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VOLUME 21 (2020), NUMBER 2

CONTENTS

Chafika Amel Aitemrar
L^p estimates for rough semiclassical Fourier integral operators 533

Md Nur Alam and Cemil Tunc
Construction of soliton and multiple soliton solutions to the longitudinal wave motion equation in a magneto-electro-elastic circular rod and the Drinfeld-Sokolov-Wilson equation 545

Esmaeil Alizadeh, Asghar Rahimi, and Mortaza Rahmani
The algebra of Bessel sequences and means of frames in Hilbert Spaces . 563

Muhammad Arif, Mohsan Raza, Miraj Ul Haq, and Gautam Srivastava
Relationship of the Bernardi integral operator with certain classes of analytic functions 575

M. U. Awan, M. A. Noor, F. Safdar, A. Islam, M. V. Mihai, and K. I. Noor
Hermite-Hadamard type inequalities with applications 593

Zs. Balogh
The structure of the unit group of some group algebras 615

H. Barsam and A. R. Sattarzadeh
Hermite-Hadamard inequalities for uniformly convex functions and its applications in means 621

O. Bazighifan, S. R. Grace, J. Alzabut, and A. Özbekler
New results for oscillatory properties of neutral differential equations with a p-Laplacian like operator 631

Omar Bazighifan and Andrea Scapellato
Oscillatory properties of even-order ordinary differential equations with variable coefficients 641

Miskolc Mathematical Notes

VOLUME 21 (2020), NUMBER 2

CONTENTS

Florina Blaga, Laura Mesaroş, Dorian Popa, Georgiana Pugna, and Ioan Raşa
Bounds for solutions of linear differential equations and Ulam stability . . 653

H. Budak, H. Kara, M. Z. Sarikaya, and M. E. Kiriş
New extensions of the Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals 665

S. I. Butt, L. Horváth, and J. Pečarić
Cyclic refinements of the discrete Hölder's inequality with applications . . 679

Saad Ihsan Butt, Ahmet Ocak Akdemir, Jamshed Nasir, and Fahd Jarad
Some Hermite-Jensen-Mercer like inequalities for convex functions through a certain generalized fractional integrals and related results 689

Saad Ihsan Butt, Ahmet Ocak Akdemir, Alper Ekinci, and Muhammad Nadeem
Inequalities of Chebyshev-Pólya-Szegö type via generalized proportional fractional integral operators 717

Mehtap Lafci Büyükkahraman, Gizem S. Oztepe, and Hüseyin Bereketoğlu
Behavior of solutions of a nonlinear impulsive system with piecewise constant mixed arguments 733

P. Chansri, P. Chansangiam, and Sorin V. Sabau
The geometry on the slope of a mountain 747

Nak Eun Cho and Virendra Kumar
Initial coefficients and fourth Hankel determinant for certain analytic functions 763

B. Daraby, N. Khosravi, and A. Rahimi
Fuzzy barrels on locally convex fuzzy topological vector spaces 781

Miskolc Mathematical Notes

VOLUME 21 (2020), NUMBER 2

CONTENTS

Ö. Duran, N. Ömür, and S. Koparal
On sums with generalized harmonic, hyperharmonic and special numbers 791

Abdullah Ergün
A half-inverse problem for the singular diffusion operator with jump conditions 805

György Gát and Anteneh Tilahun
On almost everywhere convergence of the generalized Marcienkiwicz means with respect to two dimensional Vilenkin-like systems 823

B. Ghanmi, T. Fajraoui, and I. Basdouri
On the practical $\psi^{\gamma}-$exponential asymptotic stability of nonlinear systems of differential equations 841

Rafik Guefaifia, Salah Boulaaras, Jiabin Zuo, and Praveen Agarwal
Existence and multiplicity of positive weak solutions for a new class of (p; q)-Laplacian systems 861

Özlem Ak Gümüş
Neimark-Sacker bifurcation and stability of a prey-predator system 873

Mohamed Haiour, Salah Boulaaras, Rafik Guefaifia, and Fares Kamache
Existence result for a new class of Kirchhoff elliptic system with variable parameters 887

F. Iranmanesh, M. Ghadiri, and A. Borumand Saeid
On H_vBE-algebras 897

Gamzegul Karahisarli, Huseyin Merdan, and Abdessamad Tridane
Stability and zero-Hopf bifurcation analysis of a tumour and T-helper cells interaction model in the case of HIV infection 911

Miskolc Mathematical Notes

VOLUME 21 (2020), NUMBER 2

CONTENTS

Virendra Kumar, Rekha Srivastava, and Nak Eun Cho
Sharp estimation of Hermitian-Toeplitz determinants for Janowski type starlike and convex functions 939

H. Laith, A. Mohammad, and H. Dutta
A new class of special functions in higher dimensions 953

Kirill S. Lapin and Cemil Tunç
Uniform-ultimate Poisson boundedness of solutions of P-perturbed systems of differential equations 959

Fengzhen Long and Biao Zeng
Lagrangian methods for optimal control problems governed by quasi-hemivariational inequalities 969

Dijana Mosić
Integral and limit representations of the CMP inverse 983

Celil Nebıyev and Hasan Hüseyın Ökten
Cofinitely radical supplemented and cofinitely weak radical supplemented lattices 993

Muhammad Aslam Noor, Khalida Inayat Noor, and Muhammad Uzair Awan
Generalized fractional Hermite-Hadamard inequalities 1001

Hasan Hüseyin Ökten and Ayten Pekin
Essential supplemented lattices 1013

S. Shokooh
A class of fourth-order Kirchhoff type elliptic problems with four solutions 1019

Abbas Kareem Wanas
Coefficient estimates for Bazilevič functions of bi-prestarlike functions . . 1031

Miskolc Mathematical Notes
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L^p ESTIMATES FOR ROUGH SEMICLASSICAL FOURIER INTEGRAL OPERATORS

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Abstract. In this paper the author investigates the global boundedness in Banach L^p spaces of rough semiclassical Fourier integral operators defined by generalized rough Hörmander class amplitudes and rough class phase functions which behave in the spatial variable like L^p functions.

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Keywords: semiclassical Fourier integral operators, L^p boundedness, rough amplitudes, rough phase functions

1. Introduction

A Fourier integral operator or FIO in short is a singular operator defined by

$$\left(I_{a,\phi}f\right)(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\phi(x,\xi)}a(x,\xi)\,\widehat{f}(\xi)\,d\xi,\ f\in C_0^\infty(\mathbb{R}^n),$$

where $a(x,\xi)$ and $\phi(x,\xi)$ are smooth functions called respectively the amplitude and the phase function.

The study of the theory of FIOs has a long history, many efforts have been made to study the regularity of these operators in functional spaces and there is a large body of results concerning the regularity like in [8] and [5] where some results of local L^2 boundedness have been obtained for Fourier integral operators associated to smooth amplitudes $a(x,\xi)\in S^m_{\rho,\delta}$ and phase functions $\phi(x,\xi)\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\})$, those are non degenerate and positively homogeneous of degree 1 in the frequency variable ξ. These results of local boundedness have been generalized in L^p for $1\le p\le\infty$ by Seeger-Sogge-Stein [15], Gala-Liu-Ragusa [6], Khelouki-Ikassoulene [10] and Ragusa [12].

Since 1970s, motivated by applications in microlocal analysis and hyperbolic partial differential equations, many authors extended local L^2 boundedness results to

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global $L^2(\mathbb{R}^n)$ regularity. The first result has been given in Asada-Fujiwara [2] considering Fourier integral operators associated to amplitudes in $S^0_{0,0}$ and inhomogeneous phase functions. This result was extended to a general class of amplitudes in $S^m_{\rho,\delta}$, satisfying some conditions, see [11]. For the global L^p boundedness, Cordiro, Nicola and Rodino have considered Fourier integral operators with amplitudes in $S^m_{1,0}$, see [3]. Another result of global L^p boundedness of Fourier integral operators, due to Coriasco and Ruzhanky [4], was to consider their amplitudes in some subspaces of $S^0_{1,0}$.

In the semiclassical case an h-Fourier integral operator has the following form

$$(I_h(a,\phi)f)(x) = (2\pi h)^{-n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\phi(x,\xi)}a(x,\xi)\widehat{f}(\xi)\,d\xi,\ f\in S(\mathbb{R}^n),$$

where $h\in\,]0,h_0]$ is a semiclassical parameter and S denotes the Schwartz space. These operators appear naturally in the expression of the solution of the semiclassical hyperbolic partial differential equations, and when expressing the C^∞ -solution of the associated Cauchy's problem. Comparatively with the study of FIOs, there has been a smaller amount of activity for h-FIOs concerning the investigation of the corresponding L^p boundedness properties. Some results of L^2 boundedness and L^2 compactness are obtained for h -Fourier integral operators with amplitudes introduced by Hörmander and non degenerate and homogeneous phase functions, see [7] and [16] . Recently we have establish in our paper [1] the L^p boundedness for h-Fourier integral operators with rough amplitudes in the class $L^pS^m_\rho$ introduced by [14] and non degenerate and homogeneous phase functions in Φ^2. We have obtained results of boundedness of these operators from L^q into L^r, where the numbers q,r satisfy some relations with the others parameters p, n, m, ρ.

Motivated by the lack of L^p boundedness results for h-FIOs in the literature of semiclassical analysis, the aim of this wok is to extend the aforementioned results for a general class of h-FIOs. Our investigation is to study rough h-Fourier integral operators associated to amplitudes in the class $L^\infty S^m_\rho$ and phase functions in the class $L^\infty\Phi^k$ as introduced in [9].

To summarize the paper is organized as follows. In the second section we introduce some definitions and notations of a general class of amplitudes and phase functions associated to the class of h-FIOs treated in this paper. Tools and preliminary lemmas are mentioned in the third section. The last section is devoted to establish the results of the boundedness of this class of h-FIOs.

2. Rough amplitudes and phase functions

In this section we define the classes of linear amplitudes and the class of phase functions with rough spatial behaviour that appear in the definition of operators treated here. But at first we recall some definitions and notations of classical amplitudes and phase functions introduced by Hörmander [8].

For the sake of simplicity we use the notation $\langle\xi\rangle$ for $\left(1+|\xi|^2\right)^{\frac{1}{2}}$.

Definition 1. Let $m\in\mathbb{R}$, $0\leq\rho,\delta\leq 1$. We denote $S^m_{\rho,\delta}$ the class of all functions $a(x,\xi)\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ such that

$$\forall\alpha,\beta\in\mathbb{N}^n:\ \sup_{\xi\in\mathbb{R}^n}\langle\xi\rangle^{-m+\rho|\alpha|-\delta|\beta|}\left|\partial_\xi^\alpha\partial_x^\beta a(x,\xi)\right|<+\infty.$$

Notation 1. We denote by Φ^k, $k\in\mathbb{N}$, the space of real valued functions $\varphi(x,\xi)\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\})$, such that $\varphi(x,\xi)$ is positively homogeneous of degree 1 in the frequency variable ξ, and satisfies the following condition:
$\forall\alpha,\beta\in\mathbb{N}^n$, with $|\alpha|+|\beta|\geq k$, $\exists C_{\alpha,\beta}>0$:

$$\sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}}|\xi|^{-1+|\alpha|}\left|\partial_\xi^\alpha\partial_x^\beta\varphi(x,\xi)\right|\leq C_{\alpha,\beta}.$$

Definition 2. A real valued phase $\varphi(x,\xi)\in C^2(\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\})$ satisfies the *Strong Non-Degeneracy* condition or the SND condition for short, if there exists a constant $c>0$ such that

$$\left|\det\frac{\partial^2\varphi(x,\xi)}{\partial x_j\partial\xi_k}\right|\geq c \text{ for all } (x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}.$$

Example 1. The function $\phi(x,\xi)=\langle x,\xi\rangle$ is in Φ^2 and satisfies the strong non-degenerate condition. A Fourier integral operator with such phase function is called pseudodifferential operator.

Now we introduce the definitions and conditions of rough amplitudes and phase functions which we will use for our study of the L^p boundedness for h-Fourier integral operators associated to these amplitudes and phase functions.

Definition 3. Let $m\in\mathbb{R}$ and $0\leq\rho\leq 1$. We denote by $L^\infty S^m_\rho$, the space of functions $a(x,\xi)$, $x,\xi\in\mathbb{R}^n$ such that $a(x,\xi)$ is measurable in $x\in\mathbb{R}^n$, $a(x,\xi)\in C^\infty\left(\mathbb{R}^n_\xi\right)$ a.e. $x\in\mathbb{R}^n$, and for each multi-index α, there exists a constant C_α such that

$$\left\|\partial_\xi^\alpha a(.,\xi)\right\|_{L^\infty(\mathbb{R}^n)}\leq C_\alpha\langle\xi\rangle^{m-\rho|\alpha|},\ \forall\xi\in\mathbb{R}^n.$$

Remark 1. It is clear that $S^m_{\rho,\delta}\subset L^\infty S^m_\rho$ for all $m\in\mathbb{R}$ and $0\leq\rho,\delta\leq 1$.

Notation 2. Let $k\in\mathbb{N}$, we denote by $L^\infty\Phi^k$ the space of real valued functions $\phi(x,\xi)$ such that $\phi(x,\xi)$ is smooth and homogeneous of degree 1 in the frequency variable ξ, $\phi(x,\xi)$ is measurable in $x\in\mathbb{R}^n$ and satisfies the following condition:
$\forall\alpha\in\mathbb{N}^n$ with $|\alpha|\geq k$, $\exists C_\alpha>0$:

$$\sup_{\xi\in\mathbb{R}^n\setminus\{0\}}|\xi|^{-1+|\alpha|}\left\|\partial_\xi^\alpha\phi(.,\xi)\right\|_{L^\infty(\mathbb{R}^n)}\leq C_\alpha.$$

Example 2. For each $f \in L^{\infty}(\mathbb{R}^n)$ the phase function $\phi(x,\xi) = f(x)|\xi| + \langle x,\xi\rangle$ belongs to the class $L^{\infty}\Phi^2$.

Instead of the non-degeneracy condition in case of smooth phase function, we need in our case to an analogous condition which we will call rough non-degeneracy condition.

Definition 4 (The rough non-degeneracy condition)**.** We say that a real valued phase ϕ satisfies the *Rough Non-Degeneracy condition*, if it is C^1 on $\mathbb{R}^n\setminus\{0\}$ in the frequency variable ξ, bounded measurable in the spatial variable x, and there exists a constant $c > 0$ such that

$$\left|\partial_{\xi}\phi(x,\xi) - \partial_{\xi}\phi(y,\xi)\right| \geq c|x-y|, \text{ for all } x,y \in \mathbb{R}^n \text{ and } \xi \in \mathbb{R}^n\setminus\{0\}.$$

The aim of this work is to obtain some results of L^p boundedness for h -Fourier integral operators of the form

$$(I_h(a,\phi)f)(x) = (2\pi h)^{-n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\phi(x,\xi)}a(x,\xi)\widehat{f}(\xi)\,d\xi, \tag{2.1}$$

where $a \in L^{\infty}S^m_{\rho}$, $\phi \in L^{\infty}\Phi^k$ and $h \in]0,1]$ is a parameter. Here

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-\frac{i}{h}x\xi}f(x)\,dx.$$

3. Preliminary tools

For our study we collect here the main tools in proving our boundedness results of boundedness of h-FIOs. First we need Seeger-Sogge-Stein partition of unity.

For each $s \in]0,1]$ let $\{\xi^{\nu}\}_{1\leq\nu\leq N}$ a finite collection of unit vectors of $\mathbb{S}^{n-1}$ satisfying

1) $\left|\xi^{\nu} - \xi^{\nu'}\right| \geq h^{-\frac{j}{2}}$, if $\nu \neq \nu'$,

2) $\forall\xi \in \mathbb{S}^{n-1}$, $\exists\xi^{\nu}$ so that $|\xi - \xi^{\nu}| \leq s^{\frac{1}{2}}$.

For any $\nu \in \{1,\ldots,N\}$ let Γ^{ν} the cone defined by

$$\Gamma^{\nu} = \left\{\xi \in \mathbb{R}^n : \left|\frac{\xi}{|\xi|} - \xi^{\nu}\right| \leq s^{\frac{1}{2}}\right\}.$$

In order to cover the annulus $\left\{\xi; \frac{1}{2} \leq |\xi| \leq 2\right\}$, we need $s^{-\frac{n-1}{2}}$ cones such Γ^{ν}.

Next we construct an associated partition of unity subordinated to $\{\Gamma^{\nu}\}_{1\leq\nu\leq N}$ given by functions χ^{ν}, homogenous of degree 0 in ξ with support in Γ^{ν}, i.e.

$$\sum_{\nu=1}^{N}\chi^{\nu}(\xi) = 1 \text{ for all } \xi \neq 0,$$

and such that

$$\forall\alpha \in \mathbb{N}^n, \exists C_{\alpha} > 0 : |\partial^{\alpha}\chi^{\nu}(\xi)| \leq C_{\alpha}s^{-\frac{|\alpha|}{2}}.$$

According to this partition of unity we have get the following lemma (see also [1]).

Lemma 1. *Any h-Fourier integral operator I_h of the type (2.1) with amplitude $a(x,\xi) \in L^\infty S^m_\rho$ and phase function $\phi(x,\xi) \in L^\infty \Phi^2$, can be written as a finite sum of operators of the form*

$$(2\pi h)^{-n} \int \sigma(x,\xi)\, e^{\frac{i}{h}\psi(x,\xi)+\frac{i}{h}\left\langle \nabla_\xi \phi(x,\zeta),\xi\right\rangle} \widehat{u}(\xi)\, d\xi,$$

where ζ is a point on the unit sphere S^{n-1}, $\psi(x,\xi) \in L^\infty \Phi^1$ and $\sigma(x,\xi) \in L^\infty S^m_\rho$ is localized in the ξ variable around the point ζ.

But in order to study the L^p boundedness for rough h-Fourier integral operators with amplitudes in $L^\infty S^m_\rho$ and phase functions in $L^\infty \Phi^2$ we need the Littlewood-Paley partition of unity (see [15]):

$$\Psi_0(\xi) + \sum_{j=1}^{\infty} \Psi_j(\xi) = 1,$$

where $\operatorname{supp}\Psi_0 \subset \{\xi; |\xi| \le 2\}$, and $\Psi_j(\xi) = \Psi\left(2^{-j}\xi\right)$, with $\operatorname{supp}\Psi \subset \left\{\xi; \frac{1}{2} \le |\xi| \le 2\right\}$.

So we can write

$$\begin{aligned} I_h(a,\phi) &= \Psi_0(\xi) I_h(a,\phi) + \sum_{j=1}^{\infty} \Psi_j(\xi) I_h(a,\phi) \\ &= I_{h,0}(a,\phi) + \sum_{j=1}^{\infty} I_{h,j}(a,\phi). \end{aligned}$$

Applying the Seeger-Sogge-Stein decomposition in the annulus $\left\{\xi; \frac{1}{2} \le |\xi| \le 2\right\}$ to the operator $I_{h,j}(a,\phi)$ we obtain

$$I_{h,j}(a,\phi) = \sum_{\nu=1}^{N(j)} \chi^\nu(\xi) I_{h,j}(a,\phi) = \sum_{\nu=1}^{N(j)} I^\nu_{h,j}.$$

The operator $I^\nu_{h,j}(a,\phi)$ has kernel

$$\begin{aligned} I^\nu_{h,j}(x,y) &= \frac{(2\pi h)^{-n}}{2^{jn}} \int e^{\frac{i}{h}2^j(\phi(x,\xi)-\langle x,\xi\rangle)} \Psi(\xi) \chi^\nu(\xi)\, a\left(x, 2^j\xi\right) d\xi \\ &= \frac{(2\pi h)^{-n}}{2^{jn}} \int e^{\frac{i}{h}2^j\left\langle \nabla_\xi \phi(x,\xi^\nu)-y,\xi\right\rangle} b^\nu_j(x,\xi)\, d\xi, \end{aligned} \tag{3.1}$$

where $b^\nu_j(x,\xi) = e^{\frac{i}{h}2^j\left\langle \nabla_\xi \phi(x,\xi)-\nabla_\xi \phi(x,\xi^\nu),\xi\right\rangle} \Psi(\xi) \chi^\nu(\xi)\, a\left(x, 2^j\xi\right)$.

In the sequel we will also need the following lemmas for the proof (see [9]).

Lemma 2. *Let $a \in L^\infty S^m_\rho$ and $\phi(x,\xi) \in L^\infty \Phi^2$, then the symbol $b^\nu_j(x,\xi)$ above satisfies the estimate*

$$\forall \alpha \in \mathbb{N}^n, \exists C_\alpha > 0 : \sup_\xi \left\| \partial^\alpha_\xi b^\nu_j(.,\xi) \right\|_{L^\infty} \le C_\alpha \left(2^j\right)^{m+|\alpha|(1-\rho)+\frac{|\alpha'|}{2}}.$$

Lemma 3. *Let $b(x,\xi)$ be a bounded function which is $C^{n+1}\left(\mathbb{R}^n_\xi\setminus\{0\}\right)$ and compactly supported in the frequency variable ξ and $L^\infty(\mathbb{R}^n_x)$ in the space variable x, satisfying*

$$\sup_{\xi\in\mathbb{R}^n\setminus\{0\}} |\xi|^{-1+|\alpha|}\left\|\partial_\xi^\alpha b(.,\xi)\right\|_{L^\infty(\mathbb{R}^n)} < +\infty,\ \forall |\alpha|\leq n+1.$$

Then for all $\mu\in[0,1[$ we have

$$\sup_{x,y\in\mathbb{R}^n}\langle y\rangle^{n+\mu}\left|\int e^{-i\langle y,\xi\rangle}b(x,\xi)\,d\xi\right| < +\infty.$$

Finally we give the semiclassical version of Hausdorff-Young and Minkowsky's inequality.

Hausdorff-Young inequality: For all $p,q\in\mathbb{R}$ such that $1\leq p\leq 2$ and $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\exists C>0:\left\|\widehat{f_h}\right\|_{L^q(\mathbb{R}^n)}\leq Ch^{\frac{n}{q}}\|f_h\|_{L^q(\mathbb{R}^n)},\ f\in L^p(\mathbb{R}^n).$$

Minkowsky's inequality for integrals: For all $p,q\in\mathbb{R}$ such that $1\leq p\leq\infty$ and all measurable function f on $\mathbb{R}^n_x\times\mathbb{R}^n_y$, we have

$$\left(\int_{\mathbb{R}^n_y}\left|\int_{\mathbb{R}^n_x} f(x,y)\,dx\right|^p dy\right)^{\frac{1}{p}}\leq\int_{\mathbb{R}^n_x}\left(\int_{\mathbb{R}^n_y}|f(x,y)|^p\,dy\right)^{\frac{1}{p}}dx.$$

4. Main results

In this section we establish the results of L^p boundedness for rough h-Fourier integral operators, which permits us to obtain those of smooth h-Fourier integral operators as consequences.

Let us begin with rough h-Fourier integral operators associated to amplitudes with compact support in the frequency variable ξ.

Theorem 1. *Let $m\in\mathbb{R}$ and $\rho\in[0,1]$, let $a(x,\xi)\in L^\infty S^m_\rho$ such that $supp_\xi a(x,\xi)$ is compact, and let $\phi(x,\xi)\in L^\infty\Phi^2$ a phase function satisfying the rough non-degeneracy condition. Then the h-Fourier integral operator given by (2.1) is bounded on L^p for every $p\in[1,\infty]$.*

Proof. According to Lemma 1, it suffices to prove the L^p boundedness of each Fourier integral operator of the form

$$Au(x)=\int\sigma(x,\xi)\,e^{\frac{i}{h}\psi(x,\xi)+\frac{i}{h}\left\langle\nabla_\xi\phi(x,\zeta),\xi\right\rangle}\widehat{u}(\xi)\,d\xi,$$

where $\zeta\in S^{n-1}$, $\sigma\in L^\infty S^m_\rho$ and $\psi\in L^\infty\Phi^1$.

Let us rewrite the operator A with its kernel, i.e.

$$Au(x)=\int A(x,y)\,u(y)\,dy,$$

where the kernel of A is given by

$$A(x,y)=\int\sigma(x,\xi)\,e^{\frac{i}{h}\psi(x,\xi)+\frac{i}{h}\langle\nabla_\xi\phi(x,\zeta)-y,\xi\rangle}d\xi.$$

To get an estimate of $\|Au\|_{L^p}$ we will estimate $A(x,y)$ in L^1. Set $b(x,\xi)=\sigma(x,\xi)\,e^{\frac{i}{h}\psi(x,\xi)}$, this function is bounded because σ and ψ are bounded uniformly in x and $\operatorname{supp}_\xi\sigma$ is compact, further since $\psi\in L^\infty\Phi^1$ so

$$\forall|\alpha|\geq 1:\sup_{x\in\mathbb{R}^n}\sup_{|\xi|\neq 0}|\xi|^{-1+|\alpha|}\left|\partial_\xi^\alpha b(x,\xi)\right|<\infty,$$

in particular for $|\alpha|\leq n+1$, we obtain from Lemma 3

$$\sup_{x\in\mathbb{R}^n}|A(x,y)|\leq C\left\langle\nabla_\xi\phi(x,\zeta)-y\right\rangle^{-n-\mu}\text{ for all }\mu\in[0,1[,$$

which implies (by choosing some $\mu\in]0,1[$)

$$\sup_x\int|A(x,y)\,dy|<\infty.$$

On the other hand from the rough non-degeneracy assumption we get

$$\int v\circ\partial_\xi\phi(x,\zeta)\,dx=\int v(z)\,J(z)\,dz,\ \forall v\in L^1,$$

where J is the Jacobian of $\partial_\xi\phi$. Thus we obtain

$$\int|A(x,y)|\,dx\leq C\int\left\langle\nabla_\xi\phi(x,\zeta)-y\right\rangle^{-n-\mu}dx\leq C'\int\langle z\rangle^{-n-\mu}dz<\infty,$$

uniformly in y.

Finally from this estimate and using Young's inequality we can easily obtain an estimate of $\|Au\|_{L^p}$ in term of $\|u\|_{L^p}$, and therefore we get the L^p boundedness of the operator A. □

Next let us consider the general case of rough h-Fourier integral operators, without assumption of compactness on the amplitude.

Theorem 2. *Let $0\leq\rho\leq 1$ and $m\in\mathbb{R}$ such that $m<-\frac{n-1}{2}+n(\rho-1)$. If $a\in L^\infty S^m_\rho$ and $\phi\in L^\infty\phi^2$ satisfying the rough non- degeneracy condition, then the h-Fourier integral operator given by (2.1) is bounded on $\mathcal{S}(\mathbb{R}^n)$ provided with the norm L^1, i.e.*

$$\exists C>0:\|I_h(a,\phi)\,u\|_{L^1}\leq C\|u\|_{L^1},\ \forall u\in\mathcal{S}(\mathbb{R}^n).$$

Proof. By the Littlewood-Paley partition of unity, we can decompose the operator $I_h(a,\phi)$ as a sum of operators of the form

$$\left(I_{h,j}f\right)(x)=\int_{\mathbb{R}^n}e^{\frac{i}{h}\phi(x,\xi)}\Psi_j(\xi)\,a(x,\xi)\,\widehat{f}(\xi)\,d\xi,\ j=0,1,\ldots$$

For $j=0$, the operator $I_{h,0}$ has amplitude $\Psi_0(\xi)\,a(x,\xi)$ which is supported around the origin, so $I_{h,0}$ is bounded on L^1 from Theorem 1.

For $j=1,2,\ldots$ using the Seeger-Sogge-Stein decomposition for $s=2^{-j}$, we have $I_{h,j}=\sum_{\nu=1}^{N(j)} I_{h,j}^{\nu}$, where the operator $I_{h,j}^{\nu}$ has kernel defined by (3.1). In order to estimate this kernel we consider the differential operator

$$L=1-\partial_{\xi_1}^2-2^{-j}\partial_{\xi'}^2.$$

from Lemma 2, we have for any $k\in\mathbb{N}$

$$\sup_{\xi}\left\|L^k b_j^{\nu}(.,\xi)\right\|_{L^\infty}\leq C_k\left(2^{-j}\right)^{-m-2k(1-\rho)}.$$

Using an integrations by parts in ξ we get

$$\begin{aligned}\left|I_{h,j}^{\nu}(x,y)\right| &\leq \frac{(2\pi h)^{-n}}{2^{jn}}\left(1+g\left(y-\nabla_{\xi}\phi(x,\xi^{\nu})\right)\right)^{-k}\int\left|L^k b_j^{\nu}(x,\xi)\right|d\xi\\ &\leq C_k\left(2^{-j}\right)^{-m-\frac{n+1}{2}-2k(1-\rho)}\left(1+g\left(y-\nabla_{\xi}\phi(x,\xi^{\nu})\right)\right)^{-k},\end{aligned}$$

where for simplify $g(y)=2^{2j}y_1^2+2^j|y'|^2$. This estimate remains valid if the integer k is replaced by any positive number M. We can check that by some interpolation inequality.

So for any real number $M>n$ we obtain

$$\sup_{x}\int\left|I_{h,j}^{\nu}(x,y)\right|dy\leq C_M\left(2^{-j}\right)^{-m-M(1-\rho)}.$$

On the other hand from the rough non-degeneracy assumption on $\phi(x,\xi)$ we get

$$\sup_{y}\int\left|I_{h,j}^{\nu}(x,y)\right|dx\leq C_M'\left(2^{-j}\right)^{-m-M(1-\rho)}.$$

Hence summing over ν and using Young's inequality we obtain

$$\left\|I_{h,j}u\right\|_{L^1}\leq\sum_{\nu=1}^{N(j)}\left\|I_{h,j}^{\nu}u\right\|_{L^1}\leq C_M''\left(2^{-j}\right)^{-m-\frac{n-1}{2}-M(1-\rho)}\|u\|_{L^1}.$$

(Note that $N(j)=s^{-\frac{n-1}{2}}=\left(2^{-j}\right)^{-\frac{n-1}{2}}$). Since $m<-\frac{n-1}{2}-n(1-\rho)<-\frac{n-1}{2}-M(1-\rho)$ so $\sum\left(2^{-j}\right)^{-m-\frac{n-1}{2}-M(1-\rho)}<\infty$ and therefore by summing over $j=1,\ldots$ we find

$$\|I_h(a,\phi)u\|_{L^1}\leq\|I_{h,0}u\|_{L^1}+\sum_{j=1}^{\infty}\left\|I_{h,j}u\right\|_{L^1}\leq C\|u\|_{L^1}.$$

The proof of the theorem is complete. □

Remark 2. In case of $\rho=0$, i.e. $a\in L^\infty S_0^m$, the result can be extended for all $m<-n$.

Now we will establish the result of L^2 boundedness of rough h-Fourier integral operators, but we must prove an intermediary result when the phase function $\phi(x,\xi)$ is in Φ^2 instead of $L^\infty\Phi^2$.

Theorem 3. *Let $a \in L^\infty S^m_\rho$ with $m < \frac{n}{2}(\rho - 1)$ and $\phi \in \Phi^2$ satisfying the SND condition. Then the h-Fourier integral operator $I_h(a,\phi)$ given by (2.1) is bounded from L^2 to itself.*

Proof. As in Theorem 2, we write the operator $I_h(a,\phi)$ as a sum of operators $I_{h,j}$ and the boundedness of $I_{h,0}$ follows from 1. Next for $I_{h,j}$ we deal with the operator $T_{h,j} = I_{h,j} I^*_{h,j}$ where its kernel is given by

$$T_{h,j}(x,y) = \frac{(2\pi h)^{-n}}{2^{jn}} \int e^{\frac{i}{h} 2^j (\phi(x,\xi) - \phi(y,\xi))} \chi^2(\xi)\, a(x,\xi)\, \overline{a}(y,\xi)\, d\xi.$$

From the SND condition on the function ϕ there exists a constant $C > 0$ such that

$$\left|\nabla_\xi \phi(x,\xi) - \nabla_\xi \phi(y,\xi)\right| \geq c\,|x-y|,\ \forall (x,y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\},$$

so by the non-stationary phase estimate in [8] we have for all integer k

$$\forall k \in \mathbb{N}, \exists C_k > 0 : \left|T_{h,j}(x,y)\right| \leq C_k 2^{j[2m+n+(1-\rho)k]} \left\langle 2^j (x-y)\right\rangle^{-k}.$$

This estimate remains valid for $k \in \mathbb{R}^*_+$ by using the integer part, thus

$$\forall M > n, \exists C_M > 0 : \sup_x \int \left|T_{h,j}(x,y)\right| dy \leq C_M 2^{j[2m+(1-\rho)M]},$$

hence by Cauchy-Schwarz and Young inequalities we get for all $M > n$,

$$\left\|I^*_{h,j} u\right\|^2_{L^2} \leq \left\|T_{h,j} u\right\|_{L^2} \|u\|_{L^2} \leq C_M 2^{j[2m+(1-\rho)M]} \|u\|^2_{L^2}.$$

Since $m < (\rho - 1)\frac{n}{2}$ we can choose $M > n$ such that $m < (\rho - 1)\frac{M}{2}$ in order that the series $\sum 2^{j[2m+(1-\rho)M]}$ converges and therefore by summing over $j = 0, 1, \ldots$, we obtain

$$\|I_h(a,\phi) u\|_{L^2} \leq C \|u\|_{L^2}.$$

Thus the proof of Theorem 3 is complete. □

Remark 3. Note that Theorem 3 is not valid for $m = (\rho - 1)\frac{n}{2}$, there are pseudodifferential with symbols $a \in S^m_{\rho,1}$ which are not bounded in L^2 (see [13]). Recall that $S^m_{\rho,1} \subset L^\infty S^m_\rho$ and $\phi(x,\xi) = \langle x,\xi\rangle$ is in $L^\infty \Phi^2$ and satisfies the SND condition.

Finally we prove the L^2 boundedness of rough h-Fourier integral operator, i.e. with rough amplitude and rough phase function.

Theorem 4. *Let $a \in L^\infty S^m_\rho$ with $0 \leq \rho \leq 1$ and $m < n\frac{(\rho-1)}{2} - \frac{(n-1)}{4}$, and let $\phi \in L^\infty \Phi^2$ satisfying the rough non-degeneracy condition. Then the h-Fourier integral operator given by (2.1) is bounded from L^2 to itself.*

Proof. We will follows the steps of the proof of Theorem 3. First we write the operator $I_h(a,\phi)$ as a sum of operators $I_{h,j}$ and the boundedness of $I_{h,0}$ follows from

Theorem 1. Next for $I_{h,j}$ we deal with the operator $T_{h,j} = I_{h,j} I_{h,j}^*$ where its kernel is given by

$$T_{h,j}(x,y) = \frac{(2\pi h)^{-n}}{2^{jn}} \int e^{\frac{i}{h} 2^j (\phi(x,\xi) - \phi(y,\xi))} \chi^2(\xi)\, a(x,\xi)\, \overline{a}(y,\xi)\, d\xi.$$

Here we need the Seeger-Sogge-Stein decomposition, so $T_{h,j} = \sum_{\nu=1}^{N(j)} T_{h,j}^{\nu}$ where $T_{h,j}^{\nu}$ has the kernel

$$T_{h,j}^{\nu}(x,y) = \frac{(2\pi h)^{-n}}{2^{jn}} \int e^{\frac{i}{h} 2^j \langle \nabla_\xi \phi(x,\xi^\nu) - \nabla_\xi \phi(y,\xi^\nu), \xi \rangle} b_j^{\nu}(x,\xi)\, \overline{b_j^{\nu}}(y,\xi)\, d\xi.$$

After as in the the proof of Theorem 2, we introduce the differential operator

$$L = 1 - \partial_{\xi_1}^2 - 2^{-j} \partial_{\xi'}^2,$$

from Lemma 2, we have for any $k \in \mathbb{N}$

$$\sup_{\xi} \left\| L^k b_j^{\nu}(.,\xi) \right\|_{L^\infty} \leq C_k \left(2^{-j}\right)^{-m-2k(1-\rho)}.$$

Taking account the rough non-degeneracy assumption on ϕ and using integration by parts we obtain in the same manner

$$\forall M > \frac{n}{2}, \exists C > 0 : \left\| T_{h,j}^{\nu} u \right\|_{L^2} \leq C \left(2^{-j}\right)^{-2m-2M(1-\rho)} \|u\|_{L^2},$$

which gives for the operator $I_{h,j}^*$ the estimate

$$\left\| I_{h,j}^* u \right\|_{L^2}^2 \leq \sum_{\nu=1}^{N(j)} \left\| T_{h,j}^{\nu} u \right\|_{L^2} \|u\|_{L^2} \leq C \left(2^{-j}\right)^{-2m-\frac{n+1}{2}-2M(1-\rho)} \|u\|_{L^2}^2.$$

Since $m < (\rho - 1)\frac{n}{2} - \frac{n-1}{4}$ and $M > \frac{n}{2}$ then $-2m - \frac{n+1}{2} - 2M(1-\rho) > 0$, therefore $\sum_{j=1}^{\infty} \left(2^{-j}\right)^{-2m-\frac{n+1}{2}-2M(1-\rho)}$ converges. Thus by summing over $j = 0, 1, \ldots$ we get

$$\|I_h(a,\phi)\, u\|_{L^2} \leq \widetilde{C} \|u\|_{L^2},$$

which means the boundedness of the operator $I_h(a,\phi)$ in L^2. □

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CONSTRUCTION OF SOLITON AND MULTIPLE SOLITON SOLUTIONS TO THE LONGITUDINAL WAVE MOTION EQUATION IN A MAGNETO-ELECTRO-ELASTIC CIRCULAR ROD AND THE DRINFELD-SOKOLOV-WILSON EQUATION

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Abstract. The present paper implements the novel generalized (G'/G)-expansion approach in solving the most popular nonlinear wave equations such as the longitudinal wave motion equation in a magneto-electro-elastic circular rod and the Drinfeld-Sokolov-Wilson equation. In this regard, we investigate the method to obtain new type of wave solutions of the studied models. New exact wave solutions are derived in the structures such as singular bright solition, compaction, singular bright periodic wave solitions, singular dark solition and singular dark periodic wave solition solutions of the studied models by using the novel generalized (G'/G)-expansion scheme. To draw the physical aspect of the got results, the 2D, 3D surfaces as well as the relating the contour plot surfaces of some acquired results are performed. The obtained results can assist to illustrate the physical application of the examined models and other nonlinear physical models appearing in mathematical physics.

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1. Introduction

It can be said with certainty, there are various nonlinear physical phenomena in nature that are represented by nonlinear evolution equations (NLEEs). Under deliberation prompt construction of symbolic computation operations, the exploration for the exact solutions of nonlinear evolutions has explored a lot of attention because exact solutions can explain the physical property of a natural system properly. Each exact solution of a system of NLEEs corresponds to a particular process. Therefore, methods for obtaining explicit exact solutions of nonlinear equations of partial differential equations (PDEs) implement a significant function in applied mathematics,

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mathematical physics and engineering. In the last two decades, a variety of procedures have been introduced and deployed to the nonlinear equations of partial differential equations, including the $exp(-\phi(\xi))$-expansion method [9, 10, 12], the sine-Gordon expansion method [17, 19], the Cole-Hopf transformation method [49], the improved Bernoulli sub-equation function method [15], the modified $exp(-\phi(\xi))$-expansion method [16, 41], F-expansion method [43, 59], the tanh-sech method [35, 48], sine-cosine method [47, 55], the (G'/G)-expansion method [7, 32, 46, 57, 58], the extended trial equation method [38, 44] and so on.

Naher and Abdullah [39] introduced a novel generalized (G'/G)-expansion method for the treatment of NLEEs. Afterwards, a few scientists and researchers examined some nonlinear equation of partial differential equations to construct exact solutions through this method. One might see references [3, 6, 11] for better understanding. This paper will employ the novel generalized $(\frac{G\prime}{G})$-expansion system that will show explicitly exact wave solutions as well as other solutions. The equation for the model interpreted here is a nonlinear equation. We observe that this model has not yet been examined employing the novel generalized $(\frac{G\prime}{G})$-expansion system. Hence, this paper is designed as follows: In Sections 2 and 3, application of the novel generalized $(\frac{G\prime}{G})$-expansion method to the longitudinal wave motion equation in a magneto-electro-elastic circular rod and the Drinfeld-Sokolov-Wilson equation is discussed and then we obtain soliton solutions. Subsequently, the conclusion is presented in the last section.

2. Exact solution of the longitudinal wave motion equation in a magneto-electro-elastic circular rod

With the growing acceptance of magneto-electro-elastic structures in numerous engineering fields which as sensors, actuators, and so many interested various scientists and researchers. Waved propagation in MEE media also has attracted many researchers ([22, 23, 34, 52]). Very recently, Xue et al. [53] has discovered the nonlinear longitudinal wave equation including dispersion produced through the transverse Poisson's effect in an magneto-electro-elastic circular rod, and Jacobi elliptic function approach has easily received the travelling and solitary waves. The nonlinear longitudinal wave equation has the following form:

$$\frac{\delta^2 u}{\delta t^2} - c_0^2 \frac{\delta^2 u}{\delta z^2} - \frac{\delta^2}{\delta z^2}\left(\frac{c_0^2}{2}u^2 + N\frac{\delta^2 u}{\delta t^2}\right) = 0.$$

If we choose $u = V$, $c_0 = c$, $N = p$ and $Z = x$ for the sake of simplicity, we can rewrite this model as follows:

$$\frac{\delta^2 V}{\delta t^2} - c^2 \frac{\delta^2 V}{\delta x^2} - \frac{\delta^2}{\delta x^2}\left(\frac{c^2}{2}V^2 + p\frac{\delta^2 V}{\delta t^2}\right) = 0, \tag{2.1}$$

here p is the dispersion parameter and c is the wave velocity for an magneto-electro-elastic circular rod. The dispersion parameter and the wave velocity for an magneto-electro-elastic circular rod depend on the material characteristics and also the geometry of the rod [53]. For the actual significance of equation (2.1) concluded that the infinite homogeneous magneto-electro-elastic circular rod is make of composite $BaTiO_3 - CoFe_2O_4$ with various volume fractions (v_f) of $BaTiO_3$. It might be also noted that the rod has a radius of $R = 0.05m$. The material features of the composite are obtained to make the simple rule of the mixture in accordance with the volume fraction. For more further details, the readers can refer to [33, 53, 56]. Numerous computational procedures have been employed to study the exact wave solutions of the linear longitudinal wave velocity for a MEE circular rod [16, 20, 26, 33, 36, 42, 44, 53, 56]. In this section, we firstly convert partial differential equation to the ordinary differential equation of the longitudinal wave motion equation in a magneto-electro-elastic circular rod through the travelling wave equation. Then, we apply the novel generalized $(\frac{G\prime}{G})$-expansion method to determine exact wave solution of the above model.

We assume that the equation (2.1) is in the following form:

$$V(\xi) = V(x,t), \xi = k(x - \omega t), \tag{2.2}$$

where k and ω are the wave number and wave velocity, respectively.

Equation (2.2) and equation (2.1) convert into a nonlinear equation as follows:

$$k^2 \frac{d^4V}{d\xi^4} - \frac{\omega^2 - c^2}{p\omega^2} \frac{d^2V}{d\xi^2} + \frac{c^2}{2p\omega^2} \frac{d^2(V^2)}{d\xi^2} = 0. \tag{2.3}$$

Integrating equation (2.3) twice with respect to ξ, and letting the integral constants be zero for convenience, we then have

$$k^2 \frac{d^2V}{d\xi^2} - \frac{\omega^2 - c^2}{p\omega^2} V + \frac{c^2}{2p\omega^2} V^2 = 0. \tag{2.4}$$

According to the newly generalized $(\frac{G\prime}{G})$-expansion method [3, 6, 11], we apply the balance rule in the equation (2.4), yields $n = 2$. Therefore, we have

$$V(\xi) = a_0 + a_1(d + T(\xi) + a_2(d + T(\xi)^2 + b_1(d + T(\xi))^{-1} + b_2(d + T(\xi))^{-2}, \tag{2.5}$$

where $T(\xi) = (\frac{G\prime}{G})$ and the coefficients a_0, a_1, a_2, b_1, b_2 and d are free unknown arbitrary constants to be evaluated. Plugging the values of equation (2.5) into equation (2.4) and then equating each coefficients of $(d + T(\xi))^n$ to zero, we get the following relations, respectively:

- **Cluster one**: $d = -\frac{B}{2\Psi}$, $k^2 = \frac{\omega^2 - c^2}{\omega^2} \frac{A^2}{4Ep\Psi + B^2 p}$, $\omega = \omega$, $A_0 = -\frac{3(c^2 - \omega^2)}{c^2}$, $A_1 = 0$, $A_2 = 0$, $B_1 = 0$, $B_2 = \frac{3}{4} \frac{c^2 - \omega^2}{c^2} \frac{4E\Psi + B^2}{\Psi^2}$,
- **Cluster two**: $d = -\frac{B}{2\Psi}$, $k^2 = \frac{c^2 - \omega^2}{\omega^2} \frac{A^2}{4Ep\Psi + B^2 p}$, $\omega = \omega$, $A_0 = \frac{(c^2 - \omega^2)}{c^2}$, $A_1 = 0$, $A_2 = 0$, $B_1 = 0$, $B_2 = -\frac{3}{4} \frac{c^2 - \omega^2}{c^2} \frac{4E\Psi + B^2}{\Psi^2}$,

- **Cluster three**: $d=-\frac{B}{2\Psi}$, $k^2=\frac{\omega^2-c^2}{\omega^2}\frac{A^2}{16Ep\Psi+4B^2p}$, $\omega=\omega$, $A_0=-\frac{1}{2}\frac{(c^2-\omega^2)}{c^2}$, $A_1=0$, $A_2=-\frac{(c^2-\omega^2)}{c^2}\frac{\Psi^2}{4E\Psi+B^2}$, $B_1=0$, $B_2=-\frac{3}{16}\frac{c^2-\omega^2}{c^2}\frac{4E\Psi+B^2}{\Psi^2}$,
- **Cluster four**: $d=-\frac{B}{2\Psi}$, $k^2=\frac{\omega^2-c^2}{\omega^2}\frac{A^2}{16Ep\Psi+4B^2p}$, $\omega=\omega$, $A_0=-\frac{3}{2}\frac{(c^2-\omega^2)}{c^2}$, $A_1=0$, $A_2=\frac{3(c^2-\omega^2)}{c^2}\frac{\Psi^2}{4E\Psi+B^2}$, $B_1=0$, $B_2=\frac{3}{16}\frac{c^2-\omega^2}{c^2}\frac{4E\Psi+B^2}{\Psi^2}$,
- **Cluster five**: $d=\frac{1}{6\Psi}(-3B\pm\sqrt{3}(E\Psi-B^2))$, $k^2=\frac{\omega^2-c^2}{\omega^2}\frac{A^2}{4Ep\Psi+B^2p}$, $\omega=\omega$, $A_0=-\frac{2(c^2-\omega^2)}{c^2}$, $A_1=0$, $A_2=0$, $B_1=\frac{2(4E\Psi+B^2)(c^2-\omega^2)}{\Psi c^2(\pm\sqrt{3}(E\Psi-B^2)}$, $B_2=\frac{1}{3}\frac{(4E\Psi+B^2)(c^2-\omega^2)}{\Psi^2c^2}$,
- **Cluster six**: $d=\frac{1}{6\Psi}(-3B\pm\sqrt{3}(E\Psi-B^2))$, $k^2=\frac{\omega^2-c^2}{\omega^2}\frac{A^2}{4Ep\Psi+B^2p}$, $\omega=\omega$, $A_0=0$, $A_1=0$, $A_2=0$, $B_1=-\frac{2(4E\Psi+B^2)(c^2-\omega^2)}{\Psi c^2(\pm\sqrt{3}(E\Psi-B^2)}$, $B_2=-\frac{1}{3}\frac{(4E\Psi+B^2)(c^2-\omega^2)}{\Psi^2c^2}$,
- **Cluster seven**: $d=d$, $k^2=\frac{\omega^2-c^2}{\omega^2}\frac{A^2}{4Ep\Psi+B^2p}$, $\omega=\omega$, $A_0=\frac{c^2-\omega^2}{c^2}\frac{d^2\Psi^2+Bd\Psi-E\Psi}{4E\Psi+B^2}$, $A_1=0$, $A_2=0$, $B_1=-\frac{12(c^2-\omega^2)}{c^2}\frac{(2d\Psi+B)(d^2\Psi+Bd-E)}{(4E\Psi+B^2)}$, $B_2=\frac{12(c^2-\omega^2)}{c^2}\frac{(d^2\Psi+Bd-E)^2}{(4E\Psi+B^2)}$,
  where $\Psi=A-C$, A, B, C and E are free constants.

Using the values of the cluster one into equation (2.5) and equation (2.4), we obtain a new hyperbolic function solutions, rational function solutions and trigonometric function solutions, respectively:

$$V_1(x,t)=-\frac{3(c^2-\omega^2)}{c^2}+\frac{3}{4}\frac{c^2-\omega^2}{c^2}\frac{4E\Psi+B^2}{\Psi^2}(\frac{\sqrt{\Omega}}{2\Psi}tanh(\frac{\sqrt{\Omega}\xi}{2A})^{-2}, \tag{2.6}$$

$$V_2(x,t)=-\frac{3(c^2-\omega^2)}{c^2}+\frac{3}{4}\frac{c^2-\omega^2}{c^2}\frac{4E\Psi+B^2}{\Psi^2}(\frac{\sqrt{\Omega}}{2\Psi}coth(\frac{\sqrt{\Omega}\xi}{2A})^{-2}, \tag{2.7}$$

$$V_3(x,t)=-\frac{3(c^2-\omega^2)}{c^2}+\frac{3}{4}\frac{c^2-\omega^2}{c^2}\frac{4E\Psi+B^2}{\Psi^2}(\frac{-B}{2\Psi}+\frac{\sqrt{\triangle}}{\Psi}tanh(\frac{\sqrt{\triangle}\xi}{A})^{-2}, \tag{2.8}$$

$$V_4(x,t)=-\frac{3(c^2-\omega^2)}{c^2}+\frac{3}{4}\frac{c^2-\omega^2}{c^2}\frac{4E\Psi+B^2}{\Psi^2}(\frac{-B}{2\Psi}+\frac{\sqrt{\triangle}}{\Psi}coth(\frac{\sqrt{\triangle}\xi}{A})^{-2}, \tag{2.9}$$

$$V_5(x,t)=-\frac{3(c^2-\omega^2)}{c^2}+\frac{3}{4}\frac{c^2-\omega^2}{c^2}\frac{4E\Psi+B^2}{\Psi^2}(\frac{C_2}{C_1+C_2\xi})^{-2}, \tag{2.10}$$

$$V_6(x,t)=-\frac{3(c^2-\omega^2)}{c^2}+\frac{3}{4}\frac{c^2-\omega^2}{c^2}\frac{4E\Psi+B^2}{\Psi^2}(-\frac{\sqrt{-}\Omega}{2\Psi}tan(\frac{\sqrt{-}\Omega\xi}{2A})^{-2}, \tag{2.11}$$

$$V_7(x,t)=-\frac{3(c^2-\omega^2)}{c^2}+\frac{3}{4}\frac{c^2-\omega^2}{c^2}\frac{4E\Psi+B^2}{\Psi^2}(\frac{\sqrt{-}\Omega}{2\Psi}cot(\frac{\sqrt{-}\Omega\xi}{2A})^{-2}, \tag{2.12}$$

$$V_8(x,t)=-\frac{3(c^2-\omega^2)}{c^2}+\frac{3}{4}\frac{c^2-\omega^2}{c^2}\frac{4E\Psi+B^2}{\Psi^2}(\frac{-B}{2\Psi}-\frac{\sqrt{-}\triangle}{\Psi}tant(\frac{\sqrt{-}\triangle\xi}{A})^{-2}, \tag{2.13}$$

$$V_9(x,t)=-\frac{3(c^2-\omega^2)}{c^2}+\frac{3}{4}\frac{c^2-\omega^2}{c^2}\frac{4E\Psi+B^2}{\Psi^2}(\frac{-B}{2\Psi}+\frac{\sqrt{-}\triangle}{\Psi}cot(\frac{\sqrt{-}\triangle\xi}{A})^{-2}, \tag{2.14}$$

where $\xi = \frac{\beta A}{\omega}(x - \omega t)$.

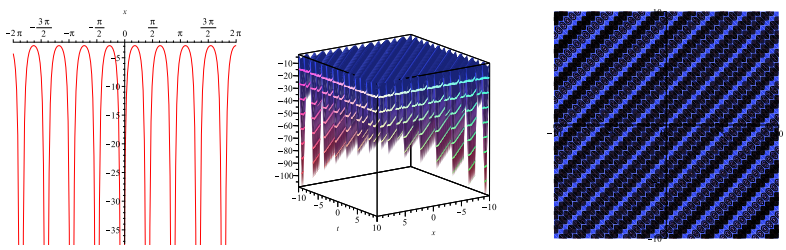
Similarly, we could provide exact solutions, including the soliton solutions of other six case of the longitudinal wave motion equation in a magneto-electro-elastic circular rod, which are omitted for convenience. Here, we compare our obtained results with Xue's exact wave solution in [53]. Xue et al. only got one wave solution in [53]. In this paper, we received total sixty three exact wave solutions of equation (2.1).

Remark 1. All the above solutions of the longitudinal wave motion equation in a magneto-electro-elastic circular rod are merely novel and identical from the published in [53].

Remark 2. All of the above solutions of this article have been verified with symbolic computer program Maple through substituting them into the main equation as the longitudinal wave motion equation in a magneto-electro-elastic circular rod.

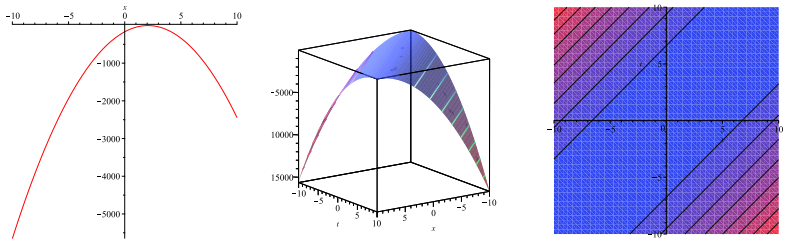
2.1. *Graphical illustrations of the obtained solutions*

For some suitable parameters, we get explicit travelling wave solution including compaction, bright solition, dark solition, and periodic wave solutions. These obtained solutions could assist in explaining the dynamics of spatiotemporal dispersion, higher-order dispersion, and full nonlinearity and also could be used to the propagation of solitons by a variety of waveguides. We illustrate some of the obtained solutions through choosing the proper values of the unknown free arbitrary parameters received. The graphical illustrations, including the three, two dimensional as well as the corresponding the contour structure surfaces of these obtained solutions are displayed in Figures 1, 2 and 3, respectively. Similarly, we could present the graphical description of the other six cases of the longitudinal wave motion equation in a magneto-electro-elastic circular rod, which are omitted for convenience.



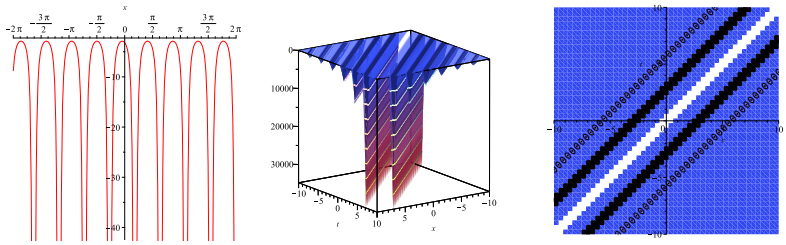
a. b. c.

FIGURE 1. $2D$, $3D$ and contour plot graphical representation of solution in $V_1(x,t)$ and its projection at $t = 2$ for the unknown perimeters $A = 2$, $B = 1$, $C = 1$, $E = 1$, $\omega = 1$, $c = 5.025$ and $p = 1.25$.



a. b. c.

FIGURE 2. $2D$, $3D$ and contour plot graphical representation of solution in $V_5(x,t)$ and its projection at $t=2$ for the unknown perimeters $A=5$, $B=2$, $C=2$, $E=1$, $C_1=1$, $C_2=2$, $\omega=1$, $c=5.025$ and $p=1.25$.



a. b. c.

FIGURE 3. $2D$, $3D$ and contour plot graphical representation of solution in $V_4(x,t)$ and its projection at $t=2$ for the unknown perimeters $A=4$, $B=0$, $C=1$, $E=1$, $\omega=1$, $c=5.025$ and $p=1.25$.

2.2. *Results and Discussion*

In this section, we present the physical meaning of the new hyperbolic, trigonometric as well as rational function solutions achieved through the new generalized $(\frac{G'}{G})$-expansion method on equation (2.1). These obtained explicit exact travelling wave solutions have proposed to the literature some relevant physical information about equation (2.1). It is considered that these explicit exact travelling solutions are linked to physical characteristics of hyperbolic functions [18, 50]. In particular, the complex hyperbolic function solutions equation (2.13) and equation (2.14) attained in this paper are prognosticated to symbolize the gravitational potential of a cylinder modelled by equation (2.1) [50]. Conclusively, when it comes to the surfaces, Figures 1, 2 and 3, respectively have been plotted including the two, three-dimensional as well as the corresponding the contour plot surfaces via Maple 13 by studying

proper values of parameters taken through the MEE material properties. When we examine all explicit exact travelling wave solutions achieved in this paper via the new generalized $(\frac{G\prime}{G})$-expansion method, it is decided that two-dimensional, three-dimensional surfaces and the corresponding contour plot surfaces are the graphical illustrations determined. Therefore, it can be announced that they are understandable physically because all the graphical illustrations dispense similar wave behaviours for proper values of parameters under the terms of magneto-electro-elastic material properties. Furthermore, it has been mentioned that all explicit exact travelling wave solutions satisfy equation (2.1) under the terms of equation (2.2) via Maple 13. To the skilled of our knowledge, the employment of the new generalized $(\frac{G\prime}{G})$-expansion method to equation (2.1) has not been published beforehand. Similarly, we can give the explicit exact travelling wave solutions and all the graphical representation of the other six cases of equation (2.1), which are omitted for convenience.

3. Exact solution of the Drinfeld-Sokolov-Wilson (DSW) equation

Drinfeld and Sokolov [25] and Wilson [51] has been proposed the DSW model for dispersive water waves and that play an important role in fluid dynamics [24, 29]. To study of the model is given by

$$\frac{\delta U}{\delta t} + \alpha_1 V \frac{\delta V}{\delta x} = 0, \tag{3.1}$$

$$\frac{\delta V}{\delta t} + \alpha_2 \frac{\delta^3 V}{\delta x^3} + \alpha_3 U \frac{\delta V}{\delta x} + \alpha_4 V \frac{\delta U}{\delta x} = 0, \tag{3.2}$$

where α_1, α_2, α_3, and α_4 are nonzero parameters. Recently, several researchers have been showed their interested on this model [1, 2, 13, 14, 21, 25, 27–32, 37, 40, 45, 51, 54, 59]. Qin and Yan [54] construct doubly periodic solutions of the coupled DSW equation by using an improve F-expansion method. Ayub et al. [14] have introduced the Exp-function method to look for solitary solutions of the generalized DSW system. Sweet and Gorder [45] applied the method of homotopy analysis to obtain analytical solutions of the generalized DSW system. Jawad [30] has introduced the traveling wave solutions and new solitary wave solutions of the DSW equation, Fornberg-Whitham equation, potential-TSF equation, Jimbo-Miwa equation, Modified Zakharov-Kuznetsov equation, and (2 + 1)-dimensional Konopelchenko-Dubrovsky equation via the tanh and Sech function methods. Cesar [21] obtained exact solutions of this model by applying the improved tanh-coth. Abdelaziz and Ibrahim [1] has been proposed the enhanced of the G'/G-expansion method combined with Liu's theorem to find new exact solutions of the nonlinear (1+1)-dimensional DSW equation. Zhang [59] established variational principles of the DSW equation Via the semi-inverse method and also obtained an exact solitary solution and exact singular periodic wave solution using the variational scheme. Gurefe

and Misirli [28] applied the Exp-function method to obtain generalized solitary solutions of the generalized (2+1)dimensional Burgers-type equation and the generalized DSW system.

Niu and Liu [40] re-examined the well known coupled system as the DSW equation. Its proper Darboux transformation is constructed with the help of a Lax operator of fourth order and some solutions are calculated and a nonlinear superposition formula is worked out for the associated Backlund transformation. Arnous et al. [13] employed two integration schemes to draw solitons, singular periodic waves and other types of solutions of the DSW equation. Jin and Lu [31] applied the variational iteration method to solve the classical DSW equation. Hirota et al. [29] present this equation a novel type of solutions called static solitons and this static solutions interact with moving solitons without deformations.

In this section, we implemented the analytical method including the newly generalized $(\frac{G\prime}{G})$-expansion method to solve exact solution of the Drinfeld-Sokolov-Wilson (DSW) equation.

Herein, we use traveling wave variable $V(\xi) = V(x,t)$, $\xi = x + \beta t$. Then equation (3.1) and (3.2) converts into the following nonlinear ODEs:

$$\beta \frac{dU}{d\xi} + \alpha_1 V \frac{dV}{d\xi} = 0, \tag{3.3}$$

$$\beta \frac{dV}{d\xi} + \alpha_2 \frac{d^3V}{d\xi^3} + \alpha_3 U \frac{dV}{d\xi} + \alpha_4 \frac{dU}{d\xi} V = 0. \tag{3.4}$$

Now integrating equation (3.3), we have

$$U = -\frac{\alpha_1 V^2}{2\beta}. \tag{3.5}$$

From equation (3.5) and equation (3.4), we obtain

$$2\beta\alpha_2 \frac{d^3V}{d\xi^3} + 2\beta^2 \frac{dV}{d\xi} - \alpha_1(\alpha_3 + 2\alpha_4)V^2 \frac{dV}{d\xi} = 0. \tag{3.6}$$

Again integrating equation (3.6), we obtain

$$2\beta\alpha_2 \frac{d^2V}{d\xi^2} + 2\beta^2 V - \frac{\alpha_1(\alpha_3 + 2\alpha_4)}{3} V^3 = 0. \tag{3.7}$$

According to the newly generalized $(\frac{G\prime}{G})$-expansion method [4, 5, 8], we obtain

$$\mathrm{u}(\xi) = t_0 + t_1 S(\xi) + r_1 S(\xi)^{-1}, \tag{3.8}$$

where $S(\xi) = (d + T(\xi))^i$, $T(\xi) = (\frac{G\prime}{G})$ and the coefficients t_0, t_1, r_1 and d are unknown costants. In view of equation (3.8) and equation (3.7), then equating each coefficients of $S(\xi)$ to zero, we have the following relations respectively:

- **Phase one**: $\beta = \frac{(B^2+4E\Psi)\alpha_2}{4A^2}$, $d = -\frac{B}{2\Psi}$, $t_0 = 0$, $t_1 = 0$, $r_1 = \frac{\alpha_5\alpha_2(B^2+4E\Psi)}{\Psi A^2}$, where $\alpha_5 = \pm\frac{3B^2+12E\Psi}{64\alpha_1\alpha_4+32\alpha_1\alpha_3}$, $\Psi = A - C$, A, B, C and E are free constants,

- **Phase two**: $\beta = \frac{(B^2+4E\Psi)\alpha_2}{A^2}$, $d = \alpha_6$, $t_0 = -\frac{\alpha_2\alpha_7(B+\alpha_6)}{3A^2}$, $t_1 = 0$, $r_1 = \frac{(B^2+4E\Psi)\alpha_2\alpha_7}{\Psi A^2}$,
  where $\alpha_6 = \frac{-B\pm\sqrt{(B^2+36E\Psi+8B^2)}}{2}$, $\alpha_7 = \pm\frac{24B^2+96E\Psi}{2\alpha_1\alpha_4+\alpha_1\alpha_3}$, $\Psi = A-C$, A, B, C and E are free constants,
- **Phase three**: $\beta = \frac{(B^2+4E\Psi)\alpha_2}{A^2}$, $d = d$, $t_0 = -\frac{12\alpha_2(8Ed\Psi^2+2\Psi B^2d+4BE\Psi+B^3)}{(\alpha_3+2\alpha_4)\alpha_7A^2\alpha_1}$, $t_1 = \frac{\alpha_7\Psi\alpha_2)}{4A^2}$, $r_1 = 0$,
  where $\Psi = A-C$, A, B, C and E are free constants.

For phase one, we get a new hyperbolic function solutions, rational function solutions and trigonometric function solutions to the DSW equation as given below, respectively:

$$V_{11}(\xi) = \frac{\alpha_5\alpha_2(B^2+4E\Psi)}{\Psi A^2}\frac{2\Psi}{\sqrt{\Omega}}tanh(\frac{\sqrt{\Omega}\xi}{2A}), \tag{3.9}$$

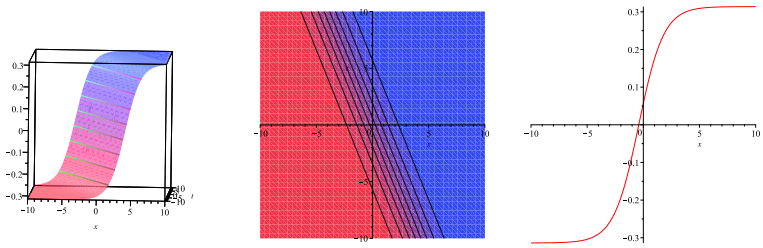


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FIGURE 4. $3D$, contour plot and $2D$ shape of the exact solution $V_{11}(x,t)$ at the particular values of $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$, $\alpha_4 = 4$, $A = 4$, $B = 1$, $C = 1$, $E = 1$ and $t = 1$.

$$V_{12}(\xi) = \frac{\alpha_5\alpha_2(B^2+4E\Psi)}{\Psi A^2}\frac{2\Psi}{\sqrt{\Omega}}coth(\frac{\sqrt{\Omega}\xi}{2A}), \tag{3.10}$$

$$V_{13}(\xi) = \frac{\alpha_5\alpha_2(B^2+4E\Psi)}{\Psi A^2}\frac{2\Psi}{\sqrt{\Omega}}tan(\frac{\sqrt{-\Omega}\xi}{2A}), \tag{3.11}$$

$$V_{14}(\xi) = \frac{\alpha_5\alpha_2(B^2+4E\Psi)}{\Psi A^2}\frac{2\Psi}{\sqrt{\Omega}}cot(\frac{\sqrt{-\Omega}\xi}{2A}), \tag{3.12}$$

$$V_{15}(\xi) = \frac{\alpha_5\alpha_2(B^2+4E\Psi)}{\Psi A^2}(\frac{C_2}{C_1+C_2\xi})^{-1}, \tag{3.13}$$

$$V_{16}(\xi) = \frac{\alpha_5\alpha_2(B^2+4E\Psi)}{\Psi A^2}(\frac{-B}{2\Psi}+\frac{\sqrt{\triangle}}{\Psi}coth(\frac{\sqrt{\triangle}\xi}{A}))^{-1}, \tag{3.14}$$

$$V_{17}(\xi) = \frac{\alpha_5\alpha_2(B^2+4E\Psi)}{\Psi A^2}\left(\frac{-B}{2\Psi} + \frac{\sqrt{\triangle}}{\Psi} tanh\left(\frac{\sqrt{\triangle}\xi}{A}\right)\right)^{-1}, \tag{3.15}$$

$$V_{18}(\xi) = \frac{\alpha_5\alpha_2(B^2+4E\Psi)}{\Psi A^2}\left(\frac{-B}{2\Psi} + \frac{\sqrt{-\triangle}}{\Psi} cot\left(\frac{\sqrt{-\triangle}\xi}{A}\right)\right)^{-1}, \tag{3.16}$$

$$V_{19}(\xi) = \frac{\alpha_5\alpha_2(B^2+4E\Psi)}{\Psi A^2}\left(\frac{-B}{2\Psi} + \frac{\sqrt{-\triangle}}{\Psi} tan\left(\frac{\sqrt{-\triangle}\xi}{A}\right)\right)^{-1}, \tag{3.17}$$

where $\xi = x + \frac{(B^2+4E\Psi)\alpha_2}{4A^2}t$.

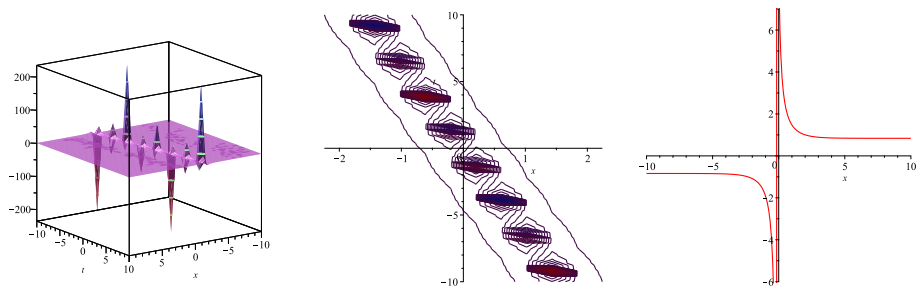

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FIGURE 5. $3D$, contour plot and $2D$ shape of the exact solution $V_{12}(x,t)$ at the particular values of $\alpha_1 = 1$, $\alpha_2 = 0.5$, $\alpha_3 = 0.3$, $\alpha_4 = 0.1$, $A = 2$, $B = 1$, $C = 1$, $E = 1$ and $t = 1$.

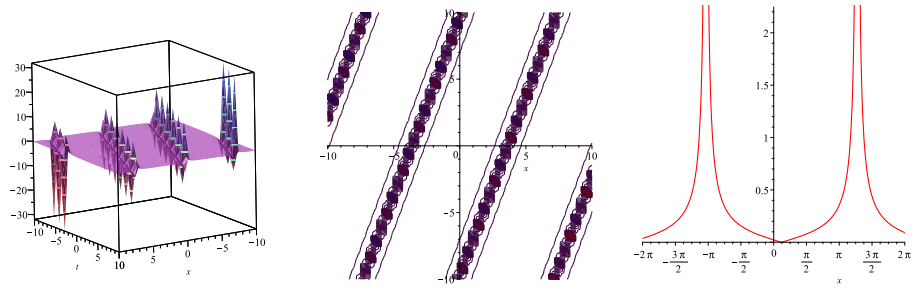

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FIGURE 6. $3D$, contour plot and $2D$ shape of the exact solution $V_{13}(x,t)$ at the particular values of $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$, $\alpha_4 = 4$, $A = 2$, $B = 1$, $C = 3$, $E = 1$ and $t = 1$.

Similarly to phase two, we have the following relations respectively:

$$V_{21}(\xi) = -\frac{\alpha_2\alpha_7(B+\alpha_6)}{3A^2} + \frac{\alpha_5\alpha_2(B^2+4E\Psi)}{\Psi A^2}\left(\alpha_6 + \frac{B}{2\Psi} + \frac{\sqrt{\Omega}}{2\Psi} coth\left(\frac{\sqrt{\Omega}\xi}{2A}\right)\right)^{-1}, \tag{3.18}$$

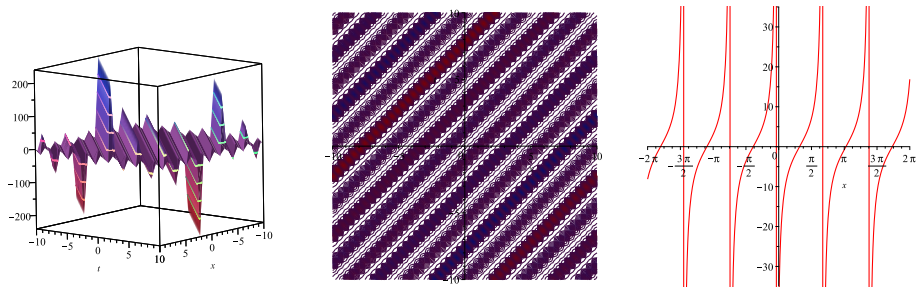

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FIGURE 7. $3D$, contour plot and $2D$ shape of the exact solution $V_{18}(x,t)$ at the particular values of $\alpha_1 = 1$, $\alpha_2 = 0.5$, $\alpha_3 = 0.3$, $\alpha_4 = 0.1$, $A = 1$, $B = 0$, $C = 2$, $E = 1$ and $t = 1$.

$$V_{22}(\xi) = -\frac{\alpha_2\alpha_7(B+\alpha_6)}{3A^2} + \frac{\alpha_5\alpha_2(B^2+4E\Psi)}{\Psi A^2}(\alpha_6 + \frac{B}{2\Psi} + \frac{\sqrt{\Omega})}{2\Psi} tanh(\frac{\sqrt{\Omega}\xi}{2A}))^{-1}, \quad (3.19)$$

$$V_{23}(\xi) = -\frac{\alpha_2\alpha_7(B+\alpha_6)}{3A^2} + \frac{\alpha_5\alpha_2(B^2+4E\Psi)}{\Psi A^2}(\alpha_6 + \frac{B}{2\Psi} + \frac{\sqrt{-\Omega}}{2\Psi} cot(\frac{\sqrt{-\Omega}\xi}{2A}))^{-1}, \quad (3.20)$$

$$V_{24}(\xi) = -\frac{\alpha_2\alpha_7(B+\alpha_6)}{3A^2} + \frac{\alpha_5\alpha_2(B^2+4E\Psi)}{\Psi A^2}(\alpha_6 + \frac{B}{2\Psi} + \frac{\sqrt{-\Omega}}{2\Psi} tan(\frac{\sqrt{-\Omega}\xi}{2A}))^{-1}, \quad (3.21)$$

$$V_{25}(\xi) = -\frac{\alpha_2\alpha_7(B+\alpha_6)}{3A^2} + \frac{\alpha_5\alpha_2(B^2+4E\Psi)}{\Psi A^2}(\alpha_6 + \frac{B}{2\Psi} + \frac{C_2}{C_1+C_2\xi})^{-1}, \quad (3.22)$$

$$V_{26}(\xi) = -\frac{\alpha_2\alpha_7(B+\alpha_6)}{3A^2} + \frac{\alpha_5\alpha_2(B^2+4E\Psi)}{\Psi A^2}(\alpha_6 + \frac{\sqrt{\triangle}}{\Psi} tanh(\frac{\sqrt{\triangle}\xi}{A}))^{-1}, \quad (3.23)$$

$$V_{27}(\xi) = -\frac{\alpha_2\alpha_7(B+\alpha_6)}{3A^2} + \frac{\alpha_5\alpha_2(B^2+4E\Psi)}{\Psi A^2}(\alpha_6 + \frac{\sqrt{\triangle}}{\Psi} coth(\frac{\sqrt{\triangle}\xi}{A}))^{-1}, \quad (3.24)$$

$$V_{28}(\xi) = -\frac{\alpha_2\alpha_7(B+\alpha_6)}{3A^2} + \frac{\alpha_5\alpha_2(B^2+4E\Psi)}{\Psi A^2}(\alpha_6 + \frac{\sqrt{-\triangle}}{\Psi} tan(\frac{\sqrt{-\triangle}\xi}{A}))^{-1}, \quad (3.25)$$

$$V_{29}(\xi) = -\frac{\alpha_2\alpha_7(B+\alpha_6)}{3A^2} + \frac{\alpha_5\alpha_2(B^2+4E\Psi)}{\Psi A^2}(\alpha_6 + \frac{\sqrt{-\triangle}}{\Psi} cot(\frac{\sqrt{-\triangle}\xi}{A}))^{-1}, \quad (3.26)$$

where $\xi = x + \frac{(B^2+4E\Psi)\alpha_2}{A^2}t$.

Again similarly to phase three, we find the following relations respectively:

$$V_{31}(\xi) = -\frac{12\alpha_2(8Ed\Psi^2 + 2\Psi B^2 d + 4BE\Psi + B^3)}{(\alpha_3+2\alpha_4)\alpha_7 A^2\alpha_1} + \frac{\alpha_7\Psi\alpha_2}{4A^2}(d + \frac{B}{2\Psi} + \frac{\sqrt{\Omega}}{2\Psi} tanh(\frac{\sqrt{\Omega}\xi}{2A})), \quad (3.27)$$

$$V_{32}(\xi) = -\frac{12\alpha_2(8Ed\Psi^2 + 2\Psi B^2 d + 4BE\Psi + B^3)}{(\alpha_3+2\alpha_4)\alpha_7 A^2\alpha_1} + \frac{\alpha_7\Psi\alpha_2}{4A^2}(d + \frac{B}{2\Psi} + \frac{\sqrt{\Omega}}{2\Psi} coth(\frac{\sqrt{\Omega}\xi}{2A})), \quad (3.28)$$

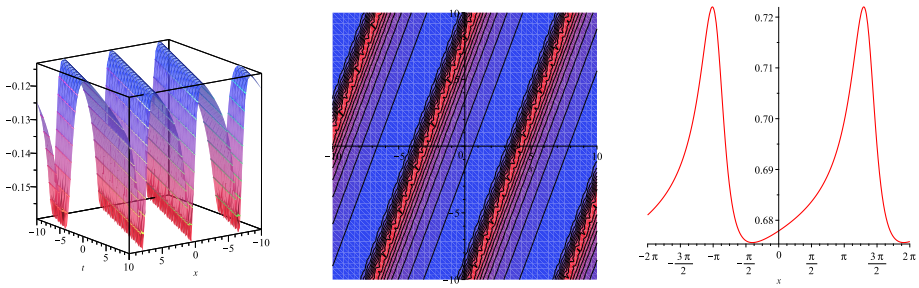

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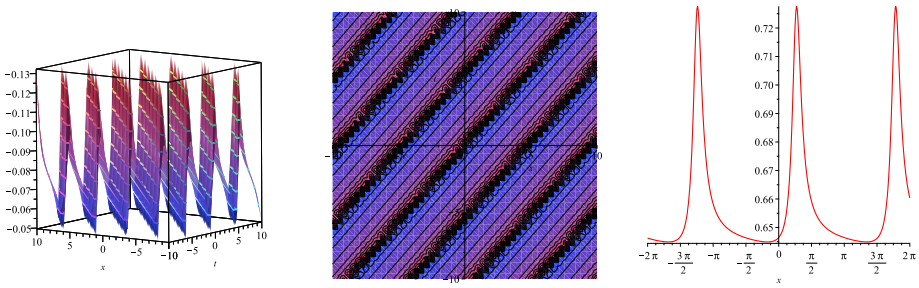
FIGURE 8. $3D$, contour plot and $2D$ shape of the exact solution $V_{23}(x,t)$ at the particular values of $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$, $\alpha_4 = 4$, $A = 2$, $B = 1$, $C = 3$, $E = 1$ and $t = 1$.



a. b. c.

FIGURE 9. $3D$, contour plot and $2D$ shape of the exact solution $V_{24}(x,t)$ at the particular values of $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$, $\alpha_4 = 4$, $A = 2$, $B = 1$, $C = 4$, $E = 1$ and $t = 1$.

$$V_{33}(\xi) = -\frac{12\alpha_2(8Ed\Psi^2 + 2\Psi B^2 d + 4BE\Psi + B^3)}{(\alpha_3 + 2\alpha_4)\alpha_7 A^2 \alpha_1} + \frac{\alpha_7 \Psi \alpha_2}{4A^2}(d + \frac{B}{2\Psi} + \frac{\sqrt{\Omega}}{2\Psi} tan(\frac{\sqrt{\Omega}\xi}{2A})), \tag{3.29}$$

$$V_{34}(\xi) = -\frac{12\alpha_2(8Ed\Psi^2 + 2\Psi B^2 d + 4BE\Psi + B^3)}{(\alpha_3 + 2\alpha_4)\alpha_7 A^2 \alpha_1} + \frac{\alpha_7 \Psi \alpha_2}{4A^2}(d + \frac{B}{2\Psi} + \frac{\sqrt{\Omega}}{2\Psi} cot(\frac{\sqrt{\Omega}\xi}{2A})), \tag{3.30}$$

$$V_{35}(\xi) = -\frac{12\alpha_2(8Ed\Psi^2 + 2\Psi B^2 d + 4BE\Psi + B^3)}{(\alpha_3 + 2\alpha_4)\alpha_7 A^2 \alpha_1} + \frac{\alpha_7 \Psi \alpha_2}{4A^2}(d + \frac{B}{2\Psi} + \frac{C_2}{C_1 + C_2\xi}), \tag{3.31}$$

$$V_{36}(\xi) = -\frac{12\alpha_2(8Ed\Psi^2 + 2\Psi B^2 d + 4BE\Psi + B^3)}{(\alpha_3 + 2\alpha_4)\alpha_7 A^2 \alpha_1} + \frac{\alpha_7 \Psi \alpha_2}{4A^2}(d + \frac{\sqrt{\triangle}}{\Psi} coth(\frac{\sqrt{\triangle}\xi}{A})), \tag{3.32}$$

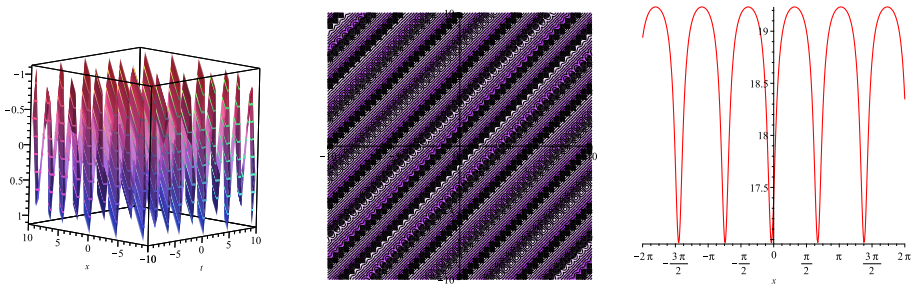


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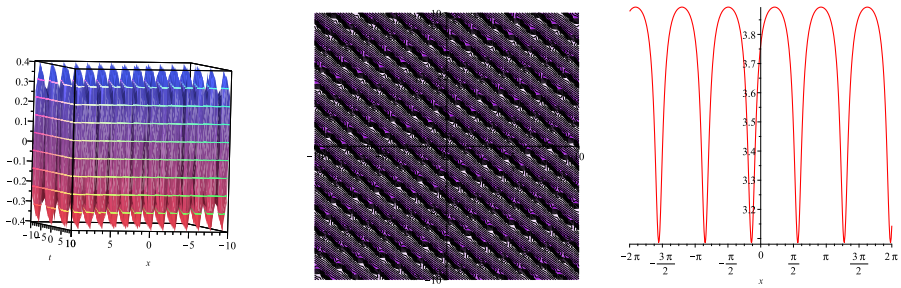
FIGURE 10. $3D$, contour plot and $2D$ shape of the exact solution $V_{28}(x,t)$ at the particular values of $\alpha_1 = 1$, $\alpha_2 = 0.5$, $\alpha_3 = 0.3$, $\alpha_4 = 0.1$, $A = 1$, $B = 0$, $C = 2$, $E = 2$ and $t = 1$.



a. b. c.

FIGURE 11. $3D$, contour plot and $2D$ shape of the exact solution $V_{29}(x,t)$ at the particular values of $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$, $\alpha_4 = 4$, $A = 1$, $B = 0$, $C = 2$, $E = 2$ and $t = 1$.

$$V_{37}(\xi) = -\frac{12\alpha_2(8Ed\Psi^2 + 2\Psi B^2 d + 4BE\Psi + B^3)}{(\alpha_3 + 2\alpha_4)\alpha_7 A^2 \alpha_1} + \frac{\alpha_7 \Psi \alpha_2}{4A^2}(d + \frac{\sqrt{\triangle}}{\Psi} tanh(\frac{\sqrt{\triangle}\xi}{A})), \tag{3.33}$$

$$V_{38}(\xi) = -\frac{12\alpha_2(8Ed\Psi^2 + 2\Psi B^2 d + 4BE\Psi + B^3)}{(\alpha_3 + 2\alpha_4)\alpha_7 A^2 \alpha_1} + \frac{\alpha_7 \Psi \alpha_2}{4A^2}(d + \frac{\sqrt{-\triangle}}{\Psi} cot(\frac{\sqrt{-\triangle}\xi}{A})), \tag{3.34}$$

$$V_{39}(\xi) = -\frac{12\alpha_2(8Ed\Psi^2 + 2\Psi B^2 d + 4BE\Psi + B^3)}{(\alpha_3 + 2\alpha_4)\alpha_7 A^2 \alpha_1} + \frac{\alpha_7 \Psi \alpha_2}{4A^2}(d + \frac{\sqrt{-\triangle}}{\Psi} tan(\frac{\sqrt{-\triangle}\xi}{A})), \tag{3.35}$$

where $\xi = x + \frac{(B^2+4E\Psi)\alpha_2}{A^2} t$.

Remark 3. All of the above solutions of this article have been verified with symbolic computer program Maple through substituting them into the main equation as the DSW equation.

3.1. *Graphical illustrations of the obtained solutions*

Similarly, some figures of the solutions are presented in the Figures 4, 5, 6, 7, 8, 9, 10 and 11, respectively.

4. Conclusion

In this research, the analytical soliton solutions to the considered model have been successfully got through the novel generalized $(\frac{G\prime}{G})$-expansion method. The solutions are obtained by rational, trigonometric and hyperbolic function by using this method. The obtained solutions showed that the studied method is more effective than the other methods (e.g., exp-function method, the Jacobi elliptic function expansion method), because it gives more new solutions. Consequently, the conclusions confirm that the above method is useful algorithms for the analytical approach of a broad spectrum of nonlinear modes of NLEEs arising in mathematical physics. We may decide that the proposed method could be continued to determine the nonlinear problems that appear in the theory of solitons and other fields.

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THE ALGEBRA OF BESSEL SEQUENCES AND MEANS OF FRAMES IN HILBERT SPACES

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Abstract. In this paper, a one-to-one correspondence between Bessel sequences and bounded linear operators is provided. This leads to an algebra structure on the set of all Bessel sequences in a separable Hilbert space. Some kinds of frames as special classes of operators are considered. Also, normal Bessel sequences and positive frames are presented. Finally, power means of positive frames are introduced. These allow researches to construct a large number of new frames from existing frames.

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1. Introduction and Preliminaries

Frames in Hilbert spaces were introduced by Duffin and Schaeffer [5] as a part of their study in non-harmonic Fourier series. Frames did not find serious attention until 1986 when Daubechies, Grossmann and Meyer [2] brought it all back to life during their fundamental work on wavelets.

There are many generalizations of frame. Most of these generalizations have been found useful applications. Some generalizations of frame significance have been presented such as generalized frames [12, 13], continuous frames [7], continuous fusion frames [6] and K-frames [8]. In the sequel, we discuss results on one such generalization of frames, called K-frames.

In recent years many researchers studied some methods for constructing new frames from existing ones. In 2009, Obeidat et al. [11] studied sum of two frames.

A basic and frequently used tool in study and characterizations of frames is the frame operator. It is known that the frame operator of a given frame is a bounded positive invertible operator. Since two different frames may have the same frame operator, a natural question arises here: Is there a way to correspond each Bessel sequence to one and only one bounded linear operator such that its properties characterizes special kinds of frames?

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In this paper, we find an appropriate correspondence between Bessel sequences (some kinds of frames) with bounded linear operators. Using this relation, we introduce an algebraic structure of the set of Bessel sequences in Hilbert spaces. Also, we study the means of frames.

Now, we recall some basic definitions and results that we need in the sequel. For background material on frame theory and the related topics, we refer to [1]. Throughout the paper $\mathcal{H}$ is a separable Hilbert space with an arbitrary fixed orthonormal basis $\{e_j\}_{j\in J}$ where $J \subseteq \mathbb{N}$. We denote by $B(\mathcal{H})$ the space of all bounded linear operators on $\mathcal{H}$. For $A \in \mathcal{H}$, we denote $R(A)$ for range of A. A bounded operator $A \in B(\mathcal{H})$ is called positive if $\langle Af, f\rangle \geq 0$ for all $f \in \mathcal{H}$. Let $B_+(\mathcal{H})$ denote the set of all bounded and positive operators on $\mathcal{H}$. Every positive operator is clearly self-adjoint. For $A, B \in B(\mathcal{H})$, we write $A \leq B$ if $B - A$ is positive. If $A \in B(\mathcal{H})$ is positive, then there exists a unique positive operator B such that $B^2 = A$. This will be denoted by $B = A^{\frac{1}{2}}$. Moreover, if an operator C commutes with A, then C commutes with every operator in the C^*-algebra generated by A and I, specially C commutes with $A^{\frac{1}{2}}$.

Definition 1. A family $F = \{f_j\}_{j\in J}$ of elements in $\mathcal{H}$ is called a *Bessel sequence* if there exists a constant $\mu > 0$ such that

$$\sum_{j\in J} |\langle f,\ f_j\rangle|^2 \leq \mu\|f\|^2, \qquad \forall f \in \mathcal{H}.$$

For a Bessel sequence $\{f_j\}_{j\in J}$, the operator $T_F : \ell_2 \to \mathcal{H}$ defined by $T_F(\{c_j\}_{j\in J}) = \sum_{j\in J} c_j f_j$, is bounded. The operator T_F is called the *synthesis operator* (or, *pre-frame operator*). The adjoint of T_F, $T_F^* : \mathcal{H} \to \ell_2$ defined by $T_F^* f = \{\langle f,\ f_j\rangle\}_{j\in J}$ is called the *analysis operator*. By composing T_F and T_F^*, we obtain the *frame operator*

$$S_F f = T_F T_F^* f = \sum_{j\in J} \langle f,\ f_j\rangle f_j \qquad \forall f \in \mathcal{H}. \tag{1.1}$$

Moreover, for each $f \in \mathcal{H}$,

$$\langle S_F f, f\rangle = \sum_{j\in J} |\langle f, f_j\rangle|^2.$$

Definition 2. The Bessel sequence $F = \{f_j\}_{j\in J}$ is a *frame* for $\mathcal{H}$ if there exists a constant $\lambda > 0$ such that

$$\lambda\|f\|^2 \leq \sum_{j\in J} |\langle f, f_j\rangle|^2, \qquad \forall f \in \mathcal{H}.$$

The constant λ is called the *lower frame bound* for the frame $\{f_j\}_{j\in J}$. Note that this bound is not unique.

For a given frame $F = \{f_j\}_{j\in J}$, the operator S_F is a bounded positive self-adjoint operator and hence S_F has a unique positive square root, denoted by $S_F^{1/2}$. The frame operator S_F of a given frame is invertible and, in fact, satisfies $\lambda I \leq S_F \leq \mu I$. For

normalized tight frames (that is, for frames with $\lambda = \mu = 1$), the frame operator is the identity operator.

The notion of K-frames has been introduced by Gavruta to study the atomic systems with respect to a bounded linear operator K in a separable Hilbert space. K-frames are more general than ordinary frames in the sense that the lower frame bound only holds for the elements in the range of K.

Definition 3 ([8])**.** Let $K \in B(\mathcal{H})$. A sequence $F = \{f_j\}_{j\in J}$ in $\mathcal{H}$ is called a *K-frame* for $\mathcal{H}$ if there exist constants $\lambda, \mu > 0$ such that

$$\lambda \|K^* f\|^2 \leq \sum_{j\in J} |\langle f, f_j\rangle|^2 \leq \mu \|f\|^2 \quad \forall f \in \mathcal{H}.$$

Definition 4 ([3])**.** Let $K \in B(\mathcal{H})$. A sequence $\{f_j\}_{j\in J}$ in $\mathcal{H}$ is said a *tight K-frame* with bound λ if

$$\lambda \|K^* f\|^2 = \sum_{j\in J} |\langle f, f_j\rangle|^2, \quad \forall f \in \mathcal{H}. \tag{1.2}$$

If $\lambda = 1$ in (1.2), then the sequence $\{f_j\}_{j\in J}$ is called Parseval *K-frame* for $\mathcal{H}$.

Theorem 1 ([13])**.** *Let $\{f_j\}_{j\in J}$ be a Bessel sequence in $\mathcal{H}$ and $K \in B(\mathcal{H})$. Then $F = \{f_j\}_{j\in J}$ is a K-frame for $\mathcal{H}$ if and only if there exists $\alpha > 0$ such that $S \geq \alpha KK^*$, where S is the frame operator for $\{f_j\}_{j\in J}$.*

In the sequel we need the following theorem, which is known as Douglas' factorization theorem.

Theorem 2 ([4])**.** *Let $\mathcal{H}$ be a Hilbert spaces and $A, B \in B(\mathcal{H})$. Then the following conditions are equivalent.*

(1) $R(A) \subseteq R(B)$.
(2) $AA^* \leq \alpha^2 BB^*$ *for some* $\alpha > 0$.
(3) *There exists* $X \in B(\mathcal{H})$ *such that* $A = BX$.

2. Bessel algebra $Bess(\mathcal{H})$

We seek a way to express each Bessel sequence (or frame) by one and only one operator. One natural way in this direction is using the frame operator.

Remark 1. Let $\{e_j\}_{j\in J}$ be an arbitrary orthonormal basis for $\mathcal{H}$ and $U \in B_+(\mathcal{H})$. It is easily seen that $F = \{f_j\}_{j\in J} = \{U^{1/2} e_j\}_{j\in J}$ is a tight K-frame with $K = U^{1/2}$ for $\mathcal{H}$. Moreover, considering (1.1), for each $f \in \mathcal{H}$, we have

$$\begin{aligned} S_F f &= \sum_{j\in J} \langle f, f_j\rangle f_j = \sum_{j\in J} \langle f, U^{1/2} e_j\rangle U^{1/2} e_j \\ &= U^{1/2} \left(\sum_{j\in J} \langle U^{1/2} f, e_j\rangle e_j \right) = U^{1/2} \left(U^{1/2} f \right) \\ &= U f. \end{aligned}$$

That is the frame operator of the frame $\{U^{1/2}e_j\}_{j\in J}$ is S_F.

The following example shows that two different Bessel sequences (or frames) may have the same frame operator.

Example 1. Let $\{e_j\}_{j\in J}$ be an arbitrary orthonormal basis for $\mathcal{H}$. Define

$$f_1 = e_1, f_2 = e_1 - e_2, \;\; f_3 = e_1 + e_2, \;\; f_j = e_{j-1} \; (j \geq 4).$$

It is easy task to show that $\{f_j\}_{j\in J}$ is a frame with the frame operator S_F defined by

$$S_F e_1 = 3e_1, \;\; S_F e_2 = 2e_2, \;\; S_F e_j = e_j \; (j \geq 3).$$

Putting $g_j = S_F^{1/2} e_j$, we have

$$g_1 = \sqrt{3}e_1, \;\; g_2 = \sqrt{2}e_2, \;\; g_j = e_j \; (j \geq 3).$$

It follows from Remark 1 that the frame operator of $\{g_j\}_{j\in J}$ is also S_F.

This example shows that the frame operator is not appropriate for our goal. In the following theorem, we correspond each Bessel sequence to one and only one operator. We denote by $Bess(\mathcal{H})$, $Fr(\mathcal{H})$ and $KFr(\mathcal{H})$ the set of all Bessel sequences, frames and K-frames in $\mathcal{H}$, respectively.

Theorem 3. *Let $\{e_j\}_{j\in J}$ be an arbitrary orthonormal basis for $\mathcal{H}$. The Bessel sequences for $\mathcal{H}$ are precisely the families $\{Ue_j\}_{j\in J}$, where U is a bounded operator on $\mathcal{H}$. Consequently, there is a one-to-one correspondence between the elements of $B(\mathcal{H})$ and $Bess(\mathcal{H})$.*

Proof. Let $\{\delta_j\}_{j\in J}$ be the canonical basis for $\ell^2(\mathbb{N})$ and $\{e_j\}_{j\in J}$ be an ortonormal basis for $\mathcal{H}$. Let $\Gamma : \mathcal{H} \to \ell^2(\mathbb{N})$ be the isometric isomorphism defined by $\Gamma e_j = \delta_j$. If $F = \{f_j\}_{j\in J}$ is a Bessel sequence, then the pre-frame operator T_F is bounded and $T_F\delta_j = f_j$. Putting $U_F = T_F\Gamma$, we have

$$\{f_j\}_{j\in J} = \{U_F e_j\}_{j\in J} \tag{2.1}$$

and U_F is bounded. If $U' \in B(\mathcal{H})$ is another operator such that $\{f_j\}_{j\in J} = \{U'e_j\}_{j\in J}$, then clearly $U' = U_F$. Therefore, U_F is unique.
On the other hand, if U is a bounded operator on $\mathcal{H}$, then for each $f \in \mathcal{H}$ we have

$$\sum_{j\in J} |\langle f, \, Ue_j\rangle|^2 = \sum_{j\in J} |\langle U^* f, \, e_j\rangle|^2 = \|U^* f\|^2 \leq \|U^*\|^2 \|f\|^2.$$

Hence $\{Ue_j\}_{j\in J}$ is a Bessel sequence for $\mathcal{H}$. Moreover, if S is the frame operator of $\{Ue_j\}_{j\in J}$, then for each $f \in \mathcal{H}$ we have

$$\begin{aligned} Sf &= \sum_{j\in J} \langle f, Ue_j\rangle Ue_j = \sum_{j\in J} \langle U^* f, e_j\rangle Ue_j \\ &= U\left(\sum_{j\in J} \langle U^* f, e_j\rangle e_j\right) = U\left(U^* f\right) \end{aligned}$$

$$= UU^* f \tag{2.2}$$

That is, the frame operator of the Bessel sequence $\{Ue_j\}_{j\in J}$ is UU^*. Now, define

$$\Theta : Bess(\mathcal{H}) \to B(\mathcal{H}) \quad \text{by} \quad \Theta(\{f_j\}_{j\in J}) = U_F \tag{2.3}$$

and

$$\Omega : B(\mathcal{H}) \to Bess(\mathcal{H}) \quad \text{by} \quad \Omega(U) = \{Ue_j\}_{j\in J}.$$

By the above mentioned argument, we see that Θ and Ω are well defined. Uniqueness of U_F together with (2.1) implies that

$$\Omega(\Theta(\{f_j\}_{j\in J})) = \Omega(U_F) = \{U_F e_j\}_{j\in J} = \{f_j\}_{j\in J}$$

and

$$\Theta(\Omega(U)) = \Theta(\{Ue_j\}_{j\in J}) = U.$$

This completes the proof. □

For a Bessel sequence $F = \{f_j\}_{j\in J}$ in $\mathcal{H}$ we denote by U_F the operator obtained in Theorem 3. As a consequence of Theorem 3 we have the following known result (see e.g. [1]).

Corollary 1. *Let $\mathcal{H}$ be a Hilbert spaces with the Bessel sequence space $Bess(\mathcal{H})$ and Θ be as in (2.3). Then,*

$$KFr(\mathcal{H}) = \Omega\{A \in B(\mathcal{H}) : R(K) \subseteq R(A)\}$$

and so

$$Fr(\mathcal{H}) = \Omega\{A \in B(\mathcal{H}) : A \text{ is surjective}\}.$$

Proof. It follows from Theorem 1 and (2.2) that a Bessel sequence $F = \{f_j\}_{j\in J}$ is K-frame if and only if there exists $\alpha > 0$ such that $U_F U_F^* = S_F \geq \alpha KK^*$. By Douglas' factorization theorem (Theorem 2), it is equivalent to $R(K) \subseteq R(U_F)$. Moreover, taking $K = I$ (the identity operator) we obtain desired conclusion. □

One-to-one correspondence in Theorem 3 motivate us to introduce the following notions.

Definition 5. Let $\{e_j\}_{j\in J}$ be an arbitrary orthonormal basis for $\mathcal{H}$ and $\{f_j\}_{j\in J}$ be a Bessel sequence in $\mathcal{H}$. Adjoint of $\{f_j\}_{j\in J}$ is a Bessel sequence $\{g_j\}_{j\in J}$ in $\mathcal{H}$ such that

$$\sum_{j\in J} \langle f, g_j\rangle\langle e_j, g\rangle = \sum_{j\in J} \langle f_j, g\rangle\langle f, e_j\rangle, \quad \forall f, g \in \mathcal{H}. \tag{2.4}$$

We denote by $\{f_j^*\}_{j\in J}$ the adjoint of $\{f_j\}_{j\in J}$. If

$$\sum_{j\in J} \langle f, f_j\rangle\langle e_j, g\rangle = \sum_{j\in J} \langle f_j, g\rangle\langle f, e_j\rangle, \quad \forall f, g \in \mathcal{H}, \tag{2.5}$$

we say that $\{f_j\}_{j\in J}$ is self-adjoint. Also, $\{f_j\}_{j\in J}$ is positive if

$$\sum_{j\in J} \langle f_j, f\rangle\langle f, e_j\rangle \geq 0, \quad \forall f \in \mathcal{H}. \tag{2.6}$$

A positive frame is a frame in which inequality (2.6) holds.

Theorem 4. *Let $\{f_j\}_{j\in J}$ be a Bessel sequence in $\mathcal{H}$. Then the followings hold:*

(i) *$\{f_j\}_{j\in J}$ has a unique adjoint in $\mathcal{H}$;*
(ii) *$\{f_j\}_{j\in J}$ is self-adjoint if and only if U_F is self-adjoint;*
(iii) *$\{f_j\}_{j\in J}$ is positive if and only if U_F is positive.*

Proof. (i) Let $\{f_j\}_{j\in J}$ be a Bessel sequence in $\mathcal{H}$. Consider the Bessel sequence $\{g_j\}_{j\in J} = \{U_F^* e_j\}_{j\in J}$. Now, we verify the equation (2.4). Let $f, g \in \mathcal{H}$. Then

$$\begin{aligned}
\sum_{j\in J}\langle f, g_j\rangle\langle e_j, g\rangle &= \sum_{j\in J}\langle f, U_F^* e_j\rangle\langle e_j, g\rangle \\
&= \sum_{j\in J}\langle U_F f, e_j\rangle\langle e_j, g\rangle \\
&= \left\langle \sum_{j\in J}\langle U_F f, e_j\rangle e_j, g\right\rangle \\
&= \langle U_F f, g\rangle \\
&= \langle f, U_F^* g\rangle \\
&= \left\langle f, \sum_{j\in J}\langle U_F^* g, e_j\rangle e_j\right\rangle \\
&= \sum_{j\in J}\langle e_j, U_F^* g\rangle\langle f, e_j\rangle \\
&= \sum_{j\in J}\langle U_F e_j, g\rangle\langle f, e_j\rangle = \sum_{j\in J}\langle f_j, g\rangle\langle f, e_j\rangle,
\end{aligned}$$

so $\{U_F^* e_j\}_{j\in J}$ is an adjoint of $F = \{f_j\}_{j\in J}$. Uniqueness follows from uniqueness of U_F in Theorem 3 and its adjoint.
(ii) By a similar computation as in (i), we get that

$$\langle U_F f, g\rangle = \langle f, U_F^* g\rangle = \sum_{j\in J}\langle f_j, g\rangle\langle f, e_j\rangle, \quad \forall f, g \in \mathcal{H}, \tag{2.7}$$

and

$$\langle U_F^* f, g\rangle = \sum_{j\in J}\langle f, f_j\rangle\langle e_j, g\rangle, \quad \forall f, g \in \mathcal{H},$$

which proves (ii).
It follows from (2.7) that

$$\langle U_F f, f\rangle = \sum_{j\in J}\langle f_j, f\rangle\langle f, e_j\rangle, \quad \forall f \in \mathcal{H},$$

which gives us (iii). □

Corollary 2. *Let $\{f_j\}_{j\in J}$ be a Bessel sequence in $\mathcal{H}$.*

(i) *If $\{f_j\}_{j\in J}$ is self-adjoint, then $\{U_F f_j\}_{j\in J}$ is positive.*

(ii) *If $V \in B(\mathcal{H})$ is self-adjoint, then $\{V^2 e_j\}_{j\in J}$ is positive.*
(iii) *If $\{f_j\}_{j\in J}$ is positive, then there exists a self-adjoint operator $V \in B(\mathcal{H})$ such that $\{f_j\}_{j\in J} = \{V^2 e_j\}_{j\in J}$.*

Proof. (i) We have

$$U_F f_j = U_F U_F e_j = U_F^2 e_j.$$

By Theorem 4, U_F is self-adjoint and so $U_F^2 = U_F U_F^* \geq 0$. Again, it follows from Theorem 4 that $\{U_F f_j\}_{j\in J}$ is positive.
(ii) Taking $\{f_j\}_{j\in J} = \{Ve_j\}_{j\in J}$, we have $U_F = V$. Since V is self-adjoint, then $\{f_j\}_{j\in J}$ is self-adjoint. Hence using the previous part, $\{V^2 e_j\}_{j\in J} = \{V f_j\}_{j\in J}$ is positive.
(iii) Since $\{f_j\}_{j\in J}$ is positive, it follows from Theorem 4 that U_F is positive. Put $V = U_F^{\frac{1}{2}}$. Then, $\{f_j\}_{j\in J} = \{U_F e_j\}_{j\in J} = \{V^2 e_j\}_{j\in J}$. □

Lemma 1. *Let $F = \{f_j\}_{j\in J}$ be a Bessel sequence with the frame operator S_F. If $\{f_j\}_{j\in J}$ is positive, then $U_F = S_F^{\frac{1}{2}}$.*

Proof. It follows from (2.2) that

$$S_F = U_F U_F^* = U_F^2.$$

By the uniqueness of positive square root we obtain the desired result. □

The following lemma provides a basic tool to define product of Bessel sequences.

Lemma 2. *Let $F = \{f_j\}_{j\in J}$ and $G = \{g_j\}_{j\in J}$ be two Bessel sequences in $\mathcal{H}$. Then, there exists a unique Bessel sequence $\{h_j\}_{j\in J}$ in $\mathcal{H}$ such that*

$$\sum_{j\in J} \langle f, h_j\rangle e_j = \sum_{j\in J}\sum_{i\in J} \langle f, f_i\rangle \langle e_i, g_j\rangle e_j, \quad \forall f \in \mathcal{H}. \tag{2.8}$$

Proof. Put $\{h_j\}_{j\in J} = \{U_F U_G e_j\}_{j\in J}$. For each $f \in \mathcal{H}$ we have

$$\begin{aligned}
\sum_{j\in J} \langle f, h_j\rangle e_j &= \sum_{j\in J} \langle f, U_F U_G e_j\rangle e_j = \sum_{j\in J} \langle U_F^* f, g_j\rangle e_j \\
&= \sum_{j\in J} \left\langle \sum_{i\in J} \langle U_F^* f, e_i\rangle e_i, g_j \right\rangle e_j \\
&= \sum_{j\in J}\sum_{i\in J} \langle U_F^* f, e_i\rangle \left\langle e_i, g_j\right\rangle e_j \\
&= \sum_{j\in J}\sum_{i\in J} \langle f, U_F e_i\rangle \left\langle e_i, g_j\right\rangle e_j \\
&= \sum_{j\in J}\sum_{i\in J} \langle f, f_i\rangle \langle e_i, g_j\rangle e_j.
\end{aligned}$$

To prove the uniqueness, let $H = \{h_j\}_{j\in J}$ and $M = \{m_j\}_{j\in J}$ be two Bessel sequences in $\mathcal{H}$ for which the equality (2.8) holds. Hence,

$$\sum_{j\in J}\langle f,h_j\rangle e_j = \sum_{j\in J}\sum_{i\in J}\langle f,f_i\rangle\langle e_i,g_j\rangle e_j = \sum_{j\in J}\langle f,m_j\rangle e_j, \quad \forall f\in\mathcal{H}. \tag{2.9}$$

On the other hand, we have

$$U_H^* f = \sum_{j\in J}\langle U_H^* f,e_j\rangle e_j = \sum_{j\in J}\langle f,U_H e_j\rangle e_j = \sum_{j\in J}\langle f,h_j\rangle e_j, \quad \forall f\in\mathcal{H}. \tag{2.10}$$

Considering (2.9) and (2.10) we see that $U_H = U_M$ and so $\{h_j\}_{j\in J} = \{m_j\}_{j\in J}$. □

For the Bessel sequences $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ in $\mathcal{H}$, we define product of them as the Bessel sequence obtained in Lemma 2.

Remark 2. By using the same argument as Theorem 4 and Lemma 2, we see that for the Bessel sequence $F = \{f_j\}_{j\in J}$ in $\mathcal{H}$, the operator U_F is normal if and only if

$$\sum_{j\in J}\sum_{i\in J}\langle f,f_i\rangle\langle e_i,f_j^*\rangle e_j = \sum_{j\in J}\sum_{i\in J}\langle f,f_i^*\rangle\langle e_i,f_j\rangle e_j, \quad \forall f\in\mathcal{H}.$$

In this case we say that the Bessel sequence $\{f_j\}_{j\in J}$ is normal.

Theorem 5. *The set $Bess(\mathcal{H})$ is an algebra with the following operations:*

$$\begin{aligned}\{f_j\}_{j\in J}+\{g_j\}_{j\in J} &:= \{f_j+g_j\}_{j\in J} = \{(U_F+U_G)e_j\}_{j\in J},\\ \alpha\{f_j\}_{j\in J} &:= \{\alpha f_j\}_{j\in J} = \{(\alpha U_F)e_j\}_{j\in J}\\ \{f_j\}_{j\in J}\cdot\{g_j\}_{j\in J} &= \{(U_F U_G)e_j\}_{j\in J}.\end{aligned}$$

The proof is straightforward.

Theorem 6. *The function*

$$\Theta : Bess(\mathcal{H}) \to B(\mathcal{H}) \qquad \textit{defined by} \qquad \Theta(\{f_j\}_{j\in J}) = U_F$$

is an algebra positive homomorphism.

3. Means of frames

Recall that power mean of two positive operators A and B is defined by

$$\left(\frac{A^r+B^r}{2}\right)^{\frac{1}{r}} \qquad r\in\mathbb{R}\backslash\{0\}.$$

We begin with Arithmetic mean of two positive frames.

Theorem 7. *Frame operator of Arithmetic mean of two positive frames is power mean of their frame operators for $r = \frac{1}{2}$.*

Proof. Let $F=\{f_j\}_{j\in J}$ and $G=\{g_j\}_{j\in J}$ be two positive frames in $\mathcal{H}$ with the frame operators S_F and S_G, respectively. We know from Lemma 1 that

$$f_j=U_Fe_j=S_F^{\frac{1}{2}}e_j,\quad g_j=U_Ge_j=S_G^{\frac{1}{2}}e_j.$$

Since Arithmetic mean of $F=\{f_j\}_{j\in J}$ and $G=\{g_j\}_{j\in J}$ is

$$\left\{\frac{f_j+g_j}{2}\right\}_{j\in J}=\left\{\left(\frac{U_F+U_G}{2}\right)e_j\right\}_{j\in J},$$

considering (2.2), its frame operator is

$$\left(\frac{U_F+U_G}{2}\right)\left(\frac{U_F+U_G}{2}\right)^*=\left(\frac{S_F^{1/2}+S_G^{1/2}}{2}\right)^2.$$

□

To extend the idea to other means we need apply functional calculus on $Bess(\mathcal{H})$. Let A be a normal operator on $\mathcal{H}$, $C(Sp(A))$ be the set of all continuous functions on the spectrum of A and $C^*(A,I)$ be the abalian C^*-algebra generated by A and I. An immediate consequence of Gelfand's Theorem [10, Theorem 2.1.10] is that there is an isometric $*$-isomorphism between $C(Sp(A))$ and $C^*(A,I)$. Hence, for any $\varphi\in C(Sp(A))$ the operator $\varphi(A)$ is in $C^*(A,I)$. Moreover, spectral mapping theorem [10, Theorem 2.1.14] guaranties that $\varphi(Sp(A))=Sp(\varphi(A))$. So, if $\varphi(Sp(A))\subseteq[0,+\infty)$, then $\varphi(A)$ is a positive operator.

Now, we define new Bessel sequences by using functional calculus. Let $F=\{f_j\}_{j\in J}$ be a Bessel sequence in $\mathcal{H}$ for which U_F is normal operator and $\varphi\in C(Sp(U_F))$. We define

$$\varphi(F)=\{\varphi(U_F)e_j\}_{j\in J}.$$

For example, if the Bessel sequence $F=\{f_j\}_{j\in J}$ is positive, then $\varphi(t)=t^{\frac{1}{2}}$ is continuous on $Sp(U_F)$ and we have

$$\varphi(F)=\{\varphi(U_F)e_j\}_{j\in J}=\{U_F^{\frac{1}{2}}e_j\}_{j\in J}.$$

We denote this new Bessel sequences by $\{f_j\}_{j\in J}^{\frac{1}{2}}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces. Recall that a linear mapping $\Phi:B(\mathcal{H})\to B(\mathcal{K})$ is called positive map if Φ maps positive elements to positive elements.

Definition 6. Let $\mathcal{H}$ and $\mathcal{K}$ be two separable Hilbert spaces. Every positive linear maps $\Phi:B(\mathcal{H})\to B(\mathcal{K})$ induces positive linear map $\widehat{\Phi}:Bess(\mathcal{H})\to Bess(\mathcal{K})$ such that

$$\widehat{\Phi}(F)=\{\Phi(U_F)\widehat{e}_j\}_{j\in J},$$

where $F=\{f_j\}_{j\in J}$ is a Bessel sequences in $\mathcal{H}$ and $\{\widehat{e}_j\}_{j\in J}$ is an arbitrary and fixed orthonormal bases for $\mathcal{K}$.

Lemma 3. *Let $\mathcal{H}$ and $\mathcal{K}$ be two separable Hilbert spaces with the Bessel sequence spaces $Bess(\mathcal{H})$ and $Bess(\mathcal{K})$, respectively. For each positive linear map $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$, the following diagram commutes:*

$$\begin{tikzcd} Bess(\mathcal{H}) \arrow[r, "\widehat{\Phi}"] \arrow[d, "\Theta"'] & Bess(\mathcal{K}) \arrow[d, "\Theta"] \\ B(\mathcal{H}) \arrow[r, "\Phi"'] & B(\mathcal{K}) \end{tikzcd}$$

where Θ is defined by (2.3).

Proof. Let $F = \{f_j\}_{j\in J}$ be a Bessel sequence in $\mathcal{H}$. Considering (2.3) we have $\Phi(\Theta\{f_j\}_{j\in J}) = \Phi(U_F)$. On the other hand,

$$\Theta(\widehat{\Phi}\{f_j\}_{j\in J}) = \Theta(\Phi(U_F)\widehat{e}_j) = \Phi(U_F).$$

□

Let $B_+(\mathcal{H})$ be the set of all positive operators of $B(\mathcal{H})$ and $P_N[B(\mathcal{H}),\ B(\mathcal{K})]$ be the set of all normalized positive linear maps from $B(\mathcal{H})$ to $B(\mathcal{K})$. Micic and Pecaric in [9] introduced the following weighted power means of positive operators and studied related inequalities. Let $\Phi_j \in P_N[B(\mathcal{H}),\ B(\mathcal{K})], A_i \in B_+(\mathcal{H})$ with $Sp(A_i) \subseteq [m,\ M]$ for some scalars $0 < m < M$ and $\omega_i \in \mathbb{R}^+$ such that $\sum_{i=1}^k \omega_i = 1$ $(i = 1,\ \ldots,\ k)$. Define

$$M_k^{[r]}(\mathrm{A},\ \Phi,\ \omega) := \left(\sum_{i=1}^k \omega_i \Phi_i(A_i^r)\right)^{1/r} \quad \text{if} \quad r \in \mathbb{R}\backslash\{0\}, \tag{3.1}$$

where $A = (A_1, \ldots, A_k)$, $\Phi = (\Phi_1, \ldots, \Phi_k)$ and $\omega = (\omega_1, \ldots, \omega_k)$. Similarly, if $F_i = \{f_{ij}\}_{j\in J}$ are positive frames in $\mathcal{H}$, we define their weighted power means as

$$\mathcal{M}_k^{[r]}(\mathrm{F},\ \Phi,\ \omega) := \left(\sum_{i=1}^k \omega_i \widehat{\Phi}_i(\{f_{ij}\}_{j\in J}^r)\right)^{1/r} \quad \text{if} \quad r \in \mathbb{R}\backslash\{0\}, \tag{3.2}$$

where $F = (F_1, \ldots, F_k)$.

Theorem 8. *Frame operator of weighted power mean of positive frames for r is weighted power mean of their frame operators for $\frac{r}{2}$.*

Proof. Let $\{e_j\}_{j\in J}$ and $\{\widehat{e}_j\}_{j\in J}$ be arbitrary and fixed orthonormal bases for $\mathcal{H}$ and $\mathcal{K}$, respectively. Let $F_i = \{f_{ij}\}_{j\in J}$ $(i = 1, \ldots, k)$ be positive frames in $\mathcal{H}$ with the frame operators S_{F_i}'s. The weighted power mean of the positive frames $F_i = \{f_{ij}\}_{j\in J}$

is a positive frame in $\mathcal{K}$ such that

$$\begin{aligned}\mathcal{M}_k^{[r]}(\mathrm{F},\ \Phi,\ \omega) &= \left(\sum_{i=1}^{k} \omega_i \widehat{\Phi}_i(\{f_{ij}\}_{j\in J}^r)\right)^{1/r} \\ &= \left(\sum_{i=1}^{k} \omega_i \Phi_i(U_{F_i}^r)\right)^{1/r} \widehat{e}_j = \left(\sum_{i=1}^{k} \omega_i \Phi_i(S_{F_i}^{\frac{r}{2}})\right)^{1/r} \widehat{e}_j := V\widehat{e}_j.\end{aligned}$$

Therefore, the frame operator of $\mathcal{M}_k^{[r]}(\mathrm{F},\ \Phi,\ \omega)$ is

$$\begin{aligned}VV^* &= \left(\sum_{i=1}^{k} \omega_i \Phi_i(S_{F_i}^{\frac{r}{2}})\right)^{1/r} \left(\sum_{i=1}^{k} \omega_i \Phi_i(S_{F_i}^{\frac{r}{2}})\right)^{1/r} \\ &= \left(\sum_{i=1}^{k} \omega_i \Phi_i(S_{F_i}^{\frac{r}{2}})\right)^{2/r} = M_k^{[\frac{r}{2}]}(S_F,\ \Phi,\ \omega),\end{aligned}$$

where $S_F = (S_{F_1}, \ldots, S_{F_k})$. □

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RELATIONSHIP OF THE BERNARDI INTEGRAL OPERATOR WITH CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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Abstract. In this article, we find a condition on α so that if $1+\alpha z\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}$, then $\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} \prec 1+\sin(z)$, where

$$\mathcal{F}(z) = \frac{\gamma+1}{z^{\beta}}\int_0^z t^{\gamma-1} f(t)\,dt,$$

is the well-known Bernardi integral operator. We also study the case $\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} \prec \cos(z)$. Some similar implications are also discussed for both functions.

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1. Introduction and Basic Notions

We recall here some basic notions and concepts of geometric function theory which are essential for clarity and understandings of the upcoming work. We start with the symbol $\mathcal{A}$ which represents the class of analytic functions in $\mathcal{D} = \{z : |z| < 1\}$ and any function f in $\mathcal{A}$ satisfies the conditions $f(0) = f'(0) - 1 = 0$. Also let $\mathcal{S}$ denote a subclass of $\mathcal{A}$ which contains univalent functions in $\mathcal{D}$. The notion of subordinations between analytic functions is defined as; If f and g are analytic functions, then f is subordinated by g, if there exists a Schwarz function w with the properties $w(0) = 0$ and $|w(z)| < |z|$ such that $f(z) = g(w(z))$. Further, if the function g is univalent in $\mathcal{D}$, then we have:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0), \tag{1.1}$$

and $f(\mathcal{D}) \subset g(\mathcal{D})$. Moreover, let $\mathcal{P}[A,B]$ denote a family of analytic functions p in $\mathcal{D}$ with $p(0) = 1$ and satisfy

$$p(z) \prec \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1.$$

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Geometrically, a function $p \in \mathcal{P}[A,B]$, if and only if $p(0)=1$ and $p(\mathcal{D}) \subset \Omega[A,B]$, where circular domain $\Omega[A,B]$ is defined as:

$$\Omega[A,B] = \begin{cases} \left\{\omega : \left|\omega - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2}\right\} & \text{for} \quad B \neq -1, \\ \{\omega : \Re\mathfrak{e}\{\omega\} > (1-A)/2\} & \text{for} \quad B = -1. \end{cases} \tag{1.2}$$

The domain $\Omega[A,B]$, $B \neq -1$, represents an open circular disk centered on the real axis with diameter end points $D_1 = \frac{1-A}{1-B}$ and $D_2 = \frac{1+A}{1+B}$ with $0 < D_1 < 1 < D_2$. This work was introduced by Janowski [9] and he further defined the class $\mathcal{S}^*[A,B]$ of Janowski starlike functions as:

$$\mathcal{S}^*[A,B] = \left\{ f : f \in \mathcal{A} \text{ and } \frac{zf'(z)}{f(z)} \in \mathcal{P}[A,B], \ (z \in \mathcal{D}) \right\}.$$

We see that the classes $\mathcal{P}[1,-1]$ and $\mathcal{S}^*[1,-1]$ coincides with the well-known classes $\mathcal{P}$ of functions with positive real part and $\mathcal{S}^*$of starlike functions respectively. Many researchers introduced and studied various subclasses of analytic functions connected with different image domains, see the work of Cho et al. [5], Dziok et al. [7], Kumar and Ravichandran [12], Mediratta et al. [15], Sokół and Stankiewicz [22], Raina and Sokół [16], Kanas and Răducanu [10], Sharma et al. [17], see also [1, 11, 18–20, 25] and the references therein.

Recently, Cho et al. [6] have studied a class of starlike functions connected with sine function and is defined as:

$$\mathcal{S}_s^* = \left\{ f : f \in \mathcal{A} \text{ and } \frac{zf'(z)}{f(z)} \prec 1 + \sin(z), \ (z \in \mathcal{D}) \right\}.$$

Analogous to the class $\mathcal{S}_s^*$, Bano and Raza [3] introduced the class $\mathcal{S}_c^*$ of starlike functions associated with cosine functions which is given by:

$$\mathcal{S}_c^* = \left\{ f : f \in \mathcal{A} \text{ and } \frac{zf'(z)}{f(z)} \prec \cos(z), \ (z \in \mathcal{D}) \right\}.$$

Integral and differential operators are very useful and are of great importance in geometric function theory, specially in univalent function theory. Certain differential and integral operators have been introduced by using convolution of certain analytic functions. It is observed that this formalism brings an ease in further mathematical exploration and also helps to understand the geometric properties of analytic and univalent functions. Their study can be traced back to 1916 by Alexander. Later, Libera and Bernardi introduced certain integral operators to study the classes of starlike, convex and close-to-convex functions. Recently, many researchers have shown great interest in studying some properties of integral and differential operators. Srivastava et al. [24] applied the fractional q-calculus operator to define and study a new class of analytic functions with complex order. Mahmood et al. [13] considered one-parameter families of integral operators and, studied the classes of uniformly starlike and uniformly convex functions with respect to symmetric points. Mehmood et al.

[14] analyzed certain classes of analytic functions defined by using q-integral operators. Arif et al. [2] introduced a class of multivalent functions by using q-analogue of the Ruscheweyh type operator. Sim et al. [21] investigated argument properties of analytic functions defined by the Srivastava-Attiya Operator. Srivastava et al. [26] found Faber polynomial coefficients for bi-univalent functions defined by using fractional derivative operator. Recently, Srivastava [23] gave brief survey of operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis.

Motivated by the above work, we study the inclusion of Bernardi integral operator in the classes of starlike functions associated with sine and cosine functions in $\mathcal{D}$. The Bernardi [4] integral operator is defined as:

$$\mathcal{F}(z) = \frac{\gamma+1}{z^{\beta}} \int_0^z t^{\gamma-1} f(t)\, dt, \quad \gamma \geq 0. \tag{1.3}$$

To prove our main results we need the following lemma.

Lemma 1 ([8]). *Let w be a non-constant analytic function in $\mathcal{D}$ with $w(0) = 0$. If*

$$|w(z_0)| = \max\{|w(z)|, \quad |z| \leq |z_0|\}, \quad z \in \mathcal{D},$$

then there exists a real number m $(m \geq 1)$ such that $z_0 w'(z_0) = m w(z_0)$.

2. Bernardi's integral operator associated with sine function

Theorem 1. *Assume that*

$$|\alpha| \geq \frac{(A-B)(1+\gamma+\sin h(1))}{(1-|B|)\cos(1) - (1+|B|)(1+\gamma+\sin h(1))(1+\sin h(1))}. \tag{2.1}$$

If

$$1 + \alpha z \left(\frac{z f'(z)}{f(z)} \right) \prec \frac{1+Az}{1+Bz}, \quad -1 \leq B < A \leq 1, \tag{2.2}$$

then

$$\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} \prec 1 + \sin(z),$$

where $\mathcal{F}$ is the Bernardi integral operator defined in (1.3).

Proof. Let us define a function w as

$$w(z) = \arcsin\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} - 1 \right), \tag{2.3}$$

where we have chosen the principle branches of the square root and logarithmic functions. Since arcsin functions is defined as

$$\arcsin z = -i \log\left[iz + (1-z^2)^{1/2} \right],$$

therefore w is analytic in $\mathcal{D}$ with $w(0)=0$. To prove our result, we need only to show that $|w(z)|<1$ in $\mathcal{D}$. From (2.3), we have

$$\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}=1+\sin(w(z)).$$

Logarithmic differentiation of above relation yields

$$1+\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}-\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}=\frac{zw'(z)\cos(w(z))}{1+\sin(w(z))}.$$

Differentiating (1.3), we obtain

$$(\gamma+1)f(z)=z\mathcal{F}'(z)+\gamma\mathcal{F}(z). \tag{2.4}$$

Differentiating (2.4) logarithmically, we have

$$\begin{aligned}\frac{zf'(z)}{f(z)}&=\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\left\{\frac{1+\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}-\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}}{\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}+\gamma}+1\right\}\\&=\frac{zw'(z)\cos(w(z))+(1+\sin(w(z))+\gamma)(1+\sin(w(z)))}{(1+\sin(w(z))+\gamma)}.\end{aligned}$$

Now, we define a function

$$p(z)=1+\alpha z\left(\frac{zf'(z)}{f(z)}\right),$$

where p is analytic in $\mathcal{D}$ and $p(0)=1$. Also

$$\left|\frac{p(z)-1}{A-Bp(z)}\right|=\left|\frac{1}{\frac{(A-B)(1+\sin(w(z))+\gamma)}{\alpha z[zw'(z)\cos(w(z))+(1+\sin(w(z))+\gamma)(1+\sin(w(z)))]}+B}\right|.$$

Suppose that there exists a point $z_0\in\mathcal{D}$ such that

$$\max_{|z|\le|z_0|}|w(z)|=|w(z_0)|=1.$$

By using Lemma 1, there exists a number $m\ge 1$ such that $z_0w'(z_0)=mw(z_0)$. We also suppose that $w(z_0)=e^{i\theta}$. Then we have

$$\begin{aligned}\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right|&=\left|\frac{1}{\frac{(A-B)(1+\sin(w(z_0))+\gamma)}{\alpha[zw'(z_0)\cos(w(z_0))+(1+\sin(w(z_0))+\gamma)(1+\sin(w(z_0)))]}+B}\right|\\&=\left|\frac{1}{\frac{(A-B)\left(1+\sin\left(e^{i\theta}\right)+\gamma\right)}{\alpha[me^{i\theta}\cos(e^{i\theta})+(1+\sin(e^{i\theta})+\gamma)(1+\sin(e^{i\theta}))]}+B}\right|.\end{aligned}$$

Let $|z| = r$, $-\pi \leq \theta \leq \pi$. Then after some simplifications, we have

$$\left|\sin\left(e^{i\theta}\right)\right|^2 = \cos^2(\cos\theta)\sin h^2(\sin\theta) + \sin^2(\cos\theta)\cos h^2(\sin\theta) = \Psi(\theta). \quad (2.5)$$

A simple computation shows that the equation $\Psi'(\theta) = 0$ has five roots in $[-\pi, \pi]$ namely 0, $\pm\pi$, $\pm\frac{\pi}{2}$. Since $\Psi'(\theta) = \Psi'(-\theta)$, it is sufficient to consider $\theta \in [0, \pi]$ and we see that

$$\Psi''(0) = \Psi''(\pi) \approx 1.090702574 \text{ and } \Psi''\left(\frac{\pi}{2}\right) \approx -1.626860410.$$

This implies that

$$\max\{\Psi(\theta)\} = \Psi\left(\frac{\pi}{2}\right) = \sin h^2(1),$$
$$\min\{\Psi(\theta)\} = \Psi(0) = \Psi(\pi) = sin^2(1).$$

Also consider

$$\left|\cos\left(e^{i\theta}\right)\right|^2 = \cos^2(\cos\theta)\cos h^2(\sin\theta) + \sin^2(\cos\theta)\sin h^2(\sin\theta) = \phi(\theta).$$

Similarly, after simple calculations the equation $\phi'(\theta) = 0$ has five roots in $[-\pi, \pi]$ namely 0, $\pm\pi$, $\pm\frac{\pi}{2}$. Since $\phi(\theta) = \phi(-\theta)$, it is sufficient to consider those roots which lies in $[0, \pi]$ and we see that

$$\phi''(0) = \phi(\pi) \approx 2.909297427 \text{ and } \phi''\left(\frac{\pi}{2}\right) \approx -5.626860410.$$

Therefore

$$\max\{\phi(\theta)\} = \phi\left(\frac{\pi}{2}\right) = \cos h^2(1),$$
$$\min\{\phi(\theta)\} = \phi(\pi) = \phi(0) = \cos^2(1).$$

This implies that

$$\cos(1) \leq \left|\cos\left(e^{i\theta}\right)\right| \leq \cosh(1) \text{ and } \sin(1) \leq \left|\sin\left(e^{i\theta}\right)\right| \leq \sin h(1). \quad (2.6)$$

Now

$$\left|\frac{p(z_0) - 1}{A - Bp(z_0)}\right| = \left|\frac{1}{\dfrac{(A-B)\left(1+\sin\left(e^{i\theta}\right)+\gamma\right)}{\alpha\left[me^{i\theta}\cos\left(e^{i\theta}\right) + (1+\sin\left(e^{i\theta}\right)+\gamma)(1+\sin\left(e^{i\theta}\right))\right]} + B}\right|$$
$$\geq \frac{|\alpha|\left[\left|me^{i\theta}\cos\left(e^{i\theta}\right)\right| - \left|\left(1+\sin\left(e^{i\theta}\right)+\gamma\right)\left(1+\sin\left(e^{i\theta}\right)\right)\right|\right]}{(A-B)\left|(1+\sin\left(e^{i\theta}\right)+\gamma)\right| + |\alpha B|\left[\left|me^{i\theta}\cos\left(e^{i\theta}\right)\right| + \left|(1+\sin\left(e^{i\theta}\right)+\gamma)(1+\sin\left(e^{i\theta}\right))\right|\right]}$$
$$\geq \frac{|\alpha|\left[m\cos(1) - (1+\gamma+\sin h(1))(1+\sin h(1))\right]}{(A-B)(1+\gamma+\sin h(1)) + |\alpha B|\left[m\cos(1) + (1+\gamma+\sin h(1))(1+\sin h(1))\right]}.$$

Now let

$$\Phi(m)=\frac{|\alpha|\left[m\cos(1)-(1+\gamma+\sin h(1))(1+\sin h(1))\right]}{(A-B)(1+\gamma+\sin h(1))+|\alpha B|\left[m\cos(1)+(1+\gamma+\sin h(1))(1+\sin h(1))\right]}.$$

Then

$$\Phi'(m)=$$
$$\frac{\left\{(A-B)(1+\gamma+\sin h(1))+2|\alpha|^2|B|\left[(1+\gamma+\sin h(1))(1+\sin h(1))\right]\right\}|\alpha|\cos(1)}{\left\{(A-B)(1+\gamma+\sin h(1))+|\alpha B|\left[m\cos(1)+(1+\gamma+\sin h(1))(1+\sin h(1))\right]\right\}^2}>0.$$

This shows that Φ is an increasing function and it has its minimum value at $m=1$ so

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right|\geq\frac{|\alpha|\left[\cos 1-(1+\gamma+\sin h(1))(1+\sin h(1))\right]}{(A-B)(1+\gamma+\sin h(1))+|\alpha B|\left[\cos 1+(1+\gamma+\sin h(1))(1+\sin h(1))\right]}.$$

Now by (2.1), we have

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right|\geq 1,$$

a contradiction to the hypothesis. Hence we have the required result. □

By choosing $A=1,\ B=0$, we have the following result.

Corollary 1. *Let*

$$|\alpha|\geq\frac{(1+\gamma+\sin h(1))}{\cos(1)-(1+\gamma+\sin h(1))(1+\sin h(1))}.$$

If

$$1+\alpha z\left(\frac{zf'(z)}{f(z)}\right)\prec 1+z,$$

then

$$\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\prec 1+\sin(z).$$

By choosing $A=1,\ B=-1$, we have the following result.

Corollary 2. *Let* $|\alpha|\geq -0.363370517$ *and*

$$1+\alpha z\left(\frac{zf'(z)}{f(z)}\right)\prec\frac{1+z}{1-z}.$$

Then

$$\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\prec 1+\sin(z).$$

Theorem 2. *Assume that*

$$|\alpha|\geq\frac{(A-B)(\gamma+1)}{(1-|B|)\cos(1)-(1+|B|)(1+\gamma)(1+\sin h(1))}. \tag{2.7}$$

If

$$1+\alpha f(z) \prec \frac{1+Az}{1+Bz}, \tag{2.8}$$

then

$$\frac{\mathcal{F}(z)}{z} \prec 1+\sin(z),$$

where $\mathcal{F}$ is the Bernardi integral operator defined in (1.3).

Proof. Let a function w be defined by

$$w(z) = \arcsin\left(\frac{\mathcal{F}(z)}{z} - 1\right), \tag{2.9}$$

where we have chosen the principle branches of the square root and logarithmic functions. Then w is analytic in $\mathcal{D}$ with $w(0)=0$. We need only to show that $|w(z)|<1$ in $\mathcal{D}$. From (2.9), we have

$$\frac{\mathcal{F}(z)}{z} = 1+\sin(w(z)). \tag{2.10}$$

Also we define a function

$$p(z) = 1+\alpha f(z), \tag{2.11}$$

where p is analytic in $\mathcal{D}$ with $p(0)=1$. Now by using (2.10), (2.4) and (2.11), we have

$$\left|\frac{p(z)-1}{A-Bp(z)}\right| = \left|\frac{\alpha z\left[zw'(z)\cos(w(z))+(1+\gamma)(1+\sin(w(z)))\right]}{(A-B)(1+\gamma)+\alpha Bz\left[zw'(z)\cos(w(z))+(1+\gamma)(1+\sin(w(z)))\right]}\right|.$$

Suppose that there exists a point $z_0 \in \mathcal{D}$ such that

$$\max_{|z|\le|z_0|} |w(z)| = |w(z_0)| = 1.$$

By using Lemma 1, there exists a number $m \ge 1$ such that $z_0 w'(z_0) = mw(z_0)$. We also suppose that $w(z_0) = e^{i\theta}$. Then we have

$$\begin{aligned}
\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| &= \frac{\alpha z_0\left[me^{i\theta}\cos\left(e^{i\theta}\right)+(1+\gamma)\left(1+\sin\left(e^{i\theta}\right)\right)\right]}{(A-B)(1+\gamma)+\alpha Bz_0\left[me^{i\theta}\cos\left(e^{i\theta}\right)+(1+\gamma)\left(1+\sin\left(e^{i\theta}\right)\right)\right]} \\
&\ge \frac{|\alpha|\left[\left|me^{i\theta}\cos\left(e^{i\theta}\right)\right|-\left|(1+\gamma)\left(1+\sin\left(e^{i\theta}\right)\right)\right|\right]}{(A-B)|(1+\gamma)|+|\alpha B|\left[\left|me^{i\theta}\cos\left(e^{i\theta}\right)\right|+\left|(1+\gamma)\left(1+\sin\left(e^{i\theta}\right)\right)\right|\right]} \\
&\ge \frac{|\alpha|\left[m\cos(1)-(1+\gamma)(1+\sinh(1))\right]}{(A-B)(1+\gamma)+|\alpha B|\left[m\cos(1)+(1+\gamma)(1+\sinh(1))\right]}.
\end{aligned}$$

Now let

$$\Theta(m) = \frac{|\alpha|\left[m\cos(1)-(1+\gamma)(1+\sinh(1))\right]}{(A-B)(1+\gamma)+|\alpha B|\left[m\cos 1+(1+\gamma)(1+\sinh(1))\right]}.$$

Then

$$\Theta'(m) = \frac{((A-B)+2|\alpha||B|(1+\sin h(1)))|\alpha|(\gamma+1)\cos(1)}{\{(A-B)(1+\gamma)+|\alpha B|[m\cos(1)+(1+\gamma)(1+\sin h(1))]\}^2} > 0,$$

which shows that Θ is an increasing function and it has its minimum value at $m=1$, so

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| \geq \frac{|\alpha|[\cos(1)-(1+\gamma)(1+\sin h(1))]}{(A-B)(1+\gamma)+|\alpha B|[\cos(1)+(1+\gamma)(1+\sin h(1))]}.$$

Now by (2.7), we have

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| \geq 1,$$

a contradiction to the hypothesis. Hence

$$1+\alpha f(z) \prec \frac{1+Az}{1+Bz}.$$

□

By choosing $A=1,\ B=0$, we have the following result.

Corollary 3. *Let* $|\alpha| \geq \dfrac{(\gamma+1)}{\cos(1)-(1+|B|)(1+\gamma)(1+\sin h(1))}$ *and*

$$1+\alpha f(z) \prec 1+z.$$

Then

$$\frac{\mathcal{F}(z)}{z} \prec 1+\sin(z).$$

By choosing $A=1,\ B=-1$, we have the following result.

Corollary 4. *Let* $|\alpha| \geq \dfrac{-1}{1+\sin h(1)}$ *and*

$$1+\alpha f(z) \prec \frac{1+z}{1-z}.$$

Then

$$\frac{\mathcal{F}(z)}{z} \prec 1+\sin(z).$$

Theorem 3. *Assume that*

$$|\alpha| \geq \frac{(A-B)(\gamma+1)}{(1-|B|)\cos(1)-(1+|B|)(1+\gamma)(1+\sin h(1))}. \tag{2.12}$$

If

$$1+\alpha z f'(z) \prec \frac{1+Az}{1+Bz}, \tag{2.13}$$

then

$$\mathcal{F}'(z) \prec 1+\sin(z),$$

where $\mathcal{F}$ is the Bernardi integral operator defined in (1.3).

Proof. Let a function w be defined by

$$w(z) = \arcsin\left(\mathcal{F}'(z) - 1\right), \tag{2.14}$$

where we have chosen the principle branches of the square root and logarithmic functions. It is clear that w is an analytic function in $\mathcal{D}$ with $w(0) = 0$. We need only to show that $|w(z)| < 1$ in $\mathcal{D}$. From (2.14), we have

$$\mathcal{F}'(z) = 1 + \sin(w(z)). \tag{2.15}$$

Differentiating (2.4), we get

$$f'(z) = \mathcal{F}'(z) + \frac{z\mathcal{F}''(z)}{\gamma+1}. \tag{2.16}$$

Also we define a function

$$p(z) = 1 + \alpha z f'(z), \tag{2.17}$$

where p is analytic with $p(0) = 1$. Now by using (2.15), (2.16) and (2.17), we have

$$\left|\frac{p(z)-1}{A-Bp(z)}\right| = \left|\frac{\alpha z\left[zw'(z)\cos(w(z)) + (1+\gamma)(1+\sin(w(z)))\right]}{(A-B)(1+\gamma) + \alpha Bz\left[zw'(z)\cos(w(z)) + (1+\gamma)(1+\sin(w(z)))\right]}\right|.$$

Suppose that there exists a point $z_0 \in \mathcal{D}$ such that

$$\max_{|z|\le|z_0|} |w(z)| = |w(z_0)| = 1.$$

By using Lemma 1, there exists a number $m \ge 1$ such that $z_0 w'(z_0) = mw(z_0)$. We also suppose that $w(z_0) = e^{i\theta}$. Then we have

$$\begin{aligned}
\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| &= \left|\frac{\alpha z_0\left[me^{i\theta}\cos\left(e^{i\theta}\right) + (1+\gamma)\left(1+\sin\left(e^{i\theta}\right)\right)\right]}{(A-B)(1+\gamma) + \alpha Bz_0\left[me^{i\theta}\cos\left(e^{i\theta}\right) + (1+\gamma)(1+\sin(e^{i\theta}))\right]}\right| \\
&\ge \frac{|\alpha|\left[\left|me^{i\theta}\cos\left(e^{i\theta}\right)\right| - \left|(1+\gamma)\left(1+\sin\left(e^{i\theta}\right)\right)\right|\right]}{(A-B)|(1+\gamma)| + |\alpha B|\left[\left|me^{i\theta}\cos\left(e^{i\theta}\right)\right| + \left|(1+\gamma)(1+\sin(e^{i\theta}))\right|\right]} \\
&\ge \frac{|\alpha|\left[m\cos(1) - (1+\gamma)(1+\sin h(1))\right]}{(A-B)(1+\gamma) + |\alpha B|\left[m\cos(1) + (1+\gamma)(1+\sin h(1))\right]}.
\end{aligned}$$

Now let

$$\Xi(m) = \frac{|\alpha|\left[m\cos(1) - (1+\gamma)(1+\sin h(1))\right]}{(A-B)(1+\gamma) + |\alpha B|\left[m\cos(1) + (1+\gamma)(1+\sin h(1))\right]}.$$

Then

$$\Xi'(m) = \frac{((A-B) + 2|\alpha||B|(1+\sin h(1)))|\alpha|(\gamma+1)\cos(1)}{\left\{(A-B)(1+\gamma) + |\alpha B|\left[m\cos(1) + (1+\gamma)(1+\sin h(1))\right]\right\}^2} > 0,$$

which shows that Ξ is an increasing function and it has its minimum value at $m=1$ so

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| \geq \frac{|\alpha|\left[\cos(1)-(1+\gamma)(1+\sin h(1))\right]}{(A-B)(1+\gamma)+|\alpha B|\left[\cos(1)+(1+\gamma)(1+\sin h(1))\right]}.$$

Now by (2.12), we have

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| \geq 1,$$

a contradiction to the hypothesis

$$1+\alpha z f'(z) \prec \frac{1+Az}{1+Bz}.$$

Hence we have the required result. □

By choosing $A=1,\ B=0$, we have the following result.

Corollary 5. *Assume that*

$$|\alpha| \geq \frac{(\gamma+1)}{\cos(1)-(1+\gamma)(1+\sin h(1))}.$$

If

$$1+\alpha z f'(z) \prec 1+z,$$

then

$$\mathcal{F}'(z) \prec 1+\sin(z).$$

By choosing $A=1,\ B=-1$, we have the following result.

Corollary 6. *Assume that* $|\alpha| \geq \dfrac{-1}{(1+\sin h(1))}$ *and if*

$$1+\alpha z f'(z) \prec \frac{1+z}{1-z},$$

then

$$\mathcal{F}'(z) \prec 1+\sin(z).$$

3. Bernardi's integral operator associated with cosine function

Theorem 4. *Assume that*

$$|\alpha| \geq \frac{(A-B)(\gamma+1)}{(1-|B|)\sin(1)-(1+|B|)(\cosh(1)+\gamma)\cosh(1)}. \tag{3.1}$$

If

$$1+\alpha z\left(\frac{zf'(z)}{f(z)}\right) \prec \frac{1+Az}{1+Bz}, \tag{3.2}$$

then

$$\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} \prec \cos(z),$$

where $\mathcal{F}$ is the Bernardi integral operator defined in (1.3).

Proof. Let a function w be defined by

$$w(z) = \cos^{-1}\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right). \tag{3.3}$$

where we have chosen the principle branches of the square root and logarithmic functions. Since arccos function is defined as

$$\arccos z = -i\log\left[z + i(1-z^2)^{1/2}\right],$$

therefore w is an analytic function in $\mathcal{D}$ with $w(0) = 0$. To prove our result, we need only to show that $|w(z)| < 1$ in $\mathcal{D}$. From (3.3), we have

$$\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} = \cos(w(z)).$$

Logarithmic differentiation of above relation yields

$$1 + \frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} - \frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} = \frac{-zw'(z)\sin(w(z))}{\cos(w(z))}.$$

Using (1.3), we have

$$(\gamma+1)f(z) = z\mathcal{F}'(z) + \gamma\mathcal{F}(z).$$

Differentiating logarithmically, we have

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= \frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\left\{\frac{1+\frac{z\mathcal{F}''(z)}{\mathcal{F}(z)}-\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}}{\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}+\gamma}+1\right\}\\ &= \frac{-zw'(z)\sin(w(z)) + (\cos(w(z))+\gamma)\cos(w(z))}{\cos(w(z))+\gamma}.\end{aligned}$$

Now we define a function

$$p(z) = 1 + \alpha z\left(\frac{zf'(z)}{f(z)}\right),$$

where p is analytic in $\mathcal{D}$ with $p(0) = 1$. Also it is easy to see that

$$\left|\frac{p(z)-1}{A-Bp(z)}\right| = \left|\frac{\alpha z\left[-zw'(z)\sin(w(z)) + (\cos(w(z))+\gamma)(\cos(w(z)))\right]}{(A-B)(\cos(w(z))+\gamma) + \alpha Bz\left[-zw'(z)\sin(w(z)) + (\cos(w(z))+\gamma)(\cos(w(z)))\right]}\right|.$$

Suppose that there exists a point $z_0 \in \mathcal{D}$ such that

$$\max_{|z|\le|z_0|}|w(z)| = |w(z_0)| = 1.$$

By using Lemma 1, there exists a number $m \geq 1$ such that $z_0 w'(z_0) = m w(z_0)$. We also suppose that $w(z_0) = e^{i\theta}$ and using (2.6), we have

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| =$$

$$\left|\frac{\alpha z_0\left[-z_0 w'(z_0)\sin(w(z_0))+(\cos(w(z_0))+\gamma)(\cos(w(z_0)))\right]}{(A-B)(\cos(w(z_0))+\gamma)+\alpha B z_0\left[-z_0 w'(z_0)\sin(w(z_0))+(\cos(w(z_0))+\gamma)(\cos(w(z_0)))\right]}\right|$$

$$= \left|\frac{\alpha z_0\left[-me^{i\theta}\sin\left(e^{i\theta}\right)+\left(\cos\left(e^{i\theta}\right)+\gamma\right)\left(\cos\left(e^{i\theta}\right)\right)\right]}{(A-B)\left(\cos\left(e^{i\theta}\right)+\gamma\right)+\alpha B z_0\left[-me^{i\theta}\sin\left(e^{i\theta}\right)+\left(\cos\left(e^{i\theta}\right)+\gamma\right)\cos\left(e^{i\theta}\right)\right]}\right|$$

$$\geq \frac{\alpha\left[\left|me^{i\theta}\sin\left(e^{i\theta}\right)\right|-\left|\left(\cos\left(e^{i\theta}\right)+\gamma\right)\cos\left(e^{i\theta}\right)\right|\right]}{(A-B)\left(\cos\left(e^{i\theta}\right)+\gamma\right)+|\alpha B|\left[\left|m\left(e^{i\theta}\right)\sin\left(e^{i\theta}\right)\right|+\left|\left(\cos\left(e^{i\theta}\right)+\gamma\right)\cos\left(e^{i\theta}\right)\right|\right]}$$

$$\geq \frac{\alpha\left[m\sin(1)-(\cosh(1)+\gamma)\cosh(1)\right]}{(A-B)(\cosh(1)+\gamma)+|\alpha B|\left[m\sin(1)+(\cosh(1)+\gamma)\cosh(1)\right]}.$$

Now let

$$\Phi_1(m) = \frac{\alpha\left[m\sin(1)-(\cosh(1)+\gamma)\cosh(1)\right]}{(A-B)(\cosh(1)+\gamma)+\alpha B\left[m\sin(1)+(\cosh(1)+\gamma)\cosh(1)\right]}.$$

Then

$$\Phi_1'(m) = \frac{|\alpha|\sin(1)(A-B)(\cosh(1)+\gamma)+2|\alpha|^2|B|(\cosh(1)+\gamma)\cosh(1)}{\left\{(A-B)(\cosh(1)+\gamma)+|\alpha||B|\left[m\sin(1)+(\cosh(1)+\gamma)\cosh(1)\right]\right\}^2} > 0,$$

which shows that Φ_1 is an increasing function and has its minimum value at $m = 1$ so

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| \geq \frac{|\alpha|\left[\sin(1)-(\cosh(1)+\gamma)\cosh(1)\right]}{(A-B)(\cosh(1)+\gamma)+|\alpha||B|\left[\sin(1)+(\cosh(1)+\gamma)\cosh(1)\right]}.$$

Now by (3.1), we have

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| \geq 1,$$

a contradiction to the hypothesis. Hence the result is complete. □

By choosing $A = 1$, $B = 0$, we have the following result.

Corollary 7. *Assume that*

$$|\alpha| \geq \frac{(\gamma+1)}{\sin(1)-(\cosh(1)+\gamma)\cosh(1)}.$$

If

$$1+\alpha z\left(\frac{zf'(z)}{f(z)}\right) \prec 1+z,$$

then

$$\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} \prec \cos(z),$$

where $\mathcal{F}$ is the Bernardi integral operator defined in (1.3).

By choosing $A=1,\ B=-1$, we have the following result.

Corollary 8. *Assume that*

$$|\alpha| \geq \frac{-(\gamma+1)}{(\cosh(1)+\gamma)\cosh(1)}.$$

If

$$1+\alpha z\left(\frac{zf'(z)}{f(z)}\right) \prec \frac{1+z}{1-z},$$

then

$$\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} \prec \cos(z).$$

Theorem 5. *Assume that*

$$\alpha \geq \frac{(A-B)(\gamma+1)}{(1-|B|)\sin(1)-(1+|B|)(\cosh(1)+\gamma)\cosh(1)}. \tag{3.4}$$

If

$$1+\alpha f(z) \prec \frac{1+Az}{1+Bz}, \tag{3.5}$$

then

$$\frac{\mathcal{F}(z)}{z} \prec \cos(z),$$

where $\mathcal{F}$ is the Bernardi integral operator defined in (1.3).

Proof. Let a function w be defined by

$$w(z) = \arccos\left(\frac{\mathcal{F}(z)}{z}\right), \tag{3.6}$$

where we have chosen the principle branches of the square root and logarithmic functions. Therefore w is an analytic function in $\mathcal{D}$ with $w(0)=0$. We need only to show that $|w(z)|<1$ in $\mathcal{D}$. From (3.6), we have

$$\frac{\mathcal{F}(z)}{z} = \cos w(z). \tag{3.7}$$

Also we define a function

$$p(z) = 1+\alpha f(z), \tag{3.8}$$

where p is analytic in $\mathcal{D}$ with $p(0)=1$. Now by using (3.7), (2.4) and (3.8) we have

$$\left|\frac{p(z)-1}{A-Bp(z)}\right| = \left|\frac{\alpha\left[-z^2w'(z)\sin(w(z))+z(1+\gamma)\cos(w(z))\right]}{(A-B)(1+\gamma)+\alpha B\left[-z^2w'(z)\sin(w(z))+z(1+\gamma)\cos(w(z))\right]}\right|.$$

Suppose that there exists a point $z_0 \in \mathcal{D}$ such that

$$\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1.$$

By using Lemma 1, there exists a number $m \geq 1$ such that $z_0 w'(z_0) = m w(z_0)$. We also suppose that $w(z_0) = e^{i\theta}$ and by using (2.6), we have

$$\begin{aligned}\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| &= \left|\frac{\alpha\left[-z_0 m e^{i\theta}\sin\left(e^{i\theta}\right)+z_0(1+\gamma)\cos\left(e^{i\theta}\right)\right]}{(A-B)(1+\gamma)+\alpha B\left[-z_0 m e^{i\theta}\sin\left(e^{i\theta}\right)+z_0(1+\gamma)\cos\left(e^{i\theta}\right)\right]}\right| \\ &\geq \frac{|\alpha|\left[\left|m e^{i\theta}\sin\left(e^{i\theta}\right)\right|-\left|(1+\gamma)\cos\left(e^{i\theta}\right)\right|\right]}{(A-B)|(1+\gamma)|+|\alpha B|\left[\left|m e^{i\theta}\sin\left(e^{i\theta}\right)\right|+\left|(1+\gamma)\cos\left(e^{i\theta}\right)\right|\right]} \\ &\geq \frac{|\alpha|\left[m\sin(1)-(1+\gamma)\cosh(1)\right]}{(A-B)(1+\gamma)+|\alpha B|\left[m\sin(1)+(1+\gamma)\cosh(1)\right]}.\end{aligned}$$

Now let

$$\Theta_1(m) = \frac{|\alpha|\left[m\sin 1-(1+\gamma)\cosh 1\right]}{(A-B)(1+\gamma)+|\alpha B|\left[m\sin 1+(1+\gamma)\cosh 1\right]}.$$

Then

$$\Theta_1'(m) = \frac{(A-B)(1+\gamma)\sin(1)+2|\alpha|^2|B|(1+\gamma)\cosh(1)\sin(1)}{\left\{(A-B)(1+\gamma)+|\alpha B|\left[m\sin(1)+(1+\gamma)\cosh(1)\right]\right\}^2} > 0,$$

which shows that Θ_1 is an increasing function and has its minimum value at $m = 1$ so

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| = \frac{|\alpha|\left[\sin(1)-(1+\gamma)\cosh(1)\right]}{(A-B)(1+\gamma)+|\alpha B|\left[\sin(1)+(1+\gamma)\cosh(1)\right]}.$$

Now by (3.4), we have

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| \geq 1,$$

a contradiction to the hypothesis. Hence it completes the proof. □

By choosing $A = 1,\ B = 0$, we have the following result.

Corollary 9. *Assume that*

$$|\alpha| \geq \frac{(\gamma+1)}{\sin(1)-(\cosh(1)+\gamma)\cosh(1)}.$$

If

$$1+\alpha f(z) \prec 1+z,$$

then

$$\frac{\mathcal{F}(z)}{z} \prec \cos(z).$$

By choosing $A = 1,\ B = -1$, we have the following result.

Corollary 10. *Assume that*

$$|\alpha| \geq \frac{-(\gamma+1)}{(\cosh(1)+\gamma)\cosh(1)}.$$

If

$$1+\alpha f(z) \prec \frac{1+z}{1-z},$$

then

$$\frac{\mathcal{F}(z)}{z} \prec \cos(z).$$

Theorem 6. *Assume that*

$$|\alpha| \geq \frac{(A-B)(\gamma+1)}{(1-|B|)\sin(1)-(1+|B|)(1+\gamma)\cosh(1)}. \tag{3.9}$$

If

$$1+\alpha z f'(z) \prec \frac{1+Az}{1+Bz}, \tag{3.10}$$

then

$$\mathcal{F}'(z) \prec \cos(z),$$

where $\mathcal{F}$ is the Bernardi integral operator defined in (1.3).

