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PREFACE

The last paper of Professor Kozák

Volume 10, Number 2 in 2015 was dedicated, among others, to Professor Kozák, a founder of JCAM. Volume 12, Number 2 informed our readers that Professor Kozák died. After his death an article devoted to some stability problems was found in electronic form on his computer. After reviewing the manuscript we decided to publish it in the present issue since it contains some valuable ideas concerning the problem of how to attack a class of stability issues by using the finite element method. We should, however, make the following remarks: (a) the manuscript is not complete since there are no numerical examples involved, (b) the reference list is are very short though the manuscript presents a reasoning for omitting a detailed citation list, (c) the text is very concise, therefore, it is not easy reading. In spite of this we are convinced that it is worth publishing his last manuscript since it might be interesting and motivating for those researchers interested in attacking non-linear stability problems by using numerical methods.

Miskolc, June 21, 2018

László Baranyi, István Páczelt and György Szeidl
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ESTIMATION OF HEAT FLOW IN CIRCULAR BARS OF VARIABLE DIAMETER

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Abstract. Upper and lower bounds for the heat flow in nonhomogeneous circular bars of variable diameter are presented. The thermal properties may depend on the axial and radial coordinates, and the boundary conditions of the considered heat conduction problem do not depend on the polar angle. The analysed steady-state heat conduction problem is axisymmetric. Equations of Fourier's theory are used to formulate the thermal boundary value problem of heat conduction in nonhomogeneous circular bars with nonuniform cross-section. The computation of the heat flow is based on the concept of overall heat transfer coefficient. The derivation of bounding formulae for the overall heat transfer coefficient is based on a minimum principle and Schwarz's inequality. Six examples illustrate the applications of the derived upper and lower bound formulae how one can use to estimate the heat flow in a nonhomogeneous circular bar with nonuniform cross section.

Mathematical Subject Classification: 80A20, 80M30
Keywords: Heat flow, lower and upper bounds, nonhomogeneous, nonuniform circular bar

1. Introduction

The overall heat transfer coefficient in the steady-state heat conduction problem is an important structural property of a solid body in which the heat is flowing between its two separated parts of its boundary surfaces. The exact (strict) value of the overall heat transfer coefficient is known only with bodies of very simple shapes; therefore principles and methods are needed for creating lower and upper bounds to the numerical value of the overall heat transfer coefficient. From the higher temperature boundary part of body to the lower temperature boundary part of the body the process of heat flow is characterized by the overall heat transfer coefficient according to the equation $Q = \Lambda(T_1 - T_2)$, $T_1 > T_2$ where Q is the heat flow in unit time, T_1 and T_2 are given temperature and Λ is the overall heat transfer coefficient. There are several papers which formulate upper and lower bounds for the heat flow in the case of steady-state heat conduction problems. In [1] the author examines the problem of planar heat conduction through an irregularly shaped body found as an inclusion in a perfectly insulating wall between two half-planes maintained at different temperatures. He obtains upper and lower bounds for the heat flow in terms of the temperature difference, conductivity and some global properties of the body.

The presented method is based on the Schwarz's inequality. Paper [2] deals with the problem of determining the temperature distribution for steady-state heat conduction in a long cylindrical pipe. The author gives upper and lower bounds of the heat flow for the case of constant parameters describing the conductivity and density. In [3] a heat conduction problem in hollow three-dimensional body is considered and the author derives some inequality relations by the application of which lower and upper bounds may be obtained for the numerical value of the overall heat transfer coefficient. A linear problem of the steady-state heat conduction is studied in isotropic inhomogeneous hollow rigid bodies in [4]. Applying Schwarz's inequality upper and lower bounds are derived for the overall heat conduction coefficient. Some mean value formula and bounds on the thermal energy for the steady-state heat conduction in anisotropic three-dimensional body are proven in [5]. The upper and lower bounds for the heat flux are derived by the application of Schwarz's inequality, avoiding the application of the minimum principles of potential thermal energy and complementary heat flux energy which were developed by Wojnar [6].

2. Governing equations

Let us consider a bar in the form of body of rotation. In cylindrical coordinates (r,φ,z) the domain under consideration is $z_1 \le z \le z_2$, $0 \le r \le R(z)$, $0 \le \varphi \le 2\pi$ and the axis of the bar is taken as the axis z (Figure 1). This body of rotation occupies the region $\overline{B} = B \cup \partial B$, where the inner points of $\overline{B}$ are denoted by B and the set of boundary points of $\overline{B}$ is denoted by ∂B. ∂B is divided into three parts as $\partial B_1 = A_1$, $\partial B_2 = A_2$ and ∂B_3. It is obvious that $\partial B = \partial B_2 \cup \partial B_2 \cup \partial B_3$.

The boundary surface ∂B_i $(i = 1,2,3)$ is defined as

$$\partial B_3 = \left\{ (r,\varphi,z) \middle| r = R(z),\ z_1 \le z \le z_2,\ 0 \le \varphi \le 2\pi \right\},$$

$$\partial B_i = \left\{ (r,\varphi,z) \middle| z = z_i,\ 0 \le r \le R_i,\ 0 \le \varphi \le 2\pi \right\} \qquad (i = 1,2),$$

$$R_1 = R(z_1), \qquad R_2 = R(z_2).$$

The temperature in the body is denoted by $T = T(r,\varphi,z)$ $(r,\varphi,z) \in \overline{B}$ and $k = k(r,z)$ $(r,\varphi,z) \in \overline{B}$ denotes the thermal conductivity of the material of the nonuniform circular bar. The local heat transfer coefficient at cross section z_i $(i = 1,2)$ is denoted by $h_i = h_i(r,z_i)$ $(r,\varphi,z) \in \partial B_i$ $(i = 1,2)$.

There is no distributed heat source in B and no heat flux across the boundary surface segment ∂B_3. The boundary surface segment ∂B_i is subjected to convective heat exchange and "fluid" temperature T_i $(i = 1,2)$. It is assumed that T_1 and T_2 are constants and $T_1 > T_2$.

With Fourier's theory of heat conduction [7–9] it can be shown that under the conditions prescribed above the temperature field of a nonuniform circular bar can be obtained as

$$T(r,z) = (T_1 - T_2)\theta + T_2, \tag{2.1}$$

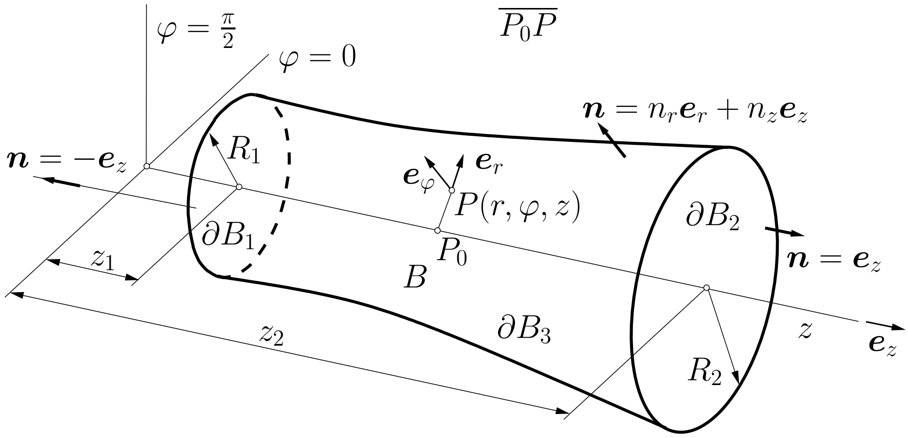

Figure 1. The body of rotation

where the function $\theta = \theta(r, z)$ is the solution to the next boundary value problem (Figure 1)

$$\nabla \cdot (k\nabla\theta) = 0 \qquad \text{in} \quad B, \tag{2.2}$$

$$\boldsymbol{n} \cdot \nabla\theta = 0 \qquad \text{on} \quad \partial B_3, \tag{2.3}$$

$$k\,\boldsymbol{n} \cdot \nabla\theta + h_1(\theta - 1) = 0 \qquad \text{on} \quad \partial B_1, \tag{2.4}$$

$$k\,\boldsymbol{n} \cdot \nabla\theta + h_2\theta = 0 \qquad \text{on} \quad \partial B_2. \tag{2.5}$$

Here, the symbol $\nabla = \frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial}{\partial\varphi}\boldsymbol{e}_\varphi + \frac{\partial}{\partial z}\boldsymbol{e}_z$ is the Hamilton-type vector differential operator in cylindrical coordinate frame, $\boldsymbol{n}$ is the unit outward normal vector on ∂B, dot between two vectors denotes their scalar product and $\boldsymbol{e}_r = \boldsymbol{e}_r(\varphi)$ is the unit vector in radial direction, $\boldsymbol{e}_z$ is the unit vector in axial direction and $\boldsymbol{e}_\varphi(\varphi) = \boldsymbol{e}_z \times \boldsymbol{e}_r(\varphi)$. The cross between two vectors denotes their vectorial product.

We note that the boundary value problem relating to the function $\theta = \theta(r, z)$ is axisymmetric and on the axis of revolution the following boundary condition can be obtained from the symmetry:

$$\frac{\partial\theta}{\partial r} = 0 \qquad r = 0, \quad 0 \le z \le L. \tag{2.6}$$

Heat flows from the cross section A_1 through the circular bar of variable diameter to the cross section A_2. This process is characterized by the equation

$$Q = \Lambda(T_1 - T_2), \qquad T_1 > T_2. \tag{2.7}$$

In equation (2.7) Λ is a constant called the overall heat transfer coefficient and its value depends on the shape and the thermal properties of the nonuniform circular bar, while Q denotes the heat conducted within unit of time through the end sections. Here, we note in the book by Carslaw and Jaeger [8] that the thermal resistance ρ is defined by the equation $\rho\Lambda = 1$. It is evident from [7–9] that

$$Q = \int_{A_1} k\,\boldsymbol{n}\cdot\nabla T \mathrm{d}A = -\int_{A_2} k\,\boldsymbol{n}\cdot\nabla T \mathrm{d}A = (T_1 - T_2)\int_{A_1} k\,\boldsymbol{n}\cdot\nabla\theta \mathrm{d}A =$$
$$(T_2 - T_1)\int_{A_2} k\,\boldsymbol{n}\cdot\nabla\theta \mathrm{d}A. \quad (2.8)$$

Starting from the equation

$$\nabla\cdot(\theta k\nabla\theta) = k\left|\nabla\theta\right|^2 + \theta\nabla\cdot(k\nabla\theta) \quad (2.9)$$

by integration and by the application of the Gaussian theorem of integral transform and equations (2.2), (2.3), (2.4), (2.5) we obtain

$$0 = \int_B \nabla\cdot(k\theta\nabla\theta)\mathrm{d}B - \int_B k\left|\nabla\theta\right|^2 \mathrm{d}B = \int_{\partial B} \theta k\boldsymbol{n}\cdot\nabla\theta \mathrm{d}\partial B - \int_B k\left|\nabla\theta\right|^2 \mathrm{d}B =$$
$$\int_{A_1} \theta k\boldsymbol{n}\cdot\nabla\theta \mathrm{d}A + \int_{A_2} \theta k\boldsymbol{n}\cdot\nabla\theta \mathrm{d}A - \int_B k\left|\nabla\theta\right|^2 \mathrm{d}B =$$
$$-\left\{\int_B k\left|\nabla\theta\right|^2 \mathrm{d}B + \int_{A_1}\frac{k^2}{h_1}(\boldsymbol{n}\cdot\nabla\theta)^2\mathrm{d}A + \int_{A_2}\frac{k^2}{h_2}(\boldsymbol{n}\cdot\nabla\theta)^2\mathrm{d}A\right\} + \int_{A_1} k\boldsymbol{n}\cdot\nabla\theta \mathrm{d}A. \quad (2.10)$$

The combination of formulae (2.7) and (2.8) with equation (2.10) yields

$$\Lambda = \int_B k\left|\nabla\theta\right|^2 \mathrm{d}B + \sum_{i=1}^{2}\int_{A_i}\frac{k^2}{h_i}(\boldsymbol{n}\cdot\nabla\theta)^2\mathrm{d}A. \quad (2.11)$$

It follows from equation (2.11) that $\Lambda > 0$.

The primary purpose of this paper is to derive such inequality relations using which lower and upper bounds may be found for Λ. The exact value of the overall heat transfer coefficient Λ may be given only with the knowledge of the solution to the boundary value problem defined by equations (2.2), (2.3), (2.4) and (2.5). The solution of the explicit form of the boundary value problem formulated in equations (2.2), (2.3), (2.4) and (2.5) is known only for bodies B of very simple shapes [7–9], therefore identifying principles and methods for producing lower and upper bounds to the numerical value of Λ is a topic of great significance. On the other hand, some of the bounding formulae of Λ may be the theoretical framework for the different types of finite element formulation of the heat conduction problem described by the equations (2.2), (2.3), (2.4) and (2.5).

3. Upper bound

We introduce the symbol $E[\phi]$ by the definition

$$E[\phi] = \int_B k\left|\nabla\phi\right|^2 \mathrm{d}B + \int_{A_1} h_1(\phi - 1)^2\mathrm{d}A + \int_{A_2} h_2\phi^2\mathrm{d}A, \quad (3.1)$$

where $\phi = \phi(r, \varphi, z)$ is a function for which the integrals appear in (3.1) exist and where they have finite values.

Theorem 1. Let $F = F\,(r, z)$ be continuous in the domain $\overline{M}$ and in the domain M at least once continuously differentiable, otherwise an arbitrary function of r and z. The inequality relation

$$\Lambda \leq E[F] \tag{3.2}$$

is valid.

The domain $\overline{M} = M \cup \partial M$ is the meridian section of the body of rotation $\overline{B}$. This means that $\overline{M} = \{(r,z)\,|\,0 \leq r \leq R(z),\ z_1 \leq z \leq z_2\}$ and $\partial M = \partial M_1 \cup \partial M_2 \cup \partial M_3 \cup \partial M_4$, where $\partial M_1 = \{(r,z)\,|\,z = z_1, 0 \leq r \leq R_1\}$, $\partial M_2 = \{(r,z)\,|\,z = z_2, 0 \leq r \leq R_2\}$, $\partial M_3 = \{(r,z)\,|\,r = R(z), z_1 \leq z \leq z_2\}$ and $\partial M_4 = \{(r,z)\,|\,r = 0, z_1 \leq z \leq z_2\}$.

Proof. Consider the function

$$\eta(r, z) = \theta(r, z) - F(r, z). \tag{3.3}$$

Using the expressions of $E[F]$ and $E[\phi]$ we obtain

$$E[F] = E[\theta] + \int\limits_B k|\nabla\eta|^2 \mathrm{d}B + \sum_{i=1}^{2} \int\limits_{A_i} h_i \eta^2 \mathrm{d}A + 2\left\{ \int\limits_B k\nabla\theta \cdot \nabla\eta \mathrm{d}B + \int\limits_{A_1} h_1(\theta - 1)\eta \mathrm{d}A + \int\limits_{A_2} h_2\theta\eta \mathrm{d}A \right\}. \tag{3.4}$$

By lengthy, but elementary calculations which involve the application of the derivation of product function and Gaussian theorem of integral transformation, the following relationship may be deduced

$$\begin{aligned}
&\int\limits_B k\nabla\theta \cdot \nabla\eta \mathrm{d}B + \int\limits_{A_1} h_1(\theta - 1)\eta \mathrm{d}A + \int\limits_{A_2} h_2\theta\eta \mathrm{d}A = \\
&\int\limits_{\partial B} k\eta\boldsymbol{n} \cdot \nabla\theta \mathrm{d}\partial B - \int\limits_B \eta\nabla \cdot (k\nabla\theta) \mathrm{d}B + \int\limits_{A_1} h(\theta - 1) \mathrm{d}A + \int\limits_{A_2} h_2\theta\eta \mathrm{d}A = \\
&\int\limits_{A_1} \eta\left[k\boldsymbol{n} \cdot \nabla\theta + h_1(\theta - 1)\right] \mathrm{d}A + \int\limits_{A_2} \eta\left[k\boldsymbol{n} \cdot \nabla\theta + h_2\theta\right] \mathrm{d}A = 0.
\end{aligned} \tag{3.5}$$

The combination of equation (3.4) with equation (3.5) leads to inequality relation (3.2). From the demonstration it follows that equality in (3.2) can be reached only if $F = \theta$.

4. Lower bound

Theorem 2. In $\overline{B}$ the continuous vector field $\boldsymbol{b} = \boldsymbol{b}(r, \varphi, z)$ differing from the identically zero vector should satisfy the differential equation

$$\nabla \cdot \boldsymbol{b} = 0 \qquad \text{in} \quad B \tag{4.1}$$

and the boundary condition

$$\boldsymbol{n}\cdot\boldsymbol{b}=0 \qquad \text{on} \quad \partial B_3. \tag{4.2}$$

The following inequality relation is valid

$$\Lambda \geq \frac{\left(\int\limits_{A_1} \boldsymbol{b}\cdot\boldsymbol{n}\mathrm{d}A\right)^2}{\int\limits_{B}\frac{\boldsymbol{b}^2}{k}\mathrm{d}B+\int\limits_{A_1}\frac{(\boldsymbol{b}\cdot\boldsymbol{n})^2}{h_1}\mathrm{d}A+\int\limits_{A_2}\frac{(\boldsymbol{b}\cdot\boldsymbol{n})^2}{h_2}\mathrm{d}A}. \tag{4.3}$$

Proof. Let us have

$$D(\boldsymbol{e},\boldsymbol{f})=\int\limits_{B}\frac{\boldsymbol{e}\cdot\boldsymbol{f}}{k}\mathrm{d}B+\sum_{i=1}^{2}\int\limits_{A_i}\frac{(\boldsymbol{e}\cdot\boldsymbol{n})(\boldsymbol{f}\cdot\boldsymbol{n})}{h_i}\mathrm{d}A, \tag{4.4}$$

where $\boldsymbol{e}=\boldsymbol{e}(r,\varphi,z)$ and $\boldsymbol{f}=\boldsymbol{f}(r,\varphi,z)$ defined in B are two arbitrary continuous vector fields. On the basis of the Schwarz inequality it may be written that

$$D(k\nabla\theta,k\nabla\theta)\,D(\boldsymbol{b},\boldsymbol{b})\geq\left(D(k\nabla\theta,\boldsymbol{b})\right)^2. \tag{4.5}$$

It can easily be understood that

$$\Lambda=D(k\nabla\theta,k\nabla\theta). \tag{4.6}$$

The relationship

$$\begin{aligned}
D(k\nabla\theta,\boldsymbol{b})&=\int\limits_{B}\nabla\theta\cdot\boldsymbol{b}\mathrm{d}B+\int\limits_{A_1}\frac{k}{h_1}(\boldsymbol{n}\cdot\nabla\theta)(\boldsymbol{n}\cdot\boldsymbol{b})\mathrm{d}A+\\
&\int\limits_{A_2}\frac{k}{h_2}(\boldsymbol{n}\cdot\nabla\theta)(\boldsymbol{n}\cdot\boldsymbol{b})\mathrm{d}A=\int\limits_{\partial B}\theta\boldsymbol{n}\cdot\boldsymbol{b}\mathrm{d}\partial B+\int\limits_{A_1}\frac{k}{h_1}(\boldsymbol{n}\cdot\nabla\theta)(\boldsymbol{n}\cdot\boldsymbol{b})\mathrm{d}A+\\
&\int\limits_{A_2}\frac{k}{h_2}(\boldsymbol{n}\cdot\nabla\theta)(\boldsymbol{n}\cdot\boldsymbol{b})\mathrm{d}A-\int\limits_{B}\theta\nabla\cdot\boldsymbol{b}\mathrm{d}B=\int\limits_{A_1}\boldsymbol{n}\cdot\boldsymbol{b}\left(\theta+\frac{k}{h_1}\boldsymbol{n}\cdot\nabla\theta\right)\mathrm{d}A+\\
&\int\limits_{A_2}\boldsymbol{n}\cdot\boldsymbol{b}\left(\theta+\frac{k}{h_2}\boldsymbol{n}\cdot\nabla\theta\right)\mathrm{d}A=\int\limits_{A_1}\boldsymbol{n}\cdot\boldsymbol{b}\mathrm{d}A
\end{aligned} \tag{4.7}$$

further, inequality (4.5) and formula (4.6) by their combination directly yield the lower bound formula (4.3) to be proven. In deriving the relationship (4.7) the rule of differentiation of the product function as well as the Gaussian integration theorem, equations (2.4), (2.5), (4.1) and (4.2) have been applied.

By some discussion it may be pointed out that in relation (4.3) the sign of equality is valid only in the case when

$$\boldsymbol{b}=\alpha k\nabla\theta, \tag{4.8}$$

where α differs from zero otherwise being an arbitrary real constant.

5. Examples

5.1. Example for upper bound. We assume

$$F(r,z) = C_1 \int_{z_1}^{z} \frac{\mathrm{d}\zeta}{K(\zeta)} + C_2, \tag{5.1}$$

where

$$C_1 = \frac{1}{I + \frac{1}{H_1} + \frac{1}{H_2}}, \qquad C_2 = \frac{1 + \frac{1}{H_2}}{I + \frac{1}{H_1} + \frac{1}{H_2}}, \tag{5.2}$$

$$K(z) = \int_0^{R(z)} rk(r,z)\,\mathrm{d}r, \quad I = \int_{z_1}^{z_2} \frac{\mathrm{d}z}{K(z)}, \quad H_i = \int_0^{R_i} rh_i(r)\mathrm{d}r \qquad (i = 1,2). \tag{5.3}$$

Inserting the function given by formula (5.1) into inequality relation (3.2) we obtain

$$\Lambda \leq \Lambda_U = 2\pi C_1 = \frac{2\pi}{I + \frac{1}{H_1} + \frac{1}{H_2}}. \tag{5.4}$$

5.2. Example for lower bound. In order to get the lower bound for Λ, we use in (4.3) the divergence free vector field

$$\boldsymbol{b} = \frac{1}{R^2}\left[\frac{r}{R}\frac{\mathrm{d}R}{\mathrm{d}z}\boldsymbol{e}_r(\varphi) + \boldsymbol{e}_z\right]. \tag{5.5}$$

This vector field satisfies boundary condition (4.2) and the condition

$$\boldsymbol{b}\cdot\boldsymbol{e}_r = 0 \qquad r = 0, \qquad z_1 \leq z \leq z_2. \tag{5.6}$$

We introduce the following function and constants

$$M_1(z) = \int_0^{R(z)} \frac{r^3}{k(r,z)}\mathrm{d}r, \qquad M_2(z) = \int_0^{R(z)} \frac{r}{k(r,z)}\mathrm{d}r, \tag{5.7}$$

$$N_1 = \int_{z_1}^{z_2} \frac{M_1}{(R(z))^6}\left(\frac{\mathrm{d}R}{\mathrm{d}z}\right)^2 \mathrm{d}z, \qquad N_2 = \int_{z_1}^{z_2} \frac{M_2}{(R(z))^4}\mathrm{d}z, \quad N = N_1 + N_2, \tag{5.8}$$

$$\frac{1}{S_i} = \frac{1}{R_i^4}\int_0^{R_i} \frac{r\mathrm{d}r}{h_i(r)} \qquad (i = 1,2). \tag{5.9}$$

Putting the vector field given by the formula (5.7) into inequality relation (4.3), we get

$$\Lambda \geq \Lambda_L = \frac{\pi}{2(N + \frac{1}{S_1} + \frac{1}{S_2})}. \tag{5.10}$$

5.3. **Example for circular bar with uniform cross section.** Let us apply formulae (5.4) and (5.10) to the circular cylindrical bar. We assume that the thermal conductivity depends only on the axial coordinate z and h_1, h_2 are constants. In this case the upper and lower bounds formulated in (5.4) and (5.10) give the same result, which is the exact value of Λ. The computations yield the following value of Λ:

$$\Lambda = \frac{c^2\pi}{\int\limits_0^L \frac{\mathrm{d}z}{k(z)} + \frac{1}{h_1} + \frac{1}{h_2}} \qquad z_1 = 0, \quad z_2 = L. \tag{5.11}$$

In equation (5.11) the constant c is the radius of the considered circular bar, that is $R(z) = c$, $0 \leq z \leq L$.

5.4. **Example for homogeneous circular cone.** In this section, we deal with the homogeneous conical bars. Setting $R(z) = a + bz$, where a and b are constants and $z_1 = 0$, $z_2 = L$. We find, from (5.4) and (5.10)

$$\Lambda_U = \frac{\pi}{\frac{L}{ka(a+bL)} + \frac{1}{h_1 a^2} + \frac{1}{h_2(a+bL)^2}}, \quad \Lambda_L = \frac{\pi}{\frac{(1+b^2/2)L}{ka(a+bL)} + \frac{1}{h_1 a^2} + \frac{1}{h_2(a+bL)^2}}. \tag{5.12}$$

It is assumed in equation (5.12) that k, h_1 and h_2 are constants.

In the case $h_i \to \infty$ at the end cross section A_i, the Robin type boundary condition will be replaced by the Dirichlet type boundary conditionmeaning that the cross section A_i is subjected to constant temperature T_i $(i = 1, 2)$.

Putting in formula (5.12) h_1, $h_2 \to \infty$ we obtain

$$\frac{\Lambda_U}{\Lambda_L} = 1 + \frac{b^2}{2}. \tag{5.13}$$

which shows that there is a significant difference between Λ_U and Λ_L for sufficiently large values of b.

Upper and lower bounds for Λ may be improved by means of the Rayleigh-Ritz method [10] and finite element method [11], which are based on minimizing (3.2) with respect to $F = F(r, z)$ and maximizing (4.3) with respect to $\boldsymbol{b} = \boldsymbol{b}(r, \varphi, z)$.

5.5. **Example for nonhomogeneous circular cylindrical bar of uniform cross section.** Let c be the radius of the boundary circle of the considered bar. The material properties are functions of the radial coordinate r. It is assumed that

$$k(r) = k_0 r, \quad h_i(r) = h_{0i} r, \quad (i = 1, 2), z_1 = 0, z_2 = L. \tag{5.14}$$

Let Λ_0 be defined as

$$\Lambda_0 = \frac{c^3\pi}{\frac{L}{k_0} + \frac{1}{h_{01}} + \frac{1}{h_{02}}}. \tag{5.15}$$

From the bounding formulae (5.4) and (5.10) the following result can be derived

$$\lambda_L = \frac{\Lambda_L}{\Lambda_0} = \frac{1}{2} \leq \lambda = \frac{\Lambda}{\Lambda_0} \leq \lambda_U = \frac{\Lambda_U}{\Lambda_0} = \frac{2}{3}. \tag{5.16}$$

Denote the mean value of λ_L and λ_U $\overline{\lambda} = 0.5(\lambda_U + \lambda_L)$. It is evident that

$$\left|\lambda - \overline{\lambda}\right| \leq \frac{1}{12}. \tag{5.17}$$

5.6. **Example for functionally graded circular cone.** The points of the meridian section of a circular cone are given by the prescription

$$\overline{M} = \left\{(r,z)\Big|\, 0 \leq r \leq az,\, z_1 \leq z \leq z_2\right\} \text{ and } \partial M = \partial M_1 \cup \partial M_2 \cup \partial M_3 \cup \partial M_4,$$

$$\partial M_1 = \left\{(r,z)\Big|\, z = z_1,\, 0 \leq r \leq az_1\right\}, \quad \partial M_2 = \left\{(r,z)\Big|\, z = z_2,\, 0 \leq r \leq az_2\right\},$$

$$\partial M_3 = \left\{(r,z)\Big|\, r = az,\, z_1 \leq z \leq z_2\right\}, \quad \partial M_4 = \left\{(r,z)\Big|\, r = 0,\, z_1 \leq z \leq z_2\right\},$$

The thermal properties are given functions of the radial coordinate according to equations

$$k(r) = k_0 \exp(\nu r),\; h_1(r) = h_2(r) = h_0 \exp(\nu r), \tag{5.18}$$

where k_0, h_0 and ν are material parameters. In the numerical example the following data are used: $a = 0.5$, $z_1 = 0.8\,\mathrm{m}$, $z_2 = 3\,\mathrm{m}$, $k_0 = 100\,\frac{\mathrm{W}}{\mathrm{mK}}$, $h_0 = 20\,\frac{\mathrm{W}}{\mathrm{m^2K}}$, $\nu = 0.5\,\frac{1}{\mathrm{m}}$.

Substitution of this data into equations (5.3), (5.4) and equations (5.7-5.10) gives

$$I = 0.05769894569\,\frac{\mathrm{K}}{\mathrm{W}},\; H_1 = 1.830223478\,\frac{\mathrm{W}}{\mathrm{K}},\; H_2 = 37.65999967\,\frac{\mathrm{W}}{\mathrm{K}},$$

$$N_1 = 0.001731450850\,\frac{\mathrm{K}}{\mathrm{W}},\; N_2 = 0.0145366169\,\frac{\mathrm{K}}{\mathrm{W}},\; N = 0.01626806776\,\frac{\mathrm{K}}{\mathrm{W}},$$

$$\frac{1}{S_1} = 0.1368991899\,\frac{\mathrm{K}}{\mathrm{W}},\; \frac{1}{S_2} = 0.006848732156\,\frac{\mathrm{K}}{\mathrm{W}},$$

$$\Lambda_U = 9.96328894\,\frac{\mathrm{W}}{\mathrm{K}},\; \Lambda_L = 9.816496019\,\frac{\mathrm{W}}{\mathrm{K}}.$$

If we approximate Λ, the mean value of Λ_U and Λ_L then the relative error is less than 0.7421 %.

6. Conclusions

Upper and lower bounds for the heat flux in nonhomogeneous circular bars of variable diameter are presented. Thermal properties may depend on the radial and axial coordinates. The axisymmetric nonhomogenity considered also includes those cases, when the bar is a composite of different homogeneous materials, so that the thermal conductivity and surface conductivity are piecewise constants. The discontinuities of the thermal properties should not affect the presented analysis. Here we note that for a compound bar the function $F = F(r,z)$ is continuous on the whole meridian section and its normal derivative computed on the curves which separate the different parts of meridian section may have jumps. Normal component of $\boldsymbol{b}$ remains continuous and the tangential component of $\boldsymbol{b}$ may have jumps across the common boundary curves of different phases. Equations of Fourier's theory of steady-state heat conduction are used to formulate the field equations and boundary conditions of the heat transfer problem analyzed. Examples illustrate the applications of the bounding formulae derived. The Rayleigh-Ritz method and finite element formulation give possibilities to improve the presented estimation of heat flux in circular bars of variable diameter.

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SENSITIVITY INVESTIGATION OF EQUILIBRIUM STATES AND THE DETERMINATION OF THE FOLD LINE OF LIMIT POINTS FOR AN ELASTIC BODY

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Abstract. The paper presents the stability theory of a 3D elastic body by setting up those equations which govern the problem in the frameworks of the finite element method. In this respect it appears partly as a review paper. Special emphasis is laid on the issue, of how to determine the fold line of the limit points.

Mathematical Subject Classification: 74G60, 74B15
Keywords: Finite deformation, follower loads, imperfections, critical points, fold line

1. Introductory remarks

A number of books have been published on the problem of stability. The first one [1] appeared in 1936. The fast development of the solution methods and deeper clarification of the theoretical background began in the sixties of the last century. In this respect a pioneering work was book [2] by Bolotin in 1964. As regards the theoretical background and the correct mathematical formulation of the stability problem it is worth referring to books [3–8] and [9]. Various numerical methods can be used to find the equilibrium path and the characteristic points (a limit point for example) on it. One of them is the finite element method (FEM) – see for instance [10] or as an example for the application of FEM the papers [11, 12]. Though we have just cited some books it is also inevitable here to refer to Subsection **68bis.** in Trusdell and Noll [13]: 'There is a vast literature on elastic stability and to little purpose. Most of it rests on upon improper, or at best unduly special formulation of the principles of elasticity. Whole volumes have been devoted to presenting ostensible solutions to particular problems by means of criteria that are never even clearly stated and morass of equations spouted forth on the subject can be regarded as little else then rhetoric. Stability theory is, necessarily, an application of some theory of finite deformations, such as elasticity, but most of the specialists of stability theory show no evidence of having troubled to learn the theory they claim to be applying.'

[1] The author died in 2016. The present paper is his last work which he could not totally complete.

In the present paper a stability investigation of an elastic body is carried out the basis of a continuum mechanical model in which the problem is considered as a 3D one by using a static procedure applying the nonlinear theory of deformation. It is assumed that the constitutive relation between the Green-Lagrange deformation tensor and the second Piola-Kirchoff stress tensor is a linear one. The load the body is subjected to is a surface load which is always perpendicular to the deformed surface and is, therefore, in general non-conservative.

A difference is made between the stress and deformation free state of the body (initial configuration) and the instantaneous equilibrated states (current configurations).

The perfect state of the body and the deviations from the perfect state are also treated. Deviations might be caused by load imperfections and/or geometrical imperfections. The imperfections in the problems considered are described by imperfection parameters.

The problems outlined in the title of the present paper include the following problems:

– finding the equilibrium states (configurations) and depicting the equilibrium paths and surfaces using the parameters of the problem considered;
– calculation of the work done by the total force system of the equilibrium state on the equilibrium path and setting up stability criteria;
– qualifying the critical points (bifurcation and limit points) on the equilibrium paths;
– determination of the fold line by a total stability analysis (joining the limit points of the equilibrium paths) or directly (departing from a limit point state).

