtain

$$T_{\overline{\phi}} T_\phi U_a = T_{\overline{\phi}} U_a T_{\phi\circ\phi_a}. \tag{3.2}$$

Since, T_ϕ is an isometry and $T_{\overline{\phi}} U_a = U_a T_{\overline{\phi}\circ\phi_a}$, equation (3.2) turns out to be

$$U_a = T_{\overline{\phi}} U_a T_{\phi\circ\phi_a} = U_a T_{\overline{\phi}\circ\phi_a} T_{\phi\circ\phi_a}.$$

Thus, $T_{\overline{\phi}\circ\phi_a} T_{\phi\circ\phi_a} = I$. Further, since $T_{\overline{\phi}\circ\phi_a}$ is hyponormal and $T_{\overline{\phi}\circ\phi_a} T_{\phi\circ\phi_a} = I$. This implies, $T_{\overline{\phi}\circ\phi_a} T_{\phi\circ\phi_a} = T_{\phi\circ\phi_a} T_{\overline{\phi}\circ\phi_a} = I$. Hence, T_ϕ is unitary. □

Theorem 5. *Let $\phi \in L^\infty(\mathbb{D})$. If $T^{*^2}_{\phi\circ\phi_a} T^2_{\phi\circ\phi_a} - 2T^*_{\phi\circ\phi_a} T_{\phi\circ\phi_a} + I = 0$ and $\widetilde{T^2_\phi}(z) = 1$ then T_ϕ is unitary.*

Proof. It is well known that $U_a T_\phi U_a = T_{\phi\circ\phi_a}$. Given that $\widetilde{T^2_\phi}(z) = 1$. Then $T_\phi^2 = I$ if and only if $T_{\overline{\phi}}^2 = I$. Again $T^{*^2}_{\phi\circ\phi_a} T^2_{\phi\circ\phi_a} - 2T^*_{\phi\circ\phi_a} T_{\phi\circ\phi_a} + I = 0$ gives $U_a T^{*^2}_\phi T_\phi U_a - 2U_a T^*_\phi T_\phi U_a + I = 0$. Thus, $2 - 2U_a T^*_\phi T_\phi U_a = 0$ or $U_a T^*_\phi T_\phi U_a = I$ by using $T_\phi^2 = I$ and $T_\phi^{*^2} = I$. Hence T_ϕ is isometry and invertible. Therefore T_ϕ is unitary. □

Theorem 6. *If ϕ is a bounded harmonic function on $\mathbb{D}$ with $\|\phi\|_\infty = 1$ and $\widetilde{T^n_\phi}(z) = 1$ then T_ϕ is unitary.*

Proof. Since ϕ is a bounded harmonic function on $\mathbb{D}$ then it follows from (Theorem 5, [4]) that $\|T_\phi\| = \|\phi\|_\infty = 1$. Now $\widetilde{T^n_\phi}(z) = 1$ that implies $\langle T^n_\phi k_z, k_z\rangle = 1 = \langle k_z, k_z\rangle$, $\forall z \in \mathbb{D}$. Thus, $T^n_\phi = I$. So T_ϕ is invertible. Then T_ϕ and T^{-1}_ϕ are power bounded and therefore similar to unitary operator (Proposition 3.8 and Corollary 1.16, [10]). Hence $\|T_\phi\| = 1$ implies that T_ϕ is unitarily equivalent to unitary operator. Therefore T_ϕ is unitary. □

Theorem 7. *Let $\phi, \phi_1, \phi_2 \in L^\infty(\mathbb{D})$ and $\|\phi\|_\infty \leq 1$. Then T_ϕ can be expressed as sum of three unitary operators.*

Proof. Let $T_\phi = T_{\phi_1} + iT_{\phi_2}$ be the Cartesian decomposition of T_ϕ. Since $\|\phi\|_\infty \leq 1$, $\|T_\phi\| \leq \|\phi\|_\infty \leq 1$. So T_ϕ is a contraction. Thus the real and imaginary parts of T_ϕ are also contractions. Let $T_1 = \frac{1}{2}(T_{\phi_1} - (I - T^2_{\phi_2})^{\frac{1}{2}})$ and notice that T_1 is a contraction. Therefore one can define the operators $W_1 = T_1 + i(I - T_1^2)^{\frac{1}{2}}$ and $W_2 = (I - T^2_{\phi_2})^{\frac{1}{2}} + iT_{\phi_2}$. Here $W_1 W_1^* = W_1^* W_1 = I$ and $W_2 W_2^* = W_2^* W_2 = I$. Hence $T_\phi = W_1 + W_1^* + W_2$, which is sum of three unitary operators. □

Theorem 8. *Let $\phi, \phi_1, \phi_2 \in L^\infty(\mathbb{D})$ and $T_\phi = T_{\phi_1} + iT_{\phi_2}$ be the Cartesian decomposition of T_ϕ. Then $T_{1-\phi}$ is unitary if and only if $T_{\phi_1} T_{\phi_2} = T_{\phi_2} T_{\phi_1}$ and $T^2_{\phi_1 - 1} = T_{1-\phi_2} T_{1+\phi_2}$.*

Proof. Suppose $T_{\phi_1} T_{\phi_2} = T_{\phi_2} T_{\phi_1}$. Then $T_\phi T_{\overline{\phi}} = T_{\overline{\phi}} T_\phi = T^2_{\phi_1} + T^2_{\phi_2}$. Again, since $T^2_{\phi_1 - 1} = T_{1-\phi_2} T_{1+\phi_2}$ then, $(T_{\phi_1} - I)^2 = (I - T_{\phi_2})(I + T_{\phi_2})$. So $T_\phi T_{\overline{\phi}} = T_{\overline{\phi}} T_\phi = T_{\phi + \overline{\phi}}$, it can be easily shown that $T_{1-\overline{\phi}} T_{1-\phi} = I = T_{1-\phi} T_{1-\overline{\phi}}$. Conversely, suppose $T_{1-\phi}$ is unitary. Then $T_\phi T_{\overline{\phi}} = T_{\overline{\phi}} T_\phi = T_{\phi+\overline{\phi}}$. Since $T_\phi T_{\overline{\phi}} = T_{\overline{\phi}} T_\phi$, we have $T_{\phi_1} T_{\phi_2} = T_{\phi_2} T_{\phi_1}$. Again, since $T_\phi T_{\overline{\phi}} = T_{\phi+\overline{\phi}}$, therefore $T^2_{\phi_1 - 1} = T_{1-\phi_2} T_{1+\phi_2}$. □

Corollary 3. *Let $\phi, \phi_1, \phi_2 \in L^\infty(\mathbb{D})$ where $\phi_1 \geq 0$ and $\phi_2 \geq 0$. If $T_\phi = T_{\phi_1} + iT_{\phi_2}$ be the Cartesian decomposition of T_ϕ with $T_{\phi_1} T_{\phi_2}$ is p-hyponormal and $T^2_{\phi_1 - 1} = T_{1-\phi_2} T_{1+\phi_2}$, then $T_{1-\phi}$ is unitary.*

Proof. If $\phi_1 \geq 0$ and $\phi_2 \geq 0$, then T_{ϕ_1}, T_{ϕ_2} are positive. Let $A = T_{\phi_1} T_{\phi_2}$, then $AT_{\phi_1} = T_{\phi_1} A^*$. So from [15], we get $A^* T_{\phi_1} = T_{\phi_1} A$, then $(T_{\phi_1} T_{\phi_2})^* T_{\phi_1} = T_{\phi_1} T_{\phi_1} T_{\phi_2}$. Thus $T_{\overline{\phi}_2} T_{\overline{\phi}_1} T_{\phi_1} = T^2_{\phi_1} T_{\phi_2}$. Therefore $T_{\phi_2} T^2_{\phi_1} = T^2_{\phi_1} T_{\phi_2}$. As T_{ϕ_1} is positive, so $T_{\phi_1} T_{\phi_2} = T_{\phi_2} T_{\phi_1}$. Hence by Theorem 8, the corollary holds. □

Corollary 4. *Let $\phi, \phi_1 \in L^\infty(\mathbb{D})$ and $T_\phi = T_{\phi_1} + iT_{\overline{\phi_1}}$ be the Cartesian decomposition of T_ϕ. If T_{ϕ_1}, $T_{\overline{\phi_1}}$ are paranormal with $ker(T_{\phi_1}) = ker(T_{\overline{\phi_1}})$ and $T^2_{\phi_1 - 1} = T_{1-\overline{\phi_1}} T_{1+\overline{\phi_1}}$. Then $T_{1-\phi}$ is unitary.*

Proof. Since T_{ϕ_1} and $T_{\overline{\phi_1}}$ are paranormal with $ker(T_{\phi_1}) = ker(T_{\overline{\phi_1}})$, then by (Theorem 5, [2]), T_{ϕ_1} is normal. That is $T_\phi T_{\overline{\phi}} = T_{\overline{\phi}} T_\phi$. Therefore by replacing $T_{\phi_2} = T_{\overline{\phi_1}}$ in Theorem 8, the assertion holds. □

Theorem 9. *Let* $\|\phi\|_\infty \leq 1$ *for* $\phi \in L^\infty(\mathbb{D})$ *and* $\|U_a - T_\phi\| < 1$ *for all* $a \in \mathbb{D}$. *If* $\|T_\phi^n\|^{-1} - \|T_\phi^{-n}\| - (|1 - \|T_\phi^n\|^{-1}| + |1 - \|T_\phi^{-n}\||) = 0$ *for all* $n \in \mathbb{N}$. *Then* T_ϕ *is unitary.*

Proof. Since $\|\phi\|_\infty \leq 1$ and $\|T_\phi\| \leq 1$, supposing that $\|U_a - T_\phi\| < 1$, it follows from [13] that T_ϕ is invertible. Then, T_ϕ^n is also invertible for $n \in \mathbb{N}$. As T_ϕ is a contraction, $\|T_\phi^n\|^{-1} \geq 1$. Thus, $1 \leq \|T_\phi^{-n}\|\|T_\phi^n\|$ and therefore, $\|T_\phi^{-n}\| \geq 1$. Since, $\|T_\phi^n\|^{-1} - \|T_\phi^{-n}\| = (|1 - \|T_\phi^n\|^{-1}| + |1 - \|T_\phi^{-n}\||)$, this implies $\|T_\phi^{-n}\| = 1$. Therefore, T_ϕ^n is unitary on $A^2(\mathbb{D})$. Further, assuming that $\|T_\phi\| \leq 1$ and T_ϕ^n is an isometry, T_ϕ is unitary. In particular, if $\|T_\phi\| \leq 1$ then $\|T_\phi^{n-1}\| \leq 1$. This implies that $T_{\overline{\phi}}^{n-1} T_\phi^{n-1} \leq I$. Therefore, $I \geq T_{\overline{\phi}} T_\phi \geq T_{\overline{\phi}}(T_{\overline{\phi}}^{n-1} T_\phi^{n-1}) T_\phi = T_{\overline{\phi}}^n T_\phi^n = I$. Hence, T_ϕ is an isometry. Further, if $\|T_\phi\| \leq 1$ and $T_{\overline{\phi}}^n$ is an isometry, $T_{\overline{\phi}}$ is an isometry. So, $\|T_\phi\| \leq 1$ and T_ϕ^n is unitary, it clearly implies that T_ϕ is a unitary operator. □

Theorem 10. *Let* ϕ *be a positive essentially bounded Lebesgue measurable function with* $\|\phi\|_\infty \leq 1$. *Then* $\mp T_\phi \pm i\sqrt{I - T_\phi^2}$ *are unitary operators on* $A^2(\mathbb{D})$.

Proof. As ϕ is positive, T_ϕ is positive on $A^2(\mathbb{D})$ and since $\|\phi\|_\infty \leq 1$, $\|T_\phi\| \leq 1$. Therefore, $I - T_\phi^2$ is a positive operator on $A^2(\mathbb{D})$. Let's define $U_1 = T_\phi + i\sqrt{I - T_\phi^2}$, $U_2 = T_\phi - i\sqrt{I - T_\phi^2}$, $U_3 = -T_\phi + i\sqrt{I - T_\phi^2}$ and $U_4 = -T_\phi - i\sqrt{I - T_\phi^2}$. One can easily observe that, $U_1^* = U_2, U_3^* = U_4$. Hence, $U_1^* U_1 = U_1 U_1^* = I, U_2^* U_2 = U_2 U_2^* = I, U_3^* U_3 = U_3 U_3^* = I$ and $U_4^* U_4 = U_4 U_4^* = I$. Therefore, U_1, U_2, U_3 and U_4 are unitary operators on $A^2(\mathbb{D})$. □

Proposition 2. *Let* $\phi \in L^\infty(\mathbb{D})$. *If* $\widetilde{T_\phi^2}(z) = 1$ *then* T_ϕ *is unitarily equivalent with* $T_{\overline{\phi}}$.

Proof. Let $T_\phi = UP$ be the polar decomposition of T where U is partial isometry and P is the positive operator. Since positive operators are self-adjoint, so $P^* = P$. Further since $\widetilde{T_\phi^2}(z) = 1$, $\langle T_\phi^2 k_z, k_z \rangle = 1 = \langle k_z, k_z \rangle, \ \ \forall z \in \mathbb{D}$. That implies $T_\phi^2 = I$ and therefore T_ϕ is invertible. Hence the partial isometry U will become a unitary operator U_a(say) $\forall a \in \mathbb{D}$. Thus $T_\phi = U_a P$ where $U_a = U_a^* = U_a^{-1}$. Then

$$U_a T_\phi U_a = U_a U_a P U_a = P U_a = T_{\overline{\phi}}.$$

Hence T_ϕ is unitarily equivalent with $T_{\overline{\phi}}$. □

Theorem 11. *Let* $\phi \in L^\infty(\mathbb{D})$ *and* $\phi \geq 0$. *Then* T_ϕ *is unitary if and only if* $\widetilde{T_\phi^2}(z) = 1$.

Proof. Let $\phi \geq 0$, that implies $T_\phi \geq 0$. Since positive operators are self adjoint, $T_\phi^* = T_\phi$. Given $\widetilde{T_\phi^2}(z) = 1$, so we get $T_\phi^2 = I$. Now

$$T_\phi^* T_\phi = T_\phi^2 = I = T_\phi T_\phi^*. \tag{3.3}$$

So T_ϕ is unitary. Conversely suppose T_ϕ is unitary, then $T_\phi^2 = I$. Thus $\langle T_\phi^2 f, g\rangle = \langle f, g\rangle \ \ \forall f, g \in A^2(\mathbb{D})$. If $f = g = k_z, \ \ \forall z \in \mathbb{D}$, then $\widetilde{T_\phi^2}(z) = 1$. □

3.2. *Little Hankle operators*

In this section, we discuss some sufficient conditions for little Hankle operators on $A^2(\mathbb{D})$ to be unitary.

Theorem 12. *Let $\phi \in L^\infty(\mathbb{D})$. If S_ϕ is power bounded and bounded below with $\|S_\phi^2\|^2 + 1 = 2\|S_\phi\|^2$ then S_ϕ is unitary.*

Proof. Given that

$$\|S_\phi^2\|^2 + 1 = 2\|S_\phi\|^2. \tag{3.4}$$

We have to show S_ϕ^n will satisfy the equation (3.4). We will prove this by method of induction. Now the above equation is true for $n = 1$. Let us assume that equation (3.4) is true for $n = k$, that is $\|S_\phi^{2k}\|^2 + 1 = 2\|S_\phi^k\|^2$. This implies

$$S_\phi^{*^{2k}} S_\phi^{2k} + I = 2S_\phi^{*^k} S_\phi^k. \tag{3.5}$$

Then we will prove that equation (3.4) is true for $n = k+1$. Now

$$\begin{aligned}
S_\phi^{*^{2(k+1)}} S_\phi^{2(k+1)} + I - 2S_\phi^{*^{k+1}} S_\phi^{k+1} &= S_\phi^{*^2}(S_\phi^{*^{2k}} S_\phi^{2k}) S_\phi^2 + I - 2S_\phi^{*^{k+1}} S_\phi^{k+1} \\
&= S_\phi^{*^2}(2S_\phi^{*^k} S_\phi^k - I) S_\phi^2 + I - 2S_\phi^{*^{k+1}} S_\phi^{k+1} \\
&= 2S_\phi^{*^{k+2}} S_\phi^{k+2} - S_\phi^{*^2} S_\phi^2 + I - 2S_\phi^{*^{k+1}} S_\phi^{k+1} \\
&= 2S_\phi^{*^k}(S_\phi^{*^2} S_\phi^2 - S_\phi^* S_\phi) S_\phi^k - S_\phi^{*^2} S_\phi^2 + I \\
&= 2S_\phi^{*^k}(S_\phi^* S_\phi - I) S_\phi^k - S_\phi^{*^2} S_\phi^2 + I \quad (by\ (3.4)) \\
&= 2S_\phi^{*^{k+1}} S_\phi^{k+1} - 2S_\phi^{*^k} S_\phi^k - S_\phi^{*^2} S_\phi^2 + I \\
&= 2S_\phi^{*^{k-1}}(S_\phi^{*^2} S_\phi^2 - S_\phi^* S_\phi) S_\phi^{k-1} - S_\phi^{*^2} S_\phi^2 + I \\
&= 2S_\phi^{*^{k-1}}(S_\phi^* S_\phi - I) S_\phi^{k-1} - S_\phi^{*^2} S_\phi^2 + I \\
&= 2S_\phi^{*^k} S_\phi^k - 2S_\phi^{*^{k-1}} S_\phi^{k-1} - S_\phi^{*^2} S_\phi^2 + I
\end{aligned}$$

continuing in this fashion

$$\begin{aligned}
&\vdots \\
&= 2S_\phi^{*^2} S_\phi^2 - 2S_\phi^* S_\phi - S_\phi^{*^2} S_\phi^2 + I \\
&= S_\phi^{*^2} S_\phi^2 - 2S_\phi^* S_\phi + I \\
&= 0,
\end{aligned}$$

where the second equality follows from (3.5). Hence S_ϕ^n is satisfying equation (3.4) that is, $\|S_\phi^{2n}\|^2+1=2\|S_\phi^n\|^2$. Therefore,

$$S_\phi^{*^{2n}}S_\phi^{2n}-2S_\phi^{*^n}S_\phi^n+I=0. \tag{3.6}$$

Again from equation (3.4),

$$\|S_\phi^2\|^2=2\|S_\phi\|^2-1.$$

So

$$\|S_\phi^4\|^2=2\|S_\phi^2\|^2-1=2(2\|S_\phi\|^2-1)-1=4\|S_\phi\|^2-3.$$

Again,

$$\|S_\phi^8\|^2=2\|S_\phi^4\|^2-1=2(4\|S_\phi\|^2-3)-1=8\|S_\phi\|^2-7.$$

In general, we get

$$\|S_\phi^{2^n}\|^2=2^n\|S_\phi\|^2-(2^n-1) \tag{3.7}$$

for every $n\in\mathbb{N}$. Further, since S_ϕ is power bounded, so $\exists$ a positive real number M such that

$$\|S_\phi^n\|\leq M \quad for\ \ n=1,2,3\ldots\ . \tag{3.8}$$

Thus, equation (3.7) and (3.8) will provide $M^2\geq\|S_\phi^{2^n}\|^2=2^n\|S_\phi\|^2-(2^n-1)$. This implies, $\frac{M^2}{2^n}\geq\|S_\phi\|^2-1+2^{-n}\geq 0$. As $n\to\infty$, we get $\|S_\phi\|^2=1$. Therefore,

$$S_\phi^*S_\phi=I. \tag{3.9}$$

Since any isometry is one-to-one, $ker(S_\phi)=\{0\}$. It follows from ([5], Lemma-2.1) that $ker(S_\phi^*)=\{0\}$. It is well known that $\overline{Range(S_\phi)}=ker(S_\phi^*)^\perp=\{0\}^\perp=A^2(\mathbb{D})$. That implies $Range(S_\phi)$ is dense in $A^2(\mathbb{D})$. Since S_ϕ is bounded below and has dance range, then by [7], S_ϕ is invertible and $Range(S_\phi)=A^2(\mathbb{D})$. Let $S_\phi=UP$ be the polar decomposition of S_ϕ where U is the partial isometry and P is the positive operator. Since S_ϕ is invertible, so the partial isometry U can be extended to a unitary operator V(say). Therefore, $S_\phi=VP$. It can be easily shown that $S_\phi=VN$, where N is a normal operator. Hence,

$$S_\phi S_\phi^*=VS_\phi^*S_\phi V^*=I. \tag{3.10}$$

Therefore, S_ϕ is unitary. □

Corollary 5. *Let $\phi\in L^\infty(\mathbb{D})$ and $\|\phi\|_\infty\leq 1$. If $\widetilde{(S_\phi^*S_\phi)}(z)\geq 1$ and S_ϕ is bounded below. Then S_ϕ is unitary.*

Proof. Now for any $f\in A^2(\mathbb{D})$,

$$\|S_\phi f\|\leq\|S_\phi\|\|f\|\leq\|\phi\|_\infty\|f\|\leq\|f\|. \tag{3.11}$$

Thus,

$$\|S_\phi\|^2\leq\|f\|^2\Rightarrow\langle S_\phi f,S_\phi f\rangle\leq\langle f,f\rangle\Rightarrow S_\phi^*S_\phi\leq I.$$

Again, since $\widetilde{(S_\phi^* S_\phi)}(z) \geq 1$, $\langle S_\phi k_z, S_\phi k_z\rangle \geq 1 = \|k_z\|^2$, $\langle k_z, k_z\rangle \leq \langle S_\phi^* S_\phi k_z, k_z\rangle$ and also $I \leq S_\phi^* S_\phi$. Hence, S_ϕ is isometry. Thus, the corollary follows from Theorem 12. □

Corollary 6. *Let $\phi \in L^\infty(\mathbb{D})$. If $\|S_\phi^2\|^2 + 1 = 2\|S_\phi\|^2$ and $\widetilde{S_\phi^2}(z) = 1$. Then S_ϕ is unitary.*

Proof. Given that $\widetilde{S_\phi^2}(z) = 1$. Then $\langle S_\phi^2 k_z, k_z\rangle = 1 = \|k_z\|^2$. That implies $\langle S_\phi^2 k_z, k_z\rangle = \langle k_z, k_z\rangle$. That is $S_\phi^2 = I$. It is easy to verify that $S_\phi^2 = I$ if and only if $S_\phi^{*^2} = I$. Since $\|S_\phi^2\|^2 + 1 - 2\|S_\phi\|^2 = 0$ then $S_\phi^{*^2} S_\phi^2 + I - 2S_\phi^* S_\phi = 0$ and an easy computation demonstrates $2I - 2S_\phi^* S_\phi = 0$ or $I = S_\phi^* S_\phi$. Thus S_ϕ is isometry. Further , Since $S_\phi^2 = I$ then S_ϕ is right invertible as well as left invertible and hence S_ϕ is invertible. Therefore, the corollary follows from Theorem 12. □

Corollary 7. *Let $\phi \in L^\infty(\mathbb{D})$. If S_ϕ is normaloid and $\widetilde{S_\phi^n}(z) = 1$, then S_ϕ is unitary.*

Proof. It is well known that $\|S_\phi^n\| \leq \|S_\phi\|^n$, for $n \in \mathbb{N}$. Since $\widetilde{S_\phi^n}(z) = 1$, we have that $\langle S_\phi^n k_z, k_z\rangle = 1 = \|k_z\|^2 = \langle k_z, k_z\rangle$. That is $S_\phi^n = I$. Again, since S_ϕ is normaloid, then $\|S_\phi^n\| = \|S_\phi\|^n = \|S_\phi\|$. Since $S_\phi^n = I$, then it is obvious that $1 = \|S_\phi^n\| = \|S_\phi\|^n = \|S_\phi\|$. Thus S_ϕ is isometry and $S_\phi^2 = I$. Therefore, the corollary follows from Corollary 6. □

Theorem 13. *Let $\phi \in L^\infty(\mathbb{D})$ and T_ϕ, S_ϕ^* are p-hyponormal operators with $T_\phi X = X S_\phi$ where $X : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ such that $ker(X) = \{0\}$ and $\overline{Range(X)} = A^2(\mathbb{D})$. If $T_{1-\phi}$ is unitary then $S_{1-\phi}$ is unitary.*

Proof. Since T_ϕ, S_ϕ^* are p-hyponormal operators with $T_\phi X = X S_\phi$ where $X : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ such that $ker(X) = \{0\}$ and $\overline{Range(X)} = A^2(\mathbb{D})$ then from [9], S_ϕ is unitarily equivalent to T_ϕ. That is $S_\phi = V^* T_\phi V$ where V is unitary. As $T_{1-\phi} = I - T_\phi$ is unitary, so $T_{\bar{\phi}} T_\phi = T_{\phi+\bar{\phi}} = T_\phi T_{\bar{\phi}}$. Thus $S_\phi + S_\phi^* = V^* T_{\phi+\bar{\phi}} V$. Now

$$\begin{aligned} S_{1-\phi}^* S_{1-\phi} &= (I - S_\phi)^*(I - S_\phi) \\ &= S_\phi^* S_\phi - S_\phi^* - S_\phi + I \\ &= V^* T_\phi^* T_\phi V - S_\phi^* - S_\phi + I \\ &= V^* T_\phi^* T_\phi V - V^* T_{\phi+\bar{\phi}} V + I = I. \end{aligned}$$

Similarly, we have $S_{1-\phi} S_{1-\phi}^* = I$. Hence the assertion holds. □

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NONEXISTENCE OF GLOBAL SOLUTIONS OF A DELAYED WAVE EQUATION WITH VARIABLE-EXPONENTS

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Abstract. This work deals with a Petrovsky equation with delay term and variable exponents. Firstly, we establish the local existence result by the Faedo-Galerkin method. Later, we prove the blow-up of solutions in a finite time. Our results are more general than the earlier results.

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1. INTRODUCTION

In this work, we study the following Petrovsky equation with variable exponents and delay term

$$\begin{cases} u_{tt}+\Delta^2 u-\Delta u_t+\mu_1 u_t(x,t)\left|u_t\right|^{m(x)-2}(x,t) & \\ \quad +\mu_2 u_t(x,t-\tau)\left|u_t\right|^{m(x)-2}(x,t-\tau)=bu\left|u\right|^{p(x)-2} & \text{in } \Omega\times R^+, \\ u(x,t)=\frac{\partial u(x,t)}{\partial v}=0 & \text{in } \partial\Omega\times[0,\infty), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x) & \text{in } \Omega, \\ u_t(x,t-\tau)=f_0(x,t-\tau) & \text{in } \Omega\times(0,\tau), \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in R^n, $n\geq 1$. Here, $\tau>0$ is a time delay term, $b\geq 0$ is a constant, μ_1 is a positive constant and μ_2 is a real number. The functions u_0, u_1, f_0 are the initial data to be specified later.

The variable exponents $p(\cdot)$ and $m(\cdot)$ are given as measurable functions on $\overline{\Omega}$ satisfy:

$$\begin{cases} 2\leq m^-\leq m(x)\leq m^+\leq m^* \\ 2\leq p^-\leq p(x)\leq p^+\leq p^* \end{cases} \tag{1.2}$$

where

$$m^-=ess\inf_{x\in\Omega} m(x), \qquad m^+=ess\sup_{x\in\Omega} m(x),$$
$$p^-=ess\inf_{x\in\Omega} p(x), \qquad p^+=ess\sup_{x\in\Omega} p(x),$$

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and

$$m^*, p^* = \frac{2(n-2)}{n-4} \qquad \text{if } n > 4.$$

The problems with variable exponents arises in many branches in sciences such as electrorheological fluids, nonlinear elasticity theory and image processing [3, 4, 20–23]. Time delay often appears in many practical problems like thermal, biological, chemical, physical and economic phenomena [7].

There has been published much work concerning the wave equations with variable exponents or time delay. Our goal is to consider the Petrovsky equation both with the delay term ($\mu_2 u_t(x,t-\tau)$) and variable exponents which make the problem more interesting than from those concerned in the literature.

Li et al. [11] considered the Petrovsky equation with strong damping term as follows

$$u_{tt} + \Delta^2 u - \Delta u_t + |u_t|^{p-2} u_t = |u|^{q-2} u. \tag{1.3}$$

The authors established the blow up of solutions, existence and decay of the problem (1.3). Then, Polat and Pişkin [19] proved the global existence and decay of solutions of (1.3).

In [12], Messaoudi studied the Petrovsky equation as follows

$$u_{tt} + \Delta^2 u + g(u_t) = \beta |u|^{r-1} u, \tag{1.4}$$

where $g(u_t) = \alpha |u_t|^{p-1} u_t$ and he investigated the blow-up result in finite time for $r > p$. In [25], for when $g(u_t) = \alpha |u_t|^{p-1} u_t$, Tsai and Wu looked into that the solution is global for equation (1.4). Moreover, they established the blow-up result in finite time for the nonnegative initial energy.

Messaoudi and Kafini [6] looked into the nonlinear wave equation with variable exponents and delay term as follows

$$u_{tt} - \Delta u + \mu_1 u_t(x,t) |u_t|^{m(x)-2}(x,t) + \mu_2 u_t(x,t-\tau) |u_t|^{m(x)-2}(x,t-\tau) = bu|u|^{p(x)-2}.$$

They proved the global nonexistence and decay estimates of the equation (1).

In recent years, some other authors investigate hyperbolic type equation with variable exponents (see [8, 13, 16, 18, 24]).

There is no research, to our best knowledge, about Petrovsky equation with delay term and variable exponents, hence, our paper is generalization of the previous ones. In this paper, our main goal is to study the local existence and blow-up result of Petrovsky equation (1.1) with variable exponents and delay term.

The plan of this paper is as follows. Firstly, in Section 2, the definition of the variable exponent Sobolev and Lebesgue spaces are introduced. In Section 3, we obtain the local existence result. Finally, in Section 4, we prove the blow-up result for negative initial energy.

2. PRELIMINARIES

In this part, we state the results related to Lebesgue $L^{p(\cdot)}(\Omega)$ and Sobolev $W^{1,p(\cdot)}(\Omega)$ spaces with variable exponents (see [1,2,4,5,10,17]).

Let $p\colon \Omega \to [1,\infty)$ be a measurable function. We define the variable exponent Lebesgue space with a variable exponent $p(\cdot)$ by

$$L^{p(\cdot)}(\Omega) = \left\{ u\colon \Omega \to R;\ \text{measurable in } \Omega\ :\ \int_\Omega |u|^{p(\cdot)}\,dx < \infty \right\},$$

with a Luxemburg-type norm

$$\|u\|_{p(\cdot)} = \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \le 1 \right\}.$$

Equipped with this norm, $L^{p(\cdot)}(\Omega)$ is a Banach space (see [4]).

Next, we define the variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ as following:

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) :\ \nabla u \text{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}.$$

Variable exponent Sobolev space with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$$

is a Banach space. $W_0^{1,p(\cdot)}(\Omega)$ is the space which is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. For $u \in W_0^{1,p(\cdot)}(\Omega)$, we can define an equivalent norm

$$\|u\|_{1,p(\cdot)} = \|\nabla u\|_{p(\cdot)}.$$

The dual of $W_0^{1,p(\cdot)}(\Omega)$ is defined as $W_0^{-1,p'(\cdot)}(\Omega)$, similar to Sobolev spaces, where

$$\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1.$$

We also assume that:

$$|p(x)-p(y)| \le -\frac{A}{\log|x-y|} \qquad \text{and} \qquad |m(x)-m(y)| \le -\frac{B}{\log|x-y|} \tag{2.1}$$

for all $x,y \in \Omega$, $A,B>0$ and $0<\delta<1$ with $|x-y|<\delta$ (log-Hölder condition).

Lemma 1 ([1] Poincare inequality)**.** *Let Ω be a bounded domain of R^n and suppose that $p(\cdot)$ satisfies (2.1). Then,*

$$\|u\|_{p(\cdot)} \le c\,\|\nabla u\|_{p(\cdot)} \qquad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega),$$

where $c = c(p^-, p^+, |\Omega|) > 0$.

Lemma 2 ([1])**.** *If $p\colon \overline{\Omega} \to [1,\infty)$ is continuous,*

$$2 \le p^- \le p(x) \le p^+ \le \frac{2n}{n-2}, \qquad n \ge 3,$$

satisfies, then the embedding $H_0^1(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ *is continuous.*

Lemma 3 ([1]). *If* $p^+ < \infty$ *and* $p\colon \Omega \to [1,\infty)$ *is a measurable function, then* $C_0^\infty(\Omega)$ *is dense in* $L^{p(\cdot)}(\Omega)$.

Lemma 4 ([1] Hölder' inequality). *Let* $p,q,s \geq 1$ *be measurable functions defined on* Ω *and*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \qquad \text{for a.e. } y \in \Omega,$$

satisfies. If $f \in L^{p(\cdot)}(\Omega)$ *and* $g \in L^{q(\cdot)}(\Omega)$, *then* $fg \in L^{s(\cdot)}(\Omega)$ *and*

$$\|fg\|_{s(\cdot)} \leq 2\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}.$$

Lemma 5 ([6, Lemma 2.5] unit ball property). *Let* $p \geq 1$ *be a measurable function on* Ω. *Then* $\|f\|_{p(\cdot)} \leq 1$ *if and only if* $\rho_{p(\cdot)}(f) \leq 1$, *where*

$$\rho_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)}\,dx.$$

Lemma 6 ([1]). *If* $p \geq 1$ *is a measurable function on* Ω, *then*

$$\min\left\{\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right\} \leq \rho_{p(\cdot)}(u) \leq \max\left\{\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right\},$$

for any $u \in L^{p(\cdot)}(\Omega)$ *and for a.e.* $x \in \Omega$.

Remark 1. Let c be various positive constants which may be different from line to line. Then, we use the embedding

$$H_0^2(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^p(\Omega)$$

which satisfies

$$\|u\|_p \leq c\|\nabla u\| \leq c\|\Delta u\|,$$

where $2 \leq p < \infty$ $(n = 1,2)$, $2 \leq p \leq \frac{2n}{n-2}$ $(n \geq 3)$.

Moreover

$$\|u\|_q \leq C\|\Delta u\|,$$

$$q = \begin{cases} \infty & \text{if } n < 4, \\ \text{any number in } [1,\infty) & \text{if } n = 4, \\ \frac{2(n-2)}{n-4} & \text{if } n > 4. \end{cases}$$

3. LOCAL EXISTENCE

In this part, our goal is to prove the local existence result for our main problem (1.1) by using Faedo-Galerkin method. We use similar arguments as in [14, 16] to get the result. Firstly, we give the lemma which we need:

Lemma 7 ([9, Lemma 3.1]). *(Lemma 3.1 in [9]) Let $x \in \Omega$ and $p(\cdot)$ satisfies*

$$2 \leq p^- \leq p(x) \leq p^+ \leq \infty,$$

then, $h(s) = b|s|^{p(x)-2}s$ is differentiable function and $|h'(s)| = b|p(x)-1||s|^{p(x)-2}$.

Suppose that μ_1 and μ_2 satisfy

$$|\mu_2| < \frac{m^-}{m^+}\mu_1, \tag{3.1}$$

where $m^- = es\sin f_{x\in\Omega} m(x)$, $m^+ = es\sup f_{x\in\Omega} m(x)$. Assume that ζ is a positive constant such that

$$\tau\left(m^+ - 1\right)\mu_2 < \zeta < \tau\left(m^-\mu_1 - |\mu_2|\right). \tag{3.2}$$

Now, similar to [15], we introduce, the new variable

$$z(x,\rho,t) = u_t(x, t-\tau\rho), \qquad x \in \Omega, \rho \in (0,1), t > 0.$$

Hence, problem (1.1) takes the form

$$\begin{cases} u_{tt} + \Delta^2 u - \Delta u_t + \mu_1 |u_t(x,t)|^{m(x)-2} u_t(x,t) & \\ \quad + \mu_2 |z(x,1,t)|^{m(x)-2} z(x,1,t) = bu|u|^{p(x)-2} & \text{in } \Omega \times R^+, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 & \text{in } \Omega \times (0,1) \times (0,\infty), \\ u(x,t) = \frac{\partial u(x,t)}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{in } \Omega, \\ z(x,\rho,0) = f_0(x, -\tau\rho) & \text{in } \Omega \times (0,1), \\ z(x,0,t) = u_t(x,t) & \text{in } \Omega \times (0,\infty). \end{cases} \tag{3.3}$$

Theorem 1. *Assume that (3.1) holds and $m(\cdot)$ satisfies (1.2), (2.1) and $p(\cdot)$ satisfies (2.1) and*

$$2 \leq p^- \leq p(x) \leq p^+ \leq \frac{2(n-2)}{n-4} \qquad \text{if } n > 4. \tag{3.4}$$

Assume further that $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$, $f_0 \in L^{m(\cdot)}(\Omega \times (0,1))$ and $T > 0$. Then, the problem (3.3) has a unique local solution

$$\begin{aligned} &u \in C\left([0,T]; H_0^2(\Omega)\right), \\ &u_t \in C\left([0,T]; L^2(\Omega)\right) \cap L^{m(\cdot)}(\Omega \times (0,T)), \\ &z \in L^{m(\cdot)}(\Omega \times (0,1)). \end{aligned}$$

Proof. **Existence:** Let $v \in L^{\infty}\left((0,T);H_0^2(\Omega)\right)$. Since

$$2\left(p^- - 1\right) \leq 2\left(p^+ - 1\right) \leq \frac{2n}{n-4},$$

then

$$\|h(v)\|^2 \leq |b|^2 \left\{ \int_\Omega |v|^{2(p^- - 1)}\, dx + \int_\Omega |v|^{2(p^+ - 1)}\, dx \right\} < \infty.$$

Hence, we have

$$h(v) \in L^{\infty}\left((0,T);L^2(\Omega)\right) \subset L^2(\Omega \times (0,T)).$$

Thus, for each $v \in L^{\infty}\left((0,T);H_0^2(\Omega)\right)$, there exists a unique solution

$$\begin{aligned} &u \in L^{\infty}\left((0,T);H_0^2(\Omega)\right), \\ &u_t \in L^{\infty}\left((0,T);L^2(\Omega)\right) \cap L^{m(\cdot)}(\Omega \times (0,T)), \\ &z \in L^{m(\cdot)}(\Omega \times (0,1)) \end{aligned}$$

satisfying the following problem

$$\begin{cases} u_{tt} + \Delta^2 u - \Delta u_t + \mu_1 |u_t(x,t)|^{m(x)-2} u_t(x,t) & \\ \quad + \mu_2 |z(x,1,t)|^{m(x)-2} z(x,1,t) = h(v) & \text{in } \Omega \times (0,T), \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 & \text{in } \Omega \times (0,1) \times (0,T), \\ u(x,t) = \frac{\partial u(x,t)}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{in } \Omega, \\ z(x,0,t) = u_t(x,t) & \text{in } \Omega \times (0,T), \\ z(x,\rho,0) = f_0(x,-\tau\rho) & \text{in } \Omega \times (0,1). \end{cases} \tag{3.5}$$

Define the following space that the sequence (u^k) is Cauchy in

$$X := C\left([0,T];H_0^2(\Omega)\right) \cap C^1\left([0,T];L^2(\Omega)\right),$$

equipped with the norm

$$\|u\|_X^2 = \max_{0 \leq t \leq T} \left\{ \|u_t\|^2 + \|\Delta u\|^2 \right\}.$$

We define the nonlinear mapping $K\colon X \to X$ by $K(v) = u$, here, u is the unique solution of (3.5). Now, we shall show that there exist $T > 0$, such that

(i) $K\colon X \to X$,
(ii) K is a contraction mapping in X.

To show (i), we multiply the first equation in (3.5) by u_t and integrate over $\Omega\times(0,t)$, to obtain

$$\begin{aligned}&\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\Delta u\|^2+\int_0^t\|\nabla u_t\|^2\,ds+\mu_1\int_0^t\int_\Omega|u_t(s)|^{m(x)}\,dxds\\&+\mu_2\int_0^t\int_\Omega|z(x,1,s)|^{m(x)-2}z(x,1,s)\,u_t(s)\,dxds\\&=\frac{1}{2}\int_\Omega u_1^2dx+\frac{1}{2}\int_\Omega|\Delta u_0|^2\,dx+b\int_0^t\int_\Omega|v|^{p(x)-2}vu_t(s)\,dxds.\end{aligned}\tag{3.6}$$

We multiply the second equation in (3.5) by $\frac{\zeta}{\tau}z^{m(x)-1}$, and integrate over $\Omega\times(0,1)\times(0,t)$, to have

$$\begin{aligned}&\int_0^1\int_\Omega\frac{\zeta}{m(x)}\left(|z(x,\rho,t)|^{m(x)}-|z(x,\rho,0)|^{m(x)}\right)dxd\rho\\&\qquad=\int_0^t\int_\Omega\frac{\zeta}{m(x)\tau}\left(|z(x,0,s)|^{m(x)}-|z(x,1,s)|^{m(x)}\right)dxds.\end{aligned}\tag{3.7}$$

Combining (3.6) and (3.7), we have

$$\begin{aligned}&\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\Delta u\|^2+\int_0^t\|\nabla u_t\|^2\,ds+\int_0^1\int_\Omega\frac{\zeta}{m(x)}|z(x,\rho,t)|^{m(x)}\,dxd\rho\\&+\mu_1\int_0^t\int_\Omega|u_t(s)|^{m(x)}\,dxds+\mu_2\int_0^t\int_\Omega|z(x,1,s)|^{m(x)-2}z(x,1,s)\,u_t(s)\,dxds\\&+\int_0^t\int_\Omega\frac{\zeta}{m(x)\tau}\left(|z(x,1,s)|^{m(x)}-|u_t(s)|^{m(x)}\right)dxds\\&=\frac{1}{2}\int_\Omega u_1^2dx+\frac{1}{2}\int_\Omega|\Delta u_0|^2\,dx+\int_0^1\int_\Omega\frac{\zeta}{m(x)}|f_0(x,-\tau\rho)|^{m(x)}\,dxd\rho\\&+b\int_0^t\int_\Omega|v|^{p(x)-2}vu_t(s)\,dxds\end{aligned}\tag{3.8}$$

Utilizing Young's inequality and (1.2), we get

$$\begin{aligned}&-\mu_2\int_\Omega|z(x,1,s)|^{m(x)-2}z(x,1,s)\,u_t(s)\,dxds\\&\qquad\le\frac{|\mu_2|}{m^-}\int_\Omega|u_t(s)|^{m(x)}\,dx+\frac{(m^+-1)\,|\mu_2|}{m^+}\int_\Omega|z(x,1,s)|^{m(x)}\,dx.\end{aligned}\tag{3.9}$$

Applying Young's inequality and Sobolev embedding $H_0^2(\Omega)\hookrightarrow L^{\frac{2n}{n-4}}(\Omega)$, we obtain

$$\begin{aligned}\left|\int_\Omega|v|^{p(x)-2}vu_t(s)\,dx\right|&\le\frac{\varepsilon}{4}\int_\Omega|u_t(s)|^2\,dx+\frac{1}{\varepsilon}\int_\Omega|v|^{2(p(x)-1)}\,dx\\&\le\frac{\varepsilon}{4}\int_\Omega|u_t(s)|^2\,dx+\frac{c_e}{\varepsilon}\left\{\|\Delta v\|^{2(p^--1)}+\|\Delta v\|^{2(p^+-1)}\right\},\end{aligned}\tag{3.10}$$

here, c_e is the embedding constant. We should inserting (3.9) and (3.10) into (3.8), then, we have

$$\begin{aligned}
&\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\Delta u\|^2+\int_0^t\|\nabla u_t\|^2\,ds+\int_0^1\int_\Omega\frac{\zeta}{m(x)}|z(x,\rho,t)|^{m(x)}\,dxd\rho\\
&+\left(\mu_1-\frac{|\mu_2|}{m^-}-\frac{\zeta}{m^-\tau}\right)\int_0^t\int_\Omega|u_t(s)|^{m(x)}\,dxds\\
&+\left(\frac{\zeta}{m^+\tau}-\frac{(m^+-1)|\mu_2|}{m^+}\right)\int_0^t\int_\Omega|z(x,1,s)|^{m(x)}\,dxds\\
\leq&\frac{1}{2}\int_\Omega u_1^2dx+\frac{1}{2}\int_\Omega|\Delta u_0|^2\,dx+\int_0^1\int_\Omega\frac{\zeta}{m(x)}|f_0(x,-\tau\rho)|^{m(x)}\,dxd\rho\\
&+\frac{\varepsilon cT}{4}\sup_{(0,T)}\int_\Omega|u_t|^2\,dx+\frac{c_ec}{\varepsilon}\left\{\int_0^T\|\Delta v\|^{2(p^--1)}\,ds+\int_0^T\|\Delta v\|^{2(p^+-1)}\,ds\right\}.
\end{aligned}$$

By (3.2), we have

$$\begin{aligned}
&\frac{1}{2}\sup_{(0,T)}\|u_t\|^2+\frac{1}{2}\sup_{(0,T)}\|\Delta u\|^2+\frac{\zeta}{m^+}\|z(x,\rho,t)\|^{m(\cdot)}_{L^{m(\cdot)}(\Omega\times(0,1))}\\
&\qquad\leq\frac{1}{2}\int_\Omega u_1^2dx+\frac{1}{2}\int_\Omega|\Delta u_0|^2\,dx+\frac{\zeta}{m^-}\int_0^1\int_\Omega|f_0(x,-\tau\rho)|^{m(x)}\,dxd\rho\\
&\qquad\qquad+\frac{\varepsilon cT}{4}\sup_{(0,T)}\|u_t\|^2+\frac{c_ecT}{\varepsilon}\left\{\|v\|_X^{2(p^--1)}+\|v\|_X^{2(p^+-1)}\right\}.
\end{aligned}$$

By taking ε such that $\varepsilon cT=1$, we get

$$\begin{aligned}
\|u\|_X^2\leq&\frac{c^*}{2}\int_\Omega u_1^2dx+\frac{c^*}{2}\int_\Omega|\Delta u_0|^2\,dx+\frac{c^*\zeta}{m^-}\int_0^1\int_\Omega|f_0(x,-\tau\rho)|^{m(x)}\,dxd\rho\\
&+c_*T\left\{\|v\|_X^{2(p^--1)}+\|v\|_X^{2(p^+-1)}\right\},
\end{aligned}$$

where $\frac{1}{c^*}=\min\left\{\frac{1}{4},\frac{\zeta}{m^+}\right\}$ and $c_*=\frac{c^*c_ec}{\varepsilon}$. Here, we choose $M>0$ large enough, such that $\|v\|_X\leq M$, then

$$c^*\int_\Omega u_1^2dx+c^*\int_\Omega|\Delta u_0|^2\,dx+\frac{2c^*\zeta}{m^-}\int_0^1\int_\Omega|f_0(x,-\tau\rho)|^{m(x)}\,dxd\rho\leq M^2$$

and T sufficiently small such that

$$T\leq\frac{1}{2c_*\left(M^{2(p^--2)}+M^{2(p^+-2)}\right)}.$$

As a result, we have

$$\|u\|_X^2\leq M^2.$$

Thus we have $K\colon Z \to Z$, where

$$Z = \{u \in X \text{ such that } \|u\|_X \leq M\}.$$

Next, we show that K is a contraction mapping. For this purpose, we let $K(v^1) = u^1$ and $K(v^2) = u^2$ and set $u = u^1 - u^2$ and $w = w^1 - w^2$ then u and w satisfy

$$\begin{cases} u_{tt} + \Delta^2 u - \Delta u_t + \mu_1 \left|u_t^1(x,t)\right|^{m(x)-2} u_t^1(x,t) & \\ -\mu_1 \left|u_t^2(x,t)\right|^{m(x)-2} u_t^2(x,t) & \\ +\mu_2 \left|z^1(x,1,t)\right|^{m(x)-2} z^1(x,1,t) & \text{in } \Omega \times (0,T), \\ -\mu_2 \left|z^2(x,1,t)\right|^{m(x)-2} z^2(x,1,t) & \\ = b\left|v^1\right|^{p(x)-2} v^1 - b\left|v^2\right|^{p(x)-2} v^2 & \\ u(x,t) = \frac{\partial u(x,t)}{\partial v} = 0 & \text{on } \partial\Omega \times (0,T), \\ z(x,0,t) = u_t(x,t) & \text{in } \Omega \times (0,T), \\ z(x,\rho,0) = 0 & \text{in } \Omega \times (0,1), \\ u(x,0) = 0,\ u_t(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{3.11}$$

We multiply equation (3.11) by u_t and integrate over $\Omega \times (0,t)$, we get

$$\begin{aligned} &\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \int_0^t \|\nabla u_t\|^2 ds \\ &\qquad + \mu_1 \int_0^t \int_\Omega \left(\left|u_t^1(s)\right|^{m(x)-2} u_t^1(s) - \left|u_t^2(s)\right|^{m(x)-2} u_t^2(s)\right) u_t(s)\, dxds \\ &+ \mu_2 \int_0^t \int_\Omega \left(\left|z^1(x,1,s)\right|^{m(x)-2} z^1(x,1,s) - \left|z^2(x,1,s)\right|^{m(x)-2} z^2(x,1,s)\right) u_t(s)\, dxds \\ &\qquad = \int_0^t \int_\Omega \left(h(v_1) - h(v_2)\right) u_t(s)\, dxds, \end{aligned}$$

where $h(v) = b|v|^{p(x)-2}v$. Since the function $u \to |u|^{m(x)-2}u$ is increasing, we conclude that

$$\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \int_0^t \|\nabla u_t\|^2 ds \leq \int_0^t \int_\Omega \left(h(v_1) - h(v_2)\right) u_t(s)\, dxds. \tag{3.12}$$

Thanks to (3.4), Young's inequality and Sobolev embedding, we obtain

$$\begin{aligned} &\int_\Omega |h(v_1) - h(v_2)|\,|u_t(s)|\, dx = \int_\Omega \left|h'(\rho)\right| \|v\|\,|u_t(s)|\, dx \\ &\leq \frac{\delta_0}{2}\int_\Omega |u_t(s)|^2 dx + \frac{1}{2\delta_0}\int_\Omega \left|h'(\rho)\right|^2 |v|^2 dx \leq \frac{\delta_0}{2}\int_\Omega |u_t(s)|^2 dx \\ &+ \frac{b^2(p^+-1)^2}{2\delta_0}\left[\left(\int_\Omega |\rho|^{\frac{n(p^- -2)}{2}} dx\right)^{\frac{4}{n}} + \left(\int_\Omega |\rho|^{\frac{n(p^+ -2)}{2}} dx\right)^{\frac{4}{n}}\right]\left(\int_\Omega |v|^{\frac{2n}{n-4}} dx\right)^{\frac{n-4}{n}} \end{aligned}$$

$$\leq \frac{\delta_0}{2}\|u_t(s)\|^2+\frac{b^2(p^+-1)^2c_e}{2\delta_0}\left[\|\Delta\rho\|^{2(p^--2)}+\|\Delta\rho\|^{2(p^+-2)}\right]\|\Delta v\|^2 \tag{3.13}$$
$$\leq \frac{\delta_0}{2}\|u_t(s)\|^2+\frac{b^2(p^+-1)^2c_e}{\delta_0}\left(M^{2(p^--2)}+M^{2(p^+-2)}\right)\|\Delta v\|^2,$$

where $v=v_1-v_2$ and $\rho=\vartheta v_1+(1-\vartheta)v_2$, $0\leq\vartheta\leq 1$.

By inserting (3.13) into (3.12) and choosing δ_0 small enough, we obtain

$$\|u\|_X^2\leq d\|v\|_X^2, \tag{3.14}$$

where $d=\frac{4b^2(p^+-1)^2c_eT}{\delta_0}\left(M^{2(p^--2)}+M^{2(p^+-2)}\right)$.

Now, we choose T small enough so that $0<d<1$. Thus, (3.14) indicates that K is a contraction. The Banach fixed theorem shows that the existence of a unique $u\in Z$ satisfying $K(u)=u$. Obviously, it is a solution of (3.3).

Uniqueness: Assume that (3.3) have two solutions (u^1,z^1), (u^2,z^2). We define $\widetilde{u}=u^1-u^2$ and $\widetilde{z}=z^1-z^2$, then $(\widetilde{u},\widetilde{z})$ satisfy

$$\begin{cases}
\widetilde{u}_{tt}+\Delta^2\widetilde{u}-\Delta\widetilde{u}_t+\mu_1\left|u_t^1(t)\right|^{m(x)-2}u_t^1(t) & \\
-\mu_1\left|u_t^2(t)\right|^{m(x)-2}u_t^2(x,t) & \\
+\mu_2\left|z^1(x,1,t)\right|^{m(x)-2}z^1(x,1,t) & \text{in } \Omega\times(0,T),\\
-\mu_2\left|z^2(x,1,t)\right|^{m(x)-2}z^2(x,1,t) & \\
=bu^1\left|u^1\right|^{p(x)-2}-bu^2\left|u^2\right|^{p(x)-2} & \\
\tau\widetilde{z}_t(x,\rho,t)+\widetilde{z}_\rho(x,\rho,t)=0 & \text{in } \Omega\times(0,1)\times(0,T),\\
\widetilde{u}(x,t)=\frac{\partial\widetilde{u}(x,t)}{\partial y}=0 & \text{in } \partial\Omega\times(0,T),\\
\widetilde{z}(x,0,t)=\widetilde{u}_t(x,t) & \text{in } \Omega\times(0,T),\\
\widetilde{z}(x,\rho,0)=0 & \text{in } \Omega\times(0,1),\\
\widetilde{u}(x,0)=0\ \widetilde{u}_t(x,0)=0 & \text{in } \Omega.
\end{cases} \tag{3.15}$$

We multiply the first equation in (3.15) by $\widetilde{u}_t$ and integrate over Ω, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega\left|\widetilde{u}_t\right|^2dx+\int_\Omega\left|\Delta\widetilde{u}\right|^2dx\right\}+\int_\Omega\left|\nabla\widetilde{u}_t\right|^2dx\\
&+\mu_1\int_\Omega\left(\left|u_t^1(t)\right|^{m(x)-2}u_t^1(t)-\left|u_t^2(t)\right|^{m(x)-2}u_t^2(x,t)\right)\widetilde{u}_t(t)\,dx\\
&+\mu_2\int_\Omega\left(\left|z^1(x,1,t)\right|^{m(x)-2}z^1(x,1,t)-\left|z^2(x,1,t)\right|^{m(x)-2}z^2(x,1,t)\right)\widetilde{u}_t(t)\,dx\\
&=b\int_\Omega\left(u^1\left|u^1\right|^{p(x)-2}-bu^2\left|u^2\right|^{p(x)-2}\right)\widetilde{u}_t(t)\,dx.
\end{aligned} \tag{3.16}$$

Multiplying the second equation in (3.15) by $\widetilde{z}$ and integrating over $\Omega \times (0,1)$, we have

$$\frac{\tau}{2}\frac{d}{dt}\int_0^1 \int_\Omega \left|\widetilde{z}(x,\rho,t)\right|^2 dxd\rho + \frac{1}{2}\left(\left\|\widetilde{z}(x,1,t)\right\|^2 - \left\|\widetilde{u}_t(t)\right\|^2\right) = 0. \tag{3.17}$$

Combining (3.16) and (3.17), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega \left|\widetilde{u}_t(t)\right|^2 dx + \int_\Omega \left|\widetilde{\Delta u}(t)\right|^2 dx + \tau\left\|\widetilde{z}(x,\rho,t)\right\|^2_{L^2(\Omega\times(0,1))}\right\} + \int_\Omega \left|\nabla \widetilde{u}_t\right|^2 dx \\
&+ \frac{1}{2}\left\|\widetilde{z}(x,1,t)\right\|^2 + \mu_1 \int_\Omega \left(\left|u_t^1(t)\right|^{m(x)-2} u_t^1(t) - \left|u_t^2(t)\right|^{m(x)-2} u_t^2(t)\right)\widetilde{u}_t(t)\,dx \\
&+ \mu_2 \int_\Omega \left(\left|z^1(x,1,t)\right|^{m(x)-2} z^1(x,1,t) - \left|z^2(x,1,t)\right|^{m(x)-2} z^2(x,1,t)\right)\widetilde{u}_t(t)\,dx \\
&= b\int_\Omega \left(u^1\left|u^1\right|^{p(x)-2} - bu^2\left|u^2\right|^{p(x)-2}\right)\widetilde{u}_t(t)\,dx + \frac{1}{2}\left\|\widetilde{u}_t(t)\right\|^2.
\end{aligned} \tag{3.18}$$

Since the function $y \to |y|^{m(\cdot)-2}y$ is increasing, we get

$$\int_\Omega \left(\left|u_t^1(t)\right|^{m(x)-2} u_t^1(t) - \left|u_t^2(t)\right|^{m(x)-2} u_t^2(t)\right)\widetilde{u}_t(t)\,dx \geq 0, \tag{3.19}$$

$$\int_\Omega \left(\left|z^1(x,1,t)\right|^{m(x)-2} z^1(x,1,t) - \left|z^2(x,1,t)\right|^{m(x)-2} z^2(x,1,t)\right)\widetilde{u}_t(t)\,dx \geq 0. \tag{3.20}$$

By using (3.18), (3.19) and (3.20), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega \left\|\widetilde{u}_t(t)\right\|^2 + \left\|\Delta\widetilde{u}(t)\right\|^2 + \tau\left\|\widetilde{z}(x,\rho,t)\right\|^2_{L^2(\Omega\times(0,1))}\right\} + \frac{1}{2}\left\|\widetilde{z}(x,1,t)\right\|^2 \\
&\leq c\left(\left\|\widetilde{u}_t(t)\right\|^2 + \left\|\Delta\widetilde{u}(t)\right\|^2\right)
\end{aligned}$$

which implies that $\widetilde{u} = 0$, $\widetilde{z} = 0$. □

4. BLOW UP

In this part, for the case $b > 0$, we establish the blow-up result for our main problem (1.1). Now we introduce, as in the work of [15], the new function

$$z(x,\rho,t) = u_t(x,t-\tau\rho), \qquad x \in \Omega, \qquad \rho \in (0,1), \qquad t > 0,$$

which implies that

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \qquad x \in \Omega, \qquad \rho \in (0,1), \qquad t > 0.$$

Consequently, problem (1.1) is equivalent to:

$$\begin{cases} u_{tt}+\Delta^2 u-\Delta u_t+\mu_1 u_t(x,t)\left|u_t(x,t)\right|^{m(x)-2} & \\ +\mu_2 z(x,1,t)\left|z(x,1,t)\right|^{m(x)-2} & \text{in } \Omega\times(0,\infty), \\ =bu\left|u\right|^{p(x)-2} & \\ \tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0 & \text{in } \Omega\times(0,1)\times(0,\infty), \\ z(x,\rho,0)=f_0(x,-\rho\tau) & \text{in } \Omega\times(0,1), \\ u(x,t)=\frac{\partial u(x,t)}{\partial \nu}=0 & \text{on } \partial\Omega\times[0,\infty), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x) & \text{in } \Omega. \end{cases} \tag{4.1}$$

We define the energy functional of (4.1) as

$$E(t)=\frac{1}{2}\left\|u_t\right\|^2+\frac{1}{2}\left\|\Delta u\right\|^2+\int_0^1\int_\Omega \frac{\xi(x)\left|z(x,\rho,t)\right|^{m(x)}}{m(x)}dxd\rho-b\int_\Omega\frac{\left|u\right|^{p(x)}}{p(x)}dx,$$

for $t\geq 0$, where ξ is a continuous function satisfies

$$\tau\left|\mu_2\right|(m(x)-1)<\xi(x)<\tau(\mu_1 m(x)-\left|\mu_2\right|), \qquad x\in\overline{\Omega}. \tag{4.2}$$

The following lemma gives that, under the condition $\mu_1>\left|\mu_2\right|$, $E(t)$ is nonincreasing.

Lemma 8. *Let (u,z) be a solution of (4.1). Then there exists some $C_0>0$ such that*

$$E'(t)\leq -C_0\int_\Omega\left(\left|u_t\right|^{m(x)}+\left|z(x,1,t)\right|^{m(x)}\right)dx\leq 0.$$

Proof. We multiply the first equation in (4.1) by u_t, integrate over Ω, then multiplying the second equation of (4.1) by $\frac{1}{\tau}\xi(x)\left|z\right|^{m(x)-2}z$ and integrate over $\Omega\times(0,1)$, summing up, we get

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\left\|u_t\right\|^2+\frac{1}{2}\left\|\Delta u\right\|^2+\int_0^1\int_\Omega\frac{\xi(x)\left|z(x,\rho,t)\right|^{m(x)}}{m(x)}dxd\rho-b\int_\Omega\frac{\left|u\right|^{p(x)}}{p(x)}dx\right]\\ &=-\left\|\nabla u_t\right\|^2-\mu_1\int_\Omega\left|u_t\right|^{m(x)}dx-\frac{1}{\tau}\int_\Omega\int_0^1\xi(x)\left|z(x,\rho,t)\right|^{m(x)-2}zz_\rho(x,\rho,t)\,d\rho dx\\ &\qquad-\mu_2\int_\Omega u_t z(x,1,t)\left|z(x,1,t)\right|^{m(x)-2}dx.\end{aligned} \tag{4.3}$$

Next, we estimate the last two terms of the right-hand side of (4.3) as following,

$$\begin{aligned}&-\frac{1}{\tau}\int_\Omega\int_0^1\xi(x)\left|z(x,\rho,t)\right|^{m(x)-2}zz_\rho(x,\rho,t)\,d\rho dx\\ &\qquad=-\frac{1}{\tau}\int_\Omega\int_0^1\frac{\partial}{\partial\rho}\left(\frac{\xi(x)\left|z(x,\rho,t)\right|^{m(x)}}{m(x)}\right)d\rho dx\\ &\qquad=\frac{1}{\tau}\int_\Omega\frac{\xi(x)}{m(x)}\left(\left|z(x,0,t)\right|^{m(x)}-\left|z(x,1,t)\right|^{m(x)}\right)dx\end{aligned}$$

$$= \int_\Omega \frac{\xi(x)}{\tau m(x)} |u_t|^{m(x)} dx - \int_\Omega \frac{\xi(x)}{\tau m(x)} |z(x,1,t)|^{m(x)}.$$

Using the Young's inequality, $q = \frac{m(x)}{m(x)-1}$ and $q' = m(x)$ for the last term to obtain

$$|u_t| |z(x,1,t)|^{m(x)-1} \leq \frac{1}{m(x)} |u_t|^{m(x)} + \frac{m(x)-1}{m(x)} |z(x,1,t)|^{m(x)}.$$

Consequently, we deduce that

$$-\mu_2 \int_\Omega u_t z |z(x,1,t)|^{m(x)-2} dx \leq |\mu_2| \left(\int_\Omega \frac{1}{m(x)} |u_t(t)|^{m(x)} dx + \int_\Omega \frac{m(x)-1}{m(x)} |z(x,1,t)|^{m(x)} dx \right).$$

So

$$\frac{dE(t)}{dt} \leq -\int_\Omega \left[\mu_1 - \left(\frac{\xi(x)}{\tau m(x)} + \frac{|\mu_2|}{m(x)} \right) \right] |u_t(t)|^{m(x)} dx - \int_\Omega \left(\frac{\xi(x)}{\tau m(x)} - \frac{|\mu_2|(m(x)-1)}{m(x)} \right) |z(x,1,t)|^{m(x)} dx.$$

As a result, for all $x \in \overline{\Omega}$, the relation (4.2) satisfies,

$$f_1(x) = \mu_1 - \left(\frac{\xi(x)}{\tau m(x)} + \frac{|\mu_2|}{m(x)} \right) > 0,$$

$$f_2(x) = \frac{\xi(x)}{\tau m(x)} - \frac{|\mu_2|(m(x)-1)}{m(x)} > 0.$$

Since $m(x)$, and hence $\xi(x)$, is bounded, we infer that $f_1(x)$ and $f_2(x)$ are also bounded. So, if we define

$$C_0(x) = \min\{f_1(x), f_2(x)\} > 0 \text{ for any } x \in \overline{\Omega},$$

and take $C_0(x) = \inf_{\overline{\Omega}} C_0(x)$, so $C_0(x) \geq C_0 > 0$. Hence,

$$E'(t) \leq -C_0 \left[\int_\Omega |u_t(t)|^{m(x)} dx + \int_\Omega |z(x,1,t)|^{m(x)} dx \right] \leq 0.$$

□

To prove the blow-up result, we assume that $E(0) < 0$ in addition to (1.2). Set $H(t) = -E(t)$, hence $H'(t) = -E'(t) \geq 0$,

$$0 < H(0) \leq H(t) \leq b \int_\Omega \frac{|u|^{p(x)}}{p(x)} dx \leq \frac{b}{p^-} \rho(u),$$

where

$$\rho(u) = \rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)} dx.$$

Lemma 9 ([6, Lemma 3.2]). *Suppose that condition (1.2) satisfies.Then, depending on Ω only, there exists a positive $C>1$, such that*

$$\rho^{s/p^-}(u) \leq C\left(\|\Delta u\|^2+\rho(u)\right).$$

Then, we have following inequalities:

$$\begin{aligned}
\|u\|_{p^-}^s &\leq C\left(\|\Delta u\|^2+\|u(t)\|_{p^-}^{p^-}\right),\\
\rho^{s/p^-}(u) &\leq C\left(|H(t)|+\|u_t\|^2+\rho(u)+\int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho\right),\\
\|u\|_{p^-}^s &\leq C\left(|H(t)|+\|u_t\|^2+\|u\|_{p^-}^{p^-}+\int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho\right), \qquad (4.4)
\end{aligned}$$

for any $u\in H_0^1(\Omega)$ and $2\leq s\leq p^-$. Let (u,z) be a solution of (4.1), then

$$\rho(u)\geq C\|u\|_{p^-}^{p^-}, \qquad \int_\Omega |u|^{m(x)}dx \leq C\left(\rho^{m^-/p^-}(u)+\rho^{m^+/p^-}(u)\right). \qquad (4.5)$$

The blow-up result is given by the following theorem:

Theorem 2. *Let conditions (1.2) and (2.1) be provided and assume that $E(0)<0$. Then, the solution (4.1) blows up in finite time T^*, and*

$$T^*\leq \frac{1-\alpha}{\Psi\alpha\left[L(0)\right]^{\alpha/(1-\alpha)}},$$

where $L(t)$ and α are given in (4.6) and (4.7), respectively.