Proof. Let a function w be defined by

$$w(z) = \arccos \mathcal{F}'(z), \tag{3.11}$$

where we have chosen the principle branches of the square root and logarithmic functions. It is clear that w is an analytic function in $\mathcal{D}$ with $w(0)=0$. We need only to show that $|w(z)|<1$ in $\mathcal{D}$. From (3.11), we have

$$\mathcal{F}'(z) = \cos(w(z)). \tag{3.12}$$

Differentiating (2.4), we get

$$f'(z) = \mathcal{F}'(z) + \frac{z\mathcal{F}''(z)}{\gamma+1}. \tag{3.13}$$

Also we define a function

$$p(z) = 1+\alpha z f'(z), \tag{3.14}$$

where p is analytic and $p(0)=1$. Now by using (3.12), (3.13) and (3.14), we have

$$\left|\frac{p(z)-1}{A-Bp(z)}\right| = \left|\frac{\alpha z[-zw'(z)\sin(w(z))+(\gamma+1)\cos(w(z))]}{(A-B)(1+\gamma)+\alpha Bz[-zw'(z)\sin(w(z))+(\gamma+1)\cos(w(z))]}\right|.$$

Suppose that there exists a point $z_0 \in \mathcal{D}$ such that

$$\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1.$$

By using Lemma 1, there exists a number $m \geq 1$ such that $z_0 w'(z_0) = m w(z_0)$. We also suppose that $w(z_0) = e^{i\theta}$. Then we have

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right| = \left|\frac{\alpha z_0\left[-me^{i\theta}\sin\left(e^{i\theta}\right)+(\gamma+1)\cos\left(e^{i\theta}\right)\right]}{(A-B)(1+\gamma)+\alpha Bz_0[-me^{i\theta}\sin(e^{i\theta})+(\gamma+1)\cos(e^{i\theta})]}\right|$$

$$\geq \frac{|\alpha|\left[\left|me^{i\theta}\sin\left(e^{i\theta}\right)\right|-\left|(\gamma+1)\cos\left(e^{i\theta}\right)\right|\right]}{(A-B)(1+\gamma)+|\alpha B|\left[\left|me^{i\theta}\sin\left(e^{i\theta}\right)\right|+\left|(\gamma+1)\cos\left(e^{i\theta}\right)\right|\right]}$$
$$\geq \frac{|\alpha|\left[m\sin(1)-(1+\gamma)\cosh(1)\right]}{(A-B)(1+\gamma)+|\alpha B|\left[m\sin(1)+(1+\gamma)\cosh(1)\right]}.$$

Now let

$$\Xi_1(m)=\frac{|\alpha|\left[m\sin(1)-(1+\gamma)\cosh(1)\right]}{(A-B)(1+\gamma)+|\alpha B|\left[m\sin[1]+(1+\gamma)\cosh(1)\right]}.$$

Then

$$\Xi_1'(m)=\frac{(A-B)(1+\gamma)|\alpha|\sin(1)+2(1+\gamma)|\alpha|^2|B|\cosh(1)\sin(1)}{\left\{(A-B)(1+\gamma)+|\alpha B|\left[m\sin(1)+(1+\gamma)\cosh(1)\right]\right\}^2}>0,$$

which shows that Ξ_1 is an increasing function and has its minimum value at $m=1$ so

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right|\geq\frac{|\alpha|\left[\sin(1)-(1+\gamma)\cosh(1)\right]}{(A-B)(1+\gamma)+|\alpha B|\left[\sin(1)+(1+\gamma)\cosh(1)\right]}.$$

Now by (3.9), we have

$$\left|\frac{p(z_0)-1}{A-Bp(z_0)}\right|\geq 1,$$

a contradiction to the hypothesis. Hence we have the required result. □

By choosing $A=1,\ B=0$, we have the following result.

Corollary 11. *Assume that*

$$|\alpha|\geq\frac{(\gamma+1)}{\sin(1)-(1+\gamma)\cosh(1)}.$$

If

$$1+\alpha zf'(z)\prec 1+z,$$

then

$$\mathcal{F}'(z)\prec\cos(z).$$

By choosing $A=1,\ B=-1$, we have the following result.

Corollary 12. *Assume that*

$$|\alpha|\geq\frac{-(\gamma+1)}{(1+\gamma)\cosh(1)}.$$

If

$$1+\alpha zf'(z)\prec\frac{1+z}{1-z},$$

then

$$\mathcal{F}'(z)\prec\cos(z).$$

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HERMITE-HADAMARD TYPE INEQUALITIES WITH APPLICATIONS

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Abstract. The main objective of this article is to obtain some new estimates related to Hermite-Hadamard-like integral inequalities essentially using the class of h-convex functions. For this we first derive a new integral identity which will serve as an auxiliary result for obtaining the main results of the article. Several new and known special cases are also discussed. In the last some applications to means of real numbers are also discussed in detail which elaborates the significance of the obtained results.

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1. Introduction and Preliminaries

Convexity due to its numerous applications has attracted considerable number of researchers. As a result several new generalizations of classical convexity have been obtained in the literature. For example, see [2, 3, 8, 9, 11, 12]. Among these new generalizations h-convexity which is due to Varosanec [14] is one of the most charming generalization of classical convexity. It not only unifies the classical convexity but also some of other its generalizations, such as it generalizes Breckner type of s-convexity [1], Godunova-Levin type of s-convexity [4], P type of functions [7], Q type of functions [10] and tgs-convex functions [13].
The class of h-convex functions reads as:

Definition 1. Let $h:(0,1)\to\mathbb{R}$ be a real function. We say that $f:I\to\mathbb{R}$ is an h-convex function, if

$$f(tx+(1-t)y)\leq h(t)f(x)+h(1-t)f(y),\forall x,y\in I,t\in(0,1).$$

Now one can easily observe that for different suitable choices of the function $h(.)$, we have different classes of convex functions. For details, see [14]. Another fascinating aspect of theory of convexity is its close link with theory of inequalities. Many inequalities are direct consequences of the applications of classical convexity. A result in this regard, which is due to Hermite and Hadamard and is commonly known as Hermite-Hadamard's integral inequality. It reads as:

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Theorem 1. *Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function. If $f \in L[a,b]$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

This result through its simplicity yet having a number applications has attracted several researchers from all over the world. Consequently many new generalizations of this result have been derived in the literature. For more information, see [5, 6, 11, 12, 15].

The main objective of this article is obtain some new refinements of Hermite-Hadamard's integral inequality. For this we derive a new integral identity for differentiable functions. This identity will serve as an auxiliary result for obtaining main results of the article. We also discuss several new special cases in detail. In the last we present applications of some of the proposed results to means of real numbers.

2. Auxiliary Result

In this section, we derive new integral identity which will play significant role in obtaining main results of the article. Before we move further, for simplicity, we let

$$\Omega_1 = \frac{n}{n+1}a + \frac{1}{n+1}b \text{ and } \Omega_2 = \frac{1}{n+1}a + \frac{n}{n+1}b.$$

Lemma 1. *Let $f : I \to \mathbb{R}$ be differentiable function on I^o , $a,b \in I$ with $a < b$. If $f' \in L_1[a,b], 0 \leq \lambda,\mu \leq 1$ and $n \in \mathbb{N}^*$, then we have*

$$\phi(n;\lambda,\mu;a,b) - \frac{1}{b-a}\left[\int_a^{\Omega_1} f(x)\mathrm{d}x + \int_{\Omega_2}^b f(x)\mathrm{d}x\right]$$
$$= \frac{b-a}{(n+1)^2} \times \int_0^1 \left[(1-\lambda-t)f'\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right) + (\mu-t)f'\left(\frac{t}{n+1}a + \frac{n+1-t}{n+1}b\right)\right]\mathrm{d}t,$$

where

$$\phi(n;\lambda,\mu;a,b)$$
$$= \frac{1}{n+1}\left[\lambda f(a) + (1-\lambda)f\left(\frac{n}{n+1}a + \frac{1}{n+1}b\right) + (1-\mu)f\left(\frac{1}{n+1}a + \frac{n}{n+1}b\right) + \mu f(b)\right].$$

Proof. Take

$$I = \frac{b-a}{(n+1)^2}\int_0^1 \left[(1-\lambda-t)f'\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right) + (\mu-t)f'\left(\frac{t}{n+1}a + \frac{n+1-t}{n+1}b\right)\right]\mathrm{d}t$$
$$= \frac{b-a}{(n+1)^2}(I_1 + I_2). \tag{2.1}$$

Now

$$I_1 = \int_0^1 (1-\lambda-t)f'\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right)\mathrm{d}t$$

$$= \frac{n+1}{b-a}\left[\lambda f(a)+(1-\lambda)f\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)-\int_0^1 f\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\mathrm{d}t\right].$$

Putting $x=\frac{n+t}{n+1}a+\frac{1-t}{n+1}b$ in the above equality, we have

$$I_1=\frac{n+1}{b-a}\left[\lambda f(a)+(1-\lambda)f\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)-\frac{n+1}{b-a}\int_a^{\Omega_1} f(x)\mathrm{d}x\right], \tag{2.2}$$

and

$$\begin{aligned} I_2 &= \int_0^1 (\mu-t)f'\left(\frac{t}{n+1}a+\frac{n+1-t}{n+1}b\right)\mathrm{d}t \\ &= \frac{n+1}{b-a}\left[(1-\mu)f\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)+\mu f(b)-\int_0^1 f\left(\frac{t}{n+1}a+\frac{n+1-t}{n+1}b\right)\right]\mathrm{d}t. \end{aligned}$$

Using $x=\frac{t}{n+1}a+\frac{n+1-t}{n+1}b$ in above equality, we have

$$I_2=\frac{n+1}{b-a}\left[(1-\mu)f\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)+\mu f(b)-\frac{n+1}{b-a}\int_{\Omega_2}^b f(x)\mathrm{d}x\right]. \tag{2.3}$$

Utilizing the values of I_1 and I_2 from (2.2) and (2.3) in (2.1), we have

$$\begin{aligned} I &= \frac{b-a}{(n+1)^2}\frac{n+1}{b-a}\left[\lambda f(a)+(1-\lambda)f\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)\right. \\ &\quad \left.-\frac{n+1}{b-a}\int_a^{\Omega_1} f(x)\mathrm{d}x+(1-\mu)f\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)+\mu f(b)-\frac{n+1}{b-a}\int_{\Omega_2}^b f(x)\mathrm{d}x\right] \\ &= \frac{1}{n+1}\left[\lambda f(a)+(1-\lambda)f\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)+(1-\mu)f\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)+\mu f(b)\right] \\ &\quad -\frac{1}{b-a}\left[\int_a^{\Omega_1} f(x)\mathrm{d}x+\int_{\frac{1}{n+1}a+\frac{n}{n+1}b}^b f(x)\mathrm{d}x\right]. \end{aligned}$$

This implies

$$\begin{aligned} &\phi(n;\lambda,\mu;a,b)-\frac{1}{b-a}\left[\int_a^{\Omega_1} f(x)\mathrm{d}x+\int_{\Omega_2}^b f(x)\mathrm{d}x\right] \\ &= \frac{b-a}{(n+1)^2}\int_0^1\left[(1-\lambda-t)f'\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)+(\mu-t)f'\left(\frac{t}{n+1}a+\frac{n+1-t}{n+1}b\right)\right]\mathrm{d}t, \end{aligned}$$

where

$$\begin{aligned} &\phi(n;\lambda,\mu;a,b) \\ &= \frac{1}{n+1}\left[\lambda f(a)+(1-\lambda)f\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)+(1-\mu)f\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)+\mu f(b)\right]. \end{aligned}$$

This completes the proof. □

3. Results and Discussions

In this section, we derive our main results.

Theorem 2. *Let $f : I \to \mathbb{R}$ be differentiable function on I^o , $a,b \in I$ with $a < b$. If $|f'|$ is h-convex function, $0 \le \lambda,\mu \le 1$ and $n \in \mathbb{N}^*$, then we have*

$$\left|\phi(n;\lambda,\mu;a,b) - \frac{1}{b-a}\left[\int_a^{\Omega_1} f(x)dx + \int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|$$
$$\le \frac{b-a}{(n+1)^2}\left[\alpha(t)\left|f'(a)\right| + \beta(t)\left|f'(b)\right|\right],$$

where

$$\alpha(t) = \int_0^1 \left[(1-\lambda-t)h\left(\frac{n+t}{n+1}\right) + (\mu-t)\left(\frac{t}{n+1}\right)\right]\mathrm{d}t,$$

and

$$\beta(t) = \int_0^1 \left[(1-\lambda-t)h\left(\frac{1-t}{n+1}\right) + (\mu-t)\left(\frac{n+1-t}{n+1}\right)\right]\mathrm{d}t.$$

Proof. Using Lemma 1, we have

$$\left|\phi(n;\lambda,\mu;a,b) - \frac{1}{b-a}\left[\int_a^{\Omega_1} f(x)\mathrm{d}x + \int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|$$
$$= \left|\frac{b-a}{(n+1)^2}\int_0^1 \left[(1-\lambda-t)f'\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right) + (\mu-t)f'\left(\frac{t}{n+1}a + \frac{n+1-t}{n+1}b\right)\right]\mathrm{d}t\right|.$$

Take

$$I = \left|\phi(n;\lambda,\mu;a,b) - \frac{1}{b-a}\left[\int_a^{\Omega_1} f(x)\mathrm{d}x + \int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|$$
$$\le \frac{b-a}{(n+1)^2}\int_0^1 \left[|1-\lambda-t|\left|f'\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right)\right| + |\mu-t|\left|f'\left(\frac{t}{n+1}a + \frac{n+1-t}{n+1}b\right)\right|\right]\mathrm{d}t.$$

Since $|f'|$ is an h-convex function, we have

$$I \le \frac{b-a}{(n+1)^2}\int_0^1 \left[|1-\lambda-t|\left\{h\left(\frac{n+t}{n+1}\right)|f'(a)| + h\left(\frac{1-t}{n+1}\right)|f'(b)|\right\}\right.$$
$$\left. + |\mu-t|\left\{h\left(\frac{t}{n+1}\right)|f'(a)| + h\left(\frac{n+1-t}{n+1}\right)|f'(b)|\right\}\right]\mathrm{d}t$$
$$= \frac{b-a}{(n+1)^2}\int_0^1 \left[\left\{|1-\lambda-t|h\left(\frac{n+t}{n+1}\right) + |\mu-t|h\left(\frac{t}{n+1}\right)\right\}|f'(a)|\right.$$
$$\left. + \left\{|1-\lambda-t|h\left(\frac{1-t}{n+1}\right) + |\mu-t|h\left(\frac{n+1-t}{n+1}\right)\right\}|f'(b)|\right]\mathrm{d}t.$$

$$\left|\phi(n;\lambda,\mu;a,b) - \frac{1}{b-a}\left[\int_a^{\Omega_1} f(x)dx + \int_{\Omega_2}^b f(x)dx\right]\right|$$

$$\leq \frac{b-a}{(n+1)^2}\left[\alpha(t)\left|f'(a)\right|+\beta(t)\left|f'(b)\right|\right], \tag{3.1}$$

where

$$\alpha(t)=\int_0^1\left[|1-\lambda-t|h\left(\frac{n+t}{n+1}\right)+|\mu-t|h\left(\frac{t}{n+1}\right)\right]\mathrm{d}t, \tag{3.2}$$

and

$$\beta(t)=\int_0^1\left[|1-\lambda-t|h\left(\frac{1-t}{n+1}\right)+|\mu-t|h\left(\frac{n+1-t}{n+1}\right)\right]\mathrm{d}t. \tag{3.3}$$

This completes the proof. □

Now we will discuss some special cases of the Theorem 2.

(i) When $h(t)=1$

Corollary 1. *Let $f:I\to\mathbb{R}$ be differentiable function on I^o , $a,b\in I$ with $a<b$. If $|f'|$ is P-function, $0\leq\lambda,\mu\leq 1$ and $n\in\mathbb{N}^*$, then we have*

$$\left|\phi(n;\lambda,\mu;a,b)-\frac{1}{b-a}\left[\int_a^{\Omega_1}f(x)dx+\int_{\Omega_2}^b f(x)dx\right]\right|$$
$$\leq\frac{b-a}{(n+1)^2}\left[\left(\lambda^2+\mu^2\right)-(\lambda+\mu)+1\right]\left(|f'(a)|+|f'(b)|\right).$$

(ii) When $h(t)=t$

Corollary 2. *Let $f:I\to\mathbb{R}$ be differentiable function on I^o , $a,b\in I$ with $a<b$. If $|f'|$ is classical convex function, $0\leq\lambda,\mu\leq 1$ and $n\in\mathbb{N}^*$, then we have*

$$\left|\phi(n;\lambda,\mu;a,b)-\frac{1}{b-a}\left[\int_a^{\Omega_1}f(x)dx+\int_{\Omega_2}^b f(x)dx\right]\right|$$
$$\leq\frac{b-a}{6(n+1)^3}\times\left[\left\{3(1+n)-3(1+2n)\lambda-3\mu+6(1+n)\lambda^2-2\left(\lambda^3-\mu^3\right)\right\}|f'(a)|\right.$$
$$\left.+\left\{3(1+n)-3(1+2n)\mu-3\lambda+6(1+n)\mu^2-2\left(\mu^3-\lambda^3\right)\right\}|f'(b)|\right].$$

(iii) When $h(t)=t^s$

Corollary 3. *Let $f:I\to\mathbb{R}$ be differentiable function on I^o , $a,b\in I$ with $a<b$. If $|f'|$ is Breckner type of s-convex function, $0\leq\lambda,\mu\leq 1$ and $n\in\mathbb{N}^*$, then we have*

$$\left|\phi(n;\lambda,\mu;a,b)-\frac{1}{b-a}\left[\int_a^{\Omega_1}f(x)\mathrm{d}x+\int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|$$
$$\leq\frac{b-a}{(n+1)^{s+2}(s+1)(s+2)}\times\left[\left\{(s+2)(\lambda-1)n^{s+1}+2(1-\lambda+n)^{s+2}\right.\right.$$
$$\left.+(s+2)(n+1)^{s+1}\lambda-n^{s+2}-(n+1)^{s+2}+2\mu^{s+2}+(s+2)(1-\mu)-1\right\}|f'(a)|^q$$

$$+\left\{2\lambda^{s+2}+(s+2)(1-\lambda)-1+(s+2)(\mu-1)n^{s+1}+2(1-\mu+n)^{s+2}\right.$$
$$\left.+\ (s+2)(n+1)^{s+1}\mu-n^{s+2}-(n+1)^{s+2}\right\}|f'(b)|^q\Big].$$

(iv) When $h(t)=t^{-s}$

Corollary 4. *Let $f:I\to\mathbb{R}$ be differentiable function on I^o , $a,b\in I$ with $a<b$. If $|f'|$ is Godunova-Levin-Dragomir type of s-convex function, $0\le\lambda,\mu\le 1$ and $n\in\mathbb{N}^*$, then we have*

$$\left|\phi(n;\lambda,\mu;a,b)-\frac{1}{b-a}\left[\int_a^{\Omega_1}f(x)\mathrm{d}x+\int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|$$
$$\le\frac{b-a}{(1-s)(2-s)(n+1)^{2-s}}\times\Big[\Big\{(2-s)(\lambda-1)n^{1-s}+2(1-\lambda+n)^{2-s}$$
$$+\ (2-s)(n+1)^{1-s}\lambda-n^{2-s}-(n+1)^{2-s}+2\mu^{2-s}+(2-s)(1-\mu)-1\Big\}\left|f'(a)\right|$$
$$+\left\{2\lambda^{2-s}+(2-s)(1-\lambda)-1+(2-s)(\mu-1)n^{1-s}+2(1-\mu+n)^{2-s}\right.$$
$$\left.+\ (2-s)(n+1)^{1-s}\mu-n^{2-s}-(n+1)^{2-s}\right\}\left|f'(b)\right|\Big].$$

(v) When $h(t)=t(1-t)$

Corollary 5. *Let $f:I\to\mathbb{R}$ be differentiable function on I^o, $a,b\in I$ with $a<b$. If $|f'|$ is tgs-convex function, $0\le\lambda,\mu\le 1$ and $n\in\mathbb{N}^*$, then we have*

$$\left|\phi(n;\lambda,\mu;a,b)-\frac{1}{b-a}\left[\int_a^{\Omega_1}f(x)dx+\int_{\Omega_2}^b f(x)dx\right]\right|$$
$$\le\left[\frac{(b-a)\left(|f'(a)|+|f'(a)|\right)}{12(n+1)^4}\right]\times\left[2(1+3n)\left(\mu-\lambda\right)+4(1+n)\left(\mu^3+\lambda^3\right)-2\left(\mu^4+\lambda^4\right)\right].$$

Theorem 3. *Let $f:I\to\mathbb{R}$ be differentiable function on I^o, $a,b\in I$ with $a<b$. If $|f'|^q$ is h-convex function, $0\le\lambda,\mu\le 1$ and $n\in\mathbb{N}^*$, then for $\frac{1}{p}+\frac{1}{q}=1$, $q>1$, we have*

$$\left|\phi(n;\lambda,\mu;a,b)-\frac{1}{b-a}\left[\int_a^{\Omega_1}f(x)dx+\int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|$$
$$\le\frac{b-a}{(n+1)^2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\lambda^{p+1}+(1-\lambda)^{p+1}\right)^{\frac{1}{p}}K_1+\left(\mu^{p+1}+(1-\mu)^{p+1}\right)^{\frac{1}{p}}K_2\right],$$

where

$$K_1=\left(|f'(a)|^q\int_0^1 h\left(\frac{n+t}{n+1}\right)\mathrm{d}t+|f'(b)|^q\int_0^1 h\left(\frac{1-t}{n+1}\right)\mathrm{d}t\right)^{\frac{1}{q}},$$

and

$$K_2=\left(|f'(a)|^q\int_0^1 h\left(\frac{t}{n+1}\right)\mathrm{d}t+|f'(b)|^q\int_0^1 h\left(\frac{n+1-t}{n+1}\right)\mathrm{d}t\right)^{\frac{1}{q}}.$$

Proof. Using Lemma 1, we have

$$\left|\phi(n;\lambda,\mu;a,b)-\frac{1}{b-a}\left[\int_a^{\Omega_1} f(x)\mathrm{d}x+\int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|$$
$$=\left|\frac{b-a}{(n+1)^2}\int_0^1\left[(1-\lambda-t)f'\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)+(\mu-t)f'\left(\frac{t}{n+1}a+\frac{n+1-t}{n+1}b\right)\right]\mathrm{d}t\right|.$$

Now using the property of modulus, we have

$$\begin{aligned}&\left|\phi(n;\lambda,\mu;a,b)-\frac{1}{b-a}\left[\int_a^{\Omega_1} f(x)\mathrm{d}x+\int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|\\&\leq\frac{b-a}{(n+1)^2}\left[\int_0^1|1-\lambda-t|\left|f'\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|\mathrm{d}t\right.\\&\quad\left.+\int_0^1|\mu-t|\left|f'\left(\frac{t}{n+1}a+\frac{n+1-t}{n+1}b\right)\right|\mathrm{d}t\right]\\&=\frac{b-a}{(n+1)^2}\left[I_1+I_2\right].\end{aligned}\tag{3.4}$$

Using Hölder's inequality, we have

$$\begin{aligned}&\int_0^1|1-\lambda-t|\left|f'\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|\mathrm{d}t\\&\leq\left(\int_0^1|1-\lambda-t|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}},\end{aligned}\tag{3.5}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

$$\int_0^1|1-\lambda-t|^p\,\mathrm{d}t=\frac{(1-\lambda)^{p+1}+(\lambda)^{p+1}}{p+1}.\tag{3.6}$$

Since $|f'|^q$ is h-convex, then

$$\begin{aligned}&\int_0^1\left|f'\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|^q\mathrm{d}t\\&\leq\int_0^1 h\left(\frac{n+t}{n+1}\right)|f'(a)|^q\,\mathrm{d}t+\int_0^1 h\left(\frac{1-t}{n+1}\right)|f'(b)|^q\,\mathrm{d}t.\end{aligned}\tag{3.7}$$

Using (3.6) and (3.7) in (3.5), we have

$$\begin{aligned}&\int_0^1|1-\lambda-t|\left|f'\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|\mathrm{d}t\\&\leq\left(\frac{(1-\lambda)^{p+1}+(\lambda)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 h\left(\frac{n+t}{n+1}\right)|f'(a)|^q\,\mathrm{d}t+\int_0^1 h\left(\frac{1-t}{n+1}\right)|f'(b)|^q\,\mathrm{d}t\right)^{\frac{1}{q}}.\end{aligned}\tag{3.8}$$

Now

$$\int_0^1 |\mu - t| \left| f'\left(\frac{t}{n+1}a + \frac{n+1-t}{n+1}b\right)\right| \mathrm{d}t$$

Using Hölder's inequality, we have

$$\begin{aligned}&\int_0^1 |\mu - t| \left| f'\left(\frac{t}{n+1}a + \frac{n+1-t}{n+1}b\right)\right| \mathrm{d}t \\ &\leq \left(\int_0^1 |\mu - t|^p \,\mathrm{d}t\right)^{\frac{1}{p}} \left(\int_0^1 \left| f'\left(\frac{t}{n+1}a + \frac{n+1-t}{n+1}b\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}},\end{aligned} \tag{3.9}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

$$\int_0^1 |\mu - t|^p \,\mathrm{d}t = \frac{(1-\mu)^{p+1} + (\mu)^{p+1}}{p+1}. \tag{3.10}$$

Since $|f'|^q$ is h-convex, then

$$\begin{aligned}&\int_0^1 \left| f'\left(\frac{t}{n+1}a + \frac{n+1-t}{n+1}b\right)\right|^q \mathrm{d}t \\ &\leq \int_0^1 h\left(\frac{t}{n+1}\right) |f'(a)|^q \,\mathrm{d}t + \int_0^1 h\left(\frac{n+1-t}{n+1}\right) |f'(b)|^q \,\mathrm{d}t.\end{aligned} \tag{3.11}$$

Using (3.10) and (3.11) in (3.9), we have

$$\begin{aligned}&\int_0^1 |\mu - t| \left| f'\left(\frac{t}{n+1}a + \frac{n+1-t}{n+1}b\right)\right| \mathrm{d}t \\ &\leq \left(\frac{(1-\mu)^{p+1} + (\mu)^{p+1}}{p+1}\right)^{\frac{1}{p}} \left(\int_0^1 h\left(\frac{t}{n+1}\right) |f'(a)|^q \,\mathrm{d}t + \int_0^1 h\left(\frac{n+1-t}{n+1}\right) |f'(b)|^q \,\mathrm{d}t\right)^{\frac{1}{q}}.\end{aligned} \tag{3.12}$$

Now using (3.8) and (3.12) in (3.4), we have

$$\begin{aligned}&\left|\phi(n;\lambda,\mu;a,b) - \frac{1}{b-a}\left[\int_a^{\Omega_1} f(x)\mathrm{d}x + \int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right| \\ &\leq \frac{b-a}{(n+1)^2}\left[\left(\frac{\lambda^{p+1} + (1-\lambda)^{p+1}}{p+1}\right)^{\frac{1}{p}} K_1 + \left(\frac{\mu^{p+1} + (1-\mu)^{p+1}}{p+1}\right)^{\frac{1}{p}} K_2\right],\end{aligned} \tag{3.13}$$

where

$$K_1 = \left(\int_0^1 h\left(\frac{n+t}{n+1}\right) |f'(a)|^q \,\mathrm{d}t + \int_0^1 h\left(\frac{1-t}{n+1}\right) |f'(b)|^q \,\mathrm{d}t\right)^{\frac{1}{q}},$$

and

$$K_2 = \left(\int_0^1 h\left(\frac{t}{n+1}\right) |f'(a)|^q \,\mathrm{d}t + \int_0^1 h\left(\frac{n+1-t}{n+1}\right) |f'(b)|^q \,\mathrm{d}t\right)^{\frac{1}{q}}.$$

This completes the proof. □

Now we will discuss some special cases of Theorem 3

(i) When $h(t)=1$

Corollary 6. *Let $f:I\to\mathbb{R}$ be differentiable function on I^o , $a,b\in I$ with $a<b$. If $|f'|^q$ is P-function, $0\le\lambda,\mu\le 1$ and $n\in\mathbb{N}^*$, then we have*

$$\left|\phi(n;\lambda,\mu;a,b)-\frac{1}{b-a}\left[\int_a^{\Omega_1}f(x)dx+\int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|$$
$$\le\frac{b-a}{(n+1)^2}\left(\left(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{p+1}\right)^{\frac{1}{p}}+\left(\frac{\mu^{p+1}+(1-\mu)^{p+1}}{p+1}\right)^{\frac{1}{p}}\right)\times\left(|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}.$$

(ii) When $h(t)=t$

Corollary 7. *Let $f:I\to\mathbb{R}$ be differentiable function on I^o , $a,b\in I$ with $a<b$. If $|f'|^q$ is classical convex function, $0\le\lambda,\mu\le 1$ and $n\in\mathbb{N}^*$, then we have*

$$\left|\phi(n;\lambda,\mu;a,b)-\frac{1}{b-a}\left[\int_a^{\Omega_1}f(x)dx+\int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|$$
$$\le\frac{b-a}{(n+1)^2}\left[\left(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left(\frac{(2n+1)|f'(a)|^q+|f'(b)|^q}{2(n+1)}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\mu^{p+1}+(1-\mu)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+(2n+1)|f'(b)|^q}{2(n+1)}\right)^{\frac{1}{q}}\right].$$

(iii) When $h(t)=t^s$

Corollary 8. *Let $f:I\to\mathbb{R}$ be differentiable function on I^o , $a,b\in I$ with $a<b$. If $|f'|^q$ is Breckner type of s-convex function, $0\le\lambda,\mu\le 1$ and $n\in\mathbb{N}^*$, then we have*

$$\left|\phi(n;\lambda,\mu;a,b)-\frac{1}{b-a}\left[\int_a^{\Omega_1}f(x)dx+\int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|$$
$$\le\frac{b-a}{(n+1)^2}\left[\left(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left(\frac{\left((n+1)^{1+s}-n^{1+s}\right)|f'(a)|^q+|f'(b)|^q}{(1+s)(n+1)^s}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\mu^{p+1}+(1-\mu)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+\left((n+1)^{1+s}-n^{1+s}\right)|f'(b)|^q}{(1+s)(n+1)^s}\right)^{\frac{1}{q}}\right].$$

(iv) When $h(t)=t^{-s}$

Corollary 9. *Let $f : I \to \mathbb{R}$ be differentiable function on I^o , $a,b \in I$ with $a < b$. If $|f'|^q$ is Godunova-Levin-Dragomir type of s-convex function, $0 \le \lambda,\mu \le 1$ and $n \in \mathbb{N}^*$, then we have*

$$\left|\phi(n;\lambda,\mu;a,b) - \frac{1}{b-a}\left[\int_a^{\Omega_1} f(x)dx + \int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|$$
$$\le \frac{b-a}{(n+1)^2}\left[\left(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left(\frac{\left((n+1)^{1-s}-n^{1-s}\right)|f'(a)|^q+|f'(b)|^q}{(1-s)(n+1)^{-s}}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\mu^{p+1}+(1-\mu)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+\left((n+1)^{1-s}-n^{1-s}\right)|f'(b)|^q}{(1-s)(n+1)^{-s}}\right)^{\frac{1}{q}}\right].$$

(v) When $h(t) = t(1-t)$

Corollary 10. *Let $f : I \to \mathbb{R}$ be differentiable function on I^o , $a,b \in I$ with $a < b$. If $|f'|^q$ is tgs-convex function, $0 \le \lambda,\mu \le 1$ and $n \in \mathbb{N}^*$, then we have*

$$\left|\phi(n;\lambda,\mu;a,b) - \frac{1}{b-a}\left[\int_a^{\Omega_1} f(x)dx + \int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|$$
$$\le \frac{b-a}{(n+1)^2}\left[\left(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{p+1}\right)^{\frac{1}{p}}+\left(\frac{\mu^{p+1}+(1-\mu)^{p+1}}{p+1}\right)^{\frac{1}{p}}\right]$$
$$\times\left[\frac{(3n+1)\left(|f'(a)|^q+|f'(b)|^q\right)}{6(n+1)^2}\right]^{\frac{1}{q}}.$$

Theorem 4. *Let $f : I \to \mathbb{R}$ be differentiable function on I^o , $a,b \in I$ with $a < b$. If $|f'|^q$ is h-convex function, $0 \le \lambda,\mu \le 1$ and $n \in \mathbb{N}^*$, then we have*

$$\left|\phi(n;\lambda,\mu;a,b) - \frac{1}{b-a}\left[\int_a^{\Omega_1} f(x)\mathrm{d}x + \int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|$$
$$\le \frac{b-a}{(n+1)^2}\left[\left(\frac{1}{2}-\lambda+\lambda^2\right)^{1-\frac{1}{q}}K_1{}^{\frac{1}{q}}+\left(\frac{1}{2}-\mu+\mu^2\right)^{1-\frac{1}{q}}K_2{}^{\frac{1}{q}}\right],$$

where

$$K_1^* = \int_0^1 |1-\lambda-t|\left(h\left(\frac{n+t}{n+1}\right)|f'(a)|^q + h\left(\frac{1-t}{n+1}\right)|f'(b)|^q\right)\mathrm{d}t,$$

and

$$K_2^* = \int_0^1 |\mu-t|\left(h\left(\frac{t}{n+1}\right)|f'(a)|^q + h\left(\frac{n+1-t}{n+1}\right)|f'(b)|^q\right)\mathrm{d}t.$$

Proof. Using Lemma 1, we have

$$\left|\phi(n;\lambda,\mu;a,b) - \frac{1}{b-a}\left[\int_a^{\Omega_1} f(x)\mathrm{d}x + \int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|$$

$$= \left| \frac{b-a}{(n+1)^2} \int_0^1 \left[(1-\lambda-t) f'\left(\frac{n+t}{n+1} a + \frac{1-t}{n+1} b \right) + (\mu - t) f'\left(\frac{t}{n+1} a + \frac{n+1-t}{n+1} b \right) \right] \mathrm{d}t \right|.$$

Now using the property of modulus, we have

$$\begin{aligned} &\left| \phi(n;\lambda,\mu;a,b) - \frac{1}{b-a} \left[\int_a^{\Omega_1} f(x)\mathrm{d}x + \int_{\Omega_2}^b f(x)\mathrm{d}x \right] \right| \\ &\leq \frac{b-a}{(n+1)^2} \left[\int_0^1 |1-\lambda-t| \left| f'\left(\frac{n+t}{n+1} a + \frac{1-t}{n+1} b \right) \right| \mathrm{d}t \right. \\ &\quad \left. + \int_0^1 |\mu - t| \left| f'\left(\frac{t}{n+1} a + \frac{n+1-t}{n+1} b \right) \right| \mathrm{d}t \right]. \end{aligned} \tag{3.14}$$

Using Hölder's integral inequality and h-convexity, we have

$$\begin{aligned} &\int_0^1 |1-\lambda-t| \left| f'\left(\frac{n+t}{n+1} a + \frac{1-t}{n+1} b \right) \right| \mathrm{d}t \\ &\leq \left(\int_0^1 |1-\lambda-t| \, \mathrm{d}t \right)^{1-\frac{1}{q}} \\ &\quad \times \left(\int_0^1 |1-\lambda-t| \left(h\left(\frac{n+t}{n+1} \right) |f'(a)|^q + h\left(\frac{1-t}{n+1} \right) |f'(b)|^q \right) \mathrm{d}t \right)^{\frac{1}{q}} \\ &= \left(\frac{1}{2} - \lambda + \lambda^2 \right)^{1-\frac{1}{q}} K_1^{*\frac{1}{q}}. \end{aligned}$$

Now

$$\begin{aligned} &\int_0^1 |\mu - t| \left| f'\left(\frac{t}{n+1} a + \frac{n+1-t}{n+1} b \right) \right| \mathrm{d}t \\ &\leq \left(\int_0^1 |\mu - t| \, \mathrm{d}t \right)^{1-\frac{1}{q}} \times \left(\int_0^1 |\mu - t| \left(h\left(\frac{t}{n+1} \right) |f'(a)|^q + h\left(\frac{n+1-t}{n+1} \right) |f'(b)|^q \right) \mathrm{d}t \right)^{\frac{1}{q}} \\ &= \left(\frac{1}{2} - \mu + \mu^2 \right)^{1-\frac{1}{q}} K_2^{*\frac{1}{q}}. \end{aligned}$$

Using above two inequalities in (3.14), we have

$$\begin{aligned} &\left| \phi(n;\lambda,\mu;a,b) - \frac{1}{b-a} \left[\int_a^{\Omega_1} f(x)\mathrm{d}x + \int_{\Omega_2}^b f(x)\mathrm{d}x \right] \right| \\ &\leq \frac{b-a}{(n+1)^2} \left[\left(\frac{1}{2} - \lambda + \lambda^2 \right)^{1-\frac{1}{q}} K_1^{\frac{1}{q}} + \left(\frac{1}{2} - \mu + \mu^2 \right)^{1-\frac{1}{q}} K_2^{\frac{1}{q}} \right], \end{aligned} \tag{3.15}$$

where

$$K_1 = \int_0^1 |1-\lambda-t| \left(h\left(\frac{n+t}{n+1} \right) |f'(a)|^q + h\left(\frac{1-t}{n+1} \right) |f'(b)|^q \right) \mathrm{d}t,$$

and

$$K_2 = \int_0^1 |\mu - t| \left(h\left(\frac{t}{n+1}\right) |f'(a)|^q + h\left(\frac{n+1-t}{n+1}\right) |f'(b)|^q \right) \mathrm{d}t.$$

This completes the proof. □

Now we will discuss some cases of Theorem 4.

(i) When $h(t) = 1$

Corollary 11. *Let $f : I \to \mathbb{R}$ be differentiable function on I^o , $a, b \in I$ with $a < b$. If $|f'|^q$ is P-convex function, $0 \le \lambda, \mu \le 1$ and $n \in \mathbb{N}^*$, then we have*

$$\left| \phi(n;\lambda,\mu;a,b) - \frac{1}{b-a}\left[\int_a^{\Omega_1} f(x)\mathrm{d}x + \int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|$$
$$\le \frac{b-a}{(n+1)^2}\left(\lambda^2+\mu^2-(\lambda+\mu)+1\right)\left(|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}.$$

(ii) When $h(t) = t$

Corollary 12. *Let $f : I \to \mathbb{R}$ be differentiable function on I^o , $a, b \in I$ with $a < b$. If $|f'|^q$ is classical convex function, $0 \le \lambda, \mu \le 1$ and $n \in \mathbb{N}^*$, then we have*

$$\left| \phi(n;\lambda,\mu;a,b) - \frac{1}{b-a}\left[\int_a^{\Omega_1} f(x)\mathrm{d}x + \int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|$$
$$\le \frac{b-a}{(n+1)^2}\left[\left(\frac{1}{2}-\lambda+\lambda^2\right)^{1-\frac{1}{q}}\left\{\frac{|f'(a)|^q}{6(n+1)}\left((3n+1)-(6n+3)\lambda\right.\right.\right.$$
$$\left.+6(n+1)\lambda^2-2\lambda^3\right)+\frac{|f'(b)|^q}{6(n+1)}\left(2-3\lambda+2\lambda^3\right)\Big\}^{\frac{1}{q}}$$
$$+\left(\frac{1}{2}-\mu+\mu^2\right)^{1-\frac{1}{q}}\left\{\frac{|f'(a)|^q}{6(n+1)}\left(2-3\mu+2\mu^3\right)\right.$$
$$\left.\left.+\frac{|f'(b)|^q}{6(n+1)}\left((3n+1)-(6n+3)\mu+6(n+1)\mu^2-2\mu^3\right)\right\}^{\frac{1}{q}}\right].$$

(iii) When $h(t) = t^s$

Corollary 13. *Let $f : I \to \mathbb{R}$ be differentiable function on I^o, $a, b \in I$ with $a < b$. If $|f'|^q$ is Breckner type of s-convex function, $0 \le \lambda, \mu \le 1$ and $n \in \mathbb{N}^*$, then we have*

$$\left| \phi(n;\lambda,\mu;a,b) - \frac{1}{b-a}\left[\int_a^{\Omega_1} f(x)\mathrm{d}x + \int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|$$
$$\le \frac{b-a}{(n+1)^2}\times\left[\left(\frac{1}{2}-\lambda+\lambda^2\right)^{1-\frac{1}{q}}\left\{\frac{|f'(a)|^q}{(n+1)^s}\left(\frac{(s+2)(\lambda-1)n^{s+1}+2(1-\lambda+n)^{s+2}}{(s+1)(s+2)}\right)\right.\right.$$

$$+\frac{(s+2)(n+1)^{s+1}\lambda-n^{s+2}-(n+1)^{s+2}}{(s+1)(s+2)}\Bigg)+\frac{|f'(b)|^q}{(n+1)^s}\left(\frac{2\lambda^{s+2}+(s+2)(1-\lambda)-1}{(s+1)(s+2)}\right)\Bigg\}^{\frac{1}{q}}$$
$$+\left(\frac{1}{2}-\mu+\mu^2\right)^{1-\frac{1}{q}}\Bigg\{\frac{|f'(a)|^q}{(n+1)^s}\left(\frac{2\mu^{s+2}+(s+2)(1-\mu)-1}{(s+1)(s+2)}\right)$$
$$+\frac{|f'(b)|^q}{(n+1)^s}\Bigg(\frac{(s+2)(\mu-1)n^{s+1}+(1-\mu+n)^{s+2}}{(s+1)(s+2)}$$
$$+\frac{(s+2)(n+1)^{s+1}\mu-n^{s+2}-(n+1)^{s+2}}{(s+1)(s+2)}\Bigg)\Bigg\}^{\frac{1}{q}}\Bigg].$$

(iv) When $h(t)=t^{-s}$

Corollary 14. *Let $f:I\to\mathbb{R}$ be differentiable function on I^o, $a,b\in I$ with $a<b$. If $|f'|^q$ is Godunova-Levin-Dragomir type of s-convex function, $0\le\lambda,\mu\le 1$ and $n\in\mathbb{N}^*$, then we have*

$$\left|\phi(n;\lambda,\mu;a,b)-\frac{1}{b-a}\left[\int_a^{\Omega_1}f(x)\mathrm{d}x+\int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|$$
$$\le\frac{b-a}{(n+1)^2}\times\Bigg[\left(\frac{1}{2}-\lambda+\lambda^2\right)^{1-\frac{1}{q}}\Bigg\{\frac{|f'(a)|^q}{(n+1)^{-s}}\left(\frac{(2-s)(\lambda-1)n^{1-s}+2(1-\lambda+n)^{2-s}}{(1-s)(2-s)}\right)$$
$$+\frac{(2-s)(n+1)^{1-s}\lambda-n^{2-s}-(n+1)^{2-s}}{(1-s)(2-s)}\Bigg)+\frac{|f'(b)|^q}{(n+1)^{-s}}\left(\frac{2\lambda^{2-s}+(2-s)(1-\lambda)-1}{(1-s)(2-s)}\right)\Bigg\}^{\frac{1}{q}}$$
$$+\left(\frac{1}{2}-\mu+\mu^2\right)^{1-\frac{1}{q}}\Bigg\{\frac{|f'(a)|^q}{(n+1)^{-s}}\left(\frac{2\mu^{2-s}+(2-s)(1-\mu)-1}{(1-s)(2-s)}\right)$$
$$+\frac{|f'(b)|^q}{(n+1)^{-s}}\Bigg(\frac{(2-s)(\mu-1)n^{1-s}+(1-\mu+n)^{2-s}}{(1-s)(2-s)}$$
$$+\frac{(2-s)(n+1)^{1-s}\mu-n^{2-s}-(n+1)^{2-s}}{(1-s)(2-s)}\Bigg)\Bigg\}^{\frac{1}{q}}\Bigg].$$

(v) When $h(t)=t(1-t)$

Corollary 15. *Let $f:I\to\mathbb{R}$ be differentiable function on I^o , $a,b\in I$ with $a<b$. If $|f'|^q$ is tgs-convex function, $0\le\lambda,\mu\le 1$ and $n\in\mathbb{N}^*$, then we have*

$$\left|\phi(n;\lambda,\mu;a,b)-\frac{1}{b-a}\left[\int_a^{\Omega_1}f(x)\mathrm{d}x+\int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|$$
$$\le\frac{b-a}{(n+1)^2}\left(\frac{|f'(a)|^q+|f'(b)|^q}{(n+1)^2}\right)^{\frac{1}{q}}\Bigg[\left(\frac{1}{2}-\lambda+\lambda^2\right)^{1-\frac{1}{q}}$$
$$\left(\frac{(1-\lambda)^2\left(1+4n+2n\lambda-\lambda^2\right)}{6}+\frac{6n\lambda+2\lambda-4n-1}{12}\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{2}-\mu+\mu^2\right)^{1-\frac{1}{q}}\left(\frac{\mu^3\left(2+2n-\mu\right)}{6}+\frac{6n\mu+2\mu-4n-1}{12}\right)^{\frac{1}{q}}\Bigg].$$

Theorem 5. *Let $f:I\to\mathbb{R}$ be differentiable function on I^o , $a,b\in I$ with $a<b$. If $|f'|^q$ is h-convex function, $0\leq\lambda,\mu\leq 1$ and $n\in\mathbb{N}^*$, then for $q>1$, we have*

$$\left|\phi(n;\lambda,\mu;a,b)-\frac{1}{b-a}\left[\int_a^{\Omega_1}f(x)\mathrm{d}x+\int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|\leq\frac{b-a}{(n+1)^2}\left[J_1+J_2\right],$$

where

$$J_1=\left[\frac{1}{n+1}\int_0^1|1-\lambda-t|^q\left\{h(n+t)|f'(a)|^q+h(1-t)|f'(b)|^q\right\}\mathrm{d}t\right]^{\frac{1}{q}},$$

and

$$J_2=\left[\frac{1}{n+1}\int_0^1|\mu-t|^q\left\{h(t)|f'(a)|^q+h(n+1-t)|f'(b)|^q\right\}\mathrm{d}t\right]^{\frac{1}{q}}.$$

Proof. By h-convexity of $|f'(x)|^q$ on $[a,b]$, Lemma 1, and Hölder's integral inequality, we have

$$\begin{aligned}
&\left|\phi(n;\lambda,\mu;a,b)-\frac{1}{b-a}\left[\int_a^{\Omega_1}f(x)\mathrm{d}x+\int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|\\
&\leq\frac{b-a}{(n+1)^2}\times\Bigg[\left(\int_0^1\mathrm{d}t\right)^{1-\frac{1}{q}}\Bigg\{\frac{1}{n+1}\int_0^1|1-\lambda-t|^q\\
&\quad\left(h(n+t)|f'(a)|^q+h(1-t)|f'(b)|^q\right)\mathrm{d}t\Bigg\}^{\frac{1}{q}}+\left(\int_0^1\mathrm{d}t\right)^{1-\frac{1}{q}}\\
&\quad\left\{\frac{1}{n+1}\int_0^1|\mu-t|^q\left(h(t)|f'(a)|^q+h(n+1-t)|f'(b)|^q\right)\mathrm{d}t\right\}^{\frac{1}{q}}\Bigg]\\
&=\frac{b-a}{(n+1)^2}\left[J_1+J_2\right].
\end{aligned}$$

This completes the proof. □

Now we will discuss special cases of Theorem 5.

(i) When $h(t)=1$

Corollary 16. *Let $f:I\to\mathbb{R}$ be differentiable function on I^o , $a,b\in I$ with $a<b$. If $|f'|^q$ is P-function, $0\leq\lambda,\mu\leq 1$ and $n\in\mathbb{N}^*$, then we have*

$$\left|\phi(n;\lambda,\mu;a,b)-\frac{1}{b-a}\left[\int_a^{\Omega_1}f(x)\mathrm{d}x+\int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|$$

$$\leq \frac{(b-a)\left(|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}}{(n+1)^3}\left[\left(\frac{(1-\lambda)^{q+1}+\lambda^{q+1}}{q+1}\right)^{\frac{1}{q}}+\left(\frac{(1-\mu)^{q+1}+\mu^{q+1}}{q+1}\right)^{\frac{1}{q}}\right].$$

(ii) When $h(t)=t$

Corollary 17. *Let $f:I\to\mathbb{R}$ be differentiable function on I^o , $a,b\in I$ with $a<b$. If $|f'|^q$ is classical convex function, $0\leq\lambda,\mu\leq 1$ and $n\in\mathbb{N}^*$, then we have*

$$\begin{aligned}
&\left|\phi(n;\lambda,\mu;a,b)-\frac{1}{b-a}\left[\int_a^{\Omega_1}f(x)\mathrm{d}x+\int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|\\
&\leq\frac{b-a}{(n+1)^3}\left[\frac{1}{(q+1)(q+2)}\right]^{\frac{1}{q}}\times\left[\left\{|f'(a)|^q\left((nq+2n+1-\lambda)(1-\lambda)^{q+1}\right.\right.\right.\\
&\quad\left.+((n+1)q+2(n+1)-\lambda)\lambda^{q+1}\right)+|f'(b)|^q\left((q+1+\lambda)(1-\lambda)^{q+1}+\lambda^{q+2}\right)\Big\}^{\frac{1}{q}}\\
&\quad+\left\{|f'(a)|^q\left(\mu^{q+2}+(q+1+\mu)(1-\mu)^{q+1}\right)\right.\\
&\quad\left.\left.+|f'(b)|^q\left(((n+1)q+2(n+1)-\mu)\mu^{q+1}+(nq+2n+1-\mu)(1-\mu)^{q+1}\right)\right\}^{\frac{1}{q}}\right].
\end{aligned}$$

Letting $\lambda=\mu=\frac{1}{2}$ in Corollary 17, we have

$$\begin{aligned}
\phi\left(n;\frac{1}{2},\frac{1}{2}a,b\right)=\frac{1}{n+1}\left[\frac{f(a)+f(b)}{2}+\frac{1}{2}\left\{f\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)\right.\right.\\
\left.\left.+f\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)\right\}\right].
\end{aligned}$$

Corollary 18. *Let $f:I\to\mathbb{R}$ be differentiable function on I^o, $a,b\in I$ with $a<b$. If $|f'|^q$ is convex function, $q\geq 1$, $\lambda=\mu=\frac{1}{2}$ and $n\in\mathbb{N}^*$, then we have*

$$\begin{aligned}
&\left|\phi\left(n;\frac{1}{2},\frac{1}{2}a,b\right)-\frac{1}{b-a}\left[\int_a^{\Omega_1}f(x)\mathrm{d}x+\int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|\\
&\leq\frac{b-a}{2(n+1)^3}\left[\frac{1}{4(q+1)(q+2)}\right]^{\frac{1}{q}}\\
&\quad\times\left[\left\{|f'(a)|^q\left((2nq+4n+1)+(2(n+1)q+4n+3)\right)+|f'(b)|^q\left((2q+3)+1\right)\right\}^{\frac{1}{q}}\right.\\
&\quad\left.+\left\{|f'(a)|^q\left(1+(2q+3)\right)+|f'(b)|^q\left((2(n+1)q+4n+3)+(2nq+4n+1)\right)\right\}^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 19. *Let $f:I\to\mathbb{R}$ be differentiable function on I^o, $a,b\in I$ with $a<b$. If $|f'|^q$ is convex function, $q\geq 1$, $\lambda=\mu=\frac{1}{2}$ and $n\in\mathbb{N}^*$ and*

$$\frac{f(a)+f(b)}{2}=f\left(\frac{a+b}{2}\right),$$

then

$$\left|\phi\left(n;\frac{1}{2},\frac{1}{2}a,b\right)-\frac{1}{b-a}\left[\int_a^{\Omega_1}f(x)\mathrm{d}x+\int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|$$
$$\leq\frac{b-a}{2(n+1)^3}\left[\frac{1}{4(q+1)(q+2)}\right]^{\frac{1}{q}}\times\Big\{\left[(2nq+4n+1)|f'(a)|^q+(2q+3)|f'(b)|^q\right]^{\frac{1}{q}}$$
$$+\left[(2(n+1)q+4n+3)|f'(a)|^q+|f'(b)|^q\right]^{\frac{1}{q}}$$
$$+\left[|f'(a)|^q+(2(n+1)q+4n+3)|f'(b)|^q\right]^{\frac{1}{q}}$$
$$+\left[(2q+3)|f'(a)|^q+(2nq+4n+1)|f'(b)|^q\right]^{\frac{1}{q}}\Big\}.$$

(iii) When $h(t)=t^s$

Corollary 20. *Let $f:I\to\mathbb{R}$ be differentiable function on I^o , $a,b\in I$ with $a<b$. If $|f'|^q$ is Breckner type of s-convex function, $0\leq\lambda,\mu\leq 1$ and $n\in\mathbb{N}^*$, then we have*

$$\left|\phi(n;\lambda,\mu;a,b)-\frac{1}{b-a}\left[\int_a^{\Omega_1}f(x)\mathrm{d}x+\int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|\leq\frac{b-a}{(n+1)^2}\left[T_1+T_2\right].$$

where

$$T_1=\left[\frac{1}{n+1}\int_0^1|1-\lambda-t|^q\left\{(n+t)^s|f'(a)|^q+(1-t)^s|f'(b)|^q\right\}\mathrm{d}t\right]^{\frac{1}{q}},$$

and

$$T_2=\left[\frac{1}{n+1}\int_0^1|\mu-t|^q\left\{t^s|f'(a)|^q+(n+1-t)^s|f'(b)|^q\right\}\mathrm{d}t\right]^{\frac{1}{q}}.$$

(iv) When $h(t)=t^{-s}$

Corollary 21. *Let $f:I\to\mathbb{R}$ be differentiable function on I^o , $a,b\in I$ with $a<b$. If $|f'|^q$ is Godunova-Levin-Dragomir type of s-convex function, $0\leq\lambda,\mu\leq 1$ and $n\in\mathbb{N}^*$, then we have*

$$\left|\phi(n;\lambda,\mu;a,b)-\frac{1}{b-a}\left[\int_a^{\Omega_1}f(x)\mathrm{d}x+\int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right|\leq\frac{b-a}{(n+1)^2}\left[T_1^*+T_2^*\right],$$

where

$$T_1^*=\left[\frac{1}{n+1}\int_0^1|1-\lambda-t|^q\left\{(n+t)^{-s}|f'(a)|^q+(1-t)^{-s}|f'(b)|^q\right\}\mathrm{d}t\right]^{\frac{1}{q}},$$

and

$$T_2^*=\left[\int_0^1|\mu-t|^q\left\{t^{-s}|f'(a)|^q+(n+1-t)^{-s}|f'(b)|^q\right\}\mathrm{d}t\right]^{\frac{1}{q}}.$$

(v) When $h(t)=t(1-t)$

Corollary 22. *Let $f : I \to \mathbb{R}$ be differentiable function on I^o , $a,b \in I$ with $a < b$. If $|f'|^q$ is tgs-convex function, $0 \le \lambda,\mu \le 1$ and $n \in \mathbb{N}^*$, then we have*

$$\begin{aligned}
&\left|\phi(n;\lambda,\mu;a,b) - \frac{1}{b-a}\left[\int_a^{\Omega_1} f(x)\mathrm{d}x + \int_{\Omega_2}^b f(x)\mathrm{d}x\right]\right| \\
&\le \left(\frac{b-a}{(n+1)^3}\right)\left(\frac{1}{(q+1)(q+2)(q+3)}\right)^{\frac{1}{q}} \\
&\quad \Big[|f'(a)|^q \times \big\{(1-\lambda)^{q+1}\left[n(1-n)(q+2)(q+3)+(1-2n)(1-\lambda)(q+3)-2(1-\lambda)^2\right] \\
&\quad -(-\lambda)^{q+1}\left[n(1+n)(q+2)(q+3)-(1+2n)(q+3)\lambda+2\lambda^2\right]\big\} \\
&\quad +|f'(b)|^q \times \big\{(1-\lambda)^{q+2}\left[q+3-2(1-\lambda)-2(1-\lambda)^2\right] \ -(-\lambda)^{q+2}\left[2\lambda-(q+3)\right]\big\}\Big]^{\frac{1}{q}} \\
&\quad + \Big[|f'(a)|^q \times \big\{\mu^{q+2}\left[(q+3)-2\mu\right](q+1)-(\mu-1)^{q+2}\left[(q+3)+2(\mu-1)\right]\big\} \\
&\quad +|f'(b)|^q \times \big\{(\mu)^{q+1}\left[-n(1+n)(q+2)(q+3)+(1+2n)(q+3)\mu-2\mu^2\right] \\
&\quad -(\mu-1)^{q+1}\left[n(n-1)(q+2)(q+3)(\mu-1)^{q+1}\right] \\
&\quad -(\mu-1)^{q+1}\left[(1-2n)(q+3)(\mu-1)+2(\mu-1)^2\right]\big\}\Big]^{\frac{1}{q}}.
\end{aligned}$$

4. Applications

For $a > 0$, $b > 0$ and $0 \le \omega < \infty$, define

$$A(a,b) = \frac{a+b}{2}, \quad G(a,b) = \sqrt{ab}, \quad H(a,b) = \frac{2ab}{a+b}.$$

$$L(a,b) = \begin{cases} \left[\dfrac{b^{s+1}-a^{s+1}}{(s+1)(b-a)}\right], & a \ne b \\ a, & a = b, \end{cases}$$

$$I(a,b) = \begin{cases} \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & a \ne b \\ a, & a = b, \end{cases}$$

$$H_{\omega,s}(a,b) = \begin{cases} \left[\dfrac{a^s+(ab)^{s/2}\omega+b^s}{\omega+2}\right], & s \ne 0 \\ \sqrt{ab}, & s = 0. \end{cases}$$

These means of two positive number a and b are, respectively, called the arithmetic, harmonic, generalized logarithmic, identric, and Heronion means respectively. Now applying Corollary 17 to $f(x) = x^s$ for $s \ne 0$ and $x > 0$ leads to the following inequality for means.

Theorem 6. *Let $a,b>0, q\geq 1$, either $s>1$ and $(s-1)q\geq 1$ or $s<0$, then*

$$\begin{aligned}
&\left|\frac{A(a^s,b^s)+A\left(\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)^s,\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)^s\right)}{n+1}\right.\\
&\quad\left.+\left(\frac{n-1}{n+1}\right)L^s\left(\frac{1}{n+1}a+\frac{n}{n+1}b,\frac{n}{n+1}a+\frac{1}{n+1}b\right)-L^s(a,b)\right|\\
&\leq\frac{b-a}{2(n+1)^3}|s|^q\left[\frac{1}{4(q+1)(q+2)}\right]^{\frac{1}{q}}\\
&\quad\times\left\{\left[(2nq+4n+1)a^{(s-1)q}+(2q+3)b^{(s-1)q}\right]^{\frac{1}{q}}\right.\\
&\quad+\left[(2(n+1)q+4n+3)a^{(s-1)q}+b^{(s-1)q}\right]^{\frac{1}{q}}\\
&\quad+\left[a^{(s-1)q}+(2(n+1)q+4n+3)b^{(s-1)q}\right]^{\frac{1}{q}}\\
&\quad\left.+\left[(2q+3)a^{(s-1)q}+(2nq+4n+1)b^{(s-1)q}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Taking $f(x)=\ln x$ for $x>0$ in Corollary 17 results in the following inequalities for means.

Theorem 7. *Let $a,b>0$ and $q\geq 1$, we get*

$$\begin{aligned}
&\left|\frac{\ln G(a,b)+\ln G\left(\frac{1}{n+1}a+\frac{n}{n+1}b,\frac{n}{n+1}a+\frac{1}{n+1}b\right)}{n+1}\right.\\
&\quad\left.+\frac{n-1}{n+1}\ln I\left(\frac{1}{n+1}a+\frac{n}{n+1}b,\frac{n}{n+1}a+\frac{1}{n+1}b\right)-\ln I(a,b)\right|\\
&\leq\frac{b-a}{2(n+1)^3}|s|^q\left[\frac{1}{4(q+1)(q+2)}\right]^{\frac{1}{q}}\\
&\quad\times\left\{\left[\frac{2nq+4n+1}{a^q}+\frac{2q+3}{b^q}\right]^{\frac{1}{q}}+\left[\frac{2(n+1)q+4n+3}{a^q}+\frac{1}{b^q}\right]^{\frac{1}{q}}\right.\\
&\quad\left.+\left[\frac{1}{a^q}+\frac{2(n+1)q+4n+3}{b^q}\right]^{\frac{1}{q}}+\left[\frac{2q+3}{a^q}+\frac{2nq+4n+1}{b^q}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Theorem 8. *For $b>a>0$, $w\geq 0$, and $s\geq 4$ or $0\neq s<1$, one has*

$$\left|\frac{1}{n+1}\left[\frac{H^s_{w,s}(a,b)}{H(a^s,b^s)}+\frac{1}{2}\left\{H^s_{w,s}\left(\frac{n}{n+1}\frac{a}{b}+\frac{1}{n+1}\frac{b}{a},1\right)\right.\right.\right.$$

$$+H_{w,s}^{s}\left(\frac{1}{n+1}\frac{a}{b}+\frac{n}{n+1}\frac{b}{a},1\right)\Bigg\}\Bigg]-H_{w,s}^{s}\left(\frac{L\left(a^2,b^2\right)}{G^2(a,b)}\right)$$
$$-\left(\frac{1-n}{1+n}\right)H_{w,s}^{s}\left(\frac{L\left(\frac{1}{n+1}a^2+\frac{n}{n+1}b^2,\frac{1}{n+1}a^2+\frac{1}{n+1}b^2\right)}{G^2(a,b)}\right)\Bigg|$$
$$\leq\frac{(b-a)|s|}{(n+1)^2}\left[\frac{A(a,b)A\left(a^{2(s-1)},b^{2(s-1)}\right)}{(w+2)G^{2s}(a,b)}+\frac{A(a,b)A\left(a^{s-2},b^{s-2}\right)}{(w+2)G^{s}(a,b)}\right].$$

Proof. Let $f(x)=\frac{x^s+wx^{s/2}+1}{w+2}$ for $s\notin(1,4)$ and $x>0$, then

$$f'(x)=\frac{s}{w+2}\left(x^{s-1}+\frac{w}{2}x^{s/2-1}\right),$$
$$\int f(x)\mathrm{d}x=\frac{1}{w+2}\left(\frac{x^{s+1}}{s+1}+\frac{wx^{s/2+1}}{s/2+1}+x\right).$$

Now

$$\frac{1}{\frac{b}{a}-\frac{a}{b}}\left[\int_{\frac{a}{b}}^{\frac{n}{n+1}\frac{a}{b}+\frac{1}{n+1}\frac{b}{a}}f(x)\mathrm{d}x+\int_{\frac{1}{n+1}\frac{a}{b}+\frac{n}{n+1}\frac{b}{a}}^{\frac{b}{a}}f(x)\mathrm{d}x\right]$$
$$=\frac{\left(\frac{n}{n+1}a^2+\frac{1}{n+1}b^2\right)^{s+1}-\left(\frac{n}{n+1}a^2+\frac{1}{n+1}b^2\right)^{s+1}}{(w+2)(s+1)(b^2-a^2)(ab)^s}$$
$$+w\frac{\left(\frac{n}{n+1}a^2+\frac{1}{n+1}b^2\right)^{\frac{s}{2}+1}-\left(\frac{n}{n+1}a^2+\frac{1}{n+1}b^2\right)^{\frac{s}{2}+1}}{(w+2)(\frac{s}{2}+1)(b^2-a^2)(ab)^{\frac{s}{2}}}$$
$$+\frac{1}{w+2}+\frac{\left(b^2\right)^{s+1}-\left(a^2\right)^{s+1}}{(w+2)(s+1)(b^2-a^2)(ab)^s}$$
$$+w\frac{\left(b^2\right)^{\frac{s}{2}}-\left(a^2\right)^{\frac{s}{2}}}{(w+2)(\frac{s}{2}+1)(b^2-a^2)(ab)^{\frac{s}{2}}}+\frac{1}{w+2}$$
$$=\left(\frac{1-n}{1+n}\right)H_{w,s}^{s}\left(\frac{L\left(\frac{1}{n+1}a^2+\frac{n}{n+1}b^2,\frac{1}{n+1}a^2+\frac{1}{n+1}b^2\right)}{G^2(a,b)}\right)+H_{w,s}^{s}\left(\frac{L\left(a^2,b^2\right)}{G^2(a,b)}\right),$$

and

$$\phi\left(n;\frac{1}{2},\frac{1}{2},a,b\right)$$
$$=\frac{1}{n+1}\left[\frac{f(a)+f(b)}{2}+\frac{1}{2}\left\{f\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)\ +f\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)\right\}\right]$$
$$=\frac{1}{n+1}\left[\frac{\left(\frac{a}{b}\right)^s+w\left(\frac{a}{b}\right)^{s/2}+1}{2(w+2)}+\frac{\left(\frac{b}{a}\right)^s+w\left(\frac{b}{a}\right)^{s/2}+1}{2(w+2)}\right.$$

$$+\frac{1}{2(w+2)}\left\{\left(\frac{n}{n+1}\frac{a}{b}+\frac{1}{n+1}\frac{b}{a}\right)^{s}+w\left(\frac{n}{n+1}\frac{a}{b}+\frac{1}{n+1}\frac{b}{a}\right)^{s/2}+1\right.$$
$$\left.+\left(\frac{1}{n+1}\frac{a}{b}+\frac{n}{n+1}\frac{b}{a}\right)^{s}+w\left(\frac{1}{n+1}\frac{a}{b}+\frac{n}{n+1}\frac{b}{a}\right)^{s/2}+1\right\}\Bigg]$$
$$=\frac{1}{n+1}\left[\frac{H^{s}_{w,s}(a,b)}{H(a^{s},b^{s})}+\frac{1}{2}\left\{H^{s}_{w,s}\left(\frac{n}{n+1}\frac{a}{b}+\frac{1}{n+1}\frac{b}{a},1\right)\right.\right.$$
$$\left.\left.+H^{s}_{w,s}\left(\frac{1}{n+1}\frac{a}{b}+\frac{n}{n+1}\frac{b}{a},1\right)\right\}\right].$$

In virtue of Corollary 19, it follows that

$$\left|\phi(n;\lambda,\mu,a,b)-\frac{1}{b-a}\left[\int_{a}^{\Omega_1}f(x)\mathrm{d}x+\int_{\Omega_2}^{b}f(x)\mathrm{d}x\right]\right|$$
$$=\left|\frac{1}{n+1}\left[\frac{H^{s}_{w,s}(a,b)}{H(a^{s},b^{s})}+\frac{1}{2}\left\{H^{s}_{w,s}\left(\frac{n}{n+1}\frac{a}{b}+\frac{1}{n+1}\frac{b}{a},1\right)\right.\right.\right.$$
$$\left.\left.+H^{s}_{w,s}\left(\frac{1}{n+1}\frac{a}{b}+\frac{n}{n+1}\frac{b}{a},1\right)\right\}\right]-H^{s}_{w,s}\left(\frac{L\left(a^{2},b^{2}\right)}{G^{2}(a,b)}\right)$$
$$\left.-\left(\frac{1-n}{1+n}\right)H^{s}_{w,s}\left(\frac{L\left(\frac{1}{n+1}a^{2}+\frac{n}{n+1}b^{2},\frac{1}{n+1}a^{2}+\frac{1}{n+1}b^{2}\right)}{G^{2}(a,b)}\right)\right|.$$

On the other hand, we have

$$\frac{\frac{b}{a}-\frac{a}{b}}{4(n+1)^{2}}\left[\left|f'\left(\frac{a}{b}\right)\right|+\left|f'\left(\frac{b}{a}\right)\right|\right]$$
$$=\frac{b^{2}-a^{2}}{4(n+1)^{2}ab}\left(\frac{|s|}{w+2}\right)\left\{\left[\left(\frac{b}{a}\right)^{s-1}+\left(\frac{a}{b}\right)^{s-1}\right]+\frac{w}{2}\left[\left(\frac{b}{a}\right)^{s/2-1}+\left(\frac{a}{b}\right)^{s/2-1}\right]\right\}$$
$$=\frac{(b-a)|s|}{(n+1)^{2}}\left[\frac{A(a,b)A\left(a^{2(s-1)},b^{2(s-1)}\right)}{(w+2)G^{2s}(a,b)}+\frac{A(a,b)A\left(a^{s-2},b^{s-2}\right)}{(w+2)G^{s}(a,b)}\right].$$

This completes the proof. □

5. Conclusion

In this article, we have derived a new auxiliary result. With the help of this result we obtained several new Hermite-Hadamard type of integral inequalities. Numerous new special cases which can be deduced from our main results are also discussed in detail. At the last we have discussed applications of our results to means of real numbers. It is expected that the ideas and techniques of this paper may stimulate further research in the field.

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THE STRUCTURE OF THE UNIT GROUP OF SOME GROUP ALGEBRAS

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Abstract. Let FM be the group algebra of the modular 2-group M over a finite field F of characteristic two. In the present note we establish the structure of the unit group of the group algebra FM and verify the question of Johnson.

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1. Introduction and Results

Let F be a field of characteristic p and let G be a group such that G contains an element of order p. Let $U(FG)$ be the group of units of the group algebra FG. It is easy to see that $U(FG) = U(F) \times V(FG)$, in which

$$V(FG) := \left\{ x = \sum_{g \in G} \alpha_g g \in U(FG) \mid \chi(x) = \sum_{g \in G} \alpha_g = 1 \right\},$$

where $\chi(x)$ is the augmentation of $x \in FG$ (see [8, Chapters 2-3, p. 194-196]).

The structure of the group of units and its subgroup $V(FG)$ has been investigated by several authors, but the complete description is known only for certain group algebras (for example, see [1, 2, 10–13, 15, 16, 18–21]). For an overview in this topic we recommend the survey paper [8].

Let $\zeta(G)$ be the center and let G' be the commutator subgroup of G, respectively. It is well known [9, Theorem 2], if FG is a modular group algebra, then $G \cap \zeta(V(FG)) = \zeta(G)$ and $G \cap V(FG)' = G'$. The question whether $G \cap V(FG)^p = G^p$ is due to Johnson [15]. The Johnson's question was affirmatively confirmed for nonabelian groups in the following cases: (i) the group of exponent p and order p^3 [15, Theorem 7]; (ii) G is a finite p-group (p is an odd prime) with Frattini subgroup of order p [4]; and (iii) G is the modular 2-group of order 16 and F is the field of 2 elements [14, Theorem 2]. The structure of elements of order two in $V(FG)$, where

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G is a 2-group of maximal class and F is the field of elements two, was described in [5].

Let

$$M_n = \langle\, a,b \mid a^{2^{n-1}} = b^2 = 1, (a,b) = a^{2^{n-2}} \,\rangle = \langle\, a \,\rangle \rtimes \langle\, b \,\rangle, \qquad (n \geq 4) \tag{1.1}$$

be the modular 2-group. The group M_n appears very frequently in the investigation of the group of units [3, 6, 7, 14, 17].

In the present note the structure of $V(FM_n)$ is established and affirmative answer for the Johnson‘s question is provided.

Theorem 1. *Let M_n be the modular group given in (1.1). If F is a field with $|F| = 2^m \geq 2$, then $V(FM_n)$ is a central extension of $C_2^{3m2^{n-3}}$ by*

$$C_{2^{n-2}}^m \times C_2^{7m2^{n-4}} \times \prod_{i=0}^{n-5} C_{2^{n-i-3}}^{2^i m}.$$

Corollary 1. *Let M_n be the modular group of order 2^n. If F is a field with $|F| = p^m \geq 2$, then*

$$M_n \cap V(FM_n)^2 = M_n^2.$$

2. Proof

Let H be a normal subgroup of a finite p-group G. The ideal of FG generated by the set $\{h-1 \mid h \in H\}$ is denoted by $I(H)$. Let $G[p^i]$ denote the subgroup of G generated by the elements of order p^i. We use the notation G^{p^i} for the subgroup $\langle\, g^{p^i} \mid g \in G \,\rangle$. Set $x^g := g^{-1}xg$, where $g \in G$ and $x \in FG$. Let $\widehat{S} = \sum_{s \in S} s \in FG$, where $S \subseteq G$ is a finite subset and let $|S|$ denote the cardinality of S. Furthermore, the order of $g \in G$ will be denoted by $|g|$.

If G is an abelian p-group, then the number of subgroups of order p^i in the decomposition of G into a direct product of cyclic groups will be denoted by $f_i(G)$.

Lemma 1. *Let F be a field with $|F| = p^m \geq p$. If G is a finite abelian p-group, then*

(i) $V(FG)^p = V(FG^p)$;
(ii) $V(FG)[p] = 1 + I(G[p])$; *and*
(iii) $f_i\big(V(FG)\big) = m\big(|G^{p^{i-1}}| - 2|G^{p^i}| + |G^{p^{i+1}}|\big)$.

Proof. (i) If $u = \sum_{g \in G} \alpha_g g \in V(FG)$, then $u^p = \sum_{g \in G} \alpha_g^p g^p \in V(FG^p)$, so $V(FG)^p \subseteq V(FG^p)$.

Let $u = \sum_{g \in G^p} \alpha_g g \in V(FG^p)$. Obviously, the mapping $\tau(\alpha) = \alpha^p$ is an automorphism of F. Therefore there exists a $\beta_g \in F$ and $h \in G$ for every $g \in G^p$ such that $\beta_g^p = \alpha_g$ and $h^p = g$. We have that $u = \sum_{g \in G^p} \alpha_g g = \sum_{h \in G} \beta_g^p h^p \in V(FG)^p$, which completes the proof.

(ii) If $u \in 1 + I(G[p])$, then $u^p = 1$ and $1 + I(G[p]) \subseteq V(FG)[p]$.

Let $u \in V(FG)[p]$. Clearly, $u-1$ can be written as $x_1h_1+x_2h_2+\cdots+x_sh_s$ for some s, where $x_i \in FG[p]$ and the set $\{h_i\}$ is a complete set of right coset representatives of $G[p]$ in G. We have that $x_1^ph_1^p+x_2^ph_2^p+\cdots+x_s^ph_s^p=(u-1)^p=u^p-1=0$. Suppose that $h_i^p=h_j^p$ for some i, j and $i \neq j$. Clearly, $h_ih_j^{-1} \in G[p]$ which is impossible. Without loss of generality we can assume that $h_1=1$ and $h_i^p \neq 1$ if $1<i\leq s$. Therefore $x_i^p=0$ if $1<i\leq s$ and $u-1 \in I(G[p])$ which proves that $V(FG)[p] \subseteq 1+I(G[p])$.

(iii) It is true when $|F|=p$ [18, Theorem 2.4]. Now we extend it to any finite field.

If $V(FG)=\langle\, a_1 \,\rangle \times \cdots \times \langle\, a_s \,\rangle$, then $V(FG)[p]=\langle\, a_1^{b_1-1} \,\rangle \times \langle\, a_2^{b_2-1} \,\rangle \times \cdots \times \langle\, a_s^{b_s-1} \,\rangle$ in which $b_j=|a_j|$. The number of elements in $1+I(G[p])$ equals $|I(G[p])|$. Evidently, $I(G[p])$ can be considered as a vector space over F with the basis $\{u(h-1) \mid u \in T(G/G[p]),\ h \in G[p]\}$ in which $T(G/G[p])$ is a complete set of right coset representatives of $G[p]$ in G. Thus

$$|I(G[p])|=p^{m\frac{|G|}{|G[p]|}(G[p]-1)}=p^{m(|G|-|G^p|)}.$$

According to part (ii), the p-rank of $V(FG)$ is $m(|G|-|G^p|)$.

The part (i) shows that the p-rank of $V(FG)^p$ is $m(|G^p|-|G^{p^2}|)$, so

$$f_1\big(V(FG)\big)=m(|G|-|G^p|)-m(|G^p|-|G^{p^2}|)=m\big(|G|-2|G^p|+|G^{p^2}|\big).$$

The proof can be easily completed using part (i) and induction on $V(FG^{p^i})$. □

Lemma 2. *If F is a field with $|F|=2^m \geq 2$, then*

$$\zeta\big(V\big)=V\big(F\zeta(M_n)\big)\times N,$$

where $N \cong C_2^{3m2^{n-3}}$ *and* $f_i\Big(V\big(F\zeta(M_n)\big)\Big)=\begin{cases} m & \text{if} \quad i=n-2; \\ m\cdot 2^{n-3-i} & \text{if} \quad i<n-2. \end{cases}$

Proof. Let C_g be the conjugacy class of $g \in M_n \setminus \zeta(M_n)$. Clearly, $|C_g|=2$, $M_n'=\{1,a^{2^{n-2}}\}$ and $\widehat{C_g}=g\widehat{M_n'}$. Let N be defined by

$$N=\langle\, 1+\beta_ia^{2i+1}\widehat{M_n'} \mid 0\leq i<2^{n-3}, \beta_i \in F \,\rangle \times \langle\, 1+\gamma_ia^ib\widehat{M_n'} \mid 0\leq i<2^{n-2}, \gamma_i \in F \,\rangle.$$

Since $(1+x\widehat{M_n'})(1+y\widehat{M_n'})=1+x\widehat{M_n'}+y\widehat{M_n'}$ and $(1+x\widehat{M_n'})^2=1$ for every $x,y \in FM_n$, the group $N \cong C_2^{3m2^{n-3}}$ is an elementary abelian 2-group and

$$\zeta\big(V\big)=V\big(F\zeta(M_n)\big)\times N.$$

Indeed, $V\big(F\zeta(M_n)\big)\times N \subseteq \zeta\big(V\big)$. Since $\zeta(M_n)=M_n^2$, each element $x \in \zeta\big(V\big)$ can be written as $x=x_1+x_2$ in which

$$x_1=\textstyle\sum_{i=0}^{2^{n-2}-1}\alpha_ia^{2i}, \quad x_2=\sum_{i=0}^{2^{n-3}-1}\beta_ia^{2i+1}\widehat{M'}+\sum_{i=0}^{2^{n-2}-1}\gamma_ia^ib\widehat{M'}, \qquad (\alpha_i,\beta_i,\gamma_i \in F).$$

It is clear, that the augmentation of x_2 equals 0 therefore x_1 is an invertible element with augmentation 1. Obviously, $1+a^{2i}\cdot x_2 \in N$ therefore $x_1^{-1}x=x_1^{-1}(x_1+x_2)=$

$1+x_1^{-1}x_2 \in N$. Since $V\big(F\zeta(M_n)\big)\cap N=\{1\}$ we have proved that $\zeta\big(V\big)\subseteq V\big(F\zeta(M_n)\big)\times N$.

Since $\zeta(M_n)=M_n^2\cong C_{2^{n-2}}$, Lemma 1(iii) ensures that

$$f_i\Big(V\big(F\zeta(M_n)\big)\Big)=m(2^{n-1-i}-2\cdot 2^{n-2-i}+2^{n-3-i})=m2^{n-i-3}$$

for $i<n-2$ and $f_i\Big(V\big(F\zeta(M_n)\big)\Big)=m$ for $i=n-2$. □

Lemma 3. *Let F be a field with $|F|=2^m\geq 2$. Then $|\zeta\big(V\big)|=2^{5m2^{n-3}-m}$ and*

$$\zeta\big(V\big)\cong C_{2^{n-2}}^m\times C_2^{7m2^{n-4}}\times\prod_{i=0}^{n-5}C_{2^{n-i-3}}^{2^i m}.$$

Proof. According to the previous lemma, $\zeta\big(V\big)\cong V\big(F\zeta(M_n)\big)\times N$. Since $|N|=2^{3m2^{n-3}}$ and $|V\big(F\zeta(M_n)\big)|=|F|^{|\zeta(M_n)|-1}=2^{m(2^{n-2}-1)}$, we can easily compute that

$$|\zeta\big(V\big)|=2^{3m2^{n-3}+m2^{n-2}-m}=2^{5m2^{n-3}-m}.$$

Finally, using Lemma 2, it is easy to check that

$$\zeta\big(V\big)\cong C_{2^{n-2}}^m\times C_{2^{n-3}}^m\times C_{2^{n-4}}^{2m}\times C_{2^{n-5}}^{2^2m}\times C_{2^{n-6}}^{2^3m}\times\cdots\times C_{2^2}^{m2^{n-5}}\times C_2^{7m2^{n-4}}.$$

□

Proof of Theorem. Each $x\in FM_n$ can be written as $x=x_1+x_2b$, where $x_1,x_2\in F\langle\, a\,\rangle$ (see (1.1)) and

$$x^2=x_1^2+x_2x_2^b+(x_1+x_1^b)x_2b.$$

Obviously, we can write $x_1=y_1+y_2a$ and $x_2=z_1+z_2a$, where $y_1,y_2,z_1,z_2\in F\langle\, a^2\,\rangle$, so

$$x_1+x_1^b=y_2a(1+a^{2^{n-2}})\in I(M_n') \quad \text{and}$$

$$x_2x_2^b=z_1^2+z_2^2a^{2^{n-2}+2}+z_1z_2a(1+a^{2^{n-2}})\in\zeta(FM_n).$$

Consequently, $(x_1+x_1^b)x_2b\in I(M_n')\subseteq\zeta(FM_n)$ and $x^2\in\zeta\big(V(FM_n)\big)$ for every $x\in V(FM_n)$. Hence $V(FM_n)/\zeta(V)$ is an elementary abelian 2-group of order $2^{3m2^{n-3}}$ and

$$\zeta(V)\cong C_{2^{n-2}}^m\times C_{2^{n-3}}^m\times C_{2^{n-4}}^{2m}\times C_{2^{n-5}}^{2^2m}\times C_{2^{n-6}}^{2^3m}\times\cdots\times C_{2^2}^{m2^{n-5}}\times C_2^{7m2^{n-4}}$$

by Lemma 3 which is the desired conclusion. □

Proof of Corollary. Since $V(FM_n)^2\subseteq\zeta(V)$ and $\zeta(M_n)=M_n^2$,

$$M_n\cap V(FM_n)^2\subseteq M_n\cap\zeta(V)\subseteq\zeta(M_n)=M_n^2.$$

□

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HERMITE-HADAMARD INEQUALITIES FOR UNIFORMLY CONVEX FUNCTIONS AND ITS APPLICATIONS IN MEANS

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This paper is dedicated to Professor Hossien Mohebi on his 60th birthday.

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Abstract. In this paper, we prove Hermite-Hadamard inequality for uniformly convex, uniformly s-convex functions. Also, we obtain Hermite Hadamard inequality for fractional integral by using these functions. Finally, some applications of these inequalities are given.

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1. Introduction and Preliminaries

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$, then the following inequality holds:

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

The above inequality is well known in the literature as the Hermite-Hadamard inequality. Recently, the generalizations, improvements, variations and applications for convexity and the Hermite-Hadamard inequality have attracted the attention of many researchers, see [4–8, 11] and the references therein.

The following definitions can be found in [2, 12] and [1].

Definition 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then f is called uniformly convex with modulus $\psi : [0, +\infty) \to [0, +\infty]$ if ψ is incresaing, ψ vanishes only at 0, and

$$f(tx + (1-t)y) + t(1-t)\psi(|x-y|) \leq tf(x) + (1-t)f(y), \tag{1.1}$$

for each $x, y \in \mathbb{R}$ and $t \in (0, 1)$.
If (1.1) holds with $\psi = \frac{\beta}{2}|.|^2$ for some $\beta > 0$, then f is called strongly convex with constant β.

In the following we give a simple example of a uniformly convex function (see [2], Corollary 2.14).

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Example 1. In view of the following equality,

$$(\alpha x+(1-\alpha)y)^2+\alpha(1-\alpha)(x-y)^2=\alpha x^2+(1-\alpha)y^2,$$

for all $\alpha\in(0,1)$ and $x,y\in\mathbb{R}$, the function $f(t)=t^2$ for $t\in\mathbb{R}$ is uniformly convex with modulus $\psi(t)=t^2$ for all $t\geq 0$.

In the following proposition, the relation between convex functions and strongly convex functions is expressed. For more details about uniformly and strongly convex functions see [2].

Proposition 1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a function and $\beta>0$. Then f is a strongly convex function with constant β if and only if $f-\frac{\beta}{2}|.|^2$ is a convex function.*

Clearly, strong convexity implies uniformly convexity, uniformly convexity implies strict convexity, and strict convexity implies convexity.
We can define the concept of uniformly s-convexity as follows:

Definition 2. Let $f:\mathbb{R}\to\mathbb{R}$ be a function. Then f is called s-uniformly convex function with modulus $\psi:[0,+\infty)\to[0,+\infty]$ if ψ is incresaing, ψ vanishes only at 0, and

$$f(tx+(1-t)y)+t^s(1-t)\psi(|x-y|)\leq t^sf(x)+(1-t)^sf(y), \tag{1.2}$$

for each $x,y\in\mathbb{R}$, $t\in(0,1)$ and $s\in(0,1)$.
If Definition (1.2) holds with $\psi=\frac{\beta}{2}|.|^2$ for some $\beta>0$, then f is called strongly s-convex with constant β.

Definition 3. Let $f\in L[a,b]$. The left-sided and right-sided Riemann-Liouville fractional integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ of order $\alpha>0$ with $a\geq 0$ are defined by

$$J^{\alpha}_{a^+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt \quad \text{with} \quad x>a$$

$$J^{\alpha}_{b^-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)dt \quad \text{with} \quad x<b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function and its defnition is

$$\Gamma(\alpha)=\int_0^{+\infty}e^{-t}t^{\alpha-1}dt.$$

It is to be noted that $J^0_{a^+}f(x)=J^0_{b^-}f(x)=f(x)$. In the case of $\alpha=1$, the fractional integral reduces to the classical integral.

In [12], M. Z. Sarikaya et al. presented the following Hermite-Hadamard's inequalities for fractional integrals.

Theorem 1 ([12]). *Let $f : I \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L[a,b]$. If f is a convex function on $[a,b]$, then the following inequality for fractional integrals holds:*

$$f(\frac{a+b}{2}) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)] \le \frac{f(a)+f(b)}{2}.$$

2. Main Results

In this section, we shall state our main results. At the first, we obtain Hermite-Hadamard type inequalities for the class of uniformly convex, uniformly s-convex and strongly convex functions.

Theorem 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly convex function. Then, the following inquality holds:*

$$f(\frac{a+b}{2})+\frac{1}{8(b-a)}\int_{a-b}^{b-a}\psi(|t|)dt \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}-\frac{1}{6}\psi(|a-b|).$$

Proof. In (1.1), set $t=\frac{1}{2}$, then one has

$$f(\frac{x+y}{2})+\frac{1}{4}\psi(|x-y|) \le \frac{f(x)+f(y)}{2}. \tag{2.1}$$

Now in (2.1), set $x = ta+(1-t)b$ and $y = (1-t)a+tb$, and integrate inequality (2.1) on $[0,1]$ with respect to t. We conclude

$$\begin{aligned} f(\frac{a+b}{2})+\frac{1}{4}\int_0^1 \psi(|(2t-1)(a-b)|)dt \\ \le \frac{1}{2}\int_0^1 f(ta+(1-t)b)dt+\frac{1}{2}\int_0^1 f((1-t)a+tb)dt. \end{aligned}$$

Also, the following equalities holds

$$\begin{aligned} \frac{1}{4}\int_0^1 \psi(|(2t-1)(a-b)|)dt &= \frac{1}{4}\int_{b-a}^{a-b}\psi(|u|)\frac{du}{2(a-b)} \\ &= \frac{1}{8(b-a)}\int_{a-b}^{b-a}\psi(|t|)dt \end{aligned}$$

and

$$\int_0^1 f((1-t)a+tb)dt = \int_0^1 f(ta+(1-t)b)dt = \frac{1}{b-a}\int_a^b f(t)dt.$$

Therefore,

$$f(\frac{a+b}{2})+\frac{1}{8(b-a)}\int_{a-b}^{b-a}\psi(|t|)dt \le \frac{1}{b-a}\int_a^b f(t)dt.$$

On the other hand, in (1.1) put $x=a$, $y=b$ and integrate on $[0,1]$ with respect to t. Hence

$$\int_0^1 f(ta+(1-t)b)dt+\int_0^1 t(1-t)\psi(|a-b|)dt \le \int_0^1 \frac{f(a)+f(b)}{2}dt,$$

and so

$$\frac{1}{b-a}\int_a^b f(t)dt+\psi(|a-b|)\frac{\Gamma(2)\Gamma(2)}{\Gamma(4)} \le \frac{f(a)+f(b)}{2}.$$

Therefore,

$$\frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}-\frac{1}{6}\psi(|a-b|),$$

which completes the proof. It is worth noting that we used the following fact:

$$\int_0^1 t(1-t)dt = B(2,2)=\frac{\Gamma(2)\Gamma(2)}{\Gamma(4)}=\frac{1}{6},$$

where

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,\ \ \Gamma(x)=\int_0^{+\infty} e^{-t}t^{x-1}dt,\ x>0,\ y>0,$$
$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

□

In order to prove the main theorems, we need the following lemma that has been proved in [3].

Lemma 1. *Let $f: I^o \to \mathbb{R}$ be a differentiable function on I^o, $a,b \in I^o$ with $a<b$. If $f' \in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt=\frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt.$$

Theorem 3. *Let $f: I^o \to \mathbb{R}$ be a differentiable function on I^o, $a,b \in I^o$ with $a<b$. If $|f'|$ is uniformly convex function on I^o, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{b-a}{8}(|f'(a)|+|f'(b)|)-\frac{b-a}{32}\psi(|a-b|).$$

Proof. In view of Lemma 1 and uniformly convexity of $|f'|$, one has

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{b-a}{2}\int_0^1 |(1-2t)||f'(ta+(1-t)b)|dt$$
$$\le \frac{b-a}{2}\int_0^1 |1-2t|(t|f'(a)|+(1-t)|f'(b)|+t(t-1)\psi(|a-b|))dt$$
$$\le \frac{b-a}{2}\int_0^1 t|1-2t||f'(a)|dt+\int_0^1 |1-2t|(1-t)|f'(b)|dt$$

$$+\int_0^1 |1-2t|t(t-1)\psi(|a-b|))dt$$
$$\leq \frac{b-a}{8}(|f'(a)|+(f'(b)))-\frac{b-a}{32}\psi(|a-b|),$$

which completes the proof. Also, note that

$$\int_0^1 t|1-2t|dt=\int_0^1 (1-t)|1-2t|dt=\frac{1}{4},$$
$$\int_0^1 |1-2t|t(t-1)\psi(|a-b|)dt=-\frac{1}{16}\psi(|a-b|).$$

□

Theorem 4. *Let $f:I^o\to\mathbb{R}$ be a differentiable mapping on I^o, $a,b\in I^o$ with $a<b$ and $p>1$. If $|f'|^q$ is uniformly convex on I^o, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}-\frac{1}{6}\psi(|a-b|)\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By Lemma 1 and Hölder's inequality, we conclude

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq \frac{b-a}{2}\int_0^1 |(1-2t)||f'(ta+(1-t)b)|dt$$
$$\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{b-a}{2}\frac{1}{(p+1)^{\frac{1}{p}}}\left(|f(a)|^q\int_0^1 tdt+|f'(b)|^q\int_0^1 (1-t)dt+\psi(|a-b|)\int_0^1 t(t-1)dt\right)^{\frac{1}{q}}$$
$$\leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}-\frac{1}{6}\psi(|a-b|)\right)^{\frac{1}{q}}.$$

Hence, the proof is complete. □

Theorem 5. *Let $f:\mathbb{R}\to\mathbb{R}$ be strongly convex function. Then*

$$f(\frac{a+b}{2})+\frac{\beta}{24}(b-a)^2\leq \frac{1}{b-a}\int_a^b f(t)dt\leq \frac{f(a)+f(b)}{2}-\frac{\beta}{12}(b-a)^2.$$

Proof. From Hermite-Hadamard inequality for convex functions, we have

$$f(\frac{a+b}{2})\leq \frac{1}{b-a}\int_a^b f(t)dt\leq \frac{f(a)+f(b)}{2}. \tag{2.2}$$

Since from Proposition 1 f is a strongly convex function, we have $f-\frac{\beta}{2}|.|^2$ is convex. Hence in (2.2) replace f by $f-\frac{\beta}{2}|.|^2$ and after some calculations the result is obtained. □

Theorem 6. *Let $f:\mathbb{R}\to\mathbb{R}$ be uniformly s-convex function. Then*

$$2^{s-1}f(\frac{a+b}{2})+\frac{1}{8(b-a)}\int_{a-b}^{b-a}\psi(|t|)dt\leq\frac{1}{b-a}\int_a^b f(t)dt$$
$$\leq\frac{f(a)+f(b)}{s+1}-\frac{1}{(s+1)(s+2)}\psi(|a-b|).$$

Proof. In (1.2), set $t=\frac{1}{2}$, then we have

$$f(\frac{x+y}{2})+\frac{1}{2^{s+1}}\psi(|x-y|)\leq\frac{f(x)+f(y)}{2^s}. \tag{2.3}$$

Now, set $x=ta+(1-t)b$ and $y=(1-t)a+tb$ in (2.5) and integrate on $[0,1]$ with respect to t. We get

$$f(\frac{a+b}{2})+\frac{1}{2^{s+1}}\int_0^1\psi(|(2t-1)(a-b)|)dt$$
$$\leq\frac{1}{2^s}\int_0^1 f(ta+(1-t)b)dt+\frac{1}{2^s}\int_0^1 f((1-t)a+tb)dt.$$

Now,

$$\frac{1}{2^{s+1}}\int_0^1\psi(|(2t-1)(a-b)|)dt=\frac{1}{2^{s+1}}\int_{b-a}^{a-b}\psi(|u|)\frac{du}{2(a-b)}$$
$$=\frac{1}{2^{s+2}(b-a)}\int_{a-b}^{b-a}\psi(|t|)dt.$$

Also, we have $\int_0^1 f((1-t)a+tb)dt=\int_0^1 f((1-t)b+ta)dt=\frac{1}{b-a}\int_a^b f(t)dt$. Therefore

$$f(\frac{a+b}{2})+\frac{1}{2^{s+2}(b-a)}\int_{a-b}^{b-a}\psi(|t|)dt\leq\frac{1}{2^{s-1}(b-a)}\int_a^b f(t)dt.$$

On the other hand, in (1.1) put $x=a$, $y=b$ and integrate on $[0,1]$ with respect to t. Then we obtain

$$\int_0^1 f(ta+(1-t)b)dt+\int_0^1 t^s(1-t)\psi(|a-b|)dt\leq\int_0^1 t^s f(a)+(1-t)^s f(b)dt$$

so,

$$\frac{1}{b-a}\int_a^b f(t)dt+\psi(|a-b|)\frac{\Gamma(s+1)\Gamma(2)}{\Gamma(s+3)}\leq\frac{f(a)+f(b)}{s+1},$$

finally,

$$\frac{1}{b-a}\int_a^b f(t)dt\leq\frac{f(a)+f(b)}{s+1}-\frac{1}{(s+1)(s+2)}\psi(|a-b|),$$

which completes the proof. □

Theorem 7. *Let $p \in [2,+\infty)$, then the following inequality holds:*

$$|\frac{a+b}{2}|^p + \frac{1}{8(b-a)}2^{1-p}\min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\}\int_{a-b}^{b-a}|t|^p dt \leq \frac{1}{b-a}\int_a^b |t|^p dt$$
$$\leq \frac{|a|^p+|b|^p}{2} - \frac{1}{6}\min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\}|a-b|^p.$$

Proof. According to ([2], Proposition 10.13), since $|.|^2$ is uniformly convex with modules of convexity $|.|^2$. Hence for $p \in [2,+\infty)$ is uniformly convex with modules of convexity ψ such that ψ satisfing

$$\psi \geq 2^{1-p}\min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\}|.|^p, \tag{2.4}$$

Hence, in view of Theorem 2 for function $f(t) = |t|^p$ and (2.4), one has

$$|\frac{a+b}{2}|^p + \frac{1}{8(b-a)}2^{1-p}\min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\}\int_{a-b}^{b-a}|t|^p dt$$
$$\leq |\frac{a+b}{2}|^p + \frac{1}{8(b-a)}\int_{a-b}^{b-a}\psi(t)dt$$
$$\leq \frac{1}{b-a}\int_a^b |t|^p dt$$
$$\leq \frac{|a|^p+|b|^p}{2} - \frac{1}{6}\psi(|a-b|)$$
$$\leq \frac{|a|^p+|b|^p}{2} - \frac{1}{6}\min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\}|a-b|^p.$$

□

Proposition 2. *Let p be an even number and let $a,b \in \mathbb{R}$ with $0 < a < b$, then the following inequality holds:*

$$(p+1)(\frac{a+b}{2})^p + \frac{(b-a)^{p+1}}{2^{p+2}(b-a)}\min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\}$$
$$\leq \frac{b^{p+1}-a^{p+1}}{b-a}$$
$$\leq \left(\frac{a^p+b^p}{2} - \frac{(b-a)^p}{6}\min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\}\right)(p+1).$$

Proof. The proof is immediate consequence of Theorem 7. □

2.1. *Hermite-Hadamard's inequalities for fractional integrals*

Theorem 8. *Let $f : [a,b] \to \mathbb{R}$ be a uniformly convex function. Then, for $\alpha > 0$ the following inquality for fractional integrals holds:*

$$f(\frac{a+b}{2}) + \frac{\Gamma(\alpha+1)}{2^{\alpha+2}(b-a)^\alpha}J^\alpha_{(a-b)^+}\psi(|a-b|) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)]$$

$$\leq \frac{f(a)+f(b)}{2} - \alpha\beta(\alpha+1,2)\psi(|a-b|).$$

Proof. In (1.1), set $t=\frac{1}{2}$, then we have

$$f(\frac{x+y}{2}) + \frac{1}{4}\psi(|x-y|) \leq \frac{f(x)+f(y)}{2}. \tag{2.5}$$

Now, set $x = ta+(1-t)b$ and $y=(1-t)a+tb$ in (2.5). Multiplying both sides of (2.5) by $t^{\alpha-1}$ and then integrating the resulting inequality with respect to t over [0,1], we obtain

$$\int_0^1 t^{\alpha-1} f(\frac{a+b}{2})dt + \frac{1}{4}\int_0^1 t^{\alpha-1}\psi(|(2t-1)(a-b)|)dt$$
$$\leq \frac{1}{2}\int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt + \frac{1}{2}\int_0^1 t^{\alpha-1} f((1-t)a+tb)dt.$$

Let $ta+(1-t)b=r$, $(1-t)a+tb=s$ and $(2t-1)(a-b)=x$, then

$$\frac{f(\frac{a+b}{2})}{\alpha} + \frac{1}{4}\int_{b-a}^{a-b}(\frac{b-a-x}{2(b-a)})^{\alpha-1}\psi(|x|)\frac{dx}{2(a-b)} \leq$$
$$\frac{1}{2}\int_b^a (\frac{b-r}{b-a})^{\alpha-1} f(r)\frac{dr}{a-b} + \frac{1}{2}\int_a^b (\frac{s-a}{b-a})^{\alpha-1} f(s)\frac{ds}{b-a}.$$

So, we have

$$\frac{f(\frac{a+b}{2})}{\alpha} + \frac{1}{2^{\alpha+2}(b-a)^{\alpha}} J^{\alpha}_{(a-b)^+}\psi(|a-b|) \leq \frac{\Gamma(\alpha)}{2(b-a)^{\alpha}}[J^{\alpha}_{a^+}f(b) + J^{\alpha}_{b^-}f(a)].$$

Conversely, since f is uniformly convex one has

$$f(tx+(1-t)y) + t(1-t)\psi(|x-y|) \leq tf(x) + (1-t)f(y). \tag{2.6}$$

Now, replacing x by y we have

$$f(ty+(1-t)x) + t(1-t)\psi(|x-y|) \leq tf(y) + (1-t)f(x). \tag{2.7}$$

Adding the two equations (2.6) and (2.7) we obtain

$$f(tx+(1-t)y) + f((1-t)x+ty) + 2t(1-t)\psi(|x-y|) \leq f(x)+f(y). \tag{2.8}$$

Set $x=a$ and $y=b$ in (2.8) and also multiplying both sides of (2.8) by $t^{\alpha-1}$ and then integrating the resulting inequality with respect to t over [0,1], we obtain

$$\int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt + \int_0^1 t^{\alpha-1} f((1-t)a+tb)dt + \int_0^1 2t^{\alpha}(1-t)\psi(|a-b|)dt$$
$$\leq \int_0^1 t^{\alpha-1} f(a)dt + \int_0^1 t^{\alpha-1} f(b)dt.$$

So,

$$\frac{\Gamma(\alpha)}{2(b-a)^{\alpha}}[J^{\alpha}_{a^+}f(b) + J^{\alpha}_{b^-}f(a)] \leq \frac{f(a)+f(b)}{2\alpha} - \beta(\alpha+1,2)\psi(|a-b|),$$

which completes the proof. □

3. Applications to special means

Consider the following special means for two nonnegative real numbers α, β with $\alpha \neq \beta$ as follows (see [3,5,9,10]):

(1) The arithmetic mean:

$$A = A(\alpha, \beta) = \frac{\alpha + \beta}{2}, \ \alpha, \beta \in \mathbb{R},$$

with $\alpha, \beta > 0$.

(2) The logarithmic mean:

$$\overline{L} = \overline{L}(\alpha, \beta) = \frac{\beta - \alpha}{\ln \beta - \ln \alpha}, \ \ \alpha \neq \beta, \alpha, \beta \in \mathbb{R},$$

with $\alpha, \beta > 0$.

(3) The generalized logarithmic mean:

$$L_n(\alpha, \beta) = [\frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta - \alpha)}]^{\frac{1}{n}}, \ n \in \mathbb{R} \backslash \{-1, 0\}, \alpha \neq \beta, \alpha, \beta \in \mathbb{R},$$

with $\alpha, \beta > 0$.

Proposition 3. *Let $a, b \in \mathbb{R}$ with $0 < a < b$ and let p be an even number. Then the following inequality holds:*

$$\left((\frac{a+b}{2})^p + \frac{(b-a)^{p+1}}{2^{p+2}(p+1)(b-a)} \min\{p2^{-\frac{p}{2}}, 1 - 2^{-\frac{p}{2}}\} \right)^{\frac{1}{p}}$$
$$\leq L_p(a, b) \leq \left((\frac{a^p + b^p}{2} - \frac{(b-a)^p}{6} \min\{p2^{-\frac{p}{2}}, 1 - 2^{-\frac{p}{2}}\}) \right)^{\frac{1}{p}}$$

Proof. Since the function $g(t) = t^{\frac{1}{p}}$ is increasing for $t \geq 0$ and $p > 0$, in view of Proposition 2, the proof is complete. □

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NEW RESULTS FOR OSCILLATORY PROPERTIES OF NEUTRAL DIFFERENTIAL EQUATIONS WITH A p-LAPLACIAN LIKE OPERATOR

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Abstract. Results reported in this paper provide a generalization for some previously obtained results. Based on comparing with the oscillatory behavior of first-order delay equations, we provide new oscillation criteria for the solutions of even-order neutral differential equations with a p-Laplacian like operator. The proposed theorems not only provide totally different approach but also essentially improve a number of results reported in the literature. To demonstrate the advantage of our results, we present two examples.

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Keywords: even-order differential equations, neutral delay, oscillation, p-Laplacian operator

1. Introduction

Recently, it has been recognized that higher order neutral differential equations can describe many real life applications; see [1, 13]. As a result of this, the qualitative behavior of solutions for these equations have been the object of many scholars during the previous years [3, 7, 10–12, 14, 15, 18, 22, 23]. Particular emphasis has been given to the study of oscillatory behavior of these equations which have been under investigation by using different methods and various techniques; we refer to the papers [4–6, 9, 17, 19–21]. The consideration of higher-order equations was motivated by the attempt to promote the work and obtain a general platform that covers all particular cases. The consideration of equations incorporating the p-Laplacian operator has been one way to generalize existing result in the literature [2, 16, 26].

The present paper deals with the investigation of the qualitative behavior of even order neutral differential equation

$$\left(b(t)\Phi_p[w^{(\kappa-1)}(t)]\right)' + q(t)\Phi_p[y(\delta(t))] = 0; \qquad t \geq t_0, \quad (\kappa = 4, 6, 8, \ldots) \tag{1.1}$$

where $\Phi_p[s] = |s|^{p-2}s$, $p > 1$ and

$$w(t) := y(t) + a(t)y(\tau(t)).$$

The main results are obtained under the following conditions:

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(i) $b(t)$ is a positive continuous function on $[t_0,\infty)$ with that $b'(t)\geq 0$ and

$$\int_{t_0}^{\infty}[b(s)]^{-1/(p-1)}\mathrm{d}s=\infty. \tag{1.2}$$

(ii) $a(t)$ and $q(t)$ are continuous functions on $[t_0,\infty)$ with $q(t)>0$, $0\leq a(t)<a_0<\infty$, and that $q(t)\not\equiv 0$ for large values of t.

(iii) $\tau\in C^1[t_0,\infty)$, $\delta\in C[t_0,\infty)$, $\tau'(t)>0$, $\tau(t)\leq t$ and that

$$\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\delta(t)=\infty.$$

By establishing a new oscillation theorem that compares the higher-order equation (1.1) with a couple of first-order delay differential equations whose oscillatory behavior is known, we improve some existing results in the literature. Examples are presented to illustrate the advantage of our results over previously obtained theorems.

For the sake of comparison, we review some previous results. In [25], Zafer proved that the even-order differential equation

$$w^{(\kappa)}(t)+q(t)y(\delta(t))=0 \tag{1.3}$$

is oscillatory if

$$\liminf_{t\to\infty}\int_{\delta(t)}^{t}Q(s)\,\mathrm{d}s>\frac{1}{e}(\kappa-1)2^{(\kappa-1)(\kappa-2)}, \tag{1.4}$$

where $Q(t):=\delta^{\kappa-1}(t)q(t)\left[1-a(\delta(t))\right]$. In a similar approach, Zhang and Yan [27] proved that (1.3) is oscillatory if

$$\liminf_{t\to\infty}\int_{\delta(t)}^{t}Q(s)\,\mathrm{d}s>\frac{1}{e}(\kappa-1)!. \tag{1.5}$$

It is easy to see that $(\kappa-1)!<(\kappa-1)2^{(\kappa-1)(\kappa-2)}$ for $\kappa>3$, and hence the results obtained in [27] improve those of Zafer [25].

For non-linear equation, Xing et al. [24] proved that Eq. (1.1) is oscillatory if

$$\left(\delta^{-1}\right)'(t)\geq\delta_0>0,\quad \tau'(t)\geq\tau_0>0,\quad \tau^{-1}(\delta(t))<t \tag{1.6}$$

and

$$\liminf_{t\to\infty}\int_{\tau^{-1}(\delta(t))}^{t}\frac{\widehat{q}(s)}{b(s)}s^{\alpha(\kappa-1)}\mathrm{d}s>\frac{1}{e\delta_0}\left(1+\frac{a_0^{\alpha}}{\tau_0}\right)[(\kappa-1)!]^{\alpha}, \tag{1.7}$$

where $\widehat{q}(t):=\min\left\{q\left(\delta^{-1}(t)\right),q\left(\delta^{-1}(\tau(t))\right)\right\}$.

2. Hypotheses and preliminaries

For our purpose, we define the following notations:

$$a_k(t):=\frac{1}{a(\tau^{-1}(t))}\left(1-\frac{\varepsilon\left[\tau^{-1}\left(\tau^{-1}(t)\right)\right]^{k-1}}{\left[\tau^{-1}(t)\right]^{k-1}a\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}\right);\qquad k=2,\ldots,\kappa,$$

$$R_0(t) := \left(\frac{1}{b(t)} \int_t^\infty q(s) \left[a_2(\delta(s)) \right]^\alpha \mathrm{d}s \right)^{1/(p-1)}$$

and

$$R_m(t) := \int_t^\infty R_{m-1}(s) \mathrm{d}s; \qquad m = 1, 2, \ldots, \kappa - 3.$$

To complete the main results, we need the following lemmas.

Lemma 1. *Let $y \in C^n([t_0, \infty), (0, \infty))$. Assume that $y^{(n)}(t)$ is of fixed sign and not identically zero on $[t_0, \infty)$, and that there exists a $t_1 \geq t_0$ such that $y^{(n-1)}(t) y^{(n)}(t) \leq 0$ for all $t \geq t_1$. If*

$$\lim_{t \to \infty} y(t) \neq 0,$$

then for every $\mu \in (0, 1)$ there exists $t_\mu \geq t_1$ such that

$$y(t) \geq \frac{\mu}{(n-1)!} \left| y^{(n-1)}(t) \right| t^{n-1}$$

for all $t \geq t_\mu$.

Lemma 2. *Assume that f, $g \geq 0$ and β is a positive real number. Then the inequalities*

$$(f+g)^\beta \leq 2^{\beta-1} \left(f^\beta + g^\beta \right); \qquad \beta \geq 1$$

and

$$(f+g)^\beta \leq f^\beta + g^\beta; \qquad \beta \leq 1$$

hold.

Lemma 3. *If the function y satisfies $y^{(i)}(t) > 0$, $i = 0, 1, \ldots, n$, and $y^{(n+1)}(t) < 0$, then we have*

$$n y(t) \geq \varepsilon \, t y'(t)$$

for $\varepsilon \in (0, 1)$.

Lemma 4. *Assume that y is an eventually positive solution of Eq. (1.1). Then, there exist two possible cases:*

(a) $w(t) > 0$, $w'(t) > 0$, $w''(t) > 0, \ldots, w^{(n-1)}(t) > 0$, $w^{(n)}(t) < 0$
(b) $w(t) > 0$, $w^{(m)}(t) > 0$, $w^{(m+1)}(t) < 0$ *for all odd integers* $m \in \{1, 3, \ldots, \kappa - 3\}$, $w^{(\kappa-1)}(t) > 0$, $w^{(\kappa)}(t) < 0$,

for $t \in [t_1, \infty)$ for some $t_1 \geq t_0$ sufficiently large.

The lemmas given above can be found in [1, Lemma 2.2.3], [3, Lemma 1, Lemma 2], [8] and [23, Lemma 1.2], respectively.

2.1. *Oscillation Criteria*

The following is the main result.

Theorem 1. *Let*

$$\frac{\left[\tau^{-1}\left(\tau^{-1}(t)\right)\right]^{\kappa-1}}{\left[\tau^{-1}(t)\right]^{\kappa-1} a\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)} \leq 1. \tag{2.1}$$

Assume that there exist positive functions ϑ, $\zeta \in C^1([t_0,\infty),\mathbb{R})$ *satisfying* $\vartheta(t) \leq \delta(t)$, $\vartheta(t) < \tau(t)$, $\zeta(t) \leq \delta(t)$, $\zeta(t) < \tau(t)$, $\zeta'(t) \geq 0$ *and*

$$\lim_{t\to\infty} \vartheta(t) = \lim_{t\to\infty} \zeta(t) = \infty.$$

If there exists a constant $\mu \in (0,1)$ *such that the differential equations*

$$\psi'(t) + \left(\frac{\left[\mu\left(\tau^{-1}(\vartheta(t))\right)\right]^{\kappa-1}}{(\kappa-1)!\left[b\left(\tau^{-1}(\vartheta(t))\right)\right]^{1/\alpha}}\right)^{p-1} q(t)\left[a_\kappa(\delta(t))\right]^{p-1}\psi\left(\tau^{-1}(\vartheta(t))\right) = 0 \tag{2.2}$$

and

$$\phi'(t) + \tau^{-1}(\zeta(t)) R_{\kappa-3}(t)\phi\left(\tau^{-1}(\zeta(t))\right) = 0 \tag{2.3}$$

are oscillatory, then Eq. (1.1) is oscillatory.

Proof. Let y be a non-oscillatory solution of (1.1) on $[t_0,\infty)$. Without loss of generality, we can assume that y is eventually positive. It follows from Lemma 4 that there exist two possible cases (a) and (b).

Assume that the case (a) holds. From the definition of $w(t)$, we see that

$$y(t) = \frac{1}{a\left(\tau^{-1}(t)\right)}\left[w\left(\tau^{-1}(t)\right) - y\left(\tau^{-1}(t)\right)\right].$$

By repeating the same process, we find that

$$\begin{aligned} y(t) &= \frac{w\left(\tau^{-1}(t)\right)}{a\left(\tau^{-1}(t)\right)} - \frac{1}{a\left(\tau^{-1}(t)\right)} \times \left\{\frac{w\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}{a\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)} - \frac{y\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}{a\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}\right\} \\ &\geq \frac{w\left(\tau^{-1}(t)\right)}{a\left(\tau^{-1}(t)\right)} - \frac{1}{a\left(\tau^{-1}(t)\right)} \times \frac{w\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}{a\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}. \end{aligned} \tag{2.4}$$

Using Lemma 3, we get $w(t) \geq \varepsilon t w'(t)/(\kappa-1)$ and hence the function $v^{1-\kappa}(t)w(t)$ is non-increasing which gives

$$\left[\tau^{-1}(t)\right]^{\kappa-1} w\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right) \leq \varepsilon\left[\tau^{-1}\left(\tau^{-1}(t)\right)\right]^{\kappa-1} w\left(\tau^{-1}(t)\right). \tag{2.5}$$

by the fact that $\tau(t) \leq t$.

Using (2.5), (2.4) turns out to

$$y(t) \geq \frac{1}{a\left(\tau^{-1}(t)\right)}\left(1 - \frac{\varepsilon\left[\tau^{-1}\left(\tau^{-1}(t)\right)\right]^{\kappa-1}}{\left[\tau^{-1}(t)\right]^{\kappa-1} a\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}\right) w\left(\tau^{-1}(t)\right)$$

$$= a_{\kappa}(t)\, w\left(\tau^{-1}(t)\right). \tag{2.6}$$

From (1.1) and (2.6), we obtain

$$\left(b(t)\left[w^{(\kappa-1)}(t)\right]^{p-1}\right)' + q(t)\left[a_{\kappa}(\delta(t))\right]^{p-1}\left[w\left(\tau^{-1}(\delta(t))\right)\right]^{p-1} \leq 0.$$

Since $\vartheta(t) \leq \delta(t)$ and $w'(t) > 0$, we get

$$\left(b(t)\left[w^{(\kappa-1)}(t)\right]^{p-1}\right)' \leq -q(t)\left[a_{\kappa}(\delta(t))\right]^{p-1}\left[w\left(\tau^{-1}(\vartheta(t))\right)\right]^{p-1}. \tag{2.7}$$

Now, by using Lemma 1, we have

$$w(t) \geq \frac{\mu}{(\kappa-1)!} t^{\kappa-1} w^{(\kappa-1)}(t). \tag{2.8}$$

for some $\mu \in (0,1)$. It follows from (2.7) and (2.8) that

$$\left(b(t)\left[w^{(\kappa-1)}(t)\right]^{p-1}\right)' + \left(\frac{\left[\mu\left(\tau^{-1}(\vartheta(t))\right)\right]^{\kappa-1}}{(\kappa-1)!}\right)^{p-1} q(t)\left[a_{\kappa}(\delta(t))\right]^{p-1} \times \left[w^{(\kappa-1)}\left(\tau^{-1}(\vartheta(t))\right)\right]^{p-1} \leq 0$$

for all $\mu \in (0,1)$.

Thus, if we set $\psi(t) = b(t)\left[w^{(\kappa-1)}(t)\right]^{p-1}$, then we see that ψ is a positive solution of the first-order delay differential inequality

$$\psi'(t) + q(t)\left(\frac{\left[\mu\left(\tau^{-1}(\vartheta(t))\right)\right]^{\kappa-1}}{(\kappa-1)!\left[b\left(\tau^{-1}(\vartheta(t))\right)\right]^{1/(p-1)}}\right)^{p-1}\left[a_{\kappa}(\delta(t))\right]^{p-1}\psi\left(\tau^{-1}(\vartheta(t))\right) \leq 0.$$

It is well known (see [22, Theorem 1]) that corresponding Eq. (2.2) also has a positive solution, which is a contradiction.

Assume that the case (b) holds. Using Lemma 3, we get that

$$w(t) \geq \varepsilon t w'(t) \tag{2.9}$$

and thus the function $w(t)/t$ is non-increasing, eventually. Since

$$\tau^{-1}(t) \leq \tau^{-1}\left(\tau^{-1}(t)\right),$$

we obtain

$$\tau^{-1}(t)\, w\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right) \leq \varepsilon \tau^{-1}\left(\tau^{-1}(t)\right) w\left(\tau^{-1}(t)\right). \tag{2.10}$$

Using (2.10), (2.4) turns out to

$$\begin{aligned} y(t) &\geq \frac{1}{a\left(\tau^{-1}(t)\right)}\left(1 - \frac{\varepsilon \tau^{-1}\left(\tau^{-1}(t)\right)}{\tau^{-1}(t)\, a\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}\right) w\left(\tau^{-1}(t)\right) \\ &= a_2(t)\, w\left(\tau^{-1}(t)\right), \end{aligned}$$

which yields with (1.1)

$$\left(b(t)\left[w^{(\kappa-1)}(t)\right]^{p-1}\right)'+q(t)\left[a_2(\delta(t))\right]^{p-1}\left[w\left(\tau^{-1}(\delta(t))\right)\right]^{p-1}\leq 0.$$

Since $\zeta(t)\leq\delta(t)$ and $w'(t)>0$, we have that

$$\left(b(t)\left[w^{(\kappa-1)}(t)\right]^{p-1}\right)'\leq -q(t)\left[a_2(\delta(t))\right]^{p-1}\left[w\left(\tau^{-1}(\zeta(t))\right)\right]^{p-1}. \tag{2.11}$$

Integrating (2.11) from t to ∞, we obtain

$$w^{(\kappa-1)}(t)\geq R_0(t)\,w\left(\tau^{-1}(\zeta(t))\right).$$

Now, integrating the above inequality from t to ∞, $\kappa-3$ times, we obtain

$$w''(t)+R_{\kappa-3}(t)\,w\left(\tau^{-1}(\zeta(t))\right)\leq 0. \tag{2.12}$$

Now, if we set $\phi(t):=w'(t)$ and using (2.9), then we conclude that ϕ is a positive solution of the differential inequality

$$\phi'(t)+\tau^{-1}(\zeta(t))R_{\kappa-3}(t)\,\phi\left(\tau^{-1}(\zeta(t))\right)\leq 0. \tag{2.13}$$

It is well known (see [22, Theorem 1]) that corresponding Eq. (2.3) also has a positive solution, which is a contradiction. The proof is complete. □

Corollary 1. *Assume that (2.1) holds and there exist positive functions ϑ, ζ satisfying the conditions given in Theorem 1. If*

$$\liminf_{t\to\infty}\int_{\tau^{-1}(\vartheta(t))}^{t}\left(\frac{\left[\tau^{-1}(\vartheta(s))\right]^{\kappa-1}}{\left[b\left(\tau^{-1}(\vartheta(s))\right)\right]^{1/(p-1)}}\right)^{p-1}q(s)\left[a_\kappa(\delta(s))\right]^{p-1}\mathrm{d}s > \frac{1}{e}\left[(\kappa-1)!\right]^{p-1} \tag{2.14}$$

and

$$\liminf_{t\to\infty}\int_{\tau^{-1}(\zeta(t))}^{t}\tau^{-1}(\zeta(s))R_{\kappa-3}(s)\,\mathrm{d}s>\frac{1}{e}, \tag{2.15}$$

then Eq. (1.1) is oscillatory.

Proof. It is well-known (see, e.g., [15, Theorem 2]) that Conditions (2.14) and (2.15) imply the oscillation of (2.2) and (2.3), respectively. □

3. Examples and discussion

We present two particular examples.

Example 1. Consider the equation

$$\left[y(t)+a_0y(\tau t)\right]^{(\kappa)}+q_0t^{-\kappa}y(\lambda t)=0;\qquad t\geq 1, \tag{3.1}$$

where $q_0>0$, $\tau\in(a_0^{-1/(\kappa-1)},1)$ and $\lambda\in(0,\tau)$. We note that $b(t)=1$, $a(t)=a_0$, $\tau(t)=\tau t$, $\delta(t)=\lambda t$ and $q(t)=q_0t^{-\kappa}$. Thus, if we choose $\vartheta(t)=\zeta(t)=\lambda t$, then

it is straightforward to see that (2.1) and all the conditions given in Theorem 1 are satisfied. Moreover, we have

$$a_k(t) = \frac{1}{a_0}\left(1 - \frac{\tau^{1-k}}{a_0}\right); \qquad k = 2, \ldots, \kappa,$$
$$R_0(t) = \frac{q_0}{(\kappa-1)a_0}\left(1 - \frac{1}{\tau a_0}\right) t^{1-\kappa},$$

and

$$R_{\kappa-3}(t) = \frac{q_0}{(\kappa-1)(\kappa-2)(\kappa-3)!a_0}\left(1 - \frac{1}{\tau a_0}\right) t^{-2}.$$

Hence, condition (2.14) and (2.15) become

$$\frac{q_0}{a_0}\left(\frac{\lambda}{\tau}\right)^{\kappa-1}\left(1 - \frac{\tau^{1-\kappa}}{a_0}\right)\ln\left(\frac{\tau}{\lambda}\right) > \frac{1}{e}(\kappa-1)! \tag{3.2}$$

and

$$\frac{\lambda q_0}{\tau a_0}\left(1 - \frac{1}{\tau a_0}\right)\ln\left(\frac{\tau}{\lambda}\right) > \frac{1}{e}(\kappa-1)!, \tag{3.3}$$

respectively. It is clear to see that (3.2) implies (3.3). We end up with the resullt that (3.1) is oscillatory if (3.2) holds by Corollary 1.

Remark 1. When $\kappa = 4$, $a_0 = 16$, $\tau = 1/2$ and $\lambda = 1/3$ in Eq. (3.1), condition (3.2) yields $q_0 > 587.93$ which is better than that is obtained in [24], i.e., $q_0 > 4850.4$. Hence, our results improve those obtained in [24].

Example 2. Consider the particular equation

$$\left[y(t) + (7/8)y(t/e)\right]^{(4)} + q_0 t^{-4} y\left(t/e^2\right) = 0; \qquad t \geq 1, \tag{3.4}$$

where $q_0 > 0$ is a constant, and

$$p = 2,\ \kappa = 4,\ b(t) = 1,\ a(t) = 7/8,\ \tau(t) = t/\mathrm{e},\ q(t) = q_0 t^{-4}$$

and $\delta(t) = t/\mathrm{e}^2$. If we apply the previous results to Eq. (3.4), then we get

(i) $q_0 > 113981.3$ by applying condition (1.4) in [25];
(ii) $q_0 > 3561.9$ by applying condition (1.5) in [27];
(iii) $q_0 > 3008.5$ by applying conditions (1.6)-(1.7) in [24].

Hence, the results of [24] improved those obtained in [25, 27]. Furthermore, one can easily see that the criteria obtained in [24, 25, 27] cannot be applied to (2.14) and (2.15) which demonstrates that our results are essentially new.

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OSCILLATORY PROPERTIES OF EVEN-ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract. The aim of this paper is to investigate oscillatory properties of even-order advanced differential equations. The key idea of our approach is to conduct a comparison with first order equations and use the Riccati transformation technique. Some new oscillation criteria are shown. Two examples are presented in order to clarify the main results.

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1. Introduction

Advanced differential equations can find application in several real world problems where the evolution rate depends not only on the present, but also on the future. Hence, an advance could be introduced into the equation in order to take into account the influence of potential future actions. For instance, it is worth pointing out that there are a lot of applications to dynamical systems, mathematics of networks, optimization, and their application in the mathematical modelling of engineering problems, such as concerning electrical power systems, materials, energy (see [13]).

In this paper, we establish new oscillation criteria for all solutions of non-linear even-order differential equations with variable coefficients and advanced term of the form

$$\left(a(t)\left(w^{(n-1)}(t)\right)^{\beta}\right)' + \sum_{i=1}^{j} q_i(t)\, g(w(\eta_i(t))) = 0, \quad t \geq t_0, \tag{1.1}$$

where $j \geq 1$ and β is a quotient of odd positive integers.

Throughout this paper, we suppose that:

- $a \in C^1([t_0,\infty),\mathbb{R})$,
- $a(t) > 0,\ a'(t) \geq 0,\ q_i, \eta_i \in C([t_0,\infty),\mathbb{R})$,
- $q_i(t) \geq 0$,
- $\eta_i(t) \geq t,\ \lim_{t\to\infty} \eta_i(t) = \infty,\ i = 1,2,..,j$,

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- $g \in C(\mathbb{R},\mathbb{R})$ such that $\frac{g(x)}{x^{\beta}} \geq k > 0$, for $x \neq 0$
- $\displaystyle\int_{t_0}^{\infty} \frac{1}{a^{1/\beta}(s)} \mathrm{d}s = \infty.$

By a *solution* of (1.1) we mean a function $w \in C^{n-1}[t_w,\infty)$, $t_w \geq t_0$, which has the property $a(t)\left(w^{(n-1)}(t)\right)^{\beta} \in C^1[t_w,\infty)$, and satisfies (1.1) on $[t_w,\infty)$. A solution of (1.1) is called *oscillatory* if it has arbitrarily large zeros on $[t_w,\infty)$; otherwise, it is called *non-oscillatory*. Equation (1.1) is said to be oscillatory if all of its solutions are oscillatory.

2. The aim of this paper

In the last decade, the interest in studying of oscillation properties for differential equations increased (see for instance [3–8, 10–12, 14–19, 23–25]). On the other hand, the study of qualitative properties of solutions to differential equations were extensively studied also in [20–22].

The aim of this paper is to complement and enrich the results contained in [1,2,9]. For this reason, it is interesting to discuss briefly the aforementioned results.

By using the comparison technique, the equation

$$\left(\left(w^{(n-1)}(t)\right)^{\beta}\right)' + q(t)\, w^{\beta}(\eta(t)) = 0, \tag{2.1}$$

have been studied by Agarwal and Grace [1] and they proved that (2.1) is oscillatory if

$$\liminf_{t\to\infty} \int_t^{\eta(t)} (\eta(s)-s)^{n-2} \left(\int_s^{\infty} q(t)\,dt\right)^{1/\beta} ds > \frac{(n-2)!}{e}. \tag{2.2}$$

Agarwal et al. in [2] extended the Riccati transformation obtaining as new oscillation criterion for (2.1), the following condition:

$$\limsup_{t\to\infty} t^{\beta(n-1)} \int_t^{\infty} q(s)\,ds > ((n-1)!)^{\beta}. \tag{2.3}$$

In [9] the authors studied the oscillatory behaviour of (2.1) for $\beta = 1$. Using the Riccati transformation, they show that (2.1) is oscillatory if there exists a function $\mu \in C^1([t_0,\infty),(0,\infty))$, such that

$$\int_{t_0}^{\infty} \left(\mu(s)\,q(s) - \frac{(n-2)!\,(\mu'(s))^2}{2^{3-2n} s^{n-2} \mu(s)}\right) ds = \infty. \tag{2.4}$$

We apply the previous results to the equation

$$w^{(4)}(t) + \frac{q_0}{t^4} w(3t) = 0.\ t \geq 1, \tag{2.5}$$

(1) Applying the condition (2.2) contained in [1], we get

$$q_0 > 13.6.$$

(2) Applying condition (2.3) contained in [2], we get

$$q_0 > 18.$$

(3) Applying condition (2.4) contained in [9], we get

$$q_0 > 576.$$

From the above results, we can infer that the results in [2] improve the results contained in [9]. Moreover, the results in [1] improve the results [2,9].

Thus, the motivation of this paper is complement and enrich the results in [1,2,9].

The key idea of our approach is to conduct a comparison with first order equations whose oscillatory behaviours are already known, using Riccati transformation technique.

The rest of the paper is organized as follows. Section 3 is devoted to the main results of the paper. We present our investigations for equation (1.1). Meanwhile, a relevant result on the existence of oscillatory oscillatory behaviour of solutions for first order differential equations is stated. The proofs rely on some mathematical inequalities and lemmas which are given for the sake of completeness. In Section 4, we provide two examples with specific parameters to illustrate the applicability of our theorems. The paper ends with some concluding remarks.

The following lemmas will be very useful in the sequel:

Lemma 1 ([16])**.** *If the function w satisfies $w^{(i)}(t) > 0$, $i = 0,1,\ldots,n$, and $w^{(n+1)}(t) < 0$, then*

$$\frac{w(t)}{t^n/n!} \geq \frac{w'(t)}{t^{n-1}/(n-1)!}.$$

Lemma 2 ([25])**.** *Suppose that $w \in C^n([t_0,\infty),(0,\infty))$, $w^{(n)}$ is of a fixed sign on $[t_0,\infty)$, $w^{(n)}$ not identically zero and there exists a $t_1 \geq t_0$ such that*

$$w^{(n-1)}(t)\, w^{(n)}(t) \leq 0,$$

for all $t \geq t_1$. If we have $\lim_{t\to\infty} w(t) \neq 0$, then there exists $t_\theta \geq t_1$ such that

$$w(t) \geq \frac{\theta}{(n-1)!} t^{n-1} \left| w^{(n-1)}(t) \right|,$$

for every $\theta \in (0,1)$ and $t \geq t_\theta$.

Lemma 3 ([3])**.** *Let β be a ratio of two odd numbers, $V > 0$ and U are constants. Then*

$$Ux - Vx^{(\beta+1)/\beta} \leq \frac{\beta^\beta}{(\beta+1)^{\beta+1}} \frac{U^{\beta+1}}{V^\beta},\ V > 0.$$

Lemma 4 ([15]). *Assume that w is an eventually positive solution of (1.1). Then, there exist two possible cases:*

$$\begin{array}{ll}(\mathbf{S}_1) & w(t)>0,\ w'(t)>0,\ w''(t)>0,\ \ w^{(n-1)}(t)>0,\ \ w^{(n)}(t)<0,\\ (\mathbf{S}_2) & w(t)>0,\ w^{(r)}(t)>0,\ w^{(r+1)}(t)<0 \textit{ for all odd integer}\\ & r\in\{1,3,\ldots,n-3\},\ w^{(n-1)}(t)>0,\ w^{(n)}(t)<0,\end{array}$$

for $t\geq t_1$, where $t_1\geq t_0$ is sufficiently large.

3. Oscillation criteria

In the following theorem, we compare the oscillatory behaviour of (1.1) with suitable first-order differential equations.

Theorem 1. *Assume that (1) holds. If the differential equations*

$$x'(t)+k\sum_{i=1}^{j}q_i(t)\left(\frac{\theta t^{n-2}}{(n-2)!a^{1/\beta}(t)}\right)^{\beta}x(\eta(t))=0 \tag{3.1}$$

and

$$z'(t)+z(t)\frac{t}{(n-4)!}\int_t^{\infty}(\varsigma-t)^{n-4}\left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty}\sum_{i=1}^{j}q_i(s)\,\mathrm{d}s\right)^{1/\beta}\mathrm{d}\varsigma=0 \tag{3.2}$$

are oscillatory, then every solution of (1.1) is oscillatory.

Proof. On the contrary, let us assume that w is a positive solution of (1.1). Then, we can suppose that $w(t)$ and $w(\eta_i(t))$ are positive for all $t\geq t_1$ sufficiently large. From Lemma 4, we have two possible cases $(\mathbf{S}_1)$ and $(\mathbf{S}_2)$.
Let us assume that $(\mathbf{S}_1)$ holds. From Lemma 2, we get

$$w(t)\geq\frac{\theta t^{n-2}}{(n-2)!a^{1/\beta}(t)}\left(a^{1/\beta}(t)\,w^{(n-1)}(t)\right),$$

for every $\theta\in(0,1)$ and for all large t. Thus, if we set

$$x(t)=a(t)\left(w^{(n-1)}(t)\right)^{\beta}>0,$$

then we see that δ is a positive solution of the inequality

$$x'(t)+k\sum_{i=1}^{j}q_i(t)\left(\frac{\theta t^{n-2}}{(n-2)!a^{1/\beta}(t)}\right)^{\beta}x(\eta(t))\leq 0. \tag{3.3}$$

From [19, Theorem 1], we conclude that the corresponding equation (3.1) also has a positive solution, which is a contradiction.
Let us assume that $(\mathbf{S}_2)$ holds. Integrating (1.1) from t to m and using $w'(t)>0$, we obtain

$$a(m)\left(w^{(n-1)}(m)\right)^{\beta}-a(t)\left(w^{(n-1)}(t)\right)^{\beta}=-\int_t^m\sum_{i=1}^{j}q_i(s)\,g(w(\eta_i(s)))\,ds.$$

By virtue of $w'(t) > 0$ and $\eta_i(t) \geq t$, we get

$$a(m)\left(w^{(n-1)}(m)\right)^{\beta} - a(t)\left(w^{(n-1)}(t)\right)^{\beta} \leq -kw^{\beta}(t)\int_t^m \sum_{i=1}^{j} q_i(s)\,ds.$$

Letting $m \to \infty$, we see that

$$a(t)\left(w^{(n-1)}(t)\right)^{\beta} \geq kw^{\beta}(t)\int_t^{\infty} \sum_{i=1}^{j} q_i(s)\,\mathrm{d}s$$

and so

$$w^{(n-1)}(t) \geq w(t)\left(\frac{k}{a(t)}\int_t^{\infty} \sum_{i=1}^{j} q_i(s)\,\mathrm{d}s\right)^{1/\beta}.$$

Integrating again from t to ∞ for a total of $n-4$ times, we get

$$w''(t) + \frac{w(t)}{(n-4)!}\int_t^{\infty} (\varsigma - t)^{n-4}\left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty} \sum_{i=1}^{j} q_i(s)\,\mathrm{d}s\right)^{1/\beta} \mathrm{d}\varsigma \leq 0. \tag{3.4}$$

Using Lemma 1, we get

$$w(t) \geq tw'(t), \tag{3.5}$$

From (3.4) and (3.5), we get

$$w''(t) + w'(t)\frac{t}{(n-4)!}\int_t^{\infty} (\varsigma - t)^{n-4}\left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty} \sum_{i=1}^{j} q_i(s)\,\mathrm{d}s\right)^{1/\beta} \mathrm{d}\varsigma \leq 0.$$

Thus, if we set

$$z(t) = w'(t),$$

then we see that δ is a positive solution of the inequality

$$z'(t) + z(t)\frac{t}{(n-4)!}\int_t^{\infty} (\varsigma - t)^{n-4}\left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty} \sum_{i=1}^{j} q_i(s)\,\mathrm{d}s\right)^{1/\beta} \mathrm{d}\varsigma \leq 0. \tag{3.6}$$

It is well known (see [19, Theorem 1]) that the corresponding equation (3.2) also has a positive solution, which is a contradiction. The proof is complete. □

Corollary 1. *Assume that* (1) *holds. If*

$$\liminf_{t\to\infty} \int_{\eta_i(t)}^{t} \sum_{i=1}^{j} q_i(s)\left(\frac{\theta t^{n-2}}{(n-2)!a^{1/\beta}(t)}\right)^{\beta} \mathrm{d}s > \frac{((n-1)!)^{\beta}}{\mathrm{e}} \tag{3.7}$$

and

$$\liminf_{t\to\infty} \int_{\eta_i(t)}^{t} \frac{s}{(n-4)!}\int_t^{\infty} (\varsigma - t)^{n-4}\left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty} \sum_{i=1}^{j} q_i(s)\,\mathrm{d}s\right)^{1/\beta} \mathrm{d}\varsigma\mathrm{d}s > \frac{1}{\mathrm{e}} \tag{3.8}$$

are oscillatory, then every solution of (1.1) *is oscillatory.*

Lemma 5. *Assume that w be an eventually positive solution of* (1.1) *and* $(\mathbf{S}_1)$ *holds. If*

$$\phi(t) := \mu(t)\left(\frac{a(t)\left(w^{(n-1)}(t)\right)^{\beta}}{w^{\beta}(t)}\right), \tag{3.9}$$

where $\mu \in C^1([t_0,\infty),(0,\infty))$, then

$$\phi'(t) \le \frac{\mu'(t)}{\mu(t)}\phi(t) - k\mu(t)\sum_{i=1}^{j} q_i(t) - \frac{\beta\theta t^{n-2}}{(n-2)!\,(\mu(t)a(t))^{\frac{1}{\beta}}}\phi(t)^{\frac{\beta+1}{\beta}}, \tag{3.10}$$

for all $t > t_1$, where t_1 large enough.

Proof. Let w is an eventually positive solution of (1.1) and let us assume that $(\mathbf{S}_1)$ holds. Thus, from Lemma 2, we get

$$w'(t) \ge \frac{\theta}{2} t^{n-2} w^{(n-1)}(t), \tag{3.11}$$

for every $\theta \in (0,1)$ and for all large t. From (3.9), we see that $\phi(t) > 0$ for $t \ge t_1$, and

$$\begin{aligned}\phi'(t) = \mu'(t)\frac{a(t)\left(w^{(n-1)}(t)\right)^{\beta}}{w^{\beta}(t)} + \mu(t)\frac{\left(a\left(w^{(n-1)}\right)^{\beta}\right)'(t)}{w^{\beta}(t)} \\ - \beta\mu(t)\frac{w^{\beta-1}(t)\,w'(t)\,a(t)\left(w^{(n-1)}(t)\right)^{\beta}}{w^{2\beta}(t)}.\end{aligned}$$

Using (3.11) and (3.9), we obtain

$$\begin{aligned}\phi'(t) &\le \frac{\mu'_+(t)}{\mu(t)}\phi(t) + \mu(t)\frac{\left(a(t)\left(w^{(n-1)}(t)\right)^{\beta}\right)'}{w^{\beta}(t)} \\ &\quad - \beta\mu(t)\frac{\theta}{(n-2)!}t^{n-2}\frac{a(t)\left(w^{(n-1)}(t)\right)^{\beta+1}}{w^{\beta+1}(t)} \\ &\le \frac{\mu'(t)}{\mu(t)}\phi(t) + \mu(t)\frac{\left(a(t)\left(w^{(n-1)}(t)\right)^{\beta}\right)'}{w^{\beta}(t)} \\ &\quad - \frac{\beta\theta t^{n-2}}{(n-2)!\,(\mu(t)a(t))^{\frac{1}{\beta}}}\phi(t)^{\frac{\beta+1}{\beta}}.\end{aligned} \tag{3.12}$$

From (1.1) and (3.12), we obtain

$$\phi'(t) \le \frac{\mu'(t)}{\mu(t)}\phi(t) - k\mu(t)\frac{\sum_{i=1}^{j} q_i(t)\,w^{\beta}(\eta_i(t))}{w^{\beta}(t)} - \frac{\beta\theta t^{n-2}}{(n-2)!\,(\mu(t)a(t))^{\frac{1}{\beta}}}\phi(t)^{\frac{\beta+1}{\beta}}.$$

Note that $w'(t) > 0$ and $\eta_i(t) \geq t$, thus, we find

$$\phi'(t) \leq \frac{\mu'(t)}{\mu(t)}\phi(t) - k\mu(t)\sum_{i=1}^{j} q_i(t) - \frac{\beta\theta t^{n-2}}{(n-2)!\,(\mu(t)\,a(t))^{\frac{1}{\beta}}}\phi(t)^{\frac{\beta+1}{\beta}}.$$

The proof is complete. □

Lemma 6. *Assume that w be an eventually positive solution of (1.1) and $(\mathbf{S}_2)$ holds. If*

$$\delta(t) := \vartheta(t)\frac{w'(t)}{w(t)}. \tag{3.13}$$

where $\vartheta \in C^1([t_0,\infty),(0,\infty))$, then

$$\delta'(t) \leq \frac{\vartheta'(t)}{\vartheta(t)}\delta(t) - \Phi(t) - \frac{1}{\vartheta(t)}\delta(t)^2, \tag{3.14}$$

for all $t > t_1$, where t_1 large enough and

$$\Phi(t) = \frac{\vartheta(t)}{(n-4)!}\int_t^{\infty}(\varsigma - t)^{n-4}\left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty}\sum_{i=1}^{j} q_i(s)\,\mathrm{d}s\right)^{1/\beta}\mathrm{d}\varsigma.$$

Proof. Let w is an eventually positive solution of (1.1) and let us assume that $(\mathbf{S}_2)$ holds. From the definition of $\delta(t)$, we see that $\delta(t) > 0$ for $t \geq t_1$. By differentiating, we find

$$\delta'(t) = \frac{\vartheta'(t)}{\vartheta(t)}\delta(t) + \vartheta(t)\frac{w''(t)}{w(t)} - \frac{1}{\vartheta(t)}\delta(t)^2. \tag{3.15}$$

From (3.4) and (3.15), we obtain

$$\delta'(t) \leq \frac{\vartheta'(t)}{\vartheta(t)}\delta(t) - \frac{\vartheta(t)}{(n-4)!}\int_t^{\infty}(\varsigma - t)^{n-4}\left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty}\sum_{i=1}^{j} q_i(s)\,\mathrm{d}s\right)^{1/\beta}\mathrm{d}\varsigma - \frac{1}{\vartheta(t)}\delta(t)^2.$$

Thus,

$$\delta'(t) \leq \frac{\vartheta'(t)}{\vartheta(t)}\delta(t) - \Phi(t) - \frac{1}{\vartheta(t)}\delta(t)^2.$$

The proof is complete. □

In this theorem, we will establish an oscillation criterion for equation (1.1).

Theorem 2. *Assume that there exist positive functions $\mu, \vartheta \in C^1([t_0,\infty),(0,\infty))$ such that*

$$\int_{t_0}^{\infty}\left(k\mu(s)\sum_{i=1}^{j} q_i(s) - \frac{((n-2)!)^{\beta}\,a(t)\,(\mu'(t))^{\beta+1}}{(\beta+1)^{\beta+1}\,(t^{n-2}\theta\mu(t))^{\beta}}\right)\mathrm{d}s = \infty, \tag{3.16}$$

for some $\theta \in (0,1)$, and either

$$\int_{t_0}^{\infty}\sum_{i=1}^{j} q_i(s)\,ds = \infty \tag{3.17}$$

or

$$\int_{t_0}^{\infty}\left(\Phi(s)-\frac{1}{4\vartheta(s)}\left(\vartheta'(s)\right)^2\right)\mathrm{d}s=\infty. \tag{3.18}$$

Then every solution of (1.1) *is oscillatory.*

Proof. Assume that w is an eventually positive solution of (1.1). Then, we can suppose that $w(t)$ and $w(\eta_i(t))$ are positive for all $t\geq t_1$ sufficiently large. From Lemma 4, we have two possible cases $(\mathbf{S}_1)$ and $(\mathbf{S}_2)$.
Assume that $(\mathbf{S}_1)$ holds. From Lemma 5, we get that (3.10) holds. Using Lemma 3 with

$$U=\frac{\mu'(t)}{\mu(t)},\ \ V=\frac{\beta\theta t^{n-2}}{(n-2)!\,(\mu(t)\,a(t))^{\frac{1}{\beta}}}\ \text{ and } x=\phi(t),$$

we get

$$\frac{\mu'(t)}{\mu(t)}\phi(t)-\frac{\beta\theta t^{n-2}}{(n-2)!\,(\mu(t)\,a(t))^{\frac{1}{\beta}}}\phi(t)^{\frac{\beta+1}{\beta}}\leq-\frac{((n-2)!)^{\beta}\,a(t)\,(\mu'(t))^{\beta+1}}{(\beta+1)^{\beta+1}\,(t^{n-2}\theta\mu(t))^{\beta}}. \tag{3.19}$$

From (3.10) and (3.19), we obtain

$$\phi'(t)\leq-k\mu(t)\sum_{i=1}^{j}q_i(t)+\frac{((n-2)!)^{\beta}\,a(t)\,(\mu'(t))^{\beta+1}}{(\beta+1)^{\beta+1}\,(t^{n-2}\theta\mu(t))^{\beta}}.$$

Integrating from t_1 to t, we get

$$\int_{t_1}^{t}\left(k\mu(s)\sum_{i=1}^{j}q_i(s)-\frac{((n-2)!)^{\beta}\,a(t)\,(\mu'(t))^{\beta+1}}{(\beta+1)^{\beta+1}\,(t^{n-2}\theta\mu(t))^{\beta}}\right)\mathrm{d}s\leq\phi(t_1),$$

for every $\theta\in(0,\ 1)$, which contradicts (3.16). Assume that $(\mathbf{S}_2)$ holds. Integrating (1.1) from m to t, we conclude that

$$-a(m)\left(w^{(n-1)}(m)\right)^{\beta}=-\int_{m}^{t}\sum_{i=1}^{j}q_i(s)\,g(w(\eta_i(s)))\,ds.$$

By virtue of $w'(t)>0$ and $\eta_i(t)\geq t$, we get

$$\int_{m}^{t}\sum_{i=1}^{j}q_i(s)\,ds\leq\frac{a(m)\left(w^{(n-1)}(m)\right)^{\beta}}{kw^{\beta}(m)},$$

which contradicts (3.17).
From Lemma 6, we get that (3.14) holds. Using Lemma 3 with

$$U=\vartheta'(t)/\vartheta(t),\ V=1/\vartheta(t),\beta=1\text{ and }x=\delta(t),$$

we get

$$\frac{\vartheta'(t)}{\vartheta(t)}\delta(t)-\frac{1}{\vartheta(t)}\delta^2(t)\leq-\frac{1}{4\vartheta(t)}\left(\vartheta'(t)\right)^2. \tag{3.20}$$

From (3.14) and (3.20), we obtain

$$\delta'(t) \leq -\Phi(t) + \frac{1}{4\vartheta(t)}\left(\vartheta'(t)\right)^2. \tag{3.21}$$

Integrating from t_1 to t, we get

$$\int_{t_1}^{t}\left(\Phi(s) - \frac{1}{4\vartheta(s)}\left(\vartheta'(s)\right)^2\right)\mathrm{d}s \leq \delta(t_1),$$

which contradicts (3.18). The proof is complete. □

Putting $\mu(t) = t^{n-1}$ and $\vartheta(t) = t$ into Theorem 2, we get the following oscillation criterion:

Corollary 2. *Let* (1.1) *hold. Assume that*

$$\int_{t_0}^{\infty}\left(s^{n-1}\sum_{i=1}^{j} q_i(s) - \frac{((n-2)!)^{\beta}(n-1)^{\beta+1}s^{-n\beta+n+\beta-2}a(s)}{(\beta+1)^{\beta+1}\theta^{\beta}}\right)\mathrm{d}s = \infty, \tag{3.22}$$

or some $\theta \in (0,1)$. *If* (3.17) *holds and*

$$\int_{t_0}^{\infty}\left(\frac{s}{(n-4)!}\int_{t}^{\infty}(\varsigma - t)^{n-4}\left(\frac{k}{a(\varsigma)}\int_{\varsigma}^{\infty}\sum_{i=1}^{j} q_i(s)\,\mathrm{d}s\right)^{1/\beta}\mathrm{d}\varsigma - \frac{1}{4s}\right)\mathrm{d}s = \infty, \tag{3.23}$$

then every solution of (1.1) *is oscillatory.*

4. Examples and concluding remarks

In this section we show two numerical examples as application of the theoretical findings discussed in the previous section. Finally, we list some perspectives for future works.

Example 1. Let us consider the differential equation

$$\left(t\left(w'''(t)\right)\right)' + \frac{d_0}{t^3}w(ct) = 0,\ t \geq 1, \tag{4.1}$$

where $c > 0$ and $d_0 > 1$ is a constant. Note that $\beta = 1$, $n = 4$, $a(t) = t$, $q(t) = d_0/t^3$. If we set $k = 1$, and $\mu(s) = s^3$,then

$$\begin{aligned}
&\int_{t_0}^{\infty}\left(k\mu(s)\sum_{i=1}^{j} q_i(s) - \frac{((n-2)!)^{\beta}a(s)\left(\mu'(s)\right)^{\beta+1}}{(\beta+1)^{\beta+1}\left(s^{n-2}\theta\mu(s)\right)^{\beta}}\right)\mathrm{d}s \\
&= \int_{t_0}^{\infty}\left(d_0\frac{s^3}{s^3} - \frac{3^2 s^5}{2\theta s^5}\right)\mathrm{d}s = \int_{t_0}^{\infty}\left(d_0 - \frac{9}{2\theta}\right)ds \\
&= \infty \quad \text{if} \quad d_0 > \frac{9}{2\theta},
\end{aligned}$$

for some constant $\theta \in (0,\ 1)$. Hence, by Theorem 2, every solution of equation (4.1) is oscillatory if

$$d_0 > \frac{9}{2\theta}.$$

Remark 1. Applying the condition (3.22) to the equation (2.5), we find

$$q_0 > 4.5.$$

Therefore, our result improves the results contained in [1,2,9].

Example 2. Let us consider the differential equation

$$w^{(4)}(t) + \frac{q_0}{t^4} w(2t) = 0,\ t \geq 1, \tag{4.2}$$

where $q_0 > 0$ is a constant. Note that $\beta = 1,\ n = 4,\ a(t) = 1,\ q(t) = q_0/t^4$ and $\eta(t) = 2t$. If we set $k = 1$, then condition (3.16) becomes

$$\begin{aligned}
&\int_{t_0}^{\infty} \left(s^{n-1} \sum_{i=1}^{j} q_i(s) - \frac{((n-2)!)^{\beta} (n-1)^{\beta+1} s^{-n\beta+n+\beta-2} a(s)}{(\beta+1)^{\beta+1} \theta^{\beta}} \right) \mathrm{d}s \\
&= \int_{t_0}^{\infty} \left(\frac{q_0}{s} - \frac{9}{2\theta s} \right) ds = \left(q_0 - \frac{9}{2\theta} \right) \int_{t_0}^{\infty} \frac{1}{s} ds \\
&= \infty \quad \text{if} \quad q_0 > 4.55 \ \ (\text{let } \theta = 99/100)
\end{aligned}$$

and condition (3.18) becomes

$$\begin{aligned}
&\left(\frac{s}{(n-4)!} \int_{t}^{\infty} (\varsigma - t)^{n-4} \left(\frac{k}{a(\varsigma)} \int_{\varsigma}^{\infty} \sum_{i=1}^{j} q_i(s)\, \mathrm{d}s \right)^{1/\beta} \mathrm{d}\varsigma - \frac{1}{4s} \right) \mathrm{d}s \\
&= \int_{t_0}^{\infty} \left(\frac{q_0}{6s} - \frac{1}{4s} \right) \mathrm{d}s \\
&= \infty, \quad \text{if} \quad q_0 > \frac{3}{2}.
\end{aligned}$$

Therefore, from Corollary 2, every solution to the equation (4.2) is oscillatory if $q_0 > 4.55$.

Remark 2. Now we compare our result with the known related criteria for oscillation of the equation under consideration.

(1) Applying condition (2.2) in [1], we get

$$q_0 > 25.5.$$

(2) Applying condition (2.3) in [2], we obtain

$$q_0 > 18.$$

(3) Applying condition (2.4) in [9], we find

$$q_0 > 1728.$$

Therefore, our result improves the results contained in [1, 2, 9].

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BOUNDS FOR SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS AND ULAM STABILITY

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Abstract. We obtain Gronwall type bounds for the solutions of a linear system of differential equations. As applications we get results on Ulam stability for linear differential equations and linear systems of differential equations.

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1. Introduction

Let $I = [a,b)$, $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$, $a < b$, be an interval, $\mathbb{K}$ one of the fields $\mathbb{R}$ or $\mathbb{C}$, $A = (a_{ij})_{1 \le i,j \le n}$, $B = (b_1, \dots, b_n)^T$, $a_{ij}, b_i \in C(I, \mathbb{K})$, $1 \le i$, $j \le n$. We consider linear systems of differential equations of the form

$$y'(x) = A(x)y(x) + B(x), \quad x \in I, \tag{1.1}$$

where the unknown is $y \in C^1(I, \mathbb{K}^n)$, $y = (y_1, \dots, y_n)^T$.

We are looking for bounds of solutions for the equation (1.1) and study its Ulam stability. In what follows we suppose that $\mathbb{K}^n$ is endowed with the Euclidean norm $\|\cdot\|$ defined as usual by

$$\|u\| = \sqrt{\langle u, u \rangle}, \quad u \in \mathbb{K}^n,$$

and the inner product $\langle , \rangle$ given by

$$\langle u, v \rangle = \sum_{k=1}^{n} u_k \cdot \overline{v_k}$$

for $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$ in $\mathbb{K}^n$.

In this paper we present a new approach to Ulam stability of linear differential equations. We obtain bounds of solutions for a linear system of differential equations by using a result on boundedness of solutions for a Bernoulli type differential inequality.

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Let $X = (x_{ij})_{1\le i,j\le n}$ be a matrix with entries in $\mathbb{K}$. The induced matrix norm $\sigma(X)$ is the spectral norm, defined as the square root of the largest eigenvalue of the matrix X^*X, where X^* denotes the conjugate transposed of X.

The Frobenius norm of X, denoted by $\|X\|_F$, is given by

$$\|X\|_F = \left(\sum_{i,j=1}^{n} |x_{ij}|^2\right)^{\frac{1}{2}}$$

and is an upper bound of $\sigma(X)$.

Recall that the equation (1.1) is called stable in Ulam sense if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $y \in C^1(I, \mathbb{K}^n)$ satisfying

$$\left\|y'(x) - A(x)y(x) - B(x)\right\| \le \varepsilon, \qquad x \in I, \tag{1.2}$$

there exists a solution $z \in C^1(I, \mathbb{K}^n)$ of (1.1) such that

$$\|y(x) - z(x)\| \le \delta, \qquad x \in I.$$

If in the previous definition ε and δ are replaced by some functions $\varphi, \psi : I \to \mathbb{R}_+$, the equation (1.1) is called generalized Ulam stable.

A function $y \in C^1(I, \mathbb{K}^n)$ which satisfies (1.2) is called approximate solution of (1.1). So, generally we say that equation (1.1) is stable (or generalized stable) in Ulam sense if for every approximate solution of the equation there exists an exact solution close to it. For more details on Ulam stability see [2–4, 8].

The first result on Ulam stability of differential equations was given by M. Obłoza, who investigated also the relations between Ulam and Lyapunov stability [9, 10]. T. Miura, S. Miyajima and S.E. Takahasi [7] and S.E. Takahasi, H. Takagi, T. Miura and S. Miyajima [14] obtained results on Ulam stability for the first order linear differential operators and the higher order linear differential operators with constant coefficients. Some extensions of the results mentioned above were obtained by D. Popa and I. Raşa for the linear differential operators of order n with constant coefficients [11–13]. Recall also the results obtained by S.M. Jung on the stability of various differential equations [5, 6].

It should be mentioned that the results presented above concern linear differential equations and operators with constant coefficients or particular forms of linear differential operators with nonconstant coefficients.

In this paper we obtain an Ulam stability result for a linear system of differential equations with nonconstant coefficients such that the difference of the approximate and the exact solution is controlled by a function depending only on the norm of the matrix of the system. We present several examples and applications of this general result.

2. Bounds of solutions and Ulam stability

Let $\mathbb{R}_+$ denote the interval $[0,+\infty)$. The following result concerning bounds of solutions of a Bernoulli type differential inequality will be useful for the main result of this paper.

Lemma 1. *Let $f,g \in C(I,\mathbb{R}_+)$ and $u \in C^1(I,\mathbb{R}_+)$ be such that*

$$u'(x) \le 2f(x)u(x) + 2g(x)\cdot\sqrt{u(x)}, \qquad x \in I.$$

Then

$$\sqrt{u(x)} \le \sqrt{u(a)}e^{F(x)-F(a)} + e^{F(x)}\int_a^x g(t)e^{-F(t)}dt, \quad x \in I, \tag{2.1}$$

where F is an antiderivative of f, i.e., $F' = f$ on I.

