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A NOTE ON SKEW LIE PRODUCT OF PRIME RING WITH INVOLUTION

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Abstract. Let R be a ring with involution. The skew Lie product of $a,b \in R$ is defined by $\nabla[a,b] = ab - ba^*$. In the present paper we study prime ring with involution satisfying identities involving skew Lie product and left centralizers.

2010 *Mathematics Subject Classification:* 16W10; 16N60; 16W25

Keywords: prime ring, left centralizer, involution, skew Lie product

1. Introduction

Through out this paper R will be a prime ring with involution $*$, $Q_m r$ and C denotes the maximal right ring of quotient and the extended centroid of R, $H(R)$ will be the set of hermitian elements and $S(R)$ will be the set of skew hermitian elements of R. If $\mathrm{char}(R) \neq 2$, involution $*$ is said to be of the first kind if $Z(R) \subset H(R)$, otherwise it is said to be of second kind. In the second case $S(R) \cap Z(R) \neq (0)$. We refer reader to [6, 13] for justification and amplification for the above mentioned notations and key definitions.

Following [22], an additive mapping $T : R \to R$ is said to be a left (resp. right) centralizer (multiplier) of R if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) for all $x, y \in R$. If T is both left as well as the right centralizer of R, it is said to be the centralizer of R. Considerable work has been done on left (resp. right) centralizers (multipliers) in prime and semiprime rings during the last few decades (see for example [2, 5, 12, 17, 21, 22]) where further references can be found. The relationship between the commutativity of the ring R and certain specific types of maps on R has been extensively studied over the last few decades. The first result in this direction is due to Divinsky [11], who proved that a simple artinian ring is commutative if it has a commuting non-trivial automorphism. Further refinements and extension has been done by a number of authors in various directions (viz., [1, 3, 4, 7–10, 14, 18]). Recently, Ali and Dar [2] proved that if a prime ring with involution of $\mathrm{char}(R) \neq 2$ admits a left centralizer $T : R \to R$ satisfying any one of the following conditions: (i)

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$T([x,x^*]) = 0$ (ii) $T(xox^*) = 0$ (iii) $T([x,x^*]) \pm [x,x^*] = 0$ (iv) $T(xox^*) \pm (xox^*) = 0$ for all $x \in R$, then R is commutative.

Let R be a ring with an involution $*$. For $a,b \in R$, denote by $\nabla[a,b] = ab - ba^*$ the skew Lie product. This kind of product is found playing a more and more important role in some research topics such as representing quadratic functionals with sesquilinear functionals, and its study has attracted many authors attention (see [19,20] and the references therein). Motivated by the theory of rings (and algebras) equipped with a Lie product or a Jordan product, Molnár [15] initiated the systematic study of this skew Lie product, and studied the relation between subspaces and ideals of $B(H)$, the algebra of all bounded linear operators acting on a Hilbert space H. Here, an additive map f on R is skew centralizing if $\nabla[a, f(a)] \in Z(R)$ holds for all $a \in R$. The purpose here is to study the skew centralizing left centralizer on prime ring with involution. Moreover some other results involving skew Lie product with respect to left centralizers in prime rings with involution have also been studied.

We shall restrict our attention on left centralizers, since all results presented in this article are also true for right centralizers because of left-right symmetry.

2. Results

We begin with the following lemmas, which are essential to prove our main results.

Lemma 1 ([16]). *Let R be a prime ring with involution of the second kind. Then $[x,x^*] \in Z(R)$ for all $x \in R$ if and only if R is commutative.*

Lemma 2 ([16]). *Let R be a prime ring with involution of the second kind. Then $x \circ x^* \in Z(R)$ for all $x \in R$ if and only if R is commutative.*

Lemma 3 ([13]). *Suppose that the elements a_i, b_i in the central closure of a prime ring R satisfy $\sum a_i x b_i = 0$ for all $x \in R$. If $b_i \neq 0$ for some i, then a_i's are C-independent.*

Lemma 4. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If $\nabla[x,x^*] \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We have $\nabla[x,x^*] \in Z(R)$ for all $x \in R$. On linearizing, we get

$$\nabla[x,y^*] + \nabla[y,x^*] \in Z(R) \text{ for all } x,y \in R. \tag{2.1}$$

Replacing y by ky, where $k \in S(R) \cap Z(R)$ in (2.1) and using (2.1), we obtain $2yx^*k \in Z(R)$ for all $x,y \in R$ and $k \in S(R) \cap Z(R)$. Replacing x by x^* and y by $h \in H(R) \cap Z(R)$, we get $2xhk \in Z(R)$ for all $x \in R$, $h \in H(R) \cap Z(R)$ and $k \in S(R) \cap Z(R)$. Since $\text{char}(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, this implies that $x \in Z(R)$ for all $x \in R$. That is, R is commutative. □

Lemma 5. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a nonzero left centralizer $T : R \to R$ such that $T(x) \circ x^* \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. By the given hypothesis we have

$$T(x)\circ x^* \in Z(R) \text{ for all } x \in R. \tag{2.2}$$

Linearizing (2.2), we get

$$T(x)\circ y^* + T(y)\circ x^* \in Z(R) \text{ for all } x,y \in R. \tag{2.3}$$

Replacing y by ky in (2.3) and use (2.3), we get

$$2(T(y)\circ x^*)k \in Z(R) \text{ for all } x,y \in R \text{ and } k \in S(R)\cap Z(R). \tag{2.4}$$

Since $\mathrm{char}(R) \neq 2$ and $S(R)\cap Z(R) \neq (0)$, this implies that $T(y)\circ x^* \in Z(R)$ for all $x,y \in R$. Replacing x by h, where $h \in H(R)\cap Z(R)$, we get $2T(y)h \in Z(R)$ for all $y \in R$ and $h \in H(R)\cap Z(R)$. Since $\mathrm{char}(R) \neq 2$ and $S(R)\cap Z(R) \neq (0)$, we get $T(y) \in R$. This can be further written as $[T(y),r] = 0$ for all $y,r \in R$. Replacing y by yw and using the last relation, since $T \neq 0$, we get R is commutative. □

Theorem 1. *Let R be a prime ring with involution of the second kind such that $\mathrm{char}(R) \neq 2$. If R admits a non zero left centralizer $T : R \to R$ such that $\nabla[x,T(x^*)] \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. By the given hypothesis

$$\nabla[x,T(x^*)] \in Z(R) \text{ for all } x \in R. \tag{2.5}$$

Linearizing (2.5), we get

$$\nabla[x,T(y^*)] + \nabla[y,T(x^*)] \in Z(R) \text{ for all } x,y \in R. \tag{2.6}$$

That is,

$$xT(y^*) - T(y^*)x^* + yT(x^*) - T(x^*)y^* \in Z(R) \text{ for all } x,y \in R.$$

This further implies that

$$[xT(y^*),r] - [T(y^*)x^*,r] + [yT(x^*),r] - [T(x^*)y^*,r] = 0 \text{ for all } x,y,r \in R.$$

Thus

$$\begin{aligned} x[T(y^*),r] + [x,r]T(y^*) - T(y^*)[x^*,r] - [T(y^*),r]x^* + y[T(x^*),r] + [y,r]T(x^*) \\ - T(x^*)[y^*,r] - [T(x^*),r]y^* = 0 \text{ for all } x,y,r \in R. \end{aligned} \tag{2.7}$$

Replacing y by ky in (2.7), where $k \in S(R)\cap Z(R)$ and using (2.7), we get

$$2(y[T(x^*),r] + [y,r]T(x^*))k = 0 \text{ for all } x,y,r \in R \text{ and } k \in S(R)\cap Z(R).$$

Since $\mathrm{char}(R) \neq 2$ and $S(R)\cap Z(R) \neq (0)$, this implies that

$$y[T(x^*),r] + [y,r]T(x^*) = 0 \text{ for all } x,y,r \in R.$$

Taking $y = z$, where $z \in Z(R)$, we get $[T(x^*),r]z = 0$. Then by the primeness of R and the fact that $S(R)\cap Z(R) \neq (0)$, we have $[T(x^*),r] = 0$ for all $x,r \in R$. Taking $x = x^*$, we obtain

$$[T(x),r] = 0 \text{ for all } x,r \in R. \tag{2.8}$$

Replacing x by xy in (2.8), where $y \in R$ and using (2.8), we get $T(x)[y,r] = 0$ for all $x,y,r \in R$. Replacing x by xw, where $w \in R$. We get $T(x)w[y,r] = 0$ for all $x,y,w,r \in R$. Thus by the primeness of R, we have either $T(x) = 0$ for all $x \in R$ or $[y,r] = 0$ for all $y,r \in R$. Since $T \neq 0$, we get R is commutative. □

Theorem 2. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a non zero left centralizer $T : R \to R$ such that $T(\nabla[x,x^*]) \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We have

$$T(\nabla[x,x^*]) \in Z(R) \text{ for all } x \in R. \tag{2.9}$$

Linearizing (2.9), we get

$$T(\nabla[x,y^*]) + T(\nabla[y,x^*]) \in Z(R) \text{ for all } x,y \in R. \tag{2.10}$$

That is,

$$T(xy^* - y^*x^*) + T(yx^* - x^*y^*) \in Z(R) \text{ for all } x,y \in R.$$

This can be further written as

$$\begin{aligned} &T(x)[y^*,r] + [T(x),r]y^* - T(y^*)[x^*,r] - [T(y^*),r]x^* + T(y)[x^*,r] \\ &+ [T(y),r]x^* - T(x^*)[y^*,r] - [T(x^*),r]y^* = 0 \text{ for all } x,y,r \in R. \end{aligned} \tag{2.11}$$

Replacing y by ky in (2.11) where $k \in S(R) \cap Z(R)$ and using (2.11), we get

$$2(T(y)[x^*,r] + [T(y),r]x^*)k = 0 \text{ for all } x,y,r \in R \text{ and } k \in S(R) \cap Z(R).$$

Since $\text{char}(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, we arrive at

$$T(y)[x^*,r] + [T(y),r]x^* = 0 \text{ for all } x,y,r \in R.$$

Taking $x = h$, where $h \in H(R) \cap Z(R)$, we get $[T(y),r] = 0$ for all $y,r \in R$. Since $S(R) \cap Z(R) \neq (0)$, primeness of R implies that $[T(y),r] = 0$ for all $y,r \in R$ which is same as (2.8). Thus proceeding as we did in the Theorem 1, we get the required result. □

Theorem 3. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a left centralizer $T : R \to R$ such that $\nabla[x,T(x^*)] \pm \nabla[x,x^*] \in Z(R)$ for all $x \in R$, then either T is centralizer or R is commutative.*

Proof. We have

$$\nabla[x,T(x^*)] \pm \nabla[x,x^*] \in Z(R) \text{ for all } x \in R. \tag{2.12}$$

If $T = 0$, then by Lemma 4, we get R is commutative. Now consider $T \neq 0$. On linearizing (2.12), we get

$$\nabla[x,T(y^*)] + \nabla[y,T(x^*)] \pm \nabla[x,y^*] \pm \nabla[y,x^*] \in Z(R) \text{ for all } x,y \in R. \tag{2.13}$$

Replacing y by ky in (2.13), where $k \in S(R) \cap Z(R)$, we have

$$-*[x,T(y^*)]k + yT(x^*)k + T(x^*)y^*k \mp *[x,y^*]k \pm yx^*k \pm x^*y^*k \in Z(R)$$

for all $x,y \in R$ and $k \in S(R) \cap Z(R)$. Making use of (2.13), we get $2(yT(x^*) \pm yx^*)k \in Z(R)$ for all $x,y \in R$ and $k \in S(R) \cap Z(R)$. Since $\mathrm{char}(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, we obtain $yT(x^*) \pm yx^* \in Z(R)$ for all $x,y \in R$. This can be further written as

$$[yT(x^*), r] \pm [yx^*, r] = 0 \text{ for all } x,y,r \in R.$$

Taking $y = z$ and $x = x^*$ and using primeness of R and the fact that $S(R) \cap Z(R) \neq (0)$, we obtain

$$[T(x), r] \pm [x, r] = 0 \text{ for all } x,r \in R.$$

Taking $r = x$, we get

$$[T(x), x] = 0 \text{ for all } x \in R. \tag{2.14}$$

Linearizing (2.14), we have

$$[T(x), y] + [T(y), x] = 0 \text{ for all } x,y \in R. \tag{2.15}$$

Replacing x by xw in (2.15), where $w \in R$ and using (2.15), we get

$$T(x)[w,y] + x[T(y), w] = 0 \text{ for all } x,y,w \in R. \tag{2.16}$$

Replacing y by ym in (2.16), where $m \in R$ and using (2.16), we obtain

$$(T(x)y - xT(y))[w,m] = 0 \text{ for all } x,y,w,m \in R.$$

Replacing m by mu where $u \in R$ and using the previous expression, we arrive at

$$(T(x)y - xT(y))m[w,u] = 0 \text{ for all } x,y,w,m,u \in R. \tag{2.17}$$

Thus by the primeness of R, we have either T is a centralizer or R is commutative. □

Theorem 4. *Let R be a prime ring with involution of the second kind such that $\mathrm{char}(R) \neq 2$. If R admits a left centralizer $T : R \to R$ such that $\nabla[x, T(x^*)] \pm x \circ x^* \in Z(R)$ for all $x \in R$, then either T is centralizer or R is commutative.*

Proof. We have

$$\nabla[x, T(x^*)] \pm x \circ x^* \in Z(R) \text{ for all } x \in R. \tag{2.18}$$

If $T = 0$, then by Lemma 2, we have R is commutative. Now consider $T \neq 0$. On linearizing (2.18), we get

$$\nabla[x, T(y^*)] + \nabla[y, T(x^*)] \pm x \circ y^* \pm y \circ x^* \in Z(R) \text{ for all } x,y \in R. \tag{2.19}$$

Replacing y by ky in (2.19) where $k \in S(R) \cap Z(R)$, we get

$$-\nabla[x, T(y^*)]k + yT(x^*)k + T(x^*)y^*k \mp (x \circ y^*)k \pm (y \circ x^*)k \in Z(R)$$

for all $x,y \in R$ and $k \in S(R) \cap Z(R)$. Making use of (2.19), we obtain

$$2(yT(x^*) \pm y \circ x^*)k \in Z(R) \text{ for all } x,y \in R \text{ and } k \in S(R) \cap Z(R).$$

Since $\mathrm{char}(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, we get

$$yT(x^*) \pm y \circ x^* \in Z(R) \text{ for all } x,y \in R.$$

Taking $x = x^*$ and $y = z$, where $z \in S(R) \cap Z(R)$, we obtain

$$(T(x) \pm 2x)z \in Z(R) \text{ for all } x \in R \text{ and } z \in Z(R).$$

Using the primeness of R, we get $T(x) \pm 2x \in Z(R)$ for all $x \in R$. This can be further written as $[T(x), r] \pm [2x, r] = 0$ for all $x, r \in R$. Taking $r = x$, we get $[T(x), x] \in R$ for all $x \in R$, which is same as equation (2.14). Thus proceeding on similar lines as in the previous theorem, we get either T is centralizer or R is commutative. □

Theorem 5. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a left centralizer $T : R \to R$ such that $T(x) \circ x^* \pm \nabla[x, x^*] \in Z(R)$, then either T is centralizer or R is commutative.*

Proof. We have

$$T(x) \circ x^* \pm \nabla[x, x^*] \in Z(R) \quad for \quad all \quad x \in R. \tag{2.20}$$

If $T = 0$ then by Lemma 4, we get R is commutative. Now consider $T \neq 0$, linearizing (2.20), we get

$$T(x) \circ y^* + T(y) \circ x^* \pm \nabla[x, y^*] \pm \nabla[y, x^*] \in Z(R) \text{ for all } x, y \in R. \tag{2.21}$$

Replacing y by yk in (2.21), where $k \in S(R) \cap Z(R)$, we get

$$(-T(x) \circ y^*)k + (T(y) \circ x^*)k \mp \nabla[x, y^*]k \pm yx^*k \pm x^*y^*k \in Z(R)$$

for all $x, y \in R$. Using (2.21), we have

$$2(T(y) \circ x^* \pm yx^*)k \in Z(R) \text{ for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

Since $\text{char}(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, we get

$$T(y) \circ x^* \pm yx^* \in Z(R) \text{ for all } x, y \in R.$$

Taking $x = h$, where $h \in H(R) \cap Z(R)$, we get

$$(2[T(y), r] \pm [y, r])h \in Z(R) \text{ for all } y, r \in R \text{ and } h \in H(R) \cap Z(R).$$

Since $S(R) \cap Z(R) \neq (0)$, this implies that

$$2[T(y), r] \pm [y, r] \in Z(R) \text{ for all } y, r \in R.$$

Thus $2[T(y), y] = 0$ for all $y \in R$. Since $\text{char}(R) \neq 2$, this implies that $[T(y), y] = 0$ for all $y \in R$, which is same as equation (2.14) and the result follows as in the Theorem 3. □

Theorem 6. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a left centralizer $T : R \to R$ such that $\nabla[x, T(x \circ x^*)] \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. we have

$$\nabla[x,T(x\circ x^*)]\in Z(R) \text{ for all } x\in R. \tag{2.22}$$

Replacing x by $x+y$ in (2.22), we get

$$\begin{aligned} &xT(x\circ y^*)+xT(y\circ x^*)+xT(y\circ y^*)+yT(x\circ x^*)+yT(x\circ y^*) \\ &+yT(y\circ x^*)-T(x\circ y^*)x^*-T(y\circ x^*)x^*-T(y\circ y^*)x^*-T(x\circ x^*)y^* \\ &-T(x\circ y^*)y^*-T(y\circ x^*)y^*\in Z(R) \text{ for all } x,y\in R. \end{aligned} \tag{2.23}$$

Replace x by $-x$ in (2.23) and adding with (2.23), we get

$$\begin{aligned} &2(xT(x\circ y^*)+xT(y\circ x^*)+yT(x\circ x^*)-T(x\circ y^*)x^* \\ &-T(y\circ x^*)x^*-T(x\circ x^*)y^*)\in Z(R) \text{ for all } x,y\in R. \end{aligned}$$

Since $\mathrm{char}(R)\neq 2$, this implies that

$$\begin{aligned} &xT(x\circ y^*)+xT(y\circ x^*)+yT(x\circ x^*)-T(x\circ y^*)x^* \\ &-T(y\circ x^*)x^*-T(x\circ x^*)y^*\in Z(R) \text{ for all } x,y\in R. \end{aligned} \tag{2.24}$$

Replacing y by yk in (2.24) and using (2.24), we get

$$2(xT(y\circ x^*)+yT(x\circ x^*)-T(y\circ x^*)x^*)k\in Z(R)$$

for all $x,y\in R$ and $k\in S(R)\cap Z(R)$. Since $\mathrm{char}(R)\neq 2$ and $S(R)\cap Z(R)\neq(0)$, this implies that

$$xT(y\circ x^*)+yT(x\circ x^*)-T(y\circ x^*)x^*\in Z(R) \text{ for all } x,y\in R. \tag{2.25}$$

Replacing x by kx in (2.25), where $k\in S(R)\cap Z(R)$ and using (2.25), we get $2(T(y\circ x^*)x^*)k^2\in Z(R)$ for all $x,y\in R$ and $k\in S(R)\cap Z(R)$. Since $\mathrm{char}(R)\neq 2$ and $S(R)\cap Z(R)\neq(0)$, we arrive at $T(y\circ x^*)x^*\in Z(R)$ for all $x,y\in R$. Taking $x=h$, where $h\in H(R)\cap Z(R)$, we get $2T(y)h^2\in Z(R)$ for all $y\in R$ and $h\in H(R)\cap Z(R)$. Since $\mathrm{char}(R)\neq 2$ and $S(R)\cap Z(R)\neq(0)$, this implies that $T(y)\in Z(R)$ for all $y\in R$. This can be further written as $[T(y),r]=0$ for all $y,r\in R$. This is same as equation (2.8). Hence following the same steps as before, we get R is commutative. □

Theorem 7. *Let R be a prime ring with involution of the second kind such that $\mathrm{char}(R)\neq 2$. If R admits a left centralizer $T:R\to R$ such that $T(\nabla[x,x^*])\pm\nabla[x,x^*]\in Z(R)$ for all $x\in R$, then either T is centralizer or R is commutative.*

Proof. We have

$$T(\nabla[x,x^*])\pm\nabla[x,x^*]\in Z(R) \text{ for all } x\in R. \tag{2.26}$$

If $T=0$, then in view of Lemma 4, we get R is commutative. Now consider $T\neq 0$. Linearizing (2.26), we get

$$T(\nabla[x,y^*])+T(\nabla[y,x^*])\pm\nabla[x,y^*]\pm\nabla[y,x^*]\in Z(R) \text{ for all } x,y\in R. \tag{2.27}$$

Replacing y by ky in (2.27), where $k\in S(R)\cap Z(R)$ and using (2.27), we get

$$2(T(y)x^*\pm yx^*)k\in Z(R) \text{ for all } x,y\in R \text{ and } k\in S(R)\cap Z(R). \tag{2.28}$$

Since $\mathrm{char}(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, we get

$$T(y)x^* \pm yx^* \in Z(R) \text{ for all } x, y \in R. \tag{2.29}$$

Taking $x = h$, where $h \in H(R) \cap Z(R)$ and using the primeness of R and the fact that $S(R) \cap Z(R) \neq (0)$, we get $T(y) \pm y \in Z(R)$ for all $y \in R$. This can be further written as $[T(y), r] \pm [y, r] = 0$ for all $y, r \in R$. Replacing r by y, we get $[T(y), y] = 0$ for all $y \in R$. Which is same as (2.14). Thus proceeding as before, we get the required result. □

Theorem 8. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits left centralizers $T_1, T_2 : R \to R$ such that $[T_1(x), T_2(x^*)] \pm \nabla[x, x^*] \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. we have

$$[T_1(x), T_2(x^*)] \pm \nabla[x, x^*] \in Z(R) \text{ for all } x \in R. \tag{2.30}$$

If either $T_1 = 0$ or $T_2 = 0$, then in view of Lemma 4, we get R be a commutative. Now consider $T_1 \neq 0$ and $T_2 \neq 0$, linearizing (2.30), we get

$$[T_1(x), T_2(y^*)] + [T_1(y), T_2(x^*)] \pm \nabla[x, y^*] \pm \nabla[y, x^*] \in Z(R) \text{ for all } x, y \in R. \tag{2.31}$$

Replacing y by yk in (2.31), where $k \in S(R) \cap Z(R)$ and using (2.31), we get

$$2([T_1(y), T_2(x^*)] \pm yx^*)k \in Z(R) \text{ for all } x, y \in R \text{ and } k \in S(R) \cap Z(R). \tag{2.32}$$

Since $\mathrm{char}(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, we get

$$[T_1(y), T_2(x^*)] \pm yx^* \in Z(R) \text{ for all } x, y \in R.$$

This can be further written as

$$[[T_1(y), T_2(x^*)], r] \pm [yx^*, r] = 0 \text{ for all } x, y, r \in R. \tag{2.33}$$

Replacing x by $xT_1(y)$ in (2.33) and using (2.33), we get

$$([T_1(y), T_2(x)] \pm yx)[T_1(y), r] = 0 \text{ for all } x, y, r \in R. \tag{2.34}$$

Replacing r by rm in (2.34) and using (2.34), we get

$$([T_1(y), T_2(x)] \pm yx)r[T_1(y), m] = 0 \text{ for all } x, y, r, m \in R. \tag{2.35}$$

Thus by the primeness of R, we have either

$$[T_1(y), T_2(x)] \pm yx = 0 \text{ for all } x, y \in R \tag{2.36}$$

or

$$[T_1(y), m] = 0 \text{ for all } y, m \in R. \tag{2.37}$$

Replace x by xr in (2.36) and using (2.36), we get $T_2(x)[T_1(y), r] = 0$ for all $x, y, r \in R$. Again replacing x by xm, where $m \in R$, we get $T_2(x)m[T_1(y), r] = 0$ for all $x, y, r, m \in R$. Hence by the primeness of R, we get either $T_2(x) = 0$ for all $x \in R$ or $[T_1(y), r] = 0$ for all $y, r \in R$. Since $T_2 \neq 0$, we have $[T_1(y), r] = 0$ for all $y, r \in R$. Replacing y by yu, where $u \in R$, we obtain $T_1(y)[u, r] = 0$ for all $y, u, r \in R$. Replacing y by yw, where $w \in R$, we get $T_1(y)w[u, r] = 0$ for all $y, w, u, r \in R$. Hence by primeness of R, we get

either $T_1(y)=0$ for all $y\in R$ or $[u,r]=0$ for all $u,r\in R$. Since $T_1\neq 0$, we get $[u,r]=0$ for all $u,r\in R$. That is, R is commutative. Similarly we can get R is commutative in case $[T_1(y),m]=0$ for all $y,m\in R$. □

Theorem 9. *Let R be a prime ring with involution of the second kind such that $char(R)\neq 2$. If R admit two left centralizers $T_1,T_2:R\to R$ such that $(\nabla[x,x^*])T_1(x)\pm T_2(x)(\nabla[x,x^*])\in Z(R)$ for all $x\in R$, then either R is commutative or $T_1(x)=\mp T_2(x)$ for all $x\in R$.*

Proof. We have

$$(\nabla[x,x^*])T_1(x)\pm T_2(x)(\nabla[x,x^*])\in Z(R)\ \text{ for all }\ x\in R. \tag{2.38}$$

If $T_1=0$ and $T_2\neq 0$ then we have

$$\pm T_2(x)(\nabla[x,x^*])\in Z(R)\ \text{ for all }\ x\in R. \tag{2.39}$$

This implies that

$$T_2(x)(\nabla[x,x^*])\in Z(R)\ \text{ for all }\ x\in R. \tag{2.40}$$

Linearizing (2.40), we get

$$\begin{aligned}T_2(x)(\nabla[x,y^*])+T_2(x)(\nabla[y,x^*])+T_2(x)(\nabla[y,y^*])+T_2(y)(\nabla[x,x^*])\\ +T_2(y)(\nabla[x,y^*])+T_2(y)(\nabla[y,x^*])\in Z(R)\ \text{ for all }\ x,y\in R.\end{aligned} \tag{2.41}$$

Replacing x by $-x$ in (2.41) and using (2.41), we get

$$2(T_2(x)(\nabla[x,y^*]+T_2(x)(\nabla[y,x^*])+T_2(y)(\nabla[x,x^*]))\in Z(R)\ \text{ for all }\ x,y\in R. \tag{2.42}$$

Since $\mathrm{char}(R)\neq 2$, this implies that

$$T_2(x)(\nabla[x,y^*])+T_2(x)(\nabla[y,x^*])+T_2(y)(\nabla[x,x^*])\in Z(R)\ \text{ for all }\ x,y\in R. \tag{2.43}$$

Replacing y by yk in (2.43), where $k\in S(R)\cap Z(R)$ and using (2.43), we get

$$2(T_2(x)yx^*+T_2(y)(\nabla[x,x^*])k\in Z(R)\ \text{ for all }\ x,y\in R\ \text{ and }\ k\in S(R)\cap Z(R).$$

Since $\mathrm{char}(R)\neq 2$ and $S(R)\cap Z(R)\neq(0)$, we get

$$T_2(x)yx^*+T_2(y)(\nabla[x,x^*])\in Z(R)\ \text{ for all }\ x,y\in R. \tag{2.44}$$

This can be further written as

$$T_2(x)yx^*+T_2(y)xx^*+T_2(y)(x^*)^2\in Z(R)\ \text{ for all }\ x,y\in R. \tag{2.45}$$

Replacing x by kx in (2.45), where $k\in S(R)\cap Z(R)$ and using (2.45), we get

$$2T_2(y)(x^*)^2k\in Z(R)\ \text{ for all }\ x,y\in R\ \text{ and }\ k\in S(R)\cap Z(R).$$

Since $\mathrm{char}(R)\neq 2$ and $S(R)\cap Z(R)\neq(0)$, this implies that $T_2(y)(x^*)^2\in Z(R)$ for all $x,y\in R$. Taking $x=h$, where $h\in H(R)\cap Z(R)$, we have $T_2(y)h^2\in Z(R)$ for all $y\in R$ and $h\in H(R)\cap Z(R)$. Since $S(R)\cap Z(R)\neq(0)$, making use of primeness of R, we get $T_2(y)\in Z(R)$ for all $y\in R$. This can be further written as $[T_2(y),r]=0$ for all $y,r\in R$. Which is same as equation (2.8). Thus proceeding as in Theorem 1, we have R is commutative.

Now suppose $T_2 = 0$ and $T_1 \neq 0$, we get $(\nabla[x,x^*])T_1(x) \in Z(R)$ for all $x \in R$. Now following the same steps as we follow after (2.40), we get R is commutative. At last consider neither $T_1 \neq 0$ nor $T_2 \neq 0$. Linearizing (2.38), we get

$$\begin{aligned}&(\nabla[x,y^*])T_1(x)+(\nabla[y,x^*])T_1(x)+(\nabla[y,y^*])T_1(x)+(\nabla[x,x^*])T_1(y)\\&+(\nabla[x,y^*])T_1(y)+(\nabla[y,x^*])T_1(y)\pm T_2(x)(\nabla[x,y^*])\pm T_2(x)(\nabla[y,x^*])\\&\pm T_2(x)(\nabla[y,y^*])\pm T_2(y)(\nabla[x,x^*])\pm T_2(y)(\nabla[x,y^*])\pm T_2(y)(\nabla[y,x^*])\\&\in Z(R)\ \text{ for all }\ x,y\in R.\end{aligned} \tag{2.46}$$

Replacing x by $-x$ in (2.46) and using (2.46), we get

$$\begin{aligned}&2((\nabla[x,y^*])T_1(x)+(\nabla[y,x^*])T_1(x)+(\nabla[x,x^*])T_1(y)\pm T_2(x)(\nabla[x,y^*])\\&\pm T_2(x)(\nabla[y,x^*])\pm T_2(y)(\nabla[x,x^*]))\in Z(R)\ \text{ for all }\ x,y\in R.\end{aligned}$$

Since $\mathrm{char}(R) \neq 2$, this implies that

$$\begin{aligned}&(\nabla[x,y^*])T_1(x)+(\nabla[y,x^*])T_1(x)+(\nabla[x,x^*])T_1(y)\pm T_2(x)(\nabla[x,y^*])\\&\pm T_2(x)(\nabla[y,x^*])\pm T_2(y)(\nabla[x,x^*])\in Z(R)\ \text{ for all }\ x,y\in R.\end{aligned} \tag{2.47}$$

Replacing x by kx in (2.47) and using (2.47), we get

$$2(xy^*T_1(x)-(x^*)^2T_1(y)\pm T_2(x)xy^*\mp T_2(y)(x^*)^2)k^2\in Z(R)$$

for all $x,y \in R$ and $k \in S(R)\cap Z(R)$. As $\mathrm{char}(R) \neq 2$ and $S(R)\cap Z(R) \neq (0)$, we get

$$xy^*T_1(x)-(x^*)^2T_1(y)\pm T_2(x)xy^*\mp T_2(y)(x^*)^2\in Z(R)\ \text{ for all }\ x,y\in R. \tag{2.48}$$

Replacing y by ky in (2.48), where $k \in S(R)\cap Z(R)$, and using (2.48), we get

$$2(-(x^*)^2T_1(y)\mp T_2(y)(x^*)^2)k\in Z(R)\ \text{ for all }\ x,y\in R\ \text{ and }\ k\in S(R)\cap Z(R).$$

Since $\mathrm{char}(R) \neq 2$ and $S(R)\cap Z(R) \neq (0)$, this implies that

$$-(x^*)^2T_1(y)\mp T_2(y)(x^*)^2\in Z(R)\ \text{ for all }\ x,y\in R.$$

Taking $x = h$, where $h \in H(R)\cap Z(R)$, we get

$$(-T_1(y)\pm T_2(y))h^2\in Z(R)\ \text{ for all }\ y\in R\ \text{ and }\ h\in H(R)\cap Z(R).$$

Since $S(R)\cap Z(R) \neq (0)$, this implies that

$$-T_1(y)\pm T_2(y)\in Z(R)\ \text{ for all }\ y\in R\ \text{ and }\ h\in H(R)\cap Z(R).$$

If we consider

$$-T_1(y)+T_2(y)\in Z(R)\ \text{ for all }\ y\in R.$$

This implies

$$T_1(y)-T_2(y)\in Z(R)\ \text{ for all }\ y\in R.$$

Hence

$$[T_1(y),r]-[T_2(y),r]=0\ \text{ for all }\ y,r\in R. \tag{2.49}$$

Replacing y by yu in (2.49) and using (2.49), we get

$$(T_1(y)-T_2(y))[u,r]=0\ \text{ for all }\ y,u,r\in R. \tag{2.50}$$

Thus by the primeness of R, we have either $T_1(y) = T_2(y)$ for all $y \in R$ or R is commutative. Similarly we get either $T_1(y) = -T_2(y)$ for all $y \in R$ or R is commutative in the case $-T_1(y) - T_2(y) \in Z(R)$ for all $y \in R$. □

Theorem 10. *Let R ba a noncommutative prime ring with involution of the second kind such that char$(R) \neq 2$. If R admits two non zero left centralizes $T_1, T_2 : R \to R$ such that $T_1(x \circ x^*) \pm \nabla[x, T_2(x^*)] \in Z(R)$ for all $x \in R$, then $T_1 = \lambda T_2$, where $\lambda \in C$.*

Proof. We have

$$T_1(x \circ x^*) \pm \nabla[x, T_2(x^*)] \in Z(R) \text{ for all } x \in R. \tag{2.51}$$

Linearizing (2.51), we get

$$T_1(x \circ y^*) + T_2(y \circ x^*) \pm \nabla[x, T_2(y^*)] \pm \nabla[y, T_2(x^*)] \in Z(R) \text{ for all } x, y \in R. \tag{2.52}$$

Replacing y by ky in (2.52), where $k \in S(R) \cap Z(R)$ and using (2.52), we get

$$2(T_1(y \circ x^*) \pm yT_2(x^*))k \in Z(R) \text{ for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

Since char$(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, this implies that

$$T_1(y \circ x^*) \pm yT_2(x^*) \in Z(R) \text{ for all } x, y \in R.$$

Taking $x = x^*$ and $y = z$, where $z \in Z(R)$, we obtain

$$(2T_1(x) \pm T_2(x))z \in Z(R) \text{ for all } x \in R \text{ and } z \in Z(R).$$

Thus by the primeness and the fact that $S(R) \cap Z(R) \neq (0)$, we have

$$2T_1(x) \pm T_2(x) \in Z(R) \text{ for all } x \in R.$$

This can be further written as

$$2[T_1(x), r] \pm [T_2(x), r] = 0 \text{ for all } x, r \in R.$$

Replacing r by $T_2(x)$, we get

$$2[T_1(x), T_2(x)] = 0 \text{ for all } x \in R.$$

Since char$(R) \neq 2$, this implies that

$$[T_1(x), T_2(x)] = 0 \text{ for all } x \in R. \tag{2.53}$$

Linearizing (2.53), we get

$$[T_1(x), T_2(y)] + [T_1(y), T_2(x)] = 0 \text{ for all } x, y \in R. \tag{2.54}$$

Replacing x by xz in (2.54) and using (2.54), we get

$$T_1(x)[z, T_2(y)] + T_2(x)[T_1(y), z] = 0 \text{ for all } x, y, z \in R. \tag{2.55}$$

Again, replacing x by xw in (2.55) and using (2.55), we get

$$T_1(x)w[z, T_2(y)] + T_2(x)w[T_1(y), z] = 0 \text{ for all } x, y, z, w \in R. \tag{2.56}$$

Thus in view of Lemma 3, we have $[z,T_2(y)]=0$ for all $y,z\in R$ or $T_1(x)=\lambda(x)T_2(x)$, where $\lambda(x)\in C$. But $[z,T_2(y)]\neq 0$ for all $z,y\in R$ since R is not commutative. Hence we get $T_1(x)=\lambda(x)T_2(x)$, where $\lambda(x)\in C$. Using this in (2.56), we get

$$\lambda(x)T_2(x)w[z,T_2(y)]+T_2(x)w[\lambda(y)T_2(y),z]=0 \text{ for all } x,y,z,w\in R.$$

$$(\lambda(x)T_2(x)-\lambda(y)T_2(x))w[z,T_2(y)]=0 \text{ for all } x,y,z,w\in R.$$

Using the primeness of R yields that either $\lambda(x)T_2(x)-\lambda(y)T_2(x)=0$ or $[z,T_2(y)]=0$. Again since R is not commutative we have $\lambda(x)T_2(x)=\lambda(y)T_2(x)$ and so $T_1=\lambda T_2$. This completes the proof. □

Theorem 11. *Let R be a noncommutative prime ring with involution of the second kind such that $char(R)\neq 2$. If R admits two non zero left centralizers $T_1,T_2:R\to R$ such that $T_1(\nabla[x,x^*])\pm\nabla[x,T_2(x^*)]\in Z(R)$, then $T_1=\lambda T_2$ where $\lambda\in C$.*

Proof. We have

$$T_1(\nabla[x,x^*])\pm\nabla[x,T_2(x^*)]\in Z(R) \text{ for all } x\in R. \tag{2.57}$$

Linearizing (2.57), we get

$$T_1(\nabla[x,y^*])+T_1(\nabla[y,x^*])\pm\nabla[x,T_2(y^*)]\pm\nabla[y,T_2(x^*)]\in Z(R) \text{ for all } x,y\in R. \tag{2.58}$$

Replacing y by ky in (2.58), where $k\in S(R)\cap Z(R)$ and using (2.58), we get

$$2(T_1(yx^*)\pm yT_2(x^*))k\in Z(R) \text{ for all } x,y\in R \text{ and } k\in S(R)\cap Z(R).$$

Since $\mathrm{char}(R)\neq 2$ and $S(R)\cap Z(R)\neq(0)$, we get

$$T_1(yx^*)\pm yT_2(x^*)\in Z(R) \text{ for all } x,y\in R.$$

Taking $y=z$, where $z\in Z(R)$ and $x=x^*$, we obtain

$$(T_1(x)\pm T_2(x))z\in Z(R) \text{ for all } x\in R \text{ and } z\in Z(R).$$

Since $S(R)\cap Z(R)\neq(0)$, we get

$$T_1(x)\pm T_2(x)\in Z(R) \text{ for all } x\in R.$$

This can be further written as $[T_1(x),r]\pm[T_2(x),r]=0$ for all $x,r\in R$. Replacing r by $T_2(x)$, we get $[T_1(x),T_2(x)]=0$ for all $x\in R$. Which is same as equation (2.53), proceeding as before, we get $T_1(x)=\lambda T_2(x)$ for all $x\in R$. □

Theorem 12. *Let R be a noncommutative prime ring with involution of the second kind such that $char(R)\neq 2$. If R admits two non zero left centralizers $T_1,T_2:R\to R$ such that $\nabla[x,T_1(x^*)]\pm\nabla[x,T_2(x^*)]\in Z(R)$ for all $x\in R$, then $T_1=\lambda T_2$.*

Proof. We have

$$\nabla[x,T_1(x^*)]\pm\nabla[x,T_2(x^*)]\in Z(R) \text{ for all } x\in R. \tag{2.59}$$

Linearizing (2.59), we get

$$\nabla[x,T_1(y^*)]+\nabla[y,T_1(x^*)]\pm\nabla[x,T_2(y^*)]\pm\nabla[y,T_2(x^*)]\in Z(R) \text{ for all } x,y\in R. \tag{2.60}$$

Replacing y by ky in (2.60) and using (2.60), we get

$$2(yT_1(x^*)\pm yT_2(x^*))k\in Z(R) \text{ for all } x,y\in R \text{ and } k\in S(R)\cap Z(R).$$

Since $\mathrm{char}(R)\neq 2$ and $S(R)\cap Z(R)\neq(0)$, we get

$$yT_1(x^*)\pm yT_2(x^*)\in Z(R) \text{ for all } x,y\in R.$$

Replacing y by z, where $z\in Z(R)$ and using primeness, $S(R)\cap Z(R)\neq(0)$ conditions, we obtain $T_1(x^*)\pm T_2(x^*)\in Z(R)$ for all $x,r\in R$. Taking $x=x^*$, we obtain $T_1(x)\pm T_2(x)\in Z(R)$ for all $x\in R$. This can be further written as $[T_1(x),r]\pm[T_2(x),r]=0$ for all $x\in R$. Replacing r by $T_2(x)$, we get $[T_1(x),T_2(x)]=0$ for all $x\in R$, which is same as equation (2.53) and hence, we get $T_1(x)=\lambda T_2(x)$ for all $x\in R$. □

Theorem 13. *Let R be a prime ring with involution of the second kind such that $char(R)\neq 2$. If R admits two left centralizers $T_1,T_2:R\to R$ such that $T_1(x)\circ x^*\pm\nabla[x,T_2(x^*)]\in Z(R)$ for all $x\in R$, then R is commutative.*

Proof. We have

$$T_1(x)\circ x^*\pm\nabla[x,T_2(x^*)]\in Z(R) \text{ for all } x\in R. \tag{2.61}$$

If $T_1=0$ and $T_2\neq 0$, then R is commutative by Theorem 1. If $T_2=0$ and $T_1\neq 0$, then by Lemma 5, we get R is commutative. Now consider $T_1\neq 0$ and $T_2\neq 0$, linearizing (2.61), we get

$$T_1(x)\circ y^*+T_1(y)\circ x^*\pm\nabla[x,T_2(y^*)]\pm\nabla[y,T_2(x^*)]\in Z(R) \text{ for all } x,y\in R. \tag{2.62}$$

Replacing y by ky in (2.62), where $k\in S(R)\cap Z(R)$ and using (2.62), we get

$$2(T_1(y)\circ x^*\pm yT_2(x^*))k\in Z(R) \text{ for all } x,y\in R \text{ and } k\in S(R)\cap Z(R).$$

Since $\mathrm{char}(R)\neq 2$ and $S(R)\cap Z(R)\neq(0)$, we get

$$T_1(y)\circ x^*\pm yT_2(x^*)\in Z(R) \text{ for all } x,y\in R.$$

Taking $x=x^*$, we obtain

$$T_1(y)\circ x\pm yT_2(x)\in Z(R) \text{ for all } x,y\in R.$$

This can be further written as

$$T_1(y)[x,r]+[T_1(y),r]x+x[T_1(y),r]+[x,r]T_1(y)\pm y[T_2(x),r]\pm[y,r]T_2(x)=0 \tag{2.63}$$

for all $x,y,r\in R$. Replacing x by xr and using of (2.63), we get

$$x[T_1(y),r]r+[x,r]T_1(y)r+xr[T_1(y),r]+[x,r]rT_1(y)=0 \text{ for all } x,y,r\in R. \tag{2.64}$$

Replacing x by $x+z$ in (2.64) and combining it with (2.64), we obtain

$$z[T_1(y),r]r+zr[T_1(y),r]=0 \text{ for all } y,r\in R \text{ and } z\in Z(R).$$

Since $S(R)\cap Z(R)\neq(0)$, then primeness of R gives us

$$[T_1(y),r]r+r[T_1(y),r]=0 \text{ for all } y,r\in R. \tag{2.65}$$

Substituting $r+z$ for r in (2.65) and using (2.65), we obtain

$$2[T_1(y),r]z=0 \text{ for all } y,z\in R \text{ and } z\in Z(R).$$

Since $\mathrm{char}(R)\neq 2$ and $S(R)\cap Z(R)\neq(0)$, primeness of R gives $[T_1(y),r]=0$ for all $y,r\in R$. Since $T_1\neq 0$, this implies that R is commutative. □

Theorem 14. *Let R be a noncommutative prime ring with involution of the second kind such that $\mathrm{char}(R)\neq 2$. If R admits two left centralizers $T_1,T_2:R\to R$ such that $T_1(\nabla[x,x^*])\pm T_2(\nabla[x,x^*])\in Z(R)$ for all $x\in R$, then $T_1=\lambda T_2$ where $\lambda\in C$.*

Proof. We have

$$T_1(\nabla[x,x^*])\pm T_2(\nabla[x,x^*])\in Z(R) \tag{2.66}$$

If either of $T_1=0$ and $T_2\neq 0$ or $T_2=0$ and $T_1\neq 0$, then by Theorem 2, we get R is commutative. Now consider $T_1\neq 0$ and $T_2\neq 0$, linearizing (2.66), we get

$$T_1(\nabla[x,y^*])+T_1(\nabla[y,x^*])\pm T_2(\nabla[x,y^*])\pm T_2(\nabla[y,x^*])\in Z(R) \text{ for all } x,y\in R. \tag{2.67}$$

Replacing y by ky in (2.67) and using (2.67), we get

$$2(T_1(y)x^*\pm T_2(y)x^*)k\in Z(R) \text{ for all } x,y\in R \text{ and } k\in S(R)\cap Z(R).$$

Since $\mathrm{char}(R)\neq 2$ and $S(R)\cap Z(R)\neq(0)$, we get

$$T_1(y)x^*\pm T_2(y)x^*\in Z(R) \text{ for all } x,y\in R. \tag{2.68}$$

Taking $x=h$, where $h\in H(R)\cap Z(R)$ and using primeness of R and the fact that $S(R)\cap Z(R)\neq(0)$, we get $T_1(y)\pm T_2(y)\in R$ for all $y\in R$. This can be further written as $[T_1(y),r]\pm[T_2(y),r]=0$ for all $y,r\in R$. Replacing r by $T_2(y)$, we get $[T_1(y),T_2(y)]=0$ for all $y\in R$, which is same as equation (2.53). Hence following the same steps as in Theorem 10, we get $T_1(x)=\lambda T_2(x)$ for all $x\in R$. □

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BIFURCATION OF POSITIVE AND NEGATIVE SOLUTIONS OF NONLINEARIZABLE STURM-LIOUVILLE PROBLEMS WITH INDEFINITE WEIGHT

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Abstract. We consider nonlinearizable Sturm-Liouville problem indefinite weight function. We show the existence of two pairs of global continua emanating from the bifurcation intervals surrounding the principal eigenvalues of the corresponding linear problem and contained in the classes of positive and negative functions.

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1. Introduction

We consider the following nonlinear Sturm-Liouville problem

$$(\ell y) \equiv -(p(x)y')' + q(x)y = \lambda\rho(x)y + f(x,y,y',\lambda),\ x \in (0,1), \tag{1.1}$$

$$\alpha_0 y(0) - \beta_0 y'(0) = 0, \tag{1.2}$$

$$\alpha_1 y(1) + \beta_1 y'(1) = 0, \tag{1.3}$$

where $\lambda \in \mathbb{R}$ is a spectral parameter, $p(x)$ is a positive continuously differentiable function on $[0,1]$, $q(x)$ is nonnegative continuous function on $[0,1]$, and $\rho(x)$ are real-valued continuous functions on $[0,1]$ such that meas $\{x \in [0,1] : \sigma\rho(x) > 0\} > 0$ for each $\sigma \in \{+,-\}$, α_i, β_i, $i = 0, 1$, are real constants such that $|\alpha_i| + |\beta_i| > 0$ and $\alpha_i\beta_i \geq 0$, $i = 0, 1$. We also assume that the nonlinear term $f \in C([0,1] \times \mathbb{R}^3)$ is a real-valued function satisfying the following conditions:

$$uf(x,u,s,\lambda) \leq 0; \tag{1.4}$$

there exists constants $K > 0$ such that

$$\left| \frac{f(x,u,s,\lambda)}{u} \right| \leq K,\ (x,u,s,\lambda) \in [0,1] \times \mathbb{R}^3. \tag{1.5}$$

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Problem (1.1)-(1.3) with $\rho > 0$ in a more general case has been considered in [6, 7, 10, 11], where it is proved that there exist global continua of nontrivial solutions in $\mathbb{R} \times C^1[0,1]$ (which is a Banach space with the norm defined by $||(\lambda,u)|| = \left\{ |\lambda|^2 + ||u||_1^2 \right\}^{\frac{1}{2}}$, where $||u||_1 = ||u||_\infty + ||u'||_\infty$, $||u||_\infty = \max\limits_{x\in[0,1]} |u(x)|$, is the usual norm in a Banach space $C^1[0,1]$ of real-valued, continuously differentiable functions on $[0,1]$) corresponding to the usual nodal properties and emanating from bifurcation points and intervals surrounding the eigenvalues of the linear problem (1.1)-(1.3) with $f \equiv 0$. Similar problems for the nonlinear eigenvalue problems of ordinary differential equations of fourth order have been considered in [1, 2].

In the case when nonlinear term f satisfies $o(|u| + |u'|)$ condition near $(u,u') = (0,0)$ problem (1.1)-(1.3) was investigated in a recent paper [5], where it was shown that there exist four families of global continua of nontrivial solutions in $\mathbb{R} \times C^1[0,1]$ corresponding to the usual nodal properties and emanating from bifurcation points corresponding to the eigenvalues of the linear problem (1.1)-(1.3) with $f \equiv 0$. Similar results in the nonlinear eigenvalue problems for ordinary differential equations of fourth order and elliptic partial differential equations with indefinite weights in the classes of positive and negative functions obtained in recent papers [3, 4].

In the present paper we show the existence of four global continua of solutions of problem (1.1)-(1.3) emanating from principal eigenvalues of linear problem obtained from (1.1)-(1.3) by setting $f \equiv 0$ and containing in the classes of positive and negative functions.

2. Preliminary

In the case of $f \equiv 0$ from (1.1)-(1.3) we obtain the corresponding linear problem

$$\begin{cases} \ell(y)(x) = \lambda\rho(x)y(x), \ x \in (0,\ 1), \\ y \in B.C., \end{cases} \tag{2.1}$$

where by $B.C.$ we denote the set of boundary conditions (1.2), (1.3). It is a classical result (see [8, 9]) that the problem (2.1) in the case $\rho(x) > 0, x \in [0,1]$, possesses infinitely many real eigenvalues

$$\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots,$$

all of which are simple and $\lim\limits_{k\to+\infty} \lambda_k = +\infty$. The eigenfunction $y_k(x)$ corresponding to eigenvalue λ_k, $k \in \mathbb{N}$, has exactly $k-1$ simple zeros in the interval $(0,1)$.

Let $E = C^1[0,1] \cap B.C.$ be a Banach space with the norm $||u||_1$. Let S_k^+ be the set of $u \in E$ which have exactly $k-1$ simple zeros in $(0,1)$ and which are positive for $0 \neq x$ near 0; then, $S_k^- = -S_k^+$ and $S_k = S_k^- \cup S_k^+$. The sets S_k^+, S_k^- and S_k are open in E. Moreover, if $u \in \partial S_k$, then u has at least one double zero in $[0,1]$.

Theorem 1 (see [9], Ch. 10, §§10·6, 10·61)**.** *The eigenvalues of problem (2.1) are all real and simple, and form a two sequences*

$$0 > \lambda_1^- > \lambda_2^- > \ldots > \lambda_k^- \mapsto -\infty \text{ and } 0 < \lambda_1^+ < \lambda_2^+ < \ldots < \lambda_k^+ \mapsto +\infty.$$

Moreover, for each $k \in \mathbb{N}$ and each $\sigma \in \{+,-\}$ the eigenfunction $y_k^\sigma(x)$ corresponding to eigenvalue λ_k^σ, has exactly $k-1$ simple zeros in the interval $(0,1)$ (more precisely, $y_k^\sigma(x) \in S_k$).

We define the linear differential operator $L : D(L) \to L_2(0,1)$ by

$$(Ly)(x) = (\ell y)(x),$$

and

$$D(L) = \{y \in L_2(0,1) : y \in W_2^2(0,1),\, \ell y \in L_2(0,1),\, y \in B.C.\}.$$

It is easy to see that the differential operator L is a densely defined self-adjoint operator on $L_2(0,1)$ whose spectrum contains only positive eigenvalues. For fixed $\lambda \in \mathbb{R}$ we consider the following eigenvalue problem

$$\begin{cases} (\ell y)(x) - \lambda\rho(x)y(x) = \mu y(x),\ x \in (0,1), \\ y \in B.C. \end{cases} \tag{2.2}$$

It is well known that for each $\lambda \in \mathbb{R}$ the eigenvalues problem (2.2) are real, simple and forms a infinitely increasing sequence

$$\mu_1(\lambda) < \mu_2(\lambda) < \cdots < \mu_k(\lambda) < \cdots.$$

Moreover, for each $k \in \mathbb{N}$ the eigenfunction $y_k(x,\lambda)$ corresponding to the eigenvalue $\mu_k(\lambda)$ has $k-1$ simple zeros in $(0,1)$ (it should be noted that $y_k(x,\lambda) \in S_k$).

For each $k \in \mathbb{N}$ the k-th eigenvalue $\mu_k(\lambda)$, $k \in \mathbb{N}$, of problem (2.2) can be characterized as

$$\mu_k(\lambda) = \max_{V^{(k-1)}} \min_{y \in B.C.} \left\{ R_\lambda[y] : \int_0^1 y(x)\,\varphi(x)dx = 0,\ \varphi \in V^{(k-1)} \right\}, \tag{2.3}$$

where

$$R_\lambda[y] = \frac{\int_0^1 \left\{ y'^2 + q(x)y^2 \right\} dx + N[y]}{\int_0^1 y^2 dx} - \frac{\lambda \int_0^1 \rho(x)\,y^2 dx}{\int_0^1 y^2 dx}, \tag{2.4}$$

$$N[y] = -y'(1)y(1) + y'(0)y(0) \geq 0,$$

and $V^{(k-1)}$ denotes any set of $(k-1)$ linearly independent functions with $\varphi_j(x) \in B.C.$, $1 \leq j \leq k-1$ (see [8]). From this max-min characterization it follows that the number λ is an eigenvalue of problem (2.1) corresponding to the eigenfunction having $k-1$ simple zeros in $(0,1)$, if and only if $\mu_k(\lambda) = 0$. Moreover for each $k \in \mathbb{N}$

the eigenvalue $\mu_k(\lambda)$ of problem (2.2) and corresponding eigenfunction $y_k(x,\lambda)$ are continuous functions of $\lambda \in \mathbb{R}$.

In order to study the local and global bifurcation of solutions of problems (1.1)-(1.3) in the classes of positive and negative functions we need the following properties of eigenvalues $\mu_1(\lambda)$ as a functions of $\lambda \in \mathbb{R}$.

Lemma 1. *Let $y_k(x,\lambda)$, $k \in \mathbb{N}$, be an eigenfunction of (2.2) corresponding to the k-th eigenvalue $\mu_k(\lambda)$. Then $\mu_k(\lambda) \in C^\infty(\mathbb{R})$ and*

$$\frac{d\mu_k(\lambda)}{d\lambda} = -\frac{\int\limits_0^1 \rho(x) y_k^2(x,\lambda)\, dx}{\int\limits_0^1 y_k^2(x,\lambda)\, dx}, \lambda \in \mathbb{R}, k \in \mathbb{N}. \tag{2.5}$$

Proof. By virtue of (2.2), for each $k \in \mathbb{N}$ and ν, $\lambda \in \mathbb{R}$, we have

$$\begin{gathered} y_k(x,\lambda)\ell y_k(x,\nu) - y_k(x,\nu)\ell y_k(x,\lambda) - (\nu-\lambda)\rho(x) y_k(x,\nu) y_k(x,\lambda) = \\ (\mu_k(\nu)-\mu_k(\lambda)) y_k(x,\nu) y_k(x,\lambda),\ x \in (0,1). \end{gathered} \tag{2.6}$$

Integrating this relation from 0 to 1, using the formula for the integration by parts, and taking into account conditions (1.2), (1.3), we obtain

$$\begin{gathered} (\mu_k(\nu)-\mu_k(\lambda)) \int\limits_0^1 y_k(x,\nu) y_k(x,\lambda) dx = \\ -(\nu-\lambda) \int\limits_0^1 \rho(x)\, y_k(x,\nu) y_k(x,\lambda) dx. \end{gathered} \tag{2.7}$$

For $\nu, \lambda \in \mathbb{R}$, $\nu \neq \lambda$, we get

$$\frac{\mu_k(\nu)-\mu_k(\lambda)}{\nu-\lambda} = -\frac{\int\limits_0^1 \rho(x)\, y_k(x,\nu) y_k(x,\lambda) dx}{\int\limits_0^1 y_k(x,\nu) y_k(x,\lambda) dx}. \tag{2.8}$$

Note that $y_k(x,\nu) \to y_k(x,\lambda)$ as $\nu \to \lambda$ uniformly in $x \in [0,1]$ (see [9], Ch. 10, § 10·72). Then passing to the limit as $\nu \to \lambda$ in (2.8) we obtain (2.5). Moreover, $\mu_k(\lambda) \in C^1(\mathbb{R})$. Next, by the principle of mathematical induction, it is proved that $\mu_k(\lambda) \in C^\infty(\mathbb{R})$. □

Lemma 2. *The function $\lambda \to \mu_1(\lambda)$ is a concave in $\mathbb{R}$.*

Proof. By virtue of Lemma 1 differentiating (2.5) we get

$$\frac{d^2\mu_1(\lambda)}{d\lambda^2} = -2\left(\int_0^1 y_1^2(x,\lambda)dx\right)^{-2}$$
$$\left\{\int_0^1 \rho(x)y_1(x,\lambda)\frac{\partial y_1(x,\lambda)}{\partial\lambda}dx\int_0^1 y_1^2(x,\lambda)\,dx-\right.$$
$$\left.-\int_0^1 \rho(x)y_1^2(x,\lambda)\,dx\int_0^1 y_1(x,\lambda)\frac{\partial y_1(x,\lambda)}{\partial\lambda}dx\right\}= \tag{2.9}$$
$$-2\frac{\int_0^1 \rho(x)y_1(x,\lambda)\frac{\partial y_1(x,\lambda}{\partial\lambda})dx+\frac{d\mu_1(\lambda)}{d\lambda}\int_0^1 y_1(x,\lambda)\frac{dy_1(x,\lambda)}{d\lambda}dx}{\int_0^1 y_1^2(x,\lambda)dx}.$$

On the other hand, by Lemma 1 from (2.2) we have the following relation

$$\ell\left(\frac{\partial y_1(x,\lambda)}{\partial\lambda}\right)-\lambda\rho(x)\frac{\partial y_1(x,\lambda)}{\partial\lambda}-\rho(x)y_1(x,\lambda)=$$
$$\mu_1(\lambda)\frac{\partial y_1(x,\lambda)}{\partial\lambda}+\frac{d\mu_1(\lambda)}{d\lambda}y_1(x,\lambda). \tag{2.10}$$

Multiplying both sides of equation (2.10) by $\frac{\partial y_1(x,\lambda)}{\partial\lambda}$ and integrating the obtained equality from 0 to 1, using the formula for the integration by parts, and taking into account boundary conditions (1.2), (1.3), we obtain

$$\int_0^1\left(\frac{\partial y_1'(x,\lambda)}{\partial\lambda}\right)^2dx+\int_0^1 q(x)\left(\frac{\partial y_1(x,\lambda)}{\partial\lambda}\right)^2dx+$$
$$N\left[\frac{\partial y_1(x,\lambda)}{\partial\lambda}\right]-\lambda\int_0^1\rho(x)\left(\frac{\partial y_1(x,\lambda)}{\partial\lambda}\right)^2dx-$$
$$\int_0^1\rho(x)\,y_1(x,\lambda)\frac{\partial y_1(x,\lambda)}{\partial\lambda}dx=\mu_1(\lambda)\int_0^1\left(\frac{\partial y_1(x,\lambda)}{\partial\lambda}\right)^2dx+ \tag{2.11}$$
$$\frac{d\mu_1(\lambda)}{d\lambda}\int_0^1 y_1(x,\lambda)\frac{\partial y_1(x,\lambda)}{\partial\lambda}dx.$$

By virtue of (2.9) it follows from (2.11) that

$$\int_0^1 \left(\frac{\partial y_1'(x,\lambda)}{\partial\lambda}\right)^2 dx + \int_0^1 q(x)\left(\frac{\partial y_1(x,\lambda)}{\partial\lambda}\right)^2 dx +$$

$$N\left[\frac{\partial y_1(x,\lambda)}{\partial\lambda}\right] - \lambda\int_0^1 \rho(x)\left(\frac{\partial y_1(x,\lambda)}{\partial\lambda}\right)^2 dx -$$

$$\mu_1(\lambda)\int_0^1 \left(\frac{\partial y_1(x,\lambda)}{\partial\lambda}\right)^2 dx = -\frac{1}{2}\frac{d^2\mu_1(\lambda)}{d\lambda^2}\int_0^1 y_1^2(x,\lambda)dx,$$

which implies that

$$\frac{d^2\mu_1(\lambda)}{d\lambda^2} =$$

$$-2\left\{\frac{\int_0^1 \left(\frac{\partial y_1'(x,\lambda)}{\partial\lambda}\right)^2 dx + \int_0^1 q(x)\left(\frac{\partial y_1(x,\lambda)}{\partial\lambda}\right)^2 dx + N[\frac{\partial y_1(x,\lambda)}{\partial\lambda}]}{\int_0^1 \left(\frac{\partial y_1(x,\lambda)}{\partial\lambda}\right)^2 dx} - \right.$$

$$\left.\frac{\lambda\int_0^1 \rho(x)\left(\frac{\partial y_1(x,\lambda)}{\partial\lambda}\right)^2 dx}{\int_0^1 \left(\frac{\partial y_1(x,\lambda)}{\partial\lambda}\right)^2 dx} - \mu_1(\lambda)\right\}\frac{\int_0^1 \left(\frac{\partial y_1(x,\lambda)}{\partial\lambda}\right)^2 dx}{\int_0^1 y_1^2(x,\lambda)dx}. \tag{2.12}$$

Hence by (2.3) and (2.4) from (2.12) we get $\frac{d^2\mu_1(\lambda)}{d\lambda^2} < 0$ for $\lambda \in \mathbb{R}$. □

3. Global bifurcation of positive and negative solutions of problem (1.1)-(1.3)

Along with (2.1) and (2.2) we consider the following linear eigenvalue problems

$$\begin{cases} (\ell y)(x) + \psi(x)y(x) = \lambda\rho(x)y(x), \ x \in (0,1), \\ y \in B.C., \end{cases} \tag{3.1}$$

$$\begin{cases} (\ell y)(x) + \psi(x)y(x) - \lambda\rho(x)y(x) = \mu y(x), \ x \in (0,1), \\ y \in B.C., \end{cases} \tag{3.2}$$

where ψ is a nonnegative continuous function on $[0,1]$.

Since $\psi(x) \geq 0$, $x \in [0,1]$, it follows from formula (4.2) of [2] that

$$0 \leq \tilde{\mu}_k(\lambda) - \mu_k(\lambda) \leq \tilde{K}, \tag{3.3}$$

where $\tilde{\mu}_k(\lambda)$ is the k-th eigenvalue of problem (3.2) and $\tilde{K} = \sup\{\,\psi(x) : x \in [0,1]\,\}$.

Remark 1. Since the function $\hat{\mu}_1(\lambda)$ is also a concave on $\mathbb{R}$ and $\tilde{\mu}_1(\lambda) \geq \mu_k(\lambda)$ for any $\lambda \in \mathbb{R}$ it follows that $\tilde{\lambda}_1^+ > \lambda_1^+$ and $\tilde{\lambda}_1^- < \lambda_1^-$, where $\tilde{\lambda}_1^+$ and $\tilde{\lambda}_1^-$ are the positive and negative principal eigenvalues of problem (3.2), respectively.

In what follows we need the following result.

Lemma 3. *For each $\sigma \in \{+,-\}$ the following relation holds:*

$$|\tilde{\lambda}_1^\sigma - \lambda_1^\sigma| \leq \frac{\sigma \int\limits_0^1 (y_1^\sigma(x,\lambda))^2\, dx}{\int\limits_0^1 \rho(x)\,(y_1^\sigma(x,\lambda))^2\, dx}\,. \tag{3.4}$$

The proof of this lemma is similar to that of Lemma 3.5 of [4].

Remark 2. Since the class of continuous functions $C[0,1]$ is dense in $L_1[0,1]$ the statement of Lemma 3 holds for $\psi \in L_1[0,1]$.

In order to study the bifurcation of solutions of problem (1.1)-(1.3) we define the sets $S_k^{\sigma,\nu}$, $k \in \mathbb{N}$, $S_{k,\sigma}$, $\sigma \in \{+,-\}$ and $\nu \in \{+,-\}$, as follows:

$$S_k^{\sigma,\nu} = \{y \in S_k^\nu : \sigma \int\limits_0^1 \rho(x) y^2(x)\, dx > 0\}\,,\; S_{k,\sigma} = S_k^{\sigma,+} \cup S_k^{\sigma,-}.$$

Remark 3. It follows immediately from the definition of the sets $S_k^{\sigma,\nu}$ and $S_{k,\sigma}$, $k \in \mathbb{N}$, $\sigma, \nu \in \{+,-\}$, that these sets are open subsets of E and $S_k^{\sigma,\nu} \cap S_m^{\rho,\theta} = \varnothing$, $S_{k,\sigma} \cap S_{m,\rho} = \varnothing$, where $(k,\sigma,\nu) \neq (m,\rho,\theta)$, $(k,\sigma) \neq (m,\rho)$, respectively. Moreover, if y lies of the boundary of some $S_{k,\sigma}$, then either y must have a double zero in $[0,1]$ or $\int\limits_0^1 \rho(x) y^2(x)\, dx = 0$.

The norm in $\mathbb{R} \times E$ is defined to be $||(\lambda,u)|| = \{\,|\lambda|^2 + ||u||_1^2\}^{\frac{1}{2}}$.

We denote by $\mathcal{C}$ the closure in $\mathbb{R} \times E$ of the set of nontrivial solutions of (1.1)-(1.3). The eigenfunction y_k^σ, $\sigma \in \{+,-\}$, corresponding to the eigenvalue λ_k^σ of problem (2.1) is made unique by requiring that $y_k^\sigma \in S_k^{\sigma,+}$ and $||y_k^{\sigma,+}||_1 = 1$.

Lemma 4. *If $(\lambda,y) \in \mathbb{R} \times E$ is a solution of (1.1)-(1.3) such that $y \in \partial S_k^{\sigma,\nu}$, $k \in \mathbb{N}$, $\sigma, \nu \in \{+,-\}$, then $y \equiv 0$.*

Proof. Let (λ,y) be a solution of (1.1)-(1.3) and let $y \in \partial S_k^{\sigma,\nu}$. Then by Remark 3 either there exists a point $\xi \in [0,1]$ such that $y(\xi) = y'(\xi) = 0$ or $\int\limits_0^1 \rho(x) y^2(x)\, dx = 0$.

Suppose that $y \in \partial S_k^{\sigma,\nu}$ and $y(\xi) = y'(\xi) = 0$ for some $\xi \in [0,1]$. Then, using growth estimate on f and linearity of ℓ and ρy, and applying Gronwall's inequality we obtain that $y \equiv 0$ on $[0,1]$ (see Lemma 3.1 of [2]).

Now let $y \in \partial S_k^{\sigma,\nu}$ and $\int\limits_0^1 \rho(x) y^2(x)\, dx = 0$. Then by (1.1)-(1.3) we have

$$\int\limits_0^1 \left(y'^2 + q(x) y^2\right) dx + N[y] = \int\limits_0^1 y(x)\, h\left(x, y(x), y'(x), \lambda\right) dx.$$

Since the left-hand side of this equality is nonnegative (see inequality after formula (2.4)) and the right-hand side is nonpositive in view of (1.4), it follows that $y \equiv 0$. $\square$

Corollary 1. *Suppose that the condition (1.4) is not satisfied. If $(\lambda, y) \in \mathbb{R} \times E$ is a solution of 1.1)-(1.3) such that $y \in \partial S_k^{\nu}$, $k \in \mathbb{N}$, $\nu \in \{+, -\}$, then $y \equiv 0$.*

We say that $(\lambda, 0)$, $\lambda \in \mathbb{R}$, is a bifurcation point of (1.1)-(1.3) with respect to the set $\mathbb{R} \times S_k^{\sigma,\nu}$ ($\mathbb{R} \times S_{k,\sigma}$) if in every small neighborhood of this point there is a solution of this problem which is contained in $\mathbb{R} \times S_k^{\sigma,\nu}$ ($\mathbb{R} \times S_k^{\sigma}$) (see [2]).

Along with (1.1)-(1.3) we consider the following approximating problem

$$\begin{cases} \ell y(x) = \lambda \rho(x) y(x) + f(x, |y|^{\varepsilon} y, y', \lambda), \ x \in (0,\ 1), \\ y \in B.C., \end{cases} \tag{3.5}$$

where $\varepsilon \in (0,1]$.

By virtue of boundary condition (1.3) we have

$$f(x, |u|^{\varepsilon} u, v, \lambda) = o\left(|u| + |s|\right) \text{ as } |u| + |s| \to 0,$$

uniformly in $x \in [0,1]$ and $\lambda \in \mathbb{R}$. Then it follows from Theorem 2 of [5] that for each $k \in \mathbb{N}$, each $\sigma \in \{+, -\}$ and each $\nu \in \{+, -\}$ there exists an unbounded continuum $D_k^{\sigma,\nu}$ of solutions of problem (3.5) in $(\mathbb{R} \times S_k^{\sigma,\nu}) \cup \{(\lambda_k^{\sigma}, 0)\}$ which contain $(\lambda_k^{\sigma}, 0)$ and is unbounded in $\mathbb{R} \times E$. Moreover, by condition (1.4) if $(\lambda, y) \in D_k^{\sigma,\nu}$, then $\sigma\lambda > 0$.

Lemma 5. *For each $\sigma \in \{+, -\}$, $\nu \in \{+, -\}$ and each $0 < r < 1$ problem (1.1)-(1.3) has a solution $(\lambda_{1,r}^{\sigma,\nu}, y_{1,r}^{\sigma,\nu})$ such that $||y_{1,r}^{\sigma,\nu}|| = r$, $y_{1,r}^{\sigma,\nu} \in S_1^{\sigma,\nu}$ and $\lambda_{1,r}^{\sigma,\nu} \in I_1^{\sigma}$, where $I_1^{+} = [\lambda_1^{+}, \lambda_1^{+} + h_1^{+}]$, $I_1^{-} = [\lambda_1^{-} - h_1^{-}, \lambda_1^{-}]$ and*

$$h_1^{+} = \frac{\int\limits_0^1 \left(y_1^{+}(x)\right)^2 dx}{\int\limits_0^1 \rho(x) \left(y_1^{+}(x)\right)^2 dx}, \ h_1^{-} = -\frac{\int\limits_0^1 \left(y_1^{-}(x)\right)^2 dx}{\int\limits_0^1 \rho(x) \left(y_1^{-}(x)\right)^2 dx}.$$

Proof. Let $(\lambda_\varepsilon, y_\varepsilon)$ be a solution of problem (3.5) such that $\lambda_\varepsilon > 0$, $y_\varepsilon \in S_1^\nu$ and $||y_\varepsilon|| < 1$. We define a function $\psi_\varepsilon(x)$, $x \in [0,1]$, as follows:

$$\psi_\varepsilon(x) = \begin{cases} -\frac{f(x,|y_\varepsilon(x)|^\varepsilon y_\varepsilon(x), y'_\varepsilon(x), \lambda_\varepsilon)}{y_\varepsilon(x)} & \text{if } y_\varepsilon(x) \neq 0, \\ 0 & \text{if } y_\varepsilon(x) = 0. \end{cases} \tag{3.6}$$

Then $(\lambda_\varepsilon, y_\varepsilon)$ is a solution of the following spectral problem

$$\begin{cases} \ell y(x) + \psi_\varepsilon(x) y(x) = \lambda \rho(x) y(x), \ x \in (0,\ 1), \\ y \in B.C. \end{cases} \tag{3.7}$$

Since $y_\varepsilon(x) \neq 0$ for $x \in (0,1)$ it follows that λ_ε is a positive principal eigenvalue of problem (3.7).

By virtue of $||y_\varepsilon||_1 < 1$ we have $|y_\varepsilon(x)| < 1$. Then by (1.4) and (1.5) from (3.6) we obtain the following relations

$$\psi_\varepsilon(x) \geq 0 \text{ and } |\psi_\varepsilon(x)| \leq K|y(x)|^\varepsilon \leq K, x \in [0,1]. \tag{3.8}$$

Note that $y_\varepsilon(x) \neq 0$ for $x \in (0,1)$ and is bounded on the closed interval $[0,1]$. Then it follows from Remark 2 that the statement of Lemma 3 holds for the problem (3.7). Hence by (3.8) from (3.4) we obtain

$$|\lambda_\varepsilon - \lambda_1^+| < h_1^+,$$

with implies that $\lambda_\varepsilon \in I_1^+$. In the case when $\lambda_\varepsilon < 0$ we have $\lambda_\varepsilon \in I_1^-$.

Let $\{\varepsilon_n\}_{n=1}^\infty$, $0 < \varepsilon_n < 1$, be a sequence converging to zero. Since the continuum $D_1^{\sigma,\nu}$ of solutions of problem (3.5) containing the point $(\lambda_1^\sigma, 0)$ is unbounded in $\mathbb{R} \times E$, it follows that for each ε_n and each $r \in (0,1)$ problem (3.5) has a solution $(\lambda_{1,r,\varepsilon_n}^{\sigma,\nu}, y_{1,r,\varepsilon_n}^{\sigma,\nu})$ such that $\lambda_{1,r,\varepsilon_n}^{\sigma,\nu} \in I_1^\sigma$, $y_{1,r,\varepsilon_n}^{\sigma,\nu} \in S_1^{\sigma,\nu}$ and $||y_{1,r,\varepsilon_n}^{\sigma,\nu}||_1 = r$. Since the sequence $\{y_{1,r,\varepsilon_n}^{\sigma,\nu}\}_{n=1}^\infty$ is bounded in $C^1[0,1]$, f is continuous in $[0,1] \times \mathbb{R}^3$, and the set $\{\lambda_{1,r,\varepsilon_n}^{\sigma,\nu}\}_{n=1}^\infty$ is bounded in $\mathbb{R}$, it follows from 3.5) that $\{y_{1,r,\varepsilon_n}^{\sigma,\nu}\}_{n=1}^\infty$ is also bounded in $C^2[0,1]$. Then by Arzelà-Ascoli theorem we can assume (without loss of generality) that $\{y_{1,r,\varepsilon_n}^{\sigma,\nu}\}_{n=1}^\infty$ converges in E to a function $y_{1,r,}^{\sigma,\nu}$ with $||y_{1,r,}^{\sigma,\nu}||_1 = r$. Without loss of generality we can also assume that $\{\lambda_{1,r,\varepsilon_n}^{\sigma,\nu}\}_{n=1}^\infty$ converges to some $\lambda_{1,r}^{\sigma,\nu} \in I_1^\sigma$. Next passing to the limit in (3.5) at $n \to \infty$, we get that $(\lambda_{1,r}^{\sigma,\nu}, y_{1,r}^{\sigma,\nu})$ is a solution of problem (1.1)-(1.3). Since $y_{1,r}^{\sigma,\nu}$ is contained in the closure of the set $S_1^{\sigma,\nu}$ and $||y_{1,r}^{\sigma,\nu}||_1 = r > 0$, it follows from Lemma 4 that $y_{1,r}^{\sigma,\nu} \in S_1^{\sigma,\nu}$. □

Corollary 2. *The set of bifurcation points of problem (1.1)-(1.3) is nonempty. If $(\lambda, 0)$ is a bifurcation point of problem (1.1)-(1.3) with respect to the set $S_1^{\sigma,\nu}$, then $\lambda \in I_1^\sigma$.*

For $\sigma \in \{+,-\}$ and $\nu \in \{+,-\}$ let $\tilde{D}_1^{\sigma,\nu}$ denote the union of connected components $D_{1,\lambda}^{\sigma,\nu}$ of the solution set of problem (1.1)-(1.3) emanating from bifurcation points $(\lambda, 0) \in I_1^\sigma \times 0$ with respect to the set $S_1^{\sigma,\nu}$. It follows from Corollary 2 that $\tilde{D}_1^{\sigma,\nu} \neq \varnothing$.

Note that $D_1^{\sigma,\nu} = \tilde{D}_1^{\sigma,\nu} \cup (I_1^{\sigma} \times \{0\})$ is a connected subset of $\mathbb{R} \times E$, but $\tilde{D}_1^{\sigma,\nu}$ is not necessarily connected in $\mathbb{R} \times E$.

The main result of this paper is the following theorem.

Theorem 2. *For each $\sigma \in \{+,-\}$ and $\nu \in \{+,-\}$ the connected component $D_1^{\sigma,\nu}$ of the solution set of problem (1.1)-(1.3) lies in the strip $I_1^{\sigma} \times S_1^{\sigma,\nu}$ and is unbounded $\mathbb{R} \times E$.*

Proof. Let $(\lambda, y) \in \mathbb{R} \times S_1^{\sigma,\nu}$ be a solution of problem (1.1)-(1.3). We define the function $\psi(x)$ as follows:

$$\psi(x) = \begin{cases} -\frac{f(x,y(x),y'(x),\lambda)}{y(x)} & \text{if } \ y(x) \neq 0, \\ 0 & \text{if } y(x) = 0. \end{cases} \tag{3.9}$$

Then (λ, y) is a solution of problem (3.1). By (1.4)-(1.5) it follows from (3.9) that

$$\psi(x) \geq 0 \text{ and } |\psi(x)| \leq K \text{ for } x \in [0,1].$$

Since $y \in S_1^{\sigma,\nu}$ it follows from Theorem 1 that λ is a principal eigenvalue of problem (3.1). Then by virtue of Lemma 3 we have $\lambda \in I_1^{\sigma}$. Thus we show that if $(\lambda, y) \in \mathbb{R} \times S_1^{\sigma,\nu}$ is a solution of problem (1.1)-(1.3), then $\lambda \in I_1^{\sigma}$. Next proof of this theorem is similar to that of Theorem 1.3 of [2]. □

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AN EXTENSION OF TYCHONOFF FIXED POINT THEOREM WITH APPLICATION TO THE SOLVABILITY OF THE INFINITE SYSTEMS OF INTEGRAL EQUATIONS IN THE FRÉCHET SPACES

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Abstract. In the present article, we introduce a new concept of contraction and prove a new type of the extension of Tychonoff fixed point theorem. Then, as an application, we study the problem of existence of solutions for the infinite systems of integral equations using the technique of measures of noncompactness in conjunction with this extension in the Fréchet spaces.

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1. Introduction

The theory of infinite systems of integral or differential equations creates an important branch of nonlinear analysis. Up to now, several papers have been published on this significant topic (see [9, 12, 16, 17, 19, 21]) . It is connected naturally with a large number of problems considered in mechanics, engineering, in the theory of branching processes, the theory of neutral nets and so on (see [14, 15, 18, 20, 22, 23]). In this paper, we investigate the existence of solutions for the infinite systems of integral equations of the forms

$$\begin{aligned} x_n(t) &= f_n(t,x_1(t),\dots,x_n(t)) \\ &\quad + q_n(t,x_1(t),\dots,x_n(t))\int_0^{\beta_n(t)} g_n(t,s,(x_j(s))_{j=1}^{\infty})ds, \quad n\in\mathbb{N}, t\in\mathbb{R}, \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} y_n(t) &= f_n(t,y_1(t),\dots,y_n(t)) \\ &\quad + q_n(t,y_1(t),\dots,y_n(t))\int_0^{\infty} k_n(t,s)h_n(s,(y_j(s))_{j=1}^{\infty})ds, \quad n\in\mathbb{N}, t\in\mathbb{R}, \end{aligned} \tag{1.2}$$

by using the measures of noncompactness μ introduced in [9] and a new type of extension of Tychonoff fixed point theorem in the Fréchet space $(BC(\mathbb{R}_+))^{\omega}$ (denote

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the countable cartesian product of $BC(\mathbb{R}_+)$ with itself). The results of this paper improve and generalize those obtained in papers [2–8, 10, 11].

2. Notation and auxiliary facts

Here, we recall some basic facts concerning measures of noncompactness. Denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0,+\infty)$. The symbol $\overline{X}$ and $ConvX$ will denote the closure and closed convex hull of a subset X of E, respectively. Moreover, let $\mathfrak{N}_E$ indicate the family of all nonempty and relatively compact subsets of E.
A topological vector space (TVS) is a vector space X over the field $\mathbb{R}$ which is endowed with a topology such that the maps $(x,y) \to x+y$ and $(\alpha,x) \to \alpha x$ are continuous from $X \times X$ and $\mathbb{R} \times X$ to X. A topological vector space is called locally convex if there is a basis for the topology consisting of convex sets (that is, set A such that if $x,y \in A$ then $tx+(1-t)y \in A$ for $0 < t < 1$).

Definition 1 ([13])**.** A Fréchet space is a locally convex space which is complete with respect to a translation-invariant metric.

Example 1. Let E_i be a Banach space for all $i \in \mathbb{N}$, then $\prod_{i\in\mathbb{N}} E_i$ is a Fréchet space with respect to the metric

$$d(x,y) = \sup\{\frac{1}{2^i}\min\{1, d_i(x_i,y_i)\} : i \in \mathbb{N}\},$$

where $x = (x_1,x_2,\ldots)$, $y = (y_1,y_2,\ldots) \in \prod_{i\in\mathbb{N}} E_i$.

Definition 2 ([9])**.** Let $\mathcal{M}$ be a class of subsets of a Fréchet space E, we say $\mathcal{M}$ is admissible class if $\mathfrak{N}_E \cap \mathcal{M} \neq \varnothing$ and if $X \in \mathcal{M}$, then $Conv(X), \overline{X} \in \mathcal{M}$.

Definition 3 ([9])**.** Let $\mathcal{M}$ be an admissible class of a Fréchet space E, we say that $\mu : \mathcal{M} \longrightarrow \mathbb{R}_+$ is a measure of noncompactness on Fréchet space E if it satisfies the following conditions:

(1°) The family $\ker\mu = \{X \in \mathcal{M} : \mu(X) = 0\}$ is nonempty and $\ker\mu \subseteq \mathfrak{N}_E$;
(2°) $X \subset Y \Longrightarrow \mu(X) \leq \mu(Y)$;
(3°) $\mu(\overline{X}) = \mu(X)$;
(4°) $\mu(ConvX) = \mu(X)$;
(5°) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$;
(6°) If $\{X_n\}$ is a sequence of closed sets from $\mathcal{M}$ such that $X_{n+1} \subset X_n$ for $n = 1,2,\cdots$, and if $\lim_{n\to\infty} \mu(X_n) = 0$, then $X_\infty = \cap_{n=1}^{\infty} X_n \neq \varnothing$.

Theorem 1 (Tychonoff fixed point theorem [1])**.** *Let E be a Hausdorff locally convex linear topological space, C be a convex subset of E and $F : C \longrightarrow E$ be a continuous mapping such that*

$$F(C) \subseteq A \subseteq C,$$

with A compact. Then F has at least one fixed point.

Theorem 2 ([9]). *Suppose μ_i be a measure of noncompactness on Banach spaces E_i for all $i \in \mathbb{N}$. If we define*

$$\mathcal{M} = \{C \subseteq \prod_{i=1}^{\infty} E_i : \sup_i \{\mu_i(\pi_i(C))\} < \infty\},$$

where $\pi_i(C)$ denotes the natural projection of $\prod_{i=1}^{\infty} E_i$ into E_i and $\mu : \mathcal{M} \longrightarrow \mathbb{R}_+$ is defined by

$$\mu(C) = \sup\{\mu_i(\pi_i(C)) : i \in \mathbb{N}\},$$

then $\mathcal{M}$ is an admissible set and μ is a measure of noncompactness on $X = \prod_{i=1}^{\infty} E_i$.

3. Main result

In this section, we introduce a new concept of contraction and prove some new extensions of Tychonoff fixed point theorem.

Theorem 3. *Let Ω be a nonempty, closed and convex subset of a Fréchet space E, $\mathcal{M}$ is admissible class such that $\Omega \in \mathcal{M}$ and $\mu : \mathcal{M} \longrightarrow \mathbb{R}_+$ is a measure of noncompactness on E. Let $F, G : \Omega \longrightarrow \Omega$ be two continuous mappings such that*

$$\mu(FX) + \psi(\mu(GY)) \leq \varphi\Big(\mu(X) + \psi(\mu(Y))\Big), \tag{3.1}$$

and $F(X), G(Y) \in \mathcal{M}$ for any nonempty subset $X, Y \in \mathcal{M}$ where $\varphi, \psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are nondecreasing and right continuous functions such that $\varphi(0) = \psi(0) = 0$ and $\varphi(t) < t$ for each $t > 0$. Then F and G have at least one fixed point in the set Ω.

Proof. By induction, we obtain sequences $\{\Omega_n\}$ and $\{\Lambda_n\}$ such that

$$\begin{cases} \Omega_0 = \Lambda_0 = \Omega, \\ \Omega_n = Conv(F\Omega_{n-1}) & n \geq 1, \\ \Lambda_n = Conv(G\Lambda_{n-1}) & n \geq 1. \end{cases} .$$

It is obvious that $\Omega_n, \Lambda_n \in \mathcal{M}$ for all $n \in \mathbb{N}$. If there exists an integer $N \geq 0$ such that $\mu(\Omega_N) = \mu(\Lambda_N) = 0$, then Ω_N and Λ_N are compact. Thus, Theorem 1 implies that F and G have a fixed point. Now assume that $\mu(\Omega_n) \neq 0$ or $\mu(\Lambda_n) \neq 0$ for $n \geq 0$. Since we have $F\Omega_0 = F\Omega \subseteq \Omega = \Omega_0, \Omega_1 = Conv(F\Omega_0) \subseteq \Omega = \Omega_0$, and by continuing this process we obtain

$$\Omega_0 \supseteq \Omega_1 \supseteq \Omega_2 \supseteq \cdots,$$

and

$$\Lambda_0 \supseteq \Lambda_1 \supseteq \Lambda_2 \supseteq \cdots,$$

so $\mu(\Omega_n)$ and $\mu(\Lambda_n)$ is a positive decreasing sequence of real numbers. Thus, there are $r_1, r_2 \geq 0$ such that $\mu(\Omega_n) \longrightarrow r_1$ and $\mu(\Lambda_n) \longrightarrow r_2$ as $n \longrightarrow \infty$. On the other hand, in view of (3.1), we obtain

$$\limsup_{n\longrightarrow\infty} \mu(\Omega_{n+1}) + \psi(\mu(\Lambda_{n+1}) \leq \limsup_{n\longrightarrow\infty} \varphi\Big(\mu(\Omega_n) + \psi(\mu(\Lambda_n))\Big).$$

This show that $r_1 + \psi(r_2) \leq \varphi\big(r_1 + \psi(r_2)\big)$. Consequently $r_1 + \psi(r_2) = 0$, so $r_1 = r_2 = 0$. Hence we deduce that $\mu(\Omega_n) \longrightarrow 0$ and $\mu(\Lambda_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Since the sequences (Ω_n) and (Λ_n) are nested, in view of axiom (6°) of Definition 3 we derive that the sets $\Omega_\infty = \bigcap_{n=1}^{\infty} \Omega_n$ and $\Lambda_\infty = \bigcap_{n=1}^{\infty} \Lambda_n$ are nonempty, closed and convex subsets of the set Ω. Moreover, the sets Ω_∞ and Λ_∞ are invariant under the operators F and G respectively, and belongs to $\ker\mu$. Now, Tychonoff fixed point theorem implies that F and G have fixed points in the set Ω. □

Corollary 1. *Let Ω be a nonempty, closed and convex subset of a Fréchet space E, $\mathcal{M}$ satisfies the hypotheses of Theorem 3 and $\Omega \in \mathcal{M}$. Let $F, G : \Omega \longrightarrow \Omega$ be two continuous mappings such that*

$$\mu(FX) + \psi(\mu(GY)) \leq k[\mu(X) + \psi(\mu(Y))],$$

and $F(X), G(Y) \in \mathcal{M}$ for any nonempty subset $X, Y \in \mathcal{M}$ where $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is nondecreasing and right continuous function such that $\psi(0) = 0$, μ is an arbitrary measure of noncompactness on $\mathcal{M}$ and $k \in [0,1)$. Then F and G have at least a fixed point in the set Ω.

Proof. Take $\varphi(t) = kt$ in Theorem 3. □

Definition 4. Let X be a Banach space. An operator $F : X \longrightarrow X$ is affine if $F(ax + (1-a)y) = aF(x) + (1-a)F(y)$ for all $x, y \in X$ and all $a \in \mathbb{R}$.

Corollary 2. *Let Ω be a nonempty, closed and convex subset of a Fréchet space E, $\mathcal{M}$ is admissible class such that $\Omega \in \mathcal{M}$ and $\mu : \mathcal{M} \longrightarrow \mathbb{R}_+$ is a measure of noncompactness on E. Let $F, G : \Omega \longrightarrow \Omega$ be two continuous operators such that*

$$\mu(FX) \leq \varphi(\mu(X)) \tag{3.2}$$

and $F(X) \in \mathcal{M}$ for any nonempty subset $X \in \mathcal{M}$ where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfies the hypotheses of Theorem 3. Moreover, $G(F(X)) \subseteq F(X)$ for all $X \in \mathfrak{N}_\Omega$ and F is affine operator. Then F and G have at least one common fixed point in the set Ω.

Proof. Take $\psi(t) = 0$ in Theorem 3. Thus, applying Theorem 3, F has a fixed point. Now suppose that $\Gamma = \{x \in \Omega \ : \ Fx = x\}$, then by (3.2) and $F(\Gamma) = \Gamma$ we have

$$\begin{aligned} \mu(\Gamma) &= \mu(F(\Gamma)) \\ &\leq \varphi(\mu(\Gamma)) \\ &< \mu(\Gamma). \end{aligned}$$

which is a contradiction. So Γ is relatively compact, and since F is a continuous and affine operator so Γ is nonempty, compact and convex subset of Ω. Also, we have

$$\begin{aligned} G(\Gamma) &= G(F(\Gamma)) \\ &\subseteq F(\Gamma) \\ &= \Gamma. \end{aligned}$$

Now, Tychonoff fixed point theorem implies that F and G have a common fixed point in the set Γ. □

We introduce the following useful corollary which will be used in Section 4.

Corollary 3. *Let Ω_i $(i \in \mathbb{N})$ be a nonempty, convex and closed subset of a Banach space E_i, μ_i an arbitrary measure of noncompactness on E_i and $\sup_i\{\mu_i(\Omega_i)\} < \infty$. Let $F_i, G_i : \prod_{i=1}^{\infty} \Omega_i \longrightarrow \Omega_i$ $(i = 1, 2, \ldots)$ be continuous operators such that*

$$\mu_i(F_i(\prod_{i=1}^{\infty} X_i)) + \psi(\mu_i(G_i(\prod_{i=1}^{\infty} Y_i))) \leq \varphi\Big(\sup_i\{\mu_i(X_i) + \psi(\mu_i(Y_i))\}\Big), \tag{3.3}$$

for any subsets X_i and Y_i of Ω_i $(i \in \mathbb{N})$ where $\varphi, \psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfies the hypotheses of Theorem 3. Then there exist $(x_j^*)_{j=1}^{\infty}, (y_j^*)_{j=1}^{\infty} \in \prod_{j=1}^{\infty} \Omega_j$ such that for all $i \in \mathbb{N}$

$$\begin{cases} F_i((x_j^*)_{j=1}^{\infty}) = x_i^*, \\ G_i((y_j^*)_{j=1}^{\infty}) = y_i^*. \end{cases} \tag{3.4}$$

Proof. Assume that $\widetilde{F}, \widetilde{G} : \prod_{i=1}^{\infty} \Omega_i \longrightarrow \prod_{i=1}^{\infty} \Omega_i$ are defined as follows

$$\widetilde{F}((x_j)_{j=1}^{\infty}) = (F_1((x_j)_{j=1}^{\infty}), F_2((x_j)_{j=1}^{\infty}), \ldots, F_i((x_j)_{j=1}^{\infty}), \ldots),$$

and

$$\widetilde{G}((y_j)_{j=1}^{\infty}) = (G_1((y_j)_{j=1}^{\infty}), G_2((y_j)_{j=1}^{\infty}), \ldots, G_i((y_j)_{j=1}^{\infty}), \ldots),$$

for all $(x_j)_{j=1}^{\infty}, (y_j)_{j=1}^{\infty} \in \prod_{i=1}^{\infty} \Omega_i$. It is obvious that F and G are continuous. It suffices to show that the hypothesis (3.1) of Theorem 3 holds where μ is defined by Theorem 2. Take arbitrary nonempty subset X and Y of $\prod_{i=1}^{\infty} \Omega_i$. Now, by (2°) and (3.3) we

obtain

$$\begin{aligned}
\mu(\widetilde{F}(X))+\psi(\mu(\widetilde{G}(Y))) &\leq \mu(\prod_{i=1}^{\infty}F_i(\prod_{j=1}^{\infty}\pi_j(X)))+\psi(\mu(\prod_{i=1}^{\infty}G_i(\prod_{j=1}^{\infty}\pi_j(Y))))\\
&= \sup_i \mu_i(F_i((\prod_{j=1}^{\infty}\pi_j(X))))+\psi(\sup_i \mu_i(G_i((\prod_{j=1}^{\infty}\pi_j(Y)))))\\
&\leq \sup_i\{\mu_i(F_i((\prod_{j=1}^{\infty}\pi_j(X))))+\psi(\mu_i(G_i((\prod_{j=1}^{\infty}\pi_j(Y)))))\}\\
&\leq \sup_i \varphi\Big(\sup_j\{\mu_j(X_j)+\psi(\mu_j(Y_j))\}\Big)\\
&\leq \varphi\Big(\sup_j \mu_j(X_j)+\psi(\sup_j \mu_j(Y_j))\Big)\\
&\leq \varphi(\mu(X)+\psi(\mu(Y))).
\end{aligned}$$

Therefore, all the conditions of Theorem 3 are satisfied, hence $\widetilde{F}$ and G have fixed points and there exist $(x_j^*)_{j=1}^{\infty},(y_j^*)_{j=1}^{\infty}\in\prod_{j=1}^{\infty}\Omega_j$ such that

$$\begin{cases}
(x_j^*)_{j=1}^{\infty}=\widetilde{F}((x_j^*)_{j=1}^{\infty})=(F_1((x_j^*)_{j=1}^{\infty}),F_2((x_j^*)_{j=1}^{\infty}),\ldots,F_j((x_j^*)_{j=1}^{\infty}),\ldots),\\
(y_j^*)_{j=1}^{\infty}=\widetilde{G}((y_j^*)_{j=1}^{\infty})=(G_1((x_j^*)_{j=1}^{\infty}),G_2((x_j^*)_{j=1}^{\infty}),\ldots,G_j((x_j^*)_{j=1}^{\infty}),\ldots),
\end{cases}$$

and that (3.4) holds. □

4. Application

In this section we are going to show how the measure of noncompactness defined in the previous section, can be applied to an infinite system of nonlinear integral equations. Here, we will work in the classical Banach space $BC(\mathbb{R}_+)$ consisting of all real functions defined, bounded and continuous on $\mathbb{R}_+$ equipped with the standard norm

$$\|x\|=sup\{|x(t)|:t\geq 0\}.$$

Now, we present the definition of a special measure of noncompactness in $BC(\mathbb{R}_+)$ which will be needed in the sequel.

To do this, let X be a fixed nonempty and bounded subset of $BC(\mathbb{R}_+)$ and fix a positive number T. For $x\in X$ and $\varepsilon>0$, denote by $\omega^T(x,\varepsilon)$ the modulus of the continuity of function x on the interval $[0,T]$, i.e.,

$$\omega^T(x,\varepsilon)=sup\{|x(t)-x(s)|:t,s\in[0,T],|t-s|\leq\varepsilon\}.$$

Further, let us put

$$\omega^T(X,\varepsilon)=sup\{\omega^T(x,\varepsilon):x\in X\},$$
$$\omega_0^T(X)=\lim_{\varepsilon\to 0}\omega^T(X,\varepsilon)$$

and

$$\omega_0(X) = \lim_{T\to\infty} \omega_0^T(X).$$

If t is a fixed number from $\mathbb{R}_+$, let us denote $X(t) = \{x(t) : x \in X\}$. Finally, consider the function μ defined on $\mathfrak{M}_{BC(\mathbb{R}_+)}$ by the formula

$$\mu(X) = \omega(X_0) + \limsup_{t\to\infty} diamX(t),$$

where,

$$diamX(t) = \sup\{|x(t) - y(t)| : x, y \in X\}.$$

It is shown [10] that the function μ is a measure of noncompactness in the space $BC(\mathbb{R}_+)$.

We will consider Equations (1.1) and (1.2) under the following assumptions:

(A_1) $f_n, q_n : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ $(n \in \mathbb{N})$ are continuous with

$$M := \sup\Big\{\max\{|f_n(t,0,\dots,0)|, |q_n(t,0,\dots,0)|\} : t \in \mathbb{R}_+, n \in \mathbb{N}\Big\} < \infty.$$

Moreover, there exists a nondecreasing, concave and upper semicontinuous function φ with $\varphi(t) < t$ for all $t > 0$ such that

$$|f_n(t,x_1,\dots,x_n) - f_n(t,y_1,\dots,y_n)| \le \varphi(\max_{1\le i\le n} |x_i - y_i|),$$

and

$$|q_n(t,x_1,\dots,x_n) - q_n(t,y_1,\dots,y_n)| \le \varphi(\max_{1\le i\le n} |x_i - y_i|);$$

(B_1) $\beta_n : \mathbb{R}_+ \longrightarrow [0,\infty)$ is continuous function for all $n \in \mathbb{N}$;

(B_2) $g_n : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^\omega \longrightarrow \mathbb{R}$ $(n \in \mathbb{N})$ is continuous and there exists a positive constant D such that

$$D := \sup\{\Big|\int_0^{\beta_n(t)} g_n(t,s,(x_j(s))_{j=1}^\infty)ds\Big| : \ t \in \mathbb{R}_+, \ x_j \in BC(\mathbb{R}_+), 1 \le n < \infty\}.$$

Moreover,

$$\lim_{t\longrightarrow\infty} \Big|\int_0^{\beta_n(t)} g_n(t,s,(x_j(s))_{j=1}^\infty) - g_n(t,s,(y_j(s))_{j=1}^\infty)ds\Big| = 0,$$

uniformly respect to $x_j, y_j \in BC(\mathbb{R}_+)$;

(C_1) $k_n : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ is continuous function for all $n \in \mathbb{N}$;

(C_2) $h_n : \mathbb{R}_+ \times \mathbb{R}^\omega \longrightarrow \mathbb{R}$ $(n \in \mathbb{N})$ is a continuous and there exists a continuous function $a_n : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and a continuous and nondecreasing function $b_n : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$|h_n(t,(x_j)_{j=1}^\infty)| \le a_n(t)b_n(\sup_{1\le j<\infty} |x_j|),$$

for all $t \in \mathbb{R}_+$ and $(x_j)_{j=1}^\infty \in \mathbb{R}^\omega$ with $\sup_{1\le j<\infty} |x_j| < \infty$. Also the function $s \longrightarrow a_n(s)k_n(t,s)$ is integrable over $\mathbb{R}_+$ for any fixed $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$;

(C_3) There exists a positive constant Q such that

$$Q := \sup\{\int_0^\infty a_n(s)|k_n(t,s)|ds : t \in \mathbb{R}_+, n \in \mathbb{N}\} < \infty,$$

and

$$\lim_{t\to\infty} \int_0^\infty a_n(s)|k_n(t,s)|ds = 0.$$

(C_4) The following equality holds:

$$\lim_{T\to\infty} \left\{ \sup \left\{ \int_T^\infty a_n(s)|k_n(t,s)|ds : t \in \mathbb{R}_+ \right\} \right\} = 0$$

for all $n \in \mathbb{N}$;

(D_1) There exists a positive solution r_0 of the inequalities

$$(1+D)\big(\varphi(r)+M\big) \le r,$$

and

$$(1+Qb_n(r))\big(\varphi(r)+M\big) \le r,$$

for all $n \in \mathbb{N}$ such that

$$\Big(\max\Big\{1+D, \sup_{n\in\mathbb{N}}\{1+Qb_n(r_0)\}\Big\}\Big)\varphi(t) < t.$$

Theorem 4. *Under above conditions Eq. (1.1) and Eq. (1.2) have at least one solution in the space* $(BC(\mathbb{R}_+))^\omega$.

Proof. Let us fix arbitrarily $n \in \mathbb{N}$. $F_n, G_n : (BC(\mathbb{R}_+))^\omega \longrightarrow BC(\mathbb{R}_+)$ $(n \in \mathbb{N})$ are defined by

$$\begin{aligned} F_n((x_j)_{j=1}^\infty)(t) &= f_n(t, x_1(t), \dots, x_n(t)) \\ &\quad + q_n(t, x_1(t), \dots, x_n(t)) \int_0^{\beta_n(t)} g_n(t, s, (x_j(s))_{j=1}^\infty) ds, \end{aligned}$$

and

$$\begin{aligned} G_n((x_j)_{j=1}^\infty)(t) &= f_n(t, x_1(t), \dots, x_n(t)) \\ &\quad + q_n(t, x_1(t), \dots, x_n(t)) \int_0^\infty k_n(t,s) h_n(s, (x_j(s))_{j=1}^\infty) ds. \end{aligned}$$

In view of imposed assumptions, we infer that the operators $F_n((x_j)_{j=1}^{\infty})$ and $G_n((x_j)_{j=1}^{\infty})$ are continuous for arbitrarily $(x_j)_{j=1}^{\infty} \in (BC(\mathbb{R}_+))^{\omega}$. Also from our assumptions, we obtain

$$\begin{aligned}
&|F_n((x_j)_{j=1}^{\infty})(t)| \leq |f_n(t,x_1(t),\dots,x_n(t))| \\
&\quad + |q_n(t,x_1(t),\dots,x_n(t))| \Big| \int_0^{\beta_n(t)} g_n(t,s,(x_j(s))_{j=1}^{\infty})ds \Big| \\
&\leq |f_n(t,x_1(t),\dots,x_n(t)) - f_n(t,0,\dots,0)| + |f_n(t,0,\dots,0)| \\
&\quad + \Big(|q_n(t,x_1(t),\dots,x_n(t)) - q_n(t,0,\dots,0)| + |q_n(t,0,\dots,0)| \Big) \\
&\quad \cdot \Big| \int_0^{\beta_n(t)} g_n(t,s,(x_j(s))_{j=1}^{\infty})ds \Big| \\
&\leq \varphi(\max_{1\leq i\leq n} |x_i(t)|) + M + D(\varphi(\max_{1\leq i\leq n} |x_i(t)|) + M) \\
&\leq (1+D)\Big(\varphi(\max_{1\leq i\leq n} |x_i(t)|) + M\Big).
\end{aligned}$$

Thus,

$$\|F_n((x_j)_{j=1}^{\infty})\| \leq (1+D)\Big(\varphi(\max_{1\leq i\leq n} \|x_i\|) + M\Big), \tag{4.1}$$

and with similar argument

$$\|G_n((x_j)_{j=1}^{\infty})\| \leq (1+Qb_n(\sup_{1\leq j<\infty} \|x_j\|))\Big(\varphi(\max_{1\leq i\leq n} \|x_i\|) + M\Big). \tag{4.2}$$

$F_n((x_j)_{j=1}^{\infty})$ and $G_n((x_j)_{j=1}^{\infty}) \in BC(\mathbb{R}_+)$ for any $(x_j)_{j=1}^{\infty} \in (BC(\mathbb{R}_+))^{\omega}$ with $\sup_{1\leq j<\infty} \|x_j\| < \infty$. Due to Inequalities (4.1), (4.2) and using (D_1), the opretors F_n and G_n maps $(\bar{B}_{r_0})^{\omega}$ into $\bar{B}_{r_0}$. Now we show that G_n is a continuous function on $(\bar{B}_{r_0})^{\omega}$. To prove this, let us fix $0 < \varepsilon < \frac{1}{2^n}$ and take arbitrary $x = (x_j)_{j=1}^{\infty}, y = (y_j)_{j=1}^{\infty} \in (\bar{B}_{r_0})^{\omega}$ such that $d(x,y) = \sup\Big\{\dfrac{1}{2^i}\min\{1, \|x_i - y_i\|\} : i \in \mathbb{N}\Big\} < \varepsilon$. Then, for $t \in \mathbb{R}_+$, we have

$$
\begin{aligned}
&\Big|G_n((x_j)_{j=1}^{\infty})(t) - G_n((y_j)_{j=1}^{\infty})(t)\Big| \\
&= \Big|f_n(t,x_1(t),\dots,x_n(t)) + q_n(t,x_1(t),\dots,x_n(t))\int_0^{\infty} k_n(t,s)h_n(s,(x_j(s))_{j=1}^{\infty})ds \\
&- f_n(t,y_1(t),\dots,y_n(t)) + q_n(t,y_1(t),\dots,y_n(t))\int_0^{\infty} k_n(t,s)h_n(s,(y_j(s))_{j=1}^{\infty})ds\Big| \\
&\le \Big|f_n(t,x_1(t),\dots,x_n(t)) - f_n(t,y_1(t),\dots,y_n(t))\Big| \\
&+ \Big|q_n(t,x_1(t),\dots,x_n(t)) - q_n(t,y_1(t),\dots,y_n(t))\Big| \\
&\cdot \Big|\int_0^{\infty} k_n(t,s)h_n(s,(y_j(s))_{j=1}^{\infty})ds\Big| + \Big|q_n(t,x_1(t),\dots,x_n(t))\Big| \\
&\cdot \Big|\int_0^{\infty} k_n(t,s)h_n(s,(x_j(s))_{j=1}^{\infty})ds - \int_0^{\infty} k_n(t,s)h_n(s,(y_j(s))_{j=1}^{\infty})ds\Big| \\
&\le \varphi(\max_{1\le i\le n}|x_i(t)-y_i(t)|) + \varphi(\max_{1\le i\le n}|x_i(t)-y_i(t)|)b_n(r_0)\int_0^{\infty} a_n(s)|k_n(t,s)|ds \\
&+ \Big(\varphi(\max_{1\le i\le n}|x_i(t)|) + M\Big)\Big|\int_0^{\infty} k_n(t,s)[h_n(s,(x_j(s))_{j=1}^{\infty}) - h_n(s,(y_j(s))_{j=1}^{\infty})]ds\Big|.
\end{aligned}
$$

So, as a result of condition (C_3), we can infer there exists $T > 0$ such that for $t > T$, we have

$$
\begin{aligned}
&\Big|G_n((x_j)_{j=1}^{\infty})(t) - G_n((y_j)_{j=1}^{\infty})(t)\Big| \\
&\le \varphi(\max_{1\le i\le n}|x_i(t)-y_i(t)|)\Big(1 + b_n(r_0)\int_0^{\infty} a_n(s)|k_n(t,s)|ds\Big) \\
&+ \Big(\varphi(\max_{1\le i\le n}|x_i(t)|) + M\Big)2b_n(r_0)\int_0^{\infty} |k_n(t,s)|a_n(s)ds \\
&\le \varphi(\varepsilon)\big(1 + b_n(r_0)Q\big) + 2\big(\varphi(r_0) + M\big)b_n(r_0)\varepsilon.
\end{aligned}
$$

Now, we assume that $t \in [0,T]$. By applying the assumptions, we have

$$\begin{aligned}
&\Big|G_n((x_j)_{j=1}^{\infty})(t) - G_n((y_j)_{j=1}^{\infty})(t)\Big| \leq \\
&\leq \varphi(\max_{1\leq i\leq n}|x_i(t)-y_i(t)|)\big(1+b_n(r_0)Q\big) \\
&+\Big(\varphi(\max_{1\leq i\leq n}|x_i(t)|)+M\Big)\Big|\int_0^T k_n(t,s)[h_n(s,(x_j(s))_{j=1}^{\infty})-h_n(s,(y_j(s))_{j=1}^{\infty})]ds\Big| \\
&+\Big(\varphi(\max_{1\leq i\leq n}|x_i(t)|)+M\Big)\Big|\int_T^{\infty} k_n(t,s)[h_n(s,(x_j(s))_{j=1}^{\infty})-h_n(s,(y_j(s))_{j=1}^{\infty})]ds\Big| \\
&\leq \varphi(\varepsilon)\big(1+b_n(r_0)Q\big)+\big(\varphi(r_0)+M\big)\Big(TK_T^n\omega_{r_0}^T(h_n,\varepsilon)+2b_n(r_0)\int_T^{\infty}|k_n(t,s)|a_n(s)ds\Big),
\end{aligned}$$

where

$$\begin{aligned}
K_T^n &= \sup\{k_n(t,s) : t,s\in[0,T]\} \\
\omega_{r_0}^T(h_n,\varepsilon) &= \sup\{|h_n(s,(x_j)_{j=1}^{\infty})-h_n(s,(y_j)_{j=1}^{\infty})| : s\in[0,T], \\
&\qquad x_i,y_i\in[-r_0,r_0],\ |x_i-y_i|\leq\varepsilon\}.
\end{aligned}$$

By using the continuity of h_n on the compact set $[0,T]\times[-r_0,r_0]^{\omega}$ (Tychonoff's theorem implies that $[-r_0,r_0]^{\omega}$ is a compact space), we have $\omega_{r_0}^T(h_n,\varepsilon)\longrightarrow 0$ as $\varepsilon\longrightarrow 0$. Moreover, in view of assumption (C_4) we can choose T in such a way that last term of the above estimate is sufficiently small. Thus G_n is a continuous function on $(\bar{B}_{r_0})^{\omega}$. Also with similar argument and using conditions $(B_1)-(B_2)$ we have F is a continuous function on $(\bar{B}_{r_0})^{\omega}$. Now we show that F_n and G_n satisfy all the conditions of Colloraly 3. Let X_j and Y_j be nonempty and bounded subsets of $\bar{B}_{r_0}$ for all $j\in\mathbb{N}$ such that $\sup_i(\mu(X_i))<\infty$ and $\sup_i(\mu(Y_i))<\infty$. Assume that $T>0$ and $\varepsilon>0$ are arbitrary constants. Also we take $t_1,t_2\in[0,T]$, with $|t_2-t_1|\leq\varepsilon$ and $x_j\in X_j$ and $y_j\in Y_j$ for all $j\in\mathbb{N}$. Then we have

$$
\begin{aligned}
&\Big|F_n((x_j)_{j=1}^{\infty})(t_2) - F_n((x_j)_{j=1}^{\infty})(t_1)\Big| \\
&= \Big| f_n(t_2, x_1(t_2), \dots x_n(t_2)) + q_n(t_2, x_1(t_2), \dots x_n(t_2)) \\
&\quad \cdot \int_0^{\beta_n(t_2)} g_n(t_2, s, (x_j(s))_{j=1}^{\infty}) ds \\
&\quad - f_n(t_1, x_1(t_1), \dots x_n(t_1)) + q_n(t_1, x_1(t_1), \dots x_n(t_1)) \\
&\quad \cdot \int_0^{\beta_n(t_1)} g_n(t_1, s, (x_j(s))_{j=1}^{\infty}) ds \Big| \\
&\leq |f_n(t_2, x_1(t_2), \dots x_n(t_2)) - f_n(t_1, x_1(t_2), \dots x_n(t_2))| \\
&\quad + |f_n(t_1, x_1(t_2), \dots x_n(t_2)) - f_n(t_1, x_1(t_1), \dots x_n(t_1))| \\
&\quad + |q_n(t_2, x_1(t_2), \dots x_n(t_2)) - q_n(t_1, x_1(t_2), \dots x_n(t_2))| \Big| \\
&\quad \cdot \int_0^{\beta_n(t_1)} g_n(t_2, s, (x_j(s))_{j=1}^{\infty}) ds \Big| \\
&\quad + |q_n(t_1, x_1(t_2), \dots x_n(t_2)) - q_n(t_1, x_1(t_1), \dots x_n(t_1))| \Big| \\
&\quad \cdot \int_0^{\beta_n(t_2)} g_n(t_2, s, (x_j(s))_{j=1}^{\infty}) ds \Big| \\
&\quad + |q_n(t_1, x_1(t_1), \dots x_n(t_1))| \Big| \int_0^{\beta_n(t_2)} g_n(t_2, s, (x_j(s))_{j=1}^{\infty}) ds \\
&\quad - \int_0^{\beta_n(t_1)} g_n(t_2, s, (x_j(s))_{j=1}^{\infty}) ds \Big| \\
&\quad + |q_n(t_1, x_1(t_1), \dots x_n(t_1))| \Big| \int_0^{\beta_n(t_1)} g_n(t_2, s, (x_j(s))_{j=1}^{\infty}) ds \\
&\quad - \int_0^{\beta_n(t_1)} g_n(t_1, s, (x_j(s))_{j=1}^{\infty}) ds \Big| \\
&\leq \omega_{r_0}^T(f_n, \varepsilon) + \varphi(\max_{1 \leq i \leq n} |x_i(t_1) - x_i(t_2)|) + D\omega_{r_0}^T(q_n, \varepsilon) \\
&\quad + D\varphi(\max_{1 \leq i \leq n} |x_i(t_1) - x_i(t_2)|) + |q_n(t_1, x_1(t_1), \dots x_n(t_1))| \Big| \\
&\quad \cdot \int_{\beta(t_1)}^{\beta(t_2)} g_n(t_2, s, (x_j(s))_{j=1}^{\infty}) ds \Big| + |q_n(t_1, x_1(t_1), \dots x_n(t_1))| \Big| \\
&\quad \cdot \int_0^{\beta_n(t_1)} [g_n(t_2, s, (x_j(s))_{j=1}^{\infty}) ds - g_n(t_1, s, (x_j(s))_{j=1}^{\infty})] ds \Big| \\
&\leq \omega_{r_0}^T(f_n, \varepsilon) + D\omega_{r_0}^T(q_n, \varepsilon) + (1 + D)\varphi(\max_{1 \leq i \leq n} \omega^T(x_i, \varepsilon)) \\
&\quad + \beta_T Q_{r_0}^T \omega_{r_0}^T(g_n, \varepsilon) + Q_{r_0}^T U_{r_0}^T \omega^T(\beta, \varepsilon).
\end{aligned}
$$

Thus, we deduce

$$\begin{aligned}\left|F_n((x_j)_{j=1}^{\infty})(t_2)-F_n((x_j)_{j=1}^{\infty})(t_1)\right| &\leq \omega_{r_0}^T(f_n,\varepsilon)\\ &+D\omega_{r_0}^T(q_n,\varepsilon)+(1+D)\varphi(\max_{1\leq i\leq n}\omega^T(x_i,\varepsilon))\\ &+\beta_T Q_{r_0}^T\omega_{r_0}^T(g_n,\varepsilon)+Q_{r_0}^T U_{r_0}^T\omega^T(\beta,\varepsilon),\end{aligned} \tag{4.3}$$

and with similar argument, we get

$$\begin{aligned}&\left|G_n((y_j)_{j=1}^{\infty})(t_2)-G_n((y_j)_{j=1}^{\infty})(t_1)\right|\\ &\leq \omega_{r_0}^T(f_n,\varepsilon)+D\omega_{r_0}^T(q_n,\varepsilon)+(1+Qb(r_0))\varphi(\max_{1\leq i\leq n}\omega^T(x_i,\varepsilon))\\ &+TQ_{r_0}^T V_{r_0}^T{\omega_{r_0}}^T(k_n,\varepsilon)+Q_{r_0}^T b_n(r_0)\int_T^{\infty}[|k_n(t_2,s)|+|k_n(t_1,s)|]a(s)ds,\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}\beta_T &= \sup\{\beta(t): t\in[0,T]\},\\ \omega_{r_0}^T(f_n,\varepsilon) &= \sup\{|f_n(t_1,x_1,\ldots,x_n)-f_n(t_2,x_1,\ldots,x_n)|\\ &\quad : t_1,t_2\in[0,T], |t_2-t_1|\leq\varepsilon, |x_i|\leq r_0\},\\ \omega_{r_0}^T(q_n,\varepsilon) &= \sup\{|q_n(t_1,x_1,\ldots,x_n)-q_n(t_2,x_1,\ldots,x_n)|\\ &\quad : t_1,t_2\in[0,T], |t_2-t_1|\leq\varepsilon, |x_i|\leq r_0\},\\ {\omega_{r_0}}^T(g_n,\varepsilon) &= \sup\{|g_n(t_1,s,(x_j)_{j=1}^{\infty})-g_n(t_2,s,(x_j)_{j=1}^{\infty})|\\ &\quad : t_1,t_2,s\in[0,T], |t_2-t_1|\leq\varepsilon, s\in[0,\beta_T], |x_j|\leq r_0\},\\ \omega^T(\beta,\varepsilon) &= \sup\{|\beta(t)-\beta(s)| : t,s\in[0,T], |t-s|\leq\varepsilon\},\\ U_{r_0}^T &= \sup\{|g_n(t,s,(x_j)_{j=1}^{\infty})| : t\in[0,T], s\in[0,\beta_T], x_i\in[-r_0,r_0]\},\\ Q_{r_0}^T &= \sup\{|q_n(t,x_1,\ldots,x_n)| : t\in[0,T], x_i\in[-r_0,r_0]\},\\ \omega_{r_0}^T(k_n,\varepsilon) &= \sup\{|k_n(t_1,s)-k_n(t_2,s)| : t_1,t_2,s\in[0,T], |t_2-t_1|\leq\varepsilon\},\\ V_{r_0}^T &= \sup\{|h_n(s,(x_j)_{j=1}^{\infty})| : s\in[0,T], x_i\in[-r_0,r_0]\}.\end{aligned}$$

Since x_i is an arbitrary element of X_i and y_i was an arbitrary element of Y_i for all $i\in\mathbb{N}$ in (4.3) and (4.4), we obtain

$$\begin{aligned}\omega^T(F_n(\prod_{i=1}^{\infty}X_i) &\leq \omega_{r_0}^T(f_n,\varepsilon)+D\omega_{r_0}^T(q_n,\varepsilon)+(1+D)\varphi(\max_{1\leq i\leq n}\omega^T(X_i,\varepsilon))\\ &+\beta_T Q_{r_0}^T\omega_{r_0}^T(g_n,\varepsilon)+Q_{r_0}^T U_{r_0}^T\omega^T(\beta,\varepsilon),\end{aligned}$$

$$\begin{aligned}&\omega^T(G_n(\prod_{i=1}^{\infty}Y_i)\leq \omega_{r_0}^T(f_n,\varepsilon)+D\omega_{r_0}^T(q_n,\varepsilon)+(1+Qb_n(r_0))\varphi(\max_{1\leq i\leq n}\omega^T(X_i,\varepsilon))\\ &+TQ_{r_0}^T V_{r_0}^T{\omega_{r_0}}^T(k_n,\varepsilon)+Q_{r_0}^T b_n(r_0)\int_T^{\infty}[|k_n(t_2,s)|+|k_n(t_1,s)|]a(s)ds,\end{aligned}$$

and by the uniform continuity of f_n, q_n, g_n and β on the compact sets $[0,T]\times[-r_0,r_0]^n$, $[0,T]\times[-r_0,r_0]^n$, $[0,T]\times[0,\beta_T]\times[-r_0,r_0]^\omega$ and $[0,T]$ respectively, we have ${\omega_{r_0}}^T(f_n,\varepsilon)\longrightarrow 0$, ${\omega_{r_0}}^T(q_n,\varepsilon)\longrightarrow 0$, ${\omega_{r_0}}^T(g_n,\varepsilon)\longrightarrow 0$ and $\omega^T(\beta,\varepsilon)\longrightarrow 0$ as $\varepsilon\longrightarrow 0$. Therefore we obtain

$$\omega_0{}^T(F_n(\prod_{i=1}^{\infty}X_i))\leq(1+D)\varphi(\max_{1\leq i\leq n}\omega_0^T(X_i)),$$

$$\begin{aligned}\omega_0^T(G_n(\prod_{i=1}^{\infty}Y_i)&\leq(1+Qb_n(r_0))\varphi(\max_{1\leq i\leq n}\omega_0^T(Y_i,\varepsilon))\\&\quad+Q_{r_0}^Tb_n(r_0)\int_T^{\infty}[|k_n(t_2,s)|+|k_n(t_1,s)|]a(s)ds.\end{aligned}$$

Now taking $T\longrightarrow\infty$ and by using of assumption (C_4), we get

$$\omega_0(F_n(\prod_{i=1}^{\infty}X_i))\leq(1+D)\varphi(\max_{1\leq i\leq n}\omega_0(X_i)),\tag{4.5}$$

and

$$\omega_0(G_n(\prod_{i=1}^{\infty}Y_i))\leq(1+Qb_n(r_0))\varphi(\max_{1\leq i\leq n}\omega_0(Y_i)).\tag{4.6}$$

On the other hand, for all $x_i,u_i\in X_i$, $y_i,v_i\in Y_i$ $(i\in\mathbb{N})$ and $t\in\mathbb{R}_+$, we get

$$\begin{aligned}&\left|F_n((x_j)_{j=1}^{\infty})(t)-F_n((u_j)_{j=1}^{\infty})(t)\right|\leq(1+D)\varphi(\max_{1\leq i\leq n}|x_i(t)-u_i(t)|)\\&+(\varphi(r_0)+M)|\int_0^{\beta(t)}|g_n(t,s,(x_j(s))_{j=1}^{\infty})-g_n(t,s,(u_j(s))_{j=1}^{\infty})|ds,\end{aligned}$$

and with similar argument

$$\begin{aligned}&\left|G_n((y_j)_{j=1}^{\infty})(t)-G_n((v_j)_{j=1}^{\infty})(t)\right|\leq(1+Qb_n(r_0))\varphi(\max_{1\leq i\leq n}|y_i(t)-v_i(t)|)\\&+(\varphi(r_0)+M)|\int_0^{\infty}k_n(t,s)[h_n(t,s,(y_j(s))_{j=1}^{\infty})-h_n(t,s,(v_j(s))_{j=1}^{\infty})]ds\\&\leq(1+Qb_n(r_0))\varphi(\max_{1\leq i\leq n}|y_i(t)-v_i(t)|)\\&+(\varphi(r_0)+M)2b_n(r_0)\int_0^{\infty}|k_n(t,s)|a(s)ds.\end{aligned}$$

Thus

$$\begin{aligned}&diam(F_n(\prod_{i=1}^{\infty}X_i)(t))\leq(1+D)\varphi(\max_{1\leq i\leq n}diam(X_i(t)))\\&+(\varphi(r_0)+M)|\int_0^{\beta(t)}[g_n(t,s,(x_j(s))_{j=1}^{\infty})-g_n(t,s,(u_j(s))_{j=1}^{\infty})]ds,\end{aligned}\tag{4.7}$$

and

$$\begin{aligned} diam(G_n(\prod_{i=1}^{\infty} Y_i)(t)) &\leq (1+Qb_n(r_0))\varphi(\max_{1\leq i\leq n} diam(Y_i(t))) \\ &+ (\varphi(r_0)+M)2b_n(r_0)\int_0^{\infty} |k_n(t,s)|a(s)ds. \end{aligned} \tag{4.8}$$

If take $t \longrightarrow \infty$ in the inequalities (4.7) and (4.8), then using (B_2) and (C_3), we have

$$\limsup_{t\longrightarrow\infty} diamF_n(\prod_{i=1}^{\infty} X_i)(t) \leq (1+D)\varphi(\max_{1\leq i\leq n} \limsup_{t\longrightarrow\infty} diam(X_i(t))), \tag{4.9}$$

and

$$\limsup_{t\longrightarrow\infty} diamG_n(\prod_{i=1}^{\infty} Y_i)(t) \leq (1+Qb_n(r_0))\varphi(\max_{1\leq i\leq n} \limsup_{t\longrightarrow\infty} diam(X_i(t))). \tag{4.10}$$

Further, combining (4.5) and (4.9), we get

$$\begin{aligned} \limsup_{t\to\infty} diamF_n(\prod_{i=1}^{\infty} X_i)(t) + \omega_0(F_n(\prod_{i=1}^{\infty} X_i)) &\leq (1+D) \\ [\varphi(\max_{1\leq i\leq n} \omega_0(X_i)) + \varphi(\max_{1\leq i\leq n} \limsup_{t\to\infty} diam(X_i(t)))], \end{aligned} \tag{4.11}$$

from (4.6) and (4.10) we have

$$\begin{aligned} \limsup_{t\to\infty} diamG_n(\prod_{i=1}^{\infty} Y_i)(t) + \omega_0(G_n(\prod_{i=1}^{\infty} Y_i)) \leq &(1+Qb_n(r_0))[\varphi(\max_{1\leq i\leq n} \omega_0(Y_i)) \\ &+ \varphi(\max_{1\leq i\leq n} \limsup_{t\to\infty} diam(Y_i(t)))]. \end{aligned} \tag{4.12}$$

Since φ is concave, (4.11) and (4.12) imply

$$\frac{1}{4}\mu(F_n(\prod_{i=1}^{\infty} X_i)) + \frac{1}{4}\mu(G_n(\prod_{i=1}^{\infty} Y_i)) \leq \varphi'\Big(\frac{1}{4}\sup_i \mu(X_i) + \frac{1}{4}\sup_i \mu(Y_i)\Big),$$

where $\varphi'(t) = \Big(\max\Big\{1+D, \sup_{n\in\mathbb{N}}\{1+Qb_n(r_0)\}\Big\}\Big)\varphi(t)$. Taking $\mu' = \frac{1}{4}\mu$ and $\psi(t) = t$. Then we get

$$\mu'(F_n(\prod_{i=1}^{\infty} X_i)) + \mu'(G_n(\prod_{i=1}^{\infty} Y_i)) \leq \varphi'\Big(\sup_i\{\mu'(X_i)+\mu'(Y_i)\}\Big).$$

Now by using Colloraly 3, there exist $(x_i)_{i=1}^{\infty}, (y_i)_{i=1}^{\infty} \in (BC(\mathbb{R}_+))^{\omega}$ such that

$$\begin{aligned} x_n(t) &= f_n(t, x_1(t), \ldots, x_n(t)) \\ &+ q_n(t, x_1(t), \ldots, x_n(t)) \int_0^{\beta_n(t)} g_n(t, s, (x_j(s))_{j=1}^{\infty})ds, \end{aligned}$$

and

$$y_n(t) = f_n(t, y_1(t), \dots, y_n(t)) + q_n(t, y_1(t), \dots, y_n(t)) \int_0^\infty k_n(t,s) h_n(s, (y_j(s))_{j=1}^\infty) ds\Big),$$

and this completes the proof. □

Example 2. Consider the following system of functional integral equations

$$x_n(t) = \frac{1}{4}\arctan(\frac{1}{n}\sum_{i=1}^n |x_i(t)|) + \int_0^{\frac{\sin t}{n}} \frac{s\cos(x_n(s^2))}{e^{t^2}} ds, \tag{4.13}$$

and

$$y_n(t) = \frac{1}{4}\arctan(\frac{1}{n}\sum_{i=1}^n |y_i(t)|) + \int_0^\infty \frac{t(e^{-ns}-1)}{(t^2+1)n} \sum_{i=1}^\infty e^{-is} \sqrt[2n-1]{y_i(s)} ds. \tag{4.14}$$

Eq. (4.13) is a special case of Eq. (1.1) and Eq. (4.14) is a special case of Eq. (1.2) where

$$\begin{aligned}
f_n(t, x_1, \dots, x_n) &= \frac{1}{4}\arctan(\frac{1}{n}\sum_{i=1}^n |x_i|),\\
q_n(t, x_1, \dots, x_n) &= 1,\\
g_n(t, s, (x_j)_{j=1}^\infty) &= \frac{s\cos x_n}{e^{t^2}},\\
k_n(t,s) &= \frac{t}{(t^2+1)n}(e^{-ns}-1),\\
h_n(s, (y_j)_{j=1}^\infty) &= \sum_{i=1}^\infty e^{-is} \sqrt[2n-1]{y_i},\\
a_n(s) &= \frac{1}{e^s-1},\ b_n(r) = \sqrt[2n-1]{r},\ \beta_n(t) = \frac{\sin t}{n}.
\end{aligned}$$

Suppose that $t \in \mathbb{R}_+$ and $|x_i| \geq |y_i|$. Now, by taking $\varphi(t) = \frac{1}{4}\arctan(t)$ we have

$$\begin{aligned}
|f_n(t, x_1, \dots, x_n) - f_n(t, y_1, \dots, y_n)| \leq& \frac{1}{4}|\arctan(\frac{1}{n}\sum_{i=1}^n |x_i|) - \arctan(\frac{1}{n}\sum_{i=1}^n |y_i|)|\\
\leq& \frac{1}{4}\arctan(\frac{1}{n}\sum_{i=1}^n |x_i - y_i|)\\
\leq& \frac{1}{4}\arctan(\max_{1\leq i\leq n} |x_i - y_i|)\\
=& \varphi(\max_{1\leq i\leq n} |x_i - y_i|).
\end{aligned} \tag{4.15}$$

The case $|y_i| \geq |x_i|$ can be treated in the same way. Moreover,

$$M := \sup\Big\{\max\{|f_n(t,0,\dots,0)|, |q_n(t,0,\dots,0)|\} : t \in \mathbb{R}_+, n \in \mathbb{N}\Big\} = 1 < \infty.$$

Thus from (4.15) we infer that condition (A_1) holds. The condition (B_1) is obvious. Also, g_n is continuous and

$$D = \sup\{\Big|\int_0^{\beta_n(t)} g_n(t,s,(x_j(s))_{j=1}^{\infty})ds\Big| : t \in \mathbb{R}_+,\ x_j \in BC(\mathbb{R}_+), 1 \le n < \infty\}$$
$$= \sup\{\Big|\int_0^{\frac{\sin t}{n}} \frac{s\cos(x_n(s^2))}{e^{t^2}}ds\Big| : t \in \mathbb{R}_+,\ x_j \in BC(\mathbb{R}_+), 1 \le n < \infty\} = \frac{1}{2},$$

$$\lim_{t\longrightarrow\infty}\Big|\int_0^{\frac{\sin t}{n}} \frac{s\cos(x_n(s^2))}{e^{t^2}} - \frac{s\cos(y_n(s^2))}{e^{t^2}})ds\Big| = 0,$$

uniformly respect to $x_j, y_j \in BC(\mathbb{R}_+)$, which implies that condition (B_2) is satisfied. The condition (C_1) clearly is evident. In order to show that condition (C_2) is satisfied, let us assume that $\sup\limits_{1\le i<\infty} |y_i| < \infty$, so we have

$$|h_n(t,(y_j)_{j=1}^{\infty})| = \sum_{i=1}^{\infty} e^{-it}\sqrt[2n-1]{y_i} \le \sum_{i=1}^{\infty} e^{-it}\sqrt[2n-1]{\sup_{1\le i<\infty}|y_i|}$$
$$\le \frac{1}{e^t-1}\sqrt[2n-1]{\sup_{1\le i<\infty}|y_i|}$$
$$= a_n(t)b_n(\sup_{1\le i<\infty}|y_i|).$$

On the other hand, the function $s \longrightarrow a_n(s)k_n(t,s)$ is integrable over $\mathbb{R}_+$ for any fixed $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$. Thus, condition (C_2) is valid. Further, we get:

$$Q = \sup\{\int_0^{\infty} a_n(s)|k_n(t,s)|ds : t \in \mathbb{R}_+, n \in \mathbb{N}\}$$
$$= \sup\{\frac{t}{(t^2+1)n}\int_0^{\infty}\frac{e^{-ns}-1}{e^s-1}ds : t \in \mathbb{R}_+, n \in \mathbb{N}\}$$
$$= \sup\{\frac{t}{(t^2+1)n}n : t \in \mathbb{R}_+, n \in \mathbb{N}\} = \frac{1}{2} < \infty,$$

and

$$\lim_{t\longrightarrow\infty}\int_0^{\infty} a_n(s)|k_n(t,s)|ds = \lim_{t\longrightarrow\infty}\frac{t}{(t^2+1)n}\int_0^{\infty}\frac{e^{-ns}-1}{e^s-1}ds = \lim_{t\longrightarrow\infty}\frac{t}{t^2+1} = 0.$$

This impliese that the condition (C_3) holds. Moreover, for arbitrarily fixed $T > 0$ we obtain:

$$\int_T^{\infty} a_n(s)|k_n(t,s)|ds = \frac{t}{(t^2+1)n}\int_T^{\infty}\frac{e^{-ns}-1}{e^s-1}ds$$
$$\le \frac{1}{2n}[e^{-nT} + e^{-(n-1)T} + e^{-(n-2)T} + \ldots + e^{-T}].$$

From the above estimate, we infer that condition (C_4) holds. It is easy to see that each number $r \geq 4$ satisfies the inequality in condition (D_1), i.e.,

$$(1+D)\big(\varphi(r)+M\big) = (1+\frac{1}{2})\big(\frac{1}{4}\arctan(r)+1\big) \leq r,$$

and

$$(1+Qb_n(r))\big(\varphi(r)+M\big) = (1+\frac{1}{2}\sqrt[2n-1]{r})\big(\frac{1}{4}\arctan(r)+1\big) \leq r,$$

for all $n \in \mathbb{N}$ such that

$$\Big(\max\Big\{1+D, \sup_{n\in\mathbb{N}}\{1+Qb_n(r_0)\}\Big\}\Big)\varphi(t) = \Big(\max\Big\{1+\frac{1}{2}, \sup_{n\in\mathbb{N}}\{1+\frac{1}{2}\sqrt[2n-1]{4}\}\Big\}\Big)\frac{1}{4}\arctan(t) < t.$$

Thus, we can take $r_0 = 4$. Consequently, all the conditions of Theorem 2 are satisfied. Hence the system of functional integral equations (4.13) and (4.14) have at least one solution put which belongs to the space $(BC(\mathbb{R}_+))^{\omega}$.

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SOME APPLICATIONS OF GENERALIZED SRIVASTAVA-ATTIYA OPERATOR TO THE BI-CONCAVE FUNCTIONS

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Abstract. In this present investigation, we are concerned with the class $\Omega^{m,k}_{\Sigma;\mu,b}C_0(\alpha)$ of bi-concave functions defined by using the generalized Srivastava-Attiya operator. Moreover, we derive some coefficient inequalities for functions in this class.

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1. Introduction

Let A indicate an analytic function family, which is normalized under the condition of $f(0) = f'(0) - 1 = 0$ in $\mathbb{D} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and given by the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Further, by S we shall denote the class of all functions in A which are univalent in $\mathbb{D}$.

It is well known that every function $f \in S$ has an inverse f^{-1}, defined by $f^{-1}(f(z)) = z,\ (z \in \mathbb{D})$ and $f\left(f^{-1}(w)\right) = w,\ \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right)$. In fact, the inverse function is given by

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots$$

(for details, see Duren [13]). A function $f \in A$ is said to be bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$. Let Σ stand for the class of bi-univalent functions defined in the unit disk $\mathbb{D}$. A brief history and interesting examples of functions in the class Σ can be found in the pioneering work on this subject by Srivastava *et al.* [34], which has apparently revived the study of bi-univalent functions in recent years. In fact, ever since the publication by Srivastava *et al.* [34], a huge flood of papers have appeared and are still appearing in the literature dealing with various subclasses of the bi-univalent and other related function classes (see, for example, [6, 7, 20, 24, 26, 32, 35, 37]). But the coefficient problem for each one of the following

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Taylor-Maclaurin coefficients

$$|a_n|, \quad n \in \mathbb{N}\backslash\{1,2\}; \quad \mathbb{N} = \{1,2,3,...\}$$

is still an open problem. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([11, 16, 33, 36, 38]).

The study of operators plays an important role in Geometric Function Theory in Complex Analysis and its related fields. Recently, the interest in this area has been increasing because it permits detailed investigations of problems with physical applications. For example, many derivative and integral operators can be written in terms of convolution of certain analytic functions. For functions

$$f_j(z) = \sum_{n=0}^{\infty} a_{n,j} z^n \quad (j = 1,2)$$

analytic in $\mathbb{D}$, we define the Hadamard product of f_1 and f_2 as

$$(f_1 * f_2)(z) = \sum_{n=0}^{\infty} a_{n,1} a_{n,2} z^n = (f_2 * f_1)(z) \quad (z \in \mathbb{D}). \tag{1.2}$$

In terms of the Hadamard product (or convolution), the Dziok-Srivastava linear convolution operator involving the generalized hypergeometric function was introduced and studied systematically by Dziok and Srivastava [14, 15] and (subsequently) by many other authors (see, for details, [17, 18, 29]). We recall here a general Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ defined in [31] by

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$$

$\left(a \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ s \in \mathbb{C},\ \text{when } |z| < 1;\ \Re(s) > 1 \text{ when } |z| = 1\right)$, where $\mathbb{Z}_0^- := \mathbb{Z}\backslash\mathbb{N}$. Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ can be found in [9], and the references stated there in (see also [20], [21, 30]). Srivastava and Attiya [30] (also see [2, 8, 19]) introduced and investigated the linear operator:

$$\Omega_b^{\mu} : A \to A$$

defined in terms of the Hadamard product by

$$\Omega_b^{\mu} f(z) = \left(G_b^{\mu} * f\right)(z), \quad \left(z \in \mathbb{D};\ b \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ \mu \in \mathbb{C};\ f \in A\right), \tag{1.3}$$

where, for convenience,

$$G_b^{\mu}(z) := (1+b)^{\mu}[\Phi(z,\mu,b) - b^{-\mu}] \quad (z \in \mathbb{D}). \tag{1.4}$$

Next, we recall the following relationships which follow easily by using (1.1), (1.3) and (1.4)

$$\Omega_b^{\mu} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b}\right)^{\mu} a_n z^n. \tag{1.5}$$

Motivated essentially by the Srivastava-Attiya operator, Murugusundaramoorthy [25] introduced the generalized integral operator $\Omega_{\mu,b}^{m,k}$ given by

$$\Omega_{\mu,b}^{m,k} f(z) = z + \sum_{n=2}^{\infty} C_n^m(b,\mu,k) a_n z^n, \tag{1.6}$$

where

$$C_n^m(b,\mu,k) = \left| \left(\frac{1+b}{n+b} \right)^{\mu} \right| \frac{m!(n+k-2)!}{(k-2)!(n+m-1)!} \tag{1.7}$$

and (throughout this paper unless otherwise mentioned) the parameters μ, b are constrained as $b \in \mathbb{C} \backslash \mathbb{Z}_0^-; \mu \in \mathbb{C}, k \geq 2$ and $m > -1$. It is of interest to note that $\Omega_{\mu,b}^{1,2}$ is the Srivastava-Attiya operator and $\Omega_{0,b}^{m,k}$ is the well-known Choi-Saigo-Srivastava operator (see [22]). Suitably specializing the parameters m, k, μ and b in $\Omega_{\mu,b}^{m,k} f(z)$, we can get various integral operators introduced by Alexander [1] and Bernardi [4], Ling and Liu [22], Livingstone [23].

2. Preliminaries

Conformal maps of the unit disk onto convex domains are a classical topic. Recently Avkhadiev and Wirths [3] discovered that conformal maps onto concave domains (the complements of convex closed sets) have some novel properties.

A function $f : \mathbb{D} \to \mathbb{C}$ is said to belong to the family $C_0(\alpha)$ if f satisfies the following conditions:

- f is analytic in $\mathbb{D}$ with the standard normalization $f(0) = f'(0) - 1 = 0$. In addition it satisfies $f(1) = \infty$,
- f maps $\mathbb{D}$ conformally onto a set whose complement with respect to $\mathbb{C}$ is convex,
- the opening angle of $f(\mathbb{D})$ at ∞ is less than or equal to $\pi\alpha$, $\alpha \in (1,2]$.

The class $C_0(\alpha)$ is referred to as the class of concave univalent functions and for a detailed discussion about concave functions, we refer to Avkhadiev and Wirths [3], Cruz and Pommerenke [10] and references there in.

In particular, the inequality

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < 0 \qquad (z \in \mathbb{D})$$

is used - sometimes also as a definition - for concave functions $f \in C_{0_O}$ (see e.g. [27] and others).

Bhowmik et al. [5] showed that an analytic function f maps $\mathbb{D}$ onto a concave domain of angle $\pi\alpha$, if and only if $\Re(P_f(z)) > 0$, where

$$P_f(z) = \frac{2}{\alpha - 1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z} - 1 - z\frac{f''(z)}{f'(z)}\right].$$

There has been a number of investigations on basic subclasses of concave univalent functions (see, for example [12], [28]).
To establish our new subclass we require the following Definition.

Definition 1. Let the functions $h, p : \mathbb{D} \to \mathbb{C}$ be so constrained that

$$\min\{\Re(h(z)), \Re(p(z))\} > 0$$

and

$$h(0) = p(0) = 1.$$

Motivated by each of the above definitions, we now define a new subclass of bi-concave analytic functions involving the generalized integral operator $\Omega_{\mu,b}^{m,k}$.

Definition 2. A function $f \in \Sigma$ given by (1.1) is said to be in the class

$$\Omega_{\Sigma;\mu,b}^{m,k} C_0(\alpha) \quad (b \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ \mu \in \mathbb{C};\ k \geq 2;\ m > -1;\ \alpha \in (1,2];\ z, w \in \mathbb{D})$$

the following conditions are satisfied:

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z} - 1 - z\frac{\left[\Omega_{\Sigma;\mu,b}^{m,k} f(z)\right]''}{\left[\Omega_{\Sigma;\mu,b}^{m,k} f(z)\right]'}\right] \in h(\mathbb{D}) \tag{2.1}$$

and

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1-w}{1+w} - 1 - w\frac{\left[\Omega_{\Sigma;\mu,b}^{m,k} g(w)\right]''}{\left[\Omega_{\Sigma;\mu,b}^{m,k} g(w)\right]'}\right] \in p(\mathbb{D}), \tag{2.2}$$

where $g(w) = f^{-1}(w)$.

3. Main results and their consequences

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\Omega_{\Sigma;\mu,b}^{m,k} C_0(\alpha)$.

Theorem 1. *Let f given by (1.1) be in the class $\Omega_{\Sigma;\mu,b}^{m,k} C_0(\alpha)$. Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{8(\alpha+1)^2+(\alpha-1)^2\left(|h'(0)|^2+|p'(0)|^2\right)}{32\left[C_2^m(b,\mu,k)\right]^2} + \frac{\left(\alpha^2-1\right)\left(|h'(0)|+|p'(0)|\right)}{8\left[C_2^m(b,\mu,k)\right]^2}},\right. \tag{3.1}$$

$$\left.\sqrt{\frac{(\alpha+1)}{2\left|2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right|} + \frac{(\alpha-1)\left(|h''(0)|+|p''(0)|\right)}{16\left|2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right|}}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{8(\alpha+1)^2+(\alpha-1)^2\left(|h'(0)|^2+|p'(0)|^2\right)+4\left(\alpha^2-1\right)\left(|h'(0)|+|p'(0)|\right)}{32\left[C_2^m(b,\mu,k)\right]^2}\right.$$

$$+\frac{(\alpha-1)(|h''(0)|+|p''(0)|)}{48C_3^m(b,\mu,k)}, \tag{3.2}$$

$$\left.\frac{(\alpha-1)\left\{\left|3C_3^m(b,\mu,k)-\left[C_2^m(b,\mu,k)\right]^2\right||h''(0)|+\left[C_2^m(b,\mu,k)\right]^2|p''(0)|\right\}+12(\alpha+1)C_3^m(b,\mu,k)}{24C_3^m(b,\mu,k)\left|2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right|}\right\}.$$

Proof. Let $f\in\Omega_{\Sigma;\mu,b}^{m,k}C_0(\alpha)$ and g be the analytic extension of f^{-1} to $\mathbb{D}$. It follows from (2.1) and (2.2) that

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z}-1-z\frac{\left[\Omega_{\Sigma;\mu,b}^{m,k}f(z)\right]''}{\left[\Omega_{\Sigma;\mu,b}^{m,k}f(z)\right]'}\right]=h(z) \tag{3.3}$$

and

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1-w}{1+w}-1-w\frac{\left[\Omega_{\Sigma;\mu,b}^{m,k}g(w)\right]''}{\left[\Omega_{\Sigma;\mu,b}^{m,k}g(w)\right]'}\right]=p(w), \tag{3.4}$$

where h and p satisfy the conditions of Definiton 1. Furthermore, the functions h and p have the following Taylor-Maclaurin series expensions:

$$h(z)=1+h_1z+h_2z^2+\cdots$$

and

$$p(w)=1+p_1w+p_2w^2+\cdots,$$

respectively. Now, equating the coefficients in (3.3) and (3.4), we get

$$\frac{2\left[(\alpha+1)-2C_2^m(b,\mu,k)a_2\right]}{\alpha-1}=h_1, \tag{3.5}$$

$$\frac{2\left[(\alpha+1)-2\left(3C_3^m(b,\mu,k)a_3-2\left[C_2^m(b,\mu,k)\right]^2a_2^2\right)\right]}{\alpha-1}=h_2 \tag{3.6}$$

and

$$-\frac{2\left[(\alpha+1)-2C_2^m(b,\mu,k)a_2\right]}{\alpha-1}=p_1, \tag{3.7}$$

$$\frac{2\left[(\alpha+1)+2\left\{2\left[C_2^m(b,\mu,k)\right]^2a_2^2-3C_3^m(b,\mu,k)\left(2a_2^2-a_3\right)\right\}\right]}{\alpha-1}=p_2. \tag{3.8}$$

From (3.5) and (3.7), we find that

$$h_1=-p_1. \tag{3.9}$$

Also, from (3.5), we can write

$$a_2=\frac{\alpha+1}{2C_2^m(b,\mu,k)}-\frac{h_1(\alpha-1)}{4C_2^m(b,\mu,k)}. \tag{3.10}$$

Next, by using (3.5), (3.7), (3.9) and (3.10), we get

$$a_2^2 = \frac{(\alpha+1)^2}{4\left[C_2^m(b,\mu,k)\right]^2} + \frac{(\alpha-1)^2\left(h_1^2+p_1^2\right)}{32\left[C_2^m(b,\mu,k)\right]^2} - \frac{\left(\alpha^2-1\right)(h_1-p_1)}{8\left[C_2^m(b,\mu,k)\right]^2}. \tag{3.11}$$

By adding (3.6) to (3.8), we get

$$a_2^2 = \frac{(\alpha-1)(h_2+p_2)}{8\left(2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right)} - \frac{\alpha+1}{2\left(2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right)}. \tag{3.12}$$

Thus, we find from the equations (3.11) and (3.12) that

$$|a_2|^2 \leq \frac{(\alpha+1)^2}{4\left[C_2^m(b,\mu,k)\right]^2} + \frac{(\alpha-1)^2\left(|h'(0)|^2+|p'(0)|^2\right)}{32\left[C_2^m(b,\mu,k)\right]^2} + \frac{\left(\alpha^2-1\right)(|h'(0)|+|p'(0)|)}{8\left[C_2^m(b,\mu,k)\right]^2}$$

and

$$|a_2|^2 \leq \frac{(\alpha+1)}{2\left|2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right|} + \frac{(\alpha-1)(|h''(0)|+|p''(0)|)}{16\left|2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right|}.$$

Similarly, subtracting (3.8) from (3.6), we have

$$a_3 = a_2^2 - \frac{(\alpha-1)(h_2-p_2)}{24C_3^m(b,\mu,k)}. \tag{3.13}$$

Then, upon substituting the value of a_2^2 from (3.11) and (3.12) into (3.13), we deduce that

$$\begin{aligned} a_3 &= \\ &\frac{(\alpha+1)^2}{4\left[C_2^m(b,\mu,k)\right]^2} + \frac{(\alpha-1)^2\left(h_1^2+p_1^2\right)}{32\left[C_2^m(b,\mu,k)\right]^2} - \frac{\left(\alpha^2-1\right)(h_1-p_1)}{8\left[C_2^m(b,\mu,k)\right]^2} - \frac{(\alpha-1)(h_2-p_2)}{24C_3^m(b,\mu,k)} \\ &= \frac{(\alpha-1)(h_2+p_2)}{8\left(2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right)} - \frac{\alpha+1}{2\left(2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right)} - \frac{(\alpha-1)(h_2-p_2)}{24C_3^m(b,\mu,k)}. \end{aligned}$$

Consequently, we have

$$|a_3| \leq \frac{8(\alpha+1)^2+(\alpha-1)^2\left(|h'(0)|^2+|p'(0)|^2\right)+4\left(\alpha^2-1\right)(|h'(0)|+|p'(0)|)}{32\left[C_2^m(b,\mu,k)\right]^2} + \frac{(\alpha-1)(|h''(0)|+|p''(0)|)}{48C_3^m(b,\mu,k)}$$

and

$$|a_3| \leq \frac{(\alpha-1)\left\{\left|3C_3^m(b,\mu,k)-\left[C_2^m(b,\mu,k)\right]^2\right||h''(0)|+\left[C_2^m(b,\mu,k)\right]^2|p''(0)|\right\}+12(\alpha+1)C_3^m(b,\mu,k)}{24C_3^m(b,\mu,k)\left|2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right|}.$$

This completes the proof. □

4. Conclusions

It is easily seen that, by specializing the functions $h(z)$ and $p(z)$ involved in Theorem 1, some results for could be expressed as illustrative examples:

Corollary 1. *If we set*

$$h(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}=1+2\gamma z+2\gamma^2z^2+\ldots \quad (0<\gamma\leq 1),$$

and

$$p(z)=\left(\frac{1-z}{1+z}\right)^{\gamma}=1-2\gamma z+2\gamma^2z^2+\ldots \quad (0<\gamma\leq 1),$$

then inequalities (3.1) and (3.2) become

$$|a_2|\leq \min\left\{\frac{(\alpha+1)+(\alpha-1)\gamma}{2C_2^m(b,\mu,k)},\sqrt{\frac{(\alpha-1)\gamma^2+(\alpha+1)}{2\left|2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right|}}\right\}$$

and

$$|a_3|\leq \min\left\{\frac{(\alpha+1)^2+(\alpha-1)^2\gamma^2+2(\alpha^2-1)\gamma}{4\left[C_2^m(b,\mu,k)\right]^2}+\frac{(\alpha-1)\gamma^2}{6C_3^m(b,\mu,k)},\right.$$

$$\left.\frac{(\alpha-1)\left\{\left|3C_3^m(b,\mu,k)-\left[C_2^m(b,\mu,k)\right]^2\right|\gamma^2+\left[C_2^m(b,\mu,k)\right]^2\gamma^2\right\}+3(\alpha+1)C_3^m(b,\mu,k)}{6C_3^m(b,\mu,k)\left|2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right|}\right\}.$$

Corollary 2. *If we let*

$$h(z)=\frac{1+(1-2\beta)z}{1-z}=1+2(1-\beta)z+2(1-\beta)z^2+\cdots \quad (0\leq\beta<1),$$

and

$$p(z)=\frac{1-(1-2\beta)z}{1+z}=1-2(1-\beta)z+2(1-\beta)z^2+\cdots \quad (0\leq\beta<1),$$

then inequalities (3.1) and (3.2) become

$$|a_2|\leq \min\left\{\frac{(\alpha+1)+(\alpha-1)(1-\beta)}{2C_2^m(b,\mu,k)},\sqrt{\frac{(\alpha-1)(1-\beta)+(\alpha+1)}{2\left|2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right|}}\right\}$$

and

$$|a_3|\leq \min\left\{\frac{(\alpha+1)^2+(\alpha-1)^2(1-\beta)^2+2(\alpha^2-1)(1-\beta)}{4\left[C_2^m(b,\mu,k)\right]^2}+\frac{(\alpha-1)(1-\beta)}{6C_3^m(b,\mu,k)},\right.$$

$$\left.\frac{(\alpha-1)\left\{\left|3C_3^m(b,\mu,k)-\left[C_2^m(b,\mu,k)\right]^2\right|(1-\beta)+\left[C_2^m(b,\mu,k)\right]^2(1-\beta)\right\}+3(\alpha+1)C_3^m(b,\mu,k)}{6C_3^m(b,\mu,k)\left|2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right|}\right\}.$$

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AN APPLICATION ON DIFFERENTIAL EQUATIONS OF ORDER m

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Abstract. In this paper we introduce the classes $\mathcal{T}_n(p,\lambda,A,B)$ and $\mathcal{K}_a(p,\lambda,\mu,m,A,B)$ and derive distortion inequalities of the functions belonging to class $\mathcal{K}_a(p,\lambda,\mu,m,A,B)$. Further we apply to the $(n,\delta)-$ neighborhoods of functios in the class $\mathcal{K}_a(p,\lambda,\mu,m,A,B)$.

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Keywords: analytic function, multivalent function, subordination, $(n,\delta)-$ neighborhoods, coefficient bounds, distortion inequalities

1. Introduction and definitions

Let $\mathcal{T}_n(p)$ denote the class of functions $f(z)$ normalized by

$$f(z)=z^p-\sum_{k=n+p}^{\infty}a_kz^k \qquad (a_k\geq 0;\ n,p\in\mathbb{N}:=\{1,2,3,\cdots\}) \tag{1.1}$$

which are analytic and p-valent in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ on the complex plane $\mathbb{C}$.

Let f and F be analytic functions in the unit disk $\mathbb{U}$. A function f is said to be subordinate to F, written as $f\prec F$ or $f(z)\prec F(z)$, if there exists a Schwarz function $\omega:\mathbb{U}\to\mathbb{U}$ with $\omega(0)=0$ such that $f(z)=F\big(\omega(z)\big)$. In particular, if F is univalent in $\mathbb{U}$, we have the following equivalence:

$$f(z)\prec F(z)\Longleftrightarrow[f(0)=F(0)\wedge f(\mathbb{U})\subseteq F(\mathbb{U})].$$

Following the earlier investigations by Goodman [11] and Ruscheweyh [15] (see also [1–3,5,6,9,13]), we define the $(n,\delta)-$neighborhoods of functions $f\in\mathcal{T}_n(p)$ by

$$\mathcal{N}_{a,\delta}(f;g)=\left\{g\in\mathcal{T}_n(p):g(z)=z^p-\sum_{k=n+p}^{\infty}b_kz^k \text{ and } \sum_{k=n+p}^{\infty}k|a_k-b_k|\leq\delta\right\}. \tag{1.2}$$

Let $\mathcal{S}^*$ and $\mathcal{C}$ be the usual subclasses of functions which members are univalent, starlike and convex in $\mathbb{U}$, respectively.

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A function $f \in \mathcal{T}_n(p)$ is called p-valently starlike of order γ if it satisfies the conditions

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma \tag{1.3}$$

and

$$\int_0^{2\pi} \Re\left(\frac{zf'(z)}{f(z)}\right) d\theta = 2p\pi \tag{1.4}$$

for $0 \leq \gamma < p$, $p \in \mathbb{N}$ and $z \in \mathbb{U}$. We denote by $\mathcal{S}_n^*(p,\gamma)$ the class of all p-valently starlike functions of order γ. Furthermore, a function $f \in \mathcal{T}_n(p)$ is called p-valently convex of order γ if it satisfies the conditions

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \gamma \tag{1.5}$$

and

$$\int_0^{2\pi} \Re\left(1+\frac{zf''(z)}{f'(z)}\right) d\theta = 2p\pi \tag{1.6}$$

for $0 \leq \gamma < p$, $p \in \mathbb{N}$ and $z \in \mathbb{U}$. We denote by $\mathcal{C}_n(p,\gamma)$ the class of all p-valently convex functions of order γ.

Clearly, $\mathcal{S}^* := \mathcal{S}_1^*(1,0)$ and $\mathcal{C} := \mathcal{C}_1(1,0)$. We note that

$$f(z) \in \mathcal{C}_n(p,\gamma) \Leftrightarrow \frac{f'(z)}{p} \in \mathcal{S}_n^*(p,\gamma) \tag{1.7}$$

The classes $\mathcal{S}_n^*(p,\gamma)$ and $\mathcal{C}_n(p,\gamma)$ were introduced by Patil and Thakare [14].

Therefore, various subclasses of p-valent functions in $\mathbb{U}$ was studied by Altıntaş et al. in [8], Nunokawa et al. in [12] and Srivastava et al. in [16, 17].

A function $f \in \mathcal{T}_n(p)$ is called Janowski p-valently starlike if it satisfies the condition

$$\frac{zf'(z)}{f(z)} \prec p\frac{1+Az}{1+Bz} \tag{1.8}$$

for $-1 \leq A < B \leq 1$, $p \in \mathbb{N}$ and $z \in \mathbb{U}$. We denote by $\mathcal{S}_n^*(p,A,B)$ the class of all Janowski p-valently starlike functions.

Also, a function $f \in \mathcal{T}_n(p)$ is called Janowski p-valently convex if it satisfies the condition

$$1+\frac{zf''(z)}{f'(z)} \prec p\frac{1+Az}{1+Bz} \tag{1.9}$$

for $-1 \leq A < B \leq 1$, $p \in \mathbb{N}$ and $z \in \mathbb{U}$. We denote by $\mathcal{C}_n(p,A,B)$ the class of all Janowski p-valently convex functions.

We note that, $\mathcal{S}_n^*(p,\gamma) := \mathcal{S}_n^*(p,1-2\gamma,-1)$, $\mathcal{S}^* := \mathcal{S}_1^*(1,1,-1)$ and $\mathcal{C}_n(p,\gamma) := \mathcal{C}_n(p,1-2\gamma,-1)$, $\mathcal{C} := \mathcal{C}_1(1,1,-1)$.

Let $\mathcal{T}_n(p,\lambda,A,B)$ denote the subclass of $\mathcal{T}_n(p)$ consisting of functions $f(z)$ which satisfy the following inequality:

$$\frac{zf'(z)+\lambda z^2 f''(z)}{\lambda zf'(z)+(1-\lambda)f(z)} \prec p\frac{1+Az}{1+Bz} \tag{1.10}$$

where $0 \leq \lambda \leq 1$, $-1 \leq A < B \leq 1$, $p \in \mathbb{N}$, $z \in \mathbb{U}$. The class $\mathcal{T}_n(p,\lambda,A,B)$ was introduced and studied by Altıntaş in [3, 7].

Clearly, we have the following relationships:

$$\mathcal{S}_n^*(p,A,B) := \mathcal{T}_n(p,0,A,B) \text{ and } \mathcal{C}_n(p,A,B) := \mathcal{T}_n(p,1,A,B).$$

We note that these classes are studied in [10].

Recently, we have defined and studied in [1, 2, 4–6] the following second order differential equation:

$$z^2\frac{d^2w}{dz^2}+2(\mu+1)z\frac{dw}{dz}+\mu(\mu+1)w=(p+\mu)(p+\mu+1)g \tag{1.11}$$

where $w=f(z)\in\mathcal{T}_n(p)$, $g=g(z)$ satisfy the following inequality:

$$\Re\frac{zg'(z)+\lambda z^2 g''(z)}{\lambda zg'(z)+(1-\lambda)g(z)} > \alpha \tag{1.12}$$

where $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$, $p \in \mathbb{N}$, $\mu > -p$, $z \in \mathbb{U}$.