Proof. Define

$$L(t)=H^{1-\alpha}(t)+\varepsilon\int_\Omega uu_t dx+\frac{\varepsilon}{2}\|\nabla u\|^2, \qquad (4.6)$$

where ε small to be chosen later and

$$0\leq\alpha\leq\min\left\{\frac{p^--2}{2p^-},\ \frac{p^--m^-}{p^-(m^+-1)},\ \frac{p^--m^+}{p^-(m^+-1)}\right\}. \qquad (4.7)$$

Differentiation $L(t)$ with respect to t, and using the first equation in (4.1), we obtain

$$\begin{aligned}
L'(t)=(1-\alpha)H^{-\alpha}(t)H'(t)+\varepsilon\int_\Omega\left[u_t^2-|\Delta u|^2\right]dx\\
+\varepsilon b\int_\Omega |u|^{p(x)}dx-\varepsilon\mu_1\int_\Omega uu_t(x,t)\left|u_t(x,t)\right|^{m(x)-2}dx\\
-\varepsilon\mu_2\int_\Omega uz(x,1,t)\left|z(x,1,t)\right|^{m(x)-2}dx.
\end{aligned}$$

By using the definition of the $H(t)$ and for $0<a<1$, such that

$$\begin{aligned}
L'(t) \geq\ & C_0(1-\alpha)H^{-\alpha}(t)\left[\int_\Omega |u_t|^{m(x)}\,dx+\int_\Omega |z(x,1,t)|^{m(x)}\,dx\right]\\
&+\varepsilon\left((1-a)p^- H(t)+\frac{(1-a)p^-}{2}\|u_t\|^2+\frac{(1-a)p^-}{2}\|\Delta u\|^2\right)\\
&+\varepsilon(1-a)p^-\int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho\\
&+\varepsilon\int_\Omega\left[u_t^2-|\Delta u|^2\right]dx+\varepsilon ab\int_\Omega |u|^{p(x)}\,dx\\
&-\varepsilon\mu_1\int_\Omega uu_t(x,t)\,|u_t(x,t)|^{m(x)-2}\,dx-\varepsilon\mu_2\int_\Omega uz(x,1,t)\,|z(x,1,t)|^{m(x)-2}\,dx.
\end{aligned}$$

Hence

$$\begin{aligned}
L'(t) \geq\ & C_0(1-\alpha)H^{-\alpha}(t)\left[\int_\Omega |u_t|^{m(x)}\,dx+\int_\Omega |z(x,1,t)|^{m(x)}\,dx\right]\\
&+\varepsilon(1-a)p^- H(t)+\varepsilon\frac{(1-a)p^-+2}{2}\|u_t\|^2+\varepsilon\frac{(1-a)p^--2}{2}\|\Delta u\|^2\\
&+\varepsilon(1-a)p^-\int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho+\varepsilon ab\rho(u)\\
&-\varepsilon\mu_1\int_\Omega uu_t(x,t)\,|u_t(x,t)|^{m(x)-2}\,dx-\varepsilon\mu_2\int_\Omega uz(x,1,t)\,|z(x,1,t)|^{m(x)-2}\,dx.
\end{aligned}$$

Utilizing Young's inequality, we get

$$\int_\Omega |u_t|^{m(x)-1}|u|\,dx\leq\frac{1}{m^-}\int_\Omega \delta^{m(x)}|u|^{m(x)}\,dx+\frac{m^+-1}{m^+}\int_\Omega \delta^{-\frac{m(x)}{m(x)-1}}|u_t|^{m(x)}\,dx \tag{4.8}$$

and

$$\begin{aligned}
&\int_\Omega |z(x,1,t)|^{m(x)-1}|u|\,dx\\
&\qquad\leq\frac{1}{m^+}\int_\Omega \delta^{m(x)}|u|^{m(x)}\,dx+\frac{m^+-1}{m^+}\int_\Omega \delta^{-\frac{m(x)}{m(x)-1}}|z(x,1,t)|^{m(x)}\,dx.
\end{aligned} \tag{4.9}$$

As in [14], estimates (4.8) and (4.9) remain valid if δ is time-dependent. Let us choose δ so that

$$\delta^{-\frac{m(x)}{m(x)-1}}=kH^{-\alpha}(t),$$

where $k\geq 1$ is specified later, we obtain

$$\int_\Omega \delta^{-\frac{m(x)}{m(x)-1}}|u_t|^{m(x)}\,dx=kH^{-\alpha}(t)\int_\Omega |u_t|^{m(x)}\,dx, \tag{4.10}$$

$$\int_\Omega \delta^{-\frac{m(x)}{m(x)-1}}|z(x,1,t)|^{m(x)}\,dx=kH^{-\alpha}(t)\,|z(x,1,t)|^{m(x)}\,dx \tag{4.11}$$

and

$$\int_\Omega \delta^{m(x)} |u|^{m(x)} dx = \int_\Omega k^{1-m(x)} H^{\alpha(m(x)-1)}(t) |u|^{m(x)} dx \leq \int_\Omega k^{1-m^-} H^{\alpha(m^+-1)}(t) \int_\Omega |u|^{m(x)} dx. \quad (4.12)$$

By using (4.5), we obtain

$$H^{\alpha(m^+-1)}(t) \int_\Omega |u|^{m(x)} dx \leq C\left[(\rho(u))^{m^-/p^-+\alpha(m^+-1)} + (\rho(u))^{m^+/p^-+\alpha(m^+-1)}\right]. \quad (4.13)$$

From (4.7), we deduce that

$$s = m^- + \alpha p^-\left(m^+ - 1\right) \leq p^- \qquad \text{and} \qquad s = m^+ + \alpha p^-\left(m^+ - 1\right) \leq p^-.$$

Then, by using Lemma 9, satisfies

$$H^{\alpha(m^+-1)}(t) \int_\Omega |u|^{m(x)} dx \leq C\left(\|\Delta u\|^2 + \rho(u)\right). \quad (4.14)$$

Combining (4.8)-(4.14), we get

$$\begin{aligned} L'(t) \geq\ & (1-\alpha) H^{-\alpha}(t)\left[C_0 - \varepsilon\left(\frac{m^+-1}{m^+}\right)ck\right]\int_\Omega |u_t|^{m(x)} dx \\ & + (1-\alpha) H^{-\alpha}(t)\left[C_0 - \varepsilon\left(\frac{m^+-1}{m^+}\right)ck\right]\int_\Omega |z(x,1,t)|^{m(x)} dx \\ & + \varepsilon\left(\frac{(p^- - 2) - ap^-}{2} - \frac{C}{m^- k^{1-m^-}}\right)\|\Delta u\|^2 \\ & + \varepsilon(1-a) p^- H(t) + \varepsilon\frac{(1-a) p^- + 2}{2}\|u_t\|^2 + \varepsilon\left(ab - \frac{C}{m^- k^{1-m^-}}\right)\rho(u) \\ & + \varepsilon(1-a) p^- \int_0^1 \int_\Omega \frac{\xi(x) |z(x,\rho,t)|^{m(x)}}{m(x)} dx d\rho. \end{aligned} \quad (4.15)$$

Let us choose a small enough such that

$$\frac{(1-a) p^- + 2}{2} > 0$$

and k large enough so that

$$\frac{(p^- - 2) - ap^-}{2} - \frac{C}{m^- k^{1-m^-}} > 0 \qquad \text{and} \qquad ab - \frac{C}{m^- k^{1-m^-}} > 0.$$

Once k and a are fixed, picking ε small enough such that

$$C_0 - \varepsilon\left(\frac{m^+-1}{m^+}\right)ck > 0, \qquad C_0 - \varepsilon\left(\frac{m^+-1}{m^+}\right)ck > 0$$

and

$$L(0)=H^{1-\alpha}(0)+\varepsilon\int_{\Omega}u_0u_1dx+\frac{\varepsilon}{2}\|\nabla u_0\|^2>0.$$

Consequently, (4.15) yields

$$L'(t)\geq\varepsilon\eta\left[H(t)+\|u_t\|^2+\|\Delta u\|^2+\rho(u)+\int_0^1\int_{\Omega}\frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho\right] \tag{4.16}$$

for a constant $\eta>0$. Thus we get $L(t)\geq L(0)>0$, $\forall t\geq 0$.

Now, for some constants σ, $\Gamma>0$ we denote $L'(t)\geq\Gamma L^{\sigma}(t)$. On the other hand, applying Hölder inequality, we obtain

$$\left|\int_{\Omega}uu_tdx\right|^{1/(1-\alpha)}\leq C\|u\|_{p^-}^{1/(1-\alpha)}\|u_t\|_2^{1/(1-\alpha)},$$

and by using Young's inequality gives

$$\left|\int_{\Omega}uu_tdx\right|^{1/(1-\alpha)}\leq C\left[\|u\|_{p^-}^{mu/(1-\alpha)}+\|u_t\|_2^{\Theta/(1-\alpha)}\right],$$

where $1/\mu+1/\Theta=1$. From (4.7), the choice of $\Theta=2(1-\alpha)$ will make $\mu/(1-\alpha)=2/(1-2\alpha)\leq p^-$. Hence,

$$\left|\int_{\Omega}uu_tdx\right|^{1/(1-\alpha)}\leq C\left[\|u\|_{p^-}^{s}+\|u_t\|^2\right],$$

where $s=\mu/(1-\alpha)$. From (4.4), we have

$$\left|\int_{\Omega}uu_tdx\right|^{1/(1-\alpha)}\leq C\left[|H(t)|+\|u_t\|^2+\rho(u)+\int_0^1\int_{\Omega}\frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho\right].$$

Hence, we get

$$\begin{aligned}L^{1/(1-\alpha)}(t)&=\left[H^{(1-\alpha)}(t)+\varepsilon\int_{\Omega}uu_tdx+\frac{\varepsilon}{2}\|\nabla u\|^2\right]^{1/(1-\alpha)}\\&\leq 2^{\alpha/(1-\alpha)}\left[H(t)+\left|\int_{\Omega}uu_tdx\right|^{1/(1-\alpha)}\right]\\&\leq C\left[|H(t)|+\|u_t\|^2+\|\Delta u\|^2+\rho(u)+\int_0^1\int_{\Omega}\frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho\right]\end{aligned}$$

So, for some $\Psi>0$, from (4.16) we arrive

$$L'(t)\geq\Psi L^{1/(1-\alpha)}(t). \tag{4.17}$$

A simple integration of (4.17) over $(0,t)$ satisfies

$$L^{\alpha/(1-\alpha)}(t)\geq\frac{1}{L^{-\alpha/(1-\alpha)}(0)-\Psi\alpha t/(1-\alpha)},$$

which implies that the solution blows up in a finite time T^*, with

$$T^* \leq \frac{1-\alpha}{\Psi\alpha\left[L(0)\right]^{\alpha/(1-\alpha)}}.$$

As a result, the proof is completed. □

5. Conclusions

In recent years, there has been published much work concerning the wave equation with constant delay or time-varying delay. However, to the best of our knowledge, there was no blow-up result for the Petrovsky equation with delay term and variable exponents. Firstly, we have been obtained the local existence result by using the Faedo-Galerkin method. Later, we have been proved that blow-up of solutions for problem (1.1) under the sufficient conditions in a bounded domain.

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LOCAL EXISTENCE AND BLOW UP OF SOLUTIONS FOR A COUPLED VISCOELASTIC KIRCHHOFF-TYPE EQUATION WITH DEGENERATE DAMPING

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Abstract. In this paper, we consider the initial boundary value problem of a coupled viscoelastic Kirchhoff-type equations with degenerate damping:

$$\begin{cases} u_{tt}-M(\|\nabla u\|^2)\Delta u+\int_0^t \mu_1(t-s)\Delta u(s)ds+\left(|u|^k+|v|^l\right)|u_t|^{p-1}u_t=f_1(u,v), \\ v_{tt}-M(\|\nabla v\|^2)\Delta v+\int_0^t \mu_2(t-s)\Delta v(s)ds+\left(|v|^\theta+|u|^\rho\right)|v_t|^{q-1}v_t=f_2(u,v). \end{cases}$$

Firstly, we prove a local existence theorem by using the Faedo-Galerkin approximations. Then, we study blow up of solutions when initial energy is positive.

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Keywords: Kirchhoff type equation, local existence, blow up, viscoelastic equation, degenerate damping, energy

1. Introduction and Preliminaries

This paper is concerned with the local existence and blow up of solutions to the following viscoelastic Kirchhoff-type equation with degenerate damping:

$$\begin{cases} u_{tt}-M(\|\nabla u\|^2)\Delta u+\int_0^t \mu_1(t-s)\Delta u(s)ds+\left(|u|^k+|v|^l\right)|u_t|^{p-1}u_t=f_1(u,v), \\ \qquad\qquad (x,t)\in\Omega\times(0,T), \\ v_{tt}-M(\|\nabla v\|^2)\Delta v+\int_0^t \mu_2(t-s)\Delta v(s)ds+\left(|v|^\theta+|u|^\rho\right)|v_t|^{q-1}v_t=f_2(u,v), \\ \qquad\qquad (x,t)\in\Omega\times(0,T), \\ u(x,t)=v(x,t)=0, \qquad (x,t)\in\partial\Omega\times(0,T), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), \qquad x\in\Omega, \\ v(x,0)=v_0(x),\ v_t(x,0)=v_1(x), \qquad x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of R^n $(n\geq 1)$ with smooth boundary $\partial\Omega$, $p,q\geq 1$, $j,k,l,\theta,\rho\geq 0$; $\mu_i(.):R^+\to R^+$, $f_i(.,.):R^2\longrightarrow R$ $(i=1,2)$ are given functions to

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be specified later. $M(s)$ is a nonnegative C^1 function satisfying

$$M(s) = b_1 + b_2 s^{\gamma},$$

where $\gamma, s \geq 0$ and $b_1 = b_2 = 1$.

For last years, the study on nonlinear evolution equations with damping terms has been an interesting topic. These problems have wide range of applications in the context of fluid flows, viscosity effects often arise as damping terms in evolution equations. In addition, in the theory of classical mechanics, the physical problems of vibrating membranes, strings or shells in elastic media, damping terms reflect the internal energy that is dissipated by the motion. There is an extensive literature on this kind of problems having damping terms. For instance [4, 11, 12, 15] and one of them is our work [9] where we investigated problem (1.1) and obtained global existence and the general decay of the global solution. Then we proved blow up of solutions of non-linear wave equations having negative initial energy.

We now state other related problem in the literature. Firstly, we mention the pioneer work of Wu [17], where he studied a general decay of the solution for the following famous system

$$\begin{cases} u_{tt} - \Delta u + \int_0^t \mu_1(t-s)\Delta u(s)ds + \left(|u|^k + |v|^l\right)|u_t|^{p-1}u_t = f_1(u,v), \\ v_{tt} - \Delta v + \int_0^t \mu_2(t-s)\Delta v(s)ds + \left(|v|^\theta + |u|^\rho\right)|v_t|^{q-1}v_t = f_2(u,v). \end{cases} \tag{1.2}$$

Then, Pişkin et al. [10] studied local existence and uniqueness results by using the Faedo-Galerkin method with a new scenario under appropriate assumptions on degenerate damping terms, the parameters and the relaxation functions w_i, $(i = 1,2)$. In addition, some author studied blow up, existence and decay of the solutions (1.2) for $k = l = \theta = \rho = 0$ (see [2, 3, 6–8, 14]).

Rammaha and Sakuntasathien [13] and Zennir et al. [1, 18] investigated the following famous system with degenerate damping

$$\begin{cases} u_{tt} - \Delta u + \left(|u|^k + |v|^l\right)|u_t|^{p-1}u_t = f_1(u,v), \\ v_{tt} - \Delta v + \left(|v|^\theta + |u|^\rho\right)|v_t|^{q-1}v_t = f_2(u,v). \end{cases} \tag{1.3}$$

The authors considered well posedness of solutions, blow up and growth properties.

Moreover, Liu et al. [5] investigated the following system with weak damping

$$\begin{cases} u_{tt} - M(\|\nabla u\|^2)\Delta u + \int_0^t \mu_1(t-s)\Delta u(s)ds - \Delta u_t = f_1(u,v), \\ v_{tt} - M(\|\nabla v\|^2)\Delta v + \int_0^t \mu_2(t-s)\Delta v(s)ds - \Delta v_t = f_2(u,v). \end{cases}$$

Under some conditions, they proved blow up and general decay result.

The content of this paper is organized as follows: In Section 1, we give necessary assumptions and notation that will be used later. In Section 2, firstly, we give definition of weak solution then we investigate the local existence of weak solutions by Galerkin's approximation. In Section 3, we obtain finite time blow up of solutions with positive initial energy.

Throughout this paper, we denote the standard $L^2(\Omega)$-norm by $\|.\| = \|.\|_{L^2(\Omega)}$ and $L^p(\Omega)$-norm by $\|.\| = \|.\|_{L^p(\Omega)}$.

We need the following assumptions to state and prove our results.

(A1) μ_i $(i = 1,2)$ are nonincreasing relaxation functions and satisfy

$$\mu_i(s) \geq 0, \ \mu_i'(s) \leq 0, \quad 1 - \int_0^\infty \mu_i(s)ds = l_i > 0, \ s \geq 0. \tag{1.4}$$

(A2) For the nonlinearity in damping, we suppose that

$$\begin{cases} 1 \leq p,q \ \text{ if } n = 1,2, \\ 1 \leq p,q \leq \frac{n+2}{n-2} \ \text{ if } n \geq 3. \end{cases}$$

We pick up $f_1(u,v)$ and $f_2(u,v)$ functions as follows

$$\begin{aligned} f_1(u,v) &= a|u+v|^{2(r+1)}(u+v) + b|u|^r u|v|^{r+2}, \\ f_2(u,v) &= a|u+v|^{2(r+1)}(u+v) + b|v|^r v|u|^{r+2}, \end{aligned} \tag{1.5}$$

where $a,b > 0$ are constants and r satisfies

$$\begin{cases} -1 < r \ \text{ if } n = 1,2, \\ -1 < r \leq \frac{3-n}{n-2} \ \text{if } n \geq 3. \end{cases} \tag{1.6}$$

One can easily verify that

$$uf_1(u,v) + vf_2(u,v) = 2(r+2)F(u,v), \ \forall (u,v) \in R^2, \tag{1.7}$$

where

$$F(u,v) = \frac{1}{2(r+2)}\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]. \tag{1.8}$$

We introduce the energy function associated to problem (1.1) by

$$\begin{aligned} E(t) &= \frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) + \frac{1}{2}\left[(\mu_1 \diamond \nabla u)(t) + (\mu_2 \diamond \nabla v)(t)\right] \\ &\quad + \frac{1}{2}\left[(1 - \int_0^t \mu_1(s)ds)\|\nabla u(t)\|^2 + (1 - \int_0^t \mu_2(s)ds)\|\nabla v(t)\|^2\right] \\ &\quad + \frac{1}{2(\gamma+1)}(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}) - \int_\Omega F(u,v)\,dx. \end{aligned} \tag{1.9}$$

where $(\mu_i \diamond \nabla w)(t) = \int_0^t \mu_i(t-s)\|\nabla w(t) - \nabla w(s)\|_2^2 ds$.

By computation, we get

$$\begin{aligned} E'(t) &= \frac{1}{2}\left[(\mu_1' \diamond \nabla u)(t) + (\mu_2' \diamond \nabla v)(t)\right] - \frac{1}{2}\left(\mu_1(t)\|\nabla u\|^2 + \mu_2(t)\|\nabla v\|^2\right) \\ &\quad - \int_\Omega \left(|u|^k + |v|^l\right)|u_t|^{p+1}dx - \int_\Omega \left(|v|^\theta + |u|^\rho\right)|v_t|^{q+1}dx. \end{aligned} \tag{1.10}$$

2. LOCAL EXISTENCE

In this section, we shall discuss the local existence of a weak solution of problem (1.1). Firstly, we give the definition of weak solutions to problem (1.1).

Definition 1. We say that (u,v) is a weak solution of (1.1) on $[0,T)$ under the assumptions (A1) and (A2) if $u,v\in L^{\infty}\left(0,T;W_0^{1,2(\gamma+1)}(\Omega)\right)$, $u_t,v_t\in L^{\infty}\left(0,T;L^2(\Omega)\right)$, and satisfies

$$\begin{aligned}
&\left\langle u'(t),\theta\right\rangle-\left\langle u^1,\theta\right\rangle+\int_0^t\left\langle\int_\Omega M(\|\nabla u\|^2)\nabla u(\alpha)d\alpha,\nabla\theta\right\rangle d\xi\\
&-\int_0^t\left\langle\int_0^s\mu_1(\xi-\alpha)\nabla u(\alpha)d\alpha,\nabla\theta\right\rangle d\xi+\int_0^t\left\langle\left(|u|^k+|v|^l\right)|u'(\xi)|^{p-1}u'(\xi),\theta\right\rangle d\xi\\
&\qquad=\int_0^t\left\langle f_1\left(u(\xi),v(\xi)\right),\theta\right\rangle d\xi,
\end{aligned}$$

$$\begin{aligned}
&\left\langle v'(t),\phi\right\rangle-\left\langle v^1,\phi\right\rangle+\int_0^t\left\langle\int_\Omega M(\|\nabla v\|^2)\nabla v(\alpha)d\alpha,\nabla\phi\right\rangle d\xi\\
&-\int_0^t\left\langle\int_0^s\mu_2(\xi-\alpha)\nabla v(\alpha)d\alpha,\nabla\phi\right\rangle d\xi+\int_0^t\left\langle\left(|v|^\theta+|u|^\rho\right)|v'(\xi)|^{q-1}v'(\xi),\phi\right\rangle d\xi\\
&\qquad=\int_0^t\left\langle f_2\left(u(\xi),v(\xi)\right),\phi\right\rangle d\xi,
\end{aligned}$$

for almost everywhere $t\in[0,T)$ and any test functions $\theta,\phi\in W_0^{1,2(\gamma+1)}(\Omega)$.

Theorem 1 (Local existence)**.** *Assume assumptions (A1), (A2), (1.6) and $n=1,2,3$ hold. Then, for some $T>0$ problem (1.1) has at least a local weak solution (u,v) on $[0,T)$.*

Proof. We follow the standard Fadeo-Galerkin approximation to show the existence of solution (1.1). The combination of the Faedo-Galerkin method and the compactness argument give us an efficient method that allows us to deal with some evolution equations with degenerate damping terms.

Let the sequence $\left\{e_j:j=1,2,...\right\}$ is an orthogonal basis for $L^2(\Omega)\cap W_0^{1,2(\gamma+1)}$. Standard results on ordinary differential equations guarantee that there exists only one local solution. We construct approximate solutions $(u_M(t),v_M(t))$ $(M=1,2,3,...)$ in

the form

$$u_M(t)=\sum_{j=1}^{M}u_{M,j}(t)e_j,\quad v_M(t)=\sum_{j=1}^{M}v_{M,j}(t)e_j.$$

Approximate system

$$\left\langle u_M''(t),e_j\right\rangle+\left\langle\int_\Omega M\left(\|\nabla u_M(t)\|^2\right)\nabla u_M(t),\nabla e_j\right\rangle-\left\langle\int_0^t\mu_1(t-s)\nabla u_M(s)ds,\nabla e_j\right\rangle$$
$$+\left\langle\left(|u_M(t)|^k+|v_M(t)|^l\right)\left|u_M'(t)\right|^{p-1}u_M'(t),e_j\right\rangle=\left\langle f_1(u_M(t),v_M(t)),e_j\right\rangle,\quad(2.1)$$

$$\left\langle v_M''(t),e_j\right\rangle+\left\langle\int_\Omega M\left(\|\nabla v_M(t)\|^2\right)\nabla v_M(t),\nabla e_j\right\rangle-\left\langle\int_0^t\mu_2(t-s)\nabla v_M(s)ds,\nabla e_j\right\rangle$$
$$+\left\langle\left(|v_M(t)|^\theta+|u_M(t)|^\rho\right)\left|v_M'(t)\right|^{q-1}v_M'(t),e_j\right\rangle=\left\langle f_2(u_M(t),v_M(t)),e_j\right\rangle,\quad(2.2)$$

with initial data

$$\begin{aligned}u_M(0)&=\sum_{j=1}^{M}u_{M,j}(0)e_j,\quad v_M(0)=\sum_{j=1}^{M}v_{M,j}(0)e_j,\\ u_M'(0)&=\sum_{j=1}^{N}u_{M,j}'(0)e_j,\quad v_M'(0)=\sum_{j=1}^{N}v_{M,j}(0)e_j,\end{aligned}\quad(2.3)$$

where

$$u_{M,j}(0)=\left\langle u^0,e_j\right\rangle,\ v_{M,j}(0)=\left\langle v^0,e_j\right\rangle,\ u_{M,j}'(0)=\left\langle u^1,e_j\right\rangle,\ v_{M,j}'(0)=\left\langle v^1,e_j\right\rangle.\quad(2.4)$$

A priori estimate

Multiply (2.1) by $u_{M,j}'(t)$, (2.2) by $v_{M,j}'(t)$, and summing with respect j from 1 to M, we get

$$\frac{1}{2}\frac{d}{dt}\left[\left\|u_M'(t)\right\|^2+\left(1-\int_0^t\mu_1(s)\,ds\right)\|\nabla u_M(t)\|^2+(\mu_1\diamond\nabla u_M)(t)+\frac{1}{\gamma+1}\|\nabla u_M(t)\|^{2(\gamma+1)}\right]$$
$$+\frac{1}{2}\mu_1(t)\|\nabla u_M(t)\|^2-\frac{1}{2}(\mu_1'\diamond\nabla u_M)(t)+\int_\Omega\left(|u_M(t)|^k+|v_M(t)|^l\right)\left|u_M'(t)\right|^{p+1}dx$$
$$=\int_\Omega f_1(u_M(t),v_M(t))u_M'(t)dx,\quad(2.5)$$

$$\frac{1}{2}\frac{d}{dt}\left[\left\|v_M'(t)\right\|^2+\left(1-\int_0^t\mu_2(s)\,ds\right)\|\nabla v_M(t)\|^2+(\mu_2\diamond\nabla v_M)(t)+\frac{1}{\gamma+1}\|\nabla v_M(t)\|^{2(\gamma+1)}\right]$$
$$+\frac{1}{2}\mu_2(t)\|\nabla v_M(t)\|^2-\frac{1}{2}(\mu_2'\diamond\nabla v_M)(t)+\int_\Omega\left(|v_M(t)|^\theta+|u_M(t)|^\rho\right)\left|v_M'(t)\right|^{q+1}dx$$
$$=\int_\Omega f_2(u_M(t),v_M(t))v_M'(t)dx.\quad(2.6)$$

Summing (2.5) and (2.6) and integrating from 0 to $t \leq T_M$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\left[\left\|u_M'(t)\right\|^2+\left\|v_M'(t)\right\|^2\right] \\
&\quad+\frac{1}{2}\left[\left(1-\int_0^t \mu_1(s)\,ds\right)\|\nabla u_M(t)\|^2+\left(1-\int_0^t \mu_2(s)\,ds\right)\|\nabla v_M(t)\|^2\right] \\
&\quad+\frac{1}{2}\left[\left(\mu_1 \diamond \nabla u_M\right)(t)+\left(\mu_2 \diamond \nabla v_M\right)(t)\right] \\
&\quad+\frac{1}{2}\int_0^t\left[\mu_1(s)\|\nabla u_M(s)\|^2+\mu_2(s)\|\nabla v_M(s)\|^2\right]ds \\
&\quad-\frac{1}{2}\int_0^t\left[\left(\mu_1' \diamond \nabla u_M\right)(s)+\left(\mu_2' \diamond \nabla v_M\right)(s)\right]ds \\
&\quad+\frac{1}{2(\gamma+1)}\left[\|\nabla u_M(t)\|^{2(\gamma+1)}+\|\nabla v_M(t)\|^{2(\gamma+1)}\right] \\
&\quad+\int_0^t\int_\Omega\left(|u_M(s)|^k+|v_M(s)|^l\right)\left|u_M'(s)\right|^{p+1}dxds \\
&\quad+\int_0^t\int_\Omega\left(|v_M(s)|^\theta+|u_M(s)|^\rho\right)\left|v_M'(s)\right|^{q+1}dxds \\
&=\frac{1}{2}\left(\left\|u_M'(0)\right\|^2+\left\|v_M'(0)\right\|^2+\|\nabla u_M(0)\|^2+\|\nabla v_M(0)\|^2\right) \\
&\quad+\frac{1}{2(\gamma+1)}\left(\|\nabla u_M(0)\|^{2(\gamma+1)}+\|\nabla v_M(0)\|^{2(\gamma+1)}\right) \\
&\quad+\int_0^t\int_\Omega\left[f_1(u_M(s),v_M(s))u_M'(s)+f_2(u_M(s),v_M(s))v_M'(s)\right]dxds \\
&\leq C_0+\int_0^t\int_\Omega\left[f_1(u_M(s),v_M(s))u_M'(s)+f_2(u_M(s),v_M(s))v_M'(s)\right]dxds, \qquad (2.7)
\end{aligned}
$$

with a positive constant

$$
C_0=C\left(\left|u^0\right|_{H^1(\Omega)},\left|v^0\right|_{H^1(\Omega)},\left|u^1\right|_{L^2(\Omega)},\left|v^1\right|_{L^2(\Omega)},\left|u^0\right|_{W^{1,2(\gamma+1)}(\Omega)},\left|v^0\right|_{W^{1,2(\gamma+1)}(\Omega)}\right).
$$

To estimate the last term in (2.7) applying (1.5) and using Young inequalities, Hölder inequalities and Sobolev embedding theorem, we have

$$
\begin{aligned}
&\left|\int_\Omega f_1(u_M,v_M)u_M'dx\right|\leq C\int_\Omega\left(|u_M+v_M|^{2r+3}\left|u_M'\right|+|v_M|^{r+2}|u_M|^{r+1}\left|u_M'\right|\right)dx \\
&\leq C\left[\left(\|u_M\|_{2(2r+3)}^{2r+3}+\|v_M\|_{2(2r+3)}^{2r+3}\right)\left\|u_M'\right\|+\|u_M\|_{4(r+1)}^{r+1}\|v_M\|_{4(r+2)}^{r+2}\left\|u_M'\right\|\right] \\
&\leq C\left[\|\nabla u_M\|^{2(2r+3)}+\|\nabla v_M\|^{2(2r+3)}+\|\nabla u_M\|^{2(r+1)}\|\nabla v_M\|^{2(r+2)}+\left\|u_M'\right\|^2\right]. \qquad (2.8)
\end{aligned}
$$

In the same way, we obtain

$$\left|\int_\Omega f_2(u_M,v_M)v_N'dx\right| \le C\left[\|\nabla u_M\|^{2(2r+3)}+\|\nabla v_M\|^{2(2r+3)} + \|\nabla u_M\|^{2(r+2)}\|\nabla v_M\|^{2(r+1)}+\left\|v_M'\right\|^2\right]. \tag{2.9}$$

Now, by putting

$$y_M(t) := \left\|u_M'(t)\right\|^2+\left\|v_M'(t)\right\|^2+\|\nabla u_M(t)\|^2+\|\nabla v_M(t)\|^2 + \frac{1}{l(\gamma+1)}\left[\|\nabla u_M(t)\|^{2(\gamma+1)}+\|\nabla v_M(t)\|^{2(\gamma+1)}\right],$$

where $l=\min\{l_1,l_2\}<1$. Then, we infer from (2.7)-(2.9)

$$\begin{aligned} &y_M(t)+\frac{1}{l}\left[(\mu_1\diamond\nabla u_M)(t)+(\mu_2\diamond\nabla v_M)(t)\right] \\ &+\frac{1}{l}\int_0^t\left[\mu_1(s)\|\nabla u_M(s)\|^2+\mu_2(s)\|\nabla v_M(s)\|^2\right]ds \\ &-\frac{1}{l}\int_0^t\left[\left(\mu_1'\diamond\nabla u_M\right)(s)+\left(\mu_2'\diamond\nabla v_M\right)(s)\right]ds \\ &+\frac{2}{l}\int_0^t\int_\Omega\left(|u_M(s)|^k+|v_M(s)|^l\right)|u_M'(s)|^{p+1}dxds \\ &+\frac{2}{l}\int_0^t\int_\Omega\left(|v_M(s)|^\theta+|u_M(s)|^\rho\right)|v_M'(s)|^{q+1}dxds \le C_0+C\int_0^t y_M{}^{(2r+3)}(s)\,ds, \end{aligned} \tag{2.10}$$

Particularly, $y_M(t)$ satisfies the inequality

$$y_M(t)\le C_0+C\int_0^t y_M^{(2r+3)}(s)\,ds. \tag{2.11}$$

By applying Grönwall inequality we have

$$y_M(t)\le C_1 \text{ for all } t\in[0,T]. \tag{2.12}$$

The estimates follow from (2.10) and (2.12):

u_M, v_M are uniformly bounded in $L^\infty\left(0,T;W_0^{2(\gamma+1)}(\Omega)\right)$;

u_M', v_M' are uniformly bounded in $L^\infty\left(0,T;L^2(\Omega)\right)$;

The sequences $\left\{\int_0^t\int_\Omega\left(|u_M(s)|^k+|v_M(s)|^l\right)|u_M'(s)|^{p+1}dxds\right\}$ and $\left\{\int_0^t\int_\Omega\left(|v_M(s)|^\theta+|u_M(s)|^\rho\right)|v_M'(s)|^{q+1}dxds\right\}$ are uniformly bounded in $L^\infty(0,T)$.

Then

$$\begin{aligned} &u_M\to u, \qquad v_M\to v \quad \text{weakly } * \text{ in } L^\infty\left(0,T,W_0^{2(\gamma+1)}(\Omega)\right), \\ &u_M'\to u', \qquad v_M'\to v' \quad \text{weakly } * \text{ in } L^\infty\left(0,T,L^2(\Omega)\right), \end{aligned}$$

By applying the techniques in [10], we obtain the sequence of approximate solutions (u_M, v_M) satisfying

$$\begin{cases} \{u_M\},\ \{v_M\} \text{ are Cauchy sequences in } L^{\infty}\left(0,T,W_0^{2(\gamma+1)}(\Omega)\right), \\ \{u'_M\},\ \{v'_M\} \text{ are Cauchy sequences in } L^{\infty}\left(0,T,L^2(\Omega)\right). \end{cases}$$

Limiting process

Integrating (2.1) and (2.2) over $[0,T]$, we get

$$\begin{aligned} &\left\langle u'_M(t),e_j\right\rangle - \left\langle u'_M(0),e_j\right\rangle + \int_0^T \left\langle M\left(\|\nabla u_M(s)\|^2\right)\nabla u_M(s),\nabla e_j\right\rangle ds \\ &- \int_0^T \left\langle \int_0^s \mu_1(s-\tau)\nabla u_M(\tau)d\tau,\nabla e_j\right\rangle ds + \int_0^T \left\langle \left(|u_M|^k + |v_M|^l\right)\left|u'_M\right|^{p-1}u'_M,e_j\right\rangle ds \\ &= \int_0^T \left\langle f_1(u_M(s),v_M(s)),e_j\right\rangle ds, \quad (2.13) \end{aligned}$$

$$\begin{aligned} &\left\langle v'_M(t),e_j\right\rangle - \left\langle v'_M(0),e_j\right\rangle + \int_0^T \left\langle M\left(\|\nabla v_M(s)\|^2\right)\nabla v_M(s),\nabla e_j\right\rangle ds \\ &- \int_0^T \left\langle \int_0^s \mu_2(s-\tau)\nabla v_M(\tau)d\tau,\nabla e_j\right\rangle ds + \int_0^T \left\langle \left(|v_M|^{\theta} + |u_M|^{\rho}\right)\left|v'_M\right|^{q-1}v'_M,e_j\right\rangle ds \\ &= \int_0^T \left\langle f_2(u_M(s),v_M(s)),e_j\right\rangle ds. \quad (2.14) \end{aligned}$$

Now, we can pass to the limit in (2.13) and (2.14) as $M \to \infty$. Therefore, this completes our proof of the local existence of a weak solution. □

3. Blow up solutions

Our main result in this section is to show the blow up result of the solution to (1.1). For this purpose, we need the following lemmas.

Lemma 1 ([6]). *There exist two positive constants c_0 and c_1 such that*

$$c_0\left(|u|^{2(r+2)} + |v|^{2(r+2)}\right) \le 2(r+2)F(u,v) \le c_1\left(|u|^{2(r+2)} + |v|^{2(r+2)}\right) \quad (3.1)$$

is satisfied.

Lemma 2. *Suppose that (1.6) holds. Then there exists $\eta > 0$ such that for the solution (u,v)*

$$\begin{aligned} &\|u+v\|_{2(r+2)}^{2(r+2)} + 2\|uv\|_{r+2}^{r+2} \\ &\le \eta\left(\frac{1}{\gamma+1}\left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}\right) + l_1\|\nabla u\|^2 + l_2\|\nabla v\|^2\right)^{r+2}. \quad (3.2) \end{aligned}$$

Proof. By applying Minkowski inequality, we get

$$\|u+v\|_{2(r+2)}^2 \leq 2\left(\|u\|_{2(r+2)}^2+\|v\|_{2(r+2)}^2\right). \tag{3.3}$$

Moreover, applying Hölder's inequality and Young's inequality we have

$$\|uv\|_{(r+2)} \leq \|u\|_{2(r+2)}\|v\|_{2(r+2)} \leq \frac{1}{2}\left(\|u\|_{2(r+2)}^2+\|v\|_{2(r+2)}^2\right). \tag{3.4}$$

A combination of (3.3), (3.4), and the embedding $H_0^1(\Omega) \hookrightarrow L^{2(r+2)}(\Omega)$, imply

$$\|u+v\|_{2(r+2)}^2 \leq C\left(l_1\|\nabla u\|^2+l_2\|\nabla v\|^2\right), \tag{3.5}$$

$$\|uv\|_{(r+2)} \leq C\left(l_1\|\nabla u\|^2+l_2\|\nabla v\|^2\right). \tag{3.6}$$

Then, we have

$$\|\nabla u\|^2+\|\nabla v\|^2 \leq \|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}. \tag{3.7}$$

Thus,

$$\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)} \geq l_1\|\nabla u\|^2+l_2\|\nabla v\|^2. \tag{3.8}$$

Combining (3.3), (3.4) and (3.8), we have

$$\|u+v\|_{2(r+2)}^2 \leq C\left[\frac{1}{\gamma+1}\left(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\right)+l_1\|\nabla u\|^2+l_2\|\nabla v\|^2\right], \tag{3.9}$$

$$\|uv\|_{r+2} \leq C\left[\frac{1}{\gamma+1}\left(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\right)+l_1\|\nabla u\|^2+l_2\|\nabla v\|^2\right]. \tag{3.10}$$

Thus, the lemma follows. □

We take $a=b=1$ for simplicity and introduce the following:

$$B=\eta^{\frac{1}{2(r+2)}},\ \alpha_1=B^{-\frac{r+2}{r+1}},\ E_1=\left(\frac{1}{2}-\frac{1}{2(r+2)}\right)\alpha_1^2,\ \ E_2=\left(\frac{1}{2(\gamma+1)}-\frac{1}{2(r+2)}\right)\alpha_1^2, \tag{3.11}$$

where η is the optimal constant in (3.2).

Let

$$\Gamma(t):=\left(1-\int_0^t \mu_1(s)\,ds\right)\|\nabla u\|^2+\left(1-\int_0^t \mu_2(s)\,ds\right)\|\nabla v\|^2+(\mu_1\diamond\nabla u)+(\mu_2\diamond\nabla v). \tag{3.12}$$

The next lemma will act an important role in our proof, and it is similar to a lemma introduced firstly by Vitillaro [16].

Lemma 3. *Suppose that assumptions (A1) and (1.6) hold. Let (u,v) be a solution of (1.1). Moreover, assume that $E(0)<E_1$ and*

$$\left(\frac{1}{\gamma+1}(\|\nabla u(0)\|^{2(\gamma+1)}+\|\nabla v(0)\|^{2(\gamma+1)})+\Gamma(0)\right)^{\frac{1}{2}}>\alpha_1. \tag{3.13}$$

Then there exists a constant $\alpha_2 > \alpha_1$ *such that*

$$\left(\frac{1}{\gamma+1}(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)})+\Gamma(t)\right)^{\frac{1}{2}} \geq \alpha_2, \text{ for } t>0, \tag{3.14}$$

$$\left(\|u+v\|_{2(r+2)}^{2(r+2)}+2\|uv\|_{r+2}^{r+2}\right)^{\frac{1}{2(r+2)}} \geq B\alpha_2, \text{ for } t>0. \tag{3.15}$$

for all $t \in [0,T)$.

Proof. We first note that by (1.9), (3.2), (3.12) and the definition of B, we have

$$\begin{aligned}
E(t) &\geq \frac{1}{2(\gamma+1)}(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)})+\frac{1}{2}\Gamma(t)-\int_\Omega F(u,v)\,dx \\
&= \frac{1}{2(\gamma+1)}(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)})+\frac{1}{2}\Gamma(t) \\
&\quad -\frac{1}{2(r+2)}(\|u+v\|_{2(r+2)}^{2(r+2)}+2\|uv\|_{r+2}^{r+2}) \\
&\geq \frac{1}{2(\gamma+1)}(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)})+\frac{1}{2}\Gamma(t) \\
&\quad -\frac{\eta}{2(r+2)}\left[\frac{1}{\gamma+1}(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)})+l_1\|\nabla u\|^2+l_2\|\nabla v\|^2\right]^{r+2} \\
&\geq \frac{1}{2}\left[\frac{1}{\gamma+1}(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)})+\Gamma(t)\right] \\
&\quad -\frac{B^{2(r+2)}}{2(r+2)}\left[\frac{1}{\gamma+1}(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)})+\Gamma(t)\right]^{r+2} \\
&= \frac{1}{2}\alpha^2-\frac{B^{2(r+2)}}{2(r+2)}\alpha^{2(r+2)}=G(\alpha),
\end{aligned} \tag{3.16}$$

where

$$\alpha=\left[\frac{1}{\gamma+1}(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)})+\Gamma(t)\right]^{1/2}.$$

It is not difficult to verify that G is increasing for $0<\alpha<\alpha_1$, decreasing for $\alpha>\alpha_1$, $G(\alpha)\to-\infty$ as $\alpha\to\infty$, and

$$G(\alpha_1)=\frac{1}{2}\alpha_1^2-\frac{B^{2(r+2)}}{2(r+2)}\alpha_1^{2(r+2)}=E_1, \tag{3.17}$$

where α_1 is given in (3.11). Since $E(0)<E_1$, there exists $\alpha_2>\alpha_1$ such that $G(\alpha_2)=E(0)$.

Set $\alpha_0=\left[\frac{1}{\gamma+1}(\|\nabla u(0)\|^{2(\gamma+1)}+\|\nabla v(0)\|^{2(\gamma+1)})+\Gamma(0)\right]^{1/2}$. Then by (3.16) we get $G(\alpha_0)\leq E(0)=G(\alpha_2)$, which implies that $\alpha_0\geq\alpha_2$.

Now, to establish (3.14), we suppose by contradiction that

$$\left[\frac{1}{\gamma+1}(\|\nabla u(t_0)\|^{2(\gamma+1)}+\|\nabla v(t_0)\|^{2(\gamma+1)})+\Gamma(t_0)\right]^{1/2}<\alpha_2,$$

for some $t_0>0$. By the continuity of $\left[\frac{1}{\gamma+1}(\|\nabla u(t_0)\|^{2(\gamma+1)}+\|\nabla v(t_0)\|^{2(\gamma+1)})+\Gamma(t_0)\right]$, we can choose t_0 such that

$$\left[\frac{1}{\gamma+1}(\|\nabla u(t_0)\|^{2(\gamma+1)}+\|\nabla v(t_0)\|^{2(\gamma+1)})+\Gamma(t_0)\right]^{1/2}>\alpha_1.$$

Again, using (3.16) we get

$$E(t_0)\geq G(\left[\frac{1}{\gamma+1}(\|\nabla u(t_0)\|^{2(\gamma+1)}+\|\nabla v(t_0)\|^{2(\gamma+1)})+\Gamma(t_0)\right]^{1/2})>G(\alpha_2)=E(0).$$

This is impossible since $E(t)\leq E(0)$ for all $t\in[0,T)$. Hence (3.14) is established.

To prove (3.15), we apply (1.9) to get

$$\begin{aligned}\frac{1}{2(\gamma+1)}(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)})+\frac{1}{2}\Gamma(t)&\\ \leq E(0)+\frac{1}{2(r+2)}(\|u+v\|_{2(r+2)}^{2(r+2)}+2\|uv\|_{r+2}^{r+2}).&\end{aligned}$$

Consequently, (3.14) yields

$$\begin{aligned}&\frac{1}{2(r+2)}(\|u+v\|_{2(r+2)}^{2(r+2)}+2\|uv\|_{r+2}^{r+2})\\ &\quad\geq\frac{1}{2(\gamma+1)}(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)})+\frac{1}{2}\Gamma(t)-E(0)\\ &\quad\geq\frac{1}{2}\alpha_2^2-G(\alpha_2)=\frac{B^{2(r+2)}}{2(r+2)}\alpha_2^{2(r+2)}.\end{aligned}\tag{3.18}$$

Therefore, (3.18) and (3.11) yield the desired result. □

Theorem 2. *Assume that (A1), (A2) and (1.6) hold. Assume further that*

$$2(r+2)>\max\{2(\gamma+1),\ k+p+1,\ l+p+1,\ \theta+q+1,\ \rho+q+1\}.$$

Then any the solution of the problem (1.1) with initial data satisfying

$$\left[\frac{1}{\gamma+1}(\|\nabla u(0)\|^{2(\gamma+1)}+\|\nabla v(0)\|^{2(\gamma+1)})+\Gamma(0)\right]^{\frac{1}{2}}>\alpha_1,\ E(0)<E_2,$$

cannot exist for all time, where α_1 and E_2 are defined in (3.11).

Proof. We suppose that the solution exists for all time and we reach to a contradiction. Set

$$H(t) = E_2 - E(t). \tag{3.19}$$

By applying (1.9) and (3.19), we have

$$\begin{aligned} 0 < H(0) \leq H(t) = E_2 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) - \frac{1}{2}\Gamma(t) \\ - \frac{1}{2(\gamma+1)}(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}) + \int_\Omega F(u,v)\,dx. \end{aligned} \tag{3.20}$$

From (3.18) and (3.1), we have

$$\begin{aligned} & E_2 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) - \frac{1}{2}\Gamma(t) \\ & \quad - \frac{1}{2(\gamma+1)}(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}) + \int_\Omega F(u,v)\,dx \\ & \leq E_1 - \frac{1}{2}\alpha_1^2 + \frac{c_1}{2(r+2)}\left(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\right) \\ & \leq -\frac{1}{2(r+2)}\alpha_1^2 + \frac{c_1}{2(r+2)}\left(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\right) \\ & \leq \frac{c_1}{2(r+2)}\left(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\right). \end{aligned} \tag{3.21}$$

By combining (3.20) and (3.21), we have

$$0 < H(0) \leq H(t) \leq \frac{c_1}{2(r+2)}\left(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\right). \tag{3.22}$$

We then define with $\varepsilon > 0$

$$\Psi(t) = H^{1-\sigma}(t) + \varepsilon\left(\int_\Omega u_t u dx + \int_\Omega v_t v dx\right). \tag{3.23}$$

The remainder of the proof is similar to the proof of Theorem 12 in [9], and then we get the desired result. □

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INTEGRAL INEQUALITIES FOR EXPONENTIALLY HARMONICALLY CONVEX FUNCTIONS VIA FRACTIONAL INTEGRAL OPERATORS

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Abstract. The main aim of this paper is to derive some new integral inequalities related to Hermite-Hadamard type by using Riemann-Liouville fractional integral operator for the class of exponentially harmonically convex functions. The formal technique of this paper may enhance further research in this field.

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1. INTRODUCTION AND PRELIMINARIES

Theory of convex functions had not only stimulated new and deep results in many branches of mathematical and engineering sciences, but also provided us a unified and general frame work to study a wide class of unrelated problems. For recent applications, generalizations and other aspects of convex functions, see [2, 4–10, 13, 15–18, 23–30].

The class of harmonic convex function was introduced by Anderson *et al.* in [2] and in [8], İşcan has proved some new integral inequalities for this class of functions. It is natural to unify these different concepts.

An important class of convex functions, which is called an exponentially convex functions, was introduced and studied by Antczak in [3], Dragomir *et al.* in [6] and Noor *et al.* in [17]. In [1], Alirezai and Mathar have investigated their mathematical properties along with their potential applications in statistics and information theory, see [1, 19]. Due to its significance, in [4], Awan et al and also in [20], Pecaric and Jaksetic defined another kind of exponential convex functions, have shown that the class of exponential convex functions unifies various concepts in different manners.

The advantages of fractional calculus have been described and pointed out in the last few decades by many authors. Fractional calculus is based on derivatives and integrals of fractional order, fractional differential equations and methods of their

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solution. The most celebrated inequality has been studied extensively since it was established by Hermite, is the Hermite-Hadamard inequality not only established for classical integrals but also for fractional integrals, see [5, 10–14, 22, 25, 26, 28].

We now recall some known basic results and concepts, which are necessary to obtain the main results.

Definition 1 ([8]). A function $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonically convex function, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (1-t)f(x)+tf(y),\ \forall x,y \in I,\ t \in [0,1].$$

We now define the concept of exponentially convex function, which is mainly due to Antczak [3], Dragomir [6] and Noor et al [17].

Definition 2 ([3, 6, 17]). Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is an exponentially convex function, if f is positive, $\forall a,b \in I$ and $t \in [0,1]$, we have

$$e^{f((1-t)a+tb)} \leq [(1-t)e^{f(a)}+te^{f(b)}], \quad a,b \in I,\ t \in [0,1]. \tag{1.1}$$

We recall the following special functions and inequality.

(1) The Gamma function:

$$\Gamma(x,y) = \int_0^\infty t^{x-1}e^{-t}dt, \quad x,y>0$$

(2) The Beta function:

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x,y>0$$

(3) The hypergeometric function (see [15]):

$${}_2F(a,b;c;z) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt, \quad c>b>0, \quad |z|<1.$$

Lemma 1 ([22, 30]). *For $0<\alpha\leq 1$ and $0\leq a<b$, we have*

$$|a^\alpha - b^\alpha| \leq (b-a)^\alpha.$$

We now give the definition of the fractional integral, which is mainly due to [21].

Definition 3 ([21]). Let $\alpha>0$ with $n-1<\alpha\leq n,\ n\in\mathbb{N}$, and $1<x<v$. The left- and right-hand side Riemann-Liouville fractional integrals of order α of function f are given by

$$J^\alpha_{u_+}f(x) = \frac{1}{\Gamma(\alpha)}\int_u^x (x-t)^{\alpha-1}f(t)dt, \tag{1.2}$$

and

$$J^{\alpha}_{v_-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^v (t-x)^{\alpha-1} f(t)dt, \tag{1.3}$$

where $\Gamma(\alpha)$ is the Gamma function.

In this article, we aim to study a new class of harmonically convex functions, which is called exponentially harmonically convex functions. We also derive some new integral inequalities via Riemann-Liouville fractional integrals for exponentially harmonically convex functions by using a new integral identity. Innovative ideas and techniques of this paper may stimulate further research in this dynamic field.

2. MAIN RESULTS

In this section, we derive our main results. Firstly, we will define the concept of exponentially harmonic convex functions, which is the main motivation of this paper.

Definition 4. A function $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be an exponentially harmonically convex function, if

$$e^{f\left(\frac{xy}{tx+(1-t)y}\right)} \leq (1-t)e^{f(x)} + te^{f(y)}, \ \forall x,y \in I, \ t \in [0,1]. \tag{2.1}$$

Also note that for $t = \frac{1}{2}$ in Definition 4, we have Jensen type exponentially harmonic convex functions with

$$e^{f\left(\frac{2xy}{x+y}\right)} \leq \frac{1}{2}[e^{f(x)} + e^{f(y)}], \ \forall x,y \in I.$$

Theorem 1. *Let $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be an exponentially harmonic convex function and $a,b \in I$ with $a < b$. If $f \in L[a,b]$, then one has the following inequalities:*

$$e^{f(\frac{2ab}{a+b})} \leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{ba}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{a}-} e^{f\circ g(\frac{1}{b})} + J^{\alpha}_{\frac{1}{b}+} e^{f\circ g(\frac{1}{a})}\right] \leq \frac{e^{f(a)} + e^{f(b)}}{2}. \tag{2.2}$$

Proof. Since f is an exponentially harmonic convex function, for $t = \frac{1}{2}$ in inequality (2.1), we have

$$e^{f(\frac{2xy}{x+y})} \leq \frac{1}{2}[e^{f(x)+e^{f(y)}}], \quad \forall x,y \in I.$$

Substituting $x = \frac{ab}{tb+(1-t)a}$, $y = \frac{ab}{ta+(1-t)b}$, we get

$$e^{f(\frac{2ab}{a+b})} \leq \frac{1}{2}[e^{f(\frac{ab}{tb+(1-t)a})} + e^{f(\frac{ab}{ta+(1-t)b})}]. \tag{2.3}$$

Multiplying both sides of (2.3) by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\frac{e^{f(\frac{2ab}{a+b})}}{\alpha} \leq \frac{1}{2}\left[\int_0^1 t^{\alpha-1} e^{f(\frac{ab}{tb+(1-t)a})}dt + \int_0^1 t^{\alpha-1} e^{f(\frac{ab}{ta+(1-t)b})}dt\right]$$

$$= \frac{\alpha}{2}\Big(\frac{ba}{b-a}\Big)^{\alpha}\Bigg[\int_{\frac{1}{b}}^{\frac{1}{a}}\Big(x-\frac{1}{b}\Big)^{\alpha-1}e^{f(\frac{1}{x})}dx+\int_{\frac{1}{a}}^{\frac{1}{b}}\Big(\frac{1}{a}-x\Big)^{\alpha-1}e^{f(\frac{1}{x})}dx\Bigg]$$

$$= \frac{\alpha\Gamma(\alpha)}{2}\Big(\frac{ba}{b-a}\Big)^{\alpha}\big[J^{\alpha}_{\frac{1}{a}-}\,e^{f\circ g(\frac{1}{b})}+J^{\alpha}_{\frac{1}{b}+}\,e^{f\circ g(\frac{1}{a})}\big]$$

$$= \frac{\Gamma(\alpha+1)}{2}\Big(\frac{ba}{b-a}\Big)^{\alpha}\big[J^{\alpha}_{\frac{1}{a}-}\,e^{f\circ g(\frac{1}{b})}+J^{\alpha}_{\frac{1}{b}+}\,e^{f\circ g(\frac{1}{a})}\big].$$

This completes the proof of the left hand side of (2.2).

Since f is an exponentially harmonic convex function, then we can write

$$e^{f\left(\frac{ab}{tb+(1-t)a}\right)} \le te^{f(a)}+(1-t)e^{f(b)}$$

and

$$e^{f\left(\frac{ab}{ta+(1-t)b}\right)} \le te^{f(b)}+(1-t)e^{f(a)}.$$

By adding these inequalities, we have

$$e^{f\left(\frac{ab}{tb+(1-t)a}\right)}+e^{f\left(\frac{ab}{ta+(1-t)b}\right)} \le e^{f(a)+e^{f(b)}}.$$

Multiplying both sides of the above inequality by $t^{\alpha-1}$, and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\int_0^1 t^{\alpha-1}e^{f\left(\frac{ab}{tb+(1-t)a}\right)}dt+\int_0^1 t^{\alpha-1}e^{f\left(\frac{ab}{ta+(1-t)b}\right)}dt \le \int_0^1 t^{\alpha-1}[e^{f(a)+e^{f(b)}}]dt.$$

Thus

$$\Gamma(\alpha+1)\Big(\frac{ba}{b-a}\Big)^{\alpha}\big[J^{\alpha}_{\frac{1}{a}-}\,e^{f\circ g(\frac{1}{b})}+J^{\alpha}_{\frac{1}{b}+}\,e^{f\circ g(\frac{1}{a})}\big] \le [e^{f(a)+e^{f(b)}}],$$

which completes the proof of the right hand side of (2.2). □

Now, we are in a position that we can discuss a special new case of Theorem 1.

Corollary 1. *If we take $\alpha = 1$, then we have a new Hadamard type result for harmonically convex functions:*

$$e^{f(\frac{2ab}{a+b})} \le \frac{ab}{b-a}\int_a^b \frac{e^{f(x)}}{x^2}dx \le \frac{e^{f(a)}+e^{f(b)}}{2}.$$

Lemma 2. *Let $f : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be an exponentially differentiable function on the interior I° of I such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $f \in L[a,b]$, then the following equality holds:*

$$\Phi_f(g;\alpha,a,b) = \frac{ab(b-a)}{2}\int_0^1 \frac{[t^{\alpha}-(1-t)^{\alpha}]}{M_t^2}e^{f\left(\frac{ab}{M_t}\right)}f'\Big(\frac{ab}{M_t}\Big)dt, \tag{2.4}$$

where

$$\Phi_f(g;\alpha,a,b)=\frac{e^{f(a)}+e^{f(b)}}{2}-\frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{a}-}e^{f\circ g(\frac{1}{b})}+J^{\alpha}_{\frac{1}{b}+}e^{f\circ g(\frac{1}{a})}\right],$$

$M_t=ta+(1-t)b$ *and* $g(x)=\frac{1}{x}$.

Proof. Consider

$$I=\frac{ab(b-a)}{2}\int_0^1\frac{[t^{\alpha}-(1-t)^{\alpha}]}{M_t^2}e^{f\left(\frac{ab}{M_t^2}\right)}f'\left(\frac{ab}{M_t^2}\right)dt \tag{2.5}$$

$$=\frac{ab(b-a)}{2}\int_0^1\frac{t^{\alpha}}{M_t^2}e^{f\left(\frac{ab}{M_t^2}\right)}f'\left(\frac{ab}{M_t^2}\right)dt-\frac{ab(b-a)}{2}\int_0^1\frac{(1-t)^{\alpha}}{M_t^2}e^{f\left(\frac{ab}{M_t^2}\right)}f'\left(\frac{ab}{M_t^2}\right)dt$$

$$=I_1+I_2.$$

By applying integration by parts, we have

$$I_1=\frac{1}{2}\left[t^{\alpha}e^{f\left(\frac{ab}{M_t}\right)}\Big|_0^1-\alpha\int_0^1 t^{\alpha-1}e^{f\left(\frac{ab}{M_t}\right)}dt\right] \tag{2.6}$$

$$=\frac{1}{2}\left[e^{f(b)}-\alpha\left(\frac{ab}{b-a}\right)^{\alpha}\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{1}{a}-x\right)^{\alpha}e^{f(\frac{1}{x})}dx\right]$$

$$=\frac{1}{2}\left[e^{f(b)}-\alpha\left(\frac{ab}{b-a}\right)^{\alpha}J^{\alpha}_{\frac{1}{b}+}e^{f\circ g(\frac{1}{a})}\right],$$

similarly, we obtain

$$I_2=\frac{1}{2}\left[t^{\alpha}e^{f\left(\frac{ab}{M_t}\right)}\Big|_0^1-\alpha\int_0^1 t^{\alpha-1}e^{f\left(\frac{ab}{M_t}\right)}dt\right] \tag{2.7}$$

$$=\frac{1}{2}\left[e^{f(b)}-\alpha\left(\frac{ab}{b-a}\right)^{\alpha}\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{1}{a}-x\right)^{\alpha}e^{f(\frac{1}{x})}dx\right]$$

$$=\frac{1}{2}\left[e^{f(b)}-\alpha\left(\frac{ab}{b-a}\right)^{\alpha}J^{\alpha}_{\frac{1}{b}+}e^{f\circ g(\frac{1}{a})}\right].$$

Using (2.6) and (2.7) in (2.5), we get (2.4). □

Theorem 2. *Let* $f:I\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ *be an exponentially differentiable function on the interior* I° *of* I *such that* $f'\in L[a,b]$, *where* $a,b\in I^{\circ}$ *with* $a<b$. *If* $|f'|^q$ *is*

harmonically convex, for some fixed $q \geq 1$, then the following inequality holds for Riemann-Liouville integral operators:

$$\Phi_f(g;\alpha,a,b) \leq \frac{ab(b-a)}{2}\left(\eta_1(\alpha;a,b)\right)^{1-\frac{1}{q}} \tag{2.8}$$
$$\left[\eta_2(\alpha;a,b)|e^{f(a)}f'(a)|^q+\eta_3(\alpha;a,b)|e^{f(b)}f'(b)|^q+\eta_3(\alpha;a,b)\Delta_1(a,b)\right]^{\frac{1}{q}},$$

where

$$\Delta_1(a,b) = |e^{f(b)}f'(a)|^q + |e^{f(a)}f'(b)|^q$$

$$\eta_1(\alpha;a,b) = \frac{1}{b^2(\alpha+1)}\left[{}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right)+{}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right)\right],$$

$$\eta_2(\alpha;a,b) = \frac{1}{b^2(\alpha+3)}$$
$$\left[\frac{2}{(\alpha+1)(\alpha+2)}{}_2F_1\left(2,3;\alpha+4;1-\frac{a}{b}\right)+2{}_2F_1\left(2,\alpha+3;\alpha+3;1-\frac{a}{b}\right)\right],$$

$$\eta_3(\alpha;a,b) = \frac{1}{b^2(\alpha+3)}$$
$$\left[\frac{2}{(\alpha+1)(\alpha+2)}{}_2F_1\left(2,\alpha+1;\alpha+4;1-\frac{a}{b}\right)+{}_2F_1\left(2,1;\alpha+4;1-\frac{a}{b}\right)\right],$$

$$\eta_4(\alpha;a,b) = \frac{1}{b^2(\alpha+3)(\alpha+4)}$$
$$\left[\frac{2}{(\alpha+1)(\alpha+2)}{}_2F_1\left(2,2;\alpha+4;1-\frac{a}{b}\right)+{}_2F_1\left(2,\alpha+2;\alpha+4;1-\frac{a}{b}\right)\right].$$

Proof. Using Lemma 2, the property of modulus, the power mean inequality and by using harmonically convexity of $|e^f f'|$, we have

$$\Phi_f(g;\alpha,a,b) \leq \frac{ab(b-a)}{2}\int_0^1 \frac{|t^\alpha-(1-t)^\alpha|}{M_t^2}\left|e^{f\left(\frac{ab}{M_t}\right)}f'\left(\frac{ab}{M_t}\right)\right|dt$$

$$\leq \frac{ab(b-a)}{2}\left(\int_0^1 \frac{|t^\alpha-(1-t)^\alpha|}{M_t^2}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{|t^\alpha-(1-t)^\alpha|}{M_t^2}\left|e^{f\left(\frac{ab}{M_t}\right)}f'\left(\frac{ab}{M_t}\right)\right|dt\right)^{\frac{1}{q}}$$

$$\leq \frac{ab(b-a)}{2}\left(\int_0^1 \frac{|t^\alpha-(1-t)^\alpha|}{M_t^2}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{|t^\alpha-(1-t)^\alpha|}{M_t^2}\left[t^2|e^{f(b)}f'(b)|^q\right.\right.$$
$$\left.\left.+(1-t)^2|e^{f(a)}f'(a)|^q+t(1-t)\left[|e^{f(b)}f'(a)|^q+|e^{f(a)}f'(b)|^q\right]\right]dt\right)^{\frac{1}{q}}.$$

Namely,

$$\Phi_f(g;\alpha,a,b) \le \frac{ab(b-a)}{2}\left(\int_0^1 \frac{|t^\alpha-(1-t)^\alpha|}{M_t^2}dt\right)^{1-\frac{1}{q}} \tag{2.9}$$

$$\left(\int_0^1 \frac{|t^\alpha-(1-t)^\alpha|}{M_t^2}\left[t^2|e^{f(b)}f'(b)|^q+(1-t)^2|e^{f(a)}f'(a)|^q+t(1-t)\Delta_1(a,b)\right]dt\right)^{\frac{1}{q}}$$

$$\le \frac{ab(b-a)}{2}\left(\eta_1(\alpha;a,b)\right)^{1-\frac{1}{q}}$$

$$\left[\eta_2(\alpha;a,b)|e^{f(a)}f'(a)|^q+\eta_3(\alpha;a,b)|e^{f(b)}f'(b)|^q+\eta_3(\alpha;a,b)\Delta_1(a,b)\right]^{\frac{1}{q}}.$$

By calculating $\eta_1(\alpha;a,b)$, $\eta_2(\alpha;a,b)$, $\eta_3(\alpha;a,b)$ and $\eta_4(\alpha;a,b)$, we obtain

$$\eta_1(\alpha;a,b) = \int_0^1 \frac{|t^\alpha-(1-t)^\alpha|}{M_t^2}dt \tag{2.10}$$

$$= \frac{1}{b^2(\alpha+1)}\left[{}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right)+{}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right)\right]$$

$$\eta_2(\alpha;a,b) = \int_0^1 \frac{|t^\alpha-(1-t)^\alpha|}{M_t^2}t^2dt \tag{2.11}$$

$$= \frac{1}{b^2(\alpha+3)}\left[\frac{2}{(\alpha+1)(\alpha+2)}{}_2F_1\left(2,3;\alpha+4;1-\frac{a}{b}\right)+2{}_2F_1\left(2,\alpha+3;\alpha+3;1-\frac{a}{b}\right)\right]$$

$$\eta_3(\alpha;a,b) = \int_0^1 \frac{|t^\alpha-(1-t)^\alpha|}{M_t^2}(1-t)^2dt \tag{2.12}$$

$$= \frac{1}{b^2(\alpha+3)}\left[\frac{2}{(\alpha+1)(\alpha+2)}{}_2F_1\left(2,\alpha+1;\alpha+4;1-\frac{a}{b}\right)+{}_2F_1\left(2,1;\alpha+4;1-\frac{a}{b}\right)\right]$$

$$\eta_4(\alpha;a,b) = \int_0^1 \frac{|t^\alpha-(1-t)^\alpha|}{M_t^2}(1-t)^2dt = \frac{1}{b^2(\alpha+3)(\alpha+4)} \tag{2.13}$$

$$\left[\frac{2}{(\alpha+1)(\alpha+2)}{}_2F_1\left(2,2;\alpha+4;1-\frac{a}{b}\right)+{}_2F_1\left(2,\alpha+2;\alpha+4;1-\frac{a}{b}\right)\right].$$

Thus, if we use (2.10), (2.11), (2.12) and (2.13) in (2.9), we obtain the required inequality (2.8). □

Theorem 3. *Let $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be an exponentially differentiable function on the interior I° of I such that $f' \in L[a,b]$, where $a,b \in I^\circ$ with $a < b$. If $|f'|^q$ is harmonically convex, for some fixed $q > 1$ and $p^{-1} + q^{-1} = 1$, then the following inequality holds for Riemann-Liouville integral operators:*

$$\Phi_f(g;\alpha,a,b) \leq \frac{a(b-a)}{2b}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left[\frac{2\left(|e^{f(a)}f'(a)|^q + |e^{f(b)}f'(b)|^q\right) + \Delta_1(a,b)}{6}\right]^{\frac{1}{q}} \times \left[{}_2F_1^{\frac{1}{p}}\left(2p,1;p\alpha+2;1-\frac{a}{b}\right) + {}_2F_1^{\frac{1}{p}}\left(2p,p\alpha+1;p\alpha+2;1-\frac{a}{b}\right)\right]. \tag{2.14}$$

Proof. Using Lemma 2, the Hölder inequality and the exponentially harmonically convexity of $|f'|$, we find

$$\begin{aligned}
&\Phi_f(g;\alpha,a,b) \qquad (2.15)\\
&\leq \frac{ab(b-a)}{2}\left[\int_0^1 \frac{(1-t)^\alpha}{M_t^2}\left|e^{f\left(\frac{ab}{M_t}\right)}f'\left(\frac{ab}{M_t}\right)\right|dt + \int_0^1 \frac{t^\alpha}{M_t^2}\left|e^{f\left(\frac{ab}{M_t}\right)}f'\left(\frac{ab}{M_t}\right)\right|dt\right]\\
&\leq \frac{ab(b-a)}{2}\left[\left(\int_0^1 \frac{(1-t)^{p\alpha}}{M_t^2}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|e^{f\left(\frac{ab}{M_t}\right)}f'\left(\frac{ab}{M_t}\right)\right|dt\right)^{\frac{1}{q}}\right.\\
&\quad\left. + \left(\int_0^1 \frac{t^{p\alpha}}{M_t^2}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|e^{f\left(\frac{ab}{M_t}\right)}f'\left(\frac{ab}{M_t}\right)\right|dt\right)^{\frac{1}{q}}\right]\\
&\leq \frac{ab(b-a)}{2}\left\{\left(\int_0^1 \frac{(1-t)^{p\alpha}}{M_t^2}dt\right)^{\frac{1}{p}} + \left(\int_0^1 \frac{t^{p\alpha}}{M_t^2}dt\right)^{\frac{1}{p}}\right)\right\}\left(\int_0^1 \left|e^{f\left(\frac{ab}{M_t}\right)}f'\left(\frac{ab}{M_t}\right)\right|dt\right)^{\frac{1}{q}}\\
&\leq \frac{ab(b-a)}{2}\left(\zeta_1^{\frac{1}{p}} + \zeta_2^{\frac{1}{p}}\right)\\
&\quad\left(\int_0^1 \left[(1-t)^2|e^{f(a)}f'(a)|^q + t^2|e^{f(b)}f'(b)|^q + t(1-t)\Delta_1(a,b)\right]dt\right)^{\frac{1}{q}}\\
&\leq \frac{ab(b-a)}{2}\left(\zeta_1^{\frac{1}{p}} + \zeta_2^{\frac{1}{p}}\right)\left[\frac{2\left(|e^{f(a)}f'(a)|^q + |e^{f(b)}f'(b)|^q\right) + \Delta_1(a,b)}{6}\right]^{\frac{1}{q}}.
\end{aligned}$$

Calculating ζ_1 and ζ_2, we have

$$\zeta_1 = \int_0^1 \frac{(1-t)^{p\alpha}}{M_t^2}dt = \frac{b^{-2p}}{1+\alpha p}\,{}_2F_1\left(2p,1;p\alpha+2;1-\frac{a}{b}\right), \tag{2.16}$$

$$\zeta_2 = \int_0^1 \frac{t^{p\alpha}}{M_t^2} dt = \frac{b^{-2p}}{1+\alpha p}\, {}_2F_1\big(2p, p\alpha+1; p\alpha+2; 1-\frac{a}{b}\big). \tag{2.17}$$

Thus, if we use (2.16) and (2.17) in (2.15), we obtain inequality (2.14).