In the present paper the equations of the 3D continuum mechanical model are affiliated with those of the finite element model.

Arbitrary curvilinear coordinate systems are assumed for the continuum mechanical models. In direct notation the (vectors) [tensors] are denoted by (boldface) [italic boldface] letters. The components of such quantities are denoted by the same letter in lightface with indices identifying the component considered. The indices have the range 1,2,3 and the subscripts preceded by a semicolon denote covariant derivations with respect to the coordinates that belong to the subscripts.

For the finite element models the matrices are denoted by boldface letters and the number of indices shows the size of the matrix. The indices have the range $1, 2, \ldots, m$ where m is the degree of freedom. Transposition is denoted by $()^{\mathrm{T}}$.

For dummy indices the summation convention should be applied both in the equations of the continuum model and in the equations of the finite element model.

2. Assumptions and notational conventions

The initial configuration and the instantaneous equilibrium configurations (current configurations) are designated by (B) and $\big(\bar{B}\big)$ or $\big(\bar{B} + \Delta\bar{B}\big)$.

Quantities in the current configuration are denoted by barred letters. However there is no a distinguishing symbol (bar) for the same quantities in the initial configuration.

Displacement and stress boundary conditions are assumed. The displacement boundary conditions are independent of the problem parameters and the deformations of the body.

Explicating the train of thought is performed in the Lagrangian description, i.e., the tensor fields that describe the physical state of the body are considered in state (B) of the body. In this configuration $\left(x^1, x^2, x^3\right)$ is an arbitrary curvilinear coordinate system for which $\mathbf{g}_k$ and $\mathbf{g}^l$ are the base vectors, g_{kl} is the metric tensor, $g = \det g_{kl}$ and δ_k^l is the Kronecker delta.

In configuration (B) of the body:

$$\mathbf{u} = u^k \mathbf{g}_k, \qquad \Delta u^k, \qquad \delta u^k = \delta\left(\Delta u^k\right) \tag{2.1}$$

are the displacement vector, its increment and the virtual displacement. The Green-Lagrange strain tensor and its increment are given by

$$E_{kl} = \frac{1}{2}\left(u_{k;l} + u_{l;k} + u^m{}_{;k} u_{m;l}\right), \tag{2.2}$$

$$\Delta E_{kl} = \Delta E_{kl}^{(1)} + \Delta E_{kl}^{(2)} \tag{2.3}$$

where the number in parentheses shows what the power of the derivatives is in the quantities considered – see for instance relations (2.6) and (2.7).

The constitutive equations in configuration (B) of the body, which is assumed to be homogeneous, relate the second Piola-Kirchoff stress tensor (called simply stress tensor) to the Green-Lagrange strain tensor

$$S^{pq} = E_{kl} C^{klpq} = C^{pqkl} E_{kl}\,. \tag{2.4}$$

where C^{klpq} is the tensor of material constants (stiffness tensor).

The increment of the stress tensor is given by

$$\Delta S^{pq} = \Delta E_{kl} C^{klpq} = \Delta E_{kl}^{(1)} C^{klpq} + \Delta E_{kl}^{(2)} C^{klpq} \tag{2.5}$$

in which according to equation (2.2)

$$\Delta E_{kl}^{(1)} C^{klpq} = \Delta u_{m;k}\left(\delta_l^m + u^m{}_{;l}\right) C^{klpq}, \tag{2.6}$$

$$\Delta E_{kl}^{(2)} C^{klpq} = \frac{1}{2} \Delta u_{m;k} \Delta u^m{}_{;l} C^{klpq}\,. \tag{2.7}$$

Let

$$\mathrm{d}\bar{\mathbf{A}} = \mathrm{d}\bar{A}_{\bar{s}}\, \bar{\mathbf{g}}^s = \mathrm{d}\bar{A}_k \mathbf{g}^k \tag{2.8}$$

be the surface element vector in the current configuration $\left(\bar{B}\right)$ where $\mathrm{d}\bar{A}_k$ denotes the components of the vector $\mathrm{d}\bar{\mathbf{A}}$ in the local basis $\mathbf{g}^k$, i.e., in the initial configuration. Further let

$$\mathrm{d}\mathbf{A} = \mathrm{d}A_p \mathbf{g}^p \tag{2.9}$$

be the surface element vector in the initial configuration. As is well known

$$\mathrm{d}\bar{A}_k = Q_k{}^p \mathrm{d}A_p \tag{2.10}$$

where

$$Q_k{}^p = \frac{1}{2} e_{klm} e^{pqr} \left(\delta_q^l + u^l{}_{;q}\right)\left(\delta_r^m + u^m{}_{;r}\right) \tag{2.11}$$

while e_{klm} and e^{pqr} are the permutation symbols.

The increment of the surface element vector of configuration $(\bar{B})$ in the local coordinate system of the initial configuration (in state (B)) is given by:

$$\Delta\left(\mathrm{d}\bar{A}_k\right) = \Delta Q_k{}^p \mathrm{d}A_p = \Delta Q^{(1)}{}_k{}^p \mathrm{d}A_p + \Delta Q^{(2)}{}_k{}^p \mathrm{d}A_p\,, \tag{2.12}$$

where

$$\Delta Q^{(1)}{}_k{}^p = e_{klm}e^{pqr}\left(\delta^l_q + u^l{}_{;q}\right)\Delta u^m{}_{;r}, \quad \Delta Q^{(2)}{}_k{}^p = \frac{1}{2}e_{klm}e^{pqr}\Delta u^l{}_{;q}\Delta u^m{}_{;r}\,. \tag{2.13}$$

The position vector is denoted by $\mathbf{r}$.

Parts $\bar{A}_{\mathrm{tI}}$ and $\bar{A}_{\mathrm{tII}}$ ($\bar{A}_{\mathrm{tI}} \cap \bar{A}_{\mathrm{tII}} = 0$) of the surface $\bar{A}$ are subjected to distributed loads which are perpendicular to the surface:

$$\widetilde{p}_{\mathrm{I}} = p_{\mathrm{I}}\,\widetilde{p}_{\mathrm{I0}}, \quad \bar{\mathbf{r}} \in \bar{A}_{\mathrm{tI}}; \qquad \widetilde{p}_{\mathrm{II}} = p_{\mathrm{II}}\,\widetilde{p}_{\mathrm{II0}}, \quad \bar{\mathbf{r}} \in \bar{A}_{\mathrm{tII}}\,. \tag{2.14}$$

Here p_{I} are p_{II} independent load parameters whereas $\widetilde{p}_{\mathrm{I0}}$ and $\widetilde{p}_{\mathrm{II0}}$ are reference loads. The corresponding elementary forces are given by

$$\mathrm{d}\bar{\mathbf{F}}_{\mathrm{N}} = \widetilde{p}_{\mathrm{N}}\,\mathrm{d}\bar{\mathbf{A}}, \qquad \mathrm{d}\bar{F}_{\mathrm{N}k} = p_{\mathrm{N}}\,\widetilde{p}_{\mathrm{N0}}\,\mathrm{d}\bar{A}_k = p_{\mathrm{N}}\,\widetilde{p}_{\mathrm{N0}}\,Q_k{}^p\,\mathrm{d}A_p, \qquad \mathrm{N} = \mathrm{I}, \mathrm{II} \tag{2.15}$$

where $\mathrm{d}\bar{F}_{\mathrm{N}k}$ stands for the components of the elementary $\mathrm{d}\bar{\mathbf{F}}_{\mathrm{N}}$ in configuration (B).

The loaded surface parts $\bar{A}_{\mathrm{tI}}$ and $\bar{A}_{\mathrm{tII}}$ may depend on the geometrical imperfection parameters.

3. Fundamental equations

3.1. **Introductory remarks.** With regard to all that has been said in Section 2 we may conclude that the state of the elastic body for given stiffness tensor C^{klpq}, reference loads $\widetilde{p}_{\mathrm{I0}}$, $\widetilde{p}_{\mathrm{II0}}$ and displacement boundary conditions is determined by the displacement field $u_k\left(x^1, x^2, x^3\right)$, the loading parameters $p_{\mathrm{I}}, p_{\mathrm{II}}$ and the geometrical imperfection parameter h. p_{I} is the parameter of the fundamental load, whereas p_{II} is the parameter of the supplementary load which depends on what nature the problem considered has, i.e., it may also be the parameter of the disturbing load (load imperfection parameter). p_{I}, p_{II} and h are referred to as problem parameters. The imperfection parameters fall into the following two categories: load imperfection parameter p_{II} and geometrical imperfection parameter h. The displacement field u_k depends naturally on the problem parameters and can, therefore, be expressed in the form $u_k\left(x^1, x^2, x^3; p_{\mathrm{I}}, p_{\mathrm{II}}, h\right)$

3.2. **Equilibrium conditions.**

3.2.1. *Principles of virtual power and virtual work.* The current configuration $(\bar{B})$ is an equilibrium state of the body under given p_I, p_{II} and h problem parameters if the principle of virtual power

$$\int_{(B)} \delta\dot{E}_{kl}C^{klpq}E_{pq}\mathrm{d}V = \sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}} \int_{(A_{\mathrm{tN}})} \widetilde{p}_{\mathrm{N0}}\delta\dot{u}^k Q_k{}^p \mathrm{d}A_p \tag{3.1}$$

regarded in the initial configuration holds for any virtual velocity field $\delta\dot{u}^k$.

Configuration $(\bar{B})$ is also an equilibrium state of the body if, equivalently to (3.1), the principle of virtual work

$$\int_{(B)} \delta\Delta E^{(1)}_{kl} C^{klpq} E_{pq} \mathrm{d}V = \sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}} \int_{(A_{\mathrm{tN}})} \widetilde{p}_{\mathrm{N0}} \delta u^k Q_k{}^p \mathrm{d}A_p \tag{3.2}$$

is fulfilled for any virtual displacement field δu^k.

According to equations (2.3), (2.6), (2.7) and $(2.1)_3$

$$\delta\dot{E}_{kl} C^{klpq} = \delta\dot{u}_{m;k} \left(\delta^m_l + u^m{}_{;l}\right) C^{klpq}, \tag{3.3}$$

$$\delta\Delta E^{(1)}_{kl} C^{klpq} = \delta u_{m;k} \left(\delta^m_l + u^m{}_{;l}\right) C^{klpq}, \qquad \delta\Delta E^{(2)}_{kl} C^{klpq} = \delta u_{m;k} \Delta u^m{}_{;l} C^{klpq} \,. \tag{3.4}$$

A comparison of (3.3) and $(3.4)_1$ shows that (3.1) and (3.2) are really equivalent to each other.

3.2.2. *The Newton-Raphson iteration.* The equilibrium conditions of the current configuration $(\bar{B})$, which is determined by the problem parameters $p_{\mathrm{I}}, p_{\mathrm{II}}, h$, can also be expressed with respect to an arbitrary but appropriately chosen initial state $(B'_{\underline{1}})$ of the body by using the principle of virtual work. Configuration $(B'_{\underline{1}})$ is not necessarily equilibrated and the description is again Lagrangian, which means that the principle of virtual work is regarded in the state B of the body.

If $u_{\underline{1}k}$ is the displacement field of the configuration $\left(B'_{\underline{1}}\right)$ the displacement field of state $\left(\bar{B}\right)$ of the body is of the form $u_k = u_{\underline{1}k} + \Delta u_k$ and the virtual work principle can be given as

$$\int_{(B)} \delta\Delta E_{kl} C^{klpq} \left(E_{pq} + \Delta E_{pq}\right) \mathrm{d}V = \sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}} \int_{(A_{\mathrm{tN}})} \widetilde{p}_{\mathrm{N0}} \delta u^k \left(Q_k{}^p + \Delta Q_k{}^p\right) \mathrm{d}A_p. \tag{3.5}$$

or by detailing the increments as

$$\int_{(B)} \left(\delta\Delta E^{(1)}_{kl} + \delta\Delta E^{(2)}_{kl}\right) C^{klpq} \left(E_{pq} + \Delta E^{(1)}_{pq} + \Delta E^{(2)}_{pq}\right) \mathrm{d}V =$$

$$= \sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}} \int_{(A_{\mathrm{tN}})} \widetilde{p}_{\mathrm{N0}} \delta u^k \left(Q_k{}^p + \Delta Q^{(1)}{}_k{}^p + \Delta Q^{(2)}{}_k{}^p\right) \mathrm{d}A_p \,. \tag{3.6}$$

Condition (3.6) is a non-linear equation for the displacement increment Δu_k as the unknown. The solution can be found by applying the Newton-Raphson iteration algorithm. According to this the displacement field u_k of the configuration $\left(\bar{B}\right)$ can be approximated by the displacement increment series

$$u_k + \Delta u_k = u_{\underline{1}k} + \Delta u_{\underline{1}k} + \Delta u_{\underline{2}k} + \cdots + \Delta u_{\underline{s}k} + \cdots$$

in which $\Delta u_{\underline{s}k}$, $(\underline{s} = \underline{1}, \underline{2}, \cdots)$ is the displacement increment that belongs to the $\underline{s}^{\mathrm{th}}$ intermediate state of the body in the series $(B'_{\underline{1}}), (B'_{\underline{2}}), \cdots, (B'_{\underline{s}}), \cdots$.

By dropping the terms

$$\int_{(B)} \left(\delta\Delta E^{(2)}_{\underline{s}kl} C^{klpq} \Delta E^{(1)}_{\underline{s}pq} + \delta\Delta E^{(2)}_{\underline{s}kl} C^{klpq} \Delta E^{(2)}_{\underline{s}pq}\right) \mathrm{d}V\,,$$

$$\sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}} \int_{(A_{\mathrm{tN}})} \widetilde{p}_{\mathrm{N}0} \delta u^k \Delta Q^{(2)}{}_{\underline{s}k}{}^{p} \mathrm{d}A_p$$

non-linear in (3.6) one can obtain a linearized equation for the unknown $\Delta u_{\underline{s}k}$:

$$\int_{(B)} \left(\delta\Delta E^{(1)}_{\underline{s}kl} C^{klpq} \Delta E^{(1)}_{\underline{s}pq} + \delta\Delta E^{(2)}_{\underline{s}kl} C^{klpq} E_{\underline{s}pq}\right) \mathrm{d}V - \\ - \sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}} \int_{(A_{\mathrm{tN}})} \widetilde{p}_{\mathrm{N}0} \delta u^k \Delta Q^{(1)}{}_{\underline{s}k}{}^{p} \mathrm{d}A_p = - \int_{(B)} \delta\Delta E^{(1)}_{\underline{s}kl} C^{klpq} E_{\underline{s}pq} \mathrm{d}V + \\ + \sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}} \int_{(A_{\mathrm{tN}})} \widetilde{p}_{\mathrm{N}0} \delta u^k Q_{\underline{s}k}{}^{p} \mathrm{d}A_p. \quad (3.7)$$

The solution for $u_{\underline{s}k}$ is given by

$$u_{\underline{s}k} = u_{\underline{1}k} + \Delta u_{\underline{1}k} + \Delta u_{\underline{2}k} + \cdots + \Delta u_{\underline{\mathrm{s}-1}k}; \qquad \Delta u_{\underline{s}k}. \quad (3.8)$$

The prescribed error limits should be satisfied at the end of the iteration. Then $\Delta u_{\underline{s}k} \to 0$, $u_{\underline{s}k} \to u_k$, $E_{\underline{s}pq} \to E_{pq}$, hence the left side of (3.7) tends to zero and the right side coincides with equation (3.2), which expresses the equilibrium of the state $(\bar{B})$.

The $\underline{s}^{\mathrm{th}}$ step of the Newton-Raphson iteration algorithm can be given in a finite element formulation as well:

$$\mathbf{K}_{\underline{s}ij} \Delta \mathbf{t}_{\underline{s}j} = -\, \mathbf{b}_{\underline{s}j} + \sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}} \mathbf{g}_{\underline{s}\mathrm{N}i} \quad (3.9)$$

where by omitting the subscript $\underline{\mathrm{s}}$ that identifies the $\underline{s}^{\mathrm{th}}$ step

$$\mathbf{K}_{ij} = \mathbf{K}^{\mathrm{L}}_{ij} + \mathbf{K}^{\mathrm{G}}_{ij} - \sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}} \mathbf{K}^{\mathrm{C}}_{\mathrm{N}ij}, \qquad \mathbf{K}^{\mathrm{L}}_{ij} + \mathbf{K}^{\mathrm{G}}_{ij} = \mathbf{K}^{\mathrm{T}}_{ij}. \quad (3.10)$$

Here utilizing the usual technical terms

$\mathbf{K}_{ij}$ is the total stiffness matrix,
$\mathbf{K}^{\mathrm{L}}_{ij}$ is the linear stiffness matrix,
$\mathbf{K}^{\mathrm{G}}_{ij}$ is the geometric stiffness matrix,
$\mathbf{K}^{\mathrm{C}}_{\mathrm{N}ij}$ denotes the load correction stiffness matrices $(N = I, II)$,
$\mathbf{K}^{\mathrm{T}}_{ij}$ is the tangent stiffness matrix,
$\mathbf{t}_j$ is the generalized node displacement matrix.

Equations of the 3D continuum mechanical model and those of the finite element model are related with each other via the following relations

$$\int_{(B)} \delta\Delta E^{(1)}_{kl} C^{klpq} \Delta E^{(1)}_{pq} \mathrm{d}V = \int_{(B)} \delta u_{m;k} \left(\delta^m_l + u^m{}_{;l}\right) C^{klpq} \left(\delta^s_p + u^s{}_{;l}\right) \Delta u_{s;q} \mathrm{d}V =$$
$$= \delta \mathbf{t}^{\mathrm{T}}_i \mathbf{K}^{\mathrm{L}}_{ij} \Delta \mathbf{t}_j, \quad (3.11)$$

$$\int_{(B)} \delta\Delta E^{(2)}_{kl} C^{klpq} E_{pq} \mathrm{d}V = \int_{(B)} \delta u_{m;k} \Delta u^m{}_{;l} C^{klpq} E_{pq} \mathrm{d}V = \delta \mathbf{t}^{\mathrm{T}}_i \mathbf{K}^{\mathrm{G}}_{ij} \Delta \mathbf{t}_j, \quad (3.12)$$

$$\int_{(A_{\mathrm{tN}})} \widetilde{p}_{\mathrm{N}0} \delta u^k \Delta Q^{(1)}_{\underline{\mathrm{s}}\ k}{}^p \mathrm{d}A_p = \int_{(A_{\mathrm{tN}})} \widetilde{p}_{\mathrm{N}0} \delta u^k e_{klm} e^{pqr} \left(\delta^l_q + u^l{}_{;q}\right) \Delta u^m{}_{;r} \mathrm{d}A_p =$$
$$= \delta \mathbf{t}^{\mathrm{T}}_i \mathbf{K}^{\mathrm{C}}_{\mathrm{N}ij} \Delta \mathbf{t}_j; \qquad \mathrm{N} = \mathrm{I}, \mathrm{II}, \quad (3.13)$$

$$\int_{(B)} \delta\Delta E^{(1)}_{kl} C^{klpq} E_{pq} \mathrm{d}V = \int_{(B)} \delta u_{m;k} \left(\delta^m_l + u^m{}_{;l}\right) C^{klpq} E_{pq} \mathrm{d}V = \delta \mathbf{t}^{\mathrm{T}}_i \mathbf{b}_i, \quad (3.14)$$

$$\int_{(A_{\mathrm{tN}})} \widetilde{p}_{\mathrm{N}0} \delta u^k Q_k{}^p \mathrm{d}A_p = \delta \mathbf{t}^{\mathrm{T}}_i \mathbf{g}_{\mathrm{N}i}; \qquad \mathrm{N} = \mathrm{I}, \mathrm{II}. \quad (3.15)$$

The load vectors $\mathbf{b}_i$ and $\mathbf{g}_{\mathrm{N}i}$ are defined by equations (3.14) and (3.15).

Remark 3.1: Here the conventional FEM is enlarged by the introduction of the load-correction stiffness matrices in (3.10). It is, however, worth mentioning that the iteration algorithm (3.7)-(3.10) also works if the load correction matrices are taken with a slight modification from the right side of the equation to the left side. If this is the case the number of iteration steps needed to achieve a given error limit is, however, significantly increased according to the numerical experiments.

Remark 3.2: Let the non-equilibrium state $(\bar{B}'_{\underline{1}})$ be a starting point. Then the Newton-Raphson iteration algorithm based on equation (3.6) results in the equilibrium state $(\bar{B})$ under the loading parameters $p_{\mathrm{I}}, p_{\mathrm{II}}$. This procedure can also be applied in the same manner for determining the equilibrium state $\bar{B} + \Delta\bar{B}$ under the loading parameters $p_{\mathrm{I}} + \Delta p_{\mathrm{I}}$, $p_{\mathrm{II}} + \Delta p_{\mathrm{II}}$ by regarding the earlier equilibrium state $\bar{B}$ as the point of departure.

Remark 3.3: The problem parameters are constant quantities in the previous part of the present section.

3.2.3. *Equilibrium paths, equilibrium surfaces and critical equilibrium states.*

3.2.3.1. By an equilibrium path/surface is meant such a curve/surface which depicts the displacement t_m as a function of $p_{\mathrm{I}}/p_{\mathrm{I}}$ and p_{II} (or h) in the interval of the investigated problem parameters. Here and in the sequel the subscript m is fixed when the matrix notation is used, i.e., it does not take any other value.

An equilibrium path is obtained in the coordinate system (p_{I}, t_m) if $h = h_0 =$ constant and $p_{\mathrm{II}} = p_{\mathrm{II}0} =$ constant.

An equilibrium surface is obtained in the coordinate system $(p_{\mathrm{II}}, p_{\mathrm{I}}, t_m)$ if $h = h_0 =$ constant. The intersections of the equilibrium surface with the coordinate planes $p_{\mathrm{II}} =$ constant or $p_{\mathrm{I}} =$ constant results in a series of equilibrium paths.

The surface $p_{\mathrm{II}} = p_{\mathrm{II}0} =$ constant is also an equilibrium surface in the coordinate system (h, p_{I}, t_m). Then the intersections with the coordinate planes $h =$ constant or $p_{\mathrm{I}} =$ constant or $t_m =$ constant) yield again a series of equilibrium paths.

If we proceed we can conclude that a system of surfaces is obtained if the parameter h is changed in the coordinate system $(p_{\mathrm{II}}, p_{\mathrm{I}}, t_m)$. In the same way a system of surfaces is obtained again if the parameter p_{II} is changed in the coordinate system (h, p_{I}, t_m).

In order to make the introduced terminology and concepts clearer some special cases will be defined in the sequel. It should, however, be remarked that the possibilities for the problem parameters are considered only partly.

3.2.3.2. First the case $h = 0$ is considered. Assuming that the problem has one load parameter only, i.e., $p_{\mathrm{II}} = 0$, the following terminology has come into general use for many problems: p_{I} as loading parameter, fundamental equilibrium path (or primary equilibrium path), which starts from the point $p_{\mathrm{I}} = 0$ that corresponds to the stress and deformation free state (B) of the body, and secondary equilibrium path (in some cases it is called the bifurcational equilibrium path) which characterizes the postcritical states. Then $p_{\mathrm{I}} \geq p_I^{\mathrm{crit}}$ where p_I^{crit} is the critical value of the one parameter load.

A point on the secondary equilibrium path can be obtained by determining the equilibrium state $(\bar{B})$ if the state $(B_{\underline{1}}')$ which belongs to the loading parameter $p_{\mathrm{I}} > p_I^{\mathrm{crit}}$ is the starting point for the Newton-Raphson iteration.

From here other points can be calculated on the secondary equilibrium path if the loading parameter is set to $p_{\mathrm{I}} + \Delta p_{\mathrm{I}}$.

The Newton-Raphson iteration can be parametrized. It is possible to select the load p_{I} as a parameter (loading parametrization) or a displacement component (for instance t_m) as parameter (displacement parametrization).

In this sense the common point of the primary path and the secondary path is a branching point (bifurcation point) which determines the value $p_{\mathrm{I}} = p_I^{\mathrm{crit}}$ of the load (path following method).

There is a further method for finding the critical load. This consists in determining the loading parameter that belongs to the smallest non-zero eigenvalue of the corresponding eigenvalue problem (determinant observing method):

$$\mathbf{K}_{ij}\Delta\mathbf{t}_j = \mathbf{0}, \qquad \Delta\mathbf{t}_j \neq \mathbf{0}, \qquad \det\mathbf{K}_{ij} = 0. \tag{3.16}$$

The equilibrium paths can also be sought one by one for the values $p_{\mathrm{II}} = \text{constant} \neq 0$ as well.

In all that has been said above the role of the parameters p_{I} and p_{II} is interchangeable, i.e., it is possible to determine equilibrium paths if $p_{\mathrm{I}} = \text{constant} \neq 0$. Then p_{II} is the loading parameter.

For problems with two loading parameters the equilibrium states can be illustrated by equilibrium surfaces. In these cases the loading parameters p_{I}, p_{II} and the displacement t_m are the variables. The equilibrium paths that belong to the values $p_{\mathrm{I}} = \text{constant}$ and $p_{\mathrm{II}} = \text{constant}$ are intersections of the equilibrium surface and the planes $p_{\mathrm{I}} = \text{constant}$ and $p_{\mathrm{II}} = \text{constant}$. Here p_{II} can be either a supplementary load or a load imperfection parameter. The equilibrium paths $p_{\mathrm{I}} = \text{constant}$ or $p_{\mathrm{II}} = \text{constant}$ may have maxima, minima and in some special cases points with vertical tangents. The point with a horizontal tangent, i.e., where the equilibrium path has, therefore, a maximum (or minimum), is called the limit point. The intersection point of the primary and secondary equilibrium paths (the primary branching point) and the limit points are called critical points, while the loads that belong to these points are the critical loads. The states of the body under these loads are referred to as critical states. A more rigorous definition for the critical state of the body is given in Subsection 4.1.

3.2.3.3. By changing the parameter h one can determine the different equilibrium paths, equilibrium surfaces, critical points and critical loads for each distinct value of $h =$ constant.

3.2.3.4. When investigating the effect of (load imperfection for which the parameter is p_{II}) [geometrical imperfection for which the parameter is h] the locus of (limit points) [branching points] is called the curve of (limit points or fold line) [branching points].

3.2.4. *Asymptotic numerical method.* Nonlinear equilibrium problems can be solved either by using the Newton-Rapshon iteration or some other method: the asymptotic numerical method, for instance. The later method is based on the idea that the characteristic variables of the problem can be expended into a Taylor series in a small neighborhood of the reference state $\bar{B}$ by selecting a scalar for the governing parameter of the problem. This scalar is regarded formally as if it were time. The approximations would consist of a few terms only, though they become more accurate if the number of terms is increased.

The governing parameter can be (a) displacement parameter: $\tau = t_m - t_{m0}$; (b) loading parameter: $\tau = p_{\mathrm{N}} - p_{\mathrm{N}0}$, $\mathrm{N} = \mathrm{I}, \mathrm{II},$; (c) geometric imperfection parameter $\tau = h - h_0$; or something else (d) the arc length of the equilibrium path for instance: $\tau = s - s_0$ where t_{m0}, $p_{\mathrm{N}0}$, h_0, s_0 belong to the equilibrium state $(\bar{B})$.

For the equilibrium state $(\bar{B})$ it holds that $\tau = 0$, whereas $\tau = \Delta\tau$ in the state $(\bar{B} + \Delta\bar{B})$ of the body. For the intermediate states, however, $0 < \tau < \Delta\tau$.

If the subscript 0, which identifies state $(\bar{B})$ of the body, is omitted the Taylor series have the following forms:

$$u^k(\tau) = u^k + \Delta u^k, \qquad \Delta u^k = \dot{u}^k\tau + \frac{1}{2}\ddot{u}^k\tau^2 + \frac{1}{6}\dddot{u}^k\tau^3 + \cdots, \tag{3.17a}$$

$$\mathbf{t}_i(\tau) = \mathbf{t}_i + \Delta\mathbf{t}_i, \qquad \Delta\mathbf{t}_i = \dot{\mathbf{t}}_i\tau + \frac{1}{2}\ddot{\mathbf{t}}_i\tau^2 + \frac{1}{6}\dddot{\mathbf{t}}_i\tau^3 + \cdots, \tag{3.17b}$$

$$E_{kl}(\tau) = E_{kl} + \Delta E_{kl}, \qquad \Delta E_{kl} = \dot{E}_{kl}\tau + \frac{1}{2}\ddot{E}_{kl}\tau^2 + \frac{1}{6}\dddot{E}_{kl}\tau^3 + \cdots, \tag{3.17c}$$

$$p_{\mathrm{N}}(\tau) = p_{\mathrm{N}} + \Delta p_{\mathrm{N}}, \qquad \Delta p_{\mathrm{N}} = \dot{p}_{\mathrm{N}}\tau + \frac{1}{2}\ddot{p}_{\mathrm{N}}\tau^2 + \frac{1}{6}\dddot{p}_{\mathrm{N}}\tau^3 \cdots \quad \mathrm{N} = \mathrm{I}, \mathrm{II}, \tag{3.17d}$$

$$Q_k^{\ p}(\tau) = Q_k^{\ p} + \Delta Q_k^{\ p}, \quad \Delta Q_k^{\ p} = \dot{Q}_k^{\ p}\tau + \frac{1}{2}\ddot{Q}_k^{\ p}\tau^2 + \frac{1}{6}\dddot{Q}_k^{\ p}\tau^3 + \cdots, \tag{3.17e}$$

$$h(\tau) = h + \Delta h, \qquad \Delta h = \dot{h}\tau + \frac{1}{2}\ddot{h}\tau^2 + \frac{1}{6}\dddot{h}\tau^3 + \cdots. \tag{3.17f}$$

where the derivatives of u^k, $\mathbf{t}_i$, E_{kl}, p_{N}, $Q_k^{\ p}$ and h with respect to τ belong to the state $(\bar{B})$.

The essence of the numerical method consists in determining the unknown derivatives in the state $(\bar{B})$ of the body [see equations (3.41) and (3.42) presented later] and then the increments are calculated by utilizing the above relations and taking the prescribed error limits also into account.

For preparing the derivation of the equilibrium equations we determine the first three derivatives of E_{kl} and $Q_k^{\ p}$ assuming that $h =$ constant:

$$\dot{E}_{kl}C^{klpq} = \dot{u}_{m;k}\left(\delta_l^m + u^m_{\ ;l}\right)C^{klpq}, \tag{3.18a}$$

$$\ddot{E}_{kl}C^{klpq} = \left[\ddot{u}_{m;k}\left(\delta_l^m + u^m_{\ ;l}\right) + \dot{u}_{m;k}\dot{u}^m_{\ ;l}\right]C^{klpq}, \tag{3.18b}$$

$$\dddot{E}_{kl}C^{klpq} = \left[\dddot{u}_{m;k}\left(\delta_l^m + u^m_{\ ;l}\right) + 2\ddot{u}_{m;k}\dot{u}^m_{\ ;l} + \dot{u}_{m;k}\ddot{u}^m_{\ ;l}\right]C^{klpq}, \tag{3.18c}$$

$$\dot{Q}_k^{\ p} = e_{klm}e^{pqr}\left(\delta_q^l + u^l_{\ ;q}\right)\dot{u}^m_{\ ;r}, \tag{3.18d}$$

$$\ddot{Q}_k^{\ p} = e_{klm}e^{pqr}\left[\left(\delta_q^l + u^l_{\ ;q}\right)\ddot{u}^m_{\ ;r} + \dot{u}^l_{\ ;q}\dot{u}^m_{\ ;r}\right], \tag{3.18e}$$

$$\dddot{Q}_k^{\ p} = e_{klm}e^{pqr}\left[\left(\delta_q^l + u^l_{\ ;q}\right)\dddot{u}^m_{\ ;r} + 2\dot{u}^l_{\ ;q}\ddot{u}^m_{\ ;r} + \ddot{u}^l_{\ ;q}\dot{u}^m_{\ ;r}\right]. \tag{3.18f}$$

According to the principle of virtual power state $(\bar{B})$ $(\tau = 0)$ is in equilibrium under the problem parameters $p_{\rm I}, p_{\rm II}, h$ if (3.1) is fulfilled. With the finite element algorithm we may write

$$\delta\dot{\mathbf{t}}_i^{\rm T}\left(\mathbf{b}_i - \sum_{\rm N=I}^{\rm II} p_{\rm N}\mathbf{g}_{{\rm N}i}\right) = 0 \quad \text{from where it follows that} \quad \mathbf{b}_i - \sum_{\rm N=I}^{\rm II} p_{\rm N}\mathbf{g}_{{\rm N}i} = \mathbf{0}\,. \tag{3.19}$$

Here in accordance with (3.14) and (3.15)

$$\delta\dot{\mathbf{t}}_i^{\rm T}\mathbf{b}_i = \int_{(B)}\delta\dot{E}_{kl}C^{klpq}E_{pq}{\rm d}V = \int_{(B)}\delta\dot{u}_{m;k}\left(\delta_l^m + u^m_{\ ;l}\right)C^{klpq}E_{pq}{\rm d}V, \tag{3.20}$$

and

$$\delta\dot{\mathbf{t}}_i^{\rm T}\mathbf{g}_{{\rm N}i} = \int_{(A_{\rm tN})}\widetilde{p}_{\rm N0}\delta\dot{u}^kQ_k^{\ p}{\rm d}A_p =$$

$$= \int_{(A_{\rm tN})}\widetilde{p}_{\rm N0}\delta\dot{u}^k\frac{1}{2}e_{klm}e^{pqr}\left(\delta_q^l + u^l_{\ ;q}\right)\left(\delta_r^m + u^m_{\ ;r}\right){\rm d}A_p; \quad {\rm N} = {\rm I}, {\rm II}. \tag{3.21}$$

Introducing the notation

$$\mathbf{A}_i = \mathbf{b}_i - \sum_{\rm N=I}^{\rm II} p_{\rm N}\mathbf{g}_{{\rm N}i}, \tag{3.22}$$

equation (3.19) can be rewritten into the following form:

$$\delta\dot{\mathbf{t}}_i^{\rm T}\mathbf{A}_i = 0 \quad \text{which means that} \quad \mathbf{A}_i = \mathbf{0}. \tag{3.23}$$

The virtual variables $\delta\dot{u}^k$ and $\delta\dot{\mathbf{t}}_i$ are independent of the governing parameter τ. Hence

$$\left(\delta\dot{u}^k\right)^{\cdot} = \left(\delta\dot{u}^k\right)^{\cdot\cdot} = \cdots = 0, \qquad \left(\delta\dot{\mathbf{t}}_i\right)^{\cdot} = \left(\delta\dot{\mathbf{t}}_i\right)^{\cdot\cdot} = \cdots = 0\,. \tag{3.24}$$

For the general case it is assumed that the geometric imperfection parameter depends also on the governing parameter τ. The following relations are, therefore, assumed for the general case:

$$\begin{array}{llll} u^k(\tau), & \mathbf{t}_i(\tau), & p_{\mathrm{N}}(\tau), & h(\tau), \\ \mathbf{b}_i(\mathbf{t}_j, h), & \mathbf{g}_{\mathrm{N}i}(\mathbf{t}_j, h), & \mathbf{A}_i(\mathbf{t}_j, p_{\mathrm{N}}, h) . & \end{array} \tag{3.25}$$

Equations (3.17a)-(3.19) are also valid for the general case. Some special cases are given below:

- ♦ $\tau = p_{\mathrm{I}} - p_{\mathrm{I}0}$; $\dot{p}_{\mathrm{I}} = 1,\ \ddot{p}_{\mathrm{I}},\ \dddot{p}_{\mathrm{I}}, \dots = 0$; $\dot{p}_{\mathrm{II}}, \ddot{p}_{\mathrm{II}}, \dots = 0$; $h = h_0,\ \dot{h}, \ddot{h}, \dots = 0$,
- ♦ $\tau = p_{\mathrm{II}} - p_{\mathrm{II}0}$; $\dot{p}_{\mathrm{II}} = 1,\ \ddot{p}_{\mathrm{II}},\ \dddot{p}_{\mathrm{II}}, \dots = 0$; $\dot{p}_{\mathrm{I}}, \ddot{p}_{\mathrm{I}}, \dots = 0$; $h = h_0,\ \dot{h}, \ddot{h}, \dots = 0$,
- ♦ $\tau = h - h_0$; $\dot{h} = 1,\ \ddot{h},\ \dddot{h}, \dots = 0$; $\dot{p}_{\mathrm{I}}, \ddot{p}_{\mathrm{I}}, \dots = 0$; $\ddot{p}_{\mathrm{II}}, \dddot{p}_{\mathrm{II}}, \dots = 0$.