Definition 1. The following non-homogenous Cauchy-Euler differential equation of order 3 is

$$\begin{aligned} z^3\frac{d^3w}{dz^3}+3(\mu+2)z^2\frac{d^2w}{dz^2}+3(\mu+1)(\mu+2)z\frac{dw}{dz}+\mu(\mu+1)(\mu+2)w \\ =(p+\mu)(p+\mu+1)(p+\mu+2)g \end{aligned} \tag{1.13}$$

where $w=f(z)\in\mathcal{T}_n(p)$, $g=g(z)\in\mathcal{T}_n(p,\lambda,A,B)$ and $\mu>-p$.

This differential equation is defined and studied in [3].

Definition 2. The following non-homogenous Cauchy-Euler differential equation of order m is

$$\begin{aligned} z^m\frac{d^mw}{dz^m}+\binom{m}{1}(\mu+m-1)z^{m-1}\frac{d^{m-1}w}{dz^{m-1}}+\ldots+\binom{m}{r}\prod_{j=r}^{m-1}(\mu+j)z^r\frac{d^rw}{dz^r}+ \\ \ldots+\binom{m}{m}\prod_{j=0}^{m-1}(\mu+j)w=\prod_{j=0}^{m-1}(p+\mu+j)g \end{aligned} \tag{1.14}$$

where $w=f(z)\in\mathcal{T}_n(p)$, $g=g(z)\in\mathcal{T}_n(p,\lambda,A,B)$, $m\in\mathbb{N}^*:=\{2,3,\cdots\}$ and $\mu>-p$.

Finally $\mathcal{K}_u(p,\lambda,\mu,m,A,B)$ denote the subclass of the class $\mathcal{T}_n(p)$ consisting of functions $f(z)$, satisfying the equation (1.14) in Definition 2.

In this paper, we obtain coefficient bounds, distortion inequalities and $(n,\delta)-$ neighborhoods of functions $f\in\mathcal{T}_n(p)$ in the class $\mathcal{K}_u(p,\lambda,\mu,m,A,B)$.

2. Coefficient Bounds and Distortion Inequalities

For proving the main results in this paper, we will use the following lemmas.

Lemma 1 ([3]). *Let the function $\mathcal{T}_n(p)$ be defined by (1.1). Then $f(z)$ is in the class $\mathcal{T}_n(p,\lambda,A,B)$ if and only if*

$$\sum_{k=n+p}^{\infty}(k-p-pA+kB)(\lambda k-\lambda+1)a_k \leq p(B-A)(\lambda p-\lambda+1) \tag{2.1}$$

where $0\leq\lambda\leq 1$, $-1\leq A<B\leq 1$, $p\in\mathbb{N}$.

The result is sharp for the function $f(z)$ given by

$$f(z)=z^p-\frac{p(B-A)(\lambda p-\lambda+1)}{[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1]}z^{n+p}. \tag{2.2}$$

Lemma 2 ([3]). *Let the function $f(z)\in\mathcal{T}_n(p)$ defined by (1.1) be in the class $\mathcal{T}_n(p,\lambda,A,B)$. Then, we have*

$$\sum_{k=n+p}^{\infty}a_k \leq \frac{p(B-A)(\lambda p-\lambda+1)}{[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1]} \tag{2.3}$$

and

$$\sum_{k=n+p}^{\infty}ka_k \leq \frac{p(B-A)(\lambda p-\lambda+1)(n+p)}{[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1]}. \tag{2.4}$$

The distortion inequalities for functions in the class $\mathcal{K}_u(p,\lambda,\mu,m,A,B)$ are given by Theorem 1 below.

Theorem 1. *If a function $f\in\mathcal{T}_n(p)$ is in the class $\mathcal{K}_u(p,\lambda,\mu,m,A,B)$, then*

$$|f(z)|\leq|z|^p+\frac{p(B-A)(\lambda p-\lambda+1)\prod_{j=0}^{m-1}(p+\mu+j)}{(m-1)[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1]\prod_{j=0}^{m-2}(n+p+\mu+j)}|z|^{n+p} \tag{2.5}$$

and

$$|f(z)|\geq|z|^p-\frac{p(B-A)(\lambda p-\lambda+1)\prod_{j=0}^{m-1}(p+\mu+j)}{(m-1)[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1]\prod_{j=0}^{m-2}(n+p+\mu+j)}|z|^{n+p}. \tag{2.6}$$

Proof. We first suppose that a function $f\in\mathcal{T}_n(p)$ is in the class $\mathcal{K}_u(p,\lambda,\mu,m,A,B)$. Let the function $g(z)=z^p-\sum_{k=n+p}^{\infty}b_kz^k\in\mathcal{T}_n(p,\lambda,A,B)$ occurring in the non-homogenous Cauchy-Euler differential equation of order m in (1.14) with, of course,

$$b_k\geq 0 \qquad (k=n+p,n+p+1,\ldots).$$

Then, we readily find from (1.14) that

$$a_k = \frac{\prod_{j=0}^{m-1}(p+\mu+j)}{\prod_{j=0}^{m-1}(k+\mu+j)} b_k \qquad (k = n+p, n+p+1, \ldots). \tag{2.7}$$

so that

$$f(z) = z^p - \sum_{k=n+p}^{\infty} \frac{\prod_{j=0}^{m-1}(p+\mu+j)}{\prod_{j=0}^{m-1}(k+\mu+j)} b_k z^k. \tag{2.8}$$

Since $g \in \mathcal{T}_n(p,\lambda,A,B)$, the first assertion (2.3) of Lemma 2 yields the following inequality:

$$b_k \leq \frac{p(B-A)(\lambda p-\lambda+1)}{[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1]}. \tag{2.9}$$

Together with (2.8) and (2.9) yields that

$$|f(z)| \leq |z|^p + |z|^{n+p} \frac{p(B-A)(\lambda p-\lambda+1)}{[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1]} \sum_{k=n+p}^{\infty} \frac{\prod_{j=0}^{m-1}(p+\mu+j)}{\prod_{j=0}^{m-1}(k+\mu+j)} \tag{2.10}$$

and using the following identity that

$$\begin{aligned} &\sum_{k=n+p}^{\infty} \frac{1}{\prod_{j=0}^{m-1}(k+\mu+j)} \\ &= \frac{1}{(m-1)!} \sum_{k=n+p}^{\infty} \left[\frac{\binom{m-1}{0}}{k+\mu} - \frac{\binom{m-1}{1}}{k+\mu+1} + \cdots + (-1)^{m-1} \frac{\binom{m-1}{m-1}}{k+\mu+m-1} \right] \\ &= \frac{1}{m-1} \frac{1}{\prod_{j=0}^{m-2}(n+p+\mu+j)} \end{aligned} \tag{2.11}$$

where $\mu \in \mathbb{R} \setminus \{-n-p, -n-p-1, \ldots\}$. The assertion (2.5) of Theorem 1 follows at once from (2.10) with (2.11). The assertion (2.6) of Theorem 1 can be proven by similarly. □

Corollary 1 ([3]). *If $f \in \mathcal{K}_a(p,\lambda,\mu,2,A,B)$, then we have*

$$|f(z)| \leq |z|^p + \frac{p(B-A)(\lambda p-\lambda+1)(p+\mu)(p+\mu+1)}{[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1](n+p+\mu)} |z|^{n+p}$$

and

$$|f(z)| \geq |z|^p - \frac{p(B-A)(\lambda p-\lambda+1)(p+\mu)(p+\mu+1)}{[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1](n+p+\mu)} |z|^{n+p}.$$

Corollary 2. *If $f \in K_n(p,\lambda,\mu,3,A,B)$, then we have*

$$|f(z)| \leq |z|^p + \frac{p(B-A)(\lambda p-\lambda+1)(p+\mu)(p+\mu+1)(p+\mu+2)}{2[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1](n+p+\mu)(n+p+\mu+1)}|z|^{n+p}$$

and

$$|f(z)| \geq |z|^p - \frac{p(B-A)(\lambda p-\lambda+1)(p+\mu)(p+\mu+1)(p+\mu+2)}{2[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1](n+p+\mu)(n+p+\mu+1)}|z|^{n+p}.$$

3. Neighborhoods for The Class $\mathcal{K}_u(p,\lambda,\mu,m,A,B)$

In this section, we determine inclusion relations for the class $\mathcal{K}_u(p,\lambda,\mu,m,A,B)$ concerning the $(n,\delta)-$neighborhoods defined by (1.2).

Theorem 2. *If $f \in \mathcal{T}_n(p)$ is in the class $\mathcal{K}_u(p,\lambda,\mu,m,A,B)$, then*

$$\mathcal{K}_u(p,\lambda,\mu,m,A,B) \subset N_{n,\delta}(g;f) \tag{3.1}$$

where $g(z)$ is given by (1.14) and

$$\delta := \frac{p(B-A)(\lambda p-\lambda+1)(n+p)}{[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1]} \left[1+\frac{\prod_{j=0}^{m-1}(p+\mu+j)}{(m-1)\prod_{j=0}^{m-2}(n+p+\mu+j)}\right]. \tag{3.2}$$

Proof. Suppose that $\mathcal{K}_u(p,\lambda,\mu,m,A,B)$. Then, upon substituting from (2.7) into the following coefficient inequality:

$$\sum_{k=n+p}^{\infty} k|b_k-a_k| \leq \sum_{k=n+p}^{\infty} kb_k + \sum_{k=n+p}^{\infty} ka_k \qquad (a_k \geq 0,\ b_k \geq 0) \tag{3.3}$$

we obtain that

$$\sum_{k=n+p}^{\infty} k|b_k-a_k| \leq \sum_{k=n+p}^{\infty} kb_k + \sum_{k=n+p}^{\infty} \frac{\prod_{j=0}^{m-1}(p+\mu+j)}{\prod_{j=0}^{m-1}(k+\mu+j)} kb_k. \tag{3.4}$$

Since $g \in \mathcal{T}_n(p,\lambda,A,B)$, the second assertion (2.4) of Lemma 2 yields that

$$kb_k \leq \frac{p(B-A)(\lambda p-\lambda+1)(n+p)}{[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1]} \quad (k=n+p,n+p+1,\ldots). \tag{3.5}$$

In the right hand side of (3.4), we obtain the assertion (3.2) using (3.5) and (2.11), respectively.

Thus, by Definition 2 with $g(z)$ interchanged by $f(z)$, we conclude that

$$f \in N_{n,\delta}(g;f).$$

This completes the proof of Theorem 2. □

Corollary 3 ([3]). *If $f \in \mathcal{K}_a(p,\lambda,\mu,2,A,B)$, then*

$$\mathcal{K}_a(p,\lambda,\mu,2,A,B) \subset N_{n,\delta}(g;f)$$

where $g(z)$ is given by (1.14) for $m=2$ and δ is given by

$$\delta := \frac{p(B-A)(\lambda p-\lambda+1)(n+p)}{[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1]}\left[1+\frac{(p+\mu)(p+\mu+1)}{n+p+\mu}\right].$$

Corollary 4. *If $f \in \mathcal{K}_a(p,\lambda,\mu,3,A,B)$, then*

$$\mathcal{K}_a(p,\lambda,\mu,3,A,B) \subset N_{n,\delta}(g;f)$$

where $g(z)$ is given by (1.14) for $m=3$ and δ is given by

$$\delta := \frac{p(B-A)(\lambda p-\lambda+1)(n+p)}{[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1]} \left[1+\frac{(p+\mu)(p+\mu+1)(p+\mu+2)}{2(n+p+\mu)(n+p+\mu+1)}\right].$$

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ON THE SOLVABILITY OF THE MODIFIED CAUCHY PROBLEM FOR LINEAR SYSTEMS OF IMPULSIVE DIFFERENTIAL EQUATIONS WITH SINGULARITIES

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Abstract. The modified Cauchy problem is considered for systems of linear impulsive differential equations with singularities. The singularity is considered in the sense that the matrix-and vector-functions corresponding to the impulsive system are generally not integrable at the initial point. The sufficient conditions are established for the unique solvability of the problem

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1. Statement of the problem and basic notation

Let $I \subset \mathbb{R}$ be a non-degenerate interval in the point, $t_0 \in I$ and

$$I_{t_0} = I \setminus \{t_0\}.$$

Consider the linear system of impulsive differential equations

$$\frac{dx}{dt} = P(t)x + q(t) \quad \text{for a.a.} \quad t \in I_{t_0} \setminus T, \tag{1.1}$$

$$x(\tau_l+) - x(\tau_l-) = G(l)x(\tau_l) + g(l) \quad (l = 1, 2, \dots), \tag{1.2}$$

where $T = \{\tau_1, \tau_2, \dots\}$, $\tau_l \in I_{t_0}$ $(l = 1, 2, \dots)$ are points of impulses actions such that $\lim\limits_{l \to +\infty} \tau_l = t_0$,

$$P = (p_{ik})_{i,k=1}^n \in L_{loc}(I_{t_0}; \mathbb{R}^{n \times n}) \quad \text{and} \quad q = (q_k)_{k=1}^n \in L_{loc}(I_{t_0}; \mathbb{R}^n)$$

are matrix- and vector-functions, respectively, and

$$G = (g_{ik})_{i,k=1}^n \in E(\mathbb{N}; \mathbb{R}^{n \times n}) \quad \text{and} \quad g = (g_k)_{k=1}^n \in E(\mathbb{N}; \mathbb{R}^n)$$

are matrix- and vector-functions of the discrete argument.

Let $H = \operatorname{diag}(h_1, \dots, h_n) : I_{t_0} \to \mathbb{R}^{n \times n}$ be a diagonal matrix-function with continuous diagonal elements

$$h_k : I_{t_0} \to]0, +\infty[\quad (k = 1, \dots, n).$$

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We consider the problem of finding of a solution $x : I_{t_o} \to R^n$ of system (1.1), (1.2), satisfying the condition

$$\lim_{t\to t_0}(H^{-1}(t)x(t)) = 0. \tag{1.3}$$

As we know firstly the initial problem for systems of ordinary differential equations with singularities have been fundamentally investigated by V.A. Chechik in [7], where the sufficient conditions for the existence and uniqueness of a solution to the problem and some related questions are given.

The modified Cauchy and some other problems for systems of ordinary differential equations with singularities, i.e. for problem (1.1), (1.3), are investigated, for example, in [7–10] (see also the references therein).

The singularity of system (1.1) is considered in the sense that the matrix-function P or vector-function q are generally not integrable at the point t_0, i.e. on some $]a,b[$ from I such that $t_0 \in]a,b[$. So that, in general, the solution of problem (1.1), (1.3) is not continuous at the point t_0 and, therefore, it is not a solution in the classical sense. But its restriction on the every interval from I_{t_0} is a solution of system (1.1) in the classical sense. To illustrate this we give the following example from [8].

Let $\alpha > 0$ and $\varepsilon \in]0,\alpha[$. Then the problem

$$\frac{dx}{dt} = -\frac{\alpha x}{t} + \varepsilon|t|^{\varepsilon-1-\alpha}, \quad \lim_{t\to 0}(t^\alpha x(t)) = 0$$

has the unique solution $x(t) = |t|^{\varepsilon-\alpha}\operatorname{sgn} t$. This function is not solution of the equation on the set $I = \mathbb{R}$, but its restrictions on $]-\infty,0[$ and $]0,+\infty[$ are solutions of the one.

As we know, such a problem for the impulsive differential system (1.1), (1.2) has not been investigated. So, the present investigation is quite topical.

The theory of the regular impulsive differential equations has been investigated in earlier papers (see, for example, [1–6, 11, 12] and references therein). As to singular case, the corresponding theory, as we know, is far enough from deep research. Some boundary value problems for linear impulsive systems with singularities are investigated in [4]).

In the present paper, we give sufficient conditions for the unique solvability of problem (1.1) – (1.3). Analogous results for the analogous problem for systems of ordinary differential equations with singularities belong to I. Kiguradze (see [8–10]).

In the paper the use will be made of the following notation and definitions.

$\mathbb{N} = \{1,2,\dots\}$.

$\mathbb{R} =]-\infty,+\infty[$, $\mathbb{R}_+ = [0,+\infty[$, $[a,b]$ and $]a,b[$ $(a,b \in \mathbb{R})$ are, respectively, closed and open intervals.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices $X = (x_{i,j})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m}\sum_{i=1}^{n}|x_{ij}|.$$

$O_{n\times m}$ (or O) is the zero $n\times m$ matrix.

If $X=(x_{ik})_{i,k=1}^{n,m}\in\mathbb{R}^{n\times m}$, then

$$|X|=(|x_{ik}|)_{i,k=1}^{n,m},\quad [X]_+=\frac{|X|+X}{2},\quad [X]_-=\frac{|X|-X}{2}.$$

$$\mathbb{R}_+^{n\times m}=\Big\{(x_{ij})_{i,j=1}^{n,m}:\ x_{ij}\geq 0\ \ (i=1,\dots,n;\ \ j=1,\dots,m)\Big\}.$$

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^n$.

If $X\in\mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X; I_n is the identity $n\times n$-matrix.

The inequalities between the matrices are understood componentwise.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its components is such.

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of the matrix-function $X:[a,b]\to\mathbb{R}^{n\times m}$ at the point t.

$AC([a,b];D)$, where $D\subset\mathbb{R}^{n\times m}$, is the set of all absolutely continuous matrix-functions $X:[a,b]\to D$.

$AC_{loc}(I;\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:I\to\mathbb{R}^{n\times m}$ for which the restriction on $[a,b]$ belongs to $AC([a,b];\mathbb{R}^{n\times m})$ for every closed interval $[a,b]$ from I;

$AC_{loc}(I_{t_0,T};D)$, where $I_{t_0,T}=I_{t_0}\setminus T$, is the set of all matrix-functions $X:I_{t_0}\to D$ whose restrictions to an arbitrary closed interval $[a,b]$ from $I_{t_0,T}$ belong to $AC([a,b];D)$.

If $\alpha<\beta$ then $N_{\alpha,\beta}=\{l\in\mathbb{N}:\alpha\leq\tau_l<\beta\}$.

$L([a,b];D)$ is the set of all integrable matrix-functions $X:[a,b]\to D$.

$L_{loc}(I_{t_0};D)$ is the set of all matrix-functions $X:I_{t_0}\to D$ whose restrictions to an arbitrary closed interval $[a,b]$ from I_{t_0} belong to $L([a,b];D)$.

$E(M;\mathbb{R}^{n\times m})$, where $M\subset\mathbb{N}$, is the set of all discrete matrix-functions from M into $\mathbb{R}^{n\times m}$.

A vector-functions $x\in AC_{loc}(I_{t_0,T};\mathbb{R}^n)$ is said to be a solution of system (1.1), (1.2) if $x'(t)=P(t)x(t)+q(t)$ for a.a. $t\in I_{t_0,T}$ and there exist the onesided limits $x(\tau_l-)$ and $x(\tau_l+)$ $(l=1,2,\dots)$ such that equalities (1.2) hold.

Without loss of generality we can assume that the solution x of the impulsive differential system (1.1), (1.2) is continuous from the left in the points of the impulses actions τ_l $(l=1,2,\dots)$, i.e. $x(\tau_l)=x(\tau_l-)$ $(l=1,2,\dots)$.

We assume that

$$\det(I_n+G(l))\neq 0\ \ (l=1,2,\dots).$$

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding nonsingular systems, i.e. for the case when $P\in L_{loc}(I;\mathbb{R}^{n\times n})$ and $q\in L_{loc}(I;\mathbb{R}^n)$ (see [1, 5, 6, 12]).

Let $P_0\in L_{loc}(I_{t_0};\mathbb{R}^{n\times n})$ and $G_0\in E(\mathbb{N};\mathbb{R}^{n\times n})$. Then a matrix-function $C_0:I_{t_0}\times I_{t_0}\to\mathbb{R}^{n\times n}$ is said to be the Cauchy matrix of the homogeneous impulsive differential

system

$$\frac{dx}{dt} = P_0(t)x, \tag{1.4}$$

$$x(\tau_l+) - x(\tau_l-) = G_0(l)x(\tau_l) \ \ (l = 1,2,\dots) \tag{1.5}$$

if for every interval $J \subset I_{t_0}$ and $\tau \in J$, the restriction of the matrix-function $C_0(.,\tau) : I_{t_0} \to \mathbb{R}^{n\times n}$ on J is the fundamental matrix of system (1.4), (1.5) satisfying the condition $C_0(\tau,\tau) = I_n$. Therefore, C_0 is the matrix Cauchy of system (1.4), (1.5) if and only if the restriction of C_0 on $J \times J$, for every interval $J \subset I_{t_0}$, is the matrix Cauchy of the system in the sense of definition given in [8].

We assume

$$I_{t_0}(\delta) = [t_0 - \delta, t_0 + \delta] \cap I_{t_0}$$

for every $\delta > 0$.

We consider problem (1.1) – (1.3) only in the case where $t_0 = \sup I$ and $\tau_l < \tau_{l+1}$ $(l = 1,2,\dots)$. Similarly, we can investigate the case where $t_0 = \inf I$ and $\tau_{l+1} < \tau_l$ $(l = 1,2,\dots)$. The general case $t_0 \in]\inf I, \sup I[$ will be reduced to the given two cases.

2. Formulation of the main results

Theorem 1. *Let there exist a matrix-function $P_0 \in L_{loc}(I_{t_0,T}; R^{n\times n})$, discrete matrix-function $G_0 \in E(\mathbb{N};\mathbb{R}^{n\times n})$ and constant matrices $B_0, B \in \mathbb{R}_+^{n\times n}$ such that the conditions*

$$\det(I_n + G_0(l)) \neq 0 \ \ (l = 1,2,\dots), \tag{2.1}$$

$$r(B) < 1 \tag{2.2}$$

and the estimates

$$|C_0(t,\tau)| \le H(t)B_0H^{-1}(\tau) \ \ \textit{for} \ \ t \le \tau, \ \ t,\tau \in [t_0 - \delta, t_0[\tag{2.3}$$

and

$$\Big| \int_t^{t_0} |C_0(t,\tau)(P(s) - P_0(s))H(s)|ds \Big| + \sum_{l\in N_{t,t_0}} \big|C_0(t,\tau_l)(I_n + G_0(l))^{-1}(G(l) - G_0(l))H(\tau_l)\big| \le H(t)B$$
$$\textit{for} \ \ t \in [t_0 - \delta, t_0[\tag{2.4}$$

hold for some $\delta > 0$, where C_0 is the Cauchy matrix of system (1.4), (1.5). Let, moreover,

$$\lim_{t\to t_0}\Bigg\|\int_t^{t_0} H^{-1}(s)C_0(t,s)q(s)ds + \sum_{l\in N_{t,t_0}} H^{-1}(\tau_l)C_0(t,\tau_l)(I_n+G_0(l))^{-1}g(l)\Bigg\| = 0. \quad (2.5)$$

Then problem (1.1) – (1.3) has a unique solution.

Theorem 2. *Let there exist a constant matrix $B=(b_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n\times n}$ such that condition (2.2) hold, and the estimates*

$$c_i(t,\tau) \le b_0\frac{h_i(t)}{h_i(\tau)} \quad \text{for} \quad t\le\tau,\ \ t,\tau\in[t_0-\delta,t_0[\ \ (i=1,\dots,n), \quad (2.6)$$

$$\Bigg|\int_t^{t_0} c_i(t,\tau)h_i(\tau)[p_{ii}(\tau)]_-d\tau + \sum_{l\in N_{t,t_0}} c_i(t_l,\tau_l)h_i(\tau_l)[g_{ii}(l)]_-\Bigg| \le b_{ii}h_i(t) \quad \text{for} \quad t\in[t_0-\delta,t_0[\ \ (i=1,\dots,n), \quad (2.7)$$

$$\Bigg|\int_t^{t_0} c_i(t,\tau)h_k(\tau)|p_{ik}(\tau)|d\tau + \sum_{l\in N_{t,t_0}} c_i(t,\tau_l)[g_{ii}(l)]_+\cdot(1+[g_{ii}(l)]_+)^{-1}h_k(\tau_l)g_{ik}(l)\Bigg| \le b_{ik}h_i(t) \quad \text{for} \quad t\in[t_0-\delta,t_0[\ \ (i\ne k;\ \ i,k=1,\dots,n) \quad (2.8)$$

hold for some $b_0>0$ and $\delta>0$. Let, moreover,

$$\lim_{t\to t_0}\Bigg(\int_t^{t_0}\frac{c_i(t,\tau)}{h(t)}q(\tau)d\tau + \sum_{l\in N_{t,t_0}}\frac{c_i(t,\tau_l)}{h_i(t)}[g_{ii}(l)]_+\cdot(1+[g_{ii}(l)]_+)^{-1}g(l)\Bigg) = 0 \quad (i=1,\dots,n), \quad (2.9)$$

where c_i is the Cauchy function of the impulsive differential equations

$$\frac{dx}{dt} = p_{0i}(t)x,$$

$$x(\tau_l+)-x(\tau_l-) = g_{0i}(l)x(\tau_l) \quad (l=1,2,\dots),$$

here

$$p_{0i}(t)\equiv[p_{ii}(t)]_+,\ \ g_{0i}(l)\equiv[g_{ii}(l)]_+ \quad (i=1,\dots,n).$$

Then problem (1.1) – (1.3) has a unique solution.

Remark 1. The Cauchy functions $c_i(t,\tau)$ $(i=1,\dots,n)$ have the form

$$c_i(t,\tau)=\begin{cases}\exp\Big(\displaystyle\int_\tau^t [p_{ii}(s)]_+ds\Big)\prod_{l\in T_{\tau,t}}(1+[g_{ii}(l)]_+) & \text{if } t>\tau,\\ \exp\Big(\displaystyle\int_\tau^t [p_{ii}(s)]_+ds\Big)\prod_{l\in T_{t,\tau}}(1+[g_{ii}(l)]_+)^{-1} & \text{if } t<\tau,\\ 1 & \text{if } t=\tau\end{cases} \tag{2.10}$$

for $t,\tau\in I$.

Remark 2. In Theorems 1 and 2, the strong inequality (2.2) can not replace by a non-strong one. We give the corresponding example from [8] for the ordinary differential case, i.e., when $G(l)\equiv O_{n\times n}$ and $g(l)\equiv 0_n$.

On the interval $]-1,0[$ consider the problem

$$\frac{dx}{dt}=\frac{x}{t}+\frac{1}{|\ln|t||}, \tag{2.11}$$

$$\lim_{t\to 0}\frac{x(t)}{t}=0. \tag{2.12}$$

Every solution of equation (2.11) has the form

$$x(t)=ct-t\ln|\ln|t||\quad (c\in\mathbb{R}).$$

So that, problem (2.11), (2.12) is not solvable. On the other hand, the Cauchy function $c(t,\tau)$ has the form $c(t,\tau)=t\tau^{-1}$ for $t\le\tau<0$ and the conditions of Theorem 1, except of condition (2.2), are fulfilled (on $]-1,0[$) for $n=1$, $P(t)\equiv t^{-1}$, $q(t)\equiv|\ln|t||^{-1}$ and $h(t)\equiv t$ only for the case where $B\ge 1$, i.e., when $r(B)\ge 1$.

3. Auxiliary propositions

We use the lemma on the a priory estimate of the solutions of system (1.1), (1.2) (see below, Lemma 2). To prove the lemma, we use a Cauchy formula to represent solutions of the impulsive differential systems.

Lemma 1 (Variation-of-constants formula)**.** *Let $G_*\in E(\mathbb{N};\mathbb{R}^{n\times n})$ be such that*

$$\det(I_n+G_*(l))\neq 0\ \ (l=1,2,\dots).$$

Then every solution of the system

$$\frac{dx}{dt}=P_*(t)x+q_*(t)\ \ \textit{for a. a.}\ \ t\in I,$$
$$x(\tau_l+)-x(\tau_l-)=G_*(l)x(\tau_l)+g_*(l)\ \ (l=1,2,\dots),$$

where $P_* \in L_{loc}(I;\mathbb{R}^{ns\times n})$, $q_* \in L_{loc}(I;\mathbb{R}^n)$ *and* $g_* \in E(\mathbb{N};\mathbb{R}^{n\times n})$, *admits the representation*

$$x(t) = C_*(t,\tau)x(\tau) + \int_\tau^t C_*(t,s)q_*(s)ds$$
$$- \sum_{l\in N_{\tau,t}} C_*(t,\tau_l)G_*(l)(I_n+G_*(l))^{-1}g_*(l) \quad \text{for} \quad \tau<t,\ \tau,t\in I, \tag{3.1}$$

where C_* *is the Cauchy matrix of the homogeneous system*

$$\frac{dx}{dt} = P_*(t)x \quad \text{for a. a.} \quad t\in I,$$
$$x(\tau_l+)-x(\tau_l-) = G_*(l)x(\tau_l) \quad (l=1,2,\dots).$$

Representation (3.1) is proved for example in [5].

Lemma 2. *Let the matrix-functions* $P_0 \in L_{loc}(I_{t_0};\mathbb{R}^{n\times n})$, $G_0 \in E(\mathbb{N};\mathbb{R}^{n\times n})$ *and the constant matrices* B_0 *and* B *from* $\mathbb{R}_+^{n\times n}$ *be such that conditions (2.1) – (2.4) hold for some* $\delta>0$, *where* C_0 *is the Cauchy matrix of system (1.4), (1.5). Let, moreover,*

$$\gamma(t) = \sup\Big\{\Big\| \int_\tau^{t_0} |H^{-1}(s)C_0(t,s)q(s)ds$$
$$+ \sum_{l\in N_{t,t_0}} H^{-1}(\tau_l)C_0(t,\tau_l)G_0(l)(I_n+G_0(l))^{-1}g(l)\Big\| : \ t\le\tau<t_0\Big\} < +\infty$$
$$\text{for} \quad t\in[t_0-\delta,t_0[. \tag{3.2}$$

Then every solution x *of system (1.1), (1.2) admits the estimate*

$$\|H^{-1}(t)x(t)\| \le \rho\big(\|B_0\|\,\|H^{-1}(\tau_0)x(\tau_0)\| + \gamma(t)\big)$$
$$\text{for} \quad t\in J, \quad t<\tau_0, \tag{3.3}$$

where $\rho = \|(I_n-B)^{-1}\|$, *and* $J\subset[t_0-\delta,t_0[$ *and* $\tau_0\in J$ *are arbitrary interval and point, respectively.*

Proof. Let $x=(x_i)_{i=1}^n$ be an arbitrary solution of system (1.1), (1.2) on the J. Then x satisfies the impulsive system

$$\frac{dx}{dt} = P_0(t)x + (P(t)-P_0(t))x + q(t) \quad \text{for a. a.} \quad t\in I_{t_0,T},$$

$$x(\tau_l+)-x(\tau_l-) = G_0(l)x(\tau_l) + (G(l)-G_0(l))x(\tau_l) + g(l) \quad (l=1,2,\dots).$$

Let the vector-function $z(t)=(z_i(t))_{i=1}^n$ is defined by

$$z(t) \equiv H^{-1}(t)x(t).$$

According to variations-of-constants formula (3.1) we have

$$x(t)=C_0(t,\tau_0)x(\tau_0)+\int\limits_{\tau_0}^{t}C_0(t,s)q_*(s)ds$$
$$-\sum_{l\in N_{t,\tau_0}}C_0(t,\tau_l)G_0(l)(I_n+G_0(l))^{-1}g_*(l)\quad\text{for}\quad t\in J,\ \ t<\tau_0,$$

where

$$q_*(t)\equiv(P(t)-P_0(t))x(t)+q(t)\ \ \textit{and}\ \ g_*(l)\equiv(G(l)-G_0(l))\,x(\tau_l)+g(l).$$

Therefore,

$$z(t)=H^{-1}(t)C_0(t,\tau_0)x(\tau_0)+H^{-1}(t)\int\limits_{\tau_0}^{t}C_0(t,s)q_*(s)ds$$
$$-H^{-1}(t)\sum_{l\in N_{t,\tau_0}}C_0(t,\tau_l)G_0(l)(I_n+G_0(l))^{-1}g_*(l)\quad\text{for}\quad t\in J,\ \ t<\tau_0. \tag{3.4}$$

Let the components of the vector-function $y(t)=(y_i(t))_{i=1}^n$ be defined by

$$y_i(t)=\sup\{|z_i(s)|:t\le s\le\tau_0\}\quad\text{for}\quad t\in J\ \ (i=1,\dots,n).$$

In view of (3.4), take into account (2.3) and (2.4), it is not difficult to verify that

$$\begin{aligned}&|z(t)|\le H^{-1}(t)|C_0(t,\tau_0)x(\tau_0)|\\&+H^{-1}(t)\int\limits_{t}^{\tau_0}\big|C_0(t,s)(P(s)-P_0(s))H(s)\big|\,|z(s)|ds\\&+H^{-1}(t)\sum_{l\in N_{t,\tau_0}}\big|C_0(t,\tau_l)\,G_0(l)(I_n+G_0(l))^{-1}(G(l)-G_0(l))H(\tau_l)\big|\,|z(\tau_l)|\\&+H^{-1}(t)f(t)\le B_0|H^{-1}(\tau_0)x(\tau_0)|+By(t)+H^{-1}(t)f(t)\\&\qquad\qquad\qquad\qquad\text{for}\quad t\in J,\ \ t<\tau_0.\end{aligned}$$

where

$$f(t)\equiv\int\limits_{t}^{\tau_0}|C_0(t,s)q(s)|ds+\sum_{l\in N_{t,\tau_0}}|C_0(t,\tau_l)G_0(l)(I_n+G_0(l))^{-1}g(l)|.$$

Therefore, thanks to (3.2) we find

$$y(t)\le B_0|H^{-1}(\tau_0)x(\tau_0)|+By(t)+\overline{\gamma}(t)\quad\text{for}\quad t\in J,\ \ t<\tau_0$$

and

$$(I_n-B)y(t)\le B_0|H^{-1}(s_0)x(s_0)|+\overline{\gamma}(t)\quad\text{for}\quad t\in J,\quad t<\tau_0,$$

where $\overline{\gamma}(t)$ is a vector-function with components $\gamma_i(t)\equiv\gamma(t)$ $(i=1,\dots,n)$.

From this, in view of (2.2) and nonnegativity of matrix B, we have

$$y(t) \le (I_n - B)^{-1}\big(B_0|H^{-1}(s_0)x(s_0)| + g(t)\big) \quad \text{for} \quad t \in J, \quad t < \tau_0.$$

Hence estimate (3.3) holds. □ □

4. Proofs of results

Proof of Theorem 1. In the first place, we note that by (2.5) estimate (3.2) holds and

$$\lim_{t \to t_0} \gamma(t) = 0, \tag{4.1}$$

where the function $\gamma(t)$ is defined as in Lemma 2.

Let $t_k \in [t_0 - \delta, t_0[$ $(k = 1, 2, \dots)$ be some increasing sequence such that

$$\lim_{k \to +\infty} t_k = t_0. \tag{4.2}$$

According to Theorem 2.1.1 from [5], for every natural k, system (1.1), (1.2) has a unique solution x_k defined on the interval $[t_0 - \delta, t_0[$ and satisfying the condition

$$x_k(t_k) = 0.$$

Moreover, due to Lemma 2 we have the estimates

$$\|H^{-1}(t)x_k(t)\| \le \rho\gamma(t) \quad \text{for} \quad t_1 \le t \le t_k \quad (k = 1, 2, \dots), \tag{4.3}$$

where $\rho = \|(I_n - B)^{-1}\|\gamma(t_1)$. In particular, from (4.3) it follows that

$$\|x_k(t_1)\| \le \rho_0 \quad (k = 1, 2, \dots),$$

where $\rho_0 = \rho\|H(t_1)\|\gamma(t_1)$. So with out of generality we can assume that the sequence $x_k(t_1)$ $(k = 1, 2, \dots)$ converges. Let

$$\lim_{k \to +\infty} x_k(t_1) = c_0.$$

By the theorems on the well-posedness of the Cauchy problem (see [5]) we conclude

$$\lim_{k \to +\infty} x_k(t) = x(t)$$

uniformly on the every closed interval from $[t_0 - \delta, t_0[$, where x is a solution of system (1.1), (1.2) under the condition

$$x(t_1) = c_0.$$

On the other hand, thanks to (4.2) and (4.3) we find that

$$\|H^{-1}(t)x(t)\| \le \rho\gamma(t) \quad \text{for} \quad t_1 \le t < t_0.$$

From this, by (4.1) we conclude that the vector-function x is a solution of problem (1.1) – (1.3) on the interval $[t_0 - \delta, t_0[$.

Now, let us show that problem (1.1) – (1.3) has a unique solution on the interval $[t_0-\delta, t_0[$. Let, x_* be an arbitrary solution of the problem. Then the vector-function $x_0(t)=x(t)-x_*(t)$ will be a solution of the homogeneous system

$$\frac{dx}{dt}=P(t)x \quad \text{for a. a.} \quad t\in I_{t_0,T}, \tag{1.1$_0$}$$

$$x(\tau_l+)-x(\tau_l-)=G(l)x(\tau_l) \quad (l=1,2,\dots) \tag{1.2$_0$}$$

under the condition

$$\lim_{t\to t_0}(H^{-1}(t)x_0(t))=0. \tag{4.4}$$

In view of estimate (3.3) of Lemma 2, for every $\tau_0\in[t_0-\delta,t_0[$, we find

$$\|x_0(t_0+\delta)\|\le\rho_0\|H(t_0+\delta)\|\cdot\|H^{-1}(\tau_0)x_0(\tau_0)\| \quad \text{for} \quad t_0+\delta\le t\le\tau_0,$$

where $\rho_0=\|(I_n-B)^{-1}\|\cdot\|B_0\|$. Passing to the limit as $\tau_0\to t_0$ in the last estimate and taking into account (4.4) we get

$$x(t_0+\delta)=0. \tag{4.5}$$

Since the matrix-function P is integrable at the point $t_0+\delta$, i.e., we have regular case, by the above-mentioned theorem from [5] problem (1.1$_0$), (1.2$_0$); (4.5) has only the trivial solution and, therefore, $x(t)\equiv x_*(t)$. □ □

Proof of Theorem 2. Let us assume

$$P_0(t)\equiv\operatorname{diag}\big([p_{11}(t)]_+,\dots,[p_{nn}(t)]_+\big)$$

and

$$G_0(t)\equiv\operatorname{diag}\big([g_{11}(l)]_+,\dots,[g_{nn}(l)]_+\big).$$

Then the Cauchy matrix of system (1.4), (1.5) has the form

$$C(t,\tau)\equiv\operatorname{diag}(c_1(t,\tau),\dots,c_n(t,\tau)),$$

where the functions $c_i(t,\tau)$ $(i=1,\dots,n)$ are defined by (2.10).

In addition, note that due to (2.10) we have

$$c_i(t,\tau)>0 \quad \text{for} \quad t\le\tau, \quad t,\tau\in[t_0-\delta.$$

hold. By this, conditions (2.6) – (2.9) we conclude that conditions (2.3), (2.4) and (2.5) of Theorem 1 are valid. Hence the theorem immediately follows from Theorem 1. □ □

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COFINITELY $\oplus$-SUPPLEMENTED LATTICES

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Abstract. In this work, cofinitely $\oplus-$supplemented and strongly cofinitely $\oplus-$supplemented lattices are defined and investigated some properties of these lattices. Let L be a lattice and $1 = \bigoplus_{i\in I} a_i$ with $a_i \in L$. If $a_i/0$ is cofinitely $\oplus-$supplemented for every $i \in I$, then L is also cofinitely $\oplus-$supplemented. Let L be a distributive lattice and $1 = a_1 \oplus a_2$ with $a_1, a_2 \in L$. If $a_1/0$ and $a_2/0$ are strongly cofinitely $\oplus-$supplemented, then L is also strongly cofinitely $\oplus-$supplemented. Let L be a lattice. If every cofinite element of L lies above a direct summand in L, then L is cofinitely $\oplus-$supplemented.

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1. Introduction

Throughout this paper, all lattices are complete modular lattices with the smallest element 0 and the greatest element 1. Let L be a lattice, $a, b \in L$ and $a \leq b$. A sublattice $\{x \in L | a \leq x \leq b\}$ is called a *quotient sublattice*, denoted by b/a. An element a' of a lattice L is called a *complement* of a if $a \wedge a' = 0$ and $a \vee a' = 1$, this case we denote $1 = a \oplus a'$ (a and a' also is called *direct summands* of L). L is called a *complemented lattice* if each element has at least one complement in L. An element c of L is said to be *compact* if for every subset X of L such that $c \leq \vee X$, there exists a finite $F \subseteq X$ such that $c \leq \vee F$. A lattice L is said to be *compactly generated* if each of its elements is a join of compact elements. A lattice L is said to be *compact* if 1 is a compact element of L. An element a of a lattice L is said to be *cofinite* if $1/a$ is compact. An element a of L is said to be *small* or *superfluous* and denoted by $a \ll L$ if $b = 1$ for every element b of L such that $a \vee b = 1$. The meet of all the maximal elements ($\neq 1$) of a lattice L is called the *radical* of L and denoted by $r(L)$. An element c of L is called a *supplement* of b in L if it is minimal for $b \vee c = 1$. a is a supplement of b in a lattice L if and only if $a \vee b = 1$ and $a \wedge b \ll a/0$. A lattice L is said to be *supplemented* if every element of L has a supplement in L. L is said to be *cofinitely supplemented* if every cofinite element of L has a supplement in L. L is said to be $\oplus-$*supplemented* if every element of L has a supplement that is a direct

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summand in L.We say that an element b of L *lies above* an element a of L if $a \leq b$ and $b \ll 1/a$. L is said to be *hollow* if every element $(\neq 1)$ is superfluous in L, and L is said to be *local* if L has the greatest element $(\neq 1)$. An element a of L is called a *weak supplement* of b in L if $a \vee b = 1$ and $a \wedge b \ll L$. A lattice L is said to be *weakly supplemented*, if every element of L has a weak supplement in L. L is said to be *cofinitely weak supplemented*, if every cofinite element of L has a weak supplement in L. An element $a \in L$ has *ample supplements* in L if for every $b \in L$ with $a \vee b = 1$, a has a supplement b' in L with $b' \leq b$. L is called an *amply supplemented lattice*, if every element of L has ample supplements in L. It is clear that every supplemented lattice is weakly supplemented and every amply supplemented lattice is supplemented. A lattice L is said to be *distributive* if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for every $a, b, c \in L$. Let L be a lattice. It is defined β_* relation on the elements of L by $a\beta_* b$ with $a, b \in L$ if and only if for each $t \in L$ such that $a \vee t = 1$ then $b \vee t = 1$ and for each $k \in L$ such that $b \vee k = 1$ then $a \vee k = 1$.

More details about (amply) supplemented lattices are in [1, 2, 7]. The definitions of cofinitely (weak) supplemented lattices and some properties of these lattices are in [1, 2]. The definition of $\oplus-$supplemented lattices and some properties of these lattices are in [5]. More results about (amply) supplemented modules are in [6, 11]. Some important properties of $\oplus-$supplemented modules are in [8, 9]. The definition of $\oplus-$cofinitely supplemented modules and some properties of these modules are in [4]. The definition of β_* relation on lattices and some properties of this relation are in [10]. The definition of β^* relation on modules and some properties of this relation are in [3].

Lemma 1. *Let L be a lattice and $a, b, c \in L$ with $a \leq b$. If c is a supplement of b in L, then $a \vee c$ is a supplement of b in $1/a$.*

Proof. Similar to proof of [7, Proposition12.2(7)]. □

Lemma 2 ([7, Lemma 7.4])**.** *Let L be a lattice, $a, b \in L$ and $a \leq b$. If $a \ll b/0$ then $a \ll L$.*

Lemma 3 ([7, Lemma 7.5])**.** *In a lattice L let $c' \ll c/0$ and $d' \ll d/0$. Then $c' \vee d' \ll (c \vee d)/0$.*

Lemma 4 ([7, Exercise 7.3])**.** *If L is a lattice and $a \in L$, then $r(a/0) \leq r(L)$.*

Lemma 5 ([7, Lemma 12.3])**.** *In any modular lattice $[(c \vee d) \wedge b] \leq [c \wedge (b \vee d)] \vee [d \wedge (b \vee c)]$ holds for every $b, c, d \in L$.*

Lemma 6 (See also [5])**.** *Let L be a lattice, $a, b \in L$ and $a \leq b$. Then b lies above a if and only if $a\beta_* b$.*

Proof. $(\Longrightarrow)$ See [10, Theorem 3].

$(\Longleftarrow)$ Let $b \vee t = 1$ with $t \in 1/a$. Since $a\beta_* b$, $a \vee t = 1$ and since $a \leq t$, $t = 1$. Hence $b \ll 1/a$ and b lies above a. □

2. COFINITELY $\oplus-$ SUPPLEMENTED LATTICES

Definition 1. Let L be a lattice. L is called a cofinitely $\oplus-$supplemented lattice, if every cofinite element of L has a supplement that is a direct summand of L.

Clearly we can see that every $\oplus-$supplemented lattice is cofinitely $\oplus-$supplemented and every cofinitely $\oplus-$supplemented lattice is cofinitely supplemented.

Proposition 1. *Let L be a lattice. Then L is cofinitely $\oplus-$supplemented if and only if for every cofinite $b \in L$, there exists a direct summand c of L such that $b \vee c = 1$ and $b \wedge c \ll c/0$.*

Proof. Clear from definition. □

Proposition 2. *Let L be a lattice. If every cofinite element of L has a weak supplement that is a direct summand of L, then L is cofinitely $\oplus-$supplemented.*

Proof. Let a be a cofinite element of L and b be a weak supplement of a in L that is a direct summand of L. Since b is a weak supplement of a in L, $a \wedge b \ll L$ and since b is a direct summand of L, $a \wedge b \ll b/0$. Hence b is a supplement of a in L and L is cofinitely $\oplus-$supplemented. □

Lemma 7 (See also [5])**.** *Let L be a lattice, and $a, b \in L$. If x is a supplement of $a \vee b$ in L and y is a supplement of $a \wedge (x \vee b)$ in $a/0$, then $x \vee y$ is a supplement of b in L. (See also [5]).*

Proof. Since x is a supplement of $a \vee b$ in L and y is a supplement of $a \wedge (x \vee b)$ in $a/0$, then $1 = a \vee b \vee x$, $(a \vee b) \wedge x \ll x/0$, $a = [a \wedge (x \vee b)] \vee y$ and $(x \vee b) \wedge y = a \wedge (x \vee b) \wedge y \ll y/0$. Here $1 = a \vee b \vee x = [a \wedge (x \vee b)] \vee y \vee b \vee x = b \vee x \vee y$. By Lemma 5, $(x \vee y) \wedge b \leq [(y \vee b) \wedge x] \vee [(x \vee b) \wedge y] \leq [(a \vee b) \wedge x] \vee [(x \vee b) \wedge y] \ll (x \vee y)/0$. Hence $x \vee y$ is a supplement of b in L. □

Lemma 8. *Let L be a lattice and $1 = \bigoplus_{i \in I} a_i$ with $a_i \in L$. If $a_i/0$ is cofinitely $\oplus-$supplemented for every $i \in I$, then L is also cofinitely $\oplus-$supplemented.*

Proof. Let x be any cofinite element of L. Since $1/x$ is compact and $1 = \bigvee_{i \in I} (x \vee a_i)$, there exists a finite subset $F = \{i_1, i_2, ..., i_n\}$ of I such that $1 = \bigvee_{t=1}^{n} (x \vee a_{i_t}) = x \vee \left(\bigvee_{t=1}^{n} a_{i_t} \right)$. Since x is a cofinite element of L, $x \vee \left(\bigvee_{t=1}^{n-1} a_{i_t} \right)$ is a cofinite element of L. Then by $\frac{1}{x \vee \left(\bigvee_{t=1}^{n-1} a_{i_t} \right)} = \frac{x \vee \left(\bigvee_{t=1}^{n-1} a_{i_t} \right) \vee a_{i_n}}{x \vee \left(\bigvee_{t=1}^{n-1} a_{i_t} \right)} \cong \frac{a_{i_n}}{a_{i_n} \wedge \left(x \vee \left(\bigvee_{t=1}^{n-1} a_{i_t} \right) \right)}$, $a_{i_n} \wedge \left(x \vee \left(\bigvee_{t=1}^{n-1} a_{i_t} \right) \right)$ is a cofinite element of $a_{i_n}/0$ and since $a_{i_n}/0$ is cofinitely $\oplus-$supplemented, $a_{i_n} \wedge \left(x \vee \left(\bigvee_{t=1}^{n-1} a_{i_t} \right) \right)$ has a supplement x_{i_n} that is a direct summand in

$a_{i_n}/0$. Since 0 is a supplement of $x \vee \left(\bigvee_{t=1}^{n} a_{i_t}\right)$ in L and x_{i_n} is a supplement of $a_{i_n} \wedge \left(x \vee \left(\bigvee_{t=1}^{n-1} a_{i_t}\right)\right)$ in $a_{i_n}/0$, by Lemma 7, $x_{i_n} = x_{i_n} \vee 0$ is a supplement of $x \vee \left(\bigvee_{t=1}^{n-1} a_{i_t}\right)$ in L. Since x is a cofinite element of L, $x \vee \left(\bigvee_{t=1}^{n-2} a_{i_t}\right) \vee x_{i_n}$ is a cofinite element of L. Then by

$$\frac{1}{x \vee \left(\bigvee_{t=1}^{n-2} a_{i_t}\right) \vee x_{i_n}} = \frac{x \vee \left(\bigvee_{t=1}^{n-2} a_{i_t}\right) \vee x_{i_n} \vee a_{i_{n-1}}}{x \vee \left(\bigvee_{t=1}^{n-2} a_{i_t}\right) \vee x_{i_n}} \cong \frac{a_{i_{n-1}}}{a_{i_{n-1}} \wedge \left(x \vee \left(\bigvee_{t=1}^{n-2} a_{i_t}\right) \vee x_{i_n}\right)},$$

$a_{i_{n-1}} \wedge \left(x \vee \left(\bigvee_{t=1}^{n-2} a_{i_t}\right) \vee x_{i_n}\right)$ is a cofinite element of $a_{i_{n-1}}/0$ and since $a_{i_{n-1}}/0$ is cofinitely $\oplus-$supplemented, $a_{i_{n-1}} \wedge \left(x \vee \left(\bigvee_{t=1}^{n-2} a_{i_t}\right) \vee x_{i_n}\right)$ has a supplement $x_{i_{n-1}}$ that is a direct summand in $a_{i_{n-1}}/0$. Since x_{i_n} is a supplement of $x \vee \left(\bigvee_{t=1}^{n-1} a_{i_t}\right)$ in L and $x_{i_{n-1}}$ is a supplement of $a_{i_{n-1}} \wedge \left(x \vee \left(\bigvee_{t=1}^{n-2} a_{i_t}\right) \vee x_{i_n}\right)$ in $a_{i_{n-1}}/0$, by Lemma 7, $x_{i_{n-1}} \vee x_{i_n}$ is a supplement of $x \vee \left(\bigvee_{t=1}^{n-2} a_{i_t}\right)$ in L. If so, x has a supplement $\bigvee_{t=1}^{n} x_{i_t}$ in L where x_{i_t} is a direct summand of $a_{i_t}/0$ for every $t = 1,2,...,n$. Since x_{i_t} is a direct summand of $a_{i_t}/0$ for every $t = 1,2,...,n$ and $1 = \bigoplus_{i \in I} a_i$, $\bigvee_{t=1}^{n} x_{i_t}$ is a direct summand of L. Hence L is cofinitely $\oplus-$supplemented. □

Corollary 1. *Let L be a lattice, $a_1, a_2, ..., a_n \in L$ and $1 = a_1 \oplus a_2 \oplus ... \oplus a_n$. If $a_i/0$ is cofinitely $\oplus-$supplemented for every $i = 1,2,\ldots,n$, then L is cofinitely $\oplus-$supplemented.*

Proof. Clear from Lemma 8. □

Lemma 9. *Let L be a lattice, $a \in L$ and $a = (a \wedge a_1) \oplus (a \wedge a_2)$ for every $a_1, a_2 \in L$ with $1 = a_1 \oplus a_2$. If L is cofinitely $\oplus-$supplemented, then $1/a$ is also cofinitely $\oplus-$supplemented.*

Proof. Let x be a cofinite element of $1/a$. Then $1/x$ is compact and x is a cofinite element of L. Since L is cofinitely $\oplus-$supplemented, there exist $y, z \in L$ such that $1 = x \vee y$, $x \wedge y \ll y/0$ and $1 = y \oplus z$. Since y is a supplement of x in L and $a \leq x$, by Lemma 1, $a \vee y$ is a supplement of x in $1/a$. Since $1 = y \oplus z$, by hypothesis, $a = (a \wedge y) \oplus (a \wedge z)$. Then $(a \vee y) \wedge (a \vee z) = [(a \wedge y) \vee (a \wedge z) \vee y] \wedge [(a \wedge y) \vee (a \wedge z) \vee z] = [y \vee (a \wedge z)] \wedge [(a \wedge y) \vee z] = (a \wedge y) \vee [(y \vee (a \wedge z)) \wedge z] = (a \wedge y) \vee [(y \wedge$

$z) \vee (a \wedge z)] = (a \wedge y) \vee (0 \vee (a \wedge z)) = (a \wedge y) \vee (a \wedge z) = a$. Hence $1/a$ is cofinitely $\oplus-$supplemented. □

Corollary 2. *Let L be a distributive lattice. If L is cofinitely $\oplus-$supplemented, then $1/a$ is also cofinitely $\oplus-$supplemented for every $a \in L$.*

Proof. Clear from Lemma 9. □

Proposition 3. *Let L be a cofinitely $\oplus-$supplemented lattice and $r(L)$ be a cofinite element of L. Then there exist $a_1, a_2 \in L$ such that $1 = a_1 \oplus a_2$, $r(a_1/0) \ll a_1/0$ and $r(a_2/0) = a_2$.*

Proof. Since L is cofinitely $\oplus-$supplemented and $r(L)$ is a cofinite element of L, there exist $a_1, a_2 \in L$ such that $1 = r(L) \vee a_1 = a_1 \oplus a_2$ and $r(L) \wedge a_1 \ll a_1/0$. Then by Lemma 4, $r(a_1/0) \leq r(L) \wedge a_1 \ll a_1/0$.

Assume x be a maximal $(\neq a_2)$ element of $a_2/0$. Since $1/(a_1 \vee x) = (a_1 \oplus a_2)/(a_1 \vee x) = (a_1 \vee x \vee a_2)/(a_1 \vee x) \cong a_2/[a_2 \wedge (a_1 \vee x)] = a_2/[(a_2 \wedge a_1) \vee x] = a_2/x$, $a_1 \vee x$ is a maximal element $(\neq 1)$ of L and since $1 = r(L) \vee a_1 \leq a_1 \vee x$, this is a contradiction. Hence $r(a_2/0) = a_2$. □

Definition 2. Let L be a cofinitely supplemented lattice. L is called a strongly cofinitely $\oplus-$supplemented lattice if every supplement element of any cofinite element in L is a direct summand of L.

Clearly we can see that every strongly cofinitely $\oplus-$supplemented lattice is cofinitely $\oplus-$supplemented and every strongly $\oplus-$supplemented lattice is strongly cofinitely $\oplus-$supplemented.

Lemma 10 (See also [5])**.** *Let a be a supplement of b in L and $x, y \in a/0$. Then y is a supplement of x in $a/0$ if and only if y is a supplement of $b \vee x$ in L.*

Proof. $(\Longrightarrow)$ Let y be a supplement of x in $a/0$ and $b \vee x \vee z = 1$ with $z \leq y$. Because of $x, y \in a/0$ and $z \leq y$, $x \vee z \leq a$. Since a is a supplement of b in L, $a = x \vee z$. Since and y is a supplement of x in $a/0$, $z = y$. Hence y is a supplement of $b \vee x$ in L.

$(\Longleftarrow)$Let y be a supplement of $b \vee x$ in L. So, $b \vee x \vee y = 1$ and $(b \vee x) \wedge y \ll y/0$. Since $x \vee y \leq a$ and a is a supplement of b in L, $x \vee y = a$ and $x \wedge y \leq (b \vee x) \wedge y \ll y/0$. Hence y is a supplement of x in $a/0$. □

Proposition 4. *Let L be a strongly cofinitely $\oplus-$supplemented lattice. Then for every direct summand a of L, the quotient sublattice $a/0$ is strongly cofinitely $\oplus-$supplemented.*

Proof. Since a is a direct summand of L, there exists $b \in L$ such that $1 = a \oplus b$. Since L is cofinitely supplemented, we can see that $1/b$ is cofinitely supplemented. Then by $1/b = (a \vee b)/b \cong a/(a \wedge b) = a/0$, $a/0$ is cofinitely supplemented. Let x be a cofinite element of $a/0$ and y be supplement of x in $a/0$. By Lemma 10, y is a supplement of $b \vee x$ in L. By $\frac{1}{b \vee x} = \frac{a \vee b \vee x}{b \vee x} \cong \frac{a}{a \wedge (b \vee x)} = \frac{a}{(a \wedge b) \vee x} = \frac{a}{x}$, $b \vee x$

is a cofinite element of L. Since L is strongly cofinitely $\oplus-$supplemented, y is a direct summand of L. Here there exists $z \in L$ such that $1 = y \oplus z$. By modularity, $a = a \wedge 1 = a \wedge (y \oplus z) = y \oplus (a \wedge z)$. Thus y is a direct summand of $a/0$. Hence $a/0$ is strongly cofinitely $\oplus-$supplemented. □

Lemma 11. *Let L be a distributive lattice and $a_1, a_2 \in L$ with $1 = a_1 \oplus a_2$. If $a_1/0$ and $a_2/0$ are strongly cofinitely $\oplus-$supplemented, then L is also strongly cofinitely $\oplus-$supplemented.*

Proof. Let b be a cofinite element of L and a be a supplement of b in L. Since L is distributive, $a = a \wedge 1 = a \wedge (a_1 \oplus a_2) = (a \wedge a_1) \oplus (a \wedge a_2)$ holds. By Lemma 10, $a \wedge a_1$ is a supplement of $(a \wedge a_2) \vee b$ in L. Then we can see that $a \wedge a_1$ is a supplement of $a_1 \wedge ((a \wedge a_2) \vee b)$ in $a_1/0$. Since b is a cofinite element of L, we can see that $a_1 \wedge ((a \wedge a_2) \vee b)$ is a cofinite element of $a_1/0$. Since $a_1/0$ is strongly cofinitely $\oplus-$supplemented, $a \wedge a_1$ is a direct summand of $a_1/0$. Similarly we can see that $a \wedge a_2$ is a direct summand of $a_2/0$. Since $1 = a_1 \oplus a_2$ and $a = (a \wedge a_1) \oplus (a \wedge a_2)$, a is a direct summand of L. Hence L is strongly cofinitely $\oplus-$supplemented. □

Corollary 3. *Let L be a distributive lattice, $a_1, a_2, ..., a_n \in L$ and $1 = a_1 \oplus a_2 \oplus ... \oplus a_n$. If $a_i/0$ is strongly cofinitely $\oplus-$supplemented for every $i = 1, 2, \ldots, n$, then L is strongly cofinitely $\oplus-$supplemented.*

Proof. Clear from Lemma 11. □

Proposition 5. *Let L be a cofinitely supplemented lattice. The following statements are equivalent.*

(i) L is strongly cofinitely $\oplus-$supplemented.

(ii) Every supplement element of a cofinite element of L lies above a direct summand in L.

(iii) (a) For every nonzero supplement element a which is a supplement of a cofinite element of L, $a/0$ contains a nonzero direct summand of L.

(b) For every nonzero supplement element a which is a supplement of a cofinite element of L, $a/0$ contains a maximal direct summand of L.

Proof. $(i) \Longrightarrow (ii)$ Clear, since every element of L lies above itself.

$(ii) \Longrightarrow (iii)$ Let a be a nonzero supplement element which is a supplement of a cofinite element of L. Assume a is a supplement of a cofinite element b of L. By hypothesis, there exists a direct summand x of L such that a lies above x in L. By Lemma 6, $a\beta_* x$ and since $a \vee b = 1$, $x \vee b = 1$. Since a is a supplement of b in L and $x \leq a$, $a = x$ and a is a nonzero direct summand of L.

$(iii) \Longrightarrow (i)$ Let a be a supplement of a cofinite element b of L and x be a maximal direct summand of L with $x \leq a$. Assume $1 = x \oplus y$ with $y \in L$. Then $a = a \wedge 1 = a \wedge (x \oplus y) = x \oplus (a \wedge y)$ and by Lemma 10, $a \wedge y$ is a supplement of $b \vee x$ in L. If $a \wedge y$ is not zero, then by hypothesis, $(a \wedge y)/0$ contains a nonzero direct summand c of L. Here $x \oplus c$ is a direct summand of L and $x \oplus c \leq a$. This contradicts the choice

of x. Hence $a \wedge y = 0$ and $a = x$. Thus a is a direct summand of L and L is strongly cofinitely $\oplus-$supplemented. $\square$

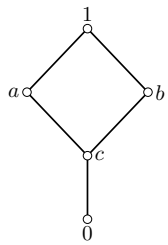
Lemma 12. *Let L be a lattice. If every cofinite element of L is β_* equivalent to a direct summand in L, then L is cofinitely $\oplus-$supplemented.*

Proof. Let a be a cofinite element of L. By hypothesis, there exist $x, y \in L$ with $x \oplus y = 1$ and $a\beta_* x$. Then $a \vee y = 1$. Let $a \vee t = 1$ with $t \leq y$. Since $a\beta_* x$, $x \vee t = 1$ and since $x \oplus y = 1$, $t = y$. Hence y is a supplement of a in L and L is cofinitely $\oplus-$supplemented. $\square$

Corollary 4. *Let L be a lattice. If every cofinite element of L lies above a direct summand in L, then L is cofinitely $\oplus-$supplemented.*

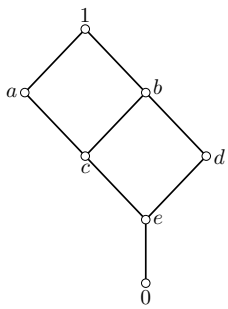
Proof. Clear from Lemma 6 and Lemma 12. $\square$

Example 1. Consider the lattice $L = \{0, a, b, c, 1\}$ given by the following diagram.



Then L is cofinitely supplemented but not cofinitely $\oplus-$supplemented.

Example 2. Consider the lattice $L = \{0, a, b, c, d, e, 1\}$ given by the following diagram.



Then L is cofinitely supplemented but not cofinitely $\oplus-$supplemented.

Example 3. Consider the interval $[0,1]$ with natural topology. Let P be the set of all closed subsets of $[0,1]$. P is complete modular lattice by the inclusion (See [1, Example 2.10]). Here $\bigwedge_{i\in I} C_i = \bigcap_{i\in I} C_i$ and $\bigvee_{i\in I} C_i = \overline{\bigcup_{i\in I} C_i}$ for every $C_i \in P$ $(i \in I)$ $\left(\overline{\bigcup_{i\in I} C_i} \text{ is the closure of } \bigcup_{i\in I} C_i\right)$. Let $X \in P$ and $X \vee Y = [0,1]$ with $Y \in P$. Then

$[0,1]-X \subset Y$ and since Y is closed $\overline{[0,1]-X} \subset Y$. Let $X^{'} = \overline{[0,1]-X}$. Then $X^{'} \in P$, $X \vee X^{'} = X \cup X^{'} = [0,1]$ and $X^{'} \subset Y$ for every $Y \in P$ with $X \vee Y = [0,1]$. Hence X has ample supplements in P (here $X^{'} = \overline{[0,1]-X}$ is the only supplement of X in P) and P is amply supplemented. Let $A = [0,a] \in P$ with $0 < a < 1$. Here $A^{'} = \overline{[0,1]-A} = [a,1]$ is the only supplement of A in P. Let $A^{'} \vee B = A^{'} \cup B = [0,1]$ with $B \in P$. Since $A^{'} \cup B = [0,1]$, $[0,a) = [0,1]-A^{'} \subset B$ and since B is closed, $[0,a] \subset B$. This case $a \in B$ and since $a \in A^{'}$, $A^{'} \wedge B = A^{'} \cap B \neq \varnothing$. Hence $A^{'}$ is not a direct summand of P and P is not $\oplus-$supplemented (See also [5]). We can see that $[0,1]$ is only a cofinite element of L. Hence P is strongly cofinitely $\oplus-$supplemented.

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ON HERMITE-HADAMARD TYPE INEQUALITIES FOR MULTIPLICATIVE FRACTIONAL INTEGRALS

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Abstract. In this study, we first establish two Hermite-Hadamard type inequality for multiplicative (geometric) Riemann-Liouville fractional integrals. Then, by using some properties of multiplicative convex function, we give some new inequalities involving multiplicative fractional integrals.

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Keywords: Hermite-Hadamard inequality, integral inequalities, bounded functions

1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g.,[8], [11], [18, p.137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality.

Over the last twenty years, the numerous studies have focused on to obtain new bound for left hand side and right and side of the inequality (1.1). For some examples, please refer to ([3], [5], [6], [7], [8], [9], [10], [12], [13], [14], [15], [16], [17], [19], [20], [21], [22], [23], [24], [25]).

1.1. *Multiplicative Calculus*

Recall that the concept of the multiplicative integral called * integral is denoted by $\int\limits_a^b (f(x))^{dx}$ which introduced by Bashirov et al. in [4]. While the sum of the terms of product is used in the definition of a classical Riemann integral of f on $[a,b]$, the

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product os terms raised to power is used in the definition multiplicative integral of f on $[a,b]$.

There is the following relation between Rimann integral and multiplicative integral [4]:

Proposition 1. *If f is Riemann integrable on $[a,b]$, then f is multiplicative integrable on $[a,b]$ and*

$$\int_a^b (f(x))^{dx} = e^{\int_a^b \ln(f(x))dx}.$$

Moreover, Bashirov et al [4] show that multiplicative integrable has the following results and properties:

Proposition 2. *If f is positive and Riemann integrable on $[a,b]$, then f is *integrable on $[a,b]$ and*

(i) $\int_a^b ((f(x))^p)^{dx} = \int_a^b ((f(x))^{dx})^p$,

(ii) $\int_a^b (f(x)g(x))^{dx} = \int_a^b (f(x))^{dx}.\int_a^b (g(x))^{dx}$,

(iii) $\int_a^b \left(\frac{f(x)}{g(x)}\right)^{dx} = \frac{\int_a^b (f(x))^{dx}}{\int_a^b (f(x))^{dx}}$,

(iv) $\int_a^b (f(x))^{dx} = \int_a^c (f(x))^{dx}.\int_c^b (f(x))^{dx}$, $a \le c \le b$.

(v) $\int_a^a (f(x))^{dx} = 1$ *and* $\int_a^b (f(x))^{dx} = \left(\int_b^a (f(x))^{dx}\right)^{-1}$.

On the other hand, Abdeljawed and Grossman [1] intoduce the following Multiplicative Riemann-Liouville fractional integrals.

Definition 1. The multiplicative left Riemann-Liouville fractional integral $({}_aI_*^\alpha f)(x)$ of order $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$ starting from α is defined by

$$({}_aI_*^\alpha f)(x) = e^{(J_{a^+}^\alpha (\ln \circ f))(x)}$$

and the multiplicative right one is defined by

$$({}_*I_b^\alpha f)(x) = e^{(J_{b^-}^\alpha (\ln \circ f))(x)},$$

Here $J_{a^+}^\alpha f(x)$ and $J_{b^-}^\alpha f(x)$ denote the left and right Riemann-Liouville fractional integral, defined by

$$J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, x > a$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, x < b$$

respectively.

1.2. *Hermite-Hadamard inequality and convexity*

For the our main results we need to following definition.

Definition 2. A non-empty set K is said to be convex, if for every $a,b \in K$ we have

$$a + \mu(b-a) \in K, \ \forall \mu \in [0,1].$$

Definition 3. A function f is said to be convex function on set K, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \ \forall t \in [0,1].$$

Definition 4. A function f is said to be *log* or multiplicatively convex function on set K, if

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t}, \forall t \in [0,1].$$

Proposition 3. *If f and g are log (multiplicatively) convex functions, then the functions fg and $\frac{f}{g}$ are log (multiplicatively) convex functions.*

The classical Hermite-Hadamard inequality for convex function given by the inequality (1.1).

Hermite-Hadamard inequality for multiplicatively convex function is proved by Ali et al. in [2] as follows:

Theorem 1. *Let f be a positive and multiplicatively convex function on interval $[a,b]$,then the following inequalities hold*

$$f\left(\frac{a+b}{2}\right) \leq \left(\int_a^b (f(x))^{dx}\right)^{\frac{1}{b-a}} \leq G(f(a), f(b)), \tag{1.2}$$

where $G(.,.)$ is a geometric mean.

The inequality (1.2) is the same result proved by Dragomir in [8, Page 197, inequality (5.3)]

On the other hand, Sarikaya et al.[22] proved following important inequality which is the Hermite-Hadamard inequality for Riemann-Liouville fractional integrals.

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

with $\alpha > 0$.

In this paper, we establish Hermite-Haddamard inequality for multiplicative Riemann-Liouville fractional integrals.

2. Main results

In this section we obtain some Hermite-Hadamard type inequalities for multiplicatively convex function via multiplicative Riemann-Liouville fractional integrals.

Theorem 3. *Let f be a positive and multiplicatively convex function on interval $[a,b]$,Then we have the following Hermite-Haddamard inequality for multiplicative Riemann-Liouville fractional integrals*

$$f\left(\frac{a+b}{2}\right) \leq \left[{}_aI_*^{\alpha} f(b) \cdot {}_*I_b^{\alpha} f(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}} \leq G(f(a), f(b)) \tag{2.1}$$

where $G(.,.)$ is a geometric mean.

Proof. Since f is multiplicatively convex function on interval $[a,b]$, then we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{at+(1-t)b+(1-t)a+tb}{2}\right) \\ &\leq [f(at+(1-t)b)]^{\frac{1}{2}} [f((1-t)a+tb)]^{\frac{1}{2}}, \end{aligned}$$

i.e.

$$\ln f\left(\frac{a+b}{2}\right) = \frac{1}{2}\ln f(at+(1-t)b) + \frac{1}{2}\ln f((1-t)a+tb) \tag{2.2}$$

Multiplying both sides of (2.2) by $t^{\alpha-1}$then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\int_0^1 t^{\alpha-1} \ln f\left(\frac{a+b}{2}\right) dt \leq \frac{1}{2}\int_0^1 t^{\alpha-1} \ln f(at+(1-t)b)dt + \frac{1}{2}\int_0^1 t^{\alpha-1} \ln f((1-t)a+tb)dt.$$

By using the change variable, we have

$$\frac{1}{\alpha}\ln f\left(\frac{a+b}{2}\right) dt \leq \frac{1}{(b-a)^{\alpha}}\int_a^b (b-x)^{\alpha-1} \ln f(x)dx + \frac{1}{2}\frac{1}{(b-a)^{\alpha}}\int_a^b (x-a)^{\alpha-1} \ln f(x)dx.$$

That is,

$$\ln f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{b^-}^{\alpha} \ln f(a) + J_{a^+}^{\alpha} \ln f(b)].$$

Thus we get,

$$f\left(\frac{a+b}{2}\right) \leq e^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a^+}^{\alpha} \ln f(b) + J_{b^-}^{\alpha} \ln f(a)]}$$

$$= \left[e^{J_{a^+}^{\alpha} \ln f(b)} . e^{J_{b^-}^{\alpha} \ln f(a)}\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}}$$

$$= \left[{}_aI_*^{\alpha} f(b) .{}_*I_b^{\alpha} f(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}}.$$

which completes the proof of the first inequality in (2.1).

As f is multiplicatively convex function on interval $[a,b]$, then we have

$$f(at+(1-t)b) \leq [f(a)]^t [f(b)]^{1-t}$$

$$f((1-t)a+tb) \leq [f(a)]^{1-t} [f(b)]^t$$

i.e.

$$\ln f(at+(1-t)b) + \ln f((1-t)a+tb) \tag{2.3}$$

$$\leq t \ln f(a) + (1-t) \ln f(b) + (1-t) \ln f(a) + t \ln f(b)$$

$$= \ln f(a) + \ln f(b)$$

Multiplying both sides of (2.3) by $t^{\alpha-1}$then integrating the resulting inequality with respect to t over $[0,1]$,we establish

$$\int_0^1 t^{\alpha-1} \ln f(at+(1-t)b)\, dt + \int_0^1 t^{\alpha-1} \ln f((1-t)a+tb) dt \leq \frac{\ln f(a) + \ln f(b)}{\alpha}.$$

Hence,

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} [J_{b^-}^{\alpha} f(a) + J_{a^+}^{\alpha} f(b)] \leq \frac{1}{2} \ln[f(a).f(b)].$$

Thus, we have the inequality

$$e^{[J_{b^-}^{\alpha} \ln f(a) + J_{a^+}^{\alpha} \ln f(b)]^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}}} \leq \sqrt{f(a).f(b)}$$

and

$$\left[I_*^{\alpha} f(b) .{}_*I_b^{\alpha} f(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}} \leq \sqrt{f(a).f(b)}.$$

The proof is completed □

Remark 1. If we choose $\alpha = 1$ in Theorem 3, then Theorem 3 reduces to the Theorem 1.

Corollary 1. *If f and g are two positive and multiplicative convex functions, then we have the following inequality*

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\leq\left[{}_aI_*^{\alpha}fg(b)\cdot{}_*I_b^{\alpha}fg(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}}\leq G(f(a),f(b)).G(g(a),g(b)). \tag{2.4}$$

Proof. Since f and g are positive and multiplicative convex functions, then fg is positive and multiplicative convex function. Thus if we apply Theorem 3 to the function fg, then we obtain the desired inequality (2.4). □

Remark 2. If we take $\alpha=1$ in Corollary 1, then we have the following inequality

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\leq\left(\int_a^b(f(x))^{dx}.\int_a^b(g(x))^{dx}\right)^{\frac{1}{b-a}}\leq G(f(a),f(b)).G(g(a),g(b)) \tag{2.5}$$

which is given by Ali et al. in [2].

Theorem 4. *Let f be a positive and multiplicatively convex function on interval $[a,b]$,Then we have the following Hermite-Hadamard inequality for multiplicative Riemann-Liouville fractional integrals*

$$f\left(\frac{a+b}{2}\right)\leq\left[{}_{\frac{a+b}{2}}I_*^{\alpha}f(b)\cdot{}_*I_{\frac{a+b}{2}}^{\alpha}f(a)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}\leq G(f(a),f(b)) \tag{2.6}$$

where $G(.,.)$ is a geometric mean.

Proof. By using the multiplicatively convexity of f, we have

$$f\left(\frac{a+b}{2}\right)=f\left[\frac{1}{2}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+\frac{1}{2}(\frac{2-t}{2}a+\frac{t}{2}b)\right]$$

$$\leq\left(f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right)^{\frac{1}{2}}.\left(f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right)^{\frac{1}{2}},$$

i.e.

$$\ln f\left(\frac{a+b}{2}\right)\leq\frac{1}{2}\left[\ln f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+\ln f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right]. \tag{2.7}$$

Multiplying both sides of (2.7) by $t^{\alpha-1}$then integrating the resulting inequality with respect to t over $[0,1]$, we get

$$\frac{1}{\alpha}\ln f(\frac{a+b}{2})\leq\frac{1}{2}\int_0^1 t^{\alpha-1}\ln f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt+\int_0^1 t^{\alpha-1}\ln f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt$$

$$= \frac{\Gamma(\alpha)}{2}\left[\int_{\frac{a+b}{2}}^{b}\left(\frac{2(b-x)}{b-a}\right)^{\alpha-1}\ln f(x)\frac{2}{b-a}dx\right.$$

$$\left.+\int_{a}^{\frac{a+b}{2}}\left(\frac{2(x-a)}{b-a}\right)^{\alpha-1}\ln f(x).\frac{2}{b-a}dx\right]$$

$$= \frac{2^{\alpha-1}\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+b}{2})^{+}}\ln f(b)+J^{\alpha}_{(\frac{a+b}{2})^{-}}\ln f(a)\right].$$

Then it follows that,

$$f\left(\frac{a+b}{2}\right)\leq e^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+b}{2})^{+}}\ln f(b)+J^{\alpha}_{(\frac{a+b}{2})^{-}}\ln f(a)\right]}$$

$$= \left[{}_{\frac{a+b}{2}}I^{\alpha}_{*}f(b).{}_{*}I^{\alpha}_{\frac{a+b}{2}}f(a)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}.$$

This completes the proof of first inequality in the inequality (2.6).

On the other hand, since f is multiplicatively convex function, we get

$$f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\leq[f(a)]^{\frac{t}{2}}+[f(b)]^{\frac{2-t}{2}}+[f(a)]^{\frac{2-t}{2}}+[f(b)]^{\frac{t}{2}}.$$

Thus, we have

$$\ln f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+\ln f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\leq\ln f(a)+\ln f(b) \tag{2.8}$$

Multiplying both sides of (2.8) by $t^{\alpha-1}$then integrating the resulting inequality with respect to t over $[0,1]$, we have

$$\frac{2^{\alpha}}{(b-\alpha)^{\alpha}}\Gamma(\alpha)\left[J^{\alpha}_{(\frac{a+b)}{2})^{+}}\ln f(b)+J^{\alpha}_{(\frac{a+b}{2})^{-}}\ln f(a)\right]\leq\frac{\ln f(a)+\ln f(b)}{\alpha}$$

i.e.

$$\frac{2^{\alpha-1}}{(b-\alpha)^{\alpha}}\Gamma(\alpha+1)\left[J^{\alpha}_{(\frac{a+b}{2})^{+}}\ln f(b)+J^{\alpha}_{(\frac{a+b}{2})^{-}}\ln f(a)\right]\leq\frac{\ln[f(a).f(b)]}{2}.$$

Hence, we get the inequality

$$\left[{}_{\frac{a+b}{2}}I^{\alpha}_{*}f(b).{}_{*}I^{\alpha}_{\frac{a+b}{2}}f(a)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}\leq\sqrt{f(a)f(b)}.$$

The proof is completed □

Corollary 2. *If f and g are positive and multiplicative convex function, then we have the following inequality*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\ & \leq \left[{}_{(\frac{a+b}{2})}I_{*}^{\alpha} f.g(b) .{}_{*}I_{(\frac{a+b}{2})}^{\alpha} f.g(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}} \leq G(f(a), f(b)) G(g(a), g(b)). \end{aligned} \tag{2.9}$$

Proof. The proof is similar to the proof of Corollary 1. □

Remark 3. If we take $\alpha = 1$ in Corollary 2, then the inequality (2.9) reduces to the inequality (2.5).

Corollary 3. *If f and g are positive and multiplicative convex function, then we get the following inequality*

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EXTENSION OF SOME CHENEY-SHARMA TYPE OPERATORS TO A TRIANGLE WITH ONE CURVED SIDE

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Abstract. We extend the definition of some Cheney-Sharma type operators to a triangle with one curved side. We construct their product and Boolean sum, we study their interpolation properties, the orders of accuracy and we give different expressions of the corresponding remainders. We also give some illustrative examples.

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Keywords: Cheney-Sharma operator, product and Boolean sum operators, modulus of continuity, degree of exactness, the Peano's theorem, error evaluation

1. Introduction

In order to match all the boundary information on a curved domain (as Dirichlet, Neumann or Robin boundary conditions for differential equation problems), there were considered interpolation operators on domains with curved sides (see, e.g., [3], [4], [5], [6], [8], [9], [10]).

The aim of this paper is to construct some Cheney-Sharma type operators defined on a triangle with one curved side and to study the interpolation properties, the orders of accuracy and the remainders of the corresponding approximation formulas.

Using the interpolation properties of such operators, blending function interpolants can be constructed, that exactly match the function on some sides of the given region. Important applications of these blending functions are in computer aided geometric design, in finite element method for differential equations problems and for construction of surfaces which satisfy some given conditions.

2. Univariate operators

Let $m \in \mathbb{N}$ and β be a nonnegative parameter. The Cheney-Sharma operator of second kind $Q_m : C[0,1] \to C[0,1]$, introduced in [7], is given by

$$(Q_m f)(x) = \sum_{i=0}^{m} q_{m,i}(x) f(\tfrac{k}{m}), \tag{2.1}$$

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with

$$q_{m,i}(x) = \binom{m}{i} \frac{x(x+i\beta)^{i-1}(1-x)[1-x+(m-i)\beta]^{m-i-1}}{(1+m\beta)^{m-1}}.$$

We recall some results regarding these Cheney-Sharma type operators, studied, for example, in [1], [14].

Remark 1. 1) Notice that for $\beta = 0$, the operator Q_m becomes the Bernstein operator.

2) In [14], it has been proved that the Cheney-Sharma operator Q_m interpolates a given function at the endpoints of the interval.

3) In [7] and [14], it has been proved that the Cheney-Sharma operator Q_m reproduces the constant and the linear functions, so its degree of exactness is 1 (denoted $\mathrm{dex}(Q_m) = 1$).

4) In [7] it is given the following result

$$\begin{aligned}(Q_m e_2)(x) = x(1+m\beta)^{1-m}[&S(2,m-2,x+2\beta,1-x) \\ &-(m-2)\beta S(2,m-3,x+2\beta,1-x+\beta)],\end{aligned} \tag{2.2}$$

where $e_i(x) = x^i$, $i \in \mathbb{N}$, and

$$S(j,m,x,y) = \sum_{k=0}^{m} \binom{m}{k} (x+k\beta)^{k+j-1}[y+(m-k)\beta]^{m-k}, \tag{2.3}$$

$j = \overline{0,m}$, $m \in \mathbb{N}$, $x,y \in [0,1]$, $\beta > 0$.

We consider the standard triangle $\tilde{T}_h$ (see Figure 1), with vertices $V_1 = (0,h)$, $V_2 = (h,0)$ and $V_3 = (0,0)$, with two straight sides Γ_1, Γ_2, along the coordinate axes, and with the third side Γ_3 (opposite to the vertex V_3) defined by the one-to-one functions f and g, where g is the inverse of the function f, i.e., $y = f(x)$ and $x = g(y)$, with $f(0) = g(0) = h$, for $h > 0$. Also, we have $f(x) \le h$ and $g(y) \le h$, for $x,y \in [0,h]$.

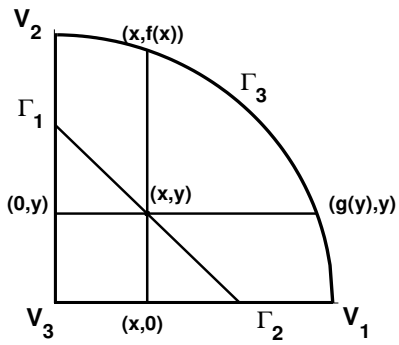

FIGURE 1. Triangle $\tilde{T}_h$.

For $m,n \in \mathbb{N}$, $\beta, b \in \mathbb{R}_+$, we consider the following extensions of the Cheney-Sharma operator given in (2.1):

$$(Q_m^x F)(x,y) = \sum_{i=0}^{m} q_{m,i}(x,y) F\left(i\frac{g(y)}{m}, y\right), \tag{2.4}$$
$$(Q_n^y F)(x,y) = \sum_{j=0}^{n} q_{n,j}(x,y) F\left(x, j\frac{f(x)}{n}\right),$$

with

$$q_{m,i}(x,y) = \binom{m}{i} \frac{\frac{x}{g(y)}(\frac{x}{g(y)}+i\beta)^{i-1}(1-\frac{x}{g(y)})[1-\frac{x}{g(y)}+(m-i)\beta]^{m-i-1}}{(1+m\beta)^{m-1}},$$
$$q_{n,j}(x,y) = \binom{n}{j} \frac{\frac{y}{f(x)}(\frac{y}{f(x)}+jb)^{j-1}(1-\frac{y}{f(x)})[1-\frac{y}{f(x)}+(n-j)b]^{n-j-1}}{(1+nb)^{n-1}},$$

where

$$\Delta_m^x = \left\{ i\frac{g(y)}{m} \,\middle|\, i = \overline{0,m} \right\} \text{ and } \Delta_n^y = \left\{ j\frac{f(x)}{n} \,\middle|\, j = \overline{0,n} \right\}$$

are uniform partitions of the intervals $[0, g(y)]$ and $[0, f(x)]$.

Remark 2. As the Cheney-Sharma operator of second kind interpolates a given function at the endpoints of the interval, we may use the operators Q_m^x and Q_n^y as interpolation operators.

Theorem 1. *If F is a real-valued function defined on $\widetilde{T}_h$ then*

(i) $Q_m^x F = F$ on $\Gamma_1 \cup \Gamma_3$,
(ii) $Q_n^y F = F$ on $\Gamma_2 \cup \Gamma_3$.

Proof. (i) We may write

$$\begin{aligned}(Q_m^x F)(x,y) = \tfrac{1}{(1+m\beta)^{m-1}} \Big\{ & (1-\tfrac{x}{g(y)})[1-\tfrac{x}{g(y)}+m\beta]^{m-1} F(0,y) \\ & + \tfrac{x}{g(y)}(1-\tfrac{x}{g(y)}) \sum_{i=1}^{m-1} \binom{m}{i} (\tfrac{x}{g(y)}+i\beta)^{i-1} \\ & \cdot [1-\tfrac{x}{g(y)}+(m-i)\beta]^{m-i-1} F\left(i\tfrac{g(y)}{m}, y\right) \\ & + \tfrac{x}{g(y)}(\tfrac{x}{g(y)}+m\beta)^{m-1} F(g(y),y) \Big\}.\end{aligned} \tag{2.5}$$

Considering (2.5), we may easily prove that

$$(Q_m^x F)(0,y) = F(0,y),$$
$$(Q_m^x F)(g(y),y) = F(g(y),y).$$

(ii) Similarly, writing

$$(Q_n^y F)(x,y) = \tfrac{1}{(1+nb)^{n-1}} \Big\{ (1-\tfrac{y}{f(x)})[1-\tfrac{y}{f(x)}+nb]^{n-1} F(x,0)$$

$$+\tfrac{y}{f(x)}(1-\tfrac{y}{f(x)})\sum_{j=1}^{n-1}\tbinom{n}{j}(\tfrac{y}{f(x)}+jb)^{j-1}$$
$$\cdot[1-\tfrac{y}{f(x)}+(n-j)b]^{n-j-1}F\left(x,j\tfrac{f(x)}{n}\right)$$
$$+\tfrac{y}{f(x)}(\tfrac{y}{f(x)}+nb)^{n-1}F\left(x,f(x)\right)\Big\},$$

we get that

$$(Q_n^yF)(x,0)=F(x,0),$$
$$(Q_n^yF)(x,f(x))=F(x,f(x)).$$

□

Theorem 2. *The operators Q_m^x and Q_n^y have the following orders of accuracy:*

(i) $(Q_m^xe_{ij})(x,y)=x^iy^j,\ \ i=0,1;\ j\in\mathbb{N}$;
(ii) $\left(Q_n^ye_{ij}\right)(x,y)=x^iy^j,\ \ i\in\mathbb{N};\ j=0,1$, where $e_{ij}(x,y)=x^iy^j,\ \ i,j\in\mathbb{N}$.

Proof. (i) We have

$$(Q_m^xe_{ij})(x,y)=y^j\sum_{i=0}^{m}q_{m,i}(x,y)[i\tfrac{g(y)}{m}]^i,$$

and since the degree of exactness of the univariate Cheney-Sharma operator is equal to 1 (see Remark 1), the result follows.

Property (ii) is proved in the same way. □

We consider the approximation formula

$$F=Q_m^xF+R_m^xF,$$

where R_m^xF denotes the approximation error.

Theorem 3. *If $F(\cdot,y)\in C[0,g(y)]$ then we have*

$$\left|(R_m^xF)(x,y)\right|\le(1+\tfrac{1}{\delta}\sqrt{A_m-x^2})\omega(F(\cdot,y);\delta),\quad\forall\delta>0,\tag{2.6}$$

where $\omega(F(\cdot,y);\delta)$ is the modulus of continuity and $A_m=x(1+m\beta)^{1-m}[S(2,m-2,x+2\beta,1-x)-(m-2)\beta S(2,m-3,x+2\beta,1-x+\beta)]$, with S given in (2.3).

Proof. By Theorem 2 we have that $\mathrm{dex}(Q_m^x)=1$, thus we may apply the following property of linear operators (see, for example, [2], [13]):

$$\left|(Q_m^xF)(x,y)-F(x,y)\right|\le[1+\delta^{-1}\sqrt{(Q_m^xe_{20})(x,y)-x^2}]\omega(F(\cdot,y);\delta),\ \ \forall\delta>0,$$

and taking into account (2.2), we get (2.6). □

Theorem 4. *If $F(\cdot,y)\in C^2[0,g(y)]$ then*

$$\begin{aligned}(R_m^xF)(x,y)=\tfrac{1}{2}F^{(2,0)}(\xi,y)\{x^2-x(1+m\beta)^{1-m}[S(2,m-2,x+2\beta,1-x)\\-(m-2)\beta S(2,m-3,x+2\beta,1-x+\beta)]\},\end{aligned}\tag{2.7}$$

for $\xi \in [0, g(y)]$ *and* $\beta > 0$.

Proof. Taking into account that $\mathrm{dex}(Q_m^x) = 1$, by Theorem 2, and applying the Peano's theorem (see, e.g., [12]), it follows

$$(R_m^x F)(x,y) = \int_0^{g(y)} K_{20}(x,y;s) F^{(2,0)}(s,y)ds,$$

where

$$K_{20}(x,y;s) = (x-s)_+ - \sum_{i=0}^{m} q_{m,i}(x,y)\left(i\tfrac{g(y)}{m} - s\right)_+.$$

For a given $\nu \in \{1,...,m\}$ one denotes by $K_{20}^{\nu}(x,y;\cdot)$ the restriction of the kernel $K_{20}(x,y;\cdot)$ to the interval $\left[(\nu-1)\frac{g(y)}{m}, \nu\frac{g(y)}{m}\right]$, i.e.,

$$K_{20}^{\nu}(x,y;\nu) = (x-s)_+ - \sum_{i=\nu}^{m} q_{m,i}(x,y)\left(i\tfrac{g(y)}{m} - s\right),$$

whence,

$$K_{20}^{\nu}(x,y;s) = \begin{cases} x - s - \sum\limits_{i=\nu}^{m} q_{m,i}(x,y)\left(i\frac{g(y)}{m} - s\right), & s < x \\ -\sum\limits_{i=\nu}^{m} q_{m,i}(x,y)\left(i\frac{g(y)}{m} - s\right), & s \geq x. \end{cases}$$

It follows that $K_{20}^{\nu}(x,y;s) \leq 0$, for $s \geq x$.

For $s < x$ we have

$$K_{20}^{\nu}(x,y;s) = x - s - \sum_{i=0}^{m} q_{m,i}(x,y)\left[i\tfrac{g(y)}{m} - s\right] + \sum_{i=0}^{\nu-1} q_{m,i}(x,y)\left[i\tfrac{g(y)}{m} - s\right].$$

Applying Theorem 2, we get

$$\sum_{i=0}^{m} q_{m,i}(x,y)\left[i\tfrac{g(y)}{m} - s\right] = (Q_m^x e_{10})(x,y) - s(Q_m^x e_{00})(x,y) = x - s,$$

whence it follows that

$$K_{20}^{\nu}(x,y;s) = \sum_{i=0}^{\nu-1} q_{m,i}(x,y)\left[i\tfrac{g(y)}{m} - s\right] \leq 0.$$

So, $K_{20}^{\nu}(x,y;\cdot) \leq 0$, for any $\nu \in \{1,...,m\}$, i.e., $K_{20}(x,y;s) \leq 0$, for $s \in [0, g(y)]$.

By the Mean Value Theorem, one obtains

$$(R_m^x F)(x,y) = F^{(2,0)}(\xi,y)\int_0^{g(y)} K_{20}(x,y;s)ds, \quad \text{for } 0 \leq \xi \leq g(y),$$

with

$$\int_0^{g(y)} K_{20}(x,y;s)ds = \tfrac{1}{2}[x^2 - (Q_m^x e_{20})(x,y)],$$

and using (2.2) we get (2.7). □

Remark 3. Analogous results with the ones in Theorems 3 and 4 could be obtained for the remainder $R_n^y F$ of the formula $F = Q_n^y F + R_n^y F$.

3. Product operators

Let $P_{mn}^1 = Q_m^x Q_n^y$, respectively, $P_{nm}^2 = Q_n^y Q_m^x$ be the products of the operators Q_m^x and Q_n^y.

We have

$$\left(P_{mn}^1 F\right)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}(x,y)\, q_{n,j}\left(i\tfrac{g(y)}{m}, y\right) F\left(i\tfrac{g(y)}{m}, j\tfrac{f(i\frac{g(y)}{m})}{n}\right),$$

respectively,

$$\left(P_{nm}^2 F\right)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}\left(x, j\tfrac{f(x)}{n}\right) q_{n,j}(x,y)\, F\left(i\tfrac{g(j\frac{f(x)}{n})}{m}, j\tfrac{f(x)}{n}\right).$$

Theorem 5. *If F is a real-valued function defined on $\widetilde{T}_h$ then*

(i) $(P_{mn}^1 F)(V_i) = F(V_i), \quad i = 1,2,3;$
$(P_{mn}^1 F)(\Gamma_3) = F(\Gamma_3),$

(ii) $(P_{nm}^2 F)(V_i) = F(V_i), \quad i = 1,2,3;$
$(P_{nm}^2 F)(\Gamma_3) = F(\Gamma_3),$

Proof. By a straightforward computation, we get the following properties

$$\begin{aligned}
&(P_{mn}^1 F)(x,0) = (Q_m^x F)(x,0),\\
&(P_{mn}^1 F)(0,y) = (Q_n^y F)(0,y),\\
&(P_{mn}^1 F)(x,f(x)) = F(x,f(x)), \qquad x,y \in [0,h]
\end{aligned}$$

and

$$\begin{aligned}
&(P_{nm}^2 F)(x,0) = (Q_m^x F)(x,0),\\
&(P_{nm}^2 F)(0,y) = (Q_n^y F)(0,y),\\
&(P_{nm}^2 F)(g(y),y) = F(g(y),y), \qquad x,y \in [0,h],
\end{aligned}$$

and, taking into account Theorem 1, they imply (i) and (ii). □

We consider the following approximation formula

$$F = P_{mn}^1 F + R_{mn}^{P^1} F,$$

where $R_{mn}^{P^1}$ is the corresponding remainder operator.

Theorem 6. *If $F \in C(\widetilde{T}_h)$ then*

$$\left|(R_{mn}^{P^1}F)(x,y)\right| \leq (A_m + B_n - x^2 - y^2 + 1)\omega(F; \tfrac{1}{\sqrt{A_m - x^2}}, \tfrac{1}{\sqrt{B_n - y^2}}), \ \forall (x,y) \in \widetilde{T}_h, \quad (3.1)$$

where

$$\begin{aligned} A_m &= x(1+m\beta)^{1-m}[S(2,m-2,x+2\beta,1-x) \\ &\quad - (m-2)\beta S(2,m-3,x+2\beta,1-x+\beta)] \\ B_n &= y(1+nb)^{1-n}[S(2,n-2,y+2b,1-y) \\ &\quad - (n-2)bS(2,n-3,y+2b,1-y+\beta)] \end{aligned} \quad (3.2)$$

and $\omega(F;\delta_1,\delta_2)$, with $\delta_1 > 0$, $\delta_2 > 0$, is the bivariate modulus of continuity.

Proof. Using a basic property of the modulus of continuity we have

$$\begin{aligned} \left|(R_{mn}^{P^1}F)(x,y)\right| \leq \Bigg[& \tfrac{1}{\delta_1}\sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right)\left|x - \tfrac{i}{m}g(y)\right| \\ & + \tfrac{1}{\delta_2}\sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right)\left|y - \tfrac{j}{n}f\left(\tfrac{i}{m}g(y)\right)\right| \\ & + \sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right)\Bigg]\omega(F;\delta_1,\delta_2), \quad \forall \delta_1,\delta_2 > 0. \end{aligned}$$

Since

$$\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right)\left|x - \tfrac{i}{m}g(y)\right| \leq \sqrt{(Q_m^x e_{20})(x,y) - x^2},$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right)\left|y - \tfrac{j}{n}f\left(\tfrac{i}{m}g(y)\right)\right| \leq \sqrt{(Q_n^y e_{02})(x,y) - y^2},$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right) = 1,$$

applying (2.2), we get

$$\begin{aligned} &\left|(R_{mn}^{P^1}F)(x,y)\right| \leq \Big\{ \tfrac{1}{\delta_1}[x(1+m\beta)^{1-m}]^{\frac{1}{2}} \\ &\cdot \{[S(2,m-2,x+2\beta,1-x) \\ &-(m-2)\beta S(2,m-3,x+2\beta,1-x+\beta)] - x^2\}^{\frac{1}{2}} + \tfrac{1}{\delta_2}[y(1+nb)^{1-n}]^{\frac{1}{2}} \\ &\cdot \{[S(2,n-2,y+2b,1-y) \\ &-(n-2)bS(2,n-3,y+2b,1-y+\beta)] - y^2\}^{\frac{1}{2}} + 1\Big\}\,\omega(F;\delta_1,\delta_2). \end{aligned}$$

Denoting

$$A_m = x(1+m\beta)^{1-m}[S(2,m-2,x+2\beta,1-x)$$

$$
\begin{aligned}
&-(m-2)\beta S(2,m-3,x+2\beta,1-x+\beta)]\\
B_n = y(1+nb)^{1-n}&[S(2,n-2,y+2b,1-y)\\
&-(n-2)bS(2,n-3,y+2b,1-y+\beta)]
\end{aligned}
$$

and, taking $\delta_1 = \frac{1}{\sqrt{A_m - x^2}}$ and $\delta_2 = \frac{1}{\sqrt{B_n - y^2}}$, we get (3.1). □

4. Boolean sum operators

We consider the Boolean sums of the operators Q_m^x and Q_n^y,

$$
\begin{aligned}
S_{mn}^1 &:= Q_m^x \oplus Q_n^y = Q_m^x + Q_n^y - Q_m^x Q_n^y,\\
S_{nm}^2 &:= Q_n^y \oplus Q_m^x = Q_n^y + Q_m^x - Q_n^y Q_m^x.
\end{aligned}
$$

Theorem 7. *If F is a real-valued function defined on $\widetilde{T}_h$, then*

$$
\begin{aligned}
S_{mn}^1 F\Big|_{\partial\widetilde{T}_h} &= F\Big|_{\partial\widetilde{T}_h},\\
S_{mn}^2 F\Big|_{\partial\widetilde{T}_h} &= F\Big|_{\partial\widetilde{T}_h}.
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
(Q_m^x Q_n^y F)(x,0) &= (Q_m^x F)(x,0),\\
(Q_n^y Q_m^x F)(0,y) &= (Q_n^y F)(0,y),\\
(Q_m^x F)(x,h-x) &= (Q_n^y F)(x,h-x)\\
&= (P_{mn}^1 F)(x,h-x) = (P_{nm}^2 F)(x,h-x) = F(x,h-x),
\end{aligned}
$$

and, taking into account Theorem 1, the conclusion follows. □

We consider the following approximation formula

$$
F = S_{mn}^1 F + R_{mn}^{s^1} F,
$$

where $R_{mn}^{s^1}$ is the corresponding remainder operator.

Theorem 8. *If $F \in C(\widetilde{T}_h)$ then*

$$
\begin{aligned}
&\left|(R_{mn}^{s^1} F)(x,y)\right| \le \\
&\le (1+A_m-x^2)\omega(F(\cdot,y);\tfrac{1}{\sqrt{A_m-x^2}}) + (1+B_n-y^2)\omega(F(x,\cdot);\tfrac{1}{\sqrt{B_n-y^2}})\\
&+(A_m+B_n-x^2-y^2+1)\omega(F;\tfrac{1}{\sqrt{A_m-x^2}},\tfrac{1}{\sqrt{B_n-y^2}}),
\end{aligned}
\tag{4.1}
$$

with A_m and B_n given in (3.2).

Proof. The identity

$$F - S^1_{mn}F = (F - Q^x_m F) + (F - Q^y_n F) - (F - P^1_{mn}F)$$

implies that

$$\left|(R^{s^1}_{mn}F)(x,y)\right| \leq \left|(R^x_m F)(x,y)\right| + \left|(R^y_n F)(x,y)\right| + \left|(R^{P^1}_{mn}F)(x,y)\right|,$$

and, applying Theorems 3 and 6, we get (4.1). □

5. Numerical examples

We consider the function:

$$\text{Gentle:} \quad F(x,y) = \tfrac{1}{3}\exp[-\tfrac{81}{16}\left((x-0.5)^2 + (y-0.5)^2\right)],$$

generally used in the literature, (see, e.g., [11]). In Figure 2 we plot the graphs of F, $Q^x_m F$, $Q^y_n F$, $P^1_{mn}F$, $S^1_{mn}F$, on $\tilde{T}_h$, considering $h = 1, m = 5$, $n = 6$, $\beta = 1$ and we can see the good approximation properties.

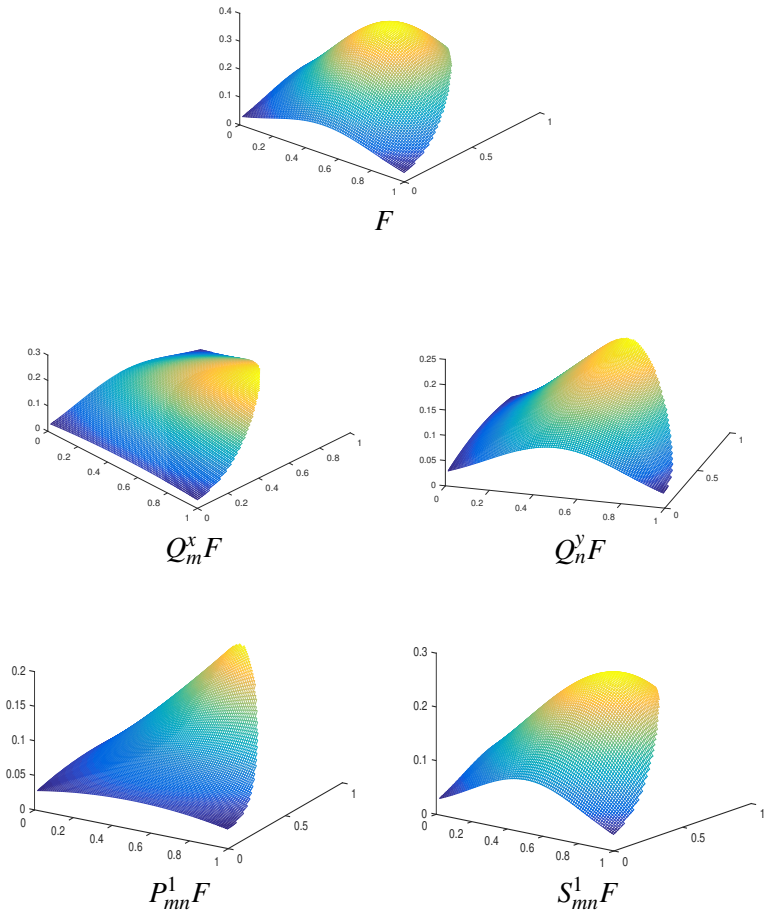

FIGURE 2. The Cheney-Sharma approximants for $\tilde{T}_h$.

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DIRECTLY DECOMPOSABLE IDEALS AND CONGRUENCE KERNELS OF COMMUTATIVE SEMIRINGS

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Abstract. As pointed out in the monographs [5, 6] on semirings, ideals play an important role despite the fact that they need not be congruence kernels as in the case of rings. Hence, having two commutative semirings $\mathbf{S}_1$ and $\mathbf{S}_2$, one can ask whether an ideal I of their direct product $\mathbf{S} = \mathbf{S}_1 \times \mathbf{S}_2$ can be expressed in the form $I_1 \times I_2$ where I_j is an ideal of $\mathbf{S}_j$ for $j = 1, 2$. Of course, the converse is elementary, namely if I_j is an ideal of $\mathbf{S}_j$ for $j = 1, 2$ then $I_1 \times I_2$ is an ideal of $\mathbf{S}_1 \times \mathbf{S}_2$. Having a congruence Θ on a commutative semiring $\mathbf{S}$, its 0-class is an ideal of $\mathbf{S}$, but not every ideal is of this form. Hence, the lattice $\mathbf{Id}\,\mathbf{S}$ of all ideals of $\mathbf{S}$ and the lattice $\mathbf{Ker}\,\mathbf{S}$ of all congruence kernels (i.e. 0-classes of congruences) of $\mathbf{S}$ need not be equal. Furthermore, we show that the mapping $\Theta \mapsto [0]\Theta$ need not be a homomorphism from $\mathbf{Con}\,\mathbf{S}$ onto $\mathbf{Ker}\,\mathbf{S}$. Moreover, the question arises when a congruence kernel of the direct product $\mathbf{S}_1 \times \mathbf{S}_2$ of two commutative semirings can be expressed as a direct product of the corresponding kernels on the factors. In the paper we present necessary and sufficient conditions for such direct decompositions both for ideals and for congruence kernels of commutative semirings. We also provide sufficient conditions for varieties of commutative semirings to have directly decomposable kernels.

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1. Introduction

The importance of the investigation of semirings is based on the fact that they form a concept which covers both distributive lattices with 0 and rings. Hence, several results known for rings can be transformed to distributive lattices and vice versa. Of course, the concept of a semiring is much more general than that of a ring and hence not all the results known for rings can be extended to semirings. In particular, rings are congruence permutable contrary to the case of semirings. Further, for rings, congruences are in a one-to-one correspondence with ideals. This does not hold for semirings. However, important results on rings motivate the investigation of similar

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results on semirings. In the present paper we study the relation between ideals and congruence kernels with respect to direct decomposability.

There exist two different versions of the concept of a semiring, namely semirings having a unit element ([5]) and those without such an element ([3, 6]). Since the second version is mostly used in applications, we define the basic concept of this paper as follows:

Definition 1 (see [6])**.** A *commutative semiring* is an algebra $\mathbf{S} = (S, +, \cdot, 0)$ of type $(2, 2, 0)$ satisfying

- $(S, +, 0)$ is a commutative monoid,
- $(S, \cdot)$ is a commutative semigroup,
- $(x + y)z \approx xz + yz$,
- $0x \approx 0$.

If $\mathbf{S}$ is a commutative semiring containing an element 1 satisfying the identity $1x \approx x$ then $\mathbf{S}$ is called *unitary*, and if it satisfies the identity $xx \approx x$ then it is called *idempotent*. A (semi-)ring $(S, +, \cdot, 0)$ is called *zero-(semi-)ring* if $xy = 0$ for all $x, y \in S$.

It is evident that every (unitary) commutative ring is a (unitary) commutative semiring and that every distributive lattice $\mathbf{L} = (L, \vee, \wedge, 0)$ with least element 0 is an idempotent commutative semiring.

In the following let $\mathbb{N}$ denote the set of non-negative integers. Then clearly, $\mathbf{N} = (\mathbb{N}, +, \cdot, 0)$ is a unitary commutative semiring. For every positive integer a let $a\mathbb{N}$ denote the set $\{0, a, 2a, 3a, \ldots\}$ of all non-negative multiples of a. It is evident that $a\mathbf{N} = (a\mathbb{N}, +, \cdot, 0)$ is again a commutative semiring which is unitary if and only if $a = 1$.

We recall the following definition from [5]:

Definition 2. Let $\mathbf{S} = (S, +, \cdot, 0)$ be a commutative semiring. An *ideal* of $\mathbf{S}$ is a subset I of S satisfying

- $0 \in I$,
- if $a, b \in I$ then $a + b \in I$,
- if $a \in I$ and $s \in S$ then $as \in I$.

Note that in case that $\mathbf{S}$ is a ring, the ideals of $\mathbf{S}$ in the sense of Definition 2 need not be ring ideals. Consider e.g. the zero-ring whose additive group is the group $(\mathbb{Z}, +, 0)$ of the integers. Then the ideals of this zero-ring in the sense of Definition 2 are the submonoids of $(\mathbb{Z}, +, 0)$, whereas the ring ideals are the subgroups of $(\mathbb{Z}, +, 0)$.

The converse is, of course, trivial for any ring $\mathbf{S}$: Every ring ideal of $\mathbf{S}$ is an ideal in the sense of Definition 2.

Let $\mathrm{Id}\,\mathbf{S}$ denote the set of all ideals of a commutative semiring $\mathbf{S} = (S, +, \cdot, 0)$. It is clear that $\mathbf{Id}\,\mathbf{S} = (\mathrm{Id}\,\mathbf{S}, \subseteq)$ is a complete lattice with smallest element $\{0\}$ and greatest

element S. Moreover, for $I,J \in \mathrm{Id}\,\mathbf{S}$ we have

$$I \vee J = I + J = \{i + j \mid i \in I, j \in J\},$$
$$I \wedge J = I \cap J.$$

For $a \in S$ let $I(a)$ denote the ideal of $\mathbf{S}$ generated by a. Obviously, $I(a) = a\mathbb{N} + aS$. The lattice $\mathbf{Id}\,\mathbf{S}$ need not be modular as the following example shows:

Example 1. The Hasse diagram of the ideal lattice of the commutative zero-semiring $\mathbf{S} = (S,+,\cdot,0)$ on $\{0,a,b,c,d,e,f,g\}$ defined by

$+$	0	a	b	c	d	e	f	g
0	0	a	b	c	d	e	f	g
a	a	b	c	0	e	f	g	d
b	b	c	0	a	f	g	d	e
c	c	0	a	b	g	d	e	f
d	d	e	f	g	d	e	f	g
e	e	f	g	d	e	f	g	d
f	f	g	d	e	f	g	d	e
g	g	d	e	f	g	d	e	f

looks as follows (see Figure 1):

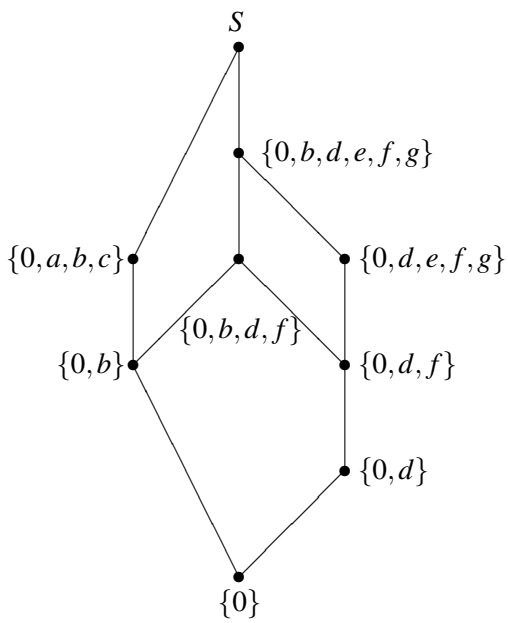

FIGURE 1.

and hence this lattice is not modular. Observe that $(S,+,0)$ is isomorphic to the direct product of its submonoids $\{0,a,b,c\}$ (four-element cyclic group) and $\{0,d\}$ (two-element join-semilattice). The ideals of the semiring $\mathbf{S}$ are the submonoids of $(S,+,0)$.

Let $\mathbf{Con}\,\mathbf{S} = (\mathrm{Con}\,\mathbf{S},\subseteq)$ denote the congruence lattice of a commutative semiring $\mathbf{S}$. A *congruence kernel* of $\mathbf{S}$ is a set of the form $[0]\Theta$ with $\Theta \in \mathrm{Con}\,\mathbf{S}$. It is well known (cf. [5]) that every congruence kernel is an ideal of $\mathbf{S}$, but not vice versa. Let

$$\mathbf{Ker}\,\mathbf{S} = (\mathrm{Ker}\,\mathbf{S},\subseteq) = (\{[0]\Theta \mid \Theta \in \mathrm{Con}\,\mathbf{S}\},\subseteq)$$

denote the (complete) *lattice of congruence kernels* of $\mathbf{S}$. In contrast to rings, $\mathbf{Con}\,\mathbf{S}$ and $\mathbf{Ker}\,\mathbf{S}$ need not be isomorphic as the following example shows, in which two different congruences have the same kernel.

Example 2. Consider the three-element lattice $\mathbf{D}_3 = (D_3,\vee,\wedge,0) = (\{0,a,1\},\ \vee,\wedge,0)$. Then the Hasse diagram of $\mathbf{Con}\,\mathbf{D}_3$ looks as follows (see Figure 2):

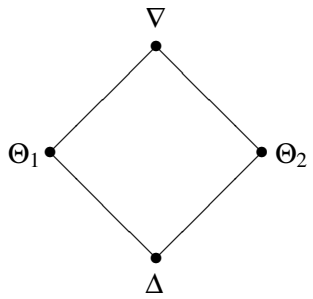

FIGURE 2.

where

$$\Theta_1 = \{0,a\}^2 \cup \{1\}^2 \text{ and } \Theta_2 = \{0\}^2 \cup \{a,1\}^2.$$

However, Δ and Θ_2 have the same kernel $\{0\}$. Hence

$$\mathbf{Ker}\,\mathbf{D}_3 = (\{\{0\},\{0,a\},D_3\},\subseteq)$$

is a three-element chain and $\mathbf{Con}\,\mathbf{D}_3 \ncong \mathbf{Ker}\,\mathbf{D}_3$. Moreover, even the mapping $\Theta \mapsto [0]\Theta$ is not a homomorphism from $\mathbf{Con}\,\mathbf{D}_3$ onto $\mathbf{Ker}\,\mathbf{D}_3$ since

$$[0](\Theta_1 \vee \Theta_2) = [0]\nabla = D_3 \neq \{0,a\} = \{0,a\} \vee \{0\} = [0]\Theta_1 \vee [0]\Theta_2.$$

2. Ideals of direct products of commutative semirings

In the following we are interested in ideals on a direct product of two commutative semirings. Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be commutative semirings. Of course, if $I_1 \in \operatorname{Id}\mathbf{S}_1$ and $I_2 \in \operatorname{Id}\mathbf{S}_2$ then $I_1 \times I_2 \in \operatorname{Id}(\mathbf{S}_1 \times \mathbf{S}_2)$. An ideal I of $\mathbf{S}_1 \times \mathbf{S}_2$ is called *directly decomposable* if there exist $I_1 \in \operatorname{Id}\mathbf{S}_1$ and $I_2 \in \operatorname{Id}\mathbf{S}_2$ with $I_1 \times I_2 = I$. If I is not directly decomposable then it is called a *skew ideal*. The aim of this paper is to characterize those commutative semirings which have directly decomposable ideals.

Example 3. If $\mathbf{R}_2 = (R_2, +, \cdot, 0) = (\{0,1\}, +, \cdot, 0)$ denotes the two-element zero-ring and $\mathbf{D}_2 = (D_2, \vee, \wedge, 0) = (\{0,1\}, \vee, \wedge, 0)$ the two-element lattice then $\mathbf{S} := \mathbf{R}_2 \times \mathbf{D}_2$ has the non-trivial ideals

$$\begin{aligned}
&\{0\} \times D_2,\\
&R_2 \times \{0\},\\
&\{(0,0),(0,1),(1,1)\}
\end{aligned}$$

and hence $\mathbf{Id}\,\mathbf{S} \cong \mathbf{N}_5$ which is not modular and, moreover, the last mentioned ideal is skew.

Example 4. If $\mathbf{R}_4 = (R_4, +, \cdot, 0) = (\{0,a,b,c\}, +, \cdot, 0)$ denotes the zero-ring whose additive group is the Kleinian 4-group defined by

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

then $\mathbf{S} := \mathbf{R}_4 \times \mathbf{D}_2$ has the non-trivial ideals

$$\begin{aligned}
&\{0,a\} \times \{0\},\\
&\{0,b\} \times \{0\},\\
&\{0,c\} \times \{0\},\\
&R_4 \times \{0\},\\
&\{0\} \times D_2,\\
&\{0,a\} \times D_2,\\
&\{0,b\} \times D_2,\\
&\{0,c\} \times D_2,\\
&\{(0,0),(0,1),(a,1)\},\\
&\{(0,0),(0,1),(b,1)\},\\
&\{(0,0),(0,1),(c,1)\},
\end{aligned}$$

$$\{(0,0),(0,1),(a,1),(b,1),(c,1)\},$$
$$\{(0,0),(a,0),(0,1),(a,1),(b,1),(c,1)\},$$
$$\{(0,0),(b,0),(0,1),(a,1),(b,1),(c,1)\},$$
$$\{(0,0),(c,0),(0,1),(a,1),(b,1),(c,1)\}$$

the last seven of which are skew.

For sets A and B let π_1 and π_2 denote the first and second projection from $A \times B$ onto A and B, respectively. Note that for any subset C of $A \times B$ we have $C \subseteq \pi_1(C) \times \pi_2(C)$. Furthermore, if C is of the form $A_1 \times B_1$ with $A_1 \subseteq A$ and $B_1 \subseteq B$ then $\pi_1(C) = A_1$ and $\pi_2(C) = B_1$.

We borrow the method from [4] (which was used also in [2]) to prove the following theorem:

Theorem 1. *Let* $\mathbf{S}_1 = (S_1,+,\cdot,0)$ *and* $\mathbf{S}_2 = (S_2,+,\cdot,0)$ *be commutative semirings and* $I \in \mathrm{Id}(\mathbf{S}_1 \times \mathbf{S}_2)$ *and consider the following assertions:*

(i) *I is directly decomposable,*
(ii) $(S_1 \times \{0\}) \cap ((\{0\} \times S_2) + I) \subseteq I$ *and* $((S_1 \times \{0\}) + I) \cap (\{0\} \times S_2) \subseteq I$,
(iii) *if* $(a,b) \in I$ *then* $(a,0),(0,b) \in I$,
(iv) $((S_1 \times \{0\}) + I) \cap ((\{0\} \times S_2) + I) = I$.

Then (iii) $\Leftrightarrow$ (i) $\Rightarrow$ (iv) $\Rightarrow$ (ii).

Proof.
(iii) $\Rightarrow$ (i): If $(a,b) \in \pi_1(I) \times \pi_2(I)$ then there exists some pair $(c,d) \in S_1 \times S_2$ with $(a,d),(c,b) \in I$, hence $(a,0),(0,b) \in I$ which shows $(a,b) = (a,0) + (0,b) \in I$.
(i) $\Rightarrow$ (iii): This is clear.
(i) $\Rightarrow$ (iv): If $I = I_1 \times I_2$ then

$$\begin{aligned}&((S_1 \times \{0\}) + I) \cap ((\{0\} \times S_2) + I) = \\ &= ((S_1 \times \{0\}) + (I_1 \times I_2)) \cap ((\{0\} \times S_2) + (I_1 \times I_2)) = \\ &= (S_1 \times I_2) \cap (I_1 \times S_2) = I_1 \times I_2 = I.\end{aligned}$$

(iv) $\Rightarrow$ (ii): This follows immediately. □

Remark 1. That (ii) does not imply (iii) can be seen by considering the ideal $I := \{(0,0),(0,1),(a,1)\}$ of $\mathbf{S}$ in Example 4. Since

$$(S_1 \times \{0\}) \cap ((\{0\} \times S_2) + I) = \{(0,0)\} \subseteq I,$$
$$((S_1 \times \{0\}) + I) \cap (\{0\} \times S_2) = \{(0,0),(0,1)\} \subseteq I,$$

(ii) holds. Because of $(a,1) \in I$ and $(a,0) \notin I$, (iii) does not hold. This shows that (ii) does not imply (iii). It is worth noticing that the implication (ii) $\Rightarrow$ (iii) holds in the case of commutative rings since in this case

$$(a,0) = (0,-b) + (a,b),$$

$$(0,b) = (-a,0) + (a,b).$$

So in this case (i) and (ii) are equivalent.

Example 5. According to (iii) of Theorem 1, the ideal $I(4,6)$ of $2\mathbf{N} \times 2\mathbf{N}$ is not directly decomposable since

$$(4,0),(0,6) \notin I(4,6) = (4,6)\mathbb{N} + (4,6)(2\mathbb{N} \times 2\mathbb{N}) = (4,6)\mathbb{N} + (8\mathbb{N} \times 12\mathbb{N}) = \\ = (8\mathbb{N} \times 12\mathbb{N}) \cup ((4+8\mathbb{N}) \times (6+12\mathbb{N})).$$

Next we present several simple sufficient conditions for direct decomposability of ideals.

Corollary 1. *Let* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *be commutative semirings such that one of the following conditions hold:*

(i) $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *are unitary,*
(ii) $\mathbf{S}_1$ *is unitary and* $\mathbf{S}_2$ *is idempotent,*
(iii) $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *are idempotent,*
(iv) $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *are rings and* $\mathbf{Id}(\mathbf{S}_1 \times \mathbf{S}_2)$ *is distributive.*

Then every ideal of $\mathbf{S}_1 \times \mathbf{S}_2$ *is directly decomposable.*

Proof. Assume $(a,b) \in I \in \mathrm{Id}(\mathbf{S}_1 \times \mathbf{S}_2)$. Then
$(a,0) = (a,b)(1,0) \in I$ and $(0,b) = (a,b)(0,1) \in I$ in case (i),
$(a,0) = (a,b)(1,0) \in I$ and $(0,b) = (a,b)(0,b) \in I$ in case (ii) and
$(a,0) = (a,b)(a,0) \in I$ and $(0,b) = (a,b)(0,b) \in I$ in case (iii)
showing direct decomposability of I according to condition (iii) of Theorem 1. In case (iv), direct decomposability of I follows from condition (ii) of Theorem 1 together with Remark 1. □

If a field $\mathbf{F} = (F,+,\cdot)$ is considered as a ring then it has no proper ideals. However, the same is valid also in the case of semiring ideals. Namely, if I is a non-zero semiring ideal in $\mathbf{F}$ and $d \in I \setminus \{0\}$, then for each $x \in F$ we have $x = dd^{-1}x \in I$ proving $I = F$.

Proposition 1. *If* $\mathbf{S}$ *is a commutative semiring and* $\mathbf{F}$ *a field then every ideal of* $\mathbf{S} \times \mathbf{F}$ *is directly decomposable.*

Proof. Assume $\mathbf{S} = (S,+,\cdot,0)$ and $\mathbf{F} = (F,+,\cdot,0)$, let $I \in \mathrm{Id}(\mathbf{S} \times \mathbf{F})$ and put $J := \pi_1(I)$. If $I \subseteq S \times \{0\}$ then $I = J \times \{0\}$. Now assume $I \not\subseteq S \times \{0\}$. Then there exists some $(a,b) \in I$ with $b \neq 0$. If $(c,d) \in J \times F$ then $c \in J = \pi_1(I)$. Thus there exists some $e \in F$ with $(c,e) \in I$ and hence

$$(c,d) = (c,e) + (a,b)(0,b^{-1}(d-e)) \in I$$

showing $I = J \times F$. Hence, $\mathbf{S} \times \mathbf{F}$ has directly decomposable ideals. □

3. Congruence kernels of direct products of commutative semirings

Now we draw our attention to congruence kernels.

If $\Theta_1 \in \operatorname{Con}\mathbf{S}_1$ and $\Theta_2 \in \mathbf{S}_2$ then

$$\Theta_1 \times \Theta_2 := \{((x_1,x_2),(y_1,y_2)) \mid (x_1,y_1) \in \Theta_1, (x_2,y_2) \in \Theta_2\} \in \operatorname{Con}(\mathbf{S}_1 \times \mathbf{S}_2)$$

and $[(0,0)](\Theta_1 \times \Theta_2) = [0]\Theta_1 \times [0]\Theta_2$. However, there may exist congruences Θ on $\mathbf{S}_1 \times \mathbf{S}_2$ such that $[(0,0)]\Theta \neq [0]\Theta_1 \times [0]\Theta_2$ for all possible $\Theta_1 \in \operatorname{Con}\mathbf{S}_1$ and $\Theta_2 \in \operatorname{Con}\mathbf{S}_2$. It should be noted that Fraser and Horn (cf. [4]) presented necessary and sufficient conditions for direct decomposability of congruences. In the following we will modify these conditions for congruence kernels.

If $\mathbf{S}_1 = (S_1,+,\cdot,0)$ and $\mathbf{S}_2 = (S_2,+,\cdot,0)$ are commutative semirings, $\Theta \in \operatorname{Con}(\mathbf{S}_1 \times \mathbf{S}_2)$ and $i \in \{1,2\}$ then we put

$$\pi_i(\Theta) := \{(\pi_i(x),\pi_i(y)) \mid (x,y) \in \Theta\},$$
$$\Pi_i := \{(x,y) \in (S_1 \times S_2)^2 \mid \pi_i(x) = \pi_i(y)\}.$$

Note that $\Theta_i := \pi_i(\Theta) \in \operatorname{Con}(\mathbf{S}_i)$, $\Pi_i \in \operatorname{Con}(\mathbf{S}_1 \times \mathbf{S}_2)$ and $\Pi_1 \cap \Pi_2 = \{(x,x) \mid x \in S_1 \times S_2\}$. Let us remark that in general $[0]\Theta_i \neq \pi_i([(0,0)]\Theta)$, namely e.g. $a \in [0]\Theta_1$ is equivalent to the fact that there exist $b,c \in S_2$ with $(a,b) \in [(0,c)]\Theta$.

We call the kernel $[(0,0)]\Theta$ *directly decomposable* if

$$[(0,0)]\Theta = \pi_1([(0,0)]\Theta) \times \pi_2([(0,0)]\Theta),$$

and furthermore, we call the kernel $[(0,0)]\Theta$ *strongly directly decomposable* if

$$[(0,0)]\Theta = [0]\Theta_1 \times [0]\Theta_2.$$

Note that

$$\pi_i([(0,0)]\Theta) \subseteq [0]\Theta_i \text{ for } i = 1,2,$$
$$[(0,0)]\Theta \subseteq \pi_1([(0,0)]\Theta) \times \pi_2([(0,0)]\Theta),$$

thus strongly direct decomposability implies direct decomposability (cf. also the following Theorems 2 and 4).

We characterize strongly directly decomposable congruence kernels as follows:

Theorem 2. *If $\mathbf{S}_1 = (S_1,+,\cdot,0)$ and $\mathbf{S}_2 = (S_2,+,\cdot,0)$ are commutative semirings and $\Theta \in \operatorname{Con}(\mathbf{S}_1 \times \mathbf{S}_2)$ then $[(0,0)]\Theta$ is strongly directly decomposable if and only if the following holds:*

$$\textit{If } (a,b)\,\Theta(0,c) \textit{ and } (d,e)\,\Theta(f,0) \textit{ then } (a,e)\,\Theta(0,0)$$

for $(a,b),(0,c),(d,e),(f,0) \in S_1 \times S_2$.

Proof. If $[(0,0)]\Theta$ is strongly directly decomposable and

$$(a,b)\,\Theta(0,c) \text{ and } (d,e)\,\Theta(f,0)$$

for $(a,b),(0,c),(d,e),(f,0) \in S_1 \times S_2$ then $(a,e) \in [0]\Theta_1 \times [0]\Theta_2 = [(0,0)]\Theta$. If, conversely, the condition of the theorem holds and $(g,h) \in [0]\Theta_1 \times [0]\Theta_2$ then there exist $i,j \in S_1$ and $k,l \in S_2$ with $(g,k)\,\Theta(0,l)$ and $(i,h)\,\Theta(j,0)$ and hence $(g,h) \in [(0,0)]\Theta$ showing $[0]\Theta_1 \times [0]\Theta_2 \subseteq [(0,0)]\Theta$. The converse inclusion is trivial. □

Using this result we can prove the following

Theorem 3. *If* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *are commutative semirings,* $\Theta \in \mathrm{Con}(\mathbf{S}_1 \times \mathbf{S}_2)$ *and*

$$[(0,0)]((\Theta \vee \Pi_1) \cap \Pi_2) \subseteq [(0,0)]\Theta,$$
$$[(0,0)]((\Theta \vee \Pi_2) \cap \Pi_1) \subseteq [(0,0)]\Theta$$

then $[(0,0)]\Theta$ *is strongly directly decomposable.*

Proof. Let $(a,b),(0,c),(d,e),(f,0) \in S_1 \times S_2$ and assume $(a,b)\,\Theta(0,c)$ and $(d,e)\,\Theta(f,0)$. Then

$$(a,0)\,\Pi_1(a,b)\,\Theta(0,c)\,\Pi_1(0,0) \text{ and } (a,0)\,\Pi_2(0,0)$$

and hence

$$(a,0) \in [(0,0)]((\Theta \vee \Pi_1) \cap \Pi_2) \subseteq [(0,0)]\Theta.$$

Analogously,

$$(0,e)\,\Pi_2(d,e)\,\Theta(f,0)\,\Pi_2(0,0) \text{ and } (0,e)\,\Pi_1(0,0)$$

and hence

$$(0,e) \in [(0,0)]((\Theta \vee \Pi_2) \cap \Pi_1) \subseteq [(0,0)]\Theta.$$

Therefore

$$((a,0)+(0,e))\,\Theta((0,0)+(0,0)).$$

Since

$$(a,e) = (a,0)+(0,e),$$
$$(0,0) = (0,0)+(0,0),$$

we obtain

$$(a,e) \in [(0,0)]\Theta.$$

Now the assertion follows from Theorem 2. □

Recall that an *algebra* $\mathbf{A}$ with 0 is called *distributive at* 0 (see e.g. [1]) if for all $\Theta,\Phi,\Psi \in \mathrm{Con}\,\mathbf{A}$ we have

$$[0]((\Theta \vee \Phi) \cap \Psi) = [0]((\Theta \cap \Psi) \vee (\Phi \cap \Psi)).$$

A *class* of algebras with 0 is called *distributive at* 0 if any of its members has this property. Applying Theorem 3 we can state:

Corollary 2. *If* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *are commutative semirings and* $\mathbf{S}_1 \times \mathbf{S}_2$ *is distributive at* $(0,0)$ *then the kernel of every congruence on* $\mathbf{S}_1 \times \mathbf{S}_2$ *is strongly directly decomposable.*

Proof. For all $\Theta \in \mathrm{Con}(\mathbf{S}_1 \times \mathbf{S}_2)$ we have

$$\begin{aligned}[(0,0)]((\Theta \vee \Pi_1) \cap \Pi_2) &= [(0,0)]((\Theta \cap \Pi_2) \vee (\Pi_1 \cap \Pi_2)) = [(0,0)](\Theta \cap \Pi_2) \subseteq \\ &\subseteq [(0,0)]\Theta, \\ [(0,0)]((\Theta \vee \Pi_2) \cap \Pi_1) &= [(0,0)]((\Theta \cap \Pi_1) \vee (\Pi_2 \cap \Pi_1)) = [(0,0)](\Theta \cap \Pi_1) \subseteq \\ &\subseteq [(0,0)]\Theta.\end{aligned}$$

Now the result follows from Theorem 3. □

Recall the Mal'cev type characterization of distributivity at 0 from [1] (Theorem 8.2.2):

Proposition 2. *A variety with* 0 *is distributive at* 0 *if and only if there exist some positive integer* n *and binary terms* $t_0, \dots, t_n$ *such that*

$$\begin{aligned}&t_0(x,y) \approx 0, \\ &t_i(0,y) \approx 0 \textit{ for } i \in \{0,\dots,n\}, \\ &t_i(x,0) \approx t_{i+1}(x,0) \textit{ for even } i \in \{0,\dots,n-1\}, \\ &t_i(x,x) \approx t_{i+1}(x,x) \textit{ for odd } i \in \{0,\dots,n-1\}, \\ &t_n(x,y) \approx x.\end{aligned}$$

We can apply Proposition 2 to idempotent commutative semirings.

Corollary 3. *The variety of idempotent commutative semirings is distributive at* 0 *and hence has strongly directly decomposable congruence kernels.*

Proof. If

$$\begin{aligned}n &:= 2, \\ t_0(x,y) &:= 0, \\ t_1(x,y) &:= xy, \\ t_2(x,y) &:= x\end{aligned}$$

then

$$\begin{aligned}&t_0(x,y) \approx 0, \\ &t_0(0,y) \approx 0, \\ &t_1(0,y) \approx 0y \approx 0, \\ &t_2(0,y) \approx 0, \\ &t_0(x,0) \approx 0 \approx x0 \approx t_1(x,0), \\ &t_1(x,x) \approx xx \approx x \approx t_2(x,x), \\ &t_2(x,y) \approx x\end{aligned}$$

and hence, by Proposition 2, we obtain the result. □

The following characterization of directly decomposable congruence kernels is similar to the characterization presented in Theorem 2.

Theorem 4. *For commutative semirings* $\mathbf{S}_1 = (S_1, +, \cdot, 0)$ *and* $\mathbf{S}_2 = (S_2, +, \cdot, 0)$ *and* $\Theta \in \mathrm{Con}(\mathbf{S}_1 \times \mathbf{S}_2)$ *the kernel* $[(0,0)]\Theta$ *is directly decomposable if and only if*

$$(a,b),(c,d) \in [(0,0)]\Theta \text{ implies } (a,d) \in [(0,0)]\Theta. \tag{3.1}$$

Proof. If the kernel $[(0,0)]\Theta$ is directly decomposable and $(a,b),(c,d) \in$ $\in [(0,0)]\Theta$ then

$$a \in \pi_1([(0,0)]\Theta) \text{ and } d \in \pi_2([(0,0)]\Theta)$$

whence

$$(a,d) \in \pi_1([(0,0)]\Theta) \times \pi_2([(0,0)]\Theta) = [(0,0)]\Theta$$

proving (3.1). Conversely, assume (3.1) to be satisfied and let

$$(a,d) \in \pi_1([(0,0)]\Theta) \times \pi_2([(0,0)]\Theta).$$

Then there exist $b \in S_2$ and $c \in S_1$ with $(a,b),(c,d) \in [(0,0)]\Theta$. Using (3.1) we conclude $(a,d) \in [(0,0)]\Theta$ proving

$$\pi_1([(0,0)]\Theta) \times \pi_2([(0,0)]\Theta) \subseteq [(0,0)]\Theta.$$

The converse inclusion is trivial. □

It is evident also from the conditions of Theorems 2 and 4 that if a direct product of semirings has strongly directly decomposable congruence kernels then it has directly decomposable congruence kernels.

We say that a class $\mathcal{C}$ of algebras of the same type containing a constant 0 has *directly decomposable congruence kernels* if for any $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{C}$ and each $\Theta \in \mathrm{Con}(\mathbf{A}_1 \times \mathbf{A}_2)$, $[(0,0)]\Theta$ is directly decomposable.

The following Mal'cev type condition was derived in [1]:

Proposition 3 (Theorem 11.0.4 in [1])**.** *A variety of algebras with* 0 *has directly decomposable congruence kernels if there exist positive integers* m *and* n*, binary terms* $s_1, \dots, s_m, t_1, \dots\dots, t_m$ *and* $(m+2)$*-ary terms* $u_1, \dots, u_n$ *satisfying the identities*

$$\begin{aligned}
u_1(x,y,s_1(x,y),\dots,s_m(x,y)) &\approx x,\\
u_1(y,x,t_1(x,y),\dots,t_m(x,y)) &\approx x,\\
u_i(y,x,s_1(x,y),\dots,s_m(x,y)) &\approx u_{i+1}(x,y,s_1(x,y),\dots,s_m(x,y)) \text{ for } i =\\
&= 1,\dots,n-1,\\
u_i(x,y,t_1(x,y),\dots,t_m(x,y)) &\approx u_{i+1}(y,x,t_1(x,y),\dots,t_m(x,y)) \text{ for } i =\\
&= 1,\dots,n-1,\\
u_n(y,x,s_1(x,y),\dots,s_m(x,y)) &\approx x,\\
u_n(x,y,t_1(x,y),\dots,t_m(x,y)) &\approx y.
\end{aligned}$$

Corollary 4. *The variety of unitary commutative semirings has directly decomposable congruence kernels.*

Proof. If

$$\begin{aligned} m &:= 3, \\ n &:= 2, \\ s_1(x,y) &:= 1, \\ s_2(x,y) &:= 0, \\ s_3(x,y) &:= 0, \\ t_1(x,y) &:= 0, \\ t_2(x,y) &:= 1, \\ t_3(x,y) &:= y, \\ u_1(x,y,z,u,v) &:= xz+yu, \\ u_2(x,y,z,u,v) &:= yz+v \end{aligned}$$

then

$$\begin{aligned} &u_1(x,y,1,0,0) \approx x1+y0 \approx x, \\ &u_1(y,x,0,1,y) \approx y0+x1 \approx x, \\ &u_1(y,x,1,0,0) \approx y1+x0 \approx y \approx y1+0 \approx u_2(x,y,1,0,0), \\ &u_1(x,y,0,1,y) \approx x0+y1 \approx y \approx x0+y \approx u_2(y,x,0,1,y), \\ &u_2(y,x,1,0,0) \approx x1+0 \approx x, \\ &u_2(x,y,0,1,y) \approx y0+y \approx y \end{aligned}$$

and hence, by Proposition 3, we obtain the result. □

4. Conclusion

Although not all ideals of a commutative semiring are congruence kernels, we obtained a characterization of direct decomposability of ideals on a direct product of commutative semirings analogous to that by Fraser-Horn for congruences. Using a Mal'cev type condition characterizing varieties which are distributive at 0, we have shown that the variety of idempotent commutative semirings has strongly directly decomposable congruence kernels. Moreover, we proved that the variety of unitary commutative semirings has directly decomposable congruence kernels. This generalizes the corresponding result for the variety of unitary commutative rings which is well known.

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ON CONTRACTIONS VIA SIMULATION FUNCTIONS ON EXTENDED b-METRIC SPACES

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Abstract. In this paper, we introduce the notion of an admissible extended $\mathcal{Z}$-contraction mapping in the setting of extended b-metric spaces. As an application, we consider Ulam stability problems based on our contractions. The presented results cover several existing results in the literature.

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1. Introduction and Preliminaries

The notion of the distance has been discussed from Euclid period. In the beginning of the nineteen century it was axiomatically formulated under the name of metric, by Fréchet and Haussdorff. Since then, the axioms of the metric notions have been relaxed several ways to generalize it. Among all, we mention the concept of a b-metric (see [12, 15]) which was also announced as "quasi-metric" (see [23]).

Definition 1 (Czerwik [15])**.** Let X be a nonempty set and $d : X \times X \to [0,\infty)$ be a function satisfying the following conditions:

$(b1)$ $d(x,y) = 0$ if and only if $x = y$;
$(b2)$ $d(x,y) = d(y,x)$ for all $x,y \in X$;
$(b3)$ $d(x,y) \leq s[d(x,z) + d(z,y)]$ for all $x,y,z \in X$, where $s \geq 1$.

The function d is called a b-metric and the space (X,d) is called a b-metric space, in short, bMS.

Example 1. Let $X = \{x_i : 1 \leq i \leq M\}$ for some $M \in \mathbb{N}$ and $s \geq 2$. Define $d : X \times X \to \infty$ as

$$d(x_i,x_j) = \begin{cases} 0 & \text{if } i = j, \\ s & \text{if } |i-j| = 1, \\ 2 & \text{if } |i-j| = 2, \\ 1 & \textit{otherwise}. \end{cases}$$

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Accordingly, we obtain that $d(x_i,x_j) \leq \frac{s}{2}[d(x_i,x_k)+d(x_k,x_j)]$, for all $i,j,k \in \{1,M\}$. As a result, the pair (X,d) forms a b-metric for $s>2$. Note that the standard triangle inequality does not hold.

Example 2 (See e.g. [23]). The space $L^p[0,1]$ (where $0<p<1$) of all real functions $x(t)$, $t \in [0,1]$ such that $\int_0^1 |x(t)|^p dt < \infty$, together with the functional

$$d(x,y) := (\int_0^1 |x(t)-y(t)|^p dt)^{1/p}, \text{ for each } x,y \in L^p[0,1],$$

is a b-metric space. Notice that $s = 2^{1/p}$.

Example 3 (See e.g. [23]). Let E be a Banach space and 0_E be the zero vector of E. Let P be a cone in E with $int(P) \neq \varnothing$ and $\preceq$ be a partial ordering with respect to P. Let X be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies:

$(M1)$ $0 \preceq d(x,y)$ for all $x,y \in X$;
$(M2)$ $d(x,y) = 0$ if and only if $x = y$;
$(M3)$ $d(x,y) \preceq d(x,z)+d(z,y)$, for all $x,y \in X$;
$(M4)$ $d(x,y) = d(y,x)$ for all $x,y \in X$.

Then d is called a cone metric on X, and the pair (X,d) is called a cone metric space (CMS).

Let E be a Banach space and P be a normal cone in E with the coefficient of normality denoted by K. Let $D : X \times X \to [0,\infty)$ be defined by $D(x,y) = ||d(x,y)||$, where $d : X \times X \to E$ is a cone metric space. Then (X,D) is a b-metric space with a constant $s := K \geq 1$.

Czerwik [15] proved the analogue of Banach fixed point theorem. For (common) fixed point results on b-metric spaces, see [1–4, 7, 9–11, 13, 14, 20, 22, 24].

Recently, Kamran [17] introduced a new type of generalized metric spaces and they proved some fixed point theorems on this space.

Definition 2 ([17])**.** Let X be a non empty set and $\theta : X \times X \to [1,\infty)$. A function $d_\theta : X \times X \to [0,\infty)$ is called an extended $b-$metric if for all $x,y,z \in X$ is satisfies

$(d_\theta 1)$ $d_\theta(x,y) = 0$ if and only if $x = y$;
$(d_\theta 2)$ $d_\theta(x,y) = d_\theta(y,x)$;
$(d_\theta 3)$ $d_\theta(x,y) \leq \theta(x,y)\,[d_\theta(x,z)+d_\theta(z,y)]$.

The pair (X,d_θ) is called an extended $b-$metric space, in short extended-bMS.

Remark 1. If $\theta(x,y) = s$, for $s \geq 1$ then we obtain the definition of bMS. Note that neither $b-$metric nor extended $b-$metric is continuous. Throughout the paper, we presume that all considered an extended $b-$metrics are continuous.

Example 4. Let $p \in (0,1)$, $q > 1$ and $X = l_p(\mathbb{R}) \cup l_q(\mathbb{R})$ equipped with the metric

$$d(x,y) = \begin{cases} d_p(x,y) & \text{if } x,y \in l_p(\mathbb{R}), \\ d_q(x,y) & \text{if } x,y \in l_q(\mathbb{R}), \\ 0 & otherwise, \end{cases}$$

where

$$l_r(\mathbb{R}) = \left\{ x = \{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^r < \infty \right\} \text{ for } r = p, q.$$

and

$$d_r(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^r \right)^{1/r}, \text{ for } r = p, q.$$

It is clear that (X,d) forms an extended b-metric with

$$\theta(x,y) = \begin{cases} 2^{1/p} & \text{if } x,y \in l_p(\mathbb{R}), \\ 2^{1/q} & \text{if } x,y \in l_q(\mathbb{R}), \\ 1 & \text{otherwise.} \end{cases}$$

Example 5 ([17]). Let $X = \{1,2,3\}$, $\theta : X \times X \to [1,\infty)$ and $d_\theta : X \times X \to [0,\infty)$ as $\theta(x,y) = 1 + x + y$ and

$$d_\theta(1,1) = d_\theta(2,2) = d_\theta(3,3) = 0, \quad \text{and} \quad d_\theta(1,2) = d_\theta(2,1) = 80,$$
$$d_\theta(1,3) = d_\theta(3,1) = 1000, \quad \text{and} \quad d_\theta(2,3) = d_\theta(3,2) = 600.$$

Example 6 ([5]). Let $X = [0,1]$, $\theta : X \times X \to [1,\infty)$ and $d_\theta : X \times X \to [0,\infty)$ as $\theta(x,y) = \dfrac{1+x+y}{x+y}$ and

$$d_\theta(x,y) = \frac{1}{xy}, x,y \in (0,1], x \neq y,$$
$$d_\theta(x,y) = 0, x,y \in [0,1], x = y,$$
$$d_\theta(x,0) = d_\theta(0,x) = \frac{1}{x}, x \in (0,1].$$

Some fundamental concepts like convergence, Cauchy sequence and completeness in a extended-bMS are defined as follows [17].

Definition 3 ([17])**.** Let (X,d_θ) be an extended-bMS.

(i) A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to converge to $x \in X$, if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $d_\theta(x_n,x) < \varepsilon$, for all $n \geq N$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(ii) A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be Cauchy if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $d_\theta(x_m,x_n) < \varepsilon$, for all $m,n \geq N$.

Definition 4. An extended-bMS (X,d_θ) is complete if every Cauchy sequence in X is convergent.

Lemma 1. *Let (X,d_θ) be an complete extended-bMS. If d_θ is continuous, then every convergent sequence has a unique limit.*

Throughout the paper, we presume that all extended-b-metrics are continuous.

The notion of comparison functions is defined by Rus [23] and it has been extensively studied by a number of authors to get more general forms of contractive mappings.

Definition 5 ([23])**.** A function $\phi : [0,\infty) \to [0,\infty)$ is called a comparison function if it is increasing and $\phi^n(t) \to 0$ as $n \to \infty$ for every $t \in [0,\infty)$, where ϕ^n is the n-th iterate of ϕ.

Properties and examples of comparison functions can be found in [23]. An important property of comparison functions is given by the following Lemma.

Lemma 2 ([23])**.** *If $\phi : [0,\infty) \to [0,\infty)$ is a comparison function, then*

(1) *each iterate ϕ^k of ϕ, $k \geq 1$ is also a comparison function;*
(2) *ϕ is continuous at 0;*
(3) *$\phi(t) < t$ for all $t > 0$.*

Definition 6 ([23])**.** Let $s \geq 1$ be a real number. A function $\phi : [0,\infty) \to [0,\infty)$ is called a (b)-comparison function if ϕ is increasing and

(*) there exist $k_0 \in \mathbb{N}$, $a \in [0,1)$ and a convergent nonnegative series $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\phi^{k+1}(t) \leq as^k\phi^k(t) + v_k$, for $k \geq k_0$ and any $t \geq 0$.

The collection of all (b)-comparison functions will be denoted by Ψ. In the literature, a (b)-comparison function is called (c)-comparison functions when $s = 1$. It can be shown that a (c)-comparison function is a comparison function, but the converse is not true in general. Berinde [23] also proved the following important property of (b)-comparison functions.

Lemma 3 ([23])**.** *Let $\phi : [0,\infty) \to [0,\infty)$ be a (b)-comparison function. Then*

(1) *the series $\sum_{k=0}^{\infty} s^k\phi^k(t)$ converges for any $t \in [0,\infty)$;*
(2) *the function $b_s : [0,\infty) \to [0,\infty)$ defined as $b_s = \sum_{k=0}^{\infty} s^k\phi^k(t)$ is increasing and is continuous at $t = 0$.*

Remark 2. Any (b)-comparison function ϕ satisfies $\phi(t) < t$ and $\lim_{n\to\infty} \phi^n(t) = 0$ for each $t > 0$.

In order to unify several existing fixed point results in the literature, Khojasteh *et al.* [19] introduced the notion of *simulation functions* and investigate the existence and uniqueness of a fixed point for different types of contractive mappings.

Definition 7. A *simulation function* is a mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:

(ζ_1): $\zeta(t,s) < s-t$ for all $t,s>0$;
(ζ_2): if $(t_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}}$ are sequences in $(0,\infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0, \text{ then } \limsup_{n\to\infty} \zeta(t_n,s_n) < 0.$$

In [19], the condition $\zeta(0,0)=0$ was added, but Argoubi et al. [8] dropped it. Let $\mathcal{Z}$ denote the family of all simulation functions $\zeta : [0,\infty)\times[0,\infty)\to\mathbb{R}$, that is, verifying (ζ_1) and (ζ_2).

Due to the axiom (ζ_1), we have

$$\zeta(t,t) < 0 \text{ for all } t>0.$$

The following example is derived from [6, 16, 19].

Example 7. Let $\phi_i : [0,\infty)\to[0,\infty)$ be continuous functions such that $\phi_i(t)=0$ if and only if, $t=0$. For $i=1,2,3,4,5,6$, we define the mappings $\zeta_i : [0,\infty)\times[0,\infty)\to\mathbb{R}$, as follows

(i): $\zeta_1(t,s) = \phi_1(s)-\phi_2(t)$, for all $t,s\in[0,\infty)$, where $\phi_1,\phi_2 : [0,\infty)\to[0,\infty)$ are two continuous functions such that $\phi_1(t)=\phi_2(t)=0$ if and only if $t=0$ and $\phi_1(t)<t\le\phi_2(t)$ for all $t>0$.

(ii): $\zeta_2(t,s) = s-\dfrac{f(t,s)}{g(t,s)}t$, for all $t,s\in[0,\infty)$, where $f,g : [0,\infty)\times[0,\infty)\to(0,\infty)$ are two continuous functions with respect to each variable such that $f(t,s)>g(t,s)$, for all $t,s>0$.

(iii): $\zeta_3(t,s) = s-\phi_3(s)-t$, for all $t,s\in[0,\infty)$.

(iv): $\zeta_4(t,s) = s\varphi(s)-t$, for all $s,t\in[0,\infty)$, where $\varphi : [0,\infty)\to[0,1)$ is a function such that $\limsup_{t\to r^+}\varphi(t)<1$, for all $r>0$.

(v): $\zeta_5(t,s) = \eta(s)-t$, for all $s,t\in[0,\infty)$, where $\eta : [0,\infty)\to[0,\infty)$ is an upper semi-continuous mapping such that $\eta(t)<t$, for all $t>0$ and $\eta(0)=0$.

(vi): $\zeta_6(t,s) = s-\int_0^t\phi(u)du$, for all $s,t\in[0,\infty)$, where $\phi : [0,\infty)\to[0,\infty)$ is a function such that $\int_0^\varepsilon\phi(u)du$ exists and $\int_0^\varepsilon\phi(u)du>\varepsilon$, for each $\varepsilon>0$.

It is clear that each function ζ_i $(i=1,2,3,4,5,6)$ forms a simulation function.

Suppose (X,d) is a metric space, T is a self-mapping on X and $\zeta\in\mathcal{Z}$. We say that T is a *$\mathcal{Z}$-contraction* with respect to ζ [19], if

$$\zeta(d(T(x),T(y)),d(x,y))\ge 0, \text{ for all } x,y\in X.$$

Theorem 1 ([17]). *Let (X,d_θ) be an extended-bMS such that d_θ is a continuous functional. Let $T : X\to X$ satisfy:*

$$d_\theta(T(x),T(y)) \le k d_\theta(x,y), \tag{1.1}$$

for all $x,y\in X$, where $k\in[0,1)$ be such that for each $x_0\in X$, $\lim_{n,m\to\infty}\theta(x_n,x_m)<\frac{1}{k}$, here $x_n=T^n(x_0)$, $n=1,2,\ldots$. Then T has precisely one fixed point u. Moreover, for each $y\in X$, $T^n(y)\to u$.

The notion of α-admissible mappings and the concept of triangular α-admissible mappings [18] are reconsidered and refined by Popescu [21] in the following way:

Definition 8 ([21])**.** Let $\alpha : X \times X \to [0,\infty)$ be a mapping and $X \neq \varnothing$. A self-mapping $T : X \to X$ is said to be an $\alpha-$orbital admissible if for all $s \in X$, we have

$$\alpha(s,T(s)) \geq 1 \Rightarrow \alpha(T(s),T^2(s)) \geq 1.$$

Furthermore, an $\alpha-$orbital admissible mapping T is called triangular α-orbital admissible if

(TO) $\alpha(s,t) \geq 1$ and $\alpha(s,T(t)) \geq 1$ implies that $\alpha(s,T(t)) \geq 1$, for all $s,t \in X$.

In this paper, we investigate the existence of a fixed point of an admissible extended $\mathcal{Z}$-contraction in the context of extended b-metric spaces.

2. Main results

We start this section by introducing nonlinear contractive mappings in the setting of extended b-metric spaces as follows.

Definition 9. Let (X,d) be an extended-bMS and $\theta : X \times X \to [1,\infty)$. A mapping $T : X \to X$ is called an admissible extended $\mathcal{Z}$-contraction mapping if there is a $\zeta \in \mathcal{Z}$ such that

$$\zeta(\alpha(x,y)d_\theta(T(x),T(y)),\phi(M(x,y))) \geq 0, \text{ for all } x,y \in X, \tag{2.1}$$

where $\phi \in \Psi$ and

$$M(x,y) = \max\{d_\theta(x,y),d_\theta(x,T(x)),d_\theta(y,T(y))\}. \tag{2.2}$$

Remark 3. Note that for admissible extended $\mathcal{Z}$-contraction mappings we have

$$\alpha(x,y)d_\theta(Tx,Ty) \leq \phi(M(x,y))) \text{ for all } x,y \in X. \tag{2.3}$$

In what follows we shall express the main theorem of this paper.

Theorem 2. *Let (X,d_θ) be a complete extended b-metric space and $T : X \to X$ a mapping. Suppose that exists a sequence $(q_n)_{n\in\mathbb{N}}$, $q_n > 1$, for all $n \in \mathbb{N}$ such that $\theta(x_n,x_m) < q_n$, for all $m > n$. If T is an admissible extended $\mathcal{Z}$-contraction mapping satisfying*

- (i) *T is triangular $\alpha-$orbital admissible;*
- (ii) *there exists $x_0 \in X$ such that $\alpha(x_0,T(x_0)) \geq 1$;*
- (iii) *T is continuous,*

then it has a fixed point u. In the case of existence of a fixed point u, we have $T^n(y) \to u$ for each $y \in X$.

Proof. Due to (ii) there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$. Define an recursive sequence $\{x_n\}$ by $x_n = T^n(x_0)$ for $n \in \mathbb{N}$. If for some $n_0 \in \mathbb{N}$, we have $x_{n_0} = x_{n_0+1} =$

$T(x_{n_0})$, then x_{n_0} is a fixed point of T. From now on, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

On the other hand, employing (ii) and regarding that T is $\alpha-$orbital admissible, we find, by iteration, that

$$\alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n = 0, 1, \ldots. \tag{2.4}$$

Furthermore, by taking (i) into account, we deduce also

$$\alpha(x_n, x_m) \geq 1, \text{ for all } m > n. \tag{2.5}$$

By letting $x = x_n$ and $y = x_{n+1}$ in (2.3) and keeping (2.4) in mind, we derive

$$\begin{aligned} d_\theta(x_n, x_{n+1}) &= d_\theta(T(x_{n-1}), T(x_n)) \\ &\leq \alpha(x_{n-1}, x_n) d_\theta(T(x_{n-1}), T(x_n)) \leq \phi(M(x_{n-1}, x_n)), \end{aligned}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d_\theta(x_{n-1}, x_n), d_\theta(x_{n-1}, T(x_{n-1})), d_\theta(x_n, T(x_n))\} \\ &= \max\{d_\theta(x_{n-1}, x_n), d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1})\} \\ &= \max\{d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1})\} \end{aligned}$$

If for some $n \in \mathbb{N}$, we get

$$M(x_{n-1}, x_n) = \max\{d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1})\} = d_\theta(x_n, x_{n+1}),$$

then

$$0 < d_\theta(x_n, x_{n+1}) \leq \phi(d_\theta(x_n, x_{n+1})) < d_\theta(x_n, x_{n+1}),$$

which is a contradiction. Thus, for all $n \geq 1$,

$$M(x_{n-1}, x_n) = \max\{d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1})\} = d_\theta(x_{n-1}, x_n).$$

Hence

$$0 < d_\theta(x_n, x_{n+1}) \leq \phi(d_\theta(x_{n-1}, x_n)) < d_\theta(x_{n-1}, x_n), \quad \forall\, n \geq 1. \tag{2.6}$$

We deduce

$$0 < d_\theta(x_n, x_{n+1}) \leq \phi^n(d_\theta(x_0, x_1)), \quad \forall\, n \geq 0. \tag{2.7}$$

Letting $n \to \infty$ in (2.7), because ϕ is a (b)-comparison function, we get

$$\lim_{n\to\infty} d_\theta(x_n, x_{n+1}) = 0.$$

We claim that (x_n) is a Cauchy sequence. For all $m > n$,

$$\begin{aligned} d_\theta(x_n, x_m) &\leq \theta(x_n, x_m)(d_\theta(x_n, x_{n+1}) + d_\theta(x_{n+1}, x_m)) \\ &\leq \theta(x_n, x_m) d_\theta(x_n, x_{n+1}) + \theta(x_n, x_m)\theta(x_{n+1}, x_m) d_\theta(x_{n+1}, x_{n+2}) \\ &\quad + \ldots + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\ldots\theta(x_{m-1}, x_m) d_\theta(x_{m-1}, x_m) \\ &\leq \theta(x_n, x_m)\phi^n(d_\theta(x_0, x_1)) + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\phi^{n+1}(d_\theta(x_0, x_1)) \\ &\quad + \ldots + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\ldots\theta(x_{m-1}, x_m)\phi^{m-1}(d_\theta(x_0, x_1)) \\ &\leq \theta(x_1, x_m)\theta(x_2, x_m)\ldots\theta(x_n, x_m)\phi^n(d_\theta(x_0, x_1)) \end{aligned}$$

$$+\theta(x_1,x_m)\theta(x_2,x_m)...\theta(x_{n+1},x_m)\phi^{n+1}(d_\theta(x_0,x_1))$$
$$+...+\theta(x_1,x_m)\theta(x_2,x_m)...\theta(x_{m-1},x_m)\phi^{m-1}(d_\theta(x_0,x_1)).$$

Choose for all $n \in \mathbb{N}$

$$S_n = \sum_{j=1}^{n} \phi^j(d_\theta(x_0,x_1)) \prod_{i=1}^{j} \theta(x_i,x_m).$$

We deduce that

$$d_\theta(x_n,x_m) \leq S_{m-1} - S_{n-1}, \quad \forall\, m > n.$$

Consider the series

$$\sum_{n=1}^{\infty} \phi^n(d_\theta(x_0,x_1)) \prod_{i=1}^{n} \theta(x_i,x_m).$$

Let $q = \max\{q_1,...,q_n\}$. We have

$$u_n = \phi^n(d_\theta(x_0,x_1)) \prod_{i=1}^{j} \theta(x_i,x_m) \leq \phi^n(d_\theta(x_0,x_1))q^n = v_n.$$

From Lemma 3, we have that the series $\sum\limits_{j=0}^{\infty} \phi^j(d_\theta(x_0,x_1))q^j$ converges. Using comparison criteria for the convergence of series, we obtain that

$$\sum_{n=1}^{\infty} \phi^n(d_\theta(x_0,x_1)) \prod_{i=1}^{n} \theta(x_i,x_m)$$

converges, and hence

$$\lim_{n,m\to\infty} d_\theta(x_n,x_m) = 0$$

that is, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Since (X,d) is a complete extended b-metric space, there exists $z \in X$ such that

$$\lim_{n\to\infty} d_\theta(x_n,z) = 0.$$

Since T is continuous, we derive that

$$\lim_{n\to\infty} d_\theta(T(x_n),T(z)) = 0 = \lim_{n\to\infty} d_\theta(x_{n+1},T(z)) = d_\theta(z,T(z)).$$

Regarding the uniqueness of the limit, we conclude that $Tz = z$. □

Theorem 3. *Let (X,d_θ) be a complete extended b-metric space and $T : X \to X$ a mapping. Suppose that exists a sequence $(q_n)_{n\in\mathbb{N}}$, $q_n > 1$, for all $n \in \mathbb{N}$ such that $\theta(x_n,x_m) < q_n$, for all $m > n$. If T is an admissible extended Z-contraction mapping satisfying*

(i) *T is triangular α-orbital admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,T(x_0)) \geq 1$;*

(iii) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x \in X$ *as* $n \to \infty$, *then there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)}, x) \geq 1$ *for all* k.

then it has a fixed point u. *In the case of existence of a fixed point* u, *we have* $T^n(y) \to u$ *for each* $y \in X$.