Proof. Suppose first that $u(x) > 0$ for all $x \in I$ and let $z(x) = \sqrt{u(x)}$, $x \in I$. Then $z'(x) = \frac{u'(x)}{2\sqrt{u(x)}}$, $x \in I$, and

$$z'(x) - f(x)z(x) \le g(x), \qquad x \in I.$$

A multiplication by $e^{-F(x)}$ leads to

$$\left(z(x)e^{-F(x)}\right)' \le g(x)e^{-F(x)}, \quad x \in I.$$

By integration on $[a,x]$, $x \in I$, we get

$$z(x)e^{-F(x)} - z(a)e^{-F(a)} \le \int_a^x g(t)e^{-F(t)}dt,$$

or

$$z(x) \le z(a)e^{F(x)-F(a)} + e^{F(x)}\int_a^x g(t)e^{-F(t)}dt, \quad x \in I,$$

which leads to (2.1). Now let $u(x) \ge 0$ for $x \in I$. Let $n \in \mathbb{N}$, and consider the function

$$u_n(x) = u(x) + \frac{1}{n}, \qquad x \in I.$$

Since $u_n'(x) = u'(x)$, $x \in I$, we get

$$u_n'(x) \le 2f(x)u_n(x) + 2g(x)\sqrt{u_n(x)}, \qquad x \in I.$$

According to the first part of this proof it follows that

$$\sqrt{u_n(x)} \le \sqrt{u_n(a)}e^{F(x)-F(a)} + e^{F(x)}\cdot\int_a^x g(t)e^{-F(t)}dt, \qquad x \in I.$$

Letting $n \to \infty$ in the previous relation we get (2.1).

The lemma is proved. □

Theorem 1. *Suppose that $\sigma(A(t))$ is locally integrable on I, and let $F(x) = \int_a^x \sigma(A(t))dt$, $x \in I$. Then for every $y \in C^1(I, \mathbb{K}^n)$ satisfying (1.1) the following relation holds:*

$$\|y(x)\| \leq \|y(a)\| e^{F(x)} + e^{F(x)} \cdot \int_a^x \|B(t)\| e^{-F(t)} dt, \quad x \in I.$$

Proof. Let $y \in C^1(I, \mathbb{K}^n)$ be such that

$$y'(x) = A(x)y(x) + B(x), \quad x \in I.$$

A scalar multiplication of the previous relation by $y(x)$ leads to

$$\left\langle y'(x), y(x) \right\rangle = \langle A(x)y(x), y(x) \rangle + \langle B(x), y(x) \rangle, \quad x \in I.$$

It follows that

$$\begin{aligned} \left| \left\langle y'(x), y(x) \right\rangle \right| &\leq |\langle A(x)y(x), y(x) \rangle| + |\langle B(x), y(x) \rangle| \\ &\leq \|A(x)y(x)\| \cdot \|y(x)\| + \|B(x)\| \cdot \|y(x)\| \\ &\leq \sigma(A(x)) \cdot \|y(x)\|^2 + \|B(x)\| \cdot \|y(x)\|, \quad x \in I. \end{aligned}$$

Taking account of

$$\left(\|y(x)\|^2 \right)' = 2\Re \left\langle y'(x), y(x) \right\rangle, \quad x \in I,$$

it follows that the function u defined by

$$u(x) = \|y(x)\|^2, \quad x \in I,$$

satisfies the Bernoulli type differential inequality

$$u'(x) \leq 2\sigma(A(x)) \cdot u(x) + 2\|B(x)\| \cdot \sqrt{u(x)}, \quad x \in I.$$

Now the conclusion is a simple consequence of Lemma 1. □

Remark 1. It is obvious that Theorem 1 holds with

$$F(x) = \int_a^x M(t)dt, \quad x \in I,$$

where $M(x)$ is any locally integrable upper bound of $\sigma(A(x))$ on I.

Remark 2. If A is a constant matrix with entries in $\mathbb{K}$, $\lambda = \sigma(A)$, $\lambda > 0$, then for every solution $y \in C^1(I, \mathbb{K}^n)$ of (1.1) the following estimation holds

$$\|y(x)\| \leq e^{\lambda(x-a)} \left(\|y(a)\| + \frac{1}{\lambda}(1 - e^{-\lambda(x-a)}) \cdot \sup_{x \in I} \|B(x)\| \right).$$

Proof. $F(x) = \lambda(x-a)$, $x \in I$, therefore in view of Theorem 1 we get

$$\|y(x)\| \leq e^{\lambda(x-a)} \left(\|y(a)\| + \int_a^x \|B(t)\| e^{-\lambda(t-a)} dt \right)$$

$$\leq e^{\lambda(x-a)}\left(\|y(a)\|+\sup_{x\in I}\|B(x)\|\cdot\int_a^x e^{-\lambda(t-a)}dt\right)$$
$$=e^{\lambda(x-a)}\left(\|y(a)\|+\frac{1}{\lambda}(1-e^{-\lambda(x-a)})\sup_{x\in I}\|B(x)\|\right),\quad x\in I.$$

□

The next result concerns generalized Ulam stability of the equation (1.1).

Theorem 2. *Let $\phi : I \to \mathbb{R}_+$ be a continuous function, $F(x) = \int_a^x M(t)dt$, where $M(x)$ is any locally integrable upper bound of $\sigma(A(x))$ on I.*
Then for every $y \in C^1(I,\mathbb{K}^n)$ satisfying

$$\left\|y'(x)-A(x)y(x)-B(x)\right\|\leq\phi(x),\quad x\in I, \tag{2.2}$$

there exists a unique solution $z \in C^1(I,\mathbb{K}^n)$ of (1.1) such that

$$\|y(x)-z(x)\|\leq e^{F(x)}\int_a^x\phi(t)e^{-F(t)}dt,\quad x\in I. \tag{2.3}$$

Proof. Existence. Let $y \in C^1(I,\mathbb{K}^n)$ satisfying (2.2) and define

$$h(x):=y'(x)-A(x)y(x)-B(x),\quad x\in I.$$

Let z be the unique solution of (1.1) with $z(a) = y(a)$. Then the function $w : I \to \mathbb{R}$ given by

$$w(x)=y(x)-z(x),\quad x\in I,$$

satisfies

$$w'(x)=A(x)w(x)+h(x),$$
$$w(a)=0.$$

Taking account of Theorem 1 and Remark 1 it follows that

$$\|w(x)\|\leq e^{F(x)}\int_a^x\|h(t)\|e^{-F(t)}dt$$
$$\leq e^{F(x)}\int_a^x\phi(t)e^{-F(t)}dt,\quad x\in I.$$

Uniqueness. Suppose that for an $y \in C^1(I,\mathbb{K}^n)$ satisfying (2.2) there exist two solutions $z_1, z_2 \in C^1(I,\mathbb{R})$ of (1.1) such that

$$\|y(x)-z_k(x)\|\leq e^{F(x)}\int_a^x\phi(t)e^{-F(t)}dt,\quad x\in I,\quad k=1,2.$$

Then

$$\|z_1(x)-z_2(x)\|\leq\|z_1(x)-y(x)\|+\|y(x)-z_2(x)\|$$
$$\leq 2e^{F(x)}\int_a^x\phi(t)e^{-F(t)}dt,\quad x\in I.$$

It follows $z_1(a) = z_2(a)$ and, taking account of the existence and uniqueness theorem for the Cauchy problem of (1.1), we get

$$z_1(x) = z_2(x), \quad x \in I.$$

The theorem is proved. □

Corollary 1. *Let $\varepsilon > 0$. Then for every $y \in C^1(I, \mathbb{K}^n)$ satisfying the relation*

$$\left\| y'(x) - A(x)y(x) - B(x) \right\| \leq \varepsilon\sigma(A(x)), \qquad x \in I,$$

there exists a unique solution $z \in C^1(I, \mathbb{K}^n)$ of (1.1) such that

$$\|y(x) - z(x)\| \leq \varepsilon(e^{\int_a^x \sigma(A(t))dt} - 1), \qquad x \in I.$$

Proof. Let $M(x) = \sigma(A(x))$ and $\phi(x) = \varepsilon\sigma(A(x))$, $x \in I$, in Theorem 2. Then

$$\begin{aligned} e^{F(x)} \int_a^x \phi(t) e^{-F(t)} dt &= \varepsilon e^{F(x)} \int_a^x F'(t) e^{-F(t)} dt \\ &= \varepsilon(e^{F(x)} - 1) = \varepsilon(e^{\int_a^x \sigma(A(t))dt} - 1), \qquad x \in I, \end{aligned}$$

and the result holds in view of Theorem 2. □

Remark 3. Concerning the upper bound in (2.3), it is easy to see that

$$\int_a^x \phi(t)dt \leq e^{F(x)} \int_a^x \phi(t) e^{-F(t)} dt \leq e^{F(x)-F(a)} \int_a^x \phi(t)dt.$$

3. Applications

In this section we give some applications of the previous results to the Ulam stability of linear differential equations with constant coefficients and linear differential equations of order two with non-constant coefficients.

Theorem 3. *Let A be a constant matrix with entries in $\mathbb{K}$, with $\lambda = \sigma(A) > 0$ and $\varepsilon > 0$. Then for every $y \in C^1(I, \mathbb{K}^n)$ satisfying*

$$\left\| y'(x) - Ay(x) - B(x) \right\| \leq \varepsilon, \quad x \in I, \tag{3.1}$$

there exists a unique $z \in C^1(I, \mathbb{K}^n)$ satisfying (1.1) such that

$$\|y(x) - z(x)\| \leq \frac{\varepsilon}{\lambda}(e^{\lambda(x-a)} - 1), \quad x \in I. \tag{3.2}$$

Proof. Let $M(x) = \sigma(A)$, $\phi(x) = \varepsilon$, $x \in I$ in Theorem 2. Then $F(x) = \lambda(x-a)$, $x \in I$. Therefore, for every $y \in C^1(I, \mathbb{K}^n)$ satisfying (3.1) there exists a unique solution z of (1.1) such that

$$\|y(x) - z(x)\| \leq e^{\lambda(x-a)} \cdot \int_a^x \varepsilon e^{-\lambda(t-a)} dt = \frac{\varepsilon}{\lambda}\left(e^{\lambda(x-a)} - 1\right), \quad x \in I.$$

□

We give in what follows a result on Ulam stability for a linear differential equation of order two with non-constant coefficients. More precisely, we consider the equation

$$y''(x)+f(x)y'(x)+g(x)y(x)=0, \quad x\in I, \tag{3.3}$$

where $f,g\in C(I,\mathbb{K})$ and $y\in C^2(I,\mathbb{K})$.

Theorem 4. *Let $\varphi: I\to\mathbb{R}_+$ be a continuous function. Then for every $y\in C^2(I,\mathbb{K})$ satisfying*

$$\left|y''(x)+f(x)y'(x)+g(x)y(x)\right|\le\varphi(x), \quad x\in I,$$

there exists a unique solution $z\in C^2(I,\mathbb{K})$ of (3.3) with the property

$$|y(x)-z(x)|\le e^{F(x)}\int_a^x\varphi(t)e^{-F(t)}dt, \quad x\in I,$$

where $F(x)=\int_a^x\sqrt{|f(t)|^2+|g(t)|^2+1}dt, x\in I$.

Proof. Let $y\in C^2(I,\mathbb{K})$ satisfying (3.3) and define

$$y''(x)+f(x)y'(x)+g(x)y(x)=h(x), \quad x\in I.$$

Let also $y_1=y$, $y_2=y'$, $u=(y_1\ y_2)^T$, $B=(0\ h)^T$.

$$A(x)=\begin{pmatrix}0 & 1\\ -g(x) & -f(x)\end{pmatrix}, \quad x\in I.$$

Then

$$u'(x)=A(x)u(x)+B(x), \quad x\in I. \tag{3.4}$$

We have

$$A^*(x)A(x)=\begin{pmatrix}|g(x)|^2 & \overline{g(x)}f(x)\\ \overline{f(x)}g(x) & 1+|f(x)|^2\end{pmatrix}.$$

The eigenvalues $\lambda_1(x)$, $\lambda_2(x)$ of $A^*(x)A(x)$ are the roots of the polynomial

$$P(\lambda)=\lambda^2-(|f(x)|^2+|g(x)|^2+1)\lambda+|g(x)|^2,$$

therefore

$$\sigma(A(x))=\max\left\{\sqrt{\lambda_1(x)},\sqrt{\lambda_2(x)}\right\}.$$

On the other hand $P(0)\ge 0$, $P(1)\le 0$, $P(|f(x)|^2+|g(x)|^2+1)\ge 0$, so it follows that

$$M(x)=\sqrt{|f(x)|^2+|g(x)|^2+1}$$

is an upper bound of $\sigma(A(x))$.

Remark that $M(x)$ is in fact the Frobenius norm of the matrix $A(x)$.

From (3.4) we get

$$\left\|u'(x)-A(x)u(x)\right\|=\|B(x)\|=|h(x)|\le\varphi(x), \quad x\in I,$$

so according to Theorem 2 and Remark 1 it follows that there exists $w = (w_1\ w_2)^T \in C^1(I, \mathbb{K}^2)$ such that

$$w'(x) = A(x)w(x), \quad x \in I, \tag{3.5}$$
$$\|u(x) - w(x)\| \leq e^{F(x)} \cdot \int_a^x \varphi(t)e^{-F(t)}dt, \quad x \in I.$$

Relation (3.5) leads to $w_1''(x) + f(x)w_1'(x) + g(x)w(x) = 0, x \in I$. Let $z(x) = w_1(x)$, $x \in I$. Then

$$\begin{aligned} |y(x) - z(x)| &\leq \|u(x) - w(x)\| \\ &\leq e^{F(x)} \int_a^x \varphi(t)e^{-F(t)}dt, \quad x \in I. \end{aligned}$$

□

Remark 4. In the setting of Theorem 4, let $f(x) = 0$, $x \in I$. Then

$$\lambda_1(x) = |g(x)|^2, \quad \lambda_2(x) = 1, \quad x \in I.$$

Consequently,

$$\sigma(A(x)) = \max\{|g(x)|, 1\}.$$

If, in addition, $|g(x)| \leq 1$, then

$$\sigma(A(x)) = 1, \quad x \in I.$$

In such a situation, the conclusion of Theorem 4 holds with

$$F(x) = \int_a^x \max\{|g(t)|, 1\}\, dt,$$

respectively with

$$F(x) = x - a, \quad x \in I.$$

We present now some applications of Theorem 4.

Example 1. Let $\varepsilon > 0$ and $y \in C^2(I, \mathbb{K})$ satisfying

$$\left| y''(x) + (\sin x)y'(x) + (\cos x)y(x) \right| \leq \varepsilon, \qquad x \in I.$$

Then there exists a $z \in C^2(I, \mathbb{K})$ with properties

$$z''(x) + (\sin x)z'(x) + (\cos x)z(x) = 0, \qquad x \in I$$

and

$$|y(x) - z(x)| \leq \frac{\varepsilon}{\sqrt{2}}(e^{\sqrt{2}(x-a)} - 1), \qquad x \in I.$$

Proof. $F(x) = \int_a^x \sqrt{\sin^2 t + \cos^2 t + 1}dt = \sqrt{2}(x-a), x \in I$, and the result is a simple consequence of Theorem 4. In fact, also in this case it is easy to find $\lambda_{1,2}(x) = 1 \pm \sin x$, so that

$$\sigma(A(x)) = \sqrt{1 + |\sin x|}, \quad x \in I.$$

□

In the case of the linear differential equation of second order with constant coefficients we have the next general stability result.

Example 2. Let $p,q \in \mathbb{K}$ and $\varepsilon > 0$. For every $y \in C^2(I,\mathbb{K})$ satisfying

$$\left|y''(x) + py'(x) + qy(x)\right| \leq \varepsilon, \qquad x \in I,$$

there exists a unique $z \in C^2(I,\mathbb{K})$ with properties

$$z''(x) + pz'(x) + qz(x) = 0, \qquad x \in I$$

and

$$|y(x) - z(x)| \leq \frac{\varepsilon}{\sqrt{|p|^2 + |q|^2 + 1}}(e^{(x-a)\sqrt{|p|^2+|q|^2+1}} - 1), \qquad x \in I.$$

Proof. Theorem 4 is applied taking account of

$$F(x) = \sqrt{|p|^2 + |q|^2 + 1}(x-a), \qquad x \in I.$$

□

In many concrete cases it is difficult to estimate the difference between the approximate solution and the exact solution in Theorem 4 because of the impossibility to calculate the integrals from those relations.

In practice one can use lower bounds or upper bounds of the function F.

Example 3. We will exemplify this in the case of Hermite's differential equation. Hermite equation is defined as

$$y''(x) - 2xy'(x) + 2ny(x) = 0, \qquad n \in \mathbb{N}, \qquad x \in \mathbb{R}$$

and its particular solution is Hermite's polynomial H_n defined by

$$H_n(x) = (-1)^n e^{x^2}(e^{-x^2})^{(n)}, \qquad x \in \mathbb{R}.$$

Let $I = [0,+\infty)$, $\varepsilon > 0$ and $y \in C^2(I,\mathbb{R})$ such that

$$\left|y''(x) - 2xy'(x) + 2ny(x)\right| \leq \varepsilon, \qquad x \in I. \tag{3.6}$$

Then there exists a unique $z \in C^2(I,\mathbb{R})$ with the properties

$$z''(x) - 2xz'(x) + 2nz(x) = 0, \qquad x \in I, \tag{3.7}$$

and

$$\begin{aligned} |y(x) - z(x)| &\leq \varepsilon e^{F(x)} \cdot \int_0^x e^{-F(t)}dt \\ &\leq \frac{\sqrt{\pi}\varepsilon}{2} e^{x^2+(2n+1)x}, \qquad x \in I. \end{aligned}$$

Proof. Let

$$F(x) = \int_0^x \sqrt{4t^2+4n^2+1}dt$$
$$= \frac{x\sqrt{4x^2+4n^2+1}}{2} + \frac{4n^2+1}{4}\ln\frac{2x+\sqrt{4x^2+4n^2+1}}{\sqrt{4n^2+1}}, \qquad x \in I.$$

Taking $\varphi(x) = \varepsilon$, $x \in I$, in Theorem 4, it follows that for an $y \in C^2(I,\mathbb{R})$ satisfying (3.6) there exists a unique z satisfying (3.7) such that

$$|y(x) - z(x)| \le \varepsilon e^{F(x)} \int_0^x e^{-F(t)}dt.$$

On the other hand taking account of the relation

$$2t \le \sqrt{4t^2+4n^2+1} \le 2t+2n+1, \qquad t \in I$$

we get by integration on $[0,x]$

$$x^2 \le F(x) \le x^2 + (2n+1)x, \qquad x \in I.$$

Finally we get

$$e^{F(x)} \cdot \int_0^x e^{-F(t)}dt \le e^{x^2+(2n+1)x} \cdot \int_0^x e^{-t^2}dt$$
$$\le e^{x^2+(2n+1)x} \cdot \int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}e^{x^2+(2n+1)x}, \qquad x \in I.$$

□

Clearly the result obtained in Theorem 4 can be extended to a linear differential equation of order n as follows.

Consider the linear differential equation

$$y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \ldots + a_n(x)y(x) = 0, \qquad x \in I, \tag{3.8}$$

where $a_1, \ldots, a_n \in C(I,\mathbb{K})$. Then we obtain:

Theorem 5. *Let $\varphi : I \to \mathbb{R}_+$ be a continuous function. Then for every $y \in C^n(I,\mathbb{K})$ satisfying*

$$\left|y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \ldots + a_n(x)y(x)\right| \le \varphi(x), \qquad x \in I,$$

there exists a unique solution $z \in C^n(I,\mathbb{K})$ of the equation (3.8) with the property

$$|y(x) - z(x)| \le e^{F(x)} \cdot \int_a^x \varphi(t)e^{-F(t)}dt, \qquad x \in I,$$

where

$$F(x) = \int_a^x \sqrt{\sum_{k=1}^n |a_k(t)|^2 + n - 1}dt, \qquad x \in I.$$

Proof. Analogous with the proof of Theorem 4. □

4. Conclusions

Ulam stability of differential equations is a topic intensively studied in the last years. In this respect, results were obtained regarding Ulam stability of the first order linear differential equations [5–7] and the linear differential equations of higher orders with constant coefficients [3, 12]. Results on the best Ulam constant for the linear differential equations of orders one and two are given in [1, 14]. We also mention that there are some results on Ulam stability for the linear differential equation with nonconstant coefficients (see [2, 3, 13]).

The main contribution of the present paper is Theorem 2, establishing the Ulam stability for a system of differential equations with nonconstant coefficients. It can be applied in order to investigate the Ulam stability for the linear differential equations with nonconstant coefficients.

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NEW EXTENSIONS OF THE HERMITE-HADAMARD INEQUALITIES INVOLVING RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS

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Abstract. In this study, we establish the above and below bounds for the left and right hand sides of fractional Hermite-Hadamard inequalities by using functions whose second derivatives are bounded. We also give some refinements of fractional Hermite-Hadamard inequalities by using the functions that have the conditions $f'(a+b-t)-f'(t)\geq 0$, $t\in\left[a,\frac{a+b}{2}\right]$.

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1. Introduction

The Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention much interest in elementary mathematics. A number of mathematicians have devoted their efforts to generalise, refine, counterpart and extend it for different classes of functions such as using convex mappings.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g., [18, p.137], [7]). These inequalities state that if $f:I\to\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$, then

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave.

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

Definition 1. Let $f\in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha>0$ with $a\geq 0$ are defined by

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\ \ x>a$$

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and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

For more information about fraction calculus please refer to [9, 14, 16, 19].

In [23], Sarikaya et al. first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \le \frac{f(a)+f(b)}{2} \tag{1.2}$$

with $\alpha > 0$.

Sarikaya and Yildirim also give the following Hermite-Hadamard type inequality for the Riemann-Lioville fractional integrals:

Theorem 2 ([24])**.** *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J_{\left(\frac{a+b}{2}\right)^+}^{\alpha} f(b) + J_{\left(\frac{a+b}{2}\right)^-}^{\alpha} f(a)\right] \le \frac{f(a)+f(b)}{2}. \tag{1.3}$$

Moreover, Dragomir give the following another version of Hermite-Hadamard inequality for Riemann-Lioville fractional integrals:

Theorem 3 ([8])**.** *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right)\right] \le \frac{f(a)+f(b)}{2} \tag{1.4}$$

Over the years several papers devoted to fractional Hermite-Hadamard inequalities. One can refer to the references [1–4, 6, 10–13, 15, 17, 20–26] for some of them.

F. Chen prove the following inequalities which give the above and below bounds for the left and right hand sides of inequality (1.2).

Theorem 4 ([5])**.** *Let $f : [a,b] \to \mathbb{R}$ be positive, twice differentiable functions with $a < b$ and $f \in L_1[a,b]$. If f'' is bounded, then we have the inequalities*

$$\frac{m\alpha}{(b-a)^{\alpha}} \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right)^2 \left[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}\right] dx$$

$$\leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]-f\left(\frac{a+b}{2}\right)$$

$$\leq \frac{M\alpha}{(b-a)^{\alpha}}\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{2}\left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right]dx$$

and

$$\frac{-M\alpha}{2(b-a)^{\alpha}}\int_{a}^{\frac{a+b}{2}}(x-a)(b-x)\left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right]dx$$

$$\leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]-\frac{f(a)+f(b)}{2}$$

$$\leq \frac{-m\alpha}{2(b-a)^{\alpha}}\int_{a}^{\frac{a+b}{2}}(x-a)(b-x)\left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right]dx$$

for $\alpha>0$, *where* $m=\inf_{t\in[a,b]}f''(t)$, $M=\sup_{t\in[a,b]}f''(t)$.

In this paper we establish extensions of inequalities (1.3) and (1.4).

2. Main Results

Firstly, we give the following inequalities which give the above and below bounds for the left and right hand sides of inequality (1.3).

Theorem 5. *Let* $f:[a,b]\to\mathbb{R}$ *be positive, twice differentiable functions with* $a<b$ *and* $f\in L_1[a,b]$. *If* f'' *is bounded, i.e.* $m\leq f''(t)\leq M$, $t\in[a,b]$, $m,M\in\mathbb{R}$, *then we have the inequalities*

$$\frac{m(b-a)^2}{4(\alpha+1)(\alpha+2)}\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)\right]-f\left(\frac{a+b}{2}\right)$$
$$\leq\frac{M(b-a)^2}{4(\alpha+1)(\alpha+2)} \tag{2.1}$$

and

$$\frac{m(b-a)^2\alpha(\alpha+3)}{8(\alpha+1)(\alpha+2)}\leq\frac{f(a)+f(b)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)\right]$$
$$\leq\frac{M(b-a)^2\alpha(\alpha+3)}{8(\alpha+1)(\alpha+2)}. \tag{2.2}$$

Proof. By using the change of variables we have

$$\begin{aligned}
&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]\\
&=\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\left[\int_{\frac{a+b}{2}}^{b}(b-x)^{\alpha-1}f(x)dx+\int_{a}^{\frac{a+b}{2}}(x-a)^{\alpha-1}f(x)dx\right]\\
&=\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\left[\int_{a}^{\frac{a+b}{2}}(b-x)^{\alpha-1}f(a+b-x)dx+\int_{a}^{\frac{a+b}{2}}(x-a)^{\alpha-1}f(x)dx\right]\\
&=\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_{a}^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)\right](x-a)^{\alpha-1}dx. \qquad (2.3)
\end{aligned}$$

By equality (2.3), we get

$$\begin{aligned}
&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]-f\left(\frac{a+b}{2}\right)\\
&=\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_{a}^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)\right](x-a)^{\alpha-1}dx-f\left(\frac{a+b}{2}\right)\\
&=\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_{a}^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right](x-a)^{\alpha-1}dx. \qquad (2.4)
\end{aligned}$$

Using the facts that

$$f(x)-f\left(\frac{a+b}{2}\right)=\int_{\frac{a+b}{2}}^{x}f'(t)dt$$

$$f\left(\frac{a+b}{2}\right)-f(a+b-x)=-\int_{\frac{a+b}{2}}^{a+b-x}f'(t)dt,$$

we have

$$f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)=\int_{\frac{a+b}{2}}^{a+b-x}f'(t)dt-\int_{x}^{\frac{a+b}{2}}f'(t)dt$$

$$= \int_x^{\frac{a+b}{2}} f'(a+b-u)du - \int_x^{\frac{a+b}{2}} f'(t)\,dt$$

$$= \int_x^{\frac{a+b}{2}} \left[f'(a+b-t) - f'(t)\right] dt. \tag{2.5}$$

We also have

$$f'(a+b-t) - f'(t) = \int_t^{a+b-t} f''(u)du. \tag{2.6}$$

By using equality (2.6) and $m < f''(u) < M$, $u \in [a,b]$, we obtain,

$$m \int_t^{a+b-t} du \leq \int_t^{a+b-t} f''(u)du \leq M \int_t^{a+b-t} du$$

i.e.

$$m(a+b-2t) \leq f'(a+b-t) - f'(t) \leq M(a+b-2t). \tag{2.7}$$

Integrating inequality (2.7) with respect to t on $\left[x, \frac{a+b}{2}\right]$, we get

$$m\left(\frac{a+b}{2} - x\right)^2 \leq \int_x^{\frac{a+b}{2}} \left[f'(a+b-t) - f'(t)\right] dt \leq M\left(\frac{a+b}{2} - x\right)^2.$$

By equality (2.5),

$$m\left(\frac{a+b}{2} - x\right)^2 \leq f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right) \leq M\left(\frac{a+b}{2} - x\right)^2. \tag{2.8}$$

Multiplying inequality (2.8) by $\frac{2^{\alpha-1}\alpha(x-a)^{\alpha-1}}{(b-a)^{\alpha}}$ and integrating the resultant inequality with respect to x on $\left[a, \frac{a+b}{2}\right]$, we establish

$$\frac{m2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right)^2 (x-a)^{\alpha-1} dx$$

$$\leq \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_a^{\frac{a+b}{2}} \left[f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right)\right] (x-a)^{\alpha-1} dx$$

$$\leq \frac{M2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right)^2 (x-a)^{\alpha-1} dx.$$

By using equality (2.4) and

$$\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-x\right)^2 (x-a)^{\alpha-1}\,dx = \left(\frac{b-a}{2}\right)^{\alpha+2} \frac{2}{\alpha(\alpha+1)(\alpha+2)},$$

then we obtain

$$\begin{aligned}\frac{m(b-a)^2}{4(\alpha+1)(\alpha+2)} &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)\right]-f\left(\frac{a+b}{2}\right)\\ &\leq \frac{M(b-a)^2}{4(\alpha+1)(\alpha+2)}\end{aligned}$$

which completes the proof of inequalities (2.1).

On the other hand, by equality (2.3), we have

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)\right]\\ &=\frac{f(a)+f(b)}{2}-\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)\right](x-a)^{\alpha-1}dx\\ &=\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}}\left[f(a)+f(b)-f(x)-f(a+b-x)\right](x-a)^{\alpha-1}dx. \qquad (2.9)\end{aligned}$$

By using the equalities

$$f(x)-f(a)=\int_a^x f'(t)dt$$

and

$$f(b)-f(a+b-x)=\int_{a+b-x}^b f'(t)dt,$$

we get

$$\begin{aligned}f(a)+f(b)-f(x)-f(a+b-x) &= \int_{a+b-x}^b f'(t)dt-\int_a^x f'(t)dt\\ &= \int_a^x f'(a+b-u)du-\int_a^x f'(t)dt\\ &= \int_a^x \left[f'(a+b-t)dt-f'(t)\right]dt. \qquad (2.10)\end{aligned}$$

By integrating inequality (2.7) with respect to t on $[a,x]$, we get

$$m\int_a^x (a+b-2t)\,dt \le \int_a^x \left[f'(a+b-t)-f'(t)\right]dt \le M\int_a^x (a+b-2t)\,dt.$$

That is,

$$m(x-a)(b-x) \le \quad f(a)+f(b)-f(x)-f(a+b-x) \le M(x-a)(b-x). \quad (2.11)$$

Multiplying inequality (2.11) by $\frac{2^{\alpha-1}\alpha(x-a)^{\alpha-1}}{(b-a)^{\alpha}}$ and integrating the resultant inequality with respect to x on $\left[a,\frac{a+b}{2}\right]$, we establish

$$\begin{aligned}
&\frac{m2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}} (b-x)\,(x-a)^{\alpha-1}\,dx \\
&\le \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}} \left[f(a)+f(b)-f(x)-f(a+b-x)\right](x-a)^{\alpha-1}\,dx \\
&\le \frac{M2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}} (b-x)\,(x-a)^{\alpha-1}\,dx,
\end{aligned}$$

i.e.

$$\begin{aligned}
\frac{m(b-a)^2\alpha(\alpha+3)}{8(\alpha+1)(\alpha+2)} &\le \frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)\right] \\
&\le \frac{M(b-a)^2\alpha(\alpha+3)}{8(\alpha+1)(\alpha+2)}.
\end{aligned}$$

which gives inequalities (2.2).

This completes the proof of the theorem. □

Now we give the following refinement of inequality (1.3).

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be positive, twice differentiable functions with $a<b$ and $f\in L_1[a,b]$. If $f'(a+b-x)\ge f'(x)$ for all $x\in\left[a,\frac{a+b}{2}\right]$, then we have the inequalities*

$$f\left(\frac{a+b}{2}\right) \le \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)\right] \le \frac{f(a)+f(b)}{2}. \quad (2.12)$$

Proof. Since $f'(a+b-x)\ge f'(x)$ for all $x\in\left[a,\frac{a+b}{2}\right]$, by equalities (2.4) and (2.5), we have

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)\right] - f\left(\frac{a+b}{2}\right)$$

$$= \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right) \right] (x-a)^{\alpha-1}\, dx$$

$$= \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[\int_{x}^{\frac{a+b}{2}} \left[f'(a+b-t) - f'(t) \right] dt \right] (x-a)^{\alpha-1}\, dx$$

$$\geq 0$$

which gives the first inequality in (2.12).

Similarly, by equalities (2.9) and (2.10), we get

$$\frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a) \right]$$

$$= \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[f(a) + f(b) - f(x) - f(a+b-x) \right] (x-a)^{\alpha-1} dx$$

$$= \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[\int_{a}^{x} \left[f'(a+b-t)dt - f'(t) \right] dt \right] (x-a)^{\alpha-1} dx$$

$$\geq 0.$$

This completes the proof. □

Now, we establish the following inequalities which give the above and below bounds for the left and right hand sides of inequality (1.4).

Theorem 7. *Let $f : [a,b] \to \mathbb{R}$ be positive, twice differentiable functions with $a < b$ and $f \in L_1[a,b]$. If f'' is bounded, i.e. $m \leq f''(t) \leq M$, $t \in [a,b]$, $m, M \in \mathbb{R}$, then we have the inequalities*

$$\frac{m(b-a)^2\alpha}{8(\alpha+2)} \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{a^+} f\left(\frac{a+b}{2}\right) + J^{\alpha}_{b^-} f\left(\frac{a+b}{2}\right) \right] - f\left(\frac{a+b}{2}\right)$$
$$\leq \frac{m(b-a)^2\alpha}{8(\alpha+2)} \tag{2.13}$$

and

$$\frac{m(b-a)^2}{4(a+2)} \leq \frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{a^+} f\left(\frac{a+b}{2}\right) + J^{\alpha}_{b^-} f\left(\frac{a+b}{2}\right) \right]$$
$$\leq \frac{m(b-a)^2}{4(a+2)}. \tag{2.14}$$

Proof. From the definition of Riemann-Liouville fractional integrals, we get

$$\begin{aligned}
&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a^+}f\left(\frac{a+b}{2}\right)+J_{b^-}f\left(\frac{a+b}{2}\right)\right]\\
&=\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{\alpha-1}f(x)dx+\int_{\frac{a+b}{2}}^{b}\left(x-\frac{a+b}{2}\right)^{\alpha-1}f(x)dx\\
&=\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}}[f(x)+f(a+b-x)]\left(\frac{a+b}{2}-x\right)^{\alpha-1}dx. \qquad (2.15)
\end{aligned}$$

By equality (2.15), we have

$$\begin{aligned}
&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f\left(\frac{a+b}{2}\right)+J^{\alpha}_{b^-}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\\
&=\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}}[f(x)+f(a+b-x)]\left(\frac{a+b}{2}-x\right)^{\alpha-1}dx-f\left(\frac{a+b}{2}\right)\\
&=\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right]\left(\frac{a+b}{2}-x\right)^{\alpha-1}dx. \qquad (2.16)
\end{aligned}$$

Moreover, using the identities

$$f\left(\frac{a+b}{2}\right)-f(x)=\int_x^{\frac{a+b}{2}}f'(t)dt$$

and

$$f(a+b-x)-f\left(\frac{a+b}{2}\right)=\int_{\frac{a+b}{2}}^{a+b-x}f'(t)dt,$$

we obtain

$$\begin{aligned}
f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)&=\int_{\frac{a+b}{2}}^{a+b-x}f'(t)dt-\int_x^{\frac{a+b}{2}}f'(t)dt\\
&=\int_x^{\frac{a+b}{2}}f'(a+b-u)du-\int_x^{\frac{a+b}{2}}f'(t)\,dt
\end{aligned}$$

$$= \int_{x}^{\frac{a+b}{2}} \left[f'(a+b-t) - f'(t)\right] dt. \tag{2.17}$$

By integrating inequality (2.7) with respect to t on $\left[x, \frac{a+b}{2}\right]$, we get

$$m\left(\frac{a+b}{2}-x\right)^2 \leq \int_{x}^{\frac{a+b}{2}} \left[f'(a+b-t) - f'(t)\right] dt \leq M\left(\frac{a+b}{2}-x\right)^2.$$

That is, by equality (2.17),

$$m\left(\frac{a+b}{2}-x\right)^2 \leq f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right) \leq M\left(\frac{a+b}{2}-x\right)^2. \tag{2.18}$$

Multiplying inequality (2.18) by $\frac{2^{\alpha-1}\alpha(x-a)^{\alpha-1}}{(b-a)^{\alpha}}$ and integrating the resultant inequality with respect to x on $\left[a, \frac{a+b}{2}\right]$, we establish

$$\begin{aligned}
&\frac{m2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left(\frac{a+b}{2}-x\right)^{\alpha+1} dx \\
&\leq \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right]\left(\frac{a+b}{2}-x\right)^{\alpha-1} dx \\
&\leq \frac{M2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left(\frac{a+b}{2}-x\right)^{\alpha+1} dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{m(b-a)^2\alpha}{8(\alpha+2)} &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f\left(\frac{a+b}{2}\right)+J_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right) \\
&\leq \frac{m(b-a)^2\alpha}{8(\alpha+2)}
\end{aligned}$$

which proves inequality (2.13).

On the other hand, using identity (2.15), we get

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f\left(\frac{a+b}{2}\right)+J_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right)\right] \\
&= \frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[f(x)+f(a+b-x)\right]\left(\frac{a+b}{2}-x\right)^{\alpha-1} dx
\end{aligned}$$

$$= \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} [f(a)+f(b)-f(x)-f(a+b-x)] \left(\frac{a+b}{2}-x\right)^{\alpha-1} dx. \tag{2.19}$$

Using the facts that

$$f(x)-f(a) = \int_{a}^{x} f'(t)dt$$

and

$$f(b)-f(a+b-x) = \int_{a+b-x}^{b} f'(t)dt,$$

we get

$$\begin{aligned} f(a)+f(b)-f(x)-f(a+b-x) &= \int_{a+b-x}^{b} f'(t)dt - \int_{a}^{x} f'(t)dt \\ &= \int_{a}^{x} f'(a+b-u)du - \int_{a}^{x} f'(t)dt \\ &= \int_{a}^{x} \left[f'(a+b-t)dt - f'(t)\right] dt. \end{aligned} \tag{2.20}$$

By integrating inequality (2.7) with respect to t on $[a,x]$, we get

$$\begin{aligned} m\left[\left(\frac{b-a}{2}\right)^2 - \left(\frac{a+b}{2}-x\right)^2\right] &\leq f(a)+f(b)-f(x)-f(a+b-x) \\ &\leq M\left[\left(\frac{b-a}{2}\right)^2 - \left(\frac{a+b}{2}-x\right)^2\right]. \end{aligned} \tag{2.21}$$

Multiplying inequality (2.21) by $\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\left(\frac{a+b}{2}-x\right)^{\alpha-1}$ and integrating the resultant inequality with respect to x on $\left[a, \frac{a+b}{2}\right]$, we establish

$$\begin{aligned} &\frac{m2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[\left(\frac{b-a}{2}\right)^2 - \left(\frac{a+b}{2}-x\right)^2\right] \left(\frac{a+b}{2}-x\right)^{\alpha-1} dx \\ &\leq \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} [f(a)+f(b)-f(x)-f(a+b-x)] \left(\frac{a+b}{2}-x\right)^{\alpha-1} dx \end{aligned}$$

$$\leq \frac{M2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[\left(\frac{b-a}{2} \right)^{2} - \left(\frac{a+b}{2} - x \right)^{2} \right] \left(\frac{a+b}{2} - x \right)^{\alpha-1}.$$

By equality (2.19) and equality

$$\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[\left(\frac{b-a}{2} \right)^{2} - \left(\frac{a+b}{2} - x \right)^{2} \right] \left(\frac{a+b}{2} - x \right)^{\alpha-1} dx = \frac{(b-a)^{2}}{4(a+2)},$$

we get the desired inequality (2.14).

This completes the proof. □

Theorem 8. *Let $f : [a,b] \to \mathbb{R}$ be positive, twice differentiable functions with $a < b$ and $f \in L_1[a,b]$. If $f'(a+b-x) \geq f'(x)$ for all $x \in \left[a, \frac{a+b}{2}\right]$, then we have the inequalities*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right) \right] \leq \frac{f(a)+f(b)}{2}. \tag{2.22}$$

Proof. From equalities (2.16) and (2.17), we have

$$\begin{aligned}
&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right) \right] - f\left(\frac{a+b}{2}\right) \\
&= \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right) \right] \left(\frac{a+b}{2} - x \right)^{\alpha-1} dx \\
&= \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[\int_{x}^{\frac{a+b}{2}} \left[f'(a+b-t) - f'(t) \right] dt \right] \left(\frac{a+b}{2} - x \right)^{\alpha-1} dx \\
&\geq 0
\end{aligned}$$

which proves the first inequality in (2.22).

Similarly, by equalities (2.19) and (2.20)

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right) \right] \\
&= \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[f(a) + f(b) - f(x) - f(a+b-x) \right] \left(\frac{a+b}{2} - x \right)^{\alpha-1} dx \\
&= \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[\int_{a}^{x} \left[f'(a+b-t)dt - f'(t) \right] dt \right] \left(\frac{a+b}{2} - x \right)^{\alpha-1} dx
\end{aligned}$$

$$\geq 0.$$

This completes the proof. □

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CYCLIC REFINEMENTS OF THE DISCRETE HÖLDER'S INEQUALITY WITH APPLICATIONS

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Abstract. In this paper we obtain refinements of the discrete Hölder's and Minkowski's inequalities for finite and infinite sequences by using cyclic refinements of the discrete Jensen's inequality.

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1. Introduction

One of the most important inequalities concerning convex functions is the following

Theorem 1 (Discrete Jensen's inequality, [5])**.** *Let C be a convex subset of a real vector space V, and let $f : C \to \mathbb{R}$ be a convex function. If $p_1, \dots, p_n$ are nonnegative numbers with $\sum_{i=1}^{n} p_i = 1$, and $v_1, \dots, v_n \in C$, then*

$$f\left(\sum_{i=1}^{n} p_i v_i\right) \leq \sum_{i=1}^{n} p_i f(v_i).$$

A number of attempts have been made to refine this inequality (see the book [5] and the references therein).

The following cyclic refinement of the discrete Jensen's inequality is a special case of Theorem 2.1 in the recent paper [6] (see also [1]). To give the result we need the following hypotheses:

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(H_1) Let $2 \le k \le n$ be integers, let $\lambda_1, \dots, \lambda_k$ represent a positive probability distribution, and let $p_1, \dots, p_n$ be positive numbers with $P_n := \sum_{i=1}^{n} p_i$.

(H_2) Let $I \subset \mathbb{R}$ be an interval, and $f : I \to \mathbb{R}$ be a convex function.

Theorem 2 (Theorem 2.1 in [6])**.** *Assume (H_1) and (H_2). If $v_1, \dots, v_n \in I$, then*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i v_i\right) \le C_{dis} = C_{dis}\left(f, \mathbf{v}, \mathbf{p}, \lambda\right)$$

$$:= \frac{1}{P_n}\sum_{i=1}^{n}\left(\sum_{j=0}^{k-1}\lambda_{j+1}p_{i+j}\right) f\left(\frac{\sum_{j=0}^{k-1}\lambda_{j+1}p_{i+j}v_{i+j}}{\sum_{j=0}^{k-1}\lambda_{j+1}p_{i+j}}\right) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(v_i\right) \tag{1.1}$$

where $i+j$ means $i+j-n$ in case of $i+j>n$.

In this paper we first obtain cyclic refinements of the discrete Hölder's inequality by using the previous assertion. Then we give some refinements of the discrete Hölder's inequality for infinite sequences. There are a lot of papers dealing with similar refinements (see e.g. [2–4, 7] and [8]). Our results fit well into the topic of refinements of inequalities corresponding to convex functions, and they give a new approach to have such refinements. Finally, we demonstrate the applicability of our results by means of some new cyclic refinements of the Minkowski's inequality.

2. Main results

Let $2 \le k \le n$, and let $i \in \{1, \dots, n\}$ and $j \in \{0, \dots, k-1\}$. In further parts of the paper $i+j$ always means $i+j-n$ in case of $i+j>n$.

In the first result we give cyclic refinements of the discrete Hölder's inequality.

Theorem 3. *Let $2 \le k \le n$ be integers, and let $\lambda_1, \dots, \lambda_k$ represent a positive probability distribution. Let $(w_i)_{i=1}^n$ be a sequence of positive numbers, $(x_i)_{i=1}^n$ be a sequence of nonnegative numbers, and let $p>0$ and $q \in \mathbb{R} \cup \{\infty\}$ be conjugate exponents that is $\frac{1}{p}+\frac{1}{q}=1$ (if $p=1$, we define its conjugate exponent to be $q=\infty$).*
(a_1) If $(y_i)_{i=1}^n$ is a sequence of nonnegative numbers and $p>1$, then

$$\begin{aligned}\sum_{i=1}^{n} w_i x_i y_i &\le \sum_{i=1}^{n}\left(\sum_{j=0}^{k-1}\lambda_{j+1}w_{i+j}x_{i+j}^p\right)^{\frac{1}{p}}\left(\sum_{j=0}^{k-1}\lambda_{j+1}w_{i+j}y_{i+j}^q\right)^{\frac{1}{q}} \\ &\le \left(\sum_{i=1}^{n} w_i x_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n} w_i y_i^q\right)^{\frac{1}{q}}.\end{aligned} \tag{2.1}$$

(a_2) If $(y_i)_{i=1}^n$ is a sequence of nonnegative numbers, then

$$\sum_{i=1}^n w_i x_i y_i \le \sum_{i=1}^n \left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} x_{i+j} \right) \cdot \max_{j\in\{0,\dots,k-1\}} y_{i+j}$$
$$\le \left(\sum_{i=1}^n w_i x_i \right) \cdot \max_{i\in\{1,\dots,n\}} y_i.$$

(a_3) If $(y_i)_{i=1}^n$ is a sequence of positive numbers and $0 < p < 1$, then the reverse inequalities hold in (2.1).

(b_1) If $(y_i)_{i=1}^n$ is a sequence of nonnegative numbers and If $p > 1$, then

$$\sum_{i=1}^n w_i x_i y_i \le \left(\sum_{i=1}^n w_i y_i^q \right)^{\frac{1}{q}} \left(\sum_{i=1}^n \left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j}^q \right)^{1-p} \left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j} x_{i+j} \right)^p \right)^{\frac{1}{p}}$$
$$\le \left(\sum_{i=1}^n w_i x_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n w_i y_i^q \right)^{\frac{1}{q}}. \tag{2.2}$$

If $y_i = y_{i+1} = \ldots = y_{i+k-1} = 0$ for some $i \in \{1,\dots,n\}$, then

$$\left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j}^q \right)^{1-p} \left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j} x_{i+j} \right)^p$$

means

$$\left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} \right)^{1-p} \left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} x_{i+j} \right)^p.$$

(b_2) If $(y_i)_{i=1}^n$ is a sequence of nonnegative numbers, then

$$\sum_{i=1}^n w_i x_i y_i \le \max_{i\in\{1,\dots,n\}} y_i \cdot \left(\sum_{i=1}^n \frac{\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j} x_{i+j}}{\max_{j\in\{0,\dots,k-1\}} y_{i+j}} \right)$$
$$\le \left(\sum_{i=1}^n w_i x_i \right) \cdot \max_{i\in\{1,\dots,n\}} y_i.$$

If $y_i = y_{i+1} = \ldots = y_{i+k-1} = 0$ for some $i \in \{1,\dots,n\}$, then

$$\frac{\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j} x_{i+j}}{\max_{j\in\{0,\dots,k-1\}} y_{i+j}}$$

means

$$\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} x_{i+j}.$$

(b_3) If $(y_i)_{i=1}^n$ is a sequence of positive numbers and $0 < p < 1$, then the reverse inequalities hold in (2.2).

Proof. Consider the power functions

$$f_s :]0,\infty[\to \mathbb{R}, \quad f_s(x) = x^s, \quad s \neq 0,1. \tag{2.3}$$

It is well known that f_s is strictly convex for each $s \in]-\infty,0[\cup]1,\infty[$, and strictly concave for each $s \in]0,1[$. By applying Theorem 2 to the function f_s and to the positive numbers $v_1,\dots,v_n$, we obtain that

$$\left(\frac{1}{P_n}\sum_{i=1}^n p_i v_i\right)^s \le \frac{1}{P_n}\sum_{i=1}^n \left(\sum_{j=0}^{k-1}\lambda_{j+1}p_{i+j}\right)^{1-s}\left(\sum_{j=0}^{k-1}\lambda_{j+1}p_{i+j}v_{i+j}\right)^s \le \frac{1}{P_n}\sum_{i=1}^n p_i v_i^s, \tag{2.4}$$

if $s \in]-\infty,0[\cup]1,\infty[$, and the reverse inequality holds if $s \in]0,1[$.
(a_1) Assume first that x_i, y_i $(i = 1,\dots,n)$ are positive numbers. By using the substitutions

$$s = \frac{1}{p}, \quad p_i = w_i y_i^q, \quad v_i = x_i^p y_i^{-q}$$

in (2.4), and by taking into account that $f_{1/p}$ is concave, we obtain

$$\left(\frac{1}{\sum_{i=1}^n w_i y_i^q}\right)^{\frac{1}{p}}\left(\sum_{i=1}^n w_i x_i^p\right)^{\frac{1}{p}} \ge \frac{1}{\sum_{i=1}^n w_i y_i^q}\cdot\sum_{i=1}^n\left(\sum_{j=0}^{k-1}\lambda_{j+1}w_{i+j}y_{i+j}^q\right)\cdot\left(\frac{\sum_{j=0}^{k-1}\lambda_{j+1}w_{i+j}x_{i+j}^p}{\sum_{j=0}^{k-1}\lambda_{j+1}w_{i+j}y_{i+j}^q}\right)^{\frac{1}{p}}$$
$$\ge \frac{1}{\sum_{i=1}^n w_i y_i^q}\sum_{i=1}^n w_i x_i y_i,$$

and this is equivalent to (2.1).

In the general case let $\varepsilon > 0$, apply the proved inequality to the positive sequences $(x_i+\varepsilon)_{i=1}^n$ and $(y_i+\varepsilon)_{i=1}^n$, and then take the limit as $\varepsilon \to 0+$.
(a_2) The inequality can be obtained by taking the limit as $q \to \infty$ in (2.1).
(a_3) The reverse inequalities in (2.1) can be proved exactly as in (a_1) by considering that $f_{1/p}$ is convex.
(b_1) and (b_2) can be proved similarly to (a_1) and (a_2) by using the substitutions

$$s = p, \quad p_i = w_i y_i^q, \quad v_i = x_i y_i^{1-q}$$

in (2.4).
(b_3) The reverse inequalities in (2.2) can be proved exactly as in (b_1) by considering that the function f_p is concave.

The proof is complete. □

The following new inequalities follow from the previous result.

Corollary 1. *Let $2 \le k \le n$ be integers, and let $\lambda_1, \dots, \lambda_k$ represent a positive probability distribution. Let $(w_i)_{i=1}^n$ and $(y_i)_{i=1}^n$ be sequences of positive numbers, $(x_i)_{i=1}^n$ be a sequence of nonnegative numbers, and let $p > 0$ and $q \in \mathbb{R} \cup \{\infty\}$ be conjugate exponents.*
(a) If $p > 1$, then

$$\sum_{i=1}^{n} \left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j}^{q} \right)^{1-p} \left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j} x_{i+j} \right)^{p} \le \sum_{i=1}^{n} w_i x_i^p,$$

while if $0 < p < 1$, then the reverse inequality holds.
(b) If $p = 1$, then

$$\sum_{i=1}^{n} \frac{\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j} x_{i+j}}{\max_{j \in \{0,\dots,k-1\}} y_{i+j}} \le \sum_{i=1}^{n} w_i x_i.$$

Proof. Theorem 3 (b_1), (b_2) and (b_3) imply these inequalities. □

As an immediate consequence of Theorem 3 we obtain cyclic refinements of the Cauchy-Schwarz-Bunyakovszkij inequality.

Corollary 2. *Let $2 \le k \le n$ be integers, let $\lambda_1, \dots, \lambda_k$ represent a positive probability distribution, and let $(x_i)_{i=1}^n$ and $(y_i)_{i=1}^n$ be sequences of real numbers. Then*
(a)

$$\begin{aligned} \left| \sum_{i=1}^{n} w_i x_i y_i \right| &\le \sum_{i=1}^{n} \left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} x_{i+j}^2 \right)^{\frac{1}{2}} \left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j}^2 \right)^{\frac{1}{2}} \\ &\le \left(\sum_{i=1}^{n} w_i x_i^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} w_i y_i^2 \right)^{\frac{1}{2}}. \end{aligned}$$

(b)

$$\begin{aligned} \left| \sum_{i=1}^{n} w_i x_i y_i \right| &\le \left(\sum_{i=1}^{n} w_i y_i^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \frac{\left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} \left| y_{i+j} \right| \left| x_{i+j} \right| \right)^2}{\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j}^2} \right)^{\frac{1}{2}} \\ &\le \left(\sum_{i=1}^{n} w_i x_i^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} w_i y_i^2 \right)^{\frac{1}{2}}. \end{aligned}$$

If $y_i = y_{i+1} = \ldots = y_{i+k-1} = 0$ *for some* $i \in \{1,\ldots,n\}$, *then*

$$\frac{\left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} \left|y_{i+j}\right| \left|x_{i+j}\right|\right)^2}{\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j}^2}$$

means

$$\frac{\left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} \left|x_{i+j}\right|\right)^2}{\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j}}.$$

Proof. Choose $p = 2$ in Theorem 3 (a_1) and (b_1). □

By Theorem 3, some refinements of the Hölder's inequality for infinite sequences can be obtained. We have to be very careful, because a change in n causes a corresponding change in $i+j$, that is $i+j$ depends on n. We illustrate this by the generalization of inequality (2.1).

Theorem 4. *Let* $k \geq 2$ *be an integer, and let* $\lambda_1,\ldots,\lambda_k$ *represent a positive probability distribution. Let* $(w_i)_{i=1}^{\infty}$ *be a sequence of positive numbers, and let* $(x_i)_{i=1}^{\infty}$ *and* $(y_i)_{i=1}^{\infty}$ *be sequences of nonnegative numbers such that*

$$\sum_{i=1}^{\infty} w_i x_i^p < \infty, \quad \sum_{i=1}^{\infty} w_i y_i^q < \infty,$$

where $p > 0$ *and* $q \in \mathbb{R} \cup \{\infty\}$ *are conjugate exponents.*
(a) If $p > 1$, *then*

$$\begin{aligned}\sum_{i=1}^{\infty} w_i x_i y_i &\leq \sum_{i=1}^{\infty} \left(\sum_{l=i}^{i+k-1} \lambda_{l-i+1} w_l x_l^p\right)^{\frac{1}{p}} \left(\sum_{l=i}^{i+k-1} \lambda_{l-i+1} w_l y_l^q\right)^{\frac{1}{q}} \\ &\quad + \sum_{m=1}^{k-1} \left(\sum_{j=k-m}^{k-1} \lambda_{j+1} w_{m-k+1+j} x_{m-k+1+j}^p\right)^{\frac{1}{p}} \left(\sum_{j=k-m}^{k-1} \lambda_{j+1} w_{m-k+1+j} y_{m-k+1+j}^q\right)^{\frac{1}{q}} \\ &\leq \left(\sum_{i=1}^{\infty} w_i x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} w_i y_i^q\right)^{\frac{1}{q}}. \end{aligned} \tag{2.5}$$

(b) If $y_i > 0$ $(i = 1,2,\ldots)$ *and* $0 < p < 1$, *then the reverse inequalities hold in (2.5).*

Proof. (a) It follows from Theorem 3 (a_1) that for each fixed $n \geq k$

$$\sum_{i=1}^{n} w_i x_i y_i \leq \sum_{i=1}^{n} \left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} x_{i+j}^p\right)^{\frac{1}{p}} \left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j}^q\right)^{\frac{1}{q}}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n-k+1} \left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} x_{i+j}^p \right)^{\frac{1}{p}} \left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j}^q \right)^{\frac{1}{q}} \\
&\quad + \sum_{i=n-k+2}^{n} \left(\sum_{j=0}^{n-i} \lambda_{j+1} w_{i+j} x_{i+j}^p + \sum_{j=n-i+1}^{k-1} \lambda_{j+1} w_{i+j} x_{i+j}^p \right)^{\frac{1}{p}} \\
&\quad \cdot \left(\sum_{j=0}^{n-i} \lambda_{j+1} w_{i+j} y_{i+j}^q + \sum_{j=n-i+1}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j}^q \right)^{\frac{1}{q}} \\
&= \sum_{i=1}^{n-k+1} \left(\sum_{l=i}^{i+k-1} \lambda_{l-i+1} w_l x_l^p \right)^{\frac{1}{p}} \left(\sum_{l=i}^{i+k-1} \lambda_{l-i+1} w_l y_l^q \right)^{\frac{1}{q}} \\
&\quad + \sum_{m=1}^{k-1} \left(\sum_{j=0}^{k-m-1} \lambda_{j+1} w_{m+n-k+1+j} x_{m+n-k+1+j}^p + \sum_{j=k-m}^{k-1} \lambda_{j+1} w_{m-k+1+j} x_{m-k+1+j}^p \right)^{\frac{1}{p}} \\
&\quad \cdot \left(\sum_{j=0}^{k-m-1} \lambda_{j+1} w_{m+n-k+1+j} y_{m+n-k+1+j}^q + \sum_{j=k-m}^{k-1} \lambda_{j+1} w_{m-k+1+j} y_{m-k+1+j}^q \right)^{\frac{1}{q}}.
\end{aligned}
$$

Now, as n tends to infinity, we obtain the result.
(b) We can prove as in (a) by using Theorem 3 (a_3).
The proof is complete. □

3. Applications

In the first result a cyclic refinement of the Minkowski's inequality is given.

Theorem 5. *Let $2 \le k \le n$ be integers, let $\lambda_1, \dots, \lambda_k$ represent a positive probability distribution, and let $(x_i)_{i=1}^n$ and $(y_i)_{i=1}^n$ be sequences of real numbers. If $p > 1$ and $q > 1$ are conjugate exponents, and there exists $i \in \{1, \dots, n\}$ such that $x_i + y_i \neq 0$, then*

$$
\begin{aligned}
\left(\sum_{i=1}^{n} w_i \left| x_i + y_i \right|^p \right)^{\frac{1}{p}} &\le \frac{1}{\left(\sum_{i=1}^{n} w_i \left| x_i + y_i \right|^p \right)^{\frac{1}{q}}} \sum_{i=1}^{n} \left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} \left| x_{i+j} + y_{i+j} \right|^p \right)^{\frac{1}{q}} \\
&\quad \cdot \left(\left(\sum_{j=0}^{k-1} \lambda_{i+j} w_{i+j} \left| x_{i+j} \right|^p \right)^{\frac{1}{p}} + \left(\sum_{j=0}^{k-1} \lambda_{i+j} w_{i+j} \left| y_{i+j} \right|^p \right)^{\frac{1}{p}} \right) \\
&\le \left(\sum_{i=1}^{n} w_i \left| x_i \right|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} w_i \left| y_i \right|^p \right)^{\frac{1}{p}}.
\end{aligned}
$$

Proof. A similar argument as in the proof of the classical Minkowski's inequality can be applied.

By Theorem 3 (a_1),

$$\begin{aligned}
&\sum_{i=1}^{n} w_i \left|x_i+y_i\right|^p \le \sum_{i=1}^{n} w_i \left|x_i\right| \left|x_i+y_i\right|^{p-1} + \sum_{i=1}^{n} w_i \left|y_i\right| \left|x_i+y_i\right|^{p-1} \\
&\le \sum_{i=1}^{n} \left(\sum_{j=0}^{k-1} \lambda_{i+j} w_{i+j} \left|x_{i+j}\right|^p\right)^{\frac{1}{p}} \left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} \left|x_{i+j}+y_{i+j}\right|^p\right)^{\frac{1}{q}} \\
&\quad + \sum_{i=1}^{n} \left(\sum_{j=0}^{k-1} \lambda_{i+j} w_{i+j} \left|y_{i+j}\right|^p\right)^{\frac{1}{p}} \left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} \left|x_{i+j}+y_{i+j}\right|^p\right)^{\frac{1}{q}} \\
&\le \left(\sum_{i=1}^{n} w_i \left|x_i\right|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} w_i \left|x_i+y_i\right|^p\right)^{\frac{1}{q}} + \left(\sum_{i=1}^{n} w_i \left|y_i\right|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} w_i \left|x_i+y_i\right|^p\right)^{\frac{1}{q}},
\end{aligned}$$

and this implies the result.

The proof is complete. □

Remark 1. By using Theorem 3 (b_1), we can obtain another cyclic refinement of the Minkowski's inequality.

Related to Theorem 4 we have the following cyclic refinement of the Minkowski's inequality for infinite sums. For simplicity and transparency we consider only the case $k = 2$.

Theorem 6. *Let λ_1 and λ_2 be positive numbers with $\lambda_1 + \lambda_2 = 1$. Let $(w_i)_{i=1}^{\infty}$ be a sequence of positive numbers, and let $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ be sequences of real numbers such that*

$$\sum_{i=1}^{\infty} w_i \left|x_i\right|^p < \infty, \quad \sum_{i=1}^{\infty} w_i \left|y_i\right|^q < \infty,$$

where $p > 1$ and $q > 1$ are conjugate exponents. If there exists $i \in \{1, 2, \ldots\}$ such that $x_i + y_i \neq 0$, then

$$\begin{aligned}
\left(\sum_{i=1}^{\infty} w_i \left|x_i+y_i\right|^p\right)^{\frac{1}{p}} &\le \frac{1}{\left(\sum_{i=1}^{\infty} w_i \left|x_i+y_i\right|^p\right)^{\frac{1}{q}}} \cdot \left(\sum_{i=1}^{\infty} (\lambda_1 w_i \left|x_i+y_i\right|^p + \lambda_2 w_{i+1} \left|x_{i+1}+y_{i+1}\right|^p)^{\frac{1}{q}}\right. \\
&\quad \left((\lambda_1 w_i \left|x_i\right|^p + \lambda_2 w_{i+1} \left|x_{i+1}\right|^p)^{\frac{1}{p}} + (\lambda_1 w_i \left|y_i\right|^p + \lambda_2 w_{i+1} \left|y_{i+1}\right|^p)^{\frac{1}{p}}\right) \\
&\quad \left. + (\lambda_2 w_1 \left|x_1+y_1\right|^p)^{\frac{1}{q}} \left((\lambda_2 w_1 \left|x_1\right|^p)^{\frac{1}{p}} + (\lambda_2 w_1 \left|y_1\right|^p)^{\frac{1}{p}}\right)\right) \\
&\le \left(\sum_{i=1}^{\infty} w_i \left|x_i\right|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} w_i \left|y_i\right|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Proof. We can follow the proof of Theorem 5, by using Theorem 4 instead of Theorem 3 (a_1). □

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SOME HERMITE-JENSEN-MERCER LIKE INEQUALITIES FOR CONVEX FUNCTIONS THROUGH A CERTAIN GENERALIZED FRACTIONAL INTEGRALS AND RELATED RESULTS

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Abstract. In this article, in light of Jensen-Mercer inequality for functions whose derivatives in the absolute values are convex, some new Hermite-Jensen-Mercer inequalities have been obtained with the help of generalized types of fractional integral operators generated recently by specified local derivatives.

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1. Introduction

Convex functions, a function class that has a very useful structure in terms of both definition and features, are a unique and important part of the theory of inequality. This function class has reinforced its importance with its use in inequality theory studies and has been used in various application areas by sourcing new types of inequality. Jensen inequality, which is one of the cornerstones of the theory of inequality, has been the focus of many researchers and many articles on different versions of this famous inequality have been brought into the literature. Jensen-Mercer inequality, a new variant of Jensen inequality introduced for convex functions, is a well known inequality proved for convex functions in theory. A famous inequality proved by using convex functions is called Jensen's inequality and is given as follows. Assume that $0 < x_1 \leq x_2 \leq ... \leq x_n$ and $\mu = \left(\mu_1, \mu_2, ..., \mu_n\right)$ are non-negative weights such that $\sum_{k=1}^{n} \mu_k = 1$. The celebrated Jensen inequality $\left(\text{see [16]}\right)$, in the literature states that τ is convex functions on the interval $\left[\theta, \vartheta\right]$, then

$$\tau\left(\sum_{k=1}^{n} \mu_k \, x_k\right) \leq \left(\sum_{k=1}^{n} \mu_k \, \tau\left(x_k\right)\right) \tag{1.1}$$

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$\forall\, x_k \in [\theta, \vartheta]$ *and all* $\mu_k \in [0,1]$, $(k = 1,2,...,n)$.

Now we will remind another important inequality that is obtained by using convex functions. Hermite-Hadamard inequality, which gives the upper and lower limits of the average value of a convex function depending on the averages of real numbers, has been the focus of researchers for nearly a hundred years with its applications in inequality theory, numerical analysis and applied mathematics. Numerous generalizations, expansions, new variants and improvements have been achieved regarding this inequality.

The following statement;

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

holds and known as Hermite-Hadamard inequality. Both inequalities hold in the reversed direction if f is concave.

There have been many studies on the Jensen inequality, and one of the most notable studies is the Jensen-Mercer inequality, a new version of the Jensen inequality given by Mercer in [17]. Later, inspired by this impressive work, Matkovic et al. Jensen-Mercer generalized the inequality to operators and enriched them with applications (see [16]). Over the years, the Jensen-Mercer inequality has attracted the attention of many researchers, and many studies have been conducted in different ways such as taking this inequality to a higher dimension, obtaining it for superquadratic functions, reverse Jensen-Mercer inequality and finding various generalizations (see the papers [1–5, 9, 10, 13, 19, 20, 22]).

Theorem 1 (see [16]). *If* τ *is a convex function on* $[\theta, \vartheta]$, *then*

$$\tau\left(\theta+\vartheta-\sum_{k=1}^{n}\mu_k\, x_k\right) \leq \tau(\theta)+\tau(\vartheta)-\sum_{k=1}^{n}\mu_k\,\tau(x_k), \tag{1.2}$$

$\forall\, x_k \in [\theta, \vartheta]$ *and all* $\mu_k \in [0,1]$, $(k = 1,2,...,n)$.

We will proceed with some concepts that will be useful for our findings. Consider the definitions of the Euler Gamma function $\Gamma(.)$ and Beta function $B(.\ ,.)$, respectively:

$$\Gamma(\theta) = \int_0^\infty e^{-\lambda}\lambda^{\theta-1}d\lambda,$$

$$B(r,s) = \int_0^1 \lambda^{r-1}\ (1-\lambda)^{s-1}\,d\lambda.$$

The solution of some differential equations where the definition of classical derivatives was insufficient remained a problem that mathematicians wondered for years. This curiosity has also pushed mathematicians into a new quest for the real-world applications where classical analysis is desperate. As a result of this search, fractional derivative and integral operators have been identified and studies in fractional analysis

have gained speed. Today, many different fractional derivative and integral operator definitions and properties are given. The functionality of these operators was revealed by making various comparisons by examining the singularity, locality and some other features of each operator. Some of these operators came to the forefront due to generalizing a few others. Now, we are in a position to recall Riemann-Liouville fractional (RLF) integrals and certain generalized fractional integral operators that generalize Riemann-Liouville, Hadamard and generalized integral operators in a single form.

Definition 1 ([15])**.** Let $\tau \in [\theta, \vartheta]$. The Riemann-Liouville integrals $J^{\alpha}_{\theta+}\tau$ and $J^{\alpha}_{\vartheta-}\tau$ of order $\alpha > 0$ with $\theta, \vartheta \geq 0$ are defined by

$$(J^{\alpha}_{\theta+})\tau(y) = \frac{1}{\Gamma(\alpha)} \int_{\theta}^{y} (y-\lambda)^{\alpha-1}\tau(\lambda)\, d\lambda; \qquad y > \theta, \tag{1.3}$$

and

$$(J^{\alpha}_{\vartheta-})\tau(y) = \frac{1}{\Gamma(\alpha)} \int_{y}^{\vartheta} (\lambda-y)^{\alpha-1}\tau(\lambda)\, d\lambda; \qquad y < \vartheta, \tag{1.4}$$

respectively, where $\Gamma(.)$ is the Euler Gamma function and $(J^{0}_{\theta+})\tau(y) = (J^{0}_{\vartheta-})\tau(y) = \tau(y)$.

In [12], Jarad et al. have defined new conformable fractional integral operator as follows:

$${}^{\beta}_{\theta}J^{\alpha}\tau(y) = \frac{1}{\Gamma(\beta)} \int_{\theta}^{y} \left(\frac{(y-\theta)^{\alpha} - (\lambda-\theta)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{\tau(\lambda)}{(\lambda-\theta)^{1-\alpha}} d\lambda, \tag{1.5}$$

and

$${}^{\beta}J^{\alpha}_{\vartheta}\tau(y) = \frac{1}{\Gamma(\beta)} \int_{y}^{\vartheta} \left(\frac{(\vartheta-y)^{\alpha} - (\vartheta-\lambda)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{\tau(\lambda)}{(\vartheta-\lambda)^{1-\alpha}} d\lambda. \tag{1.6}$$

New conformable fractional integral operator has some special cases and connections according to special values of parameters as follows:

Remark 1. i) If we set $\theta = 0$ and $\alpha = 1$ in (1.5), then the integral operator that is defined with the equation (1.5) reduces to the Riemann-Liouville integral operator that is given in (1.3).
ii) If we choose $\theta = 0$ and $\alpha \to 0$, the equation (1.5) coincides with the Hadamard fractional integral that is defined in [14]. Also, if we select $\theta = 0$, the new fractional integral operator reduces to Katugampola fractional integral operator.
iii) Similar connecitons can be established for (1.6). The operator that is defined in (1.6) reduces to the Riemann-Liouville integral operator that is given in (1.4), if we set $\vartheta = 0$ and $\alpha = 1$. It also coincide with the Hadamard fractional integral that is given in [14] when we set $\vartheta = 0$ and $\alpha \to 0$ in (1.6).

For more results in inequality theory and various operators in fractional analysis and detailed information on these operators, we recommend readers ([6–8, 11, 12, 14, 15, 18, 23–25]).

The main motivation in this study is to obtain new Jensen-Mercer-like Hermite-Hadamard type inequalities by using fractional integrals defined in (1.5) and (1.6) based on Jensen-Mercer inequality for functions whose derivatives are convex. Then, with the help of Hölder inequality and useful variants, inequalities similar to Jensen-Mercer inequality were obtained. Many special cases have been demonstrated to verify inequalities in the literature.

2. Hermite-Jensen-Mercer like inequalities for new conformable fractional integrals

With the help of the operator known as the fractional integral introduced as a new generalization in fractional analysis, an Hermite-Jensen-Mercer type inequality is given as follows:

Theorem 2. *Suppose that $\tau : [\theta, \vartheta] \to R$ is a convex mapping. Then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\tau\left(\theta+\vartheta-\frac{x+y}{2}\right) \leq \frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(y-x)^{\alpha\beta}} \\
&\times\left\{{}^{\beta}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}J^{\alpha}\tau(\theta+\vartheta-x)+{}^{\beta}J^{\alpha}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}\tau(\theta+\vartheta-y)\right\} \\
&\leq \tau(\theta)+\tau(\vartheta)-\left(\frac{\tau(x)+\tau(y)}{2}\right),
\end{aligned} \tag{2.1}$$

where $\forall x, y \in [\theta, \vartheta]$, $\alpha, \beta > 0$ and $\Gamma(.)$ is the Euler Gamma function.

Proof. Let's start proving the first part of inequality with the fact that τ is convex, then we can write

$$\begin{aligned}
&\tau\left(\theta+\vartheta-\frac{x_1+y_1}{2}\right)=\tau\left(\frac{\theta+\vartheta-x_1+\theta+\vartheta-y_1}{2}\right) \\
&\leq \frac{\tau(\theta+\vartheta-x_1)+\tau(\theta+\vartheta-y_1)}{2},
\end{aligned}$$

$\forall\, x_1, y_1 \in [\theta, \vartheta]$. Hence if we set $x_1 = \frac{\lambda}{2}x + \frac{2-\lambda}{2}y$ *and* $y_1 = \frac{2-\lambda}{2}x + \frac{\lambda}{2}y$, for $x, y \in [\theta, \vartheta]$ and $\lambda \in [0, 1]$, we get

$$\begin{aligned}
2\tau\left(\theta+\vartheta-\frac{x+y}{2}\right) &\leq \tau\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right) \\
&\quad +\tau\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right).
\end{aligned} \tag{2.2}$$

Now, to combine the above inequality with the operator, we multiply both sides of (2.2) by $\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\lambda)^{\alpha-1}$ and then integrating the resulting inequality with respect to λ over $[0,1]$, we have

$$\begin{aligned}
&2\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\lambda)^{\alpha-1}d\lambda\\
&\leq\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\lambda)^{\alpha-1}\\
&\times\left(\tau\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)+\tau\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)\right)d\lambda\\
&=\left(\frac{2}{y-x}\right)^{\alpha\beta}\left\{\Gamma(\beta)\ {}^{\beta}J^{\alpha}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}\tau(\theta+\vartheta-y)+\Gamma(\beta)\ {}^{\beta}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}J^{\alpha}\ \tau(\theta+\vartheta-x)\right\}.
\end{aligned}$$

It is obvious that

$$\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\lambda)^{\alpha-1}d\lambda=\frac{1}{\beta\alpha^{\beta}}. \tag{2.3}$$

As a consequence,

$$\begin{aligned}
&2\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\frac{1}{\beta\alpha^{\beta}}\\
&\leq\left(\frac{2}{y-x}\right)^{\alpha\beta}\left\{\Gamma(\beta)\ {}^{\beta}J^{\alpha}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}\tau(\theta+\vartheta-y)+\Gamma(\beta)\ {}^{\beta}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}J^{\alpha}\ \tau(\theta+\vartheta-x)\right\}.
\end{aligned}$$

Thus, this completes the proof of the first inequality of (2).
Again, to prove the second part, we remember that τ is convex mapping, then for $\lambda\in[0,1]$, we can write

$$\tau\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)\leq\tau(\theta)+\tau(\vartheta)-\left(\frac{\lambda}{2}\tau(x)+\frac{2-\lambda}{2}\tau(y)\right), \tag{2.4}$$

and

$$\tau\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)\leq\tau(\theta)+\tau(\vartheta)-\left(\frac{2-\lambda}{2}\tau(x)+\frac{\lambda}{2}\tau(y)\right). \tag{2.5}$$

By adding the inequalities of (2.4) and (2.5), we have

$$\begin{aligned}
&\tau\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)+\tau\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)\\
&\leq\ 2(\tau(\theta)+\tau(\vartheta))-(\tau(x)+\tau(y)).
\end{aligned} \tag{2.6}$$

Now multiplying (2.6) by $\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\lambda)^{\alpha-1}$ and then integrating the resulting inequality with respect to λ over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 \left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\lambda)^{\alpha-1} \\
&\times\left\{\tau\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)+\tau\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)\right\}d\lambda \\
&\leq\left\{2\left(\tau(\theta)+\tau(\vartheta)\right)-\left(\tau(x)+\tau(y)\right)\right\}\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\lambda)^{\alpha-1}d\lambda.
\end{aligned}$$

Namely,

$$\begin{aligned}
&\left(\frac{2}{y-x}\right)^{\alpha\beta}\left\{\Gamma(\beta)\ {}^{\beta}J^{\alpha}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}\tau(\theta+\vartheta-y)+\Gamma(\beta)\ {}^{\beta}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}J^{\alpha}\tau(\theta+\vartheta-x)\right\} \\
&\leq\frac{1}{\beta\alpha^{\beta}}\left\{2\left(\tau(\theta)+\tau(\vartheta)\right)-\left(\tau(x)+\tau(y)\right)\right\}.
\end{aligned}$$

This completes the proof. □

Remark 2. (i) If we take $x=\theta$ and $y=\vartheta$ in Theorem 2 , then we get (Theorem 2.1, [11]).
(ii) If we take $\alpha=1$, $x=\theta$ and $y=\vartheta$ in Theorem 2, then we get (Theorem 2, [24]).

Theorem 3. *Suppose that $\tau:[\theta,\vartheta]\to R$ be a convex mapping. Then the following inequality for fractional integrals holds:*

$$\begin{aligned}
\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)&\leq\tau(\theta)+\tau(\vartheta)-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(y-x)^{\alpha\beta}}\left\{{}^{\beta}J^{\alpha}_{x}\tau(y)+{}^{\beta}_{y}J^{\alpha}\,\tau(x)\right\} \\
&\leq\tau(\theta)+\tau(\vartheta)-\tau\left(\frac{x+y}{2}\right)
\end{aligned}\tag{2.7}$$

and

$$\begin{aligned}
&\tau\left(\theta+\vartheta-\frac{x+y}{2}\right) \\
&\leq\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(y-x)^{\alpha\beta}}\left\{{}^{\beta}J^{\alpha}_{\theta+\vartheta-x}\,\tau(\theta+\vartheta-y)+{}^{\beta}_{\theta+\vartheta-y}J^{\alpha}\,\tau(\theta+\vartheta-x)\right\} \\
&\leq\frac{\tau(\theta+\vartheta-x)+\tau(\theta+\vartheta-y)}{2}\leq\tau(\theta)+\tau(\vartheta)-\frac{\tau(x)+\tau(y)}{2}
\end{aligned}\tag{2.8}$$

where $\forall x,y\in[\theta,\vartheta]$, $\alpha,\beta>0$ and $\Gamma(.)$ is the Euler Gamma function.

Proof. By making use of the Jensen-Mercer inequality, we get

$$\tau\left(\theta+\vartheta-\frac{x_1+y_1}{2}\right)\leq\tau(\theta)+\tau(\vartheta)-\frac{\tau(x_1)+\tau(y_1)}{2}, \tag{2.9}$$

$\forall x_1,y_1\in[\theta,\vartheta]$. By changing of variables $x_1=\lambda x+(1-\lambda)y$ and $y_1=(1-\lambda)x+\lambda y$, for $x,y\in[\theta,\vartheta]$ and $\lambda\in[0,1]$ in (2.9), we can write

$$\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\leq\tau(\theta)+\tau(\vartheta)-\frac{\tau(\lambda x+(1-\lambda)y)+\tau((1-\lambda)x+\lambda y)}{2}. \tag{2.10}$$

Now by multiplying (2.10) by $\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\lambda)^{\alpha-1}$ and then integrating the resulting inequality with respect to λ over $[0,1]$, we success to reach the statement of new conformable fractional operator;

$$\begin{aligned}&\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}(1-\lambda)^{\alpha-1}d\lambda\\&\leq\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}(1-\lambda)^{\alpha-1}\\&\times\left\{\tau(\theta)+\tau(\vartheta)-\frac{\tau(\lambda x+(1-\lambda)y)+\tau((1-\lambda)x+\lambda y)}{2}\right\}d\lambda.\end{aligned}$$

By making some simplyfing processes, we obtain

$$\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\leq\tau(\theta)+\tau(\vartheta)-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(y-x)^{\alpha\beta}}\left\{{}^{\beta}J_x^{\alpha}\tau(y)+{}_y^{\beta}J^{\alpha}\,\tau(x)\right\}. \tag{2.11}$$

This completes the proof of the first inequality of (2.7).

Now, for the proof of second inequality of (2.7), we first note that if τ is convex function, then for $\lambda\in[0,1]$, we obtain

$$\begin{aligned}\tau\left(\frac{x+y}{2}\right)&=\tau\left(\frac{\lambda x+(1-\lambda)y+(1-\lambda)x+\lambda y}{2}\right)\\&\leq\frac{\tau(\lambda x+(1-\lambda)y)+\tau((1-\lambda)x+\lambda y)}{2}\end{aligned} \tag{2.12}$$

By multiplying (2.12) by $\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\lambda)^{\alpha-1}$ and then integrating the resulting inequality with respect to λ over $[0,1]$, we have

$$\tau\left(\frac{x+y}{2}\right)\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}(1-\lambda)^{\alpha-1}d\lambda$$
$$\leq\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}(1-\lambda)^{\alpha-1}\left\{\frac{\tau(\lambda x+(1-\lambda)y)+\tau((1-\lambda)x+\lambda y)}{2}\right\}d\lambda,$$

after simplification and change of variables, we get

$$\tau\left(\frac{x+y}{2}\right)\leq\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(y-x)^{\alpha\beta}}\left\{{}^{\beta}J_x^{\alpha}\tau(y)+{}_y^{\beta}J^{\alpha}\tau(x)\right\}$$
$$-\tau\left(\frac{x+y}{2}\right)\geq-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(y-x)^{\alpha\beta}}\left\{{}^{\beta}J_x^{\alpha}\tau(y)+{}_y^{\beta}J^{\alpha}\tau(x)\right\} \tag{2.13}$$

adding $\tau(\theta)+\tau(\vartheta)$ to both side of (2.13), we obtain

$$\tau(\theta)+\tau(\vartheta)-\tau\left(\frac{x+y}{2}\right)$$
$$\geq\tau(\theta)+\tau(\vartheta)-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(y-x)^{\alpha\beta}}\left\{{}^{\beta}J_x^{\alpha}\,\tau(y)+{}_y^{\beta}J^{\alpha}\tau(x)\right\}. \tag{2.14}$$

By combining (2.11) and (2.14), we prove (2.7). To prove (2.8), we will start with the convexity of τ,

$$\tau\left(\theta+\vartheta-\frac{x_1+y_1}{2}\right)=\tau\left(\frac{\theta+\vartheta-x_1+\theta+\vartheta-y_1}{2}\right)$$
$$\leq\frac{\tau(\theta+\vartheta-x_1)\;+\;\tau(\theta+\vartheta-y_1)}{2}, \tag{2.15}$$

$\forall\ x_1,y_1\in[\theta,\vartheta]$. By change of variables such that $x_1=\lambda x+(1-\lambda)y$ *and* $y_1=(1-\lambda)x+\lambda y$, *for* $x,y\in[\theta,\vartheta]$ and $\lambda\in[0,1]$ in (2.15), we get

$$\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)$$
$$\leq\left\{\frac{\tau(\theta+\vartheta-(\lambda x+(1-\lambda)y))+\tau(\theta+\vartheta-((1-\lambda)x+\lambda y))}{2}\right\}. \tag{2.16}$$

To combine the above inequality with the operator, we multiply (2.16) by $\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\lambda)^{\alpha-1}$ and then integrating the resulting inequality with respect to λ over $[0,1]$, we have

$$\begin{aligned}
&\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}(1-\lambda)^{\alpha-1}\,d\lambda\\
&\leq\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}(1-\lambda)^{\alpha-1}\\
&\times\left\{\frac{\tau(\theta+\vartheta-(\lambda x+(1-\lambda)y))+\tau(\theta+\vartheta-((1-\lambda)x+\lambda y))}{2}\right\}d\lambda,
\end{aligned}$$

By making use of the necessary operations, we get

$$\begin{aligned}
&\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\\
&\leq\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(y-x)^{\alpha\beta}}\left\{{}^{\beta}J^{\alpha}_{\theta+\vartheta-x}\,\tau(\theta+\vartheta-y)+{}^{\beta}_{\theta+\vartheta-y}J^{\alpha}\,\tau(\theta+\vartheta-x)\right\}
\end{aligned}\tag{2.17}$$

On the other hand, using the convexity of τ, we can write

$$\tau(\lambda(\theta+\vartheta-x)+(1-\lambda)(\theta+\vartheta-y))\leq\lambda\tau(\theta+\vartheta-x)+(1-\lambda)\tau(\theta+\vartheta-y)$$

and

$$\tau((1-\lambda)(\theta+\vartheta-x)+\lambda(\theta+\vartheta-y))\leq(1-\lambda)\tau(\theta+\vartheta-x)+\lambda\tau(\theta+\vartheta-y)$$

By adding above two inequalities and using the celebrated Jensen-Mercer inequality, we have

$$\begin{aligned}
&\tau(\lambda(\theta+\vartheta-x)+(1-\lambda)(\theta+\vartheta-y))+\tau((1-\lambda)(\theta+\vartheta-x)+\lambda(\theta+\vartheta-y))\\
&\leq\tau(\theta+\vartheta-x)+\tau(\theta+\vartheta-y)\leq2(\tau(\theta)+\tau(\vartheta))-(\tau(x)+\tau(y)).
\end{aligned}\tag{2.18}$$

By multiplying (2.18) by $\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\lambda)^{\alpha-1}$ and then integrating the resulting

inequality with respect to λ over $[0,1]$, we have

$$\int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^\beta (1-\lambda)^{\alpha-1}\Big\{\tau(\lambda(\theta+\vartheta-x)+(1-\lambda)(\theta+\vartheta-y))$$
$$+\tau((1-\lambda)(\theta+\vartheta-x)+\lambda(\theta+\vartheta-y))\Big\}d\lambda$$
$$\leq 2(\tau(\theta)+\tau(\vartheta))-(\tau(x)+\tau(y))\int_0^1\left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^\beta(1-\lambda)^{\alpha-1}d\lambda,$$

Again some basic operations, we get

$$\frac{\alpha^\beta\Gamma(\beta+1)}{2(y-x)^{\alpha\beta}}\left\{{}^{\beta}J^{\alpha}_{\theta+\vartheta-x}\,\tau(\theta+\vartheta-y)+{}^{\beta}_{\theta+\vartheta-y}J^{\alpha}\,\tau(\theta+\vartheta-x)\right\}$$
$$\leq(\tau(\theta)+\tau(\vartheta))-\left(\frac{\tau(x)+\tau(y)}{2}\right), \tag{2.19}$$

Finally, by combining (2.17) and (2.19), we get (2.8). Which completes the proof. □

Remark 3. Under the assumptions of Theorem 3 with $\alpha=\beta=1$, one has

$$(i)\ \ \tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\leq\tau(\theta)+\tau(\vartheta)-\int_0^1\tau(\lambda x+(1-\lambda)y)\,d\lambda$$
$$\leq\tau(\theta)+\tau(\vartheta)-\tau\left(\frac{x+y}{2}\right)$$

and

$$(ii)\ \ \tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\leq\frac{1}{y-x}\int_x^y\tau(\theta+\vartheta-\lambda)\,d\lambda$$
$$\leq\tau(\theta)+\tau(\vartheta)-\frac{\tau(x)+\tau(y)}{2}.$$

The Remark 3 is proved in (Theorem 2.1, [13]).

Lemma 1. *Assume that* $\tau:[\theta,\vartheta]\to R$ *is differentiable mapping on* (θ,ϑ) *with* $\theta<\vartheta$. *If* $\tau'\in L[\theta,\vartheta]$, *then the following equation holds for fractional integrals:*

$$\frac{2^{\alpha\beta-1}\,\alpha^\beta\,\Gamma(\beta+1)}{(y-x)^{\alpha\beta}}$$
$$\times\left\{{}^{\beta}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}J^{\alpha}\,\tau(\theta+\vartheta-x)+{}^{\beta}J^{\alpha}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}\,\tau(\theta+\vartheta-y)\right\}$$
$$-\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)$$

$$= \frac{(y-x)\ \ \alpha^{\beta}}{4}\int_0^1 \left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}$$
$$\times\left\{\tau'\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)-\tau'\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)\right\}d\lambda, \tag{2.20}$$

where $\forall x,y\in[\theta,\vartheta]$, $\alpha,\beta>0$ *and* $\Gamma(.)$ *is the Euler Gamma function.*

Proof. It suffices to note that,

$$I = \frac{y-x}{4}\alpha^{\beta}\{I_1-I_2\} \tag{2.21}$$

where

$$I_1 = \int_0^1 \left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}\tau'\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)d\lambda, \tag{2.22}$$

and

$$I_2 = \int_0^1 \left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}\tau'\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)d\lambda.$$

Integrating by parts, we have

$$I_1 = -\frac{2}{\alpha^{\beta}(y-x)}\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)$$
$$+\frac{2\,\alpha\beta}{\alpha^{\beta}(y-x)}\int_0^1\left(1-(1-\lambda)^{\alpha}\right)^{\beta-1}(1-\lambda)^{\alpha-1}\tau\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)d\lambda$$
$$= -\frac{2}{\alpha^{\beta}(y-x)}\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)$$
$$+\frac{2^{\alpha\beta}\,\beta}{\alpha^{\beta-1}(y-x)^{\alpha\beta+1}}\int_{\theta+\vartheta-\frac{x+y}{2}}^{\theta+\vartheta-x}\left(\left(\frac{y-x}{2}\right)^{\alpha}-\left(\lambda_1-\left(\theta+\vartheta-\frac{x+y}{2}\right)\right)^{\alpha}\right)^{\beta-1}$$
$$\times\frac{\tau(\lambda_1)}{\left(\lambda_1-\left(\theta+\vartheta-\frac{x+y}{2}\right)\right)^{1-\alpha}}d\lambda$$
$$= -\frac{2}{\alpha^{\beta}(y-x)}\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)$$
$$+\left(\frac{2}{y-x}\right)^{\alpha\beta+1}\frac{\Gamma(\beta+1)}{\alpha^{\beta-1}}\ {}^{\beta}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}J^{\alpha}\,\tau(\theta+\vartheta-x). \tag{2.23}$$

Similarly,

$$I_2 = \int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^\beta \tau'\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right) d\lambda.$$

$$\begin{aligned} I_2 &= \frac{2}{\alpha^\beta (y-x)}\tau\left(\theta+\vartheta-\frac{x+y}{2}\right) \\ &- \left(\frac{2}{y-x}\right)^{\alpha\beta+1} \frac{\Gamma(\beta+1)}{\alpha^{\beta-1}}\ {}^{\beta}J^{\alpha}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}\ \tau(\theta+\vartheta-y). \end{aligned} \tag{2.24}$$

By connecting the equations (2.23) and (2.24) with (2.21) and get (2.20). Thus the proof is completed. □

Remark 4. (i) If we take $x=\theta$ and $y=\vartheta$, then we can get (Lemma 3.1, [11]).
(ii) If we take $\alpha=1$ and $x=\theta, y=\vartheta$ in Lemma 1, it reduces to (Lemma 1.1, [25]).

Lemma 2. *Assume that* $\tau:[\theta,\vartheta]\to R$ *is differentiable mapping on* (θ,ϑ) *with* $\theta<\vartheta$. *If* $\tau'\in L[\theta,\vartheta]$, *then the following equation holds for fractional integrals:*

$$\begin{aligned} &\frac{\tau(\theta+\vartheta-x)+\tau(\theta+\vartheta-y)}{2} - \frac{\alpha^\beta\,\Gamma(\beta+1)}{2(y-x)^{\alpha\beta}} \\ &\times\left\{{}^{\beta}_{(\theta+\vartheta-y)}J^{\alpha}\,\tau(\theta+\vartheta-x)+{}^{\beta}J^{\alpha}_{(\theta+\vartheta-x)}\,\tau(\theta+\vartheta-y)\right\} \\ &= \frac{(y-x)\,\alpha^\beta}{2}\int_0^1\left[\left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^\beta-\left(\frac{1-\lambda^\alpha}{\alpha}\right)^\beta\right] \\ &\times\tau'(\theta+\vartheta-(\lambda x+(1-\lambda)y))\,d\lambda, \end{aligned}$$

where $\forall x,y\in[\theta,\vartheta]$, $\alpha,\beta>0$ *and* $\Gamma(.)$ *is the Euler Gamma function.*

Proof.

$$\begin{aligned} &\frac{(y-x)\,\alpha^\beta}{2}\int_0^1\left[\left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^\beta-\left(\frac{1-\lambda^\alpha}{\alpha}\right)^\beta\right] \\ &\times\tau'(\theta+\vartheta-(\lambda x+(1-\lambda)y))\,d\lambda. \end{aligned}$$

$$I=\frac{(y-x)\,\alpha^\beta}{2}\{I_1-I_2\}. \tag{2.25}$$

$$
\begin{aligned}
I_1 &= \int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^\beta \tau'(\theta+\vartheta-(\lambda x+(1-\lambda)y))\,d\lambda \\
&= \frac{1}{\alpha^\beta}\frac{\tau(\theta+\vartheta-x)}{y-x} - \frac{\beta}{y-x}\int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^{\beta-1} \\
&\times (1-\lambda)^{\alpha-1}\tau(\theta+\vartheta-(\lambda x+(1-\lambda)y))\,d\lambda \\
&= \frac{1}{\alpha^\beta}\frac{\tau(\theta+\vartheta-x)}{y-x} - \frac{\Gamma(\beta+1)}{(y-x)^{\alpha\beta+1}}\left\{ {}^{\beta}J^{\alpha}_{\theta+\vartheta-x}\,\tau(\theta+\vartheta-y)\right\}.
\end{aligned}
\tag{2.26}
$$

$$
\begin{aligned}
I_2 &= \int_0^1 \left(\frac{1-\lambda^\alpha}{\alpha}\right)^\beta \tau'(\theta+\vartheta-(\lambda x+(1-\lambda)y))\,d\lambda \\
&= -\frac{1}{\alpha^\beta}\frac{\tau(\theta+\vartheta-y)}{y-x} + \frac{\Gamma(\beta+1)}{(y-x)^{\alpha\beta+1}}\left\{ {}^{\beta}_{\theta+\vartheta-y}J^{\alpha}\,\tau(\theta+\vartheta-x)\right\},
\end{aligned}
\tag{2.27}
$$

combining equations (2.26) and (2.27) with (2.25) and get (2.25). Which completes the proof. □

Corollary 1. *If we take $\alpha=\beta=1$ in Lemma 2, the we have the following equality:*

$$
\begin{aligned}
&\frac{\tau(\theta+\vartheta-x)+\tau(\theta+\vartheta-y)}{2} - \frac{1}{y-x}\int_{\theta+\vartheta-y}^{\theta+\vartheta-x}\tau(\lambda)\,d\lambda \\
&= \frac{y-x}{2}\int_0^1 (2\lambda-1)\,\tau'(\theta+\vartheta-(\lambda x+(1-\lambda)y))\,d\lambda.
\end{aligned}
\tag{2.28}
$$

Remark 5. If we take $x=\theta$ and $y=\vartheta$ in Corollary 1, then the equality (2.28) reduces to the equality as following

$$
\frac{\tau(\theta)+\tau(\vartheta)}{2} - \frac{1}{\vartheta-\theta}\int_\theta^\vartheta \tau(\lambda)\,d\lambda = \frac{\vartheta-\theta}{2}\int_0^1 (2\lambda-1)\,\tau'((1-\lambda)\theta+\lambda\vartheta)\;d\lambda,
$$

which is proved in (Lemma 2.1, [6]).

Theorem 4. *$\tau:[\theta,\vartheta]\to R$ is differentiable mapping on (θ,ϑ) with $\theta<\vartheta$ and $\tau'\in L[\theta,\vartheta]$. If $|\tau'|$ is convex function on $[\theta,\vartheta]$, then the following inequality holds for fractional integrals:*

$$
\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\,\alpha^\beta\,\Gamma(\beta+1)}{(y-x)^{\alpha\beta}}\left\{ {}^{\beta}_{(\theta+\vartheta-\frac{x+y}{2})}J^{\alpha}\,\tau(\theta+\vartheta-x) + {}^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})}\,\tau(\theta+\vartheta-y)\right\}\right. \\
&\left. -\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{y-x}{4}\alpha^{\beta}\Bigg[\left(\left|\tau'(\theta)\right|+\left|\tau'(\vartheta)\right|\right)\left(\frac{1}{\alpha^{\beta+1}}B\left(\beta+1,\frac{1}{\alpha}\right)\right)\\
&-\left\{\left|\tau'(x)\right|\left(\frac{1}{2\alpha^{\beta+1}}\left(B\left(\beta+1,\frac{2}{\alpha}\right)+B\left(\beta+1,\frac{1}{\alpha}\right)\right)\right)\right\}\\
&+\left\{\left|\tau'(y)\right|\left(\frac{1}{2\alpha^{\beta+1}}\left(B\left(\beta+1,\frac{2}{\alpha}\right)-B\left(\beta+1,\frac{1}{\alpha}\right)\right)\right)\right\}\\
&+\left(\left|\tau'(\theta)\right|+\left|\tau'(\vartheta)\right|\right)\left(\frac{1}{\alpha^{\beta+1}}B\left(\beta+1,\frac{1}{\alpha}\right)\right)\\
&-\left\{\left|\tau'(x)\right|\left(\frac{1}{2\alpha^{\beta+1}}\left(B\left(\beta+1,\frac{1}{\alpha}\right)-B\left(\beta+1,\frac{2}{\alpha}\right)\right)\right)\right\}\\
&+\left\{\left|\tau'(y)\right|\left(\frac{1}{2\alpha^{\beta+1}}\left(B\left(\beta+1,\frac{2}{\alpha}\right)+B\left(\beta+1,\frac{1}{\alpha}\right)\right)\right)\right\}\Bigg]
\end{aligned}
\tag{2.29}
$$

where $\forall\ x,y\in[\theta,\vartheta]$, $\alpha,\beta>0$, $\lambda\in[0,1]$ *and* $\Gamma(.)$, $B(.,.)$ are the Euler Gamma and Beta function.

Proof. By using Lemma 1 with Jensen-Mercer inequality and the convexity of $|\tau'|$, we can write

$$
\begin{aligned}
&\Bigg|\frac{2^{\alpha\beta-1}\,\alpha^{\beta}\,\Gamma(\beta+1)}{(y-x)^{\alpha\beta}}\left\{{}^{\beta}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}J^{\alpha}\,\tau(\theta+\vartheta-x)+{}^{\beta}J^{\alpha}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}\,\tau(\theta+\vartheta-y)\right\}\\
&-\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\Bigg|\\
&\qquad\leq\frac{y-x}{4}\alpha^{\beta}\left\{\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}\left|\tau'\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)\right|d\lambda\right.\\
&\qquad\left.+\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}\left|\tau'\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)\right|d\lambda\right\}\\
&\leq\frac{y-x}{4}\alpha^{\beta}\times\left\{\left(\left|\tau'(\theta)\right|+\left|\tau'(\vartheta)\right|\right)\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}d\lambda\right.\\
&\left.-\left(\left|\tau'(x)\right|\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}\frac{2-\lambda}{2}\,d\lambda+\left|\tau'(y)\right|\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}\frac{\lambda}{2}\,d\lambda\right)\right\}\\
&+\left\{\left(\left|\tau'(\theta)\right|+\left|\tau'(\vartheta)\right|\right)\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}d\lambda\right.\\
&\left.-\left(\left|\tau'(x)\right|\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}\frac{\lambda}{2}\,d\lambda+\left|\tau'(y)\right|\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}\frac{2-\lambda}{2}\,d\lambda\right)\right\}.
\end{aligned}
$$

By making use of the necessary computations, we get (2.29). Which completes the proof. □

Remark 6. (i) If we take $x=\theta$ and $y=\vartheta$ in Theorem 4 , then we get $\big($Theorem 3.1, [11]$\big)$.
(ii) If we take $\alpha=1$, $x=\theta$ and $y=\vartheta$ in Theorem 4 and $q=1$ in Theorem 5 of [25] are found to yield the same result.

Theorem 5. $\tau:[\theta,\vartheta]\to R$ *is differentiable mapping on* (θ,ϑ) *with* $\theta<\vartheta$ *and* $\tau'\in L[\theta,\vartheta]$. *If* $|\tau'|^q$ *is convex function on* $[\theta,\vartheta]$ *for some fixed* $q\geq 1$, *then the following inequality holds for fractional integrals:*

$$\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\,\alpha^{\beta}\,\Gamma(\beta+1)}{(y-x)^{\alpha\beta}}\left\{{}^{\beta}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}J^{\alpha}\,\tau(\theta+\vartheta-x)+{}^{\beta}J^{\alpha}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}\,\tau(\theta+\vartheta-y)\right\}\right.\\
&\left.-\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\right|\\
&\leq\frac{y-x}{4}\alpha^{\beta}\left(\frac{1}{\alpha^{\beta+1}}B\left(\beta+1,\frac{1}{\alpha}\right)\right)^{1-\frac{1}{q}}\\
&\left\{\left(|\tau'(\theta)|^q+|\tau'(\vartheta)|^q\right)\left(\left(\frac{1}{\alpha^{\beta+1}}B\left(\beta+1,\frac{1}{\alpha}\right)\right)\right)\right.\\
&-\left\{|\tau'(x)|^q\left(\frac{1}{2\alpha^{\beta+1}}\left(B\left(\beta+1,\frac{1}{\alpha}\right)-B\left(\beta+1,\frac{2}{\alpha}\right)\right)\right)\right.\\
&\left.\left.+|\tau'(y)|^q\left(\frac{1}{2\alpha^{\beta+1}}\left(B\left(\beta+1,\frac{2}{\alpha}\right)+B\left(\beta+1,\frac{1}{\alpha}\right)\right)\right)\right\}\right.\\
&+\left(|\tau'(\theta)|^q+\left|\tau'(\vartheta)\right|^q\right)\left(\left(\frac{1}{\alpha^{\beta+1}}B\left(\beta+1,\frac{1}{\alpha}\right)\right)\right)\\
&-\left\{|\tau'(x)|^q\left(\frac{1}{2\alpha^{\beta+1}}\left(B\left(\beta+1,\frac{2}{\alpha}\right)-B\left(\beta+1,\frac{1}{\alpha}\right)\right)\right)\right.\\
&\left.\left.\left.+|\tau'(y)|^q\left(\frac{1}{2\alpha^{\beta+1}}\left(B\left(\beta+1,\frac{1}{\alpha}\right)+B\left(\beta+1,\frac{2}{\alpha}\right)\right)\right)\right\}\right\}^{\frac{1}{q}},
\end{aligned}\tag{2.30}$$

where $\forall\ x,y\in[\theta,\vartheta]$, $\alpha,\beta>0$, $\lambda\in[0,1]$ *and* $\Gamma(.)$, $B(.,.)$ are the Euler Gamma and Beta function, $p^{-1}+q^{-1}=1$.

Proof. By using Lemma 1 with Jensen-Mercer inequality, by using convexity of $|\tau'|^q$ and well-known power-mean inequality, we have

$$
\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\,\alpha^{\beta}\,\Gamma(\beta+1)}{(y-x)^{\alpha\beta}}\left\{{}^{\beta}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}J^{\alpha}\,\tau(\theta+\vartheta-x)+{}^{\beta}J^{\alpha}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}\tau(\theta+\vartheta-y)\right\}\right.\\
&\left.-\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\right|\\
&\quad\leq\frac{y-x}{4}\alpha^{\beta}\left\{\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}d\lambda\right)^{1-\frac{1}{q}}\right.\\
&\quad\times\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}\left|\tau'\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)\right|^{q}d\lambda\right)^{\frac{1}{q}}\\
&\quad+\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}d\lambda\right)^{1-\frac{1}{q}}\\
&\quad\left.\times\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}\left|\tau'\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)\right|^{q}d\lambda\right)^{\frac{1}{q}}\right\}\\
&\quad\leq\frac{y-x}{4}\alpha^{\beta}\left\{\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}d\lambda\right)^{1-\frac{1}{q}}\right.\\
&\quad\times\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}\right.\\
&\quad\left.\left(\left(|\tau'(\theta)|^{q}+|\tau'(\vartheta)|^{q}\right)-\left(\frac{\lambda}{2}|\tau'(x)|^{q}+\frac{2-\lambda}{2}|\tau'(y)|^{q}\right)\right)d\lambda\right)^{\frac{1}{q}}\\
&\quad+\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}d\lambda\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}\right.\\
&\quad\left.\left.\left(\left(|\tau'(\theta)|^{q}+|\tau'(\vartheta)|^{q}\right)-\left(\frac{2-\lambda}{2}|\tau'(x)|^{q}+\frac{\lambda}{2}|\tau'(y)|^{q}\right)\right)d\lambda\right)^{\frac{1}{q}}\right\}.
\end{aligned}
\tag{2.31}
$$

Making necessary simplifications, we get (2.30). The proof is completed. □

Remark 7. (i) If we take $x=\theta$ and $y=\vartheta$ in Theorem 5 , then we get (Theorem 3.2, [11]).
(ii) If we take $\alpha=1$, $x=\theta$ and $y=\vartheta$ in Theorem 5 reduces to Theorem 5 in [25].

Theorem 6. *$\tau : [\theta,\vartheta] \to R$ is differentiable mapping on (θ,ϑ) with $\theta < \vartheta$ and $\tau' \in L[\theta,\vartheta]$. If $|\tau'|^q$ is convex function on $[\theta,\vartheta]$, then the following inequality holds for fractional integrals:*

$$\left|\frac{2^{\alpha\beta-1}\,\alpha^{\beta}\,\Gamma(\beta+1)}{(y-x)^{\alpha\beta}}\left\{{}^{\beta}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}J^{\alpha}\,\tau(\theta+\vartheta-x)+\,{}^{\beta}J^{\alpha}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}\,\tau(\theta+\vartheta-y)\right\}\right.$$
$$\left.-\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\right|$$

$$\leq \frac{y-x}{4}\alpha^{\beta}\left(\frac{1}{\alpha^{\beta p+1}}B\left(\beta p+1,\frac{1}{\alpha}\right)\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q-\left(\frac{3\left|\tau'(x)\right|^q+\left|\tau'(y)\right|^q}{4}\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q-\left(\frac{\left|\tau'(x)\right|^q+3\left|\tau'(y)\right|^q}{4}\right)\right)^{\frac{1}{q}}\right\}, \tag{2.32}$$

where $\forall\ x,y\in[\theta,\vartheta]$, $\alpha,\beta>0$, $\lambda\in[0,1]$, $q>0$ and $\Gamma(.)$, $B(.,.)$ are the Euler Gamma and Beta function, $p^{-1}+q^{-1}=1$.

Proof. If we proceed a similar argument to the proof of the previous theorem, by using Lemma 1 with Jensen-Mercer inequality and familiar Hölder integral inequality, we can write

$$\left|\frac{2^{\alpha\beta-1}\,\alpha^{\beta}\,\Gamma(\beta+1)}{(y-x)^{\alpha\beta}}\left\{{}^{\beta}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}J^{\alpha}\tau(\theta+\vartheta-x)+\,{}^{\beta}J^{\alpha}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}\tau(\theta+\vartheta-y)\right\}\right.$$
$$\left.-\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\right|$$
$$\leq \frac{y-x}{4}\alpha^{\beta}\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta p}dt\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\int_0^1\left|\tau'\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)\right|^q\right)^{\frac{1}{q}}d\lambda\right.$$
$$\left.+\left(\int_0^1\left|\tau'\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)\right|^q\right)^{\frac{1}{q}}d\lambda\right\}.$$

By applying the convexity of $|\tau'|^q$, we have

$$\left|\tau'\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)\right|^q$$
$$\leq\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q-\left(\frac{2-\lambda}{2}\left|\tau'(x)\right|^q+\frac{\lambda}{2}\left|\tau'(y)\right|^q\right)$$

$$\leq\frac{y-x}{4}\alpha^\beta\left(\int_0^1\left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^{\beta p}d\lambda\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\int_0^1\left(\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q-\left(\frac{2-\lambda}{2}\left|\tau'(x)\right|^q+\frac{\lambda}{2}\left|\tau'(y)\right|^q\right)\right)\right)^{\frac{1}{q}}d\lambda\right.$$
$$\left.+\left(\int_0^1\left(\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q-\left(\frac{\lambda}{2}\left|\tau'(x)\right|^q+\frac{2-\lambda}{2}\left|\tau'(y)\right|^q\right)\right)\right)^{\frac{1}{q}}d\lambda\right\}$$

$$\leq\frac{y-x}{4}\alpha^\beta\left(\frac{1}{\alpha^{\beta p+1}}B\left(\beta p+1,\frac{1}{\alpha}\right)\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q-\left(\frac{3\left|\tau'(x)\right|^q+\left|\tau'(y)\right|^q}{4}\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q-\left(\frac{\left|\tau'(x)\right|^q+3\left|\tau'(y)\right|^q}{4}\right)\right)^{\frac{1}{q}}\right\},$$

which completes the proof. □

Corollary 2. *If we choose $\alpha=1$ in Theorem 6, then we have the following inequality*

$$\left|\frac{1}{y-x}\int_{\theta+\vartheta-y}^{\theta+\vartheta-x}\tau(\lambda)\,d\lambda-\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\right|$$
$$\leq\frac{1}{2^{\frac{1}{p}}}\times\left\{\left(\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q-\left(\frac{3\left|\tau'(x)\right|^q+\left|\tau'(y)\right|^q}{4}\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q-\left(\frac{\left|\tau'(x)\right|^q+3\left|\tau'(y)\right|^q}{4}\right)\right)^{\frac{1}{q}}\right\}.$$

Theorem 7. *$\tau:[\theta,\vartheta]\to R$ is differentiable mapping on (θ,ϑ) with $\theta<\vartheta$ and $\tau'\in L[\theta,\vartheta]$. If $|\tau'|^q$ is convex function on $[\theta,\vartheta]$, $q>1$, then the following inequality*

holds for fractional integrals:

$$\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\,\alpha^{\beta}\,\Gamma(\beta+1)}{(y-x)^{\alpha\beta}}\left\{{}^{\beta}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}J^{\alpha}\tau(\theta+\vartheta-x)+{}^{\beta}J^{\alpha}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}\tau(\theta+\vartheta-y)\right\}\right.\\
&\left.-\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\right|\\
&\leq\frac{(y-x)\,\alpha^{\beta}}{4}\left\{\left(\left|\tau'(\theta)\right|^{q}+\left|\tau'(\vartheta)\right|^{q}\right)\left(\frac{B\left(\beta+1,\frac{1}{\alpha}\right)}{\alpha^{\beta+1}}\right)\right.\\
&-\left(\left(\frac{B\left(\beta+1,\frac{2}{\alpha}\right)+B\left(\beta+1,\frac{1}{\alpha}\right)}{2\alpha^{\beta+1}}\right)\left|\tau'(x)\right|^{q}\right.\\
&\left.+\left(\frac{B\left(\beta+1,\frac{1}{\alpha}\right)-B\left(\beta+1,\frac{2}{\alpha}\right)}{2\alpha^{\beta+1}}\right)\left|\tau'(y)\right|^{q}\right)^{\frac{1}{q}}\\
&+\left(\left|\tau'(\theta)\right|^{q}+\left|\tau'(\vartheta)\right|^{q}\right)\left(\frac{B\left(\beta+1,\frac{1}{\alpha}\right)}{\alpha^{\beta+1}}\right)\\
&-\left(\left(\frac{B\left(\beta+1,\frac{1}{\alpha}\right)-B\left(\beta+1,\frac{2}{\alpha}\right)}{2\alpha^{\beta+1}}\right)\left|\tau'(x)\right|^{q}\right.\\
&\left.\left.+\left(\frac{B\left(\beta+1,\frac{2}{\alpha}\right)+B\left(\beta+1,\frac{1}{\alpha}\right)}{2\alpha^{\beta+1}}\right)\left|\tau'(y)\right|^{q}\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

where ll $x,y\in[\theta,\vartheta]$, $\alpha,\ \beta>0,\ \lambda\in[0,1]$ *and* $\Gamma(.)$, $B(.,.)$ *are the Euler Gamma and Beta function,* $p^{-1}+q^{-1}=1$.

Proof. By using Lemma 1 with Jensen-Mercer inequality, convexity of $|\tau'|^{q}$ and familiar Hölder integral inequality, we obtain

$$\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\,\alpha^{\beta}\,\Gamma(\beta+1)}{(y-x)^{\alpha\beta}}\left\{{}^{\beta}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}J^{\alpha}\tau(\theta+\vartheta-x)+{}^{\beta}J^{\alpha}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}\tau(\theta+\vartheta-y)\right\}\right.\\
&\left.-\tau\left(a+b-\frac{x+y}{2}\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(y-x)\,\alpha^{\beta}}{4}\left\{\left(\int_0^1 1d\lambda\right)^{\frac{1}{p}}\right.\\
&\times\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}\left|\tau'\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)\right|^{q}d\lambda\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 1d\lambda\right)^{\frac{1}{p}}\\
&\left.\times\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}\left|\tau'\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)\right|^{q}d\lambda\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

By making necessary changes, we get (2.33). □

Theorem 8. *$\tau:[\theta,\vartheta]\to R$ is differentiable mapping on (θ,ϑ) with $\theta<\vartheta$ and $\tau'\in L[\theta,\vartheta]$. If $|\tau'|$ is convex function on $[\theta,\vartheta]$, then the following inequality holds for fractional integrals:*

$$
\begin{aligned}
&\left|\frac{\tau(\theta+\vartheta-x)+\tau(\theta+\vartheta-y)}{2}-\frac{\alpha^{\beta}\,\Gamma(\beta+1)}{2\,(y-x)^{\alpha\beta}}\right.\\
&\left.\times\left\{{}^{\beta}_{(\theta+\vartheta-y)}J^{\alpha}\,\tau(\theta+\vartheta-x)+{}^{\beta}J^{\alpha}_{(\theta+\vartheta-x)}\,\tau(\theta+\vartheta-y)\right\}\right|\\
&\leq\frac{(y-x)\,\alpha^{\beta}}{2}\left[\left\{\left(\left|\tau'(\theta)\right|+\left|\tau'(\vartheta)\right|\right)\frac{B\left(\frac{1}{\alpha},\beta+1\right)}{\alpha^{\beta+1}}\right.\right.\\
&-\frac{|\tau'(x)|}{\alpha^{\beta+1}}\left\{B_{\frac{1}{2\alpha}}\left(\frac{1}{\alpha},\beta+1\right)+B\left(\frac{2}{\alpha},\beta+1\right)-B\left(\frac{1}{\alpha},\beta+1\right)\right\}\\
&\left.-\frac{|\tau'(y)|}{\alpha^{\beta+1}}\left\{B_{\frac{1}{2\alpha}}\left(\frac{1}{\alpha},\beta+1\right)-B\left(\frac{2}{\alpha},\beta+1\right)\right\}\right\}\\
&+\left\{\left(\left|\tau'(\theta)\right|+\left|\tau'(\vartheta)\right|\right)\frac{B\left(\frac{1}{\alpha},\beta+1\right)}{\alpha^{\beta+1}}\right.\\
&-\frac{|\tau'(x)|}{\alpha^{\beta+1}}\left\{B_{\frac{1}{2\alpha}}\left(\frac{1}{\alpha},\beta+1\right)-B\left(\frac{2}{\alpha},\beta+1\right)\right\}\\
&\left.\left.-\frac{|\tau'(y)|}{\alpha^{\beta+1}}\left\{B_{\frac{1}{2\alpha}}\left(\frac{1}{\alpha},\beta+1\right)+B\left(\frac{2}{\alpha},\beta+1\right)-B\left(\frac{1}{\alpha},\beta+1\right)\right\}\right\}\right]
\end{aligned}
\tag{2.33}
$$

where $\forall\ x,y\in[\theta,\vartheta]$, $\alpha,\ \beta>0$, $\lambda\in[0,1]$ and $\Gamma(.),B(.,.)$ are the Euler Gamma and Beta function.

Proof. By a similar way with the proof the previous theorem, we get

$$\left|\frac{\tau(\theta+\vartheta-x)+\tau(\theta+\vartheta-y)}{2}-\frac{\alpha^{\beta}\,\Gamma(\beta+1)}{2\,(y-x)^{\alpha\beta}}\right.$$
$$\left.\times\left\{{}^{\beta}_{(\theta+\vartheta-y)}J^{\alpha}\,\tau(\theta+\vartheta-x)+\,{}^{\beta}J^{\alpha}_{(\theta+\vartheta-x)}\,\tau(\theta+\vartheta-y)\right\}\right|$$
$$\leq\frac{(y-x)\,\alpha^{\beta}}{2}\int_0^1\left|\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}-\left(\frac{1-\lambda^{\alpha}}{\alpha}\right)^{\beta}\right|\left|\tau'(\theta+\vartheta-(\lambda x+(1-\lambda)\,y))\right|\,d\lambda$$
$$\leq\frac{(y-x)\,\alpha^{\beta}}{2}\int_0^1\left|\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}-\left(\frac{1-\lambda^{\alpha}}{\alpha}\right)^{\beta}\right|$$
$$\times\left\{\left|\tau'(\theta)\right|+\left|\tau'(\vartheta)\right|-\left(\lambda\left|\tau'(x)\right|+(1-\lambda)\left|\tau'(y)\right|\right)\right\}d\lambda$$
$$\leq\frac{(y-x)\,\alpha^{\beta}}{2}\left[\int_0^{\frac{1}{2}}\left[\left(\frac{1-\lambda^{\alpha}}{\alpha}\right)^{\beta}-\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}\right]\right.$$
$$\times\left\{\left|\tau'(\theta)\right|+\left|\tau'(\vartheta)\right|-\left(\lambda\left|\tau'(x)\right|+(1-\lambda)\left|\tau'(y)\right|\right)\right\}d\lambda$$
$$+\int_{\frac{1}{2}}^1\left[\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}-\left(\frac{1-\lambda^{\alpha}}{\alpha}\right)^{\beta}\right]$$
$$\left.\times\left\{\left|\tau'(\theta)\right|+\left|\tau'(\vartheta)\right|-\left(\lambda\left|\tau'(x)\right|+(1-\lambda)\left|\tau'(y)\right|\right)\right\}d\lambda\right].$$

This completes the proof. □

3. New inequalities via improved Hölder inequality

Theorem 9. *$\tau:[\theta,\vartheta]\to R$ is differentiable mapping on (θ,ϑ) with $\theta<\vartheta$ and $\tau'\in L[\theta,\vartheta]$. If $|\tau'|^q$ is convex function on $[\theta,\vartheta]$, $q>1$, then the following inequality holds for fractional integrals:*

$$\left|\frac{2^{\alpha\beta-1}\,\alpha^{\beta}\,\Gamma(\beta+1)}{(y-x)^{\alpha\beta}}\left\{{}^{\beta}_{(\theta+\vartheta-\frac{x+y}{2})}J^{\alpha}\tau(\theta+\vartheta-x)+\,{}^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})}\tau(\theta+\vartheta-y)\right\}\right.$$
$$\left.-\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\right|$$

$$
\begin{aligned}
&\leq \frac{(y-x)\,\alpha^{\beta}}{2}\left[\left\{\left(\frac{B\left(\frac{2}{\alpha},\beta p+1\right)}{\alpha^{\beta p+1}}\right)^{\frac{1}{p}}\right.\right.\\
&\times\left(\frac{|\tau'(\theta)|^{q}+\left|\tau'(\vartheta)\right|^{q}}{2}-\left(\frac{5}{12}\left|\tau'(x)\right|^{q}+\frac{1}{12}\left|\tau'(y)\right|^{q}\right)\right)^{\frac{1}{q}}\\
&+\left(\frac{B\left(\frac{1}{\alpha},\beta p+1\right)-B\left(\frac{2}{\alpha},\beta p+1\right)}{\alpha^{\beta p+1}}\right)^{\frac{1}{p}}\\
&\left.\times\left(\frac{|\tau'(\theta)|^{q}+|\tau'(\vartheta)|^{q}}{2}-\left(\frac{1}{3}\left|\tau'(x)\right|^{q}+\frac{1}{6}\left|\tau'(y)\right|^{q}\right)\right)^{\frac{1}{q}}\right\}\\
&+\left\{\left(\frac{B\left(\frac{2}{\alpha},\beta p+1\right)}{\alpha^{\beta p+1}}\right)^{\frac{1}{p}}\right.\\
&\times\left(\frac{|\tau'(\theta)|^{q}+|\tau'(\vartheta)|^{q}}{2}-\left(\frac{1}{12}\left|\tau'(x)\right|^{q}+\frac{5}{12}\left|\tau'(y)\right|^{q}\right)\right)^{\frac{1}{q}}\\
&+\left(\frac{B\left(\frac{1}{\alpha},\beta p+1\right)-B\left(\frac{2}{\alpha},\beta p+1\right)}{\alpha^{\beta p+1}}\right)^{\frac{1}{p}}\\
&\left.\left.\times\left(\frac{|\tau'(\theta)|^{q}+|\tau'(\vartheta)|^{q}}{2}-\left(\frac{1}{6}\left|\tau'(x)\right|^{q}+\frac{1}{3}\left|\tau'(y)\right|^{q}\right)\right)^{\frac{1}{q}}\right\}\right],
\end{aligned}
\tag{3.1}
$$

where $\forall\ x,y\in[\theta,\vartheta]$, $\alpha,\ \beta>0,\ \lambda\in[0,1]$ *and* $\Gamma(.)$, $B(.,.)$ *are the Euler Gamma and Beta function,* $p^{-1}+q^{-1}=1$.

Proof. By using Lemma 1 with Jensen-Mercer inequality, convexity of $|\tau'|^{q}$ and applying the Hölder-İşcan integral inequality that is given in (Theorem 1.4, [21]), we can write

$$
\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\,\alpha^{\beta}\,\Gamma(\beta+1)}{(y-x)^{\alpha\beta}}\left\{{}^{\beta}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}J^{\alpha}\tau(\theta+\vartheta-x)+{}^{\beta}J^{\alpha}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}\tau(\theta+\vartheta-y)\right\}\right.\\
&\left.-\tau\left(a+b-\frac{x+y}{2}\right)\right|
\end{aligned}
$$

$$\leq \frac{(y-x)\ \ \alpha^{\beta}}{4}\Bigg[\Bigg\{\left(\int_0^1 (1-\lambda)\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta p} d\lambda\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 (1-\lambda)\ \left[\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q-\left(\frac{2-\lambda}{2}\left|\tau'(x)\right|^q+\frac{\lambda}{2}\left|\tau'(y)\right|^q\right)\right]d\lambda\right)^{\frac{1}{q}}$$

$$\left(\int_0^1 \lambda\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta p} d\lambda\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \lambda\left[\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q-\left(\frac{2-\lambda}{2}\left|\tau'(x)\right|^q+\frac{\lambda}{2}\left|\tau'(y)\right|^q\right)\right]d\lambda\right)^{\frac{1}{q}}\Bigg\}$$

$$+\Bigg\{\left(\int_0^1 (1-\lambda)\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta p} d\lambda\ \right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 (1-\lambda)\left[\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q-\left(\frac{\lambda}{2}\left|\tau'(x)\right|^q+\frac{2-\lambda}{2}\left|\tau'(y)\right|^q\right)\right]\ d\lambda\right)^{\frac{1}{q}}$$

$$\left(\int_0^1 \lambda\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta p} d\lambda\ \right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \lambda\left[\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q-\left(\frac{\lambda}{2}\left|\tau'(x)\right|^q+\frac{2-\lambda}{2}\left|\tau'(y)\right|^q\right)\right]\ d\lambda\right)^{\frac{1}{q}}\Bigg\}\Bigg].$$

By using calculus tools, one can have the required result. □

Theorem 10. $\tau:[\theta,\vartheta]\to R$ *is differentiable mapping on* (θ,ϑ) *with* $\theta<\vartheta$ *and* $\tau'\in L[\theta,\vartheta]$. *If* $|\tau'|^q$ *is convex function on* $[\theta,\vartheta]$, *for some* $q\geq 1$, *then the following inequality holds for fractional integrals:*

$$\left|\frac{2^{\alpha\beta-1}\ \alpha^{\beta}\ \Gamma(\beta+1)}{(y-x)^{\alpha\beta}}\left\{{}^{\beta}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}J^{\alpha}\ \tau(\theta+\vartheta-x)+\ {}^{\beta}J^{\alpha}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}\ \tau(\theta+\vartheta-y)\right\}\right.$$

$$\left.-\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\right|$$

$$
\begin{aligned}
&\leq \frac{(y-x)\alpha^{\beta}}{4}\Bigg[\Bigg\{\left(\frac{B(\frac{2}{\alpha},\beta+1)}{\alpha^{\beta+1}}\right)^{1-\frac{1}{q}} \\
&\times\left(\left(\left|\tau'(\theta)\right|^{q}+\left|\tau'(\vartheta)\right|^{q}\right)\left(\frac{B(\frac{2}{\alpha},\beta+1)}{\alpha^{\beta+1}}\right)\right. \\
&-\left(\frac{1}{2\alpha^{\beta+1}}\left|\tau'(x)\right|^{q}\left(B(\frac{2}{\alpha},\beta+1)+B(\frac{3}{\alpha},\beta+1)\right)\right. \\
&\left.\left.+\frac{1}{2\alpha^{\beta+1}}\left|\tau'(y)\right|^{q}\left(B(\frac{2}{\alpha},\beta+1)-B(\frac{3}{\alpha},\beta+1)\right)\right)\right)^{\frac{1}{q}} \\
&+\left(\frac{B(\frac{1}{\alpha},\beta+1)-B(\frac{2}{\alpha},\beta+1)}{\alpha^{\beta+1}}\right)^{1-\frac{1}{q}} \\
&\times\left(\left(\left|\tau'(\theta)\right|^{q}+\left|\tau'(\vartheta)\right|^{q}\right)\left(\frac{B(\frac{1}{\alpha},\beta+1)-B(\frac{2}{\alpha},\beta+1)}{\alpha^{\beta+1}}\right)\right. \\
&-\left(\frac{1}{2\alpha^{\beta+1}}\left|\tau'(x)\right|^{q}\left(B(\frac{1}{\alpha},\beta+1)-B(\frac{3}{\alpha},\beta+1)\right)\right. \\
&+\frac{1}{2\alpha^{\beta+1}}\left|\tau'(y)\right|^{q}\left(B(\frac{1}{\alpha},\beta+1)-2B(\frac{2}{\alpha},\beta+1)\right. \\
&\left.\left.\left.+B(\frac{3}{\alpha},\beta+1)\right)\right)\right)^{\frac{1}{q}}\Bigg\} \\
&+\Bigg\{\left(\frac{B(\frac{2}{\alpha},\beta+1)}{\alpha^{\beta+1}}\right)^{1-\frac{1}{q}}\left(\left(\left|\tau'(\theta)\right|^{q}+\left|\tau'(\vartheta)\right|^{q}\right)\left(\frac{B(\frac{2}{\alpha},\beta+1)}{\alpha^{\beta+1}}\right)\right. \\
&-\left(\frac{1}{2\alpha^{\beta+1}}\left|\tau'(x)\right|^{q}\left(B(\frac{2}{\alpha},\beta+1)-B(\frac{3}{\alpha},\beta+1)\right)\right. \\
&\left.\left.+\frac{1}{2\alpha^{\beta+1}}\left|\tau'(y)\right|^{q}\left(B(\frac{2}{\alpha},\beta+1)+B(\frac{3}{\alpha},\beta+1)\right)\right)\right)^{\frac{1}{q}} \\
&+\left(\frac{B(\frac{1}{\alpha},\beta+1)-B(\frac{2}{\alpha},\beta+1)}{\alpha^{\beta+1}}\right)^{1-\frac{1}{q}} \\
&\left(\left(\left|\tau'(\theta)\right|^{q}+\left|\tau'(\vartheta)\right|^{q}\right)\left(\frac{B(\frac{1}{\alpha},\beta+1)-B(\frac{2}{\alpha},\beta+1)}{\alpha^{\beta+1}}\right)\right. \\
&-\left(\frac{1}{2\alpha^{\beta+1}}\left|\tau'(x)\right|^{q}\left(B(\frac{1}{\alpha},\beta+1)-2B(\frac{2}{\alpha},\beta+1)+B(\frac{3}{\alpha},\beta+1)\right)\right. \\
&\left.\left.\left.+\frac{1}{2\alpha^{\beta+1}}\left|\tau'(y)\right|^{q}\left(B(\frac{1}{\alpha},\beta+1)-B(\frac{3}{\alpha},\beta+1)\right)\right)\right)^{\frac{1}{q}}\Bigg\}\Bigg],
\end{aligned}
$$

(3.2)

where $\forall\ x,y \in [\theta,\vartheta]$, $\alpha,\ \beta > 0$, $\lambda \in [0,1]$ *and* $\Gamma(.)$, $B(.,.)$ *are the Euler Gamma and Beta function,* $p^{-1}+q^{-1}=1$.

Proof. By using Lemma 1 with Jensen-Mercer inequality, convexity of $|\tau'|^q$ and applying the Improved power-mean integral inequality $\big($Theorem 1.5, [21]$\big)$, we have

$$
\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\,\alpha^{\beta}\,\Gamma(\beta+1)}{(y-x)^{\alpha\beta}}\left\{{}^{\beta}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}J^{\alpha}\tau(\theta+\vartheta-x)+{}^{\beta}J^{\alpha}_{\left(\theta+\vartheta-\frac{x+y}{2}\right)}\tau(\theta+\vartheta-y)\right\}\right.\\
&\left.-\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\right|\\
&\leq\frac{(y-x)\ \ \alpha^{\beta}}{4}\left[\left\{\left(\int_0^1(1-\lambda)\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}d\lambda\right)^{1-\frac{1}{q}}\right.\right.\\
&\times\left(\int_0^1(1-\lambda)\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}\left|\tau'\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)\right|^{q}d\lambda\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\lambda\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}d\lambda\right)^{1-\frac{1}{q}}\\
&\left.\times\left(\int_0^1\lambda\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}\left|\tau'\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)\right|^{q}d\lambda\right)^{\frac{1}{q}}\right\}\\
&+\left\{\left(\int_0^1(1-\lambda)\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}d\lambda\right)^{1-\frac{1}{q}}\right.\\
&\times\left(\int_0^1(1-\lambda)\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}\left|\tau'\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)\right|^{q}d\lambda\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\lambda\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}d\lambda\right)^{1-\frac{1}{q}}\\
&\left.\left.\times\left(\int_0^1\lambda\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\beta}\left|\tau'\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)\right|^{q}d\lambda\right)^{\frac{1}{q}}\right\}\right]
\end{aligned}
$$

By computing the above integrals, one can have the required result. □

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INEQUALITIES OF CHEBYSHEV-PÓLYA-SZEGÖ TYPE VIA GENERALIZED PROPORTIONAL FRACTIONAL INTEGRAL OPERATORS

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Abstract. This study is an example of a solid connection between fractional analysis and inequality theory, and includes new inequalities of the Pólya-Szegö-Chebyshev type obtained with the help of Generalized Proportional Fractional integral operators. The results have been performed by using Generalized Proportional Fractional integral operators, some classical inequalities such as AM-GM inequality, Cauchy-Schwarz inequality and Taylor series expansion of exponential function. The findings give new approaches to some types of inequalities that have involving the product of two functions in inequality theory.

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1. Introduction and preliminaries

Inequalities are a concept that contributes to the solution of many problems with their applications in different disciplines such as engineering, physics, statistics and economics as well as being used in many branches of mathematics. With the help of convex, differentiable, integrable, continuous, limited, synchronous functions or some other specially defined functions, many different types of inequality have been proved in the inequality theory field. These inequalities were brought to the literature under different names and later became functional with their applications. We will start by expressing Chebyshev inequality, one of the major inequality types of inequality theory (see [2]):

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \quad (1.1)$$

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where f and g are two integrable functions which are synchronous on $[a,b]$, i.e.

$$(f(x)-f(y))(g(x)-g(y))\geq 0$$

for any $x,y\in[a,b]$, then the Chebyshev inequality states that $T(f,g)\geq 0$.

Chebyshev inequality has been proven for synchronous functions, and has been the focus of researchers and many different versions have been obtained. We encourage interested readers to review the following articles. Chebyshev inequality has been proven for synchronous functions and has been the focus of researchers and many different versions have been obtained. We encourage interested readers to review the following articles [3, 14, 21] and [15].

Another interesting inequality is the Pólya-Szegö inequality, which gives boundaries for two functions that can be integrated and their product. This inequality is given as follows: (see [18])

$$\frac{\int_a^b f^2(x)\,dx\int_a^b g^2(x)\,dx}{\left(\int_a^b f(x)\,g(x)\,dx\right)^2}\leq\frac{1}{4}\left(\sqrt{\frac{MN}{mn}}+\sqrt{\frac{mn}{MN}}\right)^2$$

This inequality is very useful in proving Grüss and Chebyshev type inequalities. With the help of this inequality, a Chebyshev-Grüss type inequality is expressed by Dragomir and Diamond as follows in [7]:

Theorem 1. *Let $f,g:[a,b]\to\mathbb{R}_+$ be two integrable functions so that*

$$0<m\leq f(x)\leq M<\infty$$
$$0<n\leq g(x)\leq N<\infty$$

for $x\in[a,b]$. Then we have

$$|T(f,g;a,b)|\leq\frac{1}{4}\frac{(M-m)(N-n)}{\sqrt{mnMN}}\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.2) in the sense it can not be replaced by a smaller constant.

Remark 1 (see [7]). Assume that the inequality in (1.2) holds with a constant $c>0$, i.e.,

$$|T(f,g;a,b)|\leq c\frac{(M-m)(N-n)}{\sqrt{mnMN}}\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)$$

We choose the functions as $f=g$ with

$$f(x)=\begin{cases} cm, & x\in\left[a,\dfrac{a+b}{2}\right]\\[2ex] M, & x\in[\dfrac{a+b}{2},b]\end{cases}$$

where $0 < m < M < \infty$, then

$$mM \leq c(M-m)^2 \tag{1.3}$$

for any $0 < m < M < \infty$. If in (1.3) we consider $m = 1-\varepsilon,\ M = 1+\varepsilon,\ \varepsilon \in (0,1)$, then we get $1-(\varepsilon)^2 \leq 4c$ for any $\varepsilon \in (0,1)$, which show that $c \geq \frac{1}{4}$.

Although fractional analysis origin dates back to the beginning of classical analysis, it has developed quite rapidly in recent years. Many mathematicians who researched in this field contributed to this development and made efforts to strengthen the relationship between fractional analysis and other fields. With the introduction of new fractional derivative and integral operators, the application opportunity for many real-world problems has been revealed. The majority of the new operators came to the fore with different features such as singularity, location and generalization, and gained functionality thanks to their effective use in application areas (see the papers [1, 4–6, 8–10, 12, 13, 16, 17, 19, 20]). Due to the intensive work on it, the Riemann-Liouville integral operator is a prominent operator and is defined as follows.

Definition 1. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-x)^{\alpha-1}f(x)dx, \quad t > a$$

and

$$J^{\alpha}_{b-}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (x-t)^{\alpha-1}f(x)dx, \quad t < b$$

respectively. Here $\Gamma(t)$ is the Gamma function and its definition is $\Gamma(t) = \int_0^{\infty} e^{-t}t^{x-1}dx$. It is to be noted that $J^0_{a+}f(t) = J^0_{b-}f(t) = f(t)$ in the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

We will continue with the Generalized Proportional Fractional integral operator, which has been described recently and has been the main source of motivation for many studies in the literature with its use in many areas, especially the inequality theory. In [11], Jarad et al. identified the proportional generalized fractional integrals that satisfy many important features as follows:

Definition 2. The left and right generalized proportional fracitonal integral operators are respectively defined by

$${}_{a+}\mathfrak{J}^{\alpha,\lambda}f(t) = \frac{1}{\lambda^{\alpha}\Gamma(\alpha)}\int_a^t e^{\left[\frac{\lambda-1}{\lambda}(t-x)\right]}(t-x)^{\alpha-1}f(x)dx, \quad t > a$$

and

$${}_{b-}\mathfrak{J}^{\alpha,\lambda}f(t) = \frac{1}{\lambda^{\alpha}\Gamma(\alpha)}\int_t^b e^{\left[\frac{\lambda-1}{\lambda}(x-t)\right]}(x-t)^{\alpha-1}f(x)dx, \quad t < b$$

where $\lambda \in (0,1]$ and $\alpha \in \mathbb{C}$ and $\mathbb{R}(\alpha) > 0$.

The main aim of this study is to obtain new Pólya-Szegö type inequalities by using Generalized Proportional Fractional integral operators. Taylor series expansion of exponential function is used in addition to some classical inequalities to obtain main results. The study is enriched by giving special cases of our results.

2. Main results

In this section, we prove certain Pólya-Szegö type integral inequalities for positive integral functions involving Generalized Proportional Fractional integral operator.

Lemma 1. *Assume that f and g are two positive integrable function on $[0,\infty)$. If v_1, v_2, w_1 and w_2 are positive functions such that*

$$0 < v_1(\tau) \leq f(\tau) \leq v_2(\tau) \tag{2.1}$$

$$0 < w_1(\tau) \leq g(\tau) \leq w_2(\tau)$$

for $\tau \in [0,x]$, $x > 0$, then we have the following inequality;

$$\frac{{}_0^{GPF}I^{\alpha,p_1}[w_1w_2f^2](x)\,{}_0^{GPF}I^{\alpha,p_1}[v_1v_2g^2](x)}{\left({}_0^{GPF}I^{\alpha,p_1}\left[(v_1w_1+v_2w_2)fg\right](x)\right)^2} \leq \frac{1}{4}. \tag{2.2}$$

where $\alpha \in (n, n+1]$ and $n = 1,2,3,\ldots$.

Proof. From (2.1) for $\tau \in [0,x], x > 0$, we can write

$$\left(\frac{v_2(\tau)}{w_1(\tau)} - \frac{f(\tau)}{g(\tau)}\right) \geq 0 \tag{2.3}$$

and

$$\left(\frac{f(\tau)}{g(\tau)} - \frac{v_1(\tau)}{w_2(\tau)}\right) \geq 0. \tag{2.4}$$

If we multiply (2.3) and (2.4) side by side, we have

$$\left(\frac{v_2(\tau)}{w_1(\tau)} - \frac{f(\tau)}{g(\tau)}\right)\left(\frac{f(\tau)}{g(\tau)} - \frac{v_1(\tau)}{w_2(\tau)}\right) \geq 0.$$

This implies the following inequality,

$$\Big(v_1(\tau)w_1(\tau) + v_2(\tau)w_2(\tau)\Big)f(\tau)g(\tau) \geq w_1(\tau)w_2(\tau)f^2(\tau) + v_1(\tau)v_2(\tau)g^2(\tau). \tag{2.5}$$

Since all the functions are positive, $p_1 \in (0,1]$, $x \geq \tau$ and $x > 0$, by multiplying both sides of (2.5) by $\frac{1}{p_1^{\alpha}\Gamma(\alpha)} e^{\frac{p_1-1}{p_1}(x-\tau)}(x-\tau)^{\alpha-1}$ and then integrating the resulting inequality with respect to τ over $(0,x)$, we get

$$\frac{1}{p_1^{\alpha}\Gamma(x)} \int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}(x-\tau)^{\alpha-1}\Big(v_1(\tau)w_1(\tau) + v_2(\tau)w_2(\tau)\Big)f(\tau)g(\tau)d\tau$$

$$\geq \frac{1}{p_1^{\alpha}\Gamma(x)}\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}(x-\tau)^{\alpha-1}\Big(w_1(\tau)w_2(\tau)f^2(\tau)\Big)d\tau$$
$$+\frac{1}{p_1^{\alpha}\Gamma(x)}\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}(x-\tau)^{\alpha-1}v_1(\tau)v_2(\tau)g^2(\tau)d\tau.$$

Namely,

$$ {}_0^{GPF}I^{\alpha,p_1}\Big[(v_1w_1+v_2w_2)fg\Big](x) \geq {}_0^{GPF}I^{\alpha,p_1}[w_1w_2f^2](x)+{}_0^{GPF}I^{\alpha,p_1}[v_1v_2g^2](x). \quad (2.6)$$

Applying the A.M-G.M inequality i.e $(a+b\geq 2\sqrt{ab}, a,b\in\Re^+)$, we have

$$ {}_0^{GPF}I^{\alpha,p_1}\Big[(v_1w_1+v_2w_2)fg\Big](x) \geq 2\sqrt{{}_0^{GPF}I^{\alpha,p_1}\Big[w_1w_2f^2\Big](x){}_0^{GPF}I^{\alpha,p_1}\Big[v_1v_2g^2\Big](x)}.$$

This can be written as

$$ {}_0^{GPF}I^{\alpha,p_1}\Big[w_1w_2f^2\Big](x){}_0^{GPF}I^{\alpha,p_1}\Big[v_1v_2g^2\Big](x) \leq \frac{1}{4}\Big({}_0^{GPF}I^{\alpha,p_1}\Big[(v_1w_1+v_2w_2)fg\Big](x)\Big)^2.$$

□

Corollary 1. *If we take into account $v_1=m, v_2=m, w_1=n$ and $w_2=N$ in (2.2), then we have the following new inequality;*

$$\frac{\Big({}_0^{GPF}I^{\alpha,p_1}f^2\Big)(x)\Big({}_0^{GPF}I^{\alpha,p_1}g^2\Big)(x)}{\Big(\Big({}_0^{GPF}I^{\alpha,p_1}fg\Big)(x)\Big)^2} \leq \frac{1}{4}\left(\sqrt{\frac{mn}{MN}}+\sqrt{\frac{MN}{mn}}\right)^2.$$

Lemma 2. *Let f and g be two positive integrable functions on $[0,\infty)$. Assume that there exists four positive integrable functions v_1, v_2, w_1 and w_2 satisfying condition (2.1). Then the following inequality holds:*

$$ {}_0^{GPF}I^{\alpha,p_1}[v_1v_2](x){}_0^{GPF}I^{\beta,p_2}[w_1w_2](x)\times {}_0^{GPF}I^{\alpha,p_1}[f^2](x){}_0^{GPF}I^{\beta,p_2}[g^2](x) \quad (2.7)$$
$$\leq \frac{1}{4}\left({}_0^{GPF}I^{\alpha,p_1}[v_1f](x){}_0^{GPF}I^{\beta,p_2}[w_1g](x)+{}_0^{GPF}I^{\alpha,p_1}[v_2f](x){}_0^{GPF}I^{\beta,p_2}[w_2g](x)\right)^2$$

where $\alpha\in(n,n+1]$ and $\beta\in(k,k+1]$, $n,k=0,1,2,3,\ldots$.

Proof. From (2.1), we get

$$\left(\frac{v_2(\tau)}{w_1(\xi)}-\frac{f(\tau)}{g(\xi)}\right)\geq 0$$

and

$$\left(\frac{f(\tau)}{g(\xi)}-\frac{v_1(\tau)}{w_2(\xi)}\right)\geq 0.$$

Which leads to

$$\left(\frac{v_1(\tau)}{w_2(\xi)}+\frac{v_2(\tau)}{w_1(\xi)}\right)\frac{f(\tau)}{g(\xi)}\geq\frac{f^2(\tau)}{g^2(\xi)}+\frac{v_1(\tau)v_2(\tau)}{w_1(\xi)w_2(\xi)}. \tag{2.8}$$

Multiplying both sides of (2.8) by $w_1(\xi)w_2(\xi)g^2(\xi)$, we have

$$v_1(\tau)f(\tau)w_1(\xi)g(\xi)+v_2(\tau)f(\tau)w_2(\xi)g(\xi)\geq w_1(\xi)w_2(\xi)f^2(\tau)+v_1(\tau)v_2(\tau)g^2(\xi). \tag{2.9}$$

Multiplying both sides (2.9) by

$$\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}e^{\frac{p_1-1}{p_1}(x-\tau)}(x-\tau)^{\alpha-1}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}$$

and integrating the resulting inequality with respect to τ and ξ over $(0,x)^2$, we get

$$\begin{aligned}
&\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\\
&\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}v_1(\tau)f(\tau)w_1(\xi)g(\xi)d\tau d\xi\\
&+\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\\
&\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}v_2(\tau)f(\tau)w_2(\xi)g(\xi)d\tau d\xi\\
&\geq\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\\
&\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}w_1(\xi)w_2(\xi)f^2(\tau)d\tau d\xi\\
&+\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\\
&\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}v_1(\tau)v_2(\tau)g^{(}\xi)d\tau d\xi.
\end{aligned}$$

If we re-write the above inequality with the help of the definition of operator, we get

$$\begin{aligned}
&{}_0^{GPF}I^{\alpha,p_1}[v_1f](x){}_0^{GPF}I^{\beta,p_2}[w_1g](x)+{}_0^{GPF}I^{\alpha,p_1}[v_2f](x){}_0^{GPF}I^{\beta,p_2}[w_2g](x)\\
&\geq{}_0^{GPF}I^{\alpha,p_1}[f^2](x){}_0^{GPF}I^{\beta,p_2}[w_1w_2](x)+{}_0^{GPF}I^{\alpha,p_1}[v_1v_2](x){}_0^{GPF}I^{\beta,p_2}[g^2](x).
\end{aligned}$$

Applying the A.M-G.M inequality, we have

$$\begin{aligned}
&{}_0^{GPF}I^{\alpha,p_1}[v_1f](x){}_0^{GPF}I^{\beta,p_2}[w_1g](x)+{}_0^{GPF}I^{\alpha,p_1}[v_2f](x){}_0^{GPF}I^{\beta,p_2}[w_2g](x)\\
&\geq 2\sqrt{{}_0^{GPF}I^{\alpha,p_1}[f^2](x){}_0^{GPF}I^{\beta,p_2}[w_1w_2](x)\times{}_0^{GPF}I^{\alpha,p_1}[v_1v_2](x){}_0^{GPF}I^{\beta,p_2}[g^2](x)}.
\end{aligned}$$

Which leads to the desired inequality in (2.7). The proof is completed. □

Corollary 2. *If we set $v_1 = M$, $v_2 = M$, $w_1 = n$ and $w_2 = N$ in (2.7), then we have the following inequality;*

$$ {}_0^{GPF}I^{\alpha,p_1}(x)\,{}_0^{GPF}I^{\beta,p_2}(x)\frac{\left({}_0^{GPF}I^{\alpha,p_1}f^2\right)(x)\left({}_0^{GPF}I^{\beta,p_2}g^2\right)(x)}{\left(\left({}_0^{GPF}I^{\alpha,p_1}f\right)(x)\left({}_0^{GPF}I^{\beta,p_2}g\right)(x)\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{mn}{MN}}+\sqrt{\frac{MN}{mn}}\right)^2 $$

Let $a = \frac{p_1-1}{p_1}$. The Taylor Series of $exp(a(x-\tau))$ at the point x is given by

$$ {}_0^{GPF}I^{\alpha,p_1}(x) = \frac{1}{p_1^{\alpha}\Gamma(\alpha)}\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}(x-\tau)^{\alpha-1}d\tau $$

$$ = \frac{1}{p_1\Gamma(\alpha)}\int_0^x \sum_{k=0}^{\infty}\frac{\left(a(x-\tau)\right)^{k_1}}{k_1!}(x-\tau)^{\alpha-1}d\tau $$

$$ = \frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{1}{k!}\frac{a^{k_1}x^{\alpha+k_1}}{\alpha+k_1} $$

$$ {}_0^{GPF}I^{\beta,p_2}(x) = \frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{b^{k_2}x^{\beta+k_2}}{k_2!(\beta+k_2)} $$

$$ \frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}x^{\alpha+k_1}}{(\alpha+k_1)k!}\sum_{k_2=0}^{\infty}\frac{b^{k_2}x^{\beta+k_2}}{(\beta+k_2)k_2!} $$

$$ \times\frac{\left({}_0^{GPF}I^{\alpha,p_1}f^2\right)(x)\left({}_0^{GPF}I^{\beta,p_2}g^2\right)(x)}{\left(\left({}_0^{GPF}I^{\alpha,p_1}f\right)(x)\left({}_0^{GPF}I^{\beta,p_2}g\right)(x)\right)^2} $$

$$ \leq \frac{1}{4}\left(\sqrt{\frac{mn}{MN}}+\sqrt{\frac{MN}{mn}}\right)^2. $$

Lemma 3. *Let f and g be two positive integrable function on $[0,\infty)$. Assume that there exist four positive integrable functions v_1, v_2, w_1 and w_2 satisfying condition (2.1) then the following inequality holds.*

$$ {}_0^{GPF}I^{\alpha,p_1}[f^2](x)\,{}_0^{GPF}I^{\beta,p_2}[g^2](x) \leq {}_0^{GPF}I^{\alpha,p_1}[\frac{v_2fg}{w_1}](x)\,{}_0^{GPF}I^{\beta,p_2}[\frac{w_2fg}{v_1}](x). \tag{2.10} $$

where $\alpha \in (n,n+1]$, $\beta \in (k,k+1]$, $n,k = 0,1,2,3,\ldots$.

Proof. Using the condition (2.1), we get

$$ f^2(\tau) \leq \frac{v_2(\tau)}{w_1(\tau)}f(\tau)g(\tau). \tag{2.11} $$

Multiplying both sides of (2.11) by $\frac{1}{p_1^{\alpha}\Gamma(\alpha)}e^{\frac{p_1-1}{p_1}(x-\tau)}(x-\tau)^{\alpha-1}$ and integrating the resulting inequality with respect to τ over $(0,x)$, we get

$$\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}(x-\tau)^{\alpha-1}f^2(\tau)d\tau$$

$$\leq \frac{1}{p_1^{\alpha}\Gamma(\alpha)}\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}(x-\tau)^{\alpha-1}\frac{v_2\tau}{w_1(\tau)}f(\tau)g(\tau)d\tau$$

$$ {}_0^{GPF}I^{\alpha,p_1}[f^2](x) \leq {}_0^{GPF}I^{\alpha,p}[\frac{v_2fg}{w_1}](x). \tag{2.12}$$

Similarly, we can write

$$g^2(\xi) \leq \frac{w_2(\xi)}{v_1(\xi)}f(\xi)g(\xi).$$

By a similar argument, we have

$$\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\xi)^{\beta-1}g^2(\xi)d\xi$$

$$\leq \frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\xi)^{\beta-1}\frac{w_2(\xi)}{v_1(\xi)}f(\xi)g(\xi)d\xi.$$

Which implies

$$ {}_0^{GPF}I^{\beta,p_2}[g^2](x) \leq {}_0^{GPF}I^{\beta,p_2}[\frac{w_2fg}{v_1}](x). \tag{2.13}$$

Multiplying (2.12) and (2.13), we get then (2.10). Then the desired inequality is obtained such that

$$ {}_0^{GPF}I^{\alpha,p_1}[f^2](x){}_0^{GPF}I^{\beta,p_2}[g^2](x) \leq {}_0^{GPF}I^{\alpha,p_1}[\frac{v_2fg}{w_1}](x){}_0^{GPF}I^{\beta,p_2}[\frac{w_2fg}{v_1}](x).$$

□

Corollary 3. *If we choose $v_1 = m$, $v_2 = M, w_1 = n$ and $w_2 = N$ in (2.10), then we have the following inequality;*

$$\frac{\left({}_0^{GPF}I^{\alpha,p_1}f^2\right)(x)\left({}_0^{GPF}I^{\alpha,p_1}g^2\right)(x)}{\left({}_0^{GPF}I^{\alpha,p_1}fg\right)(x)\left({}_0^{GPF}I^{\alpha,p_1}fg\right)(x)} \leq \frac{MN}{mn}.$$

Theorem 2. *Let f and g be two positive integrable function on $[0,\infty)$. Assume that there exist four positive integrable functions v_1, v_2, w_1 and w_2 satisfying condition*

(2.1) *then the following inequality holds:*

$$\left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right)\left({}_0^{GPF}I^{\beta,p_2}fg\right)(x)$$
$$+\left(\frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{b^{k_2}}{k_2!}\frac{x^{\beta+k_2}}{\beta+k_2}\right)\left({}_0^{GPF}I^{\alpha,p_1}fg\right)(x)$$
$$-\left({}_0^{GPF}I^{\alpha,p_1}f\right)(x)\left({}_0^{GPF}I^{\beta,p_2}g\right)(x)-\left({}_0^{GPF}I^{\beta,p_2}f\right)(x)\left({}_0^{GPF}I^{\alpha,p_1}g\right)(x)$$
$$\leq\left|A_1(f,v_1,v_2)(x)+A_2(f,v_1,v_2)(x)\right|^{\frac{1}{2}}\times\left|A_1(g,w_1,w_2)(x)+A_2(g,w_1,w_2)(x)\right|^{\frac{1}{2}},$$

for $\alpha\in(n,n+1]$, $\beta\in(k,k+1]$, $n,k=0,1,2,3,\ldots$, *where*

$$A_1(u,v,w)(x)=\left(\frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{b^{k_2}}{k_2!}\frac{x^{\beta+k_2}}{\beta+k_2}\right)$$
$$\times\frac{\left({}_0^{GPF}I^{\alpha,p_1}[(v+w)u](x)\right)^2}{4\,{}_0^{GPF}I^{\alpha,p_1}[vw](x)}-\left({}_0^{GPF}I^{\alpha,p_1}u\right)(x)\left({}_0^{GPF}I^{\beta,p_2}u\right)(x)$$

and

$$A_2(u,v,w)(x)=\left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right)$$
$$\times\frac{\left({}_0^{GPF}I^{\beta,p_2}[(v+w)u](x)\right)^2}{4\,{}_0^{GPF}I^{\beta,p_2}[vw](x)}-\left({}_0^{GPF}I^{\alpha,p_1}u\right)(x)\left({}_0^{GPF}I^{\beta,p_2}u\right)(x).$$

Proof. Let f and g be two positive integrable functions on $[0,\infty)$ for $\tau,\xi\in(0,x)$ with $x>0$, we define $H(\tau,\xi)$ as

$$H(\tau,\xi)=\Big(f(\tau)-f(\xi)\Big)\Big(g(\tau)-g(\xi)\Big).$$

Namely

$$H(\tau,\xi)=f(\tau)g(\tau)+f(\xi)g(\xi)-f(\tau)g(\xi)-f(\xi)g(\tau). \tag{2.14}$$

Multiplying both sides of (2.14) by

$$\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}.$$

Then by integrating the resulting inequality with respect to τ and ξ over $(0,x)^2$, we get

$$\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}H(\tau,\xi)d\tau d\xi$$

$$
\begin{aligned}
&= \frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}f(\tau)g(\tau)d\tau d\xi \\
&+\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}f(\xi)g(\xi)d\tau d\xi \\
&-\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}f(\tau)g(\xi)d\tau d\xi \\
&-\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}f(\xi)g(\tau)d\tau d\xi \\
&= \left(\frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{b^{k_2}}{k_2!}\frac{x^{\beta+k_2}}{\beta+k_2}\right)\left({}_0^{GPF}I^{\alpha,p_1}fg\right)(x) \\
&+\left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right)\left({}_0^{GPF}I^{\beta,p_2}fg\right)(x) \\
&-\left({}_0^{GPF}I^{\alpha,p_1}f\right)(x)\left({}_0^{GPF}I^{\beta,p_2}g\right)(x)-\left({}_0^{GPF}I^{\beta,p_2}f\right)(x)\left({}_0^{GPF}I^{\alpha,p_1}g\right)(x).
\end{aligned}
$$

Applying the Cauchy-Schwarz inequality, we can write

$$\left|\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}H(\tau,\xi)d\tau d\xi\right|$$

$$\leq\left[\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}f^2(\tau)d\tau d\xi\right.$$

$$+\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}f^2(\xi)d\tau d\xi$$

$$\left.-2\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}f(\tau)f(\xi)d\tau d\xi\right]^{\frac{1}{2}}$$

$$\times\left[\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}g^2(\tau)d\tau d\xi\right.$$

$$+\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}g^2(\xi)d\tau d\xi$$

$$\left.-2\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}g(\tau)g(\xi)d\tau d\xi\right]^{\frac{1}{2}}.$$

As a consequence

$$\left|\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\frac{1}{p_2^{\beta}\Gamma(\beta)}\int_0^x\int_0^x e^{\frac{p_1-1}{p_1}(x-\tau)}e^{\frac{p_2-1}{p_2}(x-\xi)}(x-\tau)^{\alpha-1}(x-\xi)^{\beta-1}H(\tau,\xi)d\tau d\xi\right|$$
$$\leq\left[\left(\frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{b^{k_2}}{k_2!}\frac{x^{\beta+k_2}}{\beta+k_2}\right)\left({}_0^{GPF}I^{\alpha,p_1}f^2\right)(x)\right.$$
$$+\left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right)\left({}_0^{GPF}I^{\beta,p_2}f^2\right)(x)$$
$$\left.-2\left({}_0^{GPF}I^{\beta,p_2}f\right)(x)\left({}_0^{GPF}I^{\alpha,p_1}f\right)(x)\right]^{\frac{1}{2}}$$
$$\left[\left(\frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{b^{k_2}}{k_2!}\frac{x^{\beta+k_2}}{\beta+k_2}\right)\left({}_0^{GPF}I^{\alpha,p_1}g^2\right)(x)\right.$$
$$+\left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right)\left({}_0^{GPF}I^{\beta,p_2}g^2\right)(x)$$
$$\left.-2\left({}_0^{GPF}I^{\beta,p_2}g\right)(x)\left({}_0^{GPF}I^{\alpha,p_1}g\right)(x)\right]^{\frac{1}{2}}.$$

Applying Lemma 1 with $w_1(\tau)=w_2(\tau)=g(\tau)=1$, we get

$$\left(\frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{b^{k_2}}{k_2!}\frac{x^{\beta+k_2}}{\beta+k_2}\right)\left({}_0^{GPF}I^{\alpha,p_1}f^2\right)(x)$$
$$\leq\left(\frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{b^{k_2}}{k_2!}\frac{x^{\beta+k_2}}{\beta+k_2}\right)\frac{\left({}_0^{GPF}I^{\alpha,p_1}[(v_1+v_2)f](x)\right)^2}{4{}_0^{GPF}I^{\alpha,p_1}[v_1v_2](x)}.$$

This implies that

$$\left(\frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{b^{k_2}}{k_2!}\frac{x^{\beta+k_2}}{\beta+k_2}\right)\left({}_0^{GPF}I^{\alpha,p_1}f^2\right)(x)-\left({}_0^{GPF}I^{\alpha,p_1}f\right)(x)\left({}_0^{GPF}I^{\beta,p_2}f\right)(x)$$
$$\leq\left(\frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{b^{k_2}}{k_2!}\frac{x^{\beta+k_2}}{\beta+k_2}\right)\frac{\left({}_0^{GPF}I^{\alpha,p_1}[(v_1+v_2)f](x)\right)^2}{4{}_0^{GPF}I^{\alpha,p_1}[v_1v_2](x)}$$
$$-\left({}_0^{GPF}I^{\alpha,p_1}f\right)(x)\left({}_0^{GPF}I^{\beta,p_2}f\right)(x)=A_1(f,v_1,v_2) \tag{2.15}$$

and

$$\left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right)\left({}_0^{GPF}I^{\beta,p_2}f^2\right)(x)-\left({}_0^{GPF}I^{\alpha,p_1}f\right)(x)\left({}_0^{GPF}I^{\beta,p_2}f\right)(x)$$
$$\leq\left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right)\frac{\left({}_0^{GPF}I^{\beta,p_2}[(v_1+v_2)f](x)\right)^2}{4{}_0^{GPF}I^{\beta,p_2}[v_1v_2](x)}$$
$$-\left({}_0^{GPF}I^{\alpha,p_1}f\right)(x)\left({}_0^{GPF}I^{\beta,p_2}f\right)(x)=A_2(f,v_1,v_2). \tag{2.16}$$

Similarly, applying Lemma 1 with $v_1(\tau)=v_2(\tau)=f(\tau)=1$, we have

$$\left(\frac{1}{p_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{b^{k_2}}{k_2!}\frac{x^{\beta+k_2}}{\beta+k_2}\right){}_0^{GPF}I^{\alpha,p_1}[g^2](x)$$
$$-\left({}_0^{GPF}I^{\alpha,p_1}g\right)(x)\left({}_0^{GPF}I^{\beta,p_2}g\right)(x)\leq A_1(g,w_1,w_2) \tag{2.17}$$

and

$$\left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right){}_0^{GPF}I^{\beta,p_2}[g^2](x)$$
$$-\left({}_0^{GPF}I^{\alpha,p_1}g\right)(x)\left({}_0^{GPF}I^{\beta,p_2}g\right)(x)\leq A_2(g,w_1,w_2). \tag{2.18}$$

Using (2.15)-(2.18), we conclude the result. □

Theorem 3. *Let f and g be two positive integrable function on $[0,\infty)$. Assume that there exist four positive integrable functions v_1,v_2,w_1 and w_2 satisfying condition (2.1) then the following inequality holds:*

$$\left|\left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right){}_0^{GPF}I^{\alpha,p_1}[fg](x)-\left({}_0^{GPF}I^{\alpha,p_1}f\right)(x)({}_0^{GPF}I^{\alpha,p_1}g)(x)\right| \tag{2.19}$$
$$\leq\left|A(f,v_1,v_2)(x)A(g,w_1,w_2)(x)\right|^{\frac{1}{2}}$$

for $\alpha\in(n,n+1]$, $\beta\in(k,k+1]$, $n,k=0,1,2,3,\ldots$, where

$$A(u,v,w)(x)=\left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right)$$

$$\times \frac{\left({}_0^{GPF}I^{\alpha,p_1}[(v+w)u](x)\right)^2}{4\,{}_0^{GPF}I^{\alpha,p_1}[vw](x)} - \left(\left({}_0^{GPF}I^{\alpha,p_1}u\right)(x)\right)^2.$$

Proof. Setting $\alpha = \theta$ in Theorem 2, we obtain (2.19). □

Corollary 4. *Assume that all the assumptions of Theorem 3 satify, then we have the following inequality;*

$$\left|\left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right){}_0^{GPF}I^{\alpha,p_1}[fg](x) - \left({}_0^{GPF}I^{\alpha,p_1}f\right)(x)\left({}_0^{GPF}I^{\alpha,p_1}g\right)(x)\right|$$

$$\leq \left(\frac{1}{p_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a^{k_1}}{k_1!}\frac{x^{\alpha+k_1}}{\alpha+k_1}\right)\frac{(M-m)(N-n)}{4\sqrt{MmNn}} \times \left({}_0^{GPF}I^{\alpha,p_1}f\right)(x)\left({}_0^{GPF}I^{\alpha,p_1}g\right)(x).$$

Proof. If we set $v_1 = m$, $v_2 = M$, $w_1 = n$ and $w_2 = N$ in (2.19), then the proof is completed. We omit the details. □

3. Conflict of interest

All authors declare no conflicts of interest in this paper.