This completes the proof. □

Theorem 4. *Let $f : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be an exponentially differentiable function on the interior I° of I such that $f' \in L[a,b]$, where $a,b \in I^\circ$ with $a<b$. If $|f'|^q$ is harmonically convex, for some fixed $q>1$ and $p^{-1}+q^{-1}=1$, then the following inequality holds for Riemann-Liouville integral operators:*

$$\begin{aligned}\big|\Phi_f(g;\alpha,a,b)\big| \le{}& \frac{a(b-a)}{2b}\frac{{}_2F_1^{\frac{1}{p}}(2p,1;2;1-\frac{a}{b})}{2^{\frac{1}{q}}\big((q\alpha+1)(q\alpha+2)(q\alpha+3)\big)} \\ &\times\left[(q\alpha+2)^2|e^{f(a)}f'(a)|^q + |e^{f(b)}f'(b)|^q + (q\alpha+2)\Delta_1(a,b)\right]^{\frac{1}{q}}.\end{aligned} \tag{2.18}$$

Proof. Using Lemma 2, the Hölder inequality and the exponentially harmonically convexity of $|f'|$, we can write

$$\begin{aligned}\big|\Phi_f(g;\alpha,a,b)\big| &\le \frac{ab(b-a)}{2}\left[\int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{M_t^2}\Big|e^{f\left(\frac{ab}{M_t}\right)}f'\Big(\frac{ab}{M_t}\Big)\Big|dt\right. \\ &\le \frac{ab(b-a)}{2}\left(\int_0^1 \frac{1}{M_t^2}dt\right)^{\frac{1}{p}}\left(\int_0^1 |(1-t)^\alpha - t^\alpha|^q\Big|e^{f\left(\frac{ab}{M_t}\right)}f'\Big(\frac{ab}{M_t}\Big)\Big|^q\right)^{\frac{1}{q}} \\ &\le \frac{ab(b-a)}{2}\left(\int_0^1 \frac{1}{M_t^2}dt\right)^{\frac{1}{p}}\left(\int_0^1 |1-2t|^{q\alpha}\big[(1-t)^2|e^{f(a)}f'(a)|^q + t^2|e^{f(b)}f'(b)|^q\right. \\ &\quad \left.+t(1-t)\{|e^{f(a)}f'(b)|^q + |e^{f(b)}f'(a)|^q\}\big]\right)^{\frac{1}{q}} \\ &\le \frac{ab(b-a)}{2}\left(\int_0^1 \frac{1}{M_t^2}dt\right)^{\frac{1}{p}} \\ &\quad \left(\int_0^1 |1-2t|^{q\alpha}\big[(1-t)^2|e^{f(a)}f'(a)|^q + t^2|e^{f(b)}f'(b)|^q + t(1-t)\Delta_1(a,b)\big]\right)^{\frac{1}{q}} \\ &\le \frac{ab(b-a)}{2}\zeta_3^{\frac{1}{p}}\left(\zeta_4|e^{f(a)}f'(a)|^q + \zeta_5|e^{f(b)}f'(b)|^q + \zeta_6\Delta_1(a,b)\right)^{\frac{1}{q}},\end{aligned} \tag{2.19}$$

where

$$\zeta_3 = \int_0^1 \frac{1}{M_t^2}dt = b^{-2p}\int_0^1 \left(1-t\left(1-\frac{a}{b}\right)\right)^{-2p}dt = b_2^{-2p}F_1\left(2p,1;2;1-\frac{a}{b}\right),$$

$$\zeta_4 = \int_0^1 |1-2t|^{q\alpha}t^2dt = \frac{q\alpha+2}{2(q\alpha+1)(q\alpha+3)},$$

$$\zeta_5 = \int_0^1 |1-2t|^{q\alpha}(1-t)^2dt = \frac{1}{2(q\alpha+1)(q\alpha+2)(q\alpha+3)},$$

and

$$\zeta_6 = \int_0^1 |1-2t|^{q\alpha}(1-t)^2dt = \frac{1}{2(q\alpha+1)(q\alpha+3)}.$$

By computing ζ_3, ζ_4, ζ_5 and ζ_6, we obtain inequality (2.18). □

Theorem 5. *Let $f : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be an exponentially differentiable function on the interior I° of I such that $f' \in L[a,b]$, where $a,b \in I^\circ$ with $a<b$. If $|f'|^q$ is harmonically convex, for some fixed $q>1$, then the following inequality holds for Riemann-Liouville integral operators:*

$$\Phi_f(g;\alpha,a,b) \le \frac{a(b-a)}{2.6^{\frac{1}{q}}b}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(2\left[{}_2F_1\left(2q,3;4;1-\frac{a}{b}\right)|e^{f(a)}f'(a)|^q\right.\right. \tag{2.20}$$

$$\left.\left. +{}_2F_1\left(2q,1;4;1-\frac{a}{b}\right)|e^{f(b)}f'(b)|^q\right] + {}_2F_1\left(2q,2;4;1-\frac{a}{b}\right)\Delta_1(a,b)\right)^{\frac{1}{q}}.$$

Proof. By using Lemma 2, the Hölder inequality and the exponentially harmonically convexity of $|f'|$, we have

$$\Phi_f(g;\alpha,a,b) \le \frac{ab(b-a)}{2}\left[\int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{M_t^2}\left|e^{f\left(\frac{ab}{M_t}\right)}f'\left(\frac{ab}{M_t}\right)\right|dt \right. \tag{2.21}$$

$$\le \frac{ab(b-a)}{2}\left(\int_0^1 |(1-t)^\alpha - t^\alpha|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \frac{1}{M_t^{2q}}\left|e^{f\left(\frac{ab}{M_t}\right)}f'\left(\frac{ab}{M_t}\right)\right|^q\right)^{\frac{1}{q}}$$

$$\le \frac{ab(b-a)}{2}\left(\int_0^1 |1-2t|^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1 \frac{1}{M_t^{2q}}\left[(1-t)^2|e^{f(a)}f'(a)|^q + t^2|e^{f(b)}f'(b)|^q\right.\right.$$

$$\left.\left. +t(1-t)\{|e^{f(a)}f'(b)|^q + |e^{f(b)}f'(a)|^q\}\right]\right)^{\frac{1}{q}}$$

$$\leq \frac{ab(b-a)}{2}\left(\int_0^1 |1-2t|^{p\alpha}dt\right)^{\frac{1}{p}}$$
$$\left(\int_0^1 \frac{1}{M_t^{2q}}\left[(1-t)^2|e^{f(a)}f'(a)|^q + t^2|e^{f(b)}f'(b)|^q + t(1-t)\Delta_1(a,b)\right]\right)^{\frac{1}{q}}$$
$$\leq \frac{ab(b-a)}{2}\zeta_7^{\frac{1}{p}}\left(\zeta_8|e^{f(a)}f'(a)|^q + \zeta_9|e^{f(b)}f'(b)|^q + \zeta_{10}\Delta_1(a,b)\right)^{\frac{1}{q}},$$

where

$$\zeta_7 = \int_0^1 |1-2t|^{p\alpha}dt = \frac{1}{p\alpha+1},$$
$$\zeta_8 = \int_0^1 t^2 M_t^{-2q}dt = b^{-2q}\int_0^1 t^2\left(1-t(1-\frac{a}{b})\right)^{-2q}dt = \frac{1}{3b^{2q}}{}_2F_1\left(2q,3;4;1-\frac{a}{b}\right),$$
$$\zeta_9 = \int_0^1 (1-t)^2 M_t^{-2q}dt = b^{-2q}\int_0^1 (1-t)^2\left(1-t(1-\frac{a}{b})\right)^{-2q}dt$$
$$= \frac{1}{3b^{2q}}{}_2F_1\left(2q,1;4;1-\frac{a}{b}\right)$$

and

$$\zeta_{10} = \int_0^1 t(1-t)M_t^{-2q}dt = b^{-2q}\int_0^1 t(1-t)\left(1-t(1-\frac{a}{b})\right)^{-2q}dt \tag{2.22}$$
$$= \frac{1}{6b^{2q}}{}_2F_1\left(2q,2;4;1-\frac{a}{b}\right). \tag{2.23}$$

Thus, if we use ζ_7, ζ_8, ζ_9 and ζ_{10} in (2.21), we obtain inequality (2.20). The proof is completed. □

3. CONCLUSION

In this paper, the definition of exponentially harmonically convex functions is given and a new integral identity that includes Riemann-Liouville fractional integral operators is established. Depending on this new definition and identity, some new Hermite-Hadamard type inequalities for exponentially harmonically convex functions are built via Riemann-Liouville fractional forms. By choosing $\alpha = 1$, one can reduce our main results to provide integral inequalities for classical integrals, we omit the details.

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AN ITERATIVE PROCESS OF GENERALIZED LIPSCHITIZIAN MAPPINGS IN UNIFORMLY CONVEX BANACH SPACES

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Abstract. In this article, three generalized Lipschitizian mappings in the uniformly convex Banach spaces are considered. By defining an iterative process, the existence of a common fixed point of this iteration is proved. An example is presented to guarantee the convergence of the iteration.

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1. INTRODUCTION

In computational mathematics, an iterative method is a mathematical procedure that uses an initial value to generate a sequence of improving approximate solutions for a class of problems, in which the $n-th$ approximation is derived from the previous ones. If an equation can be put into the form $f(x) = x$, and a solution x is an attractive fixed point of the function f, then one may begin with a point x_1 in the basin of attraction of x, and let $x_{n+1} = f(x_n)$ for $n = 1$, and the sequence $\{x_n\}_{n\geq 1}$ will converge to the solution x. Here x_n is the n-th approximation or iteration of x and x_{n+1} is the next or $n+1$ iteration of x (see [4, 7, 9–16], and reference therein).

Let C be a nonempty subset of a Banach space E, $T : C \longrightarrow C$ be a mapping and

$$F(T) = \{x \in C : Tx = x\}$$

denotes the set of fixed points of T. A mapping T is said to be asymptotically nonexpansive, if there exists a sequence $\{k_n\}$ of positive numbers with $\lim\limits_{n\longrightarrow\infty} k_n = 1$ such that for $x, y \in C$ and $n \geq 1$,

$$\|T^n x - T^n y\| \leq k_n \|x - y\|.$$

The study of iterative construction for fixed points of asymptotically nonexpansive mappings began in 1975. Baillon [3] proved, if C is a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}$ and $T : C \to C$ is a nonexpansive mapping such that

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$F(T) \neq \varnothing$, then for every $x \in C$, the Cesáro means

$$T_n x = \frac{1}{n+1} \sum_{j=0}^{n} T^j x$$

is weakly convergent to a fixed point of T. Then, Shimizu et al. [18] studied the convergence of the following approximated sequence for an asymptotically nonexpansive mapping in Hilbert space,

$$x_0 = x \in C,$$
$$x_{n+1} = \alpha_n x + (1-\alpha_n) \frac{1}{n+1} \sum_{j=0}^{n} T^j x_n,$$

where $\{\alpha_n\}$ is a real sequence satisfying $0 < \alpha_n \leq 1$ and $\alpha_n \longrightarrow 0$ as $n \longrightarrow \infty$. They proved that $\{x_n\}$ is strongly convergent to an element of $F(T)$. We recall the definition of uniformly convex space (see [2, 6] for more details).

Definition 1. A Banach space E is said to be strictly convex if

$$\|x+y\| < 2$$

for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. We recall that a Banach space E is called uniformly convex, if for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $\|x\| = \|y\| = 1$ then

$$\|\frac{(x+y)}{2}\| \leq 1 - \delta.$$

It is obvious that uniform convexity implies strict convexity.

In 1991, [23] proved the characterization of uniform convexity as follows.

Theorem 1. *A Banach space E is uniformly convex if and only if for each fixed number $r > 0$, there exists a continuous function $\varphi : [0,\infty) \to [0,\infty)$, $\varphi(s) = 0 \Longleftrightarrow s = 0$, such that*

$$\|\lambda x + (1-\lambda) y\|^2 \leq \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\varphi(\|x-y\|)$$

for all $\lambda \in [0,1]$ and all $x, y \in E$ such that $\|x\| \leq r$ and $\|y\| \leq r$.

Shioji et al. [19] studied the strongly convergence of the sequence

$$x_0 = x \in C,$$
$$x_{n+1} = \alpha_n x + (1-\alpha_n) \frac{1}{n+1} \sum_{j=0}^{n} T^j x_n,$$

in uniformly convex Banach spaces with uniformly Gâteaux differentiable norms.

Also, for an asymptotically nonexpansive mapping T, Tan et. al [22] defined the modified Ishikawa iterations process

$$x_{n+1} = t_n T^n (s_n T^n x_n + (1-s_n) x_n) + (1-t_n) x_n,$$

where $\{t_n\}$ and $\{s_n\}$ are real sequences such that $\{t_n\}$ is bounded away from 0 and 1 and $\{s_n\}$ bounded away from 1. They proved the sequence $\{x_n\}$ is weakly convergent to a fixed point of T.

Definition 2. Let E be a Banach space. E is said to satisfy Opial's condition if for each sequence $\{x_n\}$ in E the condition $x_n \rightharpoonup x$ implies

$$\limsup_{n\longrightarrow\infty} \|x_n - x\| < \limsup_{n\longrightarrow\infty} \|x_n - y\|$$

for all $y \in E$ and $y \neq x$.

Definition 3. Let E be an arbitrary real Banach space with norm $\|.\|$ and E^* be the dual space of E. The duality mapping $J : E \longrightarrow E^*$ is defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2, \|f\| = \|x\|\},$$

where $\langle x, f\rangle$ denotes the value of the continuous linear function $f \in E^*$ at $x \in E$.

Lemma 1 ([21])**.** *Let $\{\delta_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences of nonnegative numbers satisfying the recursive inequality*

$$\delta_{n+1} \leq \beta_n\delta_n + \gamma_n \ \text{ for all } n \geq 1,$$

if $\beta_n \geq 1$, $\sum_{n=1}^{\infty}(\beta_n - 1) < \infty$ and $\sum_{n=1}^{\infty}\gamma_n < \infty$, then $\lim_{n\to\infty}\delta_n$ exists.

Lemma 2 ([17])**.** *Let E be a uniformly convex Banach space. Assume $\{t_n\}$ is a sequence of real numbers in $(0,1)$ bounded away from 0 and 1. If $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that for some $a \geq 0$*

$$\limsup_{n\longrightarrow\infty} \|x_n\| \leq a,\ \limsup_{n\longrightarrow\infty} \|y_n\| \leq a \text{ and } \limsup_{n\longrightarrow\infty} \|t_nx_n + (1-t_n)y_n\| = a,$$

then $\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0$.

Lemma 3 ([1])**.** *Let $\{x_n\}$ be a bounded sequence in a uniformly convex Banach space E. If $w_w(\{x_n\}) = \{x\}$, then $x_n \rightharpoonup x$, where $w_w(\{x_n\}) = \{x \in E : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak w-limit set of $\{x_n\}$.*

In 2001, Jung et al. [8] introduced the following class of mappings.

Definition 4. A mapping $T : C \to C$, where C is a nonempty subset of Banach space E, is said to be a generalized Lipschitzian mapping if

$$\|T^nx - T^ny\| \leq a_n\|x-y\| + b_n(\|x - T^nx\| + \|T^ny - y\|) + c_n(\|x - T^ny\| + \|T^nx - y\|) \tag{1.1}$$

for each $x, y \in C$ and $n \geq 1$, where a_n, b_n and c_n are nonnegative constants such that there exists an integer n_0 such that $b_n + c_n < 1$, for all $n > n_0$.

Definition 5. A mapping $T : C \to C$, where C is a nonempty subset of Banach space E, is said to be a uniformly generalized Lipschitzian mapping if

$$\|T^n x - T^n y\| \leq a\|x-y\| + b(\|x-T^n x\| + \|T^n y - y\|) + c(\|x-T^n y\| + \|T^n x - y\|) \tag{1.2}$$

for each $x, y \in C$ where a, b and c are nonnegative constants and $b + c < 1$.

Clearly, every Lipschitzian mapping is a generalized Lipschitzian mapping. But the vice versa is not necessarily true. See the following example.

Example 1. Let $E = \mathbb{R}$ be the set of real numbers and $C = [0, \infty)$. For each $x \in C$, we define

$$Tx = \begin{cases} \dfrac{rx}{1+x} & \text{if} \quad x \in [0, \dfrac{1}{4}], \\ 0 & \text{if} \quad x \in (\dfrac{1}{4}, \infty), \end{cases} \tag{1.3}$$

where $0 < r < \frac{1}{4}$. Then $T : C \longrightarrow C$ is not continuous at $x = \frac{1}{4}$ and hence T is not a Lipschitzian mapping. Set $C_1 = [0, \frac{1}{4}]$ and $C_2 = (\frac{1}{4}, \infty)$. In order to prove $T : C \longrightarrow C$ is a generalized Lipschitzian mapping, we need the following steps: for all $x, y \in C_1$ and $n \geq 1$,

$$|Tx - Ty| = |\frac{rx}{1+x} - \frac{ry}{1+y}| = |\frac{rx(1+y) - ry(1+x)}{(1+x)(1+y)}| \leq r|x-y|$$

and

$$|T^2 x - T^2 y| = |\frac{rTx}{1+Tx} - \frac{rTy}{1+Ty}| \leq r|Tx - Ty| \leq r^2|x-y|.$$

By induction, for all $n \geq 1$

$$|T^n x - T^n y| \leq r^n|x-y| + r^n(|x-T^n x| + |y-T^n y|) + r^n(|x-T^n y| + |y-T^n x|). \tag{1.4}$$

For all $x, y \in C_2$ and $n \geq 1$,

$$|T^n x - T^n y| = 0 \leq |x-y|. \tag{1.5}$$

For $x \in C_1$ and $y \in C_2$,

$$|Tx - Ty| = |\frac{rx}{1+x} - 0|.$$

By induction, for $n \in \mathbb{N}$,

$$T^{n+1}x = \frac{rT^n x}{1+T^n x} \leq rT^n x \leq r^{n+1}x,$$

and by a computation

$$\begin{aligned} |T^n x - T^n y| &= |T^n x - 0| \\ &\leq r^n|(x-y) + (y-T^n x) + (T^n x - x) + x| \\ &\leq r^n(|x-y| + |T^n x - x| + |y-T^n x| + |x+y-0|) \\ &\leq r^n|x-y| + r^n(|x-T^n x| + |y-T^n y|) + r^n(|x-T^n y| + |y-T^n x|). \end{aligned} \tag{1.6}$$

Thus inequalities (1.4), (1.5) and (1.6) imply that $T : C \longrightarrow C$ is a generalized Lipschitzian mapping.

Lemma 4. *Let C be a bounded, closed and convex subset of a uniformly convex Banach space E and T be a uniformly generalized Lipschitzian mapping of C into itself. Then, for any $\varepsilon > 0$, there exists a positive number $\xi(\varepsilon)$ such that $\|Tx - x\| < \varepsilon$ for all $x \in [x_0, x_1]$ whenever for $x_0, x_1 \in C$, $\|Tx_0 - x_0\| \le \xi(\varepsilon)$ and $\|Tx_1 - x_1\| \le \xi(\varepsilon)$, where*

$$[x_0, x_1] = \{\lambda x_1 + (1-\lambda)x_0 : 0 \le \lambda \le 1\}.$$

Proof. Fix $\varepsilon > 0$ and $x \in [x_0, x_1]$. Then $x = \lambda x_1 + (1-\lambda)x_0$ for some λ with $0 \le \lambda \le 1$. We consider two cases, $\|x_1 - x_0\| < \varepsilon_1$ and $\|x_1 - x_0\| \ge \varepsilon_1$.

If $\|x_1 - x_0\| < \varepsilon_1$ then it is obvious that $\|x - x_0\| < \varepsilon_1$, where $\varepsilon_1 = \frac{1-(b+c)}{1+a+2c}\frac{\varepsilon}{2}$. Since

$$\begin{aligned}
\|Tx - x\| &\le \|Tx - Tx_0\| + \|Tx_0 - x_0\| + \|x_0 - x\| \\
&\le a\|x_0 - x\| + b(\|Tx - x\| + \|Tx_0 - x_0\|) \\
&\quad + c(\|x_0 - Tx\| + \|Tx_0 - x\|) + \|Tx_0 - x_0\| + \|x_0 - x\| \\
&\le a\|x_0 - x\| + b(\|Tx - x\| + \|Tx_0 - x_0\|) \\
&\quad + c(\|x - Tx\| + \|Tx_0 - x_0\| + 2\|x_0 - x\|) + \|Tx_0 - x_0\| + \|x_0 - x\|
\end{aligned}$$

and so

$$\|Tx - x\| \le \frac{1+a+2c}{1-(b+c)}\|x_0 - x\| + \frac{b+c+1}{1-(b+c)}\|x_0 - Tx_0\|. \tag{1.7}$$

Thus if $\|Tx_0 - x_0\| \le \xi(\varepsilon) < \frac{1-(b+c)}{1+b+c}\frac{\varepsilon}{2}$, then $\|Tx - x\| < \varepsilon$.

If $\|x_0 - x_1\| \ge \varepsilon_1$, let d_0 denote the diameter of C. For any nonnegative number $0 \le \lambda \le 1$, we consider three cases:

Case 1) If $0 \le \lambda < \frac{\varepsilon_1}{d_0}$, then

$$\|x_0 - x\| = \lambda\|x_1 - x_0\| < \frac{\varepsilon_1}{d_0}\|x_1 - x_0\| = \varepsilon_1\frac{\|x_1 - x_0\|}{d_0} < \varepsilon_1.$$

Thus similar to above, if $\xi(\varepsilon) < \frac{1-(b+c)}{1+b+c}\frac{\varepsilon}{2}$, then inequation (1.7) implies $\|Tx - x\| < \varepsilon$.

Case 2) If $1 - \frac{\varepsilon_1}{d_0} < \lambda \le 1$ or $0 \le (1-\lambda) < \frac{\varepsilon_1}{d_0}$, so

$$\|x_1 - x\| = (1-\lambda)\|x_1 - x_0\| < \varepsilon_1,$$

and similar to above we have $\|Tx - x\| < \varepsilon$.

Case 3) If $\frac{\varepsilon_1}{d_0} \le \lambda \le 1 - \frac{\varepsilon_1}{d_0}$, $y = Tx$ implies

$$\|y - x_0\| = \|Tx - x_0\| \le \|Tx - Tx_0\| + \|Tx_0 - x_0\|.$$

By inequalities (1.2) and (1.7),

$$\|y - x_0\| \le \frac{a+b+c}{1-(b+c)}\|x - x_0\| + \frac{1+b+c}{1-(b+c)}\|Tx_0 - x_0\| \le r\lambda\|x_1 - x_0\| + h\xi(\varepsilon), \tag{1.8}$$

where $r=\frac{a+b+c}{1-(b+c)}$ and $h=\frac{1+b+c}{1-(b+c)}$, and also

$$\begin{aligned}\|y-x_1\| &\le \frac{a+b+c}{1-(b+c)}\|x-x_1\|+\frac{1+b+c}{1-(b+c)}\|Tx_1-x_1\| \\ &\le r(1-\lambda)\|x_1-x_0\|+h\xi(\varepsilon).\end{aligned} \tag{1.9}$$

Set

$$z_0=\frac{y-x_0}{\lambda\|x_1-x_0\|} \text{ and } z_1=\frac{x_1-y}{(1-\lambda)\|x_1-x_0\|}. \tag{1.10}$$

Then

$$\|z_0\| \le \frac{r\lambda\|x_1-x_0\|+h\xi(\varepsilon)}{\lambda\|x_1-x_0\|} \le r+\frac{h\xi(\varepsilon)d_0}{\varepsilon_1^2},$$

and similarly

$$\|z_1\| \le r+\frac{h\xi(\varepsilon)d_0}{\varepsilon_1^2}.$$

On the other hand, for λ with $\frac{\varepsilon_1}{d_0} \le \lambda \le 1-\frac{\varepsilon_1}{d_0}$,

$$\|\lambda z_0+(1-\lambda)z_1\| = \frac{\|x_1-x_0\|}{\|x_1-x_0\|}=1.$$

Therefore using uniform convexity of E, we can choose $\xi(\varepsilon)$ so small that $\|z_1-z_0\|<\frac{\varepsilon}{d_0}$. Thus by $x=\lambda x_1+(1-\lambda)x_0$ and (1.10) we have

$$\begin{aligned}\|y-x\| &= \|(1-\lambda)(y-x_0)-\lambda(x_1-y)\| \\ &= \lambda(1-\lambda)\|x_1-x_0\|\|z_1-z_0\| \\ &< \varepsilon.\end{aligned} \tag{1.11}$$

Notice that $y=Tx$, and this means $\|Tx-x\|<\varepsilon$. □

Lemma 5 ([20]). *Let C be a bounded, closed and convex subset of a uniformly convex Banach space E and T be a uniformly generalized Lipschitzian mapping of C into itself. If $\{x_j\}$ is a sequence of C such that $x_j \rightharpoonup x_0$ and $x_j-Tx_j \longrightarrow 0$, then x_0 is a fixed point of T.*

Proof. Since $\|(I-T)x_j\|=\varepsilon_j \longrightarrow 0$, for each j we assume $\varepsilon_j \le \xi(\varepsilon_{j-1})<\varepsilon_{j-1}$, where $\xi(\varepsilon)$ for any $\varepsilon>0$ is the constant described in the conclusion of Lemma 4. Hence if $x \in \overline{Co}\{x_j : j \ge k\}$ then $\|x-Tx\| \le \varepsilon_{k-1}$. Since $\overline{Co}\{x_j : j \ge k\}$ is weakly compact, $x_0 \in \overline{Co}\{x_j : j \ge k\}$, $k=1,2,3,\cdots$ and hence $\|x_0-Tx_0\| \le \varepsilon_j$, $j=1,2,3,\cdots$. This implies $\|Tx_0-x_0\|=0$. □

Let C be a bounded, closed and convex subset of a Banach space E and $\{T_n\}$ be a sequence of uniformly generalized Lipschitzian self-mappings of C, such that the set F of common fixed points of $\{T_n\}$ is nonempty. Let $k_n=\frac{a_n+b_n+c_n}{1-(b_n+c_n)}$ for T_n and $k_n \ge 1$ for all $n \ge 1$. For a given $x_1 \in C$, we define the sequence $\{x_n\}$ by $x_{n+1}=T_nx_n$ for $n \ge 1$.

Lemma 6. *Let E be a normed linear space and C be a nonempty closed and convex subset of E. Let $T : C \longrightarrow C$ be a uniformly generalized Lipschitzian mapping. Let $\{x_n\}$ be defined by $x_{n+1} = T_n x_n$. If $\lim_{n\longrightarrow\infty} \|x_n - T^n x_n\| = 0$ and $\lim_{n\longrightarrow\infty} \|x_n - x_{n+1}\| = 0$, then $\lim_{n\longrightarrow\infty} \|x_n - Tx_n\| = 0$.*

Proof. Let $r_n = \|T^n x_n - x_n\|$. Since T is a uniformly generalized Lipschitzian mapping,

$$\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &\le \|x_{n+1} - T^{n+1}x_{n+1}\| + \|Tx_{n+1} - T^{n+1}x_{n+1}\| \\
&\le r_{n+1} + a\|x_{n+1} - T^n x_{n+1}\| + b(\|x_{n+1} - Tx_{n+1}\| \\
&\quad + \|T^n x_{n+1} - T^{n+1}x_{n+1}\|) + c(\|x_{n+1} - T^{n+1}x_{n+1}\| \\
&\quad + \|T^n x_{n+1} - Tx_{n+1}\|) \\
&\le r_{n+1} + a\|x_{n+1} - T^n x_{n+1}\| + b(\|T^n x_{n+1} - x_{n+1}\| \\
&\quad + \|x_{n+1} - Tx_{n+1}\| + \|T^{n+1}x_{n+1} - x_{n+1}\|) \\
&\quad + c(\|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^n x_{n+1} - x_{n+1}\| + \|x_{n+1} - Tx_{n+1}\|),
\end{aligned}$$

and

$$\|x_{n+1} - Tx_{n+1}\| \le \frac{1+b+c}{1-(b+c)} r_{n+1} + \frac{a+b+c}{1-(b+c)} \|x_{n+1} - T^n x_{n+1}\|. \tag{1.12}$$

Now we obtain $\|x_{n+1} - T^n x_{n+1}\|$ as

$$\begin{aligned}
\|x_{n+1} - T^n x_{n+1}\| &\le \|x_n - T^n x_n\| + \|x_{n+1} - x_n\| + \|T^n x_{n+1} - T^n x_n\| \\
&\le r_n + \|x_{n+1} - x_n\| + a\|x_{n+1} - x_n\| + b(\|T^n x_n - x_n\| \\
&\quad + \|T^n x_{n+1} - x_{n+1}\|) + c(\|x_{n+1} - T^n x_n\| + \|T^n x_{n+1} - x_n\|).
\end{aligned}$$

So

$$\|T^n x_{n+1} - x_{n+1}\| \le \frac{1+b+c}{1-(b+c)} r_n + \frac{1+a+2c}{1-(b+c)} \|x_{n+1} - x_n\|. \tag{1.13}$$

By relations (1.12) and (1.13),

$$\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &\le \frac{1+b+c}{1-(b+c)} r_{n+1} + \frac{a+b+c}{1-(b+c)} \|T^n x_{n+1} - x_{n+1}\| \\
&\le \frac{1+b+c}{1-(b+c)} r_{n+1} + \frac{a+b+c}{1-(b+c)} \Big(\frac{1+b+c}{1-(b+c)} r_n \\
&\quad + \frac{1+a+2c}{1-(b+c)} \|x_{n+1} - x_n\| \Big).
\end{aligned}$$

This completes the proof. □

In the next section, based on [19] and [22], a modified version of the iterative process for three generalized Lipschitzian mappings is presented. Then the existence of a common fixed point for these three maps is proved.

2. An iterative approximation of three generalized Lipschitizian mappings

In this section, we prove, if $T_i : C \longrightarrow C$, $i = 1,2,3$, are three generalized Lipschitzian mappings, then the sequence $\{x_n\}$ which is defined by equation (2.1) converges to $q \in \bigcap_{i=1}^{3} F(T_i)$ in the uniformly convex Banach space.

The result presented in this section generalizes and improves the corresponding results, in [5, 19] and [22].

Let C be a nonempty subset of real Banach space E and $T_i : C \longrightarrow C$, $i = 1,2,3$, be three generalized Lipschitizian mappings. Consider the following iterative sequence $\{x_n\}$ which is defined by

$$\begin{cases} x_1 \in C, \\ u_n = \dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n} T_1^j x_n, \\ z_n = (1-\lambda_n)x_n + \lambda_n u_n & n \geq 1, \\ y_n = (1-\beta_n)x_n + \beta_n T_2^n z_n & n \geq 1, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_3^n y_n & n \geq 1, \end{cases} \tag{2.1}$$

where $0 < \alpha_n, \beta_n, \lambda_n < 1$. We prove, if $T_i : C \longrightarrow C$, $i = 1,2,3$, are three generalized Lipschitzian mappings, then the sequence $\{x_n\}$ which is defined by equation (2.1) converges to $q \in \bigcap_{i=1}^{3} F(T_i)$, in a uniformly convex Banach space.

Theorem 2. *Let E be a uniformly convex Banach space, satisfying Opial's condition and C be a nonempty, closed and convex subset of E. $T_i : C \longrightarrow C$ are three generalized Lipschitizian mappings for $i = 1,2,3$ satisfying*

$$\|T_i^n x - T_i^n y\| \leq a_n^i \|x-y\| + b_n^i(\|x - T_i^n x\| + \|T_i^n y - y\|) + c_n^i(\|x - T_i^n y\| + \|T_i^n x - y\|)$$

for all $x, y \in C$ and $n \geq 1$, where $k_n^i = \frac{a_n^i + b_n^i + c_n^i}{1-(b_n^i + c_n^i)}$ and $\{k_n^i\} \subset [1,\infty)$ for $i = 1,2,3$ satisfy $k_n^i \longrightarrow 1$ as $n \longrightarrow \infty$. Also $\sum_{n=1}^{\infty}(\eta_n k_n^2 k_n^3 - 1) < \infty$ where $\eta_n = \max\{k_j^1, 1 \leq j \leq n\}$. Let $\{x_n\}$ be the sequence defined by (2.1). If the following statements hold

- *(I)* $F = \bigcap_{i=1}^{3} F(T_i) \neq \phi$
- *(II)* $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty} \alpha_n < 1$
- *(III)* $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$
- *(IV)* $0 < \liminf_{n\longrightarrow\infty} \lambda_n \leq \limsup_{n\longrightarrow\infty} \lambda_n < 1$

then

1. $\lim_{n\longrightarrow\infty} \|x_n - q\|$ *exists, for all $q \in F$.*
2. $\lim_{n\longrightarrow\infty} \|x_n - u_n\| = 0$ *and* $\lim_{n\longrightarrow\infty} \|x_n - T_i x_n\| = 0$ $(i = 2,3)$.
3. *The sequence $\{x_n\}$ is weakly convergent to a common fixed point of T_i $(i = 1,2,3)$.*

Proof. For any $q \in F$ and $i = 1,2,3$, we get

$$\begin{aligned}\|T_i^n x_n - q\| &\leq a_n^i \|x_n - q\| + b_n^i \|T_i^n x_n - x_n\| + c_n^i (\|T_i^n x_n - q\| + \|x_n - q\|) \\ &\leq a_n^i \|x_n - q\| + b_n^i (\|T_i^n x_n - q\| + \|x_n - q\|) + c_n^i (\|T_i^n x_n - q\| + \|x_n - q\|),\end{aligned}$$

and so

$$\|T_i^n x_n - q\| \leq \frac{a_n^i + b_n^i + c_n^i}{1 - (b_n^i + c_n^i)} \|x_n - q\| = k_n^i \|x_n - q\|. \tag{2.2}$$

In order to find a bound for $\|x_{n+1} - q\|$, we need to compute $\|u_n - q\|$, $\|z_n - q\|$ and $\|y_n - q\|$ as follows:

(i) Suppose $1 \leq j \leq n$, by inequality (2.2)

$$\|u_n - q\| \leq \frac{1}{n+1} (\|x_n - q\| + \sum_{j=1}^{n} k_j^1 \|x_n - q\|) \leq \eta_n \|x_n - q\|. \tag{2.3}$$

(ii) The definition of z_n in (2.1) implies

$$\begin{aligned}\|z_n - q\| &= \|(1 - \lambda_n) x_n + \lambda_n u_n - q\| \\ &\leq (1 - \lambda_n) \|x_n - q\| + \lambda_n \eta_n \|x_n - q\| \\ &\leq (1 + \lambda_n (\eta_n - 1)) \|x_n - q\| \\ &\leq \eta_n \|x_n - q\|.\end{aligned} \tag{2.4}$$

(iii) The definition of y_n in (2.1) implies

$$\begin{aligned}\|y_n - q\| &= \|(1 - \beta_n) x_n + \beta_n T_2^n z_n - q\| \\ &\leq (1 - \beta_n) \|x_n - q\| + \beta_n k_n^2 \|z_n - q\| \\ &\leq (1 + \beta_n (k_n^2 \eta_n - 1)) \|x_n - q\| \\ &\leq k_n^2 \eta_n \|x_n - q\|.\end{aligned} \tag{2.5}$$

By (i), (ii) and (iii) one can have

$$\begin{aligned}\|x_{n+1} - q\| &= \|(1 - \alpha_n)(x_n - q) + \alpha_n (T_3^n y_n - q)\| \\ &\leq (1 - \alpha_n) \|x_n - q\| + \alpha_n k_n^3 \|y_n - q\| \\ &\leq (1 - \alpha_n) \|x_n - q\| + \alpha_n k_n^3 k_n^2 \eta_n \|x_n - q\| \\ &\leq (1 + \alpha_n (k_n^3 k_n^2 \eta_n - 1)) \|x_n - q\|.\end{aligned} \tag{2.6}$$

Since $\sum_{n=1}^{\infty} (\eta_n k_n^2 k_n^3 - 1) < \infty$, Lemma 1 and inequality (2.6) imply $\lim_{n \longrightarrow \infty} \|x_n - q\| = r$ exists. Furthermore,

$$\lim_{n \longrightarrow \infty} \|x_{n+1} - q\| = \lim_{n \longrightarrow \infty} \|(1 - \alpha_n)(x_n - q) + \alpha_n (T_3^n y_n - q)\| = r.$$

Also, by (2.5)

$$\limsup_{n \longrightarrow \infty} \|T_3^n y_n - q\| \leq \limsup_{n \longrightarrow \infty} (k_n^3 \|y_n - q\|)$$

$$\leq \limsup_{n\longrightarrow\infty}(k_n^3k_n^2\eta_n\|x_n-q\|)$$
$$\leq \lim_{n\longrightarrow\infty}\|x_n-q\| = r.$$

Lemma 2 shows that $\lim_{n\longrightarrow\infty}\|T_3^n y_n - x_n\| = 0$. Furthermore, by (2.6)

$$\frac{\|x_{n+1}-q\| - \|x_n-q\| + \alpha_n\|x_n-q\|}{\alpha_n k_n^3} \leq \|y_n-q\|, \tag{2.7}$$

and by taking the limit $n \longrightarrow \infty$ in (2.7), we get

$$\lim_{n\longrightarrow\infty}\|x_n-q\| \leq \liminf_{n\longrightarrow\infty}\|y_n-q\|.$$

By (2.5),

$$\limsup_{n\longrightarrow\infty}\|y_n-q\| \leq \limsup_{n\longrightarrow\infty}(k_n^2\eta_n\|x_n-q\|) = \lim_{n\longrightarrow\infty}\|x_n-q\|. \tag{2.8}$$

Notice that (2.7) and (2.8) show

$$\lim_{n\longrightarrow\infty}\|x_n-q\| = \lim_{n\longrightarrow\infty}\|y_n-q\| = r.$$

Thus

$$\limsup_{n\longrightarrow\infty}\|(1-\beta_n)(x_n-q)+\beta_n(T_2^n z_n-q)\| = \limsup_{n\longrightarrow\infty}\|y_n-q\| = r.$$

On the other hand, (2.4) implies

$$\limsup_{n\longrightarrow\infty}\|T_2^n z_n-q\| \leq \limsup_{n\longrightarrow\infty}(k_n^2\|z_n-q\|)$$
$$\leq \limsup_{n\longrightarrow\infty}(k_n^2\eta_n\|x_n-q\|)$$
$$\leq \lim_{n\longrightarrow\infty}\|x_n-q\| = r,$$

and Lemma 2 shows that $\lim_{n\longrightarrow\infty}\|T_2^n z_n - x_n\| = 0$. Using the same technique, we have $\lim_{n\longrightarrow\infty}\|z_n-q\| = r$ and $\lim_{n\longrightarrow\infty}\|u_n-x_n\| = 0$.

Now, we prove $\|x_n - T_2^n x_n\| \longrightarrow 0$. From (2.1),

$$\|z_n-x_n\| = \|(1-\lambda_n)x_n+\lambda_n u_n - x_n\| = \lambda_n\|u_n-x_n\| \longrightarrow 0, \text{ as } n\longrightarrow\infty.$$

Notice that $\|x_n - T_2^n x_n\| \leq \|x_n - T_2^n z_n\| + \|T_2^n z_n - T_2^n x_n\|$, where

$$\|T_2^n z_n - T_2^n x_n\| \leq a_n^2\|z_n-x_n\| + b_n^2(\|z_n - T_2^n z_n\| + \|T_2^n x_n - x_n\|) \tag{2.9}$$
$$+ c_n^2(\|T_2^n z_n - x_n\| + \|T_2^n x_n - z_n\|)$$
$$\leq a_n^2\|x_n-z_n\| + b_n^2(\|z_n-x_n\| + \|T_2^n z_n - x_n\| + \|T_2^n x_n - x_n\|)$$
$$+ c_n^2(\|T_2^n z_n - x_n\| + \|x_n - z_n\| + \|x_n - T_2^n x_n\|)$$
$$\leq (a_n^2+b_n^2+c_n^2)\|x_n-z_n\| + (b_n^2+c_n^2)(\|T_2^n z_n - x_n\| + \|T_2^n x_n - x_n\|).$$

Since

$$\|x_n - T_2^n x_n\| \leq \|x_n - T_2^n z_n\| + (a_n^2 + b_n^2 + c_n^2)\|x_n - z_n\| + (b_n^2 + c_n^2)(\|T_2^n z_n - x_n\| + \|T_2^n x_n - x_n\|)$$

and

$$\|x_n - T_2^n x_n\| \leq k_n^2 \|z_n - x_n\| + (1 + k_n^2)\|x_n - T_2^n z_n\|, \tag{2.10}$$

by inequality (2.10), $\|x_n - T_2^n x_n\| \longrightarrow 0$.

Similarly, $\|x_n - T_3^n x_n\| \longrightarrow 0$. Therefore, by Lemma 6, $\|x_n - T_i x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, for $i = 2, 3$.

Since $\lim_{n\longrightarrow\infty} \|x_n - q\|$ exists and by the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{nk}\}$ of $\{x_n\}$ such that $x_{nk} \rightharpoonup x$. Lemma 5 shows $x \in F$. Notice $w_w(\{x_n\})$ is a singleton. To do this, suppose there exists another subsequence $\{x_{nj}\}$ of $\{x_n\}$ which is weakly convergent to some $z \neq x$ such that $z \in F$. The existence of $\lim_{n\to\infty} \|x_n - q\|$ implies the existence of $\lim_{n\to\infty} \|x_n - x\|$ and $\lim_{n\to\infty} \|x_n - z\|$. Since E satisfies the Opial's condition, we have

$$\lim_{n\to\infty} \|x_n - x\| = \lim_{k\to\infty} \|x_{nk} - x\| < \lim_{k\to\infty} \|x_{nk} - z\| = \lim_{n\to\infty} \|x_n - z\|,$$

and

$$\lim_{n\to\infty} \|x_n - z\| = \lim_{j\to\infty} \|x_{nj} - z\| < \lim_{j\to\infty} \|x_{nj} - x\| = \lim_{n\to\infty} \|x_n - x\|,$$

which leads to a contradiction, and $w_w(\{x_n\})$ is a singleton. Therefore by Lemma 3 $\{x_n\}$ is weakly convergent to x. □

With respect to Theorem 2, the following example is presented.

Example 2. Let $E = \mathbb{R}$, $C = [0, \infty)$. Assume $\{x_n\}$ is the sequence defined by (2.1), where $T_1 x = \frac{x}{100}$, $T_2 x = \frac{x}{1000}$ and $T_3 x = \frac{x}{10000}$. Also $\alpha_n = \frac{n}{3n+1}$, $\beta_n = \frac{n}{4n+1}$ and $\lambda_n = \frac{n}{5n+1}$. We have

$$\begin{cases} u_n = \dfrac{1}{n+1} \displaystyle\sum_{j=0}^{n} \frac{x_n}{100^j}, \\ z_n = \dfrac{4n+1}{5n+1} x_n + \left(\dfrac{n}{5n+1}\right) u_n, \\ y_n = \dfrac{3n+1}{4n+1} x_n + \left(\dfrac{n}{4n+1}\right)\left(\dfrac{1}{1000^n}\right) z_n, \\ x_{n+1} = \dfrac{2n+1}{3n+1} x_n + \left(\dfrac{n}{3n+1}\right)\left(\dfrac{1}{10000^n}\right) y_n. \end{cases}$$

Let $x_1 = 1$, then by iteration we can have

$$x_{10} = 0.03842, \cdots, x_{20} = 0.00074, \cdots, x_{30} = 0.00001, \cdots$$

This shows $x_n \to 0$. Thus 0 is a common fixed point of T_1, T_2 and T_3 or $\{0\} = \bigcap_{i=1}^{3} F(T_i)$.

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HYPER-DUAL HORADAM QUATERNIONS

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Abstract. This paper deals with developing a new class of quaternions, called hyper-dual Horadam quaternions which are constructed from the quaternions whose components are hyper-dual Horadam numbers. We investigate the algebraic properties of these quaternions.

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1. INTRODUCTION

Dual numbers were invented by Clifford [3] as an extension of the real numbers. The set of dual numbers is defined as

$$\mathbb{D} = \{d = a + a^*\varepsilon \mid a, a^* \in \mathbb{R}\},$$

where ε is the dual unit with $\varepsilon \neq 0, \varepsilon^2 = 0$. Dual numbers have many interesting applications on mechanics, robotics, computer graphics, geometry, and physics. Especially, they are important to represent rigid body motions in 3D-space. The addition and multiplication of two dual numbers $d_1 = a + a^*\varepsilon$ and $d_2 = b + b^*\varepsilon$ are defined as

$$d_1 + d_2 = (a+b) + (a^* + b^*)\varepsilon \quad \text{and} \quad d_1 d_2 = ab + (ab^* + a^* b)\varepsilon,$$

respectively.

Similar to the quaternions, dual quaternions are defined by taking dual numbers instead of real numbers as coefficients. A dual quaternion $\widetilde{q}$ is defined as

$$\widetilde{q} = d_0 + d_1 i + d_2 j + d_3 k,$$

where $d_0, d_1, d_2, d_3 \in \mathbb{D}$, and the elements i, j, k satisfy the following noncommutative quaternion multiplication rules:

$$i^2 = j^2 = k^2 = ijk = -1. \tag{1.1}$$

Since any dual quaternion can be written as a dual number with real quaternion coefficients, it is constructed from eight real parameters. For the detailed information related to these numbers and their applications, we refer to [3, 10, 16].

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Hyper-dual numbers can be seen as an extension of dual numbers in the same way that quaternions are an extension of complex numbers. To get an advantage on exact calculations of second (or higher) derivatives, Fike and Alanso [6, 7] introduced the hyper-dual numbers. The set of hyper-dual numbers is defined by

$$\mathbb{HD} = \{D = a_0 + a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_1\varepsilon_2 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\},$$

where the dual units ε_1 and ε_2 satisfy the following rules:

$$\varepsilon_1^2 = \varepsilon_2^2 = (\varepsilon_1\varepsilon_2)^2 = 0, \quad \varepsilon_1 \neq \varepsilon_2, \quad \varepsilon_1 \neq 0, \varepsilon_2 \neq 0, \quad \varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1 \neq 0. \tag{1.2}$$

Hyper-dual numbers form a 4-dimensional vector space over real numbers with basis $\{1, \varepsilon_1, \varepsilon_2, \varepsilon_1\varepsilon_2\}$. Also, a hyper-dual number D can be written as $D = d + d^*\varepsilon^*$ where $d = a_0 + a_1\varepsilon, d^* = a_2 + a_3\varepsilon \in \mathbb{D}$ and $\varepsilon_1 = \varepsilon, \varepsilon_2 = \varepsilon^*$. The addition and the multiplication of hyper-dual numbers $D_1 = d_1 + d_1^*\varepsilon^*$ and $D_2 = d_2 + d_2^*\varepsilon^*$ are defined, respectively, as:

$$D_1 + D_2 = (d_1 + d_2) + (d_1^* + d_2^*)\varepsilon^* \quad \text{and} \quad D_1D_2 = d_1d_2 + (d_1d_2^* + d_1^*d_2)\varepsilon^*.$$

For applications of hyper-dual numbers, see [4, 5, 8].

Recently, Aslan et al. [1] have defined the hyper-dual split quaternions as

$$\widehat{q} = D_0 + D_1 i + D_2 j + D_3 k$$

where $D_0, D_1, D_2, D_3 \in \mathbb{HD}$ and i, j, k satisfy the split quaternion multiplication rules $i^2 = -1$, $j^2 = k^2 = ijk = 1$. It is clear to see that a hyper-dual quaternion is constructed from sixteen real parameters. For the algebraic properties and the geometric interpretations of hyper-dual split quaternions, we refer to [1].

On the other hand, Horadam [11] introduced the quaternions whose components are Fibonacci numbers. More generally, by using Horadam's approach, Halıcı and Karatas [9] defined the nth Horadam quaternion as

$$Q_{w,n} = w_n + w_{n+1}i + w_{n+2}j + w_{n+3}k,$$

where $\{w_n\}$ is the Horadam sequence [12] and is defined by

$$w_n = pw_{n-1} + qw_{n-2}, \quad n \geq 2$$

with the arbitrary initial values w_0, w_1 and nonzero integers p, q. It is clear to see that the Horadam sequence $\{w_n\} := \{w_n(w_0, w_1; p, q)\}$ generalizes many well known integer sequences such as Fibonacci sequence $\{F_n\} = \{w_n(0, 1; 1, 1)\}$, Lucas sequence $\{L_n\} = \{w_n(2, 1; 1, 1)\}$, generalized Fibonacci sequence $\{u_n\} = \{w_n(0, 1; p, q)\}$, and the generalized Lucas sequence $\{v_n\} = \{w_n(2, p; p, q)\}$. The Binet formula of the Horadam sequence $\{w_n\}$ is

$$w_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}$$

where $A := w_1 - w_0\beta$, $B := w_1 - w_0\alpha$, and α, β are the roots of the characteristic polynomial $x^2 - px - q$, that is; $\alpha = \frac{(p+\sqrt{p^2+4q})}{2}$, $\beta = \frac{(p-\sqrt{p^2+4q})}{2}$. Also we have

$\alpha\beta = -q$, $\alpha+\beta = p$, $\Delta := \alpha-\beta = \sqrt{p^2+4q}$ with $p^2+4q > 0$. The Binet formula for the Horadam quaternions is

$$Q_{w,n} = \frac{A\alpha^*\alpha^n - B\beta^*\beta^n}{\alpha-\beta}, \tag{1.3}$$

where $\alpha^* = 1+\alpha i+\alpha^2 j+\alpha^3 k$ and $\beta^* = 1+\beta i+\beta^2 j+\beta^3 k$. For details, see [2,9,17].

Also, Nurkan and Guven [14] introduced the dual Fibonacci quaternions by taking dual Fibonacci numbers instead of real numbers as coefficients. These numbers can also be seen as dual numbers with Fibonacci quaternion coefficients. A generalization of dual Fibonacci quaternions can be found in [18]. In [13], the author introduced the dual Horadam quaternions as

$$\widetilde{Q}_{w,n} = \widetilde{w}_n + \widetilde{w}_{n+1} i + \widetilde{w}_{n+2} j + \widetilde{w}_{n+3} k,$$

where $\widetilde{w}_n = w_n + w_{n+1}\varepsilon$ is the nth dual Horadam number. Recently, in [15], the authors defined the hyper-dual numbers whose coefficients are from the sequences $\{U_{kn}\}$ and $\{V_{kn}\}$ which reduce to the sequences $\{w_n(0,1;p,1)\}$ and $\{w_n(2,p;p,1)\}$ for $k=1$, respectively.

In this paper, motivating the definition of hyper-dual split quaternions in [1], we consider the quaternions whose coefficients are taken from hyper-dual Horadam numbers. To do this, first we define the hyper-dual Horadam numbers, then we introduce the quaternions whose coefficients are taken from those numbers. We give the generating function and the Binet formula for hyper-dual Horadam quaternions. Some algebraic properties of these quaternions such as Vajda's identity, Catalan's identity, Cassini's identity, and d'Ocagne's identity are derived with the aid of the Binet formula. Moreover, we develop some matrix identities involving the hyper-dual Horadam quaternions which allow us to obtain some properties of these quaternions.

2. Main Results

In this section, first we define hyper-dual Horadam numbers, then by using these numbers we introduce hyper-dual Horadam quaternions, and investigate the basic properties of these quaternions.

Definition 1. The nth hyper-dual Horadam number is defined as

$$\widehat{w}_n = w_n + w_{n+1}\varepsilon_1 + w_{n+2}\varepsilon_2 + w_{n+3}\varepsilon_1\varepsilon_2,$$

where w_n is the nth Horadam number and $\varepsilon_1, \varepsilon_2$ are dual units satisfying the multiplication rules in (1.2).

Definition 2. The nth hyper-dual Horadam quaternion is defined as

$$\widehat{Q}_{w,n} = \widehat{w}_n + \widehat{w}_{n+1} i + \widehat{w}_{n+2} j + \widehat{w}_{n+3} k,$$

where $\widehat{w}_n$ is the nth hyper-dual Horadam number and i, j, k satisfy the quaternion multiplication rules in (1.1).

In the following table, we give the types of quaternions which are mentioned in this paper.

Type of quaternion	**Definition**
Horadam quaternion [9]	$Q_{w,n} = w_n + w_{n+1}i + w_{n+2}j + w_{n+3}k$, w_n is the nth Horadam number
Dual Horadam quaternion [13]	$\widetilde{Q}_{w,n} = \widetilde{w}_n + \widetilde{w}_{n+1}i + \widetilde{w}_{n+2}j + \widetilde{w}_{n+3}k$, $\widetilde{w}_n$ is the nth dual Horadam number
Hyper-dual Horadam quaternion	$\widehat{Q}_{w,n} = \widehat{w}_n + \widehat{w}_{n+1}i + \widehat{w}_{n+2}j + \widehat{w}_{n+3}k$, $\widehat{w}_n$ is the nth hyper-dual Horadam number

Table 1

Note that the nth hyper-dual Horadam quaternion can be expressed as

$$\widehat{Q}_{w,n} = Q_{w,n} + Q_{w,n+1}\varepsilon_1 + Q_{w,n+2}\varepsilon_2 + Q_{w,n+3}\varepsilon_1\varepsilon_2,$$

where $Q_{w,n}$ is the nth Horadam quaternion, and $\varepsilon_1, \varepsilon_2$ are dual units. The addition and the multiplication of two hyper-dual Horadam quaternions $\widehat{Q}_{w,n}$ and $\widehat{Q}_{w,m}$ are defined as

$$\begin{aligned}
\widehat{Q}_{w,n} + \widehat{Q}_{w,m} &= (Q_{w,n} + Q_{w,m}) + (Q_{w,n+1} + Q_{w,m+1})\,\varepsilon_1 \\
&\quad + (Q_{w,n+2} + Q_{w,m+2})\,\varepsilon_2 + (Q_{w,n+3} + Q_{w,m+3})\,\varepsilon_1\varepsilon_2, \\
\widehat{Q}_{w,n}\widehat{Q}_{w,m} &= Q_{w,n}Q_{w,m} + (Q_{w,n}Q_{w,m+1} + Q_{w,n+1}Q_{w,m})\,\varepsilon_1 \\
&\quad + (Q_{w,n}Q_{w,m+2} + Q_{w,n+2}Q_{w,m})\,\varepsilon_2 \\
&\quad + (Q_{w,n}Q_{w,m+3} + Q_{w,n+1}Q_{w,m+2} + Q_{w,n+2}Q_{w,m+1} + Q_{w,n+3}Q_{w,m})\,\varepsilon_1\varepsilon_2,
\end{aligned}$$

respectively.

The norm of a hyper-dual Horadam quaternion $\widehat{Q}_{w,n}$ is defined as

$$N(\widehat{Q}_{w,n}) := \widehat{Q}_{w,n}\overline{\widehat{Q}}_{w,n} = \overline{\widehat{Q}}_{w,n}\widehat{Q}_{w,n} = \widehat{w}_n^2 + \widehat{w}_{n+1}^2 + \widehat{w}_{n+2}^2 + \widehat{w}_{n+3}^2,$$

where $\overline{\widehat{Q}}_{w,n} := \widehat{w}_n - \widehat{w}_{n+1}i - \widehat{w}_{n+2}j - \widehat{w}_{n+3}k$ is the conjugate of $\widehat{Q}_{w,n}$. Also the norm of $\widehat{Q}_{w,n}$ can be obtained by the determinant of the matrix

$$\begin{pmatrix} \widehat{w}_n + \widehat{w}_{n+1}i & -\widehat{w}_{n+2}j - \widehat{w}_{n+3}i \\ \widehat{w}_{n+2}j - \widehat{w}_{n+3}i & \widehat{w}_n - \widehat{w}_{n+1}i \end{pmatrix}.$$

Theorem 1. *The hyper-dual Horadam quaternions satisfy the following relation:*

$$\widehat{Q}_{w,n} = p\widehat{Q}_{w,n-1} + q\widehat{Q}_{w,n-2}, \quad n \geq 2.$$

Proof. From the definition of the hyper-dual Horadam quaternions and the Horadam quaternions, we have

$$\begin{aligned} p\widehat{Q}_{w,n-1}+q\widehat{Q}_{w,n-2} &= p\left(Q_{w,n-1}+Q_{w,n}\varepsilon_1+Q_{w,n+1}\varepsilon_2+Q_{w,n+2}\varepsilon_1\varepsilon_2\right) \\ &\quad +q\left(Q_{w,n-2}+Q_{w,n-1}\varepsilon_1+Q_{w,n}\varepsilon_2+Q_{w,n+1}\varepsilon_1\varepsilon_2\right) \\ &= \left(pQ_{w,n-1}+qQ_{w,n-2}\right)+\left(pQ_{w,n}+qQ_{w,n-1}\right)\varepsilon_1 \\ &\quad +\left(pQ_{w,n+1}+qQ_{w,n}\right)\varepsilon_2+\left(pQ_{w,n+2}+qQ_{w,n+1}\right)\varepsilon_1\varepsilon_2 \\ &= Q_{w,n}+Q_{w,n+1}\varepsilon_1+Q_{w,n+2}\varepsilon_2+Q_{w,n+3}\varepsilon_1\varepsilon_2. \end{aligned}$$

□

Theorem 2. *The generating function for hyper-dual Horadam quaternions is*

$$G(x)=\frac{\widehat{Q}_{w,0}+\left(\widehat{Q}_{w,1}-p\widehat{Q}_{w,0}\right)x}{1-px-qx^2}.$$

Proof. Let

$$G(x):=\sum_{n=0}^{\infty}\widehat{Q}_{w,n}x^n=\widehat{Q}_{w,0}+\widehat{Q}_{w,1}x+\sum_{n=2}^{\infty}\widehat{Q}_{w,n}x^n.$$

From Theorem 1, we have

$$\begin{aligned} \left(1-px-qx^2\right)G(x) &= \widehat{Q}_{w,0}+\widehat{Q}_{w,1}x+\sum_{n=2}^{\infty}\widehat{Q}_{w,n}x^n \\ &\quad -p\widehat{Q}_{w,0}x-p\sum_{n=2}^{\infty}\widehat{Q}_{w,n-1}x^n-q\sum_{n=2}^{\infty}\widehat{Q}_{w,n-2}x^n \\ &= \widehat{Q}_{w,0}+\widehat{Q}_{w,1}x-p\widehat{Q}_{w,0}x+\sum_{n=2}^{\infty}\left(\widehat{Q}_{w,n}-p\widehat{Q}_{w,n-1}-q\widehat{Q}_{w,n-2}\right)x^n \\ &= \widehat{Q}_{w,0}+\left(\widehat{Q}_{w,1}-p\widehat{Q}_{w,0}\right)x. \end{aligned}$$

Thus, we get the desired result. □

Theorem 3. *The Binet formula of hyper-dual Horadam quaternions is*

$$\widehat{Q}_{w,n}=\frac{A\alpha^*\underline{\alpha}\alpha^n-B\beta^*\underline{\beta}\beta^n}{\alpha-\beta},$$

where

$$\begin{aligned} \alpha^* &= 1+\alpha i+\alpha^2 j+\alpha^3 k, & \beta^* &= 1+\beta i+\beta^2 j+\beta^3 k, \\ \underline{\alpha} &= 1+\alpha\varepsilon_1+\alpha^2\varepsilon_2+\alpha^3\varepsilon_1\varepsilon_2, & \underline{\beta} &= 1+\beta\varepsilon_1+\beta^2\varepsilon_2+\beta^3\varepsilon_1\varepsilon_2. \end{aligned}$$

Proof. From the Binet formula of Horadam quaternions in (1.3), we have

$$\begin{aligned}
\widehat{Q}_{w,n} &= Q_{w,n} + Q_{w,n+1}\varepsilon_1 + Q_{w,n+2}\varepsilon_2 + Q_{w,n+3}\varepsilon_1\varepsilon_2 \\
&= \left(\frac{A\alpha^*\alpha^n - B\beta^*\beta^n}{\alpha-\beta}\right) + \left(\frac{A\alpha^*\alpha^{n+1} - B\beta^*\beta^{n+1}}{\alpha-\beta}\right)\varepsilon_1 \\
&\quad + \left(\frac{A\alpha^*\alpha^{n+2} - B\beta^*\beta^{n+2}}{\alpha-\beta}\right)\varepsilon_2 + \left(\frac{A\alpha^*\alpha^{n+3} - B\beta^*\beta^{n+3}}{\alpha-\beta}\right)\varepsilon_1\varepsilon_2 \\
&= \frac{A\alpha^*\alpha^n}{\alpha-\beta}\left(1+\alpha\varepsilon_1+\alpha^2\varepsilon_2+\alpha^3\varepsilon_1\varepsilon_2\right) - \frac{B\beta^*\beta^n}{\alpha-\beta}\left(1+\beta\varepsilon_1+\beta^2\varepsilon_2+\beta^3\varepsilon_1\varepsilon_2\right) \\
&= \frac{A\alpha^*\underline{\alpha}\alpha^n - B\beta^*\underline{\beta}\beta^n}{\alpha-\beta}.
\end{aligned}$$

□

From Theorem 3, the Binet formulas of the hyper-dual generalized Fibonacci and Lucas quaternions can be obtained as

$$\widehat{Q}_{u,n} = \frac{\alpha^*\underline{\alpha}\alpha^n - \beta^*\underline{\beta}\beta^n}{\alpha-\beta} \quad \text{and} \quad \widehat{Q}_{v,n} = \alpha^*\underline{\alpha}\alpha^n + \beta^*\underline{\beta}\beta^n,$$

respectively.

By using (2), we obtain the following relation between the hyper-dual Fibonacci quaternions and the hyper-dual Lucas quaternions.

Theorem 4. *Let n be a positive integer. For hyper-dual Fibonacci quaternions and hyper-dual Lucas quaternions, the following equality holds:*

$$\widehat{Q}_{v,n} = \widehat{Q}_{u,n+1} + q\widehat{Q}_{u,n-1}.$$

Now we need the following lemma which allows us a remarkable simplification for obtaining the properties of hyper-dual Horadam quaternions.