Let us expend the equilibrium equation into series in terms of the governing parameter τ in a small neighborhood of the equilibrium state $(\bar{B})$ by taking equations (3.24) and relationships (3.25) into account. We have

$$\delta\dot{\mathbf{t}}_i^{\mathrm{T}}\mathbf{A}_i + \left(\delta\dot{\mathbf{t}}_i^{\mathrm{T}}\mathbf{A}_i\right)^{\cdot}\tau + \frac{1}{2}\left(\delta\dot{\mathbf{t}}_i^{\mathrm{T}}\mathbf{A}_i\right)^{\cdot\cdot}\tau^2 + \frac{1}{6}\left(\delta\dot{\mathbf{t}}_i^{\mathrm{T}}\mathbf{A}_i\right)^{\cdot\cdot\cdot}\tau^3 + \dots = 0,$$

from where it follows with regard to $(3.23)_2$ that:

$$\begin{aligned} &\delta\dot{\mathbf{t}}_i^{\mathrm{T}}\left(\frac{\partial\mathbf{A}_i}{\partial\mathbf{t}_j}\dot{\mathbf{t}}_j + \sum_{\mathrm{N=I}}^{\mathrm{II}}\frac{\partial\mathbf{A}_i}{\partial p_{\mathrm{N}}}\dot{p}_{\mathrm{N}} + \frac{\partial\mathbf{A}_i}{\partial h}\dot{h}\right)\tau + \\ &\quad + \frac{1}{2}\delta\dot{\mathbf{t}}_i^{\mathrm{T}}\left(\frac{\partial\mathbf{A}_i}{\partial\mathbf{t}_j}\ddot{\mathbf{t}}_j + \sum_{\mathrm{N=I}}^{\mathrm{II}}\frac{\partial\mathbf{A}_i}{\partial p_{\mathrm{N}}}\ddot{p}_{\mathrm{N}} + \frac{\partial\mathbf{A}_i}{\partial h}\ddot{h}\right)\tau^2 + \\ &\quad + \frac{1}{2}\delta\dot{\mathbf{t}}_i^{\mathrm{T}}\left(\frac{\partial}{\partial\mathbf{t}_k}\frac{\partial\mathbf{A}_i}{\partial\mathbf{t}_j}\dot{\mathbf{t}}_k\dot{\mathbf{t}}_j + \sum_{\mathrm{N=I}}^{\mathrm{II}}\frac{\partial^2\mathbf{A}_i}{\partial p_{\mathrm{N}}^2}\dot{p}_{\mathrm{N}}^2 + \frac{\partial^2\mathbf{A}_i}{\partial h^2}\dot{h}^2\right)\tau^2 + \\ &+ \frac{1}{2}\delta\dot{\mathbf{t}}_i^{\mathrm{T}}\left(\sum_{\mathrm{N=I}}^{\mathrm{II}}2\frac{\partial}{\partial p_{\mathrm{N}}}\frac{\partial\mathbf{A}_i}{\partial\mathbf{t}_j}\dot{\mathbf{t}}_j\dot{p}_{\mathrm{N}} + 2\frac{\partial}{\partial h}\frac{\partial\mathbf{A}_i}{\partial\mathbf{t}_j}\dot{\mathbf{t}}_j\dot{h} + \sum\nolimits_{\mathrm{N=I}}^{\mathrm{II}}2\frac{\partial}{\partial h}\frac{\partial\mathbf{A}_i}{\partial p_{\mathrm{N}}}\dot{p}_{\mathrm{N}}\dot{h}\right)\tau^2 + \dots = 0. \end{aligned} \tag{3.26}$$

REMARK 3.4: The terms in the power series for which the power of τ is higher than two contain the derivatives $\frac{\partial}{\partial\mathbf{t}_m}\frac{\partial}{\partial\mathbf{t}_k}\frac{\partial\mathbf{A}_i}{\partial\mathbf{t}_j}$ of the matrices $\mathbf{A}_i$ only. The reason for this is simple: according to equations (3.20) and (2.2) $\mathbf{A}_i$ does not contain the power of u^m and its derivatives higher than three.

Making use of equations (3.22), (3.20) and (3.21) we shall give the details for equation (3.26):

As regards the expression

$$\delta\dot{\mathbf{t}}_i^{\mathrm{T}}\frac{\partial\mathbf{A}_i}{\partial\mathbf{t}_j}\dot{\mathbf{t}}_j = \delta\dot{\mathbf{t}}_i^{\mathrm{T}}\left(\frac{\partial\mathbf{b}_i}{\partial\mathbf{t}_j} - \sum_{\mathrm{N=I}}^{\mathrm{II}}p_{\mathrm{N}}\frac{\partial\mathbf{g}_{\mathrm{N}i}}{\partial\mathbf{t}_j}\right)\dot{\mathbf{t}}_j, \tag{3.27}$$

we get

$$\delta\dot{\mathbf{t}}_i^{\mathrm{T}}\frac{\partial\mathbf{b}_i}{\partial\mathbf{t}_j}\dot{\mathbf{t}}_j = \int_{(B)}\delta\dot{u}_{m;k}\left[\left(\delta_l^m + u^m{}_{;l}\right)C^{klpq}E_{pq}\right]^{\cdot}\mathrm{d}V =$$

$$= \int_{(B)} \delta\dot{u}_{m;k}\dot{u}^{m}_{\ ;l}C^{klpq}E_{pq}\mathrm{d}V + \int_{(B)} \delta\dot{u}_{m;k}\left(\delta^{m}_{l} + u^{m}_{\ ;l}\right)C^{klpq}\left(\delta^{s}_{p} + u^{s}_{\ ;p}\right)\dot{u}_{s;q}\mathrm{d}V =$$

$$= \delta\dot{\mathbf{t}}_i^{\mathrm{T}}\left(\mathbf{K}^{\mathrm{G}}_{ij}\dot{\mathbf{t}}_j + \mathbf{K}^{\mathrm{L}}_{ij}\dot{\mathbf{t}}_j\right), \quad (3.28)$$

and

$$-\delta\dot{\mathbf{t}}_i^{\mathrm{T}}\sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}}\frac{\partial \mathbf{g}_{\mathrm{N}i}}{\partial \mathbf{t}_j}\dot{\mathbf{t}}_j =$$

$$= -\sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}}\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N}0}\delta\dot{u}^{k}\frac{1}{2}e_{klm}e^{pqr}\left[\left(\delta^{l}_{q} + u^{l}_{\ ;q}\right)\left(\delta^{m}_{r} + u^{m}_{\ ;r}\right)\right]^{\cdot}\mathrm{d}A_p =$$

$$= -\sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}}\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N}0}\delta\dot{u}^{k}e_{klm}e^{pqr}\left(\delta^{l}_{q} + u^{l}_{\ ;q}\right)\dot{u}^{m}_{\ ;r}\mathrm{d}A_p =$$

$$= -\delta\dot{\mathbf{t}}_i^{\mathrm{T}}\sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}}\mathbf{K}^{\mathrm{C}}_{\mathrm{N}ij}\dot{\mathbf{t}}_j. \quad (3.29)$$

We shall proceeded in the same way by detailing the terms in equation (3.26) one by one:

$$\delta\dot{\mathbf{t}}_i^{\mathrm{T}}\frac{\partial \mathbf{A}_i}{\partial p_{\mathrm{N}}}\dot{p}_{\mathrm{N}} = -\delta\dot{\mathbf{t}}_i\sum_{\mathrm{N=I}}^{\mathrm{II}}\frac{\partial}{\partial p_{\mathrm{N}}}\left(\sum_{\mathrm{M=I}}^{\mathrm{II}} p_{\mathrm{M}}\mathbf{g}_{\mathrm{M}i}\right)\dot{p}_{\mathrm{N}} = -\delta\dot{\mathbf{t}}_i^{\mathrm{T}}\sum_{\mathrm{N=I}}^{\mathrm{II}}\mathbf{g}_{\mathrm{N}i}\dot{p}_{\mathrm{N}}, \quad (3.30)$$

$$\delta\dot{\mathbf{t}}_i^{\mathrm{T}}\frac{\partial \mathbf{A}_i}{\partial h}\dot{h} = \delta\mathbf{t}\mathbf{t}_i^{\mathrm{T}}\left(\frac{\partial \mathbf{b}_i}{\partial h} - \sum\nolimits_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}}\frac{\partial \mathbf{g}_{\mathrm{N}i}}{\partial h}\right)\dot{h}, \quad (3.31)$$

$$\frac{1}{2}\delta\dot{\mathbf{t}}_i^{\mathrm{T}}\left(\frac{\partial \mathbf{A}_i}{\partial \mathbf{t}_j}\ddot{\mathbf{t}}_j + \sum_{\mathrm{N=I}}^{\mathrm{II}}\frac{\partial \mathbf{A}_i}{\partial p_{\mathrm{N}}}\ddot{p}_{\mathrm{N}} + \frac{\partial \mathbf{A}_i}{\partial h}\ddot{h}\right) =$$

$$= \frac{1}{2}\delta\dot{\mathbf{t}}_i^{\mathrm{T}}\left[\left(\mathbf{K}^{\mathrm{L}}_{ij} + \mathbf{K}^{\mathrm{G}}_{ij} - \sum\nolimits_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}}\mathbf{K}^{\mathrm{C}}_{\mathrm{N}ij}\right)\ddot{\mathbf{t}}_j - \sum\nolimits_{\mathrm{N=I}}^{\mathrm{II}}\mathbf{g}_{\mathrm{N}i}\ddot{p}_{\mathrm{N}} + \right.$$

$$\left. + \left(\frac{\partial \mathbf{b}_i}{\partial h} - \sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}}\frac{\partial \mathbf{g}_{\mathrm{N}i}}{\partial h}\right)\ddot{h}\right], \quad (3.32)$$

$$\frac{1}{2}\delta\dot{\mathbf{t}}_i^{\mathrm{T}}\frac{\partial}{\partial \mathbf{t}_k}\frac{\partial \mathbf{A}_i}{\partial \mathbf{t}_j}\dot{\mathbf{t}}_k\dot{\mathbf{t}}_j = \frac{1}{2}\delta\dot{\mathbf{t}}_i^{\mathrm{T}}\left(\frac{\partial}{\partial \mathbf{t}_k}\frac{\partial \mathbf{b}_i}{\partial \mathbf{t}_j}\dot{\mathbf{t}}_k - \sum\nolimits_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}}\frac{\partial}{\partial \mathbf{t}_k}\frac{\partial \mathbf{g}_{\mathrm{N}i}}{\partial \mathbf{t}_j}\dot{\mathbf{t}}_k\right)\dot{\mathbf{t}}_j, \quad (3.33)$$

$$\frac{1}{2}\delta\dot{\mathbf{t}}_i^{\mathrm{T}}\frac{\partial}{\partial \mathbf{t}_k}\frac{\partial \mathbf{b}_i}{\partial \mathbf{t}_j}\dot{\mathbf{t}}_k\dot{\mathbf{t}}_j = \frac{1}{2}2\int_{(B)}\delta\dot{u}_{m;k}\dot{u}^{m}_{\ ;l}C^{klpq}\left(\delta^{s}_{p} + u^{s}_{\ ;p}\right)\dot{u}_{s;q}\mathrm{d}V +$$

$$+ \frac{1}{2}\int_{(B)}\delta\dot{u}_{m;k}\left(\delta^{m}_{l} + u^{m}_{\ ;l}\right)C^{klpq}\dot{u}^{s}_{\ ;p}\dot{u}_{s;q}\mathrm{d}V = \frac{1}{2}\delta\dot{\mathbf{t}}_i^{\mathrm{T}}\left(2\mathbf{J}_{ikj} + \mathbf{H}_{ikj}\right)\dot{\mathbf{t}}_k\dot{\mathbf{t}}_j, \quad (3.34)$$

$$-\frac{1}{2}\delta\dot{\mathbf{t}}_i^{\mathrm{T}}\sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}}\frac{\partial}{\partial \mathbf{t}_k}\frac{\partial \mathbf{g}_{\mathrm{N}i}}{\partial \mathbf{t}_j}\dot{\mathbf{t}}_k\dot{\mathbf{t}} =$$

$$= -\frac{1}{2}\delta\dot{\mathbf{t}}_i^{\mathrm{T}} \sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}} \int_{(A_{\mathrm{tN}})} \widetilde{p}_{\mathrm{N}0} \delta\dot{u}^k e_{klm} e^{pqr} \dot{u}^l{}_{;q} \dot{u}^m{}_{;r} \mathrm{d}A_p = -\frac{1}{2}\delta\dot{\mathbf{t}}_i^{\mathrm{T}} \sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}} \mathbf{M}_{\mathrm{N}ikj} \dot{\mathbf{t}}_k \dot{\mathbf{t}}_j, \tag{3.35}$$

$$\frac{1}{2}\delta\dot{\mathbf{t}}_i^{\mathrm{T}} \sum_{\mathrm{N=I}}^{\mathrm{II}} \frac{\partial^2 \mathbf{A}_i}{\partial p_{\mathrm{N}}^2} \dot{p}_{\mathrm{N}}^2 = 0, \tag{3.36}$$

$$\frac{1}{2}\delta\dot{\mathbf{t}}_i^{\mathrm{T}} \frac{\partial^2 \mathbf{A}_i}{\partial h^2} \dot{h}^2 = \frac{1}{2}\delta\dot{\mathbf{t}}_i^{\mathrm{T}} \left(\frac{\partial^2 \mathbf{b}_i}{\partial h^2} - \sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}} \frac{\partial^2 \mathbf{g}_{\mathrm{N}i}}{\partial h^2} \right) \dot{h}^2, \tag{3.37}$$

$$\frac{1}{2}\delta\dot{\mathbf{t}}_i^{\mathrm{T}} \sum\nolimits_{\mathrm{N=I}}^{\mathrm{II}} 2 \frac{\partial}{\partial p_{\mathrm{N}}} \frac{\partial \mathbf{A}_i}{\partial \mathbf{t}_j} \dot{\mathbf{t}}_j \dot{p}_{\mathrm{N}} = -\frac{1}{2}\delta\dot{\mathbf{t}}_i^{\mathrm{T}} \sum_{\mathrm{N=I}}^{\mathrm{II}} 2 \frac{\partial}{\partial p_{\mathrm{N}}} \left(\sum_{\mathrm{M=I}}^{\mathrm{II}} p_{\mathrm{M}} \frac{\partial \mathbf{g}_{\mathrm{M}i}}{\partial \mathbf{t}_j} \dot{\mathbf{t}}_j \right) \dot{p}_{\mathrm{N}} =$$
$$= -\frac{1}{2}\delta\dot{\mathbf{t}}_i^{\mathrm{T}} \left(2 \sum\nolimits_{\mathrm{N=I}}^{\mathrm{II}} \mathbf{K}_{\mathrm{N}ij}^{\mathrm{C}} \dot{p}_{\mathrm{N}} \right) \dot{\mathbf{t}}_j, \tag{3.38}$$

$$\frac{1}{2}\delta\dot{\mathbf{t}}_i^{\mathrm{T}} 2 \frac{\partial}{\partial h} \frac{\partial \mathbf{A}_i}{\partial \mathbf{t}_j} \dot{\mathbf{t}}_j \dot{h} = \frac{1}{2}\delta\dot{\mathbf{t}}_i^{\mathrm{T}} 2 \frac{\partial}{\partial h} \left(\frac{\partial \mathbf{b}_i}{\partial \mathbf{t}_j} - \sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}} \frac{\partial \mathbf{g}_{\mathrm{N}i}}{\partial \mathbf{t}_j} \right) \dot{\mathbf{t}}_j \dot{h} =$$
$$= \frac{1}{2}\delta\dot{\mathbf{t}}_i^{\mathrm{T}} 2 \frac{\partial}{\partial h} \left(\mathbf{K}_{ij}^{\mathrm{L}} + \mathbf{K}_{ij}^{\mathrm{G}} - \sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}} \mathbf{K}_{\mathrm{N}ij}^{\mathrm{C}} \right) \dot{\mathbf{t}}_j \dot{h}, \tag{3.39}$$

$$\frac{1}{2}\delta\dot{\mathbf{t}}_i^{\mathrm{T}} \sum\nolimits_{\mathrm{N=I}}^{\mathrm{II}} 2 \frac{\partial}{\partial h} \frac{\partial \mathbf{A}_i}{\partial p_{\mathrm{N}}} \dot{p}_{\mathrm{N}} \dot{h} = -\frac{1}{2}\delta\dot{\mathbf{t}}_i^{\mathrm{T}} 2 \left(\sum\nolimits_{\mathrm{N=I}}^{\mathrm{II}} \frac{\partial \mathbf{g}_{\mathrm{N}i}}{\partial h} \dot{p}_{\mathrm{N}} \right) \dot{h}. \tag{3.40}$$

Since equation (3.26) is fulfilled for any τ and $\delta\dot{\mathbf{t}}_i$ it yields the following equations

$$\left(\mathbf{K}_{ij}^{\mathrm{L}} + \mathbf{K}_{ij}^{\mathrm{G}} - \sum_{\mathrm{N=I}}^{\mathrm{II}} \mathbf{K}_{\mathrm{N}ij}^{\mathrm{C}} p_{\mathrm{N}} \right) \dot{\mathbf{t}}_j - \sum_{\mathrm{N=I}}^{\mathrm{II}} \mathbf{g}_{\mathrm{N}i} \dot{p}_{\mathrm{N}} + \left(\frac{\partial \mathbf{b}_i}{\partial h} - \sum_{\mathrm{N=I}}^{\mathrm{II}} \frac{\partial \mathbf{g}_{\mathrm{N}i}}{\partial h} p_{\mathrm{N}} \right) \dot{h} = 0, \tag{3.41}$$

$$\left(\mathbf{K}_{ij}^{\mathrm{L}} + \mathbf{K}_{ij}^{\mathrm{G}} - \sum_{\mathrm{N=I}}^{\mathrm{II}} \mathbf{K}_{\mathrm{N}ij}^{\mathrm{C}} p_{\mathrm{N}} \right) \ddot{\mathbf{t}}_j - \sum_{\mathrm{N=I}}^{\mathrm{II}} \mathbf{g}_{\mathrm{N}i} \ddot{p}_{\mathrm{N}} + \left(\frac{\partial \mathbf{b}_i}{\partial h} - \sum_{\mathrm{N=I}}^{\mathrm{II}} \frac{\partial \mathbf{g}_{\mathrm{N}i}}{\partial h} p_{\mathrm{N}} \right) \ddot{h} =$$
$$= - \left(2\mathbf{J}_{ikj} + \mathbf{H}_{ikj} - \sum_{\mathrm{N=I}}^{\mathrm{II}} \mathbf{M}_{\mathrm{N}ikj} p_{\mathrm{N}} \right) \dot{\mathbf{t}}_k \dot{\mathbf{t}}_j - \left(\frac{\partial^2 \mathbf{b}_i}{\partial h^2} - \sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}} \frac{\partial^2 \mathbf{g}_{\mathrm{N}i}}{\partial h^2} \right) \dot{h}^2 +$$
$$+ 2 \left(\sum_{\mathrm{N=I}}^{\mathrm{II}} \mathbf{K}_{\mathrm{N}ij}^{\mathrm{C}} \dot{p}_{\mathrm{N}} \right) \dot{\mathbf{t}}_j - 2 \frac{\partial}{\partial h} \left(\mathbf{K}_{ij}^{\mathrm{L}} + \mathbf{K}_{ij}^{\mathrm{G}} - \sum_{\mathrm{N=I}}^{\mathrm{II}} \mathbf{K}_{\mathrm{N}ij}^{\mathrm{C}} p_{\mathrm{N}} \right) \dot{\mathbf{t}}_j \dot{h} +$$
$$+ 2 \left(\sum_{\mathrm{N=I}}^{\mathrm{II}} \frac{\partial \mathbf{g}_{\mathrm{N}i}}{\partial h} \dot{p}_{\mathrm{N}} \right) \dot{h}. \tag{3.42}$$

Equations $(3.19)_2$, (3.41) and (3.42) are referred to as equilibrium equations of the asymptotic method. Equation (3.26) can be continued for the higher powers of of τ. Hence, further equilibrium equations can be obtained for the higher derivatives of the problem parameters.

Two important characteristics of the equation system should be emphasized.

The first is that the coefficient matrices of the time derivatives of the problem parameters on the left side are the same in the successive equations. The second is that the order of the derivatives on the right side is always lower than that of the derivatives on the left side. This makes possible it to calculate the first, second, etc. derivatives of the problem parameters step by step.

If a displacement is the governing parameter the equilibrium equations can be transformed into another form. Let us assume that $\dot{h} = 0$ and regard equation (3.41) as an example. Then we can rewrite it into the form

$$\begin{bmatrix} & & \vdots & \\ \mathbf{K}_{\alpha\beta} & & \vdots & \mathbf{k}_{\alpha m} \\ & & \vdots & \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{k}^{\mathrm{T}}_{m\beta} & & \vdots & k_{mm} \end{bmatrix} \begin{bmatrix} \vdots \\ \dot{\mathbf{t}}_{\beta} \\ \vdots \\ \cdots \\ \dot{t}_m \end{bmatrix} - \begin{bmatrix} \vdots \\ \mathbf{g}_{\mathrm{I}\alpha} \\ \vdots \\ \cdots \\ g_{\mathrm{I}m} \end{bmatrix} \dot{p}_{\mathrm{I}} - \begin{bmatrix} \vdots \\ \mathbf{g}_{\mathrm{II}\alpha} \\ \vdots \\ \cdots \\ g_{\mathrm{II}m} \end{bmatrix} \dot{p}_{\mathrm{II}} = 0$$

where m is the number of freedom and $\alpha, \beta = 1, 2, \cdots, m-1$. The corresponding scalar equations are given by

$$\mathbf{K}_{\alpha\beta}\dot{\mathbf{t}}_{\beta} + \mathbf{k}_{\alpha m}\dot{t}_m - \mathbf{g}_{\mathrm{I}\alpha}\dot{p}_{\mathrm{I}} - \mathbf{g}_{\mathrm{II}\alpha}\dot{p}_{\mathrm{II}} = 0, \tag{3.43}$$

$$\mathbf{k}^{\mathrm{T}}_{m\beta}\dot{\mathbf{t}}_{\beta} + k_{mm}\dot{t}_m - g_{\mathrm{I}m}\dot{p}_{\mathrm{I}} - g_{\mathrm{II}m}\dot{p}_{\mathrm{II}} = 0. \tag{3.44}$$

It is clear that these equations can be modified by performing an identity transformation:

$$\begin{bmatrix} & & \vdots & \\ \mathbf{K}_{\alpha\beta} & & \vdots & -\,\mathbf{g}_{\mathrm{II}\alpha} \\ & & \vdots & \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{k}^{\mathrm{T}}_{m\beta} & & \vdots & -\,g_{\mathrm{II}m} \end{bmatrix} \begin{bmatrix} \vdots \\ \dot{\mathbf{t}}_{\beta} \\ \vdots \\ \cdots \\ \dot{p}_{\mathrm{II}} \end{bmatrix} + \begin{bmatrix} \vdots \\ \mathbf{k}_{\alpha m} \\ \vdots \\ \cdots \\ k_{mm} \end{bmatrix} \dot{t}_m - \begin{bmatrix} \vdots \\ \mathbf{g}_{\mathrm{I}\alpha} \\ \vdots \\ \cdots \\ g_{\mathrm{I}m} \end{bmatrix} \dot{p}_{\mathrm{I}} = 0. \tag{3.45}$$

The structure of the equilibrium equations within the framework of the continuum mechanical model can be surveyed in the following manner. Let us consider equation (3.1) as the point of departure:

$$\int_{(B)} \delta\dot{u}_{m;k}\left(\delta^m_l + u^m_{\ ;l}\right) C^{klpq} E_{pq} \mathrm{d}V - \sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}} \int_{(A_{t\mathrm{N}})} \widetilde{p}_{\mathrm{N}0} \delta\dot{u}^k Q_k^{\ p} \mathrm{d}A_p = 0$$

from where further equations can be obtained:

$$\int_{(B)} \delta\dot{u}_{m;k}\left[\left(\delta^m_l + u^m_{\ ;l}\right) C^{klpq} E_{pq}\right]^{\cdot} \mathrm{d}V +$$

$$+ \frac{\partial}{\partial h}\left(\int_{(B)} \delta\dot{u}_{m;k}\left(\delta^m_l + u^m_{\ ;l}\right) C^{klpq} E_{pq} \mathrm{d}V\right)\dot{h} -$$

$$-\sum_{\mathrm{N=I}}^{\mathrm{II}}\Bigg[\dot{p}_{\mathrm{N}}\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N0}}\delta\dot{u}^{k}Q_{k}{}^{p}\mathrm{d}A_{p}+p_{\mathrm{N}}\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N0}}\delta\dot{u}^{k}\dot{Q}_{k}{}^{p}\mathrm{d}A_{p}+$$
$$+p_{\mathrm{N}}\frac{\partial}{\partial h}\left(\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N0}}\delta\dot{u}^{k}Q_{k}{}^{p}\mathrm{d}A_{p}\right)\dot{h}\Bigg]=0,$$

$$\int_{(B)}\delta\dot{u}_{m;k}\left[\left(\delta_{l}^{m}+u^{m}{}_{;l}\right)C^{klpq}E_{pq}\right]^{\cdot\cdot}\mathrm{d}V+$$
$$+2\frac{\partial}{\partial h}\left(\int_{(B)}\delta\dot{u}_{m;k}\left[\left(\delta_{l}^{m}+u^{m}{}_{;l}\right)C^{klpq}E_{pq}\right]^{\cdot}\mathrm{d}V\right)\dot{h}+$$
$$+\frac{\partial^{2}}{\partial h^{2}}\left(\int_{(B)}\delta\dot{u}_{m;k}\left(\delta_{l}^{m}+u^{m}{}_{;l}\right)C^{klpq}E_{pq}\mathrm{d}V\right)\dot{h}^{2}+$$
$$+\frac{\partial}{\partial h}\left(\int_{(B)}\delta\dot{u}_{m;k}\left(\delta_{l}^{m}+u^{m}{}_{;l}\right)C^{klpq}E_{pq}\mathrm{d}V\right)\ddot{h}-$$
$$-\sum_{\mathrm{N=I}}^{\mathrm{II}}\Bigg[\ddot{p}_{\mathrm{N}}\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N0}}\delta\dot{u}^{k}Q_{k}{}^{p}\mathrm{d}A_{p}+2\dot{p}_{\mathrm{N}}\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N0}}\delta\dot{u}^{k}\dot{Q}_{k}{}^{p}\mathrm{d}A_{p}+$$
$$+p_{\mathrm{N}}\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N0}}\delta\dot{u}^{k}\ddot{Q}_{k}{}^{p}\mathrm{d}A_{p}+2\dot{p}_{\mathrm{N}}\frac{\partial}{\partial h}\left(\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N0}}\delta\dot{u}^{k}Q_{k}{}^{p}\mathrm{d}A_{p}\right)\dot{h}+$$
$$+2p_{\mathrm{N}}\frac{\partial}{\partial h}\left(\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N0}}\delta\dot{u}^{k}\dot{Q}_{k}{}^{p}\mathrm{d}A_{p}\right)\dot{h}+p_{\mathrm{N}}\frac{\partial^{2}}{\partial h^{2}}\left(\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N0}}\delta\dot{u}^{k}Q_{k}{}^{p}\mathrm{d}A_{p}\right)\dot{h}^{2}+$$
$$+p_{\mathrm{N}}\frac{\partial}{\partial h}\left(\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N0}}\delta\dot{u}^{k}Q_{k}{}^{p}\mathrm{d}A_{p}\right)\ddot{h}\Bigg]=0.\quad(3.46)$$

The following equation is valid only under the condition that the imperfection parameter is constant, i.e., $h=h_0=\text{constant}$:

$$\int_{(B)}\delta\dot{u}_{m;k}\left[\left(\delta_{l}^{m}+u^{m}{}_{;l}\right)C^{klpq}E_{pq}\right]^{\cdots}\mathrm{d}V=\sum_{\mathrm{N=I}}^{\mathrm{II}}\left(p_{\mathrm{N}}\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N0}}\delta\dot{u}^{k}Q_{k}{}^{p}\mathrm{d}A_{p}\right)^{\cdots}.$$