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BEHAVIOR OF SOLUTIONS OF A NONLINEAR IMPULSIVE SYSTEM WITH PIECEWISE CONSTANT MIXED ARGUMENTS

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Abstract. In this paper, we consider a nonlinear impulsive differential equation system with piecewise constant mixed arguments and prove the existence and uniqueness of a solution. Moreover, we obtain sufficient conditions for oscillation of the solution and give an example for it. Finally, we show that the solution of the modified version of this equation is k periodic.

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Keywords: impulsive differential equation, piecewise constant argument, oscillation, periodicity

1. Introduction

There has been increasing interest in the applications of various neural networks in many areas, such as signal processing, image processing, pattern recognition, fault diagnosis, associative memory, and combinatorial optimization ([8], [9], [11], [13], [16]).

In addition, piecewise constant systems proposed by Busenberg and Cooke ([3]) exist in widely expanded areas such as biomedicine, chemistry, mechanical engineering, physics, etc. Such systems are described as a combination of continuous and discrete (hybrid dynamical systems) and so combine properties of both differential and difference equations. Moreover, cellular neural networks with piecewise constant argument have been investigated by some authors ([7], [10], [14], [15]) and also the qualitative analysis of differential equations with piecewise constant arguments of mixed type has been considered by some authors ([2], [5], [6]).

On the other hand, in the real world, many evolutionary processes are characterized by sudden changes at certain times. These changes are said to be impulsive phenomena that take place in many areas, such as physics, chemistry, population dynamics, optimal control, etc.

Most neural networks can be classified into two categories, either continuous or discrete. However, they exhibit characteristics both continuous and discrete styles in many real world systems and natural processes. So, the impulsive differential

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equations with piecewise constant argument which are named by Wiener ([15]) in 1993 have been important. Recently, impulsive cellular neural networks models with piecewise constant argument have been studied in the papers ([1], [4], [12]).

Abbas and Xia [1] discussed the existence and uniqueness of almost automorphic solutions of the following impulsive model of neural network with piecewise constant argument. These kinds of solutions are more general than periodic and almost periodic solutions. They gave several sufficient conditions for the exponential and global attractivity of the solution.

$$\begin{aligned}
\frac{dx_i(t)}{dt} &= -a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{m} c_{ij}(t)g_j\left(x_j\left(2\left[\frac{t+1}{2}\right]\right)\right) \\
&\quad + I_i(t), \\
\triangle x_i\,|_{t=\tau_k} &= J_k(x_i(\tau_k^-)),\ \ i=1,2,\dots,n,\ k\in\mathbb{N}, \\
\triangle(x_i(\tau_k)) &= x_i(\tau_k) - x_i(\tau_k^-),\ \ x_i(\tau_k^-) = \lim_{h\to 0^-} x_i(\tau_k+h),
\end{aligned}$$

where $\tau_k = 2k-1$, $\tau_k^- = 2k-1^-$.

Chiu [4] introduced the following impulsive cellular neural network models with piecewise alternately advanced and retarded argument. Some sufficient conditions were established for the existence and global exponential stability of a unique periodic solution.

$$\begin{aligned}
\frac{dx_i(t)}{dt} &= -a_i(t)x_i(t) + \left\{\sum_{j=1}^{m} b_{ij}(t)f_j(x_j(t)) + c_{ij}(t)g_j\left(x_j\left(2\left[\frac{t+1}{2}\right]\right)\right)\right\} + d_i(t), \\
&\quad t \neq 2k-1, \\
\triangle x_i\,|_{t=2k-1} &= J_k(x_i(2k-1^-)),\ \ i=1,2,\dots,n,\ k\in\mathbb{N}.
\end{aligned}$$

Karakoc et al. [12] proved the existence and uniqueness of a solution of the following nonlinear impulsive differential equation system with piecewise constant argument and obtained sufficient conditions for the oscillation of the solution.

$$\begin{aligned}
x'(t) &= -a(t)x(t) - x([t-1])f(y([t])) + h_1(x([t])), \\
y'(t) &= -b(t)y(t) - y([t-1])g(x([t])) + h_2(y([t])),\ \ t\neq n\in\mathbb{Z}^+,\ t>0, \\
\triangle x(n) &= x(n^+) - x(n^-) = c_n x(n), \\
\triangle y(n) &= y(n^+) - y(n^-) = d_n y(n),\ \ n\in\mathbb{Z}^+.
\end{aligned}$$

In this paper, we consider the following nonlinear impulsive differential equation system with piecewise constant mixed arguments:

$$\begin{aligned}
x'(t) + a(t)x(t) + x([t-1])f_1(y([t])) + x([t])f_2(y([t])) + x([t+1])f_3(y([t])) \\
- h_1(x([t])) = 0,
\end{aligned}$$

$$y'(t)+b(t)y(t)+y([t-1])g_1(x([t]))+y([t])g_2(x([t]))+y([t+1])g_3(x([t]))$$
$$-h_2(y([t]))=0,$$
$$t\neq n\in\mathbb{Z}^+,\ t>0 \quad (1.1)$$

with the impulse conditions

$$\triangle x(n)=\alpha_n x(n)+\beta_n, \quad (1.2)$$
$$\triangle y(n)=\gamma_n y(n)+\delta_n,\ \ t=n\in\mathbb{Z}^+,$$

and the initial conditions

$$x(-1)=x_{-1},\quad x(0)=x_0,\quad y(-1)=y_{-1},\quad y(0)=y_0, \quad (1.3)$$

where a, $b:(0,\infty)\to\mathbb{R}$ are continuous functions, f_i, g_i, $h_j\in C(\mathbb{R},\mathbb{R})$, $i=1,2,3$, $j=1,2$. α_n, β_n, γ_n, and δ_n are sequences of real numbers such that $\alpha_n\neq 1$, $\gamma_n\neq 1$ for all $n\geq 1$. $\triangle u(n)=u(n^+)-u(n^-)$, $u(n^+)=\lim_{t\to n^+}u(t)$, $u(n^-)=\lim_{t\to n^-}u(t)$, $[.]$ denotes the greatest integer functions, and x_{-1}, x_0, y_{-1}, y_0 are given real numbers.

2. Existence of solutions

Definition 1. A pair of functions $(x(t),y(t))$ is said to be a solution of (1.1)-(1.2) if it satisfies the following conditions:

(i) $x:\mathbb{R}^+\cup\{-1\}\to\mathbb{R}$ and $y:\mathbb{R}^+\cup\{-1\}\to\mathbb{R}$ are continuous with a possible exception at the points $[t]\in[0,\infty)$,

(ii) $x(t)$ and $y(t)$ are right continuous and have left-hand limits at the points $[t]\in[0,\infty)$,

(iii) $x(t)$ and $y(t)$ are differentiable and satisfy (1.1) for any $t\in\mathbb{R}^+$ with a possible exception at the points $[t]\in[0,\infty)$ where one-sided derivatives exist,

(iv) $(x(n),y(n))$ satisfies (1.2) for $n\in\mathbb{Z}^+$.

Theorem 1. *If $\alpha_n\neq 1$, $\gamma_n\neq 1$ for all $n\geq 1$ then the initial value problem (1.1)-(1.3) has a unique solution $(x(t),y(t))$ on $\{-1\}\cup[0,\infty)$, which can be formulated on the interval $n\leq t<n+1$, $n\in\mathbb{N}=0,1,2,\ldots$, as*

$$x_n(t)=e^{-\int_n^t a(u)du}\{x(n)+[-x(n-1)f_1(y(n))-x(n)f_2(y(n))-x(n+1)f_3(y(n))$$
$$+h_1(x(n))]\int_n^t e^{\int_n^s a(u)du}ds\},$$
$$y_n(t)=e^{-\int_n^t b(u)du}\{y(n)+[-y(n-1)g_1(x(n))-y(n)g_2(x(n))-y(n+1)g_3(x(n)) \quad (2.1)$$
$$+h_2(y(n))]\int_n^t e^{\int_n^s b(u)du}ds\},$$

where $(x(n),y(n))$ is the unique solution of the difference equations system

$$x(n+1)=\frac{1}{1-\alpha_{n+1}}\Big\{e^{-\int_n^{n+1}a(u)du}\{x(n)+[-x(n-1)f_1(y(n))-x(n)f_2(y(n))$$

$$-x(n+1)f_3(y(n))+h_1(x(n))]\int_n^{n+1} e^{\int_n^s a(u)du}ds\}+\beta_{n+1}\Big\}, \tag{2.2}$$

$$y(n+1)=\frac{1}{1-\gamma_{n+1}}\Big\{e^{-\int_n^{n+1} b(u)du}\{y(n)+[-y(n-1)g_1(x(n))-y(n)g_2(x(n))$$
$$-y(n+1)g_3(x(n))+h_2(y(n))]\int_n^{n+1} e^{\int_n^s b(u)du}ds\}+\delta_{n+1}\Big\}$$

for $n\geq 0$ with the initial conditions (1.3).

Proof. Let $(x(n),y(n))\equiv(x(t),y(t))$ be a solution of (1.1)-(1.2) on the interval $n\leq t<n+1$. So, the system (1.1) can be written as

$$\begin{aligned} x'(t)+a(t)x(t)&=-x(n-1)f_1(y(n))-x(n)f_2(y(n))-x(n+1)f_3(y(n))+h_1(x(n)),\\ y'(t)+b(t)y(t)&=-y(n-1)g_1(x(n))-y(n)g_2(x(n))-y(n+1)g_3(x(n))+h_2(y(n)),\\ &\qquad n\leq t<n+1. \end{aligned} \tag{2.3}$$

(2.3) consists of two linear differential equations which can be solved separately. Let us solve the first equation in system (2.3), the second one will be solved similarly. Multiplying both sides of the first equation in system (2.3) with $\lambda=e^{\int_n^t a(u)du}$ and integrating both sides from n to t

$$\begin{aligned} e^{\int_n^t a(u)du}x(t)-x(n)=\int_n^t e^{\int_n^s a(u)du}[&-x(n-1)f_1(y(n))-x(n)f_2(y(n))\\ &-x(n+1)f_3(y(n))+h_1(x(n))]ds. \end{aligned}$$

Hence,

$$\begin{aligned} x_n(t)&=e^{-\int_n^t a(u)du}\{x(n)+[-x(n-1)f_1(y(n))-x(n)f_2(y(n))\\ &\quad -x(n+1)f_3(y(n))+h_1(x(n))]\int_n^t e^{\int_n^s a(u)du}ds\},\\ y_n(t)&=e^{-\int_n^t b(u)du}\{y(n)+[-y(n-1)g_1(x(n))-y(n)g_2(x(n))\\ &\quad -y(n+1)g_3(x(n))+h_2(x(n))]\int_n^t e^{\int_n^s b(u)du}ds\}. \end{aligned} \tag{2.4}$$

On the other hand, for $n-1\leq t<n$, we have

$$\begin{aligned} x_{n-1}(t)&=e^{-\int_{n-1}^t a(u)du}\{x(n-1)+[-x(n-2)f_1(y(n-1))-x(n-1)f_2(y(n-1))\\ &\quad -x(n)f_3(y(n-1))+h_1(x(n-1))]\int_{n-1}^t e^{\int_{n-1}^s a(u)du}ds\},\\ y_{n-1}(t)&=e^{-\int_{n-1}^t b(u)du}\{y(n-1)+[-y(n-2)g_1(x(n-1))-y(n-1)g_2(x(n-1)) \end{aligned} \tag{2.5}$$

$$-y(n)g_3(x(n-1))+h_2(x(n-1))]\int_{n-1}^{t} e^{\int_{n-1}^{s} b(u)du}ds\}.$$

From impulse conditions

$$\triangle x(n)=x(n^+)-x(n^-)=\alpha_n x(n)+\beta_n.$$

Since the solution is right continuous,

$$x(n^-)=(1-\alpha_n)x(n)-\beta_n \Longrightarrow x_{n-1}(n)=(1-\alpha_n)x_n(n)-\beta_n. \tag{2.6}$$

Taking $t=n$ at (2.4), (2.5) and substituting (2.6),

$$\begin{aligned}(1-\alpha_n)x(n)=e^{-\int_{n-1}^{n} a(u)du}\{&x(n-1)+[-x(n-2)f_1(y(n-1))-x(n-1)f_2(y(n-1))\\ &-x(n)f_3(y(n-1))+h_1(x(n-1))]\int_{n-1}^{n} e^{\int_{n-1}^{s} a(u)du}ds\}+\beta_n.\end{aligned}$$

Dividing both sides with $(1-\alpha_n)$ and taking $n\to n+1$, and following the same procedure above for the other equation's solution $y(t)$, we get the following difference system

$$\begin{aligned}x(n+1)=\frac{1}{1-\alpha_{n+1}}\Big\{&e^{-\int_{n}^{n+1} a(u)du}\{x(n)+[-x(n-1)f_1(y(n))-x(n)f_2(y(n))\\ &-x(n+1)f_3(y(n))+h_1(x(n))]\int_{n}^{n+1} e^{\int_{n}^{s} a(u)du}ds\}+\beta_{n+1}\Big\},\end{aligned} \tag{2.7}$$

$$\begin{aligned}y(n+1)=\frac{1}{1-\gamma_{n+1}}\Big\{&e^{-\int_{n}^{n+1} b(u)du}\{y(n)+[-y(n-1)g_1(x(n))-y(n)g_2(x(n))\\ &-y(n+1)g_3(x(n))+h_2(y(n))]\int_{n}^{n+1} e^{\int_{n}^{s} b(u)du}ds\}+\delta_{n+1}\Big\}\end{aligned}$$

for $n\geq 0$. Take $n=0$, then

$$\begin{aligned}x(1)=\frac{1}{1-\alpha_1}\Big\{&e^{-\int_{0}^{1} a(u)du}\{x(0)+[-x(-1)f_1(y(0))-x(0)f_2(y(0))\\ &-x(1)f_3(y(0))+h_1(x(0))]\int_{0}^{1} e^{\int_{0}^{s} a(u)du}ds\}+\beta_1\Big\}.\end{aligned}$$

Since we know $x(-1)$, $x(0)$, $y(0)$, we can solve $x(1)$ uniquely and the same way can be applied for $y(1)$ also. So, step by step we can find solutions. Hence, the solution of the difference system and also the solution of the system (1.1)-(1.2) is unique. □

Remark 1. In the case of $a(t)\equiv a$, $b(t)\equiv b$ in the system (1.1), the solution (2.1) of (1.1)-(1.2) and the difference equations system (2.2) are reduced to the following forms, respectively:

$$x_n(t)=e^{-a(t-n)}\{x(n)+[-x(n-1)f_1(y(n))-x(n)f_2(y(n))-x(n+1)f_3(y(n))$$

$$+h_1(x(n))](\frac{e^{a(t-n)}-1}{a})\},$$

$$y_n(t)=e^{-b(t-n)}\{y(n)+[-y(n-1)g_1(x(n))-y(n)g_2(x(n))-y(n+1)g_3(x(n))$$

$$+h_2(y(n))](\frac{e^{b(t-n)}-1}{b})\} \tag{2.8}$$

and

$$x(n+1)=\frac{1}{1-\alpha_{n+1}}\Big\{e^{-a}\{x(n)+[-x(n-1)f_1(y(n))-x(n)f_2(y(n))$$

$$-x(n+1)f_3(y(n))+h_1(x(n))](\frac{e^{a}-1}{a})\}+\beta_{n+1}\Big\},$$

$$y(n+1)=\frac{1}{1-\gamma_{n+1}}\Big\{e^{-b}\{y(n)+[-y(n-1)g_1(x(n))-y(n)g_2(x(n))$$

$$-y(n+1)g_3(x(n))+h_2(y(n))](\frac{e^{b}-1}{b})\}+\delta_{n+1}\Big\}. \tag{2.9}$$

3. Main results

Definition 2. A solution $x(t)$ and $y(t)$ of the first and second equations of system (1.1), respectively are said to be oscillatory if $x(t)$ and $y(t)$ have arbitrarily large zeros. Otherwise, it is said to be nonoscillatory.

Definition 3. The solution of system (1.1)-(1.2) is called oscillatory if $x(t)$ and $y(t)$ are oscillatory.

Definition 4. A solution $x(n)$ (or $y(n)$) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory.

Theorem 2. *Assume that there exists $M_i>0$ and $N_j>0$ such that $f_i(u)\geq M_i$ and $g_j(u)\geq N_j$, $i,j=1,3$, $f_2(u)$, $g_2(u)>0$ for all $u\in\mathbb{R}$, $uh_1(u)<0$ and $uh_2(u)<0$ for $u\neq 0$, and $\alpha_n<1$, $\gamma_n<1$ and $\beta_n=0$, $\delta_n=0$ for $n\in\mathbb{Z}^+$. If the following conditions are satisfied, then all solutions of system (2.2) are oscillatory:*

$$\lim_{n\to\infty} sup[\int_n^{n+1}(1-\alpha_n)e^{\int_{n-1}^{s}a(u)du}ds$$

$$+M_3\int_n^{n+1}e^{\int_n^s a(u)du}ds\int_{n-1}^{n}e^{\int_{n-1}^{s}a(u)du}ds]>\frac{1}{M_1}, \tag{3.1}$$

$$\lim_{n\to\infty} sup[\int_n^{n+1}(1-\gamma_n)e^{\int_{n-1}^{s}b(u)du}ds$$

$$+N_3\int_n^{n+1}e^{\int_n^s b(u)du}ds\int_{n-1}^{n}e^{\int_{n-1}^{s}b(u)du}ds]>\frac{1}{N_1}. \tag{3.2}$$

Proof. Let $(x(n),y(n))$ be a solution of difference equation system (2.2). Suppose that $x(n)>0$, $x(n-1)>0$, and $x(n-2)>0$ for $n>N$, where N is sufficiently large. From the first equation of (2.2), taking $n-1$ in place of n, we have

$$\begin{aligned}&[(1-\alpha_n)e^{\int_{n-1}^{n}a(u)du}+f_3(y(n-1))\int_{n-1}^{n}e^{\int_{n-1}^{s}a(u)du}ds]x(n)\\ &=x(n-1)+[-x(n-2)f_1(y(n-1))-x(n-1)f_2(y(n-1))\\ &\qquad+h_1(x(n-1))]\int_{n-1}^{n}e^{\int_{n-1}^{s}a(u)du}ds\\ &<x(n-1).\end{aligned}$$

Hence,

$$[(1-\alpha_n)e^{\int_{n-1}^{n}a(u)du}+f_3(y(n-1))\int_{n-1}^{n}e^{\int_{n-1}^{s}a(u)du}ds]x(n)<x(n-1).$$

Multiplying both sides of this inequality by $-f_1(y(n))\int_n^{n+1}e^{\int_n^s a(u)du}ds<0$ and adding $x(n)+h_1(x(n))\int_n^{n+1}e^{\int_n^s a(u)du}ds$ to both sides, we obtain from (2.2) that

$$\begin{aligned}&-[(1-\alpha_n)e^{\int_{n-1}^{n}a(u)du}+f_3(y(n-1))\int_{n-1}^{n}e^{\int_{n-1}^{s}a(u)du}ds]x(n)f_1(y(n))\int_n^{n+1}e^{\int_n^s a(u)du}ds\\ &+x(n)+h_1(x(n))\int_n^{n+1}e^{\int_n^s a(u)du}ds\\ &>-x(n-1)f_1(y(n))\int_n^{n+1}e^{\int_n^s a(u)du}ds+x(n)+h_1(x(n))\int_n^{n+1}e^{\int_n^s a(u)du}ds\\ &>x(n)+[-x(n-1)f_1(y(n))-x(n)f_2(y(n))+h_1(x(n))]\int_n^{n+1}e^{\int_n^s a(u)du}ds\\ &=[(1-\alpha_{n+1})e^{\int_n^{n+1}a(u)du}+f_3(y(n))\int_n^{n+1}e^{\int_{n+1}^{s}a(u)du}ds]x(n+1)\\ &>0.\end{aligned}$$

Hence,

$$\begin{aligned}&-[(1-\alpha_n)e^{\int_{n-1}^{n}a(u)du}+f_3(y(n-1))\int_{n-1}^{n}e^{\int_{n-1}^{s}a(u)du}ds]x(n)f_1(y(n))\\ &\qquad\cdot\int_n^{n+1}e^{\int_n^s a(u)du}ds+x(n)+h_1(x(n))\int_n^{n+1}e^{\int_n^s a(u)du}ds>0\end{aligned}\tag{3.3}$$

Since $x(n)>0$, $n>N$, and $h_1(x(n))<0$, and dividing to $x(n)$ to both sides of (3.3) we get

$$1>[(1-\alpha_n)\int_n^{n+1}e^{\int_{n-1}^{s}a(u)du}ds+M_3\int_{n-1}^{n}e^{\int_{n-1}^{s}a(u)du}ds\int_n^{n+1}e^{\int_n^s a(u)du}ds]M_1.$$

So, we have

$$\frac{1}{M_1} \geq \lim_{n\to\infty} sup[(1-\alpha_n)\int_n^{n+1} e^{\int_{n-1}^s a(u)du}ds$$
$$+M_3\int_{n-1}^{n} e^{\int_{n-1}^s a(u)du}ds\int_n^{n+1} e^{\int_n^s a(u)du}ds],$$

which contradicts (3.1). If $x(n)<0$, $x(n-1)<0$, and $x(n-2)<0$ for $n>N$, then we obtain the same contradiction. So, the component $x(n)$ of the solution $(x(n),y(n))$ is oscillatory. Similarly, we can show that the component $y(n)$ is oscillatory under condition (3.2). Hence, the proof is complete. □

Corollary 1. *Under the hypotheses of Theorem 2, all solutions of system (1.1)-(1.3) are oscillatory.*

Remark 2. In the case of $a(t)\equiv a$, $b(t)\equiv b$ in the system (1.1), the conditions (3.1) and (3.2) reduce to the following conditions:

$$\lim_{n\to\infty} sup[(1-\alpha_n)(\frac{e^{2a}-e^a}{a})+M_3(\frac{e^a-1}{a})^2]>\frac{1}{M_1},$$
$$\lim_{n\to\infty} sup[(1-\gamma_n)(\frac{e^{2b}-e^b}{b})+N_3(\frac{e^b-1}{b})^2]>\frac{1}{N_1}.$$

Theorem 3. *Assume that there exists K_1, $K_2>0$, $M_i>0$ and $N_i>0$ such that $f_i(u)\geq M_i$ and $g_i(u)\geq N_i$, $i=1,2,3$, for all $u\in\mathbb{R}$, $uh_j(u)<0$, $j=1,2$ for $u\neq 0$, $\alpha_n\leq 1-K_1$, $\gamma_n\leq 1-K_2$ and $\beta_n=0$, $\delta_n=0$ for $n\in\mathbb{N}$. Suppose that the following conditions are satisfied:*

$$(M_2\liminf_{n\to\infty}\int_n^{n+1} e^{\int_n^s a(u)du}ds-1)^2<4M_1\liminf_{n\to\infty}\int_n^{n+1} e^{\int_n^s a(u)du}ds(K_1\liminf_{n\to\infty}e^{\int_n^{n+1} a(u)du}$$
$$+M_3\liminf_{n\to\infty}\int_n^{n+1} e^{\int_n^s a(u)du}ds)<\infty, \tag{3.4}$$

$$(N_2\liminf_{n\to\infty}\int_n^{n+1} e^{\int_n^s b(u)du}ds-1)^2<4N_1\liminf_{n\to\infty}\int_n^{n+1} e^{\int_n^s b(u)du}ds(K_2\liminf_{n\to\infty}e^{\int_n^{n+1} b(u)du}$$
$$+N_3\liminf_{n\to\infty}\int_n^{n+1} e^{\int_n^s b(u)du}ds)<\infty, \tag{3.5}$$

then all solutions of (2.2) are oscillatory.

Proof. Let $(x(n),y(n))$ be a solution of (2.2). We need to show that under condition (3.4), $x(n)$ is oscillatory. Assume that $x(n)>0$, $x(n-1)>0$ for $n>N$, where N is sufficiently large. From the first equation of (2.2) we obtain that

$$1=\frac{x(n+1)}{x(n)}(1-\alpha_{n+1})e^{\int_n^{n+1} a(u)du}$$

$$+\left[\frac{x(n-1)}{x(n)}f_1(y(n))+f_2(y(n))+\frac{x(n+1)}{x(n)}f_3(y(n))-\frac{h_1(x(n))}{x(n)}\right]\int_n^{n+1}e^{\int_n^s a(u)du}ds.$$

Let $v_n=\dfrac{x(n)}{x(n-1)}$. Since $v_n>0$, $\lim_{n\to\infty} inf v_n\geq 0$, we have

$$1\geq v_{n+1}(1-\alpha_{n+1})e^{\int_n^{n+1}a(u)du}+[\frac{1}{v_n}M_1+M_2+v_{n+1}M_3]\int_n^{n+1}e^{\int_n^s a(u)du}ds. \tag{3.6}$$

So, we need to consider two cases.

Case 1. Let $\liminf\limits_{n\to\infty} v_n=v=+\infty$. Then, from (3.6) we get

$$\begin{aligned}1\geq &\liminf_{n\to\infty}v_{n+1}\liminf_{n\to\infty}(1-\alpha_{n+1})\liminf_{n\to\infty}e^{\int_n^{n+1}a(u)du}\\ &+M_1\liminf_{n\to\infty}\frac{1}{v_n}\liminf_{n\to\infty}\int_n^{n+1}e^{\int_n^s a(u)du}ds\\ &+M_2\liminf_{n\to\infty}\int_n^{n+1}e^{\int_n^s a(u)du}ds\\ &+M_3\liminf_{n\to\infty}v_{n+1}\liminf_{n\to\infty}\int_n^{n+1}e^{\int_n^s a(u)du}ds\end{aligned}$$

which is a contradiction. So, we consider the second case.

Case 2. Let $\liminf\limits_{n\to\infty} v_n=v<\infty$. If the first equation of (2.2) is divided by $x(n-1)$, then we have

$$\begin{aligned}\frac{x(n)}{x(n-1)}=&\frac{x(n+1)}{x(n-1)}(1-\alpha_{n+1})e^{\int_n^{n+1}a(u)du}\\ &+\left[f_1(y(n))+\frac{x(n)}{x(n-1)}f_2(y(n))+\frac{x(n+1)}{x(n-1)}f_3(y(n))-\frac{h_1(x(n))}{x(n-1)}\right]\\ &\cdot\int_n^{n+1}e^{\int_n^s a(u)du}ds.\end{aligned}$$

and then we obtain that

$$v_n\geq v_nv_{n+1}(1-\alpha_{n+1})e^{\int_n^{n+1}a(u)du}+(M_1+v_nM_2+v_nv_{n+1}M_3)\int_n^{n+1}e^{\int_n^s a(u)du}ds. \tag{3.7}$$

Taking the inferior limit on both sides of inequality (3.7), we get

$$\begin{aligned}v\geq &v^2\liminf_{n\to\infty}(1-\alpha_{n+1})\liminf_{n\to\infty}e^{\int_n^{n+1}a(u)du}+\left(M_1+vM_2+v^2M_3\right)\\ &\cdot\liminf_{n\to\infty}\int_n^{n+1}e^{\int_n^s a(u)du}ds.\end{aligned}$$

Let $\liminf_{n\to\infty} e^{\int_n^{n+1} a(u)du} = A$ and $\liminf_{n\to\infty} \int_n^{n+1} e^{\int_n^s a(u)du} ds = B$. Then the last inequality can be written as

$$v \geq v^2 \liminf_{n\to\infty}(1-\alpha_{n+1})A + \left(M_1 + vM_2 + v^2 M_3\right) B. \tag{3.8}$$

Now, we consider two subcases: (i) If $\liminf_{n\to\infty}(1-\alpha_{n+1}) = \infty$, then we have a contradiction from (3.8). (ii) Assume that $0 < K_1 \leq \liminf_{n\to\infty}(1-\alpha_{n+1}) < \infty$. Then from (3.8) we have

$$(K_1A + M_3B)v^2 + (M_2B - 1)v + M_1B \leq 0$$

or

$$(K_1A + M_3B)\left[(v + \frac{M_2B-1}{2(K_1A+M_3B)})^2 + \frac{-(M_2B-1)^2 + 4M_1B(K_1A+M_3B)}{4(K_1A+M_3B)^2}\right] \leq 0.$$

Since $A > 0$, $K_1 > 0$, $M_3 > 0$, and $B > 0$, we have

$$\frac{-(M_2B-1)^2 + 4M_1B(K_1A+M_3B)}{4(K_1A+M_3B)^2} \leq 0,$$

which contradicts condition (3.4). In the case of $x(n) < 0$, $x(n-1) < 0$ for sufficiently large $n > N$, the proof is similar, and we obtain the same contradiction. On the other hand, if we assume that $y(n)$ is a nonoscillatory sequence, then we have a contradiction to condition (3.5). Hence, $(x(n), y(n))$ is an oscillatory solution of the system (2.2). □

Corollary 2. *Under the hypothesis of Theorem 3, all solutions of system (1.1)-(1.2) are oscillatory.*

Remark 3. In the case of $a(t) \equiv a$, $b(t) \equiv b$ in the system (1.1), the conditions (3.4) and (3.5) reduce to the following conditions:

$$4M_1\frac{e^a-1}{a}(K_1e^a + M_3(\frac{e^a-1}{a})) > (M_2(\frac{e^a-1}{a}) - 1)^2,$$
$$4N_1\frac{e^b-1}{b}(K_2e^b + N_3(\frac{e^b-1}{b})) > (N_2(\frac{e^b-1}{b}) - 1)^2.$$

Theorem 4. *The solution of the system in the case of $a(t) \equiv a$, $b(t) \equiv b$, $\alpha_n \equiv \alpha$, $\beta_n \equiv \beta$, $\gamma_n \equiv \gamma$, $\delta_n \equiv \delta$ in (1.1)-(1.2) is k periodic, $k \in \mathbb{N} - \{0\}$, if and only if*

$$x(0) = x(k),\ y(0) = y(k),\ x(-1) = x(k-1),\ y(-1) = y(k-1), \tag{3.9}$$

here $x(n)$ and $y(n)$ are solutions of the equations in (2.2).

Proof. If $x_n(t)$ is periodic with period k, then

$$x_{k+n}(t) = x_n(t-k),\ \ k+n \leq t < k+n+1,\ n = 0,1,2,\ldots$$

This implies that the equalities (3.9) are true. For the proof of sufficiency case, assume that (3.9) is satisfied. So, for $n=0$, we should see

$$x_k(t)=x_0(t-k),\ \ k\le t<k+1.$$

Taking $n=0$ in (2.8),

$$\begin{aligned}x_0(t)=e^{-at}\{&x(0)\\&+[-x(-1)f_1(y(0))-x(0)f_2(y(0))-x(1)f_3(y(0))+h_1(x(0))](\frac{e^{at}-1}{a})\}\end{aligned}\tag{3.10}$$

and writing $t=t-k$ in (3.10), then

$$\begin{aligned}x_0(t-k)=e^{-a(t-k)}\{&x(0)+[-x(-1)f_1(y(0))-x(0)f_2(y(0))-x(1)f_3(y(0))\\&+h_1(x(0))](\frac{e^{a(t-k)}-1}{a})\}.\end{aligned}$$

Now taking $n=k$ in (2.8),

$$\begin{aligned}x_k(t)=e^{-a(t-k)}\{&x(k)+[-x(k-1)f_1(y(k))-x(k)f_2(y(k))\\&-x(k+1)f_3(y(k))+h_1(x(k))](\frac{e^{a(t-k)}-1}{a})\}.\end{aligned}\tag{3.11}$$

Under the assumptions (3.9), we need to show that $x(1)=x(k+1)$. Taking $t=1$ in (3.10), we get

$$x_0(1)=\frac{e^{-a}\{x(0)+[-x(-1)f_1(y(0))-x(0)f_2(y(0))](\frac{e^a-1}{a})\}}{1+e^{-a}f_3(y(0))(\frac{e^a-1}{a})}$$

and from impulse condition for $n=1$, $n=k+1$, respectively, we have

$$x_0(1)=(1-\alpha)x_1(1)-\beta,$$

$$x_k(k+1)=(1-\alpha)x_{k+1}(k+1)-\beta.$$

Writing $t=k+1$ in (3.11), we obtain

$$x_k(k+1)=\frac{e^{-a}\{x(k)+[-x(k-1)f_1(y(k))-x(k)f_2(y(k))+h_1(x(k))](\frac{e^a-1}{a})}{1+e^{-a}f_3(y(k))(\frac{e^a-1}{a})}.$$

Since $x(0)=x(k)$, $y(0)=y(k)$, $x(-1)=x(k-1)$, $y(-1)=y(k-1)$, we find $x(1)=x(k+1)$. Similarly, it can be shown that $y(1)=y(k+1)$. So, we conclude that

$$x_k(t)=x_0(t-k),\ \ k\le t<k+1.$$

Now, taking $n=1$, $t=t-k$ and $n=k+1$ in Eq. (2.8), respectively, then

$$x_1(t-k)=e^{-a(t-k-1)}\{x(0)+[-x(-1)f_1(y(0))-x(0)f_2(y(0))-x(1)f_3(y(0))$$
$$+h_1(x(0))](\frac{e^{a(t-k-1)}-1}{a})\},$$

$$x_{k+1}(t)=e^{-a(t-k-1)}\{x(k+1)+[-x(k)f_1(y(k+1))-x(k+1)f_2(y(k+1))$$
$$-x(k+2)f_3(y(k+1))+h_1(x(k+1))](\frac{e^{a(t-k-1)}-1}{a})\}.$$

Hence, we find

$$x_{k+1}(t)=x_1(t-k),\ \ k+1\leq t<k+2.$$

From (3.9) and $x(1)=x(k+1)$, if we show that $x(2)=x(k+2)$, then we can say

$$x_{k+1}(t)=x_1(t-k),\ \ k+1\leq t<k+2.$$

From impulse conditions for $n=2$,

$$x_1(2)=(1-\alpha)x_2(2)-\beta.$$

Take $n=1$ and $t=2$ in (2.8) for $x_1(2)$

$$e^{-a}\{x(1)+[-x(0)f_1(y(1))-x(1)f_2(y(1))-x(1)f_3(y(1))+h_1(x(1))](\frac{e^a-1}{a})\}$$
$$=(1-\alpha)x(2)-\beta.$$

So, we can write $x(2)$ in terms of $x(0)$, $x(1)$, $y(1)$ and similarly for $n=k+2$,

$$x_{k+1}(k+2)=(1-\alpha)x_{k+2}(k+2).$$

Taking $n=k+1$ and $t=k+2$ in (2.8), then

$$e^{-a}\{x(k+1)+[-x(k)f_1(y(k+1))-x(k+1)f_2(y(k+1))-x(k+2)f_3(y(k+1))$$
$$+h_1(x(k+1))](\frac{e^a-1}{a})\}=(1-\alpha)x(k+2)-\beta.$$

$x(k+2)$ is written in terms of $x(k)$, $x(k+1)$ and $y(k+1)$. By induction,

$$x_{k+n}(t)=x_n(t-k),\ \ k+n\leq t<k+n+1.$$

One can show that

$$y_{k+n}(t)=y_n(t-k),\ \ k+n\leq t<k+n+1$$

by using the steps above. □

Example 1. We consider the nonlinear impulsive differential equations system with piecewise constant argument and variable coefficient

$$x'(t)+\frac{3}{t}x(t)+x([t-1])(e^{-2y([t])}+3)+x([t])(e^{-2y([t])}+2)$$
$$+x([t+1])(e^{-2y([t])}+1)-x([t])=0,$$

$$y'(t)+\frac{3t^2}{t^3+1}y(t)+y([t-1])(x^2([t])+3)+y([t])(x^2([t])+2) \tag{3.12}$$
$$+y([t+1])(x^2([t])+1)-y([t])=0,$$
$$t\neq n\in\mathbb{Z}^+,\, t>0$$

with the impulse conditions

$$\triangle x(n)=3^{-n}x(n) \tag{3.13}$$
$$\triangle y(n)=5^{-n}y(n),\ \ t=n\in\mathbb{Z}^+.$$

It is clear that this satisfies all hypotheses of Theorem 2. So, all solutions of system (3.12)-(3.13) are oscillatory. The solution $(x_n(t),y_n(t))$ of system (3.12)-(3.13) with initial conditions $x(-1)=0.5$, $x(0)=-10$, $y(-1)=-0.5$, $y(0)=0$ is shown in Figure 1.

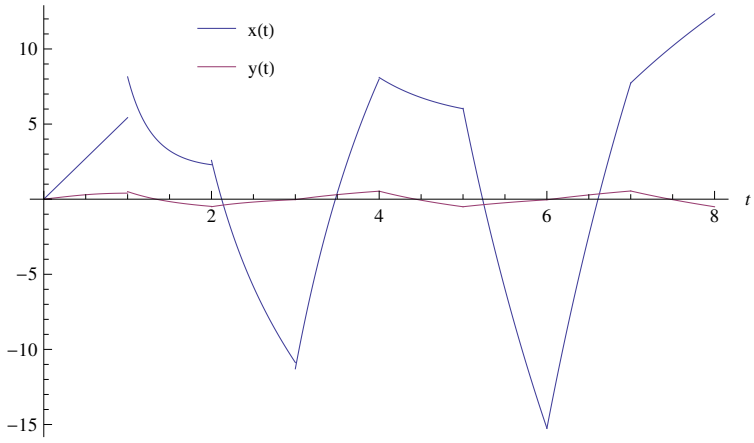

FIGURE 1. Oscillatory solutions of system (3.12)-(3.13) with the initial conditions $x(-1)=0.5$, $x(0)=-10$, $y(-1)=-0.5$, $y(0)=0$.

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THE GEOMETRY ON THE SLOPE OF A MOUNTAIN

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Abstract. The geometry on a slope of a mountain is the geometry of a Finsler metric, called here the *slope metric*. We study the existence of globally defined slope metrics on surfaces of revolution as well as the geodesic's behavior. A comparison between Finslerian and Riemannian areas of a bounded region is also studied.

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1. Introduction

Finsler manifolds, that is n-dimensional smooth manifolds endowed with Finsler metrics, are natural generalization of the well-known Riemannian manifolds. The main difference is that the metric itself and all Finsler geometric quantities depend not only on the point $x \in M$ of the manifold, but also on the direction $y \in T_xM$, where (x,y) are the canonical coordinates of the tangent bundle TM. This directional dependence reveals many hidden geometrical features that are usually obscured by the quadratic form in the y-variable of a Riemannian metric.

The Randers metrics $F = \alpha + \beta$ and the Kropina metrics $F = \frac{\alpha^2}{\beta}$ belong to a larger class of Finsler metrics called (α, β)- metrics since they are obtained by deformations of a Riemannian metric by means of a linear 1-form $\beta = b_i(x)y^i$ on TM. The common characteristic is that they are obtained by rigid translation of a Riemannian unit sphere by a vector field W. The local and global geometries of these Finslerian metrics have been extensively studied ([11]).

Another interesting but much less studied problem is the Matsumoto's slope metric $F = \frac{\alpha^2}{\alpha-\beta}$. Indeed, based on a letter of P. Finsler (1969), M. Matsumoto considered the following problem:

Suppose a person walking on a horizontal plane with velocity c, while the gravitational force is acting perpendicularly on this plane. The person is almost ignorant of the action of this force. Imagine the person walks now with same velocity on the

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inclined plane of angle ε *to the horizontal sea level. Under the influence of gravitational forces, what is the trajectory the person should walk in the center to reach a given destination in the shortest time?*

Based on this, he has formulated the following *Slope principle* ([9]).

With respect to the time measure, a plane (π) *with an angle* ε *inclination can be regarded as a Minkowski plane. The indicatrix curve of the corresponding Minkowski metric is a limaçon, contained in this plane, given by*

$$r = c + a\cos\theta,$$

in the polar coordinates (r,θ) *of* (π)*, whose pole is the origin* O *of* (π) *and the polar axis is the most steepest downhill direction, where* $a = \frac{g}{2}\sin\varepsilon$*, and* g *is the acceleration constant.*

From calculus of variations it follows that for a hiker walking the slope of a mountain under the influence of gravity, the most efficient time minimizing paths are not the Riemannian geodesics, but the geodesics of the slope metric $F = \frac{\alpha^2}{\alpha-\beta}$.

More recently, it was shown that the fire fronts evolution can be modeled by Finsler metrics of slope type and their generalizations (see [8]). In this setting the geodesics behaviour and the cut locus have real interpretations and concrete applications for the firefighters activity as well as preventing of wild fires. All these applications show that slope metrics deserve a more detalied study making in this way the motivation of the preseant paper.

Despite the quite long existence of slope metrics, their study is limited mainly to the study of their local geometrical properties, while the global existence of such metrics and other geometrical properties are conspicuously absent.

Our study leads to the following novel findings:

(1) we show that there are many examples of surfaces admitting globally defined slope metrics;
(2) we describe in some detail the geometry of a surface of revolution endowed with a slope metric. In special we study the geodesics behaviour, Clairaut relation, etc.;
(3) we compare the Finslerian areas (by using the Busemann-Hausdorff and the Holmes-Thompson volume forms, respectively) with the Riemannian one.

Here is the contents of the present paper. We recall in Section 2 the construction of the slope metric on a surface $M \to \mathbb{R}^3$ based on Matsumoto's work pointing out the strongly convexity condition such a surface must satisfy in order to admit a slope metric (Proposition 1).

Based on these we show that there exist smooth surfaces $M \to \mathbb{R}^3$ that admit globally defined slope metrics (Section 3). All the examples known until now were local one. This is for the first time the existence of global slope metrics is shown.

In Section 4 we specialize to surfaces of revolution admitting globally defined slope metrics. We study in Section 4.1 general Finsler surfaces of revolution and give a new form of the Clairaut relation in Theorem 1. This relation is very important showing that the geodesic flow of Finsler surfaces of revolution is integrable despite its highly nonlinear character. After solving the algebraic system (4.6) one can write the geodesic equations in an explicit form, however solving this system is not a trivial task. Next, in Section 4.3, we construct explicitly the slope metric on a surface of revolution and show that there are many such surfaces admitting globally defined strongly convex slope metrics, see Theorem 3 for a topological classification and examples. These are actually Finsler surfaces of revolution (see Theorem 2).

We turn to study of geodesics of slope metrics on a surface of revolution in Section 4.4 by explicitly writing the geodesic equations as second order ODEs in (4.9). Some immediate consequences are given (see Proposition 2, 3). The meridians are F-geodesics, but parallels are not. Moreover, a slope metric cannot be projectively flat or projectively equivalent to Riemannian metric α (Proposition 4). We show the concrete form of the Clairaut relation for this case in Theorem 4, and some consequence of it in Proposition 5.

Finally, we compare the area of a bounded region D on the surface of revolution M when measured by the canonical Riemannian, Busemann-Hausdorff, and Holmes-Thompson volume measures, respectively (see Theorems 5 and 6).

Other topics in the geometry of slope metrics like the study of the flag curvature, global behaviour of geodesics, and cut locus, etc. will be considered in forthcoming research.

2. The slope metric

The slope metric is obtained by the movement on a Riemannian surface under the influence of the gravity attraction force. Indeed, assume a hiker is walking on the surface M, seen now as the slope of a mountain, with speed c an level ground, along a path that makes an angle ε with the steepest downhill direction.

Let us consider the surface M embedded in the Euclidean space $\mathbb{R}^3$ with the parametrization $M \to \mathbb{R}^3, \quad (x,y) \mapsto (x,y,z = f(x,y))$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function (further conditions will be added later), that is M is the graph of $z = f(x,y)$. The tangent plane $\pi_p = T_pM$ at a point $p = (x,y,f(x,y)) \in M$ is spanned by $\partial_x := (1,0,f_x), \quad \partial_y := (0,1,f_y)$, where f_x and f_y are the partial derivatives of f with respect to x and y, respectively. The induced Riemannian metric from $\mathbb{R}^3$ to the surface M is

$$a_{ij} = \begin{pmatrix} 1+f_x^2 & f_xf_y \\ f_xf_y & 1+f_y^2 \end{pmatrix}. \tag{2.1}$$

Remark 1. Observe that at a critical point $p \in M$, i.e. a point where $(f_x(p), f_y(p)) = (0,0)$, the tangent plane T_pM is spanned by the unit vectors $(1,0,0)$ and $(0,1,0)$, while the induced Riemannian metric is just the usual flat metric.

We will construct the slope metric on the surface M by considering the plane x,y to be the sea level, the $z \geq 0$ coordinate to be the altitude above the sea level and the surface $M : z = f(x,y)$ to be the slope of the mountain.

At any point $p \in M$ we construct a Riemannian orthonormal frame $\{e_1, e_2\}$ in T_pM by choosing e_1 to point on the steepest downhill direction of T_pM. Indeed, it is elementary to see that

$$e_1 = -\frac{1}{\sqrt{(1+f_x^2+f_y^2)(f_x^2+f_y^2)}}(f_x\partial_x + f_y\partial_y), \quad e_2 = \frac{1}{\sqrt{f_x^2+f_y^2}}(-f_y\partial_x + f_x\partial_y) \tag{2.2}$$

is a such orthonormal frame.

With these notations, the Matsumoto's slope principle is telling us that the locus of unit time destinations of the hiker on the plane T_pM is given by the limaçon $r = c + a \cdot \cos\theta$, where (r,θ) are the polar coordinates in T_pM, c is the speed of the hiker on the ground level xy, and $a = \frac{g}{2} \cdot \sin\varepsilon$ is the gravity (of magnitude g) component along the steepest downhill direction. The Finsler norm F having this limaçon as indicatrix measures time travel on the surface S.

Taking into account the parametrization

$$X(t) = (c + a\cos t) \cdot \cos t, \quad Y(t) = (c + a\cos t) \cdot \sin t, \quad t \in [0, 2\pi) \tag{2.3}$$

it is easy to obtain the implicit equation of the limaçon

$$X^2 + Y^2 = c\sqrt{X^2+Y^2} + a \cdot X, \tag{2.4}$$

where X,Y are the coordinates with respect to the orthonormal frame $\{e_1, e_2\}$ in T_pM.

We get the Minkowski norm $F(X,Y) = \frac{X^2+Y^2}{c\sqrt{X^2+Y^2}+a\cdot X}$, and by converting to the canonical coordinates $(x,y,\dot{x},\dot{y})$ of TM we obtain the slope metric

$$F(x,y,\dot{x},\dot{y}) = \frac{\alpha^2}{c\alpha - \frac{g}{2}\beta},$$

where

$$\begin{cases} \alpha &= \sqrt{(1+f_x^2)\dot{x}^2 + 2f_xf_y\dot{x}\dot{y} + (1+f_y^2)\dot{y}^2} \\ \beta &= f_x\dot{x} + f_y\dot{y}. \end{cases} \tag{2.5}$$

Remark 2. Recall that a Finsler structure (M,F) is a surface (or more generally an n-dimensional manifold) M endowed with a Banach norm in each tangent space that smoothly varies with the base point all over the manifold. A Riemannian structure is the particular case when each of these norms are induced by a quadratic form.

Observe that the slope metric as the Finsler metric whose indicatrix is a limaçon, was effectively constructed only at regular points of the surface M. However, it is easy to see that the resulting Finsler metric is well defined everywhere on M, including the critical points of M, where it becomes the flat Riemannian metric. This is in

perfect accord with the physical situation initially considered, since in these points the gravitation forces have no influence.

For the sake of simplicity we can choose $c := \frac{g}{2}$ and by multiplication with c we obtain the usual form of the slope metric

$$F = \frac{\alpha^2}{\alpha - \beta} \tag{2.6}$$

(see [3], [9]). The slope metric belongs to the class of (α, β)-metrics (see [1]).

By writing $F = F(\alpha, \beta) = \alpha \cdot \phi(\mathfrak{s})$, where $\mathfrak{s} = \frac{\beta}{\alpha}$, the Hessian $g_{ij} := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ reads $g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_1(b_i\alpha_j + b_j\alpha_i) - \rho\rho_1\alpha_i\alpha_j$, where $\alpha_i := \frac{\partial \alpha}{\partial y^i}$, and $\rho = \phi^2 - \mathfrak{s}\phi\phi'$, $\rho_0 = \phi\phi'' + \phi'\phi'$, $\rho_1 = -\mathfrak{s}(\phi\phi'' + \phi'\phi') + \phi\phi'$.

It is known from Shen's work (see [1]) that (α, β) type Finsler metrics are strongly convex whenever the function $\phi(\mathfrak{s})$ satisfies

$$\phi(\mathfrak{s}) > 0, \quad \phi(\mathfrak{s}) - \mathfrak{s}\phi'(\mathfrak{s}) > 0, \quad \phi''(\mathfrak{s}) \geq 0, \quad \text{for} \quad \mathfrak{s} < b.$$

In the case of the slope metric, we have $\phi(\mathfrak{s}) = \frac{1}{1-s}$ and the relations above are clearly satisfied for $\mathfrak{s} < \frac{1}{2}$, that is $\beta < \frac{1}{2}\alpha$. It follows

Proposition 1. *A surface $M \to \mathbb{R}^3$, $(x,y) \mapsto (x,y,z = f(x,y))$ admits a strongly convex slope metric $F = \frac{\alpha^2}{\alpha-\beta}$, where α, β are given in (2.5), if and only if*

$$f_x^2 + f_y^2 < \frac{1}{3} \tag{2.7}$$

where f_x, f_y are partial derivatives of f.

This proposition is saying that $\beta < \frac{1}{2}\alpha$ is equivalent to the condition (2.7), for α, β given in (2.5).

Remark 3.
(1) This formula was obtained for the first time in [3] and the proof is based on the idea in [3].
(2) The convexity formula above is obviously equivalent to the usual convexity condition of the limaçon $c > 2a$.
(3) Taking into account the inverse matrix (a^{ij}) of (2.1), it can be seen that $b^2 := a^{ij}b_i b_j$ is given by $b^2 = \dfrac{f_x^2 + f_y^2}{1 + f_x^2 + f_y^2}$, and from (2.7) it follows that the strongly convexity of the indicatrix is equivalent to $b < \dfrac{1}{2}$.

Observe that for the slope metric (2.6) we have $\rho = \frac{2\mathfrak{s}-1}{(\mathfrak{s}-1)^3} = \frac{\alpha^2(\alpha-2\beta)}{(\alpha-\beta)^3}$, $\rho_0 = \frac{3}{(\mathfrak{s}-1)^4} = \frac{3\alpha^4}{(\alpha-\beta)^4}$, $\rho_1 = \frac{1-4\mathfrak{s}}{(\mathfrak{s}-1)^4} = \frac{\alpha^3(\alpha-4\beta)}{(\alpha-\beta)^4}$ and hence $g_{11} = \rho \cdot a_{11} + \rho_0 + 2\rho_1\alpha_1 - \rho \cdot \rho_1(\alpha_1)^2$, $g_{12} = (1 - \rho\alpha_1)\rho_1\alpha_2$, $g_{22} = \rho \cdot a_{22} - \rho \cdot \rho_1(\alpha_2)^2$.

3. Examples of slope metrics

We will consider in the following some simple examples of slope metrics. Observe that, as pointed already, the base surfaces may have critical points. At these points the slope metric reduces to a flat Riemannian metric, fact in perfect agreement with the initial setting of the problem.

One might be tempted to think that due to the convexity condition (2.7) the slope metric is strongly convex only locally. See for instance the example of the paraboloid of revolution $f(x,y) := 100 - x^2 - y^2$ in [3] where the strongly convexity condition is assured only in a circular vicinity of the hilltop. However, that is not the case. There are many Riemannian surfaces that admit globally strongly convex slope metric. We describe few such examples below.

3.1. *The plane*

The simplest surface is the plane $M : z = f(x,y) = px + qy + r$, where p,q,r are constants.

It is trivial to see that

$$(a_{ij}) = \begin{pmatrix} 1+p^2 & pq \\ pq & 1+q^2 \end{pmatrix}, \quad (b_i) = \begin{pmatrix} p \\ q \end{pmatrix},$$

thus the slope metric is actually the Minkowski metric

$$F = \frac{(1+p^2)\dot{x}^2 + 2pq\dot{x}\dot{y} + (1+q^2)\dot{y}^2}{\sqrt{(1+p^2)\dot{x}^2 + 2pq\dot{x}\dot{y} + (1+q^2)\dot{y}^2} - (p\dot{x} + q\dot{y})},$$

with the strongly convexity condition $p^2 + q^2 < \dfrac{1}{3}$. Hence, the plane $z = \lambda \cdot x$ does admit a strongly convex slope metric for any constant $\lambda^2 < \frac{1}{3}$, while $z = x$ does not (see [3]).

We recall from [12] that for a slope metric on a surface $M : z = f(x,y)$, the 1-form β is parallel with respect to α if and only if M is a plane. In this case the slope metric is a Berwald space.

3.2. *A list of surfaces*

Elementary computations show that all the following surfaces $z = f(x,y)$ admit strongly convex slope metric globally defined, where $f : \mathbb{R}^2 \to \mathbb{R}$ are given by

(1) $f(x,y) = \frac{1}{2\sqrt{6}} e^{-(x^2+y^2)}$,
(2) $f(x,y) = \frac{1}{2\sqrt{6}} e^{-(x+y)^2}$,
(3) $f(x,y) = \frac{1}{2\sqrt{6}} \arctan(x+y)$,
(4) $f(x,y) = \frac{1}{2\sqrt{6}} ((x+y) - \log(e^{x+y} + 1))$,
(5) $f(x,y) = \frac{1}{2\sqrt{6}} \log(\sqrt{(x+y)^2 + 1} + x + y)$.

Indeed, all these surfaces satisfy condition (2.7) for any $(x,y) \in \mathbb{R}^2$.

Let us remark that the surface $f(x,y) := \frac{1}{2\sqrt{6}} e^{-(x^2+y^2)}$ can be actually realized as a surface of revolution obtained by rotating the graph of the function $z = \frac{1}{2\sqrt{6}} e^{-x^2}$ around the z axis. This example suggests that surfaces of revolution are good candidates for the study of slope metrics, fact motivating the next section.

4. The slope metric of a surface of revolution

4.1. *Riemannian surface of revolution*

Let us recall some facts from the geometry of Riemannian surfaces of revolution (see [13]).

A surface of revolution $M \to \mathbb{R}^3$ can be parametrization as

$$(u,v) \mapsto (x = m(u)\cos v, y = m(u)\sin v, z = u) \tag{4.1}$$

where $u \in (0,\infty)$, $v \in \mathbb{S}^1$. Here (u,v) are the geodesic polar coordinates around the pole $p \in M$, and $m : (0,\infty) \to (0,\infty)$ is a smooth function such that $m'(0) = 1$ (see [13] for details).

Remark 4. We have defined here a classical surface of revolution by rotating the image of the curve $m : (0,\infty) \to (0,\infty)$ around the z axis. However, there is no harm in taking $m : I \to (0,\infty)$, where $I \subset \mathbb{R}$ is an open set.

It is known that a curve $u = u(t)$, $v = v_0$: constant is called a *meridian*, and $u = u_0$: constant, $v = v(t)$ is called a *parallel*. A point $p \in M$ is called *pole* if any 2 geodesics emanating from p do not meet again, in other words, the cut locus of p is empty. A unit speed geodesic is called a *ray* if $d(\gamma(0),\gamma(s)) = s$, for all $s \geq 0$. Clearly, all geodesics emanating from the pole are rays.

The induced Riemannian metric is

$$(a_{ij}) = \begin{pmatrix} 1+(m')^2(u) & 0 \\ 0 & m^2(u) \end{pmatrix} \tag{4.2}$$

and the unit speed geodesics $(u = u(t), v = v(t))$ are given by

$$\begin{cases} \frac{d^2u}{dt^2} + \frac{m'm''}{1+m'^2}\left(\frac{du}{dt}\right)^2 - \frac{mm'}{1+m'^2}\left(\frac{dv}{dt}\right)^2 = 0 \\ \frac{d^2v}{dt^2} + 2\frac{m'}{m}\frac{du}{dt}\frac{dv}{dt} = 0. \end{cases} \tag{4.3}$$

The geodesic spray coefficients of this Riemannian metric read

$$\begin{cases} 2\mathcal{G}^1_\alpha = \frac{m'm''}{1+m'^2}(y^1)^2 - \frac{mm'}{1+m'^2}(y^2)^2 \\ 2\mathcal{G}^2_\alpha = 2\frac{m'}{m}y^1y^2, \quad (m \neq 0). \end{cases}$$

From here it follows that there exists a constant ν, called the *Clairaut constant* such that

$$\frac{dv}{dt} \cdot m^2(u(t)) = \nu, \tag{4.4}$$

and hence $\frac{du}{dt} = \pm\frac{1}{m}\sqrt{\frac{m^2-\nu^2}{1+m'^2}}$, that is in the case of a Riemannian surface of revolution, the geodesic flow is integrable.

Remark 5. By changing the parameter u on the profile curve $m(u)$ we can parametrize M as $(u,v) \mapsto (m(u)\cos v, m(u)\sin v, z(u))$ such that $[m'(u)]^2 + [z'(u)]^2 = 1$. This simplifies the induced Riemannian metric (a_{ij}). We are not using this parametrization because linear form β in (2.5) is simpler when using (4.1) and this leads to simplication of computations for the slope metric.

4.2. *Finsler surfaces of revolution*

Let (M,F) be a Finsler structure defined on a surface of revolution M defined as in Section 4.1.

If $X := \frac{\partial}{\partial v}$ is a Killing vector field for F, that is $\mathcal{L}_X F = X^c(F) = 0$, where X^c is the complete lift of X to TM, or equivalently $\frac{\partial F}{\partial v} = 0$, then (M,F) is called a *Finsler surface of revolution*.

Remark 6.
(1) See [7] for a definition based on the notion of motion. Their definition is equivalent to ours.
(2) See [5] and [6] for a complete study of rotationally Randers metrics, that is Finsler metric of type $F = \alpha + \beta$ constructed on surfaces of revolution.

If we denote by $H(x,p)$ the Hamiltonian corresponding to the Finsler structure (M,F) by means of Legendre transform (see [10]), then since F is surface of revolution, it follows $\frac{\partial H}{\partial v} = 0$. Hence, Hamilton Jacobi equations $\frac{dx^i}{ds} = \frac{\partial H}{\partial p_i}$, $\frac{dp_i}{ds} = -\frac{\partial H}{\partial x^i}$ imply that $I = p_2$ is a prime integral of the geodesic flow, that is $\frac{dp_2}{ds} = 0$ along any unit speed F-geodesic.

On the other hand, recall that by the Legendre transform associated to F, we have $p_2 = g_{2i}y^i = g_{12}y^1 + g_{22}y^2$ and hence we obtain

Theorem 1. *Along any unit speed F-geodesics $\mathcal{P}(s) = (u(s), v(s))$ we have*

$$p_2(s) = g_{12}(\mathcal{P},\dot{\mathcal{P}}) \cdot \frac{du}{ds} + g_{22}(\mathcal{P},\dot{\mathcal{P}}) \cdot \frac{dv}{ds} = \nu_F = \textit{constant}. \tag{4.5}$$

That is, (4.5) is the corresponding relation to (4.4) in the Finslerian setting.

The constant ν_F plays the role of the Clairaut constant for Finslerian geodesics.

Remark 7. See [7] for an alternate proof of this formula.

It follows that, for any unit speed F-geodesic, we have

$$g_{12}(\mathcal{P},\dot{\mathcal{P}}) \cdot \frac{du}{ds} + g_{22}(\mathcal{P},\dot{\mathcal{P}}) \cdot \frac{dv}{ds} = \nu, \quad F(\mathcal{P},\dot{\mathcal{P}}) = 1 \tag{4.6}$$

and theoretically, by solving this algebraic system, we can obtain $\frac{du}{dt}$ and $\frac{dv}{dt}$ that by integration would give the trajectories of the F-geodesics. However, observe that finding an explicit solution of the system is not a trivial task.

Remark 8. (1) As far as we know, the relation (4.5) appeared for the first time in the case of the rotational Randers surface of revolution studied in [5], where the Clairaut constant for the Randers geodesics is $\frac{\nu}{1+\mu\nu}$. Here ν is the usual Clairaut constant of the corresponding Riemannian geodesic through the Zermelo navigation process.

(2) We denote by φ_t the flow of $\frac{\partial}{\partial v}$, which is a Finslerian isometry preserving the orientation of M. The Finslerian distance d_F is invariant under φ_t.

4.3. *The slope metric on a surface of revolution*

Let us consider again the surface of revolution M with the parametrization (4.1) and induced Riemannian metric (4.2).

Following again Matsumoto's slope principle, observe that the orthonormal frame in T_pM at a given $p \in M$ is $e_1 = -\frac{1}{\sqrt{(m')^2+1}} \cdot \frac{\partial}{\partial u}$, $e_2 = \frac{1}{m} \cdot \frac{\partial}{\partial v}$ and here, the relation between the coordinates (X,Y) of T_pM with respect to $\{e_1, e_2\}$ and the canonical coordinates $(\dot{u}, \dot{v})$ is $X = -\sqrt{1+(m')^2} \cdot \dot{u}, \quad Y = m \cdot \dot{v}$.

The limaçon implicit equation (2.4) reads now

$$\left[1+(m')^2\right]\dot{u}^2 + m^2 \cdot \dot{v}^2 = c\sqrt{\left[1+(m')^2\right]\dot{u}^2 + m^2\dot{v}^2} - a\sqrt{1+(m')^2} \cdot \dot{u},$$

and taking into account that $a = \sin\varepsilon = \frac{1}{\sqrt{1+(m')^2}}$ we obtain the slope metric in the form (2.6) with

$$\alpha = \sqrt{\left[1+(m')^2\right]\dot{u}^2 + m^2\dot{v}^2}, \quad \beta = \dot{u}. \tag{4.7}$$

Taking into account the strongly convexity condition $b < \frac{1}{2}$ it follows

Theorem 2. *A surface of revolution $M \to \mathbb{R}^3$, $(u,v) \mapsto (m(u)\cos v, m(u)\sin v, u)$ admits a strongly convex slope metric $F = \frac{\alpha^2}{\alpha-\beta}$, with α, β given in (4.7) if and only if*

$$(m')^2 > 3. \tag{4.8}$$

Moreover, (M,F) is a Finsler surface of revolution.

Let us recall from Poincaré -Hopf index theorem for the rotational vector filed $X = \frac{\partial}{\partial v}\Big|_p$, $p \in M$, that the strongly convexity condition (4.8) implies that number of singular points of X on M can be only 1 or 0. Indeed, otherwise X would be vanishing, or M would be homeomorphic to the sphere, and this is not possible. It is clear from (4.8) that M cannot be boundaryless compact manifold. The case of a cylinder of revolution is not possible either due (4.8), hence we obtain

Theorem 3. *The surfaces of revolution M admitting globally defined strongly convex slope metrics are homeomorphic to $\mathbb{R}^2$.*

One can now easily construct examples of surfaces of revolution satisfy condition (4.8). Here are such surfaces

(1) $m(u) = \sqrt{6u^2 - 1}$, for $u \in (\frac{1}{\sqrt{6}}, \infty)$;
(2) $m(u) = \frac{1}{2}\sqrt{-2\ln(24u^2)}$, for $u \in (0, \frac{1}{2\sqrt{6}})$.

Since the slope metric F is a Finslerian surface of revolution, the theory explained in Section 4.2 applies.

4.4. *The geodesics of a surface of revolution with the slope metric*

In order to study to geodesics of the slope metric $(M, F = \frac{\alpha^2}{\alpha-\beta})$ we need a formula for the geodesic spray of F.

We recall the general formula for an arbitrary (α, β)-metric

$$G^i = \mathcal{G}^i_\alpha + \alpha Q s^i_0 + \Theta\{-2Q\alpha s_0 + r_{00}\}\frac{y^i}{\alpha} + \Psi\{-2Q\alpha s_0 + r_{00}\}b^i$$

where G^i and $\mathcal{G}^i_\alpha$ denote the spray coefficients for F and α, respectively.

Here we use the customary notations:

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}),\ s_{ij} := \frac{1}{2}(b_{i|j} - b_{j:i}),\ s^i_j := a^{ik}s_{kj},\ s_j = b_i s^i_j,\ b^i = a^{ij}b_j,$$

and

$$Q := \frac{\phi'}{\phi - \mathfrak{s}\phi'},\ \Psi := \frac{\phi''}{2[\phi - \mathfrak{s}\phi' + (b^2 - \mathfrak{s}^2)\phi'']},$$
$$\Theta := \frac{\phi - \mathfrak{s}\phi'}{2[\phi - \mathfrak{s}\phi' + (b^2 - \mathfrak{s}^2)\phi]} \cdot \frac{\phi'}{\phi} - \mathfrak{s}\Psi = \left[\frac{\phi - \mathfrak{s}\phi'}{\phi''} \cdot \frac{\phi'}{\phi} - \mathfrak{s}\right]\Psi$$

(see [1]).

In the case of α, β given in (4.7) we obtain

$$\begin{pmatrix} b_{1|1} \\ b_{2|2} \end{pmatrix} = \begin{pmatrix} \frac{-m'm''}{1+m'^2} \\ \frac{mm'}{1+m'^2} \end{pmatrix}, \quad \begin{matrix} r_{00} = -2 \cdot \mathcal{G}^1_\alpha \\ s^i_j = s_j = 0, \end{matrix} \qquad b_{1|2} = 0,$$

and hence $G^i = \mathcal{G}^i_\alpha + r_{00}\left[\Theta\frac{y^i}{\alpha} + \Psi \cdot b^i\right]$.

By taking into account now $\phi(\mathfrak{s}) = \frac{1}{1-\mathfrak{s}}$ after some computations we get

$$\Psi = \frac{1}{2b^2 - 3\mathfrak{s} + 1} = \frac{\alpha}{(2b^2+1)\alpha - 3\beta}$$
$$\Theta = \left(\frac{1}{2} - 2\mathfrak{s}\right)\Psi = \frac{1 - 4\mathfrak{s}}{2(2b^2 - 3\mathfrak{s} + 1)} = \frac{\alpha - 4\beta}{2\alpha} \cdot \Psi = \frac{\alpha - 4\beta}{2[(2b^2+1)\alpha - 3\beta]}$$

therefore $G^i = \mathcal{G}^i_\alpha + r_{00}\left[\frac{\alpha-4\beta}{2\alpha} \cdot \frac{y^i}{\alpha} + b^i\right] \cdot \Psi$.

In particular $G^1 = \mathcal{G}^1_\alpha \cdot \frac{(\alpha-2\beta)^2}{\alpha[(2b^2+1)\alpha-3\beta]}$, $G^2 = \mathcal{G}^2_\alpha - \mathcal{G}^1_\alpha \cdot \frac{\alpha-4\beta}{\alpha[(2b^2+1)\alpha-3\beta]} \cdot y^2$, and therefore, the unit speed F-geodesic equations are

$$\begin{cases} \frac{d^2u}{ds^2} + 2\mathcal{G}^1_\alpha \cdot \frac{(\alpha-2\beta)^2}{\alpha[(2b^2+1)\alpha-3\beta]}\Big|_{(u(s),v(s))} = 0 \\ \frac{d^2v}{ds^2} + 2\mathcal{G}^2_\alpha - 2\mathcal{G}^1_\alpha \cdot \frac{\alpha-4\beta}{\alpha[(2b^2+1)\alpha-3\beta]} \cdot \frac{dv}{ds}\Big|_{(u(s),v(s))} = 0. \end{cases} \tag{4.9}$$

The geodesic equations in this form are not of much use.

However, some conclusions can be drawn.

Proposition 2. *The meridians are F-unit speed geodesics.*

Proof. If we consider an (F-unit speed) meridian $\mathcal{P}(s) = (u(s), v_0)$, then $\dot{\mathcal{P}}(s) = \left(\frac{du}{ds}, 0\right)$ and by using the F-unit speed condition the geodesic equation (4.9) are identically satisfied. □

Proposition 3. *A parallel $\mathcal{P}(s) = (u_0, v(s))$ is F-geodesic if and only if $m'(u_0) = 0$, that is a strongly convex slope metric do not admit parallels geodesics.*

Proof. ($\Rightarrow$) If the parallel $\mathcal{P}(s) = (u_0, v(s))$ is a unit speed F-geodesic, then along $\mathcal{P}(s)$, $\alpha^2\Big|_{(\mathcal{P},\dot{\mathcal{P}})} = 1$ and $\beta\Big|_{(\mathcal{P},\dot{\mathcal{P}})} = 0$, hence the conclusion follows from the same arguments as in the Riemannian case.

($\Leftarrow$) If we assume $m'(u_0) = 0$ then the conclusion follows in a similar way with the Riemannian case. □

Proposition 4. *The slope metric $F = \frac{\alpha^2}{\alpha-\beta}$ can not be projectively equivalent to the Riemannian metric of M, nor projectively flat.*

Proof. Recall that a Matsumoto metric $F = \frac{\alpha^2}{\alpha-\beta}$ is projectively equivalent to the Riemannian metric α if and only if β is parallel with respect to a_{ij}, that is $b_{i|j} = 0$, where $|$ is the covariant derivative with respect to a_{ij}.

However, observe that in the case of the slope metric we have $b_{1|1} = -\gamma^1_{11} \neq 0$, $b_{1|2} = 0$, $\quad b_{2|2} = -\gamma^1_{22} \neq 0$.

In order to be projectively flat β must be parallel and α projectively flat. Clearly, none of these conditions is true in the case of the slope metric. □

Let us consider the prime integral p_2 of the geodesic flow. A straightforward computation shows

Theorem 4. *Along the unit speed F-geodesic $\mathcal{P} : (0,a) \to M$, $\mathcal{P}(s) = (u(s), v(s))$, $\frac{du}{ds} \neq 0$ for $s \in (0,a)$ we have:*

$$p_2(s) = (g_{12}y^1 + g_{22}y^2)\big|_{(\mathcal{P},\dot{\mathcal{P}})} = \rho\big|_{(\mathcal{P},\dot{\mathcal{P}})} \cdot m^2(u(s)) \cdot \frac{dv}{ds} = \nu_F. \tag{4.10}$$

Therefore $\frac{du}{ds}$, $\frac{dv}{ds}$ are solutions of the following algebraic system:

$$\rho\big|_{(\mathcal{P},\dot{\mathcal{P}})} \cdot m^2(u(s)) \cdot \frac{dv}{ds} = \nu_F, \ \frac{\alpha^2}{\alpha-\beta}\big|_{(\mathcal{P},\dot{\mathcal{P}})} = 1, \tag{4.11}$$

where $\rho(\mathcal{P}(s),\dot{\mathcal{P}}(s)) = \frac{\alpha-2\beta}{(\alpha-\beta)^2}\big|_{(\mathcal{P}(s),\dot{\mathcal{P}}(s))}$. An explicit solution of this algebraic system can be obtained by solving a 4th order equation but the computation is too complicated to be given here.

Instead of writing the explicit solution of (4.11) we point out some consequence of (4.10).

If a unit speed F-geodesic $\mathcal{P}(0,a) \to M$ is tangent to the Killing vector field at its end points, that is

$$\dot{\mathcal{P}}(0) = \frac{1}{F\left(\mathcal{P}(0), \frac{\partial}{\partial v}\Big|_{\mathcal{P}(0)}\right)} \cdot \frac{\partial}{\partial v}\Big|_{\mathcal{P}(0)}, \ \dot{\mathcal{P}}(a) = \frac{1}{F\left(\mathcal{P}(a), \frac{\partial}{\partial v}\Big|_{\mathcal{P}(a)}\right)} \cdot \frac{\partial}{\partial v}\Big|_{\mathcal{P}(a)},$$

and $\dot{\mathcal{P}}(s)$ is linearly dependent with $\frac{\partial}{\partial v}\Big|_{\mathcal{P}(s)}$ for any $s \in (0,a)$, then Clairaut relation (4.10) implies

Proposition 5. *If $\mathcal{P} : (0,a) \to M$, $\mathcal{P}(s) = (u(s),v(s))$ is an F-unit speed geodesic such that $\dot{\mathcal{P}}(0)$ and $\dot{\mathcal{P}}(a)$ are linear dependent vectors with $\frac{\partial}{\partial v}\Big|_{\mathcal{P}(0)}$ and $\frac{\partial}{\partial v}\Big|_{\mathcal{P}(a)}$, respectively, then $m(u(0)) = m(u(a)) = \nu_F$. Moreover, $m(u(s)) > m(u(0))$ for $s \in (0,a)$.*

5. Finslerian volumes

It is known that the *Euclidean volume form* in $\mathbb{R}^n$ is the n-form $dV_{\mathbb{R}^n} := dx^1 dx^2 \dots dx^n$, and the *Euclidean volume* of a bounded open set $D \subset \mathbb{R}^n$ is given by $Vol(D) = \int_D dV_{\mathbb{R}^n} = \int_D dx^1 dx^2 \dots dx^n$.

Obviously, if $D \subset \mathbb{R}^n$ is a bounded open set, $Vol(D)$ is a finite constant.

More generally, let us consider a Riemannian manifold (M,g) with the *Riemannian volume form* $dV_g := \sqrt{g} dx^1 dx^2 \dots dx^n$, and hence the *Riemannian volume* of (M,g) can be computed as

$$Vol(M,g) = \int_M dV_g = \int_M \sqrt{g} dx^1 dx^2 \dots dx^n = \int_M \theta^1 \theta^2 \dots \theta^n,$$

where $\{\theta^1, \theta^2, \dots, \theta^n\}$ is a g-orthonormal co-frame on M, and $g = \det(g_{ij})$.

In general, a *volume form* $d\mu$ on an n-dimensional Finsler manifold (M,F) is a globally defined, non-degenerate n-form on M. In local coordinates we can always write $d\mu = \sigma(x) dx^1 \wedge \dots \wedge dx^n$, where σ is a positive function on M.

The usual Finslerian volumes are obtain by different choices of the function $\sigma(x)$. Here are two of the most well studied Finslerian volumes.

The *Busemann-Hausdorff* volume form is defined as $dV_{BH} := \sigma_{BH}(x)dx^1 \wedge \cdots \wedge dx^n$, where

$$\sigma_{BH}(x) := \frac{Vol(\mathbb{B}^n(1))}{Vol(\mathcal{B}_x^n M)}, \tag{5.1}$$

here $\mathbb{B}^n(1)$ is the Euclidean unit n-ball, $\mathcal{B}_x^n M = \{y : F(x,y) = 1\}$ is the Finslerian ball and Vol the canonical Euclidean volume.

This volume form allows us to define the *Busemann-Hausdorff* volume of the Finsler manifold (M,F) by $vol_{BH}(M,F) = \int_M dV_{BH}$.

Remark 9. Observe that the n-ball Euclidean volume is

$$Vol(\mathbb{B}^n(1)) = \frac{1}{n}Vol(\mathbb{S}^{n-1}) = \frac{1}{n}Vol(\mathbb{S}^{n-2})\int_0^\pi \sin^{n-2}(t)dt.$$

Another volume form naturally associated to a Finsler structure is the *Holmes-Thompson* volume form defined by $dV_{HT} = \sigma_{HT}(x)dx^1...dx^n$, where

$$\sigma_{HT}(x) := \frac{Vol(\mathcal{B}_x^n M, g_x)}{Vol(\mathbb{B}^n(1))} = \frac{1}{Vol(\mathbb{B}^n(1))}\int_{\mathcal{B}_x^n M}(\det g_{ij}(x,y))dy^1...dy^n, \tag{5.2}$$

and the *Holmes-Thompson* volume of the Finsler manifold (M,F) is defined as $vol_{HT}(M,F) = \int_M dV_{HT}$.

Remark 10. If (M,F) is an absolute homogeneous Finsler manifold, then the Busemann-Hausdorff volume is a Hausdorff measure of M, and therefore we have $vol_{BH}(M,F) \geq vol_{HT}(M,F)$ (see [4]). If (M,F) is not absolute homogeneous, then the inequality above is not true anymore.

In the case of an Finsler (α,β)-metric, one can compute explicitly the Finslerian volume in terms of the Riemannian volume (see [1]). Indeed, if $(M,F(\alpha,\beta))$ is an (α,β)-metric on an n-dimensional manifold M, one denotes

$$f(b) := \frac{\int_0^\pi \sin^{n-2}(t)dt}{\int_0^\pi \frac{\sin^{n-2}(t)}{\phi(b\cos(t))^n}dt}, \quad g(b) := \frac{\int_0^\pi \sin^{n-2}(t)T(b\cos t)dt}{\int_0^\pi \sin^{n-2}(t)dt}, \tag{5.3}$$

where $F = \alpha\phi(\mathfrak{s})$, $\mathfrak{s} = \beta/\alpha$, and $T(\mathfrak{s}) := \phi(\phi - \mathfrak{s}\phi')^{n-2}[(\phi - \mathfrak{s}\phi') + (b^2 - \mathfrak{s}^2)\phi'']$.

Then the Busemann-Hausdorff and Holmes-Thompson volume forms are given by $dV_{BH} = f(b)dV_\alpha$, and $dV_{HT} = g(b)dV_\alpha$, respectively, where dV_α is the Riemannian volume form.

It is remarkable that if the function $T(\mathfrak{s}) - 1$ is an odd function of $\mathfrak{s}$, then $dV_{HT} = dV_\alpha$. This is the case of Randers metrics (see [2]), but not the case of the slope metric.

The following lemma is elementary.

Lemma 1. *Let us consider the following functions*

(1) $f : (0, \frac{1}{2}) \to (\frac{8}{9}, 1)$, $f(b) := \frac{2}{2+b^2}$,

(2) $g : (0, \frac{1}{2}) \to (\frac{5\sqrt{3}}{9}, 1)$, $g(b) := \frac{(2-3b^2)}{2(1-b^2)\sqrt{1-b^2}}$,

(3) $h:(0,\frac{1}{2})\to(1,\frac{5\sqrt{3}}{8})$, $h(b):=\frac{(2+b^2)(2-3b^2)}{4(1-b^2)\sqrt{1-b^2}}$.

Then, f and g are both monotone decreasing while h is monotone increasing on the given intervals.

A direct application of this lemma is the following theorem.

Theorem 5. *Let (M,F) be a slope metric on a surface of revolution. Then*

$$Area_{BH}(D) < Area_{HT}(D) < Area_{\alpha}(D)$$

for any bounded region $D\subset M$.

Proof. Firstly, observe that in the case of a slope metric, formulas (5.3) imply

$$f(b)=\frac{\pi}{\int_0^{\pi}(1-b\cos t)^2\cdot dt}=\frac{2}{2+b^2},\quad g(b)=\frac{(2-3b^2)}{2(1-b^2)\sqrt{1-b^2}}.$$

It results

$$\begin{aligned}
dV_{BH}&=f(b)dV_{\alpha}=\frac{2}{2+b^2}\cdot dV_{\alpha},\\
dV_{HT}&=g(b)dV_{\alpha}=\frac{(2-3b^2)}{2(1-b^2)\sqrt{1-b^2}}\cdot dV_{\alpha},\\
dV_{HT}&=h(b)dV_{BH}=\frac{(2+b^2)(2-3b^2)}{4(1-b^2)\sqrt{1-b^2}}\cdot dV_{BH},
\end{aligned}$$

so the meaning of the function h in Lemma 1 is clear now.

By taking into account the monotonicity of f, g, h described in Lemma 1, the inequalities stated above hold good. □

Moreover, from Lemma 1 we have

Theorem 6. *Let (M,F) be a slope metric on a surface of revolution. Then*

(1) $\frac{8}{9}Area_{\alpha}(D)\le Area_{BH}(D)\le Area_{\alpha}(D)$,

(2) $\frac{5\sqrt{3}}{9}Area_{\alpha}(D)\le Area_{HT}(D)\le Area_{\alpha}(D)$,

(3) $Area_{BH}(D)\le Area_{HT}(D)\le\frac{5\sqrt{3}}{8}Area_{BH}(D)$,

for any bounded region $D\subset M$.

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INITIAL COEFFICIENTS AND FOURTH HANKEL DETERMINANT FOR CERTAIN ANALYTIC FUNCTIONS

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Dedicated to Professor V. Ravichandran

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Abstract. The present work is an attempt to give partial proofs of certain conjectures on the fifth coefficient of certain normalized analytic functions. Further, bounds on the sixth and seventh coefficients for the starlike functions related to a lune are also investigated. The non-sharp bound on third and fourth Hankel determinants are also obtained.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f defined on the unit disk $\mathbb{D} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and satisfying the conditions $f(0) = 0 = f'(0) - 1$. Thus the functions in the class $\mathcal{A}$ has the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. The collection of functions in $\mathcal{A}$, which are one-one, is denoted by $\mathcal{S}$. It was proved, in 1985, by de Branges that "if $f \in \mathcal{S}$, then $|a_n| \leq n$." The bound $|a_n| \leq 2(1-\alpha)$, $0 \leq \alpha < 1$, holds for the class of starlike and functions $\mathcal{S}^*(\alpha) := \{f : f \in \mathcal{S} \text{ and } \Re(zf'(z)/f(z)) > \alpha\}$ of order α. However, for the class of convex functions $\mathcal{K}(\alpha) := \{f : f \in \mathcal{S} \text{ and } 1 + \Re(zf''(z)/f'(z)) > \alpha\}$ of order α, we have the bound $|a_n| \leq 2(1-\alpha)$, $0 \leq \alpha < 1$. The classes $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{K} := \mathcal{K}(0)$ are known, respectively, as the classes of starlike and convex functions in $\mathbb{D}$. Finding bound on the coefficients of functions in a prescribed class has been one among the major problem in geometric function theory as they affects geometric properties. For example, the bound on the second coefficient gives the growth and distortion properties.

The Hankel determinant of the sequence of coefficients a_i of functions in a given class also gives the important information about the function. For given positive integers n and q, the Hankel determinant $H_{q,n}(f)$ related to the function $f \in \mathcal{A}$ is

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given by

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}, \quad a_1 = 1.$$

For some special choices of n and q; $H_{2,1}(f) = a_3 - a_2^2$ is the famed Fekete-Szegő functional; $H_{2,2}(f) = a_2a_4 - a_3^2$ is the second Hankel determinant, and $H_{3,1}(f) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$ is the third Hankel determinant. The Hankel determinant $H_{q,n}(f)$ for the class $\mathcal{S}$ was investigated by Pommerenke [21] and Hayman [6]. There has been substantial progress in finding the bounds on the Hankel determinant for subclasses of $\mathcal{S}$, see[13].

In 2007, Janteng *et al.* [7] found the sharp estimate on the second Hankel determinant for the classes of starlike and convex functions. Non-sharp estimate on third Hankel determinant for classes of starlike and convex functions were investigated by Zaprawa [33] as $|H_{3,1}(f)| \leq 1$ and $|H_{3,1}(f)| \leq 49/540$, respectively. Kowalczyk *et al.* [8], in 2018, proved that the bound $|H_{3,\,1}(f)| \leq 4/135$ is sharp for the class of convex functions. However, the best known estimate for starlike functions is $|H_{3,\,1}(f)| \leq 4/135$ due to Kwon *et al.* [11]. Later, in 2018, Lecko *et al.* [12] found the sharp bound $|H_{3,\,1}(f)| \leq 1/9$ for starlike function of order $1/2$. Kowalczyk *et al.* [9] also found sharp bound of the third kind for the class $T(\alpha) := \{f \in \mathcal{A} : \Re(f(z)/z) > \alpha; z \in \mathbb{D}\}$ for the cases when $\alpha = 0$ and $\alpha = 1/2$.

In 2015, Raina and Sokół [24] introduced the class

$$\mathcal{S}_q^* := \left\{ f \in \mathcal{S}^* : \frac{zf'(z)}{f(z)} \prec \sqrt{1+z^2} + z =: q(z) \right\}$$

and discussed several properties of the class $\mathcal{S}_q^*$. Later, Raina and Sokół [23] conjectured that $|a_4| \leq 5/12, |a_5| \leq 2/9$ and $|a_2a_4 - a_3^2| \leq 7/48$. In 2017, Cho *et al.* [Coefficient bounds for certain subclasses of starlike functions, preprint] settled the conjecture $|a_4| \leq 5/12$ posed by Raina and Sokòł [23] for functions in the class $\mathcal{S}_q^*$.

Recently, Çağlar *et al.* [3] obtained upper bound for the second Hankel determinant for certain subclasses of analytic and bi-univalent functions. Later, Srivastava *et al.* [30] also discussed several properties of a newly-constructed subclass of bi-univalent functions in $\mathbb{D}$ defined by using a symmetric basic (or q-) derivative operator. Further, they investigated bound on $|a_2a_4 - a_3^2|$. In [29] authors discussed bound on the Hankel determinant and the Toeplitz matrices for certain classes of analytic q-starlike functions. The authors [28] obtained non-sharp bound of the third Hankel determinant for a subclass of close-to-convex functions associated with the lemniscate of Bernoulli. They also investigated sharp bound on the initial coefficients. For more related results one may refer to the work [5, 16, 26, 27, 31].

Obradović and Ponnusamy [17] introduced the class

$$\mathcal{U}(\lambda) := \left\{ f \in \mathcal{A} : \left| \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < \lambda, \lambda \in (0,1], z \in \mathbb{D} \right\}.$$

The functions in the class $\mathcal{U} := \mathcal{U}(0)$ are of the form $f(z) = z/(1-bz)$, $|b| \leq 1$. Note that if $f \in \mathcal{U}(\lambda)$, then

$$\frac{f(z)}{z} \prec \frac{1}{(1+z)(1+\lambda z)}, \quad z \in \mathbb{D}.$$

Obradović and co-authors have investigated several interesting properties of the class $\mathcal{U}(\lambda)$, see the works [17–19, 22] and the references cited therein.

In 2016, Obradović *et al.* [18] conjectured that the nth coefficient of function $f \in \mathcal{U}(\lambda)$ satisfies

$$|a_n| \leq \sum_{k=0}^{n-1} \lambda^k \quad (0 < \lambda \leq 1, n \geq 2).$$

This conjecture has been verified for $n = 2$ first in [32] and a simpler proof was given in [18]. More recently, Obradović *et al.* [19] proved the conjecture for $n = 3, 4$ with an alternate proof for the case n = 2, but it remains open for all $n \geq 5$. A partial solution to this conjecture was given in [22, Theorem 1, p. 91].

$$|a_n| \leq 1 + \lambda\sqrt{n-1}\sqrt{\sum_{k=0}^{n-2} \lambda^{2k}} \quad (0 < \lambda \leq 1, n \geq 2). \tag{1.1}$$

In the present paper, we improve the existing bound on $|a_5|$ given in the above inequality for certain range of λ for $f \in \mathcal{U}(\lambda)$. We give non-sharp bound on $|a_i|$ $(i = 5, 6, 7)$, $|H_{3,1}(f)|$ and $|H_{4,1}(f)|$ for $f \in \mathcal{S}_q^*$.

2. Initial Coefficients

The following theorem gives a refinement and improvement to the bound on $|a_5|$ given in [22, Theorem 1, p. 91]:

Theorem 1. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{U}(\lambda)$. Then*

$$|a_5| \leq \frac{1}{2}\left(2 + 2\lambda + 5\lambda^2 + 6\lambda^3 + 4\lambda^4\right). \tag{2.1}$$

We need the following lemma to prove our result. Before proceeding further we recall that the class of functions $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ with $\Re\, p(z) > 0, z \in \mathbb{D}$ is denoted by $\mathcal{P}$ and they are related with the Schwarz function $w : \mathbb{D} \to \mathbb{C}$ by means of the relation

$$w(z) = \frac{p(z) - 1}{p(z) + 1}.$$

Lemma 1 ([25], Lemma 2.3, p. 507)**.** *Let $p \in \mathcal{P}$. Then for all $n, m \in \mathbb{N}$,*

$$|\mu p_n p_m - p_{m+n}| \leq \begin{cases} 2, & 0 \leq \mu \leq 1; \\ 2|2\mu - 1|, & \textit{elsewhere.} \end{cases}$$

If $0 < \mu < 1$, then the inequality is sharp for the function

$$p(z) = (1 + z^{m+n})/(1 - z^{m+n}).$$

In the other cases, the inequality is sharp for the function $\hat{p}_0$.

Proof of Theorem 1. Let $f \in \mathcal{U}(\lambda)$. Then, from [19, Eqn.(3)], we have

$$\frac{f(z)}{z} \prec \frac{1}{(1+z)(1+\lambda z)}.$$

Therefore, for $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ with $\Re\, p(z) > 0, z \in \mathbb{D}$ such that

$$\frac{f(z)}{z} = \frac{1}{\left(1 + \frac{1-p(z)}{1+p(z)}\right)\left(1 + \lambda \frac{1-p(z)}{1+p(z)}\right)}.$$

On comparing coefficients, we have

$$a_2 = -\frac{p_1}{2}(\lambda + 1),$$

$$a_3 = \frac{1}{4}\left(\left(\lambda^2 + 2\lambda + 2\right) p_1^2 - 2(\lambda + 1) p_2\right),$$

$$a_4 = \frac{1}{8}\left(-\left(\lambda^3 + 3\lambda^2 + 4\lambda + 4\right) p_1^3 + 4\left(\lambda^2 + 2\lambda + 2\right) p_1 p_2 - 4(\lambda + 1) p_3\right)$$

and

$$\begin{aligned} 16a_5 = &-8(1+\lambda)p_4 + 4(2 + 2\lambda + \lambda^2) + 8(2 + 2\lambda + \lambda^2)p_1 p_3 \\ &- 6(4 + 4\lambda + 3\lambda^2 + \lambda^3)p_1^2 p_2 + (8 + 8\lambda + 7\lambda^2 + 4\lambda^3 + \lambda^4)p_1^4, \end{aligned} \tag{2.2}$$

where $A := 8 + 8\lambda + 7\lambda^2 + 4\lambda^3 + \lambda^4$. Now rearranging the terms of (2.2), we have

$$\begin{aligned} \frac{16}{A}a_5 = &-\frac{8(1+\lambda)}{A}p_4 + \frac{4(2 + 2\lambda + \lambda^2)}{A}p_2^2 + \frac{8(2 + 2\lambda + \lambda^2)}{A}p_1 p_3 \\ &- \frac{6(4 + 4\lambda + 3\lambda^2 + \lambda^3)}{A}p_1^2 p_2 + p_1^4. \end{aligned} \tag{2.3}$$

Now upon using triangle inequality, (2.3) gives

$$\begin{aligned} \left|\frac{16}{A}a_5\right| \leq &|p_1^4 + 2p_1 p_3 + p_2^2 - 3p_1^2 p_2 - p_4| + \left|\frac{3\lambda^2(1+\lambda)^2}{A}p_1^2 p_2 - \frac{(3 + 4\lambda + \lambda^2)\lambda^2}{A}p_2^2\right| \\ &+ \left|\frac{\lambda^2(\lambda^2 + 4\lambda + 7)}{A}p_4 - \frac{2\lambda^2(3 + 4\lambda + \lambda^2)}{A}p_1 p_3\right|. \end{aligned} \tag{2.4}$$

To estimate bound on $|a_5|$, now we consider all three terms in the right hand side of (2.4) one by one and find the bound on them. It is well known form [4] (see also [15, Lemma 3, p. 227]) that

$$|p_1^4+2p_1p_3+p_2^2-3p_1^2p_2-p_4|\leq 2. \tag{2.5}$$

The following are obtained by using Lemma 1:

$$\begin{aligned}\left|\frac{3\lambda^2(1+\lambda)^2}{A}p_1^2p_2-\frac{\lambda^2(3+4\lambda+\lambda^2)}{A}p_2^2\right| &= \left|\frac{(3+4\lambda+\lambda^2)\lambda p_2}{A}\left(\frac{3(1+\lambda)^2}{3+4\lambda+\lambda^2}p_1^2-p_2\right)\right| \\ &\leq \frac{4\lambda^2(5\lambda^2+8\lambda+3)}{A}\end{aligned} \tag{2.6}$$

$$\begin{aligned}\left|\frac{\lambda^2(\lambda^2+4\lambda+7)}{A}p_4-\frac{2\lambda^2(\lambda^2+4\lambda+3)}{A}\right| &= \frac{\lambda^2}{A}\left|(\lambda^2+4\lambda+3)p_1p_3-(\lambda^2+4\lambda+7)p_4\right| \\ &= \frac{(\lambda^2+4\lambda+7)\lambda^2}{A}\left|\frac{(\lambda^2+4\lambda+3)}{(\lambda^2+4\lambda+7)}p_1p_3-p_4\right| \\ &\leq \frac{2(\lambda^2+4\lambda+7)\lambda^2}{A}.\end{aligned} \tag{2.7}$$

Now, from (2.4), (2.5), (2.6) and (2.7), we have

$$\begin{aligned}|a_5| &\leq 2\frac{A}{16}+\left(\frac{4\lambda^2(5\lambda^2+8\lambda+3)}{A}+\frac{2(\lambda^2+4\lambda+7)\lambda^2}{A}\right) \\ &= \frac{A}{8}+\frac{4\lambda^2(5\lambda^2+8\lambda+3)}{4}+\frac{2(\lambda^2+4\lambda+7)\lambda^2}{8} \\ &= \frac{2+2\lambda+5\lambda^2+6\lambda^3+4\lambda^4}{2}.\end{aligned}$$

This completes the proof. □

Remark 1. In [22, Theorem 1, p. 91], the following non-sharp bound was proved:

$$|a_5|\leq 1+2\lambda\sqrt{(\lambda^2+1)(\lambda^4+1)}. \tag{2.8}$$

Now if $x_0\approx 0.305398$ is the root of $48x^5+60x^4+76x^3+33x^2+20x-12=0$, then from (2.1) and (2.8), we have the following refined result:

$$|a_5|\leq\begin{cases}\frac{2+2\lambda+5\lambda^2+6\lambda^3+4\lambda^4}{2}, & 0<\lambda\leq x_0; \\ 1+2\lambda\sqrt{(\lambda^2+1)(\lambda^4+1)}, & x_0\leq x<1.\end{cases}$$

Thus, we see that our result improves over the result proved in [22, Theorem 1, p. 91] for the range $0<\lambda\leq x_0$.

In the following theorem non-sharp bounds on $|a_5|$, $|a_6|$ and $|a_7|$ for the class $\mathcal{S}_q^*$ are investigated.

Theorem 2. *Let $f(z)=z+a_2z^2+a_3z^3+\cdots\in\mathcal{S}_q^*$. Then*

(i) $|a_5| \leq \frac{1}{8}\left(1+2\sqrt{2}\right) \approx 0.478553$,
(ii) $|a_6| \leq \frac{5\left(711+32\sqrt{79}\right)}{8532} \approx 0.341303$,
(iii) $|a_7| \leq \frac{2665}{2304} \approx 1.15668$.

We need the following lemmas to prove the above theorem:

Lemma 2 ([14], Theorem 4(b), p. 678)**.** *A function $p \in \mathcal{P}$ if and only if*

$$\sum_{j=0}^{\infty}\left\{\left|2z_j+\sum_{k=1}^{\infty}p_k z_{k+j}\right|^2-\left|\sum_{k=0}^{\infty}p_{k+1}z_{k+j}\right|^2\right\} \geq 0$$

for every sequence $\{z_k\}_{k=1}^{\infty}$ of complex numbers that satisfies $\lim_{k\to\infty}|z_k|^{1/k} < 1$.

Lemma 3 ([20], Proposition 6, p. 7)**.** *Let $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leqq 1\}$. Also, for any real numbers a, b and c, let the quantity $Y(a,b,c) := \max_{z\in\overline{\mathbb{D}}}\left\{|a+bz+cz^2|+1-|z|^2\right\}$. If $ac \geqq 0$, then*

$$Y(a,b,c) = \begin{cases} |a|+|b|+|c| & \left(|b| \geqq 2(1-|c|)\right), \\ 1+|a|+\dfrac{b^2}{4(1-|c|)} & \left(|b| < 2(1-|c|)\right). \end{cases}$$

Furthermore, if $ac < 0$, then

$$Y(a,b,c) = \begin{cases} 1-|a|+\dfrac{b^2}{4(1-|c|)} & \left(-4ac(c^{-2}-1) \leqq b^2;\ |b| < 2(1-|c|)\right) \\ 1+|a|+\dfrac{b^2}{4(1+|c|)} & \left(b^2 < \min\left\{4(1+|c|)^2, -4ac(c^{-2}-1)\right\}\right) \\ R(a,b,c) & (\text{otherwise}), \end{cases}$$

where

$$R(a,b,c) = \begin{cases} |a|+|b|-|c| & \left(|c|(|b|+4|a|) \leqq |ab|\right) \\ -|a|+|b|+|c| & \left(|ab| \leqq |c|(|b|-4|a|)\right) \\ (|c|+|a|)\sqrt{1-\dfrac{b^2}{4ac}} & (\text{otherwise}). \end{cases}$$

Lemma 4 ([10], Lemma 1)**.** *Let $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}$. Then, for any real number μ,*

$$\left|\mu p_3 - p_1^3\right| \leq \begin{cases} 2|\mu - 4| & \left(\mu \leq \dfrac{4}{3}\right) \\ 2\mu\sqrt{\dfrac{\mu}{\mu - 1}} & \left(\dfrac{4}{3} < \mu\right). \end{cases}$$

The result is sharp.

Lemma 5 ([1], Corollary 1, p. 68)**.** *Let $p \in \mathcal{P}$. Then*

$$|p_3 - (\mu + 1)p_1 p_2 + \mu p_1^3| \leq \begin{cases} 2, & 0 \leq \mu \leq 1; \\ 2|\mu - 1|, & \textit{elsewhere.} \end{cases}$$

Proof of Theorem 2. (i) Since $f \in \mathcal{S}_q^*$, it follows that there exists a Schwarz function w, such that

$$\frac{zf'(z)}{f(z)} = w(z) + \sqrt{1 + w^2(z)}. \tag{2.9}$$

Writing

$$w(z) = \frac{p(z) - 1}{p(z) + 1}$$

with $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}$, and equating the coefficients of like power terms, we have

$$a_5 = \frac{1}{384}\left(p_1^4 - 10p_1^2 p_2 + 8p_1 p_3 + 48p_4\right) \tag{2.10}$$

and therefore,

$$\begin{aligned} |a_5| &= \frac{1}{384}\left|p_1^4 - 10p_1^2 p_2 + 8p_1 p_3 + 48p_4\right| \\ &\leq \frac{1}{384}\left(\left|48p_4 - 10p_1^2 p_2\right| + \left|p_1^4 + 8p_1 p_3\right|\right) \\ &\leq \frac{1}{384}\left(48 + \left|48p_4 - 10p_1^2 p_2\right|\right). \end{aligned} \tag{2.11}$$

We now choose a sequence

$$z_0 = \frac{5}{24}p_1 p_2,\ z_1 = 0,\ z_2 = 0,\ z_3 = -1,\ z_k = 0\ (k \geq 4)$$

to estimate the maximum value of $|48p_4 - 10p_1^2 p_2|$.

Now an application of Lemma 2, for the above choice of sequence gives

$$\left|\frac{5}{24}p_1^2 p_2 - p_4\right|^2 \leq \left|p_3 - \frac{5}{12}p_1 p_2\right|^2 + 4 \leq 8.$$

Therefore,

$$|10p_1^2p_2-48p_4|\leq 96\sqrt{2}$$

and from Eq. (2.11), we have

$$|a_5|\leq \frac{1}{8}\left(1+2\sqrt{2}\right)\approx 0.478553.$$

This was the desired bound on $|a_5|$.

(ii) Proceeding as above, we have

$$a_6=\frac{-p_1^5+42p_1^3p_2-80p_1^2p_3-80p_1p_2^2+48p_1p_4-32p_2p_3+384p_5}{3840} \tag{2.12}$$

and

$$a_7=\frac{T}{184320}, \tag{2.13}$$

where

$$\begin{aligned} T:=&-59p_1^6-392p_1^4p_2+1840p_1^3p_3+32p_1^2\left(95p_2^2-99p_4\right)\\ &-64p_1(97p_2p_3+96p_5)-320\left(3p_2^3+6p_2p_4+4p_3^2-48p_6\right). \end{aligned} \tag{2.14}$$

To find the estimate on $|a_6|$, by suitably rearranging the terms of (2.12) and using the triangle inequality, we have:

$$\begin{aligned} 3840|a_6|&=\left|-p_1^5+42p_1^3p_2-80p_1^2p_3-80p_1p_2^2+48p_1p_4-32p_2p_3+384p_5\right|\\ &=\left|-80p_1^2\left(p_3-\frac{21}{40}p_1p_2+\frac{19}{40}p_1^3\right)+(48p_1p_4-80p_1p_2^2)+(384p_5-32p_2p_3)+37p_1^5\right|\\ &\leq 80\left|p_1^2\left(p_3-\frac{21}{40}p_1p_2+\frac{19}{40}p_1^3\right)\right|+\left|48p_1p_4-80p_1p_2^2\right|+|384p_5-32p_2p_3|+37\,|p_1|^5. \end{aligned} \tag{2.15}$$

Now using Lemma 1, Lemma 5 and the fact $|p_i|\leq 2$, we have

$$80\left|p_1^2\left(p_3-\frac{21}{40}p_1p_2+\frac{19}{40}p_1^3\right)\right|\leq 640,\quad \left|48p_1p_4-80p_1p_2^2\right|\leq 320 \tag{2.16}$$

and

$$|384p_5-32p_2p_3|=384\left|\frac{1}{12}p_2p_3-p_5\right|\leq 768. \tag{2.17}$$

Now from (2.15), (2.16) and (2.17), we have

$$|a_6|\leq\frac{2272}{8532}\approx 0.341303.$$

This was the desired estimate on $|a_6|$.

(iii) From (2.14), by suitably rearranging the terms, we have

$$\begin{aligned} 184320a_7=&-59p_1^6-392p_1^4p_2+1840p_1^3p_3+32p_1^2\left(95p_2^2-99p_4\right)\\ &-64p_1(97p_2p_3+96p_5)-320\left(3p_2^3+6p_2p_4+4p_3^2-48p_6\right) \end{aligned}$$

or

$$\begin{aligned}184320|a_7| \leq 320|6p_2p_4-48p_6| &+ \left|-392p_1^4p_2+1840p_1^3p_3\right| \\ &+ \left|-3040p_1^2p_2^2+960p_2^3\right| + \left|59p_1^6+3168p_1^2p_4\right| \\ &+ |64p_1(97p_2p_3+96p_5)| + \left|2180p_3^2\right|. \end{aligned} \tag{2.18}$$

Now applications of Lemma 1 and the fact $|p_i| \leq 2$, gives

$$320|6p_2p_4-48p_6| \leq 30720, \ \left|-392p_1^4p_2+1840p_1^3p_3\right| \leq 29440, \tag{2.19}$$

$$\left|-3040p_1^2p_2^2+960p_2^3\right| \leq 40960. \tag{2.20}$$

and

$$\left|59p_1^6+3168p_1^2p_4\right| + |64p_1(97p_2p_3+96p_5)| + \left|2180p_3^2\right| \leq 108480. \tag{2.21}$$

Now from (2.18), (2.19), (2.20) and (2.21), we have

$$|a_7| \leq \frac{2665}{2304} \approx 1.15668.$$

This completes the proof. □

3. Third and Fourth Hankel Determinants

3.1. *Third Hankel determinant*

In this section, we find an estimate on the third Hankel determinant for functions in the class $\mathcal{S}_q^*$. As before, for functions in the class $\mathcal{S}_q^*$, we have

$$a_2 = \frac{p_1}{2}, \quad a_3 = \frac{1}{16}\left(p_1^2+4p_2\right) \quad \text{and} \quad a_4 = \frac{1}{96}\left(16p_3+4p_1p_2-p_1^3\right). \tag{3.1}$$

A computation using (2.10), (2.12), (2.13) and (3.1), gives

$$H_{3,1}(f) = a_3(a_2a_4-a_3^2) - a_4(a_4-a_2a_3) + a_5(a_3-a_2^2) = \frac{S}{36864}, \tag{3.2}$$

where

$$\begin{aligned} S := &-55p_1^6+128p_1^4p_2+368p_1^3p_3+1216p_1p_2p_3 \\ &-32p_1^2(11p_2^2+27p_4)-64(9p_2^3+16p_3^2-18p_2p_4). \end{aligned}$$

By rearranging terms and using the triangle inequality from (3.2), we have

$$\begin{aligned} 36864|H_{3,1}(f)| \leq &\,|-55p_1^6+128p_1^4p_2| + |1152p_2p_4-864p_1^2p_4| \\ &+ |1216p_1p_2p_3-1024p_3^2| + |368p_1^3p_3-352p_1^2p_2^2| + |-576p_2^3|. \end{aligned} \tag{3.3}$$

Using Lemma 1 and the fact $|p_i| \leq 2$, we have

$$\left|-55p_1^6+128p_1^4p_2\right| \leq 4096, \quad \left|1152p_2p_4-864p_1^2p_4\right| \leq 4608, \tag{3.4}$$

$$\left|1216p_1p_2p_3-1024p_3^2\right| \leq 5632 \ \text{ and } \ \left|-576p_2^3\right| \leq 4608. \tag{3.5}$$

Now in order to get the desired bound, we claim that $\left|368p_1^3p_3 - 352p_1^2p_2^2\right| \leq 5632$. To establish our claim, we shall prove that

$$\left|\frac{23}{22}p_1p_3 - p_2^2\right| \leq 4. \tag{3.6}$$

We use the following:

Lemma 6 ([15], Libera and Zlotkiewicz)**.** *Let* $p(z) = 1 + p_1z + p_2z^2 + \cdots \in \mathcal{P}$ *with* $p_1 \geq 0$. *Then*

$$2p_2 = p_1^2 + x(4 - p_1^2) \tag{3.7}$$

and

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)y \tag{3.8}$$

for some x *and* y *such that* $|x| \leq 1$ *and* $|y| \leq 1$.

To prove (3.6), we consider $(23/22)p_1p_3 - p_2^2$. After substituting expression for p_2, p_3 and simplifying, from Lemma 6, we have

$$\begin{aligned} I :&= \frac{23}{22}p_1p_3 - p_2^2 \\ &= \frac{23}{88}p_1(p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2) \\ &\quad - \frac{1}{4}(p_1 + x(4 - p_1^2))^2). \end{aligned} \tag{3.9}$$

Using the invariant property for the class $\mathcal{S}_q^*$ under rotation, we can assume that $s := p_1 \in [0, 2]$ and thus (3.9) becomes

$$I = \frac{1}{88}s^4 + \frac{1}{88}\left(4 - s^2\right)\frac{s^2}{2} + \frac{1}{4}\left(4 - s^2\right)\left(\frac{45}{22}s^2 - 4\right)x^2 + \frac{23}{44}s\left(4 - s^2\right)\left(1 - |x|^2\right)y. \tag{3.10}$$

Now from (3.10), we have

$$I = \begin{cases} 4, & s = 0; \\ \frac{6}{11}, & s = 2. \end{cases} \tag{3.11}$$

For $s \in (0, 2)$ consider the expression:

$$I = \frac{23s}{44}(4 - s^2)\left(a + bx + cx^2 + 1 - |x|^2\right),$$

where

$$a := \frac{s^3}{23\left(4 - s^2\right)}, \quad b := \frac{s}{23} \quad \text{and} \quad c := \frac{45s^2 - 88}{46s}.$$

For suitability, we divide the calculation in five cases:

Case (I): Here $a > 0$ always and $c \geq 0$ if and only if $1.39841 \approx (2/3)\sqrt{22/5} =:$

$s_1 \leq s < 2$. The condition $|b| > 2(1-|c|) = 2(1-|c|)$ holds if $s \geq 1.97073 \approx \left(23+\sqrt{4577}\right)/46 =: s_2$. Thus, for $s \in [s_2, 2)$, we have

$$I = \frac{23s}{44}(4-s^2)(|a|+|b|+|c|) = \frac{\left(4-s^2\right)\left(45s^4-276s^2+352\right)}{88\left(s^2-4\right)} =: f_1(s).$$

Now computation shows that f_1 attains its maximum at $s_0 = \sqrt{46/15}$ and

$$f_1(s) \leq f_1(s_0) = \frac{89}{110}, \quad s \in (s_1, 2). \tag{3.12}$$

Case (II): The condition $|b| < 2(1-|c|) = 2(1-c)$ holds for $s \in [s_1, s_2)$ and for this range, we have

$$I = \frac{s\left(91s^4-1980s^3-2024s^2+8280s+8096\right)}{88(45s+44)} = f_2(s).$$

Computation reveals that f_2 never vanish in $[s_1, s_2)$, therefore,

$$f_2(s) \leq \max\{f_2(s_1), f_2(s_2)\} = f_1(s_1) = \frac{2\left(14520+23827\sqrt{110}\right)}{334125} \approx 1.58276. \tag{3.13}$$

Case (III): Now computation shows that

$$\min\left\{4(1+c^2), -4ac(c^{-2}-1)\right\} = \begin{cases} -4ac(c^{-2}-1), & 0 < s \leq s_3; \\ 4(1+c^2), & s_3 < s < s_1, \end{cases}$$

where s_3 is a root of $95175x^6 - 443252x^4 + 859232x^2 - 681472 = 0$ in $(0, s_1)$. Further, $b^2 < -4ac(c^{-2}-1)$ holds if and only if

$$\frac{s^2}{529} \leq -\frac{2s^2\left(2025s^2-1936\right)}{529\left(45s^2-88\right)}$$

or equivalently

$$0.983378 \approx 2\sqrt{\frac{22}{91}} =: s_4 =< s < s_1.$$

Moreover, $b^2 < 4(1+c^2)$ holds for all $s > 0$. Thus, for $s \in (s_4, s_1)$, we have

$$I = \frac{s\left(91s^4-1980s^3-2024s^2+8280s+8096\right)}{88(45s+44)} = f_3(s).$$

$f_3'(s) = 0$ for $s = s^* \approx 1.1977 \in (s_4, s_1)$ and $f_3''(s^*) < 0$, so

$$f_3(s) \leq \max\{f_3(s^*), f_3(s_1), f_3(4)\} = f_3(s^*) \approx 1.65372, \quad s \in (s_4, s_1). \tag{3.14}$$

Case (IV): It can be verified that $ab \leq -c(b-4a)$ holds for $s \in (0, s_5]$, where

$$s_5 := \sqrt{\frac{2}{223}\left(155-3\sqrt{489}\right)}.$$

In this case, we have

$$I = \frac{41s^4}{88} - \frac{65s^2}{22} + 4 =: f_4(s).$$

Now a computation shows that, for $s \in (0, s_5]$,

$$\begin{aligned} f_4(s) &\leq \max\{f_4(0), f_4(s_5)\} \\ &= f_4(s_5) \\ &= \frac{4\left(11911 + 555\sqrt{489}\right)}{49729} \approx 1.94526. \end{aligned} \tag{3.15}$$

Case (V): In view of Lemma 3, for the case, $s \in (s_5, s_4]$, we have

$$I = \frac{1}{88}\sqrt{\frac{180 - 91s^2}{176 - 90s^2}}\left(47s^4 - 268s^2 + 352\right) =: f_5(s).$$

Now a computation shows that

$$\begin{aligned} f_5(s) &\leq \max\{f_5(s_4), f_5(s_5)\} = f_5(s_5) \\ &= \frac{\sqrt{4930 + 138\sqrt{489}}\left(6139 + 261\sqrt{489}\right)}{547019} \approx 1.94526, \ s \in (s_5, s_4]. \end{aligned} \tag{3.16}$$

From (3.11)-(3.16), we conclude that $I \leq 4$, that is

$$\left|\frac{23}{22}p_1p_3 - p_2^2\right| \leq 4. \tag{3.17}$$

The above inequality (3.17) is sharp for the function $p(\tilde{z}) = (1+z^3)/(1-z^3)$ and therefore,

$$|368p_1^3p_3 - 352p_1^2p_2^2| = 352|p_1^2|\left|\frac{23}{22}p_1p_3 - p_2^2\right| \leq 5632. \tag{3.18}$$

From (3.3), (3.4), (3.5) and (3.18), we have

$$|H_{3,1}(f)| \leq \frac{2}{3} \approx 0.67.$$

Thus, we have the following:

Theorem 3. *Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{S}_q^*$. Then*

$$|H_{3,1}(f)| \leq \frac{2}{3} \approx 0.67.$$

3.2. *Fourth Hankel determinant*

The fourth Hankel determinant can be expressed as, see also [2]:

$$H_{4,1}(f) = a_7 H_{3,1}(f) - a_6\Delta_1 + a_5\Delta_2 - a_4\Delta_3,$$

where

$$\Delta_1 = a_3a_6 - a_4a_5 - a_2(a_2a_6 - a_3a_5) + a_4(a_2a_4 - a_3^2),$$
$$\Delta_2 = (a_4a_6 - a_5^2) - a_2(a_3a_6 - a_4a_5) + a_3(a_3a_5 - a_4^2)$$

and

$$\Delta_3 = a_2(a_4a_6 - a_5^2) - a_3(a_3a_6 - a_4a_5) + a_4(a_3a_5 - a_4^2).$$

To compute the fourth Hankel determinant for the functions in class $\mathcal{S}_q^*$, we substitute the vales of a_i in terms of p_j as given in (2.10), (2.12), (2.13) and (3.1) and after simplification we have

$$\begin{aligned}122880\Delta_1 = {} & 31p_1^7 - 400p_1^5p_2 + 240p_1^4p_3 + 32p_1^3(18p_2^2 + 11p_4)\\ & - 64p_1(15p_2^3 - 20p_3^2 - 26p_2p_4) + 256p_1^2(2p_2p_3 - 9p_5)\\ & - 512(3p_2^2p3 + 5p_3p_4 - 6p_2p_5).\end{aligned} \tag{3.19}$$

Rearranging the terms of (3.19) and applying Lemmas 1 and 4, we have

$$\begin{aligned}122880|\Delta_1| \leq {} & |31p_1^7 - 400p_1^5p_2| + |576p_1^3p_2^2 - 960p_1p_2^3| + |3072p_2p_5 - 1536p_2^2p_3|\\ & + 256|p_1^2(2p_2p_3 - 9p_5| + |352p_1^3p_4 - 2560p_3p_4|\\ & + 64|p_1||20p_3^2p_1 + 26p_2p_4|\\ \leq {} & \frac{1024}{69}\left(5451 + 40\sqrt{345}\right).\end{aligned}$$

Therefore, we have

$$|\Delta_1| \leq \frac{5451 + 40\sqrt{345}}{8280} \approx 0.748064. \tag{3.20}$$

In a similar line, we have

$$\begin{aligned}-491520\Delta_2 = {} & (-552p_1^5p_3 + 3p_1^8) + (656p_4p_1^4 - 576p_1^2p_2p_4)\\ & + (-1920p_1^3p_2p_3 + 1216p_1p_2^2p_3) + (7680p_4^2 - 3584p_1p_3p_4)\\ & + (4096p_1^2p_2p_5 - 8192p_3p_5) + (1216p_1p_2^2p_3 - 3840p_2^2p_4).\end{aligned} \tag{3.21}$$

From (3.21) and applying Lemmas 1 and 4, we have

$$|\Delta_2| \leq \frac{118279 + 92\sqrt{8418}}{117120} \approx 1.08197. \tag{3.22}$$

Similarly rearranging the terms of

$$8847360\Delta_3 = -29p_1^9 - 78p_1^7p_2 + 768p_1^6p_3 - 72p_1^5(17p_2^2 + 74p_4)$$

$$
\begin{aligned}
&-32p_1^3(299p_2^3+180p_3^2-864p_2p_4)+288p_1^4(57p_2p_3-28p_5)\\
&-1536p_1^2(17p_2^2p_3-6p_3p_4+6p_2p_5)\\
&+512(9p_2^3p_3-80p_3^3+180p_2p_3p_4-108p_2^2p_5)\\
&+768p_1(15p_2^4-28p_2p_3^2+21p_2^2p_4-90p4^2+96p_3p_5)
\end{aligned}
$$

suitably, we have

$$
\begin{aligned}
8847360\Delta_3 = {}&(768p_1^6p_3-29p_1^9)+(27648p_1^3p_2p_4-5328p_4p_1^5)\\
&+(-78p_1^7p_2+16416p_1^4p_2p_3)+1536p_1^2(-17p_2^2p_3+6p_3p_4)\\
&+(73728p_1p_3p_5-9216p_1^2p_2p_5)+(92160p_2p_3p_4-21504p_1p_2p_3^2)\\
&+768p_1(21p_2^2p_4-90p_4^2)+512(9p_2^3p_3-108p_2^2p_5)\\
&+(11520p_1p_2^4-9568p_1^3p_2^3)\\
&+(-1224p_1^5p_2^2-5760p_3^2p_1^3-8064p_1^4p_5-40960p_3^3). \qquad (3.23)
\end{aligned}
$$

Using triangle inequality in (3.23) and applying Lemmas 1 and 4, we have

$$
|\Delta_3| \leq \frac{3720606449+33460224\sqrt{717}+2942568\sqrt{51737}}{5622886080} \approx 0.940065. \qquad (3.24)
$$

Now using the triangle inequality and substituting the bounds on initial coefficients $|a_i|$ $(i=4,5,6,7)$ and $|\Delta_j|$ $(j=1,2,3)$, we have

$$
\begin{aligned}
|H_{4,1}(f)| &\leq |a_7||H_{3,1}(f)|+|a_6||\Delta_1|+|a_5||\Delta_2|+|a_4||\Delta_3|\\
&\leq 1.93977.
\end{aligned}
$$

Thus, we have the following:

Theorem 4. *Let $f(z)=z+a_2z^2+a_3z^3+\cdots\in\mathcal{S}_q^*$. Then $|H_{4,1}(f)|\leq 1.93977$.*

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FUZZY BARRELS ON LOCALLY CONVEX FUZZY TOPOLOGICAL VECTOR SPACES

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Abstract. In this paper, we introduce the concept of fuzzy barrels on locally convex fuzzy topological vector spaces. We present some characterizations of the fuzzy locally convex spaces, which are fuzzy barrelled. Using fuzzy barrels, we prove that the same fuzzy sets are bounded in any fuzzy topology of dual pair. Finally, we prove the Banach-Steinhaus theorem for the fuzzy topological vector spaces.

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1. Introduction

In [13], the concept of fuzzy topological vector space was introduced by Katsaras and Liu. In [11], Katsaras changed the definition of fuzzy topological vector space and he considered the linear fuzzy topology on the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), which consists of all lower semi-continuous functions from $\mathbb{K}$ into $I = [0,1]$. At the first, the idea of fuzzy norm on linear spaces introduced by Katsaras. After then, in different approach many authors like Cheng and Mordeson [2], Felbin [10], Bag and Samanta [1], etc. introduced the notion of fuzzy normed linear space. Later, Xiao and Zhu [16], Fang [9], Daraby et. al. [5, 6], redefined the notion of Felbin's [10] definition of fuzzy norm. Das [7] introduced a fuzzy topology generated by fuzzy norm and studied some properties of this topology.

The concept of weak linear fuzzy topology on a fuzzy topological vector space as a generalization of usual weak topology was studied in [4]. Also, in [4] the authors have proved that this fuzzy topology consists of all weakly lower semi-continuous fuzzy sets on a given vector space when $\mathbb{K}$ endowed with its usual fuzzy topology. In [3], the concept of polar fuzzy sets on fuzzy dual spaces was studied. Also, in [3] the polar linear fuzzy topologies on fuzzy dual spaces have introduced by the help of polar fuzzy sets and a Mackey-Arens type theorem on fuzzy topological vector spaces has investigated.

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One of the most powerful theorems of functional analysis is the Banach-Steinhaus theorem. This theorem asserts that the set of continuous linear mapping that is pointwise bounded is bounded uniformly on barrelled spaces. This shows the importance of the barrelled spaces in functional analysis. In this paper we are going to study the fuzzy barrelled spaces.

2. Preliminaries

Let X be a non-empty set. A fuzzy set in X is an element of the set I^X of all functions from X into I.

Definition 1 ([7])**.** Let X and Y be any two non-empty sets, $f : X \to Y$ be a mapping and μ be a fuzzy subset of X. Then $f(\mu)$ is a fuzzy subset of Y defined by

$$f(\mu)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x) & f^{-1}(y) \neq \varnothing, \\ 0 & else, \end{cases}$$

for all $y \in Y$, where $f^{-1}(y) = \{x : f(x) = y\}$. If η is a fuzzy subset of Y, then the fuzzy subset $f^{-1}(\eta)$ of X is defined by $f^{-1}(\eta)(x) = \eta(f(x))$ for all $x \in X$.

Definition 2 ([11])**.** A fuzzy topology on a set X is a subset τ_f of I^X satisfying the following conditions:

(i) τ_f contains every constant fuzzy set in X,
(ii) if $\mu_1, \mu_2 \in \tau_f$, then $\mu_1 \wedge \mu_2 \in \tau_f$,
(iii) if $\mu_i \in \tau_f$ for each $i \in A$, then $\sup_{i \in A} \mu_i \in \tau_f$.

The pair (X, τ_f) is called a fuzzy topological space.

The elements of τ_f are called fuzzy open sets in X.

Definition 3 ([7])**.** A fuzzy topological space (X, τ_f) is said to be fuzzy Hausdorff if for $x, y \in X$ and $x \neq y$ there exist $\eta, \beta \in \tau_f$ with $\eta(x) = \beta(y) = 1$ and $\eta \wedge \beta = 0$.

A mapping f from a fuzzy topological space X into a fuzzy topological space Y is called fuzzy continuous if $f^{-1}(\mu)$ is fuzzy open in X for each fuzzy open set μ in Y. Suppose X is a fuzzy topological space and $x \in X$. A fuzzy set μ in X is called a neighborhood of $x \in X$ if there is a fuzzy open set η with $\eta \leq \mu$ and $\eta(x) = \mu(x) > 0$. Warren [15] has proved that a fuzzy set μ in X is fuzzy open if and only if μ is a neighborhood of x for each $x \in X$ with $\mu(x) > 0$. The collection O of fuzzy sets is called a fuzzy open covering of μ when $\mu \leq \sup_{\nu \in O} \nu$. The fuzzy set μ is called fuzzy compact if any fuzzy open covering of μ has finite subcovering.

Definition 4 ([7])**.** If μ_1 and μ_2 are two fuzzy subsets of a vector space E, then the fuzzy set $\mu_1 + \mu_2$ is defined by

$$(\mu_1 + \mu_2)(x) = \sup_{x = x_1 + x_2} (\mu_1(x_1) \wedge \mu_2(x_2)).$$

If $t \in \mathbb{K}$, we define the fuzzy sets $\mu_1 \times \mu_2$ and $t\mu$ as follows:

$$(\mu_1 \times \mu_2)(x_1, x_2) = \min\{\mu_1(x_1), \mu_2(x_2)\}$$

and

$$(i) \text{ for } t \neq 0,\ (t\mu)(x) = \mu(\tfrac{x}{t}) \quad \text{for all} \quad x \in E,$$

$$(ii) \text{ for } t = 0,\ (t\mu)(x) = \begin{cases} 0 & x \neq 0, \\ \sup_{y \in E} \mu(y) & x = 0. \end{cases}$$

A fuzzy set μ in vector space E is called balanced if $t\mu \leq \mu$ for each scalar t with $|t| \leq 1$. As, it has been shown in [13], μ is balanced if and only if $\mu(tx) \geq \mu(x)$ for each $x \in E$ and each scalar t with $|t| \leq 1$. Also, when μ is balanced, we have $\mu(0) \geq \mu(x)$ for each $x \in E$. The fuzzy set μ is called absorbing if and only if $\sup_{t>0} t\mu = 1$. Then a fuzzy set μ is absorbing whenever $\mu(0) = 1$. We shall say that the fuzzy set μ is convex if and only if for all $t \in I$, $t\mu + (1-t)\mu \leq \mu$, [11]. Also a fuzzy set μ is called absolutely convex if it is balanced and convex.

Definition 5 ([8])**.** A fuzzy topology τ_f on a vector space E is said to be a fuzzy linear topology, if the mappings

$$f : E \times E \to E,\ (x, y) \to x + y,$$
$$g : \mathbb{K} \times E \to E,\ (t, x) \to tx,$$

are continuous when $\mathbb{K}$ is equipped with the fuzzy topology induced by the usual topology, $E \times E$ and $\mathbb{K} \times E$ are the corresponding product fuzzy topologies. A vector space E with a fuzzy linear topology τ_f, denoted by the pair (E, τ_f), is called fuzzy topological vector space (abbreviated to FTVS).

Definition 6 ([11])**.** Let (E, τ_f) be a fuzzy topological vector space, the collection $\nu \subset \tau_f$ of neighborhoods of zero is a local base whenever for each neighborhood μ of zero and each $\theta \in (0, \mu(0))$ there is $\gamma \in \nu$ such that $\gamma \leq \mu$ and $\gamma(0) > \theta$.

Definition 7 ([12])**.** A fuzzy seminorm on E is a fuzzy set μ in E which is absolutely convex and absorbing.

We say that a fuzzy set μ, in a vector space E, absorbs the fuzzy set η if $\mu(0) > 0$ and for every $\theta < \mu(0)$ there exists $t > 0$ such that $\theta \wedge (t\eta) \leq \mu$. A fuzzy set μ in a fuzzy linear space E is called bounded if it is absorbed by every neighborhood of zero.
The Katsaras norm was defined as follows:

Definition 8 ([12])**.** A fuzzy norm on vector space E is an absolutely convex and absorbing fuzzy set ρ with $\inf_{t>0}(t\rho)(x) = 0$, for $x \neq 0$.

Definition 9 ([12])**.** A fuzzy topological vector space (E, τ_f) is called locally convex if it has a neighborhood base at zero consisting of convex fuzzy sets.

Let (E,τ) be a topological vector space. The collection of all lower semicontinuous functions from E in to $[0,1]$ is a fuzzy linear topology on E denoted by $w(\tau)$.

Proposition 1 ([3])**.** *Let (E,τ_f) be a fuzzy topological vector space. Then there is a linear topology τ on E such that $\tau_f = w(\tau)$.*

In [14], the concept of dual pair defined as follows:

We call (E,E') a dual pair, whenever E and E' are two vector spaces over the same $\mathbb{K}$ scaler field and $\langle x,x'\rangle$ is a bilinear form on E and E' satisfying the following conditions:

(D) For each $x \neq 0$ in E, there is $x' \in E'$ such that $\langle x,x'\rangle \neq 0$.
(D') For each $x' \neq 0$ in E' there is $x \in E$ such that $\langle x,x'\rangle \neq 0$.

Let (E,E') be a dual pair. We denote by $\sigma_f(E,E')$, the weakest linear fuzzy topology on E which makes all $x' \in E'$ fuzzy continuous as linear functionals from E into $(\mathbb{K},w(\tau))$, where τ is the usual topology on $\mathbb{K}$. We call $\sigma_f(E,E')$, the weak fuzzy topology (see [4]).

Lemma 1 ([4])**.** *Let (E,E') be a dual pair. If we consider the usual fuzzy topology $\omega(\tau)$ on $\mathbb{K}$, then*

$$\sigma_f(E,E') = \omega(\sigma(E,E')),$$

where $\sigma(E,E')$ usual weak topology on E.

Let E be a vector space and $\mu \in I^E$. The θ-level set of μ is defined by

$$[\mu]_\theta = \{x \in E : \mu(x) \geq \theta\},$$

where $0 < \theta \leq 1$. It is clear that for $\theta_1,\theta_2 \in (0,1]$ with $\theta_1 < \theta_2$ we have $[\mu]_{\theta_1} \supseteq [\mu]_{\theta_2}$. Therefore $\{[\mu]_\theta : \theta \in (0,1]\}$ is a decreasing collection of subsets of E.

Proposition 2 ([3])**.** *Let (E,τ_f) be a fuzzy topological vector space and $\tau_f = w(\tau)$. Then the following hold.*

(a) *The fuzzy set $\mu \in I^E$ is fuzzy compact if and only if $[\mu]_\theta$ is compact in (E,τ) for each $0 < \theta \leq 1$,*
(b) *The fuzzy set $\mu \in I^E$ is fuzzy closed if and only if $[\mu]_\theta$ is closed in (E,τ) for each $0 < \theta \leq 1$,*
(c) *The fuzzy set $\mu \in I^E$ is fuzzy absolutely convex if and only if $[\mu]_\theta$ is absolutely convex in (E,τ) for each $0 < \theta \leq 1$.*

3. Fuzzy barrelled spaces

The concept of polar of fuzzy sets was introduced in [3] as follows:

Let (E,E') be a dual pair. For the non-empty subset A of E, its polar A° defined by:

$$A^\circ = \{x' \in E' : \sup_{x\in A} |\langle x,x'\rangle| \leq 1\}.$$

Definition 10 ([3])**.** Let (E,E') be a dual pair, $\mu \in I^E$. For $x' \in E'$ and $\mu \in I^E$, we set $A_{\mu,x'} = \{\theta \in (0,1] : x' \in [\mu]^\circ_{1-\theta}\}$. For $\mu \neq 0$, we define the fuzzy set μ° on E' as

$$\mu^\circ(x') = \begin{cases} \sup A_{\mu,x'} & , \quad A_{\mu,x'} \neq \varnothing, \\ 0, & \quad A_{\mu,x'} = \varnothing, \end{cases}$$

and call it the fuzzy polar of μ in E.

Remark 1 ([3]). We note that if $\mu = 1$, then for $\theta \in (0,1]$ we have $[\mu]_{1-\theta} = E$. Therefore $[\mu]^\circ_{1-\theta} = E^\circ = \{0\}$. This shows that for $x' \neq 0$, $A_{\mu,x'} = \varnothing$. Then $\mu^\circ(x') = 0$ but clearly we have $\mu^\circ(0) = 1$.

Lemma 2 ([3])**.** *Let (E,E') be a dual pair and $\mu \in I^E$. Then we have*

$$\mu^\circ = \sup_{\theta \in (0,1]} \theta \wedge \chi_{[\mu]^\circ_{1-\theta}}.$$

Theorem 1 ([3])**.** *If (E,τ_f) is a Hausdorff locally convex fuzzy topological vector space and μ is a fuzzy neighborhood of origin, then μ° is $\sigma_f(E',E)$-compact.*

Definition 11. A fuzzy subset μ in locally convex fuzzy topological vector space E is said to be a fuzzy barrel if it is absolutely convex, absorbent and fuzzy closed.

In every Katsaras normed space (E,ρ), the fuzzy set $\overline{\rho}$ (the closure of ρ) is absolutely convex, absorbent and fuzzy closed. Therefore $\overline{\rho}$ is a fuzzy barrel.

Proposition 3. *Let (E,τ_f) be a locally convex fuzzy topological vector space. Then there is a linear fuzzy topology $\tau_{\mathfrak{B}}$ on E such that all the fuzzy barrels of (E,τ_f) are fuzzy neighborhoods in $(E,\tau_{\mathfrak{B}})$.*

Proof. Let (E,τ_f) be a locally convex fuzzy topological vector space and B be a fuzzy barrel in E. Then B is a Katsaras seminorm in E and the collection

$$U_B = \{\theta \wedge (tB) \,:\, t > 0, 0 < \theta \leq 1\}$$

is a neighborhood base for a fuzzy linear topology on E, denoted by τ_B. Now, we set

$$\tau_{\mathfrak{B}} = \bigcap_{B \text{ is a fuzzy barrel}} \tau_B.$$

The fuzzy topology $\tau_{\mathfrak{B}}$ is a linear fuzzy topology on E, since for each fuzzy barrel B, τ_B is a linear fuzzy topology. Also, it is clear that all the barrels are fuzzy neighborhoods in $(E,\tau_{\mathfrak{B}})$. □

It is clear that for the locally convex fuzzy topological vector space (E,τ_f) the fuzzy topology $\tau_{\mathfrak{B}}$ is weaker than τ_f, since for each absolutely convex absorbent fuzzy neighborhood μ in τ_f, the closure of μ is a fuzzy barrel. But the converse is not true in general.

Definition 12. A locally convex fuzzy topological vector space (E,τ_f) is said to be a fuzzy barrelled space if every fuzzy barrel in E is a fuzzy neighborhood. In other words, for each fuzzy barrel μ in E there is a fuzzy neighborhood ν such that $\nu \leq \mu$.

Corollary 1. *Let (E,τ_f) be a fuzzy barrelled locally convex space. Then the fuzzy topologies τ_f and $\tau_{\mathfrak{B}}$ are equivalent, since in the fuzzy barrelled spaces every fuzzy barrel is a fuzzy neighborhood. In fact for each neighborhood B in $\tau_{\mathfrak{B}}$ there is a fuzzy neighborhood ν in τ_f such that $\nu \leq B$.*

Lemma 3. *Let (E,τ) be a locally convex topological vector space, and B be a barrel in E. Then χ_B is a fuzzy barrel in fuzzy topological vector space $(E,\omega(\tau))$.*

Proof. Since B is a barrel, then it is absolutely convex and closed. Now Proposition 2 shows that χ_B is fuzzy absolutely convex and closed. Also since B is absorbing then for each $x \in B$ there is $\lambda > 0$ such that $x \in \lambda B$. Then $\frac{x}{\lambda} \in B$. This shows that $\chi_B(\frac{x}{\lambda}) = 1$. Then $\sup_{t>0}(t\chi_B)(x) = \sup_{t>0}\chi_B(\frac{x}{t}) = 1$. So χ_B is absorbing. □

Proposition 4. *Let E be a Hausdorff locally convex fuzzy topological vector space with dual $E^{'}$. Then the fuzzy subset μ of E is a fuzzy barrel if and only if μ is the fuzzy polar of a $\sigma_f(E',E)$-bounded fuzzy subset of $E^{'}$.*

Proof. Let μ be a fuzzy polar of a $\sigma_f(E',E)$-bounded fuzzy subset of $E^{'}$. We prove that the fuzzy subset μ of E is a fuzzy barrel. Firstly, we prove that μ° is balanced. So it is enough to show that $\mu^\circ(tx') \geq \mu^\circ(x')$ for all $x' \in E$ and $t \in \mathbb{R}$ with $|t| \leq 1$. We have

$$\begin{aligned}
\mu^\circ(tx') &= \sup A_{\mu,tx'} = \sup\{\theta \in (0,1] : tx' \in [\mu]^\circ_{1-\theta}\} \\
&= \sup\{\theta \in (0,1] : x' \in (t[\mu]_{1-\theta})^\circ\} = \sup\{\theta \in (0,1] : x' \in [t\mu]^\circ_{1-\theta}\} \\
&= \sup A_{t\mu,x'} = (t\mu)^\circ(x') = \frac{1}{|t|}\mu^\circ(x') \\
&\geq 1 \times \mu^\circ(x') = \mu^\circ(x').
\end{aligned}$$

The convexity of μ° is obvious, since $[\mu]^\circ_{1-\theta}$ is a convex set for each $\theta \in (0,1]$. We note that for each $x \in E; \varphi_x(x') =< x,x' >$ is a $\sigma_f(E',E)-$continuous linear functional on E'. Now, since $(\theta \wedge \chi_{[\mu]^\circ_{1-\theta}})$ is a closed subset of $\mathbb{K}$ and $\mu^\circ = \bigwedge_{\mu(x)>\theta} \varphi_x^{-1}(\theta \wedge \chi_{[\mu]^\circ_{1-\theta}})$ then μ° is $\sigma_f(E',E)-$closed. Conversely, suppose μ is a fuzzy barrel. Then it is absolutely convex and absorbent. Now, we prove that μ° is $\sigma_f(E',E)-$bounded. We have

$$\mu^{\circ\circ}(x) = \sup\{\theta : x \in [\mu^\circ]^\circ_{1-\theta}\}.$$

Let $\tau_f = w(\tau)$. Since μ is fuzzy closed and absolutely convex, so $[\mu]_\theta$ is closed and absolutely convex with respect to τ. Then, we have

$$[\mu^\circ]^\circ_{1-\theta} = ([\mu]^\circ_\theta)^\circ = [\mu]^{\circ\circ}_\theta = [\mu]_\theta.$$

This shows that

$$\begin{aligned}\mu^{\circ\circ}(x) &= \sup\{\theta : x \in [\mu^{\circ}]^{\circ}_{1-\theta}\}\\ &= \sup\{\theta : x \in [\mu]^{\circ\circ}_{\theta}\}\\ &= \sup\{\theta : x \in [\mu]_{\theta}\}\\ &= \mu(x).\end{aligned}$$

The fuzzy set μ° is $\sigma_f(E',E)$-compact by Theorem 1. Hence μ° is $\sigma_f(E',E)$-bounded. □

Lemma 4. *Let (E,τ_f) be a locally convex fuzzy topological vector space. Then fuzzy barrel absorbs every convex compact fuzzy set.*

Proof. Let μ be a fuzzy barrel and ν be a convex compact fuzzy set. By Proposition 1 there is a linear locally convex topology τ on E such that $\tau_f = \omega(\tau)$. Now, since ν is a compact fuzzy set, $[\nu]_{\theta}$ is compact in (E,τ) for each $0 < \theta \leq 1$ by Proposition 2. Also for each $0 < \theta \leq 1, [\mu]_{\theta}$ is a usual barrel. Then there is $\lambda > 0$ such that $[\nu]_{\theta} \subseteq \lambda[\mu]_{\theta}$ for all $0 < \theta \leq 1$. So if $\nu(x) \geq \theta$, then $\mu(\frac{x}{\lambda}) \geq \theta$. That is, $(\lambda\mu)(x) \geq \theta$. Thus $\nu \leq \lambda\mu$. □

Theorem 2. *The same fuzzy sets are bounded in every linear fuzzy topology of dual pair.*

Proof. Let τ_f be any fuzzy topology of the dual pair $(E,E^{'})$. Then τ_f is obviously finer than $\sigma_f(E,E^{'})$. Then τ_f-bounded fuzzy sets are $\sigma_f(E,E^{'})$-bounded. Conversely, let ν be a $\sigma_f(E,E^{'})$-bounded fuzzy set and μ be a closed absolutely convex τ_f-neighborhood. Then ν° be a fuzzy neighborhood in $E^{'}$ under $\sigma_f(E',E)$ and μ is absolutely convex and $\sigma_f(E^{'},E)$-compact by Theorem 1. Hence by Lemma 4, ν° absorbs μ°. Then $\mu^{\circ\circ}$ absorbs $\nu^{\circ\circ}$. Since $\mu \leq \mu^{\circ\circ}$ and $\nu^{\circ\circ} = \nu$. Then ν absorbs μ. This shows that ν is τ_f-bounded. □

Definition 13. Let (E,τ_f) be a fuzzy topological vector space. The family $\mathcal{F} \subseteq E'$ is called fuzzy equicontinuous if for each fuzzy neighborhood μ in $\mathbb{K}$, there is a fuzzy neighborhood $\nu \in \tau_f$ such that $f(\nu) \leq \mu$ for all $f \in \mathcal{F}$.

Corollary 2. *Let E be Hausdorff locally convex fuzzy topological vector space with dual E' then E is fuzzy barrelled if and only if every $\sigma_f(E',E)$-bounded fuzzy subset of E' is fuzzy equicontinuous.*

Corollary 3. *Let E be Hausdorff locally convex fuzzy topological vector space with dual E', then E has the fuzzy topology $\tau_f(E,E')$.*

Definition 14. Let E and F be locally convex fuzzy topological vector spaces and P be a collection of linear mappings from E into F. We say that P is pointwise

bounded if for every $x \in E$ and every absolutely convex closed fuzzy neighborhood ν in F we have

$$\bigvee_{t>0} \bigwedge_{T\in P} (t\nu)(T(x)) = 1.$$

One of the powerful theorems of functional analysis, the Banach-Steinhaus theorem, asserts that the set of continuous linear mapping that is bounded at each point of a Banach space is bounded uniformly on the unit ball. In the following theorem we prove the Banach-Steinhaus theorem for fuzzy barrelled spaces.

Theorem 3. *Let E be a fuzzy barrelled space and F a fuzzy convex space. Then any pointwise bounded fuzzy set of fuzzy continuous linear mappings of E into F is fuzzy equicontinuous.*

Proof. Let P be a pointwise fuzzy bounded set of continuous linear operators from E into F. Suppose ν is a fuzzy closed absolutely convex fuzzy neighborhood in F. We set $B = \bigwedge_{T\in P} T^{-1}(\nu)$. Then B absolutely convex, and closed. We prove that B is also absorbing. Since ν is absorbing for $x \in E$ we have

$$\begin{aligned}
\sup_{t>0}(tB)(x) &= \bigvee_{t>0} \bigwedge_{T\in P} T^{-1}(\nu)(\frac{x}{t}) \\
&= \bigvee_{t>0} \bigwedge_{T\in P} \nu(T(\frac{x}{t})) \\
&= \bigvee_{T\in P} \bigwedge_{t>0} \nu(\frac{T(x)}{t}) \\
&= 1.
\end{aligned}$$

Hence B is a fuzzy barrel in E. Now, since E is barrelled, then B is a fuzzy neighborhood. Also for each $T' \in P$ we have

$$\begin{aligned}
T'(B)(y) &\leqslant T'((T')^{-1}(\nu))(y) \\
&= \sup_{x\in(T')^{-1}(y)} ((T')^{-1}(\nu))(x) \\
&= \sup_{x\in(T')^{-1}(y)} \nu(T'(x)) \\
&\leqslant \nu(y).
\end{aligned}$$

This shows that P is equicontinuous. □

Example 1. Consider the fuzzy locally convex space $\mathbb{R}^n$ endowed with the fuzzy Katsaras norm

$$\rho(x_1, \ldots. x_n) = \begin{cases} 1 & \Sigma_{i=1}^n x_i^2 < 1, \\ 0 & else. \end{cases}$$

It is clear that $\mathbb{R}^n$ is complete (a fuzzy Banach space). We prove that $\mathbb{R}^n$ is barrelled. We have

$$\overline{\rho}(x_1,....x_n) = \begin{cases} 1 & \Sigma_{i=1}^n x_i^2 \leq 1, \\ 0 & else. \end{cases}$$

This show that $\overline{\rho}$ is compact (since $\mathbb{R}^n$ is complete). Now, suppose B is a fuzzy barrel in $\mathbb{R}^n$. By Lemma 4, B absorbs $\overline{\rho}$ i.e. for each $0 < \theta \leq B(0)$ there is $t > 0$ such that $\theta \wedge (t\overline{\rho}) \leq B$. This yields that $\theta \wedge (t\rho) \leq \theta \wedge (t\overline{\rho}) \leq B$. Then B absorbs ρ. Thus B is a fuzzy neighborhood in $\mathbb{R}^n$ and $\mathbb{R}^n$ is barrelled.

Example 2. Consider the fuzzy locally convex spaces $E = \mathbb{R}^n$ and $F = \mathbb{R}$ endowed with the Katsaras fuzzy norms

$$\rho(x_1,....x_n) = \begin{cases} 1 & \Sigma_{i=1}^n x_i^2 < 1, \\ 0 & else, \end{cases}$$

and

$$\nu(x) = \begin{cases} 1 & |x| < 1, \\ 0 & else. \end{cases}$$

For $i = 1,2,\cdots,n$ let $T_i(x_1,x_2,\cdots,x_n) = x_i$. Then the collection $P = \{T_i : i = 1,2, \cdots,n\}$ is pointwise bounded since

$$\begin{aligned} \bigvee_{t>0} \bigwedge_{i=1,\cdots,n} (t\nu)(T_i(x_1,\cdots,x_n)) &= \bigvee_{t>0} \bigwedge_{i=1,\cdots,n} (t\nu)(x_i) \\ &= \bigvee_{t>0} \bigwedge_{i=1,\cdots,n} \nu(\frac{x_i}{t}) \\ &= \nu(0) \\ &= 1. \end{aligned}$$

Now, since E is fuzzy barrelled, by Theorem 3, P is fuzzy equicontinuous.

4. Conclusion

The Banach-Steinhaus theorem is one of the important theorems in functional analysis, which asserts that the set of continuous linear mapping that is pointwise bounded is bounded uniformly on barrelled spaces. Theorem 3 is an extension of Banach-Steinhaus theorem to the more general case of fuzzy locally convex spaces.

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ON SUMS WITH GENERALIZED HARMONIC, HYPERHARMONIC AND SPECIAL NUMBERS

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Abstract. In this paper, we establish interesting sums including generalized harmonic numbers and special numbers by using generating functions of these numbers and some combinatorial identities.

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1. Introduction

Harmonic numbers are important in various branches of combinatorics and number theory. The harmonic numbers are defined by

$$H_0 = 0, \;\; H_n = \sum_{i=1}^{n} \frac{1}{i}, \text{ for } n = 1, 2, \cdots.$$

The first few harmonic numbers are $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \cdots$. These numbers have been generalized by some authors.

In [7], for every ordered pair $(\alpha, n) \in \mathbb{R}^+ \times \mathbb{N}$, the generalized harmonic numbers $H_n(\alpha)$ are defined by

$$H_0(\alpha) = 0, \;\; H_n(\alpha) = \sum_{i=1}^{n} \frac{1}{i\alpha^i}, \text{ for } n = 1, 2, \cdots.$$

For $\alpha = 1$, the usual harmonic numbers are $H_n(1) = H_n$ and the generating function of the generalized harmonic numbers is

$$\sum_{n=1}^{\infty} H_n(\alpha) x^n = -\frac{\ln\left(1 - \frac{x}{\alpha}\right)}{1 - x}.$$

In [10], for the generalized harmonic numbers $H_n(\alpha)$, the authors defined the generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$ as follows:

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Definition 1. For $r<0$ or $n\leq 0$, $H_n^r(\alpha)=0$ and for $n\geq 1$, the generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$ are defined by

$$H_n^r(\alpha)=\sum_{i=1}^{n}H_i^{r-1}(\alpha),\quad r\geq 1,$$

where $H_n^0(\alpha)=\frac{1}{n\alpha^n}$.

For $\alpha=1$, $H_n^r(1)=H_n^r$ are the hyperharmonic numbers of order r. The generating function of the generalized hyperharmonic numbers of order r is

$$\sum_{n=1}^{\infty}H_n^r(\alpha)x^n=-\frac{\ln\left(1-\frac{x}{\alpha}\right)}{(1-x)^r}. \tag{1.1}$$

In [8, 13], the generalized harmonic numbers $H(n,r)$ of rank r are defined as for $n\geq 1$, $r\geq 0$,

$$H(n,r)=\sum_{1\leq n_0+n_1+\cdots+n_r\leq n}\frac{1}{n_0n_1\cdots n_r}$$

or, equivalently, as

$$H(n,r)=\frac{(-1)^{r+1}}{n!}\left(\frac{d^n}{dx^n}\frac{[\ln(1-x)]^{r+1}}{1-x}\right)\Bigg|_{x=0}.$$

It is clear that $H(n,0)=H_n$.

Some special numbers in areas of number theory that we will use in the future are given as follows:

The Cauchy numbers of order r, showed by C_n^r, are defined by the generating functions to be

$$\left(\frac{x}{\ln(1+x)}\right)^r=\sum_{n=0}^{\infty}C_n^r\frac{x^n}{n!}. \tag{1.2}$$

The Daehee numbers of order r, showed by D_n^r, are defined by the generating functions to be

$$\left(\frac{\ln(1+x)}{x}\right)^r=\sum_{n=0}^{\infty}D_n^r\frac{x^n}{n!}. \tag{1.3}$$

For $r=1$, $D_n^1=D_n$ are called Daehee numbers. It is clear that

$$D_0=1,D_1=-\frac{1}{2},\cdots,D_n=(-1)^n\frac{n!}{n+1}. \tag{1.4}$$

The derangement numbers, denoted by d_n, are defined by the generating functions to be

$$\frac{e^{-x}}{1-x}=\sum_{n=0}^{\infty}d_n\frac{x^n}{n!}.$$

It is well known that Stirling numbers play an important role in combinatorial analysis. The Stirling numbers of the first kind $s(n,i)$ are given by

$$x^{\underline{n}} = \sum_{i=0}^{n} s(n,i)x^i,$$

where for $n \geq 0$, $s(n,0) = \delta_{n0}$, δ_{ni} is the Kronecker delta [1, 14]. $x^{\underline{n}}$ stands for the falling factorial defined by $x^{\underline{n}} = x(x-1)\dots(x-n+1)$.

The Stirling numbers of the first kind $s(n,k)$ satisfy the recurrence relation

$$s(n+1,k) = s(n,k-1) - ns(n,k),$$

and the generating function of these numbers is given by

$$\frac{\ln(1+x)^k}{k!} = \sum_{n=0}^{\infty} s(n,k)\frac{x^n}{n!}.$$

The signed Stirling numbers of the first kind $|s(n,k)|$ are defined such that the number of permutations of n elements which contain exactly k permutation cycles is the nonnegative number

$$|s(n,k)| = (-1)^{n-k}s(n,k).$$

This means that $s(n,k) = 0$ for $k > n$ and $s(n,n) = 1$ and the generating function of these numbers [2, 9] is given by

$$\frac{(-\ln(1-x))^k}{k!} = \sum_{n=0}^{\infty} |s(n,k)|\frac{x^n}{n!}. \tag{1.5}$$

In [2], the authors obtained many relations between the Stirling numbers of first kind and the generalized harmonic numbers $H(n,r)$ of rank r. For example, for any positive integers n,r,

$$\sum_{i=1}^{n} \frac{(-1)^{i+1}}{i!} H(n+1,i) = H_n, \qquad \sum_{i=r}^{n-1} \frac{r!}{i!} s(i,r) H_{n-i}^{i+1} = H(n,r).$$

In [12], the authors gave some identities including the hyperharmonic, the Daehee and the derangement numbers and derived some nonlinear differential equations from the generating function of the hyperharmonic numbers. For example, for any positive integers n,r,

$$D_n = n!\sum_{i=0}^{n} \binom{r}{n-i} (-1)^i H_{i+1}^r, \qquad \sum_{i=0}^{n} \frac{(-1)^{n-i}}{(n-i)!} H_i^r = \sum_{i=0}^{n} \frac{d_{n-i}}{(n-i)!} H_i^{r-1}.$$

Harmonic numbers and generalized harmonic numbers have been studied since the distant past and are involved in a wide range of diverse fields such as analysis, computer science and various branches of number theory. Recently, there are a lot of works about infinite series which are associated with the Riemann Zeta function, the Digamma (and Polygamma) functions, the harmonic (and the generalized harmonic)

numbers and also properties, generating functions including higher-order harmonic numbers by using umbral type method [3, 4, 11].