Proof. Following the proof of Theorem 1, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = T(x_n)$ for all $n \geq 0$, converges for some $u \in X$. From condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all k. Applying (2.1), for all k, we get that

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_{n(k)}, u) d_\theta(Tx_{n(k)}, Tu), \phi(M(x_{n(k)}, u))) \\ &= \zeta(\alpha(x_{n(k)}, u) d_\theta(x_{n(k)+1}, Tu), \phi(M(x_{n(k)}, u))) \\ &< M(x_{n(k)}, u) - \alpha(x_{n(k)}, u) d(x_{n(k)+1}, Tu), \end{aligned}$$

which is equivalent to

$$\begin{aligned} d_\theta(x_{n(k)+1}, T(u)) &= d_\theta(T(x_{n(k)}), T(u)) \\ &\leq \alpha(x_{n(k)}, u) d_\theta(T(x_{n(k)}), T(u)) \\ &\leq \phi(M(x_{n(k)}, u)) \\ &= \phi(\max\{d_\theta(x_{n(k)}, u), d_\theta(x_{n(k)}, T(x_{n(k)})), d_\theta(u, T(u))\}). \end{aligned}$$

Letting $\limsup$ as $k \to \infty$ in the equality above, we have $d_\theta(u, Tu) = 0$, that is, $u = Tu$. □

For the uniqueness of a fixed point of a α-admissible $\mathcal{Z}$-*contraction* with respect to ζ, we shall suggest the following hypothesis.

(U) For all $x, y \in \mathrm{Fix}(T)$, we have $\alpha(x, y) \geq 1$.

Here, $\mathrm{Fix}(T)$ denotes the set of fixed points of T.

Theorem 4. *Adding condition* (U) *to the hypotheses of Theorem 1 (resp. Theorem 3), we obtain that* u *is the unique fixed point of* T.

Proof. We shall prove that z is unique. Assume that z and w two fixed points of T with $z \neq w$. By (2.1),

$$\begin{aligned} d_\theta(z, w) &= d_\theta(T(z), T(w)) \leq \alpha(z, w) d_\theta(T(z), T(w)) \\ &\leq \phi(M(z, w)) = \phi(\max\{d_\theta(z, w), d_\theta(z, T(z)), d_\theta(w, T(w))\}) \\ &= \phi(d_\theta(z, w)) < d_\theta(z, w), \end{aligned}$$

which is a contradiction. So the fixed point of T is unique. □

3. Ulam stability and well-posedness of the fixed point problem

Definition 10. Let (X,d_θ) be an extended b-metric space and $T:X\to X$ be an operator. The fixed point problem

$$x=T(x),\ x\in X, \tag{3.1}$$

is called generalized Ulam stable if and only if there exists $\psi:\mathbb{R}_+\to\mathbb{R}_+$, increasing, continuous in 0 and $\psi(0)=0$, such that, for each $\varepsilon>0$ and for each solution $y^*\in X$ of the inequality

$$d_\theta(y,T(y))\le\varepsilon, \tag{3.2}$$

there exists a solution x^* of the fixed point problem (3.1) such that

$$d_\theta(x^*,y^*)\le\psi(\varepsilon).$$

Theorem 5. *Let (X,d_θ) be a complete extended b-metric space and $T:X\to X$ a mapping. Suppose that all the hypotheses of Theorem 2 (or Theorem 3) hold. Additionally we suppose that*

- (i) *for any solution $y^*\in X$ of (3.2), we have that $\alpha(x_n,y^*)\ge 1$ and $\lim_{n\to\infty}\theta(x_{n+1},y^*)<q=\max\{q_1,\dots,q_n\}$, where $x_n=T^n(x_0)$, $n\in\mathbb{N}$;*
- (ii) *$\beta:[0,\infty)\to[0,\infty)$, $\beta(r):=r-q\phi(r)$ is strictly increasing and onto;*
- (iii) *ϕ is continuous.*

Then, the fixed point problem (3.1) is generalized Ulam stable.

Proof. Since the conditions of Theorem 2 hold, the fixed point problem (3.1) has a unique solution $x^*\in X$. Let $y^*\in X$ be a solution of (3.2). We have

$$\begin{aligned} d_\theta(x^*,y^*) &= \lim_{n\to\infty} d_\theta(x_{n+1},y^*)=\lim_{n\to\infty} d_\theta(T(x_n),y^*) \\ &\le \lim_{n\to\infty}\theta(T(x_n),y^*)\left[d_\theta(T(x_n),T(y^*))+d_\theta(y^*,T(y^*))\right] \end{aligned} \tag{3.3}$$

Since T is an admissible extended $\mathcal{Z}$-contraction mapping, we have

$$\begin{aligned} 0 &\le \zeta(\alpha(x_n,y^*)d_\theta(T(x_n),T(y^*)),\phi(M(x_n,y^*))) \\ &< \phi(M(x_n,y^*))-\alpha(x_n,y^*)d_\theta(T(x_n),T(y^*)) \\ &\le \phi(M(x_n,y^*))-d_\theta(T(x_n),T(y^*)). \end{aligned}$$

Hence, we have

$$d_\theta(T(x_n),T(y^*))\le\phi(M(x_n,y^*)).$$

Accordingly, the inequality (3.3) becomes

$$d_\theta(x^*,y^*)\le\lim_{n\to\infty}\theta(T(x_n),y^*)\left[\phi(M(x_n,y^*))+\varepsilon\right]$$

$$d_\theta(x^*,y^*)\le q\lim_{n\to\infty}\phi(M(x_n,y^*))+q\varepsilon \tag{3.4}$$

Now, we consider

$$M(x_n,y^*)=\max\{d_\theta(x_n,y^*),d_\theta(x_n,T(x_n)),d_\theta(y^*,T(y^*))\}$$

$$M(x_n,y^*) \leq \max\{d_\theta(x_n,y^*), d_\theta(x_n,T(x_n)), \varepsilon\}$$
$$\lim_{n\to\infty} M(x_n,y^*) \leq \max\left\{\lim_{n\to\infty} d_\theta(x_n,y^*), \varepsilon\right\} = \max\{d_\theta(x^*,y^*), \varepsilon\}$$

If $\max\{d_\theta(x^*,y^*), \varepsilon\} = \varepsilon$, then (3.3) turns into

$$d_\theta(x^*,y^*) \leq q\phi(\varepsilon) + q\varepsilon = \psi(\varepsilon).$$

If $\max\{d_\theta(x^*,y^*), \varepsilon\} = d_\theta(x^*,y^*)$, then (3.3) becomes

$$d_\theta(x^*,y^*) \leq q\phi(d_\theta(x^*,y^*)) + q\varepsilon$$

That is $\beta(d_\theta(x^*,y^*)) \leq q\varepsilon$. Since β is strictly increasing and onto, we have

$$d_\theta(x^*,y^*) \leq \beta^{-1}(q\varepsilon) = \psi(\varepsilon),$$

and the proof is complete. □

Theorem 6. *Let (X,d_θ) be a complete extended b-metric space and $T : X \to X$ a mapping. Suppose that all the hypotheses of Theorem 2 (or Theorem 3) hold. Additionally we suppose that*

(i) for any solution $y^* \in X$ of (3.2), there exists a solution x^* of (3.1) such that $\alpha(x^*,y^*) \geq 1$ and $\theta(x^*,y^*) < q = \max\{q_1,\ldots,q_n\}$;
(ii) $\beta : [0,\infty) \to [0,\infty)$, $\beta(r) := r - q\phi(r)$ is strictly increasing and onto;

Then, the fixed point problem (3.1) is generalized Ulam stable.

Proof. Since the conditions of Theorem 2 hold, the fixed point problem (3.1) has a unique solution $x^* \in X$. Let $y^* \in X$ be a solution of (3.2). We have

$$\begin{aligned} d_\theta(x^*,y^*) &\leq \theta(x^*,y^*)[d_\theta(T(x^*),T(y^*)) + d_\theta(y^*,T(y^*))] \\ &\leq \theta(x^*,y^*)[d_\theta(T(x^*),T(y^*)) + \varepsilon]. \end{aligned} \tag{3.5}$$

Since T is an admissible extended $\mathcal{Z}$-contraction mapping, we have

$$d_\theta(T(x^*),T(y^*)) \leq \phi(M(x^*,y^*)).$$

Attendantly, the inequality (3.5) turns into

$$d_\theta(x^*,y^*) \leq q[\phi(M(x^*,y^*)) + \varepsilon] \tag{3.6}$$

Now, we shall consider

$$M(x^*,y^*) = \max\{d_\theta(x^*,y^*), d_\theta(x^*,T(x^*)), d_\theta(y^*,T(y^*))\}$$
$$M(x_n,y^*) \leq \max\{d_\theta(x^*,y^*), \varepsilon\}.$$

If $\max\{d_\theta(x^*,y^*), \varepsilon\} = \varepsilon$, then (3.6) becomes

$$d_\theta(x^*,y^*) \leq q\phi(\varepsilon) + q\varepsilon = \psi(\varepsilon).$$

If $\max\{d_\theta(x^*,y^*), \varepsilon\} = d_\theta(x^*,y^*)$, then (3.6) turns into

$$d_\theta(x^*,y^*) \leq q\phi(d_\theta(x^*,y^*)) + q\varepsilon$$

That is

$$\beta\left(d_\theta\left(x^*,y^*\right)\right) \leq q\varepsilon.$$

Since β is strictly increasing and onto, we have

$$d_\theta\left(x^*,y^*\right) \leq \beta^{-1}\left(q\varepsilon\right) = \psi\left(\varepsilon\right),$$

and the proof is complete. □

Definition 11. Let (X,d_θ) be an extended b-metric space with constant and $T : X \to X$ be an operator. The fixed point problem (19) is well-posed if

(i) $FixT = \{x^*\}$;

(ii) If $(x_n)_{n\in\mathbb{N}}$ is a sequence such that $d_\theta\left(x_n,T\left(x_n\right)\right) \to 0$, as $n \to \infty$, then $x_n \to x^*$, as $n \to \infty$.

Theorem 7. *Let (X,d_θ) be a complete extended b-metric space and $T : X \to X$ a mapping. Suppose that all the hypotheses of Theorem 2 (or Theorem 3) hold. Additionally we suppose that*

(i) *for any solution $x^* \in X$ of (3.1) , we have that $\alpha(x_n,x^*) \geq 1$ and $\lim\limits_{n\to\infty} \theta\left(x_n,x^*\right) < q = \max\left\{q_1,\ldots,q_n\right\}$, where $x_n = T^n\left(x_0\right)$, $n \in \mathbb{N}$;*

(ii) *$\beta : [0,\infty) \to [0,\infty)$, $\beta(r) := r - q\phi(r)$ is strictly increasing and onto;*

Then, the fixed point problem (3.1) is well-posed.

Proof. Since the conditions of Theorem 2 hold, the fixed point problem (3.1) has a unique solution $x^* \in X$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $d\left(x_n,T\left(x_n\right)\right) \to 0$, as $n \to \infty$.

$$d_\theta\left(x_n,x^*\right) \leq \theta\left(x_n,x^*\right)\left[d_\theta\left(x_n,T\left(x_n\right)\right) + d_\theta\left(T\left(x_n\right),T\left(x^*\right)\right)\right]. \tag{3.7}$$

Since T is an admissible extended $\mathcal{Z}$-contraction mapping, we have

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_n,x^*)d_\theta(T\left(x_n\right),T\left(x^*\right)),\phi(M(x_n,x^*))) \\ &< \phi(M(x_n,x^*)) - \alpha(x_n,x^*)d_\theta(T\left(x_n\right),T\left(x^*\right)) \\ &\leq \phi(M(x_n,x^*)) - d_\theta(T\left(x_n\right),T\left(x^*\right)). \end{aligned}$$

Hence

$$d_\theta(T\left(x_n\right),T\left(x^*\right)) \leq \phi(M(x_n,x^*)).$$

(3.7) yields

$$d_\theta\left(x_n,x^*\right) \leq \theta\left(x_n,x^*\right)\left[d_\theta\left(x_n,T\left(x_n\right)\right) + \phi(M(x_n,x^*))\right] \tag{3.8}$$

Now

$$M(x_n,y^*) = \max\left\{d_\theta\left(x_n,x^*\right),d_\theta\left(x_n,T\left(x_n\right)\right),d_\theta\left(x^*,T\left(x^*\right)\right)\right\} = d_\theta\left(x_n,x^*\right)$$

Hence, (3.8) implies

$$d_\theta\left(x_n,x^*\right) \leq \theta\left(x_n,x^*\right)\left[d_\theta\left(x_n,T\left(x_n\right)\right) + \phi(d_\theta(x_n,x^*))\right] \tag{3.9}$$

Letting $n \to \infty$ in (3.9) we obtain

$$\lim_{n\to\infty} [d_\theta(x_n, x^*) - q\phi(d_\theta(x_n, x^*))] \leq 0$$

which is

$$\lim_{n\to\infty} \beta(d_\theta(x_n, x^*)) = 0.$$

From here

$$\lim_{n\to\infty} d(x_n, x^*) = 0,$$

and the proof is complete. □

In what follows we shall give a data dependence result.

Theorem 8. *Let (X, d_θ) be a complete extended b-metric space and $T_1, T_2 : X \to X$ two mappings such that*

(i) *The conditions of Theorem 2 (or Theorem 3) hold for T_1;*
(ii) *$FixT_2 \neq \varnothing$;*
(iii) *For any $x^* \in FixT_1$ and $y^* \in FixT_2$ we have $\alpha(x^*, y^*) \geq 1$ and $\theta(x^*, y^*) < q = \max\{q_1, ..., q_n\}$;*
(iv) *There exits $\eta > 0$, such that $d_\theta(T_1(x), T_2(y)) \leq \eta$, for all $x, y \in X$;*

Then, we have

$$d_\theta(x^*, y^*) \leq \sup\{t \in \mathbb{R}_+ | t - q\varphi(t) \leq 2q\eta\}.$$

Proof. Since the conditions of Theorem 2 hold for T_1, there exists a unique fixed point x^* of T_1, and because $FixT_2 \neq \varnothing$, there exists y^* a fixed point of T_2. So, we have

$$\begin{aligned} d_\theta(x^*, y^*) &\leq \theta(x^*, y^*)[d_\theta(T_1(x^*), T_1(y^*)) + d_\theta(T_1(y^*), T_2(y^*))] \\ &\leq \theta(x^*, y^*)[d_\theta(T_1(x^*), T_1(y^*)) + \eta]. \end{aligned} \tag{3.10}$$

Since T_1 is an admissible extended $\mathcal{Z}$-contraction mapping, we have

$$d_\theta(T_1(x^*), T_1(y^*)) \leq \phi(M(x^*, y^*)).$$

(3.10) implies that

$$d_\theta(x^*, y^*) \leq q[\phi(M(x^*, y^*)) + \eta] \tag{3.11}$$

Consider now

$$\begin{aligned} M(x^*, y^*) &= \max\{d_\theta(x^*, y^*), d_\theta(x^*, T_1(x^*)), d_\theta(y^*, T_1(y^*))\} \\ M(x_n, y^*) &\leq \max\{d_\theta(x^*, y^*), \eta\}. \end{aligned}$$

If $\max\{d_\theta(x^*, y^*), \eta\} = \eta$, then (3.11) turns into

$$d_\theta(x^*, y^*) \leq q\phi(\eta) + q\eta < 2q\eta. \tag{3.12}$$

If $\max\{d_\theta(x^*, y^*), \eta\} = d_\theta(x^*, y^*)$, then (3.11) becomes

$$d_\theta(x^*, y^*) \leq q\phi(d_\theta(x^*, y^*)) + q\eta < q\phi(d_\theta(x^*, y^*)) + 2q\eta. \tag{3.13}$$

From (3.12) or (3.13) we reach the conclusion. □

4. Discussion and Conclusion

It wouldn't be wrong to say that our results yields several consequences most of which were announced in the literature. More precisely, by substituting the mapping ζ in a proper way like in the Example 7, we find a number of corollaries. On the other hand, by proper choice of the auxiliary function α together with the selection of ϕ in Theorem 1-Theorem 4, we are enable to derive some more existing fixed point theorems in the various settings (in the context of *partially ordered set endowed with a metric*, in the setting of *cyclic contraction* etc.). We omit the details since they are straightforward.

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FIXED POINT AND EXTENDED COUPLED FIXED POINT THEOREMS FOR MULTI-VALUED CONTRACTIONS WITH RESPECT TO THE POMPEIU FUNCTIONAL

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Abstract. In this paper, we will prove some fixed point results using a Pompeiu type metric on $P_{cl}(X)$, for multi-valued operators satisfying two conditions: contractivity and monotonicity. The approach is based on a fixed point theorem for a multi-valued operator in the setting of a b-metric space. Several qualitative properties(well-posedness, Ulam-Hyers stability) are also obtained. In the second part of this paper, we will consider the extended coupled fixed point problem for a multi-valued operator. We will consider the problem of the existence of the solutions. Data dependence and well-posedness of the extended coupled fixed point problem are also discussed.

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1. Introduction

Nadler's contraction principle [14] is an extension to the multi-valued case of the classical Banach's contraction principle. There are many applications of these results, mainly in the theory of operator equations and inclusions, see [2], [4], [6], [9] and [12]. On the other hand, several theorems were given for the so-called extended coupled fixed point problem([3], [17],[18]).

We will define first the notion of b-metric.

Definition 1 (Bakhtin [1], Czerwik [7])**.** Let X be a nonempty set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to \mathbb{R}_+$ is said to be a b-metric with constant s if all the axioms of the metric take place with the following modification of the triangle inequality property:

$$d(x,z) \leq s[d(x,y) + d(y,z)], \ for\ all\ x,y,z \in X.$$

Under the above hypotheses, the pair (X,d) is called a b-metric space with constant s.

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In the framework of a b-metric space, a set $Y \subset X$ is said to be closed if for any sequence $(x_n)_{n\in\mathbb{N}}$ in Y which converges (with respect to d) to an element x, we have $x \in Y$. The notions of bounded or compact sets are defined in a similar way to the case of usual metric spaces. For example, the set $Y \subset X$ is said to be bounded if its diameter $\delta(Y) := \sup\limits_{x,y\in Y} d(x,y)$ is finite, while Y is called compact if every sequence sequence $(x_n)_{n\in\mathbb{N}}$ in Y has a convergent subsequence in Y.

Let (X,d) be a b-metric space and $\mathcal{P}$(X) be the set of all subsets of X. We denote

$$P(X) := \{Y \in \mathcal{P}(X) \mid Y \neq \varnothing\},$$
$$P_{cl}(X) := \{Y \in \mathcal{P}(X) \mid Y \text{ is closed}\},$$
$$P_{cp}(X) := \{Y \in \mathcal{P}(X) \mid Y \text{ is compact}\}.$$

Let (X,d) be a b-metric space. If $T : X \to P(X)$ is a multi-valued operator, then $x \in X$ is called fixed point for T if and only if

$$x \in T(x). \tag{P}$$

We denote by $Fix(T)$ the fixed point set of T and by $SFix(T)$ the set of all strict fixed points of T, i.e., the elements $x \in X$ such that $T(x) = \{x\}$.

If X,Y are two nonempty sets and $T : X \to P(Y)$, then we will denote by

$$Graph(T) := \{(x,y) \in X \times Y : y \in T(x)\}$$

the graph of T.

In the context of a b-metric space (X,d) the following generalized functionals are used in the main sections of the paper.

(1) The gap functional generated by d:

$$D_d : P(X) \times P(X) \to \mathbb{R}_+, D_d(A,B) = inf\{d(a,b) \mid a \in A, b \in B\}.$$

(2) The excess generalized functional:

$$e_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}, e_d(A,B) = sup\{D_d(a,B) | a \in A\}.$$

(3) The Hausdorff-Pompeiu generalized functional :

$$H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}, H_d(A,B) = \max\{e_d(A,B), e_d(B,A)\}.$$

(4) The Pompeiu generalized functional:

$$H_d^+ : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\},\ H_d^+(A,B) := \frac{1}{2}\{e_d(A,B) + e_d(B,A)\}.$$

Definition 2 ([17])**.** Let X be a nonempty set, let " $\leq$ " be a partial order on X and d be a b-metric on X with constant $s \geq 1$. Then the triple $(X, \leq, d)$ is called an ordered b-metric space if:

i. " $\leq$ " is a partial order on X;

ii. d is a b-metric on X with constant $s \geq 1$;

iii. if $(x_n)_{n\in\mathbb{N}}$ is a monotone increasing sequence in X and $x_n \to x^* \in X$, then $x_n \leq x^*$, for all $n \in \mathbb{N}$;

iv. if $(y_n)_{n\in\mathbb{N}}$ is a monotone decreasing sequence in X, and $y_n \to y^* \in X$, then $y_n \geq y^*$, for all $n \in \mathbb{N}$.

Definition 3 ([17])**.** Let $(X,\leq)$ be a partially ordered set. Then, the partial order "$\leq$" induces on the product space $X \times X$ the following partial order:

$$\text{for } (x,y),(u,v) \in X \times X,\ (x,y) \leq_p (u,v) \Leftrightarrow x \leq u, y \geq v.$$

Definition 4 ([18])**.** Let (X,d), (Y,d) be two complete b-metric spaces, $T_1 : X \times Y \to P(X)$ and $T_2 : X \times Y \to P(Y)$ be two multi-valued operators. Then, by definition, a extended coupled fixed point for T_1 and T_2 is a pair $(x^*,y^*) \in X \times Y$ satisfying

$$\begin{cases} x^* \in T_1(x^*,y^*) \\ y^* \in T_2(x^*,y^*). \end{cases} \tag{P1}$$

We will denote by $CFix(T_1,T_2) = \{(x,y) \in X \times Y | x \in T_1(x,y), y \in T_2(x,y)\}$ the solution set for problem (P1).

In this paper, using a Pompeiu type metric on $P_{cl}(X)$, we will prove some fixed point results for multi-valued operators satisfying a combined condition (contractivity and monotonicity). Then, as applications, we will obtain some existence results for a general system of operator inclusions. Several qualitative properties(data dependence, well-posedness) are also obtained.

2. Preliminaries

We will start this section by recalling some useful notion and results.

Lemma 1 ([18])**.** *Let (X,d) be a b-metric space with constant $s \geq 1$, $A,B \in P(X)$ and $q > 1$. Then, for every $a \in A$ there exists $b \in B$ such that $e_d(A,B) \leq qd(a,b)$.*

Definition 5. Let $(X,\preceq)$ be a partially ordered set and $A,B \in P(X)$. We will denote:

a) $A \leq_{st} B \Leftrightarrow \forall a \in A,\ \forall b \in B$ we have $a \preceq b$;

b) $A \leq_{wk} B \Leftrightarrow \forall a \in A,\ \exists b \in B$ such that $a \preceq b$.

Remark 1. Notice that if $A,B,C \in P(X)$, then $A \leq_{st} B$ and $B \leq_{st} C$ implies $A \leq_{st} C$. The same property also holds for $\leq_{wk}$.

Definition 6. Let $(X,\preceq)$ a partially ordered set and $G : X \times X \to P(X)$. We say that G has the strict mixed monotone property with respect to the partial order " $\preceq$ ", if the following conditions hold:

(1) $x_0 \preceq x_1 \Rightarrow G(x_0,y) \leq_{st} G(x_1,y), \forall y \in X$;

(2) $y_0 \succeq y_1 \Rightarrow G(x,y_0) \leq_{st} G(x,y_1), \forall x \in X$.

Remark 2. Let (X,d) be a b-metric space with constant $s \geq 1$ and $Z := X \times X$. Then the functional $\tilde{d} : Z \times Z \to \mathbb{R}_+$ defined by

$$\tilde{d}((x,y),(u,v)) = d(x,u) + d(y,v), \text{ for all } (x,y),(u,v) \in Z$$

is a b-metric on Z with the same constant $s \geq 1$ and if (X,d) is a complete b-metric space, then $(Z,\tilde{d})$ is a complete b-metric space, too.
Moreover, for $x,y \in X$, $A,B,U,V \in P(X)$ we have:

$$D_{\tilde{d}}((x,y),A \times B) = D_d(x,A) + D_d(y,B),$$

$$e_{\tilde{d}}(U \times V, A \times B) = e_d(U,A) + e_d(V,B),$$

$$H_{\tilde{d}}(U \times V, A \times B) \leq H_d(U,A) + H_d(V,B),$$

$$H^+_{\tilde{d}}(U \times V, A \times B) = H^+_d(U,A) + H^+_d(V,B).$$

Additionally, by the properties of the gap functional D_d, if $(x,y) \in X \times X$ and $A,B \in P_{cl}(X)$, then

$$D_{\tilde{d}}((x,y),A \times B) = 0 \text{ if and only if } (x,y) \in A \times B.$$

Definition 7. Let (X,d) be a b-metric space with constant $s \geq 1$. A multi-valued operator $T : X \to P_{b,cl}(X)$ is called H^+-contraction with constant k if:

(1) there exists a fixed real number $0 < k < 1$ such that, for every $x,y \in X$, we have

$$H^+_d(T(x),T(y)) \leq kd(x,y),$$

(2) for every $x \in X$ and $y \in T(x)$ we have

$$D_d(y,T(y)) \leq H^+_d(T(x),T(y)).$$

In particular, a multivalued mapping $T : X \to P_{cl}(X)$ is called (H^+,α)-Lipschitz if $\alpha > 0$ and

$$H^+_d(T(x),T(y)) \leq \alpha d(x,y), \quad \forall x,y \in X.$$

Definition 8. Let $(X,\preceq)$ be a partially ordered set and $T : X \to P(X)$ be a multi-valued operator. We say that T is strong increasing (respectively strong decreasing) on X if, for every $x,y \in X$ with $x \preceq y$ we have that $T(x) \leq_{st} T(y)$ (respectively $T(x) \geq_{st} T(y)$).

Lemma 2 (R. Miculescu, A. Mihail [13])**.** *Every sequence $(x_n)_{n\in\mathbb{N}}$ of elements from b-metric space (X,d) of constant s, having the property that there exists $\gamma \in [0,1)$ such that*

$$d(x_{n+1},x_n) \leq \gamma d(x_n,x_{n-1}), \forall n \in \mathbb{N},$$

is Cauchy. Moreover

$$d(x_{n+1},x_{n+k}) \leq \gamma^n \frac{S}{1-\gamma} d(x_0,x_1),\ \forall n,k \in \mathbb{N}^*,\ \textit{where } S = \sum_{n\geq 1} \gamma^{2n\log_\gamma s + 2^{n-1}}.$$

Lemma 3. *Let (X,d) be a b-metric space with constant $s \geq 1$ and $T : X \to P_{b,cl}(X)$ be a $(H_d^+,k)-$Lipschitz. Then:*

(1) *T has closed graph in $X \times X$;*
(2) *T is H_d^+-l.s.c on X;*
(3) *T is H_d^+-u.s.c on X;*
(4) *If, additionally T has compact values, then T is l.s.c..*

Proof. (1) Let $(x_n,y_n) \subset X \times X$ such that $(x_n,y_n) \xrightarrow{d} (x,y)$, when $n \to \infty$ and $y_n \in T(x_n)$, for all $n \in \mathbb{N}$.

It follows that:

$$\begin{aligned} D(y,T(x)) &\leq s[d(y,y_n) + D(y_n,T(x))] \leq \\ &\leq s[d(y,y_n) + H_d(T(x_n),T(x))] \leq \\ &\leq s[d(y,y_n) + 2H_d^+(T(x_n),T(x))] \leq \\ &\leq s[d(y,y_n) + 2\alpha d(x_n,x)],\ \text{for all } n \in \mathbb{N}. \end{aligned}$$

Let us consider $n \to \infty$ and we obtain $D(y,T(x)) \leq 0$. Thus $y \in T(x)$ which means that T has a closed graph.

(2) Let $x \in X$ such that $x_n \to x$. We have:

$$\begin{aligned} e_d(T(x),T(x_n)) &\leq H_d(T(x),T(x_n)) \leq 2H_d^+(T(x),T(x_n)) \leq \\ &\leq 2\alpha d(x,x_n) \to 0. \end{aligned}$$

Then, T is H_d^+-l.s.c. on X.

(3) Using the relation:

$$\begin{aligned} e_d(T(x_n),T(x)) &\leq H(T(x_n),T(x)) \leq 2H_d^+(T(x_n),T(x)) \leq \\ &\leq 2\alpha d(x,x_n) \to 0, \end{aligned}$$

we obtain that T is H_d^+-u.s.c. on X.

(4) The conclusion follows by the fact that any H_d^+-l.s.c multivalued operator with compact values is l.s.c..

□

Lemma 4 (Cauchy's Lemma)**.** *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of non-negative real numbers, such that $\sum_{p=0}^{\infty} a_p < +\infty$ and $\lim_{n\to\infty} b_n = 0$. Then, $\lim_{n\to\infty}\left(\sum_{p=0}^{n} a_{n-p}b_p\right) = 0$.*

3. Fixed point theorems for multi-valued contractions with respect to a Pompeiu type metric

In this paper, we will present some fixed point theorems for a multi-valued operators in b-metric spaces. See also [5], [11], [10] and [15] for related results.

Theorem 1. *Let $(X,\preceq,d)$ be an ordered b-metric space with constant $s\geq 1$ such that, the b-metric d is complete. Let $T:X\to P_{cl}(X)$ be a multi-valued strong increasing operator with closed graph. Assume that:*

(1) *there exists $k<1$ such that*
$$H_d^+(T(x),T(y))\leq kd(x,y),\forall x,y\in X,x\preceq y, \tag{3.1}$$
(2) *for every $x\in X$ and $y\in T(x)$ we have,*
$$D_d(y,T(y))\leq H_d^+(T(x),T(y)), \tag{3.2}$$
(3) *there exists an element $x_0\in X$ such that*
$$x_0\leq_{wk} T(x_0). \tag{3.3}$$

Then,
a) $Fix(T)\neq\varnothing$.
b) there exists a sequence of successive approximation for T starting from x_0 such that
$$d(x_{n+1},x^*)\leq\frac{\beta^n Ss}{1-\beta}d(x_0,x_1),\ \forall n\in\mathbb{N},\ where\ \beta>0,\ with\ k<\beta<1.$$

Proof. a) Let $x_0\in X$ such that $x_0\leq_{wk} T(x_0)$. By (3.3) we get that there exists $x_1\in T(x_0)$ such that $x_0\preceq x_1$.

Using (3.2) we obtain that,
$$D_d(x_1,T(x_1))\leq H_d^+(T(x_0),T(x_1))\leq kd(x_0,x_1). \tag{3.4}$$
For $k<\beta<1$ the relation (3.4) becames
$$D_d(x_1,T(x_1))<\beta d(x_0,x_1). \tag{3.5}$$
Thus, there exists $x_2\in T(x_1)$ such that
$$d(x_1,x_2)<\beta d(x_0,x_1).$$
Since $x_0\preceq x_1$ and T is a multi-valued strong increasing operator, we get that
$$T(x_0)\leq_{st} T(x_1).\ Thus\ x_1\preceq x_2.$$
By (3.2) we get that,
$$D_d(x_2,T(x_2))\leq H_d^+(T(x_1),T(x_2))\leq kd(x_1,x_2)<\beta d(x_1,x_2)<\beta^2 d(x_0,x_1). \tag{3.6}$$
By induction, we obtain a sequence $(x_n)_{n\in\mathbb{N}}\in X$ with the following properties:

(1) $x_{n+1}\in T(x_n),\forall n\in\mathbb{N}$,
(2) $d(x_n,x_{n+1})<\beta d(x_{n-1},x_n)$,

(3) $d(x_n, x_{n+1}) < \beta^n d(x_0, x_1), \forall n \in \mathbb{N}$.

By Lemma 2 we get that $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X,d). Then, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Since T has closed graph in (X,d) we obtain that $x^* \in$ Fix(T).

b) By Lemma 2 we have

$$d(x_{n+1}, x_{n+k}) \leq \beta^n \frac{S}{1-\beta} d(x_0, x_1),\ \forall n,k \in \mathbb{N}^*,\ \textit{where } S = \sum_{n\geq 1} \gamma^{2n\log_\gamma s + 2^{n-1}}.$$

By (1) we have that $x_n \to x^*$, $n \to \infty$. Then,

$$\frac{1}{s} d(x_{n+1}, x^*) \leq d(x_{n+1}, x_{n+k}) + d(x_{n+k}, x^*) \leq \frac{\beta^n S}{1-\beta} d(x_0, x_1) + d(x_{n+k}, x^*).$$

Letting $k \to \infty$, we get $\frac{1}{s} d(x_{n+1}, x^*) \leq \beta^n \frac{S}{1-\beta} d(x_0, x_1)$. Hence $d(x_{n+1}, x^*) \leq \beta^n \frac{Ss}{1-\beta} d(x_0, x_1)$.

□

The next result is a corollary of the previous theorem for the global case, where it isn't necessary to suppose that the operator T has to be with closed graph.

Corollary 1. *Let (X,d) be a complete b-metric space with constant $s \geq 1$. Let $T : X \to P_{cl}(X)$ be a multi-valued operator. Assume that:*

(1) *there exists $k < 1$ such that,*

$$H_d^+(T(x), T(y)) \leq k d(x,y), \forall x,y \in X, \tag{3.7}$$

(2) *for every $x \in X$ and $y \in T(x)$ we have,*

$$D_d(y, T(y)) \leq H_d^+(T(x), T(y)), \tag{3.8}$$

Then $Fix(T) \neq \varnothing$.

The following result is about the uniqueness of the solution for the fixed point problem (P).

Theorem 2. *Suppose that all the hypotheses of Theorem 1 are satisfied. Additionally, we suppose there exists $x^* \in X$ such that $T(x^*) = \{x^*\}$ and for every $\bar{x} \in Fix(T)$ we have $\bar{x} \preceq x^*$ or $x^* \preceq \bar{x}$. Then $Fix(T) = SFix(T) = \{x^*\}$.*

Proof. Let $x^* \in X$ such that $T(x^*) = \{x^*\}$. Then, for every $\bar{x} \in Fix(T)$ we have $\bar{x} \preceq x^*$ or $x^* \preceq \bar{x}$. Then,

$$d(\bar{x}, x^*) = D_d(\bar{x}, T(x^*)) \leq H_d^+(T(\bar{x}), T(x^*)) \leq k d(\bar{x}, x^*).$$

Thus, $(1-k)d(\bar{x}, x^*) \leq 0$ which means that $d(\bar{x}, x^*) = 0$ and $\bar{x} = x^*$. Thus, $Fix(T) = SFix(T) = \{x^*\}$. □

We have the following global version of the Theorem 2.

Corollary 2. *Let $(X,\preceq,d)$ be a complete b-metric space with constant $s\geq 1$. Let $T:X\to P_{cl}(X)$ be a multi-valued operator. Assume that:*

(1) *there exists $k<1$ such that,*
$$H_d^+(T(x),T(y))\leq kd(x,y),\forall x,y\in X, \tag{3.9}$$
(2) *for every $x\in X$ and $y\in T(x)$ we have,*
$$D_d(y,T(y))\leq H_d^+(T(x),T(y)), \tag{3.10}$$
(3) *there exists $x^*\in X$ such that $T(x^*)=\{x^*\}$.*

Then $Fix(T)=SFix(T)=\{x^*\}$.

In the next part of this section, we will present some properties of that fixed point problem (P). The first one is the data dependence problem for the fixed point problem (P).

Theorem 3. *Let (X,d) be a complete b-metric space with constant $s\geq 1$ and let $T:X\to P_{cl}(X)$, $S:X\to P(X)$ be two multi-valued operators. We suppose that:*

(1) *there exists $k<\dfrac{1}{2s}$ such that*
$$H_d^+(T(x),T(y))\leq kd(x,y),\ \forall x,y\in X,$$
(2) *there exists $x^*\in X$ such that $T(x^*)=\{x^*\}$,*
(3) *there exists $\eta>0$ such that*
$$e_d(T(x),S(x))\leq\eta,\ \forall x\in X.$$

Then, $e_d(Fix(T),Fix(S))\leq\dfrac{s\eta}{1-2sk}$.

Proof. By (1) and (2) we get that $Fix(T)=SFix(T)=\{x^*\}$. Let $u^*\in X$ such that $u^*\in S(u^*)$ and $x^*\in X$ such that $T(x^*)=\{x^*\}$. We have,
$$\begin{aligned}d(u^*,x^*)=D_d(u^*,T(x^*))\leq\\ &\leq s[D_d(u^*,T(u^*))+e_d(T(u^*),T(x^*))]\\ &\leq s[e_d(S(u^*),T(u^*))+H_d(T(u^*),T(x^*))]\leq\\ &\leq s[\eta+2H_d^+(T(u^*),T(x^*))]\leq\\ &\leq s[\eta+2kd(u^*,x^*)].\end{aligned}$$

Then, $d(u^*,x^*)\leq\dfrac{s\eta}{1-2sk}$.

Since $u^*\in Fix(S)$ are arbitrary chosen, we obtain that, $e_d(Fix(T),Fix(S))\leq\dfrac{s\eta}{1-2sk}$. □

The next problem we intend to study is the well-posedness of the fixed point problem (P).

Definition 9. We consider the fixed point problem (P). By definition, (P) is well-posed for T with respect to D_d if:

(1) $Fix(T) = \{x^*\}$, $x^* \in X$.
(2) if there exists a sequence $u_n \in X$ with $D_d(u_n, T(u_n)) \to 0$, then $d(u_n, x^*) \to 0$ as $n \to \infty$.

Theorem 4. *We suppose that all the hypotheses of Corollary 2 take place. Then the fixed point problem (P) is well-posed for T with respect to D_d.*

Proof. Since the multi-valued operator T satisfies the hypotheses of Corollary 1, we get that T has a unique fixed point, i.e., $Fix(T) = \{x^*\}$ where $x^* \in X$. Then we have, $T(x^*) = \{x^*\}$.

Let $u_n \subset X$, with $D_d(u_n, T(u_n)) \to 0$ as $n \to \infty$.

Then we have,

$$\begin{aligned} d(u_n, x^*) = D_d(u_n, T(x^*)) \leq \\ &\leq s[D_d(u_n, T(u_n)) + e_d(T(u_n), T(x^*))] \leq \\ &\leq s[D_d(u_n, T(x_n)) + H_d(T(u_n), T(x^*))] \leq \\ &\leq sD_d(u_n, T(u_n)) + 2sH_d^+(T(u_n), T(x^*)) \leq \\ &\leq sD_d(u_n, T(x_n)) + 2skd(u_n, x^*). \end{aligned}$$

Hence,

$$d(u_n, x^*) \leq \frac{s}{1-2sk} D_d(u_n, T(u_n)) \to 0, \ as\ n \to \infty.$$

□

Now we will present the Ulam-Hyers property of the fixed point problem (P).

Definition 10. Let (X,d) be a b-metric space with constant $s \geq 1$ and let $T : X \to P(X)$ be a multi-valued operator. Let us to consider the inclusion $x \in T(x)$ and the inequality

$$D_d(x, T(x)) \leq \varepsilon \ where\ \varepsilon > 0\ and\ x \in X. \tag{3.11}$$

By definition, the inclusion (P) is called generalized Ulam-Hyers stable if and only if, there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 with $\psi(0) = 0$, such that for each $\varepsilon > 0$ and for each solution $u^* \in X$ of the inequality (3.11), there exists a solution $x^* \in X$ of (P) such that,

$$d(u^*, x^*) \leq \psi(\varepsilon).$$

Theorem 5. *Let $T : X \to P_{cl}(X)$ be a multi-valued operator which verifies (1) and (3) from Corollary 2 with $k < \frac{1}{2s}$. Then the fixed point inclusion (P) is Ulam-Hyers stable.*

Proof. From Corollary 2 we have that T has a unique fixed point $x^* \in X$ such that $T(x^*) = \{x^*\}$.

Let $\varepsilon > 0$ and $u^* \in X$ be a ε-solution of the fixed point problem (P), i.e., a solution of the inequality 3.11.

Then, we have $D_d(u^*, T(u^*)) \leq \varepsilon$. Hence, we get

$$\begin{aligned} d(u^*, x^*) = D_d(u^*, T(x^*)) \leq \\ \leq s[D_d(u^*, T(u^*)) + e_d(T(u^*), T(x^*))] \leq \\ \leq s[D_d(u^*, T(u^*)) + H_d(T(u^*), T(x^*))] \leq \\ \leq sD_d(u^*, T(u^*)) + 2sH_d^+(T(u^*), T(x^*)) \leq \\ \leq sD_d(u^*, T(u^*)) + 2skd(u^*, x^*)). \end{aligned}$$

Thus,

$$d(u^*, x^*) \leq \frac{s}{1-2sk}\varepsilon.$$

Therefore, if we consider $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, given by $\psi(t) = ct$(with $c = \frac{s}{1-2sk} > 0$), we can conclude that the inclusion (P) is Ulam-Hyers stable. □

Using the global existence result for the fixed point problem(see Corollary 2), we can prove the limit shadowing property of the fixed point problem (P).

Definition 11. Let (X, d) be a metric space with constant $s \geq 1$ and $T : X \to P_{cl}(X)$ be a multivalued operator. By definition, the fixed problem (P) has the limit shadowing property if, for any sequence $(x_n)_{n\in\mathbb{N}}$ in X for which,

$$D_d(x_n, T(x_n)) \to 0 \text{ as } n \to \infty,$$

there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in X such that,

$$d(x_n, u_n) \to 0 \text{ as } n \to \infty.$$

Theorem 6. *Let (X, d) be a metric space with constant $s \geq 1$ and $T : X \to P_{cl}(X)$ be a multivalued operator which verifies (1) and (3) from Corollary 2. Then, the fixed point problem (P) has the limit shadowing property.*

Proof. By Corollary 2 we get that $Fix(T) = \{x^*\}$ and for any initial point $u_0 \in X$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in X with $u_{n+1} \in T(u_n)$ for all $n \in \mathbb{N}$, such that $(u_n)_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequences in X such that,

$$D_d(x_{n+1},T(x_n)) \to 0 \ as \ n \to \infty.$$

Thus, $d(x_{n+1},u_{n+1}) \leq s[d(x_{n+1},x^*)+d(x^*,u_{n+1})]$.
Next, we can write,

$$\begin{aligned} d(x_{n+1},x^*) = D_d(x_{n+1},T(x^*)) \leq \\ &\leq s[D_d(x_{n+1},T(x_n))+e_d(T(x_n),T(x^*)] \leq \\ &\leq s[D_d(x_{n+1},T(x_n))+H_d(T(x_n),T(x^*)] \leq \\ &\leq s[D_d(x_{n+1},T(x_n))+2H_d^+(T(x_n),T(x^*)] \leq \\ &\leq sD_d(x_{n+1},T(x_n))+2ksd(x_n,x^*). \end{aligned}$$

We denote $2ks = \alpha < 1$, thus

$$\begin{aligned} d(x_{n+1},x^*) &\leq sD_d(x_{n+1},T(x_n))+\alpha d(x_n,x^*) \leq \\ &\leq sD_d(x_{n+1},T(x_n))+\alpha[sD_d(x_n,T(x_{n-1}))+\alpha d(x_{n-1},x^*)] \\ &= s[D_d(x_{n+1},T(x_n))+D_d(x_n,T(x_{n-1}))]+\alpha^2 d(x_{n-1},x^*) \\ &\leq \cdots \leq \\ &\leq s[D_d(x_{n+1},T(x_n))+\ldots+\alpha^n D_d(x_1,T(x_0))]+\alpha^n d(x_0,x^*). \end{aligned}$$

By Cauchy's Lemma 4 we obtain that,

$$D_d(x_{n+1},T(x_n))+\ldots+\alpha^n D_d(x_1,T(x_0)) \to 0 \ as \ n \to \infty.$$

Thus, we obtain that $d(x_n,x^*) \to 0 \ as \ n \to \infty$.
Then, $d(x_{n+1},u_{n+1}) \to 0 \ as \ n \to \infty$.

□

4. Extended coupled fixed point theorems for multi-valued contractions with respect to a Pompeiu type metric

The purpose of this section is to give some extended coupled fixed point theorems for multi-value operators satisfying a contraction type condition with respect to the excess functional. We will consider here the context of a b-metric space. Our results are new even for the case of metric spaces and they extend some theorems given in [8], [16], [5] and [19] and other papers in the literature.

Theorem 7. *Let $(X,\preceq,d)$, $(Y,\preceq,d)$ be two ordered complete b-metric spaces with constant $s \geq 1$. Let $T_1 : X\times Y \to P_{cl}(X)$ and $T_2 : X\times Y \to P_{cl}(Y)$ be two multi-valued operators with closed graph and having the strict mixed monotone property with respect to " $\preceq$ ". Assume that:*

(1) *there exist $k_1,k_2 > 0$ with $k_1+k_2 \in (0,1)$ such that:*

$$e_d(T_1(x,y),T_1(u,v))+e_d(T_1(u,v),T_1(x,y)) \leq k_1[d(x,u)+\rho(y,v)],$$

$\forall x \preceq u$ *and* $y \succeq v$,

$$e_\rho(T_2(x,y),T_2(u,v)) + e_\rho(T_2(u,v),T_2(x,y)) \leq k_2[d(x,u)+\rho(y,v)],$$
$\forall x \preceq u$ *and* $y \succeq v$,

(2) *there exists* $(x_0,y_0) \in X \times Y$ *such that*

$$x_0 \preceq_{wk} T_1(x_0,y_0),\ y_0 \succeq_{wk} T_2(x_0,y_0)$$

or

$$x_0 \succeq_{wk} T_1(x_0,y_0),\ y_0 \preceq_{wk} T_2(x_0,y_0).$$

Then, the following conclusions hold:

(i) there exist $(x^*,y^*) \in X \times Y$ *and the sequences* $(x_n)_{n\in\mathbf{N}}$ *in* X *and* $(y_n)_{n\in\mathbf{N}}$ *in* Y *defined by*

$$\begin{cases} x_{n+1} \in T_1(x_n,y_n) \\ y_{n+1} \in T_2(x_n,y_n) \end{cases} \quad for\ all\ n \in \mathbf{N}, \tag{4.1}$$

such that $x_n \to x^*$ *and* $y_n \to y^*$ *as* $n \to \infty$ *and*

$$\begin{cases} x^* \in T_1(x^*,y^*) \\ y^* \in T_2(x^*,y^*) \end{cases} . \tag{4.2}$$

(ii) If we denote $\tilde{d}(z_n,z^*) = d(x_n,x^*)+\rho(y_n,y^*)$ *and* $\tilde{d}(z_0,z_1) = d(x_0,x_1)+\rho(y_0,y_1)$ *then we have the following estimation holds,*

$\tilde{d}(z_{n+1},z^*) \leq \dfrac{\beta^n Ss}{1-\beta}\tilde{d}(z_0,z_1)$, *where* $\beta > 0$ *is any number with* $\beta < \dfrac{k_1+k_2}{2}$.

Proof. Let $Z = X \times Y$, $T : Z \to P_{cl}(Z)$, $T(x,y) = T_1(x,y) \times T_2(x,y)$ and consider the functional $\tilde{d} : Z \times Z \to \mathbf{R}_+$, $\tilde{d}((x,y),(u,v)) = d(x,u)+\rho(y,v)$.

Then, for $z_0 = (x_0,y_0) \in Z$, there exist $x_1 \in T_1(x_0,y_0)$ and $y_1 \in T_2(x_0,y_0)$ such that

$$x_0 \preceq x_1 \text{ and } y_0 \succeq y_1 \Rightarrow (x_0,y_0) \preceq_p (x_1,y_1) \Rightarrow z_0 \preceq_p z_1.$$

Thus, $z_0 \preceq_p T(z_0)$.

On the other hand, for $z = (x,y) \in Z$ and $w = (u,v) \in T(Z)$ with $z \preceq_p w$ we get

$$\begin{aligned} D_{\tilde{d}}(w,T(w)) &= D_{\tilde{d}}((u,v),T_1(u,v)\times T_2(u,v)) = \\ &= D_d(u,T_1(u,v)) + D_\rho(v,T_2(u,v)) \leq \\ &\leq e_d(T_1(x,y),T_1(u,v)) + e_\rho(T_2(x,y),T_2(u,v)) = \\ &= 2H_d^+(T_1(x,y)\times T_2(x,y),T_1(u,v)\times T_2(u,v)) = 2H_{\tilde{d}}^+(T(z),T(w)). \end{aligned}$$

Hence, we conclude that

$$D_{\tilde{d}}(w,T(w)) \leq 2H_{\tilde{d}}^+(T(z),T(w)),$$

for all $z \in Z$ and $w \in T(z)$ with $z \leq_p w$.

Now, we proof that:

$$H_{\tilde{d}}^{+}(T(z),T(w)) \leq k\tilde{d}(z,w)\ \forall z,w \in Z\ z \leq_p w.$$

$$\begin{aligned}
H_{\tilde{d}}^{+}(T(z),T(w)) &= H_{\tilde{d}}^{+}(T_1(z)\times T_2(z),T_1(w)\times T_2(w)) = \\
&= H_{d}^{+}(T_1(z),T_1(w)) + H_{d}^{+}(T_2(z),T_2(w)) = \\
&= \frac{1}{2}(e_d(T_1(x,y),T_1(u,v)) + e_d(T_1(u,v),T_1(x,y)))+ \\
&\quad + \frac{1}{2}(e_\rho(T_2(x,y),T_2(u,v)) + e_\rho(T_2(u,v),T_2(x,y))) \leq \\
&\leq \frac{1}{2}k_1[d(x,u)+\rho(y,v)] + \frac{1}{2}k_2[d(x,u)+\rho(y,v)] = \\
&= \frac{1}{2}(k_1+k_2)[d(x,u)+\rho(y,v)] = \frac{1}{2}(k_1+k_2)\tilde{d}(z,w).
\end{aligned}$$

Hence, there exists $k := \frac{1}{2}(k_1+k_2) \in (0,1)$ such that

$$H_{\tilde{d}}^{+}(T(z),T(w)) \leq k\tilde{d}(z,w)\ \forall z,w \in Z\ z \leq_p w.$$

We will show now that T is an increasing operator, i.e, if $z \leq_p w$ we have $T(z) \leq_s tT(w)$. This is equivalent to the following relations

$$T_1(x,y)\times T_2(x,y) \leq_{st} T_1(u,v)\times T_2(u,v)$$

or

$$T_1(x,y) \leq_{st} T_1(u,v)\ and\ T_2(x,y) \leq_{st} T_2(u,v).$$

We have

$$x \preceq u \Rightarrow \begin{cases} T_1(x,y) \leq_{st} T_1(u,y), \forall y \in Y \\ T_2(x,y) \geq_{st} T_2(u,y), \forall y \in Y \end{cases}, \tag{4.3}$$

$$y \succeq v \Rightarrow \begin{cases} T_1(x,y) \leq_{st} T_1(x,v), \forall x \in X \\ T_2(x,y) \geq_{st} T_2(x,v), \forall x \in X \end{cases}, \tag{4.4}$$

If in the relation (4.4) we put $x := u$, then we have $T_1(u,y) \leq_{st} T_1(u,v)$. Using the relation (4.3) we get $T_1(x,y) \leq_{st} T_1(u,v)$.

On the other hand, for $x := u$ in the relation (4.4) we get $T_2(u,y) \geq_{st} T_2(u,v)$. Using the relation (4.3) we obtain $T_2(x,y) \geq_{st} T_2(u,v)$. Thus $T : Z \to Z$, $T(x,y) = T_1(x,y)\times T_2(x,y)$ is an increasing operator.

Thus, the operator $T : Z \to Z$ has the following properties:

(1) T is increasing on Z.
(2) There exists $z_0 = (x_0,y_0) \in Z$ such that $z_0 \leq_p T(z_0)$.

(3) There exists $k \in (0, \frac{1}{s})$ such that

$$H_{\tilde{d}}^{+}(T(z), T(w)) \leq k\tilde{d}(z,w) \ \forall z, w \in Z \ z \leq_p w.$$

(4) For every $z = (x,y) \in Z$ and $w = (u,v) \in T(z)$ with $z \leq_p w$, we have

$$D_{\tilde{d}}(w, T(w)) \leq 2H_{\tilde{d}}^{+}(T(z), T(w)).$$

Thus, by Theorem 1 we get that $Fix(T) \neq \varnothing$ and there exists a sequence $(z_n)_{n\in\mathbb{N}} \subset Z$, $z_n = (x_n, y_n) \in X \times Y$ of successive approximation of T starting from $z_0 \in Z$ which converges to a fixed point $z^* = (x^*, y^*)$ of T.

Thus,

$$z^* \in T(z^*) \Rightarrow (x^*, y^*) \in T(x^*, y^*) \Rightarrow (x^*, y^*) \in T_1(x^*, y^*) \times T_2(x^*, y^*)$$

which leads to

$$\begin{cases} x^* \in T_1(x^*, y^*) \\ y^* \in T_2(x^*, y^*). \end{cases} \tag{4.5}$$

By Theorem 1 (ii) we get that

$$\tilde{d}(z_{n+1}, z^*) \leq \frac{\beta^n Ss}{1-\beta}\tilde{d}(z_0, z_1), \ \textit{where } \beta < \frac{k_1 + k_2}{2}.$$

□

Corollary 3. *Let $(X, \preceq, d)$, $(Y, \preceq, d)$ be two orderd complete b-metric spaces with constant $s \geq 1$. Let $T_1 : X \times Y \to P_{cl}(X)$ and $T_2 : X \times Y \to P_{cl}(Y)$ be two multi-valued operators having the strict mixed monotone property with respect to " $\preceq$ ". Assume that, there exist $k_1, k_2 > 0$ with $k_1 + k_2 \in (0,1)$ such that:*

$$e_d(T_1(x,y), T_1(u,v)) + e_d(T_1(u,v), T_1(x,y)) \leq k_1[d(x,u) + \rho(y,v)],$$
$$\forall (x,y) \in (X \times Y),$$

$$e_\rho(T_2(x,y), T_2(u,v)) + e_\rho(T_2(u,v), T_2(x,y)) \leq k_2[d(x,u) + \rho(y,v)],$$
$$\forall (x,y) \in (X \times Y).$$

Then, the following conclusions hold:

a) there exists $(x^*, y^*) \in X \times Y$ such that the sequence $(x_n)_{n\in\mathbf{N}}$ in X and $(y_n)_{n\in\mathbf{N}}$ in Y with,

$$\begin{cases} x_{n+1} \in T_1(x_n, y_n) \\ y_{n+1} \in T_2(x_n, y_n) \end{cases} \quad for\ all\ n \in \mathbf{N} \tag{4.6}$$

such that $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$ and

$$\begin{cases} x^* \in T_1(x^*, y^*) \\ y^* \in T_2(x^*, y^*) \end{cases} . \tag{4.7}$$

b) If we denote $\tilde{d}(z_n,z^*) = d(x_n,x^*)+\rho(y_n,y^*)$ *and* $\tilde{d}(z_0,z_1) = d(x_0,x_1)+\rho(y_0,y_1)$, *then we have the following estimation holds,*

$\tilde{d}(z_{n+1},z^*) \leq \dfrac{\beta^n Ss}{1-\beta}\tilde{d}(z_0,z_1)$, *where* $\beta > 0$ *is any number with* $\beta < \dfrac{k_1+k_2}{2}$.

The following result is about the uniqueness of the extended coupled fixed point.

Theorem 8. *In addition to the hypotheses of the Corollary 3 we suppose that:*

(1) *there exists* $(x^*,y^*) \in X \times Y$ *such that*

$$\begin{cases} T_1(x^*,y^*) = \{x^*\} \\ T_2(x^*,y^*) = \{y^*\} \end{cases}. \tag{4.8}$$

(2) *for every* $(\bar{x},\bar{y}) \in CFix(T_1,T_2)$ *we have* $\bar{x} \leq x^*$ *and* $\bar{y} \geq y^*$ *or* $x^* \leq \bar{x}$ *and* $y^* \geq \bar{y}$.

Then, we obtain that the extended coupled fixed point problem (P1) has an unique solution.

Proof. Let $Z := X \times Y$, $T : Z \to P_{cl}(Z)$, $T(x,y) = T_1(x,y) \times T_2(x,y)$ and consider the functional $\tilde{d} : Z \times Z \to \mathbb{R}_+$, $\tilde{d}((x,y),(u,v)) = d(x,u)+\rho(y,v)$.

Let $z^* \in Z$, $z^* = (x^*,y^*)$ such that $T(z^*) = \{z^*\}$ and $\bar{z} = (\bar{x},\bar{y}) \in Z$ such that $\bar{z} \in Fix(T)$ with $\bar{z} \leq z^*$ or $z^* \leq \bar{z}$.

By Theorem 2 we get that $Fix(T) = SFix(T) = \{z^*\}$. Hence $CFix(T_1,T_2) = \{z^*\}$. □

In the next part of this section, we will present some properties of the extended coupled fixed point problem (P1).

The first result is a data dependence problem for the extended coupled fixed point problem (P1).

Theorem 9. *Let (X,d), (Y,d) be two coupled b-metric spaces with constant* $s \geq 1$. *Let* $T_1 : X \times Y \to P_{cl}(X)$, $T_2 : X \times Y \to P_{cl}(Y)$, $S_1 : X \times Y \to P(X)$ *and* $S_2 : X \times Y \to P(Y)$ *be four multi-valued operators. We suppose that:*

(1) *there exist* $k_1,k_2 > 0$ *with* $k_1+k_2 \in (0,1)$ *such that:*

$$e_d(T_1(x,y),T_1(u,v)) + e_d(T_1(u,v),T_1(x,y)) \leq k_1[d(x,u)+\rho(y,v)],\ \forall (x,y) \in X \times Y,$$

$$e_\rho(T_2(x,y),T_2(u,v)) + e_\rho(T_2(u,v),T_2(x,y)) \leq k_2[d(x,u)+\rho(y,v)],\ \forall (x,y) \in X \times Y,$$

(2) *there exists* $(x^*,y^*) \in X \times Y$ *such that*

$$\begin{cases} T_1(x^*,y^*) = \{x^*\} \\ T_2(x^*,y^*) = \{y^*\} \end{cases}.$$

(3) *there exists* $(u^*, v^*) \in X \times Y$ *such that*

$$\begin{cases} u^* \in S_1(u^*, v^*) \\ v^* \in S_2(u^*, v^*) \end{cases}.$$

(4) *there exists* $\eta > 0$ *such that*

$$e_{\tilde{d}}(T_1(x,y) \times T_2(x,y), S_1(x,y) \times S_2(x,y)) \leq \eta, \ \forall (x,y) \in X \times Y.$$

Then, $e_{\tilde{d}}(CFix(S_1,S_2), CFix(T_1,T_2)) \leq \dfrac{s\eta}{1-2sk}$.

Proof. Let $Z := X \times Y$, $T : Z \to P_{cl}(Z)$, $T(x,y) = T_1(x,y) \times T_2(x,y)$ and consider the functional $\tilde{d} : Z \times Z \to \mathbb{R}_+$, $\tilde{d}((x,y),(u,v)) = d(x,u) + \rho(y,v)$.

Let $z^* \in Z$, $z^* = (x^*, y^*)$ such that $T(z^*) = \{z^*\}$.

We consider $S : Z \to P(Z)$, $S(x,y) = S_1(x,y) \times S_2(x,y)$. Let $w^* \in Z$, $w^* = (u^*, v^*)$ such that $w^* \in S(w^*)$.

By Theorem 3 for the multi-valued operators T and S we obtain that

$$d(w^*, z^*) \leq \frac{s\eta}{1-2sk} \text{ where } k := \frac{k_1 + k_2}{2}.$$

Since $w^* \in CFix(S)$ are arbitrary chosen, we obtain that

$$e_{\tilde{d}}(CFix(S_1,S_2), CFix(T_1,T_2)) \leq \frac{s\eta}{1-2sk}.$$

□

The second problem we intend to study is the well-posedness of the extended coupled fixed point problem.

Definition 12. We consider the extended coupled fixed point problem (P1). By definition, (P1) is well-posed for T_1 and T_2 with respect to D_d if:

(1) $CFix(T_1,T_2) = \{(u^*, v^*)\}$, $(u^*, v^*) \in X \times Y$,

(2) if there exists a sequences $(u_n, v_n) \in X \times Y$ with

$$\begin{cases} D_d(u_n, T_1(u_n, v_n)) \to 0 \\ D_d(v_n, T_2(u_n, v_n)) \to 0 \end{cases}.$$

then $d(u_n, u^*) + d(v_n, v^*) \to 0$ as $n \to \infty$.

Theorem 10. *We suppose that all the hypotheses of Theorem 9 take place. Then the extended coupled fixed problem (P1) is well-posed for* T_1 *and* T_2 *with respect to* D_d.

Proof. Let $Z := X \times Y$, $T : Z \to P_{cl}(Z)$, $T(x,y) = T_1(x,y) \times T_2(x,y)$ and consider the functional $\tilde{d} : Z \times Z \to \mathbb{R}_+$, $\tilde{d}((x,y),(u,v)) = d(x,u) + \rho(y,v)$.

Since the multi-valued operator T satisfies the hypotheses of Corollary 9, we get that T has a unique extended coupled fixed point. Let $z^* \in Z$, $z^* = (x^*, y^*)$ such that $T(z^*) = \{z^*\}$.

Let $w_n = (u_n, v_n) \in Z$, with

$$\begin{cases} D_d(u_n, T_1(u_n, v_n)) \to 0 \\ D_d(v_n, T_2(u_n, v_n)) \to 0 \end{cases}.$$

By Theorem 4 for the multi-valued operator T we obtained that $d(w_n, z^*) \to 0$, as $n \to \infty$.

□

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SOLUTION OF COMPLEX DIFFERENTIAL EQUATIONS BY USING REDUCED DIFFERENTIAL TRANSFORM METHOD

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Abstract. In this paper, complex differential equations from first and second order with constant coefficients are solved using reduced differential transform method (RDTM). Such equations were previously solved by other methods. While obtaining a solution with these methods, the solution was reached by dividing the equation into real and imaginary parts. With the RDTM used in this study, we found that the solution can be reached without dividing the real and imaginary parts of the equation.An iteration relation is given for the solution of such equations . In addition, a variable coefficient equation was solved with RDTM.

2010 *Mathematics Subject Classification:* 34B10; 34B15

Keywords: complex differential equation, reduced differential transform method

1. Introduction

The emergence of complex partial differential equations dates back to the beginning of 1900. One of the important mathematicians who made serious studies and left a name in this field was D. Pompeiu, and he described the Pompeiu integral operator, which is called by his name and which forms the basis of the theory of complex differential equations even today.

Some applications of complex partial differential equations emerged at a mechanical congress in Canada. The difficulties of some problems in real space have been overcome by the solution methods of complex equations. For instance, the $\Delta u = 0$ Laplace equation, an elliptic differential equation (see e.g. [11]), does not have a general solution in real space, but there is a general solution of this equation in complex space. One can find more information about on complex equation in [5, 9, 12].

In this study, solutions of complex differential equations from first and second order with constant coefficients are obtained. Equations which are studied are form that:

$$A\frac{\partial w}{\partial z} + B\frac{\partial w}{\partial \bar{z}} + Cw = F(z,\bar{z}) \tag{1.1}$$

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and

$$A\frac{\partial^2 w}{\partial z^2}+B\frac{\partial^2 w}{\partial z\partial \bar{z}}+C\frac{\partial^2 w}{\partial \bar{z}^2}+D\frac{\partial w}{\partial z}+E\frac{\partial w}{\partial \bar{z}}+Fw=G(z,\bar{z}) \tag{1.2}$$

where coefficients of equations are constant.

Complex differential equations were solved by using integral transforms, Adomian decomposition method [2–4]. While making the solution with these methods, the equation was divided into real and imaginary parts and the process was performed. In this article, the reduced differential transformation method can be used to solve the equation without separating it into real and imaginary parts. Therefore, the solution can be obtained by performing fewer operations. RDTM was proposed firstly by Keskin in [7]. Linear and nonlinear equations and fractional equations are solved with RDTM [1, 6, 8, 10].

This paper has been organized as follows. Basic definitions and theorems associated with RDTM and complex derivative are given in Section 2. In Section 3, for solve of 1.1 and 1.2 equations have been obtained a recursive relation and have been given some examples for validity of the method.

2. Basic Definitions and Theorems

Suppose that the two-variable $u(x,y)$ function can be written as $u(x,y)=f(x)\,g(y)$. With the help of one dimensional differential transformation, the function $u(x,y)$ can be written as follows.

$$u(x,y)=\left(\sum_{i=0}^{\infty}F(i)\,x^i\right)\left(\sum_{j=0}^{\infty}G(j)\,y^j\right)=\left(\sum_{k=0}^{\infty}U_k(x)\,y^k\right) \tag{2.1}$$

where $U_k(x)$ is called y dimensional spectrum function $u(x,y)$, and $F(i)$, $G(j)$ are differential transform of $f(x)$ and $g(y)$, respectively.

Definition 1. If a function $u(x,y)$ is analytic and differentiated continuously with respect to y and space x in the domain of interest then let

$$U_k(x)=\frac{1}{k!}\left(\frac{\partial^k}{\partial y^k}u(x,y)\right)_{y=0} \tag{2.2}$$

where the y dimensional spectrum function $U_k(x)$ is the transformed function and $u(x,y)$ is the original function.

Definition 2. The differential inverse transform of $U_k(x)$ is defined as follows:

$$u(x,y)=\sum_{k=0}^{\infty}U_k(x)\,y^k \tag{2.3}$$

From (2.2) and (2.3) ,we get

$$u(x,y)=\sum_{k=0}^{\infty}\frac{y^k}{k!}\left(\frac{\partial^k}{\partial y^k}u(x,y)\right)_{y=0} \tag{2.4}$$

Theorem 1 ([7,8]). *If* $f(x,y)=ag(x,y)+bh(x,y)$ *then* $F_k(x)=aG_k(x)+bH_k(x)$, *where a and b are constants.*

Theorem 2 ([7,8]). *If* $f(x,y)=x^m y^n$, *then* $F_k(x)=x^m\delta(k-n)$

Theorem 3 ([7, 8]). *If* $f(x,y)=\dfrac{\partial^n g(x,y)}{\partial y^n}$, *then* $F_k(x)=(k+1)(k+2)...(k+n)G_{k+n}(x)$

Theorem 4 ([7,8]). *If* $f(x,y)=\dfrac{\partial^n g(x,y)}{\partial x^n}$, *then* $F_k(x)=\dfrac{\partial^n G_k(x)}{\partial x^n}$.

Theorem 5 ([7,8]). *If* $f(x,y)=g(x,y).h(x,y)$, *then* $F_k(x)=\sum\limits_{r=0}^{k}G_r(x).H_{k-r}(x)$.

Now, let's give the equals of the first and second order derivatives of a complex function from kind of real derivatives.

Definition 3. First and second order derivatives of $w=w(z,\bar{z})$ as $z=x+iy$ are as follows.

$$\frac{\partial w}{\partial z}=\frac{1}{2}(\frac{\partial w}{\partial x}-i\frac{\partial w}{\partial y}) \tag{2.5}$$

$$\frac{\partial w}{\partial \bar{z}}=\frac{1}{2}(\frac{\partial w}{\partial x}+i\frac{\partial w}{\partial y}) \tag{2.6}$$

$$\frac{\partial^2 w}{\partial z^2}=\frac{1}{4}\left[\frac{\partial^2 w}{\partial x^2}-2i\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial^2 w}{\partial y^2}\right] \tag{2.7}$$

$$\frac{\partial^2 w}{\partial \bar{z}^2}=\frac{1}{4}\left[\frac{\partial^2 w}{\partial x^2}+2i\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial^2 w}{\partial y^2}\right] \tag{2.8}$$

$$\frac{\partial^2 w}{\partial z\partial \bar{z}}=\frac{1}{4}\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right] \tag{2.9}$$

3. Solution of complex differential equations from first and second order which is constant coeffients

In this section, two theorems for which solution of complex equations from first and second order with constant coefficients have been given and then the samples have been solved according to these theorems.

Theorem 6. *Let A,B,C are real constants and $F(z,\bar{z})$ is a function of $z,\bar{z}$. Then a special solution of equation*

$$A\frac{\partial w}{\partial z}+B\frac{\partial w}{\partial \bar{z}}+Cw=F(z,\bar{z}) \tag{3.1}$$

with condition

$$w(x,0)=f(x)$$

is

$$w(z,\bar{z})=\sum_{k=0}^{\infty}W_k(x)y^k \tag{3.2}$$

where

$$(B-A)i(k+1)W_{k+1}(x)=2F_k^*(x)-2CW_k(x)-(A+B)\frac{\partial W_k(x)}{\partial x},k\geq 0$$

$$x=\frac{z+\bar{z}}{2},y=\frac{z-\bar{z}}{2i},\ \textit{and}\ W_0(x)=f(x).$$

Proof. If (2.5), (2.6) equalities are used in equality (3.1), which is given in theorem, following equality is obtained .

$$\left(\frac{A+B}{2}\right)\frac{\partial w}{\partial x}+\left(\frac{B-A}{2}\right)i\frac{\partial w}{\partial y}+Cw=F^*(x,y) \tag{3.3}$$

where $F^*(x,y)$ is obtained by writing $x+iy$ in place of z and $x-iy$ in place of $\bar{z}$ in $F(z,\bar{z})$. If equality(3.3) is regulated then following equality is obtained.

$$(A+B)\frac{\partial w}{\partial x}+(B-A)i\frac{\partial w}{\partial y}+2Cw=2F^*(x,y) \tag{3.4}$$

If reduced differential transform is applied in equality(3.4) then following equalities is obtained.

$$(A+B)\frac{\partial W_k(x)}{\partial x}+(B-A)i(k+1)W_{k+1}(x)+2CW_k(x)=2F_k^*(x)$$

$$(B-A)i(k+1)W_{k+1}(x)=2F_k^*(x)-2CW_k(x)-(A+B)\frac{\partial W_k(x)}{\partial x} \tag{3.5}$$

In equality (3.5) $k\geq 0$, $W_0(x)=w(x,0)=f(x)$. □

Theorem 7. *Let A,B,C,D,E,F are real constants and $G(z,\bar{z})$ is a polynomial of $z,\bar{z}$. Then a special solution of equation*

$$A\frac{\partial^2 w}{\partial z^2}+B\frac{\partial^2 w}{\partial z\partial \bar{z}}+C\frac{\partial^2 w}{\partial \bar{z}^2}+D\frac{\partial w}{\partial z}+E\frac{\partial w}{\partial \bar{z}}+Fw=G(z,\bar{z}) \tag{3.6}$$

with conditions

$$w(x,0)=f(x)$$

$$\frac{\partial w}{\partial y}(x,0)=g(x)$$

is

$$w(z,\bar{z})=\sum_{k=0}^{\infty}W_k(x)\,y^k$$

where

$$W_{k+2}(x)=\frac{4\left[G_k^*(x)-FW_k(x)-\frac{(E-D)}{2}i(k+1)W_{k+1}(x)\right]}{(B-A-C)(k+1)(k+2)}$$
$$-\frac{4\left[\frac{(D+E)}{2}\frac{\partial W_k}{\partial x}+2i\frac{(C-A)}{4}(k+1)\frac{\partial W_{k+1}}{\partial x}+\frac{(A+B+C)}{4}\frac{\partial^2 W_k}{\partial x^2}\right]}{(B-A-C)(k+1)(k+2)},k\geq 0$$
$$W_0(x)=f(x),W_1(x)=g(x),\ x=\frac{z+\bar{z}}{2},y=\frac{z-\bar{z}}{2i}$$

Proof. If it is used equalities (2.5)-(2.9) in equation (3.6), which is given in theorem, following equality is obtained .

$$\left(\frac{A+B+C}{4}\right)\frac{\partial^2 w}{\partial x^2}+2i\left(\frac{C-A}{4}\right)\frac{\partial^2 w}{\partial x\partial y}+\left(\frac{B-A-C}{4}\right)\frac{\partial^2 w}{\partial y^2}$$
$$+\left(\frac{D+E}{2}\right)\frac{\partial w}{\partial x}+\left(\frac{E-D}{2}\right)i\frac{\partial w}{\partial y}+Fw=G^*(x,y) \tag{3.7}$$

where $G^*(x,y)$ is obtained by writing $x+iy$ in place of z and $x-iy$ in place of $\bar{z}$ in $G(z,\bar{z})$.

If reduced differential transform is applied in equality (3.7) then following iterative relation is obtained.

$$\left(\frac{A+B+C}{4}\right)\frac{\partial^2 W_k}{\partial x^2}+2i\left(\frac{C-A}{4}\right)(k+1)\frac{\partial W_{k+1}}{\partial x}$$
$$+\left(\frac{B-A-C}{4}\right)(k+1)(k+2)W_{k+2}+\left(\frac{D+E}{2}\right)\frac{\partial W_k}{\partial x}$$
$$+\left(\frac{E-D}{2}\right)i(k+1)W_{k+1}+FW_k=G_k^*(x) \tag{3.8}$$

Using (3.8) iterative relation, following equality can be written.

$$W_{k+2}(x)=\frac{4\left[G_k^*(x)-FW_k(x)-\frac{(E-D)}{2}i(k+1)W_{k+1}(x)\right]}{(B-A-C)(k+1)(k+2)} \tag{3.9}$$

$$-\frac{4\left[\frac{(D+E)}{2}\frac{\partial W_k}{\partial x}+2i\frac{(C-A)}{4}(k+1)\frac{\partial W_{k+1}}{\partial x}+\frac{(A+B+C)}{4}\frac{\partial^2 W_k}{\partial x^2}\right]}{(B-A-C)(k+1)(k+2)}$$

where we set $W_0(x)=f(x), W_1(x)=g(x)$ due to the conditions. □

Example 1. Solve the following problem

$$\frac{\partial w}{\partial z}+2\frac{\partial w}{\partial \bar{z}}=3z^2+2$$

$$w(x,0)=x^3+x.$$

Solution. From Theorem 6, we get $A=1, B=2, C=0, F(z,\bar{z})=3z^2+2$, $F^*(x,y)=3x^2-3y^2+2+6ixy$.

From equality (3.5)

$$i(k+1)W_{k+1}(x)=2\left(3x^2+2\right)\delta(k)-6\delta(k-2)+12ix\delta(k-1)-3\frac{\partial W_k(x)}{\partial x}$$

$$W_0(x)=x^3+x$$

$$\begin{aligned}
iW_1(x)&=6x^2+4-3\left(3x^2+1\right)=-3x^2+1, W_1(x)=i\left(3x^2-1\right)\\
2iW_2(x)&=12ix-3.6ix, W_2(x)=-3x\\
3iW_3(x)&=-6-3(-3), W_3(x)=-i\\
W_k(x)&=0, k>3.
\end{aligned}$$

$$\begin{aligned}
w(x,y)&=\sum_{k=0}^{\infty}W_k(x)y^k=W_0(x)+W_1(x).y+W_2(x)y^2+W_3(x)y^3\\
&=x^3+x+yi\left(3x^2-1\right)-3xy^2-iy^3\\
&=x^3+3ix^2y-3xy^2-iy^3+x-iy.
\end{aligned}$$

Therefore, solution is

$$w(z,\bar{z})=z^3+\bar{z}.$$

Example 2. Solve the following problem

$$2\frac{\partial w}{\partial z}-\frac{\partial w}{\partial \bar{z}}=4z+1$$

$$w(x,0)=x^2+5x$$

Solution. Coeffients of equation are $A = 2, B = -1, C = 0, F(z,\bar{z}) = 4z+1$, $F^*(x,y) = 4x+1+4iy$

From equality (3.5)

$$-3i(k+1)W_{k+1}(x) = 2(4x+1)\delta(k) + 8i\delta(k-1) - \frac{\partial W_k(x)}{\partial x}$$

$$W_0(x) = x^2 + 5x$$

$$-3iW_1(x) = 8x+2-(2x+5) = 6x-3, W_1(x) = i(2x-1)$$

$$-6iW_2(x) = 8i-2i = 6i, W_2(x) = -1,$$

$$W_k(x) = 0, \quad k > 2$$

$$w(x,y) = \sum_{k=0}^{\infty} W_k(x)y^k = W_0(x) + W_1(x)y + W_2(x)y^2$$
$$= x^2 + 5x + i(2x-1)y - y^2 = x^2 + 2ixy + 2(x+iy) + 3(x-iy).$$

Therefore, solution of the example is

$$w(z,\bar{z}) = z^2 + 2z + 3\bar{z}.$$

Example 3. Solve the following problem

$$z\frac{\partial w}{\partial z} - \bar{z}\frac{\partial w}{\partial \bar{z}} = 2z^2 + 5\bar{z}$$

with the condition

$$w(x,0) = 2x^2 - 5x$$

Solution. This equation hasn't constant coefficients. Coefficients of the equation are variable. Writing $x+iy$ in place of z, $x-iy$ in place of $\bar{z}$ in the example, from (2.5) and (2.6), the following equalities is obtained as

$$(x+iy)\frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right) - (x-iy)\frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right) = 2(x+iy)^2 + 5(x-iy) \quad (3.10)$$

$$-ix\frac{\partial w}{\partial y} + iy\frac{\partial w}{\partial x} = 2x^2 - 2y^2 + 5x + i(4xy - 5y) \quad (3.11)$$

$$-ix(k+1)W_{k+1}(x) = \left(2x^2+5x\right)\delta(k) + i(4x-5)\delta(k-1) - 2\delta(k-2)$$
$$-i\sum_{r=0}^{k}\delta(r-1)\frac{\partial W_{k-r}(x)}{\partial x}$$

$$W_0(x) = 2x^2 - 5x$$

$$-ixW_1(x) = 2x^2 + 5x, W_1(x) = i(2x+5)$$

$$-2ixW_2(x) = i(4x-5) - i(4x-5) = 0, W_2(x) = 0$$
$$-3ixW_3(x) = -2 - i2i = 0, W_k(x) = 0 (k > 3)$$
$$w(x,y) = \sum_{k=0}^{\infty} W_k(x) y^k = W_0(x) + W_1(x) y + W_2(x) y^2$$
$$= 2x^2 - 5x + i(2x+5)y.$$

Then,

$$w(z,\bar{z}) = z^2 + z.\bar{z} - 5\bar{z}.$$

Example 4. Solve the following problem

$$\frac{\partial w}{\partial z} - \frac{\partial w}{\partial \bar{z}} - w = 0$$

with the condition

$$w(x,0) = e^{3x}$$

Solution. This equation is homogeneous. Coefficients are $A = 1, B = -1$, $C = -1$.

We can write from (3.5) following equality

$$W_{k+1}(x) = i\frac{W_k(x)}{k+1}, k \geq 0$$

$$W_0(x) = e^{3x}, W_1(x) = ie^{3x}, W_2(x) = -\frac{e^{3x}}{2}, W_3(x) = -i\frac{e^{3x}}{6},$$
$$W_4(x) = \frac{e^{3x}}{24}, W_n(x) = i^n \frac{e^{3x}}{n!}$$

Solution of problem is

$$W_0(x) + W_1(x) y + W_2(x) y^2 + W_3(x) y^3 + ... = e^{3x}(1 + iy - \frac{y^2}{2} - \frac{iy^3}{6} + \frac{y^4}{24} + ...)$$
$$= e^{3x} e^{iy}.$$

So,

$$w(z,\bar{z}) = e^{2z+\bar{z}}.$$

Example 5. Solve the following problem

$$\frac{\partial^2 w}{\partial z \partial \bar{z}} = 4$$

with the conditions

$$w(x,0) = 9x^2$$
$$\frac{\partial w}{\partial y}(x,0) = 10ix$$

Solution. According to Theorem 7, coefficients of equation which in example are $B=1, A=C=D=E=F=0, G(z,\bar{z})=G^{*}(x,y)=4$.

According to (3.9)

$$W_0(x)=9x^2, W_1(x)=10ix$$

$$W_2(x)=4\left[\frac{4\delta(k)-\frac{1}{4}\frac{\partial^2 W_0(x)}{\partial x^2}}{2}\right]=-1, W_k(x)=0\ (k=3,4,5,...)$$

Solution of problem is

$$\begin{aligned}&W_0(x)+W_1(x)y+W_2(x)y^2+W_3(x)y^3+....\\&=9x^2+10ixy-y^2\\&=9\left(\frac{z+\bar{z}}{2}\right)^2+10i\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right)-\left(\frac{z-\bar{z}}{2i}\right)^2.\end{aligned}$$

Therefore,

$$w(z,\bar{z})=4z\bar{z}+5z^2.$$

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FRACTIONAL OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS OF BOUNDED VARIATON WITH TWO VARIABLES

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Abstract. We first establish some fractional equalities for functions of bounded variation with two variables. Then we derive some fractional Ostrowski and Trapezoid type inequalities for functions of bounded variation with two variables. In addition, we give some Midpoint inequalities as special cases of our main results.

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Keywords: fuctions of bounded variation, fractional integrals, Ostrowski type inequalities, midpoint type inequalities

1. Introduction

The study of various types of integral inequalities has been the focus of great attention for well over a century by a number of mathematicians, interested both in pure and applied mathematics. One of the many fundamental mathematical discoveries of A. M. Ostrowski [22] is the following classical integral inequality associated with the differentiable mappings:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)}|f'(t)|<\infty$. Then, we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\left\|f'\right\|_\infty,$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

Ostrowski inequality has applications in quadrature, probability and optimization theory, stochastic, statistics, information and integral operator theory. Until now, a large number of research papers and books have been written on Ostrowski inequalities and their numerous applications. In general, the generalizations to the Ostrowski

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type inequalities are obtained by introducing arbitrary parameters in the Peano kernels involved. The parameters can be so adjusted to recapture the previous results as well as to obtain some new estimates of such inequalities.

On the other hand, the inequalities discovered by C. Hermite and J. Hadamard for convex functions are considered significant in the literature (see, e.g.,[12], [23, p.137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Over the years, many studies have focused on to establish generalization of the inequality (1.1) and to obtain new bounds for left-hand side (Midpoint) and right-hand side (Trapezoid) of the inequality (1.1).

The remainder of this work is organized as follows: In Section 2, we present the definitions of fractional integrals and functions of bounded variation. We also mention some inequalities related fractional integrals. In Section 3, we establish three identities for double fractional integrals. Finally, some new fractional Ostrowski and Trapezoid type integral inequalities are proved for functions of bounded variation with two variables in Section 4. We also give some fractional Midpoint type inequalities in Section 4.

2. Preliminaries

Firstly, we give the definitions of Riemann-Liouville fractional integrals:

Definition 1. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

For more information and properties about fractional integrals, please refer to ([18], [21], [24]).

It is remarkable that Sarikaya et al.[30] firstly give the interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals. Whereupon several papers focus on fractional Hermite-Hadamard and Ostrowski type inequalities, for example ([1], [10], [19], [27], [29], [28]-[26], [31]).

Now, we give the definitions Riemann-Liouville fractional integrals of two variable functions:

Definition 2 ([25])**.** Let $f \in L_1([a,b]\times[c,d])$. The Riemann-Liouville fractional integrals $J^{\alpha,\beta}_{a+,c+}$, $J^{\alpha,\beta}_{a+,d-}$, $J^{\alpha,\beta}_{b-,c+}$ and $J^{\alpha,\beta}_{b-,d-}$ are defined by

$$J^{\alpha,\beta}_{a+,c+}f(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y (x-t)^{\alpha-1}(y-s)^{\beta-1}f(t,s)dsdt,\quad x>a,\ y>c,$$

$$J^{\alpha,\beta}_{a+,d-}f(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_y^d (x-t)^{\alpha-1}(s-y)^{\beta-1}f(t,s)dsdt,\quad x>a,\ y<d,$$

$$J^{\alpha,\beta}_{b-,c+}f(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_c^y (t-x)^{\alpha-1}(y-s)^{\beta-1}f(t,s)dsdt,\quad x<b,\ y>c,$$

and

$$J^{\alpha,\beta}_{b-,d-}f(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d (t-x)^{\alpha-1}(s-y)^{\beta-1}f(t,s)dsdt,\quad x<b,\ y<d.$$

Hermite-Hadamard inequality and Ostrowski inequality for fractional integrals of two-variable functions are obtained in [25] and [20], respectively. There are several papers on fractional Hermite-Hadamard and fractional Ostrowski type inequalities for two-variable functions, you can find some of them in the references.

Functions of bounded variation of one variable are of great interest and usefulness because of their valuable properties, such as particularly with respect to additivity, decomposability into monotone functions, continuity, differentiability, measurability, integrability, and so on, have been much studied. There are many of papers on inequalities for functions of bounded variation of one variable, some of them please see ([2], [6], [7], [8], [11], [13],[16], [17]). Moreover Dragomir obtained some fractional inequalities involving functions of bounded variation ([14], [15])

Functions of bounded variation with two variables are defined as follows:

Definition 3 ([9])**.** Assume that $f(x,y)$ is defined over the rectangle $Q=[a,b]\times[c,d]$. Let P be a partition of Q with

$$P: a=x_0<x_1<\ldots<x_n=b,\ \text{and}\ c=y_0<y_1<\ldots<y_m=d;$$

and for all i,j let

$$\Delta_{11}g(x_i,y_j)=g(x_{i-1},y_{j-1})-g(x_{i-1},y_j)-g(x_i,y_{j-1})+g(x_i,y_j).$$

The function $f(x,y)$ is said to be of bounded variation if the sum

$$\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left|\Delta_{11}f(x_i,y_j)\right|$$

is bounded for all nets.

Therefore, one can define the concept of total variation of a function of variables, as follows:

Let f be of bounded variation on $Q=[a,b]\times[c,d]$, and let $\sum(P)$ denote the sum $\sum_{i=1}^{n}\sum_{j=1}^{m}\left|\Delta_{11}f(x_i,y_j)\right|$ corresponding to the partition P of Q. The number

$$\bigvee_{Q}(f):=\bigvee_{c}^{d}\bigvee_{a}^{b}(f):=\sup\left\{\sum(P):P\in P(Q)\right\},$$

is called the total variation of f on Q.

There are also some paper on inequalities for functions of bounded variation with two variables ([3], [4], [5]). However, there is a few papers fractional integral inequalities for functions of bounded variation with two variables. The aim of this paper is to establish some fractional Ostrowski, Midpoint, and Trapezoid type inequalities for functions of bounded variation with two variables.

3. Some identities for double integrals

It is obtained three double integral identities involving Riemann-Liouville fractional integrals in this section. These equalities are the main material of inequalities developed throughout the article.

Lemma 1. *Let $f:\Lambda:=[a,b]\times[c,d]\to\mathbb{R}$ be an absolutely continuous function on Λ in $\mathbb{R}^2$. Then, for any $(x,y)\in[a,b]\times[c,d]$, we have*

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d Q(x,t,y,s)F(x,t,y,s)dsdt \qquad (3.1)\\
&=J_{a+,c+}^{\alpha,\beta}f(x,y)+J_{a+,d-}^{\alpha,\beta}f(x,y)+J_{b-,c+}^{\alpha,\beta}f(x,y)+J_{b-,d-}^{\alpha,\beta}f(x,y)\\
&\quad-\frac{(y-c)^{\beta}+(d-y)^{\beta}}{\Gamma(\beta+1)}\left[J_{a+}^{\alpha}f(x,y)+J_{b-}^{\alpha}f(x,y)\right]\\
&\quad-\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{\Gamma(\alpha+1)}\left[J_{c+,}^{\beta}f(x,y)+J_{d-,}^{\beta}f(x,y)\right]\\
&\quad+\frac{\left[(b-x)^{\alpha}+(x-a)^{\alpha}\right]\left[(d-y)^{\beta}+(y-c)^{\beta}\right]}{\Gamma(\alpha+1)\Gamma(\beta+1)}f(x,y)
\end{aligned}$$

where $F(x,t,y,s)$ and $Q(x,t,y,s)$ are defined by

$$F(x,t,y,s):=f(t,s)-f(t,y)-f(x,s)+f(x,y) \qquad (3.2)$$

and

$$Q(x,t,y,s) \qquad (3.3)$$

$$:= \begin{cases} (x-t)^{\alpha-1}(y-s)^{\beta-1}, & a \le t < x \text{ and } c \le s < y \\ (x-t)^{\alpha-1}(s-y)^{\beta-1}, & a \le t < x \text{ and } y \le s \le d \\ (t-x)^{\alpha-1}(y-s)^{\beta-1}, & x \le t \le b \text{ and } c \le s < y \\ (t-x)^{\alpha-1}(s-y)^{\beta-1}, & x \le t \le b \text{ and } y \le s \le d, \end{cases}$$

respectively.

Proof. From the definition of $Q(x,t,y,s)$, we write the equality

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_c^d Q(x,t,y,s)F(x,t,y,s)dsdt \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y (x-t)^{\alpha-1}(y-s)^{\beta-1} F(x,t,y,s)dsdt \\ &\quad + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_y^d (x-t)^{\alpha-1}(s-y)^{\beta-1} F(x,t,y,s)dsdt \\ &\quad + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_c^y (t-x)^{\alpha-1}(y-s)^{\beta-1} F(x,t,y,s)dsdt \\ &\quad + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_y^d (t-x)^{\alpha-1}(s-y)^{\beta-1} F(x,t,y,s)dsdt. \end{aligned}$$

Using the elementary analysis operations for the first integral in the right hand of the above identity, due to the definition of $F(x,t,y,s)$, we get the equality

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y (x-t)^{\alpha-1}(y-s)^{\beta-1} [f(t,s) - f(t,y) - f(x,s) + f(x,y)]\, dsdt \\ &= J_{a+,c+}^{\alpha,\beta} f(x,y) - \frac{(y-c)^{\beta}}{\Gamma(\beta+1)} J_{a+,}^{\alpha} f(x,y) \\ &\quad - \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)} J_{c+,}^{\beta} f(x,y) + \frac{(x-a)^{\alpha}(y-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} f(x,y). \end{aligned}$$

If we combine the resulting integrals after having been similarly analysed the other integrals, then we reach the desired equality (3.1). □

Lemma 2. *Let $f:\Lambda:=[a,b]\times[c,d]\to\mathbb{R}$ be an absolutely continuous function on Λ in $\mathbb{R}^2$. Then, for any $(x,y)\in[a,b]\times[c,d]$, we have*

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d Q(t,s)F(x,t,y,s)dsdt \qquad (3.4)\\
&=J^{\alpha,\beta}_{x+,y+}f(b,d)+J^{\alpha,\beta}_{x+,y-}f(b,c)+J^{\alpha,\beta}_{x-,y+}f(a,d)+J^{\alpha,\beta}_{x-,y-}f(a,c)\\
&\quad-\frac{(d-y)^\beta+(y-c)^\beta}{\Gamma(\beta+1)}\left[J^{\alpha}_{x+}f(b,y)+J^{\alpha}_{x-}f(a,y)\right]\\
&\quad-\frac{(b-x)^\alpha+(x-a)^\alpha}{\Gamma(\alpha+1)}\left[J^{\beta}_{y+,}f(x,d)+J^{\beta}_{y-,}f(x,c)\right]\\
&\quad+\frac{\left[(b-x)^\alpha+(x-a)^\alpha\right]\left[(d-y)^\beta+(y-c)^\beta\right]}{\Gamma(\alpha+1)\Gamma(\beta+1)}f(x,y)
\end{aligned}$$

where $F(x,t,y,s)$ is defined as in (3.2) and $Q(t,s)$ is defined by

$$\begin{aligned}
&Q(t,s) \qquad (3.5)\\
&:=\begin{cases}
(t-a)^{\alpha-1}(s-c)^{\beta-1}, & a\le t<x \text{ and } c\le s<y\\
(t-a)^{\alpha-1}(d-s)^{\beta-1}, & a\le t<x \text{ and } y\le s\le d\\
(b-t)^{\alpha-1}(s-c)^{\beta-1}, & x\le t\le b \text{ and } c\le s<y\\
(b-t)^{\alpha-1}(d-s)^{\beta-1}, & x\le t\le b \text{ and } y\le s\le d.
\end{cases}
\end{aligned}$$

Proof. By the definition of $Q(t,s)$, we find that

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d Q(t,s)F(x,t,y,s)dsdt\\
=&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d (b-t)^{\alpha-1}(d-s)^{\beta-1}F(x,t,y,s)dsdt\\
&+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_c^y (b-t)^{\alpha-1}(s-c)^{\beta-1}F(x,t,y,s)dsdt\\
&+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_y^d (t-a)^{\alpha-1}(d-s)^{\beta-1}F(x,t,y,s)dsdt\\
&+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y (t-a)^{\alpha-1}(s-c)^{\beta-1}F(x,t,y,s)dsdt.
\end{aligned}$$

Using the elementary analysis rules for the first integral in the right hand of the above equality, owing to the definition of $F(x,t,y,s)$, we obtain

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d (b-t)^{\alpha-1}(d-s)^{\beta-1}\left[f(t,s)-f(t,y)-f(x,s)+f(x,y)\right]dsdt$$
$$=J^{\alpha,\beta}_{x+,y+}f(b,d)-\frac{(d-y)^{\beta}}{\Gamma(\beta+1)}J^{\alpha}_{x+}f(b,y)$$
$$-\frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}_{y+,}f(x,d)+\frac{(b-x)^{\alpha}(d-y)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}f(x,y).$$

Should we add the finding statements side by side after having been similarly examined the other integrals, we can easily deduce the required identity (3.4). □

Lemma 3. *Let $f:\Lambda:=[a,b]\times[c,d]\to\mathbb{R}$ be an absolutely continuous function on Λ in $\mathbb{R}^2$. Then, for any $(x,y)\in[a,b]\times[c,d]$, we have*

$$\frac{1}{4}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\left\{\int_a^b\int_c^d\left[(t-a)^{\alpha-1}(s-c)^{\beta-1}+(t-a)^{\alpha-1}(d-s)^{\beta-1}\right.\right.$$
$$\left.\left.+(b-t)^{\alpha-1}(s-c)^{\beta-1}+(b-t)^{\alpha-1}(d-s)^{\beta-1}\right]F(x,t,y,s)dsdt\right\} \tag{3.6}$$
$$=\frac{J^{\alpha,\beta}_{b-,d-}f(a,c)+J^{\alpha,\beta}_{b-,c+}f(a,d)+J^{\alpha,\beta}_{a+,d-}f(b,c)+J^{\alpha,\beta}_{a+,c+}f(b,d)}{4}$$
$$-\frac{1}{2}\frac{(d-c)^{\beta}}{\Gamma(\beta+1)}\left[J^{\alpha}_{b-}f(a,y)+J^{\alpha}_{a+}f(b,y)\right]$$
$$-\frac{1}{2}\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\left[J^{\beta}_{d-,}f(x,c)+J^{\beta}_{c+,}f(x,d)\right]+\frac{(b-a)^{\alpha}(d-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}f(x,y)$$

where $F(x,t,y,s)$ is defined as in (3.2).

Proof. By the definition of $F(x,t,y,s)$ and simple calculations, for the first expression, we find that

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d (t-a)^{\alpha-1}(y-c)^{\beta-1}\left[f(t,s)-f(t,y)-f(x,s)+f(x,y)\right]dsdt$$
$$=J^{\alpha,\beta}_{b-,d-}f(a,c)-\frac{(d-c)^{\beta}}{\Gamma(\beta+1)}J^{\alpha}_{b-}f(a,y)$$
$$-\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}_{d-,}f(x,c)+\frac{(b-a)^{\alpha}(d-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}f(x,y).$$

If we similarly observe the other expressions and later we add all these identities side by side, then the desired equality can be obtained. □

Corollary 1. *Under the assumptions of Lemma 1 with $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$, then the following Midpoint type equality holds:*

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d Q\left(\frac{a+b}{2},t,\frac{c+d}{2},s\right)F\left(\frac{a+b}{2},t,\frac{c+d}{2},s\right)sdt\\
&=J^{\alpha,\beta}_{a+,c+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J^{\alpha,\beta}_{a+,d-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\quad+J^{\alpha,\beta}_{b-,c+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J^{\alpha,\beta}_{b-,d-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\quad-\frac{(d-c)^{\beta}}{2^{\beta-1}\Gamma(\beta+1)}\left[J^{\alpha}_{a+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J^{\alpha}_{b-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\\
&\quad-\frac{(b-a)^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)}\left[J^{\beta}_{c+,}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J^{\beta}_{d-,}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\\
&\quad+\frac{(b-a)^{\alpha}(d-c)^{\beta}}{2^{\alpha+\beta-2}\Gamma(\alpha+1)\Gamma(\beta+1)}\left(\frac{a+b}{2},\frac{c+d}{2}\right)
\end{aligned}$$

where $F\left(\frac{a+b}{2},t,\frac{c+d}{2},s\right)$ and $Q\left(\frac{a+b}{2},t,\frac{c+d}{2},s\right)$ are defined as in (3.2) and (3.3), respectively.

Corollary 2. *With the assumption of Lemma 2, we have the Midpoint type identity*

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d Q(t,s)F\left(\frac{a+b}{2},t,\frac{c+d}{2},s\right)dsdt\\
&=J^{\alpha,\beta}_{\frac{a+b}{2}+,\frac{c+d}{2}+}f(b,d)+J^{\alpha,\beta}_{\frac{a+b}{2}+,\frac{c+d}{2}-}f(b,c)+J^{\alpha,\beta}_{\frac{a+b}{2}-,\frac{c+d}{2}+}f(a,d)+J^{\alpha,\beta}_{\frac{a+b}{2}-,\frac{c+d}{2}-}f(a,c)\\
&\quad-\frac{(d-c)^{\beta}}{2^{\beta-1}\Gamma(\beta+1)}\left[J^{\alpha}_{\frac{a+b}{2}+}f\left(b,\frac{c+d}{2}\right)+J^{\alpha}_{\frac{a+b}{2}-}f\left(a,\frac{c+d}{2}\right)\right]\\
&\quad-\frac{(b-a)^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)}\left[J^{\beta}_{\frac{c+d}{2}+,}f\left(\frac{a+b}{2},d\right)+J^{\beta}_{\frac{c+d}{2}-,}f\left(\frac{a+b}{2},c\right)\right]\\
&\quad+\frac{(b-a)^{\alpha}(d-c)^{\beta}}{2^{\alpha+\beta-2}\Gamma(\alpha+1)\Gamma(\beta+1)}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)
\end{aligned}$$

where $F\left(\frac{a+b}{2},t,\frac{c+d}{2},s\right)$ and $Q(t,s)$ are defined as in (3.2) and (3.5), respectively.

Corollary 3. *Suppose that all the assumptions of Lemma 3 hold. If we choose* $x=\frac{a+b}{2}$ *and* $y=\frac{c+d}{2}$, *then we possess*

$$\begin{aligned}
&\frac{1}{4}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\left\{\int_a^b\int_c^d\left[(t-a)^{\alpha-1}(s-c)^{\beta-1}+(t-a)^{\alpha-1}(d-s)^{\beta-1}\right.\right.\\
&\quad\left.\left.+(b-t)^{\alpha-1}(s-c)^{\beta-1}+(b-t)^{\alpha-1}(d-s)^{\beta-1}\right]F\left(\frac{a+b}{2},t,\frac{c+d}{2},s\right)dsdt\right\}\\
&=\frac{J^{\alpha,\beta}_{b-,d-}f(a,c)+J^{\alpha,\beta}_{b-,c+}f(a,d)+J^{\alpha,\beta}_{a+,d-}f(b,c)+J^{\alpha,\beta}_{a+,c+-}f(b,d)}{4}\\
&\quad-\frac{1}{2}\frac{(d-c)^{\beta}}{\Gamma(\beta+1)}\left[J^{\alpha}_{b-}f(a,\frac{c+d}{2})+J^{\alpha}_{a+}f(b,\frac{c+d}{2})\right]\\
&\quad-\frac{1}{2}\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\left[J^{\beta}_{d-,}f(\frac{a+b}{2},c)+J^{\beta}_{c+,}f(\frac{a+b}{2},d)\right]\\
&\quad+\frac{(b-a)^{\alpha}(d-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)
\end{aligned}$$

where $F\left(\frac{a+b}{2},t,\frac{c+d}{2},s\right)$ *is defined as in (3.2).*

4. Double integral inequalities for functions of bounded variations

In this section, we establish some fractional Ostrowski and trapezoid type inequalities for function of bounded variation.

Theorem 2. *Let* $f:\Lambda\to\mathbb{R}$ *be a function of bounded variation on* Λ *in* $\mathbb{R}^2$. *Then we have the inequality*

$$\begin{aligned}
&\Big|J^{\alpha,\beta}_{a+,c+}f(x,y)+J^{\alpha,\beta}_{a+,d-}f(x,y)+J^{\alpha,\beta}_{b-,c+}f(x,y)+J^{\alpha,\beta}_{b-,d-}f(x,y)\\
&\quad-\frac{(y-c)^{\beta}+(d-y)^{\beta}}{\Gamma(\beta+1)}\left[J^{\alpha}_{a+}f(x,y)+J^{\alpha}_{b-}f(x,y)\right]\\
&\quad-\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{\Gamma(\alpha+1)}\left[J^{\beta}_{c+,}f(x,y)+J^{\beta}_{d-,}f(x,y)\right]\\
&\quad+\frac{\left[(b-x)^{\alpha}+(x-a)^{\alpha}\right]\left[(d-y)^{\beta}+(y-c)^{\beta}\right]}{\Gamma(\alpha+1)\Gamma(\beta+1)}f(x,y)\Big|\\
&\le\frac{(x-a)^{\alpha}(y-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\bigvee_a^x\bigvee_c^y(f)+\frac{(x-a)^{\alpha}(d-y)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\bigvee_a^x\bigvee_y^d(f)
\end{aligned}\tag{4.1}$$

$$
\begin{aligned}
&+\frac{(b-x)^{\alpha}(y-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\bigvee_{x}^{b}\bigvee_{c}^{y}(f)+\frac{(b-x)^{\alpha}(d-y)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\bigvee_{x}^{b}\bigvee_{y}^{d}(f)\\
&\leq\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{\alpha}\\
&\times\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]^{\beta}\bigvee_{a}^{b}\bigvee_{c}^{d}(f)
\end{aligned}
$$

for any $(x,y)\in\Lambda$.

Proof. If we take absolute value of both sides of the equality (3.1), due to the definition of $Q(x,t,y,s)$ and the well-known triangle inequality, we get

$$
\begin{aligned}
&\Big|J_{a+,c+}^{\alpha,\beta}f(x,y)+J_{a+,d-}^{\alpha,\beta}f(x,y)+J_{b-,c+}^{\alpha,\beta}f(x,y)+J_{b-,d-}^{\alpha,\beta}f(x,y)\\
&-\frac{(y-c)^{\beta}+(d-y)^{\beta}}{\Gamma(\beta+1)}\left[J_{a+}^{\alpha}f(x,y)+J_{b-}^{\alpha}f(x,y)\right]\\
&-\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{\Gamma(\alpha+1)}\left[J_{c+,}^{\beta}f(x,y)+J_{d-,}^{\beta}f(x,y)\right]\\
&+\frac{\left[(b-x)^{\alpha}+(x-a)^{\alpha}\right]\left[(d-y)^{\beta}+(y-c)^{\beta}\right]}{\Gamma(\alpha+1)\Gamma(\beta+1)}f(x,y)\Big|\\
&\leq\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{a}^{b}\int_{c}^{d}|Q(x,t,y,s)|\,|F(x,t,y,s)|\,dsdt
\end{aligned}
\tag{4.2}
$$

and

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{a}^{b}\int_{c}^{d}|Q(x,t,y,s)|\,|F(x,t,y,s)|\,dsdt\\
&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{a}^{x}\int_{c}^{y}(x-t)^{\alpha-1}(y-s)^{\beta-1}|F(x,t,y,s)|\,dsdt\\
&+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{a}^{x}\int_{y}^{d}(x-t)^{\alpha-1}(s-y)^{\beta-1}|F(x,t,y,s)|\,dsdt\\
&+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{x}^{b}\int_{c}^{y}(t-x)^{\alpha-1}(y-s)^{\beta-1}|F(x,t,y,s)|\,dsdt
\end{aligned}
$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d (t-x)^{\alpha-1}(s-y)^{\beta-1}|F(x,t,y,s)|\,dsdt$$

for any $(x,y)\in\Lambda$.
Seeing that $f:\Lambda\to\mathbb{R}$ is of bounded variation on $[a,x]\times[c,y]$, we get

$$\begin{aligned}|F(x,t,y,s)| &= |f(t,s)-f(t,y)-f(x,s)+f(x,y)|\\ &\le \bigvee_t^x\bigvee_s^y(f)\le\bigvee_a^x\bigvee_c^y(f).\end{aligned}$$

Hence, if we consider the first integral in the right hand side of the statement (4.2), then we can easily conclude that

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y (x-t)^{\alpha-1}(y-s)^{\beta-1}|F(x,t,y,s)|\,dsdt\\ &\le\frac{(x-a)^{\alpha}(y-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\bigvee_a^x\bigvee_c^y(f).\end{aligned}$$

Should the other integrals are also observed by taking account of the fact that $f:\Lambda\to\mathbb{R}$ is of bounded variation on $[a,x]\times[y,d]$, $[x,b]\times[c,y]$ and $[x,b]\times[y,d]$, one can readily attain the first inequality in (4.1).

The second inequality is obvious from the facts that

$$\max\{ac,ad,bc,bd\}=\max\{a,b\}\max\{c,d\}, \tag{4.3}$$

$$\max\{a^n,b^n\}=(\max\{a,b\})^n=\left(\frac{a+b+|a+b|}{2}\right)^n$$

for $a,b,c,d,n>0$. This completes the proof. □

Corollary 4. *Under the assumptions of Theorem 2 with $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$, we have the midpoint type inequality*

$$\begin{aligned}&\Bigg|J^{\alpha,\beta}_{a+,c+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J^{\alpha,\beta}_{a+,d-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ &+J^{\alpha,\beta}_{b-,c+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J^{\alpha,\beta}_{b-,d-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ &-\frac{(d-c)^{\beta}}{2^{\beta-1}\Gamma(\beta+1)}\left[J^{\alpha}_{a+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J^{\alpha}_{b-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\\ &-\frac{(b-a)^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)}\left[J^{\beta}_{c+,}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J^{\beta}_{d-,}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\end{aligned}$$

$$+\frac{(b-a)^{\alpha}(d-c)^{\beta}}{2^{\alpha+\beta-2}\Gamma(\alpha+1)\Gamma(\beta+1)}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\Bigg|$$
$$\leq \frac{(b-a)^{\alpha}(c-d)^{\beta}}{2^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)}\bigvee_{a}^{b}\bigvee_{c}^{d}(f).$$

Theorem 3. *Let $f:\Lambda\to\mathbb{R}$ be a function of bounded variation on Λ in $\mathbb{R}^2$. Then, one has the following inequality*

$$\begin{aligned}
&\Bigg|J^{\alpha,\beta}_{x+,y+}f(b,d)+J^{\alpha,\beta}_{x+,y-}f(b,c)+J^{\alpha,\beta}_{x-,y+}f(a,d)+J^{\alpha,\beta}_{x-,y-}f(a,c) \qquad (4.4)\\
&\quad-\frac{(d-y)^{\beta}+(y-c)^{\beta}}{\Gamma(\beta+1)}\left[J^{\alpha}_{x+}f(b,y)+J^{\alpha}_{x-}f(a,y)\right]\\
&\quad-\frac{(b-x)^{\alpha}+(x-a)^{\alpha}}{\Gamma(\alpha+1)}\left[J^{\beta}_{y+,}f(x,d)+J^{\beta}_{y-,}f(x,c)\right]\\
&\quad+\frac{\left[(b-x)^{\alpha}+(x-a)^{\alpha}\right]\left[(d-y)^{\beta}+(y-c)^{\beta}\right]}{\Gamma(\alpha+1)\Gamma(\beta+1)}f(x,y)\Bigg|\\
&\leq\frac{(x-a)^{\alpha}(y-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\bigvee_{a}^{x}\bigvee_{c}^{y}(f)+\frac{(x-a)^{\alpha}(d-y)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\bigvee_{a}^{x}\bigvee_{y}^{d}(f)\\
&\quad+\frac{(b-x)^{\alpha}(y-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\bigvee_{x}^{b}\bigvee_{c}^{y}(f)+\frac{(b-x)^{\alpha}(d-y)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\bigvee_{x}^{b}\bigvee_{y}^{d}(f)\\
&\leq\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{\alpha}\\
&\quad\times\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]^{\beta}\bigvee_{a}^{b}\bigvee_{c}^{d}(f)
\end{aligned}$$

for any $(x,y)\in\Lambda$.

Proof. Taking modulus of both sides of the equality (3.4), on account of the definition of $Q(t,s)$ and the triangle inequality, we find that

$$\begin{aligned}
&\Bigg|J^{\alpha,\beta}_{x+,y+}f(b,d)+J^{\alpha,\beta}_{x+,y-}f(b,c)+J^{\alpha,\beta}_{x-,y+}f(a,d)+J^{\alpha,\beta}_{x-,y-}f(a,c) \qquad (4.5)\\
&\quad-\frac{(d-y)^{\beta}+(y-c)^{\beta}}{\Gamma(\beta+1)}\left[J^{\alpha}_{x+}f(b,y)+J^{\alpha}_{x-}f(a,y)\right]\\
&\quad-\frac{(b-x)^{\alpha}+(x-a)^{\alpha}}{\Gamma(\alpha+1)}\left[J^{\beta}_{y+,}f(x,d)+J^{\beta}_{y-,}f(x,c)\right]
\end{aligned}$$

$$+\frac{\left[(b-x)^{\alpha}+(x-a)^{\alpha}\right]\left[(d-y)^{\beta}+(y-c)^{\beta}\right]}{\Gamma(\alpha+1)\Gamma(\beta+1)}f(x,y)\Bigg|$$

$$\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d |Q(t,s)|\,|F(x,t,y,s)|\,dsdt$$

and

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d |Q(t,s)|\,|F(x,t,y,s)|\,dsdt$$

$$=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d (b-t)^{\alpha-1}(d-s)^{\beta-1}|F(x,t,y,s)|\,dsdt$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_c^y (b-t)^{\alpha-1}(s-c)^{\beta-1}|F(x,t,y,s)|\,dsdt$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_y^d (t-a)^{\alpha-1}(d-s)^{\beta-1}|F(x,t,y,s)|\,dsdt$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y (t-a)^{\alpha-1}(s-c)^{\beta-1}|F(x,t,y,s)|\,dsdt.$$

for any $(x,y)\in\Lambda$.
Inasmuch as $f:\Lambda\to\mathbb{R}$ is of bounded variation on $[x,b]\times[y,d]$, one possess

$$|F(x,t,y,s)| = |f(t,s)-f(t,y)-f(x,s)+f(x,y)|$$
$$\leq \bigvee_t^x\bigvee_s^y(f)\leq \bigvee_x^b\bigvee_y^d(f).$$

For the first integral in the right hand side of the statement (4.5), we deduce that

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d (b-t)^{\alpha-1}(d-s)^{\beta-1}|F(x,t,y,s)|\,dsdt$$

$$\leq \frac{(b-x)^{\alpha}(d-y)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\bigvee_x^b\bigvee_y^d(f).$$

In a similar way, taking into account the other three integrals in the right hand side of (4.5) by considering that $f:\Lambda\to\mathbb{R}$ is of bounded variation on $[a,x]\times[c,y]$, $[a,x]\times$

$[y,d]$ and $[x,b]\times[c,y]$, we reach the desired first inequality in (4.4). The second inequality in (4.4) is proved above. □

Corollary 5. *With the assumption of the Theorem 3, we have the midpoint type inequality*

$$\Bigg|J^{\alpha,\beta}_{\frac{a+b}{2}+,\frac{c+d}{2}+}f(b,d)+J^{\alpha,\beta}_{\frac{a+b}{2}+,\frac{c+d}{2}-}f(b,c)+J^{\alpha,\beta}_{\frac{a+b}{2}-,\frac{c+d}{2}+}f(a,d)+J^{\alpha,\beta}_{\frac{a+b}{2}-,\frac{c+d}{2}-}f(a,c)$$
$$-\frac{(d-c)^{\beta}}{2^{\beta-1}\Gamma(\beta+1)}\left[J^{\alpha}_{\frac{a+b}{2}+}f\left(b,\frac{c+d}{2}\right)+J^{\alpha}_{\frac{a+b}{2}-}f\left(a,\frac{c+d}{2}\right)\right]$$
$$-\frac{(b-a)^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)}\left[J^{\beta}_{\frac{c+d}{2}+,}f\left(\frac{a+b}{2},d\right)+J^{\beta}_{\frac{c+d}{2}-,}f\left(\frac{a+b}{2},c\right)\right]$$
$$+\frac{(b-a)^{\alpha}(d-c)^{\beta}}{2^{\alpha+\beta-2}\Gamma(\alpha+1)\Gamma(\beta+1)}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\Bigg|$$
$$\leq\frac{(b-a)^{\alpha}(c-d)^{\beta}}{2^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)}\bigvee_a^b\bigvee_c^d(f).$$

Theorem 4. *Let $f:\Lambda\to\mathbb{R}$ be a function of bounded variation on Λ in $\mathbb{R}^2$. Then, for any $(x,y)\in\Lambda$, we have*

$$\Bigg|\frac{J^{\alpha,\beta}_{b-,d-}f(a,c)+J^{\alpha,\beta}_{b-,c+}f(a,d)+J^{\alpha,\beta}_{a+,d-}f(b,c)+J^{\alpha,\beta}_{a+,c+-}f(b,d)}{4}$$
$$-\frac{1}{2}\frac{(d-c)^{\beta}}{\Gamma(\beta+1)}\left[J^{\alpha}_{b-}f(a,y)+J^{\alpha}_{a+}f(b,y)\right]$$
$$-\frac{1}{2}\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\left[J^{\beta}_{d-,}f(x,c)+J^{\beta}_{c+,}f(x,d)\right]+\frac{(b-a)^{\alpha}(d-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}f(x,y)\Bigg|$$
$$\leq\frac{1}{4}\Bigg\{A(x)C(y)\bigvee_a^x\bigvee_c^y(f)+A(x)D(y)\bigvee_a^x\bigvee_y^d(f)$$
$$+B(x)C(y)\bigvee_x^b\bigvee_c^y(f)+B(x)D(y)\bigvee_x^b\bigvee_y^d(f)\Bigg\}$$
$$\leq\frac{1}{4\Gamma(\alpha+1)\Gamma(\beta+1)}\left[(b-a)^{\alpha}+\left|(x-a)^{\alpha}-(b-x)^{\alpha}\right|\right]$$
$$\times\left[(d-c)^{\beta}+\left|(y-c)^{\beta}-(d-y)^{\beta}\right|\right]\bigvee_a^b\bigvee_c^d(f)$$

where

$$A(x)=\frac{(b-a)^{\alpha}-(b-x)^{\alpha}+(x-a)^{\alpha}}{\Gamma(\alpha+1)},$$

$$B(x)=\frac{(b-a)^{\alpha}-(x-a)^{\alpha}+(b-x)^{\alpha}}{\Gamma(\alpha+1)},$$

$$C(y)=\frac{(d-c)^{\beta}-(d-y)^{\beta}+(y-c)^{\beta}}{\Gamma(\beta+1)},$$

$$D(y)=\frac{(d-c)^{\beta}-(y-c)^{\beta}+(d-y)^{\beta}}{\Gamma(\beta+1)}.$$

Proof. Should we take absolute value of (3.6), from the modulus property of the integral, we get the inequality

$$\begin{aligned}
&\Bigg|\frac{J^{\alpha,\beta}_{b-,d-}f(a,c)+J^{\alpha,\beta}_{b-,c+}f(a,d)+J^{\alpha,\beta}_{a+,d-}f(b,c)+J^{\alpha,\beta}_{a+,c+-}f(b,d)}{4} \\
&\quad-\frac{1}{2}\frac{(d-c)^{\beta}}{\Gamma(\beta+1)}\left[J^{\alpha}_{b-}f(a,y)+J^{\alpha}_{a+}f(b,y)\right] \\
&\quad-\frac{1}{2}\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\left[J^{\beta}_{d-,}f(x,c)+J^{\beta}_{c+,}f(x,d)\right]+\frac{(b-a)^{\alpha}(d-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}f(x,y)\Bigg| \\
&\leq\frac{1}{4}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\Bigg\{\int_a^b\int_c^d\Big[(t-a)^{\alpha-1}(s-c)^{\beta-1}+(t-a)^{\alpha-1}(d-s)^{\beta-1} \\
&\quad+(b-t)^{\alpha-1}(s-c)^{\beta-1}+(b-t)^{\alpha-1}(d-s)^{\beta-1}\Big]|F(x,t,y,s)|\,dsdt\Bigg\}.
\end{aligned} \tag{4.6}$$

Considering that $f:\Lambda\to\mathbb{R}$ is of bounded variation on Λ when calculating integral of the first of the four expressions given in the summation form in the right-hand side of the above inequality, it is easy to see that

$$\begin{aligned}
&\frac{1}{4}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(t-a)^{\alpha-1}(s-c)^{\beta-1}|F(x,t,y,s)|\,dsdt \\
&\leq\frac{1}{4}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\Bigg\{\bigvee_a^x\bigvee_c^y(f)\int_a^x\int_c^y(t-a)^{\alpha-1}(s-c)^{\beta-1}\,dsdt \\
&\quad+\bigvee_a^x\bigvee_y^d(f)\int_a^x\int_y^d(t-a)^{\alpha-1}(s-c)^{\beta-1}\,dsdt \\
&\quad+\bigvee_x^b\bigvee_c^y(f)\int_x^b\int_c^y(t-a)^{\alpha-1}(s-c)^{\beta-1}\,dsdt
\end{aligned}$$

$$+\bigvee_{x}^{b}\bigvee_{y}^{d}(f)\int_{x}^{b}\int_{y}^{d}(t-a)^{\alpha-1}(s-c)^{\beta-1}\,dsdt\Bigg\}.$$

If the above integrals are calculated, one has the inequality

$$\begin{aligned}
&\frac{1}{4}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{a}^{b}\int_{c}^{d}(t-a)^{\alpha-1}(s-c)^{\beta-1}\,|F(x,t,y,s)|\,dsdt\\
&\leq\frac{1}{4}\frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}\frac{(y-c)^{\beta}}{\Gamma(\beta+1)}\bigvee_{a}^{x}\bigvee_{c}^{y}(f)+\frac{1}{4}\frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}\frac{(d-c)^{\beta}-(y-c)^{\beta}}{\Gamma(\beta+1)}\bigvee_{a}^{x}\bigvee_{y}^{d}(f)\\
&\quad+\frac{1}{4}\frac{(b-a)^{\alpha}-(x-a)^{\alpha}}{\Gamma(\alpha+1)}\frac{(y-c)^{\beta}}{\Gamma(\beta+1)}\bigvee_{x}^{b}\bigvee_{c}^{y}(f)\\
&\quad+\frac{1}{4}\frac{(b-a)^{\alpha}-(x-a)^{\alpha}}{\Gamma(\alpha+1)}\frac{(d-c)^{\beta}-(y-c)^{\beta}}{\Gamma(\beta+1)}\bigvee_{x}^{b}\bigvee_{y}^{d}(f).
\end{aligned}$$

If we sum all the resulting inequalities side by side after having been similarly calculated integrals of the other three expressions in (4.6), the first inequality can be readily obtained.

The second inequality can be easilly proved by using the facts (4.3). The proof is thus completed. □

Corollary 6. *Suppose that all the assumptions of Theorem 4 hold. If we choose* $x=\frac{a+b}{2}$ *and* $y=\frac{c+d}{2}$, *then we have*

$$\begin{aligned}
&\Bigg|\frac{J^{\alpha,\beta}_{b-,d-}f(a,c)+J^{\alpha,\beta}_{b-,c+}f(a,d)+J^{\alpha,\beta}_{a+,d-}f(b,c)+J^{\alpha,\beta}_{a+,c+-}f(b,d)}{4}\\
&\quad-\frac{1}{2}\frac{(d-c)^{\beta}}{\Gamma(\beta+1)}\left[J^{\alpha}_{b-}f(a,\frac{c+d}{2})+J^{\alpha}_{a+}f(b,\frac{c+d}{2})\right]\\
&\quad-\frac{1}{2}\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\left[J^{\beta}_{d-,}f(\frac{a+b}{2},c)+J^{\beta}_{c+,}f(\frac{a+b}{2},d)\right]\\
&\quad+\frac{(b-a)^{\alpha}(d-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\Bigg|\\
&\leq\frac{1}{4}\frac{(b-a)^{\alpha}(c-d)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\bigvee_{a}^{b}\bigvee_{c}^{d}(f).
\end{aligned}$$

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INVESTIGATION OF SOLUTIONS TO THE FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS OF BRATU-TYPE USING LEGENDRE WAVELETS METHOD

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Abstract. In this study, Legendre wavelets has been applied to solve the fractional integro-differential equations of Bratu-type. In this method, Legendre wavelet operational matrix and numerical integration techniques have been used. Finally, this method is used for solving some examples to illustrate the simplicity of the suggested method.

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1. Introduction

Many scientific and engineering problems, including a fractional phenomenon so far are huge, and still are growing. Recently, advances of fractional derivatives and integrals are illustrated by applications in fluid mechanics, viscoelasticity, tracer transfer in underground water, mathematical biology and physics[6, 19]. Recently, fractional behavior of different kinds of dynamical problems has been represented by findings many researchers. This suggests that fractional calculus has an effective role for describing the dynamical problems[4, 5]. Several numerical techniques have been introduced for the numerical solutions of fractional differential equations(FDEs), such as Dehghan et al. [8] studied the homotopy analysis method to fractional differential equations, Odibat and Momani [16] used generalized differential transform method to solve the numerical solution of FDEs, Eslahchi et al. [9] applied the collocation method for solving nonlinear fractional integro-differential equations, the explicit and implicit Euler methods were used in the advection-diffusion equation of fractional order by Zhuang et al. [22], and many other researches [11].

Bratu's problem is often appeared in many branches of sciences, such as nanotechnology, the fuel ignition model of the thermal combustion theory, and chemical reaction theory [10]. Scientists have devoted tremendous efforts to solve Bratu's problem. Babolian et al. [3] applied the reproducing kernel Hilbert space method for

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solving Bratu-type differential equations of fractional order, Laplace Adomian decomposition method and Adomian decomposition method were applied for solving Bratu's problem by Syam and Wazwaz respectively[20]. Aksoy and Pakdemirli [1] had solved Bratu-type equation of new perturbation iteration solutions. Yiming Chen et al. [7] applied the Chebyshev wavelet method for solving fractional integral and differential equations of Bratu-type, CAS wavelets was applied to study it by Mingxu Yi et al. [21].

In recent decades, many researchers have tried to study orthogonal functions for obtaining solutions of integral equations. Using orthogonal basis, a functional integral equation can be reduced into a system of algebraic equations. Researchers used several different types of orthogonal functions for finding solutions of integral equations. Comparison between these orthogonal functions, orthogonal wavelet basis are useful and powerful basis for solving functional integral equations. Over the past thirty years, scientists have studied the fractional differential equations (FDEs), because many physical phenomena in the other sciences can be modeled using the fractional derivatives. Over the last two decades, since apply of wavelets by scientists, these functions have proved their applications in study of many problems of engineering, physics and applied mathematics. Wavelets have several useful properties, such as compact support, symmetry, orthogonality and closed form. Therefore, these functions have been widely used by many researchers.

Many types of integral equations have been solved via some wavelets such as, CAS, Haar, Legendre and Chebyshev [17, 18]. Lepik applied the Haar wavelet to obtain solutions of fractional integral equations and nonlinear integro-differential equations [12, 13]. The Haar wavelet is used to obtain numerical solutions of Fredholm integral equations. Chebyshev wavelet was used by Babolian to investigate differential equations[2]. Furthermore, a CAS wavelet has been used to FDEs in [18], and so on.

Many researchers used Legendre wavelets for their studies to solve differential equations [14] and references therein. Our purpose of this study, is to investigate approximate solutions of the following fractional integro-differential equations of Bratu-type by using Legendre wavelets.

$$\begin{aligned} &D^{\alpha}_{0_+}\zeta(x)+\lambda\int_0^x k(x,t)\exp(\zeta(t))\,dt=g(x), && m-1<\alpha\le m,\ 0\le x,t\le 1,\\ &\zeta^{(j)}(0)=C_j, && j=0,\cdots,m-1, \end{aligned} \tag{1.1}$$

where $D^{\alpha}_{0_+}$ is the fractional derivative, $\zeta(x)$ is unknown function on the interval $[0,1]$, λ and C_j, $j=0,\cdots,m-1$, are given constant, m is a positive integer number, $k(x,t)\in L^2([0,1]\times[0,1])$ and $g(x)$ is a known function. The main idea is to replace a FDE with a Volterra integral equation, and then provide an efficient numerical algorithm according to Legendre wavelets operational matrix.

This paper is organized as follows. Section 2 is given some preliminaries of the fractional calculus theory. Description of the definitions of wavelets and the Legendre polynomials are given in Section 3. In Section 4, we introduce the Legendre wavelets to approximate the functions. We presented some numerical examples to illustrate validity and simplicity of the numerical approach in Section 5. The obtained numerical solutions by this method are compared with exact solutions as well. Conclusions of this paper are summarized in Section 6.

2. Preliminaries

In this section, we illustrate fractional calculus, which are used throughout this paper [15].

Definition 1. The Riemann-Liouville fractional integration operator $I^{\alpha}_{0_+}$ of order α $(\alpha \geq 0)$ and Caputo fractional derivative operator $D^{\alpha}_{0_+}$ of order α $(\alpha \geq 0)$ on the usual Lebesgue space $L[0,T]$ are defined as:

- Riemann-Liouville fractional integration operator:
  $I^{\alpha}_{0_+}\zeta(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{(\alpha-1)}\zeta(\tau)\,d\tau, \;\; I^{0}_{0_+}\zeta(t) = \zeta(t), \;\; t > 0,$
- Caputo fractional derivative operator:
  $D^{\alpha}_{0_+}\zeta(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}\zeta^{(m)}(\tau)d\tau, \quad m-1 < \alpha \leq m, \quad t > 0.$

where m is a positive integer number and $\Gamma(.)$ is well-known Euler's gamma function.

Some of the basic and main properties of the Riemann-Liouville fractional integral and Caputo fractional derivative operators are given below:

(i) $I^{\alpha}_{0_+} I^{\mu}_{0_+}\zeta(t) = I^{\alpha+\mu}_{0_+}\zeta(t)$,
(ii) $I^{\alpha}_{0_+} I^{\mu}_{0_+}\zeta(t) = I^{\mu}_{0_+} I^{\alpha}_{0_+}\zeta(t)$,
(iii) $I^{\alpha}_{0_+} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)} t^{\alpha+\mu}$,
(iv) $I^{\alpha}_{0_+}\; D^{\alpha}_{0_+}\zeta(t) = \zeta(t) - \sum_{q=0}^{m-1}\zeta^{(q)}(0^+)\frac{t^q}{q!}, \qquad t > 0$
(v) $D^{\alpha}_{0_+} J^{\alpha}_{0_+}\zeta(t) = \zeta(t)$,
(vi) $\int_0^1 \tau^{\alpha-1}(1-\tau)^{\mu-1}\,d\tau = \frac{\Gamma(\alpha)\Gamma(\mu)}{\Gamma(\alpha+\mu)}$.

3. Legendre wavelets

Wavelets constitute of a family of functions constructed by translation and dilation of a single function called mother wavelet $\psi(x)$, which is defined as follows:

$$\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}}\psi(\frac{x-b}{a}), \quad a,b \in \mathbb{R}, \quad a \neq 0.$$

Here a and b are the dilation and translation parameters.
By restricting the parameters a and b to discrete values as $a = a_0^{-j}$, $b = kb_0a_0^{-j}$ where

$a_0 > 1$, $b_0 > 1$, $j,k \in \mathbb{N}$, we obtain a family of discrete wavelets as:

$$\psi_{j,k}(x) = \mid a_0 \mid^{\frac{j}{2}} \psi(a_0^j x - k b_0).$$

If $a_0 = 2$ and $b_0 = 1$, the above $\psi_{j,k}(x)$ may construct an orthonormal basis for certain ψ, that is

$$\left\langle \psi_{j,k}, \psi_{l,m} \right\rangle = \delta_{jl}\delta_{km},$$

where δ_{jl} indicates the Kronecker delta.
The Legendre polynomial of order m can be defined by the following recurrence formula on the interval $[-1,1]$

$$\begin{aligned} &p_0(x) = 1, \\ &p_1(x) = x, \\ &p_{m+1}(x) = \frac{2m+1}{m+1}\, x\, p_m(x) - \frac{m}{m+1} p_{m-1}(x) \quad m = 1,2,3,\ldots. \end{aligned}$$

Legendre wavelets defined on the interval $[0,1)$ as:

$$\psi_{n,m}(x) = \begin{cases} (2m+1)^{\frac{1}{2}} 2^{\frac{k}{2}} p_m(2^k x - \hat{n}), & \frac{\hat{n}-1}{2^k} \le x < \frac{\hat{n}+1}{2^k}, \\ 0, & otherwise, \end{cases}$$

where $n = 1,2,3,\ldots,2^{k-1}$, $k = 2,3,\ldots$, $\hat{n} = 2n-1$ and $m = 0,1,2,\ldots,M-1$. m is the order of Legendre polynomials and M is a positive integer.

4. Functions approximation by Legendre wavelets

For any function $u(x)$ with square integrable on $[0,1)$, we can express it in terms of the Legendre wavelet polynomials as follows:

$$u(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{n,m}(x), \tag{4.1}$$

where

$$c_{nm} = \langle u(x), \psi_{n,m}(x) \rangle .$$

If the infinite series in (4.1) is truncated, then equation (4.1) can be written as follows:

$$u(x) \cong u_{\hat{m}}(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \psi_{n,m}(x) = C^T \Psi(x), \tag{4.2}$$

where C and $\Psi(x)$ are two $\hat{m} \times 1 (\hat{m} = 2^{k-1}M)$ matrices given by

$$C = [c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, c_{21}, \ldots, c_{2M-1}, \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1}M-1}]^T,$$

$$\Psi(x) = [\psi_{10}, \psi_{11}, \ldots, \psi_{1M-1}, \psi_{20}, \ldots, \psi_{2M-1}, \ldots, \psi_{2^{k-1}0}, \ldots, \psi_{2^{k-1}M-1}]^T.$$

For simplicity the equation (4.2) can be rewritten as follows:

$$u(x) \cong u_{\hat{m}}(x) = \sum_{i=1}^{\hat{m}} c_i \psi_i(x) = C^T \Psi(x),$$

where $c_i = c_{nm}$, $\psi_i(x) = \psi_{nm}$, $i = M(n-1)+m+1$. Here, we rewrite the matrices Ψ and C as follows:

$$C = [c_1, c_2, \ldots, c_{\hat{m}}]^T, \qquad \Psi(x) = [\psi_1(x), \ldots, \psi_{\hat{m}}(x)]^T.$$

Similarly, for expanding of two variables functions such as $k(x,y) \in L^2([0,1]\times[0,1])$ in terms of Legendre wavelet functions we can write:

$$k(x,y) \cong \sum_{i=1}^{\hat{m}} \sum_{j=1}^{\hat{m}} k_{ij} \psi_i(x) \psi_j(y) = \Psi(x)^T K \Psi(y),$$

where $K_{\hat{m}\times\hat{m}}$ is given as follows:

$$K = [k_{ij}]_{\hat{m}\times\hat{m}}, \quad k_{ij} = \left\langle \psi_i(x), \left\langle k(x,y), \psi_j(y) \right\rangle \right\rangle, \quad i,j = 1,2,\ldots,\hat{m}.$$

Theorem 1. *For solving Fredholm-Volterra integral equation, the operational matrix P of integration is given as follows*

$$P_{\hat{m}\times\hat{m}} = \begin{bmatrix} L & F & F & \ldots & F \\ 0 & L & F & \ldots & F \\ 0 & 0 & L & \ldots & F \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & L \end{bmatrix}$$

where F and L are square matrices of order M given by

$$F_{M\times M} = \frac{1}{2^k} \begin{bmatrix} 2 & 0 & \ldots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L_{M\times M} = \frac{1}{2^k}\begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & 0 & \dots & 0 & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & \dots & 0 & 0 \\ 0 & -\frac{\sqrt{5}}{5\sqrt{3}} & 0 & \frac{\sqrt{5}}{5\sqrt{7}} & \dots & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{7}}{7\sqrt{5}} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} \\ 0 & 0 & 0 & 0 & \dots & -\frac{\sqrt{2M-1}}{(2M-1)\sqrt{2M-3}} & 0 \end{bmatrix}$$

Proof. First we get the operational matrix of integration for $k=2$ and $M=3$. Then we give the general matrix P. The basis functions $\psi_{nm}(x)$, $n=1,2$, $m=0,1,2$ and consequently the matrix

$$\Psi_{6\times 1}(x) = [\psi_{10}(x)\ \ \psi_{11}(x)\ \ \psi_{12}(x)\ \ \psi_{20}(x)\ \ \psi_{21}(x)\ \ \psi_{22}(x)]^T,$$

for $k=2$ and $M=3$ are computed as

$$\left.\begin{aligned} \psi_{10}(x) &= \sqrt{2}, \\ \psi_{11}(x) &= \sqrt{6}(4x-1), \\ \psi_{12}(x) &= \sqrt{10}\left[\frac{3}{2}(4x-1)^2 - \frac{1}{2}\right], \end{aligned}\right\} \qquad 0 \le x < \frac{1}{2}, \tag{4.3}$$

and

$$\left.\begin{aligned} \psi_{20}(x) &= \sqrt{2}, \\ \psi_{21}(x) &= \sqrt{6}(4x-3), \\ \psi_{22}(x) &= \sqrt{10}\left[\frac{3}{2}(4x-3)^2 - \frac{1}{2}\right], \end{aligned}\right\} \qquad \frac{1}{2} \le x < 1, \tag{4.4}$$

By integrating (4.3) and (4.4), we obtain the following relations:

$$\int_0^x \psi_{10}(t)\,dt = \begin{cases} \sqrt{2}x, & 0 \le x < \frac{1}{2}, \\ \frac{1}{\sqrt{2}}, & \frac{1}{2} \le x < 1, \end{cases}$$

$$= \frac{1}{4}\psi_{10}(x) + \frac{\sqrt{2}}{4\sqrt{6}}\psi_{11}(x) + \frac{1}{2}\psi_{20}(x) = \left[\frac{1}{4}, \frac{\sqrt{2}}{4\sqrt{6}}, 0, \frac{1}{2}, 0, 0\right]\Psi_{6\times 1}(x).$$

$$\int_0^x \psi_{11}(t)\,dt = \begin{cases} 2\sqrt{6}x^2 - \sqrt{6}x, & 0 \le x < \frac{1}{2}, \\ 0, & \frac{1}{2} \le x < 1, \end{cases}$$
$$= -\frac{\sqrt{3}}{12}\psi_{10}(x) + \frac{\sqrt{3}}{12\sqrt{5}}\psi_{12}(x) = \left[-\frac{\sqrt{3}}{12}, 0, \frac{\sqrt{3}}{12\sqrt{5}}, 0, 0, 0\right]\Psi_{6\times 1}(x).$$

Similarly, we have

$$\int_0^x \psi_{12}(t)\,dt = -\frac{\sqrt{5}}{20\sqrt{3}}\psi_{11}(x) = \left[0, -\frac{\sqrt{5}}{20\sqrt{3}}, 0, 0, 0, 0\right]\Psi_{6\times 1}(x),$$

$$\int_0^x \psi_{20}(t)\,dt = \frac{1}{4}\psi_{20}(x) + \frac{\sqrt{2}}{4\sqrt{6}}\psi_{21}(x) = \left[0, 0, 0, \frac{1}{4}, \frac{\sqrt{2}}{4\sqrt{6}}, 0\right]\Psi_{6\times 1}(x),$$

$$\int_0^x \psi_{21}(t)\,dt = -\frac{\sqrt{3}}{12}\psi_{20}(x) + \frac{\sqrt{3}}{12\sqrt{5}}\psi_{22}(x) = \left[0, 0, 0, -\frac{\sqrt{3}}{12}, 0, \frac{\sqrt{3}}{12\sqrt{5}}\right]\Psi_{6\times 1}(x),$$

$$\int_0^x \psi_{22}(t)\,dt = -\frac{\sqrt{5}}{20\sqrt{3}}\psi_{21}(x) = \left[0, 0, 0, 0, -\frac{\sqrt{5}}{20\sqrt{3}}, 0\right]\Psi_{6\times 1}(x),$$

Therefore, we have

$$\int_0^x \Psi_{6\times 1}(t)\,dt = P_{6\times 6}\Psi_{6\times 1}(x),$$

where $P_{6\times 6}$ is a square matrix of order 6 which it can be given as the following form:

$$P_{6\times 6} = \begin{bmatrix} L_{3\times 3} & F_{3\times 3} \\ 0_{3\times 3} & L_{3\times 3} \end{bmatrix}$$

where

$$L_{3\times 3} = \frac{1}{2^2}\begin{bmatrix} 1 & \frac{\sqrt{2}}{\sqrt{6}} & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} \\ 0 & -\frac{\sqrt{5}}{5\sqrt{3}} & 0 \end{bmatrix} \quad \textit{and} \quad F_{3\times 3} = \frac{1}{2^2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

If the theorem be true for M-1 and K-1, then it is simple to show for general case M and k. we have

$$\int_0^x \Psi_{\hat{m}\times 1}(t)\,dt = P_{\hat{m}\times\hat{m}}\Psi_{\hat{m}\times 1}(x), \quad (\hat{m} = 2^{k-1}M),$$

Here, $P_{\hat{m}\times\hat{m}}$ can be written as follow

$$P_{\hat{m}\times\hat{m}} = \begin{bmatrix} L & F & F & \dots & F \\ 0 & L & F & \dots & F \\ 0 & 0 & L & \dots & F \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & L \end{bmatrix}$$

where F and L are square matrices of order M given by

$$F_{M\times M} = \frac{1}{2^k}\begin{bmatrix} 2 & 0 & \dots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L_{M\times M} = \frac{1}{2^k}\begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & 0 & \dots & 0 & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & \dots & 0 & 0 \\ 0 & -\frac{\sqrt{5}}{5\sqrt{3}} & 0 & \frac{\sqrt{5}}{5\sqrt{7}} & \dots & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{7}}{7\sqrt{5}} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} \\ 0 & 0 & 0 & 0 & \dots & -\frac{\sqrt{2M-1}}{(2M-1)\sqrt{2M-3}} & 0 \end{bmatrix}$$

□

Now, we will apply our method to problem I and II.

PROBLEM I

We demonstrate this method to the following fractional integro-differential equation of Bratu type:

$$\begin{aligned} &D^{\alpha}_{0_+}\zeta(x) + \lambda\int_0^x (x-t)^p \exp(\zeta(t))\,dt = g(x), \quad m-1<\alpha\le m,\ 0\le x,t\le 1, \\ &\zeta^{(j)}(0) = C_j, \qquad\qquad j=0,\cdots,m-1. \end{aligned} \tag{4.5}$$

Using Riemann-Liouville fractional integration from Eq. (4.5), we have

$$\zeta(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left(\lambda\int_0^t (t-\tau)^p \exp(\zeta(\tau))\,d\tau\right)dt$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} g(t)\, dt. \tag{4.6}$$

We first rewrite Eq. (4) as

$$\zeta(x) = G(x) - \frac{\lambda}{\Gamma(\alpha)} \int_0^x \int_0^t (x-t)^{\alpha-1} (t-\tau)^p \exp(\zeta(\tau))\, d\tau\, dt,$$

where $G(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} g(t)\, dt$. However, by changing the order of integration in Eq. (4), we have

$$\zeta(x) = G(x) - \frac{\lambda}{\Gamma(\alpha)} \int_0^x \left(\int_\tau^x (x-t)^{\alpha-1} (t-\tau)^p\, dt \right) \exp(\zeta(\tau))\, d\tau, \tag{4.7}$$

Now, using the change of variables $s = \frac{x-t}{x-\tau}$, and according to $\int_0^1 s^{\alpha-1}(1-s)^p\, ds = \frac{\Gamma(\alpha)\Gamma(p+1)}{\Gamma(\alpha+p+1)}$, Eq. (4.7) is converted to following Volterra integral equation

$$\zeta(x) = G(x) - \frac{\lambda\, \Gamma(p+1)}{\Gamma(\alpha+p+1)} \int_0^x (x-\tau)^{\alpha+p} \exp(\zeta(\tau))\, d\tau. \tag{4.8}$$

We rewrite Eq. (4.8) as

$$\zeta(x) = F(x, \zeta(x)), \quad 0 \le x \le 1, \tag{4.9}$$

where $F(x,\zeta(x)) = G(x) - \frac{\lambda\, \Gamma(p+1)}{\Gamma(\alpha+p+1)} \int_0^x (x-\tau)^{\alpha+p} \exp(\zeta(\tau))\, d\tau$.

PROBLEM II

Consider the following fractional integro-differential equation of Bratu type:

$$\begin{aligned} &D^{\alpha}_{0_+} \zeta(x) + \lambda \textstyle\int_0^x (x-a)^p (t-b)^q \exp(\zeta(t))\, dt = g(x), && m-1 < \alpha \le m, \\ & && 0 \le x, t \le 1, \\ &\zeta^{(j)}(0) = C_j, && j = 0, \cdots, m-1. \end{aligned} \tag{4.10}$$

Using Riemann-Liouville fractional integration from Eq. (4.10), we have

$$\begin{aligned} &\zeta(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left(\lambda \int_0^t (t-a)^p (\tau-b)^q \exp(\zeta(\tau))\, d\tau \right) dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} g(t)\, dt. \end{aligned} \tag{4.11}$$

We first rewrite Eq. (4.11) as

$$\zeta(x) = G(x) - \frac{\lambda}{\Gamma(\alpha)} \int_0^x \int_0^t (x-t)^{\alpha-1} (t-a)^p (\tau-b)^q \exp(\zeta(\tau))\, d\tau\, dt,$$

where $G(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} g(t)\, dt$. However, by changing the order of integrations in Eq. (4.11), we have

$$\zeta(x) = G(x) - \frac{\lambda}{\Gamma(\alpha)} \int_0^x (\tau-b)^q \left(\int_\tau^x (x-t)^{\alpha-1} (t-a)^p\, dt \right) \exp(\zeta(\tau))\, d\tau, \quad (4.12)$$

then, using the change of variables $s = \frac{x-t}{x-\tau}$, and according to $\int_0^1 s^{\alpha-1}(1-s)^{p-k}\, ds = \frac{\Gamma(\alpha)\Gamma(p-k+1)}{\Gamma(\alpha+p-k+1)}$, Eq. (4.12) is converted to following Volterra integral equation

$$\zeta(x) = G(x) - \sum_{k=0}^{p} \frac{\lambda p!}{k!\Gamma(\alpha+p-k+1)} \int_0^x (x-\tau)^{\alpha+p-k} (\tau-a)^k (\tau-b)^q \exp(\zeta(\tau))\, d\tau. \quad (4.13)$$

We rewrite Eq. (4.13) as

$$\zeta(x) = F(x, \zeta(x)), \quad 0 \le x \le 1, \quad (4.14)$$

where
$F(x,\zeta(x)) = G(x) - \sum_{k=0}^{p} \frac{\lambda p!}{k!\Gamma(\alpha+p-k+1)} \int_0^x (x-\tau)^{\alpha+p-k} (\tau-a)^k (\tau-b)^q \exp(\zeta(\tau))\, d\tau.$

Theorem 2. *The fractional integro-differential equations of Bratu type Eq. (1.1) as well as Eqs. (4.9) and (4.14) have a unique solutions([5]).*

5. Numerical examples

In this section, in order to test the validity of our method, three examples are solved and the numerical results are compared with their exact solution.

Example 1. Assume the following fractional integro-differential equations of Bratu-type:

$$\begin{cases} D_{0_+}^{\alpha} \zeta(x) + 3 \int_0^x (x-t) \exp(\zeta(t))\, dt = \frac{1}{x+1} + \frac{1}{2}x^3 + \frac{3}{2}x^2, & 0 < x,\ t < 1, \\ \zeta(0) = 0, & 0 < \alpha \le 1. \end{cases} \quad (5.1)$$

The exact solution for $\alpha = 1$ is $\zeta(x) = \ln(x+1)$. The numerical results for the exact solution of this example for $\alpha = 1$ and approximate solutions for $\alpha = 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0$, $k = 2$, $M = 7$ are shown in Fig. 1.

Example 2. We assume the following fractional integro-differential equations of Bratu-type:

$$\begin{cases} D_{0_+}^{\alpha} \zeta(x) + \int_0^x x(t-\frac{1}{2}) \exp(\zeta(t))\, dt = & \frac{\Gamma(3)}{\Gamma(2.5)} x^{1.5} - \frac{\Gamma(2)}{\Gamma(1.5)} x^{0.5} \\ + \frac{x}{2}(\exp(x^2 - x) - 1), & 0 < x, t < 1, \\ \zeta(0) = 0, & 0 < \alpha \le 1. \end{cases} \quad (5.2)$$

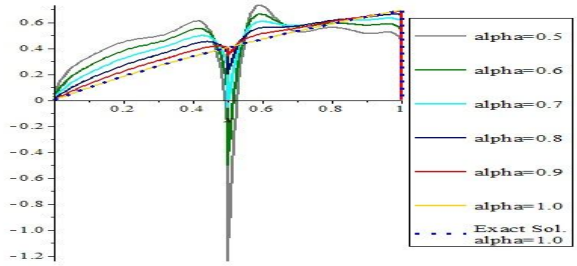

FIGURE 1. Comparison between the exact solution for $\alpha = 1$ and approximate solutions for $\alpha = 0.2,\ 0.3,\ 0.4\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ k = 2,\ M = 7$ for Example 1.

The exact solution for $\alpha = 0.5$ is $\zeta(x) = x^2 - x$. The numerical results for the exact solution of this example for $\alpha = 0.5$ and approximate solutions for $\alpha = 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ k = 2,\ M = 7$ are shown in Fig. 2.

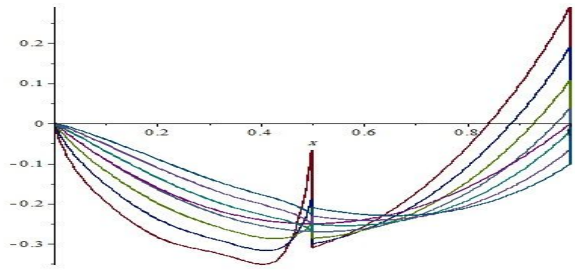

FIGURE 2. Comparison between the exact solution for $\alpha = 0.5$ and approximate solutions for $\alpha = 0.2,\ 0.3,\ 0.4\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ k = 2,\ M = 7$ for Example 2.

Example 3. We consider the following fractional integro-differential equations of Bratu-type:

$$\begin{cases} D_{0_+}^{0.75}\zeta(x) + \frac{1}{4}\int_0^x (x-t)\exp(\zeta(t))\,dt = \frac{8\sqrt[4]{x}}{\Gamma(0.25)} + \frac{\exp(2x)-2x-1}{16}, \quad 0 < x,\ t < 1, \\ \zeta(0) = 0. \end{cases} \tag{5.3}$$

The exact solution for $\alpha = 0.75$ is $\zeta(x) = 2x$. The numerical results for the exact solution of this example for $\alpha = 0.75$ and approximate solutions for $\alpha = 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0$, $k = 2,\ M = 7$ are shown in Fig. 3.

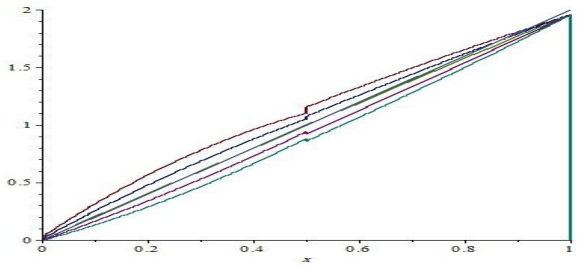

FIGURE 3. Comparison between the exact solution for $\alpha = 0.5$ and approximate solutions for $\alpha = 0.2,\ 0.3,\ 0.4\ 0.5,\ 0.6,\ 0.7,\ 0.8$, $k = 2,\ M = 7$ for Example 3

6. CONCLUDING REMARKS

The aim of this paper is to improve an efficient and precise method for solving non-linear fractional integro-differential equations of Bratu-type. In this paper, in addition to use the properties of the Legendre wavelets, we use the Gauss quadrature rules. Then, we converted the problem into a system of nonlinear algebraic equations. Finally, we applied the Newton iteration method for solving the resulted system. Examples are provided to explain and interpret the simplicity and applicability of this technique.

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REPRESENTATION OF SOLUTIONS OF A TWO-DIMENSIONAL SYSTEM OF DIFFERENCE EQUATIONS

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Abstract. In this paper we give a representation formula for the general solution to the following two-dimensional system of difference equations

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{y_n(a+by_{n-1}x_{n-2})}, \quad y_{n+1} = \frac{x_{n-1}y_{n-2}}{x_n(a+bx_{n-1}y_{n-2})}, \quad n \in \mathbb{N}_0$$

where parameters a, b and initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ are real numbers. We also give some theoretical explanations related to the representation.

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Keywords: system of difference equations, general solution, representation of solutions

1. Introduction

Solvability of difference equations and system of difference equations has attracted considerable interest recently (see, for example [1–19], and the related references therein).

The following four systems of difference equations

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{y_n(\pm 1 \pm y_{n-1}x_{n-2})}, \quad y_{n+1} = \frac{x_{n-1}y_{n-2}}{x_n(\pm 1 \pm x_{n-1}y_{n-2})}, \quad n \in \mathbb{N}_0 \tag{1.1}$$

have been studied in [5], where some closed-form formulas for their solutions are given in terms of the initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$. The closed-form formulas are given and proved by using the method of induction.

In this work we give an alternative proof in order to explain theoretically the results presented in [5], which were established through a mere application of the induction principle.

Here we consider the following extension of the systems in (1.1)

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{y_n(a+by_{n-1}x_{n-2})}, \quad y_{n+1} = \frac{x_{n-1}y_{n-2}}{x_n(a+bx_{n-1}y_{n-2})}, \quad n \in \mathbb{N}_0 \tag{1.2}$$

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where parameters a,b and initial values $x_{-2},x_{-1},x_0,y_{-2},y_{-1},y_0$ are real numbers.

Our objective is to show that system (1.2) is solvable by finding its closed-form formulas through an analytical approach, and to show that all the closed-form formulas obtained in [5] easily follow from the ones in our present paper.

2. Main results

Assume that $\{x_n,y_n\}_{n\geq -2}$ is a well-defined solution to system (1.2).

Let

$$u_n = x_n y_{n-1}, \quad v_n = y_n x_{n-1}, \tag{2.1}$$

for $n \geq -1$. Then system (1.2) can be written as

$$u_{n+1} = \frac{v_{n-1}}{a+bv_{n-1}}, \quad v_{n+1} = \frac{u_{n-1}}{a+bu_{n-1}}, \quad n \in \mathbb{N}_0. \tag{2.2}$$

To give a closed-form for the well-defined solutions of the system (2.2), we consider the system of two difference equations of first order

$$u_{n+1} = \frac{v_n}{a+bv_n}, \quad v_{n+1} = \frac{u_n}{a+bu_n}, \quad n \in \mathbb{N}_0. \tag{2.3}$$

The system (2.3) can be written as the following equation

$$u_{n+1} = \frac{u_{n-1}}{a^2+b(a+1)u_{n-1}}. \tag{2.4}$$

Let

$$u_n^{(j)} = u_{2n-j}, \quad n \in \mathbb{N}, j \in \{0,1\}. \tag{2.5}$$

Using notation (2.5), we can write (2.4) as

$$u_{n+1}^{(j)} = \frac{u_n^{(j)}}{a^2+b(a+1)u_n^{(j)}}, \tag{2.6}$$

where $j \in \{0,1\}$.

Equation (2.6) can be reduced to the equation :

$$\mathcal{H}_{n+1} = \frac{(a^2+1)\mathcal{H}_n - a^2}{\mathcal{H}_n}, \tag{2.7}$$

by using the change of variable

$$u_n^{(j)} = \frac{1}{b(a+1)}\left(\mathcal{H}_n - a^2\right). \tag{2.8}$$

Now we consider the difference equation (2.7) with the initial value $\mathcal{H}_0$ is non zero real number.

Through an analytical approach, we put

$$\mathcal{H}_n = \frac{k_n}{k_{n-1}}. \tag{2.9}$$

Then equation (2.7) becomes

$$k_{n+1}-(a^2+1)k_n-a^2k_{n-1}=0, \quad n\in\mathbb{N}_0. \tag{2.10}$$

Case $a^2\neq 1$*:*

Let $\{k_n\}_{n\geq -1}$ be the solution to equation (2.10) such that k_0 and $k_{-1}\in\mathbb{R}$. The zeros of the characteristic polynomial $P(\lambda)=\lambda^2-(a^2+1)\lambda+a^2$ are $\lambda_1=a^2$ and $\lambda_2=1$. Then the general solution to equation (2.10) can be written in the following form

$$k_n=c_1+c_2a^{2n}.$$

Using the initial values k_0 and k_{-1} with some calculations, we get

$$c_1=\frac{k_0-k_{-1}a^2}{1-a^2},$$
$$c_2=\frac{a^2(k_{-1}-k_0)}{1-a^2}.$$

So the general solution of equation (2.10) is

$$k_n=\frac{1}{1-a^2}\left[k_0\left(1-a^{2(n+1)}\right)-a^2k_{-1}\left(1-a^{2n}\right)\right]. \tag{2.11}$$

From all above mentioned we see that the following theorem holds.

Theorem 1. *Let* $\{\mathcal{H}_n\}_{n\geq 0}$ *be a well-defined solution to the equation* (2.7)*. Then, for* $n=2,3,\ldots,$

$$\mathcal{H}_n=\frac{A(1-a^{2n})-\mathcal{H}_0(1-a^{2(n+1)})}{a^2(1-a^{2(n-1)})-\mathcal{H}_0(1-a^{2n})}. \tag{2.12}$$

Then, from (2.8) we see that

$$u_n^{(j)}=\frac{1}{b(a+1)}\left(\mathcal{H}_n-a^2\right)=\frac{u_0^{(j)}}{a^{2n}+b(a+1)u_0^{(j)}\sum_{r=0}^{n-1}a^{2r}},$$

for each $j\in\{0,1\}$.

Case $a^2=1$ *:*

Then equation (2.7) becomes

$$k_{n+1}-2k_n-k_{n-1}=0, \quad n\in\mathbb{N}_0, \tag{2.13}$$

Let $\{k_n\}_{n\geq -1}$ be the solution to equation (2.13) such that k_0 and $k_{-1}\in\mathbb{R}$. The zero of the characteristic polynomial $P(\lambda)=(\lambda-1)^2$ is $\lambda_1=1$. Then the general solution to equation (2.13) can be written in the following form

$$k_n=c_1+c_2n.$$

Using the initial values k_0 and k_{-1} with some calculations we get

$$c_1 = k_0$$
$$c_2 = k_0 - k_{-1}.$$

So the general solution of equation (2.13) is

$$k_n = k_0(n+1) + k_{-1}n. \tag{2.14}$$

From all above mentioned we see that the following theorem holds.

Theorem 2. *Let $\{\mathcal{H}_n\}_{n\geq 0}$ be a well-defined solution to the equation (2.7). Then, for $n = 2, 3, \ldots,$*

$$\mathcal{H}_n = \frac{n - \mathcal{H}_0(n+1)}{(n-1) - \mathcal{H}_0 n}. \tag{2.15}$$

Then, from (2.8) we see that

$$u_n^{(j)} = \frac{u_0^{(j)}}{1 + b(a+1)u_0^{(j)}n} \tag{2.16}$$

where $j \in \{0,1\}$.

From all above mentioned with using (2.5) we see that the following corollary holds.

Corollary 1. *Let $\{u_n\}_{n\geq -1}$ be a well-defined solution to the equation (2.4). Then*

$$\textbf{\textit{if}}\ a^2 \neq 1: \qquad u_{2n-j} = \frac{u_{-j}}{a^{2n} + b(a+1)u_{-j}\sum_{r=0}^{n-1} a^{2r}}, \qquad n \in \mathbb{N}_0,$$

$$\textbf{\textit{if}}\ a^2 = 1: \qquad u_{2n-j} = \frac{u_{-j}}{1 + b(a+1)u_{-j}n},$$

where $j \in \{0,1\}$.

Corollary 2. *Let $\{u_n, v_n\}_{n\geq 0}$ be a well-defined solution to the system (2.3). Then* ***if*** $a^2 \neq 1$:

$$u_{2n} = \frac{u_0}{a^{2n} + b(a+1)u_0\sum_{r=0}^{n-1} a^{2r}}, \qquad u_{2n+1} = \frac{v_0}{a^{2n+1} + bv_0\left(a\sum_{r=0}^{n-1} a^{2r} + \sum_{r=0}^{n} a^{2r}\right)},$$

$$v_{2n} = \frac{v_0}{a^{2n} + b(a+1)v_0\sum_{r=0}^{n-1} a^{2r}}, \qquad v_{2n+1} = \frac{u_0}{a^{2n+1} + bu_0\left(a\sum_{r=0}^{n-1} a^{2r} + \sum_{r=0}^{n} a^{2r}\right)}.$$

if $a^2 = 1$:

$$u_{2n} = \frac{u_0}{1 + b(a+1)nu_0}, \qquad u_{2n+1} = \frac{v_0}{a + b((a+1)n+1)v_0},$$

$$v_{2n} = \frac{v_0}{1+b(a+1)nv_0}, \qquad v_{2n+1} = \frac{u_0}{a+b((a+1)n+1)u_0},$$

where $n \in \mathbb{N}_0$.