In the sequel we shall detail the previous equilibrium equations by assuming that the imperfection parameter $h=h_0=\text{constant}$. We get the following relations:

$$\int_{(B)}\delta\dot{u}_{m;k}\left[\left(\delta_{l}^{m}+u^{m}{}_{;l}\right)^{\cdot}C^{klpq}E_{pq}+\left(\delta_{l}^{m}+u^{m}{}_{;l}\right)C^{klpq}\left(\delta_{p}^{s}+u^{s}{}_{;p}\right)\dot{u}_{s;q}\right]\mathrm{d}V-$$
$$-\sum_{\mathrm{N=I}}^{\mathrm{II}}\left[p_{\mathrm{N}}\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N0}}\delta\dot{u}^{k}e_{klm}e^{pqr}\left(\delta_{q}^{l}+u^{l}{}_{;q}\right)\dot{u}^{m}{}_{;r}\mathrm{d}A_{p}+\dot{p}_{\mathrm{N}}\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N0}}\delta\dot{u}^{k}Q_{k}{}^{p}\mathrm{d}A_{p}\right]=0,\quad(3.47)$$

$$\int_{(B)}\delta\dot{u}_{m;k}\left[\left(\delta_{l}^{m}+u^{m}{}_{;l}\right)^{\cdot\cdot}C^{klpq}E_{pq}+2\left(\delta_{l}^{m}+u^{m}{}_{;l}\right)^{\cdot}C^{klpq}\left(\delta_{p}^{s}+u^{s}{}_{;p}\right)\dot{u}_{s;q}\right]\mathrm{d}V+$$

$$+\int_{(B)}\delta\dot{u}_{m;k}\left(\delta_l^m+u^m{}_{;l}\right)C^{klpq}\left[\left(\delta_l^m+u^m{}_{;l}\right)^{\cdot}\dot{u}_{s;q}+\left(\delta_l^m+u^m{}_{;l}\right)\ddot{u}_{s;q}\right]\mathrm{d}V-$$
$$-\sum_{\mathrm{N=I}}^{\mathrm{II}}p_{\mathrm{N}}\int_{(A_{t\mathrm{N}})}\widetilde{p}_{\mathrm{N}0}\delta\dot{u}^k e_{klm}e^{pqr}\left[\left(\delta_q^l+u^l{}_{;q}\right)^{\cdot}\dot{u}^m{}_{;r}+\left(\delta_q^l+u^l{}_{;q}\right)\ddot{u}^m{}_{;r}\right]\mathrm{d}A_p-$$
$$-\sum_{\mathrm{N=I}}^{\mathrm{II}}\left[2\dot{p}_{\mathrm{N}}\int_{(A_{t\mathrm{N}})}\widetilde{p}_{\mathrm{N}0}\delta\dot{u}^k e_{klm}e^{pqr}\left(\delta_q^l+u^l{}_{;q}\right)\dot{u}^m{}_{;r}\mathrm{d}A_p+\right.$$
$$\left.+\ddot{p}_{\mathrm{N}}\int_{(A_{t\mathrm{N}})}\widetilde{p}_{\mathrm{N}0}\delta\dot{u}^k Q_k{}^p\mathrm{d}A_p\right]=0,\quad(3.48)$$

$$\int_{(B)}\delta\dot{u}_{m;k}\left[\left(\delta_l^m+u^m{}_{;l}\right)^{\cdots}C^{klpq}E_{pq}+3\left(\delta_l^m+u^m{}_{;l}\right)^{\cdot\cdot}C^{klpq}\left(\delta_p^s+u^s{}_{;p}\right)\dot{u}_{s;q}\right]\mathrm{d}V+$$
$$+\int_{(B)}\delta\dot{u}_{m;k}3\left(\delta_l^m+u^m{}_{;l}\right)^{\cdot}C^{klpq}\left[\left(\delta_p^s+u^s{}_{;p}\right)^{\cdot}\dot{u}_{s;q}+\left(\delta_p^s+u^s{}_{;p}\right)\ddot{u}_{s;q}\right]\mathrm{d}V+$$
$$+\int_{(B)}\delta\dot{u}_{m;k}\left(\delta_l^m+u^m{}_{;l}\right)C^{klpq}\left[\left(\delta_p^s+u^s{}_{;p}\right)^{\cdot\cdot}\dot{u}_{s;q}+2\left(\delta_p^s+u^s{}_{;p}\right)^{\cdot}\ddot{u}_{s;q}+\left(\delta_p^s+u^s{}_{;p}\right)\dddot{u}_{s;q}\right]\mathrm{d}V-$$
$$-\sum_{\mathrm{N=I}}^{\mathrm{II}}p_{\mathrm{N}}\int_{(A_{t\mathrm{N}})}\widetilde{p}_{\mathrm{N}0}\delta\dot{u}^k e_{klm}e^{pqr}\left[\left(\delta_q^l+u^l{}_{;q}\right)^{\cdot\cdot}\dot{u}^m{}_{;r}+2\left(\delta_q^l+u^l{}_{;q}\right)^{\cdot}\ddot{u}^m{}_{;r}+\left(\delta_q^l+u^l{}_{;q}\right)\dddot{u}^m{}_{;r}\right]\mathrm{d}A_p-$$
$$-\sum_{\mathrm{N=I}}^{\mathrm{II}}3\dot{p}_{\mathrm{N}}\int_{(A_{t\mathrm{N}})}\widetilde{p}_{\mathrm{N}0}\delta\dot{u}^k e_{klm}e^{pqr}\left[\left(\delta_q^l+u^l{}_{;q}\right)^{\cdot}\dot{u}^m{}_{;r}+\left(\delta_q^l+u^l{}_{;q}\right)^{\cdot}\ddot{u}^m{}_{;r}\right]\mathrm{d}A_p-$$
$$-\sum_{\mathrm{N=I}}^{\mathrm{II}}3\ddot{p}_{\mathrm{N}}\int_{(A_{t\mathrm{N}})}\widetilde{p}_{\mathrm{N}0}\delta\dot{u}^k e_{klm}e^{pqr}\left(\delta_q^l+u^l{}_{;q}\right)\dot{u}^m{}_{;r}\mathrm{d}A_p-$$
$$-\sum_{\mathrm{N=I}}^{\mathrm{II}}\dddot{p}_{\mathrm{N}}\int_{(A_{t\mathrm{N}})}\widetilde{p}_{\mathrm{N}0}\delta\dot{u}^k Q_k{}^p\mathrm{d}A_p=0.\qquad(3.49)$$

3.3. **Power of the inner and external forces in the equilibrium state $(\bar{B})$ of the body.** Since $\tau=0$ in the equilibrium state $(\bar{B})$ of the body the power sought is of the form

$$P=\sum_{\mathrm{N=I}}^{\mathrm{II}}p_{\mathrm{N}}\int_{(A_{t\mathrm{N}})}\widetilde{p}_{\mathrm{N}0}\left(\dot{u}^k Q_k{}^p\right)\mathrm{d}A_p-\int_{(B)}\left(\dot{E}_{kl}C^{klpq}E_{pq}\right)\mathrm{d}V,\qquad(3.50)$$

As regards the intermediate states $0<\tau<\Delta\tau$ ($p_{\mathrm{N}}=\mathrm{const}\,\mathrm{N}=\mathrm{I},\mathrm{II}$) we have

$$P(\tau)=P+\dot{P}\tau+\frac{1}{2}\ddot{P}\tau^2+\frac{1}{6}\dddot{P}\tau^3+\cdots=$$
$$=P+\left\{\sum_{\mathrm{N=I}}^{\mathrm{II}}p_{\mathrm{N}}\int_{(A_{t\mathrm{N}})}\widetilde{p}_{\mathrm{N}0}\left(\dot{u}^k Q_k{}^p\right)^{\cdot}\mathrm{d}A_p-\int_{(B)}\left(\dot{E}_{kl}C^{klpq}E_{pq}\right)^{\cdot}\mathrm{d}V+\right.$$

$$+\left[\sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}} \frac{\partial}{\partial h}\int_{(A_{\mathrm{tN}})} \widetilde{p}_{\mathrm{N0}}\left(\dot{u}^k Q_k{}^p\right)\mathrm{d}A_p - \frac{\partial}{\partial h}\int_{(B)}\left(\dot{E}_{kl}C^{klpq}E_{pq}\right)\mathrm{d}V\right]\dot{h}\Bigg\}\tau +$$

$$+\frac{1}{2}\left\{\sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}}\int_{(A_{\mathrm{tN}})} \widetilde{p}_{\mathrm{N0}}\left(\dot{u}^k Q_k{}^p\right)^{\cdot\cdot}\mathrm{d}A_p - \int_{(B)}\left(\dot{E}_{kl}C^{klpq}E_{pq}\right)^{\cdot\cdot}\mathrm{d}V +\right.$$

$$+2\left[\sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}} \frac{\partial}{\partial h}\int_{(A_{\mathrm{tN}})} \widetilde{p}_{\mathrm{N0}}\left(\dot{u}^k Q_k{}^p\right)^{\cdot}\mathrm{d}A_p - \frac{\partial}{\partial h}\int_{(B)}\left(\dot{E}_{kl}C^{klpq}E_{pq}\right)^{\cdot}\mathrm{d}V\right]\dot{h} +$$

$$+\left[\sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}} \frac{\partial^2}{\partial h^2}\int_{(A_{\mathrm{tN}})} \widetilde{p}_{\mathrm{N0}}\left(\dot{u}^k Q_k{}^p\right)\mathrm{d}A_p - \frac{\partial^2}{\partial h^2}\int_{(B)}\left(\dot{E}_{kl}C^{klpq}E_{pq}\right)\mathrm{d}V\right]\dot{h}^2 +$$

$$\left.+\left[\sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}} \frac{\partial}{\partial h}\int_{(A_{\mathrm{tN}})} \widetilde{p}_{\mathrm{N0}}\left(\dot{u}^k Q_k{}^p\right)\mathrm{d}A_p - \frac{\partial}{\partial h}\int_{(B)}\left(\dot{E}_{kl}C^{klpq}E_{pq}\right)\mathrm{d}V\right]\ddot{h}\right\}\tau^2 + \cdots \tag{3.51}$$

3.4. The work ΔL of the inner and external forces in the equilibrium state $(\bar{B})$ of the body.

3.4.1. *Work done on the kinematically admissible increment of the displacement field.* The kinematically admissible displacement field are determined by equation $(3.17\mathrm{a})_2$. The work done by the inner and external forces can be obtained by integrating equation (3.51) with respect to time:

$$\Delta L_{kin} = \int_{\tau=0}^{\Delta\tau} P(\tau)\,\mathrm{d}\tau = \Delta L_{kin}^{(1)} + \Delta L_{kin}^{(2)} + \Delta L_{kin}^{(3)} + \cdots . \tag{3.52}$$

By using the finite element formalism we may write

$$\Delta L_{kin}^{(1)} = -\dot{\mathbf{t}}_i^{\mathrm{T}}\left(\mathbf{b}_i - \sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}}\mathbf{g}_{\mathrm{N}i}\right)\Delta\tau = 0, \tag{3.53}$$

$$\Delta L_{kin}^{(2)} = -\left[\ddot{\mathbf{t}}_i^{\mathrm{T}}\left(\mathbf{b}_i - \sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}}\mathbf{g}_{\mathrm{N}i}\right) + \dot{\mathbf{t}}_i^{\mathrm{T}}\left(\mathbf{K}_{ij}^{\mathrm{L}} + \mathbf{K}_{ij}^{\mathrm{G}} - \sum_{\mathrm{N=I}}^{\mathrm{II}}\mathbf{K}_{\mathrm{N}ij}^{\mathrm{C}} p_{\mathrm{N}}\right)\dot{\mathbf{t}}_j + \right.$$
$$\left. + \dot{\mathbf{t}}_i^{\mathrm{T}}\left(\frac{\partial \mathbf{b}_i}{\partial h} - \sum_{\mathrm{N=I}}^{\mathrm{II}}\frac{\partial \mathbf{g}_{\mathrm{N}i}}{\partial h}p_{\mathrm{N}}\right)\dot{h}\right]\frac{\Delta\tau^2}{2}, \tag{3.54}$$

$$\Delta L_{kin}^{(3)} = -\left\{\dddot{\mathbf{t}}_i^{\mathrm{T}}\left(\mathbf{b}_i - \sum_{\mathrm{N=I}}^{\mathrm{II}} p_{\mathrm{N}}\mathbf{g}_{\mathrm{N}i}\right) + 2\ddot{\mathbf{t}}_i^{\mathrm{T}}\left(\mathbf{K}_{ij}^{\mathrm{L}} + \mathbf{K}_{ij}^{\mathrm{G}} - \sum_{\mathrm{N=I}}^{\mathrm{II}}\mathbf{K}_{\mathrm{N}ij}^{\mathrm{C}} p_{\mathrm{N}}\right)\dot{\mathbf{t}}_j + \right.$$

$$+\dot{\mathbf{t}}_i^{\mathrm{T}}\left(\mathbf{K}_{ij}^{\mathrm{L}} + \mathbf{K}_{ij}^{\mathrm{G}} - \sum\nolimits_{\mathrm{N=I}}^{\mathrm{II}}\mathbf{K}_{\mathrm{N}ij}^{\mathrm{C}} p_{\mathrm{N}}\right)\ddot{\mathbf{t}}_j + \dot{\mathbf{t}}_i^{\mathrm{T}}\left(2\mathbf{J}_{ikj} + \mathbf{H}_{ijk} - \sum_{\mathrm{N=I}}^{\mathrm{II}}\mathbf{M}_{\mathrm{N}ikj}p_{\mathrm{N}}\right)\dot{\mathbf{t}}_k\dot{\mathbf{t}}_j +$$

$$+2\left[\ddot{\mathbf{t}}_i^{\mathrm{T}}\left(\frac{\partial \mathbf{b}_i}{\partial h}-\sum_{\mathrm{N=I}}^{\mathrm{II}}\frac{\partial \mathbf{g}_{\mathrm{N}i}}{\partial h}p_{\mathrm{N}}\right)\dot{h}+\dot{\mathbf{t}}_i^{\mathrm{T}}\frac{\partial}{\partial h}\left(\mathbf{K}_{ij}^{\mathrm{L}}+\mathbf{K}_{ij}^{\mathrm{G}}-\sum_{\mathrm{N=I}}^{\mathrm{II}}\mathbf{K}_{\mathrm{N}ij}^{\mathrm{C}}p_{\mathrm{N}}\right)\dot{\mathbf{t}}_j\dot{h}\right]+$$
$$+\dot{\mathbf{t}}_i^{\mathrm{T}}\left(\frac{\partial^2 \mathbf{b}_i}{\partial h^2}-\sum_{\mathrm{N=I}}^{\mathrm{II}}\frac{\partial^2 \mathbf{g}_{\mathrm{N}i}}{\partial h^2}p_{\mathrm{N}}\right)\dot{h}^2+\dot{\mathbf{t}}_i^{\mathrm{T}}\left(\frac{\partial \mathbf{b}_i}{\partial h}-\sum_{\mathrm{N=I}}^{\mathrm{II}}\frac{\partial \mathbf{g}_{\mathrm{N}i}}{\partial h}p_{\mathrm{N}}\right)\ddot{h}\Bigg\}\frac{\Delta\tau^3}{6}. \qquad (3.55)$$

Equation (3.53) reflects the equilibrium of state $(\bar{B})$ of the body.

3.4.2. *Work on the displacement increment taken on the equilibrium path.* Consider the equilibrium path to the state $(\bar{B}+\Delta\bar{B})$ determined by the load increment $(3.17\mathrm{d})_2$ and the geometrical imperfection parameter. The work of the inner and external forces on this path can be calculated by using equations (3.53)-(3.55) provided that the equilibrium equations (3.19), (3.41) and (3.42) are also taken into account:

$$\Delta L_{\mathrm{path}}=\Delta L^{(1)}+\Delta L^{(2)}+\Delta L^{(3)}+\cdots \qquad (3.56)$$

Here $\Delta L^{(1)}=0$ and

$$\Delta L^{(2)}=-\sum_{\mathrm{N=I}}^{\mathrm{II}}\left[\dot{\mathbf{t}}_i^{\mathrm{T}}\mathbf{g}_{\mathrm{N}i}\dot{p}_{\mathrm{N}}\right]\frac{\Delta\tau^2}{2}=-\left[\sum_{\mathrm{N=I}}^{\mathrm{II}}\dot{p}_{\mathrm{N}}\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N}0}\left(\dot{u}^k Q_k{}^p\right)\mathrm{d}A_p\right]\frac{\Delta\tau^2}{2} \qquad (3.57)$$

$$\Delta L^{(3)}=-\sum_{\mathrm{N=I}}^{\mathrm{II}}\left[\left(\ddot{\mathbf{t}}_i^{\mathrm{T}}\mathbf{g}_{\mathrm{N}i}+\dot{\mathbf{t}}_i^{\mathrm{T}}\mathbf{K}_{\mathrm{N}ij}^{\mathrm{C}}\dot{\mathbf{t}}_j\right)2\dot{p}_{\mathrm{N}}+\dot{\mathbf{t}}_i^{\mathrm{T}}\mathbf{g}_{\mathrm{N}i}\ddot{p}_{\mathrm{N}}+\dot{\mathbf{t}}_i^{\mathrm{T}}2\frac{\partial \mathbf{g}_{\mathrm{N}i}}{\partial h}\dot{p}_{\mathrm{N}}\dot{h}\right]\frac{\Delta\tau^3}{6}=$$
$$=-\sum_{\mathrm{N=I}}^{\mathrm{II}}\left[2\dot{p}_{\mathrm{N}}\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N}0}\left(\ddot{u}^k Q_k{}^p+\dot{u}^k\dot{Q}_k{}^p\right)\mathrm{d}A_p+\ddot{p}_{\mathrm{N}}\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N}0}\left(\dot{u}^k Q_k{}^p\right)\mathrm{d}A_p+\right.$$
$$\left.+\,2\dot{p}_{\mathrm{N}}\dot{h}\frac{\partial}{\partial h}\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N}0}\left(\dot{u}^k Q_k{}^p\right)\mathrm{d}A_p\right]\frac{\Delta\tau^3}{6}, \qquad (3.58)$$

$$\Delta L^{(4)}=-\sum_{\mathrm{N=I}}^{\mathrm{II}}\left[3\dot{p}_{\mathrm{N}}\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N}0}\left(\dddot{u}^k Q_k{}^p+2\ddot{u}^k\dot{Q}_k{}^p+\dot{u}^k\ddot{Q}_k{}^p\right)\mathrm{d}A_p+\right.$$
$$+\,3\ddot{p}_{\mathrm{N}}\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N}0}\left(\ddot{u}^k Q_k{}^p+\dot{u}^k\dot{Q}_k{}^p\right)\mathrm{d}A_p+\dddot{p}_{\mathrm{N}}\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N}0}\left(\dot{u}^k Q_k{}^p\right)\mathrm{d}A_p+$$
$$+\,6\,\dot{p}_{\mathrm{N}}\dot{h}\frac{\partial}{\partial h}\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N}0}\left(\dot{u}^k Q_k{}^p\right)^{\boldsymbol{\cdot}}\mathrm{d}A_p+3\dot{p}_{\mathrm{N}}\dot{h}^2\frac{\partial^2}{\partial h^2}\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N}0}\left(\dot{u}^k Q_k{}^p\right)\mathrm{d}A_p+$$
$$\left.+3\,\ddot{p}_{\mathrm{N}}\dot{h}\frac{\partial}{\partial h}\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N}0}\left(\dot{u}^k Q_k{}^p\right)\mathrm{d}A_p+3\dot{p}_{\mathrm{N}}\ddot{h}\frac{\partial}{\partial h}\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N}0}\left(\dot{u}^k Q_k{}^p\right)\mathrm{d}A_p\right]\frac{\Delta\tau^4}{24}. \qquad (3.59)$$

$\Delta L^{(3)}$ and $\Delta L^{(4)}$ can be given in another form as well:

$$\Delta L^{(3)}=\left(\Delta L^{(2)}\right)^{\boldsymbol{\cdot}}\frac{\Delta\tau}{3}-\sum_{\mathrm{N=I}}^{\mathrm{II}}\dot{p}_{\mathrm{N}}\left[\int_{(A_{\mathrm{tN}})}\widetilde{p}_{\mathrm{N}0}\left(\dot{u}^k Q_k{}^p\right)\mathrm{d}A_p\right]^{\boldsymbol{\cdot}}\frac{\Delta\tau^3}{6}, \qquad (3.60)$$

$$\Delta L^{(4)} = \left(\Delta L^{(3)}\right)^{\cdot} \frac{\Delta\tau}{4} -$$
$$- \sum_{\mathrm{N=I}}^{\mathrm{II}} \left\{ \dot{p}_{\mathrm{N}} \int_{(A_{\mathrm{tN}})} \widetilde{p}_{\mathrm{N0}} \left(\dot{u}^k Q_k{}^p\right)^{\cdot\cdot} \mathrm{d}A_p + \dddot{p}_{\mathrm{N}} \int_{(A_{\mathrm{tN}})} \widetilde{p}_{\mathrm{N0}} \left(\dot{u}^k Q_k{}^p\right) \mathrm{d}A_p + \right.$$
$$+ \dot{p}_{\mathrm{N}} \dot{h}^2 \frac{\partial^2}{\partial h^2} \int_{(A_{\mathrm{tN}})} \widetilde{p}_{\mathrm{N0}} \left(\dot{u}^k Q_k{}^p\right) \mathrm{d}A_p + 4\dot{p}_{\mathrm{N}} \dot{h} \frac{\partial}{\partial h} \int_{(A_{\mathrm{tN}})} \widetilde{p}_{\mathrm{N0}} \left(\dot{u}^k Q_k{}^p\right)^{\cdot} \mathrm{d}A_p +$$
$$\left. + \left(\dot{p}_{\mathrm{N}} \dot{h}\right)^{\cdot} \frac{\partial}{\partial h} \int_{(A_{\mathrm{tN}})} \widetilde{p}_{\mathrm{N0}} \left(\dot{u}^k Q_k{}^p\right) \mathrm{d}A_p \right\} \frac{\Delta\tau^4}{24} . \qquad (3.61)$$

Time derivatives $\left(\Delta L^{(3)}\right)^{\cdot}$, $\left(\Delta L^{(3)}\right)^{\cdot}$ in equations (3.60)) and (3.61) imply that the surface integrals depend on the geometrical imperfection parameter h as well.

3.5. **Stability criteria.** The equilibrium state of a solid body is stable if the work done by the inner and external forces of the equilibrium state on any kinematically admissible and sufficiently small displacement increment is negative.

This axiomatic statement is based on the following theorem of mechanics: Assume that the equilibrium state of the solid body is disturbed in such a way that it has some but low kinetic energy (this is the case of a small disturbation). If the kinetic energy is decreased on any kinematically admissible displacement increment (the work done by the inner and external forces of the equilibrium state is negative) then the motion of the solid body remains in a small neighborhood of the equilibrium state.

In the opposite case, i.e., when the work done by the inner and external forces of the equilibrium state is zero or positive the equilibrium state is not stable.

It follows from the above definition of stability that the work ΔL_{kin} is applicable to characterizing the stability of equilibrium both in pre-critical (before stability loss) and post-critical states of the body.

The numerical investigations should be carried out on the equilibrium paths.

A summary for the stability criteria can be given on the basis of all that has been said above by utilizing the work increment $\Delta L_{path} = \Delta L^{(2)} + \Delta L^{(3)} + \Delta L^{(4)} \cdots$ which can be calculated with equations (3.57)-(3.59):

The equilibrium state at a point on the equilibrium path is

$$\left\{ \begin{array}{l} \text{stable} \\ \text{critical} \\ \text{instable} \end{array} \right\} \quad \text{if} \quad \left\{ \begin{array}{l} \Delta L^{(2)} < 0 \\ \Delta L^{(2)} = 0 \\ \Delta L^{(2)} > 0 \end{array} \right\}. \qquad (3.62)$$

The critical point is a

$$\left\{ \begin{array}{l} \text{limit point} \\ \text{branching point} \end{array} \right\} \quad \text{if} \quad \left\{ \begin{array}{c} \Delta L^{(3)} \neq 0 \\ \Delta L^{(3)} = 0, \text{ but } \Delta L^{(4)} \neq 0 \end{array} \right\}. \qquad (3.63)$$

We speak about a total stability analysis if the investigations are carried out not only at a few points on the equilibrium path but in an interval of the problem parameters which includes the pre-critical and post-critical states of the body as well.

4. Characteristics of critical states

4.1. **Definitions.**

4.1.1. *Equilibrium conditions.* According to equations (3.22) $(3.23)_2$ and (3.41) in an equilibrium state it holds that

$$\mathbf{A}_i = \mathbf{b}_i - \mathbf{g}_{\mathrm{I}i}p_{\mathrm{I}} - \mathbf{g}_{\mathrm{II}i}p_{\mathrm{II}} = 0, \tag{4.1}$$

$$\dot{\mathbf{A}}_i = \mathbf{K}_{ij}\dot{\mathbf{t}}_j - \mathbf{g}_{\mathrm{I}i}\dot{p}_{\mathrm{I}} - \mathbf{g}_{\mathrm{II}i}\dot{p}_{\mathrm{II}} + \frac{\partial\mathbf{A}_i}{\partial h}\dot{h} = 0, \tag{4.2}$$

where

$$\mathbf{K}_{ij} = \frac{\partial\mathbf{A}_i}{\partial\mathbf{t}_j} = \mathbf{K}^{\mathrm{L}}_{ij} + \mathbf{K}^{\mathrm{G}}_{ij} - \mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}p_{\mathrm{I}} - \mathbf{K}^{\mathrm{C}}_{\mathrm{II}ij}p_{\mathrm{II}}.$$

In a critical equilibrium state equation (4.2) is undetermined hence

$$\det\mathbf{K}^{\mathrm{cr}}_{ij} = 0,$$

from where it follows that

$$\mathbf{K}^{\mathrm{cr}}_{ij}\mathbf{u}_j = 0, \qquad \mathbf{v}^{\mathrm{T}}_i\mathbf{K}^{\mathrm{cr}}_{ij} = 0, \tag{4.3}$$

in which $\mathbf{u}_j$ and $\mathbf{v}_i$ are the right and left eigenvectors, respectively. It is assumed that they belong to the same simple eigenvalue (to an eigenvalue of multiplicity one).

As is well known $\mathbf{u}_j = \mathbf{v}_j$ if $\mathbf{K}_{ij}$ is symmetric.

In the present case the loads are non-conservative deformation independent surface loads hence the matrix $\mathbf{K}_{ij}$ is not symmetric. Consequently $\mathbf{u}_j \neq \mathbf{v}_j$.

After multiplying equation (4.1) by $\mathbf{v}_i$ we get on the basis of $(4.3)_2$ that

$$\mathbf{v}^{\mathrm{T}}_i\mathbf{g}^{\mathrm{cr}}_{\mathrm{I}i}\dot{p}^{\mathrm{cr}}_{\mathrm{I}} + \mathbf{v}^{\mathrm{T}}_i\mathbf{g}^{\mathrm{cr}}_{\mathrm{II}i}\dot{p}^{\mathrm{cr}}_{\mathrm{II}} - \mathbf{v}^{\mathrm{T}}_i\left(\frac{\partial\mathbf{A}_i}{\partial h}\right)^{\mathrm{cr}}\dot{h}^{\mathrm{kr}} = 0\,. \tag{4.4}$$

Remark 4.1: According to Subsection 3.1 p_{II} can be either the parameter of a supplementary load or a load imperfection parameter (load disturbance).

4.1.2. *The geometrical imperfection parameter h = constant.* According to the definition of Subsection 3.2.3 equilibrium paths can be obtained in the cases h = constant or p_{II} = constant. In the sequel these two cases are considered one by one.

Assume that h = constant ($\dot{h} = 0$) and independently of each other either p_I varies (p_{II} = constant) or p_{II} varies (p_I = constant). Assume further that (4.4) is satisfied. Then the critical points on the equilibrium paths can be classified as follows:

– the critical point is a limit point if

$$\dot{p}^{\mathrm{cr}}_{\mathrm{I}} = 0,\ \dot{p}_{\mathrm{II}} = 0 \quad\text{or}\quad \dot{p}^{\mathrm{cr}}_{\mathrm{II}} = 0,\ \dot{p}_{\mathrm{I}} = 0, \tag{4.5}$$

– the critical point is a branching point if

$$\mathbf{v}^{\mathrm{T}}_i\mathbf{g}^{\mathrm{cr}}_{\mathrm{I}i} = 0,\ \dot{p}_{\mathrm{II}} = 0 \quad\text{or}\quad \mathbf{v}^{\mathrm{T}}_i\mathbf{g}^{\mathrm{cr}}_{\mathrm{II}i} = 0,\ \dot{p}_{\mathrm{I}} = 0. \tag{4.6}$$

Let us return now to the conditions the critical points should meet – these are detailed in Subsection 3.5. According to equations (3.57), (4.5), (4.6) and Remark 4.2 condition

$$\Delta L^{(2)\mathrm{cr}} = -\left(\mathbf{v}^{\mathrm{T}}_i\mathbf{g}^{\mathrm{cr}}_{\mathrm{I}i}\dot{p}^{\mathrm{cr}}_{\mathrm{I}} + \mathbf{v}^{\mathrm{T}}_i\mathbf{g}^{\mathrm{cr}}_{\mathrm{II}i}\dot{p}^{\mathrm{cr}}_{\mathrm{II}}\right)\frac{\Delta\tau^2}{2} = 0$$

for the critical point on the equilibrium path ($\dot{h} = 0$!) is satisfied at the limit point.

REMARK 4.2: Equation (4.5) is satisfied at a limit point. Hence it follows from equation (4.2) of the critical equilibrium state ($\dot{h} = 0$!) that

$$\mathbf{K}_{ij}^{\mathrm{cr}}\dot{\mathbf{t}}_j^{\mathrm{cr}} = 0\,. \tag{4.7}$$

Equation (4.7) does not determine uniquely the order of magnitude of $\dot{\mathbf{t}}_j^{\mathrm{cr}}$ since $\dot{\mathbf{t}}_j^{\mathrm{cr}}$ is obtained from $\mathbf{u}_j$ by multiplying the latter with a scalar. This follows from detailing equation 4.7:

$$\mathbf{K}_{ij}^{\mathrm{cr}}\dot{\mathbf{t}}_j^{\mathrm{cr}} = \begin{bmatrix} & & & \vdots & \\ & \mathbf{K}_{\alpha\beta}^{\mathrm{cr}} & & \vdots & \mathbf{k}_{\alpha m}^{\mathrm{cr}} \\ & & & \vdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & \mathbf{k}_{m\beta}^{\mathrm{Tcr}} & & \vdots & k_{mm}^{\mathrm{cr}} \end{bmatrix} \begin{bmatrix} \vdots \\ \dot{\mathbf{t}}_\beta^{\mathrm{cr}} \\ \vdots \\ \cdots \\ \dot{t}_m^{\mathrm{cr}} \end{bmatrix} = 0, \tag{4.8}$$

where $\alpha, \beta = 1, 2, \cdots, m-1$ and m is the number of freedom a (k_{mm}^{cr} is a scalar!). Equation (4.8) can also be given in the form

$$\begin{bmatrix} & & & \vdots & \\ & \mathbf{K}_{\alpha\beta}^{\mathrm{cr}} & & \vdots & \mathbf{k}_{\alpha m}^{\mathrm{cr}} \\ & & & \vdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & \mathbf{k}_{m\beta}^{\mathrm{Tcr}} & & \vdots & k_{mm}^{\mathrm{cr}} \end{bmatrix} \begin{bmatrix} \vdots \\ \frac{\dot{\mathbf{t}}_\beta^{\mathrm{cr}}}{\dot{t}_m^{\mathrm{cr}}} \\ \vdots \\ \cdots \\ 1 \end{bmatrix} = 0 \tag{4.9}$$

which proves the above statement.

REMARK 4.3: Though m is the number of freedom, it can be the subscript for any non-zero displacement component in the corresponding equations.

REMARK 4.4: All that has been said above is valid independently of the fact what quantity is the governing parameter: that can be either a displacement component or a load.

4.1.3. *The load imperfection parameter* $p_{\mathrm{II}} =$ *constant.* Assume that $p_{II} =$ constant ($\dot{p}_{II} = 0$) and independently of each other either p_I varies ($h =$ constant) or h varies ($p_I =$ constant). Assume further that (4.4) is satisfied. Then the critical points on the equilibrium paths can be classified as follows:

– the critical point is a limit point if

$$\dot{p}_{\mathrm{I}}^{\mathrm{cr}} = 0,\ \dot{h} = 0 \quad \text{or} \quad \dot{h}^{\mathrm{cr}} = 0,\ \dot{p}_{\mathrm{I}} = 0, \tag{4.10}$$

– the critical point is a branching point if

$$\mathbf{v}_i^{\mathrm{T}}\mathbf{g}_{\mathrm{I}i}^{\mathrm{cr}} = 0,\ \dot{h} = 0 \quad \text{or} \quad \mathbf{v}_i^{\mathrm{T}}\left(\frac{\partial\,\mathbf{A}_i}{\partial h}\right)^{\mathrm{cr}} = 0,\ \dot{p}_{\mathrm{I}} = 0. \tag{4.11}$$

4.2. **Neighborhood of critical points.** Assume that the critical equilibrium state is known, i.e., it is known what the problem parameters are for the critical equilibrium state. It is a fundamental assumption for investigating the neighborhood of the critical points that under the condition h = constant either $p_{\rm I}$ = constant and $p_{\rm II}$ varies or $p_{\rm II}$ = constant and $p_{\rm I}$ varies.

4.2.1. *Branching point.* Let s [the arc length of the branching (or secondary) equilibrium path: $\tau = s - s_0$] be the governing parameter on the branching equilibrium path. Further let $p_{\rm I}$ ($\tau = p_{\rm I} - p_{\rm I}^{cr}$) be the governing parameter on the primary equilibrium path. Let us approach the neighborhood of the branching point on the equilibrium paths assuming that $p_{\rm II}$ = constant, h = constant both on the primary equilibrium path and on the branching equilibrium path.

According to relations (3.41) and (3.42) :

$$\text{if} \quad ()^{\cdot} = \frac{\partial\,()}{\partial s}, \qquad \mathbf{L}_{ikj} = 2\mathbf{J}_{ikj} + \mathbf{H}_{ikj} - \mathbf{M}_{\mathrm{I}ikj} p_{\mathrm{I}},$$

$$\mathbf{K}_{ij}\dot{\mathbf{t}}_j - \mathbf{g}_{\mathrm{I}i}\dot{p}_{\mathrm{I}} = 0, \qquad \mathbf{K}_{ij}\ddot{\mathbf{t}}_j - \mathbf{g}_{\mathrm{I}i}\ddot{p}_{\mathrm{I}} = -\mathbf{L}_{ikj}\dot{\mathbf{t}}_k\dot{\mathbf{t}}_j + 2\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\dot{\mathbf{t}}_j\dot{p}_{\mathrm{I}}, \tag{4.12}$$

and

$$\text{if} \quad ()^{\nabla} = \frac{\partial\,()}{\partial p_{\mathrm{I}}}, \qquad p_{\mathrm{I}}^{\nabla} = 1,$$

$$\mathbf{K}_{ij}\mathbf{t}_j^{\nabla} - \mathbf{g}_{\mathrm{I}i} = 0, \qquad \mathbf{K}_{ij}\mathbf{t}_j^{\nabla\nabla} = -\mathbf{L}_{ikj}\mathbf{t}_k^{\nabla}\mathbf{t}_j^{\nabla} + 2\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\mathbf{t}_j^{\nabla} \tag{4.13}$$

are the equilibrium equations on the branching and primary equilibrium paths, respectively.