In [5], the authors derived several generating functions involving harmonic numbers by making use of an interesting approach based on the umbral calculus.

In [18], the authors obtained closed-form expressions for three families of combinatorial series associated with the Digamma (or Psi) function $\psi(z)$ and the generalized harmonic number $H_n(z) = \sum_{j=1}^{n} 1/(z+j)$. They also considered several illustrative examples and applications of their main results.

A. Sofo and H. M. Srivastava extended some results of Euler related sums [15, 16]. In [16], they investigated products of the shifted harmonic numbers and the reciprocal binomial coefficients and in [15], integral and closed-form representations of sums with products of harmonic numbers and binomial coefficients were developed in terms of Polygamma functions.

In [6], the authors gave the properties of a class of generalized harmonic numbers $H(n,r)$. By means of the method of coefficients, the authors established some identities involving $H(n,r)$. They obtained the asymptotic expansion of certain sums involving these numbers and inverse of binomial coefficients by Laplace's method.

In [17], the author obtained a set of identities for finite sums of products of harmonic numbers higher-order and reciprocal binomial coefficients.

In this paper, we establish interesting sums including the generalized harmonic numbers and special numbers by using generating functions of these numbers and some combinatorial identities. For example, for any positive integers n, r and m,

$$n!H_n^r(\alpha) = (-1)^{n-1} \sum_{i=0}^{n-1} \binom{n}{i} \frac{(n-i)(-r)^{\underline{n-i-1}}}{\alpha^{i+1}} D_i,$$

$$H(n,r) = \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{m-1}{n-i} (-1)^{n-j-r} \frac{H_j^m(\alpha)\, s(i-j,r) r!}{\alpha^{i-j}(i-j)!}.$$

2. Some identities involving generalized hyperharmonic numbers and Daehee numbers

In this section, using the generating functions of the generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$, the Daehee numbers D_n, the derangement numbers d_n and the Cauchy numbers C_n, we will give some interesting sums including these numbers.

Theorem 1. *For any positive integers n, r, we have*

$$D_n = \alpha^{n+1} n! \sum_{i=0}^{n} \binom{r}{n-i} (-1)^i H_{i+1}^r(\alpha), \tag{2.1}$$

$$H_n(\alpha) = \sum_{i=0}^{n-1} \frac{(-1)^i}{i!} \frac{D_i}{\alpha^{i+1}}. \tag{2.2}$$

Proof. From (1.3), we have

$$\sum_{n=0}^{\infty}\frac{D_n}{\alpha^{n+1}}\frac{x^n}{n!}=\frac{\ln\left(1+\frac{x}{\alpha}\right)}{x}=\left(-\frac{\ln\left(1+\frac{x}{\alpha}\right)}{(1+x)^r}\right)\left(-\frac{1}{x}\right)(1+x)^r.$$

By (1.1) and Binomial theorem, we write

$$\sum_{n=0}^{\infty}\frac{D_n}{\alpha^{n+1}}\frac{x^n}{n!}=\left(-\frac{1}{x}\right)\sum_{i=1}^{\infty}(-1)^i H_i^r(\alpha)x^i\sum_{j=0}^{\infty}\binom{r}{j}x^j.$$

and by product of generating functions, we get

$$\sum_{n=0}^{\infty}\frac{D_n}{\alpha^{n+1}}\frac{x^n}{n!}=\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{r}{n-i}(-1)^i H_{i+1}^r(\alpha)x^n.$$

Comparing the coefficients on both sides, we get the desired result.

Now, we will give second equality. Then from (1.4), we have

$$\begin{aligned}\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}\frac{D_i}{\alpha^{i+1}}&=\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}\frac{(-1)^i i!}{(i+1)\alpha^{i+1}}\\&=\sum_{i=0}^{n-1}\frac{1}{(i+1)\alpha^{i+1}}=H_n(\alpha),\end{aligned}$$

as claimed result. □

Corollary 1. *For any positive integers n,r, we have*

$$H_n(\alpha)=\sum_{i=0}^{n-1}\sum_{j=0}^{i}\binom{r}{i-j}(-1)^{i+j}H_{j+1}^r(\alpha).$$

Proof. Combining (2.1) and (2.2), the result is clearly given. □

Theorem 2. *For any positive integers n,r, we have*

$$n!H_n^r(\alpha)=(-1)^{n-1}\sum_{i=0}^{n-1}\binom{n}{i}\frac{(n-i)(-r)^{\underline{n-i-1}}}{\alpha^{i+1}}D_i,$$

$$\sum_{i=0}^{n}\frac{(-1)^{n-i}}{(n-i)!}H_i^r(\alpha)=\sum_{i=0}^{n}\frac{d_{n-i}}{(n-i)!}H_i^{r-1}(\alpha).$$

Proof. By (1.1), we write

$$\sum_{n=1}^{\infty}H_n^r(\alpha)x^n=-\frac{\ln\left(1-\frac{x}{\alpha}\right)}{(1-x)^r}=\frac{1}{\alpha}\frac{\ln\left(1-\frac{x}{\alpha}\right)}{-\frac{x}{\alpha}}\frac{x}{(1-x)^r},$$

and from (1.3) and Binomial theorem,

$$\sum_{n=1}^{\infty}H_n^r(\alpha)x^n=\frac{1}{\alpha}\sum_{i=0}^{\infty}\frac{(-1)^i D_i}{\alpha^i}\frac{x^i}{i!}\sum_{n=0}^{\infty}(-1)^n(-r)^{\underline{n}}\frac{x^{n+1}}{n!}$$

$$
\begin{aligned}
&= \sum_{i=0}^{\infty} \frac{(-1)^i D_i}{\alpha^{i+1}} \frac{x^i}{i!} \sum_{n=1}^{\infty} (-1)^{n-1} (-r)^{\underline{n-1}} \frac{x^n}{(n-1)!} \\
&= \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \frac{(-1)^i D_i}{\alpha^{i+1} i!} \frac{(-1)^{n-i-1} (n-i) (-r)^{\underline{n-i-1}}}{(n-i)!} x^n \\
&= \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \binom{n}{i} \frac{(-1)^{n-1} (n-i) (-r)^{\underline{n-i-1}}}{\alpha^{i+1} n!} D_i x^n.
\end{aligned}
$$

Thus, we have the desired result.

We will give the other sum. By multiplying the generating function of generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$ by e^{-x}, we get

$$
\begin{aligned}
-\frac{\ln\left(1-\frac{x}{\alpha}\right)}{(1-x)^r} e^{-x} &= \sum_{n=0}^{\infty} H_n^r(\alpha) x^n \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} x^i \\
&= \sum_{n=0}^{\infty} \sum_{i=0}^{n} H_i^r(\alpha) \frac{(-1)^{n-i}}{(n-i)!} x^n \qquad (2.3)
\end{aligned}
$$

and

$$
\begin{aligned}
-\frac{\ln\left(1-\frac{x}{\alpha}\right)}{(1-x)^r} e^{-x} &= \frac{-\ln\left(1-\frac{x}{\alpha}\right)}{(1-x)^{r-1}} \frac{e^{-x}}{1-x} \\
&= \sum_{n=0}^{\infty} H_n^{r-1}(\alpha) x^n \sum_{i=0}^{\infty} \frac{d_i}{i!} x^i \\
&= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{d_{n-i}}{(n-i)!} H_i^{r-1}(\alpha) x^n. \qquad (2.4)
\end{aligned}
$$

From (2.3) and (2.4), we have the desired identity. Hence we have the proof. □

Theorem 3. *For any positive integers n, r, k, we have*

$$
D_n^r = n! \alpha^{n+1} \sum_{i=0}^{n} \sum_{j=0}^{i} \frac{(-1)^j k^{\underline{n-i}}}{\alpha^{i-j} (i-j)! (n-i)!} D_{i-j}^{r-1} H_{j+1}^k(\alpha),
$$

$$
n! H_n^r(\alpha) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} \binom{n}{i} \binom{n-i}{j} \frac{(-1)^{n-1} (-r)^{\underline{n-i-j-1}} (n-i-j)}{\alpha^{i+j+1}} D_i^r C_j^{r-1}.
$$

Proof. From (1.3), we write

$$
\sum_{n=0}^{\infty} \frac{D_n^r}{\alpha^n} \frac{x^n}{n!} = \left(\frac{\ln\left(1+\frac{x}{\alpha}\right)}{\frac{x}{\alpha}} \right)^r = \alpha \frac{-\ln\left(1+\frac{x}{\alpha}\right)}{(-x)(1+x)^k} \left(\frac{\ln\left(1+\frac{x}{\alpha}\right)}{\frac{x}{\alpha}} \right)^{r-1} (1+x)^k.
$$

By (1.1), (1.3) and Binomial theorem, using product of generating functions, we have

$$
\sum_{n=0}^{\infty} \frac{D_n^r}{\alpha^n} \frac{x^n}{n!} = \alpha \sum_{n=0}^{\infty} (-1)^n H_{n+1}^k(\alpha) x^n \sum_{j=0}^{\infty} \frac{D_j^{r-1}}{\alpha^j} \frac{x^j}{j!} \sum_{i=0}^{\infty} k^{\underline{i}} \frac{x^i}{i!}
$$

$$= \alpha \sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^j H_{j+1}^k(\alpha) \frac{D_{n-j}^{r-1}}{(n-j)!\alpha^{n-j}} x^n \sum_{i=0}^{\infty} k^{\underline{i}} \frac{x^i}{i!}$$

$$= \alpha \sum_{n=0}^{\infty} \sum_{i=0}^{n} \sum_{j=0}^{i} \frac{(-1)^j k^{\underline{n-i}}}{\alpha^{i-j}(n-i)!(i-j)!} H_{j+1}^k(\alpha) D_{i-j}^{r-1} x^n.$$

Comparing the coefficients on both sides, we get the desired result. Similarly, from (1.2) and (1.3), the proof of the other result is clearly obtained. □

3. Some identities involving generalized harmonic numbers of rank r and Stirling numbers of the first kind

In this section, inspiring from the definition of $H(n,r)$ of rank r in their works [8, 13], $H(n,r,\alpha)$ are defined as for $n \geq 1$, $r \geq 0$,

$$H(n,r,\alpha) = \sum_{1 \leq n_0+n_1+\cdots+n_r \leq n} \frac{1}{n_0 n_1 \cdots n_r \alpha^{n_0+n_1+\cdots+n_r}} \tag{3.1}$$

or, equivalently, as

$$H(n,r,\alpha) = \frac{(-1)^{r+1}}{n!} \left(\frac{d^n}{dx^n} \frac{\left[\ln\left(1-\frac{x}{\alpha}\right)\right]^{r+1}}{1-x} \right)\Bigg|_{x=0}.$$

For $\alpha = 1$, $H(n,r,1) = H(n,r)$.

TABLE 1. $H(n,r,2)$

n \ r	0	1	2	3	4
1	$\frac{1}{2}$	0			
2	$\frac{5}{8}$	$\frac{1}{4}$	0		
3	$\frac{2}{3}$	$\frac{3}{8}$	$\frac{1}{8}$	0	
4	$\frac{131}{192}$	$\frac{83}{192}$	$\frac{7}{32}$	$\frac{1}{16}$	0
5	$\frac{661}{960}$	$\frac{11}{24}$	$\frac{35}{128}$	$\frac{1}{8}$	$\frac{1}{32}$

The generating function of the generalized harmonic numbers of rank r, $H(n,r,\alpha)$ is given by

$$\sum_{n=0}^{\infty} H(n,r,\alpha)x^n = \frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}{1-x}. \tag{3.2}$$

Now, we will give some sums with the help of the generating functions.

Theorem 4. *For any positive integers n,r, we have*

$$\begin{aligned} H(n,r,\alpha) &= (-1)^{n-r}\sum_{i=0}^{n}\frac{(-1)^i s(n-i,r)r!}{\alpha^{n-i}(n-i)!}H_i(\alpha) \\ &= (-1)^{r+1}\sum_{i=0}^{n}\frac{(-1)^i s(i,r+1)(r+1)!}{\alpha^i i!}. \end{aligned}$$

Proof. By (1.5) and (3.2), we consider

$$\begin{aligned} \sum_{n=0}^{\infty} H(n,r,\alpha)x^n &= \frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}{1-x} \\ &= \frac{-\ln\left(1-\frac{x}{\alpha}\right)}{1-x}\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^r \\ &= \sum_{n=0}^{\infty} H_n(\alpha)x^n \sum_{i=0}^{\infty}\frac{(-1)^{i-r}s(i,r)r!}{\alpha^i i!}x^i \\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{(-1)^{n-i-r}s(n-i,r)r!}{\alpha^{n-i}(n-i)!}H_i(\alpha)x^n. \end{aligned}$$

Comparing the coefficients on both sides, we arrive at the desired result. Similarly, using $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$, the other result is obtained. □

Theorem 5. *For any positive integers n,r,m, we have*

$$H(n,r,\alpha) = \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{m-1}{n-i}(-1)^{n-j-r}\frac{H_j^m(\alpha)s(i-j,r)r!}{\alpha^{i-j}(i-j)!},$$

$$D_n^{r+1} = n!\alpha^{n+1}\sum_{i=0}^{n}\sum_{j=0}^{i}\binom{k}{n-i}\frac{(-1)^j D_{i-j}^r H_{j+1}^k(\alpha)}{\alpha^{i-j}(i-j)!}.$$

Proof. By (1.1), (1.5), (3.2) and Binomial theorem, we write

$$\begin{aligned} \sum_{n=0}^{\infty} H(n,r,\alpha)x^n &= \frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}{1-x} \\ &= \frac{-\ln\left(1-\frac{x}{\alpha}\right)}{(1-x)^m}\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^r(1-x)^{m-1} \end{aligned}$$

$$= \sum_{n=0}^{\infty} H_n^m(\alpha) x^n \sum_{j=0}^{\infty} \frac{(-1)^{j-r} s(j,r) r!}{\alpha^j j!} x^j \sum_{i=0}^{\infty} \binom{m-1}{i} (-1)^i x^i,$$

and using product of generating functions, equals

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{(-1)^{n-j-r} s(n-j,r) r!}{\alpha^{n-j}(n-j)!} H_j^m(\alpha) x^n \sum_{i=0}^{\infty} \binom{m-1}{i} (-1)^i x^i$$
$$= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{m-1}{n-i} (-1)^{n-j-r} \frac{H_j^m(\alpha) s(i-j,r) r!}{\alpha^{i-j}(i-j)!} x^n.$$

Thus, comparing the coefficients on both sides, the desired result is given.

Similarly, considering

$$\left(\frac{\ln\left(1+\frac{x}{\alpha}\right)}{\frac{x}{\alpha}} \right)^{r+1} = \alpha \frac{-\ln\left(1+\frac{x}{\alpha}\right)}{-x(1+x)^k} \left(\frac{\ln\left(1+\frac{x}{\alpha}\right)}{\frac{x}{\alpha}} \right)^r (1+x)^k$$

and the generating function of Daehee numbers, the proof of the other equality is complete. □

Theorem 6. *For any positive integers n, r, we have*

$$(-1)^{n+1-r} s(n+1,r) r! = \alpha^{n+1}(n+1)! \left(H(n+1,r-1,\alpha) - H(n,r-1,\alpha) \right).$$

Proof. By (1.5) and (3.2), we have

$$\sum_{n=r}^{\infty} \frac{(-1)^{n-r} s(n,r) r!}{\alpha^n n!} x^{n-1} = \frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^r}{x} = \frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^r}{1-x} \left(\frac{1}{x} - 1 \right)$$
$$= \frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^r}{x(1-x)} - \frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^r}{1-x}$$
$$= \sum_{n=0}^{\infty} H(n,r-1,\alpha) x^{n-1} - \sum_{n=0}^{\infty} H(n,r-1,\alpha) x^n,$$

and from $H(0,r,\alpha) = 0$, equals

$$\sum_{n=1}^{\infty} H(n,r-1,\alpha) x^{n-1} - \sum_{n=0}^{\infty} H(n,r-1,\alpha) x^n$$
$$= \sum_{n=0}^{\infty} H(n+1,r-1,\alpha) x^n - \sum_{n=0}^{\infty} H(n,r-1,\alpha) x^n$$
$$= \sum_{n=0}^{\infty} \left(H(n+1,r-1,\alpha) - H(n,r-1,\alpha) \right) x^n. \tag{3.3}$$

With the help of $s(k,r) = 0$ for $0 \le k < r$, from here,

$$\sum_{n=r}^{\infty} \frac{(-1)^{n-r} s(n,r) r!}{\alpha^n n!} x^{n-1} = \sum_{n=r-1}^{\infty} \frac{(-1)^{n+1-r} s(n+1,r) r!}{\alpha^{n+1}(n+1)!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1-r} s(n+1,r) r!}{\alpha^{n+1}(n+1)!} x^n. \tag{3.4}$$

From (3.3) and (3.4), we have the proof. □

Theorem 7. *For any positive integers n, r, we have*

$$\sum_{i=0}^{n} \binom{n-i+r}{r} \frac{(-1)^{i-r-1} s(i,r+1)(r+1)!}{\alpha^i i!} = \sum_{i=0}^{n} \binom{n-i+r-1}{r-1} H(i,r,\alpha),$$

$$\sum_{i=0}^{n} \binom{n}{i} \frac{(-1)^{i-r} s(i,r) r!}{\alpha^i} d_{n-i} = n! \sum_{i=0}^{n} \frac{(-1)^{n-i}}{(n-i)!} H(i,r-1,\alpha).$$

Proof. Observed that

$$\left(\frac{-\ln\left(1-\frac{x}{\alpha}\right)}{1-x} \right)^{r+1} = \left(-\ln\left(1-\frac{x}{\alpha}\right) \right)^{r+1} \frac{1}{(1-x)^{r+1}}.$$

From $\sum_{n=0}^{\infty} \binom{n}{r} x^n = \frac{x^r}{(1-x)^{r+1}}$ and the generating functions of Stirling numbers of the first kind, we have

$$\begin{aligned} \left(\frac{-\ln\left(1-\frac{x}{\alpha}\right)}{1-x} \right)^{r+1} &= \sum_{n=0}^{\infty} \frac{(-1)^{n-r-1} s(n,r+1)(r+1)!}{\alpha^n n!} x^n \sum_{i=0}^{\infty} \binom{i+r}{r} x^i \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n-i+r}{r} \frac{(-1)^{i-r-1} s(i,r+1)(r+1)!}{\alpha^i i!} x^n, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \left(\frac{-\ln\left(1-\frac{x}{\alpha}\right)}{1-x} \right)^{r+1} &= \frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}{1-x} \frac{1}{(1-x)^r} \\ &= \sum_{n=0}^{\infty} H(n,r,\alpha) x^n \sum_{i=0}^{\infty} \binom{i+r-1}{r-1} x^i \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n-i+r-1}{r-1} H(i,r,\alpha) x^n. \end{aligned} \tag{3.6}$$

By (3.5) and (3.6), we have the result. The proof of the other result is similar to the proof. Thus, the proof is complete. □

Theorem 8. *For any positive integers n, r, we have*

$$\begin{aligned} &\sum_{i=0}^{n} H(i,r,\alpha) H(n-i,r,\alpha) \\ &\qquad = \sum_{i=0}^{n} \left\{ (1+i) H(i+1,r,\alpha) \frac{(-1)^{n-i-r-1} s(n-i,r+1)(r+1)!}{\alpha^{n-i}(n-i)!} \right. \end{aligned}$$

$$-\frac{r+1}{\alpha}\sum_{j=0}^{i}\frac{(-1)^{j-r}s(j,r)r!}{\alpha^{n-i+j}j!}H(i-j,r,\alpha)\Bigg\}.$$

Proof. Define a function as

$$A(x):=\sum_{n=0}^{\infty}H(n,r,\alpha)x^n=(-1)^{r+1}\left(\frac{1-x}{\left(\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}\right)^{-1}.$$

Derivating of $A(x)$ respect to x, we write

$$\begin{aligned}A'(x)&=(-1)^r(A(x))^2\left(\frac{-1}{\left(\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}+\frac{(-1)^{r+1}(r+1)(A(x))^{-1}}{(\alpha-x)\ln\left(1-\frac{x}{\alpha}\right)}\right)\\&=\frac{(-1)^{r+1}(A(x))^2}{\left(\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}-\frac{(r+1)A(x)}{(\alpha-x)\ln\left(1-\frac{x}{\alpha}\right)},\end{aligned}$$

and from here,

$$\begin{aligned}(A(x))^2&=\left(A'(x)+\frac{(r+1)A(x)}{(\alpha-x)\ln\left(1-\frac{x}{\alpha}\right)}\right)\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}\\&=A'(x)\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}-\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r}\frac{(r+1)A(x)}{(\alpha-x)}\\&=\sum_{n=0}^{\infty}(n+1)H(n+1,r,\alpha)x^n\sum_{i=0}^{\infty}\frac{(-1)^{i-r-1}s(i,r+1)(r+1)!}{\alpha^i i!}x^i\\&\quad-\frac{r+1}{\alpha}\sum_{n=0}^{\infty}\frac{(-1)^{n-r}s(n,r)r!}{\alpha^n n!}x^n\sum_{j=0}^{\infty}H(j,r,\alpha)x^j\sum_{i=0}^{\infty}\left(\frac{1}{\alpha}\right)^i x^i.\end{aligned}$$

Using product of generating functions, we get

$$\begin{aligned}(A(x))^2&=\sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{(i+1)H(i+1,r,\alpha)(-1)^{n-i-r-1}s(n-i,r+1)(r+1)!}{\alpha^{n-i}(n-i)!}x^n\\&\quad-\frac{r+1}{\alpha}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{i}\frac{(-1)^{j-r}s(j,r)r!H(i-j,r,\alpha)}{\alpha^{n-i+j}j!}x^n.\end{aligned}\tag{3.7}$$

After that, we also have

$$(A(x))^2=\sum_{n=0}^{\infty}\sum_{i=0}^{n}H(i,r,\alpha)H(n-i,r,\alpha)x^n.\tag{3.8}$$

By combining (3.7) and (3.8), the proof is over. □

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A HALF-INVERSE PROBLEM FOR THE SINGULAR DIFFUSION OPERATOR WITH JUMP CONDITIONS

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Abstract. In this paper, half inverse spectral problem for diffusion operator with jump conditions dependent on the spectral parameter and discontinuity coefficient is considered. The half inverse problems is studied of determining the coefficient and two potential functions of the boundary value problem its spectrum by Hocstadt-Lieberman and Yang-Zettl methods. We show that two potential functions on the whole interval and the parameters in the boundary and jump conditions can be determined from the spectrum.

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1. Introduction and preliminaries

We consider the boundary value problem of the form

$$l(y) := -y'' + [2\lambda p(x) + q(x)]\, y = \lambda^2 \delta(x)\, y,\ x \in [0,\pi] / \{a_1, a_2\} \tag{1.1}$$

with the boundary conditions

$$y'(0) = 0, \quad y(\pi) = 0 \tag{1.2}$$

and the jump conditions

$$y(a_1+0) = \alpha_1 y(a_1-0) \tag{1.3}$$

$$y'(a_1+0) = \beta_1 y'(a_1-0) + i\lambda\gamma_1 y(a_1-0) \tag{1.4}$$

$$y(a_2+0) = \alpha_2 y(a_2-0) \tag{1.5}$$

$$y'(a_2+0) = \beta_2 y'(a_2-0) + i\lambda\gamma_2 y(a_2-0), \tag{1.6}$$

where λ is a spectral parameter, $p(x) \in W_2^1[0,\pi]$, $q(x) \in L_2[0,\pi]$ are real valued functions, $a_1 \in \left[0, \frac{\pi}{2}\right]$, $a_2 \in \left[\frac{\pi}{2}, \pi\right]$, $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ are real numbers, $|\alpha_i - 1|^2 + \gamma_i^2 \neq 0$ $(\alpha_i > 0;\ i = 1,2)$, $\beta_i = \frac{1}{\alpha_i}$ $(i = 1,2)$ and

$$\delta(x) = \begin{cases} \alpha^2, & x \in \left(0, \frac{\pi}{2}\right) \\ \beta^2, & x \in \left(\frac{\pi}{2}, \pi\right) \end{cases}$$

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for $0<\alpha<\beta<1$, $\alpha+\beta>1$.

The inverse problems consist in recovering the coefficients of an operator from their spectral characteristics. A lot of study were done the inverse spectral problem for Sturm-Liouville operators and diffusion operators [1,2,4–26]. The first results on inverse problems theory of Sturm-Liouville operators were given by Ambarzumyan [3]. The half inverse problems for Sturm-Liouville equations; the known potential in half interval is determined by the help of a one spectrum over the interval. New results on the half inverse problem were obtained by Hochstadt and Lieberman [11]. They proved the spectrum of the problem as:

$$\begin{aligned} -y''+q(x)y &= \lambda y,\ x\in[0,1] \\ y'(0)-hy(0) &= 0 \\ y'(1)+Hy(1) &= 0 \end{aligned}$$

and potential $q(x)$ on the $\left(\frac{1}{2},1\right)$ uniquely determine the potential $q(x)$ on the whole interval $[0,1]$ almost everywhere. Hald [10] proved similar results in the case when there exists a impulsive conditions inside the interval. Many studies have been done by different authors for half inverse problems using these methods [14,19]. In [19] the authors studied the existence of the solution for the half-inverse problem of Sturm-Liouville problems and gave method of reconstructing this solution under same conditions by Sakhnovich [19]. Recently, some new uniqueness results have been given on the inverse or half inverse spectral analysis of differential operators. Koyunbakan and Panakhov [14] proved the half inverse problem for diffusion operator on the finite interval $[0,\pi]$. Ran Zhang, Xiao-Chuan Xu, Chuan-Fu Yang and Natalia Pavlovna Bondarenko proved the determination of the impulsive Sturm-Liouville operator from a set of eigenvalues [26]. The purpose of this study is to prove half inverse problem by using the Hocstadt- Lieberman and Yang-Zettl methods for the following equations

$$\tilde{l}(y):=-y''+[2\lambda\tilde{p}(x)+\tilde{q}(x)]y=\lambda^2\tilde{\delta}(x)y,\ x\in[0,\pi]/\{a_1,a_2\} \tag{1.7}$$

$$y'(0)=0, y(\pi)=0 \tag{1.8}$$

$$y(a_1+0)=\tilde{\alpha}_1 y(a_1-0) \tag{1.9}$$

$$y'(a_1+0)=\tilde{\beta}_1 y'(a_1-0)+i\lambda\tilde{\gamma}_1 y(a_1-0) \tag{1.10}$$

$$y(a_2+0)=\tilde{\alpha}_2 y(a_2-0) \tag{1.11}$$

$$y'(a_2+0)=\tilde{\beta}_2 y'(a_2-0)+i\lambda\tilde{\gamma}_2 y(a_2-0). \tag{1.12}$$

Lemma 1. *Let $p(x)\in W_2^1(0,\pi)$, $q(x)\in L_2(0,\pi)$, $M(x,t)$,$N(x,t)$ are summable functions on $[0,\pi]$ such that the representation for each $x\in[0,\pi]/\{a_1,a_2\}$.* $\varphi(x,\lambda)$

is the solution of the equations (1.1)*, providing boundary conditions* (1.2) *and discontinuity conditions* (1.3)-(1.6)*,*

$$\varphi(x,\lambda)=\varphi_0(x,\lambda)+\int_0^x M(x,t)\cos\lambda t\,dt+\int_0^x N(x,t)\sin\lambda t\,dt$$

for $0<x<\frac{\pi}{2}$*, that is given as*

$$\begin{aligned}\varphi_0(x,\lambda)=&\left(\beta_1^+ +\frac{\gamma_1}{2\alpha}\right)\cos\left[\lambda\xi^+(x)-\frac{1}{\alpha}\int_{a_1}^x p(t)\,dt\right]\\&+\left(\beta_1^- -\frac{\gamma_1}{2\alpha}\right)\cos\left[\lambda\xi^-(x)+\frac{1}{\alpha}\int_{a_1}^x p(t)\,dt\right],\end{aligned}\tag{1.13}$$

for $\frac{\pi}{2}<x\le\pi$*,*

$$\begin{aligned}\varphi_0(x,\lambda)=&\left(\beta_2^+ +\frac{\gamma_2}{2\beta}\right)\cos\left[\lambda k^+(\pi)-\frac{1}{\beta}\int_{a_2}^\pi p(t)\,dt\right]\\&+\left(\beta_2^- +\frac{\gamma_2}{2\beta}\right)\cos\left[\lambda k^-(\pi)-\frac{1}{\beta}\int_{a_2}^\pi p(t)\,dt\right]\\&+\left(\beta_2^- -\frac{\gamma_2}{2\beta}\right)\cos\left[\lambda s^+(\pi)+\frac{1}{\beta}\int_{a_2}^\pi p(t)\,dt\right]\\&+\left(\beta_2^+ -\frac{\gamma_2}{2\beta}\right)\cos\left[\lambda s^-(\pi)+\frac{1}{\beta}\int_{a_2}^\pi p(t)\,dt\right],\end{aligned}\tag{1.14}$$

where $\xi^\pm(x)=\pm\alpha x\mp\alpha a_1+a_1,\qquad k^\pm(x)=\xi^+(a_2)\pm\beta x\mp\beta a_2,$

$$s^\pm(x)=\xi^-(a_2)\pm\beta x\mp\beta a_2,\qquad \beta_1^\mp=\frac{1}{2}\left(\alpha_1\mp\frac{\beta_1}{\alpha}\right),\qquad \beta_2^\mp=\frac{1}{2}\left(\alpha_2\mp\frac{\alpha\beta_2}{\beta}\right).$$

Thus, the following relations hold:

If $p(x)\in W_2^2(0,\pi)$, $q(x)\in W_2^1(0,\pi)$

$$\begin{cases}\frac{\partial^2 M(x,t)}{\partial x^2}-\rho(x)\frac{\partial^2 M(x,t)}{\partial t^2}=2p(x)\frac{\partial N(x,t)}{\partial t}+q(x)M(x,t)\\ \frac{\partial^2 N(x,t)}{\partial x^2}-\rho(x)\frac{\partial^2 N(x,t)}{\partial t^2}=-2p(x)\frac{\partial M(x,t)}{\partial t}+q(x)N(x,t)\end{cases}$$

$$M\left(x,\varsigma^+(x)\right)\cos\frac{\beta(x)}{\alpha}+N\left(x,\varsigma^+(x)\right)\sin\frac{\beta(x)}{\alpha}=\left(\beta_1^+ +\frac{\gamma_1}{2\alpha}\right)\int_0^x\left(q(t)+\frac{p^2(t)}{\alpha^2}\right)dt$$

$$M\left(x,\varsigma^+(x)\right)\sin\frac{\beta(x)}{\alpha}-N\left(x,\varsigma^+(x)\right)\cos\frac{\beta(x)}{\alpha}=\left(\beta_1^+ +\frac{\gamma_1}{2\alpha}\right)(p(x)-p(0))$$

$$\begin{aligned}&M\left(x,k^+(x)+0\right)-M\left(x,k^+(x)-0\right)\\&=-\left(\beta_2^+ +\frac{\gamma_2}{2\beta}\right)(p(x)-p(0))\sin\frac{\omega(x)}{\beta}-\left(\beta_2^+ +\frac{\gamma_2}{2\beta}\right)\int_0^x\left(q(t)+\frac{p^2(t)}{\beta^2}\right)dt\cos\frac{\omega(x)}{\beta}\\&N\left(x,k^+(x)+0\right)-N\left(x,k^+(x)-0\right)\end{aligned}$$

$$= \left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right)(p(x) - p(0)) \cos\frac{\omega(x)}{\beta} - \left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right)\int_0^x \left(q(t) + \frac{p^2(t)}{\beta^2}\right) dt \sin\frac{\omega(x)}{\beta},$$

$$\left.\frac{\partial M(x,t)}{\partial t}\right|_{t=0} = N(x,0) = 0,$$

where $\beta(x) = \int_0^x p(t)\,dt$, $\omega(x) = \int_{a_2}^x p(t)\,dt + \int_0^{a_1} p(t)\,dt$.

The proof is done as in [6].

Definition 1. The function $\Delta(\lambda)$ is called the characteristic function of the eigenvalues $\{\lambda_n\}$ of the problem (1.1)-(1.6). $\tilde{\Delta}(\lambda)$ is called the characteristic function of the eigenvalues $\left\{\tilde{\lambda}_n\right\}$ of the problem (1.7)-(1.12).

Let $\lambda = s^2, s = \sigma + i\tau$, $\sigma, \tau \in \mathrm{R}$. The solution $\varphi(x,\lambda)$ of (1.1)-(1.6) has the following asymptotic formulas hold on for $|\lambda| \to \infty$,
for $0 < x < \frac{\pi}{2}$,

$$\varphi(x,\lambda) = \frac{1}{2}\left(\frac{\alpha_1}{2} \mp \frac{\beta_1}{2\alpha} + \frac{\gamma_1}{2\alpha}\right)\exp\left(-i\left(\lambda\xi^+(x) - \frac{v(x)}{\alpha}\right)\right)\left(1 + O\left(\frac{1}{\lambda}\right)\right),$$

for $\frac{\pi}{2} < x \leq \pi$,

$$\varphi(x,\lambda) = \frac{1}{2}\left(\frac{\alpha_2}{2} + \frac{\alpha\beta_2}{2\beta} + \frac{\gamma_2}{2\beta}\right)\exp\left(-i\left(\lambda k^+(x) - \frac{t(x)}{\beta}\right)\right)\left(1 + O\left(\frac{1}{\lambda}\right)\right),$$

where $v(x) = \int_{a_1}^x p(t)\,dt$, $t(x) = \int_{a_2}^x p(t)\,dt$.

In this study, if $q(x)$ and $p(x)$ to be known almost everywhere on $\left(\frac{\pi}{2}, \pi\right)$, it is sufficient to determine uniquely $p(x)$ and $q(x)$ on the whole interval $(0,\pi)$.

2. Main result

If $\varphi_0(x,\lambda)$ is a nontrivial solution of equation (1.1) with conditions (1.2)-(1.6), then λ_0 is called an eigenvalue. Additionally, $\varphi_0(x,\lambda)$ is called the eigenfunction of the problem corresponding to the eigenvalue λ_0. $\{\lambda_n\}$ are the eigenvalues of the problem.

Lemma 2. *If* $\lambda_n = \tilde{\lambda}_n$, $\frac{\alpha}{\tilde{\alpha}} = \frac{\beta}{\tilde{\beta}}$ *then* $\alpha = \tilde{\alpha}$ *and* $\beta = \tilde{\beta}$ *for all* $n \in \mathrm{N}$.

Proof. Since $\lambda_n = \tilde{\lambda}_n$ and $\Delta(\lambda), \tilde{\Delta}(\lambda)$are entire functions in λ of order one by Hadamard factorization theorem for $\lambda \in \mathrm{C}$

$$\Delta(\lambda) \equiv C\tilde{\Delta}(\lambda).$$

On the other hand, (1.1) can be written as

$$\Delta_0(\lambda) - C\tilde{\Delta}_0(\lambda) = C\left[\tilde{\Delta}(\lambda) - \tilde{\Delta}_0(\lambda)\right] - \left[\Delta(\lambda) - \Delta_0(\lambda)\right].$$

Hence

$$C\left[\tilde{\Delta}(\lambda) - \tilde{\Delta}_0(\lambda)\right] - \left[\Delta(\lambda) - \Delta_0(\lambda)\right]$$

$$
\begin{aligned}
&= \left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right)\cos\left[\lambda k^+(\pi) - \frac{w(\pi)}{\beta}\right] + \left(\beta_2^- + \frac{\gamma_2}{2\beta}\right)\cos\left[\lambda k^-(\pi) - \frac{w(\pi)}{\beta}\right] \\
&\quad + \left(\beta_2^- - \frac{\gamma_2}{2\beta}\right)\cos\left[\lambda s^+(\pi) + \frac{w(\pi)}{\beta}\right] + \left(\beta_2^+ - \frac{\gamma_2}{2\beta}\right)\cos\left[\lambda s^-(\pi) + \frac{w(\pi)}{\beta}\right] \\
&\quad - C\left(\tilde{\beta}_2^+ + \frac{\tilde{\gamma}_2}{2\tilde{\beta}}\right)\cos\left[\lambda k^+(\pi) - \frac{\tilde{w}(\pi)}{\tilde{\beta}}\right] - C\left(\tilde{\beta}_2^- + \frac{\tilde{\gamma}_2}{2\tilde{\beta}}\right)\cos\left[\lambda k^-(\pi) - \frac{\tilde{w}(\pi)}{\tilde{\beta}}\right] \\
&\quad - C\left(\tilde{\beta}_2^- - \frac{\tilde{\gamma}_2}{2\tilde{\beta}}\right)\cos\left[\lambda s^+(\pi) + \frac{\tilde{w}(\pi)}{\tilde{\beta}}\right] - C\left(\tilde{\beta}_2^+ - \frac{\tilde{\gamma}_2}{2\tilde{\beta}}\right)\cos\left[\lambda s^-(\pi) + \frac{\tilde{w}(\pi)}{\beta}\right].
\end{aligned}
\tag{2.1}
$$

If we multiply both sides of (2.1) by $\cos\left[\lambda k^+(\pi) - \frac{w(\pi)}{\beta}\right]$ and integrate with respect to λ in (ε, T), (ε is a sufficiently small positive number) for any positive real number T, then we get

$$
\begin{aligned}
&\int_\varepsilon^T \left(C\left[\tilde{\Delta}(\lambda) - \tilde{\Delta}_0(\lambda)\right] - \left[\Delta(\lambda) - \Delta_0(\lambda)\right]\right)\cos\left[\lambda k^+(\pi) - \frac{w(\pi)}{\beta}\right] d\lambda \\
&= \int_\varepsilon^T \left\{\left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right)\cos\left[\lambda k^+(\pi) - \frac{w(\pi)}{\beta}\right] + \left(\beta_2^- + \frac{\gamma_2}{2\beta}\right)\cos\left[\lambda k^-(\pi) - \frac{w(\pi)}{\beta}\right]\right. \\
&\quad + \left(\beta_2^- - \frac{\gamma_2}{2\beta}\right)\cos\left[\lambda s^+(\pi) + \frac{w(\pi)}{\beta}\right] + \left(\beta_2^+ - \frac{\gamma_2}{2\beta}\right)\cos\left[\lambda s^-(\pi) + \frac{w(\pi)}{\beta}\right] \\
&\quad - C\left(\tilde{\beta}_2^+ + \frac{\tilde{\gamma}_2}{2\tilde{\beta}}\right)\cos\left[\lambda k^+(\pi) - \frac{\tilde{w}(\pi)}{\tilde{\beta}}\right] - C\left(\tilde{\beta}_2^- + \frac{\tilde{\gamma}_2}{2\tilde{\beta}}\right)\cos\left[\lambda k^-(\pi) - \frac{\tilde{w}(\pi)}{\tilde{\beta}}\right] \\
&\quad \left. - C\left(\tilde{\beta}_2^- - \frac{\tilde{\gamma}_2}{2\tilde{\beta}}\right)\cos\left[\lambda s^+(\pi) + \frac{\tilde{w}(\pi)}{\tilde{\beta}}\right] - C\left(\tilde{\beta}_2^+ - \frac{\tilde{\gamma}_2}{2\tilde{\beta}}\right)\cos\left[\lambda s^-(\pi) + \frac{\tilde{w}(\pi)}{\beta}\right]\right\} d\lambda.
\end{aligned}
$$

And so

$$
\begin{aligned}
&\int_\varepsilon^T \left(C\left[\tilde{\Delta}(\lambda) - \tilde{\Delta}_0(\lambda)\right] - \left[\Delta(\lambda) - \Delta_0(\lambda)\right]\right)\cos\left[\lambda k^+(\pi) - \frac{w(\pi)}{\beta}\right] d\lambda \\
&= \int_\varepsilon^T \left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right)\cos^2\left[\lambda k^+(\pi) - \frac{w(\pi)}{\beta}\right] d\lambda \\
&\quad - C\int_\varepsilon^T \left(\tilde{\beta}_2^+ + \frac{\tilde{\gamma}_2}{2\tilde{\beta}}\right)\cos\left[\lambda k^+(\pi) - \frac{w(\pi)}{\beta}\right]\cos\left[\lambda k^+(\pi) - \frac{\tilde{w}(\pi)}{\tilde{\beta}}\right] d\lambda \\
&= \int_\varepsilon^T \frac{1}{2}\left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right) + \frac{1}{2}\left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right)\cos\left[2\lambda k^+(\pi) - \frac{2w(\pi)}{\beta}\right] d\lambda \\
&\quad - C\int_\varepsilon^T \frac{1}{2}\left(\tilde{\beta}_2^+ + \frac{\tilde{\gamma}_2}{2\tilde{\beta}}\right)\left(\cos\left[2\lambda k^+(\pi) - \frac{\tilde{w}(\pi) + w(\pi)}{\beta}\right] + \cos\left[\frac{w(\pi) - \tilde{w}(\pi)}{\tilde{\beta}}\right]\right) d\lambda,
\end{aligned}
$$

$\Delta(\lambda)-\Delta_0(\lambda)=O\left(\frac{1}{|\lambda|}e^{|Im\lambda|k^+(\pi)}\right)$, $\tilde{\Delta}(\lambda)-\tilde{\Delta}_0(\lambda)=O\left(\frac{1}{|\lambda|}e^{|Im\lambda|k^+(\pi)}\right)$ for all λ in (ε,T),

$$\frac{C}{2}\left(\tilde{\beta}_2^+ + \frac{\tilde{\gamma}_2}{2\tilde{\beta}}\right)-\frac{1}{2}\left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right)=O\left(\frac{1}{T}\right).$$

By letting T tend to infinity we see that

$$C=\frac{\tilde{\beta}_2^+ + \frac{\tilde{\gamma}_2}{2\tilde{\beta}}}{\beta_2^+ + \frac{\gamma_2}{2\beta}}.$$

Similarly, if we multiply both side of (2.1) $\cos\left[\lambda k^-(\pi)-\frac{w(\pi)}{\beta}\right]$ and integrate again with respect to λ in (ε,T) and by letting T tend to infinity, then we get

$$C=\frac{\tilde{\beta}_2^- + \frac{\tilde{\gamma}_2}{2\tilde{\beta}}}{\beta_2^- + \frac{\gamma_2}{2\beta}}.$$

But since α,β and $\tilde{\alpha},\tilde{\beta}$ are positive, and $w^+(\pi)-\tilde{w}^+(\pi)=w^-(\pi)-\tilde{w}^-(\pi)$ we conclude that $C=1$. Hence $\frac{\tilde{\beta}_2^+}{\beta_2^+}=\frac{\tilde{\beta}_2^-}{\beta_2^-}$ is obtained. We have therefore proved since $\alpha=\tilde{\alpha}$ that $\beta=\tilde{\beta}$.

The proof is completed. □

Lemma 3. *If $\lambda_n=\tilde{\lambda}_n$ then $\alpha_i=\tilde{\alpha}_i$ and $\gamma_i=\tilde{\gamma}_i$ $(i=1,2)$ for all $n\in\mathbb{N}$.*
The proof is done as in [6].

Theorem 1. *Let $\{\lambda_n\}$ be the eigenvalues of both problem* (1.1)-(1.6) *and* (1.7)-(1.12). *If $p(x)=\tilde{p}(x)$ and $q(x)=\tilde{q}(x)$ on $\left[\frac{\pi}{2},\pi\right]$, then $p(x)=\tilde{p}(x)$ and $q(x)=\tilde{q}(x)$ are almost everywhere on $[0,\pi]$.*

Proof of Theorem 1. Let function $\varphi(x,\lambda)$ be the solution of equation (1.1) under the conditions (1.2)-(1.6) and the function $\tilde{\varphi}(x,\lambda)$ the solution of equation (1.7) under the conditions (1.8)-(1.12) on $\left[0,\frac{\pi}{2}\right]$. The integral forms of the functions $\varphi(x,\lambda)$ and $\tilde{\varphi}(x,\lambda)$ can be obtained as follows

$$\begin{aligned}\varphi(x,\lambda)=&\left(\beta_1^+ + \frac{\gamma_1}{2\alpha}\right)\cos\left[\lambda\xi^+(x)-\frac{1}{\alpha}\int_{a_1}^x p(t)\,dt\right]\\&+\left(\beta_1^- - \frac{\gamma_1}{2\alpha}\right)\cos\left[\lambda\xi^-(x)+\frac{1}{\alpha}\int_{a_1}^x p(t)\,dt\right]\\&+\int_0^x M(x,t)\cos\lambda t\,dt+\int_0^x N(x,t)\sin\lambda t\,dt\end{aligned}\tag{2.2}$$

and

$$\tilde{\varphi}(x,\lambda)=\left(\tilde{\beta}_1^+ + \frac{\tilde{\gamma}_1}{2\alpha}\right)\cos\left[\lambda\xi^+(x)-\frac{1}{\alpha}\int_{a_1}^x \tilde{p}(t)\,dt\right]$$

$$+\left(\tilde{\beta}_1^- - \frac{\tilde{\gamma}_1}{2\alpha}\right)\cos\left[\lambda\xi^-(x) + \frac{1}{\alpha}\int_{a_1}^x \tilde{p}(t)\,dt\right]$$
$$+\int_0^x \tilde{M}(x,t)\cos\lambda t\,dt + \int_0^x \tilde{N}(x,t)\sin\lambda t\,dt. \tag{2.3}$$

If we multiply equations (2.2) and (2.3):

$$\begin{aligned}
\varphi(x,\lambda)\cdot\tilde{\varphi}(x,\lambda) = {} & \frac{S^+\tilde{S}^+}{2}\left[\cos\left(2\lambda\xi^+(x) - K(x)\right) + \cos L(x)\right] \\
& + \frac{S^+\tilde{S}^-}{2}\left[\cos\left(2\lambda a_1 t - L(x)\right) + \cos\left(2\lambda\alpha(x-a_1) - K(x)\right)\right] \\
& + \frac{S^-\tilde{S}^+}{2}\left[\cos\left(2\lambda a_1 + L(x)\right) + \cos\left(2\lambda\alpha(x-a_1) + K(x)\right)\right] \\
& + \frac{S^-\tilde{S}^-}{2}\left[\cos\left(2\lambda\xi^-(x) + L(x)\right) + \cos K(x)\right] \\
& + S^+\int_0^x \tilde{M}(x,t)\cos\left[\lambda\xi^+(x) - \frac{t(x)}{\alpha}\right]\cos\lambda t\,dt \\
& + S^+\int_0^x \tilde{N}(x,t)\cos\left[\lambda\xi^+(x) - \frac{t(x)}{\alpha}\right]\sin\lambda t\,dt \\
& + S^-\int_0^x \tilde{M}(x,t)\cos\left[\lambda\xi^-(x) + \frac{t(x)}{\alpha}\right]\cos\lambda t\,dt \\
& + S^-\int_0^x \tilde{N}(x,t)\cos\left[\lambda\xi^-(x) + \frac{t(x)}{\alpha}\right]\sin\lambda t\,dt \\
& + \tilde{S}^+\int_0^x M(x,t)\cos\left[\lambda\xi^+(x) - \frac{\tilde{t}(x)}{\alpha}\right]\cos\lambda t\,dt \\
& + \tilde{S}^+\int_0^x N(x,t)\cos\left[\lambda\xi^+(x) - \frac{\tilde{t}(x)}{\alpha}\right]\sin\lambda t\,dt \\
& + \tilde{S}^-\int_0^x M(x,t)\cos\left[\lambda\xi^-(x) + \frac{\tilde{t}(x)}{\alpha}\right]\cos\lambda t\,dt \\
& + \tilde{S}^-\int_0^x N(x,t)\cos\left[\lambda\xi^-(x) + \frac{\tilde{t}(x)}{\alpha}\right]\sin\lambda t\,dt \\
& + \left(\int_0^x M(x,t)\cos\lambda t\,dt\right)\left(\int_0^x \tilde{M}(x,t)\cos\lambda t\,dt\right) \\
& + \left(\int_0^x N(x,t)\sin\lambda t\,dt\right)\left(\int_0^x \tilde{N}(x,t)\sin\lambda t\,dt\right) \\
& + \left(\int_0^x M(x,t)\cos\lambda t\,dt\right)\left(\int_0^x \tilde{N}(x,t)\sin\lambda t\,dt\right)
\end{aligned}$$

$$+\left(\int_0^x \tilde{M}(x,t)\cos\lambda t\,dt\right)\left(\int_0^x N(x,t)\sin\lambda t\,dt\right),$$

$$\begin{aligned}\varphi(x,\lambda)\cdot\tilde{\varphi}(x,\lambda) = {} & \frac{S^+\tilde{S}^+}{2}\left[\cos\left(2\lambda\xi^+(x)-K(x)\right)+\cos L(x)\right] \\ & + \frac{S^+\tilde{S}^-}{2}\left[\cos\left(2\lambda a_1 t - L(x)\right)+\cos\left(2\lambda\alpha(x-a_1)-K(x)\right)\right] \\ & + \frac{S^-\tilde{S}^+}{2}\left[\cos\left(2\lambda a_1 + L(x)\right)+\cos\left(2\lambda\alpha(x-a_1)+K(x)\right)\right] \\ & + \frac{S^-\tilde{S}^-}{2}\left[\cos\left(2\lambda\xi^-(x)+L(x)\right)+\cos K(x)\right] \\ & + \frac{1}{2}\left\{\int_0^x U_c(x,t)\cos\left(2\lambda t-K(t)\right)dt - \int_0^x U_s(x,t)\sin\left(2\lambda t-K(t)\right)dt\right\}\end{aligned} \tag{2.4}$$

is obtained, being

$$S^{\pm}=\left(\beta_1^{\pm}\mp\frac{\gamma_1}{2\alpha}\right),\ \tilde{S}^{\pm}=\left(\tilde{\beta}_1^{\pm}\mp\frac{\tilde{\gamma}_1}{2\alpha}\right),\ K(x)=\frac{t(x)+\tilde{t}(x)}{2},\ L(x)=\frac{t(x)-\tilde{t}(x)}{2},$$

$$\begin{aligned}& U_c(x,t) \\ & = S^+\tilde{M}\left(x,\xi^+(x)-2t\right)\cos\left(K(t)-\frac{t(x)}{\alpha}\right)+S^-\tilde{M}\left(x,\xi^-(x)-2t\right)\cos\left(K(t)-\frac{t(x)}{\alpha}\right) \\ & \quad +\tilde{S}^+M\left(x,\xi^+(x)-2t\right)\cos\left(K(t)-\frac{\tilde{t}(x)}{\alpha}\right)+\tilde{S}^-M\left(x,\xi^-(x)-2t\right)\sin\left(K(t)-\frac{\tilde{t}(x)}{\alpha}\right) \\ & \quad -S^-\tilde{N}\left(x,\xi^+(x)-2t\right)\sin\left(K(t)-\frac{t(x)}{\alpha}\right)-S^-\tilde{N}\left(x,\xi^-(x)-2t\right)\sin\left(K(t)-\frac{t(x)}{\alpha}\right) \\ & \quad -\tilde{S}^+N\left(x,\xi^+(x)-2t\right)\sin\left(K(t)-\frac{\tilde{t}(x)}{\alpha}\right)-\tilde{S}^-N\left(x,\xi^-(x)-2t\right)\sin\left(K(t)-\frac{\tilde{t}(x)}{\alpha}\right) \\ & \quad +K_1(x,t)\cos K(t)+K_2(x,t)\cos K(t)+M_1(x,t)\sin K(t)+M_2(x,t)\sin K(t),\end{aligned}$$

$$\begin{aligned}& U_s(x,t) \\ & = S^+\tilde{M}\left(x,\xi^+(x)-2t\right)\sin\left(K(t)-\frac{t(x)}{\alpha}\right)+S^-\tilde{M}\left(x,\xi^-(x)-2t\right)\sin\left(K(t)-\frac{t(x)}{\alpha}\right) \\ & \quad +\tilde{S}^+M\left(x,\xi^+(x)-2t\right)\sin\left(K(t)-\frac{\tilde{t}(x)}{\alpha}\right)+\tilde{S}^-M\left(x,\xi^-(x)-2t\right)\sin\left(K(t)-\frac{\tilde{t}(x)}{\alpha}\right) \\ & \quad +S^+\tilde{N}\left(x,\xi^+(x)-2t\right)\cos\left(K(t)-\frac{t(x)}{\alpha}\right)+S^-\tilde{N}\left(x,\xi^-(x)-2t\right)\cos\left(K(t)-\frac{t(x)}{\alpha}\right) \\ & \quad +\tilde{S}^+N\left(x,\xi^+(x)-2t\right)\cos\left(K(t)-\frac{\tilde{t}(x)}{\alpha}\right)+\tilde{S}^-N\left(x,\xi^-(x)-2t\right)\cos\left(K(t)-\frac{\tilde{t}(x)}{\alpha}\right) \\ & \quad +K_1(x,t)\sin K(t)+K_2(x,t)\sin K(t)),\end{aligned}$$

$$K_1(x,t)=\int_{-x}^{x-2t}M(x,s)\tilde{M}(x,s+2t)\,ds+\int_{2t-x}^{x}M(x,s)\tilde{M}(x,s+2t)\,ds$$
$$K_2(x,t)=\int_{-x}^{x-2t}N(x,s)\tilde{N}(x,s+2t)\,ds+\int_{2t-x}^{x}n(x,s)\tilde{N}(x,s+2t)\,ds$$
$$M_1(x,t)=\int_{-x}^{x-2t}M(x,s)\tilde{N}(x,s+2t)\,ds-\int_{2t-x}^{x}M(x,s)\tilde{N}(x,s+2t)\,ds$$
$$M_2(x,t)=-\int_{-x}^{x-2t}N(x,s)\tilde{M}(x,s+2t)\,ds+\int_{2t-x}^{x}N(x,s)\tilde{M}(x,s+2t)\,ds.$$

Let $\varphi(x,\lambda)$ and $\tilde{\varphi}(x,\lambda)$ be substituted into (1.1) and (1.7),

$$-\varphi''(x,\lambda)+(2\lambda p(x)+q(x))\varphi(x,\lambda)=\lambda^2\rho(x)\varphi(x,\lambda) \tag{2.5}$$

$$-\tilde{\varphi}''(x,\lambda)+(2\lambda p(x)+q(x))\tilde{\varphi}(x,\lambda)=\lambda^2\rho(x)\tilde{\varphi}(x,\lambda) \tag{2.6}$$

The following equations are obtained using (2.5) and (2.6):

$$\int_0^{\frac{\pi}{2}}\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)\left[2\lambda(p(x)-\tilde{p}(x))+(q(x)-\tilde{q}(x))\right]dx$$
$$=\left[\tilde{\varphi}'(x,\lambda)\varphi(x,\lambda)-\varphi'(x,\lambda)\tilde{\varphi}(x,\lambda)\right]_0^{\frac{\pi}{2}}+|_{\frac{\pi}{2}}^{\pi},$$

$$\int_0^{\frac{\pi}{2}}\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)\left[2\lambda(p(x)-\tilde{p}(x))+(q(x)-\tilde{q}(x))\right]dx$$
$$+\tilde{\varphi}'(\pi,\lambda)\varphi(\pi,\lambda)-\varphi'(\pi,\lambda)\tilde{\varphi}(\pi,\lambda)=0. \tag{2.7}$$

Let $Q(x)=q(x)-\tilde{q}(x)$ and $P(x)=p(x)-\tilde{p}(x)$,

$$U(\lambda)=\int_0^{\frac{\pi}{2}}\left[2\lambda P(x)+Q(x)\right]\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)\,dx.$$

It is obvious that the functions $\varphi(x,\lambda)$ and $\tilde{\varphi}(x,\lambda)$ are the solutions which satisfy boundary value conditions of (1.2) and (1.8), respectively, then if we consider these facts in equation (2.7), we obtain the following equation

$$U(\lambda_n)=0 \tag{2.8}$$

for each eigenvalue λ_n. Let us marked

$$U_1(\lambda)=\int_0^{\frac{\pi}{2}}P(x)\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)\,dx,U_2(\lambda)=\int_0^{\frac{\pi}{2}}Q(x)\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)\,dx.$$

Then equations (2.7) can be rewritten as

$$2\lambda_n U_1(\lambda_n)+U_2(\lambda_n)=0.$$

From (2.4) and (2.7) we obtain

$$|U(\lambda)| \leq (C_1 + C_2|\lambda|)\exp(\tau\pi), \tag{2.9}$$

where $C_1, C_2 > 0$ are constants for all complex λ. Since $\lambda_n = \tilde{\lambda}_n$, $\Delta(\lambda) = \varphi(\pi,\lambda) = \tilde{\varphi}(\pi,\lambda)$, thus,

$$U(\lambda) = \int_0^{\frac{\pi}{2}} [2\lambda P(x) + Q(x)]\,\varphi(x,\lambda)\,\tilde{\varphi}(x,\lambda)\,dx = \Delta(\lambda)\left[\varphi(\pi,\lambda) - \tilde{\varphi}(\pi,\lambda)\right].$$

The function $\phi(\lambda) = \frac{U(\lambda)}{\Delta(\lambda)}$ is an entire function with respect to λ.

It follows from $\Delta(\lambda) \geq (|\lambda\beta| - C)\exp(\tau\xi^+(x))$ and (2.9), $\phi(\lambda) = O(1)$ for sufficient large $|\lambda|$. We obtain $\phi(\lambda) = C$, for all λ by Liouville's Theorem.

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \varphi(x,\lambda)\,\tilde{\varphi}(x,\lambda)\,[2\lambda P(x) + Q(x)]\,dx \\
&= C\left[\left(\beta_2^+ + \frac{\gamma_2}{2\beta}\right) R_1(a_2)\cos\left[\lambda k^+(\pi) - \frac{1}{\beta}\int_{a_2}^{\pi} p(t)\,dt\right]\right. \\
&\quad + \left(\beta_2^- + \frac{\gamma_2}{2\beta}\right) R_2(a_2)\cos\left[\lambda k^-(\pi) - \frac{1}{\beta}\int_{a_2}^{\pi} p(t)\,dt\right] \\
&\quad + \left(\beta_2^- - \frac{\gamma_2}{2\beta}\right) R_1(a_2)\cos\left[\lambda s^+(\pi) + \frac{1}{\beta}\int_{a_2}^{\pi} p(t)\,dt\right] \\
&\quad \left. + \left(\beta_2^+ - \frac{\gamma_2}{2\beta}\right) R_2(a_2)\cos\left[\lambda s^-(\pi) + \frac{1}{\beta}\int_{a_2}^{\pi} p(t)\,dt\right]\right] + O\left(\exp\left(\tau k^+(\pi)\right)\right).
\end{aligned}$$

By the Riemann-Lebesgue lemma, for $\lambda \to \infty$, $\lambda \in \mathrm{R}$ we get $C = 0$. Then,

$$\begin{aligned}
2U_1(\lambda) = {} & S^+\tilde{S}^+ \int_0^{\frac{\pi}{2}} P(x)\cos\left(2\lambda\xi^+(x) - K(x)\right)dx \\
& + S^+\tilde{S}^+ \int_0^{\frac{\pi}{2}} P(x)\cos L(x)\,dx \\
& + S^+\tilde{S}^- \int_0^{\frac{\pi}{2}} P(x)\cos\left(2\lambda a_1 t - L(x)\right)dx \\
& + S^+\tilde{S}^- \int_0^{\frac{\pi}{2}} P(x)\cos\left(2\lambda\alpha(x - a_1) - K(x)\right)dx \\
& + S^-\tilde{S}^+ \int_0^{\frac{\pi}{2}} P(x)\cos\left(2\lambda a_1 + L(x)\right)dx \\
& + S^-\tilde{S}^+ \int_0^{\frac{\pi}{2}} P(x)\cos\cos\left(2\lambda\alpha(x - a_1) + K(x)\right)dx \\
& + S^-\tilde{S}^- \int_0^{\frac{\pi}{2}} P(x)\cos\left(2\lambda\xi^-(x) + L(x)\right)dx
\end{aligned}$$

$$+S^{-}\tilde{S}^{-}\int_{0}^{\frac{\pi}{2}}P(x)\cos K(x)\,dx$$
$$+\int_{0}^{\frac{\pi}{2}}P(x)\left(\int_{0}^{x}U_{c}(x,t)\cos(2\lambda t-K(t))\,dt\right)dx$$
$$-\int_{0}^{\frac{\pi}{2}}P(x)\left(\int_{0}^{x}U_{s}(x,t)\sin(2\lambda t-K(t))\,dt\right)dx,$$

where $\xi^{\pm}(x)=\pm\alpha x\mp\alpha a_1+a_1$, $k^{\pm}(x)=\mu^{+}(a_2)\pm\beta x\mp\beta a_2$,

$$s^{\pm}(x)=\mu^{-}(a_2)\pm\beta x\mp\beta a_2,\ \beta_1^{\mp}=\frac{1}{2}\left(\alpha_1\mp\frac{\beta_1}{\alpha}\right),\ \beta_2^{\mp}=\frac{1}{2}\left(\alpha_2\mp\frac{\alpha\beta_2}{\beta}\right).$$

$$\begin{aligned}
2U_1(\lambda)=&\frac{S^{+}\tilde{S}^{+}}{2}\int_{0}^{\frac{\pi}{2}}P(t)\,e^{-i(K(t))}e^{i\left(2\lambda\xi^{+}(t)\right)}dt+\frac{S^{+}\tilde{S}^{+}}{2}\int_{0}^{\frac{\pi}{2}}P(t)\,e^{i(K(t))}e^{-i\left(2\lambda\xi^{+}(t)\right)}dt\\
&+\frac{S^{+}\tilde{S}^{-}}{2}\int_{0}^{\frac{\pi}{2}}P(t)\,e^{-i(L(t))}e^{i(2\lambda a_1 t)}dt+\frac{S^{+}\tilde{S}^{-}}{2}\int_{0}^{\frac{\pi}{2}}P(t)\,e^{i(L(t))}e^{-i(2\lambda a_1 t)}dt\\
&+\frac{S^{+}\tilde{S}^{-}}{2}\int_{0}^{\frac{\pi}{2}}P(t)\,e^{-i(K(t))}e^{i(2\lambda\alpha(t-a_1))}dt\\
&+\frac{S^{+}\tilde{S}^{-}}{2}\int_{0}^{\frac{\pi}{2}}P(t)\,e^{i(K(t))}e^{-i(2\lambda\alpha(t-a_1))}dt\\
&+\frac{S^{-}\tilde{S}^{+}}{2}\int_{0}^{\frac{\pi}{2}}P(t)\,e^{i(L(t))}e^{i(2\lambda a_1 t)}dt+\frac{S^{-}\tilde{S}^{+}}{2}\int_{0}^{\frac{\pi}{2}}P(t)\,e^{-i(L(t))}e^{i(2\lambda a_1 t)}dt\\
&+\frac{S^{-}\tilde{S}^{+}}{2}\int_{0}^{\frac{\pi}{2}}P(t)\,e^{i(K(t))}e^{i(2\lambda\alpha(t-a_1))}dt+\frac{S^{-}\tilde{S}^{+}}{2}\int_{0}^{\frac{\pi}{2}}P(t)\,e^{-i(K(t))}e^{i(2\lambda\alpha(t-a_1))}dt\\
&+\frac{S^{-}\tilde{S}^{-}}{2}\int_{0}^{\frac{\pi}{2}}P(t)\,e^{i(L(t))}e^{-i\left(2\lambda\xi^{-}(t)\right)}dt+\frac{S^{-}\tilde{S}^{-}}{2}\int_{0}^{\frac{\pi}{2}}P(t)\,e^{-i(L(t))}e^{i\left(2\lambda\xi^{-}(t)\right)}dt\\
&+S^{+}\tilde{S}^{+}\int_{0}^{\frac{\pi}{2}}P(x)\cos L(x)\,dx+S^{-}\tilde{S}^{-}\int_{0}^{\frac{\pi}{2}}P(x)\cos K(x)\,dx\\
&+\int_{0}^{\frac{\pi}{2}}P(x)\left(\int_{0}^{x}U_{c}(x,t)\cos(2\lambda t-K(t))\,dt\right)dx\\
&-\int_{0}^{\frac{\pi}{2}}P(x)\left(\int_{0}^{x}U_{s}(x,t)\sin(2\lambda t-K(t))\,dt\right)dx
\end{aligned}$$

if necessary operations are performed and integrals are calculated.

$$\begin{aligned}
2U_1(\lambda)=&\frac{S^{+}\tilde{S}^{+}}{2}\left[\frac{T_1(\pi/2)}{2i\lambda\alpha}e^{i\left(2\lambda\xi^{+}\left(\frac{\pi}{2}\right)\right)}-\frac{T_1(0)}{2i\lambda\alpha}e^{2i\lambda(\alpha a_1+a_1)}-\frac{1}{2i\lambda\alpha}\int_{0}^{\frac{\pi}{2}}T_1'(t)\,e^{i\left(2\lambda\xi^{+}(t)\right)}dt\right]\\
&+\frac{S^{+}\tilde{S}^{+}}{2}\left[-\frac{T_2(\pi/2)}{2i\lambda\alpha}e^{-i\left(2\lambda\xi^{+}\left(\frac{\pi}{2}\right)\right)}+\frac{T_2(0)}{2i\lambda\alpha}e^{-2i\lambda(\alpha a_1+a_1)}+\frac{1}{2i\lambda\alpha}\int_{0}^{\frac{\pi}{2}}T_2'(t)\,e^{-i\left(2\lambda\xi^{+}(t)\right)}dt\right]\\
&+\frac{S^{+}\tilde{S}^{-}}{2}\left[\frac{T_3(\pi/2)}{2i\lambda\alpha}e^{i\lambda a_1}-\frac{T_3(0)}{2i\lambda\alpha}-\frac{1}{2i\lambda\alpha}\int_{0}^{\frac{\pi}{2}}T_3'(t)\,e^{2ia_1 t}dt\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{S^{+}\tilde{S}^{-}}{2}\left[-\frac{T_4\left(\pi/2\right)}{2i\lambda\alpha}e^{i\lambda a_1}+\frac{T_4\left(0\right)}{2i\lambda\alpha}+\frac{1}{2i\lambda\alpha}\int_0^{\frac{\pi}{2}}T_4^{'}\left(t\right)e^{-2ia_1t}dt\right]\\
&+\frac{S^{+}S^{-}}{2}\left[\frac{T_1\left(\pi/2\right)}{2i\lambda\alpha}e^{2i\lambda\alpha\left(\frac{\pi}{2}-a_1\right)}-\frac{T_1\left(0\right)}{2i\lambda\alpha}e^{-2i\lambda\alpha a_1}-\frac{1}{2i\lambda\alpha}\int_0^{\frac{\pi}{2}}T_1^{'}\left(t\right)e^{2i\lambda\alpha(t-a_1)}dt\right]\\
&+\frac{S^{+}S^{-}}{2}\left[-\frac{T_2\left(\pi/2\right)}{2i\lambda\alpha}e^{-2i\lambda\alpha\left(\frac{\pi}{2}-a_1\right)}+\frac{T_2\left(0\right)}{2i\lambda\alpha}e^{2i\lambda\alpha a_1}+\frac{1}{2i\lambda\alpha}\int_0^{\frac{\pi}{2}}T_2^{'}\left(t\right)e^{-2i\lambda\alpha(t-a_1)}dt\right]\\
&+\frac{S^{-}\tilde{S}^{+}}{2}\left[-\frac{T_3\left(\pi/2\right)}{2i\lambda\alpha}e^{-i\lambda a_1\pi}+\frac{T_3\left(0\right)}{2i\lambda\alpha}+\frac{1}{2i\lambda\alpha}\int_0^{\frac{\pi}{2}}T_3^{'}\left(t\right)e^{-2ia_1t}dt\right]\\
&+\frac{S^{-}\tilde{S}^{+}}{2}\left[\frac{T_4\left(\pi/2\right)}{2i\lambda\alpha}e^{i\lambda a_1\pi}-\frac{T_4\left(0\right)}{2i\lambda\alpha}-\frac{1}{2i\lambda\alpha}\int_0^{\frac{\pi}{2}}T_4^{'}\left(t\right)e^{2ia_1t}dt\right]\\
&+\frac{S^{-}\tilde{S}^{+}}{2}\left[-\frac{T_1\left(\pi/2\right)}{2i\lambda\alpha}e^{-2i\lambda\alpha\left(\frac{\pi}{2}-a_1\right)}+\frac{T_1\left(0\right)}{2i\lambda\alpha}e^{2i\lambda\alpha a_1}+\frac{1}{2i\lambda\alpha}\int_0^{\frac{\pi}{2}}T_1^{'}\left(t\right)e^{-2i\lambda\alpha(t-a_1)}dt\right]\\
&+\frac{S^{-}\tilde{S}^{+}}{2}\left[\frac{T_2\left(\pi/2\right)}{2i\lambda\alpha}e^{2i\lambda\alpha\left(\frac{\pi}{2}-a_1\right)}-\frac{T_2\left(0\right)}{2i\lambda\alpha}e^{-2i\lambda\alpha a_1}-\frac{1}{2i\lambda\alpha}\int_0^{\frac{\pi}{2}}T_2^{'}\left(t\right)e^{2i\lambda\alpha(t-a_1)}dt\right]\\
&+\frac{S^{-}\tilde{S}^{-}}{2}\left[-\frac{T_4\left(\pi/2\right)}{2i\lambda\alpha}e^{i\left(2\lambda\xi^{-}\left(\frac{\pi}{2}\right)\right)}+\frac{T_4\left(0\right)}{2i\lambda\alpha}e^{2i\lambda(\alpha a_1+a_1)}+\frac{1}{2i\lambda\alpha}\int_0^{\frac{\pi}{2}}T_4^{'}\left(t\right)e^{i\left(2\lambda\xi^{-}(t)\right)}dt\right]\\
&+\frac{S^{-}\tilde{S}^{-}}{2}\left[\frac{T_3\left(\pi/2\right)}{2i\lambda\alpha}e^{-i\left(2\lambda\xi^{-}\left(\frac{\pi}{2}\right)\right)}-\frac{T_3\left(0\right)}{2i\lambda\alpha}e^{2i\lambda(\alpha a_1-a_1)}-\frac{1}{2i\lambda\alpha}\int_0^{\frac{\pi}{2}}T_3^{'}\left(t\right)e^{-i\left(2\lambda\xi^{+}(t)\right)}dt\right]\\
&+S^{+}\tilde{S}^{+}\int_0^{\frac{\pi}{2}}P\left(x\right)\cos L\left(x\right)dx+S^{-}\tilde{S}^{-}\int_0^{\frac{\pi}{2}}P\left(x\right)\cos K\left(x\right)dx\\
&+\left[\frac{T_5\left(\pi/2\right)}{2i\lambda}e^{i\pi\lambda}-\frac{T_5\left(0\right)}{2i\lambda}-\frac{1}{2i\lambda}\int_0^{\frac{\pi}{2}}T_1'\left(t\right)e^{2i\lambda t}dt\right]\\
&+\left[-\frac{T_6\left(\pi/2\right)}{2i\lambda}e^{-i\pi\lambda}+\frac{T_6\left(0\right)}{2i\lambda}+\frac{1}{2i\lambda}\int_0^{\frac{\pi}{2}}T_6'\left(t\right)e^{-2i\lambda t}dt\right],
\end{aligned}
$$

where

$$T_1\left(t\right)=P\left(t\right)e^{-i(K(t))},\ \ T_2\left(t\right)=P\left(t\right)e^{i(K(t))},\ \ T_3\left(t\right)=P\left(t\right)e^{-i(L(t))},$$

$$T_4\left(t\right)=P\left(t\right)e^{i(L(t))},\ \ P_1\left(t\right)=\int_t^{\frac{\pi}{2}}P\left(x\right)U_c\left(x,t\right)dx,\ \ P_2\left(t\right)=\int_t^{\frac{\pi}{2}}P\left(x\right)U_s\left(x,t\right)dx,$$

$$T_5\left(t\right)=\frac{P_1\left(t\right)+iP_2\left(t\right)}{2}e^{-iK(t)},\ \ T_6\left(t\right)=\frac{P_1\left(t\right)-iP_2\left(t\right)}{2}e^{iK(t)}.$$

By the Riemann-Lebesgue lemma $\int_0^{\frac{\pi}{2}}P\left(x\right)\cos L\left(x\right)dx=0$, $\int_0^{\frac{\pi}{2}}P\left(x\right)\cos K\left(x\right)dx=0$ and $P\left(\frac{\pi}{2}\right)=0$ for $\lambda\to\infty$.

Thus,

$$
\begin{aligned}
2U_1(\lambda) = & -\frac{S^+\tilde{S}^+}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_1'(t)\, e^{i(2\lambda\xi^+(t))}dt + \frac{S^+\tilde{S}^+}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_2'(t)\, e^{-i(2\lambda\xi^+(t))}dt \\
& -\frac{S^+\tilde{S}^-}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_3'(t)\, e^{2ia_1t}dt + \frac{S^+\tilde{S}^-}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_4'(t)\, e^{-2ia_1t}dt \\
& -\frac{S^+S^-}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_1'(t)\, e^{2i\lambda\alpha(t-a_1)}dt + \frac{S^+S^-}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_2'(t)\, e^{-2i\lambda\alpha(t-a_1)}dt \\
& +\frac{S^-\tilde{S}^+}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_3'(t)\, e^{-2ia_1t}dt - \frac{S^-\tilde{S}^+}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_4'(t)\, e^{2ia_1t}dt \\
& +\frac{S^-\tilde{S}^+}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_1'(t)\, e^{-2i\lambda\alpha(t-a_1)}dt + \frac{S^-\tilde{S}^+}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_2'(t)\, e^{2i\lambda\alpha(t-a_1)}dt \\
& +\frac{S^-\tilde{S}^-}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_4'(t)\, e^{i(2\lambda\xi^-(t))}dt - \frac{S^-\tilde{S}^-}{4i\lambda\alpha}\int_0^{\frac{\pi}{2}} T_3'(t)\, e^{-i(2\lambda\xi^-(t))}dt \\
& +\frac{i}{2\lambda}\int_0^{\frac{\pi}{2}} T_5'(t)\, e^{2i\lambda t}dt - \frac{i}{2\lambda}\int_0^{\frac{\pi}{2}} T_6'(t)\, e^{-2i\lambda t}dt, \qquad (2.10)
\end{aligned}
$$

$$
\begin{aligned}
2U_2(\lambda) = & S^+\tilde{S}^+\int_0^{\frac{\pi}{2}} Q(x)\left(\frac{e^{i(2\lambda\xi^+(x)-K(x))}+e^{-i(2\lambda\xi^+(x)-K(x))}}{2}\right)dx \\
& +S^+\tilde{S}^-\int_0^{\frac{\pi}{2}} Q(x)\left(\frac{e^{i(2\lambda a_1t-L(x))}+e^{-i(2\lambda a_1t-L(x))}}{2}\right)dx \\
& +S^+\tilde{S}^-\int_0^{\frac{\pi}{2}} Q(x)\left(\frac{e^{i(2\lambda\alpha(x-a_1)-K(x))}+e^{-i(2\lambda\alpha(x-a_1)-K(x))}}{2}\right)dx \\
& +S^-\tilde{S}^+\int_0^{\frac{\pi}{2}} Q(x)\left(\frac{e^{i(2\lambda a_1t+L(x))}+e^{-i(2\lambda a_1t+L(x))}}{2}\right)dx \\
& +S^-\tilde{S}^+\int_0^{\frac{\pi}{2}} Q(x)\left(\frac{e^{i(2\lambda\alpha(x-a_1)+K(x))}+e^{-i(2\lambda\alpha(x-a_1)+K(x))}}{2}\right)dx \\
& +S^-\tilde{S}^-\int_0^{\frac{\pi}{2}} Q(x)\left(\frac{e^{i(2\lambda\xi^-(x)+L(x))}+e^{-i(2\lambda\xi^-(x)+L(x))}}{2}\right)dx \\
& +S^+\tilde{S}^+\int_0^{\frac{\pi}{2}} Q(x)\cos L(x)\,dx + S^-\tilde{S}^-\int_0^{\frac{\pi}{2}} Q(x)\cos K(x)\,dx \\
& +\int_0^{\frac{\pi}{2}} Q(x)\left(\int_0^x U_c(x,t)\cos(2\lambda t-K(t))\,dt\right)dx \\
& -\int_0^{\frac{\pi}{2}} Q(x)\left(\int_0^x U_s(x,t)\sin(2\lambda t-K(t))\,dt\right)dx,
\end{aligned}
$$

where

$$R_1(t)=Q(t)e^{-i(K(t))},\ \ R_2(t)=Q(t)e^{i(K(t))},\ \ R_3(t)=Q(t)e^{-i(L(t))},$$

$$R_4(t)=Q(t)e^{i(L(t))},\ \ Q_1(t)=\int_t^{\frac{\pi}{2}}P(x)U_c(x,t)\,dx,\ \ Q_2(t)=\int_t^{\frac{\pi}{2}}P(x)U_s(x,t)\,dx,$$

$$R_5(t)=\frac{Q_1(t)+iQ_2(t)}{2}e^{-iK(t)},\ \ R_6(t)=\frac{Q_1(t)-iQ_2(t)}{2}e^{iK(t)}.$$

By the Riemann-Lebesgue lemma $\int_0^{\frac{\pi}{2}}Q(x)\cos L(x)\,dx=0$, $\int_0^{\frac{\pi}{2}}Q(x)\cos K(x)\,dx=0$. Thus,

$$\begin{aligned}
2U_2(\lambda)=&\frac{S^+\tilde{S}^+}{2}\int_0^{\frac{\pi}{2}}R_1(t)e^{i(2\lambda\xi^+(t))}dt+\frac{S^+\tilde{S}^+}{2}\int_0^{\frac{\pi}{2}}R_2(t)e^{-i(2\lambda\xi^+(t))}dt\\
&+\frac{S^+\tilde{S}^-}{2}\int_0^{\frac{\pi}{2}}R_3(t)e^{2ia_1t}dt+\frac{S^+\tilde{S}^-}{2}\int_0^{\frac{\pi}{2}}R_4(t)e^{-2ia_1t}dt\\
&+\frac{S^+S^-}{2}\int_0^{\frac{\pi}{2}}R_1(t)e^{2i\lambda\alpha(t-a_1)}dt+\frac{S^+S^-}{2}\int_0^{\frac{\pi}{2}}R_2(t)e^{-2i\lambda\alpha(t-a_1)}dt\\
&+\frac{S^-\tilde{S}^+}{2}\int_0^{\frac{\pi}{2}}R_3(t)e^{-2ia_1t}dt+\frac{S^-\tilde{S}^+}{2}\int_0^{\frac{\pi}{2}}R_4(t)e^{2ia_1t}dt\\
&+\frac{S^-\tilde{S}^+}{2}\int_0^{\frac{\pi}{2}}R_1(t)e^{-2i\lambda\alpha(t-a_1)}dt+\frac{S^-\tilde{S}^+}{2}\int_0^{\frac{\pi}{2}}R_2(t)e^{2i\lambda\alpha(t-a_1)}dt\\
&++\frac{S^-\tilde{S}^-}{2}\int_0^{\frac{\pi}{2}}R_4(t)e^{i(2\lambda\xi^-(t))}dt+\frac{S^-\tilde{S}^-}{2}\int_0^{\frac{\pi}{2}}R_3(t)e^{-i(2\lambda\xi^-(t))}dt\\
&+\frac{i}{2\lambda}\int_0^{\frac{\pi}{2}}R_5(t)e^{2i\lambda t}dt+\frac{i}{2\lambda}\int_0^{\frac{\pi}{2}}R_6(t)e^{-2i\lambda t}dt
\end{aligned}\tag{2.11}$$

$$2\lambda U_1(\lambda)+U_2(\lambda)=0.\tag{2.12}$$

If (2.10) and (2.11) are substituted into (2.12), we get

$$\begin{aligned}
&\frac{S^+\tilde{S}^+}{2\alpha}\int_0^{\frac{\pi}{2}}\left(R_1(t)+iT_1'(t)\right)e^{i\left(2\lambda\xi^+(t)\right)}dt+\frac{S^+\tilde{S}^+}{2\alpha}\int_0^{\frac{\pi}{2}}\left(R_2(t)-iT_2'(t)\right)e^{-i\left(2\lambda\xi^+(t)\right)}dt\\
&+\frac{S^+\tilde{S}^-}{2\alpha}\int_0^{\frac{\pi}{2}}\left(R_3(t)+iT_3'(t)\right)e^{2ia_1t}dt+\frac{S^+\tilde{S}^-}{2\alpha}\int_0^{\frac{\pi}{2}}\left(R_4(t)-iT_4'(t)\right)e^{-2ia_1t}dt\\
&+\frac{S^+S^-}{2\alpha}\int_0^{\frac{\pi}{2}}\left(R_1(t)+iT_1'(t)\right)e^{2i\lambda\alpha(t-a_1)}dt+\frac{S^+S^-}{2\alpha}\int_0^{\frac{\pi}{2}}\left(R_2(t)-iT_2'(t)\right)e^{-2i\lambda\alpha(t-a_1)}dt\\
&+\frac{S^-\tilde{S}^+}{2\alpha}\int_0^{\frac{\pi}{2}}\left(R_4(t)+iT_4'(t)\right)e^{-2ia_1t}dt+\frac{S^-\tilde{S}^+}{2\alpha}\int_0^{\frac{\pi}{2}}\left(R_3(t)-iT_3'(t)\right)e^{2ia_1t}dt\\
&+\frac{S^-\tilde{S}^+}{2}\int_0^{\frac{\pi}{2}}\left(R_2(t)+iT_2'(t)\right)e^{2i\lambda\alpha(t-a_1)}dt+\frac{S^-\tilde{S}^+}{2}\int_0^{\frac{\pi}{2}}\left(R_1(t)-iT_1'(t)\right)e^{-2i\lambda\alpha(t-a_1)}dt\\
&+\frac{S^-\tilde{S}^-}{2}\int_0^{\frac{\pi}{2}}\left(R_4(t)-iT_4'(t)\right)e^{i\left(2\lambda\xi^-(t)\right)}dt+\frac{S^-\tilde{S}^-}{2}\int_0^{\frac{\pi}{2}}\left(R_3(t)+iT_3'(t)\right)e^{-i\left(2\lambda\xi^-(t)\right)}dt
\end{aligned}$$

$$+\int_0^{\frac{\pi}{2}}\left(R_5(t)+iT_5'(t)\right)e^{2i\lambda t}dt+\int_0^{\frac{\pi}{2}}\left(R_6(t)-iT_6'(t)\right)e^{-2i\lambda t}dt=0.$$

Since the systems $\left\{e^{\pm 2i\lambda\xi^+(t)}:\ \lambda\in\mathrm{R}\right\}$, $\left\{e^{\pm 2i\lambda a_1 t}:\ \lambda\in\mathrm{R}\right\}$, $\left\{e^{\pm 2i\lambda\alpha(t-a_1)}:\ \lambda\in\mathrm{R}\right\}$ and $\left\{e^{\pm 2i\lambda t}:\ \lambda\in\mathrm{R}\right\}$ are entire in $L_2\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, it follows

$$\begin{aligned}
&R_1(t)+iT_1'(t)=0\,,\ R_2(t)-iT_2'(t)=0\,,\ R_3(t)+iT_3'(t)=0\\
&R_4(t)-iT_4'(t)=0\,,\ R_1(t)+iT_1'(t)=0\,,\ R_2(t)-iT_2'(t)=0\\
&R_4(t)+iT_4'(t)=0\,,\ R_3(t)-iT_3'(t)=0\,,\ R_2(t)+iT_2'(t)=0\\
&R_1(t)-iT_1'(t)=0\,,\ R_4(t)-iT_4'(t)=0\,,\ R_3(t)+iT_3'(t)=0\\
&R_5(t)+iT_5'(t)=0\,,\ R_6(t)-iT_6'(t)=0.
\end{aligned}$$

Then, we get the following system:

$$\begin{aligned}
R_5(t)+iT_5'(t)&=0\\
R_6(t)-iT_6'(t)&=0
\end{aligned}$$

and hence

$$\begin{cases}\left[Q_1(t)+P_1(t)K'(t)-P_2'(t)\right]+i\left[Q_2(t)+P_2(t)K'(t)+P_1'(t)\right]=0\\ \left[Q_1(t)+P_1(t)K'(t)-P_2'(t)\right]-i\left[Q_2(t)+P_2(t)K'(t)+P_1'(t)\right]=0\end{cases}$$

and

$$\begin{cases}Q_1(t)+P_1(t)K'(t)-P_2'(t)=0\\ Q_2(t)+P_2(t)K'(t)+P_1'(t)=0\end{cases}$$

$$\begin{cases}P'(t)=U_c(t,t)P(t)\\ -\int_t^{\frac{\pi}{2}}U_s(x,t)Q(x)\,dx-\int_t^{\frac{\pi}{2}}\left(K'(t)U_s(x,t)+\frac{\partial H_s(x,t)}{\partial t}\right)P(x)\,dx\\ P(t)=-\int_t^{\frac{\pi}{2}}P'(x)\,dx\\ Q(t)=-\left(K'(t)+U_s(t,t)\right)P(t)\\ -\int_t^{\frac{\pi}{2}}U_c(x,t)Q(x)\,dx-\int_t^{\frac{\pi}{2}}\left(K'(t)U_c(x,t)-\frac{\partial H_s(x,t)}{\partial t}\right)P(x)\,dx.\end{cases}\tag{2.13}$$

If we mark this

$$S(t)=\left(Q(t),P(t),P'(t)\right)^T$$

and

$$K(x,t)=\begin{pmatrix}U_c(x,t) & K'(t)U_c(x,t)-\frac{\partial U_s(x,t)}{\partial t} & -\left(K'(t)+U_s(t,t)\right)\\ 0 & 0 & 1\\ U_s(x,t) & K'(t)U_s(x,t)+\frac{\partial U_s(x,t)}{\partial t} & U_c(x,t)\end{pmatrix}$$

Equations (2.13) can be reduced to a vector from

$$S(t)+\int_t^{\frac{\pi}{2}}K(x,t)S(x)\,dx=0\tag{2.14}$$

for $0 < t < \frac{\pi}{2}$.

Since equation (2.14) is a homogenous Volterra integral equation, it only has the trivial solution. Thus, we obtain $S(t) = 0$ for $0 < t < \frac{\pi}{2}$. This gives us $Q(t) = P(t) = 0$ for $0 < t < \frac{\pi}{2}$.

Thus, we obtain $q(x) = \tilde{q}(x)$ and $p(x) = \tilde{p}(x)$ on $(0, \pi)$. The proof is completed. □

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ON ALMOST EVERYWHERE CONVERGENCE OF THE GENERALIZED MARCIENKIWICZ MEANS WITH RESPECT TO TWO DIMENSIONAL VILENKIN-LIKE SYSTEMS

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Abstract. In this paper we investigate the almost everywhere convergence of two dimensional Marcinkiewicz-like means of two variable integrable functions which is given by

$$t_n^{\alpha} f = \frac{1}{n} \sum_{k=0}^{n-1} S_{\alpha_1(|n|,k),\alpha_2(|n|,k)} f$$

$(M_{|n|} \leq n < M_{|n|+1})$ and give a sufficient condition for functions $\alpha : N^2 \mapsto N^2$ in order to have the a.e. relation $t_n^{\alpha} f \to f$ for all $f \in L^1(G_m^2)$ with respect to two dimensional bounded Vilenkin-like systems. Finally, we give an application of the main result with respect to triangular summability of Vilenkin-like-Fourier series.

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Keywords: almost everywhere convergence, generalized Marcienkiwicz means and two dimensional Vilenkin-like system

1. Introduction

In 1939 for the two-dimensional trigonometric Fourier partial sums $S_{j,j}f$ Marcinkiewicz [14] proved that for all $f \in L\log L([0,2\pi]^2)$ the a.e. relation

$$\frac{1}{n} \sum_{j=1}^{n} S_{j,j} f \to f \tag{1.1}$$

holds as $n \to \infty$. Zhizhiashvili [18] improved this result for $f \in L([0,2\pi]^2)$. Dyachenko [3] proved this result for dimensions greater than 2. In 2003 Goginava [10] proved this result with respect to the d-dimensional Walsh-Paley system. The case $d = 2$ is due to Weisz [17]. In 2012 Gát [7] proved this result for generalized Marcinkiewicz means with respect to the two dimensional Walsh system and in 2016 [8] for bounded two-dimensional Vilenkin systems. The aim of this paper is to generalize this result of Gát with respect to two-dimensional generalized Vilenkin-like systems.

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Besides, we give an application of the main result of this paper, that is, Theorem 3 with respect to triangular summability of Vilenkin-Fourier series.

Denote by $\mathbb{N}$ the set of natural numbers and $\mathbb{P}$ the set of positive integers. Denote $m := (m_k : k \in \mathbb{N})$ a sequence of positive integers such that $m_k \geq 2$, $k \in \mathbb{N}$ and G_{m_k} a set of cardinality m_k. Suppose that each (coordinate) set has the discrete topology and measure μ_k which maps every singleton of G_{m_k} to $\frac{1}{m_k}$ ($\mu_k(G_{m_k}) = 1$), $k \in \mathbb{N}$. Let G_m be the compact set formed by the complete direct product of G_{m_k} with the product of the topologies and measures (μ). Thus, each $x \in G_m$ is a sequence $x := (x_0, x_1, \ldots)$, where $x_k \in G_{m_k}$, $k \in \mathbb{N}$. G_m is called a Vilenkin space. G_m is a compact totally disconnected space, with normalized regular Borel measure μ, $\mu(G_m) = 1$. The Vilenkin space G_m is said to be bounded if the generating system m is a bounded one. *Throughout this paper the boundedness of G_m is supposed.* In this paper C, C_p denote absolute constants (the latter may depend on p), which may vary from line to line.

A base for the neighborhoods of G_m can be given as follows

$$I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbb{N}) \in G_m : y_i = x_i \,\text{for}\, i < n\}$$

for $x \in G_m, n \in \mathbb{P}$.

$$I := \{I_n(x) : n \in \mathbb{N}, x \in G_m\}$$

is the set of intervals on G_m.

Denote by $L^p(G_m)$ the usual Lebesgue spaces ($\|.\|_p$ the corresponding norms) ($1 \leq p \leq \infty$), $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x)$ ($x \in G_m$) and E_n the conditional expectation operator with respect to $\mathcal{A}_n$ ($n \in \mathbb{N}$).

Let $L^0(G_m)$ be the space of measurable functions on the Vilenkin space G_m. We say that an operator $T : L^1(G_m) \to L^0(G_m)$ is of type (L^p, L^p) (for $1 \leq p \leq \infty$) if $\|Tf\|_p \leq C_p \|f\|_p$ for all $f \in L^p(G_m)$, where the constant C_p depends only on p. We say that T is of weak type (L^1, L^1) if $\mu(|Tf| > \lambda) \leq C\|f\|_1/\lambda$ for all $f \in L^1(G_m)$ and $\lambda > 0$.

Let $M_0 := 1$ and $M_{k+1} := m_k M_k$, for $k \in \mathbb{N}$ be the so-called generalized powers. Then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, $0 \leq n_k < m_k$, $n_k \in \mathbb{N}$. This allows one to say that the sequence $(n_0, n_1, \ldots)$ is the expansion of n with respect to m. We often use the following notations. Let $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$ (that is, $M_{|n|} \leq n < M_{|n|+1}$) and $n^{(k)} = \sum_{j=k}^{\infty} n_j M_j$. Next we introduce on G_m an orthonormal system called a Vilenkin-like system.

For $k \in \mathbb{N}$ and $x \in G_m$ denote by r_k the k-th generalized Rademacher function:

$$r_k(x) := \exp(2\pi\imath \frac{x_k}{m_k}) \quad (x \in G_m,\ \imath := \sqrt{-1},\ k \in \mathbb{N}).$$

We introduce a so-called Vilenkin-like (or $\psi\alpha$) system (see [5], [2]). Let functions

$$\psi_n, \alpha_n, \alpha_j^k : G_m \to \mathbb{C}\ (n, j, k \in \mathbb{N})$$

satisfy :

$$\alpha_k^j \text{ is measurable with respect to } \mathcal{A}_k \ (j,k \in \mathbb{N}), \tag{1.2}$$

$$|\alpha_k^j| = \alpha_k^j(0) = \alpha_0^j = \alpha_k^0 = 1 \ (j,k \in \mathbb{N}), \tag{1.3}$$

$$\alpha_n := \prod_{k=0}^{\infty} \alpha_k^{n^{(k)}}, \psi_n := \prod_{k=0}^{\infty} r_k^{n_k}, \ n^{(k)} := \sum_{i=k}^{\infty} n_i M_i \ (n \in \mathbb{N}). \tag{1.4}$$

Let $\chi_n := \psi_n \alpha_n \, (n \in \mathbb{N})$. The system $\chi := \{\chi_n : n \in \mathbb{N}\}$ is called a Vilenkin-like (or $\psi\alpha$) system (see [5], [2]). We also introduce the two-variable functions:

$$\chi_n(y,x) := \chi_n(y)\bar{\chi}_n(x), \quad r_n(y,x) := r_n(y)\bar{r}_n(x) \quad (n \in \mathbb{N}, y,x \in G_m).$$

This will not cause misunderstand by clearly making a difference between $\chi_n(x)$ and $\chi_n(y,x)$. The following lemma is an easy consequence of properties (1.2), (1.3), (1.4). We also need the fact that the generalized Rademacher functions form an orthonormal system (see e.g. [1,5]).

Lemma 1 ([5]). *Let $t,n,l \in \mathbb{N}, u \in G_m$. Then we have that*

$$\int_{I_{t+1}(u)} \chi_n(x)\bar{\chi}_l(x)d\mu(x) \neq 0$$

implies $n^{(t+1)} = l^{(t+1)}$.

Example A, the Vilenkin and the Walsh system

Let $G_{m_k} := Z_{m_k}$ be the m_k-th ($2 \leq m_k \in \mathbb{N}$) discrete cyclic group ($k \in \mathbb{N}$). That is Z_{m_k} can be represented by the set $\{0,1,...,m_k-1\}$, where the group operation is the mod m_k addition and every subset is open. The group operation on G_m $(+)$ is the coordinate-wise addition. G_m is called a Vilenkin group. The Vilenkin group for which $m_k = 2$ for all $k \in \mathbb{N}$ is the Walsh-Paley group. In this case let $\alpha_k^j(x) := 1$, where $j,k \in \mathbb{N}, x \in G_m$. The system $\chi := (\chi_n : n \in \mathbb{N})$ is the Vilenkin system, where $\chi_n := \prod_{k=0}^{\infty} r_k^{n_k} \alpha_k^{n^{(k)}} = \prod_{k=0}^{\infty} r_k^{n_k}$. In the case of the Vilenkin group, $m_k = 2$ for all $k \in \mathbb{N}$, we get the Walsh-Paley system. Properties (1.2), (1.3), (1.4) are trivial fulfilled. For more on Vilenkin and Walsh system and group see e.g. [1] and [6]. For this special case the result of this paper can be found in [8].

Example B, the group of 2-adic (m-adic) integers

Let $G_{m_k} := \{0,1,...,m_k-1\}$ for all $k \in \mathbb{N}$. Define on G_m the following (commutative) addition: Let $x,y \in G_m$. Then $x+y = z \in G_m$ is defined in a recursive way. $x_0 + y_0 = t_0 m_0 + z_0$, where (of course) $z_0 \in \{0,1,...,m_0-1\}$ and $t_0 \in \mathbb{N}$. Suppose that $z_0,...,z_k$ and $t_0,...,t_k$ have been defined. Then write $x_{k+1} + y_{k+1} + t_k = t_{k+1}m_{k+1} + z_{k+1}$, where $z_{k+1} \in \{0,1,...,m_{k+1}-1\}$ and $t_{k+1} \in \mathbb{N}$. Then G_m is called

the group of m-adic integers (if $m_k = 2$ for all $k \in \mathbb{N}$, then 2-adic integers). In this case let

$$\alpha_k^j(x) := \left(\exp\left(2\pi\imath\left(\frac{x_{k-1}}{m_k m_{k-1}} + \cdots + \frac{x_0}{m_k m_{k-1} \ldots m_0}\right)\right)\right)^j.$$

Let $\chi_n := \prod_{k=0}^{\infty} r_k^{n_k} \alpha_k^{n^{(k)}}$. Then the system $\chi := (\chi_n : n \in \mathbb{N})$ is the character system of the group of m-adic (if $m_k = 2$ for each $k \in \mathbb{N}$ then 2-adic) integers. Conditions (1.2), (1.3), (1.4) are trivially fulfilled. For more on the group of m-adic (if $m_k = 2$ for each $k \in \mathbb{N}$ then 2-adic) integers see e.g. [2] or [13]. For the case when $m_k = 2\,(k \in \mathbb{N})$ the a.e. convergence of the ordinary Marcinkiewicz means were discussed by Blahota and Gát in [2]. That is, the results of this paper are new on the two-dimensional group of m-adic integers. Not only with respect to the general case $\alpha : \mathbb{N}^2 \to \mathbb{N}^2$ but also for $\alpha_1(n) = \alpha_2(n) = n$. Besides, the same can be said in the situation of the Example C below.

Example C, a system in the field of number theory

Let

$$\chi_n(x) := \exp\left(2\pi\imath \sum_{j=0}^{\infty} \frac{n_j}{M_{j+1}} \sum_{i=0}^{\infty} x_i M_i\right)$$

for $n \in \mathbb{N}$ and $x \in G_m$. Then

$$\chi_n(x) = \exp\left(2\pi\imath\left(\sum_{k=0}^{\infty} \frac{n_k x_k}{m_k} + \sum_{k=0}^{\infty} \frac{n_k}{M_{k+1}} \sum_{i=0}^{k-1} x_i M_i\right)\right) = \psi_n(x)\alpha_n(x),$$

where $\alpha_k^{n^{(k)}}(x) = \exp\left(2\pi\imath \frac{n_k}{M_{k+1}} \sum_{i=0}^{k-1} x_i M_i\right)$. Then, $\chi := (\chi_n : n \in \mathbb{N})$ is a Vilenkin-like system (introduced in [5]) which is a useful tool in the approximation theory of limit periodic, almost even arithmetical functions [5] and [4]. Again, properties (1.2), (1.3), (1.4) are trivially fulfilled. This system (on Vilenkin groups) was a new tool in order to investigate limit periodic arithmetical functions. For the definition of these arithmetical functions see also the book of Mauclaire [15].

Define the Fourier coefficients, the partial sums of the Fourier series and the Dirichlet kernels with respect to the Vilenkin-like system χ as follows

$$\hat{f}(n) := \int_{G_m} f\bar{\chi}_n d\mu, \quad S_n f := \sum_{k=0}^{n-1} \hat{f}(k)\chi_k,$$

$$D_n(y,x) := \sum_{k=0}^{n-1} \chi_k(y)\bar{\chi}_k(x) = \sum_{k=0}^{n-1} \chi_k(y,x) \quad (n \in \mathbb{N}, \quad y,x \in G_m, f \in L^1(G_m)).$$

It is well-known that

$$S_n f(y) = \int_{G_m} f(x) D_n(y,x) d\mu(x), \quad (n \in \mathbb{N}, y \in G_m, f \in L^1(G_m)).$$

It is also well-known [5] that

$$
\begin{aligned}
&D_{M_n}(y,x)=\begin{cases} M_n, & \text{if } y\in I_n(x)\\ 0, & \text{if } y\notin I_n(x)\end{cases}\\
&D_n(y,x)=\chi_n(y)\bar{\chi}_n(x)\sum_{j=0}^{\infty}D_{M_j}(y,x)\sum_{p=m_j-n_j}^{m_j-1}r_j^p(y)\bar{r}_j^p(x),\\
&S_{M_n}f(x)=M_n\int_{I_n(x)}fd\mu=E_nf(x)\quad (f\in L^1(G_m),\, n\in\mathbb{N}),\\
&D_n(y,x)=\chi_n(y)\bar{\chi}_n(x)\left(\sum_{j=0}^{t-1}n_jM_j+M_t\sum_{i=m_t-n_t}^{m_t-1}r_t^i(y)\bar{r}_t^i(x)\right),\\
&y\in I_t(x)\setminus I_{t+1}(x),\, t\in\mathbb{N}.
\end{aligned}
\tag{1.5}
$$

Next, we introduce some notation with respect to the theory of two-dimensional Vilenkin-like systems. Let $\tilde{m}$ be a sequence like m. The relation between the sequence $(\tilde{m}_n)$ and $(\tilde{M}_n)$ is the same as between sequence (m_n) and (M_n). The group $G_m\times G_{\tilde{m}}$ is called a two-dimensional Vilenkin group. The normalized Haar measure is denoted by μ, just as in the one-dimensional case. It will not cause any misunderstood. In this paper we also suppose that $m=\tilde{m}$. The two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, the Dirichlet kernels, the Marcinkiewicz means, and the Marcinkiewicz kernels with respect to the two-dimensional Vilenkin-like system are defined as follows:

$$
\begin{aligned}
&\hat{f}(n_1,n_2):=\int_{G_m\times G_{\tilde{m}}}f(x^1,x^2)\bar{\chi}_{n_1}(x^1)\bar{\chi}_{n_2}(x^2)d\mu(x^1,x^2),\\
&S_{n_1,n_2}f(y^1,y^2):=\sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}\hat{f}(k_1,k_2)\chi_{k_1}(y^1)\chi_{k_2}(y^2),\\
&D_{n_1,n_2}(y,x)=D_{n_1}(y^1,x^1)D_{n_2}(y^2,x^2)\\
&:=\sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}\chi_{k_1}(y^1)\chi_{k_2}(y^2)\bar{\chi}_{k_1}(x^1)\bar{\chi}_{k_2}(x^2),\\
&\sigma_nf:=\frac{1}{n}\sum_{j=0}^{n-1}S_{j,j}f,\\
&K_n(y,x):=\frac{1}{n}\sum_{j=0}^{n-1}D_{j,j}(y,x),\\
&(y=(y^1,y^2),\, x=(x^1,x^2)\in G_m\times G_m,\, n\in\mathbb{N}).
\end{aligned}
$$

It is also well-known that

$$\sigma_n f(y) = \int_{G_m \times G_m} f(x) K_n(y,x) d\mu(x) =: f * K_n(y).$$

2. Result

Now, turn our attention to the generalization of Marcienkiewicz means with respect to two dimensional bounded Vilenkin-like systems. The two-dimensional generalized Marcinkiewicz kernels and Marcinkiewicz means, with respect to the two-dimensional Vilenkin-like system are defined as follows: Let $\alpha = (\alpha_1, \alpha_2) : \mathbb{N}^2 \to \mathbb{N}^2$ be a function. (From now functions α_1, α_2 play the role of indices. We know that in the first part of the article the function α_n appeared in the definition of the Vilenkin-like (or $\psi\alpha$ system), but this will not cause any misunderstanding.) Define the following generalized Marcinkiewicz kernels and means respectively:

$$M_n^\alpha(y,x) := \frac{1}{n} \sum_{k=0}^{n-1} D_{\alpha_1(|n|,k)}(y^1,x^1) D_{\alpha_2(|n|,k)}(y^2,x^2),$$
$$t_n^\alpha f := f * M_n^\alpha \quad (f \in L^1(G_m^2), n \in \mathbb{P}).$$

This concept of Marcinkiewicz-like kernels and means is due to Gát [7].
The main aim of this paper is to give a class of functions α for we have the a.e. convergence relation $t_n^\alpha f \to f$ for each integrable two variable function with respect to two dimensional bounded Vilenkin-like systems. To investigate this the following properties play a prominent role (x0 $Car(B)$ denotes cardinality of the set B),

$$Car\{l \in \mathbb{N} : \alpha_j(|n|,l) = \alpha_j(|n|,k), l < n\} \le C \quad (k < n, n \in \mathbb{P}, j = 1,2), \tag{2.1}$$

$$\max\{\alpha_j(|n|,k) : k < n\} \le Cn \quad (n \in \mathbb{P}, j = 1,2). \tag{2.2}$$

We emphasize that the function α has the properties (2.1) and (2.2) everywhere in this paper. Our first aim is to prove that the operator $t_*^\alpha := \sup_{n\in\mathbb{P}} \mid t_n^\alpha \mid$ is of weak type (L^1, L^1). In order to do this we need a sequence of lemmas. The second lemma is the base of the proof of Theorems 1 and 2. The Walsh-Paley version of Theorems 1 and 2 are due to Gát [7]. That is, we generalize a result of the first author. Moreover, techniques of papers [7] and [8] will also be used in the proof of the forthcoming lemmas.

Denote for $k \in \mathbb{N}, x \in G_m$, $J_k(x) = I_k(x) \setminus I_{k+1}(x)$ and recall also that

$$n^{(s)} = \sum_{k=s}^{\infty} n_k M_k, \quad n^{(0)} = n, \quad n^{(|n|+1)} = 0 \quad (n, s \in \mathbb{N}).$$

Besides, for $x, y \in G_m^2, A, n, s, j, k \in \mathbb{N}$ let

$$\Phi(A, n^{(s+1)} + jM_s + k, y, x) = D_{\alpha_1(A, n^{(s+1)} + jM_s + k)}(y^1, x^1) D_{\alpha_2(A, n^{(s+1)} + jM_s + k)}(y^2, x^2).$$

Lemma 2. *Let $t^1, t^2, A, s \in \mathbb{N}, s \le A$ and $y \in G_m^2$. Then,*

$$\int_{J_{t^1}(y^1)\times J_{t^2}(y^2)} \sup_{|n|=A} \Big| \sum_{j=0}^{n_s-1} \sum_{k=0}^{M_s-1} \Phi(A, n^{(s+1)} + jM_s + k, y, x) \Big| d\mu(x) \le C(M_A M_{t^1})^{\frac{1}{2}}.$$

Proof. For fixed $t = (t^1, t^2)$, s, A we discuss the integral

$$\int_{J_{t^1}(y^1)\times J_{t^2}(y^2)} \sup_{|n|=A} \Big| \sum_{j=0}^{n_s-1} \sum_{k=0}^{M_s-1} \Phi(A, n^{(s+1)} + jM_s + k, y, x) \Big| d\mu(x).$$

Check the function $\sum_{j=0}^{n_s-1} \sum_{k=0}^{M_s-1} \Phi(A, n^{(s+1)} + jM_s + k, y, x)$ on the set $J_{t^1}(y^1) \times J_{t^2}(y^2)$. Since we have $x^2 \in J_{t^2}(y^2)$, then by (1.5) we have $|D_j(y^2, x^2)| \le CM_{t^2}$ for each $j \in \mathbb{N}$ (this inequality is due to the fact that the Vilenkin space is bounded) and consequently by (1.5) $|D_{\alpha_2(A, n^{(s+1)}+jM_s+k)}(y^2, x^2)| \le CM_{t^2}$.
On the other hand, again by (1.5) for $x^1 \in J_{t^1}(y^1)$ we have

$$\begin{aligned} &D_{\alpha_1(A, n^{(s+1)}+jM_s+k)}(y^1, x^1) = \chi_{[\alpha_1(A, n^{(s+1)}+jM_s+k)]^{(t^1)}}(y^1) \\ &\bar{\chi}_{[\alpha_1(A, n^{(s+1)}+jM_s+k)]^{(t^1)}}(x^1) \times \Bigg[\sum_{w=0}^{t^1-1} [\alpha_1(A, n^{(s+1)} + jM_s + k)]_w M_w \\ &+ \sum_{i=m_{t^1}-(\alpha_1(A, n^{(s+1)}+jM_s+k))_{t^1}}^{m_{t^1}-1} r_{t^1}^i(y^1) \bar{r}_{t^1}^i(x^1) M_{t^1} \Bigg]. \end{aligned}$$

The function

$$\sum_{w=0}^{t^1-1} [\alpha_1(A, n^{(s+1)} + jM_s + k)]_w M_w + \sum_{i=m_{t^1}-(\alpha_1(A, n^{(s+1)}+jM_s+k))_{t^1}}^{m_{t^1}-1} r_{t^1}^i(y^1) \bar{r}_{t^1}^i(x^1) M_{t^1}$$

depends only on $x_{t^1}^1$ as x varies (and not on the other coordinates of x^1), so its absolute value is bounded by CM_{t^1}. Besides,

$$\begin{aligned} &\Bigg| \sum_{j=0}^{n_s-1} \sum_{k=0}^{M_s-1} D_{\alpha_1(A, n^{(s+1)}+jM_s+k)}(y^1, x^1) D_{\alpha_2(A, n^{(s+1)}+jM_s+k)}(y^2, x^2) \Bigg|^2 \\ &= \sum_{j,h=0}^{n_s-1} \sum_{k,l=0}^{M_s-1} D_{\alpha_1(A, n^{(s+1)}+jM_s+k)}(y^1, x^1) \overline{D}_{\alpha_1(A, n^{(s+1)}+hM_s+l)}(y^1, x^1) \\ &\times D_{\alpha_2(A, n^{(s+1)}+jM_s+k)}(y^2, x^2) \overline{D}_{\alpha_2(A, n^{(s+1)}+hM_s+l)}(y^2, x^2). \end{aligned}$$

Therefore, apply the Cauchy-Bunyakovsky-Schwarz inequality:

$$\int_{J_{t^2}(y^2)} \left[\int_{J_{t^1}(y^1)} \sup_{|n|=A} \left| \sum_{j=0}^{n_s-1} \sum_{k=0}^{M_s-1} \Phi(A, n^{(s+1)} + jM_s + k, y^1, y^2, x^1, x^2) \right| d\mu(x^1) \right] d\mu(x^2)$$

$$\leq \int_{J_{t^2}(y^2)} M_{t^1}^{\frac{-1}{2}} \left[\int_{J_{t_1}(y^1)} \sup_{|n|=A} \left| \sum_{j=0}^{n_s-1} \sum_{k=0}^{M_s-1} \Phi(A, n^{(s+1)} + jM_s \right.\right.$$

$$\left.\left. + k, y^1, y^2, x^1, x^2) \right|^2 d\mu(x^1) \right]^{\frac{1}{2}} d\mu(x^2)$$

$$\leq \int\limits_{J_{t^2}(y^2)} M_{t^1}^{\frac{-1}{2}} \left[\int\limits_{J_{t_1}(y^1)} \sup_{|n|=A} \sum_{j,h=0}^{n_s-1} \sum_{k,l=0}^{M_s-1} D_{\alpha_1(A,n^{(s+1)}+jM_s+k)}(y^1,x^1) \overline{D}_{\alpha_1(A,n^{(s+1)}+hM_s+l)}(y^1,x^1) \right.$$

$$\left. \times D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^2,x^2) \overline{D}_{\alpha_2(A,n^{(s+1)}+hM_s+l)}(y^2,x^2) d\mu(x^1) \right] d\mu(x^2)$$

$$= \int_{J_{t^2}(y^2)} M_{t^1}^{\frac{-1}{2}} \left[\int_{J_{t^1}(y^1)} \sup_{|n|=A} \sum_{j,h=0}^{n_s-1} \sum_{k,l=0}^{M_s-1} \chi_{\left[\alpha_1(A,n^{(s+1)}+jM_s+k)\right]^{(t^1)}}(y^1,x^1) \right.$$

$$\overline{\chi}_{\left[\alpha_1(A,n^{(s+1)}+hM_s+l)\right]^{(t^1)}}(y^1,x^1)$$

$$\times \left(\sum_{w=0}^{t^1-1} \alpha_1(A, n^{(s+1)} + jM_s + k)_w M_w + \sum_{a=m_{t^1}-[\alpha_1(A,n^{(s+1)})+jM_s+k]_{t^1}}^{m_{t^1}-1} r_{t^1}^a(y^1,x^1) \right)$$

$$\times \left(\sum_{v=0}^{t^1-1} \alpha_1(A, n^{(s+1)} + hM_s + l)_v M_v + \sum_{b=m_{t^1}-[\alpha_1(A,n^{(s+1)})+hM_s+l]_{t^1}}^{m_{t^1}-1} \overline{r_{t^1}^b}(y^1,x^1) \right)$$

$$\left. D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(x^2) \overline{D}_{\alpha_2(A,n^{(s+1)}+hM_s+l)}(y^2,x^2) d\mu(x^1) \right]^{\frac{1}{2}} d\mu(x^2) =: B^1.$$

Since $n^{(s+1)}$ depends only on natural numbers n_{s+1}, ..., n_{A-1}, n_A, then the supreme $\sup_{n\in\mathbb{N}:|n|=A}$ above also depends only on n_s, n_{s+1}, ..., n_{A-1}, n_A. Thus, by $|n| = A$

$$B^1 \leq \int_{J_{t^2}(y^2)} M_{t^1}^{-\frac{1}{2}} \left[\sum_{n_A=1}^{m_A-1} \sum_{n_{A-1}=0}^{m_{A-1}-1} \cdots \sum_{n_{s+1}=0}^{m_{s+1}-1} \sum_{n_s=0}^{m_s-1} \int_{J_{t^1}(y^1)} \sum_{j,h=0}^{n_s-1} \sum_{k,l=0}^{M_s-1} \right.$$

$$\chi_{\left[\alpha_1(A,n^{(s+1)}+jM_s+k)\right]^{(t^1)}}(y^1,x^1) \overline{\chi}_{\left[\alpha_1(A,n^{(s+1)}+hM_s+l)\right]^{(t^1)}}(y^1,x^1)$$

$$\times \left(\sum_{w=0}^{t^1-1} \alpha_1(A, n^{(s+1)} + jM_s + k)_w M_w + \sum_{a=m_{t^1}-[\alpha_1(A,n^{(s+1)})+jM_s+k]_{t^1}}^{m_{t^1}-1} r_{t^1}^a(y^1,x^1) \right)$$

$$\times\left(\sum_{v=0}^{t^1-1}\alpha_1(A,n^{(s+1)}+hM_s+l)_vM_v+\sum_{b=m_{t^1}-[\alpha_1(A,n^{(s+1)})+hM_s+l]_{t^1}}^{m_{t^1}-1}\overline{r_{t^1}^b}(y^1,x^1)\right)$$

$$\times D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^1,x^1)\overline{D}_{\alpha_2(A,n^{(s+1)}+hM_s+l)}(y^2,x^2)d\mu(x^1)\Bigg]^{\frac{1}{2}}d\mu(x^2)$$

$$=\int_{J_{t^2}(y^2)}M_{t^1}^{-\frac{1}{2}}\Bigg[\sum_{n_A=1}^{m_A-1}\sum_{n_{A-1}=0}^{m_{A-1}}\cdots\sum_{n_{s+1}=0}^{m_{s+1}-1}\sum_{n_s=0}^{m_s-1}\sum_{j,h=0}^{n_s-1}\sum_{k,l=0}^{M_s-1}$$

$$\int_{J_{t^1}(y^1)}\left(\sum_{w=0}^{t^1-1}\alpha_1(A,n^{(s+1)}+jM_s+k)_wM_w+\sum_{a=m_{t^1}-[\alpha_1(A,n^{(s+1)})+jM_s+k]_{t^1}}^{m_{t^1}-1}r_{t_1}^a(y^1,x^1)\right)$$

$$\times\left(\sum_{v=0}^{t^1-1}\alpha_1(A,n^{(s+1)}+hM_s+l)_vM_v+\sum_{b=m_{t^1}-[\alpha_1(A,n^{(s+1)})+hM_s+l]_{t^1}}^{m_{t^1}-1}\overline{r_{t^1}^b}(y^1,x^1)\right)$$

$$\times\chi_{\left[\alpha_1(A,n^{(s+1)}+jM_s+k)\right]^{(t^1)}}(y^1,x^1)\overline{\chi}_{\left[\alpha_1(A,n^{(s+1)}+hM_s+l)\right]^{(t^1)}}(y^1,x^1)d\mu(x^1)$$

$$\times D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^2,x^2)\overline{D}_{\alpha_2(A,n^{(s+1)}+hM_s+l)}(y^2,x^2)\Bigg]^{\frac{1}{2}}d\mu(x^2)$$

$$=\int_{J_{t^2}(y^2)}M_{t^1}^{-\frac{1}{2}}\Bigg[\sum_{n_A=1}^{m_A-1}\sum_{n_{A-1}=0}^{m_{A-1}}\cdots\sum_{n_{s+1}=0}^{m_{s+1}-1}\sum_{n_s=0}^{m_s-1}\sum_{j,h=0}^{n_s-1}\sum_{k,l=0}^{M_s-1}$$

$$\Bigg(\sum_{w=0}^{t^1-1}\sum_{v=0}^{t^1-1}\alpha_1(A,n^{(s+1)}+jM_s+k)_wM_w\alpha_1(A,n^{(s+1)}+hM_s+l)_vM_v$$

$$\times\int_{J_{t^1}(y^1)}\chi_{\left[\alpha_1(A,n^{(s+1)}+jM_s+k)\right]^{(t^1)}}(y^1,x^1)\overline{\chi}_{\left[\alpha_1(A,n^{(s+1)}+hM_s+l)\right]^{(t^1)}}(y^1,x^1)d\mu(x^1)$$

$$+\sum_{w=0}^{t^1-1}\sum_{b=m_{t^1}-[\alpha_1(A,n^{(s+1)})+hM_s+l]_{t^1}}^{m_{t^1}-1}\alpha_1(A,n^{(s+1)}+jM_s+k)_wM_w$$

$$\times\int_{J_{t^1}(y^1)}\chi_{\left[\alpha_1(A,n^{(s+1)}+jM_s+k)\right]^{(t^1)}}(x^1)\overline{\chi}_{\left[\alpha_1(A,n^{(s+1)}+hM_s+l)\right]^{(t^1)}}(x^1)\overline{r_{t^1}^b}(y^1,x^1)d\mu(x^1)$$

$$+\sum_{a=m_{t^1}-[\alpha_1(A,n^{(s+1)})+jM_s+k]_{t^1}}^{m_{t^1}-1}\sum_{v=0}^{t^1-1}\alpha_1(A,n^{(s+1)}+hM_s+l)_vM_v$$

$$\times\int_{J_{t^1}(y^1)}\chi_{\left[\alpha_1(A,n^{(s+1)}+jM_s+k)\right]^{(t^1)}}(y^1,x^1)\overline{\chi}_{\left[\alpha_1(A,n^{(s+1)}+hM_s+l)\right]^{(t^1)}}(x^1)r_{t_1}^a(y^1,x^1)d\mu(x^1)$$

$$+\sum_{a=m_{t^1}-[\alpha_1(A,n^{(s+1)})+jM_s+k]_{t^1}}^{m_{t^1}-1}\ \sum_{b=m_{t^1}-[\alpha_1(A,n^{(s+1)})+hM_s+l]_{t^1}}^{m_{t^1}-1}$$

$$\int_{J_{t^1}(y^1)}\chi_{[\alpha_1(A,n^{(s+1)}+jM_s+k)]^{(t^1)}}(y^1,x^1)\overline{\chi}_{[\alpha_1(A,n^{(s+1)}+hM_s+l)]^{(t^1)}}(y^1,x^1)$$

$$r^a_{t_1}(y^1,x^1)\overline{r^b_{t^1}}(y^1,x^1)d\mu(x^1)\Bigg)\times D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^2,x^2)$$

$$\overline{D}_{\alpha_2(A,n^{(s+1)}+hM_s+l)}(y^2,x^2)\Bigg]^{\frac{1}{2}}d\mu(x^2)=:\sum_{q=1}^{4}B^{2,q}=:B^2.$$

Before we estimate B^2, (that is, any of $B^{2,q}$) let us determine the possible number of k, $l's$ such that the integral

$$\int_{J_{t^1}(y^1)}\chi_{[\alpha_1(A,n^{(s+1)}+jM_s+k)]^{(t^1)}}(y^1,x^1)\overline{\chi}_{[\alpha_1(A,n^{(s+1)}+hM_s+l)]^{(t^1)}}(y^1,x^1)$$
$$\times r^a_{t_1}(y^1,x^1)\overline{r^b_{t^1}}(y^1,x^1)d\mu(x^1)\neq 0.$$

(Recall that in the case of $B^{2,1}$ we have $a=b=0$, in the case of $B^{2,2}$ we have $a=0$ and in the case of $B^{2,3}$ we have $b=0$.) Suppose the integral,

$$\int_{J_{t^1}(y^1)}\chi_{[\alpha_1(A,n^{(s+1)}+jM_s+k)]^{(t^1)}}(y^1,x^1)\overline{\chi}_{[\alpha_1(A,n^{(s+1)}+hM_s+l)]^{(t^1)}}(y^1,x^1)$$
$$\times r^a_{t_1}(y^1,x^1)\overline{r^b_{t^1}}(y^1,x^1)d\mu(x^1)\neq 0$$

for some $a,b\in\{0,1,\ldots,m_{t_1}-1\}$. Then use Lemma 1 and (1.2)-(1.4). Consequently, the $(t^1+1)^{th}$, $(t^1+2)^{th}$, ..., coordinates of $\alpha_1(A,n^{(s+1)}+jM_s+k)$ and $\alpha_1(A,n^{(s+1)}+hM_s+l)$ should be equal. The absolute value of this integral is not greater than $1/M_{t^1}$. Since by (2.1) we have that for every k, there exist a bounded number of $l's$ for which $\alpha_1(A,n^{(s+1)}+jM_s+k)=\alpha_1(A,n^{(s+1)}+hM_s+l)$. This gives that for every k, there exist at most CM_{t^1} number of $l's$ for which this integral is not zero. Consequently, by

$$\sum_{w=0}^{t^1-1}[\alpha_1(A,n^{(s+1)}+jM_s+k)]_wM_w\leq CM_{t^1},\quad |D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^2,x^2)|\leq CM_{t^2}$$

we have

$$B^{2,1}\leq C\int_{J_{t^2}(y^2)}M_{t^1}^{-\frac{1}{2}}\left[\sum_{n_A=1}^{m_A-1}\sum_{n_{A-1}=0}^{m_{A-1}}\cdots\sum_{n_{s+1}=0}^{m_{s+1}-1}\sum_{n_s=0}^{m_s-1}\sum_{j,h=0}^{n_s-1}\sum_{k=0}^{M_s-1}M_{t^1}M_{t^1}^2\frac{1}{M_{t^1}}M_{t^2}^2\right]^{\frac{1}{2}}d\mu(x^2).$$

(Keep in mind that the Vilenkin space is bounded. That is, $m_j\leq C$ for all $j\in\mathbb{N}$.) By the very same steps we get the identical upper bound also for $B^{2,2},B^{2,3}$ and $B^{2,4}$. Then, also for their sum. That is,

$$B^2 \leq C \int_{J_{t^2}(y^2)} M_{t^1}^{-\frac{1}{2}} \left[\sum_{n_A=1}^{m_A-1} \sum_{n_{A-1}=0}^{m_{A-1}-1} \cdots \sum_{n_{s+1}=0}^{m_{s+1}-1} \sum_{n_s=0}^{m_s-1} ((M_{t^1}M_{t^2})^2 M_s M_{t^1} M_{t^1}^{-1}) \right]^{\frac{1}{2}} d\mu$$

$$\leq C M_{t^2}^{-1} M_{t^1}^{\frac{-1}{2}} \left[\frac{M_A}{M_s} M_{t^1}^2 M_{t^2}^2 M_s \right]^{\frac{1}{2}} \leq C(M_A M_{t^1})^{\frac{1}{2}}.$$

This means,

$$\int_{J_{t^1}(y^1)\times J_{t^2}(y^2)} \sup_{|n|=A} \left| \sum_{j=0}^{n_s-1} \sum_{k=0}^{M_s-1} \Phi(A, n^{(s+1)} + jM_s + k, y, x) \right| d\mu(x)$$

$$\leq C(M_A M_{t^1})^{\frac{1}{2}}.$$

This completes the proof of Lemma 2. □

Lemma 3. *Let $a \in \mathbb{N}, y \in G_m^2$. Then,*

$$\sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \int_{J_{t^1}(y^1)\times J_{t^2}(y^2)} \sup_{A\geq a} \sup_{|n|=A} \frac{1}{M_A} \sum_{s=t^1}^{A} \sum_{j=0}^{n_s-1} \mid \sum_{k=0}^{M_s-1} \Phi(A, n^{(s+1)} + jM_s + k, y, x) \mid d\mu(x) \leq C.$$

Proof. The inequality in Lemma 2 immediately gives the following result:

$$\sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \int_{J_{t^1}(y^1)\times J_{t^2}(y^2)} \sup_{A\geq \max(a,t^2-c)} \sup_{|n|=A} \frac{1}{M_A} \sum_{s=t^1}^{A} \sum_{j=0}^{n_s-1}$$

$$\left| \sum_{k=0}^{M_s-1} \Phi(A, n^{(s+1)} + jM_s + k, y, x) \right| d\mu(x)$$

$$\leq C \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \sum_{A=\max(a,t^2-c)}^{\infty} \sum_{s=t^1}^{A} (M_{t^1} M_A^{-1})^{\frac{1}{2}}$$

$$= C \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \sum_{A=\max(a,t^2-c)}^{\infty} (A - t^1 + 1)(M_{t^1} M_A^{-1})^{\frac{1}{2}}$$

$$\leq C \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} (\max(a,t^2) - t^1)(M_{t^1} M_{\max(a,t^2-c)}^{-1})^{\frac{1}{2}}$$

$$\leq C \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} (\max(a,t^2) - t^1)(M_{t^1} M_{\max(a,t^2)}^{-1})^{\frac{1}{2}}$$

$$\leq C \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{a-1} (a - t^1)(M_{t^1} M_a^{-1})^{\frac{1}{2}} + C \sum_{t^1=0}^{a-1} \sum_{t^2=a+1}^{\infty} (t^2 - t^1)(M_{t^1} M_{t^2}^{-1})^{\frac{1}{2}} \leq C.$$

This estimation used the fact that $M_{\max(a,t^2)} \leq CM_{\max(a,t^2-c)}$ which from the boundedness of the system (i.e. the Vilenkin space). This inequality shows that if we want to complete the proof of this lemma, then we have to discuss also the case when $\sup_{t^2-c>A\geq a}$. This follows that t^2 should be at least $a+c$. That is, we have to prove that the following integral is bounded

$$\sum_{t^1=0}^{a-1}\sum_{t^2=a+c}^{\infty}\int_{J_{t^1}(y^1)\times J_{t^2}(y^2)}\sup_{t^2-c>A\geq a}\sup_{|n|=A}\frac{1}{M_A}\sum_{s=t^1}^{A}\sum_{j=0}^{n_s-1}$$
$$\left|\sum_{k=0}^{M_s-1}\Phi(A,n^{(s+1)}+jM_s+k,x^1,x^2)\right|d\mu(x)$$
$$:= B^3.$$

The method we are going to use in order to discuss B^3 is the same as we used for the investigation of B^1. The only difference is that in the situation of B^1 we used the estimation $|D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^2,x^2)| \leq CM_{t^2}$ and in the case of B^3 we use the formula of the Dirichlet kernel D_n (1.5) and the estimation $|D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^2,x^2)| \leq CM_A$. The other steps of this process are the same. That is,

$$B^3 \leq C\sum_{t^1=0}^{a-1}\sum_{t^2=a+c}^{\infty}\int_{J_{t^2}(y^2)}\sum_{A=a}^{t^2-c}\frac{1}{M_A}\sum_{s=t^1}^{A}M_{t^1}^{-\frac{1}{2}}$$

$$\times\left[\sum_{n_A=1}^{m_A-1}\sum_{n_{A-1}=0}^{m_{A-1}-1}\cdots\sum_{n_{s+1}=0}^{m_{s+1}-1}(M_{t^1}M_A)^2M_sM_{t^1}^{-1}\right]^{\frac{1}{2}}d\mu(x^2)$$

$$= C\sum_{t^1=0}^{a-1}\sum_{t^2=a+c}^{\infty}\sum_{A=a}^{t^2-c}\sum_{s=t^1}^{A}M_{t^2}^{-1}M_A^{-1}M_{t^1}^{-\frac{1}{2}}(M_AM_s^{-1}M_{t^1}^2M_A^2M_s)^{\frac{1}{2}}$$

$$\leq C\sum_{t^1=0}^{a-1}\sum_{t^2=a+c}^{\infty}\sum_{A=a}^{t^2-c}\sum_{s=t^1}^{A}\frac{(M_AM_{t^1})^{\frac{1}{2}}}{M_{t^2}}$$

$$\leq C\sum_{t^1=0}^{a-1}\sum_{t^2=a+c}^{\infty}\sum_{A=a}^{t^2-c}(A-t^1+1)M_A^{\frac{1}{2}}M_{t^1}^{\frac{1}{2}}M_{t^2}^{-1}$$

$$\leq C\sum_{t^1=0}^{a-1}\sum_{t^2=a+c}^{\infty}(t^2-t^1+1)M_{t^1}^{\frac{1}{2}}M_{t^2}^{-\frac{1}{2}} \leq C.$$

This completes the proof.

□

In the sequel we step further and with the application of Lemma 3, we prove the main tool with respect to the maximal generalized Marcinkiwicz kernel in order to prove that the maximal operator $t_*^\alpha := \sup_{n\in\mathbb{P}} |t_n^\alpha f|$ is quasi-local (for the definition of quasi-locality, see e.g. [16, page 262]) and then it is of weak type (L^1, L^1).

Lemma 4. *Let $u \in G_m^2, a \in \mathbb{N}, y \in I_a(u^1) \times I_a(u^2)$. Then we have*

$$\int_{G_m^2(\setminus I_a(u^1)\times I_a(u^2))} \sup_{n\geq M_{a-c}} |M_n^\alpha(y,x)| d\mu(x) \leq C.$$

Proof. For $t^1 \leq a-1, t^2 \geq t^1$ and $x \in J_{t^1}(y^1) \times J_{t^2}(y^2)$ by (1.5) and (2.2) it is clear that

$$\begin{aligned}
&|\Phi(A, n^{(s+1)} + jM_s + k, y, x)| \\
&= |D_{\alpha_1(A,n^{(s+1)}+jM_s+k)}(y^1,x^1) D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^2,x^2)| \leq CM_{t^1} M_{\min(t^2,A)}.
\end{aligned}$$

This gives,

$$\begin{aligned}
&\sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \int_{J_{t^1}(y^1)\times J_{t^2}(y^2)} \sup_{A\geq a} \sup_{|n|=A} \frac{1}{M_A} \sum_{s=0}^{t^1} \sum_{j=0}^{n_s-1} \left| \sum_{k=0}^{M_s-1} \Phi(A, n^{(s+1)} + jM_s + k, y, x) \right| d\mu(x) \\
&\leq C \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \int_{J_{t^1}(y^1)\times J_{t^2}(y^2)} \sup_{A\geq a-c} \frac{1}{M_A} \sum_{s=0}^{t^1} M_s M_{t^1} M_{\min(t^2,A)} d\mu(x) \\
&\leq C \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{a-c} \frac{1}{M_{t^1} M_{t^2}} \sup_{A\geq a-c} \frac{1}{M_A} M_{t^1}^2 M_{t^2} + C \sum_{t^1=0}^{a-1} \sum_{t^2=a-c}^{\infty} \frac{1}{M_{t^1} M_{t^2}} M_{t^1}^2 \\
&\leq C \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{a-c} M_a^{-1} M_{t^1} + C \sum_{t^1=0}^{a-1} \sum_{t^2=a-c}^{\infty} M_{t^1} M_{t^2}^{-1} \leq C.
\end{aligned}$$

This by equality

$$M_n^\alpha(y,x) = \frac{1}{n} \sum_{s=0}^{A} \sum_{j=0}^{n_s-1} \sum_{k=0}^{M_s-1} \Phi(A, n^{(s+1)} + jM_s + k, y, x)$$

and by Lemma 3 immediately follows that

$$\sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \int_{J_{t^1}(y^1)\times J_{t^2}(y^2)} \sup_{\{n:|n|=A\geq a-c\}} |M_n^\alpha(y,x)| d\mu(x) \leq C.$$

Now, we prove for each $y \in I_a(u^1) \times I_a(u^2)$ the almost everywhere relation

$$\begin{aligned}
&G_m^2 \setminus \left(I_a(u^1) \times I_a(u^2)\right) \\
&\subset \left(\bigcup_{t^1=0}^{a-1} \bigcup_{t^2=t^1}^{\infty} J_{t^1}(y^1) \times J_{t^2}(y^2) \right) \cup \left(\bigcup_{t^1=0}^{a-1} \bigcup_{t^1=t^2}^{\infty} J_{t^1}(y^1) \times J_{t^2}(y^2) \right) \\
&=: J^1(y) \cup J^2(y).
\end{aligned}$$

Let $x=(x^1,x^2)\in G_m^2\setminus\big(I_a(u^1)\times I_a(u^2)\big)$. Then, either $x^1\notin I_a(u^1)$ or $x^2\notin I_a(u^2)$ (or both). Say, x^1 is not element of $I_a(u^1)$. Then, $x^1\in J_{t^1}(u^1)=J_{t^1}(y^1)$ for some $t^1<a$.

If $x^2\in I_a(u^2)=I_a(y^2)$ and $x^2\neq y^2$, then $x\in J^1(y)$. If $x^1\in J_{t^1}(u^1)=J_{t^1}(y^1)$ and x^2 is not element of $I_a(u^2)=I_a(y^2)$, then, $x^1\in J_{t^1}(u^1)=J_{t^1}(y^1)$ and $x^2\in J_{t^2}(u^2)=J_{t^2}(y^2)$ for some $t^1,t^2<a$.

For $t^2\geq t^1$ we have $x\in J^1(y)$ and for $t^1\geq t^2$ we have $x\in J^2(y)$. This procedure can be done if x^1,x^2 different from y^1 and y^2 respectively. The set of the points $x=(x^1,x^2)$, where either $x^1=y^1$ or $x^2=y^2$ is a zero measure set, so this can be supposed and the a.e. relation $G_m^2\setminus\big(I_a(u^1)\times I_a(u^2)\big)\subset J^1(y)\cup J^2(y)$ is proved for each $y\in I_a(u^1)\times I_a(u^2)$. Therefore, the proof of Lemma 4 is complete. □

Corollary 1. *Let $y\in G_m, n\in\mathbb{P}$. Then, $||M_n^\alpha(y,\cdot)||_1\leq C$.*

Proof. By Lemma 4, we have $\int_{G_m^2\setminus\left(I_{|n|}(y^1)\times I_{|n|}(y^2)\right)}|M_n^\alpha(y,x)|d\mu(x)\leq C$. Besides, by the conditions (2.1) and (2.2)

$$|M_n^\alpha(y,x)|\leq\frac{1}{n}\sum_{k=0}^{n-1}D_{\alpha_1(|n|,k)}(y^1,x^1)D_{\alpha_2(|n|,k)}(y^2,x^2)\leq C\frac{1}{n}\sum_{k=0}^{n-1}M_{|n|}M_{|n|}\leq CM_{|n|}^2.$$

Consequently,

$$\int_{I_{|n|}(y^1)\times I_{|n|}(y^2)}|M_n^\alpha(y,x)|d\mu(x)\leq C.$$

Hence, $||M_n^\alpha(y,\cdot)||\leq C$.

□

Now, let's check the quasi-locality of the maximal operator t_*^α is quasi-local (for the definition of the quasi-locality, see e.g. [16, page 262]).

Lemma 5. *Let $f\in L^1(G_m^2)$ such that* $\operatorname{supp} f\subset I_a(u^1)\times I_a(u^2)$, $\int f d\mu(x)=0$ *for some $u\in G_m^2$ and $a\in\mathbb{N}$. Then, $\int_{G_m^2\setminus I_a(u^1)\times(I_a(u^2))}t_*^\alpha f(x)d\mu(x)\leq C||f||_1$.*

Proof. If $|n|\leq a-c$ for some fixed constant $c>0$ depending only on α_1 and α_2, then we have by (4) that $\alpha_1(|n|,k),\alpha_2(|n|,k)<M_a$ for every $k<n$. Consequently, the kernel $M_n^\alpha(y,x)$, which is a linear combination of the product of Vilenkin-like functions χ_k with $k<M_a$, which is $\mathcal{A}_a$ measurable. This implies that

$$t_n^\alpha f(y)=\int_{I_a^2(u)}f(x)M_n^\alpha(x,y)d\mu(x)=M_n^\alpha(y)\int_{I_a(u^1)\times I_a(u^2)}f(x)d\mu=0.$$

That is, $|n| \geq a-c$ can be supposed. By the theorem of Fubini, by Lemma 4 and by the fact that for kernel $|M_n^\alpha(y,x)| = |M_n^\alpha(x,y)|$ we get,

$$\begin{aligned}
&\int_{G_m^2 \setminus I_a^2(u)} t_*^\alpha f \\
&= \int_{G_m^2 \setminus I_a^2(u)} \sup_{|n| \geq a-c} |t_n^\alpha f| d\mu(x) \\
&= \int_{G_m^2 \setminus I_a^2(u)} \sup_{|n| \geq a-c} \left| \int_{I_a^2(u)} f(x) M_n^\alpha(y,x) d\mu(x) \right| d\mu(y) \\
&\leq \int_{I_a^2(u)} |f(x)| \int_{G_m^2 \setminus I_a^2(u)} \sup_{|n| \geq a-c} |M_n^\alpha(y,x)| d\mu(y) | d\mu(x) \leq C \int_{I_a^2(u)} |f(x)| d\mu(x) = C||f||_1.
\end{aligned}$$

This completes the proof of Lemma 5. □

Theorem 1. *The operator t_*^α is of weak type (L^1, L^1) and it is also of type (L^p, L^p) for all $1 < p \leq \infty$.*

Proof. Now, we know that the operator t_*^α is of type (L^∞, L^∞) which is given by Corollary 1 and it is quasi-local by Lemma 5. Consequently, to prove that operator t_*^α is of weak type (L^1, L^1) is nothing else but to follow the standard argument (see e.g. [16]). Finally, the interpolation lemma of the Marcinkiwicz gives that it is also of type (L^p, L^p) for all $1 < p \leq \infty$. Consequently, by the above written by standard argument, we have the following theorem. □

Theorem 2. *Let α satisfy conditions (2.1) and (2.2). Then, we have $t_n^\alpha \to f$ for each $f \in L^1(G_m^2)$ a.e. with respect to every bounded Vilenkin-like system.*

Proof. The proof of Theorem 2 is just a standard consequence of the fact that the maximal operator t_*^α is of weak type (L^1, L^1), the fact that it holds for each two-dimensional Vilenkin-like polynomial (linear combinations of $\chi_k(x^1)\chi_n(x^2)$) and the fact that the set of two-dimensional Vilenkin-like polynomials is dense in $L^1(G_m^2)$. This density property comes from the behavior of the one dimensional kernel function D_{M_n}. That is, it is either M_n or zero. □

Finally, we give an application of Theorem 2. Before this a corollary is given:

Corollary 2. *Let (a_n) be a lacunary sequence of positive reals, i.e. $a_{n+1} \geq a_n q$ for some $q > 1$ ($n \in \mathbb{N}$) and α satisfy conditions (3) and $\alpha_j(n,k) \leq C a_n$ ($k < a_n$, $j = 1,2$) (modified version of condition (2.2). Then for every integrable function $f \in L^1(G_m^2)$ we have*

$$\frac{1}{a_n} \sum_{k=0}^{a_n - 1} S_{\alpha_1(n,k), \alpha_2(n,k)} f(x) \to f(x)$$

for a.e. $x \in G_m^2$.

Proof. The proof of this corollary runs as follows. Let b_n be defined as $M_{b_n-1} \leq a_n < M_{b_n}$ (that is, $b_n = |a_n|+1$) and

$$\tilde{\alpha}_j(b_n,k) = \begin{cases} \alpha(n,k), & \text{for } 0 \leq k < a_n, \\ k, & \text{if } a_n \leq k < M_{b_n} \end{cases} \qquad (j=1,2).$$

Then, $\tilde{\alpha}$ satisfies conditions (2.1) (trivially) and (2.2) since for $k < a_n$, $\tilde{\alpha}_j(b_n,k) = \alpha_j(n,k) \leq Ca_n \leq CM_{b_n}$. By Theorem 2 it follows that for the maximal operator $t_*^{\tilde{\alpha}} f := \sup |t_n^{\tilde{\alpha}} f|$ we have $\mu\left\{t_*^{\tilde{\alpha}} f \geq \lambda\right\} \leq C\|f\|_1/\lambda$ for all $f \in L^1(G_m^2)$ and $\lambda > 0$. Since

$$\frac{1}{a_n}\sum_{k=0}^{a_n-1} S_{\alpha_1(n,k),\alpha_2(n,k)} f = \frac{M_{b_n}}{a_n}\frac{1}{M_{b_n}}\sum_{k=0}^{M_{b_n}-1} S_{\tilde{\alpha}_1(b_n,k),\tilde{\alpha}_2(b_n,k)} f - \frac{M_{b_n}}{a_n}\frac{1}{M_{b_n}}\sum_{k=a_n}^{M_{b_n}-1} S_{k,k} f,$$

and consequently, $|t_{a_n}^{\alpha} f| \leq C|t_{M_{b_n}}^{\tilde{\alpha}} f| + C|t_{M_{b_n}}^{\mathrm{id}} f| + C|t_{a_n}^{\mathrm{id}} f|$, then (id denotes the "identical function", i.e. $\mathrm{id}(n,k) = (k,k)$) $t_*^{\alpha} f \leq Ct_*^{\tilde{\alpha}} f + Ct_* f$. The ordinary maximal Marcinkiewicz operator is of weak type (L^1, L^1) (see e.g. [6]) and this by standard argument [16] completes the proof of this corollary. □

In the sequel we give an application of the Corollary above. The triangular partial sums of the 2-dimensional Fourier series and the triangular Dirichlet kernels (with respect to the Vilenkin-like system χ) are defined as

$$S_k^{\triangle} f(x^1,x^2) := \sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1} \hat{f}(i,j)\chi_i(x^1)\chi_j(x^2), \quad D_k^{\triangle}(x^1,x^2) := \sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1} \chi_i(x^1)\chi_j(x^2).$$

The Fejér means of the triangular partial sums of the two-dimensional integrable function f (see e.g. [11]) are

$$\sigma_n^{\triangle} f := \frac{1}{n}\sum_{k=0}^{n-1} S_k^{\triangle} f.$$

For the trigonometric system Herriot proved [12] the a.e. (and norm) convergence $\sigma_n^{\triangle} f \to f$ $(f \in L^1)$. His method can not be adopted for the Vilenkin system, since for the time being there is no kernel formula available for these systems. The first result in this a.e. convergence issue of triangular means is due to Goginava and Weisz [11]. They proved for the Walsh-Paley system and each integrable function the a.e. convergence relation $\sigma_{2^n}^{\triangle} f \to f$. This result for the whole sequence of the triangular mean operators in the Walsh case is given by the first author [9]. In the Vilenkin situation there is nothing proved yet. By the corollary above, (Corollary 2) we prove for bounded Vilenkin-like systems:

Theorem 3. *For every lacunary sequence (a_n) (that is, $a_{n+1} \geq qa_n$, $q > 1$) we have the a.e. convergence $\sigma_{a_n}^{\triangle} f \to f$ for each $f \in L^1(G_m^2)$.*

To demonstrate the proof of this, see some calculations below [9] between the triangle and the one dimensional Dirichlet kernels.

$$K_n^{\triangle}(y^1,y^2,x^1,x^2) = \frac{1}{n}\sum_{k=0}^{n-1} D_k^{\triangle}(y^1,y^2,x^1,x^2) = \frac{1}{n}\sum_{k=1}^{n-1}\sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1} \chi_i(y^1,x^1)\chi_j(y^2,x^2)$$
$$= \frac{1}{n}\sum_{k=1}^{n-1}\sum_{i=0}^{k-1} \chi_i(y^1,x^1)D_{k-i}(y^2,x^2) = \frac{1}{n}\sum_{k=1}^{n-1}\sum_{i=1}^{k} \chi_{k-i}(y^1,x^1)D_i(y^2,x^2)$$
$$= \frac{1}{n}\sum_{i=1}^{n-1}\sum_{k=i}^{n-1} \chi_{k-i}(y^1,x^1)D_i(y^2,x^2) = \frac{1}{n}\sum_{i=1}^{n-1} D_{n-i}(y^1,x^1)D_i(y^2,x^2)$$
$$= \frac{1}{n}\sum_{i=1}^{n-1} D_i(y^1,x^1)D_{n-i}(y^2,x^2)$$

which is a generalized Marcienkiwicz kernel discussed in this paper (see Corollary 8:
$\alpha_1(|n|,k) = k$, $\alpha_2(|n|,k) = a_n - k$).

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ON THE PRACTICAL $\psi^{\gamma}-$EXPONENTIAL ASYMPTOTIC STABILITY OF NONLINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

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Abstract. In this paper, a new type of stability for nonlinear systems of differential equations called practical $\psi^{\gamma}-$exponential asymptotic stability, is presented. Some sufficient conditions for practical $\psi^{\gamma}-$exponential asymptotic stability are provided by using Lyapunov theory. These results generalize fundamental well known results for practical exponential asymptotic and $\psi-$exponential asymptotic stability for nonlinear time-varying systems. In addition, these results are using to investigate the practical $\psi^{\gamma}-$exponential asymptotic stability problem of nonlinear perturbed system and cascade systems. The last part is devoted to the study the problem of practical $\psi^{\gamma}-$exponential asymptotic stabilization for some classes of nonlinear systems with delayed perturbation.

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1. Introduction

The problem of stability analysis and controllability of nonlinear systems has been a topic of numerous papers and has produced a vast body of important results [4, 12, 17] and the references therein. However, in practice, disturbances often prevent the error signals from tending to the origin. Thus, the origin is not a point of equilibrium of the system. Such a situation often arises problems of guarantee the stability of the origin as an equilibrium point. For this reasons, LaSalle and Lefschetz [12] introduced the theory of practical stability when the origin is not necessary an equilibrium point. The notion of input to state practical stability introduced by [16] for to study uncertain dynamical systems. Under an output feedback controller the global uniform practical stability of the closed loop system is proved by [3]. In addition, the notion of $\psi-$stability of degree k for ordinary differential equations has been introduced by Akinyele [1]. Marchalo [13] gives a real start for the study of ψ-stability, when he introduced the notions of ψ-uniform stability for trivial solution of the nonlinear system and also obtained new sufficient conditions for the linear

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system. Recently various types of $\psi-$stability have been studied for nonlinear Lyapunov matrix differential equations have been given in many papers [6, 7, 14, 15]. In the study of stability the use of the function ψ is interpreted as a weight in the norm which ensures which ensures rate increase solutions that are not stable in the usual sense.

The main aim of the current paper, motivated by problems of generalized exponential asymptotical stability of nonlinear systems investigated by [4, 11].Combining the notion of practical stability with $\psi-$stability to ensure the study of a large number of dynamic systems, the notion of practical $\psi^{\gamma}-$exponential asymptotical stability was introduced. We investigate the preservation of this notion when considering a system with a perturbation term. This leads us to study the problem of $\psi^{\gamma}-$exponential asymptotic stability of cascade systems.

The rest of this paper is organized as follows:

First, in Section 2, basic definitions and some preliminary results about practical $\psi^{\gamma}-$exponential asymptotical stability are presented. Then, in Section 3 some sufficient conditions are given, to prove and to guarantee the main theorem about the global uniform practical $\psi^{\gamma}-$exponential asymptotical stability. However, Section 4 establishes practical $\psi^{\gamma}-$exponentially asymptotically stability for nonlinear system with delayed perturbation. finally we study the Practical $\psi^{\gamma}-$exponential asymptotic stabilization.

2. Preliminaries

Consider the nonlinear systems of differential equations

$$\dot{x} = f(t,x). \tag{2.1}$$

where the function $f \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$. Let $\psi_i : \mathbb{R}_+ \to (0,\infty), i = 1,2,...,n$ be continuous functions and

$$\psi = diag[\psi_1, \psi_2, ..., \psi_n].$$

The matrix $\psi(t)$ is invertible for each $t \geq 0$. If for all $i = 1,2,...,n$, $\psi_i(t) = 1$, then $\psi = I$ is the identity matrix. In addition, suppose

$$\|f(t,0)\| \leq f_0 \, \forall t \geq 0 \tag{2.2}$$

where $f_0 = constant \neq 0$ in general and the symbol $\|x\|$ denotes arbitrary vector norm of a vector $x \in \mathbb{R}^n$.

The condition (2.2) means that the origin $x = 0$ is not required to be an equilibrium point for the system under consideration. Indeed, this fails in many cases when studying the practical stability (see [8]). Throughout this paper, a solution of system (2.1) through a point $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ will be denoted by such a form as $x(t, t_0, x_0)$, where $x(t_0, t_0, x_0) = x_0$. In addition, suppose that the solution $x(t, t_0, x_0)$ is defined for all $t \geq t_0$. The purpose of this paper is to prove some sufficient conditions for practical $\psi^{\gamma}-$exponential asymptotic stability of solution of the system (2.1). In the first, some definitions are given below.

Definition 1 ([5]). The solution $x(.,t_0,x_0)$ of system (2.1) is said to be

i): $\psi-$bounded if there exists positive constant $\delta(t_0) < +\infty$, such that

$$\sup_{t\geq t_0} \|\psi(t)x(t,t_0,x_0)\| \leq \delta(t_0) < +\infty.$$

ii): uniformly $\psi-$bounded, if it is $\psi-$bounded with $\delta(t_0)$ is independent of t_0.

Definition 2. Let $\gamma > 0$ be given. The system (2.1) is called globally practically $\psi^{\gamma}-$exponentially asymptotically stable if there exist some constants $\lambda > 0$, $k \geq 1$ and $R \geq 0$ such that for any solution $x(t,t_0,x_0)$ of (2.1) satisfies

$$\|\psi(t)x(t,t_0,x_0)\| \leq k\|\psi(t_0)x_0\|^{\gamma}e^{-\lambda(t-t_0)} + R, t \geq t_0 \geq 0, x_0 \in \mathbb{R}^n. \tag{2.3}$$

If $\gamma = 1$, the system (2.1) is said globally practically $\psi-$exponentially asymptotically stable.

Remark 1. Note that, in the particular case where $\psi = I$, $\gamma \neq 1$ and $R \geq 0$, the notion of practical $\psi^{\gamma}-$exponential asymptotic stability is defined as follows :

a): If $R = 0$, then the system (2.1) is called globally $\gamma-$exponentially asymptotically stable.

b): If $R > 0$, then the system (2.1) is called globally practically $\gamma-$exponentially asymptotically stable.

The motivation of this paper is that the notion of practical $\psi^{\gamma}-$exponential asymptotic stability generalizes some known types of stability. In the following remark we cite some types of stability which are considered as special cases of practical $\psi^{\gamma}-$exponential asymptotic stability.

Remark 2. Definition 2 generalizes the notions of $\psi-$stability. More precisely, when $R = 0$ and $\gamma = 1$ we recover the usual definition of $\psi-$stability ([9]). Moreover, for $R \geq 0$, $\psi = I$ (identity matrix) and $\gamma = 1$, the practical $\psi^{\gamma}-$stability coincides with a known practical type of stability:

a): If $R = 0$, then the system (2.1) is globally exponentially asymptotically stable (see [10]).

b): If $R > 0$, then the system (2.1) is globally practically exponentially asymptotically stable (see [8]).

The relation between practical $\psi^{\gamma}-$exponential asymptotic stability and $\psi-$boundedness is given by the following remark.

Remark 3. By combining definition of practical $\psi^{\gamma}-$exponential asymptotic stability with that of $\psi-$boundedness, it is clear that if any solution of system (2.1) is uniformly $\psi-$bounded, then it is practically $\psi^{\gamma}-$exponentially asymptotically stable.

For some systems the insurance of practical exponential asymptotic stability might be more difficult. That is why it is sometimes more useful to look for another way

of stability, among which the practical $\psi^{\gamma}-$exponential asymptotic stability which requires appropriate sufficient conditions.

Next, we shall define a class of Lyapunov functions that will be used in the qualitative investigations of the system (2.1).

Definition 3. Consider a continuous function $V : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}_+$.

(1) The function V is said to be globally Lipschitz with respect to x, if there a constant $L > 0$ such that

$$|V(t,x) - V(t,y)| \leq L\|x-y\|, \text{ for all } t \in \mathbb{R}_+ \text{ and } x,y \in \mathbb{R}^n.$$

(2) The total derivative of V with respect to system (2.1), denoted by $D^+V_{(2.1)}$, is given by

$$D^+V_{(2.1)}(t,x) = \limsup_{\delta \longrightarrow 0^+} \frac{1}{\delta}\big[V(t+\delta, x+\delta f(t,x))) - V(t,x)\big] \tag{2.4}$$

(3) If $x(t)$ is a solution of (2.1), the upper right-hand derivative of $V(t,x(t))$, denoted by $D^+V(t,x(t))$, is given by

$$D^+V(t,x(t)) = \limsup_{\delta \longrightarrow 0^+} \frac{1}{\delta}\big[V(t+\delta, x(t+\delta)) - V(t,x(t))\big]. \tag{2.5}$$

Remark 4. Let V be a continuous function from $\mathbb{R}_+ \times \mathbb{R}^n$ to $\mathbb{R}_+$.

- If V is Lipschitz with respect to x, Yoshizawa [17] has proved that

$$D^+V_{(2.1)}(t,x) = D^+V(t,x(t)). \tag{2.6}$$

- If $V \in \mathcal{C}^1[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$, then the equalities are satisfied

$$D^+V_{(2.1)}(t,x) = D^+V(t,x(t)) = \dot{V}_{(2.1)}(t,x) := \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x).f(t,x). \tag{2.7}$$

Definition 4. A continuous function $V(t,x)$ is called a practical Lyapunov-ψ function for system (2.1), if there exist positive numbers $\lambda_1, \lambda_2, \lambda_3 > 0$, $a, b \geq 0$, $p, q, r > 0$ such that

i): $\lambda_1 \|\psi(t)x\|^p \leq V(t,x) \leq \lambda_2 \|\psi(t)x\|^q + a$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$,
ii): $D^+V_{(2.1)}(t,x) \leq -\lambda_3 \|\psi(t)x\|^r + b$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$

where $a, b \geq 0$, $\lambda_1, \lambda_2, \lambda_3 > 0$ and $p, q, r > 0$.

Without loss of generality, throughout the rest of the paper, if $V(t,x)$ is a practical Lyapunov-ψ function for system (2.1), we assume that $\lambda_1 \leq \lambda_3$ and $\lambda_1\lambda_3 < b\lambda_2$ which we need in the rest of this work.