Proof. Let $\{u_n, v_n\}_{n\geq 0}$ be a solution of system (2.3), so $\{u_n\}_{n\geq -1}$ is a solution of equation (2.6). Then, **if** $a^2 \neq 1$, let

$$u_{2n-1} = \frac{u_{-1}}{a^{2n} + b(a+1)u_{-1}\sum_{r=0}^{n-1} a^{2r}},$$

and

$$v_0 = \frac{u_{-1}}{a+bu_{-1}},$$

so

$$\begin{aligned} u_{2n+1} &= \frac{u_{-1}}{a^{2(n+1)} + b(a+1)\left(\sum_{r=0}^{n} a^{2r}\right)u_{-1}} \\ &= \frac{u_{-1}}{a^{2n+1}(a+bu_{-1}) + b\left(a\sum_{r=0}^{n-1} a^{2r} + \sum_{r=0}^{n} a^{2r}\right)u_{-1}} \\ &= \frac{v_0}{a^{2n+1} + b\left(a\left(\sum_{i=0}^{n-1} a^{2r}\right) + \left(\sum_{i=0}^{n} a^{2r}\right)\right)v_0}. \end{aligned}$$

if $a^2 = 1$, let

$$u_{2n-1} = \frac{u_{-1}}{1+b(a+1)nu_{-1}},$$

and

$$v_0 = \frac{u_{-1}}{a+bu_{-1}},$$

so

$$\begin{aligned} u_{2n+1} &= \frac{u_{-1}}{1+b(a+1)(n+1)u_{-1}} = \frac{u_{-1}}{a^2+b(a+1)(n+1)u_{-1}} \\ &= \frac{u_{-1}}{a(a+bu_{-1})+b((a+1)n+1)u_{-1}} = \frac{v_0}{a+b((a+1)n+1)v_0}. \end{aligned}$$

In the same way, after some calculation and use that

$$v_n = \frac{u_{n-1}}{a+bu_{n-1}},$$

we obtain, **if** $a^2 \neq 1$, that

$$v_{2n} = \frac{v_0}{a^{2n} + b(a+1)v_0 \sum_{r=0}^{n-1} a^{2r}}, \qquad v_{2n+1} = \frac{u_0}{a^{2n+1} + bu_0 \left(a \sum_{r=0}^{n-1} a^{2r} + \sum_{r=0}^{n} a^{2r} \right)},$$

and, **if** $a^2 = 1$, that

$$v_{2n} = \frac{v_0}{1 + b(a+1)nv_0}, \qquad v_{2n+1} = \frac{u_0}{a + b((a+1)n+1)u_0}.$$

□

Go back now to the system (2.2), we using an appropriate transformation reducing this system to the system of first-order difference equations (2.3).

The initial values with the smallest indexes are u_{-k} and v_{-k}. By using (2.2) with $n = 0$, we obtain the values of u_1 and v_1 as follows

$$u_1 = \frac{v_{-1}}{a + bv_{-1}}, \quad v_1 = \frac{u_{-1}}{a + bu_{-1}}.$$

After known the values of u_1 and v_1, by using (2.2) with $n = 2$ we get the values of u_3 and v_3. We have

$$u_3 = \frac{v_1}{a + bv_1}, \quad v_3 = \frac{u_1}{a + bu_1}.$$

The values of u_3 and v_3, by using (2.2) with $n = 4$, leads us to obtain the values of u_5 and v_5. We have

$$u_5 = \frac{v_3}{a + bv_3}, \qquad v_5 = \frac{u_3}{a + bu_3}.$$

$$u_{2m+1} = \frac{v_{2m-1}}{a + bv_{2m-1}}, \quad v_{2m+1} = \frac{u_{2m-1}}{a + bu_{2m-1}}.$$

In the same way, it is shown that the initial values u_{-i} and v_{-i}, for a fixed $i \in \{0,1\}$, determine all the values of the sequences $(u_{2(m+1)-i})_m$ and $(v_{2(m+1)-i})_m$. Also we have

$$u_{2(m+1)-i} = \frac{v_{2m-i}}{a + bv_{2m-i}}, \quad v_{2(m+1)-i} = \frac{u_{2m-i}}{a + bu_{2m-i}}. \tag{2.17}$$

Let

$$u_n^{(i)} = u_{2n-i}, \quad v_n^{(i)} = v_{2n-i}. \tag{2.18}$$

Using notation (2.18), we can write (2.2) as

$$u_{n+1}^{(i)} = \frac{v_n^{(i)}}{a + bv_n^{(i)}}, \quad v_{n+1}^{(i)} = \frac{u_n^{(i)}}{a + bu_n^{(i)}}.$$

From all above mentioned we see that the following theorem holds.

Theorem 3. *Let $\{u_n, v_n\}_{n\geq -1}$ be a well-defined solution to the system (2.2). Then, for $n = 2, 3, \ldots,$*
if $a^2 \neq 1$:

$$u_{4n-1} = \frac{u_{-1}}{a^{2n} + b(a+1)\sum_{r=0}^{n-1} a^{2r} u_{-1}}, \qquad v_{4n-1} = \frac{v_{-1}}{a^{2n} + b(a+1)\sum_{r=0}^{n-1} a^{2r} v_{-1}},$$

$$u_{4n} = \frac{u_0}{a^{2n} + b(a+1)\sum_{r=0}^{n-1} a^{2r} u_0}, \qquad v_{4n} = \frac{v_0}{a^{2n} + b(a+1)\sum_{r=0}^{n-1} a^{2r} v_0},$$

$$u_{4n+1} = \frac{v_{-1}}{a^{2n+1} + b\left(a\sum_{r=0}^{n-1} a^{2r} + \sum_{r=0}^{n} a^{2r}\right) v_{-1}}, \qquad v_{4n+1} = \frac{u_{-1}}{a^{2n+1} + b\left(a\sum_{r=0}^{n-1} a^{2r} + \sum_{r=0}^{n} a^{2r}\right) u_{-1}},$$

$$u_{4n+2} = \frac{v_0}{a^{2n+1} + b\left(a\sum_{r=0}^{n-1} a^{2r} + \sum_{r=0}^{n} a^{2r}\right) v_0}, \qquad v_{4n+2} = \frac{u_0}{a^{2n+1} + b\left(a\sum_{r=0}^{n-1} a^{2r} + \sum_{r=0}^{n} a^{2r}\right) u_0}.$$

if $a^2 = 1$:

$$u_{4n-1} = \frac{u_{-1}}{1 + b(a+1)nu_{-1}}, \qquad v_{4n-1} = \frac{v_{-1}}{1 + b(a+1)nv_{-1}},$$

$$u_{4n} = \frac{u_0}{1 + b(a+1)nu_0}, \qquad v_{4n} = \frac{v_0}{1 + b(a+1)nv_0},$$

$$u_{4n+1} = \frac{v_{-1}}{a + b((a+1)n+1)v_{-1}}, \qquad v_{4n+1} = \frac{u_{-1}}{a + b((a+1)n+1)u_{-1}},$$

$$u_{4n+2} = \frac{v_0}{a + b((a+1)n+1)v_0}, \qquad v_{4n+2} = \frac{u_0}{a + b((a+1)n+1)u_0}.$$

where $n \in \mathbb{N}_0$.

From (2.1) we have

$$x_n = \frac{u_n}{y_{n-1}}, \tag{2.19}$$

$$y_n = \frac{v_n}{x_{n-1}}. \tag{2.20}$$

Using (2.20) in (2.19), we obtain

$$x_{4n} = \frac{u_{4n}u_{4n-2}}{v_{4n-1}v_{4n-3}} x_{4n-4}. \tag{2.21}$$

Using (2.19) in (2.20), we obtain

$$y_{4n} = \frac{v_{4n}v_{4n-2}}{u_{4n-1}u_{4n-3}} y_{4n-4}. \tag{2.22}$$

For $n \in \mathbb{N}$, multiplying the equalities which are obtained form (2.21) and (2.22) from 1 to n, respectively, it follows that

$$x_{4n} = x_0 \prod_{i=0}^{n-1} \left(\frac{u_{4i} u_{4i-2}}{v_{4i-1} v_{4i-3}} \right), \tag{2.23}$$

$$y_{4n} = y_0 \prod_{i=0}^{n-1} \left(\frac{v_{4i} v_{4i-2}}{u_{4i-1} u_{4i-3}} \right). \tag{2.24}$$

Using the equalities (2.23) and (2.24) in (2.19) and (2.20), we obtain

$$x_{4n-1} = \frac{v_{6n}}{y_{6n}} = \frac{v_{4n}}{y_0} \prod_{i=0}^{n-1} \left(\frac{u_{4i-1} u_{4i-3}}{v_{4i} v_{4i-2}} \right). \tag{2.25}$$

We have

$$y_{4n-1} = \frac{u_{4n}}{x_{4n}} = \frac{u_{4n}}{x_0} \prod_{i=0}^{n-1} \left(\frac{v_{4i-1} v_{4i-3}}{u_{4i} u_{4i-2}} \right). \tag{2.26}$$

Using the equalities (2.25) and (2.26) in (2.19) and (2.20), we obtain

$$x_{4n-2} = \frac{v_{4n-1}}{y_{4n-1}} = x_0 \frac{v_{4n-1}}{u_{4n}} \prod_{i=0}^{n-1} \left(\frac{u_{4i} u_{4i-2}}{v_{4i-1} v_{4i-3}} \right), \tag{2.27}$$

and

$$y_{4n-2} = \frac{u_{4n-1}}{x_{4n-1}} = y_0 \frac{u_{4n-1}}{v_{4n}} \prod_{i=0}^{n-1} \left(\frac{v_{4i} v_{4i-2}}{u_{4i-1} u_{4i-3}} \right). \tag{2.28}$$

Using the equalities (2.27) and (2.28) in (2.19) and (2.20), we obtain

$$x_{4n+1} = \frac{u_{4n+1}}{y_{4n}} = \frac{u_{4n+1}}{y_0} \prod_{i=0}^{n-1} \left(\frac{u_{4i-1} u_{4i-3}}{v_{4i} v_{4i-2}} \right), \tag{2.29}$$

and

$$y_{4n+1} = \frac{v_{4n+1}}{x_{4n}} = \frac{v_{4n-1}}{x_0} \prod_{i=0}^{n-1} \left(\frac{v_{4i-1} v_{4i-3}}{u_{4i} u_{4i-2}} \right). \tag{2.30}$$

Using Theorem (3) we get
if $a^2 \neq 1$:

$$u_{4n-3} = \frac{v_{-1}}{a^{2n-1} + b \left(a \sum_{r=0}^{n-2} a^{2r} + \sum_{r=0}^{n-1} a^{2r} \right) v_{-1}}, \quad v_{4n-3} = \frac{u_{-1}}{a^{2n-1} + b \left(a \sum_{r=0}^{n-2} a^{2r} + \sum_{r=0}^{n-1} a^{2r} \right) u_{-1}},$$

$$u_{4n-2} = \frac{v_0}{a^{2n-1} + b \left(a \sum_{r=0}^{n-2} a^{2r} + \sum_{r=0}^{n-1} a^{2r} \right) v_0}, \quad v_{4n-2} = \frac{u_0}{a^{2n-1} + b \left(a \sum_{r=0}^{n-2} a^{2r} + \sum_{r=0}^{n-1} a^{2r} \right) u_0},$$

$$u_{4n-1}=\frac{u_{-1}}{a^{2n}+b(a+1)\sum_{r=0}^{n-1}a^{2r}u_{-1}},\qquad v_{4n-1}=\frac{v_{-1}}{a^{2n}+b(a+1)\sum_{r=0}^{n-1}a^{2r}v_{-1}},$$
$$u_{4n}=\frac{u_{0}}{a^{2n}+b(a+1)\sum_{r=0}^{n-1}a^{2r}u_{0}},\qquad v_{4n}=\frac{v_{0}}{a^{2n}+b(a+1)\sum_{r=0}^{n-1}a^{2r}v_{0}}.$$

if $a^2=1$:

$$u_{4n-3}=\frac{v_{-1}}{a+b((a+1)n-a)v_{-1}},\qquad v_{4n-3}=\frac{u_{-1}}{a+b((a+1)n-a)u_{-1}},$$
$$u_{4n-2}=\frac{v_{0}}{a+b((a+1)n-a)v_{0}},\qquad v_{4n-2}=\frac{u_{0}}{a+b((a+1)n-a)u_{0}},$$
$$u_{4n-1}=\frac{u_{-1}}{1+b(a+1)nu_{-1}},\qquad v_{4n-1}=\frac{v_{-1}}{1+b(a+1)nv_{-1}},$$
$$u_{4n}=\frac{u_{0}}{1+b(a+1)nu_{0}},\qquad v_{4n}=\frac{v_{0}}{1+b(a+1)nv_{0}}.$$

From all above mentioned and

$$u_{-1}=x_{-1}y_{-2},\quad u_0=x_0y_{-1},\quad v_{-1}=y_{-1}x_{-2},\quad v_0=y_0x_{-1}. \tag{2.31}$$

we see that the following result holds.

Theorem 4. *Let $\{x_n,y_n\}_{n\geq -2}$ be a well-defined solution to the system (1.2). Then, for $n=0,1,2,3,\ldots,$*
if $a^2\neq 1$:

$$x_{4n-2}=\prod_{i=0}^{n-1}\left(\frac{\left(a^{2n}+b(a+1)\sum_{r=0}^{n-1}a^{2r}y_{-1}x_{-2}\right)\left(a^{2n-1}+b\left(a\sum_{r=0}^{n-2}a^{2r}+\sum_{r=0}^{n-1}a^{2r}\right)x_{-1}y_{-2}\right)}{\left(a^{2n}+b(a+1)\sum_{r=0}^{i-1}a^{2r}x_0y_{-1}\right)\left(a^{2i-1}+b\left(a\sum_{r=0}^{i-2}a^{2r}+\sum_{r=0}^{i-1}a^{2r}\right)y_0x_{-1}\right)}\right)$$
$$\times\frac{x_0^ny_0^n}{y_{-2}^nx_{-2}^{n-1}}\left(\frac{a^{2n}+b(a+1)\sum_{r=0}^{n-1}a^{2r}x_0y_{-1}}{a^{2n}+b(a+1)\sum_{r=0}^{n-1}a^{2r}y_{-1}x_{-2}}\right),$$

$$x_{4n-1}=\prod_{i=0}^{n-1}\left(\frac{\left(a^{2n}+b(a+1)\sum_{r=0}^{i-1}a^{2r}y_0x_{-1}\right)\left(a^{2n-1}+b\left(a\sum_{r=0}^{i-2}a^{2r}+\sum_{r=0}^{i-1}a^{2r}\right)x_0y_{-1}\right)}{\left(a^{2n}+b(a+1)\sum_{r=0}^{i-1}a^{2r}x_{-1}y_{-2}\right)\left(a^{2n-1}+b\left(a\sum_{r=0}^{i-2}a^{2r}+\sum_{r=0}^{i-1}a^{2r}\right)y_{-1}x_{-2}\right)}\right)$$

$$\times \frac{x_{-1}y_{-2}^{n}x_{-2}^{n}}{x_0^n y_0^n}\left(\frac{1}{a^{2n}+b(a+1)\sum_{r=0}^{n-1}a^{2r}y_0x_{-1}}\right),$$

$$x_{4n}=\frac{x_0^{n+1}y_0^n}{y_{-2}^n x_{-2}^n}$$

$$\prod_{i=0}^{n-1}\left(\frac{\left(a^{2n}+b(a+1)\sum_{r=0}^{n-1}a^{2r}y_{-1}x_{-2}\right)\left(a^{2n-1}+b\left(a\sum_{r=0}^{n-2}a^{2r}+\sum_{r=0}^{n-1}a^{2r}\right)x_{-1}y_{-2}\right)}{\left(a^{2n}+b(a+1)\sum_{r=0}^{i-1}a^{2r}x_0y_{-1}\right)\left(a^{2i-1}+b\left(a\sum_{r=0}^{i-2}a^{2r}+\sum_{r=0}^{i-1}a^{2r}\right)y_0x_{-1}\right)}\right),$$

$$x_{4n+1}=\prod_{i=0}^{n-1}\left(\frac{\left(a^{2n}+b(a+1)\sum_{r=0}^{i-1}a^{2r}y_0x_{-1}\right)\left(a^{2n-1}+b\left(a\sum_{r=0}^{i-2}a^{2r}+\sum_{r=0}^{i-1}a^{2r}\right)x_0y_{-1}\right)}{\left(a^{2n}+b(a+1)\sum_{r=0}^{i-1}a^{2r}x_{-1}y_{-2}\right)\left(a^{2n-1}+b\left(a\sum_{r=0}^{i-2}a^{2r}+\sum_{r=0}^{i-1}a^{2r}\right)y_{-1}x_{-2}\right)}\right)$$

$$\times \frac{y_{-1}y_{-2}^{n}x_{-2}^{n+1}}{x_0^n y_0^{n+1}}\left(\frac{1}{a^{2n+1}+b\left(a\sum_{r=0}^{n-1}a^{2r}+\sum_{r=0}^{n}a^{2r}\right)y_{-1}x_{-2}}\right),$$

$$y_{4n-2}=\prod_{i=0}^{n-1}\left(\frac{\left(a^{2n}+b(a+1)\sum_{r=0}^{n-1}a^{2r}y_{-2}x_{-1}\right)\left(a^{2n-1}+b\left(a\sum_{r=0}^{n-2}a^{2r}+\sum_{r=0}^{n-1}a^{2r}\right)x_{-2}y_{-1}\right)}{\left(a^{2n}+b(a+1)\sum_{r=0}^{i-1}a^{2r}x_{-1}y_0\right)\left(a^{2i-1}+b\left(a\sum_{r=0}^{i-2}a^{2r}+\sum_{r=0}^{i-1}a^{2r}\right)y_{-1}x_0\right)}\right)$$

$$\times \frac{x_0^n y_0^n}{y_{-2}^{n-1}x_{-2}^n}\left(\frac{a^{2n}+b(a+1)\sum_{r=0}^{n-1}a^{2r}x_{-1}y_0}{a^{2n}+b(a+1)\sum_{r=0}^{n-1}a^{2r}y_{-2}x_{-1}}\right),$$

$$y_{4n-1}=\prod_{i=0}^{n-1}\left(\frac{\left(a^{2n}+b(a+1)\sum_{r=0}^{i-1}a^{2r}y_{-1}x_0\right)\left(a^{2n-1}+b\left(a\sum_{r=0}^{i-2}a^{2r}+\sum_{r=0}^{i-1}a^{2r}\right)x_{-1}y_0\right)}{\left(a^{2n}+b(a+1)\sum_{r=0}^{i-1}a^{2r}x_{-2}y_{-1}\right)\left(a^{2n-1}+b\left(a\sum_{r=0}^{i-2}a^{2r}+\sum_{r=0}^{i-1}a^{2r}\right)y_{-2}x_{-1}\right)}\right)$$

$$\times \frac{y_{-1}y_{-2}^{n}x_{-2}^{n}}{x_0^n y_0^n}\left(\frac{1}{a^{2n}+b(a+1)\sum_{r=0}^{n-1}a^{2r}y_{-1}x_0}\right),$$

$$y_{4n} = \frac{x_0^n y_0^{n+1}}{y_{-2}^n x_{-2}^n}$$
$$\prod_{i=0}^{n-1}\left(\frac{\left(a^{2n}+b(a+1)\sum_{r=0}^{n-1}a^{2r}y_{-2}x_{-1}\right)\left(a^{2n-1}+b\left(a\sum_{r=0}^{n-2}a^{2r}+\sum_{r=0}^{n-1}a^{2r}\right)x_{-2}y_{-1}\right)}{\left(a^{2n}+b(a+1)\sum_{r=0}^{i-1}a^{2r}x_{-1}y_0\right)\left(a^{2i-1}+b\left(a\sum_{r=0}^{i-2}a^{2r}+\sum_{r=0}^{i-1}a^{2r}\right)y_{-1}x_0\right)}\right),$$

$$y_{4n+1} = \prod_{i=0}^{n-1}\left(\frac{\left(a^{2n}+b(a+1)\sum_{r=0}^{i-1}a^{2r}y_{-1}x_0\right)\left(a^{2n-1}+b\left(a\sum_{r=0}^{i-2}a^{2r}+\sum_{r=0}^{i-1}a^{2r}\right)x_{-1}y_0\right)}{\left(a^{2n}+b(a+1)\sum_{r=0}^{i-1}a^{2r}x_{-2}y_{-1}\right)\left(a^{2n-1}+b\left(a\sum_{r=0}^{i-2}a^{2r}+\sum_{r=0}^{i-1}a^{2r}\right)y_{-2}x_{-1}\right)}\right)$$
$$\times\frac{x_{-1}y_{-2}^{n+1}x_{-2}^n}{x_0^{n+1}y_0^n}\left(\frac{1}{a^{2n+1}+b\left(a\sum_{r=0}^{n-1}a^{2r}+\sum_{r=0}^{n}a^{2r}\right)y_{-2}x_{-1}}\right).$$

if $a^2 = 1$:

$$x_{4n-2} = \prod_{i=0}^{n-1}\left(\frac{(1+b(a+1)iy_{-1}x_{-2})\,(a+b((a+1)i-a)x_{-1}y_{-2})}{(1+b(a+1)ix_0y_{-1})\,(a+b((a+1)i-a)y_0x_{-1})}\right).$$
$$\times\frac{x_0^n y_0^n}{y_{-2}^n x_{-2}^{n-1}}\left(\frac{1+b(a+1)nx_0y_{-1}}{1+b(a+1)nx_{-2}y_{-1}}\right),$$
$$x_{4n-1} = \prod_{i=0}^{n-1}\left(\frac{(1+b(a+1)iy_0x_{-1})\,(a+b((a+1)i-a)x_0y_{-1})}{(1+b(a+1)ix_{-1}y_{-2})\,(a+b((a+1)i-a)y_{-1}x_{-2})}\right)$$
$$\times\frac{x_{-1}y_{-2}^n x_{-2}^n}{x_0^n y_0^n}\left(\frac{1}{1+b(a+1)ny_0x_{-1}}\right),$$
$$x_{4n} = \frac{x_0^{n+1}y_0^n}{y_{-2}^n x_{-2}^n}\prod_{i=0}^{n-1}\left(\frac{(1+b(a+1)iy_{-1}x_{-2})\,(a+b((a+1)i-a)x_{-1}y_{-2})}{(1+b(a+1)ix_0y_{-1})\,(a+b((a+1)i-a)y_0x_{-1})}\right),$$
$$x_{4n+1} = \prod_{i=0}^{n-1}\left(\frac{(1+b(a+1)iy_0x_{-1})\,(a+b((a+1)i-a)x_0y_{-1})}{(1+b(a+1)ix_{-1}y_{-2})\,(a+b((a+1)i-a)y_{-1}x_{-2})}\right)$$
$$\times\frac{y_{-1}}{x_0^n y_0^{n+1}}\left(\frac{y_{-2}^n x_{-2}^{n+1}}{a+b((a+1)n+1)y_{-1}x_{-2}}\right),$$
$$y_{4n-2} = \frac{x_0^n y_0^n}{y_{-2}^{n-1}x_{-2}^n}\left(\frac{1+b(a+1)nx_{-1}y_0}{1+b(a+1)ny_{-2}x_{-1}}\right)$$

$$\prod_{i=0}^{n-1}\left(\frac{(1+b(a+1)iy_{-2}x_{-1})\,(a+b((a+1)i-a)x_{-2}y_{-1})}{(1+b(a+1)ix_{-1}y_0)\,(a+b((a+1)i-a)y_{-1}x_0)}\right),$$

$$y_{4n-1}=\frac{y_{-1}y_{-2}^{n}x_{-2}^{n}}{x_0^n y_0^n}\left(\frac{1}{1+b(a+1)ny_{-1}x_0}\right)$$

$$\prod_{i=0}^{n-1}\left(\frac{(1+b(a+1)iy_{-1}x_0)\,(a+b((a+1)i-a)x_{-1}y_0)}{(1+b(a+1)ix_{-2}y_{-1})\,(a+b((a+1)i-a)y_{-2}x_{-1})}\right),$$

$$y_{4n}=\frac{x_0^n y_0^{n+1}}{y_{-2}^n x_{-2}^n}\prod_{i=0}^{n-1}\left(\frac{(1+b(a+1)iy_{-2}x_{-1})\,(a+b((a+1)i-a)x_{-2}y_{-1})}{(1+b(a+1)ix_{-1}y_0)\,(a+b((a+1)i-a)y_{-1}x_0)}\right),$$

$$y_{4n+1}=\prod_{i=0}^{n-1}\left(\frac{(1+b(a+1)iy_{-1}x_0)\,(a+b((a+1)i-a)x_{-1}y_0)}{(1+b(a+1)ix_{-2}y_{-1})\,(a+b((a+1)i-a)y_{-2}x_{-1})}\right)$$

$$\times\frac{x_{-1}}{x_0^{n+1}y_0^n}\left(\frac{y_{-2}^{n+1}x_{-2}^n}{a+b((a+1)n+1)y_{-2}x_{-1}}\right).$$

3. Some applications

As some applications we show how are obtained closed-form formulas for solutions to the systems in (1.1), which were presented in [5].

First result proved in [5] is the following.

Corollary 3. *Let $\{x_n,y_n\}_{n\geq -2}$ be a well-defined solution to the following system*

$$x_{n+1}=\frac{y_{n-1}x_{n-2}}{y_n(1+y_{n-1}x_{n-2})},\quad y_{n+1}=\frac{x_{n-1}y_{n-2}}{x_n(1+x_{n-1}y_{n-2})},\quad n\in\mathbb{N}_0. \tag{3.1}$$

Then

$$x_{4n-2}=\frac{x_0^n y_0^n}{y_{-2}^n x_{-2}^{n-1}}\left(\frac{1+2nx_0y_{-1}}{1+2nx_{-2}y_{-1}}\right)\prod_{i=0}^{n-1}\left(\frac{(1+2iy_{-1}x_{-2})\,(1+(2i-1)x_{-1}y_{-2})}{(1+2ix_0y_{-1})\,(1+(2i-a)y_0x_{-1})}\right),$$

$$x_{4n-1}=\frac{x_{-1}y_{-2}^n x_{-2}^n}{x_0^n y_0^n}\left(\frac{1}{1+2ny_0x_{-1}}\right)\prod_{i=0}^{n-1}\left(\frac{(1+2iy_0x_{-1})\,(1+(2i-a)x_0y_{-1})}{(1+2ix_{-1}y_{-2})\,(1+(2i-1)y_{-1}x_{-2})}\right),$$

$$x_{4n}=\frac{x_0^{n+1}y_0^n}{y_{-2}^n x_{-2}^n}\prod_{i=0}^{n-1}\left(\frac{(1+2iy_{-1}x_{-2})\,(1+(2i-a)x_{-1}y_{-2})}{(1+2ix_0y_{-1})\,(1+(2i-a)y_0x_{-1})}\right),$$

$$x_{4n+1}=\frac{y_{-1}}{x_0^n y_0^{n+1}}\left(\frac{y_{-2}^n x_{-2}^{n+1}}{1+(2n+1)y_{-1}x_{-2}}\right)\prod_{i=0}^{n-1}\left(\frac{(1+2iy_0x_{-1})\,(1+((2i-1)x_0y_{-1})}{(1+2ix_{-1}y_{-2})\,(1+(2i-1)y_{-1}x_{-2})}\right),$$

and

$$y_{4n-2}=\frac{x_0^n y_0^n}{y_{-2}^{n-1}x_{-2}^n}\left(\frac{1+2nx_{-1}y_0}{1+2ny_{-2}x_{-1}}\right)\prod_{i=0}^{n-1}\left(\frac{(1+2iy_{-2}x_{-1})\,(1+(2i-1)x_{-2}y_{-1})}{(1+2ix_{-1}y_0)\,(1+(2i-a)y_{-1}x_0)}\right),$$

$$y_{4n-1}=\frac{y_{-1}y_{-2}^{n}x_{-2}^{n}}{x_{0}^{n}y_{0}^{n}}\left(\frac{1}{1+2iy_{-1}x_{0}}\right)\prod_{i=0}^{n-1}\left(\frac{(1+2iy_{-1}x_{0})(1+(2i-1)x_{-1}y_{0})}{(1+2ix_{-2}y_{-1})(1+(2i-1)y_{-2}x_{-1})}\right),$$

$$y_{4n}=\frac{x_{0}^{n}y_{0}^{n+1}}{y_{-2}^{n}x_{-2}^{n}}\prod_{i=0}^{n-1}\left(\frac{(1+2iy_{-2}x_{-1})(1+(2i-a)x_{-2}y_{-1})}{(1+2ix_{-1}y_{0})(1+(2i-1)y_{-1}x_{0})}\right),$$

$$y_{4n+1}=\frac{x_{-1}}{x_{0}^{n+1}y_{0}^{n}}\left(\frac{y_{-2}^{n+1}x_{-2}^{n}}{1+(2n+1)y_{-2}x_{-1}}\right)\prod_{i=0}^{n-1}\left(\frac{(1+2iy_{-1}x_{0})(1+(2i-1)x_{-1}y_{0})}{(1+2ix_{-2}y_{-1})(1+(2i-1)y_{-2}x_{-1})}\right).$$

Proof. System (3.1) is obtained from system (1.2) with $a=b=1$, so by using Theorem (4) corollary (3) follows. □

The following corollary is Theorem 2.2 in [5].

Corollary 4. *Let $\{x_n,y_n\}_{n\geq -2}$ be a well-defined solution to the following system*

$$x_{n+1}=\frac{y_{n-1}x_{n-2}}{y_n(1-y_{n-1}x_{n-2})},\quad y_{n+1}=\frac{x_{n-1}y_{n-2}}{x_n(1-x_{n-1}y_{n-2})},\quad n\in\mathbb{N}_0. \tag{3.2}$$

Then

$$x_{4n-2}=\frac{x_{0}^{n}y_{0}^{n}}{y_{-2}^{n}x_{-2}^{n-1}}\left(\frac{1-2nx_{0}y_{-1}}{1-2nx_{-2}y_{-1}}\right)\prod_{i=0}^{n-1}\left(\frac{(1-2iy_{-1}x_{-2})(1-(2i-1)x_{-1}y_{-2})}{(1-2ix_{0}y_{-1})(1-(2i-1)y_{0}x_{-1})}\right),$$

$$x_{4n-1}=\frac{x_{-1}y_{-2}^{n}x_{-2}^{n}}{x_{0}^{n}y_{0}^{n}}\left(\frac{1}{1-2ny_{0}x_{-1}}\right)\prod_{i=0}^{n-1}\left(\frac{(1-2iy_{0}x_{-1})(1-(2i-1)x_{0}y_{-1})}{(1-2ix_{-1}y_{-2})(1-(2i-1)y_{-1}x_{-2})}\right),$$

$$x_{4n}=\frac{x_{0}^{n+1}y_{0}^{n}}{y_{-2}^{n}x_{-2}^{n}}\prod_{i=0}^{n-1}\left(\frac{(1-2iy_{-1}x_{-2})(1-(2i-1)x_{-1}y_{-2})}{(1-2ix_{0}y_{-1})(1-(2i-1)y_{0}x_{-1})}\right),$$

$$x_{4n+1}=\frac{y_{-1}}{x_{0}^{n}y_{0}^{n+1}}\left(\frac{y_{-2}^{n}x_{-2}^{n+1}}{1-(2n+1)y_{-1}x_{-2}}\right)\prod_{i=0}^{n-1}\left(\frac{(1-2iy_{0}x_{-1})(1-(2i-1)x_{0}y_{-1})}{(1-ix_{-1}y_{-2})(1-(2i-1)y_{-1}x_{-2})}\right),$$

$$y_{4n-2}=\frac{x_{0}^{n}y_{0}^{n}}{y_{-2}^{n-1}x_{-2}^{n}}\left(\frac{1-2nx_{-1}y_{0}}{1-2ny_{-2}x_{-1}}\right)\prod_{i=0}^{n-1}\left(\frac{(1-2iy_{-2}x_{-1})(1-(2i-1)x_{-2}y_{-1})}{(1-2ix_{-1}y_{0})(1-(2i-1)y_{-1}x_{0})}\right),$$

$$y_{4n-1}=\frac{y_{-1}y_{-2}^{n}x_{-2}^{n}}{x_{0}^{n}y_{0}^{n}}\left(\frac{1}{1-2ny_{-1}x_{0}}\right)\prod_{i=0}^{n-1}\left(\frac{(1-2iy_{-1}x_{0})(1-(2i-1)x_{-1}y_{0})}{(1-2ix_{-2}y_{-1})(1-(2i-1)y_{-2}x_{-1})}\right),$$

$$y_{4n}=\frac{x_{0}^{n}y_{0}^{n+1}}{y_{-2}^{n}x_{-2}^{n}}\prod_{i=0}^{n-1}\left(\frac{(1-2iy_{-2}x_{-1})(1-(2i-1)x_{-2}y_{-1})}{(1-2ix_{-1}y_{0})(1-(2i-1)y_{-1}x_{0})}\right),$$

$$y_{4n+1}=\frac{x_{-1}}{x_{0}^{n+1}y_{0}^{n}}\left(\frac{y_{-2}^{n+1}x_{-2}^{n}}{1-(2n+1)y_{-2}x_{-1}}\right)\prod_{i=0}^{n-1}\left(\frac{(1-2iy_{-1}x_{0})(1-(2i-1)x_{-1}y_{0})}{(1-2ix_{-2}y_{-1})(1-(2i-1)y_{-2}x_{-1})}\right).$$

Proof. System (3.2) is obtained from system (1.2) with $a=1$ and $b=-1$, so by using Theorem (4) corollary (4) follows. □

The following corollary is Theorem 5.3 in [5].

Corollary 5. *Let $\{x_n, y_n\}_{n\geq -2}$ be a well-defined solution to the following system*

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{y_n(-1+y_{n-1}x_{n-2})}, \quad y_{n+1} = \frac{x_{n-1}y_{n-2}}{x_n(-1+x_{n-1}y_{n-2})}, \quad n \in \mathbb{N}_0. \tag{3.3}$$

Then

$$x_{4n-2} = \frac{x_0^n y_0^n (-1+x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^{n-1} (-1+y_0x_{-1})^n}, \qquad y_{4n-2} = \frac{y_0^n x_0^n (-1+y_{-1}x_{-2})^n}{x_{-2}^n y_{-2}^{n-1} (-1+x_0y_{-1})^n},$$

$$x_{4n-1} = \frac{x_{-1}y_{-2}^n x_{-2}^n (-1+x_0y_{-1})^n}{x_0^n y_0^n (-1+y_{-1}x_{-2})^n}, \qquad y_{4n-1} = \frac{y_{-1}x_{-2}^n y_{-2}^n (-1+y_0y_{-1})^n}{y_0^n x_0^n (-1+x_{-1}y_{-2})^n},$$

$$x_{4n} = \frac{x_0^{n+1} y_0^n (-1+x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^n (-1+y_0x_{-1})^n}, \qquad y_{4n} = \frac{y_0^{n+1} x_0^n (-1+y_{-1}x_{-2})^n}{x_{-2}^n y_{-2}^n (-1+x_0y_{-1})^n},$$

$$x_{4n+1} = \frac{y_{-1}y_{-2}^n x_{-2}^{n+1} (-1+x_0y_{-1})^n}{x_0^n y_0^{n+1} (-1+y_{-1}x_{-2})^{n+1}}, \qquad y_{4n+1} = \frac{x_{-1}x_{-2}^n y_{-2}^{n+1} (-1+y_0x_{-1})^n}{y_0^n x_0^{n+1} (-1+x_{-1}y_{-2})^{n+1}}.$$

Proof. System (3.3) is obtained from system (1.2) with $a = 1$ and $b = -1$, so by using Theorem (4) corollary (5) follows. □

The following corollary is Theorem 5.4 in [5].

Corollary 6. *Let $\{x_n, y_n\}_{n\geq -2}$ be a well-defined solution to the following system*

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{y_n(-1-y_{n-1}x_{n-2})}, \quad y_{n+1} = \frac{x_{n-1}y_{n-2}}{x_n(-1-x_{n-1}y_{n-2})}, \quad n \in \mathbb{N}_0. \tag{3.4}$$

Then

$$x_{4n-2} = \frac{x_0^n y_0^n (-1-x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^{n-1} (-1-y_0x_{-1})^n}, \qquad y_{4n-2} = \frac{y_0^n x_0^n (-1-y_{-1}x_{-2})^n}{x_{-2}^n y_{-2}^{n-1} (-1-x_0y_{-1})^n},$$

$$x_{4n-1} = \frac{x_{-1}y_{-2}^n x_{-2}^n (-1-x_0y_{-1})^n}{x_0^n y_0^n (-1-y_{-1}x_{-2})^n}, \qquad y_{4n-1} = \frac{y_{-1}x_{-2}^n y_{-2}^n (-1-y_0y_{-1})^n}{y_0^n x_0^n (-1-x_{-1}y_{-2})^n},$$

$$x_{4n} = \frac{x_0^{n+1} y_0^n (-1-x_{-1}y_{-2})^n}{y_{-2}^n x_{-2}^n (-1-y_0x_{-1})^n}, \qquad y_{4n-1} = \frac{y_{-1}x_{-2}^n y_{-2}^n (-1-y_0y_{-1})^n}{y_0^n x_0^n (-1-x_{-1}y_{-2})^n},$$

$$x_{4n+1} = \frac{y_{-1}y_{-2}^n x_{-2}^{n+1} (-1-x_0y_{-1})^n}{x_0^n y_0^{n+1} (-1-y_{-1}x_{-2})^{n+1}}, \qquad y_{4n+1} = \frac{x_{-1}x_{-2}^n y_{-2}^{n+1} (-1-y_0x_{-1})^n}{y_0^n x_0^{n+1} (-1-x_{-1}y_{-2})^{n+1}}.$$

Proof. System (3.4) is obtained from system (1.2) with $a = b = -1$, so by using Theorem (4) corollary (6) follows.

□

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ANALYTICAL INVESTIGATION FOR MODIFIED RIEMANN-LIOUVILLE FRACTIONAL EQUAL-WIDTH EQUATION TYPES BASED ON $(G'/G)-$ EXPANSION TECHNIQUE

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Abstract. The present investigation studies exact solutions of modified Riemann-Liouville fractional Equal-Width (MRLFEW) equation types with the help of the $(G'/G)-$ expansion method. Firstly, the MRLFEW equation is converted into an ordinary differential equation via fractional complex transform. Then, the proposed method has applied this equation to construct the exact solutions.

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1. Introduction

In recent years, real-world phenomena can be modeled successfully by the partial differential equations with fractional derivatives (FPDEs). These equations have significant applications in various areas such as fluid mechanics, viscoelastic materials, finance, control theory, fractional dynamics, biology, physics, applied mathematics and engineering [1, 3, 4, 6, 8, 10, 13, 18]. Many studies have been introduced by many researchers [5, 7, 15, 17]. To seek exact solutions of the FPDEs, many well-known methods have been proposed such as the exp-function method [16], modified simple method [21], the fractional sub-equation method [2, 23], (G'/G)-expansion method [20, 22].

In this manuscript, we are mainly concerned to explore traveling wave solutions of MRLFEW equation types. So far, various analytical methods have been suggested for the equation types by the references therein [11, 14, 19]. To our knowledge, (G'/G)-expansion technique is not used to generate the traveling wave solutions of MRLFEW equation types. Adding to this, the advantages of this technique are ease of implementation and obtaining new solutions. That's why we implement the technique to construct exact solutions to the mentioned equations. The layout of this work is as follows: In Section 1, a brief of the MRLF derivative is given. The next Section

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2 gives the description of the (G'/G)-expansion methodology. In Section 3, we construct the implementation of the suggested method for the MRLFEW equation types. The results are also illustrated in this section. Finally, we end with a short conclusion in Section 4.

1.1. *A Brief of the MRLF derivative*

Jumarie is expressed $\alpha-$th order fractional derivative in sense of MRL of a given function $y = v(t)$ as

$$D_t^{\alpha} v(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} (v(\xi) - v(0)) d\xi, & 0 < \alpha < 1, \\ \\ (v^{(m)}(t))^{\alpha - m}, & m \leq \alpha < m+1, \quad m \geq 1 \end{cases} \tag{1.1}$$

This definition has some prominent properties. They are given [12] following as

$$D_t^{\alpha} t^p = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} t^{\alpha - p}, \quad p > 0 \tag{1.2}$$

$$D_t^{\alpha}(f(t)g(t)) = f(t) D_t^{\alpha} g(t) + g(t) D_t^{\alpha} f(t). \tag{1.3}$$

2. The (G'/G)- expansion methodology

We outline the basic idea of the (G'/G)- expansion technique. Firstly, a general nonlinear fractional differential equation with two independent variables x and t is regarded as

$$R(v, D_t^{\alpha} v, D_x^{\beta} v, D_t^{\alpha} D_t^{\alpha} v, D_t^{\alpha} D_x^{\beta} v, D_x^{\beta} D_x^{\beta} v, \cdots) = 0, \quad 0 < \alpha, \beta < 1, \tag{2.1}$$

in here v is an unknown function, R is a polynomial of v and its various partial derivatives containing the highest order derivatives and nonlinear terms.

There are main steps to implement the proposed method. These are as follows.

Stage 1: A nonlinear fractional complex transformation proposed by Li and He [9] is used to reduce fractional differential equations into ordinary differential equations (ODEs). This transformation is described as

$$v(x,t) = V(\xi), \quad \xi = \frac{c_1 x^{\beta}}{\Gamma(1+\beta)} - \frac{c_2 t^{\alpha}}{\Gamma(1+\alpha)} \tag{2.2}$$

where c_1 and c_2 are arbitrary constants. We would also like to note that the chain rule can be computed as

$$D^{\alpha}{}_t v = \sigma_1 \frac{dV}{d\xi} D_t^{\alpha} \xi \tag{2.3}$$

$$D^{\beta}{}_x v = \sigma_2 \frac{dV}{d\xi} D_x^{\beta} \xi \tag{2.4}$$

where σ_1, σ_2 are fractional indexes [9]. Eqn. (2.1) can be rewritten by using Eqns. (2.2) and the chain rule (2.3)-(2.4) as the following ODE:

$$F(V, V', V'', V''', \cdots) = 0 \tag{2.5}$$

in here the prime denotes the derivation with respect to ξ.

Stage 2: Assuming the exact solution of Eqn. (2.5) can be represented by a polynomial in (G'/G) as the following form:

$$v(\xi) = \sum_{j=0}^{p} b_j \left(\frac{G'}{G}\right)^j, \quad b_p \neq 0 \tag{2.6}$$

where $b_0, b_1, \cdots, b_p$ are constants. Also, $G(\xi)$ satisfies the second-order linear ODE which is defined by

$$G''(\xi) + \lambda_1 G'(\xi) + \lambda_2 G(\xi) = 0 \tag{2.7}$$

with $\lambda_{1,2}$ being arbitrary constants. Balancing between the highest order derivatives and the nonlinear term rising in Eqn. (2.5) is used to determine the balancing number p.

Stage 3: In this step, we first substitute Eqn. (2.6) into Eqn. (2.5) and use Eqn. (2.7). Then, we gather up all the coefficients with the same power of (G'/G). Equalizing each term of the obtaining polynomial to zero yields a set of algebraic equations for $b_0, b_1, \cdots, b_p, \lambda_1, \lambda_2, c_1$, and c_2.

Stage 4: We construct the constants $b_0, b_1, \cdots, b_p, \lambda_1, \lambda_2, c_1$ and c_2 by solving the obtained system. Substituting these parameters along with the general solution of Eqn. (2.7) into Eqn. (2.6), a variety exact solutions of Eqn. (2.1) is obtained.

3. Implementation of the suggested method for MRLFEW equation types

The main aim of this part is to solve MRLFEW equation types based on the above-mentioned methodology. We consider the equation types that is defined as follows

$$D_t^{\alpha} v(x,t) + \varepsilon D_x^{\alpha} v^2(x,t) - \delta D_{xxt}^{3\alpha} v(x,t) = 0 \tag{3.1}$$

and

$$D_t^{\alpha} v(x,t) + \varepsilon D_x^{\alpha} v^3(x,t) - \delta D_{xxt}^{3\alpha} v(x,t) = 0 \tag{3.2}$$

where ε and δ are real parameters and α is the order of MRLF derivative. The first equation is known as *fractional EW equation*. The second is called *modified fractional EW equation*.

3.1. *Fractional EW equation*

By substituting the transformation (2.2) into Eqn. (3.1), the following ODE can be found

$$-cV' + \varepsilon k(V^2)' + \delta ck^2 V''' = 0 \tag{3.3}$$

where $c = -\sigma_1 c_2$ and $k = \sigma_2 c_1$. Let's integrate Eqn. (3.3) once and setting the integration constant to zero yield

$$-cV + \varepsilon kV^2 + \delta ck^2 V'' = 0 \tag{3.4}$$

Eqn. (3.4) gives the balancing number,

$$m + 2 = 2m,$$

so

$$m = 2.$$

We assume that the solution of Eqn. (3.4) can be described by a polynomial in (G'/G) as follows:

$$V(\xi) = b_0 + b_1\left(\frac{G'}{G}\right) + b_2\left(\frac{G'}{G}\right)^2, \quad b_2 \neq 0 \tag{3.5}$$

Using Eqn. (2.7) and Eqn. (3.5), we have

$$\begin{aligned} V^2(\xi) = {} & b_0^2 + b_1^2\left(\frac{G'}{G}\right)^2 + b_2^2\left(\frac{G'}{G}\right)^4 + 2b_0b_1\left(\frac{G'}{G}\right) \\ & + 2b_0b_2\left(\frac{G'}{G}\right)^2 + 2b_1b_2\left(\frac{G'}{G}\right)^3, \end{aligned} \tag{3.6}$$

$$\begin{aligned} V''(\xi) = {} & (b_1\lambda_1 + 2b_2\lambda_2)\left[\left(\frac{G'}{G}\right)^2 + \lambda_1\left(\frac{G'}{G}\right) + \lambda_2\right] \\ & + 2(b_1 + 2b_2\lambda_1)\left[\left(\frac{G'}{G}\right)^3 + \lambda_1\left(\frac{G'}{G}\right)^2 + \lambda_2\left(\frac{G'}{G}\right)\right] \\ & + 6b_2\left[\left(\frac{G'}{G}\right)^4 + \lambda_1\left(\frac{G'}{G}\right)^3 + \lambda_2\left(\frac{G'}{G}\right)^2\right]. \end{aligned} \tag{3.7}$$

Let's substitute Eqns. (3.5)-(3.6) into Eqn. (3.4). Then collecting the coefficients of $(G'/G)^j, (j = 0,1,2,3,4)$ and setting them to be zero, the system is found as form:

$$-cb_0 + ekb_0^2 + b_1\lambda_1\lambda_2\delta ck^2 + 2b_2\lambda_2^2\delta ck^2 + \chi_0 = 0, \tag{3.8}$$

$$\begin{aligned} -cb_1 + 2\varepsilon kb_0b_1 + b_1\lambda_1^2\delta ck^2 + 2b_2\lambda_1\lambda_2\delta ck^2 \\ +2b_1\lambda_2\delta ck^2 + 4b_2\lambda_1\lambda_2\delta ck^2 = 0, \end{aligned} \tag{3.9}$$

$$\begin{aligned} -cb_2 + \varepsilon kb_1^2 + 2\varepsilon kb_0b_2 + b_1\lambda_1\delta ck^2 + 2b_2\lambda_2\delta ck^2 \\ +2b_1\lambda_1\delta ck^2 + 4b_2\lambda_1^2\delta ck^2 + 6b_2\lambda_2\delta ck^2 = 0, \end{aligned} \tag{3.10}$$

$$2\varepsilon k b_1 b_2 + 2b_1\delta c k^2 + 4b_2\lambda_1\delta c k^2 + 6b_2\lambda_1\delta c k^2 = 0, \tag{3.11}$$

$$\varepsilon k b_2^2 + 6\delta c k^2 b_2 = 0. \tag{3.12}$$

This system is solved by using Maple, we obtain

$$\begin{aligned} &b_0 = -\frac{c(k^2\lambda_1^2\delta + 8k^2\lambda_2\delta - 1)}{2k\varepsilon}, \quad b_1 = -\frac{6ck\delta\lambda_1}{\varepsilon}, \quad b_2 = -\frac{6ck\delta}{\varepsilon} \\ &\delta = \delta, \quad \varepsilon = \varepsilon, \quad c = c, \quad k = k. \end{aligned} \tag{3.13}$$

where λ_1 and λ_2 are arbitrary constants. Eqn. (3.5) can be rewritten by using Eqn. (3.13) as follows:

$$V(\xi) = -\frac{c(k^2\lambda_1^2\delta + 8k^2\lambda_2\delta - 1)}{2k\varepsilon} - \frac{6ck\delta\lambda_1}{\varepsilon}\left(\frac{G'}{G}\right) - \frac{6ck\delta}{\varepsilon}\left(\frac{G'}{G}\right)^2 \tag{3.14}$$

The general solution of Eqn. (2.7) is substituted into Eqn. (3.14), we get three types of travelling wave solutions of the Eqn. (3.1) as follows:

When $\lambda_1^2 - 4\lambda_2 > 0$,

$$V_1(\xi) = -\frac{c(k^2\lambda_1^2\delta + 8k^2\lambda_2\delta - 1)}{2k\varepsilon} - \frac{6ck\delta\lambda_1}{\varepsilon}\left(\frac{G'}{G}\right) - \frac{6ck\delta}{\varepsilon}\left(\frac{G'}{G}\right)^2, \tag{3.15}$$

where

$$\left(\frac{G'}{G}\right) = -\frac{\lambda_1}{2} + \frac{\sqrt{\lambda_1^2 - 4\lambda_2}}{2}\left(\frac{C_1\cosh(\frac{\sqrt{\lambda_1^2-4\lambda_2}}{2})\xi + C_2\sinh(\frac{\sqrt{\lambda_1^2-4\lambda_2}}{2})\xi}{C_1\sinh(\frac{\sqrt{\lambda_1^2-4\lambda_2}}{2})\xi + C_2\cosh(\frac{\sqrt{\lambda_1^2-4\lambda_2}}{2})\xi}\right)$$

When $\lambda_1^2 - 4\lambda_2 < 0$,

$$V_2(\xi) = -\frac{c(k^2\lambda_1^2\delta + 8k^2\lambda_2\delta - 1)}{2k\varepsilon} - \frac{6ck\delta\lambda_1}{\varepsilon}\left(\frac{G'}{G}\right) - \frac{6ck\delta}{\varepsilon}\left(\frac{G'}{G}\right)^2, \tag{3.16}$$

where

$$\left(\frac{G'}{G}\right) = -\frac{\lambda_1}{2} + \frac{\sqrt{4\lambda_2 - \lambda_1^2}}{2}\left(\frac{C_1\cos(\frac{\sqrt{4\lambda_2-\lambda_1^2}}{2})\xi - C_2\sin(\frac{\sqrt{4\lambda_2-\lambda_1^2}}{2})\xi}{C_1\sin(\frac{\sqrt{4\lambda_2-\lambda_1^2}}{2})\xi - C_2\cos(\frac{\sqrt{4\lambda_2-\lambda_1^2}}{2})\xi}\right)$$

When $\lambda_1^2 - 4\lambda_2 = 0$,

$$V_3(\xi) = -\frac{c(k^2\lambda_1^2\delta + 8k^2\lambda_2\delta - 1)}{2k\varepsilon} - \frac{6ck\delta\lambda_1}{\varepsilon}\left(\frac{G'}{G}\right) - \frac{6ck\delta}{\varepsilon}\left(\frac{G'}{G}\right)^2, \tag{3.17}$$

where

$$\left(\frac{G'}{G}\right) = -\frac{\lambda_1}{2} + \frac{C_2}{C_1 + C_2\xi}.$$

3.2. *Modified fractional EW equation*

By substituting the transformation (2.2) into Eqn. (3.2), the following ODE can be found

$$-cV' + \varepsilon k(V^3)' + \delta c k^2 V''' = 0 \tag{3.18}$$

where $c = \sigma_1 c_2$ and $k = \sigma_2 c_1$. Let's integrate Eqn. (3.18) once, we get

$$-cV + \varepsilon k V^3 + \delta c k^2 V'' + \chi_1 = 0 \tag{3.19}$$

where χ_1 is integration constant. Eqn. (3.19) gives the balancing number,

$$m + 2 = 3m,$$

so

$$m = 1.$$

We assume that the solution of Eqn. (3.19) can be described by a polynomial in (G'/G) as follows:

$$V(\xi) = b_0 + b_1\left(\frac{G'}{G}\right), \quad b_1 \neq 0 \tag{3.20}$$

Using Eqn. (2.7) and Eqn. (3.20), we have

$$\begin{aligned} V^3(\xi) = \quad & b_0^3 + 3b_0^2 b_1\left(\frac{G'}{G}\right) + 3b_0 b_1^2\left(\frac{G'}{G}\right)^2 + b_1^3\left(\frac{G'}{G}\right)^3 \\ V''(\xi) = \quad & 2b_1\left(\frac{G'}{G}\right)^3 + 3b_1\lambda_1\left(\frac{G'}{G}\right)^2 \\ + \quad & (b_1\lambda_1^2 + 2b_1\lambda_2)\left(\frac{G'}{G}\right) + b_1\lambda_1\lambda_2 \end{aligned} \tag{3.21}$$

Let's substitute Eqns. (3.20)-(3.21) into Eqn. (3.19). Then collecting the coefficients of $(G'/G)^j, (j = 0,1,2,3)$ and setting them to be zero, the algebraic equation system is found as form:

$$-cb_0 + \varepsilon k b_0^3 + \delta c k^2 b_1\lambda_1\lambda_2 + \chi_1 = 0, \tag{3.22}$$

$$-cb_1 + 3\varepsilon k b_0^2 b_1 + 2\delta c k^2 b_1\lambda_2 + \delta c k^2 b_1\lambda_1^2 = 0, \tag{3.23}$$

$$3\varepsilon k b_0 b_1^2 + 3\delta c k^2 b_1\lambda_1 = 0, \tag{3.24}$$

$$\varepsilon k b_1^3 + 2b_1\delta c k^2 = 0. \tag{3.25}$$

This system is solved by using Maple, we obtain

$$\begin{gathered} b_0 = \pm\frac{\sqrt{2\varepsilon c k\delta}\lambda_1 i}{2\varepsilon}, \quad b_1 = \pm\frac{\sqrt{2\varepsilon c k\delta} i}{2\varepsilon}, \\ \chi_1 = 0, \quad k = \pm\frac{2}{\sqrt{-2\lambda_1^2\delta + 8\lambda_2\delta}}, \\ \delta = \delta, \quad \varepsilon = \varepsilon, \quad c = c \end{gathered} \tag{3.26}$$

where λ_1 and λ_2 are arbitrary constants. Eqn. (3.20) can be rewritten by using Eqn. (3.26) as follows:

$$V(\xi) = \pm\frac{\sqrt{2\varepsilon ck\delta}\lambda_1 i}{2\varepsilon} \pm \frac{\sqrt{2\varepsilon ck\delta}i}{2\varepsilon}\left(\frac{G'}{G}\right) \tag{3.27}$$

The general solution of Eqn. (2.7) is substituted into Eqn. (3.14), we get three types of travelling wave solutions of the Eqn. (3.1) as follows:

When $\lambda_1^2 - 4\lambda_2 > 0$,

$$V_1(\xi) = \pm\frac{\sqrt{2\varepsilon ck\delta}\lambda_1 i}{2\varepsilon} \pm \frac{\sqrt{2\varepsilon ck\delta}i}{2\varepsilon}\left(\frac{G'}{G}\right) \tag{3.28}$$

where

$$\left(\frac{G'}{G}\right) = -\frac{\lambda_1}{2} + \frac{\sqrt{\lambda_1^2 - 4\lambda_2}}{2}\left(\frac{C_1\cosh(\frac{\sqrt{\lambda_1^2-4\lambda_2}}{2})\xi + C_2\sinh(\frac{\sqrt{\lambda_1^2-4\lambda_2}}{2})\xi}{C_1\sinh(\frac{\sqrt{\lambda_1^2-4\lambda_2}}{2})\xi + C_2\cosh(\frac{\sqrt{\lambda_1^2-4\lambda_2}}{2})\xi}\right)$$

When $\lambda_1^2 - 4\lambda_2 < 0$,

$$V_2(\xi) = \pm\frac{\sqrt{2\varepsilon ck\delta}\lambda_1 i}{2\varepsilon} \pm \frac{\sqrt{2\varepsilon ck\delta}i}{2\varepsilon}\left(\frac{G'}{G}\right) \tag{3.29}$$

where

$$\left(\frac{G'}{G}\right) = -\frac{\lambda_1}{2} + \frac{\sqrt{4\lambda_2 - \lambda_1^2}}{2}\left(\frac{C_1\cos(\frac{\sqrt{4\lambda_2-\lambda_1^2}}{2})\xi - C_2\sin(\frac{\sqrt{4\lambda_2-\lambda_1^2}}{2})\xi}{C_1\sin(\frac{\sqrt{4\lambda_2-\lambda_1^2}}{2})\xi - C_2\cos(\frac{\sqrt{4\lambda_2-\lambda_1^2}}{2})\xi}\right)$$

When $\lambda_1^2 - 4\lambda_2 = 0$,

$$V_3(\xi) = \pm\frac{\sqrt{2\varepsilon ck\delta}\lambda_1 i}{2\varepsilon} \pm \frac{\sqrt{2\varepsilon ck\delta}i}{2\varepsilon}\left(\frac{G'}{G}\right) \tag{3.30}$$

where

$$\left(\frac{G'}{G}\right) = -\frac{\lambda_1}{2} + \frac{C_2}{C_1 + C_2\xi}.$$

4. Conclusion

In this study, we have obtained three traveling wave solutions of FEW and MFEW equations by using (G'/G)-expansion method. Modified Riemann-Liouville derivative is preferred for time-space fractional derivatives. Besides, fractional complex transform that is simple and effective, is implemented to convert FPDE into an ODE. We would also like to say that the method can be used for many other nonlinear fractional differential equations. Another point that is worthy of being emphasized is that the acquired solutions in this work have not been reported in the literature up to now.

As a final remark that the exactness of the obtained solutions is verified by substituting them back into the original equation. It can be said that they satisfy the FEW and MFEW equations under consideration.

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NONSMOOTH SET VARIATIONAL INEQUALITY PROBLEMS AND OPTIMALITY CRITERIA FOR SET OPTIMIZATION

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Abstract. In this work, set-valued optimization problems are considered according to an order relation, which is a partial order on the family such that contains nonempty bounded sets of the space. A generalized convexity is defined for set-valued mapping by using the partial order relation. Nonsmooth variational inequality problems are introduced with the aid of M-directionally derivative. Some optimality criteria including the necessary and sufficient optimality conditions are obtained for mentioned optimization problems.

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1. Introduction

One of the most encountered problems in our life is optimization problems (mathematical programming problems). Translating these problems into mathematical language, give us objective functions. Optimization problems are called according to objective functions. For example, a set-valued optimization problem (shortly, (SVO_P)) arises when the objective function is a set-valued mapping. (SVO_P)s are a generalization of vector and scalar optimization problems because the set-valued mappings are a generalization version of the vector-valued and real-valued functions.

The most important purpose of a mathematical optimization problem is to find the best among the suitable options. Naturally, it has been attracted the attention of scientists, who have been working in mathematics, engineering, economics, management, economic equilibria, optimal control, nonlinear optimization transportation, and many other disciplines. There are many methods to solve and obtain optimality conditions of the optimization problems such as scalarization [14], vectorization [10, 12], directional derivative [13], subdifferential [9, 11], embedding space [15, 18], variational inequality problems [1, 2, 4–8, 19, 21].

Vector variational inequality problems and their some generalizations have been used as methods to obtain the solutions and the optimality conditions of vector-valued optimization problems. Giannessi [8] started the vector variational inequalities. During the most recent decade, different kinds of variational inequality problems were

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derived as Minty variational [2, 3, 6, 7], complementarity [21], semi-monotone scalar variational inequalities [5].

Order relation is required to obtain the solution of set-valued programming and interval programming problems. Set optimization is presented by Kuroiwa [16]. Kuroiwa used six order relation such that some of them are pre-order relation and the others are not pre-order relations. Karaman et al. [14] are defined two order relations. The most important feature of these two order relations is a partial order relation on the family, which contains nonempty bounded sets of the space. These partial order relations are used in [11, 13–15] to obtain optimality criteria and solutions for set-valued optimization problems.

The point of this paper is to gain the optimality criteria for (SVO_P)s via a partial order relation introduced in [14]. Variational inequalities and convexity are used in order to achieve the aim. A new convexity concept called m-convexity, which is a generalization of known convexity in the literature, is given by using partial order relation. A relationship is obtained between m-convexity and M-directional derivative defined in [13].

The layout of this manuscript is as tracks: Some basic notations, definitions, and solution concepts are recalled for (SVO_P)s in the second section. m-convexity and some properties are obtained in Section 3. Variational inequality problems and some optimality conditions including necessary and sufficient are obtained in the last section.

2. Preliminaries

Throughout this paper, we assume that $\mathbb{R}^n$ is ordered by a convex, pointed, closed cone $C \subseteq \mathbb{R}^n$ $(n \geq 1)$ with a nonempty interior. We denote by $\mathcal{P}^n$ and $\mathcal{K}^n$ the set of all nonempty subsets of $\mathbb{R}^n$ and the set of all nonempty compact and convex subsets of $\mathbb{R}^n$, respectively. The interior of A is represented by $int(A)$ for a set $A \subseteq \mathbb{R}^n$.

Let $A, B \in \mathcal{P}^n$ and $\lambda \in \mathbb{R}$, $\lambda A := \{\lambda a \mid a \in A\}$. The algebraic sum and the algebraic difference of A and B are denoted by $A+B$ and $A-B$, respectively. Also, Minkowski (Pontryagin) difference of A and B is defined by

$$A \dot{-} B := \{x \in \mathbb{R}^n \mid x + B \subseteq A\}.$$

The algebraic sum, the algebraic difference and Minkowski difference have following properties.

It is known that C induce the following partial order relation on $\mathbb{R}^n$ for $x_1, x_2 \in \mathbb{R}^n$:

$$x_1 \leq_C x_2 \Longleftrightarrow x_2 - x_1 \in C.$$

Proposition 1 ([14, 17, 20])**.** *Let $A, B \in \mathcal{K}^n$, $a \in \mathbb{R}^n$ and $t > 0$. The following conditions are hold:*

(i) $t(A \dot{-} B) = tA \dot{-} tB$,
(ii) $(A+B) \dot{-} B = A$,

(iii) $(A \dot{-} B) + B \subseteq A$,
(iv) *if* $B = \varnothing$, *then* $A \dot{-} B = \mathbb{R}^n$,
(v) $A \dot{-} A = 0$.

Let's now define m and strictly m order relations are recalled in the next definitions.

Definition 1 ([14])**.** Let $A, B \in \mathcal{P}^n$. m order relation is defined by

$$A \preceq^m B :\Longleftrightarrow (B \dot{-} A) \cap C \neq \varnothing.$$

Note that $\preceq^m$ is not only a pre-order relation on $\mathcal{P}^n$ but also a partial order relation on $\mathcal{K}^n$ [14].

Definition 2 ([14])**.** Let $A, B \in \mathcal{P}^n$. Strictly m order relation is defined by

$$A \prec^m B :\Longleftrightarrow (B \dot{-} A) \cap int(C) \neq \varnothing.$$

We know that m and strictly m order relations are not only compatible with the nonnegative scalar multiplication but also compatible with the addition. Moreover, these order relations have the following properties, which are utilized in the next sections.

Proposition 2. *Let* $A, B, D, E \in \mathcal{P}^n$. *Then,*
(i) $A \preceq^m B \Longrightarrow A \dot{-} D \preceq^m B \dot{-} D$,
(ii) $A \preceq^m B$ *and* $D \preceq^m E \Longrightarrow A + D \preceq^m B + E$,
(iii) $A \preceq^m B \Longleftrightarrow 0 \preceq^m B \dot{-} A \Longrightarrow 0 \preceq^m B - A$,
(iv) $A \not\preceq^m B \Longleftrightarrow \alpha A \not\preceq^m \alpha B$ *for all* $\alpha > 0$.

Proof. (i) Let $A \preceq^m B$. There exists an $s \in C$ that $s \in B \dot{-} A$, it follows

$$s + A \subseteq B. \tag{2.1}$$

Then, $s + A \dot{-} D \subseteq B \dot{-} D$. Really, let $t \in A \dot{-} D$, that is

$$t + D \subseteq A. \tag{2.2}$$

Then, from (2.1) and (2.2) we have $s + t + D \subseteq s + A \subseteq B$. Hence, $s + t \in B \dot{-} D$ and we obtain $s + A \dot{-} D \subseteq B \dot{-} D$. Since $s \in C$ and $s \in (B \dot{-} D) \dot{-} (A \dot{-} D)$ we obtain $A \dot{-} D \preceq^m B \dot{-} D$.

(ii-iv) These can be proved with the aid of Proposition 1 and definitions of $\preceq^m$ and $\prec^m$.

□

Note that given all properties via m order relation in Proposition 2 are satisfied for strictly m order relation.

Some efficient sets of a family with aid of $\preceq^m$ and $\prec^m$ are remembered in the next definition.

Definition 3 ([14]). Let $\mathcal{S} \subseteq \mathcal{K}^n$ and $A \in \mathcal{S}$. Then,

(i) A is called m-minimal (resp. m-maximal) set of $\mathcal{S}$ if there isn't any $B \in \mathcal{S}$ such that $B \preceq^m A$ (resp. $A \preceq^m B$) and $A \neq B$,

(ii) A is called weakly m-minimal (resp. weakly m-maximal) set of $\mathcal{S}$ if there isn't any $B \in \mathcal{S}$ such that $B \prec^m A$ (resp. $A \prec^m B$).

Let $S \subseteq \mathbb{R}^m$ and $T : S \rightrightarrows \mathbb{R}^n$ be a nonempty, compact and convex valued set-valued mapping. Epigraph of the set-valued mapping T is described as $Epi(T) := \{(x,\alpha) \in S \times \mathbb{R}^n \mid T(x) \preceq^m \alpha\}$. We consider the following constraint (SVO_P)

$$(SVO_P) \begin{cases} \min(\max)\ T(x) \\ s.t.\ x \in S. \end{cases}$$

Set optimization criteria are derived from a comparison between the values of the set-valued mapping T. Namely, we search efficient sets of the family $\mathcal{T}(S) := \{T(x) \mid x \in S\}$ to solve a (SVO_P) via set optimization. Ordering relations defined on sets are used to investigate the efficient sets. So, we will use $\preceq^m$ and $\prec^m$ order relations to determine the efficient sets of $\mathcal{T}(S)$ in this study. If we consider the (SVO_P) with respect to m order relation, then we use the notation $(m - SVO_P)$.

We say that $\hat{x}$ is a minimal (resp. maximal) solution of $(m - SVO_P)$ if $T(\hat{x})$ is an m-minimal (resp. m-maximal) set of $\mathcal{T}(S)$. Similarly, we say that $\hat{x}$ is a weakly minimal (resp. weakly maximal) solution of $(m - SVO_P)$ if $T(\hat{x})$ is a weakly m-minimal (resp. weakly m-maximal) set of $\mathcal{T}(S)$. If $T(\hat{x}) \preceq^m T(x)$ (resp. $T(x) \preceq^m T(\hat{x})$) for all $x \in S$, then $\hat{x}$ is called strongly m-minimal (resp. strongly m-maximal) solution of $(m - SVO_P)$. $\hat{x}$ is called stritly m-minimal (resp. strictly m-maximal) solution of $(m - SVO_P)$ if $T(\hat{x}) \prec^m T(x)$ (resp. $T(x) \prec^m T(\hat{x})$) for all $x \in S$.

If $\hat{x}$ is a strongly m-minimal solution of $(m - SVO_P)$, then it is also m-minimal and weakly m-minimal solution of the problem. Also, if $\hat{x}$ is an m-minimal solution of $(m - SVO_P)$, then it is also a weakly m-minimal solution of the problem. These conditions can also be applied to maximal solutions.

Definition 4 ([13]). Let $T : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be a set-valued mapping, $S \subseteq \mathbb{R}^m$, $\hat{x} \in int(S)$ and $h \in \mathbb{R}^m$. The limit

$$T^M(\hat{x};h) := \limsup_{t \to 0^+} \frac{T(\hat{x}+th) \dot{-} T(\hat{x})}{t}$$

is called M-directional derivative of T at $\hat{x}$ in direction h where $\limsup_{x' \to x} T(x') := \{y \in \mathbb{R}^n \mid \liminf_{x' \to x} d(y, T(x')) = 0\}$ denotes the Painlevé-Kuratowski upper/outer limit of T at x. If $T^M(\hat{x};h) \neq \varnothing$ for an $\hat{x} \in int(S)$ and for all $h \in \mathbb{R}^m$, then T is called M-directionally differentiable at $\hat{x}$.

T is called M-directionally differentiable on S if $T^M(\hat{x};h) \neq \varnothing$ for all $\hat{x} \in int(S)$ and for all $h \in \mathbb{R}^m$. Besides, $T^M(\hat{x};h)$ is positively homogenous in h, that is $T^M(\hat{x};\alpha h) = \alpha T^M(\hat{x};h)$ for all $\alpha > 0$ [13].

Karaman et al. [13] obtained some optimality conditions for $(m-SVO_P)$ and existence theorems for M-directional derivative.

3. M-CONVEXITY FOR SET-VALUED MAPPINGS

A convexity notation is introduced for the set-valued mappings by using m order relation, and a relationship between convexity and M-directional differentiable is derived in this section.

Definition 5. Let $S \subseteq \mathbb{R}^m$ be a convex set and $T : S \rightrightarrows \mathbb{R}^n$ be a set-valued mapping. T is called m-convex set-valued mapping on S if

$$T(\lambda x + (1-\lambda)y) \preceq^m \lambda T(x) + (1-\lambda)T(y),$$

for all $x, y \in S$ and all $\lambda \in (0,1)$.

Note that m-convexity is reduced the C-convexity defined on $\mathbb{R}^n$ if we take $f : \mathbb{R}^m \to \mathbb{R}^n$ vector-valued function instead of the set-valued mapping $T : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$. Moreover, if we take $g : \mathbb{R}^m \to \mathbb{R}$ real-valued function instead of the set-valued map T, we obtain convexity defined on $\mathbb{R}$. So, m-convexity is a generalization of the known convexity in the literature.

Proposition 3. *Let $T : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be a set-valued mapping. The following conditions are equivalent:*

(i) *T is m-convex,*
(ii) *$Epi(T)$ is convex,*
(iii) *$T(t_1x_1 + t_2x_2 + \ldots + t_nx_n) \preceq^m t_1T(x_1) + t_2T(x_2) + \ldots + t_nT(x_n)$ for all $n \in \mathbb{N}$, for all $x_1, x_2, \ldots, x_n \in \mathbb{R}^m$ and for all $t_1, t_2, \ldots, t_n \in (0,1)$ such that $\sum_{k=1}^n t_k = 1$ (Jensen's Inequality).*

Proof. The proof is immediate from Definition 5. □

Proposition 4. *Let $T, G : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be set-valued mappings. The undermentioned assertions are satisfied:*

(i) *If T is m-convex, then $T(x) + C$ and $T(x) + int(C)$ are m-convex,*
(ii) *if T and G are m-convex, $T + G$ is also m-convex.*

Definition 6. A vector-valued mapping $T : \mathbb{R}^m \to \mathbb{R}^n$ is called affine iff

$$T(\alpha x + (1-\alpha)y) = \alpha T(x) + (1-\alpha)T(y),$$

for all $x, y \in \mathbb{R}^m$ and all $\alpha \in \mathbb{R}$.

Definition 7. Let $h : \mathbb{R}^m \to \mathbb{R}^n$ be a vector-valued mapping and $x, y \in \mathbb{R}^m$. h is called

(i) m-increasing iff $x \preceq^m y$ implies $h(x) \preceq^m h(y)$,
(ii) m-decreasing iff $x \preceq^m y$ implies $h(y) \preceq^m h(x)$.

Proposition 5. *Let $T,G:\mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be set-valued mappings and vector-valued function $h:\mathbb{R}^k \to \mathbb{R}^m$ be m-increasing. Then,*

(i) *If T is m-convex set-valued map and h is m-convex, then $T\circ h:\mathbb{R}^k \rightrightarrows \mathbb{R}^n$ is also an m-convex set-valued mapping,*

(ii) *Let G be an affine set-valued mapping. T is m-convex iff $T+G$ is m-convex set-valued mapping.*

Theorem 1. *Let $S\subseteq \mathbb{R}^m$ be a convex set and $T:\mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be compact, convex valued and M-directionally differentiable set-valued mapping on S. If T is m-convex set-valued map, $T^M(x;y-x) \preceq^m T(y)-T(x)$ for all $x,y\in S$.*

Proof. Because T is m-convex set-valued mapping. Then, we have

$$T(\alpha y+(1-\alpha)x) \preceq^m \alpha T(y)+(1-\alpha)T(x) \tag{3.1}$$

for all $x,y\in S$ and for all $\alpha\in(0,1)$. As T is convex valued map we can write the inequality (3.1) as

$$T(x+\alpha(y-x)) \preceq^m \alpha T(y)+T(x)-\alpha T(x).$$

From Proposition 2 (iii) we get

$$T(x+\alpha(y-x))\dot{-}T(x) \preceq^m [\alpha T(y)-\alpha T(x)+T(x)]\dot{-}T(x).$$

By using Proposition 1 (ii) we yield

$$T(x+\alpha(y-x))\dot{-}T(x) \preceq^m \alpha T(y)-\alpha T(x).$$

Since $\preceq^m$ is compatible with the nonnegative scalar multiplication, we obtain

$$\frac{T(x+\alpha(y-x))\dot{-}T(x)}{\alpha} \preceq^m \frac{\alpha(T(y)-T(x))}{\alpha} = T(y)-T(x).$$

By taking Painlevé-Kuratowski upper limit for $\alpha\to 0^+$, we attain $T^M(x;y-x) \preceq^m T(y)-T(x)$ for all $x,y\in S$. □

Remark 1. Let $S\subseteq\mathbb{R}^m$ be a convex set and $T:\mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be compact, convex valued and M-directionally differentiable set-valued mapping on S. If $T^M(x;y-x) \preceq^m T(y)-T(x)$ for all $x,y\in S$, T may not be an m-convex set-valued mapping. For example, set-valued mapping $T:\mathbb{R}\rightrightarrows\mathbb{R}^2$ is defined as $T(x)=\mathbb{B}((x,x^2),|1-x|)$ for all $x\in\mathbb{R}$, where $\mathbb{B}(x,r)$ denotes the open ball centered $x\in\mathbb{R}$ with radius r. Although T is not m-convex set-valued mapping, the inequality condition $T^M(x;y-x) \preceq^m T(y)-T(x)$ is satisfied for all $x,y\in S$.

Definition 8. Let $S \subseteq \mathbb{R}^m$ be a convex, $T : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be an M-directionally differentiable. Then, T is called

(i) m-pseudoconvex iff for all $x, y \in S$

$$F^M(x; y-x) \not\preceq^m 0 \implies F(y) \not\preceq^m F(x),$$

(ii) strongly m-pseudoconvex iff for all $x, y \in S$

$$0 \preceq^m F^M(x; y-x) \implies F(x) \preceq^m F(y),$$

(iii) weakly m-pseudoconvex iff for all $x, y \in S$

$$F^M(x; y-x) \not\prec^m 0 \implies F(y) \not\prec^m F(x).$$

4. Set variational inequality problems and optimality criteria for set optimization

Variational inequality problems and some optimality conditions for $(m-SVO_P)$ are introduced in this section.

Definition 9. Let $S \subseteq \mathbb{R}^n$ be a convex set and $x \in S$ be an arbitrary element. Then, the set-valued mapping $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is called

(i) m-upper sign continuous if for all $y \in S$ and all $\alpha \in (0,1)$

$$T(x+\alpha(y-x)) \not\preceq^m 0 \Rightarrow T(x) \not\preceq^m 0,$$

(ii) strongly m-upper sign continuous if for all $y \in S$ and all $\alpha \in (0,1)$

$$0 \preceq^m T(x+\alpha(y-x)) \Rightarrow 0 \preceq^m T(x),$$

(iii) weakly m-upper sign continuous if for all $y \in S$ and all $\alpha \in (0,1)$

$$T(x+\alpha(y-x)) \not\prec^m 0 \Rightarrow T(x) \not\prec^m 0.$$

The m-variational inequality problem (shortly $(m-VI_P)$): Find $x \in S$ such that

$$T^M(x; y-x) \not\preceq^m 0, \quad \forall y \in S.$$

The inverse m-variational inequality problem (shortly $(m-IVI_P)$): Find $x \in S$ such that

$$0 \not\preceq^m T^M(x; y-x), \quad \forall y \in S.$$

The strongly m-variational inequality problem (shortly $(m-SVI_P)$): Find $x \in S$ such that

$$0 \preceq^m T^M(x; y-x), \quad \forall y \in S.$$

The inverse strongly m-variational inequality problem (shortly $(m-ISVI_P)$): Find $x \in S$ such that

$$T^M(x;y-x) \preceq^m 0, \quad \forall y \in S.$$

Similarly, the weakly m-variational inequality problem (shortly $(m-WVI_P)$): Find $x \in S$ such that

$$T^M(x;y-x) \not\prec^m 0, \quad \forall y \in S.$$

The inverse weakly m-variational inequality problem (shortly $(m-IWVI_P)$): Find $x \in S$ such that

$$0 \not\prec^m T^M(x;y-x), \quad \forall y \in S.$$

We denote by $sol(m-SVI_P)$, $sol(m-VI_P)$ and $sol(m-WVI_P)$ the set of all solutions of $(m-SVI_P)$, $(m-VI_P)$ and $(m-WVI_P)$, respectively. It is obvious that $sol(m-SVI_P) \subseteq sol(m-VI_P) \subseteq sol(m-WVI_P)$. Similarly, there is the same relationship in the inverse version of the variational inequality problems. The converse implications may not be generally true.

Minty m-variational inequality problem $(m-MVI_P)$: Find $x \in S$ so that

$$T^M(y;y-x) \not\preceq^m 0, \quad \forall y \in S,$$

Minty strongly m-variational inequality problem $(m-MSVI_P)$: Find $x \in S$ so that

$$0 \preceq^m T^M(y;y-x), \quad \forall y \in S,$$

Minty weakly m-variational inequality problem $(m-MWVI_P)$: Find $x \in S$ so that

$$T^M(y;y-x) \not\prec^m 0, \quad \forall y \in S.$$

We denote by $sol(m-MSVI_P)$, $sol(m-MVI_P)$ and $sol(m-MWVI_P)$ the set of all solutions of $(m-MSVI_P)$, $(m-MVI_P)$ and $(m-MWVI_P)$, respectively. It is obvious that $sol(m-MSVI_P) \subseteq sol(m-MVI_P) \subseteq sol(m-MWVI_P)$, but the converse inclusion may not be generally true. Similarly, there is the same relationship in inverse version of variational inequality problems.

Proposition 6. *Let $S \subseteq \mathbb{R}^n$ be a nonempty convex set and set-valued mapping $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be M-directionally differentiable on S. Then*

(i) *if T^M is m-upper sign continuous, then $sol(m-MVI_P) \subseteq sol(m-VI_P)$,*
(ii) *if T^M is strongly m-upper sign continuous, then $sol(m-MSVI_P) \subseteq sol(m-SVI_P)$,*
(iii) *if T^M is weakly m-upper sign continuous, then $sol(m-MWVI_P) \subseteq sol(m-WVI_P)$.*

Proof. (i) Let $x' \in S$ is a solution of $(m-MVI_P)$. Then, $T^M(y;y-x') \not\preceq^m 0$ for all $y \in S$. Since S is a convex set, $x'+\lambda(y-x') \in S$ for all $\lambda \in (0,1)$. So, we have

$$T^M(x'+\lambda(y-x');x'+\lambda(y-x')-x') \not\preceq^m 0,$$

equivalently from positively homogenously of M-directionally differentiable

$$\lambda T^M(x'+\lambda(y-x');y-x') \not\preceq^m 0.$$

By Proposition 2 (iv), we yield $T^M(x'+\lambda(y-x');y-x') \not\preceq^m 0$. $T^M(x';y-x') \not\preceq^m 0$ yields by using m-upper sign continuity of T^M. Therefore, $x' \in sol(m-VI_P)$.

(ii-iii) These can be proven similar to (i).

□

Theorem 2. *Let $S \subseteq \mathbb{R}^m$ be a convex set and set-valued map $T : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be compact, convex valued and M-directionally differentiable on S. Then, x_0 is a maximal solution of $(m-SVO_P)$ if and only if it is also a solution of $(m-IVI_P)$.*

Proof. Let x_0 be a maximal solution of $(m-SVO_P)$. We have

$$T(x_0) \not\preceq^m T(y) \tag{4.1}$$

for all $y \in S \setminus \{x_0\}$ such that $T(y) \neq T(x_0)$. Then, we can obtain easily that

$$0 \not\preceq^m T(y) \dot{-} T(x_0). \tag{4.2}$$

Because S is convex set, we can write $\alpha y + (1-\alpha)x_0$ instead of y in (4.2) for $\alpha \in (0,1)$. Hence,

$$0 \not\preceq^m T(x_0+\alpha(y-x_0)) \dot{-} T(x_0).$$

From Proposition 2 (iv), we have

$$0 \not\preceq^m \frac{T(x_0+\alpha(y-x_0)) \dot{-} T(x_0)}{\alpha}. \tag{4.3}$$

Since T is M-directionally differentiable and $T(x_0+\alpha(y-x_0)) \dot{-} T(x_0)$ is compact, by taking Painlevé-Kuratowski upper limit $\alpha \to 0^+$ in (4.3) we yield $0 \not\preceq^m T^M(x_0;y-x_0)$. Therefore, x_0 is a solution of $(m-IVI_P)$.

For the inverse statement, let x_0 be a solution of $0 \not\preceq^m T^M(x;y-x)$ for all $y \in S$. Assume that x_0 is not a solution of $(m-SVO_P)$. There exists an $x' \in S$ such that

$$T(x_0) \preceq^m T(x'). \tag{4.4}$$

Since S is convex set and $x_0, x' \in S$, there exists a $y \in S$ and $\alpha \in [0,1]$ such that $x' = \alpha y + (1-\alpha)x_0$. From (4.4) and Proposition 2 (iii), $0 \preceq^m T(x_0+\alpha(y-x_0)) \dot{-} T(x_0)$ yield. By multiplying both sides with $\frac{1}{\alpha}$ and by taking Painlevé-Kuratowski upper limit $\alpha \to 0^+$, we obtain $0 \preceq^m T^M(x_0;y-x_0)$. This contradicts the assumption. Hence, x_0 is a maximal solution of $(m-SVO_P)$. □

Theorem 3. *Let $S \subseteq \mathbb{R}^m$ be convex set and set-valued map $T : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be compact, convex valued and M-directionally differentiable on S. Then, x_0 is a minimal solution of $(m-SVO_P)$ if and only if it is a solution of $(m-VI_P)$.*

Proof. It can be obtained similar to the proof of previous theorem. □

Theorem 4. *Let $S \subseteq \mathbb{R}^m$ be convex set and set-valued map $T : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be compact, convex valued and M-directionally differentiable on S. Then,*

(i) *x_0 is a strongly minimal (resp. strongly maximal) solution of $(m - SVO_P)$ if and only if it is also a solution of $(m - SVI_P)$ (resp. $(m - ISVI_P)$),*
(ii) *x_0 is a weakly minimal (resp. weakly maximal) solution of $(m - SVO_P)$ if and only if it is also a solution of $(m - WVI_P)$ (resp. $(m - IWVI_P)$).*

Proposition 7. *Let $S \subseteq \mathbb{R}^m$ be convex set and set-valued map $T : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be compact, convex valued and M-directionally differentiable on S. Then,*

(i) *if $\tilde{x}$ is a strongly minimal (resp. strongly maximal) solution of $(m - SVO_P)$, then $\tilde{x}$ is also not only a solution of $(m - VI_P)$ (resp. $(m - IVI_P)$) but also a solution of $(m - WVI_P)$ (resp. $(m - IWVI_P)$),*
(ii) *if $\tilde{x}$ is a minimal (resp. maximal) solution of $(m - SVO_P)$, then $\tilde{x}$ is also a solution of $(m - WVI_P)$ (resp. $(m - IWVI_P)$).*

Theorem 5. *Let $S \subseteq \mathbb{R}^m$ be convex set and $T : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be M-directionally differentiable set-valued map on $S \subseteq$. The following assertions are ture:*

(i) *if T is m-pseudoconvex, then every solution of $(m - VI_P)$ is a minimal solution of $(m - SVO_P)$,*
(i) *if T is m-pseudoconvex, then every solution of $(m - MVI_P)$ is a maximal solution of $(m - SVO_P)$,*
(i) *if T is weakly m-pseudoconvex, then every solution of $(m - WVI_P)$ is a weak minimal solution of $(m - SVO_P)$,*
(i) *if T is weakly m-pseudoconvex, then every solution of $(m - MWVI_P)$ is a weak maximal solution of $(m - SVO_P)$,*
(ii) *if T is strongly m-pseudoconvex, every solution of $(m - SVI_P)$ is also a strongly minimal solution of $(m - SVO_P)$,*
(ii) *if T is strongly m-pseudoconvex, every solution of $(m - MSVI_P)$ is also a strongly maximal solution of $(m - SVO_P)$.*

Proof. The proof can be proved easily by using the definitions. □

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EFFICIENT ESTIMATE OF THE REMAINDER FOR THE DIRICHLET FUNCTION $\eta(p)$ FOR $p \in \mathbb{R}^+$

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Abstract. For any $k,n,q \in \mathbb{N}$, where $n \geq 2k+1 \geq 3$, and $p \in \mathbb{R}^+$ the nth remainder $\rho(n,p)$ of the Dirichlet eta function $\eta(p)$, known also as the alternating Riemann zeta function,

$$\eta(p) := \sum_{i=1}^{\infty} (-1)^{i+1} \frac{1}{i^p},$$

is given in the form

$$\rho(n,p) = \Delta_q(n,p) + \delta_q(k,p),$$

where k and q are parameters controlling the magnitude of the error term $\delta_q(k,p)$. The function $\Delta_q(n,p)$ consists of $\lfloor q/2 \rfloor + 1$ simple summands and $\delta_q(k,n,p)$ is estimated as

$$\left|\delta_q(k,p)\right| < \frac{5}{3} \cdot \frac{p^{(q-1)}}{\pi^{q-1}} \cdot \frac{1}{(2k)^{p+q-1}},$$

where $p^{(q-1)}$ is the rising Pochhammer product.

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Keywords: alternating generalized harmonic number, alternating p-series, approximation, Dirichlet's eta function, estimate, remainder, slow convergence

1. Introduction

To obtain the numerical values of the sequence $n \mapsto H^*\left(n, \frac{1}{1000}\right)$,

$$\begin{aligned} H^*\left(n, \tfrac{1}{1000}\right) &:= \sum_{j=1}^{n} (-1)^{j+1} \frac{1}{j^{1/1000}} \\ &= 1.0000\ldots - 0.9993\ldots + 0.9989\ldots - 0.9986\ldots \\ &\quad + 0.9983 - 0.9982\ldots + \cdots + \frac{(-1)^{n+1}}{n^{1/1000}} \quad (n \in \mathbb{N}), \end{aligned}$$

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the Dirichlet eta function,[1]

$$\eta(p) := \sum_{j=1}^{\infty} (-1)^{j+1} \frac{1}{j^p} \quad (p > 0),$$

is not a very useful tool although $H^*\big(n, \frac{1}{1000}\big) \approx \lim_{n\to\infty} H^*\big(n, \frac{1}{1000}\big) = \eta(\frac{1}{1000}) \approx \lim_{p\downarrow 0} \eta(p) = \frac{1}{2}$ [1, Proposition 5.3]. The approximation $H^*\big(n, \frac{1}{1000}\big) \approx \eta(\frac{1}{1000})$ is good only for "very large" n due to slow convergence of $H^*\big(n, \frac{1}{1000}\big)$ illustrated[2] in Figure 1.

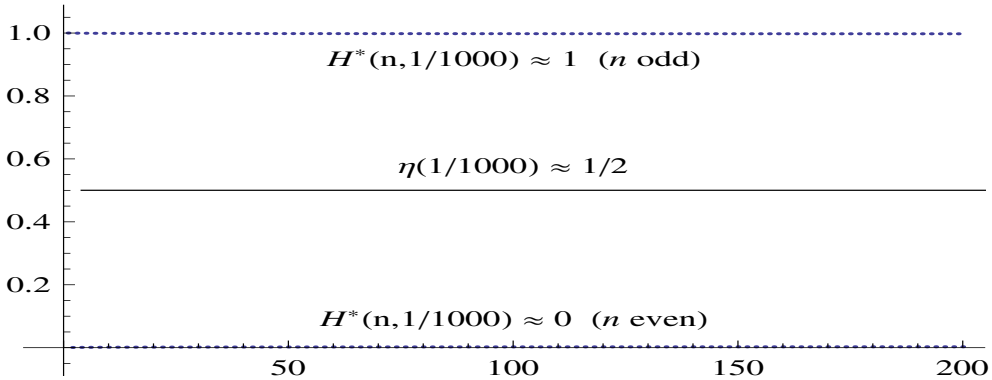

FIGURE 1. The graph of the sequence $n \mapsto H^*\big(n, \frac{1}{1000}\big)$.

We also remark that for any (fixed) $n \in \mathbb{N}$ we have

$$\lim_{p\downarrow 0} \sum_{j=1}^{n} \frac{(-1)^{j+1}}{j^p} = \lim_{p\downarrow 0} \begin{cases} 1 - \sum_{i=1}^{(n-1)/2} \left(\frac{1}{(2i)^p} - \frac{1}{(2i+1)^p} \right), & n \text{ odd} \\ \sum_{i=1}^{n/2} \left(\frac{1}{(2i-1)^p} - \frac{1}{(2i)^p} \right), & n \text{ even} \end{cases} = \begin{cases} 1, & n \text{ odd} \\ 0, & n \text{ even.} \end{cases}$$

Motivated by the results obtained by L. Tóth – J. Bukor [5], V. Timofte [4], and the Sîntămărian's papers [2, 3], we shall accurately estimate the remainder

$$\rho(n,p) := \eta(p) - H^*(n,p) \quad (n \in \mathbb{N},\ p \in \mathbb{R}^+), \tag{1.1}$$

where

$$H^*(n,p) := \sum_{j=1}^{n} (-1)^{j+1} \frac{1}{j^p} \tag{1.2}$$

is called the nth alternating generalized harmonic number of order p.

[1] known also as the alternating Riemann zeta function

[2] In this article all figures are plotted by Stephen Wolfram's Mathematica Software, Version 7.0.

2. Auxiliary results

In the sequel we shall use special sums[3]

$$\sigma_q^*(x,p) := \sum_{i=1}^{\lfloor q/2 \rfloor} \left(4^i - 1\right) \frac{B_{2i} \cdot p^{(2i-1)}}{x^{p+2i-1} \cdot (2i)!} \quad (q \in \mathbb{N},\, p,x \in \mathbb{R}), \tag{2.1}$$

where B_k is the kth Bernoulli coefficient (or Bernoulli number)[4], defined by the identity $\frac{t}{e^t-1} \equiv \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$ $(|t| < 2\pi)$. Here, for $x \in \mathbb{R}$ and $k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$ the symbol $x^{(k)}$ means the upper (rising) Pochhammer product defined as

$$x^{(0)} := 1 \quad \text{and} \quad x^{(k)} := \prod_{i=0}^{k-1} (x+i) = x(x+1) \cdot \ldots \cdot (x+k-1) \quad (k \in \mathbb{N}). \tag{2.2}$$

According to [1, Theorem 3.2] there holds the following lemma.

Lemma 1. *For $p \in \mathbb{R}^+$ and for any $k,n,q \in \mathbb{N}$, where $n \geq 2k+1 \geq 3$, we have*

$$H^*(n,p) = H_q^*(k,n,p) + r_q^*(k,p), \tag{2.3}$$

where

$$H_q^*(k,n,p) := A_q^*(k,p) + B_q^*(n,p), \tag{2.4}$$

$$A_q^*(k,p) := H^*(2k-1,p) - \frac{1}{2\left(2k\right)^p} - \sigma_q^*\left(2k,p\right), \tag{2.5}$$

$$B_q^*(n,p) := \frac{(-1)^{n+1}}{2\left(2\lfloor \frac{n+1}{2} \rfloor\right)^p} + \sigma_q^*\left(2\lfloor \tfrac{n+1}{2} \rfloor, p\right), \tag{2.6}$$

$$\left|r_q^*(k,p)\right| \leq +\frac{p^{(q-1)} \cdot \pi^{p+1}}{3(2k\pi)^{p+q-1}} \tag{2.7}$$

$$. \tag{2.8}$$

Referring to [1, Theorem 5.2] we have also the next lemma.

Lemma 2. *For $p \in \mathbb{R}^+$ and for any $k,q \in \mathbb{N}$, there holds the following relation (see (2.1)),*

$$\eta(p) = H^*(2k-1,p) - \frac{1}{2\left(2k\right)^p} - \sigma_q^*\left(2k,p\right) + \varepsilon_q(k,p), \tag{2.9}$$

where

$$\left|\varepsilon_q(k,p)\right| < \frac{p^{(q-1)}}{3\pi^{q-2}(2k)^{p+q-1}}. \tag{2.10}$$

[3] By definition, $\sum_{i=m}^{n} x_i = 0$ if $m > n$.

[4] $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = B_5 = B_7 = \ldots = 0$, $B_4 = B_8 = -\frac{1}{30}$, $B_6 = \frac{1}{42}, \ldots$

3. The main result

From Lemmas 2 and 1 immediately follows the next result.

Theorem 1. *For $p \in \mathbb{R}^+$ and every $k,n,q \in \mathbb{N}$, with $n \geq 2k+1 \geq 3$, the nth remainder $\rho(n,p) := \eta(p) - \sum_{j=1}^{n}(-1)^{j+1}\frac{1}{j^p}$ is given in the form*

$$\rho(n,p) = \Delta_q(n,p) + \delta_q(k,p), \tag{3.1}$$

where

$$\Delta_q(n,p) = \frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \sigma_q^*\left(2\lfloor\tfrac{n+1}{2}\rfloor, p\right), \tag{3.2}$$

$$\left|\delta_q(k,p)\right| < \frac{5p^{(q-1)}}{3\pi^{q-1}} \cdot \frac{1}{(2k)^{p+q-1}}. \tag{3.3}$$

Proof. We have

$$\begin{aligned}
\rho(n,p) &= \eta(p) - H^*(n,p) \\
&= \left(\cancel{H^*(2k-1,p)} - \frac{1}{2(2k)^p} - \cancel{\sigma_q^*(2k,p)} + \varepsilon_q(k,p)\right) \\
&\quad - \left(\cancel{H^*(2k-1,p)} - \frac{1}{2(2k)^p} - \cancel{\sigma_q^*(2k,p)} + \frac{(-1)^{n+1}}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p}\right. \\
&\qquad \left. + \sigma_q^*\left(2\lfloor\tfrac{n+1}{2}\rfloor, p\right) + r_q^*(k,p)\right) \\
&= \varepsilon_q(k,p) - \frac{(-1)^{n+1}}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \sigma_q^*\left(2\lfloor\tfrac{n+1}{2}\rfloor, p\right) - r_q^*(k,p),
\end{aligned}$$

where

$$\begin{aligned}
\left|\delta_q(k,p)\right| &= \left|\varepsilon_q(k,p) - r_q^*(k,p)\right| \\
&< \frac{p^{(q-1)}}{3\pi^{q-2}(2k)^{p+q-1}} + \frac{p^{(q-1)}}{3(2k\pi)^{p+q-1}} \\
&= \frac{p^{(q-1)}}{3\pi^{q-2}(2k)^{p+q-1}}\left(1 + \pi^{-(p+1)}\right) \\
&< \frac{p^{(q-1)}}{3\pi^{q-2}(2k)^{p+q-1}} \cdot \frac{5}{\pi}.
\end{aligned}$$

□

In our Theorem k and q are parameters controlling the magnitude of the error term $\delta_q(k,p) = \varepsilon_q(k,p) - r_q(k,p)$. We add that, for $0 < p \leq 1$ and $q \in \mathbb{N}$, using the Stirling factorial formula, we estimate

$$p^{(q-1)} = p(p+1)\cdots(p+q-2) \leq (q-1)! = \frac{q!}{q} < \frac{1}{q}\left(\frac{q}{e}\right)^q \sqrt{2\pi q} \cdot e^{\frac{1}{12q}}.$$

Thus, for $n \geq 2k+1 \geq 3$ and $0 < p \leq 1$, we obtain the estimates

$$\begin{aligned}\left|\delta_q(k,p)\right| &< \frac{5p^{(q-1)}}{3\pi^{q-1}} \cdot \frac{1}{(2k)^{p+q-1}} < \frac{5}{3}\sqrt{\frac{2\pi}{q}}\left(\frac{q}{e}\right)^q e^{\frac{1}{12q}} \cdot \frac{1}{(2\pi k)^{q-1}} \\ &= \frac{5}{3e}\sqrt{2\pi q}\left(\frac{q}{2e\pi k}\right)^{q-1} e^{\frac{1}{12q}} < 2\sqrt{q}\left(\frac{q}{2e\pi k}\right)^{q-1}. \end{aligned} \tag{3.4}$$

Figure 2 shows the graph of the sequence $n \mapsto \Delta_5(n, 1/1000)$.

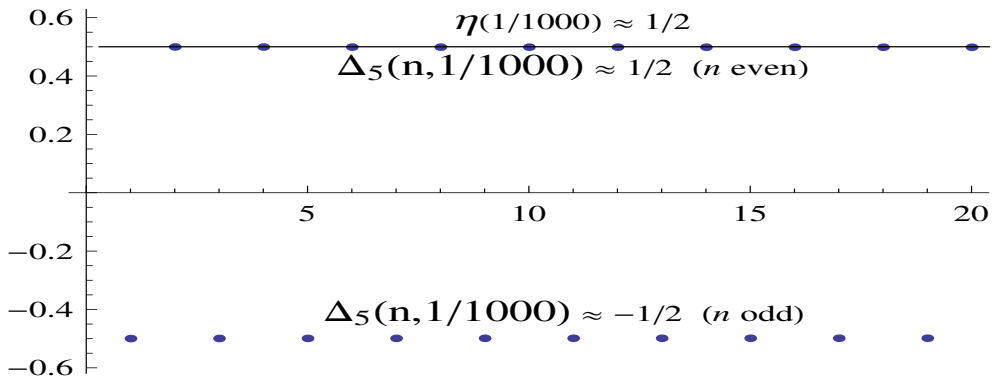

FIGURE 2. The graph of the sequence $n \mapsto \Delta_5\left(n, \frac{1}{1000}\right)$.

Since $\sigma_1(x,p) \equiv 0$, we obtain from Theorem the first corollary.

Corollary 1. *For $p \in \mathbb{R}^+$ and any $k,n \in \mathbb{N}$, with $n \geq 2k+1 \geq 3$, we have*

$$\begin{aligned}&\sum_{j=1}^{n}(-1)^{j+1}\frac{1}{j^p} + \frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \frac{5}{3(2k)^p} \\ &\qquad < \eta(p) < \sum_{j=1}^{n}(-1)^{j+1}\frac{1}{j^p} + \frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} + \frac{5}{3(2k)^p}.\end{aligned}$$

Setting $q = 5$ in Theorem we obtain the following result.

Corollary 2. *For $p \in \mathbb{R}^+$ and $k,n \in \mathbb{N}$, such that $n \geq 2k+1 \geq 3$, the following equation follows*

$$\begin{aligned}\eta(p) = &\sum_{j=1}^{n}(-1)^{j+1}\frac{1}{j^p} + \frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^p} - \frac{p}{4\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{p+1}} \\ &+ \frac{p(p+1)(p+2)}{48\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{p+3}} + \delta_5(k,p),\end{aligned}$$

where

$$\left|\delta_5(k,p)\right| < \frac{5p(p+1)(p+2)(p+3)}{3\pi^4(2k)^{p+4}}.$$

Example 1. Since $\left|\delta_5(10,0.001)\right| < 10^{-9}$ the approximation

$$\eta(0.001)\approx S_n^*:=\sum_{j=1}^{n}(-1)^{j+1}\frac{1}{j^{0.001}}+\frac{(-1)^n}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{0.001}}-\frac{0.001}{4\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{1.001}}+\frac{0.001\times1.001\times2.001}{48\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{3.001}}$$

is very accurate for every $n \geq 21$ as is seen from Figure 3 where are plotted the graphs of the sequence $n \mapsto S_n^*$ and the constant function $n \mapsto \eta(0.001) = H^*(\infty, 0.001) = 0.5002257\ldots$. The corrected term $\Delta_5(n, 0.001)$ is crucial since, for example, we have

$H^*(100000, 0.001) = 0.005949215\ldots$, $H^*(100001, 0.001) = 0.994502300\ldots$, $\Delta_5(100000, 0.001) = 0.4942765\mathbf{44}\ldots$ and $\Delta_5(100001, 0.001) = -0.4942765\mathbf{39}\ldots$.

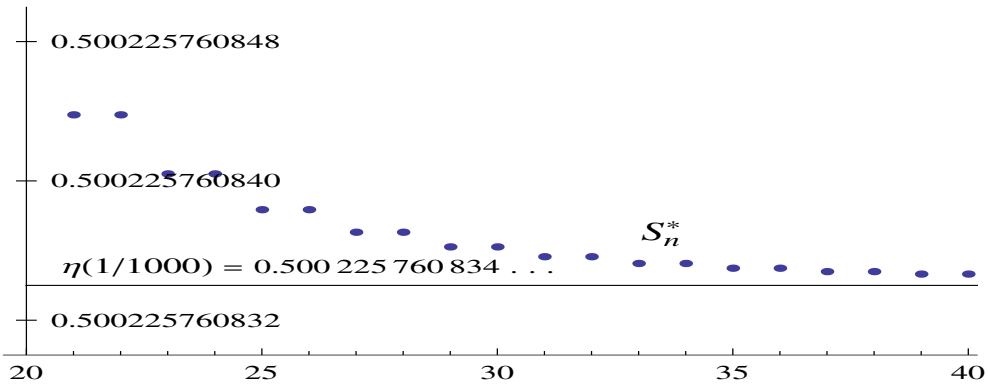

FIGURE 3. The graph of the sequence $n \mapsto S_n^*$.

Obviously, using the approximation $\eta(0.001) \approx S_n^*$ we directly obtain accurate approximation of $H^*\left(n, \frac{1}{1000}\right)$ from Introduction:

$$H^*\left(n,\frac{1}{1000}\right)\approx\eta(0.001)+\frac{(-1)^{n+1}}{2\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{0.001}}+\frac{0.001}{4\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{1.001}}-\frac{0.001\times1.001\times2.001}{48\left(2\lfloor\frac{n+1}{2}\rfloor\right)^{3.001}}\qquad(n\in\mathbb{N}).$$

CONCLUSION

The paper provides an estimate of the remainder of the series representing Dirichlet's function $\eta(p)$ for $p \in \mathbb{R}^+$. The presented estimate is effective also in the case of slowly convergent series occurring for $p \approx 0$.

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SOME PROPERTIES OF CERTAIN MEROMORPHIC MULTIVALENT CLOSE-TO-CONVEX FUNCTIONS

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Abstract. In this paper, we introduce and investigate a certain subclass of meromorphic multivalent close-to-convex functions. Such results as coefficient inequalities, and radius of meromorphic convexity are derived.

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1. Introduction

Let $\sum_p$ denote the class of functions f of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_n z^n (p \in \mathbb{N} := \{1,2,3\cdots\}), \tag{1.1}$$

which are analytic in the punctured open unit disk

$$\mathbb{U}^* := \{z : z \in \mathbb{C} \text{and} 0 < |z| < 1\} =: \mathbb{U} \setminus 0.$$

Let $\mathcal{P}$ denote the class of functions p given by

$$p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n (z \in \mathbb{U}), \tag{1.2}$$

which are analytic and convex in $\mathbb{U}$ and satisfy the condition $\mathbb{R}(p(z)) > 0 (z \in \mathbb{U})$.

A function $f \in \sum_p$ is said to be in the class $\mathcal{MS}_p^*(\alpha)$ of meromorphic multivalent starlike functions of order α if it satisfies the inequality

$$\mathbb{R}\left(\frac{zf'(z)}{f(z)}\right) < -\alpha (z \in \mathbb{U}, 0 \leq \alpha < p).$$

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Let

$$g(z)=\frac{1}{z^p}+\sum_{n=0}^{\infty}c_nz^n\in\mathcal{MS}_p^*(\alpha), \tag{1.3}$$

a function $f\in\sum_p$ is said to be in the class $\mathcal{MC}_p$ of meromorphic multivalent close-to-convex functions if it satisfies the inequality

$$\mathbb{R}\Big(\frac{zf'(z)}{g(z)}\Big)<0(z\in\mathbb{U},g\in\mathcal{MS}_p^*(0)=:\mathcal{MS}_p^*).$$

In many earlier investigations(for example [2,3,5–7,10–12,15–21,24]), various interesting subclasses of the close-to-convex functions have been studied from a number of different viewpoints. In particular, Gao and Zhou[3](see also [7, 10, 21, 24]) considered a subclass $\mathcal{K}_s$ of close-to-convex functions, which satisfy the condition

$$\mathbb{R}\Big(\frac{z^2f'(z)}{g(z)g(-z)}\Big)<0(z\in\mathbb{U},g\in\mathcal{S}^*(1/2)),$$

where $f(z)=z+a_2z^2+\cdots$, and $\mathcal{S}^*(1/2)$ denotes the usual class of starlike functions of order $1/2$.

Recently, Z.G.Wang et al.[22] introduced the meromophic close-to-convex functions class $\mathcal{MK}$, which satisfy the condition

$$\mathbb{R}\Big(\frac{f'(z)}{g(z)g(-z)}\Big)>0(z\in\mathbb{U},g\in\mathcal{MS}^*(p/2)),$$

where $f(z)=1/z+a_1z+a_2z^2+\cdots$.

Motivated essentially by the above mentioned works, we introduce a class of meoromorphic multivalent functions related to the meoromorphic multivalent starlike functions, and obtain some interesting results.

Definition 1. A function $f\in\sum_p$ is said to be in the $\mathcal{MK}_p$ if it satisfies the inequality

$$\mathbb{R}\Big(\frac{zf'(z)}{z^pg(z)g(-z)}\Big)>0(z\in\mathbb{U}), \tag{1.4}$$

where $g\in\mathcal{MS}_p^*(p/2)$.

For some recent investigation of meromorphic multivalent functions, see (for example)the works of [1, 8, 9, 13, 23, 25] and the references cited therein.

In the present paper, we prove that the class $\mathcal{MK}_p$ is a subclass of meromorphic multivalent close-to-convex functions.

Theorem 1. *Suppose that* $\mu(z)\in\mathcal{MS}_p^*(\alpha_1)$ *and* $\nu(z)\in\mathcal{MS}_p^*(\alpha_2)$ *with* $0\le\alpha_1+\alpha_2-p<p$. *Then*

$$z^p\mu(z)\nu(z)\in\mathcal{MS}_p^*(\alpha_1+\alpha_2-p).$$

Proof of Theorem 1. Let $\mu(z) \in \mathcal{MS}_p^*(\alpha_1)$ and $\nu(z) \in \mathcal{MS}_p^*(\alpha_2)$. By definition, we know that

$$\mathbb{R}\Big(\frac{z\mu'(z)}{\mu(z)}\Big) < -\alpha_1 (z \in \mathbb{U}, 0 \leq \alpha_1 < p),$$

and

$$\mathbb{R}\Big(\frac{z\nu'(z)}{\nu(z)}\Big) < -\alpha_2 (z \in \mathbb{U}, 0 \leq \alpha_2 < p).$$

Next, we assume that

$$h(z) = z^p \mu(z)\nu(z).$$

Then, we easily get

$$\frac{zh'(z)}{h(z)} = \frac{z\mu'(z)}{\mu(z)} + \frac{z\nu'(z)}{\nu(z)} + p.$$

It follows that

$$\mathbb{R}\Big(\frac{zh'(z)}{h(z)}\Big) = \mathbb{R}\Big(\frac{z\mu'(z)}{\mu(z)}\Big) + \mathbb{R}\Big(\frac{z\nu'(z)}{\nu(z)}\Big) + p < -(\alpha_1 + \alpha_2 - p).$$

Noting that $0 \leq \alpha_1 + \alpha_2 - p < p$, which implies that

$$h(z) \in \mathcal{MS}_p^*(\alpha_1 + \alpha_2 - p).$$

This completes the proof of Theorem 1. □

Theorem 2. *Let $g \in \mathcal{MS}_p^*(p/2)$. Then*

$$-z^p g(z)g(-z) \in \mathcal{MS}_p^*(0) =: \mathcal{MS}_p^*.$$

Proof of Theorem 2. Similar to the proof of Theorem 1, we can get

$$\mathbb{R}\Big(\frac{z(-z^p g(z)g(-z))'}{-z^p g(z)g(-z)}\Big) = p + \mathbb{R}\Big(\frac{zg'(z)}{g(z)}\Big) + \mathbb{R}\Big(\frac{-zg'(-z)}{g(-z)}\Big) < p - \frac{p}{2} - \frac{p}{2} = 0$$

(noting that $-z \in \mathbb{U}$). This implies the Theorem 2. □

In view of the definitions $\mathcal{MC}_p$, $\mathcal{MK}_p$ and Theorem 2, we deduce that the class $\mathcal{MK}_p$ is a subclass of the class $\mathcal{MC}_p$ of meromorphic close-to-convex functions.

To derive coefficient inequalities of $f \in \mathcal{MK}_p$, we need consider the parity of p. First, we consider the case that p is odd.

2. The case p=2k-1

In order to prove our main results, we need the following two lemmas.

Lemma 1 ([14]). *Suppose that* $h(z)=\frac{1}{z^p}+\sum_{n=0}^{\infty}c_nz^n\in\mathcal{MS}_p^*$. *Then*

$$|c_n|\le(2p)/(n+p)(n\in\mathbb{N}:=\{1,2,3\cdots\}).$$

Equality holds for the function $h(z)=z^{-p}(1+z^{n+p})^{2p/(n+p)}$.

Lemma 2. *Let* $g(z)=\frac{1}{z^p}+\sum_{n=0}^{\infty}b_nz^n\in\mathcal{MS}_p^*(p/2)$. *Then*

$$|B_{2m-1}|\le(2p)/(2m-1+p)(m\in\mathbb{N}),$$

where

$$B_{2m-1}=\begin{cases}2b_{2m-1}, & 2m-1<p,\\ 2b_{2m-1}-2b_0b_{2m-1-p}+2b_1b_{2m-2-p}-\cdots+ & \\ (-1)^{m-\frac{p+1}{2}}2b_{m-\frac{p+3}{2}}b_{m-\frac{p-1}{2}}+(-1)^{m-\frac{p-1}{2}}b^2_{m-\frac{p+1}{2}}, & 2m-1\ge p.\end{cases}\tag{2.1}$$

Equality holds for the function $g(z)=z^{-p}(1+z^{n+p})^{p/(n+p)}$.

Proof. Suppose that

$$G(z):=-z^pg(z)g(-z).\tag{2.2}$$

In view of Theorem 2, we know that $G(z)\in\mathcal{MS}_p^*$. When $p=2k-1(k=1,2,\cdots)$, it is easy to verify that

$$G(-z)=-G(z),$$

which implies that $G(z)$ is a meromorphic odd starlike multivalent function. If we set

$$G(z)=\frac{1}{z^p}+\sum_{m=1}^{\infty}B_{2m-1}z^{2m-1},\tag{2.3}$$

it follows from Lemma 1 that

$$|B_{2m-1}|\le(2p)/(2m-1+p)(m\in\mathbb{N}).\tag{2.4}$$

By substituting the series expressions of $g(z)$ and $G(z)$ into (2.2) and carefully comparing the similar items of two sides of resulting equation, we get the desired expression of B_{2m-1} given by (2.1). □

Theorem 3. *Suppose that* $f(z)=\frac{1}{z^p}+\sum_{n=0}^{\infty}a_nz^n\in\mathcal{MK}_p$. *Then*

$$|a_{2n}|\le\begin{cases}p/n, & 2n-3<p,\\ \frac{p}{n}\left(1+\frac{2p}{n-1}+\frac{2p}{n-2}+\cdots+\frac{2p}{1+p}\right), & 2n-3\ge p,\end{cases}(n\in\mathbb{N})\tag{2.5}$$

and

$$(2n-1)|a_{2n-1}| \leq \begin{cases} 2p + \frac{2p^2}{(2n-1+p)}, & 2n-3 < p, \\ 2p + \frac{2p^2}{n-1} + \frac{2p^2}{n-2} + \cdots + \frac{2p^2}{1+p} + \frac{2p^2}{2n-1+p}, & 2n-3 \geq p. \end{cases} \tag{2.6}$$

Proof of Theorem 3. Suppose that $f \in \mathcal{MK}_p$. Then, we know that $\mathbb{R}\left(\frac{zf'(z)}{G(z)}\right) < 0$, where G is given by (2.2). If we set

$$q(z) := -\frac{zf'(z)}{pG(z)}, \tag{2.7}$$

it follows that

$$q(z) = 1 + d_{p+1}z^{p+1} + d_{p+2}z^{p+2} + \cdots \in \mathcal{P}.$$

By substituting the series expressions of f, G and q into (2.7), we get

$$p(1 + d_{p+1}z^{p+1} + \cdots + d_{p+n}z^{p+n} + \cdots)\left(\frac{1}{z^p} + B_1 z + B_3 z^3 + \cdots + B_{2n-1}z^{2n-1} + \cdots\right)$$

$$= \frac{p}{z^p} - a_1 z - 2a_2 z^2 - \cdots - 2na_{2n}z^{2n} - (2n+1)a_{2n+1}z^{2n+1} - \cdots. \tag{2.8}$$

We get from (2.8) that

$$\frac{-2n}{p}a_{2n} = \begin{cases} d_{p+2n}, & 2n-3 < p, \\ d_{p+2n} + d_{p+2}B_{2n-2-p} + \cdots + d_{2n-1}B_1, & 2n-3 \geq p, \end{cases} \tag{2.9}$$

and

$$\frac{2n-1}{-p}a_{2n-1} = \begin{cases} d_{p+2n-1} + B_{2n-1}, & 2n-3 < p, \\ d_{p+2n-1} + B_{2n-1} + d_{p+1}B_{2n-p-2} & \\ +d_{p+2}B_{2n-p-3} + \cdots + d_{2n-2}B_1, & 2n-3 \geq p. \end{cases} \tag{2.10}$$

For $q(z) \in \mathcal{P}$, we know that $|d_{n+p}| \leq 2$ ([4]). Moreover, combining (2.4), (2.8), (2.9) and (2.10), we get (2.5) and (2.6). □

Theorem 4. *Let $g(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} b_n z^n \in \mathcal{MS}_p^*(p/2)$. If $f \in \sum_p$ satisfies condition*

$$\sum_{n=1}^{\infty} n|a_n| + p\sum_{n=1}^{\infty} |B_{2n-1}| \leq p, \tag{2.11}$$

where B_{2n-1} is given by (2.1), then $f \in \mathcal{MK}_p$.

Proof of Theorem 4. To prove $f \in \mathcal{MK}_p$, it needs to show that

$$\mathbb{R}\left(\frac{f'(z)}{g(z)g(-z)}\right) = \mathbb{R}\left(\frac{zf'(z)}{G(z)}\right) > 0,$$

i,e, it suffices to show that

$$\left|\frac{zf'(z)}{G(z)}+p\right|<\left|\frac{zf'(z)}{G(z)}-p\right|,$$

where G is given by (2.3). From (2.11), it is easy to know that

$$\sum_{n=1}^{\infty} n|a_n|+p\sum_{n=1}^{\infty}|B_{2n-1}|\leq 2p-\sum_{n=1}^{\infty} n|a_n|-p\sum_{n=1}^{\infty}|B_{2n-1}|. \tag{2.12}$$

Now, by the maximum principle, we deduce from (1.1) and (2.12) that

$$\left|\frac{\frac{zf'(z)}{G(z)}+p}{\frac{zf'(z)}{G(z)}-p}\right|=\left|\frac{\sum_{n=1}^{\infty} na_nz^{n+p}+\sum_{n=1}^{\infty} pB_{2n-1}z^{2n+p-1}}{\sum_{n=1}^{\infty} na_nz^{n+p}-\sum_{n=1}^{\infty} pB_{2n-1}z^{2n+p-1}-2p}\right|$$
$$<\frac{\sum_{n=1}^{\infty} n|a_n|+\sum_{n=1}^{\infty} p|B_{2n-1}|}{2p-\sum_{n=1}^{\infty} n|a_n|-\sum_{n=1}^{\infty} p|B_{2n-1}|}\leq 1.$$

This evidently complete proof of Theorem 4. □

Moreover, we consider the case that p is even.

3. The case p=2k

By similarly applying the method of proof of Lemma 3, we easily get the following Lemma.

Lemma 3. *Let $p=2k, k\in\mathbb{N}$ and*

$$g(z)=\frac{1}{z^p}+\sum_{n=0}^{\infty} b_nz^n\in\mathcal{MS}_p^*(p/2).$$

Then

$$|B_{2m}|\leq(2p)/(2m+p)(m\in\mathbb{N}), \tag{3.1}$$

where

$$B_{2m}=\begin{cases}2b_{2m}, & 2m<p,\\ 2b_{2m}+2b_0b_{2m-p}-2b_1b_{2m-1-p}+\cdots+ & \\ (-1)^{m-\frac{p+2}{2}}2b_{m-\frac{p+2}{2}}b_{m-\frac{p-2}{2}}+(-1)^{m-\frac{p}{2}}b_{m-\frac{p}{2}}^2, & 2m\geq p.\end{cases} \tag{3.2}$$

Equality holds for the function $h(z)=z^{-p}(1+z^{n+p})^{p/(n+p)}$.

Theorem 5. *Suppose that*

$$f(z)=\frac{1}{z^p}+\sum_{n=0}^{\infty} a_nz^n\in\mathcal{MK}_p.$$

Then

$$|a_{2n}| \leq \begin{cases} p/n + p^2/(2n^2+np), & 2n < p, \\ \frac{p}{n}\left(1 + \frac{2p}{n-1} + \frac{2p}{n-2} + \cdots + \frac{2p}{1+p} + 2\right) + \frac{p^2}{2n^2+np}, & 2n \geq p, \end{cases} (n \in \mathbb{N}) \tag{3.3}$$

and

$$(2n-1)|a_{2n-1}| \leq \begin{cases} 2p, & 2n-2 < p, \\ 2p + \frac{2p^2}{n-1} + \frac{2p^2}{n-2} + \cdots + \frac{2p^2}{1+p}, & 2n-2 \geq p. \end{cases} \tag{3.4}$$

Proof of Theorem 5. Suppose that $f \in \mathcal{M}\mathcal{K}_p$. Then, we know that $\mathbb{R}\left(\frac{zf'(z)}{G(z)}\right) < 0$, where G is given by (2.2). For $p = 2k, k \in \mathbb{N}$, it is easy to deduce that $G(z)$ is a meromorphic even starlike multivalent function. If we set

$$G(z) = \frac{1}{z^p} + \sum_{m=0}^{\infty} B_{2m} z^{2m}, \tag{3.5}$$

where B_{2m} is defined by (3.2) and

$$\tau(z) := -\frac{zf'(z)}{pG(z)}, \tag{3.6}$$

it follows that

$$\tau(z) = 1 + d_p z^p + d_{p+1} z^{p+1} + \cdots \in \mathcal{P}.$$

By substituting the series expressions of f, G and τ into (3.6), we get

$$p(1 + d_p z^p + d_{p+1} z^{p+1} + \cdots + d_{p+n} z^{p+n} + \cdots)\left(\frac{1}{z^p} + B_0 + B_2 z^2 + \cdots + B_{2n} z^{2n} + \cdots\right)$$

$$= \frac{p}{z^p} - a_1 z - 2a_2 z^2 - \cdots - 2na_{2n} z^{2n} - (2n+1)a_{2n+1} z^{2n+1} - \cdots. \tag{3.7}$$

We get from (3.7) that

$$\frac{-2n}{p} a_{2n} = \begin{cases} d_{p+2n} + B_{2n}, & 2n < p, \\ d_{p+2n} + B_{2n} + d_{p+2} B_{2n-2-p} + \cdots + d_{2n} B_0, & 2n \geq p, \end{cases} \tag{3.8}$$

and

$$\frac{2n-1}{-p} a_{2n-1} = \begin{cases} d_{p+2n-1}, & 2n-2 < p, \\ d_{p+2n-1} + d_{p+1} B_{2n-p-2} + \cdots + d_{2n-2} B_1, & 2n-2 \geq p. \end{cases} \tag{3.9}$$

For $\tau(z) \in \mathcal{P}$, we know that $|d_{n+p}| \leq 2$(see [4]). Moreover, combining (3.1),(3.7),(3.8) and (3.9), we get (3.3) and (3.4). □

Theorem 6. *If $f \in \sum_p$ satisfies condition*

$$\sum_{n=1}^{\infty} n|a_n| + p \sum_{n=0}^{\infty} |B_{2n}| \leq p, \tag{3.10}$$

where B_{2n} is given by (3.2), then $f \in \mathcal{MK}_p$.

Proof of Theorem 6. The proof of Theorem 6 is similar to Theorem 4, we here omit the details. □

4. On the convexity radius of the functions in $\mathcal{MK}_p$

We say a function $f(z) \in \mathcal{MK}_p$ is meromorphic convex, if $f(z)$ satisfies condition:

$$\mathbb{R}\Big(1+\frac{zf''(z)}{f'(z)}\Big) < 0 (z \in \mathbb{U}).$$

When we give the convexity radius of the functions in $\mathcal{MK}_p$, we need the following lemmas.

Lemma 4. *Let $G(z)$ is given by (2.2) and $r < 1$, then*

$$\mathbb{R}\Big(\frac{zG'(z)}{G(z)}\Big) \leq -\frac{1-r^2}{1+r^2} p(|z| = r).$$

Proof. Suppose that

$$H(z) := -\frac{zG'(z)}{pG(z)} (G(z) \in \mathcal{MS}_p^*), \tag{4.1}$$

where $G(z)$ is given by (2.2), we easily know that $G(z)$ is an odd or even meromorphic starlike function, also $H(z) \in \mathcal{P}$ and is an even function, which imply that

$$H(z) = \frac{1+[w(z)]^2}{1-[w(z)]^2}, \tag{4.2}$$

where $w(z)$ is Schwarz function with $w(0) = 0$ and $|w(z)| < 1$. Thus, we get from (4.2) that

$$[w(z)]^2 = \frac{H(z)-1}{H(z)+1}.$$

So

$$\Big|\frac{H(z)-1}{H(z)+1}\Big| = |w(z)|^2 \leq |z|^2,$$

this inequality can be written as

$$|H(z)|^2 - 2Re\{H(z)\} + 1 \leq |z|^4\{|H(z)|^2 + 2Re\{H(z)\} + 1\}.$$

From above inequality we can get

$$\Big|H(z) - \frac{1+|z|^4}{1-|z|^4}\Big|^2 \leq \Big(\frac{1+|z|^4}{1-|z|^4}\Big)^2 - 1 \leq \Big(\frac{2|z|^2}{1-|z|^4}\Big)^2,$$

that is

$$\left|H(z)-\frac{1+|z|^4}{1-|z|^4}\right| \leq \frac{2|z|^2}{1-|z|^4}.$$

From this inequality we get

$$\mathbb{R}\{-H(z)\} \leq -\frac{1-|z|^2}{1+|z|^2} = -\frac{1-r^2}{1+r^2},$$

this implies Lamma 4. □

Lemma 5 (see [3]). *Let $q(z)$ satisfy $q(0)=1, \mathbb{R}\{q(z)\}>0$, then we have*

$$\left|\frac{zq'(z)}{q(z)}\right| \leq \frac{2r}{1-r^2}(|z|=r<1).$$

Theorem 7. *Let $f(z) \in \mathcal{MK}_p$, then $f(z)$ is meromorphic convex in*

$$0<|z|<r_p = \frac{1}{2}\sqrt{4+\frac{1}{p^2}} - \frac{\sqrt{\frac{1}{\sqrt{4+\frac{1}{p^2}}p^3}+\frac{1}{p^2}+\frac{4}{\sqrt{4+\frac{1}{p^2}}p}}}{\sqrt{2}} + \frac{1}{2p}. \tag{4.3}$$

Proof of Theorem 7. When $f(z) \in \mathcal{MK}_p$, there exists $g(z) \in \mathcal{MS}_p^*(p/2)$ such that (1.4) holds, also $G(z) = -z^p g(z)g(-z)$ is an odd or even meromophic starlike multivalent function, so from (2.7) and (3.5) we have

$$zf'(z) = -pG(z)\cdot q(z),$$

where $q(z)$ satisfies the condition of Lemma 5, and

$$1+\frac{zf''(z)}{f'(z)} = \frac{zG'(z)}{G(z)} + \frac{zq'(z)}{q(z)}.$$

So using Lemma 4 and 5 we can get

$$\begin{aligned}\mathbb{R}\left\{1+\frac{zf''(z)}{f'(z)}\right\} &= \mathbb{R}\left\{\frac{zG'(z)}{G(z)}\right\} + \mathbb{R}\left\{\frac{zq'(z)}{q(z)}\right\} \\ &\leq -\frac{1-r^2}{1+r^2}p + \left|\frac{zq'(z)}{q(z)}\right| \\ &\leq -\frac{1-r^2}{1+r^2}p + \frac{2r}{1-r^2} = \frac{-pr^4+2r^3+2pr^2+2r-p}{1-r^4}.\end{aligned}$$

It is easy to know that if $-pr^4+2r^3+2pr^2+2r-p<0$, we have $\mathbb{R}\left(1+\frac{zf''(z)}{f'(z)}\right) < 0$. Let

$$T_p(r) = -pr^4+2r^3+2pr^2+2r-p<0,$$

because $T_p(0) = -p < 0, T_p(1) = 4$, and

$$T_p'(r) = -4pr^3 + 6r^2 + 4pr + 2 = 4pr(1-r^2) + 6r^2 + 2 > 0 (0 < r < 1).$$

It follows that $T_p(r)$ are strictly monotone increasing functions of r, and for very p, equation $T_p(r) = 0$ has only a root r_p in interval (0, 1), solve those equations we get the r_p in (4.3). Thus when $0 < |z| < r_p$, $\mathbb{R}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 0$, that is, $f(z)$ is meromophic convex in $0 < |z| < r_p$. □

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APPROXIMATION SOLUTIONS OF QUASI-EQUILIBRIUM PROBLEMS IN BANACH SPACES

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Abstract. In this paper, we study to approximate solutions of quasi-equilibrium problems in Banach spaces, which generalizes equilibrium problems and quasi-variational inequalities. In fact, we prove weak convergence of the sequence generated by the proximal point method to a solution of the quasi-equilibrium problem. Then we show that the generated sequence converges strongly to a solution of the quasi-equilibrium problem.

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1. Introduction

Let E be a real Banach space and $C \subset E$ be a nonempty, closed and convex set, and $K : C \to 2^C$ be a multivalued mapping such that for all $x \in C$, $K(x)$ is a nonempty, closed and convex subset of C. Suppose that $f : E \times E \to \mathbb{R}$ is a bifunction. The quasi-equilibrium problem (QEP(f,K)) is to find $x^* \in K(x^*)$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in K(x^*). \tag{1.1}$$

The set of all solutions of QEP(f,K) is denoted by $S(f,K)$. Also, the set of all fixed points of K is denoted by Fix(K). The associated Minty quasi-equilibrium problem is to find $x^* \in K(x^*)$ such that $f(y,x^*) \leq 0$ for all $y \in K(x^*)$. When $K(x) = C$ for all $x \in C$, the quasi-equilibrium problem QEP(f,K) becomes a classical equilibrium problem EP(f,C), also the associated Minty quasi-equilibrium problem becomes a classical Minty equilibrium problem (see [2] and [20]).

Equilibrium problems extend and unify many problems in optimization, variational inequalities, fixed point theory, complementarity problems, Nash equilibria and many other problems in nonlinear analysis (see [7], [9] and [10]).

Recently many researchers have studied approximation of solutions of an equilibrium problem (see, e.g. [6], [9], [10], [15], [16], [18] and references therein). Equilibrium problems with monotone and pseudo-monotone bifunctions have been studied extensively in Hilbert, Banach as well as in topological vector spaces by many

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authors (e.g. [3], [4], [5], [7], [10], [11], [14], [13]). Also, the quasi-equilibrium problems have been studied in [2] and [20]. Our aim in this paper is to approximate a solution of quasi-equilibrium problems. This type of problems arises, for example, in the electricity market (see [17]).

This paper is organized as follows. In Section 2, we introduce some preliminary material related to the geometry of Banach spaces. In Section 3, we prove weak convergence of the sequence generated by proximal point method to a solution of quasi-equilibrium problems. In Section 4, we study strong convergence of the generated sequence to a solution of the problems by imposing some additional conditions.

2. Preliminaries

Let E be a real Banach space with norm $\|.\|$ and E^* denote the dual of E. We denote the value of $v \in E^*$ at $x \in E$ by $\langle x, v\rangle$. The duality mapping J from E into 2^{E^*} is defined by:

$$Jx = \{v \in E^* : \langle x, v\rangle = \|x\|^2 = \|v\|^2\},$$

for $x \in E$. When E is a smooth Banach space, we can use the following function studied by Alber [1], Kamimura and Takahashi [12] and Reich [19]:

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \tag{2.1}$$

for all $x, y \in E$. It is obvious from the definition of ϕ that

$$0 \leq (\|x\| - \|y\|)^2 \leq \phi(x,y), \tag{2.2}$$

for all $x, y \in E$. Note that if E is a Hilbert space, then $\phi(x,y) = \|x-y\|^2$.

Proposition 1 ([12]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\lim_{n\to\infty}\phi(x_n,y_n) = 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Proposition 2 ([12]). *Let E be a reflexive, strictly convex and smooth Banach space, and let C be a nonempty closed convex subset of E and $x \in E$. Then there exists a unique element $\bar{x} \in C$ such that $\phi(\bar{x}, x) = \inf\{\phi(z,x) : z \in C\}$.*

Regarding Proposition 2, we denote the unique element $\bar{x} \in C$ by $P_C(x)$, where the mapping P_C is called the generalized projection from E onto C. It is obvious that in Hilbert spaces, P_C is coincident with the metric projection from E onto C.

Proposition 3 ([12]). *Let E be a smooth Banach space, C be a convex subset of E, $x \in E$ and $\bar{x} \in C$. Then $\phi(\bar{x}, x) = \inf\{\phi(z,x) : z \in C\}$ if and only if*

$$\langle z - \bar{x}, Jx - J\bar{x}\rangle \leq 0, \quad \forall z \in C.$$

Throughout this paper we assume that E is a real Banach space which is uniformly convex and uniformly smooth, and $C \subset E$ is nonempty closed and convex unless otherwise specified. When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$.

Definition 1. The mapping $T : C \to C$ is called quasi ϕ-nonexpansive whenever $\mathrm{Fix}(T) \neq \varnothing$ and $\phi(p,Tx) \leq \phi(p,x)$ for all $(p,x) \in \mathrm{Fix}(T) \times C$.

The following definitions are adapted from [20].

Definition 2. Suppose that $K : C \to 2^C$ is a multivalued mapping such that for every $x \in C$, $K(x)$ is nonempty, closed and convex. K is called quasi ϕ-nonexpansive whenever the mapping $T(\cdot) = P_{K(\cdot)}(\cdot)$ is quasi ϕ-nonexpansive where P is the generalized projection.

Definition 3. We say that $K : C \to 2^C$ is weak lower semicontinuous at each $\bar{x} \in C$, whenever we have $\{x_k\} \subset C$ and $x_k \rightharpoonup \bar{x}$, then for any $\bar{y} \in K(\bar{x})$, there is a sequence $\{y_k\}$ with $y_k \in K(x_k)$ for every k, such that $y_k \to \bar{y}$ as $k \to \infty$.

Definition 4. We say that $K : C \to 2^C$ is demiclosed, whenever we have $x_k \rightharpoonup \bar{x}$ and $\lim_{k\to\infty} d(x_k, K(x_k)) = 0$, then $\bar{x} \in \mathrm{Fix}(K)$.

It is well known that if T is a quasi ϕ-nonexpansive mapping, then $\mathrm{Fix}(T)$ is convex. Also, if T is demiclosed then $\mathrm{Fix}(T)$ is closed.

A bifunction f is called "monotone" if $f(x,y) + f(y,x) \leq 0$ for all $x,y \in E$, and "pseudo-monotone" if for any pair $x,y \in E$, $f(x,y) \geq 0$ implies $f(y,x) \leq 0$.

We introduce some conditions on the bifunction f and the multivalued mapping K where K is quasi ϕ-nonexpansive, which are needed in the convergence analysis.

B1: $f(x,x) = 0$ for all $x \in E$.
B2: $f(\cdot,\cdot) : E \times E \to \mathbb{R}$ is continuous.
B3: $f(x,\cdot) : E \to \mathbb{R}$ is convex for all $x \in E$.
B4: There exists $\theta \geq 0$ such that $f(x,y) + f(y,x) \leq \frac{\theta}{2}(\phi(x,y) + \phi(y,x))$, for all $x,y \in E$ (f is called θ-undermonotone).
B5: If $\{x_k\}$ and $\{y_k\}$ are two arbitrary sequences in C such that $x_k \rightharpoonup \bar{x}$, $y_k \to \bar{y}$ and also we have $\limsup_{k\to\infty} f(x_k,y_k) \geq 0$, then $f(\bar{x},\bar{y}) \geq 0$.
B6: $K : C \to 2^C$ is demiclosed and lower semicontinuous at each $x \in C$.

Finally, in order to well definedness and boundedness of the generated sequences by our algorithm in this paper, we assume that

$$S_* = \left\{ x \in \cap_{z\in C} K(z) : f(y,x) \leq 0, \ \ \forall y \in \cup_{z\in C} K(z) \right\} \neq \varnothing,$$

Note that under B1–B3, S_* is closed and convex.

3. Proximal point method and weak convergence

In this section, we study the weak convergence of the sequence generated by the following algorithm. We will assume that E is a uniformly smooth and uniformly convex Banach space, $C \subseteq E$ is nonempty closed and convex. Let $K : C \to 2^C$ be a multivalued quasi ϕ-nonexpansive mapping, and let $f : E \times E \to \mathbb{R}$ be a bifunction, $S_* \neq \varnothing$, and the assumptions B1–B6 are satisfied.

Algorithm 1.

1. **Initialization:** Take $x_0 \in E$ and consider $\beta > 0$ satisfying $\theta < \beta$ where θ is the undermonotonicity constant of f, and a sequence $\{\lambda_k\} \subset (\theta, \beta]$.
2. **Iterative step:** Given x_k, define
$$y_k = P_{K(x_k)}(x_k). \tag{3.1}$$
Find z_k such that:
$$f(z_k, y) + \lambda_k \langle y - z_k, Jz_k - Jy_k \rangle \geq 0, \qquad \forall y \in K(x_k), \tag{3.2}$$
if $z_k = x_k$ stop. Otherwise, define
$$H_k = \{y \in E : \langle y - z_k, Jz_k - Jy_k \rangle \geq 0\}, \tag{3.3}$$
$$C_k = C \cap (\cap_{i=0}^{k} H_i), \tag{3.4}$$
$$x_{k+1} = P_{C_k}(z_k). \tag{3.5}$$

The following proposition ensures the existence of the sequence $\{z_k\}$ generated by (3.2). It is easy to see that if $S_* \neq \varnothing$, then the sequences $\{x_k\}$ and $\{y_k\}$ are well defined.

Proposition 4 ([8])**.** *If f satisfies B1–B4 and $\lambda > \theta$, then $EP(\tilde{f}, C)$ has a unique solution where $\bar{x} \in E$ and $\tilde{f}(x,y) = f(x,y) + \lambda \langle y - x, Jx - J\bar{x} \rangle$.*

Proposition 5. *If Algorithm 1 stops at the k-th iteration, then z_k is a solution of QEP(f, K).*

Proof. If $z_k = x_k$, then $z_k = y_k$ by (3.1). Now the result follows from (3.2). □

Lemma 1. *Suppose that $\{x_k\}$, $\{y_k\}$ and $\{z_k\}$ are the sequences generated by the algorithm. If $S_* \neq \varnothing$, then $\lim_{k\to\infty} \phi(x^*, x_k)$ exists for all $x^* \in S_*$. Also $\{x_k\}$ is bounded and subsequently the sequences $\{y_k\}$ and $\{z_k\}$ are bounded and*
$$\lim_{k\to\infty} \|x_{k+1} - z_k\| = \lim_{k\to\infty} \|x_k - y_k\| = \lim_{k\to\infty} \|y_k - z_k\| = 0.$$

Proof. Note that since $y_k = P_{K(x_k)}(x_k)$ and $P_{K(\cdot)}(\cdot)$ is a quasi ϕ-nonexpansive mapping, we have
$$\langle x^* - y_k, Jx_k - Jy_k \rangle \leq 0, \tag{3.6}$$
where $x^* \in S_*$. Therefore, we get
$$\phi(x^*, y_k) + \phi(y_k, x_k) - \phi(x^*, x_k) \leq 0. \tag{3.7}$$
On the other hand, since $f(z_k, x^*) \leq 0$, hence (3.2) implies that $x^* \in H_k$ for all k. Therefore we have
$$\langle z_k - x^*, Jz_k - Jy_k \rangle \leq 0, \tag{3.8}$$
which implies that
$$\phi(x^*, z_k) + \phi(z_k, y_k) - \phi(x^*, y_k) \leq 0. \tag{3.9}$$

Since $x_{k+1} = P_{C_k}(z_k)$, therefore we have

$$\langle x^* - x_{k+1}, Jz_k - Jx_{k+1} \rangle \leq 0. \tag{3.10}$$

By proposition 3 which implies that

$$\phi(x^*, x_{k+1}) + \phi(x_{k+1}, z_k) - \phi(x^*, z_k) \leq 0. \tag{3.11}$$

Hence (3.7), (3.9) and (3.11) imply that

$$\begin{aligned}\phi(x^*, x_{k+1}) &\leq \phi(x^*, x_{k+1}) + \phi(x_{k+1}, z_k)\\ &\leq \phi(x^*, z_k) \leq \phi(x^*, z_k) + \phi(z_k, y_k) \leq \phi(x^*, y_k)\\ &\leq \phi(x^*, y_k) + \phi(y_k, x_k) \leq \phi(x^*, x_k).\end{aligned}$$

Therefore $\lim_{k\to\infty} \phi(x^*, x_k)$ exists. Also $\{x_k\}$ is bounded by (2.2). Passing to the limit in the above inequality as $k \to \infty$, one gets

$$\lim_{k\to\infty} \phi(x_{k+1}, z_k) = \lim_{k\to\infty} \phi(z_k, y_k) = \lim_{k\to\infty} \phi(y_k, x_k) = 0.$$

Now, by Proposition 1, we have

$$\lim_{k\to\infty} \|x_{k+1} - z_k\| = \lim_{k\to\infty} \|z_k - y_k\| = \lim_{k\to\infty} \|y_k - x_k\| = 0. \tag{3.12}$$

□

Proposition 6. *Suppose that f is a bifunction, K is a multivalued quasi ϕ-nonexpansive mapping and the conditions B1–B6 are satisfied. If there exists a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ such that $x_{k_n} \rightharpoonup p$, then*
i) $p \in \mathrm{Fix}(K)$,
ii) $\lim_{k\to\infty} \phi(q, x_k)$ exists.

Proof. i) Let $\{x_{k_n}\}$ be a subsequence of $\{x_k\}$ such that $x_{k_n} \rightharpoonup p$. Note that $\lim_{n\to\infty} \|x_{k_n} - y_{k_n}\| = 0$ by Lemma 1, where for each n, y_{k_n} is the generalized projection of x_{k_n} onto $K(x_{k_n})$. Therefore we have $\lim_{n\to\infty} d(x_{k_n}, K(x_{k_n})) = 0$. Since K is demiclosed, thus $p \in K(p)$.
ii) First we show that $p \in \cap_{k=0}^{\infty} C_k$, it is sufficient to prove that $p \in C_k$ for all integer k. Note that the sequence $\{C_k\}$ is nonincreasing, let N be a fixed integer, hence there is $j > N$ such that for all $n \geq j$, we have

$$x_{k_n+1} \in C_{k_n} \subseteq C_N,$$

where $x_{k_n+1} = P_{C_{k_n}}(z_{k_n})$. Since $x_{k_n} \rightharpoonup p$, we have $x_{k_n+1} \rightharpoonup p$ by Lemma 1. Consequently, C_N is closed and $p \in C_N$, the weak limit point of $\{x_{k_n}\}$. Since N is arbitrary, we obtain that

$$p \in \cap_{k=0}^{\infty} C_k.$$

It follows from (i) that $p \in K(p)$. Since $y_k = P_{K(x_k)}(x_k)$ and $P_{K(\cdot)}(\cdot)$ is a quasi ϕ-nonexpansive mapping, we have

$$\phi(p, y_k) \leq \phi(p, x_k). \tag{3.13}$$

On the other hand, since $p \in H_k$ for all k, we have $\langle z_k - p, Jz_k - Jy_k \rangle \leq 0$, which implies that

$$\phi(p, z_k) + \phi(z_k, y_k) - \phi(p, y_k) \leq 0. \tag{3.14}$$

Since $x_{k+1} = P_{C_k}(z_k)$, we have $\langle p - x_{k+1}, Jz_k - Jx_{k+1} \rangle \leq 0$ by Proposition 3, which implies that

$$\phi(p, x_{k+1}) + \phi(x_{k+1}, z_k) - \phi(p, z_k) \leq 0. \tag{3.15}$$

Since ϕ is non-negative by (2.2), hence by using (3.13), (3.14) and (3.15), we get $\phi(p, x_{k+1}) \leq \phi(p, x_k)$. This implies that $\lim_{k\to\infty} \phi(p, x_k)$ exists. □

In order to prove the uniqueness of the weak limit point of the sequence in the following theorem, we need the following condition on Banach space E, i.e. we assume that if $\{u_k\}$ and $\{v_k\}$ are arbitrary sequences in K that converge weakly to u and v, respectively, and $u \neq v$, then

$$\liminf_{k\to\infty} |\langle u - v, Ju_k - Jv_k \rangle| > 0. \tag{3.16}$$

For non-Hilbertian examples of such spaces, we can consider ℓ^p spaces for $1 < p < \infty$ that satisfy in the above condition. It is valuable to mention that we need this condition on E just in the following theorem and we will never use it in the sequel, i.e. when we prove the strong convergence theorems in this paper we do not need the above condition.

Theorem 1. *Suppose that f is a bifunction, K is a multivalued quasi ϕ-nonexpansive mapping and the conditions B1–B6 are satisfied. If $S_* \neq \varnothing$ and the sequence $\{x_k\}$ generated by Algorithm 1, then*
i) all cluster points of $\{x_k\}$ solve the problem,
ii) if in addition either E satisfies (3.16) or the problem has a unique solution, then the whole sequence $\{x_k\}$ is weakly convergent to an element of $S(f, K)$.

Proof. Let $x_{k_n} \rightharpoonup \bar{x}$, therefore it is enough to show that $\bar{x} \in S(f, K)$. From Lemma 1, we have

$$\lim_{n\to\infty} \|z_{k_n} - y_{k_n}\| = 0. \tag{3.17}$$

Now, uniform smoothness of E implies uniform (norm-to-norm) continuity of J on each bounded set of E. Therefore, we get from (3.17) that

$$\lim_{n\to\infty} \|Jz_{k_n} - Jy_{k_n}\| = 0. \tag{3.18}$$

On the other hand, since $x_{k_n} \rightharpoonup \bar{x}$, we have $z_{k_n} \rightharpoonup \bar{x}$ by Lemma 1. Now, take any $y \in K(\bar{x})$, since K is lower semicontinuous at $\bar{x} \in C$, therefore there is a sequence $\{\xi_{k_n}\}$ such that $\xi_{k_n} \in K(x_{k_n})$ and $\xi_{k_n} \to y$. By (3.2), we have

$$f(z_{k_n}, \xi_{k_n}) + \lambda_{k_n} \langle \xi_{k_n} - z_{k_n}, Jz_{k_n} - Jy_{k_n} \rangle \geq 0,$$

which implies that

$$f(z_{k_n}, \xi_{k_n}) + \lambda_{k_n} \|\xi_{k_n} - z_{k_n}\| \, \|Jz_{k_n} - Jy_{k_n}\| \geq 0. \tag{3.19}$$

We use (3.18) and (3.19) together with the boundedness of $\{\lambda_{k_n}\}$ and $\{\xi_{k_n}\}$ in order to obtain $\liminf_{n\to\infty} f(z_{k_n},\xi_{k_n}) \geq 0$. Now B5 implies that $f(\bar{x},y) \geq 0$. Since $y \in K(\bar{x})$ is arbitrary, hence $\bar{x} \in S(f,K)$.

ii) When the problem has a unique solution, the result is clear. Therefore we must show that there exists only one weak limit point of $\{x_k\}$ whenever E satisfies (3.16). Suppose that p is an other weak limit point of $\{x_k\}$, then there exists subsequence $\{x_{k_j}\}$ such that $x_{k_j} \rightharpoonup p$. We have already proved that p is the element of $S(f,K)$. Using Proposition 6, we get $\lim_{k\to\infty}\phi(p,x_k)$ and $\lim_{k\to\infty}\phi(\bar{x},x_k)$ exist. Note that

$$\begin{aligned} 2\langle \bar{x}-p, Jx_{k_n}-Jx_{k_j}\rangle &= 2\langle \bar{x}, Jx_{k_n}\rangle - 2\langle p, Jx_{k_n}\rangle - 2\langle \bar{x}, Jx_{k_j}\rangle + 2\langle p, Jx_{k_j}\rangle \\ &= -\phi(\bar{x},x_{k_n}) + \phi(p,x_{k_n}) + \phi(\bar{x},x_{k_j}) - \phi(p,x_{k_j}) \end{aligned}$$

(In the second equality the relation (2.1) is used). Taking limit when $n \to \infty$ and then when $j \to \infty$, we obtain $p = \bar{x}$, i.e. $\{x_k\}$ converges weakly to a point of $S(f,K)$. □

Example 1. Suppose that $C = \left\{ X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix};\ a_i \leq x_i \leq b_i,\ \forall i = 1,\ldots,n \right\}$ where a_i and b_i are real numbers, and the bifunction $f : C \times C \to \mathbb{R}$ is defined by $f(X,Y) = X^tX - X^tY$ where X^t denotes the transpose of the matrix X. Note that the conditions B1-B5 are satisfied. Now we define the multivalued mapping $K : C \to 2^C$ by

$$K(X) = \left\{ Z \in C : z_1 + \cdots + z_n \geq \max\{x_1 + \ldots + x_n, \beta\} \right\}, \tag{3.20}$$

where β is a constant number such that $\beta \leq \sum_{i=1}^n b_i$. Then, for each $x \in C$, $K(x)$ is a nonempty, closed and convex subset of C. It is also easy to see that K is demiclosed and lower semicontinuous at each $x \in C$, i.e. the condition B6 is satisfied. Since $[b_1,\ldots,b_n]^t \in S_*$, it is easy to see that the assumptions of Theorem 1 are satisfied. Therefore we can use Theorem 1, i.e. if the sequence $\{x_k\}$ is generated by Algorithm 1, it converges to an element of $S(f,K)$.

4. Strong convergence

In this section, we study the strong convergence of the sequence generated by Algorithm 1 to an element of $S(f,K)$ with some additional assumptions on the problem.

Definition 5. A bifunction f is called strongly monotone, if there exists $\alpha > 0$ such that $f(x,y) + f(y,x) \leq -\alpha\|x-y\|^2$, for all $x,y \in E$.
Also, a bifunction f is called strongly pseudo-monotone, if there exists $\beta > 0$ such that whenever $f(x,y) \geq 0$, then $f(y,x) \leq -\beta\|x-y\|^2$, for all $x,y \in E$.

Definition 6. A function $h : E \to \mathbb{R}$ is called strongly convex, whenever for each pair $x,y \in E$ and each $\lambda \in [0,1]$, we have

$$h(\lambda x + (1-\lambda)y) \leq \lambda h(x) + (1-\lambda)h(y) - \lambda(1-\lambda)\|x-y\|^2.$$

We say that h is strongly concave whenever $-h$ is strongly convex.

Example 2. Suppose that $g : E \times E \to \mathbb{R}$ is monotone. If $\alpha : E \times E \to \mathbb{R}^+$, then $f(x,y) := \alpha(x,y)g(x,y)$ is pseudo-monotone but it is not necessarily monotone. Now if $\alpha(x,y) \geq c > 0$ and g is strongly monotone, then $f(x,y) = \alpha(x,y)g(x,y)$ is strongly pseudo-monotone.

Theorem 2. *Suppose that the assumptions of Theorem 1 are satisfied. If any of the following conditions is satisfied,*
i) f is strongly pseudo-monotone,
ii) $f(x,\cdot)$ is strongly convex for all $x \in E$,
iii) $f(\cdot,y)$ is strongly concave for all $y \in E$,
then the sequence $\{x_k\}$ generated by the algorithm is strongly convergent to an element of $S(f,K)$.

Proof. Note that by Theorem 1, all cluster points of $\{x_k\}$ belong to $S(f,K)$. In the sequel, first in each item, we prove that if $x_{k_i} \rightharpoonup x^*$, then $x_{k_i} \to x^*$. Finally we prove that there is only one cluster point and hence $\{x_k\}$ converges strongly to a point of $S(f,K)$. Therefore we suppose that $x_{k_i} \rightharpoonup x^*$ and subsequently $z_{k_i} \rightharpoonup x^*$ by Lemma 1. Note that, Lemma 1 implies that

$$\lim_{i\to\infty} \|z_{k_i} - y_{k_i}\| = 0. \tag{4.1}$$

Uniform smoothness of E implies uniform norm-to-norm continuity of J on each bounded set of E. Therefore, we get from (4.1),

$$\lim_{i\to\infty} \|Jz_{k_i} - Jy_{k_i}\| = 0. \tag{4.2}$$

Now, take $x^* \in S_*$, then by (3.2), we have $f(z_{k_i},x^*) + \lambda_{k_i}\langle x^* - z_{k_i}, Jz_{k_i} - Jy_{k_i}\rangle \geq 0$, which implies that

$$f(z_{k_i},x^*) + \lambda_{k_i}\|x^* - z_{k_i}\|\,\|Jz_{k_i} - Jy_{k_i}\| \geq 0. \tag{4.3}$$

We use (4.2), (4.3) and the boundedness of $\{\lambda_k\}$ in order to obtain

$$\liminf_{i\to\infty} f(z_{k_i},x^*) \geq 0. \tag{4.4}$$

In the sequel, we continue to prove parts (i), (ii) and (iii) respectively:
i) Since $f(x^*,z_{k_i}) \geq 0$, by the definition of the strongly pseudo-monotone bifunction, there is $\beta > 0$ such that $f(z_{k_i},x^*) \leq -\beta\|z_{k_i} - x^*\|^2$. By taking liminf and using (4.4), we get

$$0 \leq \liminf_{i\to\infty} f(z_{k_i},x^*) \leq \liminf_{i\to\infty}(-\beta\|z_{k_i} - x^*\|^2) \leq -\beta\limsup_{i\to\infty}\|z_{k_i} - x^*\|^2.$$

Therefore $z_{k_i} \to x^*$ and subsequently $x_{k_i} \to x^*$.
ii) By (3.2), we have $f(z_{k_i},y) + \lambda_{k_i}\langle y - z_{k_i}, Jz_{k_i} - Jy_{k_i}\rangle \geq 0$. Now, let $\lambda \in (0,1)$ and set $p_{k_i} = \lambda z_{k_i} + (1-\lambda)x^*$, for all $i \in \mathbb{N}$. Since $f(z_{k_i},\cdot)$ is strongly convex, we have

$$\lambda_{k_i}\langle p_{k_i} - z_{k_i}, Jy_{k_i} - Jz_{k_i}\rangle \leq f(z_{k_i},p_{k_i})$$

$$\begin{aligned}&\leq \lambda f(z_{k_i}, z_{k_i}) + (1-\lambda) f(z_{k_i}, x^*)\\&-\lambda(1-\lambda)\|z_{k_i} - x^*\|^2\\&\leq (1-\lambda) f(x_{k_i}, x^*) - \lambda(1-\lambda)\|z_{k_i} - x^*\|^2.\end{aligned}$$

Hence, we have

$$\begin{aligned}\lambda(1-\lambda)\|z_{k_i} - x^*\|^2 &\leq \lambda_{k_i}\langle p_{k_i} - z_{k_i}, Jz_{k_i} - Jy_{k_i}\rangle + (1-\lambda) f(z_{k_i}, x^*)\\&\leq \lambda_{k_i}\|p_{k_i} - z_{k_i}\|\,\|Jz_{k_i} - Jy_{k_i}\| + (1-\lambda) f(z_{k_i}, x^*). \qquad (4.5)\end{aligned}$$

Taking liminf in (4.5), we use (4.2) together with the boundedness of $\{z_{k_i}\}$ and $\{p_{k_i}\}$ in order to obtain that $\|z_{k_i} - x^*\| \to 0$. Therefore we have $x_{k_i} \to x^*$.
iii) Let $\lambda \in (0,1)$ and set $p_{k_i} = \lambda z_{k_i} + (1-\lambda)x^*$, for all $i \in \mathbb{N}$. Then since $f(\cdot, x^*)$ is strongly concave, we have

$$\lambda f(z_{k_i}, x^*) + (1-\lambda) f(x^*, x^*) + \lambda(1-\lambda)\|z_{k_i} - x^*\|^2 \leq f(p_{k_i}, x^*) \leq 0.$$

Therefore, we get $f(z_{k_i}, x^*) \leq -(1-\lambda)\|z_{k_i} - x^*\|^2$. Now, by taking liminf and (4.4), we get

$$0 \leq \liminf_{i\to\infty} f(z_{k_i}, x^*) \leq -(1-\lambda)\limsup_{i\to\infty}\|z_{k_i} - x^*\|^2.$$

Therefore $z_{k_i} \to x^*$. This implies that $x_{k_i} \to x^*$.
Now, we prove that there is only one cluster point and hence $\{x_k\}$ converges strongly to a point of $S(f,K)$. Note that we proved if $x_{k_i} \rightharpoonup x^*$, then $x_{k_i} \to x^*$. Now, if $\{x_{k_i}\}$ and $\{x_{k_j}\}$ are arbitrary subsequences of $\{x_k\}$ that converge strongly to p and q, respectively, then

$$\begin{aligned}2\langle p-q, Jx_{k_i} - Jx_{k_j}\rangle &= 2\langle p, Jx_{k_i}\rangle - 2\langle q, Jx_{k_i}\rangle - 2\langle p, Jx_{k_j}\rangle + 2\langle q, Jx_{k_j}\rangle\\&= -\phi(p, x_{k_i}) + \phi(q, x_{k_i}) + \phi(p, x_{k_j}) - \phi(q, x_{k_j}).\end{aligned}$$

Since $\lim_{k\to\infty}\phi(p, x_k)$ and $\lim_{k\to\infty}\phi(q, x_k)$ exist by Lemma 1 and uniform smoothness of E implies uniform (norm-to-norm) continuity of J on each bounded set of E. Therefore, taking limit from the above equation when $i \to \infty$ and then when $j \to \infty$, we obtain $p = q$, i.e. $\{x_k\}$ converges strongly to a point of $S(f,K)$. □

Theorem 3. *Let f be a bifunction and K be a multivalued quasi ϕ-nonexpansive mapping that the conditions B1–B6 are satisfied and that* $\mathrm{int}(S_*) \neq \varnothing$. *Then the sequence $\{x_k\}$ generated by the algorithm is strongly convergent to a solution of $QEP(f,K)$.*

Proof. Since $\mathrm{int}(S_*) \neq \varnothing$, there exist $r > 0$ and $x^* \in \mathrm{int}(S_*)$ such that $\bar{B}_r(x^*) \subset \mathrm{int}(S_*)$. Note that the duality mappings $J : E \to E^*$ and $J^{-1} : E^* \to E$ are single valued, one-to-one and surjective (see [12]). Now, if $\|Jx_k - Jy_k\| \neq 0$, replacing $\tilde{x} = x^* + r\frac{J^{-1}(Jx_k - Jy_k)}{\|Jx_k - Jy_k\|}$ by x^* in (3.6), we have $\langle \tilde{x} - y_k, Jx_k - Jy_k\rangle \leq 0$ for all $k \in \mathbb{N}$, and

hence

$$\begin{aligned} r\|Jx_k - Jy_k\| &= r\langle \frac{J^{-1}(Jx_k - Jy_k)}{\|Jx_k - Jy_k\|}, Jx_k - Jy_k\rangle \\ &\leq \langle x^* - y_k, Jy_k - Jx_k\rangle \\ &= \frac{1}{2}(\phi(x^*, x_k) - \phi(x^*, y_k) - \phi(y_k, x_k)). \end{aligned} \tag{4.6}$$

Also if $\|Jx_k - Jy_k\| = 0$, then (4.6) is satisfied. In the sequel, again if $\|Jy_k - Jz_k\| \neq 0$, replacing $\tilde{x} = x^* + r\frac{J^{-1}(Jy_k - Jz_k)}{\|Jy_k - Jz_k\|}$ by x^* in (3.8), we have $\langle \tilde{x} - z_k, Jy_k - Jz_k\rangle \leq 0$ for all $k \in \mathbb{N}$, and hence

$$\begin{aligned} r\|Jy_k - Jz_k\| &= r\langle \frac{J^{-1}(Jy_k - Jz_k)}{\|Jy_k - Jz_k\|}, Jy_k - Jz_k\rangle \\ &\leq \langle x^* - z_k, Jz_k - Jy_k\rangle \\ &= \frac{1}{2}(\phi(x^*, y_k) - \phi(x^*, z_k) - \phi(z_k, y_k)). \end{aligned} \tag{4.7}$$

Also if $\|Jy_k - Jz_k\| = 0$, then (4.7) is satisfied. Once again, if $\|Jz_k - Jx_{k+1}\| \neq 0$, replacing $\tilde{x} = x^* + r\frac{J^{-1}(Jx_k - Jy^{k+1})}{\|Jx_k - Jy^{k+1}\|}$ by x^* in (3.10), we have $\langle \tilde{x} - x_{k+1}, Jz_k - Jx_{k+1}\rangle \leq 0$ for all $k \in \mathbb{N}$, and hence

$$\begin{aligned} r\|Jz_k - Jx_{k+1}\| &= r\langle \frac{J^{-1}(Jz_k - Jx_{k+1})}{\|Jz_k - Jx_{k+1}\|}, Jz_k - Jx_{k+1}\rangle \\ &\leq \langle x^* - x_{k+1}, Jx_{k+1} - Jz_k\rangle \\ &= \frac{1}{2}(\phi(x^*, z_k) - \phi(x^*, x_{k+1}) - \phi(x_{k+1}, z_k)). \end{aligned} \tag{4.8}$$

Also if $\|Jz_k - Jx_{k+1}\| = 0$, then (4.8) is satisfied. Summing up (4.6), (4.7) and (4.8), we get

$$\begin{aligned} 2r\|Jx_k - Jx_{k+1}\| &\leq 2r\|Jx_k - Jy_k\| + 2r\|Jy_k - Jz_k\| + 2r\|Jz_k - Jx_{k+1}\| \\ &\leq \phi(x^*, x_k) - \phi(x^*, x_{k+1}). \end{aligned} \tag{4.9}$$

Summing up (4.9) from $k = 0$ to $k = n$, we get

$$2r\sum_{k=0}^{n} \|Jx_k - Jx_{k+1}\| \leq \phi(x^*, x_0) - \phi(x^*, x_{n+1}). \tag{4.10}$$

Therefore we may conclude that $\sum_{k=0}^{\infty} \|Jx_k - Jx_{k+1}\| < \infty$. It follows that $\{Jx_k\}$ converges strongly to an element in E^*. Since the duality mapping J^{-1} is uniformly norm to norm continuous on each bounded subset of E^*, hence $\{x_k\}$ converges strongly to an element in E. On the other hand, since all cluster points of $\{x_k\}$ belong to $S(f, K)$ by Theorem 1, therefore $\{x_k\}$ converges strongly to an element of $S(f, K)$. □

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GENERALIZED Ψ-GERAGHTY-QUASI CONTRACTIONS IN *b*-METRIC SPACES

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Abstract. In this paper we generalize the well-known class of quasi contractions pairs, by using the so-called Geraghty's property and by controlling the contractive inequality with an altering distance function. For this class, in the setting of b-metric spaces, we prove the existence and uniqueness a of common fixed point assuming some non commutative notions. Finally, for this class of mappings we prove the b-convergence of the Jungck-Mann iterative scheme under suitable conditions on the b-metric space and on the altering distance function.

