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REVERSIBLE AND REFLEXIVE PROPERTIES FOR RINGS WITH INVOLUTION

USAMA A. ABURAWASH AND MUHAMMAD SAAD

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Abstract. In this note, we give a generalization for the class of *-IFP rings. Moreover, we introduce *-reversible and *-reflexive *-rings, which represent the involutive versions of reversible and reflexive rings and expose their properties. Nevertheless, the relation between these rings and those without involution are indicated. Moreover, a nontrivial generalization for *-reflexive *-rings is given. Finally, in *-reversible *-rings it is shown that each nilpotent element is *-nilpotent and Köthe's conjecture has a strong affirmative solution.

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Keywords: involution, quasi-*-IFP, *-reduced, *-nilpotent, *-reversible, *-reflexive and projection *-reflexive

1. Introduction

All rings considered are associative with unity. A *-*ring* R will denote a ring with involution and a self-adjoint ideal I of R; that is $I^* = I$, is called *-*ideal*. A *projection* e of R is an idempotent satisfies $e^2 = e = e^*$. Recall from [7], an idempotent $e \in R$ is left (resp. right) semicentral in R if $eRe = Re$ (resp. $eRe = eR$). Equivalently, an idempotent $e \in R$ is left (resp. right) semicentral in R if eR (resp. Re) is an ideal of R. Moreover, if R is semiprime then every left (resp. right) semicentral idempotent is central. A semicentral projection is clearly central. A ring (resp. *-ring) R is said to be Abelian (resp. *-*Abelian*) if all its idempotents (resp. projections) are central. R is *reduced* if it has no nonzero nilpotent elements. An involution * is called *proper* (resp. *semiproper*) if for every nonzero element a of R, $aa^* = 0$ (resp. $aRa^* = 0$) implies $a = 0$. Obviously, a proper involution is semiproper.

From [5], R is *semicommutative* or has *IFP* if the right annihilator $r(a) = \{x \in A | ax = 0\}$ of every element $a \in R$ is a two-sided ideal. In [1], the involutive version of IFP, that is *-*IFP*, is given as the ring in which the right annihilator of each element of R is *-ideal. Clearly, each *-ring having $*$-IFP has also IFP.

Cohn [9] called a ring R *reversible* (or *completely reflexive*) if $ab = 0$ implies $ba = 0$ for every $a, b \in R$. Clearly, the class or reversible rings contains the reduced

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rings. Moreover, each reversible ring has IFP. Moreover, in [9, Theorem 2.2], Cohn proved that for reversible rings, Köthe's conjecture has an affirmative solution. Here, we give a strong affirmative solution for Köthe's conjecture for *-reversible *-rings and show that each nilpotent element is *-nilpotent.

In [13], Mason introduced a generalization of reversible rings; namely reflexive rings. A right ideal I of a ring R is said to be *reflexive* if $aRb \subseteq I$ implies $bRa \subseteq I$, for every $a, b \in R$. A ring R is called *reflexive* if 0 is a reflexive ideal. In [10], Kim and Baik defined an *idempotent reflexive* ideal as a right ideal I satisfying $aRe \subseteq I$ if and only if $eRa \subseteq I$ for $e^2 = e, a \in R$. R is an *idempotent reflexive ring* if 0 is an idempotent reflexive ideal. Obviously, the class of idempotent reflexive rings contains reflexive rings and Abelian rings.

$$\begin{tikzcd}
\text{domain} \arrow[r, Rightarrow] & \text{reduced} \arrow[r, Rightarrow] & \text{reversible} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \text{reflexive} \arrow[r, Rightarrow] & \text{idempotent reflexive} \\
 & & & \text{IFP} \arrow[r, Rightarrow] & \text{Abelian} \arrow[u, Rightarrow]
\end{tikzcd}$$

A subring B of a *-ring R is said to be a *-*biideal*,or self adjoint biideal, of R if $BRB \subseteq B$ and $B^* = B$.

Recall from [2], a nonzero element a of a *-ring R is a *-*zero divisor* if $ab = 0$ and $a^*b = 0$ for some nonzero element $b \in R$. Obviously, a *-zero divisor element is a zero divisor, but the converse is not true (example 3 in [2]). A *-ring without *-zero divisors is said to be a *-*domain*.

Recall from [3], an element a of a *-ring R is said to be *-nilpotent if there exist two positive integers m and n such that $a^m = 0$ and $(aa^*)^n = 0$. R is a *-reduced *-ring if it has no nonzero *-nilpotent elements; equivalently $a^2 = aa^* = 0$ implies $a = 0$ for every $a \in R$. A reduced (or *-domain) *-ring with proper involution is *-reduced. Moreover, every *-reduced *-ring is semiprime.

From [4], the *-*right annihilator* of a nonempty subset S of a *-ring R is the self adjoint biideal $r_*(S) = \{x \in A | Sx = 0 = Sx^*\}$. Finally, $\mathbb{M}_n(R)$ will denote the full matrix ring of all $n \times n$ matrices over R.

2. *-RINGS WITH QUASI-*-IFP

In this section, we introduce the property of having quasi-$*$-IFP which generalizes that of having *-IFP introduced in [1].

Definition 1. A *-ring R is said to have *quasi-*-IFP* if for every $a \in R$, the *-right annihilator $r_*(a)$ is a *-ideal of R.

In view of $l_*(a) = r_*(a^*)$, we see that the *-left annihilator is also *-ideal. Thus the definition of quasi-*-IFP *-ring is left-right symmetric.

Clearly, every *-ring R having *-IFP has also quasi-*-IFP, since $r(a)$ is *-ideal implies $r_*(a) = r(a)$ for all $a \in A$. However, the converse is not true as shown by the following example.

Example 1. Consider the $*$-ring $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field and the adjoint of matrices is the involution. Since $r\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$ is not an ideal of R, then R does not have IFP and consequently does not have *-IFP. Moreover, R has quasi-*-IFP since the *-right annihilator of every nonzero noninvertible element of R takes the form $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ which is a *-ideal of R.

The following are some equivalents for a *-ring to have quasi-*-IFP.

Proposition 1. *For a *-ring R, the following conditions are equivalent:*

(1) *R has quasi-*-IFP.*
(2) *$r_*(S)$ is a *-ideal of R for every subset S of R.*
(3) *$l_*(S)$ is a *-ideal of R for every subset S of R.*
(4) *For every $a, b \in R$, $ab = ab^* = 0$ implies $aRb = 0$ (consequently $aRb^* = 0$)*

Proof. (1)⇒(2): For every $S \subseteq R$, $r_*(S) = \bigcap_{s\in S} r_*(s)$ being the intersection of *-ideals is also a *-ideal.

(2)⇒(3): From (2), $l_*(S) = r_*(S^*)$ is a *-ideal of R.

(3)⇒(4): $ab = ab^* = 0$ implies $b^*a^* = ba^* = 0$ and consequently $b, b^* \in l_*(a^*)$ which is a *-ideal of R. Hence $bR, b^*R \subseteq l_*(a^*)$ from which $bRa^* = b^*Ra^* = 0$ and therefore $aRb = aRb^* = 0$.

(4)⇒(1): Let $x \in r_*(a)$, which is a self-adjoint biideal of R, then $ax = ax^* = 0$ implies $aRx = aRx^* = 0$, form the assumption. Hence $Rx \subseteq r_*(a)$ which means that $r_*(a)$ is a left ideal of R. Therefore $r_*(a)$ is a *-ideal due to its self-adjointness. □

The following results show that quasi-*-IFP implies *-Abelian while the converse is not true.

Proposition 2. *Every *-ring with quasi-*-IFP is *-Abelian.*

Proof. Let e be a projection in R, then $(1-e)e = (1-e)e^* = 0$ implies $(1-e)Re = 0$, from Proposition 1. Hence e is a left semicentral projection and consequently is central. □

Moreover, The next example shows that the converse of Proposition 2 is not true.

Example 2. Let F be a field of characteristic 2 and consider the *-ring $R = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, a_{ij} \in F \right\}$, with involution defined as

$$\begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix}^* = \begin{pmatrix} a & a_{34} & a_{24} & a_{14} \\ 0 & a & a_{23} & a_{13} \\ 0 & 0 & a & a_{12} \\ 0 & 0 & 0 & a \end{pmatrix}.$$

Since for the matrices $x = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, we have $xy = 0 = xy^*$, while

$$xzy = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} =$$

$\begin{pmatrix} 0 & 0 & 0 & a_{23} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$, for every $z \in R$ with $a_{23} \neq 0$, it follows that R does not have quasi-$*$-IFP, by Proposition 1. Moreover R is $*$-Abelian since for any projection $e = \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix}$, $e^2 = e^* = e$ implies $a_{11} = a_{12} = a_{13} = a_{21} = a_{22} = a_{33} = 0$ and $a^2 = a$, so that R has no nontrivial projections.

Next, we answer the question of when a *-ring with quasi-*-IFP is *-reduced.

Proposition 3. *Let R be a semiprime *-ring having quasi-*-IFP, then R is *-reduced.*

Proof. Let R be a semiprime *-ring having quasi-*-IFP. Set $a^2 = aa^* = 0$ for some $a \in R$, then $aRa = aRa^* = 0$, from Proposition 1. Since R is semiprime, then $a = 0$ and R is *-reduced. □

Finally, one can easily show that the class of *-rings having quasi-*-IFP is closed under direct sums (with changeless involution) and under taking *-subrings.

Proposition 4. *The class of *-rings having quasi-*-IFP is closed under direct sums and under taking *-subrings.*

3. *-REVERSIBLE *-RINGS

Definition 2. An ideal I of a *-ring R is called *-*reversible* if $ab, ab^* \in I$ implies $ba \in I$, for every $a, b \in R$.

It is obvious that if I is *-reversible then $ab, ab^* \in I$ implies also $b^*a \in I$, for every $a, b \in R$.

We note the following:

- A one-sided *-reversible ideal must be two-sided ideal.
- The *-reversible ideal may not be self adjoint according to the following example.

Example 3. Let R be the *-ring in Example 1. The ideal $I = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ is *-reversible but not self-adjoint

Definition 3. A *-ring R is said to be *-*reversible* if 0 is a *-reversible ideal of R; that is $ab = ab^* = 0$ implies $ba = 0$ (consequently $b^*a = 0$), for every $a, b \in R$.

Example 4. Every *-domain is a *-reversible *-ring.

It is clear that every reversible ring with involution is *-reversible. But the converse is not always true as shown by the next example.

Example 5. Let R be the *-ring in Example 1. R is not reversible since the matrices $\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ satisfy $\alpha\beta = 0$ while $\beta\alpha \neq 0$. Moreover, it easy to check that R is *-reversible.

The following are some equivalents for a *-ring to be *-reversible.

Proposition 5. *For a *-ring R, the following statements are equivalent.*

(i) *R is *-reversible.*
(ii) *$r_*(S) = l_*(S)$ for every subset S of R.*
(iii) *$r_*(a) = l_*(a)$ for every element $a \in R$.*
(iv) *For any two nonempty subsets A and B of R, $AB = AB^* = 0$ implies $BA = 0$ (consequently $B^*A = 0$).*

Proof. $(i) \Rightarrow (ii)$: Let $x \in r_*(S)$, then $sx = sx^* = 0$ for every $s \in S$. Since R is *-reversible, we have $xs = x^*s = 0$ for every $s \in S$. Hence, $xS = x^*S$ implies $x \in l_*(S)$ and we get $r_*(S) \subseteq l_*(S)$. Similarly, $l_*(S) \subseteq r_*(S)$ and $r_*(S) = l_*(S)$ follows.

$(ii) \Rightarrow (iii)$ is direct by considering S as the singleton set $\{a\}$.

$(iii) \Rightarrow (iv)$: Set $AB = AB^* = 0$ for some nonempty subsets A and B of R. Then $ab = ab^* = 0$ for every $a \in A$ and $b \in B$, and hence $b \in r_*(a) = l_*(a)$ from the condition. Therefore $ba = b^*a = 0 = 0$ for every $a \in A$ and $b \in B$ which implies $BA = B^*A = 0$.

$(iv) \Rightarrow (i)$ is direct by considering A and B as the singleton sets containing a and b, respectively. □

The question when does a *-reversible *-ring become reversible has been answered in the following proposition.

Proposition 6. *Let R be a *-reversible *-ring and either*

(1) *R has *-IFP, or*
(2) ** is proper.*

Then, R is reversible.

Proof. (1) Let R have *-IFP and $ab = 0$ for some $a, b \in R$. Then, by [1, Proposition 7], $aRb^* = 0$ and hence $ab^* = 0$. The *-reversibility of R implies $ba = 0$ and R is reversible.

(2) Let the involution * be proper and $ab = 0$ for some $a, b \in R$. Then $a(bb^*) = a(bb^*)^* = 0$ and hence $bb^*a = 0$ from the *-reversibility of R. Now $(a^*b)(a^*b)^* = a^*bb^*a = 0$ implies $a^*b = b^*a = 0$, since * is proper. Finally, by the *-reversibility of R, $b^*aa^* = 0$ implies $aa^*b^* = 0$ and $(ba)(ba)^* = baa^*b^* = 0$ implies $ba = 0$. Hence R is reversible. □

Now, we see that each *-reversible *-ring has quasi-*-IFP.

Proposition 7. *Every *-reversible *-ring has quasi-*-IFP.*

Proof. Let $ab = ab^* = 0$ for some elements a, b of a *-reversible *-ring R. Using the *-reversibility of R, we have $ba = b^*a = 0$ which implies $bar = b^*ar = 0$. Again, by the *-reversibility of R, $arb = arb^* = 0$ for every $r \in R$. Therefore $aRb = aRb^* = 0$ which means that R has quasi-*-IFP, by Proposition 1. □

From Propositions 7 and 2, we get the following.

Corollary 1. *Every *-reversible *-ring is *-Abelian.*

However, the next example shows that the converse of the previous proposition and its corollary is not always true.

Example 6. Let D be a commutative domain. Then the ring

$$R = \left\{ \begin{pmatrix} a & b & d \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix} | a, b, c, d \in D \right\}$$

has IFP, by [11, Proposition 1.2]. Define an involution * on R as $\begin{pmatrix} a & b & d \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix}^* = \begin{pmatrix} a & c & -d \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}$. One can easily check that R has quasi-*-IFP and hence is *-Abelian. But R is not *-reversible since the elements $\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ of R satisfy $\alpha\beta = \alpha\beta^* = 0$ but $\beta\alpha = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$

Moreover, if the involution * is proper then the properties IFP, *-IFP, quasi-*-IFP, *-reversibility and reducedness are identical as shown in the following result.

Proposition 8. *Let R be a *-ring and the involution * is proper. Then the following conditions are equivalent:*

(1) *R is *-reversible*
(2) *R has quasi-*-IFP.*
(3) *R has IFP.*
(4) *R has *-IFP.*
(5) *R is reduced.*

Proof. (3),(4) and (5) are equivalent from [1, Proposition 9].
(1)⇒(2) is direct from Proposition 7.
(2)⇒(3): Let $ab = 0$ for some $a, b \in R$. Then $a(bb^*) = a(bb^*)^* = 0$ implies $aRbb^* = 0$ from the quasi-*-IFP of R. Now $(arb)(arb)^* = arbb^*r^*a^* = 0$ implies $arb = 0$ fore every $r \in R$ since * is proper. Therefore $aRb = 0$ and so R has IFP.
(5)⇒(1): Let $ab = ab^* = 0$ for some $a, b \in R$, then $(ba)^2 = baba = 0$ and $(b^*a)^2 = b^*ab^*a = 0$. Hence, $ba = b^*a = 0$ from the reducedness of R and so R is *-reversible. □

Next, we discuss the converse of Example 4; that is when a *-reversible *-ring is *-domain.

Proposition 9. *A *-ring is a *-domain if and only if R is *-prime and *-reversible.*

Proof. First, Suppose that R is a *-domain, hence R is obviously *-reversible. Let $IJ = 0$ for some *-ideals I and J of R, then $ab = ab^* = 0$ for every $a \in I$ and $b \in J$. Hence, either $a = 0$ or $b = 0$ which implies $I = 0$ or $J = 0$ and so R is *-prime. Conversely, let R be both *-prime and *-reversible and $ab = a^*b = 0$ for some $0 \neq a, b \in R$. We have $r^*b^*a^* = r^*b^*a = 0$ for every $r \in R$ and so $a^*r^*b^* = ar^*b^* = 0$ for every $r \in R$ from the *-reversibility of R, which gives $bRa = bRa^* = 0$. Since R is *-prime and $a \neq 0$, we get $b = 0$, by [[6], Proposition 5.4], and so R has no *-zero divisors; that is a *-domain. □

As a consequence, we get Proposition 4 in [3] as a corollary.

Corollary 2 ([3], Proposition 4)**.** *If R is a reduced *-prime *-ring, then R is *-domain.*

For a *-ring R, the trivial extension of R, denoted by $T(R, R)$, is the ring $\left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in R \right\}$. One can define the componentwise involution $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^* = \begin{pmatrix} a^* & b^* \\ 0 & a^* \end{pmatrix}$ to make $T(R, R)$ a *-ring.

Proposition 10. *Let R be a *-reduced *-ring. If R is *-reversible, then $T(R, R)$ is a *-reversible *-ring.*

Proof. Let $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} \alpha^* & \beta^* \\ 0 & \alpha^* \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then $a\alpha = a\alpha^* = 0$ and $a\beta + b\alpha = a\beta^* + b\alpha^* = 0$. Since R is *-reversible then $\alpha a = \alpha^* a = 0$. By the *-reversibility of R, it is easy to see that $aR\alpha = 0$. Now $0 = a\beta + b\alpha = \alpha(a\beta + b\alpha) = \alpha b\alpha$ and $0 = a\beta^* + b\alpha^* = a\beta^*\alpha + b\alpha^*\alpha = b\alpha^*\alpha$. Hence $(b\alpha)^2 = b\alpha b\alpha = 0$ and $(b\alpha)(b\alpha)^* = b\alpha\alpha^* b^* = 0$. Then $b\alpha = 0$ because R is *-reduced and therefore $a\beta = 0$. Similarly, one can show that $b\alpha^* = 0$ and $a\beta^* = 0$. Using the *-reversibility of R again we get $\alpha b = \alpha^* b = \beta a = \beta^* a = 0$ which implies $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} \alpha^* & \beta^* \\ 0 & \alpha^* \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Thus $T(R,R)$ is a *-reversible *-ring. □

Furthermore, one can easily show that the class of *-reversible *-rings is closed under direct sums (using changeless involution) and taking *-subrings.

Proposition 11. *The class of *-reversible *-rings is closed under direct sums and under taking *-subrings.*

4. *-REFLEXIVE *-RINGS

In this section, we introduce the involute version of reflexive ideals and rings defined by Mason [13] and study the relation between these rings and the *-reversible rings introduced in the previous section.

Definition 4. A ideal I of a *-ring R is called **-reflexive* if for every $a, b \in R$, $aRb, aRb^* \subseteq I$ implies $bRa \subseteq I$ (consequently $b^*Ra \subseteq I$). A *-ring R is said to be **-reflexive* if 0 is a *-reflexive ideal of R.

By the way, the ideal in the previous definition can not be one sided since for every $a \in I$ satisfying $aR \subseteq I$ implies $Ra \subseteq I$ by taking $b = 1$. Also, this ideal need not be self-adjoint by Example 3.

Example 7. Every *-reduced *-ring is *-reflexive.

It is evident that every reflexive *-ring is *-reflexive. However, the next example shows that the converse is not true.

Example 8. Let D be a commutative domain and $R = \{\begin{pmatrix} \alpha & \beta & \delta \\ 0 & \alpha & \gamma \\ 0 & 0 & \alpha \end{pmatrix} | \alpha, \beta, \gamma, \delta \in D\}$. R is not reflexive according to [12, Example 2.3]. Define the involution $*$: $\begin{pmatrix} \alpha & \beta & \delta \\ 0 & \alpha & \gamma \\ 0 & 0 & \alpha \end{pmatrix} \to \begin{pmatrix} \alpha & \gamma & \delta \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix}$. It is easy to check that R is $*$-reversible and in particular is $*$-reflexive.

Lemma 1. *Let R be a ring with semiproper involution *. Then $aRb = 0$ implies $aRb^* = bRa = b^*Ra = 0$.*

Proof.

$$(arb^*)R(arb^*)^* = arb^*Rbr^*a^* \subseteq aRbr^*a^* = 0,$$
$$\text{for every } r \in R \text{ implies } aRb^* = 0,$$

$$(bra)R(bra)^* = braRa^*r^*b^* \subseteq braRb^* = 0, \text{ for every } r \in R \text{ implies } bRa = 0$$

and

$$(b^*ra)R(b^*ra)^* = braRa^*r^*b \subseteq braRb = 0, \text{ for every } r \in R \text{ implies } b^*Ra = 0.$$

□

Corollary 3. *Every *-ring with semiproper involution is reflexive (and hence *-reflexive).*

The converse of the previous corollary is not necessary true as shown in the next example.

Example 9. If F is a field, then the ring $R = F \oplus F^{op}$, with the exchange involution * defined by $(a,b)^* = (b,a)$ for all $a,b \in R$, is obviously a reflexive and hence *-reflexive but * is not semiproper. Indeed, the element $0 \neq \alpha = (0,a)$ for some nonzero element a of F satisfies $\alpha R\alpha^* = 0$.

In the following proposition, we state some equivalent definitions for a *-ring to be *-reflexive .

Proposition 12. *For a *-ring R, the following statements are equivalent :*

(i) *R is *-reflexive.*
(ii) *$r_*(aR) = l_*(Ra)$ for every $a \in R$.*
(iii) *For any two nonempty subsets A and B of R , $ARB = ARB^* = 0$ implies $BRA = B^*RA = 0$.*

Proof. $(i) \Rightarrow (ii)$: Let $x \in r_*(aR)$, then $aRx = aRx^* = 0$. Hence $xRa = x^*Ra = 0$, by the *-reflexivity of R, implies $x \in l_*(Ra)$ and so $r_*(aR) \subseteq l_*(Ra)$. Similarly, $l_*(aR) \subseteq r_*(Ra)$ and we get $r_*(aR) = l_*(Ra)$.

$(ii) \Rightarrow (iii)$: Set $ARB = ARB^* = 0$ for some subsets A and B of R. Then $aRb = aRb^* = 0$ for every $a \in A$ and $b \in B$, and hence $b \in r_*(aR) \subseteq l_*(Ra)$ from the condition. Therefore $bRa = b^*Ra = 0$ for every $a \in A$ and $b \in B$ which implies $BRA = b^*RA = 0$.

$(iii) \Rightarrow (i)$ is direct by considering A and B as the singleton sets containing a and b, respectively.. □

The following proposition and example show that the class of *-reflexive *-rings generalizes strictly that of *-reversible *-rings.

Proposition 13. *Every *-reversible *-ring is *-reflexive*

Proof. Let $aRb = aRb^* = 0$, then $ab = ab^* = 0$ implies $rab = rab^* = 0$, for every $r \in R$. So that $bra = b^*ra$ for every $r \in R$, from the *-reversibility of R. Thus $bRa = b^*Ra = 0$ and hence R is *-reflexive. □

Example 10. Let $n > 2$ be an integer and $p \leq n$ be a prime number. The *-ring $R = \mathbb{M}_n(\mathbb{Z}_p)$, where * is the transpose involution, is prime and hence reflexive (in particular *-reflexive). Moreover, R is not *-reversible. Indeed , the nonzero elements

$$\alpha = e_{12} + e_{13} + \cdots + e_{1n},$$
$$\beta = e_{11} + e_{12} + \cdots + e_{1(n-1)} + 2e_{1n}$$

of R, where e_{ij} is the square matrix of order n with 1 in the (i, j)-position and 0 elsewhere, satisfy $\alpha\beta = \alpha\beta^* = 0$, while $\beta\alpha \neq 0$ and $\beta^*\alpha \neq 0$.

The question when a *-reflexive *-ring is *-reversible is answered in the following proposition.

Proposition 14. *A *-ring R is *-reversible if and only if R has quasi-*-IFP and *-reflexive.*

Proof. The necessity is obvious. For sufficiency, let $ab = ab^* = 0$ for some $a, b \in R$. Since R has quasi-*-IFP, then $aRb = aRb^* = 0$. The *-reflexivity of R implies $bRa = b^*Ra = 0$. Hence $ba = b^*a = 0$ and R is *-reversible. □

In the next result we discuss when a principal right ideal generated by a projection in a *-reflexive *-ring is *-reflexive.

Proposition 15. *Let e be a projection of a *-reflexive *-ring R. Then e is central if and only if eR is a *-reflexive *-ideal.*

Proof. Let e be central and $aRb, aRb^* \subset eR$, then $arb = earb$ and $arb^* = earb^*$ for every $r \in R$. Hence $(1-e)aRb = (1-e)aRb^* = 0$ and consequently $(1-e)bRa = (1-e)b^*Ra = 0$, since R is *-reflexive and e is central. Hence $bRa, b^*Ra \subseteq eR$ and eR is *-reflexive ideal. The converse implication is clear since eR is a *-ideal and so e is central. □

Now, we show that *-reflexive property is extended to the *-corner.

Proposition 16. *Let R be a *-reflexive *-ring, then the *-corner eRe for every projection e of R is also *-reflexive.*

Proof. Let R be *-reflexive and $a = exe, b = eye \in eRe$ such that $a(eRe)b = a(eRe)b^* = 0$. Then $exeReye = exeRey^*e = 0$ implies $eyeRexe = ey^*eRexe = 0$, since R is *-reflexive. Therefore $b(eRe)a = b^*(eRe)a = 0$ and so eRe is *-reflexive. □

Next, we illustrate by example that *-reflexivity is not closed under taking *-subrings.

Example 11. The ring $R = \mathbb{M}_2(\mathbb{Z}_2)$ is prime and hence reflexive. The upper triangular matrix ring $S = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ over $\mathbb{Z}_2$ is a *-subring of R under the involution * defined as $\begin{pmatrix} a & b \\ d & c \end{pmatrix}^* = \begin{pmatrix} c & -b \\ -d & a \end{pmatrix}$. R is clearly *-reflexive but S is not, since the elements $\alpha = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ of R satisfy $\alpha R \beta = \alpha R \beta^* = 0$ but $\beta R \alpha = \beta^* R \alpha = \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix} \neq 0$

We end this section by showing that the *-reflexivity is restricted from the full matrix ring to its underlying ring.

Proposition 17. *If $\mathbb{M}_n(R)$ is a *-reflexive *-ring for some $n \geq 1$ and with the transpose involution *, then R is also a *-reflexive *-ring .*

Proof. let $\mathbb{M}_n(R)$ be a *-reflexive *-ring for some $n \geq 1$. Since $R \cong e_{11}\mathbb{M}_n(R)e_{11}$, as *-rings, then R is *-reflexive, by Proposition 16. □

5. Projection *-reflexive rings

In this last section, we give another generalization for the class of *-reflexive rings; that is projection *-reflexive *-rings.

In [10], Kim defines an idempotent reflexive ring R as the ring satisfying $aRe = 0$ if and only if $eRa = 0$ for every idempotent $e, a \in R$.

Definition 5. An ideal I of a *-ring R satisfies $aRe \subseteq I$ if and only if $eRa \subseteq I$ for every projection $e, a \in R$, is called *projection *-reflexive*. A *-ring R is called *projection *-reflexive* if 0 is a projection *-reflexive ideal.

The ideal I of the previous definition can not be one-sided ideal, because if I is a right ideal then $aR1 \subseteq I$ for every $a \in I$ implies $1Ra \subseteq I$, since 1 is a projection. Moreover, the ideal I in the definition need not be self-adjoint; indeed, for a field F the *-ring $F \bigoplus F$ with the exchange involution, possesses the non self-adjoint projection *-reflexive ideal $(0, F)$.

It is evident from the definition that *-reflexive and idempotent reflexive *-rings are projection *-reflexive. Accordingly, we raise the following two questions.

- Is there a projection *-reflexive *-ring which is not idempotent reflexive?
- Is there a projection *-reflexive *-ring which is not *-reflexive?

The answers of these questions are in the following example.

Example 12. The *-ring $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ over a field F with the involution * defined by $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^* = \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}$, is projection *-reflexive because $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ are the only projections of R. Clearly, R is not idempotent reflexive, since the idempotent $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ of R satisfies

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

while

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} \neq 0.$$

Moreover, R is not *-reflexive, since

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

while

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$$

The proof of the following proposition, which gives an equivalent definition for projection *-reflexive *-rings, is straightforward.

Proposition 18. *A *-ring R is projection *-reflexive if and only if for any nonempty subset A and any projection e of R, $ARe = 0$ implies $eRA = 0$.*

Obviously, every *-Abelian *-ring is projection *-reflexive and consequently every *-ring having quasi-*-IFP is also projection *-reflexive, by Proposition 2. However, the converse of this statement needs additional condition, as in the next proposition.

Proposition 19. *A *-ring R is *-Abelian if and only if R is projection *-reflexive and satisfies $eR(1-e)Re = 0$ for every projection e of R.*

Proof. The necessity is obvious, For sufficiency, let e be an arbitrary projection of the projection *-reflexive *-ring R and $eR(1-e)Re = 0$. By Proposition 18, we have $eReR(1-e) = 0$ and taking involution gives $(1-e)ReRe = 0$. Hence, $(1-e)Re = 0$ which implies that e is semicentral, from [Lemma 1.1, [8]], and hence it is central. Thus R is *-Abelian □

In the next result we show when a projection in a projection *-reflexive *-ring is central.

Proposition 20. *Let R be a projection *-reflexive *-ring and e is a projection of R. Then the following are equivalent:*

(i) *e is central.*
(ii) *eR is a projection-*-reflexive *-ideal.*

Proof. $(i) \Rightarrow (ii)$: Assume that $aRf \subset eR$ for some projection f of R. So that $arf = earf$ for every $r \in R$ and hence $(1-e)aRf = 0$. Therefore $fR(1-e)a = 0 = (1-e)fRa$, since R is projection *-reflexive, and consequently $fRa = efRa \subseteq eR$. Hence eR is a projection-*-reflexive ideal.

$(ii) \Rightarrow (i)$: is clear since eR is a *-ideal and so e is central. □

Corollary 4. *If every principal *-ideal of R is projection *-reflexive, then R is *-Abelian.*

Finally, Since the only projections of the *-corner eRe is the projection e, then eRe is projection *-reflexive if R is projection *-reflexive.

Proposition 21. *Let R be a projection *-reflexive *-ring, then the *-corner eRe, for every projection e of R, is also projection *-reflexive.*

6. *-NILPOTENCY IN *-REVERSIBLE *-RINGS

According to [3], in a *-ring R every *-nilpotent element is nilpotent but the converse is not always true as shown in [3, Example 2.2]. In the next, we give a sufficient condition that makes a nilpotent element *-nilpotent.

Proposition 22. *In a *-reversible *-ring R, every nilpotent element is *-nilpotent.*

Proof. Let a be a nilpotent element of a *-reversible *-ring R. Hence $a^n = 0$, for some positive integer n, and multiplying by a^* form right, we get $a^{n-1}(aa^*) = 0$. From the *-reversibility of R, we have $(aa^*)a^{n-1} = 0$. Multiply again by a^* form right and apply the *-reversible property, we get $(aa^*)a^{n-2} = 0$. Continuing this process, we get $(aa^*)^n = 0$ and a is *-nilpotent. □

However, the *-reversibility condition in the previous proposition is sufficient but not necessary as clear from Example 6. Indeed, the elements of the *-ideal $\begin{pmatrix} 0 & D & D \\ 0 & 0 & D \\ 0 & 0 & 0 \end{pmatrix}$ are precisely all the nilpotent (which also *-nilpotent) elements of the ring R.

Corollary 5. *Every *-reduced *-reversible *-ring is reduced.*

By the definition of nilpotency, an element is nilpotent if and only if a power of it is also nilpotent. This is not the case for *-nilpotent elements as shown in the following examples.

Example 13. In the *-ring $R = \mathbb{M}_2(\mathbb{C})$ of 2×2 matrices with complex entries and transpose involution $*$, the element $a = \begin{pmatrix} \frac{\sqrt{3}+\imath}{2} & 1 \\ 1 & \frac{\sqrt{3}-\imath}{2} \end{pmatrix}$ satisfies $(aa^*)^6 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ which can not tend to zero ever with any power. Thus a is not $*$-nilpotent, while $(a^3(a^3)^*)^1 = (a^3)^2 = 0$ which means that a^3 is *-nilpotent.

In the next, a sufficient condition is given to make *-nilpotency transfers between the element and its powers.

Lemma 2. *In a *- reversible *-ring R, the element a is *-nilpotent if and only if a^2 is *-nilpotent.*

Proof. Let a be a *-nilpotent element of R, then $a^n = (aa^*)^m = 0$, for some positive integers m and n. Now, $0 = (aa^*)^m = a^*(aa^*)^m = a^*(aa^*)^{m-1}(aa^*)$ and from the *-reversibility of R, we get $0 = (aa^*)a^*(aa^*)^{m-1} = a(a^*)^2(aa^*)^{m-1}$. Multiply the last equation by a from right to get $a(a^*)^2a(a^*a)^{m-2}(a^*a) = 0$ and applying the *-reversible property again, we get $a^*a^2(a^*)^2a(a^*a)^{m-2} = 0 = a^*a^2(a^*)^2(aa^*)^{m-2}a$. Multiply again by a^* from right and apply the *-reversibility, we get $a(a^*)^2a^2(a^*)^2(aa^*)^{m-2} = 0$. Continuing, we get $(a^2(a^*)^2)^m = 0$ and a^2 is *-nilpotent.

For sufficiency, if a^2 is *-nilpotent; that is $(a^2)^n = 0 = (a^2(a^*)^2)^m$ for some positive integers m and n, we get by the same procedure as above $(a^*a)^4m = 0$ and a is *-nilpotent. □

Proposition 23. *In a *- reversible *-ring R, the element a is *-nilpotent if and only if a^k is also *-nilpotnet for every positive integer k.*

Proof. The sufficient condition is clear. For the necessity, let a be a *-nilpotent element of R, then $a^l = (aa^*)^n = 0$ for some positive integers l and n. We use induction on k to show that $a^k(a^*)^k$ is nilpotent. The case $k = 2$ is clear from Lemma 2. Now, we have to show that $a^{k+1}(a^*)^{k+1}$ is also nilpotent if $a^k(a^*)^k$ is nilpotent. Now, if $0 = (a^k(a^*)^k)^m = a^k(a^*)^k(a^k(a^*)^k)^{m-1}$, multiply by $(a^*)^{k+1}a$ from left and apply the *-reversibility, we get $(a^*)^k(a^k(a^*)^k)^{m-1}(a^*)^{k+1}a^{k+1} = 0$. Multiply by a^* from left and take involution of both sides, we obtain $(a^*)^{k+1}a^{k+1}(a^k(a^*)^k)^{m-1}a^{k+1} = 0$. The *-reversibility of R gives $a^k(a^*)^k(a^k(a^*)^k)^{m-2}a^{k+1}(a^*)^{k+1}a^{k+1} = 0$. Multiplying by $(a^*)^{k+1}a$ from left gives $(a^*)^{k+1}a^{k+1}(a^*)^k(a^k(a^*)^k)^{m-2}a^{k+1}(a^*)^{k+1}a^{k+1} = 0$ and the *-reversibility of R gives $(a^*)^k(a^k(a^*)^k)^{m-2}a^{k+1}((a^*)^{k+1}a^{k+1})^2 = 0$. Multiply again by $(a^*)^{k+1}$, we get $(a^*)^k(a^k(a^*)^k)^{m-2}(a^{k+1}(a^*)^{k+1})^3 = 0$. Continuing, we get $(a^*)^k(a^{k+1}(a^*)^{k+1})^{2m-1} = 0$ and multiplication by $a^{k+1}a^*$ gives $(a^{k+1}(a^*)^{k+1})^{2m} = 0$. □

Conjecture 1 (Köthe's conjecture)**.** *If a ring has a non-zero nil right ideal, then it has a nonzero nil ideal, is still unsolved.*

In [9, Theorem 2.2], Cohn proved that for reversible rings, Köthe's conjecture has an affirmative solution. In the next, we have a strong affirmative solution for *-reversible *-rings.

Proposition 24. *Every *-reversible *-ring which is not *-reduced, contains a nonzero nilpotent ideal.*

Proof. If R is not *-reduced and *-reversible *-ring, then R contains a nonzero *-nilpotent element, say a. So that $a^m = (aa^*)^n = 0$, for some positive integers m and n. If $n = 1$, we have $a^m = aa^* = 0$ which implies $r_1 a^m = r_1 a^{m-1} a^* = 0$ for every $r_1 \in R$. From the *-reversibility of R, we get $a r_1 a^{m-1} = 0$. Again $r_2 a r_1 a^{m-1} = r_2 a r_1 a^{m-2} a^* = 0$ implies $a r_2 a r_1 a^{m-2} = 0$ for every $r_1, r_2 \in R$. Continuing, we get $(RaR)^m = 0$; that is the ideal generated by a is a nonzero nilpotent ideal. If $n > 1$, we have $aa^* \neq 0$. Since $(aa^*)^n = 0$, then $r_1(aa^*)^n = 0$ gives $(aa^*)r_1(aa^*)^{n-1} = 0$ due to the self-adjointness of aa^* and using the *-reversible property. As before, we get $(Raa^*R)^n = 0$; that is the *-ideal generated by aa^* is a nonzero nilpotent ideal. □

Corollary 6. *In a *-reversible *-ring R, if R has a non-zero nil right ideal, then it has a nonzero nil ideal.*

Corollary 7. *Each semiprime *-reversible *-ring is *-reduced.*

CONCLUSION

We can now sate the following implications in the class of rings with involution.

$$\begin{tikzcd}
 & \text{*-reduced} & \text{proper} \arrow[l, Rightarrow] \arrow[r, Rightarrow] & \text{semiproper} \arrow[d, Rightarrow] & \text{Abelian} \arrow[d, Rightarrow] \\
\text{domain} \arrow[r, Rightarrow] & \text{reduced} \arrow[u, Rightarrow] \arrow[r, Rightarrow] & \text{reversible} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{reflexive} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{idempotent reflexive} \arrow[d, Rightarrow] \\
 & \text{*-domain} \arrow[r, Rightarrow] & \text{*-reversible} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{*-reflexive} \arrow[r, Rightarrow] & \text{projection *-reflexive} \\
\text{*-IFP} \arrow[r, Rightarrow] & \text{IFP} \arrow[r, Rightarrow] & \text{quasi-*-IFP} \arrow[rr, Rightarrow] & & \text{*-Abelian} \arrow[u, Rightarrow]
\end{tikzcd}$$

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Authors' addresses

Usama A. Aburawash
Alexandria University, Department of Mathematics and Computer Science, Alexandria, Egypt
E-mail address: aburawsh@alexu.edu.eg

Muhammad Saad
Alexandria University, Department of Mathematics and Computer Science, Alexandria, Egypt
E-mail address: muhammad.saad@alex-sci.edu.eg

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AXIOMATIC SYSTEM DEFINING AN ORDER-EMBEDDING BETWEEN INFINITE σ-ALGEBRAS

NUTEFE KWAMI AGBEKO

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Abstract. The purpose of the present paper is twofold: on the one hand to set up an axiomatic system defining an order-embedding between infinite σ-algebras to generalize the powering mapping and investigate some additional necessary and sufficient condition for the postulate of powering to hold in the system, and on the other hand to provide some theoretical applications.

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1. Introduction

It is well-known that if a σ-algebra is finite, then it is equinumerous with a power set. The question thus arises to know: *Under what condition or conditions an infinite σ-algebra would be equinumerous with a power set*?

By answering this question we constructed, in [2], an order-embedding (called powering mapping) from any given infinite σ-algebra into an appropriate power set and gave a necessary and sufficient condition, called the postulate of powering, which makes it onto (or a surjection).

The purpose of the present paper is twofold: on the one hand to set up an axiomatic system defining an order-embedding between infinite σ-algebras to generalize the powering mapping and investigate some additional necessary and sufficient condition for the postulate of powering to hold in the system, and on the other hand to provide theoretical applications.

But before that we shall recall some essential notions and results.

Definition 1 ([1, 3])**.** Let $(\Omega, \mathcal{F})$ be a measurable space. A set function $p : \mathcal{F} \to [0, 1]$ is called an *optimal measure* if it satisfies the following three axioms:

A1. p satisfies the normalization property, i.e. $p(\Omega) = 1$ and $p(\varnothing) = 0$.
A2. p is operation-preserving, i.e. $p(B \cup E) = p(B) \vee p(E)$ for all measurable sets B and E.

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A3. p is continuous from above, i.e. whenever $(E_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ is a decreasing sequence, then

$$p\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} p(E_n) = \bigwedge_{n=1}^{\infty} p(E_n).$$

The triple $(\Omega, \mathcal{F}, p)$ will be referred to as an optimal measure space.
If p is an optimal measure, by a p-atom we mean a measurable set H with $p(H) > 0$ such that whenever $B \in \mathcal{F}$ and $B \subset H$, then $p(B) = p(H)$ or $p(B) = 0$.

Definition 2 ([1], Definition 1.1)**.** A p-atom H is decomposable if there exists a subatom $B \subset H$ such that $p(B) = p(H) = p(H\backslash B)$. If no such subatom exists, the atom H is then said to be indecomposable.

The Structure Theorem ([1], Theorem 1.2). Let $(\Omega, \mathcal{F}, p)$ be an optimal measure space. Then there exists a collection $\mathcal{H}(p) = \{H_n : n \in J\}$ of disjoint indecomposable p-atoms, where J is some countable (i.e. finite or countably infinite) index set, such that for every measurable set $B \in \mathcal{F}$ with $p(B) > 0$ we have

$$p(B) = \max\{p(B \cap H_n) : n \in J\}. \tag{1.1}$$

Moreover, if J is countably infinite, then the only limit point of the set $\{p(H_n) : n \in J\}$ is 0. (The collection of all optimal measures with countably infinite generating system is denoted by $\mathcal{P}_\infty$.)

We note that the Structure Theorem (see [3]) was first proved in ZFC and some two years later a similar proof was given in ZF (cf. [7]). This, in a sense, makes relevant the above question.

In the sequel we shall be dealing with infinite measurable spaces $(\Omega, \mathcal{F})$, i.e. both Ω and the σ-algebra $\mathcal{F}$ of Ω are infinite sets. (At times this will be stressed in the statements.)

Definition 3 ([2])**.** We say that an optimal measure $p^* \in \mathcal{P}_\infty$ is of order-one if there is a unique indecomposable p^*-*atom* H such that $p^*(H) = 1$. Any such atom is referred to as an order-one-atom and the set $\widetilde{\mathcal{P}^1_\infty}$ (resp. $\mathfrak{B}$) denotes the collection of all order-one optimal measures (resp. the collection of all order-one-atoms). We also refer to order-one as unitary.

Definition 4. The binary relation $R \subset \widetilde{\mathcal{P}^1_\infty} \times \widetilde{\mathcal{P}^1_\infty}$ is called unitary and denoted by $p_1 R\, p_2$ if and only if there is some $p \in \widetilde{\mathcal{P}^1_\infty}$ with H the corresponding unitary atom such that $p_1(H) = p_2(H) = p(H) = 1$.

Remark 1. The unitary relation is an equivalence relation, i.e. it is reflexive, symmetric and transitive.

We note that the reflexivity and the symmetry are obvious. The transitivity is also evident, because a unitary optimal measure cannot possess two distinct unitary atoms.

If $p \in \widetilde{\mathcal{P}_\infty^1}$ with $H \in \mathfrak{B}$ its corresponding unitary atom and writing

$$p := \left\{ q^* \in \widetilde{\mathcal{P}_\infty^1} : q^*(H) = 1 \right\}$$

for the equivalence class induced by the unitary optimal measure p, we denote the quotient $\widetilde{\mathcal{P}_\infty^1}/R$ of $\widetilde{\mathcal{P}_\infty^1}$ modulo R by $\mathcal{P}_\infty^1$, similarly as in [2].

If A is a non-empty measurable set and $p \in \mathcal{P}_\infty^1$, the identity $p(A) = 1$ (resp. the inequality $p(A) < 1$) will simply mean that $p^*(A) = 1$ (resp. $p^*(A) < 1$) for any representing member $p^* \in p$. We shall also write $p(A) = 0$ to mean that $p^*(A) = 0$ whenever $p^* \in p$.

Lemma 1 ([2])**.** *Let $A, B \in \mathcal{F}$ and $p \in \mathcal{P}_\infty^1$ be arbitrary. In order that $p(A \cap B) = 1$ it is necessary and sufficient that $p(A) = 1$ and $p(B) = 1$.*

Remark 2 ([2]). Let $A \in \mathcal{F}$ and $p \in \mathcal{P}_\infty^1$ be arbitrary. Then the identities $p(A) = 1$ and $p\left(\overline{A}\right) = 1$ cannot hold simultaneously, i.e. for no representing member p^* of class p the identities $p^*(A) = 1$ and $p^*\left(\overline{A}\right) = 1$ hold at the same time.

Definition 5 ([2])**.** By a powering mapping we mean the following set function

$$\begin{array}{rcl} \Delta : \mathcal{F} & \to & \mathbb{P}\left(\mathcal{P}_\infty^1\right) \\ A & \mapsto & \Delta(A) := \begin{cases} \{p \in \mathcal{P}_\infty^1 : p(A) = 1\} & \text{if } A \neq \varnothing \\ \varnothing & \text{if } A = \varnothing. \end{cases} \end{array} \tag{1.2}$$

Lemma 2 ([2], Lemma 2.2)**.** *Let $(\Omega, \mathcal{F})$ be an infinite measurable space. Then $\Delta\left(\overline{B}\right) = \overline{\Delta(B)}$ for every set $B \in \mathcal{F}$.*

It is important to note that the powering mapping is an order-embedding, and it is one-to-one.

The Postulate of Powering ([2]). If $\Gamma \in \mathbb{P}\left(\mathcal{P}_\infty^1\right) \setminus \{\varnothing\}$ and $\mathcal{C}$ denotes the collection of all unitary atoms generated by the elements of Γ, then the set $\bigcup \mathcal{C}$ is measurable and $\Delta\left(\bigcup \mathcal{C}\right) \subseteq \Gamma$.

Theorem 1 ([2])**.** *The powering mapping $\Delta : \mathcal{F} \to \mathbb{P}\left(\mathcal{P}_\infty^1\right)$ is surjective if and only if the postulate of powering is valid.*

2. The axiomatic system and main results

Definition 6. Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be infinite measurable spaces. A mapping $G : \mathcal{F}_1 \to \mathcal{F}_2$ is said to be fundamental if it satisfies the normalization property:

$$G(\Omega_1) = \Omega_2, \tag{2.1}$$

the difference-preserving property:

$$G(A \setminus B) = G(A) \setminus G(B) \tag{2.2}$$

for all $A, B \in \mathcal{F}_1$ and the vanishing property:

$$G(A) = \varnothing \text{ implies } A = \varnothing \tag{2.3}$$

whenever $A \in \mathcal{F}_1$.

The following remark is an immediate consequence of Definition 6.

Remark 3. Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be infinite measurable spaces and $G : \mathcal{F}_1 \to \mathcal{F}_2$ a fundamental mapping. Then

(1) $G(\varnothing) = \varnothing$.
(2) G is a well defined mapping, i.e. if $A \in \mathcal{F}_1$, $E \in \mathcal{F}_2$ and $D \in \mathcal{F}_2$ are such that $G(A) = E$ and $G(A) = D$, then $E = D$.
(3) Identity (2.2) is equivalent to

$$G\left(A \cap \overline{B}\right) = G(A) \cap \overline{G(B)} \tag{2.4}$$

for all $A, B \in \mathcal{F}_1$, where the overline denotes the complement operation.

Proof. Making the substitutions $A = B = \varnothing$ in equation (2.2), one can easily observe that $G(\varnothing) = \varnothing$. To show that G is well defined, let us suppose $A \in \mathcal{F}_1$, $E \in \mathcal{F}_2$ and $D \in \mathcal{F}_2$ are such that $G(A) = E$ and $G(A) = D$. Then $E \setminus D = G(A) \setminus G(A) = G(A \setminus A) = G(\varnothing) = \varnothing$, and similarly we can verify that $D \setminus E = \varnothing$. The conjunction of both $E \setminus D = \varnothing$ and $D \setminus E = \varnothing$ yields the desired equality $E = D$. To end the proof, we just note that (2.4) is another way of rewriting (2.2), so it is obvious, indeed. □

The next basic properties about fundamental mappings are essential.

Proposition 1. *Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be infinite measurable spaces and $G : \mathcal{F}_1 \to \mathcal{F}_2$ be a fundamental mapping. Then for arbitrarily fixed $A, B \in \mathcal{F}_1$ we have:*

(a) *G preserves the complementation, i.e. $G\left(\overline{B}\right) = \overline{G(B)}$.*
(b) *G preserves the intersection, i.e. $G(A \cap B) = G(A) \cap G(B)$.*
(c) *G preserves the union, i.e. $G(A \cup B) = G(A) \cup G(B)$.*
(d) *G preserves the disjointion, i.e. $G(A \cap B) = \varnothing$ if and only if $A \cap B = \varnothing$.*
(e) *$G(A) = G(B)$ if and only if $A = B$.*
(f) *G is one-to-one, i.e. injective.*
(g) *G is order-embedding, i.e. $G(A) \subseteq G(B)$ if and only if $A \subseteq B$.*

Proof. Assertion **(a)** is immediate from both identities (2.2) and (2.1) if we replace in (2.2) A with Ω_1. To show **(b)** fix arbitrarily $A, B \in \mathcal{F}_1$. Then via identity (2.2) and assertion **(a)** we have

$$\begin{aligned} G(A \cap B) &= G\left(A \cap \overline{\overline{B}}\right) = G\left(A \setminus \overline{B}\right) = G(A) \setminus G\left(\overline{B}\right) \\ &= G(A) \cap \overline{\overline{G(B)}} = G(A) \cap G(B). \end{aligned}$$

Assertion **(c)** follows from assertions **(a)** and **(b)**. To show assertion **(d)** note that the sufficiency is trivial and the necessity follows from the vanishing property. The sufficiency condition of Assertion **(e)** obviously holds true. Its necessity will be shown as follows. For $A, B \in \mathcal{F}_1$ assume that $G(A) = G(B)$. Then by the difference-preserving property, $G(A \setminus B) = G(A) \setminus G(B) = \varnothing$, which implies via the vanishing property that $A \setminus B = \varnothing$. This means that $A \subseteq B$. Reversing the roles of A and B we also get $B \subseteq A$, which yields that $A = B$. Note that the fundamental mapping is one-to-one, due to assertion **(e)**. To show the necessity of assertion **(g)** let $A, B \in \mathcal{F}_1$ and assume that $G(A) \subseteq G(B)$. Then $\varnothing = G(A) \setminus G(B) = G(A \setminus B)$, via the difference-preserving property. Applying the vanishing property we obtain the desired inclusion $A \subseteq B$. For the sufficiency proof assume that $A \subseteq B$ where $A, B \in \mathcal{F}_1$. Then $\varnothing = A \setminus B$. Hence by Remark 3/(1) and the difference-preserving property, $\varnothing = G(\varnothing) = G(A \setminus B) = G(A) \setminus G(B)$ which implies that $G(A) \subseteq G(B)$. To end the proof we note that Assertion **(f)** is a consequence of Assertions **(e)** or **(g)**. □

The notion of order-embedding can be found in [4–6], for instance.

Definition 7. Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be infinite measurable spaces. A fundamental mapping $G : \mathcal{F}_1 \to \mathcal{F}_2$ is said to be tail-preserving if for every sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}_1$ and any fixed number $k \in \mathbb{N}$,

$$G\left(\bigcup_{n=k}^{\infty} A_n\right) = \bigcup_{n=k}^{\infty} G(A_n). \tag{2.5}$$

Remark 4. Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be infinite measurable spaces. If the fundamental mapping $G : \mathcal{F}_1 \to \mathcal{F}_2$. The the following two assertions are equivalent.

For every sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}_1$ and any fixed number $k \in \mathbb{N}$,

$$G\left(\bigcup_{n=k}^{\infty} A_n\right) = \bigcup_{n=k}^{\infty} G(A_n).$$

For every sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}_1$,

$$G\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} G(A_n). \tag{2.6}$$

Proof. Note that the conditional **(a)**$\Longrightarrow$**(b)** is straightforward. To show the converse conditional suppose that for every sequence in $\mathcal{F}_1$ identity (2.6) is valid. Pick arbitrarily a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}_1$ and a number $k \in \mathbb{N}$. Let $B_j = \varnothing$, $j = 1, \ldots, k-1$ and $B_j = A_j$, $j \geq k$. Since $G(\varnothing) = \varnothing$, then

$$G\left(\bigcup_{n=k}^{\infty} A_n\right) = G\left(\bigcup_{n=1}^{\infty} B_n\right) = \bigcup_{n=1}^{\infty} G(B_n) = \bigcup_{n=k}^{\infty} G(A_n).$$

Hence the conditional **(b)**$\Longrightarrow$**(a)** also holds true, which completes the proof. □

Remark 5. Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be infinite measurable spaces. If the fundamental mapping $G : \mathcal{F}_1 \to \mathcal{F}_2$ is tail-preserving, then for every sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}_1$ and any fixed number $k \in \mathbb{N}$,

$$G\left(\bigcap_{n=k}^{\infty} A_n\right) = \bigcap_{n=k}^{\infty} G(A_n). \tag{2.7}$$

Proof. Fix arbitrarily a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}_1$ and a number $k \in \mathbb{N}$. Then by Definition 7 it is also true that

$$G\left(\bigcup_{n=k}^{\infty} \overline{A_n}\right) = \bigcup_{n=k}^{\infty} G\left(\overline{A_n}\right)$$

which yields (2.5), via De Morgan infinite identity and Proposition 1. □

Proposition 2. *Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be infinite measurable spaces. If $G : \mathcal{F}_1 \to \mathcal{F}_2$ is a tail-preserving mapping, then each of the following assertions is valid.*

- **i.** *Every sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}_1$ converges increasingly if and only if $(G(A_n))_{n\in\mathbb{N}} \subset \mathcal{F}_2$ converges increasingly.*
- **ii.** *Every sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}_1$ converges decreasingly if and only if $(G(A_n))_{n\in\mathbb{N}} \subset \mathcal{F}_2$ converges decreasingly.*
- **iii.** *Every sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}_1$ is convergent (i.e. $\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$) if and only if $(G(A_n))_{n\in\mathbb{N}}$ is order-convergent (i.e. $\liminf_{n\to\infty} G(A_n) = \limsup_{n\to\infty} G(A_n)$).*

Proof. The first assertion is straightforward from the assumption and Proposition 1/**(g)**. The second assertion can also be easily derived from the assumption, Proposition 1/**(a)** and Proposition 1/**(g)**. To prove the third assertion, let us us fix an arbitrary sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}_1$. Suppose that $\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$. Then

$$\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

and taking the G map of these two equal sets we have by the first two parts (**i.** and **ii.**) that

$$\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} G(A_k) = G\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k\right) = G\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} G(A_k).$$

Consequently, $\liminf_{n\to\infty} G(A_n) = \limsup_{n\to\infty} G(A_n)$. Suppose that $\liminf_{n\to\infty} G(A_n) = \limsup_{n\to\infty} G(A_n)$. Then $\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} G(A_k) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} G(A_k)$. Hence by Definition 7 and Remark 5 it follows that $G\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k\right) = G\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right)$, which implies

$\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k$ or equivalently, $\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$. This completes the proof. □

3. Further characterization of the Postulate of powering

Proposition 3. *Let $(\Omega, \mathcal{F})$ be an infinite measurable space. Then the powering mapping Δ defined by in (1.2) is a tail-preserving fundamental mapping.*

To prove Proposition 3 we will need only some basic remarks or results such as Lemma 1, Remark 2, some fundamental notions about optimal measures, as well as the following remark.

Remark 6. Let $(\Omega, \mathcal{F})$ be an infinite measurable space and Δ the powering mapping. Then for any $A, B \in \mathcal{F}$ the following assertions are equivalent.

(1) $A \setminus B = \varnothing$.
(2) $\Delta(A \setminus B) = \varnothing$.
(3) $\Delta(A) \setminus \Delta(B) = \varnothing$.

Proof. The biconditional **(1)** $\Longleftrightarrow$ **(2)** is obvious because of Lemma 2. We show the conditional **(1)**$\Longrightarrow$**(3)**. To this end assume by the contrapositive that $\Delta(A) \setminus \Delta(B) \neq \varnothing$ and pick an arbitrary class $p \in \Delta(A) \setminus \Delta(B)$. Then $p \in \Delta(A)$ and $p \notin \Delta(B)$. Thus necessarily $p \in \overline{\Delta(B)} = \Delta\left(\overline{B}\right)$, via Lemma 2. But since also $p \in \Delta(A)$ we have that $p(A) = p\left(\overline{B}\right) = 1$ or equivalently $p\left(A \cap \overline{B}\right) = 1$ due to Lemma 1. Consequently, $p \in \Delta\left(A \cap \overline{B}\right) = \Delta(A \setminus B)$, implying that $\Delta(A \setminus B) \neq \varnothing$. One can thus easily deduce from the definition of Δ that $A \setminus B \neq \varnothing$. Finally, we show the conditional **(3)**$\Longrightarrow$**(2)**. To this end assume by the contrapositive that $\Delta(A \setminus B) \neq \varnothing$ and pick arbitrarily a class $p \in \Delta(A \setminus B)$. Then $p(A) = 1$ and $p\left(\overline{B}\right) = 1$. But $p\left(\overline{B}\right) = 1$ implies as above that $p \in \Delta\left(\overline{B}\right) = \overline{\Delta(B)}$. Consequently, $p \in \Delta(A) \cap \overline{\Delta(B)} = \Delta(A) \setminus \Delta(B)$ which proves that $\Delta(A) \setminus \Delta(B) \neq \varnothing$. This completes the proof. □

Proof of Proposition 3. First we show that Δ is a fundamental mapping. In fact, pick arbitrarily a class $p \in \mathcal{P}_{\infty}^{1}$. Then due to the fact that every optimal measure assigns to Ω the value 1 it follows that $p(\Omega) = 1$, i.e. $p \in \Delta(\Omega)$, which means that $\mathcal{P}_{\infty}^{1} \subseteq \Delta(\Omega)$. As the reverse inclusion is straightforward we conclude that the powering mapping satisfies the normalization property. Next, we show that Δ is a difference-preserving mapping, i.e. $\Delta(A \setminus B) = \Delta(A) \setminus \Delta(B)$ for all $A, B \in \mathcal{F}$. We have two cases to consider.

(a) In the first case, suppose $A \setminus B = \varnothing$. Then due to Remark 6 identity $\Delta(A \setminus B) = \Delta(A) \setminus \Delta(B)$ follows.
(b) In the second case, suppose $A \setminus B \neq \varnothing$, and fix arbitrarily $p \in \Delta(A \setminus B)$. Then $p\left(A \cap \overline{B}\right) = p(A \setminus B) = 1$, and so Lemma 1 yields that both identities $p(A) = 1$

and $p\left(\overline{B}\right) = 1$ are sastisfied. Thus $p \in \Delta(A)$ and $p \in \Delta\left(\overline{B}\right) = \overline{\Delta(B)}$. Then $p \in \Delta(A) \cap \overline{\Delta(B)}$. Consequently, $\Delta(A \setminus B) \subseteq \Delta(A) \cap \overline{\Delta(B)} = \Delta(A) \setminus \Delta(B)$. To show the reverse inclusion fix arbitrarily a class $p \in \Delta(A) \setminus \Delta(B)$. Then $p \in \Delta(A)$ and $p \notin \Delta(B)$. Thus $p \in \Delta\left(A \cap \overline{B}\right)$ and so $\Delta(A) \setminus \Delta(B) \subseteq \Delta\left(A \cap \overline{B}\right) = \Delta(A \setminus B)$. Consequently, $\Delta(A \setminus B) = \Delta(A) \setminus \Delta(B)$.

Therefore, Δ satisfies the difference-preserving property. The vanishing property is obvious from Remark 6. We have thus shown that Δ is a fundamental mapping.

Finally, we prove that Δ is tail-preserving. To this end select arbitrarily a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ and a number $k \in \mathbb{N}$. We have two cases to go into:

(i) In the first case, suppose $\bigcup_{n=k}^{\infty} A_n = \varnothing$, which is equivalent to saying that $A_n = \varnothing$ for all $n \in \mathbb{N}, n \geq k$. Hence by definition $\Delta\left(\bigcup_{n=k}^{\infty} A_n\right) = \varnothing$. Because of the above two equivalent assertions, for all $n \in \mathbb{N}, n \geq k$ we have $A_n = \varnothing$ which yields via the definition of Δ that $\Delta(A_n) = \varnothing$ whenever $n \geq k$. Consequently,

$$\Delta\left(\bigcup_{n=k}^{\infty} A_n\right) = \varnothing = \bigcup_{n=k}^{\infty} \Delta(A_n).$$

(ii) In the second case, suppose $\bigcup_{n=k}^{\infty} A_n \neq \varnothing$. Then by definition

$$\Delta\left(\bigcup_{n=k}^{\infty} A_n\right) = \left\{p \in \mathcal{P}_\infty : p\left(\bigcup_{n=k}^{\infty} A_n\right) = 1\right\} \neq \varnothing.$$

Fix arbitrarily a class $p \in \Delta\left(\bigcup_{n=k}^{\infty} A_n\right)$ and a unitary optimal measure $p^* \in p$. Then $p^*\left(\bigcup_{n=k}^{\infty} A_n\right) = 1$. Clearly, the increasing sequence $\left(p^*\left(\bigcup_{n=k}^{j} A_n\right)\right)_{j\geq k}$ tends discretely to $p^*\left(\bigcup_{n=k}^{\infty} A_n\right) = 1$, i.e. a natural number $m \geq k$ can be found such that $p^*\left(\bigcup_{n=k}^{m} A_n\right) = 1$, (see the proof of [3, Lemma 2.1]). Hence there is an index $j_0 \in \{k, k+1, \ldots, m\}$ such that $p^*\left(A_{j_0}\right) = 1$ and thus $p\left(A_{j_0}\right) = 1$,

i.e. $p \in \Delta\left(A_{j_0}\right) \subset \bigcup_{n=k}^{\infty} \Delta\left(A_n\right)$. Consequently,

$$\Delta\left(\bigcup_{n=k}^{\infty} A_n\right) \subset \bigcup_{n=k}^{\infty} \Delta\left(A_n\right).$$

The reverse inclusion can be easily derived from Proposition 1/**g**, because Δ has been proved to be a fundamental mapping.

Therefore, we conclude on the validity of the argument. □

Theorem 2. *Let $(\Omega, \mathcal{F})$ be an infinite measurable space and $\Delta : \mathcal{F} \to \mathbb{P}\left(\mathcal{P}_{\infty}^{1}\right)$ be a powering mapping. Then the following assertions are equivalent.*

C1. *The postulate of powering is a valid statement.*

C2. *The powering mapping Δ is onto.*

C3. *There is an infinite set X such that the power set $\mathbb{P}(X)$ and the σ-algebra $\mathcal{F}$ are tail-preserving order-isomorphic, i.e. there is a tail-preserving fundamental mapping $\beta : \mathbb{P}(X) \to \mathcal{F}$ which is also a bijection.*

Proof. The proof of the biconditional **C1** $\Longleftrightarrow$ **C2** can be found in [2]. The conditional **C2** $\Longrightarrow$ **C3** is trivial. In fact, since the powering mapping is one-to-one, Assertion **C2** obviously implies that the powering mapping $\Delta : \mathcal{F} \to \mathbb{P}\left(\mathcal{P}_{\infty}^{1}\right)$ is a bijection, and due to Proposition 3 it is also a tail-preserving fundamental mapping. Hence **C3** holds true if we choose X to be $\mathcal{P}_{\infty}^{1}$. It remains to check the validity of the implication **C3** $\Longrightarrow$ **C1**. In fact, for this purpose assume the existence of an infinite set X and some tail-preserving fundamental mapping $\beta : \mathbb{P}(X) \to \mathcal{F}$. We aim to prove the validity of the postulate. Pick arbitrarily a $\Gamma \in \mathbb{P}\left(\mathcal{P}_{\infty}^{1}\right) \setminus \{\varnothing\}$ with $\mathcal{C}$ its corresponding collection of unitary atoms. Let

$$X_{\mathcal{C}} := \{x \in X : \beta(\{x\}) \in \mathcal{C}\}.$$

Clearly, $X_{\mathcal{C}} \in \mathbb{P}(X)$. Hence $\beta(X_{\mathcal{C}}) \in \mathcal{F}$. Next, we show that $\beta(X_{\mathcal{C}}) = \bigcup \mathcal{C}$. In fact, since β is a bijection, for every $H \in \mathcal{C}$ there is a unique $x \in X_{\mathcal{C}}$ such that $\beta^{-1}(H) = \{x\} \subset X_{\mathcal{C}}$. As β preserves the order, the inclusion $H \subset \beta(X_{\mathcal{C}})$ follows, implying that $\bigcup \mathcal{C} \subset \beta(X_{\mathcal{C}})$. Conversely, for arbitrary $x \in X_{\mathcal{C}}$ there is a unique $H \in \mathcal{C}$ such that $\beta(\{x\}) = H \subset \bigcup \mathcal{C}$. Hence $\{x\} \subset \beta^{-1}(\bigcup \mathcal{C})$, which implies that $X_{\mathcal{C}} \subseteq \beta^{-1}(\bigcup \mathcal{C})$ or, equivalently $\beta(X_{\mathcal{C}}) \subseteq \bigcup \mathcal{C}$. We have thus proved that $\beta(X_{\mathcal{C}}) = \bigcup \mathcal{C}$. This means that $\bigcup \mathcal{C} \in \mathcal{F}$. To complete the proof pick arbitrarily some $p \in \Delta(\bigcup \mathcal{C})$. Then $p(\bigcup \mathcal{C}) = 1$, yielding $p(H) = 1$ for some $H \in \mathcal{C}$. Then $p \in \Gamma$, which implies that $\Delta(\bigcup \mathcal{C}) \subseteq \Gamma$. Therefore, the postulate holds true, indeed. This was to be proven. □

Assume that there is an infinite set X such that its power set $\mathbb{P}(X)$ and the σ-algebra $\mathcal{F}$ are tail-preserving order-isomorphic, with $\beta : \mathbb{P}(X) \to \mathcal{F}$ the acting bijective tail-preserving fundamental mapping. The question arises to know the totallity of functions $p : \mathcal{F} \to [0, 1]$ such that $p \circ \beta : \mathbb{P}(X) \to [0, 1]$ is an optimal measure on the measurable space $(X, \mathbb{P}(X))$.

Proposition 4. *Let $(\Omega, \mathcal{F})$ be an infinite measurable space. Assume there is an infinite set X such that its power set $\mathbb{P}(X)$ and the σ-algebra $\mathcal{F}$ are tail-preserving isomorphic, with $\beta : \mathbb{P}(X) \to \mathcal{F}$ the acting order-isomorphism. The following assertions hold true.*

(i) *If p is an optimal measure and $H \in \mathcal{F}$ an indecomposable p-atom, then $\beta^{-1}(H) \in \mathbb{P}(X)$ is an indecomposable $p \circ \beta$-atom.*

(ii) *If $p \circ \beta$ is an optimal measure and $H^{(X)} \in \mathbb{P}(X)$ an indecomposable $p \circ \beta$-atom, then $\beta\left(H^{(X)}\right) \in \mathcal{F}$ is an indecomposable p-atom.*

Proof. Suppose that p is an optimal measure on $\mathcal{F}$ and $H \in \mathcal{F}$ is an indecomposable p-atom. Then there uniquely exists an $H^{(X)} \in \mathbb{P}(X)$ such that $\beta\left(H^{(X)}\right) = H$. We show that $H^{(X)}$ is a $p \circ \beta$-atom. In fact, because β is order preserving, for any $B^{(X)} \subset H^{(X)}$ it is true that $\beta\left(B^{(X)}\right) \subset H$, so that either $p\left(\beta\left(B^{(X)}\right)\right) = 0$ or $p\left(\beta\left(B^{(X)}\right)\right) = p(H)$. Consequently, either $p \circ \beta\left(B^{(X)}\right) = 0$ or $p \circ \beta\left(B^{(X)}\right) = p \circ \beta\left(H^{(X)}\right)$, i.e. $H^{(X)}$ is a $p \circ \beta$-atom. Moreover, suppose H is an indecomposable p-atom, but there is some $\beta\left(B^{(X)}\right) \subset H$ with $p \circ \beta\left(B^{(X)}\right) = p \circ \beta\left(H^{(X)}\right) = p \circ \beta\left(H^{(X)} \setminus B^{(X)}\right)$, or equivalently, $p\left(\beta\left(B^{(X)}\right)\right) = p\left(\beta\left(H^{(X)}\right)\right) = p\left(\beta\left(H^{(X)}\right) \setminus \beta\left(B^{(X)}\right)\right)$. This, however, violates that H is an indecomposable p-atom, since $\beta\left(B^{(X)}\right)$ and $\beta\left(H^{(X)}\right) \setminus \beta\left(B^{(X)}\right)$ are subsets of H. We shall conclude the proof by pointing out that the second part can be similarly proved. □

4. An example of bijective tail-preserving fundamental mapping with application

Theorem 3. *Let Ω be an infinite set and $\mathcal{H} := \{H_n : n \in \mathbb{N}\}$ a countably infinite partition of Ω. Denote by $\sigma(\mathcal{H})$ the σ-algebra generated by $\mathcal{H}$. Then the correspondence $\varphi : \sigma(\mathcal{H}) \to \mathbb{P}(\mathbb{N})$, defined by*

$$\varphi(A) = \{n \in: \mathbb{N} : H_n \subset A\},\ A \in \sigma(\mathcal{H}),$$

is a bijective tail-preserving fundamental mapping.

Before we prove the theorem we first make the following obvious remark.

Remark 7. Let Ω be an infinite set and $\mathcal{H} := \{H_n : n \in \mathbb{N}\}$ a countably infinite partition of Ω. Denote by $\sigma(\mathcal{H})$ the σ-algebra generated by $\mathcal{H}$. Then the following assertions are valid.

(i)

$$\sigma(\mathcal{H}) = \left\{ \bigcup_{n \in \boldsymbol{K}} H_n : \boldsymbol{K} \subset \mathbb{N} \right\}$$

(ii) For all sets $A \in \sigma(\mathcal{H})$ and $H \in \mathcal{H}$, either $H \cap A = H$ or $H \cap A = \varnothing$.

(iii) Let sequence $(A_n)_{n\in\mathbb{N}} \subset \sigma(\mathcal{H})$ and number $k \in \mathbb{N}$ be arbitrary. Then for any given $H \in \mathcal{H}$ for which $H \subset \bigcup_{m\geq k} A_m$, there exists an index $n \geq k$ such that $H \subset A_n$.

Proof Theorem 3. The proof is split into three steps. We would like to inform the reader that Remark 7 will be used several times in the proof without always mentioning it.

Step 1. We show that φ is a bijection. In fact, let $A, B \in \sigma(\mathcal{H})$ and assume that $\varphi(A) = \varphi(B)$. Then $A = \bigcup_{n\in\varphi(A)} H_n = \bigcup_{n\in\varphi(B)} H_n = B$, i.e. φ is one-to-one. Next, let $\boldsymbol{K} \in \mathbb{P}(\mathbb{N})$ be arbitrary and denote $A := \bigcup_{n\in\boldsymbol{K}} H_n$. Clearly, $A \in \sigma(\mathcal{H})$, meaning that $\varphi(A) = \boldsymbol{K}$, i.e. φ is onto. Therefore, φ is a bijection.

Step 2. In this step, we prove that φ is a fundamental mapping. We need to verify the normalization, the difference-preserving and the vanishing properties, as follows.

(a) By definition it is obvious that $\varphi(\Omega) = \mathbb{N}$, i.e. the normalization property holds.

(b) We show that $\varphi(A \setminus B) = \varphi(A) \setminus \varphi(B)$, for all $A, B \in \sigma(\mathcal{H})$. In fact, fix arbitrarily an $n \in \varphi(A \setminus B)$. Then $H_n \subset A \setminus B = A \cap \overline{B}$, i.e. $H_n \subset A$ and $H_n \subset \overline{B}$, or equivalently, $n \in \varphi(A)$ and $n \notin \varphi(B)$, or equivalently $n \in \varphi(A) \setminus \varphi(B)$ for all $n \in \varphi(A \setminus B)$. Hence $\varphi(A \setminus B) \subset \varphi(A) \setminus \varphi(B)$. To show that $\varphi(A) \setminus \varphi(B) \subset \varphi(A \setminus B)$, pick arbitrarily an $n \in \varphi(A) \setminus \varphi(B)$. Then necessarily $H_n \subset A$ and $H_n \cap B = \varnothing$, by Remark 7. Hence $H_n \subset A$ and $H_n \subset \overline{B}$, or equivalently, $H_n \subset A \cap \overline{B}$, meaning that $n \in \varphi\left(A \cap \overline{B}\right)$, and hence $\varphi(A) \setminus \varphi(B) \subset \varphi(A \setminus B)$. Therefore, $\varphi(A \setminus B) = \varphi(A) \setminus \varphi(B)$, i.e. φ satisfies the difference-preserving property.

(c) Note that φ obviously satisfies the vanishing property.

Step 3. We are left to check that φ satisfies the tail-preserving property. To do this pick arbitrarily a sequence $(A_n)_{n\in\mathbb{N}} \subset \sigma(\mathcal{H})$ and a number $k \in \mathbb{N}$. Also fix any number

$$t \in \varphi\left(\bigcup_{m=k}^{\infty} A_m\right) = \left\{u \in \mathbb{N} : H_u \subset \bigcup_{m=k}^{\infty} A_m\right\}.$$

Then $H_t \subset \bigcup_{m=k}^{\infty} A_m$ implying the existence of an index $m_0 \geq k$ for which $H_t \subset A_{m_0}$. So that

$$t \in \varphi(A_{m_0}) \subset \bigcup_{m=k}^{\infty} \varphi(A_m).$$

We have thus proved that

$$\varphi\left(\bigcup_{m=k}^{\infty} A_m\right) \subset \bigcup_{m=k}^{\infty} \varphi(A_m).$$

To show the converse inclusion, let $t \in \bigcup_{m=k}^{\infty} \varphi(A_m)$ be arbitrarily fixed. Then there is an index $m_0 \geq k$ such that $t \in \varphi(A_{m_0})$. Note that φ, being a fundamental mapping, it is also an order-embedding. Thus, $\varphi(A_{m_0}) \subset \varphi\left(\bigcup_{m=k}^{\infty} A_m\right)$ and, as $t \in \varphi(A_{m_0})$ it follows that $t \in \varphi\left(\bigcup_{m=k}^{\infty} A_m\right)$. Hence $\bigcup_{m=k}^{\infty} \varphi(A_m) \subset \varphi\left(\bigcup_{m=k}^{\infty} A_m\right)$. Consequently,

$$\varphi\left(\bigcup_{m=k}^{\infty} A_m\right) = \bigcup_{m=k}^{\infty} \varphi(A_m)$$

which is to say that φ is tail-preserving. This completes the proof. □

Corollary 1. *Let $\mathcal{F}_0 \subset \mathbb{P}(\mathbb{N})$ be an infinite sub-σ-algebra. Then $\mathcal{F}_0$ is equinumerous with the power set $\mathbb{P}(\mathbb{N})$.*

Proof. Let $\mathcal{H} := \{H_n : n \in \mathbb{N}\}$ be a countably infinite $\mathcal{F}_0$-measurable partition of $\mathbb{N}$. Then by Theorem 3, $\sigma(\mathcal{H})$ is equinumerous with the power set $\mathbb{P}(\mathbb{N})$. But since $\sigma(\mathcal{H})$ is obviously an infinite sub-σ-algebra of $\mathcal{F}_0$, it must follow that $\mathcal{F}_0$ itself is equinumerous with the power set $\mathbb{P}(\mathbb{N})$. This was to be proven. □

We can thus remark that every sub-σ-algebra contained in the power set $\mathbb{P}(\mathbb{N})$ is equinumerous either with the power set itself or with a finite power set.

Corollary 2. *The powering mapping defined on the σ-algebra generated by a countably infinite partition of an infinite set Ω satisfies the postulate of powering and thus is onto.*

We shall end our paper with the following application involving measure theory.

Proposition 5. *Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be infinite measurable spaces. Assume $G : \mathcal{F}_1 \to \mathcal{F}_2$ is a bijective tail-preserving fundamental mapping. Then the following two assertions are valid. A function $\mu : \mathcal{F}_2 \to [0, 1]$ is a probability measure if and only if the composition mapping $\mu \circ G : \mathcal{F}_1 \to [0, 1]$ is also a probability measure.*

Proof. Assume that $\mu : \mathcal{F}_2 \to [0, 1]$ is a probability measure, i.e.

(1)

$$\mu(\Omega_2) = 1, \text{ and}$$

(2)
$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n)$$
for every sequence of pairwise disjoint measurable sets $(B_n)_{n\in\mathbb{N}}$. Then $1 = \mu(\Omega_2) = \mu(G(\Omega_1))$, i.e. $(\mu \circ G)(\Omega_1) = 1$. Next, let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}_1$ be any sequence of pairwise disjoint measurable sets. Then there uniquely exists a sequence $(B_n)_{n\in\mathbb{N}} \subset \mathcal{F}_2$ such that $B_n = G(A_n)$ for all $n \in \mathbb{N}$ (because G is a bijection). The B_n's so obtained are pairwise disjoint because G preserves the disjointion. Consequently,
$$\sum_{n=1}^{\infty} (\mu \circ G)(A_n) = \sum_{n=1}^{\infty} \mu(B_n) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \mu\left(\bigcup_{n=1}^{\infty} G(A_n)\right)$$
$$= \mu\left(G\left(\bigcup_{n=1}^{\infty} A_n\right)\right) = (\mu \circ G)\left(\bigcup_{n=1}^{\infty} A_n\right).$$
We have thus proved that $\mu \circ G$ is a probability measure on $\mathcal{F}_1$. The proof of the converse conditional can be similarly carried out, so it is left. □

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Author's address

Nutefe Kwami Agbeko
Institute of Mathematics, University of Miskolc, 3515 Miskolc-Egyetemváros, Hungary
E-mail address: matagbek@uni-miskolc.hu

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EXISTENCE OF FAST POSITIVE SEMI-WAVEFRONT SOLUTIONS TO MONOSTABLE INTEGRO-DIFFERENTIAL EQUATIONS WITH DELAY

MAITERE AGUERREA AND ROBERT HAKL

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Abstract. We establish the existence of fast positive semi-wavefront solutions to a delay integro-differential problem

$$cu'(t) = J \star u(t) - u(t) + f(u(t-h)), \quad t \in \mathbb{R}, \quad u(-\infty) = 0,$$

where the asymmetric kernel J is exponentially bounded, the nonlinearity $f \in C^1([0,+\infty);\mathbb{R})$ is monostable, $h \geq 0$, and $c > 0$.

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1. Introduction

The main object of study in this paper is a time-delayed integro-differential equation

$$cu'(t) = J \star u(t) - u(t) + f(u(t-h)) \qquad \text{for } t \in \mathbb{R}, \tag{1.1}$$

where $h \geq 0$, $c > 0$, and the non-negative averaging kernel J satisfies

$$J \star u(t) = \int_{\mathbb{R}} J(x)u(t-x)dx, \qquad \int_{\mathbb{R}} J(x)dx = 1, \qquad \int_{\mathbb{R}} J(x)|x|dx < +\infty,$$

and there exists $\lambda > 0$ such that

$$\int_{-\infty}^{0} J(x)e^{-\lambda x}dx < +\infty.$$

Further, we suppose that $f \in C^1([0,+\infty);\mathbb{R})$, $f(0) = 0$, $f(1) = 0$, $f(x) > 0$ for $x \in (0,1)$, $f(x) < 0$ for $x > 1$. By a solution to (1.1) we understand a continuously differentiable function defined on the whole real axis and satisfying (1.1) at every point $t \in \mathbb{R}$.

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Integro-differential models as (1.1) and its versions without delay are using when long distance dispersal events are considered. In ecology, long distance dispersal events are suspected to deeply modify the dynamics of a population [5]. Many works have shown, the phenomena observed in biological and ecological models depend not only on the present state but also on some past occurrences. The delay effect induces an important change in some predictions. For example, in population dynamics, the time delays effect the prediction of population expansions speed [13]. On the other hand, several studies also indicate that asymmetric kernels might appear in the population modeling in a natural way [16]. So, we are interested in understanding better the effect of delay in models as (1.1) when the kernel J is asymmetric.

If we take $h = 0$ in (1.1), we obtain an integro-differential equation without delay. Such kinds of equations appears in various biological and ecological models (e.g. population dynamics). The existence, uniqueness and propagation properties of traveling wave solutions for equation (1.1) have been investigated in a series of papers where different geometric and smoothness conditions on J and f were assumed (see, e.g. [1–4, 6–8, 18]). The problem of existence/nonexistence and existence of the minimal speed of wavefronts ($u(-\infty) = 0$ and $u(+\infty) = \kappa$) were considered in [7, 8, 18], and the propagation properties and the effect of the dispersal heavy tails in [2–4, 9], by means of different methods. In the mentioned papers, the results require several conditions on kernels (compactly supported, exponential tails or algebraic tails, symmetrical/asymmetric) and nonlinearities ($f'(0) > 0$, $0 < f(s) \leqslant f'(0)s$, $f'(1) < 0$). In such cases the propagation can occur with the constant (kernel is exponentially bounded) or accelerated (kernel with heavy tails) speed.

As far as we know, if $h > 0$, very few theoretical studies are devoted to integro-differential equations with delay [12, 14, 17, 19]. All these works have studied traveling wavefront solutions to equation

$$cu'(t) = (J \star u(t) - u(t)) - du + b(u(t-h)) \qquad \text{for } t \in \mathbb{R}, \tag{1.2}$$

where d is the death rate, b is the birth function. In these research, using different methods (weighted energy method, the comparison principles, by constructing proper upper and lower solutions), the existence/nonexistence, stability and uniqueness of traveling wavefront solutions to (1.2) were obtained, assuming the typical Fisher-KPP condition on nonlinear function b ($0 < b(s) \leq b'(0)s$), considering monotonicity or quasi-monotonicity conditions and symmetric kernels. Note that, if $h > 0$, model (1.1) cannot be deduced from equation (1.2).

In this paper, we present an analytic results on the existence of waves in delayed equations which include model (1.1). We will use the methods based on the general theory of boundary value problems for functional differential equations to get the existence of fast semi-wavefront solutions to (1.1) with asymmetric kernel, where the typical conditions on $f'(0)$ and $f'(1)$ are not required.

Below, we present our main results.

Theorem 1. *For every $c > 0$ sufficiently large there exists a positive solution u to the equation (1.1) satisfying*

$$\lim_{t\to-\infty} u(t) = 0, \qquad e^{-\lambda h} \le \liminf_{t\to+\infty} u(t) \le \sup_{t\in\mathbb{R}} u(t) \le 1 + f_0(e^{\frac{h}{c}} - 1),$$

where $f_0 = \max\{f(s) : s \in [0,1]\}$.

Corollary 1. *For every $c > 0$ sufficiently large there exists a positive solution u to the equation (1.1) with $h = 0$ satisfying*

$$\lim_{t\to-\infty} u(t) = 0, \qquad \lim_{t\to+\infty} u(t) = 1.$$

Remark 1. We observe that Corollary 1 shows that the special conditions $f'(0) > 0$ and $f'(1) < 0$ (steady state 1 is stable) are not necessary to prove the existence of fast wavefront solutions to (1.1) when $h = 0$.

The paper is organized as follows. In Section 2, some auxiliary propositions are given and the existence of fast positive semi-wavefronts is proven. In the last section, Section 3, the existence theorem is applied to some models.

2. Auxiliary propositions

In what follows we will use the following notation. $C([0,+\infty);\mathbb{R})$, $C(\mathbb{R};\mathbb{R})$, $C([-N,N];\mathbb{R})$, resp. $C^1([0,+\infty);\mathbb{R})$ are standard spaces of continuous, resp. continuously differentiable functions. If $u \in C([-N,N];\mathbb{R})$, then $\|u\|_C = \max\{|u(t)| : t \in [-N,N]\}$. Put

$$f_0 \overset{def}{=} \max\{f(s) : s \in [0,1]\}, \qquad x_c \overset{def}{=} 1 + f_0(e^{\frac{h}{c}} - 1),$$
$$f_c \overset{def}{=} \max\{|f'(s)| : s \in [0,x_c]\}.$$

Obviously, according to the assumptions laid on f, we have that $f(x) + f_c x \ge 0$ for $x \in [0,x_c]$.

Further we define auxiliar functions $\overline{f} \in C([0,+\infty);\mathbb{R})$ and $\widetilde{f} \in C(\mathbb{R};\mathbb{R})$ by

$$\overline{f}(x) = \begin{cases} f(x) & \text{for } x \in [0,x_c], \\ f(x_c) - f_c(x - x_c) & \text{for } x > x_c, \end{cases}$$

$$\widetilde{f}(x) = \begin{cases} 0 & \text{for } x < 0, \\ \overline{f}(x) + f_c x & \text{for } x \ge 0, \end{cases}$$

Note that $\widetilde{f}$ is a bounded non-negative function.

For every $N > 0$ we define an operator $\theta_N : C([-N,N];\mathbb{R}) \to C(\mathbb{R};\mathbb{R})$ by

$$\theta_N(u)(t) = \begin{cases} u(-N) & \text{for } t < -N, \\ u(t) & \text{for } t \in [-N,N], \\ u(N) & \text{for } t > N, \end{cases}$$

and let $\ell_N^+, \ell_N^- : C([-N,N];\mathbb{R}) \times \mathbb{R}_+ \to C([-N,N];\mathbb{R})$ be operators given by

$$\ell_N^+(u,\mu)(t) = \int_{\mathbb{R}} J(x)\theta_N(u)(t-x)dx + \mu\theta_N(u)(t-h),$$
$$\ell_N^-(u,\mu)(t) = u(t) + \mu\theta_N(u)(t-h).$$

It can be easily verified that the operators ℓ_N^+ and ℓ_N^- are linear positive operators in the first variable, i.e., for every fixed $\mu_0 \in \mathbb{R}_+$, the operators $\ell_N^+(\cdot,\mu_0), \ell_N^-(\cdot,\mu_0) : C([-N,N];\mathbb{R}) \to C([-N,N];\mathbb{R})$ are linear and transforms the set $C([-N,N];\mathbb{R}_+)$ into the set $C([-N,N];\mathbb{R}_+)$. Moreover, the operator $\ell_N^-(\cdot,\mu_0)$ is a Volterra operator (with respect to the point $-N$), i.e., for every $t_0 \in (-N,N]$ the equality

$$u(t) = 0 \qquad \text{for } t \in [-N,t_0]$$

implies

$$\ell_N^-(u,\mu_0)(t) = 0 \qquad \text{for } t \in [-N,t_0].$$

Lemma 1. *Let $\mu_0 \geq 0$ and let $c > 0$ satisfy*

$$c\lambda \geq \int_0^{+\infty} J(x)dx + \int_{-\infty}^0 J(x)e^{-\lambda x}dx + \mu_0. \tag{2.1}$$

Then, for every $N > h$, the function $\gamma(t) = e^{\lambda t}$ for $t \in [-N,N]$ satisfies

$$c\gamma'(t) \geq \ell_N^+(\gamma,\mu)(t) \qquad \text{for } t \in [-N,N], \quad \mu \in [0,\mu_0]. \tag{2.2}$$

Proof. According to the definition of ℓ_N^+ we have that

$$\ell_N^+(\gamma,\mu)(t) = e^{\lambda t}\left(e^{-\lambda(t+N)}\int_{t+N}^{+\infty} J(x)dx + e^{-\lambda(t-N)}\int_{-\infty}^{t-N} J(x)dx \right.$$
$$\left. + \int_{t-N}^{t+N} J(x)e^{-\lambda x}dx + p(t)\right) \qquad \text{for } t \in [-N,N],$$

where

$$p(t) = \begin{cases} \mu e^{-\lambda(t+N)} & \text{for } t \in [-N,-N+h), \\ \mu e^{-\lambda h} & \text{for } t \in [-N+h,N]. \end{cases}$$

It can be easily verified that the function in the parenthesis is non-increasing with respect to t. Therefore, we get

$$\ell_N^+(\gamma,\mu)(t) \leq e^{\lambda t}\left(\int_0^{+\infty} J(x)dx + e^{2\lambda N}\int_{-\infty}^{-2N} J(x)dx + \int_{-2N}^0 J(x)e^{-\lambda x}dx + \mu\right)$$
$$\leq e^{\lambda t}\left(\int_0^{+\infty} J(x)dx + \int_{-\infty}^0 J(x)e^{-\lambda x}dx + \mu_0\right) \quad \text{for } t \in [-N,N],\ \mu \in [0,\mu_0].$$

Now, using (2.1) in the last inequality we arrive at (2.2). □

By a direct calculation one can verify the following assertion.

Lemma 2. *Let $\mu_0 \geq 0$ and let $c > 0$ satisfy*

$$c\lambda \geq 1 + \mu_0 e^{\lambda h}. \tag{2.3}$$

Then, for every $N > h$, the function $\beta(t) = e^{-\lambda t}$ for $t \in [-N, N]$ satisfies

$$c\beta'(t) \leq -\ell_N^-(\beta, \mu)(t), \qquad \text{for } t \in [-N, N], \quad \mu \in [0, \mu_0].$$

Remark 2. Note that x_c and f_c are nonincreasing with respect to c. Therefore, there exists $c_* > 0$ such that, for every $c \geq c_*$, we have that

$$c\lambda \geq \int_0^{+\infty} J(x)dx + \int_{-\infty}^0 J(x)e^{-\lambda x}dx + f_c, \tag{2.4}$$

$$c\lambda \geq 1 + f_c e^{\lambda h}. \tag{2.5}$$

Thus, in what follows, by c_* we refer to the number described above.

Now we introduce some a priori estimates.

Lemma 3. *Let $N > h$, $c \geq c_*$, and let u be a non-negative solution to the equation*

$$cu'(t) = \ell_N^+(u, 0)(t) - u(t) + \overline{f}(\theta_N(u)(t-h)) \qquad \text{for } t \in [-N, N] \tag{2.6}$$

such that $u(-N) \leq e^{-\lambda h}$. Then

$$u(-N) = \min\{u(t) : t \in [-N, N]\}.$$

Proof. Suppose on the contrary that $u(-N) > \min\{u(t) : t \in [-N, N]\}$. Then there exists $t_0 \in (-N, N]$ such that

$$u(t_0) = \min\{u(t) : t \in [-N, N]\} < e^{-\lambda h}, \quad u(t) > u(t_0) \quad \text{for } t \in [-N, t_0).$$

Obviously,

$$\begin{aligned} cu'(t) = \ell_N^+(u, 0)(t) - \ell_N^-(u, f_c)(t) + \widetilde{f}(\theta_N(u)(t-h)) \\ \geq -\ell_N^-(u, f_c)(t) \quad \text{for } t \in [-N, t_0]. \end{aligned}$$

According to Lemma 2, the function $\beta(t) = e^{-\lambda t}$ satisfies

$$c\beta'(t) \leq -\ell_N^-(\beta, f_c)(t) \qquad \text{for } t \in [-N, t_0].$$

Consequently (see [10, Theorem 1.5]), we find that

$$u(t) \leq \frac{u(t_0)}{\beta(t_0)}\beta(t) = u(t_0)e^{-\lambda(t-t_0)} \qquad \text{for } t \in [-N, t_0].$$

In particular,

$$\theta_N(u)(t) \leq u(t_0)e^{\lambda(h+\varepsilon)} \leq 1 \qquad \text{for } t \in [t_0 - (h+\varepsilon), t_0],$$

for a suitable positive $\varepsilon \leq t_0 + N$, which implies

$$\overline{f}(\theta_N(u)(t-h)) \geq 0 \qquad \text{for } t \in [t_0 - \varepsilon, t_0].$$

Set $w(t) = u(t) - u(t_0)$ for $t \in [-N, N]$. Then, obviously, $w(t) \geq 0$ and

$$cw'(t) = \ell_N^+(w,0)(t) - w(t) + \overline{f}(\theta_N(u)(t-h)) \geq -w(t) \quad \text{for } t \in [t_0 - \varepsilon, t_0].$$

Consequently, $u(t) = u(t_0)$ for $t \in [t_0 - \varepsilon, t_0]$, a contradiction. □

Lemma 4. *Let $N > h$, $c \geq c_*$, and let u be a non-negative solution to (2.6) such that $u(-N) \leq e^{-\lambda h}$. Then*

$$\|u\|_C \leq x_c.$$

Proof. According to Lemma 3, there exists $t_0 \in (-N, N]$ such that $u(t_0) = \|u\|_C$ and

$$0 \leq cu'(t_0) = \int_{\mathbb{R}} J(x)\big[\theta_N(u)(t_0 - x) - u(t_0)\big]dx + \overline{f}(\theta_N(u)(t_0 - h)).$$

Therefore, $\overline{f}(\theta_N(u)(t_0 - h)) \geq 0$, i.e. $\theta_N(u)(t_0 - h) \leq 1$ and assuming $\|u\|_C > x_c$, there exists $t_1 \in [t_0 - h, t_0)$ such that $u(t_1) = 1$. Obviously, $t_1 \geq -N$. Then (2.6) implies

$$cu'(t) \leq u(t_0) - u(t) + f_0 \qquad \text{for } t \in [t_1, t_0],$$

whence we obtain

$$u(t_0)e^{\frac{t_0}{c}} \leq u(t_1)e^{\frac{t_1}{c}} + [u(t_0) + f_0](e^{\frac{t_0}{c}} - e^{\frac{t_1}{c}})$$

and, consequently,

$$u(t_0) \leq 1 + f_0(e^{\frac{h}{c}} - 1) = x_c,$$

a contradiction. □

Lemma 5. *Let $N > h$, $c \geq c_*$ and let u be a non-negative solution to the equation*

$$cu'(t) = \ell_N^+(u,0)(t) - u(t) + f(\theta_N(u)(t-h)) \qquad \textit{for } t \in [-N, N] \tag{2.7}$$

such that $\|u\|_C \leq x_c$. Then

$$u(0) \leq u(-N)e^{\lambda N}.$$

Proof. According to (2.7) and the assumption $f \in C^1([0, +\infty); \mathbb{R})$ we have that

$$cu'(t) \leq \ell_N^+(u, f_c)(t) \qquad \text{for } t \in [-N, N].$$

Further, according to (2.4) and Lemma 1, the function $\gamma(t) = e^{\lambda t}$ satisfies

$$c\gamma'(t) \geq \ell_N^+(\gamma, f_c)(t) \qquad \text{for } t \in [-N, N].$$

Consequently (see [10, Theorem 1.1]), we find that

$$u(t) \leq \frac{u(-N)}{\gamma(-N)}\gamma(t) = u(-N)e^{\lambda(t+N)} \qquad \text{for } t \in [-N, N].$$

In particular, for $t = 0$ we obtain the assertion of the lemma. □

Lemma 6. *Let $N > h$, $c \geq c_*$, and let u_1 and u_2 be non-negative solutions to the equation (2.7) such that*

$$\|u_i\|_C \leq x_c \quad (i = 1,2), \qquad u_1(-N) \leq u_2(-N).$$

Then

$$u_1(t) \leq u_2(t) \qquad \text{for } t \in [-N,N].$$

Proof. Set $w(t) = u_1(t) - u_2(t)$ for $t \in [-N,N]$. Then we have

$$\begin{aligned} cw'(t) &= \ell_N^+(w,0)(t) - w(t) + f(\theta_N(u_1)(t-h)) - f(\theta_N(u_2)(t-h)) \\ &= \ell_N^+(w,0)(t) - w(t) + p(t)\theta_N(w)(t-h) \quad \text{for } t \in [-N,N], \quad w(-N) \leq 0, \end{aligned}$$

where

$$p(t) = \begin{cases} \frac{f(\theta_N(u_1)(t-h)) - f(\theta_N(u_2)(t-h))}{\theta_N(u_1)(t-h) - \theta_N(u_2)(t-h)} & \text{if } \theta_N(u_1)(t-h) \neq \theta_N(u_2)(t-h), \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $|p(t)| \leq f_c$ for $t \in [-N,N]$ and according to (2.4), (2.5), and Lemmas 1 and 2, the functions $\gamma(t) = e^{\lambda t}$ and $\beta(t) = e^{-\lambda t}$ satisfy

$$\begin{aligned} c\gamma'(t) &\geq \ell_N^+(\gamma, f_c)(t) \geq \ell_N^+(\gamma,0)(t) + [p(t)]_+\theta_N(\gamma)(t-h) \quad \text{for } t \in [-N,N], \\ c\beta'(t) &\leq -\ell_N^-(\beta, f_c)(t) \leq -\beta(t) - [p(t)]_-\theta_N(\beta)(t-h) \quad \text{for } t \in [-N,N]. \end{aligned}$$

However, according to [10, Theorems 1.1, 1.2, 1.4], the latter inequalities imply that $w(t) \leq 0$ for $t \in [-N,N]$, i.e. $u_1(t) \leq u_2(t)$ for $t \in [-N,N]$. □

Remark 3. Note that according to Lemma 6, the problem

$$cu'(t) = \ell_N^+(u,0)(t) - u(t) + f(\theta_N(u)(t-h)), \qquad u(-N) = r$$

with $r \in [0,x_c]$ has at most one solution such that $\|u\|_C \leq x_c$ provided $N > h$ and $c \geq c_*$.

Lemma 7. *Let $c \geq c_*$ and $u_0 \in (0, e^{-\lambda h}]$. Then, for every $N > h$, there exists a positive solution $u \in C([-N,N];\mathbb{R})$ to the problem*

$$cu'(t) = \ell_N^+(u,0)(t) - u(t) + f(\theta_N(u)(t-h)), \qquad u(0) = u_0 \tag{2.8}$$

such that $\|u\|_C \leq x_c$.

Proof. Let $N > h$ be arbitrary but fixed. According to (2.4), (2.5), and Lemmas 1 and 2, the functions $\gamma(t) = e^{\lambda t}$ and $\beta(t) = e^{-\lambda t}$ satisfy

$$c\gamma'(t) \geq \ell_N^+(\gamma,0)(t), \qquad c\beta'(t) \leq -\ell_N^-(\beta, f_c)(t) \qquad \text{for } t \in [-N,N].$$

Consequently (see [10, Theorems 1.1, 1.2, 1.4]), the homogeneous problem

$$cu'(t) = \ell_N^+(u,0)(t) - \ell_N^-(u, f_c)(t), \qquad u(-N) = 0$$

has only the trivial solution and thus, the problem

$$cu'(t) = \ell_N^+(u,0)(t) - \ell_N^-(u, f_c)(t) + \widetilde{f}(\theta_N(u)(t-h)), \qquad u(-N) = r$$

has a solution u_r for every $r \geq 0$ (see [11, Theorem 3.1]). Moreover, since $\widetilde{f}$ is a non-negative function, the solution u_r is also non-negative on $[-N,N]$ (see [10, Theorem 1.4]). Consequently, u_r is a solution to

$$cu'(t) = \ell_N^+(u,0)(t) - u(t) + \overline{f}(\theta_N(u)(t-h)), \qquad u(-N) = r.$$

Now we will show that there exists $r_0 \in [u_0 e^{-\lambda N}, u_0]$ such that the corresponding solution u_{r_0} admits $u_{r_0}(0) = u_0$.

According to Lemmas 3 and 4, for $r \leq e^{-\lambda h}$ we have that

$$u_r(-N) \leq u_r(t) \leq x_c \qquad \text{for } t \in [-N,N] \tag{2.9}$$

and so, for $r_1 = u_0$ we get

$$u_0 = u_{r_1}(-N) \leq u_{r_1}(0).$$

Moreover, all the solutions u_r with $0 < r \leq e^{-\lambda h}$ are, obviously, also positive solutions to the equation (2.7). Consequently, according to Lemma 5, for $r_2 = u_0 e^{-\lambda N}$ we have that

$$u_{r_2}(0) \leq u_{r_2}(-N)e^{\lambda N} = u_0.$$

Furthermore, Remark 3 yields the uniqueness of every solution u_r ($r \leq e^{-\lambda h}$) and thus, the continuous dependence on parameter (see [15, Theorem 2.1]) implies the existence of $r_0 \in [u_0 e^{-\lambda N}, u_0]$ such that $u_{r_0}(0) = u_0$. □

Lemma 8. *Let $c \geq c_*$ and let u be a non-negative solution to (1.1) such that $u(t) \leq x_c$ for $t \in \mathbb{R}$. Then*

$$u(s) \leq u(t)e^{\lambda(t-s)} \qquad \textit{for } s,t \in \mathbb{R},\ s \leq t.$$

Proof. It follows from (2.5) that $\beta(t) = e^{-\lambda t}$ satisfies

$$c\beta'(t) \leq -\beta(t) - f_c\beta(t-h) \qquad \text{for } t \in \mathbb{R}.$$

Further, from (1.1) it follows that

$$\begin{aligned} cu'(t) = J \star u(t) - u(t) - f_c u(t-h) + \widetilde{f}(u(t-h)) \\ \geq -u(t) - f_c u(t-h) \quad \text{for } t \in \mathbb{R}. \end{aligned}$$

Now, let $t_0 \in \mathbb{R}$ be arbitrary but fixed. Obviously,

$$u(t) = o(\beta(t)) \qquad \text{as} \qquad t \to -\infty.$$

Therefore, there exists $t_1 \leq t_0$ such that

$$\mu \overset{def}{=} \sup\left\{\frac{u(t)}{\beta(t)} : t \leq t_0\right\} = \frac{u(t_1)}{\beta(t_1)},$$

and so

$$\mu\beta(t) - u(t) \geq 0 \qquad \text{for } t \leq t_0, \qquad \mu\beta(t_1) - u(t_1) = 0,$$

and

$$c(\mu\beta'(t) - u'(t)) \leq -(\mu\beta(t) - u(t)) - f_c(\mu\beta(t-h) - u(t-h)) \leq 0 \quad \text{for } t \leq t_0.$$

Therefore, $\mu\beta(t_0) = u(t_0)$ and so

$$u(t) \le \frac{u(t_0)}{\beta(t_0)}\beta(t) = u(t_0)e^{\lambda(t_0-t)} \qquad \text{for } t \le t_0.$$

Since t_0 was chosen arbitrarily, we get the assertion of the lemma. □

Lemma 9. *Let $c \ge c_*$. Then, for every $u_0 \in (0, e^{-\lambda h}]$, there exists a positive solution u to the equation (1.1) such that*

$$u(0) = u_0, \qquad \sup_{t\in\mathbb{R}} u(t) \le x_c. \tag{2.10}$$

Proof. According to Lemma 7 there exist an increasing sequence of real numbers N_i $(i \in \mathbb{N})$ such that

$$N_1 > h, \qquad \lim_{i\to+\infty} N_i = +\infty$$

and a sequence of positive solutions $u_{N_i} : [-N_i, N_i] \to \mathbb{R}$ to (2.8) (with $N = N_i$) satisfying $\|u_{N_i}\|_C \le x_c$ $(i \in \mathbb{N})$. Therefore, $\{\theta_{N_i}(u_{N_i})\}_{i=1}^{+\infty}$ is a sequence of positive continuous functions defined on $\mathbb{R}$, uniformly bounded by x_c, and equicontinuous on every compact interval contained in $\mathbb{R}$. Without loss of generality we can assume that there exists a continuous function u such that

$$\lim_{i\to+\infty} \theta_{N_i}(u_{N_i})(t) = u(t) \qquad \text{uniformly on every compact subinterval of } \mathbb{R}.$$

As a limit of $\theta_{N_i}(u_{N_i})$, the function u satisfies $0 \le u(t) \le x_c$ for $t \in \mathbb{R}$, which implies that

$$\lim_{i\to+\infty} \int_I |\ell^+_{N_i}(u_{N_i}, 0)(t) - J \star u(t)|dt = 0$$

for every compact interval $I \subset \mathbb{R}$. Consequently, u is a non-negative nontrivial $(u(0) = u_0 > 0)$ solution to (1.1).

To show that u is a positive function, assume on the contrary that u vanishes at some point $t_0 \in \mathbb{R}$. According to Lemma 8 we have $u(t) = 0$ for $t \le t_0$. Moreover,

$$cu'(t) = J \star u(t) - u(t) + f(u(t-h)) \le J \star u(t) + f_c u(t-h) \quad \text{for } t \in \mathbb{R}.$$

On the other hand, according to (2.4), the function $\gamma(t) = e^{\lambda t}$ satisfies

$$c\gamma'(t) \ge J \star \gamma(t) + f_c\gamma(t-h) \qquad \text{for } t \in \mathbb{R}$$

and

$$u(t) = o(\gamma(t)) \qquad \text{as} \qquad t \to +\infty.$$

Consequently, there exists $t_1 \ge t_0$ such that

$$\mu = \sup\left\{\frac{u(t)}{\gamma(t)} : t \ge t_0\right\} = \frac{u(t_1)}{\gamma(t_1)},$$

and so

$$\mu\gamma(t) - u(t) \ge 0 \qquad \text{for } t \ge t_0, \qquad \mu\gamma(t_1) - u(t_1) = 0.$$

Since $u(t) = 0$ for $t \le t_0$, we have $\mu\gamma(t) - u(t) \ge 0$ for $t \in \mathbb{R}$ and

$$c(\mu\gamma'(t) - u'(t)) \ge J \star (\mu\gamma(t) - u(t)) + f_c(\mu\gamma(t-h) - u(t-h)) \ge 0 \quad \text{for } t \ge t_0.$$

Therefore, $\mu\gamma(t_0) = u(t_0) = 0$ and so $\mu = 0$. Thus, $u \equiv 0$ on $\mathbb{R}$ that contradicts $u(0) = u_0 > 0$. □

Lemma 10. *Let $c \ge c_*$ and let u be a positive solution to (1.1) such that*

$$\inf_{t\in\mathbb{R}} u(t) < e^{-\lambda h}, \qquad \sup_{t\in\mathbb{R}} u(t) \le x_c.$$

Then

$$u(t) > \inf_{s\in\mathbb{R}} u(s) \qquad \text{for } t \in \mathbb{R}.$$

Proof. Assume on the contrary that there exists $t_0 \in \mathbb{R}$ such that

$$u(t_0) = \inf_{t\in\mathbb{R}} u(t).$$

Then, according to Lemma 8 we have that $u(t_0 - h) < 1$, and thus $f(u(t_0 - h)) > 0$. On the other hand,

$$0 = cu'(t_0) = \int_{\mathbb{R}} J(x)[u(t_0 - x) - u(t_0)]dx + f(u(t_0 - h)) \ge f(u(t_0 - h)),$$

a contradiction. □

Lemma 11. *For any non-negative bounded function u we have*

$$\liminf_{t\to\pm\infty} J \star u(t) \ge \liminf_{t\to\pm\infty} u(t).$$

Proof. Obviously, for every $N > 0$ we have that

$$\begin{aligned}\int_{\mathbb{R}} J(x)u(t-x)dx &\ge \int_{-\infty}^{N} J(x)u(t-x)dx \\ &\ge \int_{-\infty}^{N} J(x)dx \inf\{u(s) : s \in [t-N, +\infty)\} \quad \text{for } t \in \mathbb{R}.\end{aligned}$$

Consequently,

$$\liminf_{t\to+\infty} J \star u(t) \ge \int_{-\infty}^{N} J(x)dx \liminf_{t\to+\infty} u(t) \qquad \text{for } N > 0.$$

Thus, the assertion of the lemma follows as $N \to +\infty$. The case $t \to -\infty$ can be proven analogously. □

Lemma 12. *Let $c \ge c_*$ satisfy*

$$c > \int_{\mathbb{R}} J(x)|x|dx \tag{2.11}$$

and let u be a positive solution to (1.1) *such that $u(t) \le x_c$ for $t \in \mathbb{R}$ and*

$$\liminf_{t\to+\infty} u(t) < e^{-\lambda h}, \qquad \textit{resp.} \qquad \liminf_{t\to-\infty} u(t) < e^{-\lambda h}.$$

Then

$$\lim_{t\to+\infty} u(t) = 0, \qquad \textit{resp.} \qquad \lim_{t\to-\infty} u(t) = 0.$$

Proof. We will proof the lemma as $t \to +\infty$. For the case when $t \to -\infty$ the proof is similar. First we prove that there exists a limit of $u(t)$ as $t \to +\infty$. Assume on the contrary that

$$u_* \stackrel{def}{=} \liminf_{t\to+\infty} u(t) < \limsup_{t\to+\infty} u(t).$$

According to Lemma 8 we have

$$u(t-h) < 1 \qquad \text{whenever} \qquad u(t) < e^{-\lambda h}.$$

Therefore, there exist $s_n, t_n \in \mathbb{R}$ ($n \in \mathbb{N}$) and $y_0 \in (u_*, e^{-\lambda h})$ such that $s_n < t_n$,

$$\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} t_n = +\infty, \qquad \lim_{n\to+\infty} u(t_n) = u_*, \qquad u(s_n) = y_0,$$
$$u(t_n) \le u(t) \le u(s_n), \qquad u(t-h) < 1 \qquad \text{for } t \in [s_n, t_n].$$

Then

$$cu'(t) \ge \int_{\mathbb{R}} J(x)[u(t-x) - u(t)]dx \qquad \text{for } t \in [s_n, t_n]. \tag{2.12}$$

Now we show that $t_n - s_n \to +\infty$ as $n \to +\infty$. Assume on the contrary that there exists $K > 0$ such that $t_n - s_n \le K$ for $n \in \mathbb{N}$. According to Lemma 11 and (2.12), for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$cu'(t) \ge u_* - \varepsilon - u(t) \qquad \text{for } t \in [s_n, t_n], \quad n \ge n_0.$$

Consequently, we have that

$$u(t_n)e^{\frac{t_n}{c}} \ge u(s_n)e^{\frac{s_n}{c}} + (u_* - \varepsilon)(e^{\frac{t_n}{c}} - e^{\frac{s_n}{c}}),$$

whence we obtain

$$u(t_n) - u_* \ge (y_0 - u_*)e^{\frac{-K}{c}} - \varepsilon(1 - e^{\frac{-K}{c}}).$$

Passing to the limit as $n \to +\infty$ in the latter equation we find that

$$0 < y_0 - u_* \le \varepsilon(e^{\frac{K}{c}} - 1),$$

whence we get a contradiction, because ε can be chosen arbitrarily small. Consequently, we get that $t_n - s_n \to +\infty$ as $n \to +\infty$. Therefore, for every $N > 0$ we have that $N < t_n - s_n$ provided n is sufficiently large. Now

$$cu'(t) \ge \int_{\mathbb{R}} J(x)[u(t-x) - u(t)]dx$$

$$= -\int_{\mathbb{R}} J(x)x \int_0^1 u'(t-zx)dzdx \quad \text{for } t \in [s_n, t_n].$$

Integration of the last inequality over the interval $[s_n, t_n]$, using Fubini's theorem, yields

$$\begin{aligned}
c(u(t_n)-y_0) &\geq -\int_{\mathbb{R}} J(x)x \int_0^1 [u(t_n-zx)-u(s_n-zx)]dzdx \\
&\geq -\int_0^N J(x)|x| \int_0^1 u(t_n-zx)dzdx + \int_0^N J(x)|x|dx \inf\{u(\xi) : \xi \in [s_n-N, +\infty)\} \\
&\quad + \int_{-N}^0 J(x)|x|dx \inf\{u(\xi) : \xi \in [t_n, +\infty)\} - \int_{-N}^0 J(x)|x| \int_0^1 u(s_n-zx)dzdx \\
&\quad - \left(\int_N^{+\infty} J(x)|x|dx + \int_{-\infty}^{-N} J(x)|x|dx\right) x_c \\
&\geq \left(\inf\{u(\xi) : \xi \in [s_n-N, +\infty)\} - y_0\right) \int_{-N}^N J(x)|x|dx - \varepsilon(N) \\
&\geq \left(\inf\{u(\xi) : \xi \in [s_n-N, +\infty)\} - y_0\right) \int_{\mathbb{R}} J(x)|x|dx - \varepsilon(N) \qquad (2.13)
\end{aligned}$$

provided n is sufficiently large, where

$$\lim_{N\to+\infty} \varepsilon(N) = 0. \qquad (2.14)$$

However, passing to the limit as $n \to +\infty$ in (2.13) we get

$$(u_* - y_0)\left(c - \int_{\mathbb{R}} J(x)|x|dx\right) \geq -\varepsilon(N) \qquad \text{for } N > 0. \qquad (2.15)$$

Consequently, in view of (2.14), the inequality (2.15) yields

$$(u_* - y_0)\left(c - \int_{\mathbb{R}} J(x)|x|dx\right) \geq 0.$$

Thus, on account of $y_0 > u_*$ we have

$$c \leq \int_{\mathbb{R}} J(x)|x|dx,$$

which contradicts (2.11). The obtained contradiction proves that the limit of $u(t)$ as $t \to +\infty$ exists.

Now we will show that $u_* = \lim_{t\to+\infty} u(t)$ is equal to zero. Obviously, according to Lemma 11 we have that for every $\varepsilon > 0$ there exists $t_\varepsilon \in \mathbb{R}$ such that

$$cu'(t) \geq u_* - \varepsilon - u(t) + f(u(t-h)) \qquad \text{for } t \geq t_\varepsilon$$

and the integration from t to $t+1$ yields

$$cu(t+1)-cu(t) \geq u_* - \varepsilon - \int_t^{t+1} u(s)ds + \int_t^{t+1} f(u(s-h))ds \quad \text{for } t \geq t_\varepsilon.$$

Passing to the limit as $t \to +\infty$ we arrive at

$$\varepsilon \geq \lim_{t\to+\infty} \int_t^{t+1} f(u(s-h))ds = f(u_*)$$

since f is continuous. Consequently, since $\varepsilon > 0$ can be chosen arbitrarily small, $f(u_*) = 0$ and so $u_* = 0$. □

Lemma 13. *Let $c \geq c_*$ satisfy (2.11) and let u be a positive solution to (1.1) such that $u(t) \leq x_c$ for $t \in \mathbb{R}$. Then*

$$\liminf_{t\to+\infty} u(t) \geq e^{-\lambda h}. \tag{2.16}$$

Proof. Assume on the contrary that

$$\liminf_{t\to+\infty} u(t) < e^{-\lambda h}.$$

Then according to Lemma 12 we have

$$\lim_{t\to+\infty} u(t) = 0.$$

Consequently, there exists $t_0 > 0$ such that

$$u(t_0) \geq u(t), \qquad u(t-h) < 1 \qquad \text{for } t \geq t_0$$

and thus u satisfies

$$\begin{aligned} cu'(t) &\geq \int_{\mathbb{R}} J(x)[u(t-x)-u(t)]dx \\ &= -\int_{\mathbb{R}} J(x)x \int_0^1 u'(t-zx)dzdx \quad \text{for } t \geq t_0. \end{aligned}$$

Integrating the last inequality from t_0 to t and using Fubini's theorem we obtain that

$$\begin{aligned} c(u(t)-u(t_0)) &\geq -\int_{\mathbb{R}} J(x)x \int_0^1 [u(t-zx)-u(t_0-zx)]dzdx \\ &\geq -\int_0^N J(x)x \int_0^1 u(t-zx)dzdx - \int_{-\infty}^0 J(x)|x| \int_0^1 u(t_0-zx)dzdx - x_c \int_N^{+\infty} J(x)xdx \\ &\geq -\int_0^N J(x)x \int_0^1 u(t-zx)dzdx - u(t_0) \int_{-\infty}^0 J(x)|x|dx - \varepsilon(N) \quad \text{for } t \geq t_0, \end{aligned}$$

where $N > 0$ is arbitrary and ε satisfies (2.14). Passing to the limit as $t \to +\infty$, from the latter inequality it follows that

$$u(t_0)\left(c - \int_{-\infty}^{0} J(x)|x|dx\right) \le \varepsilon(N) \qquad \text{for } N > 0. \tag{2.17}$$

Consequently, in view of (2.14), the inequality (2.17) yields

$$u(t_0)\left(c - \int_{-\infty}^{0} J(x)|x|dx\right) \le 0.$$

Thus, on account of $u(t_0) > 0$ we have

$$c \le \int_{-\infty}^{0} J(x)|x|dx,$$

which contradicts (2.11). The obtained contradiction proves the assertion of the lemma. □

Proof of Theorem 1. According to Lemma 9, for every $u_0 \in (0, e^{-\lambda h})$ there exists a positive solution u to (1.1) such that (2.10) holds (recall that $x_c = 1 + f_0(e^{\frac{h}{c}} - 1)$). In view of Lemmas 10 and 12 we have that

$$\text{either} \qquad \lim_{t\to-\infty} u(t) = 0 \qquad \text{or} \qquad \lim_{t\to+\infty} u(t) = 0.$$

However, Lemma 13 assures that (2.16) is valid. Therefore, the theorem is proven. □

3. Applications and examples

In this section, we present some examples to illustrate the application of the main results of this paper about the existence of traveling wave to equation (1.1) with some specific growth term f and dispersal kernel J, which are usually used representing the dynamical population model of single species in ecology (see, e.g. [3,5,6,13,18]).

We consider the exponential kernel

$$J_\alpha(s) = \frac{e^{-\frac{(s+\rho)^2}{4\alpha}}}{\sqrt{4\pi\alpha}}, \qquad \rho \ge 0.$$

Note that $\int_{\mathbb{R}} J_\alpha(x)dx = 1$ and $\int_{-\infty}^{0} J_\alpha(x)e^{-\lambda x}dx < +\infty$ for all $\lambda > 0$. In addition, by computation we have

$$\int_{-\infty}^{+\infty} J_\alpha(x)|x|dx = \sqrt{\frac{4\alpha}{\pi}}e^{-\frac{\rho^2}{4\alpha}} + \rho\,\mathrm{erf}\left(\frac{\rho}{\sqrt{4\alpha}}\right) =: M_1 < +\infty,$$

where $\mathrm{erf}(x)$ is the Gauss error function. Moreover, we also get

$$\int_{0}^{+\infty} J_\alpha(x)dx = \frac{1}{2}\left(1 - \mathrm{erf}\left(\frac{\rho}{\sqrt{4\alpha}}\right)\right) =: M_2,$$

$$\int_{-\infty}^{0} J_\alpha(x)e^{-\lambda x}dx = \frac{e^{\lambda(\rho+\lambda\alpha)}}{2}\left(1+\operatorname{erf}\left(\frac{\rho+2\lambda\alpha}{\sqrt{4\alpha}}\right)\right) =: M_3(\lambda) \quad \text{for } \lambda > 0.$$

Let f be as in Section 1, and let f_0, x_c, and f_c be as in Section 2. Put

$$H(\lambda) := \int_0^{+\infty} J_\alpha(x)dx + \int_{-\infty}^{0} J_\alpha(x)e^{-\lambda x}dx, \quad G_c(\lambda) := c\lambda - f_c, \quad \lambda \geq 0.$$

Note that $H(0) = 1$, $H''(\lambda) > 0$ for $\lambda > 0$, $H'(0) = -\int_{-\infty}^{0} xJ_\alpha(x)dx > 0$ and $H(\lambda)$ does not depend on c. Moreover, $G_c(0) = -f_c < 0$, $G_c'(\lambda) = c > 0$ and G_c is increasing with respect to c at every positive point λ. In consequence, there exist $c_\star, \lambda_\star > 0$ such that

$$G_{c_\star}(\lambda_\star) = H(\lambda_\star), \qquad G_{c_\star}(\lambda) < H(\lambda) \quad \text{for } \lambda > 0,\ \lambda \neq \lambda_\star.$$

In addition, for each $c > c_\star$, we have

$$G_c(\lambda_\star) > H(\lambda_\star)$$

and, if $c < c_\star$, then $G_c(\lambda_\star) < H(\lambda_\star)$. By substitution, we finally have, for each speed $c \geq c_\star$,

$$c\lambda_\star \geq M_2 + M_3(\lambda_\star) + f_c. \tag{3.1}$$

We also obatin that $G_{c_\star}'(\lambda_\star) = H'(\lambda_\star)$, i.e.,

$$c_\star = -\int_{-\infty}^{0} xJ_\alpha(x)e^{-\lambda_\star x}dx.$$

Now, to estimate the right-hand term of (3.1), we will consider a particular nonlinearity. We study two types of nonlinearities, usually used in literature for nondegenerate case ($f'(0) > 0$) and degenerate case ($f'(0) = 0$).

Example 3.1 Let f be the classical Fisher-KPP nonlinearity given by $f(s) = s(1-s)$ for $s \geq 0$. Thus $f_0 = \frac{1}{4}$, $x_c = \frac{1}{4}(3 + e^{\frac{h}{c}}) > 1$, and $f_c = |f'(x_c)| = \frac{1+e^{h/c}}{2}$. According to the above-proven, there exist positive numbers $c_\star$ and $\lambda_\star$ such that

$$c_\star\lambda_\star = M_2 + M_3(\lambda_\star) + f_{c_\star} \tag{3.2}$$

and (3.1) holds for every $c \geq c_\star$. Substituting into (3.1) we obtain

$$c\lambda_\star \geq M_2 + M_3(\lambda_\star) + \frac{1+e^{h/c}}{2} \qquad \text{for } c \geq c_\star.$$

Moreover, the function

$$P_c(\lambda) := 1 + f_c e^{\lambda h} = 1 + \frac{1+e^{h/c}}{2}e^{\lambda h}$$

is increasing with respect to λ, and decreasing with respect to c at every positive point λ. Therefore, there exits $c_* \geq c_\star$ such that

$$c_*\lambda_\star = \max\left\{M_2 + M_3(\lambda_\star) + \frac{1+e^{h/c_*}}{2}, P_{c_*}(\lambda_\star)\right\}.$$

Consequently,

$$c\lambda_\star \geq \max\left\{M_2 + M_3(\lambda_\star) + \frac{1+e^{h/c}}{2}, P_c(\lambda_\star)\right\} \qquad \text{for } c \geq c_*.$$

Set

$$M = \frac{1}{\lambda_\star}\max\left\{M_2 + M_3(\lambda_\star) + \frac{1+e^{h/c_*}}{2}, P_{c_*}(\lambda_\star), \lambda_\star M_1\right\}.$$

Then the proof of Theorem 1 guarantees the following result.

Theorem 2. *For every $c > M$ there exists a positive solution u to the monostable delay equation*

$$cu'(t) = J_\alpha \star u(t) - u(t) + u(t-h)(1-u(t-h)),$$

satisfying

$$\lim_{t\to-\infty} u(t) = 0, \qquad e^{-\lambda_\star h} \leq \liminf_{t\to+\infty} u(t) \leq \sup_{t\in\mathbb{R}} u(t) \leq 1 + \frac{e^{\frac{h}{c}}-1}{4}.$$

Example 3.2 We consider the nonlinearity $f(s) = s^{p+1}(1-s)$, $p > 0$ (degenerate case). Then

$$f_0 = \frac{(p+1)^{p+1}}{(p+2)^{p+2}}, \quad x_c = 1 + \frac{(p+1)^{p+1}(e^{\frac{h}{c}}-1)}{(p+2)^{p+2}},$$

and

$$\begin{aligned} f_c &= |f'(x_c)| \\ &= \left(1 + \frac{(p+1)^{p+1}\left(e^{\frac{h}{c}}-1\right)}{(p+2)^{p+2}}\right)^p \left(1 + \frac{(p+1)^{p+1}\left(e^{\frac{h}{c}}-1\right)}{(p+2)^{p+1}}\right) =: N_1(c). \end{aligned}$$

According to the above-proven, there exist positive numbers $c_\star$ and $\lambda_\star$ such that (3.2) is fulfilled and (3.1) holds for every $c \geq c_\star$. Substituting into (3.1) we obtain

$$c\lambda_\star \geq M_2 + M_3(\lambda_\star) + N_1(c) \qquad \text{for } c \geq c_\star.$$

Moreover, the function

$$\overline{P}_c(\lambda) := 1 + f_c e^{\lambda h} = 1 + N_1(c)e^{\lambda h}$$

is increasing with respect to λ and decreasing with respect to c at every positive point λ. Therefore, there exits $c_* \geq c_\star$ such that

$$c_*\lambda_\star = \max\left\{M_2 + M_3(\lambda_\star) + N_1(c_*), \overline{P}_{c_*}(\lambda_\star)\right\}.$$

Consequently,

$$c\lambda_\star \geq \max\left\{M_2 + M_3(\lambda_\star) + N_1(c), \overline{P}_c(\lambda_\star)\right\} \qquad \text{for } c \geq c_*.$$

Now, set

$$N = \frac{1}{\lambda_\star} \max\left\{M_2 + M_3(\lambda_\star) + N_1(c_*), \overline{P}_{c_*}(\lambda_\star), \lambda_\star M_1\right\}.$$

Then we have the following result.

Theorem 3. *For every $c > N$ there exists a positive solution u to the monostable delay equation*

$$cu'(t) = J_\alpha \star u(t) - u(t) + (u(t-h))^{p+1}(1-u(t-h)),$$

satisfying

$$\lim_{t\to-\infty} u(t) = 0, \quad e^{-\lambda_\star h} \le \liminf_{t\to+\infty} u(t) \le \sup_{t\in\mathbb{R}} u(t) \le 1 + \frac{(p+1)^{p+1}\left(e^{\frac{h}{c}}-1\right)}{(p+2)^{p+2}}.$$

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Authors' addresses

Maitere Aguerrea
Facultad de Ciencias Básicas, Universidad Católica del Maule, Casilla 617, Talca, Chile
E-mail address: maguerrea@ucm.cl

Robert Hakl
Institute of Mathematics, Czech Academy of Sciences, Žižkova 22, 616 62 Brno, Czech Republic
E-mail address: hakl@ipm.cz

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CAPUTO FRACTIONAL DIFFERENTIAL INCLUSIONS OF ARBITRARY ORDER WITH NONLOCAL INTEGRO-MULTIPOINT BOUNDARY CONDITIONS

BASHIR AHMAD, DOA'A GAROUT, SOTIRIS K. NTOUYAS, AND AHMED ALSAEDI

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Abstract. We study a new class of boundary value problems of Caputo type fractional differential inclusions supplemented with nonlocal integro-multipoint boundary conditions. An existence result for the problem with convex valued (multivalued) map is obtained via nonlinear alternative of Leray-Schauder type, while the existence of solutions for the problem involving nonconvex valued map is established by means of Wegrzyk's fixed point theorem. Our results are well illustrated with examples.

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1. Introduction

In this paper, we obtain sufficient criteria for the existence of solutions for a Caputo type fractional differential inclusion

$$^{c}D^{q}x(t) \in F(t,x(t)),\ n-1<q\le n,\ \ t\in[0,1], \tag{1.1}$$

equipped with the boundary conditions

$$\begin{cases} x(0)=0,\ x'(0)=0,\ x''(0)=0,\dots,\ x^{(n-2)}(0)=0, \\ x(1)=a\displaystyle\int_0^{\xi}x(s)ds+b\sum_{i=1}^{m-2}\alpha_i x(\eta_i),\ 0<\xi<\eta_1<\eta_2<\ldots<\eta_{m-2}<1, \end{cases} \tag{1.2}$$

where $^{c}D^{q}$ denotes the Caputo fractional derivative of order q, $F:[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a multi-valued map, $\mathcal{P}(\mathbb{R})$ is a family of all nonempty subsets of $\mathbb{R}$, a and b are real constants and α_i, $i=1,\dots,m-2$, are positive real constants.

Here we remark that the last condition in (1.2) connecting the nonlocal multi-point and strip conditions can be interpreted as the linear combination of the values of the unknown function at nonlocal points $\eta_i\in(0,1)$ together with the strip contribution of the unknown function on an arbitrary segment $(0,\xi)\subset[0,1]$ is proportional to

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the value of the unknown function at $t = 1$. The multi-point-strip boundary data, occurring in certain problems of thermodynamics, elasticity and wave propagation [3, 8, 27], correspond to the situation when the controllers at the end points of the interval dissipate or add energy according to censors located at interior positions (finite many points and strip) of the domain.

In recent years, the topic of initial and boundary value problems involving fractional differential equations and inclusions has received great attention and many authors have contributed to its advancement by producing a variety of results [2, 4–7, 9, 15, 18, 23, 25, 28, 34]. The interest in this topic owes to its extensive applications in the mathematical modeling of scientific and applied problems occurring in various disciplines such as ecology, acoustics, viscoelasticity, electromagnetics, control theory and material sciences. An important characteristic, distinguishing fractional order operators from their classical counterparts, is their nonlocal nature that can take care of the past history of the processes and phenomena involved in the problem. For some specific examples, we refer the reader to the works [20, 35] and the references cited therein. In [29], the authors studied the existence of solutions for one-dimensional higher-order semi-linear Caputo type fractional differential equations supplemented with nonlocal multi-point discrete and integral boundary conditions.

Differential inclusions (generalization of differential equations and inequalities) are found to be of great utility in studying dynamical systems and stochastic processes. An important application of differential inclusions can be found in the area of sweeping processes. In fact, evolution differential inclusions appear in the mathematical modelling of sweeping processes. For a detailed account of this subject, we refer the reader to the monograph [26] and research articles [1, 16]. In [31], it has been shown that the existence and uniqueness of BV continuous sweeping processes can be easily reduced to the Lipschitz continuous case by means of a suitable reparametrization of the associated moving convex set. Differential inclusions also play a key role in the study of granular systems [30, 32], nonlinear dynamics of wheeled vehicles [10], control problems [21], etc. In [24], one can find a detailed description of pressing issues in stochastic processes, control, differential games, optimization and their application in finance, manufacturing, queueing networks, and climate control. For application of fractional differential inclusions in synchronization processes, we refer the reader to the paper [13].

The aim of the present work is to develop the existence theory for a multivalued analog of the problem addressed in [29]. The first result dealing with the convex valued maps involved in the given problem is obtained via nonlinear alternative of Leray-Schauder type, while the existence of solutions for the nonconvex valued maps is shown by applying Wegrzyk's fixed point theorem. The main results are presented in Section 3. The preliminary material needed to execute the main work is outlined in Section 2.

2. Preliminaries

First of all, we recall some definitions of fractional calculus [20, 35].

Definition 1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $g : (0,\infty) \to \mathbb{R}$ is defined by

$$J^{\alpha} g(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)ds,$$

provided the right-hand side is point-wise defined on $(0,\infty)$, where Γ is the Gamma function.

Definition 2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $g : (0,\infty) \to \mathbb{R}$ is defined by

$$D^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{g(s)}{(t-s)^{\alpha-n+1}} ds, \quad n-1 < \alpha < n,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of real number α, provided the right-hand side is point-wise defined on $(0,\infty)$.

Definition 3. The Caputo derivative of order q for a function $f : [0,\infty) \to \mathbb{R}$ can be written as

$${}^c D^q f(t) = D^q \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, \quad n-1 < q < n.$$

Remark 1. If $f(t) \in C^n[0,\infty)$, then

$${}^c D^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}} ds = I^{n-q} f^{(n)}(t),\ t > 0,\ n-1 < q < n.$$

In order to define a solution of the given problem, we need the following known result [29].

Lemma 1. *Let* $\omega \in C([0,1],\mathbb{R})$ *and*

$$\Delta = 1 - \frac{a\xi^n}{n} - b \sum_{i=1}^{m-2} \alpha_i \eta_i^{n-1} \neq 0. \tag{2.1}$$

The boundary value problem

$$\begin{cases} {}^c D^q x(t) = \omega(t), \quad t \in [0,1], \\ x(0) = 0,\ x'(0) = 0,\ x''(0) = 0, \ldots,\ x^{(n-2)}(0) = 0, \\ x(1) = a \displaystyle\int_0^{\xi} x(s)ds + b \sum_{i=1}^{m-2} \alpha_i x(\eta_i),\ 0 < \xi < \eta_1 < \eta_2 < \ldots < \eta_{m-2} < 1, \end{cases} \tag{2.2}$$

is equivalent to the following integral equation

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\omega(s)ds + \frac{t^{n-1}}{\Delta}\Bigg(a\int_0^{\xi}\frac{(\xi-s)^q}{\Gamma(q+1)}\omega(s)ds$$

$$+ b\sum_{i=1}^{m-2}\alpha_i\int_0^{\eta_i}\frac{(\eta_i-s)^{q-1}}{\Gamma(q)}\omega(s)ds - \int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\omega(s)ds\Bigg). \tag{2.3}$$

Next we fix our terminology and outline some basic concepts of multivalued analysis [14, 19].

Denote by $C([0,1],\mathbb{R})$ the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ endowed with the norm $\|x\| = \sup\{|x(t)|,\ t\in[0,1]\}$. By $L^1([0,1],\mathbb{R})$ we denote the space of Lebesgue measurable and integrable functions $x:[0,1]\to\mathbb{R}$ such that $\|x\|_{L^1} = \int_0^1 |x(t)|dt$.

For a normed space $(X,\|\cdot\|)$, let $\mathcal{P}_{cl}(X) = \{Y\in\mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y\in\mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cl,b}(X) = \{Y\in\mathcal{P}(X) : Y \text{ is closed and bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y\in\mathcal{P}(X) : Y \text{ is compact}\}$, and $\mathcal{P}_{cp,c}(X) = \{Y\in\mathcal{P}(X) : Y \text{ is compact and convex}\}$.

A multi-valued map $\mathcal{H} : X\to\mathcal{P}(X)$:

(i) is *convex (closed) valued* if $\mathcal{H}(x)$ is convex (closed) for all $x\in X$.

(ii) is *bounded* on bounded sets if $\mathcal{H}(Y) = \cup_{x\in Y}\mathcal{H}(x)$ is bounded in X for all $Y\in\mathcal{P}_b(X)$ (i.e. $\sup_{x\in Y}\{\sup\{|y| : y\in\mathcal{H}(x)\}\} < \infty$).

(iii) is called *upper semi-continuous (u.s.c.)* on X if for each $x_0\in X$, the set $\mathcal{H}(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $\mathcal{H}(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $\mathcal{H}(\mathcal{N}_0)\subseteq N$.

(iv) G is *lower semi-continuous (l.s.c.)* if the set $\{y\in X : \mathcal{H}(y)\cap Y\neq\varnothing\}$ is open for any open set Y in X.

(v) is said to be *completely continuous* if $\mathcal{H}(\mathbb{B})$ is relatively compact for every $\mathbb{B}\in\mathcal{P}_b(X)$; If the multi-valued map $\mathcal{H}$ is completely continuous with nonempty compact values, then $\mathcal{H}$ is u.s.c. if and only if $\mathcal{H}$ has a closed graph, i.e., $x_n\to x_*,\ y_n\to y_*,\ y_n\in\mathcal{H}(x_n)$ imply $y_*\in\mathcal{H}(x_*)$.

(vi) is said to be *measurable* if for every $y\in X$, the function

$$t\longmapsto d(y,\mathcal{H}(t)) = \inf\{|y-z| : z\in\mathcal{H}(t)\}$$

is measurable.

(vii) *has a fixed point* if there is $x\in X$ such that $x\in\mathcal{H}(x)$. The fixed point set of the multivalued operator $\mathcal{H}$ will be denoted by $Fix\mathcal{H}$.

3. Existence results

By Lemma 1, we can define a solution of problem (1.1)-(1.2) as follows.

Definition 4. A function $x \in C^n([0,1],\mathbb{R})$ is a solution of problem (1.1)-(1.2) if there exists a function $f \in L^1([0,1],\mathbb{R})$ such that $f(t) \in F(t,x(t))$ a.e. on $[0,1]$ and

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds + \frac{t^{n-1}}{\Delta}\Bigg(a\int_0^\xi \frac{(\xi-s)^q}{\Gamma(q+1)} f(s)ds$$

$$+ b\sum_{i=1}^{m-2}\alpha_i \int_0^{\eta_i} \frac{(\eta_i-s)^{q-1}}{\Gamma(q)} f(s)ds - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)ds\Bigg). \tag{3.1}$$

Next, we define an operator $\mathcal{K} : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ by

$$\mathcal{K}(x) = \Bigg\{ z \in C([0,1],\mathbb{R}) : z(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds$$

$$+ \frac{t^{n-1}}{\Delta}\Bigg(a\int_0^\xi \frac{(\xi-s)^q}{\Gamma(q+1)} f(s)ds + b\sum_{i=1}^{m-2}\alpha_i \int_0^{\eta_i} \frac{(\eta_i-s)^{q-1}}{\Gamma(q)} f(s)ds$$

$$- \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)ds\Bigg), f \in S_{F,x}\Bigg\}. \tag{3.2}$$

Definition 5. A multi-valued map $F : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \mapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$ and
(ii) $x \mapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,1]$.

Further, a Caratheodory function F is called L^1-Carathéodory if

(iii) for each $a > 0$, there exists $\varphi_a \in L^1([0,1],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \le \varphi_a(t)$$

for all $\|x\| \le a$ and for a.e $t \in [0,1]$.

For each $y \in C([0,1],\mathbb{R})$, we define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,1],\mathbb{R}) : v(t) \in F(t,y(t)) \ for\ a.e.\ t \in [0,1]\}.$$

3.1. *The convex-valued (Carathéodory) case*

In this subsection, we prove an existence result for problem (1.1)-(1.2), assuming that F is Carathéodory (F has convex values).

Before presenting the main result, let us state the auxiliary lemmas.

Lemma 2 (Nonlinear alternative of Leray-Schauder type [17])**.** *Let C be a convex set in a normed space, and $E \subset C$ be open subset with $0 \in E$. Then each upper semicontinuous and compact mapping $\mathcal{G} : \overline{E} \to P(C)$ with compact convex values that is fixed point free on ∂E has at least one of the following two properties:*

(a) *$\mathcal{G}$ has a fixed point in $\overline{E}$, or*
(b) *there exist $x \in \partial E$ and $\lambda \in (0,1)$ such that $x \in \lambda \mathcal{G}(x)$.*

Lemma 3 ([22]). *Let X be a separable Banach space. Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp,c}(X)$ be an L^1- Caratheodory multi-valued map, and let ϕ be a linear continuous mapping from $L^1([0,1],X)$ to $C([0,1],X)$. Then the operator*

$$\phi \circ S_F : C([0,1],X) \longrightarrow \mathcal{P}_{cp,c}(C([0,1],X)), \quad x \mapsto (\phi \circ S_F)(x) = \phi(S_{F,x}),$$

is a closed graph operator in $C([0,1],X) \times C([0,1],X)$.

Theorem 1. *Assume that*

(H_1) *$F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is L^1-Carathéodory and has compact and convex values;*

(H_2) *there exist a continuous nondecreasing function $\chi : [0,\infty) \to (0,\infty)$ and a function $k \in C([0,1],\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leq k(t)\chi(\|x\|) \ for\ all\ (t,x) \in [0,1] \times \mathbb{R};$$

(H_3) *There exists a number $N > 0$ such that*

$$\frac{N}{\chi(N)\|k\|}\left\{\frac{1}{\Gamma(q+1)}\left[1+\frac{1}{|\Delta|}\left(1+|a|\frac{\xi^{q+1}}{(q+1)}+|b|\sum_{i=1}^{m-2}\alpha_i\eta_i{}^q\right)\right]\right\}^{-1} > 1,$$

where Δ is given by (2.1).

Then the problem (1.1)-(1.2) has at least one solution on $[0,1]$.

Proof. We transform the problem (1.1)-(1.2) into a fixed point theorem by considering the operator $\mathcal{K}$ defined by (3.2). It is obvious that the fixed points of $\mathcal{K}$ are solutions of the boundary value problem (1.1)-(1.2). We will show that $\mathcal{K}$ satisfies the assumptions of Leray-Schauder nonlinear alternative. In the first step, *we will show that $\mathcal{K}(x)$ is convex for each $x \in C([0,1],\mathbb{R})$*. Let $z_1, z_2 \in \mathcal{K}(x)$, then there exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in [0,1]$, we have

$$z_i(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_i(s)ds + \frac{t^{n-1}}{\Delta}\left(a\int_0^\xi \frac{(\xi-s)^q}{\Gamma(q+1)} f_i(s)ds\right.$$
$$\left. + b\sum_{i=1}^{m-2}\alpha_i \int_0^{\eta_i} \frac{(\eta_i-s)^{q-1}}{\Gamma(q)} f_i(s)ds - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f_i(s)ds\right), \ i = 1,2.$$

Set $0 \leq \delta \leq 1$, then for each $t \in [0,1]$, we have

$$[\delta z_1 + (1-\delta)z_2](t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}[\delta f_1(s) + (1-\delta) f_2(s)]ds$$
$$+ \frac{t^{n-1}}{\Delta}\left(a\int_0^\xi \frac{(\xi-s)^q}{\Gamma(q+1)}[\delta f_1(s) + (1-\delta) f_2(s)]ds\right.$$

$$+b\sum_{i=1}^{m-2}\alpha_i\int_0^{\eta_i}\frac{(\eta_i-s)^{q-1}}{\Gamma(q)}[\delta f_1(s)+(1-\delta)f_2(s)]ds$$

$$-\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}[\delta f_1(s)+(1-\delta)f_2(s)]ds\Bigg).$$

Hence, by the convexity of $S_{F,x}$, it follows that $\delta z_1+(1-\delta)z_2\in\mathcal{K}(x)$.

Now, *we show that $\mathcal{K}$ maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$*. For $\rho>0$, let $B_\rho=\{x\in C([0,1],\mathbb{R}):\|x\|\le\rho\}$ be a bounded set in $C([0,1],\mathbb{R})$. Thus, for each $z\in\mathcal{K}(x),\ x\in B_\rho$, there exists $f\in S_{F,x}$ such that

$$z(t)=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s)ds+\frac{t^{n-1}}{\Delta}\Bigg(a\int_0^\xi\frac{(\xi-s)^q}{\Gamma(q+1)}f(s)ds$$

$$+b\sum_{i=1}^{m-2}\alpha_i\int_0^{\eta_i}\frac{(\eta_i-s)^{q-1}}{\Gamma(q)}f(s)ds-\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}f(s)ds\Bigg). \tag{3.3}$$

Then, for $x\in B_\rho$, in view of (H_2), we obtain

$$|z(t)|$$

$$\le\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}k(s)\chi(\|x\|)ds+\frac{t^{n-1}}{|\Delta|}\Bigg(|a|\int_0^\xi\frac{(\xi-s)^q}{\Gamma(q+1)}k(s)\chi(\|x\|)ds$$

$$+|b|\sum_{i=1}^{m-2}\alpha_i\int_0^{\eta_i}\frac{(\eta_i-s)^{q-1}}{\Gamma(q)}k(s)\chi(\|x\|)ds+\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}k(s)\chi(\|x\|)ds\Bigg),$$

which implies that

$$\|z\|\le\frac{\chi(\rho)\|k\|}{\Gamma(q+1)}\Bigg(1+\frac{1}{|\Delta|}\Bigg[1+|a|\frac{\xi^{q+1}}{(q+1)}+|b|\sum_{i=1}^{m-2}\alpha_i\eta_i{}^q\Bigg]\Bigg).$$

Next, *we show that $\mathcal{K}$ maps bounded sets into equicontinuous sets in $C([0,1],\mathbb{R})$*. Let $t_1,t_2\in[0,1]$ with $t_1<t_2$ and $x\in B_\rho$, then, we obtain for each $z\in\mathcal{K}(x)$

$$|z(t_2)-z(t_1)|$$

$$=\Bigg|\int_0^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}f(s)ds-\int_0^{t_1}\frac{(t_1-s)^{q-1}}{\Gamma(q)}f(s)ds+\frac{(t_2^{n-1}-t_1^{n-1})}{\Delta}\times$$

$$\times\Bigg[a\int_0^\xi\frac{(\xi-s)^q}{\Gamma(q+1)}f(s)ds+b\sum_{i=1}^{m-2}\alpha_i\int_0^{\eta_i}\frac{(\eta_i-s)^{q-1}}{\Gamma(q)}f(s)ds$$

$$
\begin{aligned}
&-\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)ds \Bigg]\Bigg| \\
&\leq \left| \int_0^{t_1} \frac{(t_2-s)^{q-1}-(t_1-s)^{q-1}}{\Gamma(q)} f(s)ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)} f(s)ds \right| \\
&+ \left| \frac{t_2^{n-1}-t_1^{n-1}}{\Delta} \right| \Bigg[|a| \int_0^{\xi} \frac{(\xi-s)^q}{\Gamma(q+1)} |f(s)|ds \\
&+ |b| \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} \frac{(\eta_i-s)^{q-1}}{\Gamma(q)} |f(s)|ds + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} |f(s)|ds \Bigg] \\
&\leq \chi(\rho)\|k\| \Bigg[\frac{|t_2^q - t_1^q| + 2(t_2-t_1)^q}{\Gamma(q+1)} \\
&+ \frac{|t_2^{n-1}-t_1^{n-1}|}{\Gamma(q+1)|\Delta|} \left(1 + |a| \frac{\xi^{q+1}}{(q+1)} + |b| \sum_{i=1}^{m-2} \alpha_i {\eta_i}^q \right) \Bigg],
\end{aligned}
$$

which tends to zero independent of $x \in B_\rho$ as $(t_2 - t_1) \to 0$. Consequently, by the Arzelá-Ascoli theorem, the operator $\mathcal{K}$ is completely continuous.

Since $\mathcal{K}$ is completely continuous, in order to prove that it is u.s.c. it is enough to prove that it has a closed graph. Thus, now, *we want to show that $\mathcal{K}$ has a closed graph.* Let $x_n \to \hat{x}$, $z_n \in \mathcal{K}(x_n)$, and $z_n \to \hat{z}$. We have to show that $\hat{z} \in \mathcal{K}(\hat{x})$. So, for $z_n \in \mathcal{K}(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for all $t \in [0,1]$, we have

$$
\begin{aligned}
z_n(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_n(s)ds + \frac{t^{n-1}}{\Delta} \Bigg(a \int_0^{\xi} \frac{(\xi-s)^q}{\Gamma(q+1)} f_n(s)ds \\
&+ b \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} \frac{(\eta_i-s)^{q-1}}{\Gamma(q)} f_n(s)ds - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f_n(s)ds \Bigg).
\end{aligned}
$$

Thus, we have to show that there exists $\hat{f} \in S_{F,\hat{x}}$ such that for each $t \in [0,1]$,

$$
\begin{aligned}
\hat{z}(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \hat{f}(s)ds + \frac{t^{n-1}}{\Delta} \Bigg(a \int_0^{\xi} \frac{(\xi-s)^q}{\Gamma(q+1)} \hat{f}(s)ds \\
&+ b \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} \frac{(\eta_i-s)^{q-1}}{\Gamma(q)} \hat{f}(s)ds - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \hat{f}(s)ds \Bigg).
\end{aligned}
$$

Consider the continuous linear operator $\Phi : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ given by

$$f \mapsto \Phi(f)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds + \frac{t^{n-1}}{\Delta}\Bigg(a\int_0^\xi \frac{(\xi-s)^q}{\Gamma(q+1)} f(s)ds$$
$$+ b\sum_{i=1}^{m-2}\alpha_i \int_0^{\eta_i} \frac{(\eta_i-s)^{q-1}}{\Gamma(q)} f(s)ds - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)ds\Bigg).$$

Observe that

$$\|z_n(t) - \hat{z}(t)\|$$
$$= \Bigg\| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}(f_n(s)-\hat{f}(s))ds + \frac{t^{n-1}}{\Delta}\Bigg(a\int_0^\xi \frac{(\xi-s)^q}{\Gamma(q+1)}(f_n(s)-\hat{f}(s))ds$$
$$+ b\sum_{i=1}^{m-2}\alpha_i \int_0^{\eta_i} \frac{(\eta_i-s)^{q-1}}{\Gamma(q)}(f_n(s)-\hat{f}(s))ds$$
$$- \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}(f_n(s)-\hat{f}(s))ds\Bigg)\Bigg\|$$

tends to 0 as $n \to \infty$. Thus, it follows by Lemma 3 that $\Phi \circ S_F$ is a closed graph operator. Moreover, we have $z_n(t) \in \Phi(S_{F,x_n})$. Since $x_n \to \hat{x}$, we have then

$$\hat{z}(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \hat{f}(s)ds + \frac{t^{n-1}}{\Delta}\Bigg(a\int_0^\xi \frac{(\xi-s)^q}{\Gamma(q+1)} \hat{f}(s)ds$$
$$+ b\sum_{i=1}^{m-2}\alpha_i \int_0^{\eta_i} \frac{(\eta_i-s)^{q-1}}{\Gamma(q)} \hat{f}(s)ds - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \hat{f}(s)ds\Bigg),$$

for some $\hat{f} \in S_{F,\hat{x}}$.

Finally, *we show there exists an open set* $\mathcal{V} \subseteq C([0,1],\mathbb{R})$ *with* $x \notin \nu\mathcal{K}(x)$ *for any* $\nu \in (0,1)$ *and all* $x \in \partial\mathcal{V}$. Let x be a solution of (1.1)-(1.2). Then there exists $f \in L^1([0,1],\mathbb{R})$ with $f \in S_{F,x}$ such that for $t \in [0,1]$, we have

$$x(t) = \nu\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds + \nu\frac{t^{n-1}}{\Delta}\Bigg(a\int_0^\xi \frac{(\xi-s)^q}{\Gamma(q+1)} f(s)ds$$
$$+ b\sum_{i=1}^{m-2}\alpha_i \int_0^{\eta_i} \frac{(\eta_i-s)^{q-1}}{\Gamma(q)} f(s)ds - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)ds\Bigg).$$

Using (H_2), we obtain

$$|x(t)| \leq \frac{\chi(\|x\|)\|k\|}{\Gamma(q+1)}\left(1+\frac{1}{|\Delta|}\left[1+|a|\frac{\xi^{q+1}}{(q+1)}+|b|\sum_{i=1}^{m-2}\alpha_i\eta_i{}^q\right]\right),$$

which implies

$$\frac{\|x\|}{\chi(\|x\|)\|k\|}\left[\frac{1}{\Gamma(q+1)}\left\{1+\frac{1}{|\Delta|}\left(1+|a|\frac{\xi^{q+1}}{(q+1)}+|b|\sum_{i=1}^{m-2}\alpha_i\eta_i{}^q\right)\right\}\right]^{-1} \leq 1.$$

By the assumption (H_3), there exists $N > 0$ such that $\|x\| \neq N$. Let us take $\mathcal{V} = \{x \in C([0,1],\mathbb{R}) : \|x\| < N+1\}$. Note that the operator $\mathcal{K} : \overline{\mathcal{V}} \to \mathcal{P}(C([0,1],\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of $\mathcal{V}$, there is no $x \in \partial\mathcal{V}$ such that $x \in \nu\mathcal{K}(x)$ for some $\nu \in (0,1)$. Therefore, by Leray-Schauder alternative, it follows that the operator $\mathcal{K}$ has a fixed point $x \in \overline{\mathcal{V}}$, which is a solution of the problem (1.1)-(1.2). This completes the proof. □

3.2. *Nonconvex-valued (Lipschitz) case*

In this part, we discuss the existence of solutions for the inclusion problem (1.1)-(1.2) with the right-hand side being nonconvex set-valued map by applying Wegrzyk's fixed point theorem.

Let (X,d) be a metric space induced from the normed space $(X;\|\cdot\|)$. Consider $Q_d : \mathcal{P}(X)\times\mathcal{P}(X) \to \mathbb{R}\bigcup\{\infty\}$ given by

$$Q_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\},$$

where $d(a,B) = \inf_{b\in B} d(a,b)$. The map Q_d is the (generalized) Pompeiu-Hausdorff functional. Clearly, $(\mathcal{P}_{b,cl}(X), Q_d)$ is a metric space and $(P_d(X), Q_d)$ is a generalized metric space.

Definition 6. A function $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a strict comparison function if it is continuous strictly increasing and $\sum_{n=1}^{\infty}\Omega^n(t) < \infty$ for all $t > 0$.

Definition 7. A multi-valued operator $\mathcal{A} : X \to \mathcal{P}_{cl}(X)$is called

(a) ϵ-Lipchitz if and only if there exists $\epsilon > 0$ such that $Q_d(\mathcal{A}(X),\mathcal{A}(Y)) \leq \epsilon d(x,y)$ for each $x,y \in X$.
(b) a contraction if and only if it is $\epsilon-$Lipschitz with $\epsilon < 1$;
(c) a generalized contraction if and only if there is a strict comparison function $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that $Q_d(\mathcal{A}(x),\mathcal{A}(y)) \leq \Omega(d(x,y))$ for each $x,y \in X$.

Lemma 4 (Wegrzyk's fixed point theorem [33])**.** *Let (X,d) be a complete metric space. If $\mathcal{A} : X \to \mathcal{P}_{cl}(X)$ is a generalized contraction, then the operator $\mathcal{A}$ has at least one fixed point.*

Theorem 2. *Assume that the following conditions hold:*

(H_4) $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ *is such that* $F(.,x) : [0,1] \to \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $x \in \mathbb{R}$.

(H_5) $Q_d(F(t,x), F(t,\bar{x})) \leq \gamma(t)\Omega(\|x-\bar{x}\|)$ *for almost all* $t \in [0,1]$ *and* $x, \bar{x} \in \mathbb{R}$ *with a function* $\gamma \in C([0,1], \mathbb{R}^+)$ *and* $d(0, F(t,0)) \leq \gamma(t)$ *for almost all* $t \in [0,1]$, *where* $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ *is strictly increasing.*

Then the problem (1.1)-(1.2) has at least one solution on $[0,1]$ *if* $\sigma\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ *is a strict comparison function, where*

$$\sigma = \frac{\|\gamma\|}{\Gamma(q+1)}\left(1 + \frac{1}{|\Delta|}\left[1 + |a|\frac{\xi^{q+1}}{(q+1)} + |b|\sum_{i=1}^{m-2}\alpha_i {\eta_i}^q\right]\right).$$

Proof. Assume that $\sigma\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ is a strict comparison function. Then, in view of (H_4) and (H_5), $F(\cdot, x(\cdot))$ is measurable and has a measurable selection $\zeta(\cdot)$ (see Theorem III.6 [11]). Also, $\gamma \in C([0,1], \mathbb{R}^+)$ and we have

$$\begin{aligned}|\zeta(t)| &\leq d(0, F(t,0)) + Q_d(F(t,0), F(t,x(t)))\\ &\leq \gamma(t) + \gamma(t)\Omega(|x(t)|) \leq (1 + \Omega(\|x\|))\gamma(t).\end{aligned}$$

Hence, the set $S_{F,x}$ is nonempty for each $x \in C([0,1], \mathbb{R})$. Now, we will show that the operator $\mathcal{K}$ which is given by (3.2), satisfies the assumptions of Wegrzyk's fixed point Theorem. First, we show that $\mathcal{K}(x) \in \mathcal{P}_{cl}(C([0,1], \mathbb{R}))$ for each $x \in C([0,1], \mathbb{R})$. So, let, $\{g_n\}_{n\geq 0} \in \mathcal{K}(x)$ be such that $g_n \to g$ as $n \to \infty$ in $C([0,1], \mathbb{R})$. Thus $g \in C([0,1], \mathbb{R})$, and there exists $\zeta_n \in S_{F,x_n}$, such that, for all $t \in [0,1]$, we have

$$\begin{aligned}g_n(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\zeta_n(s)ds + \frac{t^{n-1}}{\Delta}\Bigg(a\int_0^\xi \frac{(\xi-s)^q}{\Gamma(q+1)}\zeta_n(s)ds\\ + b\sum_{i=1}^{m-2}\alpha_i\int_0^{\eta_i}\frac{(\eta_i-s)^{q-1}}{\Gamma(q)}\zeta_n(s)ds - \int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\zeta_n(s)ds\Bigg).\end{aligned}$$

Since F has compact values, we take a subsequence to obtain that ζ_n converges to ζ in $L^1([0,1], \mathbb{R})$. Hence, $\zeta \in S_{F,x}$, and for each $t \in [0,1]$, we have,

$$\begin{aligned}g_n(t) \to g(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\zeta(s)ds + \frac{t^{n-1}}{\Delta}\Bigg(a\int_0^\xi \frac{(\xi-s)^q}{\Gamma(q+1)}\zeta(s)ds\\ + b\sum_{i=1}^{m-2}\alpha_i\int_0^{\eta_i}\frac{(\eta_i-s)^{q-1}}{\Gamma(q)}\zeta(s)ds - \int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\zeta(s)ds\Bigg).\end{aligned}$$

Thus, $g \in \mathcal{K}(x)$.