Consider equations (4.12) and (4.13) at the branching point and let

$$\dot{\mathbf{t}}_j^{\mathrm{cr}} = \xi_0\mathbf{t}_j^{\nabla\mathrm{cr}} + \xi_1\mathbf{u}_j, \qquad \xi_0 = \dot{p}_{\mathrm{I}}^{\mathrm{cr}} \tag{4.14}$$

on the branching equilibrium path. Substituting into $(4.12)_1$ and taking $(4.13)_1$ into account we have

$$\xi_1\mathbf{K}_{ij}^{\mathrm{cr}}\mathbf{u}_j = 0, \tag{4.15}$$

which means that we get back $(4.3)_1$ at the branching point.

Let us multiply equilibrium equation $(4.2)_2$ by $\mathbf{v}_i$:

$$\mathbf{v}_i^{\mathrm{T}}\mathbf{K}_{ij}^{\mathrm{cr}}\ddot{\mathbf{t}}_j^{\mathrm{cr}} - \mathbf{v}_i^{\mathrm{T}}\mathbf{g}_{\mathrm{I}i}^{\mathrm{cr}}\ddot{p}_{\mathrm{I}}^{\mathrm{cr}} = -\mathbf{v}_i^{\mathrm{T}}\mathbf{L}_{ikj}^{\mathrm{cr}}\dot{\mathbf{t}}_k^{\mathrm{cr}}\dot{\mathbf{t}}_j^{\mathrm{cr}} + 2\mathbf{v}_i^{\mathrm{T}}\mathbf{K}_{\mathrm{I}ij}^{\mathrm{C\,cr}}1\dot{p}_{\mathrm{I}}^{\mathrm{cr}}\dot{\mathbf{t}}_j^{\mathrm{cr}}.$$

Utilizing $(4.3)_2$ and (4.6) and substituting then (4.14) we arrive at the following equation:

$$0 = -\left[\mathbf{v}_i^{\mathrm{T}}\mathbf{L}_{ikj}^{\mathrm{cr}}\left(\xi_0\mathbf{t}_k^{\nabla\mathrm{cr}} + \xi_1\mathbf{u}_k\right)\left(\xi_0\mathbf{t}_j^{\nabla\mathrm{cr}} + \xi_1\mathbf{u}_j\right) - 2\mathbf{v}_i^{\mathrm{T}}\mathbf{K}_{\mathrm{I}ij}^{\mathrm{C\,cr}}\left(\xi_0\mathbf{t}_j^{\nabla\mathrm{kr}} + \xi_1\mathbf{u}_j\right)\xi_0\right], \tag{4.16}$$

or

$$a_{\mathrm{B}}\xi_1^2 + 2b_{\mathrm{B}}\xi_1\xi_0 + c_{\mathrm{B}}\xi_0^2 = 0. \tag{4.17}$$

where

$$a_{\mathrm{B}} = \mathbf{v}_i^{\mathrm{T}}\mathbf{L}_{ikj}^{\mathrm{cr}}\mathbf{u}_k\mathbf{u}_j, \tag{4.18}$$

$$b_{\mathrm{B}} = \mathbf{v}_i^{\mathrm{T}}\mathbf{L}_{ikj}^{\mathrm{cr}}\mathbf{t}_k^{\nabla\mathrm{cr}}\mathbf{u}_j - \mathbf{v}_i^{\mathrm{T}}\mathbf{K}_{\mathrm{I}ij}^{\mathrm{C\,cr}}\mathbf{u}_j, \tag{4.19}$$

$$c_{\mathrm{B}} = \mathbf{v}_i^{\mathrm{T}}\mathbf{L}_{ikj}^{\mathrm{cr}}\mathbf{t}_k^{\nabla\mathrm{cr}}\mathbf{t}_j^{\nabla\mathrm{cr}} - 2\mathbf{v}_i^{\mathrm{T}}\mathbf{K}_{\mathrm{I}ij}^{\mathrm{C\,cr}}\mathbf{t}_j^{\nabla\mathrm{cr}}. \tag{4.20}$$

With the normalization condition $2\xi_0^2+\xi_1^2=1$ equation (4.17) yields ξ_1. The coefficients in equation (4.18) make possible a further classification of the branching point. If

$$\left\{\begin{array}{l} a_{\rm B}=0,\ b_{\rm B}\neq 0\\ a_{\rm B}\neq 0,\ b_{\rm B}^2-a_{\rm B}c_{\rm B}>0\\ b_{\rm B}^2-a_{\rm B}c_{\rm B}<0\\ b_{\rm B}^2-a_{\rm B}c_{\rm B}=0\end{array}\right\}\ \text{the point is a simple}\ \left\{\begin{array}{l}\text{pitchfork bifurcation point,}\\ \text{transcritical bifurcation point,}\\ \text{isola formating point,}\\ \text{cusp point.}\end{array}\right\}$$

After multiplying relation $(4.13)_2$ by $\mathbf{v}_i$ and taking then $(4.3)_2$ into account we have

$$0=-\ \mathbf{v}_i^{\rm T}\mathbf{L}_{ikj}\mathbf{t}_k^{\nabla}\mathbf{t}_j^{\nabla}+2\mathbf{v}_i^{\rm T}\mathbf{K}_{{\rm I}ij}^{\rm C}\mathbf{t}_j^{\nabla}\,. \tag{4.21}$$

If ξ_1 is set to 1 a comparison of (4.16) and (4.21) yields

$$\xi_0=\dot p_{\rm I}^{\rm cr}=-\frac{\mathbf{v}_i^{\rm T}\mathbf{L}_{ikj}\mathbf{u}_k\mathbf{u}_j}{2\left(\mathbf{v}_i^{\rm T}\mathbf{L}_{ikj}^{\rm cr}\mathbf{t}_k^{\nabla\rm cr}\mathbf{u}_j-\mathbf{v}_i^{\rm T}\mathbf{K}_{{\rm I}ij}^{\rm C\,cr}\mathbf{u}_j\right)}\,. \tag{4.22}$$

For a symmetric branching it holds that

$$\mathbf{v}_i^{\rm T}\mathbf{L}_{ikj}^{\rm cr}\mathbf{u}_k\mathbf{u}_j=0\quad\text{hence}\quad \dot p_{\rm I}^{\rm cr}=0 \tag{4.23}$$

Let us supplement our investigations by the equilibrium equations at the branching point, which are presented here on the basis of the later equations (5.35) and (5.40).

On the bifurcation equilibrium path

$$\begin{aligned}\mathbf{K}_{ij}^{\rm cr}\dddot{\mathbf{t}}_j^{\rm cr}-\mathbf{g}_{{\rm I}i}^{\rm cr}\dddot{p}_{\rm I}^{\rm cr}+3\mathbf{L}_{ikj}^{\rm cr}\dot{\mathbf{t}}_k^{\rm cr}\ddot{\mathbf{t}}_j^{\rm cr}-3\mathbf{K}_{{\rm I}ij}^{\rm C\,cr}\ddot{\mathbf{t}}_j^{\rm cr}\dot p_{\rm I}^{\rm cr}-&\\ -3\mathbf{K}_{{\rm I}ij}^{\rm C\,cr}\dot{\mathbf{t}}_j^{\rm cr}\ddot p_{\rm I}^{\rm cr}-3\mathbf{M}_{{\rm I}ikj}^{\rm cr}\dot{\mathbf{t}}_k^{\rm cr}\dot{\mathbf{t}}_j^{\rm cr}\dot p_{\rm I}^{\rm cr}+\mathbf{N}_{ilkj}^{\rm cr}\dot{\mathbf{t}}_l^{\rm cr}\dot{\mathbf{t}}_k^{\rm cr}\dot{\mathbf{t}}_j^{\rm cr}&=0,\end{aligned} \tag{4.24}$$

whereas on the primary equilibrium path

$$\begin{aligned}\mathbf{K}_{ij}^{\rm cr}\mathbf{t}_j^{\nabla\nabla\nabla\rm cr}+3\mathbf{L}_{ikj}^{\rm cr}\mathbf{t}_k^{\nabla\rm cr}\mathbf{t}_j^{\nabla\nabla\rm cr}-3\mathbf{K}_{{\rm I}ij}^{\rm C\,cr}\mathbf{t}_j^{\nabla\nabla\rm cr}-&\\ -3\mathbf{M}_{{\rm I}ikj}^{\rm cr}\mathbf{t}_k^{\nabla\rm cr}\mathbf{t}_j^{\nabla\rm cr}+\mathbf{N}_{ilkj}^{\rm cr}\mathbf{t}_l^{\nabla\rm cr}\mathbf{t}_k^{\nabla\rm cr}\mathbf{t}_j^{\nabla\rm cr}&=0.\end{aligned} \tag{4.25}$$

are the equilibrium equations.

On the basis of (4.14) $(\xi_1=1)$

$$\ddot{\mathbf{t}}_j=\frac{\partial^2\mathbf{t}_j}{\partial p_{\rm I}^2}\left(\frac{\partial p_{\rm I}}{\partial s}\right)^2+\frac{\partial\mathbf{t}_j}{\partial p_{\rm I}}\frac{\partial^2 p_{\rm I}}{\partial s^2}+\frac{\partial\mathbf{u}_j}{\partial s},$$

which means that

$$\ddot{\mathbf{t}}_j^{\rm cr}=\mathbf{t}_j^{\nabla\nabla\rm cr}\dot p_{\rm I}^{2\rm cr}+\mathbf{t}_j^{\nabla\rm cr}\ddot p_{\rm I}^{\rm cr}+\dot{\mathbf{u}}_j \tag{4.26}$$

at the branching point

Let us substitute (4.26) and (4.14) into $(4.12)_2$ taken at the branching point:

$$\begin{aligned}\mathbf{K}_{ij}^{\rm cr}\left(\mathbf{t}_j^{\nabla\nabla\rm cr}\dot p_{\rm I}^{2\rm cr}+\mathbf{t}_j^{\nabla\rm cr}\ddot p_{\rm I}^{\rm cr}+\dot{\mathbf{u}}_j\right)-\mathbf{g}_{{\rm I}i}^{\rm cr}\ddot p_{\rm I}+&\\ +\mathbf{L}_{ikj}^{\rm cr}\left(\mathbf{t}_k^{\nabla\rm cr}\dot p_{\rm I}^{\rm cr}+\mathbf{u}_k\right)\left(\mathbf{t}_j^{\nabla\rm cr}\dot p_{\rm I}^{\rm cr}+\mathbf{u}_j\right)-2\mathbf{K}_{{\rm I}ij}^{\rm C\,cr}\left(\mathbf{t}_j^{\nabla\rm cr}\dot p_{\rm I}^{\rm cr}+\mathbf{u}_j\right)\dot p_{\rm I}^{\rm cr}&=0.\end{aligned}$$

By utilizing equations (4.13) it follows from here that

$$\mathbf{K}_{ij}^{\rm cr}\dot{\mathbf{u}}_j+2\mathbf{L}_{ikj}^{\rm cr}\mathbf{t}_k^{\nabla\rm cr}\mathbf{u}_j\dot p_{\rm I}^{\rm cr}+\mathbf{L}_{ikj}^{\rm cr}\mathbf{u}_k\mathbf{u}_j-2\mathbf{K}_{{\rm I}ij}^{\rm C\,cr}\mathbf{u}_j\dot p_{\rm I}^{\rm cr}=0\,. \tag{4.27}$$

Multiply now this equation by $\mathbf{v}_i$. In view to equation (4.15) we get

$$2\mathbf{v}_i^{\mathrm{T}}\mathbf{L}_{ikj}^{\mathrm{cr}}\mathbf{t}_j^{\nabla\mathrm{cr}}\mathbf{u}_j\,\dot{p}_{\mathrm{I}}^{\mathrm{cr}} + \mathbf{v}_i^{\mathrm{T}}\mathbf{L}_{ikj}^{\mathrm{cr}}\mathbf{u}_k\mathbf{u}_j - 2\mathbf{v}_i^{\mathrm{T}}\mathbf{K}_{\mathrm{I}ij}^{\mathrm{C\,cr}}\mathbf{u}_j\dot{p}_{\mathrm{I}}^{\mathrm{cr}} = 0\,. \tag{4.28}$$

We can find (4.22) from (4.28) and $\dot{\mathbf{u}}_j$ from 4.27.

Let us multiply equation (4.24) by $\mathbf{v}_i$ and substitute then relations (4.14) and (4.26). We have

$$\begin{aligned}
&3\mathbf{v}_i^{\mathrm{T}}\mathbf{L}_{ikj}^{\mathrm{cr}}\left(\mathbf{t}_k^{\nabla\mathrm{cr}}\dot{p}_{\mathrm{I}}^{\mathrm{cr}} + \mathbf{u}_k\right)\left(\mathbf{t}_j^{\nabla\nabla\mathrm{cr}}\dot{p}_{\mathrm{I}}^{2\,\mathrm{cr}} + \mathbf{t}_j^{\nabla\mathrm{cr}}\ddot{p}_{\mathrm{I}}^{\mathrm{cr}} + \dot{\mathbf{u}}_j\right) - \\
&\quad - 3\mathbf{v}_i^{\mathrm{T}}\mathbf{K}_{\mathrm{I}ij}^{\mathrm{C\,cr}}\left(\mathbf{t}_j^{\nabla\nabla\mathrm{cr}}\dot{p}_{\mathrm{I}}^{2\,\mathrm{cr}} + \mathbf{t}_j^{\nabla\mathrm{cr}}\ddot{p}_{\mathrm{I}}^{\mathrm{cr}} + \dot{\mathbf{u}}_j\right)\dot{p}_{\mathrm{I}}^{\mathrm{cr}} - 3\mathbf{v}_i^{\mathrm{T}}\mathbf{K}_{\mathrm{I}ij}^{\mathrm{C\,cr}}\left(\mathbf{t}_j^{\nabla\mathrm{cr}}\dot{p}_{\mathrm{I}}^{\mathrm{cr}} + \mathbf{u}_j\right)\ddot{p}_{\mathrm{I}}^{\mathrm{cr}} - \\
&\quad - 3\mathbf{v}_i^{\mathrm{T}}\mathbf{M}_{\mathrm{I}ikj}^{\mathrm{cr}}\left(\mathbf{t}_j^{\nabla\mathrm{cr}}\dot{p}_{\mathrm{I}}^{\mathrm{cr}} + \mathbf{u}_k\right)\left(\mathbf{t}_j^{\nabla\mathrm{cr}}\dot{p}_{\mathrm{I}}^{\mathrm{cr}} + \mathbf{u}_j\right)\dot{p}_{\mathrm{I}}^{\mathrm{cr}} + \\
&\qquad + \mathbf{v}_i^{\mathrm{T}}\mathbf{N}_{ilkj}^{\mathrm{cr}}\left(\mathbf{t}_l^{\nabla\mathrm{cr}}\dot{p}_{\mathrm{I}}^{\mathrm{cr}} + \mathbf{u}_l\right)\left(\mathbf{t}_k^{\nabla\mathrm{cr}}\dot{p}_{\mathrm{I}}^{\mathrm{cr}} + \mathbf{u}_k\right)\left(\mathbf{t}_j^{\nabla\mathrm{cr}}\dot{p}_{\mathrm{I}}^{\mathrm{cr}} + \mathbf{u}_j\right) = 0.
\end{aligned} \tag{4.29}$$

In view of (4.21) and equation

$$\begin{aligned}
&\mathbf{v}_i^{\mathrm{T}}3\mathbf{L}_{ikj}^{\mathrm{cr}}\mathbf{t}_k^{\nabla\mathrm{cr}}\mathbf{t}_j^{\nabla\nabla\mathrm{cr}} - 3\mathbf{v}_i^{\mathrm{T}}\mathbf{K}_{\mathrm{I}ij}^{\mathrm{C\,cr}}\mathbf{t}_j^{\nabla\nabla\mathrm{cr}} - \\
&\qquad - 3\mathbf{v}_i^{\mathrm{T}}\mathbf{M}_{\mathrm{I}ikj}^{\mathrm{cr}}\mathbf{t}_k^{\nabla\mathrm{cr}}\mathbf{t}_j^{\nabla\mathrm{cr}} + \mathbf{v}_i^{\mathrm{T}}\mathbf{N}_{ilkj}^{\mathrm{cr}}\mathbf{t}_l^{\nabla\mathrm{cr}}\mathbf{t}_k^{\nabla\mathrm{cr}}\mathbf{t}_j^{\nabla\mathrm{cr}} = 0
\end{aligned}$$

obtained from (4.25) by multiplying it with $\mathbf{v}_i$ equation (4.29) yields

$$\ddot{p}_{\mathrm{I}}^{\mathrm{cr}} = -\frac{A + B\dot{p}_{\mathrm{I}}^{\mathrm{cr}} + C\dot{p}_{\mathrm{I}}^{2\,\mathrm{cr}}}{3\left(\mathbf{v}_i^{\mathrm{T}}\mathbf{L}_{ikj}^{\mathrm{cr}}\mathbf{t}_k^{\nabla\mathrm{cr}}\mathbf{u}_j - \mathbf{v}_i^{\mathrm{T}}\mathbf{K}_{\mathrm{I}ij}^{\mathrm{C\,cr}}\mathbf{u}_j\right)}, \tag{4.30}$$

where

$$A = \mathbf{v}_i^{\mathrm{T}}\mathbf{N}_{ilkj}^{\mathrm{cr}}\mathbf{u}_l\mathbf{u}_k\mathbf{u}_j + 3\mathbf{v}_i^{\mathrm{T}}\mathbf{L}_{ikj}^{\mathrm{cr}}\mathbf{u}_k\dot{\mathbf{u}}_j, \tag{4.31}$$

$$B = 3\mathbf{v}_i^{\mathrm{T}}\mathbf{N}_{ilkj}^{\mathrm{cr}}\mathbf{t}_l^{\nabla\mathrm{cr}}\mathbf{u}_k\mathbf{u}_j + 3\mathbf{v}_i^{\mathrm{T}}\mathbf{L}_{ikj}^{\mathrm{cr}}\mathbf{t}_k^{\nabla\mathrm{cr}}\mathbf{u}_k\dot{\mathbf{u}}_j - 3\mathbf{v}_i^{\mathrm{T}}\mathbf{M}_{\mathrm{I}ikj}^{\mathrm{cr}}\mathbf{u}_k\mathbf{u}_j - 3\mathbf{v}_i^{\mathrm{T}}\mathbf{K}_{\mathrm{I}ij}^{\mathrm{C\,cr}}\dot{\mathbf{u}}_j, \tag{4.32}$$

$$C = \mathbf{v}_i\mathbf{N}_{ilkj}^{\mathrm{cr}}\mathbf{w}_l^{\mathrm{cr}}\mathbf{w}_k^{\mathrm{cr}}\mathbf{u}_j + 3\mathbf{v}_i\mathbf{L}_{ikj}^{\mathrm{cr}}\mathbf{u}_k\dot{\mathbf{w}}_j^{\mathrm{cr}} - 3\mathbf{v}_i\mathbf{M}_{\mathrm{I}ikj}^{\mathrm{cr}}\mathbf{w}_k^{\mathrm{cr}}\mathbf{u}_j. \tag{4.33}$$

For a symmetric branching point $p_{\mathrm{I}}^{\mathrm{cr}} = 0$ from (4.23). Then (4.30) is simplified into

$$\ddot{p}_{\mathrm{I}}^{\mathrm{cr}} = -\frac{A}{3\left(\mathbf{v}_i^{\mathrm{T}}\mathbf{L}_{ikj}^{\mathrm{cr}}\mathbf{t}_k^{\nabla\mathrm{cr}}\mathbf{u}_j - \mathbf{v}_i^{\mathrm{T}}\mathbf{K}_{\mathrm{I}ij}^{\mathrm{C\,cr}}\mathbf{u}_j\right)}. \tag{4.34}$$

Since $\mathbf{K}_{ij}^{\mathrm{cr}}$ is singular in equations (4.12) (4.13) which are taken at the branching point further equations are needed to ensure single valued solutions. Equation $(4.12)_1$ should be supplemented by

$$\mathbf{v}_i^{\mathrm{T}}\mathbf{L}_{ikj}\dot{\mathbf{t}}_k\dot{\mathbf{t}}_j - 2\mathbf{v}_i^{\mathrm{T}}\mathbf{K}_{\mathrm{I}ij}^{\mathrm{C}}\dot{\mathbf{t}}_j\dot{p}_{\mathrm{I}} = 0\,, \tag{4.35}$$

equation $(4.12)_2$ by

$$\begin{aligned}
&3\mathbf{v}_i^{\mathrm{T}}\mathbf{L}_{ikj}^{\mathrm{cr}}\dot{\mathrm{t}}_k^{\mathrm{cr}}\ddot{\mathbf{t}}_j^{\mathrm{cr}} - 3\mathbf{v}_i^{\mathrm{T}}\mathbf{K}_{\mathrm{I}ij}^{\mathrm{C\,cr}}\ddot{\mathbf{t}}_j^{\mathrm{cr}}\dot{p}_{\mathrm{I}}^{\mathrm{cr}} - 3\mathbf{v}_i^{\mathrm{T}}\mathbf{K}_{\mathrm{I}ij}^{\mathrm{C\,cr}}\,\dot{\mathrm{t}}_j^{\mathrm{cr}}\,\ddot{p}_{\mathrm{I}}^{\mathrm{cr}} - \\
&\qquad - 3\mathbf{v}_i^{\mathrm{T}}\mathbf{M}_{\mathrm{I}ikj}^{\mathrm{cr}}\dot{\mathrm{t}}_k^{\mathrm{cr}}\,\dot{\mathrm{t}}_j^{\mathrm{cr}}\,\dot{p}_{\mathrm{I}}^{\mathrm{cr}} + \mathbf{v}_i^{\mathrm{T}}\mathbf{N}_{ilkj}^{\mathrm{cr}}\,\dot{\mathrm{t}}_l^{\mathrm{cr}}\,\dot{\mathrm{t}}_k^{\mathrm{cr}}\,\dot{\mathrm{t}}_j^{\mathrm{cr}} = 0\,,
\end{aligned} \tag{4.36}$$

equation $(4.13)_1$ by

$$\mathbf{v}_i^{\mathrm{T}}\mathbf{L}_{ikj}\mathbf{t}_k^{\nabla}\mathbf{t}_j^{\nabla} - 2\mathbf{v}_i^{\mathrm{T}}\mathbf{K}_{\mathrm{I}ij}^{\mathrm{C}}\mathbf{t}_j^{\nabla} = 0 \tag{4.37}$$

and finally equation $(4.13)_2$ by

$$\mathbf{v}_i^{\mathrm{T}}3\mathbf{L}_{ikj}^{\mathrm{cr}}\mathbf{t}_k^{\nabla\mathrm{cr}}\mathbf{t}_j^{\nabla\nabla\mathrm{cr}} - 3\mathbf{v}_i^{\mathrm{T}}\mathbf{K}_{\mathrm{I}ij}^{\mathrm{C\,cr}}\mathbf{t}_j^{\nabla\nabla\mathrm{cr}} - 3\mathbf{v}_i^{\mathrm{T}}\mathbf{M}_{\mathrm{I}ikj}^{\mathrm{cr}}\mathbf{t}_k^{\nabla\mathrm{cr}}\mathbf{t}_j^{\nabla\mathrm{cr}} +$$

$$+\,\mathbf{v}_i^{\mathrm{T}}\mathbf{N}_{ilkj}^{\mathrm{cr}}\mathbf{t}_l^{\nabla\mathrm{cr}}\mathbf{t}_k^{\nabla\mathrm{cr}}\mathbf{t}_j^{\nabla\mathrm{cr}} = 0\,. \quad (4.38)$$

4.2.2. *Limit point.* At the limit point $\dot{p}_{\mathrm{I}}^{\mathrm{cr}} = 0$. Hence

$$\dot{\mathbf{t}}_j^{\mathrm{cr}} = \xi_1\mathbf{u}_j, \qquad \xi_0 = \dot{p}_{\mathrm{I}}^{\mathrm{cr}} = 0, \quad (4.39)$$

$$0 = -\ \left[\mathbf{v}_i^{\mathrm{T}}\mathbf{L}_{ikj}^{\mathrm{cr}}\left(\xi_1\mathbf{u}_k\right)\left(\xi_1\mathbf{u}_j\right) - \mathbf{v}_i^{\mathrm{T}}\mathbf{g}_{\mathrm{I}i}\ddot{p}_{\mathrm{I}}^{\mathrm{cr}}\right] \quad (4.40)$$

and

$$a_{\mathrm{L}}\xi_1^2 + d_{\mathrm{L}}\ddot{p}_{\mathrm{I}}^{\mathrm{cr}} = 0, \qquad \xi_1 = 1, \quad (4.41)$$

in which

$$a_{\mathrm{L}} = \mathbf{v}_i^{\mathrm{T}}\mathbf{L}_{ikj}^{\mathrm{cr}}\mathbf{u}_k\mathbf{u}_j, \qquad d_{\mathrm{L}} = -\mathbf{v}_i^{\mathrm{T}}\mathbf{g}_{\mathrm{I}i}. \quad (4.42)$$

$\ddot{p}_{\mathrm{I}}^{\mathrm{cr}}$ can be calculated from (4.41). If $\xi_1 = 1$ we have

$$\ddot{p}_{\mathrm{I}}^{\mathrm{cr}} = \frac{\mathbf{v}_i^{\mathrm{T}}\mathbf{L}_{ikj}^{\mathrm{kr}}\mathbf{u}_k\mathbf{u}_j}{\mathbf{v}_i^{\mathrm{T}}\mathbf{g}_{\mathrm{I}i}}. \quad (4.43)$$

The limit points can be classified. If

$$\left\{\begin{array}{l} a_{\mathrm{L}} \neq 0 \\ a_{\mathrm{L}} \neq 0,\ \ e_{\mathrm{L}} = \mathbf{v}_i^{\mathrm{T}} 1 \left(\frac{\partial \mathbf{L}_{ikj}}{\partial \mathbf{t}_l}\right)^{\mathrm{cr}} \mathbf{u}_l\mathbf{u}_k\mathbf{u}_j \neq 0 \end{array}\right\} \text{the limit point is} \left\{\begin{array}{l} \text{simple quadratic one,} \\ \text{simple cubic one.} \end{array}\right\}$$

Remark 4.5: The role of the problem parameters p_{I} and p_{II} can be interchanged in Subsection 4.2.

5. Determination of the fold line of limit points

5.1. **A brief survey.** According to Paragraph 3.2.3.4 in Sububsection 3.2.3 the fold line is the locus of the limit points on those equilibrium paths that belong to the different values of an imperfection parameter. In the case of the imperfection parameters introduced in the present paper we can speak about

- the fold line of the load imperfection parameter p_{II}, (then the geometrical imperfection parameter h is constant), or
- the fold line of the geometrical imperfection parameter h, (then the load imperfection parameter p_{II} is constant).

In both cases p_{I} is the fundamental load and the equilibrium paths together with the fold lines are defined as plane curves in the coordinate system (p_{I}, t_m) (in the domain of the parameters investigated).

Referring to Paragraph 3.2.3.1 we may also say that the fold line is a space curve the projections of which are the plane curves mentioned above.

We may lay emphasis on the following two ways for determining the fold line:

– The first is the application of a total stability analysis when under the condition h = constant we determine the equilibrium surfaces and the equilibrium paths as surface and coordinate plane intersections in the coordinate system $(p_{\mathrm{II}}, p_{\mathrm{I}}, t_m)$ [or under the condition p_{II} = constant in the coordinate system (h, p_{I}, t_m)]. Then by joining the limit points on the equilibrium paths we can determine the fold line.

- The second is the direct determination of the fold line. Then by the use of a previously found limit point we determine the fold line as a space curve applying the asymptotic numerical method or the Newton-Raphson iteration algorithm under the condition $h=$ constant [or $p_{\mathrm{II}}=$ constant].

Equilibrium surfaces can imagined by determining the curves p_{II} (or h) over the lines $p_{\mathrm{I}}=$ constant and $t_m=$ constant numerically (by applying, for example, the Newton-Rapson iteration algorithm on the basis of Subsection 3.2.2 or the asymptotic numerical method detailed in Subsection 3.2.4).

During the numerical investigations it is advisable to select an imperfection parameter (load or geometrical one) or a displacement parameter for governing the computations.

In the sequel we touch upon the determination of the fold line by the use of the direct method.

The equilibrium state $(\bar{B}^{\,\mathrm{cr}})$, which is the point of departure for calculating the fold line, and its problem parameters $p_{\mathrm{I}0}, p_{\mathrm{II}0}, h_0$ are assumed to be known.

Let τ be the governing parameter for the fold line starting from the state $(\bar{B}^{\mathrm{cr}})$.

On the basis of equations (3.22) and $(3.23)_2$ equilibrium of state $(\bar{B}^{\,\mathrm{cr}})$ can be expressed by equation

$$\mathbf{A}_i=\mathbf{b}_i-\mathbf{g}_{\mathrm{I}i}p_{\mathrm{I}}-\mathbf{g}_{\mathrm{II}i}p_{\mathrm{II}}=0\,. \tag{5.1}$$

For the critical equilibrium state it holds, according to (4.3) and the previous equation, that

$$\mathbf{K}_{ij}\mathbf{u}_j=0, \qquad \mathbf{v}_i^{\mathrm{T}}\mathbf{K}_{ij}=0, \qquad \det\mathbf{K}_{ij}=0, \tag{5.2}$$

where $\mathbf{v}_i$ and $\mathbf{u}_j$ are the left and right eigenvectors belonging to a simple zero eigenvalue. Equations (5.2) are referred to as constraint equations. The single valuedness of $\mathbf{u}_j$ is ensured by the normalization condition

$$\mathbf{u}_j^{\mathrm{T}}\mathbf{u}_j-1=0\,. \tag{5.3}$$

Here (and in the sequel) it is not denoted in equations (5.1)-(5.3) that the quantities they contain are all taken in the equilibrium state $(\bar{B}^{\,\mathrm{cr}})$.

Equations (5.1)-(5.3) are satisfied at each point of the space fold line – the quantities in them are taken at the point considered.

Let the state $(\bar{B}^{\,\mathrm{cr}}+\Delta\bar{B}^{\,\mathrm{cr}})$ on the fold line be in a small neighborhood of the state $(\bar{B}^{\,\mathrm{cr}})$. It is assumed that state $(\bar{B}^{\,\mathrm{cr}}+\Delta\bar{B}^{\,\mathrm{cr}})$ is determined by the value $\Delta\tau$ of the governing parameter.

For state $(\bar{B}^{\,\mathrm{cr}}+\Delta\bar{B}^{\,\mathrm{cr}})$ equations (5.1)-(5.3) assume the following forms:

$$\mathbf{A}_i+\Delta\mathbf{A}_i=\mathbf{b}_i+\Delta\mathbf{b}_i-(\mathbf{g}_{\mathrm{I}i}+\Delta\mathbf{g}_{\mathrm{I}i})\,(p_{\mathrm{I}}+\Delta p_{\mathrm{I}})-(\mathbf{g}_{\mathrm{II}i}+\Delta\mathbf{g}_{\mathrm{II}i})\,(p_{\mathrm{II}}+\Delta p_{\mathrm{II}})=0, \tag{5.4}$$

$$(\mathbf{K}_{ij}+\Delta\mathbf{K}_{ij})\,(\mathbf{u}_j+\Delta\mathbf{u}_j)=0, \tag{5.5}$$

$$\left(\mathbf{u}_j^{\mathrm{T}}+\Delta\mathbf{u}_j^{\mathrm{T}}\right)(\mathbf{u}_j+\Delta\mathbf{u}_j)-1=0\,. \tag{5.6}$$

Solution of the system of nonlinear equations (5.4)-(5.6) can be sought either by the Newton-Raphson iteration algorithm or by the asymptotic numerical method.

5.2. **Fold line of the limit points for a load imperfection.** On the limit point fold line of load imperfection $h = h_0 =$ constant and p_{II} is the imperfection parameter.

The governing parameter along the fold line can be either the load or a displacement component. In the first case $\tau = p_{\mathrm{II}} - p_{\mathrm{II}0}$, in the second one $\tau = t_m - t_{m0}$ is the governing parameter.