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1. Introduction

The Banach contraction principle (BCP) is one of the most important result in metric fixed point theory. Since its appearance, several interesting generalizations and extensions of this principle have been introduced. Typically, these extensions and generalizations are obtained by either generalizating the domain of the mappings or by extending the contractive conditions of the mappings.

In 1974, L. B. Ćirić ([5]) introduced the notion of a *quasi contraction map*, and prove its fixed point theorem.

Theorem 1 (Ćirić, [5])**.** *Let (M, d) be a metric S-orbitally complete space and let $S : M \to M$ be a quasi contraction, that is, there exists $q \in (0, 1)$ such that for all $x, y \in M$ the following inequality hold,*

$$d(Sx, Sy) \leq q \max\{d(x, y),\ d(x, Sx),\ d(y, Sy),\ (d(x, Sy),\ d(y, Sx))\}.$$

Then S has a unique fixed point in M.

In 1976, G. Jungck ([11]) generalized the BCP by using the notion of commuting mappings:

Theorem 2 (Jungck, [11])**.** *Let S, T be two selfmappings of a complete metric space (M, d) such that*

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(i) (S, T) *is a commuting pair.*
(ii) $SM \subset TM$.
(iii) T *is a continuous mapping.*
(iv) *There exists* $\alpha \in [0, 1)$ *such that*

$$d(Sx, Sy) \leq \alpha d(Tx, Ty) \quad \textit{for all } x, y \in M.$$

Then, S and T have a unique common fixed point in M.

In 1979, K. M. Das and K. V. Naik ([9]) generalized and extended the fixed point theorems of Ćirić and Jungck by proving the following result.

Theorem 3 (Das–Naik, [9])**.** *Let (M, d) be a complete metric space. Let T be a continuous selfmap on M and S be any selfmap on M that commutes with T. Assume that $SM \subset TM$ and that there exists $q \in (0, 1)$ such that*

$$d(Sx, Sy) \leq q \max\{d(Tx, Ty),\ d(Sx, Tx),\ d(Sy, Ty),\ d(Sx, Ty),\ d(Sy, Tx)\}$$

for all $x, y \in M$. Then S and T have a unique common fixed point.

In this paper we prove, in the setting of b-metric spaces, some common fixed point results that extend and generalize the Das–Naik fixed point theorem and other well-known results stated in the usual metric spaces. The approximate common fixed point can be computed by using the so-called Jungck-Mann iteration process, for which we prove its b-convergence.

2. Preliminaries

The b-metric spaces

In this section we recall some concepts, results and properties of the b-metric spaces that will be useful in the sequel. The concept of b-metric spaces (also known as *quasimetric spaces*) was reintroduced in 1989 by I. A. Bakhtin [4], and used in the metric fixed point theory by S. Czerwik ([6, 7]) in connection with some problems concerning to the convergence of measurable functions with respect to measure. In 2010, M. A. Kamsi [16] and M. A. Kamsi and N. Hussain [15] showed that each cone metric space has a b-metric structure, so they reintroduced the b-metric space with the name of *metric type space*.

Definition 1. Let M be a non empty set and $s \geq 1$ be given a real number. A function

$$\rho : M \times M \to \mathbb{R}_+ := [0, +\infty)$$

is said to be a b-metric if and only if for all $x, y, z \in M$ the following conditions hold:

(ρ_1) $\rho(x, y) = 0$ if and only if $x = y$.
(ρ_2) $\rho(x, y) = \rho(y, x)$.
(ρ_3) $\rho(x, z) \leq s\,(\rho(x, y) + \rho(y, z))$.

The pair (M, ρ) is called a b-metric space and the real number $s \geq 1$ is called the coefficient of (M, ρ).

The class of b-metric spaces is larger than the class of metric spaces, since these spaces extends the usual metric spaces for the case when $s = 1$. Even more, the classical metric spaces are properly contained in the class of b-metric spaces since there exist b-metric spaces which are not metric spaces (see, Example 1).

On the other hand, as a consequence of the Alexandroff–Urysohn Theorem, the topology induced by a given b-metric on a b-metric space is metrizable (see Definition 2 and see also, [15,18]), this metrization gives as a consequence that much of the results given in metric spaces are directly valid in b-metric spaces. Nevertheless, the convenience of prove new fixed point results on these spaces takes relevance when we are dealing with b-metric spaces which are not metric spaces.

Example 1. a) Let (M, d) be a metric space, and let the function $\varphi(t) = t^p$ for $t > 0$, and $p > 1$. Notice that this function is continuous and convex. Thus, if we define $\rho(x, y) := \varphi(d(x, y)) = (d(x, y))^p$, then ρ is a b-metric continuous and (M, ρ) is a b-metric space with $s = 2^{p-1}$. However, (M, ρ) is not necessarily a metric space.

In particular, let $M = \mathbb{R}$ be with $d(x, y) = |x - y|$ the usual euclidean metric on $\mathbb{R}$. Then $\rho(x, y) = (x - y)^2$ is a b-metric on $\mathbb{R}$, but it is not a metric since it does not satisfy the triangle inequality of the definition of metric, when $x = 0$, $y = 5$ and $z = 1$.

$$\rho(0, 5) = 25 \geq \rho(0, 1) + \rho(1, 5) = 17.$$

b) Let the sequence space $l_p^{(n)}$ $(0 < p < 1)$ defined by

$$l_p^{(n)} = \left\{ (x_n) \in \mathbb{R}^n \,/\, \sum_{k=1}^{n} |x_k|^p < \infty \right\}$$

together with the function

$$\rho : l_p^{(n)} \times l_p^{(n)} \to \mathbb{R}_+$$

defined by

$$\rho(x, y) = \left(\sum_{k=1}^{n} |x_k - y_k|^p \right)^{1/p}$$

with $x = (x_n)$, $y = (y_n) \in l_p^{(n)}$. This function ρ is a b-metric and $(l_p^{(n)}, \rho)$ is a b-metric space with $s = 2^{p-1}$, but it is not a usual metric space.

c) Let $M = C[0, 1] = \{f \,/\, f : [0, 1] \to \mathbb{R} \quad \text{continuous}\}$ be equipped with the uniform metric

$$d(x, y) = \sup_{t \in [0,1]} |x(t) - y(t)|, \quad \text{for all } x, y \in M.$$

Now, for $p > 1$ we define

$$\rho(x, y) := (d(x, y))^p = \sup_{t \in [0,1]} |x(t) - y(t)|^p, \quad \text{for all } x, y \in M.$$

Then, ρ is clearly a b-metric continuous on $C[0,1]$, consequently, (M,ρ) is a b-metric space with $s = 2^{p-1}$.

Definition 2. Let (M,ρ) be a b-metric space with $s \geq 1$. Then

(1) The open ball of center $x \in M$ and radius $r > 0$ is given by
$$B_\rho(x,r) := \{y \in M \,/\, \rho(x,y) < r\}.$$
(2) A subset $U \subset M$ is said to be an open set of M if for each $x \in M$ there exists $r > 0$ such that
$$B_\rho(x,r) \subset U.$$

Each b-metric ρ on a nonempty M generates a topology τ_p on M defined by
$$\tau_p = \{U \subset M \,/\, U \text{ is an open set in } M\}.$$
Notice that τ_p is Hausdorff.

Now, we present the notions of a convergent sequence, Cauchy sequence and complete b-metric spaces.

Definition 3. Let (M,ρ) be a b-metric space with $s \geq 1$. Then a sequence (x_n) in M is called:

(1) $b-$convergent sequence if and only if there exists $x \in M$ such that $\lim_{n\to\infty} \rho(x_n, x) = 0$. In this case we denote it by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(2) $b-$Cauchy if $\lim_{n,m\to\infty} \rho(x_n, x_m) = 0$, this is equivalent to: given $\varepsilon > 0$ there exists $n_0(\varepsilon) \in \mathbb{N}$ such that for all $n, m > n_0(\varepsilon)$ implies that $\rho(x_n, x_m) < \varepsilon$.
(3) A set $A \subset M$ is called closed if and only if for each sequence (x_n) in M which converge to x, we have $x \in A$.
(4) If every $b-$Cauchy sequence in M is convergent, then (M,ρ) is said to be a complete b-metric space.

Proposition 1. *Let (M,ρ) be a b-metric space with $s \geq 1$. The following assertions hold:*

(1) *A $b-$convergent sequence has a unique limit.*
(2) *The subsequences of a $b-$convergent sequence are also $b-$convergent to the limit of the original sequence.*

Definition 4. Let (M,ρ) be a b-metric space with $s \geq 1$. The b-metric is called continuous if $\lim_{n\to\infty} \rho(x_n, x) = 0$ and $\lim_{n\to\infty} \rho(y_n, y) = 0$ then
$$\lim_{n\to\infty} \rho(x_n, y_n) = \rho(x, y).$$

The b-metrics given in Example 2 illustrate that their respective b-metric are continuous functions, but in general a b-metric is not a continuous function in all its variables (see, [24]).

The following results proved by A. Aghajani, M. Abbas and J. R. Rushan in [2], shows some facts about $b-$convergent sequences.

Proposition 2. *Let (M,ρ) be a b-metric space with $s \geq 1$, and suppose that (x_n) and (y_n) are $b-$convergent sequences to x and y respectively. Then we have,*

$$\frac{1}{s^2}\rho(x,y) \leq \liminf_{n\to\infty} \rho(x_n,y_n) \leq \limsup_{n\to\infty} \rho(x_n,y_n) \leq s^2\rho(x,y).$$

In particular, if $x = y$ then $\lim_{n\to\infty} \rho(x_n,y_n) = 0$. Moreover, for each $z \in M$ we have

$$\frac{1}{s}\rho(x,z) \leq \liminf_{n\to\infty} \rho(x_n,z) \leq \limsup_{n\to\infty} \rho(x_n,z) \leq s\,\rho(x,z).$$

Proposition 3. *Let (M,ρ) be a b-metric space with $s \geq 1$. Let (x_n) be a sequence in M such that*

$$\lim_{n\to\infty} \rho(x_n,x_{n+1}) = 0.$$

If (x_n) is not a $b-$Cauchy sequence in M, then there exist $\varepsilon > 0$ and sequences of positive integers $(n(k))$ and $(m(k))$ with $n(k) > m(k) > k > 0$ such that

$$\rho(x_{m(k)},x_{n(k)}) \geq \varepsilon, \quad \rho(x_{m(k)},x_{n(k)-1}) < \varepsilon$$

and

$$\varepsilon \leq \liminf_{k\to\infty} \rho(x_{m(k)},x_{n(k)}) \leq \limsup_{k\to\infty} \rho(x_{m(k)},x_{n(k)}) \leq \varepsilon.$$

$$\frac{\varepsilon}{s^2} \leq \liminf_{k\to\infty} \rho(x_{m(k)},x_{n(k)-1}) \leq \limsup_{k\to\infty} \rho(x_{m(k)},x_{n(k)-1}) \leq \varepsilon.$$

$$\frac{\varepsilon}{s^2} \leq \liminf_{k\to\infty} \rho(x_{m(k)-1},x_{n(k)-1}) \leq \limsup_{k\to\infty} \rho(x_{m(k)-1},x_{n(k)-1}) \leq \varepsilon s.$$

$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} \rho(x_{m(k)-1},x_{n(k)}) \leq \limsup_{k\to\infty} \rho(x_{m(k)-1},x_{n(k)}) \leq \varepsilon s^2.$$

Non commutative properties of pair of mappings on b-metric spaces

Now, we present some well-known non commutative definitions and properties of a pair of mappings in the setting of b-metric spaces that will be useful in the proof of our main results.

Definition 5. Let (M,ρ) be a b-metric space with $s \geq 1$. The mappings $S, T : M \to M$ are said to

(1) be compatible, ([14]) if and only if

$$\lim_{n\to\infty} \rho(STx_n, TSx_n) = 0,$$

whenever (x_n) is a sequence in M such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in M,$$

(2) be non compatible if there exists at least one sequence (x_n) in M such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in M,$$

but

$$\lim_{n\to\infty} \rho(STx_n, TSx_n) \quad \text{is either non zero or non existent,}$$

(3) be weakly compatible, ([12]) if for all $x \in M$, such that $Sx = Tx$, implies that $STx = TSx$,

(4) Satisfy the b-property (EA), ([1]) if there exists a sequence in M such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t, \quad \text{for some } t \in M,$$

(5) Satisfy the $b-$limit range property with respect to T, (in short $b-CLR_T-$ property), ([23]), if there exists a sequence (x_n) in M such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Tt, \quad \text{for some } t \in M.$$

Weakly compatible selfmappings play a key role in this paper since it is a minimal requirement to prove the existence of common fixed point for mappings of contractive type, (for a discussion, see, [19, 20]). Thus, the following result will be useful in the sequel.

Proposition 4 (Jungck–Rhoades, [13])**.** *Let S and T be weakly compatible selfmaps of a nonempty set M. If S and T have a unique point of coincidence (POC); that is, a unique $u \in M$ such that $z = Su = Tu$, then z is the unique common fixed point of S and T.*

Remark 1. Note the following relations between some of the mentioned notions:

(1) If S and T are compatible, then S and T are weakly compatible.
(2) If S and T are non compatible, then S and T satisfy the b-Property (EA).
(3) The weak compatibility and the b-Property (EA) are independent to each other.

3. Generalized Ψ-Geraghty-quasi-contractions

In this section we introduce the class of generalized Ψ-Geraghty-quasi-contraction mappings in b-metric spaces, by using altering distance functions, Geraghty's property and the quasi-contraction mappings for a pair of maps.

We recall that in 1973, M. Geraghty [10] introduce a condition called *Geraghty's property* which was used to prove a fixed point theorems that generalize the (BCP) in usual metric spaces.

For its use in the framework of b-metric spaces, we consider the class of functions $\mathcal{B}_s$, where we say that $\beta \in \mathcal{B}_s$ if $\beta : \mathbb{R}_+ \to [0, 1/s)$, $s \geq 1$ and it has the Geraghty's

property if

$$\lim_{n\to\infty}\beta(t_n)=\frac{1}{s}\quad\text{implies that}\quad\lim_{n\to\infty}t_n=0.\quad\text{for any }(t_n)\subset\mathbb{R}_+. \tag{3.1}$$

In 1984, M. S. Khan, M. Swalech and S. Sessa [17] introduce a new type of contraction mapping with the help of a control function which they called *altering distance functions*.

A function $\psi:\mathbb{R}_+\to\mathbb{R}_+$ is called an altering distance function if

(i) ψ is monotonic increasing function.
(ii) ψ is a continuous mappings.
(iii) $\psi(t)=0$ if and only if $t=0$.

By Ψ we denote the set of all altering distance functions.

Definition 6. Let (M,ρ) be a b-metric space with $s\geq 1$. Two mappings $S,T:M\to M$ are called generalized $\Psi-$Geraghty-quasi-contractions, if there exist $\psi\in\Psi$ and $\beta\in\mathcal{B}_s$ such that

$$\psi[s\rho(Sx,Sy)]\leq\beta[\psi(N(x,y))]\psi(N(x,y)),\quad\text{for all }x,y\in M. \tag{3.2}$$

Where,

$$N(x,y)=\max\{\rho(Tx,Ty),\ \rho(Sx,Tx),\ \rho(Sy,Ty),\ \rho(Sx,Ty),\ \rho(Sy,Tx)\}.$$

Remark 2. Since $\beta\in\mathcal{B}_s$ are strictly smaller than $1/s$ for some $s\geq 1$, we have

$$\beta[\psi(N(x,y))]<1/s,\quad\text{for all }x,y\in M.$$

Proposition 5. *Let (M,ρ) be a b-metric space with $s\geq 1$ and let S,T be two selfmaps of M. Assume that S and T are generalized $\Psi-$Geraghty-quasi-contractions. If S and T have (POC) in M, then it is unique.*

Proof. Suppose that z and w are two (POC) of S and T. Therefore, there exists $u,v\in M$ such that $Su=Tu=z$ and $Sv=Tv=w$. Suppose $z\neq w$. From inequality (3.2), we have

$$\psi[\rho(z,w)]\leq\psi[s\rho(z,w)]=\psi[s\rho(Su,Sv)]\leq\beta[\psi(N(u,v))]\psi[N(u,v)]$$

for all $u,v\in M$, where

$$\begin{aligned}N(u,v)&=\max\{\rho(Tu,Tv),\ \rho(Su,Tu),\ \rho(Sv,Tv),\ \rho(Su,Tv),\ \rho(Su,Tu)\}\\&=\max\{\rho(z,w),\ \rho(z,z),\ \rho(w,w),\ \rho(z,w),\ \rho(w,z)\}\\&=\rho(z,w).\end{aligned}$$

Therefore,

$$\begin{aligned}\psi[\rho(z,w)]\leq\psi[s\rho(z,w)]&\leq\beta[\psi(\rho(z,w))]\psi(\rho(z,w))\\&<(1/s)\psi(\rho(z,w))\leq\psi(\rho(z,w)),\end{aligned}$$

which is a contradiction. Hence $\rho(z,w)=0$, consequently, $z=w$. □

Now, we show a condition that guarantee the existence of a unique (POC).

Theorem 4. *Let (M, ρ) be a b-metric space with $s \geq 1$ and let $S, T : M \to M$ be generalized Ψ-Geraghty-quasi-contractions. If $TM \subset M$ is a complete subspace of M and S and T satisfy the b-property (EA), then S and T have a unique (POC).*

Proof. Since S and T satisfy the b-property (EA), there exists a sequence (x_n) in M such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \quad \text{for some } z \in M.$$

Since $TM \subset M$ is complete, we have

$$\lim_{n\to\infty} Sx_n = z = Tu = \lim_{n\to\infty} Tx_n \quad \text{for some } u \in M.$$

Now we assert that $Su = Tu$. Suppose that $Su \neq Tu$. From inequality (3.2) and Proposition 2, we obtain

$$\begin{aligned}\psi[\rho(Su, Tu)] \leq \psi[s\rho(Su, Tu)] &\leq \psi\left[\frac{s^2\rho(Su, Tu)}{s}\right] \\ &\leq \limsup_{n\to\infty} \psi[s^2\rho(Sx_n, Su)] \\ &\leq s \limsup_{n\to\infty} \beta[\psi(N(x_n, u))] \limsup_{n\to\infty} \psi(N(x_n, u))\end{aligned}$$

where

$$N(x_n, u) = \max\{\rho(Tu, Tx_n),\ \rho(Sx_n, Tx_n),\ \rho(Su, Tu), \rho(Sx_n, Tu),\ \rho(Su, Tx_n)\}.$$

Taking upper limit as $k \to \infty$ and applying Proposition 2, we get

$$\begin{aligned}\limsup_{n\to\infty} N(x_n, u) &\leq \max\{s\rho(z, z),\ s^2\rho(z, z),\ \rho(Su, Tu),\ s\rho(z, z),\ s\rho(Su, Tu)\} \\ &= \max\{\rho(Su, Tu),\ s\rho(Su, Tu)\} = s\rho(Su, Tu).\end{aligned}$$

Then we have,

$$\begin{aligned}\psi(s\rho(Su, Tu)) &\leq s\beta[\psi(s\rho(Su, Tu))]\psi[s\rho(Su, Tu)] \\ &< \psi(s\rho(Su, Tu)).\end{aligned}$$

Since $\psi \in \Psi$, we have a contradiction. Therefore, $\rho(Su, Tu) = 0$, that is, $Su = Tu$. This shows that u is a coincidence point of S and T and $z = Su = Tu$ is a (POC) of S and T. From Proposition 5, z is the unique (POC) of S and T. □

4. Common fixed points

In this section we are going to prove common fixed point results for generalized Ψ-Geraghty-quasi-contractions. To attain such goal we use the notions of weakly compatible and the b-property (EA).

Theorem 5. *Let (M, ρ) be a b-metric space with $s \geq 1$ and let $S, T : M \to M$ be generalized Ψ-Geraghty-quasi-contractions. If $TM \subset M$ is a complete subspace of M and S and T satisfy the b-property (EA) and they are weakly compatible, then S and T have a unique common fixed point.*

Proof. Since S and T satisfy the b-property (EA), from Theorem 4, they have a unique (POC). Now, since S and T are weakly compatible, from Proposition 4 we obtain the conclusion. □

The following example supports our result.

Example 2. Let $M = [0, 1]$ be equipped with the b-metric defined by $\rho(x, y) = |x-y|^2 = (x-y)^2$. Consider the mappings defined by

$$Sx = \begin{cases} \dfrac{x}{9} & \text{if} \quad x \in [0, 1/2] \\ 1/9 & \text{if} \quad x \in (1/2, 1] \end{cases} \quad \text{and} \quad Tx = \begin{cases} \dfrac{x}{3} & \text{if} \quad x \in [0, 1/2] \\ 1/3 & \text{if} \quad x \in (1/2, 1]. \end{cases}$$

Let $\psi(t) = \sqrt{t}$ and $\beta(t) = \dfrac{1}{2} \quad \forall t \in \mathbb{R}_+$. Then, by considering the sequence $(x_n) = (1/n)$ in M we have that

$$\lim_{n\to\infty} \rho(Sx_n, 0) = \lim_{n\to\infty} \rho(Tx_n, 0) = 0.$$

Therefore, S and T satisfy the b-property (EA). Notice that $C(S, T) = \{x \in M \mid Sx = Tx\} = \{0\}$ and $ST0 = 0 = TS0$. Hence, S and T are weakly compatible selfmaps. Also notice that $TM \subset M$ is complete. On the other hand,

$$\begin{aligned} N(x, y) &= \max\{\rho(Tx, Ty),\ \rho(Sx, Tx),\ \rho(Sy, Ty),\ \rho(Sx, Ty),\ \rho(Sy, Tx)\} \\ &= \max\left\{\frac{1}{9}|x-y|^2,\ \frac{4}{27}x^2,\ \frac{4}{27}y^2,\ \frac{1}{81}|x-3y|^2,\ \frac{1}{8}|y-3|^2\right\} \\ &= \frac{1}{9}|x-y|^2. \end{aligned}$$

Thus, $\psi(N(x, y)) = 1/3|x-y|$, from which we obtain

$$\psi[\rho(Sx, Sy)] = \frac{1}{9}|x-y|.$$

Now, from (3.1), we have

$$\begin{aligned} \frac{1}{9}|x-y| = \psi(\rho(Sx, Sy)) &\leq \beta[1/3|x-y|](1/3|x-y|) \\ &< 1/2(1/3|x-y|) = 1/6|x-y|. \end{aligned}$$

Then all conditions of Theorem 5 holds. It is clear that $x = 0$ is the unique common fixed point of S and T.

Taking $\psi = \mathrm{Id}$, (identity map) in the generalized Ψ-Geraghty-quasi-contraction of Theorem 5, we obtain

Corollary 1. *Let (M, ρ) be a metric space with $s \geq 1$ and let $S, T : M \to M$ be mappings that satisfy the following inequality*

$$\rho(Sx, Sy) \leq \beta(N(x, y))N(x, y) \tag{4.1}$$

for all $x, y \in M$, where

$$N(x, y) = \max\{\rho(Tx, Ty),\ \rho(Sx, Sy),\ \rho(Sy, Ty),\ \rho(Sx, Ty),\ \rho(Sy, Tx)\}.$$

If S and T satisfy the b-property (EA) and if $TM \subset M$ is complete, then

(1) *S and T have a unique (POC) and*
(2) *If S and T are weakly compatible, then S and T have a unique common fixed point in M.*

Since two non compatible selfmap of a b-metric space (M, ρ) satisfy the b-property (EA), we get the following consequence.

Corollary 2. *Let (M, ρ) be a b-metric space with $s \geq 1$ and let $S, T : M \to M$ be maps that satisfy the following conditions:*

(i) *S and T are non compatible selfmaps.*
(ii) *$TM \subset M$ is complete.*
(iii) *S and T satisfy any of the following inequalities (3.2) or (4.1).*

Then,

(1) *S and T have a unique (POC) and*
(2) *If S and T are weakly compatible, then S and T have a unique common fixed point in M.*

In the next results we replace the closedness of the range of T with the b-CLR_T-property.

Theorem 6. *Let (M, ρ) be a b-metric space with $s \geq 1$ and let $S, T : M \to M$ be maps holding the following conditions:*

(i) *S and T satisfy the b-CLR_T-property.*
(ii) *S and T are generalized $\Psi-$Geraghty-quasi-contraction.*

Then,

(1) *S and T have a unique (POC).*
(2) *If S and T are weakly compatible then S and T have a unique common fixed point in M.*

Proof. Since S and T satisfy the b-CLR_T-property, then there exists a sequence (x_n) in M such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Tu \quad \text{for some } u \in M.$$

Therefore, there exists $z \in M$ such that $z = Tu$. The rest of the proof follows as in the proof of Theorem 4.1. □

Corollary 3. *Let (M,ρ) be a b-metric space with $s \geq 1$ and let $S,T : M \to M$ be maps holding the following conditions:*

(i) *S and T satisfy the b-CLR$_T$-property.*
(ii) *S and T satisfy any of following inequalities (3.2) or (4.1).*

Then,

(1) *S and T have a unique (POC), and*
(2) *if S and T are weakly compatible, then S and T have a unique common fixed point in M.*

5. On the Jungck-Mann iterative scheme for generalized Ψ-Geraghty-quasi contractions

Effective algorithms to compute approximate common fixed points for pairs of mappings is a crucial tool in the fixed point theory. In this section, we will prove the b-convergence of the so-called *Jungck-Mann iterative scheme* ([22]):

$$Tx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Tx_n, \tag{5.1}$$

where (α_n) is a sequence in $[0,1]$.

In order to define this iteration process in our setting, we are going to consider a b-metric space induced by a *quasinormed space* (M,ρ); that is, a vector space M endowed with a function $\rho : M \longrightarrow \mathbb{R}_+$ satisfying

(ρ_1) $\rho(x) \geq 0$ and $\rho(x) = 0$ iff $x = \mathbf{0}$.
(ρ_2) $\rho(\lambda x) = |\lambda|\rho(x)$, $\lambda \in \mathbb{R}$.
(ρ_3) $\rho(x+y) \leq s(\rho(x)+\rho(y))$, $s \geq 1$.

(M,ρ) is metrizable in virtue of the Aoki-Rolewicz theorem ([3,21]) and it becomes a b-metric space with the induced b-metric $\rho(x,y) := \rho(x-y)$.

The spaces given in Example 1 are b-normed spaces, as well as, some useful spaces in functional analysis as: *Lebesgue spaces with variable exponent* and scales of *weak Lebesgue spaces, Lorentz spaces, Hardy spaces, weak Hardy spaces, Lorentz-based Hardy spaces, Besov spaces, Triebel Lizorkin space* among others.

To prove the convergence of the Jungck-Mann iterative scheme, in the next result we are going to assume that the coefficient s of the b-normed space is greater or equal than 2 and $\psi \in \Psi$ satisfies the following conditions:

(i) There exists a function ϕ continuous on $[0,\infty)$ with $\psi(ab) \leq \phi(a)\psi(b)$ for $a > 0$ and $b \geq 0$.
(ii) $\psi(a+b) \leq \psi(a)+\psi(b)$ for all $a,b \geq 0$.
(iii) $\psi(s) < \frac{1}{2}$, where s is the coefficient of (M,ρ).

A nondecreasing function satisfying (i) it is said to belong to the class H. This class was introduced by F.M. Dannan in [8] in relation to the study of nonlinear generalizations of the Gronwall-Bellman inequality and its applications to the asymptotic behavior of differential equations.

Notice that the function $\psi(t) = \theta\sqrt{t}$, with $\theta < 1/(2f(s))$, for any function f satisfying $f(t) \geq \sqrt{t}$, satisfies conditions (i)–(iii) above.

Theorem 7. *Let (M, ρ) be a b-normed space with $s \geq 2$ and let $S, T : M \to M$ be generalized Ψ-Geraghty-quasi-contractions with $S(M) \subset T(M)$, and let us suppose that the pair (S,T) has a common fixed point p. Then, for any $x_0 \in M$, the iterative process (5.1) b-converges to p.*

Proof. Let $p \in M$ such that $p = Sp = Tp$ and $x_o \in M$ arbitrary. Then,

$$\begin{aligned}\rho(Tx_{n+1} - p) &= \rho((1-\alpha_n)Sx_n + \alpha_n Tx_n + (1-\alpha_n+\alpha_n)p)\\ &\leq s(1-\alpha_n)\rho(Sx_n - p) + s\alpha_n\rho(Tx_n - p).\end{aligned}$$

Since $\psi \in \Psi$ is subadditive, we conclude

$$\begin{aligned}\psi(\rho(Tx_{n+1} - p)) &\leq \psi(s(1-\alpha_n)\rho(Sx_n - p)) + \psi(s\alpha_n\rho(Tx_n - p))\\ &\leq \psi(s\rho(Sx_n - p)) + \psi(s\rho(Tx_n - p)).\end{aligned}$$

On the other hand, since S is a Ψ-Geraghty-quasi contraction, we have

$$\psi(s\rho(Sx_n - p)) = \psi(s\rho(Sx_n - Sp)) \leq \beta(\psi(N(x_n,p)))\psi(N(x_n,p)),$$

where

$$\begin{aligned}N(x_n,p) &= \max\{\rho(Tx_n - p), \rho(Sx_n - Tx_n), \rho(p-p), \rho(Sx_n - p), \rho(p - Tx_n)\}\\ &\leq \max\{\rho(Tx_n - p), s[\rho(Sx_n - p) + \rho(Tx_n - p)], \rho(Sx_n - p)\}\\ &= s[\rho(Sx_n - p) + \rho(Tx_n - p)].\end{aligned}$$

Thus,

$$\psi(N(x_n,p)) \leq \psi(s\rho(Sx_n - p)) + \psi(s\rho(Tx_n - p)).$$

From the fact that $\beta(t) < s^{-1}$ and $s \geq 2$, we obtain the following estimate:

$$\begin{aligned}\psi(s\rho(Sx_n - p)) &< \frac{1}{s}\psi(s\rho(Sx_n - p)) + \frac{1}{s}\psi(s\rho(Tx_n - p))\\ &\leq \frac{1}{2}\psi(s\rho(Sx_n - p)) + \frac{1}{2}\psi(s\rho(Tx_n - p)).\end{aligned}$$

Then, we have

$$\psi(s\rho(Sx_n - p)) < \psi(s\rho(Tx_n - p)).$$

With these estimates we obtain that

$$\begin{aligned}\psi(\rho(Tx_{n+1} - p)) &< 2\psi(s\rho(Tx_n - p))\\ &\leq 2\phi(s)(\rho(Tx_n - p))\\ &\leq (2\phi(s))^n(\rho(Tx_0 - p)).\end{aligned}$$

Therefore, we conclude that $\lim_{n\to\infty}\psi(\rho(Tx_{n+1} - p)) = 0$, since $2\phi(s) < 1$. This implies that $\lim_{n\to\infty}\rho(Tx_{n+1} - p) = 0$, as $n \to \infty$. □

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ON THE NUMBER OF SHORTEST PATHS BY NEIGHBORHOOD SEQUENCES ON THE SQUARE GRID

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This paper is dedicated to Professor Béla Vizvári on his 70th birthday.

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Abstract. In this paper we are addressing a counting problem of discrete mathematics, more precisely of digital geometry. In the Euclidean plane the shortest path between any two points is given by the straight line segment connecting the points. In discrete mathematics, the shortest path is usually not unique, e.g., in graphs there could be several shortest paths between two vertices. In this paper, a special infinite graph, the square grid, (i.e., the usual digital plane) is used. In digital geometry there are various digital, i.e., path based distance functions. A neighborhood sequence B gives the condition for each step of a B-path separately what type of neighborhood is used in that step. Therefore, the length and also the number of the shortest paths between two points depend not only on the respective positions (coordinate differences) of the points but also on the neighborhood sequence B. We give an algorithm and also closed formulae to compute the number of shortest B-paths.

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1. Introduction

When we are using a computer/digital camera/tablet a digital space, i.e., the square grid with points/pixels addressed by integer coordinate pairs is taken into consideration. Working on digital spaces, e.g., on a computer screen or on digital images, the digital, path-based distances have several advantages. The digital geometry is essentially different from the Euclidean geometry. In Euclidean space there are infinitely many points between any two distinct points. In digital geometry, the space is discrete; the regular grids can be seen as special graphs. Path-based distances are based on the concept of neighborhood relation among the points of the grid. On the square grid containing points with integer coordinates there are two basic neighborhood relations: two points are considered as 1-neighbors if one of their coordinates coincides and the difference on the other is exactly 1. A step from a point to one of its 1-neighbor points, a 1-step, may change one coordinate by exactly 1. One may

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also allow diagonal movements: two points are 2-neighbors if their coordinate differences are at most 1; consequently, in a 2-step both or any of the coordinates can be changed by at most 1. The first two digital distance functions (see [16]), from the end of 1960's, the cityblock (or Manhattan) and the chessboard distances, using these neighborhood relations in the paths, are very rough approximations of the Euclidean distance. Therefore, there are various other digital distances considered to lower the rotational dependency. Already Rosenfeld and Pfaltz, in [16], recommended to use the two types of neighborhood alternately in a path. New developments appeared approximately 20 years later. One of these approaches is the usage of neighborhood sequences giving the flexibility to change to change neighborhoods in a step of a path depending on a predefined sequence. (See [4,5,18,19], where periodic neighborhood sequences were investigated; around 2000, this periodic condition was dropped to allow general, not necessarily periodic, neighborhood sequences.) A greedy algorithm that provides a shortest path between any two points by any neighborhood sequence B can be found in [10], while a general formula to compute B-distances can be found in [12]. We should note here that digital distances, and especially B-distances, can be applied in various image processing tasks [7, 17] and also in communication networks [13], their extension for computing distances of sequences was also investigated [14, 15]. In digital geometry, similarly as in graph theory, there is usually more than one minimal (i.e., shortest) path between two points, and it is an interesting combinatorial problem to find their numbers. Already in [16] combinatorial problems based on digital distances were investigated, namely 'path counting'. Then, a matrix algorithm is provided in [2] to count the number of shortest paths between any pair of pixels of a shape in a binary image. We should note that the number of shortest paths are also important in connection to application of networks. Further, in [3], Das gave recursive formulae for the number of shortest paths between any two points of the square grid using special distance functions, e.g., the cityblock and the chessboard distances.

In this paper, we address a more general problem: we are counting the number of shortest B-paths from the point p to the point q of the square grid, where apart from the coordinates of the points, the neighborhood sequence B is also a parameter.

2. Basic notions and definitions

Let $\mathbb{N}$, $\mathbb{N}^+$ and $\mathbb{Z}$ denote the sets of nonnegative integers, positive integers and integers, respectively. Consequently, the points of the square grid are addressed by $\mathbb{Z}^2$. In digital geometry, and generally, in grids the concept neighborhood plays an essential role.

Let $p,q \in \mathbb{Z}^2$ be two points of the grid with $p = (x_p, y_p)$ and $q = (x_q, y_q)$. We say that p and q are 1-neighbors (cityblock neighbors) if $|x_p - x_q| + |y_p - y_q| = 1$ and $p \neq q$ are 2-neighbors (chessboard neighbors) if $|x_p - x_q| \leq 1$ and $|y_p - y_q| \leq 1$.

Notice that all 1-neighbors are also 2-neighbors, by definition. A movement from a point to one of its i-neighbors is called an i-step ($i \in \{1,2\}$).

Formally, a neighborhood sequence $B = (b(i))_{i=1}^{\infty}$ is an infinite sequence containing only the elements of the set $\{1,2\}$, i.e., $b(i) \in \{1,2\}$ for all $i \in \mathbb{N}^+$. A B-path, i.e., a path defined by B, from a point p to a point q of the grid is a sequence of points $p_0, p_1, \ldots, p_n$ such that $p_0 = p$ and $p_n = q$, moreover p_{i-1} and p_i are $b(i)$-neighbors for every $1 \leq i \leq n$. In this way, the neighborhood sequence B guides the path, in a sense, allowing or denying the diagonal steps for each step. The length of such a path is n, that is the number of steps in the path. Obviously, there are several B-paths between any two points and some of them (at least one) have a minimal length; this minimal length gives the B-distance $d_B(p,q)$ of the points p and q. One can see that the classical cityblock and chessboard distances can be obtained by using 'constant' neighborhood sequences $B_1 = (b_1(i))_{i=1}^{\infty}$ with $b_1(i) = 1$ for every $i \in \mathbb{N}^+$ and $B_2 = (b_2(i))_{i=1}^{\infty}$ with $b_2(i) = 2$ for every $i \in \mathbb{N}^+$, respectively. We note here, that by the original definition of neighborhood sequences, see, e.g., [4, 5] only periodic neighborhood sequences were used, and thus, their description was done by only one period, i.e., with a finite length. The general approach used here does not restrict the study to periodic neighborhood sequences, and we use their general description by infinite sequences.

In [9, 10] a greedy algorithm was shown to obtain a shortest B-path between the input points and it was also proven to be correct. Moreover, when B is fixed, the B-paths are translation invariant: for any point pair $p,q \in \mathbb{Z}^2$, any B-path from p to q has a bijective image as a B-path from the origin $(0,0)$ to the point $r = (p-q) = (x_p - x_q, y_p - y_q)$. Consequently, shortest paths are mapped to shortest paths. The B-distances are symmetric, but, depending on B, they may not fulfill the triangular inequality. In [10, 11] it is proven that B defines a metric if and only if the number of 2's in its prefix sequence with length n is not more than the number of 2's anywhere in the sequence among n consecutive elements for all $n \in \mathbb{N}^+$. A formula to compute the B-distance is provided in [11, 12], now we recall it:

Let p,q be points in $\mathbb{Z}^2$ and B be a neighborhood sequence. Let j and i denote their coordinate differences: $j = \max\{|p_x - q_x|, |p_y - q_y|\}$ and $i = \min\{|p_x - q_x|, |p_y - q_y|\}$. Then the B-distance of the points is

$$d_B(p,q) = \max\{j,k \mid j + i > \sum_{m=1}^{k-1} b(m)\}. \tag{2.1}$$

2.1. *Squaring the circles*

Digital disks (and circles) are defined based on digital distances analogously to the definition of Euclidean disks (and circles).

It is well known that the disks based on cityblock distances has a 'diamond' shape (actually, it is a square standing on one of its corner) with 'stair' sides (an example is shown in Figure 1, a). We just briefly mention here that by using the digital distance

						5					
					5	4	5				
				5	4	3	4	5			
			5	4	3	2	3	4	5		
		5	4	3	2	1	2	3	4	5	
	5	4	3	2	1	O	1	2	3	4	5
		5	4	3	2	1	2	3	4	5	
			5	4	3	2	3	4	5		
				5	4	3	4	5			
					5	4	5				
						5					

(a)

5	5	5	5	5	5	5	5	5	5	5	
5	4	4	4	4	4	4	4	4	4	5	
5	4	3	3	3	3	3	3	3	4	5	
5	4	3	2	2	2	2	2	3	4	5	
5	4	3	2	1	1	1	2	3	4	5	
5	4	3	2	1	O	1	2	3	4	5	
5	4	3	2	1	1	1	2	3	4	5	
5	4	3	2	2	2	2	2	3	4	5	
5	4	3	3	3	3	3	3	3	4	5	
5	4	4	4	4	4	4	4	4	4	5	
5	5	5	5	5	5	5	5	5	5	5	

(b)

				5	5	5	5	5			
			5	4	4	4	4	4	5		
		5	4	4	3	3	3	4	4	5	
	5	4	4	3	2	2	2	3	4	4	5
	5	4	3	2	2	1	2	2	3	4	5
	5	4	3	2	1	O	1	2	3	4	5
	5	4	3	2	2	1	2	2	3	4	5
	5	4	4	3	2	2	2	3	4	4	5
		5	4	4	3	3	3	4	4	5	
			5	4	4	4	4	4	5		
				5	5	5	5	5			

(c)

FIGURE 1. Digital disks with radius 5 based on (a) cityblock distance, (b) chessboard distance and (c) neighborhood sequence $B = (1,2,1,2,1,\ldots)$ where the elements $b(i)$ with $i \geq 6$ does not play any role in the shown disks, and therefore, their values are arbitrary. The numbers typed into the pixels shows their distance from the center (marked by 'O'). (One may also observe the disks having radius r less than 5 by considering those pixels that have value not more than r.)

based on chessboard neighbors, the digital disks are squares (with straight line sides, see Figure 1, b).

In general case, the digital disk based on a neighborhood sequence is an octagon. The previous very rough, square estimations of Euclidean circles become 'more round' (e.g., by measuring it with the isoperimetric ratio). Roughly saying, the number of the used 1's gives the length of the 'stair'-sides, while the horizontal and vertical (straight line) sides have the length double more than the used 2's (see, e.g., Figure 1, c). For this reason, (some of) the distances by neighborhood sequences on the square grid are sometimes referred as octagonal distances. Note that the size and the shape of the octagons do not depend on the order of steps.

2.2. *Known results*

In this subsection we recall some of the results about the number of shortest paths of those special cases which are described, e.g., by Das in [3].

Let p and q be two points such that their (absolute) coordinate differences are i and j in the first and in the second coordinates, respectively. Then, by cityblock distance, i.e., using the neighborhood sequence B containing only 1's, the distance is the sum of the coordinate differences, i.e., $i+j$. Further, the number of shortest paths is $\binom{i+j}{i} = \binom{i+j}{j}$, since the path contains i vertical and j horizontal moves, and their order is arbitrary. This result is also connected to shuffles of trajectories, see [8].

Das also counted the number of shortest path by the chessboard distance, and by the octagonal distance, recommended by Rosenfeld and Pfaltz, described by the neighborhood sequence $B = (1,2,1,2,...)$ with $b(i) = 1$ at each odd position ($i = 2k+1$ for $k \in \mathbb{N}$), and $b(i) = 2$ at each even position ($i = 2k$ with $k \in \mathbb{N}^+$). However, as it is written in [3] to count the the number of shortest paths in the general case left open.

We note that Das has obtained these results based on recursive formulation using recurrence relations and generating functions. Opposite to this, we will prove our formulae by enumerative combinatorial techniques. Moreover, we prove our result for more general cases, since Das and his co-authors used only periodic neighborhood sequences ([4, 5], etc.), but the concept has also been generalized to not necessarily periodic neighborhood sequences (see e.g. [9]).

In the next sections we present our results.

3. Algorithm to compute the number of shortest paths

Since the B-distances are translation invariant (they depend only on the coordinate difference of the points and on the used neighborhood sequence), further we may assume that one of the points is the origin $(0,0)$.

We give an algorithm that also shows the recursive nature of the problem. Similarly to the way one can obtain digital disks (shown, e.g., in Fig. 1), instead of increasing the distance, one can count the number of shortest paths.

Actually, the hardness of the problem is hidden in the algorithm. At step 7 there are various cases how a point gets its value:

- If $b(i) = 1$
  - There are points for which there is exactly one of its 1-neighbors had assigned value already.
  - There are points for which two of their 1-neighbors had already known value.
- If $b(i) = 2$ there are three subcases:
  - A point could have only one 2-neighbor with already known value.
  - A point may have exactly two 2-neighbors with already assigned values.
  - A point may have already three 2-neighbors with known values.

Fig. 2 shows one-one example for each of these cases.

Algorithm 1

Input: a neighborhood sequence $B = (b(i))_{i=1}^{\infty}$ and a point (x,y).
Output: the number of shortest paths from $(0,0)$ to (x,y).
1. Let the number of shortest paths to the origin $(0,0)$ be fixed as 1.
2. Let $i = 0$.
3. While the number of shortest paths is not assigned to (x,y)
   Do
4. Let $i = i+1$.
5. Let S be the set of points p with both of the following properties:
5.a p does not have any value assigned yet, and
5.b p has at least one $b(i)$-neighbor with already assigned value.
6. For each element p of S do in parallel
7. Assign the sum of the known values of its $b(i)$-neighbors to p.
8. EndFor.
9. EndWhile.

As one may observe that with $b(i) = 1$ at the straight line sides the value is copied (kept) from the previous element (Fig. 2, a), while at 'stairs' the sum of two previous values known at the neighbors are used (Fig. 2, b). In case $b(i) = 2$, roughly saying, sum of 3 previous neighbor values are used for inner points of straight line sides (Fig. 2, c), while only one value is copied at the 'corners' (Fig. 2, e) and the sum of two values is used for neighbors of the corners (Fig. 2, d). At stair sides, however, the sum of three values of neighbors with previously assigned values is used at the inner points of the stairs and one value is copied for the outer points of the stairs (see Fig. 2, f and g, respectively).

We note here that the applied algorithm can be seen as a variation of Dijkstra algorithm and the breadth-first search algorithm, the correctness of the algorithm is also based on this fact. Actually, it sums up the number of shortest paths to the points from which the target point can be reached in one step in a shortest path (and it assigns the value, the number of shortest paths to these, previous points in a similar manner based on the number of shortest paths for those points from which they can be reached in by a last step in a shortest path, etc.) Also, the distance transform algorithms used in digital image processing are closely related. Fig. 2 highlights the complexity of the problem, comparing it, e.g., to the case of cityblock distances, where the binomial coefficients are obtained based on the Pascal triangle.

The time complexity of the algorithm is quadratic in the number of steps, i.e., the distance of the points. Actually, the value for each point can be computed within constant time (Steps 6 and 7): A point has a bounded number of $b(i)$-neighbors, i.e., at most 8, but the number of those which has already value is always less than 8. The sum of those values can be computed in a fixed amount of time (assuming a type that is large enough to work with these numbers).

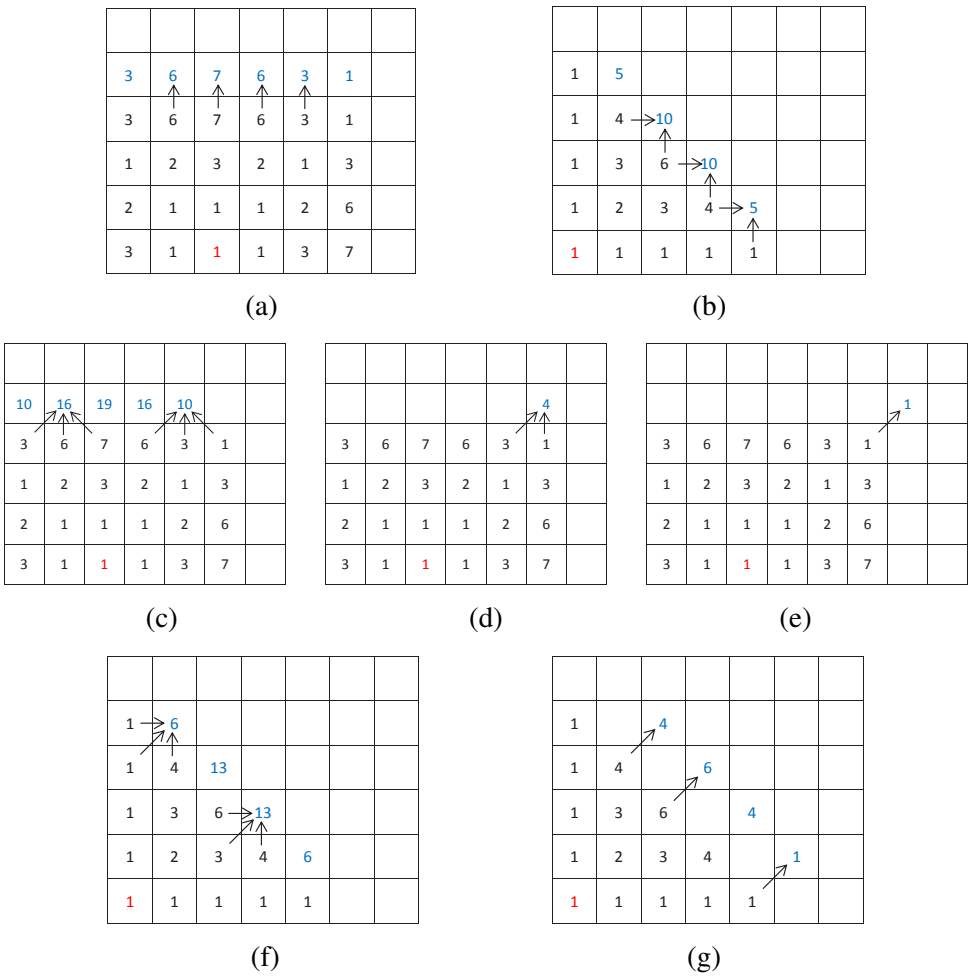

FIGURE 2. Examples for various cases from where the number of shortest paths can be summed up to a point (the origin is marked by red color in each figure, example new values are shown by blue). Cases with $b(i) = 1$ are shown in (a) straight side, (b) stair side. Cases with $b(i) = 2$ are shown in (c),(d),(e) straight side, (f),(g) stair side.

The number of points that have already assigned values is at most $(2i+1)^2$, consequently the set S has at most $4(2i+1) = 8i+4$ elements. Depending on the used neighborhood sequence B, by reaching the point (x,y), at most $(2\max\{|x|,|y|\}+1)^2$ points get their values. This shows the quadratic time and also the quadratic space complexity of the algorithm.

4. Formulae for the number of shortest paths

In this section we provide direct formulae to compute the number of shortest B-paths between any two gridpoints. The formula for the number of shortest paths can be computed in 3 different ways depending on the coordinate differences of the points and the number of 2's in the beginning part of the sequence B, i.e., in the part that is used to provide the shortest B-path(s).

The cases can be considered as follows.

Let the maximal coordinate difference of the two points (p and q) be j, and the other coordinate difference be i. Moreover, for simplicity, let $d = d_B(p,q)$ denote the B-distance of the points.

- If the number of 2's among the first j elements of B is exactly i, then this is the "fit case".
- We have the "poor case" if there are less 2's among the first d elements of B than i.
- In this case there are more 2's in the initial part of B than the number of 2's that is needed in a shortest path. This is the "rich case".

Let us use the following notation. Let k_1 and k_2 denote the number of 1's and 2's among the first k elements of B. Obviously $k_1 + k_2 = k$. (We will use this notation for $k = j$ and for $k = d$.)

Now, let us start with the simplest case.

4.1. *The fit case*

Proposition 1. *Let $p,q \in \mathbb{Z}^2$ and B be a neighborhood sequence. Denote $j = \max\{|x_p - x_q|, |y_p - y_q|\}$ and $i = \min\{|x_p - x_q|, |y_p - y_q|\}$. If the number of 2's among the first j elements of B is exactly i, then the B-distance of the points p and q is $d = j$.* □

This statement is a straightforward special case of Eq. 2.1. In a shortest B-path at each 1-step we move at the direction of the larger coordinate difference, and at each 2-step we have a diagonal movement decreasing both coordinate differences. Obviously, the conditions given in this case allow exactly the shortest paths with this property. Consequently, in this case on can establish the following result.

Theorem 1. *Let $p,q \in \mathbb{Z}^2$ and B be a neighborhood sequence. Denote $j = \max\{|x_p - x_q|, |y_p - y_q|\}$ and $i = \min\{|x_p - x_q|, |y_p - y_q|\}$. If the number of 2's among the first j elements of B is exactly i, then there is exactly 1 shortest B-path from p to q.*

Proof. By Proposition 1, the length of the shortest path(s) from p to q is exactly j. Among these j steps there could be at most i diagonal steps, and actually, it needs to be i diagonal steps to reach the endpoint q from p. Consequently, all steps decrease the coordinate difference in the direction of j and all i 2-steps decrease both

coordinate differences. This can be done exactly in one way, as it is described, the direction of each step in the path is uniquely determined by B in this case. □

In this case, $d+d_2 = i+j$ and also $d = j$, thus $i+j = j+j_2$ which leads to $i = j_2$.

Notice that there is exactly one B-path also from q to p, however, this path not necessarily built up from the same points as the one from p to q.

4.2. *The poor case*

If there are less 2's among the first d elements of B than i, then some of the "diagonal moves" (of the previous case) must be substituted by two 1-steps. Therefore, in this case, $d > j$.

We can rewrite formula Eq. 2.1 in the following form, for this case.

Proposition 2. *Let $p,q \in \mathbb{Z}^2$ and B be a neighborhood sequence. Denote $j = \max\{|x_p - x_q|, |y_p - y_q|\}$ and $i = \min\{|x_p - x_q|, |y_p - y_q|\}$. If the number of 2's among the first j elements of B is less than i, then the B-distance of the points p and q is $d = \min\left\{k \mid i+j \leq \sum_{n=1}^{k} b(n)\right\}$.* □

Roughly saying, there are not enough 2-steps for having enough diagonal steps for i, thus some 1's are needed for steps to that direction and not only for the direction of the coordinate difference j.

In this paper we are interested to count the number of shortest B-paths, and thus, it is important to recognise that there are two subcases:

- Matching case: The sum of the first d elements of B is exactly $i+j$ (with $\sum_{n=1}^{d} b(n) = j+i$).
- Mismatch case: The sum of the first d elements of B is larger than $i+j$. It can happen when the last used element of B, i.e., $b(d)$ is a 2, but 1 would suffice. In this case, $\sum_{n=1}^{d} b(n) > j+i$, i.e., with more precision, $\sum_{n=1}^{d} b(n) = j+i+1$.

4.2.1. *Matching case*

Theorem 2. *Let $p,q \in \mathbb{Z}^2$ and B be a neighborhood sequence. Denote $j = \max\{|x_p - x_q|, |y_p - y_q|\}$ and $i = \min\{|x_p - x_q|, |y_p - y_q|\}$ and let d denote the B-distance of p and q. If the number of 2's among the first j elements of B is less than i and $\sum_{n=1}^{d} b(n) = j+i$, then the number of shortest paths from p to q is $\binom{d-d_2}{i-d_2} = \binom{d_1}{i-d_2}$.*

Proof. The length of a shortest path is d, and, since $\sum_{n=1}^{d} b(n) = j+i$, every 2-steps must be a diagonal step by decreasing both the coordinate differences. All 1-steps, their number is $d_1 = d - d_2$, must also decrease one of the coordinate differences.

Actually, in the shortest paths $i-d_2$ is the number of 1-steps to the direction of the coordinate difference i (and the remaining 1-steps are used in the direction determined by j). The order of these 1-steps is arbitrary, only their number is important to reach q from p: $i-d_2$ steps corresponding to direction i and $d_1-(i-d_2)=d-i$ steps to the orthogonal direction, from the d_1 1-steps, that is, $\frac{d_1!}{(i-d_2)!(d-i)!}=\binom{d_1}{i-d_2}$. Thus the number of shortest path can be written by the binomial coefficients given in the theorem. □

Notice that in this case $i+j=2d_2+d_1$ holds, which, combined by $d=d_2+d_1$ and knowing $j<d$, gives $d=j+i-d_2$ with $i-d_2>0$.

4.2.2. *Mismatch case*

In this case the sum of the first d elements of B is $i+j+1$.

Theorem 3. *Let p,q,B be given and the notation j,i,d be the same as at Theorem 2. If the number of 2's among the first j elements of B is less than i and $\sum_{n=1}^{d} b(n) > j+i$, then the number of shortest paths from p to q is $d_2\binom{d+1-d_2}{i+1-d_2}$.*

Proof. The length of a shortest path is d, but $\sum_{n=1}^{d} b(n) > j+i$, thus, there is exactly one 2-step which is actually, a 1-step. All other 2-steps must be a diagonal step by decreasing both the coordinate differences. The number of choices to find which 2-step is a 1-step in the path, i.e., to replace a value 2 to 1 among the first d elements of B, is exactly d_2. After, the replacement of this one element, actually, the number of shortest B-paths can be computed by Theorem 2, since the sum of $i+j$ matches to the modified neighborhood sequence. Consequently, $\binom{d+1-d_2}{i+1-d_2}$ gives the number of shortest paths for a modified neighborhood sequence: $d+1-d_2$ is the number of 1-steps in the shortest paths and $i+1-d_2$ is the number of 1-steps to the direction of the coordinate difference i. Using the multiplication rule for choosing first one element 2, and then the directions of the 1-steps, the formula of the theorem follows. □

4.3. *The rich case*

This is the most complex case, although the distance is exactly $d=j$ since there are enough 2's for the diagonal steps. However, there are extra value(s) 2 in the sequence. In shortest paths each 1-step is in the direction of the coordinate difference of j (otherwise j steps would not be enough). However, the 2-steps gives more variety for the shortest paths.

Theorem 4. *Let $p,q\in\mathbb{Z}^2$ and B be a neighborhood sequence. Denote $j=\max\{|x_p-x_q|,|y_p-y_q|\}$ and $i=\min\{|x_p-x_q|,|y_p-y_q|\}$. If the number of 2's among the first j elements of B, i.e., j_2, is more than i, then the number of shortest*

B-paths from p to q can be computed as

$$\sum_{m=i}^{i+\left\lfloor \frac{j_2-i}{2} \right\rfloor} \frac{j_2!}{m!(m-i)!(j_2+i-2m)!}.$$

Proof. Let us start with a shortest path that can be obtained by the greedy algorithm ([9, 10]). In this B-path, the first i 2-steps are diagonal steps decreasing the coordinate difference in both coordinates (we call the 2-steps of this direction as 'good' 2-steps), and all other steps are 1-steps decreasing also the coordinate difference on the direction determined by j.

But now, we are looking for the number of the shortest paths without caring about if they could be generated by the greedy algorithm or not. So, definitely, the 1-steps in any shortest path must be in the direction given by j. For the 2-steps there are three possibilities:

- it is a 'good' 2-step: a diagonal step according the difference in both directions;
- it is a 1-step to the direction of j;
- it is a 'bad' 2-step, i.e., it is a diagonal step in the direction according to j, but opposite to the direction determined by i.

There cannot be other types of 2-steps which decrease the coordinate difference of j, and thus providing shortest paths.

Let m be the number of 'good' 2-steps in a shortest B-path, then there must be n 'bad' 2-steps also in the path to reach exactly q from p in j steps, such that $m-n=i$. Then, $n=m-i$. The condition (the number of diagonal steps in the path) $m+n \leq j_2$ must also be fulfilled. In such a path there are $j_2-m-n=j_2+i-2m$ 2-steps that are actually 1-steps.

The value of m must be at least i (both m,n are nonnegative) and further, at most $\left\lfloor \frac{j_2-i}{2} \right\rfloor$ pairs of 'good' and 'bad' 2-steps could be added as extra diagonal steps. The order of these 2-steps can be arbitrary (among the 2-steps allowed by B). Consequently, by fixing the value m of 'good' 2-steps, the formula $\frac{j_2!}{m!(m-i)!(j_2+i-2m)!}$ gives the number of shortest paths with that number of that type of steps. Now, by summing up the values for the possible values of m, the formula is obtained that we wanted to prove. □

Observe that the fit case can also be seen as a special case of this, rich case, as the formula of Theorem 4 gives 1 in the case $j_2=i$.

Also the case of chessboard distance (investigated already by Das) is a special case of the rich case with $j_2=j$: Thus, using the notation of Theorem 4, the number of

shortest paths from p to q by the chessboard metric is given by

$$\sum_{m=i}^{i+\lfloor \frac{j-i}{2} \rfloor} \frac{j!}{m!(m-i)!(j+i-2m)!},$$

which, as one can easily check, coincides with Das's formula [3]. Das also used the abbreviation with "trinomial coefficients", due to the fact that instead of the sum of two values (as at the usual binomial coefficients), the sum of three earlier values are used. The trinomial coefficients are defined as $\binom{i+j+k}{i,j,k} = \frac{(i+j+k)!}{i!j!k!}$ and the formula for counting the chessboard distances can be written in a somewhat shorter form:

$$\sum_{m=i}^{i+\lfloor \frac{j-i}{2} \rfloor} \binom{j}{m, m-i, j+i-2m}.$$

4.4. *Examples*

We close this section by some examples.
Let $B = (1,1,2,1,2,1,2,2,2,2,2,1,1,1,2,\dots)$ and p be the origin.

Let $q_1 = (7,3)$. Clearly, it is the fit case, since there are three 2's among the first seven elements of B. Thus, there is exactly one shortest B-path, and it is $((0,0),(1,0),(2,0),(3,1),(4,1),(5,2),(6,2),(7,3))$.

Let $q_2 = (11,12)$. We are in the poor case, $d = 15$ and it is the matching subcase (the sum of the first fifteen elements is exactly 23). We have $i = 11$ and $d_2 = 8$. Thus, by Theorem 2, the number of shortest B-paths is $\binom{7}{11-8} = \binom{7}{3} = 35$.

Now, let $q_3 = (6,3)$, and thus, $j = 6, i = 3$. This is the poor case, since there are only two 2's among the first six elements. One may check that $d = 7$, and thus we have the mismatch case, the sum of the first seven elements is 10, larger than $j+i = 6+3 = 9$. Also $d_2 = 3$. Thus, by Theorem 3, $3\binom{7+1-3}{3+1-3} = 5$.

Finally, let $q_4 = (10,4)$. Now, it can be computed according to the rich case (there are six 2's among the first ten elements of B). By our notation $j_2 = 6$ and $i = 4$. Thus, by Theorem 4, we have $\sum_{m=4}^{5} \frac{6!}{m!(m-4)!(10-2m)!} = \frac{6!}{4!0!2!} + \frac{6!}{5!1!0!} = \frac{6\cdot 5}{2} + 6 = 21$.

In Fig. 3 we show some values for the shortest paths with chessboard distance. Colors show the pixels that can be reached by various number of steps, e.g., red for one step, dark blue for two steps, green for three steps, etc. from the shaded pixel.

5. Conclusions

A combinatorial problem was considered (coined originally by Das in 1991 as an open problem for periodic neighborhood sequences) and solved (also for not necessarily periodic neighborhood sequences): Enumerative combinatorics is used to compute the number of shortest paths for digital distance functions defined by neighborhood sequences.

112	266	504	784	1016	1107	1016	784	504	266	112	36	8	1
28	77	161	266	357	393	357	266	161	77	28	7	1	8
6	21	50	90	126	141	126	90	50	21	6	1	7	36
1	5	15	30	45	51	45	30	15	5	1	6	28	112
5	1	4	10	16	19	16	10	4	1	5	21	77	266
15	4	1	3	6	7	6	3	1	4	15	50	161	504
30	10	3	1	2	3	2	1	3	10	30	90	266	784
45	16	6	2	1	1	1	2	6	16	45	126	357	1016
51	19	7	3	1	1	1	3	7	19	51	141	393	1107
45	16	6	2	1	1	1	2	6	16	45	126	357	1016
30	10	3	1	2	3	2	1	3	10	30	90	266	784
15	4	1	3	6	7	6	3	1	4	15	50	161	504

FIGURE 3. A segment of the square grid by the number of shortest chessboard paths from the shaded pixel. Our Algorithm can be applied to fill the grid, layer by layer in a parallel manner (as the colors indicate).

When the number of 2's in the prefix of the neighborhood sequence B is less than the smaller (absolute) coordinate difference, i.e., in the poor case, the number of minimal paths, can be considered as the combination based on the 1-steps: among the 1-steps some of them change the first coordinate, the others change the second coordinate, and the order of these steps can be arbitrary (among the 1-steps). Consequently, the formula can be written by binomial coefficients and closely related to the case of cityblock distances.

With sufficiently large number of 2's in the prefix of B, the formula gives summations on the possible numbers of diagonal-steps and 1-steps made with values 2 of B. These results are more related to the case of chessboard distance. As a special case of this, in the fit case (when the smaller coordinate difference gives exactly the number of 2's in the prefix of B of the length that is exactly the larger coordinate difference) there is exactly 1 shortest B-path.

The results can be seen as generalisations of binomial (and trinomial) coefficients based on the used distance functions.

We should also note that counting the number of shortest path is not only a combinatorial exercise, but also has strong image processing applications (see, e.g., [16]). We mention that in [1] a similar problem was studied for other popular digital distances, namely for the chamfer distances. Furthermore, counting the shortest paths has also been started in other grids, see e.g., [6] for the first results on the triangular grid.

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NEW DIFFERENTIAL SUBORDINATIONS OBTAINED BY USING A DIFFERENTIAL-INTEGRAL RUSCHEWEYH-LIBERA OPERATOR

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Abstract. In this paper, the operator denoted by $D^m : A \to A$, is defined by $D^m[f](z) = (1-\lambda)R^m[f](z) + \lambda L^m[f](z)$, $z \in U$, a differential-integral operator, where R^m is Ruscheweyh differential operator and L^m is Libera integral operator. By using the operator D^m the class of univalent functions denoted by $M(m,\lambda,\alpha)$, $0 \le \lambda \le 1$, $0 \le \alpha < 1$, is defined and several differential subordinations are studied.

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1. Introduction and preliminaries

The concept of differential subordination was introduced in [5, 6] by S.S. Miller and P.T. Mocanu. This theory had a great impact in the field of complex analysis and developed rapidly. Some recent published work can be seen in [7, 9, 10, 10–12].

Studying subordination properties associated with various operators is a topic of interest at this time. Many researchers continue the study of the operators and classes of univalent functions using the methods of the theory of differential subordination. In [17], the authors have generalized and improved some of the previously published results by introducing a general family of analytic and bi-univalent functions in the open unit disk defined by applying the principle of differential subordination between analytic functions and the Tremblay fractional derivative operator. By making use of the linear operator introduced and studied earlier by Srivastava and Attiya, in paper [16] the authors have investigated classes of admissible functions and have presented properties of the third-order differential subordinations establishing various sandwich-type theorems for a class of univalent analytic functions involving the celebrated Srivastava-Attiya transform. In another recent paper [15], the authors introduce and investigate the Fekete-Szegő functional associated with a new subclass of analytic functions defined by using the principle of quasi-subordination between analytic functions, some sufficient conditions for functions belonging to this class

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being also derived. The Fekete-Szegő coefficient functional for functions in a newly introduced class of normalized analytic functions in the open unit disc were obtained in paper [18]. Various integral and convolution characterizations, coefficient estimates and differential subordination results for functions belonging to that class were also investigated in the paper. By making use of the principle of differential subordination and the Dziok – Srivastava convolution operator, in paper [19] the authors introduce and investigate three interesting subclasses of analytic and univalent functions in the open unit disk. Inclusion relationships for those classes and that these classes are closed under convolution with convex functions is also proved.

In this paper a new class of univalent functions and a new operator are introduced. The new operator is used for obtaining subordination results related to this class. Similar work containing subordination results related to a class of univalent functions obtained by the use of an operator introduced by using a differential operator and an integral one has been published recently [12] and inspired the results shown in this paper. Also, following the same idea, in paper [13], the author introduces new classes obtained by using the linear operator defined with the Sălăgean differential operator and the generalized Alexander operator, studying the characteristics and other properties of these classes and obtaining Fekete-Szegő functional for these classes.

We use the following well known notations and results:

Denote by U the unit disc of the complex plane

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U and let

$$A_n = \{f \in \mathcal{H}(U) : \ f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots, \ z \in U\},$$

with $A_1 = A$.

Let $S = \{f \in A : \ f \text{ is univalent in } U\}$ be the class of holomorphic and univalent functions in the open unit disc U with the conditions $f(0) = 0$ and $f'(0) = 1$, that is the holomorphic and univalent functions with the following power series development

$$f(z) = z + a_2z^2 + \ldots, \ z \in U.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we denote by

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(U) : \ f(z) = a + a_nz^n + a_{n+1}z^{n+1} + \ldots, \ z \in U\}.$$

Denote by

$K = \left\{ f \in A : \ \text{Re}\left(\dfrac{zf''(z)}{zf'(z)} + 1\right) > 0, \ z \in U \right\}$ the class of normalized convex functions in U and let

$S^* = \left\{ f \in A : \ \text{Re}\dfrac{zf'(z)}{f(z)} > 0, \ z \in U \right\}$ denote the class of starlike functions in U.

Definition of subordination ([7, p.4])

If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there is a function w, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$ such that $f(z) = g(w(z))$ for $z \in U$. If g is univalent, then $f \prec g$ or $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Definition of second-order differential subordination ([7, p.16])

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the second-order differential subordination

(i) $\psi(p(z), zp'^2 p''(z); z) \prec h(z),\ z \in U$

then p is called a solution of the differential subordination.

The univalent function q is called a dominant of the solutions of the differential subordination, or more simply, a dominant if $p \prec q$ for all p satisfying (i).

A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (i) is said to be the best dominant of (i). (Note that the best dominant is unique up to a rotation of U).

If we require the more restrictive condition $q \in \mathcal{H}[a,n]$ then p is called an (a,n)-solution, q an (a,n)-dominant and $\widetilde{q}$ the best (a,n)-dominant.

To prove our main results, we need the following definitions and lemmas:

Definition 1 (Ruscheweyh operator [14])**.** For $f \in A$, $m \in \mathbb{N}$, the operator R^m is defined by $R^m : A \to A$,

$$\begin{aligned} &R^0[f](z) = f(z) \\ &R^1[f](z) = z \cdot [R^0 f(z)]' = zf'(z) \\ &\vdots \\ &(m+1)R^{m+1}[f](z) = z \cdot [R^m f(z)]' + mR^m f(z),\ z \in U. \end{aligned} \tag{1}$$

Remark 1. If $f \in A$, $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$R^m[f](z) = z + \sum_{k=2}^{\infty} a_k C_{m+k-1}^m z^k = z + \sum_{k=2}^{\infty} \left[\begin{array}{c} m+k-1 \\ m \end{array} \right] a_k z^k. \tag{2}$$

Definition 2 ([1, Definition 1])**.** For $f \in A$, $m \in \mathbb{N}$, $L^m : A \to A$ is defined by

$$\begin{aligned} &L^0[f](z) = f(z) \\ &L^1[f](z) = \frac{2}{z}\int_0^z L^0[f](t)dt = \frac{2}{z}\int_0^z f(t)dt \\ &\vdots \\ &L^m[f](z) = \frac{2}{z}\int_0^z L^{m-1}[f](t)dt,\ z \in U. \end{aligned} \tag{3}$$

Remark 2. a) For $m=1$,

$$L^1[f](z)=\frac{2}{z}\int_0^z f(t)dt,$$

which is Libera integral operator [5].

b) If $f\in A$, $f(z)=z+a_2z_2+a_3z^3+\ldots=z+\sum_{k=2}^{\infty}a_kz^k$, then

$$\begin{aligned}L^m[f](z)&=\frac{2}{z}\int_0^z L^{m-1}[f](t)dt\\&=\frac{2}{z}\int_0^z\left[t+\sum_{k=2}^{\infty}\frac{2^{m-1}}{(k+1)^{m-1}}\cdot a_kt^k\right]dt\\&=\frac{2}{z}\left[\frac{z^2}{2}+\sum_{k=2}^{\infty}\frac{2^{m-1}}{(k+1)^{m-1}}\cdot a_k\cdot\frac{z^{k+1}}{k+1}\right]\\&=z+\sum_{k=2}^{\infty}\frac{2^m}{(k+1)^m}\cdot a_kz^k.\end{aligned}\tag{4}$$

c) For $f\in A$, $m\in\mathbb{N}$, we obtain

$$z(L^m[f](z))'=2L^{m-1}[f](z)-L^m[f](z),\ z\in U.\tag{5}$$

Lemma A. (Hallenbeck and Ruscheweyh [7, Th. 3.1.b, p. 71], [2]) *Let h be convex in U, with $h(0)=a$, $\gamma\neq 0$ and* $\mathrm{Re}\gamma\geq 0$. *If $p\in\mathcal{H}[a,n]$ and*

$$p(z)+\frac{1}{\gamma}zp'(z)\prec h(z),\ z\in U,$$

then

$$p(z)\prec q(z)\prec h(z),\ z\in U,$$

where

$$q(z)=\frac{\frac{\gamma}{n}}{z^{\frac{\gamma}{n}}}\int_0^z h(t)t^{\frac{\gamma}{n}-1}dt,\ z\in U.$$

The function q is convex and is the best (a_n)-dominant.

Lemma B. ([7, Th. 3.4.h, p. 132]) *Let q be univalent in U and let θ and ϕ be analytic in a domain D containing $q(U)$, with $\phi(w)\neq 0$ when $w\in q(U)$. Set*

$$Q(z)=zq'(z)\phi[q(z)],\quad h(z)=\theta[q(z)]+Q(z)$$

and suppose that either

(i) h is convex, or

(ii) Q is starlike.

In addition, assume that

(iii) $\operatorname{Re}\dfrac{zh'(z)}{Q(z)} > \operatorname{Re}\left[\dfrac{\theta'[q(z)]}{\phi[q(z)]} + \dfrac{zQ'(z)}{Q(z)}\right] > 0.$

If p is analytic in U, with $p(0) = q(0)$, $p(U) \subset D$ and

$$\theta[p(z)] + zp'(z)\cdot\phi[p(z)] \prec \theta[q(z)] + zq'(z)\cdot\phi[q(z)] = h(z),$$

then $p(z) \prec q(z)$ and q is the best dominant.

2. Main results

In this paper, we define a differential-integral operator, $D^m : A \to A$, we introduce a class of holomorphic univalent functions and we study several differential subordinations.

Definition 3. Let $m \in \mathbb{N}$, $0 \le \lambda \le 1$, denote by $D^m : A \to A$,

$$D^m[f](z) = (1-\lambda)R^m[f](z) + \lambda L^m[f](z), \tag{6}$$

where R^m is given by (1) and L^m is given by (3).

Remark 3. a) If $f \in A$, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and using (2) and (4), we have

$$\begin{aligned} &D^m[f](z) \\ &= (1-\lambda)\left(z + \sum_{k=2}^{\infty}\left[\begin{array}{c} m+k-1 \\ m \end{array}\right] a_k z^k\right) + \lambda\left(z + \sum_{k=2}^{\infty}\frac{2^m}{(k+1)^m} a_k z^k\right) \\ &= z + \sum_{k=2}^{\infty} a_k z^k \left\{\left[\begin{array}{c} m+k-1 \\ m \end{array}\right](1-\lambda) + \lambda\frac{2^m}{(k+1)^m}\right\}. \end{aligned} \tag{7}$$

b) For $\lambda = 1$, differential-integral operator D^m coincides with Libera integral operator (Definition 2).

c) For $\lambda = 0$, differential-integral operator D^m coincides with differential operator Ruscheweyh (Definition 1).

Definition 4. If $0 \le \alpha < 1$, $0 \le \lambda \le 1$, $m \in \mathbb{N}$, we let $M(m, \lambda, \alpha)$ stand for the class of functions $f \in A$, which satisfy the inequality

$$\operatorname{Re}(D^m[f](z))' > \alpha,\ z \in U, \tag{8}$$

where the differential-integral operator $D^m[f]$ is given by (6).

Remark 4. a) For $m = 0$, $0 \le \lambda \le 1$, $\alpha = 0$, the operator $D^m[f]$ becomes

$$D^0[f](z) = (1-\lambda)R^0[f](z) + \lambda L^0[f](z) = (1-\lambda)f(z) + \lambda f(z) = f(z),\ z \in U,$$

then $M(m, \lambda, \alpha)$ becomes

$$M(0, \lambda, 0) = R = \{f \in A :\ \operatorname{Re} f'(z) > 0,\ z \in U\}$$

called the class of functions with bounded rotation.

This class of functions was studied by J.W. Alexander [1] and he proved that $R \subset S$, J. Kurzyz [3] and P.T. Mocanu [8] have proved that $R \not\subset S^*$. A more systematic study of the class R was done by Mac Gregor [4].

b) For $m = 0$, $0 \leq \alpha < 1$, $0 \leq \lambda \leq 1$, we have

$$M(\,0,\lambda,\alpha) = \{f \in A:\ \mathrm{Re}\, f'(z) > \alpha\} \subset R.$$

Theorem 1. *The set $M(\,m,\lambda,\alpha)$ is convex.*

Proof. Let the functions

$$f_j(z) = z + \sum_{k=2}^{\infty} C_{kj} z^j,\ j = 1,2,\ z \in U,$$

where

$$C_{kj} = a_{kj}\left\{(1-\lambda)\left[\begin{array}{c} m+k-1 \\ m \end{array}\right] + \frac{2^m}{(k+1)^m}\right\},$$

be in the class $M(\,m,\lambda,\alpha)$. It is sufficient to show that the function

$$h(z) = \mu_1 f_1(z) + \mu_2 f_2(z),\ z \in U,$$

with $\mu_1, \mu_2 \geq 0$ and $\mu_1 + \mu_2 = 1$ is in $M(\,m,\lambda,\alpha)$.

Since $h(z) = \mu_1 f_1(z) + \mu_2 f_2(z)$, $z \in U$, then

$$D^m[h](z) \tag{9}$$
$$= z + \sum_{k=2}^{\infty}\left\{(1-\lambda)\left[\begin{array}{c} m+k-1 \\ m \end{array}\right] + \lambda\frac{2^m}{(k+1)^m}\right\}(\mu_1 a_{k1} + \mu_2 a_{k2}) z^k.$$

Differentiating (9), we have

$$(D^m[h](z))' = 1 + \sum_{k=2}^{\infty}\left\{(1-\lambda)\left[\begin{array}{c} m+k-1 \\ m \end{array}\right] + \lambda\frac{2^m}{(k+1)^m}\right\}(\mu_1 a_{k-1} + \mu_2 a_{k2}) k z^{k-1}.$$

Hence

$$\mathrm{Re}\,(D^m[h](z))' \tag{10}$$
$$= 1 + \mu_1 \mathrm{Re} \sum_{k=2}^{\infty}\left\{(1-\lambda)\left[\begin{array}{c} m+k-1 \\ m \end{array}\right] + \lambda\frac{2^m}{(k+1)^m}\right\} a_{k1} k z^{k-1}$$
$$+ \mu_2 \mathrm{Re}\left\{\sum_{k=2}^{\infty}(1-\lambda)\left[\begin{array}{c} m+k-1 \\ m \end{array}\right]\lambda\frac{2^m}{(k+1)^m}\right\} a_{k2} k z^{k-1}.$$

Since $f_1, f_2 \in M(\,m,\lambda,\alpha)$, we have

$$\mu_1 \mathrm{Re} \sum_{k=2}^{\infty}\left\{(1-\lambda)\left[\begin{array}{c} m+k-1 \\ m \end{array}\right] + \lambda\frac{2^m}{(k+1)^m}\right\} a_{k1} k z^{k-1} > \mu_1(\alpha - 1) \tag{11}$$

$$\mu_2 \mathrm{Re} \sum_{k=2}^{\infty}\left\{(1-\lambda)\left[\begin{array}{c} m+k-1 \\ m \end{array}\right] + \lambda\frac{2^m}{(k+1)^m}\right\} a_{k2} k z^{k-1} > \mu_2(\alpha - 1) \tag{12}$$

Using (11) and (12), we have

$$\operatorname{Re}(D^m[h](z))' > 1+\mu_1(\alpha-1)+\mu_2(\alpha-1)$$

and since $\mu_1+\mu_2=1$, we deduce

$$\operatorname{Re}(D^m[h](z))' > \alpha,\ z\in U$$

i.e. $M(\,m,\lambda,\alpha)$ is convex.

Theorem 2. *If* $0\le\alpha<1$, $0\le\lambda\le 1$ *and* $m\in\mathbb{N}$, *then we have*

$$M(\,m,\lambda,\alpha)\subset M(\,m,\lambda,\delta),$$

where $\delta=2\alpha-1+2(1-\alpha)\ln 2$, $\delta\approx 0,62\alpha+0,38<1$.

Proof. We will prove that $\delta\approx 0,62\alpha+0,38<1$.
We know that $\ln 2\approx 0,69$. From $0\le\alpha<1$, it follows

$$0\le 0,62\alpha<0,62,\ 0,38\le 0,62\alpha+0,38<0,62+0,38,$$

$$0,38\le 0,62\alpha+0,38<1,\ 0,38\le\delta<1.$$

Consider the convex function

$$h(z)=\frac{1+(2\alpha-1)z}{1+z},\ z\in U. \tag{13}$$

For $z\in U$, $\operatorname{Re}h(z)>\alpha$ and $h(0)=1$.
We let $f\in M(\,m,\lambda,\alpha)$. From Definition 4, we have

$$\operatorname{Re}(D^m[f](z))' > \alpha,\ z\in U. \tag{14}$$

Let

$$p(z)=\frac{D^m[f](z)}{z},\ z\in U. \tag{15}$$

Using (7) in (15), we have

$$p(z)=1+\sum_{k=2}^{\infty}a_k z^{k-1}\left\{\left[\begin{array}{c} m+k-1\\ m\end{array}\right](1-\lambda)+\lambda\frac{2^m}{(k+1)^m}\right\}$$

and $p(0)=1$, $p\in\mathcal{H}[1,1]$. From (15), we have

$$D^m[f](z)=zp(z),\ z\in U. \tag{16}$$

Differentiating (16), we get

$$(D^m[f](z))'=p(z)+zp'(z),\ z\in U. \tag{17}$$

Using (17), then (14) becomes

$$\operatorname{Re}[p(z)+zp'(z)]>\alpha,\ z\in U. \tag{18}$$

Relation (18) can be written as a subordination of the form

$$p(z)+zp'(z)\prec h(z)=\frac{1+(2\alpha-1)z}{1+z},\ z\in U. \tag{19}$$

Using Lemma A, for $\gamma = 1$, $n = 1$, it follows

$$p(z) \prec q(z),$$

where

$$q(z) = \frac{1}{z}\int_0^z \frac{1+(2\alpha-1)t}{1+t}dt.$$

i.e.

$$\begin{aligned}\frac{L^m[f](z)}{z} \prec q(z) &= \frac{1}{z}\int_0^z \frac{1+(2\alpha-1)t}{1+t}dt \\ &= \frac{1}{z}\int_0^z \left[2\alpha+1+\frac{2(1-\alpha)}{1+t}\right]dt \\ &= 2\alpha-1+2(1-\alpha)\frac{\ln(1+z)}{z},\ z\in U.\end{aligned}$$

The function q is convex and is the best dominant.
Since q is a convex function and $p(z) \prec q(z)$, $z \in U$, we have

$$\operatorname{Re} p(z) > \operatorname{Re} q(1) = \delta = 2\alpha-1+2(1-\alpha)\ln 2. \tag{20}$$

Using (15), then (20) becomes

$$\operatorname{Re}\frac{D^m[f](z)}{z} > \delta = 2\alpha-1+2(1-\alpha)\ln 2. \tag{21}$$

Using (7), we have

$$\begin{aligned}&\operatorname{Re}\frac{D^m[f](z)}{z} \qquad (22)\\ &= \operatorname{Re}\frac{z+\sum_{k=2}^{\infty} a_k z^k\left\{\left[\begin{array}{c} m+k-1 \\ m \end{array}\right](1-\lambda)+\lambda\frac{2^m}{(k+1)^m}\right\}}{z} \\ &= \operatorname{Re}\left(1+\sum_{k=2}^{\infty} a_k z^{k-1}\left\{\left[\begin{array}{c} m+k-1 \\ m \end{array}\right](1-\lambda)+\lambda\frac{2^m}{(k+1)^m}\right\}\right) \\ &= \operatorname{Re}\left(z+\sum_{k=2}^{\infty} \frac{a_k}{k} z^k\left\{\left[\begin{array}{c} m+k-1 \\ m \end{array}\right](1-\lambda)+\lambda\frac{2^m}{(k+1)^m}\right\}\right)' \\ &= \operatorname{Re}\left(z+\sum_{k=2}^{\infty} b_k z^k\left\{\left[\begin{array}{c} m+k-1 \\ m \end{array}\right](1-\lambda)+\lambda\frac{2^m}{(k+1)^m}\right\}\right)' \\ &= \operatorname{Re}\left[D^m[g](z)\right]',\end{aligned}$$

where

$$g(z) = z+\sum_{k=2}^{\infty} b_k z^k.$$

Using (22) in (21) we get

$$\operatorname{Re}\frac{D^m[f](z)}{z} = \operatorname{Re}\left(D^m[g](z)\right)' > \delta.$$

Using Definition 4, we have $f \in M(\ m,\lambda,\delta)$, with δ given by (20). Since $f \in M(\ m,\lambda,\alpha)$ was taken arbitrary and we have proved that $f \in M(\ m,\lambda,\delta)$, we conclude that $M(\ m,\lambda,\alpha) \subset M(\ m,\lambda,\delta)$.

From Theorem 2 we deduce the following corollary:

Corollary 1. *If $f \in M(\ m,\lambda,\delta)$, then*

$$\operatorname{Re}\frac{D^m[f](z)}{z} > 2\alpha - 1 + 2(1-\alpha)\ln 2 = \delta.$$

Proof. From the proof of Theorem 2, we can see that

$$\begin{aligned}\frac{L^m[f](z)}{z} \prec q(z) &= \frac{1}{z}\int_0^z \frac{1+(2\alpha-1)t}{1+t}dt \\ &= 2\alpha - 1 + 2(1-\alpha)\frac{\ln(1+z)}{z},\ z \in U.\end{aligned}$$

Since q is convex function, we have that

$$\operatorname{Re}\frac{D^m[f](z)}{z} > \operatorname{Re} q(1) = 2\alpha - 1 + 2(1-\alpha)\ln 2 = \delta.$$

Theorem 3. *Let h be a convex function in U with $h(0) = 1$.*
If $f \in A$, $0 \le \lambda \le 1$, $m \in \mathbb{N}$ and satisfies the differential subordination

$$((1-\lambda)R^m[f](z) + \lambda L^m[f](z))' \prec h(z),\ z \in U, \tag{23}$$

then

$$\frac{(1-\lambda)R^m[f](z) + \lambda L^m[f](z)}{z} \prec q(z) = \frac{1}{z}\int_0^z h(t)dt$$

and this result is sharp.

Proof. We let

$$p(z) = \frac{(1-\lambda)R^m[f](z) + \lambda L^m[f](z)}{z},\ z \in U. \tag{24}$$

Using (7) in (24), we obtain

$$\begin{aligned}p(z) &= \frac{z + \sum_{k=2}^{\infty} a_k z^k \left\{ \begin{bmatrix} m+k-1 \\ m \end{bmatrix}(1-\lambda) + \lambda\frac{2^m}{(k+1)^m} \right\}}{z} \\ &= 1 + \sum_{k=2}^{\infty} a_k z^{k-1} \left\{ \begin{bmatrix} m+k-1 \\ m \end{bmatrix}(1-\lambda) + \lambda\frac{2^m}{(k+1)^m} \right\} \\ &= 1 + p_1 z + p_2 z^2 + \ldots,\ p \in \mathcal{H}[1,1].\end{aligned}$$

From (24) we have

$$(1-\lambda)R^m[f](z)+\lambda L^m[f](z)=zp(z),\ z\in U. \tag{25}$$

Differentiating (25), we obtain

$$((1-\lambda)R^m[f](z)+\lambda L^m[f](z))'=p(z)+zp'(z),\ z\in U,$$

then the differential subordination (23) becomes

$$p(z)+zp'(z)\prec h(z)=q(z)+zq'(z),\ z\in U. \tag{26}$$

By using Lemma A, for $n=1$, $\gamma=1$, we have

$$p(z)\prec q(z),$$

i.e.

$$\frac{(1-\lambda)R^m[f](z)+\lambda L^m[f](z)}{z}\prec q(z),\ z\in U,$$

where

$$q(z)=\frac{1}{z}\int_0^z h(t)dt,$$

and the function q is convex and is the best dominant.

Theorem 4. *Let q be convex function in U, with $q(0)=1$, and let θ and φ be analytic in a domain D containing $q(U)$. Set*

$$Q(z)=zq'(z)\varphi[q(z)] \text{ and } h(z)=\theta[q(z)]+Q(z).$$

Let $m\in\mathbb{N}$, $f\in A$, $0\le\lambda\le 1$ and satisfies the differential subordination

$$\frac{D^m[f](z)}{D^{m-1}[f](z)}+z\left(\frac{D^m[f](z)}{D^{m-1}[f](z)}\right)'\prec q(z)+zq'(z) \tag{27}$$

then

$$\frac{D^m[f](z)}{D^{m-1}[f](z)}\prec q(z),\ z\in U$$

and q is the best dominant. The operator $D^m[f]$ is defined in (6).

Proof. We let

$$p(z)=\frac{D^m[f](z)}{D^{m-1}[f](z)},\ z\in U. \tag{28}$$

Using (7) in (28) we obtain

$$\begin{aligned} p(z)&=\frac{z+\sum_{k=2}^{\infty}a_kz^k\left\{\left[\begin{array}{c} m+k-1\\ m\end{array}\right](1-\lambda)+\lambda\frac{2^m}{(k+1)^m}\right\}}{z+\sum_{k=2}^{\infty}a_kz^k\left\{\left[\begin{array}{c} m+k-1\\ m\end{array}\right](1-\lambda)+\lambda\frac{2^{m-1}}{(k+1)^{m-1}}\right\}}\\ &=1+p_1z+p_2z^2+\ldots,\ p\in\mathcal{H}[1,1]. \end{aligned}$$

Differentiating (28) and after a short calculation, we obtain

$$zp'(z) = z\left(\frac{D^m[f](z)}{D^{m-1}[f](z)}\right)', \ z \in U,$$

and

$$p(z) + zp'(z) = \frac{D^m[f](z)}{D^{m-1}[f](z)} + z\left(\frac{D^m[f](z)}{D^{m-1}[f](z)}\right)', \ z \in U. \tag{29}$$

Using (29), the subordination (27) becomes

$$p(z) + zp'(z) \prec q(z) + zq'(z), \ z \in U.$$

In order to prove the theorem, we shall use Lemma B.
For this, we show that the necessary conditions are satisfied.
Let the functions $\theta : \mathbb{C} \to \mathbb{C}$, and $\varphi : \mathbb{C} \to \mathbb{C}$, with

$$\theta(w) = w, \ \varphi(w) = 1. \tag{30}$$

We check the conditions from the hypothesis of Lemma B. Using (30), in the hypothesis of the theorem, we have

$$Q(z) = zq'(z)\varphi(q(z)) = zq'(z) \tag{31}$$

and

$$h(z) = \theta[q(z)] + Q(z) = q(z) + Q(z) = q(z) + zq'(z), \ z \in U. \tag{32}$$

Differentiating (31) and after short calculation, we obtain

$$\frac{zQ'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)}.$$

Since q is convex function in U, we have

$$\operatorname{Re}\frac{zQ'(z)}{Q(z)} = \operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > 0, \ z \in U,$$

hence the function Q is starlike.
Differentiating (32) and after short calculation, we obtain

$$\frac{zh'(z)}{Q(z)} = \frac{zq'(z) + zQ'(z)}{Q(z)} = \frac{zq'(z)}{zq'(z)} + \frac{zQ'(z)}{Q(z)} = 1 + \frac{zQ'(z)}{Q(z)}.$$

Since Q is starlikeness we have

$$\operatorname{Re}\frac{zh'(z)}{Q(z)} > 0, \ z \in U. \tag{33}$$

Using (30), we get $\theta[p(z)] = p(z)$, $\varphi(p(z)) = 1$, then the subordination (27) becomes

$$\theta[p(z)] + zp'(z)\varphi[p(z)] \prec \theta[q(z)] + zq'(z)\varphi[q(z)].$$

Using Lemma C, we have

$$p(z) \prec q(z),$$

i.e.

$$\frac{D^m[f](z)}{D^{m-1}[f](z)} \prec q(z),\ z \in U.$$

Theorem 5. *Let q be a convex function $q(0) = 1$, and let θ and φ be analytic in a domain D containing $q(U)$. Set*

$$Q(z) = zq'(z)\varphi[q(z)] \text{ and } h(z) = \theta[q(z)] + Q(z).$$

If $f \in A$, $\lambda \geq 0$, $m \in \mathbb{N}$, and satisfies the differential subordination

$$(D^m[f](z))' + z(D^m[f](z))'' \prec h(z),\ z \in U, \tag{34}$$

then

$$(D^m[f](z))' \prec q(z),\ z \in U,$$

and q is the best dominant.

Proof. We let

$$p(z) = (D^m[f](z))',\ z \in U. \tag{35}$$

Using (6) in (35), we get

$$\begin{aligned} p(z) &= \left(z + \sum_{k=2}^{\infty} a_k z^k \left\{ \left[\begin{array}{c} m+k-1 \\ m \end{array} \right] (1-\lambda) + \lambda \frac{2^m}{(k+1)^m} \right\}\right)' \\ &= 1 + \sum_{k=2}^{\infty} k a_k z^{k-1} \left\{ \left[\begin{array}{c} m+k-1 \\ m \end{array} \right] (1-\lambda) + \lambda \frac{2^m}{(k+1)^m} \right\}, \\ & p(0) = 1 \text{ and } p \in \mathcal{H}[1,1]. \end{aligned}$$

Multiplying (35) by z, we obtain

$$zp(z) = z(D^m[f](z))'. \tag{36}$$

Differentiating (36), we get

$$p(z) + zp'^m[f](z))'^m[f](z))'',\ z \in U. \tag{37}$$

Using (37), the differential subordination (34) becomes

$$p(z) + zp'(z) \prec h(z),\ z \in U. \tag{38}$$

In order to prove the theorem, we shall use Lemma B.
For this, we show that the necessary conditions are satisfied.
Let the function $\theta : \mathbb{C} \to \mathbb{C}$, and $\varphi : \mathbb{C} \to \mathbb{C}$, with

$$\theta(w) = w \text{ and } \varphi(w) = 1. \tag{39}$$

We check the conditions from the hypothesis of Lemma B. Using (39) in the hypothesis of the theorem, we have

$$Q(z) = zq'(z)\varphi[q(z)] = zq'(z) \tag{40}$$

and

$$h(z) = \theta[q(z)] + Q(z) = q(z) + Q(z) = q(z) + zq'(z),\ z \in U. \tag{41}$$

Differentiating (40), we have

$$\frac{zQ'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)},\ z \in U.$$

Since q is convex function,

$$\operatorname{Re}\frac{zQ'(z)}{Q(z)} > 0,\ z \in U, \tag{42}$$

hence the function Q is starlike.
Differentiating (41) and after short calculation, we obtain

$$\frac{zh'(z)}{Q(z)} = \frac{zq'(z) + zQ'(z)}{Q(z)} = \frac{zq'(z)}{zq'(z)} + \frac{zQ'(z)}{Q(z)} = 1 + \frac{zQ'(z)}{Q(z)}.$$

Since Q is starlike, we have

$$\operatorname{Re}\frac{zh'(z)}{Q(z)} = \operatorname{Re}\left(1 + \frac{zQ'(z)}{Q(z)}\right) > 0,\ z \in U. \tag{43}$$

Using (39), we get

$$\theta[p(z)] = p(z),\ \varphi[p(z)] = 1,$$

then the subordination (34) becomes

$$\theta[p(z)] + zp'(z)\varphi[p(z)] \prec \theta[q(z)] + zq'(z)\varphi[q(z)].$$

Using Lemma B, we have

$$p(z) \prec q(z),$$

i.e.

$$(D^m[f](z))' \prec q(z),\ z \in U.$$

Example 1. Let the convex function

$$h(z) = \frac{1-z}{1+z},\ h(0) = 1,$$

and

$$f(z) = z + \frac{1}{2}z + \frac{1}{3}z^3,\ \lambda = \frac{1}{4},\ m = 1,$$

then

$$\frac{\left(1 - \frac{1}{4}\right)R^1[f](z) + \frac{1}{4}L^1[f](z)}{z} = 1 + \frac{5}{6}z^2 + \frac{19}{24}z^3.$$

Using Theorem 3 we get:
If

$$\frac{5}{3}z + \frac{19}{8}z^2 \prec \frac{1-z}{1+z},$$

then

$$1+\frac{5}{6}z^2+\frac{19}{24}z^3 \prec q(z)=\frac{1}{z}\int_0^z \frac{1-t}{1+t}dt=\frac{1}{z}\int_0^z \left(-1+\frac{2}{1+t}\right)dt$$

$$=-1+\frac{2\ln(1+z)}{z}$$

and this result is sharp.

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INTERVAL ESTIMATION OF KUMARASWAMY PARAMETERS BASED ON PROGRESSIVELY TYPE II CENSORED SAMPLE AND RECORD VALUES

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Abstract. In this paper, interval estimation for the parameters of the Kumaraswamy distribution is studied based on progressively censoring scheme and the record values. Some pivotal quantities and Theorems are proposed to construct the exact confidence interval for the one-shape parameter and the joint confidence region for the two-shape parameters. Simulation study is performed to investigate the coverage probabilities of the proposed confidence under progressively censored sample and upper record data. Finally, the proposed intervals and regions have been studied based on two real data sets as the illustrative examples. The first data set contains the strength characterization of brittle material and the second data set consists of the daily average wind speeds.

2010 *Mathematics Subject Classification:* 62N02; 62P30

Keywords: brittle material, generalized spacing, joint confidence region, Kumaraswamy distribution, progressively censoring, upper record values, wind speed

1. Introduction

The Kumaraswamy (KU) distribution was introduced in the literature by Kumaraswamy ([11]) in connection with a problem in hydrology. This distribution has many similarities to the Beta distribution but it has a number of advantages. The KU distribution can be better suited than the Beta distribution for computation-intensive activities like simulation modeling and the estimation of models by simulation-based methods [15]. It has an invertible closed form cumulative distribution function which makes it more useful than the Beta distribution. Due to its practicality, it can be used for many applications, including atmospheric temperatures, wind speeds, scores obtained on a test, daily rainfall, daily stream flow, hydrological data and etc. The probability density function and the cumulative distribution function of KU distribution are given by:

$$f(x;\alpha,\beta) = \alpha\beta x^{\beta-1}(1-x^{\beta})^{\alpha-1}; \quad 0 < x < 1\,,\ \alpha > 0,\ \beta > 0, \tag{1.1}$$

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and

$$F(x;\alpha,\beta) = 1-(1-x^{\beta})^{\alpha}; \quad 0<x<1 \tag{1.2}$$

respectively. The corresponding quantile function is:

$$Q(p;\alpha,\beta) = \left(1-(1-p^{1/\alpha})\right)^{1/\beta}; \quad 0<p<1$$

The KU distribution is unimodal, uniantimodal, increasing, decreasing or constant depending (in the same way as the Beta distribution) on the values of its parameters [10]. If the parameters are greater than one, the density function is unimodal. It is uniantimodal if two parameters are less than one. The density function increases (decreases) for $\beta > 1$ and $\alpha \leq 1$ ($\beta \leq 1$ and $\alpha > 1$) and finally, it is constant for $\alpha = \beta = 1$. In Figure 1, we have displayed probability density function of the KU distribution for some arbitrarily selected parameter values.

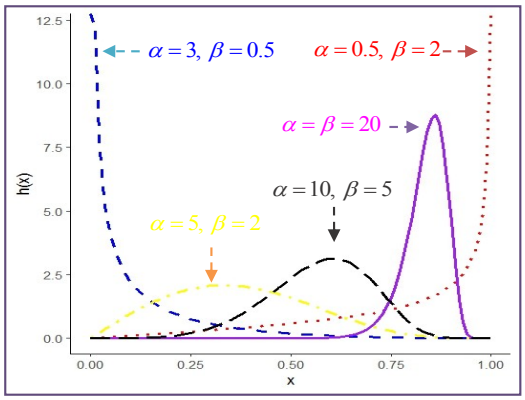

FIGURE 1. The probability density function for different values of α and β.

Several author studied different aspects of the KU distribution. For example the distribution of order statistics from the KU model has been introduced and studied by [8]. The different estimation methods for the unknown parameters of the KU distribution have been considered by [6]. The point estimations for this distribution under complete and record samples have been studied by [13, 16]. Furthermore, for more details about the other extensions of this distribution, see [7,22,25]. Although several articles have been done on the KU distribution but we have not come across any article about the exact confidence interval (ECI) and the joint confidence region (JCR) under progressively censored sample and record data. The main aim of this paper is two fold. First we try to introduce the ECI and JCR for parameter(s) on the basis of progressively Type II (PII) censoring using the appropriate pivotal quantities. Censoring is common in life tests because of time limits and other restrictions on data. Two of the commonly used censoring schemes are Type I and Type II censoring schemes.

A mixture of Type I and Type II censoring schemes is known as the hybrid censoring scheme (see, [4, 12, 19] and [18]). These censoring schemes do not allow for removal of units at points other than the terminal point of the experiment. To overcome this problem, progressively censoring scheme which enables us to use live units, removed early, in other tests has been introduced. The PII censoring scheme can be implemented as follows: Suppose n units are placed on a life test and the experimenter decides beforehand the quantity m, the number of units to be failed. Now at the time of the first failure, R_1 of the remaining $n-1$ surviving units are randomly removed from the experiment. Continuing on, at the time of the second failure, R_2 of the remaining $n-2-R_1$ units are randomly with-drawn from the experiment. Finally, at the time of the mth failure, all the remaining $R_m = n-m-R_1-R_2-\ldots-R_{m-1}$ surviving units are removed from the experiment. The $R_i's$ are fixed prior to the study. Some of the work on PII censoring was conducted by [14, 20] and [1].

Moreover, the study of record values and associated statistics are important in different multiple real life situations such as manufacturing industry, hydrology, seismology, sporting and stock market analysis. Since then, numerous papers on record values and their distributional properties have been focus of investigation for many authors, see for example,[5, 9, 23] and [21]. A record values can be explained as follows. Suppose that $\{X_i;\ i \geq 1\}$ denote a sequence of independent and identically distributed random variables with probability density function $f(x,\theta)$. Then, an observation X_i is called an upper record if its value is greater than all preceding observations, that is, if $X_i > X_j$ for every $i > j$. The lower record value can also be defined in a similar manner. Also, the sequence $\left\{X_{U(n)};\ n \geq 1\right\}$ is a sequence of upper record statistics, where

$$U(n) = \min\left\{i : i > U(n-1),\ X_i > X_{U(n-1)}\right\};\ n \geq 2,\ U(1) = 1.$$

So, the second aim of this paper is to construct the ECI and JCR based on upper record values using the two Theorems. The rest of the paper is organized as follows. In Section 2, the ECI for the parameter β and JCR for (α,β) are obtained by using some pivotal quantities. Based on record values, the exact and joint confidence are presented in Section 3. Section 4 deals with Monte Carlo simulation study. For illustration, two real data sets are discussed in Section 5 and finally we conclude the paper in Section 6.

2. The ECI and JCR based on Progressively Censored Sample

This section deals with constructing the ECI and JCR under PII censoring scheme. Let $X_{1:m:n} < X_{2:m:n} < \ldots < X_{m:m:n}$ denote the PII censored order statistics from KU distribution with censoring scheme $R = (R_1,\ldots,R_m)$. Let, $Y_i = -\log\left(1-X_{i:m:n}^{\beta}\right)^{\alpha}$; $i = 1,2,\ldots,m$, it can be observed that $Y_{1:m:n} < \ldots < Y_{m:m:n}$ are the PII censored order statistics from a standard exponential distribution. Now, we consider the following

transformation

$$\Delta_1 = nY_{1:m:n},$$

$$\Delta_2 = (n - R_1 - 1)(Y_{2:m:n} - Y_{1:m:n}),$$

$$\Delta_3 = (n - R_1 - R_2 - 2)(Y_{3:m:n} - Y_{2:m:n}),$$

$$\vdots$$

$$\Delta_m = (n - R_1 - \ldots - R_{m-1} - m + 1)(Y_{m:m:n} - Y_{m-1:m:n}).$$

It is observed that $\Delta_1, \Delta_2, \ldots, \Delta_m$ are independent and identically distributed as an exponential distribution with mean 1. We introduce the following quantities:

$$\upsilon_1 = 2\Delta_1 = 2nY_{1:m:n} = -2n\log\left(1 - X_{1:m:n}^{\beta}\right)^{\alpha} \sim \chi^2(2),$$

and

$$\upsilon_2 = 2(\Delta_2 + \Delta_3 + \ldots + \Delta_m) = 2\sum_{i=1}^{m}(R_i + 1)(Y_{i:m:n} - Y_{1:m:n}) \sim \chi^2(2m-2).$$

We can find that υ_1 and υ_2 are independent. Now, suppose that

$$W_1 = \upsilon_1 + \upsilon_2 = 2\sum_{i=1}^{m}(R_i + 1)Y_i = -2\sum_{i=1}^{m}(R_i + 1)\log\left(1 - X_{i:m:n}^{\beta}\right)^{\alpha} \sim \chi^2(2m), \quad (2.1)$$

and

$$\begin{aligned} W_2 &= \frac{2\upsilon_2}{2(m-1)\upsilon_1} \\ &= \frac{\sum_{i=1}^{m}(R_i + 1)(\log\left(1 - X_{i:m:n}^{\beta}\right)^{\alpha} - \log\left(1 - X_{1:m:n}^{\beta}\right)^{\alpha})}{n(m-1)\log\left(1 - X_{1:m:n}^{\beta}\right)^{\alpha}} \sim F(2m-2, 2). \end{aligned} \quad (2.2)$$

It is also clear that W_1 and W_2 are independent. The following Lemma and Theorems enable us to construct the ECI and JCR for the PII censored sample.

Lemma 1. *For a PII censored sample $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$, the*

$$W_2(\beta; data) = \frac{\sum_{i=1}^{m}(R_i + 1)\left(\frac{\log(1 - X_{i:m:n}^{\beta})}{\log(1 - X_{1:m:n}^{\beta})} - 1\right)}{n(m-1)}$$

is a strictly increasing function of β, where $\beta > 0$. Furthermore, there is a unique solution for the given equation $W_2(\beta; data) = t$.

Proof. The proof follows using similar arguments of [24] and therefore it is omitted. □

Theorem 1. *Suppose that $X_{i:m:n}$; $i=1,\dots,m$ are PII censored order statistics from KU distribution with censoring scheme $R=(R_1,\dots,R_m)$. Then, based on the pivotal quantities W_1 and W_2, a $100(1-\gamma)\%$ ECI for β can be written as:*

$$\varphi\left(X_{1:m:n},X_{2:m:n},\dots,X_{m:m:n},R_1,\dots,R_m,F_{1-\gamma/2}(2m-2,2)\right)<\beta$$
$$<\varphi\left(X_{1:m:n},X_{2:m:n},\dots,X_{m:m:n},R_1,\dots,R_m,F_{\gamma/2}(2m-2,2)\right)$$

where, $\varphi(X_{1:m:n},X_{2:m:n},\dots,X_{m:m:n},R_1,\dots,R_m,t)$ is the solution of β for the following equation

$$\frac{\sum_{i=1}^{m}(R_i+1)\left(\frac{\log(1-X_{i:m:n}^{\beta})}{\log(1-X_{1:m:n}^{\beta})}-1\right)}{n(m-1)}=t.$$

Proof of Theorem 1. From (2.2), we know that the quantity

$$W_2(\beta;data)=\frac{\sum_{i=1}^{m}(R_i+1)\left(\frac{\log(1-X_{i:m:n}^{\beta})}{\log(1-X_{1:m:n}^{\beta})}-1\right)}{n(m-1)},$$

has a F distribution with $2m-2$ and degrees of freedom. So, using the Lemma, we can conclude that $W_2(\beta;data)=t$, has a unique solution for any $\beta>0$ ([17]). Hence the event

$$t_1<\frac{\sum_{i=1}^{m}(R_i+1)\left(\frac{\log(1-X_{i:m:n}^{\beta})}{\log(1-X_{1:m:n}^{\beta})}-1\right)}{n(m-1)}<t_2, \tag{2.3}$$

is equivalent to

$$\varphi\left(X_{1:m:n},X_{2:m:n},\dots,X_{m:m:n},R_1,\dots,R_m,F_{1-\gamma/2}(2m-2,2)\right)<\beta$$
$$<\varphi\left(X_{1:m:n},X_{2:m:n},\dots,X_{m:m:n},R_1,\dots,R_m,F_{\gamma/2}(2m-2,2)\right)$$

□

Now we want to obtain the JCR for (α,β) using the following Theorem.

Theorem 2. *Suppose that $X_{i:m:n}$; $i=1,\dots,m$ are PII censored order statistics from KU distribution with censoring scheme $R=(R_1,\dots,R_m)$.Then, the following inequalities determine a $100(1-\gamma)\%$ JCR for (α,β).*

$$\beta_L<\beta<\beta_U$$

$$-\frac{\chi^2_{(1+\sqrt{1-\gamma})/2}(2m)}{2\sum_{i=1}^{m}(R_i+1)\log(1-X_{i:m:n}^{\beta})}<\alpha<-\frac{\chi^2_{(1-\sqrt{1-\gamma})/2}(2m)}{2\sum_{i=1}^{m}(R_i+1)\log(1-X_{i:m:n}^{\beta})}, \tag{2.4}$$

where,

$$\beta_L=\varphi(X_{1:m:n},X_{2:m:n},\dots,X_{m:m:n},R_1,\dots,R_m,F_{(1+\sqrt{1-\gamma})/2}(2m-2,2)),$$

and

$$\beta_U = \varphi(X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}, R_1, \ldots, R_m, F_{(1-\sqrt{1-\gamma})/2}(2m-2,2)).$$

Also, $\varphi(X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}, R_1, \ldots, R_m, t)$ *is introduced in Theorem 1.*

Proof of Theorem 2. The $W_1 = -2\sum_{i=1}^{m}(R_i+1)\log\left(1-X_{i:m:n}^{\beta}\right)^{\alpha}$ has a χ^2 distribution with $2m$ degrees of freedom, and it is independent of $W_2 \sim F(2m-2,2)$. Therefore,

$$P\left(\chi^2_{(1+\sqrt{1-\gamma})/2}(2m) < W_1 < \chi^2_{(1-\sqrt{1-\gamma})/2}(2m)\right) = \sqrt{1-\gamma},$$

and

$$P\left(F_{(1+\sqrt{1-\gamma})/2}(2m-2,2) < W_2 < F_{(1-\sqrt{1-\gamma})/2}(2m-2,2)\right) = \sqrt{1-\gamma}.$$

So, we can write

$$\begin{aligned}
1-\gamma &= \sqrt{1-\gamma}\times\sqrt{1-\gamma} \\
&= P\left(F_{(1+\sqrt{1-\gamma})/2}(2m-2,2) < W_2 < F_{(1-\sqrt{1-\gamma})/2}(2m-2,2)\right) \\
&\quad \times P\left(\chi^2_{(1+\sqrt{1-\gamma})/2}(2m) < W_1 < \chi^2_{(1-\sqrt{1-\gamma})/2}(2m)\right) \\
&= P\Big(F_{(1+\sqrt{1-\gamma})/2}(2m-2,2) < W_2 < F_{(1-\sqrt{1-\gamma})/2}(2m-2,2), \\
&\quad \chi^2_{(1+\sqrt{1-\gamma})/2}(2m) < W_1 < \chi^2_{(1-\sqrt{1-\gamma})/2}(2m)\Big) \\
&= P\left(\beta_L < \beta < \beta_U,\ -\frac{\chi^2_{(1+\sqrt{1-\gamma})/2}(2m)}{\kappa_i} < \alpha < -\frac{\chi^2_{(1-\sqrt{1-\gamma})/2}(2m)}{\kappa_i}\right),
\end{aligned}$$

where, $\kappa_i = 2\sum_{i=1}^{m}(R_i+1)\log(1-X_{i:m:n}^{\beta})$. This completes the proof. □

3. The ECI and JCR Under the Upper Record Values

In this section based on the upper record values, the exact confidence interval for β and the JCR for (α,β) are introduced. Suppose that $X_{U(1)} < X_{U(2)} < \ldots < X_{U(m)}$ is the first m upper record values from the KU distribution. Similarly to the previous section, let

$$Y_i = -\log\left(1-X_{U(i)}^{\beta}\right)^{\alpha};\ i = 1,\ldots,m.$$

It can be seen that $Y_1 < \ldots < Y_m$ are the first m upper record values from an exponential distribution with mean 1. We know that the following generalized spacings are independent and identically distributed as a standard exponential distribution [2].

$$Y_1^* = Y_1 = -\log\left(1-X_{U(1)}^{\beta}\right)^{\alpha}$$

$$Y_2^* = Y_2 - Y_1 = \log\left(1-X_{U(2)}^{\beta}\right)^{\alpha} - \log\left(1-X_{U(1)}^{\beta}\right)^{\alpha}$$

$$\cdots$$
$$Y_m^* = Y_m - Y_{m-1} = \log\left(1 - X_{U(m)}^{\beta}\right)^{\alpha} - \log\left(1 - X_{U(m-1)}^{\beta}\right)^{\alpha}$$

Hence, $W_1 = 2Y_1^* = 2Y_1$ and $W_2 = 2\sum_{i=2}^{m} Y_i^* = 2(Y_m - Y_1)$, have the chi-squared distribution with $2m$ and $2m-2$ degrees of freedom respectively. In order to construct the ECI of β and JCR for (α,β), we define

$$\varpi = \frac{W_2}{(m-1)W_1} = \frac{Y_m - Y_1}{(m-1)Y_1},$$

and

$$\xi = W_1 + W_2 = 2Y_m.$$

It is clear that $\varpi \sim \frac{\chi^2(2m-2)}{\chi^2(2)} \sim F(2m-2,2)$ and $\xi \sim \chi^2(2m)$ and also they are independent.

Theorem 3. *Let $X_{U(i)}$; $i = 1,2,\ldots,m$ are first i^{th} upper record values from a sequence of iid random variables following the KU distribution with probability density function in (1.1). We consider $\phi(X_{U(1)}, X_{U(2)}, \ldots, X_{U(m)}, t)$ as a solution of β for the equation*

$$\frac{\log(1 - X_{U(m)}^{\beta}) - \log(1 - X_{U(1)}^{\beta})}{(m-1)\log(1 - X_{U(1)}^{\beta})} = t, \tag{3.1}$$

Therefore, the $100(1-\gamma)\%$ ECI for β is:

$$\left[\phi\left(X_{U(1)}, X_{U(2)}, \ldots, X_{U(m)}, F_{1-\gamma/2}(2m-2,2)\right), \phi\left(X_{U(1)}, X_{U(2)}, \ldots, X_{U(m)}, F_{\gamma/2}(2m-2,2)\right)\right]$$

Proof of Theorem 3. We know that the

$$\varpi = \frac{1}{(m-1)}\left(\frac{\log(1 - X_{U(m)}^{\beta})}{\log(1 - X_{U(1)}^{\beta})} - 1\right),$$

has a F distribution with $2m-2$ and $2m$ degrees of freedom. Also, from Lemma 1, $\varpi = t$ has a unique solution. So, analogous to Theorem 1, the proof follows. □

Theorem 4. *Suppose we observe first m upper record values, $X_{U(1)}, X_{U(2)}, \ldots, X_{U(m)}$ from the KU distribution, a $100(1-\gamma)\%$ JCR for (α,β) can be obtained by considering the following inequalities:*

$$\phi(X_{U(1)}, X_{U(2)}, \ldots, X_{U(m)}, F_{(1+\sqrt{1-\gamma})/2}(2m-2,2)) < \beta < \phi(X_{U(1)}, X_{U(2)}, \ldots, X_{U(m)}, F_{(1-\sqrt{1-\gamma})/2}(2m-2,2))$$
$$-\frac{\chi^2_{(1+\sqrt{1-\gamma})/2}(2m)}{2\log(1 - X_{U(1)}^{\beta})} < \alpha < -\frac{\chi^2_{(1-\sqrt{1-\gamma})/2}(2m)}{2\log(1 - X_{U(1)}^{\beta})}. \tag{3.2}$$

Proof of Theorem 4. Using the ϖ and ξ and further similar to the Theorem 2, we can conclude that

$$P\Bigg[\phi(X_{U(1)},X_{U(2)},\ldots,X_{U(m)},F_{(1+\sqrt{1-\gamma})/2}(2m-2,2))<\beta$$

$$<\phi(X_{U(1)},X_{U(2)},\ldots,X_{U(m)},F_{(1-\sqrt{1-\gamma})/2}(2m-2,2)),$$

$$-\frac{\chi^2_{(1+\sqrt{1-\gamma})/2}(2m)}{2\log(1-X^{\beta}_{U(m)})}<\alpha<-\frac{\chi^2_{(1-\sqrt{1-\gamma})/2}(2m)}{2\log(1-X^{\beta}_{U(m)})}\Bigg]=1-\gamma.$$

□

4. Simulations

The goal of this simulation study is to examine the performance of the all proposed intervals or regions in terms of coverage probability with nominal level 0.95. The coverage probability (CP) is the probability that the interval or region contains the true parameters. All the computations were done using the *R* programming language. We reported results from two simulation studies that we carried out. First, we generated a progressively censored sample from the KU distribution with arbitrarily true values $\alpha=2,\beta=1$ and different combination of n and m. In our study we have used three different censoring schemes (C.S.), namely:
Scheme I: $R_1=n-m,R_i=0;\ i\neq 1$.
Scheme II: $R_m=n-m,R_i=0;\ i\neq m$.
Scheme III: $R_1=R_4=(n-m)/2,R_i=0;\ i\neq 1,4$.

We computed the CPs for the ECI and JCR based on 10000 progressively censored samples generated from the $KU(2,1)$ distribution of size m. The results are presented in Table 1.

It is observed that from Table 1, the CPs of the ECI for and JCR for (α,β) are all close to the desired level of 0.95. In the second simulation study, we obtained the CPs for the intervals and regions based on the upper record values. We simulated the upper record values $X_{U(i)};\ i=1,2,\ldots,m$ from the KU distribution with different values of m. We considered two cases for parameter values as:

$$\text{Case 1: } \alpha=2,\beta=1 \text{ and Case 2: } \alpha=2,\beta=2.$$

The CPs of ECI and JCR for Case 1 (ECI1 and ECI1) and Case 2 (ECI2 and ECI2) are summarized in Table 2.
From Table 2, we observe that the CPs of the exact confidence interval and region are close to the desired level of 0.95 for different parameters and different values of m.

TABLE 1. The coverage probability for ECI and JCR under different censoring schemes.

n	*m*	*C.S.*	*ECI*	*JCR*
20	16	Scheme I	0.9465	0.9414
	16	Scheme II	0.9488	0.9515
	16	Scheme III	0.9489	0.9509
30	20	Scheme I	0.9537	0.9522
	20	Scheme II	0.9531	0.9531
	20	Scheme III	0.9542	0.9493
50	30	Scheme I	0.9489	0.9541
	30	Scheme II	0.9482	0.9522
	30	Scheme III	0.9542	0.9519

TABLE 2. The coverage probability for ECIs and JCRs under different upper record values.

m	*ECI1*	*JCR1*	*ECI2*	*JCR2*
5	0.9543	0.9495	0.9475	0.9544
6	0.9475	0.9525	0.9486	0.9489
7	0.9536	0.9530	0.9532	0.9527
8	0.9518	0.9489	0.9529	0.9493
9	0.9486	0.9493	0.9482	0.9530
10	0.9538	0.9545	0.9485	0.9548
11	0.9527	0.9528	0.9537	0.9494
12	0.9514	0.9496	0.9529	0.9522

5. REAL DATA ANALYSIS

In this section, we first examine the suitability of the KU distribution to the two real data sets and then obtain the ECI for β and JCR for (α, β) under PII censored sample and record values.

Example 1. Brittle materials have many useful properties like stiffness, high strength retention at elevated temperatures, corrosion resistance associated with chemical inertness. Also, strength properties of brittle material are usually scattered; therefore, a statistical analysis is needed for the understanding of the mechanical characterization of this material. So, in this example, we consider the data of the strength property of extremely brittle solid ([3]). This dataset is reported to be recorded from the brittle glass. We have divided each data point by 100 and listed in Table 3.

TABLE 3. The strength data of the glass ceramic.

0.477, 0.502, 0.524, 0.525, 0.529, 0.538, 0.539, 0.546, 0.547, 0.549, 0.555, 0.564,0.575, 0.590, 0.600, 0.611, 0.614, 0.624, 0.627, 0.632, 0.635, 0.642, 0.654, 0.654, 0.656, 0.663, 0.666, 0.666, 0.668, 0.672, 0.675, 0.676, 0.680, 0.684, 0.696, 0.704, 0.707, 0.726, 0.744.

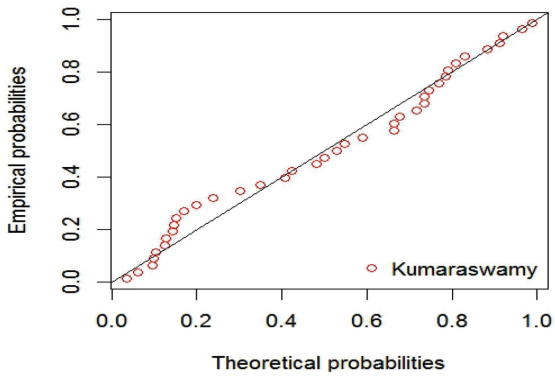

FIGURE 2. The P-P plot for the data of the strength property of brittle solid.

First we want to check whether the KU distribution fits the data or not and that we have applied only the complete data. The Kolmogorov-Smirnov (KS) distance between the empirical and the fitted distribution function and the associated p-value (in brackets) is 0.1125 (0.7065). It clearly indicates that the KU provides a good fit to this data. We also present the P-P plot for the data in Figure 2. Figure 2 shows that the data do not deviate dramatically from the line. Therefore, the KU distribution provides a reasonable fit. Moreover, the strength properties of materials are commonly characterized by Weibull, log-normal and gamma strength distribution function. So, for comparison purposes, we present fitting of three other proposed distributions. The maximum likelihood estimates, Akaike's information criterion (AIC) and

the associated second-order information criterion (AICc), Bayesian information criterion (BIC) and Kolmogorov-Smirnov (KS) statistic are presented in Table 4. From

TABLE 4. Estimated parameters, AICs, AICcs, BICs and K-S distances for different models.

Models	*MLE*	*AIC*	*AICc*	*BIC*	*KS*
Kumaraswamy	$\alpha = 104.459$ $\beta = 10.7564$	-96.81415	-96.48082	-93.48703	0.112569
Weibull	$shape = 10.8193$ $scale = 0.64867$	-96.79048	-96.45715	-93.46336	0.113163
Log-normal	$meanlog = 1.000$ $sdlog = 0.000061$	-94.33432	-94.00099	-91.00719	0.144718
Gamma	$shape = 81.8871$ $scale = 0.00756$	-94.88559	-94.55226	-91.55847	0.141601

Table 4, the KU distribution gives a better fit than other proposed distribution for this dataset. To use this data set under PII censoring schemes, we have created the PII censoring schemes as:
Censoring Scheme: $n = 39, m = 17, R_1 = R_2 = ... = R_{11} = 2, R_i = 0;\ i = 1, 2, ..., 11.$
For computing the ECI for β under complete ($x^{complete}$) and PII censored (x^{PII}) sample, we need the following percentiles;

$$F_{0.975}(2(39)-2,2) = 0.25814,\ F_{0.025}(2(39)-2,2) = 39.4847,$$
$$F_{0.975}(2(17)-2,2) = 0.24103,\ F_{0.025}(2(17)-2,2) = 39.4667.$$

Based on the above percentiles, the lower and upper bound for the 95% ECI of β under $x^{complete}$ and x^{PII} can be obtained as:

$$\beta_L = \varphi(x^{complete}, 0.25814) = 7.96316$$
$$\beta_U = \varphi(x^{complete}, 39.4847) = 21.6431$$

and

$$\beta_L = \varphi(x^{PII}, R_1 = R_2 = ... = R_{11} = 2, 0.24103) = 5.8566$$
$$\beta_U = \varphi(x^{PII}, R_1 = R_2 = ... = R_{11} = 2, 39.4667) = 20.1307.$$

respectively. Therefore under $x^{complete}$ and x^{PII} the 95% ECI of β are:

$$7.96316 < \beta < 21.6431,$$

and

$$5.8566 < \beta < 20.1307.$$

respectively. Also, to obtain the 95% JCR for (α,β), we present the following percentiles:

$$F_{0.9873}(2(39)-2,2)=0.2161,\ F_{0.0127}(2(39)-2,2)=78.2259,$$
$$F_{0.9873}(2(17)-2,2)=0.1992,\ F_{0.0127}(2(17)-2,2)=78.2079,$$
$$\chi^2_{0.0127}(78)=108.5374,\ \chi^2_{0.9873}(78)=52.7792,$$
$$\chi^2_{0.0127}(34)=55.0243,\ \chi^2_{0.9873}(34)=18.2764.$$

So, for $x^{complete}$ and x^{PII}, we have

$$\beta_L=\varphi(x^{complete},0.2161)=7.4648\ \ \&\ \ \beta_U=\varphi(x^{complete},78.2259)=23.4040,$$

and

$$\beta_L=\varphi(x^{PII},R_1=R_2=...=R_{11}=2,\ 0.1992)=\ 5.3276$$
$$\&\ \ \beta_U=\varphi(x^{PII},R_1=R_2=...=R_{11}=2,\ 78.2079)=21.912.$$

respectively. So, the 95% JCR for (α,β) under $x^{complete}$ and x^{PII} can be written as:

$$\begin{cases} 7.4648<\beta<23.4040 \\ -\dfrac{52.7792}{2\sum\limits_{i=1}^{39}\log(1-X^{\beta}_{i:39})}<\alpha<-\dfrac{108.5374}{2\sum\limits_{i=1}^{39}\log(1-X^{\beta}_{i:39})}, \end{cases}$$

and

$$\begin{cases} 5.3276<\beta<21.912 \\ -\dfrac{18.2764}{2\sum\limits_{i=1}^{17}(R_i+1)\log(1-X^{\beta}_{i:17:39})}<\alpha<-\dfrac{55.0243}{2\sum\limits_{i=1}^{17}(R_i+1)\log(1-X^{\beta}_{i:17:39})}, \end{cases}$$

respectively. Also, the confidence region for (α,β) under the complete and PII cen-

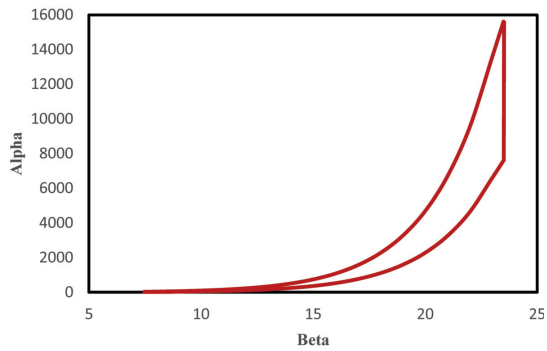

FIGURE 3. Joint confidence regions for (α,β) based on complete data in Example 1.

sored samples are displayed in Figures 3 and 4 respectively. It is easy see that the region is large when β is large.

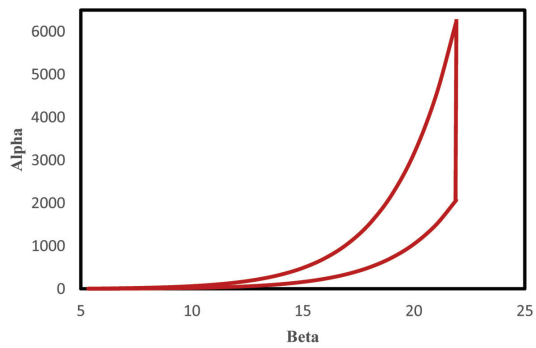

FIGURE 4. Joint confidence regions for (α, β) based on censored data in Example 1.

Example 2. The wind energy is an alternative clean energy source compared with the fossil fuels that pollute the lower layer of the atmosphere. As the wind speed is random phenomenon, the statistical distribution is required for the wind speed. For this reason, we consider the daily average wind speeds from 2001-2006 for Manjil city (http://www.chaharmahalmet.ir/stat/archive/iran/gil/MANJIL/36.asp). The data after dividing by 100 are given in Table 5.

TABLE 5. The data of the daily average wind speeds.

0.063, 0.081, 0.120, 0.125, 0.169, 0.227, 0.221, 0.207, 0.172, 0.119, 0.065,
0.067, 0.136, 0.064, 0.074, 0.096, 0.139, 0.169, 0.211, 0.211, 0.234, 0.148,
0.106, 0.052, 0.059, 0.130, 0.052, 0.059, 0.117, 0.123, 0.130, 0.214, 0.219,
0.166, 0.163, 0.085, 0.061, 0.040, 0.120, 0.042, 0.060, 0.113, 0.10, 0.111,
0.175, 0.192, 0.178, 0.152, 0.095, 0.043, 0.048, 0.110, 0.032 0.049 0.084,
0.110, 0.101, 0.194, 0.197, 0.185, 0.164, 0.100, 0.048, 0.044, 0.109, 0.046,
0.042, 0.088, 0.090, 0.127, 0.179, 0.187, 0.125, 0.143, 0.082, 0.055, 0.039, 0.100.

Here, we restricted ourselves to fitting the KU distribution to this dataset and determining the adequacy of KU distribution using the Kolmogorov-Smirnov statistic between the fitted and the empirical distributions. This statistic and the associated p-value are 0.0899 and 0.5528. Therefore, based on the p-value we can say that the KU distribution fits to this data. Now, we extract a sample of upper record values from this data set as: **0.063**, **0.081**, **0.120**, **0.125**, **0.169**, **0.227** and **0.234**. To demonstrate interval methods, we consider the following percentiles:

$$F_{0.025}(12,2) = 39.4146, F_{0.975}(12,2) = 0.1962, F_{0.9873}(12,2) = 0.1557,$$
$$F_{0.0127}(12,2) = 78.1558, \chi^2_{0.0127}(14) = 28.3704, \chi^2_{0.9873}(14) = 4.889.$$

Based on upper record values, we have:

$$\phi\left(X_{U(1)}, X_{U(2)}, \ldots, X_{U(7)}, 0.1962\right) = 0.4569$$

and

$$\phi\left(X_{U(1)}, X_{U(2)}, \ldots, X_{U(7)}, 39.4146\right) = 4.1678.$$

So by using Theorem 3, the 95% ECI of β is (0.4569, 4.1678). Also using Theorem 4,

$$\phi\left(X_{U(1)}, X_{U(2)}, \ldots, X_{U(7)}, 0.1557\right) = 0.35426$$

and

$$\phi\left(X_{U(1)}, X_{U(2)}, \ldots, X_{U(7)}, 78.1558\right) = 4.68839.$$

Therefore, the 95% JCR for (α, β) is:

$$\begin{cases} 0.35426 < \beta < 4.68839 \\ -\frac{4.889}{2\log(1-X_{U(m)}^{\beta})} < \alpha < -\frac{28.3704}{2\log(1-X_{U(m)}^{\beta})}. \end{cases}$$

Figure 5 shows the 95% confidence region for (α, β). It is easy to see that the region is large when β is large.

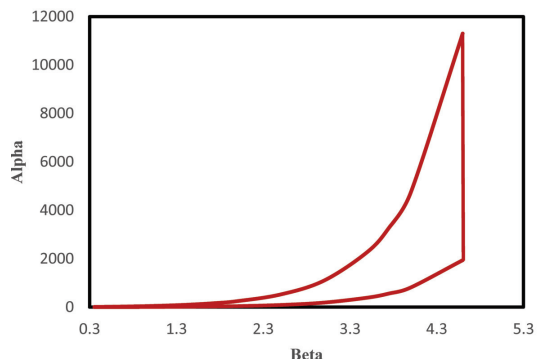

FIGURE 5. Joint confidence regions for (α, β) based on record values in Example 2.

6. CONCLUSION

The subject of progressively censoring and the record values have been received considerable attention in the past few years. The KU distribution and its new family can be applied in different sciences because of various forms of its hazard function. We computed the ECI of β and the JCR for (α, β) under the PII censored sample and upper record values using different pivotal quantities. The simulation study shows that the proposed intervals work well and the CPs of the intervals and regions are close to the desired level of 0.95. Finally, two real data sets contain the strength

of glass ceramic and the daily average wind speeds are analyzed to illustrate our methods.

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INVERSE PROBLEM FOR HIGHER ORDER ULTRAPARABOLIC EQUATION WITH UNKNOWN MINOR COEFFICIENT AND RIGHT-HAND SIDE FUNCTION

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Abstract. The problem of identifying of the several unknown time dependent parameters in the minor coefficient and in the special type right-hand side function of the semilinear higher order ultraparabolic equation from the additional initial, boundary and integral type overdetermination conditions is considered in this paper. The sufficient conditions of the unique solvability on the interval $[0,T]$, where T is determined by the coefficients of the equation, are obtained.

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Keywords: inverse problem, ultraparabolic equation, initial-boundary value problem, unique solvability

1. Introduction

The initial-boundary value problems for ultraparabolic equations often appear in mathematical modeling of population dynamics, non-isotropic processes in mechanics, Asian options in finance, etc. [3, 8, 11, 14, 15, 18, 20]. If in the equation some coefficients are unknown, then the problems of their identification together with the solution of the equation, are called inverse problems. The problems of recovering of a single time depended function in the right-hand side of the higher order PDEs from the integral type overdetermination conditions were considered in [7, 13], of several parameters in the right-hand side function of higher order ultraparabolic equations – in [16, 19]. In the works [9, 18] the conditions of the unique solvability were obtained for the inverse problems for second order ultraparabolic equations with a single unknown function in its right-hand side, and in [17] – with two unknown parameters in minor coefficient and in the right-hand side function. The problems of identifying of the minor coefficient in the parabolic or hyperbolic equations were studied in [1, 5, 6, 22], of the several coefficients in right-hand side function – in [6, 21]. The authors used the methods: of the integral equations, regularization and the Shauder principle [6, 9], of successive approximations [16–19], of continuation in a parameter [13], the generalized Fourier method [5].

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The main aim of the present paper is to find the sufficient conditions of the existence and the uniqueness of the solution for the inverse problem for higher order ultra-parabolic equation with a right-hand side function of the special type. The equation has several time dependent unknown parameters in its right-hand side function and in the minor coefficient. The initial, boundary and integral type overdetermination conditions are posed. In order to obtain the results we use the method of successive approximations and the properties of the initial-boundary value problems for the equations. Similar methods were used for the second order PDEs in [2, 17, 18].

2. Formulation of the problem

Let $\Omega \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^l$ be bounded domains with boundaries $\partial\Omega \in C^{m_0}$ and $\partial D \in C^1$ respectively; $n, l, s, m_0 \in \mathbb{N}$; $x \in \Omega,\ y \in D,\ t \in (0,T),\ T > 0$. Denote $Q_\tau = \Omega \times D \times (0,\tau)$, $\tau \in (0,T]$, $G = \Omega \times D$, $\Sigma_T = \partial\Omega \times D \times (0,T)$, $S_T = \Omega \times \partial D \times (0,T)$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1},\dots,\partial x_n^{\alpha_n}}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\gamma, \alpha \in \mathbb{N}^n$.

In the domain Q_T we consider the problem:
to find the sufficient conditions of the existence and the uniqueness of a set of functions $(u(x,y,t), c(t), q_1(t), \dots, q_s(t))$ that satisfy the equation

$$u_t + \sum_{i=1}^{l} \lambda_i(x,y,t) u_{y_i} + \sum_{0<|\alpha|=|\gamma|\le m_0} (-1)^{|\gamma|} D^\gamma (a_{\alpha\gamma}(x,y,t) D^\alpha u)$$
$$+ (c(t) + b(x,y))u + g(x,y,t,u) = \sum_{i=1}^{s} f_i(x,y,t) q_i(t) + f_0(x,y,t) \tag{2.1}$$

and the conditions

$$u(x,y,0) = u_0(x,y), \quad (x,y) \in G; \tag{2.2}$$

$$\left.\frac{\partial^i u}{\partial \nu^i}\right|_{\Sigma_T} = 0 \ \ (i = 0, 1, \dots, m_0 - 1); \qquad u|_{S_T^1} = 0; \tag{2.3}$$

$$\int_G K_i(x,y) u(x,y,t)\, dx\, dy = E_i(t), \quad t \in [0,T],\ i = 1, \dots, s+1, \tag{2.4}$$

in the sense of Definition 1 (see below). Here ν is the outward unit normal vector to the surface S_T, $S_T^1 = \{(x,y,t) \in S_T : \sum_{i=1}^{l} \lambda_i(x,y,t) \cos(\nu, y_i) < 0\}$. Let us assume that condition
1): there exists $\Gamma_1 \subset \partial D \subset \mathbb{R}^{l-1}$ such that the surface $S_T^1 = \Omega \times \Gamma_1 \times (0,T)$ holds. Denote $\Gamma_2 = \partial D \backslash \Gamma_1$, $S_T^2 = \{(x,y,t) \in S_T : \sum_{i=1}^{l} \lambda_i(x,y,t) \cos(\nu, y_i) \ge 0\}$.

We shall use the spaces $L^\infty(\cdot), L^2(\cdot), W^{k,2}(\cdot), W_0^{m_0,2}(\cdot)$, $C([0,T]; L^2(G))$, $C^k(\cdot)$, $C^1([0,T]; C^2(\overline{D}))$, $C^1(D; C^1(\overline{\Omega}))$ from [4, pp. 32, 37, 38, 44, 147].

We also introduce the following spaces:

$V_1(Q_T) := \{w : D^\alpha w \in L^2(Q_T)\,(|\alpha| \le m_0),\ \frac{\partial^i w}{\partial \nu^i}\big|_{\Sigma_T} = 0\,(i=0,1,\dots,m_0-1)\}$;

$V_2(G) := \{w : w \in L^2(D; W_0^{m_0,2}(\Omega)),\ w_{y_j} \in L^2(G)\ (j=1,\dots,l),\ w|_{\Omega\times\Gamma_1} = 0\}$;

$V_3(Q_T) := \{w :\ w \in W^{1,2}(Q_T),\ D^\alpha w \in L^2(Q_T)\ (|\alpha| \le m_0),\ w|_{S_T^1} = 0,$
$\frac{\partial^i w}{\partial \nu^i}\big|_{\Sigma_T} = 0\ (i = 0, 1, \dots, m_0-1)\}$;

$V_4(Q_T) := \{w :\ w \in V_3(Q_T),\ D^\alpha w \in L^2(Q_T)\ (|\alpha| \le 2m_0)\}$.

Definition 1. A set of functions

$$(u(x,y,t),\ c(t),\ q_1(t),\ q_2(t),\ \dots,\ q_s(t)) \tag{2.5}$$

is a solution to the problem (2.1)–(2.4) if $u \in V_4(Q_T) \cap C([0,T]; L^2(G))$, $c \in C([0,T])$, $q_i \in L^2(0,T)$, $i = 1,\dots,s$, and it satisfies (2.1) for almost all $(x,y,t) \in Q_T$ and the conditions (2.2), (2.4) hold.

Assume that following hypotheses hold:

2): $a_{\alpha\gamma} \in L^\infty(Q_T)$, $0 < |\alpha| = |\gamma| \le m_0$,

$$\sum_{0<|\alpha|=|\gamma|\le m_0} \int_\Omega a_{\alpha\gamma}(x,y,t) D^\alpha w D^\gamma w\, dx \ge a_0 \int_\Omega \sum_{|\alpha|=m_0} |D^\alpha w|^2\, dx$$

for almost all $(y,t) \in D \times (0,T)$ and for all $w \in W_0^{m_0,2}(\Omega)$, $a_0 > 0$;

3): $b \in L^\infty(G)$, $b(x,y) \ge b_0$ for almost all $(x,y) \in G$, b_0 is constant;

4): $f_i \in C([0,T]; L^2(G))$, $i = 0,\dots,s$;

5): $g(x,y,t,\xi)$ is measurable with respect to (x,y,t) in the domain Q_T for all $\xi \in \mathbb{R}^1$ and is continuous with respect to ξ for almost all $(x,y,t) \in Q_T$; moreover, there exists a positive constant g_0 such that $|g(x,y,t,\xi) - g(x,y,t,\eta)| \le g_0|\xi - \eta|$ for almost all $(x,y,t) \in Q_T$ and for all $\xi, \eta \in \mathbb{R}^1$;

6): $\lambda_i \in L^\infty(0,T; C(\overline{G}))$, $\lambda_{iy_i} \in L^\infty(Q_T)$ for all $i = 1,\dots,l$;

7): $u_0 \in V_2(G)$;

8): $K_i \in C^1(D; C^1(\overline{\Omega}))$, $K_i\big|_{\partial\Omega\times D} = 0$, $K_i|_{\Omega\times\Gamma_2} = 0$ for all $i = 1,\dots,s+1$;

9): $E_i \in W^{1,2}(0,T)$, $i = 1,\dots,s+1$.

3. Axilary results

First we assume that $c(t) = c^*(t)$, $q_i(t) = q_i^*(t)$, $i = 1,\dots,s$, in (2.1), where $c^*(t) \in C([0,T])$, $q_i^* \in L^2(0,T)$, $i = 1,\dots,s$, are known functions. The results presented in [12], [18], yield the following statements.

Theorem 1. *Suppose that the hypotheses 1)–7) hold, and, besides:*

i) $a_{\alpha\gamma}, D^{\alpha}a_{\alpha\gamma}, a_{\alpha\gamma,y_k}, b_{y_k} \in L^{\infty}(Q_T), a_{\alpha\gamma,t}, f_{iy_k} \in L^2(Q_T), q_j^* \in L^2(0,T), c^* \in C([0,T])$ $(0<|\alpha|=|\gamma|\le m_0,\ i=0,\dots,s,\ j=1,\dots,s,\ k=1,\dots,l)$;

ii) $|g_{y_i}(x,y,t,\xi)| \le g_1\ \ (i=1,\dots,l),\ \ g(x,y,t,0)|_{S_T^1}=0$ *for almost all* $(x,y,t)\in Q_T$ *and all* $\xi\in\mathbb{R}^1$, *where* g_1 *is a positive constant;*

iii) $f_i|_{S_T^1}=0\quad (i=0,\dots,s)$,

then there exists a unique function $u^*\in V_3(Q_T)\cap C([0,T];L^2(G))$, *that satisfies the condition (2.2) and for all functions* $v\in V_1(Q_T)$ *the equality*

$$\int\limits_{Q_T}\Big(u_t^*v+\sum_{i=1}^{l}\lambda_i(x,y,t)u_{y_i}^*v+\sum_{0<|\alpha|=|\gamma|\le m_0}a_{\alpha\gamma}(x,y,t)D^{\alpha}u^*D^{\gamma}v+$$
$$+(c^*(t)+b(x,y))u^*v+g(x,y,t,u^*)v\Big)\,dx\,dy\,dt$$
$$=\int\limits_{Q_T}\Big(\sum_{i=1}^{s}f_i(x,y,t)q_i^*(t)+f_0(x,y,t)\Big)v\,dx\,dy\,dt. \quad (3.1)$$

Moreover, $u^*\in V_4(Q_T)\cap C([0,T];L^2(G))$, u^* *satisfies the condition (2.2) and (2.1) for almost all* $(x,y,t)\in Q_T$ *(so,* u^* *is a solution to the problem (2.1) – (2.3)). The derivatives of* u^* *have the following estimates*

$$\int\limits_{Q_T}\sum_{i=1}^{l}(u_{y_i}^*)^2\,dx\,dy\,dt\le M_0,\qquad \int\limits_{Q_T}(u_t^*)^2\,dx\,dy\,dt\le M, \quad (3.2)$$

where the constants M_0, M *depend on* u_0, *and on the coefficients and the right-hand side function of (2.1).*

The proof is carried out according to the scheme of proving of Theorem 1, 2 and Lemma 1 [12] and Theorem 3.4.3 [18, p. 97].

Further for functions $w\in W_0^{k,2}(\Omega)$ we shall use Friedrichs' inequalities:

$$\int\limits_{\Omega}\sum_{|\alpha|=j}|D^{\alpha}w|^2\,dx\le\gamma_{k,j}\int\limits_{\Omega}\sum_{|\alpha|=k}|D^{\alpha}w|^2\,dx,\quad j=0,1,\dots,k, \quad (3.3)$$

where constant $\gamma_{k,j}$ depends on $\Omega,\ k,\ j$. Denote $\Gamma_k=\sum\limits_{j=0}^{k}\gamma_{k,j}$.

4. Reduction of the problem to the equivalent problem

Equation (2.1) and conditions (2.4) for $j=1,\dots,s+1$ imply the equalities

$$\sum_{i=1}^{s}q_i(t)\int\limits_{G}K_j(x,y)f_i(x,y,t)\,dx\,dy-c(t)E_j(t)=F_j(t),\ \ t\in[0,T], \quad (4.1)$$

where $F_j(t) := E_j'(t) - \int_G \big(K_j(x,y)(f_0(x,y,t) - b(x,y)u - g(x,y,t,u)) + \sum_{i=1}^{l}(\lambda_i(x,y,t)K_j(x,y))_{y_i}u - \sum_{0<|\alpha|=|\gamma|\le m_0}(-1)^{|\alpha|}D^{\alpha}(D^{\gamma}K_j(x,y)a_{\alpha\gamma}(x,y,t))u\big)dxdy.$

Denote $B(t) := [b_{ij}(t)]_{(s+1)\times(s+1)}$, where $b_{ij}(t) = \int_G K_i(x,y)f_j(x,y,t)\,dx\,dy$, $b_{i,s+1}(t) = -E_i(t)$, $i = 1,\dots,s+1$, $j = 1,\dots,s$, $\Delta(t) := \det B(t)$, $A_{ij}(t)$ – the algebraical complements of the elements of $B(t)$. Let

$$\Delta(t) \neq 0 \text{ for all } t \in [0,T] \tag{4.2}$$

and for $i = 1,\dots,s+1$, $j = 1,\dots,s+1$

$$\alpha_{ij}(x,y,t) := A_{ji}(t)(\Delta(t))^{-1}\big(-K_j(x,y)b(x,y) + \sum_{i=1}^{l}(\lambda_i(x,y,t)K_j(x,y))_{y_i} - \\ - \sum_{0<|\alpha|=|\gamma|\le m_0}(-1)^{|\alpha|}D^{\alpha}(D^{\gamma}K_j(x,y)a_{\alpha\gamma}(x,y,t))\big),$$

$$\beta_{ij}(t) := A_{ji}(t)(\Delta(t))^{-1}\big(E_j'(t) - \int_G K_j(x,y)f_0(x,y,t)\,dx\,dy\big).$$

Then from (4.1) for $t \in [0,T]$ and $i = 1,\dots,s$, we obtain

$$q_i(t) = \sum_{j=1}^{s+1}\left(\beta_{ij}(t) - \int_G\left(\alpha_{ij}(x,y,t)u - \frac{A_{ji}(t)}{\Delta(t)}K_j(x,y)g(x,y,t,u)\right)dxdy\right), \tag{4.3}$$

$$c(t) = \sum_{j=1}^{s+1}\left(\beta_{s+1,j}(t) - \int_G\left(\alpha_{s+1,j}(x,y,t)u - \\ - \frac{A_{j,s+1}(t)}{\Delta(t)}K_j(x,y)g(x,y,t,u)\right)dx\,dy\right). \tag{4.4}$$

From here it follows that the solution of the problem (2.1)–(2.4) satisfies (2.1) for almost all $(x,y,t) \in Q_T$ and (2.2), (4.3), (4.4). By analogous considerations as in Lemma 1 [17] it could be proved that the reverse statement is also correct. Therefore the following lemma holds

Lemma 1. *Let the assumptions of Theorem 1, hypotheses 8), 9) and (4.2) hold. The set of functions (2.5), where $u \in V_4(Q_T) \cap C([0,T];L^2(G))$, $c \in C([0,T])$, $q_i \in L^2(0,T)$, $i = 1,\dots,s$, is a solution to the problem (2.1)–(2.4) if and only if this set satisfies (2.1) for almost all $(x,y,t) \in Q_T$ and (2.2), (4.3), (4.4) hold.*

5. Main results

Further we shall use the following denotations:

$$C_{1,i} := 2(s+1)\sum_{j=1}^{s+1}\int_0^T\left(|\beta_{ij}(t)|+\int_G\left|\tfrac{A_{ji}(t)}{\Delta(t)}K_j(x,y)g(x,y,t,0)\right|dx\,dy\right)^2dt,$$

$$C_{2,i} := 2(s+1)\sum_{j=1}^{s+1}\sup_{[0,T]}\int_G\left(|\alpha_{ij}(x,y,t))|+g_0\left|\tfrac{A_{ji}(t)}{\Delta(t)}K_j(x,y)\right|\right)^2dx\,dy,$$

$$C_1 := \sum_{i=1}^{s}C_{1,i},\qquad C_2 := \sum_{i=1}^{s}C_{2,i},$$

$$C_3 := 2(s+1)\sup_{[0,T]}\sum_{j=1}^{s+1}\left(|\beta_{s+1,j}(t)|+\int_G\left|\tfrac{A_{j,s+1}(t)}{\Delta(t)}K_j(x,y)g(x,y,t,0)\right|dx\,dy\right)^2,$$

$$C_4 := 2(s+1)\sum_{j=1}^{s+1}\sup_{[0,T]}\int_G\left(|\alpha_{s+1,j}(x,y,t))|+g_0\left|\tfrac{A_{j,s+1}(t)}{\Delta(t)}K_j(x,y)\right|\right)^2dx\,dy.$$

Assume that there exist such numbers $T>0$ and $\delta>0$ that the following inequalities are fulfilled:

$$\tilde f C_2 T<\delta,\quad \varkappa>0,\quad M_7<1, \tag{5.1}$$

where $\tilde f=\max_i\sup_{[0,T]}\int_G|f_i(x,y,t)|^2dx\,dy$,

$$M_1 := \tfrac{1}{\delta}\int_{Q_T}\left(f_0(x,y,t))^2+(g(x,y,t,0))^2\right)dx\,dy\,dt+\int_G(u_0(x,y))^2dx\,dy,$$

$$M_2 := \left(C_3+C_4M_1+\tfrac{C_4\tilde f(C_1+C_2M_1T)}{\delta-C_2\tilde fT}\right)^{1/2},$$

$$\lambda^1 := \max_i\operatorname{esssup}_{Q_T}|\lambda_{iy_i}(x,y,t)|,\quad \varkappa := \tfrac{2a_0}{\gamma_{m_0,0}}+2b_0-\lambda^1 l-2M_2-2g_0-(s+2)\delta,$$

$$M_3 := \tfrac{\delta(C_1+C_2M_1T)}{\delta-C_2\tilde fT},\qquad M_4 := M_1+\tfrac{1}{\delta}\tilde fM_3,\qquad M_5 := \max\{M_4;\tilde f\},$$

$$M_6 := \frac{M_5}{\varkappa\delta},\qquad M_7 := \frac{1}{2}(C_2+C_4)M_6,\qquad M_8 := \sqrt{\tfrac{C_4M_5}{2\delta}}.$$

Theorem 2. *Let the hypotheses (4.2), (5.1) and 1)–9) hold. Then the problem (2.1)–(2.4) has at most one solution.*

If, besides, $a_{\alpha\gamma}, D^\alpha a_{\alpha\gamma}, a_{\alpha\gamma,y_k}, b_{y_k}\in L^\infty(Q_T)$, $a_{\alpha\gamma,t}, f_{iy_k}\in L^2(Q_T)$ $(0<|\alpha|=|\gamma|\le m_0,\ i=0,\dots,s,\ k=1,\dots,l)$ and the assumptions ii), iii) of Theorem 1 are true, then a solution to the problem (2.1)–(2.4) exists in Q_T.

Proof. Existence. In order to prove the existence of the solution we use the method of successive approximations. We construct the approximation of the solution to the problem (2.1)–(2.4) as follows:

$$c^1(t) := 0,\ q_i^1(t) := 0,\qquad t\in[0,T],\ i=1,\dots,s,$$

$$q_i^m(t)=\sum_{j=1}^{s+1}\Bigg(\beta_{ij}(t)-\int\limits_G\big(\alpha_{ij}(x,y,t)u^{m-1}-\frac{A_{ji}(t)}{\Delta(t)}K_j(x,y)\times$$
$$\times g(x,y,t,u^{m-1})\big)\,dx\,dy\Bigg),\ t\in[0,T],\ i=1,\dots,s,\ m\geq 2,\quad (5.2)$$

$$c^m(t)=\sum_{j=1}^{s+1}\Bigg(\beta_{s+1,j}(t)-\int\limits_G\alpha_{s+1,j}(x,y,t)u^{m-1}\,dx\,dy$$
$$+\int\limits_G\frac{A_{j,s+1}(t)}{\Delta(t)}K_j(x,y)g(x,y,t,u^{m-1})\,dx\,dy\Bigg),\ t\in[0,T],\ \ m\geq 2,\quad (5.3)$$

u^m satisfies the equality

$$\int\limits_{Q_\tau}\big(u_t^m v+\sum_{i=1}^{l}\lambda_i(x,y,t)u_{y_i}^m v+\sum_{0<|\alpha|=|\gamma|\leq m_0}a_{\alpha\gamma}(x,y,t)D^\alpha u^m D^\gamma v+$$
$$+(c^m(t)+b(x,y))u^m v+g(x,y,t,u^m)v\big)\,dx\,dy\,dt$$
$$=\int\limits_{Q_\tau}(\sum_{i=1}^{s}f_i(x,y,t)q_i^m(t)+f_0(x,y,t))v\,dx\,dy\,dt,\ \tau\in(0,T],\ m\geq 1\quad (5.4)$$

for all $v\in V_1(Q_T)$ and the condition

$$u^m(x,y,0)=u_0(x,y),\ \ (x,y)\in G.\quad (5.5)$$

It follows from (5.2), (5.3), that

$$q_i^m\in L^2(0,T),\ i=1,\dots,s,\ m\geq 2,\ \ c^m\in C([0,T]),\ m\geq 2.$$

According to Theorem 1 for each $m\in\mathbb{N}$ there exists a unique function $u^m\in V_4(Q_T)\cap C([0,T];L^2(G))$, which satisfies (5.4), (5.5).

At first we shall find some estimates for the approximations $(u^m(x,y,t),$ $c^m(t),q_1^m(t),q_2^m(t),\dots,q_s^m(t))$, and show that $c^m(t)$ for all $m\in\mathbb{N}$ and $t\in[0,T]$ could be bounded from below by the same constant. Let $c^m(t)\geq c_{0m}$ for all $t\in[0,T]$, where $c_{0m}\in\mathbb{R}$. If we choose $v=u^m$ in (5.4)

$$\int\limits_{Q_\tau}\big(u_t^m u^m+\sum_{i=1}^{l}\lambda_i(x,y,t)u_{y_i}^m u^m+\sum_{0<|\alpha|=|\gamma|\leq m_0}a_{\alpha\gamma}(x,y,t)D^\alpha u^m D^\gamma u^m+$$
$$+(c^m(t)+b(x,y))(u^m)^2+g(x,y,t,u^m)u^m\big)\,dx\,dy\,dt$$
$$=\int\limits_{Q_\tau}(\sum_{i=1}^{s}f_i(x,y,t)q_i^m(t)+f_0(x,y,t))u^m\,dx\,dy\,dt,\ \tau\in(0;T],\ \ m\geq 1,\quad (5.6)$$

and take into account the hypotheses 2)–7), then from (5.6) we get the inequalities

$$\int\limits_G (u^m(x,y,\tau))^2\,dx\,dy+\int\limits_{S^2_\tau}\sum_{i=1}^{l}\lambda_i(x,y,t)(u^m)^2\cos(\nu,y_i)\,d\sigma$$
$$+\int\limits_{Q_\tau}\Bigg(2a_0\sum_{|\alpha|=m_0}|D^\alpha u^m|^2+(2b_0-\lambda^1 l+2c_{0m}-2g_0-(s+2)\delta)(u^m)^2\Bigg)dxdydt$$
$$\le\frac{1}{\delta}\int\limits_{Q_\tau}\Bigg(\sum_{i=1}^{s}(f_i(x,y,t))^2(q_i^m(t))^2+(f_0(x,y,t))^2+(g(x,y,t,0))^2\Bigg)dx\,dy\,dt$$
$$+\int\limits_G (u_0(x,y))^2\,dx\,dy,\ \tau\in(0;T],\ \ m\ge 1. \quad (5.7)$$

After using the inequality (3.3) in the third term of (5.7), we obtain

$$\int\limits_G (u^m(x,y,\tau))^2\,dx\,dy+\int\limits_{S^2_\tau}\sum_{i=1}^{l}\lambda_i(x,y,t)(u^m)^2\cos(\nu,y_i)\,d\sigma$$
$$+\int\limits_{Q_\tau}\Bigg(\frac{2a_0}{\gamma_{m_0,0}}-\lambda^1 l+2c_{0m}+2b_0-2g_0-(s+2)\delta\Bigg)(u^m)^2\,dx\,dy\,dt$$
$$\le\frac{1}{\delta}\int\limits_{Q_\tau}\Bigg(\sum_{i=1}^{s}(f_i(x,y,t))^2(q_i^m(t))^2+(f_0(x,y,t))^2+$$
$$+(g(x,y,t,0))^2\Bigg)dx\,dy\,dt+\int\limits_G (u_0(x,y))^2\,dx\,dy,\ \tau\in(0;T],\ \ m\ge 1. \quad (5.8)$$

Using the assumption $\frac{2a_0}{\gamma_{m_0,0}}-\lambda^1 l+2c_{0m}+2b_0-2g_0-(s+2)\delta\ge 0$, from (5.8) for $\tau\in(0;T]$, $m\ge 1$, we get the estimates

$$\int\limits_G (u^m(x,y,\tau))^2\,dx\,dy\le M_1+\frac{1}{\delta}\int\limits_{Q_\tau}\sum_{i=1}^{s}(f_i(x,y,t))^2(q_i^m(t))^2\,dx\,dy\,dt. \quad (5.9)$$

Now square both sides of (5.2) and after using Hölder inequality, and integrating the result with respect to t, we get the estimates

$$\int\limits_0^T (q_i^m(t))^2\,dt\le C_{1,i}+C_{2,i}\int\limits_{Q_T}(u^{m-1})^2\,dx\,dy\,dt,\ m\ge 2,\ i=1,\dots,s, \quad (5.10)$$

rising up the both sides of (5.3) to the square and using Hölder inequality, we get the estimate

$$(c^m(t))^2\le C_3+C_4\int\limits_G (u^{m-1})^2\,dx\,dy,\qquad t\in[0;T],\,m\ge 2. \quad (5.11)$$

From the system of inequalities (5.9)–(5.11) under the condition (5.1) it is easy to proof the estimates

$$|c^m(t)| \le M_2, \qquad t \in [0,T],\ m \ge 1, \tag{5.12}$$

$$\sum_{i=1}^{s} \int_0^T |q_i^m(t)|^2\,dt \le M_3, \qquad m \ge 1, \tag{5.13}$$

$$\int_G |u^m(x,y,t)|^2\,dx\,dy \le M_4, \qquad t \in [0,T],\ m \ge 1. \tag{5.14}$$

Remark, that M_2 is independent on m and if we take $-M_2$ instead of c_{0m} for each m and take into account the condition (5.1), we get

$$\frac{2a_0}{\gamma_{m_0,0}} - \lambda^1 l + 2c_{0m} + 2b_0 - 2g_0 - (s+2)\delta = \varkappa > 0.$$

Thus, $c^m(t) \ge -M_2$ for all $m \in \mathbb{N}$.