Remark 5. Notes that, in the above definition,

i): if $V(t,x)$ is a Lyapunov-ψ function for system (2.1) ([9]), then $V(t,x)$ is also a practical Lyapunov-ψ function for system (2.1).

ii): if $p = q = r > 0$ and $\psi \equiv I$ then, the practical Lyapunov$-\psi$ function $V(t,x)$ ensure the global uniform practical exponential asymptotic stability of system (2.1)(see [8]).

The following lemmas will also be required in the investigations of our results of the paper.

Lemma 1. *Let $V(t)$ a function whose derivative $V'(t)$ exists for all $t \in \mathbb{R}_+$ and satisfies the differential inequality*

$$V'(t) \leq p(t)V(t) + q(t)$$

where $p(t)$ and $q(t)$ are two continuous functions. Then, the function V satisfies

$$V(t) \leq V(t_0)e^{\int_{t_0}^{t} p(s)ds} + \int_{t_0}^{t} e^{\int_s^t p(z)dz} q(s)ds, \ \forall t \geq t_0 \geq 0.$$

Lemma 2. *Let $\alpha, \beta \geq 0$. Then, we have the following inequalities:*

a): *For all $p \geq 1$, $(\alpha+\beta)^{\frac{1}{p}} \leq \alpha^{\frac{1}{p}} + \beta^{\frac{1}{p}}$.*
b): *For all $p \geq 1$, $(\alpha+\beta)^p \leq 2^{p-1}(\alpha^p + \beta^p)$.*
c): *If $(\alpha,\beta) \neq (0,0)$, then $0 \leq \dfrac{\alpha\beta}{\alpha+\beta} \leq \beta$.*

3. Main results

In this section, we consider practical $\psi^{\gamma}-$exponential asymptotic stability problems of solutions of system (2.1) will be discussed by the Lyapunov's second method. In the begining of this section, we give some sufficient conditions of practical $\psi^{\gamma}-$exponential asymptotic stability if the system (2.1) admits a practical Lyapunov$-\psi$ function candidate with p, q, r and a, b are arbitrary constants.

Theorem 1. *Assume that there exists a continuously differentiable function $V(t,x)$ satisfying the following properties: For all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$*

i): $\lambda_1 \|\psi(t)x\|^p \leq V(t,x) \leq \lambda_2 \|\psi(t)x\|^q + a,$
ii): $\dot{V}_{(2.1)}(t,x) \leq -\lambda_3 \|\psi(t)x\|^r + b$

where $\lambda_1, \lambda_2, \lambda_3 > 0$, $a, b \geq 0$, $p > 0$ and $r \geq q > 0$. Then, the system (2.1) is globally practically $\psi^{\frac{q}{p}}-$exponentially asymptotically stable.

Proof. The details of the proof are given in appendix. □

Now, the following corollaries can be easily obtained.
First, suppose that the system (2.1) admits a practical Lyapunov$-\psi$ function candidate such that $q = r > 0$, $p > 0$ and $a, b \geq 0$. The following corollary shows the existence of $\gamma > 0$ such that the system (2.1) is practically $\psi^{\gamma}-$exponentially asymptotically stable. In this case the practical $\psi-$stability result is obtained by the following Corollary.

Corollary 1. *Suppose there exists a continuously differentiable function $V(t,x)$ satisfying the following properties :*

i): $\lambda_1\|\psi(t)x\|^p \leq V(t,x) \leq \lambda_2\|\psi(t)x\|^q + a, \ \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,$

ii): $\dot{V}_{(2.1)}(t,x) \leq -\lambda_3\|\psi(t)x\|^q + b, \ \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$

where $\lambda_1, \lambda_2, \lambda_3 > 0$, $a, b \geq 0$ and $p, q > 0$. Then, the system (2.1) is globally practically $\psi^{\frac{q}{p}}-$exponentially asymptotically stable.

Remark 6. In Corollary 1, if $V(t,x)$ is a practical Lyapunov$-\psi$ function candidate for the system (2.1) such that $p = q > 0$ and $a, b \geq 0$, then we obtain sufficient condition for practical $\psi-$exponential asymptotic stability of system (2.1). In addition, if $\psi(t) = I$ we obtain sufficient conditions of practical exponential stability of system (2.1) (see [8]).

In the Theorem 1, we will end up taking $a = 0$ and p, q are chosen arbitrarily and $r \geq q$. The practical $\psi^\gamma-$exponential asymptotic stability of system (2.1) is given by the following corollary.

Corollary 2. *Suppose that there exists a continuously differentiable function $V(t,x)$ satisfying the following properties*

i): $\lambda_1\|\psi(t)x\|^p \leq V(t,x) \leq \lambda_2\|\psi(t)x\|^q, \ \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,$

ii): $\dot{V}_{(2.1)}(t,x) \leq -\lambda_3\|\psi(t)x\|^r + b, \ \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$

where $\lambda_1, \lambda_2, \lambda_3 > 0$, $b \geq 0$ and $p, q > 0$ such that $r \geq q$. Then, the system (2.1) is globally practically $\psi^{\frac{q}{p}}-$exponentially asymptotically stable.

Remark 7. The practical $\psi^{\frac{q}{p}}-$exponential asymptotic stability given by Corollary 2, generalizes that of practical stability given by Theorem 3.4 [2].

4. Practical $\psi^\gamma-$exponential asymptotic stability of perturbed and cascaded systems

The purpose of this section is to investigate the practical $\psi^\gamma-$exponential asymptotic stability problem for perturbed and cascaded nonlinear systems. By using practical Lyapunov$-\psi$ function, some criteria which guarantee the practical $\psi^\gamma-$exponential asymptotic stability of the addressed systems are provided.

4.1. *Practical $\psi^\gamma-$exponential asymptotic stability of perturbed systems*

A great interest is attached to the relations between the solutions of the unperturbed system (2.1) and the solutions of the perturbed system having the following form

$$\dot{y} = f(t,y) + g(t,y) \tag{4.1}$$

where $f, g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions. We suppose that functions f and g satisfy all required conditions for existence and uniqueness of the solutions of system (4.1) on the interval $[t_0, +\infty)$ for all suitable initial data $y_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}_+$.

The unique solutions of (4.1) is denoted by $y(t,t_0,y_0)$, satisfying the initial conditions $y(t_0,t_0,y_0) = y_0 \in \mathbb{R}^n$. Suppose the following assumption hold

(A1): There exists a continuously differentiable function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ satisfying the following properties:

i): $|V(t,x) - V(t,y)| \leq L\|x-y\|$, $x,y \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$,

ii): $\lambda_1\|\psi(t)x\|^p \leq V(t,x) \leq \lambda_2\|\psi(t)x\|^q + a$, $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$,

iii): $D^+V_{(2.1)}(t,x) \leq -\lambda_3\|\psi(t)x\|^r + b$, $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$

where $L > 0$, $\lambda_1,\lambda_2,\lambda_3 > 0$, $a,b \geq 0$ and $p > 0$, $r \geq q > 0$.

The practical $\psi^{\gamma}-$exponential asymptotic stability of perturbed system (4.1) is given by the following theorem.

Theorem 2. *Suppose that the assumption **(A1)** is satisfied and the perturbation term $g(t,y)$ satisfies*

$$\|g(t,y)\| \leq \lambda\|\psi(t)y\|^r + \beta \tag{4.2}$$

where $\lambda,\beta > 0$ and $r \geq q > 0$ such that $\lambda_3 - L\lambda > 0$ and . Then, the perturbed system (4.1) is globally practically $\psi^{\frac{q}{p}}-$exponentially asymptotically stable.

Proof. The total derivative $D^+V_{(4.1)}(t,y)$ of the function $V(t,y)$ with respect to system (4.1) satisfies

$$\begin{aligned}
D^+V_{(4.1)}(t,y) &= \limsup_{\delta\longrightarrow 0^+}\frac{1}{\delta}\left[V(t+\delta,y+\delta(f(t,y)+g(t,y))) - V(t,y)\right] \\
&\leq \limsup_{\delta\longrightarrow 0^+}\frac{1}{\delta}\left[V(t+\delta,y+\delta(f(t,y)+g(t,y))) - V(t+\delta,y+\delta f(t,y))\right] \\
&\quad + \limsup_{\delta\longrightarrow 0^+}\frac{1}{\delta}\left[V(t+\delta,y+\delta f(t,y)) - V(t,y)\right] \\
&\leq D^+V_{(2.1)}(t,y) + L\|g(t,y)\|, \\
&\leq -\lambda_3\|\psi(t)y\|^r + b + L\lambda\|\psi(t)y\|^r + L\beta, \\
&\leq -(\lambda_3 - L\lambda)\|\psi(t)y\|^r + b + L\beta.
\end{aligned}$$

Therefore, $V(t,y)$ is a practical Lyapunov-ψ function for the perturbed system (4.1). Then, by Theorem 1, the perturbed system (4.1) is practically $\psi^{\frac{q}{p}}-$exponentially asymptotically stable. □

4.2. *Practical $\psi^{\gamma}-$exponential asymptotic stability of cascaded systems*

Consider now cascaded system of the form:

$$(\Sigma_1): \quad \dot{x}_1 = f_1(t,x_1) + g(t,x_1,x_2) \tag{4.3}$$

$$(\Sigma_2): \quad \dot{x}_2 = f_2(t,x_2) \tag{4.4}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$ with $n_1,n_2 \in \mathbb{N}^*$ and $x := [x_1,x_2]^T \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, denotes the state of the closed-loop system. We assume that functions f_1, f_2 and g satisfy

all required conditions for existence and uniqueness of the solutions of system (4.3)-(4.4) on the interval $[t_0,+\infty)$ for all suitable initial data $x_0 \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and $t_0 \in \mathbb{R}_+$.

The main goal is to give some sufficient conditions which guarantee that if the system (4.4) and the system

$$\dot{x}_1 = f_1(t,x_1) \tag{4.5}$$

are practically $\psi^\gamma-$exponentially stable where $\gamma > 0$, then the cascaded system (4.3)-(4.4) is also practically $\psi^\gamma-$exponentially asymptotically stable, if the interconnection term $g(t,x_1,x_2)$ satisfies some condition of boundedness. The unique solution of cascaded system (4.3)-(4.4) is denoted by $x(.,t_0,x_0)$, which satisfies $x(t_0,t_0,x_0) = x(t_0) = x_0$.

In this part, we consider $\psi_1 = diag[\psi_1^1,\psi_2^1,...,\psi_{n_1}^1]$ and $\psi_2 = diag[\psi_1^2,\psi_2^2,...,\psi_{n_1}^2]$ with $\psi_i^1 : \mathbb{R}_+ \to (0,\infty), i = 1,2,...,n_1$ and $\psi_i^2 : \mathbb{R}_+ \to (0,\infty), i = 1,2,...,n_2$ be continuous functions. Before proposing our theorem, we introduce the following assumptions:

(A2): There exist two functions $V_1(t,x_1)$ and $V_2(t,x_2)$ having the following properties: For $i = 1,2$

- **i):** $V_i \in \mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}^{n_i},\mathbb{R}_+)$ and $|V_i(t,x_i) - V_i(t,y_i)| \leq L_i\|x_i - y_i\|$ for all $x_i,y_i \in \mathbb{R}^{n_i}$ and for each $t \in \mathbb{R}_+$,
- **ii):** $\lambda_i\|\psi_i(t)x_i\|^p \leq V_i(t,x_i) \leq \beta_i\|\psi_i(t)x_i\|^q + a_i$, $(t,x_i) \in \mathbb{R}_+ \times \mathbb{R}^{n_i}$,
- **iii):** $D^+V_{i(\Sigma_i)}(t,x_i) \leq -c_i\|\psi_i(t)x_i\|^r + b_i$, $(t,x_i) \in \mathbb{R}_+ \times \mathbb{R}^{n_i}$

where $a_i,b_i \geq 0$, $L_i,\lambda_i,\beta_i,c_i > 0$ for $i = 1,2$ and $p > 0$, $r \geq q > 0$.

(A3): The interconnection term $g(t,x_1,x_2)$ satisfies the following boundedness condition

$$\|g(t,x_1,x_2)\| \leq \delta(\|\psi_1(t)x_1\|^r + \|\psi_2(t)x_2\|^r + 1), t \in \mathbb{R}_+, x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2} \tag{4.6}$$

where $\delta > 0$ and $r \geq q > 0$.

Theorem 3. *We assume assumptions **(A2)** and **(A3)** are satisfied. Then, under the condition* $\min(c_1,c_2) > 2\delta L_1 \max\left\{2^{\frac{r-2}{2}},2^{\frac{2-r}{2}}\right\}$*, the cascaded system (4.3)-(4.4) is globally practically* $\psi^{\frac{q}{p}}-$*exponentially asymptotically stable.*

Proof. Let define the function $W(t,x)$ by

$$W(t,x) = V_1(t,x_1) + V_2(t,x_2),$$

where $x = [x_1,x_2]^T \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Consider the matrix function $\Psi(t)$ given by

$$\psi(t) = \begin{bmatrix} \psi_1(t) & 0 \\ 0 & \psi_2(t) \end{bmatrix}, t \in \mathbb{R}_+.$$

Then, for all $t \in \mathbb{R}_+$ and $x = [x_1,x_2]^T \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, we obtain $\psi(t)x = \left[\psi_1(t)x_1,\psi_2(t)x_2\right]^T$. By Lemma 2 we have, for all $r > 0$

$$\|\psi(t)x\|^r = \left[\|\psi_1(t)x_1\|^2 + \|\psi_2(t)x_2\|^2\right]^{\frac{r}{2}} \leq \max\left\{2^{\frac{r-2}{2}},2^{\frac{2-r}{2}}\right\}(\|\psi_1(t)x_1\|^r + \|\psi_2(t)x_2\|^r).$$

The total derivative $D^{+}W_{(\Sigma_1,\Sigma_2)}(t,x)$ of W with respect to system (4.3)-(4.4) satisfies

$$\begin{aligned}
D^{+}W_{(\Sigma_1,\Sigma_2)}(t,x) &\leq D^{+}V_{1(4.3)}(t,x_1)+D^{+}V_{2(4.4)}(t,x_2),\\
&\leq D^{+}V_{1(4.5)}(t,x_1)+D^{+}V_{2(4.4)}(t,x_2)+L_1\|g(t,x_1,x_2)\|,\\
&\leq -c_1\|\psi_1(t)x_1\|^r+b_1-c_2\|\psi_2(t)x_2\|^r+b_2+L_1\|g(t,x_1,x_2)\|,\\
&\leq -\min(c_1,c_2)\Big(\|\psi_1(t)x_1\|^r+\|\psi_2(t)x_2\|^r\Big)\\
&\quad+\delta L_1(\|\psi_1(t)x_1\|^r+\|\psi_2(t)x_2\|^r+1)+b_1+b_2,\\
&\leq -\frac{\min(c_1,c_2)}{\max\left\{2^{\frac{r-2}{2}},2^{\frac{2-r}{2}}\right\}}\|\psi(t)x\|^r+\delta L_1(\|\psi_1(t)x_1\|^r+\|\psi_2(t)x_2\|^r)\\
&\quad+b_1+b_2+\delta L_1.
\end{aligned}$$

Using the inequalities

$$\|\psi_1(t)x_1\|\leq\|\psi(t)x\|,\ \|\psi_2(t)x_2\|\leq\|\psi(t)x\|,\ \|\psi(t)x\|\leq\|\psi_1(t)x_1\|+\|\psi_2(t)x_2\| \tag{4.7}$$

yield

$$D^{+}W(t,x)\leq -c\|\psi(t)x\|^r+b,\ t\in\mathbb{R}_+,\ x\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} \tag{4.8}$$

where $b=b_1+b_2+\delta L_1$ and $c=\frac{\min(c_1,c_2)}{\max\left\{2^{\frac{r-2}{2}},2^{\frac{2-r}{2}}\right\}}-2\delta L_1>0$. By condition ii), the function $W(t,x)$ satisfies

$$\lambda_1\|\psi_1(t)x_1\|^p+\lambda_2\|\psi_2(t)x_2\|^p\leq W(t,x)\leq\beta_1\|\psi_1(t)x_1\|^q+\beta_2\|\psi_2(t)x_2\|^q+a_1+a_2.$$

Then, by Lemma 2 and (4.7), we obtain

$$\lambda\|\psi(t)x\|^p\leq W(t,x)\leq\beta\|\psi(t)x\|^q+a,\ t\in\mathbb{R}_+,\ x\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$$

with $\beta=\beta_1+\beta_2$, $\lambda=\frac{\min(\lambda_1,\lambda_2)}{2^{p-1}}$ and $a=a_1+a_2$. Therefore, $W(t,x)$ is a practical Lyapunov-ψ function for the cascaded system (4.3)-(4.4). Then, by Theorem 1, the cascaded system (4.3)-(4.4) is globally practically $\psi^{\frac{q}{p}}-$exponentially asymptotically stable. □

5. Practical $\psi^{\gamma}-$exponential asymptotic stabilization

We denote by $\mathcal{M}_{(n,m)}(\mathbb{R})$ the set of all $n\times m$ matrices with real coefficients where $m,n\in\mathbb{N}^*$.

In this section, we conclude to some practical $\psi^{\gamma}-$stabilization problem of a class nonlinear control system with delayed perturbations using feedback controls.

Consider the following nonlinear time-varying differential equation with delay.

$$\dot{x}=f(t,x)+g(t,x)\Big(\sum_{i=0}^{r}q_i(t,x(t-\tau_i(t)))+u\Big),\ r\in\mathbb{N},\ t\in\mathbb{R}_+,\ x\in\mathbb{R}^n \tag{5.1}$$

where

- $f:\mathbb{R}_+\times\mathbb{R}^n\longrightarrow\mathbb{R}^n$, $g:\mathbb{R}_+\times\mathbb{R}^n\longrightarrow\mathcal{M}_{(n,m)}(\mathbb{R})$ are continuous functions.
- $x\in\mathbb{R}^n$ is the state and $u\in\mathcal{M}_{(n,1)}(\mathbb{R})$ is the control.
- $q_i(t,x(t-\tau_i(t))\in\mathcal{M}_{(m,1)}(\mathbb{R})$ are the perturbations terms.
- $\tau_i(t)$, $i=1,2,...,r$ are the time delays positive, continuous and there exist positive constants $\overline{\tau}_i$, $i=1,2,...,r$ such that $0\leq\tau_i(t)\leq\overline{\tau}_i$.

The initial condition function is given by $x(t)=\varphi(t)$, $t\in[t_0-\overline{\tau},t_0]$ where $\overline{\tau}=\max\limits_{1\leq i\leq r}\{\overline{\tau}_i\}$ and $\varphi(t)$ is a continuous function on $[t_0-\overline{\tau},t_0]$.
The nominal system and is defined by

$$\dot{x}=f(t,x)+g(t,x)u. \tag{5.2}$$

The unforced nominal system of (5.1) is defined by

$$\dot{x}=f(t,x). \tag{5.3}$$

Our goal is to find a feedback $u=u(t,x)$ which make the closed-loop system

$$\dot{x}=f(t,x)+g(t,x)\Big(\sum_{i=0}^{r}q_i(t,x(t-\tau_i(t)))+u(t,x)\Big),\ r\in\mathbb{N}$$

practically $\psi^{\gamma}-$exponentially asymptotically stable, where $\gamma>0$.
In order to study the practical $\psi^{\gamma}-$exponential asymptotic stability of closed-loop system, we need the following assumptions

(A4): There exists a continuously differentiable function $V:\mathbb{R}_+\times\mathbb{R}^n\longrightarrow\mathbb{R}_+$ which satisfies

$$\lambda_1\|\psi(t)x\|^p\leq V(t,x)\leq\lambda_2\|\psi(t)x\|^q+a \tag{5.4}$$

$$\dot{V}_{(5.3)}(t,x)\leq-\lambda_3\|\psi(t)x\|^r+b \tag{5.5}$$

for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}^n$, where $\lambda_1,\lambda_2,\lambda_3>0$, $a,b\geq0$ and $p>0$, $r\geq q>0$.

(A5): The perturbation term $q_i(t,x(t-\tau_i(t))$ checking:

$$\|q_i(t,x(t-\tau_i(t))\|\leq\beta_i\|x(t-\tau_i(t))\|,\ \beta_i>0,\ i=1,2,...,r. \tag{5.6}$$

Theorem 4. *System (5.1) satisfying assumptions **(A4)** and **(A5)** is globally practically $\psi^{\frac{q}{p}}-$exponentially asymptotically stable under the following controller*

$$u(t,x)=\frac{-\Big[\sum\limits_{i=0}^{r}\beta_i\chi_i(t)\Big]^2\Big[g^T(t,x)\frac{\partial V}{\partial x}(t,x)\Big]}{\|\frac{\partial V}{\partial x}^T(t,x).g(t,x)\|\sum\limits_{i=0}^{r}\beta_i\chi_i(t)+\varepsilon\|\psi(t)x\|^r} \tag{5.7}$$

where $\varepsilon>0$ *and*

$$\chi_i(t)=\max_{s\in[t-\overline{\tau}_i,t]}\|x(s)\|,\ i=1,2,...,r \tag{5.8}$$

Proof. Applying (5.7) to (5.1) gives a new form of a closed-loop dynamical system:

$$\dot{x} = f(t,x) + g(t,x)\sum_{i=0}^{r} q_i(t,x(t-\tau_i(t))) + g(t,x)u(t,x). \tag{5.9}$$

By using (5.5) and (5.8), the derivative $\dot{V}_{(5.9)}(t,x)$ of the function V along the trajectories of (5.9), satisfies

$$\begin{aligned}
\dot{V}_{(5.9)}(t,x) &= \dot{V}_{(5.3)}(t,x) + \frac{\partial V}{\partial x}^T(t,x).g(t,x)\sum_{i=0}^{r} q_i(t,x(t-\tau_i(t))) \\
&\quad + \frac{\partial V}{\partial x}^T(t,x).g(t,x)u(t,x), \\
&\leq -\lambda_3\|\psi(t)x\|^r + b + \frac{\partial V}{\partial x}^T(t,x).g(t,x)\sum_{i=0}^{r} q_i(t,x(t-\tau_i(t))) \\
&\quad + \frac{\partial V}{\partial x}^T(t,x).g(t,x)u(t,x), \\
&\leq -\lambda_3\|\psi(t)x\|^r + b + \|\frac{\partial V}{\partial x}^T(t,x).g(t,x)\|\sum_{i=0}^{r}\|q_i(t,x(t-\tau_i(t)))\| \\
&\quad + \frac{\partial V}{\partial x}^T(t,x).g(t,x)u(t,x), \\
&\leq -\lambda_3\|\psi(t)x\|^r + b + \|\frac{\partial V}{\partial x}^T(t,x).g(t,x)\|\sum_{i=0}^{r}\beta_i\|x(t-\tau_i(t))\| \\
&\quad + \frac{\partial V}{\partial x}^T(t,x).g(t,x)u(t,x).
\end{aligned}$$

By using the feedback (5.7) we obtain:

$$\begin{aligned}
\dot{V}_{(5.9)}(t,x) &\leq -\lambda_3\|\psi(t)x\|^r + b + \beta - \frac{\beta^2}{\alpha+\beta} \\
&= -\lambda_3\|\psi(t)x\|^r + b + \frac{\alpha\beta}{\alpha+\beta}
\end{aligned}$$

with $\alpha = \varepsilon\|\psi(t)x\|^r$ and $\beta = \|\frac{\partial V}{\partial x}^T(t,x).g(t,x)\|\sum_{i=0}^{r}\beta_i\chi_i(t)$. Therefore, it follows from ii) of Lemma 2 that,

$$\dot{V}_{(5.9)}(t,x) \leq -\lambda_3\|\psi(t)x\|^r + b + \varepsilon\|\psi(t)x\|^r = -(\lambda_3-\varepsilon)\|\psi(t)x\|^r + b. \tag{5.10}$$

By using (5.4) and (5.10), the function $V(t,x)$ is a practical Lyapunov ψ−function for the closed loop system (5.9). Then, under the controller (5.7), the system (5.1) is globally practically $\psi^{\frac{q}{p}}$−exponentially asymptotically stable.

□

Next, consider the nonlinear uncertain system with delayed perturbations of the form

$$\dot{x} = f(t,x) + g(t,x)\sum_{i=0}^{r} q_i(t,x(t-\tau_i(t))) + g(t,x)(1+\delta(t,x))u,\ r \in \mathbb{N} \tag{5.11}$$

where $\delta(t,x)$ represents the uncertainties terms. Now, we need the following assumption:

(A6): There exists a continuous function $\phi(t,x)$ such that for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$

$$1+\delta(t,x) \geq \phi(t,x) > 0. \tag{5.12}$$

Theorem 5. *Suppose that assumptions **(A4)** - **(A6)** are satisfied. Then, the system (5.11) is globally practically $\psi^{\frac{q}{p}}-$exponentially asymptotically stable under the following controller*

$$u(t,x) = \frac{-\Big[\sum_{i=0}^{r}\beta_i\chi_i(t)\Big]^2\Big[g^T(t,x)\frac{\partial V}{\partial x}(t,x)\Big]}{\phi(t,x)\|\frac{\partial V}{\partial x}^T(t,x).g(t,x)\|\sum_{i=0}^{r}\beta_i\chi_i(t)+\varepsilon^*(t,x)\|\psi(t)x\|^r} \tag{5.13}$$

where $\varepsilon^*(t,x) = \varepsilon\phi(t,x)$ such that $0<\varepsilon<\lambda_3$.

Proof. Substituting (5.13) into (5.11) yields the following form:

$$\dot{x} = f(t,x) + g(t,x)\sum_{i=0}^{r} q_i(t,x(t-\tau_i(t))) + g(t,x)(1+\delta(t,x))u(t,x). \tag{5.14}$$

From assumption **(A5)**, the derivative $\dot{V}_{(5.14)}(t,x)$ of the practical Lyapunov $\psi-$function $V(t,x)$ along the trajectories of (5.14) satisfies

$$\begin{aligned}\dot{V}_{(5.14)}(t,x) &= \dot{V}_{(5.3)}(t,x) + \frac{\partial V}{\partial x}^T(t,x).g(t,x)\sum_{i=0}^{r} q_i(t,x(t-\tau_i(t)))\\ &\quad + \frac{\partial V}{\partial x}^T(t,x).g(t,x)(1+\delta(t,x))u(t,x),\\ &\leq -\lambda_3\|\psi(t)x\|^r + b + \|\frac{\partial V}{\partial x}^T(t,x).g(t,x)\|\sum_{i=0}^{r}\|q_i(t,x(t-\tau_i(t)))\|\\ &\quad + \frac{\partial V}{\partial x}^T(t,x).g(t,x)(1+\delta(t,x))u(t,x).\end{aligned}$$

Then, from (5.8) and (5.13) we get

$$\begin{aligned}\dot{V}_{(5.14)}(t,x) &\leq -\lambda_3\|\psi(t)x\|^r + b + \frac{\beta}{\phi(t,x)} - \frac{\beta^2}{\phi(t,x)(\alpha+\beta)}\\ &= -\lambda_3\|\psi(t)x\|^r + b + \frac{\alpha\beta}{\phi(t,x)(\alpha+\beta)}\end{aligned}$$

with $\alpha = \varepsilon^*(t,x)\|\psi(t)x\|^r$ and $\beta = \phi(t,x)\|\frac{\partial V}{\partial x}^T(t,x).g(t,x)\|\sum_{i=0}^{r}\beta_i\chi_i(t)$. By using ii) of Lemma 2 , we obtain

$$\begin{aligned}\dot{V}_{(5.14)}(t,x) &\leq -\lambda_3\|\psi(t)x\|^r + b + \frac{\varepsilon^*(t,x)\|\psi(t)x\|^r}{\phi(t,x)},\\ &= -(\lambda_3-\varepsilon)\|\psi(t)x\|^r + b.\end{aligned}$$

Similarly of the proof of Theorem 4, the function $V(t,x)$ is a practical Lyapunov $\psi-$function for the closed loop system (5.14). Then, under the controller (5.13), the system (5.11) is globally practically $\psi^{\frac{q}{p}}-$exponentially asymptotically stable.

□

6. Conclusion

In this paper, we have presented some new conditions for practical $\psi^{\gamma}-$exponential asymptotic stability of nonlinear systems of differential equations. A main theorem for practical $\psi^{\gamma}-$exponential asymptotic stability is established. In addition, one of our main interests is the study of the problem of practical $\psi^{\gamma}-$exponential asymptotic stability of the perturbed systems and cascade systems. Based on this study, we reached a novel result in practical $\psi^{\gamma}-$exponential asymptotic stabilization for dynamical systems with delayed perturbations. To guarantee that the closed-loop system is practically $\psi^{\gamma}-$exponentially asymptotically stable, a continuous controller is provided and sufficient conditions are given. Furthermore, a class of nonlinear systems with delayed perturbations and uncertain control has been considered.

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Appendix A. Proof of Theorem 1

Proof. Let $x(t) := x(t,t_0,x_0)$ be the solution of system (2.1) through $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$. In the following, the practical $\psi^{\frac{q}{p}}-$exponential asymptotic stability of the system (2.1) is shown. Four cases are presented

Case 1: : If $q = r$, then it follows from the conditions i) and ii) that, the derivative $\dot{V}_{(2.1)}(t,x)$ of the function $V(t,x)$ along the trajectories of (2.1) satisfies

$$\dot{V}_{(2.1)}(t,x) \leq -\frac{\lambda_3}{\lambda_2}V(t,x) + \frac{a\lambda_3}{\lambda_2} + b, t \in \mathbb{R}_+, x \in \mathbb{R}^n.$$

By using Lemma 1 with $p(t) = -\frac{\lambda_3}{\lambda_2}$ and $q(t) = b + \frac{a\lambda_3}{\lambda_2}$, we obtain

$$V(t,x(t)) \leq V(t_0,x_0)e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)} + (a + \frac{b\lambda_2}{\lambda_3})\big(1 - e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)}\big), t \geq t_0, x_0 \in \mathbb{R}^n.$$

Thus, for all $t \geq t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$

$$\begin{aligned} V(t,x(t)) &\leq V(t_0,x_0)e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)} + (a + \frac{b\lambda_2}{\lambda_3})\big(1 - e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)}\big) \\ &\leq \lambda_2\|\psi(t_0)x_0\|^p e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)} + ae^{-\frac{\lambda_3}{\lambda_2}(t-t_0)} + (a + \frac{b\lambda_2}{\lambda_3})\big(1 - e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)}\big) \\ &\leq \lambda_2\|\psi(t_0)x_0\|^p e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)} + \big(a + \frac{b\lambda_2}{\lambda_3}\big). \end{aligned}$$

By using the condition i) of Theorem 1 and condition a) of Lemma 2, the solution $x(t,t_0,x_0)$ satisfies

$$\|\psi(t)x(t,t_0,x_0)\| \leq \sqrt[p]{\frac{\lambda_2}{\lambda_1}}\|\psi(t_0)x_0\|^{\frac{q}{p}} e^{-\frac{\lambda_3}{p\lambda_2}(t-t_0)} + \sqrt[p]{\frac{a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}}, \forall t \geq t_0 \geq 0, \forall x_0 \in \mathbb{R}^n.$$

Case 2 : : If $r > q$ and $p = q$, we distinguish two subcases: whether the function $\psi(t)x(t)$ start from outside or inside the closed ball of $\mathbb{R}^n$, $\mathcal{B}_1 := \{x \in \mathbb{R}^n; \|x\| \leq 1\}$.

- $\|\psi(t_0)x_0\| > 1$. In this case, there exists $T_0 \in]0,+\infty]$ such that $\|\psi(t)x(t)\| > 1$ for all $t \in [t_0, t_0 + T_0[$ and $\|\psi(t_0 + T_0)x(t_0 + T_0)\| = 1$. Hence, we get that

$$\begin{aligned} \dot{V}_{(2.1)}(t,x) &\leq -\lambda_3\|\psi(t)x\|^r + b \\ &= -\lambda_3\|\psi(t)x\|^{r-q}\|\psi(t)x\|^q + b \\ &\leq -\lambda_3\|\psi(t)x\|^q + b \\ &\leq -\frac{\lambda_3}{\lambda_2}V(t,x) + \frac{a\lambda_3}{\lambda_2} + b. \end{aligned}$$

By using Lemma 1, the following inequality is obtained

$$V(t,x(t)) \leq V(t_0,x_0)e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)} + (a + \frac{b\lambda_2}{\lambda_3})\big(1 - e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)}\big), t \in [t_0, t_0 + T_0[. \quad \text{(A.1)}$$

From the condition i), it is easy to see that for all $t \in [t_0, t_0 + T_0[$, we have

$$\|\psi(t)x(t,t_0,x_0)\|^p \leq \frac{\lambda_2}{\lambda_1}\|\psi(t_0)x_0\|^p e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)} + \frac{a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}.$$

Then, by condition i) of Lemma 2, the estimation is obtained

$$\|\psi(t)x(t,t_0,x_0)\| \leq \sqrt[p]{\frac{\lambda_2}{\lambda_1}}\|\psi(t_0)x_0\| e^{-\frac{\lambda_3}{p\lambda_2}(t-t_0)} + \sqrt[p]{\frac{a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}}, t \in [t_0, t_0 + T_0[.$$

In addition, for each $t \geq t_0 + T_0$, either $\|\psi(t)x(t)\| \leq 1$, or $\|\psi(t)x(t)\| > 1$. In this second case, we can again invoke the continuity of the function $\psi(t)x(t)$ to see that there exists a nonempty time-interval $[\tau, \tau+T]$, with $T \in (0, +\infty]$, containing t and such that $\|\psi(t)x(t)\| > 1$ for all $s \in (\tau, \tau+T]$, with $\|\psi(\tau)x(\tau)\| = 1$. Hence, integrating from τ to $t \in [\tau, \tau+T]$, we obtain, whenever $\|\psi(t)x(t)\| > 1$, it holds that

$$\begin{aligned}
V(t,x(t)) \leq & V(\tau,x(\tau))e^{-\frac{\lambda_3}{\lambda_2}(t-\tau)} + (a+\frac{b\lambda_2}{\lambda_3})\big(1-e^{-\frac{\lambda_3}{\lambda_2}(t-\tau)}\big) \\
\leq & (\lambda_2\|\psi(\tau)x(\tau)\|^p + a)e^{-\frac{\lambda_3}{\lambda_2}(t-\tau)} + (a+\frac{b\lambda_2}{\lambda_3})\big(1-e^{-\frac{\lambda_3}{\lambda_2}(t-\tau)}\big) \\
\leq & \lambda_2 + a + \frac{b\lambda_2}{\lambda_3} \\
= & \frac{\lambda_2\lambda_3 + a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}.
\end{aligned}$$

Then, by condition i) of Theorem 1, the solution $x(t,t_0,x_0)$ satisfies

$$\|\psi(t)x(t,t_0,x_0)\| \leq \sqrt[p]{\frac{\lambda_2\lambda_3 + a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}}.$$

Then, the solution $x(t,t_0,x_0)$ is uniformly ψ−bounded.
To sum up, for all $t \geq t_0$, if $\|\psi(t_0)x_0\| > 1$ we have the following

$$\|\psi(t)x(t,t_0,x_0)\| \leq \sqrt[p]{\frac{\lambda_2}{\lambda_1}}\|\psi(t_0)x_0\|e^{-\frac{\lambda_3}{p\lambda_2}(t-t_0)} + \sqrt[p]{\frac{\lambda_2\lambda_3 + a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}}.$$

- $\|\psi(t_0)x_0\| \leq 1$. In this case, as long as $\|\psi(t)x(t)\| \leq 1$, we have trivially that $\|\psi(t)x(t,t_0,x_0)\| \leq \sqrt[p]{\frac{\lambda_2\lambda_3+a\lambda_3+b\lambda_2}{\lambda_1\lambda_3}}$. If $\|\psi(t)x(t)\| > 1$ at some instant $t > t_0$, then, again, there exists a nonempty time-interval $[\tau, \tau+T]$, with $T \in (0, +\infty]$ and $\tau > t_0$, containing t and such that $\|\psi(s)x(s)\| > 1$ for all $s \in (\tau, \tau+T]$, with $\|\psi(\tau)x(\tau)\| = 1$. By integration from τ to $t \in [\tau, \tau+T]$, we obtain that, whenever $\|\psi(t)x(t)\| > 1$, it holds that

  $$V(t,x(t)) \leq \frac{\lambda_2\lambda_3 + a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}.$$

  By using condition i) of Theorem 1, we have

  $$\|\psi(t)x(t,t_0,x_0)\| \leq \sqrt[p]{\frac{\lambda_2\lambda_3 + a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}}.$$

Hence, for all $t \geq t_0$ and $x_0 \in \mathbb{R}^n$, we have

$$\|\psi(t)x(t,t_0,x_0)\| \leq \sqrt[p]{\frac{\lambda_2}{\lambda_1}}\|\psi(t_0)x_0\|e^{-\frac{\lambda_3}{p\lambda_2}(t-t_0)} + \sqrt[p]{\frac{\lambda_2\lambda_3 + a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}}.$$

Then, the system (2.1) is practically $\psi-$exponentially asymptotically stable.

Case 3: : If $r > q$ and $q < p$, then from Condition i) it follows that

$$\|\psi(t)x\|^q\Big(\lambda_1\|\psi(t)x\|^{p-q} - \lambda_2\Big) \leq a. \tag{A.2}$$

If $a = 0$, then by (A.2), we get

$$\|\psi(t)x\| \leq \sqrt[p-q]{\frac{\lambda_2}{\lambda_1}}, t \in \mathbb{R}_+, x \in \mathbb{R}^n.$$

In particular, the solution $x(t,t_0,x_0)$ is uniformly $\psi-$bounded.

On the other hand, if $a > 0$, we distinguish two subcases: whether the function $\psi(t)x(t)$ start from outside or inside the ball $\mathcal{B}_1$

- $\|\psi(t_0)x_0\| > 1$. In this case, there exists $T_0 \in]0,+\infty]$ such that $\|\psi(t)x(t)\| > 1$ for all $t \in [t_0, t_0+T_0[$ and $\|\psi(t_0+T_0)x(t_0+T_0)\| = 1$. Hence, we get that

$$\|\psi(t)x(t)\|^q\big(\lambda_1\|\psi(t)x(t)\|^{p-q} - \lambda_2\big) \leq a \leq a\|\psi(t)x(t)\|^p, \forall t \in [t_0, t_0+T_0[.$$

  Then,

$$\|\psi(t)x(t)\|^q\big(\lambda_1\|\psi(t)x(t)\|^{p-q} - \lambda_2 - a\big) \leq 0, \forall t \in [t_0, t_0+T_0[.$$

  Therefore,

$$\|\psi(t)x(t)\| \leq \sqrt[p-q]{\frac{\lambda_2 + a}{\lambda_1}}, \forall t \in [t_0, t_0+T_0[.$$

  In addition, for each $t \geq t_0+T_0$, either $\|\psi(t)x(t)\| \leq 1$, or $\|\psi(t)x(t)\| > 1$. In this second case, as in the **case 2** we get, whenever if $\|\psi(t)x(t)\| > 1$, it holds that

$$\|\psi(t)x(t)\| \leq \sqrt[p]{\frac{\lambda_2\lambda_3 + a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}}.$$

- $\|\psi(t_0)x_0\| \leq 1$. In this case, as long as $\|\psi(t)x(t)\| \leq 1$, we have trivially that

$$\|\psi(t)x(t)\| \leq \sqrt[p-q]{\frac{\lambda_2 + a}{\lambda_1}} \leq \sqrt[p]{\frac{\lambda_2\lambda_3 + a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}}.$$

If $\|\psi(t)x(t)\| > 1$ at some instant $t > t_0$, then, as in the **case 2**, we obtain that, whenever $\|\psi(t)x(t)\| > 1$, it holds that

$$\|\psi(t)x(t)\| \leq \sqrt[p]{\frac{\lambda_2\lambda_3 + a\lambda_3 + b\lambda_2}{\lambda_1\lambda_3}}.$$

Therefore, the solution $x(t,t_0,x_0)$ is uniformly $\psi-$bounded

Case 4: : If $r > q$ and $p < q$, then from Condition i) it follows that

$$\|\psi(t)x\|^p\left(\lambda_1 - \lambda_2\|\psi(t)x\|^{q-p}\right) \leq a. \tag{A.3}$$

If $a = 0$, then from (A.3), we have for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$

$$\|\psi(t)x\| \geq \eta := \sqrt[q-p]{\frac{\lambda_1}{\lambda_2}}.$$

Then, for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, we have

$$\begin{aligned}
\dot{V}_{(2.1)}(t,x) &\leq -\lambda_3\|\psi(t)x\|^r + b,\\
&= -\lambda_3\|\psi(t)x\|^{r-q}\|\psi(t)x\|^q + b,\\
&\leq -\lambda_3\eta^{r-q}\|\psi(t)x\|^q + b,\\
&\leq -\frac{\lambda_3\eta^{r-q}}{\lambda_2}V(t,x) + b.
\end{aligned}$$

By Lemma 1 with $p(t) = -\frac{\lambda_3\eta^{r-q}}{\lambda_2}$ and $q(t) = b$, we obtain

$$V(t,x(t)) \leq V(t_0,x_0)e^{-\frac{\lambda_3\eta^{r-q}}{\lambda_2}(t-t_0)} + \frac{b\lambda_2}{\lambda_3\eta^{r-q}}\left(1 - e^{-\frac{\lambda_3\eta^{r-q}}{\lambda_2}(t-t_0)}\right), t \geq t_0, x_0 \in \mathbb{R}^n.$$

Therefore, from the condition i) of Theorem 1 and condition a) of Lemma 2, we have

$$\|\psi(t)x(t,t_0,x_0)\| \leq \sqrt[p]{\frac{\lambda_2}{\lambda_1}}\|\psi(t_0)x_0)\|^{\frac{q}{p}}e^{-\frac{\lambda_3\eta^{r-q}}{p\lambda_2}(t-t_0)} + \sqrt[p]{\frac{b\lambda_2}{\lambda_1\lambda_3\eta^{r-q}}}, t \geq t_0, x_0 \in \mathbb{R}^n.$$

If $a > 0$, we distinguish two subcases: whether the function $\psi(t)x(t)$ start from outside or inside the closed ball $\mathcal{B}_1$.

- $\|\psi(t_0)x_0\| > 1$: In this case, there exists $T_0 \in]0,+\infty]$ such that

$$\|\psi(t)x(t)\| \geq 1, \forall t \in [t_0, t_0 + T_0[\text{ and } \|\psi(t_0+T_0)x(t_0+T_0)\| = 1. \tag{A.4}$$

Then, we have

$$\|\psi(t)x(t)\|^p\left(\lambda_1 - \lambda_2\|\psi(t)x(t)\|^{q-p}\right) \leq a \leq a\|\psi(t)x(t)\|^q, \forall t \in [t_0, t_0 + T_0[.$$

Therefore,

$$\|\psi(t)x(t)\|^p\big(\lambda_1-(\lambda_2+a)\|\psi(t)x(t)\|^{q-p}\big)\leq 0,\ \forall t\in[t_0,t_0+T_0[.$$

Then, for all $t\in[t_0,t_0+T_0[$, we have $\|\psi(t)x(t)\|\geq\eta:=\sqrt[q-p]{\dfrac{\lambda_1}{\lambda_2+a}}$.

Hence, for all $t\in[t_0,t_0+T_0[$, we obtain

$$D^+V(t,x(t))\leq-\frac{\lambda_3\eta^{r-q}}{\lambda_2}V(t,x(t))+b.$$

By Lemma 1 with $p(t)=-\frac{\lambda_3\eta^{r-q}}{\lambda_2}$ and $q(t)=b$, we obtain

$$V(t,x(t))\leq V(t_0,x_0)e^{-\frac{\lambda_3\eta^{r-q}}{\lambda_2}(t-t_0)}+\frac{b\lambda_2}{\lambda_3\eta^{r-q}}\big(1-e^{-\frac{\lambda_3\eta^{r-q}}{\lambda_2}(t-t_0)}\big),\ t\in[t_0,t_0+T_0[.$$

Therefore, from condition i) of Theorem 1 and condition a) of Lemma 2, we have

$$\|\psi(t)x(t,t_0,x_0)\|\leq\sqrt[p]{\frac{\lambda_2}{\lambda_1}}\|\psi(t_0)x_0)\|^{\frac{q}{p}}e^{-\frac{\lambda_3\eta^{r-q}}{p\lambda_2}(t-t_0)}+\sqrt[p]{\frac{b\lambda_2}{\lambda_1\lambda_3\eta^{r-q}}},t\in[t_0,t_0+T_0[.$$

In addition, for each $t\geq t_0+T_0$, either $\|\psi(t)x\|\leq 1$ in which case $\|\psi(t)x(t)\|\leq\sqrt[p]{\frac{\lambda_2\lambda_3+a\lambda_3+b\lambda_2}{\lambda_1\lambda_3}}$, or $\|\psi(t)x\|>1$. In this second case, as in the **case 2**, we obtain, whenever $\|\psi(t)x(t)\|>1$, it holds that

$$\|\psi(t)x(t,t_0,x_0)\|\leq\sqrt[p]{\frac{\lambda_2\lambda_3+a\lambda_3+b\lambda_2}{\lambda_1\lambda_3}}.$$

- $\|\psi(t_0)x_0\|\leq 1$: As in the previous cases, we get, as long as $\|\psi(t)x(t)\|\leq 1$, we have trivially that $\|\psi(t)x(t,t_0,x_0)\|\leq\sqrt[p]{\frac{\lambda_2\lambda_3+a\lambda_3+b\lambda_2}{\lambda_1\lambda_3}}$. If $\|\psi(t)x(t)\|>1$ at some instant $t>t_0$, then, as in the **case 2**, we obtain, whenever $\|\psi(t)x(t)\|>1$, it holds that

$$\|\psi(t)x(t,t_0,x_0)\|\leq\sqrt[p]{\frac{\lambda_2\lambda_3+a\lambda_3+b\lambda_2}{\lambda_1\lambda_3}}.$$

To sum up, there exist $\lambda,R>0$, such that for all $t\geq t_0\geq 0$ and $x_0\in\mathbb{R}^n$, we have the following:

$$\|\psi(t)x(t,t_0,x_0)\|\leq\sqrt[p]{\frac{\lambda_2}{\lambda_1}}\|\psi(t_0)x_0\|^{\frac{q}{p}}e^{-\frac{\lambda_3}{p\lambda_2}(t-t_0)}+R.$$

Then, the system (2.1) practically $\psi^{\frac{q}{p}}-$exponentially asymptotically stable.

□

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EXISTENCE AND MULTIPLICITY OF POSITIVE WEAK SOLUTIONS FOR A NEW CLASS OF (P; Q)-LAPLACIAN SYSTEMS

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Abstract. The paper is concerned with the existence of positive weak solutions for a new class of (p,q)-Laplacian elliptic systems in a bounded domain by means of the method of sub-super solutions. Particularly, we do not need any sign conditions for $\gamma(0), g(0), f(0)$ and $h(0)$. Moreover, a multiplicity result is obtained when $\gamma(0) = g(0) = f(0) = h(0) = 0$. Finally, we give some examples to verify our main results.

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1. Introduction

In this paper, we deal with the existence and multiplicity of positive weak solutions for the following (p,q)-Laplacian systems

$$\begin{cases} -\triangle_p u - |u|^{p-2} u = \lambda_1 a(x) f(v) + \mu_1 \alpha(x) h(u) \text{ in } \Omega, \\ -\triangle_q v - |v|^{q-2} v = \lambda_2 b(x) g(u) + \mu_2 \beta(x) \gamma(v) \quad \text{in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\triangle_s z = \operatorname{div}\left(|\nabla z|^{s-2} \nabla z\right), s > 1, \Omega \subset \mathbb{R}^N \ (N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega, a(x), b(x), \alpha(x), \beta(x) \in C(\overline{\Omega})$, $\lambda_1, \lambda_2, \mu_1, ,\mu_2 \geq 0$, $1 < p, q < \infty$.

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The study of (p,q)-Laplacian systems is a new and interesting topic. It arises from electrorheological fluids, nonlinear elasticity theory, etc. (see [5], [15] [1]-[6]). A lot of existence results have been obtained on this class of problems, we refer to ([4], [5], [7], [10], [13], [14], [3], [2], [11], [9], [8]). These problems originate from physical models and are widely used in many fields such as combustion, mathematical biology, chemical reactions and so on. Our method is mainly focused on the method of sub-super solutions (see [10], [11] for a more detailed discussion).

As far as we know, there are very few contributions devoted to the (p,q)-Laplacian nonlinear elliptic system. Therefore, with the help of the method of sub-super solutions method, we are inspired by the paper of [12] in which a new (p,q)-Laplacian system was discussed and extended our previous results to problem (1.1) without assuming any sign conditions for $h(0)$, $g(0)$, $f(0)$, and $\gamma(0)$. Furthermore, when $h(0)=f(0)=g(0)=\gamma(0)=0$, a multiplicity result is given.

The outline of the paper is organized as follows: Sec. 2 introduces some definitions and make appropriate assumptions, which will be used in the body of the paper. In addition, we show the proof of two important results. Sec. 3, we will illustrate our main results with some interesting examples.

2. Main results

First, in order to get our main results, we will consider the following hypothesis:
$(H1)$ There exist $a(x)$, $\alpha(x)$, $b(x)$, $\beta(x) \in C(\overline{\Omega})$ such that

$$a(x) \geq a_1 > 0,\ b(x) \geq b_1 > 0,$$
$$\alpha(x) \geq \alpha_1 > 0,\ \beta(x) \geq \beta_1 > 0.$$

$(H2)$ Let f, g, h, $\gamma \in C^1([0,\infty))$ be monotone functions satisfying

$$\lim_{s\to+\infty} f(s) = \lim_{s\to+\infty} g(s) = \lim_{s\to+\infty} h(s) = \lim_{s\to+\infty} \gamma(s) = +\infty.$$

$(H3)$ $\lim_{s\to+\infty} \frac{f\left(M(g(s))^{\frac{1}{q-1}}\right)}{s^{p-1}} = 0,\ \forall M > 0.$

$(H4)$ $\lim_{s\to+\infty} \frac{h(s)}{s^{p-1}} = \lim_{s\to+\infty} \frac{\gamma(s)}{s^{q-1}} = 0.$

Next, we define weak solutions and sub-super solutions in (p,q)-Laplacian elliptic systems.

Definition 1. Let $(u,v) \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$, we say that (u,v) is a weak solution of problem (1.1), if

$$\begin{aligned}&\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\xi dx-\int_\Omega |u|^{p-2}u.\xi dx\\&=\lambda_1\int_\Omega a(x)f(v)\xi dx+\mu_1\int_\Omega \alpha(x)h(u)\xi dx \text{ in } \Omega,\end{aligned}$$

$$\begin{aligned}&\int_\Omega |\nabla v|^{q-2}\nabla v.\nabla\zeta dx-\int_\Omega |v|^{q-2}v.\zeta dx\\&=\lambda_2\int_\Omega b(x)g(u)\zeta dx+\mu_2\int_\Omega \beta(x)\gamma(v)\zeta dx \text{ in } \Omega\end{aligned}$$

for all $(\xi,\zeta)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$.

Definition 2. The nonnegative functions $(\underline{u},\underline{v}),(\overline{u},\overline{v})$ in $W^{1,p}(\Omega)\cap C(\overline{\Omega})\times W^{1,q}(\Omega)\cap C(\overline{\Omega})$ are called a weak subsolution and supersolution of problem (1.1) if they satisfy $(\underline{u},\underline{v}),(\overline{u},\overline{v})=(0,0)$ on $\partial\Omega$

$$\begin{aligned}&\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u}.\nabla\xi dx-\int_\Omega |\underline{u}|^{p-2}\underline{u}.\xi dx\\&\leq\lambda_1\int_\Omega a(x)f(\underline{v})\xi dx+\mu_1\int_\Omega \alpha(x)h(\underline{u})\xi dx \text{ in } \Omega,\end{aligned}$$

$$\begin{aligned}&\int_\Omega |\nabla \underline{v}|^{q-2}\nabla \underline{v}.\nabla\zeta dx-\int_\Omega |\underline{v}|^{q-2}\underline{v}.\zeta dx\\&\leq\lambda_2\int_\Omega b(x)g(\underline{u})\zeta dx+\mu_2\int_\Omega \beta(x)\gamma(\underline{v})\zeta dx \text{ in } \Omega\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega |\nabla \overline{u}|^{p-2}\nabla \overline{u}.\nabla\xi dx-\int_\Omega |\overline{u}|^{p-2}\overline{u}.\xi dx\\&\geq\lambda_1\int_\Omega a(x)f(\overline{v})\xi dx+\mu_1\int_\Omega \alpha(x)h(\overline{u})\xi dx \text{ in } \Omega,\end{aligned}$$

$$\begin{aligned}&\int_\Omega |\nabla \overline{v}|^{q-2}\nabla \overline{v}.\nabla\zeta dx-\int_\Omega |\overline{v}|^{q-2}\overline{v}.\zeta dx\\&\geq\lambda_2\int_\Omega b(x)g(\overline{u})\zeta dx+\mu_2\int_\Omega \beta(x)\gamma(\overline{v})\zeta dx \text{ in } \Omega\end{aligned}$$

for any $(\xi,\zeta)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$.

In what follows, we shall establish the following the existence result.

Theorem 1. *Let $(H1)-(H4)$ hold. If $\lambda_1+\mu_1$ and $\lambda_2+\mu_2$ are big enough, then problem (1.1) processes a positive weak solution.*

Proof. We will show that there exist a positive weak subsolution$(\underline{u},\underline{v})\in W^{1,p}(\Omega)\cap C(\overline{\Omega})\times W^{1,q}(\Omega)\cap C(\overline{\Omega})$ and a supersolution $(\overline{u},\overline{v})\in W^{1,p}(\Omega)\cap C(\overline{\Omega})\times W^{1,q}(\Omega)\cap C(\overline{\Omega})$ of (1.1) such that $\underline{u}\leq\overline{u}$, $\underline{v}\leq\overline{v}$. Moreover, $(\underline{u},\underline{v}),(\overline{u},\overline{v})$ satisfy $(\underline{u},\underline{v})=(0,0)=(\overline{u},\overline{v})$ on $\partial\Omega$.

Let σ_r be the first eigenvalue of $-\triangle_r$, and $\phi_r > 0$ the corresponding eigenfunction with $\|\phi_r\| = 1$ for $r = p,q$. There exist $m,\eta,\delta > 0$ such that $|\nabla\phi_r|^r - \sigma_r\,\phi_r \geq m$ on $\overline{\Omega_\delta} = \{x \in \Omega, d(x,\partial\Omega) \leq \delta\}$ and $\phi_r \geq \eta$ on $\Omega\backslash\overline{\Omega}_\delta$ for $r = p,q$. Taking $k_0 > 0$ such that $a_1 f(t), \alpha_1 h(t), b_1 g(t), \beta_1\gamma(t) > -k_0$.

First, we claim that

$$(\underline{u},\underline{v}) := \left(\left[\frac{(\lambda_1+\mu_1)k_0}{m}\right]^{1/p-1}\left(\frac{p-1}{p}\right)\phi_p^{p/p-1},\right.$$
$$\left.\left[\frac{(\lambda_2+\mu_2)k_0}{m}\right]^{1/q-1}\left(\frac{q-1}{q}\right)\phi_q^{q/q-1}\right) \tag{2.1}$$

is a subsolution of problem (1.1) when $\lambda_1+\mu_1$ and $\lambda_2+\mu_2$ are big enough. Taking the test function $\xi(x) \in W_0^{1,p}(\Omega)$ with $\xi(x) \geq 0$. Thus, from $(H1)$ we get

$$\begin{aligned}
&\int_\Omega |\nabla\underline{u}|^{p-2}\nabla\underline{u}.\nabla\xi dx - \int_\Omega |\underline{u}|^{p-2}\underline{u}.\xi dx \leq \int_\Omega |\nabla\underline{u}|^{p-2}\nabla\underline{u}.\nabla\xi dx\\
&= \left(\tfrac{(\lambda_1+\mu_1)k_0}{m}\right)\int_\Omega \left\{\sigma_p\phi_p^p - |\nabla\phi_p|^p\right\}\xi dx\\
&+ \left(\tfrac{(\lambda_1+\mu_1)k_0}{m}\right)\int_{\Omega\backslash\overline{\Omega}_\delta} \left\{\sigma_p\phi_p^p - |\nabla\phi_p|^p\right\}\xi dx.
\end{aligned}$$

We have known that $|\nabla\phi_r|^r - \sigma_r\,\phi_r \geq m$ for $s = p,q$, on $\overline{\Omega_\delta}$. Also on $\Omega\backslash\overline{\Omega}_\delta$ $\phi_r \geq \eta$ for $r = p,q$. If $\lambda_1+\mu_1$ and $\lambda_2+\mu_2$ are big enough in the definition of $\underline{u}$, $\underline{v}$, then by $(H2)$ we get

$$a_1 f(\underline{v}), \alpha_1 h(\underline{u}), b_1 g(\underline{u}), \beta_1\gamma(\underline{v}) \geq \frac{k_0}{m}\max\left\{\sigma_p,\sigma_q\right\}. \tag{2.2}$$

Therefore,

$$\begin{aligned}
&\int_\Omega |\nabla\underline{u}|^{p-2}\nabla\underline{u}.\nabla\xi dx - \int_\Omega |\underline{u}|^{p-2}\underline{u}.\xi dx\\
&\leq \left(\tfrac{(\lambda_1+\mu_1)k_0}{m}\right)\int_{\Omega_\delta} \left\{\sigma_p\phi_p^p - |\nabla\phi_p|^p\right\}\xi dx\\
&+ \left(\tfrac{(\lambda_1+\mu_1)k_0}{m}\right)\int_{\Omega\backslash\overline{\Omega}_\delta} \left\{\sigma_p\phi_p^p - |\nabla\phi_p|^p\right\}\xi dx\\
&\leq -(\lambda_1+\mu_1)k_0\int_{\Omega_\delta}\xi dx + \left(\tfrac{(\lambda_1+\mu_1)k_0}{m}\right)\int_{\Omega\backslash\overline{\Omega}_\delta}\sigma_p\xi dx\\
&\leq \int_{\Omega_\delta}[\lambda_1 a(x) f(\underline{v}) + \mu_1\alpha(x) h(\underline{u})\ ]\xi dx\\
&+ \int_{\Omega\backslash\overline{\Omega}_\delta}[\lambda_1 a(x) f(\underline{v}) + \mu_1\alpha(x) h(\underline{u})\ ]\xi dx\\
&= \int_\Omega[\lambda_1 a(x) f(\underline{v}) + \mu_1\alpha(x) h(\underline{u})\ ]\xi dx
\end{aligned}$$

Similarly,

$$\int_{\Omega} |\nabla v|^{q-2} \nabla v.\nabla \zeta dx - \int_{\Omega} |v|^{q-2} v.\zeta dx \leq \int_{\Omega} [\lambda_2 b(x) g(\underline{u}) + \mu_2 \beta(x) \gamma(\underline{v})] \zeta dx.$$

Thus $(\underline{u}, \underline{v})$ is a subsolution of problem (1.1).
Next, let ω_r be a unique positive solution of

$$\begin{cases} -\triangle_r \omega_r = 1 \text{ in } \Omega, \\ \omega_r = 0 \qquad \text{on } \partial\Omega. \end{cases}$$

for $r = p, q$. We denote

$$\overline{u} = \frac{C}{\nu_p} \left(\frac{\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty}{1 - \nu_p^{p-1}} \right)^{\frac{1}{p-1}} \omega_p, \tag{2.3}$$

$$\overline{v} = \left[\left(\frac{\lambda_2 \|b\|_\infty + \mu_2 \|\beta\|_\infty}{1 - \nu_q^{q-1}} \right) g \left(C \left(\frac{\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty}{1 - \nu_p^{p-1}} \right)^{\frac{1}{p-1}} \right)^{\frac{1}{q-1}} \right] \omega_q, \tag{2.4}$$

where $\nu_r = \|\omega_r\|_\infty$, $r = p, q$ and $C > 0$ is big enough. We claim that $(\overline{u}, \overline{v})$ is a supersolution of (1.1) such that $(\overline{u}, \overline{v}) \geq (\underline{u}, \underline{v})$.
According to $(H3) - (H4)$, we can make C big enough so that

$$\begin{aligned} \left(\frac{C}{\nu_p}\right)^{p-1} &\geq f\left(\left[\left(\frac{\lambda_2 \|b\|_\infty + \mu_2 \|\beta\|_\infty}{1 - \nu_q^{q-1}} \right) g \left(C \left(\frac{\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty}{1 - \nu_p^{p-1}} \right)^{\frac{1}{p-1}} \right)^{\frac{1}{q-1}} \right] \omega_q \right) \\ &+ \mu_1 h \left(\frac{\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty}{1 - \nu_p^{p-1}} \right)^{\frac{1}{p-1}} \omega_p. \end{aligned} \tag{2.5}$$

Hence

$$\int_{\Omega} |\nabla \overline{u}|^{p-2} \nabla \overline{u}.\nabla \xi dx - \int_{\Omega} |\overline{u}|^{p-2} \overline{u}.\xi dx = \left(\frac{C}{\nu_p} \right)^{p-1} (\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty) \int_{\Omega} \xi dx.$$

Using (2.5)

$$\begin{aligned}
&\int_\Omega |\nabla\overline{u}|^{p-2}\nabla\overline{u}.\nabla\xi dx-\int_\Omega |\overline{u}|^{p-2}\overline{u}.\xi dx\\
&\geq \lambda_1 \|a\|_\infty f\left(\left[\left(\frac{\lambda_2\|b\|_\infty+\mu_2\|\beta\|_\infty}{1-\nu_q^{q-1}}\right)g\left(C\left(\frac{\lambda_1\|a\|_\infty+\mu_1\|\alpha\|_\infty}{1-\nu_p^{p-1}}\right)^{\frac{1}{p-1}}\right)^{\frac{1}{q-1}}\right]\omega_q\right)\times\\
&\int_\Omega \xi dx+\mu_1\|\alpha\|_\infty\int_\Omega h\left(C\left(\frac{\lambda_1\|a\|_\infty+\mu_1\|\alpha\|_\infty}{1-\nu_p^{p-1}}\right)^{\frac{1}{p-1}}\right)\xi dx\\
&\geq \int_\Omega \left[\lambda_1 a(x) f(\overline{v})+\mu_1\alpha(x)h(\overline{u})\right]\xi dx.
\end{aligned}\tag{2.6}$$

Next

$$\begin{aligned}
&\int_\Omega |\nabla\overline{v}|^{q-2}\nabla\overline{v}.\nabla\zeta dx-\int_\Omega |\overline{v}|^{q-2}\overline{v}.\zeta dx\\
&=\left\{(\lambda_2\|b\|_\infty+\mu_2\|\beta\|_\infty)g\left(C\left(\frac{\lambda_1\|a\|_\infty+\mu_1\|\alpha\|_\infty}{1-\nu_p^{p-1}}\right)^{\frac{1}{p-1}}\right)\right\}\omega_q\int_\Omega \xi dx\\
&\geq\left[\lambda_2\|b\|_\infty g\left(C\left(\frac{\lambda_1\|a\|_\infty+\mu_1\|\alpha\|_\infty}{1-\nu_p^{p-1}}\right)^{\frac{1}{p-1}}\right)\right.\\
&\left.+\mu_2\|\beta\|_\infty g\left(C\left(\frac{\lambda_1\|a\|_\infty+\mu_1\|\alpha\|_\infty}{1-\nu_p^{p-1}}\right)^{\frac{1}{p-1}}\right)\right]\int_\Omega \xi dx.
\end{aligned}\tag{2.7}$$

According to $(H4)$ and choose C big enough, we obtain

$$\begin{aligned}
&g\left(C\left(\frac{\lambda_1\|a\|_\infty+\mu_1\|\alpha\|_\infty}{1-\nu_p^{p-1}}\right)^{\frac{1}{p-1}}\right)\\
&\geq\gamma\left(\left[\left(\frac{\lambda_2\|b\|_\infty+\mu_2\|\beta\|_\infty}{1-\nu_q^{q-1}}\right)g\left(C\left(\frac{\lambda_1\|a\|_\infty+\mu_1\|\alpha\|_\infty}{1-\nu_p^{p-1}}\right)^{\frac{1}{p-1}}\right)^{\frac{1}{q-1}}\right]\|\omega_q\|_\infty\right).
\end{aligned}$$

Then from (2.6) we get

$$\begin{aligned}
&\int_\Omega |\nabla\overline{v}|^{q-2}\nabla\overline{v}.\nabla\zeta dx-\int_\Omega |\overline{v}|^{q-2}\overline{v}.\zeta dx\\
&\geq \lambda_2\|b\|_\infty g\left(C\left(\frac{\lambda_1\|a\|_\infty+\mu_1\|\alpha\|_\infty}{1-\nu_p^{p-1}}\right)^{\frac{1}{p-1}}\right)\\
&+\mu_2\|\beta\|_\infty\gamma\left(\left\{\left(\frac{\lambda_2\|b\|_\infty+\mu_2\|\beta\|_\infty}{1-\nu_q^{q-1}}\right)g\left(C\left(\frac{\lambda_1\|a\|_\infty+\mu_1\|\alpha\|_\infty}{1-\nu_p^{p-1}}\right)^{\frac{1}{p-1}}\right)\right\}^{\frac{1}{q-1}}\|\omega_q\|_\infty\right)\\
&\geq \int_\Omega [b(x)g(\overline{u})+\mu_2\beta(x)\gamma(\overline{v})]\zeta dx.
\end{aligned} \tag{2.8}$$

According to (2.6) and (2.7), we can conclude that $(\overline{u},\overline{v})$ is a supersolution of (1.1). Further $\overline{u}\geq\underline{u}$ and $\overline{v}\geq\underline{v}$ for C big enough. Thus, we get a solution $(u,v)\in W^{1,p}(\Omega)\cap C(\overline{\Omega})\times W^{1,q}(\Omega)\cap C(\overline{\Omega})$ of (1.1) with $\underline{u}\leq u\leq\overline{u}$, and $\underline{v}\leq v\leq\overline{v}$. The proof of theorem 1 is complete. □

Now we show that the more general system (1.1) possesses at least two distinct positive solutions.

Theorem 2. *Suppose that the conditions $(H1)-(H4)$ hold. Let f,g,h, and γ the function be smooth enough around zero with*

$$\begin{aligned}
f(0)=h(0)=g(0)=\gamma(0)=0\\
=f^{(k)}(0)=h^{(k)}(0)=g^{(l)}(0)=\gamma^{(l)}(0)
\end{aligned}$$

for $k=1,2,....[p-1]$, $l=1,2,....[q-1]$, where $[s]$ denotes the integer part of s. Then, problem (1.1) processes at least two positive solutions when $\lambda_i+\mu_i$ are big enough; $i=1,2$.

Proof. For problem (1.1), we will look for a strict supersolution (ζ_1,ζ_2), a subsolution (ψ_1,ψ_2), a supersolution (z_1,z_2), and a strict subsolution (ω_1,ω_2), such that

$$\begin{aligned}
(\psi_1,\psi_2)\leq(\zeta_1,\zeta_2)\leq(z_1,z_2),\\
(\psi_1,\psi_2)\leq(\omega_1,\omega_2)\leq(z_1,z_2),
\end{aligned}$$

and $(\omega_1,\omega_2)\nleq(\zeta_1,\zeta_2)$. Then, problem (1.1) processes at least three distinct solutions (u_i,v_i), $i=1,2,3$, such that

$$\begin{aligned}
(u_1,v_1)\in[(\psi_1,\psi_2),(\zeta_1,\zeta_2)],\\
(u_2,v_2)\in[(\omega_1,\omega_2),(z_1,z_2)]
\end{aligned}$$

and

$$(u_3,v_3)\in[(\psi_1,\psi_2),(z_1,z_2)]\diagdown([(\psi_1,\psi_2),(\zeta_1,\zeta_2)]\cup(\omega_1,\omega_2),(z_1,z_2)).$$

It is obvious that $(\psi_1,\psi_2)=(0,0)$ is a (sub)solution. Moreover, we always can find a big supersolution $(z_1,z_2)=(\overline{u},\overline{v})$. Consider

$$\begin{cases} -\triangle_p \omega_1 - |\omega_1|^{p-2}\omega_1 = \lambda_1 a(x)\widetilde{f}(\omega_2) + \mu_1 \alpha(x)\widetilde{h}(\omega_1) \text{ in } \Omega, \\ -\triangle_q \omega_2 - |\omega_2|^{q-2}\omega_2 = \lambda_2 b(x)\widetilde{g}(\omega_1) + \mu_2 \beta(x)\widetilde{\gamma}(\omega_2) \text{ in } \Omega, \\ \omega_1 = \omega_2 = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.9}$$

where $\widetilde{g}(s) = g(s) - 1, \widetilde{\gamma}(s) = \gamma(s) - 1, \widetilde{h}(s) = h(s) - 1, \widetilde{f}(s) = f(s) - 1$. Then by Theorem 1, when $\lambda_i + \mu_i$ are big enough, we know that the problem (2.10) processes a solution $(\omega_1, \omega_2) > 0$ $i = 1, 2$. It is clear that (ω_1, ω_2) is a strict subsolution of problem (1.1).

In the end, we will find a strict supersolution (ζ_1, ζ_2).

Let ϕ_p, ϕ_q be the corresponding eigenfunction with respect to operators $\triangle_p$ and $\triangle_q$ and there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\phi_p \le C_1 \phi_q \quad \text{and} \quad \phi_q \le C_2 \phi_p. \tag{2.10}$$

Let $(\zeta_1, \zeta_2) = (\rho\phi_p, \rho\phi_q)$, where $\rho > 0$,

$$G_p(x) := (\sigma_p - 1)x^{p-1} - \lambda_1 f(C_2 x) - \mu_1 h(x)$$

and

$$G_q(x) := (\sigma_q - 1)x^{q-1} - \lambda_2 g(C_1 x) - \mu_2 \gamma(x).$$

Note that $G_p(0) = G_q(0) = 0$, $G_p^{(k)}(0) = G_q^{(l)}(0) = 0$ for $k = 1, 2, \ldots. [p-1]$ and $l = 1, 2, \ldots. [q-1]$.

$$\begin{cases} G_p^{(p-1)}(0) > 0 \text{ and } G_q^{(q-1)}(0) > 0 \text{ if } \quad p, q \in \mathbb{Z}^+, \\ \lim_{r\to+\infty} G_p^{([p])}(r) = +\infty = \lim_{r\to+\infty} G_p^{([q])}(r) \text{ if } \quad p, q \notin \mathbb{Z}^+. \end{cases}$$

Hence, there exists θ such that $G_q(x) > 0$ and $G_p(x) > 0$ for any $x \in (0, \theta]$. So, for $0 < \rho \le \theta$ we get

$$(\sigma_p - 1)\zeta_1^{p-1} = (\sigma_p - 1)(\rho\phi_p)^{p-1} > \lambda_1 f(C_2\rho\phi_p) - \mu_1 h(\rho\phi_p).$$

By (2.10) and the monotonicity of function f, we obtain

$$\begin{aligned} &(\sigma_p - 1)\zeta_1^{p-1} = (\sigma_p - 1)(\rho\phi_p)^{p-1} \\ &> \lambda_1 f(C_2\rho\phi_p) - \mu_1 h(\rho\phi_p) \\ &\ge \lambda_1 f(\rho\phi_q) - \mu_1 h(\rho\phi_p) = \lambda_1 f(\zeta_2) - \mu_1 h(\zeta_1) \end{aligned} \tag{2.11}$$

for any $x \in \Omega$. In the same way, we also have

$$\begin{aligned} &(\sigma_q - 1)\zeta_2^{q-1} = (\sigma_q - 1)(\rho\phi_q)^{q-1} \\ &> \lambda_2 g(C_1\rho\phi_q) - \mu_2 \gamma(\rho\phi_q) \\ &\ge \lambda_2 g(\rho\phi_p) - \mu_2 \gamma(\rho\phi_q) = \lambda_2 g(\zeta_1) - \mu_2 \gamma(\zeta_2), \end{aligned} \tag{2.12}$$

for any $x \in \Omega$. Making use of (2.11) and (2.12), we obtain

$$\begin{aligned}
&\int_\Omega |\nabla\zeta_1|^{p-2}\nabla\zeta_1.\nabla\xi dx - \int_\Omega |\zeta_1|^{p-2}\zeta_1.\xi dx \\
&= \rho^{p-1}\left\{\int_\Omega |\nabla\phi_p|^{p-2}\nabla\phi_p.\nabla\xi dx - \int_\Omega |\phi_p|^{p-2}\phi_p.\xi dx\right\} \\
&= \int_\Omega \left\{\sigma_p\left(\rho\phi_p\right)^{p-1} - \left(\rho\phi_p\right)^{p-1}\right\}\xi dx.
\end{aligned}$$

Since because $\phi_p > 0$, we have

$$\begin{aligned}
&\int_\Omega \left\{\sigma_p\left(\rho\phi_p\right)^{p-1} - \left(\rho\phi_p\right)^{p-1}\right\}\xi dx \\
&= \int_\Omega \left\{\left(\sigma_p - 1\right)\left(\rho\phi_p\right)^{p-1}\right\}\xi dx \\
&> \lambda_1 \int_\Omega f\left(\zeta_2\right)\xi dx - \mu_1 \int_\Omega h\left(\zeta_1\right).\xi dx,
\end{aligned}$$

Similarly we also have

$$\int_\Omega |\nabla\zeta_2|^{q-2}\nabla\zeta_2.\nabla\xi dx - \int_\Omega |\zeta_2|^{q-2}\zeta_2.\xi dx > \lambda_2 \int_\Omega g\left(\zeta_1\right)\xi dx - \mu_2 \int_\Omega \gamma(\zeta_2)\,\xi dx.$$

It follows that (ζ_1,ζ_2) is a strict supersolution. Let ρ small enough so that $(\omega_1,\omega_2) \nleq (\zeta_1,\zeta_2)$. So, we can find solutions

$$(u_1,v_1) \in \left[(\psi_1,\psi_2),(\zeta_1,\zeta_2)\right],(u_2,v_2) \in \left[(\omega_1,\omega_2),(z_1,z_2)\right]$$

and

$$(u_3,v_3) \in \left[(\psi_1,\psi_2),(z_1,z_2)\right] \diagdown \left(\left[(\psi_1,\psi_2),(\zeta_1,\zeta_2)\right] \cup (\omega_1,\omega_2),(z_1,z_2)\right).$$

This fact that $(u_1,v_1) \equiv (\psi_1,\psi_2) \equiv (0,0)$ can happen due to $(\psi_1,\psi_2) \equiv (0,0)$ is a solution. So, anyway we can find two positive solutions (u_2,v_2) and (u_3,v_3). Therefore, we conclude the proof of Theorem2. □

3. Examples

Example 1. Let

$$f(x) = \sum_{i=1}^{m} a_i x^{p_i} - c_1,\ g(x) = \sum_{j=1}^{n} b_j x^{qj} - c_2$$

$$h(x) = \sum_{k=1}^{s} \alpha_k x^{r_k} - c_3,\ \gamma(x) = \sum_{l=1}^{\tau} \beta_l x^{d_l} - c_4,$$

where

$$d_j < (q-1), r_k < (p-1), p_i q_j < (p-1)(q-1)$$

and

$$a_i, b_j, \alpha_k, \beta_l, p_i, q_j, r_k, d_j, c_1, c_2, c_3, c_4 \geq 0.$$

So, it is clear that f, g, h and γ fulfill the assumptions of Theorem 1.

Example 2. Let

$$f(x) = \begin{cases} x^{p_1}, \ x \leq 1, \\ \frac{p_1}{p_2} x^{p_2} + \left(1 - \frac{p_1}{p_2}\right), \ x > 1, \end{cases}, \ h(x) = \begin{cases} x^{p_3}, \ x \leq 1, \\ \frac{p_3}{p_4} x^{p_4} + \left(1 - \frac{p_3}{p_4}\right), \ x > 1, \end{cases},$$

$$g(x) = \begin{cases} x^{q_1}, \ x \leq 1, \\ \frac{q_1}{q_2} x^{q_2} + \left(1 - \frac{q_1}{q_2}\right), \ x > 1, \end{cases}, \ \gamma(x) = \begin{cases} x^{q_3}, \ x \leq 1, \\ \frac{q_3}{q_4} x^{q_4} + \left(1 - \frac{q_3}{q_4}\right), \ x > 1, \end{cases}$$

where we suppose that

$$\begin{cases} p_1, p_3 > p - 1 \text{ if } \ p \in \mathbb{Z}^+, \\ p_1, p_3 > [p] \ \text{ if } \ p \notin \mathbb{Z}^+, \\ q_1, q_3 > q - 1 \ \text{ if } \ q \in \mathbb{Z}^+, \\ q_1, q_3 > [q] \ \text{ if } \ q \notin \mathbb{Z}^+, \end{cases}$$

$p_4 < p - 1$, $p_2 q_2 < (p-1)(q-1)$ and $q_4 < q - 1$. Clearly, f, g, h and γ fulfill all the assumptions of Theorem 2

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AVAILABILITY OF DATA AND MATERIALS

Not applicable.

COMPETING INTERESTS

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AUTHORS' CONTRIBUTIONS

The authors contributed equally in this article. They have all read and approved the final manuscript.

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NEIMARK-SACKER BIFURCATION AND STABILITY OF A PREY-PREDATOR SYSTEM

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Abstract. In this paper, we investigate the stability and bifurcation of a discrete-time prey-predator system which is subject to an efficiency of prey conversion into predators and minimum threshold prey consumption required before predators begin to reproduce. It is concluded that the system undergoes Neimark-Sacker bifurcations in a small neighborhood of the unique positive equilibrium which depends on the number of prey-predator. Moreover, the numerical simulations are done to demonstrate the theoretical results.

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1. Introduction

Mathematical models which provide a way to design of changes in population size help to generate testable predictions. The changes in population size result from interactions between individuals of the same or different species, interactions with the environment, disease, food supply, etc. Predation, cooperative, mutualistic, commensural are different types of interactions of between individuals.

The discrete-time and continuous-time models are used for predicting the size of population. Especially, the discrete-time models are suitable to get more accurate numerical simulations for non-overlapping generations [20]. The prey-predator models have an importance at studies of mathematical biology [1, 7, 10, 25]. In recent years, studies on bifurcation and stability of systems involving population interaction have remarkable attention [2–6, 11, 13, 14, 16–19, 22, 27–29].

Bifurcation theory is a mathematical study of changes in the qualitative or topological structure of a given family. This theory is widely used in the mathematical study of dynamical systems. A bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden 'qualitative' or topological change in its behaviour [21].

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In [14], the Neimark-Sacker bifurcation of a dimensional discrete-time prey-predator model is presented as follows:

$$\begin{aligned} x_{n+1} &= ax_n(1-x_n) - x_n y_n \\ y_{t+1} &= \frac{1}{\beta} x_n y_n. \end{aligned} \tag{1.1}$$

where x_n and y_n denote the numbers of prey and predator, respectively. Moreover the parameters a, β and the initial conditions x_0, y_0 are positive real numbers.

A predator needs availability of prey as well as resources, to be able to sustain life and mature of it. Biological facts include the possibility of time spent in hunting and consumption of prey. This case associated with reproductive rates of predator is important for competitive interaction models.

In this paper, we will focus on the stability and bifurcation of a discrete-time prey-predator system which is subject to an efficiency of prey conversion into predators and minimum threshold of prey *consumption required before predators begin to reproduce* in [1, 10]. The general a prey-predator system can be given as follows:

$$\begin{aligned} x_{n+1} &= ax_n(1-x_n) - x_n y_n, \\ y_{t+1} &= \frac{1}{b} x_n y_n - e y_n, \end{aligned} \tag{1.2}$$

where x_n and y_n denote the numbers of prey and predator population in the n th generation, respectively. In this system, all the parameters a, b and e are positive real numbers. Moreover the parameter a denotes reproductive rates of prey population and the parameter b represents reproductive rates of predation depending on the growth rate of the prey. Also, the parameter e is the probability of the efficiency of prey conversion into predators and minimum threshold prey *consumption required before predators begin to reproduce* with $e < 1$.

The purpose of this work is to give the existence of the equilibrium point of the system (1.2); and to investigate the stability conditions of these equilibrium point. Also, the existence of Neimark-Sacker bifurcation of the system (1.2) is studied by choosing b as a bifurcation parameter. Moreover, we show the dynamical properties of the system (1.2), by means of trajectories, bifurcation diagrams and phase portraits.

This study consists of four section. In Section 2, we investigated the existence and local asymptotic stability of equilibrium points of the system (1.2) in $\mathbb{R}^2_+$. In Section 3, the existence of the system (1.2) undergoes a Neimark-Sacker bifurcation is discussed. Section 4 includes numerical simulations to support theoretical results. In the last section, the results are briefly presented.

2. The Existence and Stability of Equilibrium Points

In this section, we consider the prey-predator system (1.2). Firstly, we discuss the existence of equilibrium points for the system (1.2), and then study the stability

of the equilibrium points by using the characteristic polynomial or the eigenvalues of the Jacobian matrix evaluated at the equilibrium points. Now, let us give some necessary information.

Let us consider the two-dimensional discrete dynamical system of the form

$$\begin{aligned} x_{n+1} &= f(x_n, y_n), \\ y_{n+1} &= g(x_n, y_n), \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{2.1}$$

where $f : I \times J \to I$ and $g : I \times J \to J$ are continuously differentiable functions and I, J are some intervals of real numbers. Furthermore, a solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of system (2.1) is uniquely determined by initial conditions $(x_0, y_0) \in I \times J$. An equilibrium point of (2.1) is a point $(\bar{x}, \bar{y})$ that satisfies

$$\begin{aligned} \bar{x} &= f(\bar{x}, \bar{y}), \\ \bar{y} &= g(\bar{x}, \bar{y}). \end{aligned} \tag{2.2}$$

Let $(\bar{x}, \bar{y})$ be an equilibria of the map $F(x, y) = (f(x, y), g(x, y))$, where f and g are continuously differentiable functions at $(\bar{x}, \bar{y})$. The linearized system of (2.1) about the equilibria $(\bar{x}, \bar{y})$ is given by $X_{n+1} = F(X_n) = F_J X_n$, where $X_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ and F_J is a Jacobian matrix of system (2.1) about the equilibria $(\bar{x}, \bar{y})$.

Theorem 1 ([24]). *Assume that $X_{n+1} = F(X_n)$, $n = 0, 1, \ldots$ is a system of difference equations and $\bar{x}$ is an equilibrium point of F. If all eigenvalues of the Jacobian matrix F_J about the equilibria $\bar{x}$ lie inside the open unit disk $|\lambda| < 1$, then $\bar{x}$ is locally asymptotically stable. If one of them has absolute value greater than one, then $\bar{x}$ is unstable.*

Theorem 2 ([8]). *Consider the second-degree polynomial equation*

$$\lambda^2 - p\lambda - q = 0 \tag{2.3}$$

where p and q are real numbers.

(i) *If both roots of Equation (2.3) lie in the open unit disk $|\lambda| < 1$, then the equilibria $(\bar{x}, \bar{y})$ is locally asymptotically stable.*

(ii) *If at least one of the roots of Equation (2.3) has absolute value greater than one, then the equilibria $(\bar{x}, \bar{y})$ is unstable.*

(iii) *A necessary and sufficient condition for both roots of Equation (2.3) to lie inside the open disk $|\lambda| < 1$ is*

$$|p| < 1 - q < 2.$$

In this case the locally asymptotically stable equilibria $(\bar{x}, \bar{y})$ is also called a sink.

(iv) *A necessary and sufficient condition for both roots of Equation (2.3) to have absolute value greater than one is*

$$|q| > 1, |p| < |1 - q|.$$

In this case $(\bar{x},\bar{y})$ *is a repeller (source).*

(v) *A necessary and sufficient condition for one root of Equation (2.3) to have absolute value greater than one and for the other to have absolute value less than one is* $p^2+4q>0$, $|p|>|1-q|$. *In this case the unstable equilibria* $(\bar{x},\bar{y})$ *is called a saddle point.*

(vi) *A necessary and sufficient condition for a root of Equation (2.3) to have absolute value equal to one is*

$$|p|=|1-q|.$$

In this case, the steady state $(\bar{x},\bar{y})$ *is called a non-hyperbolic point.*

Theorem 3 ([18])**.** *Let* $F(x)=x^2+Bx+C$. *Suppose that* $F(1)>0$, x_1 *and* x_2 *are two roots of* $F(x)=0$. *Then*

(i) $|x_1|<1$ *and* $|x_2|<1$ *if and only if* $F(-1)>0$ *and* $C<1$;
(ii) $|x_1|<1$ *and* $|x_2|>1$ *(or* $|x_1|>1$ *and* $|x_2|<1$*) if and only if* $F(-1)<0$;
(iii) $|x_1|>1$ *and* $|x_2|>1$ *if and only if* $F(-1)>0$ *and* $C>1$;
(iv) $x_1=-1$ *and* $|x_2|\neq 1$ *if and only if* $F(-1)=0$ *and* $B\neq 0,2$;
(v) x_1 *and* x_2 *are complex and* $|x_1|=|x_2|=1$ *if and only if* $B^2-4C<0$ *and* $C=1$.

Theorem 4 ([2])**.** *The characteristic polynomial*

$$F(x)=x^2+Bx+C$$

has all its roots inside the unit open disk ($|x|<1$*) if and only if*

(i) $F(1)>0$ *and* $F(-1)>0$
(ii) $D_1^+=1+C>0$ *and* $D_1^-=1-C>0$.

Now, we will investigate the equilibrium points of system (1.2) and analyze the stability of these equilibrium points.

We easily obtain the equilibrium points of system (1.2) by using (2.2) such that

$$\bar{x}=a\bar{x}(1-\bar{x})-\overline{xy}, \tag{2.4}$$
$$\bar{y}=\frac{1}{b}\overline{xy}-e\bar{y}.$$

Then, we have the following Lemma.

Lemma 1. *For the system (1.2), the following cases hold:*

(i) *The system (1.2) has a unique trivial (extinction) equilibria* $E_0=(0,0)$ *for all parameters.*
(ii) *If* $a>1$, *then the system (1.2) has two equilibria. These are an extinction* $E_0=(0,0)$ *and an exclusion* $E_1=(\frac{a-1}{a},0)$.
(iii) *If* $a>1$ *and* $b<\frac{a-1}{a(e+1)}$, *then the system (1.2) has three equilibria. These are an extinction equilibria* $E_0=(0,0)$, *an exclusion equilibria* $E_1=(\frac{a-1}{a},0)$ *and a coexistence equilibria* $E_2=(b[e+1],\, a-ab[e+1]-1)$.

Now, let us study local dynamics of the equilibria obtaining the Jacobian matrix evaluated at E_0, E_1 and E_2. Firstly, by considering (1.2), we can get the Jacobian matrix as

$$J_{E_0} = \begin{pmatrix} a & 0 \\ 0 & -e \end{pmatrix} \tag{2.5}$$

evaluated at E_0, and the eigenvalues of J_{E_0} are

$$\lambda_1 = a, \quad \lambda_2 = -e.$$

By using Theorem 2-3, we can express the topological classification of the equilibria E_0 of the system (1.2) as follows:

Lemma 2. *For the extinction equilibrium points E_0, the following cases hold:*

(i) *If $0 < a < 1$ and $0 < e < 1$, then the equilibria E_0 is a sink point.*
(ii) *If $0 < e < 1 < a$, then the equilibria E_0 is a saddle point.*
(iii) *If $a = 1$, E_0 is non-hyperbolic point.*

Note that if $a > 1$, then the equilibria E_0 can not be a source point with $e < 1$. Secondly, we can get the Jacobian matrix as

$$J_{E_1} = \begin{pmatrix} 2-a & -1+\frac{1}{\alpha} \\ 0 & \frac{-1+a-abe}{\alpha b} \end{pmatrix} \tag{2.6}$$

evaluated at E_1 and the eigenvalues of J_{E_1} are

$$\lambda_1 = 2-a, \quad \lambda_2 = \frac{-1+a-abe}{\alpha b}.$$

Similarly, by using Theorem 2-3, we can get the topological classification of the exclusion equilibria E_1 of the system (1.2) as follows:

Lemma 3. *For the extinction equilibrium points E_1, the following cases hold:*

(i) *If $1 < a < 3$ and $\frac{a-1}{\alpha(e+1)} < b$, then the equilibria E_1 is a sink point.*
(ii) *If $(a < 1$ or $a > 3)$ and $\frac{a-1}{\alpha(e+1)} > b$, then the equilibria E_1 is a source point.*
(iii) *If $(a < 1$ or $a > 3)$ and $\frac{a-1}{\alpha(e+1)} < b$ or $1 < a < 3$ and $\frac{a-1}{\alpha(e+1)} > b$, then the equilibria E_1 is a saddle point.*
(iv) *If $a = 3$, or $b = \frac{a-1}{\alpha(e+1)}$, E_1 is non-hyperbolic point.*

Finally, we can find the Jacobian matrix as

$$J_{E_2} = \begin{pmatrix} 1-ab(e+1) & -b(e+1) \\ \frac{-1+a-ab(e+1)}{b} & 1 \end{pmatrix} \tag{2.7}$$

evaluated at E_2 and the characteristic polynomial is

$$F(\lambda) = \lambda^2 + [-2+ab(1+e)]\lambda + [-e-a(1+e)(-1+b[2+e])].$$

So, we can analyze the dynamics of a unique positive coexistence equilibrium points of the system (1.2). By using Theorem 2-3, we can get the topological classification of the coexistence equilibria E_2 of system (1.2) as follows:

Lemma 4. *Suppose that $F(1) > 0$. In this case, $0 < b < \frac{a-1}{\alpha(e+1)}$ is provided for $a > 1$, $0 < e < 1$. For the coexistence equilibrium points E_2, the following cases hold:*

(i) *If $\frac{a-1}{2a+ae} < b < \frac{3+a-e+ae}{3a+4ae+ae^2}$, for $a > 3$ or $\frac{a-1}{2a+ae} < b < \frac{a-1}{\alpha(e+1)}$ for $1 < a < 3$, then the equilibria E_2 is a sink point.*

(ii) *If $0 < b < \min\left\{\frac{a-1}{2a+ae}, \frac{3+a-e+ae}{3a+4ae+ae^2}\right\}$, then the equilibria E_2 is a source point.*

(iii) *If $b > \frac{3+a-e+ae}{3a+4ae+ae^2}$, then the equilibria E_2 is a saddle point.*

(iv) *If $0 < b < \frac{-2}{\alpha} + 2\sqrt{\frac{a+e}{a^2(e+1)}}$ and $b = b_1$ such that $b_1 = \frac{(a-1)}{(2a+ae)}$, then the eigenvalues of J_{E_2} are a pair of conjugate complex numbers whose modules are one.*

3. Neimark-Sacker Bifurcations

In this section, the direction and the existence of Neimark–Sacker bifurcation are obtained for the system (1.2) ([16, 27]). Also, if the system (1.2) provides eigenvalue assignment, transversality and non resonance conditions, then Neimark–Sacker bifurcation occurs at a bifurcation point ([9, 23]). In order to work Neimark–Sacker bifurcation in the system (1.2), we define the parameters providing non-hyperbolic conditions by

$$NSB_{E_2} = \left\{a, b, e \in \mathbb{R}^+ : 0 < b < \frac{-2}{\alpha} + 2\sqrt{\frac{a+e}{a^2(e+1)}} \text{ and } b = b_1\right\}. \tag{3.1}$$

By considering Lemma 8, the eigenvalues of J_{E_2} evaluated at a non-hyperbolic point E_2 has a pair of conjugate complex numbers whose modules are one. These eigenvalues are

$$\lambda, \bar{\lambda}\,|_{b=b_1} = \frac{1}{2}\left[2 - ab(1+e) \pm i\sqrt{(1+e)[(4a-4) - a^2b^2 - 4ab(1+e)]}\right]|_{b=b_1} \tag{3.2}$$

with

$$|\lambda| = \left|\bar{\lambda}\right| = 1.$$

For $b \in NSB_{E_2}$, we get

$$\frac{\partial|\lambda_i(b)|}{\partial b}\,|_{b=b_1} \neq 0\ ,\ i = 1, 2. \tag{3.3}$$

Also, if

$$trJ_{E_2}\,|_{b=b_1} \neq 0, -1, \tag{3.4}$$

then, we reach

$$\lambda^k(b_1) \neq 1\ , k = 1, 2, 3, 4. \tag{3.5}$$

Let $q, p \in \mathbb{C}^2$ be two eigenvectors which corresponding to the eigenvalues λ of the matrix $J(NSB_{E_2})$ and the eigenvalues $\overline{\lambda}$ of the matrix $J(NSB_{E_2})^T$, respectively. If these eigenvectors are calculated with Maple program, then we get

$$q \sim \left(\frac{T-1-i\sqrt{2T-1-T^2-4Ka}}{2a}, 1 \right), \tag{3.6}$$

and

$$p \sim \left(\frac{T-1+i\sqrt{2T-1-T^2-4Ka}}{2K}, 1 \right), \tag{3.7}$$

such that $T = \frac{3+2e-a(e+1)}{2+e}$, $K = -\frac{(-1+a)(1+e)}{a(2+e)}$.

By using the scalar product in $\mathbb{C}^2 : < p, q >= \overline{p_1} q_1 + \overline{p_2} q_2$, we define the following vector in order to normalize p according to q

$$p \sim \left(\frac{(T-1)+i\sqrt{2T-4aK-1-T^2}}{2K\overline{R}}, \frac{1}{\overline{\overline{R}}} \right). \tag{3.8}$$

where $\overline{R} = (1 + \frac{(-1+T-i\sqrt{-1-4aK+2T-T^2})^2}{4aK})$ and $< p, q >= 1$.

In order to transform the equilibrium point E_2 of the system (1.2) into the origin $(0,0)$, we take

$$u_n = x_n - b[e+1], \; v_n = y_n - [a - ab(e+1) - 1)]. \tag{3.9}$$

Then, we get the following map:

$$\begin{pmatrix} u \\ v \end{pmatrix} \to J_{E_2} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} F_1(u,v) \\ F_2(u,v) \end{pmatrix}, \tag{3.10}$$

where

$$F_1(u,v) = -au^2 - uv + O(\|U\|^3)$$
$$F_2(u,v) = \frac{1}{b} uv + O(\|U\|^3)$$

such that $U_t = (u,v)^T$. Also, the system (1.2) can be written as

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} \to J_{E_2} \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \frac{1}{2} B(u_n, v_n) + \frac{1}{6} C(u_n, v_n, w_n) + O(\|U_n\|^4), \tag{3.11}$$

with the multilinear vector functions of $u; v; w \in \mathbb{R}^2$:

$$B(u,v) = \begin{pmatrix} B_1(u,v) \\ B_2(u,v) \end{pmatrix}$$

and

$$C(u,v) = \begin{pmatrix} B_1(u,v,w) \\ B_2(u,v,w) \end{pmatrix}.$$

These vectors are expressed by

$$B_1(u,v) = \sum_{j,k=1}^{2} \frac{\partial^2 F_1}{\partial\xi_j\partial\xi_k} |_{\xi=0}\, u_{.j}v_k = -2au_1v_1 - (u_2v_1 + u_1v_2)$$

$$B_2(u,v) = \sum_{j,k=1}^{2} \frac{\partial^2 F_2}{\partial\xi_j\partial\xi_k} |_{\xi=0}\, u_{.j}v_k = \frac{1}{b}(u_2v_1 + u_1v_2)$$

$$C_1(u,v,w) = \sum_{j,k=1}^{2} \frac{\partial^3 F_1}{\partial\xi_j\partial\xi_k\xi_l} |_{\xi=0}\, u_{.j}v_kw_l = 0$$

$$C_2(u,v,w) = \sum_{j,k=1}^{2} \frac{\partial^3 F_2}{\partial\xi_j\partial\xi_k\xi_l} |_{\xi=0}\, u_{.j}v_kw_l = 0.$$

$\forall\, U \in \mathbb{R}^2$ can be uniqely offered as

$$U = zq + \overline{zq} \tag{3.12}$$

for some $z \in \mathbb{C}$. Here, $\bar{z}$ is the conjugate of that complex number z, and $z =< p, U >$. For all sufficiently small $|b|$, we can transform the system (1.2) into the form

$$z \to \lambda(b)z + g(z,\bar{z},b), \tag{3.13}$$

where $\lambda(r) = (1+\omega(b))e^{i\arctan(b)}$ with $\omega(b_1) = 0$ and $g(z,\bar{z},b)$ is a complex valued smooth function of z and $\bar{z}$. Taylor expression of g with respect to $g(z,\bar{z})$ is as follows:

$$g(z,\bar{z},b) = \sum_{k+l\geq 2} \frac{1}{k!l!} g_{kl}(b) z^k \bar{z}^l, \tag{3.14}$$

with

$$\begin{aligned} g_{20}(b_1) &=< p, B(q,q) > \\ g_{11}(b_1) &=< p, B(q,\overline{q}) > \\ g_{02}(b_1) &=< p, B(\overline{q},\overline{q}) > \\ g_{21}(b_1) &=< p, C(q,q,\overline{q}) > . \end{aligned}$$

In order to come out Neimark-Sacker bifurcation for the system (3.10), we need that the coefficient $\varphi(b_1)$ must not be zero. This coefficient is

$$\begin{aligned} \varphi(b_1) =& Re\left(\frac{e^{-iarc\tan(b_1)}}{2} g_{21}\right) - Re\left(\frac{(1-2e^{iarc\tan(b_1)})e^{-2iarc\tan(b_1)}}{2(1-e^{iarc\tan(b_1)})} g_{20}g_{11}\right) \\ &- \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2 \end{aligned} \tag{3.15}$$

where $e^{iarc\tan(b_1)} = \lambda(b_1)$. Consequently, we have the following theorem on Neimark-Sacker bifurcation:

Theorem 5. *If (3.4) holds, $\varphi(b_1) \neq 0$ and the parameter b changes its value in small vicinity of NSB_{E_2}, then the system (1.2) passes through a Neimark-Sacker bifurcation at the only equilibrium point E_2. Moreover if $\varphi(b_1) < 0$ ($\varphi(b_1) > 0$), then there exists a unique attracting (repelling) invariant closed curve which bifurcates from E_2.*

4. Numerical Simulations

In this section, theoretical results are supported with graphics by using Mathematica [15, 26] and SageMath [12] programming. Some numerical simulations is given to demonstrate existence of Neimark-Sacker bifurcation for the system (1.2). Here, trajectories, bifurcation diagrams and phase portraits are illustrated by taking b as bifurcation parameter.

Example 1. Let us consider the following system for the parameter values $e = 0.5$, $a = 2.5$ and $b = 0.24$,

$$\begin{aligned} x_{n+1} &= 2.5x_n(1-x_n) - x_n y_n \\ y_{n+1} &= \frac{1}{0.24} x_n y_n - 0.5 y_n. \end{aligned} \tag{4.1}$$

The coexistence positive equilibrium point is $(\bar{x}, \bar{y}) = (0.36, 0.6)$, and the Jacobian matrix evaluated $(\bar{x}, \bar{y})$ is obtained as follows:

$$J_{(\bar{x},\bar{y})} = \begin{pmatrix} 0.1 & -0.36 \\ 2.5 & 1 \end{pmatrix}.$$

If the eigenvalues are calculated, then we obtain

$$\lambda_{1,2} = 0.55 \mp 0.835165i$$

such that $|\lambda_{1,2}| = 1$. Let

$$q \sim (-0.3340658618 + 0.1800000000i, -i)^T,$$

and

$$p \sim (-0.3340658618 - 0.1800000000i, i)^T,$$

be complex eigenvectors corresponding to $\lambda_{1,2}$ respectively.

If the normalized vector is taken as

$$q \sim (-0.3340658618 + 0.1800000000i, -i)^T$$

to obtain the normalization $< p, q >= 1$, the vector is found

$$p \sim (0,331611 + 0.238793i, 0.139463 - 1.0678i)^T.$$

By transformation of variables

$$u_n = x_n - 0.36, \ v_n = y_n - 0.6$$

the system (4.1) can be written as follows:

$$u_{n+1} = -2.5u_n^2 + 0.1u_n - 0.36v_n - u_n v_n$$
$$y_{n+1} = 2.5u_n + v_n + 4.16667u_n v_n.$$

When the coefficient of the form (3.14) are calculated, we get

$$g_{20}(b_1) = -3.03009 + 2.14832i$$
$$g_{11}(b_1) = -0.328575 - 1.51574i$$
$$g_{02}(b_1) = 2.94708 + 1.41614i$$
$$g_{21}(b_1) = 0.$$

From (3.15), we get as $\varphi(b_{NS}) = -2.76917 < 0$ such that $\theta = 0.988432$. Consequently, the Neimark-Sacker bifurcation emerges at $b_{NS} = 0.24$.

To confirm the theoretical result, the trajectories, the bifurcation diagrams and phase portraits of the prey-predator system (1.2) are shown in Figure 1, Figure 2 and Figure 3.

5. Conclusions

This paper contains the complex dynamic behavior of the prey predator system (1.2) with an efficiency of prey conversion into predators and minimum threshold of prey *consumption required before predators begin to reproduce*. We investigate stability conditions of the equilibrium points of the system (1.2), and show that the system (1.2) displays a Neimark-Sacker bifurcations.

We find that the system (1.2) has a trivial (extinction) equilibria E_0, an exclusion equilibria E_1 and a coexistence equilibria E_2. We give asymptotic stability conditions of these equilibria by using linearization method. It is clear that there is unique positive coexistence equilibria E_2 of the system (1.2) with $a > 1$ and $b < \frac{a-1}{a(e+1)}$. Moreover, it is proved that system (1.2) undergoes Neimark-Sacker bifurcation under the condition $b = \frac{a-1}{2a+ae}$ by using mathematical techniques of bifurcation theory. It is seen that Neimark–Saker bifurcation appears when the parameters vary on the neighborhood $NSB_{E_2} = \left\{a, b, e \in \mathbb{R}^+ : 0 < b < \frac{-2}{\alpha} + 2\sqrt{\frac{a+e}{a^2(e+1)}} \text{ and } b = b_1\right\}$. Some figures present dynamical properties of system (1.2) which has an attracting invariant curve for $b < 0.24$, $e = 0.5$, $a = 2.5$ and the initial conditions $(x_0, y_0) = (0.4, 0.3)$. As the parameter b exceed 0.24, the system becomes stable. If the parameter b exceeds 0.4, the predation population disappears. We conclude that the parameter b has a different effect on the dynamics on system (1.2) with the biological effect of e.

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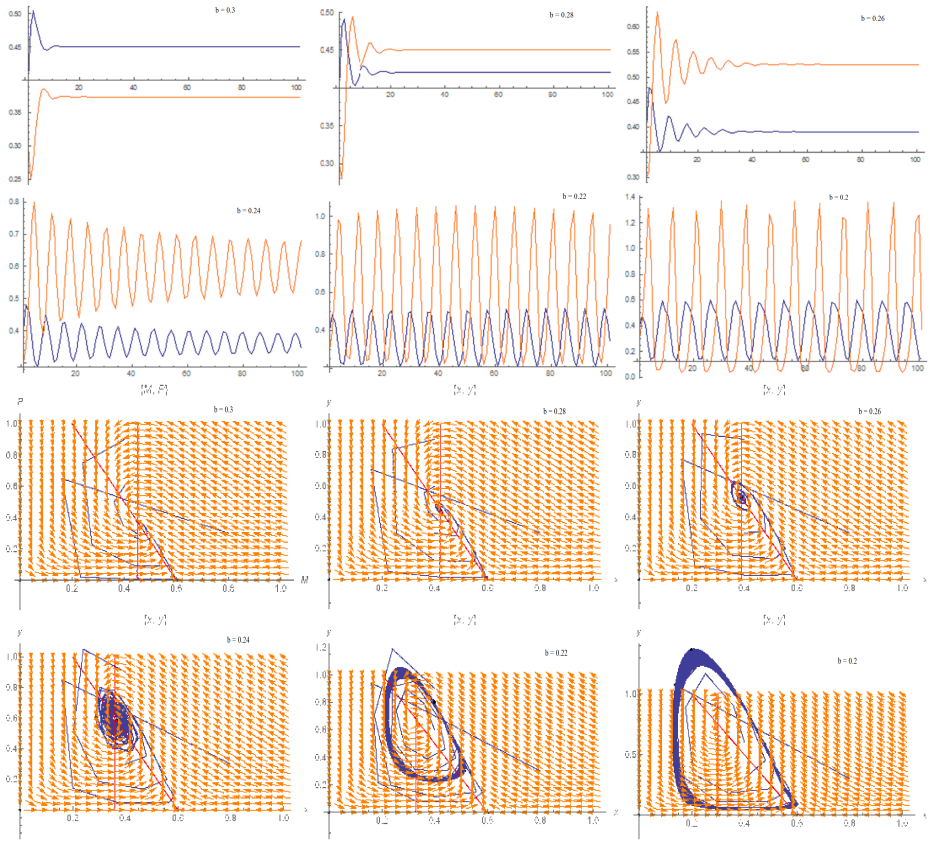

FIGURE 1. Trajectories of the prey-predator system (1.2) for different values b with the parameter values $a = 2.5$, $e = 0.5$ and the initial conditions $(x_0, y_0) = (0.4, 0.3)$.

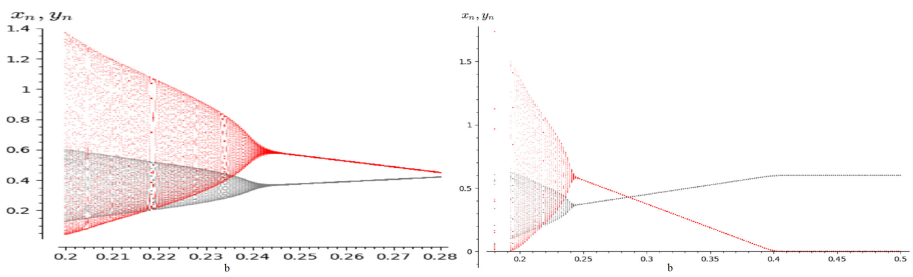

FIGURE 2. Trajectories of the prey-predator system (1.2) for different values b with the parameter values $a = 2.5$, $e = 0.5$ and the initial conditions $(x_0, y_0) = (0.4, 0.3)$.

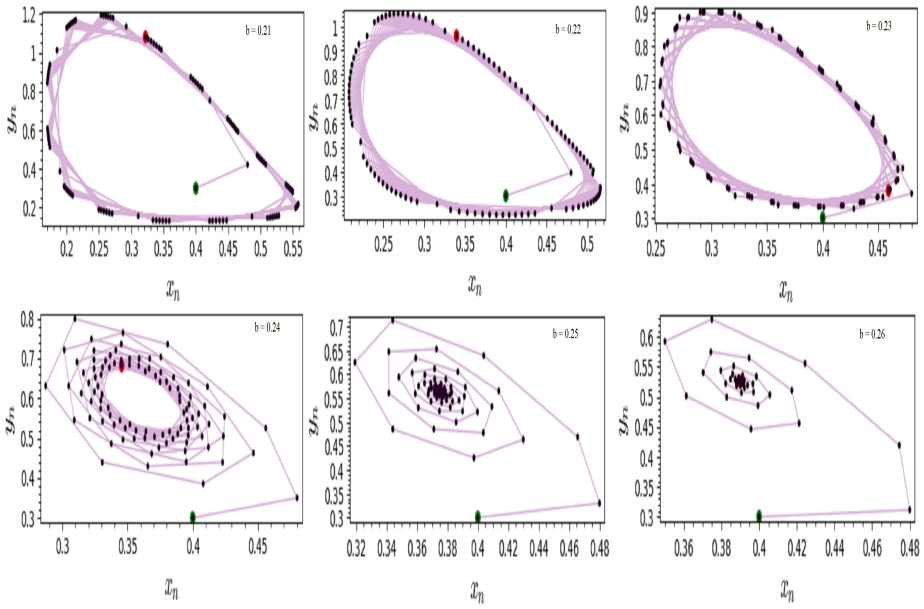

FIGURE 3. The phase portraits of the prey-predator system (1.2) with the parameter values $a = 2.5$, $e = 0.5$ and the initial conditions $(x_0, y_0) = (0.4, 0.3)$.

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EXISTENCE RESULT FOR A NEW CLASS OF KIRCHHOFF ELLIPTIC SYSTEM WITH VARIABLE PARAMETERS

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Abstract. The paper studies the existence result for a new class of Kirchhoff elliptic system with variable parameters in the right hand side. Sub-super solutions method are used for proving the main result. Our study is a natural improvement result of our previous one in (Boulaaras et al. in Math. Methods Appl. Sci. 41:5203-5210, 2018).

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1. Introduction

Consider the following system

$$\begin{cases} -A\left(\int_\Omega |\nabla u|^2 dx\right) \triangle u = \lambda\alpha(x) f(u,v) \text{ in } \Omega, \\ -B\left(\int_\Omega |\nabla v|^2 dx\right) \triangle v = \lambda\beta(x) g(u,v) \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded smooth domain with C^2 boundary $\partial\Omega$, and A, B: $\mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions with further conditions to be given later, λ is a positive parameter, and $\alpha, \beta \in C\left(\overline{\Omega}\right)$.

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This nonlocal problem originates from the stationary version of Kirchhoff's work [10] in 1883

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

where Kirchhoff extended the classical d'Alembert's wave equation by considering the effect of the changes in the length of the string during vibrations. The parameters in (1.2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

Recently, the problems associated to Laplacian operator and Kirchhoff elliptic equations have been heavily studied, we refer to [1, 3–5, 8, 9, 11–13].

In [2], Alves and Correa proved the validity of Sub-super solutions method for problems of Kirchhoff class involving a single equation and a boundary condition

$$\begin{cases} -M\left(\|u\|^2\right)\Delta u=f(x,u) \text{ in } \Omega, \\ \\ u=0 \qquad \text{on } \partial\Omega, \end{cases}$$

with $f\in C\left(\overline{\Omega}\times\mathbb{R}\right)$.

By using a comparison principle that requires M to be non-negative and non-increasing in $[0,+\infty)$, with $H(t):=M\left(t^2\right)t$ increasing and $H(\mathbb{R})=\mathbb{R}$, they managed to prove the existence of positive solutions assuming f increasing in the variable u for each $x\in\Omega$ fixed.

For systems involving similar class of equations, this result can not be used directly, i.e. the existence of a subsolution and a supersolution does not guarantee the existence of the solution. Therefore, a further construction is needed. As in [6], where we studied the system

$$\begin{cases} -A\left(\int_\Omega|\nabla u|^2dx\right)\triangle u=\lambda_1 f(v)+\mu_1 g(u) \text{ in } \Omega, \\ \\ -B\left(\int_\Omega|\nabla v|^2dx\right)\triangle v=\lambda_2 h(u)+\mu_2(x)\,l(v) \text{ in } \Omega, \\ \\ u=v=0 \text{ on } \partial\Omega. \end{cases} \tag{1.3}$$

Using a weak positive supersolution as first term of a constructed iterative sequence (u_n,v_n) in $H_0^1(\Omega)\times H_0^1(\Omega)$, and a comparison principle introduced in [2], the authors established the convergence of this sequence to a positive weak solution of the considered problem.

In this paper, we generalize the previous work in [6] by considering variable parameters α,β,γ and η in the right hand side of (1.1). We also give a better subsolution providing easier computations.

2. Existence result

Definition 1. $(u,v) \in \left(H_0^1(\Omega) \times H_0^1(\Omega)\right)$, is called a weak solution of (1.1) if it satisfies

$$A\left(\int_\Omega |\nabla u|^2\,dx\right)\int_\Omega \nabla u \nabla \phi\,dx = \lambda \int_\Omega \alpha(x) f(u,v)\phi\,dx \text{ in } \Omega,$$

$$B\left(\int_\Omega |\nabla v|^2\,dx\right)\int_\Omega \nabla v \nabla \psi\,dx = \lambda \int_\Omega \beta(x) g(u,v)\psi\,dx \text{ in } \Omega,$$

for all $(\phi,\psi) \in \left(H_0^1(\Omega) \times H_0^1(\Omega)\right)$.

Definition 2. Let $(\underline{u},\underline{v}),(\overline{u},\overline{v})$ be a pair of nonnegative functions in $\left(H_0^1(\Omega) \times H_0^1(\Omega)\right)$, they are called positive weak subsolution and positive weak supersolution (respectively) of (1.1) if they satisfy the following

$$A\left(\int_\Omega |\nabla \underline{u}|^2\,dx\right)\int_\Omega \nabla \underline{u} \nabla \phi\,dx \le \lambda \int_\Omega \alpha(x) f(\underline{u},\underline{v})\phi\,dx,$$

$$B\left(\int_\Omega |\nabla \underline{v}|^2\,dx\right)\int_\Omega \nabla \underline{v} \nabla \psi\,dx \le \lambda \int_\Omega \beta(x) g(\underline{u},\underline{v})\psi\,dx,$$

and

$$A\left(\int_\Omega |\nabla \overline{u}|^2\,dx\right)\int_\Omega \nabla \overline{u} \nabla \phi\,dx \ge \lambda \int_\Omega \alpha(x) f(\overline{u},\overline{v})\phi\,dx,$$

$$B\left(\int_\Omega |\nabla \overline{v}|^2\,dx\right)\int_\Omega \nabla \overline{v} \nabla \psi\,dx \ge \lambda \int_\Omega \beta(x) g(\overline{u},\overline{v})\psi\,dx,$$

for all $(\phi,\psi) \in \left(H_0^1(\Omega) \times H_0^1(\Omega)\right)$, with $\phi \ge 0$ and $\psi \ge 0$, and $(\underline{u},\underline{v}),(\overline{u},\overline{v}) = (0,0)$ on $\partial\Omega$.

Lemma 1 (Comparison principle [2]). *Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous nonincreasing function such that*

$$M(s) > m_0 > 0, \text{ for all } s \ge s_0, \tag{2.1}$$

and $H(t) = tM\left(t^2\right)$ increasing on $\mathbb{R}^+$.

If u_1, u_2 are two non-negative functions verifying

$$\begin{cases} -M\left(\int_\Omega |\nabla u_1|^2 dx\right) \triangle u_1 \geq -M\left(\int_\Omega |\nabla u_2|^2 dx\right) \triangle u_2 \text{ in } \Omega, \\ u_1 = u_2 = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.2}$$

then $u_1 \geq u_2$ a.e. in Ω.

Before stating and proving our main result, here are the conditions we need.

$(H1)$ $A, B : \mathbb{R}^+ \to \mathbb{R}^+$ are two continuous and increasing functions that satisfy the monotonicity conditions of Lemma 1 so that we can use the Comparison principle, and assume further that there exists $a_i, b_i > 0$, $i = 1, 2$,

$$a_1 \leq A(t) \leq a_2, \quad b_1 \leq B(t) \leq b_2 \text{ for all } t \in \mathbb{R}^+.$$

$(H2)$ $\alpha, \beta \in C(\overline{\Omega})$ and

$$\alpha(x) \geq \alpha_0 > 0, \ \beta(x) \geq \beta_0 > 0$$

for all $x \in \Omega$.

$(H3)$ f, g are continuous on $[0, +\infty[$, C^1 on $(0, +\infty)$, and increasing functions of infinite growth

$$\lim_{s,t\to+\infty} f(s,t) = +\infty, \lim_{s,t\to+\infty} g(s,t) = +\infty.$$

$(H4)$ For all $K > 0$

$$\lim_{t\to+\infty} \frac{f(t, K(g(t,t)))}{t} = 0.$$

$(H5)$

$$\lim_{t\to+\infty} \frac{g(t,t)}{t} = 0.$$

Theorem 1. *For large values of $\lambda\alpha_0$ and $\lambda\beta_0$, system (1.1) admits a large positive weak solution if conditions $(H1) - (H5)$ are satisfied.*

Proof of Theorem 1. Consider σ the first eigenvalue of $-\triangle$ with Dirichlet boundary conditions and ϕ_1 the corresponding positive eigenfunction with $\|\phi_1\| = 1$ and $\phi_1 \in C^\infty(\overline{\Omega})$ (see [7]).

Let $S = \sup_{x\in\Omega}\{\sigma\phi_1^2 - |\nabla\phi_1|^2\}$, then from growth conditions (H3)

$$f(t,t) \geq S, \ g(t,t) \geq S, \quad \text{for } t \text{ large enough.}$$

For each α_0 large, let us define

$$\underline{u} = \left(\frac{\lambda\alpha_0}{2a_2}\right)\phi_1^2$$

and

$$\underline{v} = \left(\frac{\lambda\beta_0}{2b_2}\right)\phi_1^2,$$

where a_2, b_2 are given by condition $(H1)$. Let us show that $(\underline{u}, \underline{v})$ is a subsolution of problem (1.1) for $\lambda\alpha_0$ large enough. Indeed, let $\phi \in H_0^1(\Omega)$ with $\phi \geq 0$ in Ω. By $(H1)-(H3)$, we get

$$\begin{aligned}
&A\left(\int_\Omega |\nabla \underline{u}|^2\,dx\right)\int_\Omega \nabla\underline{u}.\nabla\phi\,dx = A\left(\int_\Omega |\nabla\underline{u}|^2\,dx\right)\left(\frac{\lambda\alpha_0}{a_2}\right)\int_\Omega \phi_1\nabla\phi_1.\nabla\phi\,dx\\
&= \left(\frac{\lambda\alpha_0}{a_2}\right)A\left(\int_\Omega |\nabla\underline{u}|^2\,dx\right)\times\left\{\int_\Omega \nabla\phi_1\nabla(\phi_1.\phi)\,dx - \int_\Omega |\nabla\phi_1|^2\phi\,dx\right\}\\
&= \left(\frac{\lambda\alpha_0}{a_2}\right)A\left(\int_\Omega |\nabla\underline{u}|^2\,dx\right)\int_\Omega \left(\sigma\phi_1^2 - |\nabla\phi_1|^2\right)\phi\,dx\\
&\leq \lambda\alpha_0\int_\Omega S\phi\,dx \leq \lambda\int_\Omega \alpha(x)\,f(\underline{u},\underline{v})\,\phi\,dx
\end{aligned}$$

for $\lambda\alpha_0 > 0$ large enough, and all $\phi \in H_0^1(\Omega)$ with $\phi \geq 0$ in Ω.

Similarly,

$$B\left(\int_\Omega |\nabla\underline{v}|^2\,dx\right)\int_\Omega \nabla\underline{v}\nabla\psi\,dx \leq \lambda\int_\Omega \beta(x)\,g(\underline{u},\underline{v})\,\psi\,dx \text{ in } \Omega$$

for $\lambda\beta_0 > 0$ large enough and all $\psi \in H_0^1(\Omega)$ with $\psi \geq 0$ in Ω.
Also notice that $\underline{u} > 0$ and $\underline{v} > 0$ in Ω, $\underline{u} \to +\infty$ and $\underline{v} \to +\infty$ as $\lambda\alpha_0 \to +\infty$, $\lambda\beta_0 \to +\infty$.

For the supersolution part, consider e the solution of the following problem

$$\begin{cases} -\triangle e = 1 \text{ in } \Omega,\\ e = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.3}$$

We give the supersolution of problem (1.1) by

$$\overline{u} = \frac{C}{\mu}(\lambda\|\alpha\|_\infty)\,e, \quad \overline{v} = \left(\frac{\lambda\|\beta\|_\infty}{b_2}\right)g(C\lambda, C\lambda)\,e,$$

where $\mu = \|e\|_\infty, C > 0$ is a large positive real number to be given later.
Indeed, for all $\phi \in H_0^1(\Omega)$ with $\phi \geq 0$ in Ω, we get from (2.3) and the condition $(H1)$

$$A\left(\int_\Omega |\nabla\overline{u}|^2\,dx\right)\int_\Omega \nabla\overline{u}.\nabla\phi\,dx = A\left(\int_\Omega |\nabla\overline{u}|^2\,dx\right)\frac{C}{\mu}(\lambda\|\alpha\|_\infty)\int_\Omega \nabla e.\nabla\phi\,dx$$

$$= A\left(\int_\Omega |\nabla \overline{u}|^2\,dx\right)\frac{C\lambda}{\mu}\left(\|\alpha\|_\infty\right)\int_\Omega \phi\,dx \geq \frac{a_1 C\lambda}{\mu}\left(\|\alpha\|_\infty\right)\int_\Omega \phi\,dx.$$

By $(H4)$ and $(H5)$, we can choose C large enough so that

$$\frac{a_1 C\lambda}{\mu}\int_\Omega \phi\,dx \geq \lambda\int_\Omega f\left(C\lambda, \left(\frac{\lambda\|\beta\|_\infty}{b_2}\right) g\left(C\lambda, C\lambda\right)\mu\right)\phi\,dx.$$

Therefore,

$$\begin{aligned} A\left(\int_\Omega |\nabla \overline{u}|^2\,dx\right)\int_\Omega \nabla\overline{u}.\nabla\phi\,dx &\geq \lambda\|\alpha\|_\infty\int_\Omega f\left(C\lambda, \left(\frac{\lambda\|\beta\|_\infty}{b_2}\right) g\left(C\lambda, C\lambda\right)\mu\right)\phi\,dx \\ &\geq \lambda\|\alpha\|_\infty\int_\Omega f\left(\frac{C}{\mu}\lambda e, \left(\frac{\lambda\|\beta\|_\infty}{b_2}\right) g\left(C\lambda, C\lambda\right) e\right)\phi\,dx \\ &\geq \lambda\int_\Omega \alpha(x) f(\overline{u},\overline{v})\,\phi\,dx. \end{aligned} \tag{2.4}$$

Also,

$$\begin{aligned} B\left(\int_\Omega |\nabla \overline{v}|^2\,dx\right)\int_\Omega \nabla\overline{v}\nabla\psi dx &= \frac{\lambda\|\beta\|_\infty}{b_2} g\left(C\lambda, C\lambda\right) B\left(\int_\Omega |\nabla \overline{v}|^2\,dx\right)\int_\Omega \nabla e\nabla\psi dx \\ &\geq \lambda\|\beta\|_\infty g\left(C\lambda, C\lambda\right)\int_\Omega \psi dx. \end{aligned} \tag{2.5}$$

Using $(H4)$ and $(H5)$ again for C large enough we get

$$\frac{1}{\frac{\lambda\|\beta\|_\infty}{b_2}\mu} \geq \frac{g\left(C\lambda, C\lambda\right)}{C\lambda}$$

Hence

$$\begin{aligned} g\left(C\lambda, C\lambda\right)\int_\Omega \psi dx &\geq \int_\Omega g\left(C\lambda, g\left(C\lambda, C\lambda\right)\frac{\left(\lambda\|\beta\|_\infty\right)}{b_2}\mu\right)\psi dx \\ &\geq \int_\Omega g\left(\lambda C\left(\frac{e}{\mu}\right), g\left(C\lambda, C\lambda\right)\frac{\left(\lambda\|\beta\|_\infty\right)}{b_2} e\right)\psi dx \\ &= \int_\Omega g\left(\overline{u},\overline{v}\right)\psi dx. \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), we obtain

$$B\left(\int_{\Omega}|\nabla\overline{v}|^{2}dx\right)\int_{\Omega}\nabla\overline{v}\nabla\psi dx\geq\lambda\int_{\Omega}\beta(x)g(\overline{u},\overline{v})\psi dx. \tag{2.7}$$

By (2.4) and (2.7) we conclude that $(\overline{u},\overline{v})$ is a supersolution of problem (1.1). Furthermore, $\underline{u}\leq\overline{u}$ and $\underline{v}\leq\overline{v}$ for C choosen large enough.

Now, we use a similar argument to [6] in order to obtain a weak solution of our problem. Consider the following sequence $\{(u_n,v_n)\}\subset\left(H_0^1(\Omega)\times H_0^1(\Omega)\right)$ where: $u_0:=\overline{u},v_0=\overline{v}$ and (u_n,v_n) is the unique solution of

$$\begin{cases} -A\left(\int_{\Omega}|\nabla u_n|^{2}dx\right)\triangle u_n=\lambda\alpha(x)f(u_{n-1},v_{n-1}) \text{ in } \Omega, \\ -B\left(\int_{\Omega}|\nabla v_n|^{2}dx\right)\triangle v_n=\lambda\beta(x)g(u_{n-1},v_{n-1}) \text{ in } \Omega, \\ u_n=v_n=0 \text{ on } \partial\Omega. \end{cases} \tag{2.8}$$

Since A and B satisfy $(H1)$ and $\alpha(x)f(u_{n-1},v_{n-1})$, $\beta(x)g(u_{n-1},v_{n-1})\in L^2(\Omega)$ (in x), we deduce from a result in [2] that system (2.8) has a unique solution $(u_n,v_n)\in\left(H_0^1(\Omega)\times H_0^1(\Omega)\right)$.

Using (2.8) and the fact that (u_0,v_0) is a supersolution of (1.1), we get

$$\begin{cases} -A\left(\int_{\Omega}|\nabla u_0|^{2}dx\right)\triangle u_0\geq\lambda\alpha(x)f(u_0,v_0)=-A\left(\int_{\Omega}|\nabla u_1|^{2}dx\right)\triangle u_1, \\ -B\left(\int_{\Omega}|\nabla v_0|^{2}dx\right)\triangle v_0\geq\lambda\beta(x)g(u_0,v_0)\ =-B\left(\int_{\Omega}|\nabla v_1|dx\right)\triangle v_1 \end{cases}$$

then by Lemma 1, $u_0\geq u_1$ and $v_0\geq v_1$. Also, since $u_0\geq\underline{u}$, $v_0\geq\underline{v}$ and the monotonicity of f, g, h, and l one has

$$-A\left(\int_{\Omega}|\nabla u_1|^{2}dx\right)\triangle u_1=\lambda\alpha(x)f(u_0,v_0)\geq\lambda\alpha(x)f(\underline{u},\underline{v})\geq-A\left(\int_{\Omega}|\nabla\underline{u}|^{2}dx\right)\triangle\underline{u},$$

$$-B\left(\int_{\Omega}|\nabla v_1|^{2}dx\right)\triangle v_1=\lambda\beta(x)g(u_0,v_0)\geq\lambda\beta(x)g(\underline{u},\underline{v})\geq-B\left(\int_{\Omega}|\nabla\underline{v}|^{2}dx\right)\triangle\underline{v}$$

according to Lemma 1 again, we obtain $u_1\geq\underline{u}$, $v_1\geq\underline{v}$. Repeating the same argument for u_2,v_2, observe that

$$-A\left(\int_{\Omega}|\nabla u_1|^{2}dx\right)\triangle u_1=\lambda\alpha(x)f(u_0,v_0)\geq\lambda\alpha(x)f(u_1,v_1)=-A\left(\int_{\Omega}|\nabla u_2|^{2}dx\right)\triangle u_2,$$

$$-B\left(\int_\Omega |\nabla v_1|\,dx\right)\triangle v_1=\lambda\beta(x)\,g(u_0,v_0)\geq\lambda\alpha(x)\,g(u_1,v_1)=-B\left(\int_\Omega |\nabla v_2|^2\,dx\right)\triangle v_2,$$

then $u_1\geq u_2$, $v_1\geq v_2$. Similarly, we get $u_2\geq\underline{u}$ and $v_2\geq\underline{v}$ from

$$-A\left(\int_\Omega |\nabla u_2|^2\,dx\right)\triangle u_2=\lambda\alpha(x)\,f(u_1,v_1)\geq\lambda\alpha(x)\,f(\underline{u},\underline{v})\geq -A\left(\int_\Omega |\nabla \underline{u}|^2\,dx\right)\triangle \underline{u},$$

$$-B\left(\int_\Omega |\nabla v_2|^2\,dx\right)\triangle v_2=\lambda\beta(x)\,g(u_1,v_1)\geq\lambda\beta(x)\,g(\underline{u},\underline{v})\geq -B\left(\int_\Omega |\nabla \underline{v}|^2\,dx\right)\triangle \underline{v}.$$

By repeating these implementations we construct a bounded decreasing sequence $\{(u_n,v_n)\}\subset\left(H_0^1(\Omega)\times H_0^1(\Omega)\right)$ verifying

$$\overline{u}=u_0\geq u_1\geq u_2\geq\ldots\geq u_n\geq\ldots\geq\underline{u}>0, \tag{2.9}$$

$$\overline{v}=v_0\geq v_1\geq v_2\geq\ldots\geq v_n\geq\ldots\geq\underline{v}>0. \tag{2.10}$$

By continuity of functions f,g, h, and l and the definition of the sequences (u_n) and (v_n), there exist positive constants $C_i>0$, $i=1,\ldots,4$ such that

$$|f(u_{n-1},v_{n-1})|\leq C_1,\quad |g(u_{n-1},v_{n-1})|\ \leq C_2 \text{ for all } n. \tag{2.11}$$

From (2.11), multiplying the first equation of (2.8) by u_n, integrating, using Holder inequality and Sobolev embedding we check that

$$a_1\int_\Omega |\nabla u_n|^2\,dx\leq A\left(\int_\Omega |\nabla u_n|^2\,dx\right)\int_\Omega |\nabla u_n|^2\,dx=\lambda\int_\Omega \alpha(x)\,f(u_{n-1},v_{n-1})\,u_n dx$$
$$\leq\lambda\|\alpha\|_\infty\int_\Omega |f(u_{n-1},v_{n-1})|\ \ |u_n|\,dx\leq C_1\int_\Omega\ |u_n|\,dx\leq C_3\,\|u_n\|_{H_0^1(\Omega)}$$

or

$$\|u_n\|_{H_0^1(\Omega)}\leq C_3,\ \forall n, \tag{2.12}$$

where $C_3>0$ is a constant independent of n. Similarly, there exist $C_4>0$ independent of n such that

$$\|v_n\|_{H_0^1(\Omega)}\leq C_4,\ \ \forall n. \tag{2.13}$$

From (2.12) and (2.13), we deduce that $\{(u_n,v_n)\}$ admits a weakly converging subsequence in $H_0^1\left(\Omega,\mathbb{R}^2\right)$ to a limit (u,v) satisfying $u\geq\underline{u}>0$ and $v\geq\underline{v}>0$. Being monotone and also using a standard regularity argument, $\{(u_n,v_n)\}$ converges itself to (u,v). Now, letting $n\to+\infty$ in (2.13), we conclude that (u,v) is a positive weak solution of system (1.1). □

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ON H_vBE-ALGEBRAS

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Abstract. In this paper, we introduce the notion of H_vBE-algebra and investigate the some properties of it. Also some types of H_vBE-algebras are studied and the relationship between them are stated. We try to show that these notions are independent, by some examples. In addition we show that H_vBE-algebra is an extension of hyper BE-algebra and compute the number of H_vBE-algebras in cases $|H| = 2$ and 3. Furthermore, we study several kinds of homomorphisms on H_vBE-algebras.

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1. Introduction and preliminaries

The theory of hyper structures was introduced by Marty in 1934 during the 8th congress of the Scandinavian Mathematicians[8]. A hyper structure is a non-empty set H, together with a function $\circ : H \times H \longrightarrow P^*(H)$ called hyper operation, where $P^*(H)$ denotes the set of all non-empty subsets of H. Marty introduced hypergroups as a generalization of groups. Some basic definitions and the theorems about hyperstructures can be found in [4, 5]. The concept of H_v structures constitute a generalization of well known algebraic hyper structures where the axioms are replaced by the weak ones. H_v structures were first introduced by Vougiouklis in the forth AHA congress(1990)[14].

H. S. Kim and Y. H. Kim introduced the notation of the BE-algebra as a generalization of dual BCK algebra[7]. Using the notation of upper sets, they gave an equivalent condition of upper sets in BE-algebras and discussed some properties of them. A. Rezaei et al. in [11, 12] show that commutative implicative BE-algebra is equivalent to the commutative self distributive BE-algebra.

Recently R. A. Borzooei et al. introduced the notation of pseudo BE-algebra which is a generalization of BE-algebra[3].They defined the basic concepts of pseudo subalgebras and pseudo filters, and proved that under some conditions, pseudo subalgebra can be a pseudo filter[3].

The goal of this paper is combine the concepts H_v structure with BE-algebra and introducing the H_vBE-algebra as a generalization of hyper BE-algebra,

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defining the H_v filter and subalgebra in this structure, also it is defined the some types of H_vBE-algebras and described the relationship between them. Finally present the homomorphisms on H_vBE-algebras with considering properties of them.

Definition 1 ([7])**.** Let X be a non-empty set and let "$*$" be a binary operation on X , $1 \in X$. An algebra $(X,*,1)$ of type $(2,0)$ is called a BE-algebra if the following axioms hold:
$(BE1)$ $x*x=1$,
$(BE2)$ $x*1=1$,
$(BE3)$ $1*x=x$,
$(BE4)$ $x*(y*z)=y*(x*z)$, for all $x,y,z \in X$.

We introduce the relation "$\leq$" on X by $x \leq y$ if and only if $x*y=1$.

Proposition 1 ([7])**.** *Let X be a BE-algebra. Then*
(i) $x*(y*x)=1$.
(ii) $y*((y*x)*x)=1$, *for all* $x,y \in X$.

Example 1 ([2]). Let $X=\{1,2,...\}$ and the operation "$*$" be defined as follows:

$$x*y=\begin{cases} 1 & if \quad y \leq x \\ y & \quad otherwise \end{cases}$$

Then $(X,*,1)$ is a BE-algebra.

Definition 2 ([5])**.** Let H be a non-empty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyper operation. Then $(H,\circ)$ is called an H_v- group if it satisfies the following axioms:
$(H1)$ $x\circ(y\circ z)\bigcap(x\circ y)\circ z \neq \phi$,
$(H2)$ $a\circ H = H\circ a = H$, for all $x,y,z,a \in H$,

where $a\circ H=\bigcup\limits_{h\in H} a\circ h$, $H\circ a=\bigcup\limits_{h\in H} h\circ a$.

Definition 3 ([10])**.** Let H be a non-empty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyperoperation. Then $(H,\circ,1)$ is called a hyper BE-algebra if satisfies the following axioms:
$(HBE1)$ $x<1$ *and* $x<x$,
$(HBE2)$ $x\circ(y\circ z)=y\circ(x\circ z)$,
$(HBE3)$ $x \in 1\circ x$,
$(HBE4)$ $1<x$ implies $x=1$, for all $x,y,z \in H$,

where the relation "$<$" is defined by $x<y \iff 1 \in x\circ y$.

2. ON H_vBE-ALGEBRAS

Definition 4. Let H be a non-empty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyperoperation. Then $(H, \circ, 1)$ is called a H_vBE-algebra if satisfies the following axioms:

(H_vBE1) $x < 1$ *and* $x < x$,

(H_vBE2) $x \circ (y \circ z) \bigcap y \circ (x \circ z) \neq \phi$,

(H_vBE3) $x \in 1 \circ x$,

(H_vBE4) $1 < x$ implies $x = 1$, for all $x, y, z \in H$,

where the relation " $<$ " is defined by $x < y \iff 1 \in x \circ y$.

Also $A < B$ if and only if there exist $a \in A$ and $b \in B$ such that $a < b$.

Example 2. (i)Let $(H, *, 1)$be a BE-algebra. We know that $\circ : H \times H \longrightarrow P^*(H)$ with $x \circ y = \{x * y\}$ is a hyperoperation. Then $(H, \circ, 1)$ is a trivial hyper BE-algebra and a H_vBE-algebra.
(ii) Let $H = \{1,a,b\}$. Define a hyperoperation " $\circ$ " as follows:

$\circ$	1	a	b
1	$\{1\}$	$\{a,b\}$	$\{b\}$
a	$\{1\}$	$\{1,a\}$	$\{1,b\}$
b	$\{1\}$	$\{1,a,b\}$	$\{1\}$.

Then $(H, \circ, 1)$ is a H_vBE-algebra.
(iii) Define a hyper operation " $\circ$ " on $\mathbb{R}$ as follows:

$$x \circ y = \begin{cases} \{y\} & if \quad x = 1 \\ \mathbb{R} & otherwise \end{cases}$$

Then $(\mathbb{R}, \circ, 1)$ is a H_vBE-algebra.

Proposition 2. *Any hyper BE-algebra is a H_vBE-algebra.*

Proof. It is clear. □

In the following example we show that the converse of Proposition 2 is not true.

Example 3. Define a hyperoperation " $\circ$ " on the set $H = \{1,a,b\}$ as follows:

$\circ$	1	a	b
1	$\{1\}$	$\{a\}$	$\{b\}$
a	$\{1, b\}$	$\{1\}$	$\{1,a,b\}$
b	$\{1\}$	$\{1,b\}$	$\{1,b\}$.

Then $(H,\circ,1)$ is an H_vBE-algebra. And we have that:
$a\circ(b\circ b)=a\circ(\{1,b\})=\{1,a,b\}\neq\{1,b\}=b\circ(\{1,a,b\})=b\circ(a\circ b)$.
So $(H,\circ,1)$ does not satisfy $(HBE2)$, and $(H,\circ,1)$ is not a hyper BE-algebra.

Theorem 1. *Let $(H,\circ,1)$ be an H_vBE-algebra. Then*

(i) $A\circ(B\circ C)\bigcap B\circ(A\circ C)\neq\phi$ for every $A,B,C\in P^*(H)$,

(ii) $A<A$,

(iii) $1<A$ implies $1\in A$,

(iv) $1\in x\circ(x\circ x)$,

(v) $x<x\circ x$.

Proof. (i) Let $a\in A,b\in B,c\in C$. Then $a\circ(b\circ c)\subseteq A\circ(B\circ C)$, $b\circ(a\circ c)\subseteq B\circ(A\circ C)$, by (H_vBE2), we have $a\circ(b\circ c)\bigcap b\circ(a\circ c)\neq\phi$. therefore $A\circ(B\circ C)\bigcap B\circ(A\circ C)\neq\phi$.

(ii) Let $a\in A$. Then by (H_vBE1) $A<A$.

(iii) Let $1<A$. Then there exists an element $a\in A$ such that $1<a$ by using (H_vBE4) $a=1$ and so $1\in A$.

(iv) Let $x\in H$. Then $x<x$, by definition $1\in x\circ x$ therefore $x\circ 1\subseteq x\circ(x\circ x)$. Also by (H_vBE1), $x<1$, so by definition $1\in x\circ 1$ and then $1\in x\circ(x\circ x)$.

(v) By (iv) $1\in x\circ(x\circ x)$. Then there exists $b\in x\circ x$ such that $1\in x\circ b$ and so $x<b$. □

In the following proposition we compute the number of H_vBE-algebras in two cases.

Proposition 3. *For a set H if*
(i) $|H|=2$, there exist precisely 2^4 different H_vBE-algebras $(H,\circ,1)$,
(ii) $|H|=3$, there exist at most $4^7\times 7^2$ different H_vBE-algebras $(H,\circ,1)$.

Proof. (i) Let $|H|=2$ and $(H,\circ,1)$ be a H_vBE-algebra. Then $H=\{1,a\}$.
Consider the following table:

(I)

$\circ$	1	a
1	A	B
a	C	D

By (H_vBE1) and (H_vBE3), we have $A,C,D \in \{\{1\},\{1,a\}\}$ and $B \in \{\{a\},\{1,a\}\}$.
Thus the cardinality of A,B,C,D is at most 2.
So the number of H_vBE-algebras $(H,\circ,1)$ is at most 2^4.
Now, to determine the number of H_vBE-algebras $(H,\circ,1)$, we must consider the condition of H_v BE-algebra on table (I) for different A,B,C,D, when $A,C,D \in \{\{1\},\{1,a\}\}$ and $B \in \{\{a\},\{1,a\}\}$, that gives 2^4 different tables.One can see that every table introduce a H_vBE-algebra.
In the following we consider two cases of tables:

(1)

$\circ$	1	a
1	$\{1\}$	$\{1,a\}$
a	$\{1,a\}$	$\{1,a\}$

(2)

$\circ$	1	a
1	$\{1,a\}$	$\{1,a\}$
a	$\{1,a\}$	$\{1,a\}$.

In any table,we see that (H_vBE1), (H_vBE3) and (H_vBE4) are obvious.
In tables, we choose x,y,z from $\{1,a\}$ and conclude $x\circ(y\circ z)\bigcap y\circ(x\circ z)\neq\phi$.
For example in (1): $a\circ(1\circ 1)=1\circ(a\circ 1)=\{1\}$. Similarly in (2): $a\circ(1\circ 1)=\{1,a\}=1\circ(a\circ 1)$.
(ii): In the following, we compute the number of H_vBE-algebras in three cases:
Case 1: a and b are arbitrary,
Case 2: $a<b$,
Case 3: $b<a$.

Case 1. Let $H=\{1,a,b\}$ and $(H,\circ,1)$ be a H_vBE-algebra. Consider $a\circ a$, we have $1\in a\circ a$ and $a\circ a\in\{\{1\},\{1,a\},\{1,b\},\{1,a,b\}\}$. Therefore $|a\circ a|\leq 4$.
Similarly, we obtain:

$$max|x\circ y|=\begin{cases} 4 & for \quad x=y=a \\ 7 & for \quad x=a,y=b \\ 4 & for \quad x=a,y=1 \\ 7 & for \quad x=b,y=a \\ 4 & for \quad x=b,y=b \\ 4 & for \quad x=b,y=1 \\ 4 & for \quad x=1,y=a \\ 4 & for \quad x=1,y=b \\ 4 & for \quad x=1,y=1 \end{cases}.$$

So the number of different H_vBE-algebra is at most $4^7\times 7^2$.

Case 2. If $a<b$, then $1\in a\circ b$

$$a\circ b\in\{\{1\},\{1,a\},\{1,b\},\{1,a,b\}\}. \quad (1)$$

The following array obtained

$$max|x\circ y| = \begin{cases} 4 & for \quad x=y=a \\ 4 & for \quad x=a, y=b \\ 4 & for \quad x=a, y=1 \\ 7 & for \quad x=b, y=a \\ 4 & for \quad x=b, y=b \\ 4 & for \quad x=b, y=1 \\ 4 & for \quad x=1, y=a \\ 4 & for \quad x=1, y=b \\ 4 & for \quad x=1, y=1. \end{cases}$$

Therefore the number of H_vBE-algebras $(H,\circ,1)$ is at most $4^8\times 7$.

Case 3. If $b<a$, in a similar way, we conclude that the number of H_vBE- algebras $(H,\circ,1)$ is at most $4^8\times 7$. □

Notation 1. We see that 1 belongs to any triple combination elements of {1,a,b} in Case 2, for example: $1\in b\circ(a\circ b)\bigcap a\circ(b\circ b)$ because $1\in a\circ b$ then $b\circ 1\subseteq b\circ(a\circ b)$ and $1\in b\circ 1$ therefore $1\in b\circ(a\circ b)$. Also, $1\in b\circ b$ then $1\in a\circ 1\subseteq a\circ(b\circ b)$, therefore $1\in b\circ(a\circ b)\bigcap a\circ(b\circ b)\neq\phi$.

3. Some types of H_vBE-algebras

In this section, we introduce some types of H_vBE algebras .

Definition 5. A H_vBE-algebra is said to be
(i)a row H_vBE-algebra (briefly, $R-H_vBE$-algebra), if $1\circ x=\{x\}$, for all $x\in H$,
(ii)a column H_vBE-algebra (briefly, $C-H_vBE$-algebra), if $x\circ 1=\{1\}$, for all $x\in H$,
(iii)a diagonal H_vBE-algebra (briefly, $D-H_vBE$-algebra), if $x\circ x=\{1\}$, for all $x\in H$,
(iv)a thin H_vBE-algebra (briefly, $T-H_vBE$-algebra), if it is an $RC-H_vBE$-algebra ($RC-H_v$ means $R-H_v$ and $C-H_v$),
(v)a very thin H_vBE-algebra (briefly,$V-H_vBE$-algebra), if it is an $RCD-H_vBE$-algebra ($RCD-H_v$ means $R-H_v, C-H_v$ and $D-H_v$).

Example 4. (i) Every BE-algebra as $(H,*,1)$ with hyperoperation $x\circ y=\{x*y\}$ is an $RCD-H_vBE$-algebra.

(ii) Let $H=\{1,a\}$ and $H'=\{1,a,b\}$. Define the hyperoperations $\circ_1$ and $\circ_2$ correspond to H and hyperoperations $\circ_3$ and $\circ_4$ correspond to H' as follows:

$\circ_1$	1	a
1	$\{1\}$	$\{a\}$
a	$\{1,a\}$	$\{1\}$

$\circ_2$	1	a
1	$\{1\}$	$\{a\}$
a	$\{1\}$	$\{1,a\}$

$\circ_3$	1	a	b
1	$\{1\}$	$\{a\}$	$\{b\}$
a	$\{1,b\}$	$\{1\}$	$\{1\}$
b	$\{1,b\}$	$\{1\}$	$\{1\}$

$\circ_4$	1	a	b
1	$\{1\}$	$\{a\}$	$\{b\}$
a	$\{1\}$	$\{1\}$	$\{b\}$
b	$\{1\}$	$\{1,a\}$	$\{1\}$

Then $(H,\circ_1,1)$ is a $R-H_vBE$-algebra, $(H,\circ_2,1)$ is an $T-H_vBE$-algebra, $(H',\circ_3,1)$ is a $D-H_vBE$-algebra and $(H',\circ_4,1)$ is a $V-H_vBE$-algebra.

Theorem 2. *Let $(H,\circ,1)$ be a $D-H_vBE$- algebra. Then*
(i) $1\in x\circ a$, for some $a\in 1\circ x$,
(ii) if H be a $C-H_vBE$ algebra, then $1\in y\circ(x\circ y)$, for all $x,y\in H$.

Proof. (i) By Definition 5, $1=1\circ(x\circ x)$ and by (H_vBE2) we have $1\circ(x\circ x)\bigcap x\circ(1\circ x)\neq\phi$ and $1\circ(x\circ x)$ is singelton, then $1\in x\circ(1\circ x)=\bigcup\limits_{a\in 1\circ x} x\circ a$ and $1\in x\circ a$
for some $a\in 1\circ x$.

(ii) By (H_vBE2) and Definition 5, we obtain,
$\phi\neq y\circ(x\circ y)\bigcap x\circ(y\circ y)=y\circ(x\circ y)\bigcap x\circ 1=y\circ(x\circ y)\bigcap\{1\}$
Hence $1\in y\circ(x\circ y)$. □

Proposition 4. *Let $H=\{1,a,b\}$ and $(H,\circ,1)$ be an H_vBE-algebra. Determine the number of non-isomorphic $(H,\circ,1)$ in the following cases.*
(i) $(H,\circ,1)$ is an $R-H_vBE$-algebra,
(ii) $(H,\circ,1)$ is a $C-H_vBE$-algebra,
(iii) $(H,\circ,1)$ is a $D-H_vBE$-algebra,
(iv) $(H,\circ,1)$ is a $T-H_vBE$-algebra,
(v) $(H,\circ,1)$ is a $V-H_vBE$-algebra.

Proof. (i) By Proposition 3 and $1\circ x=\{x\}$, for all $x\in H$, we have the following array:

$$max|x\circ y| = \begin{cases} 4 & for \quad x=y=a \\ 7 & for \quad x=a, y=b \\ 4 & for \quad x=a, y=1 \\ 7 & for \quad x=b, y=a \\ 4 & for \quad x=b, y=b \\ 4 & for \quad x=b, y=1 \\ 1 & for \quad x=1, y=a \\ 1 & for \quad x=1, y=b \\ 1 & for \quad x=1, y=1. \end{cases}$$

Therefore the number of $R-H_vBE$- algebras is at most $4^4 \times 7^2$.
(iv) Since $1\circ x = \{x\}$ and $x\circ 1 = \{1\}$ for all $x \in H$. We have the following array:

$$max|x\circ y| = \begin{cases} 4 & for \quad x=y=a \\ 7 & for \quad x=a, y=b \\ 1 & for \quad x=a, y=1 \\ 7 & for \quad x=b, y=a \\ 4 & for \quad x=b, y=b \\ 1 & for \quad x=b, y=1 \\ 1 & for \quad x=1, y=a \\ 1 & for \quad x=1, y=b \\ 1 & for \quad x=1, y=1 \end{cases}$$

Hence the number of $T-H_vBE$- algebras is at most $4^2 \times 7^2$.
Similarly, for $(ii), (iii)$ and (v) we obtain the numbers $T-H_vBE$- algebras $(4^5 \times 7), (4^5 \times 7)$ and (7^2) respectively. □

In the next example we explain some relationship among $(R,C,D,T)-H_vBE-$algebras.

Example 5. (i) Every $R-H_vBE$-algebra need not be a $D-H_vBE$-algebra, because, in Example 4, $(H,\circ_2,1)$ is an $R-H_vBE$-algebra but it is not a $D-H_vBE$-algebra.

(ii) Every $RD-H_vBE$-algebra need not be a $C-H_vBE$-algebra, because, in 4 we consider that $(H',\circ_3,1)$ is an $RD-H_v$ BE-algebra, but it is not a $C-H_vBE$-algebra.

(iii) Every $T-H_vBE$- algebra need not be a $D-H_vBE$-algebra, because in 4, we see that $(H,\circ_2,1)$ is a $T-H_vBE$-algebra but it is not a $D-H_vBE$-algebra.

4. WEAK FILTERS IN H_vBE-ALGEBRAS

In [10] it is defined the concept of hyper filters in the hyper BE-algebras. In this section we introduce filters and subalgebras in H_vBE-algebras and state the relationship between them .

Definition 6. Let F be a non-empty subset of a H_vBE-algebra H and $1 \in F$. Then F is said to be

(i) a *weak H_v filter* of H if $x \circ y \subseteq F$ and $x \in F$ imply $y \in F$, for all $x, y \in H$.

(ii) a *H_v filter* of H if $x \circ y \approx F (i.e., \phi \neq (x \circ y) \bigcap F)$ and $x \in F$ imply $y \in F$, for all $x, y \in H$.

Example 6. Define hyperoperations "$\circ_1$" and "$\circ_2$" on $H = \{1,a,b\}$ as follows:

$\circ_1$	1	a	b
1	{1}	{a,b}	{b}
a	{1}	{1,a}	{1,b}
b	{1}	{1,a,b}	{1}

$\circ_2$	1	a	b
1	{1}	{a,b}	{b}
a	{1}	{1,a,b}	{b}
b	$\{1,b\}$	{1,a,b}	{1,a,b}.

We see that $(H, \circ_1, 1)$ is an H_vBE-algebra and $F_1 = \{1, a\}$ is a weak H_v filter of H. Also $(H, \circ_2, 1)$ is an H_vBE-algebra and $F_2 = \{1, a\}$ is an H_v filter of H.

In Example 6, F_1 is not an H_v filter, because $a \circ_1 b \approx F_1$ and $a \in F_1$, but $b \notin F_1$.

Theorem 3. *Every H_v filter is a weak H_v filter.*

Proof. It is straightforward. □

Notation: By Example 6, we can see that the notion of a• *weak H_v filter* and a •*H_v filter* are different in H_vBE-algebras.

Theorem 4. *Let F be a subset of an H_vBE-algebra H and $1 \in F$. If $x \circ y < F$ and $x \in F$ implies $y \in F$, for all $x, y \in H$, then F=H.*

Proof. Let x be an arbitrary element of H, by (H_vBE1) and by (H_vBE3),we obtain $x \in 1 \circ x$. Since $1 \in F$ and $x < 1$, we have $1 \circ x < F$. By hypothesis, $x \in F$, i.e. , $H \subseteq F$, This prove that $F = H$. □

Definition 7. A subset S of a H_vBE algebra H is said to be a •*subalgebra*, if $x \circ y \subseteq S$, for all $x, y \in S$.

Example 7. In Example 6, {1,b} is a subalgebra of $(H, \circ_1, 1)$, but {1,a} is not a subalgebra of $(H, \circ_1, 1)$ because $1 \circ a \nsubseteq \{1, a\}$.

Theorem 5. *Let H be an H_vBE-algebra and S be a subalgebra of H. Then*

(i) S is a weak H_v filter of H if and only if for all $x \in S$ and $y \in H \setminus S, x \circ y \nsubseteq S$.

(ii) S is an H_v filter of H if and only if for all $x \in S$ and $y \in H \setminus S$, $x \circ y \not\approx S$.

Proof. (i) Let S be a weak H_v filter of H, $x \in S$, and $y \in H \setminus S$ and $x \circ y \subseteq S$. Since S is a weak filter and $x \in H$, we have $y \in S$, which is a contradiction.

Conversely, let $x \circ y \nsubseteq S$ where $x \in S$ and $y \in H \setminus S$. Let $x \circ y \subseteq S$ and $x \in S$. If $y \notin S$, then by assumption , $x \circ y \nsubseteq S$, which is a contradiction.

(ii) Let S be an H_v filter of H, $x \in S$ and $y \in H \setminus S$, and $x \circ y \approx S$. Since S is an H_v filter and $x \in S$ we have $y \in S$ which is a contradiction.

Conversely, let $x \circ y \not\approx S$ where $x \in S$ and $y \in H \setminus S$. If $x \circ y \approx S$, $x \in S$ and $y \notin S$, then by assumption, $x \circ y \not\approx S$, which is a contradiction. □

In the next examples we show that in general every (weak) H_v filter of H is not a subalgebra and conversely.

Example 8. In Example 6, F_1 and F_2 are both weak H_v filters and H_v filters of H but these are not subalgebras of H.

Example 9. (i) Define a hyperoperation "$\circ$" on $H = \{1, a, b\}$, as follows:

$\circ$	1	a	b
1	$\{1\}$	{a}	{b}
a	$\{1, a, b\}$	$\{1\}$	{a,b}
b	$\{1, a, b\}$	{1,a,b}	$\{1\}$.

We see that $(H, \circ, 1)$ is a $D - H_v - BE$ algebra and $F_1 = \{1, a\}$ is a weak H_v filter. Since $a \circ 1 = \{1, a, b\} \nsubseteq \{1, a\}$, $\{1, a\}$ is not a subalgebra of H.

(iii) Let $H = \{1, a, b\}$ and "$\circ$" be a hyperoperation as follows:

$\circ$	1	a	b
1	$\{1\}$	{a,b}	{b}
a	$\{1\}$	{1,a,b}	{b}
b	$\{1, b\}$	{1,a,b}	{1,a,b} .

We see that $(H, \circ, 1)$ is an H_vBE-algebra and $F_2 = \{1, b\}$ is a subalgebra of H. F_2 is not an H_v filter because $b \circ a = \{1, a, b\}$ and $(b \circ a) \bigcap F_2 \neq \phi, b \in F_2$ but $a \notin F_2$.

5. Homomorphisms on H_vBE-algebras

Homomorphisms of algebraic hyperstructures are studied by Dresher, Ore, Krasner, Kuntzman, Koskas, Jantosciak, Corsini, Davvaz and many others [1, 5, 6, 9, 13]. In this section, we study several kinds of homomorphisms on Hv BE-algebras.

Definition 8. Let $(H_1, o, 1)$ and $(H_2, *, 1')$ be two H_vBE-algebras.
A map $f : H_1 \to H_2$ is said to be:
(1) a •*homomorphism* or •*inclusion homomorphism* if $f(x \circ y) \subseteq (f(x) * f(y))$ and $f(1) = 1'$, for all $x, y \in H_1$,
(2) a •*good homomorphism* if for all x, y of H_1, we have $f(x \circ y) = f(x) * f(y)$ and $f(1) = 1'$,
(3)an •*isomorphism* if it be an one to one and onto good homomorphism. If f is an •*isomorphism*, then H_1 and H_2 are said to be •*isomorphic*, which is denoted by $H_1 \cong H_2$,
(4) a •*weak homomorphism* if $f(x \circ y) \bigcap (f(x) * f(y)) \neq \phi, f(1) = 1'$, for all $x, y \in H_1$.