5.2.1. *Solution by the Newton-Rapson iteration algorithm.*

5.2.1.1. First, let us assume that the load is the governing parameter ($\tau = p_{\mathrm{II}} - p_{\mathrm{II}0},\ \Delta\tau = \Delta p_{\mathrm{II}}$). Solution of the equation system (5.4))-(5.6) is sought by the series of the increments $\Delta \mathbf{t}_{\underline{s}j},\ \Delta p_{\mathrm{I}\underline{s}},\ \Delta \mathbf{u}_{\underline{s}j};\ \underline{s} = \underline{1}, \underline{2}, \cdots$, ($p_{\mathrm{II}} = p_{\mathrm{II}0},\ \Delta p_{\mathrm{II}} =$ constant):

$$\mathbf{t}_j + \Delta\mathbf{t}_j = \mathbf{t}_j + \Delta\mathbf{t}_{\underline{1}j} + \Delta\mathbf{t}_{\underline{2}j} + \cdots + \Delta\mathbf{t}_{\underline{s}j} + \cdots\,, \tag{5.7}$$

$$p_{\mathrm{I}} + \Delta p_{\mathrm{I}} = p_{\mathrm{I}} + \Delta p_{\mathrm{I}\underline{1}} + \Delta p_{\mathrm{I}\underline{2}} + \cdots + \Delta p_{\mathrm{I}\underline{s}} + \cdots\,, \tag{5.8}$$

$$\mathbf{u}_j + \Delta\mathbf{u}_j = \mathbf{u}_j + \Delta\mathbf{u}_{\underline{1}j} + \Delta\mathbf{u}_{\underline{2}j} + \cdots + \Delta\mathbf{u}_{\underline{s}j} + \cdots \tag{5.9}$$

The linearized forms of equations (5.4)-(5.6) ($h = h_0 =$ constant, $\Delta h = 0$) are given by:

$$\mathbf{A}_i + \frac{\partial \mathbf{A}_i}{\partial \mathbf{t}_j}\Delta\mathbf{t}_j - \mathbf{g}_{\mathrm{I}i}\Delta p_{\mathrm{I}} - \mathbf{g}_{\mathrm{II}i}\Delta p_{\mathrm{II}} = 0, \tag{5.10}$$

$$\mathbf{K}_{ij}\mathbf{u}_j + \frac{\partial \mathbf{K}_{ij}}{\partial \mathbf{t}_k}\Delta\mathbf{t}_k\mathbf{u}_j + \frac{\partial \mathbf{K}_{ij}}{\partial p_{\mathrm{I}}}\mathbf{u}_j\Delta p_{\mathrm{I}} + \frac{\partial \mathbf{K}_{ij}}{\partial p_{\mathrm{II}}}\mathbf{u}_j\Delta p_{\mathrm{II}} + \mathbf{K}_{ij}\Delta\mathbf{u}_j = 0, \tag{5.11}$$

$$\mathbf{u}_j^{\mathrm{T}}\Delta\mathbf{u}_j = 0. \tag{5.12}$$

Making use of relations 5.32, 5.36, $(5.37)_2$ and (5.38) presented in the later Subsection 5.2.2, equations for the $\underline{s}^{\mathrm{th}}$ iteration step can be gathered in a simple matrix form:

$$\begin{bmatrix} \mathbf{K}_{\underline{s}ij} & 0 & -\mathbf{g}_{\mathrm{I}\underline{s}i} \\ \mathbf{L}_{\underline{s}ikj}\mathbf{u}_{\underline{s}k} & \mathbf{K}_{\underline{s}ij} & -\mathbf{K}^{\mathrm{C}}_{\mathrm{I}\underline{s}ij}\mathbf{u}_{\underline{s}j} \\ 0 & \mathbf{u}^{\mathrm{T}}_{\underline{s}j} & 0 \end{bmatrix} \begin{bmatrix} \Delta\mathbf{t}_{\underline{s}j} \\ \Delta\mathbf{u}_{\underline{s}j} \\ \Delta p_{\mathrm{I}\underline{s}} \end{bmatrix} =$$

$$= -\begin{bmatrix} \mathbf{b}_{\underline{s}i} - \left[\mathbf{g}_{\mathrm{I}\underline{s}i}p_{\mathrm{I}\underline{s}} + \mathbf{g}_{\mathrm{II}\underline{s}i}p_{\mathrm{II}}\right] \\ \mathbf{K}_{\underline{s}ij}\mathbf{u}_{\underline{s}j} \\ \mathbf{u}^{\mathrm{T}}_{\underline{s}k}\mathbf{u}_{\underline{s}k} - 1 \end{bmatrix} - \begin{bmatrix} -\ \mathbf{g}_{\mathrm{II}\underline{s}i} \\ -\ \mathbf{K}^{\mathrm{C}}_{\mathrm{II}\underline{s}ij}\mathbf{u}_{\underline{s}j} \\ 0 \end{bmatrix}\Delta p_{\mathrm{II}}. \tag{5.13}$$

If Δp_{II} is given (5.13) is a system of linear equations for calculating the unknowns $\Delta\mathbf{t}_{\underline{s}j},\ \Delta\mathbf{u}_{\underline{s}j}$ and $\Delta p_{\mathrm{I}\underline{s}}$. Introducing the notations $\mathbf{t}_j = \mathbf{t}_{\underline{1}j},\ \mathbf{u}_j = \mathbf{u}_{\underline{1}j},\ p_{\mathrm{I}} = p_{\mathrm{I}\underline{1}}$ the variables of the $\underline{s}^{\mathrm{th}}$ step can be obtained form the relations

$$\mathbf{t}_{\underline{s}j} = \mathbf{t}_{\underline{1}j} + \Delta\mathbf{t}_{\underline{1}j} + \Delta\mathbf{t}_{\underline{2}j} + \cdots + \Delta\mathbf{t}_{\underline{s-1}j}, \qquad \Delta\mathbf{t}_{\underline{s}j}, \tag{5.14}$$

$$\mathbf{u}_{\underline{s}j} = \mathbf{u}_{\underline{1}j} + \Delta\mathbf{u}_{\underline{1}j} + \Delta\mathbf{u}_{\underline{2}j} + \cdots + \Delta\mathbf{u}_{\underline{s-1}j}, \qquad \Delta\mathbf{u}_{\underline{s}j}, \tag{5.15}$$

$$p_{\mathrm{I}\underline{s}} = p_{\mathrm{I}\underline{1}} + \Delta p_{\mathrm{I}\underline{1}} + \Delta p_{\mathrm{I}\underline{2}} + \cdots + \Delta p_{\mathrm{I}\underline{s-1}}, \qquad \Delta p_{\mathrm{I}\underline{s}} \tag{5.16}$$

The right side of equation (5.13) tends to zero at the end of iteration. This means that $\Delta\mathbf{t}_{\underline{s}j}$, $\Delta\mathbf{u}_{\underline{s}j}$, $\Delta p_{\mathrm{I}\underline{s}} \to 0$ by satisfying the prescribed error limit. Thus in this manner the right side of equation (5.13) approximately coincides with equations (5.4)-(5.6) which determine the state $(\bar{B}^{\,\mathrm{cr}} + \Delta\bar{B}^{\,\mathrm{cr}})$. At the end of the iteration, that is to say, it holds that

$$\mathbf{b}_{\underline{s}i} - \mathbf{g}_{\mathrm{I}\underline{s}i} p_{\mathrm{I}\underline{s}} - \mathbf{g}_{\mathrm{II}\underline{s}i} \left(p_{\mathrm{II}} + \Delta p_{\mathrm{II}}\right) \Longleftrightarrow \mathbf{A}_i + \Delta\mathbf{A}_i \tag{5.17}$$

$$\mathbf{K}_{\underline{s}ij} = \mathbf{K}^{\mathrm{L}}_{\underline{s}ij} + \mathbf{K}^{\mathrm{G}}_{\underline{s}ij} - p_{\mathrm{I}\underline{s}}\mathbf{K}^{\mathrm{C}}_{\mathrm{I}\underline{s}ij} - p_{\mathrm{II}}\mathbf{K}^{\mathrm{C}}_{\mathrm{II}\underline{s}ij}, \qquad \text{i.e.,}$$

$$\left(\mathbf{K}_{\underline{s}ij} - \Delta p_{\mathrm{II}}\mathbf{K}^{\mathrm{C}}_{\mathrm{II}\underline{s}ij}\right)\mathbf{u}_{\underline{s}j} = \left[\mathbf{K}^{\mathrm{L}}_{\underline{s}ij} + \mathbf{K}^{\mathrm{G}}_{\underline{s}ij} - p_{\mathrm{I}\underline{s}}\mathbf{K}^{\mathrm{C}}_{\mathrm{I}\underline{s}ij} - \left(p_{\mathrm{II}} + \Delta p_{\mathrm{II}}\right)\mathbf{K}^{\mathrm{C}}_{\mathrm{II}\underline{s}ij}\right]\mathbf{u}_{\underline{s}j} \Longleftrightarrow$$
$$\Longleftrightarrow \left(\mathbf{K}_{ij} + \Delta\mathbf{K}_{ij}\right)\left(\mathbf{u}_j + \Delta\mathbf{u}_j\right), \tag{5.18}$$

$$\mathbf{u}^{\mathrm{T}}_{\underline{s}k}\mathbf{u}_{\underline{s}k} - 1 \Longleftrightarrow \left(\mathbf{u}^{\mathrm{T}}_j + \Delta\mathbf{u}^{\mathrm{T}}_j\right)\left(\mathbf{u}_j + \Delta\mathbf{u}_j\right) - 1. \tag{5.19}$$

5.2.1.2. Second, let us assume that a displacement component is the governing parameter ($\tau = t_m - t_{m0},\ \Delta\tau = \Delta t_m$). Solution of the equation system (5.4))-(5.6) is sought by the series of the increments $\Delta\mathbf{t}_{\underline{s}\alpha}$, $\Delta p_{\mathrm{I}\underline{s}}$, $\Delta p_{\mathrm{II}\underline{s}}$, $\Delta\mathbf{u}_{\underline{s}j}$; $\underline{s} = \underline{1}, \underline{2}, \ldots$; ($t_m = t_{m0} = \text{constant},\ \Delta t_m = \text{constant}$):

$$\mathbf{t}_\alpha + \Delta\mathbf{t}_\alpha = \mathbf{t}_\alpha + \Delta\mathbf{t}_{\underline{1}\alpha} + \Delta\mathbf{t}_{\underline{2}\alpha} + \cdots + \Delta\mathbf{t}_{\underline{s}\alpha} + \cdots, \quad \alpha = 1, 2, \cdots, m-1 \tag{5.20}$$

$$p_{\mathrm{I}} + \Delta p_{\mathrm{I}} = p_{\mathrm{I}} + \Delta p_{\mathrm{I}\underline{1}} + \Delta p_{\mathrm{I}\underline{2}} + \cdots + \Delta p_{\mathrm{I}\underline{s}} + \cdots, \tag{5.21}$$

$$p_{\mathrm{II}} + \Delta p_{\mathrm{II}} = p_{\mathrm{II}} + \Delta p_{\mathrm{II}\underline{1}} + \Delta p_{\mathrm{II}\underline{2}} + \cdots + \Delta p_{\mathrm{II}\underline{s}} + \cdots, \tag{5.22}$$

$$\mathbf{u}_j + \Delta\mathbf{u}_j = \mathbf{u}_j + \Delta\mathbf{u}_{\underline{1}j} + \Delta\mathbf{u}_{\underline{2}j} + \cdots + \Delta\mathbf{u}_{\underline{s}j} + \cdots \tag{5.23}$$

Here and in the sequel $\alpha, \beta = 1, 2, \cdots, m-1$, where m is the degree of freedom (independently of the previous notational convention in matrix notation, m may be the subscript of any displacement parameter; after selecting it, however, its value is fixed and can not take other values).

The linearized form of equations (5.4)-(5.6) is given by

$$\mathbf{A}_i + \frac{\partial\mathbf{A}_i}{\partial\mathbf{t}_j}\Delta\mathbf{t}_j + \frac{\partial\mathbf{A}_i}{\partial t_m}\Delta t_m - \mathbf{g}_{\mathrm{I}i}\Delta p_{\mathrm{I}} - \mathbf{g}_{\mathrm{II}i}\Delta p_{\mathrm{II}} = 0, \tag{5.24}$$

$$\mathbf{K}_{ij}\mathbf{u}_j + \frac{\partial\mathbf{K}_{ij}}{\partial\mathbf{t}_\beta}\mathbf{u}_j\Delta\mathbf{t}_\beta + \frac{\partial\mathbf{K}_{ij}}{\partial t_m}\mathbf{u}_j\Delta t_m + \frac{\partial\mathbf{K}_{ij}}{\partial p_{\mathrm{I}}}\mathbf{u}_j\Delta p_{\mathrm{I}} + \frac{\partial\mathbf{K}_{1j}}{\partial p_{\mathrm{II}}}\mathbf{u}_j\Delta p_{\mathrm{II}} + \mathbf{K}_{ij}\Delta\mathbf{u}_j = 0, \tag{5.25}$$

$$\mathbf{u}^{\mathrm{T}}_j\Delta\mathbf{u}_j = 0. \tag{5.26}$$

Equations for the $\underline{s}^{\mathrm{th}}$ iteration step can now be gathered in the following simple matrix form:

$$\begin{bmatrix} \mathbf{K}_{\underline{s}i\beta} & 0 & -\mathbf{g}_{\mathrm{I}\underline{s}i} & -\mathbf{g}_{\mathrm{II}\underline{s}i} \\ \mathbf{L}_{\underline{s}ik\beta}\mathbf{u}_{\underline{s}k} & \mathbf{K}_{\underline{s}ij} & -\mathbf{K}^{\mathrm{C}}_{\mathrm{I}\underline{s}ik}\mathbf{u}_{\underline{s}k} & -\mathbf{K}^{\mathrm{C}}_{\mathrm{II}\underline{s}ik}\mathbf{u}_{\underline{s}k} \\ 0 & \mathbf{u}^{\mathrm{T}}_{\underline{s}j} & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\mathbf{t}_{\underline{s}\beta} \\ \Delta\mathbf{u}_{\underline{s}j} \\ \Delta p_{\mathrm{I}\underline{s}} \\ \Delta p_{\mathrm{II}\underline{s}} \end{bmatrix} =$$

$$= -\begin{bmatrix} \mathbf{b}_{\underline{s}i} - \left(\mathbf{g}_{\mathrm{I}\underline{s}i}p_{\mathrm{I}\underline{s}} + \mathbf{g}_{\mathrm{II}\underline{s}i}p_{\mathrm{II}\underline{s}}\right) \\ \mathbf{K}_{\underline{s}ij}\mathbf{u}_{\underline{s}j} \\ \mathbf{u}^{\mathrm{T}}_{\underline{s}k}\mathbf{u}_{\underline{s}k} - 1 \end{bmatrix} - \begin{bmatrix} \mathbf{k}_{\underline{s}im} \\ \mathbf{L}_{\underline{s}ikm}\mathbf{u}_{\underline{s}k} \\ 0 \end{bmatrix} \Delta t_m. \quad (5.27)$$

For a given Δt_m equation (5.27) is a linear equation system for calculating the unknowns $\Delta\mathbf{t}_{\underline{s}\beta}$, $\Delta\mathbf{u}_{\underline{s}\beta}$, $\Delta\mathbf{u}_{\underline{s}m}$ and $\Delta p_{\mathrm{I}\underline{s}}$, $\Delta p_{\mathrm{II}\underline{s}}$. Introducing the notations $\mathbf{t}_j = \mathbf{t}_{\underline{1}j}$, $\mathbf{u}_j = \mathbf{u}_{\underline{1}j}$, $p_{\mathrm{I}} = p_{\mathrm{I}\underline{1}}$, $p_{\mathrm{II}} = p_{\mathrm{II}\underline{1}}$ the variables of the $\underline{s}^{\mathrm{th}}$ step can be obtained from the relations

$$\mathbf{t}_{\underline{s}\beta} = \mathbf{t}_{\underline{1}\beta} + \Delta\mathbf{t}_{\underline{1}\beta} + \Delta\mathbf{t}_{\underline{2}\beta} + \cdots + \Delta\mathbf{t}_{\underline{s-1}\beta}, \qquad \Delta\mathbf{t}_{\underline{s}\beta}, \quad (5.28)$$

$$\mathbf{u}_{\underline{s}j} = \mathbf{u}_{\underline{1}j} + \Delta\mathbf{u}_{\underline{1}j} + \Delta\mathbf{u}_{\underline{2}j} + \cdots + \Delta\mathbf{u}_{\underline{s-1}j}, \qquad \Delta\mathbf{u}_{\underline{s}j}, \quad (5.29)$$

$$p_{\mathrm{I}\underline{s}} = p_{\mathrm{I}\underline{1}} + 1\Delta p_{\mathrm{I}\underline{1}} + \Delta p_{\mathrm{I}\underline{2}} + \cdots + \Delta p_{\mathrm{I}\underline{s-1}}, \qquad \Delta p_{\mathrm{I}\underline{s}} \quad (5.30)$$

$$p_{\mathrm{II}\underline{s}} = p_{\mathrm{II}\underline{1}} + \Delta p_{\mathrm{II}\underline{1}} + \Delta p_{\mathrm{II}\underline{2}} + \cdots + \Delta p_{\mathrm{II}\underline{s-1}}, \qquad \Delta p_{\mathrm{II}\underline{s}}\,. \quad (5.31)$$

The right side of equation (5.27) tends to zero at the end of iteration. This means that $\Delta\mathbf{t}_{\underline{s}\beta}$, $\Delta\mathbf{u}_{\underline{s}j}$, $\Delta p_{\mathrm{I}\underline{s}}$, $\Delta p_{\mathrm{II}\underline{s}} \to 0$ by satisfying the prescribed error limit. The right and left sides of equation (5.27), which tend to zero, ensure together the fulfillment of equations (5.4)-(5.6) that determine the state $(\bar{B}^{\,\mathrm{cr}} + \Delta\bar{B}^{\,\mathrm{cr}})$ of the body.

5.2.2. *Solution by using the asymptotic numerical method.* For applying the asymptotic numerical method (ANM) it is worth gathering the corresponding equilibrium equations on the basis of subsection 3.2.4 ($h = h_0 =$ constant, $\dot{h} = \ddot{h} = \cdots = 0$!):

$$\mathbf{A}_i = \mathbf{b}_i - \mathbf{g}_{\mathrm{I}i}p_{\mathrm{I}} - \mathbf{g}_{\mathrm{II}i}p_{\mathrm{II}} = 0 \quad (5.32)$$

$$\dot{\mathbf{A}}_i = \mathbf{K}_{ij}\dot{\mathbf{t}}_j - \mathbf{g}_{\mathrm{I}i}\,\dot{p}_{\mathrm{I}} - \mathbf{g}_{\mathrm{II}i}\,\dot{p}_{\mathrm{II}} = 0, \quad (5.33)$$

$$\ddot{\mathbf{A}}_i = \mathbf{K}_{ij}\ddot{\mathbf{t}}_j - \mathbf{g}_{\mathrm{I}i}\ddot{p}_{\mathrm{I}} - \mathbf{g}_{\mathrm{II}i}\ddot{p}_{\mathrm{II}} + \mathbf{L}_{ikj}\dot{\mathbf{t}}_k\dot{\mathbf{t}}_j - 2\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\dot{\mathbf{t}}_j\dot{p}_{\mathrm{I}} - 2\mathbf{K}^{\mathrm{C}}_{\mathrm{II}ij}\dot{\mathbf{t}}_j\dot{p}_{\mathrm{II}} = 0, \quad (5.34)$$

$$\begin{aligned}\dddot{\mathbf{A}}_i = {}& \mathbf{K}_{ij}\dddot{\mathbf{t}}_j - \mathbf{g}_{\mathrm{I}i}\dddot{p}_{\mathrm{I}} - \mathbf{g}_{\mathrm{II}i}\dddot{p}_{\mathrm{II}} + 3\mathbf{L}_{ikj}\dot{\mathbf{t}}_k\ddot{\mathbf{t}}_j - 3\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\ddot{\mathbf{t}}_j\dot{p}_{\mathrm{I}} - 3\mathbf{K}^{\mathrm{C}}_{\mathrm{II}ij}\ddot{\mathbf{t}}_j\dot{p}_{\mathrm{II}} - \\ & - 3\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\dot{\mathbf{t}}_j\ddot{p}_{\mathrm{I}} - 3\mathbf{K}^{\mathrm{C}}_{\mathrm{II}ij}\dot{\mathbf{t}}_j\ddot{p}_{\mathrm{II}} + \mathbf{N}_{ilkj}\dot{\mathbf{t}}_l\dot{\mathbf{t}}_k\dot{\mathbf{t}}_j - 3\mathbf{M}_{\mathrm{I}ikj}\dot{\mathbf{t}}_k\dot{\mathbf{t}}_j\dot{p}_{\mathrm{I}} - 3\mathbf{M}_{\mathrm{II}ikj}\dot{\mathbf{t}}_k\dot{\mathbf{t}}_j\dot{p}_{\mathrm{II}} = 0\end{aligned} \quad (5.35)$$

where

$$\mathbf{K}_{ij} = \frac{\partial\mathbf{A}_i}{\partial\mathbf{t}_j} = \mathbf{K}^{\mathrm{L}}_{ij} + \mathbf{K}^{\mathrm{G}}_{ij} - \mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}p_{\mathrm{I}} - \mathbf{K}^{\mathrm{C}}_{\mathrm{II}ij}p_{\mathrm{II}}, \quad (5.36)$$

$$\frac{\partial\mathbf{b}_i}{\partial\mathbf{t}_j} = \mathbf{K}^{\mathrm{L}}_{ij} + \mathbf{K}^{\mathrm{G}}_{ij}, \qquad -\frac{\partial\mathbf{g}_{\mathrm{N}i}}{\partial\mathbf{t}_j} = -\mathbf{K}^{\mathrm{C}}_{\mathrm{N}ij}, \quad \mathrm{N} = \mathrm{I}, \mathrm{II}, \quad (5.37)$$

$$\mathbf{L}_{ikj} = \frac{\partial\mathbf{K}_{ij}}{\partial\mathbf{t}_k} = \frac{\partial}{\partial\mathbf{t}_k}\frac{\partial\mathbf{A}_i}{\partial\mathbf{t}_j} = 2\mathbf{J}_{ikj} + \mathbf{H}_{ikj} - \mathbf{M}_{\mathrm{I}ikj}p_{\mathrm{I}} - \mathbf{M}_{\mathrm{II}ikj}p_{\mathrm{II}}, \quad (5.38)$$

$$\frac{\partial\left(\mathbf{K}^{\mathrm{L}}_{ij} + \mathbf{K}^{\mathrm{G}}_{ij}\right)}{\partial\mathbf{t}_k} = 2\mathbf{J}_{ikj} + \mathbf{H}_{ikj}, \qquad -\frac{\partial\mathbf{K}^{\mathrm{C}}_{\mathrm{N}ij}}{\partial\mathbf{t}_k} = -\mathbf{M}_{\mathrm{N}kij}, \quad \mathrm{N} = \mathrm{I}, \mathrm{II}, \quad (5.39)$$

$$\mathbf{N}_{ilkj} = \frac{\partial\mathbf{L}_{ikj}}{\partial\mathbf{t}_l} = \frac{\partial}{\partial\mathbf{t}_l}\frac{\partial}{\partial\mathbf{t}_k}\frac{\partial\mathbf{A}_i}{\partial\mathbf{t}_j} = \frac{\partial\left(2\mathbf{J}_{ikj} + \mathbf{H}_{ikj}\right)}{\partial\mathbf{t}_l}. \quad (5.40)$$

Note that equation (5.40) is a consequence of equation

$$-\frac{\partial \mathbf{M}_{\mathrm{N}kij}}{\partial \mathbf{t}_l} = 0, \quad \mathrm{N} = \mathrm{I}, \mathrm{II}; \tag{5.41}$$

for comparison see (3.35) and equation

$$\delta \mathbf{t}_i \mathbf{N}_{ilkj} \dot{\mathbf{t}}_l \dot{\mathbf{t}}_k \dot{\mathbf{t}}_j = 3 \int_{(B)} \delta \dot{u}_{m;k} \dot{u}^m{}_{;l} C^{klpq} \dot{u}^s{}_{;p} \dot{u}_{s;q} \mathrm{d}V \tag{5.42}$$

which follows from relation (3.34).

Matrix $\mathbf{L}_{ikj}$ in equations (5.38) and (5.40) is symmetric with respect to the index pair kj. Because of its definition – see equation (5.40) – matrix $\mathbf{N}_{ilkj}$ is also symmetric with respect to the indices $l,\ k,\ j$. As regards the further derivatives it is important to mention that

$$\frac{\partial \mathbf{N}_{likj}}{\partial \mathbf{t}_h} = \frac{\partial}{\partial \mathbf{t}_h} \frac{\partial \mathbf{L}_{ikj}}{\partial \mathbf{t}_l} = 0. \tag{5.43}$$

This result is a consequence of equation (5.42).
There will be a need for the derivatives of $(5.2)_1$ and (5.3):

$$\begin{aligned}(\mathbf{K}_{ij}\mathbf{u}_j)^{\cdot} &= \left(\frac{\partial \mathbf{K}_{ij}}{\partial \mathbf{t}_k}\dot{\mathbf{t}}_k + \frac{\partial \mathbf{K}_{ij}}{\partial p_{\mathrm{I}}}\dot{p}_{\mathrm{I}} + \frac{\partial \mathbf{K}_{ij}}{\partial p_{\mathrm{II}}}\dot{p}_{\mathrm{II}}\right)\mathbf{u}_j + \mathbf{K}_{ij}\dot{\mathbf{u}}_j = \\ &= \mathbf{L}_{ikj}\mathbf{u}_j\dot{\mathbf{t}}_k - \mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\dot{p}_{\mathrm{I}}\mathbf{u}_j - \mathbf{K}^{\mathrm{C}}_{\mathrm{II}ij}\dot{p}_{\mathrm{II}}\mathbf{u}_j + \mathbf{K}_{ij}\dot{\mathbf{u}}_j = 0, \end{aligned} \tag{5.44}$$

$$\begin{aligned}(\mathbf{K}_{ij}\mathbf{u}_j)^{\cdot\cdot} = {} & \mathbf{N}_{ilkj}\mathbf{u}_j\dot{\mathbf{t}}_l\dot{\mathbf{t}}_k - 2\mathbf{M}_{\mathrm{I}ikj}\dot{p}_{\mathrm{I}}\mathbf{u}_j\dot{\mathbf{t}}_k - 2\mathbf{M}_{\mathrm{II}ikj}\dot{p}_{\mathrm{II}}\mathbf{u}_j\dot{\mathbf{t}}_k + 2\mathbf{L}_{ikj}\dot{\mathbf{u}}_j\dot{\mathbf{t}}_k + \mathbf{L}_{ikj}\mathbf{u}_j\ddot{\mathbf{t}}_k - \\ & - \mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\ddot{p}_{\mathrm{I}}\mathbf{u}_j - \mathbf{K}^{\mathrm{C}}_{\mathrm{II}ij}\ddot{p}_{\mathrm{II}}\mathbf{u}_j - 2\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\dot{p}_{\mathrm{I}}\dot{\mathbf{u}}_j - 2\mathbf{K}^{\mathrm{C}}_{\mathrm{II}ij}\dot{p}_{\mathrm{II}}\dot{\mathbf{u}}_j + \mathbf{K}_{ij}\ddot{\mathbf{u}}_j = 0, \end{aligned} \tag{5.45}$$

$$\mathbf{u}_j^{\mathrm{T}}\dot{\mathbf{u}}_j = 0, \tag{5.46}$$

$$\dot{\mathbf{u}}_j^{\mathrm{T}}\dot{\mathbf{u}}_j + \mathbf{u}_j^{\mathrm{T}}\ddot{\mathbf{u}}_j = 0. \tag{5.47}$$

5.2.2.1. First, let us find the equilibrium state $(\bar{B}^{\,\mathrm{cr}} + \Delta\bar{B}^{\,\mathrm{cr}})$ by using the load as governing parameter, i.e., in the same manner as before $(\tau = p_{\mathrm{II}} - p_{\mathrm{II}0},\ \Delta\tau = \Delta p_{\mathrm{II}})$. It is obvious that

$$\mathbf{t}_i(\tau) = \mathbf{t}_i + \dot{\mathbf{t}}_i\tau + \frac{1}{2}\ddot{\mathbf{t}}_i\tau^2 + \frac{1}{6}\dddot{\mathbf{t}}_i\tau^3 + \cdots, \tag{5.48}$$

$$p_{\mathrm{I}}(\tau) = p_{\mathrm{I}} + \dot{p}_{\mathrm{I}}\tau + \frac{1}{2}\ddot{p}_{\mathrm{I}}\tau^2 + \frac{1}{6}\dddot{p}_{\mathrm{I}}\tau^3 + \cdots, \tag{5.49}$$

$$\mathbf{u}_j(\tau) = \mathbf{u}_j + \dot{\mathbf{u}}_j\tau + \frac{1}{2}\ddot{\mathbf{u}}_j\tau^2 + \frac{1}{6}\dddot{\mathbf{u}}_j\tau^3 + \cdots \tag{5.50}$$

where

$$()^{\cdot} = \frac{\partial\,()}{\partial\tau} = \frac{\partial\,()}{\partial p_{\mathrm{II}}}, \quad \text{and} \quad \dot{p}_{\mathrm{II}} = \frac{\partial p_{\mathrm{II}}}{\partial p_{\mathrm{II}}} = 1, \quad \ddot{p}_{\mathrm{II}} = \dddot{p}_{\mathrm{II}} = \cdots = 0. \tag{5.51}$$

Equations (5.33), (5.44) and (5.46) can be used for finding the unknown first derivatives. In matrix notation they have the following form:

$$\begin{bmatrix} \mathbf{K}_{ij} & 0 & -\mathbf{g}_{\mathrm{I}i} \\ \mathbf{L}_{ikj}\mathbf{u}_k & \mathbf{K}_{ij} & -\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\mathbf{u}_j \\ 0 & \mathbf{u}_j^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{t}}_j \\ \dot{\mathbf{u}}_j \\ \dot{p}_{\mathrm{I}} \end{bmatrix} = - \begin{bmatrix} -\mathbf{g}_{\mathrm{II}i} \\ -\mathbf{K}^{\mathrm{C}}_{\mathrm{II}ij}\mathbf{u}_j \\ 0 \end{bmatrix}, \tag{5.52}$$

Equations (5.34), (5.45) and (5.47) provide the second derivatives. Their matrix form is given by

$$- \begin{bmatrix} \mathbf{K}_{ij} & 0 & -\mathbf{g}_{\mathrm{I}i} \\ \mathbf{L}_{ikj}\mathbf{u}_k & \mathbf{K}_{ij} & -\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\mathbf{u}_j \\ 0 & \mathbf{u}_j^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{t}}_j \\ \ddot{\mathbf{u}}_j \\ \ddot{p}_{\mathrm{I}} \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbf{L}_{ikj}\dot{\mathbf{t}}_k\dot{\mathbf{t}}_j - 2\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\dot{\mathbf{t}}_j\dot{p}_{\mathrm{I}} - 2\mathbf{K}^{\mathrm{C}}_{\mathrm{II}ij}\dot{\mathbf{t}}_j \\ \mathbf{N}_{ilkj}\dot{\mathbf{t}}_l\dot{\mathbf{t}}_k\mathbf{u}_j - 2\mathbf{M}_{\mathrm{I}ikj}\dot{\mathbf{t}}_k\mathbf{u}_j\dot{p}_{\mathrm{I}} - 2\mathbf{M}_{\mathrm{II}ikj}\dot{\mathbf{t}}_k\mathbf{u}_j + 2\mathbf{L}_{ikj}\dot{\mathbf{t}}_k\dot{\mathbf{u}}_j - 2\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\dot{p}_{\mathrm{I}}\dot{\mathbf{u}}_j - 2\mathbf{K}^{\mathrm{C}}_{\mathrm{II}ij}\dot{p}_{\mathrm{II}}\dot{\mathbf{u}}_j \\ \dot{\mathbf{u}}_j^{\mathrm{T}}\dot{\mathbf{u}}_j \end{bmatrix}. \tag{5.53}$$

5.2.2.2. Second, we look for the equilibrium state $(\bar{B}^{\,\mathrm{cr}} + \Delta\bar{B}^{\,\mathrm{cr}})$ by selecting a displacement component for the governing parameter ($\tau = t_m - t_{m0}$, $\Delta\tau = \Delta t_m$). Then

$$\mathbf{t}_\alpha(\tau) = \mathbf{t}_\alpha + \dot{\mathbf{t}}_\alpha\tau + \frac{1}{2}\ddot{\mathbf{t}}_\alpha\tau^2 + \frac{1}{6}\dddot{\mathbf{t}}_\alpha\tau^3 + \cdots, \tag{5.54}$$

$$p_{\mathrm{I}}(\tau) = p_{\mathrm{I}} + \dot{p}_{\mathrm{I}}\tau + \frac{1}{2}\ddot{p}_{\mathrm{I}}\tau^2 + \frac{1}{6}\dddot{p}_{\mathrm{I}}\tau^3 + \cdots, \tag{5.55}$$

$$p_{\mathrm{II}}(\tau) = p_{\mathrm{II}} + \dot{p}_{\mathrm{I}}\tau + \frac{1}{2}\ddot{p}_{\mathrm{II}}\tau^2 + \frac{1}{6}\dddot{p}_{\mathrm{II}}\tau^3 + \cdots, \tag{5.56}$$

$$\mathbf{u}_j(\tau) = \mathbf{u}_j + \dot{\mathbf{u}}_j\tau + \frac{1}{2}\ddot{\mathbf{u}}_j\tau^2 + \frac{1}{6}\dddot{\mathbf{u}}_j\tau^3 + \cdots \tag{5.57}$$

where

$$()^{\cdot} = \frac{\partial\,()}{\partial\tau} = \frac{\partial\,()}{\partial t_m} \quad \text{and} \quad \dot{t}_m = \frac{\partial t_m}{\partial t_m} = 1, \quad \ddot{t}_m = \dddot{t}_m = \cdots = 0. \tag{5.58}$$

Equations (5.33), (5.44) and (5.46) make it possible to determine the first derivatives. They can be given in a matrix form as well:

$$\begin{bmatrix} \mathbf{K}_{i\beta} & 0 & -\mathbf{g}_{\mathrm{I}i} & -\mathbf{g}_{\mathrm{II}i} \\ \mathbf{L}_{ik\beta}\mathbf{u}_k & \mathbf{K}_{ij} & -\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ik}\mathbf{u}_k & -\mathbf{K}^{\mathrm{C}}_{\mathrm{II}ik}\mathbf{u}_k \\ 0 & \mathbf{u}^{\mathrm{T}}_{\underline{s}j} & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{t}}_\beta \\ \dot{\mathbf{u}}_j \\ \dot{p}_{\mathrm{I}} \\ \dot{p}_{\mathrm{II}} \end{bmatrix} = - \begin{bmatrix} \mathbf{k}_{im} \\ \mathbf{L}_{ikm}\mathbf{u}_k \\ 0 \end{bmatrix}. \tag{5.59}$$

As regards the second derivatives (5.34), (5.45) and (5.47) yield the following matrix equation

$$-\begin{bmatrix} \mathbf{K}_{i\beta} & 0 & -\mathbf{g}_{\mathrm{I}i} & -\mathbf{g}_{\mathrm{II}i} \\ \mathbf{L}_{ik\beta}\mathbf{u}_k & \mathbf{K}_{ij} & -\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ik}\mathbf{u}_k & -\mathbf{K}^{\mathrm{C}}_{\mathrm{II}ik}\mathbf{u}_k \\ 0 & \mathbf{u}^{\mathrm{T}}_{\underline{s}\beta} & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{t}}_\beta \\ \ddot{\mathbf{u}}_j \\ \ddot{p}_{\mathrm{I}} \\ \ddot{p}_{\mathrm{II}} \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbf{L}_{ikj}\dot{\mathbf{t}}_k\dot{\mathbf{t}}_j - 2\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\dot{\mathbf{t}}_j\dot{p}_{\mathrm{I}} - 2\mathbf{K}^{\mathrm{C}}_{\mathrm{II}ij}\dot{\mathbf{t}}_j \\ \mathbf{N}_{ilkj}\dot{\mathbf{t}}_l\dot{\mathbf{t}}_k\mathbf{u}_j - 2\mathbf{M}_{\mathrm{I}ikj}\dot{\mathbf{t}}_k\mathbf{u}_j\dot{p}_{\mathrm{I}} - 2\mathbf{M}_{\mathrm{II}ikj}\dot{\mathbf{t}}_k\mathbf{u}_j + 2\mathbf{L}_{ikj}\dot{\mathbf{t}}_k\dot{\mathbf{u}}_j - 2\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\dot{p}_{\mathrm{I}}\dot{\mathbf{u}}_j - 2\mathbf{K}^{\mathrm{C}}_{\mathrm{II}ij}\dot{p}_{\mathrm{II}}\dot{\mathbf{u}}_j \\ \dot{\mathbf{u}}^{\mathrm{T}}_j\dot{\mathbf{u}}_j \end{bmatrix}. \tag{5.60}$$

It is characteristic of the asymptotic method that the left sides of the corresponding matrix equations are the same independent of the fact what quantity (load or a displacement component) is selected for the governing parameter.