Further we show that $\{(u^m(x,y,t), c^m(t), q_1^m(t), q_2^m(t), \dots, q_s^m(t))\}_{m=1}^{\infty}$ converges to the solution of the problem (2.1)–(2.4). Denote for $m \ge 2$

$$s^m(t) := c^m(t) - c^{m-1}(t), \qquad r_i^m(t) := q_i^m(t) - q_i^{m-1}(t),\ i = 1,\dots,s,$$

$$z^m := z^m(x,y,t) = u^m(x,y,t) - u^{m-1}(x,y,t).$$

From (5.5) we get $z^m(x,y,0) = 0,\ (x,y) \in G,\ m \ge 2$. Moreover, using (5.4) with $v = z^m$ we obtain the equality

$$\begin{aligned}
&\frac{1}{2}\int_G |z^m(x,y,\tau)|^2\,dx\,dy + \int_{Q_\tau} \Big(\sum_{i=1}^{l} \lambda_i(x,y,t) z^m_{y_i} z^m + b(x,y)(z^m)^2 \\
&\quad + \sum_{0<|\alpha|=|\gamma|\le m_0} a_{\alpha\gamma}(x,y,t) D^\alpha z^m D^\gamma z^m + (c^m(t)u^m - c^{m-1}(t)u^{m-1}) z^m \\
&\quad + (g(x,y,t,u^m) - g(x,y,t,u^{m-1})) z^m \Big)\,dx\,dy\,dt = \\
&\quad = \int_{Q_\tau} \sum_{i=1}^{s} f_i(x,y,t) r_i^m(t) z^m\,dx\,dy\,dt,\ \tau \in (0,T],\ m \ge 2.
\end{aligned} \tag{5.15}$$

Notice that

$$\int_{Q_\tau} \sum_{i=1}^{s} f_i(x,y,t) r_i^m(t) z^m\,dx\,dy\,dt \le \frac{s\delta}{2} \int_{Q_\tau} (z^m)^2\,dx\,dy\,dt + \frac{\tilde{f}}{2\delta} \int_0^T \sum_{i=1}^{s} (r_i^m(t))^2\,dt$$

and

$$(c^m(t)u^m - c^{m-1}(t)u^{m-1}) z^m = c^m(t)(z^m)^2 + s^m(t) u^{m-1} z^m,$$

therefore

$$\int_{Q_\tau} (c^m(t)u^m - c^{m-1}(t)u^{m-1}) z^m\,dx\,dy\,dt \ge \left(-M_2 - \frac{\delta}{2}\right) \int_{Q_\tau} (z^m)^2\,dx\,dy\,dt$$

$$-\frac{1}{2\delta}\int_0^\tau (s^m(t))^2\left(\int_{G_t}(u^{m-1})^2\,dx\,dy\right)dt \geq \left(-M_2-\frac{\delta}{2}\right)\int_{Q_\tau}(z^m)^2\,dx\,dy\,dt$$
$$-\frac{M_4}{2\delta}\int_0^\tau (s^m(t))^2\,dt, \qquad \tau\in(0,T],\ m\geq 2. \quad (5.16)$$

Then, taking into account 2) – 7) and (5.16), from (5.15) we get inequalities

$$\int_G (z^m(x,y,\tau))^2\,dx\,dy+\int_{S_\tau^2}\sum_{i=1}^l \lambda_i(x,y,t)(z^m)^2\cos(\nu,y_i)\,d\sigma$$
$$+\int_{Q_\tau}\left(2a_0\sum_{|\alpha|=m_0}|D^\alpha z^m|^2+\big(2b_0-\lambda^1 l-2g_0-2M_2-(s+2)\delta\big)(z^m)^2\right)dx\,dy\,dt$$
$$\leq \frac{M_4}{\delta}\int_0^T (s^m(t))^2\,dt+\frac{\tilde f}{\delta}\int_0^T\sum_{i=1}^s (r_i^m(t))^2\,dt,\ \tau\in(0;T],\ m\geq 2. \quad (5.17)$$

After applying (3.3) to the third term of (5.17), we get the estimates

$$\int_G (z^m(x,y,\tau))^2\,dxdy+\int_{S_\tau^2}\sum_{i=1}^l \lambda_i(x,y,t)(z^m)^2\cos(\nu,y_i)d\sigma+\varkappa\int_{Q_\tau}(z^m)^2\,dxdydt$$
$$\leq \frac{M_4}{\delta}\int_0^T (s^m(t))^2\,dt+\frac{\tilde f}{\delta}\int_0^T\sum_{i=1}^s (r_i^m(t))^2\,dt,\ \tau\in(0;T],\ m\geq 2. \quad (5.18)$$

Taking into account (5.1), from (5.18) for $m\geq 2$ we find the estimates

$$\int_G (z^m(x,y,\tau))^2\,dx\,dy\leq \frac{M_5}{\delta}\int_0^T\left((s^m(t))^2+\sum_{i=1}^s (r_i^m(t))^2\right)dt, \tau\in(0;T], \quad (5.19)$$

and

$$\int_{Q_T}(z^m)^2\,dx\,dy\,dt\leq M_6\int_0^T\left((s^m(t))^2+\sum_{i=1}^s (r_i^m(t))^2\right)dt, \qquad m\geq 2. \quad (5.20)$$

Besides, from (5.2) and (5.3) it follows that

$$\int_0^T\sum_{i=1}^s (r_i^m(t))^2\,dt\leq \frac{C_2}{2}\int_{Q_T}(z^{m-1})^2\,dx\,dy\,dt, \qquad m\geq 2, \quad (5.21)$$

$$\int_0^T (s^m(t))^2\,dt \le \frac{C_4}{2}\int_{Q_T} (z^{m-1})^2\,dx\,dy\,dt, \qquad m \ge 2, \tag{5.22}$$

$$(s^m(t))^2 \le \frac{C_4}{2}\int_G (z^{m-1}(x,y,t))^2\,dx\,dy, \qquad t \in [0,T],\, m \ge 2. \tag{5.23}$$

Then (5.21), (5.22) and (5.20) imply

$$\int_0^T \Big((s^m(t))^2 + \sum_{i=1}^s (r_i^m(t))^2\Big)\,dt \le M_7 \int_0^T \Big((s^{m-1}(t))^2 + \sum_{i=1}^s (r_i^{m-1}(t))^2\Big)\,dt$$
$$\le (M_7)^{m-2}\int_0^T \Big((s^2(t))^2 + \sum_{i=1}^s (r_i^2(t))^2\Big)\,dt, \qquad m \ge 2, \tag{5.24}$$

and with the use of (5.19), from (5.23) for $m \ge 2$ we get

$$|s^m(t)| \le M_8 \left(\int_0^T \Big((s^{m-1}(t))^2 + \sum_{i=1}^s (r_i^{m-1}(t))^2\,dt\Big)\right)^{\frac{1}{2}},\ t \in [0,T]. \tag{5.25}$$

Afterwards, using (5.24), (5.25) and the assumption $M_7 < 1$, we show that the estimates

$$|c^{m+k}(t) - c^m(t)| \le \sum_{i=m+1}^{m+k} |s^i(t)| \le M_8 \sum_{i=m+1}^{m+k} \left(\int_0^T \Big(\sum_{l=1}^s ({r_l}^{i-1}(t))^2 + \right.$$
$$\left. + (s^{i-1}(t))^2\Big)\,dt\right)^{\frac{1}{2}} \le \sum_{i=m+1}^{m+k} M_8 (M_7)^{\frac{i-3}{2}} \left(\int_0^T \Big((s^2(t))^2 + \sum_{i=1}^s (r_i^2(t))^2\Big)\,dt\right)^{\frac{1}{2}}$$
$$\le M_8 \frac{(M_7)^{\frac{m-2}{2}}}{1-(M_7)^{\frac{1}{2}}} \left(\int_0^T \Big((s^2(t))^2 + \sum_{i=1}^s (r_i^2(t))^2\Big)\,dt\right)^{\frac{1}{2}} \tag{5.26}$$

and

$$\left(\int_0^T ({q_l}^{m+k}(t) - {q_l}^m(t))^2\,dt\right)^{\frac{1}{2}} \le \sum_{i=m+1}^{m+k} \left(\int_0^T \Big(({r_l}^i(t))^2 + (s^i(t))^2\Big)\,dt\right)^{\frac{1}{2}}$$
$$\le \frac{(M_7)^{\frac{m-1}{2}}}{1-(M_7)^{\frac{1}{2}}} \left(\int_0^T \Big((s^2(t))^2 + \sum_{l=1}^s (r_l^2(t))^2\Big)\,dt\right)^{\frac{1}{2}},\ l = 1,\dots,s, \tag{5.27}$$

hold for all $k,m \in \mathbb{N}$, $m \geq 3$. From (5.26), (5.27) we conclude that for any $\varepsilon > 0$, there exists m_0 such that for all $k, m \in \mathbb{N}$, $m > m_0$, the inequalities $\|c^{m+k}(t) - c^m(t); C([0,T])\| \leq \varepsilon$ and $\sum_{i=1}^{s} \|q_i^{m+k}(t) - q_i^m(t); L^2(0,T)\| \leq \varepsilon$ are true and, hence, the sequence $\{c^m\}_{m=1}^{\infty}$ is fundamental in $C([0,T])$, and $\{q_i^m\}_{m=1}^{\infty}$, $i = 1,\dots,s$, are fundamental in $L^2(0,T)$. Therefore, (5.19) and (5.17) imply that $\{u^m\}_{m=1}^{\infty}$ is fundamental in $L^2(Q_T) \cap C([0,T]; L^2(G))$ and $\{D^\alpha u^m\}_{m=1}^{\infty}$, $|\alpha| \leq m$, is fundamental in $L^2(Q_T)$ and, hence, as $m \to \infty$

$$u^m \to u \text{ in } L^2(Q_T) \cap C([0,T]; L^2(G)),\ q_i^m \to q_i \text{ in } L^2(0,T),\ i = 1,\dots,s.$$
$$D^\alpha u^m \to D^\alpha u \text{ in } L^2(Q_T),\ |\alpha| \leq m, \quad c^m \to c \text{ in } C([0,T]). \tag{5.28}$$

Formula (3.2) implies the following estimates

$$\int_{Q_T} \sum_{i=1}^{l} (u_{y_i}^m)^2\, dx\, dy\, dt \leq M_0, \qquad \int_{Q_T} (u_t^m)^2\, dx\, dy\, dt \leq M, \tag{5.29}$$

and, by virtue of the inequalities (5.12), (5.13), the constants M_0, M are independent of m and (5.29) is true for all $m \in \mathbb{N}$. In view of (5.29), we can select a subsequence of sequence $\{u^m\}_{m=1}^{\infty}$ (we preserve the same notation for this subsequence), such that

$$u_t^m \to u_t \text{ and } u_{y_i}^m \to u_{y_i} \text{ weakly in } L^2(Q_T),\ i = 1,\dots,l, \text{ as } m \to \infty. \tag{5.30}$$

Passing to the limit in (5.2) – (5.4) as $m \to \infty$ and taking into account (5.28), (5.30), we get that the set of functions (2.5) satisfies the system of equations (4.3), (4.4) and

$$\int_{Q_\tau} \Big(u_t v + \sum_{i=1}^{l} \lambda_i(x,y,t) u_{y_i} v + \sum_{0<|\alpha|=|\gamma|\leq m_0} a_{\alpha\gamma}(x,y,t) D^\alpha u D^\gamma v$$
$$+ (c(t) + b(x,y)) uv + g(x,y,t,u) v\Big)\, dx\, dy\, dt$$
$$= \int_{Q_\tau} \Big(\sum_{i=1}^{s} f_i(x,y,t) q_i(t) + f_0(x,y,t)\Big) v\, dx\, dy\, dt \tag{5.31}$$

for all $\tau \in (0;T]$ and $v \in V_1(Q_T)$. From (5.31) we obtain that

$$\int_{\Omega} \Big(u_t w + \sum_{i=1}^{l} \lambda_i(x,y,t) u_{y_i} w + \sum_{0<|\alpha|=|\gamma|\leq m_0} a_{\alpha\gamma}(x,y,t) D^\alpha u D^\gamma w + (c(t) +$$
$$+ b(x,y)) uw + g(x,y,t,u) w\Big)\, dx = \int_{\Omega} \Big(\sum_{i=1}^{s} f_i(x,y,t) q_i(t) + f_0(x,y,t)\Big) w\, dx \tag{5.32}$$

for almost all $(y,t) \in D \times (0;T)$ and for all $w \in W_0^{m_0,2}(\Omega)$. From (5.32) we derive that u for almost all $(y,t) \in D \times (0;T)$ is a weak solution to the Dirichlet problem for the

elliptic equation

$$\sum_{0<|\alpha|=|\gamma|\le m_0} D^\gamma(a_{\alpha\gamma}(x,y,t)D^\alpha u) = F(x,y,t),\ x\in\Omega; \tag{5.33}$$

$$\left.\frac{\partial^i u}{\partial \nu^i}\right|_\Omega = 0\ (i=0,\,1,\,\dots,m_0-1),$$

where $F(x,y,t)=\sum\limits_{i=1}^{s} f_i(x,y,t)q_i(t)+f_0(x,y,t)-u_t-\sum\limits_{i=1}^{l}\lambda_i(x,y,t)u_{y_i}-(c(t)+b(x,y))u-g(x,y,t,u)$. Since function $F(x,y,t)\in L^2(\Omega)$ for almost all $(y,t)\in D\times(0,T)$, there exists a unique weak solution u of the problem (5.33). Then, from [23] and the condition $\partial\Omega\in C^{m_0}$ we conclude that $D^\alpha u^*(\cdot,y,t)\in L^2(\Omega)$, $|\alpha|\le 2m_0$. Thus, $u^*(\cdot,y,t)\in W_0^{m_0,2}(\Omega)\cap W^{2m_0,2}(\Omega)$. Using the scheme [10, p. 219] we prove that the function $u^*(x,y,t)$ satisfies the equation (2.1) for almost all $(x,y,t)\in Q_T$.

Hence, $u\in V_4(Q_T)\cap C([0,T];L^2(G))$, the set $(u(x,y,t),c(t),q_1(t),\dots,q_s(t))$ satisfies (2.1) for almost all $(x,y,t)\in Q_T$, and by virtue of Lemma 1 $(u(x,y,t),c(t),q_1(t),\dots,q_s(t))$ is a solution to the problem (2.1)–(2.4) in Q_T.

Uniqueness. Assume that $(u^{(l)}(x,y,t),c^{(l)}(t),q_1^{(l)}(t),\dots,q_s^{(l)}(t))$, $l=1,2$, are two solutions to problem (2.1)–(2.4). Denote $\tilde u(x,y,t):=u^{(1)}(x,y,t)-u^{(2)}(x,y,t)$, $\tilde c(t):=c^{(1)}(t)-c^{(2)}(t)$, $\tilde q_i(t):=q_i^{(1)}(t)-q_i^{(2)}(t)$. Then $\tilde u(x,y,0)\equiv 0$, the set of functions $(\tilde u(x,y,t),\tilde c(t),\tilde q_1(t),\dots,\tilde q_s(t))$ satisfies the equality

$$\int\limits_{Q_T}\Big(\tilde u_t v+\sum_{i=1}^{l}\lambda_i(x,y,t)\tilde u_{y_i}v+\sum_{0<|\alpha|=|\gamma|\le m_0}a_{\alpha\gamma}(x,y,t)D^\alpha\tilde u D^\gamma v+b(x,y)\tilde u v+$$
$$+(c^{(1)}(t)u^{(1)}-c^{(2)}(t)u^{(2)})v+(g(x,y,t,u^{(1)})-$$
$$-g(x,y,t,u^{(2)}))v\Big)\,dx\,dy\,dt=\int\limits_{Q_T}\sum_{i=1}^{s}f_i(x,y,t)\tilde q_i(t)v\,dx\,dy\,dt, \tag{5.34}$$

for all $v\in V_1(Q_T)$, and the system of equalities

$$\tilde c(t)=\sum_{j=1}^{s+1}\int\limits_G\Big(-\alpha_{s+1,j}(x,y,t)\tilde u+\frac{A_{j,s+1}(t)}{\Delta(t)}K_j(x,y)\big(g(x,y,t,u^{(1)})-$$
$$-g(x,y,t,u^{(2)})\big)\Big)\,dx\,dy,\ t\in[0,T], \tag{5.35}$$

$$\tilde q_i(t)=\sum_{j=1}^{s+1}\int\limits_G\Big(-\alpha_{ij}(x,y,t)\tilde u+\frac{A_{ji}(t)}{\Delta(t)}K_j(x,y)\big(g(x,y,t,u^{(1)})-$$
$$-g(x,y,t,u^{(2)})\big)\Big)\,dx\,dy,\ t\in[0,T],\ i=1,\dots,s. \tag{5.36}$$

From (5.35), (5.36) and 5) we derive the inequalities

$$\int_0^T \sum_{i=1}^s (\tilde{q}_i(t))^2\,dt \le \frac{C_2}{2}\int_{Q_T} (\tilde{u})^2\,dxdydt, \quad \int_0^T (\tilde{c}(t))^2\,dt \le \frac{C_4}{2}\int_{Q_T} (\tilde{u})^2\,dxdydt, \tag{5.37}$$

and after choosing $v = \tilde{u}$, in (5.34) we get

$$\int_{Q_T} \Big(\tilde{u}_t\tilde{u} + \sum_{i=1}^l \lambda_i(x,y,t)\tilde{u}_{y_i}\tilde{u} + \sum_{0<|\alpha|=|\gamma|\le m_0} a_{\alpha\gamma}(x,y,t)D^\alpha\tilde{u}D^\gamma\tilde{u} +$$
$$+ (c^{(1)}(t)u^{(1)} - c^{(2)}(t)u^{(2)})\tilde{u} + b(x,y)(\tilde{u})^2 + (g(x,y,t,u^{(1)}) -$$
$$- g(x,y,t,u^{(2)}))\tilde{u}\Big)\,dxdydt = \int_{Q_T} \sum_{i=1}^s f_i(x,y,t)\tilde{q}_i(t)\tilde{u}\,dxdydt.$$

From here by the same way as from (5.15) we got (5.20), we find the following estimate:

$$\int_{Q_T} (\tilde{u})^2\,dxdydt \le M_6 \int_0^T \Big((\tilde{c}(t))^2 + \sum_{i=1}^s (\tilde{q}_i(t))^2\Big)\,dt \tag{5.38}$$

and, taking into account (5.37), from (5.38) we obtain

$$(1 - M_7)\int_{Q_T} (\tilde{u})^2\,dxdydt \le 0.$$

Since $M_7 < 1$, we conclude that $u^{(1)} = u^{(2)}$ in Q_T. Then (5.37) imply $c^{(1)}(t) \equiv c^{(2)}(t)$, $q_i^{(1)}(t) \equiv q_i^{(2)}(t)$, $i = 1,\dots,s$. □

6. Conclusion

We obtained the sufficient conditions of the unique solvability for the inverse problem for higher order semilinear ultraparabolic equation with the unknown time dependent functions in the minor coefficient and in the right-hand side function of the equation.

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EULER-LAGRANGE RADICAL FUNCTIONAL EQUATIONS WITH SOLUTION AND STABILITY

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Abstract. In this article, we introduce the generalized Euler-Lagrange radical functional equations of type sextic and quintic. Also, we obtain their general solution and investigate the generalized Hyers-Ulam-Rassias stability in modular spaces using fixed point concept with suitable counter examples.

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1. Introduction

Ulam [13] raised the infamous stability problem of functional equations in 1940 at the University of Wisconsin. The solution for the Ulam problem garnered world wide attention and finally came to be identified as generalized Hyers-Ulam, generalized Hyers-Ulam-Rassias, Ulam-Găvruta-Rassias and JMR stabilities of functional equations. One can refer ([1, 7, 8, 10–12]).

In the probabilistic normed spaces, Mohammad Bagher Ghaemi et al. [6] analyzed the stability for the sextic and quintic mappings.

In the quasi-β-normed spaces via fixed point method, Tian Zhou Xu et al. [15] introduced the following functional equation of quintic type

$$\begin{aligned} g(m+3n)-5g(m+2n)+10g(m+n)-10g(m) \\ +5g(m-n)-g(m-2n)=120g(n) \end{aligned}$$

and sextic type

$$\begin{aligned} g(m+3n)-6g(m+2n)+15g(m+n)-20g(m)+15g(m-n) \\ -6g(m-2n)+g(m-3n)=720g(n) \end{aligned}$$

and also investigated their stabilities related to Ulam problem.

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In Felbin spaces, Pasupathi Narasimman et al. [9] introduced generalized sextic and quintic functional equations

$$\begin{aligned} &g(am+n)+g(am-n)+g(m+an)+g(m-an)\\ &\quad=(a^4+a^2)[g(m+n)+g(m-n)]+2(a^6-a^4-a^2+1)[g(m)+g(n)]\end{aligned}$$

$$\begin{aligned} &a[g(am+n)+g(am-n)]+g(m+an)+g(m-an)\\ &\quad=(a^4+a^2)[g(m+n)+g(m-n)]+2(a^6-a^4-a^2+1)g(m)\end{aligned}$$

with general solution and stability for $a\in\mathbb{R}-\{0,\pm1\}$.

Using fixed point theory, Zamani Eskandani and John Michael Rassias [5], Kittipong Wongkum [14] are obtained modular stability of $\gamma-$quartic and cubic functional equations.

In quasi-β-normed spaces, In Goo Cho et al. [3] analyzed the Ulam stability problem for the quintic functional equation of the form

$$\begin{aligned} &2g(2m+n)+2g(2m-n)+g(m+2n)+g(m-2n)\\ &\quad=20[g(m+n)+g(m-n)]+90g(m).\end{aligned}$$

In 2015, Abasalt Bodaghi et al.[2] analyzed the general solution and stability of a mixed type of quintic-additive functional equation of the form

$$\begin{aligned} &g(3m+n)-5g(2m+n)+g(2m-n)+10g(m+n)-5g(m-n)\\ &\quad=10g(n)+4g(2m)-8g(m)\end{aligned}$$

in real numbers.

Motivated from the above investigations on sextic and quintic functional equations, in this paper we introduce the following new generalized Euler-Lagrange radical quintic and sextic functional equations

$$\begin{aligned} &f(ax+y)+f(ax-y)+f(x+ay)+f(x-ay)\\ &\quad=(a+a^2)\{f(x+y)+f(x-y)\}-2(a+a^2-a^5-1)f(x)\\ &\quad+10(a^4-a^2)f(\sqrt[5]{xy^4})+20(a^3-a)f(\sqrt[5]{x^3y^2}),\end{aligned} \tag{1.1}$$

$$\begin{aligned} &f(ax+y)+f(ax-y)+f(x+ay)+f(x-ay)\\ &\quad=(a+a^2)\{f(x+y)+f(x-y)\}-2(a+a^2-a^6-1)\{f(x)+f(y)\}\\ &\quad+30(a^4-a)\{f(\sqrt[6]{x^4y^2})+f(\sqrt[6]{x^2y^4})\},\end{aligned} \tag{1.2}$$

for a fixed real a and $a\neq 0,\pm1$. Mainly we obtain their general solution and investigate their stabilities related to Ulam problem in modular spaces. The definitions related to modular space and fixed point theory to establish our main theorem can be referred in [4].

The paper structured as follows: In Section-2, we obtain the general solution of the functional equations (1.1) and (1.2). In Section-3 and in Section 4, authors discuss

generalized Hyers-Ulam-Rassias, Hyers-Ulam and Hyers-Ulam-Rassias stabilities of quintic and sextic functional equations respectively in modular spaces using fixed point theory. Finally the conclusion given in section-5.

2. General solution of (1.1) and (1.2)

Theorem 1. *If f satisfies (1.1), then a mapping $f : X \to Y$ is quintic and odd.*

Proof. Consider f satisfies (1.1). Setting $(x,y) = (0,0)$ in (1.1), we get $f(0) = 0$. Replacing (x,y) by $(x,0)$ in (1.1), we arrive

$$f(ax) = a^5 f(x) \tag{2.1}$$

for all $x \in X$. Therefore f is quintic. Setting (x,y) by $(0,x)$ in (1.1) and using (2.1) leads

$$f(-x) = -f(x) \tag{2.2}$$

for all $x \in X$ and hence f is odd. □

Theorem 2. *If f satisfies (1.2), then a mapping $f : X \to Y$ is sextic and even.*

Proof. Consider f satisfies (1.2). Setting $(x,y) = (0,0)$ in (1.2), we get $f(0) = 0$. Replacing (x,y) by $(x,0)$ in (1.2), we arrive

$$f(ax) = a^6 f(x) \tag{2.3}$$

for all $x \in X$. Therefore f is sextic. Setting (x,y) by $(0,x)$ in (1.2) and using (2.3) leads

$$f(-x) = f(x) \tag{2.4}$$

for all $x \in X$ and hence f is even. □

3. Stability of functional equation(1.1)

In this section, we determine the generalized Hyers-Ulam stability concerning the generalized Euler-Lagrange radical quintic functional equation (1.1) in modular spaces by using fixed point theory.

For mapping $\rho : M \to X_\xi$, consider

$$\begin{aligned} D_q f(x,y) := {} & f(x+ay) + f(x-ay) + f(ax+y) + f(ax-y) \\ & - (a+a^2)\{f(x+y) + f(x-y)\} + 2(a+a^2-a^5-1)f(x) \\ & - 10(a^4-a^2)f(\sqrt[5]{xy^4}) - 20(a^3-a)f(\sqrt[5]{x^3y^2}) \end{aligned}$$

for all $x,y \in M$ with $a \neq 0, \pm 1$

Theorem 3. *Consider a mapping $\rho : M^2 \to [0,+\infty)$ such that*

$$\lim_{n\to\infty} \frac{1}{a^{5n}} a\rho\{a^n x, a^n y\} = 0, \tag{3.1}$$

and

$$\rho\{ax, ay\} \leq a^5 \psi \rho\{x, y\}, \forall x, y \in M, \tag{3.2}$$

for $\psi < 1$. *If* $f : M \to X_\xi$ *fulfill the inequality*

$$\xi(D_q f(x, y)) \leq \rho(x, y), \tag{3.3}$$

$\forall x, y \in M$. *Then* $Q_a : M \to X_\xi$ *a unique quintic mapping exists, such that*

$$\xi(Q_a(x) - f(x)) \leq \frac{1}{2a^5(1-\psi)} \rho(x, 0), \ \forall x \in M. \tag{3.4}$$

Where M *is linear space and* X_ξ *is modular space which is complete with Fatou property.*

Proof. Consider $N = \xi'$ and define ξ' on N as,

$$\xi'(q) =: \inf\{a > 0 : \xi(f(y)) \leq a\rho(x, 0), \forall x \in M\}.$$

One can easily prove ξ' is convex modular with Fatou property on N and $N_{\xi'}$ is $\xi-$complete, see [5]. Consider the function $\sigma : N_{\xi'} \to N_{\xi'}$ defined by

$$\sigma f(x) = \frac{1}{a^5} f(ax), \tag{3.5}$$

for all $x \in M$ and $a \in N_{\xi'}$. Let $p, r \in N_{\xi'}$ and $a \in [0, 1]$ with $\xi'(p - r) < a$. By definition of ξ', we get

$$\xi(p(x) - r(x)) \leq a\rho(x, 0) \tag{3.6}$$

for all $x \in M$. By (3.2) and (3.6), we obtain

$$\xi\left(\frac{p(ax)}{a^5} - \frac{r(ax)}{a^5}\right) \leq \frac{1}{a^5}\xi(p(ax) - r(ax)) \leq \frac{1}{a^5} a\rho(ax, 0) \leq a\psi\rho(x, 0),$$

for all $x \in M$. Hence, σ is a $\xi'-$contraction. From (3.3), we obtain

$$\xi\left(\frac{f(ax)}{a^5} - f(x)\right) \leq \frac{1}{2a^5}\rho(x, 0), \tag{3.7}$$

for all $x \in M$. Substituting x by ax in (3.7), we get

$$\xi\left(\frac{f(a^2x)}{a^5} - f(ax)\right) \leq \frac{\rho(ax, 0)}{2a^5}, \ \forall x \in M. \tag{3.8}$$

We obtain from (3.7) and (3.8) that

$$\begin{aligned} &\xi\left(\frac{f(a^2x)}{a^{10}} - f(x)\right) \\ &\leq \xi\left(\frac{f(a^2x)}{a^{10}} - \frac{f(ax)}{a^5}\right) + \xi\left(\frac{f(ax)}{a^5} - f(x)\right) \\ &\leq \frac{1}{2a^{10}}\rho(ax, 0) + \frac{1}{2a^5}\rho(x, 0), \ \forall x \in M. \end{aligned} \tag{3.9}$$

We get by induction,

$$\begin{aligned}\xi\left(\frac{f(a^n x)}{a^{5n}}-f(x)\right) &\leq \sum_{i=1}^{n}\frac{1}{2a^{5i}}\rho(a^{i-1}x,0)\\ &\leq \frac{1}{2\psi a^5}\rho(x,0)\sum_{i=1}^{n}\psi^i\\ &\leq \frac{1}{2a^5(1-\psi)}\rho(x,0),\ \forall x\in M. \end{aligned}\tag{3.10}$$

We obtain from (3.10),

$$\begin{aligned}\xi\left(\frac{f(a^n x)}{a^{5n}}-\frac{f(a^s x)}{a^{5s}}\right) &\leq \frac{1}{2}\xi\left(2\frac{f(a^n x)}{a^{5n}}-2f(x)\right)+\frac{1}{2}\xi\left(2\frac{f(a^s x)}{a^{5s}}-2f(x)\right)\\ &\leq \frac{\kappa}{2}\xi\left(\frac{f(a^n x)}{a^{5n}}-f(x)\right)+\frac{\kappa}{2}\xi\left(\frac{f(a^s x)}{a^{5s}}-f(x)\right)\\ &\leq \frac{\kappa}{2a^5(1-\psi)}\rho(x,0),\ \forall x\in M \end{aligned}\tag{3.11}$$

where $n,s\in\mathfrak{N}$. Thus

$$\xi'(\sigma^n f-\sigma^s f)\leq\frac{\kappa}{2a^5(1-\psi)},$$

hence the boundedness exists of an orbit of σ at f. $\{\tau^n f\}$ is $\xi'-$converges to $Q_a\in N_{\xi'}$ by Theorem 1.5 in [5]. By $\xi'-$contractivity of σ, we get

$$\xi'(\sigma^n f-\sigma Q_a)\leq\psi\xi'(\sigma^{n-1}f-Q_a).$$

Allowing $n\to\infty$ and by Fatou property of ξ', we get

$$\begin{aligned}\xi'(\sigma Q_a-Q_a) &\leq \lim_{n\to\infty}\inf\xi'(\sigma Q_a-\sigma^n f)\\ &\leq \psi\lim_{n\to\infty}\inf\xi'(Q_a-\sigma^{n-1}f)=0.\end{aligned}$$

Hence, Q_a is a fixed point of σ. In (3.3), changing (x,y) by $(a^n x,a^n y)$, we obtain

$$\xi\left(\frac{1}{a^{5n}}D_a f(a^n x,a^n y)\right)\leq\frac{1}{a^{5n}}\rho(a^n x,a^n y),\ \forall x,y\in M.\tag{3.12}$$

By Theorem 1 and allowing $n\to\infty$, Q_a is quintic and using (3.10), we arrive (3.4). For the uniqueness of Q_a, consider another quintic mapping $Q:M\to X_\xi$ satisfying (3.4). So that, Q is fixed point of σ.

$$\xi'(Q_a-Q)=\xi'(\sigma Q_a-\sigma Q)\leq\psi\xi'(Q_a-Q).\tag{3.13}$$

From (3.13), we get $Q_a=Q$. Hence the proof. □

Proof of following Corollaries 1 and 2 follows that, all normed space implies modular space of modular $\xi(x)=\|x\|$.

Corollary 1. *Assume* ρ *is a mapping from* M^2 *to* $[0,+\infty)$ *for*

$$\lim_{n\to\infty}\frac{1}{a^{5n}}\rho\{a^n x, a^n y\} = 0, \tag{3.14}$$

and

$$\rho\{ax, ay\} \leq a^5\psi\rho\{x,y\},\ \forall x,y \in M,\ \psi < 1. \tag{3.15}$$

If $f : M \to X$ *satisfies the condition for* X *is Banach space*

$$\|D_q f(x,y)\| \leq \rho(x,y), \tag{3.16}$$

$\forall x,y \in M$. *Then a unique* $Q_a : M \to X$ *quintic mapping exists, hence*

$$\|Q_a(x) - f(x)\| \leq \frac{\rho(x,0)}{2a^5(1-\psi)}, \tag{3.17}$$

for all $x \in M$.

Theorem 4. *Assume that,*

$$\lim_{n\to\infty}\kappa^{5n}\rho\left(\frac{x}{a^n},\frac{y}{a^n}\right) = 0, \tag{3.18}$$

where ρ *is a mapping from* M^2 *to* $[0,+\infty)$ *and*

$$\rho\left(\frac{x}{a},\frac{y}{a}\right) \leq \frac{\psi}{2a^5}\rho\{x,y\},\ \forall x,y \in M, \psi < 1. \tag{3.19}$$

If $f : M \to X_\xi$ *fulfills the inequality*

$$\xi(D_q f(x,y)) \leq \rho(x,y), \tag{3.20}$$

$\forall x,y \in M$. *Then a unique* $Q_a : M \to X_\xi$ *quintic mapping exists, such that*

$$\xi(Q_a(x) - f(x)) \leq \frac{\psi}{2a^5(1-\psi)}\rho(x,0),\ \forall x \in M. \tag{3.21}$$

Proof. Considering x by $\frac{x}{a}$ in (3.5) of Theorem 3 and proceeding similar to that of Theorem 3, we complete the proof. □

Corollary 2. *Assume that,*

$$\lim_{n\to\infty}\sigma^{5n}\rho\left(\frac{x}{a^n},\frac{y}{a^n}\right) = 0, \tag{3.22}$$

where ρ *is a mapping from* M^2 *to* $[0,+\infty)$ *and*

$$\rho\left(\frac{x}{a},\frac{y}{a}\right) \leq \frac{\psi}{a^5}\rho\{x,y\},\ \forall x,y \in M, \psi < 1. \tag{3.23}$$

If $f : M \to X$ *fulfills the inequality*

$$\|D_q f(x,y)\| \leq \rho(x,y), \tag{3.24}$$

$\forall x,y \in M$. *Then a unique* $Q_a : M \to X$ *quintic mapping exists, such that*

$$\|Q_a(x) - f(x)\| \leq \frac{\psi}{2a^5(1-\psi)}\rho(x,0),\ \forall x \in M. \tag{3.25}$$

Using Corollaries 1 and 2, the Hyers-Ulam and generalized Hyers-Ulam stabilities of (1.1) are obtain in the following corollaries.

Corollary 3. *Assume ρ is a mapping from M^2 to $[0,+\infty)$, X be a Banach space and $\varepsilon \geq 0$ be a real number such that*

$$\lim_{n\to\infty} \frac{1}{a^{5n}} \rho\{a^n x, a^n y\} = 0, \tag{3.26}$$

and

$$\rho\{ax, ay\} \leq a^5 \psi \rho\{x,y\},\ \forall x,y \in M, \psi < 1. \tag{3.27}$$

If $f : M \to X$ fulfills

$$\|D_q f(x,y)\| \leq \varepsilon, \tag{3.28}$$

$\forall x,y \in M$. Then a unique $Q_a : M \to X$ quintic mapping exists and defined by $Q_a(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^{5n}}$ so that

$$\|Q_a(x) - f(x)\| \leq \frac{\varepsilon}{2(a^5-1)}, \tag{3.29}$$

for all $x \in M$ and $a \neq 0, \pm 1$.

Corollary 4. *If $f : M \to X$ fulfills the inequality for M and X are linear space and Banach space, respectively.*

$$\|D_q f(x,y)\| \leq \varepsilon \left(\|x\|^p + \|y\|^q\right), \tag{3.30}$$

$\forall x,y \in M$ with $0 \leq p,q < 5$ or $p,q > 5$ for some $\varepsilon \geq 0$. Then a unique quintic mapping $Q_a : M \to X$ exists and defined by $Q_a(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^{5n}}$, so that

$$\|Q_a(x) - f(x)\| \leq \frac{\varepsilon}{|2(a^5 - a^p)|} \|x\|^p,\ \forall x \in M,\ a \neq 0, \pm 1. \tag{3.31}$$

For $p = 5$ in Corollary 4, we will provide an counter example to prove that the functional equation (1.1) is not stable.

Example 1. If $\psi : M \to X$ fulfills the inequality for M and X are linear space and Banach space, respectively. Let $k \geq 0$ be a real number such that

$$\psi(x) = \begin{cases} kx^5, & \text{if } |x| < 1, \\ k, & \text{otherwise}, \end{cases}$$

and a function $f : M \to X$ is defined by

$$f(x) = \sum_{s=0}^{\infty} \frac{\varphi(a^s x)}{(a^5)^s}$$

for all $x \in M$. Then f satisfies the functional inequality

$$\big| f(ax+y) + f(ax-y) + f(x+ay) + f(x-ay)$$

$$
\begin{aligned}
&-(a+a^2)\{f(x+y)+f(x-y)\}+2(a+a^2-a^5-1)f(x)\\
&-10(a^4-a^2)f(\sqrt[5]{xy^4})-20(a^3-a)f(\sqrt[5]{x^3y^2})\Big|\\
&\leq\left(\frac{2a^{10}-16a^{11}-6a^{12}+20a^{13}+10a^{14}-2a^{15}}{a^5-1}\right)k\left(|x|^5+|y|^5\right) \qquad (3.32)
\end{aligned}
$$

for all $x,y\in M$. Then there do not exist a quintic mapping $Q:M\to X$ and a constant $\alpha>0$ such that

$$|f(x)-Q(x)|\leq\alpha|x|^5 \quad \text{for all } x\in M. \qquad (3.33)$$

Proof. Now

$$|f(x)|\leq\sum_{s=0}^{\infty}\frac{|\psi(a^sx)|}{|a^{5s}|}=\sum_{s=0}^{\infty}\frac{k}{a^{5s}}=\frac{a^5\,k}{a^5-1}.$$

Therefore we see that f is bounded. We are going to prove that f satisfies (3.32).

If $x=y=0$, then (3.32) is trivial. If $|x|^5+|y|^5\geq\frac{1}{a^5}$, then the left-hand side of (3.32) is less than $\left(\frac{2a^5-16a^6-6a^7+20a^8+10a^9-2a^{10}}{n^5-1}\,k\right)$. Now suppose that $0<|x|^5+|y|^5<\dfrac{1}{a^5}$. Then there exists a positive integer t such that

$$\frac{1}{(a^5)^{t+1}}\leq|x|^5+|y|^5<\frac{1}{(a^5)^t}, \qquad (3.34)$$

so that

$$(a^5)^{t-1}x^5<\frac{1}{a^5},\ (a^5)^{t-1}y^5<\frac{1}{a^5}$$

and, consequently,

$$
\begin{aligned}
a^{t-1}(x),a^{t-1}(ax+y),\ a^{t-1}(ax-y),\ a^{t-1}(x+ay),\ a^{t-1}(x-ay),\ a^{t-1}(x+y),\\
a^{t-1}(x-y),\ a^{t-1}(\sqrt[5]{xy^4}),\ a^{t-1}(\sqrt[5]{x^3y^2})\in(-1,1).
\end{aligned}
$$

Therefore for each $s=0,1,\ldots,t-1$, we have

$$
\begin{aligned}
a^s(x),a^s(ax+y),\ a^s(ax-y),\ a^s(x+ay),\ a^s(x-ay),\ a^s(x+y),\\
a^s(x-y),\ a^s(\sqrt[5]{xy^4}),\ a^s(\sqrt[5]{x^3y^2})\in(-1,1).
\end{aligned}
$$

and

$$
\begin{aligned}
&\psi\big(a^s(ax+y)\big)+\psi\big(a^s(ax-y)\big)+\psi\big(a^s(x+ay)\big)+\psi\big(a^s(x-ay)\big)\\
&\quad-(a+a^2)\big\{\psi\big(a^s(x+y)\big)+\psi\big(a^s(x-y)\big)\big\}+2(a+a^2-a^5-1)\psi\big(a^sx\big)\\
&\quad-10(a^4-a^2)\psi\big(\sqrt[5]{a^s(xy^4)}\big)-20(a^3-a)\psi\big(\sqrt[5]{a^s(x^3y^2)}\big)=0
\end{aligned}
$$

for $s=0,1,\ldots,t-1$. From the definition of f and (3.34), we obtain that

$$
\begin{aligned}
&\Big|f(ax+y)+f(ax-y)+f(x+ay)+f(x-ay)\\
&\quad-(a+a^2)\{f(x+y)+f(x-y)\}+2(a+a^2-a^5-1)f(x)
\end{aligned}
$$

$$
\begin{aligned}
&-10(a^4-a^2)f(\sqrt[5]{xy^4})-20(a^3-a)f(\sqrt[5]{x^3y^2})\Big| \\
\leq &\sum_{s=0}^{\infty}\frac{1}{a^{5s}}\Big|\psi\big(a^s(ax+y)\big)+\psi\big(a^s(ax-y)\big)+\psi\big(a^s(x+ay)\big)+\psi\big(a^s(x-ay)\big) \\
&-(a+a^2)\big\{\psi\big(a^s(x+y)\big)+\psi\big(a^s(x-y)\big)\big\}+2(a+a^2-a^5-1)\psi\big(a^sx\big) \\
&-10(a^4-a^2)\psi\big(a^s\sqrt[5]{xy^4}\big)-20(a^3-a)\psi\big(a^s\sqrt[5]{x^3y^2}\big)\Big| \\
\leq &\sum_{s=t}^{\infty}\frac{1}{a^{5s}}\Big|\psi\big(a^s(ax+y)\big)+\psi\big(a^s(ax-y)\big)+\psi\big(a^s(x+ay)\big)+\psi\big(a^s(x-ay)\big) \\
&-(a+a^2)\big\{\psi\big(a^s(x+y)\big)+\psi\big(a^s(x-y)\big)\big\}+2(a+a^2-a^5-1)\psi\big(a^sx\big) \\
&-10(a^4-a^2)\psi\big(a^s\sqrt[5]{xy^4}\big)-20(a^3-a)\psi\big(a^s\sqrt[5]{x^3y^2}\big)\Big| \\
&\leq \sum_{s=t}^{\infty}\frac{1}{a^{5s}}(-2a^5+10a^4+20a^3-6a^2-16a+2)\,k \\
&= \frac{(-2a^{10}+10a^9+20a^8-6a^7-16a^6+2a^5)k}{a^5-1}\times\frac{1}{a^{5t}} \\
&= \frac{(-2a^{15}+10a^{14}+20a^{13}-6a^{12}-16a^{11}+2a^{10})\,k}{a^5-1}\big(|x|^5+|y|^5\big).
\end{aligned}
$$

Thus f satisfies (3.32) for all $x,y\in M$ with $0<|x|^5+|y|^5<\frac{1}{a^5}$.

In Corollary 4, our claim is to prove (1.1) is not stable for $p=5$. Since f is continuous and bounded for all $x\in M$, Q is continuous and bounded at the origin. In view of Corollary 4, $Q(x)$ must have the form $Q(x)=tx^5$ for any x in M. Thus we obtain

$$|f(x)|\leq(\alpha+|t|)\times|x|^5. \tag{3.35}$$

But $nk>\alpha+|t|$, n is positive integer.

If $x\in\big(0,\frac{1}{a^{n-1}}\big)$, then $a^sx\in(0,1)$ for all $s=0,1,\ldots,n-1$. Therefore, we get

$$f(x)=\sum_{s=0}^{\infty}\frac{\psi(a^sx)}{a^{5s}}\geq\sum_{s=0}^{n-1}\frac{k(a^sx)^5}{a^{5s}}=nkx^5>(\alpha+|t|)\times x^5,$$

which contradicts (3.35). Hence (1.1) is not stable if $p=5$ in the inequality (3.30). □

4. Stability of functional equation(1.2)

In this section, we determine the generalized Hyers-Ulam stability concerning the generalized Euler-Lagrange radical sextic functional equation (1.2) in modular spaces by using fixed point theory. In this section we will not provide the proof of theorems

and corollaries, since it is similar to that of proof of Theorems and Corollaries in section 3.

For mapping $\rho : M \to X_\xi$, consider

$$\begin{aligned} D_s f(x,y) &:= f(ax+y)+f(ax-y)+f(x+ay)+f(x-ay) \\ &-(a+a^2)\{f(x+y)+f(x-y)\}+2(a+a^2-a^6-1)\{f(x)+f(y)\} \\ &-30(a^4-a)\{f(\sqrt[6]{x^4y^2})+f(\sqrt[6]{x^2y^4})\} \end{aligned}$$

for all $x,y \in M$ with $a \neq 0, \pm 1$

Theorem 5. *Consider a mapping $\rho : M^2 \to [0,+\infty)$ such that*

$$\lim_{n\to\infty} \frac{1}{a^{6n}} a\rho\{a^n x, a^n y\} = 0, \tag{4.1}$$

and

$$\rho\{ax,ay\} \leq a^6 \psi \rho\{x,y\}, \forall x,y \in M, \tag{4.2}$$

for $\psi < 1$. If $f : M \to X_\xi$ fulfill the inequality

$$\xi(D_s f(x,y)) \leq \rho(x,y), \tag{4.3}$$

$\forall x,y \in M$. Then $S_a : M \to X_\xi$ a unique sextic mapping exists, such that

$$\xi(S_a(x) - f(x)) \leq \frac{1}{2a^6(1-\psi)} \rho(x,0), \ \forall x \in M. \tag{4.4}$$

Where M is linear space and X_ξ is modular space which is complete with Fatou property.

All normed space implies modular space of modular $\xi(x) = \|x\|$ gives the following Corollaries 5 and 6

Corollary 5. *Assume ρ is a mapping from M^2 to $[0,+\infty)$ for*

$$\lim_{n\to\infty} \frac{1}{a^{6n}} \rho\{a^n x, a^n y\} = 0, \tag{4.5}$$

and

$$\rho\{ax,ay\} \leq a^6 \psi \rho\{x,y\}, \ \forall x,y \in M, \ \psi < 1. \tag{4.6}$$

If $f : M \to X$ satisfies the condition for X is Banach space

$$\|D_s f(x,y)\| \leq \rho(x,y), \tag{4.7}$$

$\forall x,y \in M$. Then a unique $S_a : M \to X$ sextic mapping exists, hence

$$\|S_a(x) - f(x)\| \leq \frac{\rho(x,0)}{2a^6(1-\psi)}, \tag{4.8}$$

for all $x \in M$.

Theorem 6. *Assume that,*

$$\lim_{n\to\infty} \kappa^{6n}\rho\left(\frac{x}{a^n},\frac{y}{a^n}\right)=0, \tag{4.9}$$

where ρ *is a mapping from* M^2 *to* $[0,+\infty)$ *and*

$$\rho\left(\frac{x}{a},\frac{y}{a}\right)\leq \frac{\psi}{2a^6}\rho\{x,y\},\ \forall x,y\in M,\psi<1. \tag{4.10}$$

If $f:M\to X_\xi$ *fulfills the inequality*

$$\xi\left(D_s f(x,y)\right)\leq \rho(x,y), \tag{4.11}$$

$\forall x,y\in M$. *Then a unique* $S_a:M\to X_\xi$ *quintic mapping exists, such that*

$$\xi(S_a(x)-f(x))\leq \frac{\psi}{2a^6(1-\psi)}\rho(x,0),\ \forall x\in M. \tag{4.12}$$

Corollary 6. *Assume that,*

$$\lim_{n\to\infty} \sigma^{6n}\rho\left(\frac{x}{a^n},\frac{y}{a^n}\right)=0, \tag{4.13}$$

where ρ *is a mapping from* M^2 *to* $[0,+\infty)$ *and*

$$\rho\left(\frac{x}{a},\frac{y}{a}\right)\leq \frac{\psi}{a^6}\rho\{x,y\},\ \forall x,y\in M,\psi<1. \tag{4.14}$$

If $f:M\to X$ *fulfills the inequality*

$$\|D_s f(x,y)\|\leq \rho(x,y), \tag{4.15}$$

$\forall x,y\in M$. *Then a unique* $S_a:M\to X$ *sextic mapping exists, such that*

$$\|S_a(x)-f(x)\|\leq \frac{\psi}{2a^6(1-\psi)}\rho(x,0),\ \forall x\in M. \tag{4.16}$$

The Hyers-Ulam and generalized Hyers-Ulam stabilities are obtain in the following corollaries using Corollaries 5 and 6.

Corollary 7. *Assume* ρ *is a mapping from* M^2 *to* $[0,+\infty)$, X *be a Banach space and* $\varepsilon\geq 0$ *be a real number such that*

$$\lim_{n\to\infty}\frac{1}{a^{6n}}\rho\{a^n x,a^n y\}=0, \tag{4.17}$$

and

$$\rho\{ax,ay\}\leq a^6\psi\rho\{x,y\},\ \forall x,y\in M,\psi<1. \tag{4.18}$$

If $f:M\to X$ *fulfills*

$$\|D_s f(x,y)\|\leq \varepsilon, \tag{4.19}$$

$\forall x, y \in M$. *Then a unique* $S_a : M \to X$ *sextic mapping exists and defined by* $S_a(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^{6n}}$ *so that*

$$\|S_a(x) - f(x)\| \leq \frac{\varepsilon}{2(a^6 - 1)}, \tag{4.20}$$

for all $x \in M$ *and* $a \neq 0, \pm 1$.

Corollary 8. *If* $f : M \to X$ *fulfills the inequality for* M *and* X *are linear space and Banach space, respectively.*

$$\|D_s f(x,y)\| \leq \varepsilon \left(\|x\|^p + \|y\|^q\right), \tag{4.21}$$

$\forall x, y \in M$ *with* $0 \leq p, q < 6$ *or* $p, q > 6$ *for some* $\varepsilon \geq 0$. *Then a unique sextic mapping* $S_a : M \to X$ *exists and defined by* $S_a(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^{6n}}$, *so that*

$$\|S_a(x) - f(x)\| \leq \frac{\varepsilon}{|2(a^6 - a^p)|} \|x\|^p, \ \forall x \in M, \ a \neq 0, \pm 1. \tag{4.22}$$

In Corollary 8, we are providing counter example for (1.2) is not stable for $p = 6$.

Example 2. If $\psi : M \to X$ fulfills the inequality for M and X are linear space and Banach space, respectively. Let $k \geq 0$ be a real number such that

$$\psi(x) = \begin{cases} kx^6, & \text{if } |x| < 1, \\ k, & \text{otherwise}, \end{cases}$$

and a function $f : M \to X$ is defined by

$$f(x) = \sum_{s=0}^{\infty} \frac{\varphi(a^s x)}{(a^6)^s}$$

for all $x \in M$. Then f satisfies the functional inequality

$$\begin{aligned} &\big| f(ax+y) + f(ax-y) + f(x+ay) + f(x-ay) \\ &\quad - (a+a^2)\{f(x+y) + f(x-y)\} + 2(a+a^2-a^6-1)\{f(x)+f(y)\} \\ &\quad - 30(a^4-a)\{f(\sqrt[6]{x^4y^2}) + f(\sqrt[6]{x^2y^4})\}\big| \\ &\quad \leq \frac{(-54a^{13} + 6a^{14} - 4a^{18} + 60a^{16})\, k}{a^6 - 1} \left(|x|^6 + |y|^6\right) \end{aligned} \tag{4.23}$$

for all $x, y \in M$. Then there do not exist a sextic mapping $S : M \to X$ and a constant $\alpha > 0$ such that

$$|f(x) - S(x)| \leq \alpha |x|^6 \quad \text{for all } \ x \in M. \tag{4.24}$$

Proof. Now

$$|f(x)| \leq \sum_{s=0}^{\infty} \frac{|\psi(a^s x)|}{|a^{6s}|} = \sum_{s=0}^{\infty} \frac{k}{a^{6s}} = \frac{a^6\, k}{a^6 - 1}.$$

Therefore we see that f is bounded. We are going to prove that f satisfies (4.23).

If $x = y = 0$, then (4.23) is trivial. If $|x|^6 + |y|^6 \geq \frac{1}{a^6}$, then the left-hand side of (4.23) is less than $\left(\frac{-54a^7+6a^8-4a^{12}+60a^{10}}{a^6-1}\, k\right)$. Now suppose that $0 < |x|^6 + |y|^6 < \dfrac{1}{a^6}$. Then there exists a positive integer t such that

$$\frac{1}{(a^6)^{t+1}} \leq |x|^6 + |y|^6 < \frac{1}{(a^6)^t}, \tag{4.25}$$

so that

$$(a^6)^{t-1}x^6 < \frac{1}{a^6},\ (a^6)^{t-1}y^6 < \frac{1}{a^6}$$

and, consequently,

$$\begin{aligned} &a^{t-1}(x), a^{t-1}(y), a^{t-1}(ax+y),\ a^{t-1}(ax-y),\ a^{t-1}(x+ay),\\ &\quad a^{t-1}(x-ay),\ a^{t-1}(x+y),\ a^{t-1}(x-y),\\ &\quad a^{t-1}(\sqrt[6]{x^4y^2}),\ a^{t-1}(\sqrt[6]{x^2y^4}) \in (-1,1). \end{aligned}$$

Therefore for each $s = 0, 1, \ldots, t-1$, we have

$$\begin{aligned} a^s(x), a^s(y), a^s(ax+y),\ a^s(ax-y),\ a^s(x+ay),\ a^s(x-ay),\ a^s(x+y),\\ a^s(x-y),\ a^s(\sqrt[6]{x^4y^2}),\ a^s(\sqrt[6]{x^2y^4}) \in (-1,1). \end{aligned}$$

and

$$\begin{aligned} &\psi\big(a^s(ax+y)\big) + \psi\big(a^s(ax-y)\big) + \psi\big(a^s(x+ay)\big) + \psi\big(a^s(x-ay)\big)\\ &\quad - (a+a^2)\big\{\psi\big(a^s(x+y)\big) + \psi\big(a^s(x-y)\big)\big\}\\ &\quad + 2(a+a^2-a^6-1)\psi\big(a^sx+a^sy\big)\\ &\quad - 30(a^4-a)\big\{\psi\big(a^s\sqrt[6]{x^4y^2}\big) + \psi\big(a^s\sqrt[6]{x^2y^4}\big)\big\} = 0 \end{aligned}$$

for $s = 0, 1, \ldots, t-1$. From the definition of f and (3.34), we obtain that

$$\begin{aligned} &\Big| f(ax+y) + f(ax-y) + f(x+ay) + f(x-ay)\\ &\quad - (a+a^2)\{f(x+y) + f(x-y)\}\\ &\quad + 2(a+a^2-a^6-1)\big\{f(x)+f(y)\big\}\\ &\quad - 30(a^4-a)\big\{f(\sqrt[6]{x^4y^2}) + f(\sqrt[6]{x^2y^4})\big\}\Big|\\ &\leq \sum_{s=0}^{\infty} \frac{1}{a^{6s}} \Big| \psi\big(a^s(ax+y)\big) + \psi\big(a^s(ax-y)\big) + \psi\big(a^s(x+ay)\big)\big)\\ &\quad + \psi\big(a^s(x-ay) - (a+a^2)\big\{\psi\big(a^s(x+y)\big) + \psi\big(a^s(x-y)\big)\big\}\\ &\quad + 2(a+a^2-a^6-1)\psi\big(a^sx+a^sy\big)\\ &\quad - 30(a^4-a)\big\{\psi\big(a^s\sqrt[6]{x^4y^2}\big) + \psi\big(a^s\sqrt[6]{x^2y^4}\big)\big\}\Big| \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{s=t}^{\infty} \frac{1}{a^{6s}} \Big| \psi\big(a^s(ax+y)\big) + \psi\big(a^s(ax-y)\big) + \psi\big(a^s(x+ay)\big) \\
&\quad + \psi\big(a^s(x-ay)\big) - (a+a^2)\big\{\psi\big(a^s(x+y)\big) + \psi\big(a^s(x-y)\big)\big\} \\
&\quad + 2(a+a^2-a^6-1)\psi\big(a^s x + a^s y\big) \\
&\quad - 30(a^4-a)\big\{\psi\big(a^s \sqrt[6]{x^4y^2}\big) + \psi\big(a^s \sqrt[6]{x^2y^4}\big)\Big| \\
&\leq \sum_{s=t}^{\infty} \frac{1}{a^{6s}} (-54a + 6a^2 - 4a^6 + 60a^4)\, k \\
&= \frac{(-54a^7 + 6a^8 - 4a^{12} + 60a^{10})k}{a^6-1} \times \frac{1}{a^{6t}} \\
&= \frac{(-54a^{13} + 6a^{14} - 4a^{18} + 60a^{16})\, k}{a^6-1} \big(|x|^6 + |y|^6\big).
\end{aligned}
$$

Thus f satisfies (4.23) for all $x, y \in M$ with $0 < |x|^6 + |y|^6 < \frac{1}{a^6}$.

The remaining part of proof is similar to that of previous Example 1. □

5. Conclusion

Mainly, we introduced new generalized Euler-Lagrange radical sextic and quintic functional equations with their general solution and investigated their generalized Hyers-Ulam stability in modular spaces by using fixed point theory with suitable counter examples.

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PATA TYPE BEST PROXIMITY POINT RESULTS IN METRIC SPACES

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Abstract. The aim of this paper is to initiate the study of best proximity point and optimal coincidence point results of some $\alpha-$Pata-proximal admissible contraction of type-I and type-II in the framework of complete metric space. Some examples are presented to support the results obtained herein. Our results unify, extend and generalize various existing results in literature.

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Keywords: metric space, $\alpha-$Pata-proximal admissible contraction of type-I and type-II, generalized $\alpha-$Pata-proximal contraction, optimal coincidence best proximity point

1. Introduction and preliminaries

Banach contraction principle states that if T is a self mapping on a complete metric space (X,d), then T has a unique fixed point in X provided that for all $x,y \in X$, there exists $k \in [0,1)$ such that the following condition hold:

$$d(Tx,Ty) \leq kd(x,y). \tag{1.1}$$

A self mapping T satisfying the above inequality is said to be a contraction mapping.

A self mapping T on (X,d) is called Kannan contraction if for all $x,y \in X$, there exists $\gamma \in [0,1)$ such that

$$d(Tx,Ty) \leq \frac{\gamma}{2}[d(x,Tx)+d(y,Ty)] \tag{1.2}$$

Every Kannan contraction mapping on a complete metric space has a unique fixed point.

A self mapping T on (X,d) is called Boyd-Wong contraction if for all $x,y \in X$, we have

$$d(Tx,Ty) \leq \phi(d(x,y)),$$

where $\phi : [0,\infty) \to [0,\infty)$ is a continuous function satisfying $\phi(t) < t$ for every $t > 0$.

Every Boyd-Wong contraction mapping on a complete metric space has a unique fixed point.

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Suppose that x_0 is an arbitrarily fixed element in (X,d). Define $\|x\| = d(x,x_0)$. Let $\Lambda \geq 0$, $\lambda \geq 1$ and $\mu \in [0,\lambda]$ be fixed constants. Let $\psi : [0,1] \to [0,\infty)$ be an increasing function which is continuous at 0 and $\psi(0) = 0$.

A self mapping T on (X,d) is called Pata contraction ([10]) if for all $x,y \in X$, we have

$$d(Tx,Ty) \leq (1-\varepsilon)d(x,y) + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+\|x\|+\|y\|]^{\mu}, \tag{1.3}$$

for every $\varepsilon \in [0,1]$.

Every Pata contraction mapping on a complete metric space has a unique fixed point ([10]).

For comparison of Banach contraction principle and Boyd-Wong contraction theorem with Pata contraction theorem, we refer to ([10]).

A self mapping T on (X,d) is called Kanan Pata type contraction ([2]), if for all $x,y \in X$, we have

$$\begin{aligned} d(Tx,Ty) \leq \frac{(1-\varepsilon)}{2}[d(x,Tx)+d(y,Ty)] \\ + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+\|x\|+\|y\|+\|Tx\|+\|Ty\|]^{\mu}, \end{aligned} \tag{1.4}$$

for every $\varepsilon \in [0,1]$.

Every Kanan Pata type contraction mapping on a complete metric space has a unique fixed point ([2]).

It is known that Kanan Pata type contraction mapping theorem extends Kanan contraction theorem.

Balasubramanian ([1]) studied fixed point results of Chakraborty-Samanta contraction mapping (1.4) in the framework of complete cone metric space.

A self mapping T on (X,d) is called Chatterjea Pata type contraction ([8]), if for all $x,y \in X$, we have

$$\begin{aligned} &d(Tx,Ty) \leq \\ &\qquad \frac{(1-\varepsilon)}{2}[d(x,Ty)+d(y,Tx)] + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+\|x\|+\|y\|+\|Tx\|+\|Ty\|]^{\mu}, \end{aligned}$$

for every $\varepsilon \in [0,1]$.

Every Chatterjea Pata type contraction mapping on a complete metric space has a unique fixed point ([8]). For related results of Chatterjea Pata type contraction mapping and their comparison with comparable existing results in the literature, we refer to ([7], [8]).

Let (A,B) be a pair of nonempty subsets of a metric space (X,d). Define

$$A_0 = \{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : d(x,y) = d(A,B) \text{ for some } x \in A\},$$

where

$$d(A,B) = \inf\{d(x,y) : x \in A \text{ and } y \in B\}.$$

If A intersects with B, then $A \cap B$ is contained in both A_0 and B_0.

A set B is said to be approximate compact with respect to A if every sequence $\{y_n\}$ in B satisfying $d(x, y_n) \longrightarrow d(x, B)$ has a convergent subsequence for some $x \in A$, where $d(x, A) = \inf_{a \in A} d(x, a)$.

It is evident that every set is approximate compact with respect to itself. Further, it can be seen that if A is compact and B is approximate compact with respect to A, then the sets A_0 and B_0 are nonempty.

The solution x^* of the following optimization problem

$$\inf_{x \in A} d(x, Tx),$$

is called *approximate fixed point* of T or *approximate solution* of an operator equation $x = Tx$.

Fan ([5]) studied the existence of approximate fixed points of a continuous mapping on *a nonempty compact convex set X in a Hausdorff locally convex topological vector space E*.

An element x^* in A is called a best proximity point of T if $d(x^*, Tx^*) = d(A, B)$, where $A \cap B = \phi$. Note that $d(A, B) = 0$, if $A \cap B \neq \phi$. Clearly, if $A = B$, then best proximity point of T becomes a fixed point of T.

Let $T : A \to B$ and $g : A \longrightarrow A$. An element x^* in A is said to be optimal coincidence point of a pair (g, T), if $d(gx^*, Tx^*) = d(A, B)$ holds.

For best proximity point results of certain mappings, we refer to ([3, 4, 9, 12]) and reference mentioned therein.

From now onward we assume that A, B are nonempty closed subsets of a complete metric space (X, d).

Definition 1 ([13])**.** A mapping $f : X \to X$ is an *expansive mapping* if

$$d(fx, fy) \geq qd(x, y),$$

for all $q > 1$ and $x, y \in X$.

Definition 2. Let $\alpha : X \times X \to [0, \infty)$. A self mapping T on X is said to be $\alpha-$admissible if

$$\alpha(x, y) \geq 1 \text{ implies that } \alpha(Tx, Ty) \geq 1.$$

Definition 3. Let $\alpha : X \times X \to [0, \infty)$. A sequence $\{x_n\}$ in X is said to be $\alpha-$regular if $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} x_n = x \in X$ imply that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k.

Definition 4. Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. A mapping T is said to be $\alpha-$proximal admissible if

$$\left.\begin{array}{r} \alpha(x, y) \geq 1 \\ d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array}\right\} \text{imply} \alpha(u, v) \geq 1,$$

for all $x, y, u, v \in A$.

Definition 5. Let $\alpha : A \times A \to [0, \infty)$. A mapping $T : A \longrightarrow B$ is said to be a $\alpha-$ Pata-proximal admissible contraction of:

(a): type-I if for any u, v, x and y in A, there exist constants $\Lambda \geq 0$, $\lambda \geq 1$ and $\mu \in [0, \lambda]$ such that the following holds:

$$\left.\begin{array}{r} \alpha(x,y) \geq 1 \\ d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{array}\right\} \text{ imply } \alpha(u,v) \geq 1$$

and

$$\alpha(x,y)d(u,v) \leq \frac{(1-\varepsilon)}{2} d(x,y) + \Lambda \varepsilon^{\lambda} \psi(\varepsilon)[1 + \|x\| + \|y\| + \|v\|]^{\mu},$$

(b): type-II if for any u, v, x and y in A, there exist constants $\Lambda \geq 0$, $\lambda \geq 1$ and $\mu \in [0, \lambda]$ such that the following holds:

$$\left.\begin{array}{r} \alpha(x,y) \geq 1 \\ d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{array}\right\} \text{ imply } \alpha(u,v) \geq 1$$

and

$$\alpha(x,y)d(Tu,Tv) \leq \frac{(1-\varepsilon)}{2} d(Tx,Ty) + \Lambda \varepsilon^{\lambda} \psi(\varepsilon)[1 + \|Tx\| + \|Ty\| + \|Tv\|]^{\mu}.$$

for every $\varepsilon \in [0,1]$, where $\psi : [0,1] \to [0, \infty)$ is an increasing function which is continuous at 0 and $\psi(0) = 0$.

Definition 6. Let $T : A \longrightarrow B$, $\alpha : A \times A \to [0, \infty)$ and $g : A \longrightarrow A$. A pair (g, T) is said to be generalized $\alpha-$Pata-proximal contraction if for any u, v, x and y in A, there exist constants $\Lambda \geq 0$, $\lambda \geq 1$ and $\mu \in [0, \lambda]$ such that the following holds

$$\left.\begin{array}{r} \alpha(x,y) \geq 1 \\ d(gu,Tx) = d(A,B) \\ d(gv,Ty) = d(A,B) \end{array}\right\} \text{ imply } \alpha(gu,gv) \geq 1$$

and

$$\alpha(x,y)d(gu,gv) \leq \frac{(1-\varepsilon)}{2} d(x,y) + \Lambda \varepsilon^{\lambda} \psi(\varepsilon)[1 + \|x\| + \|y\| + \|v\|]^{\mu}$$

for every $\varepsilon \in [0,1]$, where $\psi : [0,1] \to [0, \infty)$ is an increasing function, which is continuous at 0 and $\psi(0) = 0$.

Note that if $g = I_A$ (identity mapping on A), then generalized $\alpha-$Pata-proximal contraction becomes $\alpha-$ Pata-proximal admissible contraction of type-I.

From now and onward, we use the notation Δ for a set $\{(x_n, y_n) \in A_0 \times A_0 :$ either $x_n \preceq y_n$ or $y_n \preceq x_n$, for all $n \in \mathbb{N}$ $\}$.

We start with the following result.

Lemma 1. *Suppose that T is $\alpha-$proximal admissible and $T(A_0) \subseteq B_0$, where A_0 is a nonempty set in A. If there exists x_0 and x_1 in A_0 satisfying $d(x_1, Tx_0) = d(A,B)$ and $\alpha(x_0, x_1) \geq 1$, then there exists a sequence $\{x_n\} \subset A_0$ such that*

$$d(x_{n+1}, Tx_n) = d(A,B) \text{ and } \alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N}. \tag{1.5}$$

Proof. Since A_0 is nonempty and $x_0, x_1 \in A_0$ satisfying $d(x_1, Tx_0) = d(A,B)$ and $\alpha(x_0, x_1) \geq 1$. As $Tx_1 \in T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A,B)$. Thus

$$\begin{aligned} \alpha(x_0, x_1) &\geq 1 \\ d(x_1, Tx_0) &= d(A,B) \text{ and} \\ d(x_2, Tx_1) &= d(A,B), \end{aligned}$$

implies that $\alpha(x_1, x_2) \geq 1$. Continuing this way, we obtain a sequence $\{x_n\} \subset A_0$ and it satisfies the condition (1.5). □

Definition 7. A sequence $\{x_n\} \subset A_0$ satisfying the condition (1.5) is called *proximal admissible Picard sequence provided that* $\alpha(x_n, x_1) \geq 1$ and $\alpha(x_n, x_0) \geq 1$ for all n.

Definition 8. A set A_0 is called proximal $T-$orbitally complete if and only if every Cauchy *proximal admissible Picard sequence* in A_0 converges to an element in A_0.

We also need the following lemma in the sequel.

Lemma 2 ([6])**.** *Let A and B be nonempty closed subsets of a metric space (X,d). Suppose A_0 is a nonempty set and B is approximately compact with respect to A. Then the set A_0 is closed.*

Proposition 1. *A self mapping $g : A \to A$ is said to satisfy α_R-property if there exist a mapping $\alpha : A \times A \to [0, \infty)$ such that*

$$\alpha(gx, gy) \geq 1 \text{ implies that } \alpha(x,y) \geq 1,$$

for all $x, y \in A$.

2. Best proximity points of $\alpha-$ Pata-proximal admissible contraction mappings

We start with the following lemma.

Lemma 3. *Let $T : A \to B$ be a continuous and $\alpha-$ Pata-proximal admissible contraction of type-II mapping with $A_0 \neq \varnothing$ and $T(A_0) \subseteq B_0$. Then A_0 is proximal $T-$orbitally complete.*

Proof. Let $x_0 \in A_0$ and $\{x_n\}$ be a Cauchy *proximal admissible Picard sequence*. As (X,d) is complete and A is closed, there exist some x^* in A such that $\lim_{n\to\infty} d(x_n,x^*) = 0$. By given assumption, we have

$$\begin{aligned} &d(x_n,Tx_{n-1}) = d(A,B) \text{ and} \\ &d(x_{n+1},Tx_n) = d(A,B) \text{ with } \alpha(x_{n-1},x_n) \geq 1, \end{aligned}$$

for all $n \in \mathbb{N}$. Since T is a $\alpha-$ Pata-proximal admissible contraction of type-II, we have

$$\begin{aligned} d(Tx_n,Tx_{n+1}) &\leq \alpha(x_{n-1},x_n)d(Tx_n,Tx_{n+1}) \\ &\leq \frac{(1-\varepsilon)}{2}d(Tx_{n-1},Tx_n) \\ &\quad + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+\|Tx_{n-1}\|+\|Tx_n\|+\|Tx_{n+1}\|]^{\mu} \\ &< d(Tx_{n-1},Tx_n) \\ &\quad + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+\|Tx_{n-1}\|+\|Tx_n\|+\|Tx_{n+1}\|]^{\mu}. \end{aligned}$$

On taking limit as ε approaches to 0, we have

$$d(Tx_n,Tx_{n+1}) \leq d(Tx_{n-1},Tx_n),$$

which implies that $\{d(Tx_{n-1},Tx_n)\}$ is a decreasing sequence and hence is convergent. Let $\lim_{n\to\infty} d(Tx_{n+1},Tx_n) = r \geq 0$. We have to show that $r = 0$. First we show that $\{d(Tx_n,x_0)\}$ is bounded above by some constant c. Note that

$$\begin{aligned} d(Tx_{n+1},Tx_n) &\leq d(Tx_n,Tx_{n-1}) \\ &\leq \cdots \\ &\leq d(Tx_1,Tx_0) \\ &\leq d(Tx_1,x_0)+d(x_0,Tx_0), \end{aligned}$$

for all $n = 0,1,2,\ldots$. Let

$$\|Tx_0\| = c_0 \text{ and } c_1 = \|Tx_1\|$$

for some c_0, $c_1 \in \mathbb{R}^+$. Then

$$\begin{aligned} c_n = d(Tx_n,x_0) &\leq d(Tx_n,Tx_{n+1})+d(Tx_{n+1},Tx_1)+d(Tx_1,x_0) \\ &\leq (d(Tx_1,x_0)+d(x_0,Tx_0))+d(Tx_{n+1},Tx_1)+d(Tx_1,x_0) \\ &= 2d(Tx_1,x_0)+d(x_0,Tx_0)+d(Tx_{n+1},Tx_1) \\ &\leq 2d(Tx_1,x_0)+d(x_0,Tx_0)+\alpha(x_n,x_0)d(Tx_{n+1},Tx_1) \\ &\leq 2d(Tx_1,x_0)+d(x_0,Tx_0)+\frac{(1-\varepsilon)}{2}d(Tx_n,Tx_0) \\ &\quad + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+\|Tx_0\|+\|Tx_1\|+\|Tx_n\|]^{\mu} \end{aligned}$$

$$\leq 2c_1 + c_0 + \frac{(1-\varepsilon)}{2}c_n + \frac{(1-\varepsilon)}{2}c_0 + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+c_0+c_1+c_n]^{\mu}$$
$$\leq 2c_1 + 2c_0 + c_n - \varepsilon c_n + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)c_n^{\mu}[\frac{1+c_0+c_1}{c_n}+1]^{\mu}$$

which further implies that

$$\varepsilon c_n \leq 2c_1 + 2c_0 + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)c_n^{\lambda}[\frac{1+c_0+c_1}{c_n}+1]^{\mu},$$

as $\mu \leq \lambda$. Suppose that $a_n = \Lambda[\frac{1+c_0+c_1}{c_n}+1]^{\mu}$ and $b = 2(c_1+c_0)$. Then, the above inequality becomes

$$\varepsilon c_n \leq \varepsilon^{\lambda}\psi(\varepsilon)c_n^{\lambda}a_n + b,$$

for some $a_n, b > 0$. If there is a divergent subsequence $\{c_{n_i}\}$, correspondingly we have a subsequence $\{a_{n_i}\}$ such that $a_{n_i} \to \Lambda$. If we choose $\varepsilon_{n_i} = \dfrac{(1+b)}{c_{n_i}}$, then

$$1 \leq \psi(\varepsilon_i)(1+b)^{\lambda}a_{n_i} \to 0,$$

gives a contradiction. Now, we have to show that $\{Tx_n\}$ is a Cauchy sequence. For this, we prove that

$$d(Tx_{n+m}, Tx_n) \leq C\omega_n(\alpha),$$

where $C = \sup_{n\in\mathbb{N}} \Lambda(1+3c_n)^{\mu} < \infty$ and $\omega_n(\lambda) = (\dfrac{\lambda}{n})^{\lambda}\sum_{k=1}^{n}\psi(\dfrac{\lambda}{k})$.
Let m be fixed, define

$$p_n = n^{\lambda}d(Tx_{n+m}, Tx_n).$$

Note that

$$\begin{aligned}
p_{n+1} &= (n+1)^{\lambda}d(Tx_{n+m+1}, Tx_{n+1}) \\
&\leq (n+1)^{\lambda}\alpha(x_{n+m}, x_n)d(Tx_{n+m+1}, Tx_{n+1}) \\
&\leq (n+1)^{\lambda}\frac{(1-\varepsilon)}{2}d(Tx_{n+m}, Tx_n) \\
&\quad + (n+1)^{\lambda}\Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+\|Tx_{n+m}\|+\|Tx_n\|+\|Tx_{n+1}\|]^{\mu}] \\
&\leq (n+1)^{\lambda}(1-\varepsilon)d(Tx_{n+m}, Tx_n) \\
&\quad + (n+1)^{\lambda}\Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+\|Tx_{n+m}\|+\|Tx_n\|+\|Tx_{n+1}\|]^{\mu}].
\end{aligned}$$

Choose

$$\varepsilon = 1 - (\frac{n}{n+1})^{\lambda} = [1-(1-\frac{1}{n+1})^{\lambda}] \leq \frac{\lambda}{n+1}.$$

Then, we have

$$\begin{aligned}
p_{n+1} \leq\ & (n+1)^{\lambda}((\frac{n}{n+1})^{\lambda})d(Tx_{n+m}, Tx_n) \\
& + (n+1)^{\lambda}\Lambda(\frac{\lambda}{n+1})^{\lambda}\psi(\frac{\lambda}{n+1})[1+3c_n]^{\mu}
\end{aligned}$$

$$\leq n^{\lambda} d(Tx_{n+m}, Tx_n) + \Lambda(\lambda)^{\lambda}\psi(\frac{\lambda}{n+1})[1+3c_n]^{\mu}.$$

Now, we have

$$\begin{aligned} p_{n+1} &\leq p_n + C(\lambda)^{\lambda}\psi(\frac{\lambda}{n+1}) \\ &\leq p_{n-1} + C(\lambda)^{\lambda}\psi(\frac{\lambda}{n}) + C(\lambda)^{\lambda}\psi(\frac{\lambda}{n+1}) \\ &\cdots \\ &\leq p_0 + C(\lambda)^{\lambda}[\psi(\frac{\lambda}{1}) + \psi(\frac{\lambda}{2}) + \cdots + \psi(\frac{\lambda}{n+1})]. \end{aligned}$$

Since $p_0 = 0$, this gives

$$p_n \leq C(\lambda)^{\lambda} \sum_{k=1}^{n} \psi(\frac{\lambda}{k}).$$

After division by n^{λ}, we obtain that

$$\frac{p_n}{n^{\lambda}} = d(Tx_{n+m}, Tx_n) \leq \frac{C(\lambda)^{\lambda}}{n^{\lambda}} \sum_{k=1}^{n} \psi(\frac{\lambda}{k}).$$

On taking as limit $n \to \infty$, $\{Tx_n\}$ is a Cauchy sequence. For each $\varepsilon \in [0,1]$, we have

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq \alpha(x_{n-1}, x_n) d(Tx_n, Tx_{n+1}) \\ &\leq \frac{(1-\varepsilon)}{2} d(Tx_{n-1}, Tx_n) \\ &\quad + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + \|Tx_{n-1}\| + \|Tx_n\| + \|Tx_{n+1}\|]^{\mu}, \\ &\leq \frac{(1-\varepsilon)}{2} d(Tx_{n-1}, Tx_n) + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+3c]^{\mu}, \end{aligned}$$

for each $\varepsilon \in [0,1]$. Hence $\{Tx_n\}$ is a Cauchy sequence in a complete metric space (X,d) and B is closed, there exists $y \in B$ such that

$$\lim_{n\to\infty} d(Tx_n, y) = 0.$$

Since $\{x_n\}$ is Cauchy *proximal admissible Picard sequence* in A_0, and $x_n \to x^*$ in A and mapping T is continuous. Now, from the above inequality, we have

$$d(A,B) = \lim_{n\to\infty} d(x_{n+1}, Tx_n) = d(x^*, y),$$

which implies that $x^* \in A_0$. □

Theorem 1. *Let $T : A \to B$ be a continuous $\alpha-$ proximal admissible and $\alpha-$ Pata-proximal admissible contraction of type-II, with $A_0 \neq \phi$ and $T(A_0) \subseteq B_0$. If B is approximately compact with respect to A, then T has a unique best proximity point x^* in A_0.*

Proof. Let x_0 be a given point in A_0. From Lemma (1), sequence $\{x_n\}$ is Cauchy *proximal admissible Picard sequence* in A_0. We can obtain a sequence $\{x_n\}$ in A_0 such that it satisfies

$$d(x_n, Tx_{n-1}) = d(A,B),\ d(x_{n+1}, Tx_n) = d(A,B) \text{ and } \alpha(x_{n-1}, x_n) \geq 1 \tag{2.1}$$

for all $n \in \mathbb{N}$. Since T is $\alpha-$proximal admissible and $\alpha-$ Pata-proximal admissible contraction of type-II. Following arguments similar to those in the proof of Lemma (3), we obtain that $\{Tx_n\} \in B_0$ is a Cauchy *proximal admissible Picard sequence*. As B is a closed subset of a complete metric space, there exists some $y \in B$ such that $\lim_{n\to\infty} d(Tx_n, y) = 0$. Since A_0 is proximal $T-$orbitally complete, there exists some x^* in A_0 such that $x_n \to x^*$ in A_0 which implies that $Tx^* = y = \lim_{n\to\infty} Tx_n$ and hence

$$d(x^*, Tx^*) = d(A,B),$$

implies that x^* is the Pata-type best proximity point of the mapping T .

To show the uniqueness of best proximity point: Assume on contrary that there exists another point $y^* \neq x^*$ of T in A_0 such that

$$d(y^*, Ty^*) = d(A,B), d(x^*, Tx^*) = d(A,B) \text{ and } \alpha(x^*, y^*) \geq 1 \text{ where } x^*, y^* \in A_0.$$

Since T is $\alpha-$ Pata-proximal admissible contraction of type-II, so

$$\begin{aligned}
\alpha(x^*, y^*) d(Tx^*, Ty^*) &\leq \frac{(1-\varepsilon)}{2} d(Tx^*, Ty^*) \\
&\quad + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + \|Tx^*\| + \|Ty^*\| + \|Ty^*\|]^{\mu} \\
&\leq \frac{(1-\varepsilon)}{2} d(Tx^*, Ty^*) + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+3c]^{\mu} \\
&< d(Tx^*, Ty^*) + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+3c]^{\mu}.
\end{aligned}$$

On taking limit as $\varepsilon \to 0^+$, we have

$$\alpha(x^*, y^*) d(Tx^*, Ty^*) < d(Tx^*, Ty^*),$$

a contradiction. Hence Pata-type best proximity point of the mapping T is unique. □

Example 1. Let $X = \mathbb{R}$. We know that X is a complete metric space with respect to usual metric $d : X \times X \to [0, \infty)$ defined by $d(x,y) = |x-y|$. Let $A = \{0, \frac{2}{6}, \frac{4}{6}, \frac{6}{6}\}$ and $B = \{\frac{1}{6}, \frac{3}{6}, \frac{5}{6}, \frac{7}{6}, \frac{9}{6}, \frac{11}{6}, \frac{13}{6}\}$ be two closed subsets of X. Note that B is compact and so B is approximately compact with respect to A. Note that

$$d(A,B) = \frac{1}{6},\ A_0 = A \text{ and } B_0 = \left\{\frac{1}{6}, \frac{3}{6}, \frac{5}{6}, \frac{7}{6}\right\}.$$

Define a mapping $T : A \to B$ as:

$$Tx = \begin{cases} 2x + \frac{1}{6} \text{ when } x \in \{0, \frac{2}{6}\}, \\ 2x - \frac{1}{6} \text{ when } x = \frac{4}{6}, \\ \frac{3}{6} \quad \text{ when } x = \frac{6}{6}. \end{cases}.$$

Obviously, $T(A_0) \subseteq B_0$. Further, suppose that $\alpha(x,y) = e^{|x-y|}$ for all $x,y \in X$. Also, $\psi(t) = te^t$ for all $t \geq 0$. If $\lambda = 2$ then $\mu \in [0,2]$, take $\mu = \lambda$. For simplicity, take $\varepsilon = \frac{1}{10}$. If, we take $u = \frac{2}{6}, v = \frac{6}{6}, x = 0$ and $y = \frac{2}{6}$ in A, satisfies the following:

$$d(u,Tx) = d(A,B), \text{ and } d(v,Ty) = d(A,B),$$

Now, It is straightforward to check that mapping T satisfy the $\alpha-$ Pata-proximal admissible contraction of type-II for $50.79922618 \leq \Lambda$. Assume, $\Lambda = 51$. Moreover $x = 0$ is the only Pata type best proximity point of mapping T.

3. Optimal coincidence point of generalized $\alpha-$ Pata-proximal contraction mappings

Theorem 2. *Let $g : A \to A$ be an expansive mapping satisfying α_R-property and $T : A \to B$ be an $\alpha-$proximal admissible with $A_0 \neq \phi$, $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$ for any $t > 0$. If B is approximately compact with respect to A and the pair (g,T) is generalized $\alpha-$Pata-proximal contraction. If there exists $x_0, x_1 \in A_0$ satisfying $d(x_1, Tx_0) = d(A,B)$ and $\alpha(x_0,x_1) \geq 1$. Then the pair (g,T) has a unique optimal coincidence point x^* in A_0.*

Proof. Let x_0 be a given point in A_0. As $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, we can choose an element $x_1 \in A_0$ such that $d(gx_1, Tx_0) = d(A,B)$ and $\alpha(x_0,x_1) \geq 1$, where $x_0, x_1 \in A_0$. Also, $Tx_1 \in T(A_0) \subseteq B_0$, and $A_0 \subseteq g(A_0)$, it follows that there exists an element $x_2 \in A_0$ such that $d(gx_2, Tx_1) = d(A,B)$, since T is $\alpha-$proximal admissible mapping then $\alpha(gx_1, gx_2) \geq 1$ further g is satisfying α_R-property, which implies that $\alpha(x_1,x_2) \geq 1$, where $gx_1, gx_2 \in A_0$. Continuing this way, we can obtain a sequence $\{gx_n\}$ in A_0 satisfying:

$$d(gx_n, Tx_{n-1}) = d(A,B) \text{ and } d(gx_{n+1}, Tx_n) = d(A,B), \tag{3.1}$$

where $\alpha(x_{n-1}, x_n) \geq 1$. Note that

$$\begin{aligned} \alpha(x_{n-1},x_n)d(gx_n, gx_{n+1}) \leq \frac{(1-\varepsilon)}{2} d(x_{n-1},x_n) \\ + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + \|x_{n-1}\| + \|x_n\| + \|x_{n+1}\|]^{\mu}. \end{aligned} \tag{3.2}$$

Suppose that $x_n \neq x_{n+1}$ for all $n = 0,1,2,\ldots$. For each $n \in \mathbb{N}$ and $\varepsilon \in [0,1]$,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \alpha(x_{n-1},x_n)d(gx_n, gx_{n+1}) \\ &\leq \frac{(1-\varepsilon)}{2} d(x_{n-1},x_n) \end{aligned} \tag{3.3}$$

$$+\Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+\|x_{n-1}\|+\|x_n\|+\|x_{n+1}\|]^{\mu}.$$

Thus, for each $n\in\mathbb{N}$, letting $\varepsilon\to 0^+$ on both sides of the inequality (3.3), gives that

$$\begin{aligned} d(x_{n+1},x_n) &\le d(gx_{n+1},gx_n) \\ &\le \alpha(x_{n-1},x_n)d(gx_{n+1},gx_n) \le \frac{d(x_n,x_{n-1})}{2} < d(x_n,x_{n-1}). \end{aligned} \tag{3.4}$$

Thus $\{x_n\}$ is decreasing sequence. Continuing this way, we have

$$d(x_{n+1},x_n) \le d(x_n,x_{n-1}) \le \cdots \le d(x_1,x_0), \tag{3.5}$$

for all $n=0,1,2,\ldots$.

Now, we have to show that the sequence $d(gx_n,x_0)$ is bounded above by $\frac{c}{2}=d(x_0,gx_1)$. This is the case when $n=1$. Assume that $d(gx_{n-1},x_0)\le\frac{c}{2}$. We show that $d(gx_n,x_0)\le c$. Note that

$$\begin{aligned} d(gx_n,x_0) &\le d(gx_n,gx_{n-1})+d(gx_{n-1},x_0) \\ &\le d(x_{n-1},x_{n-2})+d(gx_{n-1},x_0) \\ &\le \cdots \\ &\le d(x_1,x_0)+\frac{c}{2} \\ &\le \frac{c}{2}+\frac{c}{2}=c. \end{aligned}$$

Thus, for all $n\in\mathbb{N}$,

$$d(gx_n,x_0)\le c.$$

Let

$$\|gx_0\|=c_0 \text{ and } c_1=\|gx_1\|,$$

for some c_0, $c_1\in\mathbb{R}^+$. Then

$$\begin{aligned} c_n=d(gx_n,x_0) &\le d(gx_n,gx_{n+1})+d(gx_{n+1},gx_1)+d(gx_1,x_0) \\ &\le d(x_{n-1},x_n)+d(gx_{n+1},gx_1)+d(gx_1,x_0) \\ &\quad\cdots \\ &\le d(x_0,x_1)+d(gx_{n+1},gx_1)+d(gx_1,x_0) \\ &\le d(x_0,x_1)+d(gx_1,x_0)+d(gx_{n+1},gx_1) \\ &\le d(x_0,x_1)+d(gx_1,x_0)+\alpha(x_n,x_0)d(gx_{n+1},gx_1) \\ &\le d(x_0,x_1)+d(gx_1,x_0)+\frac{(1-\varepsilon)}{2}d(x_n,x_0) \\ &\quad+\Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+\|x_0\|+\|x_1\|+\|x_n\|]^{\mu} \\ &\le \frac{c_1}{2}+c_0+\frac{(1-\varepsilon)}{2}c_n+\Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+c_0+c_1+c_n]^{\mu} \\ &\le \frac{c_1}{2}+c_0+(1-\varepsilon)c_n+\Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+c_0+c_1+c_n]^{\mu} \end{aligned}$$

$$\leq \frac{c_1}{2} + c_0 + c_n - \varepsilon c_n + \Lambda \varepsilon^{\lambda} \psi(\varepsilon) c_n^{\mu} [\frac{1 + c_0 + c_1}{c_n} + 1]^{\mu},$$

which further implies that

$$\varepsilon c_n \leq \frac{c_1}{2} + c_0 + \Lambda \varepsilon^{\lambda} \psi(\varepsilon) c_n^{\lambda} [\frac{1 + c_0 + c_1}{c_n} + 1]^{\mu},$$

as $\mu \leq \lambda$. Suppose that $a_n = \Lambda[\frac{1+c_0+c_1}{c_n} + 1]^{\mu}$ and $b = \frac{c_1}{2} + c_0$. Then, the above inequality becomes

$$\varepsilon c_n \leq \varepsilon^{\lambda} \psi(\varepsilon) c_n^{\lambda} a_n + b,$$

for some $a_n, b > 0$. If there is a divergent subsequence $\{c_{n_i}\}$, correspondingly, we have a subsequence $\{a_{n_i}\}$ such that $a_{n_i} \to \Lambda$. If we choose $\varepsilon_{n_i} = \dfrac{(1+b)}{c_{n_i}}$, then

$$1 \leq \psi(\varepsilon_i)(1+b)^{\lambda} a_{n_i} \to 0,$$

gives a contradiction.

Now, we have to show that $\{gx_n\}$ is Cauchy sequence. For this, we prove that

$$d(gx_{n+m}, gx_n) \leq C\omega_n(\alpha),$$

where $C = \sup_{n \in \mathbb{N}} \Lambda(1 + 3c_n)^{\mu} < \infty$ and $\omega_n(\lambda) = (\dfrac{\lambda}{n})^{\lambda} \sum_{k=1}^{n} \psi(\dfrac{\lambda}{k})$.
Let m be fixed, define

$$p_n = n^{\lambda} d(gx_{n+m}, gx_n).$$

Note that

$$\begin{aligned}
p_{n+1} &= (n+1)^{\lambda} d(gx_{n+m+1}, gx_{n+1}) \\
&\leq (n+1)^{\lambda} \alpha(x_{n+m}, x_n) d(gx_{n+m+1}, gx_{n+1}) \\
&\leq (n+1)^{\lambda} [\frac{(1-\varepsilon)}{2} d(x_{n+m}, x_n) + \Lambda \varepsilon^{\lambda} \psi(\varepsilon)[1 + \|x_{n+m}\| + \|x_n\| + \|x_{n+1}\|]^{\mu}] \\
&\leq (n+1)^{\lambda} (1-\varepsilon) d(x_{n+m}, x_n) \\
&\quad + (n+1)^{\lambda} \Lambda \varepsilon^{\lambda} \psi(\varepsilon)[1 + \|x_{n+m}\| + \|x_n\| + \|x_{n+1}\|]^{\mu}].
\end{aligned}$$

Choose

$$\varepsilon = 1 - (\frac{n}{n+1})^{\lambda} = [1 - (1 - \frac{1}{n+1})^{\lambda}] \leq \frac{\lambda}{n+1}.$$

Then, we have

$$\begin{aligned}
p_{n+1} &\leq (n+1)^{\lambda} ((\frac{n}{n+1})^{\lambda}) d(x_{n+m}, x_n) + (n+1)^{\lambda} \Lambda (\frac{\lambda}{n+1})^{\lambda} \psi(\frac{\lambda}{n+1})[1 + 3c_n]^{\mu} \\
&\leq n^{\lambda} d(x_{n+m}, x_n) + \Lambda(\lambda)^{\lambda} \psi(\frac{\lambda}{n+1})[1 + 3c_n]^{\mu} \\
&\leq n^{\lambda} d(gx_{n+m}, gx_n) + \Lambda(\lambda)^{\lambda} \psi(\frac{\lambda}{n+1})[1 + 3c_n]^{\mu}.
\end{aligned}$$

Now, we have

$$\begin{aligned} p_{n+1} &\leq p_n + C(\lambda)^\lambda \psi(\frac{\lambda}{n+1}) \\ &\leq p_{n-1} + C(\lambda)^\lambda \psi(\frac{\lambda}{n}) + C(\lambda)^\lambda \psi(\frac{\lambda}{n+1}) \\ &\cdots \\ &\leq p_0 + C(\lambda)^\lambda [\psi(\frac{\lambda}{1}) + \psi(\frac{\lambda}{2}) + \cdots + \psi(\frac{\lambda}{n+1})]. \end{aligned}$$

Now, $p_0 = 0$ gives that

$$p_n \leq C(\lambda)^\lambda \sum_{k=1}^{n+1} \psi(\frac{\lambda}{k}).$$

After division by n^λ, we will obtain

$$\frac{p_n}{n^\lambda} = d(gx_{n+m}, gx_n) \leq \frac{C(\lambda)^\lambda}{n^\lambda} \sum_{k=1}^{n} \psi(\frac{\lambda}{k}).$$

On taking limit as $n \to \infty$, we have $\{gx_n\}$, a Cauchy sequence. For each $\varepsilon \in [0,1]$, we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq \alpha(x_{n-1}, x_n) d(gx_n, gx_{n+1}) \\ &\leq \frac{(1-\varepsilon)}{2} d(x_{n-1}, x_n) + \Lambda \varepsilon^\lambda \psi(\varepsilon)[1 + \|x_{n-1}\| + \|x_n\| + \|x_{n+1}\|]^\mu, \\ &\leq \frac{(1-\varepsilon)}{2} d(x_{n-1}, x_n) + \Lambda \varepsilon^\lambda \psi(\varepsilon)[1 + 3c]^\mu, \end{aligned}$$

for each $\varepsilon \in [0,1]$. Hence $\{gx_n\}$ is a Cauchy sequence in a complete metric space (X,d). Since A_0 is closed (Lemma 2), there exists an element x^* in A_0 such that $\lim_{n\to\infty} d(gx_n, gx^*) = 0$. Now

$$\begin{aligned} d(gx^*, B) &\leq d(gx^*, Tx_n) \\ &\leq d(gx^*, gx_{n+1}) + d(gx_{n+1}, Tx_n) \\ &= d(gx^*, gx_{n+1}) + d(A, B) \\ &\leq d(gx^*, gx_{n+1}) + d(gx^*, B), \end{aligned}$$

gives that

$$d(gx^*, B) \leq d(gx^*, Tx_n) \leq d(gx^*, gx_{n+1}) + d(gx^*, B).$$

Note that $\{gx_n\}$ converges to gx^* and $d(gx^*, Tx_n) \to d(gx^*, B)$. As $\{Tx_n\} \subseteq B$ and B is approximately compact with respect to A, $\{Tx_n\}$ has a subsequence, which converges to some y in B hence $d(gx^*, y) = d(A, B)$, that is, $gx^* \in A_0$. Since $A_0 \subseteq g(A_0)$, there exist some $u \in A_0$ such that

$$d(gu, Tx^*) = d(A, B) = d(gx_{n+1}, Tx_n),$$

where $x^*, x_n \in A_0$, $\alpha(x^*,u) \geq 1$ for all $n \in \mathbb{N}$. Since g is expansive mapping, we have to show that $u = x^*$. If not, then

$$\begin{aligned} d(u,x_{n+1}) &\leq \alpha(x^*,x_n)d(gu,gx_{n+1}) \\ &\leq \frac{(1-\varepsilon)}{2}d(x^*,x_n) + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+\|x^*\|+\|x_n\|+\|x_{n+1}\|]^{\mu}, \end{aligned}$$

on taking limit as $n \to \infty$, and $\varepsilon \to 0^+$ gives

$$d(u,x^*) \leq \alpha(x^*,x^*)d(gu,gx^*) \leq 0$$

a contradiction. Hence $d(gx^*,Tx^*) = d(gu,Tx^*) = d(A,B)$, that is, x^* is the optimal coincidence point of the pair (g,T).

To prove uniqueness: Suppose that y^* be another point in A_0 such that $d(gy^*,Ty^*) = d(A,B)$, such that $\alpha(x^*,y^*) \geq 1$. Note that

$$\begin{aligned} d(gx^*,gy^*) &\leq \alpha(x^*,y^*)d(gx^*,gy^*) \\ &\leq \frac{(1-\varepsilon)}{2}d(x^*,y^*) + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+\|x^*\|+\|x^*\|+\|x^*\|]^{\mu}, \end{aligned}$$

and hence

$$d(x^*,y^*) \leq d(gx^*,gy^*) \leq \frac{(1-\varepsilon)}{2}d(x^*,y^*) + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1+\|x^*\|+\|x^*\|+\|x^*\|]^{\mu}.$$

Apply $\varepsilon \to 0^+$, then the above inequality gives a contradiction. The result follows. □

Example 2. Let $X = \mathbb{R}^2$. We know that X is a complete metric space with respect to usual metric $d : X \times X \to [0,\infty)$ defined by $d(x,y) = |x-y|$. Let $A = \{(x,2) \mid x \in \mathbb{R}\}$ and $B = \{(x,0) \mid x \in \mathbb{R} \text{and } x \geq -3\}$ be two closed subsets of X. Note that B is compact and so B is approximately compact with respect to A. Note that $d(A,B) = 2$, $A_0 = \{(x,2) \mid x \in \mathbb{R} \text{ and } x \geq -3\} \subset A$ and $B_0 = \{(x,0) \mid x \in \mathbb{R} \text{ and } x \geq -3\} \subset B$. Define a mapping $T : A \to B$ as:

$$T(x,2) = (2x+3,0) \text{ and } g(x,2) = (3x,2).$$

Obviously, $T(A_0) \subseteq B_0$. Further, suppose that $\alpha(x,y) = e^{|x-y|}$ and $\psi(t) = te^t$ for all $x,y \in X$ and $t \geq 0$. If, we choose $\lambda = 1$ then $\mu \in [0,1]$ take $\mu = \lambda$. For simplicity, take $x_0 = (0,2)$ and $\varepsilon = \frac{1}{10}$. If $u = (u_1,2), v = (v_1,2), x = (x_1,2)$ and $y = (y_1,2) \in A$ satisfies:

$$d(gu,Tx) = d(A,B) \text{ and } d(gv,Ty) = d(A,B).$$

Which implies that

$$u_1 = \frac{2}{3}x_1 + 1 \text{ and } v_1 = \frac{2}{3}y_1 + 1.$$

Now, it is straightforward to check that the pair (g,T) satisfies the generalized $\alpha-$ Pata-proximal admissible contraction for $\frac{140.2498021|x_1-y_1|}{|1+x_1+y_1+v_1|} \leq \Lambda$, where $\frac{|x_1-y_1|}{|1+x_1+y_1+v_1|} \leq 1$ for all $x_1,y_1,v_1 \in A_0$ and $x_1 \neq y_1$, hence $140.2498021 \leq \Lambda$. Assume,

$\Lambda = 141$. Moreover $x = (-3,2)$ is the only Pata-type coincidence best proximity point of pair (g,T).

Corollary 1. *If $T : A \to B$ is $\alpha-$Pata-proximal contraction of type-I, $\alpha-$proximal admissible with $A_0 \neq \phi$ and $T(A_0) \subseteq B_0$ for any $t > 0$. Then T has a unique best proximity point x^* in A_0 provided that A_0 is proximal $T-$orbitally complete.*

Proof. Take $gx = I_A$ in the proof of Theorem (2). □

Example 3. Let $X = \mathbb{R}^2$. We know X is a complete metric space with respect to usual metric $d : X \times X \to [0,\infty)$ defined as $d(x,y) = |x-y|$. Let $A = \{(x,1) \mid x \in \mathbb{R}$ and $x \geq -1\}$ and $B = \{(x,0) \mid x \in \mathbb{R}$and $x \geq -3\}$ be two closed subsets of X. Note that, B is compact and so B is approximately compact with respect to A. Note that

$$d(A,B) = 1,\ A_0 = A \text{ and } B_0 = \{(x,0) \mid x \in \mathbb{R} \text{ and } x \geq -1\} \subset B.$$

Define a mapping $T : A \to B$ as:

$$T(x,1) = (2x+1,0).$$

Obviously, $T(A_0) \subseteq B_0$. Further, suppose that $\alpha(x,y) = e^{|x-y|}$ for all $x,y \in X$ and $\psi(t) = te^t$ for all $t \geq 0$. If, we choose $\lambda = 1$ then $\mu \in [0,1]$ take $\mu = \lambda$. For simplicity, take $x_0 = (0,1)$ and $\varepsilon = \frac{1}{10}$. If $u = (u_1,1), v = (v_1,1), x = (x_1,1)$ and $y = (y_1,1) \in A$ satisfies the following

$$d(u,Tx) = d(A,B) \text{ and } d(v,Ty) = d(A,B),$$

if

$$u_1 = 2x_1 + 1 \text{ and } v_1 = 2y_1 + 1.$$

Now, It is straight forward to check that T is $\alpha-$ Pata-proximal admissible contraction of type-I for $\frac{140.2498021|x_1-y_1|}{|1+x_1+y_1+v_1|} \leq \Lambda$, where $\frac{|x_1-y_1|}{|1+x_1+y_1+v_1|} \leq 1$ for all $x_1,y_1,v_1 \in A_0$ and $x_1 \neq y_1$, hence $140.2498021 \leq \Lambda$. Assume that $\Lambda = 141$ and after simple calculation one can find that $x = (-1,1)$ is the only Pata type best proximity point of mapping T.

4. Approximation results in ordered structures

In this section, we prove results in ordered metric spaces.

From now onwards, a $3-$ tuple $(X,d,\preceq)$ is called a partially ordered metric space if $(X,\preceq)$ is a partially ordered set and (X,d) is a metric space. Unless otherwise stated, it is assumed that A, B are nonempty closed subsets of partially ordered metric space $(X,d,\preceq)$.

Definition 9 ([11])**.** A mapping $T : A \longrightarrow B$ is called (a) nondecreasing or order preserving if for any x,y in A with $x \preceq y$, we have $Tx \preceq Ty$ (b) nonincreasing or ordered reversing if for any x,y in A with $x \preceq y$, we have $Tx \succeq Ty$ (c) monotone if it is order preserving or order reversing.

Definition 10 ([11]). A mapping $T : A \longrightarrow B$ is said to be a ordered contraction if for any $x, y \in A$ with $x \preceq y$, we have $d(Tx, Ty) \leq d(x, y)$.

Definition 11 ([11]). A mapping $T : A \longrightarrow B$ is said to be a proximal order preserving if for any u, v, x and y in A, the following implication holds:

$$\left.\begin{array}{r} x \preceq y \\ d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{array}\right\} \text{imply} u \preceq v.$$

If $A = B$, then proximal order preserving mapping will become order preserving.

Definition 12 ([11]). A mapping $T : A \longrightarrow B$ is said to be a proximal order reversing if for any u, v, x and y in A, the following implication holds:

$$\left.\begin{array}{r} x \preceq y \\ d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{array}\right\} \text{imply} u \succeq v.$$

If $A = B$, then proximal order reversing mapping will become order reversing.
We start with the following result.

Lemma 4 ([11]). *If $A_0 \neq \phi$ and $T(A_0) \subseteq B_0$. Then, for $a \in A_0$, there exists a sequence $\{x_n\} \subset A_0$ such that*

$$\left.\begin{array}{l} x_0 = a, \\ d(x_{n+1}, Tx_n) = d(A,B), \textit{for all } n \in \mathbb{N} \textit{ with } (x_n, x_{n+1}) \in \Delta. \end{array}\right\} \tag{4.1}$$

Proof. As $x_0 = a \in A_0$ and $T(A_0) \subseteq B_0$, there exist $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A,B)$ where $(x_0, x_1) \in \Delta$. Also $Tx_0 \in T(A_0) \subseteq B_0$, there exist $x_2 \in A_0$, such that $d(x_2, Tx_1) = d(A,B)$ where $(x_1, x_2) \in \Delta$. Continuing this way, we can obtain a sequence $\{x_n\} \subset A_0$ that satisfies the condition (4.1). □

Definition 13. A sequence $\{x_n\} \subset A_0$ satisfying the condition (4.1) is called ordered *proximal Picard sequence* starting with $a \in A_0$.

Definition 14. A set A_0 is ordered proximal $T-$orbitally complete if and only if every Cauchy sequence $\{x_n\}$ is ordered *proximal Picard sequence starting with* some $x_0 \in A_0$ converges to an element in the set A_0.

Lemma 5. *Let $T : A \to B$ be continuous, proximally monotone and $\alpha-$ Pata-proximal admissible contraction of type-II mapping with $A_0 \neq \varnothing$ and $T(A_0) \subseteq B_0$. Suppose that each pair of elements in partially ordered complete metric spaces $(X, d, \preceq)$ has a lower and upper bound. Then A_0 is proximal $T-$orbitally complete provided that T is one to one on A_0 also there exist a function $\alpha : A \times A \to [0, \infty)$ such that $(x, y) \in \Delta$ and $\alpha(x, y) \geq 1$ for all $x, y \in A$.*

Proof. Consider a function $\alpha : A \times A \to [0,\infty)$ such that

$$\alpha(x,y) = \begin{cases} 1 \text{ if } (x,y) \in \Delta, \\ 0 \text{ otherwise.} \end{cases} \tag{4.2}$$

Let x_0 be a given point in A_0. Suppose that $\{x_n\}$ is an ordered *proximal Picard sequence* starting with x_0. As $(X,d,\preceq)$ is complete and A is closed, there exist some x^* in A such that $\lim_{n\to\infty} d(x_n,x^*) = 0$. As $\{x_n\}$ is a ordered *proximal Picard sequence* starting with x_0, we have

$$d(x_n,Tx_{n-1}) = d(A,B) \text{ and } d(x_{n+1},Tx_n) = d(A,B)$$

with (x_{n-1},x_n) belong to the set of all ordered *proximal Picard sequence* starting with x_0, for all $n \in \mathbb{N}$. Since T is an $\alpha-$ Pata-proximal admissible contraction of type-II and function α defined in (4.2) which agrees with the $\alpha-$proximal admissible function defined on $A \times A$, rest of proof follows from the arguments on the same lines as in Theorem (3). □

Theorem 3. *Let $T : A \to B$ be a continuous, proximally monotone and $\alpha-$ Pata-proximal admissible contraction of type-II with $A_0 \neq \phi$ and $T(A_0) \subseteq B_0$. Suppose that each pair of elements in partially ordered complete metric spaces $(X,d,\preceq)$ has a lower and upper bound. If B is approximately compact with respect to A, then T has a unique best proximity point x^* in A_0 for all $(x,y) \in \Delta$ such that $\alpha(x,y) \geq 1$ for all $x,y \in A$.*

Proof. Let x_0 be a given point in A_0. From Lemma (4), the set of ordered *proximal Picard sequence* starting with x_0 is nonempty and $\{x_n\}$ belong to the set of ordered *proximal Picard sequence* starting with x_0 in A_0. We can obtain a sequence $\{x_n\}$ in A_0 such that it satisfies

$$d(x_n,Tx_{n-1}) = d(A,B), d(x_{n+1},Tx_n) = d(A,B) \tag{4.3}$$

for all $n \in \mathbb{N}$. Define a function $\alpha : A \times A \to [0,\infty)$, which satisfies (4.2). Also, function α agrees with the $\alpha-$proximal admissible function defined on $A \times A$. Since T is an $\alpha-$ Pata-proximal admissible contraction of type-II. Following arguments similar to those in the proof of Lemma (3) and Theorem (1), we can show the existence and uniqueness of best proximity point in partially ordered metric spaces. □

Theorem 4. *Let $g : A \to A$ be an expansive mapping and $T : A \to B$ be an $\alpha-$ proximal admissible with $A_0 \neq \phi$, $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$ for any $t > 0$. If B is approximately compact with respect to A and the pair (g,T) is generalized $\alpha-$Pata-proximal contraction. Then the pair (g,T) has a unique optimal coincidence point x^* in A_0 provided that $\alpha(x,y) \geq 1$ for all $x,y \in A$.*

Proof. By Lemma (4), the set of ordered *proximal Picard sequence* starting with x_0 is nonempty. Let x_0 be a given point in A_0. As $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, we can choose an element $x_1 \in A_0$ such that $d(gx_1,Tx_0) = d(A,B)$, where $(x_0,x_1) \in \Delta$. Also,

$Tx_1 \in T(A_0) \subseteq B_0$, and $A_0 \subseteq g(A_0)$, it follows that there exist an element $x_2 \in A_0$ such that $d(gx_2, Tx_1) = d(A,B)$, since T is ordered, where $(gx_1, gx_2) \in \Delta$. Continuing this way, we can obtain a sequence $\{gx_n\}$ in A_0 such that it satisfies

$$d(gx_n, Tx_{n-1}) = d(A,B) \text{ and } d(gx_{n+1}, Tx_n) = d(A,B), \tag{4.4}$$

where $(x_{n-1}, x_n) \in \Delta$. Also, define a function $\alpha : A \times A \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 \text{ if } (x,y) \in \Delta, \\ 0 \text{ otherwise.} \end{cases} \tag{4.5}$$

Which agrees with $\alpha-$proximal admissible function. Following the same lines of the proof of Theorem (2), we have the result. □

Corollary 2. *If $T : A \to B$ is an $\alpha-$ Pata-proximal admissible contraction of type-I with $A_0 \neq \phi$ and $T(A_0) \subseteq B_0$ for any $t > 0$. Then T has a unique best proximity point x^* in A_0 provided that B is approximately compact with respect to A.*

Example 4. Suppose that $X = [0,1] \times \mathbb{R}$ is an orderded metric space $(X, d, \preceq)$, where $d(x,y) = |x-y|$ and $x \preceq y$ is defined as $x \leq y$. If $A = \{(0,x) : x \in \mathbb{R}\}$ and $B = \{(1,y) : y \leq 0, y \in \mathbb{R}\}$. Note that

$$d(A,B) = 1,\ A_0 = \{(0,0)\} \text{ and } B_0 = \{(1,0)\}.$$

Define $T : A \to B$ as:

$$T(0,x) = (1,2x).$$

Obviously, $T(A_0) = B_0$. Note that the points $u = (0,x_1), v = (0,x_2), x = (0,y_1)$ and $y = (0,y_2)$ in A satisfy $d(u,Tx) = d(A,B)$ and $d(v,Ty) = d(A,B)$, if $x_1 = 2y_1$ and $x_2 = 2y_2$. Also, mapping T satisfies the condition of $\alpha-$ Pata-proximal admissible contraction of type-I, where $\psi(t) = \sqrt{t}$. Thus all the conditions of the Corollary (2) are satisfied. Moreover, $(0,0)$ is the only best proximity point of T in A_0.

Corollary 3. *Let $g : A \to A$ be a isometric mapping and $T : A \to B$ with $A_0 \neq \phi$, $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$ for any $t > 0$. If B is approximately compact with respect to A and the pair (g,T) is generalized $\alpha-$Pata-proximal contraction. Then the pair (g,T) has a unique optimal coincidence point x^* in A_0.*

5. Application

As an application of our results, we prove some new fixed point theorems as follows.

We start with the following fixed point theorem:

Theorem 5. *Let (X,d) be a complete ordered metric space, and $\alpha : A \times A \to \mathbb{R}$. Let $T : X \to X$ be a continuous and $\alpha-$admissible mapping satisfying*

i): $\alpha(x,y)d(Tx,Ty) \leq \dfrac{(1-\varepsilon)}{2} d(x,y) + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + \|x\| + \|y\| + \|v\|]^{\mu}$.

ii): *There exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*.

Proof. Let $A = B = X$. We prove that T is an $\alpha-$ Pata-proximal admissible contraction of type-I. Let $x, y, u, v \in X$, satisfy the following conditions:

$$\begin{cases} (x,y) \in \Delta, \\ d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B). \end{cases}$$

Since X is complete ordered metric space, then $(Tx,Ty) \in \Delta$, Since $d(A,B) = 0$, so we have $u = Tx$ and $v = Ty$, hence $(u,v) \in \Delta$. Since T satisfies condition (i), we have

$$\begin{aligned} \alpha(x,y)d(u,v) &= \alpha(x,y)d(Tx,Ty) \\ &\leq \frac{(1-\varepsilon)}{2}d(x,y) + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + \|x\| + \|y\| + \|v\|]^{\mu}, \end{aligned}$$

which implies that

$$\alpha(x,y)d(u,v) \leq \frac{(1-\varepsilon)}{2}d(x,y) + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + \|x\| + \|y\| + \|v\|]^{\mu},$$

and hence T is an $\alpha-$ Pata-proximal admissible contraction of type-I. Let

$$\begin{cases} \alpha(x,y) \geq 1, \\ d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B). \end{cases}$$

Since T is an $\alpha-$ordered admissible mapping. Applying condition (ii), there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$. If we choose $x_1 = Tx_0$, then we obtain that

$$\alpha(x_0,x_1) \geq 1 \text{ and } d(A,B) = d(x_1,Tx_0) = d(Tx_0,Tx_0).$$

Set B is approximately compact with respect to A, the conditions of Corollary (1) are satisfied, so there exists $x^* \in X$ such that $d(x^*,Tx^*) = 0 = d(A,B)$, which implies that $Tx^* = x^*$. □

CONCLUSION

In this paper, we defined $\alpha-$ Pata-proximal admissible contraction of type-I and type-II in the framework of a complete metric space. Further, we obtained the best proximity point and optimal coincidence point results for such mappings. We proved the results which extends and generalize the existing results in [7,8,10]. In case, fixed points of mappings introduced in this paper do not exists, the results helps to obtain the approximate fixed points. To elaborate these concepts and support the results, some examples are presented.

Further, we developed some approximation results in complete ordered metric space which shows the validity of obtained results in ordered structure. Under some restriction, specially for self mapping, we obtained a fixed point result as an application of our result.

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EXPLICIT EVALUATION OF SOME OF THE THETA-FUNCTION IDENTITIES

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Abstract. In this paper, we evaluate two parameters $h_{k,n}$ and $h'_{k,n}$ of some *P-Q* type Theta functions $\varphi(q)$ for any positive real numbers k and n. During this process, we also evaluate Ramanujan's cubic continued fraction.

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1. Introduction

For any complex number z, $q = e^{2\pi i z}$, $Im(z) > 0$, define

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} =: \Theta_3(0, 2z)$$

and

$$f(-q) := (q;q)_\infty = q^{-1/24}\eta(z),$$

where Θ_3 is the classical theta-function [23, p. 464] and $\eta(z)$ denotes the Dedekind eta-function and $(a;q)_\infty$ is defined by

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1 - aq^k).$$

It is precisely assumed in the sequel that $|q| < 1$ always. Recently, J. Yi [24] evaluated many new values of $\varphi(q)$ using modular identities, transformation formula for theta-functions and are defined as follows:

Definition 1. For any positive real number k and n we have

$$h_{k,n} := \frac{\varphi(q)}{k^{1/4}\varphi(q^k)} = \frac{\Theta_3(0, i\sqrt{n/k})}{k^{1/4}\Theta_3(0, i\sqrt{nk})} \qquad q = e^{-\pi\sqrt{n/k}}, \tag{1.1}$$

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$$h'_{k,n} := \frac{\varphi(-q)}{k^{1/4}\varphi(-q^k)} = \frac{\Theta_3(0,1+2i\sqrt{n/k})}{k^{1/4}\Theta_3(0,1+2i\sqrt{nk})} \qquad q = e^{-2\pi\sqrt{n/k}}. \tag{1.2}$$

Also it is observed that

i. $h_{k,1} = 1$,
ii. $h_{k,1/n} = h_{k,n}^{-1}$,
iii. $h_{k,n} = h_{n,k}$.

The Ramanujan-Göllnitz-Gordan continued fraction $H(q)$ is defined as

$$H(q) := \frac{q^{1/2}}{1+q}\,{}_{+}\,\frac{q^2}{1+q^3}\,{}_{+}\,\frac{q^4}{1+q^5}\,{}_{+\ldots}.$$

The above continued fraction was introduced by S. Ramanujan in his second notebook [16, p. 229]. H. Göllnitz [11] and B. Gordon [12] rediscovered $H(q)$ without knowing of Ramanujan's work. Ramanujan also recorded following two identities for $H(q)$ in his second notebook [16, p. 229],

$$\frac{1}{H(q)} - H(q) = \frac{\varphi(q^2)}{q^{1/2}\psi(q^4)} \quad \text{and} \quad \frac{1}{H(q)} + H(q) = \frac{\varphi(q)}{q^{1/2}\psi(q^4)}.$$

Proofs of the above two identities can be found in [4, p. 221]. H. H. Chan and S. S. Haung [10], have established many identities for $H(q)$, which are analogues to the results of famous Roger-Ramanujan continued fraction and Ramanujan's cubic continued fraction. Chan and Haung [10] have also derived some explicit formulas for evaluating $H(e^{-\pi\sqrt{n}/2})$ in terms of Ramanujan-Weber class invariants. Recently C. Adiga et. al. [2] have established several modular relations for the Rogers-Ramanujan type functions of order eleven which are analogues to Ramanujan's forty identities and also they established certain interesting partition theoretic interpretations. H. M. Srivastava and M. P. Chaudhary [17] established a set of four new results which depict the interrelationships between q-identities, continued fraction identities and combinatorial partition identities. Also in [18, 19] H. M. Srivastava et. al. deduced some q-identities involving theta functions. These q-identities have relationship among three of the theta-type functions which arise from Jacobi's triple product identity. In [22], K. R. Vasuki and B. R. Srivatsa Kumar proved the following:

Lemma 1. *For $q = e^{-\pi\sqrt{k/2}}$, let*

$$J_k := \sqrt{2}\frac{\varphi^2(q^2)}{\varphi^2(q)},$$

then

i. $J_k J_{1/k} = 1$,
ii. $J_1 = 1$,
iii. $H(q) = \sqrt{\dfrac{\sqrt[4]{2} - \sqrt{J_k}}{\sqrt[4]{2} + \sqrt{J_k}}}$.

In his first notebook [16] Ramanujan recorded many elementary values of $\varphi(q)$. Particularly, he recorded $\varphi(e^{-n\pi})$ and $\varphi(-e^{-n\pi})$ for $n = 1,2,4,8,1/2$ and $1/4$. All these values are proved by Berndt [5, p. 325]. Ramanujan also recorded non-elementary values of $\varphi(e^{-n\pi})$ for $n = 3,5,7,9$ and 45. And all these are proved by Berndt and Chan [7] . Recently Yi [24], evaluated $\varphi(e^{-n\pi})$ for $n = 1,2,3,4,5$ and 6 and $\varphi(-e^{n\pi})$ for $n = 1,2,4,6,8,10$ and 12. Furthermore, M. S. M. Naika and Chandan kumar [13] and Naika et. al [14] established several general formulas for the explicit evaluation of $h_{2,n}$ and $h_{4,n}$ by employing modular equation of degree 2 and 4 respectively. On page 366 of his lost notebook [15, p. 248], Ramanujan recorded another continued fraction

$$G(q) := \frac{q^{1/3}}{1}\,{}_{+}\,\frac{q+q^2}{1}\,{}_{+}\,\frac{q^2+q^4}{1}\,{}_{+}\,\frac{q^3+q^6}{1}\,{}_{+\ldots}, \tag{1.3}$$

and claimed that there are many results of $G(q)$. Motivated by Ramanujan's claim, H. H. Chan [9] derived many new identities which Ramanujan vaguely referred. Recently B. C. Berndt, Chan and L. -C. Zhang [6], N. D. Baruah and Nipen Saikia [3], C. Adiga et. al. [1], S. Bhargava et. al. [8] have found several explicit values of $G(q)$. Further, as a particular case of this for $k = 3$ they proved the following:

Lemma 2. *For $q = e^{-\pi\sqrt{n/3}}$, let*

$$J_n := \frac{1}{\sqrt[4]{3}}\frac{\varphi(q)}{\varphi(q^3)},$$

then

i. $J_n J_{1/n} = 1$,
ii. $J_1 = 1$,
iii. $D_n = \sqrt{\dfrac{\sqrt{3}-J_n^2}{1+\sqrt{3}J_n^2}}$,
iv. $G(q) = \frac{1}{2}\sqrt[3]{1-3D_n^4}$.

Motivated by the above work, in this paper we find some general formulas for the explicit evaluation of $h_{2,n}$, $h_{3,n}$ and $h'_{3,n}$. Also we evaluate Ramanujan's cubic continued fraction and Ramanujan-Göllintz-Gordon continued fraction.

2. Preliminary results: P-Q type theta-function identities

In this section, we state P-Q type theta-function identities and also some $h_{k,n}$ and $h'_{k,n}$ which we need in sequel.

Theorem 1 ([8]). *If* $P := \dfrac{\varphi(q)}{\varphi(q^3)}$ *and* $Q := \dfrac{\varphi(-q)}{\varphi(-q^3)}$ *then*

$$\frac{P}{Q} + \frac{Q}{P} = \frac{3}{PQ} - PQ.$$

Theorem 2 ([8]). *If* $P := \dfrac{\varphi(-q)}{\varphi(-q^3)}$ *and* $Q := \dfrac{\varphi(-q^2)}{\varphi(-q^6)}$ *then*

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 = \frac{3}{Q^2} - Q^2.$$

Theorem 3 ([8]). *If* $P := \dfrac{\varphi(q)}{\varphi(q^3)}$ *and* $Q := \dfrac{\varphi(q^2)}{\varphi(q^6)}$ *then*

$$(PQ)^2 + \frac{9}{(PQ)^2} + 2(P^2 - Q^2) = 6\left(\frac{1}{P^2} - \frac{1}{Q^2}\right) - \left(\frac{P}{Q}\right)^2 - \left(\frac{Q}{P}\right)^2 + 12.$$

Theorem 4 ([22]). *If* $P := \dfrac{\varphi^2(q^2)}{\varphi^2(q)}$ *and* $Q := \dfrac{\varphi^2(q^4)}{\varphi^2(q^2)}$ *then*

$$4P^2Q^2 - 4P^2Q + P^2 - 2P + 1 = 0.$$

Theorem 5 ([22]). *If* $P := \dfrac{\varphi^2(q^2)}{\varphi^2(q)}$ *and* $Q := \dfrac{\varphi^2(q^6)}{\varphi^2(q^3)}$ *then*

$$64PQ + \frac{16}{PQ} - 96(P+Q) - 48\left(\frac{1}{P} + \frac{1}{Q}\right) + 138 = \left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2 - 36\left(\frac{P}{Q} + \frac{Q}{P}\right).$$

Theorem 6 ([20]). *If* $P := \dfrac{\varphi(q)}{\varphi(q^3)}$ *and* $Q := \dfrac{\varphi(q^7)}{\varphi(q^{21})}$ *then*

$$\left(\frac{P}{Q}\right)^4 - \left(\frac{Q}{P}\right)^4 + 14\left(\left(\frac{P}{Q}\right)^2 - \left(\frac{Q}{P}\right)^2\right) - 7\left(\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 - 1\right)$$
$$\times\left(PQ + \frac{3}{PQ}\right) + (PQ)^3 + \frac{27}{(PQ)^3} = 0.$$

Theorem 7 ([20]). *If* $P := \dfrac{\varphi(q^3)\varphi(q^2)}{\varphi(q)\varphi(q^6)}$ *and* $Q := \dfrac{\varphi(q^6)\varphi(q^4)}{\varphi(q^2)\varphi(q^{12})}$ *then*

$$(PQ)^2 + \frac{1}{(PQ)^2} - 8\left(PQ + \frac{1}{PQ}\right) + 6\left(\sqrt{PQ} - \frac{1}{\sqrt{PQ}}\right)\left(\sqrt{\frac{P}{Q}} + \sqrt{\frac{Q}{P}}\right)$$
$$-2\left((PQ)^{3/2} - \frac{1}{(PQ)^{3/2}}\right)\left(\sqrt{\frac{P}{Q}} + \sqrt{\frac{Q}{P}}\right) + \left(PQ + \frac{1}{PQ}\right)\left(\frac{P}{Q} + \frac{Q}{P}\right) + 10 = 0.$$

Theorem 8 ([20]). *If* $P := \dfrac{\varphi^2(q^3)}{\varphi(q)\varphi(q^9)}$ *and* $Q := \dfrac{\varphi^2(q^6)}{\varphi(q^2)\varphi(q^{18})}$ *then*

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 - 2\left(PQ - \frac{3}{PQ}\right)\left(\frac{P}{Q} + \frac{Q}{P}\right) + (PQ)^2 + \frac{9}{(PQ)^2}$$

$$-16\left(PQ+\frac{3}{PQ}\right)-44=0.$$

Theorem 9 ([20]). *If* $P:=\dfrac{\varphi(q^3)\varphi(q^5)}{\varphi(q)\varphi(q^{15})}$ *and* $Q:=\dfrac{\varphi(q^6)\varphi(q^{10})}{\varphi(q^2)\varphi(q^{30})}$ *then*

$$(PQ)^2+\frac{1}{(PQ)^2}-24\left(PQ+\frac{1}{PQ}\right)+\left(\frac{P}{Q}\right)^2+\left(\frac{Q}{P}\right)^2+\left[6\left(PQ+\frac{1}{PQ}\right)-8\right]$$
$$\times\left(\frac{P}{Q}+\frac{Q}{P}\right)-4\left(\sqrt{PQ}-\frac{1}{\sqrt{PQ}}\right)\left(\left(\frac{P}{Q}\right)^{3/2}+\left(\frac{Q}{P}\right)^{3/2}\right)$$
$$+\left[16\left(\sqrt{PQ}-\frac{1}{\sqrt{PQ}}\right)-4\left((PQ)^{3/2}-\frac{1}{(PQ)^{3/2}}\right)\right]\left(\sqrt{\frac{P}{Q}}+\sqrt{\frac{Q}{P}}\right)+36=0.$$

Lemma 3 ([24]). *If* $h_{k,n}$ *and* $h'_{k,n}$ *are as defined as in* (1.1) *and* (1.2) *then, we have*

$$h_{2,4}=\sqrt{2}+1-\sqrt{\sqrt{2}+1},\ h_{2,8}=\frac{\sqrt{2+\sqrt{2}}}{\sqrt[4]{2}+1},\ h_{3,3}=(2\sqrt{3}-3)^{1/4}=\frac{3^{1/8}\sqrt{\sqrt{3}-1}}{2^{1/4}},$$
$$h_{3,1/3}=\left(\frac{2\sqrt{3}+3}{3}\right)^{1/4}=\frac{\sqrt{\sqrt{3}+1}}{2^{1/4}3^{1/8}},\quad h_{3,5}=\frac{\sqrt{\sqrt{5}-1}}{\sqrt{2}},\quad h_{3,1/5}=\frac{\sqrt{\sqrt{5}+1}}{\sqrt{2}},$$
$$h_{3,9}=\frac{1}{\sqrt{3}}\left(1-\sqrt[3]{2}+\sqrt[3]{4}\right),\quad h_{3,1/9}=\frac{1+\sqrt[3]{2}}{\sqrt{3}},\quad h'_{3,1}=2^{-1/4}\sqrt{\sqrt{3}-1}.$$

3. Evaluation of $h_{k,n}$ and $h'_{k,n}$

Theorem 10. *We have*

i. $h_{3,6}=\sqrt[4]{603-426\sqrt{2}-348\sqrt{3}+246\sqrt{6}}=h^{-1}_{3,1/6}$,

ii. $h_{3,2/3}=\dfrac{1}{3}\sqrt[4]{603-426\sqrt{2}-348\sqrt{3}+246\sqrt{6}}\sqrt{9+6\sqrt{2}}=h^{-1}_{3,3/2}$.

Proof. On using the definition of $h_{k,n}$ in Theorem 8 and by setting $n=1/6$, we deduce

$$x^2+\frac{9}{x^2}-16\left(x+\frac{3}{x}\right)-4\left(x-\frac{3}{x}\right)+46=0,$$

where $x=\left(\dfrac{h_{3,6}}{h_{3,2/3}}\right)^2$. Now set $\dfrac{x}{\sqrt{3}}+\dfrac{\sqrt{3}}{x}=t$, $\dfrac{x^2}{3}+\dfrac{3}{x^2}=t^2-2$ and $\dfrac{x}{\sqrt{3}}-\dfrac{\sqrt{3}}{x}=\sqrt{t^2-4}$ in the above, we obtain

$$9t^4-96\sqrt{3}t^3+960t^2-1280\sqrt{3}t+1792=0.$$

Since $h_{k,n}$ is decreasing, we choose $t = \frac{4}{\sqrt{3}}(3-\sqrt{2})$. Again on solving and since $\dfrac{h_{3,6}}{h_{3,2/3}} < 1$, we have

$$\frac{h_{3,6}}{h_{3,2/3}} = \sqrt{9-6\sqrt{2}}. \tag{3.1}$$

From [8, Theorem 3.4] , if $P := \dfrac{\varphi^2(q)}{\varphi^2(q^3)}$ and $Q := \dfrac{\varphi^2(q^3)}{\varphi^2(q^9)}$ then

$$PQ + \frac{9}{PQ} = 3 + 6\frac{Q}{P} + \frac{Q^2}{P^2}.$$

On employing the definition of $h_{k,n}$ in the above, setting $n = 1/6$ and using (3.1), we deduce

$$y^2 - (70-48\sqrt{2})y + 1 = 0,$$

where $y = (h_{3,6}h_{3,2/3})^2$. On solving, we choose

$$h_{3,6}h_{3,2/3} = \sqrt{35-24\sqrt{2}-20\sqrt{3}+14\sqrt{6}}.$$

From (3.1) and the above, we obtain the desired result. □

Corollary 1. *We have*

i. $D_6 = \sqrt{\dfrac{\sqrt{3}-a}{1+\sqrt{3}a}}$, $\quad D_{1/6} = \sqrt{\dfrac{\sqrt{3}a-1}{a+\sqrt{3}}}$,

ii. $D_{2/3} = \dfrac{1}{\sqrt{3}}\sqrt{3-\sqrt{6}}$, $\quad D_{3/2} = \sqrt{\dfrac{\sqrt{3}b^2-1}{b^2+\sqrt{3}}}$,

where

$$a = \sqrt{603-426\sqrt{2}-348\sqrt{3}+246\sqrt{6}} \quad \textit{and} \quad b = \frac{1}{3}\sqrt{a(9+6\sqrt{2})}.$$

Proof. On using Theorem 10 to Lemma 2(iii), we obtain the above results. □

Corollary 2. *We have*

i. $G(e^{-\sqrt{2}\pi}) = \dfrac{1}{2}\sqrt[3]{1-3D_6^4}$, $\quad G(e^{-\pi/3\sqrt{2}}) = \dfrac{1}{2}\sqrt[3]{1-3D_{1/6}^4}$,

ii. $G(e^{-\sqrt{2}\pi/3}) = \dfrac{1}{2}\sqrt[3]{1-3D_{2/3}^4}$, $\quad G(e^{-\pi/\sqrt{2}}) = \dfrac{1}{2}\sqrt[3]{1-3D_{3/2}^4}$.

Proof. To prove the above results, we need to apply Corollary 1 to Lemma 2(iv) respectively. □

Theorem 11. *We have*

i. $h_{3,10} = \sqrt{(1+\sqrt{2})(2-\sqrt{3})}\sqrt[4]{99-40\sqrt{6}+44\sqrt{5}-18\sqrt{30}} = h_{3,1/10}^{-1}$,

ii. $$h_{3,2/5} = \frac{\sqrt[4]{99-40\sqrt{6}+44\sqrt{5}-18\sqrt{30}}}{\sqrt{(1+\sqrt{2})(2-\sqrt{3})}} = h_{3,5/2}^{-1}.$$

Proof. On using the definition of $h_{k,n}$ in Theorem 9 and by setting $n = 1/10$, we deduce

$$x^4 + \frac{1}{x^4} - 8\left(x^3 - \frac{1}{x^3}\right) - 12\left(x^2 + \frac{1}{x^2}\right) + 24\left(x - \frac{1}{x}\right) + 22 = 0,$$

where $x = \dfrac{h_{3,10}}{h_{3,2/5}}$. Setting $x - x^{-1} = t$ in the above, we obtain $t^2\left(t^2 - 8t - 8\right) = 0$. Since $h_{k,n}$ is positive and decreasing, we have $x - \dfrac{1}{x} = 4 - 2\sqrt{6}$. On solving this, we deduce

$$\frac{h_{3,10}}{h_{3,2/5}} = (1+\sqrt{2})(2-\sqrt{3}). \tag{3.2}$$

From [21, Theorem 3.1], if $P := \dfrac{\varphi(q)}{\varphi(q^3)}$ and $Q := \dfrac{\varphi(q^5)}{\varphi(q^{15})}$ then

$$(PQ)^2 + \frac{9}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3 + 5\left(\left(\frac{Q}{P}\right)^2 - \left(\frac{P}{Q}\right)^2\right) + 5\left(\frac{Q}{P} - \frac{P}{Q}\right).$$

Again using the definition of $h_{k,n}$ in the above, by setting $n = 2/5$ and then using (3.2), we deduce

$$x^2 + \frac{1}{x^2} = 198 - 80\sqrt{6},$$

where $x = (h_{3,2/5}h_{3,10})^2$. On solving this, we obtain

$$h_{3,2/5}h_{3,10} = \sqrt{99-40\sqrt{6}+44\sqrt{5}-18\sqrt{30}}.$$

On using (3.2) and the above, we obtain the desired result. □

Corollary 3. *We have*

i. $$D_{10} = \sqrt{\frac{\sqrt{3}-ab}{1+\sqrt{3}ab}}, \qquad D_{1/10} = \sqrt{\frac{\sqrt{3}ab-1}{ab+\sqrt{3}}},$$

ii. $$D_{2/5} = \sqrt{\frac{1+\sqrt{2}-a-2b}{b-\sqrt{3}a}}, \qquad D_{5/2} = \sqrt{\frac{\sqrt{3}a-b}{a+\sqrt{3}b}},$$

where

$$a = \sqrt{99-40\sqrt{6}+44\sqrt{5}-18\sqrt{30}} \quad \textit{and} \quad b = 2-\sqrt{3}+2\sqrt{2}-\sqrt{6}.$$

Proof. On using Theorem 11 to Lemma 2(*iii*), we obtain the above results. □

Corollary 4. *We have*

i. $G(e^{-\pi\sqrt{10/3}}) = \frac{1}{2}\sqrt[3]{1-3D_{10}^4}$, $\quad G(e^{-\pi\sqrt{1/30}}) = \frac{1}{2}\sqrt[3]{1-3D_{1/10}^4}$,

ii. $G(e^{-\pi\sqrt{2/15}}) = \frac{1}{2}\sqrt[3]{1-3D_{2/5}^4}$, $\quad G(e^{-\pi\sqrt{5/6}}) = \frac{1}{2}\sqrt[3]{1-3D_{5/2}^4}$.

Proof. To prove the above results, we need to apply Corollary 3 to Lemma 2(iv) respectively. □

Theorem 12. *We have*

i. $h_{2,3} = \sqrt{\sqrt{6}+\sqrt{3}-\sqrt{2}-2} = h_{2,1/3}^{-1}$,

ii. $h_{2,1/9} = \sqrt{35-24\sqrt{2}-2\sqrt{6(99-70\sqrt{2})}} = h_{2,9}^{-1}$.

Proof. On employing the definition of $h_{k,n}$ in Theorem 5 and setting $n = 1/3$, we deduce

$$x^4 + \frac{1}{x^4} - 36\left(x^2 + \frac{1}{x^2}\right) + 96\sqrt{2}\left(x + \frac{1}{x}\right) + 202 = 0.$$

where $x = h_{2,3}^2$. On setting $x + x^{-1} = t$, we obtain

$$t^4 - 40t^2 + 96\sqrt{2}t - 128 = 0.$$

On solving, we obtain $t = -2(\sqrt{6}+\sqrt{2})$, $2(\sqrt{6}-\sqrt{2})$ and $2\sqrt{2}$ as a double root. Since $h_{k,n}$ is a decreasing function, we choose

$$x + \frac{1}{x} = 2(\sqrt{6}-\sqrt{2}).$$

On solving this, we obtain the first identity. Similarly we obtain $h_{2,1/9}$ by setting $n = 1/9$ in Theorem 5. □

Corollary 5. *We have*

i. $H(e^{-\pi\sqrt{3/2}}) = \sqrt{\dfrac{\sqrt[4]{2}h_{2,3}-1}{\sqrt[4]{2}h_{2,3}+1}}$, $\quad H(e^{-\pi/\sqrt{6}}) = \sqrt{\dfrac{\sqrt[4]{2}-h_{2,1/3}}{\sqrt[4]{2}+h_{2,1/3}}}$,

ii. $H(e^{-\pi/3\sqrt{2}}) = \sqrt{\dfrac{\sqrt[4]{2}h_{2,1/9}-1}{\sqrt[4]{2}h_{2,1/9}+1}}$, $\quad H(e^{-3\pi/\sqrt{2}}) = \sqrt{\dfrac{\sqrt[4]{2}-h_{2,9}}{\sqrt[4]{2}+h_{2,9}}}$.

Proof. The above results directly follows from Lemma 1 and Theorem 12. □

Theorem 13. *We have*

$$h_{3,4} = \frac{1}{2}\left(2-\sqrt{6}+\sqrt{14-4\sqrt{6}}\right) = h_{3,1/4}^{-1}.$$

Proof. On using the definition of $h_{k,n}$ in Theorem 7 and setting $n = 1/4$, we deduce

$$x^4 + \frac{1}{x^4} - 4\left(x^3 - \frac{1}{x^3}\right) - 6\left(x^2 + \frac{1}{x^2}\right) + 12\left(x - \frac{1}{x}\right) + 10 = 0,$$

where $x = h_{3,4}^2$ and on setting $x + x^{-1} = t$, we have

$$t^2(t^2 - 4t - 2) = 0.$$

On solving, we obtain $t = 0$, $2+\sqrt{6}$ and $2-\sqrt{6}$. Since $0 < h_{k,n} < 1$, we choose $x - x^{-1} = 2-\sqrt{6}$. Further on solving this, we obtain $h_{3,4}$.

□

Corollary 6. *We have*

$$D_4 = \sqrt{\frac{4\sqrt{2}+2\sqrt{6}-2\sqrt{3}-6}{6\sqrt{3}-6\sqrt{2}-4\sqrt{6}+10}}, \quad D_{1/4} = \sqrt{\frac{6\sqrt{3}-6\sqrt{2}-4\sqrt{6}+8}{4\sqrt{3}-2\sqrt{6}-4\sqrt{2}+6}}.$$

Proof. On using Theorem 13 to Lemma 2(iii), we obtain D_4 and $D_{1/4}$. □

Corollary 7. *We have*

$$G(e^{-2\pi/\sqrt{3}}) = \frac{1}{2}\sqrt[3]{1-3D_4^4}, \qquad G(e^{-\pi/2\sqrt{3}}) = \frac{1}{2}\sqrt[3]{1-3D_{1/4}^4}.$$

Proof. To prove the above results, we need to apply Corollary 6 to Lemma 2(iv) respectively. □

Theorem 14. *We have*

$$h_{3,7} = \left(2\sqrt{7\left(7+4\sqrt{3}\right)} - 6\sqrt{3} - 9\right)^{1/4} = h_{3,1/7}^{-1}.$$

Proof. On using the definition of $h_{k,n}$ in Theorem 6 and setting $n = 1/7$, we deduce

$$\frac{1}{x^4} - x^4 + 14\left(\frac{1}{x^2} - x^2\right) - 14\sqrt{3}\left(\frac{1}{x^2} + x^2\right) + 20\sqrt{3} = 0,$$

where $x = h_{3,7}^2$ and on solving we obtain the above result. □

Corollary 8. *We have*

$$D_7 = \sqrt{\frac{\sqrt{3}-\sqrt{a}}{1+\sqrt{3a}}}, \qquad D_{1/7} = \sqrt{\frac{\sqrt{3}a-1}{a+\sqrt{3}}},$$

where

$$a = 2\sqrt{7(7+4\sqrt{3})} - 6\sqrt{3} - 9.$$

Proof. On using Theorem 14 to Lemma 2(iii), we obtain D_7 and $D_{1/7}$. □

Corollary 9. *We have*

$$G(e^{-\pi\sqrt{7/3}}) = \frac{1}{2}\sqrt[3]{1-3D_7^4}, \qquad G(e^{-\pi/\sqrt{21}}) = \frac{1}{2}\sqrt[3]{1-3D_{1/7}^4}.$$

Proof. To prove the above results, we need to apply Corollary 8 to Lemma 2(iv) respectively. □

Theorem 15. *We have*

i. $h_{3,2} = \sqrt{\sqrt{3}+\sqrt{8-4\sqrt{3}}-2} = h_{3,1/2}^{-1}$,

ii. $h_{3,12} = \sqrt{\dfrac{9+3\sqrt[4]{3}\sqrt{2}-3\sqrt{3}}{A+B}} = h_{3,1/12}^{-1}$,

iii. $h_{3,4/3} = \sqrt{\dfrac{3-\sqrt{2}\sqrt[4]{3}+3\sqrt{3}-2D}{C}} = h_{3,3/4}^{-1}$,

iv. $h_{3,20} = \sqrt{\dfrac{2\sqrt{15}+2\sqrt{3}+\sqrt{5}-9}{E}} = h_{3,1/20}^{-1}$,

v. $h_{3,4/5} = \sqrt{\dfrac{\sqrt{3}-6\sqrt{5}+\sqrt{15}+4G-6}{F}} = h_{3,5/4}^{-1}$,

vi. $h_{3,36} = \dfrac{1}{3}(H-I) = h_{3,1/36}^{-1}$,

vii. $h_{3,4/9} = \frac{1}{3}\left(3-\sqrt{3}+\sqrt[3]{4}\sqrt{3}+J\right) = h_{3,9/4}^{-1}$,

where

$$A = 9\sqrt{3}+6\sqrt{2}\sqrt[4]{3}-3\sqrt{2}\sqrt[4]{27}-9,$$

$$B = \sqrt{3\left(96-120\sqrt{2}\sqrt[4]{3}-48\sqrt{3}+72\sqrt{2}\sqrt[4]{27}\right)},$$

$$C = \sqrt{2}\sqrt[4]{27}+2\sqrt{2}\sqrt[4]{3}-\sqrt{3}-3, \qquad D = \sqrt{2\left(4-\sqrt{2}\sqrt[4]{3}+2\sqrt{3}-\sqrt{2}\sqrt[4]{27}\right)},$$

$$E = 6+\sqrt{3}-6\sqrt{5}-\sqrt{15}-4\sqrt{2(2+\sqrt{3})(3-\sqrt{5})},$$

$$F = 2\sqrt{15}-3\sqrt{5}+2\sqrt{3}-11, \qquad G = \sqrt{2(2-\sqrt{3})(3+\sqrt{5})},$$

$$H = 45+36\sqrt[3]{2}+27\sqrt[3]{4}+(25+20\sqrt[3]{2}+16\sqrt[3]{4})\sqrt{3},$$

$$I = \sqrt{11430+9072\sqrt[3]{2}+7200\sqrt[3]{4}+6597\sqrt{3}+5238\sqrt[3]{2}\sqrt{3}+4158\sqrt[3]{4}\sqrt{3}},$$

$$J = \sqrt{6+12\sqrt[3]{2}-12\sqrt[3]{4}-3\sqrt{3}-6\sqrt[3]{2}\sqrt{3}+6\sqrt[3]{4}\sqrt{3}}.$$

Proof. The above results directly follows from Theorem 3 and the definition of $h_{k,n}$, where we set $n = 1/2, 1, 3, 1/3, 5, 1/5, 9$ and $1/9$ respectively by making use of Lemma 3. □

Corollary 10. *We have*

i. $D_2 = \sqrt{\sqrt{2}-1}$, $\qquad D_{1/2} = \sqrt{\dfrac{2-2\sqrt{3}+\sqrt{3(8-4\sqrt{3})}}{2\sqrt{3}-2+\sqrt{8-4\sqrt{3}}}}$,

ii. $D_{12} = \sqrt{\dfrac{\sqrt{3}-a^2}{1+\sqrt{3}a^2}}$, $\qquad D_{1/12} = \sqrt{\dfrac{\sqrt{3}a^2-1}{a^2+\sqrt{3}}}$,

iii. $D_{4/3} = \sqrt{\dfrac{\sqrt{3}-b^2}{1+\sqrt{3}b^2}}, \qquad D_{3/4} = \sqrt{\dfrac{\sqrt{3}b^2-1}{b^2+\sqrt{3}}},$

iv. $D_{20} = \sqrt{\dfrac{\sqrt{3}-c^2}{1+\sqrt{3}c^2}}, \qquad D_{1/20} = \sqrt{\dfrac{\sqrt{3}c^2-1}{c^2+\sqrt{3}}},$

v. $D_{4/5} = \sqrt{\dfrac{\sqrt{3}-d^2}{1+\sqrt{3}d^2}}, \qquad D_{5/4} = \sqrt{\dfrac{\sqrt{3}d^2-1}{d^2+\sqrt{3}}},$

vi. $D_{36} = \sqrt{\dfrac{\sqrt{3}-e^2}{1+\sqrt{3}e^2}}, \qquad D_{1/36} = \sqrt{\dfrac{\sqrt{3}e^2-1}{e^2+\sqrt{3}}},$

vii. $D_{4/9} = \sqrt{\dfrac{\sqrt{3}-f^2}{1+\sqrt{3}f^2}}, \qquad D_{9/4} = \sqrt{\dfrac{\sqrt{3}f^2-1}{f^2+\sqrt{3}}},$

where

$$a = \sqrt{\frac{9+3\sqrt[4]{3}\sqrt{2}-3\sqrt{3}}{A+B}}, \qquad b = \sqrt{\frac{3-\sqrt{2}\sqrt[4]{3}+3\sqrt{3}-2D}{C}},$$

$$c = \sqrt{\frac{2\sqrt{15}+2\sqrt{3}+\sqrt{5}-9}{E}}, \qquad d = \sqrt{\frac{\sqrt{3}-6\sqrt{5}+\sqrt{15}+4G-6}{F}},$$

$$e = \frac{1}{3}(H-I), \qquad f = \frac{1}{3}\left(3-\sqrt{3}+\sqrt[3]{4}\sqrt{3}+J\right).$$

Here A, B, C, D, E, F, G, H, Iand J are as defined in Theorem 15.

Proof. On using Theorem 15 to Lemma 2(iii), we obtain the above results. □

Corollary 11. *We have*

i. $G(e^{-\pi\sqrt{2/3}}) = \dfrac{1}{2}\sqrt[3]{36\sqrt{2}-50}, \qquad G(e^{-\pi/\sqrt{6}}) = \dfrac{1}{2}\sqrt[3]{1-3D_{1/2}^4},$

ii. $G(e^{-2\pi}) = \dfrac{1}{2}\sqrt[3]{1-3D_{12}^4}, \qquad G(e^{-\pi/6}) = \dfrac{1}{2}\sqrt[3]{1-3D_{1/12}^4},$

iii. $G(e^{-2\pi/3}) = \dfrac{1}{2}\sqrt[3]{1-3D_{4/3}^4}, \qquad G(e^{-\pi/2}) = \dfrac{1}{2}\sqrt[3]{1-3D_{3/4}^4},$

iv. $G(e^{-\pi\sqrt{20/3}}) = \dfrac{1}{2}\sqrt[3]{1-3D_{20}^4}, \qquad G(e^{-\pi 2\sqrt{15}}) = \dfrac{1}{2}\sqrt[3]{1-3D_{1/20}^4},$

v. $G(e^{-2\pi/\sqrt{15}}) = \dfrac{1}{2}\sqrt[3]{1-3D_{4/5}^4}, \qquad G(e^{-\sqrt{5}\pi/2\sqrt{3}}) = \dfrac{1}{2}\sqrt[3]{1-3D_{5/4}^4},$

vi. $G(e^{-2\sqrt{3}\pi}) = \dfrac{1}{2}\sqrt[3]{1-3D_{36}^4}, \qquad G(e^{-\pi/6\sqrt{3}}) = \dfrac{1}{2}\sqrt[3]{1-3D_{1/36}^4},$

vii. $G(e^{-2\pi/3\sqrt{3}}) = \dfrac{1}{2}\sqrt[3]{1-3D_{4/9}^4}, \qquad G(e^{-\sqrt{3}\pi/2}) = \dfrac{1}{2}\sqrt[3]{1-3D_{9/4}^4}.$

Proof. To prove the above results, we need to apply Corollary 10 to Lemma 2(iv) respectively. □

Theorem 16. *We have*

$$h'_{3,1/3}=\sqrt{\frac{6\sqrt{3}-\sqrt{2}\sqrt[4]{27}-3\sqrt{2}\sqrt[4]{3}}{3(2+\sqrt{2}\sqrt[4]{3}+\sqrt{2}\sqrt[4]{27})}},\ h'_{3,5}=\sqrt{\frac{\sqrt{6\left(\sqrt{5}+1\right)}-\sqrt{2\left(\sqrt{5}-1\right)}}{\sqrt{2\left(\sqrt{5}+1\right)}+\sqrt{6\left(\sqrt{5}-1\right)}}},$$

$$h'_{3,1/5}=\sqrt{\frac{\sqrt{6\left(\sqrt{5}-1\right)}-\sqrt{2\left(\sqrt{5}+1\right)}}{\sqrt{2\left(\sqrt{5}-1\right)}+\sqrt{6\left(\sqrt{5}+1\right)}}},\qquad h'_{3,9}=\sqrt{\frac{\sqrt{3}-\sqrt[3]{4}\sqrt{3}+3}{\sqrt{3}+3\sqrt[3]{4}-3}},$$

$$h'_{3,1/9}=\sqrt{\frac{9-\sqrt{3}-2\sqrt[3]{2}\sqrt{3}-\sqrt[3]{4}\sqrt{3}}{3(1+2\sqrt[3]{2}+\sqrt[3]{4}+\sqrt{3})}}.$$

Proof. The above results directly follows from Theorem 1 and the definition of $h_{k,n}$, where we set $n=1/3,5,1/5,9$ and $1/9$ respectively by making use of Lemma 3. □

Theorem 17. *We have*

i. $$h_{2,16}=\frac{2}{\sqrt{2\sqrt{2}+4\sqrt{2\left((13\sqrt{2}+18)\sqrt{1+\sqrt{2}}-20\sqrt{2}-28\right)}}}=h^{-1}_{2,1/16},$$

ii. $$h_{2,32}=\sqrt{\frac{6+8\sqrt[4]{2}+12\sqrt{2}+8\sqrt[4]{8}}{A}}=h^{-1}_{2,1/32},$$

where

$$A=12+8\sqrt[4]{2}+3\sqrt{2}+4\sqrt[4]{8}+\sqrt{256+240\sqrt[4]{2}+192\sqrt{2}+144\sqrt[4]{8}}.$$

Proof. The above results directly follows from Theorem 4 and the definition of $h_{k,n}$, where we set $n=4$ and 16 respectively by making use of Lemma 3. □

Corollary 12. *We have*

i. $$H(e^{-2\sqrt{2}\pi})=\sqrt{\frac{\sqrt[4]{2}h_{2,16}-1}{\sqrt[4]{2}h_{2,16}+1}},\qquad H(e^{-\pi/4\sqrt{2}})=\sqrt{\frac{\sqrt[4]{2}-h_{2,1/16}}{\sqrt[4]{2}+h_{2,1/16}}},$$

ii. $$H(e^{-4\pi})=\sqrt{\frac{\sqrt[4]{2}h_{2,32}-1}{\sqrt[4]{2}h_{2,32}+1}},\qquad H(e^{-\pi/8})=\sqrt{\frac{\sqrt[4]{2}-h_{2,1/32}}{\sqrt[4]{2}+h_{2,1/32}}}.$$

Proof. The above results directly follows from Lemma 1 and Theorem 17. □

Theorem 18. *We have*

$$h'_{3,4}=\frac{\sqrt{3}-1}{\sqrt{2}},\qquad \textit{and}\qquad h'_{3,1/4}=\left(\frac{3-2\sqrt{2}-\sqrt{3}+\sqrt{6}}{3+\sqrt{2}-\sqrt{3}}\right)^{1/4}.$$

Proof. The above results directly follows from Theorem 2 and the definition of $h_{k,n}$, where we set $n = 1$ and $1/4$ respectively by making use of Lemma 3. □

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A FAMILY OF INCOMPLETE HURWITZ-LERCH ZETA FUNCTIONS OF TWO VARIABLES

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Abstract. Inspired essentially by the work [H. M. Srivastava, M. A. Chaudhry and R. P. Agarwal [The incomplete Pochhammer symbols and their applications to hypergeometric and related functions, *Integral Transforms Spec. Funct.* **23** (2012), 659–683] (see [16])], we introduce the families of the incomplete Hurwitz-Lerch Zeta functions of two variables. We then give the integral representations including the Mellin-Barnes contour integral representation, summation formulas, derivative formulas and recurrence relations for the incomplete Hurwitz-Lerch Zeta functions of two variables.

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1. Introduction, definitions and preliminaries

The classical incomplete Gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$ satisfy the following decomposition formula [3, 4, 7, 16]:

$$\gamma(s,x)+\Gamma(s,x)=\Gamma(s)=\int_0^\infty t^{s-1}\,\exp(-t)\,\mathrm{d}t \qquad \big(\Re(s)>0\big). \tag{1.1}$$

By using each of these classical incomplete Gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$, the incomplete Pochhammer symbols $(\lambda,x)_\nu$ and $[\lambda,x]_\nu$ were defined, for $x \geqq 0$ and $\lambda,\nu\in\mathbb{C}$, by Srivastava *et al.* (see, for details, [16]). In fact, these incomplete Pochhammer symbols $(\lambda,x)_\nu$ and $[\lambda,x]_\nu$ satisfy the decomposition formula:

$$(\lambda,x)_\nu+[\lambda,x]_\nu=(\lambda)_\nu:=\frac{\Gamma(\lambda+\nu}{\Gamma(\lambda)}=\frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda+\nu-1}\exp(-t)\,\mathrm{d}t, \tag{1.2}$$

provided that the Gamma quotient as well as the infinite integral exist. Furthermore, Srivastava *et al.* [16] investigated many properties of the generalized incomplete

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hypergeometric functions by the help of these representations (see also a closely-related recent work [23] dealing mainly with such much more general functions as the incomplete H-functions and the incomplete $\bar{H}$-functions).

Many researchers studied many different generalizations and extensions of the familiar Hurwitz-Lerch Zeta function $\Phi(z,r,\alpha)$ by inserting certain additional parameters to the series representation of the Hurwitz-Lerch Zeta function. The interested readers can refer to these earlier publications for further researches and applications [5,8,10–15,17–19,21,24]. Recently, Choi and Parmar [6] introduced and studied the following extension of the generalized Hurwitz-Lerch Zeta function in two variables:

$$\Phi_{a,b,b';c}(z,t,r,\alpha)=\sum_{k,l=0}^{\infty}\frac{(a)_{k+l}(b)_k(b')_l}{(c)_{k+l}\,k!\,l!}\frac{z^k t^l}{(k+l+\alpha)^r} \tag{1.3}$$

$$\Big(a,b,b'\in\mathbb{C};\ c,\alpha\in\mathbb{C}\setminus\mathbb{Z}_0;\ r,z,t\in\mathbb{C}\text{ when }|z|<1\text{ and }|t|<1;$$
$$\Re(r-c-a-b-b')>1\text{ when }|z|=1\text{ and }|t|=1\Big).$$

Motivated by the above-mentioned works, our aim here is to investigate an extended family of the generalized incomplete Hurwitz-Lerch Zeta functions of two variables. In particular, we obtain integral representations including the Mellin-Barnes contour integral representation, derivative formulas, summation formulas, series relations and recurrence relations. We choose to record some of our results for only one member of the above extended family of the generalized incomplete Hurwitz-Lerch Zeta functions of two variables. Analogous results can be derived fairly easily for the other member of the above extended family of the generalized incomplete Hurwitz-Lerch Zeta functions of two variables.