Lemma 1. *Let $\theta := 1 - q + q^2 - q^3$ and $\omega := (1-q)\,i + \left(1+p^2+q\right)k$. Then we have*

$$\begin{aligned}
\underline{\alpha}\underline{\beta} &= \widehat{v}_0 - (1 + pq\varepsilon_1\varepsilon_2), \\
\alpha^*\beta^* &= Q_{v,0} - \theta - \Delta q\,(Q_{u,0} - \omega), \\
\beta^*\alpha^* &= Q_{v,0} - \theta + \Delta q\,(Q_{u,0} - \omega).
\end{aligned}$$

Proof. The proof can be done by using the multiplication rules in (1.1) and (1.2). We should note that the set of hyper-dual numbers form a commutative algebra. Therefore we have $\underline{\alpha}\underline{\beta} = \underline{\beta}\underline{\alpha}$. But since the quaternion multiplication is noncommutative, $\alpha^*\beta^*$ need not be equal to $\beta^*\alpha^*$. The results for $\alpha^*\beta^*$ and $\beta^*\alpha^*$ can also be found in [2, Lemma 1]. □

Theorem 5. *(Vajda's Identity) For integers n, r, and s, we have*

$$\widehat{Q}_{w,n+r}\widehat{Q}_{w,n+s}-\widehat{Q}_{w,n}\widehat{Q}_{w,n+r+s}=AB(-q)^n(\widehat{v}_0-(1+pq\varepsilon_1\varepsilon_2))\cdot u_r((Q_{v,0}-\theta)u_s+q(Q_{u,0}-\omega)v_s).$$

Proof. From the Binet formula of hyper-dual Horadam quaternions, we have

$$\begin{aligned}
&\Delta^2\left(\widehat{Q}_{w,n+r}\widehat{Q}_{w,n+s}-\widehat{Q}_{w,n}\widehat{Q}_{w,n+r+s}\right)=\\
&=\left(A\alpha^*\underline{\alpha}\alpha^{n+r}-B\beta^*\underline{\beta}\beta^{n+r}\right)\left(A\alpha^*\underline{\alpha}\alpha^{n+s}-B\beta^*\underline{\beta}\beta^{n+s}\right)\\
&\quad-\left(A\alpha^*\underline{\alpha}\alpha^{n}-B\beta^*\underline{\beta}\beta^{n}\right)\left(A\alpha^*\underline{\alpha}\alpha^{n+r+s}-B\beta^*\underline{\beta}\beta^{n+r+s}\right)\\
&=A^2(\alpha^*\underline{\alpha})^2\alpha^{2n+r+s}-AB\alpha^*\beta^*\underline{\alpha}\underline{\beta}\alpha^{n+r}\beta^{n+s}-AB\beta^*\alpha^*\underline{\beta}\underline{\alpha}\alpha^{n+s}\beta^{n+r}\\
&\quad+B^2\left(\beta^*\underline{\beta}\right)^2\beta^{2n+r+s}-A^2(\alpha^*\underline{\alpha})^2\alpha^{2n+r+s}+AB\alpha^*\beta^*\underline{\alpha}\underline{\beta}\alpha^{n}\beta^{n+r+s}\\
&\quad+AB\beta^*\alpha^*\underline{\beta}\underline{\alpha}\beta^{n}\alpha^{n+r+s}-B^2\left(\beta^*\underline{\beta}\right)^2\beta^{2n+r+s}\\
&=AB(\alpha\beta)^n\underline{\alpha}\underline{\beta}\left(\alpha^*\beta^*\left(-\alpha^r\beta^s+\beta^{r+s}\right)+\beta^*\alpha^*\left(-\alpha^s\beta^r+\alpha^{r+s}\right)\right).
\end{aligned}$$

By using Lemma 1, we have

$$\begin{aligned}
&\widehat{Q}_{w,n+r}\widehat{Q}_{w,n+s}-\widehat{Q}_{w,n}\widehat{Q}_{w,n+r+s}=\\
&=\frac{AB}{\Delta^2}(-q)^n\underline{\alpha}\underline{\beta}(-\alpha^*\beta^*\beta^s(\alpha^r-\beta^r)+\beta^*\alpha^*\alpha^s(\alpha^r-\beta^r))\\
&=\frac{AB}{\Delta}(-q)^n\underline{\alpha}\underline{\beta}u_r(\beta^*\alpha^*\alpha^s-\alpha^*\beta^*\beta^s)\\
&=\frac{AB}{\Delta}(-q)^n\underline{\alpha}\underline{\beta}u_r(Q_{v,0}-\theta+\Delta q(Q_{u,0}-\omega))\alpha^s\\
&\quad-\frac{AB}{\Delta}(-q)^n\underline{\alpha}\underline{\beta}u_r(Q_{v,0}-\theta-\Delta q(Q_{u,0}-\omega))\beta^s\\
&=AB(-q)^n(\widehat{v}_0-(1+pq\varepsilon_1\varepsilon_2))u_r((Q_{v,0}-\theta)u_s+q(Q_{u,0}-\omega)v_s).
\end{aligned}$$

□

From Theorem 5, we have the following results:

If we set $r\to -s$, we get the following Catalan's identity for hyper-dual Horadam quaternions:

$$\widehat{Q}_{w,n-s}\widehat{Q}_{w,n+s}-\widehat{Q}_{w,n}^2=-AB(-q)^{n-s}(\widehat{v}_0-(1+pq\varepsilon_1\varepsilon_2))u_s((Q_{v,0}-\theta)u_s+q(Q_{u,0}-\omega)v_s).$$

Here note that $u_{-n}=\frac{-u_n}{(-q)^n}$.

If we set $s=-r=1$, we get the following Cassini's identity for hyper-dual Horadam quaternions:

$$\widehat{Q}_{w,n-1}\widehat{Q}_{w,n+1}-\widehat{Q}_{w,n}^{2}= \\ =-AB\left(-q\right)^{n-1}\left(\widehat{v}_{0}-\left(1+pq\varepsilon_{1}\varepsilon_{2}\right)\right)\left(Q_{v,0}-\theta+pq\left(Q_{u,0}-\omega\right)\right). \quad (2.1)$$

If we set $s\rightarrow m-n$, and fix $r=1$, we get the following d'Ocagne's identity for hyper-dual Horadam quaternions:

$$\widehat{Q}_{w,n+1}\widehat{Q}_{w,m}-\widehat{Q}_{w,n}\widehat{Q}_{w,m+1}= \\ =AB\left(-q\right)^{n}\left(\widehat{v}_{0}-\left(1+pq\varepsilon_{1}\varepsilon_{2}\right)\right)\left(\left(Q_{v,0}-\theta\right)u_{m-n}+q\left(Q_{u,0}-\omega\right)v_{m-n}\right).$$

Next, we give some summation formulas for hyper-dual Horadam quaternions.

Theorem 6. *For $n\geq 2$, we have*

$$\sum_{r=1}^{n-1}\widehat{Q}_{w,r}=\frac{\widehat{Q}_{w,n}-\widehat{Q}_{w,1}+q\left(\widehat{Q}_{w,n-1}-\widehat{Q}_{w,0}\right)}{p+q-1}.$$

Proof. From the Binet formula for hyper-dual Horadam quaternions, we have

$$\begin{aligned}\sum_{r=1}^{n-1}\widehat{Q}_{w,r}&=\sum_{r=1}^{n-1}\frac{A\alpha^{*}\underline{\alpha}\alpha^{r}-B\beta^{*}\underline{\beta}\beta^{r}}{\alpha-\beta}=\frac{A\alpha^{*}\underline{\alpha}}{\alpha-\beta}\sum_{r=1}^{n-1}\alpha^{r}-\frac{B\beta^{*}\underline{\beta}}{\alpha-\beta}\sum_{r=1}^{n-1}\beta^{r}\\&=\frac{A\alpha^{*}\underline{\alpha}}{\alpha-\beta}\left(\frac{\alpha^{n}-\alpha}{\alpha-1}\right)-\frac{B\beta^{*}\underline{\beta}}{\alpha-\beta}\left(\frac{\beta^{n}-\beta}{\beta-1}\right)\\&=\frac{1}{\left(\alpha-\beta\right)\left(1-p-q\right)}\Big(-\left(A\alpha^{*}\underline{\alpha}\alpha^{n}-B\beta^{*}\underline{\beta}\beta^{n}\right)\\&\quad-q\left(A\alpha^{*}\underline{\alpha}\alpha^{n-1}-B\beta^{*}\underline{\beta}\beta^{n-1}\right)+q\left(A\alpha^{*}\underline{\alpha}-B\beta^{*}\underline{\beta}\right)\\&\quad+\left(A\alpha^{*}\underline{\alpha}\alpha-B\beta^{*}\underline{\beta}\beta\right)\Big)\\&=\frac{-\widehat{Q}_{w,n}-q\widehat{Q}_{w,n-1}+q\widehat{Q}_{w,0}+\widehat{Q}_{w,1}}{1-p-q}.\end{aligned}$$

□

Theorem 7. *For nonnegative integers n and r, we have*

$$\sum_{m=0}^{n}\binom{n}{m}q^{n-m}p^{m}\widehat{Q}_{w,m+r}=\widehat{Q}_{w,2n+r}.$$

Proof. From the Binet formula for hyper-dual Horadam quaternions, we have

$$\sum_{m=0}^{n}\binom{n}{m}q^{n-m}p^{m}\widehat{Q}_{w,m+r}=\sum_{m=0}^{n}\binom{n}{m}q^{n-m}p^{m}\left(\frac{A\alpha^{*}\underline{\alpha}\alpha^{m+r}-B\beta^{*}\underline{\beta}\beta^{m+r}}{\alpha-\beta}\right)=$$
$$=\frac{A\alpha^{*}\underline{\alpha}\alpha^{r}}{\alpha-\beta}\sum_{m=0}^{n}\binom{n}{m}q^{n-m}\left(p\alpha\right)^{m}-\frac{B\beta^{*}\underline{\beta}\beta^{r}}{\alpha-\beta}\sum_{m=0}^{n}\binom{n}{m}q^{n-m}\left(p\beta\right)^{m}$$

$$= \frac{A\alpha^*\underline{\alpha}\alpha^r}{\alpha-\beta}(q+p\alpha)^n - \frac{B\beta^*\underline{\beta}\beta^r}{\alpha-\beta}(q+p\beta)^n$$
$$= \frac{A\alpha^*\underline{\alpha}\alpha^{2n+r} - B\beta^*\underline{\beta}\beta^{2n+r}}{\alpha-\beta} = \widehat{Q}_{w,2n+r}.$$

□

Finally, we give some matrix representations for hyper-dual Horadam quaternions, and derive some properties of hyper-dual Horadam quaternions by using matrix approach.

Let define the matrices $\mathbf{U} := \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}$, $\mathbf{W}_n := \begin{bmatrix} w_{n+2} & qw_{n+1} \\ w_{n+1} & qw_n \end{bmatrix}$. It is well-known that for the Horadam numbers, we have the matrix equality:

$$\mathbf{W}_0\mathbf{U}^n = \mathbf{W}_n. \tag{2.2}$$

For details, see [17]. Now let define the matrix $\mathbf{M}_{w,n} := \begin{bmatrix} \widehat{Q}_{w,n+2} & q\widehat{Q}_{w,n+1} \\ \widehat{Q}_{w,n+1} & q\widehat{Q}_{w,n} \end{bmatrix}$. Considering the relation (2.2), we have the matrix equalities

$$\mathbf{M}_{w,0}\mathbf{U}^n = \mathbf{M}_{w,n} \tag{2.3}$$

and

$$\mathbf{M}_{u,0}\left(\mathbf{W}_0\mathbf{U}^{n-1}\right) = \mathbf{M}_{w,n}, \tag{2.4}$$

which can be proven by using induction.

Theorem 8. *For integers $n,m \geq 1$, we have the following equalities:*

$$\widehat{Q}_{w,m+n} = \widehat{Q}_{w,m+1}u_n + q\widehat{Q}_{w,m}u_{n-1}, \tag{2.5}$$
$$\widehat{Q}_{w,m+n} = w_n\widehat{Q}_{u,m+1} + qw_{n-1}\widehat{Q}_{u,m}. \tag{2.6}$$

Proof. From the matrix equality $\mathbf{M}_{w,0}\mathbf{U}^{m+n-2} = \left(\mathbf{M}_{w,0}\mathbf{U}^{m-1}\right)\mathbf{U}^{n-1}$, we have

$$\begin{bmatrix} \widehat{Q}_{w,m+n} & q\widehat{Q}_{w,m+n-1} \\ \widehat{Q}_{w,m+n-1} & q\widehat{Q}_{w,m+n-2} \end{bmatrix} = \begin{bmatrix} \widehat{Q}_{w,m+1} & q\widehat{Q}_{w,m} \\ \widehat{Q}_{w,m} & q\widehat{Q}_{w,m-1} \end{bmatrix} \begin{bmatrix} u_n & qu_{n-1} \\ u_{n-1} & qu_{n-2} \end{bmatrix}.$$

By equating the corresponding entries of both sides of the matrix equation, we get the identity (2.5).

Now consider the matrix equality $\mathbf{M}_{u,0}\left(\mathbf{W}_0\mathbf{U}^{m+n-2}\right) = \left(\mathbf{W}_0\mathbf{U}^{n-1}\right)\mathbf{U}^{m-1}\mathbf{M}_{u,0}$. Then we have

$$\begin{bmatrix} \widehat{Q}_{w,m+n+1} & q\widehat{Q}_{w,m+n} \\ \widehat{Q}_{w,m+n} & q\widehat{Q}_{w,m+n-1} \end{bmatrix} = \begin{bmatrix} w_{n+1} & qw_n \\ w_n & qw_{n-1} \end{bmatrix} \begin{bmatrix} \widehat{Q}_{u,m+1} & q\widehat{Q}_{u,m} \\ \widehat{Q}_{u,m} & q\widehat{Q}_{u,m-1} \end{bmatrix}.$$

Similarly, by equating the corresponding entries of both sides of the above matrix equation, we get the desired result in (2.6). □

From the matrix equalities (2.3) and (2.4), one can obtain several results for the hyper-dual Horadam quaternions. For example, if we take the determinant of the both side of this matrix identity (2.3), then we get the Cassini's identity in terms of hyper-dual Horadam quaternions as:

$$\widehat{Q}_{w,n-1}\widehat{Q}_{w,n+1}-\widehat{Q}_{w,n}^{2}=(-q)^{n-1}\left(\widehat{Q}_{w,0}\widehat{Q}_{w,2}-\widehat{Q}_{w,1}^{2}\right). \tag{2.7}$$

Note that different from the identity (2.1), here the right hand side of the equation (2.7) is expressed in terms of only the hyper-dual Horadam quaternions.

3. Conclusion

In this paper, we define quaternions whose components are hyper-dual Horadam numbers. The main advantage of introducing hyper-dual Horadam quaternions is that many hyper-dual numbers with celebrated numbers such as Fibonacci, Lucas, Pell, Pell-Lucas, Jacosthal, Jacobsthal-Lucas numbers can be deduced as particular cases of hyper-dual Horadam quaternions. We give the generating function and the Binet formula for these quaternions. With the help of the Binet formula of hyper-dual Horadam quaternions, we derive many properties of these quaternions such as summation formulas, binomial sum identities, Vajda's identity, Catalan's identity, Cassini identity, and d'Ocagne's identity. Also by means of the matrix representation of hyper-dual Horadam quaternions, we examine several identities for these quaternions. The algebra of quaternions is noncommutative, whereas the algebra of hyper-dual numbers is commutative. So, it is interesting to study special type of numbers which are involving both quaternionic and hyper-dual units. For the interested readers, the results of this paper could be applied for higher order hyper-dual numbers which were given in [8].

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HYPERELASTIC CURVES ALONG IMMERSIONS

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Abstract. Hyperelastic curves in a Riemannian manifold are solutions of a constrained variational problem and characterized by Euler–Lagrange equations. We study the effect of hyperelastic curves on the geometry of isometric immersions. We investigate the relation between hyperelastic curves and umbilical submanifolds and apply the results to analyze classical elastic curves. The case of a Riemannian manifold with constant sectional curvature is also discussed and some applications are presented for illustrating the results.

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1. INTRODUCTION

A parametrized curve by its arclength is called a hyperelastic curve if it is a critical point of the following curvature energy action defined on a suitable space of curves in a Riemannian manifold:

$$\mathcal{F}_\gamma^r = \int (\kappa^r + \lambda)\, ds, \tag{1.1}$$

where κ denotes the curvature of γ [3, 21, 22]. Such curves are called as free hyperelastic curves when $\lambda = 0$. The functional $\mathcal{F}_\gamma^r$ is nothing but the classical Euler–Bernoulli's bending (or elastic) energy functional for $r = 2$. Immersed curves which are critical for the bending energy functional satisfying some boundary conditions are called elastic curves (or elastica) [16]. The existence, classification or stability problems of elastic curves or their generalizations in Riemannian manifolds have attracted the attention of many researches. Some of the most remarkable examples in the literature worked by D. Singer et al. [13, 15, 16, 23]. J. Langer and D. Singer in 1984 showed that there exist closed elastic curves of a fixed length in a compact Riemannian manifold [16]. At around the same time with the paper [16], that idea is shown in [14] by different method [10]. These curves have been characterized on the sphere in Euclidean 3-space by Brunnett and Crouch [7]. Another noteworthy

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works of elastic curves and its extensions have been studied by Garay et al. in non-Euclidean spaces [1, 3, 10]. They have found important results of this variational problem in Lorentz-Minkowski space.

The study of this topic in low dimensional cases, curves and surfaces, has important applications to Physics, Biophysics and Engineering while attention to higher dimensional submanifolds has usually shown a more purely mathematical flavor. However, the hyperelastic curves have important applications to other higher dimensional variational problems in submanifolds and physics such as Chen–Willmore submanifolds, string theory and p-branes, models of relativistic particles, capillary pressure difference, etc. [3, 10].

The special curves on the surfaces were first considered in Riemannian manifolds by Nomizu and Yano [18]. The authors described the concept of a circle and showed that if a circle on a submanifold is carried to the ambient manifold along an immersion, the submanifold is totally umbilical and the mean curvature vector field is parallel [18]. Ikawa expressed a characterization of a helix by a differential equation in a Riemannian manifold and obtained the necessary and sufficient condition as depending upon the mean curvature vector field that a helix in a Riemannian submanifold corresponds to a helix in ambient manifold [12, 17, 18]. One of the most active area of differential geometry has been the theory of isometric immersions [9] and this theory is still active area, see for instance [2] and [4]. The aim of this paper is to consider hyperelastic curves along an immersion. By considering the role of submanifold theory in geometric mechanics [8] and applications of elastic curves, we believe our result will start a fresh research area by combining these two concepts.

In section 2, we give some geometric preliminaries concerning the basic problem of the paper mentioned in the introduction. In section 3, we examine isometric immersions which carry a hyperelastic curve of the total manifold to ambient manifold as a hyperelastic curve. Then, we give a characterization of the submanifold with regard to the mean curvature vector field H by means of hyperelastic curves. We also consider elastic curves under isometric immersions. In the last section, we give some results according to constant sectional curvature of the manifold. Finally, we exemplify the findings.

2. Geometrical notations and preliminaries

Let M be a n-dimensional submanifold of a m-dimensional Riemannian manifold $\overline{M}$. $\langle\cdot,\cdot\rangle$ denotes both the metric tensor fields of $\overline{M}$ and the induced metric of M. A vector field V in $\overline{M}$ is called a normal vector field in M if $\langle X,V\rangle = 0$ for any vector field X in M and a unit normal vector field in M is known a normal section, [24]. $\overline{\nabla}$ and ∇ denote the Riemannian connection of $\overline{M}$ and induced connection of M respectively. $\chi(\overline{M})$ and $\chi(M)$ are the set of the Lie algebra of vector fields in $\overline{M}$ and M, respectively. $\chi^{\perp}(M)$, TM and $TM^{\perp}$ will express the set of all normal vector fields of M, the tangent bundle and the normal vector bundle of M, respectively.

Let X and Y be vector fields on M. The Gauss formula [24] is given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

where $\nabla_X Y$ is the tangential component, $h(X,Y)$ is the normal component of $\overline{\nabla}_X Y$. $h(X,Y)$ is the second fundamental form of M. h is a bilinear and symmetric form in X and Y.

Let V be a normal vector field in M. The Weingarten formula is given by

$$\overline{\nabla}_X V = -A_V X + D_X V, \tag{2.2}$$

where D is the connection in the normal bundle and A is the shape operator of M. A and h satisfy

$$\langle A_V X, Y\rangle = \langle h(X,Y), V\rangle. \tag{2.3}$$

From the Gauss and Weingarten formulas, we have

$$\overline{R}(X,Y)Z = R(X,Y)Z - A_{h(Y,Z)}X + A_{h(X,Z)}Y + \left(\overline{\nabla}_X h\right)(Y,Z) - \left(\overline{\nabla}_Y h\right)(X,Z), \tag{2.4}$$

where $\overline{R}$ and R are the Riemannian curvature tensor fields of $\overline{M}$ and M, respectively. In addition, $\left(\overline{\nabla}_X h\right)(Y,Z)$ and $(\nabla_X A)_V Y$ are defined as the covariant derivative of the second fundamental forms h and A as follows

$$\left(\overline{\nabla}_X h\right)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{2.5}$$

and

$$(\nabla_X A)_V Y = \nabla_X (A_V Y) - A_{D_X V} Y - A_V \nabla_X Y, \tag{2.6}$$

respectively. On the other hand, the following equality is satisfied

$$\left\langle \left(\overline{\nabla}_X h\right)(Y,Z), V \right\rangle = \langle \nabla_X (A)_V Y, Z\rangle. \tag{2.7}$$

M is called a totally geodesic submanifold in $\overline{M}$ if its second fundamental form is identically zero, that is $h = 0$ or equivalently $A = 0$. If the second fundamental form satisfies

$$h(X,Y) = \langle X, Y\rangle H,$$

then M is said to be totally umbilical. If the vector $h(x,x)$ has the same length for any unit vector $x \in T_p M$, then M is called isotropic at p. A submanifold M is isotropic iff h satisfies

$$\langle h(x,x), h(x,y)\rangle = 0$$

for any orthonormal vectors x and y of $T_p M$. Furthermore if

$$h(x,y) = 0 \tag{2.8}$$

for any orthonormal vectors x and y of $T_p M$, then M is totally umbilical at p [12, 19].

3. Characterization of the submanifold of a Riemannian manifold by hyperelastic curves

We firstly remind a characterization of the hyperelastic curve in a Riemannian manifold. We consider the family of C^∞ curves as follows

$$\Im = \{\gamma \,|\, \gamma\colon [0,\ell] \subset \mathbb{R} \to M,\ \gamma(i\ell) = p_i,\ \ p_i \in M,\ \ \dot{\gamma}(i\ell) = v_i,\ \ v_i \in T_{p_i}M\ \ i = 0,1\}.$$

For a parametrized curve $\gamma \in \Im$, κ is the geodesic curvature of γ. Then a hyperelastic curve is a critical point of the functional $\mathcal{F}_\gamma^r\colon \Im \to [0,\infty)$ defined by (1.1) for a natural number $r \geq 2$ (see [3]).

Let $\gamma\colon I \to M$ be an immersed unit speed curve in a n-dimensional Riemannian manifold M. We denote by T, N and B, the unit tangent vector field, the unit normal vector field and the binormal vector field of γ, respectively. $\kappa = \|\nabla_T T\|$ is the geodesic curvature and $\tau = -\langle \nabla_T B, N\rangle$ is the torsion of γ. γ has also curvatures $\kappa_1 = \kappa > 0$, $\kappa_2 = \tau$, κ_3, $\kappa_4, \ldots, \kappa_{n-1}$ and Frenet frame $N_0 = T$, $N_1 = N$, $N_2 = B$, $N_3, N_4, \ldots, N_{n-1}$. Then, the Frenet equations are given by

$$\nabla_T N_i = -\kappa_i N_{i-1} + \kappa_{i+1} N_{i+1}, \qquad 0 \leq i \leq n-1$$

(defining $\kappa_0 = \kappa_n = 0$) see [16]. In this case γ is called a Frenet curve of order n.

Critical points of the functional (1.1) are characterized by the Euler–Lagrange equation

$$\nabla_T^2\left(\kappa^{r-2}\nabla_T T\right) + \kappa^{r-2}R(\nabla_T T, T)T + \nabla_T(\lambda T) = 0, \tag{3.1}$$

for some constant $b \in \mathbb{R}$ and

$$\lambda = \frac{2r-1}{r}\kappa^r + b \tag{3.2}$$

see [3, 21].

Lemma 1. *Let M and $\overline{M}$ be Riemannian manifolds and $i\colon M \to \overline{M}$ an isometric immersion such that γ is a curve on M. Assume that $\overline{\gamma}(s) = i \circ \gamma(s)$ is a curve with curvature $\overline{\kappa}$ in $\overline{M}$. Then we have the following equations:*

(i)

$$\begin{aligned}\overline{\nabla}_T^2\left(\overline{\kappa}^{r-2}\overline{\nabla}_T T\right) &= \nabla_T^2(\xi\nabla_T T) + h(T, \nabla_T(\xi\nabla_T T)) - A_{\xi h(T,\nabla_T T)}T \\ &\quad + D_T\xi h(T, \nabla_T T) - \nabla_T\left(A_{\xi h(T,T)}T\right) - h\left(A_{\xi h(T,T)}T, T\right) \\ &\quad - A_{D_T\xi h(T,T)}T + D_T^2\xi h(T,T),\end{aligned}$$

(ii)

$$\begin{aligned}\overline{\kappa}^{r-2}\overline{R}\left(\overline{\nabla}_T T, T\right)T &= \xi R(\nabla_T T, T)T - \xi A_{h(T,T)}\nabla_T T + \xi A_{h(\nabla_T T,T)}T \\ &\quad + \xi\left(\overline{\nabla}_{\nabla_T T}h\right)(T,T) - \xi\left(\overline{\nabla}_T h\right)(\nabla_T T, T),\end{aligned}$$

where $\xi = (\kappa^2 + \|h(T,T)\|^2)^{\frac{r-2}{2}}$ and κ is the curvature of γ in M.

Proof. Suppose that $\bar{\gamma}(s)$ is an immersed curve with $\overline{\kappa}$ in $\overline{M}$.

(i) From Gauss and Weingarten formulas, we have

$$\begin{aligned}\overline{\nabla}_T\left(\overline{\kappa}^{r-2}\overline{\nabla}_T T\right) =&\overline{\nabla}_T\left(\xi\nabla_T T\right)+\overline{\nabla}_T\left(\xi h\left(T,T\right)\right)\\ =&\nabla_T\left(\xi\nabla_T T\right)+\xi h\left(T,\nabla_T T\right)-A_{\xi h(T,T)}T+D_T\xi h\left(T,T\right).\end{aligned} \tag{3.3}$$

Taking derivative of (3.3) and using (2.1) and (2.2), we get the desired equation.

(ii) From (2.1) and (2.4), we have

$$\begin{aligned}\overline{R}\left(\overline{\nabla}_T T,T\right)T = &R\left(\nabla_T T,T\right)T-A_{h(T,T)}\nabla_T T+A_{h(\nabla_T T,T)}T\\ &+\left(\overline{\nabla}_{\nabla_T T}h\right)\left(T,T\right)-\left(\overline{\nabla}_T h\right)\left(\nabla_T T,T\right).\end{aligned} \tag{3.4}$$

Taking inner product both sides of (3.4) with ξ, we obtain the relation (ii) of Lemma (1).

□

Now, we suppose that $\bar{\gamma}(s)$ is a hyperelastic curve in $\overline{M}$. From (3.1), we have the following equation

$$\overline{\nabla}_T^2\left(\overline{\kappa}^{r-2}\overline{\nabla}_T T\right)+\overline{\kappa}^{r-2}\overline{R}(\overline{\nabla}_T T,T)T+\overline{\nabla}_T(\lambda T)=0, \tag{3.5}$$

for some constant $b\in\mathbb{R}$ and

$$\lambda=\frac{2r-1}{r}\overline{\kappa}^r+b. \tag{3.6}$$

Taking into consideration Lemma 1, we obtain

$$\begin{aligned}&\nabla_T^2\left(\xi\nabla_T T\right)+h\left(T,\nabla_T\left(\xi\nabla_T T\right)\right)+D_T\xi h\left(T,\nabla_T T\right)-\nabla_T\left(A_{\xi h(T,T)}T\right)\\ &-\xi h\left(A_{h(T,T)}T,T\right)-A_{D_T\xi h(T,T)}T+D_T^2\xi h\left(T,T\right)+\xi R\left(\nabla_T T,T\right)T\\ &-\xi A_{h(T,T)}\nabla_T T+\xi\left(\overline{\nabla}_{\nabla_T T}h\right)\left(T,T\right)-\xi\left(\overline{\nabla}_T h\right)\left(\nabla_T T,T\right)\\ &+\nabla_T\left(\lambda T\right)+\lambda h\left(T,T\right)=0,\end{aligned} \tag{3.7}$$

where

$$\lambda=\frac{2r-1}{r}(\kappa^2+\|h(T,T\|^2)^{\frac{r}{2}}+b.$$

The tangent part of (3.7) is

$$\begin{aligned}&\nabla_T^2(\xi\nabla_T T)-\nabla_T(A_{\xi h(T,T)}T)-A_{D_T\xi h(T,T)}T\\ &+\xi R\left(\nabla_T T,T\right)T-\xi A_{h(T,T)}\nabla_T T+\nabla_T\left(\lambda T\right)=0.\end{aligned} \tag{3.8}$$

We have from (2.6) and (2.5)

$$\nabla_T\left(A_{\xi h(T,T)}T\right)=\nabla_T\left(A\right)_{\xi h(T,T)}T+A_{D_T\xi h(T,T)}T+A_{\xi h(T,T)}\nabla_T T \tag{3.9}$$

and

$$\xi\left(\overline{\nabla}_T h\right)\left(T,T\right)=\xi D_T h\left(T,T\right)-2\xi h\left(\nabla_T T,T\right), \tag{3.10}$$

respectively. Substituting (3.9) and (3.10) into (3.8), we have

$$\begin{aligned}&\nabla_T^2(\xi\nabla_T T)-\nabla_T A_{\xi h(T,T)}T-2A_{(D_T\xi)h(T,T)}T-2\xi A_{\left(\overline{\nabla}_T h\right)(T,T)}T\\&-4\xi A_{h(\nabla_T T,T)}T-2\xi A_{h(T,T)}\nabla_T T+\xi R(\nabla_T T,T)T+\nabla_T(\lambda T)=0.\end{aligned}\tag{3.11}$$

From Frenet equations of γ, (3.11) reduces to

$$\begin{aligned}&\left(\lambda_s-3\xi\kappa\kappa_s-2\kappa^2\xi_s\right)T+(\xi\kappa_{ss}-\kappa^3\xi+2\kappa_s\xi_s+\xi_{ss}\kappa-\kappa\tau^2\xi+\lambda\kappa)N\\&+(2\kappa_s\tau\xi+\kappa\tau_s\xi+2\kappa\tau\xi_s)B-4\xi\kappa A_{h(N,T)}T-2\xi\kappa A_{h(T,T)}N\\&=2A_{(D_T\xi)h(T,T)}T+2\xi A_{\left(\overline{\nabla}_T h\right)(T,T)}T+\nabla_T A_{\xi h(T,T)}T-\xi\kappa R(N,T)T.\end{aligned}\tag{3.12}$$

Taking inner product with T to (3.12) and using (3.10), we calculate

$$\begin{aligned}&\lambda_s-3\xi\kappa\kappa_s-2\kappa^2\xi_s-6\xi\kappa\langle h(T,N),h(T,T)\rangle=\\&\qquad 2\left\langle A_{(D_T\xi h)(T,T)}T,T\right\rangle-4\kappa\xi\left\langle A_{h(T,N)}T,T\right\rangle+\left\langle\nabla_T A_{\xi h(T,T)}T,T\right\rangle\end{aligned}$$

by using the fact $\langle N,T\rangle=\langle B,T\rangle=\langle R(N,T)T,T\rangle=0$, (2.3) and (3.10). By using (2.3), (2.7) and (3.10) we have

$$\begin{aligned}\lambda_s-3\xi\kappa\kappa_s-2\kappa^2\xi_s&=2\langle h(T,T),D_T\xi h(T,T)\rangle+\langle\xi h(T,T),D_T h(T,T)\rangle\\&=2(D_T\xi)\|h(T,T)\|^2+\frac{3}{2}\xi D_T\|h(T,T)\|^2\\&=\xi((r-2)(\kappa^2+\|h(T,T)\|^2)^{-1}\|h(T,T)\|^2+\frac{3}{2})D_T\|h(T,T)\|^2.\end{aligned}\tag{3.13}$$

The normal part of (3.7) is

$$\begin{aligned}&h(T,\nabla_T(\xi\nabla_T T))+D_T\xi h(T,\nabla_T T)-\xi h\left(A_{h(T,T)}T,T\right)+D_T^2\xi h(T,T)\\&+\xi\left(\overline{\nabla}_{\nabla_T T}h\right)(T,T)-\xi\left(\overline{\nabla}_T h\right)(\nabla_T T,T)+\lambda h(T,T)=0.\end{aligned}\tag{3.14}$$

We have from (2.5)

$$\begin{aligned}\xi D_T^2 h(T,T)&=\xi\left(\overline{\nabla}_T^2 h\right)(T,T)+4\xi D_T h(T,\nabla_T T)\\&\quad-2\xi h(\nabla_T T,\nabla_T T)-2\xi h\left(T,\nabla_T^2 T\right).\end{aligned}\tag{3.15}$$

Substituting (3.15) into (3.14) and using (2.5), we get

$$\begin{aligned}&h(T,\xi_s\nabla_T T)+4\xi h\left(T,\nabla_T^2 T\right)+5(D_T\xi)h(T,\nabla_T T)+4\xi\left(\overline{\nabla}_T h\right)(T,\nabla_T T)\\&+3\xi h(\nabla_T T,\nabla_T T)-\xi h\left(A_{h(T,T)}T,T\right)+(D_T^2\xi)h(T,T)+2(D_T\xi)(\overline{\nabla}_T h)(T,T)\\&+\xi\left(\overline{\nabla}_T^2 h\right)(T,T)+\xi\left(\overline{\nabla}_{\nabla_T T}h\right)(T,T)+\lambda h(T,T)=0.\end{aligned}$$

Taking into consideration the Frenet equations, we have

$$\begin{aligned}(\kappa\xi_s+4\xi\kappa_s+5(D_T\xi)\kappa)h(T,N)+4\xi\kappa\tau h(T,B)+4\xi\kappa\left(\overline{\nabla}_T h\right)(T,N)\\ +3\xi\kappa^2 h(N,N)+2(D_T\xi)(\overline{\nabla}_T h)(T,T)+\xi\kappa\left(\overline{\nabla}_N h\right)(T,T)= \qquad (3.16)\\ (4\xi\kappa^2-D_T^2\xi)h(T,T)+\xi h\left(A_{h(T,T)}T,T\right)+\lambda h(T,T)-\xi\left(\overline{\nabla}_T^2 h\right)(T,T).\end{aligned}$$

Changing B into $-B$ in (3.16) and using (3.16), we get

$$h(T,B)=0. \qquad (3.17)$$

In the following proposition, we give a characterization of the submanifold of a Riemannian manifold by using hyperelastic curves.

Proposition 1. *Let M and $\overline{M}$ be Riemannian manifolds and $i\colon M\to\overline{M}$ an isometric immersion such that γ is a curve on M. Assume that $\overline{\gamma}(s)=i\circ\gamma(s)$ is a hyperelastic curve with curvature $\overline{\kappa}$ in $\overline{M}$. Then M is an isotropic submanifold if one of the following conditions are satisfied.:*

(i) *$r=2$ that is $\overline{\gamma}(s)$ is an elastic curve,*
(ii) *curvature κ of γ is a constant when $\overline{\kappa}\neq 0$.*

Proof. If $\overline{\gamma}=\overline{\gamma}(s)$ is a hyperelastic curve with curvature $\overline{\kappa}$ and $\gamma(s)$ has the curvature κ. Then we have the following cases:

(i) If $r=2$ then (3.13) reduces to

$$D_T\|h(T,T)\|^2=0. \qquad (3.18)$$

Thus $\|h(T,T)\|$ is equal to a constant value, that is, M is an isotropic submanifold.

(ii) We suppose that κ is a constant value. From (3.13), we obtain

$$\xi((r-2)(\kappa^2+\|h(T,T)\|^2)^{-1}\|h(T,T)\|^2+\frac{3}{2})D_T\|h(T,T)\|^2=0.$$

On the other hand, since $\overline{\kappa}\neq 0$ and $r\geq 2$, then we have (3.18).

□

From (2.8), (3.17) and Proposition 1, the proof of the following corollary is clear.

Corollary 1. *We suppose that $\overline{\gamma}(s)=i\circ\gamma(s)$ is a hyperelastic curve with curvature $\overline{\kappa}$ in $\overline{M}$. In this case, M is totally umbilic submanifold if it is isotropic.*

Now, we introduce the concept of hyperelastic immersion.

Definition 1. Let i be an isometric immersion from a Riemannian manifold M to a Riemannian manifold $\overline{M}$ such that γ is a hyperelastic curve on M. If the curve $i\circ\gamma$ is also a hyperelastic curve on $\overline{M}$, then the immersion i is called a hyperelastic immersion.

From (3.1) and the Frenet equations for γ, we have the following proposition (similar to that of [5]).

Proposition 2. *Let γ be a regular curve in $\mathbb{R}^n$ with curvatures $\{\kappa,\tau,\delta,\dots\}$. Then γ be a critical for the bending energy functional (1.1) if and only if the following Euler–Lagrange equations are satisfied*

$$\begin{aligned}(r-1)\kappa_{ss}+(r-1)(r-2)\kappa^{-1}\kappa_s^2-\kappa\tau^2+\frac{r-1}{r}\kappa^3=0\\ 2(r-1)\kappa_s\tau+\kappa\tau_s=0\\ \delta=0.\end{aligned} \tag{3.19}$$

Especially, a hyperelastic curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ can be moved to a hyperelastic curve in $\mathbb{R}^n$ by a hyperelastic immersion. In the following, an example of a hyperelastic immersion is given.

Example 1. The mapping j is defined as

$$\begin{aligned}j\colon [-1,1]\times I\subset\mathbb{R}^2&\to\mathbb{R}^3\\ (u,v)&\to\left(u,\ v,\ \sqrt{\frac{r-1}{r}}\arccos(u)\right).\end{aligned}$$

Let

$$\beta(s)=\left(\cos\left(\frac{s}{\sqrt{\frac{2r-1}{r}}}\right),\ \sin\left(\frac{s}{\sqrt{\frac{2r-1}{r}}}\right)\right)\ \text{where } 0\le\left(\frac{s}{\sqrt{\frac{2r-1}{r}}}\right)\le 2\pi,$$

be a curve in $\mathbb{R}^2$. Then we find the curve

$$\overline{\beta}(s)=\left(\cos(\frac{s}{\sqrt{\frac{2r-1}{r}}}),\ \sin(\frac{s}{\sqrt{\frac{2r-1}{r}}}),\ \sqrt{\frac{r-1}{r}}\frac{s}{\sqrt{\frac{2r-1}{r}}}\right).$$

We can see from (3.19) that β and $\overline{\beta}$ are hyperelastic curves in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. This implies that j is a hyperelastic immersion.

The following theorem gives a result of hyperelastic immersions in terms of the elements of submanifold theory.

Theorem 1. *Let i be a hyperelastic immersion between the submanifold M and a Riemannian manifold $\overline{M}$. If γ is a hyperelastic curve with the constant curvature κ and the tangent vector field T in M, then M is a totally umbilical submanifold and the mean curvature vector field H satisfies*

$$D_T^2H=CH, \tag{3.20}$$

where $C=\kappa^2+\|H\|^2+\frac{\lambda}{\xi}=$ constant. Conversely if the isometric immersion is totally geodesic, then it is a hyperelastic immersion.

Proof. Let i be a hyperelastic immersion between Riemannian manifolds M and $\overline{M}$. Suppose that $\gamma(s)$ is a hyperelastic curve with a constant curvature κ and the tangent vector field T in M. Then γ satisfies the Euler–Lagrange equation (3.1) with (3.2). We can see from Proposition 1, M is an isotropic submanifold. Since $\bar{\gamma}$ is a hyperelastic curve in $\overline{M}$, we have (3.5) with (3.6). On the other hand Corollary 1 shows that M is totally umbilical submanifold.
So, (3.16) reduces to

$$\begin{aligned} &4\xi\kappa\left(\overline{\nabla}_T h\right)(T,N)+3\xi\kappa^2 h(N,N)+\xi\kappa\left(\overline{\nabla}_N h\right)(T,T)= \\ &4\xi\kappa^2 h(T,T)+\xi h\left(A_{h(T,T)}T,T\right)+\lambda h(T,T)-\xi\left(\overline{\nabla}_T^2 h\right)(T,T). \end{aligned} \tag{3.21}$$

Changing N into $-N$ in (3.21), we have

$$4\xi\kappa\left(\overline{\nabla}_T h\right)(T,N)+\xi\kappa\left(\overline{\nabla}_N h\right)(T,T)=0. \tag{3.22}$$

Substituting (3.22) into (3.21) we have

$$-\xi\kappa^2 H=\xi\left\langle A_{h(T,T)}T,T\right\rangle H+\lambda H-\xi\left(\overline{\nabla}_T^2 h\right)(T,T).$$

By using (2.3) and (3.15) we get

$$-\xi\kappa^2 H=\xi\|H\|^2 H+\lambda H-\xi D_T^2 H.$$

Then, we obtain

$$D_T^2 H=\left(\kappa^2+\|H\|^2+\frac{\lambda}{\xi}\right)H.$$

Conversely, we assume that the isotropic immersion is a totally geodesic. Since γ is a hypereastic curve with the curvature κ, $i\circ\gamma$ satisfies the Euler–Lagrange equation (3.5) with (3.6). □

Then we have the following result.

Corollary 2. *If $i\colon M\to\overline{M}$ is a hyperelastic immersion and γ is a hyperelastic curve with the constant curvature in M, then γ has constant torsion.*

Proof. Let i be a hyperelastic immersion and γ a hyperelastic curve with the constant curvature κ in M. (3.12) can be written as

$$\left(-\kappa^2-\tau^2+\tfrac{\lambda}{\xi}\right)N+\tau_s B=4A_{h(N,T)}T+2A_{h(T,T)}N. \tag{3.23}$$

Taking inner product with N in (3.23), we get

$$\|H\|^2=\frac{1}{2}\left(-\kappa^2-\tau^2+\frac{\lambda}{\xi}\right). \tag{3.24}$$

Because of $\|h(T,T)\|=constant=\|H\|$, one can see from (3.24), the torsion τ of γ is a constant, too. □

We recall the following result which will be very crucial for our theorem. A unit speed curve γ in a Riemannian manifold M is called an elastic curve (or elastica) if it satisfies (3.1) with (3.2) for $r=2$ see [16, 22]. The following theorem gives a characterization of totally umbilical submanifolds by the behaviour of elastic curves under an immersion.

Theorem 2. *Let $i\colon M\to\overline{M}$ be an immersion from a Riemannian manifold M to a Riemannian manifold $\overline{M}$ transporting each elastic curve with the tangent vector T, curvature κ and torsion τ, respectively, to another elastic curve on $\overline{M}$. Then M is totally umbilical and the mean curvature vector field H satisfies*

$$D_T^2H=\frac{1}{2}\left(\kappa^2-\tau^2+3\lambda+\frac{\kappa_{ss}}{\kappa}\right)H,\qquad \|H\|=constant. \tag{3.25}$$

Conversely, if M is an totally umbilical, its mean curvature vector satisfies (3.25) and $D_{\nabla_T T}H=2\lambda H$ then an elastic curve in M is an elastic curve in $\overline{M}$.

Proof. Suppose that $\gamma(s)$ is an elastic curve in M, with unit tangent vector field T then the following equation is satisfied

$$(\nabla_T)^3T+R(\nabla_TT,T)T+\nabla_T(\lambda T)=0$$

with (3.2) for $r=2$. Since $\bar{\gamma}$ is an elastic curve with curvature $\bar{\kappa}$ in $\overline{M}$, we have also (3.5) with (3.6) for $r=2$. Proposition 1 and Corollary 1 show that M is totally umbilical submanifold. If (3.23) is rewritten for $r=2$, and taking inner product with N, we get

$$\|H\|^2=\frac{1}{2}\left(\frac{\kappa_{ss}}{\kappa}-\tau^2+\lambda-\kappa^2\right). \tag{3.26}$$

Now we consider for the similar process for the normal part in case of $r=2$. If necessary calculations are taken, we obtain

$$D_T^2H=\left(\lambda+\kappa^2+\|H\|^2\right)H. \tag{3.27}$$

Combining (3.26) and (3.27), we find (3.25).

Conversely, we assume that M is totally umbilical and H satisfies (3.25). On the other hand, we have

$$\overline{\nabla}_T^3T=\nabla_T^3T-\nabla_T(\|H\|^2T)-(\kappa^2+\|H\|^2)H+D_T^2H. \tag{3.28}$$

If we add $\overline{\kappa}^{r-2}\overline{R}(\overline{\nabla}_TT,T)T+\overline{\nabla}_T(\lambda T)$ both sides of (3.28), then we have from (2.1), (3.4) and (3.20)

$$\begin{aligned}\overline{\nabla}_T^3(T)+\overline{R}(\overline{\nabla}_TT,T)T+\overline{\nabla}_T(\lambda T)&=\nabla_T^3(T)+\nabla_T((\lambda-2\|H\|^2)T)\\&\quad+R(\nabla_TT,T)T+2\lambda H-(D_{\nabla_TT})H.\end{aligned} \tag{3.29}$$

Since γ is an elastic curve in M, the tangent part of (3.29) is calculated as

$$(\nabla_T)^3T+\nabla_T(\lambda T)+R(\nabla_TT,T)T=0,$$

where

$$\lambda = \frac{3}{2}\kappa^2 + \bar{b},$$

for suitable choosing $\bar{b} = b + 2\|H\|^2$. Normal part of (3.29) is found zero if

$$D_{\nabla_T T} H = 2\lambda H.$$

□

4. Hyperelastic immersions in constant sectional curvature manifolds

Let M be a Riemannian manifold having constant sectional curvature c. Then we have

$$R(X,Y)Z = c(\langle Y,Z\rangle X - \langle X,Z\rangle Y).$$

From (3.1) the following result is concluded: a unit speed curve $\gamma = \gamma(s)$ parametrized by arc length s is a hyperelastic curve if it satisfies

$$\nabla_T^2\left(\kappa^{r-2}\nabla_T T\right) + c\kappa^{r-2}\nabla_T T + \nabla_T(\lambda T) = 0 \tag{4.1}$$

with (3.2) (see [11, 21]).

We know that a totally umbilical submanifold M of a Riemannian manifold $\overline{M}$ of constant sectional curvature $\bar{c}$ has also constant sectional curvature $\bar{c} + \|H\|^2$ for $\dim(M) > 2$ (see [24]). Thus, we can get the following corollary as a result of Theorem 1 and Corollary 1.

Corollary 3. *Let i be a hyperelastic immersion between the submanifold M ($\dim(M) > 2$) and a Riemannian manifold $\overline{M}$ with constant sectional curvature $\bar{c}$. If M is an isotropic submanifold of $\overline{M}$, then M has also constant sectional curvature $c = \bar{c} + \|H\|^2$, where H is the mean curvature vector field.*

We can give the following corollary in a Riemannian manifold with constant sectional curvature as a result of Theorem 2.

Corollary 4. *Let $i\colon M \to \overline{M}(c)$ be a hyperelastic immersion between a Riemannian manifold M and a Riemannian manifold $\overline{M}(c)$ of constant curvature such that γ is an elastic curve with curvature κ, the torsion τ and the unit vector field T. Then M is totally umbilical and the mean curvature vector field H satisfies*

$$D_T^2 H = \left(\frac{\kappa_{ss}}{\kappa} - \tau^2\right) H, \qquad \|H\| = \text{constant}. \tag{4.2}$$

Conversely, if M is totally umbilical submanifold of a Riemannian manifold $\overline{M}$ with constant sectional curvature c and the mean curvature vector field satisfies (4.2), then an elastic curve γ is an elastic curve in $\overline{M}$.

A Riemannian manifold of constant sectional curvature is called elliptic, hyperbolic or locally Euclidean if the sectional curvature is respectively positive, negative or zero. Now, we give an example of elastic immersion which carry elastic curves of the total manifold with 2-dimensional ($c = 0$, Euclidean plane) to ambient manifold with 3-dimensional ($c = 1$, 2-sphere). Firstly, we recall some results for elastic curves on the plane and a 2-sphere S^2. An elastic curve on Euclidean plane satisfies the following differential equation

$$\kappa'' + \frac{1}{2}\kappa^3 - \frac{1}{2}\lambda\kappa = 0 \tag{4.3}$$

see [6]. On the other hand, Brunnett and Crouch classify the forms of spherical elastic curve based on the differential equation

$$2\kappa_g'' + \kappa_g^3 + \left(\frac{2}{\rho^2} - \sigma\right)\cdot\kappa_g = 0, \tag{4.4}$$

where κ_g is the geodesic curvature, ρ is the tension parameter and σ is the radius of the sphere, [7].

Example 2. Let u and v be coordinate functions of $\mathbb{R}^2$. The mapping i is defined as

$$\begin{aligned} i\colon \mathbb{R}^2 &\to S^2 \subset \mathbb{R}^3 \\ (u,v) &\to (u^2 - v^2,\ 2uv,\ 0) \end{aligned}$$

(see [20]), where $(u^2+v^2)^2 = 1$. Let $\alpha\colon I \subset \mathbb{R} \to \mathbb{R}^2$ be a curve such that $t \to \alpha(t) = (\cos t,\ \sin t)$, where $0 \le t \le 2\pi$. We see from (4.3) that α is an elastic curve since the curvature κ of α is 1. The curve $i \circ \alpha$ is obtained as $(i \circ \alpha)(t) = (\cos 2t,\ \sin 2t,\ 0)$. One can see from (4.4) this curve is an elastica on S^2. Then i is an elastic immersion.

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HERMITE-HADAMARD TYPE INEQUALITIES AND RELATED INEQUALITIES FOR SUBADDITIVE FUNCTIONS

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Abstract. In this paper, we establish Hermite-Hadamard inequalities for subadditive functions, and we give some related inequalities according to Hermite-Hadamard inequalities, which generalized the previously published results.

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1. INTRODUCTION

The principal work on the general theory of subadditive functions is that of Hille and Phillips [6]. This reference also includes a part of the work of Rosenbaum [11] on subadditive functions of several variables. Additivity, subadditivity and superadditivity are important concepts both in measure theory and in several fields of mathematics and mathematical inequalities. Especially, there are a lot of examples of additive, subadditive and superadditive functions in various areas of mathematics such as norms, square roots, error function, growth rates, differential equations and integral means. Inequalities and especially subadditive functions theory is one of the most extensively developing fields not only in theoretical and applied mathematics but also physics and the other applied sciences. Here, we mention the results of [1, 6–8, 11] and the corresponding references cited therein.

Definition 1. A function f defined on a set H of real numbers and with range contained in the set $\mathbb{R}^+$ of all positive real numbers, is subadditive on H if, for all elements x and y of H such that $x+y$ is an element of H

$$f(x+y) \leq f(x)+f(y).$$

If equality holds, f is called additive; if the inequality is reversed, f is superadditive. A function f is convex on the (possibly infinite) interval D if, for all x and y in D and all t which satisfy $0 \leq t \leq 1$,

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y).$$

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If this inequality is reversed, f is concave on D.

Remark 1. If f is convex and subadditive on H and if $f(0)=0$, then f is additive on H.

Definition 2. The function $f:[0,b]\to\mathbb{R}$, $b>0$ is said to be starshaped if for every $x\in[0,b]$ and $t\in[0,1]$ we have $f(tx)\leq tf(x)$.

According to above definitions, if a subadditive function $f:A\subset[0,\infty)\to\mathbb{R}$ is also starshaped, then f is a convex function.

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [4], [10]):

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f:I\subset\mathbb{R}\to\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$.

The inequalities (1.1) have grown into a significant pillar for mathematical analysis and optimization, besides, by looking into a variety of settings, these inequalities are found to have a number of uses. What is more, for a specific choice of the function f, many inequalities with special means are obtainable. Hermite Hadamard's inequality (1.1), for example, is significant in its rich geometry and hence there are many studies on it to demonstrate its new proofs, refinements, extensions and generalizations. You can check ([2–5, 12–21]) and the references included there.

The most well-known inequalities related to the integral mean of a convex function are the Hermite Hadamard inequalities. Since then, some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors. Involving to the products of convex functions, Pachpatte gave two important new Hermite-Hadamard type inequalities as follows in [9]:

Theorem 1. *Let f and g be real-valued, nonnegative, and convex functions on $[a,b]$. Then,*

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$
$$\leq\frac{1}{b-a}\int_a^b f(x)g(x)dx+\frac{M(a,b)}{3}+\frac{N(a,b)}{6}$$

and

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx\leq\frac{M(a,b)}{3}+\frac{N(a,b)}{6}$$

where

$$\begin{aligned}M(a,b)&=f(a)g(a)+f(b)g(b)\\ N(a,b)&=f(a)g(b)+f(b)g(a).\end{aligned} \tag{1.2}$$

In this article, using subadditive functions, we obtained new inequalities of Hermite-Hadamard type and related. The results presented here would provide extensions of those given in earlier works.

2. Main Results

Hermite-Hadamard type inequalities for subadditive functions are given the following form:

Theorem 2. *If $f : I = [0,\infty) \to \mathbb{R}$ be a continuous subadditive function, $a,b \in I^{\circ}$ with $a < b$, then the following inequalities hold*

$$\frac{1}{2}f(a+b) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{1}{a}\int_0^a f(x)\,dx + \frac{1}{b}\int_0^b f(x)\,dx.$$

Proof. Let $x \in [a,b]$, and for $t \in [0,1]$ $x = at + (1-t)b \in [a,b]$ (or $x = a(1-t) + tb \in [a,b]$). By using subadditivity of the function f, we can write

$$f(at + (1-t)b) \leq f(at) + f((1-t)b) \tag{2.1}$$

and

$$f((1-t)a + tb) \leq f(a(1-t)) + f(tb). \tag{2.2}$$

Adding (2.1) and (2.2), and by using subadditivity of the function f, we have

$$\begin{aligned} f(a+b) &\leq f(at + (1-t)b) + f((1-t)a + tb) \\ &\leq f(at) + f((1-t)b) + f(a(1-t)) + f(tb). \end{aligned} \tag{2.3}$$

Integrating both sides of (2.3), it follows that

$$\begin{aligned} f(a+b) &\leq \int_0^1 f(at + (1-t)b)\,dt + \int_0^1 f((1-t)a + tb)\,dt \\ &\leq 2\int_0^1 f(at)\,dt + 2\int_0^1 f(tb)\,dt. \end{aligned} \tag{2.4}$$

Thus, by the change of variable we obtain that

$$\frac{1}{2}f(a+b) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{1}{a}\int_0^a f(x)\,dx + \frac{1}{b}\int_0^b f(x)\,dx$$

which completes the proof of Theorem. □

Corollary 1. *Under the conditions of Theorem 2, if we take $f(tx) \leq tf(x)$, then we get*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}f(a+b) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{2.5}$$

Proof. Since $f(tx) \leq tf(x)$, then by using (2.4) we have

$$\begin{aligned} 2f\left(\frac{a+b}{2}\right) &\leq f(a+b) \\ &\leq \int_0^1 f(at+(1-t)b)\,dt + \int_0^1 f((1-t)a+tb)\,dt \\ &\leq 2\int_0^1 tf(a)\,dt + 2\int_0^1 tf(b)\,dt = f(a)+f(b). \end{aligned}$$

By the change of variable, the desired inequalities (2.5) are achieved. □

Theorem 3. *If f, $g : I = [0,\infty) \to \mathbb{R}$ be two continuous subadditive functions, $a,b \in I^\circ$ with $a < b$, then the following inequalities hold*

$$\begin{aligned} &\frac{2}{b-a}\int_a^b f(x)g(x)dx \\ &\quad\leq \frac{2}{a}\int_0^a f(x)g(x)dx + \frac{2}{b}\int_0^b f(x)g(x)dx \\ &\qquad + 2\int_0^1 f(at)g((1-t)b)dt + 2\int_0^1 f((1-t)a)g(tb)dt \\ &\quad\leq \frac{1}{a}\int_0^a \left[(f(x))^2+(g(x))^2\right]dx + \frac{1}{b}\int_0^b \left[(f(x))^2+(g(x))^2\right]dx \\ &\qquad + 2\int_0^1 f(ta)f((1-t)b)dt + 2\int_0^1 g(ta)g((1-t)b)dt. \end{aligned} \tag{2.6}$$

Proof. By using subadditivity of the function f and g, we can write

$$f(ta+(1-t)b) \leq f(ta)+f((1-t)b) \tag{2.7}$$
$$g(ta+(1-t)b) \leq g(ta)+g((1-t)b). \tag{2.8}$$

Multiplying the above inequalities (2.7) and (2.8), we get

$$\begin{aligned} &f(ta+(1-t)b)g(ta+(1-t)b) \\ &\quad\leq (f(ta)+f((1-t)b))(g(ta)+g((1-t)b)) \\ &\quad= f(ta)g(ta)+f(ta)g((1-t)b) \\ &\qquad + f((1-t)b)g(ta)+f((1-t)b)g((1-t)b) \\ &\quad\leq \frac{1}{2}\left[(f(ta)+f((1-t)b))^2+(g(ta)+g((1-t)b))^2\right]. \end{aligned} \tag{2.9}$$

and taking the integral with respect to t on $[0,1]$ and changing the variables of integration, we obtain our desired inequalities (2.6). □

Corollary 2. *Under the conditions of Theorem 3, if we take $f(tx) \leq tf(x)$, then we get the following inequalities*

$$\begin{aligned}\frac{6}{b-a}\int_a^b f(x)g(x)dx &\leq 2M(a,b)+N(a,b)\\ &\leq (f(a)+f(b))^2+(g(a)+g(b))^2\\ &\quad -(f(a)f(b)+g(a)g(b))\end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are defined by (1.2).

Theorem 4. *If $f,\ g : I=[0,\infty)\to\mathbb{R}$ be two continuous subadditive functions, $a,b\in I^{\circ}$ with $a<b$, then the following inequalities hold*

$$\begin{aligned}&\frac{1}{2}f(a+b)g(a+b)\\ &\quad\leq \frac{1}{b-a}\int_a^b f(x)g(x)dx\\ &\qquad+\int_0^1 [f(at)g((1-t)a)+f((1-t)b)g(tb)]dt\\ &\qquad+\int_0^1 [f(at)g(tb)dt+f(tb)g(ta)]dt\end{aligned}\tag{2.10}$$

and

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)g(x)dx\\ &\quad\leq \frac{1}{a}\int_0^a f(x)g(x)dx+\frac{1}{b}\int_0^b f(x)g(x)dx\\ &\qquad+\frac{1}{2}\int_0^1 f(at)g((1-t)b)dt+\frac{1}{2}\int_0^1 f(tb)g((1-t)a)dt.\end{aligned}\tag{2.11}$$

Proof. By using subadditivity of the function f and g, we can write

$$\begin{aligned}f(a+b) &= f(at+(1-t)b+(1-t)+tb)\\ &\leq [f(at+(1-t)b)+f((1-t)a+tb)]\end{aligned}$$

and

$$g(a+b)\leq [g(at+(1-t)b)+g((1-t)a+tb)].$$

Multiplying the above inequalities, we have:

$$\begin{aligned}&f(a+b)g(a+b)\\ &\quad\leq [f(at+(1-t)b)g(at+(1-t)b)\\ &\qquad+f((1-t)a+tb)g((1-t)a+tb)]\\ &\qquad+[f(at+(1-t)b)g((1-t)a+tb)\\ &\qquad+f((1-t)a+tb)g(at+(1-t)b)]\end{aligned}$$

$$
\begin{aligned}
&\leq [f(at+(1-t)b)\,g(at+(1-t)b) \\
&\quad +f((1-t)a+tb)\,g((1-t)a+tb)] \\
&\quad +[f(at)+f((1-t)b)]\,[g((1-t)a)+g(tb)] \\
&\quad +[f((1-t)a)+f(tb)]\,[g(at)+g((1-t)b)] \\
&=[f(at+(1-t)b)\,g(at+(1-t)b) \\
&\quad +f((1-t)a+tb)\,g((1-t)a+tb)] \\
&\quad +[f(at)\,g((1-t)a)+f(at)\,g(tb) \\
&\quad +f((1-t)b)\,g((1-t)a)+f((1-t)b)\,g(tb)] \\
&\quad +[f((1-t)a)\,g(ta)+f((1-t)a)\,g((1-t)b) \\
&\quad +f(tb)\,g(ta)+f(tb)\,g((1-t)b)]. \qquad (2.12)
\end{aligned}
$$

Taking the integral of both sides in (2.12) with respect to t on $[0,1]$ and changing the variables of integration, we get

$$
\begin{aligned}
&\frac{1}{2}f(a+b)g(a+b) \\
&\qquad \leq \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \\
&\qquad\quad +\int_0^1 [f(at)\,g((1-t)a)+f(tb)\,g((1-t)b)]\,dt \\
&\qquad\quad +\int_0^1 [f(at)\,g(tb)\,dt+f(tb)\,g(ta)]\,dt
\end{aligned}
$$

which completes the inequality (2.10). Since f and g are subadditive function, we can write

$$
\begin{aligned}
f(at+(1-t)b) &\leq f(at)+f((1-t)b) \\
g(at+(1-t)b) &\leq g(at)+g((1-t)b),
\end{aligned}
$$

and

$$
\begin{aligned}
f((1-t)a+tb) &\leq f((1-t)a)+f(tb) \\
g((1-t)a+tb) &\leq g((1-t)a)+g(tb).
\end{aligned}
$$

Multiplying the above inequalities, we get

$$
\begin{aligned}
f(at\ &+(1-t)b)\,g(at+(1-t)b) \\
&\quad +f((1-t)a+tb)\,g((1-t)a+tb) \\
&\leq f(at)\,g(at)+f(at)\,g((1-t)b) \\
&\quad +f((1-t)b)\,g(at)+f((1-t)b)\,g((1-t)b) \\
&\quad +f((1-t)a)\,g((1-t)a)+f((1-t)a)\,g(tb)
\end{aligned}
$$

$$+f(tb)\,g((1-t)a)+f(tb)\,g(tb)$$

and taking the integral with respect to t on $[0,1]$ and changing the variables of integration, we obtain the inequality (2.11). □

Corollary 3. *Under the conditions of Theorem 4, if we take $f(tx)\leq tf(x)$, then we get*

$$\begin{aligned}2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)&\\ &\leq\frac{1}{2}f(a+b)\,g(a+b)\\ &\leq\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx+\frac{M(a,b)}{3}+\frac{N(a,b)}{6}\end{aligned}$$

and

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx\leq\frac{M(a,b)}{3}+\frac{N(a,b)}{6}$$

where $M(a,b)$ and $N(a,b)$ are defined by (1.2).

Proof. Since $f(tx)\leq tf(x)$, then by using this inequality in the right and left hand sides of (2.4), we have

$$\begin{aligned}\frac{1}{2}f(a+b)g(a+b)&\\ &\leq\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx\\ &\quad+\int_0^1\left[f(at)\,g((1-t)a)+f((1-t)b)\,g(tb)\right]dt\\ &\quad+\int_0^1\left[f(at)\,g(tb)\,dt+f(tb)\,g(ta)\right]dt\\ &\leq\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx\\ &\quad+\left[f(a)\,g(a)+f(b)\,g(b)\right]\int_0^1 t(1-t)\,dt\\ &\quad+\left[f(a)\,g(b)\,dt+f(b)\,g(a)\right]\int_0^1 t^2dt\\ &=\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx\\ &\quad+\frac{1}{6}\left[f(a)\,g(a)+f(b)\,g(b)\right]+\frac{1}{3}\left[f(a)\,g(b)+f(b)\,g(a)\right].\end{aligned}$$

With the same procedure as above, the other inequality is obtained. □

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SOME STEFFENSEN-TYPE ITERATIVE SCHEMES FOR THE APPROXIMATE SOLUTION OF NONLINEAR EQUATIONS

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Abstract. In this paper, we suggest some new and efficient iterative methods for solving nonlinear equations $f(x) = 0$. These methods are free from derivatives having high order of convergence. We also give some examples to illustrate the efficiency of these methods. Finally, numerical tests confirm the theoretical results and allow us to compare these variants with the classical Steffensen's method. These new methods can be considered as alternative of existing derivative-free methods.

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1. INTRODUCTION

Finding the approximate solution of the nonlinear equation $f(x) = 0$ is one of the basic problems which frequently occurs in scientific work of various fields [1–13]. Due to the higher order of the nonlinear equations and the involvement of transcendental functions, analytical methods cannot be employed and therefore, it is only possible to obtain approximate solutions by relying on numerical methods based on iteration procedure. There are several methods in the literature for the computation of the root of the nonlinear equations. Newton method is well known classical method which is described as:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots$$

This method possesses quadratic convergence in the neighborhood of simple root r. This method is not applicable when the derivative of any function is not defined at initial or any stage of computations. There are also some interval based initial methods but have slow convergence towards the solution with additional drawbacks.

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Some time we face the complicated problems for whom the calculation of derivatives are more complicated to compute. Some of the problems of such type are listed here.

(1) $f(x) = \cos(x) + \sin(2x)\sqrt{1-2x^2} + \sin(x^2) + x^{14} + x^3\ln(x) + 12$
(2) $f(x) = (1+x^3)\cos(x^2) + \sqrt{1-x^2} - 2(9\sqrt{2}+7\sqrt{3})x^4$
(3) $f(x) = \frac{1}{x} - |x|$.

Such kind of problems need derivative-free iterative methods for obtaining the approximate solutions. In the last few years, researchers have established the idea of removing derivatives from the main recurrence relations in order to escape defining new functions involving derivatives and tried to calculate iterations only by using the function that describes the problem. It is tried to preserve the order of convergence of the method after using the approximation of the involved derivatives. Steffensen [5, 13] introduced such type of method in which he used the approximation of derivative as:

$$f'(x_n) \approx \frac{f(x_n + f(x_n) - f(x_n)}{f(x_n)}$$

Using this approximation, Steffensen [5, 13] modified the Newton method as a derivative-free method:

$$x_{n+1} = x_n - \frac{[f(x_n)]^2}{f(x_n + f(x_n)) - f(x_n)}, \ n = 0, 1, 2, ...$$

This method is known as Steffensen's iterative method for solving nonlinear equations. This method has quadratic convergence. Steffensen's method is free from any derivative of the function and have great importance because sometimes the applications of iterative methods which depend upon derivatives are restricted in engineering and other fields.

The rest of this paper is organized as follows. In Section 2, we describe our methods that are free from derivatives and have better convergence rate than Steffensen's method. In Section 3, we establish the convergence order of these methods. Finally, in Section 4 different numerical tests confirm the theoretical results and allow us to compare these methods with the existing methods.

2. Construction of derivative-free iterative techniques

In this section, we derive some new iterative techniques for solving nonlinear equations without using the derivative of the function to obtain such type of iterative methods. We use approximation of the first derivatives of the function to obtain such type of iterative methods.