Now, we show that $Q_d(\mathcal{K}(x),\mathcal{K}(\bar{x}))\leq\sigma\Omega(\|x-\bar{x}\|)$, for all $x,\bar{x}\in C([0,1],\mathbb{R})$. Let $x,\ \bar{x}\in C([0,1],\mathbb{R})$ and $z_1\in\mathcal{K}(x)$, then, there exists $\zeta_1(t)\in S_{F,x}$ such that, for all $t\in[0,1]$, we have

$$z_1(t)=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\zeta_1(s)ds+\frac{t^{n-1}}{\Delta}\Bigg(a\int_0^\xi\frac{(\xi-s)^q}{\Gamma(q+1)}\zeta_1(s)ds$$
$$+b\sum_{i=1}^{m-2}\alpha_i\int_0^{\eta_i}\frac{(\eta_i-s)^{q-1}}{\Gamma(q)}\zeta_1(s)ds-\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\zeta_1(s)ds\Bigg).$$

In view of the assumption (H_5), we have $Q_d(F(t,x),F(t,\bar{x}))\leq\gamma(t)\Omega(|x(t)-\bar{x}(t)|)$. Thus, there exists $\mu\in F(t,\bar{x}(t))$such that

$$|\zeta_1(t)-\mu|\leq\gamma(t)\Omega(|x(t)-\bar{x}(t)|),\ t\in[0,1].$$

Let us define the map, $W:[0,1]\to\mathcal{P}(\mathbb{R})$ by

$$W(t)=\{\mu\in\mathbb{R}:|\zeta_1(t)-\mu|\leq\gamma(t)\Omega(|x(t)-\bar{x}(t)|)\}.$$

Because the nonempty closed set-valued operator $W(t)\bigcap F(t,\bar{x}(t))$ is measurable (Proposition III.4 [11]), there exists a function $\zeta_2(t)$ which is a measurable selection for $W(t)\bigcap F(t,\bar{x}(t))$. Hence, $\zeta_2(t)\in F(t,\bar{x}(t))$, and $|\zeta_1(t)-\zeta_2(t)|\leq\gamma(t)\Omega(|x(t)-\bar{x}(t)|)$ for each $t\in[0,1]$.

Define for each $t\in[0,1]$,

$$z_2(t)=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\zeta_2(s)ds+\frac{t^{n-1}}{\Delta}\Bigg(a\int_0^\xi\frac{(\xi-s)^q}{\Gamma(q+1)}\zeta_2(s)ds$$
$$+b\sum_{i=1}^{m-2}\alpha_i\int_0^{\eta_i}\frac{(\eta_i-s)^{q-1}}{\Gamma(q)}\zeta_2(s)ds-\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\zeta_2(s)ds\Bigg).$$

So, by the assumption (H_5), we obtain

$$|z_1(t)-z_2(t)|$$
$$\leq\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}|\zeta_1(t)-\zeta_2(t)|ds+\frac{t^{n-1}}{|\Delta|}\Bigg(|a|\int_0^\xi\frac{(\xi-s)^q}{\Gamma(q+1)}|\zeta_1(t)-\zeta_2(t)|ds$$
$$+|b|\sum_{i=1}^{m-2}\alpha_i\int_0^{\eta_i}\frac{(\eta_i-s)^{q-1}}{\Gamma(q)}|\zeta_1(t)-\zeta_2(t)|ds$$
$$+\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}|\zeta_1(t)-\zeta_2(t)|ds\Bigg),$$

which implies that

$$\|z_1 - z_2\| \leq \frac{\|\gamma\|\Omega(\|x-\bar{x}\|)}{\Gamma(q+1)}\left(1+\frac{1}{|\Delta|}\left[1+|a|\frac{\xi^{q+1}}{(q+1)}+|b|\sum_{i=1}^{m-2}\alpha_i\eta_i{}^q\right]\right).$$

Interchanging the roles of x and $\bar{x}$, we obtain

$$\begin{aligned} Q_d(\mathcal{K}(x),\mathcal{K}(\bar{x})) &\leq \sigma\Omega(\|x-\bar{x}\|) \\ &= \frac{\|\gamma\|\Omega(\|x-\bar{x}\|)}{\Gamma(q+1)}\left(1+\frac{1}{|\Delta|}\left[1+|a|\frac{\xi^{q+1}}{(q+1)}+|b|\sum_{i=1}^{m-2}\alpha_i\eta_i{}^q\right]\right), \end{aligned}$$

for all $x,\ \bar{x} \in C([0,1],\mathbb{R})$. Thus, the operator $\mathcal{K}$ is a generalized contraction. Therefore, by Wegrzyk's fixed point Theorem, the operator $\mathcal{K}$ has at least one fixed point x, which is a solution of the problem (1.1)-(1.2). This completes the proof. □

Remark 2. We emphasize that the existence result (Theorem 2) obtained by applying Wegrzyk's theorem holds for several choices of the strictly increasing function Ω involved in its hypothesis. For instance, by taking $\Omega(x) = x$, we obtain a special case which is usually obtained by applying a fixed point theorem due to Covitz and Nadler [12].

3.3. *Examples*

Example 1. (Convex-valued case) Consider the following multi-valued fractional boundary value problem:

$$\begin{cases} {}^cD^{5/4} \in F(t,x(t)), \quad t \in [0,1], \\ x(0) = 0,\ x(1) = \displaystyle\int_0^{1/8} x(s)ds + \sum_{i=1}^{4}\alpha_i x(\eta_i), \end{cases} \tag{3.4}$$

where $q = 5/4$, $a = b = 1$, $\xi = 1/8$, $\eta_1 = 1/6$, $\eta_2 = 1/3$, $\eta_3 = 1/2$, $\eta_4 = 2/3$, $\alpha_1 = 1/7$, $\alpha_2 = 2/7$, $\alpha_3 = 3/7$, $\alpha_4 = 4/7$ and

$$F(t,x(t)) = \left[\frac{|x|}{9(|x|+5)} + \frac{1}{18}\cos^2 t\ ,\ \frac{1}{2}e^{-t}\sin x\right]. \tag{3.5}$$

Using the given data, we get

$$|\Delta| = \left|1 - \frac{a\xi^2}{2} - b\sum_{i=1}^{4}\alpha_i\eta_i\right| = 0.2779,$$

$$\frac{1}{\Gamma(q+1)}\left\{1+\frac{1}{|\Delta|}\left(1+|a|\frac{\xi^{q+1}}{q+1}+|b|\sum_{i=1}^{m-2}\alpha_i\eta_i{}^q\right)\right\} = 6.0155.$$

Moreover, for $f \in F$, we have

$$|f| \leq \max\left\{\frac{|x|}{9(|x|+5)} + \frac{1}{18}\cos^2 t\ ,\ \frac{1}{2}e^{-t}\sin x\right\},\ x \in \mathbb{R},\ t \in [0,1].$$

Thus

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leq \frac{1}{2},\ x \in \mathbb{R},$$

with $k(t) = 1,\ \chi(\|x\|) = 1/2$. Next, using the condition (H_3), we find that $N > N_1 \simeq 3.0078$. Thus, all the conditions of Theorem 1 are satisfied and consequently, the problem (3.4) with $F(t,x)$ given by (3.5), has at least one solution on $[0,1]$.

Example 2. (Nonconvex-valued case) Consider the problem (3.4) that is given in Example 1, with

$$F(t,x(t)) = \left[\frac{1}{\sqrt{625+t^2}}\ ,\ \frac{\sin x + \frac{2}{\pi}\tan^{-1} x}{2(4+t(1-t))^2} + \frac{4}{81}\right]. \tag{3.6}$$

So, we find that

$$\sup\{|g| : g \in F(t,x)\} \leq \frac{1}{(4+t(1-t))^2} + \frac{4}{81},$$

$$Q_d(F(t,x),F(t,\bar{x})) \leq \frac{1}{(4+t(1-t))^2}\left[\frac{(2+\pi)}{2\pi}\|x-\bar{x}\|\right].$$

Put $\gamma(t) = \dfrac{1}{(4+t(1-t))^2}$, so, $\|\gamma\| = 1/16$. Thus, we get

$$\sigma = 6.0155\|\gamma\| = 6.0155/16 \simeq 0.37597,$$

and $Q_d(F(t,x),F(t,\bar{x})) \leq \gamma(t)\Omega(\|x-\bar{x}\|)$, where $\Omega(\|x-\bar{x}\|) = \dfrac{(2+\pi)}{2\pi}\|x-\bar{x}\|$. Hence, all the conditions of Theorem 2 are satisfied. Therefore, the problem (3.4) with $F(t,x)$ given by (3.6), has at least one solution on $[0,1]$.

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Authors' addresses

Bashir Ahmad

Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

E-mail address: bashirahmad_qau@yahoo.com

Doa'a Garout

Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

E-mail address: dgarout@kau.edu.sa

Sotiris K. Ntouyas

University of Ioannina, Department of Mathematics, 451 10 Ioannina, Greece and, Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

E-mail address: sntouyas@uoi.gr

Ahmed Alsaedi
Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: aalsaedi@hotmail.com

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ON A SYSTEM OF DIFFERENCE EQUATIONS OF SECOND ORDER SOLVED IN CLOSED FORM

Y. AKROUR, N. TOUAFEK, AND Y. HALIM

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Abstract. In this work we solve in closed form the system of difference equations

$$x_{n+1} = \frac{ay_n x_{n-1} + bx_{n-1} + c}{y_n x_{n-1}},\ y_{n+1} = \frac{ax_n y_{n-1} + by_{n-1} + c}{x_n y_{n-1}},\ n = 0, 1, \ldots,$$

where the initial values x_{-1}, x_0, y_{-1} and y_0 are arbitrary nonzero real numbers and the parameters a, b and c are arbitrary real numbers with $c \neq 0$. In particular we represent the solutions of some particular cases of this system in terms of Tribonacci and Padovan numbers and we prove the global stability of the corresponding positive equilibrium points. The results obtained here extend those obtained in some recent papers.

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Keywords: system of difference equations, closed form, stability, tribonacci numbers, Padovan numbers

1. Introduction

We find in the literature many studies that concern the representation of the solutions of some remarkable linear sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Padovan, and Perrin (see, e.g., [1, 6, 9, 10, 12–14, 20]). Solving in closed form non linear difference equations and systems is a subject that highly attract the attention of researchers (see, e.g.,[3–5, 7, 8, 11, 16–19, 21]) and the reference cited therein, where we find very interesting formulas of the solutions. A large range of these formulas are expressed in terms of famous numbers like Fibonacci and Padovan, (see, e.g., [8, 16, 18]). For solving in closed form non linear difference equations and systems generally we use some change of variables that transformed nonlinear equations and systems in linear ones. The paper of Stevic [15] has considerably motivated this line of research.

The difference equation

$$x_{n+1} = a + \frac{b}{x_{n-1}} + \frac{c}{x_n x_{n-1}}$$

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was studied by Azizi in [2]. Noting that the same equation was the subject of a very recent paper by Stevic [17].

In [21] the authors studied the system

$$x_{n+1} = \frac{1+x_{n-1}}{y_n x_{n-1}},\ y_{n+1} = \frac{1+y_{n-1}}{x_n y_{n-1}},$$

Motivated by [21], Halim et al. in [8], got the form of the solutions of the following difference equation

$$x_{n+1} = \frac{a+bx_{n-1}}{x_n x_{n-1}},$$

and the system

$$x_{n+1} = \frac{a+bx_{n-1}}{y_n x_{n-1}},\ y_{n+1} = \frac{a+by_{n-1}}{x_n y_{n-1}},$$

Here and motivated by the above mentioned papers we are interested in the following system of difference equations

$$x_{n+1} = \frac{ay_n x_{n-1}+bx_{n-1}+c}{y_n x_{n-1}},\ y_{n+1} = \frac{ax_n y_{n-1}+by_{n-1}+c}{x_n y_{n-1}},\ n=0,1,\ldots, \quad (1.1)$$

where x_{-1}, x_0, y_{-1} and y_0 are arbitrary nonzero real numbers, a, b and c are arbitrary real numbers with $c \neq 0$. Clearly our system generalized the equations and systems studied in [2, 8, 17] and [21].

2. The homogenous third order linear difference equation with constant coefficients.

Consider the homogenous third order linear difference equation

$$R_{n+1} = aR_n + bR_{n-1} + cR_{n-2},\ n=0,1,\ldots, \quad (2.1)$$

where the initial values R_0, R_{-1} and R_{-2} and the constant coefficients a, b and c are real numbers with $c \neq 0$. This equation will be of great importance for our study, so we will solve it in closed form. As it is well known, the solution $(R_n)_{n=-2}^{+\infty}$ of equation (2.1) is usually expressed in terms of the roots α, β and γ of the characteristic equation

$$\lambda^3 - a\lambda^2 - b\lambda - c = 0. \quad (2.2)$$

Here we express the solutions of the equation (2.1) using terms of the sequence $(J_n)_{n=0}^{+\infty}$ defined by the recurrent relation

$$J_{n+3} = aJ_{n+2} + bJ_{n+1} + cJ_n, \quad n \in \mathbb{N}, \quad (2.3)$$

and the special initial values

$$J_0 = 0, \quad J_1 = 1 \text{ and } J_2 = a. \quad (2.4)$$

Noting that $(R_n)_{n=-2}^{+\infty}$ and $(J_n)_{n=0}^{+\infty}$ have the same characteristic equation. Also if $a=b=c=1$, then the equation (2.3) is nothing other then the famous Tribonacci sequence $(T_n)_{n=0}^{+\infty}$.

The closed form of the solutions of $\{J_n\}_{n=0}^{+\infty}$ and many proprieties of them are well known in the literature, for the interest of the readers and for the purpose of our work, we show how we can get the formula of the solutions and we give also a result on the limit

$$\lim_{n\to\infty}\frac{J_{n+1}}{J_n}.$$

For the roots α, β and γ of the characteristic equation (2.2), we have

$$\begin{cases}\alpha+\beta+\gamma=a\\ \alpha\beta+\alpha\gamma+\beta\gamma=-b\\ \alpha\beta\gamma=c.\end{cases}\tag{2.5}$$

We have:

Case 1: If all roots are equal. In this case

$$J_n=\left(c_1+c_2n+c_3n^2\right)\alpha^n.$$

Now using (2.5) and the fact that $J_0=0$, $J_1=1$ and $J_2=a$, we obtain

$$J_n=\left(\frac{n}{2\alpha}+\frac{n^2}{2\alpha}\right)\alpha^n.\tag{2.6}$$

Case 2: If two roots are equal, say $\beta=\gamma$. In this case

$$J_n=c_1\alpha^n+(c_2+c_3n)\beta^n.$$

Using (2.5) and the fact that $J_0=0$, $J_1=1$ and $J_2=a$, we obtain

$$J_n=\frac{\alpha}{(\beta-\alpha)^2}\alpha^n+\left(\frac{-\alpha}{(\beta-\alpha)^2}+\frac{n}{\beta-\alpha}\right)\beta^n.\tag{2.7}$$

Case 3: If the roots are all different. In this case

$$J_n=c_1\alpha^n+c_2\beta^n+c_3\gamma^n.$$

Again, using (2.5) and the fact that $J_0=0$, $J_1=1$ and $J_2=a$, we obtain

$$J_n=\frac{\alpha}{(\gamma-\alpha)(\beta-\alpha)}\alpha^n+\frac{-\beta}{(\gamma-\beta)(\beta-\alpha)}\beta^n+\frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\gamma^n.\tag{2.8}$$

In this case we can get two roots of (2.2) complex conjugates say $\gamma=\overline{\beta}$ and the third one real and the formula of J_n will be

$$J_n=\frac{\alpha}{(\overline{\beta}-\alpha)(\beta-\alpha)}\alpha^n+\frac{-\beta}{(\overline{\beta}-\beta)(\beta-\alpha)}\beta^n+\frac{\overline{\beta}}{(\overline{\beta}-\alpha)(\overline{\beta}-\beta)}\overline{\beta}^n.\tag{2.9}$$

Consider the following linear third order difference equation

$$S_{n+1}=-aS_n+bS_{n-1}-cS_{n-2},\ n=0,1,\ldots,\tag{2.10}$$

the constant coefficients a, b and c and the initial values S_0, S_{-1} and S_{-2} are real numbers. As for the equation (2.1), we will express the solutions of (2.10) using terms of (2.3). To do this let us consider the difference equation

$$j_{n+3} = -aj_{n+2} + bj_{n+1} - cj_n, \quad n \in \mathbb{N}, \tag{2.11}$$

and the special initial values

$$j_0 = 0, \quad j_1 = 1 \text{ and } j_2 = -a. \tag{2.12}$$

The characteristic equation of (2.10) and (2.11) is

$$\lambda^3 + a\lambda^2 - b\lambda + c = 0. \tag{2.13}$$

Clearly the roots of (2.13) are $-\alpha$, $-\beta$ and $-\gamma$. Now following the same procedure in solving $\{J(n)\}$, we get that

$$j(n) = (-1)^{n+1} J(n).$$

Lemma 1. *Let α, β and γ be the roots of (2.2), assume that α is a real root with* $\max(|\alpha|; |\beta|; |\gamma|) = |\alpha|$. *Then,*

$$\lim_{n\to\infty} \frac{J_{n+1}}{J_n} = \alpha.$$

Proof. If α, β and γ are real and distinct then,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{J_{n+1}}{J_n} \\
&= \lim_{n\to\infty} \frac{\dfrac{\alpha}{(\gamma-\alpha)(\beta-\alpha)}\alpha^{n+1} + \dfrac{-\beta}{(\gamma-\beta)(\beta-\alpha)}\beta^{n+1} + \dfrac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n+1}}{\dfrac{\alpha}{(\gamma-\alpha)(\beta-\alpha)}\alpha^{n} + \dfrac{-\beta}{(\gamma-\beta)(\beta-\alpha)}\beta^{n} + \dfrac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n}} \\
&= \lim_{n\to\infty} \frac{\alpha^{n+1}}{\alpha^n} \frac{\dfrac{\alpha}{(\gamma-\alpha)(\beta-\alpha)}\dfrac{\alpha^{n+1}}{\alpha^{n+1}} + \dfrac{-\beta}{(\gamma-\beta)(\beta-\alpha)}\dfrac{\beta^{n+1}}{\alpha^{n+1}} + \dfrac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\dfrac{\gamma^{n+1}}{\alpha^{n+1}}}{\dfrac{\alpha}{(\gamma-\alpha)(\beta-\alpha)}\dfrac{\alpha^{n}}{\alpha^{n}} + \dfrac{-\beta}{(\gamma-\beta)(\beta-\alpha)}\dfrac{\beta^{n}}{\alpha^{n}} + \dfrac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\dfrac{\gamma^{n}}{\alpha^{n}}} \\
&= \lim_{n\to\infty} \alpha \frac{\dfrac{\alpha}{(\gamma-\alpha)(\beta-\alpha)} + \dfrac{-\beta}{(\gamma-\beta)(\beta-\alpha)}\left(\dfrac{\beta}{\alpha}\right)^{n+1} + \dfrac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\left(\dfrac{\gamma}{\alpha}\right)^{n+1}}{\dfrac{\alpha}{(\gamma-\alpha)(\beta-\alpha)} + \dfrac{-\beta}{(\gamma-\beta)(\beta-\alpha)}\left(\dfrac{\beta}{\alpha}\right)^{n} + \dfrac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\left(\dfrac{\gamma}{\alpha}\right)^{n}} \\
&= \alpha.
\end{aligned}$$

The proof of the other cases of the roots, that is when $\alpha = \beta = \gamma$ or β, γ are complex conjugate, is similar to the first one and will be omitted. □

Remark 1. If α is a real root and β, γ are complex conjugate with

$$\max(|\alpha|; |\beta|; |\overline{\beta}|) = |\beta| = |\overline{\beta}|,$$

then $\lim_{n\to\infty} \frac{J_{n+1}}{J_n}$ doesn't exist.

In the following result, we solve in closed form the equations (2.1) and (2.10) in terms of the sequence $(J_n)_{n=0}^{+\infty}$. The obtained formula will be very useful to obtain the formula of the solutions of system (1.1).

Lemma 2. *We have for all $n \in \mathbb{N}_0$,*

$$R_n = cJ_nR_{-2} + (J_{n+2} - aJ_{n+1})R_{-1} + J_{n+1}R_0, \tag{2.14}$$

$$S_n = (-1)^n\left[cJ_nS_{-2} + (-J_{n+2} + aJ_{n+1})S_{-1} + J_{n+1}S_0\right]. \tag{2.15}$$

Proof. Assume that α, β and γ are the distinct roots of the characteristic equation (2.2), so

$$R_n = c_1'\alpha^n + c_2'\beta^n + c_3'\gamma^n, n = -2, -1, 0, \ldots.$$

Using the initial values R_0, R_{-1} and R_{-2}, we get

$$\begin{cases} \frac{1}{\alpha^2}c_1' + \frac{1}{\beta^2}c_2' + \frac{1}{\gamma^2}c_3' & = R_{-2} \\ \frac{1}{\alpha}c_1' + \frac{1}{\beta}c_2' + \frac{1}{\gamma}c_3' & = R_{-1} \\ c_1' + c_2' + c_3' & = R_0 \end{cases} \tag{2.16}$$

after some calculations we get

$$\begin{aligned} c_1' &= \frac{\alpha^2\beta\gamma}{(\gamma-\alpha)(\beta-\alpha)}R_{-2} - \frac{(\gamma+\beta)\alpha^2}{(\gamma-\alpha)(\beta-\alpha)}R_{-1} + \frac{\alpha^2}{(\gamma-\alpha)(\beta-\alpha)}R_0 \\ c_2' &= -\frac{\alpha\beta^2\gamma}{(\gamma-\beta)(\beta-\alpha)}R_{-2} + \frac{(\alpha+\gamma)\beta^2}{(\gamma-\beta)(\beta-\alpha)}R_{-1} - \frac{\beta^2}{(\gamma-\beta)(\beta-\alpha)}R_0 \\ c_3' &= \frac{\alpha\beta\gamma^2}{(\gamma-\alpha)(\gamma-\beta)}R_{-2} - \frac{(\alpha+\beta)\gamma^2}{(\gamma-\alpha)(\gamma-\beta)}R_{-1} + \frac{\gamma^2}{(\gamma-\alpha)(\gamma-\beta)}R_0 \end{aligned}$$

that is,

$$\begin{aligned} R_n = &\left(\frac{\alpha^2\beta\gamma}{(\gamma-\alpha)(\beta-\alpha)}\alpha^n - \frac{\alpha\beta^2\gamma}{(\gamma-\beta)(\beta-\alpha)}\beta^n + \frac{\alpha\beta\gamma^2}{(\gamma-\alpha)(\gamma-\beta)}\gamma^n\right)R_{-2} \\ &+ \left(-\frac{(\gamma+\beta)\alpha^2}{(\gamma-\alpha)(\beta-\alpha)}\alpha^n + \frac{(\alpha+\gamma)\beta^2}{(\gamma-\beta)(\beta-\alpha)}\beta^n - \frac{(\alpha+\beta)\gamma^2}{(\gamma-\alpha)(\gamma-\beta)}\gamma^n\right)R_{-1} \\ &+ \left(\frac{\alpha^2}{(\gamma-\alpha)(\beta-\alpha)}\alpha^n - \frac{\beta^2}{(\gamma-\beta)(\beta-\alpha)}\beta^n + \frac{\gamma^2}{(\gamma-\alpha)(\gamma-\beta)}\gamma^n\right)R_0 \end{aligned}$$

$$R_n = cJ_nR_{-2} + (J_{n+2} - aJ_{n+1})R_{-1} + J_{n+1}R_0.$$

The proof of the other cases is similar and will be omitted.

Let $A := -a$ and $B := b$, $C := -c$, then equation (2.10) takes the form of (2.1) and the equation (2.11) takes the form of (2.3). Then analogous to the formula of (2.1) we obtain

$$S_n = Cj_nS_{-2} + (j_{n+2} - Aj_{n+1})S_{-1} + j_{n+1}S_0.$$

Using the fact that $j(n) = (-1)^{n+1}J(n)$, $A = -a$ and $C := -c$ we get

$$S_n = (-1)^n\left(cJ_nS_{-2} - (J_{n+2} - aJ_{n+1})S_{-1} + J_{n+1}S_0\right).$$

□

3. Closed form of well defined solutions of system (1.1)

In this section, we solve through an analytical approach the system (1.1) with $c \neq 0$ in closed form. By a well defined solution of system (1.1), we mean a solution that satisfies $x_ny_n \neq 0$, $n = -1, 0, \cdots$. Clearly if we choose the initial values and the parameters a, b and c positif, then every solution of (1.1) will be well defined.

The following result give an explicit formula for well defined solutions of the system (1.1).

Theorem 1. *Let $\{x_n, y_n\}_{n\geq -1}$ be a well defined solution of (1.1). Then, for $n = 0, 1, \ldots,$ we have*

$$x_{2n+1} = \frac{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})x_{-1} + J_{2n+2}x_{-1}y_0}{cJ_{2n} + (J_{2n+2} - aJ_{2n+1})x_{-1} + J_{2n+1}x_{-1}y_0},$$

$$x_{2n+2} = \frac{cJ_{2n+2} + (J_{2n+4} - aJ_{2n+3})y_{-1} + J_{2n+3}x_0y_{-1}}{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})y_{-1} + J_{2n+2}x_0y_{-1}},$$

$$y_{2n+1} = \frac{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})y_{-1} + J_{2n+2}x_0y_{-1}}{cJ_{2n} + (J_{2n+2} - aJ_{2n+1})y_{-1} + J_{2n+1}x_0y_{-1}},$$

$$y_{2n+2} = \frac{cJ_{2n+2} + (J_{2n+4} - aJ_{2n+3})x_{-1} + J_{2n+3}x_{-1}y_0}{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})x_{-1} + J_{2n+2}x_{-1}y_0}$$

where the initial conditions x_{-1}, x_0, y_{-1} and $y_0 \in (\mathbb{R} - \{0\}) - F$, with F is the Forbidden set of system (1.1) given by

$$F = \bigcup_{n=0}^{\infty}\{(x_{-1}, x_0, y_{-1}, y_0) \in (\mathbb{R} - \{0\}) : A_n = 0 \, or \, B_n = 0\},$$

where

$$A_n = J_{n+1}y_0x_{-1} + (J_{n+2} - aJ_{n+1})x_{-1} + cJ_n,$$
$$B_n = J_{n+1}x_0y_{-1} + (J_{n+2} - aJ_{n+1})y_{-1} + cJ_n.$$

Proof. Putting

$$x_n = \frac{u_n}{v_{n-1}}, \quad y_n = \frac{v_n}{u_{n-1}}, n = -1, 0, 1, ..., \tag{3.1}$$

we get the following linear third order system of difference equations

$$u_{n+1} = av_n + bu_{n-1} + cv_{n-2}, \quad v_{n+1} = au_n + bv_{n-1} + cu_{n-2}, \quad n = 0, 1, ..., \tag{3.2}$$

where the initial values $u_{-2}, u_{-1}, u_0, v_{-2}, v_{-1}, v_0$ are nonzero real numbers. From(3.2) we have for $n = 0, 1, ...,$

$$\begin{cases} u_{n+1} + v_{n+1} = a(v_n + u_n) + b(u_{n-1} + v_{n-1}) + c(v_{n-2} + u_{n-2}), \\ u_{n+1} - v_{n+1} = a(v_n - u_n) + b(u_{n-1} - v_{n-1}) + c(v_{n-2} - u_{n-2}). \end{cases}$$

Putting again

$$R_n = u_n + v_n, \quad S_n = u_n - v_n, n = -2, -1, 0, ..., \tag{3.3}$$

we obtain two homogenous linear difference equations of third order:

$$R_{n+1} = aR_n + bR_{n-1} + cR_{n-2}, n = 0, 1, \cdots,$$

and

$$S_{n+1} = -aS_n + bS_{n-1} - cS_{n-2}, n = 0, 1, \cdots. \tag{3.4}$$

Using (3.3), we get for $n = -2, -1, 0, ...,$

$$u_n = \frac{1}{2}(R_n + S_n), \; v_n = \frac{1}{2}(R_n - S_n).$$

From Lemma 2 we obtain,

$$\begin{cases} u_{2n-1} = \frac{1}{2}[cJ_{2n-1}(R_{-2} - S_{-2}) + (J_{2n+1} - aJ_{2n})(R_{-1} + S_{-1}) + J_{2n}(R_0 - S_0)], n = 1, 2, \cdots, \\ u_{2n} = \frac{1}{2}[cJ_{2n}(R_{-2} + S_{-2}) + (J_{2n+2} - aJ_{2n+1})(R_{-1} - S_{-1}) + J_{2n+1}(R_0 + S_0)], n = 0, 1, \cdots, \end{cases} \tag{3.5}$$

$$\begin{cases} v_{2n-1} = \frac{1}{2}[cJ_{2n-1}(R_{-2} + S_{-2}) + (J_{2n+1} - aJ_{2n})(R_{-1} - S_{-1}) + J_{2n}(R_0 + S_0)], n = 1, 2, \cdots, \\ v_{2n} = \frac{1}{2}[cJ_{2n}(R_{-2} - S_{-2}) + (J_{2n+2} - aJ_{2n+1})(R_{-1} + S_{-1}) + J_{2n+1}(R_0 - S_0)], n = 0, 1, \cdots, \end{cases} \tag{3.6}$$

Substituting (3.5) and (3.6) in (3.1), we get for $n = 0, 1, ...,$

$$\begin{cases} x_{2n+1} = \dfrac{cJ_{2n+1}(R_{-2} - S_{-2}) + (J_{2n+3} - aJ_{2n+2})(R_{-1} + S_{-1}) + J_{2n+2}(R_0 - S_0)}{cJ_{2n}(R_{-2} - S_{-2}) + (J_{2n+2} - aJ_{2n+1})(R_{-1} + S_{-1}) + J_{2n+1}(R_0 - S_0)}, \\ x_{2n+2} = \dfrac{cJ_{2n+2}(R_{-2} + S_{-2}) + (J_{2n+4} - aJ_{2n+3})(R_{-1} - S_{-1}) + J_{2n+3}(R_0 + S_0)}{cJ_{2n+1}(R_{-2} + S_{-2}) + (J_{2n+3} - aJ_{2n+2})(R_{-1} - S_{-1}) + J_{2n+2}(R_0 + S_0)}, \end{cases} \tag{3.7}$$

$$\begin{cases} y_{2n+1} = \dfrac{cJ_{2n+1}(R_{-2} + S_{-2}) + (J_{2n+3} - aJ_{2n+2})(R_{-1} - S_{-1}) + J_{2n+2}(R_0 + S_0)}{cJ_{2n}(R_{-2} + S_{-2}) + (J_{2n+2} - aJ_{2n+1})(R_{-1} - S_{-1}) + J_{2n+1}(R_0 + S_0)}, \\ y_{2n+2} = \dfrac{cJ_{2n+2}(R_{-2} - S_{-2}) + (J_{2n+4} - aJ_{2n+3})(R_{-1} + S_{-1}) + J_{2n+3}(R_0 - S_0)}{cJ_{2n+1}(R_{-2} - S_{-2}) + (J_{2n+3} - aJ_{2n+2})(R_{-1} + S_{-1}) + J_{2n+2}(R_0 - S_0)}. \end{cases} \tag{3.8}$$

Then,

$$\begin{cases} x_{2n+1} = \dfrac{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})\dfrac{R_{-1}+S_{-1}}{R_{-2}-S_{-2}} + J_{2n+2}\dfrac{R_0-S_0}{R_{-2}-S_{-2}}}{cJ_{2n} + (J_{2n+2} - aJ_{2n+1})\dfrac{R_{-1}+S_{-1}}{R_{-2}-S_{-2}} + J_{2n+1}\dfrac{R_0-S_0}{R_{-2}-S_{-2}}}, \\ x_{2n+2} = \dfrac{cJ_{2n+2} + (J_{2n+4} - aJ_{2n+3})\dfrac{R_{-1}-S_{-1}}{R_{-2}+S_{-2}} + J_{2n+3}\dfrac{R_0+S_0}{R_{-2}+S_{-2}}}{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})\dfrac{R_{-1}-S_{-1}}{R_{-2}+S_{-2}} + J_{2n+2}\dfrac{R_0+S_0}{R_{-2}+S_{-2}}}, \end{cases} \tag{3.9}$$

$$\begin{cases} y_{2n+1} = \dfrac{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})\dfrac{R_{-1}-S_{-1}}{R_{-2}+S_{-2}} + J_{2n+2}\dfrac{R_0+S_0}{R_{-2}+S_{-2}}}{cJ_{2n} + (J_{2n+2} - aJ_{2n+1})\dfrac{R_{-1}-S_{-1}}{R_{-2}+S_{-2}} + J_{2n+1}\dfrac{R_0+S_0}{R_{-2}+S_{-2}}}, \\ y_{2n+2} = \dfrac{cJ_{2n+2} + (J_{2n+4} - aJ_{2n+3})\dfrac{R_{-1}+S_{-1}}{R_{-2}-S_{-2}} + J_{2n+3}\dfrac{R_0-S_0}{R_{-2}-S_{-2}}}{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})\dfrac{R_{-1}+S_{-1}}{R_{-2}-S_{-2}} + J_{2n+2}\dfrac{R_0-S_0}{R_{-2}-S_{-2}}}. \end{cases} \tag{3.10}$$

We have

$$x_{-1} = \frac{u_{-1}}{v_{-2}} = \frac{R_{-1}+S_{-1}}{R_{-2}-S_{-2}},\ x_0 = \frac{u_0}{v_{-1}} = \frac{R_0+S_0}{R_{-1}-S_{-1}}, \tag{3.11}$$

$$y_{-1} = \frac{v_{-1}}{u_{-2}} = \frac{R_{-1}-S_{-1}}{R_{-2}+S_{-2}},\ y_0 = \frac{v_0}{u_{-1}} = \frac{R_0-S_0}{R_{-1}+S_{-1}} \tag{3.12}$$

From (3.11), (3.12) it follows that,

$$\begin{cases} \dfrac{R_0-S_0}{R_{-2}-S_{-2}} = \dfrac{R_{-1}+S_{-1}}{R_{-2}-S_{-2}} \times \dfrac{R_0-S_0}{R_{-1}+S_{-1}} = x_{-1}y_0 \\ \dfrac{R_0+S_0}{R_{-2}+S_{-2}} = \dfrac{R_0+S_0}{R_{-1}-S_{-1}} \times \dfrac{R_{-1}-S_{-1}}{R_{-2}+S_{-2}} = x_0y_{-1} \end{cases} \tag{3.13}$$

Using (3.9), (3.10), (3.11), (3.12) and (3.13), we obtain the closed form of the solutions of (1.1), that is for $n = 0, 1, \ldots$, we have

$$\begin{cases} x_{2n+1} = \dfrac{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})x_{-1} + J_{2n+2}x_{-1}y_0}{cJ_{2n} + (J_{2n+2} - aJ_{2n+1})x_{-1} + J_{2n+1}x_{-1}y_0}, \\ x_{2n+2} = \dfrac{cJ_{2n+2} + (J_{2n+4} - aJ_{2n+3})y_{-1} + J_{2n+3}x_0y_{-1}}{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})y_{-1} + J_{2n+2}x_0y_{-1}}, \end{cases}$$

$$\begin{cases} y_{2n+1} = \dfrac{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})y_{-1} + J_{2n+2}x_0y_{-1}}{cJ_{2n} + (J_{2n+2} - aJ_{2n+1})y_{-1} + J_{2n+1}x_0y_{-1}}, \\ y_{2n+2} = \dfrac{cJ_{2n+2} + (J_{2n+4} - aJ_{2n+3})x_{-1} + J_{2n+3}x_{-1}y_0}{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})x_{-1} + J_{2n+2}x_{-1}y_0}. \end{cases}$$

□

Remark 2. Writing system (1.1) in the form

$$\begin{cases} x_{n+1} = & f(x_n, x_{n-1}, y_n, y_{n-1}) = \frac{ay_nx_{n-1}+bx_{n-1}+c}{y_nx_{n-1}}, \\ y_{n+1} = & g(x_n, x_{n-1}, y_n, y_{n-1}) = \frac{ax_ny_{n-1}+by_{n-1}+c}{x_ny_{n-1}}. \end{cases}$$

So it follows that points (α,α), (β,β) and (γ,γ) are solutions of the of system

$$\begin{cases} \bar{x} = \dfrac{a\bar{y}\bar{x} + b\bar{x} + c}{\bar{y}\bar{x}}, \\ \bar{y} = \dfrac{a\bar{x}\bar{y} + b\bar{y} + c}{\bar{x}\bar{y}} \end{cases}$$

where α, β and γ are the roots of (2.2).

Theorem 2. *Under the same conditions in Lemma 1, for every well defined solution of system* (1.1)*, we have*

$$\lim_{n\to+\infty} x_{2n+1} = \lim_{n\to+\infty} x_{2n+2} = \lim_{n\to+\infty} y_{2n+1} = \lim_{n\to+\infty} y_{2n+2} = \alpha.$$

Proof. We have

$$\begin{aligned} \lim_{n\to\infty} x_{2n+1} &= \lim_{n\to\infty} \frac{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})x_{-1} + J_{2n+2}y_0x_{-1}}{cJ_{2n} + (J_{2n+2} - aJ_{2n+1})x_{-1} + J_{2n+1}y_0x_{-1}} \\ &= \lim_{n\to\infty} \frac{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})x_{-1} + J_{2n+2}y_0x_{-1}}{cJ_{2n} + (J_{2n+2} - aJ_{2n+1})x_{-1} + J_{2n+1}y_0x_{-1}} \\ &= \lim_{n\to\infty} \frac{c\frac{J_{2n+1}}{J_{2n}} + \left(\frac{J_{2n+3}}{J_{2n+2}} \times \frac{J_{2n+2}}{J_{2n+1}} \times \frac{J_{2n+1}}{J_{2n}} - a\frac{J_{2n+2}}{J_{2n+1}} \times \frac{J_{2n+1}}{J_{2n}}\right)x_{-1} + \frac{J_{2n+2}}{J_{2n+1}} \times \frac{J_{2n+1}}{J_{2n}} y_0x_{-1}}{c\frac{J_{2n}}{J_{2n}} + \left(\frac{J_{2n+2}}{J_{2n+1}} \times \frac{J_{2n+1}}{J_{2n}} - a\frac{J_{2n+1}}{J_{2n}}\right)x_{-1} + \frac{J_{2n+1}}{J_{2n}} y_0x_{-1}} \\ &= \frac{c\alpha + (\alpha^3 - a\alpha^2)x_{-1} + \alpha^2y_0x_{-1}}{c + (\alpha^2 - a\alpha)x_{-1} + \alpha y_0x_{-1}} \\ &= \alpha\,. \end{aligned}$$

In the same way we show that

$$\lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} y_{2n+1} = \alpha.$$

□

4. Particular cases

Here we are interested in some particular cases of system (1.1). Some of these particular cases were been the subject of some recent papers.

4.1. *The solutions of the equation* $x_{n+1} = \dfrac{ax_nx_{n-1} + bx_{n-1} + c}{x_nx_{n-1}}$

If we choose $y_{-1} = x_{-1}$ and $y_0 = x_0$, then system (1.1) is reduced to the equation

$$x_{n+1} = \frac{ax_nx_{n-1} + bx_{n-1} + c}{x_nx_{n-1}}, \quad n \in \mathbb{N}_0. \tag{4.1}$$

The following results are respectively direct consequences of Theorem 1 and Theorem 2.

Corollary 1. *Let* $\{x_n\}_{n\geq -1}$ *be a well defined solution of the equation* (4.1). *Then for* $n = 0, 1, \dots,$ *we have*

$$x_{2n+1} = \frac{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})x_{-1} + J_{2n+2}x_{-1}x_0}{cJ_{2n} + (J_{2n+2} - aJ_{2n+1})x_{-1} + J_{2n+1}x_{-1}x_0},$$

$$x_{2n+2} = \frac{cJ_{2n+2} + (J_{2n+4} - aJ_{2n+3})x_{-1} + J_{2n+3}x_0x_{-1}}{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})x_{-1} + J_{2n+2}x_0x_{-1}}.$$

Corollary 2. *Under the same conditions in Lemma 1, for every well defined solution of equation* (4.1), *we have*

$$\lim_{n\to+\infty} x_{2n+1} = \lim_{n\to+\infty} x_{2n+2} = \alpha.$$

The equation (4.1) was been studied by Azizi in [2] and Stevic in [17].

4.2. *The solutions of the system*
$x_{n+1} = \dfrac{y_nx_{n-1} + x_{n-1} + 1}{y_nx_{n-1}}$, $y_{n+1} = \dfrac{x_ny_{n-1} + y_{n-1} + 1}{x_ny_{n-1}}$

Consider the system

$$x_{n+1} = \frac{y_nx_{n-1} + x_{n-1} + 1}{y_nx_{n-1}}, \quad y_{n+1} = \frac{x_ny_{n-1} + y_{n-1} + 1}{x_ny_{n-1}} \quad n \in \mathbb{N}_0. \tag{4.2}$$

Clearly the system (4.2) is particular of the system (1.1) with $a = b = c = 1$. In this case the sequence $\{J_n\}$ is the famous classical sequence of Tribonacci numbers $\{T_n\}$, that is

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n, \quad n \in \mathbb{N}, \text{ where } T_0 = 0,\ T_1 = 1 \text{ and } T_2 = 1,$$

and we have

$$T_n = \frac{\alpha^{n+1}}{(\beta-\alpha)(\gamma-\alpha)} - \frac{\beta^{n+1}}{(\beta-\alpha)(\gamma-\beta)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \qquad n = 0, 1, \dots,$$

with

$$\alpha = \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3},$$

$$\beta = \frac{1+\omega\sqrt[3]{19+3\sqrt{33}}+\omega^2\sqrt[3]{19-3\sqrt{33}}}{3},$$

$$\gamma = \frac{1+\omega^2\sqrt[3]{19+3\sqrt{33}}+\omega\sqrt[3]{19-3\sqrt{33}}}{3}, \omega = \frac{-1+i\sqrt{3}}{2}.$$

Numerically we have $\alpha = 1.839286755$ and the two complex conjugate are

$$-0.4196433777+0.6062907300i, -0.4196433777-0.6062907300i$$

with $i^2 = -1$.

The following results follows respectively from Theorem 1 and Theorem 2.

Corollary 3. *Let $\{x_n, y_n\}_{n\geq -1}$ be a well defined solution of (4.2). Then, for $n = 0,1,2,3,\ldots$, we have*

$$x_{2n+1} = \frac{cT_{2n+1}+(T_{2n+3}-aT_{2n+2})x_{-1}+T_{2n+2}x_{-1}y_0}{cT_{2n}+(T_{2n+2}-aT_{2n+1})x_{-1}+T_{2n+1}x_{-1}y_0},$$

$$x_{2n+2} = \frac{cT_{2n+2}+(T_{2n+4}-aT_{2n+3})y_{-1}+T_{2n+3}x_0y_{-1}}{cT_{2n+1}+(T_{2n+3}-aT_{2n+2})y_{-1}+T_{2n+2}x_0y_{-1}},$$

$$y_{2n+1} = \frac{cT_{2n+1}+(T_{2n+3}-aT_{2n+2})y_{-1}+T_{2n+2}x_0y_{-1}}{cT_{2n}+(T_{2n+2}-aT_{2n+1})y_{-1}+T_{2n+1}x_0y_{-1}},$$

$$y_{2n+2} = \frac{cT_{2n+2}+(T_{2n+4}-aT_{2n+3})x_{-1}+T_{2n+3}x_{-1}y_0}{cT_{2n+1}+(T_{2n+3}-aT_{2n+2})x_{-1}+T_{2n+2}x_{-1}y_0}$$

Corollary 4. *For every well defined solution of system (1.1), we have*

$$\lim_{n\to+\infty} x_{2n+1} = \lim_{n\to+\infty} x_{2n+2} = \lim_{n\to+\infty} y_{2n+1} = \lim_{n\to+\infty} y_{2n+2} = \alpha.$$

For the equation

$$x_{n+1} = \frac{x_nx_{n-1}+x_{n-1}+1}{x_nx_{n-1}}, \quad n \in \mathbb{N}_0. \tag{4.3}$$

we have the following results.

Corollary 5. *Let $\{x_n\}_{n\geq -1}$ be a well defined solution of the equation (4.3). Then for $n = 0,1,\ldots$, we have*

$$x_{2n+1} = \frac{T_{2n+1}+(T_{2n+3}-T_{2n+2})x_{-1}+T_{2n+2}x_{-1}x_0}{T_{2n}+(T_{2n+2}-T_{2n+1})x_{-1}+T_{2n+1}x_{-1}x_0},$$

$$x_{2n+2} = \frac{T_{2n+2}+(T_{2n+4}-T_{2n+3})x_{-1}+T_{2n+3}x_0x_{-1}}{T_{2n+1}+(T_{2n+3}-T_{2n+2})x_{-1}+T_{2n+2}x_0x_{-1}}.$$

Corollary 6. *Under the same conditions in Lemma 1, for every well defined solution of the equation (4.3), we have*

$$\lim_{n\to+\infty} x_{2n+1} = \lim_{n\to+\infty} x_{2n+2} = \alpha.$$

Let $I = (0,+\infty)$, $J = (0,+\infty)$ and choosing x_{-1}, x_0, y_{-1} and $y_0 \in (0,+\infty)$. Then clearly the system

$$\overline{x} = f(\overline{x},\overline{y}) = \frac{\overline{xy}+\overline{x}+1}{\overline{xy}},\ \overline{y} = g(\overline{x},\overline{y}) = \frac{\overline{xy}+\overline{y}+1}{\overline{xy}}$$

has a unique solution $(\alpha,\alpha) \in I \times J$, that is (α,α) is the unique equilibrium point (fixed point) of our system

$$x_{n+1} = f(x_n, x_{n-1}, y_n, y_{n-1}) = \frac{y_n x_{n-1} + x_{n-1} + 1}{y_n x_{n-1}},$$

$$y_{n+1} = g(x_n, x_{n-1}, y_n, y_{n-1}) = \frac{x_n y_{n-1} + y_{n-1} + 1}{x_n y_{n-1}}.$$

Clearly the functions

$$f : I^2 \times J^2 \longrightarrow I \qquad \text{and} \qquad g : I^2 \times J^2 \longrightarrow I$$

defined by

$$f(u_0;u_1;v_0;v_1) = \frac{v_0u_1+u_1+1}{v_0u_1} \qquad \text{and} \qquad g(u_0;u_1;v_0;v_1) = \frac{u_0v_1+v_1+1}{u_0v_1}$$

are continuously differentiable.

In the following result we prove that the unique equilibrium point (α,α) of (4.2) is locally asymptotically stable.

Theorem 3. *The equilibrium point (α,α) is locally asymptotically stable.*

Proof. The Jacobian matrix associated to the system (4.2) around the equilibrium point (α,α), is given by

$$A = \begin{pmatrix} 0 & -\frac{1}{\alpha^3} & -\frac{\alpha+1}{\alpha^3} & 0 & \\ 1 & 0 & 0 & 0 & 0 \\ -\frac{\alpha+1}{\alpha^3} & 0 & 0 & -\frac{1}{\alpha^3} & \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then, the characteristic polynomial of A is

$$P(\lambda) = \lambda^4 + \frac{(2\alpha^3-\alpha^2-2\alpha-1)}{\alpha^6}\lambda^2 + \frac{1}{\alpha^6}$$

and the roots of $P(\lambda)$ are

$$\lambda_1 = \frac{1}{2} \times \frac{1+\alpha+\sqrt{-4\alpha^3+\alpha^2+2\alpha+1}}{\alpha^3},$$

$$\lambda_2 = -\frac{1}{2} \times \frac{1+\alpha+\sqrt{-4\alpha^3+\alpha^2+2\alpha+1}}{\alpha^3},$$

$$\lambda_3 = \frac{1}{2} \times \frac{-1-\alpha+\sqrt{-4\alpha^3+\alpha^2+2\alpha+1}}{\alpha^3},$$

$$\lambda_4 = -\frac{1}{2} \times \frac{-1-\alpha+\sqrt{-4\alpha^3+\alpha^2+2\alpha+1}}{\alpha^3}.$$

We have $|\lambda_i| < 1$, $i = 1,2,3,4$, so the equilibrium point (α,α) is locally asymptotically stable. □

The following result is a direct consequence of Theorem 3 and Corollary 4.

Theorem 4. *The equilibrium point (α,α) is globally asymptotically stable .*

Let $I = (0,+\infty)$ and choosing x_{-1}, $x_0 \in (0,+\infty)$. Writing the equation (4.3) as

$$x_{n+1} = h(x_n, x_{n-1}) = \frac{x_n x_{n-1} + x_{n-1} + 1}{x_n x_{n-1}} \tag{4.4}$$

where

$$h : I^2 \longrightarrow I$$

is defined by

$$h(u_0;u_1) = \frac{u_0 u_1 + u_1 + 1}{u_0 u_1}.$$

The function h is continuously differentiable. The equation $\overline{x} = h(\overline{x},\overline{x})$ has the unique solution $\overline{x} = \alpha$ in $(0,+\infty)$. The linear equation associated to the equation (4.4) about the equilibrium point $\overline{x} = \alpha$ is given by

$$y_{n+1} = \frac{\partial h}{\partial u_0}(\alpha,\alpha)\, y_n + \frac{\partial h}{\partial u_1}(\alpha,\alpha)\, y_{n-1},$$

the last equation has as characteristic polynomial

$$Q(\lambda) = \lambda^2 - \frac{\partial h}{\partial u_0}(\alpha,\alpha)\,\lambda - \frac{\partial h}{\partial u_1}(\alpha,\alpha).$$

In the following result we show that the unique equilibrium point $\overline{x} = \alpha$ is globally stable.

Theorem 5. *The equilibrium point $\overline{x} = \alpha$ is globally stable.*

Proof. The linear equation associated to (4.3) about the equilibrium point $\overline{x} = \alpha$ is

$$y_{n+1} = -\frac{\alpha+1}{\alpha^3} y_n - \frac{1}{\alpha^3} y_{n-1}$$

and the characteristic polynomial is

$$Q(\lambda) = \lambda^2 + \left(\frac{\alpha+1}{\alpha^3}\right)\lambda + \left(\frac{1}{\alpha^3}\right).$$

We have

$$\left|\frac{\alpha+1}{\alpha^3}\lambda + \frac{1}{\alpha^3}\right| \le \left|\frac{\alpha+1}{\alpha^3}\right| + \left|\frac{1}{\alpha^3}\right| < 1 = \left|\lambda^2\right|, \forall \lambda \in \mathbb{C} : |\lambda| = 1.$$

So, by Rouché's theorem the roots of the characteristic polynomial $Q(\lambda)$ lie in the open unit disk. Then the equilibrium point $\overline{x} = \alpha$ is locally asymptotically stable. Now, from this and Corollary 6 the result holds. □

4.3. *The system* $x_{n+1} = \dfrac{bx_{n-1}+c}{y_n x_{n-1}}$, $y_{n+1} = \dfrac{by_{n-1}+c}{x_n y_{n-1}}$

When $a = 0$, the system (1.1) takes the form

$$x_{n+1} = \frac{bx_{n-1}+c}{y_n x_{n-1}}, \; y_{n+1} = \frac{by_{n-1}+c}{x_n y_{n-1}} \quad n \in \mathbb{N}_0. \tag{4.5}$$

From Theorem (1), we get the following result.

Corollary 7. *Let* $\{x_n, y_n\}_{n\ge -1}$ *be a well defined solution of* (4.5). *Then, for* $n = 0, 1, \ldots,$ *we have*

$$x_{2n+1} = \frac{cP_{2n+1} + P_{2n+3}x_{-1} + P_{2n+2}x_{-1}y_0}{cP_{2n} + P_{2n+2}x_{-1} + P_{2n+1}x_{-1}y_0},$$

$$x_{2n+2} = \frac{cP_{2n+2} + P_{2n+4}y_{-1} + P_{2n+3}x_0y_{-1}}{cP_{2n+1} + P_{2n+3}y_{-1} + P_{2n+2}x_0y_{-1}},$$

$$y_{2n+1} = \frac{cP_{2n+1} + P_{2n+3}y_{-1} + P_{2n+2}x_0y_{-1}}{cP_{2n} + P_{2n+2}y_{-1} + P_{2n+1}x_0y_{-1}},$$

$$y_{2n+2} = \frac{cP_{2n+2} + P_{2n+4}x_{-1} + P_{2n+3}x_{-1}y_0}{cP_{2n+1} + P_{2n+3}x_{-1} + P_{2n+2}x_{-1}y_0}.$$

Here we have write $\{P_n\}_n$ instead of $\{J_n\}_n$, as in this case $\{J_n\}_n$ takes the form of a generalized (Padovan) sequence, that is

$$P_{n+3} = bP_{n+1} + cP_n, \quad n \in \mathbb{N},$$

with special values $P_0 = 0$, $P_1 = 1$ and $P_2 = 0$. The system (4.5) was been investigated by Halim et al. in [8] and by Yazlik et al. in [21] with $b = 1$ and $c = \pm 1$. The one dimensional version of system (4.5), that is the equation

$$x_{n+1} = \frac{bx_{n-1} + c}{x_n x_{n-1}}, \quad n \in \mathbb{N}_0. \tag{4.6}$$

was been also investigated by Halim et al. in [8]. Form Corollary (7), we get that the well defined solutions of equation (4.6) are given for $n = 0, 1, ...,$ by

$$x_{2n+1} = \frac{cP_{2n+1} + P_{2n+3}x_{-1} + P_{2n+2}x_{-1}x_0}{cP_{2n} + P_{2n+2}x_{-1} + P_{2n+1}x_{-1}x_0},$$

$$x_{2n+2} = \frac{cP_{2n+2} + P_{2n+4}x_{-1} + P_{2n+3}x_0x_{-1}}{cP_{2n+1} + P_{2n+3}x_{-1} + P_{2n+2}x_0x_{-1}}.$$

In [21] and [8] we can find additional results on the stability of some equilibrium points.

Remark 3. If $c = 0$, The system (1.1) become

$$x_{n+1} = \frac{ay_n + b}{y_n}, \; y_{n+1} = \frac{ax_n + b}{x_n}, n \in \mathbb{N}_0. \tag{4.7}$$

We note that if also $b = 0$, then the solutions of the system (4.7) are given by

$$\{(x_0, y_0), (a, a), (a, a), ..., \}.$$

The system (4.7) is a particular case of the more general system

$$x_{n+1} = \frac{ay_n + b}{cy_n + d}, \; y_{n+1} = \frac{\alpha x_n + \beta}{\gamma x_n + \lambda}, \; n \in \mathbb{N}_0 \tag{4.8}$$

which was been completely solved by Stevic in [16]. So, we refer to this paper for the readers interested in the form of the solutions of the system (4.8) and its particular case system (4.7). As it was proved in [16], the solutions are expressed using the terms of a corresponding generalized Fibonacci sequence. Noting that the papers [11], [18] and [19] deals also with particular cases of the system (4.8) or its one dimensional version.

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Authors' addresses

Y. Akrour

Youssouf Akrour, École Normale Supérieure de Constantine, Département des Sciences Exactes et Informatiques and LMAM Laboratory, University of Mohamed Seddik Ben Yahia, Jijel, Algeria.

E-mail address: youssouf.akrour@gmail.com

N. Touafek

Nouressadat Touafek, University of Mohamed Seddik Ben Yahia, LMAM Laboratory and Department of Mathematics, Jijel, Algeria.

E-mail address: ntouafek@gmail.com

Y. Halim

Yacine Halim, University Center of Mila, Department of Mathematics and Computer Science and LMAM Laboratory, University of Mohamed Seddik Ben Yahia, Jijel, Algeria.

E-mail address: halyacine@yahoo.fr

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FIXED POINT OF CONTINUOUS MAPPINGS DEFINED ON AN ARBITRARY INTERVAL

OSMAN ALAGOZ, BIROL GUNDUZ, AND SEZGIN AKBULUT

Dedicated to Birol Gunduz who is one of the co-authors and passed away on the 3rd of April, 2019.

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Abstract. In this work, we consider an iterative method given by Karaca and Yildirim [6] to approximate fixed point of continuous mappings defined on an arbitrary interval. Then, we give a necessary and sufficient condition for convergence theorem. We also compare the rate of convergence between the other iteration methods. Finally, we provide a numerical example which supports our theoretical results. Our findings improve corresponding results in the contemporary literature.

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1. Introduction and preliminaries

Let E be a closed interval which is a subset of reel line and let $f : E \to E$ be a continuous function. Any $p \in E$ is called a fixed point of f if $f(p) = p$. The set of all fixed points of f is denoted by $F(f)$. It is known that any linear or non-linear equation $f(x) = 0$ can be turned into a fixed point problem such as:

$$g(x) = x \tag{1.1}$$

where $g : E \to E$ is a contraction. In order to approximate fixed points of (1.1), Picard iteration can be applied. If g does not satisfy the contractive condition, then some other iteration methods can be applied to obtain a solution of (1.1). Thus, a question appears. Which iteration method should be used for approximating the fixed points of (1.1)?

In 1953, W.R. Mann defined an iteration called Mann iteration [7] to approximate fixed point of a non-linear mapping as follows: a sequence $\{x_n\}$ defined by $x_1 \in E$ and

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n f(x_n) \tag{1.2}$$

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for all $n \geq 1$, where $\{\alpha_n\}$ is a sequence in $[0,1]$. In 1991, D. Borwein and J. Borwein [2] proved a convergence theorem for a continuous function by using the iteration (1.2).

In 1974, Ishikawa [4] introduced an iteration process as follows: a sequence $\{x_n\}$ defined by $x_1 \in E$ and

$$\begin{cases} y_n &= (1-\beta_n)x_n + \beta_n f(x_n) \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n f(y_n) \end{cases} \tag{1.3}$$

for all $n \geq 1$ where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. In 2006, Qing and Qihou [11] proved a convergence theorem of the sequence generated by the iteration (1.3) for continuous function.

In 2000, Noor [8] defined the following iterative scheme by $x_1 \in E$ and

$$\begin{cases} z_n &= (1-\gamma_n)x_n + \gamma_n f(x_n) \\ y_n &= (1-\beta_n)x_n + \beta_n f(z_n) \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n f(y_n) \end{cases} \tag{1.4}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$. Clearly the Mann and the Ishikawa iterations are special cases of the Noor iteration.

Recently, Phuengrattana and Suantai [9] introduced SP-iteration as follows: $x_1 \in E$ and

$$\begin{cases} z_n &= (1-\gamma_n)x_n + \gamma_n f(x_n) \\ y_n &= (1-\beta_n)z_n + \beta_n f(z_n) \\ x_{n+1} &= (1-\alpha_n)y_n + \alpha_n f(y_n) \end{cases} \tag{1.5}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$. They proved a convergence theorem of the iteration (1.5) for continuous functions defined on an arbitrary interval in the real line.

In 2013, Kadioglu and Yildirim [5] defined the following iteration process: $x_1 \in E$ and

$$\begin{cases} z_n &= (1-a_n)x_n + a_n f(x_n) \\ y_n &= (1-b_n-c_n)x_n + b_n f(z_n) + c_n f(x_n) \\ x_{n+1} &= (1-\alpha_n-\beta_n)x_n + \alpha_n f(y_n) + \beta_n f(z_n) \end{cases} \tag{1.6}$$

for all $n \geq 1$, where $\{\alpha_n\}$,$\{\beta_n\}$,$\{a_n\}$,$\{b_n\}$ and $\{c_n\}$ are sequences in $[0,1]$. They also showed that the iteration (1.6) converges to a fixed point of f. Moreover, they showed that the iteration of (1.6) is better than Mann, Ishikawa and Noor iteration processes in the sense of Rhoades [12]. In 2006, Cholamjiak and Baiya [3] proposed a new three-step iteration process for solving a fixed point problem for continuous

functions on an arbitrary interval in the real line as follows: $p_1 \in E$ and

$$\begin{cases} r_n &= (1-a_n)p_n + a_n f(p_n) \\ q_n &= (1-b_n-c_n)r_n + b_n f(r_n) + c_n f(p_n) \\ p_{n+1} &= (1-\alpha_n-\beta_n)q_n + \alpha_n f(q_n) + \beta_n f(r_n) \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$,$\{\beta_n\}$,$\{a_n\}$,$\{b_n\}$ and $\{c_n\}$ are sequences in $[0,1)$ with $0 \le b_n + c_n < 1$ and $0 \le \alpha_n + \beta_n < 1$. They also showed that the iteration (1.7) is better than the iteration (1.6) in the sense of Rhoades [12].

On the other hand in 2014, Karaca and Yildirim [6] defined a new iteration as fallows: Let $u_1 \in E$ and the sequence $\{u_n\}$ be defined by

$$\begin{cases} w_n &= (1-a_n)u_n + a_n f(u_n) \\ v_n &= (1-b_n)w_n + b_n f(w_n) \\ u_{n+1} &= f(v_n) \end{cases} \tag{1.8}$$

where $\{a_n\}$ and $\{b_n\}$ are sequence in $(0,1)$. They showed that the iteration (1.8) process is faster than all of Picard [10], Mann [7] , Ishikawa [4] and Agarwal et al. [1] processes. They also proved a convergence theorem for nonexpansive mappings in Banach spaces.

In this paper we prove some convergence theorems by using the (1.8) iteration for continuous function defined on an arbitrary closed interval in the real line. Secondly, we compare the rate of convergence of (1.7) iteration and (1.8) iteration in the sense of Rhoades [12].

2. Convergence theorems

In this section, we propose convergence theorem for the iteration process defined by (1.8) for continuous functions on an arbitrary interval.

Theorem 1. *Let E be a closed interval in the real line and let $f : E \to E$ be a continuous mapping. For $x_1 \in E$, let the iteration $\{u_n\}$ be defined by (1.8), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$ and $\lim_{n\to\infty} |f(v_n) - v_n| = 0$. Then $\{u_n\}$ is bounded if and only if $\{u_n\}$ converges to a fixed point of f.*

Proof. If $\{u_n\}$ converges to a fixed point of f then it is obvious that $\{u_n\}$ is bounded. Now, assume that $\{u_n\}$ is bounded. Our goal is to show that $\{u_n\}$ is convergent. Assume to get a contradiction that it is not. Then there exist $a, b \in \mathbb{R}$, $a = \liminf_{n\to\infty} u_n$,$b = \limsup_{n\to\infty} u_n$ and $a < b$. First, we show that if $a < m < b$, then $f(m) = m$. Suppose that $f(m) \neq m$ Then, without loss of generality, we suppose that $f(m) - m > 0$. Since f is a continuous mapping, there exists $\delta \in (0, b-a)$ such that

$$f(x) - x > 0 \quad \text{for} \quad |x - m| \le \delta. \tag{2.1}$$

By the hypothesis of $\{u_n\}$, we have $\{u_n\}$ belongs to a bounded closed interval. The continuity of f implies that $\{f(u_n)\}$ belongs to another bounded closed interval. So, $\{f(u_n)\}$ is bounded and since

$$w_n = (1-\beta_n)u_n + \beta_n f(u_n)$$

we get that $\{w_n\}$ is bounded and so, $\{f(w_n)\}$ is bounded. Similarly, since

$$v_n = (1-\alpha_n)w_n + \alpha_n f(w_n),$$

we have $\{v_n\}$ and $\{f(v_n)\}$ are bounded. It follows by (1.8) that

$$\begin{aligned} u_{n+1}-u_n &= f(v_n)-v_n+\alpha_n(f(w_n)-w_n)+\beta_n(f(u_n)-u_n),\\ v_n-u_n &= \beta_n(f(u_n)-u_n)+\alpha_n(f(w_n)-w_n),\\ w_n-u_n &= \beta_n(f(u_n)-u_n). \end{aligned}$$

By the assumption of the theorem (1), we get $|u_{n+1}-u_n| \to 0, |v_n-u_n| \to 0$ and $|w_n-u_n| \to 0$. Thus, there exists N such that

$$|u_{n+1}-u_n| < \frac{\delta}{3}, \quad |v_n-u_n| < \frac{\delta}{3}, \quad |w_n-u_n| < \frac{\delta}{3} \tag{2.2}$$

for all $n > \mathbb{N}$. Since $b = \limsup_{n\to\infty} u_n > m$, there exists $k_1 > \mathbb{N}$ such that $u_{n_{k_1}} > m$. Let $k = n_{k_1}$, then $u_k > m$. For u_k, there exist only two cases;

Case 1: $u_k \geq m + \frac{\delta}{3}$, then by (2.2) we have $u_{k+1} - u_k > -\frac{\delta}{3}$, then $u_{k+1} > u_k - \frac{\delta}{3} \geq m$, so $u_{k+1} > m$.

Case 2: $m < u_k < m + \frac{\delta}{3}$, then by (2.2), we have $m - \frac{\delta}{3} < v_k < m + \frac{2\delta}{3}$ and $m - \frac{\delta}{3} < w_k < m + \frac{2\delta}{3}$. So, we have $|u_k - m| < \frac{\delta}{3}$, $|v_k - m| < \frac{2\delta}{3} < \delta$ and $|w_k - m| < \frac{2\delta}{3} < \delta$ Using (2.1) we get

$$f(u_k)-u_k > 0, \quad f(v_k)-v_k > 0, \quad f(w_k)-w_k > 0$$

By iteration (1.8), we have

$$u_{k+1} = u_k + f(v_k) - v_k + \alpha_k(f(w_k)-w_k) + \beta_k(f(x_k)-x_k)$$

By Case 1 and Case 2, we can conclude that $u_{k+1} > m$. By using the above argument, we obtain $u_{k+2} > m, u_{k+3} > m, u_{k+4} > m \ldots$

Thus, we get $u_n > m$ for all $n = k = n_{k_1}$. So , $a = \liminf_{n\to\infty} u_n \geq m$ which is a contradiction with $a < m$. Thus, $f(m) = m$.

For the sequence $\{u_n\}$, we consider the following two cases,

Case 1′: There exists u_m such that $a < u_m < b$. Then, $f(u_m) = u_m$. Thus,

$$\begin{aligned} w_m &= (1-\beta_m)u_m + \beta_m f(u_m) = u_m,\\ v_m &= (1-\alpha_m)u_m + \alpha_m f(w_m) = u_m, \end{aligned}$$

$$u_{m+1} = f(v_m) = f(u_m) = u_m.$$

By induction, we obtain $u_m = u_{m+1} = u_{m+2} = u_{m+3} = \dots$
So $u_n \to u_m$. This implies that $u_m = a$ and $u_n \to a$, which contradicts to our assumption.

Case $2'$**:** For all n, $u_n \le a$ or $u_n \ge b$. Because $b - a > 0$ and $|u_{n+1} - u_n| \to 0$, there exists N_0 such that $|u_{n+1} - u_n| < \dfrac{b-a}{3}$ for all $n > N_0$. This implies that either $u_n \le a$ for all $n > N_0$ or $u_n \ge b$ for all $n > N_0$. If $u_n \le a$ for $n > N_0$, then $b = \limsup_{n\to\infty} \le a$, which is a contradiction with $a < b$. Hence, we have $\{u_n\}$ is convergent.

Finally, we show that $\{u_n\}$ converges to a fixed point of f. Let $u_n \to p$ and suppose that $f(p) \ne p$. By the continuity of f, we have $\{f(u_n)\}$ is bounded. From

$$w_n = (1-\beta_n)u_n + \beta_n f(u_n)$$

and $\beta_n \to 0$, we obtain $w_n \to p$. Similarly, by

$$v_n = (1-\alpha_n)w_n + \alpha_n f(w_n)$$

and $\alpha_n \to 0$, it follows that $v_n \to p$. Let $p_k = f(v_k) - u_k$. By the continuity of f, we have $\lim_{k\to\infty} p_k = \lim_{k\to\infty}(f(v_k) - u_k) = f(p) - p \ne 0$. Put $w = f(p) - p$. Then $w \ne 0$. By the iteration (1.8), we have

$$u_{n+1} - u_n = f(v_n) - u_n$$

it follows that

$$u_n = u_1 + \sum_{k=1}^{n-1} p_k \tag{2.3}$$

By $p_k \to w \ne 0$, we have that $\sum_{k=1}^{\infty} p_k$ is divergent, which is a contradiction with $u_n \to p$. Thus, $f(p) = p$. That is, $\{u_n\}$ converges to a fixed point of f. □

3. Rate of convergence

In this section, we give conclusions on the rate of convergence of the iteration (1.8) and the iteration (1.7). We use some useful definition and lemmas to do this.

Definition 1 ([12])**.** Let E be a closed interval in the real line and let $f : E \to E$ be a continuous mapping. Suppose that $\{u_n\}$ and $\{x_n\}$ are two iterations which converge to a fixed point of p of f. We say that $\{u_n\}$ is better than $\{x_n\}$ if

$$|u_n - p| \le |x_n - p|, \quad \forall n \ge 1$$

Lemma 1 ([3, Lemma 3.3])**.** *Let E be a closed interval in the real line and let $f : E \to E$ be a continuous and non-decreasing mapping. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences in $[0,1)$ with $0 \le b_n + c_n < 1$ and $\alpha_n + \beta_n < 1$. Let $\{p_n\}$ be defined by (1.7). Then the followings hold.*

(1) *If $f(p_1) < p_1$, then $f(p_n) < p_n$ for all $n \ge 1$ and $\{p_n\}$ is non-increasing*

(2) *If* $f(p_1) > p_1$, *then* $f(p_n) > p_n$ *for all* $n \geq 1$ *and* $\{u_n\}$ *is non-decreasing*

Lemma 2. *Let* E *be a closed interval in the real line and let* $f : E \to E$ *be a continuous and non-decreasing mapping. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $[0,1)$. *Let* $\{u_n\}$ *be defined by (1.8). Then the followings hold.*

(1) *If* $f(u_1) < u_1$, *then* $f(u_n) < u_n$ *for all* $n \geq 1$ *and* $\{u_n\}$ *is non-increasing*
(2) *If* $f(u_1) > u_1$, *then* $f(u_n) > u_n$ *for all* $n \geq 1$ *and* $\{u_n\}$ *is non-decreasing*

Proof. **(1):** Let $f(u_1) < u_1$. Then $f(u_1) < w_1 \leq u_1$. Since f is non-decreasing, we have $f(w_1) \leq f(u_1) < w_1$. On the other hand, we have $f(w_1) < v_1 \leq w_1$. Since f is non-decreasing we get $f(v_1) \leq f(w_1) \leq v_1$. It follows that $f(v_1) = u_2 \leq v_1$, then $f(u_2) \leq f(v_1) = u_2$. Hence, we have $f(u_2) < u_2$. By induction, we conclude that $f(u_n) < u_n$ for all $n \geq 1$. This implies $f(u_n) < w_n \leq u_n$ for all $n \geq 1$. Since f is non-decreasing, we have $f(w_n) \leq f(u_n) < w_n \leq u_n$ for all $n \geq 1$. Thus, $f(w_n) < v_n \leq w_n$ for all $n \geq 1$. Then $f(v_n) \leq f(w_n) \leq u_n$ for all $n \geq 1$. Hence, we have $u_{n+1} \leq u_n$ for all $n \geq 1$, that is $\{u_n\}$ is non-increasing
(2): Following the line of (1), we show the desired result. □

Lemma 3. *Let* E *be a closed interval in the real line and let* $f : E \to E$ *be a continuous and non-decreasing mapping. Let* $\{a_n\}$,$\{b_n\}$,$\{c_n\}$,$\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $[0,1)$. *For* $u_1 = p_1 \in E$, *let* $\{u_n\}$ *and* $\{p_n\}$ *be the sequences defined by (1.8) and (1.7), respectively. Then, the following are satisfied:*

(1) *If* $f(p_1) < p_1$ *then* $u_n \leq p_n$ *for all* $n \geq 1$.
(2) *If* $f(p_1) > p_1$ *then* $u_n \geq p_n$ *for all* $n \geq 1$.

Proof. **(1):** Let $f(p_1) < p_1$. Then $f(u_1) < u_1$ since $u_1 = p_1$. From the iteration (1.8) we get, $f(u_1) < w_1 \leq u_1$. Since f is non-decreasing, we obtain $f(w_1) \leq f(u_1) < w_1 \leq u_1$. Hence, $f(w_1) < v_1 \leq w_1$.

Using the iteration (1.7) and the iteration (1.8), we obtain the following estimation:

$$w_1 - r_1 = (1-a_1)(u_1 - p_1) + a_1(f(u_1) - f(p_1)) = 0.$$

So, $w_1 = r_1$, and also

$$v_1 - q_1 = (1-b_1)(w_1 - r_1) + b_1(f(w_1) - f(r_1)) + c_1(r_1 - f(p_1)) \geq 0.$$

Since f is non-decreasing, we have $f(q_1) \leq f(v_1)$. On the other hand, if $f(p_1) < p_1$. Then from the iteration (1.7), we get $f(p_1) < r_1 < p_1$. Since f is non-decreasing, we get $f(r_1) < f(p_1) < r_1$. By the iteration (1.7) we also conclude $f(r_1) < q_1 < r_1$. Since f is non-decreasing, we get $f(q_1) < f(r_1) < q_1$. Next, by using the above result which is $w_1 = r_1$ in the inequality of $f(w_1) < v_1 < w_1$, we get $f(r_1) < v_1 < r_1$. Since f is non-decreasing, we get $f(v_1) < f(r_1) < v_1$ and so it follows that $f(v_1) < f(r_1) < q_1$. By using the above arguments, anyone can easily see that

$$u_2 - p_1 = f(v_1) - q_1 + \alpha_1(q_1 - f(q_1)) + \beta_1(q_1 - f(r_1)) \leq 0.$$

So, $u_2 \leq p_2$. Assume that $u_k \leq p_k$. Thus $f(u_k) \leq f(p_k)$. From Lemma 1(i) and Lemma 2(i), we get $f(p_k) < p_k$ and $f(u_k) < u_k$. It follows that $f(u_k) < w_k < u_k$ and $f(w_k) < f(u_k) < w_k$. Hence,

$w_k - r_k = (1-a_k)(u_k - p_k) + a_k(f(u_k) - f(p_k)) \leq 0.$
So, $w_k \leq r_k$. Since $f(w_k) \leq f(r_k)$,
$v_k - q_k = (1-b_k)(w_k - r_k) + b_k(f(w_k) - f(r_k)) + c_k(r_k - f(p_k)) \leq 0.$
So $v_k \leq q_k$, which yields $f(v_k) \leq f(q_k)$. This shows that
$u_{k+1} - p_{k+1} = f(v_k) - q_k + \alpha_k(q_k - f(q_k)) + \beta_k(q_k - f(r_k)) \leq 0,$
which gives, $u_{k+1} \leq w_{k+1}$. By induction, we conclude that $u_n \leq p_n$ for all $n \geq 1$.
(2): From Lemma 1(ii) and Lemma 2(ii) and the same proof as in (i), we can show that $u_n \geq p_n$ for all $n \geq 1$. □

Theorem 2. *Let E be a closed interval in the real line and let $f : E \to E$ be a continuous and non-decreasing mapping such that $F(f)$ is non-empty and bounded. Let $\{a_n\}$,$\{b_n\}$,$\{c_n\}$,$\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1)$. For $u_1 = p_1 \in E$, let $\{u_n\}$ and $\{p_n\}$ be the sequences defined by (1.8) and (1.7), respectivey, and converge to $p \in F(f)$. Then, the iteration (1.8) is better than (1.7).*

Proof. Put $L = \inf\{p \in F : \quad p = f(p)\}$ and $U = \sup\{p \in E : \quad p = f(p)\}$. For u_1 there are three cases;
Case 1: $u_1 = p_1 > U$. By [9, Proposition 3.5], we get $f(u_1) < u_1$ and $f(p_1) < p_1$. Using Lemma 3(1), we get that $u_n \leq p_n$ for all $n \geq 1$.

Following the line of the proof of [9, Theorem 3.7], we have $U \leq u_n$ for all $n \geq 1$. Then we have $0 \leq u_n - p \leq p_n - p$, so $|u_n - p| \leq |x_n - p|$ for all $n \geq 1$. We can see that the iteration (1.8) is better than the iteration (1.7).
case 2: $u_1 = p_1 < U$. By [9, Proposition 3.5], we get $f(u_1) > u_1$ and $f(p_1) > p_1$. Using Lemma 3(2), we get that $u_n \geq p_n$ for all $n \geq 1$.

Following the line of the proof of [9, Theorem 3.7], we get $u_n \leq L$ for all $n \geq 1$. So, $|u_n - p| \leq |x_n - p|$ for all $n \geq 1$. We can see that the iteration (1.8) is better than the iteration (1.7).
Case 3: $L \leq u_1 = p_1 \leq U$. Suppose that $f(u_1) \neq u_1$. If $f(u_1) < u_1$, we have by Lemma 1(1) that $\{u_n\}$ is non-decreasing with limit p. By Lemma 3(1), we have $p \leq u_n \leq p_n$ for all $n \geq 1$. It follows that $|u_n - p| \leq |p_n - p|$ for all $n \geq 1$. Hence we have that the iteration (1.8) is better than the iteration (1.7). If $f(u_1) > u_1$, we have by Lemma 1(2) that $\{u_n\}$ is nondecreasing with limit p. By Lemma 3(2), we have $p \geq u_n \geq p_n$ for all $n \geq 1$. It follows that $|u_n - p| \leq |p_n - p|$ for all $n \geq 1$. Hence, we have that the iteration (1.8) is better than the iteration (1.7). □

Now, we give a numerical example to compare the rates of convergence of the iteration (1.8) and the iteration (1.7).

Example 1. Let $f : [0,3] \to [0,3]$ be defined by $f(x) = \frac{x^2+6}{7}$. It is obvious that f is continuous and non-decreasing with a fixed point $p = 1$. Set the initial point $u_1 = x_1 = 3.0$ and control sequences be defined by $\alpha_n = \beta_n = a_n = \frac{1}{n^2+1}$, $b_n = \frac{2}{7}$ and $c_n = \frac{1}{3}$.

Remark 1. Since Karaca and Yildirim showed that the iteration (1.6) is better than the Mann, the Ishikawa and the Noor iteration under the same control conditions in Table (1) in [5] and since the iteration (1.8) is better than the iteration (1.7) ; from the example above, we see that the iteration (1.6) is also better than the Mann, the Ishikawa and the Noor iteration (see Table 1).

TABLE 1. Comparison of rates of convergence between the Mann, the Ishikawa, the Noor, the iteration (1.7) and the iteration (1.8) for the given function in Example 1.

n	Mann	Ishikawa	Noor	Iteration (1.7)	Iteration (1.8)
1	3.0000000000000	3.0000000000000	3.0000000000000	3.0000000000000	3.0000000000000
2	2.5714285714286	2.3888439436791	2.4008746355685	1.7313063612876	1.5401673015252
3	2.4174927113703	2.2254517072870	2.2376492818701	1.5181740350893	1.1412984225699
4	2.3449473117980	2.1550012729794	2.1669386104705	1.4419390309085	1.0370942630072
5	2.3036376203066	2.1162688700155	2.1279937754439	1.4035834760126	1.0098984194011
6	2.2771611529634	2.0918680685218	2.1034379658412	1.3806435160226	1.0026876820299
7	2.2588033724454	2.0751132884023	2.0865685584221	1.3654218100568	1.0007395163945
8	2.2453478554976	2.0629069049125	2.0742749466098	1.3545983144137	1.0002053712816
9	2.2350712664948	2.0536220205337	2.0649218163292	1.3465132450409	1.0000574007962
10	2.2269702990746	2.0463236735674	2.0575687715750	1.3402467219105	1.0000161162114
11	2.2204223565279	2.0404368413529	2.0516371936686	1.3352485577363	1.0000045397692
12	2.2150210906564	2.0355886384227	2.0467517462118	1.3311697883865	1.0000012819360
13	2.2104902244470	2.0315267588978	2.0426584045758	1.3277785155870	1.0000003626680
14	2.2066354643304	2.0280744244732	2.0391791535626	1.3249146769522	1.0000001027505
15	2.2033162415763	2.0251041121328	2.0361855592762	1.3224642602428	1.0000000291448
16	2.2004283710133	2.0225215278982	2.0335826431041	1.3203438702731	1.0000000082746
17	2.1978930104902	2.0202554519681	2.0312986612631	1.3184911072883	1.0000000023511
18	2.1956494058656	2.0182510914141	2.0292784132917	1.3168583563639	1.0000000006684
19	2.1936499906887	2.0164656045780	2.0274787359825	1.3154086566764	1.0000000001901
20	2.1918569955612	2.0148650125129	2.0258653933936	1.3141128814885	1.0000000000541
21	2.1902400515356	2.0134220207870	2.0244108832959	1.3129477682783	1.0000000000154
22	2.1887744636632	2.0121144533649	2.0230928598647	1.3118945144684	1.0000000000044
23	2.1874399459628	2.0109241069485	2.0218929797399	1.3109377580364	1.0000000000013
24	2.1862196801664	2.0098358997613	2.0207960445974	1.3100648253675	1.0000000000004
25	2.1850996055759	2.0088372301401	2.0197893550185	1.3092651680701	1.0000000000001
26	2.1840678764676	2.0079174869986	2.0188622173221	1.3085299356161	1.0000000000000

We also give a graphic to compare the rates of convergence of the iterations mentioned in Example 1 visually.

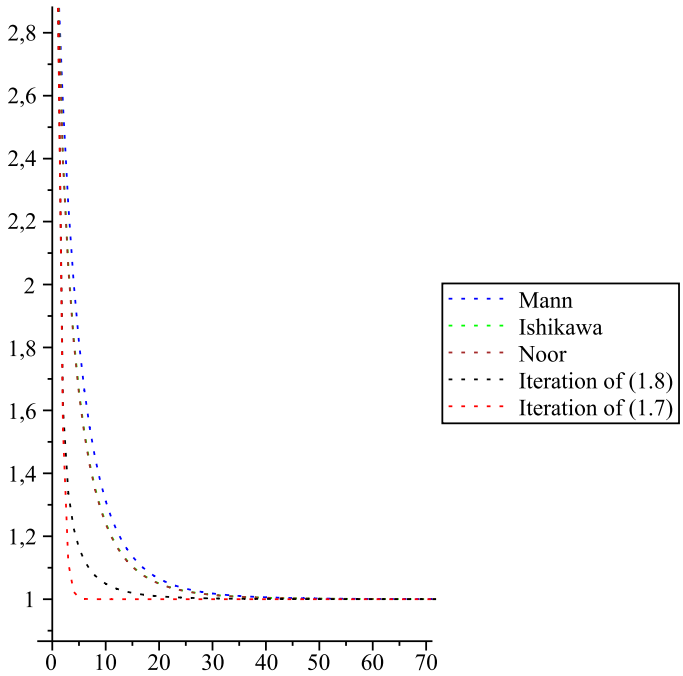

FIGURE 1. Behaviour of the iterations given in Example 1.

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Authors' addresses

Osman Alagoz
Bilecik Seyh Edebali University, Department of Mathematics, 11000, Bilecik, Turkey
E-mail address: osman.alagoz@bilecik.edu.tr

Birol Gunduz
Erzincan University, Department of Mathematics, 24000 Erzincan, Turkey
E-mail address: birolgndz@gmail.com

Sezgin Akbulut
Ataturk University, Department of Mathematics, 25000, Erzurum, Turkey
E-mail address: sezginakbulut@atauni.edu.tr

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SOME RESULTS ON HYBRID RELATIVES OF THE SHEFFER POLYNOMIALS VIA OPERATIONAL RULES

MAHVISH ALI, TABINDA NAHID, AND SUBUHI KHAN

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Abstract. The intended objective of this paper is to introduce a new class of polynomials, namely the extended Laguerre-Gould-Hopper-Sheffer polynomials. The generating function and operational rule are derived by making use of integral transform. Their quasi-monomial properties and determinant forms are also established. Examples of certain members belonging to the extended Laguerre-Gould-Hopper-Sheffer polynomials are constructed and their corresponding results are established.

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Keywords: Sheffer polynomials, Euler's integral, operational methods

1. Introduction and preliminaries

Theory of special functions performs an essential role in the formalism of mathematical physics. They provide indeed a unique tool for developing simplified yet realistic models of physical problems, thus allowing for analytic solutions and hence a deeper insight into the problem under study. The specific physical problem indeed can suggest investigating new aspects of the well-established theory of special functions as well as introducing new family of special polynomials, which usually exhibit deeper features and thereby appear many times in new roles in numerous branches of mathematics and physics. Consequently, reformulating a physical problem in terms of special functions allows a more elegant mathematical model and then for an easier reading and numerical handling of the relevant equations as well as for the discovery of unsuspected connections with other fields of Physics.

The Sheffer polynomials are one of the most important class of polynomial sequences and have been extensively studied not only due to the fact that they arise in numerous branches of mathematics but also because of their importance in applied sciences, such as physics and engineering. In view of the result [6, p.17], the Sheffer polynomials can be defined as:

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Let $f(t)$ be a delta series and let $g(t)$ be an invertible series of the following form:

$$f(t) = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!}, \qquad f_0 = 0,\ f_1 \neq 0 \tag{1.1a}$$

and

$$g(t) = \sum_{n=0}^{\infty} g_n \frac{t^n}{n!}, \qquad g_0 \neq 0, \tag{1.1b}$$

then the sequences $\{s_n(x)\}_{n\in\mathbb{N}}$ is Sheffer for the pair $(g(t), f(t))$ if and only if the following orthogonality condition holds:

$$\left\langle g(t)\ f(t)^k \,|\, s_n(x)\right\rangle = n!\ \delta_{n,k}\ , \qquad \forall\ n,k \geq 0, \tag{1.2}$$

where $\delta_{n,k}$ is the Kronecker delta.

According to Roman [6, p.18 (Theorem 2.3.4)], the polynomial sequence $s_n(x)$ is uniquely determined by two (formal) power series given by equations (1.1a) and (1.1b). The exponential generating function of the Sheffer polynomials $s_n(x)$ is then given by

$$\frac{1}{g\left(f^{-1}(t)\right)} \exp\left(x f^{-1}(t)\right) = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \tag{1.3}$$

for all x in $\mathbb{C}$, where $f^{-1}(t)$ is the compositional inverse of $f(t)$.

Let $(s_n(x))_{n\in\mathbb{N}}$ be Sheffer sequence for $(g(t), f(t))$ satisfying the following condition:

$$x^n = \sum_{k=0}^{n} a_{n,k}\ s_k(x), \tag{1.4}$$

then $s_n(x)$ can be expressed by the following determinant form [9, p.232]:

$$s_0(x) = \frac{1}{a_{0,0}},$$

$$s_n(x) = \frac{(-1)^n}{a_{0,0}\ a_{1,1}\ \dots\ a_{n,n}} \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} & x^n \\ a_{0,0} & a_{1,0} & a_{2,0} & \cdots & a_{n-1,0} & a_{n,0} \\ 0 & a_{1,1} & a_{2,1} & \cdots & a_{n-1,1} & a_{n,1} \\ 0 & 0 & a_{2,2} & \cdots & a_{n-1,2} & a_{n,2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n,n-1} \end{vmatrix}, \tag{1.5}$$

where $a_{n,k}$ is the (n,k) entry of the Riordan array $(g(t), f(t))$ [10].

Operational methods can be exploited to simplify the derivation of properties associated with ordinary and generalized special functions and to define new families of hybrid special polynomials. The combined use of integral transforms and operational methods provides a powerful tool to deal with fractional derivatives. Using the Euler's integral [7, p.218]:

$$a^{-\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty t^{\nu-1} e^{-at} dt, \qquad \min\{Re(\nu), Re(a)\} \geq 0, \tag{1.6}$$

a new family of special polynomials are introduced in [1]:

$${}_L H_{n,\nu}^{(m,r)}(x,y,z;\alpha) = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\alpha t} t^{\nu-1} {}_L H_n^{(m,r)}(x,y,zt) dt, \tag{1.7}$$

where ${}_L H_n^{(m,r)}(x,y,z)$ are the Laguerre-Gould-Hopper polynomials [4].

The polynomials ${}_L H_{n,\nu}^{(m,r)}(x,y,z)$ are also defined by the following operational rule:

$$\left(\alpha - z\frac{\partial^r}{\partial y^r}\right)^{-\nu} \exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right)\{y^n\} = {}_L H_{n,\nu}^{(m,r)}(x,y,z;\alpha). \tag{1.8}$$

Most of the properties of hybrid special polynomials recognized as quasi-monomial, can be deduced by using operational rules associated with the relevant multiplicative and derivative operators. For the multi-variable hybrid special polynomials, the use of operational techniques combined with the monomiality principle provides new means of analysis for the solution of a wide class of partial differential equations often encountered in physical problems.

According to monomiality principle [2, 8], a given polynomial set $r_n(x)$ $(n \in \mathbb{N},\ x \in \mathbb{C})$ can be considered as *quasi-monomial*, if two operators $\hat{M}$ and $\hat{P}$, called "multiplicative" and "derivative" operators respectively, can be defined in such a way that

$$\hat{M}\{r_n(x)\} = r_{n+1}(x), \tag{1.9}$$

$$\hat{P}\{r_n(x)\} = n r_{n-1}(x), \tag{1.10}$$

for all $n \in \mathbb{N}$. The operators $\hat{M}$ and $\hat{P}$ also satisfy the commutation relation

$$[\hat{P}, \hat{M}] = \hat{P}\hat{M} - \hat{M}\hat{P} = \hat{1} \tag{1.11}$$

and thus display the Weyl group structure.

If the considered polynomial set $\{r_n(x)\}_{n\in\mathbb{N}}$ is quasi-monomial, its properties can easily be derived from those of the $\hat{M}$ and $\hat{P}$ operators.

If $\hat{M}$ and $\hat{P}$ have a differential realization, then

$$\hat{M}\,\hat{P}\{r_n(x)\} = n\, r_n(x), \tag{1.12}$$

can be interpreted as the differential equation satisfied by $r_n(x)$.

The theory of hybrid special polynomials has been one of the most rapidly growing research topic in mathematical analysis. In 2016, N. Raza *et. al.* [5] introduced the Laguerre-Gould-Hopper-Sheffer polynomials
(LGHSP)${}_{L}H^{(m,r)}s_n(x,y,z)$ which are defined by the following generating function:

$$\frac{1}{g\big(f^{-1}(t)\big)}C_0\left(-x\big(f^{-1}(t)\big)^m\right)\exp\left(yf^{-1}(t)+z\big(f^{-1}(t)\big)^r\right) \tag{1.13}$$
$$= \sum_{n=0}^{\infty} {}_{L}H^{(m,r)}s_n(x,y,z)\frac{t^n}{n!}.$$

The following operational representations for the LGHSP ${}_{L}H^{(m,r)}s_n(x,y,z)$ hold [5]:

$${}_{L}H^{(m,r)}s_n(x,y,z) = \exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m} + z\frac{\partial^r}{\partial y^r}\right)s_n(y), \tag{1.14}$$

$${}_{L}H^{(m,r)}s_n(x,y,z) = \exp\left(z\frac{\partial^r}{\partial y^r}\right){}_{mL}s_n(x,y) \tag{1.15}$$

and

$${}_{L}H^{(m,r)}s_n(x,y,z) = \exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right){}_{H^{(r)}}s_n(y,z), \tag{1.16}$$

where $s_n(y)$, ${}_{mL}s_n(x,y)$ and ${}_{H^{(r)}}s_n(y,z)$ are the Sheffer, 2-variable generalized Laguerre-Sheffer and Gould-Hopper-Sheffer polynomials, respectively.

For suitable choices of the variables and indices the LGHSP ${}_{L}H^{(m,r)}s_n(x,y,z)$ reduce to certain hybrid special polynomials. These polynomials along with their notation and name are mentioned in Table 1.

This article is an attempt to further stress the importance of the operational methods, Sheffer polynomials and Laguerre-Gould-Hopper polynomials. In Section 2, the extended Laguerre-Gould-Hopper-Sheffer polynomials are introduced by operational rule and generating function. Operational rule providing the connection between the extended Laguerre-Gould-Hopper-Sheffer polynomials and Sheffer polynomials are established. These special polynomials are framed within the context of monomiality principle formalism and their determinant form is also obtained. In the last section, the corresponding results for the extended Laguerre-Gould-Hopper-Gegenbauer polynomials and extended Laguerre-Gould-Hopper-Jacobi polynomials are derived.

TABLE 1. Special cases of the LGHSP ${}_{L}H^{(m,r)}s_n(x,y,z)$.

S. No.	Values of the Indices and Variables	Relation between the LGHSP ${}_{L}H^{(m,r)}s_n(x,y,z)$ and its Special Case	Name of the hybrid special polynomials
I.	$m=1,\ r=2; x\to -x$	${}_{L}H^{(1,2)}s_n(-x,y,z) = {}_{L}H\,s_n(x,y,z)$	3-Variable Laguerre-Hermite-Sheffer polynomials
II.	$m=1,\ r=2; y=1,$ $z\to y, x\to -x$	${}_{L}H^{(1,2)}s_n(-x,1,y) = {}_{\varphi}s_n(x,y)$	Laguerre-Hermite type-Sheffer polynomials
III.	$x=0$	${}_{L}H^{(m,r)}s_n(0,y,z) = {}_{H^{(r)}}s_n(y,z)$	Gould-Hopper-Sheffer polynomials
IV.	$z=0$	${}_{L}H^{(m,r)}s_n(x,y,0) = {}_{m}L_n\,s_n(x,y)$	2-Variable generalized Laguerre-Sheffer polynomials
V.	$r=m; x=0,$ $y\to -D_x^{-1}, z\to y$	${}_{L}H^{(m,m)}s_n(0,-D_x^{-1},y) = {}_{[m]}L\,s_n(x,y)$	2-Variable generalized Laguerre type-Sheffer polynomials
VI.	$r=m-1; x=0,$ $y\to x, z\to y$	${}_{L}H^{(m,m-1)}s_n(0,x,y) = {}_{U^{(m)}}s_n(x,y)$	Generalized Chebyshev-Sheffer polynomials
VII.	$m=1; z=0,$ $x\to -x$	${}_{L}H^{(1,r)}s_n(-x,y,0) = {}_{L}s_n(x,y)$	2-Variable Laguerre-Sheffer polynomials
VIII.	$x=0,\ y\to x,$ $z\to yD_y y$	${}_{L}H^{(m,r)}s_n(0,x,yD_y y) = {}_{e^{(r)}}s_n(x,y)$	2-Variable truncated-Sheffer polynomials of order r
IX.	$r=2; x=0$	${}_{L}H^{(m,2)}s_n(0,y,z) = {}_{H}s_n(y,z)$	2-Variable Hermite-Kampé de Fériet-Sheffer polynomials
X.	$x\to yD_y y,$ $y\to x$	${}_{L}H^{(m,r)}s_n(yD_y\,y,x,z) \equiv {}_{H^{(r,m)}}s_n(x,y,z)$	3-Variable generalized Hermite-Sheffer polynomials
XI.	$m=2,\ r=3; y\to x,$ $z\to y, x\to zD_z z,$	${}_{L}H_n^{(2,3)}s_n(zD_z\,z,x,y) \equiv {}_{H^{(3,2)}}s_n(x,y,z)$	Bell-type-Sheffer polynomials

2. EXTENDED LAGUERRE-GOULD-HOPPER-SHEFFER POLYNOMIALS

In order to introduce the extended Laguerre-Gould-Hopper-Sheffer polynomials (ELGHSP) denoted by ${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$, first we prove the following theorem:

Theorem 1. *For the extended Laguerre-Gould-Hopper-Sheffer polynomials* ${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$, *the following operational rule holds true:*

$$\left(\alpha - z\frac{\partial^r}{\partial y^r}\right)^{-\nu} {}_{m}Ls_n(x,y) = {}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha). \tag{2.1}$$

Proof. Replacing a by $\left(\alpha - z\frac{\partial^r}{\partial y^r}\right)$ in relation (1.6) and then operating the resultant equation on 2-variable generalized Laguerre-Sheffer polynomials ${}_{m}Ls_n(x,y)$, it follows that

$$\left(\alpha - z\frac{\partial^r}{\partial y^r}\right)^{-\nu} {}_{m}Ls_n(x,y) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}\exp\left(zt\frac{\partial^r}{\partial y^r}\right) {}_{m}Ls_n(x,y)dt, \tag{2.2}$$

which in view of equation (1.15) gives

$$\left(\alpha - z\frac{\partial^r}{\partial y^r}\right)^{-\nu} {}_{m}Ls_n(x,y) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1} {}_{L}H^{(m,r)}s_n(x,y,zt)dt. \tag{2.3}$$

Denoting the transform on the r.h.s of equation (2.3) by a new class of extended Laguerre-Gould-Hopper-Sheffer polynomials (ELGHSP) ${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$, so that we have

$${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1} {}_{L}H^{(m,r)}s_n(x,y,zt)dt. \tag{2.4}$$

In view of equations (2.3) and (2.4), assertion (2.1) follows.

□

Next, the generating function of the ELGHSP ${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ is obtained by proving the following result:

Theorem 2. *The following generating function for the ELGHSP* ${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ *holds true:*

$$\frac{\exp(yf^{-1}(u))C_0(-x(f^{-1}(u))^m)}{g(f^{-1}(u))(\alpha - z(f^{-1}(u))^r)^\nu} = \sum_{n=0}^\infty {}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)\frac{u^n}{n!}. \tag{2.5}$$

Proof. Multiplying on both sides of equation (2.4) by $\frac{u^n}{n!}$ and summing over n, we find

$$\sum_{n=0}^\infty {}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)\frac{u^n}{n!} = \sum_{n=0}^\infty \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1} {}_{L}H^{(m,r)}s_n(x,y,zt)\frac{u^n}{n!}dt, \tag{2.6}$$

which on using equation (1.13) in the r.h.s. gives

$$\sum_{n=0}^{\infty} {}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)\frac{u^n}{n!} \tag{2.7}$$
$$= \frac{C_0(-x(f^{-1}(u))^m)\exp(yf^{-1}(u))}{g(f^{-1}(u))\Gamma(\nu)}\int_0^{\infty} e^{-(\alpha-z(f^{-1}(u))^r)t}t^{\nu-1}dt.$$

Making use of equation (1.6) in the r.h.s. of equation (2.7), yields assertion (2.5). □

Remark 1. For $\alpha = 1$, $\nu = 1$ and $z = D_z^{-1}$, the extended Laguerre-Gould-Hopper-Sheffer polynomials ${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ reduce to the Laguerre-Gould-Hopper-Sheffer polynomials ${}_{L}H^{(m,r)}s_{n}(x,y,z)$ [5].

Differentiating generating function (2.5) w.r.t α, the following recurrence relation for the ELGHSP ${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ is obtained:

$$\frac{\partial}{\partial\alpha}{}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha) = -\nu\ {}_{L}H^{(m,r)}s_{n,\nu+1}(x,y,z;\alpha). \tag{2.8}$$

In order to derive quasi-monomial properties and operational representation for the ELGHSP ${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$, the following operation will be used:

(Θ): *Replacement of z by zt, multiplication by $\frac{1}{\Gamma(\nu)}e^{-\alpha t}t^{\nu-1}$ and then integration with respect to t from $t = 0$ to $t = \infty$.*

To frame the ELGHSP ${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ within the context of monomiality principle, we prove the following result:

Theorem 3. *The ELGHSP ${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ are quasi-monomial with respect to the following multiplicative and derivative operators:*

$$\hat{M}_{LHs_\nu} = \left(y + mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}} - rz\frac{\partial^r}{\partial\alpha\partial y^{r-1}} - \frac{g'(\partial_y)}{g(\partial_y)}\right)\frac{1}{f'(\partial_y)} \tag{2.9}$$

and

$$\hat{P}_{LHs_\nu} = f(\partial_y), \tag{2.10}$$

respectively, where $\partial_y := \frac{\partial}{\partial y}$.

Proof. We recall the following equations [5]:

$$\left(y + mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}} + rz\frac{\partial^{r-1}}{\partial y^{r-1}} - \frac{g'(\partial_y)}{g(\partial_y)}\right)\frac{1}{f'(\partial_y)}\ {}_{L}H^{(m,r)}s_{n}(x,y,z) \tag{2.11}$$
$$= {}_{L}H^{(m,r)}s_{n+1}(x,y,z);$$

$$f\left(\partial_y\right) {}_{L}H^{(m,r)}s_n(x,y,z) = n\ {}_{L}H^{(m,r)}s_{n-1}(x,y,z). \tag{2.12}$$

Performing the operation (Θ) on equations (2.11) and (2.12) and using recurrence relation (2.8), we obtain assertions (2.9) and (2.10).

□

The following corollary is an immediate consequence of Theorem 3:

Corollary 1. *The following differential equation for the ELGHSP* ${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ *holds true:*

$$\left(\left(y + mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}} - rz\frac{\partial^r}{\partial\alpha\partial y^{r-1}} - \frac{g'\left(\partial_y\right)}{g\left(\partial_y\right)}\right)\frac{f\left(\partial_y\right)}{f'\left(\partial_y\right)} - n\right) {}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha) = 0 \tag{2.13}$$

Proof. Using equations (2.9) and (2.10) in equation (1.12), assertion (2.13) follows.

□

The operational representation between the ELGHSP ${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ and Sheffer polynomials $s_n(x)$ is obtained in the form of the following result:

Theorem 4. *The following operational representation between the ELGHSP* ${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ *and Sheffer polynomials* $s_n(x)$ *holds true:*

$$\left(\alpha - z\frac{\partial^r}{\partial y^r}\right)^{-\nu}\exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right)\{s_n(y)\} = {}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha). \tag{2.14}$$

Proof. Performing operation (Θ) on equation (1.14), we obtain

$$\begin{aligned}\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}\exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m} + zt\frac{\partial^r}{\partial y^r}\right)s_n(y)dt \\ = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}\ {}_{L}H^{(m,r)}s_n(x,y,zt)dt.\end{aligned} \tag{2.15}$$

Decoupling the exponential operator in the l.h.s. of above equation by using the Weyl identity [3, p. 7]

$$e^{\hat{A}+\hat{B}} = e^{-k/2}e^{\hat{A}}e^{\hat{B}}, \quad k \in \mathbb{C}, \tag{2.16}$$

we get

$$\exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right)\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-(\alpha - z\frac{\partial}{\partial y}^r)t}t^{\nu-1}dt\,s_n(y) = {}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha), \tag{2.17}$$

which in view of relation (1.6), yields assertion (2.14).

□

The determinant definition of the Sheffer sequences proposed by W. Wang [9] in 2014, provides motivation to establish the determinant forms of the new hybrid special polynomials. The determinant approach is equivalent to the corresponding approach based on operational methods. This approach is beneficial in detecting the solution of general linear interpolation problems and also suitable for computations. Inspired by the novel work on determinant approaches, the determinant definition of the ELGHSP ${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ is established by proving the following result:

Theorem 5. *The ELGHSP ${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ of degree n are defined by*

$$ {}_{L}H^{(m,r)}s_{0,\nu}(x,y,z;\alpha) = \frac{1}{a_{0,0}}\,{}_{L}H^{(m,r)}_{0,\nu}(x,y,z;\alpha), \tag{2.18} $$

$$ {}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha) = \frac{(-1)^n}{a_{0,0}\,a_{1,1}\,\dots\,a_{n,n}} \times \begin{vmatrix} {}_{L}H^{(m,r)}_{0,\nu}(x,y,z;\alpha) & {}_{L}H^{(m,r)}_{1,\nu}(x,y,z;\alpha) & \cdots & {}_{L}H^{(m,r)}_{n-1,\nu}(x,y,z;\alpha) & {}_{L}H^{(m,r)}_{n,\nu}(x,y,z;\alpha) \\ a_{0,0} & a_{1,0} & \cdots & a_{n-1,0} & a_{n,0} \\ 0 & a_{1,1} & \cdots & a_{n-1,1} & a_{n,1} \\ 0 & 0 & \cdots & a_{n-1,2} & a_{n,2} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1} & a_{n,n-1} \end{vmatrix}, \tag{2.19} $$

where $a_{n,k}$ is the (n,k) entry of the Riordan array $\big(g(t), f(t)\big)$.

Proof. Interchanging x by y and operating $\left(\alpha - z\frac{\partial^r}{\partial y^r}\right)^{-\nu}\exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right)$ on both sides of equation (1.4) and using operational rules (1.8) and (2.14) in r.h.s and l.h.s. respectively, we find

$$ {}_{L}H^{(m,r)}_{n,\nu}(x,y,z;\alpha) = \sum_{k=0}^{n} a_{n,k}\,{}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha). \tag{2.20} $$

The above equality leads to the following system of infinite equations in the unknowns ${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$, $\quad n = 0,1,\dots,$

$$\begin{cases} a_{0,0}\ {}_LH^{(m,r)}s_{0,\nu}(x,y,z;\alpha) = {}_LH_{0,\nu}^{(m,r)}(x,y,z;\alpha), \\ a_{1,0}\ {}_LH^{(m,r)}s_{0,\nu}(x,y,z;\alpha) + a_{1,1}\ {}_LH^{(m,r)}s_{1,\nu}(x,y,z;\alpha) = {}_LH_{1,\nu}^{(m,r)}(x,y,z;\alpha), \\ a_{2,0}\ {}_LH^{(m,r)}s_{0,\nu}(x,y,z;\alpha) + a_{2,1}\ {}_LH^{(m,r)}s_{1,\nu}(x,y,z;\alpha) \\ \qquad\qquad + a_{2,2}\ {}_LH^{(m,r)}s_{2,\nu}(x,y,z;\alpha) = {}_LH_{2,\nu}^{(m,r)}(x,y,z;\alpha), \\ \vdots \\ a_{n,0}\ {}_LH^{(m,r)}s_{0,\nu}(x,y,z;\alpha) + a_{n,1}\ {}_LH^{(m,r)}s_{1,\nu}(x,y,z;\alpha) + \\ \qquad\qquad \cdots + a_{n,n}\ {}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha) \\ \vdots \end{cases} \tag{2.21}$$

From first equation of system (2.21), assertion (2.18) follows. Applying the Cramer's rule to the first $n+1$ equations, it follows that

$$\begin{aligned} &{}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha) \\ &= \frac{1}{a_{0,0}\,a_{1,1}\,\dots\,a_{n,n}} \begin{vmatrix} a_{0,0} & 0 & \cdots & 0 & {}_LH_{0,\nu}^{(m,r)}(x,y,z;\alpha) \\ a_{1,0} & a_{1,1} & \cdots & 0 & {}_LH_{1,\nu}^{(m,r)}(x,y,z;\alpha) \\ a_{2,0} & a_{2,1} & \cdots & 0 & {}_LH_{2,\nu}^{(m,r)}(x,y,z;\alpha) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{n-1,0} & a_{n-1,1} & \cdots & a_{n-1,n-1} & {}_LH_{n-1,\nu}^{(m,r)}(x,y,z;\alpha) \\ a_{n,0} & a_{n,1} & \cdots & a_{n,n-1} & {}_LH_{n,\nu}^{(m,r)}(x,y,z;\alpha) \end{vmatrix}. \end{aligned} \tag{2.22}$$

Now, bringing the $(n+1)$-th column to the first place by n transpositions of adjacent columns and noting that the determinant of a square matrix is the same as that of its transpose, assertion (2.19) follows.

□

In the next section, certain members belonging to the ELGHSP ${}_LH^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ are constructed and their analogues results are established.

3. EXAMPLES

The Sheffer family contains several polynomials as their members for different choices of $g(t)$ and $f(t)$. Thus, by taking $g(t)$ and $f(t)$ of the special polynomials belonging to Sheffer family, we get the corresponding members belonging to the ELGHSP family ${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$.

Example 3.1 Taking $g(u)=\left(\frac{2}{1+\sqrt{1-u^2}}\right)^{\lambda}$, $f(u)=\frac{-u}{1+\sqrt{1-u^2}}$, $f^{-1}(u)=\frac{-2u}{1+u^2}$ and $g(f^{-1}(u))=(1+u^2)^{\lambda}$ (for which the Sheffer polynomials becomes the Gegenbauer polynomials $C_n^{(\lambda)}(x)$ [9]) in Theorem 2- Theorem 4, we find that:

The following generating function for the extended Laguerre-Gould-Hopper Gegenbauer polynomials (ELGHGnP) ${}_{L}H^{(m,r)}C_{n,\nu}^{(\lambda)}(x,y,z;\alpha)$ holds true:

$$\frac{\exp\left(y\left(\frac{-2u}{1+u^2}\right)\right)C_0\left(-x\left(\frac{-2u}{1+u^2}\right)^m\right)}{(1+u^2)^{\lambda}\left(\alpha-z\left(\frac{-2u}{1+u^2}\right)^r\right)^{\nu}}=\sum_{n=0}^{\infty}{}_{L}H^{(m,r)}C_{n,\nu}^{(\lambda)}(x,y,z;\alpha)\frac{u^n}{n!}. \tag{3.1}$$

The ELGHGnP ${}_{L}H^{(m,r)}C_{n,\nu}^{(\lambda)}(x,y,z;\alpha)$ are quasi-monomial with respect to the following multiplicative and derivative operators:

$$\hat{M}_{LHC_\nu}=\lambda\partial y+\left(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}-rz\frac{\partial^r}{\partial\alpha\partial y^{r-1}}\right)\left((\partial y)^2-1-\sqrt{1-(\partial y)^2}\right), \tag{3.2}$$

and

$$\hat{P}_{LHC_\nu}=\frac{-\partial y}{1+\sqrt{1-(\partial y)^2}}. \tag{3.3}$$

The ELGHGnP ${}_{L}H^{(m,r)}C_{n,\nu}^{(\lambda)}(x,y,z;\alpha)$ satisfy the following differential equation:

$$\left(\frac{-\lambda(\partial y)^2}{1+\sqrt{1-(\partial y)^2}}+\left(y\frac{\partial}{\partial y}+mD_x^{-1}\frac{\partial^m}{\partial y^m}-rz\frac{\partial^{r+1}}{\partial\alpha\partial y^r}\right)\frac{\left(1-(\partial y)^2+\sqrt{1-(\partial y)^2}\right)}{1+\sqrt{1-(\partial y)^2}}-n\right)$$
$$\times{}_{L}H^{(m,r)}C_{n,\nu}^{(\lambda)}(x,y,z;\alpha)=0. \tag{3.4}$$

The following relation between the Gegenbauer polynomials $C_n^{(\lambda)}(x)$ and ELGHGnP ${}_{L}H^{(m,r)}C_{n,\nu}^{(\lambda)}(x,y,z;\alpha)$ holds true:

$$\left(\alpha-z\frac{\partial^r}{\partial y^r}\right)^{-\nu}\exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right)\{C_n^{(\lambda)}(y)\}={}_{L}H^{(m,r)}C_{n,\nu}^{(\lambda)}(x,y,z;\alpha). \tag{3.5}$$

Taking

$$a_{n,k} = \begin{cases} 0, & \text{if } n-k \text{ is odd}, \\ \frac{c_n(-1)^k(\lambda+k)}{c_k 2^n(\lambda+n)}\binom{\lambda+n}{n-k/2}, & \text{if } n-k \text{ is even}, \end{cases} \tag{3.6}$$

where $c_n = 1/\binom{-\lambda}{n}$ and $a_{n,n} = \left(-\frac{1}{2}\right)^n$ in equations (2.18) and (2.19), we find that the ELGHGnP ${}_{L}H^{(m,r)}C_{n,\nu}^{(\lambda)}(x,y,z;\alpha)$ for $n=4$ are defined by

$${}_{L}H^{(m,r)}C_{0,\nu}^{(\lambda)}(x,y,z;\alpha) = {}_{L}H_{0,\nu}^{(m,r)}(x,y,z;\alpha), \tag{3.7}$$

$${}_{L}H^{(m,r)}C_{n,\nu}^{(\lambda)}(x,y,z;\alpha) = \frac{128\lambda(\lambda+1)(\lambda+2)(\lambda+3)}{3}$$

$$\times \begin{vmatrix} {}_{L}H_{0,\nu}^{(m,r)}(x,y,z;\alpha) & {}_{L}H_{1,\nu}^{(m,r)}(x,y,z;\alpha) & {}_{L}H_{2,\nu}^{(m,r)}(x,y,z;\alpha) & {}_{L}H_{3,\nu}^{(m,r)}(x,y,z;\alpha) & {}_{L}H_{4,\nu}^{(m,r)}(x,y,z;\alpha) \\ 1 & 0 & \frac{1}{2(\lambda+1)} & 0 & \frac{3}{4(\lambda+1)(\lambda+2)} \\ 0 & -\frac{1}{2} & 0 & \frac{-3}{4(\lambda+2)} & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4(\lambda+3)} \\ 0 & 0 & 0 & -\frac{1}{8} & 0 \end{vmatrix}. \tag{3.8}$$

Example 3.2 Taking $g(u) = \left(\frac{2}{1+\sqrt{1+2u}}\right)^{1+\alpha+\beta}$, $f(u) = \frac{u}{1+u+\sqrt{1+2u}}$, $f^{-1}(u) = \frac{2u}{(1-u)^2}$ and
$g(f^{-1}(u)) = (1-u)^{1+\alpha+\beta}$ (for which the Sheffer polynomials becomes the Jacobi polynomials $J_n(x)$ [9]) in Theorem 2 - Theorem 4, we find that:

The following generating function for the extended Laguerre-Gould-Hopper-Jacobi polynomials (ELGHJP) ${}_{L}H^{(m,r)}J_{n,\nu}(x,y,z;\alpha)$ holds true:

$$\frac{\exp\left(y\left(\frac{2u}{(1-u)^2}\right)\right)C_0\left(-x\left(\frac{2u}{(1-u)^2}\right)^m\right)}{(1-u)^{1+\alpha+\beta}\left(\alpha - z\left(\frac{2u}{(1-u)^2}\right)^r\right)^\nu} = \sum_{n=0}^{\infty} {}_{L}H^{(m,r)}J_{n,\nu}(x,y,z;\alpha)\frac{u^n}{n!}. \tag{3.9}$$

The ELGHJP ${}_{L}H^{(m,r)}J_{n,\nu}(x,y,z;\alpha)$ are quasi-monomial with respect to the following multiplicative and derivative operators:

$$\hat{M}_{LHJ_\nu} = \left(y + mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}} - rz\frac{\partial^r}{\partial\alpha\partial y^{r-1}} + \frac{(1+\alpha+\beta)}{1+2\partial y+\sqrt{1+2\partial y}}\right) \times\left(1+\partial y+\sqrt{1+2\partial y}\right)\sqrt{1+2\partial y} \tag{3.10}$$

and

$$\hat{P}_{LHJ_\nu} = \frac{\partial y}{1+\partial y+\sqrt{1+2\partial y}}. \tag{3.11}$$

The ELGHJP ${}_{L}H^{(m,r)}J_{n,\nu}(x,y,z;\alpha)$ satisfy the following differential equation:

$$\left(\left(y\frac{\partial}{\partial y}+mD_x^{-1}\frac{\partial^m}{\partial y^m}-rz\frac{\partial^{r+1}}{\partial y^r\partial\alpha}-\frac{(1+\alpha+\beta)\partial y}{1+2\partial y+\sqrt{1+2\partial y}}\right)\sqrt{1+2\partial y}-n\right)$$
$$\times{}_{L}H^{(m,r)}J_{n,\nu}(x,y,z;\alpha)=0. \tag{3.12}$$

The following relation between the Jacobi polynomials $J_n(x)$ and ELGHJP ${}_{L}H^{(m,r)}J_{n,\nu}(x,y,z;\alpha)$ holds true:

$$\left(\alpha-z\frac{\partial^r}{\partial y^r}\right)^{-\nu}\exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right)\{J_n(y)\}={}_{L}H^{(m,r)}J_{n,\nu}(x,y,z;\alpha). \tag{3.13}$$

Taking

$$a_{n,k}=\frac{(-1)^{n-k}}{2^n}\frac{c_n}{c_k}\frac{1+\alpha+\beta+2k}{1+\alpha+\beta+2n}\binom{1+\alpha+\beta+2n}{n-k}, \tag{3.14}$$

where $c_n=\frac{4^n(\alpha+n)_n n!}{(\alpha+\beta+2n)_{2n}}$ and $a_{n,n}=\left(\frac{1}{2}\right)^n$ in equations (2.18) and (2.19), we find that the ELGHJP ${}_{L}H^{(m,r)}J_{n,\nu}(x,y,z;\alpha)$ for $n=4$ are defined by

$${}_{L}H^{(m,r)}J_{0,\nu}(x,y,z;\alpha)={}_{L}H_{0,\nu}^{(m,r)}(x,y,z;\alpha), \tag{3.15}$$
$${}_{L}H^{(m,r)}J_{n,\nu}(x,y,z;\alpha)=1024$$
$$\times\begin{vmatrix} {}_{L}H_{0,\nu}^{(m,r)}(x,y,z;\alpha) & {}_{L}H_{1,\nu}^{(m,r)}(x,y,z;\alpha) & {}_{L}H_{2,\nu}^{(m,r)}(x,y,z;\alpha) & {}_{L}H_{3,\nu}^{(m,r)}(x,y,z;\alpha) & {}_{L}H_{4,\nu}^{(m,r)}(x,y,z;\alpha) \\ 1 & -\frac{2(\alpha+1)}{\alpha+\beta+2} & \frac{4(\alpha+2)_2}{(\alpha+\beta+3)_2} & -\frac{8(\alpha+3)_3}{(\alpha+\beta+4)_3} & \frac{16(\alpha+4)_4}{(\alpha+\beta+5)_4} \\ 0 & \frac{1}{2} & -\frac{2(\alpha+2)}{\alpha+\beta+4} & \frac{6(\alpha+3)_2}{(\alpha+\beta+5)_2} & -\frac{16(\alpha+4)_3}{(\alpha+\beta+6)_3} \\ 0 & 0 & \frac{1}{4} & -\frac{3(\alpha+3)}{2(\alpha+\beta+6)} & \frac{6(\alpha+4)_2}{(\alpha+\beta+7)_2} \\ 0 & 0 & 0 & \frac{1}{8} & -\frac{\alpha+4}{\alpha+\beta+8} \end{vmatrix}. \tag{3.16}$$

The above examples show that the operational rules provide a mechanism to obtain the results for the members belonging to the ELGHSP ${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ and prove the usefulness of the method adopted in this paper. The operational techniques can be used for a more general insight into the theory of hybrid special polynomials and for their extensions. The appropriate combination of methods relevant to generalized operational calculus and to special functions can be very useful tool to treat a large body of problems both in physics and mathematics. Thus, we conclude that the method based on the operational rules may provide powerful tools to deal with the possibilities offered by extended forms of the hybrid special polynomials.

APPENDIX

We have mentioned several special cases of the LGHSP ${}_{L}H^{(m,r)}s_n(x,y,z)$ in Table 1. Now, for the same choice of the variables and indices, the ELGHSP ${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$ reduce to the corresponding special case. These new hybrid special polynomials related to the Sheffer polynomials are given in Table 2.

TABLE 2. Special cases of the ELGHSP ${}_{L}H^{(m,r)}s_{n,\nu}(x,y,z;\alpha)$.

S. No.	Values of the Indices and Variables	Name of the hybrid special polynomials	Generating function
I.	$m=1,\ r=2;\ x\to -x$	Extended 3-variable Laguerre-Hermite-Sheffer polynomials (E3VLHSP)	$\frac{\exp(yf^{-1}(u))C_0(xf^{-1}(u))}{g(f^{-1}(u))(\alpha-z(f^{-1}(u))^2)^{\nu}} = \sum_{n=0}^{\infty} {}_{L}H s_{n,\nu}(x,y,z;\alpha)\frac{u^n}{n!}$
II.	$m=1,\ r=2;\ y=1,$ $z\to y, x\to -x$	Extended Laguerre-Hermite type-Sheffer polynomials (ELHtSP)	$\frac{\exp(f^{-1}(u))C_0(xf^{-1}(u))}{g(f^{-1}(u))(\alpha-y(f^{-1}(u))^2)^{\nu}} = \sum_{n=0}^{\infty} \varphi s_{n,\nu}(x,y;\alpha)\frac{u^n}{n!}$
III.	$x=0$	Extended Gould-Hopper-Sheffer polynomials (EGHSP)	$\frac{\exp(yf^{-1}(u))}{g(f^{-1}(u))(\alpha-z(f^{-1}(u))^r)^{\nu}} = \sum_{n=0}^{\infty} H^{(r)} s_{n,\nu}(y,z;\alpha)\frac{u^n}{n!}$
IV.	$r=1;\ y=0,$ $z\to y$	Extended 2-Variable generalized Laguerre-Sheffer polynomials (E2VGLSP)	$\frac{C_0(-x(f^{-1}(u))^m)}{g(f^{-1}(u))(\alpha-y(f^{-1}(u)))^{\nu}} = \sum_{n=0}^{\infty} m\, L_n s_{n,\nu}(x,y;\alpha)\frac{u^n}{n!}$
V.	$r=m;\ x=0,$ $y\to -D_x^{-1}, z\to y$	Extended 2-Variable generalized Laguerre type-Sheffer polynomials (E2VGLtSP)	$\frac{C_0(xf^{-1}(u))}{g(f^{-1}(u))(\alpha-y(f^{-1}(u))^m)^{\nu}} = \sum_{n=0}^{\infty} [m]L s_{n,\nu}(x,y;\alpha)\frac{u^n}{n!}$
VI.	$r=m-1;\ x=0,$ $y\to x,\ z\to y$	Extended generalized Chebyshev-Sheffer polynomials (EGCSP)	$\frac{\exp(xf^{-1}(u))}{g(f^{-1}(u))(\alpha-y(f^{-1}(u))^{m-1})^{\nu}} = \sum_{n=0}^{\infty} U^{(m)} s_{n,\nu}(x,y;\alpha)\frac{u^n}{n!}$
VII.	$m=1;\ z=0,$ $x\to -x$	Extended 2-Variable Laguerre-Sheffer polynomials (E2VLSP)	$\frac{C_0(xf^{-1}(u))}{g(f^{-1}(u))(\alpha-y(f^{-1}(u)))^{\nu}} = \sum_{n=0}^{\infty} L s_{n,\nu}(x,y;\alpha)\frac{u^n}{n!}$
VIII.	$x=0,\ y\to x,$ $z\to yD_y y$	Extended 2-Variable truncated-Sheffer polynomials of order r (E2VTSP)	$\frac{\exp(xf^{-1}(u))}{g(f^{-1}(u))(\alpha-yD_y y(f^{-1}(u))^r)^{\nu}} = \sum_{n=0}^{\infty} e^{(r)} s_{n,\nu}(x,y;\alpha)\frac{u^n}{n!}$
IX.	$r=2;\ x=0$	Extended 2-Variable Hermite-Kampé de Fériet-Sheffer polynomials (E2VHKdFSP)	$\frac{\exp(yf^{-1}(u))}{g(f^{-1}(u))(\alpha-z(f^{-1}(u))^2)^{\nu}} = \sum_{n=0}^{\infty} H s_{n,\nu}(y,z;\alpha)\frac{u^n}{n!}$
X.	$x\to yD_y y,$ $y\to x$	Extended 3-Variable generalized Hermite-Sheffer polynomials (E3VGHSP)	$\frac{\exp\left(xf^{-1}(u)+y(f^{-1}(u))^m\right)}{g(f^{-1}(u))(\alpha-z(f^{-1}(u))^r)^{\nu}} = \sum_{n=0}^{\infty} H^{(r,m)} s_{n,\nu}(x,y,z;\alpha)\frac{u^n}{n!}$
XI.	$m=2,\ r=3;\ y\to x,$ $z\to y,\ x\to zD_z z,$	Extended Bell-type-Sheffer polynomials (EBtSP)	$\frac{\exp\left(xf^{-1}(u)+z(f^{-1}(u))^2\right)}{g(f^{-1}(u))(\alpha-y(f^{-1}(u))^3)^{\nu}} = \sum_{n=0}^{\infty} H^{(3,2)} s_{n,\nu}(x,y,z;\alpha)\frac{u^n}{n!}$

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Authors' addresses

Mahvish Ali
Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University), New Delhi-110025, India,
E-mail address: mahvishali37@gmail.com

Tabinda Nahid
Department of Mathematics, Aligarh Muslim University, Aligarh, India
E-mail address: tabindanahid@gmail.com

Subuhi Khan
Department of Mathematics, Aligarh Muslim University, Aligarh, India
E-mail address: subuhi2006@gmail.com

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SOME APPLICATIONS OF GENERALIZED SRIVASTAVA-ATTIYA OPERATOR TO THE BI-CONCAVE FUNCTIONS

ŞAHSENE ALTINKAYA AND SIBEL YALÇIN

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Abstract. In this present investigation, we are concerned with the class $\Omega^{m,k}_{\Sigma;\mu,b}C_0(\alpha)$ of bi-concave functions defined by using the generalized Srivastava-Attiya operator. Moreover, we derive some coefficient inequalities for functions in this class.

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1. Introduction

Let A indicate an analytic function family, which is normalized under the condition of $f(0) = f'(0) - 1 = 0$ in $\mathbb{D} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and given by the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Further, by S we shall denote the class of all functions in A which are univalent in $\mathbb{D}$.

It is well known that every function $f \in S$ has an inverse f^{-1}, defined by $f^{-1}(f(z)) = z$, $(z \in \mathbb{D})$ and $f\left(f^{-1}(w)\right) = w$, $\left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right)$. In fact, the inverse function is given by

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots$$

(for details, see Duren [13]). A function $f \in A$ is said to be bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$. Let Σ stand for the class of bi-univalent functions defined in the unit disk $\mathbb{D}$. A brief history and interesting examples of functions in the class Σ can be found in the pioneering work on this subject by Srivastava *et al.* [34], which has apparently revived the study of bi-univalent functions in recent years. In fact, ever since the publication by Srivastava *et al.* [34], a huge flood of papers have appeared and are still appearing in the literature dealing with various subclasses of the bi-univalent and other related function classes (see, for example,

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[6,7,20,24,26,32,35,37]). But the coefficient problem for each one of the following Taylor-Maclaurin coefficients

$$|a_n|, \quad n \in \mathbb{N}\backslash\{1,2\}; \quad \mathbb{N} = \{1,2,3,...\}$$

is still an open problem. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([11,16,33,36, 38]).

The study of operators plays an important role in Geometric Function Theory in Complex Analysis and its related fields. Recently, the interest in this area has been increasing because it permits detailed investigations of problems with physical applications. For example, many derivative and integral operators can be written in terms of convolution of certain analytic functions. For functions

$$f_j(z) = \sum_{n=0}^{\infty} a_{n,j} z^n \quad (j = 1,2)$$

analytic in $\mathbb{D}$, we define the Hadamard product of f_1 and f_2 as

$$(f_1 * f_2)(z) = \sum_{n=0}^{\infty} a_{n,1,} a_{n,2} z^n = (f_2 * f_1)(z) \quad (z \in \mathbb{D}). \tag{1.2}$$

In terms of the Hadamard product (or convolution), the Dziok-Srivastava linear convolution operator involving the generalized hypergeometric function was introduced and studied systematically by Dziok and Srivastava [14, 15] and (subsequently) by many other authors (see, for details, [17,18,29]). We recall here a general Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ defined in [31] by

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$$

$\big(a \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ s \in \mathbb{C},$ when $|z| < 1;\ \Re(s) > 1$ when $|z| = 1\big)$, where $\mathbb{Z}_0^- := \mathbb{Z}\backslash\mathbb{N}$. Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ can be found in [9], and the references stated there in (see also [20], [21,30]). Srivastava and Attiya [30] (also see [2,8,19]) introduced and investigated the linear operator:

$$\Omega_b^{\mu} : A \to A$$

defined in terms of the Hadamard product by

$$\Omega_b^{\mu} f\ (z) = \left(G_b^{\mu} * f\right)\ (z), \quad \left(z \in \mathbb{D};\ b \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ \mu \in \mathbb{C};\ f \in A\right), \tag{1.3}$$

where, for convenience,

$$G_b^{\mu}(z) := (1+b)^{\mu}[\Phi(z,\mu,b) - b^{-\mu}] \quad (z \in \mathbb{D}). \tag{1.4}$$

Next, we recall the following relationships which follow easily by using (1.1), (1.3) and (1.4)

$$\Omega_b^{\mu} f\ (z) = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b}\right)^{\mu} a_n z^n. \tag{1.5}$$

Motivated essentially by the Srivastava-Attiya operator, Murugusundaramoorthy [25] introduced the generalized integral operator $\Omega_{\mu,b}^{m,k}$ given by

$$\Omega_{\mu,b}^{m,k}\ f(z) = z + \sum_{n=2}^{\infty} C_n^m(b,\mu,k) a_n z^n, \tag{1.6}$$

where

$$C_n^m(b,\mu,k) = \left|\left(\frac{1+b}{n+b}\right)^{\mu}\right| \frac{m!(n+k-2)!}{(k-2)!(n+m-1)!} \tag{1.7}$$

and (throughout this paper unless otherwise mentioned) the parameters μ, b are constrained as $b \in \mathbb{C}\backslash\mathbb{Z}_0^-; \mu \in \mathbb{C}, k \geq 2$ and $m > -1$. It is of interest to note that $\Omega_{\mu,b}^{1,2}$ is the Srivastava-Attiya operator and $\Omega_{0,b}^{m,k}$ is the well-known Choi-Saigo-Srivastava operator (see [22]). Suitably specializing the parameters m, k, μ and b in $\Omega_{\mu,b}^{m,k} f(z)$, we can get various integral operators introduced by Alexander [1] and Bernardi [4], Ling and Liu [22], Livingstone [23].

2. Preliminaries

Conformal maps of the unit disk onto convex domains are a classical topic. Recently Avkhadiev and Wirths [3] discovered that conformal maps onto concave domains (the complements of convex closed sets) have some novel properties.

A function $f : \mathbb{D} \to \mathbb{C}$ is said to belong to the family $C_0(\alpha)$ if f satisfies the following conditions:

- f is analytic in $\mathbb{D}$ with the standard normalization $f(0) = f'(0) - 1 = 0$. In addition it satisfies $f(1) = \infty$,
- f maps $\mathbb{D}$ conformally onto a set whose complement with respect to $\mathbb{C}$ is convex,
- the opening angle of $f(\mathbb{D})$ at ∞ is less than or equal to $\pi\alpha$, $\alpha \in (1,2]$.

The class $C_0(\alpha)$ is referred to as the class of concave univalent functions and for a detailed discussion about concave functions, we refer to Avkhadiev and Wirths [3], Cruz and Pommerenke [10] and references there in.

In particular, the inequality

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < 0 \qquad (z \in \mathbb{D})$$

is used - sometimes also as a definition - for concave functions $f \in C_{0_O}$ (see e.g. [27] and others).

Bhowmik et al. [5] showed that an analytic function f maps $\mathbb{D}$ onto a concave domain of angle $\pi\alpha$, if and only if $\Re(P_f(z)) > 0$, where

$$P_f(z) = \frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z} - 1 - z\frac{f''(z)}{f'(z)}\right].$$

There has been a number of investigations on basic subclasses of concave univalent functions (see, for example [12], [28]).
To establish our new subclass we require the following Definition.

Definition 1. Let the functions $h, p : \mathbb{D} \to \mathbb{C}$ be so constrained that

$$\min\{\Re(h(z)), \Re(p(z))\} > 0$$

and

$$h(0) = p(0) = 1.$$

Motivated by each of the above definitions, we now define a new subclass of bi-concave analytic functions involving the generalized integral operator $\Omega^{m,k}_{\mu,b}$.

Definition 2. A function $f \in \Sigma$ given by (1.1) is said to be in the class

$$\Omega^{m,k}_{\Sigma;\mu,b}C_0(\alpha) \quad (b \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ \mu \in \mathbb{C};\ k \geq 2;\ m > -1;\ \alpha \in (1,2];\ z, w \in \mathbb{D})$$

the following conditions are satisfied:

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z} - 1 - z\frac{\left[\Omega^{m,k}_{\Sigma;\mu,b}f(z)\right]''}{\left[\Omega^{m,k}_{\Sigma;\mu,b}f(z)\right]'}\right] \in h(\mathbb{D}) \tag{2.1}$$

and

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1-w}{1+w} - 1 - w\frac{\left[\Omega^{m,k}_{\Sigma;\mu,b}g(w)\right]''}{\left[\Omega^{m,k}_{\Sigma;\mu,b}g(w)\right]'}\right] \in p(\mathbb{D}), \tag{2.2}$$

where $g(w) = f^{-1}(w)$.

3. Main results and their consequences

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\Omega^{m,k}_{\Sigma;\mu,b}C_0(\alpha)$.

Theorem 1. *Let f given by (1.1) be in the class $\Omega^{m,k}_{\Sigma;\mu,b}C_0(\alpha)$. Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{8(\alpha+1)^2+(\alpha-1)^2\left(|h'(0)|^2+|p'(0)|^2\right)}{32\left[C_2^m(b,\mu,k)\right]^2} + \frac{(\alpha^2-1)(|h'(0)|+|p'(0)|)}{8\left[C_2^m(b,\mu,k)\right]^2}}\right., \tag{3.1}$$

$$\sqrt{\frac{(\alpha+1)}{2\left|2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right|}+\frac{(\alpha-1)(|h''(0)|+|p''(0)|)}{16\left|2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right|}}\Bigg\}$$

and

$$|a_3| \leq \min\Bigg\{\frac{8(\alpha+1)^2+(\alpha-1)^2\left(|h'(0)|^2+|p'(0)|^2\right)+4\left(\alpha^2-1\right)(|h'(0)|+|p'(0)|)}{32\left[C_2^m(b,\mu,k)\right]^2} + \frac{(\alpha-1)(|h''(0)|+|p''(0)|)}{48C_3^m(b,\mu,k)}, \tag{3.2}$$

$$\frac{(\alpha-1)\left\{\left|3C_3^m(b,\mu,k)-\left[C_2^m(b,\mu,k)\right]^2\right||h''(0)|+\left[C_2^m(b,\mu,k)\right]^2|p''(0)|\right\}+12(\alpha+1)C_3^m(b,\mu,k)}{24C_3^m(b,\mu,k)\left|2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right|}\Bigg\}.$$

Proof. Let $f \in \Omega^{m,k}_{\Sigma;\mu,b}C_0(\alpha)$ and g be the analytic extension of f^{-1} to $\mathbb{D}$. It follows from (2.1) and (2.2) that

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z}-1-z\frac{\left[\Omega^{m,k}_{\Sigma;\mu,b}f(z)\right]''}{\left[\Omega^{m,k}_{\Sigma;\mu,b}f(z)\right]'}\right]=h(z) \tag{3.3}$$

and

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1-w}{1+w}-1-w\frac{\left[\Omega^{m,k}_{\Sigma;\mu,b}g(w)\right]''}{\left[\Omega^{m,k}_{\Sigma;\mu,b}g(w)\right]'}\right]=p(w), \tag{3.4}$$

where h and p satisfy the conditions of Definiton 1. Furthermore, the functions h and p have the following Taylor-Maclaurin series expensions:

$$h(z)=1+h_1z+h_2z^2+\cdots$$

and

$$p(w)=1+p_1w+p_2w^2+\cdots,$$

respectively. Now, equating the coefficients in (3.3) and (3.4), we get

$$\frac{2\left[(\alpha+1)-2C_2^m(b,\mu,k)a_2\right]}{\alpha-1}=h_1, \tag{3.5}$$

$$\frac{2\left[(\alpha+1)-2\left(3C_3^m(b,\mu,k)a_3-2\left[C_2^m(b,\mu,k)\right]^2a_2^2\right)\right]}{\alpha-1}=h_2 \tag{3.6}$$

and

$$-\frac{2\left[(\alpha+1)-2C_2^m(b,\mu,k)a_2\right]}{\alpha-1}=p_1, \tag{3.7}$$

$$\frac{2\left[(\alpha+1)+2\left\{2\left[C_2^m(b,\mu,k)\right]^2a_2^2-3C_3^m(b,\mu,k)\left(2a_2^2-a_3\right)\right\}\right]}{\alpha-1}=p_2. \tag{3.8}$$

From (3.5) and (3.7), we find that

$$h_1 = -p_1. \tag{3.9}$$

Also, from (3.5), we can write

$$a_2 = \frac{\alpha+1}{2C_2^m(b,\mu,k)} - \frac{h_1(\alpha-1)}{4C_2^m(b,\mu,k)}. \tag{3.10}$$

Next, by using (3.5), (3.7), (3.9) and (3.10), we get

$$a_2^2 = \frac{(\alpha+1)^2}{4\left[C_2^m(b,\mu,k)\right]^2} + \frac{(\alpha-1)^2\left(h_1^2+p_1^2\right)}{32\left[C_2^m(b,\mu,k)\right]^2} - \frac{\left(\alpha^2-1\right)(h_1-p_1)}{8\left[C_2^m(b,\mu,k)\right]^2}. \tag{3.11}$$

By adding (3.6) to (3.8), we get

$$a_2^2 = \frac{(\alpha-1)(h_2+p_2)}{8\left(2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right)} - \frac{\alpha+1}{2\left(2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right)}. \tag{3.12}$$

Thus, we find from the equations (3.11) and (3.12) that

$$|a_2|^2 \le \frac{(\alpha+1)^2}{4\left[C_2^m(b,\mu,k)\right]^2} + \frac{(\alpha-1)^2\left(|h'(0)|^2+|p'(0)|^2\right)}{32\left[C_2^m(b,\mu,k)\right]^2} + \frac{(\alpha^2-1)(|h'(0)|+|p'(0)|)}{8\left[C_2^m(b,\mu,k)\right]^2}$$

and

$$|a_2|^2 \le \frac{(\alpha+1)}{2\left|2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right|} + \frac{(\alpha-1)(|h''(0)|+|p''(0)|)}{16\left|2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right|}.$$

Similarly, subtracting (3.8) from (3.6), we have

$$a_3 = a_2^2 - \frac{(\alpha-1)(h_2-p_2)}{24C_3^m(b,\mu,k)}. \tag{3.13}$$

Then, upon substituting the value of a_2^2 from (3.11) and (3.12) into (3.13), we deduce that

$$a_3 = \frac{(\alpha+1)^2}{4\left[C_2^m(b,\mu,k)\right]^2} + \frac{(\alpha-1)^2\left(h_1^2+p_1^2\right)}{32\left[C_2^m(b,\mu,k)\right]^2} - \frac{\left(\alpha^2-1\right)(h_1-p_1)}{8\left[C_2^m(b,\mu,k)\right]^2} - \frac{(\alpha-1)(h_2-p_2)}{24C_3^m(b,\mu,k)}$$
$$= \frac{(\alpha-1)(h_2+p_2)}{8\left(2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right)} - \frac{\alpha+1}{2\left(2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right)} - \frac{(\alpha-1)(h_2-p_2)}{24C_3^m(b,\mu,k)}.$$

Consequently, we have

$$|a_3| \le \frac{8(\alpha+1)^2+(\alpha-1)^2\left(|h'(0)|^2+|p'(0)|^2\right)+4\left(\alpha^2-1\right)(|h'(0)|+|p'(0)|)}{32\left[C_2^m(b,\mu,k)\right]^2} + \frac{(\alpha-1)(|h''(0)|+|p''(0)|)}{48C_3^m(b,\mu,k)}$$

and

$$|a_3| \le \frac{(\alpha-1)\left\{\left|3C_3^m(b,\mu,k)-\left[C_2^m(b,\mu,k)\right]^2\right||h''(0)|+\left[C_2^m(b,\mu,k)\right]^2|p''(0)|\right\}+12(\alpha+1)C_3^m(b,\mu,k)}{24C_3^m(b,\mu,k)\left|2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right|}.$$

This completes the proof. □

4. Conclusions

It is easily seen that, by specializing the functions $h(z)$ and $p(z)$ involved in Theorem 1, some results for could be expressed as illustrative examples:

Corollary 1. *If we set*

$$h(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}=1+2\gamma z+2\gamma^2z^2+\ldots \quad (0<\gamma\leq 1),$$

and

$$p(z)=\left(\frac{1-z}{1+z}\right)^{\gamma}=1-2\gamma z+2\gamma^2z^2+\ldots \quad (0<\gamma\leq 1),$$

then inequalities (3.1) and (3.2) become

$$|a_2|\leq\min\left\{\frac{(\alpha+1)+(\alpha-1)\gamma}{2C_2^m(b,\mu,k)},\sqrt{\frac{(\alpha-1)\gamma^2+(\alpha+1)}{2\left|2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right|}}\right\}$$

and

$$|a_3|\leq\min\left\{\frac{(\alpha+1)^2+(\alpha-1)^2\gamma^2+2(\alpha^2-1)\gamma}{4\left[C_2^m(b,\mu,k)\right]^2}+\frac{(\alpha-1)\gamma^2}{6C_3^m(b,\mu,k)},\right.$$

$$\left.\frac{(\alpha-1)\left\{\left|3C_3^m(b,\mu,k)-\left[C_2^m(b,\mu,k)\right]^2\right|\gamma^2+\left[C_2^m(b,\mu,k)\right]^2\gamma^2\right\}+3(\alpha+1)C_3^m(b,\mu,k)}{6C_3^m(b,\mu,k)\left|2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right|}\right\}.$$

Corollary 2. *If we let*

$$h(z)=\frac{1+(1-2\beta)z}{1-z}=1+2(1-\beta)z+2(1-\beta)z^2+\cdots \quad (0\leq\beta<1),$$

and

$$p(z)=\frac{1-(1-2\beta)z}{1+z}=1-2(1-\beta)z+2(1-\beta)z^2+\cdots \quad (0\leq\beta<1),$$

then inequalities (3.1) and (3.2) become

$$|a_2|\leq\min\left\{\frac{(\alpha+1)+(\alpha-1)(1-\beta)}{2C_2^m(b,\mu,k)},\sqrt{\frac{(\alpha-1)(1-\beta)+(\alpha+1)}{2\left|2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right|}}\right\}$$

and

$$|a_3|\leq\min\left\{\frac{(\alpha+1)^2+(\alpha-1)^2(1-\beta)^2+2(\alpha^2-1)(1-\beta)}{4\left[C_2^m(b,\mu,k)\right]^2}+\frac{(\alpha-1)(1-\beta)}{6C_3^m(b,\mu,k)},\right.$$

$$\left.\frac{(\alpha-1)\left\{\left|3C_3^m(b,\mu,k)-\left[C_2^m(b,\mu,k)\right]^2\right|(1-\beta)+\left[C_2^m(b,\mu,k)\right]^2(1-\beta)\right\}+3(\alpha+1)C_3^m(b,\mu,k)}{6C_3^m(b,\mu,k)\left|2\left[C_2^m(b,\mu,k)\right]^2-3C_3^m(b,\mu,k)\right|}\right\}.$$

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Authors' addresses

Şahsene Altınkaya
Bursa Uludag University, Department of Mathematics, 16059, Bursa, Turkey
E-mail address: sahsenealtinkaya@gmail.com

Sibel Yalçın
Bursa Uludag University, Department of Mathematics, 16059 Bursa, Turkey
E-mail address: syalcin@uludag.edu.tr

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ANALYSIS OF HIGHER ORDER DIFFERENCE METHOD FOR A PSEUDO-PARABOLIC EQUATION WITH DELAY

ILHAME AMIRALI

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Abstract. In this paper, the author considers the one dimensional initial-boundary problem for a pseudo-parabolic equation with time delay in second spatial derivative. To solve this problem numerically, the author constructs higher order difference method and obtain the error estimate for its solution. Based on the method of energy estimates the fully discrete scheme is shown to be convergent of order four in space and of order two in time. Some numerical examples illustrate the convergence and effectiveness of the numerical method.

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1. Introduction

In the domain $\overline{Q} = \overline{\Omega} \times [0,T]$; $\overline{\Omega} = [0,l]$, $Q = \Omega \times (0,T]$, $\Omega = (0,l)$, we consider the following pseudo-parabolic equation with delay (DPPEs)

$$\frac{\partial u(x,t)}{\partial t} - a(t)\frac{\partial^3 u(x,t)}{\partial t \partial x^2} = b(t)\frac{\partial^2 u(x,t)}{\partial x^2} + c(t)\frac{\partial^2 u(x,t-r)}{\partial x^2}$$
$$+ d(t)u(x,t) + f(x,t), (x,t) \in Q, \tag{1.1}$$
$$u(x,t) = \phi(x,t), (x,t) \in \overline{\Omega} \times [-r,0], \tag{1.2}$$
$$u(0,t) = u(l,t) = 0, t \in (0,T], \tag{1.3}$$

where $r > 0$ represents the delay parameter, $a \geqslant \alpha > 0$, b, c, d, f and ϕ are given sufficiently smooth functions satisfying certain regularity conditions to be specified.

Pseudo-parabolic or Sobolev-type differential equations appears in a variety of physical problems such as flow of fluid through fissured rocks, thermodynamics and propagation of long waves of small amplitude (see, e.g. [9, 24, 25]). The significant characteristic of these equations is that they state the conservation of a certain quantity (mass, momentum, heat, etc.) in any sub-domain. Such problems are interesting not only because they are generalizations of a standard parabolic problem, but also because they arise naturally in a large variety of applications. Various numerical schemes have been constructed to treat PPEs in [2, 3, 5, 6, 10, 12, 14, 15, 23] (see also

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the references cited in them). Not only the existence, uniqueness and nonexistence results for pseudo-parabolic equations were obtained, but also the asymptotic behavior, regularity and others properties of solutions were investigated. For example, in [6] the initial-boundary value problem for a linear PPEs with boundary layers is considered. They developed an exponentially fitted difference scheme and get its discrete energy estimation. The difference scheme is constructed by the method of integral identities with the use of exponential basis functions and interpolating quadrature rules with the weight and remainder terms in integral form. Both explicit and implicit in time discretization schemes have been developed in [10] which were based on the piecewise linear finite elements for the solution a pseudo parabolic Burgers equation. In [12] a Crank-Nicolson-Galerkin approximation with extrapolated coefficients is presented for three cases for the nonlinear PPEs along with a conjugate gradient iterative procedure which can be used efficiently to solve the different linear systems of algebraic equations arising at each step from the Galerkin method. In [23] authors presented two different schemes with respect to artificial diffusion parameter using extension of the finite difference streamline diffusion method for linear Sobolev equations with convection-dominated term. Further in [15] two difference approximation schemes to a nonlinear pseudo-parabolic equation are developed. Each of these schemes possesses a unique solution which can be obtained by an iterative procedure. The equivalence of the three different formulations for the PPEs and different time discrete (implicit or semi-implicit) numerical schemes has been discussed in [14]. The one-dimensional initial-boundary value problem for a linear PPEs with initial jump is studied in [5]. They developed a numerical method which combines a finite difference spatial discretization on uniform mesh and the implicit rule on Shishkin mesh (S-mesh) for the time variable. For a discussion of existence and uniqueness results of PPEs see [8, 13, 18]. The above mentioned papers, related with PPEs were only concerned with the cases without delay. Also delay pseudo-parabolic equations (DPPEs) frequently arise in many scientific applications. For works on existence and uniqueness results and for applications of DPPEs, see [11, 16]. In [4] for solving one dimensional initial-boundary delay PPE numerically, authors constructed high-order finite difference technique to the considered problem and obtain the error estimate for its solution. In [17] authors gave fourth order differential-difference scheme for solving one dimensional initial-boundary DPPE and obtain the error estimate for its solution. Further, the fourth order accurate Runge-Kutta method was used for the realization of acquired differential-difference problem. In [1] authors considered the explicit finite difference method for quasilinear DPPEs and proved that the fully discrete scheme is absolutely stable and convergent of order two in space and of order one in time variable. In [22] a super accurate numerical scheme to solve the one-dimensional Sobolev type partial differential equation with an initial and two nonlocal integral boundary conditions is considered. This methods are based on the shifted Standard and shifted Chebyshev Tau method. In [7] the abstract quasilinear

evolution equations of Sobolev type in a Hilbert setting are considered. Authors proposed two fully discrete schemes and proved some error estimates under minimal assumptions. In [20, 21] the authors present a regularity result for solutions of parabolic equations in the framework of mixed Morrey spaces. [19] aims at defining new spaces and to study some embeddings between them. These spaces generalize Morrey spaces and give a refinement of Lebesgue spaces. Some embeddings between these new classes are also proved. The authors apply these classes of functions to obtain regularity results for solutions of partial differential equations of parabolic type in nondivergence form.

The present study is concerned with the one dimensional pseudo-parabolic equation containing time delay in second-order spatial derivative. Our aim is to construct higher order difference method for approximation to the considered problem when the coefficients are independent of spatial variable. Based on the method of energy estimates and difference analogue of the Gronwall's inequality with delay, the fully discrete scheme is shown to be convergent of order four in space and of order two in time. Numerical example on the performance of the method is presented.

2. The mesh and difference scheme

2.1. *Notation*

Let a set of nodes that discretises Q be given by $\omega = \omega_N \times \omega_{N_0}$ with

$$\omega_N = \{x_i = ih,\, i = 1,2,...,N-1,\, h = l/N\},$$
$$\omega_{N^+} = \omega_N \cup \{x_N = l\},\ \bar{\omega}_N = \omega_N \cup \{x_0 = 0,\ x_N = l\},$$
$$\omega_{N_0} = \left\{t_j = j\tau,\ \ j = 1,2,...,N_0,\, \tau = T/N_0 = r/n_0\right\},$$
$$\bar{\omega}_{N_0} = \omega_{N_0} \cup \{t_0 = 0\},\ \bar{\omega} = \bar{\omega}_N \times \bar{\omega}_{N_0},$$
$$\omega_{n_0} = \left\{t_j = j\tau,\ \ j = 1,2,...,n_0,\ \ \tau = r/n_0\right\},$$
$$\omega_{n_0^-} = \left\{t_j = j\tau,\ \ j = -n_0,...,0,\ \ \tau = r/n_0\right\}$$

and define the following finite differences and notation

$$v^j_{\bar{x},i} = \frac{v^j_i - v^j_{i-1}}{h}, v^j_{\bar{x}x,i} = \frac{v^j_{i+1} - 2v^j_i + v^j_{i-1}}{h^2}, v^j_{\bar{t},i} = \frac{v^j_i - v^{j-1}_i}{\tau},$$
$$v^j_{\bar{t}t,i} = \frac{v^{j+1}_i - 2v^j_i + v^{j-1}_i}{\tau^2}, v^{(0.5)j}_i = \frac{v^j_i + v^{j-1}_i}{2}, v^{j-0.5}_i = v(x_i, t_j - \frac{\tau}{2})$$

for any mesh function $v^j_i = v(x_i, t_j)$ given on $\bar{\omega}$.

Introduce the following inner product and norm for the mesh functions v_i and w_i

$$(v,w) \equiv (v,w)_{\omega_N} = \sum_{i=1}^{N-1} h v_i w_i,\ (v,w)_{\omega_{N^+}} = (v,w)_{\omega_N} \sum_{i=1}^{N} h v_i w_i,$$

$$\|v\|^2 = (v,v), \quad \|v_{\bar{x}}\|^2 = (v_{\bar{x}}, v_{\bar{x}})_{\omega_{N+}}, \ (v_0 = v_N = 0).$$

2.2. *Difference scheme*

To construct the difference scheme, we will use the following relation which is valid for any $g(x) \in C^6(\overline{\Omega})$

$$\frac{1}{12}\left[g''(x_{i+1}) + 10g''(x_i) + g''(x_{i-1})\right] = g_{\bar{x}x,i} + \bar{R}_i, \tag{2.1}$$

where

$$\bar{R}_i = h^{-1}\int_{x_{i-1}}^{x_{i+1}} \frac{\partial^6 g}{\partial x^6}(\xi_i)\,\Lambda(\xi)d\xi = \frac{h^4}{240}\frac{\partial^6 g}{\partial x^6}(\xi_i),$$

$$\Lambda(\xi) = \begin{cases} \frac{h}{72}(x_{i+1}-\xi)^3 - \frac{h^{-1}}{120}(x_{i+1}-\xi)^5, \xi > x_i \\ \frac{h}{72}(\xi - x_{i-1})^3 - \frac{h^{-1}}{120}(\xi - x_{i-1})^5, \xi < x_i \end{cases}, \xi_i \in (x_{i-1}, x_{i+1}).$$

Using formula (2.1) we get

$$\frac{1}{12}\left[\frac{\partial^3 u(x_{i+1},t)}{\partial t\,\partial x^2} + 10\frac{\partial^3 u(x_i,t)}{\partial t\,\partial x^2} + \frac{\partial^3 u(x_{i-1},t)}{\partial t\,\partial x^2}\right] = u_{\bar{x}x,i}(t) + \frac{h^4}{240}\frac{\partial^7 u(\xi_i,t)}{\partial t\,\partial x^6},$$

$$\frac{1}{12}\left[\frac{\partial^2 u(x_{i+1},t)}{\partial x^2} + 10\frac{\partial^2 u(x_i,t)}{\partial x^2} + \frac{\partial^2 u(x_{i-1},t)}{\partial x^2}\right]$$

$$= u_{\bar{x}x,i}(t) + \frac{h^4}{240}\frac{\partial^6 u(\xi_i,t)}{\partial x^6},$$

$$\frac{1}{12}\left[\frac{\partial^2 u(x_{i+1},t-r)}{\partial x^2} + 10\frac{\partial^2 u(x_i,t-r)}{\partial x^2} + \frac{\partial^2 u(x_{i-1},t-r)}{\partial x^2}\right]$$

$$= u_{\bar{x}x,i}(t-r) + \frac{h^4}{240}\frac{\partial^6 u(\xi_i,t-r)}{\partial x^6}.$$

Note also that

$$\frac{1}{12}\left[\frac{\partial^3 u(x_{i+1},t)}{\partial t\,\partial x^2} + 10\frac{\partial^3 u(x_i,t)}{\partial t\,\partial x^2} + \frac{\partial^3 u(x_{i-1},t)}{\partial t\,\partial x^2}\right] = u_i'(t) + \frac{h^2}{12}u'_{\bar{x}x,i}(t)$$

and

$$\frac{1}{12}[u(x_{i+1},t) + 10u(x_i,t) + u(x_{i-1},t)] = u_i(t) + \frac{h^2}{12}u_{\bar{x}x,i}(t),$$

then we obtain the semi-discrete relation on $\omega_N \times [0,T]$ for equation (1.1)

$$u_i'(t)-(a(t)-\frac{h^2}{12})u_{\bar{x}x,i}'(t)=(b(t)+d(t)\frac{h^2}{12})u_{\bar{x}x,i}(t)+c(t)u_{\bar{x}x,i}(t-r)+d(t)u_i(t)$$
$$+\bar{f_i}(t)+R_i{}^{(0)}(t),\ i=1,2,...,N-1,\ t\in(0,T], \tag{2.2}$$

with

$$\bar{f_i}(t)=\frac{1}{12}[f_{i+1}(t)+10f_i(t)+f_{i-1}(t)],$$

$$R_i{}^{(0)}(t)=a(t)\frac{h^4}{240}\frac{\partial^7 u(\xi_i,t)}{\partial t\partial x^6}+b(t)\frac{h^4}{240}\frac{\partial^6 u(\xi_i,t)}{\partial x^6}+c(t)\frac{h^4}{240}\frac{\partial^6 u(\xi_i,t-r)}{\partial x^6},$$
$$\xi_i\in(x_{i-1},x_{i+1}).$$

Setting $t=t_{j-0.5}=t_j-\frac{\tau}{2}$ in (2.2) and taking into account there the relations

$$u_i'(t_{j-0.5})=u_{\bar{t},i}^j-\frac{\tau^2}{24}\frac{\partial^3 u_i(x_i,\eta_j^{(1)})}{\partial t^3},$$

$$u_{\bar{x}x,i}'(t_{j-0.5})=u_{\bar{t}\bar{x}x,i}^j-\frac{\tau^2}{24}\frac{\partial^5 u_i(x_i,\eta_j^{(2)})}{\partial t^3\partial x^2},$$

$$u_i(t_{j-0.5})=\frac{u_i^j+u_i^{j-1}}{2}-\frac{\tau^2}{8}\frac{\partial^2 u_i(x_i,\eta_j^{(3)})}{\partial t^2},$$

$$u_{\bar{x}x,i}(t_{j-0.5})=\frac{u_{\bar{x}x,i}^j+u_{\bar{x}x,i}^{j-1}}{2}-\frac{\tau^2}{8}\frac{\partial^4 u_i(x_i,\eta_j^{(3)})}{\partial t^2\partial x^2},$$

$$u_i(t_{j-0.5}-r)=\frac{u_i^{j-n_0}+u_i^{j-n_0-1}}{2}-\frac{\tau^2}{8}\frac{\partial^2 u_i(x_i,\bar{\eta}_j^{(1)})}{\partial t^2},$$
$$u_{\bar{x}x,i}(t_{j-0.5}-r)=\frac{u_{\bar{x}x,i}^{j-n_0}+u_{\bar{x}x,i}^{j-n_0-1}}{2}-\frac{\tau^2}{8}\frac{\partial^4 u_i(x_i,\bar{\eta}_j^{(2)})}{\partial t^2\partial x^2},$$
$$t_{j-1}<\eta_j^{(k)}<t_j, k=1,2,3,4; t_{j-n_0-1}<\bar{\eta}_j^{(k)}<t_{j-n_0}, k=1,2,$$

we get

$$u_{\bar{t},i}^j-(a(t_{j-0.5})-\frac{h^2}{12})u_{\bar{t}\bar{x}x,i}^j=(b(t_{j-0.5})+d(t_{j-0.5})\frac{h^2}{12})u_{\bar{x}x,i}^{(0.5)j}$$
$$+c(t_{j-0.5})u_{\bar{x}x,i}^{(0.5)(j-n_0)}+d(t_{j-0.5})u_i^{(0.5)j}$$
$$+\bar{f}_i^j+R_i^j, i=1,2,...,N-1; j=1,2,...,N_0, \tag{2.3}$$

where

$$R_i^j=R_i^{(0)j}+R_i^{(1)j},$$

$$R_i^{(1)j} = -\frac{\tau^2}{24}\left(\frac{\partial^3 u_i(x_i,\eta_j^{(1)})}{\partial t^3} + \frac{\partial^5 u_i(x_i^{(1)},\eta_j^{(2)})}{\partial t^3 \partial x^2}\right)$$
$$-\frac{\tau^2}{8}\left(\frac{\partial^2 u_i(x_i,\eta_j^{(3)})}{\partial t^2} + \frac{\partial^4 u_i(x_i^{(2)},\eta_j^{(4)})}{\partial t^2 \partial x^2}\right)$$
$$+\frac{\tau^2}{8}\left(\frac{\partial^2 u_i(x_i,\bar{\eta}_j^{(1)})}{\partial t^2} + \frac{\partial^4 u_i(x_i,\bar{\eta}_j^{(2)})}{\partial t^2 \partial x^2}\right).$$

Since

$$v^{(0.5)j} = v^{j-1} + \frac{\tau}{2} v_{\bar{t}}^j,$$

we then obtain

$$(1-d_{j-0.5}\frac{\tau}{2})u_{\bar{t},i}^j - (a_{j-0.5} - \frac{h^2}{12} + \frac{\tau}{2}(b_{j-0.5} + c_{j-0.5}\frac{h^2}{12}))u_{\bar{t}\bar{x}x,i}^j$$
$$= (b_{j-0.5} + d_{j-0.5}\frac{h^2}{12})u_{\bar{x}x,i}^{j-1} + c_{j-0.5}u_{\bar{x}x,i}^{(0.5)(j-n_0)} + d_{j-0.5}u_i^{j-1}$$
$$+ \bar{f}_{\hat{t}}^j + R_i^j, i = 1,2,...,N-1; j = 1,2,...,N_0. \tag{2.4}$$
$$u_i^j = \phi_i^j, i = 1,2,...,N-1; j = -n_0, -n_0+1,...,0, \tag{2.5}$$
$$u_0^j = u_N^j = 0, j = 1,2,...,N_0. \tag{2.6}$$

Neglecting the remainder term R_i^j in (2.4), we propose the following difference scheme for approximating (1.1)-(1.3):

$$E^j y_{\bar{t},i}^j - A^j y_{\bar{t}\bar{x}x,i}^j = B^j y_{\bar{x}x,i}^{j-1} + C^j y_{\bar{x}x,i}^{(0.5)(j-n_0)}$$
$$+ \bar{f}_i^j, i = 1,2,...,N-1, j = 1,2,...,N_0, \tag{2.7}$$
$$y_i^j = \phi_i^j, i = 1,2,...,N-1, j = -n_0, -n_0+1,...,0, \tag{2.8}$$
$$y_0^j = y_N^j = 0, j = 1,2,...,N_0, \tag{2.9}$$

where

$$E^j = (1-d_{j-0.5}\frac{\tau}{2}), A^j = a_{j-0.5} - \frac{h^2}{12} + \frac{\tau}{2}(b_{j-0.5} + c_{j-0.5}\frac{h^2}{12}),$$
$$B^j = b_{j-0.5} + d_{j-0.5}\frac{h^2}{12}, C^j = c_{j-0.5}.$$

For the error function $z = y - u$, from the relations (2.4)-(2.6) and (2.7)-(2.9), we have the following difference problem

$$E^j z_{\bar{t},i}^j - A^j z_{\bar{t}\bar{x}x,i}^j = B^j z_{\bar{x}x,i}^{j-1} + C^j z_{\bar{x}x,i}^{(0.5)(j-n_0)}$$
$$+ D^j z_i^{j-1} + R_i^j, i = 1,2,...,N-1, j = 1,2,...,N_0, \tag{2.10}$$

$$z_i^j = 0, i = 1,2,...,N-1, j = -n_0, -n_0+1, ..., 0, \tag{2.11}$$

$$z_0^j = z_N^j = 0, j = 1,2,...,N_0. \tag{2.12}$$

The following lemma is used to discuss the stability and convergence properties of our discrete problem (2.7)-(2.9).