2. The incomplete extended Hurwitz-Lerch zeta functions of two variables

In this section, we introduce the following family of the incomplete extended Hurwitz-Lerch Zeta functions of two variables:

$$\phi_{a,b,b';c}(z,t,r,\alpha)=\sum_{k,l=0}^{\infty}\frac{(a,x)_{k+l}(b)_k(b')_l}{(c)_{k+l}\,k!\,l!}\frac{z^k\,t^l}{(k+l+\alpha)^r} \tag{2.1}$$

and

$$\varphi_{a,b,b';c}(z,t,r,\alpha)=\sum_{k,l=0}^{\infty}\frac{[a,x]_{k+l}(b)_k(b')_l}{(c)_{k+l}\,k!\,l!}\frac{z^k\,t^l}{(k+l+\alpha)^r} \tag{2.2}$$

$$\Big(x\geqq 0;\ a,b,b'\in\mathbb{C};\ c,\alpha\in\mathbb{C}\setminus\mathbb{Z}_0;\ r,z,t\in\mathbb{C}\text{ when }|z|<1\text{ and }|t|<1;$$

$$\Re(r-c-a-b-b')>1 \text{ when } |z|=1 \text{ and } |t|=1\Big).$$

By virtue of (1.2), these families of the incomplete extended Hurwitz-Lerch Zeta functions of two variables satisfy a decomposition formula in terms of $\Phi_{a,b,b';c}(z,t,r,\alpha)$, which is the extension of the generalized Hurwitz-Lerch function of two variables in [6].

Remark 1. The limit cases of the families of the incomplete Hurwitz-Lerch Zeta functions of two variables are given by

$$\begin{aligned}\phi^*_{a,b;c}(z,t,r,\alpha) &= \lim_{b'\to\infty}\left\{\phi_{a,b,b';c}\left(z,\frac{t}{b'},r,\alpha\right)\right\}\\ &= \sum_{k,l=0}^{\infty}\frac{(a,x)_{k+l}(b)_k}{(c)_{k+l}\,k!\,l!}\,\frac{z^k\,t^l}{(k+l+\alpha)^r}\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\varphi^*_{a,b;c}(z,t,r,\alpha) &= \lim_{b'\to\infty}\left\{\varphi_{a,b,b';c}\left(z,\frac{t}{b'},r,\alpha\right)\right\}\\ &= \sum_{k,l=0}^{\infty}\frac{[a,x]_{k+l}(b)_k}{(c)_{k+l}\,k!\,l!}\,\frac{z^k\,t^l}{(k+l+\alpha)^r}\end{aligned} \tag{2.4}$$

$$\Big(x\geqq 0;\ a,b,b'\in\mathbb{C};\ c,\alpha\in\mathbb{C}\setminus\mathbb{Z}_0;\ r,z,t\in\mathbb{C} \text{ when } |z|<1 \text{ and } |t|<1;$$

$$\Re(r-c-a-b-b')>1 \text{ when } |z|=1 \text{ and } |t|=1\Big).$$

3. Integral Representations

For the parameters $a,b,b'\in\mathbb{C}$; $c\in\mathbb{C}\setminus\mathbb{Z}_0$, we make use of the following incomplete first Appell hypergeometric functions (see [7]):

$$\gamma_1\left[(a,x),b,b';c;x,y\right]=\sum_{k,l=0}^{\infty}\frac{(a,x)_{k+l}(b)_k(b')_l}{(c)_{k+l}}\,\frac{x^k\,y^l}{k!\,l!} \tag{3.1}$$

and

$$\Gamma_1\left[(a,x),b,b';c;x,y\right]=\sum_{k,l=0}^{\infty}\frac{[a,x]_{k+l}(b)_k(b')_l}{(c)_{k+l}}\,\frac{x^k\,y^l}{k!\,l!}, \tag{3.2}$$

which satisfy the following decomposition formula:

$$\gamma_1\left[(a,x),b,b';c;x,y\right]+\Gamma_1\left[(a,x),b,b';c;x,y\right]=F_1\left(a,b,b';c;x,y\right) \tag{3.3}$$

where $F_1(a,b,b';c;x,y)$ is the first Appell hypergeometric function (see, for example, [1,2,9,20,22]).

Theorem 1. *The following integral representation for the function* $\varphi_{a,b,b';c}(z,t,r,\alpha)$ *defined by* (2.2) *holds true:*

$$\varphi_{a,b,b';c}(z,t,r,\alpha) = \frac{1}{\Gamma(r)}\int_0^\infty u^{r-1}\exp(-\alpha u)\Gamma_1\left[(a,x),b,b';c;ze^{-u},te^{-u}\right]\,\mathrm{d}u \tag{3.4}$$

$$\left(x \geqq 0;\ \min\{\Re(r),\Re(\alpha)\} > 0 \textit{ when } |z| \leqq 1\ \ (z \neq 1)\right.$$
$$\left.\textit{and } |t| \leqq 1\ \ (t \neq 1);\ \Re(r) > 0 \textit{ when } z = t = 1\right).$$

Proof. Applying the following Eulerian integral (see [6]):

$$\frac{1}{(k+l+r)^\alpha} = \frac{1}{\Gamma(r)}\int_0^\infty u^{r-1}\ \exp(-\alpha u - ku - lu)\,\mathrm{d}u \tag{3.5}$$
$$\left(\min\{\Re(r),\Re(\alpha)\} > 0;\ k,l \in \mathbb{N}_0\right)$$

to the equation (2.2) and inverting the order of summation and integration, followed by using (3.2), we can get the desired result (3.4). □

Theorem 2. *The following integral representations for the function* $\varphi_{a,b,b';c}(z,t,r,\alpha)$ *defined by* (2.2) *hold true*:

$$\varphi_{a,b,b';c}(z,t,r,\alpha) = \frac{1}{\Gamma(r)\Gamma(a)}\int_x^\infty\int_0^\infty u^{r-1}\ v_1^{a-1}\ \exp(-\alpha u - v_1)$$
$$\cdot\,\Phi_2\left[b,b';c;ze^{-u}v_1,te^{-u}v_1\right]\,\mathrm{d}u\,\mathrm{d}v_1, \tag{3.6}$$

$$\varphi_{a,b,b';c}(z,t,r,\alpha)$$
$$= \frac{1}{\Gamma(r)\Gamma(b)\Gamma(b')}\int_0^\infty\int_0^\infty\int_0^\infty u^{r-1}v_2^{b-1}v_3^{b'-1}\exp(-\alpha u - v_2 - v_3)$$
$$\cdot\ {}_1\Gamma_1\left[(a,x);c;ze^{-u}v_2 + te^{-u}v_3\right]\,\mathrm{d}u\,\mathrm{d}v_2\,\mathrm{d}v_3, \tag{3.7}$$

$$\varphi_{a,b,b';c}(z,t,r,\alpha) = \frac{1}{\Gamma(r)\Gamma(a)\Gamma(b)\Gamma(b')}$$
$$\cdot\int_0^\infty\int_0^\infty\int_x^\infty\int_0^\infty u^{r-1}v_1^{a-1}v_2^{b-1}v_3^{b'-1}\exp(-\alpha u - v_1 - v_2 - v_3)$$
$$\cdot\ {}_0F_1\left[__;c;ze^{-u}v_1v_2 + te^{-u}v_1v_3\right]\,\mathrm{d}u\,\mathrm{d}v_1\,\mathrm{d}v_2\,\mathrm{d}v_3, \tag{3.8}$$

$$\varphi_{a,b,b';c+1}(-z,-t,r,\alpha) = \frac{\Gamma(c+1)}{\Gamma(r)\Gamma(a)\Gamma(b)\Gamma(b')}$$
$$\cdot\int_0^\infty\int_0^\infty\int_x^\infty\int_0^\infty u^{r-1}v_1^{a-\frac{c}{2}-1}v_2^{b-1}v_3^{b'-1}$$

$$\cdot \exp(-\alpha u - v_1 - v_2 - v_3)\left(ze^{-u}v_2 + te^{-u}v_3\right)^{-\frac{c}{2}}$$
$$\cdot J_c\left(2\sqrt{ze^{-u}v_1v_2 + te^{-u}v_1v_3}\right) \mathrm{d}u\, \mathrm{d}v_1\, \mathrm{d}v_2\, \mathrm{d}v_3 \tag{3.9}$$

and

$$\varphi_{a,b,b';c+1}(z,t,r,\alpha) = \frac{\Gamma(c+1)}{\Gamma(r)\Gamma(a)\Gamma(b)\Gamma(b')}$$
$$\cdot \int_0^\infty \int_0^\infty \int_x^\infty \int_0^\infty u^{r-1} v_1^{a-\frac{c}{2}-1} v_2^{b-1} v_3^{b'-1}$$
$$\cdot \exp(-\alpha u - v_1 - v_2 - v_3)\left(ze^{-u}v_2 + te^{-u}v_3\right)^{-\frac{c}{2}}$$
$$\cdot I_c\left(2\sqrt{ze^{-u}v_1v_2 + te^{-u}v_1v_3}\right) \mathrm{d}u\, \mathrm{d}v_1\, \mathrm{d}v_2\, \mathrm{d}v_3 \tag{3.10}$$

$$\left(x \geqq 0;\ \min\{\Re(r), \Re(\alpha)\} > 0;\ \max\{\Re(z), \Re(t)\} < 1;\ \Re(a) > 0 \textit{ when } x = 0\right),$$

provided that conditions analogous to those that are listed above with (3.10)*, are satisfied also for all the assertions* (3.6) *to* (3.9)*,* $J_\nu(z)$ *and* $I_\nu(z)$ *being the Bessel and the modified Bessel functions, respectively.*

Proof. If we use following integral representation of the incomplete first Appell hypergeometric function Γ_1 in (3.4) and interchange the order of summation and integration, we can obtain the desired result (3.6):

$$\Gamma_1\left[(a,x), b, b'; c; z, t\right] = \frac{1}{\Gamma(a)} \int_x^\infty v_1^{a-1} \exp(-v_1) \Phi_2\left[b, b'; c; zv_1, tv_1\right] \mathrm{d}v_1,$$

where, for convergence, $\Re(a) > 0$.

In the same way as above, by using the other integral representations of the incomplete first Appell hypergeometric function Γ_1 in [7], we can obtain the equations (3.7) to (3.10). □

Theorem 3. *The following integral representations for the function* $\varphi_{a,b,b';c}(z,t,r,\alpha)$ *defined by* (2.2) *hold true*:

$$\varphi_{a,b,b';c}(z,t,r,\alpha) = \frac{\Gamma(c)}{\Gamma(r)\Gamma(a)\Gamma(b)\Gamma(b')\Gamma(c-b-b')}$$
$$\cdot \int_0^1 \int_0^1 \int_x^\infty \int_0^\infty u^{r-1} v_1^{a-1} \mu^{b-1} \sigma^{b'-1}$$
$$\cdot \exp\left(-\alpha u - v_1 + ze^{-u}v_1\mu + te^{-u}v_1(1-\mu)\sigma\right)$$
$$\cdot\ (1-\mu)^{c-b-1}(1-\sigma)^{c-b-b'-1}\ \mathrm{d}u\, \mathrm{d}v_1\, \mathrm{d}\mu\, \mathrm{d}\sigma, \tag{3.11}$$

$$\varphi_{a,b,b';c}(z,t,r,\alpha) = \frac{\Gamma(\delta)}{\Gamma(r)\Gamma(a)\Gamma(b)\Gamma(b')\Gamma(\delta-b-b')}$$

$$\cdot \int_0^1 \int_0^1 \int_x^\infty \int_0^\infty u^{r-1} v_1^{a-1} \mu^{b-1} \sigma^{b'-1}$$
$$\cdot \exp\left(-\alpha u - v_1 + z e^{-u} v_1 \mu + t e^{-u} v_1 (1-\mu)\sigma\right)$$
$$\cdot \ {}_1F_1\left[c-\delta; c; -z e^{-u} v_1 \mu - t e^{-u} v_1 (1-\mu)\sigma\right]$$
$$\cdot (1-\mu)^{\delta-b-1} (1-\sigma)^{\delta-b-b'-1} \, du \, dv_1 \, d\mu \, d\sigma, \tag{3.12}$$

$$\varphi_{a,b,b';c}(z,t,r,\alpha) = \frac{\Gamma(c)}{\Gamma(r)\Gamma(a)\Gamma(\delta_1)\Gamma(b')\Gamma(c-\delta_1-b')}$$
$$\cdot \int_0^1 \int_0^1 \int_x^\infty \int_0^\infty u^{r-1} v_1^{a-1} \mu^{\delta_1-1} \sigma^{b'-1}$$
$$\cdot \exp\left(-\alpha u - v_1 + z e^{-u} v_1 \mu + t e^{-u} v_1 (1-\mu)\sigma\right)$$
$$\cdot \ {}_1F_1\left[\delta_1 - b; \delta_1; -z e^{-u} v_1 \mu\right]$$
$$\cdot (1-\mu)^{c-\delta_1-1} (1-\sigma)^{c-\delta_1-b'-1} \, du \, dv_1 \, d\mu \, d\sigma, \tag{3.13}$$

$$\varphi_{a,b,b';c}(z,t,r,\alpha) = \frac{\Gamma(c)}{\Gamma(r)\Gamma(a)\Gamma(\delta_1)\Gamma(b')\Gamma(c-\delta_1-b')}$$
$$\cdot \int_0^1 \int_0^1 \int_x^\infty \int_0^\infty u^{r-1} v_1^{a-1} \mu^{\delta_1-1} \sigma^{b'-1}$$
$$\cdot \exp\left(-\alpha u - v_1 + z e^{-u} v_1 \mu + t e^{-u} v_1 (1-\mu)\right)$$
$$\cdot \ {}_1F_1\left[b-\delta_1; c-\delta_1-b'; z e^{-u} v_1 (1-\mu)(1-\sigma)\right]$$
$$\cdot (1-\mu)^{c-\delta_1-1} (1-\sigma)^{c-\delta_1-b'-1} \, du \, dv_1 \, d\mu \, d\sigma, \tag{3.14}$$

$$\varphi_{a,b,b';c}(z,t,r,\alpha) = \frac{\Gamma(c)}{\Gamma(r)\Gamma(a)\Gamma(\delta_1)\Gamma(\delta_2)\Gamma(c-\delta_1-\delta_2)}$$
$$\cdot \int_0^1 \int_0^1 \int_x^\infty \int_0^\infty u^{r-1} v_1^{a-1} \mu^{\delta_1-1} \sigma^{\delta_2-1}$$
$$\cdot \exp\left(-\alpha u - v_1 + z e^{-u} v_1 \mu + t e^{-u} v_1 (1-\mu)\sigma\right)$$
$$\cdot \ {}_1F_1\left[\delta_1 - b; \delta_1; -z e^{-u} v_1 \mu\right] \ {}_1F_1\left[\delta_2 - b'; \delta_2; -t e^{-u} v_1 (1-\mu)\sigma\right]$$
$$\cdot (1-\mu)^{c-\delta_1-1} (1-\sigma)^{c-\delta_1-\delta_2-1} \, du \, dv_1 \, d\mu \, d\sigma \tag{3.15}$$

and

$$\varphi_{a,b,b';c}(z,t,r,\alpha) = \frac{\Gamma(c)}{\Gamma(r)\Gamma(a)\Gamma(\delta_1)\Gamma(\delta_2)\Gamma(c-\delta_1-\delta_2)}$$
$$\cdot \int_0^1 \int_0^1 \int_x^\infty \int_0^\infty u^{r-1} v_1^{a-1} \mu^{\delta_1-1} \sigma^{\delta_2-1}$$
$$\cdot \exp\left(-\alpha u - v_1 + z e^{-u} v_1 \mu + t e^{-u} v_1 (1-\mu)\sigma\right)$$

$$\cdot\,\Phi_2\left[b-\delta_1,b'-\delta_2;c;-ze^{-u}v_1\left(1-\mu\right)\left(1-\sigma\right),te^{-u}v_1\left(1-\mu\right)\left(1-\sigma\right)\right]$$
$$\cdot\left(1-\mu\right)^{c-\delta_1-1}\left(1-\sigma\right)^{c-\delta_1-\delta_2-1}\,\mathrm{d}u\,\mathrm{d}v_1\,\mathrm{d}\mu\,\mathrm{d}\sigma, \tag{3.16}$$

provided that the second member of each of the assertions (3.11) *to* (3.16) *exists.*

Proof. We recall from [2] that

$$\Phi_2\left[b,b';c;z,t\right]=\frac{\Gamma\left(c\right)}{\Gamma\left(r\right)\Gamma\left(a\right)\Gamma\left(b\right)\Gamma\left(b'\right)\Gamma\left(c-b-b'\right)}$$
$$\cdot\int_0^1\int_0^1\mu^{b-1}\sigma^{b'-1}\,\exp\left(z\mu+t\left(1-\mu\right)\sigma\right)$$
$$\cdot\left(1-\mu\right)^{c-b-1}\left(1-\sigma\right)^{c-b-b'-1}\,\mathrm{d}\mu\,\mathrm{d}\sigma. \tag{3.17}$$

If we use (3.17) in (3.6) and interchange order of summation and integration, we can get the desired result (3.11).

In the same way as above, by using the other integral representations of the Humbert function of two variables Φ_2 (see [2]), we can obtain the equations (3.12) to (3.16). □

4. Mellin-Barnes contour integral representation

In this section, we give the Mellin-Barnes contour integral representation for the families of the incomplete Hurwitz-Lerch functions defined by (2.1) and (2.2).

Theorem 4. *The Mellin-Barnes contour integral representation for* (2.2) *is given by*

$$\varphi_{a,b,b';c}\left(z,t,r,\alpha\right)=\frac{\Gamma\left(c\right)}{\Gamma\left(a\right)\Gamma\left(b\right)\Gamma\left(b'\right)}\,\frac{1}{2\pi i}\int_{\mathcal{L}_z}\int_{\mathcal{L}_t}$$
$$\cdot\,\frac{\Gamma\left(-\mu\right)\Gamma\left(-\nu\right)\Gamma\left(a+\mu+\nu,x\right)\Gamma\left(b+\mu\right)\Gamma\left(b'+\nu\right)\left[\Gamma\left(\mu+\nu+\alpha\right)\right]^r}{\Gamma\left(c+\mu+\nu\right)\left[\Gamma\left(\mu+\nu+\alpha+1\right)\right]^r}$$
$$\cdot\left(-z\right)^{\mu}\left(-t\right)^{\nu}\,\mathrm{d}\mu\,\mathrm{d}\nu\qquad\left(\left|\arg(-z)\right|<\pi;\ \left|\arg(-t)\right|<\pi\right), \tag{4.1}$$

where it is assumed that the poles of the integrand in (4.1) *are simple and the Mellin-Bernes type contours* $\mathcal{L}_z$ *and* $\mathcal{L}_t$ *of integration are so chosen that the poles of* $\Gamma\left(-\mu\right)$ *and* $\Gamma\left(-\nu\right)$ *can be separated from those of* $\Gamma\left(b+\mu\right)$ *and* $\Gamma\left(b'+\nu\right)$ *with indentations, if needed.*

Proof. Taking the sum of the residues at the poles of $\Gamma\left(-\mu\right)$ at the points $\mu=k$ $(k\in\mathbb{N}_0)$ and the poles of $\Gamma\left(-\nu\right)$ at the points $\nu=l$ $(l\in\mathbb{N}_0)$ in the equation (4.1), we readily get the following series expansion:

$$\varphi_{a,b,b';c}\left(z,t,r,\alpha\right)=\frac{\Gamma\left(c\right)}{\Gamma\left(a\right)\Gamma\left(b\right)\Gamma\left(b'\right)}$$

$$\cdot \sum_{k,l=0}^{\infty} \frac{\Gamma(a+k+l,x)\Gamma(b+k)\Gamma(b'+l)}{\Gamma(c+k+l)\,k!\,l!} \frac{z^k t^l}{(k+l+\alpha)^r},$$

which completes the proof of Theorem 4. □

5. A SUMMATION FORMULA

Theorem 5. *A summation formula for the function* $\varphi_{a,b,b';c}(z,t,r,\alpha)$ *defined by* (2.2) *is given, for* $|\beta| < \alpha$ *and* $r \neq 1$, *by*

$$\varphi_{a,b,b';c}(z,t,r,\alpha-\beta) = \sum_{p=0}^{\infty} \frac{(r)_p}{p!} \varphi_{a,b,b';c}(z,t,r+p,\alpha)\,\beta^p. \tag{5.1}$$

Proof. Using the definition (2.2) in the left-hand side of the assertion (5.1), we find that

$$\begin{aligned}
&\varphi_{a,b,b';c}(z,t,r,\alpha-\beta) \\
&= \sum_{k,l=0}^{\infty} \frac{[a,x]_{k+l}(b)_k(b')_l}{(c)_{k+l}\,k!\,l!} \frac{z^k t^l}{(k+l+\alpha)^r} \left(1-\frac{\beta}{k+l+\alpha}\right)^{-r} \\
&= \sum_{k,l=0}^{\infty} \frac{[a,x]_{k+l}(b)_k(b')_l}{(c)_{k+l}\,k!\,l!} \frac{z^k t^l}{(k+l+\alpha)^r} \left(\sum_{p=0}^{\infty} \frac{(r)_p}{p!} \frac{\beta^p}{(k+l+\alpha)^p}\right) \\
&= \sum_{p=0}^{\infty} \frac{(r)_p}{p!} \left(\sum_{k,l=0}^{\infty} \frac{[a,x]_{k+l}(b)_k(b')_l}{(c)_{k+l}\,k!\,l!} \frac{z^k t^l}{(k+l+\alpha)^{r+p}}\right) \beta^p,
\end{aligned} \tag{5.2}$$

which readily gives the desired result (5.1). □

6. DERIVATIVE FORMULAS

In this section, we consider the derivative formulas for the families of the incomplete Hurwitz-Lerch functions defined by (2.1) and (2.2). In view of the above-mentioned decomposition formula resulting from (2.1) and (2.2), we choose to record here the derivative formulas for the function $\varphi_{a,b,b';c}(z,t,r,\alpha)$ defined by (2.2). The corresponding results for the function $\phi_{a,b,b';c}(z,t,r,\alpha)$ defined by (2.1) can be derived fairly easily.

Theorem 6. *Each of the following derivative formulas for the function* $\varphi_{a,b,b';c}(z,t,r,\alpha)$ *defined by* (2.2) *holds true*:

$$\begin{aligned}
\frac{\partial^k}{\partial z^k}\{\varphi_{a,b,b';c}(z,t,r,\alpha)\} &= \frac{(a)_k(b)_k}{(c)_k} \\
&\quad \cdot \varphi_{a+k,b+k,b';c+k}(z,t,r,\alpha+k) \qquad (k \in \mathbb{N}_0),
\end{aligned} \tag{6.1}$$

$$\frac{\partial^l}{\partial t^l}\{\varphi_{a,b,b';c}(z,t,r,\alpha)\} = \frac{(a)_l(b')_l}{(c)_l}$$

$$\cdot\, \varphi_{a+l,b,b'+l;c+l}\,(z,t,r,\alpha+l) \qquad (l \in \mathbb{N}_0) \tag{6.2}$$

and

$$\frac{\partial^{k+l}}{\partial z^k \partial t^l}\left\{\varphi_{a,b,b';c}\,(z,t,r,\alpha)\right\} = \frac{(a)_{k+l}\,(b)_k\,(b')_l}{(c)_{k+l}} \cdot \varphi_{a+k+l,b+k,b'+l;c+k+l}\,(z,t,r,\alpha+k+l) \qquad (k,l \in \mathbb{N}_0)\,. \tag{6.3}$$

Proof. Differentiating both sides of the series definition (2.2) with respect to the variable z, which is valid under the conditions in (2.2), we obtain

$$\frac{\partial}{\partial z}\left\{\varphi_{a,b,b';c}\,(z,t,r,\alpha)\right\} = \sum_{l=0}^{\infty}\sum_{k=1}^{\infty} \frac{[a,x]_{k+l}\,(b)_k\,(b')_l}{(c)_{k+l}\,(k-1)!\,l!}\,\frac{z^{k-1}\,t^l}{(k+l+\alpha)^r}\,. \tag{6.4}$$

Putting $k-1=k'$ in (6.4) and dropping the prime on k, we get

$$\frac{\partial}{\partial z}\left\{\varphi_{a,b,b';c}\,(z,t,r,\alpha)\right\} = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{[a,x]_{k+l+1}\,(b)_{k+1}\,(b')_l}{(c)_{k+l+1}\,k!\,l!} \cdot \frac{z^k\,t^l}{(k+l+1+\alpha)^r}\,. \tag{6.5}$$

Next, by applying the following identities:

$$[\lambda,x]_{k+1} = \lambda\,[\lambda+1,x]_k \qquad \text{and} \qquad (\lambda)_{k+1} = \lambda\,(\lambda+1)_k \qquad (k \in \mathbb{N}_0)$$

in the last equation (6.5), we have

$$\frac{\partial}{\partial z}\left\{\varphi_{a,b,b';c}\,(z,t,r,\alpha)\right\} = \frac{ab}{c}\,\varphi_{a+1,b+1,b';c+1}\,(z,t,r,\alpha+1)\,. \tag{6.6}$$

Finally, if we successively differentiate the right-hand side of (6.6) $k-1$ times with respect to the variable z, we have the assertion (6.1) of Theorem 6.

In a similar manner, we can derive the derivative formulas (6.2) and (6.3) asserted by Theorem 6. □

Theorem 7. *The derivative formulas for the function* $\varphi_{a,b,b';c}\,(z,t,r,\alpha)$ *defined by* (2.2) *are given by*

$$D_t^n\left\{t^{b'+n-1}\,\varphi_{a,b,b';c}\,(z,t,r,\alpha)\right\} = t^{b-1}\,(b')_n\,\varphi_{a,b,b'+n;c}\,(z,t,r,\alpha)\,, \tag{6.7}$$

$$D_t^n\left\{t^{c-1}\,\varphi_{a,b,b';c}\,(zt,t,r,\alpha)\right\} = (-1)^n\,t^{c-n-1}\,(1-c)_n\,\varphi_{a,b,b';c-n}\,(zt,t,r,\alpha) \tag{6.8}$$

and

$$D_t^n\left\{t^{c-b-1}\,\varphi_{a,b,b';c}\,(z,t,r,\alpha)\right\} = (-1)^n\,t^{c-b-n-1}\,(b-c+1)_n \cdot \sum_{p=0}^{n}\binom{n}{p}\frac{(a)_p(b')_p t^p}{(c)_p(c-b-n)_p}\,\varphi_{a+p,b,b'+p;c+p}\,(z,t,r,\alpha)\,, \tag{6.9}$$

where

$$D_t^n\{f(z,t)\} = \frac{\partial^n}{\partial t^n}\{f(z,t)\}. \tag{6.10}$$

Proof. Using the definition (2.2) in the left-hand side of the assertion (6.7), we have

$$\begin{aligned}
&D_t^n\left\{t^{b'+n-1}\,\varphi_{a,b,b';c}\,(z,t,r,\alpha)\right\}\\
&\quad= D_t^n\left\{t^{b'+n-1}\sum_{k,l=0}^{\infty}\frac{[a,x]_{k+l}(b)_k(b')_l}{(c)_{k+l}\,k!\,l!}\,\frac{z^k t^l}{(k+l+\alpha)^r}\right\}\\
&\quad= \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\,\frac{z^k}{k!}\cdot D_t^n\left\{t^{b'+n-1}\sum_{l=0}^{\infty}\frac{[a+k,x]_l(b')_l}{(c+k)_l}\,\frac{t^l}{l!}\right\}\,\frac{1}{(k+l+\alpha)^r}\\
&\quad= (b')_n t^{b'-1}\sum_{k,l=0}^{\infty}\frac{[a,x]_{k+l}(b)_k(b')_l}{(c)_{k+l}\,k!\,l!}\,\frac{z^k\,t^l}{(k+l+\alpha)^r},
\end{aligned} \tag{6.11}$$

which yields the desired result (6.7) asserted by Theorem 7.

In the same way, we can get the other derivative formulas (6.8) and (6.9) asserted by Theorem 7. □

Corollary 1. *The following derivative formulas for the first Appell hypergeometric functions* Γ_1 *in* (3.2) *hold true:*

$$\begin{aligned}
&D_t^n\left\{\Gamma_1\left[(a,x),b,b';c;z,t\right]\right\}\\
&\qquad= \frac{(a)_n(b')_n}{(c)_n}\,\Gamma_1\left[(a+n,x),b,b'+n;c+n;z,t\right],
\end{aligned} \tag{6.12}$$

$$\begin{aligned}
&D_t^n\left\{t^{b'+n-1}\,\Gamma_1\left[(a,x),b,b';c;z,t\right]\right\}\\
&\qquad= t^{b'-1}\left(b'\right)_n\,\Gamma_1\left[(a,x),b,b'+n;c;z,t\right],
\end{aligned} \tag{6.13}$$

$$\begin{aligned}
&D_t^n\left\{t^{c-1}\,\Gamma_1\left[(a,x),b,b';c;zt,t\right]\right\}\\
&\qquad= (-1)^n t^{c-n-1}\,(1-c)_n\,\Gamma_1\left[(a,x),b,b';c-n;zt,t\right]
\end{aligned} \tag{6.14}$$

and

$$\begin{aligned}
D_t^n\left\{t^{c-b-1}\Gamma_1\left[(a,x),b,b';c;z,t\right]\right\} &= (-1)^n t^{c-b-n-1}\,(b-c+1)_n\\
&\cdot\sum_{p=0}^{n}\binom{n}{p}\frac{(a)_p\,(b')_p\,t^p}{(c)_p\,(c-b-n)_p}\,\Gamma_1\left[(a+p,x),b,b'+p;c+p;z,t\right],
\end{aligned} \tag{6.15}$$

where $D_t^n\{f(z,t)\}$ *is given by* (6.10).

Proof. Differentiating both sides of the equation (3.2) with respect to t for n times, we can derive the required result (6.12). By using the same operations, we can deduce the desired results (6.13), (6.14) and (6.15). □

Remark 2. The special cases of (6.12) to (6.15) when $x = 0$ are easily seen to reduce to the known derivative formulas for the Appell hypergeometric function F_1 (see, for details, [2]).

7. Recurrence relations

In this section, we will give the recurrence relation for the extended family of the incomplete Hurwitz-Lerch functions defined by (2.1) and (2.2).

Theorem 8. *The following recurrence relation for the function* $\varphi_{a,b,b';c}(z,t,r,\alpha)$ *defined by* (2.2) *holds true:*

$$\begin{aligned}\varphi_{a,b,b';c}(z,t,r,\alpha) &= \varphi_{a,b,b';c-1}(z,t,r,\alpha)\\ &+\frac{abz}{c(1-c)}\varphi_{a+1,b+1,b';c+1}(z,t,r,\alpha+1)\\ &+\frac{ab't}{c(1-c)}\varphi_{a+1,b,b'+1;c+1}(z,t,r,\alpha+1).\end{aligned} \tag{7.1}$$

Proof. Applying the following well-known contiguous formula for the function ${}_0F_1$ given in [9, 20, 24]:

$${}_0F_1[\,—\,;c-1;z] - {}_0F_1[\,—\,;c;z] - \frac{z}{c(c-1)}\,{}_0F_1[\,—\,;c+1;z] = 0 \tag{7.2}$$

to the integral representation in (3.8), we can derive the required result (7.1). □

Corollary 2. *The following recurrence relations for the function* $\varphi_{a,b,b';c}(z,t,r,\alpha)$ *defined by* (2.2) *hold true*:

$$\begin{aligned}\Gamma_1[(a,x),b+n,b';c;z,t] &= \Gamma_1[(a,x),b,b';c;z,t]\\ &+\frac{az}{c}\sum_{m=0}^{n}\Gamma_1[(a+1,x),b+m,b';c+1;z,t].\end{aligned} \tag{7.3}$$

$$\begin{aligned}\Gamma_1[(a,x),b-n,b';c;z,t] &= \Gamma_1[(a,x),b,b';c;z,t]\\ &-\frac{az}{c}\sum_{m=0}^{n-1}\Gamma_1[(a+1,x),b-m,b';c+1;z,t]\end{aligned} \tag{7.4}$$

and

$$\begin{aligned}\Gamma_1[(a,x),b,b';c-n;z,t] &= \Gamma_1[(a,x),b,b';c;z,t]\\ &+abz\sum_{m=0}^{n}\frac{\Gamma_1[(a+1,x),b+1,b';c+2-m;z,t]}{(c-p)(c-m+1)}\end{aligned}$$

$$+ab't\sum_{m=0}^{n}\frac{\Gamma_1[(a+1,x),b+1,b';c+2-m;z,t]}{(c-m)(c-m+1)}. \tag{7.5}$$

Proof. Using an obvious identity for $(b+1)_k$ in terms of $(b)_k$ in the definition (3.2) of the incomplete Appell hypergeometric function Γ_1, we obtain

$$\begin{aligned}\Gamma_1\left[(a,x),b+1,b';c;z,t\right] &= \Gamma_1\left[(a,x),b,b';c;z,t\right]\\ &\quad+\frac{az}{c}\Gamma_1\left[(a+1,x),b+1,b';c+1;z,t\right].\end{aligned} \tag{7.6}$$

Appropriate iteration of this procedure for n times on Γ_1 with the parameter $b+n$, we can get the desired result (7.3).

In the same way as above, we can obtain the recursion relations (7.4) and (7.5). □

Theorem 9. *The following recurrence relations for the function* $\varphi_{a,b,b';c}(z,t,r,\alpha)$ *defined by* (2.2) *hold true*:

$$\begin{aligned}\varphi_{a,b+n,b';c}(z,t,r,\alpha) &= \varphi_{a,b,b';c}(z,t,r,\alpha)\\ &\quad+\frac{az}{c}\sum_{m=0}^{n}\varphi_{a+1,b+m,b';c+1}(z,t,r,\alpha),\end{aligned} \tag{7.7}$$

$$\begin{aligned}\varphi_{a,b-n,b';c}(z,t,r,\alpha) &= \varphi_{a,b,b';c}(z,t,r,\alpha)\\ &\quad-\frac{az}{c}\sum_{m=0}^{n-1}\varphi_{a+1,b-m,b';c+1}(z,t,r,\alpha)\end{aligned} \tag{7.8}$$

and

$$\begin{aligned}\varphi_{a,b,b';c-n}(z,t,r,\alpha) &= \varphi_{a,b,b';c}(z,t,r,\alpha)\\ &\quad+abz\sum_{m=0}^{n}\frac{\varphi_{a+1,b+1,b';c+2-m}(z,t,r,\alpha)}{(c-m)(c-m+1)}\\ &\quad+ab't\sum_{m=0}^{n}\frac{\varphi_{a+1,b,b'+1;c+2-m}(z,t,r,\alpha)}{(c-m)(c-m+1)}.\end{aligned} \tag{7.9}$$

Proof. If we use the equation (7.3) in (3.4), after some simple rearrangements, we can get the desired result (7.7).

In the same way as above, by using the other recursion formulas (7.4) and (7.5) of the incomplete first Appell hypergeometric function Γ_1, we get the equations (7.8) and (7.9). □

Corollary 3. *The following recurrence relations for the function* $\Gamma_1[(a,x),b,b';c;z,t]$ *defined by* (3.2) *hold true*:

$$\Gamma_1\left[(a,x),b+n,b';c;z,t\right]$$

$$= \sum_{p=0}^{n} \binom{n}{p} \frac{(a)_p}{(c)_p} \Gamma_1 \left[(a+p,x), b+p, b'; c+m; z, t\right] z^p \tag{7.10}$$

and

$$\Gamma_1 \left[(a,x), b-n, b'; c; z, t\right] = \sum_{p=0}^{n} \binom{n}{p} \frac{(a)_p}{(c)_p} \Gamma_1 \left[(a+p,x), b, b'; c+p; z, t\right] (-z)^p . \tag{7.11}$$

Proof. Using the recursion relation (7.6), we can compute the incomplete first Appell hypergeometric function Γ_1 with the parameter $b+2$ as follows:

$$\begin{aligned}\Gamma_1 \left[(a,x), b+2, b'; c; z, t\right] &= \Gamma_1 \left[(a,x), b, b'; c; z, t\right] \\ &+ \frac{2az}{c} \Gamma_1 \left[(a+1,x), b+1, b'; c+1; z, t\right] \\ &+ \frac{(a)_2}{(c)_2} z^2 \Gamma_1 \left[(a+2,x), b+2, b'; c+2; z, t\right].\end{aligned}$$

Now, by iterating this procedure for n times on Γ_1 with the parameter $b+n$, we obtain the recursion formula (7.10). Similarly, by making use of the same calculations, we can deduce the desired result (7.11). □

Theorem 10. *The following recurrence relations hold true*:

$$\varphi_{a,b+n,b';c}(z,t,r,\alpha) = \sum_{m=0}^{n} \binom{n}{m} \frac{(a)_m}{(c)_m} \cdot \varphi_{a+m,b+m,b';c+m}(z,t,r,\alpha+m) z^m \tag{7.12}$$

and

$$\varphi_{a,b-n,b';c}(z,t,r,\alpha) = \sum_{m=0}^{n} \binom{n}{m} \frac{(a)_m}{(c)_m} \cdot \varphi_{a+m,b+m,b';c+m}(z,t,r,\alpha+m) (-z)^m. \tag{7.13}$$

Proof. If we use the equation (7.10) in (3.4), after some simple rearrangements, we can derive the desired result (7.12). In precisely the same way, by using the other recursion formula (7.11) for the incomplete first Appell hypergeometric functions Γ_1, we get the equation (7.13). □

Remark 3. The special cases of our results (7.3) to (7.5), (7.10) and (7.11) when $x=0$ are easily seen to reduce to the known recursion formulas for the Appell hypergeometric function F_1 (see [25]).

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A CERTAIN CLASS OF ANALYTIC FUNCTIONS OF COMPLEX ORDER CONNECTED WITH A q-ANALOGUE OF INTEGRAL OPERATORS

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Abstract. In this paper, we introduce a certain class $\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$ of normalized analytic functions of complex order connected with a q-analogue of integral operators. For this complex-order analytic function class, we determine a sufficient condition in terms of the coefficients, estimates for the coefficients and a maximization theorem concerning the coefficients. Various consequences and applications of our main results are also considered. A brief remark about the demonstrated equivalence of the q-calculus and the so-called (p,q)-calculus is also presented.

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Keywords: analytic functions, univalent functions, complex-order analytic functions, complex-order univalent functions, integral operators, q-numbers and q-factorials, q-derivative operator, maximization theorem, equivalence of the q-calculus and the so-called (p,q)-calculus

1. Introduction, Definitions and Preliminaries

The theory of q-calculus plays an important rôle in many areas of mathematical, physical and engineering sciences. Jackson (see [8] and [9]) was the first to have some applications of the q-calculus and introduced the q-analogue of the classical derivative and integral operators (see also [1]). Let $\mathcal{A}$ denote the class of functions $f(z)$ of the following normalized form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (z \in \mathbb{U}), \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}$ given by

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}).$$

We also let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of normalized analytic functions which are univalent in $\mathbb{U}$.

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For a function $f(z)$ given by (1.1) and the $g(z)$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \qquad (z \in \mathbb{U}), \tag{1.2}$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is defined here by

$$(f * g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k =: (g * f)(z). \tag{1.3}$$

We use Ω to denote the class of Schwarz functions $w(z)$, which are analytic in U and satisfy the conditions

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1 \quad (z \in \mathbb{U}).$$

We now define the integral operator $\mathcal{K}_{m,n}^{\alpha} : \mathcal{A} \to \mathcal{A}$ for $\alpha > 0$ and $m \geqq 0$ as follows:

$$\mathcal{K}_m^0 f(z) = f(z)$$

and

$$\mathcal{K}_m^{\alpha} f(z) = \frac{(m+1)^{\alpha}}{\Gamma(\alpha) z^m} \int_0^z t^{m-1} \left(\log \frac{z}{t}\right)^{\alpha-1} f(t)dt. \tag{1.4}$$

For $f \in \mathcal{A}$, it can be easily verified that

$$\mathcal{K}_m^{\alpha} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{m+1}{m+k}\right)^{\alpha} a_k z^k. \tag{1.5}$$

Next, for $0 < q < 1$, the q-derivative of the function $\mathcal{K}_m^{\alpha} f(z) \in \mathcal{A}$ is defined by

$$D_q \{\mathcal{K}_m^{\alpha} f(z)\} = \frac{\mathcal{K}_m^{\alpha} f(z) - \mathcal{K}_m^{\alpha} f(qz)}{z(1-q)} \qquad (z \neq 0), \tag{1.6}$$

so that

$$D_q \left\{ z + \sum_{k=2}^{\infty} \left(\frac{m+1}{m+k}\right)^{\alpha} a_k z^k \right\} = 1 + \sum_{k=2}^{\infty} \left(\frac{m+1}{m+k}\right)^{\alpha} [k]_q \, a_k z^{k-1},$$

where

$$[k]_q := \frac{1-q^k}{1-q} = 1 + \sum_{j=1}^{k-1} q^j \qquad \text{and} \qquad [0]_q = 0.$$

Remark 1. The first usage of the above-defined q-derivative operator D_q in Geometric Function Theory was made in 1990 by Ismail *et al.* [7] (see also [1]). Moreover, a firm footing of the usage of the q-calculus in the context of Geometric Function Theory was actually provided and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory in a 1989 book-chapter by Srivastava (see, for details, [21]). Several recent developments on various applications of the the q-derivative operator D_q in Geometric Function Theory can be found in (for example) [2, 13, 14, 16, 18, 22–24, 26].

It is easily seen from (1.6) that

$$z\,D_q\left\{\mathcal{K}_m^{\alpha} f(z)\right\} = z + \sum_{k=2}^{\infty}\left(\frac{m+1}{m+k}\right)^{\alpha}[k]_q\; a_k z^k. \tag{1.7}$$

For any non-negative integer n, the q-factorial $[n]_q!$ is given by

$$[n]_q! = \begin{cases} 1 & (k=0) \\ [1]_q\;[2]_q\;[3]_q\;\cdots\;[n]_q & (n\in\mathbb{N}), \end{cases} \tag{1.8}$$

where $\mathbb{N}$ denotes the set positive integers. Also the q-Pochhammer symbol $[\lambda]_{q,n}$ $(\nu\in\mathbb{C})$ is defined by

$$[\nu]_{q,n} = \begin{cases} 1 & (n=0) \\ [\nu]_q\;[\nu+1]_q\;\cdots\;[\nu+n-1]_q & (n\in\mathbb{N}). \end{cases} \tag{1.9}$$

For $\lambda > -1$, we define the operator $\mathcal{N}_{m,q}^{\lambda,\alpha}$ by

$$\mathcal{N}_{m,q}^{\lambda,\alpha} f(z) * \mathcal{M}_{q,\lambda+1}(z) = z\,D_q\left\{\mathcal{K}_m^{\alpha} f(z)\right\}, \tag{1.10}$$

where the function $\mathcal{M}_{q,\lambda+1}(z)$ is given by

$$\mathcal{M}_{q,\lambda+1}(z) = z + \sum_{k=2}^{\infty}\frac{[\lambda+1]_{q,k-1}}{[k-1]_q!}\,z^k.$$

We thus obtain

$$\begin{aligned}\mathcal{N}_{m,q}^{\lambda,\alpha} f(z) &= z + \sum_{k=2}^{\infty}\left(\frac{m+1}{m+k}\right)^{\alpha}\frac{[k]_q\;[k-1]_q!}{[\lambda+1]_{q,k-1}}\,a_k z^k \\ &= z + \sum_{k=2}^{\infty}\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha} a_k z^k \end{aligned} \tag{1.11}$$

$$(\alpha>0;\ \lambda>-1;\ m\geqq 0;\ 0<q<1).$$

We can easily verify from (1.11) that

$$[\lambda+1]_q\;\mathcal{N}_{m,q}^{\lambda,\alpha} f(z) = [\lambda]_q\;\mathcal{N}_{m,q}^{\lambda+1,\alpha} f(z) + q^{\lambda}\,z\,D_q\left\{\mathcal{N}_{m,q}^{\lambda+1,\alpha} f(z)\right\}. \tag{1.12}$$

We also note that

$$\lim_{q\to 1-}\mathcal{N}_{m,q}^{\lambda,\alpha} f(z) = I_m^{\lambda,\alpha} f(z) = z + \sum_{k=2}^{\infty}\frac{k!}{(\lambda+1)_{k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha} a_k z^k. \tag{1.13}$$

In the special case when $\alpha=0$, we have

$$\mathcal{N}_{m,q}^{\lambda,0} f(z) =: \mathfrak{J}_q^{\lambda} f(z).$$

The operator in $\mathfrak{J}_q^{\lambda} f(z)$ was studied by Arif *et al.* [5].

Definition 1. We say that a function $f(z)$ belonging to $\mathcal{A}$ is in the normalized complex-order analytic function class

$$\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M}) \quad \left(\zeta \in \mathbb{C}^* := \mathbb{C}\setminus\{0\};\ \mathcal{M} > \frac{1}{2}\right)$$

if and only if

$$\left|1-\frac{1}{\zeta}+\frac{z\left(\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right)'}{\zeta\,\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)}-\mathcal{M}\right| < \mathcal{M} \tag{1.14}$$

$$(\alpha>0;\ \lambda>-1;\ m\geqq 0;\ 0<q<1;\ z\in\mathbb{U}).$$

By letting $q\to 1-$, it follows from the work of Kulshrestha [11] that

$$g(z)\in\mathcal{H}_{m,q}^{1,0}(1,\mathcal{M}) = F(1,\mathcal{M})$$

if and only if

$$\frac{zg'(z)}{g(z)} = \frac{1+w(z)}{1-mw(z)} \qquad \left(m=1-\frac{1}{\mathcal{M}};\ \mathcal{M}>\frac{1}{2};\ w(z)\in\Omega\right) \tag{1.15}$$

for $z\in\mathbb{U}$.

It can easily be shown that $f(z)\in\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$ if and only if there exists a function

$$g(z)\in\lim_{q\to 1-}\mathcal{H}_{m,q}^{1,0}(1,\mathcal{M}) = F(1,\mathcal{M})$$

such that

$$\mathcal{N}_{m,q}^{\lambda,\alpha}f(z) = z\left[\frac{g(z)}{z}\right]^{\zeta}. \tag{1.16}$$

Thus, from (1.15) and (1.16), it follows that $f(z)\in\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$ if and only if

$$\frac{z\left(\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right)'}{\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)} = \frac{1+[\zeta(1+m)-m]\,w(z)}{1-mw(z)} \tag{1.17}$$

$$\left(m=1-\frac{1}{\mathcal{M}};\ \mathcal{M}>\frac{1}{2};\ w(z)\in\Omega\right)$$

for $z\in\mathbb{U}$.

By giving specific values to the parameters λ, α and ζ, we obtain the following interesting subclasses:

(i) $\lim_{q\to 1-}\mathcal{H}_{m,q}^{1,0}(\zeta,\mathcal{M}) = F(\zeta,\mathcal{M})$ (see Nasr and Aouf [17]);

(ii) $\lim_{q\to 1-}\mathcal{H}_{m,q}^{1,0}(1,\mathcal{M}) = F(1,\mathcal{M})$ (see Singh and Singh [19]);

(iii) $\lim_{q\to 1-}\mathcal{H}_{m,q}^{1,0}(\cos\lambda e^{-i\lambda},\mathcal{M}) = F_{\lambda,\mathcal{M}} \quad \left(|\lambda|<\frac{\pi}{2}\right)$ (see Kulshrestha [11]);

(iv) $\lim\limits_{q\to 1-} \mathcal{H}_{m,q}^{1,0}((1-\alpha)\cos\lambda e^{-i\lambda},\infty) = S^{\lambda}(\alpha) \quad \left(|\lambda| < \frac{\pi}{2};\ 0 \leqq \alpha < 1\right)$
(see Libera [12]; see also Chichra [6] and Sižuk [20]);

(v) $\lim\limits_{q\to 1-} \mathcal{H}_{m,q}^{1,0}\big((1-\alpha)\cos\lambda e^{-i\lambda},\mathcal{M}\big) = F_{\mathcal{M}}(\lambda,\alpha) \quad \left(|\lambda| < \frac{\pi}{2};\ 0 \leqq \alpha < 1\right)$
(see Aouf [3] and Aouf [4]).

We also have the following presumably new function classes:

(i) $\lim\limits_{q\to 1-} \mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M}) =: \mathcal{S}_m^{\lambda,\alpha}(\zeta,\mathcal{M})$, where

$$\mathcal{S}_m^{\lambda,\alpha}(\zeta,\mathcal{M}) := \left\{ f : f(z) \in \mathcal{A} \quad \text{and} \quad \left| 1 - \frac{1}{\zeta} + \frac{z\big(I_m^{\lambda,\alpha} f(z)\big)'}{\zeta\, I_m^{\lambda,\alpha} f(z)} - \mathcal{M} \right| < \mathcal{M} \right.$$
$$\left. \left(M > \frac{1}{2};\ \zeta \in \mathbb{C}^*;\ \alpha > 0;\ \lambda > -1;\ m \geqq 0;\ z \in \mathbb{U} \right) \right\};$$

(ii) $\mathcal{H}_{m,q}^{\lambda,0}(\zeta,\mathcal{M}) =: \mathcal{F}_q^{\lambda}(\zeta,\mathcal{M})$, where

$$\mathcal{F}_q^{\lambda}(\zeta,\mathcal{M}) := \left\{ f : f(z) \in \mathcal{A} \quad \text{and} \quad \left| 1 - \frac{1}{\zeta} + \frac{z\big(\mathfrak{J}_q^{\lambda} f(z)\big)'}{\zeta\, \mathfrak{J}_q^{\lambda} f(z)} - \mathcal{M} \right| < \mathcal{M} \right.$$
$$\left. \left(\mathcal{M} > \frac{1}{2};\ \zeta \in \mathbb{C}^*;\ \lambda > -1;\ m \geqq 0;\ 0 < q < 1;\ z \in \mathbb{U} \right) \right\}.$$

From the above definitions of the function classes $F(\zeta,\mathcal{M})$ and $\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$, we note that

$$f(z) \in \mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M}) \quad \Longleftrightarrow \quad \mathcal{N}_{m,q}^{\lambda,\alpha} f(z) \in F(\zeta,\mathcal{M}). \tag{1.18}$$

The purpose of the present paper is to determine a sufficient condition in terms of the coefficients for functions belonging to the normalized complex-order analytic function class $\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$, estimates for the coefficients and a maximization theorem involving $\left|a_3 - \mu a_2^2\right|$ for the class $\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$ for complex values of the parameter μ.

2. Sufficient Condition for a Function to be in the Class $\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$

Unless otherwise mentioned, we assume throughout this paper that

$$\alpha > 0,\ \ \lambda > -1,\ \ m \geqq 0,\ \ 0 < q < 1,\ \ \zeta \in \mathbb{C}^*,$$

$$m = 1 - \frac{1}{\mathcal{M}},\ \ \mathcal{M} > \frac{1}{2} \quad \text{and} \quad z \in \mathbb{U}.$$

Theorem 1. *Let the function $f(z)$ be defined by (1.1). Also let the following inequality holds true:*

$$\sum_{k=2}^{\infty}\left\{(k-1)+|\zeta(1+m)+m(k-1)|\right\}|a_k|\,\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}|\zeta(1+m)|. \tag{2.1}$$

Then $f(z)$ belongs to the class normalized complex-order analytic function class $\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$.

Proof. Suppose that the inequality (2.1) holds true. Then we find for $z\in\mathbb{U}$ that

$$\begin{aligned}
&\left|z\left(\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right)'-\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right|-\left|\zeta(1+m)\,\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)+m\left\{z\left(\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right)'-\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right\}\right|\\
&=\left|\sum_{k=2}^{\infty}(k-1)\,\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}a_kz^k\right|\\
&\quad-\left|\zeta(1+m)\left\{z+\sum_{k=2}^{\infty}\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}a_kz^k\right\}\right.\\
&\quad\left.+m\left\{\sum_{k=2}^{\infty}(k-1)\,\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}a_kz^k\right\}\right|\\
&=\left|\sum_{k=2}^{\infty}(k-1)\,\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}a_kz^k\right|\\
&\quad-\left|\zeta(1+m)z+\sum_{k=2}^{\infty}\{\zeta(1+m)+m(k-1)\}\cdot\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}a_kz^k\right|\\
&\leqq\sum_{k=2}^{\infty}(k-1)\,\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}|a_k|\,r^k\\
&\quad-\left\{|\zeta(1+m)|\,r-\sum_{k=2}^{\infty}\{|\zeta(1+m)|+m(k-1)\}\cdot\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}|a_k|\,r^k\right\}\\
&=\sum_{k=2}^{\infty}\{(k-1)+|\zeta(1+m)|+m(k-1)\}\cdot\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}|a_k|\,r^k-|\zeta(1+m)|\,r.
\end{aligned}$$

Letting $r\to1-$ in the above equation, we get

$$\begin{aligned}
&\left|z\left(\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right)'-\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right|-\left|\zeta(1+m)\,\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)+m\left\{z\left(\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right)'-\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right\}\right|\\
&\leqq\sum_{k=2}^{\infty}\{(k-1)+|\zeta(1+m)|+m(k-1)\}\,|a_k|\cdot\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}-|\zeta(1+m)|\leqq0,
\end{aligned}$$

where we have made use of the assertion (2.1) of Theorem 1. Consequently, we obtain

$$\left|\frac{\dfrac{z\left(\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right)'}{\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)}-1}{\zeta(1+m)+m\left(\dfrac{z\left(\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right)'}{\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)}-1\right)}\right|<1 \qquad (z\in\mathbb{U}).$$

If we now set

$$w(z)=\frac{\dfrac{z\left(\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right)'}{\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)}-1}{\zeta(1+m)+m\left(\dfrac{z\left(\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right)'}{\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)}-1\right)},$$

then $w(0)=0$, $w(z)$ is analytic in the open unit disk $\mathbb{U}$ and

$$|w(z)|<1 \qquad (z\in\mathbb{U}).$$

Hence we have

$$\frac{z\left(\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right)'}{\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)}=\frac{1+[\zeta(1+m)-m]\,w(z)}{1-mw(z)},$$

which shows that the function $f(z)$ belongs to the class $\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$. □

In the limit when $q\to 1-$ in Theorem 1, we obtain the following corollary.

Corollary 1. *Let the function $f(z)$ be defined by (1.1). Also let the following inequality holds true*:

$$\sum_{k=2}^{\infty}\{(k-1)+|\zeta(1+m)+m(k-1)|\}\,|a_k|\cdot\frac{k!}{(\lambda+1)_{k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}\leqq|\zeta(1+m)|. \tag{2.2}$$

Then the function $f(z)$ belongs to the class $\mathcal{S}_m^{\lambda,\alpha}(\zeta,\mathcal{M})$.

If we set $\alpha=0$ in Theorem 1, we obtain the following corollary.

Corollary 2. *Let the function $f(z)$ be defined by (1.1). Also let the following inequality holds true*:

$$\sum_{k=2}^{\infty}\{(k-1)+|\zeta(1+m)+m(k-1)|\}\,|a_k|\,\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\leqq|\zeta(1+m)|. \tag{2.3}$$

Then the function $f(z)$ belongs to the class $\mathcal{F}_q^{\lambda}(\zeta,\mathcal{M})$.

3. Coefficient Estimates

In this section, we first state and prove the following result.

Theorem 2. *Let the function $f(z)$ given by (1.1) be in the normalized complex-order analytic function class $\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$.*
(a) *If*

$$2m(k-1)\Re(\zeta) > (k-1)^2(1-m) - |\zeta|^2(1+m),$$

let

$$\mathcal{G} = \left[\frac{2m(k-1)\Re(\zeta)}{(k-1)^2(1-m) - |\zeta|^2(1+m)}\right] \qquad (k = 2,3,4,\cdots,j-1),$$

where $\mathcal{N} = [\mathcal{G}]$ (the Gaussian symbol) and $[\mathcal{G}]$ is the greatest integer not greater than $\mathcal{G}$. Then

$$|a_j| \leqq \frac{[\lambda+1]_{q,j-1}}{[j]_q!\left(\frac{m+1}{m+j}\right)^{\alpha}(j-1)!}\prod_{k=2}^{j}|\zeta(1+m)+m(k-2)| \tag{3.1}$$

$$(j = 2,3,4,\cdots,\mathcal{N}+2)$$

and

$$|a_j| \leqq \frac{[\lambda+1]_{q,j-1}}{[j]_q!\,(j-1)\left(\frac{m+1}{m+j}\right)^{\alpha}(\mathcal{N}+1)!}\cdot\prod_{k=2}^{\mathcal{N}+3}|\zeta(1+m)+m(k-2)|. \tag{3.2}$$

$$(j > \mathcal{N}+2)$$

(b) *If*

$$2m(k-1)\Re(\zeta) \leqq (k-1)^2(1-m) - |\zeta|^2(1+m),$$

then

$$|a_j| \leqq \frac{[\lambda+1]_{q,j-1}(1+m)|\zeta|}{\left(\frac{m+1}{m+j}\right)^{\alpha}[j]_q!\,(j-1)} \qquad (j \geqq 2). \tag{3.3}$$

The inequalities (3.1) and (3.3) are sharp.

Proof. Let us assume that $f(z) \in \mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$. Then we find from (1.16) that

$$\sum_{k=2}^{\infty}(k-1)\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}a_kz^k$$
$$= \left\{\zeta(1+m)z + \sum_{k=2}^{\infty}\{\zeta(1+m)+m(k-1)\}\cdot\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}a_kz^k\right\}w(z), \tag{3.4}$$

which is equivalent to

$$\sum_{k=2}^{j}(k-1)\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}a_kz^k+\sum_{k=j+1}^{\infty}c_kz^k$$
$$=\left\{\zeta(1+m)z+\sum_{k=2}^{j-1}\{\zeta(1+m)+m(k-1)\}\cdot\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}a_kz^k\right\}w(z),$$

where the coefficients c_j are some complex numbers and the series $\sum\limits_{k=j+1}^{\infty}c_kz^k$ converges when $z\in\mathbb{U}$. Then, since

$$|w(z)|<1 \qquad (z\in\mathbb{U}),$$

we have

$$\left|\sum_{k=2}^{j}(k-1)\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}a_kz^k+\sum_{k=j+1}^{\infty}c_kz^k\right|$$
$$\leqq\left|\zeta(1+m)z+\sum_{k=2}^{j-1}\{\zeta(1+m)+m(k-1)\}\cdot\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\frac{m+1}{m+k}\right)^{\alpha}a_kz^k\right|. \tag{3.5}$$

Squaring both sides of (3.5), we get

$$\sum_{k=2}^{j}(k-1)^2\left(\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\right)^2\left(\frac{m+1}{m+k}\right)^{2\alpha}|a_k|^2r^{2k}+\sum_{k=j+1}^{\infty}|c_k|^2r^{2k}$$
$$\leqq\left\{(1+m)^2|\zeta|^2r^2+\sum_{k=2}^{j-1}|\zeta(1+m)+m(k-1)|^2\left(\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\right)^2\right.$$
$$\left.\cdot\left(\frac{m+1}{m+k}\right)^{2\alpha}|a_k|^2r^{2k}\right\}.$$

We now let $r\to1-$. Then, on some simplification, we obtain

$$(j-1)^2|a_j|^2\left(\frac{[j]_q!}{[\lambda+1]_{q,j-1}}\right)^2\left(\frac{m+1}{m+j}\right)^{2\alpha}$$
$$\leqq(1+m)^2|\zeta|^2+\sum_{k=2}^{j-1}\left\{|\zeta(1+m)+m(k-1)|^2-(k-1)^2\right\}$$
$$\cdot|a_k|^2\left(\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\right)^2\left(\frac{m+1}{m+k}\right)^{2\alpha}. \tag{3.6}$$

The following two cases arise:
(a) Let

$$2m(k-1)\Re(\zeta)>(k-1)^2(1-m)-|\zeta|^2(1+m).$$

Suppose also that $j \leqq \mathcal{N}+2$. Then, for $j=2$, the equation (3.6) gives

$$|a_2| \leqq \frac{(1+m)\,[\lambda+1]_{q,1}\,|\zeta|}{[2]_q!\left(\dfrac{m+1}{m+2}\right)^{\alpha}},$$

which yields (3.1) for $j=2$. We establish the assertion (3.1) by appealing to the principle of mathematical induction. Suppose (3.1) is valid for $k=2,3,4,\cdots,j-1$. Then, clearly, it follows from (3.6) that

$$\begin{aligned}
&(j-1)^2\,|a_j|^2\left(\frac{[j]_q!}{[\lambda+1]_{q,j-1}}\right)^2\left(\frac{m+1}{m+j}\right)^{2\alpha}\\
&\leqq (1+m)^2\,|\zeta|^2+\sum_{k=2}^{j-1}\left(\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\right)^2\left(\frac{m+1}{m+k}\right)^{2\alpha}\\
&\quad\cdot\Big\{|\zeta(1+m)+m(k-1)|^2-(k-1)^2\Big\}\\
&\quad\cdot\frac{\Big([\lambda+1]_{q,k-1}\Big)^2}{\Big([k]_q!\Big)^2\left(\dfrac{m+1}{m+k}\right)^{2\alpha}((k-1)!)^2}\prod_{p=2}^{k}|\zeta(1+m)+m(p-2)|^2\\
&=\frac{1}{((j-2)!)^2}\prod_{k=2}^{j}|\zeta(1+m)+m(k-2)|^2.
\end{aligned}$$

We thus find that

$$|a_j| \leqq \frac{[\lambda+1]_{q,j-1}}{[j]_q!\left(\dfrac{m+1}{m+j}\right)^{\alpha}(j-1)!}\prod_{k=2}^{j}|\zeta(1+m)+m(k-2)|,$$

which completes the proof of the assertion (3.1) of Theorem 2.

We next suppose that $j>\mathcal{N}+2$. Then (3.6) gives

$$\begin{aligned}
&(j-1)^2\,|a_j|^2\left(\frac{[j,q]!}{[\lambda+1]_{q,j-1}}\right)^2\left(\frac{m+1}{m+j}\right)^{2\alpha}\\
&\leqq (1+m)^2\,|\zeta|^2+\sum_{k=2}^{\mathcal{N}+2}\left(\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\right)^2\left(\frac{m+1}{m+k}\right)^{2\alpha}\\
&\quad\cdot\Big\{|\zeta(1+m)+m(k-1)|^2-(k-1)^2\Big\}\,|a_k|^2\\
&\quad+\sum_{k=\mathcal{N}+3}^{j-1}\left(\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\right)^2\left(\frac{m+1}{m+k}\right)^{2\alpha}
\end{aligned}$$

$$\cdot\left\{|\zeta(1+m)+m(k-1)|^2-(k-1)^2\right\}|a_k|^2$$
$$\leqq (1+m)^2|\zeta|^2+\sum_{k=2}^{\mathcal{N}+2}\left(\frac{[k]_q!}{[\lambda+1]_{q,k-1}}\right)^2\left(\frac{m+1}{m+k}\right)^{2\alpha}$$
$$\cdot\left\{|\zeta(1+m)+m(k-1)|^2-(k-1)^2\right\}|a_k|^2.$$

Upon substituting the above-derived upper estimates for $a_2, a_3, \cdots, a_{\mathcal{N}+2}$ if we simplify the resulting equations, we obtain the assertion (3.2) of Theorem 2.

(b) If we let

$$2m(k-1)\Re(\zeta) \leqq (k-1)^2(1-m)-|\zeta|^2(1+m),$$

then it follows from (3.6) that

$$\left(\frac{[j]_q!}{[\lambda+1,]_{q,j-1}}\right)^2\left(\frac{m+1}{m+j}\right)^{2\alpha}(j-1)^2|a_j|^2 \leqq (1+m)^2|\zeta|^2 \qquad (j \geqq 2),$$

which proves the assertion (3.3) of Theorem 2. □

Taking $q \to 1-$ in Theorem 2, we obtain the following corollary.

Corollary 3. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{S}_m^{\lambda,\alpha}(\zeta,\mathcal{M})$.*

(a) *If*

$$2m(k-1)\Re(\zeta) > (k-1)^2(1-m)-|\zeta|^2(1+m),$$

let

$$\mathcal{G}=\left[\frac{2m(k-1)\Re(\zeta)}{(k-1)^2(1-m)-|\zeta|^2(1+m)}\right] \qquad (k=2,3,4,\cdots,j-1),$$

where $\mathcal{N}=[\mathcal{G}]$ (the Gaussian symbol) and $[\mathcal{G}]$ is the greatest integer not greater than $\mathcal{G}$. Then

$$|a_j| \leqq \frac{(\lambda+1)_{k-1}}{\left(\frac{m+1}{m+j}\right)^{\alpha}k!(j-1)!}\prod_{k=2}^{j}|\zeta(1+m)+m(k-2)| \tag{3.7}$$
$$(j=2,3,4,\cdots,\mathcal{N}+2)$$

and

$$|a_j| \leqq \frac{(\lambda+1)_{k-1}}{(j-1)k!\left(\frac{m+1}{m+j}\right)^{\alpha}(\mathcal{N}+1)!}\prod_{k=2}^{\mathcal{N}+3}|\zeta(1+m)+m(k-2)| \tag{3.8}$$
$$(j>\mathcal{N}+2).$$

(b) *If*

$$2m(k-1)\Re(\zeta) \leqq (k-1)^2(1-m) - |\zeta|^2(1+m),$$

then

$$|a_j| \leqq \frac{(\lambda+1)_{k-1}(1+m)|\zeta|}{\left(\frac{m+1}{m+j}\right)^{\alpha} k!(j-1)} \qquad (j \geqq 2). \tag{3.9}$$

The inequalities (3.7) and (3.9) are sharp.

If we set $\alpha = 0$ in Theorem 2, then we obtain the following corollary.

Corollary 4. *Let the function $f(z)$ be defined by (1.1) be in the class $\mathcal{F}_q^{\lambda}(\zeta, \mathcal{M})$.*
(a) *If*

$$2m(k-1)\Re(\zeta) > (k-1)^2(1-m) - |\zeta|^2(1+m),$$

let

$$\mathcal{G} = \left[\frac{2m(k-1)\Re(\zeta)}{(k-1)^2(1-m) - |\zeta|^2(1+m)}\right] \qquad (k = 2,3,4,\cdots,j-1),$$

where $\mathcal{N} = [\mathcal{G}]$ (the Gaussian symbol) and $[\mathcal{G}]$ is the greatest integer not greater than $\mathcal{G}$. Then

$$|a_j| \leqq \frac{[\lambda+1]_{q,j-1}}{[j]_q!\,(j-1)!} \prod_{k=2}^{j} |\zeta(1+m) + m(k-2)| \tag{3.10}$$

$$(j = 2,3,4,\cdots,\mathcal{N}+2)$$

and

$$|a_j| \leqq \frac{[\lambda+1]_{q,j-1}}{[j]_q!(j-1)\,(\mathcal{N}+1)!} \prod_{k=2}^{\mathcal{N}+3} |\zeta(1+m) + m(k-2)| \tag{3.11}$$

$$(j > \mathcal{N}+2).$$

(b) *If*

$$2m(k-1)\Re(\zeta) \leqq (k-1)^2(1-m) - |\zeta|^2(1+m),$$

then

$$|a_j| \leqq \frac{[\lambda+1]_{q,j-1}(1+m)|\zeta|}{[j]_q!\,(j-1)} \qquad (j \geqq 2). \tag{3.12}$$

The inequalities (3.10) and (3.12) are sharp.

4. Maximization of $\left|a_3-\mu a_2^2\right|$

In this section, we shall need the following lemma in our discussion.

Lemma 1 ([10]). *Let*

$$w(z)=\sum_{k=1}^{\infty} c_k z^k \in \Omega.$$

If μ is any complex number, then

$$\left|c_2-\mu c_1^2\right| \leqq \max\{1,|\mu|\} \tag{4.1}$$

for any complex number μ. Equality in (4.1) may be attained with the functions $w(z)=z^2$ and $w(z)=z$ for $|\mu|<1$ and $|\mu| \geqq 1$, respectively.

We now state and prove our main result in this section.

Theorem 3. *Let the function $f(z)$ defined by (1.1) be in the normalized complex-order analytic function class $\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$. Suppose also that μ is any complex number. Then*

$$\left|a_3-\mu a_2^2\right| \leqq \frac{|\zeta(1+m)|}{2\dfrac{[3]_q!}{[\lambda+1]_{q,2}}\left(\dfrac{m+1}{m+3}\right)^{\alpha}} \max\{1,|\delta|\}, \tag{4.2}$$

where

$$\delta=\frac{2\dfrac{[3]_q!}{[\lambda+1]_{q,2}}\left(\dfrac{m+1}{m+3}\right)^{\alpha}\mu\zeta(1+m)}{\left(\dfrac{[2]_q!}{[\lambda+1]_q}\right)^2\left(\dfrac{m+1}{m+2}\right)^{2\alpha}}-[\zeta(1+m)+m]. \tag{4.3}$$

The result is sharp.

Proof. Since $f(z)\in\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$, we have

$$\begin{aligned}
w(z)&=\frac{z\left(\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right)'-\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)}{[\zeta(1+m)-m]\,\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)+mz\left(\mathcal{N}_{m,q}^{\lambda,\alpha}f(z)\right)'}\\
&=\frac{\sum\limits_{k=2}^{\infty}(k-1)\dfrac{[3]_q!}{[\lambda+1]_{q,2}}\left(\dfrac{m+1}{m+3}\right)^{\alpha}a_k z^{k-1}}{\zeta(1+m)+\sum\limits_{k=2}^{\infty}[\zeta(1+m)+m(k-1)]\dfrac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\dfrac{m+1}{m+k}\right)^{\alpha}a_k z^{k-1}}\\
&=\frac{\sum\limits_{k=2}^{\infty}(k-1)\dfrac{[k]_q!}{[\lambda+1]_{q,k-1}}\left(\dfrac{m+1}{m+k}\right)^{\alpha}a_k z^{k-1}}{\zeta(1+m)}
\end{aligned}$$

$$\cdot \left(1 + \frac{\sum_{k=2}^{\infty} \left[\zeta(1+m) + m(k-1)\right] \frac{[k]_q!}{[\lambda+1]_{q,k-1}} \left(\frac{m+1}{m+k}\right)^{\alpha} a_k z^{k-1}}{\zeta(1+m)} \right)^{-1} . \tag{4.4}$$

We now compare the coefficients of z and z^2 on both sides of the last equation (4.4). We thus obtain

$$a_2 = \frac{\zeta(1+m)\,[\lambda+1]_q\, c_1}{[2]_q! \left(\frac{m+1}{m+2}\right)^{\alpha}} \tag{4.5}$$

and

$$a_3 = \frac{\zeta(1+m)}{\frac{2\,[3]_q!}{[\lambda+1]_{q,2}} \left(\frac{m+1}{m+3}\right)^{\alpha}} \left\{c_2 + \left[\zeta(1+m)+m\right] c_1^2\right\}. \tag{4.6}$$

Hence

$$a_3 - \mu a_2^2 = \frac{\zeta(1+m)}{\frac{2\,[3]_q!}{[\lambda+1]_{q,2}} \left(\frac{m+1}{m+3}\right)^{\alpha}} \left\{c_2 - \phi c_1^2\right\}, \tag{4.7}$$

where

$$\phi = \frac{\frac{2\,[3]_q!}{[\lambda+1]_{q,2}} \left(\frac{m+1}{m+3}\right)^{\alpha} \mu\zeta(1+m)}{\left(\frac{[2]_q!}{[\lambda+1]_q}\right)^2 \left(\frac{m+1}{m+2}\right)^{2\alpha}} - \left[\zeta(1+m)+m\right]. \tag{4.8}$$

Taking the modulus on both sides of (4.7), we have

$$\left|a_3 - \mu a_2^2\right| \leqq \left| \frac{\zeta(1+m)}{\frac{2\,[3]_q!}{[\lambda+1]_{q,2}} \left(\frac{m+1}{m+3}\right)^{\alpha}} \right| \cdot \left|c_2 - \phi c_1^2\right|. \tag{4.9}$$

Now, by using the above lemma in (4.9), we have

$$\left|a_3 - \mu a_2^2\right| \leqq \left| \frac{\zeta(1+m)}{\frac{2\,[3]_q!}{[\lambda+1]_{q,2}} \left(\frac{m+1}{m+3}\right)^{\alpha}} \right| \max\left\{1, |\phi|\right\},$$

where ϕ is given by (4.8).

Finally, the assertion (4.2) of Theorem 3 is sharp in view of the fact that the assertion (4.1) of the above lemma is known to be sharp. □

5. Concluding Remarks and Observations

In our present investigation, we have introduced and systematically studied the general class $\mathcal{H}_{m,q}^{\lambda,\alpha}(\zeta,\mathcal{M})$ of normalized analytic functions of complex order, which are connected with a q-analogue of integral operators. For this complex-order analytic function class, we have successfully determined a sufficient condition in terms of the coefficients and the estimates for the coefficients and a maximization theorem concerning the coefficients. Our main results are stated and proved as theorems (see Theorems 1, 2 and 3). Various interesting consequences and applications of our main results are stated as corollaries.

In conclusion, it seems to worthwhile to reiterate the now well-understood fact that the results for the q-calculus, which we have considered in this presentation for $0<q<1$, can easily be translated into the corresponding results for the so-called (p,q)-calculus (with $0<q<p\leqq 1$) by applying some obviously trivial parametric and argument variations, the additional parameter p being redundant. As a matter of fact, the so-called (p,q)-number $[n]_{p,q}$ is given (for $0<q<p\leqq 1$) by

$$[n]_{(p,q)} := \begin{cases} \dfrac{p^n-q^n}{p-q} & (n\in\{1,2,3,\cdots\}) \\ 0 & (n=0) \end{cases} \tag{5.1}$$

$$=: p^{n-1}\,[n]_{\frac{q}{p}}, \tag{5.2}$$

where, for the classical q-number $[n]_q$, we have (see also Section 1 above)

$$[n]_q := \frac{1-q^n}{1-q} \tag{5.3}$$

$$= p^{1-n}\left(\frac{p^n-(pq)^n}{p-(pq)}\right)$$

$$= p^{1-n}\,[n]_{(p,pq)}. \tag{5.4}$$

Furthermore, the so-called (p,q)-derivative or the so-called (p,q)-difference of a suitable function $f(z)$ is denoted by $(D_{p,q}\,f)(z)$ and defined, in a given subset of $\mathbb{C}$, by

$$(D_{p,q}\,f)(z) = \begin{cases} \dfrac{f(pz)-f(qz)}{(p-q)z} & (z\in\mathbb{C}\setminus\{0\};\ 0<q<p\leqq 1) \\ f'(0) & (z=0;\ 0<q<p\leqq 1), \end{cases} \tag{5.5}$$

so that, clearly, we have the following connection with the familiar q-derivative $(D_q f)(z)$ used in (1.6):

$$(D_{p,q}\,f)(z) = \left(D_{\frac{q}{p}}\,f\right)(pz) \quad \text{and} \quad (D_q\,f)(z) = (D_{p,pq}\,f)\left(\frac{z}{p}\right) \tag{5.6}$$

$$(z \in \mathbb{C};\ 0 < q < p \leqq 1).$$

Remarkably, therefore, any claimed extensions of at least some investigations involving the classical q-calculus to the corresponding obviously straightforward investigations involving the (p,q)-calculus are somewhat inconsequential. The interested reader will find a recent investigation [25] which is intended here to provide an illustration of such transitions from the classical q-calculus to the (p,q)-calculus.

Further investigations on the applications of the q-calculus to meromorphic univalent and meromorphic multivalent functions along the lines of a recent work [15] may be worthy of consideration.

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ON THE LOCAL TRANSFORMED BASED METHOD FOR PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

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Abstract. In the present work, a localized Laplace transform based numerical scheme is constructed for fractional partial integro-differential equations with weakly singular kernels. By application of the Laplace transform the time variable is eliminated and the spatial derivatives are approximated by RBF which transformed the problem into a linear system of equations. The resultant system lead to differentiation matrices which are sparse contrary to large global collocation matrices. The inverse Laplace transform is then computed numerically using contour integration to recover the solution. The efficiency and accuracy of the method is demonstrated by several numerical experiments. The results obtained by the present method are compared with analytical solutions and other numerical methods.

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1. Introduction

In this work, we consider the following type of fractional order partial integro-differential equations with memory term of convolution type

$$D_t^{\alpha} v(x,t) = \gamma v_{xx}(x,t) + \int_0^t v_{xx}(x,z)\beta(t-z)dz + f(x,t), \tag{1.1}$$

where $0 < \alpha < 1$, $0 \le t \le T$, $f(x,t)$ is given function and $\beta(t) = t^{-\frac{1}{2}}$ is the weakly singular kernel, with the following initial condition

$$v(x,0) = v_0,\ x \in \Omega, \tag{1.2}$$

and with the RBC boundary condition

$$\mathcal{B}v(x,t) = 0,\ x \in \partial\Omega. \tag{1.3}$$

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In recent years, researchers have shown growing interest in the study of fractional integro-differential equations. In the last decades such type of equations have numerous applications in the fields of science and engineering like as finance, hydrology, physics and control theory, cosmology, solid mechanics, chemistry, economics, bioengineering, statistical mechanics, viscoelastic materials, and control theory [1, 11, 29, 30]. Fractional order integro-differential equations successfully modeled many physical phenomenon. The analytical solution of these equations are not possible or are hard to find. Therefore researchers have solved these problems numerically. For the solution of fractional integro-differential equations (FIDEs) various numerical method have been used over the years, some of these may be found in the references [9, 25, 26, 32, 33].

Brunner and co-authors have developed very efficient collocation methods for the solution of Volterra integral equations, some of these can be found in the references [2, 3, 7], and some other accurate numerical methods are referred in [8, 27, 31, 42]. Several mathematical models and simulations of real world problems usually expressed in terms of functional equations. These models include PDEs and FIDEs. However, some practical problems such as modeling and simulations in science and engineering can be formulated in terms of fractional order partial integro-differential equations. These equations are often used to model heat conduction in material with memory term, chemical, nuclear reactor design and fluid dynamics [4, 6, 12, 15, 17–19, 23, 35, 36, 38, 40].

Analytical solution of many integral equations are hard to find or either do not exist so we need an efficient and accurate numerical method. Many numerical methods have been used over the last ten decades, including finite element methods [4, 13], finite difference methods [35] and Laplace transform methods [39], Spectral collocation method and Galerkin methods [12, 14, 41] for approximating integrals and integral equations. The authors in [16, 21] have been considered numerical methods for partial integro-differential equations.

Recently applications of radial basis functions (RBFs) is an active area of research. These RBF based meshfree techniques are employed for finding approximate solution of partial differential equations (PDEs) in irregular shaped domains. The detail of such methods can be found [10]. The researchers have shown interest in derivations and implementations of such methods for solving the various types of problems. The mesh-based methods such as the finite difference and finite element methods are not well suited for problems in complex geometry. In the present work, a new technique Laplace transform based local radial basis functions is applied for solving problems (1.1) – (1.3) numerically. By application of Laplace transform the time variable is eliminated and the resulting equation is solved by local radial basis functions method. For getting high accuracy some contours like parabolic and hyperbolic are also used in the numerical inversion of the Laplace transform.

2. Preliminaries

In this section, we provide some definitions and notations of the fractional calculus [22, 28].

Definition 1. The Caputo derivative of fractional order is defined by

$$D_t^{\alpha} v(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{1}{(t-z)^{\alpha+1-n}} \frac{d^n}{ds^n} v(z) dz, \tag{2.1}$$

where, $n-1 < \alpha < n \in z^+$.

Definition 2. The Laplace transform of a given function $v(t)$, where $t \geq 0$ is defined as

$$\hat{v}(z) = \mathcal{L}\{v(t)\} = \int_0^{\infty} e^{-zt} v(t) dt, \tag{2.2}$$

if this improper integral converges.

Lemma 1. *Assume the function $v(t)$ is continuous over $0 \leq t \leq t_n$, if there are some constants K_1, K_2, with the property,*

$$| e^{-K_2 t} \ v(t) | < K_1, \forall \ t > t_n, \tag{2.3}$$

then the Laplace transform of $v(t)$ exists.

Lemma 2. *If $v(t) \in C^p[0,\infty)$, with $\alpha \in (n-1,n) \in z^+$, then the Laplace transform of the Caputo fractional derivative is defined as*

$$\mathcal{L}\{D_t^{\alpha} v(t)\}(z) = z^{\alpha} \hat{v} - \sum_{i=0}^{n-1} z^{\alpha-i-1} v^{(i)}(0). \tag{2.4}$$

Theorem 1 (Lerch's theorem in [5])**.** *If*

$$\hat{v}(z) = \int_0^{\infty} e^{-zt} v(t) dt, \ z > \gamma, \tag{2.5}$$

is satisfied by a continuous function $v(t)$, there is no other continuous function which satisfies the equation (2.5).

Theorem 2 (The Bromwich Inversion Theorem in [5])**.** *Let v(t) have a continuous derivative and let $| v(t) | < Ke^{\gamma t}$, where K and γ are positive constants. Define $\hat{v}(z) = \int_0^{\infty} e^{-zt} v(t) dt, \ \ Re(z) > \gamma$, then*

$$v(t) = \frac{1}{2\pi i} \int_{\xi - i\infty}^{\xi + i\infty} \hat{v}(z) e^{zt} dz. \tag{2.6}$$

3. Description of the Method

3.1. *Time discretization via Laplace transform*

Now the Laplace transform is applied to eliminate the time variable and the resultant equations are solved using local meshless method. Application of Laplace transform to problem (1.1) – (1.3) gives

$$z^{\alpha}\hat{v}(x,z) - z^{\alpha-1}v_0 = \hat{\beta}(z)\mathcal{D}\hat{v}_{xx}(x,z) + \hat{f}(x,z),\ x \in \Omega \subset \mathbb{R}^d, \tag{3.1}$$

$$\mathcal{B}\hat{v}(x,z) = \hat{h}(z),\ x \in \partial\Omega. \tag{3.2}$$

The following system of equations is obtained

$$(z^{\alpha}I - \hat{\beta}(z)\mathcal{D})\hat{v}(x,z) = z^{\alpha-1}v_0 + \hat{f}(x,z),\ x \in \Omega \subset \mathbb{R}^d, \tag{3.3}$$

$$\mathcal{B}\hat{v}(x,z) = \hat{h}(z),\ x \in \partial\Omega. \tag{3.4}$$

The solution $\hat{v}(x,z)$ of the transformed problem (3.3)-(3.4) will be approximated using local RBF method. Consequently by using numerical inversion we get the original solution $v(x,t)$ of the given problem (1.1) – (1.3).

3.2. *Spatial discretization via radial kernels*

In this section we need to discretize the transform problem (3.3)-(3.4) for the linear operators $\mathcal{B}$ and $\mathcal{L}$ by using local radial basis functions [37]. Consider the centers $\{x_1,...,x_m\} \subset \Omega \subset \mathbb{R}^d, d \geq 1$, where Ω is bounded domain. To approximate the unknown function $\hat{v}(x)$ at the functional values $\{\hat{v}(x_j), j = 1,2,...,m\}$, the local interpolant at each $x_j \in \Omega$ takes the form

$$\hat{v}(x_j) = \sum_{x_k \in \Omega_j} \sigma_k^j \varphi^j(\|x_j - x_k\|), \tag{3.5}$$

where, $\sigma^j = [\sigma_1^j, \sigma_2^j, ..., \sigma_n^j]$ is vector of unknown coefficients, and $r_{jk} = \|x_j - x_k\|$ is the norm between nodes x_j and x_k, $\varphi(r)$, $r \geq 0$ is a radial kernel (Multi-quadric radial basis function) and $\Omega_j \subset \Omega$ is a local domain for around each x_j, contains n neighboring nodes around the node x_j. So we have m small size linear systems each of order $n \times n$ given by

$$\begin{pmatrix} \hat{v}_1^j \\ \hat{v}_2^j \\ \vdots \\ \hat{v}_n^j \end{pmatrix} = \begin{pmatrix} \varphi_{11}^j & \varphi_{12}^j & \cdots & \varphi_{1n}^j \\ \varphi_{21}^j & \varphi_{22}^j & \cdots & \varphi_{2n}^j \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{n1}^j & \varphi_{n2}^j & \cdots & \varphi_{nn}^j \end{pmatrix} \begin{pmatrix} \sigma_1^j \\ \sigma_2^j \\ \vdots \\ \sigma_n^j \end{pmatrix},\ j = 1,2,...,m,$$

which can be denoted by

$$\hat{\mathbf{V}}^j = \mathbf{P}^j\sigma^j, j = 1,2,...,m, \tag{3.6}$$

where $\varphi^j_{lk} = \varphi^j(\|x_l - x_k\|), x_l, x_k \in \Omega_j$, the matrix $\mathbf{P}^j$ is called system matrix.

Next applying the operator $\mathcal{L}$ to (3.5), we get

$$\mathcal{L}\hat{v}(x_j) = \sum_{x_k \in \Omega_j} \sigma^j_k \mathcal{L}\varphi^j(\|x_j - x_k\|), \tag{3.7}$$

In vector form this gives,

$$\mathcal{L}\hat{v}(x_j) = \mathbf{W}^j \cdot \sigma^j, \tag{3.8}$$

where $\mathbf{W}^j$ is given by

$$\mathbf{W}^j = \mathcal{L}\varphi^j(\|x_j - x_k\|), x_j, x_k \in \Omega_j, \tag{3.9}$$

the unknown coefficients can be eliminated from equation (3.6)

$$\sigma^j = (\mathbf{P}^j)^{-1}\hat{\mathbf{V}}^j, \tag{3.10}$$

by inserting the values of σ^j in (3.8) we have,

$$\mathcal{L}\hat{v}(x_j) = \mathbf{W}^j(\mathbf{P}^j)^{-1}\hat{\mathbf{V}}^j = \mathbf{N}^j\hat{\mathbf{V}}^j \tag{3.11}$$

where,

$$\mathbf{N}^j = \mathbf{W}^j(\mathbf{P}^j)^{-1}. \tag{3.12}$$

Hence the localized discretized form of the linear operator at each center x_j gives

$$\mathcal{L}\hat{v} = \mathbf{H}\hat{\mathbf{V}}, \tag{3.13}$$

where $\mathbf{H}$ is the sparse differentiation matrix of order $m \times m$ with n non-zeros values while $m - n$ zeros values in each row, and n denote the nodes in local sub-domain Ω_j for each j.

4. Numerical inversion technique

In this section, the solution $v(x,t)$ of the problem (1.1) – (1.3) is obtained using the following Laplace inversion form

$$v(x,t) = \frac{1}{2\pi\iota}\int_{\xi-\iota\infty}^{\xi+\iota\infty} \hat{v}(x,z)e^{zt}dz = \frac{1}{2\pi\iota}\int_{\Upsilon} e^{zt}\hat{v}(x,z)dz, \xi > \xi_0, \tag{4.1}$$

where Υ is suitable path joining $\xi - \iota\infty$ to $\xi + \iota\infty$. The solution of problem (1.1) – (1.3) is obtained from the complex contour integral (4.1) which is called the Bromwich integral along the chosen path. The hyperbolic and parabolic contours are used in the present work [20, 40]. The hyperbolic path [20] is given by

$$z(\eta) = \omega + \lambda(1 - \sin(\sigma - \iota\eta)), \text{ for } \eta \in \mathbb{R}, \tag{4.2}$$

with $\lambda > 0$, $\omega \geq 0$, $0 < \sigma < \beta - \frac{1}{2}\pi$, and $\frac{1}{2}\pi < \beta < \pi$ (for detail see [20]). The optimal parameters corresponding to the hyperbolic path are given as $l(x) = \max(1, \log(1/x))$, $r > 0$, $\mu = \bar{r}(1-\theta)/b$, $\bar{r} = 2\pi r$, $0 < \theta < 1$, $\rho_r = \theta\bar{r}\tau\sin(\sigma - r)/b$, $b = cosh^{-1}(1/(\theta\tau\sin(\sigma)))$, $\tau = t_0/T$, $0 < t_0 < T$, $t_0 \leq t \leq T$, $\lambda = \theta\bar{r}M/(bT)$, $k = b/M$

and error bound is $(l(\rho_r M))$ (see Theorem 3.1, [20]). Using the paths hyperbolic path defined the integral (4.1) becomes

$$v(x,t) = \frac{1}{2\pi\imath}\int_{-\infty}^{\infty} e^{z(\eta)t}\hat{v}(x,z(\eta))\acute{z}(\eta)d\eta, \tag{4.3}$$

and equal width trapezoidal rule is used for approximation of (4.3) in the following equation

$$v_k(x,t) = \frac{k}{2\pi\imath}\sum_{j=-M}^{M}\hat{v}(x,z_j)e^{z_j t}\acute{z}_j, z_j = z(\eta_j),\ \eta_j = jk, \tag{4.4}$$

where k is step size.

5. Stability

In this section, we discuss stability of the system (3.3)-(3.4), and can be expressed as

$$A\hat{\mathbf{v}} = \mathbf{b}, \tag{5.1}$$

where the matrix A is known as the differentiation matrix of order $N \times N$ obtained in using local RBF method. From system (5.1) the stability constant is given by

$$C = \sup_{\hat{\mathbf{v}}\neq 0}\frac{\|\hat{\mathbf{v}}\|}{\|A\hat{\mathbf{v}}\|}, \tag{5.2}$$

where the constant C is finite which can be obtained by using some discrete norms $\|.\|$ on R^N. Equation (5.2) gives

$$\|A\|^{-1} \leq \frac{\|\hat{\mathbf{v}}\|}{\|A\hat{\mathbf{v}}\|} \leq C, \tag{5.3}$$

Next in case of pseudoinverse $A^\dagger$ of A

$$\|A^\dagger\| = \sup_{w\neq 0}\frac{\|A^\dagger w\|}{\|w\|}. \tag{5.4}$$

Now we have

$$\|A^\dagger\| \geq \sup_{w=A\hat{\mathbf{v}}\neq 0}\frac{\|A^\dagger A\hat{\mathbf{v}}\|}{\|A\hat{\mathbf{v}}\|} = \sup_{\hat{\mathbf{v}}\neq 0}\frac{\|\hat{\mathbf{v}}\|}{\|A\hat{\mathbf{v}}\|} = C. \tag{5.5}$$

Hence equations (5.3) and (5.5) assures the boundedness of the stability constant C. To ensure numerical stability in solving system (5.1), Pseudoinverse can be used, but calculation may be computationally expansive. In case of square systems, we use the MATLAB built in function condest estimates

$$C = \frac{condest(A')}{\|A\|_\infty} \tag{5.6}$$

6. Application of the method

In this section, we applied our numerical technique which is the combination of Laplace transform and localized RBF based meshfree method to approximate the time fractional partial integro-differential equations. The accuracy of the method is shown in the form of tables and graphs.

Example 1. Here we consider the model defined in equations (1.1) – (1.3) as our first test problem with the following function and boundary conditions

$$f(x,t) = \frac{1}{\Gamma(3-\alpha)}(x^2-1)t^{2-\alpha} - 2t^2 - \frac{32}{15}t^{\frac{5}{2}}, \tag{6.1}$$

$$v(x,0) = 0,\ x \in (-1,1),\ v(0,t) = v(1,t) = 0,\ 0 < t \ \leq \ T, \tag{6.2}$$

where the actual solution [24] of above problem is $v(x,t) = t^2(x^2-1)$. The contour mentioned in (4.2) and the following optimal values of the parameter are used in the solution of the above problem.

$$\omega = 2, \sigma = 0.3812, \lambda = \frac{\theta r_b M}{bT}, x_k = hk, \theta = 0.1, r_b = 2\pi r, r = 0.3431,$$

$$h = b/M, b = \cosh^{-1}(\frac{1}{\theta\tau\sin(\sigma)}), \tau = t_0/T, t_0 = 0.1, t = 1, T = 5.$$

The l_∞ errors at various quadrature points M, collocation points $m \in \Omega$, stencils points $n \in \Omega_i$ and fractional order α, and the error estimate $\varepsilon(c,M) = (e^{\frac{-cM}{log(M)}})$, $c = 1$ of the proposed numerical scheme are shown in Table 1. The error estimate $\varepsilon(c,M)$ decreasing exponentially for $11 \leq M \leq 91$, while at $M = 50$ and $4 \leq n \leq 15$ the error estimate remain unchanged are shown in Table 1 and Figure 1 respectively. The numerical solutions obtained by Laplace transform based local meshless method are better than the results obtained by the method in [24].

Example 2. In this example, we apply the proposed numerical method to solve model (1.1) – (1.3) as a second test problem in [24] with

$$f(x,t) = \cos(\pi x)(\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha} + t^2 + \frac{16}{15}t^{\frac{5}{2}}), \tag{6.3}$$

with the following initial and boundary conditions

$$v(x,0) = 0,\ x \in (0,\pi),\ v(0,t) = v(1,t) = 0,\ 0 < t \ \leq \ T. \tag{6.4}$$

The exact solution of the above problem is $v(x,t) = t^2\cos(x)$. The same set of parameters are used for the solution of Example 2 as well. The l_∞ error norm is steadily decreasing from top to bottom in each column for $10 \leq M \leq 60$ and similarly the error is decaying slowly from left to right in each row by increasing value of α and are shown in Table 2 and Figure 2 respectively. The results obtained by the present method are more accurate than the results in [24].

TABLE 1. Numerical results using Laplace transform based local RBF method corresponding to Example 1.

$(m,n=45,5)$	$\alpha=0.2$	$\alpha=0.4$	$\alpha=0.6$	$\alpha=0.8$	
M	l_∞	l_∞	l_∞	l_∞	$\varepsilon(c,M)$
11	5.29e-006	5.31e-006	5.34e-006	5.37e-006	0.0102
21	5.62e-006	5.39e-006	5.14e-006	4.89e-006	0.0010
41	1.62e-005	1.56e-005	1.49e-005	1.41e-005	1.60e-005
51	2.73e-005	2.63e-005	2.52e-005	2.40e-005	2.32e-006
71	3.99e-005	3.85e-005	3.69e-005	3.52e-005	5.83e-008
91	5.61e-005	5.41e-005	5.20e-005	4.97e-005	1.73e-009
$(M,m=50,45)$					
n					
4	5.41e-004	5.24e-004	5.05e-004	4.84e-004	2.81e-006
5	3.00e-005	2.89e-005	2.77e-005	2.63e-005	2.81e-006
7	8.38e-005	8.13e-005	7.85e-005	7.55e-005	2.81e-006
9	6.58e-005	6.37e-005	6.14e-005	5.89e-005	2.81e-006
11	1.51e-004	1.46e-004	1.41e-004	1.35e-004	2.81e-006
15	2.13e-004	2.06e-004	1.98e-004	1.90e-004	2.81e-006
$m=100$	$L_\infty=7.4715e^{-5}$	[24]			

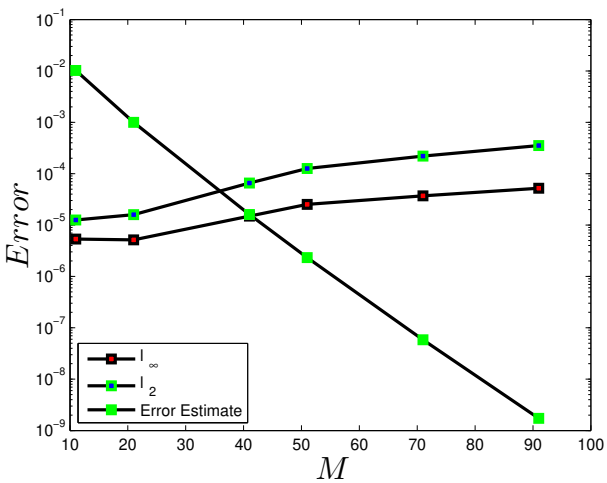

FIGURE 1. Approximate solution: error versus quadrature points corresponding to Example 1.

TABLE 2. Numerical results of fractional partial-integro-differential equation at time $t = 1 \in [t_0 = 0.1, T = 5]$ by present method to Example 2.

$m = 45, n = 5$	$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.7$	$\alpha = 0.8$	
M	l_∞	l_∞	l_∞	l_∞	$\varepsilon(c,M)$
10	1.18e-004	1.17e-004	1.14e-004	1.13e-004	0.0130
20	1.33e-005	1.33e-005	1.31e-005	1.31e-005	0.0013
30	8.10e-006	8.10e-006	8.09e-006	8.09e-006	1.47e-004
40	7.03e-006	7.03e-006	7.03e-006	7.03e-006	1.95e-005
50	6.58e-006	6.58e-006	6.58e-006	6.58e-006	2.81e-006
60	6.88e-006	6.88e-006	6.88e-006	6.88e-006	4.32e-007
$M = 50, n = 5$					
m	l_∞	l_∞	l_∞	l_∞	$\varepsilon(c,M)$
10	0.0068	0.0068	0.0068	0.0068	0.0130
20	9.94e-005	9.94e-005	9.94e-005	9.94e-005	0.0013
30	1.44e-004	1.44e-004	1.44e-004	1.44e-004	1.47e-004
40	3.44e-005	3.44e-005	3.44e-005	3.44e-005	1.95e-005
50	6.65e-006	6.61e-006	6.46e-006	6.43e-006	2.81e-006
60	4.22e-006	4.18e-006	3.99e-006	3.95e-006	4.32e-007

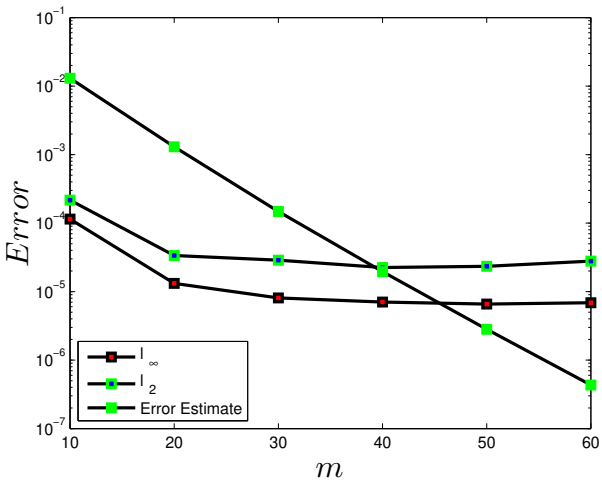

FIGURE 2. Approximate solution: error versus quadrature points corresponding to Example 2

Example 3. In the third test problem, we apply the proposed method for model problem given in equations (1.1) – (1.3), subject to the initial and boundary conditions and function f

$$f(x,t) = \sin(\pi x)\left[\frac{1}{\Gamma(2-\alpha)}t^{1-\alpha} + \pi^2 t + \frac{4}{3}\pi^2 t^{\frac{3}{2}}\right], \tag{6.5}$$

$$v(x,0) = 0,\ x \in (0,1],\ v(0,t) = v(1,t) = 0,\ 0 < t \leq T, \tag{6.6}$$

where the exact solution $v(x,t) = t\sin(\pi x)$ is taken for the purpose of comparison [24]. Various quadrature points M along the hyperbolic Γ contour (4.2) are used in approximating the solution. The l_∞ errors at various $m \in \Omega$, stencils points n in Ω_i and α are shown in Table 3 and Figure 3. From Table 4, we observed that the l_∞ error is decaying algebraically for $20 \leq M \leq 60$ quadrature points. Hence the present method is highly convergent.

TABLE 3. Numerical results using Laplace transform method corresponding to Example 3.

$M = 70, n = 5$	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.6$	$\alpha = 0.75$
N	l_∞	l_∞	l_∞	l_∞
11	9.63e-005	9.63e-005	9.65e-005	9.68e-005
21	1.42e-005	1.42e-005	1.42e-005	1.42e-005
41	2.99e-005	2.99e-005	3.00e-005	3.00e-005
51	3.81e-005	3.81e-005	3.82e-005	3.83e-005
71	5.48e-005	5.48e-005	5.48e-005	5.50e-005
91	7.43e-005	7.43e-005	7.44e-005	7.46e-005
$N = 50, M = 70$				
n				
4	3.03e-004	3.03e-004	3.04e-004	3.05e-004
5	4.10e-005	4.10e-005	4.11e-005	4.12e-005
7	7.87e-005	7.86e-005	7.88e-005	7.90e-005
9	1.07e-004	1.07e-004	1.07e-004	1.07e-004
11	1.67e-004	1.67e-004	1.68e-004	1.68e-004
15	2.06e-004	2.06e-004	2.06e-004	2.07e-004

Example 3. Finally we consider the following two dimensional initial-boundary value partial integro-differential equation

$$\frac{\partial v}{\partial t}(x,y,t) - \int_0^t \beta_1(t-z)\Delta v(x,y,s)dz = f(x,y,t),\ x,y \in \Omega,\ t \geq 0, \tag{6.7}$$

where $\beta_1(t) = (\pi t)^{\frac{-1}{2}}$, and initial and boundary conditions are chosen as

$$v(x,y,0) = 0,\ x,y \in \Omega,\ v(x,y,t) = v(x,y,t) = 0,\ x,y \in \partial\Omega, t \geq 0, \tag{6.8}$$

TABLE 4. Approximate solution corresponding to Example 3.

$N=50, n=5$	$\alpha=0.1$	$\alpha=0.25$	$\alpha=0.6$	$\alpha=0.75$
M	l_∞	l_∞	l_∞	l_∞
20	0.0112	0.0112	0.01	0.0112
30	0.0019	0.0019	0.01	0.0019
45	1.99e-004	1.99e-004	1.99e-004	1.99e-004
60	1.46e-005	1.46e-005	1.46e-005	1.47e-005
70	4.10e-005	4.10e-005	4.11e-005	4.12e-005
90	3.70e-005	3.69e-005	3.70e-005	3.71e-005

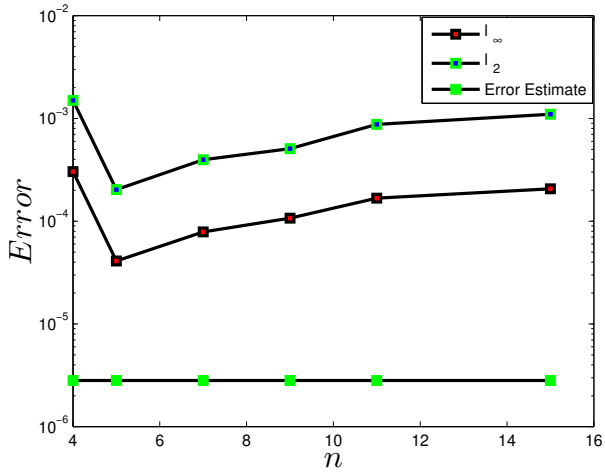

FIGURE 3. Approximate solution: error versus quadrature points corresponding to Example 3

where the exact solution $v(x,y,t) = t^{\frac{3}{2}}(1-x^2)(1-y^2)$ of this problem is selected from [34]. The parabolic contour $z = \nu(iw+1)^2$ for the strip $w = \eta + ic$, where $-\infty < \eta < \infty$ which reduces to

$$z(\eta) = \nu((1-c)^2 - \eta^2) + 2i\nu\eta(1-c), \tag{6.9}$$

For the optimal values of the parameters corresponding to this path we used [40]. This problem is approximated over the domain $[-1,1]^2$. The error estimate $\varepsilon(c,M)$, l_∞ error norm and l_2 error norm are displayed in Table 5. For various points m, $M=30$ quadrature points, $n=20$ stencil points the l_∞ error norm and l_2 error norm decreasing steadily for $20 \le M \le 60$, while error estimate $\varepsilon(c,M) = e^{\frac{-cM}{ln(M)}}$ for $c=1$ is decaying algebraically. The results are also computed for fixed collocation points $m=50$, quadrature pints $M=30$ and various stencil points n in Ω_i. The actual and

numerical results are also displayed in Figure 4. Hence our numerical scheme is highly convergent for approximation of multi-dimensional partial integro-differential equations.

TABLE 5. Numerical results using Laplace transform based local RBF method corresponding to Example 4.

$(M,n) = (30,20)$				
m	l_∞	l_2	$\varepsilon(c,M)$	RMS
20	1.77e-004	0.0016	0.0013	7.92e-005
30	1.71e-004	9.08e-004	1.47e-004	3.02e-005
40	1.31e-004	8.45e-004	1.95e-005	2.11e-005
50	5.01e-005	7.51e-004	2.81e-006	1.50e-005
60	6.72e-005	6.97e-004	4.32e-007	1.16e-005
$(M,m) = (30,50)$				
n				
10	5.41e-004	0.0616	2.81e-006	0.0586
20	5.01e-004	7.51e-004	2.81e-006	1.50e-005
25	1.07e-004	0.0016	2.81e-006	3.13e-005
30	6.39e-005	0.0015	2.81e-006	3.01e-005
40	4.84e-005	0.0015	2.81e-006	2.97e-005
	$l_2 = 1.679e-004$ [34]			

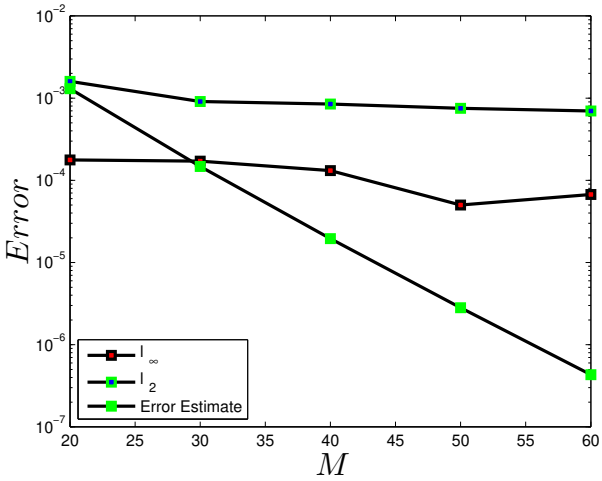

FIGURE 4. Numerical results corresponding to Example 4.

7. Conclusion

In this work a numerical scheme is constructed which is based on Laplace transform and local radial basis functions method. The proposed numerical scheme efficiently approximated fractional order partial integro-differential equations with weakly singular kernel as well as memory term of convolution type. By application of Laplace transform time independent PDEs are obtained, after which the space variable is discretized using local meshless method. Finally the solution is obtained with inverse Laplace transform using quadrature rule along a contour with high accuracy. The accuracy and efficiency is of the numerical scheme is demonstrated and the present numerical method is an alternative for such type similar problems.

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A SHARP IMPROVEMENT OF FIXED POINT RESULTS FOR QUASI-CONTRACTIONS IN b-METRIC SPACES

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Abstract. In this paper, a general fixed point theorem for quasi-contractions in b-metric spaces, which is a sharp improvement of Amini-Harandi's result, Mitrovic and Hussain's result, and is a generalization of many b-metric fixed point theorems in the literature, is proved. The technique overcomes some limits in b-metric fixed point theory compared to metric fixed point theory. The obtained results are also supported by examples.

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Keywords: quasi-contraction, b-metric, fixed point

1. Introduction and preliminaries

There have been many types of contraction conditions in metric spaces and generalized metric spaces [12], [19]. One of the most interesting types is the quasi-contraction [7]. Quasi-contractions have been studied and many nice results have been proved. In [4], Bessenyei studied nonlinear quasicontractions in complete metric spaces. In [5], Bessenyei studied weak φ-quasi-contractions and presented an elementary proof for known fixed point results in the literature. In [2], Amini-Harandi proved a fixed point theorem for quasi-contraction maps in b-metric spaces. Recently, Mitrović and Hussain [17] established fixed point results for weak φ-quasi-contractions involving comparison function in b-metric spaces.

Recall that the b-metric space is a generalization of a metric space. One of the main differences between a b-metric space and a metric space is that the modulus of concavity $\kappa \geq 1$ in the generalized triangle inequality, see Definition 1. (3) below. It implies that a b-metric is not necessarily continuous, see [3, Example 3.10] for example. It also implies that the contraction constants in certain b-metric fixed point theorems are in $[0; \frac{1}{\kappa})$ instead of $[0; 1)$, see [9, Remark 2.7] and [17, Corollary 3.5] for example. So, in many b-metric fixed point theorems, certain additional assumptions have been added to overcome the above difference such as the Fatou property in [2],

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the bounded orbit in [5] and [17]. For some recent improvements in b-metric fixed point theory, the reader may refer to [9], [18], [10], [14], [16], [15], [21].

In this paper, we are interested to improve the main results of [2] and [17]. By using a technical calculation that $\lambda^n \in [0; \frac{1}{\kappa})$ for all $\lambda \in [0;1)$ and all n large enough, we prove a fixed point theorem for quasi-contractions in b-metric spaces which is a sharp improvement of the main results in [2] and [17], and is a generalization of many b-metric fixed point theorems in the literature. We also construct examples to support the obtained results.

Now we recall notions and results which will be useful in the next.

Definition 1 ([8], page 263)**.** Let X be a nonempty set, $\kappa \geq 1$ and $D : X \times X \to [0;\infty)$ be a function such that for all $x, y, z \in X$,

(1) $D(x,y) = 0$ if and only if $x = y$.
(2) $D(x,y) = D(y,x)$.
(3) $D(x,z) \leq \kappa[D(x,y) + D(y,z)]$.

Then

(1) D is called a *b-metric* on X and (X,D,κ) is called a *b-metric space*. Without loss of generality we may assume that κ is the smallest possible value, and it is called the *modulus of concavity* of the given b-metric.
(2) The sequence $\{x_n\}$ is called *convergent* to x if $\lim_{n\to\infty} D(x_n,x) = 0$, written by $\lim_{n\to\infty} x_n = x$.
(3) The sequence $\{x_n\}$ is called *Cauchy* if $\lim_{n,m\to\infty} D(x_n,x_m) = 0$.
(4) The b-metric space (X,D,κ) is called *compete* if every Cauchy sequence is a convergent sequence.

Definition 2 ([12], Definition 12.7)**.** Let X be a nonempty set, $\kappa \geq 1$ and $D : X \times X \to [0,\infty)$ be a function such that for all $x, y, z \in X$,

(1) $D(x,y) = 0$ if and only if $x = y$.
(2) $D(x,y) = D(y,x)$.
(3) $D(x,z) \leq D(x,y) + \kappa D(y,z)$.

Then D is called a *strong b-metric* on X and (X,D,κ) is called a *strong b-metric space*.

Note that every strong b-metric is continuous, and the convergence and completeness in strong b-metric spaces are defined as in b-metric spaces.

Definition 3 ([2], Definition 2.4)**.** A b-metric space (X,D,κ) is called to have *Fatou property* if for all $x, y \in X$ and $\lim_{n\to\infty} x_n = x$ we have

$$D(x,y) \leq \liminf_{n\to\infty} D(x_n,y).$$

Theorem 1 ([2], Theorem 2.8)**.** *Let (X,D,κ) be a complete b-metric space having Fatou property and $f:X\to X$ be a map such that for some $\lambda\in[0;\frac{1}{\kappa})$ and all $x,y\in X$,*

$$\begin{aligned}&D(f(x),f(y))\\ \le&\lambda\max\{D(x,y),D(x,f(x)),D(y,f(y)),D(x,f(y)),D(y,f(x))\}.\end{aligned}\tag{1.1}$$

Then f has a unique fixed point x^* and $\lim_{n\to\infty}f^n(x)=x^*$ for all $x\in X$.

Theorem 2 ([5], Theorem on page 289)**.** *Assume that*

(1) *(X,D,κ) is a complete metric space and $f:X\to X$ is a map such that for all $x,y\in X$,*

$$D(f(x),f(y))\le\varphi(\mathrm{diam}O(x,y))\tag{1.2}$$

where $\varphi:[0;\infty)\to[0;\infty)$ is an increasing, upper semicontinuous function, $\varphi(0)=0$ and $\varphi(t)<t$ for all $t>0$, and

$$O(x,y)=\left\{f^n(x),f^n(y):n\in\mathbb{N}\cup\{0\}\right\}.$$

(2) *Each orbit of f is bounded.*

Then f has unique fixed point x^* and $\lim_{n\to\infty}f^n(x)=x^*$ for all $x\in X$.

Theorem 3 ([17], Theorem 3.3)**.** *Assume that*

(1) *(X,D,κ) is a complete b-metric space and $f:X\to X$ is a map such that for some $\lambda\ge0$ and all $x,y\in X$,*

$$D(f(x),f(y))\le\lambda\mathrm{diam}O(x,y).\tag{1.3}$$

(2) *$\lambda\in[0;1)$ and each orbit of f is bounded.*

Then we have

(1) *There exists $x^*\in X$ such that $\lim_{n\to\infty}f^n(x)=x^*$ for all $x\in X$.*

(2) *f has unique fixed point x^* if one of the following holds*
   (a) *f is continuous at x^*.*
   (b) *D is continuous.*

Theorem 4 ([17], Corollary 3.5)**.** *Let (X,D,κ) be a complete b-metric space and $f:X\to X$ be a map such that for some $\lambda\in[0;\frac{1}{\kappa})$ and all $x,y\in X$,*

$$\begin{aligned}&D(f(x),f(y))\\ \le&\lambda\max\left\{D(x,y),D(x,f(x)),D(y,f(y)),\frac{D(x,f(y))}{2\kappa},\frac{D(y,f(x))}{2\kappa}\right\}.\end{aligned}\tag{1.4}$$

Then f has a unique fixed point.

Theorem 5 ([17], Corollary 3.6)**.** *Let (X,D,κ) be a complete strong b-metric space and $f:X\to X$ be a map such that for some $\lambda\in[0;1)$ and all $x,y\in X$,*

$$D(f(x),f(y))\le\lambda\max\left\{D(x,y),D(x,f(x)),D(y,f(y)),D(x,f(y)),D(y,f(x))\right\}.$$

Then f has a unique fixed point.

2. The main result

The main result is Theorem 6 below. Note that

(1) Theorem 6 is an improvement of Theorem 1 in the sense that the assumption of Fatou property is omitted, and the contraction constant $\lambda \in [0;1)$.
(2) Theorem 6 is an improvement of Theorem 4 in the sense that the right side of (2.1) is greater than that of (1.4), and the contraction constant $\lambda \in [0;1)$.
(3) Theorem 6 is an improvement of Theorem 5 in the sense that the strong b-metric is replaced by a continuous b-metric.
(4) Theorem 6 is a generalization of many b-metric fixed point theorems in the literature such as [1, Theorem 2.1], [1, Theorem 3.1], [11, Corollary 3.12], [20, Corollary 2.6].
(5) Recently, an analogue of Reich contraction in b-metric spaces was proved [13, Theorem 3.1]. In the proof on [13, page 85], the author claimed $\lim_{n\to\infty} d(x_{n+1}, Tx^*) = d(x^*, Tx^*)$ provided that $\lim_{n\to\infty} x_n = x^*$. Unfortunately, this claim does not hold since the b-metric d is not necessarily continuous . In fact, the conclusion in [13, Theorem 3.1] does not hold which was proved in [9, Remark 2.7].

Theorem 6. *Assume that*

(1) (X, D, κ) *is a complete b-metric space and* $f : X \to X$ *is a map such that for some* $\lambda \geq 0$ *and all* $x, y \in X$,

$$\begin{aligned} &D(f(x), f(y)) \\ \leq &\lambda \max\left\{D(x,y), D(x,f(x)), D(y,f(y)), D(x,f(y)), D(y,f(x))\right\}. \end{aligned} \tag{2.1}$$

(2) *One of the following holds*
   (a) D *is continuous and* $\lambda \in [0;1)$.
   (b) $\lambda \in [0; \frac{1}{\kappa})$.

Then f *has a unique fixed point* x^* *and* $\lim_{n\to\infty} f^n(x) = x^*$ *for all* $x \in X$.

Proof. For $m \leq i \leq n-1$ and $m \leq j \leq n$, from (2.1) we find that

$$\begin{aligned} &D(f^i(x), f^j(x)) \\ =&D(ff^{i-1}(x), ff^{j-1}(x)) \\ \leq&\lambda \max\left\{D(f^{i-1}(x), f^{j-1}(x)), D(f^{i-1}(x), ff^{i-1}(x)), D(f^{j-1}(x), ff^{j-1}(x)),\right. \\ &\left. D(f^{i-1}(x), ff^{j-1}(x)), D(f^{j-1}(x), ff^{i-1}(x))\right\} \\ =&\lambda \max\left\{D(f^{i-1}(x), f^{j-1}(x)), D(f^{i-1}(x), f^i(x)), D(f^{j-1}(x), f^j(x)),\right. \\ &\left. D(f^{i-1}(x), f^j(x)), D(f^{j-1}(x), f^i(x))\right\}. \end{aligned} \tag{2.2}$$

From (2.2), we get

$$\max\{D(f^i(x), f^j(x)) : m \leq i, j \leq n\}$$

$$\begin{aligned}
&\leq \lambda \max\{D(f^i(x), f^j(x)) : m-1 \leq i, j \leq n\} \\
&\leq \dots \\
&\leq \lambda^m \max\{D(f^i(x), f^j(x)) : 0 \leq i, j \leq n\}. \qquad (2.3)
\end{aligned}$$

It implies that

$$\max\{D(f^i(x), f^j(x)) : 1 \leq i, j \leq n\} \leq \lambda \max\{D(f^i(x), f^j(x)) : 0 \leq i, j \leq n\}.$$

Since $0 \leq \lambda < 1$, we see that

$$\max\{D(f^i(x), f^j(x)) : 0 \leq i, j \leq n\} = \max\{D(x, f^i(x)) : 1 \leq i \leq n\}.$$

So there exists $1 \leq k_n(x) \leq n$ such that

$$D(x, f^{k_n(x)}(x)) = \max\{D(f^i(x), f^j(x)) : 0 \leq i, j \leq n\}.$$

Since $\lambda \in [0;1)$, there exists n_0 such that $\lambda^{n_0} < \frac{1}{\kappa}$. If $k_n(x) \leq n_0$, then

$$D(x, f^{k_n(x)}(x)) \leq \max\{D(x, f^i(x)) : 0 \leq i \leq n_0\}. \qquad (2.4)$$

If $k_n(x) > n_0$, then using (2.3) we find that

$$\begin{aligned}
&D(x, f^{k_n(x)}(x)) \\
&\leq \kappa[D(x, f^{n_0}(x)) + D(f^{n_0}(x), f^{k_n(x)}(x))] \\
&\leq \kappa[D(x, f^{n_0}(x)) + \lambda^{n_0} \max\{D(f^i(x), f^j(x)) : 0 \leq i, j \leq k_n(x)\}] \\
&\leq \kappa[D(x, f^{n_0}(x)) + \lambda^{n_0} \max\{D(f^i(x), f^j(x)) : 0 \leq i, j \leq n\}] \\
&= \kappa[D(x, f^{n_0}(x)) + \lambda^{n_0} D(x, f^{k_n(x)}(x))].
\end{aligned}$$

Note that $\lambda^{n_0} < \frac{1}{\kappa}$. So we get

$$D(x, f^{k_n(x)}(x)) \leq \frac{\kappa}{1-\kappa\lambda^{n_0}} D(x, f^{n_0}(x)). \qquad (2.5)$$

It follows from (2.4) and (2.5) that

$$\max\{D(f^i(x), f^j(x)) : 0 \leq i, j \leq n\} \leq \frac{\kappa}{1-\kappa\lambda^{n_0}} \max\{D(x, f^i(x)) : 0 \leq i \leq n_0\}$$

for all n. So

$$\sup\{D(f^i(x), f^j(x)) : 0 \leq i, j \leq \infty\} \leq M < \infty$$

where

$$M = \frac{\kappa}{1-\kappa\lambda^{n_0}} \max\{D(x, f^i(x)) : 0 \leq i \leq n_0\}.$$

By (2.3) we have

$$\begin{aligned}
\sup\{D(f^i(x), f^j(x)) : m \leq i, j \leq \infty\} &\leq \lambda \sup\{D(f^i(x), f^j(x)) : m-1 \leq i, j \leq \infty\} \\
&\leq \dots \leq \lambda^m \sup\{D(f^i(x), f^j(x)) : 0 \leq i, j \leq \infty\} \leq \lambda^m M.
\end{aligned}$$

Then

$$\lim_{m\to\infty} \sup\{D(f^i(x), f^j(x)) : m \leq i, j \leq \infty\} = 0.$$

Therefore the sequence $\{f^n(x)\}$ is a Cauchy sequence. Since X is complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} f^n(x) = x^*. \tag{2.6}$$

By (2.1) we get

$$\begin{aligned} &D(f^{n+1}(x), f(x^*)) \\ =&D(ff^n(x), f(x^*)) \\ \leq&\lambda \max\{D(f^n(x), x^*), D(f^n(x), f^{n+1}(x)), D(x^*, f(x^*)), \\ &D(f^n(x), f(x^*)), D(x^*, f^{n+1}(x))\}. \end{aligned} \tag{2.7}$$

We consider two following cases.

Case 1. *D is continuous and* $\lambda \in [0;1)$.

Using (2.7) and the continuity of D, we obtain

$$\begin{aligned} D(x^*, f(x^*)) \leq&\lambda \max\{0, 0, D(x^*, f(x^*)), D(x^*, f(x^*)), 0\} \\ =&\lambda D(x^*, f(x^*)). \end{aligned}$$

Since $\lambda \in [0;1)$, we have $D(x^*, f(x^*)) = 0$. So x^* is a fixed point of f.

Case 2. $\lambda \in [0; \frac{1}{\kappa})$.

It follows from (2.7) that

$$\begin{aligned} &\liminf_{n\to\infty} D(f^{n+1}(x), f(x^*)) \\ \leq&\lambda \max\{0, 0, D(x^*, f(x^*)), \liminf_{n\to\infty} D(f^n(x), f(x^*)), 0\} \\ =&\lambda \max\{D(x^*, f(x^*)), \liminf_{n\to\infty} D(f^n(x), f(x^*))\}. \end{aligned} \tag{2.8}$$

By (2.8), we consider two following subcases.

Subcase 2.1. $\liminf_{n\to\infty} D(f^{n+1}(x), f(x^*)) \leq \lambda D(x^*, f(x^*))$

We find that

$$D(x^*, f(x^*)) \leq \kappa[D(x^*, f^{n+1}(x)) + D(f^{n+1}(x), f(x^*))]. \tag{2.9}$$

From (2.6) and (2.9) we deduce that

$$\liminf_{n\to\infty} D(f^{n+1}(x), f(x^*)) \geq \frac{1}{\kappa} D(x^*, f(x^*)). \tag{2.10}$$

On the contrary, suppose that $x^* \neq f(x^*)$. Note that $0 \leq \lambda < \frac{1}{\kappa}$. Then

$$\liminf_{n\to\infty} D(f^{n+1}(x), f(x^*)) \leq \lambda D(x^*, f(x^*)) < \frac{1}{\kappa} D(x^*, f(x^*)).$$

This is a contradiction with (2.10). Therefore $x^* = f(x^*)$.

Subcase 2.2. $\liminf_{n\to\infty} D(f^{n+1}(x), f(x^*)) \leq \lambda \liminf_{n\to\infty} D(f^n(x), f(x^*))$.

From $\liminf_{n\to\infty} D(f^n(x), f(x^*)) = \liminf_{n\to\infty} D(f^{n+1}(x), f(x^*))$ and $0 \le \lambda < \frac{1}{\kappa}$ we have $\liminf_{n\to\infty} D(f^n(x), f(x^*)) = 0$. So there exists a subsequence $\{f^{k_n}(x)\}$ of $\{f^n(x)\}$ such that

$$\lim_{n\to\infty} f^{k_n}(x) = f(x^*). \tag{2.11}$$

Note that

$$D(x^*, f(x^*)) \le \kappa[D(x^*, f^{k_n}(x)) + D(f^{k_n}(x), f(x^*))]. \tag{2.12}$$

Letting $n \to \infty$ in (2.12) and using (2.11), (2.6) we obtain $D(x^*, f(x^*)) = 0$. Then $x^* = f(x^*)$.

By the conclusions of Case 1 and Case 2, we find that f has a fixed point x^* and by (2.6), $\lim_{n\to\infty} f^n(x) = x^*$.

Finally, we prove the uniqueness of the fixed point of f. Indeed, let x^*, y^* be two fixed points of f. From (2.1) we have

$$\begin{aligned}
&D(x^*, y^*)\\
=&D(f(x^*), f(y^*))\\
\le&\lambda \max\left\{D(x^*, y^*), D(x^*, f(x^*)), D(y^*, f(y^*)), D(x^*, f(y^*)), D(y^*, f(x^*))\right\}\\
=&\lambda D(x^*, y^*).
\end{aligned}$$

Since $\lambda \in [0;1)$, we obtain $D(x^*, y^*) = 0$, that is, $x^* = y^*$. Then the fixed point of f is unique. □

Next we present some examples to illustrate the obtained result. The following example shows there exists the map $f : X \to X$ so that Theorem 6 is applicable but Theorem 1, Theorem 4 and Theorem 5 are not.

Example 1. Let $X = \mathbb{R}$, and $D(x, y) = |x - y|^2$ for all $x, y \in X$, and the map $f : X \to X$ be defined by $f(x) = \frac{3}{4}x$ for all $x \in X$. Then

(1) (X, D, κ) is a complete b-metric space with the modulus of concavity $\kappa = 2$, D is continuous, and the condition (2.1) holds for all $\lambda \in [\frac{3}{4}, 1)$. Then Theorem 6 is applicable to f.
(2) The conditions (1.1) and (1.4) do not hold for all $\lambda \in [0, \frac{1}{\kappa})$. Then Theorem 1 and Theorem 4 are not applicable to f.
(3) D is not a strong b-metric. Then Theorem 5 is not applicable to f.

Proof. (1). It is easy to check that (X, D, κ) is a complete b-metric space with the modulus of concavity $\kappa = 2$, D is continuous, and the condition (2.1) holds for all $\lambda \in [\frac{3}{4}, 1)$. Then Theorem 6 is applicable to f.

(2). For $x = 0, y = 1$ and $\lambda \in [0, \frac{1}{\kappa}) = [0, \frac{1}{2})$, we find that

$$\begin{aligned}
&D(f(0), f(1))\\
=&\frac{9}{16}
\end{aligned}$$

$$\begin{aligned}&\geq \lambda\\&=\lambda \max\left\{D(0,1),D(0,f(0)),D(1,f(1)),D(0,f(1)),D(1,f(0))\right\}.\end{aligned}$$

This proves that conditions (1.1) and (1.4) do not hold for all $\lambda \in [0,\frac{1}{\kappa})$. Then Theorem 1 and Theorem 4 are not applicable to f.

(3). On the contrary, suppose that D is a strong b-metric. Then there exists $K \geq 1$ such that for all $x,y,z \in X$,

$$|x-y|^2 \leq |x-z|^2 + K|z-y|^2. \tag{2.13}$$

For $n \in \mathbb{N}$, and $x_0 = \frac{1}{n}$, $y_0 = 1+\frac{1}{n}$, $z_0 = 1$ we have

$$|x_0 - y_0|^2 = 1$$

$$|x_0 - z_0|^2 + K|z_0 - y_0|^2 = (\frac{1}{n}-1)^2 + \frac{K}{n^2} = 1 + \frac{1-2n+K}{n^2}.$$

So for $n > K$ we have

$$|x_0 - z_0|^2 + K|z_0 - y_0|^2 < 1 = |x_0 - y_0|^2.$$

It is a contradiction to (2.13). Then D is not a strong b-metric, and Theorem 5 is not applicable to f. □

The following example shows that the continuity of D in Theorem 6. (2a) and the condition $\lambda \in [0;\frac{1}{\kappa})$ in Theorem 6. (2b) are essential.

Example 2. Let $X = \left\{0,1,\frac{1}{2},\dots,\frac{1}{n},\dots\right\}$, and

$$D(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \in \{0,1\} \\ |x-y| & \text{if } x \neq y \in \{0\} \cup \left\{\frac{1}{2n} : n = 1,2,\dots\right\} \\ \frac{1}{4} & \text{otherwise,} \end{cases}$$

and let $f : X \to X$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{10n} & \text{if } x = \frac{1}{n}, n = 1,2,\dots \end{cases}$$

Then

(1) (X,D,κ) is a complete b-metric space with the modulus of concavity $\kappa = 4$.
(2) There exist $\lambda \geq 0$ such that the contraction condition (2.1) holds for all $x,y \in X$.
(3) D is not continuous and $\lambda \in [\frac{1}{\kappa};1)$.
(4) f is fixed point free.

Proof. By [9, Example 2.6], (X,D,κ) is a complete metric-type space with the modulus of concavity $\kappa = 4$. Then (X,D,κ) is also a complete b-metric on X with the modulus of concavity $\kappa = 4$. The remaining conclusions were proved in [9, Example 2.6] and [9, Remark 2.7]. □

The following example shows that the assumption of bounded orbit in Theorem 2 and Theorem 3 is essential. Moreover, for the case of unbounded orbit, the value $\operatorname{diam} O(x,y)$ cannot be replaced by $\max\{d(x,f(y)),d(y,f(x))\}$. However, the value $\operatorname{diam} O(x,y)$ can be replaced by $d(x,y)$ in the class of complete regular semimetric spaces, which is a generalization of the class of complete b-metric spaces, see [6, Theorem 1].

Example 3. Let $X=\{1,2,3,\dots\}$, $d(x,y)=|x-y|$ for all $x,y\in X$, $f(x)=x+2$ for all $x\in X$ and $\varphi:[0;\infty)\to[0;\infty)$ be defined by

$$\varphi(t)=\begin{cases}\frac{2}{3}t & \text{if } t\in[0;3)\\ t-1 & \text{if } t\geq 3.\end{cases}$$

Then we have

(1) (X,d) is a complete metric space, and φ is an increasing, upper semicontinuous function, $\varphi(0)=0$ and $\varphi(t)<t$ for all $t>0$. In particular, (X,d) is also a complete b-metric space.
(2) Every orbit of f is unbounded. So $\operatorname{diam} O(x,y)=\infty$ and then (1.3) holds for all $x,y\in X$ and all $\lambda\in(0;1)$.
(3) $d(f(x),f(y))\leq\varphi(\max\{d(x,f(y)),d(y,f(x))\})$ for all $x,y\in X$. Then (1.2) holds for all $x,y\in X$.
(4) f is fixed point free.

Proof. (1). It is clear that (X,d) is a complete metric space, and φ is an increasing, upper semicontinuous function, $\varphi(0)=0$ and $\varphi(t)<t$ for all $t>0$.

(2). For all $x\in X$ we have $O(x)=\{x,x+2,x+4,\dots\}$ which is an unbounded orbit of f.

(3). Let $x,y\in X$. We may assume that $x<y$. The we have

$$d(f(x),f(y))=|x-y|=y-x$$

and

$$d(x,f(y))=y-x+2\geq 3.$$

Then we have

$$\varphi(\max\{d(x,f(y)),d(y,f(x))\})=\varphi(d(x,f(y)))=y-x+1>d(f(x),f(y)).$$

This proves that (1.2) holds for all $x,y\in X$.

(4). Since $f(x)=x+2$ for all $x\in X$, f is fixed point free. □

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EXPONENTIAL MONOMIALS ON HYPERGROUP JOINS

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Abstract. Exponential monomials and polynomials are the basic building blocks of spectral synthesis. Recently a systematic study of exponential polynomials has been started on hypergroups. In this paper we join these investigations and describe exponential polynomials on hypergroup joins.

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1. Introduction

We started the study of basic function classes on hypergroup joins in our paper [14]. The basic function classes we are studying play a fundamental role in spectral analysis and spectral synthesis (see [9, 12]). In this paper we continue this work and describe further basic function classes, called *moment functions* on hypergroup joins (see [1]). In the sequel $\mathbb{C}$ denotes the set of complex numbers. By a *hypergroup* we always mean a locally compact hypergroup. For basics about hypergroups see the monograph [1]. A comprehensive monograph on functional equations on hypergroups is [7].

Let K be a hypergroup with identity e and involution $^\vee$. The non-identically zero continuous function m is called an *exponential* on K if $m : K \to \mathbb{C}$ satisfies $m(x * y) = m(x)m(y)$ for each x, y in K. The description of exponentials on some types of commutative hypergroups can be found in [7]. In [11] (see also [10]) the author defined the concept of exponential monomial on commutative hypergroups: let K be a commutative hypergroup and $f : K \to \mathbb{C}$ a continuous function. We say that f is a *generalized exponential monomial*, if there exists an exponential $m : K \to \mathbb{C}$ and a natural number n such that

$$\Delta_{m;y_1,y_2,\dots,y_{n+1}} * f = 0$$

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holds for every $y_1, y_2, \ldots, y_{n+1}$ in K. We recall that $\Delta_{m;y_1,y_2,\ldots,y_{n+1}}$ is the convolution product

$$\Delta_{m;y_1,y_2,\ldots,y_{n+1}} = \Pi_{k=1}^{n+1}(\delta_{\check{y}} - m(y)\delta_o),$$

where, in general, δ_x denotes the point mass supported at x in K, and o denotes the identity in K. It is known that if f is nonzero, then the exponential m is unique, and the smallest n with the above property is called the *degree* of f. In this case we say that f is a *generalized m-exponential monomial*, or f is *associated with m*. We call f simply an *exponential monomial*, if the linear space generated by all translates $\delta_{\check{y}} * f$ is finite dimensional. Clearly, any constant multiple of the exponential m is an m-exponential monomial, and if the constant is nonzero, then it has degree 0. Another simple example is that of the m-sine functions: f is called an *m-sine function*, if it satisfies

$$f(x * y) = f(x)m(y) + f(y)m(x)$$

for each x, y in K. If f is nonzero, then it is an m-exponential monomial of degree 1. Linear combinations of generalized exponential monomials are called *generalized exponential polynomials*, and linear combinations of exponential monomials are called *exponential polynomials*. We note that quadratic functions (see [14]) are not necessarily exponential polynomials, but if the hypergroup structure on K arises from a group structure, then every quadratic function is a generalized exponential monomial associated with the exponential identically 1.

Exponential polynomials have fundamental importance in spectral analysis and synthesis. Spectral synthesis on hypergroups has been investigated in the works [7, 9]. For more about spectral analysis and spectral synthesis see the monograph [12]. Characterization of these functions classes and related functional equations on different types of hypergroups have been studied in several papers(see e.g [2, 8]). In this paper we describe generalized exponential classes on hypergroup joins. The hypergroup join construction is a special way to unite two hypergroups, one of them is compact and the other is discrete. Hypergroup joins and their duals have been studied in [17]. Hypergroup joins were generalized by M. Voit [15] and by H. Heyer and S. Kawakami [3], based on exact sequences.

Another important class is presented by the class of *moment functions*. According to the terminology in [7] we say that the continuous functions $f_n : K \to \mathbb{C}$ ($n = 0, 1, \ldots, N$, N is a natural number) on the hypergroup K form a *(generalized) moment function sequence* of order N if f_0 is non-identically zero and

$$f_n(x * y) = \sum_{k=0}^{n} \binom{n}{k} f_k(x) f_{n-k}(y) \tag{1.1}$$

holds for each x, y in K and for $k = 0, 1, \ldots, N$(see e.g [4–6]). Clearly, f_0 is an exponential and we say that the sequence $(f_n)_{n\in\mathbb{N}}$ is *associated with the exponential* f_0. For the sake of simplicity we omit the adjective "generalized". The function f_k in this sequence is called a *moment function* of order k. Hence moment functions of

order 0 are exactly the exponentials and moment functions of order 1 associated with the exponential $m = f_0$ are exactly the m-sine functions. In [13] generalized moment functions on hypergroup joins were characterized and described. We shall see below that generalized moment functions are exponential monomials.

2. Hypergroup join

The definition of hypergroup join can be found in [1], p. 59. Here we recall the construction. Let $(C,*)$ be a compact hypergroup with normalized Haar measure ω_C and $(D,\cdot)$ a discrete hypergroup with $C \cap D = \{e\}$, the identity of both hypergroups. The *hypergroup join* $C \vee D$ is the set $C \cup D$ with the unique topology for which both C and D are closed subspaces. Involution on $C \cup D$ is defined in the way that its restriction to C and to D, respectively, coincides with the involution on C and on D, respectively. Convolution on $C \vee D$ is defined in the following way:

(1) For x, y in C the convolution of δ_x and δ_y is $\delta_x * \delta_y$.
(2) For x, y in D and $x \neq \check{y}$ the convolution of δ_x and δ_y is $\delta_x \cdot \delta_y$.
(3) For x in C and $y \neq e$ in D the convolution of δ_x and δ_y and also the convolution of δ_y and δ_x is δ_y.
(4) For $y \neq e$ in D we have the unique representation
$$\delta_y \cdot \delta_{\check{y}} = \sum_{w \in D} c_w \delta_w$$
with some complex numbers c_w for w in D. Then the convolution of δ_y and $\delta_{\check{y}}$ and also the convolution of $\delta_{\check{y}}$ and δ_y is
$$c_e \omega_C + \sum_{w \in D, w \neq e} c_w \delta_w = \delta_y \cdot \delta_{\check{y}} + c_e(\omega_C - \delta_e).$$

For the sake of simplicity, by virtue of 1. above, we denote the convolution in $C \vee D$ with $*$, too. We note that commutativity is not assumed in C nor in D. In fact, $C \vee D$ is commutative if and only if C and D is commutative. Clearly, C is a compact subhypergroup of $C \vee D$, D is a discrete subset of $C \vee D$, but D is not necessarily a subhypergroup of $C \vee D$. For further information about hypergroup joins and their applications see [1].

In what follows we shall always assume that $D \neq \{e\}$ and we denote $D \backslash \{e\}$ with D_e. It follows that D_e is nonempty.

3. Moment functions as exponential monomials

A special case of exponential monomials is presented by the generalized moment functions as it is shown in the following theorem.

Theorem 1. *Let K be a commutative hypergroup and $(\varphi_n)_{n \in \mathbb{N}}$ a generalized moment function sequence. Then φ_n is an exponential monomial of degree at most n for each n.*

Proof. By definition, the sequence satisfies

$$\varphi_n(x*y) = \sum_{k=0}^{n} \varphi_k(x)\varphi_{n-k}(y)$$

for each x,y in K ($n = 0,1,\dots$). We prove the statement by induction on n and it is obvious for $n = 0$. Clearly $m = \varphi_0$ is an exponential. Suppose that $n \geq 1$ and we have proved our statement for $k = 0,1,\dots,n-1$. Now we prove it for $k = n$. Let $y_1,y_2,\dots,y_{n+1}$ be arbitrary in K. We have

$$\begin{aligned}
&\Delta_{m;y_1,y_2,\dots,y_{n+1}}\varphi_n(x) = \Delta_{m;y_1,y_2,\dots,y_n}(\varphi_n(x*y_{n+1}) - m(y_{n+1})\varphi_n(x))\\
&= \Delta_{m;y_1,y_2,\dots,y_n}\Big(\sum_{k=0}^{n}\binom{n}{k}\varphi_k(x)\varphi_{n-k}(y_{n+1})\Big) - m(y_{n+1})\Delta_{m;y_1,y_2,\dots,y_n}\varphi_n(x)\\
&= \Delta_{m;y_1,y_2,\dots,y_n}\varphi_n(x).m(y_{n+1}) - m(y_{n+1})\Delta_{m;y_1,y_2,\dots,y_{n+1}}\varphi_n(x) = 0,
\end{aligned}$$

which proves the statement. □

Theorem 2. *Let K be a commutative hypergroup and $\Phi : K \times \mathbb{C}^n \to \mathbb{C}$ an exponential family for K. Let N be a nonnegative integer and $1 \leq i \leq n$ an integer. Then for every polynomial $P : \mathbb{C}^n \to \mathbb{C}$ of degree N the function $x \mapsto P(\partial_\lambda)\Phi(x,\lambda)$ is an exponential monomial of degree at most N.*

Proof. The proof can be found in [10, Theorem 3]. □

4. Exponential monomials on compact hypergroups

Theorem 3. *On a compact commutative hypergroup every nonzero generalized exponential monomial is of degree zero.*

Proof. We prove the statement by induction on the degree of the generalized m-exponential monomial $f \neq 0$, and it is obvious if $\deg f = 0$. First we note that every nonzero generalized m-exponential polynomial of degree 0 is a constant multiple of m. Indeed, we have

$$0 = \Delta_{m;y} * f(x) = f(x*y) - m(y)f(x)$$

for each x,y in K. Interchanging x and y we obtain $f(x)m(y) = f(y)m(x)$, hence the substitution $y = o$ gives $f(x) = f(o)m(x)$.

We introduce the function $g(x) = f(x) - f(o)m(x)$. Then, by the obvious property

$$\Delta_{m;y} * g = \Delta_{m;y} * f,$$

it follows that g is a generalized m-exponential monomial of degree at most n. In addition, $g(o) = 0$.

We assume that we have proved that every nonzero generalized m-exponential polynomial of degree at most $n-1$ is of degree zero, and now we prove it for $\deg g \leq n$. By assumption,

$$\Delta_{m;y_1,y_2,\dots,y_n} * (\Delta_{m;y} * g)(x) = \Delta_{m;y,y_1,y_2,\dots,y_n} * g(x) = 0,$$

that is, $\Delta_{m;y} * g$ is a generalized m-exponential polynomial of degree at most $n-1$, hence $\Delta_{m;y} * g = c(y)m(x)$ holds for each y in K, where $c : K \to \mathbb{C}$ is some continuous function. In other words, we have

$$g(x*y) = m(y)g(x) + c(y)m(x)$$

for each x, y in K. Putting $x = o$ we have $g = c$, hence g is an m-sine function, and by the results in [16], g is identically zero. It follows that $f = f(o)m$ and our theorem is proved. □

In [16] M. Voit proved that on commutative compact hypergroups every m-sine function is zero. It is not known if this statement is true on non-commutative compact hypergroups. The following theorem shows that it is true at least for 1-sine functions, that is, for *additive functions*. In fact, we prove a stronger statement.

Theorem 4. *On a compact hypergroup every generalized* 1*-exponential monomial is constant.*

Proof. Let $f : K \to \mathbb{C}$ be a generalized 1-exponential monomial on the compact hypergroup K. If $\deg f = 0$, then we have $f(x*y) - f(x) = 0$, hence with $x = o$ it follows $f(y) = f(o)$ for each y in K.

Assume that we have proved our statement for $\deg f \leq n-1$ and now we let $\deg f = n \geq 1$. We have for $y, y_1, y_2, \ldots, y_n$ in K

$$0 = \Delta_{1;y_1,y_2,\ldots,y_n,y} * f(x) = \Delta_{1;y_1,y_2,\ldots,y_n} * \left(\Delta_{1;y} * f\right)(x) = 0,$$

hence, by assumption, $\Delta_{1;y} * f$ is a constant:

$$\Delta_{1;y} * f(x) = f(x*y) - f(x) = c(y)$$

holds for each x, y in K, where $c : K \to \mathbb{C}$ is a continuous function. We have

$$f(x*y) = f(x) + c(y)$$

whenever x, y is in K. Using associativity we have

$$f(x*y*z) = f(x*y) + c(z) = f(x) + c(y) + c(z),$$

and

$$f(x*y*z) = f(x) + c(y*z),$$

hence we infer $c(y*z) = c(y) + c(z)$ for each y, z in K. As K is compact and c is continuous, the range of c is compact in $\mathbb{C}$. If there is an x_0 such that $c(x_0) \neq 0$, then $c(n \cdot x_0) = c(x_0 * x_0 * \ldots x_0) = n \cdot c(x_0)$, which implies that the range of c is unbounded, a contradiction. Hence $c \equiv 0$ and f is constant. □

In [14] we proved that at least the integral of every m-sine function is zero on any compact hypergroup. Now we prove the analogous result for generalized m-exponential functions on compact hypergroups. We note that, clearly, on a compact hypergroup the integral of a constant is the constant itself, as the integral always refers to the integral with respect to the unique normalized Haar measure.

Theorem 5. *On a compact hypergroup the integral of every non-constant generalized m-exponential monomial is zero.*

Proof. Let K be a compact hypergroup with normalized Haar measure dx. We can show easily, by induction on n, that for any continuous function $g : K \to \mathbb{C}$ and exponential m on K we have

$$\int_K \Delta_{m;y_1,y_2,\dots,y_{n+1}} * g(x)\,dx = \Pi_{k=1}^{n+1}\big(1-m(y_k)\big)\int_K g(x)\,dx$$

for each $y_1, y_2, \dots, y_{n+1}$ in K. Indeed, th statement is true for $n = 0$.

Clearly, this equation implies the statement. Indeed, if g is a generalized m-exponential monomial of degree at most n, then the left side is zero for each $y_1, y_2, \dots, y_{n+1}$ in K. Hence the right side is zero, too, consequently if $\int_K g(x)\,dx \neq 0$, then $m \equiv 1$, that is g is a generalized 1-exponential monomial, and, by the previous theorem, it is constant, which proves our statement. □

We note that generalized 1-exponential polynomials are called *generalized polynomials*.

5. Exponential monomials on hypergroup joins

Here we recall the theorem about the description of exponentials on hypergroup joins (see [14, Theorem 1]).

Theorem 6. *Let C, D be as above. The continuous function $m : C \cup D \to \mathbb{C}$ is an exponential on the hypergroup join $C \vee D$ if and only if one of the following possibilities holds:*

i) $m|_C \neq 1$ is an exponential on C and $m|_{D_e}$ is identically zero;
ii) $m|_C$ is identically 1 and $m|_D$ is an exponential on D.

Our main theorem follows.

Theorem 7. *Let C be a compact hypergroup and D a discrete commutative hypergroup. Then the continuous function $f : C \cup D \to \mathbb{C}$ is a generalized exponential monomial of degree at most n on the hypergroup join $C \vee D$ if and only if any of the following cases holds:*

i) $f\big|_C$ is a generalized exponential monomial of degree at most n associated with an exponential $m_C \neq 1$ on C, and $f\big|_{D_e}$ is zero.
ii) $f\big|_C$ is constant, and $f\big|_D$ is a generalized exponential monomial of degree at most n on D.

Proof. In this proof we shall denote the convolution on C and on $C \vee D$ by $x * y$, and on D by $x \cdot y$, further ω_C denotes the unique normalized Haar measure on the compact hypergroup C. We note that $f\big|_C$ is always a generalized exponential monomial of degree at most n on C, as the convolution on $C \vee D$ coincides with the convolution on C and the restriction of any exponential on $C \vee D$ to C is an exponential on C.

Let f be a generalized m-exponential monomial of degree at most n on $C \vee D$. By the previous theorem, we have two possibilities for m. In the first case $m\big|_C$ is an exponential on C, and $m\big|_{D_e} \equiv 0$. Then clearly, $f\big|_C$ is a generalized $m\big|_C$-exponential monomial of degree at most n on C. On the other hand, let $y \neq e$ be in D, then we have for each x in D_e:

$$0 = \Delta_{m;y}^{n+1} * f(x) = f(x),$$

hence f is identically zero on D_e, which is case i) above.

In the second case $m\big|_C$ is identically 1, and $m\big|_D$ is an exponential on D. Clearly, $f\big|_C$ is a generalized 1-exponential monomial on C, hence it is constant, by Theorem 4. We claim that $f\big|_D$ is a generalized $m\big|_D$-exponential monomial of degree at most n on D. We have to prove the equality

$$\Delta_{m|_D;y_1,y_2,\dots,y_{n+1}} \cdot f(x) = 0$$

whenever $x, y_1, y_2, \dots, y_{n+1}$ are in D. Here Δ is formed using the convolution in D which is the same as in $C \vee D$ if there is no pair among the elements $x, y_1, y_2, \dots,$ y_{n+1} which are involutive to each other. But if there are such pairs $z, \check{z}$, then we have

$$f(z * \check{z}) = f(z \cdot \check{z}) + c_e\Big(\int_C f(t)\, d\omega_C(t) - f(e)\Big) = f(z \cdot \check{z}),$$

as $f\big|_C$ is constant, that is $f\big|_C = f(e)$, which implies our statement.

For the converse we suppose first that i) holds, and $f\big|_C$ is a generalized m_C-exponential monomial of degree at most n on C, where $m_C \neq 1$ is an exponential on C, further $f\big|_{D_e}$ is zero. First we note, that $f\big|_C$ is non-constant, as $m_C \neq 1$. We define $m : C \cup D \to \mathbb{C}$ as $m(x) = m_C(x)$ for x in C, and $m(x) = 0$ for x in D_e. Then, by Theorem 1 in [14], m is an exponential on $C \vee D$. We show that f is a generalized m-exponential monomial of degree at most n on K.

If $x, y_1, y_2, \dots, y_{n+1}$ are in C, then clearly we have

$$\Delta_{m;y_1,y_2,\dots,y_{n+1}} * f(x) = \Delta_{m_C;y_1,y_2,\dots,y_{n+1}} * f\big|_C(x) = 0.$$

Let y be in D and x in C. Then $f(x * y)$ is either $f(x)$ or $f(y) = 0$ depending on if $y = e$ or $y \neq e$, by the definition of the convolution on K. Hence

$$\Delta_{m;y} * f(x) = \begin{cases} f(x) - f(x) = 0 & \text{if } y = e \\ f(y) - m(y)f(y) = 0 & \text{if } y \neq e. \end{cases}$$

It follows that

$$\Delta_{m;y_1,y_2,\dots,y_{n+1}} * f(x) = 0$$

if x is in C and at least one of the y_i's is in D.

Now let x, y be in D, $x \neq \check{y}$. Then we have

$$\Delta_{m;y} * f(x) = f(x * y) - m(y)f(x) = \begin{cases} f(x * y) = f(x \cdot y) = 0 & \text{if } y \neq e \\ 0 & \text{if } y = e. \end{cases}$$

The first part follows form the fact that if $x \neq \check{y}$, then e is not in the support of $x \cdot y$, hence $f(x \cdot y) = \sum_{w \in D_e} c_w f(w) = 0$, as f vanishes on D_e. On the other hand, if $x = \check{y}$, then we have

$$\Delta_{m;y} * f(x) = f(\check{y} * y) - m(y) f(\check{y}) = \begin{cases} f(\check{y} * y) & \text{if } y \neq e \\ 0 & \text{if } y = e. \end{cases}$$

We recall that, by the definition of the convolution on $C \vee D$, we have

$$\delta_{\check{y}} * \delta_y = \delta_y \cdot \delta_{\check{y}} + c_e(\omega_C - \delta_e),$$

where c_e is the coefficient of δ_e in the expansion

$$\delta_y \cdot \delta_{\check{y}} = \sum_{w \in D} c_w \delta_w$$

on the hypergroup D. It follows

$$\begin{aligned} f(\check{y} * y) &= f(\check{y} \cdot y) + c_e\Big(\int_C f(t)\, d\omega_C(t) - f(e)\Big) \\ &= \sum_{w \in D} c_w f(w) - c_e f(e) + c_e \int_C f(t)\, d\omega_C(t) \\ &= \sum_{w \in D, w \neq e} c_w f(w) + c_e \int_C f(t)\, d\omega_C(t) \\ &= c_e \int_C f(t)\, d\omega_C(t), \end{aligned}$$

as f vanishes on D_e. Since f is non-constant on C, this integral is zero, by Theorem 5.

We conclude that

$$\Delta_{m;y_1,y_2,\ldots,y_{n+1}} * f(x) = 0$$

holds for each $x, y_1, y_2, \ldots, y_{n+1}$ in $C \cup D$, hence f is a generalized m-exponential monomial on $C \vee D$ of degree at most n.

Now we assume that *ii*) holds, that is, $f\big|_C$ is constant, and $f\big|_D$ is a generalized m_D-exponential monomial of degree at most n on D with some exponential m_D on D. Now we define $m : C \cup D \to \mathbb{C}$ as $m(x) = 1$ for x in C and $m(x) = m_D(x)$ for x in D. Then m_D is an exponential on $C \vee D$, by Theorem 6. We claim that f is a generalized m-exponential monomial of degree at most n on $C \vee D$.

As $f\big|_C$ is constant and $m\big|_C = 1$, we clearly have

$$\Delta_{m;y} * f(x) = 0$$

for each x, y in C. It follows that

$$\Delta_{m;y_1,y_2,\ldots,y_{n+1}} * f(x) = 0$$

holds for each $x, y_1, y_2, \ldots, y_{n+1}$ in C.

For x in C and y in D we have

$$\Delta_{m;y} * f(x) = f(x*y) - m(y)f(x) = f(y) - f(e)m_D(y) = \Delta_{m_D;y} \cdot f(e).$$

By iteration, we have

$$\Delta_{m;y_1,y_2,\dots,y_{n+1}} * f(x) = \Delta_{m_D;y_1,y_2,\dots,y_{n+1}} \cdot f(e) = 0,$$

for each x in C and $y_1, y_2, \dots, y_{n+1}$ in D, as, by assumption, $f\big|_D$ is a generalized m_D-exponential monomial of degree at most n on D. We note that on the right side Δ is formed by using convolution in D.

For x in D and y in C we have

$$\Delta_{m;y} * f(x) = f(x*y) - m(y)f(x) = f(x) - f(x) = 0,$$

hence by iteration, we have again

$$\Delta_{m;y_1,y_2,\dots,y_{n+1}} * f(x) = 0$$

whenever x is in D and there is a y_i which is in C.

If x, y are in D and $x \neq \check{y}$, then

$$\Delta_{m;y} * f(x) = f(x*y) - m(y)f(x) = f(x \cdot y) - m_D(y)f(x).$$

On the other hand, if $x = \check{y}$, then

$$\begin{aligned}\Delta_{m;\check{y}} * f(y) &= f(y*\check{y}) - m_D(\check{y})f(y)\\ &= f(y\cdot\check{y}) + c_e\Big(\int_C f(t)\,d\omega_C(t) - f(e)\Big) - m_D(\check{y})f(y)\\ &= f(y\cdot\check{y}) - m_D(\check{y})f(y),\end{aligned}$$

as $f(t) = f(e)$ for t in C. By iteration, we conclude, that if $x, y_1, y_2, \dots, y_{n+1}$ are in D then

$$\Delta_{m;y_1,y_2,\dots,y_{n+1}} * f(x) = \Delta_{m_D;y_1,y_2,\dots,y_{n+1}} \cdot f(x) = 0,$$

where Δ on the right side formed by using the convolution in D, and, by assumption, $f\big|_D$ is a generalized m_D-exponential monomial of degree at most n on D. The proof is complete. □

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BASIC INVARIANTS OF GEOMETRIC MAPPINGS

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Abstract. This study is motivated by the researches in the field for invariants of geodesic and conformal mappings presented in (T. Y. Thomas, [17]) and (H. Weyl, [20]). The Thomas projective parameter and the Weyl projective tensor are generalized in this article. Generators for vector spaces of invariants of geometric mappings are obtained in here.

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1. Introduction

An N-dimensional manifold $\mathcal{M}_N$ equipped with an affine connection ∇ (with torsion) is called the non-symmetric affine connection space $\mathbb{GA}_N$ (see [6, 12–15, 18, 19, 21]). As a special case, the manifold $\mathcal{M}_N$ equipped with a torsion-free affine connection $\overset{0}{\nabla}$ is called the symmetric affine connection space $\mathbb{A}_N$. More details about the theory of symmetric affine connection spaces may be found in [9–11, 16].

T. Y. Thomas [17] and H. Weyl [20] started the research about invariants of special diffeomorphisms between symmetric affine connection spaces for different applications in physics. Many authors have continued the Thomas's and Weyl's works. J. Mikeš [1, 2, 8–11], I. Hinterleitner [10, 11], N. S. Sinyukov [16], are some of them. Some of invariant geometrical object for diffeomorphisms of non-symmetric affine connection spaces are obtained in [15, 18, 19, 21].

In this paper, as in the previous articles, books and monographs, the spaces $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$ will be the manifold $\mathcal{M}_N$ equipped with the affine connections ∇ and $\overline{\nabla} = f(\nabla)$. A diffeomorphism $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ which the affine connection ∇ of the space $\mathbb{GA}_N$ transforms to the affine connection $\overline{\nabla}$ of the space $\mathbb{G}\overline{\mathbb{A}}_N$ is *the mapping* of the space $\mathbb{GA}_N$.

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In this paper, we will obtain sets of families of invariants for diffeomorphisms defined on affine connection spaces with and without torsion. Moreover, we will prove that several of these families of invariants are linearly independent.

1.1. *Affine connection spaces*

For different applications in physics, for example in the Theory of Relativity [3–5], affine connection spaces with torsion have been studied.

Let $\mathbb{GA}_N$ be a non-symmetric affine connection space. The affine connection coefficients L^i_{jk} of this space are non-symmetric in the indices j and k. The symmetric and anti-symmetric part of the coefficient L^i_{jk} are respectively

$$L^i_{\underline{jk}} = \tfrac{1}{2}\left(L^i_{jk} + L^i_{kj}\right) \quad \text{and} \quad L^i_{\underset{\vee}{jk}} = \tfrac{1}{2}\left(L^i_{jk} - L^i_{kj}\right). \tag{1.1}$$

The symmetric part $L^i_{\underline{jk}}$ is the affine connection coefficient for a torsion-free affine connection $\overset{0}{\nabla}$. The manifold $\mathcal{M}^N$ equipped with the affine connection $\overset{0}{\nabla}$ $\left(\text{whose coefficients are } L^i_{\underline{jk}}\right)$ is the associated space $\mathbb{A}_N$ (of the space $\mathbb{GA}_N$).

The covariant derive of a tensor a^i_j of the type $(1,1)$ with respect to the affine connection of the associated space $\mathbb{A}_N$ is (see [9–11, 16])

$$a^i_{j|k} = a^i_{j,k} + L^i_{\underline{\alpha k}} a^\alpha_j - L^\alpha_{\underline{jk}} a^i_\alpha, \tag{1.2}$$

for the partial derivatives $\partial/\partial x^i$ denoted by comma.

With respect to the affine connection $\overset{0}{\nabla}$ and the corresponding covariant derivative $|$, one Ricci-type identity is obtained. The corresponding curvature tensor of the associated space $\mathbb{A}_N$ is (see [9–11, 16])

$$R^i_{jmn} = L^i_{\underline{jm},n} - L^i_{\underline{jn},m} + L^\alpha_{\underline{jm}} L^i_{\underline{\alpha n}} - L^\alpha_{\underline{jn}} L^i_{\underline{\alpha m}}. \tag{1.3}$$

Based on the definitions and results from L. P. Eisenhart [6, 7], A. Einstein [3–5], S. M. Minčić defined four kinds of covariant derivaties with respect to the affine connection ∇ of the space $\mathbb{GA}_N$ [12–14]

$$\begin{aligned} a^i_{\underset{1}{j|k}} &= a^i_{j,k} + L^i_{\alpha k} a^\alpha_j - L^\alpha_{jk} a^i_\alpha, \quad a^i_{\underset{2}{j|k}} = a^i_{j,k} + L^i_{k\alpha} a^\alpha_j - L^\alpha_{kj} a^i_\alpha, \\ a^i_{\underset{3}{j|k}} &= a^i_{j,k} + L^i_{\alpha k} a^\alpha_j - L^\alpha_{kj} a^i_\alpha, \quad a^i_{\underset{4}{j|k}} = a^i_{j,k} + L^i_{k\alpha} a^\alpha_j - L^\alpha_{jk} a^i_\alpha. \end{aligned} \tag{1.4}$$

With respect to these generalizations of the covariant derivative (1.2), S. M. Minčić got four curvature tensors, eight derived curvature tensors and fifteen curvature pseudotensors of the space $\mathbb{GA}_N$ [12–14]. Curvature tensors and derived curvature tensors of the space $\mathbb{GA}_N$ are elements of the family [21]

$$K^i_{jmn} = R^i_{jmn} + uL^i_{\underset{\vee}{jm}|n} + u' L^i_{\underset{\vee}{jn}|m} + vL^\alpha_{\underset{\vee}{jm}} L^i_{\underset{\vee}{\alpha n}} + v' L^\alpha_{\underset{\vee}{jn}} L^i_{\underset{\vee}{\alpha m}} + wL^\alpha_{\underset{\vee}{mn}} L^i_{\underset{\vee}{\alpha j}}, \tag{1.5}$$

for the curvature tensor R^i_{jmn} of the associated space $\mathbb{A}_N$ and real coefficients u, u', v, v', w. Five of twelve curvature tensors from the family (1.5) are linearly independent [13], while the rest can be expressed in terms of these five tensors and the curvature tensor R^i_{jmn} of the associated space $\mathbb{A}_N$.