Further matrix equations can be devised for the higher (third, fourth etc.) derivatives to attain the desired accuracy.

5.3. **Fold line of the limit points for a geometrical imperfection.** On the limit point fold line of geometrical imperfection $p_{\mathrm{II}} = p_{\mathrm{II}0} =$ constant and h is the imperfection parameter.

In this case equations (5.1), (5.2), (5.3) are also satisfied in state $(\bar{B}^{\,\mathrm{cr}})$ of the body and in the same manner equations (5.4), (5.5), (5.6) should be satisfied in state $(\bar{B}^{\,\mathrm{cr}} + \Delta\bar{B}^{\,\mathrm{kr}})$.

5.3.1. *Solution by the Newton-Rapson iteration algorithm.*

5.3.1.1. First, the geometrical imperfection parameter is selected for governing parameter $(\tau = h - h_0,\ \Delta\tau = \Delta h)$. Solutions to equations (5.4)-(5.6) are sought in the form of the series

$$\mathbf{t}_j + \Delta\mathbf{t}_j = \mathbf{t}_j + \Delta\mathbf{t}_{\underline{1}j} + \Delta\mathbf{t}_{\underline{2}j} + \cdots + \Delta\mathbf{t}_{\underline{s}j} + \cdots \tag{5.61}$$

$$p_{\mathrm{I}} + \Delta p_{\mathrm{I}} = p_{\mathrm{I}} + \Delta p_{\mathrm{I}\underline{1}} + \Delta p_{\mathrm{I}\underline{2}} + \cdots + \Delta p_{\mathrm{I}\underline{s}} + \cdots, \tag{5.62}$$

$$\mathbf{u}_j + \Delta\mathbf{u}_j = \mathbf{u}_j + \Delta\mathbf{u}_{\underline{1}j} + \Delta\mathbf{u}_{\underline{2}j} + \cdots + \Delta\mathbf{u}_{\underline{s}j} + \cdots \tag{5.63}$$

constituted by the increments $\Delta\mathbf{t}_{\underline{s}j},\ \Delta p_{\mathrm{I}\underline{s}},\ \Delta\mathbf{u}_{\underline{s}j};\quad \underline{s} = \underline{1}, \underline{2}, \cdots,\ (h = h_0, \Delta h =$ constant).

Equations (5.4)-(5.6) can now be linearized in the following manner ($p_{\mathrm{II}} = p_{\mathrm{II}0} =$ constant, $\Delta p_{\mathrm{II}} = 0$):

$$\mathbf{A}_i + \frac{\partial\mathbf{A}_i}{\partial\mathbf{t}_j}\Delta\mathbf{t}_j + \frac{\partial\mathbf{A}_i}{\partial h}\Delta h - \mathbf{g}_{\mathrm{I}i}\Delta p_{\mathrm{I}} = 0 \tag{5.64}$$

$$\mathbf{K}_{ij}\mathbf{u}_j + \frac{\partial\mathbf{K}_{ij}}{\partial\mathbf{t}_k}\Delta\mathbf{t}_k\mathbf{u}_j + \frac{\partial\mathbf{K}_{ij}}{\partial p_{\mathrm{I}}}\mathbf{u}_j\Delta p_{\mathrm{I}} + \frac{\partial\mathbf{K}_{ij}}{\partial h}\mathbf{u}_j\Delta h + \mathbf{K}_{ij}\Delta\mathbf{u}_j = 0, \tag{5.65}$$

$$\mathbf{u}_j^{\mathrm{T}} \Delta \mathbf{u}_j = 0. \tag{5.66}$$

In view of relation (5.36), (5.38)) and the notational convention

$$\overset{*}{()} = \frac{\partial\,()}{\partial h} \tag{5.67}$$

applied in the sequel, the equations that are valid for the $\underline{\mathrm{s}}^{\mathrm{th}}$ iteration step can be gathered into a single matrix equation:

$$\begin{bmatrix} \mathbf{K}_{\underline{s}ij} & 0 & -\,\mathbf{g}_{\mathrm{I}\underline{s}i} \\ \mathbf{L}_{\underline{s}ikj}\mathbf{u}_{\underline{s}k} & \mathbf{K}_{\underline{s}ij} & -\,\mathbf{K}^{\mathrm{C}}_{\mathrm{I}\underline{s}ij}\mathbf{u}_{\underline{s}j} \\ 0 & \mathbf{u}^{\mathrm{T}}_{\underline{s}j} & 0 \end{bmatrix} \begin{bmatrix} \Delta \mathbf{t}_{\underline{s}j} \\ \Delta \mathbf{u}_{\underline{s}j} \\ \Delta p_{\mathrm{I}\underline{s}} \end{bmatrix} = - \begin{bmatrix} \mathbf{A}_{\underline{s}i} \\ \mathbf{K}_{\underline{s}ij}\mathbf{u}_{\underline{s}j} \\ \mathbf{u}^{\mathrm{T}}_{\underline{s}k}\mathbf{u}_{\underline{s}k} - 1 \end{bmatrix} - \begin{bmatrix} \overset{*}{\mathbf{A}}_{\underline{s}i} \\ \overset{*}{\mathbf{K}}_{\underline{s}ij}\mathbf{u}_{\underline{s}i} \\ 0 \end{bmatrix} \Delta h. \tag{5.68}$$

Under a given Δh equation system (5.68) is a linear one for calculating $\Delta \mathbf{t}_{\underline{s}j}$, $\Delta \mathbf{u}_{\underline{s}j}$, and $\Delta p_{\mathrm{I}\underline{s}}$. The iteration can be continued till the prescribed error limit is reached.

5.3.1.2. Second, a displacement component is selected as a governing parameter for solving equation system (5.4)-(5.6) ($\tau = t_m - t_{m0},\ \Delta\tau = \Delta t_m$). The solution is sought in the form of the following series

$$\mathbf{t}_\alpha + \Delta \mathbf{t}_\alpha = \mathbf{t}_\alpha + \Delta \mathbf{t}_{\underline{1}\alpha} + \Delta \mathbf{t}_{\underline{2}\alpha} + \cdots + \Delta \mathbf{t}_{\underline{s}\alpha} + \cdots, \quad \alpha = 1, 2, \ldots, m-1 \tag{5.69}$$

$$p_{\mathrm{I}} + \Delta p_{\mathrm{I}} = p_{\mathrm{I}} + \Delta p_{\mathrm{I}\underline{1}} + \Delta p_{\mathrm{I}\underline{2}} + \cdots + \Delta p_{\mathrm{I}\underline{s}} + \cdots, \tag{5.70}$$

$$h + \Delta h = h + \Delta p h_{\underline{1}} + \Delta h_{\underline{2}} + \cdots + \Delta h_{\underline{s}} + \cdots, \tag{5.71}$$

$$\mathbf{u}_j + \Delta \mathbf{u}_j = \mathbf{u}_j + \Delta \mathbf{u}_{\underline{1}j} + \Delta \mathbf{u}_{\underline{2}j} + \cdots + \Delta \mathbf{u}_{\underline{s}j} + \cdots \tag{5.72}$$

constituted by the increments $\Delta \mathbf{t}_{\underline{s}\alpha}$, $\Delta p_{\mathrm{I}\underline{s}}$, $\Delta h_{\underline{s}}$, $\Delta \mathbf{u}_{\underline{s}j}$; $\underline{\mathrm{s}} = \underline{1}, \underline{2}, \ldots$; ($t_m = t_{m0} =$ constant, $\Delta t_m =$ constant, $p_{\mathrm{II}} = p_{\mathrm{II}0}$, $\Delta p_{\mathrm{II}} = 0$).

The linearized form of equations (5.4)-(5.6) is given by:

$$\mathbf{A}_i + \frac{\partial \mathbf{A}_i}{\partial \mathbf{t}_\beta}\Delta \mathbf{t}_\beta + \frac{\partial \mathbf{A}_i}{\partial t_m}\Delta t_m + \frac{\partial \mathbf{A}_i}{\partial h}\Delta h - \mathbf{g}_{\mathrm{I}i}\Delta p_{\mathrm{I}} = 0 \tag{5.73}$$

$$\mathbf{K}_{ij}\mathbf{u}_j + \frac{\partial \mathbf{K}_{ij}}{\partial \mathbf{t}_\beta}\mathbf{u}_j\Delta \mathbf{t}_\beta + \frac{\partial \mathbf{K}_{ij}}{\partial t_m}\mathbf{u}_j\Delta t_m + \frac{\partial \mathbf{K}_{ij}}{\partial h}\mathbf{u}_j\Delta h + \frac{\partial \mathbf{K}_{ij}}{\partial p_{\mathrm{I}}}\mathbf{u}_j\Delta p_{\mathrm{I}} + \mathbf{K}_{ij}\Delta \mathbf{u}_j = 0, \tag{5.74}$$

$$\mathbf{u}_j^{\mathrm{T}} \Delta \mathbf{u}_j = 0. \tag{5.75}$$

By taking relations (5.36)) and (5.38) into account the equation to be solved in the $\underline{\mathrm{s}}^{\mathrm{th}}$ iteration step is of form

$$\begin{bmatrix} \mathbf{K}_{\underline{s}i\beta} & 0 & -\mathbf{g}_{\mathrm{I}\underline{s}i} & -\overset{*}{\mathbf{A}}_{\underline{s}i} \\ \mathbf{L}_{\underline{s}ik\beta}\mathbf{u}_{\underline{s}k} & \mathbf{K}_{\underline{s}ij} & -\mathbf{K}^{\mathrm{C}}_{\mathrm{I}\underline{s}ik}\mathbf{u}_{\underline{s}k} & -\overset{*}{\mathbf{K}}_{\underline{s}ik}\mathbf{u}_{\underline{s}k} \\ 0 & \mathbf{u}^{\mathrm{T}}_{\underline{s}j} & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta \mathbf{t}_{\underline{s}\beta} \\ \Delta \mathbf{u}_{\underline{s}j} \\ \Delta p_{\mathrm{I}\underline{s}} \\ \Delta h_{\underline{s}} \end{bmatrix} =$$

$$= -\begin{bmatrix} \mathbf{A}_{\underline{s}i} \\ \mathbf{K}_{\underline{s}ij}\mathbf{u}_{\underline{s}j} \\ \mathbf{u}^{\mathrm{T}}_{\underline{s}k}\mathbf{u}_{\underline{s}k} - 1 \end{bmatrix} - \begin{bmatrix} \mathbf{k}_{\underline{s}im} \\ \mathbf{L}_{\underline{s}ikm}\mathbf{u}_{\underline{s}k} \\ 0 \end{bmatrix} \Delta t_m \quad (5.76)$$

in matrix notation. If Δt_m is given $\Delta \mathbf{t}_{\underline{s}\alpha}$, $\Delta p_{\mathrm{I}\underline{s}}$, $\Delta h_{\underline{s}}$ and $\Delta \mathbf{u}_{\underline{s}j}$ are the unknowns in the above linear equitation system. The iteration should be continued till the prescribed error limit is satisfied.

5.3.2. *Solution by the asymptotic numerical method.* For applying the asymptotic numerical method it is worth gathering the corresponding equilibrium equations on the basis of Subsection 5.2.2 ($p_{\mathrm{II}} = p_{\mathrm{II}0}$ = constant, $\dot{p}_{\mathrm{II}} = \ddot{p}_{\mathrm{II}} = \cdots = 0!$). Regarding equilibrium equation (5.32) as a point of departure we get

$$\dot{\mathbf{A}}_i = \mathbf{K}_{ij}\dot{\mathbf{t}}_j + \overset{*}{\mathbf{A}}_i\dot{h} - \mathbf{g}_{\mathrm{I}i}\dot{p}_{\mathrm{I}} = 0, \quad (5.77)$$

$$\ddot{\mathbf{A}}_i = \mathbf{K}_{ij}\ddot{\mathbf{t}}_j - \mathbf{g}_{\mathrm{I}i}\ddot{p}_{\mathrm{I}} + \overset{*}{\mathbf{A}}_i\ddot{h} + \mathbf{L}_{ikj}\dot{\mathbf{t}}_k\dot{\mathbf{t}}_j - 2\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\dot{\mathbf{t}}_j\dot{p}_{\mathrm{I}} + 2\overset{*}{\mathbf{K}}_{ij}\dot{\mathbf{t}}_j\dot{h} - 2\overset{*}{\mathbf{g}}_{\mathrm{I}i}\dot{p}_{\mathrm{I}}\dot{h} + \overset{**}{\mathbf{A}}_i\dot{h}^2 = 0, \quad (5.78)$$

$$\dddot{\mathbf{A}}_i = \mathbf{K}_{ij}\dddot{\mathbf{t}}_j - \mathbf{g}_{\mathrm{I}i}\dddot{p}_{\mathrm{I}} + 3\mathbf{L}_{ikj}\dot{\mathbf{t}}_k\ddot{\mathbf{t}}_j - 3\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\ddot{\mathbf{t}}_j\dot{p}_{\mathrm{I}} - 3\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\dot{\mathbf{t}}_j\ddot{p}_{\mathrm{I}} + \mathbf{N}_{ilkj}\dot{\mathbf{t}}_l\dot{\mathbf{t}}_k\dot{\mathbf{t}}_j -$$
$$- 3\mathbf{M}_{\mathrm{I}ikj}\dot{\mathbf{t}}_k\dot{\mathbf{t}}_j\dot{p}_{\mathrm{I}} + +3\overset{*}{\mathbf{K}}_{ij}\ddot{\mathbf{t}}_j\dot{h} + 3\overset{*}{\mathbf{K}}_{ij}\dot{\mathbf{t}}_j\ddot{h} - 3\overset{*}{\mathbf{g}}_{\mathrm{I}i}\ddot{p}_{\mathrm{I}}\dot{h} - 3\overset{*}{\mathbf{g}}_{\mathrm{I}i}\dot{p}_{\mathrm{I}}\ddot{h} + 3\overset{*}{\mathbf{L}}_{ikj}\dot{\mathbf{t}}_k\dot{\mathbf{t}}_j\dot{h} -$$
$$- 6\overset{*}{\mathbf{K}}{}^{C}_{\mathrm{I}ij}\dot{\mathbf{t}}_j\dot{p}_{\mathrm{I}}\dot{h} + 3\overset{**}{\mathbf{K}}_{ij}\dot{\mathbf{t}}_j\dot{h}^2 - 3\overset{**}{\mathbf{g}}_{\mathrm{I}i}\dot{p}_{\mathrm{I}}\dot{h}^2 + 3\overset{**}{\mathbf{A}}_i\ddot{h}\dot{h} + \overset{***}{\mathbf{A}}_i\dot{h}^3 = 0. \quad (5.79)$$

Determine now the derivatives of equations $(5.2)_1$ and (5.3):

$$(\mathbf{K}_{ij}\mathbf{u}_j)^{\cdot} = \mathbf{L}_{ikj}\mathbf{u}_j\dot{\mathbf{t}}_k - \mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\mathbf{u}_j\dot{p}_{\mathrm{I}} + \overset{*}{\mathbf{K}}_{ij}\mathbf{u}_j\dot{h} + \mathbf{K}_{ij}\dot{\mathbf{u}}_j = 0, \quad (5.80)$$

$$(\mathbf{K}_{ij}\mathbf{u}_j)^{\cdot\cdot} = \mathbf{N}_{ilkj}\mathbf{u}_j\dot{\mathbf{t}}_l\dot{\mathbf{t}}_k - \mathbf{M}_{\mathrm{I}ikj}\dot{p}_{\mathrm{I}}\mathbf{u}_j\dot{\mathbf{t}}_k + \mathbf{L}_{ikj}\dot{\mathbf{u}}_j\dot{\mathbf{t}}_k + \mathbf{L}_{ikj}\mathbf{u}_j\ddot{\mathbf{t}}_k + 2\overset{*}{\mathbf{L}}_{ikj}\mathbf{u}_j\dot{\mathbf{t}}_k\dot{h} +$$
$$+ \overset{**}{\mathbf{K}}_{ij}\mathbf{u}_j\dot{h}^2 - \mathbf{M}_{\mathrm{I}kij}\mathbf{u}_j\dot{\mathbf{t}}_k\dot{p}_{\mathrm{I}} - 2\overset{*}{\mathbf{K}}{}^{C}_{\mathrm{I}ij}\mathbf{u}_j\dot{p}_{\mathrm{I}}\dot{h} + \mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\mathbf{u}_j\ddot{p}_{\mathrm{I}} - \mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\dot{\mathbf{u}}_j\dot{p}_{\mathrm{I}} + 2\overset{*}{\mathbf{K}}_{ij}\dot{\mathbf{u}}_j\dot{h} +$$
$$+ \overset{*}{\mathbf{K}}_{ij}\mathbf{u}_j\ddot{h} + \mathbf{L}_{ikj}\dot{\mathbf{u}}_j\dot{\mathbf{t}}_k - \mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\dot{p}_{\mathrm{I}}\dot{\mathbf{u}}_j + \mathbf{K}_{ij}\ddot{\mathbf{u}}_j = 0, \quad (5.81)$$

$$\mathbf{u}^{\mathrm{T}}_j\dot{\mathbf{u}}_j = 0, \quad (5.82)$$

$$\dot{\mathbf{u}}^{\mathrm{T}}_j\dot{\mathbf{u}}_j + \mathbf{u}^{\mathrm{T}}_j\ddot{\mathbf{u}}_j = 0. \quad (5.83)$$

5.3.2.1. First, let us determine the equilibrium state ($\bar{B}^{\,\mathrm{cr}} + \Delta\bar{B}^{\,\mathrm{cr}}$) by using the geometrical imperfection parameter as governing parameter in the following manner ($\tau = h - h_0$, $\Delta\tau = \Delta h$):

$$\mathbf{t}_i(\tau) = \mathbf{t}_i + \dot{\mathbf{t}}_i\tau + \frac{1}{2}\ddot{\mathbf{t}}_i\tau^2 + \frac{1}{6}\dddot{\mathbf{t}}_i\tau^3 + \cdots, \quad (5.84)$$

$$p_{\mathrm{I}}(\tau) = p_{\mathrm{I}} + \dot{p}_{\mathrm{I}}\tau + \frac{1}{2}\ddot{p}_{\mathrm{I}}\tau^2 + \frac{1}{6}\dddot{p}_{\mathrm{I}}\tau^3 \cdots, \quad (5.85)$$

$$\mathbf{u}_j(\tau) = \mathbf{u}_j + \dot{\mathbf{u}}_j\tau + \frac{1}{2}\ddot{\mathbf{u}}_j\tau^2 + \frac{1}{6}\dddot{\mathbf{u}}_j\tau^3 + \cdots \quad (5.86)$$

where

$$()^{\cdot} = \frac{\partial\,()}{\partial\tau} = \frac{\partial\,()}{\partial h}, \quad \text{and} \quad \dot{h} = \frac{\partial h}{\partial h} = 1, \quad \ddot{h} = \dddot{h} = \cdots = 0. \tag{5.87}$$

Equations (5.77), (5.80) and (5.82) set up for the first derivatives can be given in matrix notation as well:

$$\begin{bmatrix} \mathbf{K}_{ij} & 0 & -\mathbf{g}_{\mathrm{I}i} \\ \mathbf{L}_{ikj}\mathbf{u}_k & \mathbf{K}_{ij} & -\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\mathbf{u}_j \\ 0 & \mathbf{u}_j^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{t}}_j \\ \dot{\mathbf{u}}_j \\ \dot{p}_{\mathrm{I}} \end{bmatrix} = - \begin{bmatrix} \overset{*}{\mathbf{A}}_i \\ \overset{*}{\mathbf{K}}_{ij}\mathbf{u}_j \\ 0 \end{bmatrix}. \tag{5.88}$$

As regards the second derivatives the matrix form of equilibrium equations (5.78), (5.81) and (5.83) is as follows:

$$\begin{bmatrix} \mathbf{K}_{ij} & 0 & -\mathbf{g}_{\mathrm{I}i} \\ \mathbf{L}_{ikj}\mathbf{u}_k & \mathbf{K}_{ij} & -\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\mathbf{u}_j \\ 0 & \mathbf{u}_j^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{t}}_j \\ \ddot{\mathbf{u}}_j \\ \ddot{p}_{\mathrm{I}} \end{bmatrix} =$$

$$= - \begin{bmatrix} \mathbf{L}_{ikj}\dot{\mathbf{t}}_k\dot{\mathbf{t}}_j - 2\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\dot{\mathbf{t}}_j\dot{p}_{\mathrm{I}} + 2\overset{*}{\mathbf{K}}_{ij}\dot{\mathbf{t}}_j - 2\overset{*}{\mathbf{g}}_{\mathrm{I}i}\dot{p}_{\mathrm{I}}\dot{h} + \overset{**}{\mathbf{A}} \\ \mathbf{N}_{ilkj}\dot{\mathbf{t}}_l\dot{\mathbf{t}}_k\mathbf{u}_j - 2\mathbf{M}_{\mathrm{I}ikj}\dot{\mathbf{t}}_k\mathbf{u}_j\dot{p}_{\mathrm{I}} + 2\mathbf{L}_{ikj}\dot{\mathbf{t}}_k\dot{\mathbf{u}}_j - 2\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\dot{p}_{\mathrm{I}}\dot{\mathbf{u}}_j \\ \dot{\mathbf{u}}_j^{\mathrm{T}}\dot{\mathbf{u}}_j \end{bmatrix}$$

$$- \begin{bmatrix} 0 \\ 2\overset{*}{\mathbf{L}}_{ikj}\mathbf{u}_j\dot{\mathbf{t}}_k - 2\overset{*}{\mathbf{K}}{}^{C}_{\mathrm{I}ij}\mathbf{u}_j\dot{p}_{\mathrm{I}} + 2\overset{*}{\mathbf{K}}_{ij}\dot{\mathbf{u}} + \overset{**}{\mathbf{K}}_{ij}\mathbf{u}_j \\ 0 \end{bmatrix}. \tag{5.89}$$

5.3.2.2. Second, it is assumed that the governing parameter for finding the equilibrium state $\left(\bar{B}^{\,\mathrm{cr}} + \Delta\bar{B}^{\,\mathrm{cr}}\right)$ is the displacement component t_m ($\tau = t_m - t_{m0}$, $\Delta\tau = \Delta t_m$). Then

$$\mathbf{t}_\alpha(\tau) = \mathbf{t}_\alpha + \dot{\mathbf{t}}_\alpha\tau + \frac{1}{2}\ddot{\mathbf{t}}_\alpha\tau^2 + \frac{1}{6}\dddot{\mathbf{t}}_\alpha\tau^3 + \cdots, \tag{5.90}$$

$$p_{\mathrm{I}}(\tau) = p_{\mathrm{I}} + \dot{p}_{\mathrm{I}}\tau + \frac{1}{2}\ddot{p}_{\mathrm{I}}\tau^2 + \frac{1}{6}\dddot{p}_{\mathrm{I}}\tau^3\cdots, \tag{5.91}$$

$$\mathbf{u}_j(\tau) = \mathbf{u}_j + \dot{\mathbf{u}}_j\tau + \frac{1}{2}\ddot{\mathbf{u}}_j\tau^2 + \frac{1}{6}\dddot{\mathbf{u}}_j\tau^3 + \cdots \tag{5.92}$$

$$h(\tau) = h + \dot{h}\tau + \frac{1}{2}\ddot{h}\tau^2 + \frac{1}{6}\dddot{h}\tau^3 + \cdots \tag{5.93}$$

where

$$()^{\cdot} = \frac{\partial\,()}{\partial\tau} = \frac{\partial\,()}{\partial t_m}, \quad \text{and} \quad \dot{t}_m = \frac{\partial t_m}{\partial t_m} = 1, \quad \ddot{t}_m = \dddot{t}_m = \cdots = 0. \tag{5.94}$$

In the present case

$$\begin{bmatrix} \mathbf{K}_{i\beta} & 0 & -\,\mathbf{g}_{\mathrm{I}i} & \overset{*}{\mathbf{A}}_i \\ \mathbf{L}_{ik\beta}\mathbf{u}_k & \mathbf{K}_{ij} & -\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ik}\mathbf{u}_k & \overset{*}{\mathbf{K}}_{ik}\mathbf{u}_k \\ 0 & \mathbf{u}^{\mathrm{T}}_{\underline{s}j} & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{t}}_\beta \\ \dot{\mathbf{u}}_j \\ \dot{p}_{\mathrm{I}} \\ \dot{h} \end{bmatrix} = -\begin{bmatrix} \mathbf{k}_{im} \\ \mathbf{L}_{\alpha km}\mathbf{u}_k \\ 0 \end{bmatrix}, \tag{5.95}$$

is the matrix form of equations (5.77), (5.80) and (5.82) derived for the first derivatives. As regards the second derivatives equations (5.78), (5.81) and (5.83) can also be presented in a matrix form:

$$\begin{bmatrix} \mathbf{K}_{i\beta} & 0 & -\,\mathbf{g}_{\mathrm{I}i} & -\mathbf{g}_{\mathrm{II}i} \\ \mathbf{L}_{ik\beta}\mathbf{u}_k & \mathbf{K}_{ij} & -\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ik}\mathbf{u}_k & \overset{*}{\mathbf{K}}_{ik}\mathbf{u}_k \\ 0 & \mathbf{u}^{\mathrm{T}}_{\underline{s}j} & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{t}}_\beta \\ \ddot{\mathbf{u}}_j \\ \ddot{p}_{\mathrm{I}} \\ \ddot{h} \end{bmatrix} =$$

$$= -\begin{bmatrix} \mathbf{L}_{ikj}\dot{\mathbf{t}}_k\dot{\mathbf{t}}_j - 2\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\dot{\mathbf{t}}_j\dot{p}_{\mathrm{I}} \\ \mathbf{N}_{ilkj}\dot{\mathbf{t}}_l\dot{\mathbf{t}}_k\mathbf{u}_j - 2\mathbf{M}_{\mathrm{I}ikj}\dot{\mathbf{t}}_k\mathbf{u}_j\dot{p}_{\mathrm{I}} + 2\mathbf{L}_{ikj}\dot{\mathbf{t}}_k\dot{\mathbf{u}}_j - 2\mathbf{K}^{\mathrm{C}}_{\mathrm{I}ij}\dot{p}_{\mathrm{I}}\dot{\mathbf{u}}_j \\ \dot{\mathbf{u}}_j^{\mathrm{T}}\dot{\mathbf{u}}_j \end{bmatrix} - $$

$$- \begin{bmatrix} 2\overset{*}{\mathbf{K}}_{ij}\dot{\mathbf{t}}_j - 2\overset{*}{\mathbf{g}}_{\mathrm{I}i}\dot{p}_{\mathrm{I}} + \overset{**}{\mathbf{A}}_i \\ 2\overset{*}{\mathbf{L}}_{ikj}\mathbf{u}_j\dot{\mathbf{t}}_k + \overset{**}{\mathbf{K}}_{ij}\mathbf{u}_j - 2\overset{*}{\mathbf{K}}{}^{C}_{\mathrm{I}ij}\mathbf{u}_j\dot{p}_{\mathrm{I}} + 2\overset{*}{\mathbf{K}}_{ij}\dot{\mathbf{u}}_j \\ 0 \end{bmatrix}. \tag{5.96}$$

Equations for the third, fourth etc. derivatives can also be derived in case they are needed to attain the required accuracy.

6. Concluding remarks

The present paper investigated the following issues if a 3D body is subjected to a distributed load perpendicular to the deformed surface:

– what the characteristics are of the critical states ,
– how to determine the fold line of limit points under various conditions.

In each case a FEM based numerical algorithm has been developed. Unfortunately there are no solved numerical examples at the moment though these could provide the final proof for the effective applicability of the methods presented in the paper. The author truly hopes that his former students shall be able to code the procedures suggested and then they will perform the necessary computations.

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HOMOTOPY PERTURBATION METHOD FOR ANALYSIS OF SQUEEZING AXISYMMETRIC FLOW OF NEWTONIAN FLUID UNDER THE EFFECTS OF SLIP AND MAGNETIC FIELD

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Abstract. In this paper, a steady two-dimensional axisymmetric flow of an incompressible viscous fluid under the influence of a uniform transverse magnetic field with slip boundary condition is studied. Using suitable similarity variables, the developed nonlinear partial differential equation of the flow phenomena is converted to a nonlinear ordinary differential equation which is solved analytically using homotopy perturbation method. By comparing the results of approximate analytical methods in this work with the results of numerical method using Runge-Kutta coupled with shooting method, the verification and the accuracy of approximate analytical solution are established. Thereafter, through the developed analytical solutions, the effects of pertinent flow, magnetic field and slip parameters on the steady two-dimensional axisymmetric flow of the viscous fluid are investigated, graphically illustrated and discussed. It is observed from the results that the velocity of the fluid increases with increase in the magnetic parameter under slip condition while the velocity decreases with increase in the magnetic field parameter under the no slip condition. By increasing the slip parameter, the velocity of the fluid increases and the fluid velocity decrease as the Reynolds number increases. It is hoped that this study will enhance and advance the understanding of axisymmetric squeezing flow of viscous fluid under no-slip and slip conditions.

Mathematical Subject Classification: 05C38, 15A15
Keywords: First-grade fluid, squeezing flow, magnetic field, slip boundary, homotopy perturbation method

1. Introduction

The analysis of squeezing flow between two parallel surfaces has received considerable and appreciable research attention as its applications increase continuously in various industrial and engineering processes. The numerous applications of such flow processes is evident in industrial and engineering applications such as moving pistons, chocolate fillers, hydraulic lifts, electric motors, flow inside syringes and nasogastric tubes, compression, and injection, power transmission squeezed film and polymers processing. In such applications, the flow of fluid is performed as a result of the moving apart or the coming together of two parallel plates. Following the pioneer work and the basic formulations of squeezing flows under lubrication assumption by Stefan

[1], there have been increasing research interests and many scientific studies on these types of flow. In past work Reynolds [2] analyzed the squeezing flow between elliptic plates while Archibald [3] investigated the same problem for rectangular plates. Earlier studies on squeezing flows were based on the Reynolds equation whose insufficiency for some cases has been shown by Jackson [4] and Usha and Sridharan [5]. Therefore, there have been several attempts and renewed research interests by different researchers to properly analyze and understand squeezing flows [6–15]. In the past efforts to analyze such flow process, Rashidi et al. [16] used the homotopy analysis method (HAM) to develop analytical approximate solutions to study the unsteady two-dimensional axisymmetric squeezing flow between parallel plates while Duwairi et al. [17] investigated effects of squeezing on heat transfer of a viscous fluid between parallel plates. Qayyum et al. [18] studied the squeezing flow of non-Newtonian second grade fluids and micro-polar models, presenting the effect on velocity profiles. Hamdan [13] analyzed the effect of squeezing flow on dusty fluids discussing squeeze effect on fluid flow. Mahmood et al. [19] investigated the effects of Prandtl's number and Nusselt number on the squeezed flow and heat transfer over a porous surface for viscous fluids. Hatami and Jing [20] applied the differential transformation method to study the natural convection of a non–Newtonian nanofluid between two vertical plates and Newtonian nanofluid between horizontal plates. Mohyud-Din et al. [21] investigated heat and mass transfer for the flow of a nanofluid between rotating parallel plates while Mohyud-Din and Khan [22] analyzed the nonlinear radiation effects on squeezing flow of a Casson fluid between parallel disks. Qayyum et al. [23] modeled and applied the homotopy perturbation method to analyze the unsteady axisymmetric squeezing fluid flow through the porous medium channel with slip boundary. Qayyum and Khan [24] presented the behavioral study of unsteady squeezing flow through porous medium using the homotopy perturbation method. Mustafa et al. [25] presented their study on the heat and mass transfer in unsteady fluid flow under squeezed flow between two parallel plates using the homotopy analysis method.