Example 10. Let $H_1 = \{1, a, b\}, H_2 = \{1', a', b'\}$. Define hyperoperations "$\circ_1$" and "$\circ_2$" as follows:

$\circ_1$	1	a	b
1	$\{1\}$	{a,b}	{b}
a	$\{1\}$	{1,a}	{1,b}
b	$\{1\}$	{1,a,b}	$\{1\}$

$\circ_2$	$1'$	a'	b'
$1'$	$\{1'\}$	$\{a',b'\}$	$\{b'\}$
a'	$\{1',b'\}$	$\{1,a',b'\}$	$\{1',b'\}$
b'	$\{1',b\}$	$\{1',a',b'\}$	$\{1,a',b'\}$.

We see that $(H_1, \circ_1, 1)$ and $(H_2, \circ_2, 1')$ are H_vBE-algebras.
Let $f : H_1 \to H_2$ be defined by $f(1) = 1', f(a) = a', f(b) = b'$. Clearly, f is an inclusion homomorphism, but it is not a good homomorphism, because $f(a \circ_1 1) = f(\{1\}) = \{1'\}, f(a) \circ_2 f(1) = a' \circ_2 1' = \{1', b'\}$.

Proposition 5. *Let $f : H_1 \to H_2$ be a one to one and onto map, $(H_1, \circ, 1)$ and $(H_2, *, 1')$ are H_vBE-algebras.*
If we have $f(x \circ y) = f(x) * f(y)$, then $f(1) = 1'$.

Proof. By (H_vBE4) we know that the element 1 in every H_vBE-algebra is unique. We must prove that :
(i) $f(1) \in x' * f(1), f(1) \in x' * x'$,
(ii) $x' \in f(1) * x'$,
(iii) $f(1) < x'$ implies $x' = f(1)$, for all $x' \in H_2$.

Since $x' \in H_2$ and f is onto, there exists $x \in H_1$ such that $f(x) = x'$.
By $(HvBE1)$, $x < 1$ and hence $1 \in x \circ 1$ .Moreover

$$f(1) \in f(x \circ 1) = f(x) * f(1) = x' * f(1).$$

Therefore $f(1) \in x' * f(1)$. The proof of other parts is similar. □

Notation 2. We can see that any *homomorphism* is a *weak homomorphism*, but conversely need not be true.

Example 11. Let $H_1 = \{1, a, b\}, H_2 = \{1', a', b'\}$. Define hyperoperations "$\circ$" and "$*$" as follows:

$\circ$	1	a	b
1	$\{1\}$	{a,b}	{b}
a	$\{1\}$	{1,a,b}	{b}
b	$\{1, b\}$	{1,a,b}	{1,a,b}

$*$	$1'$	a'	b'
$1'$	$\{1'\}$	$\{a', b'\}$	$\{b'\}$
a'	$\{1'\}$	$\{1', a'\}$	$\{1', b'\}$
b'	$\{1'\}$	$\{1', a', b'\}$	$\{1'\}$.

We see that $(H_1, \circ, 1), (H_2, *, 1')$ are H_vBE algebras.
Let $f : H_1 \to H_2$ be defined by $f(1) = 1', f(a) = a', f(b) = b'$. Then f is a *weak homomorphism*, but it is not an *inclusion homomorphism*, because $f(b \circ 1) = f(\{1, b\}) = \{1', b'\}$, Then $f(b) * f(1) = b' * 1' = \{1'\}$, therefore $f(b \circ 1) \bigcap (f(b) * f(1)) \neq \phi$, But $f(b \circ 1) \nsubseteq f(b) * f(1)$.

6. Conclusion

In this present paper, we have introduced the concept of H_vBE-algebras and investigated some of their useful properties.

This work focused on some types of H_vBE-algebras. Also we discuss on H_v filters in this structure and present some fundamental properties that compute number of particular H_vBE-algebras.
In our future work, we will get more results in H_vBE-algebras with applications, and we will define concepts as a quotient, a center in H_vBE-algebras and construct new BE-algebra or H_vBE-algebra.

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STABILITY AND ZERO-HOPF BIFURCATION ANALYSIS OF A TUMOUR AND T-HELPER CELLS INTERACTION MODEL IN THE CASE OF HIV INFECTION

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Abstract. In this paper, we present a mathematical model governing the dynamics of tumour-immune cells interaction under HIV infection. The interactions between tumour cells, helper T-cells, infected helper T-cells and virus cells are explained by using delay differential equations including two different discrete time delays. In the model, these time lags describe the time needed by the helper T-cells to find (or recognize) tumour cells and virus, respectively. First, we analyze the dynamics of the model without delays. We prove the positivity of the solution, analyze the local and global stabilities of the steady states of the model. Second, we study the effects of two discrete time delays on the stability of the endemically infected equilibrium point. We determine the conditions on parameters at which the system undergoes a zero-Hopf bifurcation. Choosing one of the delay terms as a bifurcation parameter and fixing the other, we show that a zero-Hopf bifurcation arises as the bifurcation parameter passes through a critical value. Finally, we perform numerical simulations to support and extend our theoretical results. The results concluded help to better understand the links between the immune system and the tumour development in the case of HIV infection.

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1. Introduction

Human immunodeficiency virus (HIV) is a retrovirus that attacks and destroys the immune system, so that it causes the HIV infection [10] and the disease called AIDS (Acquired Immune Deficiency Syndrome [8, 24, 30]). AIDS is a condition in humans in which the progressive collapse of the immune system allows infections and cancers which threat human life. According to data published by the World Health Organization (WHO), it is presumed that more than 35 million people died due to AIDS since it was first discovered in 1981, and 37.9 million people were living with HIV at the end of 2018. There is no cure for AIDS but there are certain medicines that are used to slow down this disease.

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The main target of HIV is the immune system, especially the $CD4^+$ T-cells, which are the most sufficient white blood cells, or lymphocytes. $CD4^+$ T lymphocytes play a central role in orchestrating the beginning and maintenance of the adaptive immune response [2,13]. These cells are more commonly referred to as helper T-cells which is the term we will use in this paper. The helper T-cells can be categorized as effector and non-effector helper T-cells. The effector helper T-cells are triggered by the tumour cell proliferation, and they have a death rate which is considerably small compare to the normal helper T-cells. These cells are mediated by the cytokines secreted by the differentiated cells. Therefore, they are specific to certain disease [2]. HIV causes the destruction of helper T-cells; as a result, it decreases the body's ability to fight the infection. The count of helper T-cell is normally around 1000 mm^{-3} in a human being. However, when it decreases to 200 mm^{-3} or below in an HIV infected individual, then that person is classified as having AIDS [15,26].

HIV infected individuals have a high risk of developing certain tumours. The connection between HIV/AIDS and certain tumours has not been understood completely, but it likely depends on a weakened immune system [4, 6, 33, 36]. When normal cells begin to change and grow uncontrollably, a mass called tumour forms. A tumour can be benign, also called non-cancerous, or malignant which is cancerous (see the reference [1] for more details). Malignancies that are found to have a high incidence of HIV-infected individuals are Kaposi's sarcoma (KS), Hodgkin's disease (HD), non-Hodgkin's lymphoma (NHL), squamous cell carcinomas, plasmacytomas and leiomyosarcoma in children [36]. In the tumour cells of HIV infected patients, no viral sequence in the DNA was found; therefore, it seems that the virus doesn't include the tumour itself [20]. Furthermore, the immune surveillance hypothesis explains that the immune system patrols the body to recognize and destroy invading pathogens [6,7].

There is a variety of mathematical models that studied and analyzed the tumour growth [19, 27]. Also, in recent years, modeling, analysis and control of the HIV infection have attracted the attention of researchers in bio-mathematics (see, for example, [11, 25, 26] and references cited therein). However, the interaction between both the HIV-immune system and the tumour-immune system has not been fully understood, since several types of tumours are associated to the HIV occurrence. Therefore, it is important to investigate the dynamics of the immune system in this situation. A few of such studies can be found in [5, 12, 21, 22, 29].

The aim of this work is to better understand the interaction between tumour and immune system in an HIV infected individual. More precisely, we want to analyze how the existence of HIV infection affects the dynamics of the immune system and tumour cells. To do this, we use a system of delay differential equations where the time lag describes the time needed by helper T-cells to find (or it can be said 'recognize') tumour cells and virus. Inspired by the studies in [12, 29], we present a mathematical model involving four populations: tumour cells, uninfected effector

helper T-cells, infected helper T-cells and free virus. The models presented in this manuscript and also [12, 29] are improved from the model introduced in the papers by Lou et al. [21, 22]. In [22], Lou et al. presented a model of cell-to-cell spread of HIV together with tumour in tissue cultures (in vitro). They aimed at explaining some properties concerning tumour occurring during the HIV infection. The same model is studied with the delay in [21].

In [5], Bodnar et al. proposed a model to describe the HIV related tumour-immune system interactions in vitro. Moreover, in [12], Forys and Poleszczuk considered a similar model. However, they considered the issue of immune reaction against tumour and the second way for HIV to disseminate in vivo (circulating free viral particles to T-cells directly). In [29], Rihan and Rahman studied a model which describes the interaction between tumour cells, the general population of the helper T-cells, infected helper T-cells and virus where the time lag is considered to represent the time needed by the healthy effector cells to recognize the tumour cells and virus. Also, they consider that HIV disseminates in vivo by circulating free viral particles to T-cells directly.

The difference between the study in [29] and the present paper is that the model we examined here involves two different discrete delays, and its stability and bifurcation analysis is given for the full model, i.e., without reducing the dimension of the model proposed. In addition, the difference between the study in [12] and this work is that the helper T-cells that we consider are exactly the effector helper T-cells, not general ones. HIV attacks the helper T-cells but then each helper T-cell is target to the virus. Namely, we care more about the cells that are specific to tumour and what happened to them is more important for us.

In this paper, the existence, uniqueness and non-negativity of solutions, and also both local and global stabilities of steady states of the model without delay are first studied. And then, the existence of a zero-Hopf bifurcation for the model with delay is given. Apart from Section 1, the paper is organized in the following aspect: In Section 2, we introduce a mathematical model of HIV infection with tumour cells; the mathematical analysis of the ODE model is performed. The theoretical results about steady states and their stabilities are presented in Section 3. Later, the analysis is presented for the DDE model in Section 4. The existence of a zero-Hopf bifurcation is investigated in this section. Numerical simulations that support and extend the theoretical results are given in Section 5. Section 6 is devoted to conclusions and predictions of the models.

2. THE MODEL

We consider the following delay differential equation system describing the tumour-immune system interactions in the case of HIV:

$$
\begin{aligned}
\frac{dT}{dt} &= r_1 T(t) - k_1 T(t) E(t-\tau_1), \\
\frac{dE}{dt} &= r_2 T(t) - \mu_1 E(t) - \theta k_1 T(t) E(t-\tau_1) - k_2 E(t-\tau_2) V(t), \\
\frac{dI}{dt} &= k_2 E(t-\tau_2) V(t) - c I(t), \\
\frac{dV}{dt} &= N\delta I(t) - \mu_2 V(t),
\end{aligned}
\tag{2.1}
$$

where $T(t), E(t), I(t), V(t)$ denote the concentration of tumour cells, healthy effector helper T-cells, helper T-cells infected by free HIVs and free HIV particles at time t, respectively, and all parameters are positive. The time lags τ_1 and τ_2 describe the time needed by the helper T-cells to recognize tumour cells and virus, respectively.

We assume that the tumour cells grow exponentially with a constant proliferation rate; we do not consider resource limitation. Such type of tumour growth is experimentally observed at the beginning of the tumour development [35]. Also, we assume the linear response of the immune system to tumour cell presence. In the model, we take this response as proportional to the multiplication of both tumour and immune system cell concentrations.

Healthy effector helper T-cells are reproduced as a result of the presence of tumour. The parameter r_2 indicates the antigenicity of tumour. Antigenicity can be thought of as a measure of how different the tumour cells are from normal cells [16]. Parameters μ_1, c and μ_2 are natural death rates of the healthy T-cells, the infected T-cells and the HIV particles, respectively, because cells have a finite life span. Also, the parameter θ represents the small percentage of T-cells that do not survive after killing the tumour cells (the inequality $\theta \leq 1$ is obvious because of the definition of this parameter).

HIV can spread out in vivo either by transmission of cell-free virus or directly from cell-to-cell via the formation of virological synapses as stated in [9, 17, 18]. In this model, similar to [11], we assume that the transition of the healthy T-cells into the infected ones is due to direct interaction with the virus. Accordingly, the infection rate is given by k_2 which increases the count of the infected helper T-cells.

Finally, according to [26], the virus is produced by the productively infected T-cells. Here, we have assumed that on average each productively infected helper T-cell produces N virus during its lifetime. Since the average lifetime of a productively infected cell is $\frac{1}{\delta}$, the average rate of virus production is $N\delta$. Therefore, $N\delta$ represents the source for free viruses. In this derivation, we have ignored the loss of virus due to the infection of a cell.

3. Analysis of the ODE model

In this section, we study the model without delay. By taking $\tau_1 = \tau_2 = 0$ in system (2.1) we obtain the following ODE model:

$$\begin{aligned}
\frac{dT}{dt} &= r_1 T(t) - k_1 T(t)E(t),\\
\frac{dE}{dt} &= r_2 T(t) - \mu_1 E(t) - \theta k_1 T(t)E(t) - k_2 E(t)V(t),\\
\frac{dI}{dt} &= k_2 E(t)V(t) - cI(t),\\
\frac{dV}{dt} &= N\delta I(t) - \mu_2 V(t),
\end{aligned} \tag{3.1}$$

where all parameters and variables are the same as described in the former section.

One can write system (3.1) as a vector equation form as follows:

$$\frac{dX(t)}{dt} = F(t, X(t)), \tag{3.2}$$

where $X(t) = (T(t), E(t), I(t), V(t))^T$ and $F(t) = (f_1(t), f_2(t), f_3(t), f_4(t))^T$ in which

$$\begin{aligned}
f_1(t) &= r_1 T(t) - k_1 T(t)E(t),\\
f_2(t) &= r_2 T(t) - \mu_1 E(t) - \theta k_1 T(t)E(t) - k_2 E(t)V(t),\\
f_3(t) &= k_2 E(t)V(t) - cI(t),\\
f_4(t) &= N\delta I(t) - \mu_2 V(t).
\end{aligned}$$

3.1. *Positivity of solutions*

The following Lemma underlines that for positive initial data, the solution of system (3.1) uniquely exists and remains in $\mathbb{R}^4_+$. From biological point of view, it means that the model is reasonable in the sense that no population becomes negative. Therefore, there is no need analyzing of the trivial steady state of system (3.1).

Lemma 1. *The solution of system (3.1) with non-negative initial conditions T_0, E_0, I_0 and V_0 uniquely exists and remains in $\mathbb{R}^4_+$.*

Proof. Note that $F(t, X(t))$ in Eq. (3.2) is continuous and also Lipschitz with respect to $X(t)$ on any four-dimensional box D. Then system (3.1) has the unique solution on $[0, b)$ where b can be determined as $t \to b^-$ at which either the solution becomes unbounded or the solution approaches to the boundary of D. In addition, We assume that $T(t)$, $E(t)$, $I(t)$ and $V(t)$ initially have positive values. Recall that all constants in the system are positive. For positive initial conditions T_0, E_0, I_0 and V_0, from the first and the second equations of system (3.1) we have the following (where $A(t)$ is the integrating factor):

$$T(t) = T_0 e^{\int_0^t r_1 - k_1 E(s)ds} \geq 0, \qquad \forall t \geq 0.$$

$$E(t) = \frac{A(0)E_0 + r_2 \int_0^t A(s)T(s)ds}{A(t)} \geq 0, \qquad \forall t \geq 0.$$

From the third and the fourth equations of the system, we have

$$I(t) = I_0 e^{-ct} + k_2 e^{-ct} \int_0^t e^{cs} E(s)V(s)ds,$$
$$V(t) = V_0 e^{-\mu_2 t} + N\delta e^{-\mu_2 t} \int_0^t e^{\mu_2 s} I(s)ds.$$

Let us denote by t_* the first time for which one of the populations $I(t)$ and $V(t)$ become zero, or more precisely $\min\{I(t_*), V(t_*)\} = 0$. Without loss of generality, let $V(t_*) = 0$. So, $I(t_*) > 0$ for $t \in [0, t_*]$ since t_* is the first time for which one of the populations $I(t)$ and $V(t)$ become zero. Also $V(t) > 0$ for $t \in [0, t_*)$ since we assume that $T(t)$, $E(t)$, $I(t)$ and $V(t)$ initially have non-negative values. Therefore, $V(t)$ must be non-increasing on $[0, t_*]$, or more precisely

$$\left.\frac{dV}{dt}\right|_{t=t_*} \leqslant 0.$$

On the other hand, one can see that from the last equation of system (3.1)

$$\left.\frac{dV}{dt}\right|_{t=t_*} = N\delta I(t_*) - \mu_2 V(t_*) = N\delta I(t_*) > 0$$

since the equation $V(t_*) = 0$ holds. Consequently, this leads to a contradiction. Thus, there cannot be found a t_* such that $V(t_*) = 0$. So, for $\forall t > 0$, $V(t) > 0$ and $I(t) > 0$. This completes the proof. □

3.2. *Steady states*

In order to fully understand the dynamics of the model, first we must establish the values of steady states. The steady states of system (3.1) can be obtained by setting the equations $f_1(t)$, $f_2(t)$, $f_3(t)$, $f_4(t)$ simultaneously equal to zero. The following lemma explains the steady states of the model.

Lemma 2. *Let*

$$\Re_1 = r_2 - \theta r_1 \text{ and } \Re_0 = \sqrt{\frac{N\delta k_2 r_1}{c k_1 \mu_2}}.$$

If $\Re_1 > 0$, then system (3.1) has two non-negative steady states other than the trivial one:

(1) *If $\Re_0 \neq 1$, then one obtains the non-infected steady state $S_0 = \left(\frac{\mu_1 r_1}{k_1(r_2 - \theta r_1)}, \frac{r_1}{k_1}, 0, 0\right)$.*

(2) *If $\Re_0 = 1$, then one obtains the steady state $S^* = \left(\frac{\mu_1 r_1 + k_2 r_1 \vartheta}{k_1(r_2 - \theta r_1)}, \frac{r_1}{k_1}, \frac{r_1 k_2}{k_1 c}\vartheta, \vartheta\right)$, where $\vartheta \in \mathbb{R}^+ \cup \{0\}$.*

3.3. *Local stability analysis*

The fact that the stability properties depend on the eigenvalues of the system is well-known for linear ODEs. However, our model is nonlinear, and thus we must use linearization. We will investigate the local stability properties of the steady states by approximating the nonlinear system of differential equations with a linear system at the points S_0 and S^*. The local stability analysis of these steady states is given below.

Theorem 1. *For system (3.1), if $\mathfrak{R}_1 > 0$ and $\mathfrak{R}_0 < 1$, then the steady state S_0 is locally asymptotically stable. Furthermore, the steady state S^* is always L-stable.*

Proof. First, we linearize system (3.1) around its steady states and then find its Jacobian matrices as follows:

$$J_{S_0} = \begin{bmatrix} 0 & -\frac{\mu_1 r_1}{r_2-\theta r_1} & 0 & 0 \\ r_2-\theta r_1 & -\mu_1 - \frac{\theta\mu_1 r_1}{r_2-\theta r_1} & 0 & -\frac{r_1 k_2}{k_1} \\ 0 & 0 & -c & \frac{r_1 k_2}{k_1} \\ 0 & 0 & N\delta & -\mu_2 \end{bmatrix}, \tag{3.3}$$

and

$$J_{S^*} = \begin{bmatrix} 0 & -r_1\frac{\mu_1+k_2\vartheta}{r_2-\theta r_1} & 0 & 0 \\ r_2-\theta r_1 & -\mu_1 - k_2\vartheta - \theta r_1\left(\frac{\mu_1+k_2\vartheta}{r_2-\theta r_1}\right) & 0 & -\frac{r_1 k_2}{k_1} \\ 0 & k_2\vartheta & -c & \frac{r_1 k_2}{k_1} \\ 0 & 0 & N\delta & -\mu_2 \end{bmatrix}. \tag{3.4}$$

If all eigenvalues of the Jacobian matrix have negative real parts, then the steady state is locally asymptotically stable. Characteristic equation of J_{S_0} is given by

$$P(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4,$$

where

$$a_1 = \frac{1}{r_2-\theta r_1}\left((c+\mu_2)(r_2-\theta r_1)+\mu_1 r_2\right),$$

$$a_2 = \frac{1}{k_1\,(r_2-\theta r_1)}\left((c\mu_2 k_1 - N\delta k_2 r_1)(r_2-\theta r_1)+\mu_1 k_1 r_2(c+\mu_2)+\mu_1 k_1 r_1(r_2-\theta r_1)\right),$$

$$a_3 = \frac{\mu_1}{k_1\,(r_2-\theta r_1)}(k_1 r_1(c+\mu_2)(r_2-\theta r_1)+r_2(c\mu_2 k_1 - N\delta k_2 r_1)),$$

$$a_4 = \frac{\mu_1 r_1}{k_1}(c\mu_2 k_1 - N\delta k_2 r_1).$$

Also, one can calculate that

$$a_1a_2a_3 - a_3^2 - a_1^2a_4 = \frac{\mu_1 r_2(c+\mu_2)}{k_1^2(r_2-\theta r_1)^3}\cdot \begin{pmatrix} (r_2-\theta r_1)^2(c\mu_2k_1 - N\delta k_2r_1 - \mu_1k_1r_1)^2 \\ +(c+\mu_2)(r_2-\theta r_1)(c\mu_2k_1 - N\delta k_2r_1)\mu_1k_1r_2 \\ +(c+\mu_2)(r_2-\theta r_1)\mu_1^2k_1^2r_1r_2 + (c\mu_2k_1 - N\delta k_2r_1)\mu_1^2k_1r_2^2 \\ +(c+\mu_2)^2(r_2-\theta r_1)^2\mu_1k_1^2r_1 \end{pmatrix}.$$

Here, we know all parameters are positive. If $\Re_0 < 1$, then we obtain $a_1a_2a_3 - a_3^2 - a_1^2a_4 > 0$. So, because of the Routh-Hurwitz criteria all eigenvalues of the Jacobian matrix for S_0 have negative real parts. Thus, the steady state S_0 is locally asymptotically stable.

However, for the steady state S^* the characteristic equation has the form of

$$P(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4,$$

where

$$\begin{aligned} a_1 &= c + \mu_1 + \mu_2 + k_2\vartheta + \theta r_1\frac{\mu_1 + k_2\vartheta}{r_2 - \theta r_1}, \\ a_2 &= \theta r_1(c+\mu_2)\frac{\mu_1 + \vartheta k_2}{r_2 - \theta r_1} + (\vartheta k_2 + \mu_1)(c + \mu_2 + r_1), \\ a_3 &= c\mu_1r_1 + \mu_1\mu_2r_1 + c\vartheta\mu_2k_2 + c\vartheta k_2r_1 + \vartheta\mu_2k_2r_1, \\ a_4 &= 0. \end{aligned}$$

Here, one of the eigenvalues is equal to zero. Applying the Routh-Hurwitz criteria to the reduced characteristic equation below:

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 + a_4$$

we get three remaining eigenvalues of the Jacobian matrix that have negative real parts. Therefore, the steady state S^* is Lyapunov stable (L-stable) (see the reference [3] for the definition of Lyapunov Stability). □

3.4. *Global stability analysis*

In this section, our aim is to investigate the long time behavior of the given system by analyzing global stability. Let's start this section with a remark. Next, we prove the global stability of the disease-free steady state.

Remark 1. The steady state S^* cannot be globally stable since it is L-stable. Some numerical simulations that support this observation will be given later.

Theorem 2. *For system (3.1), if $\Re_1 > 0$ and $\Re_0 < 1$, then the disease free steady state S_0 is globally asymptotically stable on Γ where*

$$\Gamma = \left\{ (T, E, I, V) \in \mathbb{R}_+^4 : T + E + I + V \le \frac{r_1}{k_1} \right\}.$$

Proof. First, system (3.1) can be thought as a compartmental model, i.e., the system can be written as follows:

$$\begin{aligned} \frac{dx}{dt} &= \mathcal{F}(x,y) - \mathcal{V}(x,y) \\ \frac{dy}{dt} &= g(x,y) \end{aligned} \tag{3.5}$$

with $g = (g_1, g_2)^T$ where $g_1 = r_1T - k_1TE$ and $g_2 = r_2T - \mu_1E - \theta k_1TE - k_2EV$. Here, $x = (I, V)^T$ and $y = (T, E)^T$ represent the populations in disease compartments and non-disease compartments, respectively. In addition, $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)^T$ and $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2)^T$ where $\mathcal{F}_1 = k_2EV$, $\mathcal{F}_2 = N\delta I$ represent the rates of new infections in the disease compartments, and $\mathcal{V}_1 = cI$, $\mathcal{V}_2 = \mu_2V$ represent the transition terms in the disease compartments.

One can easily check that following assumptions are satisfied in order to ensure the model (3.5) is well posed:

(1) $\mathcal{F}_i(0,y) = 0$ and $\mathcal{V}_i(0,y) = 0$ for all $y \geq 0$ and $i = 1,2$.
(2) $\mathcal{F}_i(x,y) \geq 0$ for all non-negative x and y and $i = 1,2$.
(3) $\mathcal{V}_i(x,y) \leq 0$ whenever $x_i = 0$, $i = 1,2$.
(4) $\mathcal{V}_1(x,y) + \mathcal{V}_2(x,y) \geq 0$ for all non-negative x and y.
(5) The disease free system $\frac{dy}{dt} = g(0,y)$ has a unique equilibrium that is locally asymptotically stable. This equilibrium point is $y_0 = (T_0, E_0) = \left(\frac{\mu_1 r_1}{k_1(r_2 - \Theta r_1)}, \frac{r_1}{k_1}\right)$.

Therefore, the linearized equations for the disease compartments can be written as follows:

$$\frac{dx}{dt} = (F - V)x,$$

where F and V are 2×2 matrices with entries

$$F = \frac{\partial \mathcal{F}_i}{\partial x_j}(0,y_0) = \begin{bmatrix} 0 & \frac{k_2r_1}{k_1} \\ N\delta & 0 \end{bmatrix} \text{ and } V = \frac{\partial \mathcal{V}_i}{\partial x_j}(0,y_0) = \begin{bmatrix} c & 0 \\ 0 & \mu_2 \end{bmatrix}.$$

Hence, the next generation matrix is

$$K = FV^{-1} = \begin{bmatrix} 0 & \frac{k_2r_1}{k_1} \\ N\delta & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{c} & 0 \\ 0 & \frac{1}{\mu_2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{k_2r_1}{\mu_2k_1} \\ \frac{N\delta}{c} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{k_2E_0}{\mu_2} \\ \frac{N\delta}{c} & 0 \end{bmatrix},$$

and the reproduction number $\mathcal{R}_0$ can be defined as the spectral radius of K (see the references [14], [31], [34]), that can be calculated as

$$\mathcal{R}_0 = \sqrt{\frac{N\delta k_2r_1}{c\mu_2k_1}} = \sqrt{\frac{N\delta k_2E_0}{c\mu_2}}.$$

One can easily show that the biologically feasible region

$$\Gamma = \left\{ (T, E, I, V) \in \mathbb{R}^4 : T + E + I + V \leq \frac{r_1}{k_1} \right\}$$

is positively invariant. On the other hand, set

$$f(x,y) = \begin{bmatrix} k_2 V (E_0 - E) \\ 0 \end{bmatrix}.$$

Then for the disease compartments can be written as

$$\frac{dx}{dt} = (F - V)x - f(x,y).$$

So, the Lyapunov function on Γ can be constructed as in [14] and [31]. $Q = W^T V^{-1} x$ is a Lyapunov function where W^T be the left eigenvector of $V^{-1}F$. A straightforward calculation gives

$$Q = W^T V^{-1} x = \begin{bmatrix} 1 & \dfrac{k_2 r_1 \sqrt{c\mu_2 k_1}}{k_1 c \sqrt{N\delta k_2 r_1}} \end{bmatrix} \begin{bmatrix} \dfrac{1}{c} & 0 \\ 0 & \dfrac{1}{\mu_2} \end{bmatrix} \begin{bmatrix} I \\ V \end{bmatrix} = \frac{1}{c} I + \frac{k_2 r_1 \sqrt{c\mu_2 k_1}}{k_1 c \mu_2 \sqrt{N\delta k_2 r_1}} V.$$

One can now easily verify that Q is a Lyapunov function on Γ provided $\mathcal{R}_0 < 1$ as follows:

$$\begin{aligned} \frac{dQ}{dt} &= \frac{1}{c}(k_2 E V - cI) + \frac{k_2 r_1 \sqrt{c\mu_2}}{k_1 c \mu_2 \sqrt{N\delta k_2 E_0}} (N\delta I - \mu_2 V) \\ &= V \left(\frac{k_2 E}{c} - \frac{k_2 r_1 \sqrt{c\mu_2}}{k_1 c \sqrt{N\delta k_2 E_0}} \right) + I \left(\frac{N\delta k_2 r_1 \sqrt{c\mu_2}}{k_1 c \mu_2 \sqrt{N\delta k_2 E_0}} - 1 \right) \\ &= V \left(\frac{k_2 E}{c} - \frac{k_2 r_1 \sqrt{c\mu_2}}{k_1 c \sqrt{N\delta k_2 E_0}} \right) + I \left(\frac{\sqrt{N\delta k_2 r_1}}{\sqrt{k_1 c \mu_2}} - 1 \right) < 0. \end{aligned} \tag{3.6}$$

Hence Q is a Lyapunov function on Γ, and the largest compact invariant set in $\left\{ (T, E, I, V) \in \Gamma : \dfrac{dQ}{dt} = 0 \right\}$ is $\{S_0\}$. Thus, by LaSalle's invariance principle, every solution of system (3.1) with initial conditions in Γ approaches S_0 as $t \to \infty$ whenever $\mathcal{R}_0 < 1$. Therefore, the disease-free equilibrium point S_0 is globally asymptotically stable on Γ for $\mathcal{R}_0 < 1$. □

4. Analysis of the DDE model

In this section, we study the local stability of the equilibria and the existence of zero-Hopf bifurcation in system (2.1) by dividing it into the following cases due to the delay:

(1) $\tau_1 = \tau_2 = \tau$,
(2) $\tau_1 > 0$ and $\tau_2 = 0$,
(3) $\tau_1 > 0$, and $\tau_2 > 0$ but $\tau_1 \neq \tau_2$.

Note that the case without delay ($\tau_1 = \tau_2 = 0$) is already analyzed in the former section. Also, note that the steady states of system (2.1) is the same as those of system (3.1). Therefore, Lemma 2 still holds for system (2.1). We now want to analyze the system for the endemic equilibrium point S^* which exists under the condition $\mathfrak{R}_0 = 1$.

4.1. *Bifurcation analysis of system (2.1) when* $\tau_1 = \tau_2 = \tau$

Under the condition $\mathfrak{R}_0 = 1$, the corresponding linearized system at S^* as follows:

$$\begin{aligned}
\frac{dT}{dt} &= a_{13}E(t-\tau),\\
\frac{dE}{dt} &= a_{21}T(t) + a_{22}E(t) + a_{23}E(t-\tau) + a_{25}V(t),\\
\frac{dI}{dt} &= a_{33}E(t-\tau) + a_{34}I(t) + a_{35}V(t),\\
\frac{dV}{dt} &= a_{44}I(t) + a_{45}V(t),
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
a_{13} &= -\frac{r_1(\mu_1 + k_1\vartheta)}{r_2 - \theta r_1}, & a_{21} &= r_2 - \theta r_1,\\
a_{22} &= -\mu_1, & a_{23} &= -\frac{r_2 k_2 \vartheta - \theta r_1 \mu_1}{r_2 - \theta r_1},\\
a_{25} &= -\frac{k_2 r_1}{k_1}, & a_{33} &= k_2\vartheta,\\
a_{34} &= -c, & a_{35} &= \frac{k_2 r_1}{k_1},\\
a_{43} &= N\delta, & a_{45} &= -\mu_2.
\end{aligned}$$

Therefore, for this case corresponding characteristic equation of system (4.1) becomes:

$$P(\lambda,\tau) \equiv \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + e^{-\lambda\tau}(b_1\lambda^3 + b_2\lambda^2 + b_3\lambda) = 0, \tag{4.2}$$

where

$$\begin{aligned}
a_1 &= c + \mu_1 + \mu_2,\\
a_2 &= \mu_1(c + \mu_2),\\
b_1 &= \frac{k_2\vartheta r_2 + \mu_1\theta r_1}{r_2 - \theta r_1},\\
b_2 &= \left(\frac{k_2\vartheta r_2 + \mu_1\theta r_1}{r_2 - \theta r_1}\right)(c + \mu_2) + r_1(\mu_1 + k_2\vartheta),\\
b_3 &= r_1(\mu_1 k_2\vartheta)(c + \mu_2) + c\mu_2 k_2\vartheta.
\end{aligned}$$

$P(\lambda,\tau)$ in Eq. (4.2) is a transcendental polynomial and has infinitely many roots. First, it is easy to see that one of its roots is zero. This is a simple root since a_2 and b_3 are positive because of the positivity of the parameters of the model. Notice that when $\tau = 0$ in Eq. (4.2), one obtains the characteristic equation of the ODE system in which the distribution of its eigenvalues is already studied in the former section.

In this section, our aim is to investigate the existence of the zero-Hopf bifurcation. In other words, we will determine the conditions on the parameters for the occurrence of a zero-Hopf bifurcation. To do this, we first need to show the existence of a pair of purely imaginary roots for Eq. (4.2).

Let us denote $\lambda = \eta(\tau) + iw(\tau)$. We look for a pair of purely imaginary roots such that $\lambda(\tau^*) = i\omega(\tau^*) = i\omega_0$, $\omega_0 > 0$ (without loss of generality). Notice that if such an $\omega(\tau^*) = \omega_0$ does not exist, then the steady state S^* stays stable forever because of the continuity. It is clear that $\lambda = i\omega$, $\omega > 0$, is a root of Eq. (4.2) if

$$(i\omega)^3 + a_1(i\omega)^2 + a_2(i\omega) + e^{-(i\omega)\tau}(b_1(i\omega)^3 + b_2(i\omega) + b_3) = 0.$$

Separating the real and imaginary parts, we obtain the following equations:

$$\begin{aligned} -a_1\omega^2 - b_1 cos(\omega\tau)\omega^2 + b_3 cos(\omega\tau) + b_2 sin(\omega\tau)\omega &= 0, \\ -\omega^3 + a_2\omega + b_2 cos(\omega\tau)\omega + b_1 sin(\omega\tau)\omega^2 - b_3 sin(\omega\tau) &= 0. \end{aligned} \tag{4.3}$$

Squaring first both sides of these equations and then adding them up, one reaches to the following equation:

$$\omega^6 + ({a_1}^2 - 2a_2 - {b_1}^2)\omega^4 + ({a_2}^2 + 2b_1b_3 - {b_2}^2)\omega^2 - {b_3}^2 = 0. \tag{4.4}$$

Now, let us take $\omega^2 = z$, then we can rewrite Eq. (4.4) as follows:

$$z^3 + ({a_1}^2 - 2a_2 - {b_1}^2)z^2 + ({a_2}^2 + 2b_1b_3 - {b_2}^2)z - {b_3}^2 = 0. \tag{4.5}$$

It is obvious that $-{b_3}^2 < 0$. Also, by the Vieta's Theorem [23], it is known that the expression ${b_3}^2$ is equal to the product of the roots. Therefore, the product of the roots of Eq. (4.5) is positive. Then, it is clear that at least one of them is positive since Eq. (4.5) has at most three roots. As a result of this, it is assured that there is at least one positive root ω_0 of Eq. (4.4), too. Thus, the characteristic equation (4.2) has a pair of purely imaginary roots $\pm i\omega_0$ at τ^*. Now, if we solve $sin(\omega\tau)$ and $cos(\omega\tau)$ from Eq. (4.3) simultaneously, we obtain the following equations:

$$\begin{aligned} sin(\omega\tau) &= \frac{b_2a_1\omega^3 - (b_3 - b_1\omega^2)(\omega^3 - a_2\omega)}{{b_2}^2\omega^2 + (b_1\omega^2 - b_3)^2}, \\ cos(\omega\tau) &= \frac{a_1\omega^2(b_3 - {b_1}^2) + b_2\omega^2(\omega^2 - a_2)}{{b_2}^2\omega^2 + (b_1\omega^2 - b_3)^2}. \end{aligned} \tag{4.6}$$

Finally, utilizing these equations we solve τ_n for $n = 0, 1, 2, \ldots$ as follows:

$$\tau_n = \frac{1}{\omega_0}\arccos\left(\frac{a_1{\omega_0}^2(b_3 - b_1{\omega_0}^2) + b_2{\omega_0}^2({\omega_0}^2 - a_2)}{{b_2}^2{\omega_0}^2 + (b_1{\omega_0}^2 - b_3)^2}\right) + \frac{2\pi n}{\omega_0}. \tag{4.7}$$

Let us denote $\tau^* = \min\{\tau_n : n = 0,1,2,..\} > 0$ such that $i\omega_0 = i\omega(\tau^*)$ is a root of Eq. (4.2). Now, differentiating both sides of Eq. (4.2) with respect to τ, and then utilizing Eq. (4.2), we derive the following equation:

$$\begin{aligned}\left(\frac{d\lambda}{d\tau}\right)^{-1} &= -\frac{(3\lambda^2+2a_1\lambda+a_2)-\tau e^{-\lambda\tau}(b_1\lambda^2+b_2\lambda+b_3)+e^{-\lambda\tau}(2b_1\lambda+b_2)}{-\lambda e^{-\lambda\tau}(b_1\lambda^2+b_2\lambda+b_3)}\\ &= \frac{(3\lambda^2+2a_1\lambda+a_2)e^{\lambda\tau}}{\lambda(b_1\lambda^2+b_2\lambda+b_3)} - \frac{2b_1\lambda+b_2}{\lambda(b_1\lambda^2+b_2\lambda+b_3)} - \frac{\tau}{\lambda}\\ &= -\frac{3\lambda^2+2a_1\lambda+a_2}{\lambda^4+a_1\lambda^3+a_2\lambda^2} - \frac{2b_1\lambda+b_2}{b_1\lambda^3+b_2\lambda^2+b_3\lambda} - \frac{\tau}{\lambda}.\end{aligned} \tag{4.8}$$

Thus, its reel part has the form of

$$Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau^*} = \frac{\omega_0^2\left(3\omega_0^4+(2a_1^2-4a_2)\omega_0^2+a_2^2\right)}{(\omega_0^4-a_2\omega_0^2)^2+(a_1\omega_0^3)^2} - \frac{-2b_1^2\omega_0^4+(2b_1b_3-b_2^2)\omega_0^2}{(b_2\omega_0^2)^2+(b_3\omega_0-b_1\omega_0^3)^2} > 0$$

since $2a_1^2-4a_2 > 0$ and $2b_1b_3-b_2^2 < 0$. Hence, the transversality condition holds.

Combining all derivations above, we have concluded the following Theorem.

Theorem 3. *Assume that $\Re_1 > 0$ and $\Re_0 = 1$ hold. Let us denote $\tau^* = \min\{\tau_n : n = 0,1,2,..\} > 0$ such that $\omega(\tau^*) = \omega_0$. Then Eq. (4.2) has a simple root zero, and also its all other roots have negative real parts for $\tau \in [0,\tau^*)$. Therefore, S^* is stable for $\tau \in [0,\tau^*)$. Moreover, system (2.1) undergoes a zero-Hopf bifurcation at S^* when τ passes through τ^*.*

4.2. *Bifurcation analysis of system (2.1) when $\tau_1 > 0$ and $\tau_2 = 0$*

Under the condition $\Re_0 = 1$, the corresponding characteristic equation of system (2.1) is:

$$P(\lambda,\tau_1) \equiv \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + e^{-\lambda\tau_1}(b_1\lambda^3+b_2\lambda^2+b_3\lambda) = 0, \tag{4.9}$$

where

$$\begin{aligned}a_1 &= c+\mu_1+\mu_2+k_2\vartheta,\\ a_2 &= (\mu_1+k_2\vartheta)(c+\mu_2),\\ a_3 &= k_2\vartheta c\mu_2,\\ b_1 &= \frac{\theta r_1(\mu_1+k_2\vartheta)}{r_2-\theta r_1},\\ b_2 &= \frac{\theta r_1}{r_2-\theta r_1}(c+\mu_2)(\mu_1+k_2\vartheta)+r_1(\mu_1+k_2\vartheta),\\ b_3 &= r_1(\mu_1+k_2\vartheta)(c+\mu_2).\end{aligned}$$

$P(\lambda,\tau_1)$ in Eq. (4.9) is a transcendental polynomial and has infinitely many roots. First, it is easy to see that one of its roots is zero. This is a simple root since a_3 and b_3 are positive because of the positivity of the parameters of the model.

In this section, we will determine the conditions on the parameters for the occurrence of a zero-Hopf bifurcation. To do this, we first need to show the existence of a pair of purely imaginary roots for Eq. (4.9).

Let us denote $\lambda = \eta(\tau_1) + iw(\tau_1)$. We look for a pair of purely imaginary roots such that $\lambda(\tau_1^*) = i\omega(\tau_1^*) = i\omega_1$, $\omega_1 > 0$ (without loss of generality). It is clear that $\lambda = i\omega$, $\omega > 0$, is a root of Eq. (4.9) if

$$(i\omega)^3 + a_1(i\omega)^2 + a_2(i\omega) + a_3 + e^{-(i\omega)\tau_1}(b_1(i\omega)^2 + b_2(i\omega) + b_3) = 0$$

Separating the real and imaginary parts, we obtain the following equations:

$$\begin{aligned} -a_1\omega^2 + a_3 &= b_1\omega^2 cos(\omega\tau_1) - b_3 cos(\omega\tau_1) - b_2\omega sin(\omega\tau_1), \\ \omega^3 - a_2\omega &= b_2\omega cos(\omega\tau_1) + b_1\omega^2 sin(\omega\tau_1) - b_3 sin(\omega\tau_1). \end{aligned} \tag{4.10}$$

Squaring first both sides of these equations and then adding them up, one reaches to the following equation:

$$\omega^6 + ({a_1}^2 - 2a_2 - {b_1}^2)\omega^4 + ({a_2}^2 - 2a_1a_3 + 2b_1b_3 - {b_2}^2)\omega^2 + {a_3}^2 - {b_3}^2 = 0 \tag{4.11}$$

Now, let us take $\omega^2 = z$, then we can rewrite Eq. (4.11) as follows:

$$z^3 + ({a_1}^2 - 2a_2 - {b_1}^2)z^2 + ({a_2}^2 - 2a_1a_3 + 2b_1b_3 - {b_2}^2)z + {a_3}^2 - {b_3}^2 = 0. \tag{4.12}$$

It is known that the expression ${b_3}^2 - {a_3}^2$ is equal to the product of the roots by the Vieta's Theorem [23]. Therefore, the product of the roots of Eq. (4.12) must be positive. Then, at least one of them must be positive since Eq. (4.12) has at most three roots. Let us define the following:

$$\Re_2 = {b_3}^2 - {a_3}^2 = r_1(\mu_1 + k_2\vartheta)(c + \mu_2) - k_2\vartheta c\mu_2$$

then, assume that $\Re_2 > 0$. This guarantees that there is at least one positive root of Eq. (4.12). Following that, there exist at least one positive root o (4.11), too. Thus, the characteristic equation (4.9) has a pair of purely imaginary roots $\pm i\omega_1$ at τ_1^*. Now, if we solve $sin(\omega\tau)$ and $cos(\omega\tau)$ from Eq. (4.10) simultaneously, we obtain the following equations:

$$\begin{aligned} cos(\omega\tau) &= \frac{b_2\omega^2(\omega^2 - a_2) + (b_1\omega^2 - b_3)(a_3 - a_1\omega^2)}{{b_2}^2\omega^2 + (b_1\omega^2 - b_3)^2}, \\ sin(\omega\tau) &= \frac{(a_2\omega - \omega^3)(b_3 - b_1\omega^2) - b_2\omega(a_3 - a_1\omega^2)}{{b_2}^2\omega^2 + (b_1\omega^2 - b_3)^2}. \end{aligned} \tag{4.13}$$

Finally, utilizing these equations we solve τ_1^n for $n = 0, 1, 2, \ldots$ as follows:

$$\tau_1^n = \frac{1}{\omega}\arccos\left(\frac{b_2\omega^2(\omega^2 - a_2) + (b_1\omega^2 - b_3)(a_3 - a_1\omega^2)}{{b_2}^2\omega^2 + (b_1\omega^2 - b_3)^2}\right) + \frac{2\pi n}{\omega} \tag{4.14}$$

Let us denote $\tau_1^* = \min\{\tau_1^n : n = 0,1,2,..\} > 0$ such that $i\omega_1 = i\omega(\tau_1^*)$ is a root of Eq. (4.9). Now, differentiating both sides of Eq. (4.9) with respect to τ_1, one can show that the transversality condition holds.

Theorem 4. *Assume that $\Re_1 > 0$, $\Re_2 > 0$ and $\Re_0 = 1$ hold. Let us denote $\tau_1^* = \min\{\tau_1^n : n = 0,1,2,..\} > 0$ such that $\omega(\tau_1^*) = \omega_1$. Then Eq. (4.9) has a simple root zero, and also its all other roots have negative real parts for $\tau_1 \in [0,\tau_1^*)$. Therefore, S^* is stable for $\tau_1 \in [0,\tau_1^*)$. Moreover, system (2.1) undergoes a zero-Hopf bifurcation at S^* when τ_1 passes through τ_1^*.*

4.3. *Bifurcation analysis of system (2.1) when $\tau_1 > 0$, $\tau_2 > 0$ but $\tau_1 \neq \tau_2$*

Under the condition $\Re_0 = 1$, the corresponding characteristic equation of linearized system at S^* as follows:

$$P(\lambda,\tau_1,\tau_2) \equiv \lambda\left(P_0(\lambda) + P_1(\lambda)e^{-\lambda\tau_1} + P_2(\lambda)e^{-\lambda\tau_2}\right) = 0, \tag{4.15}$$

where

$$\begin{aligned} P_0(\lambda) &= \lambda^3 + a_1\lambda^2 + a_2\lambda, \\ P_1(\lambda) &= b_1\lambda^2 + b_2\lambda + b_3, \\ P_2(\lambda) &= c_1\lambda^2 + c_2\lambda + c_3. \end{aligned}$$

The coefficients can be found in the following

$$\begin{aligned} a_1 &= c + \mu_1 + \mu_2, \\ a_2 &= \mu_1(c + \mu_2), \\ b_1 &= \frac{\theta r_1(\mu_1 + k_2\vartheta)}{r_2 - \theta r_1}, \\ b_2 &= \frac{\theta r_1}{r_2 - \theta r_1}(c + \mu_2)(\mu_1 + k_2\vartheta) + r_1(\mu_1 + k_2\vartheta), \\ b_3 &= r_1(\mu_1 + k_2\vartheta)(c + \mu_2), \\ c_1 &= k_2\vartheta, \\ c_2 &= k_2\vartheta(c + \mu_2), \\ c_3 &= k_2\vartheta c\mu_2. \end{aligned}$$

$P(\lambda,\tau_1,\tau_2)$ in Eq. (4.15) is a transcendental polynomial and has infinitely many roots. Again, it is easy to see that one of its roots is zero. This is a simple root since b_3 and c_3 are positive because of the positivity of the parameters of the model.

In this section, we investigate the existence of the zero-Hopf bifurcation with τ_1 in its interval of stability, regarding τ_2 as a parameter. So, we consider the system under the previous case.

Let us denote $\lambda = \eta(\tau_2) + iw(\tau_2)$. We look for a pair of purely imaginary roots such that $\lambda(\tau_2^*) = i\omega(\tau_2^*) = i\omega_2$, $\omega_2 > 0$. Choosing τ_2 as a parameter, we obtain that

$\lambda = i\omega$, $\omega > 0$, is a root of Eq. (4.15) if

$$P(i\omega, \tau_1, \tau_2) \equiv i\omega\left(P_0(i\omega) + P_1(i\omega)e^{-i\omega\tau_1} + P_2(i\omega)e^{-i\omega\tau_2}\right) = 0$$

$P_i(i\omega)$ are given in the following:

$$\begin{aligned} P_0(i\omega) &= -a_1\omega^2 + i(-\omega^3 + a_2\omega), \\ P_1(i\omega) &= (b_3 - b_1\omega^2) + i(b_2\omega), \\ P_2(i\omega) &= (c_3 - c_1\omega^2) + i(c_2\omega). \end{aligned}$$

Since $\mid e^{-i\omega\tau_1} \mid = 1$ we have the following equations:

$$\begin{aligned} \mid P_0(i\omega) + P_2(i\omega)e^{-i\omega\tau_2} \mid &= \mid P_1(i\omega)e^{-i\omega\tau_1} \mid \\ &= \mid P_1(i\omega) \mid\mid e^{-i\omega\tau_1} \mid \\ &= \mid P_1(i\omega) \mid \end{aligned}$$

which equals to

$$\begin{aligned} \mid P_0(i\omega) + P_2(i\omega)e^{-i\omega\tau_2} \mid^2 &= \mid P_1(i\omega) \mid^2 \\ (P_0(i\omega) + P_2(i\omega)e^{-i\omega\tau_2})(\bar{P}_0(i\omega) + \bar{P}_2(i\omega)e^{i\omega\tau_2}) &= \mid P_1(i\omega) \mid^2 . \end{aligned}$$

After some simplifications, the last equation becomes

$$\begin{aligned} &\mid P_0(i\omega) \mid^2 + \mid P_2(i\omega) \mid^2 + 2Re(P_0(i\omega)\bar{P}_2(i\omega))cos(\omega\tau_2) \\ &- 2Im(P_0(i\omega)\bar{P}_2(i\omega))sin(\omega\tau_2) = \mid P_1(i\omega) \mid^2 . \end{aligned} \tag{4.16}$$

Note that for $\omega > 0$, $P_0(i\omega) = -a_1\omega^2 + i(-\omega^3 + a_2\omega) \neq 0$ and $P_2(i\omega) = (c_3 - c_1\omega^2) + i(c_2\omega) \neq 0$. So the folllowing inequality holds.

$$[Re(P_0(i\omega)\bar{P}_2(i\omega))]^2 + [Im(P_0(i\omega)\bar{P}_2(i\omega))]^2 = \mid P_0(i\omega)\bar{P}_2(i\omega) \mid^2 > 0.$$

Finally, the Eqn. (4.16) can be written as

$$\begin{aligned} &\frac{\mid P_1(i\omega) \mid^2 - \mid P_0(i\omega) \mid^2 - \mid P_2(i\omega) \mid^2}{2 \mid P_0(i\omega)\bar{P}_2(i\omega) \mid} = \frac{Re(P_0(i\omega)\bar{P}_2(i\omega))}{\mid P_0(i\omega)\bar{P}_2(i\omega) \mid} cos(\omega\tau_2) \\ &- \frac{Im(P_0(i\omega)\bar{P}_2(i\omega))}{\mid P_0(i\omega)\bar{P}_2(i\omega) \mid} sin(\omega\tau_2). \end{aligned}$$

On the other hand, one can show that there is a continous $\phi(\omega)$ function such that:

$$cos(\phi(\omega)) = \frac{Re(P_0(i\omega)\bar{P}_2(i\omega))}{\mid P_0(i\omega)\bar{P}_2(i\omega) \mid} \text{ and } sin(\phi(\omega)) = \frac{Im(P_0(i\omega)\bar{P}_2(i\omega))}{\mid P_0(i\omega)\bar{P}_2(i\omega) \mid}$$

holds. Therefore, we have the following:

$$\left| \frac{\mid P_1(i\omega) \mid^2 - \mid P_0(i\omega) \mid^2 - \mid P_2(i\omega) \mid^2}{2 \mid P_0(i\omega)\bar{P}_2(i\omega) \mid} \right| = \mid cos(\phi(\omega) + \omega\tau_2) \mid \leq 1$$

and so,

$$\left| \mid P_1(i\omega) \mid^2 - \mid P_0(i\omega) \mid^2 - \mid P_2(i\omega) \mid^2 \right| \leq 2 \mid P_0(i\omega)\bar{P}_2(i\omega) \mid .$$

Finally, if we define

$$\Omega = \left\{\omega > 0 : || P_1(i\omega) |^2 - | P_0(i\omega) |^2 - | P_2(i\omega) |^2 | \leq 2 | P_0(i\omega)\bar{P}_2(i\omega) |\right\}$$

then we have the following lemma:

Lemma 3. *Let* $\Re_2 = r_1(\mu_1 + k_2 v)(c + \mu_2) - k_2 v c \mu_2$. *If* $\Re_2 > 0$ *then the set* Ω *is not empty.*

The above lemma means that for $n = 0, 1, 2, ...$ and $\omega \in \Omega$ the characteristic equation has purely imaginary roots when

$$\tau_2^n = \frac{\psi(\omega_2) - \phi(\omega_2) + 2\pi n}{\omega_2} \tag{4.17}$$

where

$$cos(\psi(\omega_2)) = \frac{| P_1(i\omega) |^2 - | P_0(i\omega) |^2 - | P_2(i\omega) |^2}{2 | P_0(i\omega)\bar{P}_2(i\omega) |}.$$

On the other hand, we can write the characteristic equation as the form of

$$P(\lambda) + Q(\lambda)e^{-\lambda\tau_2} = 0$$

where

$$\begin{aligned} P(\lambda) &= P_0(\lambda) + P_1(\lambda)e^{-\lambda\tau_1}, \\ Q(\lambda) &= P_2(\lambda). \end{aligned}$$

So, let's define

$$F(\omega) = | P(i\omega) |^2 - | Q(i\omega) |^2 .$$

After some calculations, one can get the following inequality

$$\begin{aligned} w_2 F'(w_2) &= A w_2^6 + B w_2^5 + C w_2^4 + D w_2^3 + E w_2^2 + F w_2 + G \\ &= w_2^4(A w_2^2 + C) + w_2^3(B w_2^2 + D) + w_2(E w_2 + F) + G \end{aligned}$$

where the coefficients are

$$\begin{aligned} A &= (2 - 2b_1 \sin\omega_2 T_1) \\ B &= (T_1 (2a_1 b_1 + 2b_2) \sin\omega_2 T_1 - 2b_1 \sin\omega_2 T_1) \\ C &= (T_1 2b_3 \cos\omega_2 T_1 - T_1 2a_1 b_2 \cos\omega_2 T_1 + T_1 2a_2 b_1 \cos\omega_2 T_1) \\ D &= (-T_1 2a_2 b_2 \sin\omega_2 T_1 + T_1 2a_1 b_3 \sin\omega_2 T_1) \\ E &= \big(-T_1 2a_2 b_3 \cos\omega_2 T_1 - 2a_2^2 - 4a_2 b_2 \cos\omega_2 T_1 - 2b_2^2 - 2c_2^2 + 4b_1 b_3 \\ &\quad -4a_1 b_3 \cos\omega_2 T_1 - 4c_1 c_3\big) \\ F &= (6a_2 b_3 \sin\omega_2 T_1) \\ G &= 4(c_3^2 - b_3^2) \end{aligned}$$

Under the following conditions

H1): $m_1 = \dfrac{-C}{A}$. $m_1 \geq 0$ and $A \neq 0$ and $\omega_2 < \sqrt{m_1}$,

H2): $m_2 = \dfrac{-D}{B}$. $m_2 \geq 0$ and $B \neq 0$ and $\omega_2 < \sqrt{m_2}$,

H3): $m_3 = \dfrac{-F}{E}$. $E \neq 0$ and $\omega_2 < m_3$,

it is clear that $F'(w_2) < 0$ holds. This inequality also means that the transversality condition holds. Hence, we have the following result.

Theorem 5. *Assume that the conditions (H1),(H2) and (H3) defined above are satisfied and also $\mathfrak{R}_0 = 1$, $\mathfrak{R}_1 > 0$ and $\mathfrak{R}_2 > 0$ hold. Let us denote $\tau_2^* = \min\{\tau_2^n : n = 0,1,2,..\} > 0$ such that $\omega(\tau_2^*) = \omega_2$. Then,*

(1) *Eq. (4.15) has a simple root zero, and also its all other roots have negative real parts for $\tau_1 \in [0,\tau_1^*)$ and $\tau_2 \in [0,\tau_2^*)$ such that $i\omega = i\omega(\tau_2^*)$, $\tau_2^* > 0$. Therefore, S^* is stable for $\tau_1 \in [0,\tau_1^*)$ and $\tau_2 \in [0,\tau_2^*)$.*

(2) *For $\tau_1 \in [0,\tau_1^*)$, system (2.1) undergoes a zero-Hopf bifurcation at S^* when τ_2 passes through τ_2^*.*

5. Numerical simulations

In this section, we perform numerical simulations that support the analytic results proved in former sections. For each simulation, we use either the ODE package (ode45) or the DDE package (dde23) in MATLAB. To illustrate the theoretical results, we study numerically the dynamics of system (2.1) with the parameter values which are chosen from Table 1 below. Choosing parameter values characteristic of the in vivo situation is difficult; many of the parameters in our model have not been measured, or, if measurements have been attempted, they may not be as accurate as we need for quantitative predictions [25]. Thefore we choose these parameters for the reason that either they are used in other models describing a similar phenomenon or they are based on experimental data in the corresponding references.

5.1. *Numerical simulations when $\tau_1 = \tau_2 = 0$*

First, we use the following parameter values for simulations: $r_1 = 0.3$, $k_1 = 0.001$, $r_2 = 0.05$, $\mu_1 = 0.03$, $\theta = 0.1$, $k_2 = 2.4 \times 10^{-5}$, $c = 0.3$, $N = 275$, $\delta = 0.3$ and $\mu_2 = 2.1$. With respect to these parameters, the corresponding basic reproduction number and the disease-free steady state are $\mathfrak{R}_0 = 0.942857$ and $S_0 = (T_0, E_0, I_0, V_0) = (450, 300, 0, 0)$, respectively. Figure 1 illustrates the numerical solutions of the ODE model (3.1) for different initial values. It shows that as stated in Theorem 1 and Theorem 2, the disease-free steady state is globally asymptotically stable. These figures underline the ability of the immune system to eliminate the infection and to prevent more tumour cell growth. So, the immune system eradicates virus perfectly but has low persistence of the tumour.

TABLE 1. The range of parameter values with corresponding references

Parameters	Range Values	Unit	References
r_1	$0.05-0.5$	day^{-1}	[28]
k_1	$10^{-3}-10^{-5}$	$\text{mm}^3\text{day}^{-1}$	[25]
r_2	$0-0.05$	day^{-1}	[25]
μ_1	0.03	$\text{mm}^3\text{day}^{-1}$	[11]
θ	0.1	day^{-1}	[29]
k_2	2.4×10^{-5}	$\text{mm}^3\text{day}^{-1}$	[32]
c	0.3	day^{-1}	[29]
N	$100-2000$		[11]
δ	$0.3-0.7$	day^{-1}	[26]
μ_2	$2.1-3.8$	day^{-1}	[29]

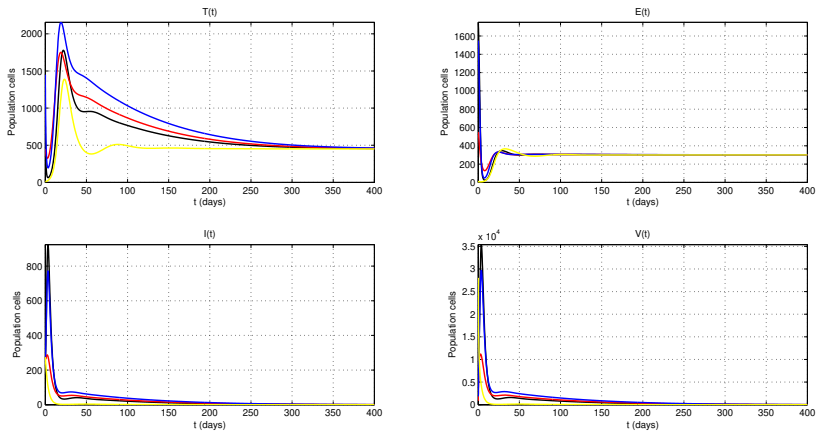

FIGURE 1. The numerical solutions of the ODE model (2.1) with different initial values. Each color represents a different solution with different initial values.

Second, Figure 3 presents the behaviour of solutions of system (3.1) for the parameter values: $r_1 = 0.1$, $k_1 = 0.0002$, $r_2 = 0.04$, $\mu_1 = 0.03$, $\theta = 0.1$, $k_2 = 2.4\times10^{-5}$, $c = 0.3$ $N = 275$, $\delta = 0.3$ and $\mu_2 = 2.1$. The corresponding basic reproduction number is equal to $\Re_0 = 1.571429$, and the disease free steady state is $S_0 = (T_0, E_0, I_0, V_0) = (500, 500, 0, 0)$ for this case. The uncontrolled growth of the tumour cells can be seen in Figure 3. The weakness caused by HIV may advance to uncontrolled tumour growth and spread in this case.

Third, we now solve the system with the parameter values that are $r_1 = 0.1$, $k_1 = 0.0003$, $r_2 = 0.03$, $\mu_1 = 0.03$, $\theta = 0.1$, $k_2 = 2.4\times10^{-5}$, $c = 0.3$, $N = 275$, $\delta =$

0.3 and $\mu_2 = 2.2$. The corresponding basic reproduction number for this case is equal to $\Re_0 = 1$, so the endemic steady state is $S^* = (540, 333.333, 2.66, 100)$. The coexistence of the tumour cells, the infected cells, and the virus can be seen from Figure 2 which illustrates the local stability of the disease-free steady state proven in Theorem 1. This means that the underlying HIV related tumour develops in the HIV infected individual because of the existence of virus; the tumour can escape from the surveillance of the immune system. As it can be seen from the simulations in Figure 2, the steady state could not be globally stable which was underlined by the remark in the former section.

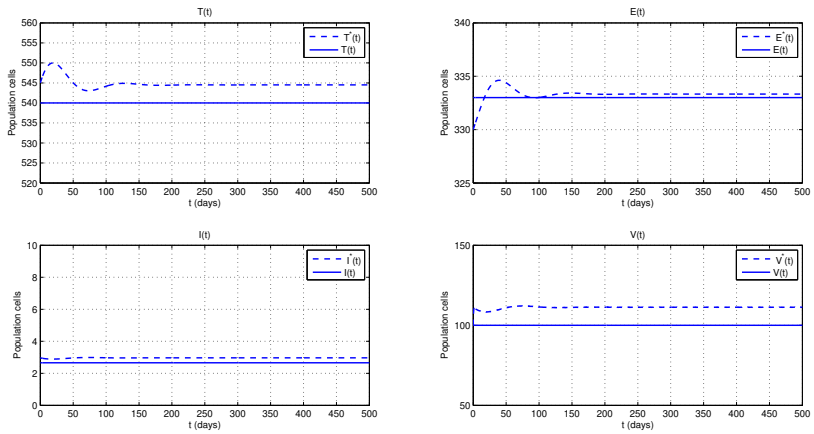

FIGURE 2. The numerical solutions of system (2.1) with initial values $(T_0, E_0, I_0, V_0) = (250, 750, 275, 250)$. The trajectories of the system are close enough to the steady state of the system, which is denoted by the solid line, however, not approach to it.

5.2. *Numerical simulations when* $\tau_1 = \tau_2 = \tau$

We numerically solve the DDE model (2.1) with the parameter values: $r_1 = 0.3$, $k_1 = 0.0032$, $r_2 = 0.05$, $\mu_1 = 0.03$, $\theta = 0.1$, $k_2 = 2.4 \times 10^{-5}$, $c = 0.3$, $N = 400$, $\delta = 0.7$, and $\mu_2 = 2.1$. Figures 4, 5 and 6 present the numerical solutions of the DDE system (2.1) when $\tau = 1.95$, $\tau = 1.962$ and $\tau = 1.97$, respectively. These figures show that τ plays a crucial role in the oscillations of the solutions around steady state. Clearly, delay causes the appearance of oscillations, and affects the stability of steady state.

Also, Figures 5 and 7 show the periodic solutions arising as the bifurcation parameter τ passes through the critical value $\tau^* \approx 1.962$. Here, one observes the occurrence of limit cycles due to the zero-Hopf bifurcation at $\tau = 1.95$. Figure 6 displays that the system becomes unstable for the value of τ which is grater than τ^*.

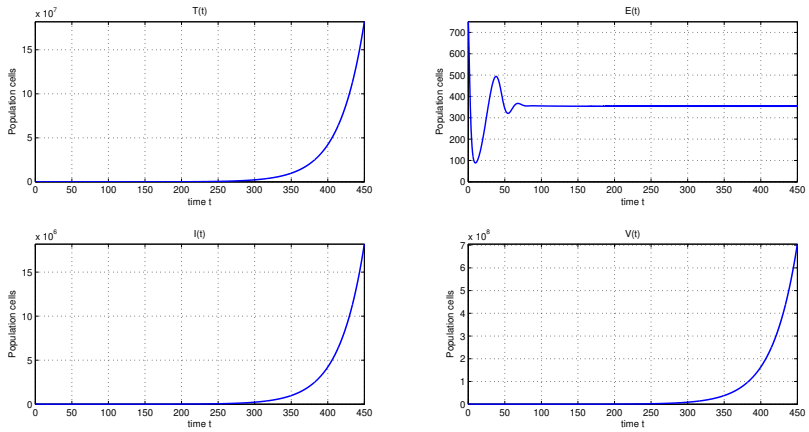

FIGURE 3. The numerical solutions of system (2.1) with initial values $(T_0, E_0, I_0, V_0) = (250, 750, 275, 250)$. Note that the trajectories of the system move away.

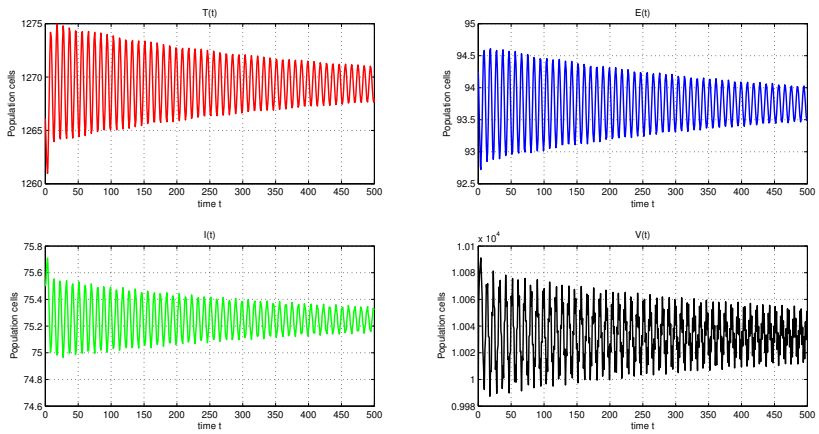

FIGURE 4. The numerical solutions of the DDE model (2.1) for $\tau = 1.95 < \tau^* \approx 1.962$. Simulations present stability.

5.3. *Numerical simulations when* $\tau_1 > 0, \tau_2 > 0, \tau_1 \neq \tau_2$

We numerically solve the DDE model (2.1) with the parameter values: $r_1 = 0.3$, $k_1 = 0.0032$, $r_2 = 0.05$, $\mu_1 = 0.03$, $\theta = 0.1$, $k_2 = 2.4 \times 10^{-5}$, $c = 0.3$, $N = 400$, $\delta = 0.7$, and $\mu_2 = 2.1$. Figures 8 presents the numerical solutions of the DDE system

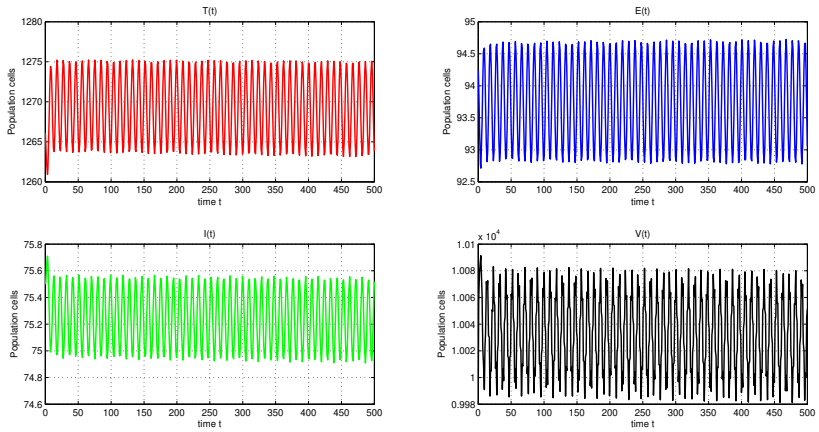

FIGURE 5. The numerical solutions of the DDE model (2.1) for $\tau = \tau^* \approx 1.962$. They present the cyclic behaviour.

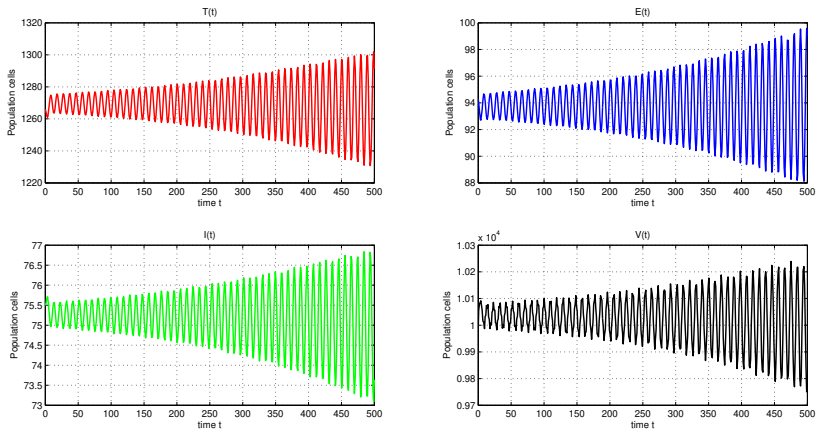

FIGURE 6. The numerical solutions of the DDE model (2.1) for $\tau = 1.97 > \tau^* \approx 1.962$. Simulations show unstability.

(2.1) when $\tau_1 = 1.8 < \tau_1^*$ and $\tau_2 = 2.3415$. These figures show that τ_2 plays a crucial role in the oscillations of the solutions around steady state. Clearly, delay causes the appearance of oscillations, and affects the stability of steady state.

Also, Figures 8 and 9 show the periodic solutions arising as the bifurcation parameter τ passes through the critical value $\tau_2^* \approx 2.3415$. Here, one observes the occurrence of limit cycles due to the zero-Hopf bifurcation at $\tau_2 = 2.3415$.

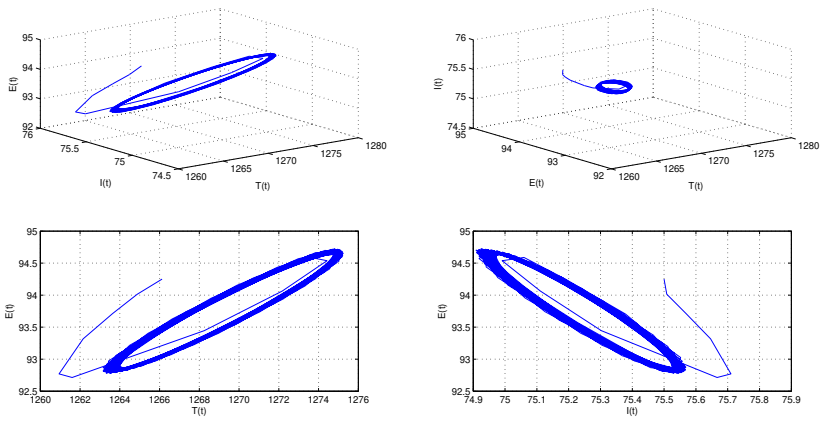

FIGURE 7. The numerical solutions of the DDE model (2.1) for $\tau = \tau^* \approx 1.962$.

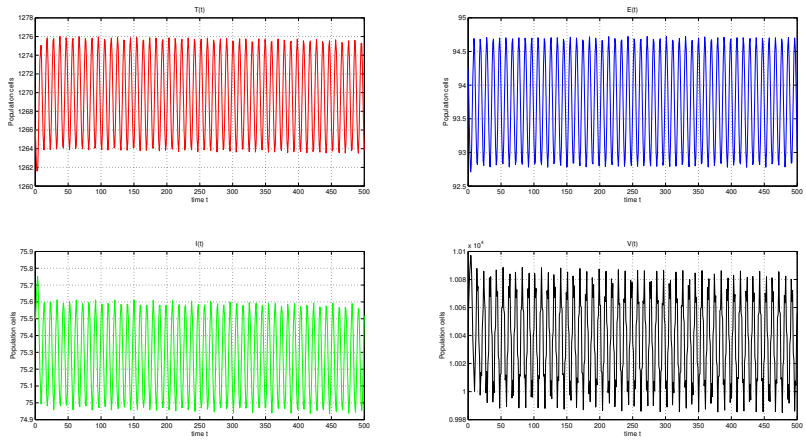

FIGURE 8. The numerical solutions of the DDE model (2.1) for $\tau_1 = 1.8 < \tau_1^*$ and $\tau_2 \approx \tau_2^* = 2.3415$. They present the cyclic behaviour.

6. DISCUSSION

HIV infected individuals suffer several types tumours during their infection, which creates a burden on the treatment. For this reason, it is very crucial to understand the relationship between HIV infection and certain tumours. In this paper, we have shown how HIV affects the immune system's ability to fight tumours. Since HIV is

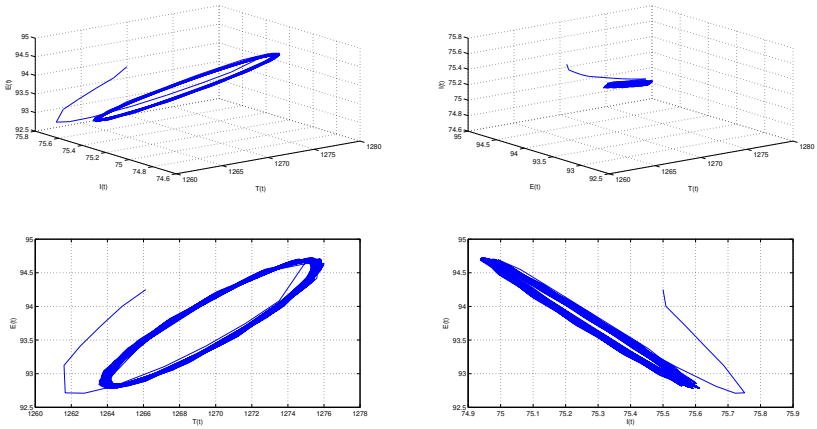

FIGURE 9. The numerical solutions of the DDE model (2.1) for $\tau_1 = 1.8 < \tau_1^*$ and $\tau_2 \approx \tau_2^* = 2.3415$.

present in vivo, the surveillance of the immune system is very weak. Accordingly, the growth of tumour is uncontrolled.

The model we introduced involves four components: tumour cells, uninfected helper T-cells, infected helper T-cells and free virus. Under the condition $\Re_1 > 0$, system (2.1) has two non-negative steady states. This condition means that the ratio of the rate of growth of the T-helper cells (triggered by tumour cells) with respect to the rate of growth of tumour cells is larger than the percentage of the helper cell loss due to killing tumour cells. First, the existence and positiveness of the solutions of the model without delay are studied. By utilizing the Routh-Hurwitz criteria, we have determined the conditions on the parameters for the stability of steady states of the model. When the condition $\Re_0 < 1$, we have proved that the non-infected steady state is both locally and globally asymptotically stable. This means that the ability of the immune system to eliminate the virus is strong enough but also there is low persistence of tumour. Also, we obtained that uncontrolled tumour growth and spread can be possible because of the exhaustion of the immune system response caused by infection. On the other hand, the infected steady state is L-stable. This biologically means that tumour can escape from surveillance of the immune system.

Second, the zero-Hopf bifurcation of the delay differential equation (DDE) model is studied. We study the effects of two discrete time delays on the stability of the endemically infected equilibrium point. We determine the conditions on parameters at which the system undergoes a zero-Hopf bifurcation. The time lags in the DDE model describe the time needed by the helper T-cells to find (or recognize) tumour cells and virus. Choosing one of the delay terms as a bifurcation parameter and fixing

the other, we have shown that a zero-Hopf bifurcation occurs as the bifurcation parameter passes through a critical value. In other words, the stability of the steady state S^* changes from stable to unstable, and periodic solutions with increasing periods arise. For the larger values of the bifurcation parameter, solutions become unstable. The performed numerical simulations support and extend our analytical results.

The results concluded underline that as the immune system gets weaker, it becomes difficult to keep up of the T-helper cells which leads to the compute exhaustion of tumour cells.

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SHARP ESTIMATION OF HERMITIAN-TOEPLITZ DETERMINANTS FOR JANOWSKI TYPE STARLIKE AND CONVEX FUNCTIONS

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Abstract. The sharp upper and lower bounds for the third-order Hermitian-Toeplitz determinant are investigated for the classes of Janowski type starlike and convex functions. The results presented in this paper generalize several recent works in this direction.

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1. Introduction

Investigating the sharp bound for the coefficient functionals of normalised analytic functions defined over the unit disk has gained much focus after the Bieberbach conjecture (1916) came into the picture. Even after the historical proof of the Bieberbach conjecture, in 1984, by L. de Branges, investigation of the bounds for coefficients of various classes of analytic functions did not stop as the bounds on coefficient also unfold many geometric properties of analytic functions. For example, the growth and distortion of an analytic function can be estimated using the bound on the second coefficient of an analytic univalent function [8]. The Hankel determinants have been among the most studied topics in Geometric Function Theory (GFT) in recent years; such studies can be back to the 1960s (see [10,18]). In many of the recently-published works dealing extensively with the Hankel and Toeplitz determinants, use is made also of the basic (or q-) calculus (see, for example, [9, 17, 20, 22, 23]; see also a survey-cum-expository review article by Srivastava [21]).

In this line, using Hankel determinant of coefficients of analytic functions, Cantor [4] gave a criterion for rationality of such functions. The results of this kind

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makes the estimation of coefficient functionals much interesting. The following few paragraphs give a short review of basic notations that we will be using in this paper.

From now on, let $\mathcal{A}$ be the class of normalised holomorphic functions defined on the open unit disk

$$\mathbb{D} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Also a subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{D}$ shall be denoted by $\mathcal{S}$. Because of the normalisation $f(0) = 0 = f'(0) - 1$, the functions in the class $\mathcal{A}$ have the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

and the same for functions in the class $\mathcal{S}$ too. For α $(0 \leqq \alpha < 1)$, let

$$p_\alpha(z) := \frac{1 + (1 - 2\alpha)z}{1 - z}$$

be an analytic function in the unit disk $\mathbb{D}$ which maps the unit disk $\mathbb{D}$ onto the right of the line $x = \alpha$ of the complex plane. Robertson [19], in 1936, considered the classes of starlike and convex functions of order α $(0 \leqq \alpha < 1)$ defined as follows:

$$\mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{S} \quad \text{and} \quad \frac{zf'(z)}{f(z)} \prec p_\alpha(z) \right\}$$

and

$$\mathcal{K}(\alpha) := \left\{ f : f \in \mathcal{S} \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} \prec p_\alpha(z) \right\}.$$

Let $\mathcal{S}^*(0) =: \mathcal{S}^*$ and $\mathcal{K}(0) =: \mathcal{K}$ be the classes of starlike and convex functions, respectively. Janowski [11] gave a generalisation to the classes of starlike and convex functions of order α by replacing the function p_α with

$$p_{A,B}(z) := \frac{1 + Az}{1 + Bz} \qquad (-1 \leqq B < A \leqq 1).$$

The classes of Janowski type starlike and Janowski type convex functions are defined, respectively, by

$$\mathcal{S}^*[A,B] := \left\{ f : f \in \mathcal{S} \quad \text{and} \quad \frac{zf'(z)}{f(z)} \prec p_{A,B}(z) \right\}$$

and

$$\mathcal{K}[A,B] := \left\{ f : f \in \mathcal{S} \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} \prec p_{A,B}(z) \right\}.$$

Janowski investigated growth and distortion rates for functions in the classes $\mathcal{S}^*[A,B]$ and $\mathcal{K}[A,B]$. Several recent investigations involving Janowski type analytic and univalent functions include, for example, those published in [2, 24–26] (see also the aforementioned review article by Srivastava [21]), in each of which the basic (or q-) calculus is also used.

Janteng *et al*. [12] computed the sharp bound on the second Hankel determinant for the classes $\mathcal{S}^*$ and $\mathcal{K}$. Later, Babalola [3] investigated the non-sharp bounds for third Hankel determinant for these classes. In 2018, Kowalczyk *et al.* [13] obtained the sharp bound of third Hankel determinant to be $4/135$ for the class of convex functions. However, the best known estimate for starlike functions till date is $8/9$ (see [14]). For more details, we refer the reader to [6, 15]. In line of the investigation on Hankel determinants in recent years, Ali *et al.* [1] investigated the sharp bound for the second and third order symmetric Toeplitz determinants. Cudna *et al.* [7] considered the qth-order Hermitian-Toeplitz determinants with its entries as coefficients of the function $f(z)$ given by (1.1) as follows:

$$T_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ \bar{a}_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ \bar{a}_{n+q-1} & \bar{a}_{n+q-2} & \cdots & a_n \end{vmatrix}.$$

There it is reported that $T_{q,n}(f)$ is rotationally invariant and if a_n's are real, then $T_{q,n}(f)$ is Hermitian and, therefore, the determinant $T_{q,n}(f)$ is a real number. They obtained the sharp lower and upper bound for the third-order Hermitian-Toeplitz determinants for the classes of starlike and convex functions of order α $(0 \leqq \alpha < 1)$. Inspired by their work, Cho and Kumar [5] investigated the sharp lower and upper bounds of the third-order Hermitian-Toeplitz determinants for the class of analytic functions with bounded turning.

In this paper, the authors aim to generalise the work of Cudna *et al.* [7] for the class of Janowski type starlike and convex functions.

2. Janowski type starlike functions

This section gives the estimation for lower and upper bounds of the third-order Hermitian-Toeplitz determinant $T_{3,1}(f)$ for Janowski type starlike functions. From the definition, it is easy to verify that $T_{2,1}(f) = 1 - |a_2|^2$ and

$$T_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ \bar{a}_2 & 1 & a_2 \\ \bar{a}_3 & \bar{a}_2 & 1 \end{vmatrix} = 2\,\Re(a_2^2\bar{a}_3) - 2|a_2|^2 - |a_3|^2 + 1.$$

Estimating the coefficient bounds in many cases, in our case too, depends on the comparison coefficients of functions under consideration with that of the functions with positive real part. For this reason, let $\mathcal{P}$ denotes the class of analytic functions $p : \mathbb{D} \to \mathbb{C}$ with $p(0) = 1$ and $\Re\, p(z) > 0$.

To keep the proof brief and avoid repetitions, let us note that for any function $f \in \mathcal{S}^*[A,B]$, there is a function with positive real part $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$ such

that

$$\frac{zf'(z)}{f(z)} = \frac{(1+A)p(z)+1-A}{(1+B)p(z)+1-B}. \tag{2.1}$$

On expanding in the Taylors series and comparing the coefficients, we get

$$a_2 = \frac{A-B}{2}p_1, \text{ and } a_3 = \frac{A-B}{8}\left[(A-2B-1)p_1^2+2p_2\right]. \tag{2.2}$$

Theorem 1. *Let $f \in \mathcal{S}^*[A,B]$. Then, the following sharp estimation holds*

$$1-(A-B)^2 \leqq T_{2,1}(f) \leqq 1.$$

Proof. Since $f \in \mathcal{S}^*[A,B]$, in view of the known fact $|p_n| \leqq 2$, we get $|a_2| \leqq A-B$ and thus $T_{2,1}(f) = 1-|a_2|^2 \geqq 1-(A-B)^2$ and $T_{2,1}(f) = 1-|a_2|^2 \leqq 1$. The extremal functions for lower and upper bounds are, respectively

$$\tilde{h}_0(z) = \begin{cases} z(1+Bz)^{\frac{A-B}{B}}, & B \neq 0; \\ ze^{Az}, & B = 0 \end{cases} \tag{2.3}$$

and

$$\tilde{h}_1(z) = \begin{cases} z(1+Bz^2)^{\frac{A-B}{2B}}, & B \neq 0; \\ ze^{Az^2/2}, & B = 0. \end{cases} \tag{2.4}$$

This ends the proof. □

Before we proceed further in this section, let Ω_1 be the set of points (A,B) such that

$$\left(-1 \leqq B \leqq -\frac{1}{2} \text{ and } 0 < A \leqq \frac{4B+2\sqrt{B^2+6}}{3}\right) \text{ or } \left(-\frac{1}{2} < B \leqq 0 \text{ and } 0 < A \leqq 1\right).$$

Further, let Ω_2, Ω_3 and τ be the set of points (A,B) satisfying

$$\Omega_2 := \left\{-1 \leqq B \leqq -\frac{1}{2} \text{ and } \frac{4B+2\sqrt{B^2+6}}{3} < A \leqq 1\right\},$$

$$\Omega_3 := \left\{-1 \leqq B < 0 \text{ and } \delta_1 := \frac{\sqrt{16B^2-16B+49}+8B-1}{6} < A \leqq 1\right\}$$

and

$$\tau := \frac{\left(3A^2-5AB+2B^2-2\right)\left(A^2-3AB+2B^2-2\right)}{4}.$$

Theorem 2. *Let $f \in \mathcal{S}^*[A,B]$. Then, the following best possible estimations hold:*

$$T_{3,1}(f) \leqq \begin{cases} \tau, & (A,B) \in \Omega_1; \\ 1, & (A,B) \in \Omega_2 \text{ or } -1 \leqq B < A \leqq 0 \end{cases}$$

and

$$T_{3,1}(f) \geqq \begin{cases} -\frac{\left(A^2-2AB+A+B^2-1\right)^2}{(A-2B+1)(3A-2B-1)}, & (A,B) \in \Omega_3; \\ \tau, & -1 \leqq B \leqq 0 < A \leqq \delta_1 \text{ or } -1 \leqq B < A \leqq 0. \end{cases}$$

We owe to Libera and Zlotkiewicz for a result related to the class $\mathcal{P}$:

Lemma 1 ([16, Lemma 3, p. 254]). *Let* $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}$. *Then* $2p_2 = p_1^2 + (4 - p_1^2)\zeta$ *for some* $\zeta \in \overline{\mathbb{D}}$.

Proof of Theorem 2. The well-known fact about the class of functions with positive real part is that it is rotationally invariant and for this reason, we shall limit ourselves to a consideration of non–negative value of p_1 and since $|p_1| \leqq 2$, we can let $0 \leqq p_1 \leqq 2$ and hereafter we use $x := p_1^2 \in [0,4]$ and $y := |\zeta| \in [0,1]$. Keeping these in mind from (2.2), a computation gives

$$\begin{aligned} T_{3,1}(f) &= 1 + \frac{1}{64}(A-2B)(3A-2B)(A-B)^2 p_1^4 - \frac{(A-B)^2}{2} p_1^2 \\ &\quad - \frac{(A-B)^2}{64}(4-p_1^2)^2|\zeta|^2 + \frac{1}{32}A(A-B)^2(4-p_1^2)p_1^2 \Re\zeta \qquad (2.5) \\ &=: \Psi\left(p_1^2, |\zeta|, \Re\zeta\right). \qquad (2.6) \end{aligned}$$

We proceed in the proof in various cases.
Case I. If $-1 \leqq B < A = 0$, then from (2.5), we have

$$\begin{aligned} T_{3,1}(f) &= \frac{1}{16}(4-B^2p_1^2)^2 - \frac{B^2}{64}(4-p_1^2)^2|\zeta|^2 \\ &\leqq \frac{1}{16}(4-B^2p_1^2)^2 \\ &\leqq 1 \end{aligned}$$

and the minimum is given by

$$\begin{aligned} T_{3,1}(f) &\geqq \frac{1}{16}(4-B^2p_1^2)^2 - \frac{B^2}{64}(4-p_1^2)^2 \\ &\geqq \frac{4-B^2}{4}. \end{aligned}$$

Case II. If $-1 \leqq B \leqq 0 < A \leqq 1$, then with the settings $p_1^2 =: x \in [0,4]$ and $|\zeta| =: y \in [0,1]$, we have

$$T_{3,1}(f) = \Psi(p_1^2, |\zeta|, \Re\zeta) \leqq \Psi(p_1^2, |\zeta|, |\zeta|) = \Psi(p_1^2, |\zeta|) = G(x,y).$$

Note that G is a continuously differentiable function of two variables x and y defined over the rectangular region $[0,4] \times [0,1]$ and is given by

$$G(x,y) := 1 + \frac{1}{64}(A-2B)(3A-2B)(A-B)^2x^2 - \frac{(A-B)^2}{2}x$$

$$-\frac{(A-B)^2}{64}(4-p_1^2)^2y^2+\frac{1}{32}A(A-B)^2(4-x)xy. \tag{2.7}$$

On the boundary lines of the rectangular region $[0,4]\times[0,1]$, we see that

$$G(0,y)=1-\frac{1}{4}y^2(A-B)^2\leqq 1,$$

$$G(4,y)=\frac{1}{4}\left(3A^2-5AB+2B^2-2\right)\left(A^2-3AB+2B^2-2\right)=:\tau$$

and

$$G(x,0)=1-\frac{1}{2}x(A-B)^2+\frac{1}{64}x^2(A-2B)(3A-2B)(A-B)^2=:f_1(x). \tag{2.8}$$

Now the positivity of the second derivative $f_1''(x)>0$ assures that there is no maxima of f_1 for $0<x<4$. Therefore, the possibilities left are the end points and we have

$$f_1(x)\leqq \max\{f_1(0),f_1(4)\}=f_1(4)=\tau.$$

Thus, for $-1\leqq B\leqq 0<A\leqq 1$, we have

$$G(x,y)=\max\{1,\tau\}=\begin{cases}1, & (A,B)\in\Omega_1;\\ \tau, & (A,B)\in\Omega_2.\end{cases}$$

To find the lower bound, we write

$$T_{3,1}(f)=\Psi(p_1^2,|\zeta|,\Re\zeta)\geqq F(p_1^2,|\zeta|,-|\zeta|)\geqq F(p_1^2,1,-1)=f_2(x),$$

where the function $f_2:[0,4]\to\mathbb{R}$ is defined by

$$f_2(x):=1-\frac{(A-B)^2}{4}-\frac{(A+3)(A-B)^2}{8}x+\frac{(A-2B+1)(3A-2B-1)(A-B)^2}{64}x^2. \tag{2.9}$$

At the end points, we see that

$$f_2(0)=1-\frac{1}{4}(A-B)^2$$

and

$$f_2(4)=\frac{1}{4}\left(3A^2-5AB+2B^2-2\right)\left(A^2-3AB+2B^2-2\right)=\tau.$$

Now $f_2''(x)>0$ holds for

$$\left(-1\leqq B\leqq -\frac{1}{2}\text{ and }0<A\leqq 1\right)\text{ or }\left(-\frac{1}{2}<B\leqq 0\text{ and }\frac{1}{3}(2B+1)<A\leqq 1\right)$$

and the only critical point of f_2 is

$$x=x^*=\frac{4(A+3)}{(A-2B+1)(3A-2B-1)}.$$

A computation shows that $x^* \in (0,4)$ if and only if $(A,B) \in \Omega_3$. But, since

$$2(2B+1) \leqq \sqrt{16B^2-16B+49}+8B-1$$

holds for all $-1 \leqq B \leqq 0 < A \leqq 1$, it follows that the minimum of f_2 attains at x^* for $(A,B) \in \Omega_3$ and

$$f_2(x^*) = -\frac{\left(A^2-2AB+A+B^2-1\right)^2}{(A-2B+1)(3A-2B-1)}.$$

From the above analysis, we conclude that

$$T_{3,1}(f) \geqq \begin{cases} -\frac{\left(A^2-2AB+A+B^2-1\right)^2}{(A-2B+1)(3A-2B-1)}, & (A,B) \in \Omega_3; \\ \tau, & -1 \leqq B \leqq 0 \text{ and } 0 < A \leqq \delta_1. \end{cases}$$

Case III. Let $-1 \leqq B < A < 0$. Then, using (2.5) and the notations $p_1^2 =: x \in [0,4]$ and $|\zeta| =: y \in [0,1]$, we can write

$$T_{3,1}(f) = F(p_1^2, |\zeta|, \Re\zeta) \leqq F(p_1^2, |\zeta|, -|\zeta|) = H(x,y),$$

where $H : [0,4] \times [0,1] \to \mathbb{R}$ is defined by

$$\begin{aligned} H(x,y) := 1 &+ \frac{1}{64}(A-2B)(3A-2B)(A-B)^2x^2 - \frac{(A-B)^2}{2}x \\ &- \frac{(A-B)^2}{64}(4-p_1^2)^2y^2 - \frac{1}{32}A(A-B)^2(4-x)xy. \end{aligned} \tag{2.10}$$

Now, at the boundary lines, we see that

$$H(0,y) = 1 - \frac{1}{4}(A-B)^2y^2 \leqq 1, \;\; H(4,y) = \tau,$$

$$H(x,0) = 1 - \frac{1}{2}(A-B)^2x + \frac{1}{64}(A-2B)(3A-2B)(A-B)^2x^2 = f_3(x)$$

and $H(x,1) = f_2(x)$, where f_2 is as defined in (2.9). The first derivative of f_3 vanishes for $x = x_2 = 16/(A-2B)(3A-2B)$ and since $(A-2B)(3A-2B) < 4$ for $-1 \leqq B < A < 0$, it follows that $x_2 \notin (0,4)$ and therefore it is sufficient to consider the values of f_3 at the end points of the interval $(0,4)$. Further, as previous, we see that $f_2'(x) = 0$ holds for $x = x_1$ and further, for $-1 \leqq B < A < 0$, it can be verified that $x_1 \notin (0,4)$. Thus the above discussion make us to write

$$T_{3,1}(f) \leqq \max\{1,\tau\} = 1.$$

Now it remains to look for the claimed lower bound in the case $-1 \leqq B < A < 0$. For this purpose, we write

$$T_{3,1}(f) = F(p_1^2, |\zeta|, \Re\zeta) \geqq F(p_1^2, |\zeta|, |\zeta|) \geqq F(p_1^2, 1, 1) = f_1(x),$$

where f_1 is defined by (2.8). As before, we see at the end points of $[0,4]$ that

$$f_1(0) = 1 - \frac{1}{4}(A-B)^2 \text{ and } \;\; f_1(4) = \tau.$$

Also f_1 has no critical points inside the interval $(0,4)$ and therefore,

$$T_{3,1}(f) \geqq \min\left\{1-\frac{1}{4}(A-B)^2, \tau\right\} = \tau.$$

Case IV. In this case, we consider the situation when $-1 \leqq B < A = 0$. Upon setting $A = 0$ in (2.5), we have

$$\begin{aligned} T_{3,1}(f) &= \frac{(4-Bp_1^2)^2}{16} - \frac{B^2(4-p_1^2)^2}{64}|\zeta|^2 \\ &\leqq \frac{(4-Bp_1^2)^2}{16} \\ &\leqq 1 \end{aligned}$$

and

$$\begin{aligned} T_{3,1}(f) &= \frac{(4-Bp_1^2)^2}{16} - \frac{B^2(4-p_1^2)^2}{64}|\zeta|^2 \\ &\geqq \frac{(4-Bp_1^2)^2}{16} - \frac{B^2(4-p_1^2)^2}{64} \\ &\geqq \frac{(4-B)^2}{4}. \end{aligned}$$

The functional $T_{3,1}(f)$ equals τ in case of the function $\tilde{h_0}$ defined in (2.3). Further $T_{3,1}(f)$ equals τ for the function

$$\tilde{h}_2(z) = \begin{cases} z(1+Bz^3)^{\frac{A-B}{3B}}, & B \neq 0; \\ ze^{Az^3/3}, & B = 0. \end{cases} \tag{2.11}$$

For $(A,B) \in \Omega_3$, the extremal function $\tilde{h}_3$ satisfying (2.1) with the function p is replace by $\tilde{p}$ defined as

$$\tilde{p}(z) = \frac{1-z^2}{1-2\sqrt{t}z+z^2}, \quad t := x^*/4.$$

The Taylor series of $\tilde{h}_3$ is given by

$$\tilde{h}_3(z) = z + \sqrt{t}(A-B)z^2 + \frac{(A-B)((A-2B+1)t-1)}{2}z^3 + \cdots$$

and thus we have

$$T_{3,1}(f) = 1 + 2a_2^2a_3 - |a_2|^2 - |a_3|^2 = -\frac{\left(A^2-2AB+A+B^2-1\right)^2}{(A-2B+1)(3A-2B-1)}.$$

Cases I to IV, together, bring the proof to an end. □

Remark 1. For $A = 1-2\alpha$ and $B = -1$, Theorems 1 and 2 reduce to the results [7, Theorem 2] and [7, Theorem 3], respectively.

3. JANOWSKI TYPE CONVEX FUNCTIONS

In this section, the sharp bounds for the Hermitian-Toeplitz determinant of the third order for the classes of Janowski type convex functions are investigated.

For any function $f \in \mathcal{K}[A,B]$, there is a function with positive real part $p(z) = 1+\sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$ such that

$$1+\frac{zf''(z)}{f'(z)} = \frac{(1+A)p(z)+1-A}{(1+B)p(z)+1-B}. \tag{3.1}$$

Expanding both sides of the above in Taylor series and comparing the coefficients of similar terms, we get

$$a_2 = \frac{A-B}{4}p_1 \text{ and } a_3 = \frac{A-B}{4}[(A-2B-1)p_1^2+2p_2]. \tag{3.2}$$

Clearly, since $|a_2| \leqq (A-B)/2$, it follows that $(4-(A-B)^2)/4 \leqq T_{3,1}(f) \leqq 1$. These lower bound is sharp in case of the function k_0 satisfying (3.1) with $p(z) = (1+z)/(1-z)$ and that the upper bound is sharp in case of the function k_1 satisfying (3.1) with $p(z) = (1+z^2)/(1-z^2))$. Thus, we have the following:

Theorem 3. *Let $f \in \mathcal{K}[A,B]$. Then, the following best possible estimations hold:*

$$\frac{4-(A-B)^2}{4} \leqq T_{2,1}(f) \leqq 1.$$

Theorem 4. *Let $f \in \mathcal{K}[A,B]$. Then, the following best possible estimations hold:*

$$\sigma \leqq |T_{3,1}(f)| \leqq 1,$$

where

$$\sigma := \frac{1}{36}\left(2A^2-3AB+B^2-6\right)\left(A^2-3AB+2B^2-6\right).$$

Proof. Using (3.2) computations give

$$\begin{aligned} T_{3,1}(f) = 1 &+ \frac{(A-B)^2(A-2B)(2A-B)}{576}p_1^4 - \frac{(A-B)^2}{8}p_1^2 - \frac{(A-B)^2}{576}(4-p_1^2)^2|\zeta|^2 \\ &+ \frac{(A-B)^2(A+B)}{576}(4-p_1^2)p_1^2\Re\zeta =: \Upsilon(p_1^2,|\zeta|,\Re\zeta). \end{aligned} \tag{3.3}$$

Now the proof will be completed in a few cases.

Case I. Let $A+B>0$. Then, since the coefficient of $\Re\zeta$ is positive or equal to zero, it follows that

$$\Upsilon(p_1^2,|\zeta|,\Re\zeta) \leqq \Upsilon(p_1^2,|\zeta|,|\zeta|) = G(x,y),$$

where $x = p_1^2, y = |\zeta|$ and the function $G : [0,4]\times[0,1] \to \mathbb{R}$ is defined by

$$G(x,y) := 1 - \frac{(A-B)^2}{8}x + \frac{(2A-B)(A-2B)(A-B)^2}{576}x^2$$

$$+\frac{(A+B)(A-B)^2}{576}(4-x)xy-\frac{(A-B)^2}{576}(4-x)^2y^2.$$

Now on the boundary lines of the rectangular region $[0,4]\times[0,1]$, we see that

$$G(0,y)=1-\frac{1}{36}y^2(A-B)^2,$$

$$G(4,y)=\frac{1}{36}\left(2A^2-3AB+B^2-6\right)\left(A^2-3AB+2B^2-6\right)=:\sigma,$$

$$G(x,0)=1-\frac{(A-B)^2}{8}x+\frac{(2A-B)(A-2B)(A-B)^2}{576}x^2=:g_1(x) \tag{3.4}$$

and

$$G(x,1)=1-\frac{1}{36}(A-B)^2+\frac{1}{144}(A+B-16)(A-B)^2x$$
$$+\frac{1}{576}(2A-B+1)(A-2B-1)(A-B)^2x^2=:g_2(x). \tag{3.5}$$

Consider the functions g_1 and g_2. The first derivative of these functions, namely $g_1'(x)$ and $g_2'(x)$ vanish for $x=x_3=36/(A-2B)(2A-B)$ and $x=x_4=-2(A+B-16)/(A-2B-1)(2A-B+1)$, respectively. Since $(A-2B)(2A-B)<9$ and $-(A-2B-1)(2A-B+1)<9$, it folows that none of the x_3 and x_4 lie inside the interval $(0,4)$. At the end points, we have $g_1(0)=1$, $g_1(4)=\sigma=g_2(4)$ and $g_2(0)=1-(A-B)^2/36$. Therefore, we conclude that

$$T_{3,1}(f)\leqq\max\left\{1,\sigma,1-\frac{(A-B)^2}{36}\right\}=1.$$

Next we find the minimum for $T_{3,1}(f)$. For $A+B>0$, with the setting $p_1^2=x$, we can write

$$\Upsilon(p_1^2,|\zeta|,\Re\zeta)\geqq\Upsilon(p_1^2,|\zeta|,-|\zeta|)\geqq\Upsilon(p_1^2,1,-1)=g_3(x),$$

where the function $g_2:[0,4]\to\mathbb{R}$ is defined by

$$g_3(x):=1-\frac{(A-B)^2}{36}-\frac{(A+B+16)(A-B)^2}{144}x$$
$$+\frac{(2A-B-1)(A-2B+1)(A-B)^2}{576}x^2. \tag{3.6}$$

The first derivative of g_3 vanishes for $x=x_5=2(A+B+16)/(A-2B+1)(2A-B-1)$. But since $2(A+B+16)>32$ and $(A-2B+1)(2A-B-1)<8$, we conclude that $x_5\notin(0,4)$ and therefore, it is suffices to consider the values of g_3 at the end points of the interval $[0,4]$, and we get

$$g_3(0)=1-\frac{(A-B)^2}{36}\ \text{ and }\ g_3(4)=\sigma.$$

From the above discussion, we conclude that

$$T_{3,1}(f) \geqq \min\left\{\sigma, 1-\frac{(A-B)^2}{36}\right\} = \sigma.$$

Case II. Let $A+B<0$. Then, since the coefficient of $\Re\zeta$ is negative or equal to zero, it follows that

$$\Upsilon(p_1^2, |\zeta|, \Re\zeta) \leqq \Upsilon(p_1^2, |\zeta|, -|\zeta|) = H(x,y),$$

where $x = p_1^2, y = |\zeta|$ and the function $H : [0,4]\times[0,1] \to \mathbb{R}$ is defined by

$$H(x,y) = 1 - \frac{(A-B)^2}{8}x + \frac{(2A-B)(A-2B)(A-B)^2}{576}x^2 - \frac{(A+B)(A-B)^2}{576}(4-x)xy - \frac{(A-B)^2}{576}(4-x)^2y^2.$$

Now on the boundary of the rectangular region $[0,4]\times[0,1]$, we have

$$H(0,y) = 1-\frac{(A-B)^2}{36}y^2 \leqq 1,\ H(4,y) = \sigma,\ H(x,0) = g_1(x) \text{ and } H(x,1) = g_3(x),$$

where the functions g_1 and g_3 are defined by (3.4) and (3.6), respectively. Note that at the end point of $[0,4]$, $g_1(0) = 1 = g_3(0)$, $g_1(4) = \sigma = g_3(4)$ and $g_1'(x) = 0$ has no critical point inside the interval $(0,4)$ as $(A-2B)(2A-B) < 9$ for all $-1 \leqq B < A \leqq 1$. Similarly, it can be proved that the function $g_3'(x) = 0$ has no root inside the interval $(0,4)$. From the above discussion, for $A+B<0$, we conclude that

$$T_{3,1}(f) \leqq \max\left\{1, \sigma, 1-\frac{(A-B)^2}{36}\right\} = 1.$$

We now proceed to find the minimum of $T_{3,1}(f)$ for the case $-1 \leqq B < A < 0$. For this, with the setting $x = p_1^2$, we can write

$$\Upsilon(p^2, |\zeta|, \zeta) \geqq \Upsilon(p^2, |\zeta|, |\zeta|) \geqq \Upsilon(p^2, 1, 1) = g_2(x),$$

where the function g_3 is defined by (3.6). As in the Case II, it can be established that g_2 has no root in the interval $(0,4)$ and $g_2(0) = 1-(A-B)^2/36$ and $g_2(4) = \sigma$. Thus, for $-1 \leqq B < A < 0$, we have

$$|T_{3,1}(f)| \geqq \min\left\{1-\frac{(A-B)^2}{36}, \sigma\right\} = \sigma.$$

Case III. Let $A+B=0$. Then, from (3.3), we get

$$\begin{aligned} T_{3,1}(f) &= 1 - \frac{A^2}{2}x + \frac{A^4}{16}x^2 - \frac{A^2}{144}(4-x)^2y^2 \\ &\leqq \frac{1}{16}\left(A^2x-4\right)^2 \\ &\leqq 1. \end{aligned}$$

We also have

$$\begin{aligned} T_{3,1}(f) &= 1 - \frac{A^2}{2}x + \frac{A^4}{16}x^2 - \frac{A^2}{144}(4-x)^2y^2 \\ &\geqq 1 - \frac{A^2}{9} - \frac{4A^2}{9}x + \frac{\left(9A^2-1\right)A^2}{144}x^2. \\ &=: h(x). \end{aligned}$$

The first derivative of h vanishes only at $x = x_5 = 32/(9A^2-1)$ and $x_5 \notin (0,4)$. At the end points of the interval $(0,4)$, we have $h(0) = 1 - A^2/9$ and $h(4) = (A^2-1)^2$. Thus for $A+B=0$, we conclude that

$$(A^2-1)^2 \leqq |T_{3,1}(f)| \leqq 1, \text{ or equivalently } (B^2-1)^2 \leqq |T_{3,1}(f)| \leqq 1.$$

We put together the results discussed in the above three cases to conclude the assertion of the theorem. These lower bound is sharp in case of the function k_0 satisfying (3.1) with $p(z) = (1+z)/(1-z)$ and that the upper bound is sharp in case of the function k_1 satisfying (3.1) with $p(z) = (1+z^2)/(1-z^2)$). This ends the proof. □

Remark 2. For $A = 1-2\alpha$ and $B = -1$, Theorems 3 and 4 reduce to the results [7, Theorem 4] and [7, Theorem 5], respectively.

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A NEW CLASS OF SPECIAL FUNCTIONS IN HIGHER DIMENSIONS

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Abstract. We introduce a new class of functions $\mathcal{G}_f^{\nu}$ that are defined via integrals over σ-finite measure spaces and their kernels are defined in terms of Bessel functions and measurable functions. We prove that for certain values of ν, the function $\mathcal{G}_f^{\nu}$ is the Fourier transform of the distribution function d_f.

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1. Introduction

Bessel function of first kind, J_ν, is defined to be the solution of the Bessel differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\nu^2)y=0$$

see [6], a series formula of Bessel function can be given by

$$J_\nu(z)=\left(\frac{z}{2}\right)^\nu\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{2k}$$

In this paper we study a higher dimensional analog of the function

$$g_\nu(x)=\Gamma(\nu+1)\left(\frac{2}{x}\right)^\nu J_\nu(x),$$

where $x\in\mathbb{R}$, $\nu>\frac{-1}{2}$ and J_ν stands for the Bessel function of the first kind, see [1]. Inequalities involving the function g_ν have been studied by many authors. For instance, Askey [2] wrote Grünbaum's inequality [4] in terms of the function g_ν. we refer the readers to [1,5] for more background information.

For $n\geq 2$, let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n$ equipped with the surface measure $d\sigma$. Let (X,μ) be a σ-finite measure space, that is, there exists a sequence of measurable subsets $(A_n)_{n\in\mathbb{N}}$ such that $X=$

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$\bigcup_{n\in\mathbb{N}} A_n$ and $\mu(A_n) < \infty$ for all n. Let $f : X \to \mathbb{C}$ be a measurable function. Now, if $f \in L^{\nu+1}(X,\mu)$ consider the function

$$\mathcal{G}_f^{\nu}(\xi) = \frac{1}{|\xi|^{\nu}} \int_X |f(x)| J_{\nu}(2\pi|f(x)||\xi|)\, \mathrm{d}\mu(x),$$

where $\xi \in \mathbb{R}^n$, $\nu \geq \frac{-1}{2}$ and

$$J_{\nu}(t) = \frac{(t/2)^{\nu}}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} e^{its}(1-s^2)^{\nu} \frac{\mathrm{d}s}{\sqrt{1-s^2}}.$$

The function $\mathcal{G}_f^{\nu}(\xi)$ is well defined. To see this, we need to recall the following properties of Bessel functions.

$$|J_{\nu}(t)| \leq C(\nu) t^{\nu}, \tag{1.1}$$

$$|J_{\nu}(t)| \leq C(\nu) t^{-1/2}, \tag{1.2}$$

where $t > 0$ and the constant $C(\nu)$ depends only on ν. Thus, by using (1.1) we have

$$\begin{aligned} |\mathcal{G}_f^{\nu}(\xi)| &= \left| \frac{1}{|\xi|^{\nu}} \int_X |f(x)| J_{\nu}(2\pi|f(x)||\xi|)\, \mathrm{d}\mu(x) \right| \\ &\leq \frac{1}{|\xi|^{\nu}} \int_X |f(x)|\, |J_{\nu}(2\pi|f(x)||\xi|)|\, \mathrm{d}\mu(x) \\ &\leq (2\pi)^{\nu}\, ||f||_{\nu+1}^{\nu+1} < \infty, \end{aligned}$$

from which we have that $\mathcal{G}_f^{\nu}$ is bounded above. Notice also that if $f \equiv \frac{1}{2\pi}$ and X is a finite measure space with $\mu(X) = 2^{\nu+1}\pi\Gamma(\nu+1)$, then $\mathcal{G}_f^{\nu}(x) = g_{\nu}(x)$ for $x > 0$.

In the remaining part of this paper we clarify how this class of functions came into being and how it is related to some other well-known functions.

2. Main result

Functions are central objects in most fields of mathematics and science. One of the most important tools that we use to investigate the size of these objects, not its point-wise behavior is a function itself which is known as the distribution function. In the following, we review the main tools that we shall use to conclude our main result, see [3, 7–9].

We start by recalling the definition of the distribution function.

Definition 1. Let (X,μ) be a σ-finite measure space and $f : X \to \mathbb{C}$ be a measurable function. The distribution function d_f is given by

$$d_f(\alpha) = \mu\{x \in X : |f(x)| > \alpha\},$$

where $\alpha \geq 0$.

If $f \in L^p(X,\mu)$ for $0 < p < \infty$, then we have the following integral formula which correlates between the L^p-norm of f and its distribution function.

$$||f||_{L^p}^p = p\int_0^\infty d_f(\alpha)\alpha^{p-1}\,\mathrm{d}\alpha. \tag{2.1}$$

We know that if η is a radial function, then

$$\int_{\mathbb{R}^n} \eta(x)\,\mathrm{d}x = \sigma(\mathbb{S}^{n-1})\int_0^\infty \eta(r)r^{n-1}\,\mathrm{d}r, \tag{2.2}$$

where $\sigma(\mathbb{S}^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$. Thus, in view of (2.1) and (2.2) we can think about d_f as an integrable radial function on $\mathbb{R}^p$. To be more precise, we have

$$||f||_{L^p}^p = \frac{p}{\sigma(\mathbb{S}^{p-1})}\int_{\mathbb{R}^p} d_f(|y|)\,\mathrm{d}y. \tag{2.3}$$

Equation (2.3) says that $d_f \in L^1(\mathbb{R}^p)$ whenever $f \in L^p(X,\mu)$ and so its Fourier transform exists. The following result shows how the class $\mathcal{G}_f^\gamma$ did arise.

Theorem 1. *Let X be a σ-finite measure space and $f \in L^2(X,\mu)$. Then $\mathcal{G}_f^1(\xi) = \hat{d}_f(\xi)$, $\xi \in \mathbb{R}^2$.*

Proof. By using the fact that the Fourier transform of a radial function η is given by

$$\hat{\eta}(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}}\int_0^\infty \eta(r)J_{\frac{n}{2}-1}(2\pi r|\xi|)\,r^{\frac{n}{2}}\,\mathrm{d}r,$$

we have

$$\begin{aligned}\hat{d}_f(\xi) &= \int_{\mathbb{R}^2} e^{-2\pi i y\cdot\xi} d_f(|y|)\,\mathrm{d}y \\ &= 2\pi\int_0^\infty d_f(r)J_0(2\pi r|\xi|)\,r\,\mathrm{d}r.\end{aligned} \tag{2.4}$$

The definition of distribution function and (2.4) give

$$\begin{aligned}\hat{d}_f(\xi) &= 2\pi\int_0^\infty m\{x \in X : |f(x)| > r\}\,J_0(2\pi r|\xi|)\,r\,\mathrm{d}r \\ &= 2\pi\int_0^\infty \left(\int_{\{z\in X:|f(z)|>r\}} \mathrm{d}\mu\right) J_0(2\pi r|\xi|)\,r\,\mathrm{d}r.\end{aligned} \tag{2.5}$$

Noting that $\mathbb{1}_{\{z\in X:|f(z)|>r\}}(x) \equiv \mathbb{1}_{(0,|f(x)|)}(r)$ and applying Fubini's theorem we obtain

$$\hat{d}_f(\xi) = 2\pi\int_0^\infty \left(\int_X \mathbb{1}_{\{z\in X:|f(z)|>r\}}(x)\,\mathrm{d}\mu(x)\right) J_0(2\pi r|\xi|)\,r\,\mathrm{d}r$$

$$= 2\pi \int_X \left(\int_0^\infty \mathbb{1}_{(0,|f(x)|)}(r) J_0(2\pi r|\xi|)\, r\, \mathrm{d}r \right) \mathrm{d}\mu(x)$$
$$= 2\pi \int_X \left(\int_0^{|f(x)|} J_0(2\pi r|\xi|)\, r\, \mathrm{d}r \right) \mathrm{d}\mu(x). \tag{2.6}$$

Now, by using the change of variables, $s = r/|f(x)|$, we have

$$\hat{d}_f(\xi) = 2\pi \int_X |f(x)|^2 \left(\int_0^1 J_0(2\pi s|f(x)||\xi|)\, s\, \mathrm{d}s \right) \mathrm{d}\mu(x). \tag{2.7}$$

Next, applying the integral identity

$$\int_0^1 J_\alpha(ts) s^{\alpha+1}(1-s^2)^\beta\, \mathrm{d}s = \frac{\Gamma(\beta+1)2^\beta}{t^{\beta+1}} J_{\alpha+\beta+1}(t) \tag{2.8}$$

with $\alpha = \beta = 0$, we get

$$\hat{d}_f(\xi) = 2\pi \int_X |f(x)|^2 \left(\frac{1}{2\pi|f(x)||\xi|} J_1(2\pi|f(x)||\xi|) \right) \mathrm{d}\mu(x)$$
$$= \frac{1}{|\xi|} \int_X |f(x)| J_1(2\pi|f(x)||\xi|)\, \mathrm{d}\mu(x)$$
$$= \mathcal{G}_f^1(\xi). \tag{2.9}$$

□

Remark 1. If p is an integer greater than 2, then using the integral identity

$$\int t^\nu J_{\nu-1}(t)\, \mathrm{d}t = t^\nu J_\nu(t) + C.$$

in place of (2.8) and applying the same technique that was used in the proof of Theorem 1 we have $\mathcal{G}_f^{p/2}(\xi) = \hat{d}_f(\xi)$, $\xi \in \mathbb{R}^p$.

Remark 2. Theorem 1 and Remark 1 show that $\mathcal{G}_f^{p/2}(\xi)$ enjoys the property that it is uniformly continuous on $\mathbb{R}^p$. Moreover, $\mathcal{G}_f^{p/2}(\xi) \to 0$ as $|\xi| \to \infty$.

Theorem 2. *If X is a finite measure space and $f \in L^2(X,\mu)$, then $\mathcal{G}_f^1 \in L^2(\mathbb{R}^2)$.*

Proof. Notice that

$$\left(\int_{\mathbb{R}^2} |\mathcal{G}_f^1(\xi)|^2\, \mathrm{d}\xi \right)^{1/2} = \left(\int_{|\xi|<1} |\mathcal{G}_f^1(\xi)|^2\, \mathrm{d}\xi + \int_{|\xi|>1} |\mathcal{G}_f^1(\xi)|^2\, \mathrm{d}\xi \right)^{1/2}$$
$$= \left(\int_{|\xi|<1} |\mathcal{G}_f^1(\xi)|^2\, \mathrm{d}\xi \right)^{1/2} + \left(\int_{|\xi|>1} |\mathcal{G}_f^1(\xi)|^2\, \mathrm{d}\xi \right)^{1/2}$$
$$:= I + II. \tag{2.10}$$

By using (1.1), we have

$$\begin{aligned} I &= \left(\int_{|\xi|<1} |\mathcal{G}_f^1(\xi)|^2 \, \mathrm{d}\xi\right)^{1/2} \\ &\leq \left(\int_{|\xi|<1} \left(\frac{1}{|\xi|}\int_X |f(x)|\,|J_1(2\pi|f(x)||\xi|)|\,\mathrm{d}\mu(x)\right)^2 \mathrm{d}\xi\right)^{1/2} \\ &\leq 2\pi\left(\int_{|\xi|<1} \left(\frac{1}{|\xi|}\int_X |f(x)|^2\,|\xi|\,\mathrm{d}\mu(x)\right)^2 \mathrm{d}\xi\right)^{1/2} \\ &\leq 2\pi^{3/2}||f||_{L^2(X,\mu)}. \end{aligned} \tag{2.11}$$

Again, applying (1.2) we obtain

$$\begin{aligned} II &= \left(\int_{|\xi|>1} |\mathcal{G}_f^1(\xi)|^2 \, \mathrm{d}\xi\right)^{1/2} \\ &\leq \left(\int_{|\xi|>1} \left(\frac{1}{|\xi|}\int_X |f(x)|\,|J_1(2\pi|f(x)||\xi|)|\,\mathrm{d}\mu(x)\right)^2 \mathrm{d}\xi\right)^{1/2} \\ &\leq \frac{1}{\sqrt{2\pi}}\left(\int_{|\xi|>1} \frac{1}{|\xi|^3}\left(\int_X |f(x)|^{1/2}\,\mathrm{d}\mu(x)\right)^2 \mathrm{d}\xi\right)^{1/2} \\ &\leq \frac{1}{\sqrt{2\pi}}\left(\int_X |f(x)|^{1/2}\,\mathrm{d}\mu(x)\right)\left(\int_{|\xi|>1} \frac{1}{|\xi|^3}\mathrm{d}\xi\right)^{1/2} \\ &= (\mu(X))^{3/4}\sqrt{||f||_{L^2(X,\mu)}}. \end{aligned} \tag{2.12}$$

Now, combining (2.10), (2.11) and (2.12) we get the desired result. □

3. Conclusion

We have introduced a new class of functions which are defined via integrals. We have proved that the function $\mathcal{G}_f^\nu$ is uniformly continuous and can be an element of the Lebesgue spaces for some values of ν. Fourier analysis techniques were used to show that Bessel functions are related to the statistical terminology "distribution function". The main findings of this paper should open the door for studying these functions extensively.

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UNIFORM-ULTIMATE POISSON BOUNDEDNESS OF SOLUTIONS OF P-PERTURBED SYSTEMS OF DIFFERENTIAL EQUATIONS

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Abstract. The concept of P-perturbed systems with respect to given system of ordinary differential equations is introduced. Certain sufficient conditions for uniform-ultimate Poisson boundedness of solutions of P-perturbed systems are obtained by using the method of Lyapunov functions. In conclusion, an interesting application of the above sufficient conditions to the standard closed electric RLC circuit with nonlinear elements is given.

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1. Introduction and preliminaries

The theory of boundedness of solutions in the Poisson sense was developed in [2–5]. It generalizes the usual theory of boundedness of solutions for systems of differential equations [7]. The concept of Poisson boundedness of solutions is characterized by the fact that the solution may not be completely contained in some ball in the phase space, but has the property of a countable number of times of recurrence into this ball.

The paper is devoted to finding sufficient conditions for uniform-ultimate Poisson boundedness of solutions of P-perturbed systems. By a P-perturbed system we mean a system whose right-hand side is equal to the right-hand side of a given system multiplied by the function $1/(g(t)+\alpha(t))$, where $g(t)$ is any continuously differentiable periodic function, for which $\min g(t) = 0$ and $\max g(t) > 0$, and $\alpha(t)$ is any positive decreasing differentiable function satisfying the condition $\alpha(t) \to 0$ for $t \to +\infty$. The need to research P-perturbed systems with respect to given systems occurs when, under the influence of external forces, the modulus of the vector of the phase velocity of a given system begins to perform forced oscillations, the amplitude of which increases with time indefinitely.

The paper begins with consideration of the necessary information related to the concept of Poisson uniform-ultimate boundedness of solutions. Next, we introduce the concept of a P-perturbed system with respect to a given system. A sufficient

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condition is obtained for uniform-ultimate Poisson boundedness of solutions of a P-perturbed system with respect to any linear system with constant coefficients. Then, using this sufficient condition and the method of Lyapunov functions, we obtain a sufficient condition for the uniform-ultimate Poisson boundedness of solutions of a P-perturbed system with respect to an arbitrary system. In the conclusion of this paper we give an interesting application of the above sufficient conditions to the standard closed electric RLC circuit with nonlinear elements. Now let us pass to the exact definitions and formulations.

We consider a system of differential equations in $n \geqslant 1$ variables

$$\frac{dx}{dt} = F(t,x), \quad F(t,x) = (F_1(t,x), \dots, F_n(t,x))^T, \tag{1.1}$$

where $F : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is any continuous vector function. Here $\mathbb{R}^+$ denotes the set $\{t \in \mathbb{R} \mid t \geqslant 0\}$. It is assumed that all solutions of system (1.1) are extendable to the whole semi-axis $\mathbb{R}^+$. The uniqueness of the solution of the Cauchy problem for system (1.1) is not required.

Further, under $\|\cdot\|$ everywhere, we mean the usual Euclidean norm for vectors and matrices. The solution $x = x(t)$ of the system (1.1) passing through the point $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ will be written as $x = x(t, t_0, x_0)$. For any $t_0 \in \mathbb{R}^+$, we will further denote by $\mathbb{R}^+(t_0)$ the set $\{t \in \mathbb{R} \mid t \geqslant t_0\}$. Any non-negative increasing sequence $\tau = (\tau_i)_{i \geqslant 1}$ of real numbers for which the condition $\lim\limits_{i \to +\infty} \tau_i = +\infty$ is satisfied will be called a P-sequence. For each P-sequence $\tau = (\tau_i)_{i \geqslant 1}$, we will further denote by $M(\tau)$ the set $\bigcup\limits_{i=1}^{\infty} [\tau_{2i-1}, \tau_{2i}]$.

Recall [3] that the solutions of the system (1.1) are said to be uniform-ultimately Poisson bounded if there exist a number $B > 0$ and a P-sequence $\tau = (\tau_i)_{i \geqslant 1}$, such that for any $\alpha \geqslant 0$ there exist number $T \geqslant 0$ such that for any solution $x = x(t, t_0, x_0)$ of system (1.1), where $t_0 \in M(\tau)$ and $\|x_0\| \leqslant \alpha$, the condition $\|x(t, t_0, x_0)\| < B$ is satisfied for all $t \in \mathbb{R}^+(t_0 + T) \bigcap M(\tau)$. In the case where it is required to specify the corresponding P-sequence $\tau = (\tau_i)_{i \geqslant 1}$, it says that the solutions of the system (1.1) are uniform-ultimately Poisson bounded with respect to the P-sequence $\tau = (\tau_i)_{i \geqslant 1}$.

For comparison, it is also useful to recall [7] that the solutions of the system (1.1) are said to be uniform-ultimately bounded if for the system (1.1) there is a number $B > 0$, and for any $\alpha \geqslant 0$ there exists a number $T \geqslant 0$ such that for any solution $x = x(t, t_0, x_0)$ of system (1.1), where $t_0 \in \mathbb{R}^+$ and $\|x_0\| \leqslant \alpha$, the condition $\|x(t, t_0, x_0)\| < B$ is satisfied for all $t \in \mathbb{R}^+(t_0 + T)$.

It can be seen that if the solutions of the system (1.1) are uniform-ultimately bounded, then they are uniform-ultimately Poisson bounded.

In [5], based on the Lyapunov vector function method [6], the following sufficient condition for the uniform-ultimate Poisson boundedness of solutions of the system (1.1) was obtained.

Theorem 1. *Suppose that for system (1.1) there exist a P-sequence* $\tau = (\tau_i)_{i\geqslant 1}$, *continuously differentiable function* $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$, *an increasing function* $a : \mathbb{R}^+ \to \mathbb{R}^+$, *a non-decreasing function* $b : \mathbb{R}^+ \to \mathbb{R}^+$, *which has the property* $b(r) \to +\infty$ *for* $r \to +\infty$, *and continuous function* $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *that satisfy the following conditions:*

(1) $b(\|x\|) \leqslant V(t,x) \leqslant a(\|x\|)$ *for all* $(t,x) \in M(\tau) \times \mathbb{R}^n$.
(2) $\dot{V}(t,x) \leqslant f(t,V(t,x))$ *for all* $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$.
(3) *The solutions of the equation* $\dot{\xi} = f(t,\xi)$, *called the comparison equation for the system (1.1), are uniform-ultimately Poisson bounded with respect to the P-sequence* $\tau = (\tau_i)_{i\geqslant 1}$.

Then the solutions of the system (1.1) are uniform-ultimately Poisson bounded.

2. Results

We now introduce the concept of a P-perturbed system with respect to the system (1.1). Let $g(t)$ be any continuously differentiable periodic function defined on $\mathbb{R}^+$ satisfying the conditions $\min g(t) = 0$ and $\max g(t) > 0$. From properties of the function $g(t)$, it follows that $g'(t)$ is a continuous periodic function for which the conditions $\max g'(t) > 0$ and $\min g'(t) < 0$ are satisfied. Moreover, let $\alpha(t) > 0$ be any differentiable decreasing function defined on $\mathbb{R}^+$ for which the condition $\alpha(t) \to 0$ is fulfilled as $t \to +\infty$. Simple examples of such functions are $g(t) = 1 + \sin(t)$ and $\alpha(t) = e^{-t}$.

We will call a system that has the form

$$\frac{dx}{dt} = \frac{1}{g(t)+\alpha(t)} F(t,x), \tag{2.1}$$

where $g(t)$ and $\alpha(t)$ are any of the above functions, a P-perturbed system with respect to the system (1.1) or, if misunderstandings are excluded, just a P-perturbed system.

Consider a P-perturbed system with respect to an arbitrary linear system $\dot{x} = Ax$, $x \in \mathbb{R}^n$, $n \geqslant 1$, i.e., consider the system

$$\frac{dx}{dt} = \frac{1}{g(t)+\alpha(t)} Ax, \tag{2.2}$$

where A is a constant real $(n \times n)$ - matrix.

Theorem 2. *If the eigenvalues* $\lambda_1, \dots, \lambda_n$ *of the matrix A on the right-hand side of the P-perturbed system (2.2) satisfy the condition* $\mathrm{Re}(\lambda_i) < -\max g'(t)$ *for all* $1 \leqslant i \leqslant n$, *then solutions of this P-perturbed system are uniform-ultimately Poisson bounded.*

Proof. First we define the Lyapunov function $V(t,x)$. To do this, we will make the change of variables $x = Sy$, where $y = (y_1, \dots, y_n)^T \in \mathbb{C}^n$, S is a complex $(n \times n)$-matrix for which $S^{-1}AS = B$ is an upper triangular complex matrix, on the main diagonal of which are the eigenvalues $\lambda_1, \dots, \lambda_n$ of matrix A. Note that the above

change of variables $x = Sy$ is equivalent to changing the variables $x = \overline{S}\overline{y}$, where $\overline{y}$ is a complex number conjugate to y and $\overline{S} = (\overline{s}_{ij})$ is the matrix conjugate to $S = (s_{ij})$. It is clear that for the above replacement of variables $x = \overline{S}\overline{y}$, the equality $(\overline{S})^{-1}A\overline{S} = \overline{B}$ is true, where $\overline{B} = (\overline{b}_{ij})$ is an upper triangular matrix, conjugate to the matrix $B = (b_{ij})$. It is known (see, for example, [1]), that for any real number $b > 0$ there exists a change of variables $x = Sy$ such that for all $1 \leqslant i < j \leqslant n$ the inequality $|b_{ij}| < b$ is valid, where $|b_{ij}|$ is the modulus of the complex number b_{ij}. Define the function $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ by

$$\begin{aligned} V(t,x) &= (g(t)+\alpha(t)) \cdot (S^{-1}x, (\overline{S})^{-1}x) = (g(t)+\alpha(t)) \cdot (y,\overline{y}) \\ &= (g(t)+\alpha(t)) \sum_{i=1}^{n} y_i\overline{y}_i = (g(t)+\alpha(t))\|y\|^2 = (g(t)+\alpha(t)) \sum_{i=1}^{n} |y_i|^2 \geqslant 0, \end{aligned}$$

where $(\ ,\ )$ is a scalar multiplication in $\mathbb{C}^n$. Consider, for any fixed $0 < \gamma < \max g(t)$, an increasing sequence $\vartheta_1 < \vartheta_2 < \cdots < \vartheta_i < \ldots$ of all roots of the equation $g(t) + \alpha(t) = \gamma$. Let ϑ_k be the first root for which on the segment $[\vartheta_k, \vartheta_{k+1}]$ the condition $g(t) + \alpha(t) \geqslant \gamma$ is satisfied. Define the increasing sequence $\tau_1 < \tau_2 < \cdots < \tau_i < \ldots$, assuming $\tau_i = \vartheta_{i+k-1}$, $i \geqslant 1$. From the properties of the functions $g(t)$ and $\alpha(t)$, it follows that $\lim\limits_{i\to+\infty} \tau_i = +\infty$ and, therefore, $\tau = (\tau_i)_{i\geqslant 1}$ is a P-sequence. It is easy to check that for any $t \in \mathbb{R}^+(\tau_1)$ and $x \in \mathbb{R}^n$, the inequality $V(t,x) \leqslant a(\|x\|)$ holds, where $a(r) = (\max g(t) + \max \alpha(t))(\|S^{-1}\|r)^2$. Since on each closed interval $[\tau_{2i-1}, \tau_{2i}]$, $i \geqslant 1$, $\gamma \leqslant g(t) + \alpha(t)$, then for all $(t,x) \in M(\tau) \times \mathbb{R}^n$ we get $b(\|x\|) \leqslant V(t,x)$, where $b(r) = \gamma \cdot (r/\|S\|)^2$. It follows that the double inequality $b(\|x\|) \leqslant V(t,x) \leqslant a(\|x\|)$ holds for all $(t,x) \in M(\tau) \times \mathbb{R}^n$. For the derivative $\dot{V}(t,x)$ of the function $V(t,x)$ along solutions of the of the system (2.2), we have the following equalities:

$$\begin{aligned} \dot{V}(t,x) &= (g'(t)+\alpha'(t))\|y\|^2 + (S^{-1}Ax, (\overline{S})^{-1}x) + (S^{-1}x, (\overline{S})^{-1}Ax) \\ &= (g'(t)+\alpha'(t))\|y\|^2 + (S^{-1}ASy, (\overline{S})^{-1}\overline{S}\overline{y}) + (S^{-1}Sy, (\overline{S})^{-1}A\overline{S}\overline{y}) \\ &= (g'(t)+\alpha'(t))\|y\|^2 + (By,\overline{y}) + (y,\overline{B}\overline{y}) = (g'(t)+\alpha'(t))\|y\|^2 \\ &+ \left(\sum_{i=1}^{n} \lambda_i y_i\overline{y}_i + \sum_{1\leqslant i<j\leqslant n} b_{ij}y_j\overline{y}_i\right) + \left(\sum_{i=1}^{n} \overline{\lambda}_i y_i\overline{y}_i + \sum_{1\leqslant i<j\leqslant n} \overline{b}_{ij}\overline{y}_j y_i\right) \\ &= (g'(t)+\alpha'(t))\|y\|^2 + \sum_{i=1}^{n} (\lambda_i + \overline{\lambda}_i) y_i\overline{y}_i + \sum_{1\leqslant i<j\leqslant n} (b_{ij}y_j\overline{y}_i + \overline{b}_{ij}\overline{y}_j y_i). \end{aligned}$$

Since $\alpha'(t) < 0$ for all $t \geqslant 0$ and $\max g'(t) > 0$, we have $g'(t) + \alpha'(t) < \max g'(t)$ for all $t \geqslant 0$. Under the conditions of the theorem, $\mathrm{Re}(\lambda_i) < -\max g'(t)$ for any $1 \leqslant i \leqslant n$. Therefore, we have $\lambda_i + \overline{\lambda}_i = 2\mathrm{Re}(\lambda_i) < -2\max g'(t)$. Because $\overline{b_{ij}y_j\overline{y}_i} = \overline{b}_{ij}\overline{y}_j y_i$, we get

$$\sum_{1\leqslant i<j\leqslant n} (b_{ij}y_j\overline{y}_i + \overline{b}_{ij}\overline{y}_j y_i) = \sum_{1\leqslant i<j\leqslant n} 2\mathrm{Re}(b_{ij}y_j\overline{y}_i) \leqslant \sum_{1\leqslant i<j\leqslant n} 2|b_{ij}y_j\overline{y}_i|$$

$$= \sum_{1\leqslant i<j\leqslant n} 2|b_{ij}|\,|y_j|\,|\bar{y}_i| \leqslant \sum_{1\leqslant i<j\leqslant n} |b_{ij}|(|y_j|^2+|y_i|^2) \leqslant \sum_{1\leqslant i<j\leqslant n} |b_{ij}|\,\|y\|^2 \leqslant$$

$$\leqslant \sum_{1\leqslant i<j\leqslant n} \mu\|y\|^2 = \frac{1}{2}n(n-1)\mu\|y\|^2,$$

where $\mu>0$ is a preassigned number for which $|b_{ij}|<\mu$. From the above inequalities we get $\dot{V}(t,x)\leqslant(-\max g'(t)+(1/2)n(n-1)\mu)\|y\|^2$. If we choose the number $\mu>0$ so that $\mu<(2\max g'(t))/(n(n-1))$, then we get $\dot{V}(t,x)\leqslant-\psi\|y\|^2$, where $\psi=\max g'(t)-(1/2)n(n-1)\mu>0$. From the inequality $\dot{V}(t,x)\leqslant-\psi\|y\|^2$, it follows that

$$\dot{V}(t,x)\leqslant-\frac{\psi}{g(t)+\alpha(t)}V(t,x).$$

Because $-\psi/(g(t)+\alpha(t))\leqslant-\psi/\max(g(t)+\alpha(t))$, we obtain the inequality

$$\dot{V}(t,x)\leqslant-cV(t,x),$$

where $c=\psi/\max(g(t)+\alpha(t))>0$. This inequality suggests that the comparison equation for the system (2.2) is $\dot{\xi}=-c\xi$. We can establish by the Lyapunov functions method that the solutions of equation $\dot{\xi}=-c\xi$ are uniform-ultimately bounded. For example, we can take the function $W(\xi)=\xi^2$ as a Lyapunov function for the equation $\dot{\xi}=-c\xi$. Since the solutions of the equation $\dot{\xi}=-c\xi$ are uniform-ultimately bounded, then they are uniform-ultimately Poisson bounded with respect to the above P-sequence $\tau=(\tau_i)_{i\geqslant1}$. Thus, for the P-perturbed system (2.2), all conditions of Theorem1 are fulfilled and, therefore, the solutions of this P-perturbed system are uniform-ultimately bounded. □

Obviously, if in the conditions of Theorem 1 we take $n=1$, then we obtain the following statement.

Corollary 1. *Suppose that for P-perturbed equation*

$$\frac{dx}{dt}=\frac{1}{g(t)+\alpha(t)}ax$$

with respect to linear equation $\dot{x}=ax$, where $a\in\mathbb{R}$ satisfies condition $a<-\max g'(t)$. Then solutions of this P-perturbed equation is uniform-ultimately Poisson bounded.

We obtain, using the method of Lyapunov functions and Corollary 1, the sufficient condition for uniform-ultimate Poisson boundedness of solutions of P-perturbed system (2.1).

Theorem 3. *Suppose that for the P-perturbed system (2.1) there exists a differentiable function $V:\mathbb{R}^+\times\mathbb{R}^n\to\mathbb{R}$, constants $c_1>0$, $c_2>0$, $c_3>0$, a continuously differentiable periodic function $g:\mathbb{R}^+\to\mathbb{R}^+$, satisfying the requirements $\min g(t)=0$, $\max g(t)>0$, and a differentiable decreasing function $\alpha:\mathbb{R}^+\to\mathbb{R}^+$, $\alpha(t)>0$, $t\in\mathbb{R}^+$ satisfying $\alpha(t)\to0$ with $t\to+\infty$. For all $t\in\mathbb{R}^+$ and $x\in\mathbb{R}^n$ the following conditions are assumed:*

(1) $c_1\|x\|^2 \leqslant V(t,x) \leqslant c_2\|x\|^2$;

(2) $(g(t)+\alpha(t))\dfrac{\partial V(t,x)}{\partial t} + \left(\dfrac{\partial V(t,x)}{\partial x}, F(t,x)\right) \leqslant -c_3\|x\|^2$;

(3) $\max g'(t) < \dfrac{c_3}{2c_2}$.

Then solutions of the P-perturbed system (2.1) *are uniform-ultimately Poisson bounded.*

Proof. We define the function $W : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ by

$$W(t,x) = (g(t)+\alpha(t))V(t,x)$$

. For any fixed $0 < \gamma < \max g(t)$ we consider an increasing sequence $\vartheta_1 < \vartheta_2 < \cdots < \vartheta_i < \ldots$ of all roots of the equation $g(t)+\alpha(t) = \gamma$. Let ϑ_k be the first of the above roots, for which on the segment $[\vartheta_k, \vartheta_{k+1}]$ the condition $g(t)+\alpha(t) \geqslant \gamma$ is satisfied. We define an increasing sequence $\tau_1 < \tau_2 < \cdots < \tau_i < \ldots$ by setting $\tau_i = \vartheta_{i+k-1}$, $i \geqslant 1$. From the properties of functions $g(t)$ and $\alpha(t)$, it follows that $\lim\limits_{i\to+\infty} \tau_i = +\infty$ and, therefore, $\tau = (\tau_i)_{i\geqslant 1}$ is a P-sequence. Condition (1) of the theorem follows that the inequality $W(x,t) \leqslant a(\|x\|)$ holds for any $t \in [\tau_1, \infty]$ and $x \in \mathbb{R}^n$, where $a(r) = \max(g(t)\alpha(t))c_2 r^2$. It is easy to see that on each closed interval $[\tau_{2i-1}, \tau_{2i}]$, $i \geqslant 1$, the inequality $\gamma \leqslant g(t)\alpha(t)$ holds. Therefore, by using condition (1) of the theorem, we obtain that the inequality $b(\|x\|) \leqslant W(t,x)$ is true for all $(t,x) \in M(\tau) \times \mathbb{R}^n$, where $b(r) = \gamma c_1 r^2$, it follows that the double inequality

$$b(\|x\|) \leqslant W(t,x) \leqslant a\|x\|)$$

holds for all $(t,x) \in M(\tau) \times \mathbb{R}^n$. The derivative $\dot{W}(t,x)$ of $W(t,x)$ along solutions of the system (2.1) becomes

$$\begin{aligned}\dot{W}(t,x) &= \frac{\partial W(t,x)}{\partial t} + \left(\frac{\partial W(t,x)}{\partial x}, \frac{F(t,x)}{g(t)+\alpha(t)}\right)\\ &= (g'(t)+\alpha'(t))V(t,x) + (g(t)+\alpha(t))\frac{\partial V(t,x)}{\partial t} + \left(\frac{\partial V(t,x)}{\partial x}, F(t,x)\right).\end{aligned}$$

Using conditions (1) and (2) of the theorem, and also considering that $\alpha'(t) < 0$, we obtain the inequality

$$\dot{W}(t,x) \leqslant (\max g'(t))V(t,x) - c_3\|x\|^2 \leqslant \frac{1}{g(t)+\alpha(t)}(\max g'(t)) - \frac{c_3}{c_2})W(t,x).$$

This inequality means that the equation

$$\dot{\xi} = \frac{1}{g(t)+\alpha(t)}(\max g'(t)) - \frac{c_3}{c_2})\xi \tag{2.3}$$

is the comparison equation for system (2.1). Since $\max g'(t)) - (c_3/c_2) < -\max g'(t))$ holds by condition (3) of the theorem, using Corollary 1 we find that solutions of the

comparison equation (2.3) are uniform-ultimately Poisson bounded. Thus, all conditions of Theorem 1 are fulfilled and, therefore, solutions of P-perturbed system (2.1) are uniform-ultimately Poisson bounded. □

Let us consider an interesting application of Theorem 3 to the standard closed electric *RLC* circuit with nonlinear elements. The equation of the charge x (with x' being the current) is the following nonlinear ordinary differential equation of second order:

$$Lx'' + Rx' + \frac{1}{C}x + f(x) = 0, \tag{2.4}$$

where $R > 0$ is the resistance, $L > 0$ is the inductance, $C > 0$ is the capacitance, and f is a continuously differentiable function with respect to the charge x. Equation (2.4) can be transformed to the following equivalent system

$$\begin{cases} x_1' = x_2, \\ x_2' = -\dfrac{1}{LC}x_1 - \dfrac{R}{L}x_2 - \dfrac{1}{L}f(x_1). \end{cases} \tag{2.5}$$

We now make the linear change of the variables as

$$\begin{cases} y = x_1, \\ z = x_2 + x_1. \end{cases} \tag{2.6}$$

Then, by using (2.6), the system (2.5) can be written as

$$\begin{cases} y' = z - y, \\ z' = \left(-\dfrac{1}{LC} + \dfrac{R}{L} - 1\right)y + \left(-\dfrac{R}{L} + 1\right)z - \dfrac{1}{L}f(y). \end{cases} \tag{2.7}$$

Now we assume that the condition $0 < C(R-L) < 1$ holds. Then, it can be checked that

$$\frac{1}{LC} - \frac{R}{L} + 1 > 0 \text{ and } 1 - \frac{R}{L} < 0.$$

Moreover, we also assume that

$$f(0) = 0, \text{ and } 0 < \frac{f(y)}{y} < 2AL, \quad \text{if } y \neq 0,$$

where $A > 0$ is a constant. Then we have

$$0 \leqslant \frac{1}{L}\int_0^y f(\eta)d\eta = \frac{1}{L}\int_0^y \frac{f(\eta)}{\eta}\eta d\eta \leqslant \frac{2}{L}\int_0^y (AL)\eta d\eta = Ay^2.$$

Define a Lyapunov function by

$$V(y,z) = \frac{1}{2}\left(\frac{1}{LC} - \frac{R}{L} + 1\right)y^2 + \frac{1}{2}z^2 + \frac{1}{L}\int_0^y f(\eta)d\eta.$$

It is clear that

$$\frac{1}{3}\left(\frac{1}{LC}-\frac{R}{L}+1\right)y^2+\frac{1}{3}z^2 \leqslant V(y,z) \leqslant \left(\frac{1}{LC}-\frac{R}{L}+A+1\right)y^2+z^2.$$

If we put

$$c_1=\min\left(\frac{1}{3}\left(\frac{1}{LC}-\frac{R}{L}+1\right),\frac{1}{3}\right), \quad c_2=\max\left(\left(\frac{1}{LC}-\frac{R}{L}+A+1\right),1\right),$$

then we have

$$c_1\|(y,z)^T\|^2 \leqslant V(t,y,z) \leqslant c_2\|(y,z)^T\|^2.$$

For the time derivative $\dot{V}(y,z)$ of the Lyapunov function $V(y,z)$ along the solutions of system (2.7), we obtain

$$\dot{V}(y,z)=-\left(\frac{1}{LC}-\frac{R}{L}+1\right)y^2+\left(1-\frac{R}{L}\right)z^2+\frac{1}{L}(-g(y)y).$$

Since $g(y)y \geqslant 0$ and $L>0$, we have

$$\dot{V}(t,y,z) \leqslant -\left(\frac{1}{LC}-\frac{R}{L}+1\right)y^2-\left(\frac{R}{L}-1\right)z^2 \leqslant -c_3(y^2+z^2)=-c_3\|(y,z)^T\|^2,$$

where

$$c_3=\min\left(\left(\frac{1}{LC}-\frac{R}{L}+1\right),\left(\frac{R}{L}-1\right)\right).$$

Now, for the system (2.7) and the functions $g(t)=1+\sin(t)$ and $\alpha(t)=e^{-t}$, we consider the P-perturbed system

$$\begin{cases} y'=\dfrac{1}{1+\sin(t)+e^{-t}}(z-y), \\ z'=\dfrac{1}{1+\sin(t)+e^{-t}}\left(\left(-\dfrac{1}{LC}+\dfrac{R}{L}-1\right)y+\left(-\dfrac{R}{L}+1\right)z-\dfrac{1}{L}f(y)\right). \end{cases} \tag{2.8}$$

If we assume that $c_3/(2c_2)>1=\max(\cos(t))=g'(t)$, then by using Theorem 3 we can conclude the solutions of the system (2.8) are uniform-ultimately Poisson bounded. Since (2.6) is a linear change of the variables, the solutions of the P-perturbed system

$$\begin{cases} x_1'=\dfrac{1}{1+\sin(t)+e^{-t}}x_2, \\ x_2'=\dfrac{1}{1+\sin(t)+e^{-t}}\left(-\dfrac{1}{LC}x_1-\dfrac{R}{L}x_2-\dfrac{1}{L}f(x_1)\right) \end{cases}$$

with respect to the system (2.5) are uniform-ultimately Poisson bounded.

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LAGRANGIAN METHODS FOR OPTIMAL CONTROL PROBLEMS GOVERNED BY QUASI-HEMIVARIATIONAL INEQUALITIES

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Abstract. The aim of this paper is to study an optimal control problem governed by a quasi-hemivariational inequality by using nonlinear Lagrangian methods. We first show the existence of solutions to the inequality problem, and then, we establish several sufficient conditions for the zero duality gap property between the optimal control problem and its nonlinear dual problem.

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1. Introduction

Optimal control problems governed variational inequalities have attracted great attention in recent years and have many applications in engineering and mechanics, see, for instance, [1, 5, 20, 25, 27]. Lagrangian methods are useful tools to study constrained optimization problems and optimal control problems. The zero duality gap property in the study of optimization problems, where the optimal values of the prime and dual problems equal, is an important property to be used in the development of primal-dual methods. For more details on the zero duality gap property for nonconvex optimization problems, we refer to [21, 32] and the references therein. Recently, Zhou et al. [31] investigated the zero duality gap property for an optimal control problem governed by a variational inequality and gave several sufficient conditions for the zero duality gap property between the optimal control problem and its nonlinear Lagrangian dual problem. Later, Wang et al. [24] generalized the results in [31] and studied an optimal control problem where the state of the system is defined by a mixed quasi-variational inequality.

In this work, we study a class of quasi-hemivariational inequalities. The hemivariational inequalities were introduced by Panagiotoulos [17, 18] for describing a variety of mechanical problems such as unilateral contact in nonlinear elasticity, adhesive and friction effects, nonconvex semipermeability, masonry structures, and delamination in multilayered composites. Quasi-hemivariational inequalities are important

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and useful generalization of variational inequalities and hemivariational inequalities, and have significant applications in mechanics problems (see [12, 13]). Very recently, many authors studied the existence results for some types of hemivariational inequalities (see [8, 15, 23, 26, 29, 30]). For some related optimal control problem for hemivariational inequalities, we refer to [4, 6, 7, 9–11, 14, 19, 22].

The purpose of this paper is to study the optimal control by a quasi-hemivariational inequality. We first show the set of the solutions to the variational inequality is nonempty. Then we obtain several sufficient conditions for the zero duality gap property between the optimal control problem and its nonlinear dual problem. The main novelties of the paper are following. Instead of equilibrium problems and variational inequality problems widely used in the literature, we deal with a quasi-hemivariational inequality problem. The results obtained in this paper improve and extend many results in equilibrium problems and variational inequalities. To our best knowledge, it is the first work to study the zero duality gap property between the optimal control problem and its nonlinear dual problem for hemivariational inequalities. Several hemivariational inequality problems can also be further studied.

The rest of this paper is organized as follows. In the next section, we will introduce some useful preliminaries and necessary materials. In Section 3, we show the existence of solutions to the quasi-hemivariational inequality. In Section 4, we provide a result for the zero duality gap property between the optimal control problem and its nonlinear dual problem. Theorem 2 and Theorem 3 are main results in this part. We also give some deduced theorems in which we consider a special case of quasi-hemivariational inequalities in the last section.

2. Preliminaries

Let $(X, \|\cdot\|_X)$ be a Banach space, X^* denote its dual space and $\langle\cdot,\cdot\rangle_X$ be the duality pairing between X^* and X. We denote by " $\to$ " the strong convergence and by " $\rightharpoonup$ " the weak convergence.

Definition 1. A function $f\colon X \to \mathbb{R}$ is said to be

(i) (weakly) upper semicontinuous (u.s.c.) at u_0, if any sequence $\{u_n\} \subset X$ with $(u_n \rightharpoonup u_0)$ $u_n \to u_0$, we have $\limsup f(u_n) \le f(u_0)$.

(ii) (weakly) lower semicontinuous (l.s.c.) at u_0, if any sequence $\{u_n\} \subset X$ with $(u_n \rightharpoonup u_0)$ $u_n \to u_0$, we have $f(u_0) \le \liminf f(u_n)$.

(iii) f is said to be (weakly) u.s.c. (l.s.c.) on X, if f is (weakly) u.s.c. (l.s.c.) at u for all $u \in X$.

Definition 2 ([28])**.** Let D be a nonempty subset of X and let $F : D \to X^*$ be a single-valued mapping. F is said to be

(1) continuous if for every $u_0 \in D$ and any sequence $\{u_n\} \subset D$ with $u_n \to u_0$, we have $Fu_n \to Fu_0$.

(2) hemicontinuous if for all $u, v, w \in D$ the functional, $t \to \langle F(u+tv), z\rangle_X$ is continuous on $[0,1]$.

(3) strongly continuous if for every $u_0 \in D$ and any $\{u_n\} \subset D$ with $u_n \rightharpoonup u_0$ in D, we have $Fu_n \to Fu_0$.

(4) monotone if for any $u_1, u_2 \in D$, we have

$$\langle Fu_1 - Fu_2, u_1 - u_2\rangle_X \geq 0.$$

(5) pseudomonotone if for every $u_0 \in D$ and any $\{u_n\} \subset D$ with $u_n \rightharpoonup u_0$ and $\limsup_{n\to\infty}\langle Fu_n, u_n - u_0\rangle_X \leq 0$, we have

$$\langle Fu_0, u_0 - v\rangle_X \leq \liminf_{n\to\infty}\langle Fu_n, u_n - v\rangle_X$$

for all $v \in D$.

Remark 1 ([28]). (1) If F is hemicontinuous and monotone, then F is pseudomonotone.

(2) If $F : D \subset X \to X^*$ and is hemicontinuous if and only if F is continuous from the topology of X to the weakly* topology of X^*.

(3) Let $F_1, F_2 : D \to X^*$ be operators. If F_1 is pseudomonotone and F_2 is strongly continuous, then $F_1 + F_2$ is pseudomonotone. In addition, if F_1 and F_2 are pseudomonotone, then $F_1 + F_2$ is also pseudomonotone.

Definition 3. Let $G : X_1 \to 2^{X_2}$ be a multivalued mapping from a topological space X_1 into a topological space X_2. G is said to be bounded if $G(D)$ is a bounded set for any bounded subset D of X_1.

Definition 4 ([16])**.** Let D be a nonempty subset of X and $K : D \to 2^X$ be a multivalued mapping. For any $\{w_n\} \subset D$ with $w_n \rightharpoonup w_0 \in D$, we say that the sequence of sets $K(w_n)$ Mosco-converges to $K(w)$ if the following two conditions hold:

(i) for every sequence $\{u_n\}$, where $u_n \in K(w_n)$, such that $u_n \rightharpoonup u_0$, then $u_0 \in K(w_0)$;

(ii) for every $u_0 \in K(w_0)$, there exists $u_n \in K(w_n)$ (for n large enough) such that $u_n \to u_0$.

Lemma 1 ([3])**.** *Let D be a nonempty subset of a reflexive Banach space X, and let $G : D \to 2^X$ be a multivalued mapping satisfying*

(a) G is a KKM mapping, that is, for any $\{v_1, v_2, ..., v_n\} \subset D$, the convex hull $co\{v_1, v_2, \cdots, v_n\}$ is contained in $\bigcup_{i=1}^{n} G(v_i)$,

(b) $G(v)$ is closed in X for every $v \in D$,

(c) $G(v_0)$ is compact in X for some $v_0 \in D$.

Then $\bigcap_{v\in D} G(v) \neq \varnothing$.

Next, we present some needed elements of subdifferential calculus for locally Lipschitz functions (see [2, 12]).

Given a locally Lipschitz function $J : X \to \mathbb{R}$ on a Banach space X, we denote by $J^\circ(u,v)$ the generalized directional derivative of J at the point $u \in X$ in the direction $v \in X$, that is

$$J^\circ(u;v) := \limsup_{\lambda \to 0^+; w \to u} \frac{J(w+\lambda v)+J(w)}{\lambda}.$$

The generalized gradient of $J : X \to \mathbb{R}$ at $u \in X$ is defined by

$$\partial J(u) := \{\xi \in X^* : J^\circ(u;v) \geq \langle \xi; v\rangle_X \ for\ all\ v \in X\}.$$

Proposition 1 ([2]). *Let $J : X \to \mathbb{R}$ be locally Lipschitz of rank $L_u > 0$ near $u \in X$. Then, there hold the following:*

(a) the function $v \to J^\circ(u;v)$ is positively homogeneous, subadditive and satisfies $|J^\circ(u;v)| \leq L_u \|v\|_X$ for all $v \in X$;

(b) $J^\circ(u;v)$ is u.s.c. as a function of (u,v);

(c) $\partial J(u)$ is a nonempty, convex, weak* compact subset of X^* with $\|\xi\|_{X^*} \leq L_u$ for all $\xi \in \partial J(u)$;

(d) for each $v \in X$, we have $J^\circ(u;v) = \max\{\langle \xi, v\rangle_X : \xi \in \partial J(u)\}$.