Special kind of non-symmetric affine connection spaces are N-dimensional differentiable manifolds equipped with the non-symmetric metric tensor g_{ij} of the type $(0,2)$. The symmetric and anti-symmetric part of the metric g_{ij} are

$$g_{\underline{ij}} = \tfrac{1}{2}\big(g_{ij} + g_{ji}\big) \quad \text{and} \quad g_{\underset{\vee}{ij}} = \tfrac{1}{2}\big(g_{ij} - g_{ji}\big). \tag{1.6}$$

These spaces are *the generalized Riemannian spaces* $\mathbb{GR}_N$ (see [7]). The affine connection coefficients of the space $\mathbb{GR}_N$ are the generalized Christoffel symbols of the second kind

$$\Gamma^i_{jk} = \frac{1}{2} g^{\underline{i\alpha}}\big(g_{j\alpha,k} - g_{jk,\alpha} + g_{\alpha k,j}\big), \tag{1.7}$$

for $\big[g^{\underline{ij}}\big] = \big[g_{\underline{ij}}\big]^{-1}$. After symmetrizing the symbols Γ^i_{jk} in the indices j and k, we get that they reduce to the corresponding Christoffel symbols $\Gamma^i_{\underline{jk}}$ obtained from the symmetric metric $g_{\underline{ij}}$ of the associated space $\mathbb{R}_N$.

2. About invariants

Many invariants of mappings for torsion-free spaces are obtained. Some of them are the Thomas projective parameter, the Weyl projective tensor, the Weyl conformal curvature, and many others. These invariants may be found in the next monographs, books and papers: Mikeš [1, 2, 8–11], Sinyukov [16], Hinterleitner [10], Berezovski [1, 2, 10], etc.

In this paper, the author's main purpose is to search inceptive invariants of different mappings defined on affine connection spaces. We will start the generalization of the basic invariants in here. The next aim of this article is to discover how many of the basic invariants are linearly independent.

Let $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be a mapping between non-symmetric affine connection spaces $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$.

The deformation tensor $P^i_{jk} = \overline{L}^i_{jk} - L^i_{jk}$ of this mapping is

$$P^i_{jk} = \overline{\omega}^i_{jk} - \omega^i_{jk} + \overline{\tau}^i_{jk} - \tau^i_{jk}, \tag{2.1}$$

for geometrical objects $\omega^i_{jk}, \overline{\omega}^i_{jk}, \tau^i_{jk}, \overline{\tau}^i_{jk}$ of the type $(1,2)$ such that $\omega^i_{jk} = \omega^i_{kj}$, $\overline{\omega}^i_{jk} = \overline{\omega}^i_{kj}$, $\tau^i_{jk} = -\tau^i_{kj}$, $\overline{\tau}^i_{jk} = -\overline{\tau}^i_{kj}$. If the mapping f is equitorsion [15, 19, 21], i.e. if $\overline{L}^i_{\underset{\vee}{jk}} = L^i_{\underset{\vee}{jk}}$, the equation (2.1) reduces to

$$P^i_{jk} = \overline{\omega}^i_{jk} - \omega^i_{jk}. \tag{2.2}$$

After symmetrizing the equations (2.1, 2.2) in the indices j and k, we get

$$P^i_{\underline{jk}} = \overline{\omega}^i_{jk} - \omega^i_{jk}. \tag{2.3}$$

The deformation tensor $\overline{P}^i_{jk}$ of the inverse map $f^{-1} : \mathbb{G}\overline{\mathbb{A}}_N \to \mathbb{G}\mathbb{A}_N$ is

$$\overline{P}^i_{jk} = L^i_{jk} - \overline{L}^i_{jk} = -P^i_{jk}.$$

So, the following equalities hold

$$P^i_{\underline{jk}} = \overline{L}^i_{\underline{jk}} - L^i_{\underline{jk}} = \overline{\omega}^i_{jk} - \omega^i_{jk} = \Big(-\frac{1}{2}\overline{P}^i_{\underline{jk}}\Big) - \Big(-\frac{1}{2}P^i_{\underline{jk}}\Big). \tag{2.4}$$

Hence, the next equation is satisfied

$$P^i_{\underline{jk}} = \overline{\omega}^i_{(1).jk} - \omega^i_{(1).jk} = \overline{\omega}^i_{(2).jk} - \omega^i_{(2).jk} = \overline{\omega}^i_{(3).jk} - \omega^i_{(3).jk}, \tag{2.5}$$

for the geometrical objects

$$\omega^i_{(1).jk} = L^i_{\underline{jk}}, \quad \omega^i_{(2).jk} = \omega^i_{jk}, \quad \omega^i_{(3).jk} = -\tfrac{1}{2}P^i_{\underline{jk}}, \tag{2.6}$$

and the corresponding $\overline{\omega}^i_{(1).jk}, \overline{\omega}^i_{(2).jk}, \overline{\omega}^i_{(3).jk}$.

Here and after, the symbol (p) means that the equal-index summation convention does not apply to the index p.

After anti-symmetrizing the equation (2.1) in the indices j and k, we get

$$\underset{\vee}{P^i_{jk}} = \xi^i_{jk} = \underset{\vee}{\overline{L}^i_{jk}} - \underset{\vee}{L^i_{jk}} = \overline{\tau}^i_{jk} - \tau^i_{jk}. \tag{2.7}$$

2.1. *Invariants in symmetric affine connection space*

With respect to the equation (2.5), we get

$$\overline{\widetilde{\mathcal{T}}}^i_{(1).jk} = \widetilde{\mathcal{T}}^i_{(1).jk}, \quad \overline{\widetilde{\mathcal{T}}}^i_{(2).jk} = \widetilde{\mathcal{T}}^i_{(2).jk}, \quad \overline{\widetilde{\mathcal{T}}}^i_{(3).jk} = \widetilde{\mathcal{T}}^i_{(3).jk},$$

for the geometrical objects

$$\widetilde{\mathcal{T}}^i_{(1).jk} = 0, \quad \widetilde{\mathcal{T}}^i_{(2).jk} = L^i_{\underline{jk}} - \omega^i_{jk}, \quad \widetilde{\mathcal{T}}^i_{(3).jk} = \tfrac{1}{2}\big(\overline{L}^i_{\underline{jk}} + L^i_{\underline{jk}}\big), \tag{2.8}$$

and the corresponding $\overline{\widetilde{\mathcal{T}}}^i_{(1).jk}, \overline{\widetilde{\mathcal{T}}}^i_{(2).jk}, \overline{\widetilde{\mathcal{T}}}^i_{(3).jk}$.

From the equation

$$\begin{aligned}\overline{\widetilde{\mathcal{T}}}^i_{(p).jm,n} - \overline{\widetilde{\mathcal{T}}}^i_{(p).jn,m} + \overline{\widetilde{\mathcal{T}}}^\alpha_{(p).jm}\overline{\widetilde{\mathcal{T}}}^i_{(p).\alpha n} - \overline{\widetilde{\mathcal{T}}}^\alpha_{(p).jn}\overline{\widetilde{\mathcal{T}}}^i_{(p).\alpha m}\\ = \widetilde{\mathcal{T}}^i_{(p).jm,n} - \widetilde{\mathcal{T}}^i_{(p).jn,m} + \widetilde{\mathcal{T}}^\alpha_{(p).jm}\widetilde{\mathcal{T}}^i_{(p).\alpha n} - \widetilde{\mathcal{T}}^\alpha_{(p).jn}\widetilde{\mathcal{T}}^i_{(p).\alpha m},\end{aligned}$$

$p = 1,2,3$, one obtains that the following equalities are satisfied

$$\overline{\widetilde{\mathcal{W}}}^i_{(p).jmn} = \widetilde{\mathcal{W}}^i_{(p).jmn},$$

for the geometrical objects

$$\widetilde{\mathcal{W}}^i_{(p).jmn} = R^i_{jmn} - \omega^i_{(p).jm|n} + \omega^i_{(p).jn|m} + \omega^\alpha_{(p).jm}\omega^i_{(p).\alpha n} - \omega^\alpha_{(p).jn}\omega^i_{(p).\alpha m}, \tag{2.9}$$

and the corresponding $\widetilde{\overline{\mathcal{W}}}^i_{(p).jmn}$.

Therefore, the following lemma holds.

Lemma 1. *Let $f:\mathbb{GA}_N\to\mathbb{G}\overline{\mathbb{A}}_N$ be a mapping of the associated space $\mathbb{A}_N$ characterized by the deformation tensor* (2.5).

The geometrical objects $\widetilde{\mathcal{T}}^i_{(p).jk}, p=1,2,3,$ given by the equation (2.8) *are invariants of the mapping f.*

The geometrical objects $\widetilde{\mathcal{W}}^i_{(p).jmn}, p=1,2,3,$ given by the equation (2.9) *are invariants of the mapping f.*

An invariant $\widetilde{\mathcal{T}}^i_{(p).jk}$, for $p=1,2,3$, of the mapping $f:\mathbb{GA}_N\to\mathbb{G}\overline{\mathbb{A}}_N$ is *the* (*basic*) *p-th class associated invariant of the Thomas type*. An invariant $\widetilde{\mathcal{W}}^i_{(p).jk}$, for $p=1,2,3$, of the mapping $f:\mathbb{GA}_N\to\mathbb{G}\overline{\mathbb{A}}_N$ is *the basic p-th class associated invariant of the Weyl type*.

Remark 1. The equalities

$$\overline{\omega}^\alpha_{(3).jm}\overline{\omega}^i_{(3).\alpha n}=\frac{1}{4}\overline{P}^\alpha_{\underline{jm}}\overline{P}^i_{\underline{\alpha n}}=\frac{1}{4}P^\alpha_{\underline{jm}}P^i_{\underline{\alpha n}}=\omega^\alpha_{(3).jm}\omega^i_{(3).\alpha n},$$

are satisfied. Thus, the invariant $\widetilde{\mathcal{W}}^i_{(3).jmn}$ reduces to

$$\widetilde{\mathcal{W}}^i_{(3).jmn}=R^i_{jmn}-\omega^i_{(3).jm|n}+\omega^i_{(3).jn|m}. \tag{2.10}$$

This invariant is important for researches about invariants of mappings characterized by deformation tensors $P^i_{\underline{jk}}$ which are not expressed in the form (2.3). The almost geodesic mappings of the first kind are an example of maps such that $\big($see [1, 2, 10]$\big)$.

Corollary 1. *Let in the equation* (2.6) *be $\omega^i_{(2).jk}=\delta^i_j\rho_k+\delta^i_k\rho_j+\sigma^i_{jk}$, for a* 1*-form ρ_j and a geometrical object σ^i_{jk} of the type* $(1,2)$ *symmetric in the indices j and k.*

The geometrical objects

$$\widetilde{T}^i_{(2).jk}=L^i_{\underline{jk}}-\sigma^i_{jk}-\frac{1}{N+1}\Big(\big(L^\alpha_{\underline{j\alpha}}-\sigma^\alpha_{\underline{j\alpha}}\big)\delta^i_k+\big(L^\alpha_{\underline{k\alpha}}-\sigma^\alpha_{\underline{k\alpha}}\big)\delta^i_j\Big), \tag{2.11}$$

$$\begin{aligned}\widetilde{W}^i_{(2).jmn}=&\,R^i_{jmn}-\sigma^i_{jm|n}+\sigma^i_{jn|m}+\sigma^\alpha_{jm}\sigma^i_{\alpha n}-\sigma^\alpha_{jn}\sigma^i_{\alpha m}\\&+\frac{1}{N+1}\delta^i_j\big(R_{[mn]}+\sigma^\alpha_{\alpha[m|n]}\big)+\frac{N}{N^2-1}\delta^i_{[m}R_{jn]}+\frac{1}{N^2-1}\delta^i_{[m}R_{n]j}\\&-\frac{1}{N^2-1}\delta^i_m\Big(\sigma^\alpha_{\alpha[j|n]}+(N+1)\big(\sigma^\alpha_{jn|\alpha}-\sigma^\alpha_{j\alpha|n}-\sigma^\alpha_{jn}\sigma^\beta_{\alpha\beta}+\sigma^\alpha_{j\beta}\sigma^\beta_{n\alpha}\big)\Big)\\&+\frac{1}{N^2-1}\delta^i_n\Big(\sigma^\alpha_{\alpha[j|m]}+(N+1)\big(\sigma^\alpha_{jm|\alpha}-\sigma^\alpha_{j\alpha|m}-\sigma^\alpha_{jm}\sigma^\beta_{\alpha\beta}+\sigma^\alpha_{j\beta}\sigma^\beta_{\alpha m}\big)\Big)\end{aligned} \tag{2.12}$$

are the invariants of the mapping f of the Thomas and the Weyl type respectively.

Proof. The geometrical objects

$$\widetilde{\mathcal{T}}^{i}_{(2).jk} = L^{i}_{\underline{jk}} - \delta^{i}_{k}\rho_{j} - \delta^{i}_{j}\rho_{k} - \sigma^{i}_{jk} \tag{2.13}$$

$$\begin{aligned}\widetilde{\mathcal{W}}^{i}_{(2).jmn} = {} & R^{i}_{jmn} - \delta^{i}_{j}\rho_{[m|n]} - \delta^{i}_{m}\big(\rho_{j|n} + \rho_{j}\rho_{n} + \sigma^{\alpha}_{jn}\rho_{\alpha}\big) \\ & + \delta^{i}_{n}\big(\rho_{j|m} + \rho_{j}\psi_{m} + \sigma^{\alpha}_{jm}\rho_{\alpha}\big) \\ & - \sigma^{i}_{jm|n} + \sigma^{i}_{jn|m} + \sigma^{\alpha}_{jm}\sigma^{i}_{\alpha n} - \sigma^{\alpha}_{jn}\sigma^{i}_{\alpha m},\end{aligned} \tag{2.14}$$

are the second class basic associated invariants of the mapping f of the Thomas and the Weyl type.

If we contract the identity $\widetilde{\overline{\mathcal{T}}}^{i}_{(2).jk} - \widetilde{\mathcal{T}}^{i}_{(2).jk} = 0$ over i and k, we get

$$(N+1)\big(\overline{\rho}_{j} - \rho_{j}\big) = \overline{L}^{\alpha}_{\underline{j\alpha}} - \overline{\sigma}^{\alpha}_{j\alpha} - L^{\alpha}_{\underline{j\alpha}} + \sigma^{\alpha}_{j\alpha}. \tag{2.15}$$

After substituting the equation (2.15) into the equality $\widetilde{\overline{\mathcal{T}}}^{i}_{(2).jk} - \widetilde{\mathcal{T}}^{i}_{(2).jk} = 0$, one confirms that the following equality holds

$$\widetilde{\overline{T}}^{i}_{(2).jk} = \widetilde{T}^{i}_{(2).jk},$$

for $\widetilde{T}^{i}_{(2).jk}$ from the equation (2.11) and the corresponding $\widetilde{\overline{T}}^{i}_{(2).jk}$.

Let be $\rho_{ij} = \rho_{j|n} + \rho_{j}\psi_{n} + \sigma^{\alpha}_{jn}\rho_{\alpha}$ and $\overline{\rho}_{ij} = \overline{\rho}_{j\|n} + \overline{\rho}_{j}\overline{\rho}_{n} + \overline{\sigma}^{\alpha}_{jn}\overline{\rho}_{\alpha}$, for the covariant derivative with respect to the affine connection of the torsion-free space $\overline{\mathbb{A}}_{N}$ denoted by $\|$.

With respect to this substitution, the equality $0 = \widetilde{\overline{\mathcal{W}}}^{i}_{(2).jmn} - \widetilde{\mathcal{W}}^{i}_{(2).jmn}$ transforms to

$$\begin{aligned}0 = {} & \overline{R}^{i}_{jmn} - R^{i}_{jmn} - \delta^{i}_{j}\big(\overline{\rho}_{[mn]} - \rho_{[mn]}\big) - \delta^{i}_{m}\big(\overline{\rho}_{jn} - \rho_{jn}\big) + \delta^{i}_{n}\big(\overline{\rho}_{jm} - \rho_{jm}\big) \\ & - \overline{\sigma}^{i}_{jm\|n} + \overline{\sigma}^{i}_{jn\|m} + \overline{\sigma}^{\alpha}_{jm}\overline{\sigma}^{i}_{\alpha n} - \overline{\sigma}^{\alpha}_{jn}\overline{\sigma}^{i}_{\alpha m} + \sigma^{i}_{jm|n} - \sigma^{i}_{jn|m} - \sigma^{\alpha}_{jm}\sigma^{i}_{\alpha n} + \sigma^{\alpha}_{jn}\sigma^{i}_{\alpha m}.\end{aligned} \tag{2.16}$$

After contracting the equation (2.16) in the indices i and j and using the relations $R^{\alpha}_{\alpha mn} = -\big(R_{mn} - R_{nm}\big) \equiv -R_{[mn]}$, for the alternation in the indices m and n denoted by the square brackets, we get

$$(N+1)\big(\overline{\rho}_{[mn]} - \rho_{[mn]}\big) = -\overline{R}_{[mn]} + R_{[mn]} - \overline{\sigma}^{\alpha}_{\alpha m\|n} + \overline{\sigma}^{\alpha}_{\alpha n\|m} + \sigma^{\alpha}_{\alpha m|n} - \sigma^{\alpha}_{\alpha n|m},$$

i.e.

$$\begin{aligned}0 = {} & \overline{R}^{i}_{jmn} + \frac{1}{N+1}\delta^{i}_{j}\big(\overline{R}_{[mn]} + \overline{\sigma}^{\alpha}_{\alpha[m\|n]}\big) + \delta^{i}_{n}\big(\overline{\rho}_{jm} - \rho_{jm}\big) - \delta^{i}_{m}\big(\overline{\rho}_{jn} - \rho_{jn}\big) \\ & - \overline{\sigma}^{i}_{jm\|n} + \overline{\sigma}^{i}_{jn\|m} + \overline{\sigma}^{\alpha}_{jm}\overline{\sigma}^{i}_{\alpha n} - \overline{\sigma}^{\alpha}_{jn}\overline{\sigma}^{i}_{\alpha m} - R^{i}_{jmn} - \frac{1}{N+1}\delta^{i}_{j}\big(R_{[mn]} + \sigma^{\alpha}_{\alpha[m|n]}\big) \\ & + \sigma^{i}_{jm|n} - \sigma^{i}_{jn|m} - \sigma^{\alpha}_{jm}\sigma^{i}_{\alpha n} + \sigma^{\alpha}_{jn}\sigma^{i}_{\alpha m}.\end{aligned} \tag{2.17}$$

If we contract the last equation over i and n, we obtain that the following equation is satisfied

$$\begin{aligned}(N-1)\left(\overline{\rho}_{jm}-\rho_{jm}\right) = &-\overline{R}_{jm}+\frac{1}{N+1}\left(\overline{R}_{[jm]}+\overline{\sigma}^{\alpha}_{\alpha[j\|m]}\right)\\ &+R_{jm}-\frac{1}{N+1}\left(R_{[jm]}+\sigma^{\alpha}_{\alpha[j|m]}\right)\\ &+\overline{\sigma}^{\alpha}_{jm\|\alpha}-\overline{\sigma}^{\alpha}_{j\alpha\|m}-\overline{\sigma}^{\alpha}_{jm}\overline{\sigma}^{\beta}_{\alpha\beta}+\overline{\sigma}^{\alpha}_{j\beta}\overline{\sigma}^{\beta}_{\alpha m}\\ &-\sigma^{\alpha}_{jm|\alpha}+\sigma^{\alpha}_{j\alpha|m}+\sigma^{\alpha}_{jm}\sigma^{\beta}_{\alpha\beta}-\sigma^{\alpha}_{j\beta}\sigma^{\beta}_{\alpha m}.\end{aligned} \tag{2.18}$$

Based on the equations (2.17, 2.18), one gets

$$\widetilde{\overline{W}}^{i}_{(2).jmn}=\widetilde{W}^{i}_{(2).jmn},$$

for the geometrical object $\widetilde{W}^{i}_{(2).jmn}$ from the equation (2.12) and the corresponding $\widetilde{\overline{W}}^{i}_{(2).jmn}$. □

The invariants (2.11, 2.12) are *the* (*second kind*) *derived associated invariants of the mapping f of the Thomas and Weyl type*, respectively.

Remark 2. If the deformation tensor P^{i}_{jk} of a studied mapping is expressed in the form (2.1) all invariants of the Weyl type reduce to the corresponding invariants of the second class. If the deformation tensor P^{i}_{jk} of a mapping satisfies a differential equation, it may be obtained just the Weyl type invariants of the third class. The invariants of the first and the second class of the Thomas type produce the multiplied families of invariants for mappings. Hence, it is enough to obtain invariants $\mathcal{W}^{i}_{(p).jmn}$ for one p. All other invariants $\widetilde{\mathcal{W}}$ of the Weyl type reduce to the obtained one.

2.2. *Invariants in non-symmetric affine connection space*

We will generalize the invariants (2.8, 2.9) in this part of the paper.

With respect to the equation (2.7), one gets

$$\widehat{\overline{\mathcal{T}}}^{i}_{jk}=\hat{\mathcal{T}}^{i}_{jk},$$

for the geometrical object

$$\hat{\mathcal{T}}^{i}_{jk}=L^{i}_{\underset{\vee}{jk}}-\tau^{i}_{jk}, \tag{2.19}$$

and the corresponding $\widehat{\overline{\mathcal{T}}}^{i}_{jk}$. Based on the equations (2.8, 2.19), one obtains that it is satisfied the equalities

$$\overline{\mathcal{T}}^{i}_{(1).jk}=\mathcal{T}^{i}_{(1).jk},\quad \overline{\mathcal{T}}^{i}_{(2).jk}=\mathcal{T}^{i}_{(2).jk},\quad \overline{\mathcal{T}}^{i}_{(3).jk}=\mathcal{T}^{i}_{(3).jk},$$

for the geometrical objects

$$\mathcal{T}^{i}_{(1).jk}=L^{i}_{\underset{\vee}{jk}}-\tau^{i}_{jk},\quad \mathcal{T}^{i}_{(2).jk}=L^{i}_{jk}-\omega^{i}_{jk}-\tau^{i}_{jk},\quad \mathcal{T}^{i}_{(3).jk}=L^{i}_{jk}+\frac{1}{2}P^{i}_{\underline{jk}}-\tau^{i}_{jk}, \tag{2.20}$$

and the corresponding $\overline{\mathcal{T}}^i_{(1).jk}, \overline{\mathcal{T}}^i_{(2).jk}, \overline{\mathcal{T}}^i_{(3).jk}$.

Lemma 2. *Let $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be a mapping of the non-symmetric affine connection space $\mathbb{GA}_N$. The geometrical objects $\hat{\mathcal{T}}^i_{jk}$ and $\mathcal{T}^i_{(p).jk}$, respectively given by the equations (2.19, 2.20), are invariants of the mapping f.*

Corollary 2. *The invariants (2.8, 2.19, 2.20) satisfy the equation*

$$\mathcal{T}^i_{(p).jk} = \widetilde{\mathcal{T}}^i_{(p).jk} + \hat{\mathcal{T}}^i_{jk}, \tag{2.21}$$

for $p = 1,2,3$.

The invariant $\mathcal{T}^i_{(p).jk}$, for $p = 1,2,3$, is *the p-th class general invariant of the Thomas type*. The invariant $\hat{\mathcal{T}}^i_{jk}$ is *the anti-symmetric invariant of the Thomas type*.

From the difference $\hat{\overline{\mathcal{T}}}^i_{jm\|n} - \hat{\mathcal{T}}^i_{jm|n}$ and the equality $\hat{\overline{\mathcal{T}}}^\alpha_{jm}\hat{\overline{\mathcal{T}}}^i_{\alpha n} - \hat{\mathcal{T}}^\alpha_{jm}\hat{\mathcal{T}}^i_{\alpha n} = 0$, we obtain the following transformation rules

$$\begin{aligned}
\overline{L}^i_{\underset{\vee}{jm}\|n} - L^i_{\underset{\vee}{jm}|n} = {} & \overline{\tau}^i_{jm\|n} + \overline{\omega}^i_{(p_1).\alpha n}\big(\overline{L}^\alpha_{\underset{\vee}{jm}} - \overline{\tau}^\alpha_{jm}\big) \\
& - \overline{\omega}^\alpha_{(p_2).jn}\big(\overline{L}^i_{\underset{\vee}{\alpha m}} - \overline{\tau}^i_{\alpha m}\big) - \overline{\omega}^\alpha_{(p_3).mn}\big(\overline{L}^i_{\underset{\vee}{j\alpha}} - \overline{\tau}^i_{j\alpha}\big) \\
& - \tau^i_{jm|n} - \omega^i_{(p_1).\alpha n}\big(L^\alpha_{\underset{\vee}{jm}} - \tau^\alpha_{jm}\big) \\
& + \omega^\alpha_{(p_2).jn}\big(L^i_{\underset{\vee}{\alpha m}} - \tau^i_{\alpha m}\big) + \omega^\alpha_{(p_3).mn}\big(L^i_{\underset{\vee}{j\alpha}} - \tau^i_{j\alpha}\big),
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
\overline{L}^\alpha_{\underset{\vee}{jm}}\overline{L}^i_{\underset{\vee}{\alpha n}} - L^\alpha_{\underset{\vee}{jm}}L^i_{\underset{\vee}{\alpha n}} = {} & \overline{L}^\alpha_{\underset{\vee}{jm}}\overline{\tau}^i_{\alpha n} + \overline{L}^i_{\underset{\vee}{\alpha n}}\overline{\tau}^\alpha_{jm} - \overline{\tau}^\alpha_{jm}\overline{\tau}^i_{\alpha n} \\
& - L^\alpha_{\underset{\vee}{jm}}\tau^i_{\alpha n} - L^i_{\underset{\vee}{\alpha n}}\tau^\alpha_{jm} + \tau^\alpha_{jm}\tau^i_{\alpha n},
\end{aligned} \tag{2.23}$$

for $p_1, p_2, p_3 = 1,2$.

Remark 3. The following equalities are satisfied

$$\begin{aligned}
\overline{\omega}^i_{(3).jk}\hat{\overline{\mathcal{T}}}^l_{mn} - \omega^i_{(3).jk}\hat{\mathcal{T}}^l_{mn} &= -\frac{1}{2}\overline{P}^i_{\underline{jk}}\hat{\overline{\mathcal{T}}}^l_{mn} + \frac{1}{2}P^i_{\underline{jk}}\hat{\mathcal{T}}^l_{mn} \\
&= \overline{\omega}^i_{(1).jk}\hat{\overline{\mathcal{T}}}^l_{mn} - \omega^i_{(1).jk}\hat{\mathcal{T}}^l_{mn}.
\end{aligned}$$

Hence, the geometrical objects $\omega^i_{(1).jk}, \omega^i_{(2).jk}$, are enough to express all transformation rules (2.22) with respect to a mapping $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$.

Let us currently express the invariants (2.9) in the form

$$\widetilde{\mathcal{W}}^i_{(p).jmn} = R^i_{jmn} - \widetilde{\mathcal{D}}^i_{(p).jmn},$$

for the corresponding geometrical objects $\widetilde{\mathcal{D}}^i_{(p).jmn}$. From this expression, and the equalities $\widetilde{\overline{\mathcal{W}}}^i_{(p).jmn} - \widetilde{\mathcal{W}}^i_{(p).jmn} = 0, p = 1,2,3$, we get

$$\overline{R}^i_{jmn} - R^i_{jmn} = \widetilde{\overline{\mathcal{D}}}^i_{(p).jmn} - \widetilde{\mathcal{D}}^i_{(p).jmn}. \tag{2.24}$$

With respect to the equation (1.5), we obtain that the following equation holds

$$\begin{aligned}\overline{K}^i_{jmn} - K^i_{jmn} = \overline{R}^i_{jmn} - R^i_{jmn} &+ u\big(\overline{L}^i_{jm\underset{\vee}{\|}n} - L^i_{jm\underset{\vee}{|}n}\big)\\ &+ u'\big(\overline{L}^i_{jn\underset{\vee}{\|}m} - L^i_{jn\underset{\vee}{|}m}\big) + v\big(\overline{L}^\alpha_{jm\underset{\vee}{|}}\overline{L}^i_{\alpha n\underset{\vee}{|}} - L^\alpha_{jm\underset{\vee}{|}}L^i_{\alpha n\underset{\vee}{|}}\big)\\ &+ v'\big(\overline{L}^\alpha_{jn\underset{\vee}{|}}\overline{L}^i_{\alpha m\underset{\vee}{|}} - L^\alpha_{jn\underset{\vee}{|}}L^i_{\alpha m\underset{\vee}{|}}\big) + w\big(\overline{L}^\alpha_{mn\underset{\vee}{|}}\overline{L}^i_{\alpha j\underset{\vee}{|}} - L^\alpha_{mn\underset{\vee}{|}}L^i_{\alpha j\underset{\vee}{|}}\big).\end{aligned} \tag{2.25}$$

After substituting the results (2.22, 2.23, 2.24) into the equation (2.25), we get

$$\overline{\mathcal{W}}^i_{(p).(p^1).(p^2).jmn} = \mathcal{W}^i_{(p).(p^1).(p^2).jmn},$$

for $p = 1,2,3, p^1_1, \ldots, p^2_3 = 1,2$, the family geometrical objects

$$\begin{aligned}\mathcal{W}^i_{(p).(p^1).(p^2).jmn} = K^i_{jmn} &- \omega^i_{(p).jm|n} + \omega^i_{(p).jn|m} + \omega^\alpha_{(p).jm}\omega^i_{(p).\alpha n} - \omega^\alpha_{(p).jn}\omega^i_{(p).\alpha m}\\ &- u\Big(\tau^i_{jm|n} + \omega^i_{(p^1_1).\alpha n}\big(L^\alpha_{jm\underset{\vee}{|}} - \tau^\alpha_{jm}\big) - \omega^\alpha_{(p^1_2).jn}\big(L^i_{\alpha m\underset{\vee}{|}} - \tau^i_{\alpha m}\big)\Big)\\ &- u'\Big(\tau^i_{jn|m} + \omega^i_{(p^2_1).\alpha m}\big(L^\alpha_{jn\underset{\vee}{|}} - \tau^\alpha_{jn}\big) - \omega^\alpha_{(p^2_2).jm}\big(L^i_{\alpha n\underset{\vee}{|}} - \tau^i_{\alpha n}\big)\Big)\\ &+ \big(u\omega^\alpha_{(p^1_3).mn} + u'\omega^\alpha_{(p^2_3).mn}\big)\big(L^i_{j\alpha\underset{\vee}{|}} - \tau^i_{j\alpha}\big),\end{aligned} \tag{2.26}$$

the real coefficients u, u', v, v', w and the corresponding $\widetilde{\overline{\mathcal{W}}}^i_{(p).(p^1).(p^2).jmn}$.

The following theorem holds.

Theorem 1. *Let $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be a mapping of the non-symmetric affine connection space $\mathbb{GA}_N$. The set $\mathcal{W}^i_{(p).(p^1).(p^2).jmn}, p = 1,2,3, p^k = (p^k_1, p^k_2, p^k_3)$, $k = 1,2, p^k_r = 1,2$, of families of geometrical objects given in the equation (2.26) is the set of families of invariants of the Weyl type for the mapping f.*

Corollary 3. *The invariants* (2.9, 2.26) *of the mapping f satisfy the equation*

$$\begin{aligned}\mathcal{W}^i_{(p).(p^1).(p^2).jmn} &= \widetilde{\mathcal{W}}^i_{(p).jmn}\\ &+u\Big(L^i_{\underset{\vee}{jm|n}}-\tau^i_{jm|n}-\omega^i_{(p^1_1).\alpha n}\big(L^\alpha_{\underset{\vee}{jm}}-\tau^\alpha_{jm}\big)+\omega^\alpha_{(p^1_2).jn}\big(L^i_{\underset{\vee}{\alpha m}}-\tau^i_{\alpha m}\big)\Big)\\ &+u'\Big(L^i_{\underset{\vee}{jn|m}}-\tau^i_{jn|m}-\omega^i_{(p^2_1).\alpha m}\big(L^\alpha_{\underset{\vee}{jn}}-\tau^\alpha_{jn}\big)-\omega^\alpha_{(p^2_2).jm}\big(L^i_{\underset{\vee}{\alpha n}}-\tau^i_{\alpha n}\big)\Big)\\ &+\big(u\omega^\alpha_{(p^1_3).mn}+u'\omega^\alpha_{(p^2_3).mn}\big)\big(L^i_{\underset{\vee}{j\alpha}}-\tau^i_{j\alpha}\big)+v\big(L^\alpha_{\underset{\vee}{jm}}-\tau^\alpha_{jm}\big)\big(L^i_{\underset{\vee}{\alpha n}}-\tau^i_{\alpha n}\big)\\ &+v'\big(L^\alpha_{\underset{\vee}{jn}}-\tau^\alpha_{jn}\big)\big(L^i_{\underset{\vee}{\alpha m}}-\tau^i_{\alpha m}\big)+w\big(L^\alpha_{\underset{\vee}{mn}}-\tau^\alpha_{mn}\big)\big(L^i_{\underset{\vee}{\alpha j}}-\tau^i_{\alpha j}\big),\end{aligned} \tag{2.27}$$

for $p=1,2,3,p^1_1,\dots,p^2_3=1,2$.

Corollary 4. *Let $f:\mathbb{GA}_N\to\mathbb{G}\overline{\mathbb{A}}_N$ be an equitorsion mapping between non-symmetric affine connection spaces $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$. The invariants* (2.20) *of this mapping reduce to the corresponding invariants* (2.8). *The invariant* (2.19) *coincides with the anti-symmetric part $L^i_{\underset{\vee}{jk}}$ of the affine connection coefficient L^i_{jk}.*

The set (2.26) *of families of invariants of the mapping f reduces to*

$$\begin{aligned}\mathcal{W}^i_{(p).(p^1).(p^2).jmn} &= K^i_{jmn}-\omega^i_{(p).jm|n}+\omega^i_{(p).jn|m}+\omega^\alpha_{(p).jm}\omega^i_{(p).\alpha n}-\omega^\alpha_{(p).jn}\omega^i_{(p).\alpha m}\\ &-u\Big(\omega^i_{(p^1_1).\alpha n}L^\alpha_{\underset{\vee}{jm}}-\omega^\alpha_{(p^1_2).jn}L^i_{\underset{\vee}{\alpha m}}\Big)-u'\Big(\omega^i_{(p^2_1).\alpha m}L^\alpha_{\underset{\vee}{jn}}-\omega^\alpha_{(p^2_2).jm}L^i_{\underset{\vee}{\alpha n}}\Big)\\ &+\big(u\omega^\alpha_{(p^1_3).mn}+u'\omega^\alpha_{(p^2_3).mn}\big)L^i_{\underset{\vee}{j\alpha}},\end{aligned} \tag{2.28}$$

for $p=1,2,3,p^1_1,\dots,p^2_3=1,2$.

The equation (2.27) *reduces to*

$$\begin{aligned}\mathcal{W}^i_{(p).(p^1).(p^2).jmn} &= \widetilde{\mathcal{W}}^i_{(p).jmn}+\big(u\omega^\alpha_{(p^1_3).mn}+u'\omega^\alpha_{(p^2_3).mn}\big)L^i_{\underset{\vee}{j\alpha}}\\ &+u\Big(L^i_{\underset{\vee}{jm|n}}-\omega^i_{(p^1_1).\alpha n}L^\alpha_{\underset{\vee}{jm}}+\omega^\alpha_{(p^1_2).jn}L^i_{\underset{\vee}{\alpha m}}\Big)\\ &+u'\Big(L^i_{\underset{\vee}{jn|m}}-\omega^i_{(p^2_1).\alpha m}L^\alpha_{\underset{\vee}{jn}}-\omega^\alpha_{(p^2_2).jm}L^i_{\underset{\vee}{\alpha n}}\Big)\\ &+vL^\alpha_{\underset{\vee}{jm}}L^i_{\underset{\vee}{\alpha n}}+v'L^\alpha_{\underset{\vee}{jn}}L^i_{\underset{\vee}{\alpha m}}+wL^\alpha_{\underset{\vee}{mn}}L^i_{\underset{\vee}{\alpha j}},\end{aligned} \tag{2.29}$$

for $p=1,2,3,p^1_1,\dots,p^2_3=1,2$.

The sets of invariants $\mathcal{W}^i_{(p).(p^1).(p^2).jmn}$, given by the equations (2.26, 2.28), are *the sets of the $(p.p^1.p^2)$-th class invariants of the Weyl type*.

The families of invariants in the set (2.26), and in the set (2.28) as well, are equivalent invariants of the mapping $f:\mathbb{GA}_N\to\mathbb{G}\overline{\mathbb{A}}_N$. Because these families are different in general, we are aimed to find how many of these families are linearly independent.

The set $\mathcal{W}^i_{(p).(p^1).(p^2).jmn}$ of invariants may be expressed as

$$\begin{aligned}
&\mathcal{W}^i_{(p).(p^1).(p^2).jmn}\\
&= \widetilde{\mathcal{W}}^i_{(p).jmn} + u\hat{\mathcal{T}}^i_{jm|n} + u'\hat{\mathcal{T}}^i_{jn|m} + v\hat{\mathcal{T}}^\alpha_{jm}\hat{\mathcal{T}}^i_{\alpha n} + v'\hat{\mathcal{T}}^\alpha_{jn}\hat{\mathcal{T}}^i_{\alpha m} + w\hat{\mathcal{T}}^\alpha_{mn}\hat{\mathcal{T}}^i_{\alpha j}\\
&\quad - c^1_p u L^i_{\underline{\alpha n}}\hat{\mathcal{T}}^\alpha_{jm} - c^2_p u\omega^i_{(p).\alpha n}\hat{\mathcal{T}}^\alpha_{jm} + c^3_p u L^\alpha_{\underline{jn}}\hat{\mathcal{T}}^i_{\alpha m} + c^4_p u\omega^\alpha_{(p).jn}\hat{\mathcal{T}}^i_{\alpha m}\\
&\quad - c^7_p u' L^i_{\underline{\alpha m}}\hat{\mathcal{T}}^\alpha_{jn} - c^8_p u'\omega^i_{(p).\alpha m}\hat{\mathcal{T}}^\alpha_{jn} + c^9_p u' L^\alpha_{\underline{jm}}\hat{\mathcal{T}}^i_{\alpha n} + c^{10}_p u'\omega^\alpha_{(p).jm}\hat{\mathcal{T}}^i_{\alpha n}\\
&\quad + \left(c^5_p u + c^{11}_p u'\right) L^\alpha_{\underline{mn}}\hat{\mathcal{T}}^i_{j\alpha} + \left(c^6_p u + c^{12}_p u'\right)\omega_{(p).mn}\hat{\mathcal{T}}^i_{j\alpha},
\end{aligned} \tag{2.30}$$

for the corresponding coefficients $c^k_p \in \{0,1\}, k = 1,\dots,12$.

There are 64 families of invariants in the set (2.30) characterized by the corresponding 11-tuples

$$\begin{aligned}
c_k = \big(&1, -c^1_{pk}u, -c^2_{pk}u, c^3_{pk}u, c^4_{pk}u, -c^7_{pk}u', -c^8_{pk}u',\\
&c^9_{pk}u', c^{10}_{pk}u', c^5_{pk}u + c^{11}_{pk}u', c^6_{pk}u + c^{12}_{pk}u'\big),
\end{aligned}$$

$k = 1,\dots,64$, for $c^r_{pk} = c^r_p$ in the k-th of families of invariants in the set $\widetilde{\mathcal{W}}^i_{(p).(p^1).(p^2).jmn}$.

The rank of the matrix $\begin{bmatrix} c_1 \\ \vdots \\ c_{64} \end{bmatrix}$ of the type 64×13 is 6.

The following theorem holds.

Theorem 2. *The set* (2.26) *of invariants of a mapping* $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$, *generates the* 6*-dimensional vector space.*

3. Applications and examples

From the above obtained results, we will search invariants of equitorsion geodesic mappings defined on a generalized Riemannian space $\mathbb{GR}_N$. Furthermore, we will obtain an associated basic invariant of Weyl type of an almost geodesic mapping [1, 2, 10] defined on a Riemannian space $\mathbb{R}_N$. These invariants will be applied in the examples after theoretical researches.

Equitorsion geodesic mappings. Let $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ be an equitorsion geodesic mapping. This mapping is characterized by the following equation

$$P^i_{jk} = \psi_j\delta^i_k + \psi_k\delta^i_j, \tag{3.1}$$

for the deformation tensor $P^i_{jk} = \overline{L}^i_{jk} - L^i_{jk}$ and a 1-form ψ_j.

After symmetrizing this equation in the indices j and k, we get

$$P^i_{\underline{jk}} = \psi_j\delta^i_k + \psi_k\delta^i_j. \tag{3.2}$$

If we contract the last equation over i and k, we will obtain the following expression of the 1-form ψ_j

$$\psi_j = \frac{1}{N+1}\left(\overline{\Gamma}^{\alpha}_{\underline{j\alpha}} - \Gamma^{\alpha}_{\underline{j\alpha}}\right).$$

After substituting this expression into the equation (3.2) and recalling the equation (2.3), we get

$$\omega^i_{(2).jk} = \frac{1}{N+1}\delta^i_j\Gamma^{\alpha}_{\underline{k\alpha}} + \frac{1}{N+1}\delta^i_k\Gamma^{\alpha}_{\underline{j\alpha}}. \tag{3.3}$$

If we substitute this expression into the equations (2.8, 2.9), we obtain that the geometrical objects

$$\widetilde{\mathcal{T}}^i_{(2).jk} = \Gamma^i_{\underline{jk}} - \frac{1}{N+1}\left(\delta^i_j\Gamma^{\alpha}_{\underline{k\alpha}} + \delta^i_k\Gamma^{\alpha}_{\underline{j\alpha}}\right), \tag{3.4}$$

$$\begin{aligned}\widetilde{\mathcal{W}}^i_{(2).jmn} = R^i_{jmn} &- \frac{1}{(N+1)^2}\delta^i_m\left((N+1)\Gamma^{\alpha}_{\underline{j\alpha}|n} + \Gamma^{\alpha}_{\underline{j\alpha}}\Gamma^{\alpha}_{\underline{n\alpha}}\right)\\ &+ \frac{1}{(N+1)^2}\delta^i_n\left((N+1)\Gamma^{\alpha}_{\underline{j\alpha}|m} + \Gamma^{\alpha}_{\underline{j\alpha}}\Gamma^{\alpha}_{\underline{m\alpha}}\right),\end{aligned} \tag{3.5}$$

are the basic associated invariants of the Thomas and the Weyl type of the mapping f.

Example 1. Let $\mathbb{GR}_3$ be a generalized Riemannian space equipped with the non-symmetric metric

$$g_{ij} = \begin{bmatrix} (x^1)^2 & x^1 & x^2 \\ -x^1 & (x^2)^2 & x^3 \\ -x^2 & -x^3 & (x^3)^2 \end{bmatrix}. \tag{3.6}$$

The symmetric and anti-symmetric part of this metric are

$$g_{\underline{ij}} = \begin{bmatrix} (x^1)^2 & 0 & 0 \\ 0 & (x^2)^2 & 0 \\ 0 & 0 & (x^3)^2 \end{bmatrix} \quad \text{and} \quad g_{\underset{\vee}{ij}} = \begin{bmatrix} 0 & x^1 & x^2 \\ -x^1 & 0 & x^3 \\ -x^2 & -x^3 & 0 \end{bmatrix}. \tag{3.7}$$

The contravariant metric tensor $\left[g^{\underline{ij}}\right] = \left[g_{\underline{ij}}\right]^{-1}$ of the space $\mathbb{GR}_3$ is

$$g^{\underline{ij}} = \begin{bmatrix} (x^1)^{-2} & 0 & 0 \\ 0 & (x^2)^{-2} & 0 \\ 0 & 0 & (x^3)^{-2} \end{bmatrix}. \tag{3.8}$$

To obtain the invariants (2.29), we need to search the corresponding invariant (2.9), the geometrical objects $\omega^i_{(1).jk}$ and $\omega^i_{(2).jk}$ and the anti-symmetric part $\Gamma^i_{\underset{\vee}{jk}}$ of the generalized Christoffel symbol Γ^i_{jk} given by the equation (1.7).

The affine connection coefficients of the associated space $\mathbb{R}_3$ are the Chrisfoffel symbols

$$\Gamma^i_{\underline{jk}} = \omega^i_{(1).jk} = \begin{cases} \left(x^i\right)^{-1}, & i=j=k \\ 0, & \text{otherwise.} \end{cases} \tag{3.9}$$

Furthermore, based on the equation (3.3) we get

$$\omega^i_{(2).jk} = \frac{1}{N+1}\left(\delta^i_j\left(x^k\right)^{-1} + \delta^i_k\left(x^j\right)^{-1}\right). \tag{3.10}$$

For this reason, the curvature tensor of the associated space $\mathbb{R}_3$ is $R^i_{jmn} = 0$. Moreover, the associated invariant of Weyl type (3.5) is

$$\begin{aligned} \widetilde{\mathcal{W}}^i_{(2).jmn} = &-\frac{1}{(N+1)^2}\delta^i_m\left((N+1)\left(\left(x^j\right)^{-1}\right)_{|n} + \left(x^j\right)^{-1}\left(x^n\right)^{-1}\right) \\ &+ \frac{1}{(N+1)^2}\delta^i_n\left((N+1)\left(\left(x^j\right)^{-1}\right)_{|m} + \left(x^j\right)^{-1}\left(x^m\right)^{-1}\right). \end{aligned} \tag{3.11}$$

Because $\Gamma^i_{\underset{\vee}{jk}} = \frac{1}{2}g^{i\underline{\alpha}}\left(g_{\underset{\vee}{j\alpha},k} - g_{\underset{\vee}{jk},\alpha} + g_{\underset{\vee}{\alpha k},j}\right)$, we get

$$\Gamma^1_{\underset{\vee}{23}} = \frac{1}{2}\left(x^1\right)^{-2} = -\Gamma^1_{\underset{\vee}{32}}, \quad \Gamma^2_{\underset{\vee}{31}} = \frac{1}{2}\left(x^2\right)^{-2} = -\Gamma^2_{\underset{\vee}{13}}, \quad \Gamma^3_{\underset{\vee}{12}} = \frac{1}{2}\left(x^3\right)^{-2} = -\Gamma^3_{\underset{\vee}{21}}, \tag{3.12}$$

and $\Gamma^i_{\underset{\vee}{jk}} = 0$ in all other cases.

After substituting the expression (3.9) into the equation (2.29) and with respect to the equations (3.10 - 3.12), for $p^1_1, \ldots, p^2_3 \in \{1,2\}$, we get that the set of invariants of Weyl type of the mapping f is

$$\begin{aligned} \widetilde{\mathcal{W}}^i_{(2).(p^1).(p^2).jmn} = &-\frac{1}{(N+1)^2}\delta^i_m\left((N+1)\left(\left(x^j\right)^{-1}\right)_{|n} + \left(x^j\right)^{-1}\left(x^n\right)^{-1}\right) \\ &+ \frac{1}{(N+1)^2}\delta^i_n\left((N+1)\left(\left(x^j\right)^{-1}\right)_{|m} + \left(x^j\right)^{-1}\left(x^m\right)^{-1}\right) \\ &+ \left(u\omega^\alpha_{(p^1_3).mn} + u'\omega^\alpha_{(p^2_3).mn}\right) \\ &+ u\left(\Gamma^i_{\underset{\vee}{jm}|n} - \omega^i_{(p^1_1).\alpha n}\Gamma^\alpha_{\underset{\vee}{jm}} + \omega^\alpha_{(p^1_2).jn}\Gamma^i_{\underset{\vee}{\alpha m}}\right) \\ &+ u'\left(\Gamma^i_{\underset{\vee}{jn}|m} - \omega^i_{(p^2_1).\alpha m}\Gamma^\alpha_{\underset{\vee}{jn}} - \omega^\alpha_{(p^2_2).jm}\Gamma^i_{\underset{\vee}{\alpha n}}\right) \\ &+ v\Gamma^\alpha_{\underset{\vee}{jm}}\Gamma^i_{\underset{\vee}{\alpha n}} + v'\Gamma^\alpha_{\underset{\vee}{jn}}\Gamma^i_{\underset{\vee}{\alpha m}} + w\Gamma^\alpha_{\underset{\vee}{mn}}\Gamma^i_{\underset{\vee}{\alpha j}}\Gamma^i_{\underset{\vee}{j\alpha}}. \end{aligned}$$

Almost geodesic mappings. N. S. Sinyukov [16], J. Mikeš [1, 2, 8, 10] and many other authors have developed the concept of geodesics. We will search an associated basic invariant of the mapping $f : \mathbb{R}_N \to \overline{\mathbb{R}}_N$ characterized by the equation

$$P^i_{\underline{nm}|j} + P^i_{\underline{jm}|n} + P^\alpha_{\underline{jm}}P^i_{\underline{\alpha n}} + P^\alpha_{\underline{nm}}P^i_{\underline{\alpha j}} = \delta^i_j a_{mn} + \delta^i_n a_{mj}, \tag{3.13}$$

for the tensor a_{ij} of the type $(0,2)$ symmetric by the indices i and j. This mapping is called *the almost geodesic mapping* of the type $\tilde{\pi}_1$.

After replacing the indices $m \leftrightarrow n$, we get

$$P^i_{\underline{jn}|m} + P^i_{\underline{mn}|j} + P^\alpha_{\underline{jn}} P^i_{\underline{\alpha m}} + P^\alpha_{\underline{mn}} P^i_{\underline{\alpha j}} = \delta^i_j a_{nm} + \delta^i_m a_{nj}. \tag{3.14}$$

If we subtract the equations (3.13) and (3.14), we will obtain that the following equation is satisfied

$$P^i_{\underline{jm}|n} - P^i_{\underline{jn}|m} = -P^\alpha_{\underline{jm}} P^i_{\underline{\alpha n}} + P^\alpha_{\underline{jn}} P^i_{\underline{\alpha m}} + \delta^i_n a_{jm} - \delta^i_m a_{jn}. \tag{3.15}$$

Based on the equations (2.10, 3.15) and the invariance $\overline{P}^\alpha_{\underline{jm}} \overline{P}^i_{\underline{\alpha n}} = P^\alpha_{\underline{jm}} P^i_{\underline{\alpha n}}$, we obtain that the geometrical object

$$\widetilde{\mathcal{W}}^i_{(3).jmn} = R^i_{jmn} + \frac{1}{2}\delta^i_m a_{jn} - \frac{1}{2}\delta^i_n a_{jm}, \tag{3.16}$$

is the associated basic invariant of the Weyl type of the mapping f.

After contracting the equality $\widetilde{\overline{\mathcal{W}}}^i_{(3).jmn} - \widetilde{\mathcal{W}}^i_{(3).jmn} = 0$ by the indices i and n, we obtain that the curvature tensor R^i_{jmn} is the derived invariant of this mapping.

Example 2. Let $\mathbb{R}_3$ be a Riemannian space equipped with the symmetric metric g_{ij} given in the equation (3.7). Let also $f : \mathbb{R}_3 \to \overline{\mathbb{R}}_3$ be an almost geodesic mapping of the type $\tilde{\pi}_1$.

As we obtained in the previous example, the curvature tensor of the space $\mathbb{R}_3$ is $R^i_{jmn} = 0$. For this reason, the geometrical objects

$$\widetilde{\mathcal{W}}^i_{(3).jmn} = \delta^i_m a_{jn} - \delta^i_n a_{jm} \quad \text{and} \quad \widetilde{W}^i_{(3).jmn} = 0, \tag{3.17}$$

are the associated basic and the associated derived invariant of the Weyl type of the mapping f.

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FINITE APPROXIMATE CONTROLLABILITY OF HILFER FRACTIONAL SEMILINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate the finite approximate controllability of fractional semilinear differential equations involving the Hilfer derivative. We show that if the linear part is approximate controllable, then under suitable conditions the semilinear system is finite approximate controllable.

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1. Introduction

An important qualitative behaviour of a dynamic system is controllability. The problem of exact controllability or controllability is to show the existence of a control function, which steers the solution of the system from its initial state to the final state, approximate controllability means that a system can be steered to an arbitrary small neighbourhood of the final state, and finite approximate controllability means that this system is approximate controllability and the chosen control function satisfies simultaneously a finite number of constraints.

Fractional differential equations and inclusions are important because of their applications in various fields, such as physics, mechanics and engineering [3, 11, 14]. For existence results for fractional differential equations and inclusions involving the Riemann-Liouville or the Caputo derivative we refer the reader to [4,12,23,27,30,32] and the references therein. Note that Hilfer [11] proposed a generalized Riemann-Liouville fractional derivative, for short, the Hilfer fractional derivative, which includes the Riemann-Liouville and the Caputo fractional derivatives. Furati et al. [1] established the existence and uniqueness of a global solution in the space of weighted continuous functions for a fractional differential equation involving the Hilfer derivative. Abbas et al. [20] adopted the weakly Picard operators theory to give Ulam

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stability results for Hilfer type fractional differential inclusions. Wang et al. [26] considered a Hilfer-type fractional differential switched inclusion with noninstantaneous impulsive and nonlocal conditions. Gu and Trujillo [9] obtained some sufficient conditions to ensure the existence of mild solutions of evolution equation with the Hilfer fractional derivative. Yang and Wang [28, 29] investigated existence of mild solutions of Hilfer evolution equations and the approximate controllability of Hilfer fractional differential inclusions with nonlocal conditions and Du et al. [7] established the approximate controllability of impulsive Hilfer fractional differential inclusions. Kamocki [13] derived an equivalent definition of the Hilfer derivative and showed that such a derivative is very useful in practical applications. Wang et al. [24, 25] studied the controllability of Caputo fractional noninstantaneous impulsive differential inclusions without compactness in reflexive Banach spaces.

For works on controllability, approximate controllability, and finite approximate controllability of differential and inclusions we refer the reader to [2, 5, 6, 8, 10, 15–19, 21, 22, 31]. It is worth mentioning that Menezes et al. [19] investigated the finite approximate controllability for the semilinear heat equation, Mahmudov [18] studied the finite approximate controllability of a semilinear evolution equation in a Hilbert space under the assumption that the linear part of the system is approximately controllability and Mahmudov [16] established, under the assumption that the linear part of the system is approximately controllability, the finite approximate controllability of a nonlocal Sobolev-type evolution equation involving the Caputo fractional derivative.

In this paper, we study the finite approximate controllability of the following Hilfer fractional evolution equation:

$$\begin{cases} D_{0^+}^{\alpha,\beta}x(t) = Ax(t) + f(t,x(t)) + Bu(t), \ a.e.\ t \in (0,b], \\ I_{0^+}^{1-\gamma}x(0^+) = x_0, \end{cases} \tag{1.1}$$

where $\frac{1}{2} < \alpha < 1$, $0 \le \beta \le 1$, $\gamma = \alpha + \beta - \alpha\beta$, $J = [0,b]$, $b > 0$, $D_{0^+}^{\alpha,\beta}x(t)$ is the left-sided Hilfer derivative (see [11]) with lower limit at 0 of order α and type β, E is a real Hilbert space, A is the infinitesimal generator of a strongly continuous semigroup $T(t), t > 0$ and $I_{0^+}^{1-\gamma}$ is the left-sided Riemann-Liouville integral of order $1-\gamma$ with lower limit at 0. Moreover, $f : J \times E \to E$ is a function, the control function u is given in $U = L^p(J,X)$, $p > \frac{1}{\alpha}$ a Banach space of admissible control functions, with X being a real Hilbert space, B is a bounded linear operator from X into E and x_0 is a fixed point of E.

To the best of our knowledge, up to now, no work was reported on finite approximate controllability of fractional differential equation involving the Hilfer derivative, and this is the main aim of this paper. The paper is organized as follows. In Section 2, we collect some background material about multifunctions and fractional calculus to be used later. We introduce a measure of noncompactness on the space of weighted

continuous functions. In Section 3, we establish the finite approximate controllability of (1.1). Finally, an example is provided to illustrate our results.

2. Preliminaries and notation

Let $J=[0,b]$, $C(J,E)$ be the Banach space of all E valued continuous functions from J to E with the norm $\|x\|_{C(J,E)}=\sup_{t\in J}\|x(t)\|$. For $a\in[0,b)$ and $0\le\gamma\le 1$, consider the weighted spaces of continuous functions

$$C_\gamma(J,E)=\{x\in C((a,b],E):(t-a)^\gamma x(t)\in C(J,E)\}.$$

Now $C_\gamma(J,E)$ is a Banach space with the norm

$$\|x\|_{C_\gamma(J,E)}=\sup_{t\in(a,b]}\|(t-a)^\gamma x(t)\|.$$

Let $K_\alpha(t)=t^{\alpha-1}P_\alpha(t)$, $P_\alpha(t)=\int_0^\infty \alpha\theta M_\alpha(\theta)T(t^\alpha\theta)d\theta$, and $S_{\alpha,\beta}(t)=I_{0^+}^{\beta(1-\alpha)}K_\alpha(t)$, $t\ge 0$, where $M_\mu(\theta)=\sum\limits_{n=1}^{\infty}\frac{(-\theta)^{n-1}}{(n-1)\Gamma(1-\mu n)},\mu\in(0,1),\theta\in\mathbb{C}$ satisfying $\int_0^\infty\theta^\tau M_\mu(\theta)d\theta=\frac{\Gamma(1+\tau)}{\Gamma(1+\tau\mu)}$ for $\theta\ge 0$.

Definition 1 ([9, Definition 2.13])**.** Let $f:[0,b]\times E\to E$ be a suitable function. By a mild solution of

$$\begin{cases} D_{a^+}^{\alpha,\beta}x(t)=Ax(t)+f(t,x(t)),t\in(0,b],\\ I_{0^+}^{1-\gamma}x(0^+)=x_0,\end{cases}\tag{2.1}$$

we mean a function $x\in C((0,b],E)$ which satisfies

$$x(t)=S_{\alpha,\beta}(t)x_0+\int_0^t K_\alpha(t-s)f(s,x(s))ds,t\in(0,b].$$

Remark 1. From [9, Remark 2.14], we have (i) $D_{0^+}^{\beta(1-\alpha)}S_{\alpha,\beta}(t)=K_\alpha(t)$, $t\in(0,b]$; (ii) When $\beta=0$, the fractional equation (2.1) reduces to the classical Riemann-Liouville fractional equation which was studied by Zhou et al. [31]. Note $S_{\alpha,0}(t)=K_\alpha(t)=t^{\alpha-1}P_\alpha(t)$; (iii) When $\beta=1$, the fractional equation (2.1) reduces to the classical Caputo fractional equation which has studied by Zhou et al. [28]. Note $S_{\alpha,1}(t)=S_\alpha(t)$, where $S_\alpha(t)$ is defined in [28].

In the following we collect the properties for $S_{\alpha,\beta}(t)$ and $K_\alpha(t)$ (see [9, Propositions 2.15, 2.16]).

We need the following condition.

(A_1) $T(t)$ is continuous for the uniform operator topology for $t>0$, and there is a $M>1$ such that $\sup_{t\ge 0}\|T(t)\|\le M$.

Lemma 1. *If (A_1) holds, then we have*

(i) *$P_\alpha(t)$ is continuous for the uniform operator topology for $t>0$.*

(ii) *For any fixed* $t > 0, S_{\alpha,\beta}(t)$ *and* $K_\alpha(t)$ *are linear bounded operators and for any fixed* $x \in E$,

$$\|S_{\alpha,\beta}(t)x\| \leq \frac{Mt^{\gamma-1}}{\Gamma(\gamma)}\|x\|, \ \gamma = \alpha + \beta - \alpha\beta, \tag{2.2}$$

and

$$\|K_\alpha(t)x\| \leq \frac{Mt^{\alpha-1}}{\Gamma(\alpha)}\|x\|. \tag{2.3}$$

(iii) $\{K_\alpha(t) : t > 0\}$ *and* $\{S_{\alpha,\beta}(t) : t > 0\}$ *are strongly continuous, which means that, for any* $x \in E$ *and* $0 < t_1 < t_2 \leq b$, *we have* $\|K_\alpha(t_1)x - K_\alpha(t_2)x\| \to 0$ *and* $\|S_{\alpha,\beta}(t_1)x - S_{\alpha,\beta}(t_2)x\| \to 0$, *as* $t_1 \to t_2$.

(iv) $S_{\alpha,\beta}(t)$ *and* $K_\alpha(t)$ *are compact for* $t > 0$ *if* $T(t), t > 0$ *is compact.*

In the text we denote the mild solution which corresponds to the control function u by $x(.,u)$ and to the value of it at the terminal point b by $x(b,u)$.

Definition 2. Now (1.1) is said to be approximate controllable on J if for every $x_b \in E$, and any $\varepsilon > 0$, there exists a control function $u_\varepsilon \in L^p(J,X)$ such that the corresponding mild solution of (1.1), $x(.,u_\varepsilon)$ satisfies $\|x(b,u_\varepsilon) - x_b\| \leq \varepsilon$.

Definition 3. Now (1.1) is said to be finite approximate controllable on J if for every $x_b \in E$, and any $\varepsilon > 0$, there exists a control function $u_\varepsilon \in L^p(J,X)$ such that the corresponding mild solution of (1.1), $x(.,u_\varepsilon)$ satisfies $\|x(b,u_\varepsilon) - x_b\| \leq \varepsilon$ and $\pi_\Upsilon x(b,u_\varepsilon) = \pi_\Upsilon x_b$, where Υ is a given finite dimensional subspace of E and π_Υ is the orthogonal projection on Υ.

3. Finite approximate controllability for (1.1)

Let $\varepsilon > 0$ and $x_b \in E$ be a fixed point. In order to investigate the finite approximate controllable for (1.1) we have the following assumptions:

(H_1) $f : J \times E \to E$ is a continuous and there is a function $\varphi \in L^p(J,\mathbb{R}^+), p > \frac{1}{\alpha}$ and a continuous nondecreasing function $\Omega : [0,\infty) \to [0,\infty)$ such that for every $x \in C_{1-\gamma}(J,E), \|f(t,x(t)\| \leq \varphi(t)\Omega(t^{1-\gamma}\|x(t)\|)$, for $t \in J$ and $\liminf_{n\to\infty} \frac{\|\Omega(n)\|}{n} = \upsilon < \infty$.

(H_2) $\{T(t) : t > 0\}$ is compact semigroup.

(H_3) The linear system associated with (1.1) is approximately controllable in J.

Remark 2. Similarly to that in [16, 18], we can show that the linear system associated by (1.1) is approximate controllable in J if and only if the condition $B^*K_\alpha^*(b-s)z = 0$, for all $s \in J$ implies that $z = 0$.

To prove our main result we first present some preliminary lemmas.

Lemma 2. *Let $r > 0$ and $B(0,r) = \{\psi \in C_{1-\gamma}(J,E) : \|\psi\|_{C_{1-\gamma}(J,E)} \leq r\}$. The function*

$$h : B(0,r) \to E,$$
$$h(\psi) = x_b - S_{\alpha,\beta}(b)x_0 - \int_0^b K_\alpha(b-s)f(s,\psi(s))ds, \tag{3.1}$$

is continuous and maps $B(0,r)$ into a relatively compact subset of E.

Proof. Let (ψ_n) be a sequence in $B(0,r)$ such that $\psi_n \to \psi$ in $B(0,r)$. It follows from (3.1) that

$$\lim_{n\to\infty} \|h(\psi_n) - h(\psi)\| \leq \frac{M}{\Gamma(\alpha)} \lim_{n\to\infty} \int_0^b \|(b-s)^{\alpha-1} f(s,\psi_n(s)) - (b-s)^{\alpha-1} f(s,\psi(s))\| ds.$$

Note that for any $s \in J$,

$$\begin{aligned}
&\|(b-s)^{\alpha-1} f(s,\psi_n(s)) - (b-s)^{\alpha-1} f(s,\psi(s))\| \\
&\leq (b-s)^{\alpha-1}\varphi(s)\Omega(t^{1-\gamma}\psi_n(s)) + (b-s)^{\alpha-1}\varphi(s)\Omega(t^{1-\gamma}\psi(s)) \\
&\leq 2(b-s)^{\alpha-1}\varphi(s)\Omega(r) \in L^1(J,\mathbb{R}^+).
\end{aligned}$$

Then the continuity of $f(s,.)$ and the Lebesgue dominated convergence theorem implies $\lim_{n\to\infty} \|h(\psi_n) - h(\psi)\| = 0$, which proves the continuity of h.

Now we show that h maps $B(0,r)$ into a relatively compact subset of E. Let $\sigma \in (0,b)$. From the compactness of $K_\alpha(s), s > 0$ and the continuity of $f(s,.)$, the set $\{\int_0^{b-\sigma} K_\alpha(b-s)f(s,\psi(s))ds : \psi \in B(0,r)\}$ is relatively compact in E. Furthermore, for any $\psi \in B(0,r)$ we have

$$\begin{aligned}
&\lim_{\sigma\to 0} \| \int_0^b K_\alpha(b-s)f(s,\psi(s))ds - \int_0^{b-\sigma} K_\alpha(b-s)f(s,\psi(s))ds\| \\
&\leq \lim_{\sigma\to 0} \| \int_{b-\sigma}^b K_\alpha(b-s)f(s,\psi(s))ds\| \\
&\leq \lim_{\sigma\to 0} \frac{M}{\Gamma(\alpha)} \int_{b-\sigma}^b (b-s)^{\alpha-1}\varphi(s)\Omega(r)ds \\
&\leq \lim_{\sigma\to 0} \frac{\Omega(r)M}{\Gamma(\alpha)} (\frac{p-1}{\alpha p-1})^{\frac{p-1}{p}} \sigma^{\alpha-\frac{1}{p}} \|\varphi\|_{L^p(J,\mathbb{R}^+)} = 0.
\end{aligned}$$

This means that the set $\{\int_0^b K_\alpha(b-s)f(s,\psi(s))ds : \psi \in B(0,r)\}$ is arbitrarily close to a relatively compact set, and hence it is a relatively compact set. □

Next we define the functional

$$J_\varepsilon : E \times C_{1-\gamma}(J,E) \to \mathbb{R},$$
$$J_\varepsilon(z,\psi) = \frac{1}{2}\int_0^b \|B^* K_\alpha^*(b-s)z\|^2 ds + \varepsilon\|(I-\pi_\Upsilon)z\| - < z, h(\psi) >,$$

where B^* and $K_\alpha^*(b-\cdot)$ are the Hilbert adjoint of B and $K_\alpha(b-\cdot)$.

In the following lemma we prove some properties for the functional J_ε.

Lemma 3. *For any $\psi \in C_{1-\gamma}(J,E)$ the map $z \to J_\varepsilon(z,\psi)$ is strictly convex, continuous and for any $r > 0$,*

$$\lim \inf_{\|z\|\to\infty} \inf_{\psi\in B(0,r)} \frac{J_\varepsilon(z,\psi)}{\|z\|} \geq \varepsilon. \tag{3.2}$$

Proof. Clearly J_ε is strictly convex and the map $z \to J_\varepsilon(z,\psi)$ is continuous (see [16, 18]). Assume that (3.2) is not true. Then there are two sequence (z_n) of E and (ψ_n) of $B(0,r)$ such that $\lim_{n\to\infty}\|z_n\| = \infty$ and

$$\lim \inf_{n\to\infty} \frac{J_\varepsilon(z_n,\psi_n)}{\|z_n\|} < \varepsilon. \tag{3.3}$$

Next, for any $n \geq 1$ set $\widetilde{z_n} = \frac{z_n}{\|z_n\|}$. From the reflexivity of E, there is a subsequence, denoted again by $\widetilde{z_n}$, which converges weakly to an element $\widetilde{z} \in E$, and hence it follows from the compactness of $K^*_\alpha(b-s)$ and the continuity of B, that

$$\lim_{n\to\infty} B^* K^*_\alpha(b-s)\widetilde{z_n} = B^* K^*_\alpha(b-s)\widetilde{z}. \tag{3.4}$$

Moreover, according to Lemma 2, $h(\psi_n)$ has a convergent subsequence, denoted again by $h(\psi_n)$, which converges strongly to $z_0, z_0 \in E$. Now

$$\begin{aligned}\frac{J_\varepsilon(z_n,\psi_n)}{\|z_n\|} &= \frac{1}{2\|z_n\|}\int_0^b \|B^*K^*_\alpha(b-s)z_n\|^2 ds + \varepsilon\|(I-\pi_\Upsilon)\frac{z_n}{\|z_n\|}\| - < \frac{z_n}{\|z_n\|}, h(\psi_n) > \\ &= \frac{\|z_n\|}{2}\int_0^b \|B^*K^*_\alpha(b-s)\widetilde{z_n}\|^2 ds + \varepsilon\|(I-\pi_\Upsilon)\widetilde{z_n}\| - < \widetilde{z_n}, h(\psi_n) > .\end{aligned}$$

We distinguish the following two cases:

Case 1. $\liminf_{n\to\infty}\int_0^b \|B^*K^*_\alpha(b-s)\widetilde{z_n}\|^2 ds > 0$. When this holds we have $\liminf_{n\to\infty}\frac{J_\varepsilon(z_n,\psi_n)}{\|z_n\|} = \infty$, which contradicts (3.3).

Case 2. $\liminf_{n\to\infty}\int_0^b \|B^*K^*_\alpha(b-s)\widetilde{z_n}\|^2 ds = 0$. When this holds we have from (3.4) and Fatou's lemma that

$$\int_0^b \|B^*K^*_\alpha(b-s)\widetilde{z}\|^2 ds \leq \lim \inf_{n\to\infty}\int_0^b \|B^*K^*_\alpha(b-s)\widetilde{z_n}\|^2 ds = 0. \tag{3.5}$$

This together assumption (H_3) implies that $\widetilde{z} = 0$. On the other hand, since Υ is of finite dimensional π_Υ is compact, we have $\lim_{n\to\infty}\pi_\Upsilon\widetilde{z_n} = \pi_\Upsilon 0 = 0$. Therefore, $\lim_{n\to\infty}(I-\pi_\Upsilon)\widetilde{z_n} = \lim_{n\to\infty}\widetilde{z_n}$, which implies that $\lim_{n\to\infty}\|(I-\pi_\Upsilon)\widetilde{z_n}\|^2 = \lim_{n\to\infty}\|\widetilde{z_n}\|^2 = 1$.

Notice that since $h(\psi_n) \to z_0$ strongly and $\widetilde{z_n}, \to 0$ weakly, $\lim_{n\to\infty} < \widetilde{z_n}, h(\psi_n) >= 0$. Then $\liminf_{n\to\infty}\frac{J_\varepsilon(z_n,\psi_n)}{\|z_n\|} = \varepsilon$, which contradicts (3.3). □

Remark 3. According to Lemma 2, for any $\psi \in C_{1-\gamma}(J,E)$ the map $z \to J_\varepsilon(z,\psi)$ is strictly convex and continuous, and hence for any $\psi \in C_{1-\gamma}(J,E)$ the functional

$J_\varepsilon(,\psi)$ admits a unique minimum $z_{\varepsilon,\psi}$. Then we can define a map $\Phi_\varepsilon : C_{1-\gamma}(J,E) \to E, \Phi_\varepsilon(\psi) = z_{\varepsilon,\psi}$, that is $z_{\varepsilon,\psi} = \min_{y\in E} J_\varepsilon(y,\psi)$.

Next, we present some properties for Φ_ε.

Lemma 4. *For any $r > 0$, there is a positive real number $R(\varepsilon,r)$ such that*

$$\|\Phi_\varepsilon(\psi)\| \le R(\varepsilon,r), \forall \psi \in B(0,r). \tag{3.6}$$

Moreover Φ_ε is continuous on $B(0,r)$.

Proof. Now (3.2) implies the existence of a real number $R(\varepsilon,r)$ such that

$$\|z\| > R(\varepsilon,r) \text{ implies } \inf_{\psi\in B(0,r)} \frac{J_\varepsilon(z,\psi)}{\|z\|} \ge \frac{\varepsilon}{2}. \tag{3.7}$$

From the definition of Φ_ε it follows that $J_\varepsilon(\Phi_\varepsilon(\psi),\psi) \le J_\varepsilon(0,\psi) = 0, \forall \psi \in C_{1-\gamma}(J,E)$, and hence

$$\inf_{\psi\in B(0,r)} \frac{J_\varepsilon(\Phi_\varepsilon(\psi),\psi)}{\|z\|} = 0. \tag{3.8}$$

From (3.7) and (3.8) we get (3.6).

To prove the continuity of Φ_ε on $B(0,r)$, let $\psi_n \to \psi$ in $B(0,r)$. In view of (3.6) the sequence $\Phi_\varepsilon(\psi_n)$ is bounded in the Hilbert space E which is reflexive, so $\Phi_\varepsilon(\psi_n)$ has a subsequence, denoted again by $\Phi_\varepsilon(\psi_n)$, which converges weakly to z^*. Therefore, from the definitions of J_ε and Φ_ε we get

$$J_\varepsilon(\Phi_\varepsilon(\psi),\psi) \le J_\varepsilon(z^*,\psi) \le \lim\inf_{n\to\infty} J_\varepsilon(\Phi_\varepsilon(\psi_n),\psi_n). \tag{3.9}$$

and

$$\lim\sup_{n\to\infty} J_\varepsilon(\Phi_\varepsilon(\psi_n),\psi_n) \le \lim_{n\to\infty} J_\varepsilon(z^*,\psi_n) = J_\varepsilon(z^*,\psi). \tag{3.10}$$

It follows from (3.9) and (3.10) that $\lim_{n\to\infty} J_\varepsilon(\Phi_\varepsilon(\psi_n),\psi_n) = J_\varepsilon(z^*,\psi)$.

This means that z^* is a minimum for $J_\varepsilon(,\psi)$. By the uniqueness of the minimum we conclude that $\Phi_\varepsilon(\psi) = z^*$. It remains to show that $\Phi_\varepsilon(\psi_n)$ converges strongly to $\Phi_\varepsilon(\psi)$. Note that from the facts that $\psi_n \to \psi$ in $B(0,r)$, $\Phi_\varepsilon(\psi_n) \to \Phi_\varepsilon(\psi)$ weakly, $K^*_\alpha(t), t > 0$ is compact and h is continuous we get

$$\lim_{n\to\infty} J_\varepsilon(\Phi_\varepsilon(\psi_n),\psi_n) = J_\varepsilon(\Phi_\varepsilon(\psi),\psi), \tag{3.11}$$

$$\lim_{n\to\infty} \int_0^b \|B^* K^*_\alpha(b-s)\Phi_\varepsilon(\psi_n)\|^2 ds = \int_0^b \|B^* K^*_\alpha(b-s)\Phi_\varepsilon(\psi)\|^2 ds, \tag{3.12}$$

$$\lim_{n\to\infty} < \Phi_\varepsilon(\psi_n), h(\psi_n) > = < \Phi_\varepsilon(\psi), h(\psi) >, \tag{3.13}$$

and

$$\|(I-\pi_\Upsilon)\psi\| \le \lim_{n\to\infty} \|(I-\pi_\Upsilon)\Phi_\varepsilon(\psi_n)\|. \tag{3.14}$$

Now (3.11), (3.12), (3.13) and (3.14) imply

$$\|(I-\pi_\Upsilon)\psi\| = \lim_{n\to\infty} \|(I-\pi_\Upsilon)\Phi_\varepsilon(\psi_n)\|. \tag{3.15}$$

Since E is a Hilbert space then (3.15) with the fact that $\Phi_\varepsilon(\psi_n) \to \Phi_\varepsilon(\psi)$ weakly assures us that $\Phi_\varepsilon(\psi_n)$ converges strongly to $\Phi_\varepsilon(\psi)$. □

In the following theorem we investigate the finite approximate controllability of (1.1).

Theorem 1. *Assume that $\alpha > \frac{1}{2}$ and conditions (A_1), (H_1), (H_2) and (H_3) hold. Then (1.1) is finite approximately controllable provided that*

$$\lim_{n\to\infty} \frac{R(\varepsilon,n)}{n} = \sigma, \tag{3.16}$$

$$\frac{M\eta\upsilon b^{1-\gamma}}{\Gamma(\alpha)}\|\varphi\|_{L^p(J,\mathbb{R}^+)} + \frac{M^2\|B\|^2 b^{1-\gamma+\alpha}}{\alpha[\Gamma(\alpha)]^2}\sigma < 1, \tag{3.17}$$

where $\eta = b^{\alpha-\frac{1}{p}}\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}$.

Proof. According to Remark 3, for any $x \in C_{1-\gamma}(J,E)$ there is a unique element $\Phi_\varepsilon(x) \in E$ such that $J_\varepsilon(\Phi_\varepsilon(x),x) \le J_\varepsilon(z,x), \forall z \in E$, so we can define a function

$$N_\varepsilon : C_{1-\gamma}(J,E) \to C_{1-\gamma}(J,E),$$
$$(N_\varepsilon x)(t) = S_{\alpha,\beta}(t)x_0 + \int_0^t K_\alpha(t-s)f(s,x(s))ds + \int_0^t K_\alpha(t-s)Bu_{\varepsilon,x}(s)ds,$$

where

$$u_{\varepsilon,x}(s) = B^* K_\alpha^*(b-s)\Phi_\varepsilon(x). \tag{3.18}$$

We now show that using the control function, defined by (3.18), any fixed point, x_ε, for N_ε is a mild solution for (1.1) and satisfies $\|x_\varepsilon(b) - x_b\| \le \varepsilon$ and $\pi_\Upsilon x_\varepsilon(b) = \pi_\Upsilon x_b$. In fact, for $t \in (0,b]$

$$\begin{aligned} x_\varepsilon(t) = S_{\alpha,\beta}(t)x_0 &+ \int_0^t K_\alpha(t-s)f(s,x_\varepsilon(s))ds \\ &+ \int_0^t K_\alpha(t-s)BB^*K_\alpha^*(b-s)\Phi_\varepsilon(x_\varepsilon)ds. \end{aligned} \tag{3.19}$$

Let $z \in E$ and λ be a nonzero real number. From the definition of $\Phi_\varepsilon(x_\varepsilon)$ we get for $t \in (0,b]$, $J_\varepsilon(\Phi_\varepsilon(x_\varepsilon),x_\varepsilon) \le J_\varepsilon(\Phi_\varepsilon(x_\varepsilon)+\lambda z, x_\varepsilon)$.

It follows from the definition of J_ε that

$$\begin{aligned} &\frac{1}{2}\int_0^b \|B^*K_\alpha^*(b-s)\Phi_\varepsilon(x_\varepsilon)\|^2 ds + \varepsilon\|(I-\pi_\Upsilon)\Phi_\varepsilon(x_\varepsilon)\| - <\Phi_\varepsilon(x_\varepsilon), h(x_\varepsilon)> \\ &\le \frac{1}{2}\int_0^b \|B^*K_\alpha^*(b-s)(\Phi_\varepsilon(x_\varepsilon)+\lambda z)\|^2 ds + \varepsilon\|(I-\pi_\Upsilon)(\Phi_\varepsilon(x_\varepsilon)+\lambda z)\| \\ &\quad - <\Phi_\varepsilon(x_\varepsilon)+\lambda z, h(x_\varepsilon)>. \end{aligned}$$

Now

$$\lambda < z, h(x_\varepsilon) > \le \frac{\lambda^2}{2}\int_0^b \|B^*K_\alpha^*(b-s)z\|^2 ds$$

$$+\lambda\int_0^b < B^*K_\alpha^*(b-s)\Phi_\varepsilon(x_\varepsilon), B^*K_\alpha^*(b-s)z > ds + \varepsilon|\lambda|\|(I-\pi_\Upsilon)z)\|. \tag{3.20}$$

If $\lambda > 0$, then (3.20) becomes

$$< z, h(x_\varepsilon) > \leq \frac{\lambda}{2}\int_0^b \|B^*K_\alpha^*(b-s)z\|^2 ds$$
$$+\int_0^b < B^*K_\alpha^*(b-s)\Phi_\varepsilon(x_\varepsilon), B^*K_\alpha^*(b-s)z > ds + \varepsilon\|(I-\pi_\Upsilon)z)\|.$$

Let $\lambda \to 0^+$ and one obtains

$$-\varepsilon\|(I-\pi_\Upsilon)z)\| \leq \int_0^b < B^*K_\alpha^*(b-s)\Phi_\varepsilon(x_\varepsilon), B^*K_\alpha^*(b-s)z > ds - < z, h(x_\varepsilon) > . \tag{3.21}$$

If $\lambda < 0$, then (3.20) becomes

$$< z, h(x_\varepsilon) > \geq \frac{\lambda}{2}\int_0^b \|B^*K_\alpha^*(b-s)z\|^2 ds$$
$$+\int_0^b < B^*K_\alpha^*(b-s)\Phi_\varepsilon(x_\varepsilon), B^*K_\alpha^*(b-s)z > ds - \varepsilon\|(I-\pi_\Upsilon)z)\|.$$

Let $\lambda \to 0^-$ and one obtains

$$\int_0^b < B^*K_\alpha^*(b-s)\Phi_\varepsilon(x_\varepsilon), B^*K_\alpha^*(b-s)z > ds - < z, h(x_\varepsilon) > \leq \varepsilon\|(I-\pi_\Upsilon)z)\|. \tag{3.22}$$

Combining (3.21) and (3.22) and we have

$$|\int_0^b < B^*K_\alpha^*(b-s)\Phi_\varepsilon(x_\varepsilon), B^*K_\alpha^*(b-s)z > ds - < z, h(x_\varepsilon) > | \leq \varepsilon\|(I-\pi_\Upsilon)z)\|.$$

Notice that $< z, h(x_\varepsilon) > = < h(x_\varepsilon), z >$. Then

$$|\int_0^b < K_\alpha(b-s)BB^*K_\alpha^*(b-s)\Phi_\varepsilon(x_\varepsilon), z > ds - < h(x_\varepsilon), z > | \leq \varepsilon\|(I-\pi_\Upsilon)z)\|. \tag{3.23}$$

From (3.23) and (3.1) (the definition of h) we get

$$|\int_0^b < K_\alpha(b-s)BB^*K_\alpha^*(b-s)\Phi_\varepsilon(x_\varepsilon) + K_\alpha(b-s)f(s,x_\varepsilon(s)), z > ds$$
$$- < x_b, z > + < S_{\alpha,\beta}(b)x_0, z > | \leq \varepsilon\|(I-\pi_\Upsilon)z)\|.$$

This inequality with (3.19) gives

$$| < x_\varepsilon(b) - x_b, z > | \leq \varepsilon\|(I-\pi_\Upsilon)z)\|, \ \forall z \in E. \tag{3.24}$$

Observe that $\|(I-\pi_\Upsilon)z)\| = \|z - \pi_\Upsilon z\| = \inf_{\theta\in\Upsilon} d(z,\theta)$.

Since $0 \in \Upsilon$ one obtains $\|(I-\pi_\Upsilon)z)\| \le \|z\|$. Hence (3.24) yields $< x_\varepsilon(b) - x_b, z > \le \varepsilon\|z\|$, $\forall z \in E$.

Now taking $z = x_\varepsilon(b) - x_b$, one obtains $\|x_\varepsilon(b) - x_b\| \le \varepsilon$.

On the other hand if we take in (3.24) z in Υ, then it follows that $< x_\varepsilon(b) - x_b, z >= 0$, $\forall z \in \Upsilon$, which means that $\pi_\Upsilon x_\varepsilon(b) = \pi_\Upsilon x_b$.

Therefore the control function, defined by (3.18) is the desired control function, so it is sufficient to show that N_ε has a fixed point. We use Schauder's fixed point theorem and divide the proof into steps.

Step 1. In this step we show that there is a natural number n_0 such that N_ε maps $B(0,n_0)$ into itself. Suppose the contrary. Then there are two sequence (x_n) and (y_n) in $C_{1-\gamma}(J,E)$ such that $y_n = N_\varepsilon(x_n)$, $\|x_n\|_{C_{1-\gamma}(J,E)} \le n$ and $\|y_n\|_{C_{1-\gamma}(J,E)} > n$. Recall the definition of N_ε and we have

$$\begin{aligned} y_n(t) = S_{\alpha,\beta}(t)x_0 &+ \int_0^t K_\alpha(t-s)f(s,x_n(s))ds \\ &+ \int_0^t K_\alpha(t-s)BB^*K_\alpha(b-s)\Phi_\varepsilon(x_n)(s)ds,\ t \in (0,b]. \end{aligned}$$

Then, if $t \in (0,b]$, we get from (H_1), (2.2), (2.3) and Hölder's inequality that

$$\begin{aligned} t^{1-\gamma}\|y_n(t)\| &\le t^{1-\gamma}\|S_{\alpha,\beta}(t)x_0\| + \frac{Mt^{1-\gamma}\Omega(\|x_n\|_{C_{1-\gamma}(J,E)})}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi(s)ds \\ &\quad + \frac{M^2\|B\|^2 b^{1-\gamma}\|\Phi_\varepsilon(x_n)\|}{[\Gamma(\alpha)]^2}\int_0^t (t-s)^{\alpha-1}(b-s)^{\alpha-1}ds \\ &\le \frac{M}{\Gamma(\gamma)}\|x_0\| + \frac{Mb^{1-\gamma}}{\Gamma(\alpha)}\Omega(n)\|\varphi\|_{L^p(J,\mathbb{R}^+)}\eta \\ &\quad + \frac{M^2\|B\|^2 b^{1-\gamma}\|\Phi_\varepsilon(x_n)\|}{[\Gamma(\alpha)]^2}\int_0^t (t-s)^{2\alpha-2}ds. \end{aligned}$$

Notice that $t^{1-\gamma}\|y_n(0)\| = \lim_{t\to 0} t^{1-\gamma}\|y_n(t)\|$. Then, from (3.6) and (3.17) we have

$$n < \|y_n\|_{C_{1-\gamma}(J,E)} \le \frac{M}{\Gamma(\gamma)}\|x_0\| + \frac{Mb^{1-\gamma}}{\Gamma(\alpha)}\Omega(n)\|\varphi\|_{L^p(J,\mathbb{R}^+)}\eta + \frac{M^2\|B\|^2 b^{2\alpha-\gamma}R(\varepsilon,n)}{(2\alpha-1)[\Gamma(\alpha)]^2}.$$

Divide both sides by n and let $n \to \infty$, and we obtain

$$1 \le \frac{M\eta\upsilon b^{1-\gamma}}{\Gamma(\alpha)}\|\varphi\|_{L^p(J,\mathbb{R}^+)} + \frac{M^2\|B\|^2 b^{2\alpha-\gamma}}{(2\alpha-1)[\Gamma(\alpha)]^2}\sigma,$$

which contradicts (3.17). Thus we deduce that there is a natural number n_0 such that $N_\varepsilon(B(0,n_0)) \subseteq B(0,n_0)$.

Step 2. N_ε is continuous on $B(0,n_0)$.

Consider a sequence $(x_n)_{n\ge 1}$ in $B(0,n_0)$ with $x_n \to x$ in $B(0,n_0)$ and let $y_n = N_\varepsilon(x_n)$ and $y = N_\varepsilon(x)$. We show that $y_n \to y$ in $C_{1-\gamma}(J,E)$. Recall the definition of N_ε and

we get for $t \in (0,b]$,

$$\|t^{1-\gamma}y_n(t) - t^{1-\gamma}y(t)\| = \frac{Mt^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x_n(s)) - f(s,x(s)\|ds$$
$$+t^{1-\gamma}\int_0^t K_\alpha(t-s)BB^*K_\alpha(b-s)\|\Phi_\varepsilon(x_n) - \Phi_\varepsilon(x_n)\|ds.$$

Note that for any $t \in J$, and any $s \in (0,t)$,

$$\begin{aligned}&\|(t-s)^{\alpha-1}f(s,x_n(s)) - (t-s)^{\alpha-1}f(s,x(s))\|\\ &\leq (t-s)^{\alpha-1}\varphi(s)\Omega(t^{1-\gamma}x_n(s)) + (t-s)^{\alpha-1}\varphi(s)\Omega(t^{1-\gamma}x(s))\\ &\leq 2(t-s)^{\alpha-1}\varphi(s)\Omega(n_0) \in L^1(J,\mathbb{R}^+).\end{aligned}$$

Then the continuity of $f(s,.)$ and the Lebesgue dominated convergence theorem gives

$$\lim_{n\to\infty}\frac{Mt^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x_n(s)) - f(s,x(s)\|ds = 0.$$

Moreover, in view of Lemma 1, for any $t \in J$, and any $s \in (0,t)$,

$$\begin{aligned}\|K_\alpha(t-s)BB^*K_\alpha(b-s)\|\Phi_\varepsilon(x_n) - \Phi_\varepsilon(x_n)\| &\leq 2\frac{M^2\|B\|^2R(\varepsilon,n_0)}{[\Gamma(\alpha)]^2}(t-s)^{\alpha-1}(b-s)^{\alpha-1}\\ &\leq 2\frac{M^2\|B\|^2R(\varepsilon,n_0)}{[\Gamma(\alpha)]^2}(t-s)^{2\alpha-2} \in L^1(J,\mathbb{R}^+).\end{aligned}$$

Again by Lemma 4, Φ_ε is continuous, and hence by using the Lebesgue dominated convergence theorem we have

$$\lim_{n\to\infty}\int_0^t K_\alpha(t-s)BB^*K_\alpha(b-s)\|\Phi_\varepsilon(x_n) - \Phi_\varepsilon(x_n)\|ds = 0,$$

so, N_ε is continuous on $B(0,n_0)$.

Step 3. For any $t \in J$, the set $\{N_\varepsilon(x) : x \in B(0,n_0)\}$ is a relatively compact subset in E.

Let $t \in J$ be fixed and $\delta \in (0,b)$ be enough small and put

$$W_{1,\delta} = \{\int_0^{t-\delta} K_\alpha(t-s)f(s,x(s))ds : x \in B(0,n_0)\},$$

and

$$W_{2,\delta} = \{\int_0^{t-\delta} K_\alpha(t-s)Bu_{\varepsilon,x}(s)ds : x \in B(0,n_0)\}.$$

Notice that for any $x \in B(0,n_0)$, $t \in (0,b]$ and $s \in (0,t-\delta]$, $\|f(s,x(s))\| \leq \varphi(s)\Omega(n_0)$. This inequality with the compactness of $K(\lambda), \lambda > 0$, implies that $W_{1,\delta}$ and $W_{2,\delta}$ are relatively compact subset in E.

Next,

$$\|\int_0^t K_\alpha(t-s)f(s,x(s))ds - \int_0^{t-\delta} K_\alpha(t-s)f(s,x(s))ds\|$$

$$\leq \| \int_{t-\delta}^{t} K_\alpha(t-s) f(s,x(s)) ds \|$$
$$\leq \frac{M\Omega(n_0)}{\Gamma(\alpha)} \int_{t-\delta}^{t} (t-s)^{\alpha-1} \varphi(s) ds.$$

The last term tends to zero as $\delta \to 0$. Hence the set $\{\int_0^t K_\alpha(t-s) f(s,x(s)) ds : x \in B(0,n_0)\}$ is arbitrarily close to a relatively compact subset in E. Similarly,

$$\begin{aligned}
&\| \int_0^t K_\alpha(t-s) B u_{\varepsilon,x}(s) ds - \int_0^{t-\delta} K_\alpha(t-s) B u_{\varepsilon,x}(s) ds \| \\
&\leq \| \int_{t-\delta}^{t} K_\alpha(t-s) B u_{\varepsilon,x}(s) ds \| \\
&\leq \frac{M^2 \|B\|^2 R(\varepsilon,n_0)}{[\Gamma(\alpha)]^2} \int_{t-\delta}^{t} (t-s)^{\alpha-1} (b-s)^{\alpha-1} ds \\
&\leq \frac{M^2 \|B\|^2 R(\varepsilon,n_0)}{[\Gamma(\alpha)]^2} \int_{t-\delta}^{t} (t-s)^{2\alpha-2} ds \\
&= \frac{M^2 \|B\|^2 R(\varepsilon,n_0)}{[\Gamma(\alpha)]^2} \frac{\delta^{2\alpha-1}}{2\alpha-1}.
\end{aligned}$$

Since $\alpha \in (\frac{1}{2},1)$ the last term tends to zero as $\delta \to 0$. Hence the set $\{\int_0^t K_\alpha(t-s) B u_{\varepsilon,x}(s) ds : x \in B(0,n_0)\}$ is arbitrarily close to a relatively compact subset in E. Hence our claim follows.

Step 3. Set

$$K = \{z \in C(J,E) : z(t) = t^{1-\gamma} N_\varepsilon x(t), t \in (0,b], z(0) = z(0^+), x \in B(0,n_0)\}.$$

We check that K is equicontinuous.

Let $z \in K$. Then there is $x \in B(0,n_0)$ such that for $t \in (0,b]$, $z(t) = t^{1-\gamma} y(t)$, where

$$y(t) = S_{\alpha,\beta}(t) x_0 + \int_0^t K_\alpha(t-s)(f(s,x(s)) + B u_{\varepsilon,x}(s)) ds,$$

and $z(0) = \lim_{t\to 0+} t^{1-\gamma} y(t)$. It follows for $t = 0, \delta \in (0,b]$ that

$$\lim_{\delta\to 0+} z(\delta) = \lim_{\delta\to 0+} \delta^{1-\gamma} y(\delta) = \lim_{t\to 0+} t^{1-\gamma} y(t) = z(0)$$

Let $t, t+\delta$ be two points in $(0,b]$. Then

$$\begin{aligned}
\|z(t+\delta) - z(t)\| &\leq \|(t+\delta)^{1-\gamma} S_{\alpha,\beta}(t+\delta) x_0 - t^{1-\gamma} S_{\alpha,\beta}(t) x_0\| \\
&\quad + \|(t+\delta)^{1-\gamma} \int_0^{t+\delta} K_\alpha(t+\delta-s)(f(s,x(s)) + B u_{\varepsilon,x}(s)) ds \\
&\quad - t^{1-\gamma} \int_0^t K_\alpha(t-s)(f(s,x(s)) + B u_{\varepsilon,x}(s)) ds\| \\
&\leq (t+\delta)^{1-\gamma} \|S_{\alpha,\beta}(t+\delta) - S_{\alpha,\beta}(t)\| \|x_0\|
\end{aligned}$$

$$
\begin{aligned}
&+|(t+\delta)^{1-\gamma}-t^{1-\gamma}|\|S_{\alpha,\beta}(t)x_0\| \\
&+\|(t+\delta)^{1-\gamma}\int_t^{t+\delta}K_\alpha(t+\delta-s)f(s,x(s))ds\| \\
&+\|\int_0^t(t+\delta)^{1-\gamma}K_\alpha(t+\delta-s)f(s)-t^{1-\gamma}(t-s)^{\alpha-1}P_\alpha(t+\delta-s)f(s,x(s))]ds\| \\
&+\|\int_0^t[t^{1-\gamma}(t-s)^{\alpha-1}P_\alpha(t+\delta-s)-t^{1-\gamma}K_\alpha(t-s)]f(s,x(s))ds\| \\
&+\|(t+\delta)^{1-\gamma}\int_t^{t+\delta}K_\alpha(t+\delta-s)Bu_{\varepsilon,x}(s))ds\| \\
&+\|\int_0^t[(t+\delta)^{1-\gamma}K_\alpha(t+\delta-s)-t^{1-\gamma}(t-s)^{\alpha-1}P_\alpha(t+\delta-s)]Bu_{\varepsilon,x}(s)ds\| \\
&+\|\int_0^t t^{1-\gamma}(t-s)^{\alpha-1}P_\alpha(t+\delta-s)-t^{1-\gamma}K_\alpha(t-s)]Bu_{\varepsilon,x}(s)ds\| \\
=&\sum_{i=1}^{i=8}I_i.
\end{aligned}
$$

In view of Lemma 1 we have

$$\lim_{\delta\to 0}I_1=\lim_{\delta\to 0}(t+\delta)^{1-\gamma}\|S_{\alpha,\beta}(t+\delta)-S_{\alpha,\beta}(t)\|\|x_0\|=0,$$

and

$$
\begin{aligned}
\lim_{\delta\to 0}I_2&=\lim_{\delta\to 0}|(t+\delta)^{1-\gamma}-t^{1-\gamma}|\|S_{\alpha,\beta}(t)x_0\| \\
&\leq\frac{M\,t^{\gamma-1}}{\Gamma(\gamma)}\|x_0\|\lim_{\delta\to 0}|(t+\delta)^{1-\gamma}-t^{1-\gamma}|=0.
\end{aligned}
$$

Again by Lemma 1, (H_1) and the absolute integral of the Lebesgue integral we get

$$
\begin{aligned}
\lim_{\delta\to 0}I_3&=\lim_{\delta\to 0}\|(t+\delta)^{1-\gamma}\int_t^{t+\delta}K_\alpha(t+\delta-s)f(s,x(s))ds\| \\
&\leq\frac{M\,\Omega(n_0)}{\Gamma(\alpha)}\lim_{\delta\to 0}(t+\delta)^{1-\gamma}\int_t^{t+\delta}(t+\delta-s)^{\alpha-1}\varphi(s)ds=0.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\lim_{\delta\to 0}I_4&\leq\lim_{\delta\to 0}\|\int_0^t(t+\delta)^{1-\gamma}K_\alpha(t+\delta-s)f(s)-t^{1-\gamma}(t-s)^{\alpha-1}P_\alpha(t+\delta-s)f(s,x(s))]ds\| \\
&=\lim_{\delta\to 0}\|\int_0^t(t+\delta)^{1-\gamma}(t+\delta-s)^{\alpha-1}P_\alpha(t+\delta-s)f(s,x(s)) \\
&\qquad-t^{1-\gamma}(t-s)^{\alpha-1}P_\alpha(t+\delta-s)f(s,x(s))]ds\| \\
&\leq\frac{M\,\Omega(n_0)}{\Gamma(\alpha)}\lim_{\delta\to 0}\int_0^t|(t+\delta)^{1-\gamma}(t+\delta-s)^{\alpha-1}-t^{1-\gamma}(t-s)^{\alpha-1}|\varphi(s)ds.
\end{aligned}
$$

Since $\varphi \in L^p(J,\mathbb{R}^+)$, $\int_0^t [(t+\delta)^{1-\gamma}(t+\delta-s)^{\alpha-1} - t^{1-\gamma}(t-s)^{\alpha-1}]\varphi(s)ds$ exists, then by the Lebesgue dominated convergence theorem, we obtain that $\lim_{\delta\to 0} I_4 = 0$.

Next,

$$\begin{aligned}\lim_{\delta\to 0} I_5 &= \lim_{\delta\to 0} \| \int_0^t t^{1-\gamma}(t-s)^{\alpha-1}P_\alpha(t+\delta-s) - t^{1-\gamma}K_\alpha(t-s)]f(s,x(s))ds\| \\ &= \lim_{\delta\to 0} \| \int_0^t t^{1-\gamma}(t-s)^{\alpha-1}P_\alpha(t+\delta-s) - t^{1-\gamma}(t-s)^{\alpha-1}P_\alpha(t-s)]f(s,x(s))ds\|.\end{aligned}$$

To find this limit, let $\varepsilon > 0$ be enough small. We have

$$\begin{aligned}\lim_{\delta\to 0} I_5 &\le \Omega(n_0)t^{1-\gamma}\lim_{\delta\to 0}\int_0^{t-\varepsilon}(t-s)^{\alpha-1}\varphi(s)\sup_{s\in[0,t-\varepsilon]}\|P_\alpha(t+\delta-s)-P_\alpha(t-s)\|ds \\ &\quad + \lim_{\delta\to 0}\int_{t-\varepsilon}^t t^{1-\gamma}(t-s)^{\alpha-1}\|P_\alpha(t+\delta-s)f(s)-P_\alpha(t-s)f(s)ds\| \\ &\le \Omega(n_0)t^{1-\gamma}\lim_{\delta\to 0}\int_0^{t-\varepsilon}(t-s)^{\alpha-1}\varphi(s)\sup_{s\in[0,t-\varepsilon]}\|P_\alpha(t+\delta-s)-P_\alpha(t-s)\|ds \\ &\quad + \frac{2M\,\Omega(n_0)}{\Gamma(\alpha)}\lim_{\delta\to 0}\int_{t-\varepsilon}^t t^{1-\gamma}(t-s)^{\alpha-1}\varphi(s)ds \\ &\le \Omega(n_0)t^{1-\gamma}\lim_{\delta\to 0}\int_0^{t-\varepsilon}(t-s)^{\alpha-1}\varphi(s)\sup_{s\in[0,t-\varepsilon]}\|P_\alpha(t+\delta-s)-P_\alpha(t-s)\|ds \\ &\quad + \frac{2M\,\Omega(n_0)t^{1-\gamma}}{\Gamma(\alpha)}\|\varphi\|_{L^p(J,\mathbb{R}^+)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}\varepsilon^{\frac{\alpha p-1}{p}}.\end{aligned}$$

From Lemma 1, $\lim_{\delta\to 0}\sup_{s\in[0,t-\varepsilon]}\|P_\alpha(t+\delta-s)-P_\alpha(t-s)\| = 0$, and since $\varphi \in L^p(J,\mathbb{R}^+)$, then by the Lebesgue dominated convergence theorem, we obtain that $I_5 \to 0$ as $\delta \to 0$ and $\varepsilon \to 0$.

For I_6, it follows from (3.6) that

$$\begin{aligned}\lim_{\delta\to 0} I_6 &= \lim_{\delta\to 0}(t+\delta)^{1-\gamma}\|\int_t^{t+\delta} K_\alpha(t+\delta-s)Bu_{\varepsilon,x}(s)ds\| \\ &\le \lim_{\delta\to 0}\frac{M(t+\delta)^{1-\gamma}\|B\|}{\Gamma(\alpha)}\int_t^{t+\delta}(t+\delta-s)^{\alpha-1}\|B^*K_\alpha(b-s)\Phi_\varepsilon(x)\|ds \\ &\le \lim_{\delta\to 0}\frac{M^2(t+\delta)^{1-\gamma}\|B\|^2R(\varepsilon,n_0)}{[\Gamma(\alpha)]^2}\int_t^{t+\delta}(t+\delta-s)^{\alpha-1}(b-s)^{\alpha-1}ds \\ &\le \lim_{\delta\to 0}\frac{M^2(t+\delta)^{1-\gamma}\|B\|^2R(\varepsilon,n_0)}{[\Gamma(\alpha)]^2}\int_t^{t+\delta}(t+\delta-s)^{2\alpha-2}ds = 0.\end{aligned}$$

For I_7 from the Lebesgue dominated convergence theorem we have that

$$\lim_{\delta\to 0} I_7 \le \lim_{\delta\to 0}\|\int_0^t [(t+\delta)^{1-\gamma}K_\alpha(t+\delta-s) - t^{1-\gamma}(t-s)^{\alpha-1}P_\alpha(t+\delta-s)]Bu_{\varepsilon,x}ds\|$$

$$
\begin{aligned}
&= \lim_{\delta\to 0}\int_0^t |(t+\delta)^{1-\gamma}(t+\delta-s)^{\alpha-1} - t^{1-\gamma}(t-s)^{\alpha-1}|\|P_\alpha(t+\delta-s)Bu_{\varepsilon,x})\|ds \\
&\le \frac{\|B\|M}{\Gamma(\alpha)}\lim_{\delta\to 0}\int_0^t |(t+\delta)^{1-\gamma}(t+\delta-s)^{\alpha-1} - t^{1-\gamma}(t-s)^{\alpha-1}|\|B^*K_\alpha(b-s)\Phi_\varepsilon(x)\|ds \\
&\le \frac{\|B\|^2M^2\, R(\varepsilon,n_0)}{[\Gamma(\alpha)]^2}\lim_{\delta\to 0}\int_0^t |(t+\delta)^{1-\gamma}(t+\delta-s)^{\alpha-1} - t^{1-\gamma}(t-s)^{\alpha-1}|(b-s)^{\alpha-1}ds \\
&= 0.
\end{aligned}
$$

Next,

$$
\begin{aligned}
\lim_{\delta\to 0} I_8 &= \lim_{\delta\to 0}\|\int_0^t t^{1-\gamma}(t-s)^{\alpha-1}P_\alpha(t+\delta-s) - t^{1-\gamma}K_\alpha(t-s)]Bu_{\varepsilon,x}(s)ds\| \\
&\le \frac{\|B\|^2\, MR(\varepsilon,n_0)}{\Gamma(\alpha)} \\
&\quad\times \lim_{\delta\to 0}\|\int_0^t t^{1-\gamma}(t-s)^{\alpha-1}P_\alpha(t+\delta-s) - t^{1-\gamma}(t-s)^{\alpha-1}P_\alpha(t-s)](b-s)^{\alpha-1}ds\|.
\end{aligned}
$$

To find this limit, let $\varepsilon > 0$ be enough small, and one has

$$
\begin{aligned}
\lim_{\delta\to 0} I_8 &\le \frac{\|B\|^2M\, R(\varepsilon,n_0)}{\Gamma(\alpha)}t^{1-\gamma}\lim_{\delta\to 0}\int_0^{t-\varepsilon}(t-s)^{2\alpha-2}\sup_{s\in[0,t-\varepsilon]}\|P_\alpha(t+\delta-s) - P_\alpha(t-s)\|ds \\
&\quad + \frac{\|B\|^2M\, R(\varepsilon,n_0)}{\Gamma(\alpha)}\lim_{\delta\to 0}\int_{t-\varepsilon}^t t^{1-\gamma}(t-s)^{2\alpha-2}\|P_\alpha(t+\delta-s) - P_\alpha(t-s)\|ds \\
&\le \frac{\|B\|^2M\, R(\varepsilon,n_0)}{\Gamma(\alpha)}t^{1-\gamma}\lim_{\delta\to 0}\int_0^{t-\varepsilon}(t-s)^{2\alpha-2}\sup_{s\in[0,t-\varepsilon]}\|P_\alpha(t+\delta-s) - P_\alpha(t-s)\|ds \\
&\quad + \frac{2M^2\|B\|^2\, R(\varepsilon,n_0)}{[\Gamma(\alpha)]^2}\lim_{\delta\to 0}\int_{t-\varepsilon}^t t^{1-\gamma}(t-s)^{2\alpha-2}ds \\
&\le \frac{\|B\|^2M\, R(\varepsilon,n_0)}{\Gamma(\alpha)}t^{1-\gamma}\lim_{\delta\to 0}\int_0^{t-\varepsilon}(t-s)^{2\alpha-2}\sup_{s\in[0,t-\varepsilon]}\|P_\alpha(t+\delta-s) - P_\alpha(t-s)\|ds \\
&\quad + \frac{2M^2\|B\|^2\, R(\varepsilon,n_0)}{[\Gamma(\alpha)]^2}\Big(\frac{p-1}{2\alpha-p-1}\Big)^{\frac{p-1}{p}}\varepsilon^{\frac{2\alpha p-p-1}{p}}.
\end{aligned}
$$

From Lemma 1, $\lim_{\delta\to 0}\sup_{s\in[0,t-\varepsilon]}\|P_\alpha(t+\delta-s) - P_\alpha(t-s)\| = 0$, and since $2\alpha p - p - 1 > 0$, then by the Lebesgue dominated convergence theorem, we obtain that $I_8 \to 0$ as $\delta \to 0$ and $\varepsilon \to 0$.

As a consequence of Step 1, Step 2, Step 3, and from Schauder's fixed point theorem we see that N_ε has a fixed point. □

Remark 4. If the function f is uniformly bounded, then (3.16) and (3.17) hold. In fact, suppose that $\|f(t,z)\| \le \xi$, $\forall (t,z) \in J\times E$. Then $\upsilon = 0$. Moreover, for any $r > 0$ and any $\psi \in B(0,r)$ we have

$$
\|h(\psi)\| \le \|x_b\| + \frac{Bb^{\gamma-1}}{\Gamma(\gamma)}\|x_0\| + \frac{M\xi}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}ds
$$

$$\leq \|x_b\| + \frac{Bb^{\gamma-1}}{\Gamma(\gamma)}\|x_0\| + \frac{M\xi b^{\alpha}}{\Gamma(\alpha+1)} = \zeta.$$

Then for any $r > 0$, any $\psi \in B(0,r)$ and any $z \in E$,

$$|J_\varepsilon(z,\psi)| \leq \frac{\|B\|^2 M^2 \|z\|^2}{2[\Gamma(\alpha)]^2}\int_0^b (b-s)^{2\alpha-2}ds + \varepsilon\|z\| + \|z\|\zeta.$$

Therefore, $J_\varepsilon(z,\psi)$ takes its minimum at $z = 0$, and hence $\Phi_\varepsilon(\psi) = 0, \forall \psi \in B(0,r)$. Consequently, $R(\varepsilon,r) = 0$, for any $r > 0$, so (3.16) is satisfied with $\sigma = 0$. From this and the fact that $\upsilon = 0$, (3.17) holds.

We now give an example to illustrate our theory.

Example 1. Take $\alpha = \frac{2}{3}$, $0 \leq \beta \leq 1$, $\gamma = \alpha + \beta - \alpha\beta$, $J = [0,1]$ and $E = U = L^2[0,1]$. Now E is a separable Hilbert space. Let $B : E \to E$, $B = I_d$, where I_d is the identity operator. Define $A : D(A) \subseteq L^2[0,1] \to L^2[0,1]$ by $Ax = x_{yy}$, where the domain of A is given by: $D(A) = \{x \in L^2[0,1] : x, x_y$ are absolutely continuous, $x_{yy} \in L^2[0,1], x(t,0) = x(t,1) = 0\}$.

Consider the following fractional control system:

$$\begin{cases} D_{s_i^+}^{\frac{2}{3},\beta} x(t,y) = x_{yy}(t,y) + f(t,x(t,y)) + B(u(t)),\ a.e.\ t \in (0,\ b], \\ I_{0^+}^{1-\gamma} x(0,y) = x_0 \end{cases} \tag{3.25}$$

where $u \in L^2(J, L^2(J))$.

Notice that A can be written as $Ax = \sum_{n=1}^{\infty} n^2 < x, x_n > x_n, x \in D(A)$, where $x_n(y) = \sqrt{2}\sin ny, n = 1,2,\cdots$ is the orthonormal basis of E. Moreover, for any $x \in L^2[0,1]$, we have $T(t)(x) = \sum_{n=1}^{\infty} e^{-n^2 t} < x, x_n > x_n$.

Clearly, A is the infinitesimal generator of the compact C_0-semigroup $\{T(t) : t \geq 0\}$.

Moreover, for any $x \in E$

$$\begin{aligned} P_\alpha(1-s)x &= \int_0^\infty \alpha\theta M_\alpha(\theta) T((1-s)^{\frac{2}{3}}\theta) x d\,\theta \\ &= \frac{2}{3}\int_0^\infty \theta M_\alpha(\theta) \sum_{n=1}^{\infty} e^{-n^2\theta(1-s)^{\frac{2}{3}}} < x, x_n > x_n d\theta. \end{aligned}$$

Consider the Mittag-Leffler function $E_{\frac{2}{3}}(m) = \frac{2}{3}\int_0^\infty \theta M_{\frac{2}{3}}(\theta) e^{-m\theta} d\,\theta, m \in \mathbb{R}$. Then

$$P_\alpha(1-s)x = \sum_{n=1}^{\infty} E_{\frac{2}{3}}(-n^2(1-s)^{\frac{2}{3}}) < x,\ x_n > x_n.$$

It follows that

$$K_\alpha(1-s)x = (1-s)^{\alpha-1}\sum_{n=1}^{\infty} E_{\frac{2}{3}}(-n^2(1-s)^{\frac{2}{3}}) < x, x_n > x_n,$$

which leads to

$$K_\alpha^*(1-s)x = (1-s)^{\alpha-1}\sum_{n=1}^{\infty} E_{\frac{2}{3}}(-n^2(1-s)^{\frac{2}{3}}) < x, x_n > x_n.$$

Next assume that for some $x \in E$, $K_\alpha^*(1-s)x = 0, \forall s \in [0,1]$. This means that for any natural number n, $E_{\frac{2}{3}}(-n^2(1-s)^{\frac{2}{3}}) < x, x_n >= 0, \forall s \in (0,1)$. If $x \neq 0$, then there is a $n_0 \in \mathbb{N}$ such that $< x, x_{n_0} > \neq 0$, and hence $E_{\frac{2}{3}}(-n_0^2(1-s)^{\frac{2}{3}}) = 0, \forall s \in (0,1)$. Thus $0 = \lim_{s\to 1^-} E_{\frac{2}{3}}(-n_0^2(1-s)^{\frac{2}{3}}) = E_{\frac{2}{3}}(0) = \frac{1}{\Gamma(\frac{2}{3})}$, a contradiction. Therefore (H_3) holds.

Let $f : J \times E \to E$ be a continuous function satisfying (H_1). Then (3.25) is finite approximate if (3.16) and (3.17) hold.

4. Conclusion

Motivated by finite approximate controllability for memory problems in mechanics and engineering, we study finite approximate controllability for a class of Hilfer fractional semilinear evolution equations, which is an extension of the work in [16–18]. We establish sufficient conditions to guarantee that the semilinear problem is finite approximate controllable by requiring that the associated linear system is approximately controllable via semigroup theory. For future research, it is of interest to extend the results to non-instantaneous impulsive evolution equation.

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ON THE STABILITY OF SOLUTIONS TO CONFORMABLE STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study the stability of solutions to conformable stochastic differential equations. Firstly, we show the trivial solution are stochastially stable, stochastically asymptotically stable and almost surely exponentially stable, respectively. Secondly, we show the nontrivial solution are Ulam's type stable in the sense of probabilities. Finally, two examples are given to present the theoretically results.

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1. Introduction

Conformable derivative, an extension of the classical limit definitions of the derivatives of a function, has been proposed in Khalil et al. [1]. The details of the basic theory are reported in [2, 4] and the application are reported in [3, 9]. The theory of stochastic differential equations (SDE) and Itô integrals have been studied extensively, one can refer to the classical results in [5, 6].

Consider the stability of solutions to the following conformable stochastic differential equations

$$\begin{cases} D_{\rho}^{\alpha}X(t) = b(X(t),t) + \sigma(X(t),t)\dfrac{dW(t)}{dt}, \quad \rho \in (0,1],\ t \in [\alpha,\infty), \\ X(\alpha) = X_{\alpha}, \end{cases} \tag{1.1}$$

where D_{ρ}^{α} is conformable derivative, $b : \mathbb{R}^n \times [\alpha, \alpha+h] \to \mathbb{R}^n, h > 0$, and $\sigma : \mathbb{R}^n \times [\alpha, \alpha+h] \to \mathbb{R}^{n\times m}$ are continuous and $W(\cdot)$ is a standard scalar Brownian motion on an underlying complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := \{\mathcal{F}_t\}_{t\in[\alpha,\infty)}, P)$. $\mathcal{F}$ is a σ-algebra and P is a probability measure. For each $t \in [\alpha,\infty)$, $\mathbb{L}^2(\Omega, \mathcal{F}_t, P)$ denotes the space of all $\mathcal{F}_t$-measurable, mean square integrable functions endowed with the standard norm.

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Existence and uniqueness of solution to (1.1) is given in [7] by using Picard's iterative technique and the Itô formula of the conformable version is also established. In the present paper, we study the stability of solutions to (1.1). In Section 3, we prove the trivial solutions are stochastically stable, stochastically asymptotically stable, almost surely exponentially stable by setting suitable conditions on the constructed Lyapunov function. In Section 4, we propose the concepts of Ulam's type stability for (1.1) in the sense of probabilities and show the nontrivial solutions are almost surely Ulam's type stable. Two numerical simulation examples are presented in final section.

2. Preliminaries

Definition 1 ([2, Definition 2.1])**.** The conformable derivative with low index ρ of a function $f:[\alpha,\infty)\to\mathbb{R}$ is defined as

$$D_\rho^\alpha f(x)=\lim_{\varepsilon\to 0}\frac{f(x+\varepsilon(x-\alpha)^{1-\rho})-f(x)}{\varepsilon},\quad x>\alpha,\quad 0<\rho\le 1.$$

Remark 1 ([2, Theorem 2]). Fix $0<\rho\le 1$ and $x>\alpha$. A function $f:[\alpha,\infty)\to\mathbb{R}$ has a conformable derivative $D_\rho^\alpha f(x)$ if and only if it is differentiable at x and $D_\rho^\alpha f(x)=(x-\alpha)^{1-\rho}f'(x)$ holds.

Definition 2 ([7, Definition 4.1])**.** We say that an $\mathbb{R}^n$-valued stochastic process $X(\cdot)$ is a solution of (1.1), if $X(\cdot)$ is continuous and $\mathbb{F}_t$-adapted and

$$\begin{aligned}X(t)&=X_\alpha+\int_\alpha^t b(X(\tau),\tau)(\tau-\alpha)^{\rho-1}d\tau\\&+\int_\alpha^t\sigma(X(\tau),\tau)(\tau-\alpha)^{\rho-1}dW(\tau),\ t\in[\alpha,\alpha+h].\end{aligned}\tag{2.1}$$

Lemma 1 ([5, Theorem 3.8])**.** *(Doob's martingale inequalities) Let $\{M_t\}_{t\ge a}$ be an $\mathbb{R}^n$-valued martingale and $[a,b]$ an interval in $\mathbb{R}^+$. If $p\ge 1$ and $M_t\in\mathbb{L}^p(\Omega,\mathbb{R}^n)$, then*

$$P\{\omega:\ \sup_{a\le t\le b}|M_t(\omega)|\ge c\}\le\frac{1}{c^p}E|M_b|^p,\quad c>0.$$

Lemma 2 ([5, Lemma 5.4])**.** *Denote $\mathcal{M}^2([a,b];\mathbb{R})$ the space of all real-valued measurable $\{\mathcal{F}_t\}$-adapted processes $g=\{g(t)_{a\le t\le b}\}$ such that*

$$E(\int_a^b|g(t)|^2dt)<\infty.$$

If $g\in\mathbf{M}_0([a,b];\mathbb{R})\subset\mathcal{M}^2([a,b];\mathbb{R})$, then, for $a\le t\le b$, we have

$$E\left(\int_a^b g(t)dW(t)\right)=0,$$

$$E\left(\left|\int_a^b g(t)dW(t)\right|^2\right) = E\left(\int_a^b |g(t)|^2 dt\right).$$

Theorem 1 ([7, Theorem 2.8]). *Let $0 < h < \infty, X(t), t \in [\alpha, \alpha+h]$ be an Itô process for*

$$D_\rho^\alpha X(t) = f(t) + g(t)\frac{dW(t)}{dt},\ \rho \in (0,1],$$

$Y(\cdot) := Y(X(\cdot),\cdot) \in C^{2,1}(\mathbb{R}^n \times [\alpha, \alpha+h], \mathbb{R}^n)$. Then, $Y(t), t \in [\alpha, \alpha+h]$ is an Itô process given by

$$dY(t) = \frac{\partial Y(X(t),t)}{\partial t}dt + \frac{\partial Y(X(t),t)}{\partial X}f(t)(t-\alpha)^{\rho-1}dt$$
$$+ \frac{\partial Y(X(t),t)}{\partial X}g(t)(t-\alpha)^{\rho-1}dW(t) + \frac{1}{2}\frac{\partial Y^2(X(t),t)}{\partial X^2}g^2(t)(t-\alpha)^{2\rho-2}dt.$$

Definition 3 ([5, Definition 2.1]). (i) The trivial solution of (1.1) is said to be stochastically stable or stable in probability if for every pair of $\varepsilon \in (0,1)$ and $r > 0$, there exists a $\delta = \delta(\varepsilon, r, t_0) > 0$, for all $t \geq t_0$ such that $P\{|X(t)| < r\} \geq 1-\varepsilon$, whenever $|X_0| < \delta$. Otherwise it is said to be stochastically unstable.

(ii) The trivial solution of (1.1) is said to be stochastically asymptotically stable if it is stochastically stable and, moreover, for every $\varepsilon \in (0,1)$, there exists a $\delta_0 = \delta_0(\varepsilon, t_0) > 0$ such that $P\{\lim_{t\to\infty} X(t) = \mathbf{0}\} \geq 1-\varepsilon$, whenever $|X_0| < \delta_0$ and $\mathbf{0}$ denotes n-dimensional zero vector.

Definition 4 ([5, Definition 3.1]). The trivial solution of (1.1) is said to be almost surely exponentially stable, if $\lim_{t\to\infty}\sup\frac{1}{t}\ln|X(t)| < 0,\ a.s.$ for all $X_0 \in \mathbb{R}^n$, where a.s. means almost surely.

Lemma 3 ([5, Lemma 2.4])). *(Borel-Cantelli's lemma) If $\{A_k\} \subset \mathcal{F}$ and $\sum_{k=1}^\infty P(A_k) < \infty$. Then $P\{\lim_{k\to\infty}\sup A_k\ \ i.o.\} = 0$, where i.o. means infinitely often.*

Lemma 4 ([8, Theorem 1]). *Set $x(\cdot), g(\cdot)$ is real continue function on $[t_0, t_1]$, $f(\cdot) \geq 0$ is the integrable function over interval $[t_0, t_1]$, $g(\cdot) \geq 0$ is nondecreasing. If*

$$x(t) \leq g(t) + \int_{t_0}^t f(\tau)x(\tau)d\tau,\ \ t \in [t_0, t_1].$$

Then

$$x(t) \leq g(t)\exp(\int_{t_0}^t f(\tau)d\tau),\ \ t \in [t_0, t_1].$$

3. Stability of trivial solutions

We first introduce the following assumptions and denote $|\cdot|$ by the norm of $\mathbb{R}^n$.

[H1] There exists a constant $L > 0$ such that b and σ satisfying $|b(X,t) - b(\hat{X},t)| \leq L|X-\hat{X}|$, $|\sigma(X,t) - \sigma(\hat{X},t)| \leq L|X-\hat{X}|$, $t \in [\alpha, \alpha+h]$, $X, \hat{X} \in \mathbb{R}^n$.

[H2] There exists a constant $L>0$ such that b and σ satisfying $|b(X,t)|\le L|1+X|$, $|\sigma(X,t)|\le L|1+X|$, $t\in[\alpha,\alpha+h]$, $X\in\mathbb{R}^n$.

[H3] $E(|X_\alpha|^2)<\infty$ where X_α is independent of $W^+(\alpha)$.

Lemma 5 ([7, Theorem 4.3]). *Suppose that [H1], [H2] and [H3] hold. Then,* (1.1) *has a unique solution* $X(\cdot)\in\mathbb{L}_n^2[\alpha,\alpha+h]$ *give by* (2.1) *provided that* $\rho\in(\frac{1}{2},1]$.

Next, we discuss the stability of solutions to (1.1) under [H1], [H2] and [H3], which includes stability, asymptotical stability and almost surely exponentially stability.

For any $t\ge\alpha$, denote

$$D_\rho V(X(t),t)=V_t(X(t),t)(t-\alpha)^{\rho-1}+V_X(X(t),t)b(X(t),t)(t-\alpha)^{\rho-1}$$
$$+\frac{1}{2}trace[\sigma^T(X(t),t)V_{XX}(X(t),t)\sigma(X(t),t)(t-\alpha)^{2\rho-2}].$$

Set $k>0$ be arbitrary, denotes $S_k:=\{X(\cdot)\in\mathbb{R}^n\mid|X(\cdot)|<k\}$. $V(X(\cdot),\cdot)\in C^{2,1}(S_k\times[\alpha,\infty);\mathbb{R}^+)$ denote the family of all real-valued function $V(X(\cdot),\cdot)$ defined on $S_k\times[\alpha,\infty)$ such that they are continuously twice differentiable in X and once in t. $a\wedge b$: the minimum of a and b, $a\vee b$: the maximum of a and b. $I_{\{\cdot\}}$ is indicative function. Let $\mathbb{K}$ be the family of all continuous nondecreasing functions. $\mathbb{L}_n^2[\alpha,\alpha+h]$ is the space of n dimensional $2th$ integrable functions defined on $[\alpha,\alpha+h]$.

We introduce assumptions for positive-definite function $V(X(\cdot),\cdot)\in C^{2,1}(S_k\times[\alpha,\infty);\mathbb{R}^+)$.

[H4] $D_\rho V(X(\cdot),\cdot)$ is negative-definite, i.e. $D_\rho V(X(\cdot),\cdot)\le 0$.

[H5] $V(X(\cdot),\cdot)\ge c_1|X(\cdot)|^p$, $c_1>1, p\ge 1$.

[H6] $D_\rho V(X(\cdot),\cdot)\le c_2V(X(\cdot),\cdot)$, $c_2\in\mathbb{R}$.

[H7] $|V_X(X(\cdot),\cdot)\sigma(X(\cdot),\cdot)(\cdot-\alpha)^{\rho-1}|^2\ge c_3V^2(X(\cdot),\cdot)$, $c_3>0$.

Theorem 2. *If [H1], [H2], [H3], [H4] hold and* $\rho\in(\frac{1}{2},1]$, *then the trivial solution of* (1.1) *is stochastically stable.*

Proof. From Lemma 5, (1.1) has a unique solution $X(\cdot)\in\mathbb{L}_n^2[\alpha,\alpha+h]$. Following the same procedure as the proof of [5, p.111, Theorem 2.2], one can check the result. Here we present more details for the reader's convenience. Note that $V(X(t),t)$ is positive-definite and $V(\mathbf{0},t)\equiv 0$, Then, there exists a function $\mu(\cdot)\in\mathbb{K}$, such that $V(X(t),t)\ge\mu(|X(t)|)$, for all $(X(t),t)\in(S_k\times[\alpha,\infty))$.

Let $\varepsilon\in(0,1)$ and $r>0$ be arbitrary. Assume that $r<k$, since $V(X(\cdot),\cdot)$ is continuity and $V(\mathbf{0},\alpha)=0$, we can find a $\delta=\delta(\varepsilon,r,\alpha)>0$ such that

$$\frac{1}{\varepsilon}\sup_{X\in S_\delta}V(X(t),\alpha)\le\mu(r). \tag{3.1}$$

Obviously, $\delta<r$, we fix initial value $X_\alpha\in S_\delta$ arbitrary, let τ be the first exit time of $X(t)$ from S_r, i.e. $\tau=\inf\{t>\alpha:X(t)\notin S_r\}$.

By Theorem 1, for any $t > \alpha$, we obtain

$$\begin{aligned}
&V(X(\tau\wedge t),\tau\wedge t)\\
&= V(X_\alpha,\alpha) + \int_\alpha^{\tau\wedge t} V_s(X(s),s)(s-\alpha)^{\rho-1}ds\\
&+ \int_\alpha^{\tau\wedge t} V_X(X(s),s)b(X(s),s)(s-\alpha)^{\rho-1}ds\\
&+ \int_\alpha^{\tau\wedge t} V_X(X(s),s)\sigma(X(s),s)(s-\alpha)^{\rho-1}dW(s)\\
&+ \frac{1}{2}trace\int_\alpha^{\tau\wedge t}[\sigma^T(X(s),s)V_{XX}(X(s),s)\sigma(X(s),s)(s-\alpha)^{2\rho-2}]ds\\
&= V(X_\alpha,\alpha) + \int_\alpha^{\tau\wedge t} D_\rho V(X(s),s)ds\\
&+ \int_\alpha^{\tau\wedge t} V_X(X(s),s)\sigma(X(s),s)(s-\alpha)^{\rho-1}dW(s).
\end{aligned}$$

By [H4], $D_\rho V(X(\cdot),\cdot) \le 0$. Following the same procedure in the proof of [5, p.111, Theorem 2.2], taking the expectation on both sides, we have

$$EV(X(\tau\wedge t),\tau\wedge t) \le V(X_\alpha,\alpha).$$

Clearly $|X(\tau\wedge t)| = |X(\tau)| = r$ if $\tau < t$. Hence, we obtain

$$EV(X(\tau\wedge t),\tau\wedge t) \ge E\left[I_{\{\tau\le t\}}V(X(\tau),\tau)\right] \ge \mu(r)P\{\tau\le t\}.$$

Then, from (3.1), we get $P\{\tau\le t\}\le\varepsilon$. Letting $t\to\infty$, which implies $P\{\tau<\infty\}\le\varepsilon$, i.e.

$$P\{|X(t)|\le r\}\ge 1-\varepsilon,\ t\ge\alpha.$$

By Definition 3 (i), the trivial solution is stochastically stable. The proof is completed. □

Theorem 3. *If [H1], [H2], [H3] and [H4] hold, $V(X(\cdot),\cdot)$ is a decreasing function and $\rho\in(\frac{1}{2},1]$, then the trivial solution of (1.1) is stochastically asymptotically stable.*

Proof. From Theorem 2, the trivial solution of (1.1) is stochastic stable. Similarly, following the procedure of proof in [5, p.112, Theorem 2.3] and using Lemma 1, one can derive for any $\varepsilon\in(0,1)$, there exist a $\delta_0=\delta_0(\varepsilon,\alpha)>0$ such that $P\{\lim_{t\to\infty}X(t)=\mathbf{0}\}\ge 1-\varepsilon$, whenever $|X_\alpha|<\delta_0$.

By Definition 3 (ii), the trivial solution is stochastically asymptotically stable. □

Remark 2. For $\rho=1,\alpha=t_0$, Theorems 2 and 3 are the same as [5, p.111, Theorem 2.2] and [5, p.112, Theorem 2.3].

Theorem 4. *If [H1], [H2], [H3], [H5], [H6] and [H7] hold and* $\rho \in (\frac{1}{2}, 1]$*, then, for all* $X(t) \neq \mathbf{0}$ *and* $X_\alpha \neq \mathbf{0}$*,*

$$\limsup_{t\to\infty} \frac{1}{t} \ln |X(t)| \leq \frac{c_3 + 2c_2}{2p}, \quad a.s.$$

holds for all $X_\alpha \in \mathbb{R}^n$*. In particular, if* $c_3 + 2c_2 < 0$*, the trivial solution of equation* (1.1) *is almost surely exponentially stable.*

Proof. By Theorem 1, we can deduce that for $\rho \in (0,1]$, $t \geq \alpha$

$$\begin{aligned}
&\ln V(X(t),t) \\
&\leq \ln V(X_\alpha,\alpha) + \int_\alpha^t \frac{(V_s(X(s),s) + V_X(X(s),s)b(X(s),s))}{V(X(s),s)}(s-\alpha)^{\rho-1}ds \\
&+ \int_\alpha^t \frac{V_X(X(s),s)\sigma(X(s),s)}{V(X(s),s)}(s-\alpha)^{\rho-1}dW(s) \\
&+ \frac{1}{2}\int_\alpha^t \frac{|V_{XX}(X(s),s)V(X(s),s) - V_X(X(s),s)V_X(X(s),s)|}{V^2(X(s),s)}(dX(t))^2 \\
&\leq \ln V(X_\alpha,\alpha) + \int_\alpha^t \frac{D_\rho V(X(s),s)}{V(X(s),s)}ds \\
&+ \int_\alpha^t \frac{V_X(X(s),s)\sigma(X(s),s)}{V(X(s),s)}(s-\alpha)^{\rho-1}dW(s) \\
&- \frac{1}{2}\int_\alpha^t \frac{|V_X(X(s),s)\sigma(X(s),s)(s-\alpha)^{\rho-1}|^2}{V^2(X(s),s)}ds.
\end{aligned}$$

Set $M(t) := \int_\alpha^t \frac{V_X(X(s),s)\sigma(X(s),s)}{V(X(s),s)}(s-\alpha)^{\rho-1}dW(s)$. Then, for any $\varepsilon \in (0,1)$ is arbitrarily, let $n = 1,2...$, by [H7] and Lemma 1, we can get

$$\begin{aligned}
&P\left\{\sup_{\alpha\leq t\leq\alpha+n}\left[M(t) + \frac{\varepsilon}{2}\int_\alpha^t \frac{|V_X(X(s),s)\sigma(X(s),s)(s-\alpha)^{\rho-1}|^2}{V^2(X(s),s)}ds\right] > c_3(t-\alpha)\right\} \\
&\leq \frac{\varepsilon}{2c_3(t-\alpha)}.
\end{aligned} \tag{3.2}$$

Using Lemma 3, we have

$$\begin{aligned}
&P\left\{\sup_{\alpha\leq t\leq\alpha+n}\left[M(t) + \frac{\varepsilon}{2}\int_\alpha^t \frac{|V_X(X(s),s)\sigma(X(s),s)(s-\alpha)^{\rho-1}|^2}{V^2(X(s),s)}ds\right] > c_3(t-\alpha)\ i.o.\right\} \\
&= 0,
\end{aligned}$$

which implies

$$M(t) \leq c_3(t-\alpha) - \frac{\varepsilon}{2}\int_\alpha^t \frac{|V_X(X(s),s)\sigma(X(s),s)|^2}{V^2(X(s),s)}(s-\alpha)^{2(\rho-1)}ds. \quad a.s.$$

From [H6] and [H7], for $t \geq \alpha$, we have

$$\ln V(X(t),t) \leq \ln V(X_\alpha,\alpha) + c_2(t-\alpha) + \frac{1}{2}\left[(1-\varepsilon)c_3(t-\alpha)\right].$$

Then, for $t \geq \alpha$

$$\frac{1}{t}\ln V(X(t),t) \leq \frac{(t-\alpha)}{2t}\left[(1-\varepsilon)c_3 + 2c_2\right] + \frac{\ln V(X_\alpha,\alpha)}{t}.$$

Thus, letting $t \to \infty$, we can get

$$\limsup_{t\to\infty} \frac{1}{t}\ln V(X(t),t) \leq \frac{1}{2}\left[(1-\varepsilon)c_3 + 2c_2\right].$$

Using [H5], we have

$$\limsup_{t\to\infty} \frac{1}{t}\ln(c_1|X(t)|^p) \leq \lim_{t\to\infty} \frac{1}{t}\ln V(X(t),t) \leq \frac{1}{2}\left[(1-\varepsilon)c_3 + 2c_2\right].$$

Note that $c_1 > 1, p \geq 1$, then

$$\limsup_{t\to\infty} \frac{1}{t}\ln|X(t)|^p = p\lim_{t\to\infty}\frac{1}{t}\ln|X(t)| \leq \frac{1}{2}\left[(1-\varepsilon)c_3 + 2c_2\right].$$

Finally, we obtain

$$\limsup_{t\to\infty} \frac{1}{t}\ln|X(t)| \leq \frac{(1-\varepsilon)c_3 + 2c_2}{2p}.$$

Since ε is arbitrarily, we have

$$\limsup_{t\to\infty} \frac{1}{t}\ln|X(t)| \leq \frac{c_3 + 2c_2}{2p} \quad a.s.$$

Then, if $c_3 + 2c_2 < 0$, by Definition 4, the trivial solution of (1.1) is almost surely exponentially stable. The proof is completed. □

Remark 3. Concerning on Theorem 4 for $\rho = 1, \alpha = t_0$, [5, p.121, Theorem 3.3] uses different technique (exponential martingale inequality) in (3.2), i.e.

$$P\left\{\sup_{t_0\leq t\leq t_0+n}\left[M(t) - \frac{\varepsilon}{2}\int_{t_0}^{t}\frac{|V_X(X(s),s)\sigma(X(s),s)|^2}{V^2(X(s),s)}ds\right] > \frac{2}{\varepsilon}\ln n\right\} \leq \frac{1}{n^2},$$

then derives the following different inequality

$$\limsup_{t\to\infty} \frac{1}{t}\ln|X(t)| \leq -\frac{c_3 - 2c_2}{2p}. \quad a.s.$$

4. Ulam's type stability

Denote $C([\alpha,\alpha+h],\mathbb{R}^+)$ by the family of all continuous functions from $[\alpha,\alpha+h]$ to $\mathbb{R}^+$. For $\forall\varepsilon>0$, $\varphi(\cdot)\in C([\alpha,\alpha+h],\mathbb{R}^+)$, $t\in[\alpha,\alpha+h]$, consider

$$\left|D_\rho^\alpha Y(t)-b(Y(t),t)-\sigma(Y(t),t)\frac{dW(t)}{dt}\right|\le\varepsilon,\ \ \frac{1}{2}<\rho\le 1, \tag{4.1}$$

and

$$\left|D_\rho^\alpha Y(t)-b(Y(t),t)-\sigma(Y(t),t)\frac{dW(t)}{dt}\right|\le\varepsilon\varphi(t),\ \ \frac{1}{2}<\rho\le 1. \tag{4.2}$$

Definition 5. (1.1) is called almost surely Ulam-Hyers stable in the sense of probability, if there exists a constant $N>0$ such that for $\forall\varepsilon>0, r\in(0,1)$ and for each process $Y(\cdot)\in\mathbb{L}_n^2[\alpha,\alpha+h]$ satisfy

$$P\{|Y(t)-X(t)|\le N\varepsilon\}\ge 1-r,\ \ t\in[\alpha,\alpha+h].$$

Remark 4. A process $Y(\cdot)\in\mathbb{L}_n^2[\alpha,\alpha+h]$ is a solution of (4.1), iff for $\forall\varepsilon>0$, there exists a function $G(\cdot)\in\mathbb{L}_n^2[\alpha,\alpha+h]$ such that (i) $|G(\cdot)|<\varepsilon^2$; (ii) $D_\rho^\alpha Y(t)=b(Y(t),t)+\sigma(Y(t),t)\frac{dW(t)}{dt}+G(t)$, $t\in[\alpha,\alpha+h]$.

Definition 6. (1.1) is called almost surely Ulam-Hyers-Rassias stable in the sense of probability, if there exists a constant $\tilde{N}>0$ such that for $\forall\varepsilon>0$, $r\in(0,1)$, and $\varphi(\cdot)\in C([\alpha,\alpha+h],\mathbb{R}^+)$. for each process $Y(\cdot)\in\mathbb{L}_n^2[\alpha,\alpha+h]$ satisfy

$$P\left\{|Y(t)-X(t)|\le\tilde{N}\varepsilon\varphi(t)\right\}\ge 1-r,\quad t\in[\alpha,\alpha+h].$$

Remark 5. A process $Y(\cdot)\in\mathbb{L}_n^2[\alpha,\alpha+h]$ is a solution of (4.2), iff there exists a constant M and function $\bar{G}(\cdot)\in\mathbb{L}_n^2[\alpha,\alpha+h]$ such that (i) $|\bar{G}(t)|\le\varepsilon^2M\le\varepsilon^2\varphi(t)$, $t\in[\alpha,\alpha+h]$; (ii) $D_\rho^\alpha Y(t)=b(Y(t),t)+\sigma(Y(t),t)\frac{dW(t)}{dt}+\bar{G}(t)$, $t\in[\alpha,\alpha+h]$.

Lemma 6. *Let process $Y(\cdot)$ be a solution of equation (4.1) provided $\rho\in(\frac{1}{2},1]$. Then*

$$E\left(\left|Y(t)-Y_\alpha-\int_\alpha^t b(Y(\tau),\tau)(\tau-\alpha)^{\rho-1}d\tau\right.\right.$$
$$\left.\left.-\int_\alpha^t\sigma(Y(\tau),\tau)(\tau-\alpha)^{\rho-1}dW(\tau)\right|^2\right)\le\frac{\varepsilon^4h^{2\rho}}{2\rho-1}$$

hold for $t\in[\alpha,\alpha+h]$.

Proof. We know that $D_\rho^\alpha Y(t)=b(Y(t),t)+\sigma(Y(t),t)\frac{dW(t)}{dt}+G(t)$, $t\in[\alpha,\alpha+h]$, with initial value $Y(\alpha)=Y_\alpha$, for $t\in[\alpha,\alpha+h]$, the solution can express as

$$Y(t)=Y_\alpha+\int_\alpha^t b(Y(\tau),\tau)(\tau-\alpha)^{\rho-1}d\tau$$

$$+\int_\alpha^t \sigma(Y(\tau),\tau)(\tau-\alpha)^{\rho-1}dW(\tau)+\int_\alpha^t G(\tau)(\tau-\alpha)^{\rho-1}d\tau.$$

Then, by Hölder inequality, for $t\in[\alpha,\alpha+h]$, we can get

$$\begin{aligned}
&E\Bigg(|Y(t)-Y_\alpha-\int_\alpha^t b(\tau,Y(\tau))(\tau-\alpha)^{\rho-1}d\tau\\
&-\int_\alpha^t \sigma(\tau,Y(\tau))(\tau-\alpha)^{\rho-1}dW(\tau)|^2\Bigg)\\
&=E\left|\int_\alpha^t G(\tau)(\tau-\alpha)^{\rho-1}d\tau\right|^2\\
&\le E\left|\int_\alpha^t |G(\tau)|^2 d\tau\int_\alpha^t(\tau-\alpha)^{2(\rho-1)}d\tau\right|\\
&\le E\left|\varepsilon^4(t-\alpha)\frac{(t-\alpha)^{2\rho-1}}{2\rho-1}\right|\\
&\le E\left|\frac{\varepsilon^4h^{2\rho}}{2\rho-1}\right|=\frac{\varepsilon^4h^{2\rho}}{2\rho-1},\ \rho\in(\frac{1}{2},1].
\end{aligned}$$

The proof is completed. □

Lemma 7. *Let the process $Y(\cdot)$ be a solution of equation (4.2) provided $\rho\in(\frac{1}{2},1]$. Then*

$$\begin{aligned}
&E\Bigg(\Bigg|Y(t)-Y_\alpha-\int_\alpha^t b(Y(\tau),\tau)(\tau-\alpha)^{\rho-1}d\tau\\
&-\int_\alpha^t \sigma(Y(\tau),\tau)(\tau-\alpha)^{\rho-1}dW(\tau)\Bigg|^2\Bigg)\\
&\le\frac{\varepsilon^4h^{2\rho}}{2\rho-1}\varphi^2(t)
\end{aligned}$$

hold for $t\in[\alpha,\alpha+h]$.

Proof. We know that $D_\rho^\alpha Y(t)=b(t,Y(t))+\sigma(t,Y(t))\frac{dW(t)}{dt}+\bar{G}(t),\ t\in[\alpha,\alpha+h]$, with initial value $Y(\alpha)=Y_\alpha$, for $t\in[\alpha,\alpha+h]$, the solution can expressed as

$$\begin{aligned}
Y(t)&=Y_\alpha+\int_\alpha^t b(Y(\tau),\tau)(\tau-\alpha)^{\rho-1}d\tau\\
&+\int_\alpha^t \sigma(Y(\tau),\tau)(\tau-\alpha)^{\rho-1}dW(\tau)+\int_\alpha^t \bar{G}(\tau)(\tau-\alpha)^{\rho-1}d\tau.
\end{aligned}$$

Then, by Hölder inequality, for $t\in[\alpha,\alpha+h]$, we can get

$$E\Bigg(|Y(t)-Y_\alpha-\int_\alpha^t b(Y(\tau),\tau)(\tau-\alpha)^{\rho-1}d\tau$$

$$
\begin{aligned}
&-\int_\alpha^t \sigma(Y(\tau),\tau)(\tau-\alpha)^{\rho-1}dW(\tau)|^2\Big)\\
&=E\left(\left|\int_\alpha^t \bar{G}(\tau)(\tau-\alpha)^{\rho-1}d\tau\right|^2\right)\\
&\le E\left|\int_\alpha^t |G(\tau)|^2 d\tau\cdot\int_\alpha^t (\tau-\alpha)^{2(\rho-1)}d\tau\right|\\
&\le E\left|\varepsilon^4 M^2(t-\alpha)\frac{(t-\alpha)^{2\rho-1}}{2\rho-1}\right|\\
&\le E\left|\frac{\varepsilon^4 h^{2\rho}}{2\rho-1}M^2\right|=\frac{\varepsilon^4 h^{2\rho}}{2\rho-1}M^2.
\end{aligned}
$$

Noting $M\le\varphi(\cdot)$, then $M^2\le\varphi^2(\cdot)$. For $\alpha\le t\le\alpha+h$, we obtain

$$
\begin{aligned}
&E\Bigg(\Bigg|Y(t)-Y_\alpha-\int_\alpha^t b(Y(\tau),\tau)(\tau-\alpha)^{\rho-1}d\tau\\
&-\int_\alpha^t \sigma(Y(\tau),\tau)(\tau-\alpha)^{\rho-1}dW(\tau)\Bigg|^2\Bigg)\le\frac{\varepsilon^4 h^{2\rho}}{2\rho-1}\varphi^2(t).
\end{aligned}
$$

The proof is completed. □

Theorem 5. *Assume [H1], [H2] and [H3] hold and* $\rho\in(\frac{1}{2},1]$. *Then,* (1.1) *is almost surely Ulam-Hyers stable on* $[\alpha,\alpha+h]$.

Proof. Let $Y(\cdot)\in\mathbb{L}_n^2[\alpha,\alpha+h]$ is a solution of (4.1), and $X(\cdot)$ is a solution of (1.1) give by (2.1). By Lemma 2, Lemma 6 and [H1], for $t\in[\alpha,\alpha+h]$, we can get

$$
\begin{aligned}
&E(|Y(t)-X(t)|^2)\\
&=E\Bigg(\Bigg|Y(t)-Y_\alpha-\int_\alpha^t b(Y(\tau),\tau)(\tau-\alpha)^{\rho-1}d\tau\\
&-\int_\alpha^t \sigma(Y(\tau),\tau)(\tau-\alpha)^{\rho-1}dW(\tau)\\
&+\int_\alpha^t (b(Y(\tau),\tau)-b(Y(\tau),\tau))(\tau-\alpha)^{\rho-1}d\tau\\
&+\int_\alpha^t (\sigma(Y(\tau),\tau)-\sigma(Y(\tau),\tau))(\tau-\alpha)^{\rho-1}dW(\tau)\Bigg|^2\Bigg)\\
&\le 3\varepsilon^4\frac{h^{2\rho}}{2\rho-1}+3L^2(t-\alpha)E\left(\int_\alpha^t |Y(\tau)-X(\tau)|^2(\tau-\alpha)^{2(\rho-1)}d\tau\right)\\
&+3L^2E\left(\int_\alpha^t |Y(\tau)-X(\tau)|^2(\tau-\alpha)^{2(\rho-1)}d\tau\right)\\
&\le 3\varepsilon^4\frac{h^{2\rho}}{2\rho-1}+3L^2(1+t-\alpha)\int_\alpha^t E\left(|Y(\tau)-X(\tau)|^2\right)(\tau-\alpha)^{2(\rho-1)}d\tau
\end{aligned}
$$

$$\leq 3\varepsilon^4 \frac{h^{2\rho}}{2\rho-1} + 3L^2(1+h)\int_\alpha^t E\big(|Y(\tau)-X(\tau)|^2\big)(\tau-\alpha)^{2(\rho-1)}d\tau.$$

Next, using Lemma 4, for $t \in [\alpha, \alpha+h]$, we have

$$\begin{aligned}
&E(|Y(t)-X(t)|^2) \leq 3\varepsilon^4 \frac{h^{2\rho}}{2\rho-1}\exp\left(3L^2(1+h)\frac{(t-\alpha)^{2\rho-1}}{2\rho-1}\right)\\
&\leq 3\varepsilon^4 \frac{h^{2\rho}}{2\rho-1}\exp\left(3L^2(1+h)\frac{h^{2\rho-1}}{2\rho-1}\right)\\
&\leq N\varepsilon^4,
\end{aligned}$$

where $N = 3\frac{h^{2\rho}}{2\rho-1}\exp\left(3L^2(1+h)\frac{h^{2\rho-1}}{2\rho-1}\right)$.

Therefore, from Lemma 1, we can get

$$P\big\{\sup_{\alpha\leq t\leq\alpha+h}|Y(t)-X(t)| \geq N\varepsilon\big\} \leq \frac{N\varepsilon^4}{(N\varepsilon)^2} = \frac{\varepsilon^2}{N}.$$

By Lemma 3, we can get

$$P\{\sup_{\alpha\leq t\leq\alpha+h}|Y(t)-X(t)| \geq N\varepsilon \ \ i.o.\} = 0. \quad a.s.$$

Then, for $\forall r \in (0,1)$, we have

$$P\{\sup_{\alpha\leq t\leq\alpha+h}|Y(t)-X(t)| \leq N\varepsilon\} \geq 1-r.$$

From Definition 5, (1.1) is almost surely Ulam-Hyers stable. The proof is completed. □

Theorem 6. *Assume [H1], [H2], [H3] hold, $\varphi(\cdot)$ is nondecreasing and $\rho \in (\frac{1}{2}, 1]$. Then (1.1) is almost surely Ulam-Hyers-Rassias stable on $[\alpha, \alpha+h]$.*

Proof. Let $Y(\cdot) \in \mathbb{L}_n^2[\alpha, \alpha+h]$ is a solution of (4.2), $X(\cdot)$ is a solution of (1.1) given by (2.1). By Lemma 2, Lemma 7 and [H1], for $t \in [\alpha, \alpha+h]$, we have

$$\begin{aligned}
&E(|Y(t)-X(t)|^2)\\
&= E(\Big|Y(t)-Y_\alpha - \int_\alpha^t b(Y(\tau),\tau)(\tau-\alpha)^{\rho-1}d\tau\\
&- \int_\alpha^t \sigma(Y(\tau),\tau)(\tau-\alpha)^{\rho-1}dW(\tau)\\
&+ \int_\alpha^t (b(Y(\tau),\tau)-b(Y(\tau),\tau))(\tau-\alpha)^{\rho-1}d\tau\\
&+ \int_\alpha^t (\sigma(Y(\tau),\tau)-\sigma(Y(\tau),\tau))(\tau-\alpha)^{\rho-1}dW(\tau)\Big|^2)\\
&\leq 3\frac{\varepsilon^4 h^{2\rho}}{2\rho-1}\varphi^2(t) + 3L^2(t-\alpha)\int_\alpha^t E|Y(\tau)-X(\tau)|^2(\tau-\alpha)^{2(\rho-1)}d\tau)
\end{aligned}$$

$$
\begin{aligned}
&+3L^2\int_\alpha^t E|Y(\tau)-X(\tau)|^2(\tau-\alpha)^{2(\rho-1)}d\tau\\
&\le 3\frac{\varepsilon^4h^{2\rho}}{2\rho-1}\varphi^2(t)+3L^2(1+t-\alpha)\int_\alpha^t E|Y(\tau)-X(\tau)|^2(\tau-\alpha)^{2(\rho-1)}d\tau\\
&\le 3\frac{\varepsilon^4h^{2\rho}}{2\rho-1}\varphi^2(t)+3L^2(1+h)\int_\alpha^t E|Y(\tau)-X(\tau)|^2(\tau-\alpha)^{2(\rho-1)}d\tau.
\end{aligned}
$$

Note $\varphi(\cdot)$ is nondecreasing and $(\varphi^2)'=2\varphi\varphi'\ge 0$. From Lemma 4, for $t\in[\alpha,\alpha+h]$, we have

$$
\begin{aligned}
&E(|Y(t)-X(t)|^2)\le 3\frac{\varepsilon^4h^{2\rho}}{2\rho-1}\varphi^2(t)\exp\left(3L^2(1+h)\int_\alpha^t(\tau-\alpha)^{2(\rho-1)}d\tau\right)\\
&\le 3\frac{\varepsilon^4h^{2\rho}}{2\rho-1}\varphi^2(t)\exp\left(3L^2(1+h)\frac{(t-\alpha)^{2\rho-1}}{2\rho-1}\right)\\
&\le \tilde{N}\varepsilon^4\varphi^2(t),
\end{aligned}
$$

where $\tilde{N}=3\frac{h^{2\rho}}{2\rho-1}\exp\left(3L^2(1+h)\frac{h^{2\rho-1}}{2\rho-1}\right)$. Therefore, by Lemma 1, we have

$$
P\{\sup_{\alpha\le t\le\alpha+h}|Y(t)-X(t)|\ge\tilde{N}\varepsilon\varphi(t)\}\le\frac{\tilde{N}\varepsilon^4\varphi^2(t)}{\tilde{N}^2\varepsilon^2\varphi^2(t)}=\frac{\varepsilon^2}{\tilde{N}}.
$$

By Lemma 3, we get

$$
P\{\sup_{\alpha\le t\le\alpha+h}|Y(t)-X(t)|\ge\tilde{N}\varepsilon\varphi(t)\ \ i.o.\}=0.\quad a.s.
$$

Then, for $\forall r\in(0,1)$, we have

$$
P\{\sup_{\alpha\le t\le\alpha+h}|Y(t)-X(t)|\le\tilde{N}\varepsilon\varphi(t)\}\ge 1-r.
$$

From Definition 6, (1.1) is almost surely Ulam-Hyers stable. The proof is completed. □

5. Examples

Example 1. Consider an one-dimensional stochastic differential equation

$$
\begin{cases}
D_\rho^\alpha X(t)=b(X(t),t)+\sigma(X(t),t)\dfrac{dW(t)}{dt},\ t\ge\alpha,\ \rho\in(\dfrac{1}{2},1],\\
X(\alpha)=X_\alpha,
\end{cases}
\tag{5.1}
$$

where $X_\alpha\in\mathbb{R}$ and $X_\alpha\ne\pm\infty$, $b:\mathbb{R}\times[\alpha,\infty)\to\mathbb{R},\sigma:\mathbb{R}\times[\alpha,\infty)\to\mathbb{R}$.

Let b,σ satisfying [H1], [H2] and

$$
b(X,\cdot)=a(\cdot)X(\cdot),\quad \sigma(X,\cdot)=b(\cdot)X(\cdot),
\tag{5.2}
$$

in a neighbourhood of $X(\cdot)=0$, where $a(\cdot),b(\cdot)$ are bounded Borel-measurable functions. Assume there is a pair of positive constants λ and K such that

$$-K \leq \int_{\alpha}^{t}\left(a(\tau)+\frac{\varepsilon-1}{2}b^2(\tau)(\tau-\alpha)^{\rho-1}+\lambda\right)d\tau \leq K,\ t\geq\alpha. \tag{5.3}$$

We define the stochastic Lyapunov function

$$V(X,t)=|X(t)|^{\varepsilon}\exp\left[-\varepsilon\int_{\alpha}^{t}(a(\tau)+\frac{\varepsilon-1}{2}b^2(\tau)(\tau-\alpha)^{\rho-1}+\lambda)d\tau\right],\ t\geq\alpha. \tag{5.4}$$

From (5.3), for $t\geq\alpha$, we have

$$|X(t)|^{\varepsilon}e^{-\varepsilon K}\leq V(X,t)\leq|X(t)|^{\varepsilon}e^{\varepsilon K}.$$

Hence, V is positive-define. Combing (5.2) and (5.4), for $t\geq\alpha$, we have

$$\begin{aligned}
&D_{\rho}V(X(t),t)=|X(t)|^{\varepsilon}\exp\left[-\varepsilon\int_{\alpha}^{t}(a(\tau)-\frac{\varepsilon-1}{2}b^2(\tau)(\tau-\alpha)^{\rho-1}+\lambda)d\tau\right]\\
&\cdot-\varepsilon\left(a(t)+\frac{\varepsilon-1}{2}b^2(t)(t-\alpha)^{\rho-1}+\lambda\right)(t-\alpha)^{\rho-1}\\
&+\varepsilon|X(t)|^{\varepsilon-1}\exp\left[-\varepsilon\int_{\alpha}^{t}(a(\tau)+\frac{\varepsilon-1}{2}b^2(\tau)(\tau-\alpha)^{\rho-1}+\lambda)d\tau\right]\\
&\cdot\left[a(t)X(t)\right](t-\alpha)^{\rho-1}\\
&+\frac{1}{2}\left(b^T(t)X(t)\varepsilon(\varepsilon-1)|X(t)|^{\varepsilon-2}X(t)b(t)\right)\\
&\cdot\exp\left[-\varepsilon\int_{\alpha}^{t}(a(\tau)+\frac{\varepsilon-1}{2}b^2(\tau)(\tau-\alpha)^{\rho-1}+\lambda)d\tau\right](t-\alpha)^{2(\rho-1)}\\
&\leq-\varepsilon\lambda e^{-\varepsilon K}|X(t)|^{\varepsilon}(t-\alpha)^{\rho-1}\\
&<0.
\end{aligned}$$

From above $[H4]$ holds. By Theorem 2, the trivial solution of (5.1) is stochastically stable under (5.2) and (5.3).

Example 2. Consider the following conformable stochastic differential equations and give the simulation results of the Ulam's type stable of the solution.

$$\begin{cases} D_{\rho}^{\alpha}X(t)=rX(t)+\beta(t-\alpha)^{1-\rho}X(t)\dfrac{dW(t)}{dt},\ t\in[\alpha,\alpha+h],\\ X(\alpha)=X_{\alpha}\neq\pm\infty. \end{cases} \tag{5.5}$$

The solution of (5.5) can express as

$$X(t)=X_{\alpha}e^{\left(r\frac{(t-\alpha)^{\rho}}{\rho}+\beta W(t)-\frac{1}{2}\beta^2(t-\alpha)\right)},\ \ t\in[\alpha,\alpha+h]. \tag{5.6}$$

Set $X_0 = 10, r = 0.12, \beta = 0.18, \alpha = 0, h = 500$, Brownian motion $W(t)$ and its white noise $\frac{dW(t)}{dt}$ when $t \in [0,500]$. Then (5.6) reduces to

$$X(t) = 10e^{(0.12\frac{t^\rho}{\rho}+0.18W(t)-0.0072t)},\ t \in [0,500].$$

Let $\varphi(t) = \exp(0.5\frac{t^\rho}{\rho}), t \in [0,500]$, we consider

$$\left| D_\rho^\alpha X(t) - rX(t) - \beta t^{1-\rho} X(t)\frac{dW(t)}{dt} \right| \le \varepsilon,\ \ \frac{1}{2} < \rho \le 1,\ t \in [0,500],$$

and

$$\left| D_\rho^\alpha X(t) - rX(t) - \beta t^{1-\rho} X(t)\frac{dW(t)}{dt} \right| \le \varepsilon\varphi(t),\ \ \frac{1}{2} < \rho \le 1,\ t \in [0,500].$$

Clearly, (5.5) is Ulam-Hyers stable and Ulam-Hyers-Rassias stable from Theorems 5 and 6, respectively.

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