Let us approximate the first derivative of the function $f(x)$ by

$$f'(x) = \frac{f(x + bf(x)) - f(x)}{bf(x)}, \ \because b \in \mathbb{R} \text{ and } b \neq 0.$$

Now we use this approximation in well known Newton method [3, 13] and obtain the following derivative-free iterative method for solving nonlinear equations as:

Algorithm 2.1. For a given x_0, find the approximate solution x_{n+1} by the iterative scheme:

$$x_{n+1} = x_n - \frac{b[f(x_n)]^2}{f(x_n + bf(x_n)) - f(x_n)}, \ n = 0, 1, 2, \ldots$$

For different values of b, we obtain classes of iterative schemes without involving derivatives. If we select, $b = 1$, then Algorithm 2.1 reduces to the well known Steffensen's method [5, 13]. Our aim is to modify higher order methods as derivative-free method preserving their order of convergence. Halley method is a well known cubic convergent method [5, 13] involving higher derivatives. This method has cubic convergence and can be derived by various techniques [5, 12, 13]. Halley method is described as:

Algorithm 2.2. For a given x_0, find the approximate solution x_{n+1} by the iterative scheme:

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)}, \ n = 0, 1, 2, \ldots$$

This method involves second derivative of the function for solving nonlinear equations. We will use the approximation of the first and second derivatives of function $f(x)$ as:

$$f'(x_n) \approx \frac{f(x_n + bf(x_n)) - f(x_n - bf(x_n))}{2b[f(x_n)]},$$

$$f''(x_n) \approx \frac{f(x_n + bf(x_n)) - 2f(x) + f(x_n - bf(x_n))}{[bf(x_n)]^2}.$$

Using the above approximations of derivatives in Algorithm 2.2, we obtain the following derivative-free iterative method as:

Algorithm 2.3. For a given x_0, find the approximate solution x_{n+1} by the iterative scheme:

$$x_{n+1} = x_n - \frac{b[f(x_n)]^2 A(x_n)}{[A(x_n)]^2 - 2f(x_n)[f(x_n + bf(x_n)) - 2f(x_n) + f(x_n - bf(x_n))}},$$

where

$$A(x_n) = [f(x_n + bf(x_n)) - f(x_n - bf(x_n))] \because b \in \mathbb{R}, \text{ and } b \neq 0.$$

Algorithm 2.3 is a newly derived third order convergent derivative-free iterative method for solving nonlinear equations. Shah and Noor [12] suggested a fourth-order iterative method for solving nonlinear equations as:

Algorithm 2.4. For a given x_0, find the approximate solution x_{n+1} by the iterative schemes:

$$y_n = x_n - \frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)},$$

$$x_{n+1} = y_n - \frac{2f(y_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)}, \ n = 0, 1, 2, \ldots$$

Algorithm 2.4 involves the second derivative of the function. We use the approximations of the first and the second derivative of function $f(x_n)$ and obtain the new iterative scheme by this modification as:

Algorithm 2.5. For a given x_0, find the approximate solution x_{n+1} by the iterative schemes:

$$y_n = x_n - \frac{2bA(x_n)}{[A(x_n)]^2 - 2f(x_n)[f(x_n + bf(x_n)) - 2f(x_n) + f(x_n - bf(x_n))}},$$

$$x_{n+1} = y_n - \frac{2bf(x_n)f(y_n)[A(x_n)]}{[A(x_n)]^2 - 2f(x_n)[f(x_n + bf(x_n)) - 2f(x_n) + f(x_n - bf(x_n))}}.$$

Algorithm 2.5 is a fourth order convergent derivative-free iterative method for solving nonlinear equations.

Shah [11] suggested the following fourth-order iterative method for solving nonlinear equations.

Algorithm 2.6. For a given x_0, find the approximate solution x_{n+1} by the iterative schemes:

$$y_n = x_n - \frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n))},$$

$$x_{n+1} = y_n - \frac{4f(y_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n))} - \frac{4f(y_n)f'(y_n)[f'(x_n)]^2}{[2[f'(x_n)]^2 - f(x_n)f''(x_n))]^2}.$$

This method also involves the derivatives of the function. We modify this method by using approximation of derivatives and obtain the following derivative-free iterative method.

Algorithm 2.7. For a given x_0, find the approximate solution x_{n+1} by the iterative schemes:

$$y_n = x_n - \frac{2b[f(x_n)]^2[A(x_n)]}{[A(x_n)]^2 - 2f(x_n)[f(x_n + bf(x_n)) - 2f(x_n) + f(x_n - bf(x_n))}},$$

$$x_{n+1} = y_n - \frac{4bf(x_n)f(y_n)[A(x_n)]}{[A(x_n)]^2 - 2f(x_n)[f(x_n + bf(x_n)) - 2f(x_n) + f(x_n - bf(x_n))} - \frac{4[bf(x_n)]^3[A(x_n)][f(y_n + bf(y_n)) - f(y - bf(y_n))]}{[A(x_n)]^2 - 2f(x_n)[f(x_n + bf(x_n)) - 2f(x_n) + f(x_n - bf(x_n))]^2}.$$

Algorithm 2.7 is a fourth order convergent derivative-free iterative method for solving nonlinear equations.

Noor [10] suggested the following two iterative methods for solving nonlinear equations which involve the first derivative of the function as:

Algorithm 2.8. For a given x_0, find the approximate solution x_{n+1} by the iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \frac{f(x_n) + hf(y_n)}{f'(y_n)}.$$

Now we use the suggested approximation in Algorithm 2.8 and obtain a derivative-free iterative method as:

Algorithm 2.9. For a given x_0, find the approximate solution x_{n+1} by the iterative schemes:

$$x_{n+1} = x_n - \frac{2b[f(x_n)]^2}{f(x_n + bf(x_n)) - f(x_n - bf(x_n))},$$
$$x_{n+1} = y_n - \frac{2bf(y_n)[f(x_n) + hf(y_n)]}{f(y_n + bf(y_n)) - f(y_n - bf(y_n))}.$$

Similarly Noor [10] also suggested a method which involves derivatives:

Algorithm 2.10. For a given x_0, find the approximate solution x_{n+1} by the iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = y_n - \frac{f(x_n) + 2hf(y_n)}{f'(x_n)} + \frac{hf(y_n)f'(y_n)}{[f(x)]^2}.$$

Using the approximation in Algorithm 2.10 we obtain a derivative-free method as:

Algorithm 2.11. For a given x_0, find the approximate solution x_{n+1} by the iterative schemes:

$$y_n = x_n - \frac{2b[f(x_n)]^2}{f(x_n + bf(x_n)) - f(x_n - bf(x_n))},$$
$$x_{n+1} = y_n - \frac{2bf(x_n)[f(x_n) + 2hf(y_n)]}{f(x_n + bf(x_n)) - f(x_n - bf(x_n))} + \frac{h[f(y_n + bf(y_n)) - f(y_n - bf(y_n))]}{2b[f(x_n)]^2}.$$

3. Convergence analysis

In this section, we consider the convergence criteria of Algorithm 2.5 developed in Section 2. All other methods can be studied for the convergence in a similar manner.

Theorem 1. *Let α be a simple zero of a sufficiently differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ in an open interval D. If x_0 is sufficiently close to α, then Algorithm 2.5 has fourth order convergence and satisfies the error equation*

$$e_{n+1} = [(c_2^2 - c_3 + c_3 b^2 (c_1)^2) e_n^4 + O(e_n^5)].$$

Proof. Let α be the simple root of $f(x)$. Using Taylor series and expanding $f(x_n)$, $f(x_n - bf(x_n))$ and $f(x_n + bf(x_n))$, we obtain

$$f(x_n) = c_1[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + O(e_n^6)], \tag{3.1}$$

$$f(x_n + bf(x_n)) = c_1(1 + bc_1)e_n + 3(c_1)^2 bc_2 + c_1 c_2 + (c_1)^3 b^2 c_2) e_n^2 + O(e_n^3) \tag{3.2}$$

and

$$f(x_n - bf(x_n)) = [-c_1(-1+bc_1)e_n + (3c_1)^2bc_2 + c_1c_2 + (c_1)^3b^2c_2)e_n^2 + O(e_n^3)], \tag{3.3}$$

where

$$c_k = \frac{f^{(k)}(p)}{f'(p)},\ k = 2,3,...,\ c_1 = f'(p),\ e_n = x_n - p.$$

Now using (3.2) and (3.3), we obtain

$$\begin{aligned} &f(x_n + bf(x_n)) + f(x_n - bf(x_n)) \\ &= [2b(c_1)^2e_n + 6(c_1)^2bc_2e_n^2 + 4b(c_1)^2c_2^2 + 8b(c_1)^2c_3 + 2c_3(c_1)^4b^3)e_n^3 + O(e_n^4)]. \end{aligned} \tag{3.4}$$

Using (3.1), (3.2), (3.3) and (3.4) in Algorithm 2.5, we have

$$\begin{aligned} &x_n - \\ &\frac{2b[f(x_n)]^2[f(x_n + bf(x_n)) - f(x_n - bf(x_n))]}{[f(x_n + bf(x_n)) - f(x_n - bf(x_n))]^2 - 2f(x_n)[f(x_n + bf(x_n)) - 2f(x_n) + f(x_n - bf(x_n))} \\ &= \alpha + [(c_2^2 - c_3 + c_3b^2(c_1)^2)e_n^3 + O(e_n^4)]. \end{aligned}$$

Using above all equations and simple manipulations yield the following error equations

$$e_{n+1} = [(c_2^2 - c_3 + c_3b^2(c_1)^2)e_n^4 + O(e_n^5)].$$

This shows that Algorithm 2.5 has fourth order convergence. □

4. Numerical Results

In this section, we exhibit computational comparison of newly derived methods for some examples. All the examples are tested by using Maple 11. Stopping criteria used for the computational work are given as:

- $|x_n - x_{n-1}| < \varepsilon$,
- $|f(x_n)| < \varepsilon$, where $\varepsilon = 10^{-64}$.

Example 1. We consider the nonlinear equation $f(x) = \cos(x) - x$, and the initial guess for all the methods is $x_0 = 2$. Results and comparison is shown in Table 1 and Table 2 for $b = 1$ and $b = 0.25$ respectively.

TABLE 1. Comparison of methods for $b = 1$.

Method	IT	x_n	$\lvert x_n - x_{n-1}\rvert$	$\lvert f(x_n)\rvert$
SM	6	0.7390848856	$5.08170e^{-22}$	0.0000000
Alg 2.5	3	0.7390848856	$1.12139e^{-3}$	$6.8890e^{-12}$
Alg 2.7	3	0.7390848856	$4.63400e^{-5}$	$7.9041e^{-9}$
Alg 2.9	5	0.7390848856	$1.19850e^{-20}$	0.0000000
Alg 2.11	3	0.7390848856	$2.53515e^{-23}$	0.0000000

TABLE 2. Comparison of methods for $b = \frac{1}{4}$.

Method	IT	x_n	$\lvert x_n - x_{n-1}\rvert$	$\lvert f(x_n)\rvert$
SM	6	0.7390848856	$5.08170e^{-22}$	0.0000000
Alg 2.5	3	0.7390848856	$1.29387e^{-05}$	$1.45143e^{-8}$
Alg 2.7	3	0.7390848856	$1.5601833e^{-05}$	$5.16302e^{-6}$
Alg 2.9	6	0.7390848856	$3.40895e^{-16}$	0.0000000
Alg 2.11	6	0.7390848856	$8.68285e^{-23}$	0.0000000

TABLE 3. Comparison of methods for $b = 1$.

Method	IT	x_n	$\lvert x_n - x_{n-1}\rvert$	$\lvert f(x_n)\rvert$
SM	11	1.36523001478	0.00000000	$2.31492e^{-14}$
Alg 2.5	6	1.36523001478	$2.6400e^{-7}$	$2.45754e^{-17}$
Alg 2.7	4	1.36523001478	$9.823e^{-6}$	$1.609915e^{-14}$
Alg 2.9	8	1.36523001478	0.00000000	$2.2193e^{-24}$
Alg 2.11	7	1.36523001478	0.00000000	$6.08207e^{-27}$

TABLE 4. Comparison of methods for $b = \frac{1}{4}$.

Method	IT	x_n	$\lvert x_n - x_{n-1}\rvert$	$\lvert f(x_n)\rvert$
SM	11	1.36523001478	0.0000000	$2.31492e^{-14}$
Alg 2.5	3	1.36523001478	$8.17300e^{-9}$	$6.136728e^{-14}$
Alg 2.7	3	1.36523001478	$2.103500e^{-7}$	$6.671405e^{-5}$
Alg 2.9	8	1.36523001478	0.0000000	$2.5083e^{-17}$
Alg 2.11	7	1.36523001478	0.0000000	$5.0634e^{-32}$

Example 2. We consider the nonlinear equation $f(x) = x^3 + 4x - 10$.

Table 3 and Table 4 depict the numerical results of Example 2. We use the initial guess $x_0 = 1.5$ for the computer program.

Example 3. Consider the nonlinear equation $f(x) = 2x^3 - 3x^2$. We observe that its derivative becomes zero on the interval $[\frac{1}{2}, 2]$, so for this function we can use only derivative-free methods.

Table 5 and Table 6 depict the numerical results of Example 3. We use the initial guess $x_0 = 0.5$ for the computer program.

TABLE 5. Comparison of methods for $b = 1$.

Method	IT	x_n	$\lvert x_n - x_{n-1}\rvert$	$\lvert f(x_n)\rvert$
SM	23	1.499999335	$1.0449181e^{-18}$	0.0000000
Alg 2.5	5	1.499999335	$5.5190301e^{-4}$	$-3.4115e^{-5}$
Alg 2.7	4	1.499999335	$2.6258151e^{-5}$	$2.566e^{-6}$
Alg 2.9	9	1.499999335	$7.5387141e^{-15}$	0.0000000
Alg 2.11	9	1.499999335	$1.7219124e^{-16}$	0.0000000

TABLE 6. Comparison of methods for $b = \frac{1}{4}$.

Method	IT	x_n	$\lvert x_n - x_{n-1}\rvert$	$\lvert f(x_n)\rvert$
SM	23	1.499999335	$1.0449181e^{-18}$	0.0000000
Alg 2.5	4	1.499999335	$2.3284121e^{-6}$	$-4.569e^{-12}$
Alg 2.7	3	1.499999335	$1.0355809e^{-6}$	$4.04686e^{-9}$
Alg 2.9	7	1.499999335	$4.6836141e^{-16}$	0.0000000
Alg 2.11	7	1.499999335	$2.6909900e^{-16}$	0.0000000

Example 4. Consider the nonlinear equation

$$f(x) = \cos(x) + \sin(2x)\sqrt{1-x^2} + \sin(x^2) + x^{14} + x^3 + 1/2x$$

with $x_0 = -0.92$.

Table 7 and Table 8 show the numerical results of Example 4.

Example 5. Consider the nonlinear equation $f(x) = \dfrac{1}{x} - \mid x \mid$ with $x_0 = 0.7$.

Table 9 and Table 10 show the numerical results of Example 5.

TABLE 7. Comparison of methods for $b = 1$.

Method	IT	x_n	$\lvert x_n - x_{n-1}\rvert$	$\lvert f(x_n)\rvert$
SM	5	-0.92577268	$7.6591100e^{-20}$	0.0000000
Alg 2.5	2	-0.92577268	$3.0216400e^{-4}$	$2.634e^{-6}$
Alg 2.7	2	-0.92577268	$3.2210000e^{-4}$	$2.566e^{-6}$
Alg 2.9	4	-0.92577268	$1.7934900e^{-17}$	0.0000000
Alg 2.11	4	-0.92577268	$1.6139720e^{-22}$	0.0000000

TABLE 8. Comparison of methods for $b = \frac{1}{2}$.

Method	IT	x_n	$\lvert x_n - x_{n-1}\rvert$	$\lvert f(x_n)\rvert$
SM	5	-0.92577268	$7.6591100e^{-20}$	0.0000000
Alg 2.5	2	-0.92577268	$2.2412000e^{-4}$	$5.647e^{-7}$
Alg 2.7	2	-0.92577268	$5.0631000e^{-4}$	$4.327e^{-8}$
Alg 2.9	4	-0.92577268	$1.0470550e^{-4}$	0.0000000
Alg 2.11	5	-0.92577268	$8.3425000e^{-27}$	0.0000000

TABLE 9. Comparison of methods for $b = 1$.

Method	IT	x_n	$\lvert x_n - x_{n-1}\rvert$	$\lvert f(x_n)\rvert$
SM	6	0.9999999932	$3.2474947e^{-18}$	0.0000000
Alg 2.5	4	1.00000027	$5.9794800e^{-4}$	$1.5650e^{-10}$
Alg 2.7	7	1.00000027	$1.2535300e^{-5}$	$7.01990e^{-7}$
Alg 2.9	16	1.00000027	$1.2615400e^{-25}$	0.0000000
Alg 2.11	13	1.00000027	$2.7432108e^{-21}$	0.0000000

TABLE 10. Comparison of methods for $b = \frac{1}{2}$.

Method	IT	x_n	$\lvert x_n - x_{n-1}\rvert$	$\lvert f(x_n)\rvert$
SM	6	0.9999999932	$3.2474947e^{-18}$	0.000000
Alg 2.5	3	0.9999999932	$8.8873000e^{-5}$	$9.88077e^{-8}$
Alg 2.7	4	0.9999999932	$5.4401360e^{-5}$	$7.25509e^{-6}$
Alg 2.9	5	0.9999999932	$7.8798800e^{-14}$	0.0000000
Alg 2.11	6	0.9999999932	$8.7039110e^{-21}$	0.0000000

CONCLUSION

In this article, we have used approximations with a parameter $b \in R$ for the first and higher derivatives in various known methods. These approximations generated

the new derivative-free methods and also these newly derived methods preserved the same order of convergence. The theoretical results have been checked with some numerical examples. Comparison is also given to check the performance and efficiency of these iterative schemes.

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ON PERFECT POWERS WHICH ARE SUM OR DIFFERENCE OF TWO LUCAS NUMBERS

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Abstract. In this paper, we consider the Diophantine equation $L_n \pm L_m = kx^2$ with $k \in \{1,2\}$ and we find all solutions of this equation in nonnegative integers n, m, and x when $n \equiv m(mod 2)$. With the help of these solutions, we solve the equation $L_n - L_m = 2^a$. In order to solve the last equation, we also use lower bounds for linear forms in logarithms and a version of the Baker-Davenport reduction method in diophantine approximation.

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1. INTRODUCTION

The Fibonacci sequence (F_n) is defined as $F_0 = 0,\ F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The Lucas sequence (L_n), which is similar to the Fibonacci sequence, is defined by the same recursive pattern with initial conditions $L_0 = 2,\ L_1 = 1$. Binet's formulas

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$$

and

$$L_n = \alpha^n + \beta^n$$

are well known, where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$, which are the roots of the characteristic equation $x^2 - x - 1 = 0$. It can be seen that $1 < \alpha < 2$ and $-1 < \beta < 0$. The following relations given for Fibonacci and Lucas numbers will be useful for us:

$$\alpha^{n-1} \leq L_n \leq 2\alpha^n \quad \text{for } n \geq 0, \tag{1.1}$$

$$(F_m, F_n) = F_{(m,n)}, \tag{1.2}$$

$$L_{n+m} + L_{n-m} = \begin{cases} 5F_mF_n \text{ if } n \text{ is odd}, \\ L_nL_m \text{ otherwise.} \end{cases}, \tag{1.3}$$

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and

$$L_{n+m} - L_{n-m} = \begin{cases} L_n L_m \text{ if } n \text{ is odd}, \\ 5F_m F_n \text{ otherwise.} \end{cases} . \tag{1.4}$$

The relations (1.1), (1.3), and (1.4) can be proved using Binet's formulae. The Fibonacci and Lucas sequences are the most important among the second order linear recussive sequences and have been investigated by many researchers. The problem to find all perfect powers in the Fibonacci sequence and in the Lucas sequence have been an interesting problem for many years. Finally, this problem was solved by Bugeaud, Mignotte and Siksek in [8]. They stated that the only perfect powers in the Fibonacci sequence are $0, 1, 8, 144$ and the only perfect powers in the Lucas sequence are 1 and 4. Then, in [7], the authors found all the nonnegative integer solutions (n, y, p) of the equations

$$F_n \pm 1 = y^p, \ p \geq 2.$$

Later, in [12], Luca and Patel have tackled the problem to find the integer solutions (n, m, y, p) of the Diophantine equation

$$F_n \pm F_m = y^p, \ p \geq 2. \tag{1.5}$$

They found that if $n \equiv m(mod 2)$, this equation has solution either
$\max\{|n|, |m|\} \leq 36$ or $y = 0$ and $|n| = |m|$.

Recently, many mathematicians, in [2–5], have handled some exponential Diophantine equations such as

$$u_n + u_m = 2^a \text{ and } u_n + u_m + u_r = 2^a,$$

where (u_n) is the Fibonacci sequence or the Lucas sequence or the Pell sequence. Furthermore, In [15] , we determined the nonnegative integer solutions (n, m, a) of the equation $F_n - F_m = 2^a$. In this study, we solve the equation $L_n \pm L_m = kx^2$ with $k = 1, 2$ when $n \equiv m(mod 2)$. With help of these solutions, and motivated by the studies of Bravo and Luca [4] , we find all nonnegative integers solutions (n, m, a) of the equation

$$L_n - L_m = 2^a. \tag{1.6}$$

In this paper, we follow the approach and the method presented in [4]. Therefore, in section 2, we introduce necessary lemmas and theorems related to this method. Then in section 3, we prove our main theorems.

2. Auxiliary results

Lately, in many articles, to solve Diophantine equations such as the equation (1.6), authors have used Baker's theory lower bounds for a nonzero linear form in logarithms of algebraic numbers. Since such bounds are of crucial importance in effectively solving of Diophantine equations , we start with recalling some basic notions from algebraic number theory.

Let η be an algebraic number of degree d with minimal polynomial

$$a_0x^d + a_1x^{d-1} + ... + a_d = a_0\prod_{i=1}^{d}\left(X - \eta^{(i)}\right) \in \mathbb{Z}[x],$$

where the a_i's are relatively prime integers with $a_0 > 0$ and $\eta^{(i)}$'s are conjugates of η. Then

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d}\log\left(\max\left\{|\eta^{(i)}|, 1\right\}\right)\right) \tag{2.1}$$

is called the logarithmic height of η. In particularly, if $\eta = a/b$ is a rational number with $\gcd(a,b) = 1$ and $b > 1$, then $h(\eta) = \log(\max\{|a|, b\})$.

The following properties of logarithmic height are found in many works stated in references:

$$h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2, \tag{2.2}$$

$$h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma), \tag{2.3}$$

$$h(\eta^s) = |s|h(\eta). \tag{2.4}$$

The following theorem, is deduced from Corollary 2.3 of Matveev [13], provides a large upper bound for the subscript n in the equation (1.6) (also see Theorem 9.4 in [8]).

Theorem 1. *Assume that $\gamma_1, \gamma_2, ..., \gamma_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, b_2, ..., b_t$ are rational integers, and*

$$\Lambda := \gamma_1^{b_1}...\gamma_t^{b_t} - 1$$

is not zero. Then

$$|\Lambda| > \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1+\log D)(1+\log B)A_1A_2...A_t\right),$$

where

$$B \geq \max\{|b_1|, ..., |b_t|\},$$

and $A_i \geq \max\{Dh(\gamma_i), |\log\gamma_i|, 0.16\}$ for all $i = 1, ..., t$.

The following lemma, was proved by Dujella and Pethő [10], is a variation of a lemma of Baker and Davenport [1]. And this lemma will be used to reduce the upper bound for the subscript n in the equation (1.6). In the following lemma, the function $||\cdot||$ denotes the distance from x to the nearest integer. That is, $||x|| = \min\{|x-n| : n \in \mathbb{Z}\}$ for a real number x.

Lemma 1. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational number γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\varepsilon := ||\mu q|| - M||\gamma q||$. If $\varepsilon > 0$, then there exists no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w},$$

in positive integers u, v, and w with

$$u \leq M \text{ and } w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

From (1.3) and (1.4), we can give the following lemma.

Lemma 2. *Assume that $n \equiv m(mod 2)$. Then*

$$L_n + L_m = \begin{cases} 5F_{(n+m)/2}F_{(n-m)/2} \text{ if } n \equiv m+2(mod 4), \\ L_{(n+m)/2}L_{(n-m)/2} \text{ if } n \equiv m(mod 4). \end{cases},$$

and

$$L_n - L_m = \begin{cases} L_{(n+m)/2}L_{(n-m)/2} \text{ if } n \equiv m+2(mod 4), \\ 5F_{(n+m)/2}F_{(n-m)/2} \text{ if } n \equiv m(mod 4). \end{cases}.$$

The following two theorems and lemma are given in [9] and [11].

Theorem 2. *If $F_n = x^2$, then $n = 0, 1, 2, 12$. If $F_n = 2x^2$, then $n = 0, 3, 6$. If $L_n = x^2$, then $n = 1, 3$ and if $L_n = 2x^2$, then $n = 0, 6$.*

Theorem 3. *Let $m > 3$ be an integer and $F_n = F_m x^2$ for some $x \in \mathbb{Z}$. Then $n = m$.*

Lemma 3. *There is no integer x such that $F_n = 5F_m x^2$ for $m \geq 3$.*

The proof of following theorem is similar to that of the above lemma. Therefore, we omit its proof.

Theorem 4. *There is no integer x such that $F_n = 10F_m x^2$ for $m \geq 1$.*

From Theorem 4.1 and Theorem 4.2 given in [14], we can deduce the following results.

Theorem 5. *If $L_n L_m = x^2$ for $1 \leq m \leq n$, then $n = m$ or $(n,m) = (3,1)$.*

Theorem 6. *If $L_n L_m = 2x^2$ for $1 \leq m \leq n$, then $(n,m) = (3,1), (6,1)$, or $(6,3)$.*

3. Main theorems

Theorem 7. *Let $m \leq n$ and $n \equiv m(mod 2)$. Then all solutions of the equation $L_n + L_m = kx^2$ with $k = 1, 2$ in nonnegative integers n, m, are given by*

$$(n,m,x) = (6,6,6), (17,7,60), (6,4,5), \text{ and } (4r,0,L_{2r}) \text{ with } r \geq 0$$

when $k = 1$, and

$$(n,m,x) = (1,1,1), (3,3,2), (11,1,10), \text{ and } (8,2,5)$$

when $k = 2$.

Proof. Assume that $n \equiv m(mod2)$. If $n = m$, we have $L_n = 2z^2$ for $k = 1$, which implies that $n = 0,6$, or $L_n = x^2$ for $k = 2$, which implies that $n = 1,3$ by Theorem 2. Then, assume that $n > m$.

Firstly, let $n \equiv m+2(mod4)$. Then we can write $5F_{(n+m)/2}F_{(n-m)/2} = kx^2$ by Lemma 2. Thus we get the equations $F_{(n+m)/2} = aF_du^2$, $F_{(n-m)/2} = bF_dv^2$ for $k = 1$ and $F_{(n+m)/2} = aF_du^2$, $F_{(n-m)/2} = 2bF_dv^2$ or $F_{(n+m)/2} = 2aF_du^2$, $F_{(n-m)/2} = bF_dv^2$ for $k = 2$, where $ab = 5$ and $\left(F_{(n+m)/2}, F_{(n-m)/2}\right) = F_d$ by (1.2). Assume that $a = 1, b = 5$. Then we have $F_{(n+m)/2} = u^2$, $F_{(n-m)/2} = 5v^2$ for $k = 1$ by Lemma 3, and $F_{(n+m)/2} = 2F_du^2$, $F_{(n-m)/2} = 5F_dv^2$ for $k = 2$ by Theorem 4. If $F_{(n+m)/2} = u^2$, $F_{(n-m)/2} = 5v^2 = F_5v^2$, then it can be easily seen that $n = 17$ and $m = 7$ by Theorems 2 and 3. If $F_{(n+m)/2} = 2F_du^2$, $F_{(n-m)/2} = 5F_dv^2$, then $d \leq 2$ by Lemma 3, i.e., $F_{(n+m)/2} = 2u^2$, $F_{(n-m)/2} = 5v^2$. Thus we get $n = 11$ and $m = 1$ by Theorems 2 and 3. Assume that $a = 5$ and $b = 1$. Then we have the equation $F_{(n+m)/2} = 5u^2$, $F_{(n-m)/2} = v^2$ for $k = 1$ by Lemma 3, and $F_{(n+m)/2} = 5F_du^2$, $F_{(n-m)/2} = 2F_dv^2$ for $k = 2$ by Theorem 4. Similarly, it can be seen that $n = 6, m = 4$ for $k = 1$ and $n = 8, m = 2$ for $k = 2$.

Secondly, let $n \equiv m(mod4)$. Then we can write $L_{(n+m)/2}L_{(n-m)/2} = kx^2$ by Lemma 2. Using Theorems 5 and 6, we get $m = 0$ or $n = 4, m = 2$ for $k = 1$, and $(n,m) = (4,2),(7,5),(9,3)$ for $k = 2$. If $m = 0$, then $n \equiv 0(mod4)$. This implies that $n = 4r$ for some integer r. Besides, if $m = 0$, we have $x^2 = L_n + 2 = L_{4r} + 2 = L_{2r}^2$. Hence $(4r, 0, L_{2r})$ is a solution of the equation $L_n + L_m = x^2$. Also, it is clear that $L_4 + L_2 \neq kx^2$ for $k = 1,2$, and $L_7 + L_5 \neq 2x^2$, $L_9 + L_3 \neq 2x^2$. □

Since the proof of the following theorem is similar, we omit its proof.

Theorem 8. *Let $n \equiv m(mod2)$. Then all solutions of the equation $L_n - L_m = kx^2$ with $k = 1,2$, in nonnegative integers n, m, and x, are given by*

$$(n,m,x) = (4,2,2),(7,3,5) \text{ or } (4r+2,0,L_{2r+1}) \text{ with } r \geq 0$$

when $k = 1$, and

$$(n,m,x) = (7,5,3),\ (9,3,6)$$

when $k = 2$.

The following lemma, which will be used in the next theorem, gives a sufficient condition for a rational number to be a convergent of a given real number.

Lemma 4. ([6])*Let γ be a real number. Any non-zero rational number $\frac{a}{b}$ with*

$$\left|\gamma - \frac{a}{b}\right| < \frac{1}{2b^2}$$

is a convergent of γ.

Theorem 9. *The only solutions of the Diophantine equation $L_n - L_m = 2^a$ in non-negative integers $m < n$, and a, are given by*

$$(n,m,a) \in \{(2,0,0),(3,0,1),(6,0,4),(2,1,1)\}$$

and

$$(n,m,a) \in \{(3,2,0),(4,2,2),(5,2,3),(5,4,2)\}.$$

Proof. Assume that the equation $L_n - L_m = 2^a$ holds in nonnegative integers $m < n$, and a . Firstly, let $n \equiv m(mod2)$. Then by Theorem 8, it is possible only for $(n,m,2^{a/2}) = (4,2,2)$, which implies that $(n,m,a) = (4,2,2)$ or $(n,m,2^{a/2}) = (4r+2,0,L_{2r+1})$. This yields to $(n,m,a) = (2,0,0),(6,0,4)$. Secondly, let $n \not\equiv m(mod2)$. If $n-m = 1$, then we get the equation $L_{n-2} = 2^a$. This equation implies that $(n,m,a) = (3,2,0),(5,4,2),(2,1,1)$ by Theorem 2. If $n-m = 3$, then we have the equation $L_{n-2} = 2^{a-1}$, which shows that $(n,m,a) = (3,0,1),(5,2,3)$ by Theorem 2. Thus we may assume that $n-m \geq 5$. With the help of Mathematica program, one can see that the equation (1.6) has no solutions for $0 \leq m < n \leq 200$ and $n-m \geq 5$. From now on, assume that $n > 200$ and $n-m \geq 5$. Now, let us show that $a < n$. Using (1.1), we get the inequality

$$2^a = L_n - L_m < L_n \leq 2\alpha^n < 2^{n+1},$$

which yields to $a \leq n$.

On the other hand, rearranging the equation (1.6) as $\alpha^n - 2^a = L_m - \beta^n$ and taking absolute values, we obtain

$$|\alpha^n - 2^a| = |L_m - \beta^n| \leq L_m + |\beta|^n < 2\alpha^m + \frac{1}{2}$$

by (1.1). If we divide both sides of the above inequality by α^n, we get

$$\left|1 - 2^a\alpha^{-n}\right| < \frac{3}{\alpha^{n-m}}, \tag{3.1}$$

where we have used the facts that $\frac{1}{2}\alpha^{-m} < 1$ and $n > m$. Now, let us apply Theorem 1 with $\gamma_1 := 2, \gamma_2 := \alpha$ and $b_1 := a, b_2 := -n$. Note that the numbers γ_1 and γ_2 are positive real numbers and elements of the field $K = Q(\sqrt{5})$, so $D = 2$. It can be shown that the number $\Lambda_1 := 2^a\alpha^{-n} - 1$ is nonzero. Indeed, if $\Lambda_1 = 0$, then we get

$$2^a = \alpha^n = L_n - \beta^n > L_n - 1 \geq L_n - L_m = 2^a,$$

which is impossible. Since $h(\gamma_1) = \log 2 = 0.6931...$ and $h(\gamma_2) = \dfrac{\log\alpha}{2} = \dfrac{0.4812...}{2}$ by (2.1), we can take $A_1 := 1.4$, $A_2 := 0.5$. Also, since $a \leq n$, we can take $B := \max\{|a|,|-n|,1\} = n$. Thus, taking into account the inequality (3.1) and using Theorem 1, we obtain

$$\frac{3}{\alpha^{n-m}} > |\Lambda_1| > \exp\left(-1.4\cdot 30^5\cdot 2^{4.5}\cdot 2^2(1+\log 2)(1+\log n)(1.4)(0.5)\right)$$

and so

$$(n-m)\log\alpha - \log 3 < 1.4\cdot 30^5\cdot 2^{4.5}\cdot 2^2(1+\log 2)(1+\log n)(1.4)(0.5) \tag{3.2}$$

Now, we try to apply Theorem 1 a second time. Rearranging the equation (1.6) as $\alpha^n - \alpha^m - 2^a = -\beta^n + \beta^m$ and taking absolute values in here, we obtain

$$\left|\alpha^n(1-\alpha^{m-n}) - 2^a\right| = \left|-\beta^n + \beta^m\right| \leq |\beta|^n + |\beta|^m < 4/3,$$

where we used the fact that $|\beta|^n + |\beta|^m < 4/3$ for $n > 200$ and $m \geq 0$. Dividing both sides of the above inequality by $\alpha^n(1-\alpha^{m-n})$, we get

$$\left|1 - 2^a\alpha^{-n}(1-\alpha^{m-n})^{-1}\right| < \frac{4/3}{\alpha^n(1-\alpha^{m-n})}. \tag{3.3}$$

Since

$$\alpha^{m-n} = \frac{1}{\alpha^{n-m}} < \frac{1}{\alpha} < \frac{2}{3},$$

it is seen that

$$1 - \alpha^{m-n} > 1 - \frac{2}{3} = \frac{1}{3},$$

and therefore

$$\frac{1}{1-\alpha^{m-n}} < 3.$$

Then from (3.3), it follows that

$$\left|1 - 2^a\alpha^{-n}(1-\alpha^{m-n})^{-1}\right| < \frac{4}{\alpha^n}. \tag{3.4}$$

Thus, taking $\gamma_1 := 2$, $\gamma_2 := \alpha$, $\gamma_3 := (1-\alpha^{m-n})^{-1}$ and $b_1 := a$, $b_2 := -n$, $b_3 := 1$, we can apply Theorem 1. One can see that the numbers γ_1, γ_2, and γ_3 are positive real numbers and elements of the field $K = Q(\sqrt{5})$, so $D = 2$. Put $\Lambda_2 := 2^a\alpha^{-n}(1-\alpha^{m-n})^{-1} - 1$. Since

$$\alpha^n(1-\alpha^{m-n}) = \alpha^n - \alpha^m = L_n - \beta^n - L_m + \beta^m \neq 2^a$$

for $n > m$, the number $\Lambda_2 := 2^a\alpha^{-n}(1-\alpha^{m-n})^{-1} - 1$ is nonzero. Also, since $h(\gamma_1) = \log 2 = 0.6931...$, and $h(\gamma_2) = \dfrac{\log\alpha}{2} = \dfrac{0.4812...}{2}$ by (2.1), we can take $A_1 := 1.4$ and $A_2 := 0.5$. Besides, using (2.2), (2.3), and (2.4), we get that $h(\gamma_3) \leq \log 2 + (n-m)\dfrac{\log\alpha}{2}$. A simple computation shows that $|\log\gamma_3| \leq \log 4 + (n-m)\log\alpha$. So we can take $A_3 := \log 4 + (n-m)\log\alpha$. Also, since $a \leq n$, it follows that $B := \max\{|a|, |-n|, 1\} = n$. Thus, taking into account the inequality (3.4) and using Theorem 1, we obtain

$$\frac{4}{\alpha^n} > |\Lambda_2| > \exp(-C)(1+\log 2)(1+\log n)\,(1.4)\,(0.5)\,(\log 4 + (n-m)\log\alpha)$$

or

$$n\log\alpha - \log 4 < C(1+\log 2)(1+\log n)\,(1.4)\,(0.5)\,(\log 4 + (n-m)\log\alpha), \tag{3.5}$$

where $C = 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2$. Inserting the inequality (3.2) into the last inequality, a computer search with Mathematica gives us that $n < 1.81 \cdot 10^{25}$.

Now, let us try to reduce the upper bound on n applying Lemma 1. Let

$$z_1 := a\log 2 - n\log\alpha.$$

Then

$$|1-e^{z_1}| < \frac{3}{\alpha^{n-m}}$$

by (3.1). The inequality

$$\alpha^n = L_n - \beta^n > L_n - 1 \geq L_n - L_m = 2^a$$

implies that $z_1 < 0$. In that case, since $\dfrac{3}{\alpha^{n-m}} < \dfrac{1}{2}$ for $n-m \geq 5$, it follows that $e^{|z_1|} < 2$. Hence, we get

$$0 < |z_1| < e^{|z_1|} - 1 = e^{|z_1|}\,|1-e^{z_1}| < \frac{6}{\alpha^{n-m}},$$

or

$$0 < |a\log 2 - n\log\alpha| < \frac{6}{\alpha^{n-m}}.$$

Dividing this inequality by $\log\alpha$ and a, we get

$$0 < \left|\frac{\log 2}{\log\alpha} - \frac{n}{a}\right| < \frac{13}{a\cdot\alpha^{n-m}}. \tag{3.6}$$

Assume that $n-m \geq 137$. Then it can be seen that $\dfrac{\alpha^{n-m}}{26} > 1.6\cdot 10^{27} > n \geq a$, and so we have $\left|\frac{\log 2}{\log\alpha} - \frac{n}{a}\right| < \dfrac{13}{a\cdot\alpha^{n-m}} < \dfrac{1}{2a^2}$. This implies by Lemma 4 that the rational number $\dfrac{n}{a}$ is a convergent to $\gamma = \frac{\log 2}{\log\alpha}$. Now let $[a_0,a_1,a_2,...] = [1,2,3,1,2,3,2,4,...]$ be the continued fraction expression of γ and let $\dfrac{p_k}{q_k}$ be its k th convergent. Assume that $\dfrac{n}{a} = \dfrac{p_t}{q_t}$ for some t. Then we have $1.88\cdot 10^{25} > q_{56} > 1.81\cdot 10^{25} > a$. Thus $t \in \{0,1,2,...,55\}$. Let $a_M = \max\{a_i | i = 0,1,2,...,56\}$. Then we find that $a_M = 134$. From the known properties of continued fraction, we know that

$$\left|\gamma - \frac{p_t}{q_t}\right| = \frac{1}{(\gamma_{t+1}q_t + q_{t-1})q_t} = \frac{1}{(\gamma_{t+1} + \frac{q_{t-1}}{q_t})q_t^2} > \frac{1}{(a_{t+1}+2)q_t^2} > \frac{1}{(a_M+2)q_t^2}$$

and this shows that

$$\left|\gamma - \frac{p_t}{q_t}\right| > \frac{1}{(a_M+2)a^2},$$

where we have used the facts that $a_t = \lfloor\gamma_t\rfloor$ and $q_{t-1} < q_t$. Thus, from (3.6), we obtain

$$\frac{13}{a\cdot\alpha^{n-m}} > \frac{1}{(a_M+2)a^2},$$

which implies that

$$\frac{1}{3.2\cdot 10^{27}} > \frac{13}{\alpha^{n-m}} > \frac{1}{136\cdot q_{56}} > \frac{1}{2.55\cdot 10^{27}},$$

a contradiction. Therefore $n-m<137$. Substituting this upper bound for $n-m$ into (3.5), we obtain $n<3.5\cdot 10^{15}$.

Now, let

$$z_2 := a\log 2 - n\log\alpha - \log\left((1-\alpha^{m-n})\right).$$

In this case,

$$|1-e^{z_2}| < \frac{4}{\alpha^n}$$

by (3.4). It is seen that $\frac{4}{\alpha^n} < \frac{1}{2}$. If $z_2>0$, then $0<z_2<e^{z_2}-1<\frac{4}{\alpha^n}$. If $z_2<0$, then $|1-e^{z_2}| = 1-e^{z_2} < \frac{4}{\alpha^n} < \frac{1}{2}$. From this, we get $e^{|z_2|}<2$ and therefore

$$0<|z_2|<e^{|z_2|}-1 = e^{|z_2|}\,|1-e^{z_2}| < \frac{8}{\alpha^n}.$$

In both case, the inequality

$$0<|z_2|<\frac{8}{\alpha^n}$$

is true. That is,

$$0<\left|a\log 2 - n\log\alpha - \log(1-\alpha^{m-n})\right| < \frac{8}{\alpha^n}.$$

Dividing both sides of the above inequality by $\log\alpha$, we get

$$0<\left|a\left(\frac{\log 2}{\log\alpha}\right) - n + \frac{\log(1-\alpha^{m-n})^{-1}}{\log\alpha}\right| < 17\cdot\alpha^{-n}. \tag{3.7}$$

Let $\gamma := \frac{\log 2}{\log\alpha}$ and $M := 3.5\cdot 10^{15}$. Then the denominator of the 63^{rd} convergent of γ exceeds $6M$. Also, taking

$$\mu := \frac{\log(1-\alpha^{m-n})^{-1}}{\log\alpha}$$

a quick computation using Mathematica gives us that

$$\varepsilon : \varepsilon(\mu) = ||\mu q|| - M||\gamma q|| > 0$$

for every odd $n-m\in[5,137]$. Let $A:=17$, $B:=\alpha$, and $w:=n$ in Lemma 1. Then, it follows from Lemma 1, applied to the inequality (3.7) in order to reduce a little bit the upper bound on n, that

$$n < \frac{\log(Aq_{64}/\varepsilon)}{\log B} \le 155.$$

Thus we can say that if (n,m,a) is a solution of the equation (1.6), then $n<155$. This contradicts our assumption that $n>200$. This completes the proof.

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NOVEL INVARIANTS FOR ALMOST GEODESIC MAPPINGS OF THE THIRD TYPE

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Abstract. Two kinds of invariance for geometrical objects under transformations are involved in this paper. Based on these kinds, we obtained new invariants for almost geodesic mappings of the third type of a non-symmetric affine connection space in this paper. Our results are presented in two sections. In the Section 3, we obtained the invariants for the equitorsion almost geodesic mappings which do not have the property of reciprocity.

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1. INTRODUCTION

Many authors have obtained invariants for different mappings between symmetric and non-symmetric affine connection spaces. Some of them are J. Mikeš and his research group [1–3, 5, 15], N. S. Sinyukov [14], M. S. Stanković [13, 18, 19, 21], M. Lj. Zlatanović [22, 23], M. Z. Petrović [12, 13] and many others.

1.1. *Affine connection spaces*

An N-dimensional manifold equipped with non-symmetric affine connection ∇ (see Eisenhart [4]) is the non-symmetric affine connection space $\mathbb{GA}_N$ (in Eisenhart's sense).

Affine connection coefficients of the space $\mathbb{GA}_N$ are L^i_{jk}, $L^i_{jk} \not\equiv L^i_{kj}$ for at least one pair $(j,k) \in \{1,\dots,N\} \times \{1,\dots,N\}$. For this reason, symmetric and anti-symmetric parts of affine connection coefficients L^i_{jk} are

$$L^i_{\underline{jk}} = \frac{1}{2}\left(L^i_{jk} + L^i_{kj}\right) \quad \text{and} \quad L^i_{\underset{\vee}{jk}} = \frac{1}{2}\left(L^i_{jk} - L^i_{kj}\right). \tag{1.1}$$

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The N-dimensional manifold equipped with torsion-free affine connection $\overset{0}{\nabla}$ whose coefficients are $L^i_{\underline{jk}}$ is associated space $\mathbb{A}_N$ $\left(\text{of space } \mathbb{GA}_N\right)$ [20, 21].

Based on affine connection $\overset{0}{\nabla}$, one kind of covariant derivative exists [2,3,5,14,15]

$$a^i_{j|k} = a^i_{j,k} + L^i_{\underline{\alpha k}} a^\alpha_j - L^\alpha_{\underline{jk}} a^i_\alpha, \tag{1.2}$$

for partial derivative $\partial/\partial x^k$ denoted by comma.

In this case, one Ricci identity $a^i_{j|mn} - a^i_{j|nm} = a^\alpha_j R^i_{\alpha mn} - a^i_\alpha R^\alpha_{jmn}$ exists. Curvature tensor R^i_{jmn} of associated space $\mathbb{A}_N$ is

$$R^i_{jmn} = L^i_{\underline{jm},n} - L^i_{\underline{jn},m} + L^\alpha_{\underline{jm}} L^i_{\underline{\alpha n}} - L^\alpha_{\underline{jn}} L^i_{\underline{\alpha m}}. \tag{1.3}$$

Geometrical object

$$R_{ij} = R^\alpha_{ij\alpha}, \tag{1.4}$$

is the Ricci tensor.

Based on non-symmetric affine connection ∇, S. M. Minčić defined four kinds of covariant derivatives [7, 8]

$$\begin{aligned} a^i_{j|k} &= a^i_{j,k} + L^i_{\alpha k} a^\alpha_j - L^\alpha_{jk} a^i_\alpha, \quad a^i_{\underset{1}{j|k}} = a^i_{j,k} + L^i_{k\alpha} a^\alpha_j - L^\alpha_{kj} a^i_\alpha, \\ a^i_{\underset{3}{j|k}} &= a^i_{j,k} + L^i_{\alpha k} a^\alpha_j - L^\alpha_{kj} a^i_\alpha, \quad a^i_{\underset{4}{j|k}} = a^i_{j,k} + L^i_{k\alpha} a^\alpha_j - L^\alpha_{jk} a^i_\alpha. \end{aligned} \tag{1.5}$$

Explanation of covariant derivatives [10]:

Two kinds of parallel displacement of vectors in $\mathbb{GA}_N$ exist. For a vector field $v^i = v^i(t)$, defined along a curve $C\colon x^i = x^i(t)$, v^i is a field of k-th, $k = 1,2$, parallel transport if for the differential one has

$$\underset{k}{dv^i} = -\left(L^i_{\underline{\alpha\beta}} - (-1)^k L^i_{\underset{\vee}{\alpha\beta}}\right) v^\alpha dx^\beta. \tag{1.6}$$

After considering a surface element at the tangent space determined by two infinitesimal vectors whose origin is the point $P(x^i)$. The ends of these vectors are $Q(x^i + dx^i)$ and $R(x^i + \delta x^i)$.

If we make the parallel displacement of the first kind of the vector dx^i along δx^i and δx^i along dx^i, we obtain different points S and T for the ends. The coordinates of these points are

$$\underset{T}{x^i} - \underset{S}{x^i} = \underset{1}{d}(\delta x^i) - \underset{1}{\delta}(dx^i) = (L^i_{\alpha\beta} - L^i_{\beta\alpha}) dx^\alpha \delta x^\beta = 2 L^i_{\underset{\vee}{\alpha\beta}} dx^\alpha \delta x^\beta.$$

From the last equation, we can conclude that the points S and T are different if $L^i_{\underset{\vee}{jk}} \neq 0$.

Based on Ricci identities $a^i_{j\underset{p}{|}m\underset{q}{|}n} - a^i_{j\underset{r}{|}n\underset{s}{|}m}$, $p,q,r,s \in \{0,\dots,4\}$, $a^i_{j\underset{0}{|}k} = a^i_{j|k}$, the family of curvature tensors for the space $\mathbb{GA}_N$ is obtained [6–8]

$$K^i_{jmn} = R^i_{jmn} + uL^i_{jm\underset{\vee}{|}n} + u'L^i_{jn\underset{\vee}{|}m} + vL^\alpha_{jm\underset{\vee}{}}L^i_{\alpha\underset{\vee}{}n} + v'L^\alpha_{jn\underset{\vee}{}}L^i_{\alpha\underset{\vee}{}m} + wL^\alpha_{mn\underset{\vee}{}}L^i_{\alpha\underset{\vee}{}j}, \tag{1.7}$$

for curvature tensor R^i_{jmn} of associated space $\mathbb{A}_N$ and scalars u, u', v, v', w.

Family of Ricci-curvatures $K_{ij} = K^\alpha_{ij\alpha}$ of space $\mathbb{GA}_N$ is

$$K_{ij} = R_{ij} + uL^\alpha_{ij\underset{\vee}{|}\alpha} + u'L^\alpha_{i\alpha\underset{\vee}{|}j} + vL^\alpha_{ij\underset{\vee}{}}L^\beta_{\alpha\beta\underset{\vee}{}} - (v'+w)L^\alpha_{i\beta\underset{\vee}{}}L^\beta_{j\alpha\underset{\vee}{}}. \tag{1.8}$$

1.2. *Geodesic and almost geodesic lines and almost geodesic mappings*

Curve $\ell = \ell(t)$ in associated space $\mathbb{A}_N$ whose tangential vector $\lambda = \dfrac{d\ell}{dt}$ satisfies the system of differential equations [5, 14]

$$\frac{d\lambda^i}{dt} + L^i_{\underline{\alpha\beta}}\lambda^\alpha\lambda^\beta = \zeta\lambda^i, \tag{1.9}$$

for scalar ζ, is the geodesic line of space $\mathbb{A}_N$.

Definition 1. [2, 3, 5, 14, 15] A curve $\tilde{\bar{\ell}} = \tilde{\bar{\ell}}(t)$ in associated space $\overline{\mathbb{A}}_N$ whose tangential vector $\tilde{\bar{\lambda}} = \dfrac{d\tilde{\bar{\ell}}}{dt}$ satisfies the system of equations

$$\tilde{\bar{\lambda}}^i_2 = \bar{a}(t)\tilde{\lambda}^i + \bar{b}(t)\tilde{\bar{\lambda}}^i_1, \qquad \tilde{\bar{\lambda}}^i_1 = \tilde{\lambda}^i_{\|\alpha}\tilde{\lambda}^\alpha, \qquad \tilde{\bar{\lambda}}^i_2 = \tilde{\lambda}^i_{(1)\|\alpha}\tilde{\lambda}^\alpha, \tag{1.10}$$

for functions $\bar{a}(t)$, $\bar{b}(t)$ and covariant derivative with respect to the affine connection of the space $\overline{\mathbb{A}}_N$ denoted by $\|$, is *almost geodesic line* of space $\overline{\mathbb{A}}_N$.

Mapping $f\colon \mathbb{A}_N \to \overline{\mathbb{A}}_N$ is an almost geodesic mapping if any geodesic line of $\mathbb{A}_N$ transforms to an almost geodesic line in $\overline{\mathbb{A}}_N$ (see [2, 3, 5, 14, 15]).

Sinjukov proved [14] that three types of almost geodesic mappings of a space $\mathbb{A}_N$ exist. These types are π_1, π_2, π_3.

It was proved [14] that the inverse mapping of almost geodesic one $f\colon \mathbb{A}_N \to \overline{\mathbb{A}}_N$ of the type π_3 is almost geodesic mapping of the type π_3. Almost geodesic mappings of the third type whose inverse transformations are almost geodesic mappings of the third type have the property of reciprocity (see [14], page 191).

M. S. Stanković [16–18] generalized the Sinyukov's concept of almost geodesic mappings. A curve $\tilde{\bar{\ell}} = \tilde{\bar{\ell}}(t)$ whose tangential vector $\tilde{\bar{\lambda}} = \dfrac{d\tilde{\bar{\ell}}}{dt}$ is solution of the system [12, 13, 16–19, 21]

$$\tilde{\tilde{\lambda}}^i_{k(2)} = \underset{k}{\overline{a}}(t)\tilde{\lambda}^i + \underset{k}{\overline{b}}(t)\tilde{\tilde{\lambda}}^i_{k(1)}, \qquad \tilde{\tilde{\lambda}}^i_{k(1)} = \tilde{\lambda}^i_{\|\alpha}\tilde{\lambda}^\alpha, \qquad \tilde{\tilde{\lambda}}^i_{k(2)} = \tilde{\tilde{\lambda}}^i_{k(1)\|\alpha}\tilde{\lambda}^\alpha, \tag{1.11}$$

$k \in \{1,2\}$, of differential equations is an almost geodesic line of the k-th kind of space $\mathbb{G}\overline{\mathbb{A}}_N$.

Mapping $f\colon \mathbb{G}\mathbb{A}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ that any geodesic line of space $\mathbb{G}\mathbb{A}_N$ transforms to an almost geodesic line of the k-th type, $k = 1,2$, of space $\mathbb{G}\overline{\mathbb{A}}_N$ is almost geodesic mapping of space $\mathbb{G}\mathbb{A}_N$.

A mapping $f\colon \mathbb{G}\mathbb{A}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ is equitorsion [9] if anti-symmetric parts $\underset{\vee}{L}{}^i_{jk}$ and $\underset{\vee}{\overline{L}}{}^i_{jk}$ of affine connection coefficients L^i_{jk} and $\overline{L}^i_{jk} \overset{f}{\leftarrow} L^i_{jk}$ of spaces $\mathbb{G}\mathbb{A}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$ are equal. In this paper, we will stay focused on equitorsion almost geodesic mappings of the third type and first kind.

Motivated by the Sinyukov's research [14], M. S. Stanković [16–18] determined two types of almost geodesic lines. Any of three types of almost geodesic mappings can be divided into two subtypes. These subtypes of almost geodesic mappings are $\underset{1}{\pi}_r$, $\underset{2}{\pi}_r$, $r = 1,2,3$.

An almost geodesic mapping of a subtype $\underset{k}{\pi}_3$, $k = 1,2$, has the property of reciprocity if its inverse mapping is almost geodesic mapping of the same subtype. Unlike almost geodesic mappings of symmetric affine connection spaces, almost geodesic mappings of a non-symmetric affine connection space of the third type do not have the property of reciprocity in general.

We will obtain invariants for almost geodesic mappings of the third type of space $\mathbb{G}\mathbb{A}_N$ which have or do not have the property of reciprocity below.

1.3. *Two kinds of invariants for transformations*

Invariants for mappings between symmetric and non-symmetric affine connection spaces are such geometric objects whose values and forms do not change under the acting of the corresponding mapping. If almost geodesic mappings $f\colon \mathbb{G}\mathbb{A}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ or $f\colon \mathbb{A}_N \to \overline{\mathbb{A}}_N$ do not have the property of reciprocity, invariants for these mappings of the common values and forms cannot be obtained. For this reason, to obtain the invariants for almost geodesic mappings based on changes of the curvature tensors under almost geodesic mappings, authors assume that these mappings have the property of reciprocity.

Two kinds of invariance are important in physics (taken from the textbook Đ. Mušicki, B. Milić, *Mathematical Foundations of Theoretical Physics With a Collection of Solved Problems* [11], page 103):

– *The invariant* of a mapping f is an object whose value stays saved but form changes under transformation of coordinates under this mapping.

– *The total invariant* of a mapping f is an object whose value and form stay saved under the transformation of coordinates under this mapping.

Invariants for geometric mappings that have been obtained so far are analogies to the total invariants from physics. To debilitate this condition for invariance of geometrical objects, we give the following definition.

Definition 2. Let $f\colon \mathbb{G}\mathbb{A}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be a mapping and let $U^{i_1\dots i_p}_{j_1\dots j_q}$ be a geometrical object of the type (p,q).

– If the transformation f preserves the value of the object $U^{i_1\dots i_p}_{j_1\dots j_q}$ but changes its form to $\overline{V}^{i_1\dots i_p}_{j_1\dots j_q}$, then the invariance for geometrical object $U^{i_1\dots i_p}_{j_1\dots j_q}$ under transformation f is *valued*.
– If the transformation f preserves both the value and the form of the geometrical object $U^{i_1\dots i_p}_{j_1\dots j_q}$, then the invariance for the geometrical object $U^{i_1\dots i_p}_{j_1\dots j_q}$ under transformation f is *total*.

Example 1. An invariant for geodesic mapping $f\colon \mathbb{A}_N \to \overline{\mathbb{A}}_N$ is the Weyl projective tensor

$$W^i_{jmn} = R^i_{jmn} + \frac{1}{N+1}\delta^i_j R_{[mn]} + \frac{N}{N^2-1}\delta^i_{[m}R_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}R_{n]j}, \tag{1.12}$$

for $R_{[mn]} = R_{mn} - R_{nm}$, $\delta^i_{[m}R_{jn]} = \delta^i_m R_{jn} - \delta^i_n R_{jm}$, $\delta^i_{[m}R_{n]j} = \delta^i_m R_{nj} - \delta^i_n R_{mj}$.

The geodesic mapping f transforms Weyl projective tensor W^i_{jmn} to

$$\overline{W}^i_{jmn} = \overline{R}^i_{jmn} + \frac{1}{N+1}\delta^i_j \overline{R}_{[mn]} + \frac{N}{N^2-1}\delta^i_{[m}\overline{R}_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}\overline{R}_{n]j}. \tag{1.13}$$

It holds $W^i_{jmn} = \overline{W}^i{}_{jmn}$ and the forms of the tensors W^i_{jmn} and $\overline{W}^i{}_{jmn}$ are equal. For this reason, the invariant W^i_{jmn} is total.

If $R_{ij} \not\equiv R_{ji}$ and $\overline{R}_{ij} \equiv \overline{R}_{ji}$ as it is the case in [23], the form of invariant W^i_{jmn} stays same but the invariant $\overline{W}^i{}_{jmn}$ changes to

$$\overline{\overline{W}}^i{}_{jmn} = \overline{R}^i_{jmn} + \frac{1}{N-1}\left(\delta^i_m R_{jn} - \delta^i_n R_{jm}\right). \tag{1.14}$$

The equality $W^i_{jmn} = \overline{\overline{W}}^i{}_{jmn}$ holds (the values of geometrical objects W^i_{jmn} and $\overline{\overline{W}}^i{}_{jmn}$ are equal) but their forms are different. For this reason, the invariant W^i_{jmn} is valued.

The transformations which are the main subject of research in this paper are almost geodesic mappings of the third type.

1.4. *Motivation*

The transformation rules for affine connection coefficients under the third type almost geodesic mappings of symmetric and non-symmetric affine connection spaces are

$$\begin{aligned} L^i_{\underline{jk}} &= L^i_{\underline{jk}} + \psi_j \delta^i_k + \psi_k \delta^i_j + \sigma_j k \varphi^i, \\ \overline{L}^i_{jk} &= L^i_{jk} + \psi_j \delta^i_k + \psi_k \delta^i_j + \sigma_j k \varphi^i + \xi^i_{jk}, \end{aligned} \tag{1.15}$$

for the 1-form ψ_i, the contravariant vector φ^i and the tensors σ_{jk} and ξ^i_{jk} symmetric and anti-symmetric in the covariant indices j and k, respectively.

To generalize the Weyl projective tensor by an invariant for a third type almost geodesic mapping of symmetric affine connection space $\mathbb{A}_N$, N. S. Sinyukov involved (see [14], page 193) geometrical object q_i so that $q_\alpha \varphi^\alpha = e$, $e = \pm 1$. After some computations, Sinyukov generalized the Thomas projective parameter and the Weyl projective tensor as invariants for almost geodesic mapping f.

M. S. Stanković (see [18]) continued Sinyukov's research about invariants for almost geodesic mappings of the third type. Motivated by the results presented in [18], N. O. Vesić, Lj. S. Velimirović and M. S. Stanković [21] obtained the family of invariants for the equitorsion third type almost geodesic mappings of a non-symmetric affine connection space. We will generalize this result below.

N. S. Sinyukov [14] generalized the Weyl projective tensor as invariant for the third type almost geodesic mappings of a symmetric affine connection space. M. S. Stanković [18] obtained one generalization of the Weyl projective tensor as invariant for the third type almost geodesic mapping of a non-symmetric affine connection space based on the change of the curvature tensor of the corresponding associated space. N. O. Vesić, Lj. S. Velimirović, M. S. Stanković [21] obtained one family of invariants for the third type almost geodesic mappings of a non-symmetric affine connection space which generalizes the Weyl projective tensor.

In [20], two invariants for mappings of an associated space analogue to the Weyl projective tensor (called the invariants of the Weyl type) are obtained. That motivated us to obtain invariants for almost geodesic mappings of the third type of a non-symmetric affine connection space.

The formulae of invariants for mappings between symmetric and non-symmetric affine connection spaces are obtained in [20]. We will use these formulae to meet the main goals of this paper. These goals are:

(1) To obtain invariants for equitorsion almost geodesic mappings of a non-symmetric affine connection space.
(2) To obtain necessary and sufficient conditions for these invariants to be total.

2. Review of basic invariants

In [20], invariants for mappings $f\colon \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ are obtained. If deformation tensor $P^i_{jk} = \overline{L}^i_{jk} - L^i_{jk}$ of mapping f is

$$P^i_{jk} = \overline{L}^i_{jk} - L^i_{jk} = \overline{\omega}^i_{jk} - \omega^i_{jk} + \overline{\tau}^i_{jk} - \tau^i_{jk}, \tag{2.1}$$

for $\omega^i_{jk} = \omega^i_{kj}$, $\overline{\omega}^i_{jk} = \overline{\omega}^i_{kj}$, $\tau^i_{jk} = -\tau^i_{kj}$, $\overline{\tau}^i_{jk} = -\overline{\tau}^i_{kj}$, basic associated invariants of Thomas and Weyl type for mapping f are

$$\widetilde{T}^i_{jk} = L^i_{\underline{jk}} - \omega^i_{jk}, \tag{2.2}$$

$$\widetilde{W}^i_{jmn} = R^i_{jmn} - \omega^i_{jm|n} + \omega^i_{jn|m} + \omega^\alpha_{jm}\omega^i_{\alpha n} - \omega^\alpha_{jn}\omega^i_{\alpha m}. \tag{2.3}$$

Remark 1. The geometrical object ω^i_{jk} depends of a mapping f. If the mapping $f\colon \mathbb{GA}_N \to \mathbb{G\overline{A}}_N$ is geodesic, this object is $\omega^i_{jk} = L^\alpha_{\underline{j\alpha}}\delta^i_k + L^\alpha_{\underline{k\alpha}}\delta^i_j$. If the mapping $f\colon \mathbb{GA}_N \to \mathbb{GR}_N$ is almost geodesic of the third type, the object ω^i_{jk} will be different as we will se below.

To simplify these formulae, the next geometrical object is used [20]

$$L^i_{\underline{jm}|n} = L^i_{\underline{jm},n} + L^i_{\underline{\alpha n}}L^\alpha_{\underline{jm}} - L^\alpha_{\underline{jn}}L^i_{\underline{\alpha m}} - L^\alpha_{\underline{mn}}L^i_{\underline{j\alpha}}. \tag{2.4}$$

In case of

$$\omega^i_{jk} = \delta^i_k\rho_j + \delta^i_j\rho_k + \sigma^i_{jk}, \tag{2.5}$$

for $\sigma^i_{jk} = \sigma^i_{kj}$, the invariants for the mapping f given by the equations (2.2, 2.3) transform to

$$\widetilde{T}^i_{jk} = L^i_{\underline{jk}} - \sigma^i_{jk} - \frac{1}{N+1}\Big(\big(L^\alpha_{\underline{j\alpha}} - \sigma^\alpha_{j\alpha}\big)\delta^i_k + \big(L^\alpha_{\underline{k\alpha}} - \sigma^\alpha_{k\alpha}\big)\delta^i_j\Big), \tag{2.2'}$$

$$\begin{aligned}\widetilde{W}^i_{jmn} = {} & R^i_{jmn} - \delta^i_{[m}\rho_{j|n]} - \delta^i_j\rho_{[m|n]} - \sigma^i_{j[m|n]} - \delta^i_{[m}\rho_j\rho_{n]} \\ & + \delta^i_{[m}\rho_\alpha\sigma^\alpha_{jn]} + \sigma^\alpha_{j[m}\sigma^i_{\alpha n]},\end{aligned} \tag{2.6}$$

The derived invariant of Weyl type for mapping f is [20]

$$\begin{aligned}\widetilde{W}^i_{jmn} = {} & R^i_{jmn} + \frac{1}{N+1}\delta^i_j\big(R_{[mn]} + \sigma^\alpha_{\alpha[m|n]}\big) + \frac{N}{N^2-1}\delta^i_{[m}R_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}R_{n]j} \\ & - \sigma^i_{jm|n} + \sigma^i_{jn|m} + \sigma^\alpha_{jm}\sigma^i_{\alpha n} - \sigma^\alpha_{jn}\sigma^i_{\alpha m} \\ & - \frac{1}{N^2-1}\delta^i_m\Big(\sigma^\alpha_{\alpha[j|n]} + (N+1)\big(\sigma^\alpha_{jn|\alpha} - \sigma^\alpha_{j\alpha|n} - \sigma^\alpha_{jn}\sigma^\beta_{\alpha\beta} + \sigma^\alpha_{j\beta}\sigma^\beta_{n\alpha}\big)\Big) \\ & + \frac{1}{N^2-1}\delta^i_n\Big(\sigma^\alpha_{\alpha[j|m]} + (N+1)\big(\sigma^\alpha_{jm|\alpha} - \sigma^\alpha_{j\alpha|m} - \sigma^\alpha_{jm}\sigma^\beta_{\alpha\beta} + \sigma^\alpha_{j\beta}\sigma^\beta_{m\alpha}\big)\Big).\end{aligned} \tag{2.3'}$$

The basic invariant for equitorsion mapping $f\colon \mathbb{GA}_N \to \mathbb{G\overline{A}}_N$ obtained from the transformation of anti-symmetric part $\overset{\vee}{L}{}^i_{jk}$ of affine connection coefficient L^i_{jk} is

$$\hat{T}^i_{jk} = \overset{\vee}{L}{}^i_{jk}. \tag{2.7}$$

Let be $\omega^i_{(1).jk} = L^i_{\underline{jk}}$, $\overline{\omega}^i_{(1).jk} = \overline{L}^i_{\underline{jk}}$, $\omega^i_{(2).jk} = \omega^i_{jk}$, $\overline{\omega}^i_{(2).jk} = \overline{\omega}^i_{jk}$. Based on equalities $\hat{\widetilde{T}}^i_{jm\|n} - \hat{T}^i_{jm|n} = P^i_{\underline{\alpha n}}\hat{T}^\alpha_{jm} - P^\alpha_{\underline{jn}}\hat{T}^i_{\alpha n} - P^\alpha_{\underline{mn}}\hat{T}^i_{j\alpha}$, $P^i_{\underline{jk}} = \overline{\omega}^i_{(1).jk} - \omega^i_{(1).jk}$, $P^i_{\underline{jk}} =$

$\overline{\omega}^i_{(2).jk} - \omega^i_{(2).jk}$, we obtain [20]

$$\overline{\theta}^i_{(p).jmn} = \theta^i_{(p).jmn},$$

where $p = (p_1, p_2, p_3)$, $p_1, p_2, p_3 \in \{1,2\}$, and

$$\theta^i_{(p).jmn} = L^i_{\underset{\vee}{jm}|n} - \tau^i_{jm|n} - \omega^i_{(p_1).\alpha n}\hat{T}^\alpha_{jm} + \omega^\alpha_{(p_2).jn}\hat{T}^i_{\alpha m} + \omega^\alpha_{(p_3).mn}\hat{T}^i_{j\alpha}. \tag{2.8}$$

The family of invariants of for equitorsion mapping f obtained based on the transformation of family K^i_{jmn} of curvature tensors for space $\mathbb{GA}_N$ is [20]

$$W^i_{(p^1).(p^2).jmn} = \widetilde{W}^i_{jmn} + u\theta^i_{(p^1).jmn} + u'\theta^i_{(p^2).jmn}, \tag{2.9}$$

for $p^1 = (p^1_1, p^1_2, p^1_3)$, $p^2 = (p^2_1, p^2_2, p^2_3)$, $p^i_j \in \{1,2\}$ and the corresponding invariants $\theta^i_{(p^1).jmn}$, $\theta^i_{(p^2).jmn}$, given by (2.8).