Lemma 1 ([4]). *Let the mesh function $\delta \geqslant 0$, defined on ω_{N_0}, satisfies*

$$\delta_j \leqslant \alpha + \tau \sum_{k=1}^{j} \{a\delta_k + b\delta_{k-1} + c\delta_{k-N} + d\delta_{k-N-1} + f_k\}, j \geqslant 1 \tag{2.13}$$

$$\delta_j \leqslant \phi_j, -N \leqslant j \leqslant 0, \delta_0 \leqslant \alpha,$$

where $\alpha, a, b, c, d, f_j \geqslant 0$, ϕ_j given, $N \geqslant 0$ integer, $1 - \tau a > 0$.

Then

$$\delta_j \leqslant \tilde{\alpha} e^{\gamma t_j} + \frac{\tau}{1-\tau a} \sum_{k=1}^{j} f_k e^{\gamma t_{j-k}}, \tag{2.14}$$

where

$$\tilde{\alpha} = \alpha + (c+d)\|\phi\|_1, \gamma = \frac{a+b+c+d}{1-\tau a}, \|\phi\|_1 = \sum_{j=-N}^{0} \tau \mid \phi_j \mid.$$

3. Error analysis and convergence

Now we give the main result of this paper.

Theorem 1. *Let the derivatives $\frac{\partial^7 u}{\partial t \partial x^6}, \frac{\partial^6 u}{\partial x^6}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial^4 u}{\partial t^2 \partial x^2}$ are bounded on the $\overline{Q}$ and $E^j \geqslant \beta_* > 0$, $A^j \geqslant \alpha_* > 0$.*

Then the error of the problem (2.7)-(2.9) satisfies

$$\|y-u\| + \|y_{\overline{x}} - u_{\overline{x}}\| \leqslant C(h^4 + \tau^2), \tag{3.1}$$

where C is a constant which is independent of h and τ.

Proof. Consider the following identity

$$(E^j z_{\bar{t}}^j, z_{\bar{t}}^j) - (A^j z_{\bar{t}\bar{x}x}^j, z_{\bar{t}}^j) = (B^j z_{\bar{x}x}^{j-1}, z_{\bar{t}}^j)$$
$$+ (C^j z_{\bar{x}x}^{(0.5)(j-n_0)}, z_{\bar{t}}^j) + (D^j z^{j-1}, z_{\bar{t}}^j) + (R^j, z_{\bar{t}}^j).$$

After some manipulations, we get

$$\beta_* \left\| z_{\bar{t}}^j \right\|^2 + \alpha_* \left\| z_{\bar{t}\bar{x}}^j \right\|^2 \leq B^* \left\| z_{\bar{t}\bar{x}}^j \right\| \left\| z_{\bar{x}}^{j-1} \right\| + C^* \left\| z_{\bar{t}\bar{x}}^j \right\| \left\| z_{\bar{x}}^{(0.5)(j-n_0)} \right\|$$
$$+ D_* \left\| z_{\bar{t}}^j \right\| \left\| z^{j-1} \right\| + \left\| z_{\bar{t}}^j \right\| \left\| R^j \right\|,$$

where $\mid B^j \mid \leq B^*$, $\mid C^j \mid \leq C^*$ and $\mid D^j \mid \leq D^*$.

From here, using the inequality $|ab| \leq \mu a^2 + (1/4\mu)b^2$ $(\mu > 0)$ we have

$$(\beta_* - \mu_1 - \mu_2)\left\|z_{\bar{t}}^j\right\|^2 + (\alpha_* - \mu_3 - \mu_4)\left\|z_{\bar{t}\bar{x}}^j\right\|^2 \leq \frac{1}{4\mu_3}(B^*)^2\left\|z_{\bar{x}}^{j-1}\right\|^2$$

$$+\frac{1}{4\mu_4}(C^*)^2\left\|z_{\bar{x}}^{(0.5)(j-n_0)}\right\|^2 + \frac{1}{4\mu_1}(D^*)^2\left\|z^{j-1}\right\|^2 + \frac{1}{4\mu_2}\left\|R^j\right\|^2.$$

After choosing, $\mu_1 = \mu_2 = \frac{\beta_*}{4}$ and $\mu_3 = \mu_4 = \frac{\alpha_*}{2}$ this inequality reduces to

$$\beta_*\left\|z_{\bar{t}}^j\right\|^2 + \alpha_*\left\|z_{\bar{t}\bar{x}}^j\right\|^2 \leq \frac{(B^*)^2}{\alpha_*}\left\|z_{\bar{x}}^{j-1}\right\|^2 + \frac{(C^*)^2}{\alpha_*}\left\|z_{\bar{x}}^{(0.5)(j-n_0)}\right\|^2$$

$$+\frac{(D^*)^2}{\beta_*}\left\|z^{j-1}\right\|^2 + \frac{1}{\beta_*}\left\|R^j\right\|^2.$$

Multiplying this inequality by $T\tau$ and summing it up from $k = 1$ to $k = j$, using the inequality

$$v_j^2 \leqslant t_j\tau\sum_{k=1}^{j} v_{\bar{t},k}^2 \leqslant T\tau\sum_{k=1}^{j} v_{\bar{t},k}^2, (v_0 = 0)$$

we obtain

$$\beta_*\left\|z^j\right\|^2 + \alpha_*\left\|z_{\bar{x}}^j\right\|^2 \leqslant T\tau\sum_{k=1}^{j}\left\{(B^*)^2\alpha_*^{-1}\left\|z_{\bar{x}}^{j-1}\right\|^2 + (C^*)^2\alpha_*^{-1}\left\|z_{\bar{x}}^{(0.5)(j-n_0)}\right\|^2\right.$$

$$\left.+(D^*)^2\beta_*^{-1}\left\|z^{j-1}\right\|^2 + \beta_*^{-1}\left\|R^j\right\|^2\right\}.$$

Denoting now

$$\delta_j = \beta_*\left\|z^j\right\|^2 + \alpha_*\left\|z_{\bar{x}}^j\right\|^2,$$

we have

$$\delta_j \leqslant \sum_{k=1}^{j}\tau\left\{c_1\delta_{k-1} + c_2\delta_{k-n_0} + c_2\delta_{k-n_0-1} + \rho_k\right\}, j \geqslant 1,$$

with

$$c_1 = T\max\left\{(B^*)^2\alpha_*^{-2}, (D^*)^2\beta_*^{-2}\right\},$$

$$c_2 = T(C^*)^2\alpha_*^{-2}.$$

$$\rho_k = T\beta_*^{-1}\left\|R^k\right\|^2.$$

Applying now Lemma 1 we obtain

$$\beta_* \left\| z^j \right\|^2 + \alpha_* \left\| z_{\bar{x}}^j \right\|^2 \leqslant T\beta_*^{-1}\tau \sum_{k=1}^{j} e^{(c_1+c_2)t_{j-k}} \left\| R^k \right\|^2. \tag{3.2}$$

It is not difficult to see that, under the assumed smoothness

$$(\tau \sum_{k=1}^{N_0} \left\| R^k \right\|^2)^{1/2} = O(h^4 + \tau^2),$$

which together with (3.2), completes the proof of the theorem. □

4. Numerical results

In this section we present some numerical results for the scheme discussed in this paper. Consider the problem

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^3 u(x,t)}{\partial t \partial x^2} = \frac{\partial^2 u(x,t)}{\partial x^2} + 2\frac{\partial^2 u(x,t-1)}{\partial x^2} - u(x,t) = 50e^{1-t} sinh(x).$$

$$u(x,t) = e^{-t}(x\sinh(1) - \sinh(x)), (x,t) \in [0,1] \times [-1,0],$$

$$u(0,t) = u(1,t) = 0, t \in (0,2].$$

The exact solution is given by

$$u(x,t) = 25e^{-t}(x\sinh(1) - \sinh(x)).$$

The computational results are presented in Table 1 and Table 2.

TABLE 1. The numerical results on $(0,1) \times (0,1)$

Nodes (x,t)	Exact Solution	Numerical Solution $h = 0.1, \tau = 0.05$	Pointwise Error $\lvert y - u \rvert$
(0.1,0.1)	0.392565	0.392566	0.47440E-05
(0.2,0.2)	0.689985	0.690003	0.74700E-04
(0.3,0.3)	0.889722	0.889745	0.94742E-04
(0.4,0.4)	0.994252	0.994276	0.99745E-04
(0.5,0.5)	1.008508	1.008538	0.12070E-03
(0.6,0.6)	0.939428	0.939452	0.96937E-04
(0.7,0.7)	0.795281	0.795303	0.90725E-04
(0.8,0.8)	0.584801	0.584818	0.69740E-04
(0.9,0.9)	0.316850	0.316851	0.46582E-05

We see from the above tables that these results display a well agreement with our theoretical analysis.

TABLE 2. The numerical results on $(0,1)\times(1,2)$

Nodes (x,t)	Exact Solution	Numerical Solution $h=0.1, \tau=0.05$	Pointwise Error $\lvert y-u\rvert$
(0.1,1.1)	0.144415	0.144418	0.14730E-04
(0.2,1.2)	0.253786	0.253794	0.32855E-04
(0.3,1.3)	0.327310	0.327327	0.70685E-04
(0.4,1.4)	0.365754	0.365770	0.64967E-04
(0.5,1.5)	0.371012	0.371033	0.86997E-04
(0.6,1.6)	0.345599	0.345623	0.98410E-04
(0.7,1.7)	0.292567	0.292588	0.85730E-04
(0.8,1.8)	0.215136	0.215154	0.73182E-04
(0.9,1.9)	0.116528	0.116535	0.28105E-04

5. CONCLUSIONS

In this paper, the higher order difference method is applied to the problem (1.1)-(1.3). Based on the method of energy estimates, the fully discrete scheme was shown to be convergent of order four in space and of order two in time. To demonstrate the accuracy and usefulness of this method, numerical example has been presented.

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Author's address

Ilhame Amirali
Duzce University, Faculty of Arts and Sciences, Department of Mathematics, 81620, Duzce, Turkey
E-mail address: ailhame@gmail.com

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A FAMILY OF LACUNARY RECURRENCES FOR FIBONACCI NUMBERS

CRISTINA BALLANTINE AND MIRCEA MERCA

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Abstract. We introduce an infinite family of lacunary recurrences for the Fibonacci numbers and give a combinatorial proof. The first entry in the family was proved by Lucas in 1876.

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1. Introduction

A lacunary recurrence for a given sequence is a recurrence relation involving only terms of the sequence with indices in an arithmetic progression. We refer to the common difference in the indices in arithmetic progression as the gap of the lacunary recurrence. Lacunary recurrences allow for faster computation of sequence terms. The speed of the computation increases with the size of the gap. Lacunary recurrences have been studied for many types of sequences including Bernoulli numbers [1, 8, 9, 11], Euler numbers [8], k-Fibonacci numbers (which are Fibonacci polynomials at positive integer values) [4], Eisenstein series [10], Tribonacci numbers [6], more general sequences that include Bernoulli, Euler, Fibonacci and Genocchi numbers [5], and sequences satisfying an arbitrary linear recurrence [3, 12, 13].

In this article we provide an infinite family of lacunary recurrences for Fibonacci numbers.

By definition, the first two Fibonacci numbers are 0 and 1, and each subsequent number is the sum of the previous two. Thus, the sequence $\{F_n\}_{n\geq 0}$ of Fibonacci numbers is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

with seed values $F_0 = 0$ and $F_1 = 1$. In this article, we set $F_n = 0$ for $n < 0$.

Lucas proved in 1876 that for every positive integer n the Fibonacci numbers of even indices can be expressed in terms of the previous Fibonacci numbers of odd

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indices

$$F_{2n} = \sum_{k=1}^{n} F_{2k-1} \tag{1.1}$$

and vice versa

$$F_{2n+1} = 1 + \sum_{k=1}^{n} F_{2k}. \tag{1.2}$$

After easy manipulations, one can rewrite the relations (1.1) and (1.2) as

$$F_n = \frac{1+(-1)^n}{2} + F_{n-2} + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} F_{n-2k}. \tag{1.3}$$

This identity involves only indices in an arithmetic progression of length 2 and is thus a lacunary recurrence of gap 2.

We generalize Lucas's identity (1.3) to lacunary recurrences with gap of any size for the Fibonacci numbers.

Theorem 1. *Given a positive integer $N \geqslant 2$, we have*

$$F_n = F_N \cdot F_{N-1}^{\lfloor \frac{n-1}{N} \rfloor - 1} \cdot F_{(n-1) \bmod N} + F_{N+1} \cdot F_{n-N} + F_N^2 \cdot \sum_{k=2}^{\lfloor \frac{n-1}{N} \rfloor} F_{N-1}^{k-2} \cdot F_{n-kN},$$

for all $n \geqslant N$.

As a consequence of this theorem, we have the following congruence identity.

Corollary 1. *For a fixed integer $N \geqslant 2$ we have*

$$F_n - F_{N+1} \cdot F_{n-N} \equiv 0 \pmod{F_N}$$

for all $n \geqslant N$.

Using the relation [2, p. 4, Identity 3]

$$F_n - F_{N+1} \cdot F_{n-N} = F_N \cdot F_{n-N-1},$$

Theorem 1 can be written as follows.

Corollary 2. *For $n \geqslant 0$, $N \geqslant 2$,*

$$F_{n+N} = F_{n \bmod N} \cdot F_{N-1}^{\lfloor n/N \rfloor + 1} + F_N \cdot \sum_{k=0}^{\lfloor n/N \rfloor} F_{N-1}^{k} \cdot F_{n+1-kN}.$$

If $N = 2$, we recover Lucas's identities in the concise form (1.3).

If $N = 3$, we recover the result of [4] (case $k = 1$ of Corollary 5, part (2)):

$$F_{3n} = 2 \sum_{k=1}^{n} F_{3k-2},$$

$$F_{3n+1} = 1 + 2\sum_{k=1}^{n} F_{3k-1},$$

$$F_{3n+2} = 1 + 2\sum_{k=1}^{n} F_{3k}.$$

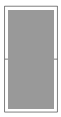
For $N = 4$ the statement of Theorem 1 becomes

$$F_n = 3 \cdot 2^{\lfloor \frac{n-1}{4} \rfloor - 1} \cdot F_{(n-1) \bmod 4} + 5 \cdot F_{n-4} + 3^2 \cdot \sum_{k=2}^{\lfloor \frac{n-1}{4} \rfloor} 2^{k-2} \cdot F_{n-4k}$$

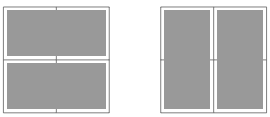
and thus $F_n - 5F_{n-4} \equiv 0 \pmod 3$.

2. A COMBINATORIAL PROOF OF THEOREM 1

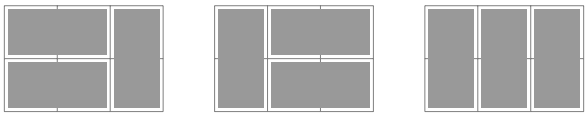
In [7], Fibonacci numbers are interpreted combinatorially in terms of tilings of a $2 \times n$ grid with dominos. The author then provides combinatorial proofs for several identities, including Lucas's identities mentioned in the introduction. In particular, for $n \geqslant 1$, the $(n+1)$st Fibonacci number, F_{n+1}, equals the number of distinct domino tilings of the $2 \times n$ grid. The empty 2×0 grid has, by convention, one tiling. For example, if $n = 1$ there is only $F_2 = 1$ such tiling.



If $n = 2$, there are $F_3 = 2$ tilings.



If $n = 3$, there are $F_4 = 3$ tilings.

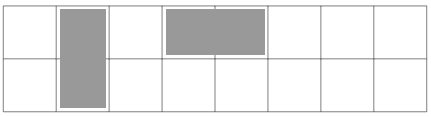


Obviously, the first row in a tiled $2 \times n$ grid completely determines the second row and the combinatorial interpretation given above is equivalent to the interpretation in [2] given in terms of tilings of the $1 \times n$ grid by (horizontal) dominos and squares. For our proof we will use the above interpretation in terms of domino tilings of the $2 \times n$ grid.

We denote by a_n the number of distinct domino tilings of the $2 \times n$ grid. As mentioned above, we set $a_0 = 1$. Then $a_n = F_{n+1}$.

Definition 1. The position of a domino in a tiling is defined as the column number i of its most left box.

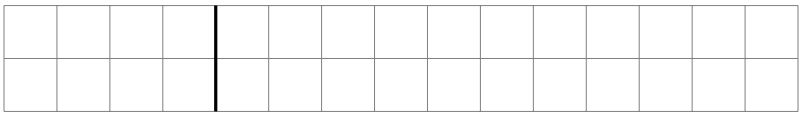
For example, the position of the shaded horizontal domino in the figure below is 4 while the position of the vertical domino is 2.



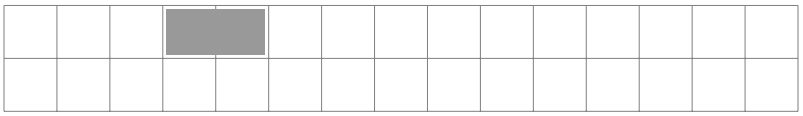
We will never have to refer to the position of horizontal dominos in the second row as they are determined by the horizontal dominos in the first row.

To prove Theorem 1, we count the domino tilings of the $2\times n$ grid according to whether there is a horizontal domino with position N or not.

The number of tilings of the $2\times n$ grid with no horizontal domino with position N equals $a_N\cdot a_{n-N}$ since in this case the left $2\times N$ grid can be tiled independently of the right $2\times(n-N)$ grid.



If there is a horizontal domino with position N, by the same argument, there are $a_{N-1}\cdot a_{n-N-1}$ such tilings.



Thus,

$$a_n = a_N\cdot a_{n-N} + a_{N-1}\cdot a_{n-N-1}.$$

Next, we count the tilings of the $2\times(n-N-1)$ grid in a similar manner according to whether there is a horizontal domino with position $N-1$ or not. We have

$$a_{n-N-1} = a_{N-1}\cdot a_{n-2N} + a_{N-2}\cdot a_{n-2N-1}$$

and thus

$$a_n = a_N a_{n-N} + a_{N-1}(a_{N-1}a_{n-2N} + a_{N-2}a_{n-2N-1}).$$

Repeating the procedure, and using the fact that

$$a_{n-kN-1} = a_{N-1}a_{n-(k+1)N} + a_{N-2}a_{n-(k+1)N-1},$$

for all $1\leqslant k\leqslant \lfloor\frac{n}{N}\rfloor-1$, we have

$$\begin{aligned} a_n = {} & a_N a_{n-N} + a_{N-1}(a_{N-1}a_{n-2N} + a_{N-2}(a_{N-1}a_{n-3N} + a_{N-2}(a_{N-1}a_{n-4N} \\ & + \cdots + a_{N-2}(a_{N-1}a_{n-N\lfloor\frac{n}{N}\rfloor} + a_{N-2}a_{n-1-N\lfloor\frac{n}{N}\rfloor})\cdots))) \end{aligned}$$

$$= a_N a_{n-N} + \sum_{k=2}^{\lfloor \frac{n}{N} \rfloor} a_{N-1}^2 a_{N-2}^{k-2} a_{n-kN} + a_{N-1} a_{N-2}^{\lfloor \frac{n}{N} \rfloor - 1} a_{n-1-N\lfloor \frac{n}{N} \rfloor}.$$

3. Concluding remarks

An infinite family of lacunary recurrence formulas for Fibonacci numbers has been introduced in this paper using a combinatorial approach. Some specializations of this result are also presented. The first entry in this family is well known as Lucas's identity (1.3). The second entry is also known [4].

We conclude with an open problem. The Lucas numbers are defined by $L_0 = 2$, $L_1 = 1$ and for $n > 1$,

$$L_n = L_{n-1} + L_{n-2}.$$

These numbers are related to the Fibonacci numbers by the identity

$$L_n = F_{n-1} + F_{n+1}.$$

It is natural to ask for a general lacunary recurrence for Lucas numbers in the spirit of Theorem 1.

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Authors' addresses

Cristina Ballantine
College of The Holy Cross, Department of Mathematics and Computer Science, Worcester, MA 01610, USA
E-mail address: cballant@holycross.edu

Mircea Merca
Academy of Romanian Scientists, Ilfov 3, Sector 5, Bucharest, Romania
E-mail address: mircea.merca@profinfo.edu.ro

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$\oplus-$SUPPLEMENTED LATTICES

ÇIĞDEM BIÇER AND CELIL NEBIYEV

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Abstract. In this work, $\oplus-$supplemented and strongly $\oplus-$supplemented lattices are defined and investigated some properties of these lattices. Let L be a lattice and $1 = a_1 \oplus a_2 \oplus ... \oplus a_n$ with $a_1, a_2, ..., a_n \in L$. If $a_i/0$ is $\oplus-$supplemented for each $i = 1, 2, ..., n$, then L is also $\oplus-$supplemented. Let L be a distributive lattice and $1 = a_1 \oplus a_2 \oplus ... \oplus a_n$ with $a_1, a_2, ..., a_n \in L$. If $a_i/0$ is strongly $\oplus-$supplemented for each $i = 1, 2, ..., n$, then L is also strongly $\oplus-$supplemented. A lattice L has $(D1)$ property if and only if L is amply supplemented and strongly $\oplus-$supplemented.

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1. Introduction

Throughout this paper, all lattices are complete modular lattices with the smallest element 0 and the greatest element 1. Let L be a lattice, $a, b \in L$ and $a \leq b$. A sublattice $\{x \in L | a \leq x \leq b\}$ is called a *quotient sublattice*, denoted by b/a. An element a' of a lattice L is called a *complement* of a in L if $a \wedge a' = 0$ and $a \vee a' = 1$, this case we denote $1 = a \oplus a^{'}$ (a and $a^{'}$ also is called *direct summands* of L). L is called a *complemented lattice* if each element has at least one complement in L. An element a of L is said to be *small* or *superfluous* and denoted by $a \ll L$ if $b = 1$ for every element b of L such that $a \vee b = 1$. The meet of all the maximal elements ($\neq 1$) of a lattice L is called the *radical* of L and denoted by $r(L)$. An element c of L is called a *supplement* of b in L if it is minimal for $b \vee c = 1$. a is a supplement of b in a lattice L if and only if $a \vee b = 1$ and $a \wedge b \ll a/0$. A lattice L is said to be *supplemented* if every element of L has a supplement in L. We say that an element b of L *lies above* an element a of L if $a \leq b$ and $b \ll 1/a$. L is said to be *hollow* if every element ($\neq 1$) is superfluous in L, and L is said to be *local* if L has the greatest element ($\neq 1$). An element a of L is called a *weak supplement* of b in L if $a \vee b = 1$ and $a \wedge b \ll L$. A lattice L is said to be *weakly supplemented*, if every element of L has a weak supplement in L. We say that an element $a \in L$ has *ample supplements* in L if for every $b \in L$ with $a \vee b = 1$, a has a supplement b' in L with $b' \leq b$. L is called an *amply supplemented lattice*, if every element of L has ample supplements

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in L. It is clear that every supplemented lattice is weakly supplemented and every amply supplemented lattice is supplemented. A lattice L is said to be *distributive* if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for every $a, b, c \in L$. Let L be a lattice. It is defined β_* relation on the elements of L by $a\beta_* b$ with $a, b \in L$ if and only if for each $t \in L$ such that $a \vee t = 1$ then $b \vee t = 1$ and for each $k \in L$ such that $b \vee k = 1$ then $a \vee k = 1$.

Let L be a lattice. Consider the following conditions.

$(D1)$ For every element a of L, there exist $a_1, a_2 \in L$ such that $1 = a_1 \oplus a_2$, $a_1 \in a/0$ and $a_2 \wedge a \ll a_2/0$.

$(D3)$ If a_1 and a_2 are direct summands of L and $1 = a_1 \vee a_2$, then $a_1 \wedge a_2$ is also a direct summand of L.

More details about (amply) supplemented lattices are in [1,2] and [5]. More results about (amply) supplemented modules are in [4] and [9]. Some important properties of $\oplus-$supplemented modules are in [6] and [7]. The definition of β_* relation on lattices and some properties of this relation are in [8]. The definition of β^* relation on modules and some properties of this relation are in [3].

In this paper, we generalize some properties of $\oplus-$supplemented modules to lattices. We constitute relationships between $\oplus-$supplemented quotient sublattices and $\oplus-$supplemented lattices by Lemma 11 and Corollary 2. We also constitute relationships between lattices which has $(D1)$ property and strongly $\oplus-$supplemented lattices by Proposition 4. We give some examples at the end of this paper.

Lemma 1. *Let L be a lattice and $a, b, c \in L$ with $a \leq b$. If c is a supplement of b in L, then $a \vee c$ is a supplement of b in $1/a$.*

Proof. Similar to proof of [5, Proposition 12.2(7)]. □

Lemma 2 ([5, Lemma 7.4])**.** *Let L be a lattice, $a, b \in L$ and $a \leq b$. If $a \ll b/0$ then $a \ll L$.*

Lemma 3 ([5, Lemma 7.5])**.** *In a lattice L let $c' \ll c/0$ and $d' \ll d/0$. Then $c' \vee d' \ll (c \vee d)/0$.*

Lemma 4 ([5, Lemma 7.6])**.** *If $a \ll L$, then $a \leq r(L)$.*

Lemma 5 ([5, Exercise 7.3])**.** *If L is a lattice and $a \in L$, then $r(a/0) \leq r(L)$.*

Lemma 6 ([5, Lemma 12.3])**.** *In any modular lattice $[(c \vee d) \wedge b] \leq [c \wedge (b \vee d)] \vee [d \wedge (b \vee c)]$ holds for every $b, c, d \in L$.*

Lemma 7. *Let L be a lattice, $a, b \in L$ and $a \leq b$. Then b lies above a if and only if $a\beta_* b$.*

Proof. ($\Longrightarrow$) See [8, Theorem 3].

($\Longleftarrow$) Let $b \vee t = 1$ with $t \in 1/a$. Since $a\beta_* b$, $a \vee t = 1$ and since $a \leq t$, $t = 1$. Hence $b \ll 1/a$ and b lies above a. □

Lemma 8 ([8, Lemma 2])**.** *Let L be a lattice and $a, b, c \in L$. If $a \vee b = 1$ and $(a \wedge b) \vee c = 1$, then $a \vee (b \wedge c) = b \vee (a \wedge c) = 1$.*

2. $\oplus-$SUPPLEMENTED LATTICES

Definition 1. Let L be a lattice. L is called a $\oplus-$supplemented lattice, if every element of L has a supplement that is a direct summand of L.

Clearly we see that every $\oplus-$supplemented lattice is supplemented and every complemented lattice is $\oplus-$supplemented. We also clearly see that hollow and local lattices are $\oplus-$supplemented.

Proposition 1. *Let L be a lattice. Then L is $\oplus-$supplemented if and only if for every $b \in L$, there exists a direct summand c of L such that $b \vee c = 1$ and $b \wedge c \ll c/0$.*

Proof. Clear from definition. □

Proposition 2. *Let L be a lattice. If every element of L has a weak supplement that is a direct summand of L, then L is $\oplus-$supplemented.*

Proof. Let a be a weak supplement of b in L and a be a direct summand of L. Since a is a weak supplement of b in L, $a \wedge b \ll L$ and since a is a direct summand of L, $a \wedge b \ll a/0$. Hence a is a supplement of b in L and L is $\oplus-$supplemented. □

Lemma 9. *Let L be a lattice, and $a, b \in L$. If x is a supplement of $a \vee b$ in L and y is a supplement of $a \wedge (x \vee b)$ in $a/0$ then $x \vee y$ is a supplement of b in L.*

Proof. Since x is a supplement of $a \vee b$ in L and y is a supplement of $a \wedge (x \vee b)$ in $a/0$, then $1 = a \vee b \vee x$, $(a \vee b) \wedge x \ll x/0$, $a = [a \wedge (x \vee b)] \vee y$ and $(x \vee b) \wedge y = a \wedge (x \vee b) \wedge y \ll y/0$. Here $1 = a \vee b \vee x = [a \wedge (x \vee b)] \vee y \vee b \vee x = b \vee x \vee y$. By Lemma 6, $(x \vee y) \wedge b \leq [(y \vee b) \wedge x] \vee [(x \vee b) \wedge y] \leq [(a \vee b) \wedge x] \vee [(x \vee b) \wedge y] \ll (x \vee y)/0$. Hence $x \vee y$ is a supplement of b in L. □

Lemma 10. *Let L be a lattice and $a_1, a_2 \in L$ where $a_1/0$ and $a_2/0$ are $\oplus-$supplemented and $1 = a_1 \oplus a_2$. Then L is $\oplus-$supplemented.*

Proof. Let x be any element of L. Then $1 = a_1 \vee a_2 \vee x$ and $a_1 \vee a_2 \vee x$ has a supplement 0 in L. Since $a_2/0$ is $\oplus-$supplemented, $a_2 \wedge (a_1 \vee x)$ has a supplement y that is a direct summand in $a_2/0$. By Lemma 9, y is a supplement of $a_1 \vee x$ in L. Since $a_1/0$ is $\oplus-$supplemented, $a_1 \wedge (x \vee y)$ has a supplement z that is a direct summand in $a_1/0$. By Lemma 9, $y \vee z$ is a supplement of x in L. Since y is a direct summand of $a_2/0$ and z is a direct summand of $a_1/0$, by $1 = a_1 \oplus a_2$, $y \vee z = y \oplus z$ is a direct summand of L. Finally, L is $\oplus-$supplemented. □

Corollary 1. *Let L be a lattice, $a_1, a_2, ..., a_n \in L$ and $1 = a_1 \oplus a_2 \oplus ... \oplus a_n$. If $a_i/0$ is $\oplus-$supplemented for every $i = 1, 2, \ldots, n$, then L is $\oplus-$supplemented.*

Proof. Clear from Lemma 10. □

Lemma 11. *Let L be a lattice, $a \in L$ and $a = (a \wedge a_1) \oplus (a \wedge a_2)$ for every $a_1, a_2 \in L$ with $1 = a_1 \oplus a_2$. If L is $\oplus-$supplemented, then $1/a$ is also $\oplus-$supplemented.*

Proof. Let $x \in 1/a$. Since L is $\oplus-$supplemented, there exist $y, z \in L$ such that $1 = x \vee y$, $x \wedge y \ll y/0$ and $1 = y \oplus z$. Since y is a supplement of x in L and $a \leq x$, by Lemma 1, $a \vee y$ is a supplement of x in $1/a$. Since $1 = y \oplus z$, by hypothesis, $a = (a \wedge y) \oplus (a \wedge z)$. Then $(a \vee y) \wedge (a \vee z) = [(a \wedge y) \vee (a \wedge z) \vee y] \wedge [(a \wedge y) \vee (a \wedge z) \vee z] = [y \vee (a \wedge z)] \wedge [(a \wedge y) \vee z] = (a \wedge y) \vee [(y \vee (a \wedge z)) \wedge z] = (a \wedge y) \vee [(y \wedge z) \vee (a \wedge z)] = (a \wedge y) \vee (0 \vee (a \wedge z)) = (a \wedge y) \vee (a \wedge z) = a$. Hence $1/a$ is $\oplus-$supplemented. □

Corollary 2. *Let L be a distributive lattice. If L is $\oplus-$supplemented, then $1/a$ is $\oplus-$supplemented for every $a \in L$.*

Proof. Clear from Lemma 11. □

Lemma 12. *Let L be a supplemented lattice and $a/0$ is a quotient sublattice such that $a \wedge r(L) = 0$. Then every element of $a/0$ is a direct summand of $a/0$.*

Proof. Let $x \in a/0$. Since L is supplemented, there exists an element y of L with $1 = x \vee y$ and $x \wedge y \ll y/0$. Since $1 = x \vee y$ and $x \leq a$, $a = x \vee (a \wedge y)$. Since $x \wedge y \ll y/0$, by Lemma 4, $x \wedge y \leq r(L)$. Then $x \wedge (a \wedge y) = a \wedge x \wedge y \leq a \wedge r(L) = 0$ and $a \wedge x \wedge y = 0$. Hence $a = x \oplus (a \wedge y)$ in $a/0$ and x is a direct summand of $a/0$. □

Corollary 3. *Let L be a supplemented lattice and $a/0$ is a quotient sublattice such that $a \wedge r(L) = 0$. Then $a/0$ is complemented.*

Proof. Clear from Lemma 12. □

Proposition 3. *Let L be a $\oplus-$supplemented lattice. Then there exist $a_1, a_2 \in L$ such that $1 = a_1 \oplus a_2$, $r(a_1/0) \ll a_1/0$ and $r(a_2/0) = a_2$.*

Proof. Since L is $\oplus-$supplemented, there exist $a_1, a_2 \in L$ such that $1 = r(L) \vee a_1 = a_1 \oplus a_2$ and $r(L) \wedge a_1 \ll a_1/0$. Then by Lemma 5, $r(a_1/0) \leq r(L) \wedge a_1 \ll a_1/0$.

Assume x be a maximal ($\neq a_2$) element of $a_2/0$. Since $1/(a_1 \vee x) = (a_1 \oplus a_2)/(a_1 \vee x) = (a_1 \vee x \vee a_2)/(a_1 \vee x) \cong a_2/[a_2 \wedge (a_1 \vee x)] = a_2/[(a_2 \wedge a_1) \vee x] = a_2/x$, $a_1 \vee x$ is a maximal element ($\neq 1$) of L and since $1 = r(L) \vee a_1 \leq a_1 \vee x$, this is a contradiction. Hence $r(a_2/0) = a_2$. □

Definition 2. Let L be a lattice. L is called a completely $\oplus-$supplemented lattice, if every quotient sublattice $a/0$ such that a is a direct summand of L is $\oplus-$supplemented.

Theorem 1. *Let L be a $\oplus-$supplemented lattice with $(D3)$. Then L is completely $\oplus-$supplemented.*

Proof. Let u be a direct summand of L and $x \in u/0$. Since L is $\oplus-$supplemented, then there exists a direct summand y of L such that $1 = x \vee y$ and $x \wedge y \ll y/0$.

Because of $1 = x \vee y$, $u \vee y = 1$ and because of L has $(D3)$, $u \wedge y$ is a direct summand of L and hence $u \wedge y$ is a direct summand of $u/0$. Since $1 = x \vee y$ and $x \leq u$, $u = x \vee (u \wedge y)$. By $x \wedge u \wedge y = x \wedge y \ll y/0$, $x \wedge u \wedge y \ll L$. By $x \wedge u \wedge y \leq u \wedge y$ and $u \wedge y$ is a direct summand of L, $x \wedge u \wedge y \ll u \wedge y$. Thus $u/0$ is $\oplus-$supplemented. □

Definition 3. Let L be a supplemented lattice. L is called a strongly $\oplus-$supplemented lattice if every supplement element in L is a direct summand of L.

Clearly we see that every strongly $\oplus-$supplemented lattice is $\oplus-$supplemented and every complemented lattice is strongly $\oplus-$supplemented. Hollow and local lattices are strongly $\oplus-$supplemented.

Lemma 13. *Let a be a supplement of b in L and $x, y \in a/0$. Then y is a supplement of x in $a/0$ if and only if y is a supplement of $b \vee x$ in L.*

Proof. ($\Longrightarrow$) Let y be a supplement of x in $a/0$ and $b \vee x \vee z = 1$ with $z \leq y$. Because of $x, y \in a/0$ and $z \leq y$, $x \vee z \leq a$. Since a is a supplement of b in L, $a = x \vee z$. Since y is a supplement of x in $a/0$, $z = y$. Hence y is a supplement of $b \vee x$ in L.

($\Longleftarrow$)Let y be a supplement of $b \vee x$ in L. So, $b \vee x \vee y = 1$ and $(b \vee x) \wedge y \ll y/0$. Since $x \vee y \leq a$ and a is a supplement of b in L, $x \vee y = a$ and $x \wedge y \leq (b \vee x) \wedge y \ll y/0$. Hence y is a supplement of x in $a/0$. □

Lemma 14. *Let L be a strongly $\oplus-$supplemented lattice. Then for every direct summand a of L, the quotient sublattice $a/0$ is strongly $\oplus-$supplemented.*

Proof. Let $1 = a \oplus b$ with $b \in L$, $x, y \in a/0$ and y be supplement of x in $a/0$. By Lemma 13, y is a supplement of $b \vee x$ in L. Since L is strongly $\oplus-$supplemented, every supplement element is a direct summand of L and y is a direct summand of L. Here there exists $z \in L$ such that $1 = y \oplus z$. By modularity, $a = a \wedge 1 = a \wedge (y \oplus z) = y \oplus (a \wedge z)$. Thus y is a direct summand of $a/0$. □

Corollary 4. *Every strongly $\oplus-$supplemented lattice is completely $\oplus-$supplemented.*

Proof. Clear from Lemma 14. □

Lemma 15. *Let L be a distributive lattice and $a_1, a_2 \in L$ with $1 = a_1 \oplus a_2$. If $a_1/0$ and $a_2/0$ are strongly $\oplus-$supplemented, then L is also strongly $\oplus-$supplemented.*

Proof. Let a be a supplement of b in L. Since L is distributive, $a = a \wedge 1 = a \wedge (a_1 \oplus a_2) = (a \wedge a_1) \oplus (a \wedge a_2)$ holds. By Lemma 13, $a \wedge a_1$ is a supplement of $(a \wedge a_2) \vee b$ in L. We can also see that $a \wedge a_1$ is a supplement of $a_1 \wedge ((a \wedge a_2) \vee b)$ in $a_1/0$. Since $a_1/0$ is strongly $\oplus-$supplemented, $a \wedge a_1$ is a direct summand of $a_1/0$. Similarly we can see that $a \wedge a_2$ is a direct summand of $a_2/0$. Since $1 = a_1 \oplus a_2$ and $a = (a \wedge a_1) \oplus (a \wedge a_2)$, a is a direct summand of L. Hence L is strongly $\oplus-$supplemented. □

Corollary 5. *Let L be a distributive lattice, $a_1, a_2, ..., a_n \in L$ and $1 = a_1 \oplus a_2 \oplus ... \oplus a_n$. If $a_i/0$ is strongly $\oplus-$supplemented for every $i = 1, 2, \ldots, n$, then L is strongly $\oplus-$supplemented.*

Proof. Clear from Lemma 15. □

Lemma 16. *Let L be a supplemented lattice. The following statements are equivalent.*

(i) L is strongly $\oplus-$supplemented.

(ii) Every supplement element of L lies above a direct summand in L.

(iii) (a) For every nonzero supplement element a in L, $a/0$ contains a nonzero direct summand of L.

(b) For every nonzero supplement element a in L, $a/0$ contains a maximal direct summand of L.

Proof. $(i) \Longrightarrow (ii)$ Clear, since every element of L lies above itself.

$(ii) \Longrightarrow (iii)$ Let a be a nonzero supplement element in L. Assume a is a supplement of b in L. By hypothesis, there exists a direct summand x of L such that a lies above x in L. By Lemma 7, $a\beta_* x$ and since $a \vee b = 1$, $x \vee b = 1$. Since a is a supplement of b in L and $x \leq a$, $a = x$ and a is a nonzero direct summand of L.

$(iii) \Longrightarrow (i)$ Let a be a supplement of b in L and x be a maximal direct summand of L with $x \leq a$. Assume $1 = x \oplus y$ with $y \in L$. Then $a = a \wedge 1 = a \wedge (x \oplus y) = x \oplus (a \wedge y)$ and by Lemma 13, $a \wedge y$ is a supplement of $b \vee x$ in L. If $a \wedge y$ is not zero, then by hypothesis, $(a \wedge y)/0$ contains a nonzero direct summand c of L. Here $x \oplus c$ is a direct summand of L and $x \oplus c \leq a$. This contradicts the choice of x. Hence $a \wedge y = 0$ and $a = x$. Thus a is a direct summand of L and L is strongly $\oplus-$supplemented. □

Proposition 4. *Let L be a lattice. The following statements are equivalent.*

(i) L has $(D1)$ property.

(ii) Every element of L lies above a direct summand in L.

(iii) L is amply supplemented and strongly $\oplus-$supplemented.

Proof. $(i) \Longrightarrow (ii)$ Let $a \in L$. Since L has $(D1)$ property, there exist $a_1, a_2 \in L$ such that $1 = a_1 \oplus a_2$, $a_1 \leq a$ and $a_2 \wedge a \ll a_2/0$. Let $a \vee t = 1$ with $t \in 1/a_1$. Since $a_1 \leq a$ and $1 = a_1 \oplus a_2$, $a = a \wedge 1 = a \wedge (a_1 \oplus a_2) = a_1 \oplus (a \wedge a_2)$. Then $1 = a \vee t = a_1 \vee (a \wedge a_2) \vee t = (a \wedge a_2) \vee t$ and since $a \wedge a_2 \ll L$, $t = 1$. Hence $a \ll 1/a_1$ and a lies above a_1.

$(ii) \Longrightarrow (iii)$ Let $a \vee b = 1$ with $a, b \in L$. By hypothesis, $a \wedge b$ lies above a direct summand in L. Here there exist $x, y \in L$ such that $1 = x \oplus y$ and $a \wedge b$ lies above x. Since $1 = x \oplus y$ and $x \leq b$, $b = b \wedge 1 = b \wedge (x \vee y) = x \vee (b \wedge y)$. Then $1 = a \vee b = a \vee x \vee (b \wedge y) = a \vee (b \wedge y)$. By hypothesis, $b \wedge y$ lies above a direct summand in L. Here there exist $x_1, y_1 \in L$ such that $1 = x_1 \oplus y_1$ and $b \wedge y$ lies above x_1. By Lemma 7, $(b \wedge y)\beta_* x_1$ and since $1 = a \vee (b \wedge y)$, $1 = a \vee x_1$ holds.

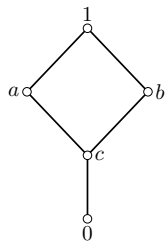
Let $(a \wedge x_1) \vee t = 1$ with $t \in L$. By $a \wedge x_1 \leq a \wedge b \wedge y$, $(a \wedge b \wedge y) \vee t = 1$ holds. Here $y = y \wedge 1 = y \wedge ((a \wedge b \wedge y) \vee t) = (a \wedge b \wedge y) \vee (y \wedge t)$ and $1 = x \vee y = x \vee (a \wedge b \wedge y) \vee (y \wedge t) = x \vee (a \wedge b) \vee (y \wedge t)$. Since $a \wedge b$ lies above x, by Lemma 7, $(a \wedge b)\, \beta_* x$. Then $1 = x \vee (a \wedge b) \vee (y \wedge t) = x \vee (y \wedge t)$ and since y is a supplement of x in L and $y \wedge t \leq y$, $y \wedge t = y$ and $y \leq t$. Hence $1 = (a \wedge b \wedge y) \vee t = t$ and $a \wedge x_1 \ll L$. Since x_1 a direct summand of L, $a \wedge x_1 \ll x_1/0$ and x_1 is a supplement of a in L. Moreover, $x_1 \leq b$. Hence L is amply supplemented. By Lemma 16, L is strongly $\oplus-$supplemented.

$(iii) \Longrightarrow (i)$ Let a be any element of L. By hypothesis, a has a supplement b in L. Here $1 = a \vee b$ and $a \wedge b \ll b/0$. Since L is amply supplemented, b has a supplement x in L with $x \leq a$. By hypothesis, x is a direct summand of L and there exists an element y of L such that $1 = x \oplus y$. Let $(a \wedge y) \vee t = 1$ with $t \in L$. Since $1 = x \vee y = a \vee y$, by Lemma 8, $a \vee (y \wedge t) = 1$. Since $1 = x \vee b$ and $x \leq a$, $a = a \wedge 1 = a \wedge (x \vee b) = x \vee (a \wedge b)$. Then $1 = a \vee (y \wedge t) = x \vee (a \wedge b) \vee (y \wedge t)$ and since $a \wedge b \ll L$, $1 = x \vee (y \wedge t)$. Since $1 = x \vee (y \wedge t)$ and y is a supplement of x in L, $y \wedge t = y$ and $y \leq t$. Then $1 = (a \wedge y) \vee t = t$ and $a \wedge y \ll L$. Since y is a direct summand of L, $a \wedge y \ll y/0$. Hence L has $(D1)$ property. □

Corollary 6. *Let L be a lattice with $(D1)$ property. Then L is $\oplus-$supplemented.*

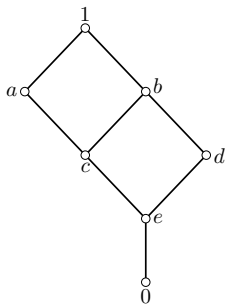
Proof. Clear from Proposition 4 and Corollary 4. □

Example 1. Consider the lattice $L = \{0, a, b, c, 1\}$ given by the following diagram.



Then L is supplemented but not $\oplus-$supplemented.

Example 2. Consider the lattice $L = \{0, a, b, c, d, e, 1\}$ given by the following diagram.



Then L is supplemented but not $\oplus-$supplemented.

Example 3. Consider the interval $[0,1]$ with natural topology. Let P be the set of all closed subsets of $[0,1]$. P is complete modular lattice by the inclusion (See [1, Example 2.10]). Here $\underset{i\in I}{\wedge} C_i = \underset{i\in I}{\cap} C_i$ and $\underset{i\in I}{\vee} C_i = \overline{\underset{i\in I}{\cup} C_i}$ for every $C_i \in P$ $(i \in I)$ $\left(\overline{\underset{i\in I}{\cup} C_i}\right.$ is the closure of $\left.\underset{i\in I}{\cup} C_i\right)$. Let $X \in P$ and $X \vee Y = [0,1]$ with $Y \in P$. Then $[0,1]-X \subset Y$ and since Y is closed $\overline{[0,1]-X} \subset Y$. Let $X^{'} = \overline{[0,1]-X}$. Then $X^{'} \in P$, $X \vee X^{'} = X \cup X^{'} = [0,1]$ and $X^{'} \subset Y$ for every $Y \in P$ with $X \vee Y = [0,1]$. Hence X has ample supplements in P (here $X^{'} = \overline{[0,1]-X}$ is the only supplement of X in P) and P is amply supplemented. Let $A = [0,a] \in P$ with $0 < a < 1$. Here $A^{'} = \overline{[0,1]-A} = [a,1]$ is the only supplement of A in P. Let $A^{'} \vee B = A^{'} \cup B = [0,1]$ with $B \in P$. Since $A^{'} \cup B = [0,1]$, $[0,a) = [0,1]-A^{'} \subset B$ and since B is closed, $[0,a] \subset B$. This case $a \in B$ and since $a \in A^{'}$, $A^{'} \wedge B = A^{'} \cap B \neq \varnothing$. Hence $A^{'}$ is not a direct summand of P and P is not $\oplus-$supplemented.

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Authors' addresses

Çiğdem Biçer
Ondokuz Mayıs University, 55270, Kurupelit, Atakum, Samsun, Turkey
E-mail address: cigdem_bicer184@hotmail.com

Celil Nebiyev
Ondokuz Mayıs University, 55270, Kurupelit, Atakum, Samsun, Turkey
E-mail address: cnebiyev@omu.edu.tr

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NEW GENERALIZED MIDPOINT TYPE INEQUALITIES FOR FRACTIONAL INTEGRAL

H. BUDAK AND P. AGARWAL

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Abstract. Here, our first aim to establish a new identity for differentiable function involving Riemann-Liouville fractional integrals. Then, we obtain same generalized midpoint type inequalities utilizing convex and concave function.

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1. Introduction

In recent years, the Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention much interest in elementary mathematics.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g.,[17, p.137], [7]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave.

In [12], U. S. Kırmacı give the following identity and using this identiy, obtain some bounds for the left hand side of the inequality (1.1)

Lemma 1. *Let $f : I^* \to \mathbb{R}$ be differentiable function on I^*, $a, b \in I^*$ (I^* is interior of I) with $a < b$. If $f' \in L[a,b]$, then we have*

$$\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \tag{1.2}$$

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$$=(b-a)\left[\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt+\int_{\frac{1}{2}}^{1}(1-t)\,f'(ta+(1-t)b)dt\right].$$

Over the last twenty years, the numerous studies have focused on to obtain new bound for left hand side and right and side of the inequality (1.1). For some examples, please refer to ([2,4,6–8,14,15,18,19,24])

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

Definition 1. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\,f(t)dt,\ \ x>a$$

and

$$J^{\alpha}_{b-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\,f(t)dt,\ \ x<b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J^{0}_{a+}f(x)=J^{0}_{b-}f(x)=f(x)$.

It is remarkable that Sarikaya et al.[22] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\leq a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)\right]\leq\frac{f(a)+f(b)}{2} \tag{1.3}$$

with $\alpha>0$.

Sarıkaya and Yıldırım also give the following Hermite-Hadamard type inequality for the Riemann-Lioville fractional integrals in [23].

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)\right]\leq\frac{f(a)+f(b)}{2}. \tag{1.4}$$

Whereupon Sarikaya et al. obtain the Hermite-Hadamard inequality for Riemann-Lioville fractional integrals, many authors have studied to generalize this inequality and establish Hermite-Hadamard inequality other fractional integrals such as k-fractional integral, Hadamard fractional integrals, Katugampola frtactional integrals, Conformable fractional integrals, etc. For some of them, please see ([1,3,5,9,11,13,16,20,21,25–27]). For more information about fraction calculus please refer to [10].

In the following section, we establish some new generalized midpoint type inequalities for Riemann-Liouville fractional integrals.

2. Generalized midpoint inequalities for Riemann-Liouville fractional integral operators

First, we give the following lemma which will be used frequently later.

Lemma 2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then for all $x\in[a,b]$ the following equality for fractional integrals holds:*

$$\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha}J_{b-}^{\alpha}f(a+b-x)+(b-x)^{1-\alpha}J_{a+}^{\alpha}f(a+b-x)\right] \tag{2.1}$$
$$-f(a+b-x)$$

$$=\frac{(x-a)^2}{b-a}\int_0^1\left(1-t^{\alpha}\right)f'(tb+(1-t)(a+b-x))\,dt$$

$$+\frac{(b-x)^2}{b-a}\int_0^1\left(t^{\alpha}-1\right)f'(ta+(1-t)(a+b-x))\,dt.$$

Proof. Integrating the by parts, we have

$$\int_0^1\left(1-t^{\alpha}\right)f'(tb+(1-t)(a+b-x))\,dt \tag{2.2}$$
$$=\left(1-t^{\alpha}\right)\frac{f'(tb+(1-t)(a+b-x))}{x-a}\Bigg|_0^1$$
$$+\frac{\alpha}{x-a}\int_0^1 t^{\alpha-1}f'(tb+(1-t)(a+b-x))\,dt$$

$$= -\frac{f(a+b-x)}{x-a} + \frac{\Gamma(1+\alpha)}{(x-a)^{\alpha+1}} \int_{a+b-x}^{b} (u-(a+b-x))^{\alpha-1} f(u)du$$
$$= -\frac{f(a+b-x)}{x-a} + \frac{\Gamma(1+\alpha)}{(x-a)^{\alpha+1}} J_{b-}^{\alpha} f(a+b-x).$$

Similarly, we get

$$\int_0^1 \left(t^{\alpha}-1\right) f'(ta+(1-t)(a+b-x))\,dt \tag{2.3}$$

$$= -\frac{f(a+b-x)}{b-x} + \frac{\Gamma(1+\alpha)}{(b-x)^{\alpha+1}} J_{a+}^{\alpha} f(a+b-x).$$

By the identities (2.2) and (2.3), we obtain the required result (2.1). □

Theorem 3. *$f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0\le a<b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q>1$, then for all $x\in[a,b]$ following inequality for fractional integrals holds:*

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha} J_{b-}^{\alpha} f(a+b-x) + (b-x)^{1-\alpha} J_{a+}^{\alpha} f(a+b-x)\right] - f(a+b-x)\right| \tag{2.4}$$

$$\le \frac{1}{b-a}\left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}} \left[(x-a)^2\left[\frac{|f'(b)|^q+|f'(a+b-x)|^q}{2}\right]^{\frac{1}{q}} + (b-x)^2\left[\frac{|f'(a)|^q+|f'(a+b-x)|^q}{2}\right]^{\frac{1}{q}}\right]$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By the Lemma 2, we have

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha} J_{b-}^{\alpha} f(a+b-x) + (b-x)^{1-\alpha} J_{a+}^{\alpha} f(a+b-x)\right] - f(a+b-x)\right| \tag{2.5}$$

$$\le \frac{(x-a)^2}{b-a}\int_0^1 |1-t^{\alpha}|\left|f'(tb+(1-t)(a+b-x))\right| dt$$

$$+\frac{(b-x)^2}{b-a}\int_0^1 |t^\alpha-1|\left|f'(ta+(1-t)(a+b-x))\right|dt.$$

Using the Hölder's inequality and convexity of $|f'|^q$, we obtain

$$\begin{aligned}
&\int_0^1 |1-t^\alpha|\left|f'(tb+(1-t)(a+b-x))\right|dt \qquad (2.6)\\
&\le\left(\int_0^1 |1-t^\alpha|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(tb+(1-t)(a+b-x))\right|^q dt\right)^{\frac{1}{q}}\\
&\le\left(\int_0^1 \left(1-t^{p\alpha}\right)dt\right)^{\frac{1}{p}}\left(\int_0^1 \left[t\left|f'(b)\right|^q+(1-t)\left|f'(a+b-x)\right|^q\right]dt\right)^{\frac{1}{q}}\\
&=\left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{|f'(b)|^q+|f'(a+b-x)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

Here we use

$$(A-B)^q\le A^q-B^q,$$

for any $A>B\ge 0$ and $q\ge 1$.

Similarly we have

$$\begin{aligned}
&\int_0^1 |t^\alpha-1|\left|f'(ta+(1-t)(a+b-x))\right|dt \qquad (2.7)\\
&\le\left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(a+b-x)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

If we substitute the inequalities (2.6) and (2.7) in (2.5), then we obtain the desired result. □

Corollary 1. *Under assumption of Theorem 3 with $x=\frac{a+b}{2}$, we have the following inequality*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{b-}^\alpha f\left(\frac{a+b}{2}\right)+J_{a+}^\alpha f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right| \qquad (2.8)\\
&\le\frac{b-a}{4}\left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(b)|^q+|f'(a)|^q}{4}\right)^{\frac{1}{q}}\right]
\end{aligned}$$

$$\leq \frac{b-a}{4}\left(\frac{4\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

Proof. The proof of the first inequality in (2.8) is obvious from the convexity of $|f'|^q$. For the proof of second inequality, let $a_1 = 3|f'(a)|^q$, $b_1 = |f'(b)|^q$, $a_2 = |f'(a)|^q$ and $b_2 = 3|f'(b)|^q$. Using the fact that,

$$\sum_{k=1}^{n}(a_k+b_k)^s \leq \sum_{k=1}^{n}a_k^s+\sum_{k=1}^{n}b_k^s,\ 0\leq s<1$$

the desired result can be obtained straightforwardly. □

Theorem 4. *$f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0\leq a<b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q\geq 1$, then for all $x\in[a,b]$ following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha}J_{b-}^{\alpha}f(a+b-x)+(b-x)^{1-\alpha}J_{a+}^{\alpha}f(a+b-x)\right]\right.\\&\left.-f(a+b-x)\right|\\&\leq\frac{1}{b-a}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\\&\times\left[(x-a)^2\left(\frac{\alpha}{2(\alpha+2)}\left|f'(b)\right|^q+\left[\frac{(\alpha+1)(\alpha+2)-2}{2(\alpha+1)(\alpha+2)}\right]\left|f'(a+b-x)\right|^q\right)^{\frac{1}{q}}\right.\\&\left.+(b-x)^2\left(\frac{\alpha}{2(\alpha+2)}\left|f'(a)\right|^q+\left[\frac{(\alpha+1)(\alpha+2)-2}{2(\alpha+1)(\alpha+2)}\right]\left|f'(a+b-x)\right|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

Proof. By the Lemma 2 and the power mean inequality, we have

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha}J_{b-}^{\alpha}f(a+b-x)+(b-x)^{1-\alpha}J_{a+}^{\alpha}f(a+b-x)\right]\right.\\&\left.-f(a+b-x)\right|\\&\leq\frac{(x-a)^2}{b-a}\int_0^1|1-t^{\alpha}|\left|f'(tb+(1-t)(a+b-x))\right|dt\\&+\frac{(b-x)^2}{b-a}\int_0^1|t^{\alpha}-1|\left|f'(ta+(1-t)(a+b-x))\right|dt.\end{aligned}\tag{2.9}$$

$$\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 |1-t^\alpha|\,dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 |1-t^\alpha|\left|f'(tb+(1-t)(a+b-x))\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\int_0^1 |t^\alpha-1|\,dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 |t^\alpha-1|\left|f'(ta+(1-t)(a+b-x))\right|^q dt\right)^{\frac{1}{q}}.$$

Using the convexity of $|f'|^q$, we obtain

$$\int_0^1 |1-t^\alpha|\left|f'(tb+(1-t)(a+b-x))\right|^q dt$$
$$\leq \int_0^1 \left(1-t^\alpha\right)\left[t\left|f'(b)\right|^q+(1-t)\left|f'(a+b-x)\right|^q\right]dt$$
$$=\frac{\alpha}{2(\alpha+2)}\left|f'(b)\right|^q+\left[\frac{(\alpha+1)(\alpha+2)-2}{2(\alpha+1)(\alpha+2)}\right]\left|f'(a+b-x)\right|^q$$

and similarly we have

$$\int_0^1 |t^\alpha-1|\left|f'(ta+(1-t)(a+b-x))\right|^q dt$$
$$\leq \frac{\alpha}{2(\alpha+2)}\left|f'(a)\right|^q+\left[\frac{(\alpha+1)(\alpha+2)-2}{2(\alpha+1)(\alpha+2)}\right]\left|f'(a+b-x)\right|^q$$

which completes the proof. □

Corollary 2. *Under assumption of Theorem 4 with $x=\frac{a+b}{2}$, we have the following inequality*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{b-}^\alpha f\left(\frac{a+b}{2}\right)+J_{a+}^\alpha f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{b-a}{4}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\frac{\alpha(3\alpha+5)}{4(\alpha+1)(\alpha+2)}\left|f'(b)\right|^q+\left[\frac{(\alpha+1)(\alpha+2)-2}{4(\alpha+1)(\alpha+2)}\right]\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\alpha(3\alpha+5)}{4(\alpha+1)(\alpha+2)}\left|f'(a)\right|^q+\left[\frac{(\alpha+1)(\alpha+2)-2}{4(\alpha+1)(\alpha+2)}\right]\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right].$$

Theorem 5. *$f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0\leq a<b$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $q>1$, then for all $x\in[a,b]$ following inequality for fractional integrals holds:*

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha}J^{\alpha}_{b-}f(a+b-x)+(b-x)^{1-\alpha}J^{\alpha}_{a+}f(a+b-x)\right]\right.$$
$$\left.-f(a+b-x)\right|$$

$$\leq\frac{1}{b-a}\left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\left[(x-a)^2\left|f'\left(\frac{a+2b-x}{2}\right)\right|+(b-x)^2\left|f'\left(\frac{2a+b-x}{2}\right)\right|\right]$$

where $\frac{1}{q}+\frac{1}{p}=1$.

Proof. By the Lemma 2 and the Hölder inequality, we have

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha}J^{\alpha}_{b-}f(a+b-x)+(b-x)^{1-\alpha}J^{\alpha}_{a+}f(a+b-x)\right]\right.$$
$$\left.-f(a+b-x)\right| \tag{2.10}$$

$$\leq\frac{(x-a)^2}{b-a}\int_0^1|1-t^{\alpha}|\left|f'(tb+(1-t)(a+b-x))\right|dt$$
$$+\frac{(b-x)^2}{b-a}\int_0^1|t^{\alpha}-1|\left|f'(ta+(1-t)(a+b-x))\right|dt$$
$$\leq\frac{(x-a)^2}{b-a}\left(\int_0^1\left(1-t^{\alpha}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tb+(1-t)(a+b-x))\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^2}{b-a}\left(\int_0^1 \left(1-t^{\alpha}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(ta+(1-t)(a+b-x))\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is concave on $[a,b]$, by using Jensen integral inequality, we obtain

$$\begin{aligned}
&\int_0^1 \left|f'(tb+(1-t)(a+b-x))\right|^q dt \qquad (2.11)\\
&=\int_0^1 t^0\left|f'(tb+(1-t)(a+b-x))\right|^q dt\\
&\le\left(\int_0^1 t^0 dt\right)\left|f'\left(\frac{1}{\int_0^1 t^0 dt}\int_0^1 t^0(tb+(1-t)(a+b-x))\,dt\right)\right|^q\\
&=\left|f'\left(\frac{a+2b-x}{2}\right)\right|^q
\end{aligned}$$

and similarly,

$$\int_0^1 \left|f'(ta+(1-t)(a+b-x))\right|^q dt \le \left|f'\left(\frac{2a+b-x}{2}\right)\right|^q. \qquad (2.12)$$

By substituting the inequalities (2.11) and (2.12) in (2.10) and using the fact that

$$\int_0^1 \left(1-t^{\alpha}\right)^p dt \le \int_0^1 \left(1-t^{p\alpha}\right) dt = \frac{\alpha p}{\alpha p+1},$$

we obtain the desired result. □

Corollary 3. *Under assumptions of Theorem 5, if we choose $x=\frac{a+b}{2}$, then we have the inequality*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\le\left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{b-a}{4}\right)\left[\left|f'\left(\frac{a+3b}{4}\right)\right|+\left|f'\left(\frac{3a+b}{4}\right)\right|\right].$$

Theorem 6. *$f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0\le a<b$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $q\ge 1$, then for all $x\in[a,b]$ following inequality for fractional integrals holds:*

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha}J^{\alpha}_{b-}f(a+b-x)+(b-x)^{1-\alpha}J^{\alpha}_{a+}f(a+b-x)\right]\right.$$
$$\left.-f(a+b-x)\right|$$

$$\le\frac{1}{b-a}\left(\frac{\alpha}{\alpha+1}\right)^{2-\frac{1}{q}}\left[(x-a)^2\left|f'\left(\frac{2\alpha(\alpha+2)b+[(\alpha+1)(\alpha+2)-2](a-x)}{2(\alpha+2)}\right)\right|^q\right.$$
$$\left.+(b-x)^2\left|f'\left(\frac{2\alpha(\alpha+2)a+[(\alpha+1)(\alpha+2)-2](b-x)}{2(\alpha+2)}\right)\right|^q\right].$$

Proof. From the ineqauality (2.9) we have

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(x-a)^{1-\alpha}J^{\alpha}_{b-}f(a+b-x)+(b-x)^{1-\alpha}J^{\alpha}_{a+}f(a+b-x)\right]\right. \tag{2.13}$$
$$\left.-f(a+b-x)\right|$$

$$\le\frac{(x-a)^2}{b-a}\left(\int_0^1|1-t^{\alpha}|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1|1-t^{\alpha}|\,|f'(tb+(1-t)(a+b-x))|^q\,dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\int_0^1|t^{\alpha}-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1|t^{\alpha}-1|\,|f'(ta+(1-t)(a+b-x))|^q\,dt\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is concave on $[a,b]$, by using Jensen integral inequality, we obtain

$$\int_0^1|1-t^{\alpha}|\,|f'(tb+(1-t)(a+b-x))|^q\,dt$$
$$\le\left(\int_0^1(1-t^{\alpha})\,dt\right)f'\left(\frac{1}{\int_0^1(1-t^{\alpha})\,dt}\int_0^1(1-t^{\alpha})(tb+(1-t)(a+b-x))\,dt\right)$$
$$=\frac{\alpha}{\alpha+1}\left|f'\left(\frac{2\alpha(\alpha+2)b+[(\alpha+1)(\alpha+2)-2](a-x)}{2(\alpha+2)}\right)\right|^q$$

and similarly,

$$\int_0^1|t^{\alpha}-1|\,|f'(ta+(1-t)(a+b-x))|^q\,dt$$

$$\leq \frac{\alpha}{\alpha+1}\left|f'\left(\frac{2\alpha(\alpha+2)a+[(\alpha+1)(\alpha+2)-2](b-x)}{2(\alpha+2)}\right)\right|^q.$$

This completes the proof. □

Corollary 4. *Under assumptions of Theorem 5 with* $x=\frac{a+b}{2}$, *then we have the inequality*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{b-}^\alpha f\left(\frac{a+b}{2}\right)+J_{a+}^\alpha f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|\\ &\leq\left(\frac{\alpha}{\alpha+1}\right)^{2-\frac{1}{q}}\left(\frac{b-a}{4}\right)\left[\left|f'\left(\frac{\alpha(3\alpha+5)b+[(\alpha+1)(\alpha+2)-2]a}{4(\alpha+2)}\right)\right|^q\right.\\ &\quad\left.+\left|f'\left(\frac{\alpha(3\alpha+5)a+[(\alpha+1)(\alpha+2)-2]b}{4(\alpha+2)}\right)\right|^q\right].\end{aligned}$$

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Authors' addresses

H. Budak
Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce,Turkey
E-mail address: hsyn.budak@gmail.com

P. Agarwal
Department of Mathematics, Anand International College of Engineering, Near Kanota, Agra Road, Jaipur-303012, Rajasthan, INDIA
E-mail address: goyal.praveen2011@gmail.com

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A DISCONTINUOUS q-FRACTIONAL BOUNDARY VALUE PROBLEM WITH EIGENPARAMETER DEPENDENT BOUNDARY CONDITIONS

F. AYCA CETINKAYA

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Abstract. This work aims to examine a boundary value problem which consists of a second order q-differential equation and eigenvalue dependent boundary conditions. We introduce a modified inner product in a suitable direct sum space and define a symmetric operator. We investigate the properties of eigenvalues and eigenfunctions and construct the Green's function.

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1. Introduction

Recently, there has been a considerable attention on q-calculus and many papers subject to the boundary value problems consisting a Jackson q-derivative in the differential equation have appeared ([1–12, 14, 17, 18]). In [6, 8] the authors studied a q-analogue of the Sturm-Liouville eigenvalue problems and formulated a self-adjoint q-difference operator in a Hilbert space. Their results are applied and developed in different aspects. In [1, 5], for instance, sampling theory associated with q-difference equations of the Sturm-Liouville type is considered. In [3, 17] a regular q-fractional Sturm-Liouville problem which includes the left-sided Riemann-Liouville and right-sided Caputo q-fractional derivatives of the same order is formulated and the properties of eigenvalues and eigenfunctions are investigated. In [4] a Parseval equality and an expansion formula in eigenfunctions for a singular q-Sturm-Liouville operator on the whole line are established. In [2] the eigenvalues and the spectral singularities of non-selfadjoint q-difference equations of second order are investigated. In [11] a boundary value problem consisting a second order q-difference equation together with Dirichlet boundary conditions is reduced to an eigenvalue problem for a second order Euler q-difference equation by separation of variables and in [12] a q-Sturm-Liouville boundary value problem with a spectral parameter in the boundary condition is considered. For further studies related to the spectral analysis of q-differential

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equations and the Green's function, the readers are directed to [7, 10, 14, 16, 19–21] and the references therein.

Enlightened by these literature, in this paper, we study the boundary value problem

$$l(\upsilon) := -\frac{1}{q} D_{q^{-1}} D_q \upsilon(x) + r(x)\upsilon(x) = \eta w(x)\upsilon(x), \tag{1.1}$$

$$U_1(\upsilon) := \alpha_1 \upsilon(0) + \alpha_2 D_{q^{-1}} \upsilon(0) + \eta\left[\alpha_3 \upsilon(0) + \alpha_4 D_{q^{-1}} \upsilon(0)\right] = 0, \tag{1.2}$$

$$U_2(\upsilon) := \beta_1 \upsilon(\pi) + \beta_2 D_{q^{-1}} \upsilon(\pi) + \eta\left[\beta_3 \upsilon(\pi) + \beta_4 D_{q^{-1}} \upsilon(\pi)\right] = 0, \tag{1.3}$$

where $q \in (0,1]$ is fixed, $r(\cdot)$ is a real valued function which is continuous at zero, η is a complex parameter, $\alpha_i \neq 0$, $\beta_j \neq 0$ $(i, j = 1,2,3,4)$ are arbitrary real numbers and the function $w(x)$ is a positive piecewise continuous function such as

$$w(x) = \begin{cases} w_1, & 0 \leq x < a, \\ w_2, & a < x \leq \pi. \end{cases}$$

The structure of the paper is as follows. In Section 2, we introduce notations, definitions and preliminary facts which are used throughout the paper. In Section 3, we establish an operator-theoretic formulation for the boundary value problem (1.1)-(1.3) in the Hilbert space $L^2_{q,w}(0,\pi) \oplus \mathbb{C}^2$ and we give some of the virtues of eigenvalues and eigenfunctions. Section 4 is devoted to construct the Green's function for the boundary value problem (1.1)-(1.3) and to mention some of its properties. And, Section 5 gives the concluding remarks of the paper.

2. Preliminaries

In this section, we introduce some of the q-notations which will be used throughout the paper. We use the standard notations found in [6, 9, 17].

The set of non-negative integers is denoted by $\mathbb{N}_0$, and the set of positive integers is denoted by $\mathbb{N}$. For $t > 0$,

$$A_{q,t} := \{tq^n : n \in \mathbb{N}_0\}, \quad A^*_{q,t} := A_{q,t} \cup \{0\},$$

and

$$\mathfrak{A}_{q,t} := \{\mp tq^n : n \in \mathbb{N}_0\}.$$

When $t = 1$, we simply use A_q, A^*_q and $\mathfrak{A}$ to denote $A_{q,1}$, $A^*_{q,1}$ and $\mathfrak{A}_{q,1}$, respectively.

A set $S \subseteq \mathbb{R}$ is called a q-geometric set if, for every $x \in S$, $qx \in S$. Let υ be a real or complex valued function defined on a q-geometric set S. The q-difference operator is defined by

$$D_q \upsilon(x) := \frac{\upsilon(x) - \upsilon(qx)}{x(1-q)}, \quad x \neq 0.$$

If $0 \in S$, the q-derivative of a function υ at zero is defined as

$$D_q \upsilon(0) := \lim_{n\to\infty} \frac{\upsilon(xq^n) - \upsilon(0)}{xq^n}, \quad x \in S,$$

if the limit exists and does not depend on x. Since the formulation of self-adjoint eigenvalue problems requires D_q^{-1}, we define it for $x \in S$ to be

$$D_{q^{-1}}\upsilon(x) = \begin{cases} \frac{\upsilon(x)-\upsilon(q^{-1}x)}{x(1-q^{-1})}, & x \neq 0, \\ D_q\upsilon(0), & x = 0 \end{cases}$$

provided that $D_q\upsilon(0)$ exists. A right inverse, q-integration of the q-difference operator D_q is defined by Jackson [15] as

$$\int_0^x \upsilon(t)d_qt := x(1-q)\sum_{n=0}^{\infty} q^n\upsilon(xq^n),\ x \in S,$$

if the series converges. In general, the below equation is valid:

$$\int_a^b \upsilon(t)d_q(t) := \int_0^b \upsilon(t)d_qt - \int_0^a \upsilon(t)d_qt,\ a,b \in S.$$

There is no unique canonical choice for the q-integration over $[0,\infty)$. Hahn [13] defined the q-integration for a function υ over $[0,\infty)$ by

$$\int_0^{\infty} \upsilon(t)d_qt = (1-q)\sum_{n=-\infty}^{\infty} q^n\upsilon(q^n),$$

while Matsuo [18] defined q-integration on the interval $[0,\infty)$ with

$$\int_0^{\infty} \upsilon(t)d_qt : b(1-q)\sum_{n=-\infty}^{\infty} q^n\upsilon(bq^n),\ b > 0,$$

provided that the series converges.

Consequently, the q-integration of a function υ defined on $\mathbb{R}$ can be defined as

$$\int_{-\infty/b}^{\infty/b} \upsilon(t)d_qt = \frac{1-q}{b}\sum_{n=-\infty}^{\infty} q^n\left(\upsilon(q^n/b)+\upsilon(-q^n/b)\right),\ b > 0$$

provided that the series converges absolutely.

Definition 1. Let υ be a function defined on a q-geometric set S. We say that υ is q-integrable on S if and only if $\int_0^x \upsilon(t)d_qt$ exists for all $x \in S$.

Let S^* be a q-geometric set containing zero. A function υ defined on S^* is called q-regular at zero if

$$\lim_{n\to\infty} \upsilon(xq^n) = \upsilon(0)$$

holds for all $\upsilon \in S^*$.

Let $C(S^*)$ denote the space of all q-regular at zero functions defined on S^* with values in $\mathbb{R}$. $C(S^*)$, associated with the norm function

$$\|\upsilon\| = \sup\left\{|\upsilon(xq^n)| : x \in S^*, n \in \mathbb{N}_0\right\},$$

is a normed space. The q-integration by parts rule [8] is

$$\int_a^b g(t)D_q f(t)d_q t = (fg)|_a^b + \int_a^b D_q g(t) f(qt)d_q t, \quad a,b \in S^* \tag{2.1}$$

as $f,g \in C(S^*)$. While $p > 0$, and X is equivalent to $A_{q,t}$ or $A^*_{q,t}$, the space $L^q_p(X)$ is the normed space of all functions defined on X such that

$$\|\upsilon\|_p := \left(\int_0^x |\upsilon|^p \, d_q t\right)^{1/p} < \infty.$$

If $p = 2$, then $L^2_q(X)$ associated with the inner product

$$\langle \varphi, \psi \rangle := \int_0^x \varphi(t)\overline{\psi(t)} d_q t \tag{2.2}$$

is a Hilbert space. By a weighted $L^2_{q,w}(X)$ space we mean the space of all functions φ on X such that

$$\int_0^x |\varphi(t)|^2 w(t) d_q t < \infty,$$

where w is a positive function defined on X. The space $L^2_{q,w}(X)$ with the inner product

$$\langle \varphi, \psi \rangle := \int_0^x \varphi(t)\overline{\psi(t)} w(t) d_q t,$$

is a Hilbert space.

The space of all q-absolutely functions on $A^*_{q,t}$ is denoted by $AC_q(A^*_{q,t})$ and defined as the space of all q-regular at zero functions υ satisfying

$$\sum_{j=0}^{\infty} \left| \upsilon(tq^j) - \upsilon(tq^{j+1}) \right| \leq K$$

for all $t \in A^*_{q,t}$, and K is a constant depending on the function υ (c.f. [6], pg. 118, Definition 4.3.1), i. e. $AC_q(A^*_{q,t}) \subseteq C_q(A^*_{q,t})$. The space $AC^{(n)}_q(A^*_{q,t})$ ($n \in \mathbb{N}$) is the space of all functions defined on S^* such that υ, $D_q\upsilon, \cdots, D^{n-1}_q\upsilon$ are q-regular at zero and $D^{n-1}_q\upsilon \in AC_q(A^*_{q,t})$ (c.f. [6], pg. 119, Definition 4.3.2).

The following lemma which is needed in the sequel indicates that unlike the classical differential operator $\frac{d}{dx}$, D_q is neither self adjoint nor skew self adjoint. Equation (2.4) below shows that the adjoint of D_q is $-\frac{1}{q}D_{q^{-1}}$.

Lemma 1. *Let $f(\cdot)$, $g(\cdot)$ in $L^2_{q,w}(0,\pi)$ be defined on $[0,q^{-1}\pi]$. Then for $x \in (0,\pi]$, we have*

$$(D_q g)(xq^{-1}) = D_{q,xq^{-1}} g(xq^{-1}) = D_{q^{-1}} g(x), \tag{2.3}$$

$$\begin{aligned}\langle D_q f, g\rangle &= f(\pi)w(\pi)\overline{g(\pi q^{-1})}\\ &\quad - \lim_{n\to\infty} f(\pi q^n)w((\pi q^n)\overline{g(\pi q^{n-1})} + \left\langle f, -\frac{1}{q}D_{q^{-1}}g\right\rangle,\end{aligned} \tag{2.4}$$

$$\begin{aligned}\left\langle -\frac{1}{q}D_{q^{-1}}f, g\right\rangle &= \lim_{n\to\infty} f(\pi q^{n-1})w(\pi q^{n-1})\overline{g(\pi q^n)}\\ &\quad - f(\pi q^{-1})w(\pi q^{-1})\overline{g(\pi)} + \langle f, D_q g\rangle.\end{aligned} \tag{2.5}$$

Proof. The proof can be done similar to [8]. Indeed, relation (2.3) follows from

$$D_{q^{-1}}g(x) = \frac{g(x)-g(q^{-1}x)}{x(1-q^{-1})} = \frac{g(xq^{-1})-g(x)}{xq^{-1}(1-q)} = \left(D_q g\left(xq^{-1}\right)\right).$$

Using the q-integration by parts formula (2.1), we obtain

$$\begin{aligned}\langle D_q f, g\rangle &= \int_0^\pi D_q f(x)w(x)\overline{g(x)} = f(\pi)w(\pi)\overline{g(\pi)}\\ &\quad - \lim_{n\to\infty} f(\pi q^n)w(\pi q^n)\overline{g(\pi q^n)} - \int_0^\pi f(qt)w(qt)\overline{D_q g(t)}d_q t\\ &= f(\pi)w(\pi)\overline{g(\pi)} - \lim_{n\to\infty} f(\pi q^n)w(\pi q^n)\overline{g(\pi q^n)}\\ &\quad - \int_0^{q\pi} f(t)w(t)\frac{1}{q}\overline{D_{q^{-1}}g(t)}d_q t\\ &= f(\pi)w(\pi)\overline{g(\pi)} - \lim_{n\to\infty} f(\pi q^n)w(\pi q^n)\overline{g(\pi q^n)}\\ &\quad + \pi q^{-1}(1-q)f(\pi)w(\pi)\overline{D_{q^{-1}}g(\pi)} + \int_0^\pi f(t)w(t)\overline{-\frac{1}{q}D_{q^{-1}}g(t)}d_q t\\ &= f(\pi)w(\pi)\overline{g(\pi q^{-1})} - \lim_{n\to\infty} f(\pi q^n)w(\pi q^n)\overline{g(\pi q^n)} + \left\langle f, -\frac{1}{q}D_{q^{-1}}g\right\rangle\end{aligned}$$

proving (2.4). Equation (2.5) can be proven by using (2.4). □

3. Operator theoretic formulation

In this section, we introduce a modified inner product in a suitable direct sum space, we define a symmetric operator on this space and we investigate some properties of eigenvalues and eigenfunctions.

In the Hilbert space $\mathcal{H} = L^2_{q,w}(0,\pi)\oplus\mathbb{C}^2$ let an inner product defined by

$$(f,g) := \int_0^\pi f_1(x)g_1(x)w(x)d_q x + \frac{f_2\overline{g_2}}{\chi_1} + \frac{f_3\overline{g_3}}{\chi_2}$$

where

$$f=\begin{pmatrix} f_1(x)\\ f_2\\ f_3\end{pmatrix}\in\mathcal{H},\quad g=\begin{pmatrix} g_1(x)\\ g_2\\ g_3\end{pmatrix}\in\mathcal{H},\quad \chi_1:=\alpha_1\alpha_4-\alpha_2\alpha_3>0,$$

$$\chi_2:=\beta_1\beta_4-\beta_2\beta_3>0.$$

We define the operator A

$$A(f):=\begin{pmatrix} -\frac{1}{q}D_{q^{-1}}D_q f_1(x)+r(x)f_1(x)\\ \alpha_1 f_1(0)+\alpha_2 D_{q^{-1}}f_1(0)\\ \beta_1 f_1(\pi)+\beta_2 D_{q^{-1}}f_1(\pi)\end{pmatrix}$$

with the domain $D(A)$ which consists all the functions $f(x)\in\mathcal{H}$ that satisfy (1.2), (1.3) such that $f_1(x), D_{q^{-1}}f_1(x)\in AC_q[0,\pi q^n]$ and $l(f_1)\in L^2_{q,w}(0,\pi)$. Thus A is the operator generated by the differential expression $l(f)=\eta w(x)f$ and the boundary conditions (1.2), (1.3).

Theorem 1. *The operator A is symmetric in the Hilbert space $\mathcal{H}$.*

Proof. For each $f,g\in D(A)$ we have

$$\begin{aligned}(Af,g)-(f,Ag)&=\int_0^\pi Af_1(x)\overline{g_1(x)}w(x)d_qx+\frac{Af_2\overline{g_2}}{\chi_1}+\frac{Af_3\overline{g_3}}{\chi_2}\\ &\quad-\int_0^\pi f_1(x)\overline{Ag_1(x)}w(x)d_qx-\frac{f_2\overline{Ag_2}}{\chi_1}-\frac{f_3\overline{Ag_3}}{\chi_2}\\ &=\int_0^\pi\left(-\frac{1}{q}D_{q^{-1}}D_qf_1(x)+r(x)f_1(x)\right)\overline{g_1(x)}w(x)d_q(x)\\ &\quad-\int_0^\pi f_1(x)\overline{\left(-\frac{1}{q}D_{q^{-1}}D_qg_1(x)+r(x)g_1(x)\right)}w(x)d_q(x)\\ &\quad+\frac{Af_2\overline{g_2}}{\chi_1}+\frac{Af_3\overline{g_3}}{\chi_2}-\frac{f_2\overline{Ag_2}}{\chi_1}-\frac{f_3\overline{Ag_3}}{\chi_2}.\end{aligned}$$

Applying equation (2.4) in Lemma 1 with $f(x)=D_qf_1(x)$, $g(x)=g_1(x)$ to the first integral and keeping the realness of the function $r(x)$ in mind gives us

$$\begin{aligned}(Af,g)-(f,Ag)&=\lim_{n\to\infty}(D_qf_1)(\pi q^{n-1})w(\pi q^{n-1})\overline{g_1(\pi q^n)}\\ &\quad-(D_qf_1)(\pi q^{-1})r(\pi q^{-1})\overline{g_1(\pi)}\\ &\quad+\langle D_qf_1,D_qg_1\rangle-\int_0^\pi f_1(x)\left(-\frac{1}{q}D_{q^{-1}}D_qg_1(x)\right)w(x)d_qx\\ &\quad+\frac{Af_2\overline{g_2}}{\chi_1}+\frac{Af_3\overline{g_3}}{\chi_2}-\frac{f_2\overline{Ag_2}}{\chi_1}-\frac{f_3\overline{Ag_3}}{\chi_2}\end{aligned}\tag{3.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $L_q^2(0,\pi)$ which is defined in (2.2). Using (2.5) with $f(x)=f_1(x)$, $g(x)=D_q g_1(x)$ to the term $\langle D_q f_1, D_q g_1\rangle$ in (3.1) we have

$$(Af,g)-(f,Ag)=[f_1,g_1](\pi)-\lim_{n\to\infty}[f_1,g_1](\pi q^n)$$
$$+\frac{Af_2\overline{g_2}}{\chi_1}+\frac{Af_3\overline{g_3}}{\chi_2}-\frac{f_2\overline{Ag_2}}{\chi_1}-\frac{f_3\overline{Ag_3}}{\chi_2}$$

where

$$[f,g](x):=f(x)w(x)\overline{D_{q^{-1}}g(x)}-D_{q^{-1}}f(x)w(x)\overline{g(x)}.$$

If we use the definition for the domain of the operator A we conclude that

$$\frac{Af_2\overline{g_2}}{\chi_1}+\frac{Af_3\overline{g_3}}{\chi_2}-\frac{f_2\overline{Ag_2}}{\chi_1}-\frac{f_3\overline{Ag_3}}{\chi_2}=0.$$

Thus, we have

$$(Af,g)-(f,Ag)=[f_1,g_1](\pi)-\lim_{n\to\infty}[f_1,g_1](\pi q^n).$$

Due to the fact that $f_1(x),g_1(x)\in C_q^2(0)$ satisfy the boundary conditions (1.2), (1.3) we have

$$\begin{cases}\alpha_1 f_1(0)+\alpha_2 D_{q^{-1}}f_1(0)+\eta\left[\alpha_3 f_1(0)+\alpha_4 D_{q^{-1}}f_1(0)\right]=0,\\ \alpha_1 g_1(0)+\alpha_2 D_{q^{-1}}g_1(0)+\eta\left[\alpha_3 g_1(0)+\alpha_4 D_{q^{-1}}g_1(0)\right]=0.\end{cases}\tag{3.2}$$

The continuity of $f_1(x),g_1(x)$ at zero implies that

$$\lim_{n\to\infty}[f_1,g_1](\pi q^n)=[f_1,g_1](0)$$

and we have

$$(Af,g)-(f,Ag)=[f_1,g_1](\pi)-[f_1,g_1](0).$$

Validity of the equations below follows from the fact that the functions $f(\cdot)$ and $g(\cdot)$ satisfy the boundary conditions (1.2), (1.3) and from formula (3.2):

$$[f_1,g_1](0)=f_1(0)w(0)\overline{D_{q^{-1}}g_1(0)}-D_{q^{-1}}f_1(0)w(0)\overline{g_1(0)}=0,$$

$$[f_1,g_1](\pi)=f_1(\pi)w(\pi)\overline{D_{q^{-1}}g_1(\pi)}-D_{q^{-1}}f_1(\pi)w(\pi)\overline{g_1(\pi)}=0.$$

Hence, the equation $(Af,g)-(f,Ag)=0$ is satisfied and this completes the proof. □

Definition 2. An η which the boundary value problem (1.1)-(1.3) has a nontrivial solution is called an eigenvalue, and the corresponding solution, an eigenfunction. The multiplicity of an eigenvalue is defined to be the number of linearly independent solutions corresponding to it. In particular an eigenvalue is simple if and only if it has only one linearly independent solution.

The eigenfunctions of the operator A are in the form of

$$\Phi(x,\eta_n) = \Phi_n := \begin{pmatrix} \varphi(x,\eta_n) \\ \alpha_3\varphi(0,\eta_n) + \alpha_4 D_{q^{-1}}\varphi(0,\eta_n) \\ \beta_3\varphi(\pi,\eta_n) + \beta_4 D_{q^{-1}}\varphi(\pi,\eta_n) \end{pmatrix}.$$

Having the self-adjointness of the operator A proven we have the following corollaries.

Corollary 1. *The eigenfunctions Φ_1 and Φ_2 corresponding to the different eigenvalues are orthogonal.*

Corollary 2. *The eigenvalues of the boundary value problem (1.1)-(1.3) are real.*

Now, let us denote

$$\Delta(\eta) := \begin{vmatrix} U_1(\Phi_1) & U_1(\Phi_2) \\ U_2(\Phi_1) & U_2(\Phi_2) \end{vmatrix}$$

where $\Phi_i(\cdot,\eta)$ are determined by the initial conditions

$$D_q^{j-1}\Phi_i(\cdot,\eta) = \delta_{ij}, \qquad (i = 1,2,\ \eta \in \mathbb{C})$$

as δ_{ij} refers to the Kronecker delta. The function $\Delta(\eta)$ is the characteristic function of the boundary value problem (1.1)-(1.3). It is an entire function with respect to η and thus the eigenvalues of the boundary value problem (1.1)-(1.3) has an at most countable set of $\{\eta_n\}$ with no finite limit points.

In the following theorem, we prove that the eigenvalues of the boundary value problem (1.1)-(1.3) are the simple zeros of the characteristic function $\Delta(\eta)$.

Theorem 2. *The eigenvalues of the boundary value problem (1.1)-(1.3) coincide with the simple zeros of $\Delta(\eta)$.*

Proof. The proof follows similar steps as shown in Theorem 3.6 in [6] (pg. 83). Indeed, let us define the functions $\theta_1(\cdot,\eta)$ and $\theta_2(\cdot,\eta)$ as

$$\begin{cases} \theta_1(x,\eta) := U_1(\phi_1)\phi_1(x,\eta) - U_1(\phi_1)\phi_2(x,\eta), \\ \theta_2(x,\eta) := U_2(\phi_2)\phi_1(x,\eta) - U_2(\phi_1)\phi_2(x,\eta). \end{cases} \tag{3.3}$$

Hence it can easily be seen that the functions $\theta_1(\cdot,\eta)$ and $\theta_2(\cdot,\eta)$ are the two solutions of equation (1.1) which satisfy the initial conditions

$$\begin{cases} \theta_1(0,\eta) = \alpha_2 + \eta\alpha_4, \quad D_{q^{-1}}\theta_1(0,\eta) = -(\alpha_1 + \eta\alpha_3), \\ \theta_2(\pi,\eta) = \beta_2 + \eta\beta_4, \quad D_{q^{-1}}\theta_2(\pi,\eta) = -(\beta_1 + \eta\beta_3). \end{cases} \tag{3.4}$$

Now, let η_0 be an eigenvalue of the boundary value problem (1.1)-(1.3). The equation below

$$W_q(\theta_1(\cdot,\eta),\theta_2(\cdot,\eta)) = \Delta(\eta) W_q(\phi_1(\cdot,\eta),\phi_2(\cdot,\eta))(x) = \Delta(\eta)$$

leads us to the fact that the real valued functions $\theta_i(x,\eta_0)$ $(i = 1,2)$ are linearly dependent:

$$\theta_1(x,\eta_0) = k_0\theta_2(x,\eta_0),\ (k_0 \neq 0). \tag{3.5}$$

Using (3.4) and (3.5) implies

$$\begin{cases} \theta_1(\pi,\eta_0) = k_0\theta_2(\pi,\eta_0) = k_0(\beta_2+\eta\beta_4), \\ D_{q^{-1}}\theta_1(\pi,\eta) = k_0 D_{q^{-1}}\theta_2(\pi,\eta_0) = -k_0(\beta_1+\eta\beta_3). \end{cases}$$

By applying q-Lagrange identity to the functions $\theta_1(x,\eta)$ and $\theta_1(x,\eta_0)$ we obtain

$$\begin{aligned} (\eta-\eta_0)\int_0^\pi \theta_1(x,\eta)\theta_1(x,\eta_0)d_qx \\ &= \theta_1(\pi,\eta)D_{q^{-1}}\theta_1(\pi,\eta_0) - D_{q^{-1}}\theta_1(\pi,\eta)\theta_1(\pi,\eta) \\ &= k_0\left(\theta_1(\pi,\eta)D_{q^{-1}}\theta_2(\pi,\eta_0) - \theta_2(\pi,\eta_0)D_{q^{-1}}\theta_1(\pi,\eta)\right) \\ &= k_0 W_q\left(\theta_1(\cdot,\eta),\theta_2(\cdot,\eta_0)\right)(q^{-1}\pi) \\ &= k_0\Delta(\eta). \end{aligned}$$

Since $\Delta(\eta)$ is an entire function of η we have the opportunity to write the expression below:

$$\frac{d}{d\eta}\Delta(\eta) = \lim_{\eta\to\eta_0}\frac{\Delta(\eta)-\Delta_0(\eta)}{\eta-\eta_0} = \frac{1}{k_0}\int_0^\pi \theta_1^2(x,\eta_0)d_qx \neq 0. \tag{3.6}$$

The simplicity of the zeros of the function $\Delta(\eta)$ is the direct result of (3.6). □

4. Construction of the q-type Green's function

The q-type Green's function arises when we pursue a solution of the inhomogeneous boundary value problem

$$l(\upsilon) := -\frac{1}{q}D_{q^{-1}}D_q\upsilon(x) + \{-\eta w(x) + r(x)\}\upsilon(x) = f(x), \quad x\in[0,\pi] \tag{4.1}$$

$$U_1(\upsilon) := \alpha_1\upsilon(0) + \alpha_2 D_{q^{-1}}y(0) + \eta\left[\alpha_3\upsilon(0) + \alpha_4 D_{q^{-1}}\upsilon(0)\right] = f_2, \tag{4.2}$$

$$U_2(\upsilon) := \beta_1\upsilon(\pi) + \beta_2 D_{q^{-1}}\upsilon(\pi) + \eta\left[\beta_3\upsilon(\pi) + \beta_4 D_{q^{-1}}\upsilon(\pi)\right] = f_3, \tag{4.3}$$

where $f(x)\in L^2_{q,w}(0,\pi)$.

Theorem 3. *Assume that η is not an eigenvalue of the boundary value problem (1.1)-(1.3). Let $\phi(\cdot,\eta)$ satisfy the q-difference equation (4.1) and the boundary conditions (4.2)-(4.3) where $f(x)\in L^2_{q,w}(0,\pi)$. Then*

$$\begin{aligned} \phi(x,\eta) = \int_0^\pi G(x,t,\eta)f(t)d_qt + \frac{f_2\left(\alpha_3 G(0,\cdot,\eta) + \alpha_4 D_{q^{-1}}G(0,\cdot,\eta)\right)}{\chi_1} \\ + \frac{f_3\left(\beta_3 G(\pi,\cdot,\eta) + \beta_4 D_{q^{-1}}G(\pi,\cdot,\eta)\right)}{\chi_2} \end{aligned} \tag{4.4}$$

where $G(x,t;\eta)$ is the Green's function of the inhomogeneous boundary value problem (4.1)-(4.3) defined by

$$G(x,t;\eta) = -\frac{1}{\Delta(\eta)}\begin{cases} \theta_2(x,\eta)\theta_1(t,\eta), & t\le x, \\ \theta_1(x,\eta)\theta_2(t,\eta), & x\le t. \end{cases}$$

Conversely, the function $\phi(x,\eta)$ defined by (4.4) satisfies (4.1) and (4.2), (4.3).

Proof. We shall search the solution of the inhomogeneous boundary value problem (4.1)-(4.3) as

$$\phi(x,\eta) = c_1(x)\theta_1(x,\eta) + c_2(x)\theta_2(x,\eta) \tag{4.5}$$

where the functions $c_1(x)$ and $c_2(x)$ are the solutions of the system of equations

$$\begin{cases} D_{q,x}c_1(x)\theta_1(x,\eta) + D_{q,x}c_2(x)\theta_2(x,\eta) = 0, \\ D_{q,x}c_1(x)D_{q,x}\theta_1(x,\eta) + D_{q,x}c_2(x)D_{q,x}\theta_2(x,\eta) = f(x). \end{cases} \tag{4.6}$$

If the functions $D_{q,x}c_i(x)$ $(i=1,2)$ are q-integrable on $[0,t]$ then

$$\lim_{n\to\infty} tq^n\theta_i\left(tq^{n+1},\eta\right) f(tq^{n+1}) = 0, (i=1,2).$$

Now, let us define the q-geometric set S_f by

$$S_f := \left\{x \in [0,\pi] : \lim_{n\to\infty} xq^n \left|f(xq^n)\right|^2 = 0\right\}.$$

Since, $f \in L^2_{q,w}(0,\pi)$ the set S_f is a q-geometric set containing $\{\pi q^m : m \in \mathbb{N}_0\}$. Hence, $D_q c_i(\cdot)$, $(i=1,2)$ are q-integrable on $[0,x]$ for all $x \in S_f$ and the solutions of (4.6) are

$$\begin{cases} c_1(x) = \tilde{c_1} + \frac{q}{\Delta(\eta)}\int_0^x \theta_2(qt,\eta)f(qt)w(qt)d_qt, \\ c_2(x) = \tilde{c_2} + \frac{q}{\Delta(\eta)}\int_x^\pi \theta_1(qt,\eta)f(qt)w(qt)d_qt, \end{cases} \tag{4.7}$$

where $\tilde{c_1}$, $\tilde{c_2}$ are unknown constants and $x \in S_f$. Substituting (4.7) into (4.5) and taking (4.2), (4.3) into consideration leads us to (4.4). Conversely, if $\phi(x,\eta)$ is given by (4.4), then it is a solution of (4.1) which satisfies the boundary conditions (4.2), (4.3) and this completes the proof. □

The following theorem lists a number of properties of the Green's function.

Theorem 4. *Green's function has the following properties:*

(1) *$G(x,t,\eta)$ is continuous at the point $(0,0)$.*
(2) *$G(x,t,\eta) = G(t,x,\eta)$.*
(3) *For each fixed $t \in (0,q\pi]$, $G(x,t,\eta)$ satisfies the q-difference equation (4.1) in the intervals $[0,t)$, $(t,\pi]$ and it also satisfies the boundary conditions (4.2)-(4.3).*

Proof. The proof can easily be obtained by using a similar procedure to [6]. □

5. Conclusion

In this work we investigate a boundary value problem which consists of a second order q-differential equation and eigenvalue dependent boundary conditions. We introduce a modified inner product in a suitable direct sum space $L^2_{q,w}(0,\pi)\oplus\mathbb{C}^2$. We define the operator A on this space and prove that it is symmetric. We provide some

of the properties of eigenvalues and eigenfunctions and we construct the Green's function.

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Author's address

F. Ayca Cetinkaya
Mersin University, Department of Mathematics, 33343 Mersin, Turkiye
E-mail address: faycacetinkaya@mersin.edu.tr

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ON THE GENERALIZED BI-PERIODIC FIBONACCI AND LUCAS QUATERNIONS

YOUNSEOK CHOO

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Abstract. In this paper we introduce the generalized bi-periodic Fibonacci and Lucas quaternions which are the further generalizations of the bi-periodic Fibonacci and Lucas quaternions considered in the literature. For those quaternions, we derive the generating functions, Binet's formulas and Catalan's identities.

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1. Introduction

As is well known, the Fibonacci sequence $\{F_n\}$ is generated from the recurrence relation $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$ with $F_0 = 0,\ F_1 = 1$, and the Lucas sequence $\{L_n\}$ is generated from the recurrence relation $L_n = L_{n-1} + L_{n-2}$ $(n \geq 2)$ with $L_0 = 2,\ L_1 = 1$. The Binet's formulas for $\{F_n\}$ and $\{L_n\}$ are respectively given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$
$$L_n = \alpha^n + \beta^n,$$

where $\alpha(> 0)$ and $\beta(< 0)$ are roots of the equation $x^2 - x - 1 = 0$.

Many authors generalized the Fibonacci and Lucas sequences by changing initial conditions and/or recurrence relations. In particular, Edson and Yayenie [5] introduced the bi-periodic Fibonacci sequence $\{p_n\}$ defined by

$$p_0 = 0,\ p_1 = 1,\ \ p_n = \begin{cases} ap_{n-1} + p_{n-2}, & \text{if } n \text{ is even} \\ bp_{n-1} + p_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad (n \geq 2). \tag{1.1}$$

The Binet's formula for $\{p_n\}$ is given by [5]

$$p_n = \frac{a^{\zeta(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \Big(\frac{\alpha^n - \beta^n}{\alpha - \beta} \Big), \tag{1.2}$$

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where $\alpha(>0)$ and $\beta(<0)$ are roots of the equation $x^2-abx-ab=0$, and $\zeta(\cdot)$ is the parity function such that $\zeta(n)=0$ if n is even and $\zeta(n)=1$ if n is odd.

The bi-periodic Fibonacci sequence $\{p_n\}$ given in (1.1) includes many sequences as special cases. For $a=b=1$, $\{p_n\}$ becomes the Fibonacci sequence. For $a=b=2$, $\{p_n\}$ becomes the Pell sequence. If $a=b=k$, then $\{p_n\}$ denotes the k-Fibonacci sequence defined in [8], etc.

On the other hand, Bilgici [2] generalized the Lucas sequence by introducing the bi-periodic Lucas sequence $\{u_n\}$ defined by

$$u_0=2,\ u_1=a,\ \ u_n=\begin{cases} bu_{n-1}+u_{n-2}, & \text{if } n \text{ is even} \\ au_{n-1}+u_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad (n\geq 2). \tag{1.3}$$

If $a=b=1$, then $\{u_n\}$ becomes the Lucas sequence $\{L_n\}$. If $a=b=k$, then $\{u_n\}$ becomes the k-Lucas sequence in [7].

The Binet's formula for $\{u_n\}$ is given by [2]

$$u_n=\frac{a^{\zeta(n)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor}}(\alpha^n+\beta^n), \tag{1.4}$$

where α and β are as defined in (1.2).

A quaternion q is defined by

$$q=q_0e_0+q_1e_1+q_2e_2+q_3e_3,$$

where $q_0,q_1,q_2,q_3\in\mathbb{R}$, $e_0=1$, and e_1,e_2 and e_3 are the standard basis in $\mathbb{R}^3$ such that $e_i^2=-1$, $i=1,2,3$, and

$$e_1e_2=-e_2e_1=e_3,\ e_2e_3=-e_3e_2=e_1,\ e_3e_1=-e_1e_3=e_2.$$

As noted in the literature [1, 6, 9, 20], quaternions are widely used in the fields of engineering and physics as well as mathematics, and attracted sustained attention from many researchers. In particular, a variety of results are available in the literature on the properties of quaternions related to the sequences described earlier. Horadam [13] defined the Fibonacci quaternion sequence $\{G_n\}$ and Lucas quaternion sequence $\{H_n\}$ as

$$\begin{aligned} G_n&=F_ne_0+F_{n+1}e_1+F_{n+2}e_2+F_{n+3}e_3,\\ H_n&=L_ne_0+L_{n+1}e_1+L_{n+2}e_2+L_{n+3}e_3, \end{aligned}$$

where F_n and L_n are respectively the nth Fibonacci and Lucas numbers.

Following the work of Horadam [13], diverse results have appeared in the literature. Halici [10] obtained the generating functions, Binet's formulas and some combinatorial properties of the Fibonacci and Lucas quaternions. Halici [11] also introduced the complex Fibonacci quaternions. Ramirez [15] studied the properties of the k-Fibonacci and k-Lucas quaternions. Çimen and İpek [4], Szynal-Liana and Włoch [17] investigated the Pell and Pell-Lucas quaternions. Szynal-Liana and Włoch [17]

introduced the Jacobsthal and Jacobsthal-Lucas quaternions also. Catarino [3] considered the modified Pell and modified k-Pell quaternions. Halici and Karataş [12] defined a general quaternion which includes several quaternions mentioned above as special cases.

Recently Tan et al. [19] introduced the bi-periodic Fibonacci quaternion sequence $\{P_n\}$ defined by

$$P_n = p_n e_0 + p_{n+1} e_1 + p_{n+2} e_2 + p_{n+3} e_3, \tag{1.5}$$

where p_n is the nth bi-periodic Fibonacci number.

The Binet's formula for $\{P_n\}$ is given by [19]

$$P_n = \begin{cases} \frac{1}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^* \alpha^n - \beta^* \beta^n}{\alpha - \beta} \right), & \text{if } n \text{ is even} \\ \frac{1}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^{**} \alpha^n - \beta^{**} \beta^n}{\alpha - \beta} \right), & \text{if } n \text{ is odd} \end{cases} \tag{1.6}$$

where α and β are as defined in (1.2), and

$$\alpha^* = \sum_{l=0}^{3} \frac{a^{\zeta(l+1)}}{(ab)^{\lfloor \frac{l}{2} \rfloor}} \alpha^l e_l,$$

$$\beta^* = \sum_{l=0}^{3} \frac{a^{\zeta(l+1)}}{(ab)^{\lfloor \frac{l}{2} \rfloor}} \beta^l e_l,$$

$$\alpha^{**} = \sum_{l=0}^{3} \frac{a^{\zeta(l)}}{(ab)^{\lfloor \frac{l+1}{2} \rfloor}} \alpha^l e_l,$$

$$\beta^{**} = \sum_{l=0}^{3} \frac{a^{\zeta(l)}}{(ab)^{\lfloor \frac{l+1}{2} \rfloor}} \beta^l e_l.$$

Tan et al. [18] also introduced the bi-periodic Lucas quaternion sequence $\{U_n\}$ as follows:

$$U_n = u_n e_0 + u_{n+1} e_1 + u_{n+2} e_2 + u_{n+3} e_3, \tag{1.7}$$

where u_n is the nth bi-periodic Lucas number.

The Binet's formula for $\{U_n\}$ is given by [18]

$$U_n = \begin{cases} \frac{1}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} (\alpha^{**} \alpha^n + \beta^{**} \beta^n), & \text{if } n \text{ is even} \\ \frac{1}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} (\alpha^* \alpha^n + \beta^* \beta^n), & \text{if } n \text{ is odd} \end{cases} \tag{1.8}$$

where α, β are as defined in (1.2), and α^*, β^*, α^{**} and β^{**} are as defined in (1.6).

If we use the initial condition $P_0 = e_1 + e_2 + 2e_3$ and $P_1 = e_0 + e_1 + 2e_2 + 3e_3$ in (1.5), then $\{P_n\}$ is the same as the generalized Fibonacci quaternion sequence considered in [14]. Also if we set $P_0 = 2e_0 + e_1 + 3e_2 + 4e_3$ and $P_1 = e_0 + 3e_1 + 4e_2 +$

$7e_3$ in (1.5), then $\{P_n\}$ is the same as the generalized Lucas quaternion sequence considered in [14].

In this paper we introduce the generalized bi-periodic Fibonacci and Lucas quaternion sequences which include $\{P_n\}$ and $\{U_n\}$ as special cases. For those quaternions, we derive the generating functions, Binet's formulas and Catalan's identities.

2. Main results

2.1. *Generalized bi-periodic Fibonacci quaternion*

Consider the generalized bi-perioodic Fibonacci sequence $\{q_n\}$ defined by Sahin [16] and Yayenie [21] as

$$q_0 = 0,\ q_1 = 1,\ \ q_n = \begin{cases} aq_{n-1} + cq_{n-2}, & \text{if } n \text{ is even} \\ bq_{n-1} + dq_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad (n \geq 2). \tag{2.1}$$

The Binet's formula for $\{q_n\}$ is given by [21]

$$q_n = \frac{a^{\zeta(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^{\lfloor \frac{n}{2} \rfloor}(\alpha + d - c)^{n - \lfloor \frac{n}{2} \rfloor} - \beta^{\lfloor \frac{n}{2} \rfloor}(\beta + d - c)^{n - \lfloor \frac{n}{2} \rfloor}}{\alpha - \beta} \right), \tag{2.2}$$

where $\alpha(> 0)$ and $\beta(< 0)$ are roots of the equation $x^2 - (ab + c - d)x - abd = 0$.

Definition 1. We define the generalized bi-periodic Fibonacci quaternion sequence $\{Q_n\}$ by

$$Q_n = q_n e_0 + q_{n+1} e_1 + q_{n+2} e_2 + q_{n+3} e_3, \tag{2.3}$$

where q_n is the nth generalized bi-periodic Fibonacci number.

If $c = d = 1$, then $\{Q_n\}$ becomes the bi-periodic Fibonacci quaternion sequence given in (1.5).

If $a = b = 1$ and $c = d = 2$, then $\{Q_n\}$ becomes the Jacobsthal quaternion sequence defined in [17].

In the rest of the paper, we will use the following identities [21] whenever necessary: (i) $\alpha + \beta = ab + c - d$, (ii) $\alpha\beta = -abd$, (iii) $\alpha(\alpha + d - c) = ab(\alpha + d)$, (iv) $\beta(\beta + d - c) = ab(\beta + d)$, (v) $(\alpha + d)(\beta + d) = cd$.

Theorem 1 (Generating function)**.** *The generating function for the generalized bi-periodic Fibonacci quaternion sequence is*

$$G(x) = \frac{\left(1 - (ab + d)x^2 + bcx^3\right) Q_0 + x(1 + ax - cx^2) Q_1}{1 - (ab + c + d)x^2 + cdx^4}. \tag{2.4}$$

Proof. We can show that $\{Q_n\}$ satisfies the same recurrence relation as $\{q_n\}$ with the initial condition

$$\begin{aligned} Q_0 &= q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 \\ &= e_1 + a e_2 + (ab + d) e_3, \end{aligned}$$

$$\begin{aligned} Q_1 &= q_1e_0 + q_2e_1 + q_3e_2 + q_4e_3 \\ &= e_0 + ae_1 + (ab+d)e_2 + a(ab+c+d)e_3, \end{aligned}$$

and

$$\begin{aligned} Q_{2n} &= (ab+c+d)Q_{2n-2} - cdQ_{2n-4}, \\ Q_{2n+1} &= (ab+c+d)Q_{2n-1} - cdQ_{2n-3}. \end{aligned}$$

Then, proceeding as in the proof of [21, Theorem 7], we can obtain (2.4). □

If $a = b$ and $c = d$, then

$$\begin{aligned} G(x) &= \frac{(1-ax)Q_0 + xQ_1}{1-ax-cx^2} \\ &= \frac{xe_0 + e_1 + (a+x)e_2 + (a^2+c+acx)e_3}{1-ax-cx^2}. \end{aligned}$$

Hence, for $a = b = c = d = 1$, we get the generating function for the Fibonacci quaternion

$$G(x) = \frac{xe_0 + e_1 + (1+x)e_2 + (2+x)e_3}{1-x-x^2},$$

as in [10], and, for $a = b = k$ and $c = d = 1$, we obtain the generating function for the k-Fibonacci quaternion

$$G(x) = \frac{xe_0 + e_1 + (k+x)e_2 + (k^2+1+kx)e_3}{1-kx-x^2},$$

which is given in [15].

Theorem 2 (Binet's formula)**.** *The Binet's formula for the generalized bi-periodic Fibonacci quaternion sequence is*

$$Q_n = \begin{cases} \frac{1}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha_e \alpha^{\lfloor \frac{n}{2} \rfloor}(\alpha+d-c)^{n-\lfloor \frac{n}{2} \rfloor} - \beta_e \beta^{\lfloor \frac{n}{2} \rfloor}(\beta+d-c)^{n-\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta} \right), & \text{if } n \text{ is even} \\ \frac{1}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha_o \alpha^{\lfloor \frac{n}{2} \rfloor}(\alpha+d-c)^{n-\lfloor \frac{n}{2} \rfloor} - \beta_o \beta^{\lfloor \frac{n}{2} \rfloor}(\beta+d-c)^{n-\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta} \right), & \text{if } n \text{ is odd} \end{cases} \tag{2.5}$$

where

$$\begin{aligned} \alpha_e &= \sum_{l=0}^{3} \frac{a^{\xi(l+1)}}{(ab)^{\lfloor \frac{l}{2} \rfloor}} \alpha^{\lfloor \frac{l}{2} \rfloor}(\alpha+d-c)^{\lfloor \frac{l+1}{2} \rfloor} e_l, \\ \beta_e &= \sum_{l=0}^{3} \frac{a^{\xi(l+1)}}{(ab)^{\lfloor \frac{l}{2} \rfloor}} \beta^{\lfloor \frac{l}{2} \rfloor}(\beta+d-c)^{\lfloor \frac{l+1}{2} \rfloor} e_l, \\ \alpha_o &= \sum_{l=0}^{3} \frac{a^{\xi(l)}}{(ab)^{\lfloor \frac{l+1}{2} \rfloor}} \alpha^{\lfloor \frac{l+1}{2} \rfloor}(\alpha+d-c)^{\lfloor \frac{l}{2} \rfloor} e_l, \end{aligned}$$

$$\beta_o = \sum_{l=0}^{3} \frac{a^{\zeta(l)}}{(ab)^{\lfloor \frac{l+1}{2} \rfloor}} \beta^{\lfloor \frac{l+1}{2} \rfloor} (\beta + d - c)^{\lfloor \frac{l}{2} \rfloor} e_l.$$

Proof. Firstly we note that $\lfloor \frac{n}{2} \rfloor = \frac{n-\zeta(n)}{2}$.
¿From (2.2) and (2.3), we have

$$\begin{aligned}
Q_n = {} & \frac{a^{\zeta(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^{\lfloor \frac{n}{2} \rfloor}(\alpha+d-c)^{n-\lfloor \frac{n}{2} \rfloor} - \beta^{\lfloor \frac{n}{2} \rfloor}(\beta+d-c)^{n-\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta} \right) e_0 \\
& + \frac{a^{\zeta(n)}}{(ab)^{\zeta(n)}(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^{\zeta(n)}(\alpha+d-c)^{\zeta(n+1)}\alpha^{\lfloor \frac{n}{2} \rfloor}(\alpha+d-c)^{n-\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta} \right. \\
& \qquad \left. - \frac{\beta^{\zeta(n)}(\beta+d-c)^{\zeta(n+1)}\beta^{\lfloor \frac{n}{2} \rfloor}(\beta+d-c)^{n-\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta} \right) e_1 \\
& + \frac{a^{\zeta(n+1)}}{ab(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha(\alpha+d-c)\alpha^{\lfloor \frac{n}{2} \rfloor}(\alpha+d-c)^{n-\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta} \right. \\
& \qquad \left. - \frac{\beta(\beta+d-c)\beta^{\lfloor \frac{n}{2} \rfloor}(\beta+d-c)^{n-\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta} \right) e_2 \\
& + \frac{a^{\zeta(n)}}{(ab)^{1+\zeta(n)}(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^{1+\zeta(n)}(\alpha+d-c)^{1+\zeta(n+1)}\alpha^{\lfloor \frac{n}{2} \rfloor}(\alpha+d-c)^{n-\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta} \right. \\
& \qquad \left. - \frac{\beta^{1+\zeta(n)}(\beta+d-c)^{1+\zeta(n+1)}\beta^{\lfloor \frac{n}{2} \rfloor}(\beta+d-c)^{n-\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta} \right) e_3,
\end{aligned}$$

or

$$Q_n = \frac{1}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha_n \alpha^{\lfloor \frac{n}{2} \rfloor}(\alpha+d-c)^{n-\lfloor \frac{n}{2} \rfloor} - \beta_n \beta^{\lfloor \frac{n}{2} \rfloor}(\beta+d-c)^{n-\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta} \right),$$

where

$$\begin{aligned}
\alpha_n = {} & a^{\zeta(n+1)} e_0 + \frac{a^{\zeta(n)}\alpha^{\zeta(n)}(\alpha+d-c)^{\zeta(n+1)}}{(ab)^{\zeta(n)}} e_1 \\
& + \frac{a^{\zeta(n+1)}\alpha(\alpha+d-c)}{ab} e_2 + \frac{a^{\zeta(n)}\alpha^{1+\zeta(n)}(\alpha+d-c)^{1+\zeta(n+1)}}{(ab)^{1+\zeta(n)}} e_3, \\
\beta_n = {} & a^{\zeta(n+1)} e_0 + \frac{a^{\zeta(n)}\beta^{\zeta(n)}(\beta+d-c)^{\zeta(n+1)}}{(ab)^{\zeta(n)}} e_1 \\
& + \frac{a^{\zeta(n+1)}\beta(\beta+d-c)}{ab} e_2 + \frac{a^{\zeta(n)}\beta^{1+\zeta(n)}(\beta+d-c)^{1+\zeta(n+1)}}{(ab)^{1+\zeta(n)}} e_3.
\end{aligned}$$

If n is even, then

$$\begin{aligned}\alpha_n &= ae_0 + (\alpha+d-c)e_1 + \frac{a\alpha(\alpha+d-c)}{ab}e_2 + \frac{\alpha(\alpha+d-c)^2}{ab}e_3\\ &= \sum_{l=0}^{3}\frac{a^{\zeta(l+1)}}{(ab)^{\lfloor\frac{l}{2}\rfloor}}\alpha^{\lfloor\frac{l}{2}\rfloor}(\alpha+d-c)^{\lfloor\frac{l+1}{2}\rfloor}e_l,\\ \beta_n &= ae_0 + (\beta+d-c)e_1 + \frac{a\beta(\beta+d-c)}{ab}e_2 + \frac{\beta(\beta+d-c)^2}{ab}e_3\\ &= \sum_{l=0}^{3}\frac{a^{\zeta(l+1)}}{(ab)^{\lfloor\frac{l}{2}\rfloor}}\beta^{\lfloor\frac{l}{2}\rfloor}(\beta+d-c)^{\lfloor\frac{l+1}{2}\rfloor}e_l.\end{aligned}$$

Similarly, if n is odd, then

$$\begin{aligned}\alpha_n &= \sum_{l=0}^{3}\frac{a^{\zeta(l)}}{(ab)^{\lfloor\frac{l+1}{2}\rfloor}}\alpha^{\lfloor\frac{l+1}{2}\rfloor}(\alpha+d-c)^{\lfloor\frac{l}{2}\rfloor}e_l,\\ \beta_n &= \sum_{l=0}^{3}\frac{a^{\zeta(l)}}{(ab)^{\lfloor\frac{l+1}{2}\rfloor}}\beta^{\lfloor\frac{l+1}{2}\rfloor}(\beta+d-c)^{\lfloor\frac{l}{2}\rfloor}e_l,\end{aligned}$$

and the proof is completed. □

If $c = d = 1$, then (2.5) becomes the Binet's formula for the bi-periodic Fibonacci quaternion given in (1.6).

Theorem 3 (Catalan's identity)**.** *The Catalan's identity for the generalized bi-periodic Fibonacci quaternion sequence is*

$$Q_n^2 - Q_{n+2r}Q_{n-2r} = \begin{cases}(cd)^{\frac{n-2r}{2}}\left(\frac{\alpha_e\beta_e\left((\alpha+d)^{2r}-(cd)^r\right)+\beta_e\alpha_e\left((\beta+d)^{2r}-(cd)^r\right)}{(\alpha-\beta)^2}\right), & if\ n\ is\ even,\\ abc(cd)^{\frac{n-2r-1}{2}}\left(\frac{\alpha_o\beta_o\left((cd)^r-(\alpha+d)^{2r}\right)+\beta_o\alpha_o\left((cd)^r-(\beta+d)^{2r}\right)}{(\alpha-\beta)^2}\right), & if\ n\ is\ odd.\end{cases} \tag{2.6}$$

Proof. Firstly, assume that n is even, and let

$$\begin{aligned}X_1 &= (\alpha-\beta)^2(ab)^n Q_n^2,\\ X_2 &= (\alpha-\beta)^2(ab)^n Q_{n+2r}Q_{n-2r}.\end{aligned}$$

Then

$$\begin{aligned}X_1 &= \left(\alpha_e\alpha^{\frac{n}{2}}(\alpha+d-c)^{\frac{n}{2}} - \beta_e\beta^{\frac{n}{2}}(\beta+d-c)^{\frac{n}{2}}\right)^2\\ &= \alpha_e^2\alpha^n(\alpha+d-c)^n + \beta_e^2\beta^n(\beta+d-c)^n\end{aligned}$$

$$
\begin{aligned}
&-(\alpha_e\beta_e+\beta_e\alpha_e)\alpha^{\frac{n}{2}}(\alpha+d-c)^{\frac{n}{2}}\beta^{\frac{n}{2}}(\beta+d-c)^{\frac{n}{2}},\\
&=\alpha_e^2\alpha^n(\alpha+d-c)^n+\beta_e^2\beta^n(\beta+d-c)^n\\
&-(\alpha_e\beta_e+\beta_e\alpha_e)(ab)^n(\alpha+d)^{\frac{n}{2}}(\beta+d)^{\frac{n}{2}},\\
&=\alpha_e^2\alpha^n(\alpha+d-c)^n+\beta_e^2\beta^n(\beta+d-c)^n\\
&-(\alpha_e\beta_e+\beta_e\alpha_e)(ab)^n(cd)^{\frac{n}{2}},
\end{aligned}
$$

and

$$
\begin{aligned}
X_2=&\left(\alpha_e\alpha^{\frac{n+2r}{2}}(\alpha+d-c)^{\frac{n+2r}{2}}-\beta_e\beta^{\frac{n+2r}{2}}(\beta+d-c)^{\frac{n+2r}{2}}\right)\\
&\times\left(\alpha_e\alpha^{\frac{n-2r}{2}}(\alpha+d-c)^{\frac{n-2r}{2}}-\beta_e\beta^{\frac{n-2r}{2}}(\beta+d-c)^{\frac{n-2r}{2}}\right)\\
&=\alpha_e^2\alpha^n(\alpha+d-c)^n+\beta_e^2\beta^n(\beta+d-c)^n\\
&-\alpha_e\beta_e\alpha^{\frac{n+2r}{2}}(\alpha+d-c)^{\frac{n+2r}{2}}\beta^{\frac{n-2r}{2}}(\beta+d-c)^{\frac{n-2r}{2}}\\
&-\beta_e\alpha_e\alpha^{\frac{n-2r}{2}}(\alpha+d-c)^{\frac{n-2r}{2}}\beta^{\frac{n+2r}{2}}(\beta+d-c)^{\frac{n+2r}{2}},\\
&=\alpha_e^2\alpha^n(\alpha+d-c)^n+\beta_e^2\beta^n(\beta+d-c)^n\\
&-\alpha_e\beta_e(ab)^n(\alpha+d)^{\frac{n+2r}{2}}(\beta+d)^{\frac{n-2r}{2}}\\
&-\beta_e\alpha_e(ab)^n(\alpha+d)^{\frac{n-2r}{2}}(\beta+d)^{\frac{n+2r}{2}},\\
&=\alpha_e^2\alpha^n(\alpha+d-c)^n+\beta_e^2\beta^n(\beta+d-c)^n\\
&-\alpha_e\beta_e(ab)^n(cd)^{\frac{n-2r}{2}}(\alpha+d)^{2r}\\
&-\beta_e\alpha_e(ab)^n(cd)^{\frac{n-2r}{2}}(\beta+d)^{2r}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
X_1-X_2=&\,\alpha_e\beta_e(ab)^n(cd)^{\frac{n-2r}{2}}\left((\alpha+d)^{2r}-(cd)^r\right)\\
&+\beta_e\alpha_e(ab)^n(cd)^{\frac{n-2r}{2}}\left((\beta+d)^{2r}-(cd)^r\right),
\end{aligned}
$$

and the proof is completed for the case where n is even.

When n is odd, we can proceed similarly, and details are omitted. □

If $c=d=1$, then $\alpha^2=ab(\alpha+1)$ and

$$
\begin{aligned}
(\alpha+1)^{2r}-1&=\frac{(ab)^{2r}(\alpha+1)^{2r}-(ab)^{2r}}{(ab)^{2r}}\\
&=\frac{\alpha^{4r}-(ab)^{2r}}{(ab)^{2r}}.
\end{aligned}
$$

Similarly

$$(\beta+1)^{2r}-1=\frac{\beta^{4r}-(ab)^{2r}}{(ab)^{2r}},$$

and Theorem 3 reduces to [19, Theorem 5].

2.2. *Generalized bi-periodic Lucas quaternion*

Consider the generalized bi-periodic Lucas sequence $\{v_n\}$ defined by Bilgici [2] as

$$v_0=\frac{d+1}{d},\ v_1=a,\ \ v_n=\begin{cases} bv_{n-1}+dv_{n-2}, & \text{if } n \text{ is even} \\ av_{n-1}+cv_{n-2}, & \text{if } n \text{ is odd} \end{cases}\quad (n\geq 2). \tag{2.7}$$

The Binet's formula for $\{v_n\}$ is given by [2]

$$v_n=\frac{a^{\zeta(n)}}{(ab)^{\lfloor\frac{n-1}{2}\rfloor}}\left(\frac{(\alpha+d+1)\alpha^{\lfloor\frac{n-1}{2}\rfloor}(\alpha+d-c)^{\lfloor\frac{n}{2}\rfloor}-(\beta+d+1)\beta^{\lfloor\frac{n-1}{2}\rfloor}(\beta+d-c)^{\lfloor\frac{n}{2}\rfloor}}{\alpha-\beta}\right), \tag{2.8}$$

where α and β are as defined in (2.2).

Definition 2. The generalized bi-periodic Lucas quaternion sequence $\{V_n\}$ is defined by

$$V_n=v_ne_0+v_{n+1}e_1+v_{n+2}e_2+v_{n+3}e_3, \tag{2.9}$$

where v_n is the nth generalized bi-periodic Lucas number.

If $c=d=1$, $\{V_n\}$ becomes the bi-periodic Lucas quaternion sequence given in (1.7).

Theorem 4 (Generating function)**.** *The generating function for the generalized bi-periodic Lucas quaternion sequence is*

$$H(x)=\frac{\big(1-(ab+c)x^2+adx^3\big)V_0+x(1+bx-dx^2)V_1}{1-(ab+c+d)x^2+cdx^4}. \tag{2.10}$$

Proof. Replacing Q_0, Q_1, a, b, c and d by V_0, V_1, b, a, d and c in (2.4), we obtain (2.10). □

If $a=b$ and $c=d$, then

$$\begin{aligned} H(x)&=\frac{(1-ax)V_0+xV_1}{1-ax-cx^2}\\ &=\frac{\frac{c+1-ax}{c}e_0+\big(a+(c+1)x\big)e_1+(a^2+c+1+acx)e_2}{1-ax-cx^2}\\ &\quad+\frac{\big(a^3+2ac+a+(a^2+c^2+1)x\big)e_3}{1-ax-cx^2}. \end{aligned}$$

Hence, for $a = b = k$ and $c = d = 1$, we obtain the generating function for the k-Lucas quaternion

$$H(x) = \frac{(2-kx)e_0+(k+2x)e_1+(k^2+2+kx)e_2+\big(k^3+3k+(k^2+2)x\big)e_3}{1-kx-x^2},$$

as in [15].

Theorem 5 (Binet's formula)**.** *The Binet's formula for the generalized bi-periodic Lucas quaternion sequence is*

$$V_n = \begin{cases} \frac{1}{(ab)^{\lfloor \frac{n-1}{2} \rfloor}} V_{ne}, & \text{if } n \text{ is even} \\ \frac{1}{(ab)^{\lfloor \frac{n-1}{2} \rfloor}} V_{no}, & \text{if } n \text{ is odd} \end{cases} \tag{2.11}$$

where

$$V_{ne} = \frac{\alpha_o(\alpha+d+1)\alpha^{\lfloor \frac{n-1}{2} \rfloor}(\alpha+d-c)^{\lfloor \frac{n}{2} \rfloor}-\beta_o(\beta+d+1)\beta^{\lfloor \frac{n-1}{2} \rfloor}(\beta+d-c)^{\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta},$$

$$V_{no} = \frac{\alpha_e(\alpha+d+1)\alpha^{\lfloor \frac{n-1}{2} \rfloor}(\alpha+d-c)^{\lfloor \frac{n}{2} \rfloor}-\beta_e(\beta+d+1)\beta^{\lfloor \frac{n-1}{2} \rfloor}(\beta+d-c)^{\lfloor \frac{n}{2} \rfloor}}{\alpha-\beta},$$

with α_e, β_e, α_o and β_o as defined in (2.5).

Proof. Using the Binet's formula for $\{v_n\}$ and proceeding as in the proof of Theorem 2, we can easily obtain (2.11). □

If $c = d = 1$, then

$$\begin{aligned}(\alpha+2)\alpha^{\lfloor \frac{n-1}{2} \rfloor+\lfloor \frac{n}{2} \rfloor} &= (\alpha+2)\alpha^{n-1} \\ &= \Big(1+\frac{2}{\alpha}\Big)\alpha^n \\ &= \Big(1-\frac{2\beta}{ab}\Big)\alpha^n \\ &= \frac{(\alpha-\beta)\alpha^n}{ab},\end{aligned}$$

and

$$\begin{aligned}(\beta+2)\beta^{\lfloor \frac{n-1}{2} \rfloor+\lfloor \frac{n}{2} \rfloor} &= (\beta+2)\beta^{n-1} \\ &= \Big(1+\frac{2}{\beta}\Big)\beta^n \\ &= \Big(1-\frac{2\alpha}{ab}\Big)\beta^n \\ &= \frac{(\beta-\alpha)\beta^n}{ab}.\end{aligned}$$

Hence (2.11) reduces to the Binet's formula for the bi-periodic Lucas quaternion given in (1.8).

We verify (2.11) for $n = 1$. From (2.7) and the definition of $\{V_n\}$, we have

$$\begin{aligned} V_1 &= v_1e_0 + v_2e_1 + v_3e_2 + v_4e_3 \\ &= ae_0 + (ab + d + 1)e_1 + a(ab + c + d + 1)e_2 \\ &\quad + (a^2b^2 + abc + 2abd + ab + d^2 + d)e_3. \end{aligned}$$

On the other hand, if $n = 1$, then (2.11) becomes

$$V_1 = \frac{\alpha_e(\alpha + d + 1) - \beta_e(\beta + d + 1)}{\alpha - \beta}.$$

In this case, α_e and β_e respectively can be written as

$$\begin{aligned} \alpha_e &= ae_o + (\alpha + d - c)e_1 + a(\alpha + d)e_2 + (\alpha + d)(\alpha + d - c)e_3, \\ \beta_e &= ae_o + (\beta + d - c)e_1 + a(\beta + d)e_2 + (\beta + d)(\beta + d - c)e_3. \end{aligned}$$

Let

$$\alpha_e(\alpha + d + 1) - \beta_e(\beta + d + 1) = E_0e_0 + E_1e_1 + E_2e_2 + E_3e_3.$$

Then

$$\begin{aligned} E_0 &= a(\alpha - \beta), \\ E_1 &= (\alpha + d + 1)(\alpha + d - c) - (\beta + d + 1)(\beta + d - c) \\ &= (\alpha + d)^2 - (\beta + d)^2 - (c - 1)(\alpha - \beta) \\ &= (\alpha + \beta + 2d)(\alpha - \beta) - (c - 1)(\alpha - \beta) \\ &= (ab + d + 1)(\alpha - \beta), \\ E_2 &= a\big((\alpha + d)(\alpha + d - c) - (\beta + d)(\beta + d - c)\big) \\ &= a\big((\alpha + d)^2 - (\beta + d)^2 + (\alpha - \beta)\big) \\ &= a\big((\alpha + \beta + 2d)(\alpha - \beta) + (\alpha - \beta)\big) \\ &= a(ab + c + d + 1)(\alpha - \beta), \\ E_3 &= (\alpha + d)(\alpha + d + 1)(\alpha + d - c) - (\beta + d)(\beta + d + 1)(\beta + d - c) \\ &= (\alpha + d)^3 - (\beta + d)^3 - (c - 1)\big((\alpha + d)^2 - (\beta + d)^2\big) - c(\alpha - \beta) \\ &= \big((\alpha + \beta + 2d)^2 - (\alpha + d)(\beta + d)\big)(\alpha - \beta) \\ &\quad - (c - 1)(\alpha + \beta + 2d)(\alpha - \beta) - c(\alpha - \beta) \\ &= \big((ab + c + d)^2 - cd\big)(\alpha - \beta) - (c - 1)(ab + c + d)(\alpha - \beta) - c(\alpha - \beta) \\ &= (a^2b^2 + abc + 2abd + ab + d^2 + d)(\alpha - \beta). \end{aligned}$$

Hence (2.11) is true for $n = 1$.

Theorem 6 (Catalan's identity)**.** *The Catalan's identity for the generalized bi-periodic Lucas quaternion sequence is*

$$V_n^2 - V_{n+2r}V_{n-2r} = \begin{cases} \frac{(cd)^{\frac{n-2r}{2}}(d\alpha-\beta-cd+d^2)(d\beta-\alpha-cd+d^2)}{d^2} \\ \times\left(\frac{\alpha_o\beta_o\left((\alpha+d)^{2r}-(cd)^r\right)+\beta_o\alpha_o\left((\beta+d)^{2r}-(cd)^r\right)}{(\alpha-\beta)^2}\right), & if\ n\ is\ even, \\ \frac{(cd)^{\frac{n-2r-1}{2}}(d\alpha-\beta-cd+d^2)(d\beta-\alpha-cd+d^2)}{abd} \\ \times\left(\frac{\alpha_e\beta_e\left((cd)^r-(\alpha+d)^{2r}\right)+\beta_e\alpha_e\left((cd)^r-(\beta+d)^{2r}\right)}{(\alpha-\beta)^2}\right), & if\ n\ is\ odd. \end{cases} \tag{2.12}$$

Proof. Assume that n is even, and let

$$Y_1 = (\alpha-\beta)^2(ab)^{n-2}V_n^2,$$
$$Y_2 = (\alpha-\beta)^2(ab)^{n-2}V_{n+2r}V_{n-2r}.$$

Then

$$\begin{aligned} Y_1 &= \left(\alpha_o(\alpha+d+1)\alpha^{\frac{n-2}{2}}(\alpha+d-c)^{\frac{n}{2}} - \beta_o(\beta+d+1)\beta^{\frac{n-2}{2}}(\beta+d-c)^{\frac{n}{2}}\right)^2 \\ &= \alpha_o^2(\alpha+d+1)^2\alpha^{n-2}(\alpha+d-c)^n + \beta_o^2(\beta+d+1)^2\beta^{n-2}(\beta+d-c)^n \\ &\quad -(\alpha_o\beta_o+\beta_o\alpha_o)(\alpha+d+1)(\beta+d+1)\alpha^{\frac{n-2}{2}}(\alpha+d-c)^{\frac{n}{2}}\beta^{\frac{n-2}{2}}(\beta+d-c)^{\frac{n}{2}}, \end{aligned}$$

and

$$\begin{aligned} Y_2 &= \left(\alpha_o(\alpha+d+1)\alpha^{\frac{n+2r-2}{2}}(\alpha+d-c)^{\frac{n+2r}{2}} - \beta_o(\beta+d+1)\beta^{\frac{n+2r-2}{2}}(\beta+d-c)^{\frac{n+2r}{2}}\right) \\ &\quad \times\left(\alpha_o(\alpha+d+1)\alpha^{\frac{n-2r-2}{2}}(\alpha+d-c)^{\frac{n-2r}{2}} - \beta_o(\beta+d+1)\beta^{\frac{n-2r-2}{2}}(\beta+d-c)^{\frac{n-2r}{2}}\right), \\ &= \alpha_o^2(\alpha+d+1)^2\alpha^{n-2}(\alpha+d)^n + \beta_o^2(\beta+d+1)^2\beta^{n-2}(\beta+d-c)^n \\ &\quad -\alpha_o\beta_o(\alpha+d+1)(\beta+d+1)\alpha^{\frac{n+2r-2}{2}}(\alpha+d-c)^{\frac{n+2r}{2}}\beta^{\frac{n-2r-2}{2}}(\beta+d-c)^{\frac{n-2r}{2}}, \\ &\quad -\beta_o\alpha_o(\alpha+d+1)(\beta+d+1)\alpha^{\frac{n-2r-2}{2}}(\alpha+d-c)^{\frac{n-2r}{2}}\beta^{\frac{n+2r-2}{2}}(\beta+d-c)^{\frac{n+2r}{2}}. \end{aligned}$$

Hence

$$Y_1 - Y_2 = \alpha_o\beta_o A_1 + \beta_o\alpha_o A_2,$$

where

$$\begin{aligned} A_1 &= (\alpha+d+1)(\beta+d+1)\Big(\alpha^{\frac{n+2r-2}{2}}(\alpha+d-c)^{\frac{n+2r}{2}}\beta^{\frac{n-2r-2}{2}}(\beta+d-c)^{\frac{n-2r}{2}} \\ &\quad -\alpha^{\frac{n-2}{2}}(\alpha+d-c)^{\frac{n}{2}}\beta^{\frac{n-2}{2}}(\beta+d-c)^{\frac{n}{2}}\Big) \\ A_2 &= (\alpha+d+1)(\beta+d+1)\Big(\alpha^{\frac{n-2r-2}{2}}(\alpha+d-c)^{\frac{n-2r}{2}}\beta^{\frac{n+2r-2}{2}}(\beta+d-c)^{\frac{n+2r}{2}} \\ &\quad -\alpha^{\frac{n-2}{2}}(\alpha+d-c)^{\frac{n}{2}}\beta^{\frac{n-2}{2}}(\beta+d-c)^{\frac{n}{2}}\Big). \end{aligned}$$

Since $\alpha\beta = -abd$ and $\alpha + \beta = ab + c - d$, we have

$$\begin{aligned}(\alpha + d + 1) &= \left(1 + \frac{d+1}{\alpha}\right)\alpha \\ &= \left(1 - \frac{(d+1)\beta}{abd}\right)\alpha \\ &= \left(\frac{(\alpha+\beta)d - d(c-d) - (d+1)\beta}{abd}\right)\alpha \\ &= \frac{(d\alpha - \beta - cd + d^2)\alpha}{abd}.\end{aligned}$$

Similarly

$$(\beta + d + 1) = \frac{(d\beta - \alpha - cd + d^2)\beta}{abd}.$$

Then

$$\begin{aligned}A_1 &= \frac{(d\alpha - \beta - cd + d^2)(d\beta - \alpha - cd + d^2)\alpha^{\frac{n}{2}}(\alpha + d - c)^{\frac{n}{2}}\beta^{\frac{n}{2}}(\beta + d - c)^{\frac{n}{2}}}{(abd)^2} \\ &\quad \times \left(\frac{\alpha^r(\alpha + d - c)^r}{\beta^r(\beta + d - c)^r} - 1\right) \\ &= \frac{(ab)^{n-2}(cd)^{\frac{n}{2}}(d\alpha - \beta - cd + d^2)(d\beta - \alpha - cd + d^2)}{d^2} \\ &\quad \times \frac{\alpha^{2r}(\alpha + d - c)^{2r} - \alpha^r(\alpha + d - c)^r\beta^r(\beta + d - c)^r}{\alpha^r(\alpha + d - c)^r\beta^r(\beta + d - c)^r} \\ &= \frac{(ab)^{n-2}(cd)^{\frac{n-2r}{2}}(d\alpha - \beta - cd + d^2)(d\beta - \alpha - cd + d^2)\big((\alpha + d)^{2r} - (cd)^r\big)}{d^2}.\end{aligned}$$

Similarly we have

$$A_2 = \frac{(ab)^{n-2}(cd)^{\frac{n-2r}{2}}(d\alpha - \beta - cd + d^2)(d\beta - \alpha - cd + d^2)\big((\beta + d)^{2r} - (cd)^r\big)}{d^2},$$

and the proof is completed for the case where n is even.

Using the same procedure, we can also prove (2.12) for the case where n is odd. □

If $c = d = 1$, then Theorem 6 reduces to [18, Theorem 5].

3. Conclusions

In this paper we introduced the generalized bi-periodic Fibonacci and Lucas quaternions which are the further generalizations of the bi-periodic Fibonacci and Lucas quaternions considered in the literature. For those quaternions, we obtained the generating functions, Binet's formulas and Catalan's identities.

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Author's address

Younseok Choo
Hongik University, Department of Electronic and Electrical Engineering, 2639 Sejong-Ro, 30016 Sejong, Republic of Korea
E-mail address: yschoo@hongik.ac.kr

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COMPARISONS OF THE EXACT AND THE APPROXIMATE SOLUTIONS OF SECOND-ORDER FUZZY LINEAR BOUNDARY VALUE PROBLEMS

HÜLYA GÜLTEKİN ÇİTİL

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Abstract. In this paper, the approximate solutions by using the undetermined fuzzy coefficients method and the exact solutions by using the Hukuhara differentiability of second-order fuzzy linear boundary value problems with constant coefficients are investigated. Thus, comparisons of the found solutions are given.

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1. Introduction

Many researchers study fuzzy differential equations. Because, fuzzy differential equations form the mathematical model of real world problems in which uncertainty. Fuzzy differential equations can be solved with several approach. The first approach is Hukuhara differentiability. Firstly, the existence and uniqueness of the solution of fuzzy differential equation were examined [7, 14]. The second approach is generalized differentiability. Khastan and Nieto [15] found new solutions for some fuzzy boundary value problems using the generalized differentiability. Using the generalized differentiability, they found solutions for larger class of fuzzy boundary value problems than using the Hukuhara differentiability. The third approach generate the fuzzy solution from the crips solution. This can be three ways. The first one is extension principle and here the initial value is taken as a real constant, crips problem is solved and then the real constant in the solution is replaced with the fuzzy initial value [7, 8]. The second way is the concept of differential inclusion [13]. Here, the alpha-cut of the initial value is taken and the differential equation is converted to a differential inclusion. The third way consider the fuzzy problem to be a set of crips problem [10]. Howewer, many fuzzy initial and boundary value problems can not be solved as analytical every time. Thus, to find approximate solutions of these

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problems is important. The numeric methods were introduced in [1, 2, 4]. Allahviranloo [3] found the approximate solution of nth-order linear differential equations with fuzzy initial conditions by the collocation method. Guo at al. [11] found the approximate solution of a class of-second-order linear differential equation with fuzzy boundary value conditions by the undetermined fuzzy coefficients method.

In this paper, the approximate solutions by using the undetermined fuzzy coefficients method and the exact solutions by using the Hukuhara differentiability of second-order fuzzy linear boundary value problems constant coefficients are investigated . Thus, comparison results of the found solutions are given. The aim of this study is to investigate how close the exact solutions to of the approximate solutions by the undetermined fuzzy coefficients method of fuzzy boundary value problems for fuzzy differential equations with the positive and negative constant coefficients.

2. Preliminaries

Definition 1 ([16])**.** A fuzzy number is a mapping $u : \mathbb{R} \to [0,1]$ satisfying the following properties:

u is normal: $\exists x_0 \in \mathbb{R}$ for which $u(x_0) = 1$,

u is convex fuzzy set: $u(\lambda x + (1-\lambda) y) \geqslant \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}, \lambda \in [0,1]$,

u is upper semi-continuous on $\mathbb{R}$,

$cl\{x \epsilon \mathbb{R} \mid u(x) > 0\}$ is compact, where cl denotes the closure of a subset.

Let $\mathbb{R}_F$ denote the set of all fuzzy numbers.

Definition 2 ([15])**.** Let $u \epsilon \mathbb{R}_F$. The α-level set of u, denoted , $[u]^\alpha$, $0 < \alpha \leq 1$, is $[u]^\alpha = \{x \epsilon \mathbb{R} \mid u(x) \geq \alpha\}$. If $\alpha = 0$, $[u]^0 = cl\{\text{suppu}\} = cl\{x \epsilon \mathbb{R} \mid u(x) > 0\}$.The notation, $[u]^\alpha = \left[\underline{u}_\alpha, \overline{u}_\alpha\right]$ denotes explicitly the α-level set of u, where $\underline{u}_\alpha$ and $\overline{u}_\alpha$ denote the left-hand endpoint and the right-hand endpoint of $[u]^\alpha$, respectively.

The following remark shows that $\left[\underline{u}_\alpha, \overline{u}_\alpha\right]$ is a valid α-level set.

Remark 1 ([9, 15]). The sufficient and necessary conditions for $\left[\underline{u}_\alpha, \overline{u}_\alpha\right]$ to define the parametric form of a fuzzy number as follows:

$\underline{u}_\alpha$ is bounded monotonic increasing (non-decreasing) left-continuous function on $(0,1]$ and right-continuous for $\alpha = 0$,

$\overline{u}_\alpha$ is bounded monotonic decreasing (non-increasing) left-continuous function on $(0,1]$ and right-continuous for $\alpha = 0$,

$\underline{u}_\alpha \leq \overline{u}_\alpha$, $0 \leq \alpha \leq 1$.

Definition 3 ([16])**.** If A is a symmetric triangular fuzzy number with support $[\underline{a}, \overline{a}]$, the $\alpha-$level set of A is $[A]^\alpha = \left[\underline{A}_\alpha, \overline{A}_\alpha\right] = \left[\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right]$, $(\underline{A}_1 = \overline{A}_1, \underline{A}_1 - \underline{A}_\alpha = \overline{A}_\alpha - \overline{A}_1)$.

Definition 4 ([11, 12]). For arbitrary $u = [\underline{u}, \overline{u}], v = [\underline{v}, \overline{v}] \in \mathbb{R}_F$, the quantity

$$D(u,v) = \left(\int_0^1 \left(\underline{u}_\alpha - \underline{v}_\alpha\right)^2 d\alpha + \int_0^1 \left(\overline{u}_\alpha - \overline{v}_\alpha\right)^2 d\alpha \right)^{\frac{1}{2}}$$

is the distance between fuzzy numbers u and v.

Definition 5 ([11, 15, 17]). Let $u, v \in \mathbb{R}_F$. If there exists $w \in \mathbb{R}_F$ such that $u = v + w$, then w is called the Hukuhara difference of fuzzy numbers u and v, and it is denoted by $w = u \ominus v$.

Definition 6 ([5, 11, 15]). Let $f : [a,b] \to \mathbb{R}_F$ and $t_0 \in [a,b]$. We say that f is Hukuhara differential at t_0, if there exists an element $f'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists f(t_0 + h) \ominus f(t_0)$, $f(t_0) \ominus f(t_0 - h)$ and the limits (in the metric D)

$$\lim_{h \to 0} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \to 0} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0).$$

Lemma 1 ([6]). *If $g : [a,b] \to \mathbb{R}$ is differential on $[a,b]$ such that g', g'' are non-negative and monotonic increasing on $[a,b]$, then $\forall c \in \mathbb{R}_F$, $f(x) = cg(x)$ is differential on $[a,b]$ and*

$$f'(x) = cg'(x), f''(x) = cg''(x), \forall x \in [a,b].$$

Definition 7 ([11]). The undetermined fuzzy coefficients method is to seek an approximate solution as

$$\widetilde{y}_N(t) = \sum_{k=0}^{N} \widetilde{\theta}_k \phi_k(t),$$

where, $\phi_k(t)$, $k = 0, 1, ..., N$ are positive basic functions whose all differentiations are positive.

Lemma 2 ([3, 11]). *Let the basis functions $\phi_k(t), k = 0, 1, ..., N$ and all of their differentiations be positive, without loss of generality. Then $\left(\underline{y}_N\right)^{(i)}(t) = \underline{y_N^{(i)}}(t)$ and $\left(\overline{y}_N^{(i)}\right)(t) = \overline{y_N^{(i)}}(t), i = 0, 1, 2.$*

3. Second-order fuzzy linear boundary value problems

3.1. *The case of positive constant coefficient*

Consider the fuzzy boundary value problem

$$y''(t) = \lambda y(t),\ t \in [0, \ell] \tag{3.1}$$

$$y(0) = A,\ \ y(\ell) = B, \tag{3.2}$$

where $\lambda > 0$, A and B are symmetric triangular fuzzy numbers with supports $[\underline{a},\overline{a}]$ and $[\underline{b},\overline{b}]$, respectively,

$$[A]^{\alpha} = \left[\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right], \; [B]^{\alpha} = \left[\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha, \overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right].$$

3.1.1. *The Exact Solution with Hukuhara Differentiability*

From the fuzzy differential equation (3.1), we have differential equations

$$\underline{Y}''_{\alpha}(t) = \lambda \underline{Y}_{\alpha}(t),$$
$$\overline{Y}''_{\alpha}(t) = \lambda \overline{Y}_{\alpha}(t)$$

by using the Hukuhara differentiability. Then, the lower solution and the upper solution of the fuzzy differential equation (3.1) are obtained as

$$\underline{Y}_{\alpha}(t) = \underline{c}_1(\alpha)e^{\sqrt{\lambda}t} + \underline{c}_2(\alpha)e^{-\sqrt{\lambda}t},$$
$$\overline{Y}_{\alpha}(t) = \overline{c}_1(\alpha)e^{\sqrt{\lambda}t} + \overline{c}_2(\alpha)e^{-\sqrt{\lambda}t}.$$

Using the boundary conditions (3.2), coefficients $\underline{c}_1(\alpha)$, $\underline{c}_2(\alpha)$, $\overline{c}_1(\alpha)$, $\overline{c}_2(\alpha)$ are solved as

$$\underline{c}_1(\alpha) = \frac{\left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right) - \left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)e^{-\sqrt{\lambda}\ell}}{e^{\sqrt{\lambda}\ell} - e^{-\sqrt{\lambda}\ell}},$$
$$\underline{c}_2(\alpha) = \frac{\left(\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)e^{\sqrt{\lambda}\ell} - \left(\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)}{e^{\sqrt{\lambda}\ell} - e^{-\sqrt{\lambda}\ell}},$$
$$\overline{c}_1(\alpha) = \frac{\left(\overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right) - \left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)e^{-\sqrt{\lambda}\ell}}{e^{\sqrt{\lambda}\ell} - e^{-\sqrt{\lambda}\ell}},$$
$$\overline{c}_2(\alpha) = \frac{\left(\overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)e^{\sqrt{\lambda}\ell} - \left(\overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)}{e^{\sqrt{\lambda}\ell} - e^{-\sqrt{\lambda}\ell}}.$$

3.1.2. *The Approximate Solution with the Undetermined Fuzzy Coefficients Method*

An approximate solution with the undetermined fuzzy coefficients method is

$$\widetilde{y}_N(t) = \sum_{k=0}^{N} \widetilde{\theta}_k \phi_k(t), \tag{3.3}$$

where, $\phi_k(t)$, $k = 0, 1, ..., N$ are positive basic functions whose all differentiations are positive and the lower solution and upper solution are

$$\underline{y}_{\alpha}(t) = \sum_{k=0}^{N} \underline{\theta}_k(\alpha)\phi_k(t), \qquad \overline{y}_{\alpha}(t) = \sum_{k=0}^{N} \overline{\theta}_k(\alpha)\phi_k(t),$$

respectively. Substituting the expression (3.3) in the fuzzy differential equation (3.1) yields

$$\sum_{k=0}^{N}\underline{\theta}_k(\alpha)\phi_k''(t)-\lambda\sum_{k=0}^{N}\underline{\theta}_k(\alpha)\phi_k(t)=0,$$

$$\sum_{k=0}^{N}\overline{\theta}_k(\alpha)\phi_k''(t)-\lambda\sum_{k=0}^{N}\overline{\theta}_k(\alpha)\phi_k(t)=0.$$

Using the boundary conditions (3.2),

$$\sum_{k=0}^{N}\underline{\theta}_k(\alpha)\phi_k(0)=\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,$$

$$\sum_{k=0}^{N}\underline{\theta}_k(\alpha)\phi_k(\ell)=\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha,$$

$$\sum_{k=0}^{N}\overline{\theta}_k(\alpha)\phi_k(0)=\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,$$

$$\sum_{k=0}^{N}\overline{\theta}_k(\alpha)\phi_k(\ell)=\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha$$

are obtained. Taking

$$\phi_k''(t)+\lambda\phi_k(t)=\rho_k,\quad \phi_k(0)=\zeta_{0k},\quad \phi_k(\ell)=\xi_{\ell k},$$

we obtain

$$\sum_{k=0}^{N}\underline{\theta}_k(\alpha)\rho_k=0, \tag{3.4}$$

$$\sum_{k=0}^{N}\underline{\theta}_k(\alpha)\zeta_{0k}=\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \tag{3.5}$$

$$\sum_{k=0}^{N}\underline{\theta}_k(\alpha)\xi_{\ell k}=\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha, \tag{3.6}$$

$$\sum_{k=0}^{N}\overline{\theta}_k(\alpha)\rho_k=0, \tag{3.7}$$

$$\sum_{k=0}^{N}\overline{\theta}_k(\alpha)\zeta_{0k}=\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \tag{3.8}$$

$$\sum_{k=0}^{N}\overline{\theta}_k(\alpha)\,\xi_{\ell k}=\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha. \tag{3.9}$$

Let us write the equations (3.4) - (3.9) as

$$S(t)\,X(\alpha)=Y(\alpha),$$

where

$$S=\begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix},$$

$$S_1=\begin{pmatrix} \rho_1 & \rho_2 & \dots & \rho_N \\ \zeta_{00} & \zeta_{01} & \dots & \zeta_{0N} \\ \xi_{\ell 0} & \xi_{\ell 1} & \dots & \xi_{\ell N} \end{pmatrix},\quad S_2=\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

$$X(\alpha)=\left(\underline{\theta}_0\ \underline{\theta}_1\ \dots\underline{\theta}_N\ \overline{\theta}_0\ \overline{\theta}_1\dots\overline{\theta}_N\right)^T,\quad Y(\alpha)=\left(0\ \underline{A}_\alpha\ \underline{B}_\alpha\ 0\ \overline{A}_\alpha\ \overline{B}_\alpha\right)^T.$$

From this, $\underline{\theta}_0$, $\underline{\theta}_1$, ...$\underline{\theta}_N$, $\overline{\theta}_0$, $\overline{\theta}_1$,...,$\overline{\theta}_N$ are solved and the approximate solution is obtained [11].