In order to study the influence of magnefield on the sqeezing flow of non-Newtonian fluid, Siddiqui et al. [26] adopted the homotopy perturbation method to the magnetic effect of squeezing viscous magnetohydrodynamics (MHD) fluid flow. A few years later, Domairry and Aziz [27] used the homotopy perturbation method (HPM) to study the MHD squeezed flow between two parallel disks with suction or injection. Also, the effect of squeeze on copper-water and copper-kerosene nanofluid between two parallel plates subjected to magnetic field was studied by Acharya et al. [28] using the differential transformation method (DTM). Ahmed et al. [29] analyzed magneto hydrodynamic (MHD) squeezing flow of a Casson fluid between parallel disks. A year later, Ahmed et al. [30] investigated MHD flow of an incompressible fluid through porous medium between dilating and squeezing permeable walls. The same year, Khan et al. [31] studied unsteady two-dimensional and axisymmetric squeezing flow between parallel plates. The same authors [32] studied MHD squeezing flow between two infinite plates while Hayat et al. [33] had earlier investigated the effect of squeezing flow of second grade fluid between two parallel disks. Khan et al. [34] analyzed unsteady squeezing flow of Casson fluid with magnetohydrodynamic effect and passing through porous medium while Ullah et al. [35] used homotopy perturbation method to present an analytical solution of squeezing flow in porous medium with MHD effect. Thin Newtonian liquid films squeezing between two plates were studied by Grimm [10]. Squeezing flow under the influence of magnetic field is widely applied to bearing with liquid-metal lubrication [36–39].

Islam et al [40] studied squeezing fluid flow between the two infinite parallel plates in a porous medium channel. In case of many polymeric liquids when the weight of molecule is high, they show slip at the boundary. The no-slip boundary condition is not applicable in this case. In many cases such as thin film problems, rarefied fluid problems, fluids containing concentrated suspensions, and flow on multiple interfaces, the no-slip boundary condition fails to work. Navier [41], for the first time, proposed the general boundary condition which demonstrates the fluid slip at the surface. The difference of fluid velocity and velocity of the boundary is proportional to the shear stress at that boundary. The proportionality constant is named the slip parameter with length as its dimension. The slip condition is of great importance especially when fluids with elastic character are under consideration [42]. Newtonian fluid was considered by Ebaid [43] to study the effects of magnetic field and wall slip conditions on the peristaltic transport in an asymmetric channel. This has great importance in medical sciences, particularly in polishing artificial heart valves and internal cavities in many manufactured parts achieved by embedding such fluids with abrasives [44]. The influence of slip on the peristaltic motion of third-order fluid in asymmetric channel is studied by Hayat et al. [45]. The effects of slip condition on the rotating flow of a third-grade fluid in a nonporous medium are investigated by Hayat and Abelman [46]. Their work was extended to a porous medium and obtaining the numerical solutions for the steady magnetohydrodynamics flow of a third grade fluid in a rotating frame is presented by Abelman et al. [47]. Ullah et al. [48] presented an approximation of first-grade MHD squeezing fluid flow with slip boundary condition using differential transformation method and optimal homotopy asymptotic method.

The past efforts in analyzing the squeezing flow problems have been largely based on the applications of approximate analytical methods such as the homotopy analysis method, differential transformation method, domian decomposition method, variation iteration method, optimal homotopy asymptotic method etc. In this paper, magnetohydrodynamic squeezing flow of first-grade fluid with slip boundary condition between two infinite plates is analyzed using the homotopy perturbation method. The study is carried out to further study and analyze the applications and limitations of the homotopy perturbation method to the fluid flow problem. Also, effects of pertinent flow, magnetic field and slip parameters are studied. By comparing the results of approximate analytical methods in this work with the numerical method using Runge-Kutta coupled with the shooting method, the verification and the accuracy of this approximate analytical solution is established.

2. Problem formulation

Consider a squeezing flow of an incompressible Newtonian fluid with constant density ρ and viscosity μ, squeezed between two large planar parallel plates separated

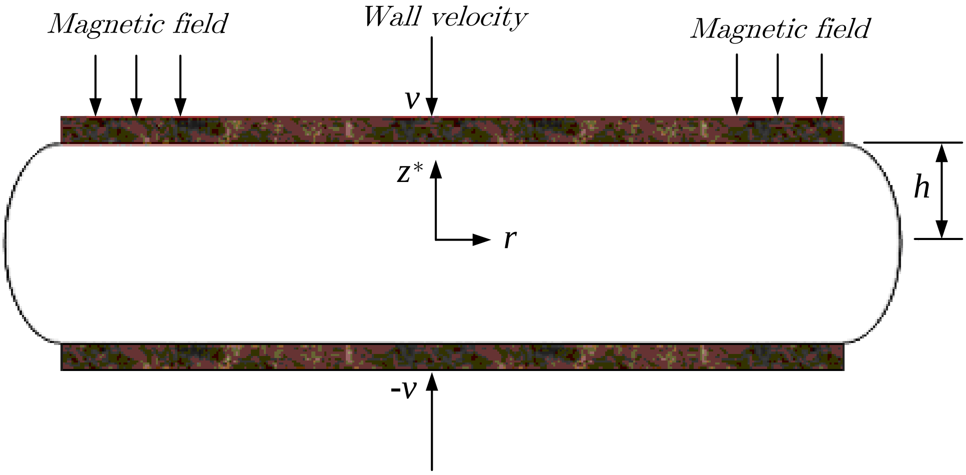

Figure 1. Model of the squeezing flow of viscous fluid under transverse uniform magnetic field

by a small distance $2h$ approaching each other with a low constant wall velocity v in the presence of a magnetic field, as shown in Figure 1. Assume that the flow is quasi steady. Then the Navier-Stokes equations governing such flow if the inertial terms are retained are:

$$\nabla \cdot \mathbf{v} = 0 \tag{1}$$

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\nabla \cdot \mathbf{v})\mathbf{v} \right] = \nabla \cdot \mathbf{T} + (\mathbf{v} \times \mathbf{B}) \times \mathbf{B} \tag{2}$$

where ∇ denotes the gradient operator, $\mathbf{v}$ is the fluid velocity, ρ is the fluid density, $\mathbf{T} = \mu\mathbf{A} - p\mathbf{I}$ is the Cauchy stress tensor, p is the fluid pressure $\mathbf{I}$ is the identity tensor μ the fluid dynamic viscosity, $\mathbf{A} = \frac{1}{2}(\mathbf{v}\nabla + \nabla\mathbf{v})$, $\mathbf{B} = \mathbf{B}_o + b\mathbf{B}_o$ is the total magnetic field given in terms of the the magnetic field intensity $\mathbf{B}_o$ and the imposed and induced magnetic fields parameter b. The modified Ohm's law and Maxwell's equations. in the absence of displacement currents, are:

$$\mathbf{J} = \sigma\left[\mathbf{E} + \mathbf{v} \times \mathbf{B}\right], \quad \nabla \cdot \mathbf{B} = 0, \tag{3}$$

$$\nabla \times \mathbf{B} = \mu_m \mathbf{J}, \quad \nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}\,. \tag{4}$$

Here $\mathbf{J}$ is the electric current density, σ represents the electrical conductivity, $\mathbf{E}$ is the electric field, and μ_m is the magnetic permeability. If ρ and μ_m are constants, b is negligible as compared to $\mathbf{B}_o$, $\mathbf{B}$ is perpendicular to $\mathbf{v}$ so that the Reynolds number is small with no electric field in the fluid flow region and then the magneto hydrodynamic force involved can be written as

$$\mathbf{J} \times \mathbf{B} = -\sigma \mathbf{B}_o^2 \mathbf{v}\,. \tag{5}$$

Assuming that the plates are non-conducting and the magnetic field is applied along the z-axis the gap distance $2h$ between the plates changes slowly with time t for small values of the velocity $\mathbf{v}$ so that it can be taken as constant. The flow is axisymmetric and is investigated in the cylindrical coordinates (r, θ, z^*) chosen in such a way that the z^*-axis is perpendicular plates for which $z^* = \pm h$ are the vertical coordinates. It is clear that $\mathbf{v}$ can now be given in the form $\mathbf{v} = (v_r, 0, v_z)$. If the body forces are negligible – this is an assumption – the Navier-Stokes equations in cylindrical coordinates are [1, 6, 7]:

$$\frac{\partial p}{\partial r} - \rho\Omega v_r = -\mu\frac{\partial \Omega}{\partial z^*} - \sigma B_o^2 \mathrm{v}_r\,, \tag{6}$$

$$\frac{\partial p}{\partial z^*} - \rho\Omega v_r = \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\Omega\right)\,, \tag{7}$$

where

$$\Omega\left(r, z^*\right) = \frac{\partial \mathrm{v}_z}{\partial r} - \frac{\partial \mathrm{v}_r}{\partial z^*}\,. \tag{8}$$

Introducing the stream function $\psi(r, z^*)$ we get

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial z^*}\,, \quad v_z = -\frac{1}{r}\frac{\partial \psi}{\partial r} \tag{9a}$$

and a generalized pressure for the cylindrical coordinate system as follows

$$p = \frac{\rho}{2}\left(v_r^2 + v_z^2\right)\,. \tag{9b}$$

Eliminating the pressure term from equations (6) and (7) we have

$$\rho\left[\frac{\partial\left(\psi, E^2\psi/r^2\right)}{\partial\left(r, z^*\right)}\right] = -\frac{\mu}{r}E^2\psi + \frac{\sigma B_0^2}{r}\frac{\partial^2\psi}{\partial z^{*2}} \tag{10}$$

where

$$E^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^{*2}}\,. \tag{11}$$

Using the transformation $\psi\left(r, z^*\right) = r^2 f\left(z^*\right)$ equation (10) can be written as

$$f^{(4)}\left(z^*\right) - \frac{\sigma B_o^2}{\mu} f^{(2)}\left(z^*\right) + 2\frac{\rho}{\mu} f\left(z^*\right) f^{(3)}\left(z^*\right) = 0\,. \tag{12}$$

This equation is associated with the following slip boundary conditions:

$$\begin{aligned} &f(0) = 0\,, && f^{(2)}(0) = 0\,,\\ &f(h) = \frac{v_z}{2}\,, && f^{(1)}(h) = \beta f^{(2)}(h) \end{aligned} \tag{13}$$

where β is the slip parameter.
Applying the following dimensionless variables

$$F = \frac{2f}{v_z}\,, \quad z = \frac{z^*}{h}\,, \quad R = \frac{\rho h v_z}{\mu} \quad \text{and} \quad m = B_o h\sqrt{\frac{\sigma}{\mu}}\,, \tag{14}$$

equation (12) can be manipulated into the following form

$$F^{(4)}(z) - m^2 F^{(2)}(z) + RF(z)\,F^{(3)}(z) = 0. \tag{15}$$

The boundary conditions can be obtained in the same way from (13):

$$F(0) = 0\,, \qquad F^{(2)}(0) = 0\,,$$
$$f(1) = 1\,, \qquad F^{(1)}(1) = \gamma F^{(2)}(1)\,. \tag{16}$$

Here $\gamma = \beta/h$ while R and m are the Reynolds and Hartmann numbers respectively. It should be noted that γ is a dimensionless slip parameter.

3. Problem formulation

3.1. **The basic idea of homotopy perturbation method.** It is very difficult to develop a closed-form solution for the above non-linear equation (15). Therefore, recourse has to be made to either the approximation analytical method, semi-numerical method or numerical method of solution. In this work, the homotopy perturbation method is used to solve the equation. In order to establish the basic idea for the homotopy perturbation method, consider a system of nonlinear differential equations given as

$$A(U) - f(r) = 0, \quad r \in \Omega, \tag{17}$$

with the boundary conditions

$$B\left(u, \frac{\partial u}{\partial \eta}\right) = 0, \qquad r \in \Gamma \tag{18}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω.

The operator A can be divided into two parts denoted by L and N where L is a linear operator and N is a non-linear operator. Equation (17) can therefore be rewritten as follows

$$L(u) + N(u) - f(r) = 0\,. \tag{19}$$

By the homotopy technique, a homotopy $U(r,p) : \Omega \times [0,1] \to R$ can be constructed, which satisfies the equation

$$H(U,p) = (1-p)\left[L(U) - L(U_o)\right] + p\left[A(U) - f(r)\right] = 0, \quad p \in [0,1]\,, \tag{20}$$

or the equation

$$H(U,p) = L(U) - L(U_o) + pL(U_o) + p\left[N(U) - f(r)\right] = 0\,. \tag{21}$$

In equations (20) and (21) $p \in [0,1]$ is an embedding parameter and U_o is an initial approximation of U in (17) which should satisfy the boundary conditions.

Also, from equations (20) and (21) we will have

$$H(U,0) = L(U) - L(U_o) = 0, \tag{22}$$

$$H(U,0) = A(U) - f(r) = 0. \tag{23}$$

The changing process of p from zero to unity is just that of $U(r,p)$ from $U_o(r)$ to $U(R)$. This is referred to homotopy in topology. Using the embedding parameter p as a small one, solution of equations (20) and (21) can be represented by a power series in p as is shown in equation (24):

$$U = U_o + pU_1 + p^2 U_2 + \cdots \tag{24}$$

It should be pointed out that the values of p are between 0 and 1; $p = 1$ produces the best result. Therefore, setting P to 1 results in the approximative solution of equation (17):

$$u = \lim_{p \to 1} U = U_o + U_1 + U_2 + \cdots \tag{25}$$

The basic idea expressed above is a combination of the homotopy and perturbation methods. Hence, the method is called homotopy perturbation method (HPM), designed to the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantages of the traditional perturbation techniques. The series in equation (25) is convergent for most cases.

3.2. **Application of the homotopy perturbation method to the present problem.** According to the homotopy perturbation method (HPM), one can construct a homotopy for equation (15) as

$$H(z,p) = (1-p)\,\tilde{F}^{(4)} + p\left[\tilde{F}^{(4)} - m^2\tilde{F}^{(2)} + R\tilde{F}\tilde{F}^{(3)}\right] \tag{26}$$

Using the embedding parameter p as a small parameter, the approximative solution of equation (15) can be represented in the form of a power series in p as is given by equation (24):

$$\tilde{F} = \tilde{F}_o + p\tilde{F}_1 + p^2\tilde{F}_2 + p^3\tilde{F}_3 + \cdots \tag{27}$$

Upon substituting (27) into equation (26), expanding the equation obtained and collecting then all terms with the same order together, the resulting equation appears in the form of a polynomial in p. After equating each coefficient of the polynomial in p to zero, we arrive at a set of differential equations and the corresponding boundary conditions as:

$$\begin{aligned} p^0 :\ & \tilde{F}_0^{(4)} = 0, \\ & \tilde{F}_0(0) = 0,\ \ \tilde{F}_0^{(2)}(0) = 0,\ \ \tilde{F}_0(1) = 1,\ \ \tilde{F}_0^{(1)}(1) = \gamma\tilde{F}_0^{(2)}(1); \end{aligned} \tag{28}$$

$$\begin{aligned} p^1 :\ & \tilde{F}_1^{(4)} - m^2\tilde{F}_0^{(2)} + R\tilde{F}_0\tilde{F}_0^{(3)} = 0, \\ & \tilde{F}_1(0) = 0,\ \ \tilde{F}_1^{(2)}(0) = 0,\ \ \tilde{F}_1(1) = 1,\ \ \tilde{F}_1^{(1)}(1) = \gamma\tilde{F}_1^{(2)}(1); \end{aligned} \tag{29}$$

$$\begin{aligned} p^2 :\ & \tilde{F}_2^{(4)} - m^2\tilde{F}_1^{(2)} + R\tilde{F}_1\tilde{F}_0^{(3)} + R\tilde{F}_0\tilde{F}_1^{(3)} = 0, \\ & \tilde{F}_2(0) = 0,\ \ \tilde{F}_2^{(2)}(0) = 0,\ \ \tilde{F}_2(1) = 1,\ \ \tilde{F}_2^{(1)}(1) = \gamma\tilde{F}_1^{(2)}(1); \end{aligned} \tag{30}$$

$$\begin{aligned} p^3 :\ & \tilde{F}_3^{(4)} - m^2\tilde{F}_2^{(2)} + R\tilde{F}_2\tilde{F}_0^{(3)} + R\tilde{F}_1\tilde{F}_1^{(3)} + R\tilde{F}_0\tilde{F}_2^{(3)} = 0, \\ & \tilde{F}_3(0) = 0,\ \ \tilde{F}_3^{(2)}(0) = 0,\ \ \tilde{F}_3(1) = 1,\ \ \tilde{F}_3^{(1)}(1) = \gamma\tilde{F}_1^{(2)}(1); \end{aligned} \tag{31}$$

$$\begin{aligned} p^4 :\ & \tilde{F}_4^{(4)} - m^2\tilde{F}_3^{(2)} + R\tilde{F}_3\tilde{F}_0^{(3)} + R\tilde{F}_3\tilde{F}_0^{(3)} + R\tilde{F}_2\tilde{F}_1^{(3)} + R\tilde{F}_1\tilde{F}_2^{(3)} + R\tilde{F}_0\tilde{F}_3^{(3)} = 0 \\ & \tilde{F}_4(0) = 0,\ \ \tilde{F}_4^{(2)}(0) = 0,\ \ \tilde{F}_4(1) = 1,\ \ \tilde{F}_4^{(1)}(1) = \gamma\tilde{F}_1^{(2)}(1); \end{aligned} \tag{32}$$

$$\begin{aligned} p^5 :\ & \tilde{F}_5^{(4)} - m^2\tilde{F}_4^{(2)} + R\tilde{F}_3\tilde{F}_0^{(3)} + R\tilde{F}_4\tilde{F}_0^{(3)} + R\tilde{F}_3\tilde{F}_1^{(3)} + R\tilde{F}_2\tilde{F}_2^{(3)} + \\ & \qquad\qquad + R\tilde{F}_1\tilde{F}_3^{(3)} + R\tilde{F}_0\tilde{F}_4^{(3)} = 0 \\ & \tilde{F}_5(0) = 0,\ \ \tilde{F}_5^{(2)}(0) = 0,\ \ \tilde{F}_5(1) = 1,\ \ \tilde{F}_5^{(1)}(1) = \gamma\tilde{F}_1^{(2)}(1). \end{aligned} \tag{33}$$

After solving equations (28)-(33) we have

$$\tilde{F}_0(z) = \frac{3(2\gamma - 1)z + z^3}{2(3\gamma - 1)}, \tag{34}$$

$$\begin{aligned}\tilde{F}_1(z) = \frac{3R}{2(3\gamma-1)^2} z^7 + \left\{ \frac{3m^2}{3\gamma - 1} + \frac{9R(2\gamma-1)}{2(3\gamma-1)^2} \right\} z^5 - \\ - \frac{1}{3(2\gamma+1)} \left\{ \gamma \left(\frac{60m^2}{3\gamma-1} + \frac{90R(2\gamma-1)}{(3\gamma-1)^2} - \frac{63R}{(3\gamma-1)^2} \right) - \right. \\ \left. - \frac{12m^2}{3\gamma-1} - \frac{36R(2\gamma-1)}{(3\gamma-1)^2} - \frac{9R}{(3\gamma-1)^2} \right\} z^3 - \\ - \left\{ \frac{3m^2}{3\gamma-1} + \frac{9R(2\gamma-1)}{(3\gamma-1)^2} + \frac{3R}{2(3\gamma-1)^2} \right\} z. \end{aligned} \tag{35}$$

In the same manner, expressions such as $\tilde{F}_2(z)$, $\tilde{F}_3(z)$, $\tilde{F}_4(z)$, $\tilde{F}_5(z)$, $\tilde{F}_6(z)$ can be obtained. However, they are too complicated expressions to be included in this paper.

Setting p to 1 results in the approximative solution of equation (15):

$$F(z) = \lim_{p \to 1} \tilde{F}(z) = \tilde{F}_o(z) + \tilde{F}_1(z) + \tilde{F}_2(z) + \tilde{F}_3(z) + \tilde{F}_4(z) + \cdots \tag{36}$$

4. Results and discussion

The above analysis shows the development of approximate analytical methods of differential transformation and homotopy perturbation methods for the analysis of a quasi steady two-dimensional axisymmetric flow of an incompressible viscous fluid under the influence of a uniform transverse magnetic field with slip boundary condition. Using HPM, a series solution (10 terms) is obtained as it provides excellent approximation to the solution of the non-linear equation with good accuracy. Although the other approximative analytical methods such as differential transformation method (DTM), homotopy analysis method (HAM), Adomiam decomposition method (ADM) and variational iterative method (VIM) might likely present more accurate results than HPM in some nonlinear problems, the search for included unknown parameter(s) that will satisfy the end boundary condition(s) leads to additional computational cost in the generation of the analytical solutions to the problems using the approximate analytical methods (DTM, HAM, ADM, VIM, etc). Moreover, they have their own operational restrictions that severely narrow their functioning domain as they are limited to a small domain. Using DTM, HAM, ADM, VIM for large or infinite domains is done with either the application of before-treatment techniques such as domain transformation techniques, domain truncation techniques and conversion of the boundary value problems to initial value problems or the use of after-treatment techniques such as Pade-approximants, basis functions, cosine after-treatment technique, sine after-treatment technique and domain decomposition technique. This is because they were initially established for initial value problems. Amending the methods to boundary value problems especially for large or infinite domains boundary value problems leads us to search for unknown parameter(s) that will satisfy the end boundary condition (s). This drawback in the other approximation analytical methods is not experienced in HPM as such tasks of before- and after-treatment techniques are not necessarily required in HPM as it easily applied to the boundary value problems without any included unknown parameter in the solution as found in DTM, HAM, ADM, VIM.

Table 1. Comparison of results

$F(z)$		
z	NM	HPM
0.00	0.000000	0.000000
0.10	0.075739	0.075738
0.20	0.152935	0.152935
0.30	0.233046	0.233045
0.40	0.317540	0.317540
0.50	0.407893	0.407892
0.60	0.505591	0.505592
0.70	0.612134	0.612134
0.80	0.729034	0.729035
0.90	0.857813	0.857813
1.00	1.000000	1.000000

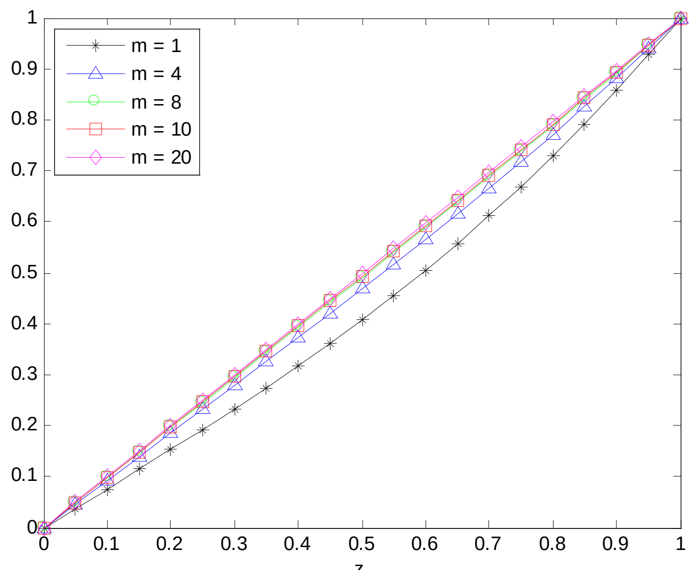

Figure 2. Effects of magnetic parameter on the flow behavior of the fluid under the influence of slip condition, $\gamma = 0.5$

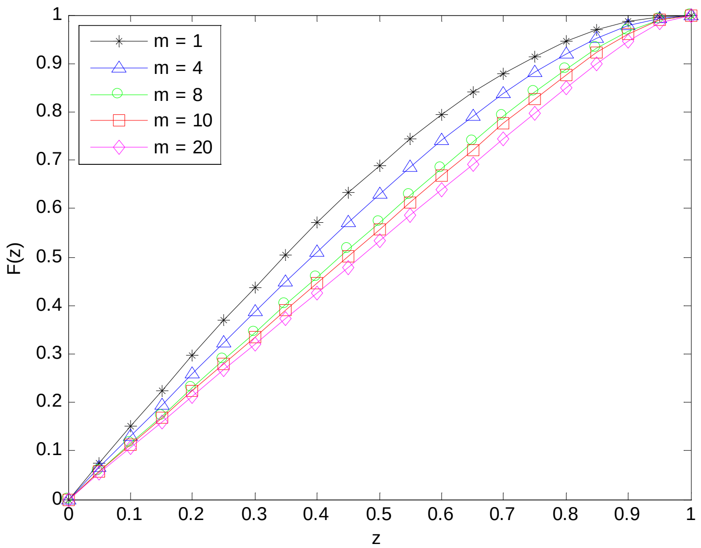

Figure 3. Effects of magnetic field parameter on the flow behavior of the fluid for no-slip condition

In order to get an insight into the problem, the effects of pertinent flow, magnetic field and slip parameters on the velocity profile of the fluid are investigated. Figure 2 shows the effects of the magnetic field parameter, Hartmann number m on the velocity of the fluid under the influence of slip condition, while Figure 3 depicts the influence of the magnetic field parameter on the velocity of the fluid under no-slip condition. It can be inferred from the figures that the velocity of the fluid increases with increase in the magnetic parameter under slip condition while an opposite trend was recorded during no-slip condition, as the velocity of the fluid decreases with increase in the magnetic field parameter under the no slip condition. The magnetic field plays the role of a resistance contributed to by the magnetic pressure field component of Lorentz force. The observed decrease in the velocity of the fluid as magnetic field increases under no-slip condition is due to the fact that the applied transverse magnetic field produces a damping or retarding force in the form of Lorentz force. As the value of magnetic parameter M increases, the retarding body force enhances and consequently the velocity reduces. The physical significance of this behavior is that the Lorentz force is a frictional resistive force which opposes the fluid motion and consequently, reduces the velocity of fluid flow. Under this scenario, the boundary layer thickness becomes thicker for a stronger magnetic field.

Figure 4 shows the influence of the slip parameter γ on the fluid velocity. By increasing γ, it is observed that the velocity of the fluid increases. The impact of slip conditions significantly enhances the velocity profile in the presence and absence of Hartmann number.

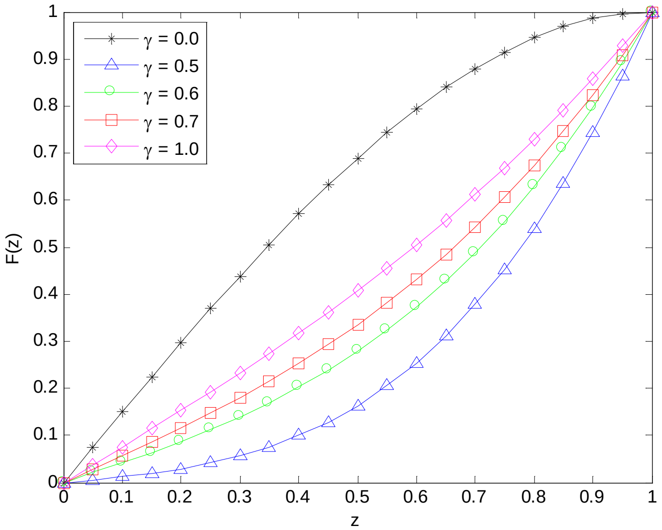

Figure 4. Effects of slip parameter on the flow behavior of the fluid

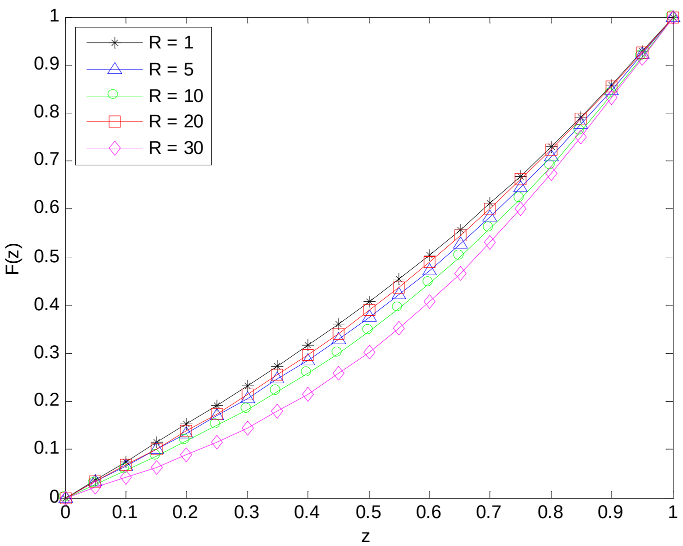

Figure 5. Effects of Reynolds number on the flow behavior of the fluid under the influence of slip condition

Figure 5 presents the effects of Reynolds number on the velocity of the fluid. It is observed from the figure that by increasing the value R, the velocity of the fluid decreases. Also, the flow increases significantly towards the center of the pipe, as depicted in the figure. The observed behavior of fluid velocity for different Reynolds number is because the flow toward the center becomes greater to make up for the space and consequently the fluid velocity also becomes greater near the center.

5. Conclusion

In this work, the homotopy perturbation method has been used to analyze steady two-dimensional axisymmetric flow of an incompressible viscous fluid under the influence of a uniform transverse magnetic field with slip boundary condition. Effects of pertinent flow, magnetic field and slip parameters have been investigated. It was established from the results that the velocity of the fluid increases with increase in the magnetic parameter under slip condition while the velocity of the fluid decreases with increase in the magnetic field parameter under the no slip condition. By increasing the slip parameter, the velocity of the fluid increases and the fluid velocity decreases as the Reynolds number increases. The approximate analytical solution has been verified by comparing the results of the approximate analytical methods with the numerical method using Runge-Kutta coupled with shooting method. It is hoped that this study will enhance and advance the understanding of axisymmetric squeezing flow of viscous fluid under no-slip and slip conditions.

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A Short History of the Publications of the University of Miskolc

The University of Miskolc (Hungary) is an important center of research in Central Europe. Its parent university was founded by the Empress Maria Teresia in Selmecbánya (today Banska Štiavnica, Slovakia) in 1735. After the first World War the legal predecessor of the University of Miskolc moved to Sopron (Hungary) where, in 1929, it started the series of university publications with the title *Publications of the Mining and Metallurgical Division of the Hungarian Academy of Mining and Forestry Engineering* (Volumes I.-VI.). From 1934 to 1947 the Institution had the name Faculty of Mining, Metallurgical and Forestry Engineering of the József Nádor University of Technology and Economic Sciences at Sopron. Accordingly, the publications were given the title *Publications of the Mining and Metallurgical Engineering Division* (Volumes VII.-XVI.). For the last volume before 1950 – due to a further change in the name of the Institution – *Technical University, Faculties of Mining, Metallurgical and Forestry Engineering, Publications of the Mining and Metallurgical Divisions* was the title.

For some years after 1950 the Publications were temporarily suspended.

After the foundation of the Mechanical Engineering Faculty in Miskolc in 1949 and the movement of the Sopron Mining and Metallurgical Faculties to Miskolc, the Publications restarted with the general title *Publications of the Technical University of Heavy Industry* in 1955. Four new series - Series A (Mining), Series B (Metallurgy), Series C (Machinery) and Series D (Natural Sciences) - were founded in 1976. These came out both in foreign languages (English, German and Russian) and in Hungarian.

In 1990, right after the foundation of some new faculties, the university was renamed to University of Miskolc. At the same time the structure of the Publications was reorganized so that it could follow the faculty structure. Accordingly three new series were established: Series E (Legal Sciences), Series F (Economic Sciences) and Series G (Humanities and Social Sciences). The latest series, i.e., the series H (European Integration Studies) was founded in 2001. The eight series are formed by some periodicals and such publications which come out with various frequencies.

Papers on computational and applied mechanics were published in the

Publications of the University of Miskolc, Series D, Natural Sciences.

This series was given the name Natural Sciences, Mathematics in 1995. The name change reflects the fact that most of the papers published in the journal are of mathematical nature though papers on mechanics also come out.

The series

Publications of the University of Miskolc, Series C, Fundamental Engineering Sciences

founded in 1995 also published papers on mechanical issues. The present journal, which is published with the support of the Faculty of Mechanical Engineering and Informatics as a member of the Series C (Machinery), is the legal successor of the above journal.

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