3. Invariants for equitorsion almost geodesic mappings

Let $f\colon \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be an equitorsion almost geodesic mapping of the third type. Its basic equations are [18]

$$\begin{cases} \overline{L}^i_{\underset{\wedge}{jk}} = L^i_{\underset{\wedge}{jk}} + \psi_j\delta^i_k + \psi_k\delta^i_j + 2\sigma_{jk}\varphi^i, \\ \varphi^i_{\underset{1}{|j}} = \underset{1}{\nu}_j\varphi^i + \underset{1}{\mu}\delta^i_j. \end{cases} \tag{3.1}$$

Let us rewrite the first of last basic equations as

$$\overline{L}^i_{\underset{\wedge}{jk}} = L^i_{\underset{\wedge}{jk}} + \psi_j\delta^i_k + \psi_k\delta^i_j + D^i_{jk}, \tag{3.1'}$$

for tensor D^i_{jk}, $D^i_{jk} = D^i_{kj} = 2\sigma_{jk}\varphi^i$. In case of inverse mapping $f^{-1}\colon \mathbb{G}\overline{\mathbb{A}}_N \to \mathbb{GA}_N$, tensor $\overline{D}^i_{jk}$, $\overline{D}^i_{jk} = \overline{D}^i_{kj}$, $\overline{D}^i_{jk} = -D^i_{jk}$, exists, so that

$$L^i_{\underset{\wedge}{jk}} = \overline{L}^i_{\underset{\wedge}{jk}} - \psi_j\delta^i_k - \psi_k\delta^i_j - D^i_{jk} = \overline{L}^i_{\underset{\wedge}{jk}} - \psi_j\delta^i_k - \psi_k\delta^i_j + \overline{D}^i_{jk}. \tag{3.2}$$

Therefore, the equation (3.1') transforms to

$$\overline{L}^i_{\underset{\wedge}{jk}} = L^i_{\underset{\wedge}{jk}} + \psi_j\delta^i_k + \psi_k\delta^i_j - \frac{1}{2}\left(\overline{D}^i_{jk} - D^i_{jk}\right). \tag{3.1''}$$

After contracting the last equation by i and k, we obtain

$$\psi_j = \frac{1}{N+1}\left(\overline{L}^\alpha_{\underset{\wedge}{j\alpha}} + \frac{1}{2}\overline{D}^\alpha_{j\alpha}\right) - \frac{1}{N+1}\left(L^\alpha_{\underset{\wedge}{j\alpha}} + \frac{1}{2}D^\alpha_{j\alpha}\right). \tag{3.3}$$

If we substitute the expression (3.3) in (3.1'') and use expression $D^i_{jk} = 2\sigma_{jk}\varphi^i$, we will obtain

$$\omega^i_{jk} = \frac{1}{N+1}\delta^i_k\left(L^\alpha_{\underset{\wedge}{j\alpha}} + \sigma_{j\alpha}\varphi^\alpha\right) + \frac{1}{N+1}\delta^i_j\left(L^\alpha_{\underset{\wedge}{k\alpha}} + \sigma_{k\alpha}\varphi^\alpha\right) - \sigma_{jk}\varphi^i. \tag{3.4}$$

The second of basic equations (3.1) is equivalent to

$$\varphi^i_{|j} = -\underset{\vee}{L^i_{\alpha j}}\varphi^\alpha + \underset{1}{\nu}_j\varphi^i + \underset{1}{\mu}\delta^i_j. \tag{3.5}$$

After comparing the equations (2.5) and (3.5), we read

$$\rho_j = \frac{1}{N+1}\left(\underline{L^\alpha_{j\alpha}} + \sigma_{j\alpha}\varphi^\alpha\right) \quad \text{and} \quad \sigma^i_{jk} = -\sigma_{jk}\varphi^i. \tag{3.6}$$

Therefore, we obtain

$$\begin{cases} -\sigma^i_{jm|n} = \left(\sigma_{jm}\varphi^i\right)_{|n} = \sigma_{jm|n}\varphi^i - \sigma_{jm}\underset{\vee}{L^i_{\alpha n}}\varphi^\alpha + \sigma_{jm}\underset{1}{\nu}_n\varphi^i + \sigma_{jm}\underset{1}{\mu}\delta^i_n, \\ -\sigma^\alpha_{ij|\alpha} = \left(\sigma_{ij}\varphi^\alpha\right)_{|\alpha} = \sigma_{ij|\alpha}\varphi^\alpha - \sigma_{ij}\underset{\vee}{L^\beta_{\alpha\beta}}\varphi^\alpha + \sigma_{ij}\underset{1}{\nu}_\alpha\varphi^\alpha + N\underset{1}{\mu}\sigma_{ij}, \\ -\sigma^\alpha_{\alpha i|j} = \left(\sigma_{\alpha i}\varphi^\alpha\right)_{|j} = \left(\sigma_{i\alpha}\varphi^\alpha\right)_{|j} = \\ \sigma_{\alpha i|j}\varphi^\alpha - \sigma_{\beta i}\underset{\vee}{L^\beta_{\alpha j}}\varphi^\alpha + \sigma_{\alpha i}\underset{1}{\nu}_j\varphi^\alpha + \underset{1}{\mu}\sigma_{ij}. \end{cases} \tag{3.7}$$

After substituting the expressions (3.4, 3.6, 3.7) in (2.2', 2.3', 2.9), we obtain

$$\underset{1}{\widetilde{\mathcal{T}}}{}^i_{jk} = L^i_{\underline{jk}} - \frac{1}{N+1}\delta^i_k\left(\underline{L^\alpha_{j\alpha}} + \sigma_{j\alpha}\varphi^\alpha\right) - \frac{1}{N+1}\delta^i_j\left(\underline{L^\alpha_{k\alpha}} + \sigma_{k\alpha}\varphi^\alpha\right) + \sigma_{jk}\varphi^i, \tag{3.8}$$

$$\begin{aligned} \underset{1}{\widetilde{\mathcal{W}}}{}^i_{jmn} &= R^i_{jmn} + \frac{1}{N+1}\delta^i_j\Big(R_{[mn]} - \left(\sigma_{[m\alpha|n]} - \sigma_{[m\beta}\underset{\vee}{L^\beta_{\alpha n]}} + \sigma_{[m\alpha}\underset{1}{\nu}_{n]}\right)\varphi^\alpha\Big) \\ &+ \sigma_{j[m|n]}\varphi^i + \sigma_{j[m}\sigma_{\alpha n]}\varphi^\alpha\varphi^i - \sigma_{j[m}\underset{\vee}{L^i_{\alpha n]}}\varphi^\alpha + \sigma_{j[m}\underset{1}{\nu}_{n]}\varphi^i - \delta^i_{[m}\underset{1}{\mu}\sigma_{jn]} \\ &- \frac{1}{N+1}\Big(\delta^i_{[m}\underline{L^\alpha_{j\alpha|n]}} + \left(\delta^i_{[m}\sigma_{j\alpha|n]} - \delta^i_{[m}\sigma_{j\beta}\underset{\vee}{L^\beta_{\alpha n]}} + \delta^i_{[m}\sigma_{j\alpha}\underset{1}{\nu}_{n]}\right)\varphi^\alpha + \delta^i_{[m}\underset{1}{\mu}\sigma_{jn]}\Big) \\ &+ \frac{1}{N+1}\delta^i_{[m}\sigma_{jn]}\left(\underline{L^\beta_{\alpha\beta}} + \sigma_{\alpha\beta}\varphi^\beta\right)\varphi^\alpha \\ &- \frac{1}{(N+1)^2}\left(\underline{L^\alpha_{j\alpha}} + \sigma_{j\alpha}\varphi^\alpha\right)\left(\delta^i_{[m}\underline{L^\beta_{n]\beta}} + \delta^i_{[m}\sigma_{n]\beta}\varphi^\beta\right), \end{aligned} \tag{3.9}$$

$$\begin{aligned} \widetilde{W}{}^i_{jmn} &= R^i_{jmn} + \frac{1}{N+1}\delta^i_j R_{[mn]} + \frac{N}{N^2-1}\delta^i_{[m}R_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}R_{n]j} \\ &+ \sigma_{j[m|n]}\varphi^i - \sigma_{j[m}\underset{\vee}{L^i_{\alpha n]}}\varphi^\alpha + \sigma_{j[m}\underset{1}{\nu}_{n]}\varphi^i + \sigma_{j[m}\sigma_{\alpha n]}\varphi^\alpha\varphi^i \\ &- \frac{1}{N+1}\delta^i_j\left(\sigma_{\alpha[m|n]} - \sigma_{\beta[m}\underset{\vee}{L^\beta_{\alpha n]}} + \sigma_{\alpha[m}\underset{1}{\nu}_{n]}\right)\varphi^\alpha \\ &+ \frac{1}{N-1}\left(\delta^i_{[m}\sigma_{jn]|\alpha} - \delta^i_{[m}\sigma_{jn]}\underset{\vee}{L^\beta_{\alpha\beta}} + \delta^i_{[m}\sigma_{jn]}\underset{1}{\nu}_\alpha\right)\varphi^\alpha \\ &+ \frac{1}{N-1}\left(\delta^i_{[m}\sigma_{jn]}\sigma_{\alpha\beta} - \delta^i_{[m}\sigma_{j\alpha}\sigma_{n]\beta}\right)\varphi^\alpha\varphi^\beta \\ &- \frac{N}{N^2-1}\left(\delta^i_{[m}\sigma_{\alpha j|n]} - \delta^i_{[m}\sigma_{\beta j}\underset{\vee}{L^\beta_{\alpha n]}} + \delta^i_{[m}\sigma_{\alpha j}\underset{1}{\nu}_{n]}\right)\varphi^\alpha \end{aligned} \tag{3.10}$$

$$-\frac{1}{N^2-1}\left(\delta^i_{[m}\sigma_{\alpha n]|j}-\delta^i_{[m}\sigma_{\beta n]}\overset{\beta}{\underset{\vee}{L}}_{\alpha j}+\delta^i_{[m}\sigma_{\alpha n]}\underset{1}{\nu}_j\right)\varphi^\alpha.$$

Let us express invariant $\underset{1}{\widetilde{W}}{}^i{}_{jmn}$ in the form

$$\begin{aligned}\underset{1}{\widetilde{W}}{}^i{}_{jmn}&=R^i{}_{jmn}+\frac{1}{N+1}\delta^i_jR_{[mn]}+\frac{N}{N^2-1}\delta^i_{[m}R_{jn]}+\frac{1}{N^2-1}\delta^i_{[m}R_{n]j}\\&\quad+\delta^i_j\underset{1}{X}_{[mn]}+\delta^i_{[m}\underset{1}{Y}_{jn]}+\underset{1}{Z}{}^i{}_{jmn},\end{aligned}\tag{3.10'}$$

for tensors

$$\underset{1}{X}_{ij}=-\frac{1}{N+1}\left(\sigma_{\alpha i|j}-\sigma_{\beta i}\overset{\beta}{\underset{\vee}{L}}_{\alpha j}+\sigma_{\alpha i}\underset{1}{\nu}_j\right)\varphi^\alpha,\tag{3.11}$$

$$\begin{aligned}\underset{1}{Y}_{ij}&=\frac{1}{N-1}\left(\sigma_{ij|\alpha}-\sigma_{ij}\overset{\beta}{\underset{\vee}{L}}_{\alpha\beta}+\sigma_{ij}\underset{1}{\nu}_\alpha\right)\varphi^\alpha\\&\quad+\frac{1}{N-1}\left(\sigma_{ij}\sigma_{\alpha\beta}-\sigma_{i\alpha}\sigma_{j\beta}\right)\varphi^\alpha\varphi^\beta\\&\quad-\frac{N}{N^2-1}\left(\sigma_{\alpha i|j}-\sigma_{\beta i}\overset{\beta}{\underset{\vee}{L}}_{\alpha j}+\sigma_{\alpha i}\underset{1}{\nu}_j\right)\varphi^\alpha\\&\quad-\frac{1}{N^2-1}\left(\sigma_{\alpha j|i}-\sigma_{\beta j}\overset{\beta}{\underset{\vee}{L}}_{\alpha i}+\sigma_{\alpha j}\underset{1}{\nu}_i\right)\varphi^\alpha,\end{aligned}\tag{3.12}$$

$$\underset{1}{Z}{}^i{}_{jmn}=\sigma_{j[m|n]}\varphi^i-\sigma_{j[m}\overset{i}{\underset{\vee}{L}}_{\alpha n]}\varphi^\alpha+\sigma_{j[m}\underset{1}{\nu}_{n]}\varphi^i+\sigma_{j[m}\sigma_{\alpha n]}\varphi^\alpha\varphi^i.\tag{3.13}$$

After contracting equality $0=\underset{1}{\widetilde{\overline{W}}}{}^i{}_{jmn}-\underset{1}{\widetilde{W}}{}^i{}_{jmn}$ by i and j, we obtain

$$\underset{1}{\overline{X}}_{[mn]}-\underset{1}{X}_{[mn]}=-\frac{1}{N}\left(\underset{1}{\overline{Y}}_{[mn]}+\underset{1}{\overline{Z}}{}^\alpha_{\alpha mn}\right)+\frac{1}{N}\left(\underset{1}{Y}_{[mn]}+\underset{1}{Z}{}^\alpha_{\alpha mn}\right),\tag{3.14}$$

where

$$\underset{1}{Y}_{[ij]}=-\frac{1}{N+1}\left(\sigma_{\alpha[i|j]}-\sigma_{\beta[i}\overset{\beta}{\underset{\vee}{L}}_{\alpha j]}+\sigma_{\alpha[i}\underset{1}{\nu}_{j]}\right)\varphi^\alpha.\tag{3.15}$$

After substituting the equations (3.12, 3.13, 3.14, 3.15) in

$$\begin{aligned}0&=\underset{1}{\widetilde{\overline{W}}}{}^i{}_{jmn}-\underset{1}{\widetilde{W}}{}^i{}_{jmn}\\&=\overline{R}^i{}_{jmn}-R^i{}_{jmn}+\frac{1}{N+1}\delta^i_j\left(\overline{R}_{[mn]}-R_{[mn]}\right)+\frac{N}{N^2-1}\left(\delta^i_{[m}\overline{R}_{jn]}-\delta^i_{[m}R_{jn]}\right)\\&\quad+\frac{1}{N^2-1}\left(\delta^i_{[m}\overline{R}_{n]j}-\delta^i_{[m}R_{n]j}\right)+\delta^i_j\left(\underset{1}{\overline{X}}_{[mn]}-\underset{1}{X}_{[mn]}\right)\\&\quad+\left(\delta^i_{[m}\underset{1}{\overline{Y}}_{jn]}-\delta^i_{[m}\underset{1}{Y}_{jn]}\right)+\underset{1}{\overline{Z}}{}^i{}_{jmn}-\underset{1}{Z}{}^i{}_{jmn},\end{aligned}\tag{3.16}$$

we obtain $\overset{\approx}{\underset{1}{W}}{}^{i}_{jmn} = \underset{1}{\widetilde{W}}{}^{i}_{jmn}$, where

$$\begin{aligned}
\overset{\approx}{\underset{1}{W}}{}^{i}_{jmn} &= R^{i}_{jmn} + \frac{1}{N+1}\delta^{i}_{j}R_{[mn]} + \frac{N}{N^2-1}\delta^{i}_{[m}R_{jn]} + \frac{1}{N^2-1}\delta^{i}_{[m}R_{n]j} \\
&+ \sigma_{j[m|n]}\varphi^{i} - \sigma_{j[m}\underset{\vee}{L}{}^{i}_{\alpha n]}\varphi^{\alpha} + \sigma_{j[m}\underset{1}{\nu}_{n]}\varphi^{i} + \sigma_{j[m}\sigma_{\alpha n]}\varphi^{\alpha}\varphi^{i} \\
&- \frac{N}{N+1}\delta^{i}_{j}\bigl(\sigma_{\alpha[m|n]} - \sigma_{\beta[m}\underset{\vee}{L}{}^{\beta}_{\alpha n]} + \sigma_{\alpha[m}\underset{1}{\nu}_{n]}\bigr)\varphi^{\alpha} \\
&+ \frac{1}{N-1}\bigl(\delta^{i}_{[m}\sigma_{jn]|\alpha} - \delta^{i}_{[m}\sigma_{jn]}\underset{\vee}{L}{}^{\beta}_{\alpha\beta} + \delta^{i}_{[m}\sigma_{jn]}\underset{1}{\nu}_{\alpha}\bigr)\varphi^{\alpha} \\
&+ \frac{1}{N-1}\bigl(\delta^{i}_{[m}\sigma_{jn]}\sigma_{\alpha\beta} - \delta^{i}_{[m}\sigma_{j\alpha}\sigma_{n]\beta}\bigr)\varphi^{\alpha}\varphi^{\beta} \\
&- \frac{N}{N^2-1}\bigl(\delta^{i}_{[m}\sigma_{\alpha j|n]} - \delta^{i}_{[m}\sigma_{\beta j}\underset{\vee}{L}{}^{\beta}_{\alpha n]} + \delta^{i}_{[m}\underset{1}{\nu}_{n]}\bigr)\varphi^{\alpha} \\
&- \frac{1}{N^2-1}\bigl(\delta^{i}_{[m}\sigma_{\alpha n]|j} - \delta^{i}_{[m}\sigma_{\beta n]}\underset{\vee}{L}{}^{\beta}_{\alpha j} + \delta^{i}_{[m}\sigma_{\alpha n]}\underset{1}{\nu}_{j}\bigr)\varphi^{\alpha}.
\end{aligned} \tag{3.17}$$

Based on invariants $\underset{1}{\widetilde{W}}{}^{i}_{jmn}$ and $\overset{\approx}{\underset{1}{W}}{}^{i}_{jmn}$, we can conclude that geometrical object $\bigl(\sigma_{\alpha[i|j]} - \sigma_{\beta[i}\underset{\vee}{L}{}^{\beta}_{\alpha j]} + \sigma_{\alpha[i}\underset{1}{\nu}_{j]}\bigr)\varphi^{\alpha}$ is an invariant for mapping f. For this reason, invariant $\underset{1}{\widetilde{W}}{}^{i}_{jmn}$ given by (3.10) reduces to

$$\begin{aligned}
\underset{1}{\widetilde{W}}{}^{i}_{jmn} &= R^{i}_{jmn} + \frac{1}{N+1}\delta^{i}_{j}R_{[mn]} + \frac{N}{N^2-1}\delta^{i}_{[m}R_{jn]} + \frac{1}{N^2-1}\delta^{i}_{[m}R_{n]j} \\
&+ \sigma_{j[m|n]}\varphi^{i} - \sigma_{j[m}\underset{\vee}{L}{}^{i}_{\alpha n]}\varphi^{\alpha} + \sigma_{j[m}\underset{1}{\nu}_{n]}\varphi^{i} + \sigma_{j[m}\sigma_{\alpha n]}\varphi^{\alpha}\varphi^{i} \\
&+ \frac{1}{N-1}\bigl(\delta^{i}_{[m}\sigma_{jn]|\alpha} - \delta^{i}_{[m}\sigma_{jn]}\underset{\vee}{L}{}^{\beta}_{\alpha\beta} + \delta^{i}_{[m}\sigma_{jn]}\underset{1}{\nu}_{\alpha}\bigr)\varphi^{\alpha} \\
&+ \frac{1}{N-1}\bigl(\delta^{i}_{[m}\sigma_{jn]}\sigma_{\alpha\beta} - \delta^{i}_{[m}\sigma_{j\alpha}\sigma_{n]\beta}\bigr)\varphi^{\alpha}\varphi^{\beta} \\
&- \frac{N}{N^2-1}\bigl(\delta^{i}_{[m}\sigma_{\alpha j|n]} - \delta^{i}_{[m}\sigma_{\beta j}\underset{\vee}{L}{}^{\beta}_{\alpha n]} + \delta^{i}_{[m}\underset{1}{\nu}_{n]}\bigr)\varphi^{\alpha} \\
&- \frac{1}{N^2-1}\bigl(\delta^{i}_{[m}\sigma_{\alpha n]|j} - \delta^{i}_{[m}\sigma_{\beta n]}\underset{\vee}{L}{}^{\beta}_{\alpha j} + \delta^{i}_{[m}\sigma_{\alpha n]}\underset{1}{\nu}_{j}\bigr)\varphi^{\alpha}.
\end{aligned} \tag{3.18}$$

If contract equality $0 = \overset{\approx}{\underset{1}{W}}{}^{i}_{jmn} - \underset{1}{\widetilde{W}}{}^{i}_{jmn}$ $\bigl($equivalent to the equation (3.16)$\bigr)$ by i and n and anti-symmetrize the contracted equation by j and m, we will obtain

$$\overset{\approx}{\underset{1}{X}}{}_{[jm]} - \underset{1}{X}{}_{[jm]} = -\frac{N-1}{2}\bigl(\overset{\approx}{\underset{1}{Y}}{}_{[jm]} - \underset{1}{Y}{}_{[jm]}\bigr) + \frac{1}{2}\bigl(\overset{\approx}{\underset{1}{Z}}{}^{\alpha}_{[jm]\alpha} - \underset{1}{Z}{}^{\alpha}_{[jm]\alpha}\bigr). \tag{3.19}$$

After substituting the expression (3.19) in (3.16), we obtain $\underset{1}{\overset{\approx}{\overset{\approx}{W}}}{}^{i}_{jmn} = \underset{1}{\overset{\approx}{\overset{\approx}{W}}}{}^{i}_{jmn}$, for

$$\begin{aligned}
\underset{1}{\overset{\approx}{\overset{\approx}{W}}}{}^{i}_{jmn} &= R^{i}_{jmn} + \frac{1}{N+1}\delta^{i}_{j}R_{[mn]} + \frac{N}{N^2-1}\delta^{i}_{[m}R_{jn]} + \frac{1}{N^2-1}\delta^{i}_{[m}R_{n]j} \\
&\quad + \sigma_{j[m|n]}\varphi^{i} - \sigma_{j[m}\underset{\vee}{L}{}^{i}_{\alpha n]}\varphi^{\alpha} + \sigma_{j[m}\underset{1}{\nu}_{n]}\varphi^{i} + \sigma_{j[m}\sigma_{\alpha n]}\varphi^{\alpha}\varphi^{i} \\
&\quad + \frac{N-1}{2(N+1)}\delta^{i}_{j}\big(\sigma_{\alpha[m|n]} - \sigma_{\beta[m}\underset{\vee}{L}{}^{\beta}_{\alpha n]} + \sigma_{\alpha[m}\underset{1}{\nu}_{n]}\big)\varphi^{\alpha} \\
&\quad - \frac{1}{2}\delta^{i}_{j}\big(\sigma_{[m\alpha|n]} - \sigma_{[m\beta}\underset{\vee}{L}{}^{\beta}_{\alpha n]} + \sigma_{[m\alpha}\underset{1}{\nu}_{n]}\big)\varphi^{\alpha} \\
&\quad + \frac{1}{N-1}\big(\delta^{i}_{[m}\sigma_{jn]|\alpha} - \delta^{i}_{[m}\sigma_{jn]}\underset{\vee}{L}{}^{\beta}_{\alpha\beta} + \delta^{i}_{[m}\sigma_{jn]}\underset{1}{\nu}_{\alpha}\big)\varphi^{\alpha} \\
&\quad + \frac{1}{N-1}\big(\delta^{i}_{[m}\sigma_{jn]}\sigma_{\alpha\beta} - \delta^{i}_{[m}\sigma_{j\alpha}\sigma_{n]\beta}\big)\varphi^{\alpha}\varphi^{\beta} \\
&\quad - \frac{N}{N^2-1}\big(\delta^{i}_{[m}\sigma_{\alpha j|n]} - \delta^{i}_{[m}\sigma_{\beta j}\underset{\vee}{L}{}^{\beta}_{\alpha n]} + \delta^{i}_{[m}\sigma_{\alpha j}\underset{1}{\nu}_{n]}\big)\varphi^{\alpha} \\
&\quad - \frac{1}{N^2-1}\big(\delta^{i}_{[m}\sigma_{\alpha n]|j} - \delta^{i}_{[m}\sigma_{\beta n]}\underset{\vee}{L}{}^{\beta}_{\alpha j} + \delta^{i}_{[m}\sigma_{\alpha n]}\underset{1}{\nu}_{j}\big)\varphi^{\alpha}.
\end{aligned} \tag{3.20}$$

After comparing invariants $\underset{1}{\overset{\approx}{\overset{\approx}{W}}}{}^{i}_{jmn}$ and $\underset{1}{\widetilde{W}}{}^{i}_{jmn}$ for mapping f, we obtain that invariant $\underset{1}{\overset{\approx}{\overset{\approx}{W}}}{}^{i}_{jmn}$ given by (3.20) reduces to invariant $\underset{1}{\widetilde{W}}{}^{i}_{jmn}$ given by (3.18).

Based on the transformation of family (1.7) of curvature tensors of space $\mathbb{GA}_N$ under mapping f, we obtain:

$$\begin{aligned}
\underset{1}{\mathcal{W}}{}^{i}_{0.(p^1).(p^2).jmn} &= \underset{1}{\widetilde{\mathcal{W}}}{}^{i}_{jmn} + u\underset{\vee}{L}{}^{i}_{jm|n} + u'\underset{\vee}{L}{}^{i}_{jn|m} \\
&\quad - u\big(\omega^{i}_{(p^1_1).\alpha n}\underset{\vee}{L}{}^{\alpha}_{jm} - \omega^{\alpha}_{(p^1_2).jn}\underset{\vee}{L}{}^{i}_{\alpha m} - \omega^{\alpha}_{(p^1_3).mn}\underset{\vee}{L}{}^{i}_{j\alpha}\big) \\
&\quad - u'\big(\omega^{i}_{(p^2_1).\alpha m}\underset{\vee}{L}{}^{\alpha}_{jn} - \omega^{\alpha}_{(p^2_2).jm}\underset{\vee}{L}{}^{i}_{\alpha n} - \omega^{\alpha}_{(p^2_3).mn}\underset{\vee}{L}{}^{i}_{j\alpha}\big),
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
\underset{1}{W}{}^{i}_{0.(p^1).(p^2).jmn} &= \underset{1}{\widetilde{W}}{}^{i}_{jmn} + u\underset{\vee}{L}{}^{i}_{jm|n} + u'\underset{\vee}{L}{}^{i}_{jn|m} \\
&\quad - u\big(\omega^{i}_{(p^1_1).\alpha n}\underset{\vee}{L}{}^{\alpha}_{jm} - \omega^{\alpha}_{(p^1_2).jn}\underset{\vee}{L}{}^{i}_{\alpha m} - \omega^{\alpha}_{(p^1_3).mn}\underset{\vee}{L}{}^{i}_{j\alpha}\big) \\
&\quad - u'\big(\omega^{i}_{(p^2_1).\alpha m}\underset{\vee}{L}{}^{\alpha}_{jn} - \omega^{\alpha}_{(p^2_2).jm}\underset{\vee}{L}{}^{i}_{\alpha n} - \omega^{\alpha}_{(p^2_3).mn}\underset{\vee}{L}{}^{i}_{j\alpha}\big),
\end{aligned} \tag{3.22}$$

for $p^1_1,\dots,p^2_3 \in \{1,2\}$, $\omega^{i}_{(1).jk} = \underset{\wedge}{L}{}^{i}_{jk}$ and $\omega^{i}_{(2).jk} = \omega^{i}_{jk}$, for geometrical object ω^{i}_{jk} given in (3.4).

The next theorem holds.

Theorem 1. *Let $f\colon \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be an equitorsion almost geodesic mapping of the type $\underset{1}{\pi}_3$.*

The geometrical object $\widetilde{T}^i_{jk}$ given by (3.8) is basic invariant of Thomas type for mapping f. The invariance of this geometrical object is total.

Geometrical object $\widetilde{W}^i_{jmn}$ given by (3.9) is basic associated invariant of Weyl type for mapping f. The invariance of this geometrical object is valued. It is total if and only if mapping f has the property of reciprocity.

The geometrical object $\widetilde{W}^i_{jmn}$ given by (3.18) is associated derived invariant of the Weyl type for mapping f. The invariance of this geometrical object is valued. It is total if and only if the mapping f has the property of reciprocity.

Geometrical objects $\underset{1}{\mathcal{W}}{}^i_{jmn}$, $\underset{1}{W}{}^i_{jmn}$, given by (3.21, 3.22), are invariants for the equitorsion third type almost geodesic mapping f. The invariance of these geometrical objects is valued. This invariance is total if and only if mapping f has the property of reciprocity. □

4. Conclusion

We have obtained new invariants for the almost geodesic mappings of the third type of a non-symmetric affine connection space in this paper.

In Section 3, the invariants for equitorsion almost geodesic mappings of the third type are presented. The method used for obtaining these invariants (see [20]), simplified the corresponding method presented by Sinyukov [14] and used latter in [19,21].

The results obtained in this paper motivate the authors to continue their research about invariants for almost geodesic mappings of non-symmetric affine connection spaces.

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FIXED POINT THEOREM AND SELF-SIMILARITY ON MIXED VICSEK PATTERNS

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Abstract. The purpose if this paper is to present a fixed point result constructed by finite sequences. Using iterated function systems and related fractal operators, a mixed patterns generated by the a finite sequence patterns construct the sets of patterns built by black and white squares. A complete metric space related to a mixed pattern sequence is defined using the distance based on difference of the black squares' area.

The main result of the paper highlights that these fractal operators has unique fixed points for the sets generated by the mixed patterns. Moreover, the main theorem is also applied for Vicsek fractals such that results also hold for mixed Vicsek patterns. Motivated by various studies on growing graph sequences and related large structures, this paper underlines a new connection between fixed point theory and network science. Using circle patterns, the paper also interprets the main result on sets mixed patterns based on touching circles. Thus, the paper focuses a fixed point theorem on the sets mixed patterns built by iterated function systems and the distances calculated between the areas of these geometric shapes.

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Keywords: fixed point theorem, mixed patterns, interated function systems, Vicsek fractal, circle pattern

1. INTRODUCTION

Fractals are well known for describing growing processes in biology, physics and many other interdisciplinary topics too. Fixed point theorems [6] are strong and elegant tools for constructing self-similar sets and fractals, via the fractal operator theory and some iteration methods.

Growing mixed patterns [1–3] are self-similar geometrical sequences driven by a finite set of fixed patterns. An n-size pattern is constructed on the $n \times n$ sized grid with filled squares. Special cases of parameterized patterns are well known as the initial iteration of the Sierpiński carpet or the Vicsek fractal [8]. Thus, the growing pattern sequences based on these single patterns output widely known fractals.

Using iterated function systems and fractal operators, we interpret the limit of mixed pattern sequences using fixed point theory. We construct growing mixed sequences based on parameterized patterns built by squares.

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In operator theory, a bounded operator $T\colon X \to Y$ from the vector space X to the vector space Y is said to be a contraction if its norm is less or equal then 1.

In complete metric spaces the Banach fixed point theorem guarantees the existence and uniqueness of fixed points for contraction operators. So, a map $T\colon X \to X$ is called a contraction mapping on the complete metric space (X,d) if there exists $q \in [0,1)$ such that $d(T(x),T(y)) \leq qd(x,y)$.

Using the contraction principle we can construct fractals (or more precisely, prefractals), by an iterative procedure.

Mixed pattern can be constructed on other geometrical shapes too. As example, we confirm the results based on circle-driven patterns too. Moreover, specific growing mixed sequences are interpreted as mixed patterns based on the Vicsek fractal.

2. MIXED PATTERNS AND ITERATED FUNCTION SYSTEMS

Mixed patterns are constructed using simple patterns interpreted on unit squares. For any integer $m \geq 1$ let define $S_{i,j,m} = \{(x,y) | \frac{i}{m} \leq x \leq \frac{i+1}{m}$ and $\frac{j}{m} \leq y \leq \frac{j+1}{m}\}$ and $\mathcal{S}_m = \{S_{i,j,k} | 0 \leq i \leq m-1$ and $0 \leq j \leq m-1\}$.

Let define the nonempty $A \in \mathcal{S}_m$ as an m-pattern with width m. Thus, an m-pattern is constructed by little squares $S_{i,j,m}$ corresponding with the ordered integer pair (i,j) such that $A = S_{i_1,j_1,m} \cup S_{i_2,j_2,m} \cup \ldots$. Let also name the $S_{i,j,m}$ sets as black squares. Moreover, we also exclude the trivial case when A is the whole unit square.

One way to construct mixed patters is with the help of iterated function systems. For any non-empty $A \in \mathcal{S}_m$ we define the mapping $f_{i,j,m}\colon [0,1]^2 \to [0,1]^2$ as

$$f_{i,j,m}(x,y) = \left(\frac{i}{m}, \frac{j}{m}\right) + \frac{1}{m}(x,y),$$

for all $S_{i,j,m}$.

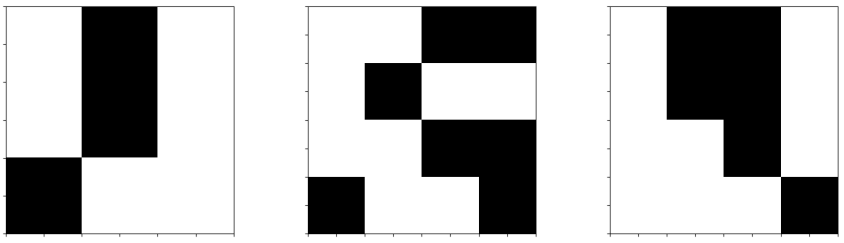

FIGURE 1. m-patterns: $\hat{f}_{A_1}([0,1]^2)$ (a 3-pattern), $\hat{f}_{A_2}([0,1]^2)$ and $\hat{f}_{A_3}([0,1]^2)$ (4-patterns).

Definition 1. Let define the iterated function system $f_A = \{f_{i,j,m} | S_{i,j,m} \in A\}$ and the pattern-driven fractal operator $\hat{f}_A : P_{cp}([0,1]^2) \to P_{cp}([0,1]^2)$ as

$$\hat{f}_A(Y) = \bigcup_{S_{i,j,m} \in A} f_{i,j,m}(Y), \qquad \text{for all} \qquad Y \in P_{cp}([0,1]^2)$$

such that Y is a compact square in the from the compact subsets of the unit square $P_{cp}([0,1]^2)$.

Let $\{A_i\}_{i=1}^k$ be a finite sequence of non-empty patterns with the width-sequence $\{m_i\}_{i=1}^k$. $\{\hat{f}_{A_i}\}_{i=1}^k$ marks the corresponding iterated function system sequence. Let also use the notation for the multiplication $m(k) = m_1 m_2 \cdots m_k$.

For any fixed A_1 and A_2 the IFS $\hat{f}_{A_1}$ constructs the set $\hat{f}_{A_1}([0,1]^2) \subseteq [0,1]^2$. This pattern contains squares with side lengths equal with $\frac{1}{m_1}$ such that each mapping of the IFS generates a smaller black square from the unit square. Using the same construction, $\hat{f}_{A_2}(\hat{f}_{A_1}([0,1]^2))$ is built by $m_1 m_2$ little squares such that each element of the fractal operator $\hat{f}_{A_2}$ constructs squares with side length $\frac{1}{m_1 m_2}$. Thus, $(\hat{f}_{A_2} \circ \hat{f}_{A_1})([0,1]^2))$ generates a mixed pattern width value $m_1 m_2$.

This implies that a sequence of $\{A_i\}_{i=1}^k$ patterns generates a mixed $m(k)$-pattern $(\hat{f}_{A_k} \circ \hat{f}_{A_{k-1}} \circ \cdots \circ \hat{f}_{A_2} \circ \hat{f}_{A_1})$. Let us note the composed fractal operator as $\hat{F}_k$.

Definition 2. Let define the $m(k)$-pattern M_k built by the different patterns $A_1, A_2, \ldots, A_k$ as a mixed pattern generated by a finite sequence of patterns.

Moreover, all of the patterns are defined on the same set $[0,1]^2$ and this brings the associativity on the composition of the corresponding fractal operator sequence $\{\hat{f}_{A_i}\}_{i=1}^k$.

In this paper we apply finite sequences of non-empty patterns on the set $[0,1]^2$ infinite times such that the end of a sequence is always followed by the first pattern. Thus, there exist the self-similar subsequence $[0,1]^2$, $\hat{F}_k([0,1]^2) = (\hat{f}_{A_k} \circ \hat{f}_{A_{k-1}} \circ \cdots \circ \hat{f}_{A_2} \circ \hat{f}_{A_1})([0,1]^2)$, $\hat{F}_k^2([0,1]^2) = (\hat{f}_{A_k} \circ \hat{f}_{A_{k-1}} \circ \cdots \circ \hat{f}_{A_2} \circ \hat{f}_{A_1})^2([0,1]^2)$, $\ldots$ constructed by the patterns.

3. Metric spaces constructed by mixed patterns

We construct the set of mixed pattern M_k based on the elements $\{A_i\}_{i=1}^k$. Using the iterated function systems let define the set as

$$\mathcal{F}_k = \{[0,1]^2,\ (\hat{f}_{A_k} \circ \hat{f}_{A_{k-1}} \circ \cdots \circ \hat{f}_1)([0,1]^2) = \hat{F}_k([0,1]^2),$$
$$(\hat{f}_{A_k} \circ \hat{f}_{A_{k-1}} \circ \cdots \circ \hat{f}_1)^2([0,1]^2) = \hat{F}_k^2([0,1]^2), \hat{F}_k^3([0,1]^2),\ \ldots\}.$$

The construction based on the set $[0,1]^2$ implies that all elements are built by little squares and it is easy to check that $\mathcal{A}_k$ is constructed as a unique sequence based on a set of mixed patterns. Thus, let us note the elements as the sequence

$$\mathcal{F}_k = \{F_0 = [0,1]^2, F_1 = \hat{F}_k([0,1]^2), F_2, \ldots\}.$$

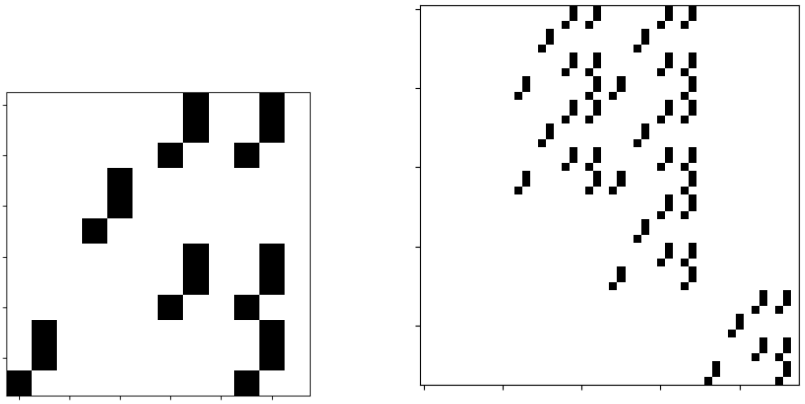

FIGURE 2. Mixed patters: $\hat{f}_{A_2}(\hat{f}_{A_1}([0,1]^2))$ (a 12-pattern) and $\hat{f}_{A_3}(\hat{f}_{A_2}(\hat{f}_{A_1}([0,1]^2)))$ (a 48-pattern).

Let us define the distance based the areas of the black squares.

Definition 3. Let note the construction of the element F_i by using the black squares f_i. The distance between two elements is

$$d(F_u,F_v)=\left|\sum_{f_u\in F_u}[[f_u]]-\sum_{f_v\in F_v}[[f_v]]\right|,$$

where $k\in\mathbb{N}^*$ and indexes $u,v\in\mathbb{N}$. The operators $|\cdot|$ and $[[\cdot]]$ refer to absolute value and area of black squares, respectively.

We show that this distance constructs a metric space on the set $\mathcal{A}_k$.

Theorem 1. *$(\mathcal{F}_k,d)$ is a metric space, where $\mathcal{A}_k=\{A_i\}_{i=1}^k$, $k\in\mathbb{N}^*$ is the set of mixed pattern and d denotes the distance between the element of the pattern sequence generated by $\mathcal{A}_k$.*

Proof. Let use the notation that F_i is an $m(i)$-pattern built by the $n(i)$ number of squares. This means that $[[F(i)]]=n(i)\frac{1}{m(i)^2}$ such that $n(i)\leq m(i)^2$.

We calculate the difference between the areas of F_i and F_{i+1} for $i\in\mathbb{N}^*$ and $F_i,F_{i+1}\in\mathcal{F}_k$. Based on the construction, let be the m_{i+1}-pattern constructed by n_{j+1} number of squares A_{i+1} and the fractal operator $\hat{f}_{A_{i+1}}$ such that $F_{i+1}=\hat{f}_{A_{i+1}}(F_i), j=1,2,\ldots,k$. This means that $[[F_{i+1}]]=(n(i)n_{i+1})\frac{1}{m_i^2m_{i+1}^2}$. Based on the notation we get that $m(i+1)=m(i)m_{i+1}$, $n(i+1)=n(i)n_{i+1}$ and $[[F_{i+1}]]=n(i+1)\frac{1}{m(i+1)^2}$.

Thus

$$d(F_i,F_{i+1}) = \left| n(i)\frac{1}{m(i)^2} - n(i)n_{i+1}\frac{1}{(m(i)m_{i+1})^2} \right| = \\ = \left| \frac{n(i)}{m(i)^2} - \frac{n(i+1)}{m(i+1)^2} \right| = \frac{n(i)(m_{i+1}^2 - n_{i+1})}{m(i+1)^2},$$

where $m_{i+1}^2 \geq n_{i+1}$ and this implies that $[[F_i]] \geq [[F_{i+1}]]$ and $d(F_i,F_{i+1}) = \frac{n(i)}{m(i)^2} - \frac{n(i+1)}{m(i+1)^2}$.

As the distance is defined, it is automatically verified that $d(F_i,F_j) \geq 0$ and $d(F_i,F_j) = d(F_j,F_i)$, for any $i,j \in \mathbb{N}, i \neq j$.

Last, but not least, let verify the triangle inequality such that $d(F_i,F_j) \geq d(F_i,F_k) + d(F_j,F_k)$, where $i,j,k \in \mathbb{N}$, $i \neq j \neq k$ and $[[F_i]] \geq [[F_{i+1}]]$.

If $i \leq j \leq k$ then

$$d(F_i,F_j) \leq d(F_i,F_k) + d(F_j,F_k)$$
$$\left| \frac{n(i)}{m(i)^2} - \frac{n(j)}{m(j)^2} \right| \leq \left| \frac{n(i)}{m(i)^2} - \frac{n(k)}{m(k)^2} \right| + \left| \frac{n(j)}{m(j)^2} - \frac{n(k)}{m(k)^2} \right|$$
$$\frac{n(i)}{m(i)^2} - \frac{n(j)}{m(j)^2} \leq \frac{n(i)}{m(i)^2} + \frac{n(j)}{m(j)^2} - 2\frac{n(k)}{m(k)^2}$$
$$\frac{n(i)}{m(i)^2} - \frac{n(j)}{m(j)^2} \leq \frac{n(i)}{m(i)^2} + \frac{n(j)}{m(j)^2} - 2\frac{n(k)}{m(k)^2}$$
$$0 \leq 2\frac{n(j)}{m(j)^2} - 2\frac{n(k)}{m(k)^2}.$$

Thus, we get the highlighted inequality $[[F_j]] \geq [[F_k]]$, where $j \leq k$.

Moreover, the case $i \leq k \leq j$ implies that

$$d(F_i,F_j) \leq d(F_i,F_k) + d(F_j,F_k)$$
$$\frac{n(i)}{m(i)^2} - \frac{n(j)}{m(j)^2} \leq \frac{n(i)}{m(i)^2} - \frac{n(k)}{m(k)^2} + \frac{n(k)}{m(k)^2} - \frac{n(j)}{m(j)^2}$$

and this is equivalent with $d(F_i,F_j) = d(F_i,F_j)$. If $j \leq i \leq k$, then

$$d(F_i,F_j) \leq d(F_i,F_k) + d(F_j,F_k)$$
$$\frac{n(j)}{m(j)^2} - \frac{n(i)}{m(i)^2} \leq \frac{n(i)}{m(i)^2} - \frac{n(k)}{m(k)^2} + \frac{n(j)}{m(j)^2} - \frac{n(k)}{m(k)^2}$$
$$2\frac{n(i)}{m(i)^2} \geq 2\frac{n(k)}{m(k)^2}$$

equivalent with the showed inequality $[[F_i]] \geq [[F_k]]$

If $j \le k \le i$, then the inequality becomes equality

$$d(F_i,F_j) \le d(F_i,F_k)+d(F_j,F_k)$$
$$\frac{n(j)}{m(j)^2}-\frac{n(i)}{m(i)^2} \le \frac{n(k)}{m(k)^2}-\frac{n(i)}{m(i)^2}+\frac{n(j)}{m(j)^2}-\frac{n(k)}{m(k)^2}$$

and this implies equality.

The case $k \le i \le j$ implies the inequality

$$d(F_i,F_j) \le d(F_i,F_k)+d(F_j,F_k)$$
$$\frac{n(i)}{m(i)^2}-\frac{n(j)}{m(j)^2} \le 2\frac{n(k)}{m(k)^2}-\frac{n(j)}{m(j)^2}-\frac{n(i)}{m(i)^2}$$
$$2\frac{n(i)}{m(i)^2} \le 2\frac{n(k)}{m(k)^2}$$

and the inequality $[[F_k]] \ge [[F_j]]$, $k \le i$ is checked.

If $k \le j \le i$ then the previous case can be followed as

$$d(F_i,F_j) \le d(F_i,F_k)+d(F_j,F_k)$$
$$\frac{n(j)}{m(j)^2}-\frac{n(i)}{m(i)^2} \le 2\frac{n(k)}{m(k)^2}-\frac{n(j)}{m(j)^2}-\frac{n(i)}{m(i)^2}$$
$$2\frac{n(j)}{m(j)^2} \le 2\frac{n(k)}{m(k)^2}$$

and we get the inequality $[[F_k]] \ge [[F_i]]$, where $k \le i$. □

Thus, the distance constructs the metric space $(\mathcal{F}_k,d)$.

4. Fixed point result

A mixed pattern sequence $\{A_k\}$ and the distance d construct the fractal operator $\hat{F}_k$. The operator has the fixed point in $(\mathcal{F}_k,d)$ if and only if there exists $F^* \in \mathcal{F}$ such that $\hat{F}_k(F^*) = F^*$.

As the lemma confirms, the metric space is complete.

Lemma 1. *$(\mathcal{F}_k,d)$ is a complete metric space for any finite mixed pattern sequence $\{A_k\}$.*

Proof. Based on Theorem 1, we need to show that arbitrary Cauchy sequence is convergent in the metric space $(\mathcal{F}_k,d)$. The construction implies that $[[\hat{f}_{A_i}(F)]] \le [[F]]$ for every mixed pattern $F \in \mathcal{F}_k$ and for every pattern-driven fractal operator $\hat{f}_{A_i}$.

Based on the distance

$$d(F_u,F_v) = \left| \sum_{f_u \in F_u} [[f_u]] - \sum_{f_v \in F_v} [[f_v]] \right|,$$

where $F_u,F_v \in \mathcal{F}_k$ and these are built by the little squares $f_i \in F_i$ and $f_j \in F_j$, respectively.

The construction implies one of the two cases: F_u is an $m(u)$-pattern and F_v is an $m(v)$-pattern such that $m(v) = m(u)m_{u+1}m_{u+2}\cdots m_v$ or F_v is an $m(v)$-pattern and F_u is an $m(v)$-pattern such that $m(u) = m(v)m_{v+1}m_{v+2}\cdots m_u$.

As the metric symmetry implies, we need to check only the first case. If $u < v$, F_u is an $m(u)$-pattern, F_v is an $m(v)$-pattern and $m(v) = m(u)m_{u+1}m_{u+2}\cdots m_v$ then

$$d(F_u, F_v) = \frac{n_1 n_2 \cdots n_u}{m_1^2 m_2^2 \cdots m_u^2} - \frac{n_1 n_2 \cdots n_u n_{u+1} \cdots n_v}{m_1^2 m_2^2 \cdots m_u^2 m_{u+1}^2 \cdots m_v},$$

where $n_i < m_i^2, i \in \{1,2,\ldots,u,u+1,\ldots,v\}$. Thus, for all $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}^*$ such that $[[F_N]] = \frac{n_1 n_2 \ldots n_N}{m_1^2 m_2^2 \ldots m_N^2} < \varepsilon$. This implies, that for all $N(\varepsilon) \le u \le v$ hold the inequality $0 \le [[F_v]] \le [[F_u]] \le \varepsilon$ and

$$d(F_u, F_v) \le [[F_u]] - [[F_v]] \le [[F_u]] < \varepsilon.$$

This and the symmetry implies that arbitrary Cauchy sequence is convergent in the metric space $(\mathcal{F}_k, d)$ and the metric space is complete. □

Let consider the metric space $(\mathcal{F}_k, d)$ and the fractal operator $\hat{F}_k$ constructed by a sequence of patterns $\{A_1, A_2, \ldots, A_k\}$.

As it is defined, the space $\mathcal{F}_k$ is build such that the fractal operator transforms the element F_i to $\hat{F}_k(F_i) = F_{i+1}$, where F_i is an $m(i)$-pattern, F_{i+1} is an $m(i+k)$-pattern, $m(i+k) = m(i)m_1 m_2 \cdots m_k$ and $i \in \mathbb{N}$.

The contraction principle implies the following theorem.

Theorem 2. *The pattern-driven fractal operator $\hat{F}_k$ has an unique fixed point on the metric space $(\mathcal{F}_k, d)$.*

Proof. As the composition implies $\hat{F}_k(F) = \frac{n_1}{m_1^2}\frac{n_2}{m_2^2}\cdots\frac{n_k}{m_k^2}[[F]]$ for all $S \in \mathcal{F}_k$.

Let consider the elements F_i, and F_j from the complete metric space $\mathcal{F}_k$ such that $i \le j$ and $i, j \in \mathbb{N}$. Using Theorem 1, we get that

$$d(\hat{F}_k(F_i), \hat{F}_k(F_j)) = \left| \frac{n_1 n_2 \cdots n_k}{m_1^2 m_2^2 \cdots m_k^2}[[F_i]] - \frac{n_1 n_2 \cdots n_k}{m_1^2 m_2^2 \cdots m_k^2}[[F_j]] \right| =$$

$$= \left| \frac{n_1 n_2 \cdots n_k}{m_1^2 m_2^2 \cdots m_k^2} \right| d(F_i, F_j),$$

where $i, j \in \mathbb{N}$ and $\frac{n_1 n_2 \ldots n_k}{m_1^2 m_2^2 \ldots m_k^2} < 1$ is Lipschitz parameter with the conditions $n_{i+1} < m_{i+1}^2$ and $n_{j+1} < m_{j+1}^2$.

Thus, fractal operator $\hat{F}_k$ is a Lipschitz-function on $(\mathcal{F}_k, d)$ and the contraction principle implies that it has an unique fixed point. Furthermore, the fixed point F^* can be found as follows: start with an arbitrary element $F_i \in \mathcal{F}_k$ and use the sequence defined as $F_i, \hat{F}_k(F_i) = F_{i+1}, \hat{F}_k(F_{i+1}) = F_{i+2}, \ldots$, then $i \to +\infty$ implies that $F_i \to F^*$. □

As a conclusion, we highlighted the fractal operator $\hat{F}_k$ generated by the pattern sequence has an unique fixed point.

5. MIXED CIRCLE PATTERNS

Mixed circle patterns start from the compact unit circle $circ(0,1)$ and an example can be constructed using three patterns: the pattern C_1 constructs with 5 little circles, C_2 operates with 9 and C_3 works with 13.

Thus, let also define the functions

$$f_{C_1} = \Big\{ f_1(x,y) = \frac{1}{5}(x,y),\ f_2(x,y) = \Big(0,\frac{2}{3}\Big) + \frac{1}{3}(x,y),$$
$$f_3(x,y) = \Big(0,-\frac{2}{3}\Big) + \frac{1}{3}(x,y),\ f_4(x,y) = \Big(\frac{2}{3},0\Big) + \frac{1}{3}(x,y),$$
$$f_5(x,y) = \Big(-\frac{2}{3},0\Big) + \frac{1}{3}(x,y)\Big\},$$

the iterated function system $f_{C_1} = \{f_1, f_2, f_3, f_4, f_5\}$ and the corresponding fractal operator

$$\hat{f}_{C_1}(Y) = f_1(Y) \cup f_2(Y) \cup f_3(Y) \cup f_4(Y) \cup f_5(Y), \quad \text{for all} \quad Y \in P_{cp}(circ(0,1)).$$

On the other hand, the circle pattern C_2 and C_3 are based on the mappings

$$f_{C_2} = \Big\{ f_1(x,y) = \frac{1}{5}(x,y),\ f_2(x,y) = \Big(0,\frac{2}{5}\Big) + \frac{1}{5}(x,y),$$
$$f_3(x,y) = \Big(0,-\frac{2}{5}\Big) + \frac{1}{5}(x,y),\ f_4(x,y) = \Big(\frac{2}{5},0\Big) + \frac{1}{5}(x,y),$$
$$f_5(x,y) = \Big(-\frac{1}{5},0\Big) + \frac{1}{5}(x,y),\ f_6(x,y) = \Big(0,\frac{2}{5}\Big) + \frac{1}{5}(x,y),$$
$$f_7(x,y) = \Big(0,-\frac{2}{5}\Big) + \frac{1}{5}(x,y),\ f_8(x,y) = \Big(-\frac{1}{5},0\Big) + \frac{1}{5}(x,y),$$
$$f_9(x,y) = \Big(-\frac{2}{5},0\Big) + \frac{1}{5}(x,y)\Big\}.$$

These construct the iterated function system $f_{C_2} = \{f_1, f_2, \ldots, f_{13}\}$ and the corresponding fractal operator

$$\hat{f}_{C_2}(Y) = f_1(Y) \cup f_2(Y) \cup \cdots \cup f_{13}(Y), \qquad \text{for all} \qquad Y \in P_{cp}(circ(0,1)).$$

and

$$f_{C_3} = \Big\{ f_1(x,y) = \frac{1}{5}(x,y),\ f_2(x,y) = \Big(0,\frac{2}{5}\Big) + \frac{1}{5}(x,y),$$
$$f_3(x,y) = \Big(0,-\frac{2}{5}\Big) + \frac{1}{5}(x,y),\ f_4(x,y) = \Big(\frac{2}{5},0\Big) + \frac{1}{5}(x,y),$$
$$f_5(x,y) = \Big(-\frac{1}{5},0\Big) + \frac{1}{5}(x,y),\ f_6(x,y) = \Big(0,\frac{2}{5}\Big) + \frac{1}{5}(x,y),$$

$$f_7(x,y)=\Big(0,-\frac{2}{5}\Big)+\frac{1}{5}(x,y),\ f_8(x,y)=\Big(\frac{2}{5},0\Big)+\frac{1}{5}(x,y),$$
$$f_9(x,y)=\Big(-\frac{2}{5},0\Big)+\frac{1}{5}(x,y),\ f_{10}(x,y)=\Big(-\frac{1}{5},\frac{1}{5}\Big)+\frac{1}{5}(x,y),$$
$$f_{11}(x,y)=\Big(\frac{1}{5},\frac{1}{5}\Big)+\frac{1}{5}(x,y),\ f_{12}(x,y)=\Big(\frac{1}{5},-\frac{1}{5}\Big)+\frac{1}{5}(x,y),$$
$$f_{13}(x,y)=\Big(-\frac{1}{5},-\frac{1}{5}\Big)+\frac{1}{5}(x,y)\Big\}.$$

These construct the iterated function system $f_{C_2}=\{f_1,f_2,\dots,f_{13}\}$ and the corresponding fractal operator

$$\hat{f}_{C_2}(Y)=f_1(Y)\cup f_2(Y)\cup\cdots\cup f_{13}(Y),\qquad \text{for all}\qquad Y\in P_{cp}(circ(0,1)).$$

Based on the multipliers, we interpret C_1 as a 3-pattern, C_2 and C_3 as a 5-pattern. We base the mixed circle pattern C on the elements C_1,C_2 and C_3 such that

$$\mathcal{C}=\{circ(0,1),\ (\hat{C}_3\circ\hat{C}_2\circ\hat{C}_1)(circ(0,1))=\hat{C}([0,1]^2),$$
$$(\hat{C}_3\circ\hat{C}_2\circ\hat{C}_1)^2(circ(0,1))=\hat{C}^2(circ(0,1)),\ \hat{C}^3(circ(0,1)),\ \dots\}$$

Thus, the operators $\hat{C}_1,\hat{C}_2$ and $\hat{C}_3$ form the fractal operator $\hat{C}$. The construction based on the unit square $[0,1]^2$ implies also for this case that the mixed circle pattern constructs the sequence

$$\mathcal{C}=\{C_0=circ(0,1),\ C_1=\hat{C}(circ(0,1)),\ C_2=\hat{C}^2(circ(0,1)),\ \dots\}.$$

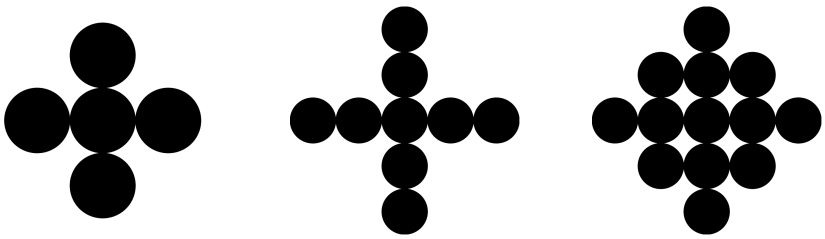

FIGURE 3. m-patterns: $\hat{f}_{C_1}(circ(0,1))$ (3-pattern), $\hat{f}_{C_2}(circ(0,1))$ (5-pattern) and $\hat{f}_{C_3}(circ(0,1))$ (5-pattern).

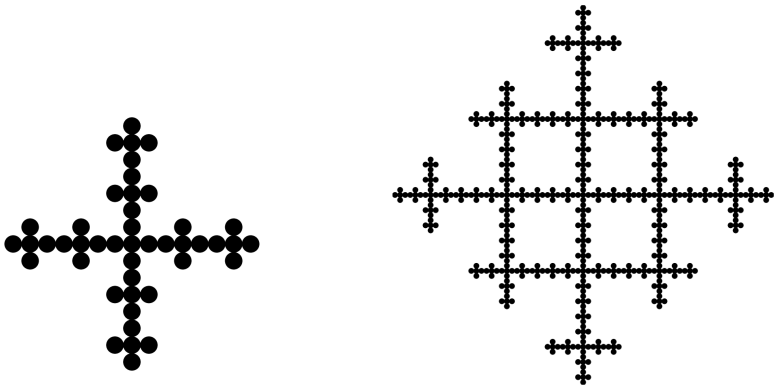

FIGURE 4. Mixed patters: $\hat{f}_{C_2}(\hat{f}_{C_1}(circ(0,1)))$ (a 15-pattern) and $\hat{f}_{C_3}(\hat{f}_{C_2}(\hat{f}_{C_1}(circ(0,1))))$ (a 75-pattern).

Let us define the distance based on the areas of circles.

Definition 4. Let note the construction of the element C_i by using the black circles c_i. The distance between two elements is

$$d_C(C_u, C_v) = \left| \sum_{c_u \in C_u} [[c_u]] - \sum_{c_v \in C_v} [[c_v]] \right|,$$

where $k \in \mathbb{N}^*$ and the indexes $u, v \in \mathbb{N}$. The operators $|\cdot|$ and $[[\cdot]]$ refer to absolute value and area, respectively.

Our results imply that this distance builds a metric space on $\mathcal{C}$.

Theorem 3. *$(\mathcal{C}, d_C)$ is a metric space, where d_C denotes the distance between two elements of the pattern sequence generated by $\mathcal{C}$.*

Moreover, we also confirm that the metric space is complete.

Lemma 2. *$(\mathcal{C}, d_C)$ is a complete metric space for any finite mixed pattern sequence $\{C_k\} \subseteq \mathcal{C}$.*

The metric space $(\mathcal{C}, d_C)$ and the fractal operator $\hat{C}$ constructs the set $\mathcal{C}$ such that the fractal operator transforms the element C_i to $\hat{C}(C_i) = C_{i+1}$. Using the contraction principle and Theorem 2 we obtain the following theorem.

Theorem 4. *The fractal operator $\hat{C}$ has an unique fixed point on the metric space $\mathcal{C}$.*

Thus, the fractal operator $\hat{C}$ extends the results for circle patterns.

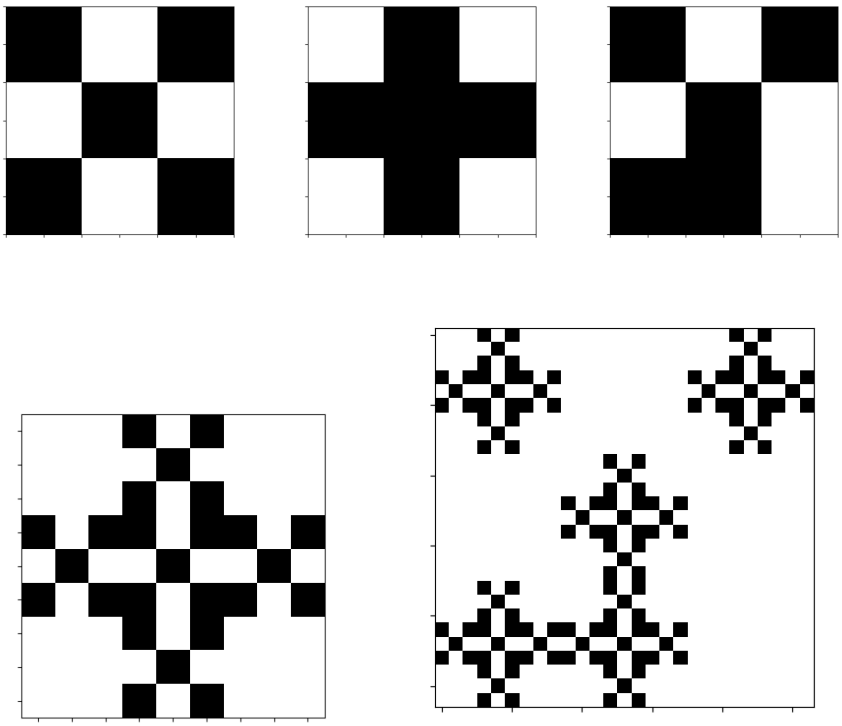

FIGURE 5. Vicsek patters: $\hat{f}_{V_1}([0,1]^2)$, $\hat{f}_{V_2}([0,1]^2)$, $\hat{f}_{V_3}([0,1]^2)$, $\hat{f}_{V_2}(\hat{f}_{V_1}([0,1]^2))$ (a 9-pattern) and $\hat{f}_{V_3}(\hat{f}_{V_2}(\hat{f}_{V_1}([0,1]^2)))$ (a 27-pattern).

6. INTERPRETATION ON VICSEK PATTERNS

Studies on Vicsek fractals [7, 8] and related growing patterns mainly focus on spectral [4, 9] or network coherence [5]. We highlight a fixed point result based on a finite sequence of Vicsek fractals.

The Vicsek fractal arise from a construction based on the Sierpinski carpet. It started from the $3-$pattern $V_1 = \{S_{0,0,3}, S_{0,2,3}, S_{1,1,3}, S_{2,0,3}, S_{2,2,3}\}$. Its cross form $V_2 = \{S_{0,1,3}, S_{1,0,3}, S_{1,1,3}, S_{1,2,3}, S_{1,1,3}\}$ is also named as antenna fractal.

Let us define the sequence of Vicsek fractals $\mathcal{V}_k = \{V_1, V_2, \ldots, V_k\}$ and the corresponding operators $\hat{f}_{V_1}, \hat{f}_{V_2}, \ldots, \hat{f}_{V_k}$. The sequence generates a mixed Vicsek pattern and the corresponding fractal operator $\hat{F}_{V_k}$. We also define the set $\mathcal{V}_k$ generated by the fractal operated of the mixed pattern.

By the fixed point theorem 2, we also get the following corollary for mixed Vicsek patterns.

Corollary 1. *The fractal operator $\hat{F}_{V_k}$ based on the mixed pattern constructed by a finite sequence of Vicsek fractals has an unique fixed point on the metric space $(\mathcal{V}_{\|}, d)$*

Moreover, let us change elements of the Vicsek pattern with circles. It is easy to check that we can interpret the corollary with circles too. Thus, we define the set of Vicsek 3-size circle patterns $\mathcal{C}_k = \hat{C}_1, \hat{C}_2, \ldots, \hat{C}_k$ and the fractal operator $\hat{F}_{C_k}$. This implies the corollary corresponding with the circle based construction.

Corollary 2. *The fractal operator $\hat{F}_{C_k}$ based on the mixed pattern constructed by a finite sequence of Vicsek circle fractals has an unique fixed point on the metric space $(\mathcal{C}_k, d_C)$*

Thus, there is gotten a fixed point for the fractal operator corresponding with the mixed Vicsek patterns and mixes Vicsek circle patterns too.