Example 1. Consider the fuzzy boundary value problem

$$y''(t)=y(t), t\in\left(0,\frac{3}{2}\pi\right), \tag{3.10}$$

$$y(0)=\left[1+\frac{1}{2}\alpha,2-\frac{1}{2}\alpha\right],\quad y(\frac{3\pi}{2})=\left[3+\frac{1}{2}\alpha,4-\frac{1}{2}\alpha\right]. \tag{3.11}$$

Using the Hukuhara differentiability, the lower exact solution and the upper exact solution of the fuzzy differential equation (3.10) are obtained as

$$\underline{Y}_\alpha(t)=\underline{c}_1(\alpha)e^t+\underline{c}_2(\alpha)e^{-t}, \tag{3.12}$$

$$\overline{Y}_\alpha(t)=\overline{c}_1(\alpha)e^t+\overline{c}_2(\alpha)e^{-t}. \tag{3.13}$$

Using the boundary conditions (3.11),

$$\underline{c}_1(\alpha)=\frac{\left(3+\frac{1}{2}\alpha\right)-\left(1+\frac{1}{2}\alpha\right)e^{-\frac{3\pi}{2}}}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}},$$

$$\underline{c}_2(\alpha)=\frac{\left(1+\frac{1}{2}\alpha\right)e^{\frac{3\pi}{2}}-\left(3+\frac{1}{2}\alpha\right)}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}},$$

$$\overline{c}_1(\alpha)=\frac{\left(4-\frac{1}{2}\alpha\right)-\left(2-\frac{1}{2}\alpha\right)e^{-\frac{3\pi}{2}}}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}},$$

$$\overline{c}_2(\alpha)=\frac{\left(2-\frac{1}{2}\alpha\right)e^{\frac{3\pi}{2}}-\left(4-\frac{1}{2}\alpha\right)}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}}$$

are obtained. Now, let us find the approximate solution. Let

$$\phi_k(t)=t^k, k=0,1,2.$$

Then, the lower approximate solution and the upper approximate solution are

$$\underline{y}_\alpha(t) = \underline{\theta}_0 + \underline{\theta}_1 t + \underline{\theta}_2 t^2, \tag{3.14}$$

$$\overline{y}_\alpha(t) = \overline{\theta}_0 + \overline{\theta}_1 t + \overline{\theta}_2 t^2. \tag{3.15}$$

From the fuzzy differential equation (3.10),

$$\underline{y}''_\alpha(t) - \underline{y}_\alpha(t) = 0 \Rightarrow -\underline{\theta}_0 - \underline{\theta}_1 t + \underline{\theta}_2 \left(2 - t^2\right) = 0, \tag{3.16}$$

$$\overline{y}''_\alpha(t) - \overline{y}_\alpha(t) = 0 \Rightarrow -\overline{\theta}_0 - \overline{\theta}_1 t + \overline{\theta}_2 \left(2 - t^2\right) = 0 \tag{3.17}$$

and from the boundary conditions (3.11),

$$\underline{y}_\alpha(0) = \underline{\theta}_0 = 1 + \frac{1}{2}\alpha, \tag{3.18}$$

$$\overline{y}_\alpha(0) = \overline{\theta}_0 = 2 - \frac{1}{2}\alpha, \tag{3.19}$$

$$\underline{y}_\alpha(\frac{3\pi}{2}) = \underline{\theta}_0 + \underline{\theta}_1 \frac{3\pi}{2} + \underline{\theta}_2 \frac{9\pi^2}{4} = 3 + \frac{1}{2}\alpha, \tag{3.20}$$

$$\overline{y}_\alpha(\frac{3\pi}{2}) = \overline{\theta}_0 + \overline{\theta}_1 \frac{3\pi}{2} + \overline{\theta}_2 \frac{9\pi^2}{4} = 4 - \frac{1}{2}\alpha. \tag{3.21}$$

Taking $t = \frac{1}{2}$, from the equations (3.16), (3.18) and (3.20),

$$-\underline{\theta}_1 \frac{1}{2} + \underline{\theta}_2 \frac{7}{4} = 1 + \frac{1}{2}\alpha,$$

$$\underline{\theta}_1 \frac{3\pi}{2} + \underline{\theta}_2 \frac{9\pi^2}{4} = 2$$

are obtained. From here, the coefficients $\underline{\theta}_1$, $\underline{\theta}_2$ are obtained as

$$\underline{\theta}_1 = -0.9667505819 - 0.57381463442\alpha,$$

$$\underline{\theta}_2 = 0.29521411946 + 0.12176724731\alpha.$$

Also, from the equations (3.17), (3.19) and (3.21),

$$-\overline{\theta}_1 \frac{1}{2} + \overline{\theta}_2 \frac{7}{4} = 3 + \frac{1}{2}\alpha$$

$$\overline{\theta}_1 \frac{3\pi}{2} + \overline{\theta}_2 \frac{9\pi^2}{4} = 2$$

are obtained. From here, the coefficients $\overline{\theta}_1$, $\overline{\theta}_2$ are obtained as

$$\overline{\theta}_1 = -2.1143798507 + 0.57381463442\alpha,$$

$$\overline{\theta}_2 = 0.53874861409 - 0.12176724731\alpha.$$

Substituting these values in (3.14) and (3.15), the lower approximate solution and the upper approximate solution are obtained as

$$\underline{y}_\alpha(t) = (1 + 0.5\alpha) + (-0.9667505819 - 0.57381463442\alpha)\, t$$

$$+(0.29521411946+0.12176724731\alpha)t^2,$$
$$\overline{y}_\alpha(t)=(2-0.5\alpha)+(-2.1143798507+0.57381463442\alpha)t$$
$$+(0.53874861409-0.12176724731\alpha)t^2.$$

The approximate lower and upper solutions for $t=0.01$ are

$$\underline{y}_\alpha(t)=0.99036201559+0.49426307132\alpha,$$
$$\overline{y}_\alpha(t)=1.97891007635-0.49427403038\alpha.$$

From the equations (3.12) and (3.13), the exact lower and upper solution for $t=0.01$ are

$$\underline{Y}_\alpha(t)=0.9905872695+0.4951139515\alpha,$$
$$\overline{Y}_\alpha(t)=1.9808151724-0.49511395143\alpha.$$

Comparison the results of the lower exact solution and the lower approximate solution

α	$\underline{Y}_\alpha(t)$	$\underline{y}_\alpha(t)$	$Error$
0	0.9905872695	0.99036201559	0.00022525391
0.1	1.04009866465	1.03978832272	0.00031034193
0.2	1.0896100598	1.08921462985	0.00039542995
0.3	1.13912145495	1.13864093698	0.00048051797
0.4	1.1886328501	1.18806724411	0.00056533599
0.5	1.23814424525	1.23749355125	0.000650694
0.6	1.2876556404	1.28691985838	0.00073578202
0.7	1.33716703555	1.33634616551	0.00082087004
0.8	1.3866784307	1.38577247264	0.00090595806
0.9	1.43618982585	1.43519877977	0.00099104608
1	1.485701221	1.48462508691	0.00107613409

Comparison the results of the upper exact solution and the upper approximate solution

α	$\overline{Y}_\alpha(t)$	$\overline{y}_\alpha(t)$	$Error$
0	1.9808151724	1.97891007635	0.00190509605
0.1	1.93130377726	1.92948267332	0.00182110394
0.2	1.88179238212	1.88005527028	0.00173711184
0.3	1.83228098698	1.83062786724	0.00165311974
0.4	1.1886328501	1.7812004642	0.00156912763
0.5	1.78276959183	1.73177306116	0.00148513553
0.6	1.73325819669	1.68234565813	0.00140114342
0.7	1.68374680155	1.63291825509	0.00131715131
0.8	1.58472401126	1.58349085205	0.00123315921
0.9	1.53521261612	1.53406344901	0.00114916711
1	1.48570122097	1.48463604597	0.001065175

3.2. *The case of negative constant coefficient*

Consider the fuzzy boundary value problem

$$y''(t) = -\lambda y(t), \tag{3.22}$$

$$y(0) = A, \quad y(\ell) = B, \tag{3.23}$$

where $\lambda > 0$.

3.2.1. *The Exact Solution with Hukuhara Differentiability*

From the fuzzy differential equation (3.22), we have differential equations

$$\underline{Y}''_{\alpha}(t) = -\lambda \overline{Y}_{\alpha}(t),$$

$$\overline{Y}''_{\alpha}(t) = -\lambda \underline{Y}_{\alpha}(t)$$

by using the Hukuhara differentiability and the fuzzy arithmetic property $-\left[\underline{Y}_{\alpha}, \overline{Y}_{\alpha}\right] = \left[-\overline{Y}_{\alpha}, -\underline{Y}_{\alpha}\right]$. Then, the lower solution and the upper solution of the fuzzy differential equation (3.22) are

$$\underline{Y}_{\alpha}(t) = -c_1(\alpha)e^{\sqrt{\lambda}t} - c_2(\alpha)e^{-\sqrt{\lambda}t} + c_3(\alpha)\ \sin\left(\sqrt{\lambda}t\right) + c_4(\alpha)\cos\left(\sqrt{\lambda}t\right),$$

$$\overline{Y}_{\alpha}(t) = c_1(\alpha)e^{\sqrt{\lambda}t} + c_2(\alpha)e^{-\sqrt{\lambda}t} + c_3(\alpha)\sin\left(\sqrt{\lambda}t\right) + c_4(\alpha)\cos\left(\sqrt{\lambda}t\right).$$

Using the boundary conditions (3.23), the coefficient $c_1(\alpha)$, $c_2(\alpha)$, $c_3(\alpha)$, $c_4(\alpha)$ are obtained as

$$c_1(\alpha) = \left(\frac{1-\alpha}{2}\right)\left(\frac{\left(\overline{b}-\underline{b}\right) - (\overline{a}-\underline{a})\,e^{-\sqrt{\lambda}\ell}}{e^{\sqrt{\lambda}\ell} - e^{-\sqrt{\lambda}\ell}}\right),$$

$$c_2(\alpha) = \left(\frac{1-\alpha}{2}\right)\left((\overline{a}-\underline{a}) - \frac{\left(\overline{b}-\underline{b}\right) - (\overline{a}-\underline{a})\,e^{-\sqrt{\lambda}\ell}}{e^{\sqrt{\lambda}\ell} - e^{-\sqrt{\lambda}\ell}}\right),$$

$$c_3(\alpha) = \frac{\left(\overline{b}+\underline{b}\right) - (\overline{a}+\underline{a})\cos\left(\sqrt{\lambda}\ell\right)}{2\sin\left(\sqrt{\lambda}\ell\right)},$$

$$c_4(\alpha) = \frac{\overline{a}+\underline{a}}{2}.$$

3.2.2. *The Approximate Solution with the Undetermined Fuzzy Coefficients Method*

The undetermined fuzzy coefficients method is to seek an approximate solution as

$$\widetilde{y}_N(t) = \sum_{k=0}^{N} \widetilde{\theta}_k \phi_k(t), \tag{3.24}$$

where, $\phi_k(t)$, $k = 0, 1, ..., N$ are positive basic functions whose all differentiations are positive and and the lower solution and upper solution are

$$\underline{y}_\alpha(t) = \sum_{k=0}^{N} \underline{\theta}_k(\alpha)\phi_k(t), \qquad \overline{y}_\alpha(t) = \sum_{k=0}^{N} \overline{\theta}_k(\alpha)\phi_k(t),$$

respectively. Substituting the equation (3.24) in the fuzzy differential equation (3.22) and using the fuzzy arithmetic property $-\left[\underline{y}_\alpha, \overline{y}_\alpha\right] = \left[-\overline{y}_\alpha, -\underline{y}_\alpha\right]$ yields

$$\sum_{k=0}^{N} \underline{\theta}_k(\alpha)\phi_k''(t) + \lambda \sum_{k=0} \overline{\theta}_k(\alpha)\phi_k(t) = 0,$$

$$\sum_{k=0}^{N} \overline{\theta}_k(\alpha)\phi_k''(t) + \lambda \sum_{k=0}^{N} \underline{\theta}_k(\alpha)\phi_k(t) = 0.$$

Using the boundary conditions (3.23),

$$\sum_{k=0}^{N} \underline{\theta}_k(\alpha)\phi_k(0) = \underline{a} + \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha,$$

$$\sum_{k=0}^{N} \underline{\theta}_k(\alpha)\phi_k(\ell) = \underline{b} + \left(\frac{\overline{b} - \underline{b}}{2}\right)\alpha,$$

$$\sum_{k=0}^{N} \overline{\theta}_k(\alpha)\phi_k(0) = \overline{a} - \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha,$$

$$\sum_{k=0}^{N} \overline{\theta}_k(\alpha)\phi_k(\ell) = \overline{b} - \left(\frac{\overline{b} - \underline{b}}{2}\right)\alpha$$

are obtained. Taking

$$\phi_k''(t) = \rho_k, \quad \lambda\phi_k(t) = \varepsilon_k, \quad \phi_k(0) = \zeta_{0k}, \quad \phi_k(\ell) = \xi_{\ell k},$$

we obtain

$$\sum_{k=0}^{N} \underline{\theta}_k(\alpha)\rho_k + \sum_{k=0}^{N} \overline{\theta}_k(\alpha)\varepsilon_k = 0, \tag{3.25}$$

$$\sum_{k=0}^{N} \underline{\theta}_k(\alpha)\,\zeta_{0k} = \underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \tag{3.26}$$

$$\sum_{k=0}^{N} \underline{\theta}_k(\alpha)\,\xi_{\ell k} = \underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha, \tag{3.27}$$

$$\sum_{k=0}^{N} \overline{\theta}_k(\alpha)\,\rho_k + \sum_{k=0}^{N} \underline{\theta}_k(\alpha)\,\varepsilon_k = 0, \tag{3.28}$$

$$\sum_{k=0}^{N} \overline{\theta}_k(\alpha)\,\zeta_{0k} = \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \tag{3.29}$$

$$\sum_{k=0}^{N} \overline{\theta}_k(\alpha)\,\xi_{\ell k} = \overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha. \tag{3.30}$$

Let us write the equations (3.25) - (3.30) as

$$S(t)\,X(\alpha) = Y(\alpha),$$

where

$$S = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix},$$

$$S_1 = \begin{pmatrix} \rho_1 & \rho_2 & \dots & \rho_N \\ \zeta_{00} & \zeta_{01} & \dots & \zeta_{0N} \\ \xi_{\ell 0} & \xi_{\ell 1} & \dots & \xi_{\ell N} \end{pmatrix}, \quad S_2 = \begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \dots & \varepsilon_N \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

$$X(\alpha) = \left(\underline{\theta}_0\ \underline{\theta}_1 \dots \underline{\theta}_N, \overline{\theta}_0\ \overline{\theta}_1 \dots \overline{\theta}_N\right)^T, Y(\alpha) = \left(0\ \underline{A}_\alpha \underline{B}_\alpha 0 . \overline{A}_\alpha \overline{B}_\alpha\right)^T.$$

From this, $\underline{\theta}_0$, $\underline{\theta}_1$, ...$\underline{\theta}_N$, $\overline{\theta}_0$, $\overline{\theta}_1$,...,$\overline{\theta}_N$ are solved and the approximate solution is obtained.

Example 2. Consider the fuzzy boundary value problem

$$y''(t) = -y(t), t \in \left(0, \frac{3}{2}\pi\right), \tag{3.31}$$

$$y(0) = \left[1 + \frac{1}{2}\alpha, 2 - \frac{1}{2}\alpha\right],\ y(\frac{3\pi}{2}) = \left[3 + \frac{1}{2}\alpha, 4 - \frac{1}{2}\alpha\right]. \tag{3.32}$$

Using the Hukuhara differentiability and using the fuzzy arithmetic property $-\left[\underline{y}_\alpha, \overline{y}_\alpha\right] = \left[-\overline{y}_\alpha, -\underline{y}_\alpha\right]$ in the fuzzy differential equation (3.31), we have

$$\underline{Y}''_\alpha(t) + \overline{Y}_\alpha(t) = 0,$$

$$\overline{Y}''_\alpha(t) + \underline{Y}''_\alpha(t) = 0.$$

Then, using the boundary conditions (3.32), the lower exact solution and the upper exact solution of the fuzzy differential equation (3.31) are obtained as

$$\underline{Y}_\alpha(t) = \left(\frac{1-\alpha}{2}\right)\left(\frac{e^{-\frac{3\pi}{2}}-1}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}}\right)e^t + \left(\frac{1-\alpha}{2}\right)\left(\frac{1-e^{\frac{3\pi}{2}}}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}}\right)e^{-t} - \frac{7}{2}\sin(t) + \frac{3}{2}\cos(t)\ , \tag{3.33}$$

$$\overline{Y}_\alpha(t) = \left(\frac{1-\alpha}{2}\right)\left(\frac{1-e^{-\frac{3\pi}{2}}}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}}\right)e^t + \left(\frac{1-\alpha}{2}\right)\left(\frac{e^{\frac{3\pi}{2}}-1}{e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}}}\right)e^{-t} - \frac{7}{2}\sin(t) + \frac{3}{2}\cos(t)\,. \tag{3.34}$$

Let

$$\phi_k(t) = t^k, k = 0,1,2.$$

Then, the the lower approximate solution and the upper approximate solution are

$$\underline{y}_\alpha(t) = \underline{\theta}_0 + \underline{\theta}_1 t + \underline{\theta}_2 t^2, \tag{3.35}$$

$$\overline{y}_\alpha(t) = \overline{\theta}_0 + \overline{\theta}_1 t + \overline{\theta}_2 t^2. \tag{3.36}$$

Substituting these equations in (3.31) and using $-\left[\underline{y}_\alpha, \overline{y}_\alpha\right] = \left[-\overline{y}_\alpha, -\underline{y}_\alpha\right]$,

$$\underline{y}''_\alpha(t) + \overline{y}_\alpha(t) = 0 \Rightarrow \overline{\theta}_0 + \overline{\theta}_1 t + \overline{\theta}_2 t^2 + 2\underline{\theta}_2 = 0,$$

$$\overline{y}''_\alpha(t) + \underline{y}_\alpha(t) = 0 \Rightarrow \underline{\theta}_0 + \underline{\theta}_1 t + \underline{\theta}_2 t^2 + 2\overline{\theta}_2 = 0$$

are obtained. From the boundary conditions (3.32), it yields

$$\underline{y}_\alpha(0) = \underline{\theta}_0 = 1 + \frac{1}{2}\alpha,$$

$$\underline{y}_\alpha(\frac{3\pi}{2}) = \underline{\theta}_0 + \underline{\theta}_1\frac{3\pi}{2} + \underline{\theta}_2\frac{9\pi^2}{4} = 3 + \frac{1}{2}\alpha,$$

$$\overline{y}_\alpha(0) = \overline{\theta}_0 = 2 - \frac{1}{2}\alpha,$$

$$\overline{y}_\alpha(\frac{3\pi}{2}) = \overline{\theta}_0 + \overline{\theta}_1\frac{3\pi}{2} + \overline{\theta}_2\frac{9\pi^2}{4} = 4 - \frac{1}{2}\alpha.$$

Taking $t = \frac{1}{2}$, the equations

$$\overline{\theta}_1\frac{1}{2} + \overline{\theta}_2\frac{1}{4} + 2\underline{\theta}_2 = -1 - \frac{1}{2}\alpha,$$

$$\underline{\theta}_1\frac{3\pi}{2} + \underline{\theta}_2\frac{9\pi^2}{4} = 2,$$

$$\underline{\theta}_1\frac{1}{2} + \underline{\theta}_2\frac{1}{4} + 2\overline{\theta}_2 = -2 + \frac{1}{2}\alpha,$$

$$\overline{\theta}_1\frac{3\pi}{2} + \overline{\theta}_2\frac{9\pi^2}{4} = 2$$

are obtained. From this, we have the coefficients $\underline{\theta}_1, \underline{\theta}_2, \overline{\theta}_1, \overline{\theta}_2$ as

$$\underline{\theta}_1 = -74.981076422 - 0.5738146367\alpha,$$

$$\underline{\theta}_2 = 16.001541876 + 0.1217672478\alpha,$$

$$\overline{\theta}_1 = -75.653915728 + 0.5738146324\alpha,$$

$$\overline{\theta}_2 = 16.144322811 - 0.1217672469\alpha.$$

Substituting these values in the equations (3.35) and (3.36), the approximate lower and upper solutions

$$\begin{aligned}\underline{y}_\alpha(t) &= (1+0.5\alpha) + (-74.981076422 - 0.5738146367\alpha)\,t \\ &\quad + (16.001541876 + 0.1217672478\alpha)\,t^2, \\ \overline{y}_\alpha(t) &= (2-0.5\alpha) + (-75.653915728 + 0.5738146324\alpha)\,t \\ &\quad + (16.144322811 - 0.1217672469\alpha)\,t^2\end{aligned}$$

are obtained. The approximate lower and upper solutions for $t = 0.01$ are

$$\underline{y}_\alpha(t) = 0.2517893899 + 0.4942740304\alpha,$$

$$\overline{y}_\alpha(t) = 1.245075275 - 0.4942740304\alpha.$$

From the equations (3.33) and (3.34), the exact lower and upper solution for $t = 0.01$ are

$$\underline{Y}_\alpha(t) = 0.9698116325 + 0.4951139514\alpha,$$

$$\overline{Y}_\alpha(t) = 1.9600395353 - 0.4951139514\alpha.$$

Comparison the results of the lower exact solution and the lower approximate solution

α	$\underline{Y}_\alpha(t)$	$\underline{y}_\alpha(t)$	$Error$
0	0.9698116325	0.2517893899	0.7180222426
0.1	1.0193230276	0.3012167929	0.7181062347
0.2	1.0688344228	0.350644196	0.7181902268
0.3	1.1183458179	0.400071599	0.7182742189
0.4	1.1678572131	0.4494990021	0.718358211
0.5	1.2173686082	0.4989264051	0.7184422031
0.6	1.2668800033	0.5483538081	0.7185261952
0.7	1.3163913985	0.5977812112	0.7186101873
0.8	1.3659027936	0.6472086142	0.7186941794
0.9	1.4154141888	0.6966360173	0.7187781715
1	1.4649255839	0.7460634203	0.7188621636

Comparison the results of the upper exact solution and the upper approximate solution

α	$\overline{Y}_\alpha(t)$	$\overline{y}_\alpha(t)$	$Error$
0	1.9600395353	1.245075275	0.7149642603
0.1	1.9105281402	1.195647872	0.7148802682
0.2	1.861016745	1.1462204689	0.7147962761
0.3	1.8115053499	1.0967930659	0.714712284
0.4	1.7619939547	1.0473656628	0.7146282919
0.5	1.7124825596	0.9979382598	0.7145442998
0.6	1.6629711645	0.9485108568	0.7144603077
0.7	1.6134597693	0.8990834537	0.7143763156
0.8	1.5639483742	0.8496560507	0.7142923235
0.9	1.514436979	0.8002286476	0.7142083314
1	1.4649255839	0.7508012446	0.7141243393

4. Conclusions

In this paper, the approximate solutions by using the undetermined fuzzy coefficients method and the exact solutions by using the Hukuhara differentiability of second-order fuzzy linear boundary value problems with positive and negative constant coefficients are investigated. The values of the exact and approximate solutions are computed for each $\alpha = 0, 0.1, 0,2, 0,3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$. The error values between the exact and approximate solutions are found. Thus, comparison results of the found solutions are given. While the error values increase for the lower solutions of the worked problems, the error values decrease for the upper solutions. But, the error values for negative constant coefficient case are the more than the errors values for the positive constant coefficient case.

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Author's address

Hülya Gültekin Çitil

Giresun University, Department of Mathematics, Faculty of Arts and Sciences, 28100 Giresun, Turkey

E-mail address: hulya.citil@giresun.edu.tr

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A NOTE ON LATTICES WITH MANY SUBLATTICES

GÁBOR CZÉDLI AND ESZTER K. HORVÁTH

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Abstract. For every natural number $n \geq 5$, we prove that the number of subuniverses of an n-element lattice is 2^n, $13 \cdot 2^{n-4}$, $23 \cdot 2^{n-5}$, or less than $23 \cdot 2^{n-5}$. Also, we describe the n-element lattices with exactly 2^n, $13 \cdot 2^{n-4}$, or $23 \cdot 2^{n-5}$ subuniverses.

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Keywords: finite lattice, sublattice, number of sublattices, subuniverse

1. Introduction and our result

For a lattice L, $\mathrm{Sub}(L)$ will denote its *sublattice lattice*. In spite of this standard terminology, $\mathrm{Sub}(L)$ consists of all *subuniverses* of L. That is, a subset X of L is in $\mathrm{Sub}(L)$ iff X is closed with respect to join and meet. In particular, $\emptyset \in \mathrm{Sub}(L)$. All lattices occurring in this paper will be assumed to be *finite* even if this is not always emphasized. For a natural number $n \in \mathbb{N}^+ := \{1, 2, 3, \dots\}$, let

$$\mathrm{NS}(n) := \{|\mathrm{Sub}(L)| : L \text{ is a lattice of size } |L| = n\}.$$

That is, $k \in \mathrm{NS}(n)$ if and only if some n-element lattice has exactly k subuniverses. Although the acronym NS comes from Number of Sublattices, L has only $|\mathrm{Sub}(L)| - 1$ *sublattices*. If K and L are finite lattices, then their *glued sum* $K +_{glu} L$ is the ordinal sum of the posets $K \setminus 1_K$, the singleton lattice, and $L \setminus \{0_L\}$, in this order. In other words, we put L atop K and identify the elements 1_K and 0_L; see Figure 1. For example, if each of K and L is the two-element chain, then $K +_{glu} L$ is the three-element chain. Note that $+_{glu}$ is an associative but not a commutative operation.

The following fact is trivial:

$$\begin{aligned}&\text{The largest number in } \mathrm{NS}(n) \text{ is } 2^n = 32 \cdot 2^{n-5}.\\ &\text{Furthermore, an } n\text{-element lattice } L \text{ has exactly } 2^n\\ &\text{subuniverses if an only if } L \text{ is a chain.}\end{aligned} \tag{1.1}$$

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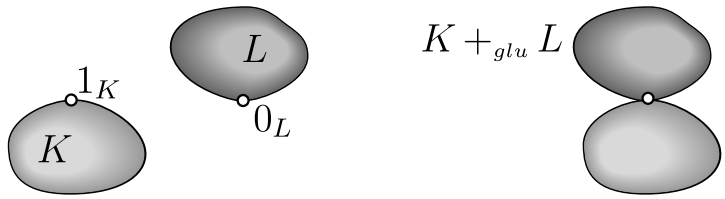

FIGURE 1. The glued sum $K +_{glu} L$ of K and L

The *four-element boolean lattice* B_4 and the *pentagon* (lattice) N_5 are given in Figure 2, together with other important lattices. For $k \in \mathbb{N}^+$, the k-element chain will be denoted by $C^{(k)}$. Our goal is to prove the following result:

Theorem 1. *If $5 \leq n \in \mathbb{N}^+$, then the following three assertions hold.*

(1) *The second largest number in* $\mathrm{NS}(n)$ *is* $26 \cdot 2^{n-5}$. *Furthermore, an n-element lattice L has exactly $26 \cdot 2^{n-5}$ subuniverses if and only if $L \cong C_1 +_{glu} B_4 +_{glu} C_2$, where C_1 and C_2 are chains.*

(2) *The third largest number in* $\mathrm{NS}(n)$ *is* $23 \cdot 2^{n-5}$. *Furthermore, an n-element lattice L has exactly $23 \cdot 2^{n-5}$ subuniverses if and only if $L \cong C_0 +_{glu} N_5 +_{glu} C_1$, where C_0 and C_1 are chains.*

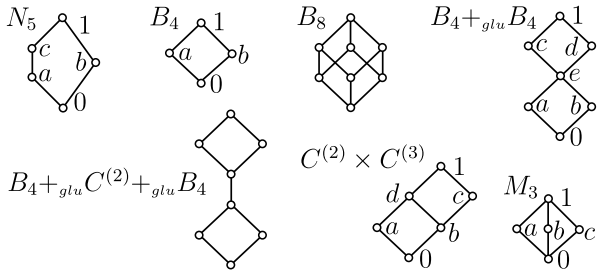

FIGURE 2. Lattices for Theorem 1 and Lemma 2

Since $\mathrm{NS}(n) = \{2^n\}$ for $n \in \{1,2,3\}$ and $\mathrm{NS}(4) = \{13,16\}$, we have formulated this theorem only for $n \geq 5$. To make the comparison of the numbers occurring in the paper easier, we often give $|\mathrm{Sub}(L)|$ as a multiple of $2^{|L|-5}$. Next, we repeat the first sentence of the Abstract, which is a trivial consequence of Theorem 1.

Corollary 1. *For $5 \leq n \in \mathbb{N}^+$, the number of subuniverses of an n-element lattice is 2^n, $13 \cdot 2^{n-4}$, $23 \cdot 2^{n-5}$, or less than $23 \cdot 2^{n-5}$.*

Remark 1. Let $\mathrm{Con}(L)$ and $\mathrm{NC}(n)$ stand for the lattice of congruences of a lattice L and $\{|\mathrm{Con}(L)| : L \text{ is a lattice with } |L| = n\}$, respectively. For $n \geq 5$, the five largest numbers in $\mathrm{NC}(n)$ are $16 \cdot 2^{n-5}$, $8 \cdot 2^{n-5}$, $5 \cdot 2^{n-5}$, $4 \cdot 2^{n-5}$, and $3.5 \cdot 2^{n-5}$ by Freese [4], Czédli [2] and, mainly, Kulin and Mureşan [6].

Remark 2. Interestingly, the first three of the five numbers mentioned in Remark 1 are witnessed exactly by the same lattices that occur in Theorem 1 and (1.1). However, we will show at the end of Section 2 that

$$\begin{aligned} |\mathrm{Sub}(N_5 +_{glu} C^{(3)})| &= 23 \cdot 2^{7-5} > 21.25 \cdot 2^{7-5} \\ &= |\mathrm{Sub}(B_4 +_{glu} B_4)| > 19 \cdot 2^{7-5} \\ &= |\mathrm{Sub}((C^{(2)} \times C^{(3)}) +_{glu} C^{(2)})|, \end{aligned} \tag{1.2}$$

which indicates that $|\mathrm{Sub}((C^{(2)} \times C^{(3)}) +_{glu} C^{(2)})|$ is not the fourth largest number in NS(7), although we know from Kulin and Mureşan [6] that $|\mathrm{Con}((C^{(2)} \times C^{(3)}) +_{glu} C^{(2)})|$ is the fourth largest number in NC(7).

While there are powerful tools to determine the first few large numbers in NC(n), see the above-mentioned papers and, for additional tools, Czédli [3], the analogous task for NS(n) seems to be more tedious. This together with Remark 2 are our excuses that we do not determine the fourth and fifth largest numbers in NS(n).

The rest of the paper is devoted to the proof of Theorem 1.

2. Two preparatory lemmas

Our notation and terminology is standard, see, for example, Grätzer [5]. However, we recall some notation and introduce some auxiliary concepts. For elements u, v in a lattice L, the *interval* $[u, v] := \{x \in L : u \leq x \leq v\}$ is defined only if $u \leq v$, but the *sublattice* $[\{u, v\}]$ generated by $\{u, v\}$ always makes sense. In order to avoid confusion, the curly brackets are never omitted from $[\{a_1, \dots, a_k\}]$ when a generated sublattice is mentioned. For $u \in L$, the principal ideal and the principal filter generated by u are $\downarrow u := \{x \in L : x \leq u\}$ and $\uparrow u := \{x \in L : u \leq x\}$, respectively. We can also write $\downarrow_L u$ and $\uparrow_L v$ to specify the lattice L. For $u, v \in L$, we write $u \parallel v$ if u and v are *incomparable*, that is, $u \not\leq v$ and $v \not\leq u$. We say that u is *join-irreducible* if u has at most one lower cover; note that $0 = 0_L$ is join-irreducible by our convention. *Meet-irreducibility* is defined dually, and an element is *doubly irreducible* if it is both join-irreducible and meet-irreducible. Next, let us call an element $u \in L$ *isolated* if u is doubly irreducible and $L = \downarrow u \cup \uparrow u$. That is, if u is doubly irreducible and $x \parallel u$ holds for no $x \in L$. Finally, an interval $[u, v]$ will be called an *isolated edge* if it is a prime interval, that is, $u \prec v$, and $L = \downarrow u \cup \uparrow v$.

Lemma 1. *If K is a sublattice and H is a subset of a finite lattice L, then the following three assertions hold.*

(1) *With the notation $t := |\{H \cap S : S \in \mathrm{Sub}(L)\}|$, we have that $|\mathrm{Sub}(L)| \leq t \cdot 2^{|L|-|H|}$.*
(2) $|\mathrm{Sub}(L)| \leq |\mathrm{Sub}(K)| \cdot 2^{|L|-|K|}$.

(3) *Assume, in addition, that K has neither an isolated element, nor an isolated edge. Then* $|\mathrm{Sub}(L)| = |\mathrm{Sub}(K)| \cdot 2^{|L|-|K|}$ *if and only if L is (isomorphic to)* $C_0 +_{glu} K +_{glu} C_1$ *for some chains C_0 and C_1.*

Proof. With respect to the map $\varphi\colon \mathrm{Sub}(L) \to \{H \cap S : S \in \mathrm{Sub}(L)\}$, defined by $X \mapsto H \cap X$, each $Y \in \{H \cap S : S \in \mathrm{Sub}(L)\}$ has at most $2^{|L|-|H|}$ preimages. This yields part (1). Clearly, (1) implies (2). The argument above yields a bit more than stated in (1) and (2); namely, for later reference, note the following.

$$\text{If } |\mathrm{Sub}(L)| = |\mathrm{Sub}(K)| \cdot 2^{|L|-|K|}, \text{ then for every } S \in \mathrm{Sub}(K) \text{ and every subset } X \text{ of } L \setminus K, \text{ we have that } S \cup X \in \mathrm{Sub}(L). \tag{2.1}$$

Next, we claim that for an element $u \in L$,

$$u \text{ is isolated if and only if for every } X \in \mathrm{Sub}(L), \text{ we have that } X \cup \{u\} \in \mathrm{Sub}(L) \text{ and } X \setminus \{u\} \in \mathrm{Sub}(L). \tag{2.2}$$

Assume that u is isolated and $X \in \mathrm{Sub}(L)$. Since u is doubly irreducible, $X \setminus \{u\} \in \mathrm{Sub}(L)$. Since u is comparable with all elements of X, $X \cup \{u\} \in \mathrm{Sub}(L)$, proving the "only if" part of (2.2). To show the converse, assume that u is not isolated. If u is not doubly irreducible, then $u = a \vee b$ with $a, b < u$ or dually, and $X := \{a, b, u, a \wedge b\} \in \mathrm{Sub}(L)$ but $X \setminus \{u\} \notin \mathrm{Sub}(L)$. If $u \parallel v$ for some $v \in L$, then $\{v\} \in \mathrm{Sub}(L)$ but $\{v\} \cup \{u\} \notin \mathrm{Sub}(L)$. This proves the "if" part, and (2.2) has been verified.

Next, to prove part (3), assume that K has neither an isolated element, nor an isolated edge. First, let $L = C_1 +_{glu} K +_{glu} C_2$. Since every u in $L \setminus K$ is clearly an isolated element of L, it follows from a repeated application of (2.2) that whenever $X \subseteq L \setminus K$ and $S \in \mathrm{Sub}(K)$, then $S \cup X \in \mathrm{Sub}(L)$. Since $L \setminus K$ has $2^{|L|-|K|}$ subsets, $|\mathrm{Sub}(L)| \geq |\mathrm{Sub}(K)| \cdot 2^{|L|-|K|}$, and we obtain the required equality by the converse inequality given in part (2).

Conversely, assume the equality given in (3). Let x be an arbitrary element of $L \setminus K$. Applying (2.1) to $\{0_K\} \in \mathrm{Sub}(K)$ and $\{1_K\} \in \mathrm{Sub}(K)$, we obtain that both $\{0_K, x\}$ and $\{1_K, x\}$ are in $\mathrm{Sub}(L)$, whence neither $x \parallel 0_K$, nor $x \parallel 1_K$. So exactly one of the cases $0_K < x < 1_K$, $x < 0_K$, and $1_K < x$ holds; we are going to exclude the first one. Suppose for a contradiction that $0_K < x < 1_K$. Then x is comparable to every $y \in K$, because otherwise $S := \{y\}$ and $\{x\}$ would violate (2.1). By finiteness, we can take $u := \bigvee(K \cap {\downarrow}x)$ and $v := \bigwedge(K \cap {\uparrow}x)$. Now if $y \in K$, then either $y > x$ and so $y \in {\uparrow}_K v$, or $y < x$ and so $y \in {\downarrow}_K u$, which means that $K = {\downarrow}_K u \cup {\uparrow}_K v$. Hence, $[u, v]_K$ is a prime interval of K, and so it is an isolated edge of K. This is a contradiction, which excludes that $0_K < x < 1_K$. Therefore, with the notation $C_0 := {\downarrow}_L 0_K$ and $C_1 := {\uparrow}_L 1_K$, we obtain that L is (isomorphic to) $C_0 +_{glu} K +_{glu} C_1$. Consequently, in order to show that C_0 and C_1 are chains and to complete the proof, it suffices to show that every $u \in L \setminus K$ is an isolated element of L. Suppose the contrary. Then (2.2) yields a subuniverse $Y \in \mathrm{Sub}(L)$ such that

$$Y \cup \{u\} \notin \mathrm{Sub}(L) \quad \text{or} \quad Y \setminus \{u\} \notin \mathrm{Sub}(L). \tag{2.3}$$

Since $u \notin K$, we have that $Y \cap K = (Y \cup \{u\}) \cap K = (Y \setminus \{u\}) \cap K$; we denote this set by S. Then $S \in \mathrm{Sub}(K)$ since $Y \in \mathrm{Sub}(L)$. It follows from (2.1) that

$$Y \cup \{u\} = S \cup ((Y \cup \{u\}) \setminus K) \in \mathrm{Sub}(L) \text{ and}$$
$$Y \setminus \{u\} = S \cup ((Y \setminus \{u\}) \setminus K) \in \mathrm{Sub}(L),$$

which contradicts (2.3) and completes the proof of Lemma 1 □

The following lemma is easier and even a computer program could prove it. For the reader's convenience, we give its short proof.

Lemma 2. *For the lattices given in Figure 2, the following seven assertions hold.*

(1) $|\mathrm{Sub}(B_4)| = 13 = 26 \cdot 2^{4-5}$.
(2) $|\mathrm{Sub}(N_5)| = 23 = 23 \cdot 2^{5-5}$.
(3) $|\mathrm{Sub}(C^{(2)} \times C^{(3)})| = 38 = 19 \cdot 2^{6-5}$.
(4) $|\mathrm{Sub}(B_4 +_{glu} B_4)| = 85 = 21.25 \cdot 2^{7-5}$.
(5) $|\mathrm{Sub}(B_4 +_{glu} C^{(2)} +_{glu} B_4)| = 169 = 21.125 \cdot 2^{8-5}$.
(6) $|\mathrm{Sub}(M_3)| = 20 = 20 \cdot 2^{5-5}$.
(7) $|\mathrm{Sub}(B_8)| = 74 = 9.25 \cdot 2^{8-5}$.

Proof. The notation given by Figure 2 will extensively be used.

Among all subsets of B_4, only $\{a,b\}$, $\{a,b,0\}$, and $\{a,b,1\}$ are *not* subuniverses; this proves (1). Implicitly, Lemma 1(3) will often be used below. Observe that

$$|\{S \in \mathrm{Sub}(N_5) : \{a,c\} \cap S = \varnothing\}| = 8, \qquad \text{by (1.1)},$$
$$|\{S \in \mathrm{Sub}(N_5) : \{a,c\} \cap S \neq \varnothing,\ b \notin S\}| = 3 \cdot 4 = 12, \text{ and}$$
$$|\{S \in \mathrm{Sub}(N_5) : \{a,c\} \cap S \neq \varnothing,\ b \in S\}| = 3,$$

whereby $|\mathrm{Sub}(N_5)| = 8 + 12 + 3 = 23$ proves (2). Next, S will belong to $\mathrm{Sub}(C^{(2)} \times C^{(3)})$ even if this is not indicated. Let us compute:

$$|\{S : a \notin S\}| = 26, \qquad \text{by Lemmas 1(3) and 2(1)},$$
$$|\{S : a, b \in S\}| = 3, \text{ since } 0, d \in S \text{ and } c \in S \Rightarrow 1 \in S,$$
$$|\{S : a \in S,\ b \notin S,\ c \in S\}| = 1, \text{ since } 0, 1 \in S \text{ and } d \notin S,$$
$$|\{S : a \in S,\ b \notin S,\ c \notin S\}| = 8, \text{by (1.1)}.$$

Hence, $26 + 3 + 1 + 8 = 38$ proves (3). Next, S will automatically belong to $\mathrm{Sub}(B_4 +_{glu} B_4)$. We have that $|\{S : \{a,b\} \subseteq S|\} = 7$, because then $\{c,d\} \subseteq S \Rightarrow 1 \in S$ and $0, e \in S$. Also, $|\{S : \{a,b\} \not\subseteq S|\} = 13 \cdot 3 \cdot 2 = 78$, because Lemma 2(1) applies to the upper B_4, there are 3 possibilities for a and b, and two for 0. Hence, 78+7=85 proves (4). For $S \in \mathrm{Sub}(B_4 +_{glu} C^{(2)} +_{glu} B_4)$, the intersection of S with the lower B_4 and that with the upper B_4 can independently be chosen. Therefore, (5) follows from (1).

Next, we count the subuniverses S of M_3. There are 4 with the property $|\{a,b,c\}\cap S|\geq 2$, because they contain 0 and 1. There are $3\cdot 4=12$ with $|\{a,b,c\}\cap S|=1$, and 4 with $|\{a,b,c\}\cap S|=0$. Thus, $|\mathrm{Sub}(M_3)|=4+12+4=20$, proving (6).

The argument for B_8 is more tedious. It has 9 at most one-element subuniverses. There are 12 edges. We have 6 two-element subuniverses in which the heights of the two elements differ by two and 1 in which this difference is three. We have 12 three-element covering chains and 6 non-covering ones. The number of four-element (necessarily covering) chains is $3\cdot 2=6$, B_4 is embedded in 6 cover-preserving ways and (thinking of pairs of complementary elements) in 3 additional ways. The five-element sublattices are obtained from cover-preserving B_4-sublattices by adding the unique one of 0_{B_8} or 1_{B_8} that is missing; their number is 6. To each of the B_4-sublattices at the bottom we can glue a B_4-sublattice at the top in two ways, whence there are exactly 6 six-element subuniverses. In absence of doubly irreducible elements, there is no seven-element sublattice, and there is 1 eight-element one. The sum of the numbers we have listed is 74, proving (7) and Lemma 2 □

Now, we are in the position to prove (1.2), mentioned in Remark 2.

Proof of Remark 2. We obtain the statement by combining Lemma 1(3) with parts (2), (3), and (4) of Lemma 2. □

3. The rest of the proof

For brevity, a k-element antichain will be called a k-*antichain*. First, we recall two well-known facts from the folklore.

Lemma 3. *For every join-semilattice S generated by $\{a,b,c\}$, there is a unique surjective homomorphism φ from the free join-semilattice $F_{\mathrm{jsl}}(\tilde{a},\tilde{b},\tilde{c})$, given in Figure 3, onto S such that $\varphi(\tilde{a})=a$, $\varphi(\tilde{b})=b$, and $\varphi(\tilde{c})=c$.*

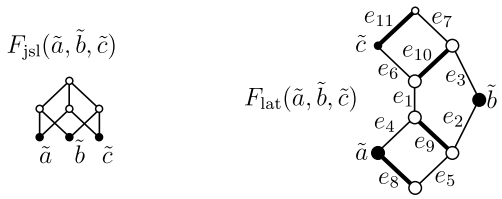

FIGURE 3. $F_{\mathrm{jsl}}(\tilde{a},\tilde{b},\tilde{c})$ and $F_{\mathrm{lat}}(\tilde{a},\tilde{b},\tilde{c})$

Lemma 4 (Rival and Wille [7, Figure 2])**.** *For every lattice K generated by $\{a,b,c\}$ such that $a<c$, there is a unique surjective homomorphism φ from the finitely presented lattice $F_{\mathrm{lat}}(\tilde{a},\tilde{b},\tilde{c})$, given in Figure 3, onto K such that $\varphi(\tilde{a})=a$, $\varphi(\tilde{b})=b$, and $\varphi(\tilde{c})=c$.*

The subscripts in the notations are explained by the facts that $F_{\mathrm{jsl}}(\tilde{a},\tilde{b},\tilde{c})$ is the free join-semilattice on 3 generators, while $F_{\mathrm{lat}}(\tilde{a},\tilde{b},\tilde{c})$ is the free lattice generated by its subposet $\{a,b,c\}$.

We are going to use the two lemmas above in the proof of the following lemma. Implicitly, we will often use the well-known Homomorphism Theorem; see, e.g., Burris and Sankappanavar [1, Theorem 6.12].

Lemma 5. *If an n-element lattice L has a 3-antichain, then we have that* $|\mathrm{Sub}(L)| \leq 20\cdot 2^{n-5}$.

Proof. Let $\{a,b,c\}$ be a 3-antichain in L. Lemma 3 yields a unique join-homomorphism from $F_{\mathrm{jsl}}((\tilde{a},\tilde{b},\tilde{c}))$ to $S := \{a,b,c,a\vee b,a\vee c,b\vee c,a\vee b\vee c\}$ such that φ maps to $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ to a, b, and c, respectively. Since $\{a,b,c\}$ is an antichain, none of the six lower edges of $F_{\mathrm{jsl}}((\tilde{a},\tilde{b},\tilde{c}))$ is collapsed by the kernel $\Theta := \ker(\varphi)$ of φ. Hence, there are only four cases for the join-subsemilattice $S \cong F_{\mathrm{jsl}}((\tilde{a},\tilde{b},\tilde{c}))/\Theta$ of L, depending on the number the upper edges collapsed by Θ.

Case 1: [none of the three upper edges is collapsed by Θ] Then S is isomorphic to $F_{\mathrm{jsl}}((\tilde{a},\tilde{b},\tilde{c}))$, whereby $\{a\vee b,a\vee c,b\vee c\}$ is a 3-antichain. We know from, say, Grätzer [5, Lemma 73], that this 3-antichain generates a sublattice isomorphic to B_8. Hence, $|\mathrm{Sub}(L)| \leq 9.25\cdot 2^{n-5} \leq 20\cdot 2^{n-5}$ by Lemmas 1(2) and 2(7), as required.

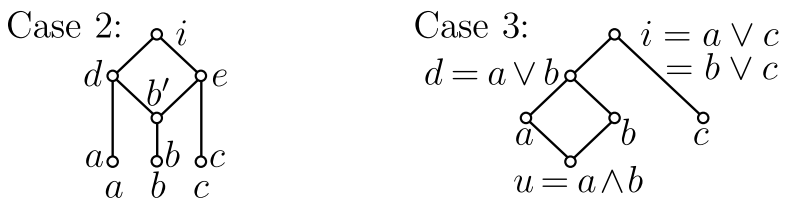

FIGURE 4. Cases 2 and 3

Case 2: [Θ collapses exactly one upper edge] Apart from notation, we have that $d := a\vee b < a\vee c =: i$ and $e := b\vee c < i$; see Figure 4 on the left. Letting $b' := d\wedge e$, we have that $a\vee b' = d$ and $b'\vee c = e$. Since $b\leq b'$ and $b\not\leq a$, we have that $b'\not\leq a$. If we had $a\leq b'$, then $i = d\vee e = a\vee b'\vee e = b'\vee e = e$ would be a contradiction. Hence, $a \parallel b'$, and $\{a,b',c\}$ is a 3-antichain by a–c symmetry. We can count the subuniverses T of the join-semilattice $H := \{a,b',c,d,e,i\}$ as follows. We have that $|\{T : b'\notin T\}| \leq 3\cdot 7 = 21$, because $\{d,e\}\not\subseteq T$ allows only three possibilities for $T\cap\{d,e\}$ and $a\vee c = i$ at most seven possibilities for $T\cap\{a,c,i\}$. Similarly,

$$|\{T : b'\in T, a\notin T, c\notin T\}| \leq 7, \quad \text{because } \{d,e\}\subseteq T \Rightarrow i\in T,$$

$$|\{T : b'\in T, a\in T, c\notin T\}| \leq 3, \quad \text{since } d\in T, \text{ so } e\in T \Rightarrow i\in T,$$

$$|\{T : b' \in T, a \notin T, c \in T\}| \leq 3, \quad \text{by } a\text{–}c \text{ symmetry, and}$$
$$|\{T : b' \in T, a \in T, c \in T\}| = 1, \quad \text{because } T = H.$$

Note that some of the inequalities above are equalities, but we do not need this fact. Forming the sum of the above numbers, the join-semilattice H has at most $35 = 17.5 \cdot 2^{6-5}$ subuniverses. Hence, Lemma 1(1) yields that $\mathrm{Sub}(L) \leq 17.5 \cdot 2^{n-5} \leq 20 \cdot 2^{n-5}$, as required.

Case 3: [Θ collapses two of the upper edges] Apart from notation, we have that $d := a \vee b < a \vee c = b \vee c =: i$. Let $u := a \wedge b$; see Figure 4. We focus on possible intersections of subuniverses of L with $H := \{a, b, c, d, u, i\}$. Denoting such an intersection by S, we can compute as follows.

$$|\{S : c \notin S\}| \leq 26, \quad \text{by Lemmas 1(2) and 2(1)},$$
$$|\{S : c \in S, a \in S, b \in S\}| \leq 1 \quad \text{since } H \subseteq [\{a, b, c\}],$$
$$|\{S : c \in S, a \in S, b \notin S\}| \leq 4 \quad \text{since } i \in [\{a, c\}],$$
$$|\{S : c \in S, a \notin S, b \in S\}| \leq 4, \quad \text{by } a\text{–}b\text{-symmetry},$$
$$|\{S : c \in S, a \notin S, b \notin S, d \in S\}| \leq 2, \quad \text{because } i \in S,$$
$$|\{S : c \in S, a \notin S, b \notin S, d \notin S\}| \leq 3 \quad \text{since } u \in S \Rightarrow i \in S.$$

Since the sum of these numbers is 40, we obtain from Lemma 1(1) that $|\mathrm{Sub}(L)| \leq 40 \cdot 2^{n-6} = 20 \cdot 2^{n-5}$, as required.

Case 4: [all the three upper edges are collapsed] Clearly, $a \vee b = a \vee c = b \vee c = a \vee b \vee c =: i$. If $a \wedge b = a \wedge c = b \wedge c = a \wedge b \wedge c$ failed, then the dual of one of the previous three cases would apply. Hence, we can assume that the sublattice $[\{a, b, c\}]$ generated by $\{a, b, c\}$ is isomorphic to M_3; see Figure 2. Therefore, $|\mathrm{Sub}(L)| \leq 20 \cdot 2^{n-5}$ by Lemmas 1(2) and 2(6), completing the proof of Case 4 and that of Lemma 5. □

Proof of Theorem 1. From Lemmas 1 and 2(1), we conclude part (1). So, we are left only with part (2).

In what follows, let L be an n-element lattice. We obtain from Lemmas 1(3) and 2(2) that if

$$L \cong C_0 +_{glu} N_5 +_{glu} C_1 \text{ for finite chains } C_0 \text{ and } C_1, \tag{3.1}$$

then $|\mathrm{Sub}(L)| = 23 \cdot 2^{n-5}$. In order the complete the proof of Theorem 1, it suffices to exclude the existence of a lattice L such that

$$\begin{aligned} &|L| = n,\ 23 \cdot 2^{n-5} \leq |\mathrm{Sub}(L)| < 26 \cdot 2^{n-5}, \text{ but } L \text{ is} \\ &\text{not of the form given in (3.1).} \end{aligned} \tag{3.2}$$

Suppose, for a contradiction, that L is a lattice satisfying (3.2). Then, by (1.1) and Theorem 1 (1) and Lemma 5,

$$L \text{ has at least two 2-antichains but it has no 3-antichain.} \tag{3.3}$$

We claim that

$$L \text{ cannot have two } \textit{non-disjoint} \text{ 2-antichains.} \tag{3.4}$$

Suppose the contrary. Then we can pick two distinct antichains $\{a,b\}$ and $\{c,b\}$ in L. Since there is no 3-antichain in L, we can assume that $a < c$. With $K := [\{a,b,c\}]$, let $\varphi\colon F_{\mathrm{lat}}(\tilde{a},\tilde{b},\tilde{c}) \to K$ be the unique lattice homomorphism from Lemma 4, and let Θ be the kernel of φ. We claim that Θ collapses e_1; see Figure 3. Suppose to the contrary that Θ does not collapse e_1. Since e_1 generates the monolith congruence, that is, the smallest nontrivial congruence of the N_5 sublattice of $F_{\mathrm{lat}}(\tilde{a},\tilde{b},\tilde{c})$, no other edge of this N_5 sublattice is collapsed. Hence, N_5 is a sublattice of L, and it follows from Lemmas 1(2) and 2(2) that $|\operatorname{Sub}(L)| \leq 23\cdot 2^{n-5}$. Thus, (3.2) yields that $|\operatorname{Sub}(L)| = 23\cdot 2^{n-5}$. Applying Lemma 1(3) for $K := N_5$ and L, we obtain that L is of the form (3.1). This contradicts (3.2), and we have shown that Θ does collapse e_1. On the other hand, since $a \parallel b$ and $c \parallel b$, none of the thick edges $e_8,\dots,e_{11}$ is collapsed by Θ. Observe that at least one of e_4 and e_6 is *not* collapsed by Θ, since otherwise $\langle\tilde{a},\tilde{c}\rangle$ would belong to $\Theta = \ker(\varphi)$ by transitivity and $a = c$ would be a contradiction. By duality, we can assume that e_4 is not collapsed by Θ. Since e_2, e_3, and e_5 are perspective to e_{10}, e_9, and e_4, respectively, these edges are not collapsed either. So, with the exception of e_1, no edge among the "big" elements in Figure 3 is collapsed. Thus, the φ-images of the elements denoted by big circles form a sublattice (isomorphic to) $C^{(2)} \times C^{(3)}$ in L. Hence, $|\operatorname{Sub}(L)| \leq 19\cdot 2^{n-5}$ by Lemmas 1(2) and 2(3), which contradicts our assumption that L satisfies (3.2). This proves (3.4).

To provide a convenient tool to exploit (3.3) and (3.4), we claim that

$$\begin{aligned}&\text{if } x,y,z \in L \text{ such that } |\{x,y,z\}| = 3 \text{ and } x \parallel y, \text{ then either}\\ &\{x,y\} \subseteq {\downarrow}z, \text{ or } \{x,y\} \subseteq {\uparrow}z.\end{aligned} \tag{3.5}$$

To see this, assume the premise. Since L has no 3-antichain, z is comparable to one of x and y. By duality and symmetry, we can assume that $x < z$. Since $z < y$ would imply $x < y$ and $z \parallel y$ together with $x \parallel y$ would contradict (3.4), we have that $y < z$. This proves (3.5).

Next, by (3.3) and (3.4), we have a four-element subset $\{a,b,c,d\}$ of L such that $a \parallel b$ and $c \parallel d$. By duality and (3.5), we can assume that $\{a,b\} \subseteq {\downarrow}c$. Applying (3.5) also to $\{a,b,d\}$, we obtain that $\{a,b\}$ is included either in ${\uparrow}d$, or in ${\downarrow}d$. Since the first alternative would lead to $d < a < c$ and so it would contradict $c \parallel d$, we have that $\{a,b\} \subseteq {\downarrow}d$. Thus, $\{a,b\} \subseteq {\downarrow}c \cap {\downarrow}d = {\downarrow}(c \wedge d)$, and we obtain that $u := a \vee b \leq c \wedge d =: v$. Let $S := \{a \wedge b, a, b, u, v, c, d, c \vee d\}$. Depending on $u = v$ or $u < v$, S is a sublattice isomorphic to $B_4 +_{glu} B_4$ or $B_4 +_{glu} C^{(2)} +_{glu} B_4$. Using Lemma 2.1 together with (4) and (5) of Lemma 2, we obtain that $|\operatorname{Sub}(L)| \leq 21.25\cdot 2^{n-5}$. This inequality contradicts (3.2) and completes the proof of Theorem 1. $\square$

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Authors' addresses

Gábor Czédli
University of Szeged, Bolyai Institute, Aradi vértanúk tere 1, 6720 Szeged, Hungary
E-mail address: czedli@math.u-szeged.hu

Eszter K. Horváth
University of Szeged, Bolyai Institute, Aradi vértanúk tere 1, 6720 Szeged, Hungary
E-mail address: horeszt@math.u-szeged.hu

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NEW BOUNDS FOR HERMITE-HADAMARD'S TRAPEZOID AND MID-POINT TYPE INEQUALITIES VIA FRACTIONAL INTEGRALS

M. ROSTAMIAN DELAVAR

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Abstract. Some trapezoid and mid-point type inequalities with new bounds for Hermite-Hadamard inequality related to Riemann-Liouville integrals of order $\alpha > 0$ are obtained. Also a refinement of Hermite-Hadamard inequality for nonnegative monotone convex functions is presented. Furthermore some applications in connection with special means are given.

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Keywords: Hermite-Hadamard inequality, trapezoid and mid-point type inequalities, fractional integrals

1. Introduction

The following inequality is known in literature as Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f : [a;b] \to \mathbb{R}$ is convex on $[a,b]$. For historical information about inequality (1.1), see [6].

In inequality (1.1), we may deal with two issues:

(i) Estimation of the difference between left and middle terms which we call it *trapezoid type estimation*

(ii) Estimation of the difference between right and middle terms which we call it *mid-point (rectangle) type estimation*.

At first S. S. Dragomir et al, in [3], obtained the trapezoid type inequality related to (1.1) as well:

Theorem 1. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{8}\Big(|f'(a)|+|f'(b)|\Big). \tag{1.2}$$

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The striped area shown in Figure 1, is equivalent to the difference between the area of trapezoid $abcd$ and the area under the graph of f which is estimated by $\frac{(b-a)^2}{8}\big(|f'(a)|+|f'(b)|\big)$. Also U. S. Kirmaci in [5], obtained the mid-point type

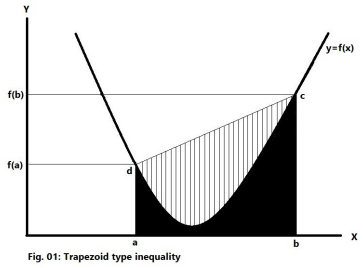

FIGURE 1.

inequality related to (1.1) as the following:

Theorem 2. *Consider I^* as the interior of interval $I \subset \mathbb{R}$. Let $f : I^* \to \mathbb{R}$ be a differentiable mapping on I^*, $a,b \in I^*$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then we have*

$$\Big|\int_a^b f(x)dx-(b-a)f\Big(\frac{a+b}{2}\Big)\Big| \le \frac{(b-a)^2}{8}\Big(|f'(a)|+|f'(b)|\Big). \tag{1.3}$$

In Figure 2, it is shown that the difference between the area under the graph of f and the area of rectangle $abcd$ can be estimated by $\frac{(b-a)^2}{8}\big(|f'(a)|+|f'(b)|\big)$. Recently In [8], the authors obtained Hermite-Hadamard's inequality related to frac-

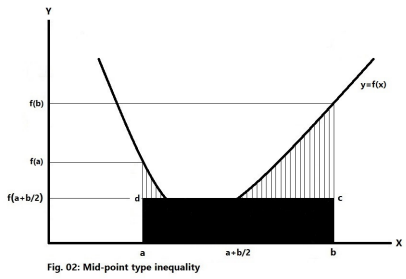

FIGURE 2.

tional integrals as the following:

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be convex function with $0 \le a < b$ and $f \in L[a,b]$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds.*

$$f\Big(\frac{a+b}{2}\Big) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\big[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)\big] \le \frac{f(a)+f(b)}{2}, \tag{1.4}$$

where $J^{\alpha}_{a+}f$ *and* $J^{\alpha}_{b-}f$ *are the Riemann-Liouville integrals of order* $\alpha > 0$ *defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \qquad x > a,$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \qquad x < b,$$

such that

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt,$$

is Gamma function and $J^{0}_{a+}f(x) = J^{0}_{b-}f(x) = f(x)$.

The trapezoid and mid-point type inequalities related to (1.4) have been obtained in [8] and [4] respectively.

Theorem 4. *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a < b$. *If* $|f'|$ *is convex function on* $[a,b]$, *then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right]\right| \tag{1.5}$$
$$\leq \frac{b-a}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\Big[|f'(a)| + |f'(b)|\Big].$$

Theorem 5. *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a < b$. *If* $|f'|$ *is convex function on* $[a,b]$, *then the following inequality for Riemann-Liouville fractional integrals holds for* $0 < \alpha \leq 1$:

$$\left|f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right]\right| \leq \frac{b-a}{2^{\alpha+1}(\alpha+1)}\Big[|f'(a)| + |f'(b)|\Big]. \tag{1.6}$$

On the other hand in [7], we can find two results related to the convexity of a function as the following:

(a) Any convex function defined on a closed interval $[a,b]$ is bounded.

(b) If a real valued function defined on the interval I is convex, then it satisfies a Lipschitz condition on any closed interval $[a,b]$ $\big($there is a constant K so that for any two points $x, y \in [a,b]$, $|f(x) - f(y)| \leq K|x-y|\big)$ contained in the interior I° of I.

Motivated by above works and results, we obtain some trapezoid and mid-point type inequalities related to (1.4) where the convexity condition for the absolute value of the derivative of considered function is replaced by boundedness and a Lipschitzian condition for the derivative. In fact we obtain new bounds for the left side of inequalities (1.5) and (1.6) which give some refinements for these inequalities. Also by

the use of fractional integrals we present a refinement of Hermite-Hadamard inequality for nonnegative monotone convex functions. Finally we give some applications of our results in connection with special means.

2. Trapezoid and mid-point type inequalities

In this section, we obtain some Hermite-Hadamard's trapezoid and mid-point type inequalities via fractional integrals where the derivative of considered function is bounded and satisfies a Lipschitz condition. The following result has been obtained in [8] and we use it to obtain trapezoid type inequalities.

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then the following equality for fractional integral holds:*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)\right]\\&=\frac{b-a}{2}\int_0^1\left[t^{\alpha}-(1-t)^{\alpha}\right]f'\big(ta+(1-t)b\big)dt.\end{aligned}$$

In the following theorem we consider that the derivative of considered function is bounded.

Theorem 6. *Suppose that $f:I\to\mathbb{R}$ is a differentiable function on I°. Consider $a,b\in I^{\circ}$ with $a<b$ such that $f'\in L[a,b]$. If there exist constants $l<L$ such that $-\infty<l\le f'(x)\le L<\infty\quad for\ all\ x\in[a,b]$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\Big[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)\Big]\right|\\&\le\frac{(b-a)(L-l)}{2(\alpha+1)}\Big(1-\frac{1}{2^{\alpha}}\Big).\end{aligned}\tag{2.1}$$

Proof. From Lemma 1 we have

$$\begin{aligned}J&=\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)\right]\right|\\&=\frac{b-a}{2}\int_0^1 h(t)\Big[f'\big(ta+(1-t)b\big)-\frac{l+L}{2}+\frac{l+L}{2}\Big]dt,\end{aligned}$$

where $h(t)=t^{\alpha}-(1-t)^{\alpha}$, for all $t\in[0,1]$. Since

$$\int_0^1 h(t)dt=0,$$

$$\int_0^1|h(t)|dt=\frac{2}{\alpha+1}(1-\frac{1}{2^{\alpha}}),$$

and

$$\left|f'\big(ta+(1-t)b\big)-\frac{l+L}{2}\right|\le\frac{L-l}{2},$$

then

$$\begin{aligned}|J|&\le\frac{b-a}{2}\int_0^1|h(t)|\left|f'\big(ta+(1-t)b\big)-\frac{l+L}{2}\right|dt\\&\le\frac{(b-a)(L-l)}{4}\int_0^1|h(t)|dt=\frac{(b-a)(L-l)}{2(\alpha+1)}(1-\frac{1}{2^\alpha}).\end{aligned}$$

□

Remark 1. If $|f'|$ is convex on $[a,b]$, then there exist l,L with

$$0\le l=2\left|f'\big(\frac{a+b}{2}\big)\right|-L\le|f'(x)|\le L=\max\{|f'(a)|,|f'(b)|\}<\infty$$

for all $x\in[a,b]$ which implies that $L-l\le|f'(a)|+|f'(b)|$, since

$$\begin{cases}|f'(b)|>|f'(a)|\to L=|f'(b)|\ and\ -l\le l\le|f'(a)|\to L-l\le|f'(a)|+|f'(b)|,\\|f'(a)|>|f'(b)|\to L=|f'(a)|\ and\ -l\le l\le|f'(b)|\to L-l\le|f'(a)|+|f'(b)|.\end{cases}$$

So (2.1) gives a refinement for (1.5).

In the following theorem we consider that the derivative of considered function satisfies a Lipschitz condition.

Theorem 7. *Suppose that $f:I\to\mathbb{R}$ is a differentiable function on I°. Consider $a,b\in I^\circ$ with $a<b$ such that $f'\in L[a,b]$ and satisfies a Lipschitz condition for some $K>0$. Then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\Big[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)\Big]\right|\\&\le\frac{K\alpha(b-a)^2}{2(\alpha+1)(\alpha+2)}.\end{aligned}$$

Proof. From Lemma 1 we have

$$\begin{aligned}J&=\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\big[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)\big]\right|\\&=\frac{b-a}{2}\int_0^1h(t)\Big[f'\big(ta+(1-t)b\big)-f'\Big(\frac{a+b}{2}\Big)+f'\Big(\frac{a+b}{2}\Big)\Big]dt\\&=\frac{b-a}{2}\int_0^1h(t)\Big[f'\big(ta+(1-t)b\big)-f'\Big(\frac{a+b}{2}\Big)\Big]dt,\end{aligned}$$

where $h(t)=t^\alpha-(1-t)^\alpha$, for all $t\in[0,1]$ and

$$\int_0^1h(t)f'\Big(\frac{a+b}{2}\Big)dt=0.$$

Since f' satisfies a Lipschitz condition for some $K > 0$, then

$$|J| \leq \frac{K(b-a)^2}{2} \int_0^1 |h(t)||t - \frac{1}{2}|dt.$$

The definition of $h(t)$ implies that

$$|J| \leq \frac{K(b-a)^2}{2} \int_0^1 \big(t^\alpha - (1-t)^\alpha\big)(t - \frac{1}{2})dt = \frac{K\alpha(b-a)^2}{2(\alpha+1)(\alpha+2)}.$$

□

The following lemma has been proved in [4] and we use it to obtain mid-point type inequalities.

Lemma 2. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b). If $f' \in L[a,b]$, then the following identity for Riemann-Liouville fractional integrals holds:*

$$f\Big(\frac{a+b}{2}\Big) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\Big[J^\alpha_{a+} f\big(b\big) + J^\alpha_{b-} f(a)\Big] = \frac{b-a}{2}\sum_{k=1}^{4} I_k,$$

where

$$I_1 = \int_0^{\frac{1}{2}} t^\alpha f'\big(tb + (1-t)a\big)dt, \qquad I_2 = \int_0^{\frac{1}{2}} (-t^\alpha) f'\big(ta + (1-t)b\big)dt,$$

$$I_3 = \int_{\frac{1}{2}}^1 (t^\alpha - 1) f'\big(tb + (1-t)a\big)dt, \qquad I_4 = \int_{\frac{1}{2}}^1 (1-t^\alpha) f'\big(ta + (1-t)b\big)dt.$$

If we consider the boundedness of the derivative of considered function we get the following mid-point type inequality.

Theorem 8. *Suppose that $f : I \to \mathbb{R}$ is a differentiable function on I°. Consider $a,b \in I^\circ$ with $a < b$ such that $f' \in L[a,b]$. If there exist constants $l < L$ such that $-\infty < l \leq f'(x) \leq L < \infty \quad for\ all\ x \in [a,b]$, then*

$$\Big|f\Big(\frac{a+b}{2}\Big) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\Big[J^\alpha_{a+} f\big(b\big) + J^\alpha_{b-} f(a)\Big]\Big| \leq \frac{M(b-a)}{2(\alpha+1)}\Big[\alpha - 1 + \big(\frac{1}{2}\big)^{\alpha-1}\Big],$$

where $M = \max\{-l, L\}$.

Proof. If we consider

$$I_1 = \int_0^{\frac{1}{2}} t^\alpha \Big[f'\big(ta + (1-t)b\big) - \frac{l+L}{2} + \frac{l+L}{2}\Big]dt,$$

then

$$\Big|I_1 - \frac{l+L}{2}\int_0^{\frac{1}{2}} t^\alpha\Big| \leq \int_0^{\frac{1}{2}} t^\alpha \Big|f'\big(ta + (1-t)b\big) - \frac{l+L}{2}\Big|dt$$

$$\leq \frac{L-l}{2}\Big(\frac{1}{\alpha+1}(\frac{1}{2})^{\alpha+1}\Big),$$

which implies that

$$|I_1| \leq \Big[\frac{L-l}{2}+\Big|\frac{l+L}{2}\Big|\Big]\Big(\frac{1}{\alpha+1}(\frac{1}{2})^{\alpha+1}\Big).$$

Moreover it is not hard to see that

$$M = \max\{-l, L\} = \Big[\frac{L-l}{2}+\Big|\frac{l+L}{2}\Big|\Big].$$

So

$$|I_1| \leq M\Big(\frac{1}{\alpha+1}(\frac{1}{2})^{\alpha+1}\Big).$$

Similarly we can obtain that

$$|I_2| \leq M\Big(\frac{1}{\alpha+1}(\frac{1}{2})^{\alpha+1}\Big).$$

It follows that

$$|I_3| \leq M\Big[\frac{1}{2}-\frac{1}{\alpha+1}\Big(1-(\frac{1}{2})^{\alpha+1}\Big)\Big],$$

and

$$|I_4| \leq M\Big[\frac{1}{2}-\frac{1}{\alpha+1}\Big(1-(\frac{1}{2})^{\alpha+1}\Big)\Big],$$

Now by adding all of above inequalities we get

$$\sum_{i=1}^{4}|I_i| \leq \frac{M}{\alpha+1}\Big(\alpha-1+(\frac{1}{2})^{\alpha-1}\Big),$$

which implies the desired result. □

Remark 2. In proof of Theorem 8 if we consider $0<\alpha\leq 1$, then by the use of the fact that

$$|t_1^\alpha - t_2^\alpha| \leq |t_1-t_2|^\alpha,$$

for any $t_1, t_2 \in [0,1]$, we obtain that

$$\Big|f\Big(\frac{a+b}{2}\Big)-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\Big[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\Big]\Big| \leq \frac{M(b-a)}{2^\alpha(\alpha+1)}. \tag{2.2}$$

where $M = \max\{-l, L\}$. Furthermore in the case that

$$2M \leq |f'(a)|+|f'(b)|,$$

inequality (2.2) gives a refinement for (1.6).

If the derivative of considered function satisfies a Lipschitz condition, then the following inequality holds.

Theorem 9. *Suppose that $f : I \to \mathbb{R}$ is a differentiable function on I°. Consider $a, b \in I^\circ$ with $a < b$ such that $f' \in L[a,b]$ and satisfies a Lipschitz condition for some $K > 0$. Then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{a+} f(b) + J^\alpha_{b-} f(a)\right]\right| \\ \leq \frac{b-a}{2^\alpha(\alpha+1)}\left[\frac{K(b-a)}{\alpha+2} + \left|f'\left(\frac{a+b}{2}\right)\right|\right],$$

for $0 < \alpha \leq 1$.

Proof. With some calculations similar to the proof of Theorem 7 we deduce that $|I_1| = |I_2| = |I_3| = |I_4| \leq \frac{1}{2^{\alpha+1}(\alpha+1)}\left[\frac{K(b-a)}{\alpha+2} + \left|f'(\frac{a+b}{2})\right|\right]$, which along with Lemma 2 imply the result. Note that in $|I_3|$ and $|I_4|$ we used the fact that

$$|t_1^\alpha - t_2^\alpha| \leq |t_1 - t_2|^\alpha,$$

for any $t_1, t_2 \in [0,1]$ and $0 < \alpha \leq 1$. □

Corollary 1. *If we consider $\alpha = 1$ in*
(i) Theorem 6, then we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{(b-a)}\int_a^b f(x)dx\right| \leq \frac{(b-a)(L-l)}{8},$$

If there exist constants $l < L$ such that $-\infty < l \leq f'(x) \leq L < \infty \quad for\ all\ x \in [a,b]$.
(ii) Theorem 7, then we obtain

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{(b-a)}\int_a^b f(x)dx\right| \leq \frac{K(b-a)^2}{12},$$

if f' satisfies a Lipschitz condition for some $K > 0$ on $[a,b]$. Comparing this inequality with inequality (2.6) in [2], shows that the existence of a Lipschitz condition for f' gives a better estimation rather than the existence of a Lipschitz condition for f.
(iii) Theorem 8, then we have

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)}\int_a^b f(x)dx\right| \leq \frac{M(b-a)}{4},$$

if there exist constants $l < L$ such that $-\infty < l \leq f'(x) \leq L < \infty \quad for\ all\ x \in [a,b]$ and $M = \max\{-l, L\}$.
(iv) Theorem 9, then we deduce that

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)}\int_a^b f(x)dx\right| \leq \frac{K(b-a)^2}{12} + \frac{1}{4}\left|f'\left(\frac{a+b}{2}\right)\right|.$$

if f' satisfies a Lipschitz condition for some $K > 0$ on $[a,b]$.

3. Refinement of Hermite-Hadamard inequality

In this section we give a refinement of Hermite-Hadamard inequality and some new inequalities in connection with fractional integrals related to the nonnegative monotone convex functions.

Theorem 10. *Let $f : [a,b] \to \mathbb{R}$ be a convex function.*
(i) For $\alpha \geq 1$, the following refinement of Hermite-Hadamard inequality holds if f is nonnegative and increasing.

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] \leq \frac{f(a)+f(b)}{2}. \tag{3.1}$$

(ii) For any $\alpha > 0$ we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] \leq \frac{\alpha}{b-a}\int_a^b f(x)dx. \tag{3.2}$$

(iii) If $\alpha \geq 1$ and f is nonnegative and increasing, then

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]dx \leq \frac{\alpha}{b-a}\int_a^b f(x)dx. \tag{3.3}$$

Proof. From convexity of f we have

$$f\big(tb+(1-t)a\big) \leq tf(b)+(1-t)f(a),$$

and

$$f\big(ta+(1-t)b\big) \leq tf(a)+(1-t)f(b).$$

By adding these inequalities we get

$$f\big(tb+(1-t)a\big) + f\big(ta+(1-t)b\big) \leq f(a)+f(b). \tag{3.4}$$

Multiplying both sides of (3.4) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$ and using Theorem 1 in [1], we obtain

$$\begin{aligned}
&\int_0^1 t^{\alpha-1}dt \cdot \int_0^1 f\big(tb+(1-t)a\big)dt + \int_0^1 t^{\alpha-1}dt \cdot \int_0^1 f\big(ta+(1-t)b\big)dt \\
&\leq \int_0^1 t^{\alpha-1} f\big(tb+(1-t)a\big)dt + \int_0^1 t^{\alpha-1} f\big(ta+(1-t)b\big)dt \\
&\leq \big[f(a)+f(b)\big]\int_0^1 t^{\alpha-1}dt.
\end{aligned}$$

Using the changes of variable $x = tb + (1-t)a$ and $x = ta + (1-t)b$, respectively, in above integrals we have

$$\frac{1}{b-a}\int_0^1 t^{\alpha-1}dt.\left[\int_a^b f(x)dx + \int_a^b f(a+b-x)dx\right]$$
$$\leq \frac{\Gamma(\alpha)}{(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b)-}^\alpha f(a)\right] \leq \left[f(a)+f(b)\right]\int_0^1 t^{\alpha-1}dt.$$

Since

$$\int_0^1 t^{\alpha-1}dt = \frac{1}{\alpha},$$

and

$$\int_a^b f(x)dx = \int_a^b f\big(a+b-x\big)dx,$$

then we obtain

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] \leq \frac{f(a)+f(b)}{2}. \tag{3.5}$$

On the other hand since f is convex, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx,$$

which along with (3.5), implies the inequality (3.1).
To obtain inequality (3.2), we consider that

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\Big[f\big(ta+(1-t)b\big) + f\big(tb+(1-t)a\big)\Big].$$

Multiplying both sides by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$ we get

$$\frac{2}{\alpha}f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha)}{(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]$$
$$\leq \int_0^1 f\big(ta+(1-t)b\big)dt + \int_0^1 f\big(tb+(1-t)a\big)dt$$

which implies the inequality (3.2). Finally inequality (3.3) comes from (3.1) and (3.2). □

Remark 3. If f and g are nonnegative decreasing functions defined on $[0,1]$ and B is an upper bound for them, then $B-f$ and $B-g$ are nonnegative increasing functions and so

$$\int_0^1 (B-f(x))dx \int_0^1 (B-g(x))dx \leq \int_0^1 (B-f(x))(B-g(x))dx,$$

which gives again

$$\int_0^1 f(x)dx \int_0^1 g(x)dx \leq \int_0^1 f(x)g(x)dx.$$

This implies that inequalities (3.1) and (3.3) of Theorem 10 can be obtained if $f : [a,b] \to \mathbb{R}$ be a decreasing nonnegative convex function and $0 < \alpha \leq 1$.

4. Application to special means

The following means for real numbers $a, b \in \mathbb{R}$ are known:

$$A(a,b) = \frac{a+b}{2} \qquad arithmetic\ mean,$$

$$L_n(a,b) = \left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}} \qquad generalized\ log{-}mean,\ n \in \mathbb{N},\ a < b.$$

Consider $f(x) = x^n$ for $x \geq 0$, $n \in \mathbb{N}$. If $x \in [a,b]$,

$$l = na^{n-1} \leq f'(x) = nx^{n-1} \leq nb^{n-1} = L.$$

So from Theorem 6, we obtain

$$\left|\frac{a^n+b^n}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\right]\right|$$
$$\leq \frac{n(b-a)(b^{n-1}-a^{n-1})}{2(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right),$$

where

$$J^\alpha_{a+}f(b) = \int_a^b (b-t)^{\alpha-1}t^n dt = \sum_{k=0}^n \frac{a^{n-k}(b-a)^{\alpha+k}P(n,k)}{\prod_{i=0}^k(\alpha+i)},$$

$$J^\alpha_{b-}f(a) = \int_a^b (t-a)^{\alpha-1}t^n dt = \sum_{k=0}^n \frac{(-1)^k b^{n-k}(b-a)^{\alpha+k}P(n,k)}{\prod_{i=0}^k(\alpha+i)},$$

and

$$P(n,k) = \frac{n!}{(n-k)!},$$

which is the number of possible permutations of k objects from a set of n.
In special case if we consider $\alpha = 1$, then we have

$$J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a) = \frac{2(b^{n+1}-a^{n+1})}{n+1}.$$

So

$$\left|\frac{a^n+b^n}{2}-\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right|\leq\frac{n(b-a)(b^{n-1}-a^{n-1})}{8},$$

or equivalently

$$\begin{aligned}\left|A(a^n,b^n)-L_n^n(a,b)\right|&\leq\frac{n(b-a)(b^{n-1}-a^{n-1})}{8}\\&\leq\frac{n(b-a)(b^{n-1}+a^{n-1})}{8}=\frac{n(b-a)A(a^{n-1},b^{n-1})}{4}.\end{aligned}\tag{4.1}$$

Inequality (4.1) gives a refinement for inequality (1.5) $\big($in the case that $\alpha=1$ and $f(x)=x^n\big)$ that turns to the inequality obtained in Proposition 3.1 in [3], where $0\leq a<b$, $n\in\mathbb{N}$ and $n\geq 2$.

It follows that $f'(x)=nx^{n-1}$, satisfies a Lipschitz condition for

$$K=\sup_{x\in[a,b]}n(n-1)x^{n-2}=n(n-1)b^{n-2}.$$

So from Theorem 7 we have

$$\left|A(a^n,b^n)-L_n^n(a,b)\right|\leq\frac{n(n-1)b^{n-2}(b-a)^2}{12}.$$

At last, Theorem 8 and 9 imply the following inequalities.

$$\left|f\big(A(a,b)\big)-L_n^n(a,b)\right|\leq\frac{nb^{n-1}(b-a)}{4},$$

and

$$\left|f\big(A(a,b)\big)-L_n^n(a,b)\right|\leq\frac{n(n-1)b^{n-1}(b-a)^2}{12}+\frac{n}{4}A^{n-1}(a,b).$$

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Author's address

M. Rostamian Delavar
Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, P. O. Box 1339, Bojnord 94531, Iran
E-mail address: m.rostamian@ub.ac.ir

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ON THE MONOID OF MONIC BINARY QUADRATIC FORMS

JEROME T. DIMABAYAO, VADIM PONOMARENKO,
AND ORLAND JAMES Q. TIGAS

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Abstract. We consider the quadratic form $x^2+mxy+ny^2$, where $|m^2-4n|$ is a prime number. Under the assumption that a particular, small, finite set of integers is representable, we determine all integers representable by this quadratic form.

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1. Introduction

Representation of integers by quadratic forms is a classical problem, with major contributions by Fermat, Euler, Lagrange, and Gauss. We consider those forms that are binary, quadratic, monic, and with a cross term. Specifically, given $m,n \in \mathbb{Z}$ with associated monic quadratic form $f_{m,n}(x,y) := x^2+mxy+ny^2$, we define $\tau = \tau(m,n) = |m^2-4n|$, the absolute value of the discriminant of this form. We will study τ to determine which integers $f_{m,n}(x,y)$ represents.

Modern knowledge about quadratic forms comes from the genus theory of Gauss and also involves the geometry of numbers and factorization over rings of integers of number fields. But it is still interesting to ask how far elementary methods can go for the study of such forms. For instance, although it is well-known that the product of two sums of squares is a sum of squares, the fact that it extends to products of arbitrary monic binary quadratic forms seems to be less-known (cf. [1]).

This problem was revisited by Nair [4] for the form $\tau(1,1) = 3$. More recently, Bahmanpour [1] determined the primes represented by forms $\tau(1,1) = 3$ and $\tau(1,-1) = 5$. We extend these results to all forms with prime τ, provided Condition P holds (as defined below). All the above-mentioned results were obtained by elementary arguments using the fact that the set of integers represented by τ forms a monoid under multiplication of integers (see Lemma 1 below).

We have verified condition P, computationally, for the following values:

$$\tau = 3,5,7,11,13,17,19,23,29,37,41,43,53,61,67,101,163,173,197.$$

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It likely holds for other τ as well, as only $\tau < 200$ were tested. It appears that $\tau = 31$ is the first prime for which Condition P fails.

Our results are restricted to the case where τ is prime. Note that if τ is prime, then m must be odd, and hence τ is also odd. We classify τ into cases via the following.

Definition 1. Let $\tau \in \mathbb{N}$ be an odd prime. We say that τ is *of Type I* if $\tau \equiv 3 \pmod 4$; we say that τ is *of Type II* if $\tau \equiv 1 \pmod 4$.

Note that by some simple case analysis, if $\tau = 4n - m^2$ (i.e. $4n > m^2$), then τ is of Type I. If instead, $\tau = m^2 - 4n$ (i.e. $m^2 > 4n$), then τ is of type II. For $m, n \in \mathbb{Z}$, let $\mathfrak{K}_{m,n} := \{x^2 + mxy + ny^2 : x, y \in \mathbb{Z}\}$ denote the set of representable integers. This set is a monoid under multiplication.

Lemma 1 ([1])**.** *Let $m, n \in \mathbb{Z}$. Then $(\mathfrak{K}_{m,n}, \times)$ is a monoid.*

Proof. Closure follows from the observation that $(a^2 + mab + nb^2)(c^2 + mcd + nd^2) = (ac - nbd)^2 + m(ac - nbd)(bc + ad + mbd) + n(bc + ad + mbd)^2$. Identity follows from $1 = 1^2 + m(1)(0) + n(0)^2$. □

We call $\mathfrak{K}_{m,n}$ *of type I/II*, based on whether $\tau = |m^2 - 4n|$ is of type I/II. Note that the type of $\mathfrak{K}_{m,n}$ is determined solely by τ, independently of choice of m and n. For example, $5 = \tau(1, -1) = \tau(3, 1)$.

The following lemma shows that we may assume that the coefficient of the cross term is one for the quadratic forms that we want to study.

Lemma 2. *Let $m, n \in \mathbb{Z}$ such that $\tau = |m^2 - 4n|$ is odd. We have $\mathfrak{K}_{m,n} = \mathfrak{K}_{1, \frac{1-(m^2-4n)}{4}}$.*

Proof. If τ is odd, then m is also odd. Now suppose $t = a^2 + mab + nb^2 \in \mathfrak{K}_{m,n}$. If we put $c = a - \left(\frac{1-m}{2}\right) b \in \mathbb{Z}$, then we see that $t = c^2 + cb + \frac{1-(m^2-4n)}{4} b^2 \in \mathfrak{K}_{1, \frac{1-(m^2-4n)}{4}}$. Conversely, if $s = x^2 + xy + \frac{1-(m^2-4n)}{4} y^2 \in \mathfrak{K}_{1, \frac{1-(m^2-4n)}{4}}$, then $s = z^2 + mzy + ny^2 \in \mathfrak{K}_{m,n}$, where $z = x + \left(\frac{1-m}{2}\right) y$. □

In view of the above result, we put $\mathfrak{K}_n := \mathfrak{K}_{1,n} = \{x^2 + xy + ny^2 : x, y \in \mathbb{Z}\}$. We define $\mathfrak{K}'_n := \{x^2 + xy + ny^2 : x, y \in \mathbb{Z}, \gcd(x, y) = 1\}$, a subset of $\mathfrak{K}_n$. Note that all nonzero squarefree elements of $\mathfrak{K}_n$ are in $\mathfrak{K}'_n$ (in particular, all primes in $\mathfrak{K}_n$). Theorem 3 will prove that $\mathfrak{K}_n$ depends only on τ and Condition P, to be defined below.

Let τ be an odd prime. We define the set $\mathfrak{P}_\tau$ using Legendre symbols (for this and other standard notation, see [2]), as follows:

$$\mathfrak{P}_\tau := \left\{ p \text{ prime} : p \le \sqrt{\frac{\tau}{3}}, \left(\frac{p}{\tau}\right) = 1 \right\}.$$

Note that $\mathfrak{P}_\tau$ is a finite (possibly empty) set of integers, all prime. Most of our results require the following condition. It states that all elements of $\mathfrak{P}_\tau$ must be

representable:

$$\mathfrak{P}_\tau \subseteq \mathfrak{K}_n. \tag{Condition P}$$

We determine all representable primes (in Theorems 1 and 2). We then find all representable integers (in Theorem 3). A prime turns out to be representable in $\mathfrak{K}_n$ if and only if it is equal to τ or is a quadratic residue modulo τ; a positive integer turns out to be representable if and only if each quadratic nonresidue prime in its factorization appears to an even power.

Computational evidence suggests that our results hold when τ is not prime. However we have been unable to remove either this restriction, or Condition P.

2. Preliminaries

Our first result proves non-representability for roughly half of $\mathbb{Z}$. If $t \in \mathbb{Z}$ is a quadratic nonresidue, then it is not representable.

Theorem 1. *Let $n, t \in \mathbb{Z}$ such that $\tau = |4n-1|$ is prime and $\tau \nmid t$. Suppose that t is a quadratic nonresidue modulo τ. Then $t \notin \mathfrak{K}_n$.*

Proof. By way of contradiction, let us assume the existence of $a, b \in \mathbb{Z}$ with $t = a^2 + ab + nb^2$. Multiplying both sides by 4 and working modulo τ, we have $4t \equiv 4a^2 + 4ab + 4nb^2 \equiv (2a+b)^2 + b^2(4n-1) \equiv (2a+b)^2$. Hence $1 = \left(\frac{4t}{\tau}\right) = \left(\frac{t}{\tau}\right)\left(\frac{2}{\tau}\right)^2 = \left(\frac{t}{\tau}\right) = -1$, a contradiction. □

We next consider the special case of representing prime τ itself. If τ is of Type I, then τ can always be represented as $f_{1,n}(-1,2) = (-1)^2 + (-1)2 + 4n = -1 + 4n = \tau$. If τ is of Type II, there is no such immediate formula. For example, $37 = \tau(1,-9) \in \mathfrak{K}_{-9}$ as $37 = f_{1,-9}(31,12)$. For the slightly larger prime $97 = \tau(1,-24) \in \mathfrak{K}_{-24}$ as $97 = f_{1,-24}(-60797, 11208)$. But the same expression $f_{1,n}(-1,2)$ as above shows that $-\tau$ is represented when τ is of Type II. We thus obtain representability for τ in this case if we can verify representability for -1. We attain this using the following known result, whose proof will be included for completeness.

Lemma 3. *Let p be a prime with $p \equiv 1 \pmod 4$. Then there exist $x, y \in \mathbb{N}$ such that $x^2 - py^2 = -1$.*

Proof. By a theorem of Lagrange (cf. e.g. [3, Thm. 1, p. 53]), there exist infinitely many pairs of positive integers $(X, Y) \neq (1, 0)$ that satisfies the Pell equation

$$X^2 - pY^2 = 1. \tag{2.1}$$

Consider such a pair (X_0, Y_0) with minimal X_0. Note that X_0 and Y_0 are relatively prime. Since $p \equiv 1 \pmod 4$, X_0 must be odd and Y_0 is even. Write $Y_0 = 2W$. Then we may write equation (2.1) as

$$\frac{X_0+1}{2} \cdot \frac{X_0-1}{2} = pW^2.$$

Since $\frac{X_0+1}{2} - \frac{X_0-1}{2} = 1$, the integers $\frac{X_0+1}{2}$ and $\frac{X_0-1}{2}$ must be relatively prime. Hence, there exist relatively prime positive integers a and b such that

$$\begin{cases} \frac{X_0+1}{2} = a^2 \\ \frac{X_0-1}{2} = pb^2 \end{cases} \qquad \text{or} \qquad \begin{cases} \frac{X_0+1}{2} = pa^2 \\ \frac{X_0-1}{2} = b^2. \end{cases}$$

In the first case we have $X_0 = 2a^2 - 1 = 2pb^2 + 1$, which gives $a^2 - pb^2 = 1$. But we also have

$$1 \le a \le a^2 \le 2a^2 - 1 = X_0.$$

This contradicts the minimality of X_0 and the assumption that $X_0 \neq 1$. Hence we must have $X_0 = 2pa^2 - 1 = 2b^2 + 1$, which gives $b^2 - pa^2 = -1$. □

Lemma 4. *Let $n \in \mathbb{Z}$ such that $\tau = 1 - 4n$ is a prime of type II. Then $-1 \in \mathfrak{K}_n$.*

Proof. Let $x, y \in \mathbb{N}$ such that $x^2 - \tau y^2 = -1$. Then we find $f_{1,n}(-x-y, 2y) = (x+y)^2 - (x+y)(2y) + 4ny^2 = x^2 - (1-4n)y^2 = -1$. □

Using Lemma 4, we see that monoids of type II are nicely symmetric around 0.

Lemma 5. *Let $m, n \in \mathbb{Z}$ such that $\tau = 1 - 4n$ is a prime of type II. Then for every $t \in \mathbb{Z}$, $t \in \mathfrak{K}_n$ if and only if $-t \in \mathfrak{K}_n$. In particular, $\tau \in \mathfrak{K}_n$.*

Proof. Apply Lemma 1 to $t = (-1)(-t)$ and $-t = (-1)(t)$. □

If instead, the monoid is of type I, then it contains no negative integers at all.

Lemma 6. *Let $n \in \mathbb{Z}$ such that $\tau = 4n - 1$ is a prime of type I. Then $\mathfrak{K}_n \subseteq \mathbb{N}_0$.*

Proof. Let $a, b \in \mathbb{Z}$. Since $4n > 1$, we must have $n > 0$. Set $s = \frac{1}{\sqrt{n}}$, and $c = b\sqrt{n}$. We have $a^2 + ab + nb^2 = a^2 + sac + c^2 = \frac{2+s}{4}(a+c)^2 + \frac{2-s}{4}(a-c)^2$. Since $4n > 1$, $|s| < 2$ and hence both $\frac{2+s}{4}$ and $\frac{2-s}{4}$ are positive. Thus $a^2 + ab + nb^2 \ge 0$, with equality only for $a = b = 0$. □

3. Representing quadratic residues

We turn now to the question of representing quadratic residues. This is less straightforward than Theorem 1, as not all quadratic residues are representable. In several steps we prove Theorem 2, which resolves representation of primes that are quadratic residues.

The next lemma, relying on the law of quadratic reciprocity, is the starting point toward Theorem 2. It will be needed for all primes except 2 and τ.

Lemma 7. *Let $n \in \mathbb{Z}$ such that $\tau = |1 - 4n|$ is a prime. Let p be any odd prime different from τ. Then p is a quadratic residue modulo τ if and only if $1 - 4n$ is a quadratic residue modulo p.*

Proof. Suppose first that $1-4n>0$. By quadratic reciprocity,

$$1=\left(\frac{1-4n}{p}\right)\left(\frac{p}{1-4n}\right),$$

since $1-4n\equiv 1 \pmod 4$. On the other hand, if $1-4n<0$, then

$$(-1)^{(p-1)/2}=\left(\frac{\tau}{p}\right)\left(\frac{p}{\tau}\right),$$

since $\tau\equiv -(1-4n)\equiv -1 \pmod 4$. But also

$$\left(\frac{\tau}{p}\right)=\left(\frac{-1}{p}\right)\left(\frac{1-4n}{p}\right)=(-1)^{(p-1)/2}\left(\frac{1-4n}{p}\right).$$

This implies

$$1=\left(\frac{1-4n}{p}\right)\left(\frac{p}{\tau}\right).$$

□

Our approach to prove that some $p\in\mathfrak{K}_n$ will be to start with $pt\in\mathfrak{K}_n$ for some integer t. The following strong lemma shows that if p is a quadratic nonresidue modulo τ, then not only is p not in $\mathfrak{K}'_n$, but no multiple of p is in $\mathfrak{K}'_n$ either. It also gives examples of nonrepresentible quadratic residues. If p and q are distinct primes which are quadratic nonresidues modulo τ, then $pq\notin\mathfrak{K}'_n$. It follows that $pq\notin\mathfrak{K}_n$, even though pq is a quadratic residue.

Lemma 8. *Let $n\in\mathbb{Z}$ such that $\tau=|1-4n|$ is a prime. Let p be any odd prime different from τ and let $t\in\mathbb{Z}$. If p is a quadratic nonresidue modulo τ, then $pt\notin\mathfrak{K}'_n$.*

Proof. We assume by way of contradiction the existence of $a,b\in\mathbb{Z}$ with $pt=a^2+ab+nb^2$ and $\gcd(a,b)=1$. If $p|b$, then $p|(pt-ab-nb^2)$, so $p|a$, which contradicts $\gcd(a,b)=1$. Hence $p\nmid b$, and we can choose an integer c so that $cb\equiv 1 \pmod p$. Working modulo p, we have $0\equiv a^2+ab+nb^2\equiv b^2((ac)^2+(ac)+n)$. Hence $0\equiv 4((ac)^2+(ac)+n)\equiv(2ac+1)^2+4n-1$, and thus $(2ac+1)^2\equiv 1-4n \pmod p$. Thus $1-4n$ is a quadratic residue, modulo p. By Lemma 7, p is a quadratic residue modulo τ; this contradicts the hypothesis. □

Since every odd prime τ has quadratic nonresidues, we can apply Dirichlet's theorem on arithmetic progressions to find some odd prime $p\neq\tau$ that is a quadratic nonresidue modulo τ. Applying Lemma 8 to this p and to $t=0$, implies that $0\notin\mathfrak{K}'_n$.

We now present an analogue of Lemma 8 for $p=2$.

Lemma 9. *Let $n,t\in\mathbb{Z}$ such that $\tau=|1-4n|$ is a prime. If 2 is a quadratic nonresidue modulo τ, then $4t\notin\mathfrak{K}'_n$.*

Proof. Since $\left(\frac{2}{\tau}\right) = -1$, by the second supplement to the law of quadratic reciprocity, we must have $|1-4n| = \tau \equiv \pm 3 \pmod 8$. A simple case analysis shows that n is odd. Assume by way of contradiction the existence of $a, b \in \mathbb{Z}$ with $4t = a^2 + ab + nb^2$ and $\gcd(a,b) = 1$. In particular, a, b cannot both be even. If a, b are both odd, then $a^2 + ab + nb^2$ is also odd, a contradiction. If b is odd and $a = 2k$ is even, we have $0 \equiv (2k)^2 + (2k)b + nb^2 \equiv b(2k + nb) \pmod 4$. But now $4|(2k + nb)$, so nb is even and hence b is even, a contradiction. Lastly, if a is odd and $b = 2j$ is even, we have $0 \equiv a^2 + a(2j) + n(2j)^2 \equiv a(a + 2j) \pmod 4$. But now $4|(a + 2j)$, so a is even, a contradiction. □

We now represent, not yet an arbitrary prime, but some integer multiple thereof. The condition $p > \sqrt{\frac{\tau}{3}}$ is why most of Condition P is imposed. An improvement here would equally improve Condition P.

Lemma 10. *Let $n \in \mathbb{Z}$ such that $\tau = |1-4n|$ is a prime. Let p be any odd prime different from τ. If p is a quadratic residue modulo τ, then $pt \in \mathfrak{K}'_n$, for some $t \in \mathbb{Z}$. If $p > \sqrt{\frac{\tau}{3}}$, then we may assume that $0 < |t| < p$.*

Proof. By Lemma 7, there is some $r \in \mathbb{Z}$ such that $r^2 \equiv 1 - 4n \pmod p$. Choose $s \in \mathbb{Z}$ such that $2s + 1 \equiv r \pmod p$. We have $4s^2 + 4s + 4n \equiv 0 \pmod p$, and hence $s^2 + s + n \equiv 0 \pmod p$. Hence, for some $t' \in \mathbb{Z}$, there is a representation $t'p = f_{1,n}(s, 1)$.

We return now to the choice of s, and try to find a different choice (but still equivalent modulo p), which will make $|t'|$ small. Consider the quadratic real polynomial $g(x) = (s + xp)^2 + (s + xp) + n$. For all $x \in \mathbb{Z}$, $g(x)$ will not only be integer-valued, but a multiple of p. The graph of $g(x)$ has vertex at $k' = -\frac{2s+1}{2p}$. We calculate $g(k') = \frac{4n-1}{4}$, and $g(k' + \frac{1}{2}) = g(k' - \frac{1}{2}) = \frac{4n-1}{4} + \frac{p^2}{4}$. Choose an integer $k \in [k' - \frac{1}{2}, k' + \frac{1}{2}]$. We have $g(k) \in [\frac{4n-1}{4}, \frac{4n-1}{4} + \frac{p^2}{4}]$. Hence, $|g(k)| \leq |\frac{4n-1}{4}| + |\frac{p^2}{4}| = \frac{\tau}{4} + \frac{p^2}{4} < \frac{3p^2}{4} + \frac{p^2}{4} = p^2$. Hence, for some t with $|t| < p$, we have $tp = g(k) = f_{1,n}(s + kp, 1)$. We have $t \neq 0$ since $0 \notin \mathfrak{K}'_n$. □

This next lemma is a generalization of a result found in [4]. It shows that if a prime p and pt, for some $t \in \mathbb{Z}$, are both representable, then t is also representable.

Lemma 11. *Let $n \in \mathbb{Z}$, $t \in \mathbb{N}$ and p be a prime. If $tp, p \in \mathfrak{K}_n$, then $t \in \mathfrak{K}_n$.*

Proof. By hypothesis, there are integers a, b, c, d with $tp = a^2 + ab + nb^2$ and $p = c^2 + cd + nd^2$. In fact we have $\gcd(c, d) = 1$, since p is prime. We calculate $b^2p - d^2tp = b^2(c^2 + cd + nd^2) - d^2(a^2 + ab + nb^2) = (bc - ad)(bd + bc + ad)$. Hence either $p|(bc - ad)$ or $p|(bd + bc + ad)$, which splits the proof into two cases.

Suppose first that $p|(bc - ad)$. There is some $r \in \mathbb{Z}$ with $rp = bc - ad$. Set $y = a + rnd$ and $x = b - rc$. We substitute for a, b to get $rp = r(c^2 + cd + nd^2) - rcd -$

$yd + xc$, so $0 = -rcd - yd + xc$ and hence $c(x - rd) = dy$. Since $\gcd(c,d) = 1$, there is some $w \in \mathbb{Z}$ with $y = cw$. Substituting, we get $x = d(w + r)$. Hence $a = cw - rnd$ and $b = d(w + r) + rc$. Since $(w^2 + wr + nr^2)(c^2 + cd + nd^2) = (cw - rnd)^2 + (cw - rnd)(d(w + r) + rc) + n(d(w + r) + rc)^2 = a^2 + ab + nb^2 = tp$, we find that $t = w^2 + wr + nr^2 \in \mathfrak{K}_n$.

Suppose now that $p|(bd + bc + ad)$. There is some $r \in \mathbb{Z}$ with $rp = bd + bc + ad$. Set $y = a - rnd$ and $x = b - rc$. We substitute for a, b to get $rp = r(c^2 + cd + nd^2) + xd + xc + yd$, so $0 = xd + xc + yd$ and hence $d(x + y) = c(-x)$. Since $\gcd(c,d) = 1$, there is some $w \in \mathbb{Z}$ with $-x = dw$. Substituting, we get $y = w(d + c)$. Hence $a = w(d + c) + rnd$ and $b = -dw + rc$. Since $(w^2 + wr + nr^2)(c^2 + cd + nd^2) = (w(d + c) + rnd)^2 + (w(d + c) + rnd)(-dw + rc) + n(-dw + rc)^2 = a^2 + ab + nb^2 = tp$, we find that $t = w^2 + wr + nr^2 \in \mathfrak{K}_n$. □

We now prove that all primes that are quadratic residues are representable, subject to Condition P.

Theorem 2. *Let $n \in \mathbb{Z}$ such that $\tau = |1 - 4n|$ is a prime. Suppose that Condition P holds. Then $p \in \mathfrak{K}_n$ for every prime p that is a quadratic residue modulo τ.*

Proof. By way of contradiction, let p be the smallest prime with $\left(\frac{p}{\tau}\right) = 1$ and $p \notin \mathfrak{K}_n$. If $p \leq \sqrt{\frac{\tau}{3}}$, then $p \in \mathfrak{P}_\tau$, which contradicts Condition P.

We now choose $t \in \mathbb{Z}$ to have minimal absolute value to satisfy both $0 < |t| < p$ and $pt \in \mathfrak{K}'_n$. Note that such a t exists by Lemma 10.

If $t = 1$ we contradict $p \notin \mathfrak{K}_n$. If τ is of Type I, $t = -1$ is impossible by Lemma 6. Suppose τ is of Type II and $t = -1$. From Lemma 4 we have $-1 \in \mathfrak{K}_n$. Thus Lemma 1 gives $p = (-1)(tp) \in \mathfrak{K}_n$, a contradiction. Hence we may assume that $|t| > 1$ and write $|t| = p_1 p_2 \cdots p_k$, a product of (not necessarily distinct) primes, each less than p.

Suppose that some $p_i \in \mathfrak{K}_n$; we will show that this is impossible. By Lemma 11, $p\frac{t}{p_i} \in \mathfrak{K}_n$. Hence $p\frac{t}{p_i} = a^2 + ab + nb^2$ for some $a, b \in \mathbb{Z}$. We have $\gcd(a,b)^2 | p\frac{t}{p_i}$. If $\gcd(a,b) = p$, then $p^2 | pt$, a contradiction. Hence $\gcd(a,b)^2 | \frac{t}{p_i}$. We now have $p\frac{t}{p_i \gcd(a,b)^2} = (\frac{a}{\gcd(a,b)})^2 + (\frac{a}{\gcd(a,b)})(\frac{b}{\gcd(a,b)}) + n(\frac{b}{\gcd(a,b)})^2 \in \mathfrak{K}'_n$. This contradicts our choice of t. Hence each $p_i \notin \mathfrak{K}_n$ and in particular $p_i \neq \tau$.

Consequently each p_i is a quadratic nonresidue modulo τ, otherwise by our choice of p we must have $p_i \in \mathfrak{K}_n$. If any p_i were odd, we would obtain a contradiction to Lemma 8. Hence $t = 2^c$ for some $c \in \mathbb{N}$, where 2 is a quadratic nonresidue modulo τ. If $c = 1$, then $tp = 2p$ is the product of a quadratic nonresidue and a quadratic residue. Hence tp is a quadratic nonresidue, and by Theorem 1, $tp \notin \mathfrak{K}_n$, a contradiction. Hence $c \geq 2$, but by Lemma 9, $tp = 4(2^{c-2}p) \notin \mathfrak{K}'_n$, a contradiction. □

We can now reproduce the known results. For the known $\tau(1,1) = 3$ of Type I and $\tau(-1,-1) = \tau(1,-1) = 5$ of type II, we have $\mathfrak{P}_3 = \varnothing$ and $\mathfrak{P}_5 = \varnothing$, respectively. Thus, condition P holds vacuously for these two examples. For another example, take $17 = \tau(3,-2) = \tau(1,-4)$. We have $\mathfrak{P}_{17} = \{2\}$. Then condition P is verified since $2 = f_{1,-4}(2,1)$. We also have $2 = f_{3,-2}(1,1)$ by inspection (or using the proof of Lemma 2).

4. Complete determination of representable integers

We turn now to the question of characterizing irreducibles in the monoid $\mathfrak{K}_n$. Due to Lemmas 5 and 6, we concern ourselves only with irreducibles in $\mathfrak{K}_n \cap \mathbb{N}$, itself a monoid.

Lemma 12. *Let $n \in \mathbb{Z}$ such that $\tau = |1-4n|$ is a prime. Suppose that Condition P holds. Then the irreducibles in the monoid $\mathfrak{K}_n \cap \mathbb{N}$ are exactly those integers of the form:*

(1) τ,
(2) *p, where p is prime and a quadratic residue modulo τ; and*
(3) *q^2, where q is prime and a quadratic nonresidue modulo τ.*

Proof. We have $\tau \in \mathfrak{K}_n$ from Section 2 and $p \in \mathfrak{K}_n$ by Theorem 2. We have $q^2 = f_{1,n}(q,0) \in \mathfrak{K}_n$; it is irreducible by Theorem 1. Now let $t \in \mathfrak{K}_n$ be some other irreducible. Write $t = p_1 p_2 \cdots p_k$, for not necessarily distinct primes p_i. We must have $k \geq 2$ by Theorem 1 again. If any $p_i \in \mathfrak{K}_n$, then by Lemma 11, $\frac{t}{p_i} \in \mathfrak{K}_n$, which contradicts irreducibility. In particular, by Condition P, no p_i can be τ. If any p_i is odd, then by Lemma 8, $t \notin \mathfrak{K}'_n$. But then, writing $t = a^2 + ab + nb^2$, there is some prime r dividing $\gcd(a,b)$. We have $r^2 = f_{1,n}(r,0)$ and $\frac{t}{r^2} = (\frac{a}{r})^2 + (\frac{a}{r})(\frac{b}{r}) + n(\frac{b}{r})^2$. But $\frac{t}{r^2} > 1$ since t is not among the two types of irreducibles already described. Hence t is reducible, which is a contradiction. The remaining possibility is that t is a power of 2, where 2 is a quadratic nonresidue modulo τ. An even power of 2 may be written as a product of irreducibles 2^2, while an odd power of 2 is not in $\mathfrak{K}_n$ by Theorem 1. Hence no such t can exist. □

With the irreducibles we may easily determine the full monoid $\mathfrak{K}_n$. The statement of Theorem 3 is similar to a well-known theorem on representing integers as the sum of two squares, i.e. the quadratic form $4 = \tau(0,1)$. We recall also Lemmas 5 and 6, which combine with Theorem 3 to resolve the membership question for negative integers.

Theorem 3. *Let $n \in \mathbb{Z}$ such that $\tau = |1-4n|$ is a prime. Suppose that Condition P holds. Let $t \in \mathbb{N}$. Then $t \in \mathfrak{K}_n$ if and only if the prime decomposition of t contains no prime, that is a quadratic nonresidue modulo τ, raised to an odd power.*

Proof. Immediate from Lemma 12. □

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Authors' addresses

Jerome T. Dimabayao
Institute of Mathematics, College of Science, University of the Philippines Diliman, C.P. Garcia St., U.P.Campus, Diliman, 1101 Quezon City, Philippines
E-mail address: jdimabayao@math.upd.edu.ph

Vadim Ponomarenko
Department of Mathematics and Statistics, San Diego State University, 5500 Campanile Drive, San Diego California, USA
E-mail address: vponomarenko@mail.sdsu.edu

Orland James Q. Tigas
Institute of Mathematics, College of Science, University of the Philippines Diliman, C.P. Garcia St., U.P.Campus, Diliman, 1101 Quezon City, Philippines
E-mail address: oqtigas@upd.edu.ph

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ON INTEGRAL INEQUALITIES OF HERMITE–HADAMARD TYPE FOR COORDINATED r-MEAN CONVEX FUNCTIONS

DAN-DAN GAO, BO-YAN XI, YING WU, AND BAI-NI GUO

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Abstract. In the paper, the authors first introduce a concept "r-mean convex function on coordinates" and then establish several integral inequalities of the Hermite–Hadamard type for r-convex functions and r-mean convex functions on coordinates.

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1. Introduction

The following definition is well known in the literature.

Definition 1. A function $h : I \subseteq \mathbb{R} \to \mathbb{R}$ is called to be convex if the inequality

$$h(\lambda x + (1-\lambda)y) \le \lambda h(x) + (1-\lambda)h(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$. If the above inequality is reversed, then we call f a concave function.

If $h : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on $[a,b]$ with $a, b \in I$ and $a < b$, then

$$h\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t \le \frac{h(a)+h(b)}{2}. \tag{1.1}$$

This inequality is called Hermite-Hardamard's integral inequality in the literature.

Hermite–Hadamard's integral inequality (1.1) has been refined, generalized, and applied by a number of mathematicians. For more information, please refer to [7, 12, 25], for example.

Definition 2 ([2, 3])**.** A real function h defined on a convex set $\mathbb{D} \subseteq \mathbb{R}^n$ is said to be r-convex if

$$\phi(q_1 x + q_2 y) \le \begin{cases} \ln\left(q_1 e^{rh(x)} + q_2 e^{rh(y)}\right)^{1/r}, & r \ne 0 \\ q_1 h(x) + q_2 h(y), & r = 0 \end{cases}$$

for all $x, y \in \mathbb{D}$ and $q_1, q_2 \ge 0$ with $q_1 + q_2 = 1$.

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Definition 3 ([14]). For $r \in \mathbb{R}$, a function $h : I \subseteq \mathbb{R} \to \mathbb{R}_+ = (0,\infty)$ is called to be r-convex if

$$h(\lambda x + (1-\lambda)y) \le \begin{cases} \{\lambda[h(x)]^r + (1-\lambda)[h(y)]^r\}^{1/r}, & r \ne 0 \\ [h(x)]^\lambda [h(y)]^{1-\lambda}, & r = 0 \end{cases} \tag{1.2}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$. If the inequality (1.2) is reversed, then we call h an r-concave function.

Definition 4 ([22, 24]). For $r \in \mathbb{R}$, a function $h : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ is called to be r-mean convex if

$$h\big([\lambda x^r + (1-\lambda)y^r]^{1/r}\big) \le \{\lambda[h(x)]^r + (1-\lambda)[h(y)]^r\}^{1/r}, \quad r \ne 0$$

and

$$h\big(x^\lambda y^{1-\lambda}\big) \le [h(x)]^\lambda [h(y)]^{1-\lambda}, \quad r = 0$$

hold for $x, y \in I$ and $\lambda \in [0,1]$. If the above inequality is reversed, then we call h an r-mean concave function.

Definition 5 ([27]). For $r \in \mathbb{R}$, a function $h : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ is called to be geometrically r-convex if

$$h\big(x^\lambda y^{1-\lambda}\big) \le \begin{cases} \{\lambda[h(x)]^r + (1-\lambda)[h(y)]^r\}^{1/r}, & r \ne 0 \\ [h(x)]^\lambda [h(y)]^{1-\lambda}, & r = 0 \end{cases}$$

holds for $x, y \in I$ and $\lambda \in [0,1]$.

For properties and inequalities of the Hermite–Hadamard type relating to r-convex functions, r-mean convex functions, and geometrically r-convex functions, please read the papers [8, 11, 14, 17, 18, 21, 22, 24, 27, 29, 30] and closely related references.

Definition 6 ([5, 7]). A function $h : \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is called to be convex on coordinates on Δ with $a < b$ and $c < d$ if the partial functions

$$h_y : [a,b] \to \mathbb{R}, \quad h_y(u) = h(u,y) \quad \text{and} \quad h_x : [c,d] \to \mathbb{R}, \quad h_x(v) = h(x,v)$$

are convex for all $x \in [a,b]$ and $y \in [c,d]$.

Definition 7 ([5, 7]). A function $h : \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is called to be convex on coordinates on Δ with $a < b$ and $c < d$ if the inequality

$$\begin{aligned} &h(tx + (1-t)z, \lambda y + (1-\lambda)w) \\ &\qquad \le t\lambda h(x,y) + t(1-\lambda)h(x,w) + (1-t)\lambda h(z,y) + (1-t)(1-\lambda)h(z,w) \end{aligned}$$

holds for all $t, \lambda \in [0,1]$ and $(x,y), (z,w) \in \Delta$.

Definition 8 ([1]). A function $h : \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}_+$ is called coordinated logarithmically convex on Δ with $a < b$ and $c < d$ for all $t, \lambda \in [0,1]$ and $(x,y), (z,w) \in \Delta$ if

$$h(tx+(1-t)z,\lambda y+(1-\lambda)w) \le [h(x,y)]^{t\lambda}[h(x,w)]^{t(1-\lambda)}[h(z,y)]^{(1-t)\lambda}[h(z,w)]^{(1-t)(1-\lambda)}.$$

Definition 9 ([13])**.** For $r \in \mathbb{R}$, a function $h : \Delta = [a,b]\times[c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}_+$ is called coordinated r-convex on Δ with $a<b$ and $c<d$ for all $t,\lambda \in [0,1]$ and $(x,y),(z,w) \in \Delta$ if

$$h(tx+(1-t)z,\lambda y+(1-\lambda)w) \le \begin{cases} \{t\lambda[h(x,y)]^r + t(1-\lambda)[h(x,w)]^r + (1-t)\lambda[h(z,y)]^r \\ \qquad +(1-t)(1-\lambda)[h(z,w)]^r\}^{1/r}, & r \neq 0; \\ [h(x,y)]^{t\lambda}[h(x,w)]^{t(1-\lambda)}[h(z,y)]^{(1-t)\lambda}[h(z,w)]^{(1-t)(1-\lambda)}, & r=0. \end{cases}$$

Remark 1. Obviously, if putting $r=0$ in Definition 9, then h is just the ordinary coordinated logarithmically convex function on Δ.

Stolarsky's mean $E(u,v;r,s)$ for $(u,v;r,s) \in \mathbb{R}_+^2 \times \mathbb{R}^2$ is defined by

$$\begin{aligned} E(u,v;r,s) &= \left[\frac{r(v^s-u^s)}{s(v^r-u^r)}\right]^{1/(s-r)}, && rs(r-s)(u-v) \neq 0; \\ E(u,v;0,s) &= \left[\frac{v^s-u^s}{s(\ln v-\ln u)}\right]^{1/s}, && s(u-v) \neq 0; \\ E(u,s;r,r) &= \frac{1}{e^{1/r}}\left(\frac{u^{u^r}}{v^{v^r}}\right)^{1/(u^r-v^r)}, && r(u-v) \neq 0; \\ E(u,v;0,0) &= \sqrt{uv}, && u \neq v; \\ E(u,u;r,s) &= u, && u=v. \end{aligned}$$

The quantities $L(u,v)=E(u,v;0,1)$ and $L_r(u,v)=E(u,v;r,r+1)$ are respectively called the logarithmic mean and generalized logarithmic mean of two real positive numbers u,v. For more information on Stolarsky's mean, please refer to the papers [9, 10, 15, 16, 19, 20] and closely related references therein.

Theorem 1 ([8, Theorem 2.1])**.** *Suppose that $h : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_+$ is a logarithmically convex function with $a<b$. Then*

$$\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t \le L(h(a),h(b)),$$

where $L(x,y)$ is the logarithmic mean.

Theorem 2 ([8, Theorem 3.1])**.** *Suppose that $h : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_+$ is an r-convex function for $r \in \mathbb{R}$ with $a<b$. Then*

$$\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t \le L_r\big(h(a),h(b)\big),$$

where $L_r(x,y)$ is the generalized logarithmic mean.

Theorem 3 ([5, 7])**.** *Let $h:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2$ be convex on coordinates on Δ with $a<b$ and $c<d$. Then*

$$\begin{aligned}
h\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b h\left(x,\frac{c+d}{2}\right)\mathrm{d}x+\frac{1}{d-c}\int_c^d h\left(\frac{a+b}{2},y\right)\mathrm{d}y\right]\\
&\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d h(x,y)\,\mathrm{d}y\,\mathrm{d}x\\
&\le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b [h(x,c)+h(x,d)]\,\mathrm{d}x+\frac{1}{d-c}\int_c^d [h(a,y)+h(b,y)]\,\mathrm{d}y\right]\\
&\le \frac{1}{4}[h(a,c)+h(b,c)+h(a,d)+h(b,d)].
\end{aligned}$$

Theorem 4 ([1, Theorem 3.3])**.** *Suppose that $h:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}_+$ is logarithmically convex on coordinates Δ for $a<b$ and $c<d$. Let*

$$A=\frac{h(a,c)h(b,d)}{h(b,c)h(a,d)},\quad B=\frac{h(a,d)}{h(b,d)},\quad C=\frac{h(b,c)}{h(b,d)}.$$

Then

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d h(x,y)\,\mathrm{d}y\,\mathrm{d}x\le M_h(\Delta),$$

where

$$M_h(\Delta)=\begin{cases}
1, & A=B=C=1;\\
\frac{B-1}{\ln B}\frac{C-1}{\ln C}, & A=1;\\
H(C), & B=1;\\
H(B), & C=1;\\
\frac{C-1}{\ln C}, & A=B=1;\\
\frac{B-1}{\ln B}, & A=C=1;\\
\frac{\gamma+\ln(-\ln A)+Ei(1,-\ln A)}{\ln A}, & B=C=1;\\
\frac{1}{2}\left[\frac{B-1}{\ln B}+\frac{AB-1}{\ln(AB)}\right], & A,B,C>0;\\
\int_0^1 C^\beta\frac{AB-1}{\ln(AB)}\,\mathrm{d}\beta, & \textit{otherwise},
\end{cases}\tag{1.3}$$

$$\begin{aligned}
H(x)&=\frac{Ei(1,-\ln A)+\ln\ln x-Ei(1,-\ln(Ax))-\ln\ln(Ax)}{\ln A}\\
&\quad+\begin{cases}\frac{2\ln(\ln A)-\ln(-\ln A)}{\ln A}, & -1<\frac{\ln x}{\ln A}<0;\\ 0, & \textit{otherwise},\end{cases}
\end{aligned}$$

$$Ei(x)=V.P.\int_{-x}^{\infty}\frac{e^{-t}}{t}\,\mathrm{d}t,$$

is the exponential integral function, and γ is the Euler constant.

In very recent years, some other kinds of inequalities of the Hermite–Hadamard type were created in, for example, [1,4,6,13,23,26,28,30] and closely related references therein.

In this paper, by combining the definition of convex functions with the definition of coordinated convex functions, we introduce the concept "r-mean convex function on coordinates" and establish integral inequalities of the Hermite–Hadamard type for r-mean convex functions on coordinates.

2. A DEFINITION AND A LEMMA

In this section, we define a concept "r-mean convex function on coordinates" and prepare a lemma necessary for establishing new inequalities of the Hermite–Hadamard type for r-mean convex function on coordinates.

Definition 10. For $r \in \mathbb{R}$, a function $h : \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}_+^2 \to \mathbb{R}_+$ is called a coordinated r-mean convex on Δ with $a < b$ and $c < d$ if

$$h\big([tx^r + (1-t)z^r]^{1/r}, [\lambda y^r + (1-\lambda)w^r]^{1/r}\big) \le \big\{t\lambda[h(x,y)]^r + t(1-\lambda)[h(x,w)]^r + (1-t)\lambda[h(z,y)]^r + (1-t)(1-\lambda)[h(z,w)]^r\big\}^{1/r}$$

for $r \neq 0$ and

$$h\big(x^t z^{1-t}, y^\lambda w^{1-\lambda}\big) \le [h(x,y)]^{t\lambda}[h(x,w)]^{t(1-\lambda)}[h(z,y)]^{(1-t)\lambda}[h(z,w)]^{(1-t)(1-\lambda)}$$

for $r = 0$, where $t, \lambda \in [0,1]$ and $(x,y), (z,w) \in \Delta$.

Remark 2. In Definition 10, if $r = 0$, then we call h a coordinated geometrically convex function on Δ; if $r = -1$, then we call f a coordinated harmonically convex function on Δ.

In order to prove our main theorems, we need the following lemma.

Lemma 1. *Let $r, R, T, S \in \mathbb{R}$ and $r \neq 0$ such that $R + T + S > 0$, $R + S > 0$, $T + S > 0$, and $S > 0$. Then*

$$F(R,T,S,r) \triangleq \int_0^1 \int_0^1 \big(Rt + T\lambda + S\big)^{1/r} \, \mathrm{d}\lambda \, \mathrm{d}t$$

$$
= \begin{cases}
\dfrac{r^2\big[(R+T+S)^{1/r+2}-(R+S)^{1/r+2} \\ \quad -(T+S)^{1/r+2}+S^{1/r+2}\big]}{(r+1)(2r+1)RT}, \quad (r+1)\left(r+\frac{1}{2}\right)RT \neq 0; \\
\frac{r\left[(R+T+S)^{1/r+1}-S^{1/r+1}\right]}{(r+1)(R+T)}, \quad (r+1)(R+T) \neq 0, \quad RT = 0; \\
\dfrac{(R+T+S)\ln(R+T+S)-(R+S)\ln(R+S) \\ \quad -(T+S)\ln(T+S)+S\ln S}{RT}, \quad r=-1, \quad RT \neq 0; \\
\frac{\ln(R+T+S)-\ln S}{R+T}, \quad r=-1, \quad RT=0, \quad R+T \neq 0; \\
\frac{\ln(R+S)+\ln(T+S)-\ln(R+T+S)-\ln S}{RT}, \quad r=-\frac{1}{2}, \quad RT \neq 0; \\
\frac{1}{S(R+T+S)}, \quad r=-\frac{1}{2}, \quad RT=0; \\
S^{1/r}, \quad r \neq 0, \quad R=T=0.
\end{cases}
$$

Proof. This follows from a straightforward computation. □

3. Main results

In this section, we establish some integral inequalities of the Hermite–Hadamard type for r-mean convex functions on coordinates.

Theorem 5. *Suppose that $h : [a,b] \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ is an r-mean convex function on $[a,b]$ for all $r \in \mathbb{R}$ with $r \neq 0$. If $h \in L_1[a,b]$, then*

$$
h\left(\left(\frac{a^r+b^r}{2}\right)^{1/r}\right) \leq \left(\frac{r}{b^r-a^r}\int_a^b \frac{[h(x)]^r}{x^{1-r}}\,\mathrm{d}x\right)^{1/r} \tag{3.1}
$$

and

$$
\frac{r}{b^r-a^r}\int_a^b \frac{h(x)}{x^{1-r}}\,\mathrm{d}x \leq L_r\big(h(a),h(b)\big),
$$

where $L_r(x,y)$ is the generalized logarithmic mean.

Proof. For $t \in [0,1]$, by the r-mean convexity of h on $[a,b]$, we have

$$
\begin{aligned}
h\left(\left(\frac{a^r+b^r}{2}\right)^{1/r}\right) &= h\left(\left[\frac{ta^r+(1-t)b^r+(1-t)a^r+tb^r}{2}\right]^{1/r}\right) \\
&\leq \left(\frac{1}{2}\right)^{1/r}\left\{\big[h([ta^r+(1-t)b^r]^{1/r})\big]^r+\big[h([(1-t)a^r+tb^r]^{1/r})\big]^r\right\}^{1/r}.
\end{aligned}
$$

If $r > 0$, then

$$
\begin{aligned}
&\left[h\left(\left(\frac{a^r+b^r}{2}\right)^{1/r}\right)\right]^r \\
&\qquad \leq \frac{1}{2}\left\{\big[h\big([ta^r+(1-t)b^r]^{1/r}\big)\big]^r+\big[h\big([(1-t)a^r+tb^r]^{1/r}\big)\big]^r\right\}.
\end{aligned}
$$

Integrating with respect to t over $[0,1]$ and putting $x = ta^r + (1-t)b^r$ for $t \in [0,1]$ yield

$$h\left(\left(\frac{a^r+b^r}{2}\right)^{1/r}\right) \le \left(\frac{r}{b^r-a^r}\int_a^b \frac{[h(x)]^r}{x^{1-r}}\,\mathrm{d}x\right)^{1/r}. \tag{3.2}$$

Similarly, if $r<0$, then we obtain the inequality (3.1). Taking $x^r = ta^r + (1-t)b^r$ for $t \in [0,1]$ and utilizing the r-mean convexity of f on $[a,b]$ lead to

$$\begin{aligned}\frac{r}{b^r-a^r}\int_a^b \frac{h(x)}{x^{1-r}}\,\mathrm{d}x &= \int_0^1 h\big([ta^r+(1-t)b^r]^{1/r}\big)\,\mathrm{d}t \\ &\le \int_0^1 \big\{t[h(a)]^r + (1-t)[h(b)]^r\big\}^{1/r}\,\mathrm{d}t = L_r\big(h(a),h(b)\big).\end{aligned}$$

Theorem 5 is thus proved. □

Theorem 6. *Let $h : \Delta = [a,b]\times[c,d] \subseteq \mathbb{R}_+^2 \to \mathbb{R}_+$ and let $a<b$, $r \in \mathbb{R}$ with $r \ne 0$, and $h \in L_1(\Delta)$. If h is a coordinated r-mean convex function on Δ, then*

$$\begin{aligned}&h\left(\left(\frac{a^r+b^r}{2}\right)^{1/r},\left(\frac{c^r+d^r}{2}\right)^{1/r}\right) \\ &\qquad\le \left[\frac{r^2}{(b^r-a^r)(d^r-c^r)}\int_a^b\int_c^d \frac{[h(x,y)]^r}{(xy)^{1-r}}\,\mathrm{d}y\,\mathrm{d}x\right]^{1/r}\end{aligned}$$

and

$$\begin{aligned}&\frac{r^2}{(b^r-a^r)(d^r-c^r)}\int_a^b\int_c^d \frac{h(x,y)}{(xy)^{1-r}}\,\mathrm{d}y\,\mathrm{d}x \\ &\qquad\le \begin{cases} \frac{F(R,T,S,r)}{2^{1/r}}, & r>0; \\ \dfrac{\min\big\{L_r\big(h(a,c),h(a,d)\big)+L_r\big(h(b,c),h(b,d)\big), \\ \quad L_r\big(h(a,c),h(b,c)\big)+L_r\big(h(a,d),h(b,d)\big)\big\}}{2}, & r<0,\end{cases}\end{aligned}$$

where $L_r(x,y)$ is the generalized logarithmic mean, $M_h(\Delta)$ and $F(R,T,S,r)$ are respectively defined as in (1.3) and Lemma 1, and

$$\begin{aligned}R &= h^r(a,c)+h^r(a,d)-h^r(b,c)-h^r(b,d),\\ T &= h^r(a,c)+h^r(b,c)-h(a,d)]^r-h^r(b,d),\\ S &= h^r(b,c)+h^r(a,d)+2h^r(b,d).\end{aligned}$$

Proof. From the coordinated r-mean convexity of h on Δ, we have

$$\begin{aligned}h\left(\left(\frac{a^r+b^r}{2}\right)^{1/r},\left(\frac{c^r+d^r}{2}\right)^{1/r}\right) &= h\left(\left[\frac{ta^r+(1-t)b^r+(1-t)a^r+tb^r}{2}\right]^{1/r},\right.\\ &\qquad\left.\left[\frac{\lambda c^r+(1-\lambda)d^r+(1-\lambda)c^r+\lambda d^r}{2}\right]^{1/r}\right)\end{aligned}$$

$$\leq \left(\frac{1}{4}\right)^{1/r} \{[h([ta^r+(1-t)b^r]^{1/r}, [\lambda c^r+(1-\lambda)d^r]^{1/r})]^r$$
$$+[h([ta^r+(1-t)b^r]^{1/r}, [(1-\lambda)c^r+\lambda d^r]^{1/r})]^r$$
$$+[h([(1-t)a^r+tb^r]^{1/r}, [\lambda c^r+(1-\lambda)d^r]^{1/r})]^r$$
$$+[h([(1-t)a^r+tb^r]^{1/r}, [(1-\lambda)c^r+\lambda d^r]^{1/r})]^r\}^{1/r}$$

for all $t,\lambda \in [0,1]$. As did in the proof of the inequality (3.2), we can obtain

$$h\left(\left[\frac{a^r+b^r}{2}\right]^{1/r}, \left[\frac{c^r+d^r}{2}\right]^{1/r}\right) \leq \left[\frac{r^2}{(b^r-a^r)(d^r-c^r)}\int_a^b\int_c^d \frac{[h(x,y)]^r}{(xy)^{1-r}}\,\mathrm{d}y\,\mathrm{d}x\right]^{1/r}.$$

Letting $x^r = ta^r+(1-t)b^r$ and $y^r = tc^r+(1-t)d^r$ for $t,\lambda \in [0,1]$ and using the coordinated r-mean convexity of h on Δ lead to

$$\frac{r^2}{(b^r-a^r)(d^r-c^r)}\int_a^b\int_c^d \frac{h(x,y)}{(xy)^{1-r}}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\int_0^1 h([ta^r+(1-t)b^r]^{1/r},$$
$$[\lambda c^r+(1-\lambda)d^r]^{1/r})\,\mathrm{d}t\,\mathrm{d}\lambda \leq \int_0^1\int_0^1 \{t\lambda h^r(a,c)+t(1-\lambda)h^r(a,d)$$
$$+(1-t)\lambda h^r(b,c)+(1-t)(1-\lambda)h^r(b,d)\}^{1/r}\,\mathrm{d}t\,\mathrm{d}\lambda. \tag{3.3}$$

(1) When $r>0$, since $2\lambda t \leq \lambda+t$ for all $\lambda,t \in [0,1]$, by Lemma 1 and the inequality (3.3), we have

$$\frac{r^2}{(b^r-a^r)(d^r-c^r)}\int_a^b\int_c^d \frac{h(x,y)}{(xy)^{1-r}}\,\mathrm{d}y\,\mathrm{d}x \leq \int_0^1\int_0^1 \{t\lambda h^r(a,c)$$
$$+t(1-\lambda)h^r(a,d)+(1-t)\lambda h^r(b,c)+(1-t)(1-\lambda)h^r(b,d)\}^{1/r}\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$\leq \frac{1}{2^{1/r}}\int_0^1\int_0^1 \{(t+\lambda)h^r(a,c)+(t+(1-\lambda))h^r(a,d)+((1-t)+\lambda)h^r(b,c)$$
$$+((1-t)+(1-\lambda))h^r(b,d)\}^{1/r}\,\mathrm{d}t\,\mathrm{d}\lambda = \frac{1}{2^{1/r}}F(R,T,S,r).$$

(2) When $r<0$, for $x,y>0$, using the convexity of $g(\lambda) = [\lambda x+(1-\lambda)y]^{1/r}$ on $[0,1]$ and by the inequality (3.3), we obtain

$$\frac{r^2}{(b^r-a^r)(d^r-c^r)}\int_a^b\int_c^d \frac{h(x,y)}{(xy)^{1-r}}\,\mathrm{d}y\,\mathrm{d}x \leq \int_0^1\int_0^1 \{t[\lambda h^r(a,c)$$
$$+(1-\lambda)h^r(a,d)]+(1-t)[\lambda h^r(b,c)+(1-\lambda)h^r(b,d)]\}^{1/r}\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$\leq \int_0^1\int_0^1 \{t[\lambda h^r(a,c)+(1-\lambda)h^r(a,d)]^{1/r}+(1-t)[\lambda h^r(b,c)$$

$$+(1-\lambda)h^r(b,d)]^{1/r}\}\,\mathrm{d}t\,\mathrm{d}\lambda = \frac{1}{2}\int_0^1 \big[\{\lambda h^r(a,c)+(1-\lambda)h^r(a,d)\}^{1/r}$$
$$+\{\lambda h^r(b,c)+(1-\lambda)h^r(b,d)\}^{1/r}\big]\,\mathrm{d}\lambda$$
$$= \frac{L_r(h(a,c),h(a,d))+L_r(h(b,c),h(b,d))}{2}.$$

Similarly, we have

$$\frac{r^2}{(b^r-a^r)(d^r-c^r)}\int_a^b\int_c^d \frac{h(x,y)}{(xy)^{1-r}}\,\mathrm{d}y\,\mathrm{d}x \le \int_0^1\int_0^1 \{\lambda[th^r(a,c)$$
$$+(1-t)h^r(b,c)]+(1-\lambda)[th^r(a,d)+(1-t)h^r(b,d)]\}^{1/r}\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$\le \frac{L_r(h(a,c),h(b,c))+L_r(h(a,d),h(b,d))}{2}.$$

Theorem 6 is thus proved. □

Theorem 7. *Suppose that $h:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}_+^2\to\mathbb{R}_+$ is geometrically convex on coordinates Δ for $a<b$ and $c<d$. Then*

$$h\big(\sqrt{ab},\sqrt{cd}\big) \le \frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\int_a^b\int_c^d \frac{h(x,y)}{xy}\,\mathrm{d}y\,\mathrm{d}x \le M_h(\Delta),$$

where $M_h(\Delta)$ is defined as in (1.3).

Proof. Since h is geometrically convex on coordinates Δ, we have

$$h\big(\sqrt{ab},\sqrt{cd}\big) = \int_0^1\int_0^1 h\big((a^tb^{1-t})^{1/2}(a^{1-t}b^t)^{1/2},$$
$$(c^\lambda d^{1-\lambda})^{1/2}(c^{1-\lambda}d^\lambda)^{1/2}\big)\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$\le \int_0^1\int_0^1 \big[h\big(a^tb^{1-t},c^\lambda d^{1-\lambda}\big)h\big(a^tb^{1-t},c^{1-\lambda}d^\lambda\big)$$
$$\times h\big(a^{1-t}b^t,c^\lambda d^{1-\lambda}\big)h\big(a^{1-t}b^t,c^{1-\lambda}d^\lambda\big)\big]^{1/4}\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$\le \frac{1}{4}\int_0^1\int_0^1 \big[h(a^tb^{1-t},c^\lambda d^{1-\lambda})+h\big(a^tb^{1-t},c^{1-\lambda}d^\lambda\big)$$
$$+h\big(a^{1-t}b^t,c^\lambda d^{1-\lambda}\big)+h\big(a^{1-t}b^t,c^{1-\lambda}d^\lambda\big)\big]\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$= \frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\int_a^b\int_c^d \frac{h(x,y)}{xy}\,\mathrm{d}y\,\mathrm{d}x.$$

Putting $x=a^tb^{1-t}$ and $y=c^\lambda d^{1-\lambda}$ for $0\le t$ and $\lambda\le 1$, utilizing the coordinated geometric convexity of h, and employing Theorem 4 result in

$$\frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\int_a^b\int_c^d \frac{h(x,y)}{xy}\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^1 \int_0^1 h\big(a^t b^{1-t}, c^\lambda d^{1-\lambda}\big)\,\mathrm{d}\lambda\,\mathrm{d}t$$

$$\le \int_0^1 \int_0^1 [h(a,c)]^{t\lambda}[h(a,d)]^{t(1-\lambda)}[h(b,c)]^{(1-t)\lambda}[h(b,d)]^{(1-t)(1-\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}t$$

$$= M_h(\Delta).$$

Theorem 7 is thus proved. □

Corollary 1. *Suppose that $h : \Delta = [a,b]\times[c,d] \subseteq \mathbb{R}_+^2 \to \mathbb{R}_+$ is coordinated geometrically convex on Δ for $a < b$ and $c < d$. Then*

$$h\big(\sqrt{ab}, \sqrt{cd}\big) \le \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_a^b \int_c^d \frac{h(x,y)}{xy}\,\mathrm{d}y\,\mathrm{d}x$$
$$\le \frac{h(a,c)+h(a,d)+h(b,c)+h(b,d)}{4}.$$

Theorem 8. *Suppose that $h : \Delta = [a,b]\times[c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}_+$ is logarithmically convex on coordinates Δ for $a < b$ and $c < d$. Then*

$$h\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \le \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d h(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

Proof. Since h is geometrically convex on coordinates Δ, we have

$$h\left(\frac{a+b}{2}, \frac{c+d}{2}\right) = \int_0^1 \int_0^1 h\left(\frac{ta+(1-t)b+(1-t)a+tb}{2}, \frac{\lambda c+(1-\lambda)d+(1-\lambda)c+\lambda d}{2}\right)\mathrm{d}t\,\mathrm{d}\lambda$$

$$\le \int_0^1 \int_0^1 \big[h\big(ta+(1-t)b, \lambda c+(1-\lambda)d\big)h\big(ta+(1-t)b, (1-\lambda)c+\lambda d\big)$$
$$\times h\big((1-t)a+tb, \lambda c+(1-\lambda)d\big)h\big((1-t)a+tb, (1-\lambda)c+\lambda d\big)\big]^{1/4}\,\mathrm{d}t\,\mathrm{d}\lambda$$

$$\le \frac{1}{4}\int_0^1 \int_0^1 \big[h\big(ta+(1-t)b, \lambda c+(1-\lambda)d\big)+h\big(ta+(1-t)b, (1-\lambda)c+\lambda d\big)$$
$$+h\big((1-t)a+tb, \lambda c+(1-\lambda)d\big)+h\big((1-t)a+tb, (1-\lambda)c+\lambda d\big)\big]\,\mathrm{d}t\,\mathrm{d}\lambda$$

$$= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d h(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

Theorem 8 is thus proved. □

Corollary 2. *Under conditions of Theorems 4 and 8, we have*

$$h\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \le \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d h(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

$$\leq M_h(\Delta) \leq \frac{h(a,c)+h(a,d)+h(b,c)+h(b,d)}{4}.$$

where $M_h(\Delta)$ is defined as in (1.3).

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Authors' addresses

Dan-Dan Gao
College of Mathematics, Inner Mongolia University for Nationalities, Tongliao 028043, China
E-mail address: Dan-DanGao@hotmail.com

Bo-Yan Xi
College of Mathematics, Inner Mongolia University for Nationalities, Tongliao 028043, China
E-mail address: baoyintu78@qq.com

Ying Wu
College of Mathematics, Inner Mongolia University for Nationalities, Tongliao 028043, China
E-mail address: wuying19800920@qq.com

Bai-Ni Guo
(Corresponding author) School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo 454010, China
E-mail address: bai.ni.guo@gmail.com

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A PERIODIC SOLUTION OF THE COUPLED MATRIX RICCATI DIFFERENTIAL EQUATIONS

ZAHRA GOODARZI, ABDOLRAHMAN RAZANI, AND M.R. MOKHTARZADEH

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Abstract. Here, we generalize the martix Riccati differential equation to the coupled matrix Riccati differential equation. Using Schauder's fixed point theorem, the existence of at least one periodic solution of the coupled matrix Riccati equation with $n \times n$ matrix coefficients is proved. Finally, two numerical examples are presented.

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Keywords: coupled matrix Riccati differential equation, Riccati differential equation, periodic solution, compact operator, Schauder's fixed point theorem

1. Introduction

Generally, Riccati differential equation with real valued functions on $\mathbb{R}$ coefficients is $y' + p(t)y^2 + q(t)y = r(t)$. This equation has many applications in different field of sciences such as quantum mechanics in physics, control theory, biomathematics, fluid mechanics, the theory of elastic vibration and econometrics [5, 12, 27]. An extensive studies of the set of periodic solutions and nonexistence of periodic solutions are investigated. Also, there are some papers where stability and asymptotic behaviour of solutions were considered. Recently, Ni [17] study the existence and stability of two periodic solutions on a class of Riccati differential equation. Also, Guillot [9], exhibit some families of Riccati differential equations in the complex domain having elliptic coefficients and study the problem of understanding the cases where there are no multi-valued solutions. For more historical background and the existence of solution for this kind of equation see [3, 4, 8, 11, 18].

The term Riccati equation is used to refer to matrix equations with an analogous quadratic term, which occur in both continuous-time and discrete-time linear-quadratic-Gaussian control. The steady-state (non-dynamic) version of it is referred to as the algebraic Riccati equation. In a standard manner Riccati equation can be reduced to a second-order linear ordinary differential equations or to a Schrodinger equation of quantum mechanics. In fact, Riccati equation naturally arises in many

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fields of quantum mechanics, in particular, in quantum chemistry, the Wentzel-Kramers-Brillouin approximation and SUSY theories. Also, methods for solving the Gross-Pitaevskii equation arising in Bose-Einstein condensates based on Riccati equation are introduced. Hacched et al. [10] consider large scale nonsymmetric differential matrix Riccati equations with low rank right hand sides. These matrix equations appear in many applications such as control theory, transport theory, applied probability and others. Koskela [13] describe the finite dimensional linear quadratic regulator problem. The differential Riccati equation arises in the finite horizon case, i.e., when a finite time integral cost functional is considered. They analysis Krylov subspace approximation to large scale differential Riccati equations. Finally, for more details of application of matrix Riccati equations see [1].

The Riccati differential equation may be generalized as matrix Riccati differential equations (see [6, 7, 20]) and it is formulated by $X'(t) = A(t)X + XB(t)X + C(t)$, where A, B and C are $n \times n$-real matrix valued function on $\mathbb{R}$. This generalization applies in many areas such as optimal control problem, stochastic control problem and etc. [2, 14, 19].

In this paper, by Green's function's technique [15, 16, 20–26] and Schauder's fixed point theorem, the existence of at least one periodic solution of the coupled matrix Riccati differential equation

$$\begin{cases} X_1'(t) = A(t)X_1(t) + X_1(t)B_{11}^{(1)}(t)X_1(t) + X_1(t)B_{12}^{(1)}(t)X_2(t) \\ \qquad\qquad + X_2(t)B_{21}^{(1)}(t)X_1(t) + X_2(t)B_{22}^{(1)}(t)X_2(t) + E_1(t), \\ X_2'(t) = A(t)X_2(t) + X_1(t)B_{11}^{(2)}(t)X_1(t) + X_1(t)B_{12}^{(2)}(t)X_2(t) \\ \qquad\qquad + X_2(t)B_{21}^{(2)}(t)X_1(t) + X_2(t)B_{22}^{(2)}(t)X_2(t) + E_2(t), \end{cases} \tag{1.1}$$

where $A, B_{ij}^{(k)}, E_k$ for $i, j, k = 1, 2$ are ω-periodic continuous matrix valued functions on $\mathbb{R}$, is proved.

In Section 2, by using a suitable Green's function, we can construct a system of integral equation and we can prove the solution of the system of integral equation (2.3) is a solution of the coupled Riccati differential equation (1.1). In Section 3, we construct a compact operator on a Banach space and by applying Schauder's fixed point theorem, it is proved that the system of integral equation (2.3) has at least one periodic solution.

2. Green's function

Assume $A, B_{ij}^{(k)}, E_k$ for $i, j, k = 1, 2$ are ω-periodic continuous matrix valued functions on $\mathbb{R}$, the coupled matrix Riccati equation (1.1) can be written as

$$\begin{cases} X_1' = A(t)X_1 + \sum_{i,j=1}^{2} X_i(t)B_{ij}^{(1)}(t)X_j(t) + E_1(t), \\ X_2' = A(t)X_2 + \sum_{i,j=1}^{2} X_i(t)B_{ij}^{(2)}(t)X_j(t) + E_2(t). \end{cases} \tag{2.1}$$

Let $X(t) = (X_1(t), X_2(t))^T$, and $\mathbf{B}^{(k)} \in \mathbf{C}([0,T_f], \mathbf{M}_{2n\times 2n}(\mathbb{R}))$ be ω-periodic real valued matrix functions for $k = 1,2$ which are defined by

$$\mathbf{B}^{(k)}(t) = \begin{pmatrix} B_{11}^{(k)}(t) & B_{12}^{(k)}(t) \\ B_{21}^{(k)}(t) & B_{22}^{(k)}(t) \end{pmatrix}.$$

Set $M = \exp(\int_0^\omega A(s)ds)$, $M_1 = (I_n - M)^{-1}$, and $M_2 = MM_1$ where I_n is $n\times n$ identity matrix. Notice that $I_n - M$ is nonsingular. Define the Green's function G by

$$G(t,s) = \begin{cases} M_1 \exp(\int_s^t A(\tau)d\tau), & 0 \le s \le t \le \omega, \\ M_2 \exp(\int_s^t A(\tau)d\tau), & 0 \le t \le s \le \omega. \end{cases} \tag{2.2}$$

Lemma 1. *Suppose $B_{ij}^{(k)}, E_k$ for $i,j,k = 1,2$ are ω-periodic continuous functions on $\mathbb{R}$. Also, suppose for all $t,s \in [0,\omega]$, $\int_0^t A(\xi)d\xi$ commutes with $\int_0^s A(\xi)d\xi$ and $A(t)$, and $A(t)$ commutes with M. Let X be a solution of the system of integral equations*

$$\begin{cases} X_1(t) = \int_0^\omega G(t,s)(\sum_{i,j=1}^{2} X_i(s)B_{ij}^{(1)}(s)X_j(s) + E_1(s))ds, \\ X_2(t) = \int_0^\omega G(t,s)(\sum_{i,j=1}^{2} X_i(s)B_{ij}^{(2)}(s)X_j(s) + E_2(s))ds, \end{cases} \tag{2.3}$$

then $X(t)$ is a periodic solution of (1.1).

Proof. Since $\int_0^t A(\xi)d\xi$ commutes with $\int_0^s A(\xi)d\xi$, one may write

$$(\int_0^t A(\xi)d\xi)(-\int_0^s A(\xi)d\xi) = (-\int_0^s A(\xi)d\xi)(\int_0^t A(\xi)d\xi).$$

Therefore

$$\begin{aligned} \exp(\int_s^t A(\xi)d\xi) &= \exp(\int_0^t A(\xi)d\xi - \int_0^s A(\xi)d\xi) = \exp(\int_0^t A(\xi)d\xi)\exp(-\int_0^s A(\xi)d\xi) \\ &= \exp(\int_0^t A(\xi)d\xi)\left(\exp(\int_0^s A(\xi)d\xi)\right)^{-1}. \end{aligned}$$

Set $\alpha(t) := \exp(\int_0^t A(\xi)d\xi)$, thus

$$\begin{aligned}
X_1(t) &= \int_0^{\omega} G(t,s)\Big(\sum_{i,j=1}^{2} X_i(s)B_{ij}^{(1)}(s)X_j(s) + E_1(s)\Big)ds \\
&= \int_0^{t} G(t,s)\Big(\sum_{i,j=1}^{2} X_i(s)B_{ij}^{(1)}(s)X_j(s) + E_1(s)\Big)ds \\
&\quad + \int_t^{\omega} G(t,s)\Big(\sum_{i,j=1}^{2} X_i(s)B_{ij}^{(1)}(s)X_j(s) + E_1(s)\Big)ds \\
&= M_1\int_0^{t} \exp\Big(\int_s^t A(\xi)d\xi\Big)\Big(\sum_{i,j=1}^{2} X_i(s)B_{ij}^{(1)}(s)X_j(s) + E_1(s)\Big)ds \\
&\quad - M_2\int_{\omega}^{t} \exp\Big(\int_s^t A(\xi)d\xi\Big)\Big(\sum_{i,j=1}^{2} X_i(s)B_{ij}^{(1)}(s)X_j(s) + E_1(s)\Big)ds \\
&= M_1\alpha(t)\int_0^{t} \alpha(s)^{-1}\Big(\sum_{i,j=1}^{2} X_i(s)B_{ij}^{(1)}(s)X_j(s) + E_1(s)\Big)ds \\
&\quad - M_2\alpha(t)\int_{\omega}^{t} \alpha(s)^{-1}\Big(\sum_{i,j=1}^{2} X_i(s)B_{ij}^{(1)}(s)X_j(s) + E_1(s)\Big)ds.
\end{aligned}$$

Since $\int_0^t A(\xi)d\xi$ commutes with $A(t)$, the chain rule is satisfied and

$$\begin{aligned}
X_1'(t) &= M_1A(t)\alpha(t)\int_0^{t} \alpha(s)^{-1}\Big(\sum_{i,j=1}^{2} X_i(s)B_{ij}^{(1)}(s)X_j(s) + E_1(s)\Big)ds \\
&\quad + M_1\Big(\sum_{i,j=1}^{2} X_i(t)B_{ij}^{(1)}(t)X_j(t) + E_1(t)\Big) \\
&\quad - M_2A(t)\alpha(t)\int_{\omega}^{t} \alpha(s)^{-1}\Big(\sum_{i,j=1}^{2} X_i(s)B_{ij}^{(1)}(s)X_j(s) + E_1(s)\Big)ds \\
&\quad - M_2\Big(\sum_{i,j=1}^{2} X_i(t)B_{ij}^{(1)}(t)X_j(t) + E_1(t)\Big).
\end{aligned}$$

Therefore

$$
\begin{aligned}
X_1'(t) &= A(t)\int_0^t M_1 \exp(\int_s^t A(\xi)d\xi)(\sum_{i,j=1}^{2} X_i(s)B_{ij}^{(1)}(s)X_j(s)+E_1(s))ds \\
&\quad + A(t)\int_t^\omega M_2 \exp(\int_s^t A(\xi)d\xi)(\sum_{i,j=1}^{2} X_i(s)B_{ij}^{(1)}(s)X_j(s)+E_1(s))ds \\
&\quad + (M_1-M_2)(\sum_{i,j=1}^{2} X_i(t)B_{ij}^{(1)}(t)X_j(t)+E_1(t)) \\
&= A(t)\int_0^t G(t,s)(\sum_{i,j=1}^{2} X_i(s)B_{ij}^{(1)}(s)X_j(s)+E_1(s))ds \\
&\quad + A(t)\int_t^\omega G(t,s)(\sum_{i,j=1}^{2} X_i(s)B_{ij}^{(1)}(s)X_j(s)+E_1(s))ds \\
&\quad + \sum_{i,j=1}^{2} X_i(t)B_{ij}^{(1)}(t)X_j(t)+E_1(t) \\
&= A(t)\int_0^\omega G(t,s)(\sum_{i,j=1}^{2} X_i(s)B_{ij}^{(1)}(s)X_j(s)+E_1(s))ds \\
&\quad + \sum_{i,j=1}^{2} X_i(t)B_{ij}^{(1)}(t)X_j(t)+E_1(t) \\
&= A(t)X_1(t)+\sum_{i,j=1}^{2} X_i(t)B_{ij}^{(1)}(t)X_j(t)+E_1(t).
\end{aligned}
$$

With the similar method, we have the same for $X_2(t)$. So, we conclude that $X(t)$ is a solution of equation (1.1). □

3. Periodic solution

Here, we prove the existence of at least one periodic solution of the system (1.1).

Theorem 1. *Suppose the coefficients of the system (1.1) are ω-periodic continuous functions on $\mathbb{R}$ and the assumptions of Lemma 1 are are satisfied. For $j=1,2$, set*

$$\mu = \sup_{0\le t,s\le\omega} \|G(t,s)\|,\ \nu = \max_{0\le t\le\omega, j=1,2} \|\int_0^\omega G(t,s)E_j(s)ds\|. \tag{3.1}$$

If

$$\max\{\int_0^\omega ||B^{(k)}||ds|k=1,2\} \leq \frac{1}{4\mu\nu}, \tag{3.2}$$

where $||\boldsymbol{B}^{(k)}||$ *is operator norm of* $2n\times 2n$ *matrix* $\boldsymbol{B}^{(k)}$. *Then system (1.1) admits at least one* ω*-periodic solution.*

Remark 1. For convenience, set $\underline{R} = \max\{\int_0^\omega ||B_{ij}^{(k)}||ds|i,j,k=1,2\}$ and $\frac{1}{4\mu\nu} = \overline{R}$.

Proof. Suppose $\phi(t) = (\phi_1(t),\phi_2(t))^T$ and

$$\mathbf{X} = \{\phi(t)|\ \phi_1(t),\phi_2(t) \text{ are } \omega\text{-periodic continuous functions from } \mathbb{R} \text{ to } \mathbb{M}_n(\mathbb{R})\},$$

which is equipped with the norm $\|\phi\|_{\mathbf{X}} = \max_{j=1,2}\|\phi_j\|_\omega$, where $\|\phi_j\|_\omega$ is operator matrix norm for $n\times n$ matrices. $(X,\|.\|_X)$ is a Banach space. Set $\psi(t) = \left(\int_0^\omega G(t,s)E_1(s)ds, \int_0^\omega G(t,s)E_2(s)ds\right)^T$ and define a set $\mathbf{F} = \{\phi\in\mathbf{X}|\ \|\phi-\psi\|_{\mathbf{X}} \leq \nu\}$. Notice that $\mathbf{F}$ is closed, bounded and convex subset of $\mathbf{X}$. Define $P:\mathbf{F}\to\mathbf{X}$ by

$$P(\phi)(t) = \begin{pmatrix} \int_0^\omega G(t,s)\sum_{i,j=1}^2 \phi_i(s)B_{ij}^{(1)}\phi_j(s) + E_1(s)ds \\ \\ \int_0^\omega G(t,s)\sum_{i,j=1}^2 \phi_i(s)B_{ij}^{(2)}\phi_j(s) + E_2(s)ds \end{pmatrix}.$$

It's easy to see that $\|\phi(t)\|_{\mathbf{X}} \leq \nu + \|\psi(t)\|_{\mathbf{X}} \leq 2\nu$ for all $t\in[0,\omega]$. By using the sub multiplicative property of the operator norm, for all $t\in[0,\omega]$ we have

$$\begin{aligned}
\|P(\phi)(t)-\psi(t)\|_{\mathbf{X}} &\leq ||\int_0^\omega G(t,s)\phi^T\mathbf{B}^{(k)}\phi ds|| \\
&\leq \int_0^\omega \|G(t,s)\phi^T(s)\mathbf{B}^{(k)}(s)\phi(s)\|ds \\
&\leq \int_0^\omega \|G(t,s)\|\|\mathbf{B}^{(k)}(s)\|\|\phi(s)\|^2 ds \\
&\leq 4\mu\nu^2\int_0^\omega ||\mathbf{B}^k||ds \\
&\leq \nu.
\end{aligned}$$

Thus for all $\phi\in\mathbf{F}$ we have $\|P(\phi)-\psi\|_{\mathbf{X}} \leq \nu$ and so $P(\phi)\in\mathbf{F}$. This shows P is an operator from $\mathbf{F}$ into $\mathbf{F}$.

Now, we recall the weak version of Ascoli-Arzel$\grave{a}$ theorem to prove the compactness of P.

Lemma 2 (Ascoli - Arzela)**.** *Let* $\{\Phi_n(t)\}_{n\in\mathbb{N}}$ *be a sequence of functions from* $[a,b]$ *to* $\mathbb{R}^2$ *which is uniformly bounded and equicontinuous. Then* $\{\Phi_n(t)\}_{n\in\mathbb{N}}$ *has a uniformly convergent subsequence.*

Suppose $\{\phi_n\} = (\{(\phi_1)_n\},\{(\phi_2)_n\})^T$, where ^{T} means transpose, is a sequence on $\mathbf{F}$. This sequence is bounded. So, there exists $\lambda > 0$ such that for all $n \in \mathbb{N}$ and for all $t \in [0,\omega]$, we have $\|\phi_n(t)\|_{\mathbf{X}} \leq \lambda$. At following, we have to show $\{\phi_n\}$ has a subsequence, $\{\phi_{n_i}\}$, such that $\{P(\phi_{n_i})\}$ is convergent on $\mathbf{F}$.