Next, we introduce the framework of this paper. We assume that X, V and E are reflexive Banach spaces, U is a nonempty closed convex subset of V and $K : U \to 2^X$ is a multivalued mapping with nonempty values. Let $\mathcal{J} : X \times V \to \mathbb{R}$ and $A : X \to X^*, B : U \to X^*, J : E \to \mathbb{R}, \gamma : X \to E$ be given mappings, $f : X \times X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ be a function such that $f(u,u) = 0$ for every $u \in X$. Consider the following optimal control problem governed by a quasi-variational inequality: Find $w^* \in U$ and $u \in S(w^*)$ such that

$$\begin{cases} \min \mathcal{J}(u,w) \\ \text{subject to } (w,u) \in U \times S(w), \end{cases} \tag{2.1}$$

where $S(w)$ is the set of solutions to the following quasi-hemivariational inequality corresponding to w: find $u \in K(w)$ such that

$$\langle Au + Bw, v-u\rangle_X + f(u,v) + J^\circ(\gamma u; \gamma v - \gamma u) \geq 0, \qquad \forall v \in K(w). \tag{2.2}$$

For any $w \in U$, we define

$$g_w(u) = \sup_{v \in K(w)} \langle Au + Bw, v-u\rangle_X + f(u,v) + J^\circ(\gamma u; \gamma v - \gamma u), \tag{2.3}$$

$$K_w(y) = \{u \in K(w) : g_w(u) \leq y\}. \tag{2.4}$$

Thus, $g_w(u) \geq 0$ and $K_w(0) = \{u \in K(w) : g_w(u) = 0\} = S(w)$. Define the perturbation function β as

$$\beta(y) = \inf_{w \in U, u \in K_w(y)} \mathcal{J}(w,u). \tag{2.5}$$

It is easy to see that $\beta(0)$ is the optimal value of problem (2.1).

Definition 5 ([1]). Let $\mathbb{R}_+ = [0,+\infty)$ and $P : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be a function. A nonlinear Lagrangian function $L_p : U \times K(U) \times \mathbb{R}_+ \to \mathbb{R}$ for problem (2.1) is defined as

$$L_P(w,u,d) = P(\mathcal{J}(w,u), dg_w(u)).$$

The function

$$F_P(d) = \inf_{w\in U, u\in K(w)} L_P(w,u,d)$$

is called its nonlinear Lagrangian dual function. The equality

$$\beta(0) = \sup_{d\in\mathbb{R}_+} F_P(d) \tag{2.6}$$

is called the zero duality gap property.

Assumption 1. (P_i) *If* $y_1 \le y_2$, *then* $P(y_1,z) \le P(y_2,z)$, $\forall z \in \mathbb{R}_+$.
(P_{ii}) $p(y,0) = y$, $\forall y \in \mathbb{R}$.
(P_{iii}) *There exists a number* $a > 0$, *such that* $P(y,z) \ge \max\{y, az\}$, $\forall (y,z) \in \mathbb{R} \times \mathbb{R}_+$.

Lemma 2 ([21]). *Suppose that P is a continuous function satisfying P_i and P_{ii}. If the zero duality gap property (2.6) holds, then the perturbation function β is l.s.c. at the origin.*

Lemma 3 ([21]). *Suppose that P satisfies $(P_i), (P_{ii}), (P_{iii})$. If $-\infty < \beta(0) < +\infty$ and the perturbation function β is l.s.c. at the origin, then the zero duality gap property (2.6) holds.*

3. Existence of solutions

In this section, we give an existence result for the quasi-hemivariational inequality (2.2).

Consider the following hypotheses on the data of (2.2).
(H_K) : $K : U \to 2^X$ is such that for each $w \in U$, $0 \in K(w)$ and the set $K(w)$ is closed and convex in X.
(H_A) : $A : K(U) \to X^*$ is hemicontinuous and monotone.
(H_f) : $f : X \times X \to \overline{\mathbb{R}}$ is a mapping satisfying the following conditions:
(i) $\mathcal{D}(f) = \{u \in K(U) : f(u,v) \ne -\infty,\ \forall v \in K(U)\}$ is nonempty,
(ii) $f(u,u) = 0$ for all $u \in K(U)$,
(iii) $f(u,v) + f(v,u) = 0$ for all $u, v \in K(U)$,
(iv) for every $v \in K(U)$, $f(\cdot, v)$ is weakly u.s.c.,
(v) for every $u \in K(U)$, $f(u,\cdot)$ is convex.
(H_J) : $J : E \to \mathbb{R}$ is a locally Lipschitz function.
(H_γ) : $\gamma : X \to E$ is a linear, bounded and compact operator.
(H_0) :

$$\lim_{u\in X, \|u\|_X\to\infty} \frac{\langle Au,u\rangle_X + \inf_{\eta_u\in\partial J(\gamma u)} \langle \eta_u, \gamma u\rangle_E + f(0,u)}{\|u\|_X} = +\infty. \tag{3.1}$$

Theorem 1. *Assume that* $(H_K),(H_A),(H_f),(H_J),(H_\gamma),(H_0)$ *are satisfied. Then* $S(w)\neq\varnothing$.

Proof. (i) At first, we suppose that $K(w)$ is bounded in X. Introduce the multivalued mapping $G:K(w)\to 2^{K(w)}$ as follows:

$$G(v):=\{u\in K(w):\langle Av+Bw,v-u\rangle_X+f(u,v)+J^\circ(\gamma u;\gamma v-\gamma u)\geq 0\},\ \ \forall v\in K(w).$$

We claim that $G(v)$ is weakly closed in V. If for $\{u_n\}\subset G(v)$ one has $u_n\rightharpoonup u$, then $u\in K(w)$ and, for each $n\in\mathbb{N}$

$$\langle Av+Bw,v-u_n\rangle_X+f(u_n,v)+J^\circ(\gamma u_n;\gamma v-\gamma u_n)\geq 0.$$

Proposition 1(b) and (H_f)(iii) imply

$$\begin{aligned}0\leq&\limsup_{n\to\infty}\langle Av+Bw,v-u_n\rangle_X+f(u_n,v)+J^\circ(\gamma u_n;\gamma v-\gamma u_n)\\ \leq&\langle Av+Bw,v-u\rangle_X+f(u,v)+J^\circ(\gamma u;\gamma v-\gamma u),\end{aligned}$$

which proves that $u\in G(v)$, thus $G(v)$ is weakly closed in V. We continue with the proof by arguing separately in two cases: (a) G is a KKM mapping. (b) G is not a KKM mapping. Assume case (a). Since $K(w)$ is bounded, closed and convex in the reflexive Banach space V, it is weakly compact. Due to the assertion above, for every $v\in K(w)$ $G(v)$ is weakly compact too. Lemma 1 with respect to the weak topology of X yields

$$\bigcap_{v\in K(w)}G(v)\neq\varnothing.$$

Taking $u_0\in\bigcap_{v\in K(w)}G(v)$, we have

$$\langle Av+Bw,v-u_0\rangle_X+f(u_0,v)+J^\circ(\gamma u_0;\gamma v-\gamma u_0)\geq 0,\qquad\forall v\in K(w).$$

Let an arbitrary $v'\in K(w)$ and let $v_n:=\frac{1}{n}v'+(1-\frac{1}{n})u_0\in K(w)$. Then we have

$$\langle Av_n+Bw,v_n-u_0\rangle_X+f(u_0,v_n)+J^\circ(\gamma v_n;\gamma v_n-\gamma u_0)\geq 0.$$

Then

$$\begin{aligned}0\leq&\langle Av_n+Bw,\frac{1}{n}v'+(1-\frac{1}{n})u_0-u_0\rangle_X+f(u_0,\frac{1}{n}v'+(1-\frac{1}{n})u_0)\\ &+J^\circ(\gamma v_n;\gamma\frac{1}{n}v'+(1-\frac{1}{n})u_0-\gamma u_0)\\ \leq&\frac{1}{n}[\langle Av_n+Bw,v'-u_0\rangle_X+f(u_0,v')+J^\circ(\gamma v_n;\gamma v'-\gamma u_0)].\end{aligned}$$

Multiplying the last inequality by n and letting $n\to\infty$, we have

$$\langle Au_0+Bw,v'-u_0\rangle_X+f(u_0,v')+J^\circ(\gamma u_0;\gamma v'-\gamma u_0)>0,\qquad\forall v'\in K(w),$$

which concludes $u_0\in S(w)$.

Now, admit case (b). Then, one can find $v_1, v_2, ..., v_N \in K(w)$ and $u_0 = \sum_{j=1}^{N} \lambda_j v_i$ with $\lambda_i \in [0,1]$ and $\sum_{j=1}^{N} \lambda_i = 1$ such that

$$u_0 \nsubseteq \bigcup_{j=1}^{N} G(v_j),$$

which expresses that

$$\langle Av_j + Bw, v_j - u_0\rangle_X + f(u_0, v_j) + J^\circ(\gamma u_0; \gamma v_j - \gamma u_0) < 0.$$

Claim. There exists a neighborhood O of u_0 in X such that whenever $v \in O \bigcap K(w)$, there holds

$$\langle Av_j + Bw, v_j - v\rangle_X + f(v, v_j) + J^\circ(\gamma v; \gamma v_j - \gamma v) < 0. \tag{3.2}$$

Arguing by contradiction, let us assume that there exist $u_n \in K(w)$ and $j_n \in \{1, 2, ..., N\}$ such that $u_n \to u_0$ and, for each n,

$$\langle Av_{j_n} + Bw, v_{j_n} - u_n\rangle_X + f(u_n, v_{j_n}) + J^\circ(\gamma u_n; \gamma v_{j_n} - \gamma u_n) \geq 0.$$

Then, there exists $j_0 \in \{1, 2, ..., N\}$ such that for all n

$$\langle Av_{j_n} + Bw, v_{j_0} - u_n\rangle_X + f(u_n, v_{j_0}) + J^\circ(\gamma u_n; \gamma v_{j_0} - \gamma u_n) \geq 0.$$

Passing to the limit gives

$$\begin{aligned} 0 \leq & \limsup_{n\to\infty} \langle Av_{j_0} + Bw, v_{j_0} - u_n\rangle_X + f(u_n, v_{j_0}) + J^\circ(\gamma u_n; \gamma v_{j_0} - \gamma u_n) \\ \leq & \langle Av_{j_0} + Bw, v_{j_0} - u_0\rangle_X + f(u_0, v_{j_0}) + J^\circ(\gamma u_n; \gamma v_{j_0} - \gamma u_0), \end{aligned}$$

which contradicts (3.2). The claim is proven.

Using **Claim**, we have for each $j \in \{1, 2, ..., N\}$ that

$$\langle Av_j + Bw, v_j - v\rangle_X + f(v, v_j) + J^\circ(\gamma v; \gamma v_j - \gamma v) < 0, \qquad \forall v \in O \bigcap K(w).$$

It follows from the monotonicity of A that

$$\langle Av + Bw, v_j - v\rangle_X + f(v, v_j) + J^\circ(\gamma v; \gamma v_j - \gamma v) \leq 0.$$

Furthermore, on the basis of Proposition 2.13(d) and the convexity of $f(v, \cdot)$, we note that

$$\begin{aligned} & \langle Av + Bw, v - u_0\rangle_X + f(v, u_0) + J^\circ(\gamma v; \gamma u_0 - \gamma v) \\ = & \langle Av + Bw, \sum_{j=1}^{N} \lambda_j v_j - v\rangle_X + f(v, \sum_{j=1}^{N} \lambda_j v_j) + J^\circ(\gamma v; \gamma u_0 - \gamma v) \\ \leq & \sum_{j=1}^{N} \lambda_j [\langle Av + Bw, v_j - v\rangle_X + f(v, v_j) + J^\circ(\gamma v; \gamma v_j - \gamma v)] \\ \leq & 0 \end{aligned}$$

for all $v \in O \bigcap K(w)$. Since

$$J^\circ(\gamma v; \gamma u_0 - \gamma v) + J^\circ(\gamma v; \gamma v - \gamma u_0) \geq J^\circ(\gamma v; \gamma 0) = 0,$$

we obtain

$$\langle Av + Bw, v - u_0 \rangle_X + f(u_0, v) + J^\circ(\gamma v; \gamma v - \gamma u_0) > 0, \quad \forall v \in O \bigcap K(w). \tag{3.3}$$

To prove that $u_0 \in S(w)$, let an arbitrary $y \in K(w)$ and consider $y_n := \frac{1}{n} y + (1 - \frac{1}{n}) u_0 \in O \bigcap K(w)$. Hence, (3.3) can be used to obtain

$$\langle Ay_n + Bw, y_n - u_0 \rangle_X + f(u_0, y_n) + J^\circ(\gamma y_n; \gamma y_n - \gamma u_0) \geq 0.$$

Then

$$\begin{aligned} 0 \leq & \langle Ay_n + Bw, \frac{1}{n} y + (1 - \frac{1}{n}) u_0 - u_0 \rangle_X + f(u_0, \frac{1}{n} y + (1 - \frac{1}{n}) u_0) \\ & + J^\circ(\gamma y_n; \gamma(\frac{1}{n} y + (1 - \frac{1}{n}) u_0) - \gamma u_0) \\ \leq & \frac{1}{n} [\langle Ay_n + Bw, y - u_0 \rangle_X + f(u_0, y) + J^\circ(\gamma y_n; \gamma y - \gamma u_0)]. \end{aligned}$$

Multiplying the last inequality by n and letting $n \to \infty$, we have

$$\langle Au_0 + Bw, y - u_0 \rangle_X + f(u_0, y) + J^\circ(\gamma u_0; \gamma y - \gamma u_0) > 0, \qquad \forall y \in K(w),$$

which concludes $u_0 \in S(w)$.

(ii) Now, we assume that $K(w)$ is unbounded. For any given $w \in U$, letting $B_r = \{v \in X : \|v\| \leq r\}$ and $K^r = B_r \bigcap K(w)$, we get that K^r is a bounded, closed and convex subset of X. According to the assumptions, Remark 1 implies that $A + B$ is demicontinuous and pseudomonotone. By (i), we know there exists $u_r \in K^r$ such that

$$\langle Au_r + Bw, v - u_r \rangle + f(u_r, v) + J^\circ(\gamma u_r; \gamma v - \gamma u_r) \geq 0, \qquad \forall v \in K^r. \tag{3.4}$$

In particular, taking $v = 0$ in (3.4), there exists $u_r \in K^r$ such that

$$\langle Au_r + Bw, u_r \rangle - f(u_r, 0) - J^\circ(\gamma u_r; \gamma 0 - \gamma u_r) \leq 0, \qquad \forall v \in K^r.$$

it follows from (H_0) that $\{u_r\}$ is bounded. So $\|u_r\| \leq M$ for some real number $M > 0$. Let $r = M + 1$. For each $v \in K(w)$, we can choose $t \in (0, 1)$ small enough such that $v_r = u_r + t(v - u_r) \in K^r$. Substituting v_r into (3.4), we obtain that

$$\begin{aligned} & \langle Au_r + Bw, u_r + t(v - u_r) - u_r \rangle + f(u_r, u_r + t(v - u_r)) \\ & \quad + J^\circ(\gamma u_r; \gamma(u_r + t(v - u_r)) - \gamma u_r) \rangle \geq 0. \end{aligned}$$

Then

$$\begin{aligned} 0 \leq & \langle Au_r + Bw, u_r + t(v - u_r) - u_r \rangle + f(u_r, u_r + t(v - u_r)) \\ & + J^\circ(\gamma u_r; \gamma(u_r + t(v - u_r)) - \gamma u_r) \rangle \\ \leq & t[\langle Au_r + Bw, v - u_r \rangle + f(u_r, v) + J^\circ(\gamma u_r; \gamma v - \gamma u_r)]. \end{aligned}$$

Dividing by t, it follows that

$$\langle Au_r + Bw, v - u_r\rangle + f(u_r, v) + J^\circ(\gamma u_r; \gamma v - \gamma u_r) \geq 0, \qquad \forall v \in K(w),$$

and hence $u_r \in S(w)$. The proof is completed. □

Remark 2. From the theorem above, it is clear that hypothesis (H_0) can be omitted if K is bounded.

4. Optimal control

In this section, we establish several conditions to guarantee the zero duality gap property for the optimal control problem governed by quasi-hemivariational inequalities and its dual problem.

Theorem 2. *Assume that $(H_K), (H_A), (H_f), (H_J), (H_\gamma)$ are satisfied. Suppose that, in addition, $K : U \to 2^X$ is bounded, $A : K(U) \to X^*$ is bounded, $f(u,\cdot)$ is u.s.c. for every $u \in K(U)$, $\mathcal{J} : U \times K(U) \to \mathbb{R}$ is weakly l.s.c. function, $B : U \to X^*$ is strongly continuous from the weak topology of V to the topology of X^* and the following conditions are satisfied*

(i) for any $w \in U$, $u \in K(w)$, $lim_{\|w\|\to+\infty}\mathcal{J}(w,u) = +\infty$,

(ii) for all $w_n \in U$ with $w_n \rightharpoonup w$, $K(w_n)$ Mosco-converges to $K(w)$.

Then $-\infty < \beta(0) < +\infty$.

Proof. From Theorem 1, it follows that for each $w \in U$, $S(w) \neq \varnothing$. Let $\{(w_n, u_n)\} \subset U \times K(U)$ be a sequence satisfying $u_n \in S(w_n)$ such that

$$\mathcal{J}(w_n, u_n) \leq \inf_{w\in U, u\in S(w)} \mathcal{J}(w,u) + \frac{1}{n} = \beta(0) + \frac{1}{n}, \quad n = 1, 2, \ldots.$$

Condition (i) implies that $\{w_n\}$ is bounded. Since K is bounded and $u_n \in S(w_n) \subset K(w_n)$, $\{u_n\}$ is a bounded sequence, and hence $\{(w_n, u_n)\}$ is bounded.

By the reflexivity of V and X, there exists a weakly convergent subsequence of $\{(w_n, u_n)\}$. Without loss of generality, we can assume that $w_n \rightharpoonup w_0$ in V and $u_n \rightharpoonup u_0$ in X as $n \to +\infty$.

Since U is closed and convex, U is a weakly closed set and so $w_0 \in U$. By the assumptions, we know that $K(w_n)$ Mosco-converges to $K(w_0)$ and so $u_0 \in K(w_0)$. According to Definition 4, there exists $u_n' \in K(w_n)$ such that $u_n' \to u_0$. By $u_n \in S(w_n)$, we know that $u_n \in K(w_n)$ such that

$$\langle Au_n + Bw_n, v - u_n\rangle_X + f(u_n, v) + J^\circ(\gamma u_n; \gamma v - \gamma u_n) \geq 0, \qquad \forall v \in K(w_n),$$

and so

$$\langle Au_n + Bw_n, u_n' - u_n\rangle_X + f(u_n, u_n') + J^\circ(\gamma u_n; \gamma u_n' - \gamma u_n) \geq 0. \tag{4.1}$$

Since B is strongly continuous from the weak topology of V to the topology of X^*, $Bw_n \to Bw_0$. Moreover, note that A is bounded. Without loss of generality, we can

assume that $Au_n \rightharpoonup u^*$. By the hypotheses, (4.1) implies that

$$\begin{aligned}\limsup_{n\to\infty}\langle Au_n, u_n-u_0\rangle_X =& \limsup_{n\to\infty}\langle Au_n, u_n-u_0\rangle_X + \lim_{n\to\infty}\langle Au_n, u_0-u_n'\rangle_X\\ &+\lim_{n\to\infty}\langle Bw_n, u_n-u_n'\rangle_X\\ =&\limsup_{n\to\infty}\langle Au_n+Bw_n, u_n-u_n'\rangle_X\\ \le&\limsup_{n\to\infty}(f(u_n,u_n')+J^\circ(\gamma u_n;\gamma u_n'-\gamma u_n))\\ \le&\lim_{n\to\infty}f(u_n,u_n')+\limsup_{n\to\infty}J^\circ(\gamma u_n;\gamma u_n'-\gamma u_n)\\ \le&0.\end{aligned}$$

It follows from (H_A) and Remark 1(i) that A is pseudomonotone. Fixing any $v'\in K(w_0)$, we have

$$\langle Au_0, u_0-v'\rangle_X \le \liminf_{n\to\infty}\langle Au_n, u_n-v'\rangle_X.$$

According to $w_n \rightharpoonup w_0$ and condition (ii), there exists $v_n\in K(w_n)$ such that $v_n\to v'$. Since $Au_n\rightharpoonup u^*$, it follows that

$$\begin{aligned}\liminf_{n\to\infty}\langle Au_n, u_n-v'\rangle_X =&\liminf_{n\to\infty}\langle Au_n, u_n\rangle_X + \lim_{n\to\infty}\langle Au_n, -v'\rangle_X\\ =&\liminf_{n\to\infty}\langle Au_n, u_n\rangle_X + \langle u^*, -v'\rangle_X\\ =&\liminf_{n\to\infty}\langle Au_n, u_n\rangle_X + \lim_{n\to\infty}\langle Au_n, -v_n\rangle_X\\ =&\liminf_{n\to\infty}\langle Au_n, u_n-v_n\rangle_X.\end{aligned}$$

Thus,

$$\begin{aligned}&\langle Au_0, u_0-v'\rangle_X + \langle Bw_0, u_0-v'\rangle_X + f(v',u_0) + J^\circ(\gamma u_0;\gamma u_0-\gamma v')\\ \le&\liminf_{n\to\infty}\langle Au_n, u_n-v'\rangle_X + \lim_{n\to\infty}\langle Bw_n, u_n-v_n\rangle_X\\ &+\liminf_{n\to\infty}f(v_n,u_n)+\lim_{n\to\infty}J^\circ(\gamma u_n;\gamma u_n-\gamma v')\\ =&\liminf_{n\to\infty}\langle Au_n, u_n-v_n\rangle_X + \lim_{n\to\infty}\langle Bw_n, u_n-v_n\rangle_X\\ &+\liminf_{n\to\infty}f(v_n,u_n)+\lim_{n\to\infty}J^\circ(\gamma u_n;\gamma u_n-\gamma v_n)\\ \le&\liminf_{n\to\infty}\langle Au_n+Bw_n, u_n-v_n\rangle_X + f(v_n,u_n)+J^\circ(\gamma u_n;\gamma u_n-\gamma v_n).\end{aligned}$$

This implies that

$$\langle Au_0+Bw_0, v'-u_0\rangle_X + f(u_0,v') + J^\circ(\gamma u_0;\gamma v'-u_0)\ge 0, \qquad \forall v'\in K(w_0).$$

Therefore, $u_0\in S(w_0)$.

Since $\mathcal{J}(w,u)$ is a weakly l.s.c. function, it follows from (2.5) that

$$\mathcal{J}(w_0,u_0)\le\liminf_{n\to\infty}\mathcal{J}(w_n,u_n)=\beta(0),$$

and so

$$\mathcal{J}(w_0,u_0)=\beta(0).$$

Therefore, $-\infty<\beta(0)<+\infty$. This completes the proof. □

Theorem 3. *Assume that all the hypotheses of Theorem 2 are satisfied. If P satisfies $(P_i),(P_{ii}),(P_{iii})$, then the zero duality gap property (2.6) holds.*

Proof. From Theorem 2, $-\infty<\beta(0)<+\infty$, By Lemma 3, we only need to prove that the perturbation function β is l.s.c. at the origin.

On the contrary, assume that there exists an $\varepsilon_0>0$ such that

$$\liminf_{y\to 0}\beta(y)\le\beta(0)-\varepsilon_0.$$

Then, there exists a sequence $\{y_k\}\subset\mathbb{R}$, $w_k\in U$ and $u_k\in K_{w_k}(y_k)$ such that

$$\mathcal{J}(w_k,u_k)\le\beta(0)-\frac{1}{2}\varepsilon_0,\quad k=1,\dots$$

hold.

From the proof of Theorem 2, the sequence $\{(w_k,u_k)\}$ is bounded and we can assume that $w_k\rightharpoonup w_0\in V$ and $u_k\rightharpoonup u_0\in X$ as $k\to+\infty$. Similarly, we can also get $u_0\in S(w_0)$, and so $u_0\in K_{w_0}(0)$. Since $\mathcal{J}(w,u)$ ia weakly l.s.c. function, it follows that

$$\beta(0)\le\mathcal{J}(w_0,u_0)\le\liminf_{k\to\infty}\mathcal{J}(w_k,u_k)\le\beta(0)-\frac{1}{2}\varepsilon_0,$$

which is impossible. Therefore, β is l.s.c. at the origin. The proof is complete. □

5. Corollaries

In this section, we consider a special case in which $f(u,v)=\varphi(v)-\varphi(u)$ for all $u,v\in X$.

Find $w^*\in U$ and $u\in S(w^*)$ such that

$$\begin{cases}\min\mathcal{J}(u,w)\\ \text{subject to }(w,u)\in U\times S'(w),\end{cases}\tag{5.1}$$

where $S'(w)$ is the set of solutions to the following quasi-hemivariational inequality corresponding to w: find $u\in K(w)$ such that

$$\langle Au+Bw,v-u\rangle_X+\varphi(v)-\varphi(u)+J^\circ(\gamma u;\gamma v-\gamma u)\ge 0,\qquad\forall v\in K(w),$$

where $\varphi:X\to\mathbb{R}\cup\{+\infty\}$ is a convex and l.s.c. function such that $K_\varphi=K(U)\cap\operatorname{dom}\varphi\neq\varnothing$.

For any $w\in U$, we define

$$g'_w(u)=\sup_{v\in K(w)}\langle Au+Bw,v-u\rangle_X+\varphi(v)-\varphi(u)+J^\circ(\gamma u;\gamma v-\gamma u),$$

$$K'_w(y)=\{u\in K(w):g'_w(u)\le y\}.$$

Thus, $g'_w(u) \geq 0$ and $K'_w(0) = \{u \in K(w) : g'_w(u) = 0\} = S'(w)$. Define the perturbation function β' as

$$\beta'(y) = \inf_{w \in U, u \in K'_w(y)} \mathcal{J}(w,u).$$

It is easy to see that $\beta'(0)$ is the optimal value of problem (5.1).

Definition 6. Let $P : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be a function. A nonlinear Lagrangian function $L_p : U \times K(U) \times \mathbb{R}_+ \to \mathbb{R}$ for problem (5.1) is defined as

$$L'_P(w,u,d) = P(\mathcal{J}(w,u), dg'_w(u)).$$

The function

$$F'_P(d) = \inf_{w \in U, u \in K(w)} L'_P(w,u,d)$$

is called its nonlinear Lagrangian dual function. The equality

$$\beta'(0) = \sup_{d \in \mathbb{R}_+} F'_P(d) \tag{5.2}$$

is called the zero duality gap property.

Consider the following hypothesis on functional φ.
(H_φ) : $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is a convex and l.s.c. function such that $K_\varphi \neq \varnothing$.
(H_1) :

$$\lim_{u \in X, \|u\|_X \to \infty} \frac{\langle Au, u\rangle_X + \inf_{\eta_u \in \partial J(\gamma u)} \langle \eta_u, \gamma u\rangle_E + \varphi(u) - \varphi(0)}{\|u\|_X} = +\infty.$$

We have the following results.

Theorem 4. *Assume that $(H_K), (H_A), (H_J), (H_\gamma), (H_\varphi), (H_1)$ are satisfied. Then $S'(w) \neq \varnothing$.*

Theorem 5. *Assume that $(H_K), (H_A), (H_J), (H_\gamma), (H_\varphi)$ are satisfied. Suppose that, in addition, φ is continuous, $K : U \to 2^X$ is bounded, $\mathcal{J} : U \times K(U) \to \mathbb{R}$ is weakly l.s.c. function, $B : U \to X^*$ is strongly continuous from the weak topology of V to the topology of X^* and the following conditions are satisfied*
(i) for any $w \in U$, $u \in K(w)$, $lim_{\|w\| \to +\infty} \mathcal{J}(w,u) = +\infty$,
(ii) for all $w_n \in U$ with $w_n \rightharpoonup w$, $K(w_n)$ Mosco-converges to $K(w)$.
Then $-\infty < \beta'(0) < +\infty$.

Theorem 6. *Assume that all the hypotheses of Theorem 5 are satisfied. If P satisfies $(P_i), (P_{ii}), (P_{iii})$, then the zero duality gap property (5.2) holds.*

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INTEGRAL AND LIMIT REPRESENTATIONS OF THE CMP INVERSE

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Abstract. We develop various integral and limit representations for the CMP inverse of a complex square matrix, which do not require any restriction on the spectrum of a corresponding matrix. Also, we present integral and limit representations for the DMP and MPD inverses.

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1. Introduction

Let $\mathbb{C}^{m\times n}$ be the set of all $m\times n$ complex matrices. We use $rank(A)$, A^*, $R(A)$ and $N(A)$ to denote the rank, the conjugate transpose, the range (column space) and the null space of $A\in\mathbb{C}^{m\times n}$, respectively. The index of $A\in\mathbb{C}^{n\times n}$, denoted by $\mathrm{ind}(A)$, is the smallest nonnegative integer k for which $rank(A^k)=rank(A^{k+1})$. By I will be denoted the identity matrix of corresponding size. If M and N are two complementary subspaces of $\mathbb{C}^{m\times 1}$ (that is, $\mathbb{C}^{m\times 1}$ is direct sum of M and N), we denote by $P_{M,N}$ the projector onto M along N. In the case that N is the subspace orthogonal to M, this notation will be reduced to P_M.

The Drazin inverse of $A\in\mathbb{C}^{n\times n}$ is the unique matrix $A^D=X\in\mathbb{C}^{n\times n}$ such that

$$A^{k+1}X=A^k,\quad XAX=X,\quad AX=XA.$$

where $k=\mathrm{ind}(A)$. If $\mathrm{ind}(A)=1$, then A^D is the group inverse of A, which is denoted by $A^{\#}$. For basic properties of the Drazin inverse and its various applications see [1,3].

The Moore–Penrose inverse of $A\in\mathbb{C}^{m\times n}$ is the unique matrix $A^\dagger=X\in\mathbb{C}^{n\times m}$ which satisfies the Penrose equations

$$AXA=A,\qquad XAX=X,\qquad (AX)^*=AX,\qquad (XA)^*=XA.$$

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The Moore-Penrose inverse is a powerful tool in computing polar decomposition, the areas of electrical networks, control theory, filtering, estimation theory and pattern recognition.

Let $A \in \mathbb{C}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{C}^n$ of dimension $s \le r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m-s$. If a matrix $X \in \mathbb{C}^{n\times m}$ satisfies

$$XAX = X, \quad R(X) = T, \quad N(X) = S,$$

then X is called the outer inverse of A with the range T and the null-space S, and the notation $X = A^{(2)}_{T,S}$ is commonly used. Drazin [6] introduced a new class of outer inverses, called the (B,C)-inverses. For $A \in \mathbb{C}^{m\times n}$ and $B,C \in \mathbb{C}^{n\times m}$, if a matrix $X \in \mathbb{C}^{n\times m}$ satisfies $XAB = B$, $CAX = C$, $R(X) \subseteq R(B)$ and $N(C) \subseteq N(X)$, then X is called the (B,C)-inverse of A. In the case when X exists, it is unique and denoted by $X = A^{||(B,C)}$ [2, 6]. By [2, Theorem 7.1], it follows that $A^{||(B,C)} = A^{(2)}_{R(B),N(C)}$.

Using the Drazin inverse and the Moore–Penrose inverse, Malik and Thome [9] defined a new generalized inverse of a square matrix of an arbitrary index, which is called the DMP inverse and defined as $A^{D,\dagger} = A^D AA^\dagger$, for $A \in \mathbb{C}^{n\times n}$. The DMP inverse for a Hilbert space operator was investigated in [13, 17, 19] as a generalization of the DMP inverse for a square matrix. For $A \in \mathbb{C}^{n\times n}$, the MPD inverse, as the dual DMP inverse, was given by $A^{\dagger,D} = A^\dagger AA^D$ [9].

Mehdipour and Salemi [10] introduced a new inverse of a square matrix A named CMP inverse, since they used the core part AA^DA of A and the Moore–Penrose inverse of A. The CMP inverse of $A \in \mathbb{C}^{n\times n}$ is defined as $A^{c,\dagger} = A^\dagger AA^DAA^\dagger$ and it is the unique solution of the following equations:

$$XAX = X, \qquad AXA = AA^DA, \qquad AX = AA^DAA^\dagger, \qquad XA = A^\dagger AA^DA.$$

For more details about the CMP inverse see [12, 18].

It is well-known that if the eigenvalues of $A \in \mathbb{C}^{n\times n}$ lie in the open right halfplane, then the inverse of A can be presented by

$$A^{-1} = \int_0^\infty \exp(-tA)dt.$$

Many integral representations of various generalized inverse such as Moore-Penrose inverse, Drazin inverse and DMP inverse were presented in papers [4, 5, 7, 20]. Several of these integral representations have some restriction on the eigenvalues of A and the other holds without any restrictions on the eigenvalues.

Notice that investigation of the limit representations of different kinds of generalized inverses are hot topics many years. One limit representation of the Drazin inverse was proved by Meyer [11] in 1974. Some limit representations of the outer inverse are given in [8, 15, 16].

The above mentioned results motivate us to investigate the integral and limit representations of the CMP inverse of a square matrix, without any restriction on the spectrum of a certain matrix. Firstly, we develop these representations based on

the full-rank decomposition of a given matrix. Then we establish integral and limit representations of the CMP inverse which depend on corresponding projections and expressions for the Moore-Penrose, Drazin and outer inverses. Various integral and limit representations of the DMP and MPD inverses are also derived.

2. Integral representations of the CMP inverse

In this section, we will establish integral representations of the CMP inverse for a square complex matrix without any restriction on the spectrum of matrix. If $A \in \mathbb{C}^{n\times n}$ is nilpotent, then $A^D = 0$ and so $A^{c,\dagger} = 0$. Since this case is trivial, we consider the matrix A to be non-nilpotent in this paper.

Lemma 1 ([1]). *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. If $A = B_1G_1$ is a full-rank decomposition and $G_iB_i = B_{i+1}G_{i+1}$ are also full-rank decompositions, $i = 1,2,\dots,k-1$. Then the following statements hold:*

(i) *G_kB_k is invertible;*
(ii) *$A^k = B_1B_2\dots B_kG_k\dots G_2G_1$;*
(iii) *$A^D = B_1B_2\dots B_k(G_kB_k)^{-k-1}G_k\dots G_2G_1$;*
(iv) *$A^\dagger = G_1^*(G_1G_1^*)^{-1}(B_1^*B_1)^{-1}B_1^*$.*

In particular, for $k = 1$, then G_1B_1 is invertible and $A^\# = B_1(G_1B_1)^{-2}G_1$.

Lemma 2. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and the full-rank decomposition of A as in Lemma 1. Then*

$$G_1A^{c,\dagger}B_1 = B_2\dots B_k(G_kB_k)^{-(k-1)}G_k\dots G_2.$$

Proof. By [20, Lemma 3.1], we have that $A^{D,\dagger}B_1 = B_1B_2\dots B_k(G_kB_k)^{-k}G_k\dots G_2$ which implies

$$A^{c,\dagger}B_1 = A^\dagger AA^{D,\dagger}B_1 = A^\dagger AB_1\dots B_k(G_kB_k)^{-k}G_k\dots G_2.$$

Therefore, by Lemma 1,

$$\begin{aligned}
G_1A^{c,\dagger}B_1 &= G_1A^\dagger AB_1\dots B_k(G_kB_k)^{-k}G_k\dots G_2\\
&= G_1G_1^*(G_1G_1^*)^{-1}(B_1^*B_1)^{-1}B_1^*B_1G_1B_1\dots B_k(G_kB_k)^{-k}G_k\dots G_2\\
&= G_1B_1\dots B_k(G_kB_k)^{-k}G_k\dots G_2 = B_2G_2B_2\dots B_k(G_kB_k)^{-k}G_k\dots G_2\\
&= B_2\dots G_kB_k(G_kB_k)^{-k}G_k\dots G_2 = B_2\dots B_k(G_kB_k)^{-(k-1)}G_k\dots G_2.
\end{aligned}$$

□

Theorem 1. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and the full-rank decomposition of A as in Lemma 1. Then*

$$A^{c,\dagger} = \int_0^\infty G_1^*\exp(-G_1G_1^*t)dt\int_0^\infty MB_1^*\exp(-B_1B_1^*u)du,$$

where $M = B_2\dots B_k(G_kB_k)^{-(k-1)}G_k\dots G_2$.

Proof. Set $X = G_1^{\dagger} M B_1^{\dagger}$. Recall that, by [7],

$$A^{\dagger} = \int_0^{\infty} A^* \exp(-AA^* t)dt. \tag{2.1}$$

It is enough to prove that $X = A^{c,\dagger}$. Because B_1 is a full-column rank matrix, then $B_1^{\dagger} = (B_1^* B_1)^{-1} B_1^*$ and so $B_1^{\dagger} B_1 = I$. Similarly, we have that $G_1^{\dagger} = G_1^*(G_1 G_1^*)^{-1}$ and $G_1 G_1^{\dagger} = I$. Notice that, using

$$G_k \dots G_2 B_2 \dots B_k = G_k \dots G_3 B_3 G_3 B_3 \dots B_k = \dots = (G_k B_k)^{k-1},$$

we get

$$\begin{aligned}
XAX &= G_1^{\dagger} B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 B_1^{\dagger} B_1 G_1 G_1^{\dagger} M B_1^{\dagger} \\
&= G_1^{\dagger} B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 M B_1^{\dagger} \\
&= G_1^{\dagger} B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 B_1^{\dagger} \\
&= G_1^{\dagger} B_2 \dots B_k (G_k B_k)^{-(k-1)} (G_k B_k)^{k-1} (G_k B_k)^{-(k-1)} G_k \dots G_2 B_1^{\dagger} \\
&= G_1^{\dagger} B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 B_1^{\dagger} \\
&= X.
\end{aligned}$$

Applying Lemma 1, we observe that

$$\begin{aligned}
AA^D A &= B_1 G_1 B_1 B_2 \dots B_k (G_k B_k)^{-k-1} G_k \dots G_2 G_1 B_1 G_1 \\
&= B_1 B_2 G_2 B_2 \dots B_k (G_k B_k)^{-k-1} G_k \dots G_2 B_2 G_2 G_1 \\
&= B_1 B_2 \dots B_k G_k B_k (G_k B_k)^{-k-1} G_k B_k G_k \dots G_2 G_1 \\
&= B_1 B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 G_1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
XA &= G_1^{\dagger} B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 G_1 \\
&= G_1^*(G_1 G_1^*)^{-1} (B_1^* B_1)^{-1} B_1^* B_1 B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 G_1 \\
&= A^{\dagger} A A^D A
\end{aligned}$$

and

$$\begin{aligned}
AX &= B_1 B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 B_1^{\dagger} \\
&= B_1 B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 G_1 G_1^*(G_1 G_1^*)^{-1} (B_1^* B_1)^{-1} B_1^* \\
&= A A^D A A^{\dagger}.
\end{aligned}$$

By [12, Corollary 2.2], we deduce that $X = A^{c,\dagger}$. □

Notice that we represent the CMP inverse by two integrals in Theorem 1. In order to simplify integral representation of the CMP inverse, we firstly use the DMP inverse, MPD inverse and orthogonal projections.

Theorem 2. *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$ *and the full-rank decomposition of A as in Lemma 1. Then*

$$A^{c,\dagger} = \int_0^\infty P_{R(A^*)} M_1 B_1^* \exp(-B_1 B_1^* u)\,du = \int_0^\infty G_1^* \exp(-G_1 G_1^* t) M_2 P_{R(A)}\,dt,$$

where $M_1 = B_1 B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2$ and $M_2 = B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 G_1$.

Proof. Based on $A^{c,\dagger} = P_{R(A^*)} A^{D,\dagger} = A^{\dagger,D} P_{R(A)}$ and [20, Theorem 3.2], we obtain this result. □

Applying an integral representation for the Drazin inverse showed in [4], which does not require any restriction on its eigenvalues, we give the following integral representations for the CMP inverse.

Theorem 3. *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$. *Then*

$$A^{c,\dagger} = \int_0^\infty P_{R(A^*)} \exp\left[-tA^k (A^{2k+1})^* A^{k+1}\right] A^k (A^{2k+1})^* A^k P_{R(A)}\,dt.$$

Proof. It follows by the equality $A^{c,\dagger} = P_{R(A^*)} A^D P_{R(A)}$ and the next integral representation for the Drazin inverse proved in [4, Theorem 2.1]:

$$A^D = \int_0^\infty \exp\left[-tA^k (A^{2k+1})^* A^{k+1}\right] A^k (A^{2k+1})^* A^k\,dt.$$

□

As Theorem 3, new integral representations for the DMP and MPD inverses are obtained.

Corollary 1. *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$. *Then*

$$A^{D,\dagger} = \int_0^\infty P_{R(A^*)} \exp\left[-tA^k (A^{2k+1})^* A^{k+1}\right] A^k (A^{2k+1})^* A^k\,dt$$

and

$$A^{\dagger,D} = \int_0^\infty \exp\left[-tA^k (A^{2k+1})^* A^{k+1}\right] A^k (A^{2k+1})^* A^k P_{R(A)}\,dt.$$

We present more expressions for the CMP inverse involving one integral.

Theorem 4. *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$. *Then*

$$A^{c,\dagger} = \int_0^\infty A^* \exp(-AA^* t) P_{R(A^k),N(A^k)} P_{R(A)}\,dt = \int_0^\infty P_{R(A^*)} P_{R(A^k),N(A^k)} A^* \exp(-AA^* t)\,dt.$$

Proof. The equalities $A^{c,\dagger} = A^\dagger P_{R(A^k),N(A^k)} P_{R(A)} = P_{R(A^*)} P_{R(A^k),N(A^k)} A^\dagger$ and (2.1) yield these formulae. □

Similarly as Theorem 4, we show some formulae for the DMP inverse and MPD inverse.

Corollary 2. *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$. *Then*

$$A^{D,\dagger} = \int_0^\infty A^* \exp(-AA^*t) P_{R(A^k),N(A^k)} dt$$

and

$$A^{\dagger,D} = \int_0^\infty P_{R(A^*)} P_{R(A^k),N(A^k)} A^* \exp(-AA^*t) dt.$$

Theorem 5. *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$. *If $G \in \mathbb{C}^{n\times n}$ such that $R(G) = R(A^\dagger A^k)$ and $N(G) = N(A^k A^\dagger)$, then*

$$A^{c,\dagger} = \int_0^\infty \exp\left[-G(GAG)^* GAt\right] G(GAG)^* G dt.$$

Proof. Using [14, Corollary 3.7], we have $A^{c,\dagger} = A^{(2)}_{R(A^\dagger A^D),N(A^D A^\dagger)} = A^{(2)}_{R(A^\dagger A^k),N(A^k A^\dagger)}$. By [17, Theorem 2.2] (or [2, Corollary 7.6]), we obtain

$$A^{(2)}_{R(A^\dagger A^k),N(A^k A^\dagger)} = \int_0^\infty \exp\left[-G(GAG)^* GAt\right] G(GAG)^* G dt.$$

□

Using the integral representation for the (B,C)-inverse proved in [2], we obtain the next integral representation for the CMP inverse based on some restriction on the eigenvalues of corresponding matrix.

Theorem 6. *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$ *and let $G \in \mathbb{C}^{n\times n}$ such that $R(G) = R(A^\dagger A^k)$ and $N(G) = N(A^k A^\dagger)$. If the nonzero spectrum of GA lies in the open left half plane, then*

$$A^{c,\dagger} = -\int_0^\infty \exp(GAt) G dt.$$

Proof. It follows by $A^{c,\dagger} = A^{(2)}_{R(A^\dagger A^k),N(A^k A^\dagger)} = A^{||(A^\dagger A^k, A^k A^\dagger)}$ and [2, Corollary 7.7]. □

3. Limit representations of the CMP inverse

In the beginning of this section, we present the limit representation of the CMP inverse based on the full-rank decomposition of A given in Lemma 1.

Theorem 7. *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$ *and the full-rank decomposition of A as in Lemma 1. Then*

$$A^{c,\dagger} = \lim_{\lambda\to 0} G_1^*(\lambda I + G_1 G_1^*)^{-1} \lim_{t\to 0} M(tI + B_1^* B_1)^{-1} B_1^*,$$

where $M = B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2$.

Proof. We have, by [15],

$$A^{\dagger} = \lim_{\lambda\to 0} A^*(\lambda I + AA^*)^{-1} = \lim_{\lambda\to 0}(\lambda I + A^*A)^{-1}A^*.$$

For $X = G_1^{\dagger} M B_1^{\dagger}$, we check that $X = A^{c,\dagger}$ as in the proof of Theorem 1. □

To avoid two limits, we included orthogonal projections in limit representations of CMP inverse. Similarly as Theorem 7 and Theorem 2, we verify the following result.

Theorem 8. *Let $A \in \mathbb{C}^{n\times n}$ with* $\mathrm{ind}(A) = k$ *and the full-rank decomposition of A as in Lemma 1. Then*

$$A^{c,\dagger} = \lim_{\lambda\to 0} P_{R(A^*)} M_1 B_1^*(\lambda I + B_1 B_1^*)^{-1} = \lim_{\lambda\to 0} G_1^*(\lambda I + G_1 G_1^*)^{-1} M_2 P_{R(A)} dt,$$

where $M_1 = B_1 B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2$ and $M_2 = B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 G_1$.

Analogously, we can prove the limit representations of DMP and MPD inverses.

Corollary 3. *Let $A \in \mathbb{C}^{n\times n}$ with* $\mathrm{ind}(A) = k$ *and the full-rank decomposition of A as in Lemma 1. Then*

$$A^{D,\dagger} = \lim_{\lambda\to 0} M_1 B_1^*(\lambda I + B_1 B_1^*)^{-1}$$

and

$$A^{\dagger,D} = \lim_{\lambda\to 0} G_1^*(\lambda I + G_1 G_1^*)^{-1} M_2 dt,$$

where $M_1 = B_1 B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2$ and $M_2 = B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 G_1$.

By the limit representation for the Drazin inverse proved in [11], we get the next limit representation for the CMP inverse.

Theorem 9. *Let $A \in \mathbb{C}^{n\times n}$ with* $\mathrm{ind}(A) = k$*. If $k \le l$, then*

$$A^{c,\dagger} = \lim_{\lambda\to 0} P_{R(A^*)} A^l (A^{l+1} + \lambda I)^{-1} P_{R(A)}.$$

Proof. This expressions can be verified using the following limit representation for the Drazin inverse presented in [11]:

$$A^D = \lim_{\lambda\to 0} A^l (A^{l+1} + \lambda I)^{-1}.$$

□

Also, the corresponding limit representations of DMP and MPD inverses can be showed.

Corollary 4. *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$. *If $k \le l$, then*

$$A^{D,\dagger} = \lim_{\lambda\to 0} P_{R(A^*)} A^l (A^{l+1} + \lambda I)^{-1}$$

and

$$A^{\dagger,D} = \lim_{\lambda\to 0} A^l (A^{l+1} + \lambda I)^{-1} P_{R(A)}.$$

As Theorem 4 and Theorem 5, we obtain some limit representations of CMP inverse which involve one limit.

Theorem 10. *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$. *Then*

$$A^{c,\dagger} = \lim_{\lambda\to 0} A^*(\lambda I + AA^*)^{-1} P_{R(A^k),N(A^k)} P_{R(A)} = \lim_{\lambda\to 0} P_{R(A^*)} P_{R(A^k),N(A^k)} A^*(\lambda I + AA^*)^{-1}.$$

For DMP and MPD inverses, the following limit representations hold.

Corollary 5. *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$. *Then*

$$A^{D,\dagger} = \lim_{\lambda\to 0} A^*(\lambda I + AA^*)^{-1} P_{R(A^k),N(A^k)}$$

and

$$A^{\dagger,D} = \lim_{\lambda\to 0} P_{R(A^k),N(A^k)} A^*(\lambda I + AA^*)^{-1}.$$

We need one auxiliary result to prove new expressions for the CMP, DMP and MPD inverses.

Lemma 3 ([16])**.** *Let $A \in \mathbb{C}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{C}^n$ of dimension $s \le r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. In addition, suppose that $G \in \mathbb{C}^{n\times m}$ satisfies $R(G) = T$ and $N(G) = S$. If $A^{(2)}_{T,S}$ exists, then it possesses the limit representations*

$$A^{(2)}_{T,S} = \lim_{\lambda\to 0} (GA + \lambda I)^{-1} G = \lim_{\lambda\to 0} G (AG + \lambda I)^{-1}. \tag{3.1}$$

Theorem 11. *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$. *If $G \in \mathbb{C}^{n\times n}$ such that $R(G) = R(A^\dagger A^k)$ and $N(G) = N(A^k A^\dagger)$, then*

$$A^{c,\dagger} = \lim_{\lambda\to 0} (GA + \lambda I)^{-1} G = \lim_{\lambda\to 0} G (AG + \lambda I)^{-1}.$$

Proof. By Lemma 3 (or [2, Corollary 7.5]), we have

$$A^{c,\dagger} = A^{(2)}_{R(A^\dagger A^k),N(A^k A^\dagger)} = \lim_{\lambda\to 0} (GA + \lambda I)^{-1} G = \lim_{\lambda\to 0} G (AG + \lambda I)^{-1}.$$

□

Theorem 12. *Let $A \in \mathbb{C}^{n\times n}$ be of rank r and* $\operatorname{ind}(A) = k$, *$B \in \mathbb{C}^{n\times s}_s$ and $C \in \mathbb{C}^{s\times n}_s$.*

(i) *Suppose that $R(B) = R(A^\dagger A^k)$ is a subspace of $\mathbb{C}^n$ of dimension $s \le r$ and $N(C) = N(A^k A^\dagger)$ is a subspace of $\mathbb{C}^n$ of dimension $n - s$. Then*

$$A^{c,\dagger} = \lim_{t\to 0} B(tI + CAB)^{-1} C.$$

(ii) *Suppose that* $R(B_1) = R(A^k)$ *is a subspace of* $\mathbb{C}^n$ *of dimension* $s \leq r$ *and* $N(C_1) = N(A^k A^\dagger)$ *is a subspace of* $\mathbb{C}^n$ *of dimension* $n-s$. *If* $(C_1)^{(2)}_{R(AB_1),N(AB_1)}$ *exists, then*

$$A^{D,\dagger} = \lim_{t\to 0} B_1(tI + C_1AB_1)^{-1}C_1$$

and

$$A^{c,\dagger} = A^\dagger (C_1)^{(2)}_{R(AB_1),N(AB_1)} C_1.$$

(iii) *Suppose that* $R(B_2) = R(A^\dagger A^k)$ *is a subspace of* $\mathbb{C}^n$ *of dimension* $s \leq r$ *and* $N(C_2) = N(A^k)$ *is a subspace of* $\mathbb{C}^n$ *of dimension* $n-s$. *If* $(B_2)^{(2)}_{R(C_2A),N(C_2A)}$ *exists, then*

$$A^{\dagger,D} = \lim_{t\to 0} B_2(tI + C_2AB_2)^{-1}C_2$$

and

$$A^{c,\dagger} = B_2 (B_2)^{(2)}_{R(C_2A),N(C_2A)} A^\dagger.$$

Proof. (i) Applying [8, Theorem 7], we have that

$$A^{(2)}_{R(A^\dagger A^k),N(A^k A^\dagger)} = \lim_{t\to 0} B(tI + CAB)^{-1}C.$$

(ii) We firstly observe that $A^{D,\dagger} = A^{(2)}_{R(A^k),N(A^k A^\dagger)}$ and then, by [8, Theorem 7],

$$A^{D,\dagger} = \lim_{t\to 0} B_1(tI + C_1AB_1)^{-1}C_1.$$

Therefore, by Lemma 3,

$$\begin{aligned} A^{c,\dagger} &= A^\dagger A A^{D,\dagger} = A^\dagger \lim_{t\to 0} AB_1(tI + C_1AB_1)^{-1}C_1 \\ &= A^\dagger (C_1)^{(2)}_{R(AB_1),N(AB_1)} C_1. \end{aligned}$$

(iii) This part can be proved in an analogy way as part (ii). □

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COFINITELY RADICAL SUPPLEMENTED AND COFINITELY WEAK RADICAL SUPPLEMENTED LATTICES

CELIL NEBIYEV AND HASAN HÜSEYIN ÖKTEN

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Abstract. In this work, cofinitely radical supplemented and cofinitely weak radical supplemented lattices are defined and some properties of them are investigated. Let L be a lattice, I be a nonempty index set and $a_i \in L$ for every $i \in I$. If $1 = \bigvee_{i\in I} a_i$ and $a_i/0$ is cofinitely (weak) radical supplemented for every $i \in I$, then L is also cofinitely (weak) radical supplemented. Let L be a cofinitely (weak) radical supplemented lattice and $a \in L$. Then $1/a$ is also cofinitely (weak) radical supplemented. Let L be a lattice. Then L is cofinitely weak radical supplemented if and only if every cofinite element of $1/r(L)$ is a direct summand of $1/r(L)$.

2010 *Mathematics Subject Classification:* 06C05; 06C15

Keywords: lattices, small elements, supplemented lattices, generalized (radical) supplemented lattices

1. Introduction

Throughout this paper, all lattices are complete modular lattices with the smallest element 0 and the greatest element 1. Let L be a lattice, $a, b \in L$ and $a \leq b$. A sublattice $\{x \in L | a \leq x \leq b\}$ is called a *quotient sublattice*, denoted by b/a. An element a' of a lattice L is called a *complement* of a if $a \wedge a' = 0$ and $a \vee a' = 1$ (in this case a and a' are said to be *direct summands* of L and denoted by $1 = a \oplus a'$). A lattice L is said to be *complemented* if each element of L has at least one complement in L. An element c of L is said to be *compact* if for every subset X of L such that $c \leq \vee X$ there exists a finite $F \subset X$ such that $c \leq \vee F$. A lattice L is said to be *compactly generated* if each of its elements is a join of compact elements. A lattice L is said to be *compact* if 1 is a compact element of L. An element a of a lattice L is said to be *cofinite* if $1/a$ is compact. An element a of L is said to be *small* or *superfluous* if $a \vee b \neq 1$ holds for every $b \neq 1$ and denoted by $a \ll L$. The meet of all the maximal $(\neq 1)$ elements of a lattice L is called the *radical* of L and denoted by $r(L)$. An element c of L is called a *supplement* of b in L if it is minimal for $b \vee c = 1$. a is a supplement of b in a lattice L if and only if $a \vee b = 1$ and $a \wedge b \ll a/0$. L is called a *supplemented lattice* if every element of L has a supplement in L. We say that an element b of L *lies above* an element a of L if $a \leq b$ and $b \ll 1/a$. L is said to be *hollow* if every element $(\neq 1)$

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is superfluous in L and L is said to be *local* if L has the greatest element $(\neq 1)$. An element a of L is called a *weak supplement* of b in L if $a \vee b = 1$ and $a \wedge b \ll L$. L is called a *weakly supplemented lattice*, if every element of L has a weak supplement in L. It is clear that every supplemented lattice is weakly supplemented. An element a of L is called a *generalized* (*radical*) *supplement* (or briefly, *Rad-supplement*) of b in L if $a \vee b = 1$ and $a \wedge b \leq r(a/0)$. L is said o be *radical* (*generalized*) *supplemented* if every element of L has a Rad-supplement in L.

More information about supplemented lattices are in [1, 2] and [5]. More results about supplemented modules are in [9] and [10]. The definitions of generalized supplemented modules and some properties of them are in [8]. More information about cofinitely Rad-supplemented modules are in [7]. We generalize cofinitely Rad-supplemented modules to lattices.

2. Cofinitely Radical Supplemented Lattices

In this part, cofinitely radical supplemented lattices are defined and some properties of them are given.

Definition 1. Let L be a lattice. If every cofinite element of L has a Rad-supplement in L, then L is called a cofinitely radical supplemented (or cofinitely Rad-supplemented) lattice.

Clearly we can see that every cofinitely supplemented lattice is cofinitely Rad-supplemented. Hollow and local lattices are cofinitely Rad-supplemented.

Proposition 1. *Let L be a compact lattice. Then L is cofinitely Rad-supplemented if and only if L is Rad-supplemented.*

Proof. Clear, since every element of L is cofinite. □

Lemma 1. *Let L be a lattice, $a \in L$ and x be a cofinite element of L. If $x \vee a$ has a Rad-supplement in L and $a/0$ cofinitely Rad-supplemented, then x has a Rad-supplement in L.*

Proof. Let b be a Rad-supplement of $x \vee a$ in L. Then $x \vee a \vee b = 1$ and $(x \vee a) \wedge b \leq r(b/0)$. Since x is a cofinite element of L, we clearly see that $x \vee b$ is a cofinite element of L. Then by $\frac{1}{x \vee b} = \frac{x \vee a \vee b}{x \vee b} \cong \frac{a}{a \wedge (x \vee b)}$, $a \wedge (x \vee b)$ is a cofinite element of $a/0$. Since $a/0$ is cofinitely Rad-supplemented, $a \wedge (x \vee b)$ has a Rad-supplement c in $a/0$. Here $(a \wedge (x \vee b)) \vee c = a$ and $c \wedge (x \vee b) = c \wedge a \wedge (x \vee b) \leq r(c/0)$. Hence $1 = x \vee a \vee b = x \vee b \vee (a \wedge (x \vee b)) \vee c = x \vee b \vee c$ and $x \wedge (b \vee c) \leq (b \wedge (x \vee c)) \vee (c \wedge (x \vee b)) \leq (b \wedge (x \vee a)) \vee r(c/0) \leq r(b/0) \vee r(c/0) \leq r((b \vee c)/0)$. Thus $b \vee c$ is a Rad-supplement of x in L. □

Corollary 1. *Let L be a lattice, $a_1, a_2, \ldots, a_n \in L$ and x be a cofinite element of L. If $x \vee a_1 \vee a_2 \vee \cdots \vee a_n$ has a Rad-supplement in L and $a_i/0$ is cofinitely Rad-supplemented for every $i = 1, 2, \ldots, n$, then x has a Rad-supplement in L.*

Proof. Clear from Lemma 1. □

Lemma 2. *Let L be a lattice, I be a nonempty index set and $a_i \in L$ for every $i \in I$. If $1 = \bigvee_{i\in I} a_i$ and $a_i/0$ is cofinitely Rad-supplemented for every $i \in I$, then L is also cofinitely Rad-supplemented.*

Proof. Let a be any cofinite element of L. By hypothesis, $1 = \bigvee_{i\in I} a_i = \bigvee_{i\in I}(a \vee a_i)$. Since $1/a$ is compact and $a \vee a_i \in 1/a$ for every $i \in I$, there exist $i_1, i_2, ..., i_n \in I$ such that $1 = a \vee a_{i_1} \vee a_{i_2} \vee ... \vee a_{i_n}$. Since 0 is a Rad-supplement of $1 = a \vee a_{i_1} \vee a_{i_2} \vee ... \vee a_{i_n}$ and $a_{i_t}/0$ is cofinitely Rad-supplemented for every $t = 1, 2, ..., n$, by Corollary 1, a has a Rad-supplement in L. Hence L is cofinitely Rad-supplemented. □

Corollary 2. *Let L be a lattice and $1 = a_1 \vee a_2 \vee ... \vee a_n$ in L. If $a_i/0$ is cofinitely Rad-supplemented for every $i = 1, 2, ..., n$, then L is also cofinitely Rad-supplemented.*

Proof. Clear from Lemma 2. □

Proposition 2. *Let L be a cofinitely Rad-supplemented lattice and $a \in L$. Then $1/a$ is also cofinitely Rad-supplemented.*

Proof. Let x be any cofinite element of $1/a$. Then $1/x$ is compact and x is a cofinite element of L. Since L is cofinitely Rad-supplemented, x has a Rad-supplement y in L. Since $a \leq x$, by [3, Lemma 5], $a \vee y$ is a Rad-supplement of x in $1/a$. Hence $1/a$ is cofinitely Rad-supplemented. □

Proposition 3. *Let L be a cofinitely Rad-supplemented lattice. Then every cofinite element of $1/r(L)$ is a direct summand of $1/r(L)$.*

Proof. Let x be any cofinite element of $1/r(L)$. Then $1/x$ is compact and x is a cofinite element of L. Since L is cofinitely Rad-supplemented, x has a Rad-supplement y in L. Here $1 = x \vee y$ and $x \wedge y \leq r(y/0) \leq r(L)$. Then $1 = x \vee y \vee r(L)$ and since $r(L) \leq x$, $x \wedge (y \vee r(L)) = (x \wedge y) \vee r(L) = r(L)$. Hence $1 = x \oplus (y \vee r(L))$ in $1/r(L)$ and x is a direct summand of $1/r(L)$. □

3. Cofinitely Weak Radical Supplemented Lattices

In this part, cofinitely weak radical supplemented lattices are defined and some properties of them are given.

Definition 2. Let L be a lattice and $a, b \in L$. If $a \vee b = 1$ and $a \wedge b \leq r(L)$, then b is called a weak radical supplement (or briefly, weak Rad-supplement) of a in L.

Definition 3. Let L be a lattice. If every element of L has a weak Rad-supplement in L, then L is called a weakly radical supplemented (or weakly Rad-supplemented) lattice. If every cofinite element of L has a weak Rad-supplement in L, then L is called a cofinitely weak radical supplemented (or cofinitely weak Rad-supplemented) lattice.

It is clear that every cofinitely weak supplemented lattice is cofinitely weak Rad-supplemented. It is also clear that every cofinitely Rad-supplemented lattice is cofinitely weak Rad-supplemented.

Proposition 4. *Let L be a cofinitely weak Rad-supplemented lattice. If $r(L) \ll L$, then L is cofinitely weak supplemented.*

Proof. Clear from definitions. □

Proposition 5. *Let L be a compact lattice. Then L is cofinitely weak Rad-supplemented if and only if L is weakly Rad-supplemented.*

Proof. Clear, since every element of L is cofinite. □

Lemma 3. *Let L be a lattice, $a \in L$ and x be a cofinite element of L. If $x \vee a$ has a weak Rad-supplement in L and $a/0$ is cofinitely weak Rad-supplemented, then x has a weak Rad-supplement in L.*

Proof. Let b be a weak Rad-supplement of $x \vee a$ in L. Then $x \vee a \vee b = 1$ and $(x \vee a) \wedge b \leq r(L)$. Since x is a cofinite element of L, we clearly see that $x \vee b$ is a cofinite element of L. Then by $\frac{1}{x \vee b} = \frac{x \vee a \vee b}{x \vee b} \cong \frac{a}{a \wedge (x \vee b)}$, $a \wedge (x \vee b)$ is a cofinite element of $a/0$. Since $a/0$ is cofinitely weak Rad-supplemented, $a \wedge (x \vee b)$ has a weak Rad-supplement c in $a/0$. Here $(a \wedge (x \vee b)) \vee c = a$ and $c \wedge (x \vee b) = c \wedge a \wedge (x \vee b) \leq r(a/0) \leq r(L)$. Hence $1 = x \vee a \vee b = x \vee b \vee (a \wedge (x \vee b)) \vee c = x \vee b \vee c$ and $x \wedge (b \vee c) \leq (b \wedge (x \vee c)) \vee (c \wedge (x \vee b)) \leq (b \wedge (x \vee a)) \vee r(L) \leq r(L) \vee r(L) = r(L)$. Thus $b \vee c$ is a weak Rad-supplement of x in L. □

Corollary 3. *Let L be a lattice, $a_1, a_2, \ldots, a_n \in L$ and x be a cofinite element of L. If $x \vee a_1 \vee a_2 \vee \cdots \vee a_n$ has a weak Rad-supplement in L and $a_i/0$ is cofinitely weak Rad-supplemented for every $i = 1, 2, \ldots, n$, then x has a weak Rad-supplement in L.*

Proof. Clear from Lemma 3. □

Lemma 4. *Let L be a lattice, I be a nonempty index set and $a_i \in L$ for every $i \in I$. If $1 = \bigvee_{i \in I} a_i$ and $a_i/0$ is cofinitely weak Rad-supplemented for every $i \in I$, then L is also cofinitely weak Rad-supplemented.*

Proof. Let a be any cofinite element of L. By hypothesis, $1 = \bigvee_{i \in I} a_i = \bigvee_{i \in I} (a \vee a_i)$. Since $1/a$ is compact and $a \vee a_i \in 1/a$ for every $i \in I$, there exist $i_1, i_2, \ldots, i_n \in I$ such that $1 = a \vee a_{i_1} \vee a_{i_2} \vee \ldots \vee a_{i_n}$. Since 0 is a weak Rad-supplement of $1 = a \vee a_{i_1} \vee a_{i_2} \vee \ldots \vee a_{i_n}$ and $a_{i_t}/0$ is cofinitely weak Rad-supplemented for every $t = 1, 2, \ldots, n$, by Corollary 3, a has a weak Rad-supplement in L. Hence L is cofinitely weak Rad-supplemented. □

Corollary 4. *Let L be a lattice and $1 = a_1 \vee a_2 \vee \ldots \vee a_n$ in L. If $a_i/0$ is cofinitely weak Rad-supplemented for every $i = 1, 2, \ldots, n$, then L is also cofinitely weak Rad-supplemented.*

Proof. Clear from Lemma 4. □

Lemma 5. *Let L be a lattice, $a,b,x \in L$ and $x \leq a$. If b is a weak Rad-supplement of a in L, then $x \vee b$ is a weak Rad-supplement of a in $1/x$.*

Proof. Since b is a weak Rad-supplement of a in L, $a \vee b = 1$ and $a \wedge b \leq r(L)$. Then $a \vee x \vee b = 1$ and $a \wedge (x \vee b) \leq x \vee (a \wedge b) \leq x \vee r(L) \leq r(1/x)$. Hence $x \vee b$ is a weak Rad-supplement of a in $1/x$. □

Proposition 6. *Let L be a cofinitely weak Rad-supplemented lattice and $a \in L$. Then $1/a$ is also cofinitely weak Rad-supplemented.*

Proof. Let x be any cofinite element of $1/a$. Then $1/x$ is compact and x is a cofinite element of L. Since L is cofinitely weak Rad-supplemented, x has a weak Rad-supplement y in L. Since $a \leq x$, by Lemma 5, $a \vee y$ is a weak Rad-supplement of x in $1/a$. Hence $1/a$ is cofinitely weak Rad-supplemented. □

Proposition 7. *Let L be a lattice. Then L is cofinitely weak Rad-supplemented if and only if every cofinite element of $1/r(L)$ is a direct summand of $1/r(L)$.*

Proof. ($\Longrightarrow$) Let x be any cofinite element of $1/r(L)$. Then $1/x$ is compact and x is a cofinite element of L. Since L is cofinitely weak Rad-supplemented, x has a weak Rad-supplement y in L. Here $1 = x \vee y$ and $x \wedge y \leq r(L)$. Then $1 = x \vee y \vee r(L)$ and since $r(L) \leq x$, $x \wedge (y \vee r(L)) = (x \wedge y) \vee r(L) = r(L)$. Hence $1 = x \oplus (y \vee r(L))$ in $1/r(L)$ and x is a direct summand of $1/r(L)$.

($\Longleftarrow$) Let x be any cofinite element of L. Here clearly we can see that $x \vee r(L)$ is a cofinite element of $1/r(L)$. By hypothesis, $x \vee r(L)$ is a direct summand of $1/r(L)$. Then there exists $y \in 1/r(L)$ such that $1 = x \vee r(L) \vee y = x \vee y$ and $(x \vee r(L)) \wedge y = r(L)$. Since $r(L) \leq y$, by modularity, $r(L) = (x \vee r(L)) \wedge y = (x \wedge y) \vee r(L)$ and $x \wedge y \leq r(L)$. Hence y is a weak Rad-supplement of x in L. Therefore, L is cofinitely weak Rad-supplemented. □

Proposition 8. *Let L be a lattice and $a \ll L$. If $1/a$ is cofinitely weak Rad-supplemented, then L is also cofinitely weak Rad-supplemented.*

Proof. Let x be any cofinite element of L. Clearly we see that $x \vee a$ is a cofinite element of $1/a$. Since $1/a$ is cofinitely weak Rad-supplemented, $x \vee a$ has a weak Rad-supplement y in $1/a$. Here $x \vee a \vee y = 1$ and $(x \vee a) \wedge y \leq r(1/a)$. Since $x \vee a \vee y = 1$ and $a \leq y$, $x \vee y = 1$. Since $a \ll L$, clearly we see that $r(1/a) = r(L)$. Hence $x \vee y = 1$ and $x \wedge y \leq (x \vee a) \wedge y \leq r(1/a) = r(L)$. Thus L is cofinitely weak Rad-supplemented. □

Let $x, y \in L$. It is defined a relation β_* on the elements of L by $x\beta_* y$ if and only if for every $t \in L$ with $x \vee t = 1$ then $y \vee t = 1$ and for every $k \in L$ with $y \vee k = 1$ then $x \vee k = 1$. The definition of β_* relation and some properties of this relation are in [6]. The definition of β^* relation on modules and some properties of this relation are in [4].

Lemma 6. *Let L be a lattice. If every cofinite element of L is β_* equivalent to a weak Rad-supplement element in L, then L is cofinitely weak Rad-supplemented.*

Proof. Let x be a cofinite element of L. By hypothesis, there exists a weak Rad-supplement element y in L such that $x\beta_* y$. Let y is a weak Rad-supplement of a in L. Here $y \vee a = 1$ and $y \wedge a \leq r(L)$. Since $x\beta_* y$ and $y \vee a = 1$, $x \vee a = 1$. Assume $x \wedge a \nleq r(L)$. Then there exists a maximal $(\neq 1)$ element t of L with $x \wedge a \nleq t$. Here $(x \wedge a) \vee t = 1$. By [6, Lemma 2], $x \vee (a \wedge t) = 1$ and since $x\beta_* y$, $y \vee (a \wedge t) = 1$. Since $a \vee t = 1$, by [6, Lemma 2], $(y \wedge a) \vee t = 1$. Since $y \wedge a \leq r(L) \leq t$, $t = (y \wedge a) \vee t = 1$. This contradicts with $t \neq 1$. Hence $x \wedge a \leq r(L)$. Therefore, a is a weak Rad-supplement of x in L and L is cofinitely weak Rad-supplemented. □

Corollary 5. *Let L be a lattice. If every cofinite element of L lies above a weak Rad-supplement element in L, then L is cofinitely weak Rad-supplemented.*

Proof. Clear from Lemma 6. □

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GENERALIZED FRACTIONAL HERMITE-HADAMARD INEQUALITIES

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Abstract. In this paper, we derive some Hermite-Hadamard type inequalities via s-convex functions of first and second sense respectively. These inequalities involve k-Riemann-Liouville fractional integrals. We also discuss some special cases.

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Keywords: convex, s-convex functions, Hermite-Hadamard inequality, k-Riemann-Liouville fractional integrals

1. Introduction

Theory of convexity has many applications in different fields of pure and applied sciences. Due to this many researchers have investigated it in different directions. Resultantly the concept of convexity has been extended and generalized in numerous ways using novel and innovative ideas, see [1–9, 18].

Theory of convexity also plays significant role in mathematical inequalities. One of the most famous inequality which provides necessary and sufficient condition for a function to be convex is Hermite-Hadamard's inequality. For some recent investigations on Hermite-Hadamard type inequalities interested readers are referred to [4, 6–8, 11–17, 19–24].

Recently Sarikaya et al. [16] has introduced the notion of k-Riemann-Liouville fractional integrals and discussed some of its interesting properties. It is worth to mention here that k-Riemann-Liouville fractional integral is the generalized form of classical Riemann-Liouville fractional integrals as when $k \to 1$ we get the classical Riemann-Liouville fractional.

In this paper, we consider the k-Riemann-Liouville fractional integrals and derive several new Hermite-Hadamard type inequalities via s-convex functions of first and second sense which involve k-Riemann-Liouville fractional integrals. The ideas and techniques of this paper may inspire interested readers to explore applications and

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other aspects of Hermite-Hadamard inequalities in other fields of pure and applied sciences. This is the main motivation of this paper.

2. PRELIMINARIES

In this section we recall some previously known concepts. First of all let $I = [a,b] \subseteq \mathbb{R}$ be the interval and $\mathbb{R}$ be the set of real numbers.
Let us suppose that $s \in [0,1]$.

Definition 1 ([18])**.** A function $f : I \to \mathbb{R}$ is said to be a s-convex function in the first sense, if

$$f(tx+(1-t)y) \leq t^s f(x) + (1-t^s) f(y), \quad \forall x,y \in I, t \in [0,1]. \tag{2.1}$$

Definition 2 ([1])**.** A function $f : I \to \mathbb{R}$ is said to be a s-convex function in the second sense (also called Breckner convex), if

$$f(tx+(1-t)y) \leq t^s f(x) + (1-t)^s f(y), \quad \forall x,y \in I, t \in [0,1]. \tag{2.2}$$

Now, we recall the concept of k-Riemann-Liouville fractional integrals. Let f be piecewise continuous on $I^* = (0,\infty)$ and integrable on any finite subinterval of $I = [0,\infty]$. Then for $t > 0$, we consider k-Riemann-Liouville fractional integral of f of order α

$${}_kJ_a^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)\mathrm{d}t, \quad x > a, k > 0.$$

Note that

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} \mathrm{d}t,$$

is the k-Gamma function. Clearly

$$\Gamma(x) = \lim_{k\to 1} \Gamma_k(x),$$

and

$$\Gamma_k(x+k) = x\Gamma_k(x).$$

For more details readers are referred to [16]. Note that when $k \to 1$ k-Riemann-Liouville fractional integrals become classical Riemann-Liouville fractional integral [10]. k-Beta function is defined as:

$$B_k(x,y) = \frac{1}{k} \int_0^1 t^{\frac{x}{k}-1} (1-t)^{\frac{y}{k}-1} \mathrm{d}t = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \quad x > 0, y > 0.$$

Now, we prove Hermite-Hadmard inequality via k-Riemann-Liouville fractional integrals.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be positive function with $0\le a<b$ and $f\in L_1[a,b]$. If f is convex on $[a,b]$, then, we have*

$$f\left(\frac{a+b}{2}\right)\le\frac{\Gamma_k(\alpha+1)}{2(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f(b)+{}_kJ^{\alpha}_{b^-}f(a)\right]\le\frac{f(a)+f(b)}{2}. \tag{2.3}$$

Proof. Since f is convex function, so, we have

$$f\left(\frac{x+y}{2}\right)\le\frac{f(x)+f(y)}{2}.$$

Let $x=ta+(1-t)b$ and $y=(1-t)a+tb$, we have

$$2f\left(\frac{a+b}{2}\right)\le f(ta+(1-t)b)+f((1-t)a+tb).$$

Multiplying both sides of above inequality by $t^{\frac{\alpha}{k}-1}$ and then integrating with respect to t on $[0,1]$, we have

$$2f\left(\frac{a+b}{2}\right)\int_0^1 t^{\frac{\alpha}{k}-1}\mathrm{d}t\le\int_0^1 t^{\frac{\alpha}{k}-1}f(ta+(1-t)b)\mathrm{d}t+\int_0^1 t^{\frac{\alpha}{k}-1}f((1-t)a+tb)\mathrm{d}t.$$

Now substituting $u=ta+(1-t)b$ and $v=(1-t)a+tb$, we have

$$\frac{2k}{\alpha}f\left(\frac{a+b}{2}\right)\le\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f(b)+{}_kJ^{\alpha}_{b^-}f(a)\right]. \tag{2.4}$$

Also

$$f(ta+(1-t)b)+f((1-t)a+tb)\le f(a)+f(b)$$

Multiplying both sides of above inequality $t^{\frac{\alpha}{k}-1}$ and then integrating with respect to t on $[0,1]$, we have

$$\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f(b)+{}_kJ^{\alpha}_{b^-}f(a)\right]\le\frac{k[f(a)+f(b)]}{\alpha}. \tag{2.5}$$

Combining (2.4) and (2.5) completes the proof. □

Note that when $k\to 1$ in Theorem 1, we have fractional Hermite-Hadamard inequality, see [17].

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f\in L_1[a,b]$, then, we have*

$$\begin{aligned}&\frac{b-a}{2}\int_0^1[t^{\frac{\alpha}{k}}-(1-t)^{\frac{\alpha}{k}}]f'((1-t)a+tb)\mathrm{d}t\\&=\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+1)}{2(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f(a)+{}_kJ^{\alpha}_{b^-}f(b)\right].\end{aligned}$$

Proof. Let

$$I=\int_0^1[t^{\frac{\alpha}{k}}-(1-t)^{\frac{\alpha}{k}}]f'((1-t)a+tb)\mathrm{d}t. \tag{2.6}$$

Now

$$\begin{aligned}
&\int_0^1 t^{\frac{\alpha}{k}}f'((1-t)a+tb)\mathrm{d}t\\
&=\left[\frac{t^{\frac{\alpha}{k}}f((1-t)a+tb)\mathrm{d}t}{b-a}\right]_0^1-\frac{\alpha}{k(b-a)}\int_0^1 t^{\frac{\alpha}{k}-1}f((1-t)a+tb)\mathrm{d}t\\
&=\frac{f(b)}{b-a}-\frac{k\alpha\Gamma_k(\alpha)}{k(b-a)^{\frac{\alpha}{k}+1}k\Gamma_k(\alpha)}\int_a^b(u-a)^{\frac{\alpha}{k}-1}f(u)\mathrm{d}u\\
&=\frac{f(b)}{b-a}-\frac{\Gamma_k(\alpha+1)}{(b-a)^{\frac{\alpha}{k}+1}}\,{}_kJ_{b^-}^{\alpha}f(a).
\end{aligned} \tag{2.7}$$

Similarly

$$\int_0^1(1-t)^{\frac{\alpha}{k}}f'((1-t)a+tb)\mathrm{d}t=-\frac{f(a)}{b-a}+\frac{\Gamma_k(\alpha+1)}{(b-a)^{\frac{\alpha}{k}+1}}\,{}_kJ_{a^+}^{\alpha}f(b). \tag{2.8}$$

Using (2.7) and (2.8) in (2.6) and then multiplying both sides by $\frac{b-a}{2}$, completes the proof. □

Now using Lemma 1 we prove our next result, which plays a key role in the development of our next results.

Lemma 2. *Let $f:I\to\mathbb{R}$ be twice differentiable function. If $f''\in L[a,b]$, then following equality for fractional integrals hold:*

$$\begin{aligned}
&\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+1)}{(b-a)^{\frac{\alpha}{k}}}[{}_kJ_{a^+}^{\alpha}f(a)+{}_kJ_{b^-}^{\alpha}f(b)]\\
&=\frac{k(b-a)^2}{2(\alpha+k)}\int_0^1[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}]f''((1-t)a+tb)\mathrm{d}t.
\end{aligned}$$

Proof. Now from Lemma 1, we have

$$\int_0^1(t^{\frac{\alpha}{k}}-(1-t)^{\frac{\alpha}{k}})f'((1-t)a+tb)\mathrm{d}t$$

$$
\begin{aligned}
&= k\int_0^1 f'((1-t)a+tb)d\frac{(1-t)^{\frac{\alpha}{k}+1}+t^{\frac{\alpha}{k}+1}}{\alpha+k}\\
&= k\left\{\left|\frac{(1-t)^{\frac{\alpha}{k}+1}+t^{\frac{\alpha}{k}+1}}{\alpha+k}f'((1-t)a+tb)\right|_0^1\right.\\
&\qquad \left.-(b-a)\int_0^1 \frac{(1-t)^{\frac{\alpha}{k}+1}+t^{\frac{\alpha}{k}+1}}{\alpha+k}f''((1-t)a+tb)\mathrm{d}t\right\}\\
&= \frac{k}{\alpha+k}\left\{f'(b)-f'(a)-(b-a)\int_0^1 [(1-t)^{\frac{\alpha}{k}+1}+t^{\frac{\alpha}{k}+1}]f''((1-t)a+tb)\mathrm{d}t\right\}.
\end{aligned}
\tag{2.9}
$$

Also

$$
f'(b)-f'(a)=\int_a^b f''(x)\mathrm{d}x=(b-a)\int_0^1 f''((1-t)a+tb)\mathrm{d}t. \tag{2.10}
$$

Utilizing (2.9), (2.10) and Lemma 1 completes the proof. □

Note that when $k\to 1$ in Lemma 2, we get Lemma 2.1 [21].

3. k-FRACTIONAL HERMITE-HADAMARD INEQUALITIES VIA s-CONVEX FUNCTIONS OF FIRST SENSE

In this section, we derive some k-fractional estimates of Hermite-Hadamard type inequalities via s-convex function of first sense.

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be twice differentiable on (a,b) with $a<b$ and $f''\in L_1[a,b]$. If $|f''|$ is s-convex function in the first sense, then*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+1)}{(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f(a)+{}_kJ^{\alpha}_{b^-}f(b)\right]\right|\\
&\le \frac{k(b-a)^2}{2(\alpha+k)}\Big[\mathcal{G}_1(\alpha,k,s)|f''(a)|+\mathcal{G}_2(\alpha,k,s)|f''(b)|\Big],
\end{aligned}
$$

where

$$
\mathcal{G}_1(\alpha,k,s)=\frac{s}{s+1}-\frac{2k}{\alpha+2}+\frac{k}{\alpha+sk+2k}+kB_k(k(s+1),\alpha+2k); \tag{3.1}
$$

$$
\mathcal{G}_2(\alpha,k,s)=\frac{1}{s+1}-\frac{k}{\alpha+sk+2k}-kB_k(k(s+1),\alpha+2k). \tag{3.2}
$$

Proof. Using Lemma 2, property of the modulus and the fact that $|f''|$ is s-convex in the first sense, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+1)}{(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f(a)+{}_kJ^{\alpha}_{b^-}f(b)\right]\right| \\
&\leq \frac{k(b-a)^2}{2(\alpha+k)}\int_0^1[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}]|f''((1-t)a+tb)|\mathrm{d}t \\
&\leq \frac{k(b-a)^2}{2(\alpha+k)}\int_0^1[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}]\left[(1-t^s)|f''(a)|+t^s|f''(b)|\right]\mathrm{d}t \\
&= \frac{k(b-a)^2}{2(\alpha+k)}\left[\left\{\frac{s}{s+1}-\frac{2k}{\alpha+2}+\frac{k}{\alpha+sk+2k}+kB_k(k(s+1),\alpha+2k)\right\}|f''(a)|\right. \\
&\qquad\left.+\left\{\frac{1}{s+1}-\frac{k}{\alpha+sk+2k}-kB_k(k(s+1),\alpha+2k)\right\}|f''(b)|\right].
\end{aligned}$$

This completes the proof. □

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be twice differentiable on (a,b) with $a<b$ and $f''\in L_1[a,b]$. If $|f''|^q$ is s-convex function in the first sense, where $\frac{1}{p}+\frac{1}{q}=1$, $q>1$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+1)}{(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f(a)+{}_kJ^{\alpha}_{b^-}f(b)\right]\right| \\
&\leq \frac{k(b-a)^2}{2(\alpha+k)}\left(\frac{p(\alpha+k)-k}{p(\alpha+k)+k}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{s+1}\right)^{\frac{1}{q}}.
\end{aligned}$$

Proof. Using Lemma 2, property of modulus, well-known Holder's inequality and the fact that $|f''|^q$ is s-convex in the first sense, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+1)}{(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f(a)+{}_kJ^{\alpha}_{b^-}f(b)\right]\right| \\
&= \left|\frac{k(b-a)^2}{2(\alpha+k)}\int_0^1[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}]f''((1-t)a+tb)\mathrm{d}t\right| \\
&\leq \frac{k(b-a)^2}{2(\alpha+k)}\left(\int_0^1(1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1})^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\left|f''((1-t)a+tb)\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&\leq \frac{k(b-a)^2}{2(\alpha+k)}\left(\int_0^1\left(1-(1-t)^{p(\frac{\alpha}{k}+1)}-t^{p(\frac{\alpha}{k}+1)}\right)\mathrm{d}t\right)^{\frac{1}{p}}
\end{aligned}$$

$$\times\left(\int_0^1[(1-t^s)|f''(a)|^q+t^s|f''(b)|^q]\mathrm{d}t\right)^{\frac{1}{q}}$$

$$=\frac{k(b-a)^2}{2(\alpha+k)}\left(\frac{p(\alpha+k)-k}{p(\alpha+k)+k}\right)^{\frac{1}{p}}\left(\frac{s|f''(a)|^q+|f''(b)|^q}{s+1}\right)^{\frac{1}{q}}.$$

This completes the proof. □

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be twice differentiable on (a,b) with $a<b$ and $f''\in L_1[a,b]$. If $|f''|^q$ is s-convex function in the first sense, where $q>1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+1)}{(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ_{a^+}^{\alpha}f(a)+{}_kJ_{b^-}^{\alpha}f(b)\right]\right|$$

$$\leq\frac{k(b-a)^2}{2(\alpha+k)}\left(\frac{\alpha+2(k-1)}{\alpha+2k}\right)^{1-\frac{1}{q}}\left(\mathcal{G}_1(\alpha,k,s)|f''(a)|^q+\mathcal{G}_2(\alpha,k,s)|f''(b)|^q\right)^{\frac{1}{q}},$$

where $\mathcal{G}_1(\alpha,k,s)$ and $\mathcal{G}_2(\alpha,k,s)$ are given by (3.1) *and* (3.2) *respectively.*

Proof. Using Lemma 2, property of modulus, well-known power-mean inequality and the fact that $|f''|^q$ is s-convex in the first sense, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+1)}{(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ_{a^+}^{\alpha}f(a)+{}_kJ_{b^-}^{\alpha}f(b)\right]\right|$$

$$=\left|\frac{k(b-a)^2}{2(\alpha+k)}\int_0^1[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}]f''((1-t)a+tb)\mathrm{d}t\right|$$

$$\leq\frac{k(b-a)^2}{2(\alpha+k)}\left(\int_0^1[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}]\mathrm{d}t\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1(1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1})\left|f''((1-t)a+tb)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}$$

$$\leq\frac{k(b-a)^2}{2(\alpha+k)}\left(\frac{\alpha+2(k-1)}{\alpha+2k}\right)^{1-\frac{1}{q}}\left(\mathcal{G}_1(\alpha,k,s)|f''(a)|^q+\mathcal{G}_2(\alpha,k,s)|f''(b)|^q\right)^{\frac{1}{q}}.$$

This completes the proof. □

4. k-FRACTIONAL HERMITE-HADAMARD INEQUALITIES VIA s-CONVEX FUNCTIONS OF SECOND SENSE

In this section, we derive some k-fractional estimates of Hermite-Hadamard type inequalities via s-convex function of second sense.

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be twice differentiable on (a,b) with $a<b$ and $f''\in L_1[a,b]$. If $|f''|$ is s-convex function in the second sense, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+1)}{(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f(a)+{}_kJ^{\alpha}_{b^-}f(b)\right]\right|$$
$$\leq\frac{k(b-a)^2}{2(\alpha+k)}\mathcal{H}(\alpha,k,s)[|f''(a)|+|f''(b)|],$$

where

$$\mathcal{H}(\alpha,k,s)=\frac{1}{s+1}-\frac{k}{\alpha+sk+2k}-kB_k(\alpha+2k,k(s+1)). \tag{4.1}$$

Proof. Using Lemma 2, property of modulus and the fact that $|f''|$ is s-convex in the second sense, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+1)}{(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f(a)+{}_kJ^{\alpha}_{b^-}f(b)\right]\right|$$
$$\leq\frac{k(b-a)^2}{2(\alpha+k)}\int_0^1\left[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}\right]|f''((1-t)a+tb)|\mathrm{d}t$$
$$\leq\frac{k(b-a)^2}{2(\alpha+k)}\int_0^1\left[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}\right]\left[(1-t)^s|f''(a)|+t^s|f''(b)|\right]\mathrm{d}t$$
$$=\frac{k(b-a)^2}{2(\alpha+k)}\left[\frac{1}{s+1}-\frac{k}{\alpha+sk+2k}-kB_k(\alpha+2k,k(s+1))\right]\left[|f''(a)|+|f''(b)|\right].$$

This completes the proof. □

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be twice differentiable on (a,b) with $a<b$ and $f''\in L_1[a,b]$. If $|f''|^q$ is s-convex function in the second sense, where $\frac{1}{p}+\frac{1}{q}=1$, $q>1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+1)}{(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f(a)+{}_kJ^{\alpha}_{b^-}f(b)\right]\right|$$
$$\leq\frac{k(b-a)^2}{2(\alpha+k)}\left(\frac{p(\alpha+k)-k}{p(\alpha+k)+k}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{s+1}\right)^{\frac{1}{q}}.$$

Proof. Using Lemma 2, property of modulus, well-known Holder's inequality and the fact that $|f''|^q$ is s-convex in the second sense, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+1)}{(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f(a)+{}_kJ^{\alpha}_{b^-}f(b)\right]\right|$$
$$=\left|\frac{k(b-a)^2}{2(\alpha+k)}\int_0^1[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}]f''((1-t)a+tb)\mathrm{d}t\right|$$

$$\leq \frac{k(b-a)^2}{2(\alpha+k)}\left(\int_0^1 (1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1})^p \mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''((1-t)a+tb)\right|^q \mathrm{d}t\right)^{\frac{1}{q}}$$

$$\leq \frac{k(b-a)^2}{2(\alpha+k)}\left(\int_0^1 \left(1-(1-t)^{p(\frac{\alpha}{k}+1)}-t^{p(\frac{\alpha}{k}+1)}\right)\mathrm{d}t\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 [(1-t)^s|f''(a)|^q+t^s|f''(b)|^q]\mathrm{d}t\right)^{\frac{1}{q}}$$

$$=\frac{k(b-a)^2}{2(\alpha+k)}\left(\frac{p(\alpha+k)-k}{p(\alpha+k)+k}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{s+1}\right)^{\frac{1}{q}}.$$

This completes the proof. □

Theorem 7. *Let $f:[a,b]\to\mathbb{R}$ be twice differentiable on (a,b) with $a<b$ and $f''\in L_1[a,b]$. If $|f''|^q$ is s-convex function in the second sense where $q>1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+1)}{(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f(a)+{}_kJ^{\alpha}_{b^-}f(b)\right]\right|$$
$$\leq \frac{k(b-a)^2}{2(\alpha+k)}\mathcal{H}^{\frac{1}{q}}(\alpha,k,s)[|f''(a)|^q+|f''(b)|^q]^{\frac{1}{q}},$$

where $\mathcal{H}(\alpha,k,s)$ is given by (4.1).

Proof. Using Lemma 2, property of modulus, well-known power-mean inequality and the fact that $|f''|^q$ is s-convex in the second sense, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+1)}{(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f(a)+{}_kJ^{\alpha}_{b^-}f(b)\right]\right|$$
$$=\left|\frac{k(b-a)^2}{2(\alpha+k)}\int_0^1[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}]f''((1-t)a+tb)\mathrm{d}t\right|$$
$$\leq \frac{k(b-a)^2}{2(\alpha+k)}\left(\int_0^1[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}]\mathrm{d}t\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1(1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1})\left|f''((1-t)a+tb)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}$$

$$\leq \frac{k(b-a)^2}{2(\alpha+k)}\left(\frac{\alpha+2(k-1)}{\alpha+2k}\right)^{1-\frac{1}{q}}$$
$$\times\left(\frac{1}{s+1}-\frac{k}{\alpha+sk+2k}-kB_k(\alpha+2k,k(s+1))\right)^{\frac{1}{q}}\left(|f''(a)|^q+|f''(b)|^q]\right)^{\frac{1}{q}}$$
$$=\frac{(b-a)^2}{2(\alpha+1)}\left(\frac{\alpha}{\alpha+2}\right)^{1-\frac{1}{q}}\mathcal{H}^{\frac{1}{q}}(\alpha,k,s)\left(|f''(a)|^q+|f''(b)|^q]\right)^{\frac{1}{q}}.$$

This completes the proof. □

Remark 1. We would like to point out that for $s=1$ results obtained in section 3 and 4 reduces to the result for classical convexity. The results for classical convexity also appear to be new in the literature.

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ESSENTIAL SUPPLEMENTED LATTICES

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Abstract. Let L be a complete modular lattice. If every essential element of L has a supplement in L, then L is called an essential supplemented (or briefly e-supplemented) lattice. In this work some properties of these lattices are investigated. Let L be a complete modular lattice and $1 = a_1 \vee a_2 \vee \ldots \vee a_n$ with $a_i \in L$ $(1 \leq i \leq n)$. If $a_i/0$ is e-supplemented for every $i = 1, 2, \ldots, n$, then L is also e-supplemented. If L is e-supplemented, then $1/a$ is also e-supplemented for every $a \in L$.

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Keywords: lattices, essential elements, small elements, supplemented lattices

1. Introduction

In this paper, every lattice is a complete modular lattice with the smallest element 0 and the greatest element 1. Let L be a lattice, $x, y \in L$ and $x \leq y$. A sublattice $\{a \in L | x \leq a \leq y\}$ is called a *quotient sublattice* and denoted by y/x. An element a of a lattice L is called a *complement* of b in L if $a \vee b = 1$ and $a \wedge b = 0$, in this case we denote $1 = a \oplus b$ (here we also call a and b are *direct summands* of L). L is said to be *complemented* if each element has at least one complement in L. An element x of L is said to be *small* or *superfluous* and denoted by $x \ll L$ if $y = 1$ for every $y \in L$ such that $x \vee y = 1$. Let L be a lattice and $k \in L$. If $t = 0$ for very $t \in L$ with $k \wedge t = 0$, then k is called an *essential element* of L and denoted by $k \trianglelefteq L$. The meet of all maximal $(\neq 1)$ elements of a lattice L is called the *radical* of L and denoted by $r(L)$. If $a \ll L$, then $a \leq r(L)$ holds. An element a of L is called a *supplement* of b in L if it is minimal for $1 = b \vee a$. a is a supplement of b in a lattice L if and only if $1 = b \vee a$ and $b \wedge a \ll a/0$. A lattice L is called a *supplemented lattice* if every element of L has a supplement in L. We say that an element y of L *lies above* an element x of L if $x \leq y$ and $y \ll 1/x$. L is said to be *hollow* if every element distinct from 1 is superfluous in L, and L is said to be *local* if L has the greatest element $(\neq 1)$. We say an element $x \in L$ has *ample supplements* in L if for every $y \in L$ with $x \vee y = 1$, x has a supplement z in L with $z \leq y$. L is said to be *amply supplemented* if every element of L has ample supplements in L. Let L be a lattice. It is defined β_* relation on the elements of L by

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$a\beta_* b$ with $a,b \in L$ if and only if for each $t \in L$ such that $a \vee t = 1$ then $b \vee t = 1$ and for each $k \in L$ such that $b \vee k = 1$ then $a \vee k = 1$.

More informations about (amply) supplemented lattices are in [1, 2, 5, 9]. More results about (amply) supplemented modules are in [8, 12]. The definition of β_* relation on lattices and some properties of this relation are in [10]. This relation is a generalization of β^* relation on modules. The definition of β^* relation on modules and some properties of this relation are in [4].

Lemma 1. *Let L be a lattice and $a,b,c \in L$ with $a \leq b$. If c is a supplement of b in L, then $a \vee c$ is a supplement of b in $1/a$.*

Proof. See [3, Lemma 1]. □

Definition 1. Let L be a lattice and k be a maximal $(\neq 1)$ element of L. If $k \trianglelefteq L$, then k is called a g-maximal element of L. The meet of all g-maximal elements of L is called the g-radical of L and denoted by $r_g(L)$. If L have not any g-maximal elements, then we call $r_g(L) = 1$.

The g-radical of any lattice is a generalization of the generalized radical of any module. The definition of the generalized radical of any module and some properties of this concept are in [6, 7].

Corollary 1. *Let L be a lattice. Then $r(L) \leq r_g(L)$.*

Proof. Clear from definitions. □

Lemma 2. *Let L be a lattice and $x \in L$. Then $r_g(x/0) \leq r_g(L)$.*

Proof. Let k be any g-maximal element of L. If $x \leq k$, then $r_g(x/0) \leq k$. If $x \nleq k$, we can easily see that $x \wedge k$ is a g-maximal element of $x/0$ and hence $r_g(x/0) \leq k$. Therefore, $r_g(x/0) \leq r_g(L)$. □

2. Essential supplemented lattices

Definition 2. Let L be a lattice. If every essential element of L has a supplement in L, then L is called an essential supplemented (or briefly e-supplemented) lattice.

This concept is a generalization of an essential supplemented module. The definition of essential supplemented modules and some properties of these modules are in [11].

It is clear that every supplemented lattice is e-supplemented. Hollow and local lattices are e-supplemented.

Proposition 1. *Let L be an e-supplemented lattice. If every element of L with distinct from 0 is essential in L, then L is supplemented.*

Proof. Clear from definitions. □

Definition 3. Let L be a lattice and $a \in L$. If a is a supplement of an essential element in L, then a is called an e-supplement element in L.

Proposition 2. *Let L be a lattice. If every essential element of L is e-supplement in L, then L is e-supplemented.*

Proof. Let $x \trianglelefteq L$. By hypothesis x is an e-supplement element in L and there exists $y \trianglelefteq L$ such that x is a supplement of y in L. Here $1 = y \vee x$ and $y \wedge x \ll x/0$. Since $y \trianglelefteq L$, by hypothesis, y is an e-supplement element in L. Since $x \wedge y \ll L$ and y is a supplement element in L, by [9, Lemma 10], $x \wedge y \ll y/0$ and y is a supplement of x in L. Hence L is e-supplemented. □

Proposition 3. *Let L be a lattice and a be an e-supplement element in L. Then $r_g(a/0) = a \wedge r_g(L)$.*

Proof. By Lemma 2, $r_g(a/0) \leq a \wedge r_g(L)$ holds. Since a is a e-supplement element in L, there exists an essential element x of L such that a is a supplement of x in L. Here $1 = x \vee a$ and $x \wedge a \ll a/0$. By Corollary 1, $x \wedge a \leq r(a/0) \leq r_g(a/0)$ hold. Let k be any g-maximal element of $a/0$. Here $x \wedge a \leq r_g(a/0) \leq k$. Since $\frac{1}{x \vee k} = \frac{x \vee k \vee a}{x \vee k} \cong \frac{a}{a \wedge (x \vee k)} = \frac{a}{(x \wedge a) \vee k} = \frac{a}{k}$ and $x \vee k \trianglelefteq L$, $x \vee k$ is a g-maximal element of L and hence $r_g(L) \leq x \vee k$. This case $a \wedge r_g(L) \leq a \wedge (x \vee k) = (x \wedge a) \vee k = k$. Therefore, $a \wedge r_g(L) \leq r_g(a/0)$ and since $r_g(a/0) \leq a \wedge r_g(L)$, $r_g(a/0) = a \wedge r_g(L)$. □

Proposition 4. *Let L be an essential supplemented lattice. Then $1/r(L)$ doesn"t have any essential element with distinct from* 1.

Proof. Let x be any essential element of $1/r(L)$. Then we clearly see that x is an essential element of L. Since L is essential supplemented, x has a supplement y in L. Here $1 = x \vee y$ and $x \wedge y \ll y/0$. Since $x \wedge y \ll y/0$, $x \wedge y \leq r(L)$. Here $1 = x \vee y = x \vee y \vee r(L)$ and $x \wedge (y \vee r(L)) = (x \wedge y) \vee r(L) = r(L)$. Hence $1 = x \oplus (y \vee r(L))$ in $1/r(L)$ and since $x \trianglelefteq 1/r(L)$, $x = 1$. □

Lemma 3. *Let L be a lattice, $x \trianglelefteq L$ and $a \in L$. If $x \vee a$ has a supplement in L and $a/0$ essential supplemented, then x has a supplement in L.*

Proof. Let y be a supplement of $x \vee a$ in L. Then $1 = x \vee a \vee y$ and $(x \vee a) \wedge y \ll y/0$. Since $x \trianglelefteq L$, $x \vee y \trianglelefteq L$ and hence $(x \vee y) \wedge a \trianglelefteq a/0$. Since $a/0$ is essential supplemented, $(x \vee y) \wedge a$ has a supplement z in $a/0$. Here $a = [(x \vee y) \wedge a] \vee z$ and $(x \vee y) \wedge z = (x \vee y) \wedge a \wedge z \ll z/0$. Following these we have $1 = x \vee a \vee y = x \vee y \vee [(x \vee y) \wedge a] \vee z = x \vee y \vee z$ and $x \wedge (y \vee z) \leq [(x \vee y) \wedge z] \vee [(x \vee z) \wedge y] \leq [(x \vee y) \wedge z] \vee [(x \vee a) \wedge y] \ll (y \vee z)/0$. Hence $y \vee z$ is a supplement of x in L. □

Lemma 4. *Let L be a lattice and $1 = a \vee b$ with $a, b \in L$. If $a/0$ and $b/0$ are e-supplemented, then L is also e-supplemented.*

Proof. Let $x \trianglelefteq L$. Then 0 is a supplement of $x \vee a \vee b$ in L. Since $b/0$ is e-supplemented and $x \vee a \trianglelefteq L$, by Lemma 3, $x \vee a$ has a supplement in L and since

$a/0$ is e-supplemented and $x \trianglelefteq L$, again by Lemma 3, x has a supplement in L. Hence L is e-supplemented. □

Corollary 2. *Let L be a lattice and $1 = a_1 \vee a_2 \vee \ldots \vee a_n$ with $a_i \in L$ $(1 \leq i \leq n)$. If $a_i/0$ is e-supplemented for every $i = 1, 2, \ldots, n$, then L is also e-supplemented.*

Proof. Clear from Lemma 4. □

Lemma 5. *Let L be an e-supplemented lattice and $a \in L$. Then $1/a$ is also e-supplemented.*

Proof. Let x be any essential element of $1/a$. Then $x \trianglelefteq L$ and since L is e-supplemented, x has a supplement y in L. By Lemma 1, $y \vee a$ is a supplement of x in $1/a$. Hence $1/a$ is e-supplemented. □

Corollary 3. *Let L be an e-supplemented lattice. Then $a/0$ is e-supplemented for every direct summand a of L.*

Proof. Since a is a direct summand of L, there exists $b \in L$ such that $1 = a \oplus b$. By Lemma 5, $1/b$ is e-supplemented. Then by $\frac{1}{b} = \frac{a \vee b}{b} \cong \frac{a}{a \wedge b} = \frac{a}{0}$, $a/0$ is also e-supplemented. □

Lemma 6. *Let L be a lattice. If every essential element of L is β_* equivalent to an e-supplement element in L, then L is e-supplemented.*

Proof. Let $x \trianglelefteq L$. By hypothesis, there exists an e-supplement element y in L such that $x\beta_* y$. Since y is an e-supplement element in L, there exists $z \trianglelefteq L$ such that y is a supplement of z in L. By hypothesis, there exists an e-supplement element a in L such that $z\beta_* a$. Since y is a supplement of z in L and $z\beta_* a$, by [10, Theorem 4 (1)], y is a supplement of a in L. Here $1 = a \vee y$ and $a \wedge y \ll y/0$. Since a is a supplement element in L, by [9, Lemma 10], $a \wedge y \ll a/0$ and a is a supplement of y in L. Since $x\beta_* y$, by [10, Theorem 4 (1)], a is a supplement of x in L. Hence L is e-supplemented. □

Corollary 4. *Let L be a lattice. If every essential element of L lies above an e-supplement element in L, then L is e-supplemented.*

Proof. Clear from [10, Theorem 3] and Lemma 6. □

Definition 4. Let L be a lattice. If every essential element of L with distinct from 1 is small in L or L have no essential elements with distinct from 1, then L is called a e-hollow lattice. If L has an essential element $c \neq 1$ such that $k \leq c$ for every $1 \neq k \trianglelefteq L$, then L is called an e-local lattice (here k is called the greatest essential element $(\neq 1)$ of L).

Clearly we can see that every hollow lattice is e-hollow. But the converse is not true in general (See Example 1).

Proposition 5. *Every e-hollow lattice is e-supplemented.*

Proof. Let L be an e-hollow lattice. Then 1 is a supplement of every essential element of L with distinct from 1 and 0 is a supplement of 1 in L. Hence L is e-supplemented. □

Proposition 6. *Let L be a lattice with $r(L) \trianglelefteq L$ and $r(L) \neq 1$. If L is e-hollow, then L is e-local.*

Proof. Let $x \trianglelefteq L$ and $x \neq 1$. Since L is e-hollow, $x \ll L$ and $x \leq r(L)$. By hypothesis, $r(L) \neq 1$. Hence $r(L)$ is the greatest essential element $(\neq 1)$ of L and L is e-local. □

Example 1. Consider the lattice $L = \{0, a, b, 1\}$ given by the following diagram.

$$\begin{tikzcd}
 & 1 & \\
a \arrow[ur] & & b \arrow[ul] \\
 & 0 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Then L is e-hollow but not hollow. Here $r(L) = 0 \neq 1 = r_g(L)$ hold.

Example 2. Consider the lattice $L = \{0, a, b, c, 1\}$ given by the following diagram.

$$\begin{tikzcd}
 & 1 & \\
b \arrow[ur] & & c \arrow[ul] \\
 & a \arrow[ul] \arrow[ur] & \\
 & 0 \arrow[u] &
\end{tikzcd}$$

Then L is e-supplemented but not e-hollow.

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A CLASS OF FOURTH-ORDER KIRCHHOFF TYPE ELLIPTIC PROBLEMS WITH FOUR SOLUTIONS

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Abstract. In this paper, we establish the existence of at least four solutions for a perturbed fourth-order Kirchhoff type elliptic problem by using a recent variational principle due to Ricceri.

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Keywords: nonlinear sciences, Kirchhoff type elliptic problem, four solutions

1. Introduction

This paper is concerned with the existence of weak solutions to the following fourth-order Kirchhoff type elliptic problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) - \left[M\left(\int_\Omega |\nabla u|^p\,dx\right)\right]^{p-1}\Delta_p u + \eta|u|^{p-2}u \\ \quad = \lambda a(x)f(u) + b(x)g(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 1)$, $p > \max\left\{1, \frac{N}{2}\right\}$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $\eta > 0$, $f, g : \mathbb{R} \to \mathbb{R}$ are continuous non-constant functions, $\lambda \in \mathbb{R}$, $a, b \in L^1(\Omega)$ are non-negative non-constant functions and the function $M : [0, +\infty[\to \mathbb{R}$ is a continuous function.

The fourth-order equation furnishes a model to study travelling waves in suspension bridges, so it is important to physics. Due to this, many researchers have discussed the existence of at least one solution, or multiple solutions, or even infinitely many solutions for fourth-order boundary value problems by using lower and upper solution methods, Morse theory, the mountain-pass theorem, constrained minimization and concentration-compactness principle, fixed point theorems and degree theory, and variational methods and critical point theory. We refer the reader to [1, 8, 10, 13, 22, 23].

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This problem models several physical and biological systems where u describes a process which depends on the average of itself, such as the population density, see [2].

The problem (1.1) is a generalization of the stationary problem of a model introduced by Kirchhoff [9]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

which L, E, h, ρ, P_0 are constants, which extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations.

However, to be accurate, the problem (1.1) is related to the models of extensible beam and plates by Woinowsky-Krieger [20] and Berger [4]. In addition, the first stationary study of such fourth order nonlocal boundary value problem was given by Ma [11]. In [13], it was named first time *fourth order problem of Kirchhoff type*. The problem is also related to the so-called p-Kirchhoff problems.

In recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer interested readers to [2, 3, 7, 8, 12, 16, 19, 21, 24]. For instance, B. Ricceri in an interesting paper [16] established the existence of at least three weak solutions to a class of Kirchhoff-type doubly eigenvalue boundary value problems using Theorem A of [14].

The starting point of our approach to problem (1.1) has been [15], in which the author considers the following ordinary Neumann problem

$$\begin{cases} -u''+u=\lambda a(x)f(u)+b(x)g(u) & \text{in } [0,1],\\ u'(0)=u'(1)=0, \end{cases} \tag{1.3}$$

where $\lambda\in\mathbb{R}$ and $f,g:\mathbb{R}\to\mathbb{R}$, $a,b:[0,1]\to[0,+\infty[$ are four continuous non-constant functions. Ricceri has proved that the above problem admits at least three non-zero solutions.

Our results here are motivated by the recent papers [5, 15, 17, 18]. We establish some sufficient conditions under which the problem (1.1) possesses four weak solutions.

The paper is arranged as follows. In Section 2, we give preliminaries and our main tool, that is, Theorem 1, while in Section 3, we present our main results.

2. Functional setting

As already said in the introduction, our approach in facing problem (1.1) is based on the multiplicity result established in [15], that we recall below for the reader's convenience (see also Theorem 1 of [17] for a similar result).

Theorem 1 ([15, Theorem 1]). *Let X be a reflexive real Banach space; $\Phi:X\to\mathbb{R}$ be a coercive and sequentially weakly lower semicontinuous C^1 functional whose*

derivative admits a continuous inverse on X^*, $\Psi_1, \Psi_2 : X \to \mathbb{R}$ two C^1 functionals with compact derivative. Assume that there exist two points $u_0, v_0 \in X$ with the following properties:

(i) *u_0 is a strict local minimum of Φ and $\Phi(u_0) = \Psi_1(u_0) = \Psi_2(u_0) = 0$;*
(ii) *$\Phi(v_0) \leq \Psi_1(v_0)$ and $\Psi_2(v_0) > 0$.*

Moreover, suppose that, for some $\rho \in \mathbb{R}$, one has either

$$\sup_{\lambda>0} \inf_{u\in X} (\lambda(\Phi(u) - \Psi_1(u) - \rho) - \Psi_2(u)) < \inf_{u\in X} \sup_{\lambda>0} (\lambda(\Phi(u) - \Psi_1(u) - \rho) - \Psi_2(u)) \tag{2.1}$$

or

$$\sup_{\lambda>0} \inf_{u\in X} (\Phi(u) - \Psi_1(u) - \lambda(\rho + \Psi_2(u))) < \inf_{u\in X} \sup_{\lambda>0} (\Phi(u) - \Psi_1(u) - \lambda(\rho + \Psi_2(u))). \tag{2.2}$$

Finally, assume that

$$\max\left\{ \limsup_{\|u\|\to+\infty} \frac{\Psi_1(u)}{\Phi(u)}, \limsup_{u\to u_0} \frac{\Psi_1(u)}{\Phi(u)} \right\} < 1 \tag{2.3}$$

and

$$\max\left\{ \limsup_{\|u\|\to+\infty} \frac{\Psi_2(u)}{\Phi(u)}, \limsup_{u\to u_0} \frac{\Psi_2(u)}{\Phi(u)} \right\} \leq 0. \tag{2.4}$$

Under such hypotheses, there exists $\lambda^* > 0$ such that the equation $\Phi'(u) = \Psi_1'(u) + \lambda^*\Psi_2'(u)$ has at least four solutions in X. More precisely, among them, one is u_0 as a strict local, not global minimum and two are global minima of the functional $\Phi - \Psi_1 - \lambda^*\Psi_2$.

Remark 1. It is important to remark that, in view of Theorem 1 of [6], condition (2.1) is equivalent to the existence of $u_1, v_1 \in X$ satisfying

$$\Phi(u_1) - \Psi_1(u_1) < \rho < \Phi(v_1) - \Psi_1(v_1)$$

and

$$\frac{\sup_{(\Phi-\Psi_1)^{-1}(]-\infty,\rho])} \Psi_2 - \Psi_2(u_1)}{\rho - \Phi(u_1) + \Psi_1(u_1)} < \frac{\sup_{(\Phi-\Psi_1)^{-1}(]-\infty,\rho])} \Psi_2 - \Psi_2(v_1)}{\rho - \Phi(v_1) + \Psi_1(v_1)}.$$

Likewise, condition (2.2) is equivalent to the existence of $u_1, v_1 \in X$ satisfying

$$\Psi_2(v_1) < \rho < \Psi_2(u_1)$$

and

$$\frac{\Phi(u_1) - \Psi_1(u_1) - \inf_{\Psi_2^{-1}([\rho,+\infty[)}(\Phi - \Psi_1)}{\Psi_2(u_1) - \rho} < \frac{\Phi(v_1) - \Psi_1(v_1) - \inf_{\Psi_2^{-1}([\rho,+\infty[)}(\Phi - \Psi_1)}{\Psi_2(v_1) - \rho}.$$

Here and in the sequel, X will denote the space $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| := \left(\int_\Omega (|\Delta u(x)|^p + |\nabla u(x)|^p + |u(x)|^p)\, dx\right)^{\frac{1}{p}}.$$

Put

$$c = \sup_{u\in X\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}} |u(x)|}{\|u\|}.$$

For $p > \max\left\{1, \frac{N}{2}\right\}$, since the embedding $X \hookrightarrow C^0(\overline{\Omega})$ is compact, one has $c < +\infty$. Let $M : [0,+\infty[\to \mathbb{R}$ be a continuous function such that there is a positive constant m with $M(t) \geq m$ for all $t \geq 0$. Set

$$\tilde{M}(t) = \int_0^t (M(s))^{p-1}\, ds \quad \text{for all } t \geq 0,$$
$$M^- := \min\{1, m^{p-1}, \eta\}.$$

The following lemma is useful in the proof of our results.

Lemma 1. *If $\gamma : \Omega \to [0,+\infty[$ be a non-zero function in $L^1(\Omega)$ and $h : \mathbb{R} \to \mathbb{R}$ a continuous non-zero function, we denote by $T_{\gamma,h}$ the functional defined on X by putting*

$$T_{\gamma,h}(u) = \int_\Omega \gamma(x)h(u(x))dx$$

for all $u \in X$. Then, one has

$$\limsup_{u\to 0} \frac{T_{\gamma,h}(u)}{\|u\|^p} \leq c^p \|\gamma\|_{L^1(\Omega)} \max\left\{0, \limsup_{\xi\to 0} \frac{h(\xi)}{|\xi|^p}\right\} \tag{2.5}$$

and

$$\limsup_{\|u\|\to+\infty} \frac{T_{\gamma,h}(u)}{\|u\|^p} \leq c^p \|\gamma\|_{L^1(\Omega)} \max\left\{0, \limsup_{|\xi|\to+\infty} \frac{h(\xi)}{|\xi|^p}\right\}. \tag{2.6}$$

Also, the functional $T_{\gamma,H}$, where H is defined by $H(\xi) = \int_0^\xi h(t)\, dt$ for all $\xi \in \mathbb{R}$, turns out to be in $C^1(X,\mathbb{R})$ and its derivative is given by

$$T'_{\gamma,H}(u)(v) = \int_\Omega \gamma(x)h(u(x))\, v(x)dx$$

for all $u, v \in X$. Moreover, the compact embedding of X in $C^0(\bar{\Omega})$ implies that $T'_{\gamma,H} : X \to X^*$ is compact.

Proof of the previous lemma is similar to proof of Lemma 2.1 in [5]. The following lemma is Proposition 2.3 in [8], it is needed in the proof of Theorem 2.

Lemma 2. *Let $\Lambda : X \to X^*$ be the operator defined by*

$$\Lambda(u)v = \int_\Omega |\Delta u(x)|^{p-2}\Delta u(x)\Delta v(x)\, dx$$

$$+\left[M\left(\int_\Omega |\nabla u|^p\,dx\right)\right]^{p-1}\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\,dx$$
$$+\eta\int_\Omega |u(x)|^{p-2}u(x)v(x)\,dx$$

for every $u,v\in X$. Then Λ admits a continuous inverse on X^*.

We can deduce that $u\in X$ is a weak solution to problem (1.1) if and only if u is a critical point of the functional

$$\frac{1}{p}\int_\Omega |\Delta u(x)|^p dx+\frac{1}{p}\tilde{M}\left(\int_\Omega |\nabla u|^p\,dx\right)+\frac{\eta}{p}\int_\Omega |u(x)|^p\,dx-\lambda T_{a,F}-T_{b,G}$$

which represents, therefore, the energy functional related to problem (1.1).

3. Main results

Now, we present our main result.

Theorem 2. *Let $f,g:\mathbb{R}\to\mathbb{R}$ be two continuous non-constant functions and let $a,b:\Omega\to[0,+\infty[$ be two non-constant functions in $L^1(\Omega)$. Assume that*

$$(f_1)\quad \max\left\{\limsup_{|\xi|\to+\infty}\frac{\int_0^\xi f(t)\,dt}{|\xi|^p},\limsup_{\xi\to 0}\frac{\int_0^\xi f(t)\,dt}{|\xi|^p}\right\}\le 0,$$

$$(g_1)\quad \sup_{\xi\in\mathbb{R}}\int_0^\xi g(t)\,dt<+\infty,\qquad \limsup_{\xi\to 0}\frac{\int_0^\xi g(t)\,dt}{|\xi|^p}<\frac{M^-}{pc^p\|b\|_{L^1(\Omega)}}.$$

Finally, suppose that there exist $\sigma>c\sqrt[p]{\frac{p}{M^-}}\max\left\{1,(\|b\|_{L^1(\Omega)}\sup_{\mathbb{R}}G)^{\frac{1}{p}}\right\}$ and $\xi_1\in\mathbb{R}$ such that

$$(f_2)\quad 0<\int_0^{\xi_1}f(t)\,dt=\sup_{|\xi|\le\sigma}\int_0^\xi f(t)\,dt<\sup_{\xi\in\mathbb{R}}\int_0^\xi f(t)\,dt,$$

$$(g_2)\quad |\xi_1|^p\le\frac{p}{\eta|\Omega|}\|b\|_{L^1(\Omega)}\int_0^{\xi_1}g(t)\,dt.$$

Under such hypotheses, there exists $\lambda^*>0$ such that the problem

$$\begin{cases}\Delta(|\Delta u|^{p-2}\Delta u)-\left[M\left(\int_\Omega|\nabla u|^p\,dx\right)\right]^{p-1}\Delta_p u+\eta|u|^{p-2}u & \\ \quad=\lambda^* a(x)f(u)+b(x)g(u) & \text{in }\Omega,\\ u=\Delta u=0 & \text{on }\partial\Omega,\end{cases}$$

has at least three non-zero solutions, two of which are global minima of the associated energy functional.

Proof. Our aim is to apply Theorem 1. Take Ψ_1, Ψ_2 equal, respectively, to $T_{b,G}, T_{a,F}$ defined in Section 2. For each $u \in X$, let the functional $\Phi : X \to \mathbb{R}$ be defined by

$$\Phi(u) := \frac{1}{p}\int_\Omega |\Delta u(x)|^p dx + \frac{1}{p}\tilde{M}\left(\int_\Omega |\nabla u|^p\, dx\right) + \frac{\eta}{p}\int_\Omega |u(x)|^p\, dx.$$

The function Φ is continuously differentiable whose differential at the point $u \in X$ is

$$\begin{aligned}\Phi'(u)h = &\int_\Omega |\Delta u(x)|^{p-2}\Delta u(x)\Delta v(x)\, dx \\ &+ \left[M\left(\int_\Omega |\nabla u|^p\, dx\right)\right]^{p-1}\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\, dx \\ &+ \eta\int_\Omega |u(x)|^{p-2}u(x)v(x)\, dx\end{aligned}$$

for every $v \in X$, while Lemma 2 gives that Φ' admits a continuous inverse on X^*. Furthermore, Φ is sequentially weakly lower semicontinuous.
Now, take $u_0 = 0$ and $v_0 = \xi_1$; of course, assumption (i) of Theorem 1 is evident. Moreover, thanks to (f_2) and (g_2), one has

$$\Phi(v_0) = \frac{\eta}{p}|\xi_1|^p|\Omega| \le \|b\|_{L^1(\Omega)}\int_0^{\xi_1} g(t)\, dt = T_{b,G}(v_0)$$

and

$$T_{a,F}(v_0) = \|a\|_{L^1(\Omega)}\int_0^{\xi_1} f(t)\, dt > 0,$$

so assumption (ii) of Theorem 1 is satisfied. Now, since $\frac{M^-}{p}\|u\|^p \le \Phi(u)$ for each $u \in X$, we see that

$$\frac{T_{b,G}(u)}{\Phi(u)} \le \frac{pT_{b,G}(u)}{M^-\|u\|^p},$$

and by Lemma 1, we get

$$\limsup_{\|u\|\to+\infty}\frac{T_{b,G}(u)}{\Phi(u)} \le \frac{pc^p}{M^-}\|b\|_{L^1(\Omega)}\max\left\{0, \limsup_{|\xi|\to+\infty}\frac{\int_0^\xi g(t)\, dt}{|\xi|^p}\right\} \le 0.$$

Also, in view of Lemma 1 and (g_1), we have

$$\limsup_{u\to 0}\frac{T_{b,G}(u)}{\Phi(u)} \le \frac{pc^p}{M^-}\|b\|_{L^1(\Omega)}\max\left\{0, \limsup_{\xi\to 0}\frac{\int_0^\xi g(t)\, dt}{|\xi|^p}\right\} < 1.$$

So, (2.3) is fulfilled. Likewise, from Lemma 1 and (f_1), one has

$$\limsup_{\|u\|\to+\infty}\frac{T_{a,F}(u)}{\Phi(u)} \le \frac{pc^p}{M^-}\|a\|_{L^1(\Omega)}\max\left\{0, \limsup_{|\xi|\to+\infty}\frac{\int_0^\xi f(t)\, dt}{|\xi|^p}\right\} \le 0$$

and

$$\limsup_{u\to 0}\frac{T_{a,F}(u)}{\Phi(u)}\leq\frac{pc^p}{M^-}\|a\|_{L^1(\Omega)}\max\left\{0,\limsup_{\xi\to 0}\frac{\int_0^\xi f(t)\,dt}{|\xi|^p}\right\}\leq 0,$$

which satisfy (2.4). Finally, let us check that (2.1) holds. Since $\sigma > c\sqrt[p]{\frac{p}{M^-}}$, we see that

$$1-\|b\|_{L^1(\Omega)}\sup_{\mathbb{R}}G<\frac{\sigma^p M^-}{pc^p}-\|b\|_{L^1(\Omega)}\sup_{\mathbb{R}}G$$

and so it is possible to choose $\rho\in\mathbb{R}$ such that

$$\max\left\{0,1-\|b\|_{L^1(\Omega)}\sup_{\mathbb{R}}G\right\}<\rho<\frac{\sigma^p M^-}{pc^p}-\|b\|_{L^1(\Omega)}\sup_{\mathbb{R}}G.$$

Let $u\in X$ such that $\Phi(u)-T_{b,G}(u)\leq\rho$. For each $u\in X$, by the definition of σ and the choice of ρ, we get

$$\frac{M^-}{p}\|u\|^p\leq\Phi(u)\leq\rho+\|b\|_{L^1(\Omega)}\sup_{\mathbb{R}}G$$

and then, being $\rho+\|b\|_{L^1(\Omega)}\sup_{\mathbb{R}}G>1$, we obtain

$$\|u\|\leq\left(\frac{p}{M^-}\left(\rho+\|b\|_{L^1(\Omega)}\sup_{\mathbb{R}}G\right)\right)^{\frac{1}{p}}\leq\frac{\sigma}{c}.$$

Thus, owing to the embedding of X in $C^0(\bar{\Omega})$ we have the inclusion

$$\left\{u\in X:\Phi(u)-T_{b,G}(u)\leq\rho\right\}\subseteq\left\{u\in X:\sup_{x\in\bar{\Omega}}|u(x)|\leq\sigma\right\}. \tag{3.1}$$

Now, in order to fulfil the equivalent formulation of (2.1) recalled in Remark 1, choose $u_1=v_0$ and take as v_1 any constant d such that $F(d)>\sup_{[-\sigma,\sigma]}F$. Such a d exists by (f_2). Thanks to (g_2), we have

$$\Phi(u_1)-T_{b,G}(u_1)\leq\frac{\eta|\Omega|}{p}|\xi_1|^p-\|b\|_{L^1(\Omega)}\int_0^{\xi_1}g(t)\,dt\leq 0<\rho.$$

Moreover, $\Phi(v_1)-T_{b,G}(v_1)$ has to be necessarily strictly greater than ρ, otherwise, by (3.1), we would have $|d|\leq\sigma$ and $F(d)\leq\sup_{[-\sigma,\sigma]}F$, a contradiction. Then, due to (f_2) and to the choice of d, we easily obtain that

$$\sup_{(\Phi-T_{b,G})^{-1}(]-\infty,\rho])}T_{a,F}\leq T_{a,F}(u_1)$$

and

$$\sup_{(\Phi-T_{b,G})^{-1}(]-\infty,\rho])}T_{a,F}\leq T_{a,F}(v_1).$$

Thus, the following inequalities hold

$$\frac{\sup_{(\Phi-T_{b,G})^{-1}(]-\infty,\rho])} T_{a,F}-T_{a,F}(u_1)}{\rho-\Phi(u_1)+T_{b,G}(u_1)}<0<\frac{\sup_{(\Phi-T_{b,G})^{-1}(]-\infty,\rho])} T_{a,F}-T_{a,F}(v_1)}{\rho-\Phi(v_1)+T_{b,G}(v_1)}$$

and, each assumption of Theorem 1 being satisfied, our problem admits at least three non-zero weak solutions, two of which are global minima of the energy functional. □

Remark 2. Suppose that $\alpha \geq 0$ and $\beta > 0$ be two real numbers and suppose $M : [\alpha,\beta] \subseteq [0,+\infty[\to \mathbb{R}$ be a function defined by $M(t) = \theta_1 + \theta_2 t$ for each $t \in [\alpha,\beta]$ where $\theta_1,\theta_2 > 0$. So, $M^- = \min\{1,(\theta_1+\theta_2\alpha)^{p-1},\eta\}$ and Theorem 2 holds for the following problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u)-(\theta_1+\theta_2\displaystyle\int_\Omega |\nabla u|^p)^{p-1}\Delta_p u+\eta|u|^{p-2}u & \\ \quad =\lambda a(x)f(u)+b(x)g(u) & \text{in } \Omega, \\ u=\Delta u=0 & \text{on } \partial\Omega, \end{cases}$$

where f, g, a and b satisfy in the conditions of Theorem 2.

Remark 3. It is worth pointing out that, if assumption (f_2) of Theorem 2 is replaced by the existence of

$$\sigma > c\sqrt[p]{\frac{p}{M^-}}\max\left\{1,(\|b\|_{L^1(\Omega)}\sup_{\mathbb{R}} G)^{\frac{1}{p}}\right\}$$

and $\xi_1,\xi_2 \in \mathbb{R}$, with $\xi_1\xi_2 > 0$, such that

$$0<\int_0^{\xi_1} f(t)\,dt=\sup_{|\xi|\leq\sigma}\int_0^{\xi} f(t)\,dt<\int_0^{\xi_2} f(t)\,dt, \tag{3.2}$$

under the additional assumption $|\Omega| \geq \frac{M^-}{\eta c^p}$, we can ensure that the three non-zero solutions of the thesis of Theorem 2 are non-negative (respectively non-positive) provided $\xi_1 > 0$ (respectively $\xi_2 < 0$). To see this, it suffices to apply Theorem 2 to the functions $f_0, g_0 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_0(\xi)=\begin{cases} f(\xi) & \text{if } \xi\geq 0, \\ 0 & \text{if } \xi<0, \end{cases}$$

$$g_0(\xi)=\begin{cases} g(\xi) & \text{if } \xi\geq 0, \\ 0 & \text{if } \xi<0, \end{cases}$$

when $\xi_1 > 0$ or by

$$f_0(\xi)=\begin{cases} f(\xi) & \text{if } \xi\leq 0, \\ 0 & \text{if } \xi>0, \end{cases}$$

$$g_0(\xi) = \begin{cases} g(\xi) & \text{if } \xi \le 0, \\ 0 & \text{if } \xi > 0, \end{cases}$$

when $\xi_1 < 0$. In fact, conditions (f_1) and (g_1) ensure that $f(0) = 0$ and $g(0) = 0$, namely, that f_0 and g_0 are continuous functions. Therefore, condition $|\Omega| \ge \frac{M^-}{\eta c^p}$, guarantees that $|\xi_1| \le \sigma$, in fact by (g_2) one has

$$|\xi_1|^p \le \frac{p\|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G}{\eta|\Omega|} \le \frac{M^-\sigma^p}{\eta c^p|\Omega|} \le \sigma^p.$$

Hence, the function f_0 satisfies in the assumption (f_2).

From Theorem 2 and Remark 3, we get:

Corollary 1. *Let $|\Omega| \ge \frac{M^-}{\eta c^p}$, and $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$(f_3)\ \ \sup_{\xi\in\mathbb{R}} \int_0^\xi f(t)\,dt < +\infty, \quad \limsup_{\xi\to 0} \frac{\int_0^\xi f(t)\,dt}{|\xi|^p} \le 0.$$

Moreover, suppose that there exist $\sigma > 0$ and $\xi_1, \xi_2 \in \mathbb{R}$, with $\xi_1\xi_2 > 0$, such that

$$(f_4)\ \ 0 < \int_0^{\xi_1} f(t)\,dt = \sup_{|\xi|\le\sigma} \int_0^\xi f(t)\,dt < \int_0^{\xi_2} f(t)\,dt$$

and

$$(f_5)\ \ \frac{\eta|\Omega||\xi_1|^p}{\int_0^{\xi_1} f(t)\,dt} < \frac{M^-\sigma^p}{c^p \sup_{\xi\in\mathbb{R}} \int_0^\xi f(t)\,dt}.$$

Under such hypotheses, for every non-constant function $a : \Omega \to [0, +\infty[$ in $L^1(\Omega)$ satisfying

$$(f_6)\ \ \max\left\{\frac{1}{\sup_{\mathbb{R}} F}, \frac{\eta|\Omega||\xi_1|^p}{p\int_0^{\xi_1} f(t)\,dt}\right\} \le \|a\|_{L^1(\Omega)} < \frac{M^-\sigma^p}{pc^p \sup_{\xi\in\mathbb{R}} \int_0^\xi f(t)\,dt},$$

there exists $\widehat{\lambda} > 1$ such that the problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) - \left[M\left(\int_\Omega |\nabla u|^p\,dx\right)\right]^{p-1} \Delta_p u + \eta|u|^{p-2}u \\ \quad = \widehat{\lambda} a(x) f(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least three non-zero solutions which are non-negative or non-positive according to whether $\xi_1 > 0$ or $\xi_1 < 0$.

Another consequence of Theorem 2 is as follows:

Proposition 1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function and z_1,z_2,z_3,z_4 four positive constants, with $z_1<z_2<z_3<z_4$ such that $f(\xi)\geq 0$ for all $\xi\in]-\infty,-z_4]\cup[-z_3,0]\cup[z_1,z_2]$, while $f(\xi)\leq 0$ for all $\xi\in]-z_4,-z_3]\cup[0,z_1]\cup[z_2,+\infty[$, and*

$$0<\int_0^{z_2} f(t)\,dt<\int_0^{-z_4} f(t)\,dt.$$

Moreover, let $g:\mathbb{R}\to\mathbb{R}$ be a continuous function and let $b:\Omega\to[0,+\infty[$ be a function in $L^1(\Omega)$ such that

$$\int_0^{z_2} g(t)\,dt>0,\quad \sup_{\xi\in\mathbb{R}}\int_0^{\xi} g(t)\,dt<+\infty,\quad \limsup_{\xi\to 0}\frac{\int_0^{\xi} g(t)\,dt}{|\xi|^p}<\frac{M^-}{pc^p\|b\|_{L^1(\Omega)}}$$

and

$$\max\left\{\frac{1}{\sup_{\mathbb{R}} G},\frac{\eta|\Omega||z_2|^p,}{p\int_0^{z_2} g(t)\,dt}\right\}\leq\|b\|_{L^1(\Omega)}<\frac{M^- z_3^p}{pc^p\sup_{\xi\in\mathbb{R}}\int_0^{\xi} g(t)\,dt}.$$

Under such hypotheses, for each non-zero function $a:\Omega\to[0,+\infty[$ in $L^1(\Omega)$, there exists $\lambda^*>0$ such that the problem

$$\begin{cases}\Delta(|\Delta u|^{p-2}\Delta u)-\left[M\left(\displaystyle\int_\Omega|\nabla u|^p\,dx\right)\right]^{p-1}\Delta_p u+\eta|u|^{p-2}u & \\ \quad=\lambda^* a(x)f(u)+b(x)g(u) & \text{in } \Omega,\\ u=\Delta u=0 & \text{on } \partial\Omega\end{cases}$$

has at least three non-zero solutions, two of which are global minima of the associated energy functional.

Proof. The same assumption also implies that $\int_0^{\xi} f(t)\,dt\leq 0$ for all $\xi\in[-z_3,z_1]$, and so (f_1) holds. Consequently, if we take $\xi_1=z_2$, the assumptions of Theorem 2 are satisfied and the conclusion follows. □

We now exhibit an example in which the hypotheses of Proposition 1 are satisfied.

Example 1. Suppose that $M:[0,+\infty[\to\mathbb{R}$ is a continuous function such that $M^->162$ and suppose $\Omega=\{(x,y)\in\mathbb{R}^2;x^2+y^2<9\}$. Define $f,g:\mathbb{R}\to\mathbb{R}$ as follows:

$$f(t)=-t^5-7t^4+7t^3+43t^2-42t,$$

$$g(t)=\begin{cases}0 & \text{if } t<0,\\ t^2 & \text{if } 0\leq t\leq 2,\\ -2t+8 & \text{if } 2<t\leq 4,\\ 0 & \text{if } t>4.\end{cases}$$

Also, put $a(x,y)=b(x,y)=x^2+y^2$ for all $(x,y)\in\Omega$.

Then, by choosing $z_1 = 1$, $z_2 = 2$, $z_3 = 3$, $z_4 = 7$, $p = 3$, and since $c = \sqrt[3]{\frac{4}{\pi}}$, the conclusion of Proposition 1 holds for the following problem

$$\begin{cases} \Delta(|\Delta u|\Delta u) - \left[M\left(\int_\Omega |\nabla u|^3\,dxdy\right)\right]^2 \Delta_3 u + \eta|u|u \\ \quad = \lambda^* a(x,y) f(u) + b(x,y) g(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

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COEFFICIENT ESTIMATES FOR BAZILEVIČ FUNCTIONS OF BI-PRESTARLIKE FUNCTIONS

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Abstract. In the present article, we introduce and study two certain classes of holomorphic prestarlike and bi-univalent functions associated with Bazilevič function. We determinate upper bounds for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belonging to these classes. Further we point out certain special cases for our results.

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1. Introduction

Let $\mathcal{A}$ indicate the collection of all holomorphic functions in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ that have the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

We also denote by S the sub-collection of the set $\mathcal{A}$ containing of functions in U satisfying (1.1) which are univalent in U.

A function $f \in \mathcal{A}$ is called starlike of order δ $(0 \leq \delta < 1)$, if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \delta, \quad (z \in U).$$

For $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ defined by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

the "Hadamard product" of f and g is defined (as usual) by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad (z \in U).$$

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Ruscheweyh [8] defined and considered the family of "prestarlike functions" of order δ, which are the function f such that $f * I_\delta$ is a starlike function of order δ, where

$$I_\delta(z) = \frac{z}{(1-z)^{2(1-\delta)}}, \quad (0 \leq \delta < 1, z \in U).$$

The function I_δ can be written in the form:

$$I_\delta(z) = z + \sum_{k=2}^{\infty} \varphi_k(\delta) z^k,$$

where

$$\varphi_k(\delta) = \frac{\prod_{i=2}^{k}(i-2\delta)}{(k-1)!}, \quad k \geq 2.$$

We note that $\varphi_k(\delta)$ is a decreasing function in δ and satisfies

$$\lim_{k\to\infty} \varphi_k(\delta) = \begin{cases} \infty, & if\, \delta < \frac{1}{2} \\ 1, & if\, \delta = \frac{1}{2} \\ 0, & if\, \delta > \frac{1}{2} \end{cases}.$$

Singh [9] (also see Kim and Srivastava [4]) introduced and studied the family of Bazilevič functions $f \in \mathcal{A}$ satisfying the condition:

$$Re\left\{\frac{z^{1-\gamma} f'(z)}{(f(z))^{1-\gamma}}\right\} > 0, \quad (z \in U, \gamma \geq 0).$$

According to the Koebe one-quarter theorem (see [3]) "every function $f \in S$ has an inverse f^{-1} which satisfies $f^{-1}(f(z)) = z$, $(z \in U)$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$", where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots. \quad (1.2)$$

For $f \in \mathcal{A}$, if both f and f^{-1} are univalent in U, we say that f bi-univalent function in U. We indicate by Σ the family of bi-univalent functions in U given by (1.1). In fact, Srivastava et al. [18] have actually revived the study of holomorphic and bi-univalent functions in recent years. Some examples of functions in the family Σ are

$$\frac{z}{1-z}, \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \quad \text{and} \quad -\log(1-z)$$

with the corresponding inverse functions

$$\frac{w}{1+w}, \quad \frac{e^{2w}-1}{e^{2w}+1} \quad \text{and} \quad \frac{e^w - 1}{e^w},$$

respectively. Other common examples of functions is not a member of Σ are

$$z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}.$$

Recently, many authors introduced various subfamilies of the bi-univalent functions family Σ and investigated upper bounds for the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1.1) (see, for example [1,5,10–17,19–24]).

We require the following lemma that will be used to prove our main results.

Lemma 1 ([3]). *If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each $k \in \mathbb{N}$, where $\mathcal{P}$ is the family of all functions h holomorphic in U for which*

$$Re\,(h(z)) > 0, \quad (z \in U),$$

where

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots, \quad (z \in U).$$

2. Coefficient estimates for the functions family $\Omega_\Sigma(\lambda,\gamma,\delta;\alpha)$

Definition 1. A function $f \in \Sigma$ given by (1.1) is called in the family $\Omega_\Sigma(\lambda,\gamma,\delta;\alpha)$ if it fulfils the conditions:

$$\left| arg \left[\frac{1}{2} \left(\frac{z^{1-\gamma}(f*I_\delta)'(z)}{((f*I_\delta)(z))^{1-\gamma}} + \left(\frac{z^{1-\gamma}(f*I_\delta)'(z)}{((f*I_\delta)(z))^{1-\gamma}} \right)^{\frac{1}{\lambda}} \right) \right] \right| < \frac{\alpha\pi}{2}, \quad (z \in U) \tag{2.1}$$

and

$$\left| arg \left[\frac{1}{2} \left(\frac{w^{1-\gamma}(g*I_\delta)'(w)}{((g*I_\delta)(w))^{1-\gamma}} + \left(\frac{w^{1-\gamma}(g*I_\delta)'(w)}{((g*I_\delta)(w))^{1-\gamma}} \right)^{\frac{1}{\lambda}} \right) \right] \right| < \frac{\alpha\pi}{2}, \quad (w \in U), \tag{2.2}$$

$$(0 < \alpha \leq 1,\, 0 < \lambda \leq 1,\, \gamma \geq 0,\, 0 \leq \delta < 1),$$

where the function $g = f^{-1}$ is given by (1.2).

Remark 1. It should be remarked that the family $\Omega_\Sigma(\lambda,\gamma,\delta;\alpha)$ is a generalization of well-known families consider earlier. These families are:

(1) For $\lambda = 1$ and $\delta = \frac{1}{2}$, the family $\Omega_\Sigma(\lambda,\gamma,\delta;\alpha)$ reduce to the family $P_\Sigma(\alpha,\gamma)$ which was introduced by Prema and Keerthi [7];
(2) For $\lambda = 1$, $\gamma = 0$ and $\delta = \frac{1}{2}$, the family $\Omega_\Sigma(\lambda,\gamma,\delta;\alpha)$ reduce to the family $S^*_\Sigma(\alpha)$ which was given by Brannan and Taha [2];
(3) For $\lambda = \gamma = 1$ and $\delta = \frac{1}{2}$, the family $\Omega_\Sigma(\lambda,\gamma,\delta;\alpha)$ reduce to the family $\mathcal{H}^\alpha_\Sigma$ which was investigated by Srivastava et al. [18].

Theorem 1. *Let $f \in \Omega_\Sigma(\lambda,\gamma,\delta;\alpha)$ $(0 < \alpha \leq 1,\, 0 < \lambda \leq 1,\, \gamma \geq 0,\, 0 \leq \delta < 1)$ be given by (1.1). Then*

$$|a_2| \leq \frac{2\alpha\lambda}{\sqrt{\alpha\lambda(\gamma+2)(\lambda+1)(1-\delta)\,(2\gamma(1-\delta)+1) + \Upsilon(\alpha,\lambda)\,(\gamma+1)^2\,(1-\delta)^2}}$$

and

$$|a_3| \leq \frac{4\alpha\lambda}{(\lambda+1)(1-\delta)}\left[\frac{\alpha\lambda}{(\gamma+1)^2(\lambda+1)(1-\delta)}+\frac{1}{(\gamma+2)(3-2\delta)}\right],$$

where

$$\Upsilon(\alpha,\lambda)=2\alpha(1-\lambda)+(1-\alpha)(\lambda+1)^2. \tag{2.3}$$

Proof. It follows from conditions (2.1) and (2.2) that

$$\frac{1}{2}\left(\frac{z^{1-\gamma}(f*I_\delta)'(z)}{((f*I_\delta)(z))^{1-\gamma}}+\left(\frac{z^{1-\gamma}(f*I_\delta)'(z)}{((f*I_\delta)(z))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right)=[p(z)]^\alpha \tag{2.4}$$

and

$$\frac{1}{2}\left(\frac{w^{1-\gamma}(g*I_\delta)'(w)}{((g*I_\delta)(w))^{1-\gamma}}+\left(\frac{w^{1-\gamma}(g*I_\delta)'(w)}{((g*I_\delta)(w))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right)=[q(w)]^\alpha, \tag{2.5}$$

where $g=f^{-1}$ and p, q in $\mathcal{P}$ have the following series representations:

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots \tag{2.6}$$

and

$$q(w)=1+q_1w+q_2w^2+q_3w^3+\cdots. \tag{2.7}$$

Comparing the corresponding coefficients of (2.4) and (2.5) yields

$$\frac{(\gamma+1)(\lambda+1)(1-\delta)}{\lambda}a_2=\alpha p_1, \tag{2.8}$$

$$\frac{(\gamma+2)(\lambda+1)(1-\delta)(3-2\delta)}{2\lambda}a_3 + \frac{\left[\lambda(\gamma+2)(\gamma-1)(\lambda+1)+(\gamma+1)^2(1-\lambda)\right](1-\delta)^2}{\lambda^2}a_2^2=\alpha p_2+\frac{\alpha(\alpha-1)}{2}p_1^2, \tag{2.9}$$

$$-\frac{(\gamma+1)(\lambda+1)(1-\delta)}{\lambda}a_2=\alpha q_1 \tag{2.10}$$

and

$$\frac{(\gamma+2)(\lambda+1)(1-\delta)(3-2\delta)}{2\lambda}\left(2a_2^2-a_3\right) + \frac{\left[\lambda(\gamma+2)(\gamma-1)(\lambda+1)+(\gamma+1)^2(1-\lambda)\right](1-\delta)^2}{\lambda^2}a_2^2=\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2. \tag{2.11}$$

In view of (2.8) and (2.10), we conclude that

$$p_1=-q_1 \tag{2.12}$$

and

$$\frac{2(\gamma+1)^2(\lambda+1)^2(1-\delta)^2}{\lambda^2}a_2^2=\alpha^2(p_1^2+q_1^2). \tag{2.13}$$

Also, by using (2.9) and (2.11), together with (2.13), we find that

$$\left(\frac{(\gamma+2)(\lambda+1)(1-\delta)(3-2\delta)}{\lambda}+\frac{2\left[\lambda(\gamma+2)(\gamma-1)(\lambda+1)+(\gamma+1)^2(1-\lambda)\right](1-\delta)^2}{\lambda^2}\right)a_2^2$$
$$=\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}\left(p_1^2+q_1^2\right)=\alpha(p_2+q_2)+\frac{(\alpha-1)(\gamma+1)^2(\lambda+1)^2(1-\delta)^2}{\alpha\lambda^2}a_2^2.$$

Further computations show that

$$a_2^2=\frac{\alpha^2\lambda^2(p_2+q_2)}{\alpha\lambda(\gamma+2)(\lambda+1)(1-\delta)(2\gamma(1-\delta)+1)+\Upsilon(\alpha,\lambda)(\gamma+1)^2(1-\delta)^2}, \tag{2.14}$$

where $\Upsilon(\alpha,\lambda)$ is given by (2.3).
By taking the absolute value of (2.14) and applying Lemma 1 for the coefficients p_2 and q_2, we have

$$|a_2|\leq\frac{2\alpha\lambda}{\sqrt{\alpha\lambda(\gamma+2)(\lambda+1)(1-\delta)(2\gamma(1-\delta)+1)+\Upsilon(\alpha,\lambda)(\gamma+1)^2(1-\delta)^2}}.$$

To determinate the bound on $|a_3|$, by subtracting (2.11) from (2.9), we get

$$\frac{(\gamma+2)(\lambda+1)(1-\delta)(3-2\delta)}{\lambda}\left(a_3-a_2^2\right)=\alpha(p_2-q_2)+\frac{\alpha(\alpha-1)}{2}\left(p_1^2-q_1^2\right). \tag{2.15}$$

Now, substituting the value of a_2^2 from (2.13) into (2.15) and using (2.12), we deduce that

$$a_3=\frac{\alpha^2\lambda^2\left(p_1^2+q_1^2\right)}{2(\gamma+1)^2(\lambda+1)^2(1-\delta)^2}+\frac{\alpha\lambda(p_2-q_2)}{(\gamma+2)(\lambda+1)(1-\delta)(3-2\delta)}. \tag{2.16}$$

Taking the absolute value of (2.16) and applying Lemma 1 once again for the coefficients p_1, p_2, q_1 and q_2, it follows that

$$|a_3|\leq\frac{4\alpha\lambda}{(\lambda+1)(1-\delta)}\left[\frac{\alpha\lambda}{(\gamma+1)^2(\lambda+1)(1-\delta)}+\frac{1}{(\gamma+2)(3-2\delta)}\right],$$

□

Remark 2. In Theorem 1, if we choose

1) $\lambda=1$ and $\delta=\frac{1}{2}$, then we have the results which was given by Prema and Keerthi [7, Theorem 2.2];
(2) $\lambda=1$, $\gamma=0$ and $\delta=\frac{1}{2}$, then we have the results obtained by Murugusundaramoorthy et al. [6, Corollary 6];

(3) $\lambda=\gamma=1$ and $\delta=\frac{1}{2}$, then we obtain the results obtained by Srivastava et al. [18, Theorem 1].

3. Coefficient estimates for the functions family $\Omega_{\Sigma}^{*}(\lambda,\gamma,\delta;\beta)$

Definition 2. A function $f\in\Sigma$ given by (1.1) is called in the family $\Omega_{\Sigma}^{*}(\lambda,\gamma,\delta;\beta)$ if it fulfills the conditions:

$$Re\left\{\frac{1}{2}\left(\frac{z^{1-\gamma}(f*I_{\delta})'(z)}{((f*I_{\delta})(z))^{1-\gamma}}+\left(\frac{z^{1-\gamma}(f*I_{\delta})'(z)}{((f*I_{\delta})(z))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right)\right\}>\beta,\quad (z\in U) \tag{3.1}$$

and

$$Re\left\{\frac{1}{2}\left(\frac{w^{1-\gamma}(g*I_{\delta})'(w)}{((g*I_{\delta})(w))^{1-\gamma}}+\left(\frac{w^{1-\gamma}(g*I_{\delta})'(w)}{((g*I_{\delta})(w))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right)\right\}>\beta,\quad (w\in U), \tag{3.2}$$

$$(0\leq\beta<1,\,0<\lambda\leq 1,\,\gamma\geq 0,\,0\leq\delta<1),$$

where the function $g=f^{-1}$ is given by (1.2).

Remark 3. It should be remarked that the family $\Omega_{\Sigma}^{*}(\lambda,\gamma,\delta;\beta)$ is a generalization of well-known families consider earlier. These families are:

(1) For $\lambda=1$ and $\delta=\frac{1}{2}$, the family $\Omega_{\Sigma}^{*}(\lambda,\gamma,\delta;\beta)$ reduce to the family $P_{\Sigma}(\beta,\gamma)$ which was introduced by Prema and Keerthi [7];
(2) For $\lambda=1$, $\gamma=0$ and $\delta=\frac{1}{2}$, the family $\Omega_{\Sigma}^{*}(\lambda,\gamma,\delta;\beta)$ reduce to the family $S_{\Sigma}^{*}(\beta)$ which was given by Brannan and Taha [2];
(3) For $\lambda=\gamma=1$ and $\delta=\frac{1}{2}$, the family $\Omega_{\Sigma}^{*}(\lambda,\gamma,\delta;\beta)$ reduce to the family $\mathcal{H}_{\Sigma}(\beta)$ which was investigated by Srivastava et al. [18].

Theorem 2. *Let $f\in\Omega_{\Sigma}^{*}(\lambda,\gamma,\delta;\beta)$ $(0\leq\beta<1,\,0<\lambda\leq 1,\,\gamma\geq 0,\,0\leq\delta<1)$ be given by (1.1). Then*

$$|a_2|\leq\frac{2\lambda\sqrt{1-\beta}}{\sqrt{\lambda(\gamma+2)(\lambda+1)(1-\delta)\left(2\gamma(1-\delta)+1\right)+2(1-\lambda)\left(\gamma+1\right)^{2}\left(1-\delta\right)^{2}}}$$

and

$$|a_3|\leq\frac{4\lambda\left(1-\beta\right)}{(\lambda+1)(1-\delta)}\left[\frac{\lambda\left(1-\beta\right)}{(\gamma+1)^{2}\left(\lambda+1\right)\left(1-\delta\right)}+\frac{1}{(\gamma+2)(3-2\delta)}\right].$$

Proof. In the light of the conditions (3.1) and (3.2), there are $p,q\in\mathcal{P}$ such that

$$\frac{1}{2}\left(\frac{z^{1-\gamma}(f*I_{\delta})'(z)}{((f*I_{\delta})(z))^{1-\gamma}}+\left(\frac{z^{1-\gamma}(f*I_{\delta})'(z)}{((f*I_{\delta})(z))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right)=\beta+(1-\beta)p(z) \tag{3.3}$$

and

$$\frac{1}{2}\left(\frac{w^{1-\gamma}(g*I_\delta)'(w)}{((g*I_\delta)(w))^{1-\gamma}}+\left(\frac{w^{1-\gamma}(g*I_\delta)'(w)}{((g*I_\delta)(w))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right)=\beta+(1-\beta)q(w), \tag{3.4}$$

where $p(z)$ and $q(w)$ have the forms (2.6) and (2.7), respectively. Comparing the corresponding coefficients in (3.3) and (3.4) yields

$$\frac{(\gamma+1)(\lambda+1)(1-\delta)}{\lambda}a_2=(1-\beta)p_1, \tag{3.5}$$

$$\frac{(\gamma+2)(\lambda+1)(1-\delta)(3-2\delta)}{2\lambda}a_3 + \frac{\left[\lambda(\gamma+2)(\gamma-1)(\lambda+1)+(\gamma+1)^2(1-\lambda)\right](1-\delta)^2}{\lambda^2}a_2^2=(1-\beta)p_2, \tag{3.6}$$

$$-\frac{(\gamma+1)(\lambda+1)(1-\delta)}{\lambda}a_2=(1-\beta)q_1 \tag{3.7}$$

and

$$\frac{(\gamma+2)(\lambda+1)(1-\delta)(3-2\delta)}{2\lambda}\left(2a_2^2-a_3\right) + \frac{\left[\lambda(\gamma+2)(\gamma-1)(\lambda+1)+(\gamma+1)^2(1-\lambda)\right](1-\delta)^2}{\lambda^2}a_2^2=(1-\beta)q_2. \tag{3.8}$$

From (3.5) and (3.7), we get

$$p_1=-q_1 \tag{3.9}$$

and

$$\frac{2(\gamma+1)^2(\lambda+1)^2(1-\delta)^2}{\lambda^2}a_2^2=(1-\beta)^2(p_1^2+q_1^2). \tag{3.10}$$

Adding (3.6) and (3.8), we obtain

$$\left(\frac{(\gamma+2)(\lambda+1)(1-\delta)(3-2\delta)}{\lambda} + \frac{2\left[\lambda(\gamma+2)(\gamma-1)(\lambda+1)+(\gamma+1)^2(1-\lambda)\right](1-\delta)^2}{\lambda^2}\right)a_2^2=(1-\beta)(p_2+q_2). \tag{3.11}$$

Hence, we find that

$$a_2^2=\frac{\lambda^2(1-\beta)(p_2+q_2)}{\lambda(\gamma+2)(\lambda+1)(1-\delta)(2\gamma(1-\delta)+1)+2(1-\lambda)(\gamma+1)^2(1-\delta)^2}.$$

By applying Lemma 1 for the coefficients p_2 and q_2, we deduce that

$$|a_2| \leq \frac{2\lambda\sqrt{1-\beta}}{\sqrt{\lambda(\gamma+2)(\lambda+1)(1-\delta)\left(2\gamma(1-\delta)+1\right)+2(1-\lambda)\left(\gamma+1\right)^2\left(1-\delta\right)^2}}.$$

To determinate the bound on $|a_3|$, by subtracting (3.8) from (3.6), we get

$$\frac{(\gamma+2)(\lambda+1)(1-\delta)(3-2\delta)}{\lambda}\left(a_3-a_2^2\right)=(1-\beta)\left(p_2-q_2\right),$$

or equivalently

$$a_3=a_2^2+\frac{\lambda(1-\beta)\left(p_2-q_2\right)}{(\gamma+2)(\lambda+1)(1-\delta)(3-2\delta)}. \tag{3.12}$$

Substituting the value of a_2^2 from (3.10) into (3.12), it follows that

$$a_3=\frac{\lambda^2\left(1-\beta\right)^2\left(p_1^2+q_1^2\right)}{2\left(\gamma+1\right)^2\left(\lambda+1\right)^2\left(1-\delta\right)^2}+\frac{\lambda\left(1-\beta\right)\left(p_2-q_2\right)}{(\gamma+2)(\lambda+1)(1-\delta)(3-2\delta)}.$$

By applying Lemma 1 once again for the coefficients p_1, p_2, q_1 and q_2, we deduce that

$$|a_3| \leq \frac{4\lambda\left(1-\beta\right)}{(\lambda+1)(1-\delta)}\left[\frac{\lambda\left(1-\beta\right)}{\left(\gamma+1\right)^2\left(\lambda+1\right)\left(1-\delta\right)}+\frac{1}{(\gamma+2)(3-2\delta)}\right].$$

□

Remark 4. In Theorem 2, if we choose

(1) $\lambda=1$ and $\delta=\frac{1}{2}$, then we have the results which was given by Prema and Keerthi [7, Theorem 3.2];
(2) $\lambda=1$, $\gamma=0$ and $\delta=\frac{1}{2}$, then we have the results obtained by Murugusundaramoorthy et al. [6, Corollary 7];
(3) $\lambda=\gamma=1$ and $\delta=\frac{1}{2}$, then we obtain the results obtained by Srivastava et al. [18, Theorem 2].

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