7. Conclusion

Our research highlighted fixed point theorems on mixed patterns. Using iterated function systems, we built growing sequences based on square size grids and we extended the results for circle-driven patterns too. We also constructed complete metric spaces on the sets driven by the mixed patters.

As underlined result, we showed that the fractal operators corresponding with the iterated function systems have unique fixed point on the sets build by the mixed patterns.

We spotlighted the result on mixed pattern driven by Vicsek fractals too such that we showed that specific mixed patters may be interpreted as mixed Vicsek patterns.

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ASYMPTOTIC PROPERTIES OF KNESER TYPE SOLUTIONS FOR THIRD ORDER HALF-LINEAR NEUTRAL DIFFERENCE EQUATIONS

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Abstract. The authors examine properties of positive solutions of the third order half-linear neutral difference equation

$$\Delta(a_n\Delta(b_n(\Delta z_n)^\alpha))+q_n y_{n+1}^\alpha=0,$$

where $z_n=y_n+p_n y_{\sigma(n)}$. They show that the positive solutions are in fact Kneser type solutions and they provide upper and lower bounds that yield the rate of convergence to zero for such solutions. Examples are provided to illustrate the main results.

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1. INTRODUCTION

In this paper, we are concerned with the asymptotic properties of solutions of the third order neutral difference equation

$$\Delta(a_n\Delta(b_n(\Delta z_n)^\alpha))+q_n y_{n+1}^\alpha=0,\quad n\ge n_0\ge 0, \tag{1.1}$$

where $z_n=y_n+p_n y_{\sigma(n)}$, α is the ratio of odd positive integers, and the following conditions are assumed to hold throughout:

- (H_1) $\{a_n\}$, $\{b_n\}$, and $\{q_n\}$ are positive real sequences for all $n\ge n_0$;
- (H_2) $\{p_n\}$ is a nonnegative real sequence with $0\le p_n\le p<1$;
- (H_3) $\{\sigma(n)\}$ is a sequence of integers such that $\sigma(n)\ge n$ for all $n\ge n_0$;
- (H_4) $\sum_{n=n_0}^{\infty}\frac{1}{a_n}=+\infty$ and $\sum_{n=n_0}^{\infty}\frac{1}{b_n^{1/\alpha}}=+\infty$.

By a *solution* of equation (1.1), we mean a nontrivial real sequence $\{y_n\}$ defined for all $n\ge n_0$ and satisfying equation (1.1). A solution $\{y_n\}$ of equation (1.1) is said to be *oscillatory* if it is neither eventually positive nor eventually negative, and *nonoscillatory* otherwise.

Oscillatory and asymptotic properties of solution of equation (1.1) and its special cases have been an active area of investigation in recent years; see for example [2–5, 7–19] and the references contained therein. The well known discrete version of

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Kiguradze's theorem [1] can be used to describe the structure of the solution space for the nonoscillatory solutions. For example, for the ordinary difference equation (see [13])

$$\Delta^2((\Delta y_n)^\alpha) + q_n y_{n+1}^\alpha = 0, \quad n \geq n_0, \tag{1.2}$$

the set K of all positive solutions has the decomposition

$$K = K_0 \cup K_2$$

where

$$y_n \in K_0 \text{ if and only if } y_n > 0,\ \Delta y_n < 0,\ \Delta((\Delta y_n)^\alpha) > 0,\ \Delta^2((\Delta y_n)^\alpha) < 0, \tag{1.3}$$

and

$$y_n \in K_2 \text{ if and only if } y_n > 0,\ \Delta y_n > 0,\ \Delta((\Delta y_n)^\alpha) > 0,\ \Delta^2((\Delta y_n)^\alpha) < 0. \tag{1.4}$$

A positive solution $\{y_n\}$ of equation (1.2) is said to be of the *Kneser type* if $\{y_n\} \in K_0$. In the study of the asymptotic behavior of the nonoscillatory solutions, many results were directed at obtaining criteria for $K_2 = \varnothing$, that is, showing that the only possible nonoscillatory solutions are the Kneser type ones. We will also say that equation (1.2) has property (A) if every positive solution $\{y_n\}$ belongs to K_0 and $\lim_{n\to\infty} y_n = 0$.

The aim of this paper is to conduct an analogous study for equation (1.1). That is, we wish to give sufficient conditions for the nonoscillatory solutions to be of the Kneser type. In addition, we wish to obtain upper and lower bounds for such solutions. These estimates allow us to determine the rate of convergence of Kneser type solutions to zero. The results presented in this paper are new and complement those in [4–8, 11–19].

2. Main Results

We begin with the following lemma that gives the basic properties of nonoscillatory solutions of equation (1.1). We present it for the case of positive solutions but clearly analogous statements hold for the negative solutions. As a part of this lemma, we define classes of solutions of equation (1.1) that are analogous to the sets K_0 and K_2 for equation (1.2).

Lemma 1. *Assume that $\{y_n\}$ is a positive solution of equation (1.1). Then the corresponding sequence $\{z_n\}$ belongs to one of the following classes:*

$$S_0 = \left\{z_n : z_n > 0,\ \Delta z_n < 0,\ \Delta(b_n(\Delta z_n)^\alpha) > 0,\ \Delta^2(a_n\Delta(b_n(\Delta z_n)^\alpha)) < 0\right\}$$
$$S_2 = \left\{z_n : z_n > 0,\ \Delta z_n > 0,\ \Delta(b_n(\Delta z_n)^\alpha) > 0,\ \Delta^2(a_n\Delta(b_n(\Delta z_n)^\alpha)) < 0\right\}$$

eventually.

Proof. The proof is similar to that of [16, Lemma 2.1] and so we omit the details. □

We will say that the neutral difference equation (1.1) has property (A), if every positive solution $\{y_n\}$ satisfies $\lim_{n\to\infty} y_n = 0$ and the corresponding sequence $\{z_n\} \in S_0$.

In the following, we first present sufficient conditions for the equation (1.1) to have only Kneser type solutions. Our approach will be to first obtain some properties that solutions of equation (1.1) that belong to the class S_2 must satisfy, and use these facts to obtain sufficient conditions for the class S_2 to in fact be empty. Then we give upper and lower estimate for such solutions.

To simplify our notation we let:

$$A_n = \sum_{s=N}^{n-1} \frac{1}{a_s}, \qquad B_n = \sum_{s=N}^{n-1} \frac{1}{b_s^{1/\alpha}},$$

$$C_n = \sum_{s=N}^{n-1} \frac{A_s}{b_s^{1/\alpha}}, \qquad E_n = \left(1 - p_n \frac{C_{\sigma(n)}}{C_n}\right) > 0,$$

$$Q_n = \left[\frac{1}{b_n} \sum_{s=n}^{\infty} \frac{1}{a_s} \sum_{t=s}^{\infty} q_t\right]^{1/\alpha}, \qquad \phi_n = \prod_{s=N}^{n-1} (1+Q_s),$$

$$R_n = (1-p)\frac{\phi_n}{b_n^{1/\alpha}} \left(\sum_{s=n}^{\infty} \frac{1}{a_s} \sum_{t=s}^{\infty} \frac{q_t}{\phi_{t+1}^{\alpha}}\right)^{1/\alpha},$$

where $N \geq n_0$.

Lemma 2. *Let $\{y_n\}$ be a positive solution of equation (1.1) with the corresponding sequence $\{z_n\} \in S_2$ for $n \geq N \geq n_0$ and assume that*

$$\sum_{n=N}^{\infty} \frac{1}{a_n} \sum_{s=n}^{\infty} q_s E_{s+1}^{\alpha} B_{s+1}^{\alpha} = \infty. \tag{2.1}$$

Then:

(i) $\{\frac{z_n}{C_n}\}$ *is decreasing for all* $n \geq N$;

(ii) $\{\frac{b_n^{1/\alpha} \Delta z_n}{A_n^{1/\alpha}}\}$ *is decreasing for all* $n \geq N$;

(iii) $\{\frac{z_n}{B_n}\}$ *is increasing for all* $n \geq N$.

Proof. Let $\{y_n\}$ be a positive solution of equation (1.1) with the corresponding sequence $\{z_n\} \in S_2$ for all $n \geq N$. Since $a_n \Delta(b_n(\Delta z_n)^{\alpha})$ is decreasing, we have

$$b_n(\Delta z_n)^{\alpha} \geq \sum_{s=N}^{n-1} \frac{a_s \Delta(b_s(\Delta z_s)^{\alpha})}{a_s} \geq A_n a_n \Delta(b_n(\Delta z_n)^{\alpha}), \quad n \geq N.$$

From the last inequality, we obtain

$$\Delta\left(\frac{b_n(\Delta z_n)^{\alpha}}{A_n}\right) = \frac{A_n \Delta(b_n(\Delta z_n)^{\alpha}) - b_n(\Delta z_n)^{\alpha} \frac{1}{a_n}}{A_n A_{n+1}} \leq 0$$

for all $n \geq N \geq n_0$. Thus, $\{\frac{b_n(\Delta z_n)^\alpha}{A_n}\}$ is decreasing for all $n \geq N$, so (ii) holds and

$$z_n \geq \sum_{s=N}^{n-1} \frac{A_s^{1/\alpha} b_s^{1/\alpha} \Delta z_s}{A_s^{1/\alpha}} \geq \frac{b_n^{1/\alpha} \Delta z_n}{A_n^{1/\alpha}} C_n, \quad n \geq N. \tag{2.2}$$

Hence,

$$\Delta\left(\frac{z_n}{C_n}\right) = \frac{C_n \Delta z_n - z_n \frac{A_n^{1/\alpha}}{b_n^{1/\alpha}}}{C_n C_{n+1}} \leq 0,$$

which implies that $\{\frac{z_n}{C_n}\}$ is decreasing for all $n \geq N$, so (i) holds.

Since $b_n^{1/\alpha}\Delta z_n$ is positive and strictly increasing for any $n \geq N$, it is easy to see that for all $n \geq N_1 \geq N$,

$$\begin{aligned} z_n &\leq z_{N_1} + b_n^{1/\alpha} \Delta z_n \sum_{s=N_1}^{n-1} \frac{1}{b_s^{1/\alpha}} \\ &= z_{N_1} - b_n^{1/\alpha} \Delta z_n \sum_{s=N}^{N_1-1} \frac{1}{b_s^{1/\alpha}} + b_n^{1/\alpha} \Delta z_n \sum_{s=N}^{n-1} \frac{1}{b_s^{1/\alpha}}. \end{aligned} \tag{2.3}$$

We claim that $b_n^{1/\alpha}\Delta z_n \to \infty$ as $n \to \infty$. If this is not the case, then $b_n^{1/\alpha}\Delta z_n \to 2d < \infty$ as $n \to \infty$ From the definition of z_n and using the fact that $\{\frac{z_n}{C_n}\}$ is decreasing, we have

$$y_n \geq z_n \left(1 - p_n \frac{C_{\sigma(n)}}{C_n}\right) = E_n z_n.$$

Summing equation (1.1) from n to ∞ and using the last inequality, we obtain

$$\Delta(b_n(\Delta z_n)^\alpha) \geq \frac{1}{a_n} \sum_{s=n}^{\infty} q_s E_{s+1}^\alpha z_{s+1}^\alpha.$$

Now $b_n^{1/\alpha}\Delta z_n \to 2d$ as $n \to \infty$ implies $b_n^{1/\alpha}\Delta z_n > d$ for n large enough, which in turn implies $z_n > dB_n$. Combining the last two inequalities and summing once more, we obtain

$$(2d)^\alpha \geq d^\alpha \sum_{n=N_1}^{\infty} \frac{1}{a_n} \sum_{s=n}^{\infty} q_s E_{s+1}^\alpha B_{s+1}^\alpha,$$

which contradicts (2.1). Thus, $b_n^{1/\alpha}\Delta z_n \to \infty$ as $n \to \infty$ as we claimed.

Hence, in view of (2.3) and (H_4), we obtain

$$z_n \leq b_n^{1/\alpha} \Delta z_n B_n, \quad n \geq N.$$

From the last inequality, we see that

$$\Delta\left(\frac{z_n}{B_n}\right) = \frac{B_n \Delta z_n - \frac{z_n}{b_n^{1/\alpha}}}{B_n B_{n+1}} \geq 0$$

eventually, and we conclude that $\{\frac{z_n}{B_n}\}$ is increasing. Therefore (iii) holds and this completes the proof of the lemma. □

Lemma 3. *Assume that* (2.1) *holds and let* $\{y_n\}$ *be a positive solution of equation* (1.1) *with the corresponding sequence* $\{z_n\} \in S_2$ *for all* $n \geq N$. *Then*

$$E_n z_n \leq y_n \leq z_n \quad \text{for } n \geq N. \tag{2.4}$$

Proof. From the definition of z_n, we have $z_n \geq y_n$ for all $n \geq N$. Furthermore,

$$y_n \geq z_n - p_n z_{\sigma(n)} \geq E_n z_n$$

where we have used the fact that $\{\frac{z_n}{C_n}\}$ is decreasing for all $n \geq N$. This completes the proof of the lemma. □

Next, we obtain sufficient condition for all nonoscillatory solutions of (1.1) to be of the Kneser type.

Theorem 1. *If condition* (2.1) *holds and*

$$\limsup_{n\to\infty} \left\{ \frac{1}{A_{n+1}} \sum_{s=N}^{n} q_s E_{s+1}^{\alpha} C_{s+1}^{\alpha} A_{s+1} + \frac{C_{n+1}^{\alpha}}{B_{n+1}^{\alpha}} \sum_{s=n+1}^{\infty} q_s E_{s+1}^{\alpha} B_{s+1}^{\alpha} \right\} > 1, \tag{2.5}$$

then $S_2 = \varnothing$.

Proof. Let $\{y_n\}$ be a positive solution of equation (1.1), with the corresponding sequence $\{z_n\} \in S_2$ for all $n \geq N$. Using (2.4) in equation (1.1), we obtain

$$\Delta(a_n \Delta(b_n (\Delta z_n)^{\alpha})) + q_n E_{n+1}^{\alpha} z_{n+1}^{\alpha} \leq 0, \quad n \geq N. \tag{2.6}$$

Summing inequality (2.6) from n to ∞, we have

$$\Delta(b_n (\Delta z_n)^{\alpha}) \geq \sum_{s=N}^{n-1} \frac{1}{a_s} \left(\sum_{t=s}^{\infty} q_t E_{t+1}^{\alpha} z_{t+1}^{\alpha} \right). \tag{2.7}$$

From (2.7), it follows that

$$\begin{aligned} b_n (\Delta z_n)^{\alpha} &\geq \sum_{s=N}^{n-1} \frac{1}{a_s} \left(\sum_{t=s}^{\infty} q_t E_{t+1}^{\alpha} z_{t+1}^{\alpha} \right) \\ &= \sum_{s=N}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t E_{t+1}^{\alpha} z_{t+1}^{\alpha} + \sum_{s=N}^{n-1} \frac{1}{a_s} \sum_{t=n}^{\infty} q_t E_{t+1}^{\alpha} z_{t+1}^{\alpha} \\ &= \sum_{s=N}^{n-1} q_s E_{s+1}^{\alpha} A_{s+1} z_{s+1}^{\alpha} + A_n \sum_{s=n}^{\infty} q_s E_{s+1}^{\alpha} z_{s+1}^{\alpha}. \end{aligned}$$

Using (2.2), in the last inequality, we obtain

$$\frac{A_{n+1} z_{n+1}^{\alpha}}{C_{n+1}^{\alpha}} \geq \sum_{s=N}^{n} q_s A_{s+1} E_{s+1}^{\alpha} z_{s+1}^{\alpha} + A_{n+1} \sum_{s=n+1}^{\infty} q_s E_{s+1}^{\alpha} z_{s+1}^{\alpha}. \tag{2.8}$$

In view of the monotonicity properties (i) and (iii) in Lemma 2, (2.8) yields

$$1 \geq \frac{1}{A_{n+1}} \sum_{s=N}^{n} q_s A_{s+1} E_{s+1}^{\alpha} C_{s+1}^{\alpha} + \frac{C_{n+1}^{\alpha}}{B_{n+1}^{\alpha}} \sum_{s=n+1}^{\infty} q_s E_{s+1}^{\alpha} B_{s+1}^{\alpha}.$$

Taking the lim sup as $n \to \infty$ on both sides of the last inequality leads to a contradiction of (2.5). This completes the proof of the theorem. □

Next, we wish to obtain estimates for the Kneser solutions. We begin with a lemma.

Lemma 4. *Let $\{y_n\}$ be a positive solution of equation (1.1) with the corresponding sequence $\{z_n\} \in S_0$ for $n \geq N$. Then:*

(i) $(1-p)z_n \leq y_n \leq z_n$ *for all* $n \geq N$;
(ii) $\{z_n \phi_n\}$ *is increasing for all* $n \geq N$.

Proof. Assume that $\{y_n\}$ is a positive solution of equation (1.1) with the corresponding sequence $\{z_n\} \in S_0$. Then z_n is positive, $z_n \geq y_n$, and

$$y_n = z_n - p_n y_{\sigma(n)} \geq (1-p)z_n, \quad n \geq N \geq n_0,$$

so (i) is proved.

It easy to see that $z_n \in S_0$ implies

$$\lim_{n\to\infty} b_n(\Delta z_n)^{\alpha} = 0;$$

otherwise we would eventually have $\Delta z_n > 0$ contradicting $z_n \in S_0$. Similarly,

$$\lim_{n\to\infty} a_n \Delta(b_n(\Delta z_n)^{\alpha}) = 0.$$

A summation of equation (1.1) then yields

$$a_n \Delta(b_n(\Delta z_n)^{\alpha}) = \sum_{s=n}^{\infty} q_s y_{s+1}^{\alpha} \leq \sum_{s=n}^{\infty} q_s z_{s+1}^{\alpha} \leq z_{n+1}^{\alpha} \sum_{s=n}^{\infty} q_s.$$

Summing once more, we obtain

$$b_n(\Delta z_n)^{\alpha} \geq -z_{n+1}^{\alpha} \sum_{s=n}^{\infty} \frac{1}{a_s} \sum_{t=s}^{\infty} q_t,$$

or

$$\Delta z_n \geq -z_{n+1} Q_n.$$

Hence,

$$\Delta(z_n \phi_n) = \phi_n \Delta z_n + z_{n+1} \Delta \phi_n \geq z_{n+1}(\Delta \phi_n - \phi_n Q_n) = 0$$

since $\{\phi_n\}$ is a solution of the difference equation $(\Delta \phi_n - Q_n \phi_n) = 0$. Therefore, $\{z_n \phi_n\}$ is increasing and this completes the proof of the lemma. □

Theorem 2. *Assume that conditions (2.1) and (2.5) hold. If $\{y_n\}$ is a positive solution of equation (1.1), then there are positive constants C_1 and C_2 such that*

$$C_1\psi_1(n) \le y_n \le C_2\psi_2(n), \tag{2.9}$$

where

$$\psi_1(n) = \frac{1}{\phi_n} \quad \text{and} \quad \psi_2(n) = \prod_{s=N}^{n-1}(1-R_s).$$

Proof. Assume that $\{y_n\}$ is a positive solution of equation (1.1). Then, by Theorem 1, $\{y_n\}$ is a Kneser type solution. From Lemma 4, we have that $\{z_n\phi_n\}$ is increasing for all $n \ge N$. Therefore,

$$y_n \ge \frac{(1-p)}{\phi_n} z_n\phi_n \ge \frac{(1-p)}{\phi_n} z_N\phi_N = C_1\psi_1(n).$$

On the other hand, summing equation (1.1) from n to ∞ and applying Lemma 4 (i), we have

$$a_n\Delta(b_n(\Delta z_n)^\alpha) = \sum_{s=n}^{\infty} q_s y_{s+1}^\alpha \ge (1-p)^\alpha \sum_{s=n}^{\infty} q_s z_{s+1}^\alpha \ge (1-p)^\alpha \phi_{n+1}^\alpha z_{n+1}^\alpha \sum_{s=n}^{\infty} \frac{q_s}{\phi_{s+1}^\alpha}.$$

Again summing the last inequality and applying Lemma 4 (ii) gives

$$-b_n(\Delta z_n)^\alpha \ge (1-p)^\alpha \phi_n^\alpha z_n^\alpha \sum_{s=n}^{\infty} \frac{1}{a_s} \sum_{t=s}^{\infty} \frac{q_t}{\phi_{t+1}^\alpha}$$

or

$$\frac{\Delta z_n}{z_n} \le -R_n,\ n \ge N.$$

Summing the last inequality from N to $n-1$, we obtain

$$y_n \le z_n \le C_2\psi_2(n).$$

This proves the theorem. □

From Theorem 2 we deduce the following result for the half-linear difference equation (1.2).

Corollary 1. *Assume that*

$$\sum_{n=N}^{\infty} \left(\sum_{s=n}^{\infty} s^\alpha q_s \right) = \infty \tag{2.10}$$

and

$$\limsup_{n\to\infty} \left\{ \frac{1}{n+1} \sum_{s=N}^{n} (s+1)^\alpha (s+2)^\alpha q_s + (n+2)^\alpha \sum_{s=n+1}^{\infty} (s+1)^\alpha q_s \right\} > 2^\alpha. \tag{2.11}$$

Then every positive solution $\{y_n\}$ of equation (1.2) is decreasing and satisfies estimates of the form (2.9) with $p = 0$ in R_n.

3. Examples

In this section, we present two examples to illustrate the significance of our main results.

Example 1. Consider the third order difference equation

$$\Delta^3 y_n + \frac{10}{n(n+1)(n+2)} y_{n+1} = 0, \quad n \geq 1. \tag{3.1}$$

A simple calculation shows that conditions (2.10) and (2.11) are satisfied, so by Corollary 1, positive nonoscillatory solutions of equation (3.1) are of the Kneser type and a positive solution $\{y_n\}$ satisfies the estimates

$$\frac{C_1}{n} \leq y_n \leq \frac{C_2}{n}$$

for some positive constants C_1 and C_2. Therefore, equation (3.1) has property (A).

Example 2. Consider the third order neutral difference equation

$$\Delta\left(\frac{1}{n+2}\Delta^2\left(y_n + \frac{1}{2}y_{2n}\right)\right) + \frac{3\beta}{n(n+1)(n+2)(n+3)} y_{n+1} = 0, \quad n \geq 1. \tag{3.2}$$

Again calculations show that conditions (2.1) and (2.5) hold if $\beta > 6$. Therefore, all positive nonoscillatory solutions of equation (3.2) are of the Kneser type and Theorem 2 provides estimates for a positive solution $\{y_n\}$ to be of the form

$$\frac{C_1}{n^\beta} \leq y_n \leq \frac{C_2}{n}$$

for some positive constants C_1 and C_2. We also see that (3.2) has property (A).

4. Conclusions

In this paper, we give conditions for all positive solutions of a third order half-linear neutral difference equation to be of the Kneser type and we derive upper and lower bounds for them. We also wish to point out that the results obtained in Lemmas 2 and 3 are new and complement those in [4, 5, 7, 8, 11–19].

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A NOTE ON CONVERGENCE THEOREMS FOR INTEGRALS

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Abstract. In this study we investigate the convergence theorems such as bounded convergence theorem, Fatou's lemma, monotone convergence theorem and dominated convergence theorem for measurable functions depending upon the concept of statistical convergence.

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1. Introduction

When we have a sequence of integrable functions, "the convergence theorems" depend on this sequence state that the integrability is preserved under the operation of limit. More clearly, let (f_n) be a sequence of integrable functions and the function f be some kind of limit of the sequence. Then we naturally desire integralibility of the function f and the relation

$$\int f = \lim \int f_n$$

holds. We call this type of results as convergence theorems. These theorems are used to show that a given function is integrable or to construct an integrable function. The best known theorem of among the convergence theorems in measure the theory is the Lebesgue dominated convergence theorem.

In this work, we prove the certain convergence theorems in $\sigma-$ finite measurable spaces, when a given sequence of measurable functions is statistically converges to a function.

The concept of statistical convergence was firstly used as "almost convergence" by Zygmund [26] in a monograph in 1935 and in a few years later related works had been published by Steinhaus [25] and Fast [5]. There are a large number of papers related with the statistical convergence and its generalisations in different areas of mathematics have been published in recent years, see [3, 4, 6, 8, 11, 18–20, 23].

In [24], Srivastava *et al.* introduced a new concept of the deferred Nörlund equi-statistical convergence to prove a Korovkin type approximation theorem and demonstrated that a theorem is a non-trivial extension of some well-known Korovkin

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type approximation theorems which were proven by earlier authors. Then, in [14] Parida *et al.* introduced the notion of equi-statistical convergence, statistical point-wise convergence and statistical uniform convergence in conjunction with the deferred statistical convergence and established a inclusion relation between them.

Srivastava and Et [20] introduced the concepts of lacunary strong summability of order α and lacunary statistical convergence of order α of real-valued functions which are measurable (in the Lebesgue sense) in the interval $(1,\infty)$, and they get some relations between them.

Recently, statistical probability convergence has been a dynamic research area due to the fact that it is more general than the statistical convergence as well as the classical convergence. Moreover, such theory is discussed in the study of Fourier Analysis and Approximation Theory. For more details, see [9, 21], and references therein. In [21], Srivastava *et al.* introduced the notion of statistical probability convergence for sequences of random variables and proved a new Korovkin-type approximation theorem with periodic test functions for a sequence of random variables on a Banach space. In [9], Jena *et al.* introduced various aspects of statistical convergence for sequences of random variables as well as for sequences of real numbers via deferred Cesàro summability mean and proposed different new Korovkin-type approximation results with algebraic test functions for a sequence of random variables on a Banach space. In [22], Srivastava *et al.* introduced the notion of statistical probability convergence via deferred Nörlund summability mean and proved a new Korovkin-type approximation theorem with algebraic test functions for a sequence of random variables on a Banach space.

In [12] Nath and Tripathy introduced the statistical convergence concepts of complex uncertain sequences: statistical convergence almost surely, statistical convergence in measure, statistical convergence in mean, statistical convergence in distribution and statistical convergence uniformly almost surely sequences of complex uncertain sequences defined by Orlicz function.

Throughout the paper, it is convenient to begin with the definitions and some well-known results which are needed. The set of positive integers will denote, as usual, by $\mathbb{N}$.

Definition 1. For $A \subseteq \mathbb{N}$, we denote the density of A by

$$d(A) = \lim_{n\to\infty} \frac{|\{k \in A : k \leq n\}|}{n}$$

if the limit exists.

Definition 2. Let (a_n) be a sequence of real numbers and $a \in \mathbb{R}$. We say that (a_n) is statistically converges to a if for all $\varepsilon > 0$

$$d(\{n \in \mathbb{N} : |a_n - a| \geq \varepsilon\}) = 0$$

and denoted by $a_n \xrightarrow{st} a$ or $st\lim a_n = a$.

We will use $a_n \to a$ or $\lim a_n = a$ for usual limit. It is clear that each convergent real number sequence (in ordinary sense) is statistical convergent, but the inverse is not true. Actually, the relation between the usual convergence and statistical convergence of number sequences can be given as follows. Let

$$\mathcal{K} := \{K \subset \mathbb{N} : d(K) = 1\}.$$

Proposition 1 ([2]). *If $K_1, K_2 \in \mathcal{K}$ then $K_1 \cap K_2 \in \mathcal{K}$.*

Proposition 2 ([17]). *Let (a_n) be a sequence of numbers and $a \in \mathbb{R}$. $a_n \xrightarrow{st} a$ iff there is a set $K = \{k_1 < k_2 < \cdots\} \in \mathcal{K}$ such that $a_{k_n} \to a$.*

There is also a relation between the statistically convergence and arithmetic means of bounded sequences of non-negative real numbers.

Proposition 3 ([5]). *Let (a_n) be a bounded sequence of non-negative real numbers. Then*

$$a_n \xrightarrow{st} 0 \qquad \text{iff} \qquad \frac{1}{n}\sum_{k=1}^{n} a_k \to 0.$$

From this proposition we can obtain following corollary.

Corollary 1 ([3]). *Let (a_n) be a bounded sequence of real numbers and $a \in \mathbb{R}$. Then*

$$a_n \xrightarrow{st} a \qquad \text{iff} \qquad \frac{1}{n}\sum_{k=1}^{n} |a_n - a| \to 0.$$

Definition 3 ([7]). Let (a_n) be a sequence of real numbers and $L_{(a_n)}$ and $U_{(a_n)}$ be subsets of $\mathbb{R}$ defined as

$$L_{(a_n)} = \{x \in \mathbb{R} : d(\{n : a_n < x\}) \neq 0\}$$

and

$$U_{(a_n)} = \{x \in \mathbb{R} : d(\{n : a_n > x\}) \neq 0\},$$

respectively, then the statistical limit inferior and the statistical limit superior of the sequence (a_n) are defined by

$$st\liminf a_n = \begin{cases} \inf L_{(a_n)} & L_{(a_n)} \neq \varnothing, \\ +\infty & L_{(a_n)} = \varnothing, \end{cases}$$

and

$$st\limsup a_n = \begin{cases} \sup U_{(a_n)} & U_{(a_n)} \neq \varnothing, \\ -\infty & U_{(a_n)} = \varnothing, \end{cases}$$

respectively.

Theorem 1 ([7]). *Let (a_n) be a real number sequence. If st $\liminf a_n = \alpha$ is finite then for all $\varepsilon > 0$*

$$d(\{n : a_n < \alpha + \varepsilon\}) \neq 0 \quad \text{and} \quad d(\{n : a_n < \alpha - \varepsilon\}) = 0 \tag{1.1}$$

hold. Conversely, if (1.1) hold for all $\varepsilon > 0$ then st $\liminf a_n = \alpha$. Similarly, If st $\limsup a_n = \beta$ is finite, then for all $\varepsilon > 0$

$$d(\{n : a_n > \beta + \varepsilon\}) = 0 \quad \text{and} \quad d(\{n : a_n > \beta - \varepsilon\}) \neq 0 \tag{1.2}$$

hold. Conversely, if (1.2) hold for all $\varepsilon > 0$ then st $\limsup a_n = \beta$.

It is clear that the inequailities

$$\liminf a_n \leq st \liminf a_n \leq st \limsup a_n \leq \limsup a_n$$

hold for any real number sequence (a_n).

Definition 4 ([7]). Let (a_n) be a real number sequence. If there exists a positive real number M satisfying $d(\{k : |x_k| > M\}) = 0$ then (a_n) is said to be statistically bounded sequence.

Theorem 2 ([7]). *Let (a_n) be a statistically bounded real number sequence. Then (a_n) is statistically convergent iff st $\limsup a_n = st \liminf a_n$.*

Note that, for any sequence of real numbers (a_n), if there exists a set $K = \{k_1 < k_2 < \cdots\} \in \mathcal{K}$ such that $st \limsup a_{k_n} = \alpha$ then $st \limsup a_n = \alpha$. Similarly, if there exists a set $K = \{k_1 < k_2 < \cdots\} \in \mathcal{K}$ such that $st \liminf a_{k_n} = \alpha$ then $st \liminf a_n = \alpha$.

Now let us give the definition of the statistical convergence of sequence of functions by using the definition of statistical convergence of real number sequences. The statistical convergence of sequence of functions was first examined in 1951 by Fast [5]. In recent years, there are many studies on the statistical convergence of function sequences (see [1, 2, 8, 10, 13, 16] and references therein).

Definition 5. For a set X, a sequence (f_n) of real valued sequence of functions on X is said to be statistically convergent to f on X if $f_n(x) \xrightarrow{st} f(x)$ for all $x \in X$ and denoted by $f_n \xrightarrow{st} f$.

Many features about the statistical convergence of the function sequences can be obtained directly with the help of the statistical convergence of the number sequences. However, the Proposition 2, which is very useful in number sequences, is unfortunately not valid in the statistical convergence of function sequences. For a counterexample, it can examined the Example 3.6 in [13]. Thus, the Corollary 3.7 in [2] is unfortunately not valid.

In this work, we investigate the convergence theorems such as bounded convergence theorem, Fatou's lemma, monotone convergence theorem and dominated convergence theorem for measurable functions depending the upon the concept of statistical convergence.

2. The Statistical Convergence Theorems

Let $(X,\mathcal{B},\mu)$ be a σ-finite measure space and complete, that is $\mathcal{B}$ is a σ-algebra of subsets of X, $\mu\colon \mathcal{B}\to[0,\infty]$ is a countably additive function with $\mu(\varnothing)=0$ and $\mathcal{B}$ contains all subsets of all sets of measure zero (for details see [15]). For example the Lebesgue measure space $(\mathbb{R},\mathcal{M},m)$ is such a space where $\mathcal{M}$ is family of all Lebesgue measurable subsets of $\mathbb{R}$. All examples in this section will be selected in the Lebesgue measure space $(\mathbb{R},\mathcal{M},m)$.

Definition 6. For $E\in\mathcal{B}$, a sequence (f_n) of μ-measurable real valued functions on E is said to be statistically convergent to f almost everywhere if $\mu(\{x\in X : st\lim f_n(x)\neq f(x)\})=0$ and denoted by $f_n \xrightarrow{st-a.e.} f$.

Proposition 4 ([13]). *If (f_n) is a sequence of real-valued μ-measurable functions and $f_n \xrightarrow{st-a.e.} f$ then f is a $\mu-$measurable function .*

Let us begin with statement of the statistical version of the bounded convergence theorem.

Theorem 3 (The Bounded Statistical Convergence Theorem). *Let $E\in\mathfrak{B}$, $\mu(E)<\infty$ and (f_n) be a sequence of μ-measurable functions that uniformly bounded on the set E (i.e. there exists a number $M\geq 0$ such that $|f_n(x)|\leq M$ holds for all $x\in E$ and $n\in\mathbb{N}$). If $f_n \xrightarrow{st-a.e.} f$ then*

$$st\lim\int_E f_n d\mu=\int_E f d\mu \tag{2.1}$$

holds.

Proof. Without loss of generality assume that $f_n \xrightarrow{st} f$ on E. From Proposition 4, the function f is μ-measurable. For fixed $x\in E$, since $f_n(x)\xrightarrow{st} f(x)$ then there is a set $K=\{k_1<k_2<\cdots\}\in\mathcal{K}$ such that $\lim_{k\to\infty} f_{n_k}(x)=f(x)$. Hence $|f(x)|\leq M$ holds. Therefore $|f_n(x)-f(x)|\leq 2M$ for all $x\in E$ and $n\in\mathbb{N}$. By using Corollary 1, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|f_n(x)-f(x)|=0 \qquad \text{for all} \quad x\in E.$$

If we take $g_n:=\frac{1}{n}\sum_{k=1}^{n}|f_n-f|$, then it is clear that g_n is $\mu-$measurable and $|g_n|\leq 2M$ for all $n\in\mathbb{N}$, and $g_n\to 0$. By the Lebesgue Convergence Theorem [15]

$$\lim_{n\to\infty}\int_E g_n d\mu=0.$$

Therefore, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\int_E|f_n-f|d\mu=0.$$

Considering $\int_E |f_n - f| d\mu \geq 0$ for all $n \in \mathbb{N}$ then, by Proposition 3, we obtain

$$st \lim_{n\to\infty} \int_E |f_n - f| d\mu = 0.$$

From the inequality

$$\left| \int_E f_n d\mu - \int_E f d\mu \right| \leq \int_E |f_n - f| d\mu$$

we get (2.1) as desired. □

Remark 1. In the equatiliy (2.1) we can not substitue *st* lim with ordinary limit.

Example 1. Let $E = [0,1] \subset \mathbb{R}$ and $f_n : E \to \mathbb{R}$, $n \in \mathbb{N}$, be defined as

$$f_n(x) = \begin{cases} 0 & \text{if } n \text{ is square,} \\ 1 & \text{if } n \text{ is non-square.} \end{cases}$$

It is clear that

$$f_n \xrightarrow{st} f \equiv 1 \qquad \text{and} \qquad st \lim \int_E f_n dm = \int_E f dm = 1$$

but $\lim_{n\to\infty} \int_E f_n dm$ does not exist.

Let f and g be a real-valued functions which are defined on μ-measurable set E. Let us define the function

$$(f \wedge g)(x) = \min\{f(x), g(x)\}, \qquad x \in E.$$

It is easy to see that if f and g are μ-measurable functions then $f \wedge g$ is a $\mu-$measurable function. By using the relation

$$f \wedge g = \frac{f + g - |f - g|}{2}$$

we have the following proposition.

Proposition 5. *$f_n \xrightarrow{st} f$ and $g_n \xrightarrow{st} g$ then $f_n \wedge g_n \xrightarrow{st} f \wedge g$.*

Let $E \in \mathcal{B}$. Let us denote the family of measurable functions by $\mathcal{L}_0(E;\mu)$ and the family of positive measurable functions by $\mathcal{L}_0^+(E;\mu)$.

Definition 7. For a sequence $(f_n) \subset \mathcal{L}_0(E;\mu)$, if there exists a set $K = \{k_1 < k_2 < \cdots\} \in \mathcal{K}$ such that $(f_{k_n})_n \subset \mathcal{L}_0^+(E;\mu)$ then the sequence (f_n) is said to be statistically positive on E.

For example the sequence (f_n) defined on $\mathbb{R}$ as

$$f_n(x) = \begin{cases} -1 & \text{if } n \text{ is prime,} \\ x^2/n & \text{otherwise,} \end{cases}$$

is statistically positive on $\mathbb{R}$.

The statistical version of the Fatou's Lemma which is handy to use when proving the convergence theorems is given below.

Theorem 4. *Let $(f_n) \subset \mathcal{L}_0^+(E;\mu)$ be a statistically positive sequence of functions and $f_n \xrightarrow{st-a.e.} f$ then*

$$\int_E f d\mu \leq st \liminf_n \int_E f_n d\mu \tag{2.2}$$

holds.

Proof. Without loss of generality, assume that $f_n \xrightarrow{st} f$ and $(f_n) \subset \mathcal{L}_0^+(E;\mu)$. It can be easily seen that $f \in \mathcal{L}_0^+(E;\mu)$. Take an arbitrary function $h \in \mathcal{L}_0(E;\mu)$ satisfying the following conditions:

(i) $h \leq f$,
(ii) there exists a number $M \geq 0$ such that $|h| \leq M$,
(iii) for $E_h = \{x \in E : h(x) \neq 0\}$, let $\mu(E_h) < \infty$.

Let us consider $h_n := f_n \wedge h$, $n \in \mathbb{N}$. It is clear that, the function (h_n) satisfies the properities $(i), (ii)$ and (iii) for all $n \in \mathbb{N}$. In addition, by Proposition 5, $h_n \xrightarrow{st} h$ holds. According to the bounded statistical convergence theorem (Theorem 2.1), we have

$$\int_{E_h} h d\mu = st \lim \int_{E_h} h_n d\mu. \tag{2.3}$$

Since $h_n \leq f_n$ for all $n \in \mathbb{N}$ then the inequalities

$$\int_{E_h} h_n d\mu \leq \int_{E_h} f_n d\mu \leq \int_E f_n d\mu$$

are valid. Thus, by using (2.3), we get

$$\int_E h d\mu = st \lim \int_E h_n d\mu = st \liminf_n \int_E h_n d\mu \leq st \liminf_n \int_E f_n d\mu.$$

By taking supremum on the functions h that satisfy the properities $(i), (ii)$ and (iii), we obtain

$$\int_E f d\mu = \sup_h \int_E h d\mu \leq st \liminf_n \int_E f_n d\mu$$

as desired. □

Remark 2. The strict inequality in (2.2) exists and it is not allowed to substitue the *st* liminf with ordinary liminf.

Example 2. Let $E = [0,\infty)$ and $f_n : E \to \mathbb{R}$ be defined as

$$f_n(x) = \begin{cases} 0 & \text{if } n \text{ is square,} \\ \frac{\chi_0^{n-1}(x)}{(x+1)^2} + \frac{n+2}{2n}\chi_{n-1}^n(x) & \text{if } n \text{ is even but non-square,} \\ \frac{\chi_0^{n-1}(x)}{(x+1)^2} + \frac{n+1}{n}\chi_{n-1}^n(x) & \text{if } n \text{ is odd but non-square,} \end{cases}$$

where $\chi_a^b := \chi_{[a,b]}$ is the characteristic function of the interval $[a,b]$. It is clear that $f_n \geq 0$ for all $n \in \mathbb{N}$ and $f_n(x) \xrightarrow{st} f(x) = (x+1)^{-2}$ for all $x \in E$. It can be easily obtained the following strict inequalities

$$\liminf \int_E f_n dm = 0 < \int_E f dm = 1 < st\liminf \int_E f_n dm = \frac{3}{2}.$$

Definition 8. A sequence (f_n) of functions is called statistically increasing if there exists a set $K = \{k_1 < k_2 < \cdots\} \in \mathcal{K}$ such that $f_{k_{n+1}} - f_{k_n} \geq 0$ for all $n \in \mathbb{N}$.

The following theorem is the statistical version of the monotone convergence theorem for measurable functions.

Theorem 5 (Monotone Statistical Convergence Theorem)**.** *Let $(f_n) \subset \mathcal{L}_0(E;\mu)$ be a statistically increasing and statistically positive sequence of functions. If $f_n \xrightarrow{st-a.e.} f$ then*

$$\int_E f d\mu = st \lim_{n\to\infty} \int_E f_n d\mu \tag{2.4}$$

holds.

Proof. Since the sequence (f_n) is statistically positive and $f_n \xrightarrow{st-a.e.} f$, by using Theorem 4, we have

$$\int_E f d\mu \leq st\liminf \int_E f_n d\mu. \tag{2.5}$$

Considering Proposition 1, there is a set $K \in \mathcal{K}$ such that the subsequence $(f_n)_{n\in K}$ is positive and increasing on the set E. Let $K = \{k_1 < k_2 < \cdots\}$. Then $f_{k_n} \leq f$ for all $n \in \mathbb{N}$. Thus

$$\int_E f_{k_n} d\mu \leq \int_E f d\mu$$

Thereby

$$st\limsup \int_E f_n d\mu = st\limsup \int_E f_{k_n} d\mu \leq \int_E f d\mu \tag{2.6}$$

holds. By gathering the inequalities (2.5) and (2.6), the inequality (2.4) can be obtained as desired. □

Remark 3. In the inequality (2.4) it is not allowed to substitue the *st* lim with usual limit.

Example 3. Let $E = [0,\infty)$ and $f_n : E \to \mathbb{R}$ be defined as

$$f_n(x) = \begin{cases} -\chi_0^n(x)/n & \text{if } n \text{ is square,} \\ \chi_0^n(x)/(x+1)^2 & \text{otherwise.} \end{cases}$$

It is clear that $f_n \in \mathcal{L}_0[0,\infty)$ for all n, and the sequence (f_n) is statistically positive and statistically increasing. In addition, $st\lim f_n(x) = (1+x)^{-2} =: f(x)$ for all $x \in [0,\infty)$.

Let $a_n = \int_E f_n dm$, $n \in \mathbb{N}$. Then

$$a_n = \begin{cases} -1 & \text{if } n \text{ is square,} \\ \frac{n}{(n+1)} & \text{otherwise.} \end{cases}$$

Thus, $st\lim a_n = 1$ while $\lim a_n$ does not exist.

Theorem 6 (Dominated Statistical Convergence Theorem)**.** *Let g be a μ-integrable function on E and $(f_n) \subset \mathcal{L}_0(E;\mu)$. If $|f_n(x)| \leq g(x)$ for all $n \in \mathbb{N}$ and $x \in E$ and $f_n \xrightarrow{st} f$ then f is μ-integrable on E and*

$$st\lim \int_E f_n d\mu = \int_E f d\mu \tag{2.7}$$

holds.

Proof. It is clear that f_n is μ-integrable function for all $n \in \mathbb{N}$. From assumption, for arbitrary fixed number $x \in E$, $st\lim f_n(x) = f(x)$ implies existness of a set $K = \{k_1 < k_2 < \cdots\} \in \mathcal{K}$ such that $\lim f_{k_n}(x) = f(x)$. By the fact that $|f_{k_n}(x)| \leq g(x)$, we get $|f(x)| \leq g(x)$ for all $x \in E$. From assumption g is a μ-integrable function then f is μ-ingtegrable function.

Let us consider the functions $h_n = g + f_n$ and $g_n = g - f_n$, $n \in \mathbb{N}$. Then both sequences (h_n) and (g_n) consist from non-negative $\mu-$integrable functions. Since $h_n \xrightarrow{st} g + f$, by Theorem 4, we have

$$\int_E (g+f)d\mu \leq st\liminf \int_E (g+f_n)d\mu$$

hence

$$\int_E f d\mu \leq st\liminf \int_E f_n d\mu \tag{2.8}$$

and since $h_n \xrightarrow{st} g + f$, by Theorem 4, we have

$$\int_E (g-f)d\mu \leq st\liminf \int_E (g-f_n)d\mu$$

hence

$$\int_E f d\mu \geq st\limsup \int_E f_n d\mu. \tag{2.9}$$

Combining (2.8) and (2.9) we obtain (2.7) as desired. □

Definition 9 ([16])**.** Let (f_n) be a sequence of functions and g be a function defined on E. If

$$d(\{n : |f_n(x)| \leq g(x), \forall x \in E\}) = 0$$

then g is called a statistically majorant function for the sequence (f_n).

Corollary 2. *$g \in \mathcal{L}_1(E;\mu)$ and $(f_n) \in \mathcal{L}_0(E;\mu)$ is given. If the function g is a statistically majorant function for the sequence (f_n) and $f_n \xrightarrow{st-a.e.} f$ then*

$$st\lim \int_E f_n d\mu = \int_E f d\mu. \tag{2.10}$$

It can not be applied the statistical versions of bounded, monotone and uniform convergence theorems to the following example, but Corollary 2 can be applied. This example also shows that it can not be replaced the *st* lim with usual limit in (2.10).

Example 4. Let $E = [0,2]$ and $f_n \colon E \to \mathbb{R}$ be defined as

$$f_n(x) = \begin{cases} n & \text{if } n \text{ is square,} \\ \sqrt{n} & \text{if } n \text{ is non-square and } 1/n \leq x \leq 2/n, \\ 0 & \text{otherwise.} \end{cases}$$

The function g defined as $g(x) = \sqrt{2/x}$ is statistically mojorant for the sequence (f_n).

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POSITIVE SOLUTIONS FOR HIGHER-ORDER MULTI-POINT FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we consider the higher-order multi-point fractional boundary value problem. We establish the criteria for the existence of at least one and three positive solutions for higher order nonlinear m-point fractional boundary value problem by using the Krasnosel'skii fixed point theorem and the Legget-Williams fixed point theorem, respectively.

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1. INTRODUCTION

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary (non-integer) order. Fractional differential equations arise in many engineering and scientific disciplines as the mathematical models of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, [1, 2, 6, 7, 13, 17–21].

Among all the researches on the theory of the fractional differential equations, the study of the boundary value problems for fractional differential equations recently has attracted a great deal of attention from many researchers. Some results have been obtained on the existence of positive solutions of the boundary value problems for some specific fractional differential equations [3–5, 8–11, 15, 22, 23].

In [16], Nyamoradi and Javidi were interested in the fractional order multi-point boundary value problem

$$\begin{cases} D_{0+}^{\sigma}(\phi_p(u''(t))) - g(t)f(u(t)) = 0, \ t \in [0,1], \ 1 < \sigma \leq 2, \\ \phi_p(u''(0)) = \phi_p(u''(1)) = 0, \\ au(0) - bu'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \\ cu(1) + du'(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \end{cases}$$

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where $D_{0^+}^{\sigma}$ is the standard Riemann-Liouville fractional derivative of order σ. Some existence results for at least one positive solutions were established by using Krasnosel'skii fixed point theorem.

In this paper, we study the existence of positive solutions to multi-point boundary value problem (BVP) for higher order fractional differential equations:

$$\begin{cases} -D_{0^+}^{\eta-2}(u''(t)) + f(t,u(t)) = 0, \ t \in [0,1], \\ u''(0) = u'''(0) = ... = u^{(n-2)}(0) = 0, \ u'''(1) = 0, \\ \alpha u(0) - \beta u'(0) = \sum_{p=1}^{m-2} a_p u'(\xi_p), \\ \gamma u(1) + \delta u'(1) = \sum_{p=1}^{m-2} b_p u'(\xi_p), \end{cases} \tag{1.1}$$

where $D_{0^+}^{\eta-2}$ is the Riemann-Liouville fractional derivative of order $\eta-2$. Throughout the paper, we suppose that $m,n \geq 3$ and $n-1 < \eta \leq n$, where $n,m \in \mathbb{N}$ and $\beta > \alpha > 1$, $\gamma, \delta > 0$, $a_p, b_p \geq 0$ are given constants and $0 < \xi_1 < \ldots < \xi_{m-2} < 1$. We assume that $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

We have organized the paper as follows. First, we provide some definitions and preliminary lemmas which are key tools for our main results. Second, we obtained the existence of at least one positive solution for the BVP (1.1) by using the Krasnosel'skii fixed point theorem. Finally, we use the Legget-Williams fixed-point theorem to show that the existence of at least three positive solutions to the BVP (1.1).

We assume that the following conditions are satisfied:

(H1) If $m \geq 3$, then $\gamma \sum_{k=1}^{m-2} a_k \geq \alpha \sum_{k=1}^{m-2} b_k$ and

if $m > 3$, then $\alpha\delta > \gamma \sum_{k=1}^{j-1} a_k \geq \alpha \sum_{k=1}^{j-1} b_k > \beta\gamma$, where $2 \leq j \leq m-2$.

(H2) $\alpha\delta > \alpha \sum_{p=1}^{m-2} b_p + \gamma \sum_{p=1}^{m-2} a_p$.

2. Preliminaries

To state the main results of this paper, we will need the following lemmas and we present some notation.

Definition 1. The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a function $u : (0,\infty) \to \mathbb{R}$ is defined by

$$D_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds,$$

where $n = [\alpha] + 1$.

Definition 2. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0,\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Lemma 1 ([13]). *The equality $D_{0^+}^{\gamma} I_{0^+}^{\gamma} f(t) = f(t)$, $\gamma > 0$ holds for $f \in L(0,1)$.*

Lemma 2 ([13]). *Let $\alpha > 0$. Then the differential equation*

$$D_{0^+}^{\alpha} u = 0 \tag{2.1}$$

has a unique solution $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1,\ldots,n$, where $n-1 < \alpha \le n$.

Lemma 3 ([13]). *Let $\alpha > 0$. Then the following equality holds for $u \in L(0,1)$, $D_{0^+}^{\alpha} u \in L(0,1)$:*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n}, \tag{2.2}$$

$c_i \in \mathbb{R}$, $i = 1,\ldots,n$, where $n-1 < \alpha \le n$.

If $-u''(t) = y(t)$ and $\eta - 2 = \sigma$, then the problem

$$\begin{cases} -D_{0^+}^{\eta-2}(u''(t)) + f(t,u(t)) = 0, \ t \in [0,1], \\ u''(0) = u'''(0) = \ldots = u^{(n-2)}(0) = 0, \ u'''(1) = 0 \end{cases} \tag{2.3}$$

is turned into the problem

$$\begin{cases} D_{0^+}^{\sigma} y(t) + f(t,u(t)) = 0, \ t \in [0,1], \\ y(0) = y'(0) = \ldots = y^{(n-4)}(0) = 0, \ y'(1) = 0. \end{cases} \tag{2.4}$$

Lemma 4. *The boundary value problem (2.4) has a unique solution*

$$y(t) = \int_0^1 H(t,s) f(s,u(s)) ds, \tag{2.5}$$

where

$$H(t,s) = \begin{cases} \dfrac{(1-s)^{\sigma-2} t^{\sigma-1}}{\Gamma(\sigma)}, & t \le s, \\ \dfrac{(1-s)^{\sigma-2} t^{\sigma-1} - (t-s)^{\sigma-1}}{\Gamma(\sigma)}, & t \ge s. \end{cases} \tag{2.6}$$

Proof. According to Lemma 3, we obtain

$$y(t) = -\frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1} f(s,u(s))ds + c_1 t^{\sigma-1} + c_2 t^{\sigma-2} + ... + c_{n-2}t^{\sigma-n+2}. \quad (2.7)$$

By boundary conditions of (2.4) we get $c_2 = c_3 = ... = c_{n-2} = 0$ and

$$c_1 = \frac{1}{\Gamma(\sigma)}\int_0^1 (1-s)^{\sigma-2} f(s,u(s))ds.$$

Thus, the unique solution of problem (2.4) is

$$\begin{aligned} y(t) &= \int_0^t \frac{(1-s)^{\sigma-2}t^{\sigma-1} - (t-s)^{\sigma-1}}{\Gamma(\sigma)} f(s,u(s))ds + \int_t^1 \frac{(1-s)^{\sigma-2}t^{\sigma-1}}{\Gamma(\sigma)} f(s,u(s))ds \\ &= \int_0^1 H(t,s) f(s,u(s))ds. \end{aligned}$$

The proof is complete. □

Lemma 5. *If* (H1) *and* (H2) *hold and*

$$K := \alpha\gamma + \alpha\delta - \alpha\sum_{p=1}^{m-2} b_p + \gamma\beta + \gamma\sum_{p=1}^{m-2} a_p,$$

then for $y \in C[0,1]$*, the boundary value problem*

$$\begin{cases} -u''(t) = y(t), \ t \in [0,1], \\ \alpha u(0) - \beta u'(0) = \sum_{p=1}^{m-2} a_p u'(\xi_p), \\ \gamma u(1) + \delta u'(1) = \sum_{p=1}^{m-2} b_p u'(\xi_p) \end{cases} \quad (2.8)$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)ds, \quad (2.9)$$

where

$$G(t,s)=\frac{1}{K}\begin{cases}(\alpha s+\beta)(\gamma(1-t)+\delta-\sum_{p=1}^{m-2}b_p), 0\le s\le\xi_1, t\ge s\\ (\gamma(1-s)+\delta-\sum_{p=1}^{m-2}b_p)(\alpha t+\beta)+\gamma\sum_{p=1}^{m-2}a_p(t-s), 0\le s\le\xi_1, t\le s\\ (\alpha s+\beta+\sum_{k=1}^{j-1}a_k)(\gamma(1-t)+\delta-\sum_{p=j}^{m-2}b_p)+\sum_{k=1}^{j-1}b_k(\alpha(t-s)+\sum_{p=j}^{m-2}a_p),\\ \qquad \xi_{j-1}<s\le\xi_j, t\ge s, 2\le j\le m-2\\ (\gamma(1-s)+\delta-\sum_{p=j}^{m-2}b_p)(\alpha t+\beta+\sum_{k=1}^{j-1}a_k)+\sum_{p=j}^{m-2}a_p(\gamma(t-s)+\sum_{k=1}^{j-1}b_k),\\ \qquad \xi_{j-1}<s\le\xi_j, t\le s, 2\le j\le m-2\\ (\alpha s+\beta+\sum_{k=1}^{m-2}a_k)(\gamma(1-t)+\delta)+\alpha\sum_{k=1}^{m-2}b_k(t-s), \xi_{m-2}<s\le 1, t\ge s\\ (\gamma(1-s)+\delta)(\alpha t+\beta+\sum_{k=1}^{m-2}a_k), \xi_{m-2}<s\le 1, t\le s.\end{cases}\tag{2.10}$$

Proof. A direct calculation gives that if $y\in C[0,1]$, then the boundary value problem (2.8) has the unique solution

$$\begin{aligned}u(t)=&-\int_0^t(t-s)y(s)\Delta s+\frac{t}{K}\left\{\alpha\int_0^1(\gamma(1-s)+\delta)y(s)\Delta s+\sum_{p=1}^{m-2}(\gamma a_p-\alpha b_p)\int_0^{\xi_p}y(s)\Delta s\right\}\\ &+\frac{1}{K}\left\{(\beta+\sum_{p=1}^{m-2}a_p)\int_0^1(\gamma(1-s)+\delta)y(s)\Delta s+(-(\beta+\sum_{p=1}^{m-2}a_p))\sum_{p=1}^{m-2}b_p\int_0^{\xi_p}y(s)\Delta s\right.\\ &\left.+\left(\frac{\gamma(\beta+\sum_{p=1}^{m-2}a_p)}{\alpha}-\frac{K}{\alpha}\right)\sum_{p=1}^{m-2}a_p\int_0^{\xi_p}y(s)\Delta s\right\}.\end{aligned}$$

Hence, we obtain (2.9). □

Lemma 6. *The Green's function $G(t,s)$ in (2.10) satisfies*

$$0<G(t,s)\le G(s,s)$$

for $(t,s)\in[0,1]\times[0,1]$.

Proof. From (H1), (H2) and (2.10), $G(t,s) > 0$. Now, we will show that $G(t,s) \leq G(s,s)$.

(i) Let $s \in [0,\xi_1]$ and $t \geq s$. Since $G(t,s)$ is decreasing in t, we get $G(t,s) \leq G(s,s)$.
(ii) Let $s \in [0,\xi_1]$ and $t \leq s$. Since $G(t,s)$ is increasing in t, we have $G(t,s) \leq G(s,s)$.
(iii) Take $s \in (\xi_{j-1},\xi_j]$, $2 \leq j \leq m-2$ and $t \geq s$. From (H1), $G(t,s)$ is decreasing in t. So we obtain $G(t,s) \leq G(s,s)$.
(iv) Take $s \in (\xi_{j-1},\xi_j]$, $2 \leq j \leq m-2$ and $t \leq s$. Since $G(t,s)$ is increasing in t, we get $G(t,s) \leq G(s,s)$.
(v) Let $s \in (\xi_{m-2},\ 1]$ and $t \geq s$. From (H1), $G(t,s)$ is decreasing in t. So we have $G(t,s) \leq G(s,s)$.
(vi) Let $s \in (\xi_{m-2},\ 1]$ and $t \leq s$. Since $G(t,s)$ is increasing in t, we obtain $G(t,s) \leq G(s,s)$.

□

Lemma 7. *Green's function $G(t,s)$ in (2.10) satisfies*

$$\min_{t\in[0,1]} G(t,s) \geq z\|G(.,s)\|,$$

with

$$z = \min\{z_1, z_2, z_3, z_4, z_5, z_6\}, \tag{2.11}$$

where

$$z_1 = \frac{\delta - \sum_{p=1}^{m-2} b_p}{\gamma+\delta-\sum_{p=1}^{m-2} b_p}, \quad z_2 = \frac{(\alpha\delta - \alpha\sum_{p=1}^{m-2} b_p - \gamma\sum_{p=1}^{m-2} a_p)}{(\gamma+\delta-\sum_{p=1}^{m-2} b_p)(\alpha+\beta)}, \quad z_3 = \frac{\delta - \sum_{p=j}^{m-2} b_p}{\gamma+\delta-\sum_{p=j}^{m-2} b_p},$$

$$z_4 = \frac{\frac{\beta}{\alpha}-1}{\alpha+\beta+\sum_{k=1}^{j-1} a_k}, \quad z_5 = \frac{\delta}{\gamma+\delta}, \quad z_6 = \frac{\frac{\beta}{\alpha}-1}{\alpha+\beta+\sum_{k=1}^{m-2} a_k}$$

and $\|.\|$ is defined by $\|x\| = \max_{t\in[0,1]} |x(t)|$.

Proof. (i) Take $s \in [0,\xi_1]$ and $t \geq s$. Since $G(t,s)$ is decreasing in t and $0 < z_1 < 1$, we get $\min_{t\in[0,1]} G(t,s) = G(1,s)$ and $\min_{t\in[0,1]} G(t,s) \geq z_1 G(s,s) = z_1\|G(.,s)\|$.
(ii) Take $s \in [0,\xi_1]$ and $t \leq s$. Since $G(t,s)$ is increasing in t and $0 < z_2 < 1$, we have $\min_{t\in[0,1]} G(t,s) = G(0,s)$ and $\min_{t\in[0,1]} G(t,s) \geq z_2 G(s,s) = z_2\|G(.,s)\|$.

(iii) Let $s \in (\xi_{j-1}, \xi_j]$, $2 \le j \le m-2$ and $t \ge s$. From (H1), $G(t,s)$ is decreasing in t. It is clear that $0 < z_3 < 1$. So $\min_{t\in[0,1]} G(t,s) = G(1,s)$ and $\min_{t\in[0,1]} G(t,s) \ge z_3 G(s,s) = z_3\|G(.,s)\|$.

(iv) Let $s \in (\xi_{j-1}, \xi_j]$, $2 \le j \le m-2$ and $t \le s$. Since $G(t,s)$ is increasing in t and $0 < z_4 < 1$, we have $\min_{t\in[0,1]} G(t,s) = G(0,s)$ and $\min_{t\in[0,1]} G(t,s) \ge z_4 G(s,s) = z_4\|G(.,s)\|$.

(v) Take $s \in (\xi_{m-2},\ 1]$ and $t \ge s$. From (H1), $G(t,s)$ is decreasing in t. It is clear that $0 < z_5 < 1$. So we get $\min_{t\in[0,1]} G(t,s) = G(1,s)$ and $\min_{t\in[0,1]} G(t,s) \ge z_5 G(s,s) = z_5\|G(.,s)\|$.

(vi) Take $s \in (\xi_{m-2},\ 1]$ and $t \le s$. Since $G(t,s)$ is increasing in t and $0 < z_6 < 1$, we have $\min_{t\in[0,1]} G(t,s) = G(0,s)$ and $\min_{t\in[0,1]} G(t,s) \ge z_6 G(s,s) = z_6\|G(.,s)\|$.

Thus $\min_{t\in[0,1]} G(t,s) \ge z\|G(.,s)\|$, where $z = \min\{z_1, z_2, z_3, z_4, z_5, z_6\}$. □

Lemma 8. *For $t,s \in [0,1]$, we have $0 \le H(t,s) \le H(1,s)$.*

Proof. From (2.6), we obtain $0 \le H(t,s)$. Now we will show that $H(t,s) \le H(1,s)$.

(i) Let $t \le s$. $H(t,s) = \frac{(1-s)^{\sigma-2}t^{\sigma-1}}{\Gamma(\sigma)} \le \frac{(1-s)^{\sigma-2}}{\Gamma(\sigma)} = H(1,s)$.

(ii) Let $t \ge s$. Since $H(t,s)$ is increasing in t, we have $H(t,s) \le H(1,s)$.

□

Lemma 9. $\min_{t\in[\xi_{m-2},1]} H(t,s) \ge k^{\sigma-1}H(1,s)$ *for $0 \le t,s \le 1$, where $k \in (0,\xi_{m-2})$ is a constant.*

Proof. (i) Take $t \le s$. Since $H(t,s)$ is an increasing function, we get

$$\min_{t\in[\xi_{m-2},1]} H(t,s) = \frac{(1-s)^{\sigma-2}\xi_{m-2}^{\sigma-1}}{\Gamma(\sigma)} \ge \frac{(1-s)^{\sigma-2}(k)^{\sigma-1}}{\Gamma(\sigma)} = k^{\sigma-1}H(1,s).$$

(ii) For $s \le t$, we have

$$\begin{aligned}
\min_{t\in[\xi_{m-2},1]} H(t,s) &= \frac{(1-s)^{\sigma-2}\xi_{m-2}^{\sigma-1} - (\xi_{m-2}-s)^{\sigma-1}}{\Gamma(\sigma)} \\
&> \frac{(1-s)^{\sigma-2}\xi_{m-2}^{\sigma-1} - (\xi_{m-2}-\xi_{m-2}s)^{\sigma-1}}{\Gamma(\sigma)} \\
&= \frac{\xi_{m-2}^{\sigma-1}((1-s)^{\sigma-2} - (1-s)^{\sigma-1})}{\Gamma(\sigma)} \\
&= \xi_{m-2}^{\sigma-1}H(1,s) \\
&> k^{\sigma-1}H(1,s).
\end{aligned}$$

Thus $\min_{t\in[\xi_{m-2},1]} H(t,s) \ge k^{\sigma-1}H(1,s)$. □

From Lemma 4 and Lemma 5, we know that $u(t)$ is a solution of the problem (1.1) if and only if

$$u(t) = \int_0^1 G(t,s) \int_0^1 H(s,\tau) f(\tau, u(\tau)) d\tau ds. \tag{2.12}$$

Let E denote the Banach space $C[0,1]$ with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. Define the cone $P \subset E$ by

$$P = \{u \in E : u(t) \geq 0, \min_{t\in[0,1]} u(t) \geq z\|u\|\} \tag{2.13}$$

where z is given in (2.11).

We can define the operator $A : P \to E$ by

$$Au(t) = \int_0^1 G(t,s) \int_0^1 H(s,\tau) f(\tau, u(\tau)) d\tau ds, \tag{2.14}$$

where $u \in P$. Therefore solving (2.12) in P is equivalent to finding fixed points of the operator A.

Lemma 10. *If the conditions* (H1) *and* (H2) *hold, then* $AP \subset P$.

Proof. If $u \in P$, then $Au(t) \geq 0$ on $[0,1]$ by using Lemma 6 and Lemma 8. On the other hand, we have

$$\begin{aligned} \min_{t\in[0,1]} Au(t) &= \int_0^1 \min_{t\in[0,1]} G(t,s) \int_0^1 H(s,\tau) f(\tau, u(\tau)) d\tau ds \\ &\geq z \int_0^1 \max_{t\in[0,1]} G(t,s) \int_0^1 H(s,\tau) f(\tau, u(\tau)) d\tau ds \\ &= z\|Au\|, \end{aligned}$$

by Lemma 7. Thus $Au \in P$ and therefore $AP \subset P$. □

In order to follow the main results of this paper easily, now we state the fixed point theorems which we applied to prove main theorems.

Theorem 1 (Krasnosel'skii Fixed Point Theorem, [12])**.** *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1 and Ω_2 are open bounded subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let*

$$A : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|Au\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_1$, $\|Au\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_2$ *or*
(ii) $\|Au\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_1$, $\|Au\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_2$

hold. Then A has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Theorem 2 (Leggett-Williams Fixed Point Theorem, [14]). *Let P be a cone in the real Banach space E. Set*

$$P_r := \{x \in P : \|x\| < r\},$$

$$P(\psi, a, b) := \{x \in P : a \leq \psi(x),\ \|x\| \leq b\}.$$

Suppose $A : \overline{P_r} \to \overline{P_r}$ be a completely continuous operator and ψ be a nonnegative continuous concave functional on P with $\psi(u) \leq \|u\|$ for all $u \in \overline{P_r}$. If there exists $0 < p < q < l \leq r$ such that the following conditions hold,

(i) $\{u \in P(\psi, q, l) : \psi(u) > q\} \neq \varnothing$ *and* $\psi(Au) > q$ *for all* $u \in P(\psi, q, l)$,
(ii) $\|Au\| < p$ *for* $\|u\| \leq p$,
(iii) $\psi(Au) > q$ *for* $u \in P(\psi, q, r)$ *with* $\|Au\| > l$,

then A has at least three fixed points u_1, u_2 and u_3 in $\overline{P_r}$ satisfying

$$\|u_1\| < p,\ \psi(u_2) > q,\ p < \|u_3\| \text{ with } \psi(u_3) < q.$$

3. Main Results

For convenience, we introduce the following notations. Let

$$M = \int_0^1 H(1, \tau)d\tau, \tag{3.1}$$

$$L = \int_0^1 G(s, s)ds, \tag{3.2}$$

$$I = \int_{\xi_{m-2}}^1 G(s, s)ds. \tag{3.3}$$

Now, we will give the sufficient conditions to have at least one positive solution for the BVP (1.1). Krasnosel'skii fixed point theorem will be used to prove the next theorem.