According to Lemma 1, the function $P(\phi_n)$ is differentiable and for all $t \in [0,\omega]$, we have

$$P(\phi_n)'(t) = \begin{pmatrix} A(t)(\phi_1)_n(t) + \sum_{i,j=1}^{2}(\phi_i)_n(t)B_{ij}^{(1)}(\phi_j)_n(t) + E_1(t) \\ A(t)(\phi_2)_n(t) + \sum_{i,j=1}^{2}(\phi_i)_n(t)B_{ij}^{(2)}(\phi_j)_n(t) + E_2(t) \end{pmatrix}.$$

Since $\mathbf{F}$ is bounded for all $n \in \mathbb{N}$ and for all $t \in [0,\omega]$, we get $\|P(\phi_n)'(t)\|_{\mathbf{X}} \leq \rho_1\lambda + \rho_2\lambda^2 + \rho_3$, where ρ_1, ρ_2 and ρ_3 are $\|A\|_\omega, \max\{||B^{(k)}|||k = 1,2, 0 \leq t \leq \omega\}$ and $\max_{k=1,2}\|E_k\|_\omega$ on $[0,\omega]$, respectively. For given $\varepsilon > 0$, let $\delta = \varepsilon/(\rho_1\lambda + \rho_2\lambda^2 + \rho_3)$, then for all $n \in \mathbb{N}$ and for all $t_1, t_2 \in [0,\omega]$, $|t_1 - t_2| < \delta$ implies that

$$\|P(\phi_n)(t_1) - P(\phi_n)(t_2)\|_{\mathbf{X}} \leq (\rho_1\lambda + \rho_2\lambda^2 + \rho_3)|t_1 - t_2| < \varepsilon. \tag{3.3}$$

So $\{P(\phi_n(t))\}$ is equicontinuous and Theorem 2 implies that there exists a subsequence of $\{P(\phi_{n_i}(t))\}$ of $\{P(\phi_n(t))\}$ which is uniformly convergent on $[0,\omega]$. We conclude that $\{P(\phi_{n_i})\}$ is convergent on $\mathbf{F}$ and so P is compact.
Thus Schauder's fixed point theorem implies there exists $X(t) = (X_1(t), X_2(t))^T \in \mathbf{F}$ such that $P(X(t)) = X(t)$, i.e. for all $t \in [0,\omega]$

$$X(t) = \begin{pmatrix} X_1(t) \\ \\ X_2(t) \end{pmatrix} = \begin{pmatrix} \int_0^\omega G(t,s)(\sum_{i,j=1}^{2} X_i(s)B_{ij}^{(1)}X_j(s) + E_1(s))ds \\ \\ \int_0^\omega G(t,s)(\sum_{i,j=1}^{2} X_i(s)B_{ij}^{(2)}X_j(s) + E_2(s))ds \end{pmatrix}.$$

Thus, by Lemma 1, $X(t)$ is a solution of equation (1.1). □

Before ending this section, we would like to generalize the system (2.1).

Remark 2. Notice that the developments in this paper may be done in the following more general case

$$\begin{cases} X_1' = A_1(t)X_1 + X_1D_1(t) + \displaystyle\sum_{i,j=1}^{2} X_i(t)B_{ij}^{(1)}(t)X_j(t) + E_1(t), \\ \\ X_2' = A_2(t)X_2 + X_2D_2(t) + \displaystyle\sum_{i,j=1}^{2} X_i(t)B_{ij}^{(2)}(t)X_j(t) + E_2(t), \end{cases} \tag{3.4}$$

where $A_k, D_k, B_{ij}^{(k)}, E_k$ for $i,j,k = 1,2$ are ω-periodic continuous matrix valued functions on $\mathbb{R}$. A good question is the existence of at least one period solution of (3.4).

Note that the system (3.4) is the system (2.1) when $D_1(t) = D_2(t) = 0$ and $A_1(t) = A_2(t) = A(t)$.

4. Numerical examples

In this section, we bring two numerical examples that show the novelty of the results.

Example 1. Consider the family of the coupled matrix Riccati equation (2.3) with the coefficients

$$A(t) = \begin{pmatrix} \frac{3}{16} & \frac{-1}{16} \\ \frac{1}{16} & \frac{1}{16} \end{pmatrix},$$

$$B^1_{11}(t) = \begin{pmatrix} \frac{\cos t}{4} & \frac{\sin t}{4} \\ \frac{-\sin t}{4} & \frac{\cos t}{4} \end{pmatrix}, \quad B^1_{12}(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$B^1_{21}(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B^1_{22}(t) = \begin{pmatrix} \frac{\cos t}{4} & \frac{-\sin t}{4} \\ \frac{\sin t}{4} & \frac{\cos t}{4} \end{pmatrix},$$

$$B^2_{11}(t) = \begin{pmatrix} \frac{\cos t}{4} & \frac{\sin t}{4} \\ \frac{-\sin t}{4} & \frac{\cos t}{4} \end{pmatrix}, \quad B^2_{12}(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$B^2_{21}(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B^1_{22}(t) = \begin{pmatrix} \frac{\cos t}{4} & \frac{-\sin t}{4} \\ \frac{\sin t}{4} & \frac{\cos t}{4} \end{pmatrix},$$

$$E_1(t) = \begin{pmatrix} \frac{511\cos t - 65\sin t}{4096} & \frac{-3(32\cos t + 171\sin t)}{4096} \\ \frac{1022\cos t - 129\sin t + \sin 3t}{5192} & \frac{-64\cos t - \sin t - \sin 3t}{5192} \end{pmatrix},$$

and

$$E_2(t) = \begin{pmatrix} \frac{-8257\cos t - 63\cos 3t - 65599\sin t - 65\sin 3t}{524288} & \frac{65536\cos t - 122225\sin t - 65\sin 3t}{524288} \\ \frac{-16577\cos t - 63\cos 3t - 131133\sin t - 65\sin 3t}{1048576} & \frac{-8257\sin t - 65\sin 3t}{1048576} \end{pmatrix}.$$

Then

$$M = \begin{pmatrix} \frac{1}{8}\exp \pi/4(8+\pi) & \frac{-1}{8}\exp \pi/4\pi \\ \frac{-1}{8}\exp \pi/4\pi & \frac{-1}{8}\exp \pi/4(-8+\pi) \end{pmatrix}.$$

Therefore, $\underline{R} < \overline{R}$ and the equation (1.1) has a periodic solution.

Remark 3. We can easily show that

$$X_1 = \begin{pmatrix} \frac{\sin t}{8} & \frac{\cos t}{8} \\ \frac{\sin t}{8} & 0 \end{pmatrix} \text{ and } X_2 = \begin{pmatrix} \frac{\cos t}{8} & \frac{\sin t}{8} \\ \frac{\sin t}{8} & 0 \end{pmatrix}.$$

is a periodic solution whose existence is guaranteed.

Remark 4. Consider the initial value problem consist of the Riccati equation in example 1 together with initial condition $X(0) = X_0$. Let

$$\begin{cases} T_1 X(t) = \int_0^t [A(s)X_1(s) + \sum_{i,j=1}^{2} X_i(s)B^{(1)}_{ij}(s)X_j(s) + E_1(s)]ds + \xi_1, \\ T_2 X(t) = \int_0^t [A(s)X_2(s) + \sum_{i,j=1}^{2} X_i(s)B^{(2)}_{ij}(s)X_j(s) + E_2(s)]ds + \xi_2, \end{cases} \tag{4.1}$$

then $T : \mathbf{C}(\mathbb{R}, \mathbf{M}_{2\times 2}(\mathbb{R})) \times \mathbf{C}(\mathbb{R}, \mathbf{M}_{2\times 2}(\mathbb{R})) \to \mathbf{C}(\mathbb{R}, \mathbf{M}_{2\times 2}(\mathbb{R})) \times \mathbf{C}(\mathbb{R}, \mathbf{M}_{2\times 2}(\mathbb{R}))$, where $T = (T_1, T_2,)^T$ is the corresponding Picard's operator that relates the unique solution of the initial value problem to a fixed point of an operator. The fixed point iteration method

$$\begin{cases} X_n(t) = TX_{n-1}(t), \\ X_0(t) = \xi, \end{cases} \tag{4.2}$$

where $\xi = (\xi_1, \xi_2)^T$ constitutes a numerical technique for generating the unique solution of the initial value problem, provided that the convergence is guaranteed. Let $ER_n = \|X_n - X\| = \max_{0\le t\le 2\pi} \|X_n(t) - X(t)\|$, be the error of approximation, where

$$X_1 = \begin{pmatrix} \frac{\sin t}{8} & \frac{\cos t}{8} \\ \frac{\sin t}{8} & 0 \end{pmatrix} \text{ and } X_2 = \begin{pmatrix} \frac{\cos t}{8} & \frac{\sin t}{8} \\ \frac{\sin t}{8} & 0 \end{pmatrix},$$

where

$$\xi_1 = \begin{pmatrix} 0 & \frac{1}{8} \\ 0 & 0 \end{pmatrix} \text{ and } \xi_2 = \begin{pmatrix} \frac{1}{8} & 0 \\ \frac{1}{8} & 0 \end{pmatrix}.$$

Numerical values for the error are given in the Table 1.

TABLE 1.

n	ER_1	ER_2
1	0.19635	0.19635
2	0.11539	0.0774063
3	0.0336085	0.0190035
4	0.00657322	0.00521541
5	0.000982148	0.000993719
6	0.000118411	0.000147843
7	0.0000203268	0.0000202704
8	$3.00827 \times 10^{-}6$	$2017528 \times 10^{-}6$

Example 2. Consider the Riccati equation (1.1) with coefficients

$$A(t) = \begin{pmatrix} \frac{1}{5}(1+\cos t) & \frac{1}{5}(1+\sin t) \\ \frac{1}{5}(-1-\sin t) & \frac{1}{5}(1+\cos t) \end{pmatrix},$$

$$B^1_{11}(t) = \begin{pmatrix} \frac{\cos t}{25} & \frac{\sin t}{25} \\ \frac{-\sin t}{25} & \frac{\cos t}{25} \end{pmatrix}, \qquad B^1_{12}(t) = \begin{pmatrix} \frac{1}{25} & \frac{1}{25} \\ \frac{-1}{25} & \frac{1}{25} \end{pmatrix},$$

$$B^1_{21}(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B^1_{22}(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$B^2_{11}(t) = \begin{pmatrix} \frac{\cos t}{25} & \frac{\sin t}{25} \\ \frac{-\sin t}{25} & \frac{\cos t}{25} \end{pmatrix}, \qquad B^2_{12}(t) = \begin{pmatrix} \frac{1}{25} & \frac{1}{25} \\ \frac{-1}{25} & \frac{1}{25} \end{pmatrix},$$

$$B^2_{21}(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B^2_{22}(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$E_1 = \begin{pmatrix} \frac{1-25\cos t-25\cos 2t-\cos 3t-100\sin t}{625} & \frac{-1-25\cos t-100\cos t-25\sin t-25\sin 2t-\sin 3t}{625} \\ \frac{1+25\cos t+100\cos t+25\sin t+25\sin 2t+\sin 3t}{625} & \frac{1-25\cos t-25\cos 2t-\cos 3t-100\sin t}{625} \end{pmatrix}$$

and

$$E_2 = \begin{pmatrix} \frac{1-26\cos t-25\cos 2t-\cos 3t-150\sin t}{625} & \frac{-1-25\cos t-150\cos t+25\sin t+\sin 3t}{625} \\ \frac{1+25\cos t+150\cos t-25\sin t-\sin 3t}{625} & \frac{1-26\cos t-25\cos 2t-\cos 3t-150\sin t}{625} \end{pmatrix}.$$

Then

$$M = \begin{pmatrix} \frac{1}{4}(-1+\sqrt{5})\exp 2\pi/5 & \sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}\exp 2\pi/5 \\ \sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}\exp 2\pi/5 & \frac{1}{4}(-1+\sqrt{5})\exp 2\pi/5 \end{pmatrix}.$$

Remark 5. For consistency of our assumptions, we can easily show that

$$X_1 = \begin{pmatrix} \frac{\cos t}{5} & \frac{\cos t}{5} \\ \frac{-\cos t}{5} & \frac{\cos t}{5} \end{pmatrix} \text{ and } X_2 = \begin{pmatrix} \frac{\cos t}{5} & \frac{-\cos t}{5} \\ \frac{\cos t}{5} & \frac{\cos t}{5} \end{pmatrix},$$

is a periodic solution. Consider the fixed point iteration corresponding to this example (see example (1)). Numerical results are shown in the Table 2.

TABLE 2.

n	E_1	E_2
1	0.502655	0.389338
2	0.403075	0.380026
3	0.176354	0.184883
4	0.0797213	0.0770404
5	0.0180976	0.0244803
6	0.00618601	0.00610193
7	0.00118407	0.0013605
8	0.000367802	0.000436594
9	0.000219501	0.0001916484
10	0.0000643376	0.0000556998
11	0.0000296998	5.65144×10^{-6}
12	$4.17076 \times 10^{-}6$	0.0000127965

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Authors' addresses

Zahra Goodarzi
Imam Khomeini International University, Faculty of Science, Department of Pure Mathematics, Postal code: 34149-16818, Qazvin, Iran
E-mail address: z.goodarzi@edu.ikiu.ac.ir

Abdolrahman Razani
Imam Khomeini International University, Faculty of Science, Department of Pure Mathematics, Postal code: 34149-16818, Qazvin, Iran and, School of Mathematics, Institute for Research in Fundamental Sciences (IPM),, P.O. Box 19395-5746, Tehran, Iran.
E-mail address: razani@sci.ikiu.ac.ir

M.R. Mokhtarzadeh
School of Mathematics, Institute for Research in Fundamental Sciences (IPM),, P.O. Box 19395-5746, Tehran, Iran.
E-mail address: mrmokhtarzadeh@ipm.ir

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OSCILLATORY BEHAVIOR OF SECOND ORDER NONLINEAR DIFFERENCE EQUATIONS WITH A NONLINEAR NONPOSITIVE NEUTRAL TERM

SAID R. GRACE AND JOHN R. GRAEF

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Abstract. The authors present some new oscillation criteria for second order nonlinear difference equations with a nonlinear nonpositive neutral term of the form

$$\Delta\left(a(t)\left(\Delta\left(x(t)-p(t)x^{\alpha}(t-k)\right)\right)^{\gamma}\right)+q(t)x^{\beta}(t+1-m)=0,$$

with positive coefficients. Examples are given to illustrate the main results.

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1. Introduction

This paper deals with the oscillatory behavior of solutions of the nonlinear second order difference equations with a nonlinear nonpositive neutral term

$$\Delta\left(a(t)\left(\Delta\left(x(t)-p(t)x^{\alpha}(t-k)\right)\right)^{\gamma}\right)+q(t)x^{\beta}(t+1-m)=0, \quad t\geq t_0, \tag{1.1}$$

where $\Delta x(t)=x(t+1)-x(t)$ and:

(i) α, γ, and β are the ratios of positive odd integers with $\gamma\geq\beta$ and $0<\alpha\leq 1$;
(ii) $\{a(t)\}$, $\{p(t)\}$ and $\{q(t)\}$ are positive real sequences for $t\geq t_0$, and $0<p(t)<p_0<1$;
(iii) k is a positive integer and m is a nonnegative integer;
(iv) $h(t)=t-m+k+1\leq t$, i.e., $m\geq k+1$.

We set

$$A(v,u)=\sum_{s=u}^{v}\frac{1}{a^{1/\gamma}(s)} \quad \text{for} \quad v\geq u\geq t_0,$$

and assume that

$$A(t,t_0)\to\infty \ \text{ as } \ t\to\infty. \tag{1.2}$$

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Let $\theta = \max\{k, m-1\}$. By a *solution* of equation (1.1), we mean a real sequence $\{x(t)\}$ defined for all $t \geq t_0 - \theta$ that satisfies equation (1.1) for all $t \geq t_0$. A solution of equation (1.1) is called *oscillatory* if its terms are neither eventually positive nor eventually negative; otherwise, it is called *nonoscillatory*. If all solutions of the equation are oscillatory, then the equation itself is called oscillatory.

In recent years there has been a great deal of research activity on the oscillation and asymptotic behavior of solutions of various classes of difference equations; for example, see the monographs [1, 2, 5, 6], and the papers listed below. There are numerous results for second order neutral functional difference equations due to their increasing use as models in the natural sciences and in theoretical studies. Some such recent results on the oscillatory and asymptotic behavior of second order difference equations can be found in [3, 4, 7–22]. However, there does not appear to be any known results on the oscillation of second order difference equations of the type (1.1). Our aim here is to present some new sufficient conditions that ensure all solutions of (1.1) are oscillatory.

2. Main results

For $t \geq T$ for any $T \geq t_0$, we let

$$\mu(t) = a^{1/\gamma}(t)A(t,T) \quad \text{and} \quad Q(t) = \sum_{s=t}^{\infty} q(s).$$

For any constant $c > 0$, we set

$$g_c(t) = \begin{cases} 1, & \text{if } \beta = \gamma, \\ cA^{(\beta-\gamma)/\gamma}(t), & \text{if } \beta < \gamma. \end{cases} \tag{2.1}$$

We begin with the following new result.

Theorem 1. *Let conditions (i)–(iv) and (1.2) hold. Assume there exists a positive nondecreasing sequence $\{\rho(t)\}$ such that for any constant $c > 0$,*

$$\limsup_{t\to\infty}\left(\rho(t)Q(t) + \sum_{s=t_2}^{t}\left[\rho(s)q(s) - \frac{\gamma^\gamma}{(1+\gamma)^{\gamma+1}}\frac{a(t-m)}{(\beta g_c(s))^\gamma}\left(\frac{(\Delta\rho(s))^{\gamma+1}}{\rho^\gamma(s)}\right)\right]\right) = \infty, \tag{2.2}$$

$$\limsup_{t\to\infty}\sum_{s=h(t)}^{t} A^{\beta/\alpha}(h(t),h(s))q(s) > 1 \quad \textit{if } \beta = \alpha\gamma, \tag{2.3}$$

and

$$\limsup_{t\to\infty}\sum_{s=h(t)}^{t} A^{\beta/\alpha}(h(t),h(s))q(s) = \infty \quad \textit{if } \beta < \alpha\gamma. \tag{2.4}$$

Then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(t-m+1) > 0$, and $x(t-k) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Then with $y(t) = x(t) - p(t)x^{\alpha}(t-k)$, it follows from (1.1) that

$$\Delta\left(a(t)\left(\Delta y(t)\right)^{\gamma}\right) = -q(t)x^{\beta}(t-m+1) \leq 0. \tag{2.5}$$

Hence, $a(t)\left(\Delta y(t)\right)^{\gamma}$ is nonincreasing and eventually of one sign. That is, there exists $t_2 \geq t_1$ such that $\Delta y(t) > 0$ or $\Delta y(t) < 0$ for $t \geq t_2$. We claim that $\Delta y(t) > 0$ for $t \geq t_2$. To see this, assume that $\Delta y(t) < 0$ for $t \geq t_2$. Then,

$$a(t)\left(\Delta y(t)\right)^{\gamma} \leq -c < 0 \quad \text{for } t \geq t_2,$$

where $c = -a(t_2)\left(\Delta y(t_2)\right)^{\gamma} < 0$, so

$$y(t) \leq y(t_2) - c^{1/\gamma} \sum_{s=t_2}^{t} a^{-1/\gamma}(s).$$

In view of (1.2), $\lim_{t\to\infty} y(t) = -\infty$. Now, we consider the following two cases.

Case 1. If $x(t)$ is unbounded, then there exists an increasing sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} x(t_n) = \infty$ where $x(t_k) = \max\{x(s) : t_0 \leq s \leq t_k\}$. This implies

$$x(t_n - m + 1) \leq \max\{x(s) : t_0 \leq s \leq t_n\} = x(t_n).$$

Therefore, since $\{x(t_n)\} \to \infty$ and (ii) holds for all large n,

$$\begin{aligned} y(t_n) = x(t_n) - p(t_n)x^{\alpha}(t_n - k) &\geq x(t_n) - p(t_n)x^{\alpha}(t_n) \\ &\geq \left(1 - \frac{p(t_n)}{x^{1-\alpha}(t_n)}\right) x(t_n) > 0. \end{aligned}$$

which contradicts the fact that $\lim_{t\to\infty} y(t) = -\infty$.

Case 2. If $x(t)$ is bounded, then $y(t)$ is also bounded, which contradicts $\lim_{t\to\infty} y(t) = -\infty$. This completes the proof of the claim so we conclude that $\Delta y(t) > 0$ for $t \geq t_2$.

Next, we have two possibilities to consider: (I) $y(t) > 0$ or (II) $y(t) < 0$ for $t \geq t_2$. If (I) holds, then in view of (2.5) and the fact that $x(t) \geq y(t)$, we have

$$\Delta\left(a(t)\left(\Delta y(t)\right)^{\gamma}\right) \leq -q(t)y^{\beta}(t-m+1) \leq 0. \tag{2.6}$$

Summing Δy from t_2 to t gives

$$\begin{aligned} y(t) &= y(t_2) + \sum_{s=t_2}^{t} \frac{\left(a(s)\left(\Delta y(s)\right)^{\gamma}\right)^{1/\gamma}}{a^{1/\gamma}(s)} \\ &\geq a^{1/\gamma}(t)\Delta y(t) \sum_{s=t_2}^{t} a^{-1/\gamma}(s) := \mu(t)\Delta y(t). \end{aligned} \tag{2.7}$$

Summing (2.6) from t to u, letting $u \to \infty$, and using the fact that $y(t)$ is increasing, we have

$$\begin{aligned} a(t)\,(\Delta y(t))^{\gamma} &\geq \sum_{s=t}^{\infty} q(s) y^{\beta}(s-m+1) \\ &\geq y^{\beta}(t-m+1) \sum_{s=t}^{\infty} q(s) := Q(t) y^{\beta}(t-m+1) \\ &\geq Q(t) y^{\beta}(t-m). \end{aligned} \tag{2.8}$$

Define

$$w(t) = \rho(t) \frac{a(t)\,(\Delta y(t))^{\gamma}}{y^{\beta}(t-m)} > 0 \quad \text{for } t \geq t_2. \tag{2.9}$$

Then, it follows that $w(t) > 0$ and

$$w(t) = \rho(t) \frac{a(t)\,(\Delta y(t))^{\gamma}}{y^{\beta}(t-m)} \geq \rho(t) \sum_{s=t}^{\infty} q(s). \tag{2.10}$$

Now,

$$\begin{aligned} \Delta w(t) &= \Delta \left(\frac{\rho(t)}{y^{\beta}(t-m)} \right) a(t+1)\,(\Delta y(t+1))^{\gamma} \\ &\quad + \Delta \left(a(t)\,(\Delta y(t))^{\gamma} \right) \left(\frac{\rho(t)}{y^{\beta}(t-m)} \right) \\ &\leq -\rho(t) q(t) + \left(\frac{\Delta \rho(t)}{\rho(t+1)} \right) w(t+1) \\ &\quad - \left(\frac{\rho(t)}{\rho(t+1)} \right) \frac{\Delta y^{\beta}(t-m)}{y^{\beta}(t-m)} w(t+1). \end{aligned} \tag{2.11}$$

By the Generalized Mean Value Theorem for Derivatives,

$$\beta y^{\beta-1}(t-m+1) \Delta y(t-m) \geq \Delta y^{\beta}(t-m) \geq \beta y^{\beta-1}(t-m) \Delta y(t-m).$$

Using this in (2.11) gives

$$\begin{aligned} \Delta w(t) &\leq -\rho(t) q(t) + \left(\frac{\Delta \rho(t)}{\rho(t+1)} \right) w(t+1) \\ &\quad - \beta \left(\frac{\rho(t)}{\rho(t+1)} \right) \frac{y^{\beta-1}(t-m) \Delta y(t-m)}{y^{\beta}(t-m)} w(t+1). \end{aligned} \tag{2.12}$$

Since $a(t)\,(\Delta y(t))^{\gamma}$ is decreasing and $y(t)$ is increasing, we have

$$\frac{\Delta y(t-m)}{\Delta y(t)} \geq \left(\frac{a(t)}{a(t-m)} \right)^{1/\gamma} \quad \text{and} \quad \frac{w(t+1)}{\rho(t+1)} \leq \frac{w(t)}{\rho(t)}. \tag{2.13}$$

Using (2.13) in (2.12), we obtain

$$\Delta w(t) \le -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) w(t+1) - \beta\left(\frac{\rho(t)}{\rho(t+1)}\right)\left(\frac{a(t)}{a(t-m)}\right)^{1/\gamma} \frac{\Delta y(t)}{y(t-m)} w(t+1).$$

Now,

$$\frac{\Delta y(t)}{y^{\beta/\gamma}(t-m)} = \rho^{-1/\gamma}(t)a^{-1/\gamma}(t)w^{1/\gamma}(t) \ge \rho^{-1/\gamma}(t)a^{-1/\gamma}(t)\left(\frac{\rho(t)}{\rho(t+1)}\right)^{1/\gamma} w^{1/\gamma}(t+1).$$

Thus,

$$\Delta w(t) \le -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) w(t+1) - \frac{\beta}{a^{1/\gamma}(t-m)}\left(\frac{\rho(t)}{\rho^{1+1/\gamma}(t+1)}\right) w^{1+1/\gamma}(t+1)y^{(\beta-\gamma)/\gamma}(t-m),$$

and so,

$$\Delta w(t) \le -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) w(t+1) - \frac{\beta\rho(t)}{a^{1/\gamma}(t-m)\rho^{1+1/\gamma}(t+1)} w^{1+1/\gamma}(t+1)y^{(\beta-\gamma)/\gamma}(t-m).$$

For the case $\beta = \gamma$, we see that $y^{(\beta-\gamma)/\gamma}(t) = 1$ while for the case $\beta < \gamma$, since $a(t)\,(\Delta y(t))^{\gamma}$ is decreasing, there exists a constant $c_1 > 0$ such that

$$a(t)\,(\Delta y(t))^{\gamma} \le c_1 \quad \text{for } t \ge t_2.$$

Summing this inequality from t_2 to t, we have

$$y(t) \le y(t_2) + A(t,t_2) \le c_2 A(t,t_2)$$

for $t \ge t_3$ for some $c_2 > 0$ and $t_3 \ge t_2$. Thus,

$$y^{(\beta-\gamma)/\gamma}(t) \ge c_2^{(\beta-\gamma)/\gamma} A^{(\beta-\gamma)/\gamma}(t,t_2) := c^* A^{(\beta-\gamma)/\gamma}(t,t_2),$$

where $c^* = c_2^{(\beta-\gamma)/\gamma}$. Combining the two cases on β and the definition of $g_{c^*}(t)$ gives

$$\Delta w(t) \le -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) w(t+1) - \frac{\beta\rho(t)}{a^{1/\gamma}(t-m)\rho^{1+1/\gamma}(t+1)} g_{c^*}(t)w^{1+1/\gamma}(t+1). \tag{2.14}$$

Setting

$$B := \frac{\Delta\rho(t)}{\rho(t+1)} \quad \text{and} \quad C := \frac{\beta\rho(t)g_{c^*}(t)}{a^{1/\gamma}(t-m)\rho^{1+1/\gamma}(t+1)},$$

and using the inequality (see [7])

$$Bu - Cu^{(1+\gamma)/\gamma} \le \frac{\gamma^\gamma}{(1+\gamma)^{\gamma+1}}\left(\frac{B^{\gamma+1}}{C^\gamma}\right),$$

with $u = w(t+1)$, we have

$$\Delta w(t) \le -\rho(t)q(t) + \frac{\gamma^\gamma}{(1+\gamma)^{\gamma+1}}\frac{a(t-m)}{(\beta g_{c^*}(t))^\gamma}\left(\frac{(\Delta\rho(t))^{\gamma+1}}{\rho^\gamma(t)}\right).$$

Summing this inequality from t_2 to t gives

$$w(t) \le w(t_2) - \sum_{s=t_2}^{t}\left[\rho(s)q(s) - \frac{\gamma^\gamma}{(1+\gamma)^{\gamma+1}}\frac{a(t-m)}{(\beta g_{c^*}(s))^\gamma}\left(\frac{(\Delta\rho(s))^{\gamma+1}}{\rho^\gamma(s)}\right)\right].$$

Taking into account (2.8), we see that

$$w(t_2) \ge \rho(t)Q(t) + \sum_{s=t_2}^{t}\left[\rho(s)q(s) - \frac{\gamma^\gamma}{(1+\gamma)^{\gamma+1}}\frac{a(t-m)}{(\beta g_{c^*}(s))^\gamma}\left(\frac{(\Delta\rho(s))^{\gamma+1}}{\rho^\gamma(s)}\right)\right].$$

Taking the lim sup of both sides in the above inequality as $t \to \infty$, we obtain a contradiction to condition (2.2).

Now consider Case (II). If we set $z(t) = -y(t) > 0$ for $t \ge t_2$, then $\Delta z(t) = -\Delta y(t) < 0$, and from equation (1.1),

$$\Delta\left(a(t)\left(\Delta z(t)\right)^\gamma\right) = q(t)x^\beta(t-m+1) \ge 0. \tag{2.15}$$

Moreover,

$$z(t) = -y(t) = p(t)x^\alpha(t-k) - x(t) \le p(t)x^\alpha(t-k),$$

so

$$x^\alpha(t-k) \ge z(t) \quad \text{or} \quad z^{1/\alpha}(t+k) \le x(t).$$

Using this inequality in (1.1), we have

$$\Delta\left(a(t)\left(\Delta z(t)\right)^\gamma\right) \ge q(t)z^{\beta/\alpha}(t-m+k+1) := q(t)z^{\beta/\alpha}(h(t)). \tag{2.16}$$

For $t_2 \le u \le v$, we may write

$$\begin{aligned} z(u) - z(v) &= -\sum_{s=u}^{v} a^{-1/\gamma}(s)\left(a(s)(\Delta z(s))^\gamma\right)^{1/\gamma} \\ &\ge A(v,u)\left(-\left(a(v)(\Delta z(v))^\gamma\right)^{1/\gamma}\right) \end{aligned}$$

for $t \geq s \geq t_2$. Setting $u = h(s)$ and $v = h(t)$ in the above inequality gives

$$z(h(s)) \geq A(h(t), h(s)) \left(-\left(a(h(t))(\Delta z(h(t)))^{\gamma} \right)^{1/\gamma} \right).$$

Summing inequality (2.16) from $h(t) \geq t_2$ to t, we find that

$$\begin{aligned} Z(t) &:= -a(h(t))(\Delta z(h(t)))^{\gamma} \\ &\geq \left(-a(h(t))(\Delta z(h(t)))^{\gamma} \right)^{\beta/\alpha\gamma} \sum_{s=h(t)}^{t} A^{\beta/\alpha}(h(t), h(s)) q(s) \\ &= Z^{\beta/\alpha\gamma}(t) \sum_{s=h(t)}^{t} A^{\beta/\alpha}(h(t), h(s)) q(s), \end{aligned}$$

and hence

$$Z^{1-\beta/\alpha\gamma}(t) \geq \sum_{s=h(t)}^{t} A^{\beta/\alpha}(h(t), h(s)) q(s).$$

Taking the lim sup of both sides of this inequality as $t \to \infty$, we arrive at a contradiction to (2.3) if $\beta = \alpha\gamma$. Since $\Delta Z(t) \leq 0$ by (2.15), $Z(t)$ is bounded, and so we obtain a contradiction to (2.4) if $\beta < \alpha\gamma$. This completes the proof of the theorem. □

Remark 1. We note that Theorem 1 holds if $Q(t) < \infty$ so the presence of the additional term $\rho(t)Q(t)$ in condition (2.2) may improve some of well-known existing results in the literature.

In case $Q(t)$ does not exist as $t \to \infty$, we see that condition (2.2) can be replaced by

$$\limsup_{t\to\infty} \sum_{s=t_2}^{t} \left[\rho(s)q(s) - \frac{\gamma^{\gamma}}{(1+\gamma)^{\gamma+1}} \frac{a(t-m+1)}{(\beta g_c(s))^{\gamma}} \left(\frac{(\Delta\rho(s))^{\gamma+1}}{\rho^{\gamma}(s)} \right) \right] = \infty \tag{2.17}$$

and the conclusion of Theorem 1 still holds.

For the non-neutral equation, i.e., equation (1.1) with $p(t) \equiv 0$, and $q(t)$ is either nonnegative or nonpositive for all large t, equation (1.1) reduces to

$$\Delta\left(a(t)(\Delta(x(t)))^{\gamma} \right) + \delta q(t) x^{\beta}(t+1-m) = 0, \tag{2.18}$$

where $\delta = \pm 1$. From Theorem 1, we extract the following immediate results.

Corollary 1. *Let conditions (i)–(iii) and (1.2) hold. If there exists a positive sequence $\{\rho(t)\}$ with $\Delta\rho(t) \geq 0$ such that condition (2.2) holds, then equation (2.18) with $\delta = \pm 1$ is oscillatory.*

Proof. The proof is contained in the proof of Case (I) in Theorem 1 and hence is omitted. □

We note that Corollary 1 is related to some of the results in [3–5, 12–16, 19] and the references cited therein.

Corollary 2. *Let conditions (i)–(iv) and (1.2) hold. If condition (2.3) or (2.4) holds, then every bounded solution of equation (2.18) with $\delta = \pm 1$ is oscillatory.*

Proof. The proof is contained in the proof of Case (II) of Theorem 1 and hence is omitted. □

The following example illustrates the above theorem.

Example 1. Consider the neutral equation

$$\Delta\left(\Delta\left(x(t)-\frac{1}{2}x^{1/3}(t-3)\right)^3\right)+8x(t-7)=0. \tag{2.19}$$

Here, $k=3$ and $m=8$, so $h(t)=t-4$. All conditions of Theorem 1 with $\rho(t)\equiv 1$ and condition (2.2) replaced by (2.17) are satisfied, so equation (2.19) is oscillatory.

Our next result follows directly from Theorem 1.

Theorem 2. *Let the hypotheses of Theorem 1 hold with $\Delta\rho(t)\le 0$ for $t\ge t_0$ and condition (2.2) replaced by*

$$\limsup_{t\to\infty}\left[\rho(t)Q(t)+\sum_{s=t_0}^{t}\rho(s)q(s)\right]=\infty. \tag{2.20}$$

Then equation (1.1) is oscillatory.

In the following theorem we employ a different approach to replacing condition (2.2) in Theorem 1.

Theorem 3. *Let the hypotheses of Theorem 1 hold with condition (2.2) replaced by*

$$\limsup_{t\to\infty}\left[\rho(t)Q(t)+\sum_{s=t_0}^{t}\rho(s)q(s)-\frac{a^{1/\gamma}(s-m)(\Delta\rho(s))^2}{4\beta g_c(s)\rho(s)Q^{(1/\gamma)-1}(s+1)}\right]=\infty. \tag{2.21}$$

Then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t)>0$, $x(t-m+1)>0$, and $x(t-k)>0$ for $t\ge t_1$ for some $t_1\ge t_0$. Proceeding as in the proof of Theorem 1, we conclude that $\Delta y(t)>0$ for $t\ge t_2$ and $y(t)$ satisfies either Case (I) or Case (II) for $t\ge t_2$. If (I) holds, then as in the proof of Theorem 1, we again obtain (2.12). Since $a(t)(\Delta y(t))^\gamma$ is nonincreasing and $y(t)$ is nondecreasing, we have

$$a^{1/\gamma}(t-m)\Delta y(t-m)\ge a^{1/\gamma}(t+1)\Delta y(t+1) \quad\text{and}\quad 1/y(t-m)\ge 1/y(t-m+1),$$

so

$$\begin{aligned}\Delta w(t) &\le -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) w(t+1)\\ &\quad - \frac{\beta\rho(t)}{\rho(t+1)} \frac{a^{1/\gamma}(t+1)\Delta y(t+1)}{a^{1/\gamma}(t-m)y(t-m+1)} w(t+1)\\ &\le -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) w(t+1)\\ &\quad - \frac{\beta\rho(t)}{\rho^{1+1/\gamma}(t+1)} \frac{y^{\frac{\beta-\gamma}{\gamma}}(t-m+1)}{a^{1/\gamma}(t-m)} w^{1+1/\gamma}(t+1)\\ &\le -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) w(t+1)\\ &\quad - \frac{\beta\rho(t)}{\rho^{1+1/\gamma}(t+1)} \frac{g_{c^*}(t)}{a^{1/\gamma}(t-m)} w^{1+1/\gamma}(t+1).\end{aligned}$$

From (2.10) we see that $w^{1-1/\gamma}(t+1)/\rho^{1-1/\gamma}(t+1) \ge Q^{1-1/\gamma}(t+1)$, so

$$\begin{aligned}\Delta w(t) &\le -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) w(t+1)\\ &\quad - \frac{\beta\rho(t)}{a^{1/\gamma}(t-m)\rho^2(t+1)} g_{c^*}(t) Q^{1/\gamma-1}(t+1) w^2(t+1).\end{aligned}$$

Completing the square on the second and third terms on the right gives

$$\Delta w(t) \le -\rho(t)q(t) + + \frac{a^{1/\gamma}(t-m)(\Delta\rho(t))^2}{4\beta g_{c^*}(t)\rho(t)Q^{(1/\gamma)-1}(t+1)}.$$

The remainder of the proof is similar to that of Theorem 1 and is omitted. □

Example 2. Consider the neutral equation

$$\Delta\left(t^3\Delta\left(x(t) - \frac{1}{3}x^{1/3}(t-2)\right)^3\right) + \frac{1}{\ln t}x(t-3) = 0,\ t > 1. \tag{2.22}$$

Here, $k = 2$, $m = 4$, $\alpha = 1/3$, and $\gamma = 3$. All conditions of Theorem 3 are satisfied with $\rho \equiv 1$ and hence equation (2.22) is oscillatory.

Next, we present some new and easily verifiable oscillation criteria for equation (1.1).

Theorem 4. *Let $\alpha = 1$ and conditions (i)–(iv) and (1.2) hold. Assume that condition (2.3) holds and*

$$\limsup_{t\to\infty} A^{\beta}(t-m,t_0)Q(t) > 1 \tag{2.23}$$

if $\beta = \gamma$, and condition (2.4) holds and

$$\limsup_{t\to\infty} A^{\beta}(t-m,t_0)Q(t) = \infty \tag{2.24}$$

if $\beta < \gamma$. Then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(t-m+1) > 0$, and $x(t-k) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Proceeding as in the proof of Theorem 1, we conclude that $\Delta y(t) > 0$ for $t \geq t_2$ and $y(t)$ satisfies either (I) or (II) for $t \geq t_2$. If (I) holds, then as in the proof of Theorem 1, we obtain (2.7) and (2.8). Using the fact that $a(t)(\Delta y(t))^{\gamma}$ is decreasing, we have

$$\begin{aligned} w(t) &:= a(t)(\Delta y(t))^{\gamma} \geq Q(t)\mu^{\beta}(t-m)(\Delta y(t-m))^{\beta} \\ &= Q(t)\mu^{\beta}(t-m)\left(a^{-\beta/\gamma}(t-m)\right)\left(a(t-m)(\Delta y(t-m))^{\gamma}\right)^{\beta/\gamma} \\ &\geq Q(t)\mu^{\beta}(t-m)\left(a^{-\beta/\gamma}(t-m)\right)\left(a(t)(\Delta y(t))^{\gamma}\right)^{\beta/\gamma} \\ &= Q(t)\mu^{\beta}(t-m)\left(a^{-\beta/\gamma}(t-m)\right)w^{\beta/\gamma}(t), \end{aligned}$$

or

$$\begin{aligned} w^{1-\beta/\gamma}(t) &\geq Q(t)\mu^{\beta}(t-m)\left(a^{-\beta/\gamma}(t-m)\right) \\ &= Q(t)\left(\sum_{s=t_2}^{t-m} a^{-1/\gamma}(s)\right)^{\beta} = A^{\beta}(t-m,t_2)Q(t). \end{aligned}$$

Taking lim sup of both sides of this inequality as $t \to \infty$, we arrive at a contradiction to condition (2.23) if $\beta = \gamma$ and to condition (2.24) and the boundedness of $w(t)$ if $\beta < \gamma$. The proof of Case (II) is similar to that in the proof of Theorem 1 and is omitted. □

Remark 2. We may note that corollaries similar to Corollaries 1 and 2 can be also drawn from Theorems 2–4. The details are left to the reader.

In conclusion, we would like to point out that our results in this paper can be extended to higher order equations of the form

$$\Delta\left(a(t)\left(\Delta^{n-1}(x(t)-p(t)x(t-k))\right)^{\gamma}\right) + q(t)x^{\beta}(t+1-m) = 0,$$

where n is a positive integer. Also, it would be of interest to study equation (1.1) in the case where $\beta > \gamma$.

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Authors' addresses

Said R. Grace
Cairo University, Department of Engineering Mathematics, Faculty of Engineering, Orman, Giza 12221, Egypt
E-mail address: saidgrace@yahoo.com

John R. Graef
Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, USA
E-mail address: John-Graef@utc.edu

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COINCIDENCE POINT RESULTS IN B-METRIC SPACES VIA C_F-s-SIMULATION FUNCTION

ANURADHA GUPTA AND MANU ROHILLA

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Abstract. The notion of C_F-s-simulation function is introduced and the existence and uniqueness of coincidence point of two self mappings in the framework of b-metric spaces is investigated. An example with a corresponding numerical simulation is also provided to support the obtained result.

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1. Introduction and preliminaries

Fixed point theory is a widely used tool in mathematical analysis and its applications. Over the decades this field has intrigued researchers and has developed extensively. Numerous authors have generalized metric spaces and contraction principle. Bakhtin [3] and Czerwik [6] generalized the notion of metric space and introduced the concept of b-metric space. Many mathematicians have obtained fixed point and coincidence point results in various generalizations of metric spaces. Mleşniţe [11] and Falset and Mleşniţe [7] studied the existence, uniqueness and Ulam-Hyers stability for the coincidence point problem of a pair of single-valued mappings. Also, Petruşel et al. [12] investigated the existence and uniqueness of coincidence points of a pair of operators satisfying contraction and expansion type conditions in the setting of b-metric spaces.

Recently, Khojasteh [9] introduced the notion of simulation function and unified several known fixed point theorems in the setting of metric spaces. In fact, Hierro et al. [15] obtained coincidence point of two self mappings in the framework of metric spaces by using simulation functions. Also, Yamaod and Sintunavarat [17] studied the existence and uniqueness of fixed point of nonlinear mappings in the context of b-metric spaces involving s-simulation functions.

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Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the set of natural numbers and real numbers, respectively. The following terminologies and definitions will be used in the sequel:

Definition 1 ([6])**.** A b-metric on a non-empty set X is a function $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$ and a constant $s \geq 1$, the following conditions hold:

(b_1) $d(x, y) = 0$ if and only if $x = y$,
(b_2) $d(x, y) = d(y, x)$,
(b_3) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a b-metric space. The number s is called the coefficient of (X, d).

Definition 2 ([5])**.** Let (X, d) be a b-metric space. Then
(i) A sequence $\{x_n\} \subseteq X$ *converges* to $x \in X$ if and only if $\lim_{n\to\infty} d(x_n, x) = 0$.
(ii) A sequence $\{x_n\} \subseteq X$ is called a *Cauchy sequence* if and only if

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

(iii) A b-metric space (X, d) is said to be *complete* if every Cauchy sequence $\{x_n\} \subseteq X$ converges to a point $x \in X$ such that $\lim_{n,m\to\infty} d(x_n, x_m) = 0 = \lim_{n\to\infty} d(x_n, x)$.
(iv) A mapping $T : X \to X$ is said to be *b-continuous* if for $\{x_n\} \subseteq X$, $x_n \to x$ in (X, d) implies that $Tx_n \to Tx$ in (X, d).

Remark 1. In a b-metric space (X, d) a convergent sequence has a unique limit.

Let $T, S : X \to X$ be self mappings on a b-metric space (X, d). If $y = Tx = Sx$ for some $x \in X$, then x is called a coincidence point of T and S and y is called a point of coincidence of T and S. If $Tx = Sx = x$ for some $x \in X$, then x is called a common fixed point of T and S. We say that the pair (T, S) is compatible if $\lim_{n\to\infty} d(TSx_n, STx_n) = 0$ for every sequence $\{x_n\} \subseteq X$ such that the sequences $\{Tx_n\}$ and $\{Sx_n\}$ are convergent and have the same limit. We say that T and S are weakly compatible if T and S commute at their coincidence points. A sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a Picard-Jungck sequence of the pair (T, S) (based at x_0) if $y_n = Tx_n = Sx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. If $T(X) \subseteq S(X)$ then there exists a Picard-Jungck sequence of (T, S) based at any point $x_0 \in X$. The following result of Abbas and Jungck [1] establishes the relationship between point of coincidence and common fixed point of T and S.

Proposition 1. *Let T and S be weakly compatible self mappings on a set X. If T and S have a unique point of coincidence $y = Tx = Sx$, then y is the unique common fixed point of T and S.*

Definition 3 ([2])**.** A mapping $F : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is called a *C-class function* if it satisfies the following conditions:

(i) F is continuous,
(ii) $F(v,u) \leq v$ for all $u,v \in [0,\infty)$,
(iii) $F(v,u) = v$ implies that either $u = 0$ or $v = 0$ for all $u,v \in [0,\infty)$.

Definition 4 ([10])**.** A mapping $F : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is said to satisfy *property* C_F, if there exists $C_F \geq 0$ such that
(i) $F(v,u) > C_F$ implies that $v > u$,
(ii) $F(u,u) \leq C_F$ for all $u \in [0,\infty)$.

Some examples of C-class functions having property C_F are:
(i) $F(v,u) = v - u$, $C_F = r$, where $r \in [0,\infty)$,
(ii) $F(v,u) = \frac{v}{1+u}$, $C_F = 1,2$,
(iii) $F(v,u) = \frac{kv}{1+u}$, $0 < k < 1$, $C_F = 1,k$.
For more examples of C-class functions having property C_F see [10, 14]. Liu et al. [10] generalized the simulation function introduced by Khojasteh et al. [9] using C-class function as follows:

Definition 5. A C_F-simulation function is a mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:
(i) $\zeta(0,0) = 0$,
(ii) $\zeta(u,v) < F(v,u)$ for all $u,v > 0$, where $F : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is a C-class function satisfying property C_F,
(iii) if $\{u_n\}$ and $\{v_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n > 0$ and $u_n < v_n$, then $\limsup_{n\to\infty} \zeta(u_n,v_n) < C_F$.

Yamaod and Sintunavarat [17] defined the concept of s-simulation function as follows:

Definition 6. Let $s \geq 1$ be a given real number. A function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is said to be an *s-simulation function* if it satisfies the following conditions:
(i) $\zeta(u,v) < v - u$ for all $u,v > 0$,
(ii) if $\{u_n\}$ and $\{v_n\}$ are sequences in $(0,\infty)$ such that

$$0 < \liminf_{n\to\infty} u_n \leq s\Big(\limsup_{n\to\infty} v_n\Big) \leq s^2\Big(\liminf_{n\to\infty} u_n\Big) < \infty$$

and

$$0 < \liminf_{n\to\infty} v_n \leq s\Big(\limsup_{n\to\infty} u_n\Big) \leq s^2\Big(\liminf_{n\to\infty} v_n\Big) < \infty,$$

then $\limsup_{n\to\infty} \zeta(u_n,v_n) < 0$.

If we take $s = 1$ then ζ is a simulation function in the sense of Khojasteh [9] if and only if ζ is an s-simulation function.

Liu et al. [10] and Radenović and Chandok [14] generalized the simulation function defined by Khojasteh et al. [9]. Motivated by them we have generalized the

s-simulation function introduced by Yamaod and Sintunavarat [17] using C-class functions having property C_F. We have generalized the fixed point results proved in [17] to the coincidence point case. We introduce the notion of C_F-s-simulation function. It is observed that every s-simulation function is a C_F-s-simulation function but the converse is not true in general. Our concept has broadened the family of s-simulation functions. The main objective of the paper is to establish the existence and uniqueness of point of coincidence of a pair of self mappings in the setting of b-metric spaces via C_F-s-simulation function, covering the case of commuting and compatible mappings. This approach enables us to study several coincidence point and fixed point problems from a common perspective. The purpose is to unify, generalize and improve several existing results in b-metric spaces. We underline that our approach has generalized the main results of [13, 17]. An example with a corresponding numerical simulation is also provided to demonstrate the utility of the results.

2. Main results

In this section, we establish the existence and uniqueness of coincidence point and common fixed point in the context of b-metric spaces. We begin with the following definition:

Definition 7. Let $s \geq 1$ be a given real number. A C_F-*s-simulation function* is a mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:

(i) $\zeta(u,v) < F(v,u)$ for all $u,v > 0$, where $F : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is a C-class function satisfying property C_F,

(ii) if $\{u_n\}$ and $\{v_n\}$ are sequences in $(0,\infty)$ such that

$$0 < \liminf_{n\to\infty} u_n \leq s\Big(\limsup_{n\to\infty} v_n\Big) \leq s^2\Big(\liminf_{n\to\infty} u_n\Big) < \infty$$

and

$$0 < \liminf_{n\to\infty} v_n \leq s\Big(\limsup_{n\to\infty} u_n\Big) \leq s^2\Big(\liminf_{n\to\infty} v_n\Big) < \infty,$$

then $\limsup_{n\to\infty} \zeta(u_n, v_n) < C_F$.

Let $\mathscr{Z}_{F_s}$ be the family of all C_F-s-simulation functions. Every s-simulation function is a C_F-s-simulation function but the converse may not be true in general. This can be illustrated by taking $F(v,u) = v - u$ and $C_F = 0$ in Example 3.3 of [15] in which $k \in \mathbb{R}$ be such that $k < 1$ and $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined as

$$\zeta(u,v) = \begin{cases} 2(v-u), & \text{if } v < u, \\ kv - u, & \text{otherwise.} \end{cases}$$

Definition 8. Let (X,d) be a b-metric space and $T, S : X \to X$ be self mappings. Then T is called a $(\mathscr{Z}_{F_s}, S)$-contraction if there exists $\zeta \in \mathscr{Z}_{F_s}$ such that

$$\zeta(d(Tx,Ty), d(Sx,Sy)) \geq C_F \tag{2.1}$$

for all $x, y \in X$ with $Sx \neq Sy$.

Theorem 1. *Let (X,d) be a b-metric space with coefficient $s \geq 1$, $T, S : X \to X$ be self mappings and T be a $(\mathscr{Z}_{F_s}, S)$-contraction. Assume that $T(X) \subseteq S(X)$ and atleast one of the following conditions hold:*
(i) $(T(X), d)$ or $(S(X), d)$ is complete,
(ii) (X,d) is complete, S is b-continuous and (T,S) is compatible,
(iii) (X,d) is complete, S is b-continuous and T and S are commuting.
Then T and S have a unique point of coincidence.

Proof. Since $T(X) \subseteq S(X)$, there exists a Picard-Jungck sequence $\{x_n\}$ such that $y_n = Tx_n = Sx_{n+1}$, where $n \in \mathbb{N} \cup \{0\}$. If $y_{n_0} = y_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$, then $Tx_{n_0} = Sx_{n_0+1} = y_{n_0} = y_{n_0+1} = Tx_{n_0+1} = Sx_{n_0+2}$. This implies that T and S have a point of coincidence. Therefore, suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Putting $x = x_{n+1}$ and $y = x_{n+2}$ in (2.1) we get,

$$\begin{aligned} C_F &\leq \zeta(d(y_{n+1}, y_{n+2}), d(y_n, y_{n+1})) \\ &< F(d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2})). \end{aligned}$$

By (i) of Definition 4, $d(y_n, y_{n+1}) > d(y_{n+1}, y_{n+2})$. Then $\{d(y_n, y_{n+1})\}$ is a decreasing sequence of non-negative real numbers therefore, it is convergent. Let $\lim_{n\to\infty} d(y_n, y_{n+1}) = L \geq 0$. Suppose that $L > 0$ then $0 < L \leq sL \leq s^2 L < \infty$. This implies that

$$\begin{aligned} 0 < \liminf_{n\to\infty} d(y_{n+1}, y_{n+2}) &\leq s\Big(\limsup_{n\to\infty} d(y_n, y_{n+1})\Big) \\ &\leq s^2\Big(\liminf_{n\to\infty} d(y_{n+1}, y_{n+2})\Big) < \infty \end{aligned}$$

and

$$\begin{aligned} 0 < \liminf_{n\to\infty} d(y_n, y_{n+1}) &\leq s\Big(\limsup_{n\to\infty} d(y_{n+1}, y_{n+2})\Big) \\ &\leq s^2\Big(\liminf_{n\to\infty} d(y_n, y_{n+1})\Big) < \infty. \end{aligned}$$

Using (ii) of Definition 7 we have $C_F \leq \zeta(d(y_{n+1}, y_{n+2}), d(y_n, y_{n+1})) < C_F$, a contradiction. Therefore, $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. Now we prove that $\{y_n\}$ is a Cauchy sequence in (X,d). Assume that $\{y_n\}$ is not Cauchy in (X,d). Then there exists $\epsilon_0 > 0$ for which we can find two subsequences $\{y_{n_i}\}$ and $\{y_{m_i}\}$ of $\{y_n\}$ such that n_i is the smallest integer for which

$$n_i > m_i > i, \qquad d(y_{m_i}, y_{n_i}) \geq \epsilon_0. \tag{2.2}$$

This means

$$d(y_{m_i}, y_{n_i-1}) < \epsilon_0. \tag{2.3}$$

Since $\epsilon_0 \leq d(y_{m_i}, y_{n_i}) \leq sd(y_{m_i}, y_{n_i-1}) + sd(y_{n_i-1}, y_{n_i}) < s\epsilon_0 + sd(y_{n_i-1}, y_{n_i})$. Taking limit superior as $i \to \infty$ we get,

$$\epsilon_0 \leq \limsup_{i\to\infty} d(y_{m_i}, y_{n_i}) \leq s\epsilon_0. \tag{2.4}$$

Similarly, we have

$$\epsilon_0 \leq \liminf_{i\to\infty} d(y_{m_i}, y_{n_i}) \leq s\epsilon_0. \tag{2.5}$$

Putting $x = x_{m_i}$ and $y = x_{n_i}$ in (2.1) we get,

$$C_F \leq \zeta(d(y_{m_i}, y_{n_i}), d(y_{m_i-1}, y_{n_i-1})) < F(d(y_{m_i-1}, y_{n_i-1}), d(y_{m_i}, y_{n_i})).$$

By (i) of Definition 4 we have $d(y_{m_i}, y_{n_i}) < d(y_{m_i-1}, y_{n_i-1})$. Therefore, $\epsilon_0 \leq d(y_{m_i}, y_{n_i}) < d(y_{m_i-1}, y_{n_i-1}) \leq sd(y_{m_i-1}, y_{m_i}) + sd(y_{m_i}, y_{n_i-1}) < sd(y_{m_i-1}, y_{m_i}) + s\epsilon_0$. Taking limit superior as $i \to \infty$ we get,

$$\epsilon_0 \leq \limsup_{i\to\infty} d(y_{m_i-1}, y_{n_i-1}) \leq s\epsilon_0. \tag{2.6}$$

Similarly, we have

$$\epsilon_0 \leq \liminf_{i\to\infty} d(y_{m_i-1}, y_{n_i-1}) \leq s\epsilon_0. \tag{2.7}$$

Using (2.4), (2.5), (2.6) and (2.7) we have

$$\begin{aligned} 0 < \liminf_{i\to\infty} d(y_{m_i}, y_{n_i}) \leq s\epsilon_0 &\leq s\Big(\limsup_{i\to\infty} d(y_{m_i-1}, y_{n_i-1})\Big) \\ &\leq s^2\epsilon_0 \leq s^2\Big(\liminf_{i\to\infty} d(y_{m_i}, y_{n_i})\Big) < \infty \end{aligned}$$

and

$$\begin{aligned} 0 < \liminf_{i\to\infty} d(y_{m_i-1}, y_{n_i-1}) \leq s\epsilon_0 &\leq s\Big(\limsup_{i\to\infty} d(y_{m_i}, y_{n_i})\Big) \\ &\leq s^2\epsilon_0 \leq s^2\Big(\liminf_{i\to\infty} d(y_{m_i-1}, y_{n_i-1})\Big) < \infty. \end{aligned}$$

By (ii) of Definition 7 we have $C_F \leq \zeta(d(y_{m_i}, y_{n_i}), d(y_{m_i-1}, y_{n_i-1})) < C_F$, a contradiction. Therefore, $\{y_n\}$ is a Cauchy sequence in (X, d).

Suppose that (i) holds. Assume that $(S(X), d)$ (or $(T(X), d)$) is complete. Then there exists $w \in S(X)$ such that $\lim_{n\to\infty} d(Sx_n, w) = 0$. Since $Tx_n = Sx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, $\lim_{n\to\infty} d(Tx_n, w) = 0$. Let $z \in X$ such that $Sz = w$. We shall show that z is a coincidence point of T and S. We have $C_F \leq \zeta(d(Tx_n, Tz), d(Sx_n, Sz)) < F(d(Sx_n, Sz), d(Tx_n, Tz))$. Therefore, $d(Tx_n, Tz) < d(Sx_n, Sz)$ which implies that $\lim_{n\to\infty} d(Tx_n, Tz) = 0$. Since limit of a convergent sequence in a b-metric space is unique, $Tz = Sz = w$. Thus, w is a point of coincidence of T and S.

Suppose that (ii) holds. Since (X,d) is complete, there exists $w' \in X$ such that $\lim_{n\to\infty} d(Sx_n, w') = 0$. Therefore, $\lim_{n\to\infty} d(Tx_n, w') = 0$. As S is b-continuous, $\lim_{n\to\infty} d(STx_n, Sw') = 0$ and $\lim_{n\to\infty} d(SSx_n, Sw') = 0$. We have

$$C_F \le \zeta(d(TSx_n, Tw'), d(SSx_n, Sw')) < F(d(SSx_n, Sw'), d(TSx_n, Tw')).$$

Therefore, $d(TSx_n, Tw') < d(SSx_n, Sw')$ which gives $\lim_{n\to\infty} d(TSx_n, Tw') = 0$. Consider

$$\begin{aligned} d(Tw', Sw') &\le sd(Tw', TSx_n) + sd(TSx_n, Sw') \\ &\le sd(Tw', TSx_n) + s^2 d(TSx_n, STx_n) + s^2 d(STx_n, Sw'). \end{aligned}$$

Taking limit as $n \to \infty$ and using (T,S) is compatible we have $Tw' = Sw'$. Therefore, w' is a coincidence point of T and S.

Finally, suppose that (iii) holds. Since T and S are commuting then $\lim_{n\to\infty} d(TSx_n, STx_n) = 0$. The proof is similar to the case when (ii) holds. Let w_1 and w_2 be two distinct point of coincidence of T and S. Then there exists z_1, $z_2 \in X$ such that $w_1 = Tz_1 = Sz_1$ and $w_2 = Tz_2 = Sz_2$. We have $C_F \le \zeta(d(Tz_1, Tz_2), d(Sz_1, Sz_2)) < F(d(w_1, w_2), d(w_1, w_2))$. Using (i) of Definition 4 we have $d(w_1, w_2) < d(w_1, w_2)$, a contradiction. Hence, T and S have a unique point of coincidence. □

The following example illustrates the efficiency of Theorem 1 by establishing the existence and uniqueness of the solution of a nonlinear equation.

Example 1. Let $X = [0,\infty)$ and $d : X \times X \to [0,\infty)$ be defined as

$$d(x,y) = \begin{cases} (x+y)^2, & \text{if } x \ne y, \\ 0, & \text{if } x = y, \end{cases}$$

for all $x, y \in X$. Then (X,d) is a complete b-metric space with coefficient $s = 2$. Define $T, S : X \to X$ as $Tx = 2x$ and $Sx = e^x + 6x - 1$. Take $\zeta(u,v) = \frac{1}{3}(v - 7u)$, $F(v,u) = v - u$ and $C_F = 0$. Consider

$$\begin{aligned} \zeta(d(Tx,Ty), d(Sx,Sy)) &= \frac{1}{3}\{(e^x + 6x + e^y + 6y - 2)^2 - 7(2x+2y)^2\} \\ &\ge \frac{1}{3}\{(6x+6y)^2 - 28(x+y)^2\} > 0. \end{aligned}$$

Therefore, T is a $(\mathscr{Z}_{F_s}, S)$-contraction. Also, we observe that $T(X) \subseteq S(X)$ and both $(T(X), d)$ and $(S(X), d)$ are complete. Hence, by Theorem 1 T and S have a unique coincidence point 0. For an initial point $x_0 = 0.2, 0.5, 1, 1.5$, the Picard-Jungck iterations are listed below. Also, the behavior of the process is shown by a graph.

TABLE 1. Picard-Jungck iterations

y_i	$x_0 = 0.2$	$x_0 = 0.5$	$x_0 = 1.0$	$x_0 = 1.5$
y_0	0.4000000000	1.0000000000	2.0000000000	3.0000000000
y_1	0.1138141813	0.2827202137	0.5591451924	0.8288366299
y_2	0.0324804548	0.0805423810	0.1588304694	0.2347627942
y_3	0.0092770515	0.0229931547	0.0453062682	0.0669133785
y_4	0.0026503351	0.0065679304	0.0129386562	0.0191050307
y_5	0.0007572181	0.0018764257	0.0036962706	0.0054575155
y_6	0.0002163463	0.0005361113	0.0010560375	0.0015592033
y_7	0.0000618131	0.0001531738	0.0003017217	0.0004454795
y_8	0.0000176608	0.0000437638	0.0000862059	0.0001272792
y_9	0.0000050459	0.0000125039	0.0000246302	0.0000363654
y_{10}	0.0000014417	0.0000035725	0.0000070372	0.0000103901
y_{11}	0.0000004119	0.0000010207	0.0000020106	0.0000029686
y_{12}	0.0000001176	0.0000002916	0.0000005744	0.0000008481
y_{13}	0.0000000336	0.0000000833	0.0000001641	0.0000002423
y_{14}	0.0000000096	0.0000000238	0.0000000468	0.0000000692
y_{15}	0.0000000027	0.0000000068	0.0000000133	0.0000000197
y_{16}	0.0000000007	0.0000000019	0.0000000038	0.0000000056
y_{17}	0.0000000002	0.0000000005	0.0000000010	0.0000000016
y_{18}	0.0000000000	0.0000000001	0.0000000003	0.0000000004
y_{19}	0.0000000000	0.0000000000	0.0000000000	0.0000000001
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

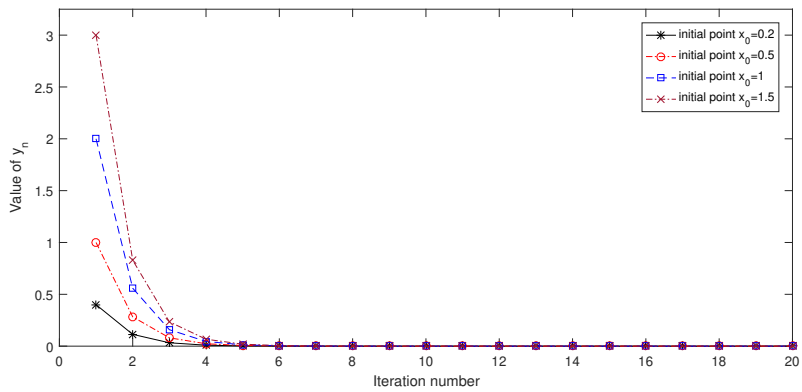

FIGURE 1. Behavior of iteration process

In the following results we see that several existing results in the literature can be obtained via the C_F-s-simulation function. We observe that the main result of [17, Theorem 4.4] can be easily deduced from Theorem 1.

Corollary 1. *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a mapping satisfying*

$$\zeta(d(Tx,Ty),d(x,y)) \geq 0$$

for all $x,y \in X$, where $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is an s-simulation function. Then T has a unique fixed point in X.

Proof. Take $F(v,u) = v-u$, $C_F = 0$ and $S = I$, where $I : X \to X$ is the identity mapping in Theorem 1. Then the results follows. □

The well-known Banach contraction principle in the framework of b-metric spaces [8, Theorem 3.3] can be deduced as follows:

Corollary 2. *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a mapping satisfying*

$$d(Tx,Ty) \leq \lambda d(x,y)$$

for all $x,y \in X$, where $\lambda \in \left[0,\frac{1}{2}\right)$. Then T has a unique fixed point in X.

Proof. Define the mappings $\zeta_1, F : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $\zeta_1(u,v) = \lambda v - u$ and $F(v,u) = v-u$. Take $C_F = 0$ and S be the identity mapping on X then $\zeta_1 \in \mathscr{Z}_{F_s}$. The desired result follows by taking $\zeta = \zeta_1$ in Theorem 1. □

Corollary 3 (Rhoades Type)**.** *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a mapping satisfying*

$$d(Tx,Ty) \leq d(x,y) - \phi(d(x,y))$$

for all $x,y \in X$, where $\phi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous function such that $\phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point in X.

Proof. Define the mappings $\zeta_2, F : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $\zeta_2(u,v) = v - \phi(v) - su$ and $F(v,u) = v-u$. Take $C_F = 0$ and S be the identity mapping on X then $\zeta_2 \in \mathscr{Z}_{F_s}$. Taking $\zeta = \zeta_2$ in Theorem 1 we get that T has a unique fixed point in X. □

Berinde [4] introduced the notion of b-comparison function as follows:

Definition 9. Let $s \geq 1$ be a given real number. A function $\psi : [0,\infty) \to [0,\infty)$ is called a b-comparison function if it satisfies

(i) ψ is monotonically increasing,

(ii) there exists $n_0 \in \mathbb{N}$, $\kappa \in (0,1)$ and a convergent series of non-negative terms $\sum_{n=1}^{\infty} a_n$ such that for all $n \geq n_0$ and $t > 0$ we have

$$s^{n+1}\psi^{n+1}(t) \leq \kappa s^n \psi^n(t) + a_n.$$

Remark 2 ([13]). If ψ is a b-comparison function, then $\psi(t) < t$ for all $t > 0$.

Păcurar in [13] obtained fixed point results of contraction mappings involving b-comparison function. We observe that the main result of [13, Theorem 4] can be inferred in the following way:

Corollary 4. *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a mapping satisfying*

$$d(Tx,Ty) \leq \psi(d(x,y))$$

for all $x,y \in X$, where ψ is a b-comparison function. Then T has a unique fixed point in X.

Proof. Define the mappings $\zeta_3, F : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $\zeta_3(u,v) = \psi(v) - su$ and $F(v,u) = v - u$. Take $C_F = 0$ and S be the identity mapping on X then $\zeta_3 \in \mathscr{Z}_{F_s}$. Taking $\zeta = \zeta_3$ in Theorem 1 we get the desired result. □

The following theorem is a direct consequence of Theorem 1 and Proposition 1.

Theorem 2. *Let (X,d) be a b-metric space with coefficient $s \geq 1$, $T,S : X \to X$ be self mappings and T be a $(\mathscr{Z}_{F_s}, S)$-contraction. Assume that $T(X) \subseteq S(X)$ and atleast one of the following conditions hold:*
(i) $(T(X),d)$ or $(S(X),d)$ is complete,
(ii) (X,d) is complete, S is b-continuous and (T,S) is compatible,
(iii) (X,d) is complete, S is b-continuous and T and S are commuting.
Moreover, assume that T and S are weakly compatible. Then T and S have a unique common fixed point in X.

In the sequel, we generalize several known results in the context of b-metric spaces via C_F-s-simulation functions.

Theorem 3. *Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $T,S : X \to X$ be self mappings. Suppose that $\zeta \in \mathscr{Z}_{F_s}$ and satisfies*

$$\zeta\Big(d(Tx,Ty), \max\Big\{d(Sx,Sy), d(Tx,Sx), d(Ty,Sy), \frac{d(Tx,Sy)+d(Sx,Ty)}{2s}\Big\}\Big) \geq C_F \tag{2.8}$$

for all $x,y \in X$ with $Sx \neq Sy$. Assume that $T(X) \subseteq S(X)$ and atleast one of the following conditions hold:
(i) $(T(X),d)$ or $(S(X),d)$ is complete,

(ii) (X,d) is complete, S is b-continuous and (T,S) is compatible,
(iii) (X,d) is complete, S is b-continuous and T and S are commuting.
Then T and S have a unique point of coincidence. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X.

Proof. Proceeding similar to Theorem 1 we get a Picard-Jungck sequence $\{x_n\}$ such that $y_n = Tx_n = Sx_{n+1}$, where $n \in \mathbb{N} \cup \{0\}$. Putting $x = x_{n+1}$ and $y = x_{n+2}$ in (2.8) we get,

$$\begin{aligned} C_F &\leq \zeta\Big(d(y_{n+1}, y_{n+2}), \max\Big\{d(y_n, y_{n+1}), d(y_{n+2}, y_{n+1}), \frac{d(y_n, y_{n+2})}{2s}\Big\}\Big) \\ &< F\Big(\max\Big\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2}), \frac{d(y_n, y_{n+2})}{2s}\Big\}, d(y_{n+1}, y_{n+2})\Big). \end{aligned}$$

Therefore, $d(y_{n+1}, y_{n+2}) < \max\Big\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2}), \frac{d(y_n, y_{n+2})}{2s}\Big\}$.
If $d(y_{n+1}, y_{n+2}) < d(y_{n+1}, y_{n+2})$, a contradiction. If $d(y_{n+1}, y_{n+2}) < \frac{d(y_n, y_{n+2})}{2s} \leq \frac{d(y_n, y_{n+1}) + (y_{n+1}, y_{n+2})}{2}$, then $d(y_{n+1}, y_{n+2}) < d(y_n, y_{n+1})$. Following the lines in the proof of Theorem 1 and Theorem 2 we get the desired result. □

Corollary 5. *Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $T, S : X \to X$ be self mappings satisfying*

$$s^3 d(Tx, Ty) \leq k \max\Big\{d(Sx, Sy), d(Tx, Sx), d(Ty, Sy), \frac{d(Tx, Sy) + d(Sx, Ty)}{2s}\Big\}$$

for all $x, y \in X$, where $k \in [0,1)$. Assume that $T(X) \subseteq S(X)$ and atleast one of the following conditions hold:
(i) $(T(X), d)$ or $(S(X), d)$ is complete,
(ii) (X,d) is complete, S is b-continuous and (T,S) is compatible,
(iii) (X,d) is complete, S is b-continuous and T and S are commuting.
Then T and S have a unique point of coincidence. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X.

Proof. Define the mappings $\zeta_4, F : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $\zeta_4(u,v) = kv - s^3 u$ and $F(v,u) = v - u$. Take $C_F = 0$ then $\zeta_4 \in \mathscr{Z}_{F_s}$. By taking $\zeta = \zeta_4$ in Theorem 3 we get the result. □

Indeed, a result of Yamaod and Sintunavarat [16, Corollary 3.6] can be obtained by considering S to be the identity mapping on X in the above result.

The following theorem can be proved on the similar lines of Theorem 1.

Theorem 4. *Let (X,d) be a b-metric space with coefficient $s \geq 1$, $T, S : X \to X$ be self mappings and $G : [0,\infty) \to [0,\infty)$ be a mapping satisfying $G(0) = 0$ and $0 < G(t) \leq t$ for all $t > 0$. Suppose that $\zeta \in \mathscr{Z}_{F_s}$ and satisfies*

$$\zeta(d(Tx, Ty), G(d(Sx, Sy))) \geq C_F$$

for all $x, y \in X$ with $Sx \neq Sy$. Assume that $T(X) \subseteq S(X)$ and at least one of the following conditions hold:

(i) $(T(X), d)$ or $(S(X), d)$ is complete,

(ii) (X, d) is complete, S is b-continuous and (T, S) is compatible,

(iii) (X, d) is complete, S is b-continuous and T and S are commuting.

Then T and S have a unique point of coincidence. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X.

Theorem 5. *Let (X, d) be a b-metric space with coefficient $s \geq 1$ and $T, S : X \to X$ be self mappings. Suppose that $\zeta \in \mathscr{Z}_{F_s}$ and satisfies*

$$\begin{aligned}\zeta(d(Tx, Ty), \lambda \max\{d(Sx, Sy), d(Tx, Sx), d(Ty, Sy),\\ d(Tx, Sy), d(Ty, Sx)\}) \geq C_F\end{aligned} \tag{2.9}$$

for all $x, y \in X$ with $Sx \neq Sy$ and $\lambda \in \left(0, \frac{1}{2s}\right)$. Assume that $T(X) \subseteq S(X)$ and at least one of the following conditions hold:

(i) $(T(X), d)$ or $(S(X), d)$ is complete,

(ii) (X, d) is complete, S is b-continuous and (T, S) is compatible,

(iii) (X, d) is complete, S is b-continuous and T and S are commuting.

Then T and S have a unique point of coincidence. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X.

Proof. Following the lines in the proof of Theorem 1 we get a Picard-Jungck sequence $\{x_n\}$ such that $y_n = Tx_n = Sx_{n+1}$, where $n \in \mathbb{N} \cup \{0\}$. Putting $x = x_{n+1}$ and $y = x_{n+2}$ in (2.9) we get,

$$\begin{aligned}C_F &\leq \zeta(d(y_{n+1}, y_{n+2}), \lambda \max\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2}), d(y_n, y_{n+2})\})\\ &< F(\lambda \max\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2}), d(y_n, y_{n+2})\}, d(y_{n+1}, y_{n+2})).\end{aligned}$$

Therefore, $d(y_{n+1}, y_{n+2}) < \lambda \max\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2}), d(y_n, y_{n+2})\}$. If $d(y_{n+1}, y_{n+2}) < \lambda d(y_{n+1}, y_{n+2}) < d(y_{n+1}, y_{n+2})$, a contradiction.
If $d(y_{n+1}, y_{n+2}) < \lambda d(y_n, y_{n+2}) \leq \lambda s d(y_n, y_{n+1}) + \lambda s d(y_{n+1}, y_{n+2})$. Then $d(y_{n+1}, y_{n+2}) \leq \frac{\lambda s}{1-\lambda s} d(y_n, y_{n+1}) < d(y_n, y_{n+1})$. Proceeding as in the proof of Theorem 1 we establish the existence of coincidence point of T and S. Let w_1 and w_2 be two distinct point of coincidence of T and S. Then there exists z_1, $z_2 \in X$ such that $w_1 = Tz_1 = Sz_1$ and $w_2 = Tz_2 = Sz_2$. We have

$$\begin{aligned}C_F &\leq \zeta(d(Tz_1, Tz_2), \lambda \max\{d(Sz_1, Sz_2),\\ &\quad d(Tz_1, Sz_1), d(Tz_2, Sz_2), d(Tz_1, Sz_2), d(Tz_2, Sz_1)\})\\ &= \zeta(d(w_1, w_2), \lambda d(w_1, w_2)).\end{aligned}$$

Therefore, $d(w_1, w_2) < d(w_1, w_2)$, a contradiction. Hence, T and S have a unique point of coincidence. By Theorem 2, it follows that T and S have a unique common fixed point. □

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Authors' addresses

Anuradha Gupta
Department of Mathematics,, Delhi College of Arts and Commerce,, University of Delhi, Netaji Nagar, New Delhi-110023,, India
E-mail address: dishna2@yahoo.in

Manu Rohilla
Department of Mathematics,, University of Delhi, New Delhi-110007,, India
E-mail address: manurohilla25994@gmail.com

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ANALYSIS OF PRIORITY QUEUE WITH REPEATED ATTEMPTS USING GENERALIZED STOCHASTIC PETRI NETS

SEDDA HAKMI, OUIZA LEKADIR, AND DJAMIL AÏSSANI

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Abstract. The paper deals with the modeling and analysis of a single server queue with repeated attempts and two priority customers involving generalized stochastic Petri nets. Indeed, the consideration of the recalls and priority introduces great analytical difficulties. Therefore, by using the GSPN, we show how this high level formalism allows us to cope with the complexity of this system. The paper extends previous works on this topic and evaluate different performance characteristics of the system. The Markov chain is obtained and some numerical results are presented to illustrate the effect of the system parameters on the developed performance measures.

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1. Introduction

In recent years there have been significant contributions concerning retrial queues which are characterized by the feature for which arriving customers who find all servers not available (blocked), will join the retrial group (Orbit) to insist their demand after a random period.

Retrial queues have been widely used to model many problems in telephone switching systems, telecommunication networks and computer systems. For detailed overviews of the main results and methods on retrial queues, the reader can consult [2,9]. There are many applications of retrial queueing models for which it's necessary to assign priority to customers. This queuing system and its variants are used to model disk memory systems, star-like local area networks, and recently in wireless communication networks where resources are to be shared between many users in this case, introduce priority can cater to the necessity of some classes of customers (see [1,10]).

Several authors have studied retrial queues with priorities, high priority customers are queued and served according to a specific discipline. In the case of blocking, low priority customers leave the system and retry until they find the server free. In these systems the high priority customers have either preemptive or non-preemptive priority over the low priority customers. The presence of the repeated customers and

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priority on the queueing systems makes their analysis more difficult. Hence, different approximating algorithms and approaches are proposed. Krishna Kumar et al. [13] considered an $M/G/1$ retrial queueing system with two-phase service and possible preemptive resume service discipline. Drekic [8] extended the classical preemptive resume model by enabling previously rendered service to be time-constrained. The $M_1, M_2/G/1$ retrial queue with two types of customers, infinite priority queue for type-1 calls and infinite retrial group of type-2 calls, is investigated by Choi and Park [6]. They obtained the joint generating function of queue lengths by a supplementary variable method. Later Falin et al. [9] extended Choi and Park's model to the case where two types of calls may have different service time distributions. The case where the priority queue has finite capacity has been investigated by Bocharov et al. [5]. Choi et al. in [7] investigated an $M_1, M_2/G/l/K+1$ retrial queue and obtained the joint generating function of queue lengths. Moutzoukis and Langaris [15] extended Choi and Park's model in which there are multiple types of calls, structured batch arrivals and single vacation. They obtained the joint generating function of queue lengths. For a detailed review of the main results and the literature on this topic the reader can consult [3, 7, 9, 10, 15, 18].

A review of the literature reveals the remarkable fact that the non-homogeneity caused by the flow of repeated attempts, the consideration of the priorities and the finiteness of the sources of customers are the keys to understanding the analytical difficulties that arise in the study of the system that we consider. It is important to indicate that most works cited in the literature about the retrial systems with priorities are analyzed only by the queueing theory. Even, though using this theory, it is difficult to obtain analytical results and the last resort remains approximation methods, and simulation (see [1, 10, 17]). However, using generalized stochastic Petri nets (GSPN), it is possible to obtain performance indices either with analytical means or by numerical algorithms.

To the best of the authors' knowledge, and after a bibliographic research, there is no such study of finite sources retrial queueing systems with priority customers using the GSPN formalism. Nevertheless, the existing works in the area of retrial queues with priorities concern systems with infinite sources. In general, the queueing analysis of finite-source systems is more difficult than the infinite ones. However, in many practical systems, the number of users who access the system is finite, and it is often important to take into account the fact that the rate of customers arrivals decreases, as the number of customers in the system increases. This can be done with the help of finite source or quasi-random input models.

The difficulty in the queueing analysis of finite sources queues arises on the obtaining of the markov chain associated to theses systems. In this case, the GSPNs formalism are convenient for generating these corresponding Markov chains and consider features that may be hard to obtain directly by queueing theory. Moreover, the behavior of complex systems can be easily and efficiently represented by using GSPN,

rather than using Markov chains directly [11]. Thus, in this paper we analyze the $M_2/M_2/1//(N_1, N_2)$ retrial queueing system with a finite population and non preemptive priority. Gomez in [12] studied the finite population retrial queue with quasi-random input and Non-preemptive Priority and obtain some performances indices. However, these performance characteristics have been provided through supplementary variable method which have made the expressions cumbersome and the obtained results cannot be put into practice.

In the present paper, we prove how we apply a simple method, based on generalized stochastic Petri nets, to obtain easily the different performance indices of this system. GSPN are a high-level graphical formalism, which allows an easier description of complex systems behavior [11].

Let's outline the structure of the paper. After the Introduction in Section 1, Section 2 is devoted to a brief introduction to GSPN formalism. In Section 3, we describe the basic model of retrial queueing system with non preemptive priority. In Section 5, we present the associated GSPN model for $M_2/M_2/1//(N_1, N_2)$ retrial queueing system, and we detail the proposed stochastic analysis approach. Furthermore, we derive the computational formulas for evaluating exact performance indices of our system. Finally, Section 5 provides some numerical examples and concluding remarks.

2. An overview of Petri Nets

Petri Nets (PN), in their various shapes and sizes, have been used for the study of the qualitative properties of systems exhibiting concurrency and synchronization characteristics. They are directed bipartite graph [14]consisting of places (used to represent conditions) and transitions (used to describe events that occur in the system), that are connected by directed arcs. The places of the net can contain tokens moving from place to place by the firing of the transition representing an event in the system [4, 16]. When a transition fires tokens from input places are absorbed and tokens are created on each of the output places.

Many extensions to Petri Nets have been proposed to enlarge the class of problems that can be represented and, in order to model real systems. Among these extensions we have Generalized Stochastic Petri nets (GSPN), introduced by Marsan et al are the modeling formalism that can be conveniently used for the analysis of complex models of Discrete Event Dynami Systems (DEDS) and for their performance and reliability evaluation [4, 16]. They have been widely and successfully used as a good modeling tool for the qualitative and quantitative analysis of asynchronous concurrent systems with synchronization, nondeterminism, conflicts and sequencing. In GSPN some transitions are timed, while others are immediate. The immediate transitions are drawn as thin bars and fire without delay. The timed transitions are drawn as empty bars fire after an exponentially distributed delay. A marking in which at least one immediate transition is enabled, is called vanishing marking while a marking in which only timed transitions are enabled is known as a tangible marking. Molloy [16]

shows that the stochastic Petri nets are isomorphic to continues time Markov process with discrete space states. Thus, the aspect which contributed significantly to the development of Generalized Stochastic Petri nets is the fact that their performance analysis is based upon Markov theory. The states of the CTMC are the markings in the tangible reachability graph, and the state transition rates are the exponential firing rates of timed transitions in the GSPN. In this case, the performance measures of this GSPN model can be evaluated by computing the steady-state distribution π, using the following linear system:

$$\begin{cases} \pi.Q = 0; \\ \sum\limits_{i \in E} \pi_i = 1; \end{cases} \tag{2.1}$$

where: E is the set of the tangible markings; π_i denotes the steady-state probability that the process is in state M_i, Q is the infinitesimal generator of the Markov process and its elements are given as a function of the timed transitions firing rates.

3. Description of $M_2/M_2/1//(N_1, N_2)$ retrial queueing system

We consider retrial queueing system with two priority classes in which two classes of customers arrive independently from two independent finite-sources of sizes N_1 and N_2, following a quasi-random input rate. Customers from the first source are called high priority customers and customers from second sources are called low priority customers. The time between the service completion and the next service requirement of a customer in class i is an independent exponentially distributed random variable with rate λ_i, i=$\overline{1,2}$. The service facility has a single server that serves customers of all the priority classes. The service times of class i are exponential, independent and identically distributed random variables with rate μ_i, i=$\overline{1,2}$. If the server is free at the time of arrival of a primary customer, then the customer starts to be served. Any high priority customer which, upon arrival, finds the server busy is queued up in an ordinary queue. Upon blocking, low priority customers immediately join a pool of unsatisfied customers, called the orbit. Any orbiting customer try to connect with the server with classical retrial policy after an exponential time period with rate $\nu > 0$, independently to the other customers in orbit, until it finds the server free.

4. The $M_2/M_2/1//(N_1, N_2)$ retrial queueing system GSPN model

In what follows, we present the GSPN model describing $M_2/M_2/1//(N_1, N_2)$ retrial queueing systems (Fig. 1).

- The place $P._{Sour1}$ (respectively $P._{Sour2}$) contains the high priority (respectively low priority) customers, represented by N_1 (respectively N_2) tokens, which represents the condition that none of the N_1 and N_2 customers has arrived for service;

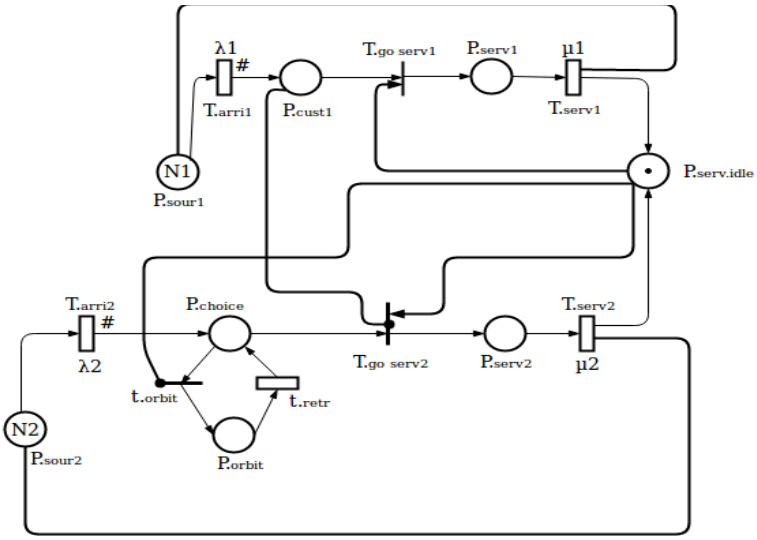


FIGURE 1. GSPN model for the $M_2/M_2/1//(N_1, N_2)$ retrial queueing system.

- The place $P_{\cdot Cust1}$ contains the high priority customers;
- The place $P_{\cdot Choice}$ represents the condition that a primary or a repeated call is ready for service;
- The place $P_{\cdot Orbit}$ represents the orbit;
- The place $P_{\cdot serv1}$ (respectively $P_{\cdot serv2}$) represents the condition that 'the server is busy by the high priority (respectively low priority) customer';
- The place $P_{\cdot serv.Idle}$ represents the condition 'the server is idle', represented by one token.
- Then, the initial marking of the net is :
  $M_o = M(P_{\cdot Sour1}), M(P_{\cdot Cust1}), M(P_{\cdot serv1}), M(P_{\cdot Sour2}), M(P_{\cdot Choice}),$
  $M(P_{\cdot Orbit}), M(P_{\cdot serv2}), M(P_{\cdot serv.Idle})$
  $M_o = (N_1, 0, 0, N_2, 0, 0, 0, 1).$

When the transition t_{Arri1} fires, one token is taken from P_{Sour1} and is deposited in P_{Cust1}. The firing of t_{Arri1} indicates the arrival of a high priority customer. This firing is marking dependent. Thus, the firing rate of t_{Arri1} depends on the number of tokens in P_{Sour1}. If we have N_1 tokens in P_{Sour1}, the firing rate is $N_1\lambda_1$. The condition of marking dependent firing is represented by the symbol # placed next to the transition t_{Arr1}.

If the arrived customer is a low priority one, the transition t_{Arri2} will fire, then the place P_{Choice} receives a token. Because the transition t_{Arri2} is a marking dependent, so the firing rate is $N_2\lambda_2$.

The immediate transition $tgo.serv1$ is enabled when the place $P_{Serv.Idle}$ contains one token (i.e. the server is idle), and the place P_{Cust1} is not empty (i.e there is at least one priority customer). Once the transition $tgo.serv1$ is fired, a token is removed from each of the two places $P_{Serv.Idle}$ and P_{Cust1}, then it's deposited in P_{Serv1}. This token represents a high priority customer in service.

The immediate transition $t.Orbit$ fires at the arrival of a low priority customer who finds no operational free server i.e. $P_{Serv.idle}$ is empty. Hence, he joins immediately the orbit represented by the place P_{Orbit}. Once in orbit, the customer starts generation of a flow of repeated calls exponentially distributed with rate ν. The firing of transition t_{Retr} represents the arrival of a repeated call from the orbit.

The immediate transition $t_{go.serv2}$ fires if the place P_{Cus1} is empty (This condition is expressed by the inhibitor arc from place P_{Cust1} to the transition $t_{go.serv2}$.), the place $P_{Serv.idle}$ contains one token which represents the idle server and the place P_{Choice} contains one token. So, the place P_{Serv2} receives a token representing a low priority customer in service.

The timed transition t_{Serv2} (respectively, the timed transition t_{Serv1}) is fired to determine the end of the low priority customer period service (respectively, the high priority customer period service). So, one token is deposited in P_{sour2} (respectively in P_{sour1}) which represents the condition that a low priority customer or a priority one will be returned to be idle and a second token is deposited in $P_{Srv.idle}$ which represents the condition that the server is ready to serve another customer.

4.1. *Stochastic analysis and performance measures*

The obtained GSPN model of Figure 2 is bounded and the initial marking is a home states then their steady-state probability distributions exist. In this case, several performance indices can be computed by the formulas given in the following subsections.

► **The mean arrival rate of the high priority requests $\bar{\lambda_1}$ (resp. low priority requests $\bar{\lambda_2}$) are:**

$$\bar{\lambda_1} = \sum_{j \in (SM_j)_1} \lambda_1(M_j)\pi_j, \quad \bar{\lambda_2} = \sum_{j \in (SM_j)_2} \lambda_2(M_j)\pi_j;$$

with : $(SM_j)_k$ is the set of markings where the transition t_{Arri_k} is enabled, and $\lambda_k(M_j)$ is the firing rate associated with the transition t_{Arri_k} in the marking M_j, with $k = \overline{1,2}$.

► **The mean retrial rate of low priority requests:**

The throughput of the transition t_{Retr} gives the mean retrial rate of low priority requests:

$$\bar{\eta} = \sum_{j \in (SM_j)o} \nu(M_j).\pi_j;$$

with : $(SM_j)_o$ is the set of markings where the transition t_{Aretr} is enabled, and $\nu(M_j)$ is the firing rate associated with the transition t_{retr} in the marking M_j.

► **The mean number of the high priority requests η_1 (resp. low priority requests η_2) in the queue:**

$$\eta_1 = \sum_j M_j(P_{Cust1})\pi_j;, \quad \eta_2 = \sum_j M_j(P_{Orbit}).\pi_j;$$

where, $M_j(P_{Cust1})$ is the number of tokens in place P_{Cust1} in the marking M_j and $M_j(P_{Orbit})$ is the number of tokens in place P_{Orbit} in the marking M_j. The sum in this formula is made on all the accessible markings.

► **The mean number of high priority requests η_{S1} (resp. low priority requests η_{S2}) in the system:**

$$\eta_{S1} = \sum_j [M_j(P_{Cust1}) + M_j(P_{Serv1})]\pi_j;$$

$$\eta_{S2} = \sum_j [M_j(P_{Orbit}) + M_j(P_{Serv2})]\pi_j.$$

The sum in this formula is made on all the accessible markings.

► **The mean waiting time of high priority W_1 (resp. low priority W_2) the requests:**

$$W_1 = \frac{\eta_1}{\bar{\lambda}_1}; \qquad W_2 = \frac{\eta_2}{\bar{\lambda}_2}.$$

► **The mean response time of high priority τ_1 (resp. low priority τ_2) requests:**

$$R_1 = \frac{\eta_{S1}}{\bar{\lambda}_1}; \qquad R_2 = \frac{\eta_{S2}}{\bar{\lambda}_2};.$$

► **The blocking probability of low priority customers** The blocking probability of low priority customers is:

$$B_p = \sum_i Prob\{M(P_{Orbit}) \geq 1 \ \ and \ \ M(P_{\cdot serv.Idle}) = 0\}.$$

► **The probability that a low priority customers is being served** This corresponds to the probability that the server is busy by low priority customer, it's given by P_s:

$$P_s = \sum_i Prob\{M(P_{serv2}) = 1\}.$$

As a numerical example for the non-preemptive priority queueing system with a single finite population, we consider the $M_2/M_2/1//(2,2)$ retrial queueing system with $N_1 = 2$ and $N_2 = 2$. The reachability graphs of these models are finite and strongly connected. Hence, underlying Markov process is ergodic and the steady-state distribution exists and is unique. We construct the continuous-time Markov

chain (CTMC) (see Figure 2) where the states of this CTMC are the tangible markings of the tree. The vanishing markings are merged with their successor tangible markings since it takes zero time to go through a vanishing marking. Hence, the in-

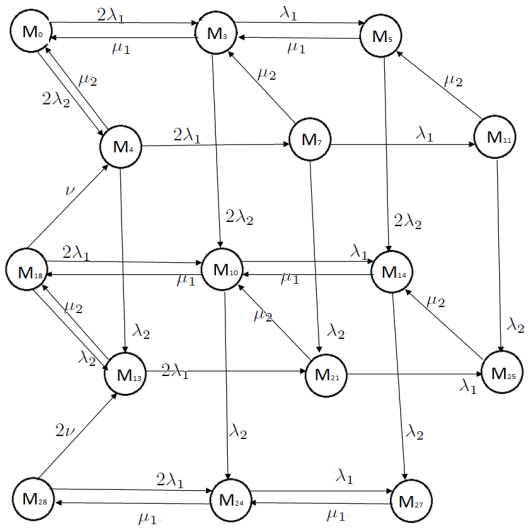


FIGURE 2. The CTMC for the $M_2/M_2/1//(2,2)$ retrial queueing GSPN model.

finitesimal generator Q of underlying Markov chain is derived and the steady-state marking probability distribution

$$\pi = (\pi_0, \pi_3, \pi_4, \pi_5, \pi_7, \pi_{10}, \pi_{11}, \pi_{13}, \pi_{14}, \pi_{15}, \pi_{18}, \pi_{21}, \pi_{24}, \pi_{25}, \pi_{27}, \pi_{28})$$

is the solution of the system 2.1.

- **The mean arrival rate of the high priority customers**

$$\bar{\lambda_1} = 2\lambda_1(\pi_0 + \pi_4 + \pi_{13} + \pi_{18} + \pi_{28}) + \lambda_1(\pi_3 + \pi_{10} + \pi_{24} + \pi_7 + \pi_{21}).$$

- **The mean arrival rate of the low priority customers**

$$\bar{\lambda_2} = 2\lambda_2(\pi_0 + \pi_3 + \pi_5) + \lambda_2(\pi_4 + \pi_7 + \pi_{11} + \pi_{18} + \pi_{10} + \pi_{14}).$$

- **The mean number of the high priority customers in the queue**

$$\eta_1 = 2(\pi_{11} + \pi_{25}) + \pi_5 + \pi_7 + \pi_{14} + \pi_{21} + \pi_{27}.$$

- **The mean number of the low priority customers in the orbit**

$$\eta_2 = 2(\pi_{24} + \pi_{27} + \pi_{28}) + \pi_{10} + \pi_{13} + \pi_{14} + \pi_{18} + \pi_{21} + \pi_{25}.$$

- **The mean number of the high priority customers in the system**

$$\eta_{S1} = 2(\pi_5 + \pi_{11} + \pi_{14} + \pi_{25} + \pi_{27}) + \pi_3 + \pi_7 + \pi_{10} + \pi_{21} + \pi_{14} + \pi_{24}.$$

- **The mean number of the low priority customers in the system**

$$\eta_{S2} = 2(\pi_{13} + \pi_{21} + \pi_{24} + \pi_{25} + \pi_{27}) + \pi_{28} + \pi_4 + \pi_7 + \pi_{11} + \pi_{10} + \pi_{14} + \pi_{18}.$$

- **The mean waiting time of the customers**

$$W_1 = \frac{2(\pi_{11} + \pi_{25}) + \pi_5 + \pi_7 + \pi_{14} + \pi_{21} + \pi_{27}}{2\lambda_1(\pi_0 + \pi_4 + \pi_{13} + \pi_{18} + \pi_{28}) + \lambda_1(\pi_3 + \pi_{10} + \pi_{24} + \pi_7 + \pi_{21})}.$$

$$W_2 = \frac{2(\pi_{24} + \pi_{27} + \pi_{28}) + \pi_{10} + \pi_{13} + \pi_{14} + \pi_{18} + \pi_{21} + \pi_{25}}{2\lambda_2(\pi_0 + \pi_3 + \pi_5) + \lambda_2(\pi_4 + \pi_7 + \pi_{11} + \pi_{18} + \pi_{10} + \pi_{14})};$$

- **The mean response time of the customers**

$$R_1 = \frac{2(\pi_5 + \pi_{11} + \pi_{14} + \pi_{25} + \pi_{27}) + \pi_3 + \pi_7 + \pi_{10} + \pi_{21} + \pi_{14} + \pi_{24}}{2\lambda_1(\pi_0 + \pi_4 + \pi_{13} + \pi_{18} + \pi_{28}) + \lambda_1(\pi_3 + \pi_{10} + \pi_{24} + \pi_7 + \pi_{21})}.$$

$$R_2 = \frac{2(\pi_{13} + \pi_{21} + \pi_{24} + \pi_{25} + \pi_{27}) + \pi_{28} + \pi_4 + \pi_7 + \pi_{11} + \pi_{10} + \pi_{14} + \pi_{18}}{2\lambda_2(\pi_0 + \pi_3 + \pi_5) + \lambda_2(\pi_4 + \pi_7 + \pi_{11} + \pi_{18} + \pi_{10} + \pi_{14})};$$

5. Numerical examples

In this section, we present the results of numerical experiments the goal is to demonstrate the feasibility of the proposed approach and to give some insight into quantitative behavior of the system.

In Figures 3 and 4 we show the influence of the arrival rates λ_1, λ_2 with $\lambda_2 = \frac{\lambda_1}{2}$ on the blocking probability B_p. We have presented three curves which correspond to the different values of service rates, (μ_1, μ_2). In all the numerical experiments, we assume $\nu = 3.5$. From Figures 3 and 4 it is shown that the blocking probability B_p increases as the rates λ_1 increases and approaches one. The increasing of B_p is rapidly for a small value of (μ_1, μ_2) and for $(N_1, N_2) = (10, 5)$. We see also that the blocking probability B_p increases rapidly when $\lambda_2 = 2\lambda_1$. In Figure 4 the blocking

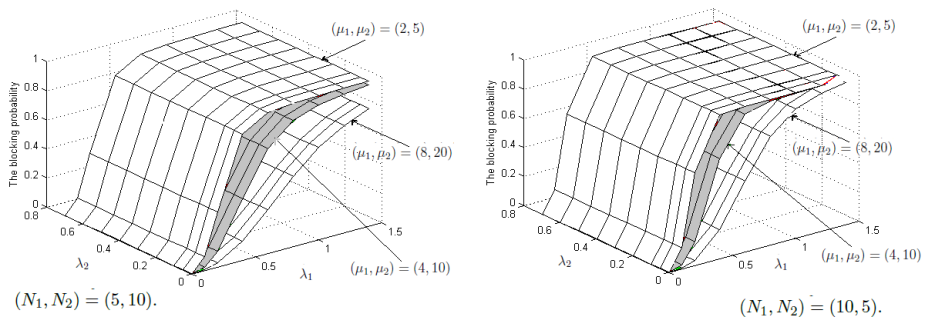
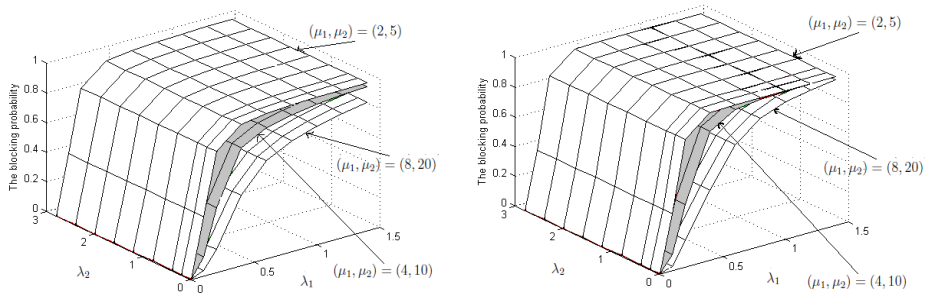


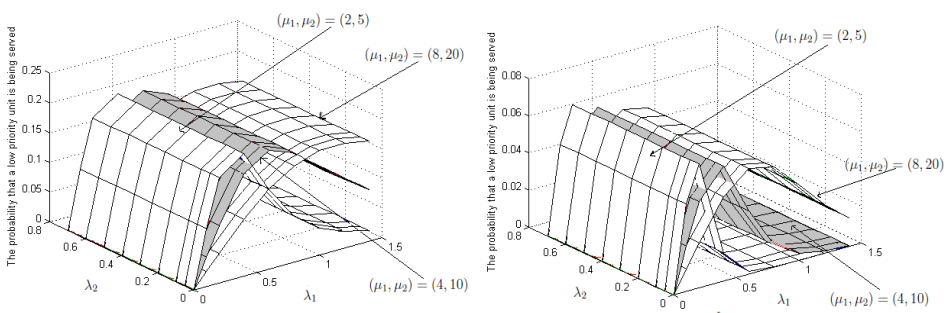
FIGURE 3. B_p versus the arrival rates λ_1 and λ_2 with $\lambda_2 = \frac{\lambda_1}{2}$.

probability B_p curves are plotted versus the arrival rate λ_1, λ_2 with $\lambda_2 = 2\lambda_1$ for



$(N_1, N_2) = (5, 10)$. $(N_1, N_2) = (10, 5)$

FIGURE 4. B_p versus the arrival rates λ_1 and λ_2 with $\lambda_2 = 2\lambda_1$.



$(N_1, N_2) = (5, 10)$. $(N_1, N_2) = (10, 5)$

FIGURE 5. P_s versus the arrival rate λ_1 and λ_2 with $\lambda_2 = \frac{\lambda_1}{2}$.

different values of (μ_1, μ_2) and (N_1, N_2). Figure 5 illustrates the behavior of the probability that a low priority customer is being served P_s versus the arrival rate λ_1, λ_2 with $\lambda_2 = \frac{\lambda_1}{2}$. We have presented three curves which correspond to the different values of service rates, (μ_1, μ_2).

Figures 6 shows the effect of the arrival rate λ_1, λ_2 with $\lambda_2 = \frac{\lambda_1}{2}$ on the probability that a low priority customer is being served P_s. We have presented three curves which correspond to the different values of service rates, (μ_1, μ_2). From these figures, it is shown that we see the probability that a low priority customer is being served P_s increases until the maximum and decreases to approach zero. We notice that P_s approaches zero with the increases of λ_1. The zero is reached rapidly for lower values of (μ_1, μ_2) and for a large number of high priority customers $(N_1, N_2) = (10, 5)$.

In the following we show the effect of of the arrival rates λ_1, λ_2 on the mean number of the high priority customers and low priority customers in the queue and in orbit respectively. Figures 7 displays $\bar{Q}_i$ for $i \in \{1, 2\}$ versus the arrival rates λ_1 and λ_2. We see that the mean number of customers in the queue and orbit is

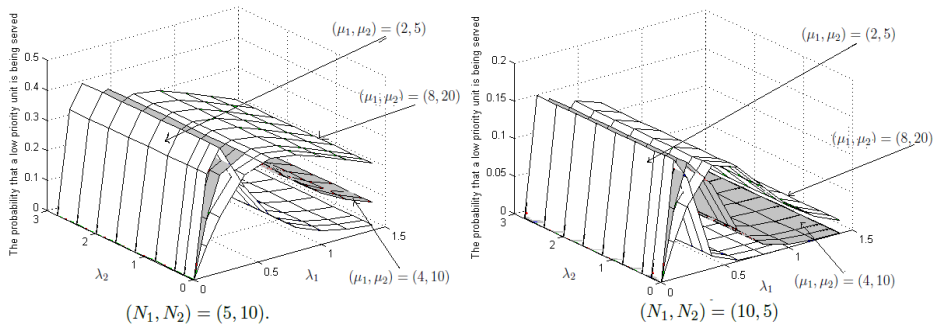


FIGURE 6. P_s versus the arrival rate λ_1 and $\lambda_2 = 2\lambda_1$

an increasing function of arrival rate λ_1. However, when more and more arrival customers arrive, Q_2 approaches N_2 ie the low priority customers are blocked. All

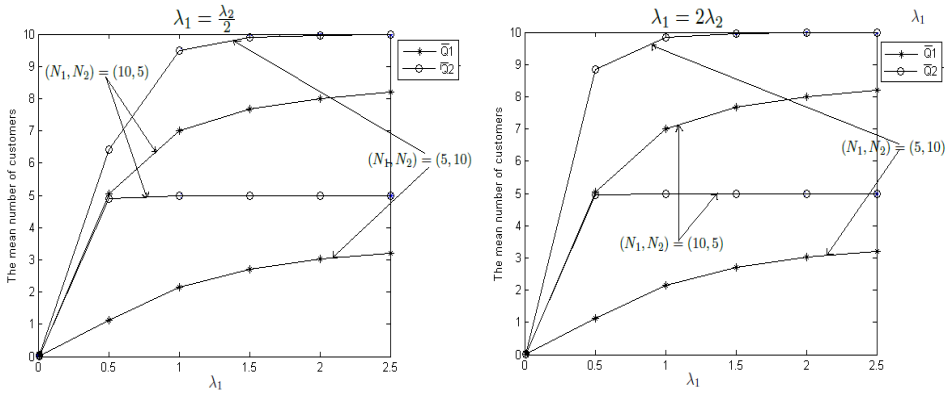


FIGURE 7. $\bar{Q}_1$, $\bar{Q}_2$ versus λ_1.

the above remarks are obtained by considering the parameters $(N_1 = 5, N_2 = 10)$ or $(N_1 = 10, N_2 = 5)$, we checked them with other different values of N_1 and N_2. However, we gave the similar values used by Gomez in order to compare our results obtained via our approach Petri nets and those obtained by him in [12].

In Figure 8 the mean response time is spotted versus the retrial rate ν. We have presented several curves which correspond to the different values of (N_1, N_2) and we have assumed that $(\lambda_1, \lambda_2) = (0.4, 0.8)$, $(\mu_1, \mu_2) = (8, 20)$. From these figures, we see that the mean response time is a decreasing function of retrial rate, for low priority customers and slightly increasing for high priority customers. Moreover, the high priority customers have the best mean response times in all cases. The worst mean response times are given by a high values of N_1 and for a small value of ν.

Furthermore, Figure 8 shows also that the retrial rate has a significant influence on the mean response time for low retrial rate values. This result is logical because the increasing of the retrial rate describes the fact that the repeated requests intensity increases. Hence, customers in orbit have more chance to be served compared to new arrivals,which explains the amelioration of the mean response time. In Tables 1

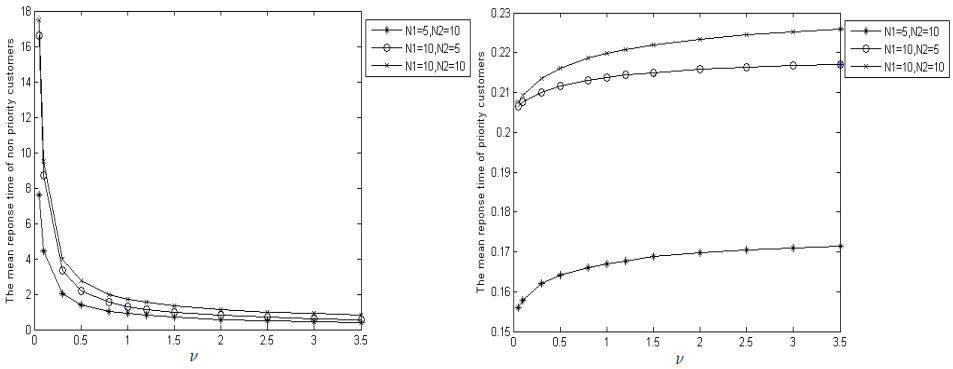


FIGURE 8. Mean response time of customers versus retrial rate.

and 2 we check respectively, the effect of arrival rates λ_1 and λ_2 with $\lambda_2 = \frac{\lambda_1}{2}$ on the mean response time for respectively $(\mu_1, \mu_2) = (8, 20)$ and $(\mu_1, \mu_2) = (4, 10)$.

TABLE 1. The effect of arrival rates on the mean response time for $(\mu_1, \mu_2) = (8, 20)$.

	$(N_1, N_2) = (5, 10)$		$(N_1, N_2) = (10, 5)$	
λ_1	High priority customers	Low priority customers	High priority customers	Low priority customers
0.01	0.1250	0.0533	0.1265	0.0555
0.1	0.1327	0.0861	0.1411	0.1140
0.2	0.1409	0.1288	0.1602	0.2016
0.4	0.1583	0.2384	0.2093	0.4916
0.6	0.1769	0.3889	0.2735	1.0549
0.8	0.1961	0.5898	0.3497	2.1343
1	0.2151	0.8482	0.4313	4.1395
1.2	0.2334	1.1672	0.5114	7.8098

From Tables 1 and 2 we can see that the mean response time is an increasing function of the arrival rates which has a significant influence when they are high. The mean response time of high priority customers is better. Furthermore, the mean response

TABLE 2. The effect of arrival rates on the mean response time for $(\mu_1, \mu_2) = (4, 10)$.

	$(N_1, N_2) = (5, 10)$		$(N_1, N_2) = (10, 5)$	
λ_1	High priority customers	Low priority customers	High priority customers	Low priority customers
0.01	0.2530	0.10856	0.2560	0.1146
0.1	0.2817	0.2013	0.3204	0.3028
0.2	0.3168	0.3450	0.4187	0.7087
0.4	0.3938	0.8246	0.7019	3.0869
0.6	0.4716	1.6716	1.0268	11.3301
0.8	0.5417	2.9668	1.3057	38.3670
1	0.6029	4.6891	1.5171	121.1
1.2	0.6566	7.0179	1.6723	350.57

time of low priority customers increases rapidly with a high values of N_1 and lower values of (μ_1, μ_2).

Finally, we have concluded that for retrial queueing systems with high priority the optimal results are obtained when the arrival rates, service rates are low and the retrial rate is high with a small number of low priority customers.

6. CONCLUSION

The paper presents a technique allowing to obtain exact performance indices, based on Generalized Stochastic Petri nets (GSPN), to analyze $M_2/M_2/1//(N_1, N_2)$ retrial queueing system with finite population. Generalized Stochastic Petri nets (GSPN) have proven to be a powerful and enduring graphically oriented framework for modeling and performance analysing of complex systems. First, We have developed formulas of the main stationary performance indices based on stationary probabilities and network parameters. Furthermore, we have presented numerical examples to illustrate the efficiency of the proposed approach where we study the effect of network parameters on performance indices. Finally, we conclude that the arrival rate of high priority customers, the arrival rate of low priority customers, retrial rate and the number of customers in the sources are the major factors affecting the performance of this system. From these results, we conclude that the Petri net theory can be used to add powerful analysis capabilities to high priority and retrial queueing systems.

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Authors' addresses

Sedda Hakmi
Research Unit LaMOS (Modeling and Optimization of Systems, Bejaia University 06000, Algeria
E-mail address: sed.hakmi@gmail.com

Ouiza Lekadir
Research Unit LaMOS (Modeling and Optimization of Systems, Bejaia University 06000, Algeria
E-mail address: ouizalekadir@gmail.com

Djamil Aïssani
Research Unit LaMOS (Modeling and Optimization of Systems, Bejaia University 06000, Algeria
E-mail address: lamos_bejaia@hotmail.com

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A MODIFIED SHRINKING PROJECTION METHODS FOR NUMERICAL RECKONING FIXED POINTS OF G-NONEXPANSIVE MAPPINGS IN HILBERT SPACES WITH GRAPHS

H.A. HAMMAD, W. CHOLAMJIAK, D. YAMBANGWAI, AND H. DUTTA

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Abstract. In this paper, we introduce four new iterative schemes by modifying the shrinking projection method with Ishikawa iteration and S-iteration. The strong convergence theorems are given for obtaining a common fixed point of two G-nonexpansive mappings in a Hilbert space with a directed graph. We also give some numerical experiments for supporting our main theorems and compare convergence rate between them.

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Keywords: strong convergence, shrinking projection, G-nonexpansive mappings, modified iterations, numerical discussion

1. Introduction

Let C be a nonempty, closed and convex subset of a normed space X. A mapping $T : C \to C$ is said to be

1. *contraction* if there exists $\alpha \in (0,1)$ such that $\|Tx - Ty\| \leq \alpha\|x - y\|$ for all $x, y \in C$;
2. *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

The fixed point set of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : x = Tx\}$.

The first important result on fixed points for contractive-type mapping was the well known Banach's contraction principle appeared in explicit form in Banach's thesis in 1922, where it was used to establish the existence of a solution for an integral equation [4]. Since this date, many articles studied and considered fixed point theorems and the existence of fixed points of a single-valued nonlinear mapping (see, for examples [2, 6, 22]).

In 1953, Mann [11] introduced the famous iteration procedure as follows:

$$\begin{aligned} &x_1 \in C \\ &x_{n+1} = \delta_n x_n + (1-\delta_n)Tx_n, \end{aligned}$$

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for all $n \in \mathbb{N}$ where $\{\delta_n\} \subset [0,1]$ and $\mathbb{N}$ the set of all positive integers. This iteration is used to obtain weak convergence theorem (see for example [16, 18]).

In 1974, Ishikawa [8] generalized the Mann's iterative algorithm by introduce the following iteration:

$$\begin{aligned} x_0 &\in C \\ x_{n+1} &= \delta_n x_n + (1-\delta_n)Ty_n, \\ y_n &= \alpha_n x_n + (1-\alpha_n)Tx_n, \end{aligned}$$

for all $n \in \mathbb{N}$ where $\{\alpha_n\}$ and $\{\delta_n\}$ are sequences in [0,1].

In 2007, Agarwal et al. [1] introduced and studied the S-iteration process for a class of nearly asymptotically nonexpansive mappings in Banach spaces and this scheme has a better convergence rate than Ishikawa iteration for a class of contractions in metric spaces.

In 2008, Takahashi et al. [23] just involved one closed convex set for a family of nonexpansive mappings $\{T_n\}$ and obtaining another modification of the Mann's iteration method:

$$\begin{aligned} u_0 \in H, u_1 &= P_{C_1}x_0 \text{ with } C_1 = C, \\ y_n &= \alpha_n u_n + (1-\alpha_n)T_n u_n, \\ C_{n+1} &= \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} &= P_{C_{n+1}}x_0. \end{aligned}$$

They proved that if $\alpha_n \leq a$ for all $n \geq 1$ and for some $0 < a < 1$, then the sequence $\{u_n\}$ converges strongly to $P_{Fix(T)}x_0$.

In 2008, by combination of the concepts in fixed point and graph theory, Jachymski [9] generalized the Banach's contraction principle in a complete metric space endowed with a directed graph. Many papers dealt with this point for existence of fixed points of monotone nonexpansive, G-nonexpansive and G-contraction mappings on a hyperbolic metric, Banach and Hilbert spaces endowed with graph and directed graph. Also these articles discussed Browders convergence theorem for G-nonexpansive mapping in a Hilbert space with a directed graph, weak and strong convergence of the Ishikawa iteration for G-nonexpansive mappings (see for example [3, 13, 24, 25]).

Motivated by the work of [1, 25], Suparatulatorn et al. [20] studied the following modified S-iteration process:

$$\begin{aligned} x_0 &\in C, \\ y_n &= (1-\sigma_n)x_n + \sigma_n S_1 x_n, \\ x_{n+1} &= (1-\delta_n)S_1 x_n + (1-\delta_n)S_2 y_n, \ n \geq 0, \end{aligned}$$

where $\{\delta_n\}$ and $\{\sigma_n\}$ are sequences in (0, 1) and $S_1, S_2 : C \to C$ are G-nonexpansive mappings. Also they proved weak and strong convergence for approximating common fixed points of two G-nonexpansive mappings in a uniformly convex Banach space X endowed with a graph under this iteration.

Motivated and inspired by the above works, we introduce the four different iterative schemes by using the shrinking projection method for approximating a common fixed point of two G-nonexpansive mappings in Hilbert spaces. We then obtain strong convergence theorems. Finally, we discuss some important numerical results to illustrate the rate convergence of the four iterations.

2. Preliminaries and lemmas

In this section, we give some known definitions and lemmas which will be used in the later sections.

Let C be a nonempty subset of a real Banach space X. Let $\triangle$ denote the diagonal of the cartesian product $C \times C$, i.e., $\triangle = \{(x,x) : x \in \triangle\}$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with C, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \triangle$. We assume G has no parallel edge. So we can identify the graph G with the pair $(V(G), E(G))$. A mapping $S : G \to G$ is said to be

• $G-$contraction if S satisfies the conditions:

(i) S is edge-preserving, i.e.,

$$(x,y) \in E(G) \Rightarrow (Sx, Sy) \in E(G),$$

(ii) S decreases weights of edges of G, i.e., there exists $\delta \in (0,1)$ such that

$$(x,y) \in E(G) \Rightarrow \|Sx - Sy\| \leq \delta \|x - y\|.$$

• $G-$nonexpansive if S satisfies the condition (i) and

(iii) S non-increases weights of edges of G, i.e.,

$$(x,y) \in E(G) \Rightarrow \|Sx - Sy\| \leq \|x - y\|.$$

Definition 1. The symbol G^{-1} is called the conversion of a graph G and it is a graph obtained from G by reversing the direction of edges as:

$$E(G^{-1}) = \{(x,y) \in X \times X : (y,x) \in E(G)\}.$$

Definition 2. The sequence $\{x_j\}_{j=0}^{N}$ of $N+1$ vertices is called a path in G from x to y of length N ($N \in \mathbb{N} \cup \{0\}$), where $x_0 = x$, $x_N = y$ and $(x_j, x_{j+1}) \in E(G)$ for $j = 0, 1, ..., N-1$.

Definition 3. If there is a path between any two vertices of the graph G, then a graph G is said to be connected.

Definition 4. If (x,y) and $(y,z) \in V(G)$, then $(x,z) \in V(G)$. This property is called the transitivity of a directed graph $G = (V(G), E(G))$ for all $x, y, z \in V(G)$.

Definition 5. Let $G = (V(G), E(G))$ be a directed graph. The set of edges $E(G)$ is said to be *convex* if for any $(x, y), (z, w) \in E(G)$ and for each $t \in (0, 1)$, then $(tx + (1-t)z, ty + (1-t)w) \in E(G)$.

Definition 6. Let $x_0 \in V(G)$ and A subset of $V(G)$. We say that
(i) A is dominated by x_0 if $(x_0, x) \in E(G)$ for all $x \in A$.
(ii) A dominates x_0 if for each $x \in A$, $(x, x_0) \in E(G)$.

Lemma 1 ([19])**.** *Let C be a nonempty, closed and convex subset of a Hilbert space H and $G = (V(G), E(G))$ a directed graph such that $V(G) = C$. Let $T : C \to C$ be a G-nonexpansive mapping and $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup x$ for some $x \in C$. If, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$ and $\{x_n - Tx_n\} \to y$ for some $y \in H$. Then $(I - T)x = y$.*

Let C be a nonempty, closed and convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the *metric projection* of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$$

for all $x, y \in H$. Furthermore, $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [21].
We know that the following result.

Lemma 2. *Let H be a real Hilbert space. Then*

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2,$$

for all $t \in [0, 1]$ and $x, y \in H$.

Lemma 3 ([10])**.** *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Given $x, y, z \in H$ and also given $a \in \mathbb{R}$, the set*

$$\{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\}$$

is convex and closed.

Lemma 4 ([12])**.** *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $P_C : H \to C$ be the metric projection from H onto C. Then the following inequality holds:*

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2, \ \ \forall x \in H, \ \forall y \in C.$$

3. Main results

In this section, by using the shrinking projection method, we obtain four different strong convergence theorems for finding the same common fixed point of two G-nonexpasive mappings in real Hilbert spaces.

Theorem 1. *Let C be a nonempty closed and convex subset of a real Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $E(G)$ is convex. Let $S_1, S_2 : C \to C$ be G-nonexpansive mappings such that $F := F(S_1) \cap F(S_2) \neq \varnothing$, F is closed and $F(S_i) \times F(S_i) \subseteq E(G)$ for all $i = 1,2$. Let $\{s_n\}$ be a sequence generated by*

$$\begin{aligned} s_1 &\in C, \text{ with } C_1 = C, \\ y_n &= (1-\beta_n)s_n + \beta_n S_1 s_n, \\ z_n &= (1-\alpha_n)s_n + \alpha_n S_2 y_n, \\ C_{n+1} &= \{z \in C_n : \|z_n - z\| \leq \|s_n - z\|\}, \\ s_{n+1} &= P_{C_{n+1}} s_1,\ n \geq 1, \end{aligned} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. Assume that the following conditions hold:
(i) $\{s_n\}$ dominates p for all $p \in F$ and if there exists a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $s_{n_k} \rightharpoonup w \in C$, then $(s_{n_k}, w) \in E(G)$;
(ii) $\liminf_{n\to\infty} \alpha_n > 0$; (iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
Then the sequence $\{s_n\}$ converges strongly to $P_F s_1$.

Proof. We split the proof into five steps.
Step 1. Show that $P_{C_{n+1}} s_1$ is well-defined for each $s_1 \in C$. As shown in Theorem 3.2 of Tiammee et al. [24], $F(S_i)$ is convex for all $i = 1,2$. It follows from the assumption that F is closed and convex. Hence, $P_F s_1$ is well-defined. We see that $C_1 = C$ is closed and convex. Assume that C_n is closed and convex. From the definition of C_{n+1} and Lemma 3, we get C_{n+1} is closed and convex. Let $p \in F$. Since $\{s_n\}$ dominates p and S_1 is edge-preserving, we have $(S_1 s_n, p) \in E(G)$. This implies that $(y_n, p) = ((1-\beta_n)s_n + \beta_n S_1 s_n, p) \in E(G)$ by $E(G)$ is convex. Since S_2 is edge-preserving, we have

$$\begin{aligned} \|z_n - p\| &\leq (1-\alpha_n)\|s_n - p\| + \alpha_n \|S_2 y_n - p\| \\ &\leq (1-\alpha_n)\|s_n - p\| + \alpha_n((1-\beta_n)\|s_n - p\| + \beta_n \|S_1 s_n - p\|) \\ &\leq \|s_n - p\|. \end{aligned} \tag{3.2}$$

We can conclude that $p \in C_{n+1}$. Thus $F \subset C_{n+1}$. This implies that $P_{C_{n+1}} s_1$ is well-defined.
Step 2. Show that $\lim_{n\to\infty} \|s_n - s_1\|$ exists. Since F is a nonempty, closed and convex subset of H, there exists a unique $v \in F$ such that $v = P_F s_1$. From $s_n = P_{C_n} s_1$ and $s_{n+1} \in C_n$, $\forall n \in \mathbb{N}$, we get

$$\|s_n - s_1\| \leq \|s_{n+1} - s_1\|, \quad \forall n \in \mathbb{N}. \tag{3.3}$$

On the other hand, as $F \subset C_n$, we obtain

$$\|s_n - s_1\| \leq \|v - s_1\|, \quad \forall n \in \mathbb{N}. \tag{3.4}$$

It follows from (3.3) and (3.4) that the sequence $\{s_n\}$ is bounded and nondecreasing. Therefore $\lim_{n\to\infty} \|s_n - s_1\|$ exists.

Step 3. Show that $s_n \to w \in C$ as $n \to \infty$. For $m > n$, by the definition of C_n, we see that $s_m = P_{C_m} s_1 \in C_m \subset C_n$. From Lemma 4, we have

$$\|s_m - s_n\|^2 \leq \|s_m - s_1\|^2 - \|s_n - s_1\|^2.$$

From Step 3, we obtain that $\{s_n\}$ is a Cauchy sequence. Hence, there exists $w \in C$ such that $s_n \to w$ as $n \to \infty$. In particular, we have

$$\lim_{n\to\infty} \|s_{n+1} - s_n\| = 0. \tag{3.5}$$

Step 4. Show that $w \in F$. Since $s_{n+1} \in C_n$, it follows from (3.5) that

$$\|z_n - s_n\| \leq \|z_n - s_{n+1}\| + \|s_{n+1} - s_n\| \leq 2\|s_{n+1} - s_n\| \to 0 \tag{3.6}$$

as $n \to \infty$. Since $\liminf_{n\to\infty} \alpha_n > 0$ and (3.6), we have

$$\|S_2 y_n - s_n\| = \frac{1}{\alpha_n}\|z_n - s_n\| \to 0, \tag{3.7}$$

as $n \to \infty$. From $\{s_n\}$ dominates p for all $p \in F$ and Lemma 2, we get

$$\begin{aligned}
\|z_n - p\|^2 &\leq (1-\alpha_n)\|s_n - p\|^2 + \alpha_n \|S_2 y_n - p\|^2 \\
&\leq (1-\alpha_n)\|s_n - p\|^2 + \alpha_n\big((1-\beta_n)\|s_n - p\|^2 \\
&\quad + \beta_n \|S_1 s_n - p\|^2 - (1-\beta_n)\beta_n \|S_1 s_n - s_n\|^2\big) \\
&\leq \|s_n - p\|^2 - \alpha_n(1-\beta_n)\beta_n \|S_1 s_n - s_n\|^2.
\end{aligned} \tag{3.8}$$

This implies that

$$\alpha_n(1-\beta_n)\beta_n \|S_1 s_n - s_n\|^2 \leq \|s_n - p\|^2 - \|z_n - p\|^2. \tag{3.9}$$

From our assumptions and (3.6), we have

$$\lim_{n\to\infty} \|S_1 s_n - s_n\| = 0. \tag{3.10}$$

This implies that

$$\lim_{n\to\infty} \|y_n - s_n\| = \lim_{n\to\infty} \beta_n \|S_1 s_n - s_n\| = 0. \tag{3.11}$$

It follows from (3.7) and (3.11) that

$$\|S_2 y_n - y_n\| \leq \|S_2 y_n - s_n\| + \|s_n - y_n\| \to 0, \tag{3.12}$$

as $n \to \infty$. By Lemma 1, (3.10), (3.11) and (3.12), we have $w \in F$.

Step 5. Show that $w = v = P_F s_1$. Since $s_n = P_{C_n} s_1$, we have

$$\langle s_1 - s_n, s_n - p\rangle \geq 0, \quad \forall p \in C_n. \tag{3.13}$$

By taking the limit in (3.13), we obtain

$$\langle s_1 - w, w - p\rangle \geq 0, \quad \forall p \in C_n. \tag{3.14}$$

Since $F \subset C_n$, so $w = P_F s_1$. This completes the proof. □

Theorem 2. *Let C be a nonempty closed and convex subset of a real Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $E(G)$ is convex. Let $S_1, S_2 : C \to C$ be G-nonexpansive mappings such that $F := F(S_1) \cap F(S_2) \neq \varnothing$, F is closed and $F(S_i) \times F(S_i) \subseteq E(G)$ for all $i = 1, 2$. Let $\{t_n\}$ be a sequence generated by*

$$\begin{aligned} t_1 &\in C, \text{ with } C_1 = C, \\ y_n &= (1-\beta_n)t_n + \beta_n S_1 t_n, \\ z_n &= (1-\alpha_n)y_n + \alpha_n S_2 y_n, \\ C_{n+1} &= \{z \in C_n : \|z_n - z\| \leq \|t_n - z\|\}, \\ t_{n+1} &= P_{C_{n+1}} t_1, \ n \geq 1, \end{aligned} \tag{3.15}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. Assume that the following conditions hold:
(i) $\{t_n\}$ dominates p for all $p \in F$ and if there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $t_{n_k} \rightharpoonup w \in C$, then $(t_{n_k}, w) \in E(G)$;
(ii) $\liminf_{n\to\infty} \alpha_n > 0$; (iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
Then the sequence $\{t_n\}$ converges strongly to $P_F t_1$.

Proof. We set $t_n = s_n$, by the same proof of Step 1 in Theorem 1, we have $P_F t_1$ well-defined, C_{n+1} is closed convex for all $n \in \mathbb{N}$ and $(y_n, p) \in E(G)$ for each $p \in F$. Since S_1, S_2 are edge-preserving, we have

$$\begin{aligned} \|z_n - p\| &\leq (1-\alpha_n)\|y_n - p\| + \alpha_n \|S_2 y_n - p\| \\ &\leq (1-\beta_n)\|t_n - p\| + \beta_n \|S_1 t_n - p\| \leq \|t_n - p\|. \end{aligned}$$

We can conclude that $p \in C_{n+1}$. Thus $F \subset C_{n+1}$. This implies that $P_{C_{n+1}} t_1$ is well-defined. By the same proof of Step 2-3 in Theorem 1, we obtain $t_n \to w \in C$ as $n \to \infty$. We next show that $w \in F$. Since $t_{n+1} \in C_n$, it follows from (3.5) that

$$\|z_n - t_n\| \leq \|z_n - t_{n+1}\| + \|t_{n+1} - t_n\| \leq 2\|t_{n+1} - t_n\| \to 0 \tag{3.16}$$

as $n \to \infty$. Since $\{t_n\}$ dominates p for all $p \in F$, we get

$$\begin{aligned} \|z_n - p\|^2 &\leq (1-\alpha_n)\|y_n - p\|^2 + \alpha_n \|S_2 y_n - p\|^2 \\ &\leq (1-\beta_n)\|t_n - p\|^2 + \beta_n \|S_1 t_n - p\|^2 - (1-\beta_n)\beta_n \|S_1 t_n - t_n\|^2 \\ &\leq \|t_n - p\|^2 - (1-\beta_n)\beta_n \|S_1 t_n - t_n\|^2. \end{aligned} \tag{3.17}$$

This implies that

$$(1-\beta_n)\beta_n \|S_1 t_n - t_n\|^2 \leq \|t_n - p\|^2 - \|z_n - p\|^2. \tag{3.18}$$

From our assumption (ii) and (3.18), we have

$$\lim_{n\to\infty} \|S_1 t_n - t_n\| = 0. \tag{3.19}$$

This implies that

$$\lim_{n\to\infty} \|y_n - t_n\| = \lim_{n\to\infty} \beta_n \|S_1 t_n - t_n\| = 0. \tag{3.20}$$

It follows from (3.16) and (3.20) that

$$\|y_n - z_n\| \leq \|y_n - t_n\| + \|t_n - z_n\| \to 0, \tag{3.21}$$

as $n \to \infty$. From $\liminf_{n\to\infty} \alpha_n > 0$ and (3.21), we have

$$\lim_{n\to\infty} \|S_2 y_n - y_n\| = \lim_{n\to\infty} \frac{1}{\alpha_n} \|z_n - y_n\| = 0. \tag{3.22}$$

By Lemma 1, (3.19), (3.20) and (3.22), we have $w \in F$. From Step 5 in Theorem 1, we obtain $w = P_F t_1$. This completes the proof. □

Theorem 3. *Let C be a nonempty closed and convex subset of a real Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $E(G)$ is convex. Let $S_1, S_2 : C \to C$ be G-nonexpansive mappings such that $F := F(S_1) \cap F(S_2) \neq \varnothing$, F is closed and $F(S_i) \times F(S_i) \subseteq E(G)$ for all $i = 1, 2$. Let $\{u_n\}$ be a sequence generated by*

$$\begin{aligned} u_1 &\in C, \text{ with } C_1 = C, \\ y_n &= (1-\beta_n)u_n + \beta_n S_1 u_n, \\ z_n &= (1-\alpha_n)S_1 u_n + \alpha_n S_2 y_n, \\ C_{n+1} &= \{z \in C_n : \|z_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} &= P_{C_{n+1}} u_1, \ n \geq 1, \end{aligned} \tag{3.23}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. Assume that the following conditions hold:
(i) $\{u_n\}$ dominates p for all $p \in F$ and if there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \rightharpoonup w \in C$, then $(u_{n_k}, w) \in E(G)$;
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
Then the sequence $\{u_n\}$ converges strongly to $P_F u_1$.

Proof. We set $u_n = s_n$, by the same proof of Step 1 in Theorem 1, we have $P_F u_1$ well-defined, C_{n+1} is closed convex for all $n \in \mathbb{N}$ and $(y_n, p) \in E(G)$ for each $p \in F$. Since S_1, S_2 are edge-preserving, we have

$$\begin{aligned} \|z_n - p\| &\leq (1-\alpha_n)\|S_1 u_n - p\| + \alpha_n \|S_2 y_n - p\| \\ &\leq (1-\alpha_n)\|u_n - p\| + \alpha_n((1-\beta_n)\|u_n - p\| + \beta_n \|S_1 u_n - p\|) \\ &\leq \|u_n - p\|. \end{aligned} \tag{3.24}$$

We can conclude that $p \in C_{n+1}$. Thus $F \subset C_{n+1}$. This implies that $P_{C_{n+1}} u_1$ is well-defined. By the same proof of Step 2-3 in Theorem 1, we obtain $u_n \to w \in C$ as $n \to \infty$. We next show that $w \in F$. Since $u_{n+1} \in C_n$, it follows from (3.5) that

$$\|z_n - u_n\| \leq \|z_n - u_{n+1}\| + \|u_{n+1} - u_n\| \leq 2\|u_{n+1} - u_n\| \to 0 \tag{3.25}$$

as $n \to \infty$. Since $\{u_n\}$ dominates p for all $p \in F$, we get

$$\|z_n - p\|^2 \leq (1-\alpha_n)\|S_1 u_n - p\|^2 + \alpha_n \|S_2 y_n - p\|^2$$

$$\begin{aligned} &\leq (1-\alpha_n)\|u_n-p\|^2+\alpha_n\big((1-\beta_n)\|u_n-p\|^2 \\ &\quad +\beta_n\|S_1u_n-p\|^2-(1-\beta_n)\beta_n\|S_1u_n-u_n\|^2\big) \\ &\leq \|u_n-p\|^2-\alpha_n(1-\beta_n)\beta_n\|S_1u_n-u_n\|^2. \end{aligned} \tag{3.26}$$

This implies that

$$(1-\beta_n)\beta_n\|S_1u_n-u_n\|^2 \leq \|u_n-p\|^2-\|z_n-p\|^2. \tag{3.27}$$

From our assumption (ii) and (3.27), we have

$$\lim_{n\to\infty}\|S_1u_n-u_n\| = 0. \tag{3.28}$$

This implies that

$$\lim_{n\to\infty}\|y_n-u_n\| = \lim_{n\to\infty}\beta_n\|S_1u_n-u_n\| = 0. \tag{3.29}$$

It follows from (3.28) and (3.29) that

$$\|S_1u_n-y_n\| \leq \|S_1u_n-u_n\|+\|u_n-y_n\| \to 0, \tag{3.30}$$

as $n\to\infty$. For $p\in F$, it follows from (3.24) that

$$\begin{aligned} \|z_n-p\|^2 &= (1-\alpha_n)\|S_1u_n-p\|^2+\alpha_n\|S_2y_n-p\|^2-(1-\alpha_n)\alpha_n\|S_1u_n-S_2y_n\|^2 \\ &\leq (1-\alpha_n)\|u_n-p\|^2+\alpha_n\|y_n-p\|^2-(1-\alpha_n)\alpha_n\|S_1u_n-S_2y_n\|^2 \\ &\leq \|u_n-p\|^2-(1-\alpha_n)\alpha_n\|S_1u_n-S_2y_n\|^2. \end{aligned}$$

This implies that

$$(1-\alpha_n)\alpha_n\|S_1u_n-S_2y_n\|^2 \leq \|u_n-p\|^2-\|z_n-p\|^2. \tag{3.31}$$

From the assumption (i) and (3.25), we have

$$\lim_{n\to\infty}\|S_1u_n-S_2y_n\| = 0. \tag{3.32}$$

It follows from (3.30) and (3.32) that

$$\|S_2y_n-y_n\| \leq \|S_2y_n-S_1u_n\|+\|S_1u_n-y_n\| \to 0, \tag{3.33}$$

as $n\to\infty$.

$$\lim_{n\to\infty}\|S_2y_n-y_n\| = \lim_{n\to\infty}\frac{1}{\alpha_n}\|z_n-y_n\| = 0. \tag{3.34}$$

By Lemma 1, (3.28), (3.29) and (3.34), we have $w\in F$. From Step 5 in Theorem 1, we obtain $w = P_F u_1$. This completes the proof. □

Theorem 4. *Let C be a nonempty closed and convex subset of a real Hilbert space H and let $G=(V(G),E(G))$ be a directed graph such that $V(G)=C$, $E(G)$ is convex and G is transitive. Let $S_1,S_2:C\to C$ be G-nonexpansive mappings such that $F:=F(S_1)\cap F(S_2)\neq\emptyset$, F is closed and $F(S_i)\times F(S_i)\subseteq E(G)$ for all $i=1,2$. Let $\{v_n\}$ be a sequence generated by*

$$v_1\in C, \text{ with } C_1=C,$$

$$\begin{aligned} y_n &= (1-\beta_n)v_n + \beta_n S_1 v_n, \\ z_n &= (1-\alpha_n)S_1 y_n + \alpha_n S_2 y_n, \\ C_{n+1} &= \{z \in C_n : \|z_n - z\| \le \|v_n - z\|\}, \\ v_{n+1} &= P_{C_{n+1}} v_1,\ n \ge 1, \end{aligned} \tag{3.35}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. *Assume that the following conditions hold:*
(i) (v_n, p), $(p, v_n) \in E(G)$ *for all* $p \in F$ *and* $n \in \mathbb{N}$ *and if there exists a subsequence* $\{v_{n_k}\}$ *of* $\{v_n\}$ *such that* $v_{n_k} \rightharpoonup w \in C$, *then* $(v_{n_k}, w) \in E(G)$;
(ii) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.
Then the sequence $\{x_n\}$ *converges strongly to* $P_F v_1$.

Proof. We set $v_n = s_n$, by the same proof of Step 1 in Theorem 1, we have $P_F v_1$ well-defined, C_{n+1} is closed convex for all $n \in \mathbb{N}$ and $(y_n, p) \in E(G)$ for each $p \in F$. Since S_1, S_2 are edge-preserving, we have

$$\begin{aligned} \|z_n - p\| &\le (1-\alpha_n)\|S_1 y_n - p\| + \alpha_n \|S_2 y_n - p\| \\ &\le (1-\beta_n)\|v_n - p\| + \beta_n \|S_1 v_n - p\| \\ &\le \|v_n - p\|. \end{aligned} \tag{3.36}$$

We can conclude that $p \in C_{n+1}$. Thus $F \subset C_{n+1}$. This implies that $P_{C_{n+1}} v_1$ is well-defined. By the same proof of Step 2-3 in Theorem 1, we obtain $v_n \to w \in C$ as $n \to \infty$. We next show that $w \in F$. Since $v_{n+1} \in C_n$, it follows from (3.5) that

$$\|z_n - v_n\| \le \|z_n - v_{n+1}\| + \|v_{n+1} - v_n\| \le 2\|v_{n+1} - v_n\| \to 0 \tag{3.37}$$

as $n \to \infty$. Since $\{v_n\}$ dominates p for all $p \in F$, we get

$$\begin{aligned} \|z_n - p\|^2 &\le (1-\alpha_n)\|S_1 y_n - p\|^2 + \alpha_n \|S_2 y_n - p\|^2 \\ &\le (1-\beta_n)\|v_n - p\|^2 + \beta_n \|S_1 v_n - p\|^2 - (1-\beta_n)\beta_n \|S_1 v_n - v_n\|^2) \\ &\le \|v_n - p\|^2 - (1-\beta_n)\beta_n \|S_1 v_n - v_n\|^2. \end{aligned} \tag{3.38}$$

This implies that

$$(1-\beta_n)\beta_n \|S_1 v_n - v_n\|^2 \le \|v_n - p\|^2 - \|z_n - p\|^2. \tag{3.39}$$

From our assumption (ii) and (3.39), we have

$$\lim_{n\to\infty} \|S_1 v_n - v_n\| = 0. \tag{3.40}$$

This implies that

$$\lim_{n\to\infty} \|y_n - v_n\| = \lim_{n\to\infty} \beta_n \|S_1 v_n - v_n\| = 0. \tag{3.41}$$

It follows from (3.37) and (3.41) that

$$\|y_n - z_n\| \le \|y_n - v_n\| + \|v_n - z_n\| \to 0, \tag{3.42}$$

as $n \to \infty$. We next show that $(v_n, y_n) \in E(G)$ for all $n \in \mathbb{N}$. Let $p \in F$. Since $(p, v_n) \in E(G)$ and S_1 is edge-preserving, so $(p, S_1 v_n) \in E(G)$ for all $n \in \mathbb{N}$. Then, $(p, y_n) = (p, (1-\beta_n)v_n + \beta_n S_1 v_n) \in E(G)$ by $E(G)$ is convex. Since G is transitive, $(v_n, y_n) \in E(G)$. This implies that

$$\begin{aligned}\|S_1 y_n - y_n\| &\le \|S_1 y_n - S_1 v_n\| + \|S_1 v_n - v_n\| + \|v_n - y_n\| \\ &\le 2\|y_n - S_1 v_n\| + \|S_1 v_n - v_n\|.\end{aligned}$$

It follows from (3.40), (3.41) and (3.43) that

$$\lim_{n\to\infty} \|S_1 y_n - y_n\| = 0. \tag{3.43}$$

It follows from (3.42) and (3.43) that

$$\|S_1 y_n - z_n\| \le \|S_1 y_n - y_n\| + \|y_n - z_n\| \to 0, \tag{3.44}$$

as $n \to \infty$. From $\liminf_{n\to\infty} \alpha_n > 0$ and (3.44), we have

$$\lim_{n\to\infty} \|S_2 y_n - z_n\| = \lim_{n\to\infty} \frac{1}{\alpha_n}\|z_n - S_1 y_n\| = 0. \tag{3.45}$$

It follows from (3.42) and (3.46) that

$$\|S_2 y_n - y_n\| \le \|S_2 y_n - z_n\| + \|z_n - y_n\| \to 0, \tag{3.46}$$

as $n \to \infty$. By Lemma 1, (3.41), (3.43) and (3.46), we have $w \in F$. From Step 5 in Theorem 1, we obtain $w = P_F v_1$. This completes the proof. □

4. Convergence rate

In this section, we give examples and numerical results for supporting our main theorem. Moreover, we compare convergence rate of all iterations in Theorem 1-4. In 1976, Rhoades [17] gave the idea how to compare the rate of convergence between two iterative methods as follows:

Definition 7 ([17])**.** Let C be a nonempty closed convex subset of a Banach space X and $S : C \to C$ be be a mapping. Suppose $\{x_n\}$ and $\{y_n\}$ are two iterations which converge to a fixed point p of S. Then $\{x_n\}$ is said to converge faster than $\{y_n\}$ if

$$\|x_n - p\| \le \|y_n - p\|,$$

for all $n \ge 1$.

In 2011, Phuengrattana and Suantai [14] showed that the Ishikawa iteration converges faster than the Mann iteration for a class of continuous functions on the closed interval in a real line. In order to study, the order of convergence of a real sequence $\{x_n\}$ converging to p, we usually use the well-known terminology in numerical analysis, see [7], for example.

Definition 8 ([7])**.** Suppose $\{x_n\}$ is a sequence that converges to p, with $x_n \neq p$ for all n. If positive constants a and b exist with

$$\lim_{n\to\infty} \frac{|x_{n+1}-p|}{|x_n-p|^b} = a,$$

then $\{x_n\}$ converges to p of order a, with asymptotic error constant b. If $b = 1$ (and $a < 1$), the sequence is linearly convergent and if $b = 2$, the sequence is quadratically convergent.

In 2002, Berinde [5] employed above concept for comparing the rate of convergence between the two iterative methods as follows:

Definition 9 ([5])**.** Let $\{x_n\}$ and $\{y_n\}$ be two sequences of positive numbers that converge to p, q,respectively. Assume there exists

$$\lim_{n\to\infty} \frac{|x_n-p|}{|y_n-q|} = \ell.$$

(i) If $\ell = 0$, then it is said that the sequence $\{x_n\}$ converges to p faster than the sequence $\{y_n\}$ to q.
(ii) If $0 < \ell < \infty$, then we say that the sequence $\{x_n\}$ and $\{y_n\}$ have the same rate of convergence.

Definition 10 ([5, 15])**.** Let C be a nonempty closed convex subset of a Banach space X and $S : C \to C$ be a mapping. Suppose $\{x_n\}$ and $\{y_n\}$ are two iterations which converge to p fixed point q of S. We say that $\{x_n\}$ converges faster than $\{y_n\}$ to q if

$$\lim_{n\to\infty} \frac{\|x_n-p\|}{\|y_n-q\|} = 0.$$

We now give an example in Euclidian space $\mathbb{R}^3$ which shows numerical experiment for supporting our main results and comparing the rate of convergence of all iterations in Theorem 1-4.

Example 1. Let $H = \mathbb{R}^3$ and $C = [-2,0]^3$. Assume that $(x,y) \in E(G)$ if and only if $-1.5 \leq x_i, y_i \leq -0.5$ or $x = y$ for all $x = (x_1,x_2,x_3), y = (y_1,y_2,y_3) \in C$. Define a mappings $S_1, S_2 : C \to C$ by $S_1x = (\frac{\arcsin(x_1+1)}{2} - 1, \log(x_2+2) - 1, -1)$ and $S_2x = (-1,-1,\frac{\cot(x_3-\frac{\pi}{2}+1)}{2} - 1)$ for all $x = (x_1,x_2,x_3) \in C$. It is easy to check that S_1 and S_2 are G-nonexpansive such that $F(S_1) \cap F(S_2) = \{(-1,-1,-1)\}$. On the other hand, S_1 is not nonexpansive since for $x = (-2,-1.46,-1)$ and $y = (-1.49,-1.82,-1)$. This implies that $\|S_1(x) - S_1(y)\| > 0.70 > \|x-y\|$. Moreover, S_2 is not nonexpansive since for $x = (-1,-1,-1.55)$ and $y = (-1,-1,-1.97)$, we have $\|S_2(x) - S_2(y)\| > 0.42 > \|x-y\|$.

We provide a numerical test of a comparison of all iterations in Theorem 1-4 and choose $\alpha_n = \frac{n+1}{5n+3}$, $\beta_n = \frac{n+3}{10n+5}$. The stoping criterion is defined by $\|x_{n+1} - x_n\| < 10^{-7}$. The different choices of x_0 are given in Table 1.

TABLE 1. Comparison the methods in Theorem 1-4 of Example 1

Iterations	Choice 1: (-1.25,-0.9,-0.65) Iterations Number	CPU Time (sec)	Choice 2: (-1.45,-1.2,-0.7) Iterations Number	CPU Time (sec)
(3.1)	118	2.344643e-03	124	1.618338e-03
(3.15)	97	2.896767e-03	102	1.452308e-03
(3.23)	39	2.051223e-03	41	1.233451e-03
(3.35)	38	3.913172e-03	39	3.129209e-03

By computing, we obtain the sequences $\{x_n\}$ generated in Theorem 1-4 converge to (-1,-1,-1). We next show the following error plots of $\|x_{n+1} - x_n\|$.

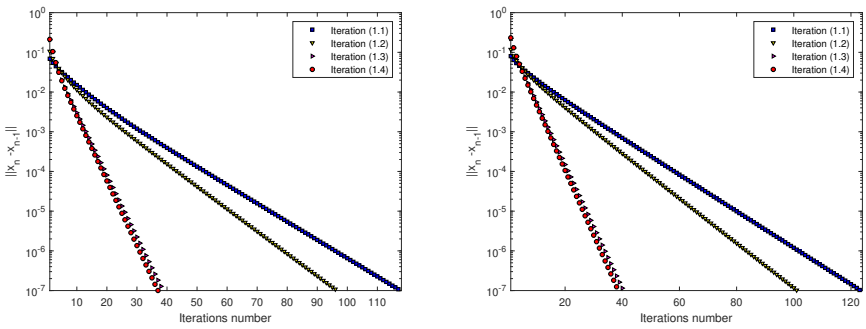

FIGURE 1. Error plots for sequences $\{x_n\}$ in Table 1 of choice 1 and choice 2, respectively.

We note that $p = (-1, -1, -1)$ is a common fixed point of S_1 and S_2. We compare the rate of convergence of $\{s_n\}$, $\{t_n\}$, $\{u_n\}$ and $\{v_n\}$ for Choice 1: $x_0 = (-1.25, -0.9, -0.65)$ and Choice 2: $x_0 = (-1.45, -1.2, -0.7)$.

TABLE 2. Comparison the rate of convergence of all iterations in Theorem 1-4 of Example 1 by choosing $x_0 = (-1.25, -0.9, -0.65)$

n	$\frac{\|v_n-p\|}{\|s_n-p\|}$	$\frac{\|v_n-p\|}{\|t_n-p\|}$	$\frac{\|v_n-p\|}{\|u_n-p\|}$	$\frac{\|u_n-p\|}{\|s_n-p\|}$	$\frac{\|u_n-p\|}{\|t_n-p\|}$	$\frac{\|t_n-p\|}{\|s_n-p\|}$
0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
1	0.637538	0.979056	0.704657	0.651176	0.719731	0.904749
2	0.434675	0.952151	0.515409	0.456519	0.541310	0.843360
3	0.314266	0.924612	0.394249	0.339890	0.426395	0.797125
4	0.236078	0.899949	0.310695	0.262324	0.345236	0.759841
5	0.181039	0.878591	0.248470	0.206056	0.282805	0.728615
6	0.140193	0.859748	0.199756	0.163063	0.232342	0.701821
7	0.095220	0.842647	0.160677	0.113001	0.190681	0.592617
8	0.074299	0.826834	0.129054	0.089860	0.156083	0.575721
9	0.057963	0.811696	0.103409	0.071409	0.127399	0.560516

TABLE 3. Comparison the rate of convergence of all iterations in Theorem 1-4 of Example 1 by choosing $x_0 = (-1.45, -1.2, -0.7)$

n	$\frac{\|v_n-p\|}{\|s_n-p\|}$	$\frac{\|v_n-p\|}{\|t_n-p\|}$	$\frac{\|v_n-p\|}{\|u_n-p\|}$	$\frac{\|u_n-p\|}{\|s_n-p\|}$	$\frac{\|u_n-p\|}{\|t_n-p\|}$	$\frac{\|t_n-p\|}{\|s_n-p\|}$
0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
1	0.713514	0.966823	0.768432	0.737998	0.794801	0.928531
2	0.527376	0.937880	0.598780	0.562306	0.638440	0.880750
3	0.397672	0.913037	0.471501	0.435549	0.516410	0.843417
4	0.302854	0.891493	0.372931	0.339716	0.418322	0.812092
5	0.231693	0.872302	0.295256	0.265611	0.338479	0.784719
6	0.177594	0.854743	0.233630	0.207774	0.273334	0.760149
7	0.136232	0.838398	0.184673	0.162490	0.220269	0.737690
8	0.104523	0.822910	0.145806	0.127016	0.177183	0.716861
9	0.080194	0.808107	0.114999	0.099237	0.142307	0.697345

Remark 1. From Figure 1, Table 1-3, it is shown that the iteration (3.35) has a good convergence speed and requires small number of iterations than the other three iterations for each of the choices.

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Authors' addresses

H.A. Hammad
Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt
E-mail address: h_elmagd89@yahoo.com

W. Cholamjiak
Department of Mathematics, School of Science, University of Phayao, Phayao 56000, Thailand
E-mail address: watcharaporn.ch@up.ac.th

D. Yambangwai
Department of Mathematics, School of Science, University of Phayao, Phayao 56000, Thailand
E-mail address: damrongsak.ya@up.ac.th

H. Dutta
Department of Mathematics, Faculty of Science, Gauhati University, Guwahati-781014, India
E-mail address: hemen_dutta08@rediffmail.com

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SOLUTIONS OF HOMOGENEOUS FRACTIONAL p-KIRCHHOFF EQUATIONS IN $\mathbb{R}^N$

VU HO, NHAT VY HUYNH, AND PHUONG LE

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Abstract. In this note, we furnish a transformation such that solutions of the fractional p-Kirchhoff equation in $\mathbb{R}^N$ are easily obtained from known solutions of the corresponding fractional p-Laplace equation. As an application, we classify all positive solutions of some (fractional) p-Kirchhoff equations with sub-critical or critical nonlinearities and Hénon-Hardy potentials. Similar results for Kirchhoff type systems are also discussed.

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1. Introduction and main results

Throughout this note, we always assume that $N \geq 1$ and $M : [0,+\infty) \to [0,+\infty)$. We also suppose that $0 < s \leq 1 < p$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is measurable and homogeneous in the first variable. More precisely,

(F) $f(x,u) = |x|^\alpha f\left(\frac{x}{|x|}, u\right)$ for some $\alpha \neq -ps$ and all $(x,u) \in \mathbb{R}^N \setminus \{0\} \times \mathbb{R}$.

We study solutions of the following nonlocal equation

$$M\left([u]_{s,p}^p\right)(-\Delta)_p^s u = f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where

$$[u]_{1,p} = \left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx\right)^{1/p},$$

$$(-\Delta)_p^1 u(x) = -\mathrm{div}(|\nabla u(x)|^{p-2}\nabla u(x))$$

and for $s \in (0,1)$,

$$[u]_{s,p} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \, dxdy\right)^{1/p},$$

$$(-\Delta)_p^s u(x) = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}[u(x)-u(y)]}{|x-y|^{N+ps}} \, dy,$$

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up to normalization factors, here $B_\varepsilon(x)$ is the ball centered at $x \in \mathbb{R}^N$ with radius $\varepsilon > 0$. We also denote

$$\langle u, \varphi \rangle_{1,p} = \int_{\mathbb{R}^N} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla \varphi(x)\, dx$$

and

$$\langle u, \varphi \rangle_{s,p} = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}[u(x)-u(y)][\varphi(x)-\varphi(y)]}{|x-y|^{N+ps}}\, dxdy$$

for $s \in (0,1)$. Note that $[u]_{s,p}^p = \langle u, u \rangle_{s,p}$. For the sake of simplicity and usual convention, we also denote $-\Delta_p u = (-\Delta)_p^1 u$ and $(-\Delta)^s u = (-\Delta)_2^s u$.

A general functional framework for (1.1) is the local (fractional) Sobolev space $W_{\mathrm{loc}}^{s,p}(\mathbb{R}^N)$ consisting of all measurable function u on $\mathbb{R}^N$ such that $u\varphi \in W^{s,p}(\mathbb{R}^N)$ for all $\varphi \in C_c^\infty(\mathbb{R}^N)$, where $W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\}$ and $C_c^\infty(\mathbb{R}^N)$ is the space of smooth functions with compact support in $\mathbb{R}^N$.

A function $u \in W_{\mathrm{loc}}^{s,p}(\mathbb{R}^N)$ is called a solution of (1.1) if $[u]_{s,p} < \infty$, $f(x,u) \in L_{\mathrm{loc}}^1(\mathbb{R}^N)$ and

$$M\left([u]_{s,p}^p\right) \langle u, \varphi \rangle_{s,p} = \int_{\mathbb{R}^N} f(x,u(x))\varphi(x)\, dx \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^N).$$

Equation (1.1) is analogous to the stationary problem of a physical model which were first introduced by Kirchhoff [10] to describe the transversal oscillations of elastic strings. This type of problems received much attention of several researchers after the work of Lions [13], where a functional analysis framework was proposed to attack it.

Due to the presence of the nonlocal term $M([u]_{s,p}^p)$, problem (1.1) is no longer a pointwise identity even if $s = 1$. This phenomenon causes some mathematical difficulties which make the study of such problems particularly interesting. In the last decade, working directly with nonlocal term via variational methods, several authors have established many interesting results about the existence and nonexistence of positive solutions, sign-changing solutions, ground state solutions, least energy nodal solutions, multiplicity of solutions, semi-classical limit and concentrations of solutions to Kirchhoff and p-Kirchhoff problems, see e.g. [5, 8, 11] and the references therein. Recently, many authors also study fractional p-Kirchhoff problems (1.1) when $s \in (0,1)$. Variational results for these and related problems are established in [6, 18, 19, 27] and references therein.

There is, however, another simple method that helps eliminate nonlocal term of some Kirchhoff equations in $\mathbb{R}^N$. In [8], the authors used a transformation that allows them to obtain solutions of the autonomous Kirchhoff equation

$$-M\left(\int_{\mathbb{R}^N} |\nabla u|^2\, dx\right) \Delta u = h(u) \quad \text{in } \mathbb{R}^N$$

from the corresponding local equation. In this note we will extends this transformation to fractional p-Kirchhoff equation (1.1) with homogeneous nonlinearity $f(x,u)$. This method allows us to reduce (1.1) to its corresponding fractional p-Laplace equation

$$(-\Delta)_p^s u = f(x,u) \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

For any function u on $\mathbb{R}^N$ and $\lambda > 0$, we denote $u^\lambda(x) \equiv u(\frac{x}{\lambda})$. Our first result is the following theorem, which extends [8, Theorem 1-3].

Theorem 1. *Assume that $0 < s \le 1 < p$ and f satisfies (F). Then u is a solution of (1.1) if and only if $M\left([u]_{s,p}^p\right) = 0$ and $f(x,u) = 0$ a.e. in $\mathbb{R}^N$, or $u = u_0^\lambda$, where u_0 is a solution of (1.2) and $\lambda > 0$ is a solution of the algebraic equation $M(\lambda^{N-ps}[u_0]_{s,p}^p) = \lambda^{\alpha+ps}$.*

Our second theorem is an analogous result for the Kirchhoff type system

$$\begin{cases} M_1\left([u]_{s_1,p_1}^{p_1}\right)(-\Delta)_{p_1}^{s_1} u = f_1(x,u,v) & \text{in } \mathbb{R}^N, \\ M_2\left([v]_{s_2,p_2}^{p_2}\right)(-\Delta)_{p_2}^{s_2} v = f_2(x,u,v) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.3}$$

We say that $(u,v) \in W_{\text{loc}}^{s_1,p_1}(\mathbb{R}^N) \times W_{\text{loc}}^{s_2,p_2}(\mathbb{R}^N)$ is a solution of (1.3) if $[u]_{s_1,p_1} < \infty$, $[v]_{s_2,p_2} < \infty$, $f_1(x,u,v) \in L_{\text{loc}}^1(\mathbb{R}^N)$, $f_2(x,u,v) \in L_{\text{loc}}^1(\mathbb{R}^N)$ and

$$\begin{cases} M_1\left([u]_{s_1,p_1}^{p_1}\right)\langle u,\varphi\rangle_{s_1,p_1} = \int_{\mathbb{R}^N} f_1(x,u(x),v(x))\varphi(x)\,dx, \\ M_2\left([v]_{s_2,p_2}^{p_2}\right)\langle v,\varphi\rangle_{s_2,p_2} = \int_{\mathbb{R}^N} f_2(x,u(x),v(x))\varphi(x)\,dx, \end{cases}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^N)$.

The corresponding system of (1.3) when $M_1 \equiv M_2 \equiv 1$ is

$$\begin{cases} (-\Delta)_{p_1}^{s_1} u = f_1(x,u,v) & \text{in } \mathbb{R}^N, \\ (-\Delta)_{p_2}^{s_2} v = f_2(x,u,v) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.4}$$

Theorem 2. *Assume that $0 < s_1, s_2 \le 1 < p_1, p_2$, $M_1, M_2 : [0,+\infty) \to [0,+\infty)$, f_1, f_2 are measurable and $f_1(x,u,v) = |x|^{\alpha_1} f_1\left(\frac{x}{|x|},u,v\right)$, $f_2(x,u,v) = |x|^{\alpha_2} f_2\left(\frac{x}{|x|},u,v\right)$ for some $\alpha_1,\alpha_2 \in \mathbb{R}$ and all $(x,u,v) \in \mathbb{R}^N \setminus \{0\} \times \mathbb{R} \times \mathbb{R}$. Then $(u_0^\lambda, v_0^\lambda)$ is a solution of (1.3) if (u_0,v_0) is a solution of (1.4) and $\lambda > 0$ is a solution of the algebraic system*

$$\begin{cases} M_1(\lambda^{N-p_1s_1}[u_0]_{s_1,p_1}^{p_1}) = \lambda^{\alpha_1+p_1s_1}, \\ M_2(\lambda^{N-p_2s_2}[v_0]_{s_2,p_2}^{p_2}) = \lambda^{\alpha_2+p_2s_2}. \end{cases}$$

We cannot obtain all solutions of (1.3) from known solutions of (1.4). However, when f_1 and f_2 have power growth in u and v, we are able to do that by using another

transformation. Indeed, let us consider the following Kirchhoff type system

$$\begin{cases} M_1\left([u]^{p_1}_{s_1,p_1}\right)(-\Delta)^{s_1}_{p_1}u = w_1(x)u^{q_1}v^{r_1} & \text{in } \mathbb{R}^N, \\ M_2\left([v]^{p_2}_{s_2,p_2}\right)(-\Delta)^{s_2}_{p_2}v = w_2(x)u^{r_2}v^{q_2} & \text{in } \mathbb{R}^N \end{cases} \tag{1.5}$$

and its corresponding local system

$$\begin{cases} (-\Delta)^{s_1}_{p_1}u = w_1(x)u^{q_1}v^{r_1} & \text{in } \mathbb{R}^N, \\ (-\Delta)^{s_2}_{p_2}v = w_2(x)u^{r_2}v^{q_2} & \text{in } \mathbb{R}^N. \end{cases} \tag{1.6}$$

We have the following.

Theorem 3. *Assume that* $0 < s_1, s_2 \le 1 < p_1, p_2$, $M_1, M_2 : [0,+\infty) \to [0,+\infty)$, $w_1, w_2 : \mathbb{R}^N \to (0,+\infty)$ *are measurable and* $q_1, q_2, r_1, r_2 \in \mathbb{R}$ *such that* $(q_1 - p_1 + 1)(q_2 - p_2 + 1) \neq r_1 r_2$. *Then* (u,v) *is a positive solution of* (1.5) *if and only if* $(u,v) = (\lambda_1 u_0, \lambda_2 v_0)$, *where* (u_0, v_0) *is a positive solution of* (1.6) *and* (λ_1, λ_2) *is a positive solution of the algebraic system*

$$\begin{cases} M_1(\lambda_1^{p_1}[u_0]^{p_1}_{s_1,p_1}) = \lambda_1^{q_1-p_1+1}\lambda_2^{r_1}, \\ M_2(\lambda_2^{p_2}[v_0]^{p_2}_{s_2,p_2}) = \lambda_1^{r_2}\lambda_2^{q_2-p_2+1}. \end{cases} \tag{1.7}$$

Remark 1. It should be noted that in our results we do not assume any condition on M, M_1, M_2 except their non-negativity. Our method can also be applied to fractional p-Kirchhoff type inequalities and poly-harmonic equations of p-Kirchhoff type.

The proofs of our theorems are given in Section 3. In the next section, we utilize these theorems to classify all positive solutions of some fractional p-Kirchhoff equations and systems having power growth and Hénon-Hardy potentials.

2. Classification of positive solutions

We consider the p-Laplace equation of Hénon-Lane-Emden type

$$-\Delta_p u = |x|^\alpha u^q \quad \text{in } \mathbb{R}^N. \tag{2.1}$$

Before stating our classification results, let us introduce the following exponents

$$q_S(p,\alpha) = \begin{cases} \frac{(p-1)N+p\alpha+p}{N-p} & \text{if } N > p, \\ \infty & \text{if } N \le p \end{cases}$$

and

$$q_S(p) = q_S(p,0).$$

When $N > p > -\alpha$ and $q = q_S(p,\alpha)$, solutions of (2.1) may be found by minimizing $\int_{\mathbb{R}^N} |\nabla u|^p\,dx$ over the manifold $\mathcal{N}(p,\alpha) = \{u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |x|^\alpha |u|^{\frac{p(N+\alpha)}{N-p}}\,dx = 1\}$. Let us denote

$$S(p,\alpha) = \inf_{u\in\mathcal{N}(p,\alpha)} \left(\int_{\mathbb{R}^N} |\nabla u|^p\,dx\right)^{\frac{N+\alpha}{p+\alpha}}$$

and

$$S(p) = S(p,0).$$

By a standard argument, solution w of (2.1) obtained from minimizers of $S(p,\alpha)$ satisfies $\int_{\mathbb{R}^N} |\nabla w|^p \, dx = \int_{\mathbb{R}^N} |x|^\alpha |w|^{\frac{p(N+\alpha)}{N-p}} \, dx = S(p,\alpha)$.

We consider equation (2.1) in the autonomous case $\alpha = 0$. Serrin and Zou [23, Corollary II] proved that (2.1) has no positive solution if $\alpha = 0$, $p > 1$ and $q < q_S(p)$. Therefore, utilizing Theorem 1, we have the following nonexistence result.

Proposition 1. *If $p > 1$ and $q < q_S(p)$, then the equation*

$$-M\left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx\right) \Delta_p u = u^q \quad \text{in } \mathbb{R}^N$$

has no positive $C^1(\mathbb{R}^N)$ solution.

In critical case $q = q_S(p) = \frac{(p-1)N+p}{N-p}$ where $1 < p < N$ and $\alpha = 0$, all positive $\mathcal{D}^{1,p}(\mathbb{R}^N)$ solutions of (2.1) have been recently classified by Vétois [26] (for $1 < p < 2$) and Sciunzi [20] (for $2 < p < N$). See also [2] for classical result in the case $p = 2$. They proved that all positive solutions of (2.1) under these assumptions are of the form

$$U_{p,\mu,x_0}(x) = \left(\frac{\mu^{\frac{1}{p-1}} N^{\frac{1}{p}} \left(\frac{N-p}{p-1}\right)^{\frac{p-1}{p}}}{\mu^{\frac{p}{p-1}} + |x-x_0|^{\frac{p}{p-1}}}\right)^{\frac{N-p}{p}},$$

for $\mu > 0$ and $x_0 \in \mathbb{R}^N$, see [20, Theorem 1.1]. We also know that $\int_{\mathbb{R}^N} |\nabla U_{p,\mu,x_0}|^p \, dx = S(p)$. From this fact and Theorem 1, we have the following result, which extends [14, Theorem 1.1 and 1.2] to p-Kirchhoff case.

Proposition 2. *If $1 < p < N$, then every positive $\mathcal{D}^{1,p}(\mathbb{R}^N)$ solution u of the equation*

$$-M\left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx\right) \Delta_p u = u^{\frac{(p-1)N+p}{N-p}} \quad \text{in } \mathbb{R}^N$$

must have the form $u(x) = U_{p,\mu,x_0}(\frac{x}{\lambda})$, where $\mu > 0, x_0 \in \mathbb{R}^N$ and $\lambda > 0$ is a solution of

$$M(\lambda^{N-p} S(p)) = \lambda^p. \tag{2.2}$$

Therefore, if (2.2) has no positive solution, then the above p-Kirchhoff equation has no positive solution.

We now turn our attention to the Hénon case $\alpha \geq 0$. When $p = 2$ and $1 < q < q_S(2,\alpha)$, the nonexistence of positive $H^1_{loc}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ solutions of (2.1) was recently obtained in [7, Theorem 1]. The optimal nonexistence result for the case $p \neq 2$ is still an open problem. There are some partial results anyway. This type of nonexistence result was proved in [9, Section 3] for radial positive solutions and

$q < q_S(p,\alpha)$, in [16, Theorem 12.4] for $q \le \frac{(p-1)(N+\alpha)}{N-p}$ ($q < \infty$ if $N \le p$), in [17, Theorem 1.2] for $q < q_S(p,\alpha)$, $N < p+1$ and in [17, Theorem 1.3] for $q < q_S(p)$. Combining these facts with Theorem 1, we have the following.

Proposition 3. *Let $\alpha \ge 0$ and $q > p-1$, the p-Kirchhoff equation*

$$-M\left(\int_{\mathbb{R}^N} |\nabla u|^p\,dx\right)\Delta_p u = |x|^\alpha u^q \quad \text{in } \mathbb{R}^N$$

has no positive $C^1(\mathbb{R}^N)$ solution u under one of the following assumptions

(i) *$p = 2$ and $q < q_S(2,\alpha)$,*
(ii) *$q \le \frac{(p-1)(N+\alpha)}{N-p}$ ($q < \infty$ if $N \le p$),*
(iii) *$q < q_S(p,\alpha)$ and u is radial,*
(iv) *$q < q_S(p,\alpha)$ and $N < p+1$,*
(v) *$q < q_S(p)$.*

Recently, all positive solutions of class $H^1_{loc}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ to equation (2.1) when $p = 2$, $-2 < \alpha < 0$, $N \ge 3$ and $q = q_S(2,\alpha) = \frac{N+2\alpha+2}{N-2}$ was classified in [7, Theorem 2]. It was proved that all such solutions must have the form

$$V_{\alpha,\mu}(x) = \left(\frac{N(N-2)(\alpha+2)^2}{4}\right)^{\frac{N-2}{4}} \left(\frac{\mu}{\mu^2 + |x|^{\alpha+2}}\right)^{\frac{N-2}{\alpha+2}}$$

for $\mu > 0$. It is also clear that $\int_{\mathbb{R}^N} |\nabla V_{\alpha,\mu}|^2\,dx = S(2,\alpha)$. From this and Theorem 1, we have the following proposition, which extends [14, Theorem 1.1 and 1.2] to Hardy case.

Proposition 4. *If $N \ge 3$ and $-2 < \alpha < 0$, then every positive $H^1_{loc}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ solution u of the equation*

$$-M\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)\Delta u = |x|^\alpha u^{\frac{N+2\alpha+2}{N-2}} \quad \text{in } \mathbb{R}^N$$

must have the form $u(x) = V_{\alpha,\mu}(\frac{x}{\lambda})$ where $\mu > 0$ and $\lambda > 0$ is a solution of

$$M(\lambda^{N-2} S(2,\alpha)) = \lambda^{\alpha+2}. \tag{2.3}$$

Therefore, if (2.3) has no positive solution, then the above Kirchhoff equation has no positive solution.

By similar arguments, we may obtain some classification results for fractional Kirchhoff problems. Let us denote

$$\mathcal{L}_\alpha = \left\{ u : \mathbb{R}^n \to \mathbb{R} \,\middle|\, \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+s}}\,dx < \infty \right\}.$$

The nonexistence of nontrivial nonnegative $C^{1,1}_{\text{loc}} \cap \mathcal{L}_\alpha$ solutions of $(-\Delta)^s u = u^q$ in $\mathbb{R}^N$ when $0 < s < 1$ and $1 < q < \frac{N+2s}{N-2s}$ was proved recently in [3, Theorem 4].

Moreover, if $q = \frac{N+2s}{N-2s}$, then nonnegative $C^{1,1}_{\text{loc}} \cap \mathcal{L}_\alpha$ solutions of this equation are radially symmetric and hence must assume an explicit form. We may utilize these results and Theorem 1 to obtain

Proposition 5. *Assume $0 < s < 1 < q$ and $u \in C^{1,1}_{\text{loc}} \cap \mathcal{L}_\alpha$ is a nonegative solution of the equation*

$$M\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy\right)(-\Delta)^s u = u^q \quad \text{in } \mathbb{R}^N.$$

Then

(i) *In the critical case $q = \frac{N+2s}{N-2s}$, $u \equiv 0$ or u assumes the form*

$$u(x) = c\left(\frac{\mu}{\mu^2+|x-x_0|^2}\right)^{\frac{N-2s}{2}}, \quad c,\mu > 0, x_0 \in \mathbb{R}^N.$$

(ii) *In the subcritical case $1 < q < \frac{N+2s}{N-2s}$, $u \equiv 0$.*

To demonstrate an application of Theorem 3, let us now consider the Lane-Emden system

$$\begin{cases} -\Delta u = v^p & \text{in } \mathbb{R}^N, \\ -\Delta v = u^q & \text{in } \mathbb{R}^N. \end{cases} \tag{2.4}$$

The famous Lane-Emden conjecture states that: *If the positive pair (p,q) lies below the Sobolev critical hyperbola, i.e. if*

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}, \tag{2.5}$$

then there is no classical positive solution to (2.4).

Up to now, the conjecture is proved to be true for radial functions by Mitidieri [15] and Serrin-Zou [21]. For the full conjecture, Souto [25], Mitidieri [15] and Serrin-Zou [22] proved that there is no supersolution to (2.4) if $pq \le 1$ or $\max\{\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\} \ge N-2$. This solves the Lane-Emden conjecture in dimensions $N = 1,2$. More recently, the conjecture is proved in dimensions $N = 3,4$ by Souplet and his collaborators, see [24]. For $N \ge 5$, the conjecture is prove to be true for (p,q) verifying (2.5) and one of the following extra conditions:

- If $p,q < \frac{N+2}{N-2}$, see Felmer-de Figuereido [4].
- If $\max(p,q) \ge N-3$, see Souplet [24].
- If $\min\{\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\} \ge \frac{N-2}{2}$, see Busca-Manásevich [1].
- If $p = 1$ or $q = 1$, see Lin [12].

We may utilize Theorem 3 and above facts to obtain an analogous result for the Kirchhoff-Lane-Emden system.

Proposition 6. *Assume that $p,q>0$ verify (2.5) and $M_1,M_2:[0,+\infty)\to[0,+\infty)$. Then the Kirchhoff-Lane-Emden system*

$$\begin{cases} -M_1(\int_{\mathbb{R}^N}|\nabla u|^2\,dx)\Delta u = v^p & \text{in } \mathbb{R}^N,\\ -M_2(\int_{\mathbb{R}^N}|\nabla v|^2\,dx)\Delta v = u^q & \text{in } \mathbb{R}^N \end{cases}$$

has no classical positive solution (u,v) under one of the following extra conditions

(i) *u,v are radial,*
(ii) $N\le 4$,
(iii) $p,q<\frac{N+2}{N-2}$,
(iv) $\max\{p,q\}\ge N-3$,
(v) $\min\{\frac{2(p+1)}{pq-1},\frac{2(q+1)}{pq-1}\}\ge\frac{N-2}{2}$,
(vi) *$p=1$ or $q=1$.*

Certainly, we can state the following open problem which is equivalent to the Lane-Emden conjecture by Theorem 3.

Open problem. Assume that $p,q>0$ verify (2.5) and $M_1,M_2:[0,+\infty)\to[0,+\infty)$. Then the Kirchhoff-Lane-Emden system has no classical positive solution.

3. Proofs of main results

Proof of Theorem 1. For any u,φ and $\lambda>0$, we have

$$\langle u^\lambda,\varphi\rangle_{s,p}=\lambda^{N-ps}\langle u,\varphi^{1/\lambda}\rangle_{s,p}. \tag{3.1}$$

Indeed, if $s=1$, then

$$\begin{aligned}\langle u^\lambda,\varphi\rangle_{1,p}&=\lambda^{-p+1}\int_{\mathbb{R}^N}\left|\nabla u\left(\frac{x}{\lambda}\right)\right|^{p-2}\nabla u\left(\frac{x}{\lambda}\right)\cdot\nabla\varphi(x)\,dx\\&=\lambda^{N-p+1}\int_{\mathbb{R}^N}|\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\varphi(\lambda x)\,dx\\&=\lambda^{N-p}\langle u,\varphi^{1/\lambda}\rangle_{1,p},\end{aligned}$$

while if $0<s<1$, then

$$\begin{aligned}\langle u^\lambda,\varphi\rangle_{s,p}&=\iint_{\mathbb{R}^{2N}}\frac{|u\left(\frac{x}{\lambda}\right)-u\left(\frac{y}{\lambda}\right)|^{p-2}[u\left(\frac{x}{\lambda}\right)-u\left(\frac{y}{\lambda}\right)][\varphi(x)-\varphi(y)]}{|x-y|^{N+ps}}\,dxdy\\&=\lambda^{N-ps}\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p-2}[u(x)-u(y)][\varphi(\lambda x)-\varphi(\lambda y)]}{|x-y|^{N+ps}}\,dxdy\\&=\lambda^{N-ps}\langle u,\varphi^{1/\lambda}\rangle_{s,p}.\end{aligned}$$

Note that (3.1) also implies $[u^\lambda]^p_{s,p}=\lambda^{N-ps}[u]^p_{s,p}$.

Assume that u_0 is a solution of (1.2) and $\lambda > 0$ satisfies $M(\lambda^{N-ps}[u_0]^p_{s,p}) = \lambda^{\alpha+ps}$. Then $u = u_0^\lambda$ verifies $[u]^p_{s,p} = \lambda^{N-ps}[u_0]^p_{s,p}$ and for all $\varphi \in C_c^\infty(\mathbb{R}^N)$,

$$\begin{aligned} M\left([u]^p_{s,p}\right)\langle u,\varphi\rangle_{s,p} &= M\left(\lambda^{N-ps}[u_0]^p_{s,p}\right)\langle u_0^\lambda,\varphi\rangle_{s,p} \\ &= M\left(\lambda^{N-ps}[u_0]^p_{s,p}\right)\lambda^{N-ps}\langle u_0,\varphi^{1/\lambda}\rangle_{s,p} \\ &= M\left(\lambda^{N-ps}[u_0]^p_{s,p}\right)\lambda^{N-ps}\int_{\mathbb{R}^N} f(x,u_0(x))\varphi(\lambda x)\,dx \\ &= \lambda^\alpha\int_{\mathbb{R}^N} f\left(\frac{x}{\lambda},u_0\left(\frac{x}{\lambda}\right)\right)\varphi(x)\,dx \\ &= \int_{\mathbb{R}^N} f(x,u(x))\varphi(x)\,dx, \end{aligned}$$

where we have used the definition of λ and assumption (F) in the last two lines. Therefore, u is a solution of (1.1). On the other hand, if $M\left([u]^p_{s,p}\right) = 0$ and $f(x,u) = 0$ a.e. in $\mathbb{R}^N$, then clearly u is a solution of (1.1).

Conversely, assume that u is a solution of (1.1) and either $M\left([u]^p_{s,p}\right) > 0$ or $f(x,u) \neq 0$ in a subset of $\mathbb{R}^N$ with positive measure. Clearly, $M\left([u]^p_{s,p}\right) > 0$ in both cases. We define $u_0 = u^{1/\lambda}$ where $\lambda = M([u]^p_{s,p})^{1/(\alpha+ps)}$. Then $u = u_0^\lambda$ and $[u]^p_{s,p} = \lambda^{N-ps}[u_0]^p_{s,p}$. Therefore, λ satisfies $M(\lambda^{N-ps}[u_0]^p_{s,p}) = \lambda^{\alpha+ps}$. For all $\varphi \in C_c^\infty(\mathbb{R}^N)$,

$$\begin{aligned} \langle u_0,\varphi\rangle_{s,p} &= \langle u^{1/\lambda},\varphi\rangle_{s,p} = \lambda^{-N+ps}\langle u,\varphi^\lambda\rangle_{s,p} \\ &= \lambda^{-N+ps}M^{-1}\left([u]^p_{s,p}\right)\int_{\mathbb{R}^N} f(x,u(x))\varphi\left(\frac{x}{\lambda}\right)dx \\ &= \lambda^{\alpha+ps}M^{-1}\left([u]^p_{s,p}\right)\int_{\mathbb{R}^N} f(x,u_0(x))\varphi(x)\,dx \\ &= \int_{\mathbb{R}^N} f(x,u_0(x))\varphi(x)\,dx, \end{aligned}$$

which means that u_0 is a solution of (1.2). □

Proof of Theorem 2. For all $\varphi \in C_c^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} M_1\left([u_0^\lambda]^{p_1}_{s_1,p_1}\right)\langle u_0^\lambda,\varphi\rangle_{s_1,p_1} &= M_1\left(\lambda^{N-p_1s_1}[u_0]^{p_1}_{s_1,p_1}\right)\lambda^{N-p_1s_1}\langle u_0,\varphi^{1/\lambda}\rangle_{s_1,p_1} \\ &= \lambda^{N+\alpha_1}\int_{\mathbb{R}^N} f_1(x,u_0(x),v_0(x))\varphi(\lambda x)\,dx \\ &= \lambda^{\alpha_1}\int_{\mathbb{R}^N} f_1\left(\frac{x}{\lambda},u_0\left(\frac{x}{\lambda}\right),v_0\left(\frac{x}{\lambda}\right)\right)\varphi(x)\,dx \\ &= \int_{\mathbb{R}^N} f_1(x,u_0^\lambda(x),v_0^\lambda(x))\varphi(x)\,dx. \end{aligned}$$

Similarly,

$$M_2\left([v_0^\lambda]_{s_2,p_2}^{p_2}\right)\langle v_0^\lambda,\varphi\rangle_{s_2,p_2}=\int_{\mathbb{R}^N} f_2(x,u_0^\lambda(x),v_0^\lambda(x))\varphi(x)\,dx.$$

Therefore, $(u_0^\lambda, v_0^\lambda)$ is a solution of (1.3). □

Proof of Theorem 3. Assume that $(u,v)=(\lambda_1 u_0,\lambda_2 v_0)$, where (u_0,v_0) is a positive solution of (1.6) and $\lambda_1,\lambda_2>0$ satisfy (1.7). For all $\varphi\in C_c^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}M_1\left([u]_{s_1,p_1}^{p_1}\right)\langle u,\varphi\rangle_{s_1,p_1}&=M_1\left(\lambda_1^{p_1}[u_0]_{s_1,p_1}^{p_1}\right)\lambda_1^{p_1-1}\langle u_0,\varphi\rangle_{s_1,p_1}\\&=\lambda_1^{q_1}\lambda_2^{r_1}\int_{\mathbb{R}^N} w_1u_0^{q_1}v_0^{r_1}\varphi\,dx\\&=\int_{\mathbb{R}^N} w_1u^{q_1}v^{r_1}\varphi\,dx.\end{aligned}$$

Similarly,

$$M_2\left([v]_{s_2,p_2}^{p_2}\right)\langle v,\varphi\rangle_{s_2,p_2}=\int_{\mathbb{R}^N} w_2u^{r_2}v^{q_2}\varphi\,dx.$$

Therefore, (u,v) is a solution of (1.5).

Conversely, assume that (u,v) is a positive solution of (1.5). We define $(u_0,v_0)=(u/\lambda_1,v/\lambda_2)$, where

$$\lambda_1=\left(\frac{M_1^{q_2-p_2+1}([u]_{s_1,p_1}^{p_1})}{M_2^{r_1}([v]_{s_2,p_2}^{p_2})}\right)^{\frac{1}{(q_1-p_1+1)(q_2-p_2+1)-r_1r_2}},$$

$$\lambda_2=\left(\frac{M_2^{q_1-p_1+1}([v]_{s_2,p_2}^{p_2})}{M_1^{r_2}([u]_{s_1,p_1}^{p_1})}\right)^{\frac{1}{(q_1-p_1+1)(q_2-p_2+1)-r_1r_2}},$$

Then $(u,v)=(\lambda_1u_0,\lambda_2v_0)$ and $[u]_{s_1,p_1}^{p_1}=\lambda_1^{p_1}[u_0]_{s_1,p_1}^{p_1}$, $[v]_{s_2,p_2}^{p_2}=\lambda_2^{p_2}[v_0]_{s_2,p_2}^{p_2}$. Therefore, λ_1,λ_2 satisfy (1.7). For all $\varphi\in C_c^\infty(\mathbb{R}^N)$,

$$\begin{aligned}\langle u_0,\varphi\rangle_{s_1,p_1}&=\lambda_1^{1-p_1}\langle u,\varphi\rangle_{s_1,p_1}\\&=\lambda_1^{1-p_1}M_1^{-1}\left([u]_{s_1,p_1}^{p_1}\right)\int_{\mathbb{R}^N} w_1u^{q_1}v^{r_1}\varphi\,dx\\&=\lambda_1^{-q_1}\lambda_2^{-r_1}\int_{\mathbb{R}^N} w_1u^{q_1}v^{r_1}\varphi\,dx\\&=\int_{\mathbb{R}^N} w_1u_0^{q_1}v_0^{r_1}\varphi\,dx.\end{aligned}$$

Similarly,

$$\langle v_0,\varphi\rangle_{s_2,p_2}=\int_{\mathbb{R}^N} w_2u_0^{r_2}v_0^{q_2}\varphi\,dx.$$

Hence, (u_0,v_0) is a solution of (1.2). □

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Authors' addresses

Vu Ho

Division of Computational Mathematics and Engineering, Institute for Computational Science, Ton Duc Thang University, Ho Chi Minh City, Vietnam; Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam

E-mail address: hovu@tdtu.edu.vn

Nhat Vy Huynh

Department of Fundamental Sciences, Ho Chi Minh City University of Transport, Ho Chi Minh City, Vietnam; Faculty of Mathematics and Computer Science, University of Science, Vietnam National University-Ho Chi Minh City, Ho Chi Minh City, Vietnam

E-mail address: hmnguon@gmail.com

Phuong Le

Department of Mathematical Economics, Banking University of Ho Chi Minh City, Vietnam

E-mail address: phuongl@buh.edu.vn

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PRODUCT OF STATISTICAL PROBABILITY CONVERGENCE AND ITS APPLICATIONS TO KOROVKIN-TYPE THEOREM

BIDU BHUSAN JENA AND SUSANTA KUMAR PAIKRAY

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Abstract. In the present work, we introduce and study the notion of statistical probability convergence for sequences of random variables as well as the concept of statistical convergence for sequences of real numbers, which are defined over a Banach space via product of deferred Cesàro and deferred Nörlund summability means. We first establish a theorem presenting a connection between them. Based upon our proposed method, we then prove a Korovkin-type approximation theorem with algebraic test functions for a sequence of random variables on a Banach space and demonstrate that our theorem effectively extends and improves most (if not all) of the previously existing results (in classical as well as statistical versions). Finally, an illustrative example is presented here by means of the generalized Meyer-König and Zeller operators for a positive sequence of random variables in order to demonstrate that our established theorem is stronger than its traditional and statistical versions.

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1. Introduction and motivation

In the study of sequence spaces, the classical convergence has got numerous applications where the convergence of a sequence requires that almost all elements are to satisfy the convergence condition. This means that, all of the elements of the sequence need to be in an arbitrarily small neighborhood of the limit. However, such restriction is relaxed in statistical convergence, where the validity of the convergence condition is achieved only for a majority of elements. Subsequently, the notion of statistical convergence was introduced by Fast [6] and Steinhaus [27]. Recently, statistical probability convergence has been a dynamic research area due to the fact that it is more general than the statistical convergence as well as the classical convergence. Moreover, such theory is discussed in the study of Fourier Analysis, Number Theory and Approximation Theory. For more details, see [9], [7] and [19].

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Let $\mathbb{N}$ be the set of natural numbers and let $K \subseteq \mathbb{N}$. Also let

$$K_n = \{k : k \leqq n \quad \text{and} \quad k \in K\}$$

and suppose that $|K_n|$ is the cardinality of K_n. Then the natural density $d(K)$ of K is defined by

$$d(K) = \lim_{n\to\infty} \frac{|K_n|}{n} = \lim_{n\to\infty} \frac{1}{n}|\{k : k \leqq n \quad \text{and} \quad k \in K\}|,$$

provided that the limit exists.

A given real sequence (x_n) is said to be statistically convergent to L if, for each $\epsilon > 0$, the set

$$K_\epsilon = \{k : k \in \mathbb{N} \quad \text{and} \quad |x_k - L| \geqq \epsilon\}$$

has zero natural density (see [6] and [27]). Thus, for each $\epsilon > 0$, we have

$$d(K_\epsilon) = \lim_{n\to\infty} \frac{|K_\epsilon|}{n} = \lim_{n\to\infty} \frac{1}{n}|\{k : k \leqq n \quad \text{and} \quad |x_k - L| \geqq \epsilon\}| = 0.$$

In this case, we write

$$\text{stat} \lim_{n\to\infty} x_n = L.$$

In the year 2002, Móricz [13] introduced the fundamental idea of statistical Cesàro summability. Later on, Mohiuddine *et al.* [12] established some Korovkin-type approximation theorems based upon statistical Cesàro summability. Subsequently, Karakaya and Chishti [10] introduced and studied the concept of weighted statistical convergence and their definition was later modified by Mursaleen *et al.* [14]. Furthermore, the fundamental concept of the deferred Cesàro statistical convergence as well as of the statistically-deferred Cesàro summability and associated approximation theorems was introduced by Jena *et al.* [8]. Recently, Srivastava *et al.* [22] introduced the notion of deferred weighted statistical convergence and proved analogous approximation theorems. For several other recent developments in this direction, see (for example) [4], [15], [17], [22], [24], [23] and [25].

Recalling the probability theory, let $X_n \quad (n \in \mathbb{N})$ be a random variable defined on an event space S with respect to a given class of events Δ. Let $P : \Delta \to \mathbb{R}$ (where $\mathbb{R}$ is the set of real numbers) be a probability density function. Then we denote the sequence $X_1, X_2, X_3, \cdots$ of random variables by $\{X_n\}_{n\in\mathbb{N}}$.

From the practical point of view, the discussion of a random variable X will be highly significant if it is known that there exists a real constant c for which

$$P(|X - c| < \epsilon) = 1,$$

where $\epsilon > 0$ is sufficiently small, that is, it is nearly certain that the values of X lie in a very small neighborhood of c.

For a sequence $\{X_n\}_{n\in\mathbb{N}}$ of random variables, where each X_n may not have the above property, it may happen that the aforesaid property (with respect to a real constant c) becomes more and more distinguished as n gradually increases and then the question of existence of such a real constant c will be answered by a notion of probability convergence (that is, convergence in probability) of the sequence $\{X_n\}_{n\in\mathbb{N}}$.

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables, where each X_n is defined on the same event space S, with respect to a given class of subsets (of S) as the class Δ of events and a given probability function $P : \Delta \to \mathbb{R}$. The sequence $\{X_n\}_{n\in\mathbb{N}}$ is said to be statistically probability convergent (or statistically convergent in probability) to a random variable X (where $X : S \to \mathbb{R}$) if, for any $\epsilon > 0$ and $\delta > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n}|k : k \leqq n \quad \text{and} \quad P(|X_n - X| \geqq \epsilon) \geqq \delta| = 0$$

or, equivalently,

$$\lim_{n\to\infty} \frac{1}{n}|k : k \leqq n \quad \text{and} \quad 1 - P(|X_n - X| \leqq \epsilon) \geqq \delta| = 0.$$

In this case, we write

$$\text{stat}_{\text{P}} \lim_{n\to\infty} P(|X_n - X| \geqq \epsilon) = 0 \qquad \text{or} \qquad \text{stat}_{\text{P}} \lim_{n\to\infty} P(|X_n - X| \leqq \epsilon) = 1.$$

We now show by means of the following example that every statistically convergent sequence is statistically probability convergent, but the converse is not necessarily true.

Example 1. Consider a probability density function of X_n of the following form:

$$f_n(x) = \begin{cases} \frac{1}{3} & (0 < x < 1;\ n = m^2\ \forall\ m \in \mathbb{N}) \\ 0 & (\text{otherwise}) \end{cases}$$

and

$$f_n(x) = \begin{cases} \dfrac{nx^{n-1}}{4^n} & (0 < x < 4;\ n \neq m^2\ \forall\ m \in \mathbb{N}) \\ 0 & (\text{otherwise}). \end{cases}$$

Let $0 < \epsilon, \delta < 1$. Then

$$P(|X_n - 4| \geqq \epsilon) = \begin{cases} 1 & (n = m^2\ \forall\ m \in \mathbb{N}) \\ 1 - P(|X_n - 4| < \epsilon) & \\ \quad = \left(1 - \dfrac{\epsilon}{4}\right)^n & (n \neq m^2\ \forall\ m \in \mathbb{N}). \end{cases}$$

This implies that

$$\lim_{n\to\infty}\frac{1}{n}|\{k : k \leqq n \quad \text{and} \quad P(|X_n - 4| \geqq \epsilon) \geqq \delta\}|$$
$$\leqq \lim_{n\to\infty}\frac{1}{n}|\{1^2, 2^2, 3^2, \cdots, n^2\}| = 0.$$

Clearly, it is neither statistically convergent nor ordinarily convergent, while it is statistically probability convergent to 4.

Quite recently, Srivastava *et al.* ([20] and [21]) first introduced and studied the fundamental idea of deferred Cesàro statistical probability convergence as well as deferred weighted statistical probability convergence of a sequence of random variables as follows.

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables, where each (X_n) is defined on the same event space S with respect to a given class of subsets (of S) as the class Δ of events and a given probability function $P : \Delta \to \mathbb{R}$. A given sequence $\{X_n\}_{n\in\mathbb{N}}$ is said to be deferred Cesàro statistically probability convergent to a random variable X (where $X : S \to \mathbb{R}$), if for every $\delta > 0$ and $\epsilon > 0$, the set

$$\{k : a_n < k \leqq b_n \quad \text{and} \quad P(|X_n - X| \geqq \epsilon) \geqq \delta\}$$

has natural density zero, that is,

$$\lim_{n\to\infty}\frac{1}{b_n - a_n}|\{k : a_n < k \leqq b_n \quad \text{and} \quad P(|X_n - X| \geqq \epsilon) \geqq \delta\}| = 0,$$

In this case, we write

$$\text{stat}_{\text{DCP}} \lim_{n\to\infty} P(|X_n - X| \geqq \epsilon) = 0.$$

Similarly, a given sequence $\{X_n\}_{n\in\mathbb{N}}$ is said to be deferred weighted statistically probability convergent to a random variable X (where $X : S \to \mathbb{R}$), if for every $\delta > 0$ and $\epsilon > 0$, the set

$$\{m : m \leqq \mathcal{R}_n \quad \text{and} \quad p_m q_m P(|X_n - X| \geqq \epsilon) \geqq \delta\}$$

has natural density zero, that is,

$$\lim_{n\to\infty}\frac{1}{\mathcal{R}_n}|\{m : m \leqq \mathcal{R}_n \quad \text{and} \quad p_m q_m P(|X_n - X| \geqq \epsilon) \geqq \delta\}| = 0.$$

In this case, we write

$$\text{stat}_{\text{D}\bar{\text{N}}\text{P}} \lim_{n\to\infty} p_m q_m P(|X_n - X| \geqq \epsilon) = 0.$$

Several researchers have worked on extending or generalizing the Korovkin-type approximation theorems in many different ways and under several different settings, including function spaces, abstract Banach lattices, Banach algebras, Banach spaces, and so on. This theory is very useful in Real Analysis, Functional Analysis, Harmonic

Analysis, Measure Theory, Probability Theory and Summability Theory. In the year 2018, Jena *et al.* [8] introduced statistically-deferred Cesàro summability for single sequences in Korovkin-type approximation theorems. Recently, Paikray *et al.* [16] established a Korovkin-type theorem based upon the (p,q)-integers for statistically-deferred Cesàro summability mean. Subsequently, Dutta *et al.* [5] demonstrated the Korovkin theorem on $\mathcal{C}[0,\infty)$ by using the test functions 1, e^{-x} and e^{-2x} via the deferred Cesàro mean. In another recent work, Srivastava *et al.* [24] made use of the notion of the deferred Nörlund statistical convergence and accordingly proved a Korovkin-type approximation theorem.

Motivated essentially by the above-mentioned investigations and results, we first introduce here the notion of the product of deferred Cesàro and deferred Nörlund statistical convergence of real sequences, and then for the statistical probability convergence of sequences of random variables. Moreover, we establish an inclusion relation between them. Furthermore, based upon our proposed methods, we prove a new Korovkin-type approximation theorem with algebraic test functions for positive sequences of random variables over a Banach space and demonstrate that our result is a non-trivial extension of some well-established ordinary and statistical versions of several known results.

2. Preliminaries and definitions

Let (a_n) and (b_n) be sequences of non-negative integers such that, (i) $a_n < b_n$ and (ii) $\lim_{n\to\infty} b_n = \infty$, then the deferred Cesàro (DC) mean is defined by (see, Agnew [1], p. 414),

$$\begin{aligned}\sigma_n &= \frac{x_{a_n+1} + x_{a_n+2} + x_{a_n+3} + \ldots + x_{b_n}}{b_n - a_n}\\ &= \frac{1}{b_n - a_n}\sum_{k=a_n+1}^{b_n} x_k. \end{aligned} \tag{2.1}$$

It is well known that, deferred Cesàro (DC) mean is regular under conditions (i) and (ii) (see, Agnew [1]).

Similarly, suppose that (p_n) be a sequence of non-negative real numbers such that

$$\mathcal{P}_n = \sum_{m=a_n+1}^{b_n} p_{b_n-m},$$

then the deferred Nörlund mean $D(N,p_n)$ is defined by (see [24])

$$t_n = \frac{1}{\mathcal{P}_n}\sum_{m=a_n+1}^{b_n} p_{b_n-m}x_m.$$

Also, it is well known that, $D(N, p_n)$ is regular under the above-mentioned conditions (i) and (ii) (see, for details, Agnew [1]).

We now define the product of deferred Cesàro (DC) and deferred Nörlund (DN) means

$$\begin{aligned}\Lambda_n = (\sigma t)_n &= \frac{1}{(b_n - a_n)} \sum_{m=a_n+1}^{b_n} (t_m) \\ &= \frac{1}{(b_n - a_n)} \sum_{m=a_n+1}^{b_n} \frac{1}{\mathcal{P}_m} \sum_{v=a_m+1}^{b_m} p_{b_m - v} x_v,\end{aligned}$$

and that a sequence (Λ_n) is summable to L by the deferred Cesàro (DC) and deferred Nörlund (DN) product [D(CN)] means if,

$$\lim_{n\to\infty} \Lambda_n = L.$$

Also, assume that the [D(CN)] product mean is regular.

Let us now introduce the following definitions which will be needed in connection with our proposed investigation here.

Definition 1. Let (a_n) and (b_n) be sequences of non-negative integers, and let (p_n) be the sequence of non-negative real numbers. A real sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be deferred Cesàro and deferred Nörlund [D(CN)] statistically convergent to L if, for each $\epsilon > 0$, the set given by

$$\{m : m \leqq (b_n - a_n)\mathcal{P}_n \quad \text{and} \quad p_{b_n - m}|x_m - L| \geqq \epsilon\}$$

has its natural density zero, that is,

$$\lim_{n\to\infty} \frac{1}{(b_n - a_n)\mathcal{P}_n} |\{m : m \leqq (b_n - a_n)\mathcal{P}_n \quad \text{and} \quad p_{b_n - m}|x_m - L| \geqq \epsilon\}| = 0.$$

In this case, we write

$$\operatorname{stat}_{\mathrm{D(CN)}} \lim x_n = L.$$

Definition 2. Let (a_n) and (b_n) be sequences of non-negative integers and let (p_n) be the sequence of non-negative real numbers. Suppose also that $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of random variables, where each (X_n) is defined on the same event space S with respect to a given class Δ of subsets of the event space S and a given probability density function $P : \Delta \to \mathbb{R}$. A given sequence $\{X_n\}_{n\in\mathbb{N}}$ is said to be deferred Cesàro and deferred Nörlund [D(CN)P] statistically probability convergent to a random variable X (where $X : S \to \mathbb{R}$) if, for every $\delta > 0$ and $\epsilon > 0$, the set

$$\{m : m \leqq (b_n - a_n)\mathcal{P}_n \quad \text{and} \quad p_{b_n - m} P(|X_n - X| \geqq \epsilon) \geqq \delta\}$$

has its natural density equal to zero, that is,

$$\lim_{n\to\infty}\frac{1}{(b_n-a_n)\mathcal{P}_n}|\{m : m \leqq (b_n-a_n)\mathcal{P}_n \quad \text{and} \quad p_{b_n-m}P(|X_n-X| \geqq \epsilon) \geqq \delta\}| = 0$$

or, equivalently,

$$\lim_{n\to\infty}\frac{1}{(b_n-a_n)\mathcal{P}_n}|\{m : m \leqq (b_n-a_n)\mathcal{P}_n \\ \text{and} \quad 1-p_{b_n-m}P(|X_n-X| \leqq \epsilon) \geqq \delta\}| = 0.$$

In this case, we write

$$\text{stat}_{\text{D(CN)P}}\lim_{n\to\infty} p_{b_n-m}P(|X_n-X| \geqq \epsilon) = 0$$

or

$$\text{stat}_{\text{D(CN)P}}\lim_{n\to\infty} p_{b_n-m}P(|X_n-X| \leqq \epsilon) = 1.$$

We next present a theorem demonstrating that every deferred Cesàro and deferred Nörlund [D(CN)] statistically convergent sequence is deferred Cesàro and deferred Nörlund [D(CN)P] statistically probability convergent. However, the converse is not true.

Theorem 1. Let the sequence $\{x_n\}$ of constants be such that $\text{stat}_{\text{D(CN)}}x_n \to x$. Then, assuming it to be a random variable having a one-point distribution at that point, the sequence $\{X_n\}$ of random variables is such that

$$\text{stat}_{\text{D(CN)P}}X_n \to X.$$

Proof. Let $\epsilon > 0$ be any arbitrarily small positive real number. Then, by Definition 1, we obtain

$$\lim_{n\to\infty}\frac{1}{(b_n-a_n)\mathcal{P}_n}|\{m : m \leqq (b_n-a_n)\mathcal{P}_n \quad \text{and} \quad p_{b_n-m}|x_m-L| \geqq \epsilon\}| = 0.$$

We now let $\delta > 0$, so that the set

$$\{m : m \leqq (b_n-a_n)\mathcal{P}_n \quad \text{and} \quad p_{b_n-m}P(|X_n-X| \geqq \epsilon) \geqq \delta\} \subseteq \mathcal{K},$$

where

$$\mathcal{K} = \{m : m \leqq (b_n-a_n)\mathcal{P}_n \quad \text{and} \quad p_{b_n-m}|x_m-L| \geqq \epsilon\}.$$

Thus, by Definition 2, we may write

$$\text{stat}_{\text{D(CN)P}}X_n \to X.$$

□

We now present below an example to show that a sequence of random variables is deferred Cesàro and deferred Nörlund [D(CN)P] statistically probability convergent, whenever it is not deferred Cesàro and deferred Nörlund [D(CN)] statistically convergent.

Example 2. Let $a_n = 2n-1$, $b_n = 4n-1$ and $p_n = n$. Suppose that the probability density function of X_n is given by

$$f_n(x) = \begin{cases} 1 & (0 < x < 1;\ n = m^2\ \forall\ m \in \mathbb{N}) \\ 0 & (\text{otherwise}) \end{cases}$$

and

$$f_n(x) = \begin{cases} \dfrac{(n+1)x^n}{4^{n+1}} & (0 < x < 4;\ n \neq m^2\ \forall\ m \in \mathbb{N}) \\ 0 & (\text{otherwise}). \end{cases}$$

Let $0 < \epsilon, \delta < 1$. Then

$$P(|X_n - 4| \geqq \epsilon) = \begin{cases} 1 & \text{when } n = m^2 \\ 1 - P(|X_n - 4| < \epsilon) = \left(1 - \dfrac{\epsilon}{4}\right)^{n+1} & \text{when } n \neq m^2. \end{cases}$$

Consequently, we have

$$\lim_{n\to\infty} \frac{1}{2n^2} |\{m : m \leqq 2n^2 \quad \text{and} \quad nP(|X_n - 4| \geqq \epsilon) \geqq \delta\}| = 0,$$

Clearly, we observe that (X_n) is neither convergent nor deferred Cesàro and deferred Nörlund [D(CN)] statistically convergent; however, it is deferred Cesàro and deferred Nörlund [D(CN)P] statistically probability convergent to 4.

3. A NEW KOROVKIN-TYPE THEOREM

In this section, we extend here the result of Jena *et al.* [8] and Srivastava *et al.* [24] by using the product deferred Cesàro and deferred Nörlund [D(CN)P] statistical probability convergence of sequences of random variables over a Banach space.

Let $\mathcal{C}(\mathbb{R})$, be the space of all real valued continuous probability functions defined on $\mathbb{R}$ under the norm $\|.\|_\infty$. Also let $\mathcal{C}(\mathbb{R})$ is a Banach space. Then, for $f \in \mathcal{C}(\mathbb{R})$, the norm of f denoted by $\|f\|$ is given by,

$$\|f\|_\infty = \sup_{x\in\mathbb{R}} \{|f(x)|\}.$$

We say that the operator $\mathfrak{L}$ is a sequence of random variables of positive linear operator provided that

$$\mathfrak{L}(f;x) \geqq 0 \quad \text{whenever} \quad f \geqq 0.$$

Now we prove the following theorem by using the product deferred Cesàro and deferred Nörlund [D(CN)P] statistical probability convergence.

Theorem 2. *Let*

$$\mathfrak{L}_m : \mathcal{C}(\mathbb{R}) \to \mathcal{C}(\mathbb{R})$$

be a sequence of random variables of positive linear operators. Then, for all $f \in \mathcal{C}(\mathbb{R})$,

$$\text{stat}_{\text{D(CN)P}} \lim_{m\to\infty} \|\mathfrak{L}_m(f;x) - f(x)\|_\infty = 0, \tag{3.1}$$

if and only if

$$\text{stat}_{\text{D(CN)P}} \lim_{m\to\infty} \|\mathfrak{L}_m(1;x) - 1\|_\infty = 0, \tag{3.2}$$

$$\text{stat}_{\text{D(CN)P}} \lim_{m\to\infty} \|\mathfrak{L}_m(x;x) - x\|_\infty = 0, \tag{3.3}$$

and

$$\text{stat}_{\text{D(CN)P}} \lim_{m\to\infty} \|\mathfrak{L}_m(x^2;x) - x^2\|_\infty = 0. \tag{3.4}$$

Proof. Since each of the following functions

$$f_0(x) = 1, \quad f_1(x) = x \quad \text{and} \quad f_2(x) = x^2$$

belonging to $\mathcal{C}(\mathbb{R})$ and are continuous, the implication given by (3.1) implies (3.2) to (3.4) is obvious. In order to complete the proof of the Theorem 2, we first assume that the conditions (3.2) to (3.4) hold true. If $f \in \mathcal{C}(\mathbb{R})$, then there exists a constant $\mathcal{M} > 0$ such that

$$|f(x)| \leqq \mathcal{M} \quad (\forall\, x \in \mathbb{R}).$$

We thus find that

$$|f(s) - f(x)| \leqq 2\mathcal{M} \quad (s, x \in I). \tag{3.5}$$

Clearly, for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(s) - f(x)| < \epsilon \tag{3.6}$$

whenever

$$|s - x| < \delta, \quad \text{for all} s, x \in I.$$

Let us choose

$$\varphi_1 = \varphi_1(s,x) = (s-x)^2.$$

If $|s - x| \geqq \delta$, then we obtain

$$|f(s) - f(x)| < \frac{2\mathcal{M}}{\delta^2}\varphi_1(s,x). \tag{3.7}$$

From equation (3.6) and (3.7), we get

$$|f(s) - f(x)| < \epsilon + \frac{2\mathcal{M}}{\delta^2}\varphi_1(s,x),$$

which implies that

$$-\epsilon-\frac{2\mathcal{M}}{\delta^2}\varphi_1(s,x)\leqq f(s)-f(x)\leqq\epsilon+\frac{2\mathcal{M}}{\delta^2}\varphi_1(s,x). \tag{3.8}$$

Now since $\mathfrak{L}_m(1;x)$ is monotone and linear, so by applying the operator $\mathfrak{L}_m(1;x)$ to this inequality, we have

$$\begin{aligned}\mathfrak{L}_m(1;x)\left(-\epsilon-\frac{2\mathcal{M}}{\delta^2}\varphi_1(s,x)\right)&\leqq\mathfrak{L}_m(1;x)(f(s)-f(x))\\&\leqq\mathfrak{L}_m(1;x)\left(\epsilon+\frac{2\mathcal{M}}{\delta^2}\varphi_1(s,x)\right).\end{aligned}$$

We note that x is fixed and so $f(x)$ is a constant number. Therefore, we have

$$\begin{aligned}-\epsilon\mathfrak{L}_m(1;x)-\frac{2\mathcal{M}}{\delta^2}\mathfrak{L}_m(\varphi_1;x)&\leqq\mathfrak{L}_m(f;x)-f(x)\mathfrak{L}_m(1;x)\\&\leqq\epsilon\mathfrak{L}_m(1;x)+\frac{2\mathcal{M}}{\delta^2}\mathfrak{L}_m(\varphi_1;x).\end{aligned} \tag{3.9}$$

But

$$\mathfrak{L}_m(f;x)-f(x)=[\mathfrak{L}_m(f;x)-f(x)\mathfrak{L}_m(1;x)]+f(x)[\mathfrak{L}_m(1;x)-1]. \tag{3.10}$$

Using (3.9) and (3.10), we have

$$\mathfrak{L}_m(f;x)-f(x)<\epsilon\mathfrak{L}_m(1;x)+\frac{2\mathcal{M}}{\delta^2}\mathfrak{L}_m(\varphi_1;x)+f(x)[\mathfrak{L}_m(1;x)-1]. \tag{3.11}$$

We now estimate $\mathfrak{L}_m(\varphi_1;x)$ as follows:

$$\begin{aligned}\mathfrak{L}_m(\varphi_1;x)&=\mathfrak{L}_m((s-x)^2;x)=\mathfrak{L}_m(s^2-2xs+x^2;x)\\&=\mathfrak{L}_m(s^2;x)-2x\mathfrak{L}_m(s;x)+x^2\mathfrak{L}_m(1;x)\\&=[\mathfrak{L}_m(s^2;x)-x^2]-2x[\mathfrak{L}_m(s;x)-x]\\&\quad+x^2[\mathfrak{L}_m(1;x)-1].\end{aligned}$$

Using (3.11), we obtain

$$\begin{aligned}\mathfrak{L}_m(f;x)-f(x)&<\epsilon\mathfrak{L}_m(1;x)+\frac{2\mathcal{M}}{\delta^2}\{[\mathfrak{L}_m(s^2;x)-x^2]\\&\quad-2x[\mathfrak{L}_m(s;x)-e^{-x}]+x^2[\mathfrak{L}_m(1;x)-1]\}\\&\quad+f(x)[\mathfrak{L}_m(1;x)-1].\\&=\epsilon[\mathfrak{L}_m(1;x)-1]+\epsilon+\frac{2\mathcal{M}}{\delta^2}\{[\mathfrak{L}_m(s^2;x)-x^2]\\&\quad-2x[\mathfrak{L}_m(s;x)-x]+x^2[\mathfrak{L}_m(1;x)-1]\}\\&\quad+f(x)[L_m(1;x)-1].\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we can write

$$\begin{aligned}|\mathfrak{L}_m(f;x)-f(x)| &\leqq \epsilon + \left(\epsilon + \frac{2\mathcal{M}}{\delta^2} + \mathcal{M}\right)|\mathfrak{L}_m(1;x)-1| \\ &\quad + \frac{4\mathcal{M}}{\delta^2}|\mathfrak{L}_m(s;x)-x| + \frac{2\mathcal{M}}{\delta^2}|\mathfrak{L}_m(s^2;x)-x^2| \\ &\leqq \mathcal{K}(|\mathfrak{L}_m(1;x)-1| + |\mathfrak{L}_m(s;x)-x| \\ &\quad + |\mathfrak{L}_m(s^2;x)-x^2|), \end{aligned} \tag{3.12}$$

where

$$\mathcal{K} = \max\left(\epsilon + \frac{2\mathcal{M}}{\delta^2} + \mathcal{M}, \frac{4\mathcal{M}}{\delta^2}, \frac{2\mathcal{M}}{\delta^2}\right).$$

Now, for a given $r > 0$, there exists $\delta, \epsilon > 0$, such that $\epsilon < r$. Then, by setting

$$\Theta_m(x;r) = \left\{m : m \leqq (b_n - a_n)\mathcal{P}_n \text{ and } p_{b_n-m}P\left(|\mathfrak{L}_m(f;x)-f(x)| \geqq r\right)\right\} \geqq \delta.$$

Also, for $i = 0, 1, 2$, we have

$$\begin{aligned}\Theta_{i,m}(x;r) &= \{m : m \leqq (b_n - a_n)\mathcal{P}_n \\ \text{and} \quad p_{b_n-m}P&\left(|\mathfrak{L}_m(f_i;x)-f_i(x)| \geqq \frac{r-\epsilon}{3\mathcal{K}}\right)\Big\} \geqq \delta,\end{aligned}$$

so that,

$$\Theta_m(x;r) \leqq \sum_{i=0}^{2} \Theta_{i,m}(x;r).$$

Clearly, we have

$$\frac{\|\Theta_m(x;r)\|_{\mathcal{C}(\mathbb{R})}}{(b_n-a_n)\mathcal{P}_n} \leqq \sum_{i=0}^{2} \frac{\|\Theta_{i,m}(x;r)\|_{\mathcal{C}(\mathbb{R})}}{(b_n-a_n)\mathcal{P}_n}. \tag{3.13}$$

Now, using the above assumption about the implications in (3.2) to (3.4) and by Definition 2, the right-hand side of (3.13) is seen to tend to zero as $n \to \infty$. Consequently, we get

$$\lim_{n\to\infty} \frac{\|\Theta_m(x;r)\|_{\mathcal{C}(\mathbb{R})}}{(b_n-a_n)\mathcal{P}_n} = 0 \ (\delta, r > 0).$$

Therefore, the implication (3.1) holds true. This completes the proof of Theorem 2. □

Now, by using the Definition 1, we present the following corollary as the consequence of Theorem 2.

Corollary 1. *Let $\mathfrak{L}_m : \mathcal{C}(\mathbb{R}) \to \mathcal{C}(\mathbb{R})$ be a sequence of positive linear operators. Also let $f \in \mathcal{C}(\mathbb{R})$. Then*

$$\text{stat}_{\text{D(CN)}} \lim_{m\to\infty} \|\mathfrak{L}_m(f;x)-f(x)\|_\infty = 0$$

if and only if

$$\operatorname{stat}_{\mathrm{D(CN)}} \lim_{m\to\infty} \|\mathfrak{L}_m(1;x)-1\|_\infty = 0,$$
$$\operatorname{stat}_{\mathrm{D(CN)}} \lim_{m\to\infty} \|\mathfrak{L}_m(x;x)-x\|_\infty = 0,$$

and

$$\operatorname{stat}_{\mathrm{D(CN)}} \lim_{m\to\infty} \|\mathfrak{L}_m(x^2;x)-x^2\|_\infty = 0.$$

We present below an illustrative example for the sequence of random variables of positive linear operators that does not satisfy the conditions of the Korovkin approximation theorems proved earlier by Jena *et al.* [8], Srivastava *et al.* [24] and Paikray *et al.* [15], but which satisfies the conditions of our Theorem 2. Consequently, our Theorem 2 is stronger than the results established earlier by both Jena *et al.* [8] and Srivastava *et al.* [24].

We now recall the operator

$$x(1+xD) \quad \left(D = \frac{d}{dx}\right)$$

which was used by Al-Salam [2] and, more recently, by Viskov and Srivastava [28] (see also [18] the monograph by Srivastava and Manocha [26] for various general families of operators of this kind). Here, in our Example 3 below, we use this operator in conjunction with the Meyer-König and Zeller operators.

Example 3. Let $X = [0,1]$ and we consider Meyer-König and Zeller operators $\mathfrak{M}_n(f;x)$ on $\mathcal{C}[0,1]$ given by (see [3]),

$$\mathfrak{M}_n(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{k+n+1}\right)\binom{n+k}{k} x^k.(1+x)^{n+1}.$$

Also let $\mathfrak{L}_m : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ be sequence of operators defined as follows:

$$\mathfrak{L}_m(f;x) = [1+X_m]x(1+xD)\mathfrak{M}_m(f) \quad (f \in \mathcal{C}(\mathbb{R}), \tag{3.14}$$

where (X_m) is a sequence of random variables defined in Example 2.

Now,

$$\mathfrak{L}_m(1;x) = [1+X_m]x(1+xD)1 = [1+X_m]x,$$

$$\mathfrak{L}_m(s;x) = [1+X_m]x(1+xD)x = [1+X_m]x(1+x),$$

and

$$\mathfrak{L}_m(s^2;x) = [1+X_n]x(1+xD)\left\{x^2\left(\frac{n+2}{n+1}\right) + \frac{x}{n+1}\right\}$$

$$= [1 + X_n(x)] \left\{ x^2 \left[\left(\frac{n+2}{n+1} \right) x + 2 \left(\frac{1}{n+1} \right) + 2x \left(\frac{n+2}{n+1} \right) \right] \right\},$$

so that we have

$$\operatorname{stat}_{\mathrm{D(CN)P}} \lim_{m\to\infty} \|\mathfrak{L}_m(1;x) - 1\|_\infty = 0,$$
$$\operatorname{stat}_{\mathrm{D(CN)P}} \lim_{m\to\infty} \|\mathfrak{L}_m(x;x) - x\|_\infty = 0,$$

and

$$\operatorname{stat}_{\mathrm{D(CN)P}} \lim_{m\to\infty} \|\mathfrak{L}_m(x^2;x) - x^2\|_\infty = 0,$$

that is, the sequence $\mathfrak{L}_m(f;x)$ satisfies the conditions (3.2) to (3.4). Therefore by Theorem 2, we have

$$\operatorname{stat}_{\mathrm{D(CN)P}} \lim_{m\to\infty} \|\mathfrak{L}_m(f;x) - f\|_\infty = 0.$$

Hence, it is deferred Cesàro and deferred Nörlund [D(CN)P] statistically probability convergent. However, since (X_m) is neither Cesàro nor Nörlund statistically convergent, so it is neither deferred Cesàro (DC) statistically convergent nor deferred Nörlund (DN) statistically convergent. Thus, we conclude that earlier works in [8] and [24] are not valid for the operators defined by (3.14), where as our Theorem 2 still works for the operators defined by (3.14).

4. Concluding remarks and observations

In this concluding section of our investigation, we present several further remarks and observations concerning to various results which we have proved here.

Remark 1. Let $(X_m)_{m\in\mathbb{N}}$ be a sequence of random variables given in Example 2. Then, since

$$\operatorname{stat}_{\mathrm{D(CN)P}} \lim_{m\to\infty} X_m = 4 \text{ on } [0,1],$$

we have

$$\operatorname{stat}_{\mathrm{D(CN)P}} \lim_{m\to\infty} \|\mathfrak{L}_m(f_i;x) - f_i(x)\|_\infty = 0 \quad (i = 0,1,2). \tag{4.1}$$

Thus, by applying by Theorem 2, we can write

$$\operatorname{stat}_{\mathrm{D(CN)P}} \lim_{m\to\infty} \|\mathfrak{L}_m(f;x) - f(x)\|_\infty = 0, \quad (i = 0,1,2), \tag{4.2}$$

where

$$f_0(x) = 1, \quad f_1(x) = x \quad \text{and} \quad f_2(x) = x^2.$$

However, since (X_m) is neither statistically convergent nor converges uniformly in the ordinary sense, thus, the classical and statistical Korovkin-type theorems do not work here for the operators defined by (3.14). Hence, clearly, this application indicates that our Theorem 2 is a non-trivial generalization of the classical as well as statistical Korovkin-type theorems (see [6] and [11]).

Remark 2. Let $(X_m)_{m\in\mathbb{N}}$ be a sequence of random variables as given in Example 2. Then, since

$$\mathrm{stat}_{\mathrm{D(CN)P}} \lim_{m\to\infty} X_m = 4 \text{ on } [0,1],$$

so (4.1) holds true. Now by applying (4.1) and Theorem 2, the condition (4.2) holds true. However, since the sequence (X_m) of random variables is not deferred Cesàro [8] and deferred Nörlund statistically convergent, the results of Jena *et al.* [8] and Srivastava *et al.* (see [24]) do not work for our operator defined in (3.14). Thus, naturally, our Theorem 2 is also a non-trivial extension of the result of Jena *et al.* [8] and Srivastava *et al.* [24] (see also [15] and [17]). Based upon the above results, it is concluded here that our proposed method has successfully worked for the operators defined in (3.14) and, therefore, it is stronger than the classical and statistical versions of the Korovkin-type approximation theorems (see [15], [17] and [24]) which were established earlier.

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Authors' addresses

Bidu Bhusan Jena
Department of Mathematics, Veer Surendra Sai University of Technology, Burla 768018, Odisha, India
E-mail address: bidumath.05@gmail.com

Susanta Kumar Paikray
Department of Mathematics, Veer Surendra Sai University of Technology, Burla 768018, Odisha, India
E-mail address: skpaikray_math@vssut.ac.in

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A NOTE ON SUMS OF A CLASS OF SERIES

SUNGTAE JUN, GRADIMIR V. MILOVANOVIĆ, INSUK KIM,
AND ARJUN K. RATHIE

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Abstract. The aim of this note is to provide sums of a unified class of series of the form

$$S_i(a) = \sum_{k=0}^{\infty} (-1)^k \binom{a-i}{k} \frac{1}{2^k (a+k+1)}$$

in the most general form for any $i \in \mathbb{Z}$. For each $\nu \in \mathbb{N}$, in four cases when $i = \pm 2\nu$ and $i = \pm(2\nu - 1)$, simple explicit expressions for $S_i(a)$ are obtained, e.g.

$$S_{2\nu}(a) = \frac{2^{2\nu-1-a}}{(a-2\nu+1)_\nu} \left[\frac{\sqrt{\pi}\,\Gamma(a+1)}{\Gamma\left(a+\frac{3}{2}-\nu\right)} - P_{\nu-1}(a) \right],$$

where $P_\nu(a)$ is an algebraic polynomial in a of degree ν.

For $i = 1$ and $a = n$ ($\in \mathbb{N}$), we recover the well known sum of the series due to Vowe and Seiffert. Several other known results due to Srivastava and Kim *et al.* can be considered as special cases of our result.

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1. Introduction and preliminaries

In 1812, Gauss [6] defined his famous infinite series as follows:

$$1 + \frac{a \cdot b}{c} \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2!} + \cdots . \tag{1.1}$$

The series (1.1) is denoted by the notation

$${}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix}; z\right], \text{ or } \quad {}_2F_1\left[\begin{matrix} a,\ b\ ; \\ c\ ; \end{matrix} z\right], \text{ or } \quad {}_2F_1[a,b;c;z],$$

or simply F and is popularly known as the Gauss's function or hypergeometric function.

The quantities a, b and c are known as the parameters (real or complex) of the series with $c \neq 0, -1, -2, \ldots$ and z is termed as the variable of the series. This is

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called the 'Hypergeometric series' because either $a = 1$ and $b = c$ or $b = 1$ and $a = c$, it reduces to the well known 'Geometric series'.

In terms of Pochhammer's symbol $(a)_n$, defined by

$$(a)_n = \begin{cases} a(a+1)\cdots(a+n-1), & n \in \mathbb{N}, \\ 1, & n = 0, \end{cases} \tag{1.2}$$

the series (1.1) is represented as

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}.$$

Also, in term of Gamma function, $(a)_n$ can be represented as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Thus, from (1.1), we have

$${}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}. \tag{1.3}$$

Here, we verify that the series (1.1)

(i) is convergent for all values of z provided $|z| < 1$ and divergent when $|z| > 1$.
(ii) is convergent for $z = 1$ provided $\mathrm{Re}(c-a-b) > 0$ and divergent for $\mathrm{Re}(c-a-b) \le 0$.
(iii) is absolutely convergent for $z = -1$ provided $\mathrm{Re}(c-a-b) > 0$ and convergent but not absolutely for $-1 < \mathrm{Re}(c-a-b) \le 0$ and divergent for $\mathrm{Re}(c-a-b) < -1$.

The limiting case of (1.1) is worth mentioning here. For this, if we replace z by z/b in (1.3) and take the limit as $b \to \infty$, then since

$$\frac{(b)_n}{b^n} z^n \to z^n,$$

we arrive at the following series which is in the literature known as the Kummer's series or the confluent hypergeometric series [9]

$${}_1F_1\left[\begin{matrix} a \\ c \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}. \tag{1.4}$$

Further, it is interesting to mention here that almost all elementary functions of mathematics and mathematical physics are special cases or limiting cases of Gauss's hypergeometric function or the confluent hypergeometric function, see [3, 15].

It is well known that whenever a ${}_2F_1$ hypergeometric function reduces to a gamma function, then the obtained result is very useful for application. Thus, for example,

Gauss summation theorem, Gauss second, Kummer, Vandermonde and Bailey's theorems for the series ${}_2F_1$ play a vital role in the theory of hypergeometric function, especially in applied mathematics, mathematical physics and engineering.

However, in this note, we are interested in the following Bailey's summation theorem [1, 2] viz.

$$ {}_2F_1\left[\begin{matrix} a,\ 1-a \\ b \end{matrix};\frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}b+\frac{1}{2}a)\Gamma(\frac{1}{2}b-\frac{1}{2}a+\frac{1}{2})}. \tag{1.5} $$

During 1992–96, Lavoie *et al.* [10–12] have successfully investigated the generalizations of the above mentioned classical summation theorems and also obtained the generalization of the classical summation theorems for the series ${}_3F_2$ with unit argument such as those of Watson, Dixon and Whipple.

In particular, they have generalized Bailey's summation theorem (1.5) by obtaining a single formula containing eleven results in the form

$$ {}_2F_1\left[\begin{matrix} a,\ 1-a+i \\ b \end{matrix};\frac{1}{2}\right] \tag{1.6} $$

for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$. Later, Rakha and Rathie [17] and Kim *et al.* [8] have further generalized and extended respectively, the above mentioned classical summation theorems in the most general form.

One of such kind of summation theorems will be used in this note in order to provide sums of a unified class of the series of the form

$$ \sum_{k=0}^{\infty}(-1)^k\binom{a-i}{k}\frac{1}{2^k(a+k+1)} \tag{1.7} $$

in the most general form for any $i \in \mathbb{Z}$. Otherwise, the first result of this type was appeared in 1987, when Vowe and Seiffert [20] evaluated the following very interesting and useful sum, viz.

$$ \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{1}{2^k(n+k+1)} = \frac{2^n(n-1)!n!}{(2n)!} - \frac{1}{2^n n} \quad (n \in \mathbb{N}) \tag{1.8} $$

by identifying it with an Eulerian integral of the type

$$ \int_0^1 \left(1-\frac{t}{2}\right)^{n-1} t^n \,\mathrm{d}t. \tag{1.9} $$

In recent years, the sum (1.8) has gained a fair amount of attention. In 1988, Srivastava [18] established the generalization of (1.8) in the form

$$ \sum_{k=0}^{\infty}(-1)^k\binom{a-1}{k}\frac{1}{2^k(a+k+1)} = \frac{2^a\Gamma(a)\Gamma(a+1)}{\Gamma(2a+1)} - \frac{1}{2^a a} \tag{1.10} $$

by employing the classical Bailey's summation theorem (1.5) and also discussed a few more series of this type.

In 1989, Srivastava [19] gave a basic (or $q-$) extension of (1.10).

In 1999, Choi *et al.* [4] have re-derived the result (1.10) and obtained the following two results closely related to (1.10) by utilizing certain contiguous function relations for ${}_2F_1$ due to Gauss [6, 16],

$$\sum_{k=0}^{\infty}(-1)^k\binom{a}{k}\frac{k}{2^k(a+k)(a+k+1)}=\frac{2^{a-1}\{\Gamma(a+1)\}^2}{\Gamma(2a+2)}-\frac{1}{2^{a+1}} \tag{1.11}$$

and

$$\sum_{k=0}^{\infty}(-1)^k\binom{a-2}{k}\frac{k}{2^k(a+k)(a+k+1)}=\frac{3\cdot 2^a\{\Gamma(a+1)\}^2}{(a-1)\Gamma(2a+1)}-\frac{a+2}{(a-1)^{2a-1}}. \tag{1.12}$$

In 2009, Prodinger [14] established a more general sum of the form

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{2^k(m+k)} \qquad (m,k\in\mathbb{N})$$

and obtained as special cases, the results (1.11) and (1.12) for $a=n$ by completely elementary tools.

In 2010, Dahlberg *et al.* [5] have presented an elegant and elementary proof of the identites (1.11) and (1.12) for $a=n$ using Zeilberger algorithm and the Wilf-Zeilberger proof style.

In 2012, Kim et al. [7] generalized the sum (1.10) (and of course, Vowe and Seiffert's sum (1.8)) and obtained the explicit expressions of (1.7) for $i=0,\pm1,\pm2,\pm3,\pm4,\pm5$ by utilizing the generalizations of the Bailey's summation formulas (1.6) obtained earlier by Lavoie *et al.* [11] and deduced a large number of very interesting results including (1.8) and (1.10).

In our present investigation, we will use the generalization of the Bailey's summation theorem (1.5) in the following two forms [17]:

$$\begin{aligned}{}_2F_1\left[\begin{matrix}a,\ 1-a+i\\ b\end{matrix};\frac{1}{2}\right]&=\frac{2^{1+i-b}\Gamma(\frac12)\Gamma(b)\Gamma(a-i)}{\Gamma(a)\Gamma(\frac12b-\frac12a)\Gamma(\frac12b-\frac12a+\frac12)}\\&\quad\times\sum_{r=0}^{i}\binom{i}{r}\frac{(-1)^r\Gamma(\frac12b-\frac12a+\frac12r)}{\Gamma(\frac12b+\frac12a+\frac12r-i)}\end{aligned} \tag{1.13}$$

for $i=0,1,2,\ldots$ and

$${}_2F_1\left[\begin{matrix}a,\ 1-a-i\\ b\end{matrix};\frac{1}{2}\right]=\frac{2^{1-i-b}\Gamma(\frac12)\Gamma(b)}{\Gamma(\frac12b-\frac12a)\Gamma(\frac12b-\frac12a+\frac12)}$$

$$\times \sum_{r=0}^{i} \binom{i}{r} \frac{\Gamma(\frac{1}{2}b-\frac{1}{2}a+\frac{1}{2}r)}{\Gamma(\frac{1}{2}b+\frac{1}{2}a+\frac{1}{2}r)} \tag{1.14}$$

for $i = 0,1,2,\ldots$, as well as a general formula (containing (1.13) and (1.14)), obtained recently by Milovanović *et al.* [13],

$$\begin{aligned} &{}_2F_1\left[\begin{matrix} a, 1-a+i \\ b \end{matrix}; \frac{1}{2}\right] \\ &= \frac{\Gamma(\frac{1}{2})\Gamma(b)\Gamma(1-a)}{2^{b-i-1}\Gamma(1-a+\frac{1}{2}(i+|i|))} \left\{ \frac{C_i(a,b)}{\Gamma(\frac{1}{2}b-\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2}a-\lfloor\frac{1+i}{2}\rfloor)} \right. \\ &\qquad \left. + \frac{D_i(a,b)}{\Gamma(\frac{1}{2}b-\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2}a-\frac{1}{2}-\lfloor\frac{i}{2}\rfloor)} \right\}, \end{aligned} \tag{1.15}$$

with coefficients $C_i(a,b)$ and $D_i(a,b)$ given by

$$C_{2\nu}(a,b) = \sum_{j=0}^{\nu} \binom{2\nu}{2j} \left(\frac{b-a}{2}\right)_j \left(\frac{b+a}{2}-(2\nu-j)\right)_{\nu-j},$$

$$D_{2\nu}(a,b) = -\sum_{j=0}^{\nu-1} \binom{2\nu}{2j+1} \left(\frac{b-a+1}{2}\right)_j \left(\frac{b+a+1}{2}-(2\nu-j)\right)_{\nu-j-1},$$

$$C_{2\nu+1}(a,b) = -\sum_{j=0}^{\nu} \binom{2\nu+1}{2j} \left(\frac{b-a}{2}\right)_j \left(\frac{b+a}{2}-(2\nu-j+1)\right)_{\nu-j},$$

$$D_{2\nu+1}(a,b) = \sum_{j=0}^{\nu} \binom{2\nu+1}{2j+1} \left(\frac{b-a+1}{2}\right)_j \left(\frac{b+a+1}{2}-(2\nu-j+1)\right)_{\nu-j},$$

$$C_{-2\nu}(a,b) = \sum_{j=0}^{\nu} \binom{2\nu}{2j} \left(\frac{b-a}{2}\right)_j \left(\frac{b+a}{2}+j\right)_{\nu-j},$$

$$D_{-2\nu}(a,b) = \sum_{j=0}^{\nu-1} \binom{2\nu}{2j+1} \left(\frac{b-a+1}{2}\right)_j \left(\frac{b+a+1}{2}+j\right)_{\nu-j-1},$$

$$C_{-(2\nu+1)}(a,b) = \sum_{j=0}^{\nu} \binom{2\nu+1}{2j} \left(\frac{b-a}{2}\right)_j \left(\frac{b+a}{2}+j\right)_{\nu-j},$$

$$D_{-(2\nu+1)}(a,b)=\sum_{j=0}^{\nu}\binom{2\nu+1}{2j+1}\left(\frac{b-a+1}{2}\right)_j\left(\frac{b+a+1}{2}+j\right)_{\nu-j}.$$

Clearly, for $i=0$, the results (1.13), (1.14) and (1.15) reduce to the Bailey's summation theorem (1.5).

The main objective of this note is to provide sums of a unified class of the series of the form

$$S_i(a)=\sum_{k=0}^{\infty}(-1)^k\binom{a-i}{k}\frac{1}{2^k(a+k+1)} \tag{1.16}$$

in the most general form for any $i\in\mathbb{Z}$. The results obtained earlier by Vowe and Seiffert [20] and Srivastava [18] follows as special cases of our main findings.

2. Main results

The following theorem gives a result on sums of a unified class of series.

Theorem 1. *For $i=0,1,2,\cdots$, the following results hold true.*

$$\sum_{k=0}^{\infty}(-1)^k\binom{a-i}{k}\frac{1}{2^k(a+k+1)}$$
$$=\frac{\Gamma(a+1-i)}{2^{a-i+1}}\sum_{r=0}^{i}(-1)^r\binom{i}{r}\frac{\Gamma(\frac{1}{2}r+\frac{1}{2})}{\Gamma(a+\frac{3}{2}+\frac{1}{2}r-i)} \tag{2.1}$$

and

$$\sum_{k=0}^{\infty}(-1)^k\binom{a+i}{k}\frac{1}{2^k(a+k+1)}=\frac{\Gamma(a+1)}{2^{a+i+1}}\sum_{r=0}^{i}\binom{i}{r}\frac{\Gamma(\frac{1}{2}r+\frac{1}{2})}{\Gamma(a+\frac{3}{2}+\frac{1}{2}r)}. \tag{2.2}$$

Proof. The proofs of our results are quite straight forward. For this, in order to establish the result (2.1), we consider (1.16) for $i\in\mathbb{N}_0$.

If we use the elementary identities

$$\binom{n}{r}=\frac{n!}{(n-r)!r!},\quad (a)_n=\frac{\Gamma(a+n)}{\Gamma(a)},\quad \Gamma(\alpha-n)=\frac{(-1)^n\Gamma(\alpha)}{(1-\alpha)_n},$$

then, after some simplification, the left-hand side in (2.1) becomes

$$S_i(a)=\frac{1}{a+1}\sum_{k=0}^{\infty}\frac{(-a+i)_k(a+1)_k}{(a+2)_k}\cdot\frac{1}{k!2^k}$$

Summing up the series with the help of (1.3), we get

$$S_i(a)=\frac{1}{a+1}\,{}_2F_1\left[\begin{matrix}-a+i,\ a+1\\ a+2\end{matrix};\frac{1}{2}\right]. \tag{2.3}$$

We now observe here that the right-hand side of (2.1) can be verified with the result (1.13) to ${}_2F_1$ above and some calculation.

In exactly the same manner, the second result (2.2) for $S_{-i}(a)$, $i \in \mathbb{N}$, can be proven with the help of the known result (1.14). □

Alternatively, the results given in previous theorem can be expressed in the following explicit form:

Theorem 2. *For each $\nu \in \mathbb{N}$ the sum (1.16) can be expressed as*

$$S_{2\nu}(a) = \frac{2^{2\nu-1-a}}{(a-2\nu+1)_\nu}\left[\frac{\sqrt{\pi}\,\Gamma(a+1)}{\Gamma\left(a+\frac{3}{2}-\nu\right)} - P_{\nu-1}(a)\right] \tag{2.4}$$

and

$$S_{2\nu-1}(a) = \frac{2^{2\nu-2-a}}{(a-2\nu+2)_\nu}\left[\frac{\sqrt{\pi}\,\Gamma(a+1)}{\Gamma\left(a+\frac{3}{2}-\nu\right)} - Q_{\nu-1}(a)\right], \tag{2.5}$$

where P_ν and Q_ν are polynomials of degree ν, defined by the recurrence relations

$$\left.\begin{aligned} P_\nu(a) &= 2Q_\nu(a-1) - (a-2\nu-1)P_{\nu-1}(a-1), \\ Q_\nu(a) &= 2(a-\nu)P_{\nu-1}(a-1) - (a-2\nu)Q_{\nu-1}(a-1), \end{aligned}\right\} \tag{2.6}$$

with $P_0(a) = 2$ and $Q_0(a) = 1$.

For negative indices we have

$$S_{-2\nu}(a) = \frac{2^{-a-2\nu-1}}{(a+1)_\nu}\left[\frac{\sqrt{\pi}\,\Gamma(a+2\nu+1)}{\Gamma\left(a+\frac{3}{2}+\nu\right)} + R_{\nu-1}(a)\right] \tag{2.7}$$

and

$$S_{-(2\nu-1)}(a) = \frac{2^{-a-2\nu}}{(a+1)_\nu}\left[\frac{\sqrt{\pi}\,\Gamma(a+2\nu)}{\Gamma\left(a+\frac{1}{2}+\nu\right)} + T_{\nu-1}(a)\right], \tag{2.8}$$

where R_ν and T_ν are polynomials of degree ν, defined by the recurrence relations

$$\left.\begin{aligned} T_\nu(a) &= 2(a+\nu+1)R_{\nu-1}(a) - (a+1)T_{\nu-1}(a+1), \\ R_\nu(a) &= 2T_\nu(a) - (a+1)R_{\nu-1}(a+1), \end{aligned}\right\} \tag{2.9}$$

with $R_0(a) = 2$ and $T_0(a) = 1$.

The expressions (2.6) and (2.9) hold also for $\nu = 0$ if we take $P_{-1}(a) = R_{-1}(a) = 0$.

In proving Theorem 2.2 we need the following auxiliary result:

Lemma 1. *For each $\nu \in \mathbb{N}_0$ the sum (1.16) satisfies the following recurrence relations*

$$\left.\begin{aligned} S_{2\nu+1}(a) &= 2\left[S_{2\nu}(a-1) - S_{2\nu-1}(a-1)\right], \\ S_{2\nu+2}(a) &= 2\left[S_{2\nu+1}(a-1) - S_{2\nu}(a-1)\right], \\ S_{-(2\nu+1)}(a) &= S_{-2\nu}(a) - \frac{1}{2}S_{-(2\nu-1)}(a+1), \\ S_{-(2\nu+2)}(a) &= S_{-(2\nu+1)}(a) - \frac{1}{2}S_{-2\nu}(a+1). \end{aligned}\right\} \tag{2.10}$$

Proof. Starting from (1.16) for $i = 2\nu+1$ and using the identity

$$\binom{a-2\nu-1}{k} = \binom{a-2\nu}{k+1} - \binom{a-2\nu-1}{k+1},$$

we can easily prove the first recurrence relation in (2.10). In a similar way we get the other ones. We omit the details. □

Proof of Theorem 2. In order to prove explicit expressions (2.4), (2.5), (2.7), and (2.8), we apply (1.15) to (2.3) for $i = 2\nu$, $2\nu-1$, -2ν, and $-(2\nu-1)$, respectively, and use the corresponding coefficients $C_i(a,b)$ and $D_i(a,b)$.

Finally, using Lemma 1 we obtain the recurrence relations (2.6) and (2.9) for the polynomials P_ν and Q_ν and R_ν and T_ν, respectively. □

Remark 1. The polynomials P_ν and Q_ν which satisfy the recurrence relations (2.6) are given, for $\nu = 0, 1, \ldots, 9$, by

$$\begin{aligned}
P_0(a) &= 2, \quad P_1(a) = 4(a-1), \quad P_2(a) = 2\left(3a^2 - 11a + 12\right), \\
P_3(a) &= 8(a-3)\left(a^2 - 5a + 10\right), \\
P_4(a) &= 2\left(5a^4 - 70a^3 + 427a^2 - 1322a + 1680\right), \\
P_5(a) &= 4(a-5)\left(3a^4 - 50a^3 + 401a^2 - 1698a + 3024\right), \\
P_6(a) &= 2\left(7a^6 - 217a^5 + 3227a^4 - 28967a^3 + 159750a^2\right. \\
&\qquad \left. -496680a + 665280\right), \\
P_7(a) &= 16(a-7)\left(a^6 - 35a^5 + 623a^4 - 6889a^3 + 47256a^2\right. \\
&\qquad \left. -183516a + 308880\right), \\
P_8(a) &= 6\left(3a^8 - 164a^7 + 4494a^6 - 79280a^5 + 953387a^4 - 7765436a^3\right. \\
&\qquad \left. +40967236a^2 - 126332400a + 172972800\right), \\
P_9(a) &= 4(a-9)\left(5a^8 - 300a^7 + 9402a^6 - 193944a^5 + 2747829a^4\right. \\
&\qquad \left. -26422764a^3 + 164805772a^2 - 602206800a + 980179200\right),
\end{aligned}$$

and

$$
\begin{aligned}
Q_0(a) &= 1, \quad Q_1(a) = 3a-2, \quad Q_2(a) = 5a^2-15a+12,\\
Q_3(a) &= 7a^3-49a^2+126a-120,\\
Q_4(a) &= 3\left(3a^4-38a^3+201a^2-526a+560\right),\\
Q_5(a) &= 11a^5-220a^4+1969a^3-9812a^2+26532a-30240,\\
Q_6(a) &= 13a^6-377a^5+5109a^4-41119a^3+202046a^2\\
&\quad -558792a+665280,\\
Q_7(a) &= 15a^7-595a^6+11361a^5-134185a^4+1032120a^3-5031580a^2\\
&\quad +14102064a-17297280,\\
Q_8(a) &= 17a^8-884a^7+22610a^6-367880a^5+4060433a^4-30316916a^3\\
&\quad +146566860a^2-414018000a+518918400,\\
Q_9(a) &= 19a^9-1254a^8+41382a^7-886692a^6+13256091a^5\\
&\quad -139709166a^4+1017764108a^3-4878321288a^2\\
&\quad +13847306400a-17643225600,
\end{aligned}
$$

respectively.

Remark 2. The polynomials R_ν and T_ν which satisfy the recurrence relations (2.9) are given, for $\nu = 0, 1, \ldots, 9$, by

$$
\begin{aligned}
R_0(a) &= 2, \quad R_1(a) = 4(a+3), \quad R_2(a) = 2\left(3a^2+25a+54\right),\\
R_3(a) &= 8(a+5)\left(a^2+11a+34\right),\\
R_4(a) &= 2\left(5a^4+130a^3+1327a^2+6218a+11160\right),\\
R_5(a) &= 4(a+7)\left(3a^4+94a^3+1193a^2+7062a+16200\right),\\
R_6(a) &= 2(7a^6+371a^5+8617a^4+110585a^3+817288a^2\\
&\quad +3270524a+5504688),\\
R_7(a) &= 16(a+9)(a^6+61a^5+1663a^4+25303a^3+222952a^2\\
&\quad +1067812a+2158800),\\
R_8(a) &= 6(3a^8+268a^7+11046a^6+269992a^5+4228307a^4\\
&\quad +43076572a^3+277195204a^2+1026169008a+1668885120),
\end{aligned}
$$

$$
R_9(a) = 4(a+11)(5a^8+500a^7+23402a^6+654296a^5+11765429a^4
$$

$$+ 137947556a^3 + 1023509532a^2 + 4376435280a + 8236080000),$$

and

$$\begin{aligned}
T_0(a) &= 1, \;\; T_1(a) = 3a+7, \;\; T_2(a) = 5a^2+35a+62,\\
T_3(a) &= 7a^3+98a^2+469a+762,\\
T_4(a) &= 3\left(3a^4+70a^3+633a^2+2606a+4088\right),\\
T_5(a) &= 11a^5+385a^4+5599a^3+41855a^2+159390a+245640,\\
T_6(a) &= 13a^6+637a^5+13559a^4+158639a^3+1065636a^2\\
&\quad +3868332a+5897520,\\
T_7(a) &= 15a^7+980a^6+28686a^5+481640a^4+4958895a^3\\
&\quad +31072820a^2+109138164a+165145680,\\
T_8(a) &= 17a^8+1428a^7+54978a^6+1250520a^5+18186753a^4\\
&\quad +171842052a^3+1024655212a^2+3512174160a+5284782720,\\
T_9(a) &= 19a^9+1995a^8+97698a^7+2888646a^6+56212203a^5\\
&\quad +740755755a^4+6574624112a^3+37758440004a^2\\
&\quad +126997604208a+190253266560,
\end{aligned}$$

respectively.

3. Concluding remark

In this note, an attempt has been made to provide a unified sum of the series of the form

$$\sum_{k=0}^{\infty}(-1)^k\binom{a-i}{k}\frac{1}{2^k(a+k+1)}$$

in the most general form for any $i = 0, \pm 1, \pm 2, \ldots$.

We believe that the results obtained in this note may be potentially useful in combinatorics, applied mathematics, mathematical physics and engineering.

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Authors' addresses

Sungtae Jun
General Education Institute, Konkuk University, Chungju 380-701, Korea
E-mail address: sjun@kku.ac.kr

Gradimir V. Milovanović
Serbian Academy of Sciences and Arts, 11000 Belgrade, Serbia, &, Faculty of Sciences and Mathematics, University of Niš, 18000 Niš, Serbia
E-mail address: gvm@mi.sanu.ac.rs

Insuk Kim
Department of Mathematics Education, Wonkwang University, Iksan 570-749, Korea
E-mail address: iki@wku.ac.kr

Arjun K. Rathie
Department of Mathematic, Vedant College of Engineering and Technology (Rajasthan Technical University), Bundi, Rajasthan, India
E-mail address: arjunkumarrathie@gmail.com

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SOME NEW INTEGRAL INEQUALITIES FOR N-TIMES DIFFERENTIABLE R-CONVEX AND R-CONCAVE FUNCTIONS

HURIYE KADAKAL, MAHIR KADAKAL, AND İMDAT İŞCAN

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Abstract. In this work, by using an integral identity together with both the Hölder and the Power-Mean integral inequality we establish several new inequalities for n-time differentiable r-convex and concave functions.

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1. Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \varnothing$. This definition is well known in the literature. Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences. Many articles have been written by a number of mathematicians on convex functions and inequalities for their different classes, using, for example, the last articles [3, 8–15] and the references in these papers.

$f : [a,b] \to \mathbb{R}$ be a convex function, then the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

is known as the Hermite-Hadamard inequality (see [6] for more information). Since then, some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors (*e.g.*, [3, 4, 15]). In [20], the first author obtained a new refinement of the Hermite-Hadamard inequality for convex functions. The Hermite-Hadamard inequality was generalized in [17] to an r-convex positive function which is defined on an interval $[a,b]$.

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Definition 1. A positive function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is called r-convex function on $[a,b]$, if for each the $x, y \in [a,b]$ and $t \in [0,1]$,

$$f(tx+(1-t)y) \leq \begin{cases} [tf^r(x)+(1-t)f^r(y)]^{\frac{1}{r}}, & r \neq 0, \\ [f(x)]^t[f(y)]^{1-t}, & r = 0. \end{cases}$$

If the equality is reversed, then the function f is said to be r-concave.

It is obvious 0-convex functions are simply log-convex functions, 1-convex functions are ordinary convex functions and -1-convex functions are arithmetically harmonically convex. One should note that if f is r-convex on $[a,b]$, then the function f^r is a convex function for $r > 0$ and f^r is a concave function for $r < 0$. We note that if f and g are convex and g is increasing, then gof is convex; moreover, since $f = exp(log\ f)$, it follows that a log-convex function is convex.

The definition of r-convexity naturally complements the concept of r-concavity, in which the inequality is reversed [18] and which plays an important role in statistics.

It is easily seen that if f is r-convex on $[a,b]$,

$$f^r\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f^r(x)\,dx \leq \frac{f^r(a)+f^r(b)}{2}, \quad r > 0 \tag{1.1}$$

$$f^r\left(\frac{a+b}{2}\right) \geq \frac{1}{b-a}\int_a^b f^r(x)\,dx \geq \frac{f^r(a)+f^r(b)}{2}, \quad r < 0 \tag{1.2}$$

Some refinements of the Hadamard inequality for r-convex functions could be found in [2,7,16,19,21]. In [1], Bessenyei studied Hermite-Hadamard-type inequalities for generalized 3-convex functions. In [16], the authors showed that if f is r-convex in $[a,b]$ and $0 < r \leq 1$, then

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{r}{r+1}\left[f^r(a)+f^r(b)\right]^{\frac{1}{r}}. \tag{1.3}$$

Theorem 1 ([5]). *Suppose that f is a positive r-convex function on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(t)dt \leq L_r(f(a), f(b)).$$

If f is a positive r-concave function, then the inequality is reversed, where

$$L_r(f(a), f(b)) = \begin{cases} \frac{r}{r+1}\frac{f^{r+1}(a)-f^{r+1}(b)}{f^r(a)-f^r(b)}, & r \neq 0,-1, \quad f(a) \neq f(b) \\ \frac{f(a)-f(b)}{lnf(a)-lnf(b)}, & r = 0, \quad f(a) \neq f(b) \\ f(a)f(b)\frac{lnf(a)-lnf(b)}{f(a)-f(b)}, & r = -1, \quad f(a) \neq f(b) \\ f(a), & f(a) = f(b). \end{cases}$$

Theorem 2. *Let $f:[a,b]\to(0,\infty)$ be r-convex function and $r\geq 1$. Then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \left[\frac{f^r(a)+f^r(b)}{2}\right]^{\frac{1}{r}}.$$

Lemma 1. *Let $a\geq 0,\ b\geq 0$. Then $(a+b)^\lambda \leq a^\lambda + b^\lambda,\ 0<\lambda\leq 1$.*

Let $0<a<b$, throughout this paper we will use

$$A(a,b)=\frac{a+b}{2},\ \ G(a,b)=\sqrt{ab}$$

$$L_p(a,b)=\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}},\ \ a\neq b,\ p\in\mathbb{R},\ \ p\neq -1,0$$

for the arithmetic, geometric, generalized logarithmic mean, respectively. Also for shortness we will use the following notation:

$$I(a,b,n,f)=\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx$$

where an empty sum is understood to be nil.

2. Main results

We will use the following Lemma for obtain our main results.

Lemma 2 ([14])**.** *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be n-times differentiable mapping on I° for $n\in\mathbb{N}$ and $f^{(n)}\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$, we have the identity*

$$I(a,b,n,f)=\frac{(-1)^{n+1}}{n!}\int_a^b x^n f^{(n)}(x)\,dx.$$

where an empty sum is understood to be nil.

Theorem 3. *For $n\in\mathbb{N}$; let $f:I\subset(0,\infty)\to\mathbb{R}$ be n-times differentiable function on I°, $r>0$ and $a,b\in I^\circ$ with $a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^q$ for $q>1$ is r-convex function on $[a,b]$, then the following inequality holds:*

$$|I(a,b,n,f)|\leq\frac{b-a}{n!}L_{np}^n(a,b)L_{\frac{1}{r}}^{\frac{1}{qr}}\left(\left|f^{(n)}(a)\right|^{qr},\left|f^{(n)}(b)\right|^{qr}\right)$$

Proof. If $\left|f^{(n)}\right|^q$ for $q>1$ is r-convex function on $[a,b]$ and $r>0$, using Lemma 2, the Hölder integral inequality and

$$\left|f^{(n)}(x)\right|^q=\left|f^{(n)}\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)\right|^q$$

$$\leq \left[\frac{x-a}{b-a}\left|f^{(n)}(b)\right|^{qr}+\frac{b-x}{b-a}\left|f^{(n)}(a)\right|^{qr}\right]^{\frac{1}{r}},$$

we have

$$\begin{aligned}
&|I(a,b,n,f)| \\
\leq &\frac{1}{n!}\int_a^b x^n\left|f^{(n)}(x)\right|dx \\
\leq &\frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}} \\
\leq &\frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b\left[\frac{x-a}{b-a}\left|f^{(n)}(b)\right|^{qr}+\frac{b-x}{b-a}\left|f^{(n)}(a)\right|^{qr}\right]^{\frac{1}{r}}dx\right)^{\frac{1}{q}} \\
= &\frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_{\left|f^{(n)}(a)\right|^{qr}}^{\left|f^{(n)}(b)\right|^{qr}} u^{\frac{1}{r}}\frac{b-a}{\left|f^{(n)}(b)\right|^{qr}-\left|f^{(n)}(a)\right|^{qr}}du\right)^{\frac{1}{q}} \\
= &\frac{1}{n!}(b-a)\left(\frac{b^{np+1}-a^{np+1}}{(np+1)(b-a)}\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n)}(b)\right|^{qr\left(\frac{1}{r}+1\right)}-\left|f^{(n)}(a)\right|^{qr\left(\frac{1}{r}+1\right)}}{\left(\frac{1}{r}+1\right)\left(\left|f^{(n)}(b)\right|^{qr}-\left|f^{(n)}(a)\right|^{qr}\right)}\right)^{\frac{1}{q}} \\
= &\frac{1}{n!}(b-a)L_{np}^{n}(a,b)\left(\frac{\left|f^{(n)}(b)\right|^{qr\left(\frac{1}{r}+1\right)}-\left|f^{(n)}(a)\right|^{qr\left(\frac{1}{r}+1\right)}}{\left(\frac{1}{r}+1\right)\left(\left|f^{(n)}(b)\right|^{qr}-\left|f^{(n)}(a)\right|^{qr}\right)}\right)^{\frac{1}{q}} \\
= &\frac{1}{n!}(b-a)L_{np}^{n}(a,b)L_{\frac{1}{r}}^{\frac{1}{qr}}\left(\left|f^{(n)}(a)\right|^{qr},\left|f^{(n)}(b)\right|^{qr}\right).
\end{aligned}$$

This completes the proof of theorem. □

Remark 1. The results obtained in this paper reduces to the results of [14] in case of $r=1$.

Corollary 1. *Under the conditions Theorem 3 for $n=1$ we have the following inequality:*

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq L_p(a,b)L_{\frac{1}{r}}^{\frac{1}{qr}}\left(\left|f'(a)\right|^{qr},\left|f'(b)\right|^{qr}\right).$$

Proposition 1. *Let $a,b\in(0,\infty)$ with $a<b$, $q>1$ and $m\geq 1$, $r\geq 1$ we have*

$$L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a,b)\leq L_p(a,b)L_{\frac{1}{r}}^{\frac{1}{qr}}\left(a^{mr},b^{mr}\right).$$

Proof. Under the assumption of the Proposition, let $f(x) = \frac{q}{m+q} x^{\frac{m}{q}+1}$, $x \in (0,\infty)$. Then $|f'(x)|^q = x^m$ is r-convex on $(0,\infty)$ and the result follows directly from Corollary 1. □

Remark 2. Under the assumption of the Proposition 2.1, If $r = 1$, $m = 1$, then the results obtained in this paper reduces to the results of [14].

Theorem 4. *For $n \in \mathbb{N}$; let $f : I \subset (0,\infty) \to \mathbb{R}$ be n-times differentiable function on I°, $r > 0$ and $a,b \in I^\circ$ with $a < b$. If $f^{(n)} \in L[a,b]$ and $\left|f^{(n)}\right|^q$ for $q \geq 1$ is r-convex function on $[a,b]$, then the following inequality holds:*

$$|I(a,b,n,f)| \leq \begin{cases} \frac{1}{n!}(b-a)^{1-\frac{1}{q}-\frac{1}{qr}} L_n^{n\left(\frac{q-1}{q}\right)}(a,b)\left[C_1\left|f^{(n)}(b)\right|^q + C_2\left|f^{(n)}(a)\right|^q\right]^{\frac{1}{q}}, & r \geq 1 \\ \frac{1}{n!}(b-a)^{1-\frac{1}{q}-\frac{1}{qr}} L_n^{n\left(\frac{q-1}{q}\right)}(a,b)\left[C_1^r\left|f^{(n)}(b)\right|^{qr} + C_1^r\left|f^{(n)}(a)\right|^{qr}\right]^{\frac{1}{qr}}, & r \leq 1 \end{cases}$$

where

$$C_1 = C_1(a,b,r,n) = \int_a^b x^n (x-a)^{\frac{1}{r}} dx, C_2 = C_2(a,b,r,n) = \int_a^b x^n (b-x)^{\frac{1}{r}} dx.$$

Proof. From Lemma 2 and Power-Mean integral inequality, we get

$$\begin{aligned} &|I(a,b,n,f)| \\ \leq &\frac{1}{n!}\int_a^b x^n \left|f^{(n)}(x)\right| dx \\ \leq &\frac{1}{n!}\left(\int_a^b x^n dx\right)^{1-\frac{1}{q}} \left(\int_a^b x^n \left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}} \\ \leq &\frac{1}{n!}\left(\int_a^b x^n dx\right)^{1-\frac{1}{q}} \left(\int_a^b x^n \left[\frac{x-a}{b-a}\left|f^{(n)}(b)\right|^{qr} + \frac{b-x}{b-a}\left|f^{(n)}(a)\right|^{qr}\right]^{\frac{1}{r}} dx\right)^{\frac{1}{q}}. \end{aligned}$$

Here, using Lemma 1 we obtain respectively,

For $r \geq 1$

$$\begin{aligned} &|I(a,b,n,f)| \\ \leq &\frac{1}{n!}\left(\int_a^b x^n dx\right)^{1-\frac{1}{q}} \times \left(\int_a^b x^n \left\{\left[\frac{x-a}{b-a}\left|f^{(n)}(b)\right|^{qr}\right]^{\frac{1}{r}} + \left[\frac{b-x}{b-a}\left|f^{(n)}(a)\right|^{qr}\right]^{\frac{1}{r}}\right\} dx\right)^{\frac{1}{q}} \\ = &\frac{1}{n!}\left(\frac{1}{b-a}\right)^{\frac{1}{qr}} \left(\frac{b^{n+1}-a^{n+1}}{n+1}\right)^{1-\frac{1}{q}} \end{aligned}$$

$$\times\left(\int_a^b x^n(x-a)^{\frac{1}{r}}\left|f^{(n)}(b)\right|^q dx+\int_a^b x^n(b-x)^{\frac{1}{r}}\left|f^{(n)}(a)\right|^q dx\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}\left(\frac{1}{b-a}\right)^{\frac{1}{qr}}(b-a)^{1-\frac{1}{q}}\left(\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right)^{1-\frac{1}{q}}$$

$$\times\left[C_1(a,b,r,n)\left|f^{(n)}(b)\right|^q+C_2(a,b,q,n)\left|f^{(n)}(a)\right|^q\right]^{\frac{1}{q}}$$

$$=\frac{1}{n!}(b-a)^{1-\frac{1}{q}-\frac{1}{qr}}L_n^{n\left(\frac{q-1}{q}\right)}(a,b)\left[C_1\left|f^{(n)}(b)\right|^q+C_2\left|f^{(n)}(a)\right|^q\right]^{\frac{1}{q}},$$

For $r\leq 1$, using Minkowski inequality, we have

$$|I(a,b,n,f)|$$

$$\leq\frac{1}{n!}\left(\int_a^b x^n dx\right)^{1-\frac{1}{q}}\left(\frac{1}{b-a}\right)^{\frac{1}{qr}}$$

$$\times\left(\int_a^b\left[x^{nr}(x-a)\left|f^{(n)}(b)\right|^{qr}+x^{nr}(b-x)\left|f^{(n)}(a)\right|^{qr}\right]^{\frac{1}{r}}dx\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}\left(\frac{b^{n+1}-a^{n+1}}{n+1}\right)^{1-\frac{1}{q}}$$

$$\times\left(\left\{\left[\int_a^b x^n(x-a)^{\frac{1}{r}}\left|f^{(n)}(b)\right|^q dx\right]^r+\left[\int_a^b x^n(b-x)^{\frac{1}{r}}\left|f^{(n)}(a)\right|^q dx\right]^r\right\}^{\frac{1}{r}}\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}(b-a)^{1-\frac{1}{q}-\frac{1}{qr}}L_n^{n\left(\frac{q-1}{q}\right)}(a,b)\left[C_1^r\left|f^{(n)}(b)\right|^{qr}+C_2^r\left|f^{(n)}(a)\right|^{qr}\right]^{\frac{1}{qr}}.$$

This completes the proof of theorem. □

Corollary 2. *Under the conditions Theorem 4 for $n=1$ we have the following inequalities:*

$$|J(a,b,f)|$$

$$\leq\begin{cases} A^{1-\frac{1}{q}}(a,b)\left\{\frac{r(b-a)}{2r+1}\left[|f'(b)|^q-|f'(a)|^q\right]+\frac{r\left[a|f'(b)|^q+b|f'(a)|^q\right]}{r+1}\right\}^{\frac{1}{q}}, & r\geq 1\\ A^{1-\frac{1}{q}}(a,b)\left\{\left(\frac{r^2(a+b)+br}{(r+1)(2r+1)}\right)^r|f'(b)|^{qr}+\left(\frac{r^2(a+b)+ar}{(r+1)(2r+1)}\right)^r|f'(a)|^{qr}\right\}^{\frac{1}{qr}}, & r\leq 1\end{cases}$$

where $J(a,b,f)=\frac{I(a,b,1,f)}{b-a}$.

Proposition 2. *Let $a,b \in (0,\infty)$ with $a<b$, $q \geq 1$ and $m \geq 1$, we have the following inequalities:*

$$L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a,b) \leq \begin{cases} A^{1-\frac{1}{q}}(a,b)\left[\frac{2rA(a^{m+1},b^{m+1})}{2r+1}+\frac{2r^2G^2(a,b)A(a^{m-1},b^{m-1})}{(r+1)(2r+1)}\right]^{\frac{1}{q}}, & r \geq 1 \\ A^{1-\frac{1}{q}}(a,b)\left\{\left(\frac{r^2(a+b)+br}{(r+1)(2r+1)}\right)^r b^{mr}+\left(\frac{r^2(a+b)+ar}{(r+1)(2r+1)}\right)^r a^{mr}\right\}^{\frac{1}{qr}}, & r \leq 1. \end{cases}$$

Proof. The result follows directly from Corollary 2 for function $f(x)=\frac{q}{m+q}x^{\frac{m}{q}+1}$, $x \in (0,\infty)$. □

Corollary 3. *Using Proposition 2 for $m=1$, we have following inequalities:*

$$L_{\frac{1}{q}+1}^{\frac{1}{q}+1}(a,b) \leq \begin{cases} A^{1-\frac{1}{q}}(a,b)\left[\frac{2r}{2r+1}A\left(a^2,b^2\right)+\frac{2r^2}{(r+1)(2r+1)}G^2(a,b)\right]^{\frac{1}{q}}, & r \geq 1 \\ A^{1-\frac{1}{q}}(a,b)\left\{\left(\frac{r^2(a+b)+br}{(r+1)(2r+1)}\right)^r b^{r}+\left(\frac{r^2(a+b)+ar}{(r+1)(2r+1)}\right)^r a^{r}\right\}^{\frac{1}{qr}}, & r \leq 1. \end{cases}$$

Corollary 4. *Using Proposition 2 for $q=1$, we have following inequalities:*

$$L_{m+1}^{m+1}(a,b) \leq \begin{cases} \frac{2rA(a^{m+1},b^{m+1})}{2r+1}+\frac{2r^2G^2(a,b)A(a^{m-1},b^{m-1})}{(r+1)(2r+1)}, & r \geq 1 \\ \left\{\left(\frac{r^2(a+b)+br}{(r+1)(2r+1)}\right)^r b^{mr}+\left(\frac{r^2(a+b)+ar}{(r+1)(2r+1)}\right)^r a^{mr}\right\}^{\frac{1}{r}}, & r \leq 1. \end{cases}$$

Corollary 5. *Using Corollary 4. for $m=1$, we have following inequalities:*

$$L_2^2(a,b) \leq \begin{cases} \frac{2r}{2r+1}A\left(a^2,b^2\right)+\frac{2r^2}{(r+1)(2r+1)}G^2(a,b), & r \geq 1 \\ \left\{\left(\frac{r^2(a+b)+br}{(r+1)(2r+1)}\right)^r b^{r}+\left(\frac{r^2(a+b)+ar}{(r+1)(2r+1)}\right)^r a^{r}\right\}^{\frac{1}{r}}, & r \leq 1. \end{cases}$$

Corollary 6. *Under the conditions Theorem 4 for $q=1$ we have the following inequalities:*

$$|I(a,b,n,f)| \leq \begin{cases} \frac{1}{n!}(b-a)^{-\frac{1}{r}}\left[C_1\left|f^{(n)}(b)\right|+C_2\left|f^{(n)}(a)\right|\right], & r \geq 1 \\ \frac{(b-a)^{-\frac{1}{r}}}{n!}\left[C_1^r\left|f^{(n)}(b)\right|^r+C_2^r\left|f^{(n)}(a)\right|^r\right]^{\frac{1}{r}}, & r \leq 1 \end{cases}$$

Theorem 5. *For $n \in \mathbb{N}$; let $f: I \subset (0,\infty) \to \mathbb{R}$ be n-times differentiable function on I°, $r>0$ and $a,b \in I^\circ$ with $a<b$. If $f^{(n)} \in L[a,b]$ and $\left|f^{(n)}\right|^q$ for $q>1$ is r-convex function on $[a,b]$, then the following inequality holds:*

$$|I(a,b,n,f)| \leq \begin{cases} \frac{1}{n!}(b-a)^{\frac{1}{p}-\frac{1}{qr}}\left(\left|f^{(n)}(b)\right|^q D_1+\left|f^{(n)}(a)\right|^q D_2\right)^{\frac{1}{q}}, & r \geq 1 \\ \frac{1}{n!}(b-a)^{\frac{1}{p}-\frac{1}{qr}}\left(\left|f^{(n)}(b)\right|^{qr} D_1^r+\left|f^{(n)}(a)\right|^{qr} D_2^r\right)^{\frac{1}{qr}}, & r \leq 1 \end{cases}$$

where

$$D_1 = D_1(a,b,r,n,q) = \int_a^b x^{nq}(x-a)^{\frac{1}{r}}dx$$

$$D_2 = D_2(a,b,r,n,q) = \int_a^b x^{nq}(b-x)^{\frac{1}{r}}dx.$$

Proof. Since $\left|f^{(n)}\right|^q$ for $q>1$ is r-convex function on $[a,b]$, using Lemma 2 and the Hölder integral inequality, we have the following inequality:

$$\begin{aligned}
&|I(a,b,n,f)| \\
&\leq \frac{1}{n!}\int_a^b 1.x^n \left|f^{(n)}(x)\right| dx \\
&\leq \frac{1}{n!}\left(\int_a^b 1^p dx\right)^{\frac{1}{p}}\left(\int_a^b x^{nq}\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}} \\
&\leq \frac{1}{n!}\left(\int_a^b 1^p dx\right)^{\frac{1}{p}}\left(\int_a^b x^{nq}\left[\frac{x-a}{b-a}\left|f^{(n)}(b)\right|^{qr}+\frac{b-x}{b-a}\left|f^{(n)}(a)\right|^{qr}\right]^{\frac{1}{r}} dx\right)^{\frac{1}{q}}.
\end{aligned}$$

Here, using Lemma 1 we obtain respectively,

For $r \geq 1$,

$$\begin{aligned}
&|I(a,b,n,f)| \\
&\leq \frac{1}{n!}\left(\int_a^b 1^p dx\right)^{\frac{1}{p}}\left(\int_a^b x^{nq}\left[\left(\frac{x-a}{b-a}\right)^{\frac{1}{r}}\left|f^{(n)}(b)\right|^q+\left(\frac{b-x}{b-a}\right)^{\frac{1}{r}}\left|f^{(n)}(a)\right|^q\right] dx\right)^{\frac{1}{q}} \\
&= \frac{(b-a)^{-\frac{1}{qr}}}{n!}(b-a)^{\frac{1}{p}}\left(\left|f^{(n)}(b)\right|^q D_1+\left|f^{(n)}(a)\right|^q D_2\right)^{\frac{1}{q}} \\
&= \frac{1}{n!}(b-a)^{\frac{1}{p}-\frac{1}{qr}}\left(\left|f^{(n)}(b)\right|^q D_1+\left|f^{(n)}(a)\right|^q D_2\right)^{\frac{1}{q}},
\end{aligned}$$

For $r \leq 1$, using Minkowski inequality, we have

$$\begin{aligned}
&|I(a,b,n,f)| \\
&\leq \frac{1}{n!}(b-a)^{\frac{1}{p}-\frac{1}{qr}}\left(\int_a^b\left[x^{nqr}(x-a)\left|f^{(n)}(b)\right|^{qr}+x^{nqr}(b-x)\left|f^{(n)}(a)\right|^{qr}\right]^{\frac{1}{r}}\right)^{\frac{1}{q}} \\
&= \frac{1}{n!}(b-a)^{\frac{1}{p}-\frac{1}{qr}} \\
&\quad \times\left\{\left(\left|f^{(n)}(b)\right|^q\int_a^b x^{nq}(x-a)^{\frac{1}{r}}dx\right)^r+\left(\left|f^{(n)}(a)\right|^q\int_a^b x^{nq}(b-x)^{\frac{1}{r}}dx\right)^r\right\}^{\frac{1}{qr}} \\
&= \frac{1}{n!}(b-a)^{\frac{1}{p}-\frac{1}{qr}}\left(\left|f^{(n)}(b)\right|^{qr}D_1^r+\left|f^{(n)}(a)\right|^{qr}D_2^r\right)^{\frac{1}{qr}}.
\end{aligned}$$

This completes the proof of theorem. □

Corollary 7. *Under the conditions Theorem 5 for $n = 1$ we have the following inequalities:*

$$\left|\frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \begin{cases} (b-a)^{\frac{1}{p}-\frac{1}{qr}-1}\left(|f'(b)|^q D_1 + |f'(a)|^q D_2\right)^{\frac{1}{q}}, & r \geq 1 \\ (b-a)^{\frac{1}{p}-\frac{1}{qr}-1}\left(|f'(b)|^{qr} D_1^r + |f'(a)|^{qr} D_2^r\right)^{\frac{1}{qr}}, & r \leq 1. \end{cases}$$

Proposition 3. *Let $a, b \in (0,\infty)$ with $a < b$, $q > 1$ and $m \geq 1$, we have*

$$L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a,b) \leq \begin{cases} (b-a)^{\frac{1}{p}-\frac{1}{qr}-1}(b^m D_1 + a^m D_2)^{\frac{1}{q}}, & r \geq 1 \\ (b-a)^{\frac{1}{p}-\frac{1}{qr}-1}\left(b^{mr} D_1^r + a^{mr} D_2^r\right)^{\frac{1}{q}}, & r \leq 1. \end{cases}$$

Proof. The result follows directly from Corollary 7 for $f(x) = \frac{q}{m+q}x^{\frac{m}{q}+1}$, $x \in (0,\infty)$. □

Corollary 8. *For $m = 1$ from Proposition 3, we obtain the following inequality:*

$$L_{\frac{1}{q}+1}^{\frac{1}{q}+1}(a,b) \leq \begin{cases} (b-a)^{\frac{1}{p}-\frac{1}{qr}-1}(bD_1 + aD_2)^{\frac{1}{q}}, & r \geq 1 \\ (b-a)^{\frac{1}{p}-\frac{1}{qr}-1}\left(b^r D_1^r + a^r D_2^r\right)^{\frac{1}{q}}, & r \leq 1. \end{cases}$$

Theorem 6. *For $n \in \mathbb{N}$; let $f : I \subset (0,\infty) \to \mathbb{R}$ be n-times differentiable function on I°(interior of I), $r > 0$ and $a, b \in I^\circ$ with $a < b$. If $f^{(n)} \in L[a,b]$ and $\left|f^{(n)}\right|^q$ for $q > 1$ is r-convex function on $[a,b]$, then the following inequalities holds:*

$$|I(a,b,n,f)|$$
$$\leq \begin{cases} 2^{\frac{1}{qr}}\frac{b-a}{n!}\left(\frac{r}{r+1}\right)^{\frac{1}{q}} L_{np}^n(a,b)\, A^{\frac{1}{qr}}\left(\left|f^{(n)}(a)\right|^{qr}, \left|f^{(n)}(b)\right|^{qr}\right), & 0 < r \leq 1, \\ \frac{b-a}{n!} L_{np}^n(a,b)\, A^{\frac{1}{qr}}\left(\left|f^{(n)}(a)\right|^{qr}, \left|f^{(n)}(b)\right|^{qr}\right), & r \geq 0, \\ \frac{1}{n!}(b-a)\, L_{np}^n(a,b)\left(L_r\left(\left|f^{(n)}(a)\right|^q, \left|f^{(n)}(b)\right|^q\right)\right)^{\frac{1}{q}}, & r > 0, \end{cases}$$

Proof. For $0 < r \leq 1$, since $\left|f^{(n)}\right|^q$ for $q > 1$ is r-convex function on $[a,b]$, with respect to Hermite-Hadamard inequality we have

$$\int_a^b \left|f^{(n)}(x)\right|^q dx \leq (b-a)\frac{r}{r+1}\left[\left|f^{(n)}(a)\right|^{qr} + \left|f^{(n)}(b)\right|^{qr}\right]^{\frac{1}{r}}.$$

Using Lemma 2 and the Hölder integral inequality we have

$$|I(a,b,n,f)|$$

$$\leq \frac{1}{n!}\int_a^b x^n \left|f^{(n)}(x)\right| dx$$
$$\leq \frac{1}{n!}\left(\int_a^b x^{np} dx\right)^{\frac{1}{p}} \left(\int_a^b \left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{n!}\left(\int_a^b x^{np} dx\right)^{\frac{1}{p}} \left((b-a)\frac{r}{r+1}\left[\left|f^{(n)}(a)\right|^{qr} + \left|f^{(n)}(b)\right|^{qr}\right]^{\frac{1}{r}}\right)^{\frac{1}{q}}$$
$$= 2^{\frac{1}{qr}} \frac{b-a}{n!}\left(\frac{r}{r+1}\right)^{\frac{1}{q}} \left[\frac{b^{np+1}-a^{np+1}}{(np+1)(b-a)}\right]^{\frac{1}{p}} \left[\frac{\left|f^{(n)}(a)\right|^{qr} + \left|f^{(n)}(b)\right|^{qr}}{2}\right]^{\frac{1}{qr}}$$
$$= 2^{\frac{1}{qr}} \frac{b-a}{n!}\left(\frac{r}{r+1}\right)^{\frac{1}{q}} L_{np}^n(a,b)\, A^{\frac{1}{qr}}\left(\left|f^{(n)}(a)\right|^{qr}, \left|f^{(n)}(b)\right|^{qr}\right).$$

For $r \geq 1$, since $\left|f^{(n)}\right|^q$ for $q > 1$ is r-convex function on $[a,b]$, with respect to Theorem 2 we get

$$\int_a^b \left|f^{(n)}(x)\right|^q dx \leq (b-a)\left[\frac{\left|f^{(n)}(a)\right|^{qr} + \left|f^{(n)}(b)\right|^{qr}}{2}\right]^{\frac{1}{r}}.$$

Using Lemma 2 and the Hölder integral inequality we have

$$\begin{aligned}
|I(a,b,n,f)| &\leq \frac{1}{n!}\int_a^b x^n \left|f^{(n)}(x)\right| dx \\
&\leq \frac{1}{n!}\left(\int_a^b x^{np} dx\right)^{\frac{1}{p}} \left(\int_a^b \left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}} \\
&= \frac{b-a}{n!}\left[\frac{b^{np+1}-a^{np+1}}{(np+1)(b-a)}\right]^{\frac{1}{p}} \left[\frac{\left|f^{(n)}(a)\right|^{qr} + \left|f^{(n)}(b)\right|^{qr}}{2}\right]^{\frac{1}{qr}} \\
&= \frac{b-a}{n!} L_{np}^n(a,b)\, A^{\frac{1}{qr}}\left(\left|f^{(n)}(a)\right|^{qr}, \left|f^{(n)}(b)\right|^{qr}\right).
\end{aligned}$$

For $r > 0$, using Lemma 2, Theorem 1 and the Hölder integral inequality we have

$$|I(a,b,n,f)| \leq \frac{1}{n!}\int_a^b x^n \left|f^{(n)}(x)\right| dx$$

$$\leq \frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b \left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left((b-a)\,L_r\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right)\right)^{\frac{1}{q}}$$

$$= \frac{1}{n!}(b-a)L_{np}^n(a,b)\left(L_r\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right)\right)^{\frac{1}{q}}.$$

This completes the proof of theorem. □

Corollary 9. *Under the conditions Theorem 6 for $n=1$ we have the following inequalities:*

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \begin{cases} 2^{\frac{1}{qr}}\left(\frac{r}{r+1}\right)^{\frac{1}{q}}L_p(a,b)\,A^{\frac{1}{qr}}\left(|f'(a)|^{qr},|f'(b)|^{qr}\right), 0<r\leq 1 \\ L_p(a,b)\,A^{\frac{1}{qr}}\left(|f'(a)|^{qr},|f'(b)|^{qr}\right), \qquad r\geq 1 \\ L_p(a,b)\left(L_r\left(|f'(a)|^q,|f'(b)|^q\right)\right)^{\frac{1}{q}}, \qquad r>0. \end{cases}$$

Proposition 4. *Let $a,b\in(0,\infty)$ with $a<b$, $q>1$ and $m\in[0,1]$, we have*

$$L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a,b)\leq \begin{cases} 2^{\frac{1}{qr}}\left(\frac{r}{r+1}\right)^{\frac{1}{q}}L_p(a,b)\,A^{\frac{1}{qr}}(a^{mr},b^{mr}), \quad 0<r\leq 1 \\ L_p(a,b)\,A^{\frac{1}{qr}}(a^{mr},b^{mr}), \qquad r\geq 1 \\ L_p(a,b)\,(L_r(a^m,b^m))^{\frac{1}{q}}, \qquad r>0 \end{cases}$$

Under the assumption of the Proposition, let $f(x)=\frac{q}{m+q}x^{\frac{m}{q}+1}$, $x\in(0,\infty)$. Then $|f'(x)|^q=x^m$ is r-convex on $(0,\infty)$ and the result follows directly from Corollary 9.

Corollary 10. *For $m=1$ from Proposition 4, we obtain the following inequalities:*

$$L_{\frac{1}{q}+1}^{\frac{1}{q}+1}(a,b)\leq \begin{cases} 2^{\frac{1}{qr}}\left(\frac{r}{r+1}\right)^{\frac{1}{q}}L_p(a,b)\,A^{\frac{1}{qr}}(a^r,b^r), \quad 0<r\leq 1 \\ L_p(a,b)\,A^{\frac{1}{qr}}(a^r,b^r), \qquad r\geq 1 \\ L_p(a,b)\,(L_r(a,b))^{\frac{1}{q}}, \qquad r>0 \end{cases}$$

Theorem 7. *For $n\in\mathbb{N}$; let $f:I\subset(0,\infty)\to\mathbb{R}$ be n-times differentiable function on I°, $r>0$ and $a,b\in I^\circ$ with $a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^{\frac{q}{r}}$ for $q>1$ is r-convex function on $[a,b]$, then the following inequality holds:*

$$|I(a,b,n,f)|\leq \frac{1}{n!}(b-a)\,L_{np}^n(a,b)\,A^{\frac{1}{q}}\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right).$$

Proof. If $\left|f^{(n)}\right|^{\frac{q}{r}}$ for $q>1$ is r-convex function on $[a,b]$ and $r>0$, using (1.1) inequality, Lemma 2 and the Hölder integral inequality respectively, we have

$$\int_a^b \left|f^{(n)}(x)\right|^q dx = \int_a^b \left(\left|f^{(n)}(x)\right|^{\frac{q}{r}}\right)^r dx \leq (b-a)\frac{\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(b)\right|^q}{2}$$

and

$$\begin{aligned}
|I(a,b,n,f)| &\leq \frac{1}{n!}\int_a^b x^n \left|f^{(n)}(x)\right| dx \\
&\leq \frac{1}{n!}\left(\int_a^b x^{np} dx\right)^{\frac{1}{p}} \left(\int_a^b \left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}} \\
&= \frac{1}{n!}\left(\frac{b^{np+1}-a^{np+1}}{np+1}\right)^{\frac{1}{p}} \left((b-a)\frac{\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(b)\right|^q}{2}\right)^{\frac{1}{q}} \\
&= \frac{1}{n!}(b-a)\, L_{np}^n(a,b)\, A^{\frac{1}{q}}\left(\left|f^{(n)}(a)\right|^q, \left|f^{(n)}(b)\right|^q\right).
\end{aligned}$$

This completes the proof of theorem. □

Corollary 11. *Under the conditions Theorem 7 for $n=1$ we have the following inequality:*

$$\left|\frac{f(b)b-f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq L_p(a,b)\, A^{\frac{1}{q}}\left(\left|f'(a)\right|^q, \left|f'(b)\right|^q\right).$$

Proposition 5. *Let $a,b\in(0,\infty)$ with $a<b$, $q>1$ and $m\in[0,1]$, we have*

$$\left|\frac{f(b)b-f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq L_p(a,b)\, A^{\frac{1}{q}}\left(a^{mr}, b^{mr}\right).$$

Proof. Under the assumption of the Proposition, let $f(x)=\frac{q}{mr+q}x^{\frac{mr}{q}+1}$, $x\in(0,\infty)$. Then $|f'(x)|^{\frac{q}{r}}=x^m$ is r-convex on $(0,\infty)$ and the result follows directly from Corollary 11. □

Corollary 12. *For $m=1$ from Proposition 5, we obtain the following inequality:*

$$\left|\frac{f(b)b-f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq L_p(a,b)\, A^{\frac{1}{q}}\left(a^{r}, b^{r}\right).$$

Theorem 8. *For $n\in\mathbb{N}$; let $f: I\subset(0,\infty)\to\mathbb{R}$ be n-times differentiable function on I°, $r>0$ and $a,b\in I^\circ$ with $a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^{\frac{q}{r}}$ for $q>1$ is*

r-concave function on $[a,b]$, then the following inequality holds:

$$|I(a,b,n,f)| \leq \frac{b-a}{n!} L_{np}^{n}(a,b) \left| f^{(n)}\left(\frac{a+b}{2}\right)\right|.$$

Proof. If $\left|f^{(n)}\right|^{\frac{q}{r}}$ for $q>1$ is r-concave function on $[a,b]$ and $r>0$, using Lemma 2, the Hölder integral inequality and

$$\int_a^b \left|f^{(n)}(x)\right|^q dx = \int_a^b \left(\left|f^{(n)}(x)\right|^{\frac{q}{r}}\right)^r dx \leq (b-a)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q$$

we have

$$\begin{aligned}
|I(a,b,n,f)| \leq & \frac{1}{n!}\int_a^b x^n \left|f^{(n)}(x)\right| dx \\
\leq & \frac{1}{n!}\left(\int_a^b x^{np} dx\right)^{\frac{1}{p}} \left(\int_a^b \left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}} \\
\leq & \frac{1}{n!}\left(\int_a^b x^{np} dx\right)^{\frac{1}{p}} \left((b-a)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} \\
= & \frac{1}{n!}(b-a) L_{np}^{n}(a,b)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|.
\end{aligned}$$

This completes the proof of theorem. □

Corollary 13. *Under the conditions Theorem 8 for $n=1$ we have the following inequality:*

$$\left|\frac{f(b)b-f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq L_p(a,b)\left|f'\left(\frac{a+b}{2}\right)\right|.$$

Proposition 6. *Let $a,b \in (0,\infty)$ with $a<b$, $q>1$ and $m \in [0,1]$, we have*

$$\left|\frac{f(b)b-f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq L_p(a,b) A^{\frac{mr}{q}}(a,b).$$

Proof. Under the assumption of the Proposition, let $f(x) = \frac{q}{mr+q} x^{\frac{mr}{q}+1}$, $x \in (0,\infty)$. Then $|f'(x)|^{\frac{q}{r}} = x^m$ is r-concave on $(0,\infty)$ and the result follows directly from Corollary 13. □

Corollary 14. *For $m=1$ from Proposition 6, we obtain the following inequality:*

$$\left|\frac{f(b)b-f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq L_p(a,b) A^{\frac{r}{q}}(a,b).$$

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Authors' addresses

Huriye Kadakal
Ministry of Education, Giresun Hamdi Bozbağ Anatolian High School Giresun-TÜRKİYE
E-mail address: huriyekadakal@hotmail.com

Mahir Kadakal
Department of Mathematics, Faculty of Sciences and Arts, Giresun University, Giresun-TÜRKİYE
E-mail address: mahirkadakal@gmail.com

İmdat İşcan
Department of Mathematics, Faculty of Sciences and Arts, Giresun University, Giresun-TÜRKİYE
E-mail address: imdat.iscan@giresun.edu.tr

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INTERMEDIATE REGULARITY RESULTS FOR THE SOLUTION OF A HIGH ORDER PARABOLIC EQUATION

AREZKI KHELOUFI

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Abstract. In this work we give new intermediate regularity results for the solution of the following $2m$-th order parabolic equation

$$\partial_t u + (-1)^m \sum_{i=1}^{n} \partial_{x_i}^{2m} u = 0,$$

where m is a positive integer, subject to Dirichlet condition on the lateral boundary of a cylindrical domain and to a non-homogeneous initial Cauchy data.

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1. Introduction

Let Ω be an open bounded set of $\mathbb{R}^n$ with boundary Γ and Q the cylinder $\mathbb{R}_+ \times \Omega$ with lateral boundary $\Sigma = \mathbb{R}_+ \times \Gamma$. We assume that Ω is of class C^{2m}. Consider in Q the following boundary value problem:

$$\begin{cases} \partial_t u + (-1)^m \sum_{i=1}^{n} \partial_{x_i}^{2m} u = 0 \ \text{ in } Q, \\ \partial_\nu^j u = 0 \ \text{ on } \Sigma,\ j = 0, 1, ..., m-1, \\ u(0,x) = u_0(x),\ x \in \Omega, \end{cases} \tag{1.1}$$

where m is a positive integer and ∂_ν^j is the derivative of order j throughout the normal vector ν on Σ.

Classical results on the resolution of Problem (1.1) when the initial Cauchy data u_0 belongs to $L^2(\Omega)$ or to the usual Sobolev space $H_0^m(\Omega)$ can be found in [6], see also [1], [2], [8] and [9] and the references therein.

Our interest in this work is the regularity of the solution u of (1.1) in terms of the regularity of the initial data u_0. More precisely, we are interested on the question of the regularity of the solution of (1.1) when u_0 is "between" $H_0^m(\Omega)$ and $L^2(\Omega)$. The

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results obtained here complement those obtained for the heat equation, i.e. $m = 1$, in [4].

The organization of this paper is as follows. In Section 2, we begin by preliminaries where we define the basic functional spaces, in which we will work and we give some of their properties needed for our study. Then, we recall a classical result for Problem (1.1) and we prove a fundamental lemma which will allow us to prove our main result in Section 3.

2. Preliminaries

2.1. *Function spaces*

In this subsection, we recall the definitions of the basic functional spaces, in which we will work. We will need some anisotropic Sobolev spaces (see [6]), which we recall in the following definitions

$$H^{r,s}\left(\mathbb{R}^n\right) = \left\{u \in L^2\left(\mathbb{R}^n\right) : \left[\left(1+\zeta^2\right)^{r/2} + \left(1+\tau^2\right)^{s/2}\right]\widehat{u} \in L^2\left(\mathbb{R}^n\right)\right\}$$

where $\widehat{u}$ is the Fourier transform of u and r, s are two non-negative numbers. We put

$$H^{r,s}(\Omega) = \left\{u|_{\Omega} : u \in H^{r,s}\left(\mathbb{R}^n\right)\right\},$$

with Ω is an open subset of $\mathbb{R}^n$. For $0 \leq r \leq 1$ and a positive integer m, we recall that the space $H^{r,2mr}(Q)$, where $Q = \mathbb{R}_+ \times \Omega$, can be defined by

$$H^{r,2mr}(Q) = L^2\left(\mathbb{R}_+, H^{2mr}(\Omega)\right) \cap H^r\left(\mathbb{R}_+, L^2(\Omega)\right)$$

and the space $H_0^r(\Omega)$ is defined by

$$H_0^r(\Omega) = \{u \in H^r(\Omega); u = 0 \text{ on } \Gamma\}$$

for $\frac{1}{2} < r \leq 1$, Γ is the boundary of Ω.

$$H_0^r(\Omega) = H_{00}^{\frac{1}{2}}(\Omega)$$

for $r = \frac{1}{2}$,

$$H_0^r(\Omega) = H^r(\Omega)$$

for $0 \leq r < \frac{1}{2}$.

We will need also some interpolation spaces in Hilbert spaces (see [7] and [3]), which we recall in the following definition.

Let X, Y be two Hilbert spaces with

$$X \subset Y \text{ continuously.}$$

Here, we give one of the usual methods, namely, that of Lions-Peetre [7] which allow us to build spaces

$$[X, Y]_\theta \;\; 0 < \theta < 1,$$

"intermediate" between X and Y.

Definition 1. The space $[X,Y]_\theta$ $0<\theta<1$, is a sub-space of Y consisting of elements a which can be written in the form

$$a=\int_0^\infty u(t)\frac{dt}{t} \tag{2.1}$$

with

$$t^\theta u(t)\in L_*^2(X),\, t^{\theta-1}u(t)\in L_*^2(Y). \tag{2.2}$$

This space is endowed with the norm

$$a\longmapsto \inf\left[\left(\int_0^\infty t^{2\theta}|u(t)|_X^2\frac{dt}{t}\right)^{\frac{1}{2}}+\left(\int_0^\infty t^{2(\theta-1)}|u(t)|_Y^2\frac{dt}{t}\right)^{\frac{1}{2}}\right];$$

the inf is taken with respect to u verifying (2.1) and (2.2).

Here $L_*^2(V)$ denotes the space of functions from $t>0$ with values in V, which are square integrable for the Haar measure dt/t.

Example 1. $H^{r,s}(\Omega)$ can also be defined as a real interpolation space between $H^{r/(1-\theta),s/(1-\theta)}(\Omega)$ and $L^2(\Omega)$, $\theta\in]0,1[$, (see [10])

$$H^{r,s}(\Omega)=\left[H^{r/(1-\theta),s/(1-\theta)}(\Omega),L^2(\Omega)\right]_\theta. \tag{2.3}$$

In this work, we consider the case $s=2mr$, $\theta=1-r$,

$$H^{r,2mr}(\Omega)=\left[H^{1,2m}(\Omega),L^2(\Omega)\right]_{1-r}\quad \forall r\in]0,1[. \tag{2.4}$$

The space $H^{r,2mr}(\Omega)$ is well defined by Relationship (2.4) because the right hand side term of (2.4) is well defined as an interpolation space between two well defined spaces $H^{1,2m}(\Omega)$ and $L^2(\Omega)$.

Example 2. The usual Sobolev spaces $H^s(\Omega)$ $(s\geq 0)$ can be defined by interpolation

$$H^s(\Omega)=\left[H^{s/(1-\theta)}(\Omega),L^2(\Omega)\right]_\theta,$$

$\theta\in]0,1[$.

Hereafter some interpolation theory properties on spaces $[.,.]_\theta$ (see Triebel [10]), needed for proving our main result in the following section.

Theorem 1 ([10])**.** *Let A_0, A_1 be two Hilbert spaces with*

$$A_0\subset A_1 \text{ continuously.}$$

Then

$$\left[L^2(A_0),L^2(A_1)\right]_\theta=L^2([A_0,A_1]_\theta),$$

$0<\theta<1$.

A direct consequence of Theorem 1 is

Corollary 1. *For each* $0 \leq r \leq 1$ *and any positive integer* m, *we have*

$$\begin{aligned}\left[L^2\left(\mathbb{R}_+, H^{2m}(\Omega)\right), L^2\left(\mathbb{R}_+, H^m(\Omega)\right)\right]_{1-r} &= L^2\left(\mathbb{R}_+, \left[H^{2m}(\Omega), H^m(\Omega)\right]_{1-r}\right) \\ &= L^2\left(\mathbb{R}_+, H^{m(1+r)}(\Omega)\right).\end{aligned}$$

with Ω *an open bounded set of* $\mathbb{R}^n$.

2.2. *Lemmas*

The next result is well known (cf. Lions and Magenes [6]).

Lemma 1. *1) For given* u_0 *in* $H_0^m(\Omega)$, *Problem (1.1) has a unique solution* u *in* $H^{1,2m}(Q)$ *defined by*

$$H^{1,2m}(Q) = L^2\left(\mathbb{R}_+, H^{2m}(\Omega)\right) \cap H^1\left(\mathbb{R}_+, L^2(\Omega)\right).$$

2) For given u_0 *in* $L^2(\Omega)$, *Problem (1.1) has a unique weak solution* u *in*

$$L^2\left(\mathbb{R}_+, H_0^m(\Omega)\right) \cap H^1\left(\mathbb{R}_+, H^{-m}(\Omega)\right) \cap L^\infty\left(\mathbb{R}_+, L^2(\Omega)\right).$$

We will need the following lemma for proving our main result in the following section.

Lemma 2. *Let* $u_0 \in L^2(\Omega)$. *Then the solution* u *of Problem (1.1) associated to* u_0 *is in* $H^{\frac{1}{2}}\left(\mathbb{R}_+, L^2(\Omega)\right)$. *Moreover, there exists a positive constant* C *(independent of* u_0*) such that*

$$\|u\|_{H^{\frac{1}{2}}\left(\mathbb{R}_+, L^2(\Omega)\right)} \leq C \|u_0\|_{L^2(\Omega)}.$$

Proof. Consider a sequence of spectral elements (λ_k, φ_k), $k \in \mathbb{N}$ of the Dirichlet problem for the operator $(-1)^m \sum_{i=1}^n \partial_{x_i}^{2m}$

$$\begin{cases} (-1)^m \sum_{i=1}^n \partial_{x_i}^{2m} \varphi_k = \lambda_k \varphi_k \\ \\ \varphi_k \in H_0^m(\Omega) \\ \\ \|\varphi_k\|_{L^2(\Omega)} = 1. \end{cases}$$

The sequence $(\varphi_k)_{k\in\mathbb{N}}$ is a basis of $L^2(\Omega)$. If $u_0 \in L^2(\Omega)$ we may write

$$u_0(x) = \sum_{k\in\mathbb{N}} a_k \varphi_k(x)$$

with $\|u_0\|_{L^2(\Omega)}^2 = \sum_{k\in\mathbb{N}} a_k^2$. The solution associated to u_0 is

$$u(t,x) = \sum_{k\in\mathbb{N}} a_k \exp(-\lambda_k t) \varphi_k(x).$$

Note $\widetilde{u}$ the extension of u to $\mathbb{R}$, i.e.,

$$\widetilde{u}(t,x) = \sum a_k \exp|-\lambda_k t| \varphi_k(x).$$

By the Fourier transform

$$\widehat{\widetilde{u}}(\zeta,x)=C\sum\frac{a_k\lambda_k}{\zeta^2+\lambda_k^2}\varphi_k(x),$$

where C is a constant, from which

$$\left\|\widehat{\widetilde{u}}(\zeta,.)\right\|^2=C^2\sum a_k^2\frac{\lambda_k^2}{\left(\zeta^2+\lambda_k^2\right)^2}$$

and by elementary calculations, we check easily that

$$\int_{\mathbb{R}}|\zeta|\left\|\widehat{\widetilde{u}}(\zeta,.)\right\|^2d\zeta=C'\sum a_k^2,$$

where C' is a constant. Consequently

$$\widetilde{u}\in H^{\frac{1}{2}}\left(\mathbb{R},L^2(\Omega)\right),$$

then

$$u\in H^{\frac{1}{2}}\left(\mathbb{R}_+,L^2(\Omega)\right)$$

by restriction of $\widetilde{u}$ to $t>0$. □

3. Main result

In the sequel, we will assume that $u_0\in H_0^{rm}(\Omega)$, $0\le r\le 1$. Thus $H_0^{rm}(\Omega)$ is the interpolation space of order $1-r$ between $H_0^m(\Omega)$ and $L^2(\Omega)$. Indeed, it suffices to take $\theta=1-r$ and $s=rm$ in Example 2. We look for the regularity of u in term of r. Our main result in this work is

Theorem 2. *For given u_0 in $H_0^{rm}(\Omega)$ $0\le r\le 1$, Problem (1.1) has a unique weak solution u in $H^{\frac{1+r}{2},m(1+r)}(Q)$.*

Proof. Let $u_0\in H_0^{rm}(\Omega)$ $0\le r\le 1$ then $u_0\in L^2(\Omega)$ and consequently (1.1) admits a unique weak solution (see Lemma 1) u in $L^2\left(\mathbb{R}_+,H_0^m(\Omega)\right)$. In order to show that this solution is in $L^2\left(\mathbb{R}_+,H^{m(1+r)}(\Omega)\right)$ it suffices to interpolate the operator S which associates u to u_0. Indeed $S:u_0\mapsto u$ is linear continuous from $H_0^m(\Omega)$ to $L^2\left(\mathbb{R}_+,H^{2m}(\Omega)\right)$ and from $L^2(\Omega)$ to $L^2\left(\mathbb{R}_+,H^m(\Omega)\right)$. By interpolation, it is linear continuous from

$$\left[H_0^m(\Omega),L^2(\Omega)\right]_{1-r}\text{ into }\left[L^2\left(\mathbb{R}_+,H^{2m}(\Omega)\right),L^2\left(\mathbb{R}_+,H^m(\Omega)\right)\right]_{1-r}.$$

Thanks to Corollary 1, S is linear continuous from

$$H_0^{rm}(\Omega)\text{ into }L^2\left(\mathbb{R}_+,H^{m(1+r)}(\Omega)\right).$$

We can interpolate again S for proving that $u \in H^{\frac{r+1}{2}}\left(\mathbb{R}_+, L^2(\Omega)\right)$. Indeed, S is linear continuous from $L^2(\Omega)$ into $H^{\frac{1}{2}}\left(\mathbb{R}_+, L^2(\Omega)\right)$ (see Lemma 2) and from $H_0^m(\Omega)$ into $H^1\left(\mathbb{R}_+, L^2(\Omega)\right)$ (see Theorem 1). By interpolation, it is linear continuous from

$$\left[H_0^m(\Omega), L^2(\Omega)\right]_{1-r} \text{ into } \left[H^{\frac{1}{2}}\left(\mathbb{R}_+, L^2(\Omega)\right), H^1\left(\mathbb{R}_+, L^2(\Omega)\right)\right]_{1-r}.$$

But, (see Triebel [10])

$$\left[H^{\frac{1}{2}}\left(\mathbb{R}_+, L^2(\Omega)\right), H^1\left(\mathbb{R}_+, L^2(\Omega)\right)\right]_{1-r} = H^{\frac{r+1}{2}}\left(\mathbb{R}_+, L^2(\Omega)\right).$$

Then, S is linear continuous from $H_0^{rm}(\Omega)$ into $H^{\frac{r+1}{2}}\left(\mathbb{R}_+, L^2(\Omega)\right)$. This ends the proof of Theorem 2. □

Remark 1. Note that the anisotropic Hilbert spaces used in this work may be extended to the case of parabolic Sobolev spaces built on L^p, $p \neq 2$. An idea for this extension can be found in [5].

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Author's address

Arezki Kheloufi
Bejaia University, Department of Technology, Lab. of Applied Mathematics, 06000 Bejaia, Algeria
E-mail address: arezkinet2000@yahoo.fr

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OPTIMAL CONTROL PROBLEMS FOR SOME CLASSES OF FUNCTIONAL-DIFFERENTIAL EQUATIONS ON THE SEMI-AXIS

O. KICHMARENKO AND O. STANZHYTSKYI

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Abstract. In this paper we study functional-differential equations on the semi-axis, which are non-linear with respect to the phase variables and linear with respect to the control. Sufficient conditions for existence of optimal control in terms of the right-hand side and the quality criterion are obtained. Relation between the solutions of the problems on infinite and finite intervals is studied and results that about these connections are proven.

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1. Introduction

Let $h > 0$ be a constant, describing the delay. By $|\cdot|$ we denote a vector norm in R^d, and by $||\cdot||$ the norm of $d \times m$–matrices, which agrees with the vector norm. We introduce the necessary functional spaces which we use in this paper. Let $C = C([-h,0]; R^d)$ be the Banach space of continuous functions from $[-h,0]$ into R^d with the uniform norm $||\varphi||_C = \max\limits_{\theta\in[-h,0]} |\varphi(\theta)|$, and let $L_p = L_p([-h,0]; R^m)$, $p > 1$ be the Banach space of p-integrable m-dimensional vector-valued functions with the norm $||\varphi||_{L_p} = \left(\int_{-h}^{0} |\varphi(s)|^p ds\right)^{1/p}$.

Let x be continuous function on $[0,\infty)$ and let $\varphi \in C$. If $x(0) = \varphi(0)$, then the function

$$x(t,\varphi) = \begin{cases} \varphi(t), & t \in [-h,0] \\ x(t), & t \geq 0 \end{cases}$$

is continuous for $t \geq 0$. In the standard way (see [11]) for each $t \geq 0$ we can introduce an element $x_t(\varphi) \in C$ by the expression $x_t(\varphi) = x(t+\theta,\varphi)$, $\theta \in [-h,0]$. Further, instead of $x_t(\varphi)$ we write x_t.

Let $t \in [0,\infty)$, and D be a domain in $[-h,\infty) \times C$ with boundary ∂D.

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In this paper, we study optimal control problems for systems of functional-differential equations ($\dot{x} = dx(t)/dt$)

$$\dot{x}(t) = f_1(t, x_t) + \int_{-h}^{0} f_2(t, x_t, y)\, u(t, y) dy,\ t \in [0, \tau],\ x(t) = \varphi_0(t),\ t \in [-h, 0], \quad (1.1)$$

with one of the next cost criterion

$$J[u] = \int_0^{\tau} \left(e^{-\gamma t} A(t, x_t) + B(t, u(t, \cdot))\right) dt \ \to inf, \quad (1.2)$$

$$J[u] = \int_0^{\tau} \left(e^{-\gamma t} A(t, x_t) + \int_{-h}^{0} |u(t, y)|^2 dy\right) \to inf. \quad (1.3)$$

These problems are considered on the infinite horizon $t \geq 0$. Here $\varphi_0 \in C$ is a fixed element such that $(0, \varphi_0) \in D$, $x(t)$ is the phase vector in R^d, and x_t is the corresponding phase vector in C, τ is the moment when (t, x_t) reaches the boundary ∂D for the first time or $\tau = \infty$ otherwise. Also, $f_1 : D \to R^d$, $f_2 : D \times [-h, 0] \to M^{d \times m}$ – $d \times m$-dimensional matrix, such that for each $(t, \varphi) \in D$ $f_2(t, \varphi, \cdot)$ belongs to the space $L_q([-h, 0]; M^{d \times m})$ with the norm

$$||f_2(t, \varphi)||_{L_q} = \left(\int_{-h}^{0} ||f_2(t, \varphi, y)||^q dy\right)^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1$$

$A : D \to R^+$, $B : [0, \infty) \times L_p \to R^+$ are given mappings.

The control parameter $u \in L_p([0, \infty) \times [-h, 0])$ is m-dimensional vector function such that for almost all $(t, y), u(t, y) \in W$, $0 \in W$, where W is a convex and closed set in R^m.

For each control function, we define corresponding solution (trajectory) of (1.1). A continuous function $x(t)$ is a solution of (1.1) on the interval $[-h, T]$, if it satisfies the following conditions: $x(t) = \varphi_0(t), t \in [-h, 0]$; $(t, x_t) \in D$ for $t \in [0, T]$; for $t \in [0, T]$ $x(t)$ satisfies the integral equation

$$x(t) = \varphi_0(0) + \int_0^t [f_1(s, x_s) + \int_{-h}^{0} f_2(s, x_s, y)\, u(s, y)\, dy] ds.$$

The control function $u(t, \cdot)$ is considered admissible for the problems (1.1)-(1.2) and (1.1), (1.3) if $u(t, y) \in L_p([0, \infty) \times [-h, 0]$; $u(t, y) \in W$ for almost all $t \geq 0$, $y \in [-h, 0]$; the solution $x(t)$ corresponding to the control $u(t, \cdot)$ exists on the interval $[-h, \tau], \tau > 0$; $|J[u]| < \infty$.

Let $V(\varphi_0)$ denote the Bellman function for the problem on the infinite horizon and let $V_T(\varphi_0)$ be the Bellman function for the corresponding problem on some finite interval $[0, T]$.

In [13] it was shown that system (1.1) includes as particular cases the usual optimal control problem for functional-differential equations

$$\dot{x}(t) = f(t, x_t) + g(t, x_t) u(t), \quad u \in L_p\left([0,\infty); R^m\right), \tag{1.4}$$

for equations with maximum and for system of ordinary differential equations.

The choice of the control $u(t,\cdot) \in L_p([0,\infty);[-h,0])$ for each t as an element of the function space is justified (determined) by two reasons:

1) the given problem to be similar to the general functional-operator form of an optimal control problem where $u(t) \in W$ and W is a topological space (see, for example, [2]).

2) the given class of problems includes some problems with applications to economics (see [4, 6]).

A great number of publications is devoted to the study functional differential systems [1, 5, 8, 11, 14] and also a lot of publications is devoted to the study of optimal control problems of type (1.4). In the monograph [14], optimal control problems for functional-differential equations are studied and method for dynamic programming and maximum principle are developed. However, in most of the cases, these methods give only necessary conditions for optimality. In the cases when they give sufficient conditions, the verification of those conditions is quite complicate and requires involving new objects, which were not presented in the initial problem. That is why, it is desirable to have the sufficient conditions for existence of optimal control in terms of the right-hand side of the system and the cost criterion. In this direction, we can mention the work [3], where under condition of compactness for the set of values of the admissible controls is obtained analogue of Filippov theorem. In the case when the set of the control values is unbounded, it is obtained analogue of the Cessari theorem. Note that if the condition for compactness of the control set is removed, then this leads to condition of the growth, which connect the right-hand side of the system and the function of quality criterion. In [9, 10], the authors study the problem for optimal control of the system

$$\dot{x}(t) = \tau x(t) + f_0\left(x(t), \int_{-T}^{0} a(y) x(t+y)\, dy\right) - u(t).$$

In [10] for some cost criteria Hamilton-Jacobi-Bellman equations are obtained and in the terms of their solutions sufficient conditions for optimality are obtained. In [7], similar questions are considered for problem with phase space restrictions. In [9], under the condition that the function $\tau x + f_0(x,y)$ is non-decreasing in both variables for quality criterion

$$J[u] = \int_0^{\infty} e^{-\rho t} u^{1-\sigma}(t) x(t)\, dt, \quad \sigma \in (0;1)$$

sufficient conditions for optimality are obtained. In [13], problem of type (1.1)-(1.2) is considered in more general settings, but only on finite interval $[0,T]$.

The goal of this work is to generalize the results obtained in [13] to the infinite horizon $[0,\infty)$ and to clarify the relation between problems on finite and infinite intervals. It turns out that by means of optimal control for finite interval, it is possible to construct easily minimizers for the problem on infinite horizon.

This paper is organized in the following way. In Section 2 we give a rigorous statement of the considered problem and the main result. Section 3 is devoted to the proof of the main result of this paper. In Subsection 3.1 a theorem for existence of optimal control for the problems (1.1), (1.2) and (1.1), (1.3) is proven. In Subsection 3.2 a theorem about the connection between the solution of the problem on infinite horizon $[0,\infty)$ and the solution of same problem on finite intervals is proven. In Subsection 3.3 existence of optimal control in the case when the domain D is unbounded is proven.

2. Statement of the problems and main results

We give rigorous statement of the problem and statement of the main result of this work. In this paper, we assume that the following conditions are satisfied. Let D be a domain in $[-h,\infty)\times C$, and ∂D be its boundary (see, for example [12] p. 18). We introduce the notations $D_t=\{\varphi\in C,(t,\varphi)\in D\}$, $D_c=\bigcup_{t\geq 0}D_t$, where D_c is bounded in C.

Assumption 1. The admissible controls are m-dimensional vector functions $u(t,y)\in L_p([0,\infty)\times[-h,0];R^m)$, such that for almost all $t\geq 0$ and $y\in[-h,0]$ we have $u(t,y)\in W$, where W is a convex closed set in R^m and $0\in W$ and there exists $J[u]$.

The set of admissible controls is denoted by $\mathcal{U}$.

Assumption 2. The mappings $f_1(t,\varphi):D\to R^d$ and $f_2(t,\varphi,y):D\times[-h,0]\to M^{d\times m}$ are defined and measurable with respect to all arguments in the domain D and $D_1=\{(t,\varphi)\in D,y\in[-h,0]\}$, respectively. Moreover, these functions satisfy in D and D_1, with respect to φ the condition for linear growth and the Lipchitz condition, i.e., there exists constant $K>0$, such that

$$|f_1(t,\varphi)|+\|f_2(t,\varphi,y)\|\leq K(1+\|\varphi\|_C), \tag{2.1}$$

for $(t,\varphi)\in D,\ y\in[-h,0]$,

$$|f_1(t,\varphi_1)-f_1(t,\varphi_2)|+\|f_2(t,\varphi_1,y)-f_2(t,\varphi_2,y)\|\leq K\|\varphi_1-\varphi_2\|_C, \tag{2.2}$$

for $(t,\varphi_1),(t,\varphi_2)\in D$.

Assumption 3. 1) The mapping $A:D\to R$, $A(t,\varphi)\geq 0$ for $(t,\varphi)\in D$ is defined and continuous in D and for $(t,\varphi)\in D$ there is a constant $K_A>0$, such that $A(t,\varphi)\leq K_A(1+\|\varphi\|_C)$;

2) The mapping $B : [0,\infty) \times L_p \to R$ is measurable with respect to all its arguments and there are constants $a > 0,\ a_1 > 0$, such that $a_1 \|z\|_{L_p}^p \geq B(t,z) \geq a\|z\|_{L_p}^p$ if $t \geq 0$;

3) For each $t \geq 0,\ B(t,z)$ is strongly differentiable with respect to z and for $t \geq 0$ and $z \in L_p$ the Frechet derivative $\frac{\partial B}{\partial z}$ satisfies the estimate

$$\left\| \frac{\partial B}{\partial z} \right\|_{\mathcal{L}(L_p;R^1)} \leq a_2 \|z\|_{L_p}^{p-1}$$

for some constant $a_2 > 0$, independently of t and z. Here $\|\cdot\|_{\mathcal{L}(L_p;R^1)}$ is the uniform operator norm in the space of linear continuous functionals over L_p.

The main results of this work are given by the following theorems.

Theorem 1. *Suppose that Assumptions 1-3 are satisfied. Then there exists a solution* (x^*, u^*) *of the problems* (1.1), (1.2) *and* (1.1), (1.3).

Let $T > 0$ be fixed. By (x_T^*, u_T^*) we denote the solution of the problems (1.1), (1.2) or (1.1), (1.3) on $[0,T]$.

For the problem on infinite horizon, we define

$$u_{T,\infty}(t,\cdot) = \begin{cases} u_T^*(t,\cdot), & t \in [0,T] \\ 0, & t > T \end{cases} \tag{2.3}$$

and $x^{T,\infty}(t)$ is the corresponding trajectory.

It is obvious that the given control is admissible for the original problem. Again, $(u^*(t,\cdot), x^*(t))$ is an optimal pair for the problem (1.1)-(1.2), τ – the time at which the solution x_t^* reaches the boundary ∂D.

Theorem 2. *Suppose that Assumptions 1-3 are satisfied, then we have:*

1)
$$V_T(\varphi_0) \to V(\varphi_0), \quad T \to \infty;$$

2) *there is a sequence* $T_n \to \infty,\ n \to \infty$, *such that the sequence* $\{u_{T_n,\infty}\}$ *is minimizer for the problem* (1.1), (1.2) *i.e.*

$$J\left[u_{T_n,\infty}\right] \to V, \quad n \to \infty, \tag{2.4}$$

3) *there is a sequence* $T_n \to \infty,\ n \to \infty$, *such that*

$$u_{T_n,\infty} \xrightarrow{w} u^*, \quad n \to \infty \tag{2.5}$$

weekly in $L_p([0,\infty) \times [-h,0]; R^m)$

4) *pointwise on* $[0,\tau^*]$, *uniformly on each finite interval*

$$x^{T_n,\infty}(t) \to x^*(t), \quad n \to \infty.$$

If the problem (1.1)-(1.2) *has unique solution, then the convergence in* (2.4), (2.5) *occurs for all* $T \to \infty$.

Proposition 1. *In the conditions of Theorem 2 for the functional (1.3) all statements of Theorem 2 are valid, and the weak convergence of optimal controls (2.5) is replaced with strong convergence in* $L_2([0,\infty)\times[-h,0];R^m)$.

The next theorem is about the case when the domain D_c in the statement of the problem is unbounded. As it is shown in [13], the solution of the original problem cannot go to infinity in finite time. However, it can increase without bound in such a way, that the integrals in (1.2) and (2.2) become divergent for all admissible controls. Now we give a theorem, which guarantees existence of optimal control in this case. So, we assume that it is possible that D is unbounded domain in $[-h,\infty)\times C$ but the set of control values W is bounded in R^m. Without loss of generality, we can assume that W is a ball with radius r.

Theorem 3. *If the conditions of Theorem 1 are satisfied and* $\gamma<(hr+1)K$, *then the problems (1.1), (1.2) and (1.1), (1.3) have solutions.*

3. Proofs of the theorems

3.1. *Proof of Theorem 1*

Proof. First, we note that in the conditions of Theorem 1 imply that the conditions of Theorem 2.1 in the work [13] are satisfied. Therefore, the solution of the problem (1.1) exists, it is unique, and it can be extended to the boundary of the domain D. Further, we note that the set $\mathcal{U}$ of admissible controls is nonempty, since $0\in\mathcal{U}$. Moreover, if $x(t,0)$ is a solution of the system (1.1), which exists for such control, and $x_t(0)$ is the corresponding element in C. Then by the condition 1) of Assumption 3 and the boundedness of D_c we have $J[0]<\infty$ here r is the radius of the ball, which contains D_c.

Since the quality functionals is non-negative quantity, then there is non-negative lower bound m of the values of $J[u]$ and therefore there is a sequence of admissible controls $\{u_n(t)\}$. Let $u^{(n)}(t,y)$ be minimizing sequence, such that $J\left[u^{(n)}\right]\to m$, $n\to\infty$ monotonically. Also, let $x^{(n)}(t)$ be a sequence of solutions of the equation (1.1), for which there exists controls $u^{(n)}$, and let $[-h,\tau_n]$ be the maximal intervals of their existence.