Theorem 3. *Suppose* (H1) *and* (H2) *hold. In addition, let there exist numbers $0 < r < R < \infty$ such that the function f satisfies the following conditions:*

(i) $f(t, u) < \frac{1}{LM}u(t)$ *for* $(t, u) \in [0, 1] \times [0, r]$,
(ii) $f(t, u) > \frac{1}{k^{\sigma-1}z^2IM}u(t)$ *for* $(t, u) \in [0, 1] \times [R, \infty)$.

Then the BVP (1.1) has at least one positive solution.

Proof. Define the open bounded subsets of E by $\Omega_1 = \{u \in P : \|u\| < r\}$ and $\Omega_2 = \left\{u \in P : \|u\| < \frac{R}{z}\right\}$. It is easy to check that $A : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is completely continuous operator.

If $u \in P \cap \partial\Omega_1$, then $\|u\| = r$. Therefore, by using the hypothesis (i), Lemma 6 and Lemma 8,

$$\begin{aligned} Au(t) &= \int_0^1 G(t,s) \int_0^1 H(s,\tau) f(\tau, u(\tau)) d\tau ds \\ &< \frac{1}{LM} \int_0^1 G(t,s) \int_0^1 H(s,\tau) u(\tau) d\tau ds \\ &\leq \frac{1}{LM} \|u\| \int_0^1 G(t,s) \int_0^1 H(s,\tau) d\tau ds \\ &\leq \frac{1}{LM} \|u\| \int_0^1 G(s,s) \int_0^1 H(1,\tau) d\tau ds \\ &= \|u\|, \end{aligned}$$

for all $t \in [0,1]$. Thus $\|Au\| \leq \|u\|$ for $u \in P \cap \partial\Omega_1$. On the other hand, $u \in P \cap \partial\Omega_2$ implies

$$u(t) \geq z\|u\| = R,$$

for $t \in [0,1]$ and

$$\begin{aligned} Au(t) &= \int_0^1 G(t,s) \int_0^1 H(s,\tau) f(\tau, u(\tau)) d\tau ds \\ &> \frac{1}{k^{\sigma-1} z^2 IM} \int_0^1 G(t,s) \int_0^1 H(s,\tau) u(\tau) d\tau ds \\ &\geq \frac{1}{k^{\sigma-1} z^2 IM} z\|u\| \int_0^1 G(t,s) \int_0^1 H(s,\tau) d\tau ds \\ &\geq \frac{1}{k^{\sigma-1} z^2 IM} z\|u\| z k^{\sigma-1} \int_{\xi_{m-2}}^1 G(s,s) \int_0^1 H(1,\tau) d\tau ds \\ &= \|u\|, \end{aligned}$$

from (ii), Lemma 7 and Lemma 9. Consequently, $\|Au\| \geq \|u\|$ for $u \in P \cap \partial\Omega_2$.

By the first part of Theorem 1, A has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$, such that $r \leq \|u\| \leq \frac{R}{z}$. Therefore BVP (1.1) has at least one positive solution. □

Now we will use the Legget-Williams fixed point theorem to prove the next theorem.

Theorem 4. *Assume that* (H1) *and* (H2) *hold. Suppose that there exist numbers* $0 < p < q < \frac{q}{z} \leq r$ *such that the function* f *satisfies the following conditions:*

(i) $f(t,u) \leq \frac{r}{ML}$ *for* $(t,u) \in [0,1] \times [0,r]$,
(ii) $f(t,u) < \frac{p}{ML}$ *for* $(t,u) \in [0,1] \times [0,p]$,
(iii) $f(t,u) > \frac{q}{k^{\sigma-1}zIM}$ *for* $(t,u) \in [0,1] \times [q,\frac{q}{z}]$,

where z, M, L *and* I *are as in* (2.11), (3.1), (3.2) *and* (3.3), *respectively and* k *is defined in Lemma* 9. *Then the BVP* (1.1) *has at least three positive solutions* u_1, u_2 *and* u_3 *satisfying*

$$\max_{t\in[0,1]} u_1(t) < p, \ \min_{t\in[0,1]} u_2(t) > q$$
$$\max_{t\in[0,1]} u_3(t) > p \text{ with } \min_{t\in[0,1]} u_3(t) < q.$$

Proof. Define the nonnegative, continuous, concave functional $\psi : P \to [0,\infty)$ to be $\psi(y) = \min_{t\in[0,1]} u(t)$ and the cone P as in (2.13). For all $u \in P$, we have $\psi(u) \leq \|u\|$. Now we show that $A : \overline{P_r} \to \overline{P_r}$ is completely continuous operator. If $u \in \overline{P_r}$, then $0 \leq u(t) \leq r$ for all $t \in [0,1]$. We get,

$$\begin{aligned}
\|Au\| &= \max_{t\in[0,1]} |Au(t)| \\
&= \int_0^1 \max_{t\in[0,1]} G(t,s) \int_0^1 H(s,\tau) f(\tau,u(\tau)) d\tau ds \\
&\leq \int_0^1 G(s,s) \int_0^1 H(1,\tau) f(\tau,u(\tau)) d\tau ds \\
&\leq \frac{r}{ML} \int_0^1 G(s,s) \int_0^1 H(1,\tau) d\tau ds \\
&\leq r
\end{aligned}$$

by hypothesis (i), Lemma 6 and Lemma 8. Thus $A : \overline{P_r} \to \overline{P_r}$. It easy to check that $A : \overline{P_r} \to \overline{P_r}$ is completely continuous.

Since $z < 1$, $u(t) = \frac{q}{z} \in P(\psi, q, \frac{q}{z})$ and $\psi(\frac{q}{z}) > q$. Then, we have

$$\{u \in P(\psi, q, \frac{q}{z}) : \psi(u) > q\} \neq \varnothing.$$

On the other hand, for all $u \in P(\psi, q, \frac{q}{z})$, we have $q \leq u(t) \leq \frac{q}{z}$ for $t \in [0,1]$. Using assumption (iii), Lemma 7 and Lemma 9, we find

$$\begin{aligned}
\psi(Au) &= \min_{t\in[0,1]} \int_0^1 G(t,s) \int_0^1 H(s,\tau) f(\tau, u(\tau)) d\tau ds \\
&\geq z \int_0^1 G(s,s) \int_0^1 H(s,\tau) f(\tau, u(\tau)) d\tau ds \\
&\geq z \int_{\xi_{m-2}}^1 G(s,s) \int_0^1 H(s,\tau) f(\tau, u(\tau)) d\tau ds \\
&\geq k^{\sigma-1} z \int_{\xi_{m-2}}^1 G(s,s) \int_0^1 H(1,\tau) f(\tau, u(\tau)) d\tau ds \\
&\geq k^{\sigma-1} z \frac{q}{k^{\sigma-1} z I M} \int_{\xi_{m-2}}^1 G(s,s) \int_0^1 H(1,\tau) d\tau ds = q.
\end{aligned}$$

Thus condition (i) of Theorem 2 holds.

For $\|u\| < p$, we have $0 \leq u(t) \leq p$ for $t \in [0,1]$. Then from assumption (ii), Lemma 6 and Lemma 8, we obtain

$$\begin{aligned}
\|Au\| &= \max_{t\in[0,1]} \int_0^1 G(t,s) \int_0^1 H(s,\tau) f(\tau, u(\tau)) d\tau ds \\
&\leq \int_0^1 G(s,s) \int_0^1 H(1,\tau) f(\tau, u(\tau)) d\tau ds \\
&< \frac{p}{ML} \int_0^1 G(s,s) \int_0^1 H(1,\tau) d\tau ds = p.
\end{aligned}$$

It follows that condition (ii) of Theorem 2 is satisfied.

Finally, we will check condition (iii) of Theorem 2. We suppose that $u \in P(\psi, q, r)$ with $\|Au\| > \frac{q}{z}$. Then we obtain

$$\psi(Au) = \min_{t\in[0,1]} Au(t) \geq z\|Au\| > q. \tag{3.4}$$

Since all conditions of the Legget-Williams fixed point theorem are satisfied, the BVP (1.1) has at least three positive solutions u_1, u_2 and u_3 such that

$$\max_{t\in[0,1]} u_1(t) < p, \quad \min_{t\in[0,1]} u_2(t) > q$$

$$\max_{t\in[0,1]} u_3(t) > p \text{ with } \min_{t\in[0,1]} u_3(t) < q.$$

□

Example 1. Taking $n=5, m=3, \xi_1=\frac{1}{2}, \alpha=\gamma=\delta=3, \beta=4, a_1=b_1=1, k=\frac{1}{4}$ and $\sigma=\frac{5}{2}$, we consider the following boundary value problem:

$$\begin{cases} -D_{0_+}^{\frac{5}{2}}(u''(t)) = \dfrac{100000u^2}{u^2+1}, \ t\in[0,1], \\ u''(0)=u'''(0)=0, u'''(1)=0, \\ 3u(0)-4u'(0)=u'(\dfrac{1}{2}), \\ 3u(1)+3y'(1)=u'(\dfrac{1}{2}). \end{cases} \tag{3.5}$$

Then we get $K=30, M=\frac{16}{45\sqrt{\pi}}, L=\frac{47}{60}, I=0,45, z=\frac{3}{35}$ and $k^{\sigma-1}=k^{\frac{3}{2}}=0.125$.

If we take $p=7.10^{-7}, q=10$ and $r=130000$, then $0<p<q<\frac{q}{z}<r$ and all the conditions in Theorem 4 are fulfilled. Hence, by Theorem 4, the BVP (3.5) has at least three positive solutions u_1, u_2 and u_3 satisfying

$$\max_{t\in[0,1]} u_1(t) < p, \ \min_{t\in[0,1]} u_2(t) > q$$
$$\max_{t\in[0,1]} u_3(t) > p \text{ with } \min_{t\in[0,1]} u_3(t) < q.$$

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NUMERICAL SOLUTION OF THE MULTI-TERM VARIABLE-ORDER SPACE FRACTIONAL NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. A numerical approach for solving the multi-term variable-order space fractional nonlinear partial differential equations is proposed. The fractional derivatives are described in the Caputo sense. The numerical approach is based on generalized Laguerre polynomials and finite difference method. The proposed scheme transforms the main problem to a system of nonlinear algebraic equations. The nonlinear system is solved by using Newton's method. The validity and the applicability of the proposed technique are shown by numerical examples.

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Keywords: nonlinear multi-term fractional partial differential equation, the variable-order Caputo fractional derivative, finite difference method, Newton's method, generalized Laguerre polynomials

1. INTRODUCTION

Fractional differential equations play a significant role in various fields of science and engineering such as fluid mechanics, mechanics of materials, biology, plasma physics, finance, chemistry, image processing (see, for example, [1–4,14,18–20,26]). Variable-order fractional calculus is an extension of the constant-order fractional calculus. In variable-order fractional calculus, the orders of fractional integrals and derivatives are functions in time and space variables. Variable-order fractional derivatives have important applications in various fields of science and engineering such as physics, mechanics [6], viscoelasticity oscillators [21], signal processing [24] and optimization control [27]. Since variable-order fractional nonlinear partial differential equations are much more complicated than other fractional differential equations and have an important place in the application areas, researchers have recently started to work on the solution of such equations.

The solutions of the nonlinear variable-order time fractional partial differential equations have been investigated in [8, 11]. An optimization method based on the

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generalized polynomials has been proposed for solving the variable-order time fractional Burgers'equation [8]. Stable three-level explicit scheme for a class of nonlinear variable-order time fractional partial differential equations has been studied [15]. Solving two-dimensional variable-order time fractional optimal control problems, a numerical approach has been introduced based on generalized polynomials and the Lagrange multipliers method [16]. Variable-order time fractional nonlinear cable equation and Klein Gordon equation have been solved by using the Chebyshev cardinal operational matrix method and nonstandard finite difference method, respectively [12, 25]. The moving least squares meshless approach has been applied to the two dimensional variable-order time fractional nonlinear diffusion-wave equation [22]. Variable-order time fractional nonlinear partial differential equations have been solved by using collocation method based on the Legendre - Laguerre polynomials and an optimization method based on the generalized shifted Chebyshev polynomials, respectively [7, 9]. The solution of the nonlinear variable-order time fractional two-dimensional Klein-Gordon equation has obtained by using a semi-discrete method based on the 2D Legendre wavelets [11]. Variable-order space-time fractional nonlinear diffusion-wave equation has been solved based on the Chebyshev cardinal functions coupling with the tau and collocation methods [10].

In this paper, we consider multi-term nonlinear variable-order space fractional partial differential equation of the form:

$$\frac{\partial u(x,t)}{\partial t} = \sum_{k=1}^{r} h_k(x,t).{}^{C}D_x^{\alpha_k(x,t)}u(x,t) + N(u(x,t)),\ 0<x<L,\ 0<t\leq T, \tag{1.1}$$

with the initial condition

$$u(x,0) = f(x),\ 0<x<L, \tag{1.2}$$

and the boundary conditions:

$$\begin{aligned} u(b_j,t) = d_j(t),\ b_0 = 0 < b_1 < ... < b_{n_r-2} < b_{n_r-1} = L, \\ j = 0,1,...,n_r-1,\ 0<t\leq T, \end{aligned} \tag{1.3}$$

where x is a space variable, t is a time variable; the parameters $\alpha_k(x,t), k = 1,...,r$, refer to the order of the Caputo variable-order fractional derivative with respect to the space variable. $0<\alpha_1(x,t)<...<\alpha_r(x,t)$, $n_i-1<\alpha_i(x,t)\leq n_i, n_i\in\mathbb{N}$, $i=1,2,...,r$, $h_k(x)$, $k=1,2,...,r$, $f(x)$ and $d_j(t)$, $j=0,1,...,n_r-1$ are known continuous functions. $N(u(x,t))$ represents the nonlinear term.

In the present paper, we obtain an approximate solution of the initial and boundary value problem (1.1)-(1.3) in terms of the Laguerre polynomials. Firstly, unknown function is written in terms of the Laguerre polynomials. Substituting Laguerre polynomial solution into equation (1.1), system of ordinary differential equations for appropriate collocation points is obtained. The finite difference method and Newton's method are applied to the obtained system, respectively. This paper is structured as follows: in Section 2, some basic preliminaries of variable-order Caputo fractional

derivative and Laguerre polynomials are presented. Spectral method for solving the initial and boundary value problem (1.1)-(1.3) is presented in Section 3. The efficiency and validity of the proposed method is demonstrated by some numerical examples in Section 4. Finally, conclusion is given in Section 5.

2. Preliminaries and notations

2.1. *The Variable-Order Caputo Fractional Derivative*

Definition 1. The Caputo variable-order fractional differential operator is given by (see, for example, [5, 23])

$$ {}^{C}D_{x}^{\alpha(x,t)}u(x,t) = \frac{1}{\Gamma(n-\alpha(x,t))}\int_{0}^{t}(t-\tau)^{n-\alpha(x,t)-1}\frac{\partial^{n}u(x,\tau)}{\partial\tau^{n}}d\tau. $$

where $n-1<\alpha(x,t)<n$ and $\Gamma(.)$ is a Gamma function.

Using the above definition, the fractional derivative of a polynomial $x^{m}, m\in\mathbb{N}$ can be obtained as follows:

$$ {}^{C}D_{x}^{\alpha(x,t)}x^{m} = \begin{cases} \frac{\Gamma(m+1)}{\Gamma(m-\alpha(x,t)+1)}x^{m-\alpha(x,t)}, & m\geq n \\ 0, & m<n. \end{cases} \tag{2.1} $$

2.2. *Some Properties of the Generalized Laguerre Polynomials*

The generalized Laguerre polynomials $L_{n}^{a}(x)$, $(a>-1)$ are orthogonal polynomials of degree n in x defined on the interval the $(0,\infty)$. The orthogonality relation is given as follows [13]:

$$ <L_{n}^{a}(x),L_{m}^{a}(x)> = \frac{1}{\Gamma(a+1)}\int_{0}^{\infty}x^{a}e^{-x}L_{n}^{a}(x)L_{m}^{a}(x)dx = \begin{pmatrix} n+a \\ n \end{pmatrix}\delta_{nm}. $$

The polynomials $L_{n}^{a}(x)$ may be generated by using the recurrence relations

$$ L_{n+1}^{a}(x) = \frac{1}{n+1}[(2n+a+1-x)L_{n}^{a}(x)-(n+a)L_{n-1}^{a}(x)],\ \ n=1,2,3,... $$

with $L_{0}^{a}(x)(x)=1$, $L_{1}^{a}(x)(x)=a+1-x$.

The analytical form of the generalized Laguerre polynomials $L_{n}^{a}(x)$ is given by

$$ L_{n}^{a}(x) = \sum_{k=0}^{n}(-1)^{k}\frac{\Gamma(n+a+1)x^{k}}{(n-k)!k!\Gamma(k+a+1)}. \tag{2.2} $$

The function $y(x)$ which belongs to the space of square integrable function in $[0,\infty)$ may be expressed in terms of the generalized Laguerre polynomials as

$$ y(x) = \sum_{i=0}^{\infty}c_{i}L_{i}^{a}(x). $$

$$c_i = \frac{\Gamma(i+1)}{\Gamma(i+a+1)} \int_0^\infty y(x) x^a e^{-x} L_i^a(x) dx. \tag{2.3}$$

Consider only the first $(m+1)-$ terms of the generalized Laguerre polynomials, so we can write

$$y_m(x) = \sum_{i=0}^{m} c_i L_i^a(x). \tag{2.4}$$

Theorem 1. *Let $y_m(x)$ be approximated function in terms of the generalized Laguerre polynomials $L_i^a(x)$ given by equation (2.4), then we have*

$${}^C D_x^{\alpha(x,t)} y_m(x) = \sum_{i=n}^{m} \sum_{k=n}^{i} c_i A_{i,k}^{\alpha} x^{k-\alpha(x,t)}, \quad n-1 < \alpha(x,t) \leq n, \tag{2.5}$$

where

$$A_{i,k}^{\alpha} = (-1)^k \frac{\Gamma(i+a+1)}{\Gamma(k+1-\alpha)\Gamma(i-k+1)\Gamma(a+k+1)}. \tag{2.6}$$

3. NUMERICAL SCHEME

In this section, we apply Laguerre collocation method to the problem (1.1)-(1.3). In the Laguerre collocation method, the unknown function is written as a finite series expansion via generalized Laguerre polynomials. Then, the finite difference method is applied to the obtained system of ordinary differential equation. Finally, the nonlinear system is solved by Newton's method and the unknown coefficients of the finite series are obtained.

Let $u_m(x,t)$ be the truncated Laguerre approximation of a function $u(x,t)$ as follows:

$$u_m(x,t) = \sum_{i=0}^{m} c_i(t) L_i^a(x). \tag{3.1}$$

Using equations (1.1) and (2.5), we obtain

$$\sum_{i=0}^{m} \frac{dc_i(t)}{dt} L_i^a(x) = \sum_{j=1}^{r} h_j(x,t) \sum_{i=n_j}^{m} \sum_{k=n_j}^{i} c_i(t) A_{i,k}^{\alpha_j} x^{k-\alpha_j(x,t)} + N(\sum_{i=0}^{m} c_i(t) L_i^a(x)). \tag{3.2}$$

Now, we collocate (3.2) at $m-n_r+1$ points x_p as follows:

$$\sum_{i=0}^{m} \frac{dc_i(t)}{dt} L_i^a(x_p) = \sum_{j=1}^{r} h_j(x_p,t) \sum_{i=n_j}^{m} \sum_{k=n_j}^{i} c_i(t) A_{i,k}^{\alpha_j} x_p^{k-\alpha_j(x_p,t)} + N(\sum_{i=0}^{m} c_i(t) L_i^a(x_p)), \tag{3.3}$$

where collocation points x_p are the roots of shifted Chebyshev polynomials of the second kinds $U_{m-1}^*(x)$.

For a positive integer N, $\Delta t = \frac{T}{N}$ denotes the step size of time variable t. So we define $t_q = q\Delta t$, $0 \leq t_q \leq T$, in which $q = 0,1,...,N$. And we introduce the

following notations: $u(b_j,t_q) = d_j^q, j = 0,1,...,n_r-1,\ c_i(t_q) = c_i^q\ ,i = 0,1,...,m,$ $V^q = \left(c_0^q, c_1^q, ..., c_m^q\right)^*$ and $*$ is the sign of transposition of the matrix.

Substituting equation (3.1) into the boundary condition (1.3) , we have

$$u_m(b_j,t) = \sum_{i=0}^{m} c_i(t) L_i^a(b_j) = d_j(t),\ \ j = 0,1,..,n_r-1. \tag{3.4}$$

By using equations (2.3) and (3.1) for initial condition (1.2), we can compute the unknown constants $c_i(t_0), i = 0,1,...,m$, i.e. V^0.

Applying the finite difference method to the system (3.3), we have

$$\begin{aligned}\sum_{i=0}^{m}(c_i^q - c_i^{q-1}) L_i^a(x_p) = \Delta t.\sum_{j=1}^{r} h_j(x_p,t_q) \sum_{i=n_j}^{m}\sum_{k=n_j}^{i} c_i(t_q) A_{i,k}^{\alpha_j} x_p^{k-\alpha_j(x_p,t_q)} \\ + \Delta t.N(\sum_{i=0}^{m} c_i(t_q) L_i^a(x_p)),\ \ q = 1,2,...,N.\end{aligned} \tag{3.5}$$

Combining equations (3.4) and (3.5), we have a system of non-linear equations as follows

$$\mathbf{F} = \mathbf{0}, \tag{3.6}$$

where $\mathbf{0}$ is zero vector with $m+1$ components and $\mathbf{F}$ is a vector whose components are given by

$$\begin{aligned}\mathbf{F}_p &= \sum_{i=0}^{m}(c_i^q - c_i^{q-1}) L_i^a(x_p) = \Delta t.\sum_{j=1}^{r} h_j(x_p,t_q) \sum_{i=n_j}^{m}\sum_{k=n_j}^{i} c_i(t_q) A_{i,k}^{\alpha_j} x_p^{k-\alpha_j(x_p,t_q)} \\ &\quad + \Delta t.N(\sum_{i=0}^{m} c_i(t_q) L_i^a(x_p)),\ \ p = 1,2,...,m-n_r+1, \\ \mathbf{F}_{m-n_r+2} &= \sum_{i=0}^{m} c_i(t_q) L_i^a(b_0) - d_0(t_q), \\ \mathbf{F}_{m-n_r+3} &= \sum_{i=0}^{m} c_i(t_q) L_i^a(b_1) - d_1(t_q), \\ &\ldots \\ \mathbf{F}_{m+1} &= \sum_{i=0}^{m} c_i(t_q) L_i^a(b_{n_r-1}) - d_{n_r-1}(t_q).\end{aligned} \tag{3.7, 3.8}$$

For the non-linear system (3.6) with unknown V^q $(q = 1,...,N)$, the following iteration formula can be written by using Newton iteration method

$$\begin{aligned}V^{q,k+1} &= V^{q,k} - J^{-1}(V^{q,k}).\mathbf{F}(V^{q,k}), \\ V^{q,0} &= V^{q-1},\ \ q = 1,...,N,\ \ k = 0,1,...\end{aligned} \tag{3.9}$$

where $J^{-1}(V^{q,k})$ is the inverse of the Jacobian matrix, $V^{q,k}$ is approximate solution of V^q. Using iteration formula (3.9), approximate solution of the problem (1.1)-(1.3) is obtained.

4. Applications

Example 1. Consider nonlinear space fractional partial differential equation with initial and boundary conditions

$$\begin{aligned} u_t(x,t) &= x^{\alpha(x,t)}\Gamma(5-\alpha(x,t))\,{}^{C}D_x^{\alpha(x,t)}u(x,t) - x^{\beta(x,t)}\Gamma(5-\beta(x,t)) \\ &\quad {}^{C}D_x^{\beta(x,t)}u(x,t) + u^3(x,t) - (x^4-x^2+1)\sin(t)u^2(x,t) + \cos(t)(x^4-x^2+1) \\ &\quad - \frac{\Gamma(5-\beta(x,t))}{\Gamma(3-\beta(x,t))}2x^2\sin(t),\ 0<x<2,\ 0<t\le 1, \end{aligned} \tag{4.1}$$

$$u(x,0) = 0,\ 0<x<2, \tag{4.2}$$

$$u(0,t) = \sin(t),\ u(1,t) = \sin(t),\ u(2,t) = 13\sin(t),\ 0<t\le 1, \tag{4.3}$$

where $\alpha(x,t) = \frac{6+\sin(x+t)}{3}$, $\beta(x,t) = \frac{4+\sin(x+t)}{3}$.

The exact solution is $u_e(x,t) = \sin(t)(x^4-x^2+1)$. Assume that the solution of the problem (4.1)-(4.3) can be written as follows

$$u_4(x,t) = \sum_{i=0}^{4} c_i(t)L_i^0(x). \tag{4.4}$$

Substituting equation (4.4) into the boundary conditions (4.3) at the points t_q, we have

$$\begin{aligned} \mathbf{F}_3 &= c_0(t_q) + c_1(t_q) + c_2(t_q) + c_3(t_q) + c_4(t_q) - \sin(t_q), \\ \mathbf{F}_4 &= c_0(t_q) - 1/2c_2(t_q) - 2/3c_3(t_q) - 5/8c_4(t_q) - \sin(t_q), \\ \mathbf{F}_5 &= c_0(t_q) - c_1(t_q) - c_2(t_q) - 1/3c_3(t_q) + 1/3c_4(t_q) - 13\sin(t_q). \end{aligned}$$

For $m=4$, collocation points are taken as $x_1 = 0.25,\ x_2 = 0.75$. $\mathbf{F}_1$ and $\mathbf{F}_2$ can be written from (3.7) for the points $x_1 = 0.25,\ x_2 = 0.75$. By using the iteration formula (3.9) with the initial data $V^0 = (0,0,0,0,0)^*$, the absolute errors for the approximate solution $u_4(x,t)$ are computed for $N = 10^3; 10^4; 10^5; 10^6; 10^7$ at $t = 0.25; 0.75$. Table 1 shows that the absolute error goes to the zero as the value N increases. Figure 1 shows 3D plot of the absolute error for the approximate solution $u_4(x,t)$ for $N = 10^6$, $0<x<2$ and $0<t<1$.

Example 2. Consider nonlinear space fractional partial differential equation with initial and boundary conditions

$$\begin{aligned} u_t(x,t) &= x^{\alpha(x,t)}\Gamma(6-\alpha(x,t))\,{}^{C}D_x^{\alpha(x,t)}u(x,t) - x^{\beta(x,t)}\Gamma(6-\beta(x,t))\,{}^{C}D_x^{\beta(x,t)}u(x,t) \\ &\quad + (x^5-2)(t^2-t)u(x,t) - u^2(x,t) + (x^5-2)(2t-1),\ 0<x<5,\ 0<t\le 1, \end{aligned}$$

TABLE 1. The absolute errors for $u_4(x,t)$, $N = 10^3;\ 10^4;10^5;\ 10^6;\ 10^7$ at $t = 0.25;0.75$ in Example 1.

x	$N=10^3$		$N=10^4$		$N=10^5$		$N=10^6$		$N=10^7$	
	$t=0.25$	$t=0.75$	$t=0.25$	$t=0.75$	$t=0.25$	$t=0.75$	$t=0.25$	$t=0.75$	$t=0.25$	$t=0.75$
0.2	10^{-5}	6.10^{-5}	10^{-6}	6.10^{-6}	10^{-7}	6.10^{-7}	10^{-8}	6.10^{-8}	10^{-9}	6.10^{-9}
0.4	2.10^{-5}	10^{-4}	2.10^{-6}	10^{-5}	2.10^{-7}	10^{-6}	2.10^{-8}	10^{-7}	2.10^{-9}	10^{-8}
0.6	2.10^{-5}	10^{-4}	2.10^{-6}	10^{-5}	2.10^{-7}	10^{-6}	2.10^{-8}	10^{-7}	2.10^{-9}	10^{-8}
0.8	2.10^{-5}	10^{-4}	2.10^{-6}	10^{-5}	2.10^{-7}	10^{-6}	2.10^{-8}	10^{-7}	2.10^{-9}	10^{-8}
1.2	2.10^{-5}	10^{-4}	2.10^{-6}	10^{-5}	2.10^{-7}	10^{-6}	2.10^{-8}	2.10^{-7}	2.10^{-9}	10^{-8}
1.4	4.10^{-5}	3.10^{-4}	4.10^{-6}	3.10^{-5}	4.10^{-7}	3.10^{-6}	4.10^{-8}	3.10^{-7}	4.10^{-9}	3.10^{-8}
1.6	5.10^{-5}	4.10^{-4}	5.10^{-6}	4.10^{-5}	5.10^{-7}	4.10^{-6}	5.10^{-8}	4.10^{-7}	5.10^{-9}	4.10^{-8}
1.8	4.10^{-5}	3.10^{-4}	4.10^{-6}	3.10^{-5}	4.10^{-7}	3.10^{-6}	4.10^{-8}	3.10^{-7}	4.10^{-9}	3.10^{-8}

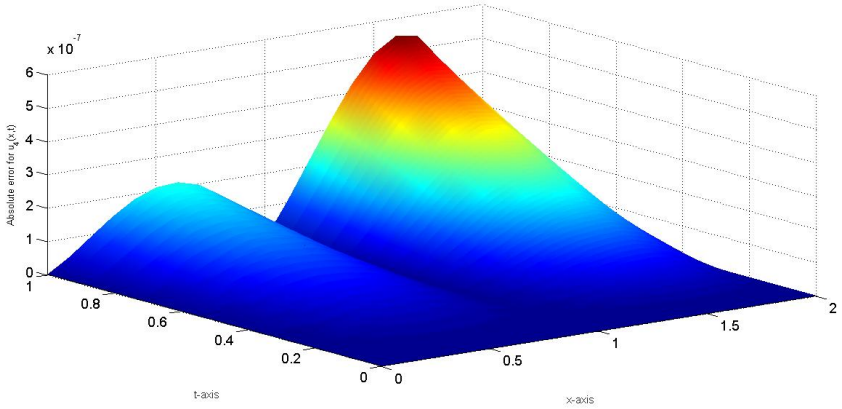

FIGURE 1. The behavior of the absolute error for the approximate solution $u_4(x,t)$ for $N = 10^6$, $0 < x < 2$ and $0 < t < 1$ in Example 1.

$$u(x,0) = 0,\ 0 < x < 5, \quad u(0,t) = 2t - 2t^2,\ u(5,t) = 3123t^2 - 3123t,\ 0 < t \le 1,$$

where $\alpha(x,t) = \frac{5+\sin(x+t)}{3}$, $\beta(x,t) = \frac{1+\sin(x+t)}{4}$.

The exact solution is $u_e(x,t) = (t^2-t)(x^5-2)$. By using the iteration formula (3.9) for $m = 5,\ a = 0$ with the initial data $V^0 = (0,0,0,0,0,0)^*$, the absolute errors for the approximate solution $u_5(x,t)$ are computed for $N = 10^4;10^5;10^6;10^7$ at $t = 0.5$. The influence of the value N on the absolute error is presented in Table 2. Table 2 shows that the value of the absolute error goes to zero as the value N increases. Figure 2 shows 3D plot of the approximate solution $u_5(x,t)$ for $N = 10^4$, $3 < x < 5$ and $0 < t < 1$.

TABLE 2. The absolute errors for $u_5(x,0.5)$ in Example 2.

x	$N=10^4$	$N=10^5$	$N=10^6$	$N=10^7$
0.4	3.10^{-6}	3.10^{-7}	4.10^{-8}	4.10^{-9}
1.2	10^{-4}	10^{-5}	10^{-6}	10^{-7}
2	10^{-3}	10^{-4}	10^{-5}	10^{-6}
2.4	2.10^{-3}	2.10^{-4}	2.10^{-5}	2.10^{-6}
3.2	5.10^{-3}	5.10^{-4}	5.10^{-5}	5.10^{-6}
3.6	7.10^{-3}	7.10^{-4}	7.10^{-5}	7.10^{-6}
4	7.10^{-3}	7.10^{-4}	7.10^{-5}	7.10^{-6}
4.4	7.10^{-3}	7.10^{-4}	7.10^{-5}	7.10^{-6}
4.8	3.10^{-3}	3.10^{-4}	3.10^{-5}	3.10^{-6}

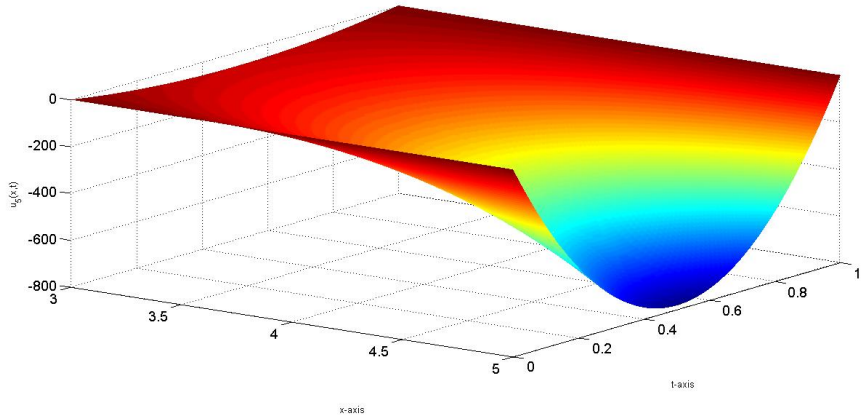

FIGURE 2. The behavior of the approximate solution $u_5(x,t)$ for $N=10^4$, $3<x<5$ and $0<t<1$ in Example 2.

Example 3. Let us consider the following nonlinear space fractional partial differential equation with initial and boundary conditions

$$\begin{aligned} u_t(x,t) &= {}^C D_x^{\alpha(x,t)} u(x,t) + u^2(x,t) - \exp(x)t^2 u(x,t) \\ &\quad + 2t\exp(x) - t^2 x^{2-\alpha(x,t)} E_{1,3-\alpha(x,t)}(x),\ 0<x<1,\ 0<t<1, \\ u(x,0) &= 0,\ 0<x<1, \quad u(0,t)=t^2,\ u(1,t)=\exp(1)t^2,\ 0<t\le 1, \end{aligned}$$

where $\alpha(x,t)=\frac{4+\sin(x+t)}{3}$ and $E_{1,3-\alpha(x,t)}(x)$ is the Mittag-Leffler function (see, for example, [17]). Note that the exact solution to this problem is $u_e(x,t)=\exp(x)t^2$. By using the iteration formula (3.9) with zero initial condition , we obtain approximate solution of the problem for $m=3,\ a=0;\ 4;\ 5$. In Table 3, the absolute errors

between the exact solution $u_e(x,t)$ and the approximate solution $u_m(x,t)$, $m = 3,4,5$, at $t = 0.5$ for the values $N = 10^6$ are given. Table 3 shows how the change of the values m effects the obtained results. As shown in Table 3, as the values m increase, the approximate solution approaches to the exact solution. Figure 3 presents 3D plot of the absolute error for the approximate solution $u_5(x,t)$ for $N = 10^5$, $0 < x < 1$ and $0 < t < 1$.

TABLE 3. The absolute errors for $u_m(x,t)$, $m = 3;4;5;10^6$ at $t = 0.2;0.8$ in Example 3.

x	$m=3$		$m=4$		$m=5$	
	$t=0.2$	$t=0.8$	$t=0.2$	$t=0.8$	$t=0.2$	$t=0.8$
0.2	3.10^{-5}	6.10^{-4}	2.10^{-6}	3.10^{-5}	7.10^{-8}	7.10^{-7}
0.4	6.10^{-5}	10^{-3}	10^{-6}	2.10^{-6}	3.10^{-7}	2.10^{-6}
0.6	4.10^{-5}	5.10^{-4}	2.10^{-6}	5.10^{-5}	2.10^{-7}	6.10^{-8}
0.8	3.10^{-5}	6.10^{-4}	2.10^{-7}	10^{-5}	5.10^{-8}	2.10^{-6}

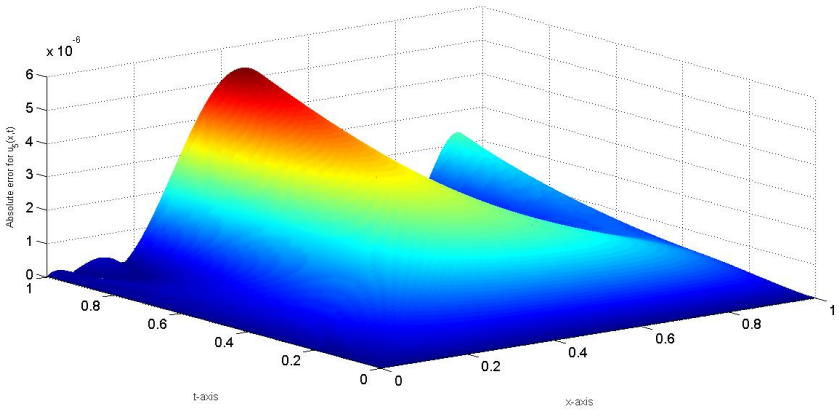

FIGURE 3. The behavior of the absolute value of the approximate solution $u_5(x,t)$ for $N = 10^5$, $0 < x < 1$ and $0 < t < 1$ in Example 3.

Example 4. Let us consider the following nonlinear space fractional partial differential equation with initial and boundary conditions

$$u_t(x,t) = \exp(-t)x\cdot {}^{C}D_x^{\alpha(x,t)}u(x,t) + u^3(x,t) - \exp(t)\cos(t)(x^3-x^2)u^2(x,t) + u(x,t) + \cos(t)x^{3-\alpha(x,t)}\frac{2}{\Gamma(3-\alpha(x,t))} - \cos(t)x^{4-\alpha(x,t)}\frac{6}{\Gamma(4-\alpha(x,t))}$$

$$-\exp(t)\sin(t)(x^3-x^2),\ 0<x<1,\ 0<t<1,$$
$$u(x,0)=x^3-x^2,\ 0<x<1,\quad u(0,t)=0,\ u(1,t)=0,\ 0<t\leq 1,$$

where $\alpha(x,t)=\frac{5+\sin(xt)}{4}$. Note that the exact solution to this problem is $u_e(x,t)=\exp(t)\cos(t)(x^3-x^2)$. By using the iteration formula (3.9) for $m=3,\ a=0$ with the initial data $V^0=(4,-14,16,-6)^*$, the absolute errors for the approximate solution $u_3(x,t)$ are computed for $N=10^3;10^4;10^5;10^6$ at $t=0.2;0.9$. The influence of the value N on the absolute error is presented in Table 4. Table 4 shows that the value of the absolute error goes to zero as the value N increases. Figure 4 shows 3D plot of the approximate solution $u_3(x,t)$ for $N=10^4$, $0<x<1$ and $0<t<1$.

TABLE 4. The absolute errors for $u_3(x,t)$, $N=10^3;10^4;10^5;10^6$ at $t=0.2;0.9$ in Example 4.

x	$N=10^3$		$N=10^4$		$N=10^5$		$N=10^6$	
	$t=0.2$	$t=0.9$	$t=0.2$	$t=0.9$	$t=0.2$	$t=0.9$	$t=0.2$	$t=0.9$
0	9.10^{-16}	2.10^{-15}	9.10^{-16}	4.10^{-15}	3.10^{-15}	0	9.10^{-16}	4.10^{-15}
0.2	9.10^{-7}	4.10^{-5}	9.10^{-8}	4.10^{-6}	9.10^{-9}	3.10^{-7}	9.10^{-10}	4.10^{-8}
0.4	2.10^{-6}	7.10^{-5}	2.10^{-7}	7.10^{-6}	2.10^{-8}	6.10^{-7}	2.10^{-9}	7.10^{-8}
0.6	3.10^{-6}	8.10^{-5}	3.10^{-7}	8.10^{-6}	3.10^{-8}	8.10^{-7}	3.10^{-9}	8.10^{-8}
0.8	3.10^{-6}	6.10^{-5}	3.10^{-7}	6.10^{-6}	3.10^{-8}	6.10^{-7}	3.10^{-9}	6.10^{-8}
1	9.10^{-16}	0	9.10^{-16}	9.10^{-16}	9.10^{-16}	0	0	2.10^{-15}

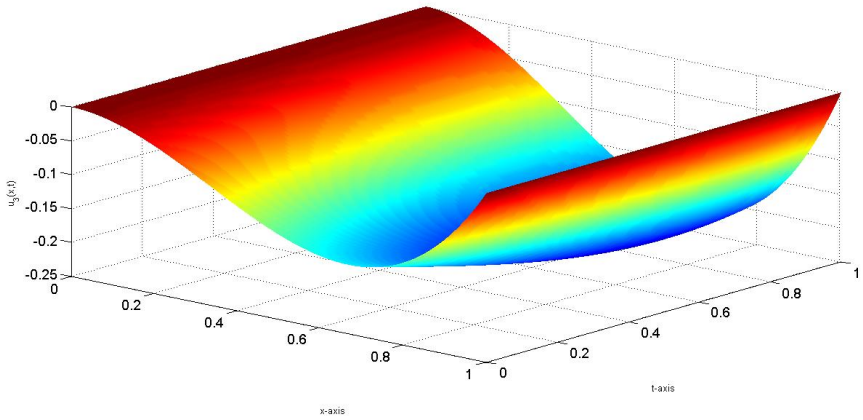

FIGURE 4. The behavior of the approximate solution $u_3(x,t)$ for $N=10^4$, $0<x<1$ and $0<t<1$ in Example 4.

5. CONCLUSION

In this paper, numerical solutions of the multi-term variable-order space fractional nonlinear partial differential equations with initial and boundary conditions have been obtained. The space variable order fractional derivative is defined in the Caputo sense. Spectral Laguerre collocation method and finite difference method have been applied to the problem and the problem has been transformed into an algebraic system of the nonlinear equations. The nonlinear system has been solved by using Newton iteration. Some numerical examples have been given to clarify the validity and accuracy of the proposed technique. The obtained numerical results are compared with exact solutions. The proposed scheme leads to results that are in excellent agreement with the exact solutions and show that the algorithm is efficient. All numerical computations were performed in MATLAB.

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SOME SPECIAL VECTOR FIELDS ON A COSYMPLECTIC MANIFOLD ADMITTING A RICCI SOLITON

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Abstract. In this paper, we deal with the geometric properties of cosymplectic manifold. We give some classifications for a cosymplectic manifold endowed with some special vector fields, such as contact, concircular, recurrent, torse-forming and some characterizations for such a manifold admitting a Ricci soliton given as to be Einstein, $\eta-$Einstein, almost quasi-Einstein and nearly quasi-Einstein.

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1. INTRODUCTION

The notion of Ricci soliton, which is a natural generalization of Einstein manifold, was appeared by Hamilton in 1988 [14]. This notion is a fixed point of Hamilton's Ricci flow defined by

$$\frac{\partial}{\partial t}g = -2S,$$

viewed as a dynamical system on the space of Riemannian metrics modulo diffeomorphims and scaling. Also, it models the formation of singularities in the Ricci flow.

In the framework of the contact geometry, Ricci solitons have been studied many mathematicians in some different classes of contact geometry since Sharma applied Ricci solitons to K-contact manifolds [17]. For example, Ghosh proved that if a Kenmotsu 3-manifold is a Ricci soliton then, its constant curvature is -1 in [9]. Turan, De and Yildiz showed that if a 3-dimensional trans-Sasakian manifold with constant scalar curvature admits a gradient Ricci soliton, then it is either a β-Kenmotsu or an Einstein manifold in [18]. Later, Blaga obtained that there is no Ricci soliton on a Lorentzian para-Sasakian manifold with the potential vector field ξ satisfying conditions $R(\xi,X).S = 0$ and $S.R(\xi,X) = 0$ in [1]. For the recent works on Ricci solitons, we refer to [6, 10–12, 15, 16, 19, 21, 22].

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A Riemannian manifold (M,g) is called a Ricci soliton if the following condition is satisfied for arbitrary vector fields X,Y on M

$$(\pounds_V g)(X,Y)+2S(X,Y)+2\lambda g(X,Y)=0, \tag{1.1}$$

where $\pounds_V g$ denotes the Lie-derivative of the metric tensor g along vector field V, S is the Ricci tensor of M and λ is a constant. A Ricci soliton is denoted by (M,g,V,λ). The vector field V is called the potential vector field of Ricci soliton. If $\pounds_V g=0$ and $\pounds_V g=\rho g$, then potential vector field V is said to be Killing and conformal Killing, respectively, where ρ is a function. Also, when V is zero or Killing in (1.1), then Ricci soliton reduces to Einstein manifold. Therefore, it is considered as a natural generalization of Einstein metric. In addition, a Ricci soliton is called a gradient if the potential vector field V is the gradient of a potential function $-f$ (i.e., $V=-\nabla f$) and is called shrinking, steady or expanding depending on $\lambda<0,\lambda=0$ or $\lambda>0$, respectively.

On the other hand, vector fields have been used for studying diferential geometry of manifolds since they determine most geometric properties of the related object. Also, they have many rich properties and play an important role in the study of Riemannian geometry. That is why, geometric vector fields have been studied extensively by many mathematicians.

Motivated by these circumstances, we study cosymplectic manifolds admitting Ricci solitons with some special geometric vector fields such as concurrent, concircular, recurrent, torse-forming and torqued.

The present paper is organized as follows. In section 2, we give some basic definitions, notations and formulas about almost contact metric manifolds. In section 3, we deal with cosymplectic manifold endowed with some special vector fields and give some relationships between these vector fields. In last section, we investigate cosymplectic manifolds admitting Ricci solitons with concurrent, concircular, recurrent and torse-forming vector field, respectively. Finally, we obtain an important result about cosymplectic manifold which is a gradient Ricci soliton.

2. PRELIMINARIES

In this section, we shall review some fundamental definitions and formulas of almost contact metric manifolds from [2], [13] and [20].

A $(2n+1)$-dimensional differentiable manifold M is called an almost contact metric manifold if there exists an almost contact metric structure (φ,ξ,η,g) on M and the Riemannian metric g satisfies the following relations:

$$\varphi^2X=-X+\eta(X)\xi,\quad \eta(\xi)=1,\quad \varphi\xi=0,\quad \eta\circ\varphi=0,\quad \eta(X)=g(X,\xi) \tag{2.1}$$

and

$$g(\varphi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),\qquad g(\varphi X,Y)=-g(X,\varphi Y) \tag{2.2}$$

for any $X,Y \in \Gamma(TM)$, where ξ is a vector field of type $(0,1)$, (which is so-called the characteristic vector field), 1-form η is the g-dual of ξ of type $(1,0)$ and φ is a tensor field of type $(1,1)$ on M.

On the other hand, in [2], D.E. Blair defined the fundamental 2-form of M as follows:

$$\Phi(X,Y) = g(X,\varphi Y)$$

for any $X,Y \in \Gamma(TM)$. Moreover, an almost contact metric structure (φ,ξ,η,g) on M becomes contact metric structure if the relation

$$\Phi(X,Y) = d\eta(X,Y)$$

holds for all $X,Y \in \Gamma(TM)$, where

$$d\eta(X,Y) = \frac{1}{2}\Big\{X\eta(Y) - Y\eta(X) - \eta([X,Y])\Big\}.$$

The Nijenhuis tensor field of φ is defined by

$$N_\varphi(X,Y) = [\varphi X,\varphi Y] + \varphi^2[X,Y] - \varphi[X,\varphi Y] - \varphi[\varphi X,Y]$$

for all $X,Y \in \Gamma(TM)$. If M is an almost contact metric manifold and the Nijenhuis tensor of φ satisfies

$$N_\varphi + 2d\eta \otimes \xi = 0$$

then, M is called a normal contact metric manifold. A normal contact metric manifold (M,φ,ξ,η,g) is said to be cosymplectic if the following relations hold:

$$d\eta = 0, \qquad\qquad d\Phi = 0.$$

Equivalently,

$$(\nabla_X \varphi)Y = 0 \tag{2.3}$$

for any $X,Y \in \Gamma(TM)$, where ∇ is the Levi-Civita connection on M. For a cosymplectic manifold, we also have

$$\nabla_X \xi = 0, \tag{2.4}$$

$$S(X,\xi) = 0, \tag{2.5}$$

where S stands for the Ricci tensor of M.

Now, we recall some definitions from [3], [4], [7], [8] and [11] and as follows:

A Riemannian manifold (M,g) is called almost quasi-Einstein and nearly quasi-Einstein manifold, respectively if its Ricci tensor field S satisfies

$$S = cg + d(\beta \otimes \gamma + \gamma \otimes \beta)$$

and

$$S = ag + bE$$

where a,b,c,d are functions, β,γ are 1-forms and E is a non-vanishing symmetric $(0,2)$-tensor on M.

Also, a Riemannian manifold (M,g) is called η-Einstein if there exists two real constants a and b such that the Ricci tensor field S satisfies

$$S = ag + b\eta \otimes \eta.$$

If the constant b is equal to zero, then M becomes Einstein.

On the other side, a vector field V on a contact manifold M is called a contact vector field if it preserves the contact form η, that is,

$$\pounds_V \eta = \mu\eta \tag{2.6}$$

for some smooth function μ on M. If μ vanishes identically in (2.6), then V is called a strict contact vector field. Also, a vector field V on a Riemannian manifold M is called a $\varphi-$analytic if it satisfies

$$(\pounds_V \varphi)(X) = 0$$

for any $X \in \Gamma(TM)$.

A vector field v on a Riemannian manifold (M,g) is called torse-forming if it satisfies the following condition

$$\nabla_X v = fX + \phi(X)v \tag{2.7}$$

for any $X \in \Gamma(TM)$, where ∇ is the Levi-Civita connection on M, ϕ is a 1-form and f is a smooth function on M. If the 1-form ϕ vanishes identically in (2.7), the vector field v is called a concircular [3]. If $\phi = 0$ and $f = 1$ in (2.7), then v is called a concurrent vector field [5]. Also, the vector field v is called a recurrent if it satisfies (2.7) with $f = 0$.

A nowhere zero vector field $\mathcal{T}$ on a Riemannian manifold (M,g) is called torqued vector field if it satisfies the following two conditions

$$\nabla_X \mathcal{T} = \rho X + \alpha(X)\mathcal{T} \qquad \text{and} \qquad \alpha(\mathcal{T}) = 0 \tag{2.8}$$

for any $X \in \Gamma(TM)$, where ∇ is the Levi-Civita connection on M. The function ρ is called the torqued function and 1-form α is called the torqued form of $\mathcal{T}$.

3. Cosymplectic Manifold Endowed with Some Special Vector Fields

In this section, we deal with a cosymplectic manifold endowed with a contact, concircular and recurrent vector field and obtain some relationships.

Now, we begin to this section with the following.

Proposition 1. *Let M be a cosymplectic manifold endowed with a concircular vector field v. Then, the vector field v is a φ-analytic on M.*

Proof. Since v is a concircular vector field, it follows from the definition of the Lie-derivative and from (2.3) we have

$$\begin{aligned}(\pounds_v\varphi)(X) &= \pounds_v\varphi(X) - \varphi(\pounds_v X) = [v,\varphi(X)] - \varphi([v,X]) \\ &= \nabla_v\varphi(X) - \nabla_{\varphi(X)}v - \varphi(\nabla_v X - \nabla_X v) = (\nabla_v\varphi)X - f\varphi(X) + \varphi(fX) = 0\end{aligned}$$

for any $X \in \Gamma(TM)$, which completes the proof. □

Next, we shall give the following theorem.

Theorem 1. *Let M be a cosymplectic manifold endowed with a vector field v. Then, the followings are equivalent to each other:*

(i) The vector field v is concircular.
(ii) The vector field v is contact.

Proof. By definition of the Lie-derivative and from (2.4), (2.7), for any $X \in \Gamma(TM)$ one has

$$\begin{aligned}(\pounds_v\eta)(X) &= \pounds_v\eta(X) - \eta(\pounds_v X) = v(\eta(X)) - \eta([v,X]) \\ &= g(\nabla_v X,\xi) + g(X,\nabla_v\xi) - \eta(\nabla_v X - \nabla_X v) \\ &= \eta(\nabla_v X) - \eta(\nabla_v X) + \eta(\nabla_X v) = \eta(\nabla_X v)\end{aligned} \tag{3.1}$$

Let us assume that v is a contact vector field. Then, using linearity of η and from (2.6), (3.1) we have

$$\eta(\nabla_X v) = \eta(fX) \tag{3.2}$$

for a smooth function f, which implies that v is a concircular vector field.
The converse of the theorem is straightforward. □

The next theorem gives the relation between strict contact vector field and concircular vector field.

Theorem 2. *Let M be a cosymplectic manifold endowed with a concircular vector field v. Then, the vector field $\varphi(v)$ is a strict contact on M.*

Proof. Since v is a concircular vector field on M, using the equalities (2.1), (2.3) and (2.4), we get

$$\begin{aligned}(\pounds_{\varphi(v)}\eta)(X) &= \pounds_{\varphi(v)}\eta(X) - \eta(\pounds_{\varphi(v)}X) = \varphi(v)(\eta(X)) - \eta([\varphi(v),X]) \\ &= g(\nabla_{\varphi(v)}X,\xi) + g(X,\nabla_{\varphi(v)}\xi) - \eta(\nabla_{\varphi(v)}X - \nabla_X\varphi(v)) \\ &= \eta(\nabla_{\varphi(v)}X) - \eta(\nabla_{\varphi(v)}X) + \eta(\nabla_X\varphi(v)) = \eta(\nabla_X\varphi(v)) = 0\end{aligned}$$

for any $X \in \Gamma(TM)$, which gives that the vector field $\varphi(v)$ is a strict contact on M and the proof is completed. □

Considering the above theorems, we have the following:

Remark 1. If M is a cosymplectic manifold endowed with a contact vector field v, then the vector field $\varphi(v)$ is also strict contact on M.

By applying the same method as given in the proof of Theorem 2, we also have the following result.

Theorem 3. *Let M be a cosymplectic manifold. Then, the characteristic vector field ξ is a strict contact on M.*

Using the relations (2.3), (2.4) and (2.7) we get the following propositions.

Proposition 2. *Let M be a cosymplectic manifold endowed with a concircular vector field v. Then, we have the followings:*

(1) If f vanishes identically, then ξ is also concircular on M.
(2) $\varphi(v)$ is never concircular on M.

Proposition 3. *Let M be a cosymplectic manifold endowed with a recurrent vector field v. Then, we have the followings:*

(1) The vector field $\varphi(v)$ is also recurrent on M.
(2) The vector field $\varphi(v)$ is a strict contact on M.

4. Cosymplectic Manifolds Admitting Ricci Solitons

In this section, we study a cosymplectic manifold admitting a Ricci soliton endowed with some special vector fields such as concurrent, recurrent, torse-forming and torqued vector field, respectively.

The first result of this section is the following.

Proposition 4. *Let M be a cosymplectic manifold admitting a Ricci soliton (M,g,v,λ) with the potential vector is a concurrent v. Then, the followings are satisfied:*

(i) v is a conformal Killing on M.
(ii) M is an Einstein.
(iii) The Ricci soliton (M,g,v,λ) is shrinking.

Proof. For any vector fields $X,Y \in \Gamma(TM)$, it follows from the definition of the Lie-derivative and using (2.7), we have

$$(\pounds_v g)(X,Y) = g(\nabla_X v,Y) + g(\nabla_Y v,X) = g(X,Y) + g(Y,X) = 2g(X,Y) \tag{4.1}$$

which implies that v is a conformal Killing vector field on M.

On the other hand, since M is a Ricci soliton and from (1.1), we get

$$(\pounds_v g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0. \tag{4.2}$$

Using the relations (4.1) and (4.2), the Ricci tensor of M,

$$S(X,Y) = -(\lambda+1)g(X,Y) \tag{4.3}$$

is found which implies (ii).

Furthermore, if we choose $X = Y = \xi$ in (4.3) and using (2.5), one has

$$\lambda = -1$$

which proves (iii). □

By applying the same method as given in the proof of Proposition 4, we also have the following result.

Proposition 5. *Let M be a cosymplectic manifold admitting a Ricci soliton $(M,g,\varphi(v),\lambda)$ such that v is a concurrent vector field on M. Then, the following properties are satisfied:*

(i) The vector field $\varphi(v)$ is Killing on M.
(ii) M is an Einstein.
(iii) The Ricci soliton $(M,g,\varphi(v),\lambda)$ is steady.

Proposition 6. *If (M,g,ξ,λ) is a cosymplectic manifold admitting a Ricci soliton, then we have the followings:*

(i) The characteristic vector field ξ is Killing on M.
(ii) M is an Einstein.
(iii) The Ricci soliton (M,g,ξ,λ) is steady.

The next two theorem presents a characterization for a cosymplectic manifold admitting a Ricci soliton to be an almost quasi-Einstein.

Theorem 4. *Let M be a cosymplectic manifold admitting a Ricci soliton. If the potential vector field v is a pointwise collinear with ξ, then (M,g,v,λ) is an almost quasi-Einstein manifold.*

Proof. Since that (M,g,v,λ) is a Ricci soliton whose potential vector field v is a pointwise collinear with ξ, that is, $v=b\xi$, from (2.1) and (2.4), we get

$$\begin{aligned}(\pounds_v g)(X,Y) &= g(\nabla_X v,Y)+g(\nabla_Y v,X) = g(X(b)\xi,Y)+g(Y(b)\xi,X)\\ &= X(b)\eta(Y)+Y(b)\eta(X) = g(\nabla b,X)\eta(Y)+g(\nabla b,Y)\eta(X)\end{aligned} \tag{4.4}$$

for any $X,Y\in\Gamma(TM)$, where ∇b is the gradient of a smooth fuction b on M.

By using (1.1) and (4.4), we have

$$S(X,Y) = -\lambda g(X,Y)-\frac{1}{2}\Big\{g(\nabla b,X)\eta(Y)+g(\nabla b,Y)\eta(X)\Big\}. \tag{4.5}$$

If we denote the the dual 1-form of ∇b by γ, then (4.5) becomes

$$S = -\lambda g-\frac{1}{2}(\gamma\otimes\eta+\eta\otimes\gamma). \tag{4.6}$$

Hence, (M,g) is an almost quasi-Einstein manifold. □

Theorem 5. *Let M be a cosymplectic manifold endowed with a recurrent vector field v such that v is a pointwise collinear with ξ. If M admits a Ricci soliton, then M is an almost quasi-Einstein manifold.*

Proof. Since v is a recurrent vector field on M, we have

$$\nabla_X v = \phi(X)v \tag{4.7}$$

for any $X \in \Gamma(TM)$, where ϕ is a 1-form. Also, since the potential vector field v of Ricci soliton is equal to $b\xi$, if we use (2.1), (2.4) and (4.7) then, we get

$$X(b) = b\phi(X) \tag{4.8}$$

for any smooth function b on M.

On the other hand, it follows from the definition of Lie-derivative and from (2.1), (4.8), one has

$$(\pounds_v g)(X,Y) = b\phi(X)\eta(Y) + b\phi(Y)\eta(X). \tag{4.9}$$

Thus, by combining the equation (1.1) with (4.9) we find

$$S(X,Y) = -\lambda g(X,Y) - \frac{b}{2}(\phi(X)\eta(Y) + \phi(Y)\eta(X))$$

equivalently,

$$S = -\lambda g - \frac{b}{2}(\phi \otimes \eta + \phi \otimes \eta)$$

which proves that M is an almost quasi-Einstein manifold. □

Now, we have an another theorem.

Theorem 6. *Let M be a cosymplectic manifold endowed with a concircular vector field v such that v is a pointwise collinear with ξ. If M admits a Ricci soliton, then M is an η-Einstein manifold.*

Proof. Since v is a concircular vector field, using (2.4), one has

$$fX = \nabla_X v = \nabla_X b\xi = X(b)\xi + b\nabla_X \xi = X(b)\xi \tag{4.10}$$

for any $X \in \Gamma(TM)$. If we take the inner product of (4.10) with ξ, we have

$$X(b) = f\eta(X) \tag{4.11}$$

for a smooth function b on M.

On the other hand, from definition of the Lie-derivative and using (2.1), (4.4), (4.10) and (4.11), we get

$$(\pounds_v g)(X,Y) = 2f\eta(X)\eta(Y). \tag{4.12}$$

Therefore, from (1.1) and (4.12) we find

$$S(X,Y) = -\lambda g(X,Y) - f\eta(X)\eta(Y)$$

which shows that M is an η-Einstein manifold. □

Using Theorem 6, we have the following remark:

Remark 2. Let M be a cosymplectic manifold endowed with a concurrent vector field v such that v is a pointwise collinear with ξ. If M admits a Ricci soliton, then M is an η-Einstein manifold.

Theorem 7. *Let M be a cosymplectic manifold endowed with a torse-forming vector field v such that v is a pointwise collinear with ξ. If M admits a Ricci soliton, then M is a nearly quasi-Einstein manifold.*

Proof. Assume that (M,g,v,λ) is a Ricci soliton whose potential vector field v is a pointwise collinear with ξ. Then, we have

$$\nabla_X v = X(b)\xi \tag{4.13}$$

for any $X \in \Gamma(TM)$. Because v is a torse-forming vector field on M, using (2.7) and (4.13), we get

$$X(b) = f\eta(X) + b\phi(X) \tag{4.14}$$

where ϕ is a 1-form and b and f are smooth functions on M. Also, from (2.1), (4.13) and (4.14), we obtain

$$(\pounds_v g)(X,Y) = 2f\eta(X)\eta(Y) + b\phi(Y)\eta(X) + b\phi(Y)\eta(X). \tag{4.15}$$

So, if we combine (1.1) with (4.15) we find

$$S(X,Y) = -\lambda g(X,Y) - f\eta(X)\eta(Y) - \frac{b}{2}(\phi(X)\eta(Y) + \phi(Y)\eta(X)). \tag{4.16}$$

If we take a non-vanishing symmetric $(0,2)$-tensor E in (4.16) such that

$$E(X,Y) = -f\eta(X)\eta(Y) - \frac{b}{2}(\phi(X)\eta(Y) + \phi(Y)\eta(X))$$

then, the equation (4.16) reduces to

$$S = -\lambda g + E.$$

which gives that M is a nearly quasi-Einstein manifold. Therefore, the proof is completed. □

Now, we are ready to give our main theorem.

Theorem 8. *Let M be a cosymplectic manifold admitting a Ricci soliton with torqued vector field $\mathcal{T}$ such that $\mathcal{T}$ is a pointwise collinear with ξ. Then, either $\mathcal{T}$ is a Killing vector field on M and M is an η-Einstein manifold or $\mathcal{T}$ is a concircular vector field on M and the Ricci soliton is steady.*

Proof. Let $\mathcal{T}$ be a pointwise collinear with the structure vector field ξ. That is, $\mathcal{T} = b\xi$, where b is a smooth function on M. Then, from (2.4) and (2.8) and from the linearity of the torqued form α we find

$$X(b)\xi = \rho X + b\alpha(X)\xi, \tag{4.17}$$

$$b\alpha(\xi) = 0 \tag{4.18}$$

for any $X \in \Gamma(TM)$. Taking the inner product of (4.17) with ξ, we have

$$X(b) = \rho\eta(X) + b\alpha(X) \tag{4.19}$$

Also, putting $X = \xi$ in (4.19) and using (4.18) gives

$$\xi(b) = \rho. \tag{4.20}$$

Since M is a Ricci soliton with torqued vector field $\mathcal{T}$, from (1.1) and (4.17) one has

$$X(b)\eta(Y) + Y(b)\eta(X) + 2S(X,Y) + 2\lambda g(X,Y) = 0. \tag{4.21}$$

By setting $Y = \xi$ in (4.21) and using (2.5), (4.20) implies

$$X(b) + \rho\eta(X) + 2\lambda\eta(X) = 0. \tag{4.22}$$

Again, setting $X = \xi$ in (4.22) and from (4.20) we derive

$$\rho + \lambda = 0, \tag{4.23}$$

From (4.22) and (4.23), we write

$$X(b) = \rho\eta(X). \tag{4.24}$$

Combining (4.19) with (4.24) yields

$$b\alpha(X) = 0.$$

This means that either $b = 0$ or $\alpha(X) = 0$. If $\alpha(X) = 0$, from (2.8) we get that $\mathcal{T}$ is a concircular vector field on M. Using the fact that and from (4.21) we find

$$S(X,Y) = -\lambda g(X,Y) - \rho\eta(X)\eta(Y),$$

which implies that M is an η-Einstein manifold.

If $b = 0$, from (4.24) we have $\rho = 0$. As a result of this, we get $\lambda = 0$. Therefore, we get the requested result. □

The next theorem gives us a characterization for a cosymplectic manifold endowed with a concurrent vector field v to be a gradient Ricci soliton.

Theorem 9. *Let M be a cosymplectic manifold endowed with a concurrent vector field v. Then, the characteristic vector field ξ is the gradient of a smooth function f on M.*

Proof. For any $X \in \Gamma(TM)$ and concurrent vector field v, we have

$$g(v,\xi) = f, \tag{4.25}$$

where f is a smooth function on M. If we differentiate (4.25) covariantly along arbitrary vector field X and use (2.4) and (2.7) then, we get

$$g(\nabla f, X) = Xg(v,\xi) = g(\nabla_X v, \xi) + g(v, \nabla_X \xi) = g(X,\xi). \tag{4.26}$$

Since the Riemannian metric g is non-degenere, we have

$$\nabla f = \xi.$$

This is the desired result. □

As an immediate consequence of Theorem 9, we have the following.

Corollary 1. *Every Ricci soliton* (M, g, ξ, λ) *on a cosymplectic manifold endowed with a concurrent vector field v is gradient.*

5. CONCLUSIONS

Ricci soliton is a natural generalization of Einstein manifold. This notion corresponds to the self-similar solution of Hamilton's Ricci flow. Over the last decades, the geometry of Ricci solitons has been studied intensively and extensively by many mathematicians on many context. In this paper, we study Ricci solitons on a cosymplectic manifold endowed with some geometric vector fields and give some relationships between these vector fields. Also, we give some necessary conditions for a Ricci soliton to be steady, shrinking. Finally, we give some important characterizations which classifies cosymleptic manifold.

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