Note that $(\tau_n,x^{(n)}_{\tau_n})\in\partial D$. It is easy to check

$$m+1\geq a\int_0^{\tau_n}\int_{-h}^0\left|u^{(n)}(t,y)\right|^p dy\,dt\geq a\int_0^{\infty}\int_{-h}^0\left|u^{(n)}(t,y)\right|^p dy\,dt, \tag{3.1}$$

for n large enough. Hence, the sequence $u^{(n)}(t,y)$ is weakly compact in $L_p([0,\infty)\times[-h,0])$. This means that it contains a weakly convergent subsequence.

Without loss of generality, we can assume that $u^{(n)}(t,y)$ itself is weakly convergent to $u^*\in L_p([0,\infty)\times[-h,0])$. By the Mazur Lemma ([15], ch. V), some

convex combination $b_k(t,y) = \sum_{i=1}^{n(k)} \alpha_i(k) u^{(i)}(t,y)$ of elements $u^{(i)}(t,y)$ converges strongly to u^* in $L_p([0,\infty)\times[-h,0])$. Therefore, there exists a subsequence $b_{k_j}(t,y)$ of the sequence $b_k(t,y)$, such that it converges to $u^*(t,y)$ almost everywhere in $[0,\infty)\times[-h,0]$. Since W is convex and closed, then $b_{k_j}(t,y)\in W$, and for this reason $u^*(t,y)\in W$, therefore, the control $u^*(t,y)$ is admissible.

Now we consider the sequence of solutions $x^{(n)}(t)$ of the system (1.1), which correspond to the controls $u^{(n)}(t,y)$. When for $t\in[0,\tau_n]$ we have

$$x^{(n)}(t) = \varphi_0(0) + \int_0^t \left[f_1\left(s, x_s^{(n)}\right) + \int_{-h}^0 f_2(s,x_s,y)\, u^{(n)}(s,y)\, dy \right] ds. \tag{3.2}$$

Using the functions $x^{(n)}(t)$ we construct the functions $z^{(n)}(t)$, which are determined on the semi-axis in the following way:

$$z^{(n)}(t) = \begin{cases} x^{(n)}(t), & t\in[0,\tau_n] \\ x^{(n)}(\tau_n), & t>\tau_n. \end{cases},$$

Since D_C is bounded, then there is $C>0$, such that

$$\left| z^{(n)}(t) \right| \le C, \quad t\ge 0. \tag{3.3}$$

We choose an arbitrary $T>0$ and fixed it. We are going to show that the family of functions $\left\{z^{(n)}(t)\right\}$ is compact on $[0,T]$. To do that, by (3.3), it is enough to prove that they are equicontinuous. For $t_1,t_2\in[0,\tau_n]$ from (3.2) and by (2.1) we have the estimate

$$\begin{aligned} &\left| x^{(n)}(t_2) - x^{(n)}(t_1) \right| \\ &\qquad \le K(1+C)(t_2-t_1) + K(1+C)h^{1/q}(t_2-t_1)^{1/q}\left(\frac{m+1}{a}\right)^{1/p}. \end{aligned}$$

Therefore, for $t_1\le t_2\le \tau_n$ and for some positive C_1, C_2 we get

$$\left| z^{(n)}(t_2) - z^{(n)}(t_1) \right| \le C_1(t_2-t_1) + C_2(t_2-t_1)^{1/q}. \tag{3.4}$$

It is easy to check if $t_1<\tau_n<t_2$, then estimate (3.4) holds. From here the equicontinuity of the family of the functions $z^{(n)}(t)$ on $[0,T]$, and therefore their compactness follows. In this way, there exists a subsequence $z_k^{(n)}(t)$ of the sequence $z^{(n)}(t)$ such that $z_k^{(n)}(t)$ converges uniformly to $z^*(t)$ on $[0,T]$. The function $z^*(t)$ is defined and continuous on $[0,T]$, and hence z_t^* exists as an element of the space C for all $t\in[0,T]$. Therefore, on each interval $[0,T]$ from the sequence $\left\{z^{(n)}(t)\right\}$ it is possible to take uniformly convergent subsequence. We show, that there exists subsequence of the sequence $\left\{z^{(n)}(t)\right\}$, which converges point-wise on $[0,\infty)$ to some continuous

function. To do that, we use the diagonal method. For $T = 1$ there exists a subsequence $\left\{z_1^{(n)}(t)\right\}$ of the sequence $\left\{z^{(n)}(t)\right\}$ such that $z_1^{(n)}(t) \rightrightarrows z_1^*(t)$ for $n \to \infty$ on $[0,1]$. For $T = 2$ there exists a subsequence $\left\{z_2^{(n)}(t)\right\}$ of the sequence $\left\{z_1^{(n)}(t)\right\}$ such that $z_2^{(n)}(t) \rightrightarrows z_2^*(t)$ for $t \in [0,2]$. We observe, that $z_2^*(t) = z_1^*(t)$ for $t \in [0,1]$. Continue this process, we obtain, for each natural number N existence of a subsequence $\left\{z_N^{(n)}(t)\right\}$ of the sequence $\left\{z_{N-1}^{(n)}(t)\right\}$ such that $z_N^{(n)}(t) \rightrightarrows z_N^*(t)$ on $[0,N]$ and $z_N^*(t) = z_{N-1}^*(t)$ for $t \in [0,N-1]$. Applying the diagonal method, from these sequences we choose the subsequences $\left\{z_n^{(n)}(t)\right\}$, such that, they obviously converge point-wise for $t \in [0,\infty)$ to the continuous function $z^*(t)$, determined in the following way: $z^*(t) = z_N^*(t)$ on $[0,N]$, where N is a natural number. For convenience, in our next considerations, we again denote the sequence $\left\{z_n^{(n)}(t)\right\}$ by $\{z^n(t)\}$, and the corresponding sequence of controls as $\left\{u^{(n)}(t)\right\}$. Since $z^*(t)$ is defined and continuous in $[0,\infty)$, then z_t^* exists as an element of the space C for all $t \geq 0$. Also, we note that $z_N^*(t)$ converges uniformly to $z^*(t)$ on each interval $[0,\tau]$.

Let τ^* be the moment when (t, z_t^*) reaches the boundary ∂D for the first time, then

$$\tau^* = \begin{cases} inf\ \{t \geq 0 : (t, z_t^*) \in \partial D \\ \infty, \ \left(t, z_t^*\right) \in D,\ t \geq 0 \end{cases}$$

Notice that $z_{\tau_n}^{(n)} = z^{(n)}(\tau_n + \theta) = x^{(n)}(\tau_n + \theta) = x_{\tau_n}^{(n)}$, and hence τ_n is the moment when $(t, z_t^{(n)})$ reaches ∂D for the first time.

We are going to show that $\tau^* \leq \lim\limits_{n\to\infty} inf\, \tau_n$.We consider two cases:

1) Let $\tau^* < \infty$. Then $\tau^* > \lim\limits_{n\to\infty} inf\, \tau_n = \tau$.

We choose an arbitrary $T \in [0,\infty)$ such that $T \geq \tau^*$. Obviously, there is a subsequence $\left\{\tau_{n_k}\right\}$ of the sequence $\{\tau_n\}$, such that $\tau_{n_k} \to \tau$ for $n_k \to \infty$. So, for n_k large enough, we have $\tau_{n_k} < \tau^*$ and

$$(\tau_{n_k}, z_{\tau_{n_k}}^*) \in D, \ \left(\tau, z_\tau^*\right) \in D, \tag{3.5}$$

but $(\tau_{n_k}, z_{\tau_{n_k}}^{(n_k)}) \in \partial D$.

On the other hand, taking into account the uniform convergence on $[-h,T]$ of the sequences $z^{(n)}(t)$ to $z^*(t)$ and the uniform continuity of $z^*(t)$ on $[-h,T]$ we have $z_{\tau_{n_k}}^{(n_k)} \to z_\tau^*$, in C. Indeed,

$$\sup_{-h\leq\theta\leq 0} \left|z^{(n_k)}(\tau_{n_k}+\theta) - z^*(\tau+\theta)\right| \leq \sup_{-h\leq\theta\leq 0} \left|z^{(n_k)}(\tau_{n_k}+\theta) - z^*(\tau_{n_k}+\theta)\right|$$

$$+ \sup_{-h\leq\theta\leq 0} \left|z^*(\tau_{n_k}+\theta) - z^*(\tau+\theta)\right| \to 0,\ n_k \to \infty,$$

Since the set ∂D is closed, then $(\tau, z^*_\tau) \in \partial D$. This contradicts (3.5).

2) Let $\tau^* = \infty$, and $\lim\limits_{n\to\infty} inf\, \tau_n < \infty$. Choose $T_2 > 0$, $T_2 > \lim\limits_{n\to\infty} inf\, \tau_n$. Now applying analogous reasoning on $[0, T_2]$, we obtain, that this case can be reduced to the previous one. Hence $\tau^* \leq \lim\limits_{n\to\infty} inf\, \tau_n = \tau$.

Set $x^*(t) = z^*(t)$ for $t \in [0, \tau^*]$ in the case of finite τ^* and $t \in [0, \infty)$ for $\tau^* = \infty$. We will show that $x^*(t)$ is a solution of the equation (1.1), to which corresponds the control $u^*(t, y)$ for all considered t.

We consider three cases.

1. If $\tau < \infty$, then the proof is analogous to the proof of the corresponding fact in [13], Theorem 2.2.

2. Let $\tau = \infty$, but $\tau^* < \infty$. In this case, either there exists subsequence $\{\tau_{n_k}\}$ of the sequence $\{\tau_n\}$ such that, $\tau_{n_k} \to \infty$, $n_k \to \infty$ or $\tau_n < \infty$ only for finite numbers. Then for large enough n_k, we have $z^{(n_k)}(t) = x^{(n_k)}(t)$ for $t \in [0, \tau^*]$ and $x^{(n_k)}(t) \rightrightarrows x^*(t)$ for $n_k \to \infty$ in $[0, \tau^*]$. From here the proof is similar to the proof of [13], Theorem 2.2.

3. Let $\tau^* = \infty$. We choose an arbitrary $T > 0$ and consider the interval $[0, T]$. Analogically to the previous case, there exists subsequence $\{\tau_{n_k}\}$ such that $\tau_{n_k} \to \infty$ for $n_k \to \infty$. Then on $[0, T]$ for large enough n_k, we have $z^{(n_k)}(t) = x^{(n_k)}(t)$ on $[0, T]$, and therefore $x^{(n_k)}(t) \rightrightarrows x^*(t)$ in $[0, T]$ for $n \to \infty$. After that, the proof is similar to the previous.

It remains to show that the control $u^*(t, y)$ is optimal. Again, we consider two cases.

1. Let $\tau^* < \tau$. In this case either there exists a subsequence $\{\tau_{n_k}\}$ of the sequence $\{\tau_n\}$ such that $\tau_{n_k} \to \tau$ for $n_k \to \infty$, or there exist only not more than finite number of finite $\{\tau_n\}$ (in the case $\tau = \infty$). Then for n_k large enough, again we have $z^{(n_k)}(t) = x^{(n_k)}(t)$ for $t \in [0, \tau^*]$ and $x^{(n_k)}(t) \rightrightarrows x^*(t)$ for $n_k \to \infty$ on $[0, \tau^*]$.

Also, obviously, we have that for all $t \in [0, \tau^*]$.

$$\left\| x_t^{(n_k)} - x_t^* \right\|_C \to 0, \quad n_k \to \infty, \tag{3.6}$$

Then

$$\int_0^{\tau_{n_k}} e^{-\gamma t} A\left(t, x_t^{(n_k)}\right) dt + \int_0^{\tau_{n_k}} B\left(t, u^{(n_k)}(t, \cdot)\right) dt$$
$$\geq \int_0^{\tau^*} e^{-\gamma t} A\left(t, x_t^{(n_k)}\right) dt + \int_0^{\tau^*} B\left(t, u^{(n_k)}(t, \cdot)\right) dt. \tag{3.7}$$

The integrand of the first summand in (3.7) for each t tends to $A(t, x_t^*)$ by (3.6) and condition 1) from Assumption 3. From the fact that D_c is bounded and the condition of linear growth $A(t, \varphi)$(condition 1) of Assumption 3), it follows for some constant $K_1 > 0$ we have the inequality $A\left(t, x_t^{(n_k)}\right) \leq K_A(1 + K_1)$. Now using Lebesgue

dominated convergence theorem, we get that

$$\int_0^{\tau^*} e^{-\gamma t} A\left(t, x_t^{(n_k)}\right) dt \to \int_0^{\tau^*} e^{-\gamma t} A\left(t, x_t^*\right) dt, n_k \to \infty. \tag{3.8}$$

Since $B(t,u)$ is convex with respect to u, then

$$B\left(t, \vartheta\left(t,\cdot\right)\right) \geq B\left(t, u^*\left(t,\cdot\right)\right) + \left\langle B_u^{'}\left(t, u^*\left(t,\cdot\right)\right), \vartheta\left(t,\cdot\right) - u^*\left(t,\cdot\right)\right\rangle, \tag{3.9}$$

for each admissible control $\vartheta\left(t,y\right) \in \mathcal{U}$. Here $\left\langle L_u^{'}, \vartheta - u^*\right\rangle$ is the action of the linear continuous functional $L_u^{'}$ on the element $\vartheta\left(t,\cdot\right) - u^*\left(t,\cdot\right) \in L_p$. So, using condition 3) from Assumption 3 we have

$$\int_0^{\infty}\int_{-h}^{0}\left|\frac{\partial B}{\partial u}(t, u^*(t,y)\right|^q dy dt \leq a_3 \int_0^{\infty}\int_{-h}^{0}\left|u^*(t,y)\right|^p dy dt < \infty.$$

Therefore, by Riesz theorem the expression

$$\int_0^{\infty} \langle B_u^{'}(t, u^*(t,\cdot), \vartheta(t,\cdot)) - u^*(t,\cdot)\rangle dt$$

defines a linear continuous functional on $L_p([0,\infty)\times[-h,0])$. Now, let $\vartheta\left(t,\cdot\right) = u^{n_k}\left(t,\cdot\right)$ in (3.9) and using the weak convergence of $u^{n_k}\left(t,y\right)$ to $u^*\left(t,y\right)$, for the second summand in (3.7) we get the inequality

$$\lim_{n_k\to\infty} inf \int_0^{\tau^*} B\left(t, u^{(n_k)}(t,\cdot)\right) dt \geq \int_0^{\tau^*} B\left(t, u^*(t,\cdot)\right) dt\ . \tag{3.10}$$

From (3.7), (3.8) and (3.10) we have

$$\begin{aligned} m &= \lim_{n\to\infty}\left(\int_0^{\tau_{n_k}} e^{-\gamma t} A\left(t, x_t^{(n_k)}\right) dt + \int_0^{\tau_{n_k}} B\left(t, u^{(n_k)}(t,\cdot)\right) dt\right) \\ &\geq \int_0^{\tau^*} e^{-\gamma t} A\left(t, x_t^*\right) dt + \int_0^{\tau^*} B\left(t, u^*(t,\cdot)\right) dt, \end{aligned}$$

therefore, in this case the control $u^*(t,y)$ is optimal.

2. Let $\tau^* = \tau$. We choose an arbitrary $t_1 = \tau^*$ and consider the interval $[0,t_1]$. On this interval $x^{(n_k)}(t) \rightrightarrows x^*(t)$, when $n_k \to \infty$, and hence $x_t^{(n_k)} \to x_t^*$ in C, $n_k \to \infty$.

By the theorem for characterization of the lower bound, the set $\{\, n \in N \,|\, \tau_n < t_1 \,\}$ is finite, and the open interval (t_1, τ^*) can contain infinite number of points τ_n (if they are finite). We consider this sequence. Then by analogy as in the previous case, we obtain

$$m \geq \int_0^{t_1} e^{-\gamma t} A\left(t, x_t^*\right) dt + \int_0^{t_1} B\left(t, u^*(t,\cdot)\right) dt$$

From here by taking limit for $t_1 \to \tau^*$ we obtain that $J[u^*] = m$.

For the functional (1.3) the proof is similar. In this case, the condition 3) from Assumption 3 is satisfied automatically, since $\frac{\partial B}{\partial z} = 2u(t,\cdot)$. This proves the theorem. □

3.2. *Proof of Theorem 2*

Proof. First, we consider the problem (1.1)-(1.2). We choose an arbitrary $T > 0$ and fix it. As before, $V(\varphi_0)$ denotes the Bellman function for the given problem and $V_T(\varphi_0)$ denotes the Bellman function for the corresponding problem on $[0,T]$. From Theorem 1 and the corresponding theorem from [13], it follows that these problems have solutions $(x^*(t), u^*(t,\cdot))$ and $(x^{*,T}(t), u^*_T(t,\cdot))$ respectively. Note that the set of admissible controls on $[0,\infty)$ is a subset of admissible controls on $[0,T]$. From the admissible controls $u(t,\cdot)$ on $[0,T]$, which are not admissible on $[0,\infty)$ we construct the following controls

$$u_{T,\infty}(t,\cdot) = \begin{cases} u(t,\cdot), & t \in [0,T] \\ 0, & t > T. \end{cases} \tag{3.11}$$

Let $\mathcal{U}_T$ denote the union of the set of admissible controls on $[0,\infty)$ with the set of controls of type (3.11). Then on $[0,\infty)$ this set of admissible controls coincides with the set $\mathcal{U}$, and on the interval $[0,T]$ it coincides with the set of all admissible controls for the problems of type (1.1)-(1.2) on $[0,T]$. Indeed, from an arbitrary admissible control $u(t,\cdot)$ on $[0,T]$, which is not admissible on $[0,\infty)$ by the rule (3.11) we construct an admissible control on $[0,\infty)$. On the other side, $L_p([0,\infty)) \subset L_p([0,T])$.

Let $D_T = D \cap [-h,T]$, and we denote by τ^*_T the moment when $x^{*,T}_t$ reaches the boundary of domain D_T. Note that in the case $\tau^*_T < T$ the control $u^*_{T,\infty}(t,\cdot)$ will be optimal for the problem (1.1), (1.2) on $[0,\infty)$. Then we conclude that $V = V_T$.

Consider now the case $\tau^*{}_T = T$. Denote by $x(t) = x(t, u^*_{T,\infty}(t,\cdot))$ the solution of the initial problem (1.1), which corresponding to the control $u^*_{T,\infty}(t,\cdot)$. We note that if $t \in [0,T]$ $u^*(t,\cdot) = u^*_T(t,\cdot)$, then by uniqueness of the solution of the initial value problem (1.1) $x(t) = x^{*,T}(t)$ for $t \in [0,T]$. Thus, from the definition of the Bellman function and using condition 2) from Assumption 3, we have

$$\begin{aligned} V \le J\left(u^*_{T,\infty}\right) &= \int_0^T (e^{-\gamma t} A(t, x^{*,T}_t) + B\left(t, u^*_T(t,\cdot)\right) dt + \int_T^{\tau_T} e^{-\gamma t} A(t, x_t)\, dt \\ &= V_T + \int_T^{\tau_T} e^{-\gamma t} A(t, x_t)\, dt. \end{aligned} \tag{3.12}$$

Here τ_T denotes the moment when x_t reaches the boundary of the domain D. The second term in (3.12) goes to zero for $T \to \infty$ by the boundedness of D_c and by the Lebesgue dominated convergence theorem.

We recall, that τ^* is the moment when the optimal trajectory x^*_t of the problem (1.1)-(1.2) on $[0,\infty)$, reaches the boundary of D. Also, note that if $\tau^* \le T$, then the

pair $(x^*(t), u^*(t,\cdot))$ will be optimal for the problem on the interval $[0,T]$ and in this case again we have $V = V_T$. Let $\tau^* > T$, then

$$V = J[u^*] \geq V_T + \int_T^{\tau^*} (e^{-\gamma t} A(t, x_t) + B(t, u^*(t,\cdot)))dt.$$

But for $T \to \infty$

$$\int_T^{\tau^*} (e^{-\gamma t} A(t, x_t^*) + B(t, u^*(t,\cdot)))dt \leq \int_T^{\infty} e^{-\gamma t} K_A (1 + K_1)\, dt + \int_T^{\infty} a_1 \|u^*(t,\cdot)\|_{L_p}^p dt$$

$$= \int_T^{\infty} e^{-\gamma t} K_A (1 + K_1)\, dt + \int_T^{\infty} a_1 \int_{-h}^{0} |u^*(t,y)|^p dy dt \to 0. \tag{3.13}$$

Then, on one side, from (3.12) we have $V - V_T \leq \int_T^{\tau_T} e^{-\gamma t} A(t, x_t)\, dt$, and on the other side, we obtain $V - V_T \geq \int_T^{\tau_T} (e^{-\gamma t} A(t, x_t^*) + B(t, u^*(t,\cdot)))dt$, from here, taking into account (3.13), we get the statement 1) of the Theorem 2, namely (2).

Further, for convenience, without loss of generality, we can assume that $T = n \in N$ is a natural number. Let $u_n^*(t,\cdot)$ be an optimal control on $[0,n]$, and let $u_{n,\infty}^*(t,\cdot)$ be an admissible control of the problem (1.1), (1.2) on $[0,\infty)$, which is determined by the formula (2.3).

Again, if $\tau_n^* < n$ for some n, then $u_{n,\infty}^*$ is optimal for the problem on infinite horizon. Here τ_n^* is the moment when $x_t^{n,\infty}$ reaches the boundary of the domain D.

Now let $\tau_n^* = n$ for all n. Since $V_n = J[u_n^*] \to V$, then there exists a constant L, such that $V_n \leq L$. But from the conditions 2) of Assumption 3 and from (2.3), we have:

$$L \geq V_n = I[u_n^*] \geq a \int_0^n \int_{-h}^0 |u_n^*(t,y)|^p dy dt = a \int_0^{\infty} \int_{-h}^0 |u_{n,\infty}^*(t,y)|^p dy dt,$$

and therefore, the sequence of admissible controls $\{u_{n,\infty}^*\}$ is weekly compact in $L_p([0,\infty)\times[-h,0])$. Therefore, there exists a weekly convergent subsequence, which without loss of generality, we again denote by $\{u_{n,\infty}^*\}$. Moreover, we have

$$u_{n,\infty}^* \xrightarrow{\omega} u^*, \quad n \to \infty, \quad L_p([0,\infty)\times[-h,0]). \tag{3.14}$$

Analogically to Theorem 1, by using the Mazur lemma, we get that $u^*(t,y) \in W$ for almost all (t,y).

We denote by $x^{n,\infty}(t)$ the solution of our original problem (1.1), corresponding to the control $u_{n,\infty}^*(t,\cdot)$. Let τ_n be the moment when the solution $x_t^{n,\infty}$ reaches the boundary of the domain D. It is obvious, that $\tau_n > n$. Then, we have

$$J[u_{n,\infty}] = V_n + \int_n^{\tau_n} (e^{-\gamma t} A(t, x_t^{n,\infty}) + B(t, u_{n,\infty}^*(t,\cdot)))dt.$$

From the construction of the sequence $u_{n,\infty}$ and condition 2) from Assumption 3, we have $B\left(t, u^*_{n,\infty}(t,\cdot)\right) = 0$ for $t \geq n$. By condition 1) of Assumption 3, we have

$$J\left[u_{n,\infty}\right] = V_n + \int_n^{\tau_n} e^{-\gamma t} A\left(t, x_t^{n,\infty}\right) dt, \tag{3.15}$$

but $\int_n^{\tau_n} e^{-\gamma t} A\left(t, x_t^{n,\infty}\right) dt \leq \int_n^{\tau_n} e^{-\gamma t} K_A \left(1 + \left\|x_t^{n,\infty}\right\|_c\right) dt$. Since the set D_c is bounded, then, as it was shown before, the estimation $\left\|x_t^{n,\infty}\right\| \leq K_1$ for $t \leq \tau_n$ holds. Then, we get

$$\int_n^{\tau_n} e^{-\gamma t} K_A (1 + K_1)\, dt \leq \int_n^{\infty} e^{-\gamma t} K_A (1 + K_1)\, dt \to 0, \;\; n \to \infty. \tag{3.16}$$

From (3.15) and (3.16) we have that

$$J\left[u_{n,\infty}\right] \to V, \;\; n \to \infty. \tag{3.17}$$

Therefore, $\left\{u^*_{n,\infty}\right\}$ is a minimizing sequence for the problem (1.1), (1.2). This proves the statement 2) of the Theorem 2. We denote by $x^*(t)$ the solution of the initial problem (1.1), with corresponding control $u^*(t,\cdot)$ from (3.14). It follows, from theorem 2.1 in [13], that such solution exists and it is unique. The statement that the pair $(u^*(t,\cdot), x^*(t))$ is optimal for the problem (1.1), (1.2) can be proven in the similar way as in the corresponding proof of Theorem 1. The statement 3) of this theorem, now becomes obvious. The proof of the statement 4) can be carried out in the similar way as the proof of the corresponding fact of the Theorem 1. If the problem (1.1), (1.2) has unique solution, then the convergence in (2.4) and (2.5) holds for all $T \to \infty$. Obviously, the later follows from the fact that from each subsequence $\left\{u^*_{n_k,\infty}\right\}$ of the sequence $\left\{u^*_{n,\infty}\right\}$ in (3.14) we can choose a sequence that weekly converges to the optimal control $u^*(t,\cdot)$ and this control is unique. This proves the theorem. □

3.3. *Proof of Proposition 1*

Proof. We consider the optimal control problem (1.1), (1.3). Obviously, the proof only requires to establish the fact that the sequence $u^*_{T_n,\infty}$ converges strongly to u^*. In the similar way as in the Theorem 1, we have

$$V \geq \int_0^{\tau^*} A\left(t, x_t^*\right) e^{-\gamma t} dt \; + \lim_{n\to\infty} \int_0^{\tau_n} \int_{-h}^{0} \left|u^*_{n,\infty}(t,y)\right|^2 dy\, dt, \tag{3.18}$$

where the last limit in (3.18) exists, and therefore coincides with its lower bound. From the construction of $u^*_{n,\infty}$ it follows that

$$\int_0^{\tau_n} \int_{-h}^{0} \left|u^*_{n,\infty}(t,y)\right|^2 dy\, dt = \int_0^{\infty} \int_{-h}^{0} \left|u^*_{n,\infty}(t,y)\right|^2 dy\, dt.$$

Thus, from (3.18) we have

$$V \geq \int_0^{\tau^*} e^{-\gamma t} A\left(t, x_t^*\right) dt + \int_0^{\tau^*} \int_{-h}^0 \left|u^*(t,y)\right|^2 dy dt = V$$

In this way, we get

$$\lim_{n\to\infty} \int_0^\infty \int_{-h}^0 \left|u_{n,\infty}^*(t,y)\right|^2 dt = \int_0^\infty \int_{-h}^0 \left|u^*(t,y)\right|^2 dy dt.$$

From this, taking into account (3.14) the strong convergence of $u_{n,\infty}^*$ to u^* in $L_2([0,\infty)\times(-h,0])$ follows, which proves Proposition 1. □

3.4. *Proof of Theorem 3*

Proof. Assuming the conditions of this theorem, we consider the problem (1.1), (1.3). First, we show that the set of admissible controls $\mathcal{U}$ is non-empty. To do that, as in Theorem 1, we show that $0 \in \mathcal{U}$.

Indeed, let $x(t,0)$ be a solution of the system (1.1), corresponding to the control $u = 0$, let x_t be the corresponding element from C, τ be the moment of its exit on the boundary of D.

For $t \in [0,\tau)$ we write the integral representation of $x(t,0)$

$$x(t,0) = \varphi_0(0) + \int_0^t f_1(s, x_s) ds. \tag{3.19}$$

From here, taking into account (2.1), we obtain:

$$|x(t)| \leq |\varphi_0(0)| + \int_0^t K(1 + \|x_s\|_c) ds \leq |\varphi_0(0)| + K\int_0^t (1 + \max_{s_1\in[-h,s]} |x(s_1)|) ds. \tag{3.20}$$

From (3.20) we have, that

$$\max_{s\in[0,t]} |x(s)| \leq |\varphi_0(0)| + K\int_0^t (1 + \|x_s\|)\, ds,$$

and

$$\max_{s\in[-h,t]} |x(s)| \leq 2 \max_{s\in[-h,0]} |\varphi_0(s)| + Kt + K\int_0^t \max_{s_1\in[-h,s]} |x(s_1)|\, ds,$$

and therefore, also

$$\max_{t\in[0,\tau]} \|x_t(0)\|_c \leq \left(2 \max_{s\in[-h,0]} |\varphi(t)| + Kt\right) e^{Kt}. \tag{3.21}$$

From here, using the condition 1) of Theorem 3, we have

$$J(0) = \int_0^\tau e^{-\gamma t} A(t, x_t(0)) dt \leq \int_0^\tau e^{-\gamma t} K_A \left(1 + 2 \max_{t\in[-h,0]} \varphi_0(t) + Kt\right) e^{Kt} dt < \infty,$$

by virtue of (2.5).

Since the functional of quality is non-negative, then again there exists a non-negative lower bound m for the value of $J[u]$ and let $\left\{u^{(n)}(t,\cdot)\right\}$ be minimizing sequence such that

$$J\left(u^{(n)}\right) \to m, \quad n \to \infty. \tag{3.22}$$

Let $\left\{x^{(n)}(t)\right\}$ be a sequence of solutions of the equation (1.1), with corresponding controls $u^{(n)}$, and let $[-h,\tau_n)$ be the maximal intervals of their existence, $\left(\tau_n, x^n_{\tau_n}\right) \in \partial D$. Analogously to Theorem 1, we again obtain that the sequence $\left\{u^{(n)}(t,\cdot)\right\}$ is weekly compact in $L_p\left([0,\infty)\times(-h,0]\right)$. Moreover, the estimation (3.1) holds. Without loss of generality, we can assume that the sequence $\left\{u^{(n)}(t,\cdot)\right\}$ converges weekly to $u^*(t,\cdot) \in L_p\left([0,\infty)\times[-h,0]\right)$. As in the proof of Theorem 1, we have that $u^*(t,\cdot) \in \mathcal{U}$ – the set of admissible controls for the problem (1.1)-(1.2).

Analogously to (3.2) for $x^{(n)}(t)$ we have the integral representation

$$x^{(n)}(t) = \varphi_0(0) + \int_0^t \left[f_1(s,x_s) + \int_{-h}^0 f_2(s,x_s,y)u^{(n)}(s,y)dy \right] ds,$$

from which, taking into account (2.1) and the boundedness of the set W, similarly to (3.21), we have

$$\max_{s\in[-h,]} \left|x^{(n)}(s)\right| \le b_1 + b_2 t + (hRK+K)\int_0^t \max_{s_1\in[-h,s]} \left|x^{(n)}(s_1)\right| ds$$

for some positive constant b_1, b_2. Then, using the Gronwall's inequality, we obtain $\max\limits_{s\in[-h,t]} \left|x^{(n)}(s)\right| \le (b_1+b_2t)\,e^{(hRK+K)t}$, from which we get

$$\max_{s\in[0,t]} \left\|x^{(n)}_s\right\| \le (b_1+b_2t)\,e^{(hRk+K)t}. \tag{3.23}$$

So, we conclude that $x^{(n)}_t$ cannot reach the infinite boundary ∂D for finite time. In other words, $x^{(n)}_t$ can go to infinity only when $t\to\infty$. The later allows us, to construct on an arbitrary interval $[0,T]$ the sequence of functions

$$z^{(n)}(t) = \begin{cases} x^{(n)}(t), t\in[0,\tau_n] \\ x^{(n)}(\tau_n), t\in[\tau_n,T], \end{cases}$$

if $\tau_n \le T$, and if $\tau_n > T$, then $z^{(n)}(t) = x^{(n)}(t)$.

Similarly to Theorem 1, we can show that the sequence $\left\{z^{(n)}(t)\right\}$ contains a subsequence, which converges point-wise on $[0,\infty)$ to some continuous function $z^*(t)$, and this convergence is uniform on each finite interval $[0,T]$. Again, we can assume that the sequence $\left\{z^{(n)}(t)\right\}$ itself has this property. We denote by τ^* – the moment when z_t reaches ∂D. Let $x^*(t) = z^*(t)$ then for $t\in[0,\tau^*]$, we can show (as in

Theorem 1), that $x^*(t)$ is a solution of the equation (1.1), to which corresponds the control $u^*(t,\cdot)$. The proof that the pair $(u^*(t,\cdot), x^*(t))$ is optimal can be done in the similar way as the proof of Theorem 1, where the passage of the limit under the integral of type (3.8) is possible by Lebesgue dominated convergence theorem. From the estimation (3.23) and the condition for linear growth of $A(t,\varphi)$ we have

$$e^{-\gamma t} A\left(t, x_t^{(n_k)}\right) \leq K_A e^{-\gamma t}\left(1 + \|x_t\|_c\right) \leq K_A e^{-\gamma t}\left(b_1 + b_2 t\right) e^{(hRK+K)t}.$$

If the condition (2.5) of the Theorem is taken into account, then the quantity $K_A e^{-\gamma t}\left(b_1 + b_2 t\right) e^{(hRK+K)t}$ is now integrable upper bound. This proves the theorem. □

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Authors' addresses

O. Kichmarenko

Odesa National I.I.Mechnikov University, Optimal Control and Economical Cybernatics Department, 2, Dvoryanskaya Str., Odesa, 65082, Ukraine

E-mail address: olga.kichmarenko@gmail.com

O. Stanzhytskyi

Taras Shevchenko National University of Kyiv, Department of General Mathematics, 64/13, Volodymyrska Str., Kyiv, 01601, Ukraine

E-mail address: ostanzh@gmail.com

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NEW RECIPROCAL SUMS INVOLVING FINITE PRODUCTS OF SECOND ORDER RECURSIONS

EMRAH KILIÇ AND DIDEM ERSANLI

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Abstract. In this paper, we present new kinds of reciprocal sums of finite products of general second order linear recurrences. In order to evaluate explicitly them by q-calculus, first we convert them into their q-notation and then use the methods of partial fraction decomposition and creative telescoping.

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1. Introduction

For $n > 1$, define the second order linear recurrences $\{U_n\}$ and $\{V_n\}$ with

$$U_n = pU_{n-1} + U_{n-2} \text{ and } V_n = pV_{n-1} + V_{n-2},$$

where $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = p$, respectively.

If $p = 1$, then $U_n = F_n$ (nth Fibonacci number) and $V_n = L_n$ (nth Lucas number). If $p = 2$, then $U_n = P_n$ (nth Pell number) and $V_n = Q_n$ (nth Pell-Lucas number).

The Binet forms are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^{n-1}(1-q^n)}{(1-q)} \text{ and } V_n = \alpha^n + \beta^n = \alpha^n(1+q^n),$$

where $\alpha, \beta = (p \pm \sqrt{p^2+4})/2$, $q = \beta/\alpha$ and $\mathbf{i} = \sqrt{-1}$.

Throughout this paper we will use the following notations: the q-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1})$. When $x = q$, we denote $(q;q)_n$ by $(q)_n$.

Many authors [1–11] have studied both finite or infinite and alternating or non-alternating reciprocal sums including terms of certain integer sequences. More recently, Frontczak [3] evaluated various reciprocal sums of the Fibonacci numbers. For example, he showed that for $m, n \geq 1$

$$\sum_{i=1}^{N} (-1)^{m(i+1)} \frac{F_{mi+n\mp 1}}{F_{m(i-1)+n} F_{mi+n} F_{m(i+1)+n}} = \frac{F_{m\mp 1}}{F_n F_m F_{2m}}$$

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$$\times\left(\frac{F_{m(N+1)}}{F_{m(N+1)+n}}+\frac{F_{mN}}{F_{mN+n}}-\frac{F_m}{F_{m+n}}\right)-(-1)^m\frac{F_{mN}}{F_m^2F_{m+n}F_{m(N+1)+n}}.$$

Kılıç and Prodinger [5] consider some classes of reciprocal sums of general Fibonacci numbers, which were computed in closed form in an earlier work and then they evaluate the same sums by a different approach: First they convert them into their q-forms and then explicitly evaluate q-versions of the sums by using partial fraction decomposition method and creative telescoping idea.

In this paper, we shall consider new kinds of reciprocal sums including finite products of the general Fibonacci and Lucas numbers. We shall summarize what we will present in this paper below.

- We consider three kinds of *alternating* reciprocal sums including finite products of the general Fibonacci and Lucas numbers. All of them have an integer parameter to increase or decrease the indices of the general Fibonacci or Lucas factors in both numerator and denominator in the sums. Two of these sums are of the forms

$$\sum_{k=0}^{n}(-1)^k\frac{U_{k-d}}{U_{k+d}U_{k+d+1}U_{k+d+2}} \text{ and } \sum_{k=0}^{n}(-1)^k\frac{V_{k+d+1}}{U_{k+d}U_{k+d+1}U_{k+d+2}}.$$

  The other kind sums has an additional parameter m which determines the number of the general Fibonacci or Lucas factors in the numerator or denominator. It will be of the form

$$\sum_{k=0}^{n}(-1)^k\frac{U_{k+c}U_{k+c+1}\cdots U_{k+c+m-1}}{X_{k+d}X_{k+d+1}\cdots X_{k+d+m+1}},$$

  where X_n is either U_n or V_n.
- Before all, we convert all the claimed results into their q-forms. After this, we will use partial fraction decomposition (pfd) method and creative telescoping idea to prove the q-version of the claimed results.
- By the q-versions of three kinds of reciprocal sums, we present general cases of the original three kinds of sums with an additional integer parameter by taking a special choosing of q. By the way, we could able to derive *non-alternating* reciprocal sums where the indices of the general Fibonacci or Lucas numbers are in the arithmetic progressions.
- Finally we shall give some applications of our results to certain alternating and non-alternating reciprocal sums with finite products of the Pell or Pell-Lucas numbers.

2. The main results

Now we are going to present our main results. We start with our first main result.

Theorem 1. *For $n,m \geq 0$ and $c \in \{-m+1,\dots,-1,0,1\}$,*

$$\sum_{k=0}^{n}(-1)^k \frac{U_{k+c}U_{k+c+1}\dots U_{k+c+m-1}}{X_{k+d}X_{k+d+1}\dots X_{k+d+m+1}} = (-1)^{c+1}\frac{U_{n+c}U_{n+c+1}\dots U_{n+c+m}}{U_{m+1}X_{d-c+1}\left[X_{n+d+1}X_{n+d+2}\dots X_{n+d+m+1}\right]},$$

where X_n is either U_n or V_n. Note that when $X_n = V_n$, we assume that d is any integer. When $X_n = U_n$, we assume that $d \geq 1$.

Proof. Let $X_n = U_n$. First we convert the LHS of the claim into its q-notation as shown

$$\sum_{k=0}^{n}(-1)^k \frac{\prod_{t=0}^{m-1} U_{k+t+c}}{\prod_{t=0}^{m+1} U_{k+t+d}} = \alpha^{mc-(m+2)d-2m+1}(1-q)^2 \sum_{k=0}^{n} \frac{q^k \prod_{t=0}^{m-1}\left(1-q^{k+c+t}\right)}{\prod_{t=0}^{m+1}(1-q^{k+d+t})}.$$

Second we convert the RHS of the claim into its q-notation as shown

$$(-1)^{c+1}\frac{\prod_{t=0}^{m} U_{n+t+c}}{U_{m+1}U_{d-c+1}\prod_{t=1}^{m+1} U_{n+t+d}}$$

$$= (-1)^{c+1}\alpha^{(m+2)c-(m+2)d-2m-1}\frac{(1-q)^2\prod_{t=0}^{m}\left(1-q^{n+c+t}\right)}{(1-q^{m+1})(1-q^{d-c+1})\prod_{t=1}^{m+1}(1-q^{n+d+t})}.$$

After some simplifications, the q-version of the claimed result is

$$\sum_{k=0}^{n} \frac{q^k \prod_{t=0}^{m-1}\left(1-q^{k+c+t}\right)}{\prod_{t=0}^{m+1}(1-q^{k+d+t})} = \frac{q^{1-c}\prod_{t=0}^{m}\left(1-q^{n+c+t}\right)}{(1-q^{m+1})(1-q^{d-c+1})\prod_{t=1}^{m+1}(1-q^{n+d+t})}$$

or, in terms of the q-Pochhammer notation,

$$\sum_{k=0}^{n} \frac{q^k\left(q^{k+c};q\right)_m}{\left(q^{k+d};q\right)_{m+2}} = \frac{q^{1-c}\left(q^{n+c};q\right)_{m+1}}{(1-q^{m+1})(1-q^{d-c+1})\left(q^{n+d+1};q\right)_{m+1}}.$$

Define

$$S_n := \sum_{k=0}^{n} \frac{q^k \left(q^{k+c};q\right)_m}{\left(q^{k+d};q\right)_{m+2}}.$$

Denote the summand term of S_n by T_k, that is,

$$T_k = \frac{q^k \left(q^{k+c};q\right)_m}{\left(q^{k+d};q\right)_{m+2}}.$$

The partial fraction decomposition of T_k reads as

$$\frac{q^k \left(q^{k+c};q\right)_m}{\left(q^{k+d};q\right)_{m+2}} = \sum_{t=1}^{m+2} \frac{q^{-d-t+1}\left(q^{c-d-t+1};q\right)_m}{(1-q^{k+d+t-1}) \prod\limits_{\substack{i=1\\ i\neq t}}^{m+2} (1-q^{i-t})}.$$

From [5], we could obtain the following identity comes from creative telescoping idea:

$$\sum_{k=0}^{n} \left(\frac{1}{1-q^{k+d+m}} - \frac{1}{1-q^{k+d+t}} \right) = \sum_{k=0}^{m-t-1} \left(\frac{1}{1-q^{k+d+n+t+1}} - \frac{1}{1-q^{k+d+t}} \right). \tag{2.1}$$

In that case, by using the equality (2.1), we write

$$\begin{aligned}
S_n &= \sum_{t=1}^{m+1} \left(\frac{1}{1-q^{d+t-1}} - \frac{1}{1-q^{d+n+t}} \right) \sum_{r=1}^{t} \frac{q^{-d-r+1}\left(q^{c-d-r+1};q\right)_m}{\left(q^{1-r};q\right)_{r-1} (q)_{m-r+2}} \\
&= (-1)^m \frac{1}{1-q^{m+1}} \sum_{t=1}^{m+1} \left(\frac{1}{1-q^{d+t-1}} - \frac{1}{1-q^{d+n+t}} \right) \\
&\quad \times q^{(m-t+1)c-(m-t+2)d+t(t-1)+\frac{m(m-2t+1)}{2}} \frac{\left(q^{c-d-2};q\right)_{3-t} \left(q^{d-c};q\right)_{t-m+1}}{(q)_{t-1}(q)_{m-t+1}} \\
&= (-1)^m \frac{1-q^{n+1}}{1-q^{m+1}} \sum_{t=1}^{m+1} \frac{q^{t(-c+d-m+t)}}{(1-q^{d+t-1})(1-q^{d+n+t})} \\
&\quad \times q^{c-1-d+\frac{1}{2}m(m+1+2c-2d)} \frac{\left(q^{c-d-2};q\right)_{3-t} \left(q^{d-c};q\right)_{t-m+1}}{(q)_{t-1}(q)_{m-t+1}} \\
&= \frac{1-q^{n+1}}{(1-q^{c-d-1})(1-q^{m+1})}
\end{aligned}$$

$$\times \sum_{t=1}^{m+1} (-1)^{t+1} \frac{q^{\binom{t+1}{2}-1} \left(q^{c-d-1};q\right)_{2-t} \left(q^{c-d};q\right)_{m-t+1}}{(1-q^{d+t-1})(1-q^{d+n+t})\,(q)_{t-1}\,(q)_{m-t+1}}$$

$$= \frac{1-q^{n+1}}{(1-q^{c-d-1})(1-q^{m+1})}$$

$$\times \sum_{t=0}^{m} (-1)^{t} \frac{q^{\binom{t+2}{2}-1} \left(q^{c-d-1};q\right)_{1-t} \left(q^{c-d};q\right)_{m-t}}{(1-q^{d+t})(1-q^{d+n+t+1})\,(q)_{t}\,(q)_{m-t}}$$

$$= \frac{1-q^{n+1}}{(1-q^{m+1})(1-q^{c-d-1})} \sum_{t=0}^{m} (-1)^{t} \frac{q^{\binom{t}{2}}}{(q^{-t}-q^{d})(q^{-t}-q^{d+n+1})}$$

$$\times \frac{(1-q^{c-d-t-1})(1-q^{c-d-t})\dots(1-q^{c-d+m-t-2})(1-q^{c-d+m-t-1})}{(q)_{t}\,(q)_{m-t}}.$$

Now we define

$$h(z) := \frac{(1-zq^{c-d-1})\dots(1-zq^{c-d+m-1})}{(z-q^{d})(z-q^{d+n+1})(1-z)(1-zq)\dots(1-zq^{m})}.$$

The partial fraction decomposition of $h(z)$ is

$$\begin{aligned} h(z) \;=\; & \frac{(1-q^{c-1})\dots(1-q^{c+m-1})}{(z-q^{d})(q^{d}-q^{d+n+1})(1-q^{d})(1-q^{d+1})\dots(1-q^{d+m})} \\ & + \frac{(1-q^{c+n})\dots(1-q^{c+n+m})}{(q^{d+n+1}-q^{d})(z-q^{d+n+1})(1-q^{d+n+1})(1-q^{d+n+2})\dots(1-q^{d+n+m+1})} \\ & + \sum_{t=0}^{m} (-1)^{t} q^{\binom{t+1}{2}} \frac{(1-q^{-t+c-d-1})\dots(1-q^{-t+c-d+m-1})}{(q^{-t}-q^{d})(q^{-t}-q^{d+n+1})\,(q)_{t}\,(q)_{m-t}\,(1-zq^{t})}. \end{aligned}$$

If we multiply $h(z)$ by z and let $z \to \infty$, then we get

$$\begin{aligned} 0 = & \frac{q^{-d}(1-q^{c-1})\dots(1-q^{c+m-1})}{(1-q^{n+1})(1-q^{d})(1-q^{d+1})\dots(1-q^{d+m})} \\ & - \frac{q^{-d}(1-q^{c+n})\dots(1-q^{c+n+m})}{(1-q^{n+1})(1-q^{d+n+1})(1-q^{d+n+2})\dots(1-q^{d+n+m+1})} \\ & + \sum_{t=0}^{m} (-1)^{t+1} q^{\binom{t}{2}} \frac{(1-q^{-t+c-d-1})\dots(1-q^{-t+c-d+m-1})}{(q^{-t}-q^{d})(q^{-t}-q^{d+n+1})\,(q)_{t}\,(q)_{m-t}}. \end{aligned}$$

Since $c \in \{-m+1,\dots,-1,0,1\}$, we write

$$\sum_{t=0}^{m} (-1)^{t} q^{\binom{t}{2}} \frac{(1-q^{-t+c-d-1})\dots(1-q^{-t+c-d+m-1})}{(q^{-t}-q^{d})(q^{-t}-q^{d+n+1})\,(q)_{t}\,(q)_{m-t}}$$

$$= -\frac{q^{-d}(1-q^{n+c})\dots(1-q^{c+m+n})}{(1-q^{n+1})(1-q^{d+n+1})(1-q^{d+n+2})\dots(1-q^{d+m+n+1})}.$$

Taking into the constant factor, we write

$$\begin{aligned}
&\frac{1-q^{n+1}}{(1-q^{m+1})(1-q^{c-d-1})}\\
&\quad\times\sum_{t=0}^{m}(-1)^t q^{\binom{t}{2}+1}\frac{(1-zq^{c-d-1})\dots(1-zq^{c-d+m-1})}{(z-q^d)(z-q^{d+n+1})\,(q)_t\,(q)_{m-t}}\\
&= -\frac{q^{-d}(1-q^{n+c})\dots(1-q^{c+m+n})}{(1-q^{m+1})(1-q^{c-d-1})(1-q^{d+n+1})\dots(1-q^{d+m+n+1})}\\
&= -\frac{q^{-d}\left(q^{n+c};q\right)_{m+1}}{(1-q^{m+1})(1-q^{c-d-1})\left(q^{n+d+1};q\right)_{m+1}}\\
&= \frac{q^{1-c}\left(q^{n+c};q\right)_{m+1}}{(1-q^{m+1})(1-q^{d-c+1})\left(q^{n+d+1};q\right)_{m+1}},
\end{aligned}$$

which completes the proof.

Also, when $X_n = V_n$, the proof is similarly obtained. □

Now we will present some interesting corollaries of Theorem 1. When $m = 2$, $X_n = L_n$, $U_n = F_n$, $d = 3$ and $c = 0$ in Theorem 1, it gives us

$$\sum_{k=0}^{n}(-1)^k\frac{F_kF_{k+1}}{L_{k+3}L_{k+4}L_{k+5}L_{k+6}} = -\frac{F_nF_{n+1}F_{n+2}}{F_3L_4L_{n+4}L_{n+5}L_{n+6}}.$$

When $m = 3$, $X_n = U_n = P_n$, $d = 5$ and $c = 1$ in Theorem 1, we get

$$\sum_{k=0}^{n}(-1)^k\frac{P_{k+1}P_{k+2}P_{k+3}}{P_{k+5}P_{k+6}P_{k+7}P_{k+8}P_{k+9}} = \frac{P_{n+1}P_{n+2}P_{n+3}P_{n+4}}{P_4P_5P_{n+6}P_{n+7}P_{n+8}P_{n+9}}.$$

When $m = 4$, $X_n = Q_n$, $U_n = P_n$, $d = 4$ and $c = -2$ in Theorem 1, we get

$$\sum_{k=0}^{n}(-1)^k\frac{P_{k-2}P_{k-1}P_kP_{k+1}}{Q_{k+4}Q_{k+5}Q_{k+6}Q_{k+7}Q_{k+8}Q_{k+9}} = -\frac{P_{n-2}P_{n-1}P_nP_{n+1}P_{n+2}}{P_5Q_7Q_{n+5}Q_{n+6}Q_{n+7}Q_{n+8}Q_{n+9}}.$$

When $m = 5$, $X_n = U_n = F_n$, $d = 5$ and $c = -3$ in Theorem 1, we get

$$\sum_{k=0}^{n}(-1)^k\frac{F_{k-3}F_{k-2}F_{k-1}F_kF_{k+1}}{F_{k+5}F_{k+6}F_{k+7}F_{k+8}F_{k+9}F_{k+10}F_{k+11}}$$

$$= \frac{F_{n-3}F_{n-2}F_{n-1}F_nF_{n+1}F_{n+2}}{F_6F_9F_{n+6}F_{n+7}F_{n+8}F_{n+9}F_{n+10}F_{n+11}}.$$

Now we shall give our second result. The first main result stands for the finite product of the general Fibonacci or Lucas numbers. But the next two results are valid for special cases.

Theorem 2. *For $d > 0$,*

$$\sum_{k=0}^{n}(-1)^k \frac{U_{k-d}}{U_{k+d}U_{k+d+1}U_{k+d+2}} = (-1)^{d+1}\frac{U_{2n+2}}{U_2U_{n+d+1}U_{n+d+2}}. \tag{2.2}$$

Proof. First we convert the LHS of (2.2) into its q-notation as shown

$$\sum_{k=0}^{n}(-1)^k \frac{U_{k-d}}{U_{k+d}U_{k+d+1}U_{k+d+2}}$$
$$= \alpha^{-4d-1}(1-q)^2\sum_{k=0}^{n}\frac{q^k(1-q^{k-d})}{(1-q^{k+d})(1-q^{k+d+1})(1-q^{k+d+2})}.$$

Second we convert the RHS of (2.2) into its q-notation as shown

$$(-1)^{d+1}\frac{U_{2n+2}}{U_2U_{n+d+1}U_{n+d+2}} = \alpha^{-2d-1}(1-q)^2\frac{(-1)^{d+1}(1-q^{2n+2})}{(1-q^2)(1-q^{d+n+1})(1-q^{d+n+2})}.$$

After some simplifications, q-version of the claimed result is

$$\sum_{k=0}^{n}\frac{q^k(1-q^{k-d})}{(1-q^{k+d})(1-q^{k+d+1})(1-q^{k+d+2})} = -\frac{q^{-d}(1-q^{2n+2})}{(1-q^2)(1-q^{d+n+1})(1-q^{d+n+2})}.$$

Define

$$S_n := \sum_{k=0}^{n}\frac{z(1-zq^{-d})}{(1-zq^d)(1-zq^{d+1})(1-zq^{d+2})}.$$

Denote the summand term of S_n by $T(z)$, that is,

$$T(z) = \frac{z(1-zq^{-d})}{(1-zq^d)(1-zq^{d+1})(1-zq^{d+2})}.$$

The partial fraction decomposition of $T(z)$ reads as

$$\frac{z(1-zq^{-d})}{(1-zq^d)(1-zq^{d+1})(1-zq^{d+2})}$$
$$= \frac{1}{q^{3d+1}(1-q)^2(1+q)}\left(-\frac{q(1-q^{2d})}{1-zq^d} + \frac{(1+q)(1-q^{2d+1})}{1-zq^{d+1}} - \frac{1-q^{2d+2}}{1-zq^{d+2}}\right)$$
$$= \frac{1}{q^{3d+1}(1-q)^2(1+q)}\left[q(1-q^{2d})\left(\frac{1}{1-zq^{d+2}} - \frac{1}{1-zq^d}\right)\right.$$

$$-(1+q)(1-q^{2d+1})\left(\frac{1}{1-zq^{d+2}}-\frac{1}{1-zq^{d+1}}\right)\Bigg].$$

And so

$$S_n=\frac{1}{q^{3d+1}(1-q)^2(1+q)}\Bigg[q(1-q^{2d})\sum_{k=0}^{n}\left(\frac{1}{1-zq^{d+2}}-\frac{1}{1-zq^{d}}\right) \tag{2.3}$$
$$-(1+q)(1-q^{2d+1})\sum_{k=0}^{n}\left(\frac{1}{1-zq^{d+2}}-\frac{1}{1-zq^{d+1}}\right)\Bigg].$$

If we take $m=2$, $t=0$, and, $m=2$, $t=1$ in (2.1), respectively, then we rewrite S_n given in (2.3) as

$$\frac{1}{q^{3d+1}(1-q)^2(1+q)}\Bigg[q(1-q^{2d})\sum_{k=0}^{1}\left(\frac{1}{1-q^{k+d+1+n}}-\frac{1}{1-q^{k+d}}\right)$$
$$-(1+q)(1-q^{2d+1})\left(\frac{1}{1-q^{d+n+2}}-\frac{1}{1-q^{d+1}}\right)\Bigg]$$
$$=-\frac{\left(1+q^{n+1}\right)\left(1-q^{d+1}\right)(1-q^{n+1})}{q^{d}\left(1-q^{d+1}\right)\left(1-q^{d+n+1}\right)\left(1-q^{d+n+2}\right)(1-q)(1+q)}$$
$$=-\frac{q^{-d}(1-q^{2n+2})}{\left(1-q^{2}\right)(1-q^{d+n+1})(1-q^{d+n+2})}.$$

Thus, we have the conclusion. □

When $U_n=F_n$ and $d=3$ in (2.2), we obtain

$$\sum_{k=0}^{n}(-1)^k\frac{F_{k-3}}{F_{k+3}F_{k+4}F_{k+5}}=\frac{F_{2n+2}}{F_{n+4}F_{n+5}}.$$

Now we going to give our third result:

Theorem 3. *For $d>0$,*

$$\sum_{k=0}^{n}(-1)^k\frac{V_{k+d+1}}{U_{k+d}U_{k+d+1}U_{k+d+2}}=\frac{U_{n+1}U_{n+2(d+1)}}{U_1U_dU_{d+1}U_{n+d+1}U_{n+d+2}}. \tag{2.4}$$

Proof. After required converting and simplifications, we find the q-version of the claimed result as follows

$$\sum_{k=0}^{n}\frac{q^k(1+q^{k+d+1})}{(1-q^{k+d})(1-q^{k+d+1})(1-q^{k+d+2})}$$
$$=\frac{(1-q^{n+1})(1-q^{n+2(d+1)})}{(1-q)\left(1-q^{d}\right)\left(1-q^{d+1}\right)(1-q^{n+d+1})(1-q^{n+d+2})}.$$

Define

$$S_n := \sum_{k=0}^{n} \frac{z(1+zq^{d+1})}{(1-zq^d)(1-zq^{d+1})(1-zq^{d+2})}.$$

Denote the summand term of S_n by $T(z)$, that is,

$$T(z) = \frac{z(1+zq^{d+1})}{(1-zq^d)(1-zq^{d+1})(1-zq^{d+2})}.$$

The partial fraction decomposition of $T(z)$ reads as

$$\begin{aligned}
&\frac{z(1+zq^{d+1})}{(1-zq^d)(1-zq^{d+1})(1-zq^{d+2})} \\
&= \frac{1}{q^d(1+q)(1-q)^2}\left(\frac{1+q}{1-zq^d} - \frac{2(1+q)}{1-zq^{d+1}} + \frac{1+q}{1-zq^{d+2}}\right) \\
&= \frac{-1}{q^d(1+q)(1-q)^2}\left[(1+q)\left(\frac{1}{1-zq^{d+2}} - \frac{1}{1-zq^d}\right)\right. \\
&\left.-2(1+q)\left(\frac{1}{1-zq^{d+2}} - \frac{1}{1-zq^{d+1}}\right)\right].
\end{aligned}$$

And so

$$\begin{aligned}
S_n =& \frac{-1}{q^d(1+q)(1-q)^2}\left[(1+q)\sum_{k=0}^{n}\left(\frac{1}{1-zq^{d+2}} - \frac{1}{1-zq^d}\right)\right. \\
&\left.-2(1+q)\sum_{k=0}^{n}\left(\frac{1}{1-zq^{d+2}} - \frac{1}{1-zq^{d+1}}\right)\right].
\end{aligned}$$

If we take $m = 2$, $t = 0$, and, $m = 2$, $t = 1$ in (2.1), respectively, then we rewrite the last equality as

$$\begin{aligned}
S_n =& \frac{-1}{q^d(1+q)(1-q)^2}\left[(1+q)\sum_{k=0}^{1}\left(\frac{1}{1-q^{k+d+1+n}} - \frac{1}{1-q^{k+d}}\right)\right. \\
&\left.-2(1+q)\left(\frac{1}{1-q^{d+n+2}} - \frac{1}{1-q^{d+1}}\right)\right] \\
=& \frac{(1-q^{n+1})(1-q^{n+2d+2})}{(1-q)(1-q^d)(1-q^{d+1})(1-q^{n+d+1})(1-q^{n+d+2})}.
\end{aligned}$$

Thus, we have the conclusion. □

When $U_n = F_n$, $V_n = L_n$ and $d = 5$ in (2.4), we obtain

$$\sum_{k=0}^{n}(-1)^k \frac{L_{k+6}}{F_{k+5}F_{k+6}F_{k+7}} = \frac{F_{n+1}F_{n+12}}{F_5F_6F_{n+6}F_{n+7}}.$$

3. General cases

In the previous section, we found the q-versions of the claimed results in Theorems 1-3 while proving them. Now we shall present more general cases of Theorems 1-3 by taking $q=\beta^s/\alpha^s$ for any integer s in the q-forms of them without proof, respectively.

Theorem 4. *For $n,m\geq 0$ and $c\in\{-m+1,\dots,-1,0,1\}$,*

$$\sum_{k=0}^{n}(-1)^{sk}\frac{\prod_{t=0}^{m-1}U_{s(k+c+t)}}{\prod_{t=0}^{m+1}X_{s(k+d+t)}}=(-1)^{s(c+1)}\frac{\prod_{t=0}^{m}U_{s(n+t+c)}}{U_{s(m+1)}X_{s(d-c+1)}\prod_{t=1}^{m+1}X_{s(n+t+d)}},\tag{3.1}$$

where X_n is either U_n or V_n. Note that when $X_n=V_n$, we assume that d is any integer. When $X_n=U_n$, we assume that $d\geq 1$.

In (3.1), if we take $m=3$, $s=2$, $U_n=P_n$, $X_n=Q_n$, $d=4$ and $c=1$, then we obtain

$$\sum_{k=0}^{n}\frac{P_{2k+2}P_{2k+4}P_{2k+6}}{Q_{2k+8}Q_{2k+10}Q_{2k+12}Q_{2k+14}Q_{2k+16}}=\frac{P_{2n+2}P_{2n+4}P_{2n+6}P_{2n+8}}{P_8Q_8Q_{2n+10}Q_{2n+12}Q_{2n+14}Q_{2n+16}}.$$

When $m=4$, $s=-1$, $U_n=X_n=F_n$, $d=6$ and $c=0$ in Theorem 4, then, by $F_{-n}=(-1)^{n+1}F_n$, we obtain

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\frac{F_{-k}F_{-k-1}F_{-k-2}F_{-k-3}}{F_{-k-6}F_{-k-7}F_{-k-8}F_{-k-9}F_{-k-10}F_{-k-11}}\\&=\sum_{k=0}^{n}(-1)^k\frac{F_kF_{k+1}F_{k+2}F_{k+3}}{F_{k+6}F_{k+7}F_{k+8}F_{k+9}F_{k+10}F_{k+11}}\\&=-\frac{F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}}{F_5F_7F_{n+7}F_{n+8}F_{n+9}F_{n+10}F_{n+11}}.\end{aligned}$$

When $m=5$, $s=3$, $U_n=F_n$, $X_n=L_n$, $d=2$ and $c=-3$ in Theorem 4, then we obtain

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\frac{F_{3k-9}F_{3k-6}F_{3k-3}F_{3k}F_{3k+3}}{L_{3k+6}L_{3k+9}L_{3k+12}L_{3k+15}L_{3k+18}L_{3k+21}L_{3k+24}}\\&=\frac{F_{3n-9}F_{3n-6}F_{3n-3}F_{3n}F_{3n+3}F_{3n+6}}{F_{18}L_{18}L_{3n+9}L_{3n+12}L_{3n+15}L_{3n+18}L_{3n+21}L_{3n+24}}.\end{aligned}$$

Theorem 5. *For $d > 0$,*

$$\sum_{k=0}^{n}(-1)^{sk}\frac{U_{s(k-d)}}{U_{s(k+d)}U_{s(k+d+1)}U_{s(k+d+2)}} = (-1)^{sd+1}\frac{U_{s(2n+2)}}{U_{2s}U_{s(n+d+1)}U_{s(n+d+2)}}. \tag{3.2}$$

If we take $U_n = F_n$, $d = 2$ and $s = 5$ in (3.2), then we obtain

$$\sum_{k=0}^{n}(-1)^{k}\frac{F_{5k-10}}{F_{5k+10}F_{5k+15}F_{5k+20}} = -\frac{F_{10n+10}}{F_{10}F_{5n+15}F_{5n+20}}.$$

Theorem 6. *For $d > 0$,*

$$\sum_{k=0}^{n}(-1)^{sk}\frac{V_{s(k+d+1)}}{U_{s(k+d)}U_{s(k+d+1)}U_{s(k+d+2)}} = \frac{U_{s(n+1)}U_{s(n+2d+2)}}{U_sU_{sd}U_{s(d+1)}U_{s(n+d+1)}U_{s(n+d+2)}}. \tag{3.3}$$

If we take $U_n = P_n$, $V_n = Q_n$, $d = 4$ and $s = -3$ in (3.3), then, by $P_{-n} = (-1)^{n+1}P_n$ and $Q_{-n} = (-1)^n Q_n$, we obtain

$$\begin{aligned}\sum_{k=0}^{n}(-1)^{k}\frac{Q_{-3k-15}}{P_{-3k-12}P_{-3k-15}P_{-3k-18}} &= \sum_{k=0}^{n}(-1)^{k}\frac{Q_{3k+15}}{P_{3k+12}P_{3k+15}P_{3k+18}}\\ &= \frac{P_{3n+3}P_{3n+30}}{P_3P_{12}P_{15}P_{3n+15}P_{3n+18}}.\end{aligned}$$

4. Conclusions

In the last section, we derived more general results, where the sign functions and the indices of Fibonacci or Lucas factors are depend on the integer parameter s. By the way, one can see that these generalized reciprocal sums are alternating for odd integer s, and, non-alternating for even integer s while the original sums in Theorems 1-3 are always alternating sums.

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Authors' addresses

Emrah Kılıç
TOBB University of Economics and Technology, Mathematics Department, 06560, Ankara, Turkey
E-mail address: ekilic@etu.edu.tr

Didem Ersanlı
TOBB University of Economics and Technology, Mathematics Department, 06560, Ankara, Turkey
E-mail address: didemersanli@gmail.com

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ON PELL HYBRINOMIALS

MIROSŁAW LIANA, ANETTA SZYNAL-LIANA, AND IWONA WŁOCH

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Abstract. Hybrid numbers generalize complex, hyperbolic and dual numbers, simultaneously. Special kinds of hybrid numbers, related to numbers of Fibonacci type, among others Pell numbers, were introduced quite recently. In this paper we introduce and study polynomials, which are a generalization of Pell hybrid numbers and so called Pell hybrinomials.

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1. Introduction

Pell numbers are well-known numbers in the number theory and they belong to the wide class of numbers of the Fibonacci type. The nth Pell number P_n is defined recursively by the second order linear recurrence relation $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$ with initial conditions $P_0 = 0$, $P_1 = 1$. A special version of Pell numbers is Pell-Lucas numbers Q_n (also named as companion Pell numbers). Then $Q_n = 2Q_{n-1} + Q_{n-2}$, for $n \geq 2$ with $Q_0 = Q_1 = 2$.

Distinct properties of Pell and Pell-Lucas numbers can be found for example in [1, 2, 5]. In [3] Horadam and Mahon introduced Pell and Pell-Lucas polynomials as follows.

For any variable quantity x, the Pell polynomial $P_n(x)$ is defined as $P_n(x) = 2x \cdot P_{n-1}(x) + P_{n-2}(x)$ for $n \geq 2$ with $P_0(x) = 0$, $P_1(x) = 1$.

The Pell-Lucas polynomial $Q_n(x)$ is defined as $Q_n(x) = 2x \cdot Q_{n-1}(x) + Q_{n-2}(x)$ for $n \geq 2$ with initial terms $Q_0(x) = 2$, $Q_1(x) = 2x$.

For $x = 1$ we obtain Pell and Pell-Lucas numbers, respectively.

For any x let $\alpha(x) = x + \sqrt{x^2+1}$ and $\beta(x) = x - \sqrt{x^2+1}$. Then solving second–order linear recurrence relations, for $P_n(x)$ and $Q_n(x)$, respectively, we have

$$P_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} \tag{1.1}$$

and

$$Q_n(x) = \alpha^n(x) + \beta^n(x). \tag{1.2}$$

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One of the generalizations of the Pell polynomial is the Horadam polynomial, whose properties can be found in [4].

Hybrid numbers were introduced by Özdemir in [6] as a new generalization of complex, hyperbolic and dual numbers.

Let $\mathbb{K}$ be the set of hybrid numbers $\mathbf{Z}$ of the form

$$\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h},$$

where $a, b, c, d \in \mathbb{R}$ and $\mathbf{i}$, ε, $\mathbf{h}$ are operators such that

$$\mathbf{i}^2 = -1,\ \varepsilon^2 = 0,\ \mathbf{h}^2 = 1 \tag{1.3}$$

and

$$\mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i}. \tag{1.4}$$

If $\mathbf{Z}_1 = a_1 + b_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h}$, and $\mathbf{Z}_2 = a_2 + b_2\mathbf{i} + c_2\varepsilon + d_2\mathbf{h}$, are any two hybrid numbers then equality, addition, substraction and multiplication by scalar are defined.

Equality: $\mathbf{Z}_1 = \mathbf{Z}_2$ only if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$,
addition: $\mathbf{Z}_1 + \mathbf{Z}_2 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\varepsilon + (d_1 + d_2)\mathbf{h}$,
substraction: $\mathbf{Z}_1 - \mathbf{Z}_2 = (a_1 - a_2) + (b_1 - b_2)\mathbf{i} + (c_1 - c_2)\varepsilon + (d_1 - d_2)\mathbf{h}$,
multiplication by scalar $s \in \mathbb{R}$: $s\mathbf{Z}_1 = sa_1 + sb_1\mathbf{i} + sc_1\varepsilon + sd_1\mathbf{h}$.

The hybrid numbers multiplication is defined using (1.3) and (1.4). Note that using formulas (1.3) and (1.4) we can find the product of any two hybrid units. The following Table presents products of $\mathbf{i}$, ε, and $\mathbf{h}$ Using rules given in Table 1. the

TABLE 1. The hybrid number multiplication.

$\cdot$	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	-1	$1-\mathbf{h}$	$\varepsilon+\mathbf{i}$
ε	$\mathbf{h}+1$	0	$-\varepsilon$
$\mathbf{h}$	$-\varepsilon-\mathbf{i}$	ε	1

multiplication of hybrid numbers can be made analogously as multiplications of algebraic expressions. For hybrid numbers details, see [6].

A special kind of hybrid numbers, namely Pell hybrid numbers and Pell-Lucas hybrid numbers, were introduced in [7] as follows.

The nth Pell hybrid number PH_n and the nth Pell-Lucas hybrid number QH_n are defined as

$$PH_n = P_n + \mathbf{i}P_{n+1} + \varepsilon P_{n+2} + \mathbf{h}P_{n+3}, \tag{1.5}$$

$$QH_n = Q_n + \mathbf{i}Q_{n+1} + \varepsilon Q_{n+2} + \mathbf{h}Q_{n+3}, \tag{1.6}$$

respectively.

Interesting results of Pell and Pell-Lucas hybrid numbers obtained recently can be found in [8].

In this paper we introduce Pell and Pell-Lucas hybrinomials, i.e. polynomials, which are a generalization of Pell hybrid numbers and Pell-Lucas hybrid numbers, respectively.

For $n \geq 0$ Pell and Pell-Lucas hybrinomials are defined by

$$PH_n(x) = P_n(x) + \mathbf{i}P_{n+1}(x) + \varepsilon P_{n+2}(x) + \mathbf{h}P_{n+3}(x) \tag{1.7}$$

and

$$QH_n(x) = Q_n(x) + \mathbf{i}Q_{n+1}(x) + \varepsilon Q_{n+2}(x) + \mathbf{h}Q_{n+3}(x), \tag{1.8}$$

where $P_n(x)$ is the nth Pell polynomial, $Q_n(x)$ is the the n-th Pell-Lucas polynomial and $\mathbf{i}$, ε, $\mathbf{h}$ are hybrid units satisfy (1.3) and (1.4).

For $x = 1$ we obtain Pell hybrid numbers and Pell-Lucas hybrid numbers, respectively.

2. Properties of Pell and Pell-Lucas hybrinomials

Theorem 1. *Let $n \geq 0$ be an integer. For any variable quantity x, we have*

$$PH_n(x) = 2x \cdot PH_{n-1}(x) + PH_{n-2}(x) \text{ for } n \geq 2 \tag{2.1}$$

with $PH_0(x) = \mathbf{i} + \varepsilon \cdot (2x) + \mathbf{h} \cdot (4x^2 + 1)$
and $PH_1(x) = 1 + \mathbf{i} \cdot (2x) + \varepsilon \cdot (4x^2 + 1) + \mathbf{h} \cdot (8x^3 + 4x)$.

Proof. If $n = 2$ we have

$$\begin{aligned}
PH_2(x) &= 2x \cdot PH_1(x) + PH_0(x) \\
&= 2x \cdot (1 + \mathbf{i} \cdot (2x) + \varepsilon \cdot (4x^2 + 1) + \mathbf{h} \cdot (8x^3 + 4x)) \\
&\quad + \mathbf{i} + \varepsilon \cdot (2x) + \mathbf{h} \cdot (4x^2 + 1) \\
&= 2x + \mathbf{i} \cdot (4x^2 + 1) + \varepsilon \cdot (8x^3 + 4x) + \mathbf{h} \cdot (16x^4 + 12x^2 + 1) \\
&= P_2(x) + \mathbf{i}P_3(x) + \varepsilon P_4(x) + \mathbf{h}P_5(x).
\end{aligned}$$

If $n \geq 3$ then using the definition of Pell polynomials we have

$$\begin{aligned}
PH_n(x) &= P_n(x) + \mathbf{i}P_{n+1}(x) + \varepsilon P_{n+2}(x) + \mathbf{h}P_{n+3}(x) \\
&= (2x \cdot P_{n-1}(x) + P_{n-2}(x)) + \mathbf{i}(2x \cdot P_n(x) + P_{n-1}(x)) \\
&\quad + \varepsilon(2x \cdot P_{n+1}(x) + P_n(x)) + \mathbf{h}(2x \cdot P_{n+2}(x) + P_{n+1}(x)) \\
&= 2x\,(P_{n-1}(x) + \mathbf{i} \cdot P_n(x) + \varepsilon \cdot P_{n+1}(x) + \mathbf{h} \cdot P_{n+2}(x)) \\
&\quad + P_{n-2}(x) + \mathbf{i} \cdot P_{n-1}(x) + \varepsilon \cdot P_n(x) + \mathbf{h} \cdot P_{n+1}(x) \\
&= 2x \cdot PH_{n-1}(x) + PH_{n-2}(x),
\end{aligned}$$

which ends the proof. □

In the same way one can easily prove the next theorem.

Theorem 2. *Let $n \geq 0$ be an integer. For any variable quantity x, we have*

$$QH_n(x) = 2x \cdot QH_{n-1}(x) + QH_{n-2}(x) \text{ for } n \geq 2 \tag{2.2}$$

with $QH_0(x) = 2 + \mathbf{i} \cdot (2x) + \varepsilon \cdot (4x^2 + 2) + \mathbf{h} \cdot (8x^3 + 6x)$ *and* $QH_1(x) = 2x + \mathbf{i} \cdot (4x^2 + 2) + \varepsilon \cdot (8x^3 + 6x) + \mathbf{h} \cdot (16x^4 + 16x^2 + 2)$.

Now we give so called Binet formulas for Pell and Pell-Lucas hybrinomials.

Theorem 3. *Let $n \geq 0$ be an integer. Then*

$$\begin{aligned} PH_n(x) = &\frac{\alpha^n(x)}{\alpha(x) - \beta(x)} \left(1 + \mathbf{i}\alpha(x) + \varepsilon\alpha^2(x) + \mathbf{h}\alpha^3(x)\right) \\ &- \frac{\beta^n(x)}{\alpha(x) - \beta(x)} \left(1 + \mathbf{i}\beta(x) + \varepsilon\beta^2(x) + \mathbf{h}\beta^3(x)\right), \end{aligned} \tag{2.3}$$

where $\alpha(x) = x + \sqrt{x^2 + 1}$ *and* $\beta(x) = x - \sqrt{x^2 + 1}$.

Proof. Using (1.1), (1.5) and (1.7) we have

$$\begin{aligned} PH_n(x) &= P_n(x) + \mathbf{i}P_{n+1}(x) + \varepsilon P_{n+2}(x) + \mathbf{h}P_{n+3}(x) \\ &= \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} + \mathbf{i}\frac{\alpha^{n+1}(x) - \beta^{n+1}(x)}{\alpha(x) - \beta(x)} \\ &\quad + \varepsilon\frac{\alpha^{n+2}(x) - \beta^{n+2}(x)}{\alpha(x) - \beta(x)} + \mathbf{h}\frac{\alpha^{n+3}(x) - \beta^{n+3}(x)}{\alpha(x) - \beta(x)} \end{aligned}$$

and after calculations the result follows. □

In the same way, using (1.2), (1.6) and (1.8), one can easily prove the next theorem.

Theorem 4. *Let $n \geq 0$ be an integer. Then*

$$\begin{aligned} QH_n(x) = &\alpha^n(x) \left(1 + \mathbf{i}\alpha(x) + \varepsilon\alpha^2(x) + \mathbf{h}\alpha^3(x)\right) \\ &+ \beta^n(x) \left(1 + \mathbf{i}\beta(x) + \varepsilon\beta^2(x) + \mathbf{h}\beta^3(x)\right), \end{aligned} \tag{2.4}$$

where $\alpha(x) = x + \sqrt{x^2 + 1}$ *and* $\beta(x) = x - \sqrt{x^2 + 1}$.

Now we will give some identities related to the well-known identities for classical Pell numbers

$$\begin{aligned} &\text{(Catalan identity)}\, P_{n-r} \cdot P_{n+r} - (P_n)^2 = (-1)^{n-r+1} P_r^2, \\ &\text{(Cassini identity)}\, P_{n-1} \cdot P_{n+1} - (P_n)^2 = (-1)^n, \\ &\text{(d'Ocagne identity)}\, P_m \cdot P_{n+1} - P_{m+1} \cdot P_n = (-1)^n P_{m-n}. \end{aligned}$$

We give their versions for Pell and Pell-Lucas hybrinomials. These identities can be proved using Binet formulas.

For simplicity of notation let

$\Delta(x) = \alpha(x) - \beta(x)$,

$\hat{\alpha}(x) = 1 + \mathbf{i}\alpha(x) + \varepsilon\alpha^2(x) + \mathbf{h}\alpha^3(x)$,
$\hat{\beta}(x) = 1 + \mathbf{i}\beta(x) + \varepsilon\beta^2(x) + \mathbf{h}\beta^3(x)$.

Then we can write (2.3) and (2.4) as

$$PH_n(x) = \frac{\alpha^n(x)}{\Delta(x)}\hat{\alpha}(x) - \frac{\beta^n(x)}{\Delta(x)}\hat{\beta}(x)$$

and

$QH_n(x) = \alpha^n(x)\hat{\alpha}(x) + \beta^n(x)\hat{\beta}(x)$, respectively.
Moreover, $\alpha(x)\cdot\beta(x) = -1$ and $\Delta^2(x) = 4x^2 + 4$.

Theorem 5 (Catalan identity for Pell hybrinomials)**.** *Let $n \geq 0$, $r \geq 0$ be integers such that $n \geq r$. Then*

$$\begin{aligned}
&PH_{n-r}(x)\cdot PH_{n+r}(x) - (PH_n(x))^2 \\
&= \frac{(-1)^n}{4x^2+4}\hat{\alpha}(x)\hat{\beta}(x)\left(1 - \frac{\beta^r(x)}{\alpha^r(x)}\right) + \frac{(-1)^n}{4x^2+4}\hat{\beta}(x)\hat{\alpha}(x)\left(1 - \frac{\alpha^r(x)}{\beta^r(x)}\right).
\end{aligned}$$

Proof. For integers $n \geq 0$, $r \geq 0$ and $n \geq r$ we have

$$\begin{aligned}
&PH_{n-r}(x)\cdot PH_{n+r}(x) - (PH_n(x))^2 \\
&= \left(\frac{\alpha^{n-r}(x)}{\Delta(x)}\hat{\alpha}(x) - \frac{\beta^{n-r}(x)}{\Delta(x)}\hat{\beta}(x)\right)\cdot\left(\frac{\alpha^{n+r}(x)}{\Delta(x)}\hat{\alpha}(x) - \frac{\beta^{n+r}(x)}{\Delta(x)}\hat{\beta}(x)\right) \\
&\quad - \left(\frac{\alpha^{n}(x)}{\Delta(x)}\hat{\alpha}(x) - \frac{\beta^{n}(x)}{\Delta(x)}\hat{\beta}(x)\right)\cdot\left(\frac{\alpha^{n}(x)}{\Delta(x)}\hat{\alpha}(x) - \frac{\beta^{n}(x)}{\Delta(x)}\hat{\beta}(x)\right) \\
&= -\frac{\alpha^{n-r}(x)}{\Delta(x)}\hat{\alpha}(x)\frac{\beta^{n+r}(x)}{\Delta(x)}\hat{\beta}(x) - \frac{\beta^{n-r}(x)}{\Delta(x)}\hat{\beta}(x)\frac{\alpha^{n+r}(x)}{\Delta(x)}\hat{\alpha}(x) \\
&\quad + \frac{\alpha^{n}(x)}{\Delta(x)}\hat{\alpha}(x)\frac{\beta^{n}(x)}{\Delta(x)}\hat{\beta}(x) + \frac{\beta^{n}(x)}{\Delta(x)}\hat{\beta}(x)\frac{\alpha^{n}(x)}{\Delta(x)}\hat{\alpha}(x) \\
&= -\frac{\alpha^{n-r}(x)\beta^{n+r}(x)}{\Delta^2(x)}\hat{\alpha}(x)\hat{\beta}(x) - \frac{\beta^{n-r}(x)\alpha^{n+r}(x)}{\Delta^2(x)}\hat{\beta}(x)\hat{\alpha}(x) \\
&\quad + \frac{\alpha^{n}(x)\beta^{n}(x)}{\Delta^2(x)}\hat{\alpha}(x)\hat{\beta}(x) + \frac{\beta^{n}(x)\alpha^{n}(x)}{\Delta^2(x)}\hat{\beta}(x)\hat{\alpha}(x) \\
&= \frac{\alpha^{n}(x)\beta^{n}(x)}{\Delta^2(x)}\hat{\alpha}(x)\hat{\beta}(x)\left(1 - \frac{\beta^r(x)}{\alpha^r(x)}\right) + \frac{\alpha^{n}(x)\beta^{n}(x)}{\Delta^2(x)}\hat{\beta}(x)\hat{\alpha}(x)\left(1 - \frac{\alpha^r(x)}{\beta^r(x)}\right) \\
&= \frac{(-1)^n}{4x^2+4}\hat{\alpha}(x)\hat{\beta}(x)\left(1 - \frac{\beta^r(x)}{\alpha^r(x)}\right) + \frac{(-1)^n}{4x^2+4}\hat{\beta}(x)\hat{\alpha}(x)\left(1 - \frac{\alpha^r(x)}{\beta^r(x)}\right),
\end{aligned}$$

which ends the proof. □

In the same way one can easily prove the next theorem, which gives Catalan identity for Pell-Lucas hybrinomials.

Theorem 6 (Catalan identity for Pell-Lucas hybrinomials)**.** *Let $n \geq 0$, $r \geq 0$ be integers such that $n \geq r$. Then*

$$\begin{aligned}&QH_{n-r}(x)\cdot QH_{n+r}(x)-(QH_n(x))^2\\&=(-1)^n\hat{\alpha}(x)\hat{\beta}(x)\left(\frac{\beta^r(x)}{\alpha^r(x)}-1\right)+(-1)^n\hat{\beta}(x)\hat{\alpha}(x)\left(\frac{\alpha^r(x)}{\beta^r(x)}-1\right).\end{aligned}$$

Note that for $r=1$ we get Cassini identities for Pell and Pell-Lucas hybrinomials. Moreover, for $r=1$ we have
$1-\frac{\beta(x)}{\alpha(x)}=\frac{\alpha(x)-\beta(x)}{\alpha(x)}=\frac{\Delta(x)}{\alpha(x)}$ and $1-\frac{\alpha(x)}{\beta(x)}=\frac{\beta(x)-\alpha(x)}{\beta(x)}=-\frac{\Delta(x)}{\beta(x)}$.

Corollary 1 (Cassini identities for Pell and Pell-Lucas hybrinomials)**.** *Let $n \geq 0$ be an integer. Then*

$$\begin{aligned}&PH_{n-1}(x)\cdot PH_{n+1}(x)-(PH_n(x))^2\\&=\frac{(-1)^{n-1}\beta(x)}{\Delta(x)}\hat{\alpha}(x)\hat{\beta}(x)-\frac{(-1)^{n-1}\alpha(x)}{\Delta(x)}\hat{\beta}(x)\hat{\alpha}(x).\end{aligned}$$

$$\begin{aligned}&QH_{n-1}(x)\cdot QH_{n+1}(x)-(QH_n(x))^2\\&=(-1)^n\hat{\alpha}(x)\hat{\beta}(x)\left(\frac{\beta(x)}{\alpha(x)}-1\right)+(-1)^n\hat{\beta}(x)\hat{\alpha}(x)\left(\frac{\alpha(x)}{\beta(x)}-1\right).\end{aligned}$$

Theorem 7 (d'Ocagne identity for Pell hybrinomials)**.** *Let $m \geq 0$, $n \geq 0$ be integers such that $m \geq n$. Then*

$$\begin{aligned}&PH_m(x)\cdot PH_{n+1}(x)-PH_{m+1}(x)\cdot PH_n(x)\\&=\frac{(-1)^n\alpha^{m-n}(x)}{\Delta(x)}\hat{\alpha}(x)\hat{\beta}(x)-\frac{(-1)^n\beta^{m-n}(x)}{\Delta(x)}\hat{\beta}(x)\hat{\alpha}(x).\end{aligned}$$

Proof. Let m,n be as in the statement of the Theorem. Then

$$\begin{aligned}&PH_m(x)\cdot PH_{n+1}(x)-PH_{m+1}(x)\cdot PH_n(x)\\&=\left(\frac{\alpha^m(x)}{\Delta(x)}\hat{\alpha}(x)-\frac{\beta^m(x)}{\Delta(x)}\hat{\beta}(x)\right)\cdot\left(\frac{\alpha^{n+1}(x)}{\Delta(x)}\hat{\alpha}(x)-\frac{\beta^{n+1}(x)}{\Delta(x)}\hat{\beta}(x)\right)\\&\quad-\left(\frac{\alpha^{m+1}(x)}{\Delta(x)}\hat{\alpha}(x)-\frac{\beta^{m+1}(x)}{\Delta(x)}\hat{\beta}(x)\right)\cdot\left(\frac{\alpha^n(x)}{\Delta(x)}\hat{\alpha}(x)-\frac{\beta^n(x)}{\Delta(x)}\hat{\beta}(x)\right)\\&=\frac{\alpha^{m+n+1}(x)}{\Delta^2(x)}\hat{\alpha}^2(x)-\frac{\alpha^m(x)\beta^{n+1}(x)}{\Delta^2(x)}\hat{\alpha}(x)\hat{\beta}(x)-\frac{\alpha^{n+1}(x)\beta^m(x)}{\Delta^2(x)}\hat{\beta}(x)\hat{\alpha}(x)\\&\quad+\frac{\beta^{m+n+1}(x)}{\Delta^2(x)}\hat{\beta}^2(x)-\frac{\alpha^{m+1+n}(x)}{\Delta^2(x)}\hat{\alpha}^2(x)+\frac{\alpha^{m+1}(x)\beta^n(x)}{\Delta^2(x)}\hat{\alpha}(x)\hat{\beta}(x)\\&\quad+\frac{\alpha^n(x)\beta^{m+1}(x)}{\Delta^2(x)}\hat{\beta}(x)\hat{\alpha}(x)-\frac{\beta^{m+1+n}(x)}{\Delta^2(x)}\hat{\beta}^2(x)\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha^{m+1}(x)\beta^{n}(x)-\alpha^{m}(x)\beta^{n+1}(x)}{\Delta^2(x)}\hat{\alpha}(x)\hat{\beta}(x)\\
&\quad+\frac{\alpha^{n}(x)\beta^{m+1}(x)-\alpha^{n+1}(x)\beta^{m}(x)}{\Delta^2(x)}\hat{\beta}(x)\hat{\alpha}(x)\\
&= \frac{\alpha^{m}(x)\beta^{n}(x)(\alpha(x)-\beta(x))}{\Delta^2(x)}\hat{\alpha}(x)\hat{\beta}(x)\\
&\quad+\frac{\alpha^{n}(x)\beta^{m}(x)(\beta(x)-\alpha(x))}{\Delta^2(x)}\hat{\beta}(x)\hat{\alpha}(x)\\
&= \frac{\alpha^{m}(x)\beta^{n}(x)}{\Delta(x)}\hat{\alpha}(x)\hat{\beta}(x)-\frac{\alpha^{n}(x)\beta^{m}(x)}{\Delta(x)}\hat{\beta}(x)\hat{\alpha}(x)\\
&= \frac{(-1)^n\alpha^{m-n}(x)}{\Delta(x)}\hat{\alpha}(x)\hat{\beta}(x)-\frac{(-1)^n\beta^{m-n}(x)}{\Delta(x)}\hat{\beta}(x)\hat{\alpha}(x).
\end{aligned}
$$

Thus the Theorem is proved. □

In the same way we can prove next theorems.

Theorem 8 (d'Ocagne identity for Pell-Lucas hybrinomials)**.** *Let $m \geq 0$, $n \geq 0$ be integers such that $m \geq n$. Then*

$$
\begin{aligned}
&QH_m(x)\cdot QH_{n+1}(x)-QH_{m+1}(x)\cdot QH_n(x)\\
&=(-1)^n\beta^{m-n}(x)\Delta(x)\hat{\beta}(x)\hat{\alpha}(x)-(-1)^n\alpha^{m-n}(x)\Delta(x)\hat{\alpha}(x)\hat{\beta}(x).
\end{aligned}
$$

Theorem 9. *Let $m \geq 0$, $n \geq 0$ be integers. Then*

$$
\begin{aligned}
&PH_m(x)\cdot QH_n(x)-QH_m(x)\cdot PH_n(x)\\
&=\frac{2(-1)^n\alpha^{m-n}(x)}{\Delta(x)}\hat{\alpha}(x)\hat{\beta}(x)-\frac{2(-1)^n\beta^{m-n}(x)}{\Delta(x)}\hat{\beta}(x)\hat{\alpha}(x).
\end{aligned}
$$

Some identities for Pell and Pell-Lucas hybrinomials can be found by analogy to well–known identities for the Pell and Pell-Lucas polynomials. In the next part of this paper we indicate such identities.

Theorem 10 ([3])**.** *Let $n \geq 1$ be an integer. Then*

$$
P_{n+1}(x)+P_{n-1}(x)=Q_n(x)=2x\cdot P_n(x)+2P_{n-1}(x). \tag{2.5}
$$

Theorem 11. *Let $n \geq 1$ be an integer. Then*

$$
PH_{n+1}(x)+PH_{n-1}(x)=QH_n(x)=2x\cdot PH_n(x)+2PH_{n-1}(x).
$$

Proof. Using (2.5) we have

$$\begin{aligned}
&PH_{n+1}(x)+PH_{n-1}(x)\\
&=P_{n+1}(x)+\mathbf{i}P_{n+2}(x)+\varepsilon P_{n+3}(x)+\mathbf{h}P_{n+4}(x)\\
&\quad+P_{n-1}(x)+\mathbf{i}P_{n}(x)+\varepsilon P_{n+1}(x)+\mathbf{h}P_{n+2}(x)\\
&=(P_{n+1}(x)+P_{n-1}(x))+\mathbf{i}(P_{n+2}(x)+P_{n}(x))\\
&\quad+\varepsilon(P_{n+3}(x)+P_{n+1}(x))+\mathbf{h}(P_{n+4}(x)+P_{n+2}(x))\\
&=Q_n(x)+\mathbf{i}Q_{n+1}(x)+\varepsilon Q_{n+2}(x)+\mathbf{h}Q_{n+3}(x)\\
&=QH_n(x).
\end{aligned}$$

On the other hand

$$\begin{aligned}
&2x\cdot PH_n(x)+2PH_{n-1}(x)\\
&=2x\cdot(P_n(x)+\mathbf{i}P_{n+1}(x)+\varepsilon P_{n+2}(x)+\mathbf{h}P_{n+3}(x))\\
&\quad+2(P_{n-1}(x)+\mathbf{i}P_{n}(x)+\varepsilon P_{n+1}(x)+\mathbf{h}P_{n+2}(x))\\
&=(2x\cdot P_n(x)+P_{n-1}(x))+\mathbf{i}(2x\cdot P_{n+1}(x)+P_{n}(x))\\
&\quad+\varepsilon(2x\cdot P_{n+2}(x)+P_{n+1}(x))+\mathbf{h}(2x\cdot P_{n+3}(x)+P_{n+2}(x))\\
&=Q_n(x)+\mathbf{i}Q_{n+1}(x)+\varepsilon Q_{n+2}(x)+\mathbf{h}Q_{n+3}(x)\\
&=QH_n(x),
\end{aligned}$$

so the result follows. □

Theorem 12 ([3]). *Let $n\geq 1$ be an integer. Then*

$$Q_{n+1}(x)+Q_{n-1}(x)=4(x^2+1)P_n(x). \tag{2.6}$$

Theorem 13. *Let $n\geq 1$ be an integer. Then*

$$QH_{n+1}(x)+QH_{n-1}(x)=4(x^2+1)PH_n(x).$$

Proof. Using (2.6) and proceeding in the same way as in the Theorem 11 the result follows. □

Theorem 14 ([3]). *Let $n\geq 2$ be an integer. Then*

$$\sum_{l=1}^{n-1}P_l(x)=\frac{P_n(x)+P_{n-1}(x)-1}{2x}. \tag{2.7}$$

Theorem 15. *Let $n\geq 2$ be an integer. Then*

$$\sum_{l=1}^{n-1}PH_l(x)=\frac{PH_n(x)+PH_{n-1}(x)-PH_0(x)-PH_1(x)}{2x}.$$

Proof. For an integer $n \geq 2$ we have

$$\begin{aligned}
&\sum_{l=1}^{n-1} PH_l(x) = PH_1(x) + PH_2(x) + \ldots + PH_{n-1}(x) \\
&= P_1(x) + \mathbf{i}P_2(x) + \varepsilon P_3(x) + \mathbf{h}P_4(x) \\
&\quad + P_2(x) + \mathbf{i}P_3(x) + \varepsilon P_4(x) + \mathbf{h}P_5(x) \\
&\quad + \cdots \\
&\quad + P_{n-1}(x) + \mathbf{i}P_n(x) + \varepsilon P_{n+1}(x) + \mathbf{h}P_{n+2}(x) \\
&= P_1(x) + P_2(x) + \cdots + P_{n-1}(x) \\
&\quad + \mathbf{i}(P_2(x) + P_3(x) + \cdots + P_n(x) + P_1(x) - P_1(x)) \\
&\quad + \varepsilon(P_3(x) + P_4(x) + \cdots + P_{n+1}(x) + P_1(x) + P_2(x) \\
&\quad - P_1(x) - P_2(x)) \\
&\quad + \mathbf{h}(P_4(x) + P_5(x) + \cdots + P_{n+2}(x) + P_1(x) + P_2(x) + P_3(x) \\
&\quad - P_1(x) - P_2(x) - P_3(x)).
\end{aligned}$$

Using (2.7) we obtain

$$\begin{aligned}
\sum_{l=1}^{n-1} PH_l(x) &= \frac{P_n(x) + P_{n-1}(x) - 1}{2x} \\
&\quad + \mathbf{i}\left(\frac{P_{n+1}(x) + P_n(x) - 1}{2x} - P_1(x)\right) \\
&\quad + \varepsilon\left(\frac{P_{n+2}(x) + P_{n+1}(x) - 1}{2x} - P_1(x) - P_2(x)\right) \\
&\quad + \mathbf{h}\left(\frac{P_{n+3}(x) + P_{n+2}(x) - 1}{2x} - P_1(x) - P_2(x) - P_3(x)\right).
\end{aligned}$$

Bringing to the common denominator we have

$$\begin{aligned}
\sum_{l=1}^{n-1} PH_l(x) &= \frac{P_n(x) + P_{n-1}(x) - 1}{2x} \\
&\quad + \mathbf{i}\left(\frac{P_{n+1}(x) + P_n(x) - 1 - 2x}{2x}\right) \\
&\quad + \varepsilon\left(\frac{P_{n+2}(x) + P_{n+1}(x) - 1 - 2x - 4x^2}{2x}\right) \\
&\quad + \mathbf{h}\left(\frac{P_{n+3}(x) + P_{n+2}(x) - 1 - 2x - 4x^2 - 2x(4x^2+1)}{2x}\right)
\end{aligned}$$

and finally

$$\sum_{l=1}^{n-1} PH_l(x) = \frac{P_n(x)+\mathbf{i}P_{n+1}(x)+\varepsilon P_{n+2}(x)+\mathbf{h}P_{n+3}(x)}{2x}$$

$$+\frac{P_{n-1}(x)+\mathbf{i}P_n(x)+\varepsilon P_{n+1}(x)+\mathbf{h}P_{n+2}(x)}{2x}$$

$$+\frac{-(0+1)-\mathbf{i}(1+2x)-\varepsilon(2x+(4x^2+1))-\mathbf{h}((4x^2+1)+(8x^3+4x))}{2x}$$

$$=\frac{PH_n(x)+PH_{n-1}(x)-PH_0(x)-PH_1(x)}{2x}.$$

Thus the Theorem is proved. □

Theorem 16 ([3]). *Let $n \geq 2$ be an integer. Then*

$$\sum_{l=1}^{n-1} Q_l(x) = \frac{Q_n(x)+Q_{n-1}(x)-2-2x}{2x}. \tag{2.8}$$

Theorem 17. *Let $n \geq 2$ be an integer. Then*

$$\sum_{l=1}^{n-1} QH_l(x) = \frac{QH_n(x)+QH_{n-1}(x)-QH_0(x)-QH_1(x)}{2x}.$$

Proof. Using (2.8) and proceeding in the same way as in the Theorem 15 the result follows. □

Next we shall give the generating function for Pell hybrinomials.

Theorem 18. *The generating function for Pell hybrinomial sequence $\{PH_n(x)\}$ is*

$$G(t) = \frac{\mathbf{i}+\varepsilon\cdot(2x)+\mathbf{h}\cdot(4x^2+1)+(1+\varepsilon+\mathbf{h}\cdot(2x))t}{1-2xt-t^2}.$$

Proof. Assume that the generating function of the Pell hybrinomial sequence $\{PH_n(x)\}$ has the form $G(t) = \sum_{n=0}^{\infty} PH_n(x)t^n$. Then

$$G(t) = PH_0(x)+PH_1(x)t+PH_2(x)t^2+\dots$$

Multiply the above equality on both sides by $-2xt$ and then by $-t^2$ we obtain

$$-G(t)\cdot(2x)t = -PH_0(x)\cdot(2x)t-PH_1(x)\cdot(2x)t^2-PH_2(x)\cdot(2x)t^3-\dots$$

$$-G(t)t^2 = -PH_0(x)t^2 - PH_1(x)t^3 - PH_2(x)t^4 - \dots$$

By adding these three equalities above, we will get

$$G(t)(1-2xt-t^2) = PH_0(x) + (PH_1(x) - PH_0(x)\cdot(2x))t$$

since $PH_n(x) = 2x\cdot PH_{n-1}(x) + PH_{n-2}(x)$ (see (2.1)) and the coefficient of t^n, for $n \geq 2$, are equal to zero. Moreover, $PH_0(x) = \mathbf{i} + \varepsilon\cdot(2x) + \mathbf{h}\cdot(4x^2+1)$, $PH_1(x) - PH_0(x)\cdot(2x) = 1 + \varepsilon + \mathbf{h}\cdot(2x)$. □

In the same way we obtain the generating function $g(t)$ for Pell-Lucas hybrinomials.

Theorem 19. *The generating function for the Pell-Lucas hybrinomial sequence* $\{QH_n(x)\}$ *is*

$$g(t) = \frac{QH_0(x) + (QH_1(x) - QH_0(x)\cdot(2x))t}{1-2xt-t^2},$$

where $QH_0(x) = 2 + \mathbf{i}\cdot(2x) + \varepsilon\cdot(4x^2+2) + \mathbf{h}\cdot(8x^3+6x)$,
and $QH_1(x) - QH_0(x)\cdot(2x) = -2x + 2\mathbf{i} + \varepsilon\cdot(2x) + \mathbf{h}\cdot(4x^2+2)$.

We will give the matrix representation of Pell hybrinomials.

Theorem 20. *Let* $n \geq 0$ *be an integer. Then*

$$\begin{bmatrix} PH_{n+2}(x) & PH_{n+1}(x) \\ PH_{n+1}(x) & PH_n(x) \end{bmatrix} = \begin{bmatrix} PH_2(x) & PH_1(x) \\ PH_1(x) & PH_0(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}^n.$$

Proof. (by induction on n)
If $n = 0$ then assuming that the matrix to the power 0 is the identity matrix the result is obvious. Now suppose that for any $n \geq 0$ holds

$$\begin{bmatrix} PH_{n+2}(x) & PH_{n+1}(x) \\ PH_{n+1}(x) & PH_n(x) \end{bmatrix} = \begin{bmatrix} PH_2(x) & PH_1(x) \\ PH_1(x) & PH_0(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}^n.$$

We shall show that

$$\begin{bmatrix} PH_{n+3}(x) & PH_{n+2}(x) \\ PH_{n+2}(x) & PH_{n+1}(x) \end{bmatrix} = \begin{bmatrix} PH_2(x) & PH_1(x) \\ PH_1(x) & PH_0(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}^{n+1}.$$

By simple calculation using induction's hypothesis we have

$$\begin{aligned} &\begin{bmatrix} PH_2(x) & PH_1(x) \\ PH_1(x) & PH_0(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} PH_{n+2}(x) & PH_{n+1}(x) \\ PH_{n+1}(x) & PH_n(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2x\cdot PH_{n+2}(x) + PH_{n+1}(x) & PH_{n+2}(x) \\ 2x\cdot PH_{n+1}(x) + PH_n(x) & PH_{n+1}(x) \end{bmatrix} \\ &= \begin{bmatrix} PH_{n+3}(x) & PH_{n+2}(x) \\ PH_{n+2}(x) & PH_{n+1}(x) \end{bmatrix}, \end{aligned}$$

which ends the proof. □

In the same way we obtain the matrix representation for Pell-Lucas hybrinomials.

Theorem 21. *Let $n \geq 0$ be an integer. Then*

$$\begin{bmatrix} QH_{n+2}(x) & QH_{n+1}(x) \\ QH_{n+1}(x) & QH_n(x) \end{bmatrix} = \begin{bmatrix} QH_2(x) & QH_1(x) \\ QH_1(x) & QH_0(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}^n.$$

COMPLIANCE WITH ETHICAL STANDARDS

Conflict of Interest: The authors declare that they have no conflict of interest.

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Authors' addresses

Mirosław Liana
Rzeszow University of Technology, The Faculty of Management, al. Powstańców Warszawy 10, 35-959 Rzeszów, Poland
E-mail address: mliana@prz.edu.pl

Anetta Szynal-Liana
Rzeszow University of Technology, The Faculty of Mathematics and Applied Physics, al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland
E-mail address: aszynal@prz.edu.pl

Iwona Włoch
Rzeszow University of Technology, The Faculty of Mathematics and Applied Physics, al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland
E-mail address: iwloch@prz.edu.pl

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A STUDY ON THE UNIFORM CONVERGENCE OF SPECTRAL EXPANSIONS FOR CONTINUOUS FUNCTIONS ON A STURM-LIOUVILLE PROBLEM

EMIR ALI MARIS AND SERTAC GOKTAS

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Abstract. The paper is about investigating the uniform convergence conditions of spectral expansions of continuous functions in terms of root functions of a spectral problem with the same eigenparameter in the second-order differential equation and depending on quadratically in one of the boundary conditions on a closed interval.

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1. Introduction

In this paper, the spectral problem

$$-y'' + q(x)y = \lambda y, \qquad 0 < x < 1, \tag{1.1}$$

$$y'(0)\sin\beta = y(0)\cos\beta, \qquad 0 \le \beta < \pi, \tag{1.2}$$

$$y'(1) = (a\lambda^2 + b\lambda + c)y(1) \tag{1.3}$$

is considered and studied the uniform convergence of Fourier series expansions in terms of root functions of this problem for continuous functions. Here, λ is a spectral parameter; $q(x)$ is real-valued continuous function on the interval $[0,1]$; $a \neq 0$, b and c are real constants.

Lineer differantial operators with the second order differential equation can be addressed by some authors [19–21] and their significant properties have been developed using novel techniques in some functional spaces. Sturm-Liouville operators with the boundary condition (or conditions) depending on the eigenparameter, one of a concrete classes of this linear differential operators, have been studied to research their various properties by many authors. For example, problems of proving existence of eigenvalues, obtaining oscillation of eigenfunctions, giving asymptotic formulae of

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eigenvalues and eigenfunctions have been studied in papers [3,5]; problems of investigating the basis properties of the system of root functions in $L_p(0,1), 1 < p < \infty$ have been investigated in papers [1, 13]; problems of obtaining the convergence conditions of the spectral expansions in terms of root functions in given some linear spaces has been proved in papers [6, 9, 10, 13–16] for one boundary value problem.

In applications, the spectral problems [11, 12], which are investigated the uniform convergence conditions of the spectral expansions, that underline an important class of mathematical physics problems appears on a model of a transrelaxation heat process, torsional vibrations of a rod with a pulley at one end, the current in a cable grounded at one end through a concentrated capacitance or inductance and vibrations of a homogeneous loaded string.

Firstly, in 2005, Code and Browne [5] studied existence and asymptotics of eigenvalues of this problem (see also [4, Section 4.1]) for the problem (1.1)-(1.3). They proved that the eigenvalues of the boundary value problem (1.1)-(1.3) form an infinite sequence λ_n $(n = 0, 1, 2, \dots)$ without finite limit points and only following cases are possible:

- all the eigenvalues are real and simple,
- all the eigenvalues are real and all, except one double, are simple,
- all the eigenvalues are real and all, except one triple, are simple,
- all the eigenvalues are simple and all, except a conjugate pair of non-real, are real.

Note that the eigenvalues λ_n $(n = 0, 1, 2, \dots)$ were considered to be listed according to non-decreasing real part and repeated according to algebraic multiplicity. Therefore, the results of the article [5] cannot be directly applied to the problem (1.1)-(1.3).

Secondly, in 2008, Aliyev and Kerimov [1] studied basisness of root functions of the problem (1.1)-(1.3). Namely, they determined the explicit form of the biorthogonal system and they proved that the root functions system is a basis of $L_p(0,1), 1 < p < \infty$; moreover, if $p = 2$, then this basis is unconditional.

Finally, solutions that obtained by using the Fourier method of partial differential equations are represented by a series. Therefore, the investigation of the properties (such as convergence or divergence) of these series is of great importance [7, 21]. We aim to investigate the uniform convergence conditions of spectral expansions of continuous functions in terms of root functions of the problem (1.1)-(1.3)

Let us give a brief description of the structure of our study. In Section 2, we express some fundamental notations and some auxiliary results to prove our hypotheses. In Section 3, we give three theorems: Theorem 1 relates to the sharpening the asymptotic formulae for eigenvalues and eigenfunctions of the problem (1.1)-(1.3). Moreover, Theorem 2, the main theorem of the paper, gives the uniform convergence conditions of the spectral expansions in terms of root functions for this problem and Theorem 3 gives the uniform convergence conditions of the spectral expansions in

terms of the systems that is biorhogonally conjugate to root functions for this problem. In Section 4, an example is given in accordance with the main theorem.

2. Preliminaries

Some properties of eigenvalues, eigenfunctions and associated functions of the problem (1.1)-(1.3) should be needed and explicit forms of the root functions system of this problem should be expressed to use in the hypothesis of main theorem to reach the desired results in this article. As follows:

Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ denote the solutions of the equation (1.1) which satisfy the initial conditions

$$\varphi(0,\lambda) = 1, \varphi'(0,\lambda) = h, \tag{2.1}$$

$$\psi(0,\lambda) = 0, \psi'(0,\lambda) = 1, \tag{2.2}$$

where $h = \cot\beta$ $(0 < \beta < \pi)$.

Let λ_n $(n = 0,1,2,\dots)$ be eigenvalues and $y_n(x)$ $(n = 0,1,2,\dots)$ be eigenfunctions corresponding to λ_n $(n = 0,1,2,\dots)$ of the problem (1.1)-(1.3).

If λ_k is the multiple eigenvalue $(\lambda_k = \lambda_{k+1})$, then the following relations hold for the first order associated function y_{k+1} corresponding to eigenfunction y_k [18, p28]:

$$-y''_{k+1} + q(x)y_{k+1} = \lambda_k y_{k+1} + y_k,$$

$$y'_{k+1}(0)\sin\beta = y_{k+1}(0)\cos\beta,$$

$$y'_{k+1}(1) = (a\lambda_k^2 + b\lambda_k + c)y_{k+1}(1) + (2a\lambda_k + b)y_k(1).$$

If λ_k is the triple eigenvalue $(\lambda_k = \lambda_{k+1} = \lambda_{k+2})$, then there exist the second order associated function y_{k+2} for which following relations hold together with the first order associated function y_{k+1} [18, p28]:

$$-y''_{k+2} + q(x)y_{k+2} = \lambda_k y_{k+2} + y_{k+1},$$

$$y'_{k+2}(0)\sin\beta = y_{k+2}(0)\cos\beta,$$

$$y'_{k+2}(1) = (a\lambda_k^2 + b\lambda_k + c)y_{k+2}(1) + (2a\lambda_k + b)y_{k+1}(1) + 2ay_k.$$

Note that the functions $y_{k+1} + dy_k$ and $y_{k+2} + ey_k$, d and e are arbitrary constants, are also associated functions of the first and second order respectively.

We shall denote the solution of the equation (1.1) which satisfies the initial condition (2.1), if $0 < \beta < \pi$ or (2.2), if $\beta = 0$ by $y(x,\lambda)$. Then, the eigenvalues of the problem (1.1)-(1.3) are the roots of the characteristic function

$$\omega(\lambda) = y'(1,\lambda) - (a\lambda_k^2 + b\lambda_k + c)y(1,\lambda). \tag{2.3}$$

In [1], it has been proven that if λ_k is a multiple (double or triple) eigenvalue of the problem (1.1)-(1.3), then

$$\begin{aligned} &y(x,\lambda) \to y_k(x), \quad y'(x,\lambda) \to y'_k(x), \\ &y_\lambda(x,\lambda) \to \tilde{y}_{k+1}(x), \quad y'_\lambda(x,\lambda) \to \tilde{y}'_{k+1}(x) \end{aligned} \tag{2.4}$$

uniformly according to $x \in [0,1]$, as $\lambda \to \lambda_k$, where $\tilde{y}_{k+1}$ is one of the associated functions of the first order. It is clear that $\tilde{y}_{k+1} = y_{k+1} + \tilde{c}\, y_k$. Furthermore, if λ_k is a triple eigenvalue of the problem (1.1)-(1.3), then

$$y_{\lambda\lambda}(x,\lambda) \to 2\tilde{y}_{k+2}(x), \qquad y'_{\lambda\lambda}(x,\lambda) \to 2\tilde{y}'_{k+2}(x) \tag{2.5}$$

uniformly according to $x \in [0,1]$, as $\lambda \to \lambda_k$, where $\tilde{y}_{k+2}$ is one of the associated functions of the second order corresponding to the first associated function $\tilde{y}_{k+1}$. It is obvious that $\tilde{y}_{k+2} = y_{k+2} + \tilde{c}\, y_{k+1} + \tilde{d}\, y_k$ (see also [8]).

By (2.4) and (2.5), it is easily seen that

$$\tilde{c} = \begin{cases} -y'_{k+1}(0), & \text{if } \beta = 0\,, \\ -y_{k+1}(0), & \text{if } 0 < \beta < \pi, \end{cases}$$

$$\tilde{d} = \begin{cases} \left(y'_{k+1}(0)\right)^2 - y'_{k+2}(0), & \text{if } \beta = 0\,, \\ y^2_{k+1}(0) - y_{k+2}(0), & \text{if } 0 < \beta < \pi\,. \end{cases}$$

Additionally, the following auxiliary associated functions were considered in [1]:

$$y^{*}_{k+1} = y_{k+1} + c_1 y_k, \tag{2.6}$$

$$y^{**}_{k+1} = y_{k+1} + c_2 y_k, \tag{2.7}$$

$$y^{\#\#}_{k+2} = y_{k+2} + c_2 y_{k+1} + c_3 y_k, \tag{2.8}$$

where

$$c_1 = -\frac{\omega'''(\lambda_k)}{3\omega''(\lambda_k)} - \frac{y_{k+1}(1)}{y_k(1)} + \tilde{c},$$

$$c_2 = -\frac{\omega^{IV}(\lambda_k)}{4\omega'''(\lambda_k)} - \frac{y_{k+1}(1)}{y_k(1)} + \tilde{c},$$

$$c_3 = -\frac{\omega^{V}(\lambda_k)}{20\omega'''(\lambda_k)} - \frac{\omega^{IV}(\lambda_k)}{4y_k(1)\omega'''(\lambda_k)}\left(y_{k+1}(1) - \tilde{c}\, y_k(1)\right) - \frac{y_{k+2}(1)}{y_k(1)} + \frac{\tilde{c}\, y_{k+1}(1)}{y_k(1)} - \tilde{c}^2 + \widetilde{d} + c_2^2,$$

Let i and j be arbitrary non-negative different integers. The following systems were investigated in [1]:*

(i) $y_n(x)$ $(n = 0,1,\ldots; n \neq i, j)$, if all of eigenvalues of the problem (1.1)-(1.3) are real and simple or all eigenvalues of the problem (1.1)-(1.3), except a conjugate pair of non-real, are real and simple.

(ii) $y_n(x)$ $(n = 0,1,\ldots; n \neq k, k+1)$, if λ_k is double eigenvalue $(\lambda_k = \lambda_{k+1})$ of the problem (1.1)-(1.3).

(iii) $y_n(x)$ $(n = 0,1,\ldots; n \neq k+1, j)$, if λ_k is double eigenvalue $(\lambda_k = \lambda_{k+1})$ of the problem (1.1)-(1.3), where $j \neq k, k+1$.

*(i) - (xii) will be used also the numbering of systems in these cases.

(iv) $y_n(x)$ $(n = 0, 1, ...; n \neq k, j)$, if λ_k is double eigenvalue $(\lambda_k = \lambda_{k+1})$ of the problem (1.1)-(1.3) and $y^*_{k+1}(1)\left(\lambda_j - \lambda_k\right) \neq y_k(1)$, where $j \neq k, k+1$.

(v) $y_n(x)$ $(n = 0, 1, ...; n \neq i, j)$, if λ_k is double eigenvalue $(\lambda_k = \lambda_{k+1})$ of the problem (1.1)-(1.3), where $i, j \neq k, k+1$.

(vi) $y_n(x)$ $(n = 0, 1, ...; n \neq k+1, k+2)$, if λ_k is triple eigenvalues $(\lambda_k = \lambda_{k+1} = \lambda_{k+2})$ of the problem (1.1)-(1.3).

(vii) $y_n(x)$ $(n = 0, 1, ...; n \neq k, k+2)$, if λ_k is triple eigenvalues $(\lambda_k = \lambda_{k+1} = \lambda_{k+2})$ of the problem (1.1)-(1.3) and $y^{\#}_{k+1}(1) \neq 0$.

(viii) $y_n(x)$ $(n = 0, 1, ...; n \neq k, k+1)$, if λ_k is triple eigenvalues $(\lambda_k = \lambda_{k+1} = \lambda_{k+2})$ of the problem (1.1)-(1.3) and $\left(y^{\#}_{k+1}(1)\right)^2 \neq y_k(1)\, y^{\#}_{k+2}(1)$.

(ix) $y_n(x)$ $(n = 0, 1, ...; n \neq k+2, j)$, if λ_k is triple eigenvalues $(\lambda_k = \lambda_{k+1} = \lambda_{k+2})$ of the problem (1.1)-(1.3), where $j \neq k, k+1, k+2$.

(x) $y_n(x)$ $(n = 0, 1, ...; n \neq k+1, j)$, if λ_k is triple eigenvalue $(\lambda_k = \lambda_{k+1} = \lambda_{k+2})$ of the problem (1.1)-(1.3) and $y^{\#}_{k+1}(1)\left(\lambda_j - \lambda_k\right) \neq y_k(1)$, where $j \neq k, k+1, k+2$.

(xi) $y_n(x)$ $(n = 0, 1, ...; n \neq k, j)$, if λ_k is triple eigenvalue $(\lambda_k = \lambda_{k+1} = \lambda_{k+2})$ of the problem (1.1)-(1.3) and $y^{\#}_{k+2}(1)\left(\lambda_j - \lambda_k\right) \neq y^{\#}_{k+1}(1)$, where $j \neq k, k+1, k+2$.

(xii) $y_n(x)$ $(n = 0, 1, ...; n \neq i, j)$, if λ_k is triple eigenvalues $(\lambda_k = \lambda_{k+1} = \lambda_{k+2})$ of the problem (1.1)-(1.3), where $i, j \neq k, k+1, k+2$.

The given conditions $y^*_{k+1}(1)(\lambda_j - \lambda_k) \neq y_k(1)$, $y^{\#}_{k+1}(1) \neq 0$, $(y^{\#}_{k+1}(1))^2 \neq y_k(1)\, y^{\#}_{k+2}(1)$, $y^{\#}_{k+1}(1)\left(\lambda_j - \lambda_k\right) \neq y_k(1)$, $y^{\#}_{k+1}(1)\left(\lambda_j - \lambda_k\right) \neq y_k(1)$ and $y^{\#}_{k+2}(1)\left(\lambda_j - \lambda_k\right) \neq y^{\#}_{k+1}(1)$ for the above-mentioned system are sufficient and necessary conditions for the basisness of the systems (iv), (vii), (viii), (x) and (xi) respectively. Moreover, for example, if $y^*_{k+1}(1)(\lambda_j - \lambda_k) = y_k(1)$, the system (iv) is neither complete nor minimal in $L_p(0,1), 1 < p < \infty$.

Denote the systems that is biorhogonally conjugate to each of the systems (i)-(xii) by the system $\{u_n(x)\}$. For example, the system $u_n(x)$ $(n = 0, 1, ...; n \neq k, j)$ is biorhogonally conjugate to the system (iv) or the system (xi) if λ_k is double or triple respectively.

3. Main results

Firstly, we shall give the sharpened asymptotics for eigenvalues and eigenfunctions, since such asymptotics are very important for the prove of main theorem.

We here note that the following asymptotic formulae are valid for sufficiently large n [5, Section 2, Theorem 1]

$$\lambda_n = \begin{cases} (n-1)^2\pi^2 + O(1), & \text{if } \beta = 0, \\ \left(n - \frac{3}{2}\right)^2 \pi^2 + O(1), & \text{if } 0 < \beta < \pi. \end{cases} \tag{3.1}$$

The following theorem aims to sharp the asymptotic formula (3.1).

Theorem 1. *Let $\lambda_n = s_n^2$ ($\operatorname{Re} s_n \geq 0$). The following asymptotic formulae are valid for sufficiently large n:*

(a) *If $\beta = 0$, then*

$$s_n = (n-1)\pi + \frac{A_1}{n\pi} + O\left(\frac{\delta_{n,1}}{n}\right), \tag{3.2}$$

$$\begin{aligned} y_n(x) = \psi(x,\lambda_n) = {} & \frac{\sin(n-1)\pi x}{(n-1)\pi} + \frac{\alpha_1(x)\cos(n-1)\pi x}{(n\pi)^2} \\ & + \frac{\alpha_{n,1}(x)\cos(n-1)\pi x}{2(n\pi)^2} + \frac{\beta_{n,1}(x)\sin(n-1)\pi x}{2(n\pi)^2} \\ & + O\left(\frac{\delta_{n,1}}{n^2}\right), \end{aligned} \tag{3.3}$$

where

$$A_1 = \frac{1}{2}\int_0^1 q(\tau)d\tau, \quad \alpha_1(x) = A_1 x - \frac{1}{2}\int_0^x q(\tau)d\tau,$$

$$\alpha_{n,1}(x) = \int_0^x q(\tau)\cos 2(n-1)\pi\tau d\tau,$$

$$\beta_{n,1}(x) = \int_0^x q(\tau)\sin 2(n-1)\pi\tau d\tau \quad \textit{an} \quad \delta_{n,1} = \left|\int_0^1 q(\tau)\cos(2n-1)\pi\tau d\tau\right| + \frac{1}{n}.$$

(b) *If $0 < \beta < \pi$, then*

$$s_n = \left(n - \frac{3}{2}\right)\pi + \frac{A_2}{n\pi} + O\left(\frac{\delta_{n,2}}{n}\right), \tag{3.4}$$

$$\begin{aligned} y_n(x) = \varphi(x,\lambda_n) = \cos\left(n-\frac{3}{2}\right)\pi x &+ \frac{\alpha_2(x)\sin\left(n-\frac{3}{2}\right)\pi x}{n\pi} \\ &+ \frac{\alpha_{n,2}(x)\sin\left(n-\frac{3}{2}\right)\pi x}{2n\pi} - \frac{\beta_{n,2}(x)\cos\left(n-\frac{3}{2}\right)\pi x}{2n\pi} \\ &+ O\left(\frac{\delta_{n,2}}{n}\right), \end{aligned} \tag{3.5}$$

where $A_2 = h + \frac{1}{2}\int_0^1 q(\tau)d\tau$, $h = \cot\beta$, $\alpha_2(x) = h - A_2 x + \frac{1}{2}\int_0^x q(\tau)d\tau$, $\alpha_{n,2}(x) = \int_0^x q(\tau)\cos(2n-3)\pi\tau d\tau$, $\beta_{n,2}(x) = \int_0^x q(\tau)\sin(2n-3)\pi\tau d\tau$ *and* $\delta_{n,2} = \left|\int_0^1 q(\tau)\cos(2n-3)\pi\tau d\tau\right| + \frac{1}{n}$.

Proof. We will only give the proof of case (b). The other case is proven similarly. Suppose that, $\lambda = s^2$. By virtue of (2.1), the equality

$$\varphi(x,\lambda) = \cos sx + \frac{h}{s}\sin sx + \frac{1}{s}\int_0^x \sin(s(x-\tau))q(\tau)\varphi(\tau,\lambda)d\tau \tag{3.6}$$

holds [17, Chapter I, Section 1.2, Lemma 1.2.1].

Let $\lambda = \sigma + it$. There exists $s_0 > 0$ such that for $|s| > s_0$, the estimate

$$\varphi(x,\lambda) = \cos sx + O(e^{|t|x}|s|^{-1}) \tag{3.7}$$

is valid [17, Chapter I, Section 1.2, Lemma 1.2.2], where the function $O(e^{|t|x}|s|^{-1})$ is the entire function of s for any fixed x in $[0,1]$. Moreover, the function (3.7) is uniform with respect to x for $0 \le x \le 1$.

Because of $0 < \beta < \pi$ in assumption of case (b), the formulae

$$s_n = \sqrt{\lambda_n} = \left(n-\frac{3}{2}\right)\pi + O(n^{-1}) \tag{3.8}$$

is satisfied by (3.1). The formulae (3.6)-(3.8) yield the following:

$$\varphi(x,\lambda_n) = \cos s_n x + \frac{h}{s_n}\sin s_n x + \frac{\sin s_n x}{2s_n}\int_0^x q(\tau)d\tau$$
$$+ \frac{\sin s_n x}{2s_n}\int_0^x q(\tau)\cos 2s_n\tau d\tau \tag{3.9}$$
$$- \frac{\cos s_n x}{2s_n}\int_0^x q(\tau)\sin 2s_n\tau d\tau + O(n^{-2}).$$

In addition, by differentiating the equality (3.6) with respect to x and substituting the formulae (3.8), we obtain

$$\varphi'(x,\lambda_n) = -s_n \sin s_n x + O(1). \tag{3.10}$$

Consequently, the roots of the equation

$$\varphi'(1,\lambda_n) = \left(a\lambda_n^2 + b\lambda_n + c\right)\varphi(1,\lambda_n) \tag{3.11}$$

are the eigenvalues of the problem (1.1)-(1.3).

Let $s_n = \left(n - \frac{3}{2}\right)\pi + \varepsilon_n$. Since the formulas

$$\sin s_n = (-1)^n + O(n^{-2}),$$
$$\cos s_n = (-1)^{n-1}\varepsilon_n + O(n^{-3})$$

by using (3.9) and (3.10), we obtain the estimates

$$\varphi(1,\lambda_n) = (-1)^{n-1}\varepsilon_n + \frac{(-1)^n h}{n\pi} + \frac{(-1)^n}{2n\pi}\int_0^1 q(\tau)d\tau + O\left(\frac{\delta_{n,2}}{n}\right), \tag{3.12}$$

$$\varphi'(1,\lambda_n) = (-1)^{n-1}n\pi + O(1). \tag{3.13}$$

By substituting (3.12) and (3.13) in equation (3.11), the equality

$$(-1)^{n-1}\varepsilon_n + \frac{(-1)^n h}{n\pi} + \frac{(-1)^n}{2n\pi}\int_0^1 q(\tau)d\tau + O\left(\frac{\delta_{n,2}}{n}\right) = 0$$

is obtained, where $\delta_{n,2} = \left|\int_0^1 q(\tau)\cos(2n-3)\pi\tau d\tau\right| + \frac{1}{n}$. The last equation shows that the asymptotic formulae (3.4) is valid.

By using (3.4), we obtain

$$\cos s_n x = \cos\left(n - \frac{3}{2}\right)\pi x - \frac{A_2 x}{n\pi}\sin\left(n - \frac{3}{2}\right)\pi x + O\left(\frac{\delta_{n,2}}{n}\right). \tag{3.14}$$

The asymptotic formulae (3.5) follows from (3.9) and (3.14) and the proof of theorem 1 is completed. □

The trigonometric system $\{\theta_n(x)\}_{n=1}^{\infty}$ are defined as follows:

$$\theta_n(x) = \begin{cases} \sqrt{2}\sin n\pi x, & \text{if } \beta = 0, \\ \sqrt{2}\cos\left(n - \frac{1}{2}\right)\pi x, & \text{if } 0 < \beta < \pi. \end{cases}$$

The following theorems are related to uniformly convergent spectral expansions in terms of root functions and the functions which are biorthogonally conjugate to root functions of the problem (1.1)-(1.3), respectively.

Now, we are ready to give main theorem, as follows:

Theorem 2. *Suppose that $f \in C[0,1]$ and $f(x)$ has a uniformly convergent Fourier expansion in the system $\{\theta_n(x)\}_{n=1}^{\infty}$ on the interval $[0,1]$. Then, this function can be expanded in Fourier series in each of the systems* (i)-(xii) *and these expansions are uniformly convergent on every interval $[0,r]$ $(0 < r < 1)$. Moreover, the Fourier series of $f(x)$ in the systems* (i)-(xii) *are uniformly convergent on $[0,1]$ if and only if*

$(f, y_i)\, y_j(1) = 0$ *for the systems* (i), (v) *and* (xii)*;*

$(f, y_k)\, y_{k+1}(1) = (f, y_{k+1})\, y_k(1)$ *for the systems* (ii) *and* (vi)*;*

$(f, y_k)\, y_j(1) = (f, y_i)\, y_k(1)$ *for the systems* (iii) *and* (ix)*;*

$\left(f, y_{k+1}^*\right) y_j(1) = \left(f, y_j\right) y_{k+1}^*(1)$ *for the system* (iv)*;*

$(f, y_k)\, y_{k+2}^{\#}(1) = \left(f, y_{k+2}^{\#}\right) y_k(1)$ *for the system* (vii)*;*

$\left(f, y_{k+1}^{\#}\right) y_{k+2}^{\#}(1) = \left(f, y_{k+2}^{\#}\right) y_{k+1}^{\#}(1)$ *for the system* (viii)*;*

$\left(f, y_{k+1}^{\#}\right) y_j(1) = \left(f, y_j\right) y_{k+1}^{\#}(1)$ *for the system* (x) *and*

$\left(f, y_{k+2}^{\#}\right) y_j(1) = \left(f, y_j\right) y_{k+2}^{\#}(1)$ *for the system* (xi).

Proof. The theorem will only be proven for the system (iv). The theorem for other systems can be proved similarly.

Consider the Fourier series of $f(x)$ in the system $y_n(x)$ $(n = 0, 1, \ldots; n \neq k, j)$ on the interval $[0,1]$:

$$F(x) = \sum_{n=0; n\neq k,j}^{\infty} (f, u_n) y_n(x). \tag{3.15}$$

The elements of the $u_n(x)$ $(n = 0, 1, \ldots; n \neq k, j)$ biorthogonal system defined as biorthogonally conjugate to the system (iv) can be represented in the following form [1]:

$$u_n(x) = \frac{1}{B_n \Delta^*_{k_j}} \begin{vmatrix} y_n(x) & y_n(1) & \lambda_n y_n(1) \\ y^*_{k+1}(x) & y^*_{k+1}(1) & \lambda_k y^*_{k+1}(1) + y_k(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix} \quad (n \neq k, k+1, j), \tag{3.16}$$

$$u_{k+1}(x) = \frac{1}{B_{k+1} \Delta^*_{k_j}} \begin{vmatrix} y_k(x) & y_k(1) & \lambda_k y_k(1) \\ y^*_{k+1}(x) & y^*_{k+1}(1) & \lambda_k y^*_{k+1}(1) + y_k(1) \\ y_j(x) & y_j(1) & \lambda_j y_j(1) \end{vmatrix},$$

where

$$B_n = \|y_n\|^2 + (2a\lambda_n + b)|y_n(1)|^2 \quad (n \neq k, k+1, j), \tag{3.17}$$

$$B_{k+1} = -\frac{y_k(1)\omega''(\lambda_k)}{2},$$

$$\Delta^*_{k_j} = y_j(1)\left[(\lambda_j - \lambda_k) y^*_{k+1}(1) - y_k(1)\right]$$

and y^*_{k+1} is defined by (2.6).

Note that, the eigenvalues λ_n are real and simple because of $n \neq k, k+1$. From this reason, $B_n \neq 0$ $(n \neq k, k+1)$ [1, Corollary 1.3]. In addition, $\Delta^*_{k_j} \neq 0$ from the condition $y^*_{k+1}(1)(\lambda_j - \lambda_k) \neq y_k(1)$ for the system (iv).

Let $r = 1 + \max\{k, j\}$. Examining the uniform convergence of the series (3.15) on the interval $[0, 1]$ is equivalent to that of the series

$$F_1(x) = \sum_{n=r}^{\infty} (f, u_n) y_n(x). \tag{3.18}$$

Therefore, let's investigate the uniform convergence of the series (3.18):

Firstly, suppose that $\beta = 0$ and the sequence $S_{N,1}(x)$ is defined by the partial sum of the series (3.18):

$$S_{N,1}(x) = \sum_{n=r}^{N} (f, u_n) y_n(x). \tag{3.19}$$

From the equality (3.16), we obtain

$$u_n(x) = \frac{y_n(x)}{B_n} + \frac{y_n(1)}{B_n} C^{(1)}_{k_j}(x) + \frac{\lambda_n y_n(1)}{B_n} C^{(2)}_{k_j}(x), \tag{3.20}$$

where

$$C^{(1)}_{k_j}(x) = -\frac{1}{\Delta^*_{k_j}} \begin{vmatrix} y^*_{k+1}(x) & \lambda_k y^*_{k+1}(1) + y_k(1) \\ y_j(x) & \lambda_j y_j(1) \end{vmatrix},$$

$$C^{(2)}_{k_j}(x) = \frac{1}{\Delta^*_{k_j}} \begin{vmatrix} y^*_{k+1}(x) & y^*_{k+1}(1) \\ y_j(x) & y_j(1) \end{vmatrix}. \tag{3.21}$$

The inequality

$$\max_{0\le x\le 1}\left|y'_n(x)\right| \le const.\left(1+\sqrt{\lambda_n}\right)\max_{0\le x\le 1}|y_n(x)|. \tag{3.22}$$

is valid for sufficiently large numbers of n [22, Theorem 2].

From the expressions (1.3), (3.1), (3.3) and (3.22), the inequality

$$\left|(a\lambda_n^2+b\lambda_n+c)y_n(1)\right| = \left|y_n'(1)\right| \le const.\left(1+\sqrt{\lambda_n}\right)\max_{0\le x\le 1}|y_n(x)| = O(1)$$

holds. Also,

$$y_n(1) = O(n^{-4}), \tag{3.23}$$

is obtained from the last inequality. By virtue of (3.2), (3.3) and (3.23) in the equality (3.17), we give

$$B_n = \frac{1}{2(n-1)^2\pi^2} + O(n^{-3}) \tag{3.24}$$

and from this,

$$\frac{1}{B_n} = 2(n-1)^2\pi^2 + O(n) \tag{3.25}$$

hods. By using the equalities (3.23) and (3.24) in the equality (3.20), we have

$$u_n(x) = \frac{y_n(x)}{B_n} + \frac{\lambda_n y_n(1)}{B_n}C_{k_j}^{(2)}(x) + O(n^{-2}). \tag{3.26}$$

By considering the equality (3.26),

$$\begin{aligned} S_{N,1}(x) = &\sum_{n=r}^{N}\frac{1}{B_n}(f,y_n)y_n(x) + \left(f,C_{k_j}^{(2)}\right)\sum_{n=r}^{N}\frac{\lambda_n y_n(1)}{B_n}y_n(x) \\ &+\sum_{n=r}^{N}O(n^{-2}) \end{aligned} \tag{3.27}$$

is obtained from the sequence (3.19). Consequently, the equality (3.27) by considering the equality (3.3) and (3.25) shows that the formula

$$\begin{aligned} S_{N,1}(x) = &\sum_{n=r}^{N}\left(f,\sqrt{2}\sin(n-1)\pi x\right)\sqrt{2}\sin(n-1)\pi x \\ &+\sum_{n=1}^{N}K_n(x) + \left(f,C_{k_j}^{(2)}\right)\sum_{n=r}^{N}\frac{\lambda_n y_n(1)}{B_n}y_n(x) \\ &+\sum_{n=r}^{N}O(n^{-2}) \end{aligned} \tag{3.28}$$

is also valid. Here $K_n(x)$ is given by

$$\begin{aligned} K_n(x) &= (f, \sin(n-1)\pi x)\, O(n^{-1}) + (f\alpha_1, \cos(n-1)\pi x)\, O(n^{-1}) \\ &\quad + \big(f, \alpha_{n,1}(x)\cos(n-1)\pi x\big)\, O(n^{-1}) \\ &\quad + \big(f, \beta_{n,1}(x)\sin(n-1)\pi x\big)\, O(n^{-1}) \\ &\quad + O\left(\frac{\delta_{n,1}}{n}\right). \end{aligned} \tag{3.29}$$

The first series to the right of the equality (3.28) is uniformly convergent in the interval $[0,1]$ due to assumption of the theorem while the last series is absolutely and uniformly convergent in same interval. On the other hand, from the equality (3.29) the following inequality is obtained for sufficiently large n:

$$\begin{aligned} |K_n(x)| &\le \frac{const.}{n}\{|(f, \sin(n-1)\pi x)| + |(f\alpha_1, \cos(n-1)\pi x)| \\ &\quad + \big|\big(f, \alpha_{n,1}(x)\cos(n-1)\pi x\big)\big| + \big|\big(f, \beta_{n,1}(x)\sin(n-1)\pi x\big)\big| \\ &\quad + O\left(\delta_{n,1}\right)\} \\ &\le const.\Big\{|(f, \sin(n-1)\pi x)|^2 + |(f\alpha_1, \cos(n-1)\pi x)|^2 \\ &\quad + \big|\big(f, \alpha_{n,1}(x)\cos(n-1)\pi x\big)\big|^2 + \big|\big(f, \beta_{n,1}(x)\sin(n-1)\pi x\big)\big|^2\Big\} \\ &\quad + O\left(\frac{\delta_{n,1}}{n}\right). \end{aligned}$$

It is easily seen that,

$$\sum_{n=r}^{\infty} |(f, \sin(n-1)\pi x)|^2 < +\infty, \qquad \sum_{n=r}^{\infty} |(f\alpha_1, \cos(n-1)\pi x)|^2 < +\infty,$$

$$\sum_{n=r}^{\infty} \left| O\left(\frac{\delta_{n,1}}{n}\right)\right| < +\infty.$$

And, by virtue of Bessel inequality, we obtain

$$\begin{aligned}\sum_{n=r}^{\infty}\left|\left(f,\alpha_{n,1}(x)\cos(n-1)\pi x\right)\right|^2 &= \sum_{n=r}^{\infty}\left|\int_0^1 f(x)\alpha_{n,1}(x)\cos(n-1)\pi x dx\right|^2\\ &\le \sum_{n=r}^{\infty}\left[\int_0^1 |f(x)\alpha_{n,1}(x)|\,dx\right]^2\\ &\le \|f\|^2\int_0^1\left[\sum_{n=r}^{\infty}\left|\int_0^x q(\tau)\cos(n-1)\pi\tau d\tau\right|^2\right]dx\\ &\le const.\|f\|^2\int_0^1\int_0^x |q(\tau)|^2 d\tau dx\\ &\le const.\|f\|^2\|q\|^2.\end{aligned}$$

Similarly, we also obtain

$$\sum_{n=r}^{\infty}\left|\left(f,\beta_{n,1}(x)\sin(n-1)\pi x\right)\right|^2 \le const.\|f\|^2\|q\|^2.$$

Therefore, the series $\sum_{n=1}^{N} K_n(x)$ in the equality (3.28) is absolutely and uniformly convergent on the interval $[0,1]$.

At now, let us investigate the uniform convergence of the third series to the right of the equality (3.28):

By (3.21), we have

$$\left(f,C_{k_j}^{(2)}\right)=\frac{1}{\Delta_{k_j}^*}\left[y_j(1)(f,y_{k+1}^*)-y_{k+1}^*(1)(f,y_j)\right].$$

If $(f,C_{k_j}^{(2)})=0$, $y_j(1)(f,y_{k+1}^*)=y_{k+1}^*(1)(f,y_j)$. Hence, this completes the proof of the second part of the theorem.

Suppose that, $(f,C_{k_j}^{(2)})\neq 0$. For the sufficiently large of n, the zeros of the functions $y_{n+1}(x)$ and $y_n(x)$ in the whole of $(0,1)$ are ranked one after the other. Namely, the kth zero of $y_{n+1}(x)$ is less than the kth zero of $y_n(x)$ [17, p.14, Comparison Theorem]. Furthermore, $y_n(x)>0$ with the initial condition (2.2) in $(0,x_n^{(1)})$. Here, $x_n^{(1)}>0$ is the zero nearest to $x=0$ of the function $y_n(x)$. According to these discussions, the inequality

$$y_{n+1}(1)y_n(1)<0 \tag{3.30}$$

can be verified. From (3.23) and (3.30)

$$y_n(1) = \frac{(-1)^n k_n}{n^4}, \tag{3.31}$$

holds, and using (3.1), (3.3) and (3.25) together with (3.31)

$$\frac{\lambda_n y_n(1)}{B_n} y_n(x) = \frac{\widetilde{k}_n}{n} \sin((n-1)\pi(1+x)) + O(n^{-2})$$

is obtained, where k_n and $\widetilde{k}_n$ are a positive (or negative) and bounded sequence of all terms. On the other hand, the third series to the right of the equality (3.28) is uniformly convergent in the interval $[0,r]$ $(0<r<1)$ since $\frac{\widetilde{k}_n}{n} \to 0$ and the relation

$$\begin{aligned}
\left|\sum_{n=r}^{N} \sin((n-1)\pi(1+x))\right| &= \frac{1}{2\left|\sin\frac{\pi(1+x)}{2}\right|}\left|\sum_{n=r}^{N} \sin((n-1)\pi(1+x)) \sin\frac{\pi(1+x)}{2}\right| \\
&= \frac{\left|\cos\frac{(2r-3)\pi(1+x)}{2} - \cos\frac{(2N-1)\pi(1+x)}{2}\right|}{2\left|\sin\frac{\pi(1+x)}{2}\right|} \\
&\le \frac{1}{\sin\frac{\pi(1+r)}{2}} \qquad (0 \le x \le r < 1)
\end{aligned}$$

is verify [2, Abel's Lemma].

Finally, suppose that $\beta \neq 0$. Then, if we consider the expressions (3.4), (3.5), (3.17), (3.22) and the partial sum of the series (3.27), we have the following equalities

$$y_n(1) = O(n^{-3}), \tag{3.32}$$

$$\frac{1}{B_n} = 2 + O(n^{-1}), \tag{3.33}$$

$$\begin{aligned}
S_{N,1}(x) = &\sum_{n=r}^{N} \left(f, \sqrt{2}\cos\left(n-\frac{3}{2}\right)\pi x\right)\sqrt{2}\cos\left(n-\frac{3}{2}\right)\pi x \\
&+ \sum_{n=1}^{N} R_n(x) + \left(f, C_{k_j}^{(2)}\right) \sum_{n=r}^{N} \frac{\lambda_n y_n(1)}{B_n} y_n(x) \\
&+ \sum_{n=r}^{N} O(n^{-2}),
\end{aligned} \tag{3.34}$$

where

$$\begin{aligned} R_n(x) = &\left(f, \cos\left(n-\frac{3}{2}\right)\pi x\right) O(n^{-1}) + \left(f\alpha_2, \sin\left(n-\frac{3}{2}\right)\pi x\right) O(n^{-1}) \\ &+ \left(f, \alpha_{n,2}(x)\sin\left(n-\frac{3}{2}\right)\pi x\right) O(n^{-1}) \\ &+ \left(f, \beta_{n,2}(x)\cos\left(n-\frac{3}{2}\right)\pi x\right) O(n^{-1}) \\ &+ O\left(\frac{\delta_{n,2}}{n}\right). \end{aligned}$$

The first series to the right of the equality (3.34) is uniformly convergent in the interval $[0,1]$ due to assumption of the theorem while the last series is absolutely and uniformly convergent in same interval. On the other hand, showing that the series $\sum_{n=1}^{N} R_n(x)$ in (3.34) is absolutely and uniformly convergent in the interval $[0,1]$ is completely similar to that of the series $\sum_{n=1}^{N} K_n(x)$.

If $(f, C_{k_j}^{(2)}) = 0$, then the second part of the theorem was proven.

Suppose that, $(f, C_{k_j}^{(2)}) \neq 0$. From (3.1), (3.5), (3.30), (3.32) and (3.33), the equality

$$\frac{\lambda_n y_n(1)}{B_n} y_n(x) = \frac{\widetilde{r}_n}{n} \sin\left(\left(n-\frac{3}{2}\right)\pi(1+x)\right) + O(n^{-2})$$

is obtained, where $\widetilde{r}_n$ is a positive (or negative) and bounded sequence of all terms.

Together with this last equality, we obtain that the third series to the right of the equality (3.34) is uniformly convergent in the interval $[0,r]$ $(0<r<1)$ with similar calculations as in the above discussion.

So, the proof of the theorem 2 is completed. □

Theorem 3. *Suppose that $f \in C[0,1]$ and $f(x)$ has a uniformly convergent Fourier expansion in the system $\{\theta_n(x)\}_{n=1}^{\infty}$ on the interval $[0,1]$, then this function can be expanded in Fourier series in each of the systems $\{u_n(x)\}$ which are biorthogonally conjugates to the systems* (i)-(xii) *and these expansions are uniformly convergent on the interval* $[0,1]$.

Proof. The theorem will only be proven for the system (3.16) which is biorthogonally conjugate to the system (iv). The proof of the theorem is similar for other systems.

Consider the Fourier series of $f(x)$ in the system (3.16) on the interval $[0,1]$:

$$G(x) = \sum_{n=0;n\neq k,j}^{\infty} (f, y_n)u_n(x). \tag{3.35}$$

The series (3.35) is uniformly convergent on the interval $[0,1]$ if and only if the series

$$G_1(x) = \sum_{n=r}^{\infty} (f, y_n)u_n(x) \tag{3.36}$$

is uniformly convergent on the interval $[0,1]$, where $r = 1 + \max\{k, j\}$.

Let $\beta = 0$ and the sequence $S_{N,2}(x)$ is defined by the partial sum of the series (3.36):

$$S_{N,2}(x) = \sum_{n=r}^{N} (f, y_n)u_n(x). \tag{3.37}$$

By using the equality (3.26), we obtain

$$\begin{aligned} S_{N,2}(x) = &\sum_{n=r}^{N} \frac{1}{B_n}(f, y_n)y_n(x) + C_{k_j}^{(2)}(x) \sum_{n=r}^{N} \frac{\lambda_n y_n(1)}{B_n}(f, y_n) \\ &+ \sum_{n=r}^{N} O(n^{-3}). \end{aligned} \tag{3.38}$$

The first sequences to the right of the equality (3.28) and (3.38) are the same. Therefore, the first sequence to the right of the equality (3.38) is uniformly convergent on the interval $[0,1]$. On the other hand, the equality

$$\sum_{n=r}^{N} \frac{\lambda_n y_n(1)}{B_n}(f, y_n) = \sum_{n=r}^{N} (f, \sin(n-1)\pi x)\, O(n^{-1})$$

is valid from (3.1), (3.3), (3.23) and (3.26). From here, the inequality

$$|(f, \sin(n-1)\pi x)O(n^{-1})| \leq const. \left\{ |(f, \sin(n-1)\pi x)|^2 + \frac{1}{n^2} \right\}$$

holds. The numerical series $\sum_{n=r}^{\infty} |(f, \sin(n-1)\pi x)|^2$ is convergent. Namely, the second sequences to the right of the equality (3.38) is absolutely and uniformly convergent on $[0,1]$.

On the other hand, the proof of theorem can be also proven similar in the case $0 < \beta < \pi$.

So, the proof of the theorem 3 is completed. □

4. Example

In this section, an example is given to make the theorem 2 more understandable. This is an example of the root functions systems (iv) in the case of $0 < \beta < \pi$.

Example 1. *Consider the spectral problem*

$$-y'' = \lambda y, \quad 0 < x < 1, \tag{4.1}$$

$$y'(0) = 0, \quad y'(1) = \left(\frac{\lambda^2}{\pi^2} - \lambda\right) y(1), \tag{4.2}$$

where λ is a spectral parameter.

For this problem, $\lambda_0 = \lambda_1 = 0$ is double eigenvalue corresponding eigenfunction $y_0(x) \equiv 1$ and associated function $y_1(x) \equiv -\frac{x^2}{2} + c$ respectively, $\lambda_2 = \pi^2$ is a simple eigenvalue corresponding eigenfunction $y_2(x) \equiv \cos \pi x$ and all other simple eigenvalues $\lambda_3 < \lambda_4 < \ldots$ are the solutions of the equation

$$\tan \sqrt{\lambda} = \sqrt{\lambda}\left(1 - \frac{\lambda}{\pi^2}\right)$$

corresponding to eigenfunctions $y_n(x) = \cos \sqrt{\lambda_n} x$ $(n \geq 3)$, where c is an arbitrary constant (see also [1]).

Since $y(x,\lambda) = \cos \sqrt{\lambda} x$, then $\widetilde{y_1}(x) = \lim\limits_{\lambda \to 0} y_\lambda(x,\lambda) = -\frac{x^2}{2}$. Note that $\widetilde{c} = -y_1(0) = -c$, $\beta = -\frac{\pi}{2} \neq 0$. Therefore, by (2.6), $y_1^*(x) = -\frac{x^2}{2} + \frac{3\pi^2+15}{5\pi^2+15} - c$.

Consider the system

$$\left\{\frac{x^2}{2} + c, \cos \sqrt{\lambda_n} x \ (n \geq 3)\right\},$$

that is the system of root functions (2.3) without removed functions $y_0(x) \equiv 1$ and $y_2(x) = \cos \pi x$. This system is a basis in $L_p(0,1), 1 < p < \infty$ if and if only $y_1^*(1)(\lambda_2 - \lambda_0) \neq y_0(1)$ or $c \neq \frac{\pi^2+15}{10(\pi^2+3)} - \frac{1}{\pi^2}$.

Let $f(x) = x - 1$. Since $\left(f, \cos\left(n - \frac{1}{2}\right)\pi x\right) = O\left(n^{-2}\right)$, then the function $f(x)$ can be expanded in Fourier series of the system $\left\{\sqrt{2}\cos(n - \frac{1}{2})\pi x\right\}_{n=1}^{\infty}$. It is easily calculated that $(f, y_1^*) = \frac{1}{24} - \frac{c}{2}$ and $(f, y_2) = -\frac{2}{\pi^2}$. From here, if $c = -\frac{2\pi^2+30}{(5\pi^2+15)(\pi^2-4)}$, then $(f, y_1^*) y_2(1) = (f, y_2) y_1^*(1)$.

Consequently, from theorem 2, if $c \neq \frac{\pi^2+15}{10(\pi^2+3)} - \frac{1}{\pi^2}, -\frac{2\pi^2+30}{(5\pi^2+15)(\pi^2-4)}$, then Fourier series of $f(x)$ is uniformly convergent on every interval $[0,r], 0 < r < 1$; $c = -\frac{2\pi^2+30}{(5\pi^2+15)(\pi^2-4)}$, then Fourier series of $f(x)$ is uniformly convergent on $[0,1]$.

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Authors' addresses

Emir Ali Maris
Vocational School of Technical Sciences, Mersin University, 33343, Mersin, TURKEY
E-mail address: ealimaris@gmail.com

Sertac Goktas
Department of Mathematics, Mersin University, 33343, Mersin, TURKEY
E-mail address: srtcgoktas@gmail.com

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CERTAIN RESULTS ON KENMOTSU PSEUDO-METRIC MANIFOLDS

DEVARAJA MALLESHA NAIK, VENKATESHA, AND D.G. PRAKASHA

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Abstract. In this paper, a systematic study of Kenmotsu pseudo-metric manifolds are introduced. After studying the properties of this manifolds, we provide necessary and sufficient condition for Kenmotsu pseudo-metric manifold to have constant φ-sectional curvature, and prove the structure theorem for ξ-conformally flat and φ-conformally flat Kenmotsu pseudo-metric manifolds. Next, we consider Ricci solitons on this manifolds. In particular, we prove that an η-Einstein Kenmotsu pseudo-metric manifold of dimension higher than 3 admitting a Ricci soliton is Einstein, and a Kenmotsu pseudo-metric 3-manifold admitting a Ricci soliton is of constant curvature $-\varepsilon$.

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1. Introduction

The study of contact metric manifolds with associated pseudo-Riemannian metrics were first started by Takahashi [10] in 1969. Since then, such structures were studied by several authors mainly focusing on the special case of Sasakian manifolds. The case of contact Lorentzian structures (η, g), where η is a contact one-form and g is a Lorentzian metric associated to it, has a particular relevance for physics and was considered in [4] and [1]. A systematic study of almost contact semi-Riemannian manifolds was undertaken by Calvaruso and Perrone [3] in 2010, introducing all the technical apparatus which is needed for further investigations.

On the other hand, in 1972 Kenmotsu [9] investigated a class of contact metric manifolds satisfying some special conditions, and after onwards such manifolds are came to known as Kenmotsu manifolds. Recently, Wang and Liu [11] investigated almost Kenmotsu manifolds with associated pseudo-Riemannian metric. These are called almost Kenmotsu pseudo-metric manifolds. In this paper, we undertake the systematic study of Kenmotsu pseudo-metric manifolds.

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The present paper is organized as follows: In Section 2, we give the basics of Kenmotsu pseudo-metric manifold (M,g). Certain properties of Kenmotsu pseudo-metric manifolds are provided in Section 3. We devote Section 4 to the study of curvature properties of Kenmotsu pseudo-metric manifold (M,g) and gave necessary and sufficient condition for (M,g) to have constant φ-sectional curvature. In Section 5, we prove necessary and sufficient condition for Kenmotsu pseudo-metric manifold to be ξ-conformally flat (and φ-conformally flat). The last section is focused on the study of Kenmotsu pseudo-metric manifolds whose metric is a Ricci soliton. We show that if (M,g) is a Kenmotsu pseudo-metric manifold admitting a Ricci soliton, then the soliton constant $\lambda = 2n\varepsilon$, where $\varepsilon = \pm 1$. Moreover, we show that if M is an η-Einstein manifold of dimension higher than 3 admitting Ricci soliton, then M is Einstein. Further we show that, a Kenmotsu pseudo-metric manifold (M,g) of dimension 3 admitting Ricci soliton is of constant curvature $-\varepsilon$, where $\varepsilon = \pm 1$. Finally, an illustrative example is constructed which verifies our results.

2. Preliminaries

Let M be a $(2n+1)$ dimensional smooth manifold. We say that M has an *almost contact structure* if there is a tensor field φ of type $(1,1)$, a vector field ξ (called the *characteristic vector field* or *Reeb vector field*), and a 1-form η such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0. \tag{2.1}$$

If M with (φ,ξ,η)-structure is endowed with a pseudo-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y), \tag{2.2}$$

where $\varepsilon = \pm 1$, for all $X, Y \in TM$, then M is called an *almost contact pseudo-metric manifold*. The relation (2.2) is equivalent to

$$\eta(X) = \varepsilon g(X,\xi) \text{ along with } g(\varphi X, Y) = -g(X, \varphi Y). \tag{2.3}$$

In particular, in an almost contact pseudo-metric manifold, it follows that $g(\xi,\xi) = \varepsilon$ and so, the characteristic vector field ξ is a unit vector field, which is either space-like or time-like, but cannot be light-like.

The *fundamental* 2*-form* of an almost contact pseudo-metric manifold is defined by

$$\Phi(X,Y) = g(X, \varphi Y),$$

which satisfies $\eta \wedge \Phi^n \neq 0$. An almost contact pseudo-metric manifold is said to be a *contact pseudo-metric manifold* if $d\eta = \Phi$. The Riemannian curvature tensor R is given by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, which is opposite to the one used in [3]. The Ricci operator Q is determined by

$$Ric(X,Y) = g(QX,Y),$$

where Ric denotes the Ricci tensor. In an almost contact pseudo-metric manifold there always exists a special kind of local pseudo-orthonormal basis $\{e_i, \varphi e_i, \xi\}_{i=1}^n$, called a local pseudo φ-basis.

Consider the manifold $M \times \mathbb{R}$, where M is an almost contact pseudo-metric manifold. Denoting the vector field on $M \times \mathbb{R}$ by $(X, f\frac{d}{dt})$, where $X \in TM$, $t \in \mathbb{R}$, and f is a smooth function $M \times \mathbb{R}$, we define the structure J on $M \times \mathbb{R}$ by

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

which defines an almost complex structure. If J is integrable, we say that the almost contact pseudo-metric structure (φ, ξ, η) is normal. Necessary and sufficient condition for integrability of J is [3]

$$[\varphi, \varphi](X, Y) + 2d\eta(X, Y)\xi = 0.$$

The following can be easily obtained.

Proposition 1. *An almost contact pseudo-metric manifold is normal if and only if*

$$(\nabla_{\varphi X}\varphi)Y - \varphi(\nabla_X \varphi)Y + (\nabla_X \eta)(Y)\xi = 0, \tag{2.4}$$

where ∇ is the Levi-Civita connection.

An *almost Kenmotsu pseudo-metric manifold* is an almost contact pseudo-metric manifold with $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. A normal almost Kenmotsu pseudo-metric manifold is called a Kenmotsu pseudo-metric manifold [11]. Equivalently, from (2.4) we have the following:

Definition 1. Almost contact pseudo-metric manifold is said to be *Kenmotsu pseudo-metric manifold* if

$$(\nabla_X \varphi)Y = -\eta(Y)\varphi X - \varepsilon g(X, \varphi Y)\xi. \tag{2.5}$$

From (2.5), we see

$$\nabla\xi = I - \eta \otimes \xi. \tag{2.6}$$

A straight forward calculation gives the following:

Proposition 2. *On Kenmotsu pseudo-metric manifold (M, g), we have*

$$(\nabla_X \eta)Y = \varepsilon g(X, Y) - \eta(X)\eta(Y), \tag{2.7}$$

$$\pounds_\xi g = 2g - \varepsilon\eta \otimes \eta, \tag{2.8}$$

$$\pounds_\xi \varphi = 0, \tag{2.9}$$

$$\pounds_\xi \eta = 0, \tag{2.10}$$

where £ denotes the Lie derivative.

3. Some properties of Kenmotsu pseudo metric manifolds

For $X \in \operatorname{Ker}\eta$, either space-like or time-like, the *ξ-sectional curvature* $K(\xi, X)$ and *φ-sectional curvature* $K(X, \varphi X)$ are defined respectively as

$$K(\xi, X) = \frac{g(R(\xi, X)X, \xi)}{\varepsilon g(X, X)},$$
$$K(X, \varphi X) = \frac{g(R(\varphi X, X)X, \varphi X)}{g(X, X)^2}.$$

Now we prove:

Proposition 3. *If (M, g) is a Kenmotsu pseudo-metric manifold, then we have*

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{3.1}$$
$$\eta(R(X, Y)Z) = \eta(Y)g(X, Z) - \eta(X)g(Y, Z), \tag{3.2}$$
$$R(X, \xi)Y = \varepsilon g(X, Y)\xi - \eta(Y)X, \tag{3.3}$$
$$Ric(X, \xi) = -2n\eta(X) \quad (\Rightarrow Q\xi = -2n\varepsilon\xi), \tag{3.4}$$
$$K(\xi, \cdot) = -\varepsilon, \tag{3.5}$$
$$(\nabla_Z R)(X, Y, \xi) = \varepsilon\{g(X, Z)Y - g(Y, Z)X\} - R(X, Y)Z. \tag{3.6}$$

Proof. Equations (2.6) and (2.7) give (3.1). Equations (3.2), (3.3), (3.4) and (3.5) are consequences of (3.1). Equation (3.6) follows from (2.6), (2.7) and (3.1). □

Definition 2. An almost contact pseudo-metric manifold for which

$$\varphi^2(\nabla_W R)(X, Y, Z) = 0,$$

for all $X, Y, Z, W \in TM$ is said to be *globally φ-symmetric*.

Using (3.2) and (3.6), we have the following:

Corollary 1. *A globally φ-symmetric Kenmotsu pseudo-metric manifold is of constant curvature $-\varepsilon$.*

A Kenmotsu pseudo-metric manifold M is said to be η-Einstein if the Ricci tensor satisfies

$$Ric(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{3.7}$$

where a and b are certain smooth functions on M. If $b = 0$, then M is called an *Einstein* manifold.

From (3.4), we have

$$\varepsilon a + b = -2n. \tag{3.8}$$

Contracting (3.7) and using (3.8), we get

$$a = \left(\frac{r}{2n} + \varepsilon\right), \quad b = -\left(\frac{\varepsilon r}{2n} + 2n + 1\right), \tag{3.9}$$

where r is the scalar curvature. Thus, we have:

Proposition 4. *A Kenmotsu pseudo-metric manifold (M,g) is η-Einstein if and only if*

$$Ric(X,Y) = \left(\frac{r}{2n} + \varepsilon\right) g(X,Y) - \left(\frac{\varepsilon r}{2n} + 2n + 1\right) \eta(X)\eta(Y). \tag{3.10}$$

In particular, we have the following:

Corollary 2. *A Kenmotsu pseudo-metric manifold (M,g) is Einstein if and only if*

$$Ric(X,Y) = -2n\varepsilon g(X,Y). \tag{3.11}$$

Proposition 5. *If the Kenmotsu pseudo-metric manifold (M,g) is η-Einstein, then*

$$X(b) + 2b\eta(X) = 0, \tag{3.12}$$

for $n > 1$, and for any vector field $X \in TM$.

Proof. Equation (3.10) is equivalent to

$$QY = aY + b\varepsilon\eta(Y)\xi, \tag{3.13}$$

where a and b are as in (3.9). It is well known that

$$\operatorname{div} Q = \frac{1}{2} Dr, \tag{3.14}$$

where D denotes the gradient. Equations (3.13) and (3.14) yields to

$$(n-1)Y(a) = \varepsilon\{\xi(b)\eta(Y) + 2nb\eta(Y)\}.$$

For $Y = \xi$, it gives $\xi(b) = -2b$, and so we get (3.12) for $n > 1$. □

Corollary 3. *If b (or a) is constant in an η-Einstein Kenmotsu pseudo-metric manifold, then it is Einstein.*

4. Curvature properties of Kenmotsu pseudo metric manifolds

First we prove the following Lemma which is very useful in subsequent sections.

Lemma 1. *On Kenmotsu pseudo-metric manifold (M,g), we have the following identities:*

$$\begin{aligned} R(X,Y)\varphi Z - \varphi R(X,Y)Z =& \varepsilon\{g(Y,Z)\varphi X - g(X,Z)\varphi Y \\ &+ g(X,\varphi Z)Y - g(Y,\varphi Z)X\}, \end{aligned} \tag{4.1}$$

$$\begin{aligned} R(\varphi X,\varphi Y)Z = R(X,Y)Z &+ \varepsilon\{g(Y,Z)X - g(X,Z)Y \\ &+ g(Y,\varphi Z)\varphi X - g(X,\varphi Z)\varphi Y\}. \end{aligned} \tag{4.2}$$

Proof. The Ricci identity shows that

$$\nabla_X\nabla_Y\varphi - \nabla_Y\nabla_X\varphi - \nabla_{[X,Y]}\varphi = R(X,Y)\varphi - \varphi R(X,Y).$$

Computing the left-hand side using (2.5) yields (4.1). The equation (4.2) is a result of (4.1). □

Note that the necessary and sufficient condition for a Sasakian pseudo-metric manifold to have constant φ-sectional curvature c is [10]

$$\begin{aligned}4R(X,Y)Z =&(c+3\varepsilon)\{g(Y,Z)X-g(X,Z)Y\}\\&+(\varepsilon c-1)\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X\}\\&+(c-\varepsilon)\{\eta(Y)g(X,Z)\xi-\eta(X)g(Y,Z)\xi+g(X,\varphi Z)\varphi Y\\&-g(Y,\varphi Z)\varphi X+2g(X,\varphi Y)\varphi Z\}.\end{aligned}$$

Here we prove:

Theorem 1. *The necessary and sufficient condition for a Kenmotsu pseudo-metric manifold M to have constant φ-sectional curvature c is*

$$\begin{aligned}4R(X,Y)Z =&(c-3\varepsilon)\{g(Y,Z)X-g(X,Z)Y\}\\&+(c+\varepsilon)\{\varepsilon\eta(X)\eta(Z)Y-\varepsilon\eta(Y)\eta(Z)X\\&+\eta(Y)g(X,Z)\xi-\eta(X)g(Y,Z)\xi+g(X,\varphi Z)\varphi Y\\&-g(Y,\varphi Z)\varphi X+2g(X,\varphi Y)\varphi Z\}.\end{aligned}\tag{4.3}$$

Proof. Suppose that M has constant φ-sectional curvature c. Then for all vector fields $U,V\in$ Ker η, we have

$$R(U,\varphi U,U,\varphi U)=-cg(U,U)^2.\tag{4.4}$$

Using (4.1), we get

$$\begin{aligned}R(U,\varphi V,U,\varphi V)=&R(U,\varphi V,V,\varphi U)+\varepsilon\{g(U,U)g(V,V)\\&-g(U,V)^2-g(U,\varphi V)^2\},\end{aligned}\tag{4.5}$$

$$R(U,\varphi U,V,\varphi U)=R(U,\varphi U,U,\varphi V),\tag{4.6}$$

for all $U,V\in$ Ker η. Putting $U+V$ in (4.4), and using(4.2), (4.5), (4.6) and the first Bianchi identity, we obtain

$$\begin{aligned}&2R(U,\varphi U,U,\varphi V)+2R(V,\varphi V,V,\varphi U)+3R(U,\varphi V,V,\varphi U)+R(U,V,U,V)\\&=-c\{2g(U,V)^2+2g(U,U)g(U,V)+2g(U,V)g(V,V)+g(U,U)g(V,V)\}.\end{aligned}$$

Replacing V by $-V$ and then summing the resulting equation to the above equation gives

$$3R(U,\varphi V,V,\varphi U)+R(U,V,U,V)=-c\{2g(U,V)^2+g(U,U)g(V,V)\}.\tag{4.7}$$

Replacing V by φV in (4.7) and then using the identities (4.2) and (4.5), we get

$$4R(U,V,U,V)=(c-3\varepsilon)\{g(U,V)^2-g(U,U)g(V,V)\}-3(c+\varepsilon)g(U,\varphi V)^2.\tag{4.8}$$

For $U,V,Z,W\in$ Ker η, we determine $R(U+Z,V+W,U+Z,V+W)$ and then using (4.8) we obtain

$$4R(U,V,Z,W)+4R(U,W,Z,V)=(c-3\varepsilon)\{g(U,V)g(Z,W)$$

$$+g(U,W)g(V,Z)-2g(U,Z)g(V,W)\}-3(c+\varepsilon)\{g(U,\varphi V)g(Z,\varphi W) \\ +g(U,\varphi W)g(Z,\varphi V)\}. \tag{4.9}$$

Interchanging V and Z in (4.9), and then subtracting the resulting equation with (4.9) and by virtue of the first Bianchi identity we obtain

$$4R(U,W,Z,V)=(c-3\varepsilon)\{g(U,V)g(Z,W)-g(U,Z)g(V,W)\} \\ -(c+\varepsilon)\{g(U,\varphi V)g(Z,\varphi W)-g(U,\varphi Z)g(V,\varphi W)+2g(U,\varphi W)g(Z,\varphi V)\}. \tag{4.10}$$

Now if $X,Y,Z,W\in TM$, then replacing U,V,Z,W by $\varphi X,\varphi Y,\varphi Z,\varphi W$ in (4.10), and using (4.1), (4.2), and $\eta(R(X,Y)Z)=\eta(Y)g(X,Z)-\eta(X)g(Y,Z)$ we get (4.3). The converse is trivial. □

Theorem 2. *If a Kenmotsu pseudo-metric manifold has constant φ-sectional curvature c, then it is a space of constant curvature and $c=-\varepsilon$.*

Proof. From (4.3), it is easy to obtain (3.7), where $a=\frac{1}{2}(n(c-3\varepsilon)+(c+\varepsilon))$ and $b=\frac{-1}{2}\varepsilon(n+1)(c+\varepsilon)$. Since a and b are constants, from Corollary 3 it follows that $c=-\varepsilon$. □

5. Some structure theorems

The tangent space T_pM of an almost contact pseudo-metric manifold M can be decomposed as $T_pM=\varphi(T_pM)\oplus L(\xi_p)$, where $L(\xi_p)$ is the linear subspace of T_pM generated by ξ_p. Thus the conformal curvature tensor C is defined as a map

$$C:T_pM\times T_pM\times T_pM\to\varphi(T_pM)\oplus L(\xi_p),\qquad p\in M,$$

such that

$$C(X,Y)Z=R(X,Y)Z-\frac{1}{2n-1}\{Ric(Y,Z)X+g(Y,Z)QX-Ric(X,Z)Y \\ -g(X,Z)QY\}+\frac{r}{2n(2n-1)}\{g(Y,Z)X-g(X,Z)Y\}. \tag{5.1}$$

Then there arise three cases:

- The projection of the image of C in $\varphi(T_pM)$ is zero, that is,
$$C(X,Y,Z,\varphi W)=0,\qquad \text{for any } X,Y,Z,W\in T_pM. \tag{5.2}$$
- Projection of the image of C in $L(\xi_p)$ is zero, that is,
$$C(X,Y)\xi=0,\qquad \text{for all } X,Y\in T_pM. \tag{5.3}$$
- Projection of the image of $C\mid_{\varphi(T_pM)\times\varphi(T_pM)\times\varphi(T_pM)}$ in $\varphi(T_pM)$ is zero, that is,
$$\varphi^2C(\varphi X,\varphi Y)\varphi Z=0,\qquad \text{for all } X,Y,Z\in T_pM. \tag{5.4}$$

An almost contact pseudo-metric manifold satisfying the cases (5.2), (5.3) and (5.4) are said to be conformally symmetric [14], ξ-conformally flat [13] and φ-conformally flat [2], respectively.

We begin with the following:

Theorem 3. *Let M be a ξ-conformally flat Kenmotsu pseudo-metric manifold of dimension higher than 3. Then the scalar curvature r of M satisfies*

$$Dr = \varepsilon\xi(r)\xi, \tag{5.5}$$

where D denotes gradient.

Proof. Since M is ξ-conformally flat, from (5.3) the equation (5.1) becomes

$$\begin{aligned} R(U,V)\xi =& \frac{1}{2n-1}\{Ric(V,\xi)U + \varepsilon\eta(V)QU - Ric(U,\xi)V - \varepsilon\eta(U)QV\} \\ &- \frac{\varepsilon r}{2n(2n-1)}\{\eta(V)U - \eta(U)V\}, \end{aligned} \tag{5.6}$$

for any $U, V \in TM$, and this further gives

$$\begin{aligned} R(U,\xi)V =& \frac{1}{2n-1}\{g(V,Q\xi)U + \varepsilon\eta(V)QU - g(QU,V)\xi - g(U,V)Q\xi\} \\ &- \frac{r}{2n(2n-1)}\{\varepsilon\eta(V)U - g(U,V)\xi\}. \end{aligned} \tag{5.7}$$

Putting $V = \xi$ in (5.6), then differentiating it covariently along W and using (5.7), we get:

$$\begin{aligned} (\nabla_W R)(U,\xi)\xi =& \frac{1}{2n-1}\{g((\nabla_W Q)\xi,\xi)U + \varepsilon(\nabla_W Q)U - g((\nabla_W Q)U,\xi)\xi \\ &- \varepsilon\eta(U)(\nabla_W Q)\xi\} - \frac{Wr}{2n(2n-1)}\{\varepsilon U - \varepsilon\eta(U)\xi\}. \end{aligned}$$

Taking the inner product of the above equation with Y and contracting with respect to U and W yield

$$\begin{aligned} \sum_{i=1}^{2n+1} \varepsilon_i g((\nabla_{e_i} R)(e_i,\xi)\xi, Y) =& \frac{1}{2n-1}\{g((\nabla_Y Q)\xi - (\nabla_\xi Q)Y,\xi)\} \\ &+ \frac{\varepsilon(2n-2)}{4n(2n-1)}\{Yr - \eta(Y)\xi(r)\}, \end{aligned} \tag{5.8}$$

where $\{e_i\}$ is a pseudo-orthonormal basis in M and $\varepsilon_i = g(e_i, e_i)$. From the second Bianchi identity we easily obtain

$$\sum_{i=1}^{2n+1} \varepsilon_i g((\nabla_{e_i} R)(Y,\xi)\xi, e_i) = g((\nabla_Y Q)\xi - (\nabla_\xi Q)Y,\xi). \tag{5.9}$$

Then from (5.8) and (5.9), noting that $n > 1$ we get

$$g((\nabla_Y Q)\xi - (\nabla_\xi Q)Y, \xi) = \frac{\varepsilon}{4n}\{Yr - \eta(Y)\xi(r)\}.$$

Since ∇Q is symmetric, the above equation becomes

$$g((\nabla_Y Q)\xi, \xi) - g((\nabla_\xi Q)\xi, Y) = \frac{\varepsilon}{4n}\{Yr - \eta(Y)\xi(r)\}. \tag{5.10}$$

From (3.4), the left hand side of above equation vanishes. Then (5.10) leads to $Yr = \eta(Y)\xi(r)$ which gives (5.5). □

Theorem 4. *A Kenmotsu pseudo-metric manifold M is ξ-conformally flat if and only if it is an η-Einstein manifold.*

Proof. If M is ξ-conformally flat, then

$$\begin{aligned} R(X,\xi)\xi =& \frac{1}{2n-1}\{Ric(\xi,\xi)X + \varepsilon QX - Ric(X,\xi)\xi - \varepsilon\eta(X)Q\xi\} \\ & - \frac{\varepsilon r}{2n(2n-1)}\{X - \eta(X)\xi\}. \end{aligned}$$

Making use of equations (3.1) and (3.4) in above gives

$$Q = \left(\frac{r}{2n} + \varepsilon\right) I - \left(\frac{\varepsilon r}{2n} + 2n + 1\right)\varepsilon\eta \otimes \xi,$$

which is equivalent to (3.10).

Conversely, suppose that M is η-Einstein. Formula (5.1) gives

$$\begin{aligned} C(X,Y)\xi =& R(X,Y)\xi - \frac{1}{2n-1}\{Ric(Y,\xi)X + \varepsilon\eta(Y)QX - Ric(X,\xi)Y \\ & - \varepsilon\eta(X)QY\} + \frac{\varepsilon r}{2n(2n-1)}\{\eta(Y)X - \eta(X)Y\}. \end{aligned}$$

Now using identities (3.1), (3.4) and (3.13) results in

$$\begin{aligned} C(X,Y)\xi =& R(X,Y)\xi - \frac{1}{2n-1}\left\{(2n - \varepsilon a) + \frac{\varepsilon r}{2n}\right\}(\eta(X)Y - \eta(Y)X) \\ =& R(X,Y)\xi - (\eta(X)Y - \eta(Y)X) = 0, \end{aligned}$$

and this concludes the proof. □

Theorem 5. *A Kenmotsu pseudo-metric manifold of dimension higher than 3 is φ-conformally flat if and only if it is a space of constant cuvature $-\varepsilon$.*

Proof. Note that the φ-conformally flat condition $\varphi^2 C(\varphi X, \varphi Y)\varphi Z = 0$ is equivalent to $C(\varphi X, \varphi Y, \varphi Z, \varphi W) = 0$, and so from (5.1) we get

$$\begin{aligned} & R(\varphi X, \varphi Y, \varphi Z, \varphi W) \\ =& \frac{1}{2n-1}\{Ric(\varphi Y, \varphi Z)g(\varphi X, \varphi W) + g(\varphi Y, \varphi Z)Ric(\varphi X, \varphi W) \\ & - Ric(\varphi X, \varphi Z)g(\varphi Y, \varphi W) - g(\varphi X, \varphi Z)Ric(\varphi Y, \varphi W)\} \end{aligned}$$

$$-\frac{r}{2n(2n-1)}\{g(\varphi Y,\varphi Z)g(\varphi X,\varphi W)-g(\varphi X,\varphi Z)g(\varphi Y,\varphi W)\}. \tag{5.11}$$

Let $\{E_i = e_i, E_{n+i} = \varphi e_i, E_{2n+1} = \xi\}_{i=1}^{n}$ be a local pseudo-orthonormal φ-basis. Taking $X = W = E_i$ in (5.11) and summing, we get

$$\begin{aligned}
&\sum_{i=1}^{2n}\varepsilon_i R(\varphi E_i,\varphi Y,\varphi Z,\varphi E_i)\\
&\quad=\sum_{i=1}^{2n}\varepsilon_i\Bigg[\frac{1}{2n-1}\{Ric(\varphi Y,\varphi Z)g(\varphi E_i,\varphi E_i)+g(\varphi Y,\varphi Z)Ric(\varphi E_i,\varphi E_i)\\
&\qquad-Ric(\varphi E_i,\varphi Z)g(\varphi Y,\varphi E_i)-g(\varphi E_i,\varphi Z)Ric(\varphi Y,\varphi E_i)\}\\
&\qquad-\frac{r}{2n(2n-1)}\{g(\varphi Y,\varphi Z)g(\varphi E_i,\varphi E_i)-g(\varphi E_i,\varphi Z)g(\varphi Y,\varphi E_i)\}\Bigg]\\
&\quad=\left(\frac{2n-2}{2n-1}\right)Ric(\varphi Y,\varphi Z)+\frac{1}{2n-1}\left(\frac{r}{2n}+\varepsilon 2n\right)g(\varphi Y,\varphi Z),
\end{aligned} \tag{5.12}$$

where $\varepsilon_i = g(E_i, E_i)$. It can be easily verified that

$$\begin{aligned}
\sum_{i=1}^{2n}\varepsilon_i R(\varphi E_i,\varphi Y,\varphi Z,\varphi E_i) &= Ric(\varphi Y,\varphi Z)-\varepsilon R(\xi,\varphi Y,\varphi Z,\xi)\\
&= Ric(\varphi Y,\varphi Z)+\varepsilon g(\varphi Y,\varphi Z).
\end{aligned}$$

So that equation (5.12) becomes

$$Ric(\varphi Y,\varphi Z)=\left(\varepsilon+\frac{r}{2n}\right)g(\varphi Y,\varphi Z).$$

Substituting this in (5.11), one obtains

$$\begin{aligned}
&R(\varphi X,\varphi Y,\varphi Z,\varphi W)\\
&=\frac{r+4n\varepsilon}{2n(2n-1)}\{g(\varphi Y,\varphi Z)g(\varphi X,\varphi W)-g(\varphi X,\varphi Z)g(\varphi Y,\varphi W)\}.
\end{aligned} \tag{5.13}$$

From (4.2), (4.1), (3.2) and (2.2), we get

$$\begin{aligned}
R(\varphi X,\varphi Y,\varphi Z,\varphi W) =& R(X,Y,Z,W)+\eta(Y)\eta(Z)g(X,W)-\eta(X)\eta(Z)g(Y,W)\\
&-\eta(Y)\eta(W)g(X,Z)+\eta(X)\eta(W)g(Y,Z).
\end{aligned} \tag{5.14}$$

Now (5.13) and (5.14) imply

$$\begin{aligned}
R(X,Y,Z,W) =& \frac{r+4n\varepsilon}{2n(2n-1)}\{g(\varphi Y,\varphi Z)g(\varphi X,\varphi W)-g(\varphi X,\varphi Z)g(\varphi Y,\varphi W)\}\\
&-\eta(Y)\eta(Z)g(X,W)+\eta(X)\eta(Z)g(Y,W)\\
&+\eta(Y)\eta(W)g(X,Z)-\eta(X)\eta(W)g(Y,Z).
\end{aligned} \tag{5.15}$$

Now taking the scalar product of (4.1) with W and by virtue of (5.15) we get an equation and then contracting the resulting equation with respect to X and W gives

$$(2n-2)\left(\frac{r+4n\varepsilon}{2n(2n-1)}+\varepsilon\right)g(Y,\varphi Z)=0.$$

Since $n>1$, it follows that

$$r=-\varepsilon 2n(2n+1). \tag{5.16}$$

Using (5.16) and (2.2) in (5.15), we get

$$R(X,Y,Z,W)=-\varepsilon\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\},$$

and so that the manifold is of constant curvature $-\varepsilon$.

The converse is trivial. □

Corollary 4. *A conformally flat Kenmotsu pseudo-metric manifold of dimension higher than 3 is a space of constant cuvature $-\varepsilon$.*

The above corollary for Riemannian case has been proved in [9].

Now contracting (5.15), we obtain (3.10). Thus we can state the following:

Corollary 5. *A φ-conformally flat Kenmotsu pseudo-metric manifold is an η-Einstein manifold.*

In view of Theorem 4 and Corrollary 5, we have the following:

Corollary 6. *A φ-conformally flat Kenmotsu pseudo-metric manifold is always ξ-conformally flat.*

6. Ricci soliton on Kenmotsu pseudo-metric manifolds

A *Ricci soliton* on a pseudo-Riemannian manifold (M,g) is defined by

$$(\pounds_V g)(X,Y)+2Ric(X,Y)+2\lambda g(X,Y)=0, \tag{6.1}$$

where λ is a constant. Ricci soliton is a natural generalization of the Einstein metric (that is, $Ric(X,Y)=ag(X,Y)$, for some constant a), and is a special self similar solution of Hamilton's Ricci flow (see [8]) $\frac{\partial}{\partial t}g(t)=-2Ric(t)$ with initial condition $g(0)=g$. We say that the Ricci soliton is *steady* when $\lambda=0$, *expanding* when $\lambda>0$ and *shrinking* when $\lambda<0$.

Before producing the main results, we prove the following:

Lemma 2. *A Kenmotsu pseudo-metric manifold (M,g) satisfies*

$$(\nabla_X Q)\xi=-QX-2n\varepsilon X, \tag{6.2}$$

$$(\nabla_\xi Q)X=-2QX-4n\varepsilon X. \tag{6.3}$$

Proof. Differentiating $Q\xi = -2n\varepsilon\xi$, and recalling (2.6) provide (6.2).
Now differentiating (3.1) along W leads to

$$(\nabla_W R)(X,Y)\xi = -R(X,Y)W + \varepsilon g(X,W)Y - \varepsilon g(Y,W)X.$$

Contracting this with respect to X and W gives us

$$\sum_{i=1}^{2n+1} \varepsilon_i g((\nabla_{e_i} R)(e_i,Y)\xi, Z) = Ric(Y,Z) + 2ng(Y,Z). \tag{6.4}$$

From the second Bianchi identity, one can easily obtain

$$\sum_{i=1}^{2n+1} \varepsilon_i g((\nabla_{e_i} R)(Z,\xi)Y, e_i) = g((\nabla_Z Q)\xi, Y) - g((\nabla_\xi Q)Z, Y). \tag{6.5}$$

Fetching (6.5) into (6.4) and with the aid of (6.2), we infer that

$$g((\nabla_\xi Q)Z, Y) = -2Ric(Y,Z) - 4ng(Y,Z),$$

which proves (6.3). □

Theorem 6. *Let (M,g) be a Kenmotsu pseudo-metric manifold. If (g,V) is a Ricci soliton, then the soliton constant $\lambda = 2n\varepsilon$, and so the soliton is either expanding or shrinking depending on the casual character of the Reeb vector field ξ.*

Proof. Differentiating (6.1) covariantly along Z gives

$$(\nabla_Z \pounds_V g)(X,Y) = -2(\nabla_Z Ric)(X,Y). \tag{6.6}$$

From Yano [12], we know the following well known commutation formula:

$$\begin{aligned}(\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y,Z)\\ = -g((\pounds_V \nabla)(X,Y), Z) - g((\pounds_V \nabla)(X,Z), Y),\end{aligned}$$

for all $X,Y,Z \in TM$. Since $\nabla g = 0$, the previous equation gives

$$(\nabla_X \pounds_V g)(Y,Z) = g((\pounds_V \nabla)(X,Y), Z) + g((\pounds_V \nabla)(X,Z), Y), \tag{6.7}$$

for all $X,Y,Z \in TM$. As $\pounds_V \nabla$ is a symmetric, it follows from (6.7) that

$$\begin{aligned}&g((\pounds_V \nabla)(X,Y), Z)\\ &= \frac{1}{2}(\nabla_X \pounds_V g)(Y,Z) + \frac{1}{2}(\nabla_Y \pounds_V g)(Z,X) - \frac{1}{2}(\nabla_Z \pounds_V g)(X,Y).\end{aligned} \tag{6.8}$$

Making use of (6.6) in (6.8) we have

$$g((\pounds_V \nabla)(X,Y), Z) = (\nabla_Z Ric)(X,Y) - (\nabla_X Ric)(Y,Z) - (\nabla_Y Ric)(Z,X). \tag{6.9}$$

Putting $Y = \xi$ in (6.9) and using (6.2) and (6.3), we obtain

$$(\pounds_V \nabla)(X,\xi) = 2QX + 4n\varepsilon X.$$

Differentiating the preceding equation along Y and using (2.6), we obtain

$$(\nabla_Y \pounds_V \nabla)(X,\xi) = -(\pounds_V \nabla)(X,Y) + 2\eta(Y)\{QX + 2n\varepsilon X\} + 2(\nabla_Y Q)X.$$

Feeding the above obtained expression into the following relation (see [12])

$$(\pounds_V R)(X,Y)Z = (\nabla_X \pounds_V \nabla)(Y,Z) - (\nabla_Y \pounds_V \nabla)(X,Z), \tag{6.10}$$

and using the symmetry of $\pounds_V \nabla$, we immediately obtain

$$\begin{aligned}(\pounds_V R)(X,Y)\xi =& 2\eta(X)\{QY + 2n\varepsilon Y\} - 2\eta(Y)\{QX + 2n\varepsilon X\} \\ &+ 2\{(\nabla_X Q)Y - (\nabla_Y Q)X\}.\end{aligned} \tag{6.11}$$

Setting $Y = \xi$ in the foregoing equation, we get

$$(\pounds_V R)(X,\xi)\xi = 0. \tag{6.12}$$

Now taking the Lie-derivative of $R(X,\xi)\xi = -X + \eta(X)\xi$ along V gives

$$(\pounds_V R)(X,\xi)\xi - 2\eta(\pounds_V \xi)X + \varepsilon g(X, \pounds_V \xi)\xi = (\pounds_V \eta)(X)\xi,$$

which by virtue of (6.12) becomes

$$(\pounds_V \eta)(X)\xi = -2\eta(\pounds_V \xi)X + \varepsilon g(X, \pounds_V \xi)\xi. \tag{6.13}$$

With the help of (3.4), the equation (6.1) takes the form

$$(\pounds_V g)(X,\xi) = -2\lambda\varepsilon\eta(X) + 4n\eta(X). \tag{6.14}$$

Changing X to ξ in the aforementioned equation gives

$$\eta(\pounds_V \xi) = \lambda - 2n\varepsilon. \tag{6.15}$$

Now Lie-differentiating $\eta(X) = \varepsilon g(X,\xi)$ yields $(\pounds_V \eta)(X) = \varepsilon(\pounds_V g)(X,\xi) + \varepsilon g(X, \pounds_V \xi)$. Using this and (6.15) in (6.13) provides $(\lambda - 2n\varepsilon)(X - \eta(X)\xi) = 0$. Tracing the previous equation yield $\lambda = 2n\varepsilon$. □

Corollary 7. *A Kenmotsu manifold admitting the Ricci soliton is always expanding with $\lambda = 2n$.*

Lemma 3. *Let (M,g) be a Kenmotsu pseudo-metric manifold. If (g,V) is a Ricci soliton, then the Ricci tensor satisfies*

$$(\pounds_V Ric)(X,\xi) = -X(r) + \xi(r)\eta(X). \tag{6.16}$$

Proof. Contracting equation (6.11) with respect to X and recalling the well-known formulas

$$\operatorname{div} Q = \frac{1}{2} Dr \quad \text{and} \quad \operatorname{trace} \nabla Q = Dr,$$

we easily obtain

$$(\pounds_V Ric)(Y,\xi) = -Y(r) - 2\eta(Y)\{r + \varepsilon 2n(2n+1)\}. \tag{6.17}$$

Substituting $Y = \xi$, we have $(\pounds_V Ric)(\xi,\xi) = -\xi(r) - 2\{r + \varepsilon 2n(2n+1)\}$. On the other hand, contracting (6.12) gives $(\pounds_V Ric)(\xi,\xi) = 0$. Using this in the previous equation leads to

$$\xi(r) = -2(r + \varepsilon 2n(2n+1)). \tag{6.18}$$

Hence (6.18) and (6.17) give (6.16). □

Combining Theorem 3 and 4, we state the following:

Lemma 4. *An η-Einstein Kenmotsu pseudo-metric manifold M of dimension higher than 3 satisfies*

$$Dr = \varepsilon\xi(r)\xi. \tag{6.19}$$

Now we prove:

Theorem 7. *Let (M,g) be an η-Einstein Kenmotsu pseudo-metric manifold of dimension higher than 3. If (g,V) is a Ricci soliton, then M is Einstein.*

Proof. Making use of (6.19) in (6.16), we have $(\pounds_V Ric)(X,\xi) = 0$. Now, Lie-differentiating the first relation of (3.4) along V, using (3.10), (6.14), $\lambda = 2n\varepsilon$ and $\eta(\pounds_V \xi) = 0$, we obtain

$$(r + \varepsilon 2n(2n+1))\pounds_V \xi = 0.$$

If $r = -\varepsilon 2n(2n+1)$, then (3.10) shows that M is Einstein.

So we assume $r \neq -\varepsilon 2n(2n+1)$ in some open set $\mathcal{O}$ of M. Hence $\pounds_V \xi = 0$ on $\mathcal{O}$, and so it follows from (2.6) that

$$\nabla_\xi V = V - \eta(V)\xi. \tag{6.20}$$

Clearly, (6.14) shows that $(\pounds_V g)(X,\xi) = 0$. This together with (6.20), we have

$$g(\nabla_X V,\xi) = -g(\nabla_\xi V, X) = -g(X,V) + \eta(X)\eta(V). \tag{6.21}$$

From Duggal and Sharma [5], we know that

$$(\pounds_V \nabla)(X,Y) = \nabla_X \nabla_Y - \nabla_{\nabla_X Y} V + R(V,X)Y.$$

Setting $Y = \xi$ in the above equation and by virtue of (2.6), (3.1), (6.20) and (6.21), we have $r = -\varepsilon 2n(2n+1)$. This leads to a contradiction as $r \neq -\varepsilon 2n(2n+1)$ on $\mathcal{O}$ and completes the proof. □

Now we consider Kenmotsu pseudo-metric 3-manifolds which admits Ricci solitons.

Theorem 8. *Let (M,g) be a Kenmotsu pseudo-metric 3-manifold. If (g,V) is a Ricci soliton, then M is of constant curvature $-\varepsilon$.*

Proof. The Riemannian curvature tenor of pseudo-Riemannian 3-manifold is given by

$$\begin{aligned} R(X,Y)Z =& g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y \\ &- \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{6.22}$$

Taking $Y = Z = \xi$ in (6.22) and using (3.1) and (3.4) gives

$$Q = \left(\frac{r}{2}+1\right) I - \left(\frac{r}{2}+3\right) \eta \otimes \xi. \tag{6.23}$$

Making use of this in (6.11) gives

$$\begin{aligned}(\pounds_V R)(X,Y)\xi =& X(r)\{Y - \eta(Y)\xi\} + Y(r)\{-X + \eta(X)\xi\} \\ & - (r+6\varepsilon)\{\eta(Y)X - \eta(X)Y\}.\end{aligned} \tag{6.24}$$

Replacing Y by ξ in the above equation and comparing it with (6.12), we obtain

$$\{\xi(r) + (r+6\varepsilon)\}\{-X + \eta(X)\xi\} = 0.$$

Contracting the above equation gives $\xi(r) + (r+6\varepsilon) = 0$, and consequently it follows from (6.18) that $r = -6\varepsilon$. Then from (6.23) we have $QX = -2\varepsilon X$, and substituting this into (6.22) shows that M is of constant curvature $-\varepsilon$. □

Corollary 8. *There does not exist a Kenmotsu pseudo-metric manifold* (M,g) *admitting the Ricci soliton* $(g, V = \xi)$.

Proof. If $V = \xi$, then from (2.8) the Ricci soliton equation (6.1) would become

$$Ric = -(1+\lambda)g + \varepsilon\eta \otimes \eta, \tag{6.25}$$

which means M is η-Einstein. Then due to Theorem 7 and 8, M must be Einstein, and this will be a contradiction to equation (6.25) as $\varepsilon \neq 0$. □

Remark 1. Clearly, Theorem 7 and 8 are generalizations of the results of Ghosh proved in [6] and [7]. Note that our approach and technique to obtain the result is different to that of Ghosh.

Now we provide an example of a Kenmotsu pseudo-metric 3-manifold which admits a Ricci soliton and verify our results.

Example 1. Let $M = N \times I$, where N is an open connected subset of $\mathbb{R}^2$ and I is an open interval in $\mathbb{R}$. Let (x,y,z) be the Cartesian coordinates in M. Define the structure (φ,ξ,η,g) on M as follows:

$$\varphi\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}, \quad \varphi\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial x}, \quad \varphi\left(\frac{\partial}{\partial z}\right) = 0,$$

$$\xi = \frac{\partial}{\partial z}, \quad \eta = dz,$$

$$(g_{ij}) = \begin{pmatrix} e^{2z} & 0 & 0 \\ 0 & e^{2z} & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}.$$

Now from Koszul's formula, the Levi-Civita connection ∇ is given by

$$\begin{aligned} &\nabla_{\partial_1}\partial_1 = -\varepsilon e^{2z}\partial_3, && \nabla_{\partial_1}\partial_2 = 0, && \nabla_{\partial_1}\partial_3 = \partial_1, \\ &\nabla_{\partial_2}\partial_1 = 0, && \nabla_{\partial_2}\partial_2 = -\varepsilon e^{2z}\partial_3, && \nabla_{\partial_2}\partial_3 = \partial_2, \\ &\nabla_{\partial_3}\partial_1 = \partial_1, && \nabla_{\partial_3}\partial_2 = \partial_2, && \nabla_{\partial_3}\partial_3 = 0, \end{aligned} \tag{6.26}$$

where $\partial_1 = \frac{\partial}{\partial x}, \partial_2 = \frac{\partial}{\partial y}$ and $\partial_3 = \frac{\partial}{\partial z}$. From (6.26), one can easily verify

$$(\nabla_{\partial_i}\varphi)\partial_j = -\eta(\partial_j)\varphi\partial_i - \varepsilon g(\partial_i, \varphi\partial_j)\xi, \tag{6.27}$$

for all $i, j = 1, 2, 3$, and so M is a Kenmotsu pseudo-metric manifold with the above (φ, ξ, η, g) structure.

With the help of (6.26), we find that:

$$\begin{gathered} R(\partial_1, \partial_2)\partial_3 = R(\partial_2, \partial_3)\partial_1 = R(\partial_1, \partial_3)\partial_2 = 0, \\ R(\partial_1, \partial_3)\partial_1 = R(\partial_2, \partial_3)\partial_2 = \varepsilon e^{2z}\partial_3, \\ R(\partial_1, \partial_2)\partial_1 = \varepsilon e^{2z}\partial_2, \quad R(\partial_2, \partial_3)\partial_3 = -\partial_2, \\ R(\partial_1, \partial_3)\partial_3 = -\partial_1, \quad R(\partial_1, \partial_2)\partial_2 = -\varepsilon e^{2z}\partial_1. \end{gathered} \tag{6.28}$$

Let $e_1 = e^{-z}\partial_1, e_2 = e^{-z}\partial_2$ and $e_3 = \xi = \partial_3$. Clearly, $\{e_1, e_2, e_3\}$ forms an orthonormal φ-basis of vector fields on M. Making use of (6.28) one can easily show that M is Einstein, that is, $Ric(Y, Z) = -2\varepsilon g(Y, Z)$, for any $Y, Z \in TM$.

Let us consider the vector field

$$V = \alpha\frac{\partial}{\partial y}, \tag{6.29}$$

where α is a non-zero constant. Making use of (6.26) one can easily show that V is Killing with respect to g, that is, we have

$$(\pounds_V g)(X, Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X) = 0,$$

for any $X, Y \in TM$. Hence g is a Ricci soliton, that is, (6.1) holds true with V as in (6.29) and $\lambda = 2\varepsilon$. Further (6.28) shows that

$$R(X, Y)Z = -\varepsilon\{g(Y, Z)X - g(X, Z)Y\},$$

for any $X, Y \in TM$, which means M is of constant curvature $-\varepsilon$ and so Theorem 8 is verified.

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Authors' addresses

Devaraja Mallesha Naik
Department of Mathematics, Kuvempu University, Shankaraghatta, 577-451 Shivamogga, India
E-mail address: devarajamaths@gmail.com

Venkatesha
Kuvempu University, Department of Mathematics, Shankaraghatta, 577-451 Shivamogga, India
E-mail address: vensmath@gmail.com

D.G. Prakasha
Department of Mathematics, Karnatak University, Dharwad, Karnataka, India and Department of Mathematics, Davangere University, Davangere, India
E-mail address: prakashadg@gmail.com

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A WEIGHTED COMPANION OF OSTROWSKI'S INEQUALITY USING THREE STEP WEIGHTED KERNEL

S. OBEIDAT, M. A. LATIF, AND A. QAYYUM

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Abstract. A weighted version of Ostrowski type integral inequalities is established . We use a newly developed special type of three steps kernel. Our findings give some new error bounds for various quadrature rules. We apply our results to cumulative distributive functions.

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1. Introduction

The importance of mathematical inequalities is due to its applications in several branches of Mathematics such as numerical integration, optimization theory, and integral operator theory. During the past few decades, many researchers worked on inequalities and their applications (see for instance [4]-[5], [7]-[8]).

In 1938, Ostrowski [6] introduced an interesting integral inequality which measures the deviation between a function and its integral mean. It is stated as follows:

Theorem 1. *Let $f:[c,d]\to R$ be continuous on $[c,d]$ and differentiable on (c,d), whose derivative $f':(c,d)\to R$ is bounded on (c,d), i.e.*

$$\|f'\|_\infty = \sup_{t\in[c,d]} |f'(t)| < \infty$$

then for all $x\in[c,d]$

$$\left| f(x) - \frac{1}{d-c}\int_c^d f(t)dt \right| \le \left[\frac{1}{4} + \left(\frac{x-\frac{c+d}{2}}{d-c} \right)^2 \right] (d-c)\|f'\|_\infty . \tag{1.1}$$

Another important inequality is so-called Grüss inequality [5] which links the integral mean of a product of two functions with the product of their integral means. It

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is stated as follows:

$$\left|\frac{1}{d-c}\int_c^d f(x)g(x)dx-\frac{1}{d-c}\int_c^d f(x)dx.\frac{1}{d-c}\int_c^d g(x)dx\right| \tag{1.2}$$
$$\leq \frac{1}{4}(\alpha_2-\alpha_1)(\theta_2-\theta_1),$$

where

$$\alpha_1 \leq f(x) \leq \alpha_2 \text{ and } \theta_1 \leq g(x) \leq \theta_2,$$

for all $x \in [c,d]$. The constant $\frac{1}{4}$ is sharp in (1.2) .

In [3], Dragomir and Wang combined Ostrowski and Grüss inequality and obtained a new inequality known in the literature as Ostrowski-Grüss type inequalities. Dragomir [2] established some companions of Ostrowski type integral inequalities.

In [1], Alomari proved the following companion inequality of Ostrowski's type using Grüss inequality.

Theorem 2. *Let $f:[c,d]\to\mathbb{R}$ be a differentiable function on (c,d), $-\infty<c<d<\infty$. If $f'\in L^1[c,d]$ and $B_1\leq f'(t)\leq B_2$ for all $t\in[c,d]$, where B_1, B_2 are constants, then the inequality*

$$\left|\frac{f(x)+f(c+d-x)}{2}-\frac{1}{d-c}\int_c^d f(t)dt\right| \leq \frac{1}{8}(d-c)(B_2-B_1) \tag{1.3}$$

holds for all $x\in\left[c,\frac{c+d}{2}\right]$.

In the proof of Theorem 2, Alomari defined the following mapping

$$p(x,t)=\begin{cases} t-c & t\in[c,x] \\ t-\frac{a+b}{2} & t\in(x,c+d-x] \\ t-b & t\in(c+d-x,d] \end{cases}$$

for all $x\in\left[c,\frac{c+d}{2}\right]$. The upper bound of Inequality (1.3) is obtained using the fact that

$$x-\frac{c+d}{2}\leq p(x,t)\leq x-c \tag{1.4}$$

for each $t\in[c,d]$ and each $x\in\left[c,\frac{c+d}{2}\right]$. **However, this fact is not correct**. For example, if $x=\frac{c+d}{2}-\frac{d-c}{100}$ and $t=\frac{c+d}{2}+\frac{d-c}{50}$, then

$$p(x,t)=t-d=\frac{-12(d-c)}{25},$$

and

$$x-\frac{c+d}{2}=\frac{-(d-c)}{100}>p(x,t),$$

which contradics Inequality (1.4). The correct property of $p(x,t)$ is that

$$\min\left\{x-\frac{c+d}{2},c-x\right\}\leq p(x,t)\leq\max\left\{\frac{c+d}{2}-x,x-c\right\}. \tag{1.5}$$

for each $t\in[c,d]$ and each $x\in\left[c,\frac{c+d}{2}\right]$. Using this property of $p(x,t)$, the correction of Inequality (1.3) becomes

$$\left|\frac{f(x)+f(c+d-x)}{2}-\frac{1}{d-c}\int_c^d f(t)dt\right|\leq\frac{1}{4}(d-c)(B_2-B_1). \tag{1.6}$$

In this paper, we will present a weighted version of Inequality (1.6) using a new three step kernel, and discuss some applications of our results. Throughout the present paper, a weight function (or density function) over some interval $[c,d]$, where $-\infty<c<d<\infty$, is a function $w:[c,d]\longrightarrow[0,\infty)$ with $0<\int_c^d w(t)dt<\infty$.

2. Main results

Definition 1. Let $-\infty<c<d<\infty$. Let w be a weight function over $[c,d]$. The 3-step linear kernel with respect to w is denoted by G_w, and is defined as follows:

$$G_w(x,t)=\begin{cases}\int_c^t w(u)\,du, & t\in[c,x]\\ \int_c^t w(u)\,du-\frac{1}{2}\int_c^d w(u)\,du, & t\in(x,c+d-x]\\ \int_d^t w(u)\,du, & t\in(c+d-x,d]\end{cases}, \tag{2.1}$$

for $x\in\left[c,\frac{c+d}{2}\right]$ and $t\in[0,1]$.

The following lemma will be used repeatedly throughout the present paper.

Lemma 1. *Let $-\infty<c<d<\infty$. For a weight function w over $[c,d]$, the identity*

$$\begin{aligned}&\int_c^d G_w(x,t)f'(t)dt \qquad (2.2)\\ &=\frac{1}{2}\left[\int_c^d w(t)\,dt\right][f(x)+f(c+d-x)]\\ &\quad-\int_c^d w(t)f(t)\,dt,\end{aligned}$$

holds for all $x \in \left[c, \frac{c+d}{2}\right]$ *and* $t \in [0,1]$.

Proof. Using Definition 1, we have

$$\int_c^d G_w(x,t) f'(t)dt = \int_c^x \left[\int_c^t w(u)\,du\right] f'(t)dt + \int_x^{c+d-x} \left[\int_c^t w(u)\,du - \frac{1}{2}\int_c^d w(u)\,du\right] f'(t)dt + \int_{c+d-x}^d \left[\int_b^t w(u)\,du\right] f'(t)dt.$$

Applying integration by parts on each integral, we get

$$\int_c^x \left[\int_c^t w(u)\,du\right] f'(t)dt = \left[\int_c^x w(u)\,du\right] f(x) - \int_c^x w(t) f(t)\,dt,$$

$$\int_x^{c+d-x} \left[\int_c^t w(u)\,du - \frac{1}{2}\int_c^d w(u)\,du\right] f'(t)dt = \left(\int_c^{c+d-x} w(u)\,du - \frac{1}{2}\int_c^d w(u)\,du\right) f(c+d-x) - \left(\int_c^x w(u)\,du - \frac{1}{2}\int_c^d w(u)\,du\right) f(x) - \int_x^{c+d-x} w(t) f(t)\,dt,$$

and

$$\int_{c+d-x}^d \left[\int_d^t w(u)\,du\right] f'(t)dt = \left[\int_{c+d-x}^d w(u)\,du\right] f(c+d-x) - \int_{c+d-x}^d w(t) f(t)\,dt.$$

Thus,

$$\begin{aligned}
&\int_c^d G_w(x,t) f'(t)dt \\
&= \frac{1}{2}\left[\int_c^d w(t)\,dt\right][f(x)+f(c+d-x)] \\
&\quad -\int_c^d w(t)\,f(t)\,dt.
\end{aligned}$$

□

Remark 1. If we set $f(t)=t$ in Identity (2.2), we have

$$\int_c^d G_w(x,t)dt = \frac{c+d}{2}\left[\int_c^d w(t)\,dt\right] - \int_c^d w(t)\,dt, \tag{2.3}$$

for $x \in \left[c, \frac{c+d}{2}\right]$.

Lemma 2. *Let $-\infty < c < d < \infty$. If w is a weight function over $[c,d]$, and w is symmetric about $\frac{c+d}{2}$, then*

$$\int_c^d w(t)\,dt = \frac{c+d}{2}\left[\int_c^d w(t)\,dt\right],$$

and hence

$$\int_c^d G_w(x,t)dt = 0,$$

for $x \in \left[c, \frac{c+d}{2}\right]$.

Proof. Since w is symmetric about $\frac{c+d}{2}$,

$$\int_c^d t\,w(t)\,dt = \int_c^d (c+d-t)\,w(t)\,dt,$$

which implies that

$$\int_c^d t\,w(t)\,dt = \frac{c+d}{2}\int_c^d w(t)\,dt.$$

Using Identity (2.3), we get that

$$\int_c^d G_w(x,t)dt = 0.$$

□

Theorem 3. *Let $-\infty < c < d < \infty$ and $f:[c,d] \to \mathbb{R}$ be a differentiable function on (c,d). Suppose that w is a weight function over $[c,d]$. If $f' \in L^1[c,d]$ and $B_1 \le f'(t) \le B_2$, for all $t \in [c,d]$, where B_1, B_2 are constants, then the inequality*

$$\left| \left[\frac{1}{2}\left(\int_c^d w(t)\,dt\right)(f(x)+f(c+d-x)) - I(c,d,x)\cdot H(c,d) \right] - \int_c^d w(t)\,f(t)\,dt \right| \le \frac{1}{4}V(c,d,x)(B_2-B_1)(d-c) \tag{2.4}$$

holds for all $x \in \left[c, \frac{c+d}{2}\right]$, where

$$I(c,d,x) = \int_c^d G_w(x,t)dt, H(c,d) = \frac{f(d)-f(c)}{d-c},$$

and

$$V(c,d,x) = \max\left\{ \int_x^{c+d-x} w(t)\,dt, \int_c^d w(t)\,dt - \int_x^{c+d-x} w(t)\,dt \right\}$$

Proof. Note that for $t \in [0,1]$ and $x \in \left[c, \frac{c+d}{2}\right]$, we have

$$L(c,d,x) \le G_w(x,t) \le U(c,d,x), \tag{2.5}$$

where

$$U(c,d,x) = \max\left\{ \int_c^x w(t)\,dt, \int_c^{c+d-x} w(t)\,dt - \frac{1}{2}\int_c^d w(t)\,dt \right\},$$

and

$$L(c,d,x) = \min\left\{ \int_c^x w(t)\,dt - \frac{1}{2}\int_c^d w(t)\,dt, \int_d^{c+d-x} w(t)\,dt \right\}.$$

By Lemma 1, we have

$$\frac{1}{d-c}\int_c^d G_w(x,t)f'(t)dt-\frac{1}{(d-c)^2}\int_c^d G_w(x,t)dt\int_c^d f'(t)dt$$
$$=\frac{1}{2(d-c)}\left[\int_c^d w(t)\,dt\right][f(x)+f(c+d-x)]$$
$$-\frac{f(d)-f(c)}{(d-c)^2}\int_c^d G_w(x,t)dt-\frac{1}{d-c}\int_c^d w(t)\,f(t)\,dt.$$

Applying Grüss Inequality and using Identity (2.5), we get that

$$\left|\frac{1}{d-c}\int_c^d G_w(x,t)f'(t)dt-\frac{1}{(d-c)^2}\int_c^d G_w(x,t)dt\int_c^d f'(t)dt\right|$$
$$\leq\frac{1}{4}(U(c,d,x)-L(c,d,x))(B_2-B_1).$$

But

$$U(c,d,x)-L(c,d,x)=V(c,d,x),$$

which implies that

$$\left|\left[\frac{1}{2}\left(\int_c^d w(t)\,dt\right)(f(x)+f(a+b-x))-I(c,d,x)\cdot H(c,d)\right]\right.$$
$$\left.-\int_a^b w(t)\,f(t)\,dt\right|$$
$$\leq\frac{1}{4}V(c,d,x)(B_2-B_1)(d-c).$$

□

Corollary 1. *Let $-\infty<c<d<\infty$ and $f:[c,d]\to\mathbb{R}$ be a differentiable function on (c,d). Suppose that w is a weight function over $[c,d]$ and w is symmetric about $\frac{c+d}{2}$. If $f'\in L^1[c,d]$ and $B_1\leq f'(t)\leq B_2$, for all $t\in[c,d]$, where B_1, B_2 are constants, then the inequality*

$$\left|\frac{1}{2}\left(\int_c^d w(t)\,dt\right)(f(x)+f(c+d-x))\right.$$

$$
\begin{aligned}
&\left. -\int_c^d w(t) f(t)\, dt \right| \\
&\leq \frac{1}{2} \max\left\{ \int_c^x w(t)\, dt, \int_x^{\frac{c+d}{2}} w(t)\, dt \right\} (B_2 - B_1)(d-c) \\
&\leq \frac{1}{2} \left(\int_c^{\frac{c+d}{2}} w(t)\, dt \right) (B_2 - B_1)(d-c). \qquad (2.6)
\end{aligned}
$$

holds for all $x \in \left[c, \frac{c+d}{2}\right]$.

Proof. Note that for $t \in [0,1]$ and $x \in \left[c, \frac{c+d}{2}\right]$, we have

$$
\min\left\{ \int_x^c w(t)\, dt, \int_{\frac{c+d}{2}}^x w(t)\, dt \right\} \leq G_w(x,t) \leq \max\left\{ \int_c^x w(t)\, dt, \int_x^{\frac{c+d}{2}} w(t)\, dt \right\}. \qquad (2.7)
$$

Applying Lemma 2, and following same argument used in the proof of Theorem 3, the result follows. □

Remark 2. If we set $w(t) = 1$, Inequality 2.6 becomes same as Inequality 1.6 .

Theorem 4. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I^0, *the interior of the interval* I, *and let* $c, d \in I$ *with* $c < d$. *Let* w *be a weight function over* $[c,d]$. *If* $f' \in L^1[c,d]$ *with* $B_1 \leq f'(t) \leq B_2$ *for all* $t \in [c,d]$, *where* B_1, B_2 *are constants, then for each* $x \in \left[c, \frac{c+d}{2}\right]$, *we have*

$$
\begin{aligned}
&\left| \left[\frac{1}{2} \left(\int_c^d w(t)\, dt \right) (f(x) + f(a+b-x)) \right.\right. \\
&\qquad \left.\left. - \frac{(B_1 + B_2)}{2} \cdot I(c,d,x) \right] - \int_c^d w(t) f(t)\, dt \right| \\
&\leq \frac{B_2 - B_1}{2} \int_c^d |G_w(x,t)|\, dt. \qquad (2.8)
\end{aligned}
$$

where $I(c,d,x) = \int_c^d G_w(x,t)dt$.

Proof. Using Lemma 1, we have

$$\int_c^d G_w(x,t)\left(f'(t) - \frac{B_1+B_2}{2}\right)dt$$
$$= \frac{1}{2}\left[\int_c^d w(t)\,dt\right][f(x) + f(c+d-x)]$$
$$- \frac{(B_1+B_2)}{2}\int_c^d G_w(x,t)dt - \int_c^d f(t)\,w(t)\,dt.$$

Note that, for each $t \in [c,d]$,

$$\frac{B_1-B_2}{2} \le f'(t) - \frac{B_1+B_2}{2} \le \frac{B_2-B_1}{2}$$

which implies that

$$\max_{t\in[c,d]}\left|f'(t) - \frac{B_1+B_2}{2}\right| \le \frac{B_2-B_1}{2}.$$

Thus,

$$\left|\int_c^d G_w(x,t)\left(f'(t) - \frac{B_1+B_2}{2}\right)dt\right|$$
$$\le \max_{t\in[c,d]}\left|f'(t) - \frac{(B_1+B_2)}{2}\right|\int_c^d |G_w(x,t)|\,dt$$
$$\le \frac{(B_2-B_1)}{2}\int_c^d |G_w(x,t)|\,dt,$$

which implies that

$$\left|\left[\frac{1}{2}\left(\int_c^d w(t)\,dt\right)(f(x)+f(c+d-x)) - \frac{(B_1+B_2)}{2}\cdot I(c,d,x)\right]\right.$$
$$\left. - \int_c^d w(t)\,f(t)\,dt\right| \le \frac{(B_2-B_1)}{2}\int_c^d |G_w(x,t)|\,dt.$$

□

Corollary 2. *Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^0, the interior of the interval I, and let $c, d \in I$ with $c < d$. Suppose that w is a weight function over $[c,d]$ and w is symmetric about $\frac{c+d}{2}$. If $f' \in L^1[c,d]$ with $B_1 \le f'(t) \le B_2$ for all $t \in [c,d]$, where B_1, B_2 are constants, then for each $x \in [c,d]$, we have*

$$\left| \frac{1}{2}\left[\int_c^d w(t)\,dt\right][f(x)+f(c+d-x)] - \int_c^d f(t)\,w(t)\,dt \right|$$
$$\le \frac{B_2 - B_1}{2}\int_c^d |G_w(x,t)|\,dt. \tag{2.9}$$

Proof. Using Lemma 2 and applying Theorem 4, the result follows. □

Theorem 5. *Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $(c,d) \subset I$. Suppose that w is a weight function over $[c,d]$ and w is symmetric about $\frac{c+d}{2}$. If $f' \in L^1[c,d]$ with $B_1 \le f'(t) \le B_2$ for all $t \in [c,d]$, where B_1, B_2 are constants, then for each $x \in \left[c, \frac{c+d}{2}\right]$, we have*

$$\left| \frac{1}{2}\left[\int_c^d w(t)\,dt\right][f(x)+f(c+d-x)] - \int_c^d f(t)\,w(t)\,dt \right|$$
$$\le (d-c)\left(\frac{f(d)-f(c)}{d-c} - B_1\right)\sup_{t\in[c,d]} |G_w(x,t)|, \tag{2.10}$$

and

$$\left| \left[\frac{1}{2}\int_c^d w(t)\,dt\right][f(x)+f(a+b-x)] - \int_c^d f(t)\,w(t)\,dt \right|$$
$$\le (d-c)\left(B_2 - \frac{f(c)-f(d)}{d-c}\right)\sup_{t\in[c,d]} |G_w(x,t)|. \tag{2.11}$$

Proof. Using Lemma 2

$$\int_a^b G_w(x,t)dt = 0,$$

which implies that

$$\int_c^d G_w(x,t) f'(t)\,dt = \int_c^d G_w(x,t)dt\left(f'(t) - B_1\right)dt,$$

and

$$\int_c^d G_w(x,t) f'(t)\,dt = \int_c^d G_w(x,t)dt\left(f'(t)-B_2\right)dt.$$

Using Lemma 1 and the triangle inequality we get

$$\left|\frac{1}{2}\left[\int_c^d w(t)\,dt\right][f(x)+f(c+d-x)]-\int_c^d f(t)\,w(t)\,dt\right|$$
$$\leq \int_c^d \left|G_w(x,t)\left(f'(t)-B_1\right)\right|dt,$$

and

$$\left|\frac{1}{2}\left[\int_c^d w(t)\,dt\right][f(x)+f(c+d-x)]-\int_c^d f(t)\,w(t)\,dt\right|$$
$$\leq \int_c^d \left|G_w(x,t)\left(f'(t)-B_2\right)\right|dt.$$

Note that

$$\int_c^d \left|G_w(x,t)dt\left(f'(t)-B_1\right)\right|dt$$
$$\leq \sup_{t\in[c,d]}|G_w(x,t)|\int_c^d \left|f'(t)-B_1\right|dt$$

and

$$\begin{aligned}\int_c^d \left|f'(t)-B_1\right|dt &= \int_c^d \left(f'(t)-B_1\right)dt\\ &= f(d)-f(c)-B_1(d-c)\\ &= (d-c)\left[\frac{f(d)-f(c)}{d-c}-B_1\right].\end{aligned}$$

Similarly,

$$\int_c^d \left|G_w(x,t)\left(f'(t)-B_2\right)\right|dt$$

$$\leq \sup_{t\in[c,d]} |G_w(x,t)| \int_c^d \left| f'(t) - B_2 \right| dt.$$

and

$$\begin{aligned}\int_c^d \left| f'(t) - D_2 \right| dt &= \int_c^d B_2 - f'(t)\, dt \\ &= B_2(d-c) - (f(d) - f(c)) \\ &= (d-c)\left[B_2 - \frac{f(d)-f(c)}{d-c}\right].\end{aligned}$$

Therefore,

$$\begin{aligned}&\left| \frac{1}{2}\left[\int_c^d w(t)\, dt\right][f(x) + f(c+d-x)] - \int_c^d f(t)\, w(t)\, dt \right| \\ &\leq (d-c)\left(\frac{f(d)-f(c)}{d-c} - B_1\right) \sup_{t\in[c,d]} |G_w(x,t)|,\end{aligned}$$

and

$$\begin{aligned}&\left| \frac{1}{2}\left[\int_c^d w(t)\, dt\right][f(x) + f(c+d-x)] - \int_c^d f(t)\, w(t)\, dt \right| \\ &\leq (d-c)\left(B_2 - \frac{f(d)-f(c)}{d-c}\right) \sup_{t\in[c,d]} |G_w(x,t)|.\end{aligned}$$

□

3. Some applications

Before introducing the first two applications, recall that a tagged partition P of a finite interval $[c,d]$ is a finite sequence of numbers $c = x_0 < x_1 < \cdots < x_n = d$, with corresponding tags $t_i \in [x_{i-1}, x_i]$, for $i = 1, \ldots, n$.

Theorem 6. *Let $-\infty < c < d < \infty$ and $f : [c,d] \to \mathbb{R}$ be a differentiable function on (c,d), and $P : c = x_0 < x_1 < \cdots < x_n = d$ be a tagged partition with corresponding values $t_i \in [x_{i-1}, \frac{x_{i-1}+x_i}{2}]$, for $i = 1, \ldots, n$. Suppose that w is a weight function over $[c,d]$. If $f' \in L^1[c,d]$ and $B_1 \leq f'(t) \leq B_2$, for all $t \in [c,d]$, where B_1, B_2 are constants, then we have the quadrature formula*

$$\int_c^d w(t)\, f(t)\, dt = A(f,P) + R(f,P),$$

where

$$A(f,P)=\sum_{i=0}^{n-1}\left[\frac{1}{2}W(x_i,x_{i+1})(f(t_{i+1})+f(x_i+x_{i+1}-t_{i+1}))\right.$$
$$\left.-I(x_i,x_{i+1},t_{i+1})\cdot H(x_i,x_{i+1})\right],$$

$$W(x_i,x_{i+1})=\int_{x_i}^{x_{i+1}}w(t)dt,0\le i\le n-1,$$

$$I(x_i,x_{i+1},t_{i+1})=\int_{x_i}^{x_{i+1}}G_w(t_{i+1},t)dt,0\le i\le n-1,$$

$$H(x_i,x_{i+1})=\frac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i},0\le i\le n-1,$$

and the remainder satisfies the inequality

$$|R(f,P)|\le\frac{(B_2-B_1)}{4}\sum_{i=0}^{n-1}V(x_i,x_{i+1},t_{i+1})(x_{i+1}-x_i),$$

where

$$V(x_i,x_{i+1},t_{i+1})=\max\left\{\int_{t_{i+1}}^{x_i+x_{i+1}-t_{i+1}}w(t)\,dt,W(x_i,x_{i+1})-\int_{t_{i+1}}^{x_i+x_{i+1}-t_{i+1}}w(t)\,dt\right\}$$

Proof. For each $0\le i\le n-1$, applying Theorem 3 on $[x_i,x_{i+1}]$ with $x=t_{i+1}$, we get that

$$\left|\int_{x_i}^{x_{i+1}}w(t)f(t)\,dt-\left[\begin{array}{c}\frac{1}{2}W(x_i,x_{i+1})(f(t_{i+1})+f(x_i+x_{i+1}-t_{i+1}))\\-I(x_i,x_{i+1},t_{i+1})\cdot H(x_i,x_{i+1})\end{array}\right]\right|$$
$$\le\frac{(B_2-B_1)}{4}V(x_i,x_{i+1},t_{i+1})(x_{i+1}-x_i).$$

Using the triangle inequality, we find that

$$\left|\sum_{i=0}^{n-1}\left(\int_{x_i}^{x_{i+1}}w(t)f(t)\,dt-\left[\begin{array}{c}\frac{1}{2}W(x_i,x_{i+1})(f(t_{i+1})+f(x_i+x_{i+1}-t_{i+1}))\\-I(x_i,x_{i+1},t_{i+1})\cdot H(x_i,x_{i+1})\end{array}\right]\right)\right|$$
$$\le\frac{(B_2-B_1)}{4}\sum_{i=0}^{n-1}V(x_i,x_{i+1},t_{i+1})(x_{i+1}-x_i).$$

But

$$\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} w(t) f(t)\, dt = \int_c^d w(t) f(t)\, dt,$$

which implies that

$$\int_c^d w(t) f(t)\, dt = A(f,P) + R(f,P)$$

and

$$|R(f,P)| \leq \frac{(B_2 - B_1)}{4} \sum_{i=0}^{n-1} V(x_i, x_{i+1}, t_{i+1})(x_{i+1} - x_i).$$

□

Theorem 7. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^0, the interior of the interval I, $c, d \in I$ with $c < d$, and $P : c = x_0 < x_1 < \cdots < x_n = d$ be a tagged partition with corresponding values $t_i \in [x_{i-1}, \frac{x_{i-1}+x_i}{2}]$, for $i = 1, \ldots, n$. Suppose that w is a weight function over $[c,d]$. If $f' \in L^1[c,d]$ with $B_1 \leq f'(t) \leq B_2$ for all $t \in [c,d]$, where B_1, B_2 are constants, then we have the quadrature formula*

$$\int_c^d w(t) f(t)\, dt = A(f,P) + R(f,P),$$

where

$$A(f,P) = \sum_{i=0}^{n-1} \left[\frac{1}{2} W(x_i, x_{i+1})(f(t_{i+1}) + f(x_i + x_{i+1} - t_{i+1})) - B \cdot I(x_i, x_{i+1}, t_{i+1}) \right],$$

$$W(x_i, x_{i+1}) = \int_{x_i}^{x_{i+1}} w(t) dt, , 0 \leq i \leq n-1,$$

$$i(x_i, x_{i+1}, t_{i+1}) = \int_{x_i}^{x_{i+1}} G_w(t_{i+1}, t) dt, , 0 \leq i \leq n-1,$$

$$B = \frac{B_1 + B_2}{2},$$

and the remainder satisfies the inequality

$$|R(f,P)| \leq \frac{(B_2 - B_1)}{2} \sum_{i=0}^{n-1} Z(x_i, x_{i+1}, t_{i+1}),$$

where

$$Z(x_i, x_{i+1}, t_{i+1}) = \int_{x_i}^{x_{i+1}} |G_w(t_{i+1}, t)|\, dt.$$

Proof. For each $0 \le i \le n-1$, applying Theorem 4 on $[x_{i-1}, x_i]$ with $x = t_{i+1}$, we get that

$$\left| \int_{x_i}^{x_{i+1}} w(t) f(t)\, dt - \left[\begin{array}{c} \frac{1}{2} W(x_i, x_{i+1}) (f(t_{i+1}) + f(x_i + x_{i+1} - t_{i+1})) \\ -B \cdot I(x_i, x_{i+1}, t_{i+1}) \end{array} \right] \right|$$

$$\le \frac{(B_2 - B_1)}{2} \int_{x_i}^{x_{i+1}} |G_w(t_{i+1}, t)|\, dt$$

$$= \frac{(B_2 - B_1)}{2} Z(x_i, x_{i+1}, t_{i+1}).$$

Using the triangle inequality, we find that

$$\left| \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} w(t) f(t)\, dt - \left[\begin{array}{c} \frac{1}{2} W(x_i, x_{i+1}) (f(t_{i+1}) + f(x_i + x_{i+1} - t_{i+1})) \\ -B \cdot I(x_i, x_{i+1}, t_{i+1}) \end{array} \right] \right|$$

$$\le \frac{(B_2 - B_1)}{2} \sum_{i=0}^{n-1} Z(x_i, x_{i+1}, t_{i+1}).$$

But

$$\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} w(t) f(t)\, dt = \int_c^d w(t) f(t)\, dt$$

and

$$\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} |G_w(x, t)|\, dt = \int_c^d |G_w(x, t)|\, dt,$$

which implies that

$$\int_c^d w(t) f(t)\, dt = A(f, P) + R(f, P)$$

and

$$|R(f, P)| \le \frac{(B_2 - B_1)}{2} \sum_{i=0}^{n-1} Z(x_i, x_{i+1}, t_{i+1}).$$

□

Before we introduce the next application, recall that if X is a random variable with values in a finite interval $[c,d]$, $c<d$, and $f:[c,d]\to[0,1]$ is a probability density function, then the cumulative distribution function with respect to f is denoted by F and is defined as:

$$F(x)=\int_c^x f(t)\,dt \text{ for each } x\in[c,d].$$

Since f satisfies the condition $\int_c^d f(x)\,dx=1, F(c)=1$.

Theorem 8. *Let X be a random variable with values in a finite interval $[c,d]$, $c<d$, and $f:[c,d]\to[0,1]$ be a probability density function. Let w be a differentiable weight function over $[c,d]$ such that $w(d)=1$ and w be symmetric about $\frac{c+d}{2}$. Let F be the cumulative distribution function with respect to f. If $f\in L^1[c,d]$ and $B_1\le f(t)\le B_2$, for all $t\in[c,d]$, where B_1, B_2 are constants, then the inequality*

$$\begin{aligned}&\left|\frac{1}{2}\left(\int_c^d w(t)\,dt\right)(F(x)+F(c+d-x))-(d-E_G)\right|\\&\le\frac{1}{4}\left[\int_c^{\frac{c+d}{2}} w(t)\,dt\right](B_2-B_1)(d-c).\end{aligned}\tag{3.1}$$

holds for all $x\in\left[c,\frac{c+d}{2}\right]$, where

$$E_G=d-\int_c^d w(t)F(t)\,dt.$$

Proof. Define the function G over $[c,d]$ as follows:

$$G(x)=\int_c^x \frac{d}{dt}(wF)\,dt, x\in[c,d].$$

Note that

$$G(c)=0$$

and

$$\begin{aligned}G(d)&=\int_c^d \frac{d}{dt}(wF)\,dt\\&=w(d)F(d)-w(c)F(c)\\&=1.\end{aligned}$$

Let

$$E_G=\int_c^d t\frac{d}{dt}(wF)\,dt.$$

Using integration by parts, we get that

$$\begin{aligned} E_G &= dw(d)F(d) - cw(c)F(c) - \int_c^d w(t)F(t)\,dt \\ &= d - \int_c^d w(t)F(t)\,dt, \end{aligned}$$

which implies that

$$\int_c^d w(t)F(t)\,dt = d - E_G.$$

Applying Corollary 1 on F, we get that

$$\begin{aligned} &\left| \frac{1}{2}\left(\int_a^b w(t)\,dt\right)(F(x)+F(c+d-x)) - \int_c^d w(t)F(t)\,dt \right| \\ &\leq \frac{1}{4}\left[\int_c^{\frac{c+d}{2}} w(t)\,dt\right](B_2-B_1)(d-c), \end{aligned}$$

which implies that

$$\begin{aligned} &\left| \frac{1}{2}\left(\int_c^d w(t)\,dt\right)(F(x)+F(c+d-x)) - (d-E_G) \right| \\ &\leq \frac{1}{4}\left[\int_c^{\frac{c+d}{2}} w(t)\,dt\right](B_2-B_1)(d-c). \end{aligned} \tag{3.2}$$

□

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Authors' addresses

S. Obeidat
Department of Basic Sciences, University of Hai'l, Saudi Arabia
E-mail address: obeidatsofian@gmail.com

M. A. Latif
Department of Basic Sciences, University of Hai'l, Saudi Arabia
E-mail address: muhammad.mlatif@gmail.com

A. Qayyum
Department of Mathematics, Institute of Southern Punjab, Pakistan
E-mail address: atherqayyum@gmail.com

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QUALITATIVE PROPERTIES OF SOLUTIONS FOR MIXED TYPE FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH MAXIMA

DIANA OTROCOL

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Abstract. In this paper we study some properties of the solutions of a second order system of functional-differential equations with maxima, of mixed type, with "boundary" conditions. We use the weakly Picard operator technique.

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Keywords: weakly Picard operator, equations of mixed type, equations with maxima

1. Introduction

Differential equations with maxima are often met in the applications, for instance in the theory of automatic control. Numerous results on existence and uniqueness, asymptotic stability as well as numerical solutions have been obtained. To name a few, we refer the reader to [1–3, 11] and the references therein.

The main goal of the presented paper is to study a second order functional-differential equations with maxima, of mixed type, using the theory of weakly Picard operators. The theory of Picard operators was introduced by I. A. Rus (see [7] and [9]) to study problems related to fixed point theory. This abstract approach is used by many mathematicians and it seemed to be a very useful and powerful method in the study of integral equations and inequalities, ordinary and partial differential equations (existence, uniqueness, differentiability of the solutions), etc.

We consider the following functional-differential equation

$$-x''(t) = f(t, x(t), \max_{t-h_1 \le \xi \le t+h_2} x(\xi)),\ t \in [a,b] \tag{1.1}$$

with the "boundary" conditions

$$\begin{cases} x(t) = \varphi(t),\ t \in [a-h_1, a], \\ x(t) = \psi(t),\ t \in [b, b+h_2]. \end{cases} \tag{1.2}$$

The novelty of this paper lies in the fact that 'maxima' is taken on interval $[t-h_1, t+h_2]$, where $h_1,\ h_2 > 0$. Our results extend and improve corresponding

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theorems in the existing literature (see, e.g. [4–6, 8]). Also, in the end an extremal principle for the solution is given.

We suppose that:

(C_1) h_1, h_2, a and $b \in \mathbb{R}$, $a < b$, $h_1 > 0$, $h_2 > 0$;
(C_2) $f \in C([a,b] \times \mathbb{R}^2, \mathbb{R})$;
(C_3) there exists $L_f > 0$ such that

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \leq L_f \max_{i=1,2} |u_i - vi|,$$

for all $t \in [a,b]$ and $u_i, v_i \in \mathbb{R}, i = 1,2$;
(C_4) $\varphi \in C([a-h_1,a], \mathbb{R})$ and $\psi \in C([b,b+h_2], \mathbb{R})$.

Let G be the Green function of the following problem

$$-x'' = \chi,\ x(a) = 0,\ x(b) = 0,\ \chi \in C[a,b].$$

The problem (1.1)–(1.2), $x \in C[a-h_1, b+h_2] \cap C^2[a,b]$ is equivalent with the following fixed point equation

$$x(t) = \begin{cases} \varphi(t),\ t \in [a-h_1,a], \\ w(\varphi,\psi)(t) + \\ \quad + \int_a^b G(t,s) f(s, x(s), \max\limits_{s-h_1 \leq \xi \leq s+h_2} x(\xi)) ds,\ t \in [a,b], \\ \psi(t),\ t \in [b, b+h_2], \end{cases} \tag{1.3}$$

$x \in C[a-h_1, b+h_2]$, where

$$w(\varphi,\psi)(t) := \frac{t-a}{b-a}\psi(b) + \frac{b-t}{b-a}\varphi(a).$$

The equation (1.1) is equivalent with

$$x(t) = \begin{cases} x(t),\ t \in [a-h_1,a], \\ w(x|_{[a-h_1,a]}, x|_{[b,b+h_2]})(t) + \\ \quad + \int_a^b G(t,s) f(s, x(s), \max\limits_{s-h_1 \leq \xi \leq s+h_2} x(\xi)) ds,\ t \in [a,b], \\ x(t),\ t \in [b, b+h_2], \end{cases} \tag{1.4}$$

$x \in C[a-h_1, b+h_2]$.

In what follow we consider the operators $B_f, E_f : C[a-h_1, b+h_2] \to C[a-h_1, b+h_2]$ defined by

$$B_f(x)(t) := \begin{cases} \varphi(t),\ t \in [a-h_1,a], \\ w(\varphi,\psi)(t) + \\ \quad + \int_a^b G(t,s) f(s, x(s), \max\limits_{s-h_1 \leq \xi \leq s+h_2} x(\xi)) ds,\ t \in [a,b], \\ \psi(t),\ t \in [b, b+h_2], \end{cases}$$

and

$$E_f(x)(t) := \begin{cases} x(t),\ t \in [a-h_1, a], \\ w(x|_{[a-h_1,a]}, x|_{[b,b+h_2]})(t) + \\ \quad + \int_a^b G(t,s) f(s, x(s), \max\limits_{s-h_1 \le \xi \le s+h_2} x(\xi)) ds,\ t \in [a,b], \\ x(t),\ t \in [b, b+h_2]. \end{cases}$$

Let $X := C[a-h_1, b+h_2]$ and $X_{\varphi,\psi} := \{x \in X \mid x|_{[a-h_1,a]} = \varphi,\ x|_{[b,b+h_2]} = \psi\}$. It is clear that

$$X = \bigcup_{\substack{\varphi \in C[a-h_1,a] \\ \psi \in C[b,b+h_2]}} X_{\varphi,\psi}$$

is a partition of X. We have

Lemma 1. *We suppose that the conditions* (C_1), (C_2) *and* (C_4) *are satisfied. Then*

(a) $B_f(X) \subset X_{\varphi,\psi}$ *and* $B_f(X_{\varphi,\psi}) \subset X_{\varphi,\psi}$;

(b) $B_f|_{X_{\varphi,\psi}} = E_f|_{X_{\varphi,\psi}}$.

In this paper we shall prove that, if L_f is small enough, then the operator E_f is weakly Picard operator and we shall study the equation (1.1) in the terms of this operator.

2. Picard and Weakly Picard operators

In this paper we use the terminologies and notations from [7–9]. Let us recall now some essential definitions and fundamental results.

Let (X,d) be a metric space and $A : X \to X$ an operator. We denote by $A^0 = 1_X$, $A^1 = A$, $A^{n+1} := A \circ A^n, n \in \mathbb{N}$ the iterates of the operator A;

We also use the following notations: $F_A := \{x \in X \mid A(x) = x\}$ - the fixed point set of A;

$I(A) := \{Y \subset X \mid A(Y) \subset Y, Y \neq \varnothing\}$ - the family of the nonempty invariant subset of A;

We begin with the definitions of a Picard and weakly Picard operator.

Definition 1. Let (X,d) be a metric space. An operator $A: X \to X$ is a Picard operator (PO) if there exists $x^* \in X$ such that:

(i) $F_A = \{x^*\}$;

(ii) the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.

Definition 2. Let (X,d) be a metric space. An operator $A: X \to X$ is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$, and its limit (which may depend on x) is a fixed point of A.

Definition 3. If A is weakly Picard operator then we consider the operator A^∞ defined by

$$A^\infty: X \to X,\ A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Remark 1. It is clear that $A^\infty(X) = F_A = \{x \in X \mid A(x) = x\}$.

The following results are very useful in the sequel.

Lemma 2. *Let $(X,d,\leq)$ be an ordered metric space and $A: X \to X$ an operator. We suppose that A is WPO and increasing. Then, the operator A^∞ is increasing.*

Lemma 3. *(Abstract Gronwall lemma) Let $(X,d,\leq)$ be an ordered metric space and $A: X \to X$ an operator. We suppose that A is WPO and increasing. Then:*

(a) $x \leq A(x) \Rightarrow x \leq A^\infty(x)$;
(b) $x \geq A(x) \Rightarrow x \geq A^\infty(x)$.

Lemma 4. *(Abstract comparison lemma) Let $(X,d,\leq)$ an ordered metric space and $A,B,C: X \to X$ be such that:*

(i) *the operators A,B,C are WPOs;*
(ii) $A \leq B \leq C$;
(iii) *the operator B is increasing.*

Then $x \leq y \leq z$ implies that $A^\infty(x) \leq B^\infty(y) \leq C^\infty(z)$.

Theorem 1. *Let (X,d) be a complete metric space and $A,B: X \to X$ two operators. We suppose that*

(i) *the operator A is an α-contraction;*
(ii) $F_B \neq \varnothing$;
(iii) *there exists $\eta > 0$ such that*

$$d(A(x),B(x)) \leq \eta, \ \forall x \in X.$$

Then, if $F_A = \{x_A^*\}$ and $x_B^* \in F_B$, we have

$$d(x_A^*, x_B^*) \leq \frac{\eta}{1-\alpha}.$$

Another important notion is the following

Definition 4. Let A be a weakly Picard operator and $c > 0$. The operator A is c-weakly Picard operator if

$$d(x, A^\infty(x)) \leq c\,d(x, A(x)), \ \forall x \in X.$$

For the c-PO$_s$ and c-WPO$_s$ we have the following lemma.

Theorem 2. *Let (X,d) be a metric space and $A_i: X \to X$, $i = 1,2$. Suppose that*

(i) *the operator A_i is c_i-WPO$_s$, $i = 1,2$;*
(ii) *there exists $\eta > 0$ such that*

$$d(A_1(x), A_2(x)) \leq \eta, \ \forall x \in X.$$

Then $H(F_{A_1}, F_{A_2}) \leq \eta \max(c_1, c_2)$.

Example 1. Let (X,d) be a complete metric space and $A: X \to X$ an α-contraction. Then A is $\frac{1}{1-\alpha}$-PO.

Example 2. Let (X,d) be a complete metric space and $A: X \to X$ continuous and α-graphic contraction. Then A is $\frac{1}{1-\alpha}$-WPO.

For more details on WPOs theory see [7,9] and [10].

3. Existence and uniqueness

Our first result is the following

Theorem 3. *We suppose that:*

(a) *the conditions* (C_1)–(C_4) *are satisfied;*

(C_5) $\frac{L_f}{8}(b-a)^2 < 1$.

Then the problem (1.1)–(1.2) has a unique solution which is the uniform limit of the successive approximations.

Proof. Consider the Banach space $(C[a-h_1,b+h_2],\|\cdot\|)$ where $\|\cdot\|$ is the Chebyshev norm, $\|\cdot\| := \max\limits_{a-h_1\le t\le b+h_2} |x(t)|$.

The problem (1.1)–(1.2) is equivalent with the fixed point equation

$$B_f(x) = x,\ x \in C[a-h_1,b+h_2].$$

From the condition (C_3) we have, for $t \in [a,b]$

$$\begin{aligned}
&\left|B_f(x)(t) - B_f(y)(t)\right| \\
&\le L_f \int_a^b G(t,s)\left[\max|x(s)-y(s)|, \left|\max_{a-h_1\le\xi\le b+h_2} x(\xi) - \max_{a-h_1\le\xi\le b+h_2} y(\xi)\right|\right] ds \\
&\le L_f \int_a^b G(t,s) \max_{a-h_1\le\xi\le b+h_2} |x(s)-y(s)|\, ds \\
&\le \frac{L_f}{8}(b-a)^2 \|x-y\|.
\end{aligned}$$

This implies that B_f is an α-contraction, with $\alpha = \frac{L_f}{8}(b-a)^2$. The proof follows from the contraction principle. □

Remark 2. From the proof of Theorem 3, it follows that the operator B_f is PO. Since

$$B_f|_{X_{\varphi,\psi}} = E_f|_{X_{\varphi,\psi}}$$

and

$$X := C[a-h_1,b+h_2] = \bigcup_{\varphi,\psi} X_{\varphi,\psi},\ E_f(X_{\varphi,\psi}) \subset X_{\varphi,\psi}$$

hence, the operator E_f is WPO and

$$F_{E_f} \cap X_{\varphi,\psi} = \{x^*_{\varphi,\psi}\}, \forall\varphi \in C[a-h_1,a],\ \forall\psi \in C[b,b+h_2],$$

where $x^*_{\varphi,\psi}$ is the unique solution of the problem (1.1)–(1.2).

Remark 3. E_f is α-graphic contraction, i.e.

$$\left\| E_f^2(x) - E_f(x) \right\| \leq \alpha \left\| x - E_f(x) \right\|, \ \forall x \in C[a-h_1, b+h_2].$$

4. Inequalities of Čaplygin type

Now we consider the operators E_f and B_f on the ordered Banach space $(C[a-h_1, b+h_2], \|\cdot\|, \leq)$. We have

Theorem 4. *We suppose that:*

(a) *the conditions* $(C_1)-(C_4)$ *are satisfied;*
(b) $\frac{L_f}{8}(b-a)^2 < 1$*;*
(c) $f(t,\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ *is increasing,* $\forall t \in [a,b]$.

Let x *be a solution of equation (1.1) and* y *a solution of the inequality*

$$-y''(t) \leq f(t, y(t), \max_{t-h_1 \leq \xi \leq t+h_2} y(\xi)), \ t \in [a,b].$$

Then if $y(t) \leq x(t), \forall t \in [a-h_1, a] \cup [b, b+h_2]$ *we obtain that* $y \leq x$.

Proof. Let us consider the operator $\widetilde{w} : C[a-h_1, b+h_2] \to C[a-h_1, b+h_2]$ defined by

$$\widetilde{w}(z)(t) := \begin{cases} z(t), \ t \in [a-h_1, a], \\ w(z|_{[a-h_1,a]}, z|_{[b,b+h_2]})(t), \ t \in [a,b], \\ z(t), \ t \in [b, b+h_2]. \end{cases}$$

First of all we remark that

$$w(y|_{[a-h_1,a]}, y|_{[b,b+h_2]}) \leq w(x|_{[a-h_1,a]}, x|_{[b,b+h_2]})$$

and

$$\widetilde{w}(y) \leq \widetilde{w}(x).$$

In the terms of the operator E_f, we have

$$x = E_f(x) \text{ and } y \leq E_f(y).$$

From the conditions $(C_1), (C_2)$ and (C_3) follows that the operator E_f is WPO. Also, from condition (c) we have that E_f is an increasing operator. Applying Lemma 2, we have that the operator E_f^∞ is increasing. From Theorem 3 we have that $E_f(X_{\varphi,\psi}) \subset X_{\varphi,\psi}$, $\forall \varphi, \psi \in \mathbb{R}$. $E_f|_{X_{\varphi,\psi}}$ is a contraction and since $\widetilde{w}(z) \in X_{\varphi,\psi}$ then

$$E_f^\infty(\widetilde{w}) = E_f^\infty(y), \ \forall y \in X_{\varphi,\psi}.$$

Let $y \le E_f(y)$, since E_f is increasing, from the Gronwall lemma (Lemma 3) we get $y \le E_f^\infty(y)$. Also $y, \widetilde{w}(y) \in X_{w(y)}$, so $E_f^\infty(y) = E_f^\infty(\widetilde{w}(y))$. But $w(y) \le w(x)$, E_f^∞ is increasing and $E_f^\infty(\widetilde{w}(x)) = E_f^\infty(x) = x$. So

$$y \le E_f^\infty(y) = E_f^\infty(\widetilde{w}(y)) \le E_f^\infty(\widetilde{w}(x)) = E_f^\infty(x) = x.$$

□

5. Data dependence: monotony

In this section we study the monotony of the solution of the problem (1.1)–(1.2) with respect to φ, ψ and f.

Theorem 5. *Let $f_i \in C([a,b]\times\mathbb{R}^2,\mathbb{R}), i = 1,2,3$, be as in Theorem 3. We suppose that:*

(i) $f_1 \le f_2 \le f_3$;
(ii) $f_2(t,\cdot,\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ *is monotone increasing;*

Let x_i be a solution of the equation

$$-x_i''(t) = f_i(t, x(t), \max_{t-h_1\le\xi\le t+h_2} x(\xi)),\ t \in [a,b] \text{ and } i = 1,2,3.$$

Then, $x_1(t) \le x_2(t) \le x_3(t)$, $\forall t \in [a-h_1,a]\cup[b,b+h_2]$, implies that $x_1 \le x_2 \le x_3$, i.e. the unique solution of the problem (1.1)–(1.2) is increasing with respect to f, φ and ψ.

Proof. From Theorem 3, the operators $E_{f_i}, i = 1,2,3$, are WPOs. From the condition (ii) the operator E_{f_2} is monotone increasing. From the condition (i) it follows that

$$E_{f_1} \le E_{f_2} \le E_{f_3}.$$

On the other hand we remark that

$$\widetilde{w}(x_1) \le \widetilde{w}(x_2) \le \widetilde{w}(x_3)$$

and

$$x_i = E_{f_i}^\infty(\widetilde{w}(x_i)),\ i = 1,2,3.$$

So, the proof follows from Lemma 4. □

6. Data dependence: continuity

Consider the boundary value problem (1.1)–(1.2) and suppose the conditions of the Theorem 3 are satisfied. Denote by $x^*(\cdot;\varphi,\psi,f)$, the solution of this problem. We state the following result:

Theorem 6. *Let $\varphi_i, \psi_i, f_i, i = 1,2$ be as in the Theorem 3. Furthermore, we suppose that there exists $\eta_i > 0, i = 1,2$ such that*

(i) $|\varphi_1(t)-\varphi_2(t)| \le \eta_1$, $\forall t \in [a-h_1,a]$ *and* $|\psi_1(t)-\psi_2(t)| \le \eta_1$, $\forall t \in [b,b+h_2]$;

(ii) $|f_1(t,u_1,u_2)-f_2(t,u_1,u_2)|\le \eta_2, \forall t\in C[a,b], u_1,u_2\in\mathbb{R}$.

Then

$$\left\|x_1^*(t;\varphi_1,\psi_1,f_1)-x_2^*(t;\varphi_2,\psi_2,f_2)\right\|\le \frac{8\eta_1+(b-a)^2\eta_2}{8-L_f(b-a)^2},$$

where $x_i^*(t;\varphi_i,\psi_i,f_i)$ *are the solution of the problem (1.1)–(1.2) with respect to* φ_i,ψ_i,f_i, $i=1,2$, *and* $L_f=\max(L_{f_1},L_{f_2})$.

Proof. Consider the operators $B_{\varphi_i,\psi_i,f_i}, i=1,2$. From Theorem 3 these operators are contractions.

Additionally

$$\left\|B_{\varphi_1,\psi_1,f_1}(x)-B_{\varphi_2,\psi_2,f_2}(x)\right\|\le \eta_1+\eta_2\frac{(b-a)^2}{8},$$

$\forall x\in C[a-h_1,b+h_2]$.

Now the proof follows from Theorem 1, with $A:=B_{\varphi_1,\psi_1,f_1}, B=B_{\varphi_2,\psi_2,f_2}, \eta=\eta_1+\eta_2\frac{(b-a)^2}{8}$ and $\alpha:=\frac{L_f}{8}(b-a)^2$ where $L_f=\max(L_{f_1},L_{f_2})$. □

In what follow we shall use the c-WPOs techniques to give some data dependence results using Theorem 2.

Theorem 7. *Let* f_1 *and* f_2 *be as in the Theorem 3. Let* $S_{E_{f_1}}, S_{E_{f_2}}$ *be the solution sets of system (1.1) corresponding to* f_1 *and* f_2. *Suppose that there exists* $\eta>0$, *such that*

$$|f_1(t,u_1,u_2)-f_2(t,u_1,u_2)|\le \eta \tag{6.1}$$

for all $t\in[a,b], u_1,u_2\in\mathbb{R}$.

Then

$$H_{\|\cdot\|_C}(S_{E_{f_1}},S_{E_{f_2}})\le \frac{(b-a)^2\eta}{8-L_f(b-a)^2},$$

where $L_f=\max(L_{f_1},L_{f_2})$ *and* $H_{\|\cdot\|_C}$ *denotes the Pompeiu-Housdorff functional with respect to* $\|\cdot\|_C$ *on* $C[a,b]$.

Proof. In the condition of Theorem 3, the operators E_{f_1} and E_{f_2} are c_1-WPO and c_2-weakly Picard operators.

Let

$$X_{\varphi,\psi}:=\{x\in X\mid x|_{[a-h_1,a]}=\varphi,\ x|_{[b,b+h_2]}=\psi\}.$$

It is clear that $E_{f_1}|_{X_{\varphi,\psi}}=B_{f_1},\ E_{f_2}|_{X_{\varphi,\psi}}=B_{f_2}$. Therefore,

$$\left|E_{f_1}^2(x)-E_{f_1}(x)\right|\le \tfrac{1}{8}L_{f_1}(b-a)^2\left|E_{f_1}(x)-x\right|,$$

$$\left|E_{f_2}^2(x)-E_{f_2}(x)\right|\le \tfrac{1}{8}L_{f_2}(b-a)^2\left|E_{f_2}(x)-x\right|,$$

for all $x\in C[a-h_1,b+h_2]$.

Now, choosing

$$\alpha_i=\tfrac{1}{8}L_{f_i}(b-a)^2, i=1,2,$$

we get that E_{f_1} and E_{f_2} are c_1-weakly Picard operators and c_2-weakly Picard operators with $c_1 = (1-\alpha_1)^{-1}$ and $c_2 = (1-\alpha_2)^{-1}$. From (6.1) we obtain that

$$\left\| E_{f_1}(x) - E_{f_2}(x) \right\|_C \leq (b-a)^2 \eta,$$

$\forall x \in C[a-h_1, b+h_2]$. Applying Theorem 2 we have that

$$H_{\|\cdot\|_C}(S_{E_{f_1}}, S_{E_{f_2}}) \leq \frac{(b-a)^2 \eta}{8 - L_f (b-a)^2},$$

where $L_f = \max(L_{f_1}, L_{f_2})$ and $H_{\|\cdot\|_C}$ is the Pompeiu-Housdorff functional with respect to $\|\cdot\|_C$ on $C[a-h_1, b+h_2]$. □

7. Extremal principle

We consider the following differential equation with maxima

$$x''(t) + p(t)x(t) + q(t) \max_{t-h_1 \leq \xi \leq t+h_2} x(\xi) = 0 \tag{7.1}$$

where $p, q \in C([a,b], \mathbb{R})$.

Remark 4. The function $x(t)$ is an eventually negative solution of equation (7.1) if and only if $y(t) = -x(t)$ is an eventually positive solution of the equation

$$y''(t) + p(t)y(t) + q(t) \min_{t-h_1 \leq \xi \leq t+h_2} y(\xi) = 0. \tag{7.2}$$

From the above remark we see that the positive and the negative solution of equation (7.1) need to be discussed separately.

Theorem 8. *(Extremal principle) Let $p(t), q(t) < 0$, for all $t \in]a,b[$ and x be a solution of (7.1). Then*

(a) *if* $\max\{x(t) |\ t \in [a,b]\} = x(t_0)$ *and* $x(t_0) > 0$ *then* $t_0 \in \{a,b\}$;
(b) *if* $min\{x(t) |\ t \in [a,b]\} = x(t_0)$ *and* $x(t_0) < 0$ *then* $t_0 \in \{a,b\}$.

Proof. (a) Let $t_0 \in]a,b[$ be such that, $x(t_0) > 0$ is the maximum value of t_0 on $[a,b]$. Since $x \in C^2[a,b]$ we have that $x(t_0) > 0, x'(t_0) = 0, x''(t_0) \leq 0$. From (7.1) we have

$$0 = x''(t_0) + p(t_0)x(t_0) + q(t_0) \max_{t_0-h_1 \leq \xi \leq t_0+h_2} x(\xi) < 0.$$

This is a contradiction, and therefore our assumption is wrong. So $t_0 \in \{a,b\}$.

(b) Let $t_0 \in]a,b[$ be such that, $x(t_0) < 0$ is the minimum value of t_0 on $[a,b]$. Since $x \in C^2[a,b]$ we have that $x(t_0) < 0, x'(t_0) = 0, x''(t_0) \geq 0$. From (7.1) we have

$$0 = x''(t_0) + p(t_0)x(t_0) + q(t_0) \max_{t_0-h_1 \leq \xi \leq t_0+h_2} x(\xi) > 0.$$

This is a contradiction, and therefore our assumption is wrong. So $t_0 \in \{a,b\}$. □

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Author's address

Diana Otrocol

Current address: Department of Mathematics, Technical University of Cluj-Napoca, Memorandumului Street 28, 400114 Cluj-Napoca, Romania

Tiberiu Popoviciu Institute of Numerical Analysis, Romanian Academy, P.O.Box. 68-1, 400110, Cluj-Napoca, Romania

E-mail address: dotrocol@ictp.acad.ro

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NOTES ON EXPLICIT AND INVERSION FORMULAS FOR THE CHEBYSHEV POLYNOMIALS OF THE FIRST TWO KINDS

FENG QI, DA-WEI NIU, AND DONGKYU LIM

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Abstract. In the paper, starting from the Rodrigues formulas for the Chebyshev polynomials of the first and second kinds, by virtue of the Faà di Bruno formula, with the help of two identities for the Bell polynomials of the second kind, and making use of a new inversion theorem for combinatorial coefficients, the authors derive two nice explicit formulas and their corresponding inversion formulas for the Chebyshev polynomials of the first and second kinds.

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1. Introduction

It is well known [5–7, 31] that the Chebyshev polynomials of the first and second kinds T_n and $U_n(x)$ are very important in mathematical sciences and that, in the study of ordinary differential equations [5, pp. xxxv and 1004], they arise as solutions to the Chebyshev differential equations

$$(1-x^2)y''-xy'+n^2y=0 \quad \text{and} \quad (1-x^2)y''-3xy'+n(n+2)y=0$$

for the Chebyshev polynomials of the first and second kinds T_n and U_n respectively.

In [6, Eqs. (4.30) and (4.31)], the Rodrigues formulas for the Chebyshev polynomials of the first and second kinds T_n and U_n read that

$$T_n(x)=(-1)^n\frac{2^n n!}{(2n)!}\bigl(1-x^2\bigr)^{1/2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\bigl[\bigl(1-x^2\bigr)^{n-1/2}\bigr] \tag{1.1}$$

and

$$U_n(x)=(-1)^n\frac{2^n(n+1)!}{(2n+1)!}\bigl(1-x^2\bigr)^{-1/2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\bigl[\bigl(1-x^2\bigr)^{n+1/2}\bigr]. \tag{1.2}$$

For variants of the Rodrigues formulas for the Chebyshev polynomials of the first and second kinds T_n and U_n, please refer to, for example, [5, pp. 1003–1004], [7, p. 442], [19, Section 4], and [31, pp. 432–433].

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In [5, p. 1003], the Rodrigues formulas for $T_n(x)$ and $U_n(x)$ are written in the forms

$$T_n(x)=(-1)^n\frac{\sqrt{1-x^2}}{(2n-1)!!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\bigl[\bigl(1-x^2\bigr)^{n-1/2}\bigr] \tag{1.3}$$

and

$$U_n(x)=\frac{(-1)^n(n+1)}{\sqrt{1-x^2}\,(2n+1)!!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\bigl[\bigl(1-x^2\bigr)^{n+1/2}\bigr]. \tag{1.4}$$

In [5, p. 1004] and [31, pp. 432–433], the Rodrigues formulas for $T_n(x)$ and $U_n(x)$ are formulated as

$$T_n(x)=\frac{(-1)^n\sqrt{\pi}}{2^n\Gamma(n+1/2)}\bigl(1-x^2\bigr)^{1/2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\bigl[\bigl(1-x^2\bigr)^{n-1/2}\bigr] \tag{1.5}$$

and

$$U_n(x)=\frac{(-1)^n\sqrt{\pi}\,(n+1)}{2^{n+1}\Gamma(n+3/2)}\bigl(1-x^2\bigr)^{-1/2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\bigl[\bigl(1-x^2\bigr)^{n+1/2}\bigr], \tag{1.6}$$

where $\Gamma(z)$ stands for the classical gamma function which can be defined [8, 16] by

$$\Gamma(z)=\lim_{n\to\infty}\frac{n!n^z}{\prod_{k=0}^{n}(z+k)},\quad z\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$$

or by

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t,\quad \Re(z)>0.$$

In [7, p. 442], the Rodrigues formulas for $T_n(x)$ and $U_n(x)$ are arranged as

$$T_n(x)=\frac{\bigl(1-x^2\bigr)^{1/2}}{(-2)^n(1/2)_n}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\bigl[\bigl(1-x^2\bigr)^{n-1/2}\bigr] \tag{1.7}$$

and

$$U_n(x)=\frac{(n+1)\bigl(1-x^2\bigr)^{-1/2}}{(-2)^n(3/2)_n}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\bigl[\bigl(1-x^2\bigr)^{n+1/2}\bigr], \tag{1.8}$$

where $(x)_n$ for $n\ge0$ and $x\in\mathbb{R}$ denotes the rising factorial which can be defined [22] by

$$(x)_n=\prod_{\ell=0}^{n-1}(x+\ell)=\frac{\Gamma(x+n)}{\Gamma(x)}=\begin{cases}x(x+1)\cdots(x+n-1), & n\ge1;\\ 1, & n=0.\end{cases}$$

By virtue of the recurrence relation $\Gamma(x+1)=x\Gamma(x)$, we have

$$\Gamma\left(n+\frac{1}{2}\right)=\prod_{\ell=0}^{n-1}\left(n-\ell-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)_n\sqrt{\pi}=\frac{(2n-1)!!}{2^n}\sqrt{\pi}$$

and

$$\Gamma\left(n+\frac{3}{2}\right)=\prod_{\ell=0}^{n}\left(n-\ell+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)=\left(\frac{3}{2}\right)_n\frac{\sqrt{\pi}}{2}=\frac{(2n+1)!!}{2^{n+1}}\sqrt{\pi}\,.$$

Substituting these into (1.5) and (1.6) respectively leads to (1.3), (1.4), (1.7), and (1.8) which are equivalent to (1.1) and (1.2) respectively.

In [31, pp. 432–433], it was listed that

$$T_n(x)=\frac{n}{2}\sum_{m=0}^{\lfloor n/2\rfloor}(-1)^m\frac{(n-m-1)!}{m!(n-2m)!}(2x)^{n-2m} \tag{1.9}$$

and

$$U_n(x)=\sum_{m=0}^{\lfloor n/2\rfloor}(-1)^m\frac{(n-m)!}{m!(n-2m)!}(2x)^{n-2m}, \tag{1.10}$$

where $n\in\mathbb{N}$ and $\lfloor t\rfloor$ denotes the floor function whose value equals the largest integer less than or equal to t.

In this paper, starting from the four formulas (1.1), (1.2), (1.9), and (1.10), by virtue of the Faà di Bruno formula, with the help of two identities for the Bell polynomials of the second kind, and making use of a new inversion theorem [28, Theorem 4.3] for combinatorial coefficients, we will derive the following two nice explicit formulas and their corresponding inversion formulas for the Chebyshev polynomials T_n and U_n.

2. Four lemmas

For proving our main results, Theorems 1 and 2 below, we need the following four lemmas.

Lemma 1 ([4, pp. 134 and 139])**.** *For $n\ge k\ge 0$, the Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ by*

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}f\circ h(t)=\sum_{k=0}^{n}f^{(k)}(h(t))\,\mathrm{B}_{n,k}\bigl(h'(t),h''(t),\dots,h^{(n-k+1)}(t)\bigr). \tag{2.1}$$

Lemma 2 ([4, p. 135])**.** *For $n\ge k\ge 0$, we have*

$$\mathrm{B}_{n,k}\bigl(abx_1,ab^2x_2,\dots,ab^{n-k+1}x_{n-k+1}\bigr)=a^kb^n\,\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}), \tag{2.2}$$

where a and b are any complex numbers.

Lemma 3 ([13, Theorem 4.1] and [27, Section 3]). *For $n \ge k \ge 0$, the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ satisfy*

$$\mathrm{B}_{n,k}(x,1,0,\dots,0) = \frac{1}{2^{n-k}} \frac{n!}{k!} \binom{k}{n-k} x^{2k-n}, \tag{2.3}$$

where $\binom{0}{0} = 1$ and $\binom{p}{q} = 0$ for $q > p \ge 0$.

Lemma 4 ([28, Theorem 4.3]). *For $n \ge k \ge 1$, let s_k and S_k be two sequences independent of n. Then*

$$\frac{s_n}{n!} = \sum_{k=1}^{n} (-1)^k \binom{k}{n-k} S_k$$

if and only if

$$n S_n = \sum_{k=1}^{n} \frac{(-1)^k}{(k-1)!} \binom{2n-k-1}{n-1} s_k.$$

3. Main results and their proofs

Now we begin to state and prove our main results, Theorems 1 and 2 below.

Theorem 1. *For $n \ge 0$, the Chebyshev polynomials T_n and U_n can be explicitly computed by*

$$T_n(x) = x^n \sum_{\ell=0}^{\lfloor n/2 \rfloor} \binom{n}{2\ell} \left(1 - \frac{1}{x^2}\right)^{\ell} \tag{3.1}$$

and

$$U_n(x) = x^n \sum_{\ell=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2\ell+1} \left(1 - \frac{1}{x^2}\right)^{\ell}. \tag{3.2}$$

Proof. By virtue of the formuals (2.1), (2.2), and (2.3), we have

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\left(1-x^2\right)^{n-1/2}\right] &= \sum_{k=1}^{n} \frac{\mathrm{d}^k u^{n-1/2}}{\mathrm{d}u^k} \mathrm{B}_{n,k}(-2x,-2,0\dots,0) \\
&= \sum_{k=1}^{n} \prod_{\ell=0}^{k-1} \left(n-\ell-\frac{1}{2}\right) u^{n-k-1/2} (-2)^k \mathrm{B}_{n,k}(x,1,0\dots,0) \\
&= \sum_{k=1}^{n} \frac{1}{2^k} \prod_{\ell=0}^{k-1} (2n-2\ell-1)\left(1-x^2\right)^{n-k-1/2} (-2)^k \frac{1}{2^{n-k}} \frac{n!}{k!} \binom{k}{n-k} x^{2k-n} \\
&= \frac{n!}{(2x)^n} \left(1-x^2\right)^{n-1/2} \sum_{k=1}^{n} (-1)^k \binom{k}{n-k} \frac{(2n-1)!!}{[2(n-k)-1]!!} \frac{2^k}{k!} \left(\frac{x^2}{1-x^2}\right)^k
\end{aligned}$$

$$= \frac{n!(2n-1)!!}{(2x)^n}\left(1-x^2\right)^{n-1/2}\sum_{k=1}^{n}(-1)^k\binom{k}{n-k}\frac{2^k}{k![2(n-k)-1]!!}\left(\frac{x^2}{1-x^2}\right)^k,$$

where $n \in \mathbb{N}$, $u = u(x) = 1-x^2$, and the double factorial of negative odd integers $-2n-1$ is defined by

$$(-2n-1)!! = \frac{(-1)^n}{(2n-1)!!} = (-1)^n\frac{2^n n!}{(2n)!}, \quad n \ge 0.$$

Substituting the above established equality into (1.1) and simplifying lead to

$$T_n(x) = \sum_{k=1}^{n}\frac{(-1)^{n-k}}{4^{n-k}}\binom{k}{n-k}\binom{n}{k}\frac{[2(n-k)]!!}{[2(n-k)-1]!!}x^{2k-n}\left(1-x^2\right)^{n-k}$$

which can be rearranged, by replacing $n-k$ by ℓ, as

$$T_n(x) = x^n\sum_{\ell=0}^{n-1}\frac{(-1)^\ell}{4^\ell}\binom{n}{n-\ell}\binom{n-\ell}{\ell}\frac{(2\ell)!!}{(2\ell-1)!!}\left(\frac{1}{x^2}-1\right)^\ell.$$

Since

$$\frac{1}{4^\ell}\binom{n}{n-\ell}\binom{n-\ell}{\ell}\frac{(2\ell)!!}{(2\ell-1)!!} = \binom{n}{2\ell},$$

we arrives at the identity (3.1).

Repeating the above process, we can obtain

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\left(1-x^2\right)^{n+1/2}\right] = \sum_{k=1}^{n}\frac{\mathrm{d}^k u^{n+1/2}}{\mathrm{d}u^k}\mathrm{B}_{n,k}(-2x,-2,0\dots,0)$$

$$= \sum_{k=1}^{n}\prod_{\ell=0}^{k-1}\left(n-\ell+\frac{1}{2}\right)u^{n-k+1/2}(-2)^k\,\mathrm{B}_{n,k}(x,1,0\dots,0)$$

$$= \sum_{k=1}^{n}\frac{1}{2^k}\prod_{\ell=0}^{k-1}(2n-2\ell+1)\left(1-x^2\right)^{n-k+1/2}(-2)^k\frac{1}{2^{n-k}}\frac{n!}{k!}\binom{k}{n-k}x^{2k-n}$$

$$= \frac{n!}{(2x)^n}\left(1-x^2\right)^{n+1/2}\sum_{k=1}^{n}(-1)^k\binom{k}{n-k}\frac{(2n+1)!!}{[2(n-k)+1]!!}\frac{2^k}{k!}\left(\frac{x^2}{1-x^2}\right)^k$$

$$= \frac{n!(2n+1)!!}{(2x)^n}\left(1-x^2\right)^{n+1/2}\sum_{k=1}^{n}(-1)^k\binom{k}{n-k}\frac{2^k}{k![2(n-k)+1]!!}\left(\frac{x^2}{1-x^2}\right)^k.$$

Substituting this into (1.2) and simplifying lead to

$$U_n(x) = \sum_{k=1}^{n}\frac{(-1)^{n-k}}{2^{2n-2k+1}}\binom{k}{n-k}\binom{n+1}{k}\frac{[2(n-k+1)]!!}{[2(n-k)+1]!!}x^{2k-n}\left(1-x^2\right)^{n-k}.$$

Replacing $n-k$ by ℓ reveals that

$$U_n(x)=\sum_{\ell=0}^{n-1}\frac{(-1)^\ell}{2^{2\ell+1}}\binom{n+1}{n-\ell}\binom{n-\ell}{\ell}\frac{[2(\ell+1)]!!}{(2\ell+1)!!}x^{n-2\ell}\bigl(1-x^2\bigr)^\ell.$$

Due to

$$\frac{1}{2^{2\ell+1}}\binom{n+1}{n-\ell}\binom{n-\ell}{\ell}\frac{[2(\ell+1)]!!}{(2\ell+1)!!}=\binom{n+1}{2\ell+1},$$

we derive (3.2). The proof of Theorem 1 is complete. □

Theorem 2. *For $n\in\mathbb{N}$, we have*

$$\sum_{k=1}^{n}\binom{2n-k-1}{n-1}(2x)^kT_k(x)=\frac{1}{2}(2x)^{2n} \tag{3.3}$$

and

$$\sum_{k=1}^{n}k\binom{2n-k-1}{n-1}(2x)^kU_k(x)=n(2x)^{2n}. \tag{3.4}$$

Proof. We notice that the formulas (1.9) and (1.10) can be rearranged as

$$T_n(x)=\frac{n}{2}\sum_{m=0}^{\lfloor n/2\rfloor}(-1)^m\binom{n-m}{m}\frac{(2x)^{n-2m}}{n-m} \tag{3.5}$$

and

$$U_n(x)=\sum_{m=0}^{\lfloor n/2\rfloor}(-1)^m\binom{n-m}{m}(2x)^{n-2m}. \tag{3.6}$$

The inversion theorem in Lemma 4 can be restated as

$$(-1)^n\frac{s_n}{n!}=\sum_{\ell=0}^{n-1}(-1)^\ell\binom{n-\ell}{\ell}S_{n-\ell}=\sum_{\ell=0}^{\lfloor n/2\rfloor}(-1)^\ell\binom{n-\ell}{\ell}S_{n-\ell}$$

if and only if

$$nS_n=\sum_{\ell=1}^{n}\frac{(-1)^\ell}{(\ell-1)!}\binom{2n-\ell-1}{n-1}s_\ell.$$

The formulas (3.5) and (3.6) can be rearranged as

$$\frac{2}{n}(2x)^nT_n(x)=\sum_{\ell=0}^{\lfloor n/2\rfloor}(-1)^\ell\binom{n-\ell}{\ell}\frac{(2x)^{2(n-\ell)}}{n-\ell}$$

and

$$(2x)^nU_n(x)=\sum_{\ell=0}^{\lfloor n/2\rfloor}(-1)^\ell\binom{n-\ell}{\ell}(2x)^{2(n-\ell)}.$$

Consequently, we obtain

$$n\frac{(2x)^{2n}}{n}=\sum_{k=1}^{n}\frac{(-1)^k}{(k-1)!}\binom{2n-k-1}{n-1}(-1)^k2(k-1)!(2x)^kT_k(x)$$

and

$$n(2x)^{2n}=\sum_{k=1}^{n}\frac{(-1)^k}{(k-1)!}\binom{2n-k-1}{n-1}(-1)^kk!(2x)^kU_k(x)$$

which can be simplified as (3.3) and (3.4). The proof of Theorem 2 is complete. □

4. Remarks

In this section, we will list several remarks to explain more about the formula (2.3), Lemma 4, our main results, and other things.

Remark 1. To the best of our knowledge, the nice formula (2.3) was first concluded in [13] and has been extensively applied in the papers [9–15, 17, 19, 21, 23–25, 27, 29, 30] and closely related references therein. The formula (2.3) has been generalized in the papers [15, 17, 20] and closely related references therein.

Remark 2. To the best of our knowledge, Lemma 4 is a new inversion theorem and has been applied in the paper [10, 18, 19, 26].

Remark 3. Because both the formula (2.3) and Lemma 4 are new, our main results stated in Theorems 1 and 2, or at least their proofs, are also new.

Remark 4. The Chebyshev polynomials are classical, but their study is still very active. As examples, we recommend three newly-published papers [1–3] to readers. Considering the length of this paper, we would not like to detail main results in these three papers and the closely-related references therein.

Remark 5. This paper is a slightly revised version of the preprint [19].

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Authors' addresses

Feng Qi

Institute of Mathematics, Henan Polytechnic University, Jiaozuo 454010, Henan, China; College of Mathematics, Inner Mongolia University for Nationalities, Tongliao 028043, Inner Mongolia, China; School of Mathematical Sciences, Tianjin Polytechnic University, Tianjin 300387, China

E-mail address: qifeng618@gmail.com

URL: https://qifeng618.wordpress.com

Da-Wei Niu

Department of Science, Henan University of Animal Husbandry and Economy, Zhengzhou 450046, Henan, China

E-mail address: nnddww@gmail.com

URL: https://orcid.org/0000-0003-4033-7911

Dongkyu Lim

Department of Mathematics Education, Andong National University, Andong 36729, Republic of Korea

E-mail address: dgrim84@gmail.com

URL: https://orcid.org/0000-0002-0928-8480

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SZEGED INDEX OF A CLASS OF UNICYCLIC GRAPHS

XULI QI

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Abstract. The Szeged index is a modification of the Wiener index to cyclic molecules. The Szeged index of a connected graph G is defined as $Sz(G) = \sum_{e\in E(G)} n_1(e|G)n_2(e|G)$, where $E(G)$ is the edge set of G, and for any $e = uv \in E(G)$, $n_1(e|G)$ is the number of vertices of G lying closer to vertex u than to vertex v, and $n_2(e|G)$ is the number of vertices of G lying closer to vertex v than to vertex u. In this paper, we determine the n-vertex unicyclic graphs whose vertices on the unique cycle have degree at least three with the first, the second and the third smallest as well as largest Szeged indices for $n \geq 6$, $n \geq 7$ and $n \geq 8$, respectively.

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Keywords: distance, Szeged index, Wiener index, unicyclic graph, fully loaded unicyclic graph

1. Introduction

A topological index is a numerical quantity derived from the hydrogen-depleted graph of a molecule as a tool for compact and effective description of structural formula which allows one to study and predict the structure-property correlations of organic compounds. The Wiener index is one of the oldest topological indices, which was introduced by Wiener [17] in 1947 to study boiling points of paraffins. Since then, it has been used to explain various chemical and physical properties of molecules and to correlate the structure of molecules to their biological activity.

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Let $d_G(x, y)$ be the distance between vertices x and y in G. Recall that the Wiener index of G is defined as $W(G) = \sum\limits_{\{x,y\}\subseteq V(G)} d_G(x, y)$ [5, 8].

If e is an edge of G connecting the vertices u and v, then we write $e = uv$ or $e = vu$.

Let $e = uv \in E(G)$. Let $n_1(e|G)$ be the number of vertices of G lying closer to vertex u than to vertex v, and $n_2(e|G)$ be the number of vertices of G lying closer to vertex v than to vertex u, i.e.,

$$n_1(e|G) = |\{x \in V(G) : d_G(x,u) < d_G(x,v)\}|$$

and

$$n_2(e|G) = |\{x \in V(G) : d_G(x,v) < d_G(x,u)\}|.$$

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If T is a tree, then $W(T) = \sum_{e\in E(T)} n_1(e|T)n_2(e|T)$, see [5, 8]. Motivated by this equality for Wiener index of trees, Gutman [7] put forward to the Szeged index. For a connected graph G, it is defined as [7]

$$Sz(G) = \sum_{e\in E(G)} n_1(e|G)n_2(e|G).$$

The Szeged index found applications for modeling physicochemical properties as well as physiological activities of organic compounds acting as drugs or possessing pharmacological activity [11], and its mathematical properties may be found, e.g.,[1–4, 6, 9, 12–16, 19].

A unicyclic graph is a connected graph with a unique cycle. The n-vertex unicyclic graphs with the smallest and the largest Szeged indices have been determined [7, 16]. Zhou *et al.* [19] determined the n-vertex unicyclic graphs of cycle length r with the smallest and the largest Szeged indices for $3 \le r \le n$, the n-vertex unicyclic graphs with the second, the third and the fourth smallest Szeged indices, and the n-vertex unicyclic graphs with the kth largest Szeged indices for all k up to $\frac{n}{2}+2$ if $n \ge 6$ is even, to four if $n = 7$, to five if $n = 9$, to $\frac{n+13}{4}$ if $n \equiv 3 \pmod 4$ with $n \ge 11$, and to $\frac{n+15}{4}$ if $n \equiv 1 \pmod 4$ with $n \ge 13$.

A unicyclic graph is fully loaded if vertices on its unique cycle all have degree at least three. In the present paper, in continuation of the study on the Szeged index, we determine the n-vertex fully loaded unicyclic graphs with the first, the second and the third smallest as well as largest Szeged indices for $n \ge 6$, $n \ge 7$ and $n \ge 8$, respectively.

2. Preliminaries

Let S_n and P_n be the n-vertex star and path, respectively.

Lemma 1 ([5, 8])**.** *Let T be an n-vertex tree different from S_n and P_n. Then*

$$(n-1)^2 = W(S_n) < W(T) < W(P_n) = \frac{n^3-n}{6}.$$

For a connected graph G with $u \in V(G)$, let $W_u(G) = \sum_{v\in V(G)} d_G(u,v)$.

Lemma 2 ([18])**.** *Let T be an n-vertex tree with $u \in V(T)$, where u is not the center if $T = S_n$ and u is not a terminal vertex if $T = P_n$. Let x and y be the center of the star S_n and a terminal vertex of the path P_n, respectively. Then*

$$n-1 = W_x(S_n) < W_u(T) < W_y(P_n) = \frac{n(n-1)}{2}.$$

Let S_n' be the graph formed by attaching a pendent vertex to one pendent vertex of S_{n-1} for $n \ge 4$. Let P_n' be the graph formed by attaching a pendent vertex to the neighbor of a terminal vertex of P_{n-1} for $n \ge 4$.

Lemma 3 ([10, 18]). *Among the n-vertex trees with $n \geq 4$, S'_n is the unique graph with the second smallest Wiener index, which is equal to $n^2 - n - 2$, and P'_n is the unique graph with the second largest Wiener index, which is equal to $\frac{1}{6}(n^3 - 7n + 18)$.*

The following lemma may be easily checked.

Lemma 4. *Let T be an n-vertex tree with $n \geq 5$, $u \in V(T)$, $T \neq S_n, P_n$, where u is not a vertex of maximal degree if $T = S'_n$, and not the terminal vertex whose neighbor has degree two if $T = P'_n$. Let x be the vertex of maximal degree of S'_n, and y be the terminal vertex whose neighbor has degree two in P'_n. Then $n = W_x(S'_n) < W_u(T) < W_y(P'_n) = \frac{(n+1)(n-2)}{2}$.*

Let $C_r(T_1, T_2, \ldots, T_r)$ be the graph constructed as follows. Let the vertices of the cycle C_r be labeled consecutively by $v_1, v_2, \ldots, v_r$. Let $T_1, T_2, \ldots, T_r$ be vertex–disjoint trees such that T_i and the cycle C_r share exactly one common vertex v_i for $i = 1, 2, \ldots, r$. Denote $|G| = |V(G)|$ for a graph G. Then any n-vertex unicyclic graph G with a cycle on r vertices is of the form $C_r(T_1, T_2, \ldots, T_r)$, where $\sum_{i=1}^{r} |T_i| = n$. For the graph $G = C_r(T_1, T_2, \ldots, T_r)$, let $d_{ij} = d_{C_r}(v_i, v_j)$ and $t_i = |T_i|$ for $i = 1, 2, \ldots, r$. Then let $N_s = \sum_{i \neq s} t_i d_{is}$.

For a subset M of $E(G)$, $G - M$ denotes the graph obtained from G by deleting the edges in M. For a subset M^* of the edge set of the complement of G, $G + M^*$ denotes the graph obtained from G by adding the edges in M^*.

Lemma 5 ([19]). *Let $G = C_r(T_1, T_2, \ldots, T_r)$ and T_i be a star with center v_i for $i = 1, 2, \ldots, r$. Suppose that trees T_k and T_l are nontrivial. Let $w \in V(T_l)$ with $w \neq v_l$. If r is odd and $N_k + \frac{1}{2}t_k \leq N_l + \frac{1}{2}t_l$, or r is even and $N_k \leq N_l$, then $Sz(G - \{v_l w\} + \{v_k w\}) < Sz(G)$.*

Let $\varepsilon_n = 1$ if n is odd and $\varepsilon_n = 0$ if n is even.

Lemma 6 ([9]). *Let $G = C_r(T_1, T_2, \ldots, T_r)$. Then*

$$Sz(G) = \sum_{i=1}^{r} W(T_i) + \sum_{i=1}^{r} (|G| - t_i) W_{v_i}(T_i) + \sum_{i=1}^{r} \sum_{j=1}^{r} t_i t_j d_{ij} - \varepsilon_r \sum_{i<j} t_i t_j .$$

For $3 \leq r \leq \lfloor \frac{n}{2} \rfloor$, let $\mathbb{U}_{n,r}$ be the set of all fully loaded unicyclic graphs with n vertices and unique cycle C_r. For $n \geq 6$, let $\mathbb{U}_n$ be the set of n-vertex fully loaded unicyclic graphs.

3. Fully loaded unicyclic graphs with small Szeged indices

For $3 \leq r \leq \lfloor \frac{n}{2} \rfloor$, let $S_{n,r} = C_r(T_1, T_2, \ldots, T_r)$ with $T_1 = \cdots = T_{r-1} = P_2$ and $T_r = S_{n-2(r-1)}$ with center v_r.

Proposition 1. *Let $G \in \mathbb{U}_{n,r}$ with $3 \le r \le \lfloor \frac{n}{2} \rfloor$. Then*

$$Sz(G) \ge \begin{cases} n^2 + (r^2 - 3r)n - r^3 + 2r^2 & \text{if } r \text{ is odd} \\ n^2 + (r^2 - r - 1)n - r^3 + r & \text{if } r \text{ is even} \end{cases}$$

with equality if and only if $G = S_{n,r}$.

Proof. By the definition of the Szeged index, if r is odd, then

$$\begin{aligned} Sz(S_{n,r}) &= 1 \cdot (n-1) \cdot (n-r) + (r-1) \cdot (r-1) \\ &\quad + (r-1) \cdot (n-r+1-2) \cdot (r-1) \\ &= n^2 + (r^2 - 3r)n - r^3 + 2r^2, \end{aligned}$$

and if r is even, then

$$\begin{aligned} Sz(S_{n,r}) &= 1 \cdot (n-1) \cdot (n-r) + r \cdot (n-r) \cdot r \\ &= n^2 + (r^2 - r - 1)n - r^3 + r. \end{aligned}$$

Let $G = C_r(T_1, T_2, \ldots, T_r)$ be a graph with the smallest Szeged index among graphs in $\mathbb{U}_{n,r}$. We need only to show that $G = S_{n,r}$.

By Lemmas 1, 2 and 6, T_i is a star with center v_i for $i = 1, 2, \ldots, r$. Suppose that $t_k, t_l \ge 3$ with $k \ne l$. Let $w \in V(T_l)$ with $w \ne v_l$. Suppose without loss of generality that $N_k + \frac{1}{2}t_k \le N_l + \frac{1}{2}t_l$ if r is odd and that $N_k \le N_l$ if r is even. By Lemma 5, $Sz(G - \{v_l w\} + \{v_k w\}) < Sz(G)$, a contradiction. Thus there can not be two trees of $T_1, T_2, \ldots, T_r$ with at least three vertices in G, i.e., $G = S_{n,r}$. □

Let Γ_n be the set of graphs $C_3(T_1, T_2, T_3)$ in $\mathbb{U}_n$ with $|T_1| = |T_2| = 2$. Let Ψ_n be the set of graphs $C_3(T_1, T_2, T_3)$ in $\mathbb{U}_n$ with $|T_3| \ge |T_2| \ge \max\{|T_1|, 3\}$. Let Φ_n be the set of graphs in $\mathbb{U}_n$ with cycle length at least four. Then $\mathbb{U}_n = \Gamma_n \cup \Psi_n \cup \Phi_n$.

For $n \ge 7$, let B'_n be the graph in Γ_n formed by attaching a path P_2 and $n-7$ pendent vertices to the vertex of degree two in $C_3(P_2, P_2, P_1)$.

Lemma 7. *Among the graphs in Γ_n with $n \ge 7$, B'_n is the unique graph with the second smallest Szeged index, which is equal to $n^2 + n - 12$.*

Proof. The case $n = 7$ is trivial. Let $G = C_3(T_1, T_2, T_3) \in \Gamma_n$ with $n \ge 8$. Note that $|T_1| = |T_2| = 2$. By Lemma 6, we have

$$\begin{aligned} Sz(G) &= 1 + 1 + W(T_3) + n - 2 + n - 2 + 4W_{v_3}(T_3) \\ &\quad + 2[2 \cdot 2 + 2 \cdot (n-4) + 2 \cdot (n-4)] \\ &\quad - [2 \cdot 2 + 2 \cdot (n-4) + 2 \cdot (n-4)] \\ &= 6n - 14 + W(T_3) + 4W_{v_3}(T_3), \end{aligned}$$

which, together with Lemmas 3 and 4, implies that B'_n is the unique graph in Γ_n with the second smallest Szeged index, which is equal to $n^2 + n - 12$. □

Let $S_n(a,b,c) = C_3(T_1,T_2,T_3)$ with $|T_1| = a$, $|T_2| = b$, $|T_3| = c$, $a+b+c = n$, and $a,b,c \geq 2$, where T_1 (T_2, T_3, respectively) is a star with center v_1 (v_2, v_3, respectively). Let G'_n for $n \geq 8$ be the graph obtained from $S_{n-1}(2,3,n-6)$ by attaching a pendent vertex to a pendent vertex at v_3, and G''_n for $n \geq 9$ the graph obtained from $S_{n-1}(2,2,n-5)$ by attaching a pendent vertex to the pendent vertex at v_2.

Lemma 8. *Among the graphs in Ψ_n, $S_n(2,3,n-5)$ for $n \geq 8$ is the unique graph with the smallest Szeged index, which is equal to n^2+n-16; G'_8 for $n = 8$, $S_9(3,3,3)$ for $n = 9$, and $S_n(2,4,n-6)$ for $n \geq 10$ are the unique graphs with the second smallest Szeged index, which is equal to 61 for $n = 8$, 75 for $n = 9$, and $n^2+2n-25$ for $n \geq 10$.*

Proof. For $n \geq 8$, let $G = C_3(T_1,T_2,T_3) \in \Psi_n$ with $c \geq b \geq \max\{a,3\}$, where $|T_1| = a$, $|T_2| = b$, $|T_3| = c$ and $a+b+c = n$.

If in G, $T_1 = P_2$ and $T_2 = S_3$ with v_2 its center, then by Lemma 6, $Sz(G) = 8n - 22 + W(T_3) + 5W_{v_3}(T_3)$, which, together with Lemmas 1–4, implies that, among the graphs $C_3(T_1,T_2,T_3) \in \Psi_n$ with $T_1 = P_2$ and $T_2 = S_3$ with v_2 its center, $S_n(2,3,n-5)$ and G'_n are the unique graphs with the smallest Szeged index n^2+n-16 and the second smallest Szeged index $n^2+2n-19$, respectively.

For $n = 8$, the graphs in Ψ_8 are $S_8(2,3,3)$, G'_8 and $C_3(P_2,P_3,P_3)$ with v_2 being a terminal vertex of one P_3 and v_3 a terminal vertex of the other P_3, where $Sz(S_8(2,3,3)) = 56 < Sz(G'_8) = 61 < Sz(C_3(P_2,P_3,P_3)) = 66$. Thus $S_8(2,3,3)$ and G'_8 are the unique graphs in Ψ_8 respectively with the smallest and the second smallest Szeged indices, which are equal to 56 and 61, respectively.

For $n = 9$, by Lemmas 1, 2 and 6, the graphs in Ψ_9 with the smallest Szeged index are just the graphs $S_9(2,3,4)$ and $S_9(3,3,3)$ with smaller Szeged index, which is equal to $\min\{74,75\} = 74$. Thus $S_9(2,3,4)$ is the unique graph in Ψ_9 with the smallest Szeged index, which is equal to 74. By the proof at the beginning, the graphs in Ψ_9 with the second smallest Szeged index are just the graphs in G'_9, G''_9 and $S_9(3,3,3)$ with the smallest Szeged index, which is equal to $\min\{80,80,75\} = 75$. Thus $S_9(3,3,3)$ is the unique graph in Ψ_9 with the second smallest Szeged index, which is equal to 75.

For $n \geq 10$, suppose first that $G = S_n(a,b,c)$ and $G \neq S_n(2,3,n-5)$, $S_n(2,4,n-6)$. Since $N_1 = \sum_{i\neq 1} t_i d_{i1} = b+c$, $N_2 = \sum_{i\neq 2} t_i d_{i2} = a+c$ and $N_3 = \sum_{i\neq 3} t_i d_{i3} = a+b$, it is easily seen that $N_3 + \frac{1}{2}c \leq N_2 + \frac{1}{2}b \leq N_1 + \frac{1}{2}a$. By Lemmas 5 and 6, if $a = 2$, then

$$\begin{aligned} Sz(S_n(2,b,c)) &> Sz(S_n(2,4,n-6)) = n^2+2n-25 \\ &> Sz(S_n(2,3,n-5)) = n^2+n-16, \end{aligned}$$

and if $a \geq 3$, then

$$Sz(S_n(a,b,c)) \geq Sz(S_n(3,3,n-6)) = n^2+2n-24 > n^2+2n-25.$$

If $G \neq S_n(a,b,c)$, then by the proof at the beginning and applying Lemmas 1, 2 and 6, we have either $Sz(G) \geq Sz(G'_n) = Sz(G''_n) = n^2+2n-19 > n^2-2n-25$ for $a=2$, or $Sz(G) > Sz(S_n(a,b,c)) \geq Sz(S_n(3,3,n-6)) > n^2+2n-25$ for $a \geq 3$. Thus, for $n \geq 10$, $S_n(2,3,n-5)$ and $S_n(2,4,n-6)$ are the unique graphs with the smallest and the second smallest Szeged indices, which are equal to n^2+n-16 and $n^2+2n-25$, respectively. The result follows. □

Theorem 1. *Among the graphs in $\mathbb{U}_n$,*
(i) $S_{n,3}$ for $n \geq 6$ is the unique graph with the smallest Szeged index, which is equal to n^2-9;
(ii) B'_7 for $n=7$, and $S_n(2,3,n-5)$ for $n \geq 8$ are the unique graphs with the second smallest Szeged index, which is equal to 44 for $n=7$, and n^2+n-16 for $n \geq 8$;
(iii) B'_8 for $n=8$, $S_9(3,3,3)$ for $n=9$, $S_n(2,4,n-6)$ for $10 \leq n \leq 12$, $S_{13}(2,4,7)$ and B'_{13} for $n=13$, and B'_n for $n \geq 14$ are the unique graphs with the third smallest Szeged index, which is equal to 60 for $n=8$, 75 for $n=9$, $n^2+2n-25$ for $10 \leq n \leq 12$, 170 for $n=13$, and n^2+n-12 for $n \geq 14$.

Proof. By Proposition 1, if r is odd, then

$$\begin{aligned} Sz(S_{n,r+1}) - Sz(S_{n,r}) &= (4r-1)n - 5r^2 - 2r \\ &\geq (4r-1)\cdot 2(r+1) - 5r^2 - 2r \\ &= 3r^2+4r-2 > 0 \end{aligned}$$

for $3 \leq r \leq \lfloor \frac{n}{2} \rfloor - 1$, implying that $Sz(S_{n,r+1}) > Sz(S_{n,r})$, and

$$\begin{aligned} Sz(S_{n,r+2}) - Sz(S_{n,r}) &= (4r-2)n - 6r^2 - 4r \\ &\geq (4r-2)\cdot 2(r+2) - 6r^2 - 4r \\ &= 2r^2+8r-8 > 0 \end{aligned}$$

for $3 \leq r \leq \lfloor \frac{n}{2} \rfloor - 2$, implying that $Sz(S_{n,r+2}) > Sz(S_{n,r})$. If r is even, then

$$Sz(S_{n,r+1}) - Sz(S_{n,r}) = -n - r^2 + 1 < 0$$

for $4 \leq r \leq \lfloor \frac{n}{2} \rfloor - 1$, implying that $Sz(S_{n,r+1}) < Sz(S_{n,r})$, and

$$\begin{aligned} Sz(S_{n,r+2}) - Sz(S_{n,r}) &= (4r+2)n - 6r^2 - 12r - 6 \\ &\geq (4r+2)\cdot 2(r+2) - 6r^2 - 12r - 6 \\ &= 2r^2+8r+2 > 0 \end{aligned}$$

for $4 \leq r \leq \lfloor \frac{n}{2} \rfloor - 2$, implying that $Sz(S_{n,r+2}) > Sz(S_{n,r})$. It follows that $Sz(S_{n,r+1}) > Sz(S_{n,r})$ for odd r with $3 \leq r \leq \lfloor \frac{n}{2} \rfloor - 1$, $Sz(S_{n,r+1}) < Sz(S_{n,r})$ for even r with $4 \leq r \leq \lfloor \frac{n}{2} \rfloor - 1$, and $Sz(S_{n,r})$ is increasing with respect to odd

$r \in \{3,5,\ldots,\lfloor \frac{n}{2} \rfloor\}$ and even $r \in \{4,6,\ldots,\lfloor \frac{n}{2} \rfloor\}$. Then by Proposition 1, $S_{n,3}$ for $n \geq 6$ is the unique graph with the smallest Szeged index, which is equal to n^2-9, proving (i). Moreover, $S_{n,4}$ for $n = 8,9$ and $S_{n,5}$ for $n \geq 10$ are the unique graphs in Φ_n with the smallest Szeged index, which are equal to $n^2+11n-60$ and $n^2+10n-75$, respectively.

Now we prove (ii). The case $n = 7$ is trivial. For $n \geq 8$, the graphs in $\mathbb{U}_n$ with the second smallest Szeged index are just the graphs in $\mathbb{U}_n \setminus \{S_{n,3}\} = (\Gamma_n \setminus \{S_{n,3}\}) \cup \Psi_n \cup \Phi_n$ with the smallest Szeged index, which, by Lemmas 7 and 8, is equal to

$$\begin{aligned}&\min\{Sz(B_n'), Sz(S_n(2,3,n-5)), Sz(S_{n,4})\}\\ &= \min\{n^2+n-12, n^2+n-16, n^2+11n-60\} = n^2+n-16\end{aligned}$$

for $n = 8,9$, and

$$\begin{aligned}&\min\{Sz(B_n'), Sz(S_n(2,3,n-5)), Sz(S_{n,5})\}\\ &= \min\{n^2+n-12, n^2+n-16, n^2+10n-75\} = n^2+n-16\end{aligned}$$

for $n \geq 10$. Then (ii) follows.

From (i) and (ii) and by Lemmas 7 and 8, we find that the graphs in $\mathbb{U}_n$ for $n \geq 8$ with the third smallest Szeged index are just the graphs in

$$\mathbb{U}_n \setminus \{S_{n,3}, S_n(2,3,n-5)\} = (\Gamma_n \setminus \{S_{n,3}\}) \cup (\Psi_n \setminus \{S_n(2,3,n-5)\} \cup \Phi_n$$

for $n \geq 8$, which is equal to

$$\min\{Sz(B_8'), Sz(G_8'), Sz(S_{8,4})\} = \min\{60, 61, 92\} = 60$$

for $n = 8$,

$$\min\{Se(B_9'), Sz(S_9(3,3,3)), Sz(S_{9,4})\} = \min\{78, 75, 120\} = 75$$

for $n = 9$, and

$$\begin{aligned}&\min\{Sz(B_n'), Sz(S_n(2,4,n-6)), Sz(S_{n,5})\}\\ &= \min\{n^2+n-12, n^2+2n-25, n^2+10n-75\}\end{aligned}$$

i.e., $n^2+2n-25$ for $10 \leq n \leq 12$, $n^2+2n-25 = n^2+n-12 = 170$ for $n = 13$, and n^2+n-12 for $n \geq 14$. Now (iii) follows easily. □

4. Fully loaded unicyclic graphs with large Szeged indices

For $3 \leq r \leq \lfloor \frac{n}{2} \rfloor$, let $P_{n,r} = C_r(T_1, T_2, \ldots, T_r)$ with $T_1 = \cdots = T_{r-1} = P_2$ and $T_r = P_{n-2(r-1)}$ with a terminal vertex v_r.

Proposition 2. *Let $G \in \mathbb{U}_{n,r}$ with $3 \leq r \leq \lfloor \frac{n}{2} \rfloor$. Then*

$$Sz(G) \leq \begin{cases} \frac{1}{6}[n^3+(-6r^2+12r-7)n+10r^3-24r^2+14r] & \text{if } r \text{ is odd} \\ \frac{1}{6}[n^3+(-6r^2+24r-13)n+10r^3-36r^2+20r] & \text{if } r \text{ is even} \end{cases}$$

with equality if and only if $G = P_{n,r}$.

Proof. By the definition of the Szeged index, if r is odd, then

$$\begin{aligned} Sz(P_{n,r}) &= 1\cdot(n-1)\cdot r + 2\cdot(n-2) + \cdots + (n-2r+1)\cdot(2r-1) \\ &\quad + (r-1)\cdot(r-1) + (r-1)\cdot(n-r+1-2)\cdot(r-1) \\ &= \frac{1}{6}[n^3 + (-6r^2+12r-7)n + 10r^3 - 24r^2 + 14r], \end{aligned}$$

and if r is even, then

$$\begin{aligned} Sz(P_{n,r}) &= 1\cdot(n-1)\cdot r + 2\cdot(n-2) + \cdots + (n-2r+1)\cdot(2r-1) + r\cdot(n-r)\cdot r \\ &= \frac{1}{6}[n^3 + (-6r^2+24r-13)n + 10r^3 - 36r^2 + 20r]. \end{aligned}$$

Let $G = C_r(T_1, T_2, \ldots, T_r)$ be a graph with the largest Szeged index among graphs in $\mathbb{U}_{n,r}$. We need only to show that $G = P_{n,r}$.

By Lemmas 1, 2 and 6, T_i is a path with a terminal vertex v_i for each $i = 1, 2, \ldots, r$. Then by Lemma 6, we have

$$\begin{aligned} Sz(G) &= \sum_{i=1}^{r}\frac{1}{6}(t_i^3 - t_i) + \sum_{i=1}^{r}\frac{1}{2}(n-t_i)t_i(t_i-1) + \sum_{i=1}^{r}\sum_{j=1}^{r} t_i t_j d_{ij} - \varepsilon_r \sum_{i<j} t_i t_j \\ &= -\frac{1}{3}\sum_{i=1}^{r} t_i^3 + \frac{1}{2}(n+1)\sum_{i=1}^{r} t_i^2 - \frac{1}{2}n^2 - \frac{1}{6}n + \sum_{i=1}^{r}\sum_{j=1}^{r} t_i t_j d_{ij} - \varepsilon_r \sum_{i<j} t_i t_j. \end{aligned}$$

Suppose that there exist distinct k, l, m with $1 \le k, l, m \le r$, such that t_k, t_l, $t_m \ge 3$ and $t_m \ge \max\{t_k, t_l\}$.

Case 1. r is odd. Assume that $t_k^2 + (n+1)t_l + 2N_l \le t_l^2 + (n+1)t_k + 2N_k$. Let G' be the graph formed from G by deleting the pendent vertex in T_l and attaching it to the pendent vertex in T_k. Obviously, $G' \in \mathbb{U}_{n,r}$. Note that $2t_l + 2d_{kl} < t_l + t_m + r \le n < n+2$. Then

$$\begin{aligned} &Sz(G) - Sz(G') \\ &= (t_k^2 + nt_l + 2N_l) - (t_l^2 + nt_k + 2N_k) + 2t_l + 2d_{kl} - n - 1 + (-t_k + t_l - 1) \\ &= [t_k^2 + (n+1)t_l + 2N_l] - [t_l^2 + (n+1)t_k + 2N_k] + 2t_l + 2d_{kl} - (n+2) < 0, \end{aligned}$$

and thus $Sz(G) < Sz(G')$, which is a contradiction. Thus $r-2$ of $t_1, t_2, \ldots, t_r$ are equal to 2, say $t_i = 2$ for $i \ne k, l$. Let $t_k = a$ and $t_l = b$. We write $G = G_{a,b}$, where $a, b \ge 2$ and $a + b = n - 2(r-2)$. Note that $N_k = 2W_{v_k}(C_r) + (b-2)d_{kl}$ and $N_l = 2W_{v_l}(C_r) + (a-2)d_{kl}$. If $a \ge b \ge 3$, then since that $a + b + 2d_{kl} < a + b + r \le n+1$ and $2b + 2d_{kl} < a + b + r < n+2$, we have

$$\begin{aligned} &Sz(G_{a,b}) - Sz(G_{a+1,b-1}) \\ &= [a^2 + (n+1)b + 4W_{v_l}(C_r) + 2(a-2)d_{kl}] \\ &- [b^2 + (n+1)a + 4W_{v_k}(C_r) + 2(b-2)d_{kl}] + 2b + 2d_{kl} - (n+2) \end{aligned}$$

$$= (a-b)[a+b+2d_{kl}-(n+1)]+2b+2d_{kl}-(n+2) < 0,$$

implying that $Sz(G_{a,b})$ is maximum for $a \geq b$ and $a+b = n-2(r-2)$ if and only if $a = n-2(r-1)$ and $b = 2$. It follows that $G = P_{n,r}$.

Case 2. r is even. Assume that $t_k^2+nt_l+2N_l \leq t_l^2+nt_k+2N_k$. Let G' be the graph formed from G by deleting the pendent vertex in T_l and attaching it to the pendent vertex in T_k. Obviously, $G' \in \mathbb{U}_{n,r}$. Note that $2t_l+2d_{kl} \leq t_l+t_m+r \leq n < n+1$. Then

$$Sz(G)-Sz(G') = (t_k^2+nt_l+2N_l)-(t_l^2+nt_k+2N_k)+2t_l+2d_{kl}-(n+1) < 0,$$

and thus $Sz(G) < Sz(G')$, which is a contradiction. Thus $r-2$ of $t_1,t_2,\ldots,t_r$ are equal to 2, say $t_i = 2$ for $i \neq k,l$. Let $t_k = a$ and $t_l = b$. We write $G = G_{a,b}$, where $a,b \geq 2$ and $a+b = n-2(r-2)$. Note that $N_k = 2W_{v_k}(C_r)+(b-2)d_{kl}$ and $N_l = 2W_{v_l}(C_r)+(a-2)d_{kl}$. If $a \geq b \geq 3$, then since that $a+b+2d_{kl} \leq a+b+r \leq n$ and $2b+2d_{kl} \leq a+b+r < n+1$, we have

$$\begin{aligned}
&Sz(G_{a,b})-Sz(G_{a+1,b-1}) = [a^2+nb+4W_{v_l}(C_r)+2(a-2)d_{kl}]\\
&-[b^2+na+4W_{v_k}(C_r)+2(b-2)d_{kl}]+2b+2d_{kl}-(n+1)\\
&= (a-b)(a+b+2d_{kl}-n)+2b+2d_{kl}-(n+1) < 0,
\end{aligned}$$

implying that $Sz(G_{a,b})$ is maximum for $a \geq b$ and $a+b = n-2(r-2)$ if and only if $a = n-2(r-1)$ and $b = 2$. It follows that $G = P_{n,r}$.

By combining Cases 1 and 2, the result follows. □

Let $P_n(a,b,c,d) = C_4(T_1,T_2,T_3,T_4)$ with $|T_1| = a$, $|T_2| = b$, $|T_3| = c$, $|T_4| = d$, $a+b+c+d = n$, and $a,b,c,d \geq 2$, where T_1 (T_2, T_3, T_4, respectively) is a path with a terminal vertex v_1 (v_2, v_3, v_4, respectively). Let $P'_{n,4}$ be the graph formed by attaching a pendent vertex to the neighbor of the pendent vertex of T_4 in $P_{n-1,4}$.

Lemma 9. *Among the graphs in $\mathbb{U}_{n,4}$ with $n \geq 9$, $S_{9,4}$ for $n = 9$ and $P_n(2,3,2,n-7)$ for $n \geq 10$ are the unique graphs with the second largest Szeged index, which is equal to 120 for $n = 9$, and $\frac{1}{6}(n^3-19n+198)$ for $n \geq 10$; $P_n(2,2,3,n-7)$ for $n = 10,11$, $P_n(2,4,2,n-8)$ for $12 \leq n \leq 16$, $P_{17}(2,4,2,9) = P'_{17,4}$ for $n = 17$, and $P'_{n,4}$ for $n \geq 18$ are the unique graphs with the third largest Szeged index, which is equal to $\frac{1}{6}(n^3-31n+306)$ for $n = 10,11$, $\frac{1}{6}(n^3-25n+264)$ for $12 \leq n \leq 16$, 792 for $n = 17$ and $\frac{1}{6}(n^3-19n+162)$ for $n \geq 18$.*

Proof. The case $n = 9$ is trivial. Let $G = C_4(T_1,T_2,T_3,T_4) \in \mathbb{U}_{n,4}$ with $n \geq 10$.

Suppose first that there is exactly one tree, say T_4 with more than two vertices. Note that $t_1 = t_2 = t_3 = 2$. By Lemma 6, we have $Sz(G) = 19n-67+W(T_4)+6W_{v_4}(T_4)$. In this case, by Lemmas 3 and 4, we know that $P'_{n,4}$ is the unique graph with the second largest Szeged index, which is equal to $\frac{1}{6}(n^3-19n+162)$.

Suppose that there are two trees (T_3 and T_4, or T_2 and T_4) with more than two vertices. By the proof of Proposition 2, we have

$$Sz(G) \le Sz(P_n(2,2,t_3,t_4)) \le Sz(P_n(2,2,3,n-7)) = \frac{1}{6}(n^3-31n+306)$$

or

$$Sz(G) \le Sz(P_n(2,t_2,2,t_4)) \le Sz(P_n(2,3,2,n-7)) = \frac{1}{6}(n^3-19n+198).$$

If there are three or four trees with more than two vertices, then by the proof of Proposition 2, we have

$$Sz(G) \le Sz(P_n(t_1,t_2,t_3,t_4)) \le \max\{Sz(P_n(2,2,3,n-7)), Sz(P_n(2,3,2,n-7))\}.$$

It follows that the graphs in $\mathbb{U}_{n,4}$ with the second largest Szeged index are just the graphs in $\mathbb{U}_{n,4} \setminus \{P_{n,4}\}$ with the largest Szeged index, which is equal to

$$\begin{aligned}
&\max\{Sz(P'_{n,4}), Sz(P_n(2,2,3,n-7)), Sz(P_n(2,3,2,n-7))\}\\
&= \max\left\{\frac{1}{6}(n^3-19n+162), \frac{1}{6}(n^3-31n+306), \frac{1}{6}(n^3-19n+198)\right\}\\
&= \frac{1}{6}(n^3-19n+198).
\end{aligned}$$

Thus $P_n(2,3,2,n-7)$ is the unique graph in $\mathbb{U}_{n,4}$ for $n \ge 10$ with the second largest Szeged index, which is equal to $\frac{1}{6}(n^3-19n+198)$.

Suppose that T_2 and T_4 have more than two vertices. If $t_2 = 3$ and $G \ne P_n(2,3,2,n-7)$, then

$$\begin{aligned}
Sz(G) &\le \max\{Sz(C_4(P_2,S_3,P_2,P_{n-7})), Sz(C_4(P_2,P_3,P_2,P'_{n-7}))\}\\
&= \max\left\{\frac{1}{6}(n^3-25n+216), \frac{1}{6}(n^3-25n+216)\right\}\\
&= \frac{1}{6}(n^3-25n+216),
\end{aligned}$$

where v_2 is the center of S_3 and v_4 is a terminal vertex of P_{n-7} in $C_4(P_2,S_3,P_2,P_{n-7})$, and v_2 is a terminal vertex of P_3 and v_4 is a terminal vertex of P'_{n-7} for $n = 11$ and a terminal vertex of P'_{n-7} whose neighbor has degree two for $n \ge 12$ in $C_4(P_2,P_3,P_2,P'_{n-7})$. And if $t_2 \ge 4$, then we have

$$Sz(G) \le Sz(P_n(2,t_2,2,t_4)) \le Sz(P_n(2,4,2,n-8)) = \frac{1}{6}(n^3-25n+264).$$

Thus the graphs in $\mathbb{U}_{n,4}$ with the third largest Szeged index are just the graphs in $\mathbb{U}_{n,4} \setminus \{P_{n,4}, P_n(2,3,2,n-7)\}$ with the largest Szeged index, which is equal to

$$\begin{aligned}
&\max\{Sz(P'_{n,4}), Sz(P_n(2,2,3,n-7)), Sz(C_4(P_2,S_3,P_2,P_{n-7})),\\
&\quad Sz(C_4(P_2,P_3,P_2,P'_{n-7})), Sz(P_n(2,4,2,n-8))\}
\end{aligned}$$

$$= \max\left\{\frac{1}{6}(n^3-19n+162), \frac{1}{6}(n^3-31n+306), \frac{1}{6}(n^3-25n+216), \frac{1}{6}(n^3-25n+264)\right\}$$

i.e., $\frac{1}{6}(n^3-31n+306)$ for $n=10,11$, $\frac{1}{6}(n^3-25n+264)$ for $12 \le n \le 16$, $\frac{1}{6}(n^3-25n+264) = \frac{1}{6}(n^3-19n+162) = 792$ for $n=17$, and $\frac{1}{6}(n^3-19n+162)$ for $n \ge 18$. Thus $P_n(2,2,3,n-7)$ for $n=10,11$, $P_n(2,4,2,n-8)$ for $12 \le n \le 16$, $P_{17}(2,4,2,9) = P'_{17,4}$ for $n=17$, and $P'_{n,4}$ for $n \ge 18$ are the unique graphs with the third largest Szeged index, which is equal to $\frac{1}{6}(n^3-31n+306)$ for $n=10,11$, $\frac{1}{6}(n^3-25n+264)$ for $12 \le n \le 16$, 792 for $n=17$ and $\frac{1}{6}(n^3-19n+162)$ for $n \ge 18$. The result follows. □

Theorem 2. *Among the graphs in $\mathbb{U}_n$,*
(i) $P_{n,3}$ for $n=6,7$ and $P_{n,4}$ for $n \ge 8$ are the unique graphs with the largest Szeged index, which is equal to $\frac{1}{6}(n^3-25n+96)$ for $n=6,7$, and $\frac{1}{6}(n^3-13n+144)$ for $n \ge 8$;
(ii) $S_{7,3}$ for $n=7$, $P_{8,3}$ for $n=8$, $S_{9,4}$ for $n=9$, and $P_n(2,3,2,n-7)$ for $n \ge 10$ are the unique graphs with the second largest Szeged index, which is equal to 40 for $n=7$, 68 for $n=8$, 120 for $n=9$, and $\frac{1}{6}(n^3-19n+198)$ for $n \ge 10$;
(iii) $C_3(P_2,P_3,P_3)$ with v_2 a terminal vertex of P_3 and v_3 a terminal vertex of P_3 for $n=8$, $P_{9,3}$ for $n=9$, $P_n(2,2,3,n-7)$ for $n=10,11$, $P_n(2,4,2,n-8)$ for $12 \le n \le 16$, $P_{17}(2,4,2,9) = P'_{17,4}$ for $n=17$, and $P'_{n,4}$ for $n \ge 18$ are the unique graphs with the third largest Szeged index, which is equal to 66 for $n=8$, 100 for $n=9$, $\frac{1}{6}(n^3-31n+306)$ for $n=10,11$, $\frac{1}{6}(n^3-25n+264)$ for $12 \le n \le 16$, 792 for $n=17$, and $\frac{1}{6}(n^3-19n+162)$ for $n \ge 18$.

Proof. The cases $n=6,7$ are obvious. Suppose that $n \ge 8$. Let $f_1(r) = Sz(P_{n,r})$ if r is odd, and $f_2(r) = Sz(P_{n,r})$ if r is even. If r is odd with $3 \le r \le \lfloor \frac{n}{2} \rfloor - 1$, then by Proposition 2,

$$\begin{aligned} f_2(r+1) - f_1(r) &= 2n + 3r^2 - 6r - 1 \ge 2\cdot 2(r+1) + 3r^2 - 6r - 1 \\ &= 3r^2 - 2r + 3 > 0, \end{aligned}$$

implying that $f_2(r+1) > f_1(r)$. If r is even with $4 \le r \le \lfloor \frac{n}{2} \rfloor - 1$, then by Proposition 2,

$$\begin{aligned} f_1(r+1) - f_2(r) &= (-4r+2)n + 7r^2 - 4r \\ &\le (-4r+2)\cdot 2(r+1) + 7r^2 - 4r \\ &= -r^2 - 8r + 4 < 0, \end{aligned}$$

implying that $f_1(r+1) < f_2(r)$.

For fixed n, taking the derivatives for $f_i(r)$ where $i=1,2$, whose expressions are given in Proposition 2, we have

$$f_1'(r) = \frac{1}{3}[15r^2 - (6n+24)r + 6n + 7],$$

$$f_2'(r) = \frac{1}{3}[15r^2 - (6n+36)r + 12n + 10].$$

The two roots of $f_1'(r) = 0$ are r_1 and r_2, where

$$r_1 = \frac{3n+12-\sqrt{9(n-1)^2+30}}{15} < \frac{3n+12-3(n-1)}{15} = 1 < 3,$$

$$r_2 = \frac{3n+12+\sqrt{9(n-1)^2+30}}{15} > \frac{3n+12+3(n-1))}{15} = \frac{2}{5}n + \frac{3}{5} > 3,$$

and $r_2 < \lfloor \frac{n}{2} \rfloor$. Hence, for fixed $n \geq 8$, $f_1'(r) < 0$ for $3 \leq r < r_2$ and $f_1'(r) \geq 0$ for $r_2 \leq r \leq \lfloor \frac{n}{2} \rfloor$. Then, for fixed $n \geq 8$, $f_1(r)$ is decreasing for $3 \leq r < r_2$ and increasing for $r_2 \leq r \leq \lfloor \frac{n}{2} \rfloor$. The two roots of $f_2'(r) = 0$ are r_3 and r_4, where

$$r_3 = \frac{3n+18-\sqrt{9(n-4)^2+30}}{15} < \frac{3n+18-3(n-4)}{15} = 2 < 4,$$

$$r_4 = \frac{3n+18+\sqrt{9(n-4)^2+30}}{15} > \frac{3n+18+3(n-4)}{15} = \frac{2}{5}n + \frac{2}{5} \geq 4$$

for $n \geq 9$, and $r_4 < \lfloor \frac{n}{2} \rfloor$ for $n \geq 8$. Hence, for fixed $n \geq 9$, $f_2'(r) < 0$ for $4 \leq r < r_4$ and $f_2'(r) \geq 0$ for $r_4 \leq r \leq \lfloor \frac{n}{2} \rfloor$. Then, for fixed $n \geq 9$, $f_2(r)$ is decreasing for $4 \leq r < r_4$ and increasing for $r_4 \leq r \leq \lfloor \frac{n}{2} \rfloor$. Note that $\frac{2}{5}n < r_4 < r_2 < \frac{2}{5}n + 1$ for $n \geq 8$. Let $G \in \mathbb{U}_n$.

By Proposition 2 and the properties of $f_1(r)$ and $f_2(r)$, we have

$$\begin{aligned} Sz(G) \leq Sz(P_{n,r}) &\leq \max\left\{ f_2(4), f_2\left(\frac{n}{2}\right)\right\} \\ &= \max\left\{ \frac{1}{6}(n^3 - 13n + 144), \frac{1}{8}(n^3 + 4n^2 - 4n)\right\} \\ &= \frac{1}{6}(n^3 - 13n + 144) \end{aligned}$$

if $n \geq 8$ and $n \equiv 0 \pmod 4$,

$$\begin{aligned} Sz(G) \leq Sz(P_{n,r}) &\leq \max\left\{ f_2(4), f_2\left(\frac{n-1}{2}\right)\right\} \\ &= \max\left\{ \frac{1}{6}(n^3 - 13n + 144), \frac{1}{8}(n^3 + 3n^2 + 7n - 27)\right\} \\ &= \frac{1}{6}(n^3 - 13n + 144) \end{aligned}$$

if $n \geq 9$ and $n \equiv 1 \pmod 4$,

$$\begin{aligned} Sz(G) \leq Sz(P_{n,r}) &\leq \max\left\{ f_2(4), f_2\left(\frac{n-2}{2}\right)\right\} \\ &= \max\left\{ \frac{1}{6}(n^3 - 13n + 144), \frac{1}{8}(n^3 + 2n^2 + 24n - 88)\right\} \end{aligned}$$

$$= \frac{1}{6}(n^3 - 13n + 144)$$

if $n \geq 10$ and $n \equiv 2 \pmod 4$, and

$$\begin{aligned} Sz(G) \leq Sz(P_{n,r}) &\leq \max\left\{ f_2(4), f_2\left(\frac{n-3}{2}\right)\right\} \\ &= \max\left\{ \frac{1}{6}(n^3 - 13n + 144), \frac{1}{8}(n^3 + n^2 + 47n - 193)\right\} \\ &= \frac{1}{6}(n^3 - 13n + 144) \end{aligned}$$

if $n \geq 11$ and $n \equiv 3 \pmod 4$. Thus $P_{n,4}$ is the unique graph in $\mathbb{U}_n$ with the largest Szeged index, and then the graphs in $\mathbb{U}_n$ with the second largest Szeged index are just the graphs in $\mathbb{U}_n \setminus \{P_{n,4}\}$ with the largest Szeged index.

Case 1. $n \geq 8$ and $n \equiv 0 \pmod 4$. The largest Szeged index of graphs in $\mathbb{U}_n \setminus \{P_{n,4}\}$ is equal to $f_1(3) = 68$ for $n = 8$ and

$$\begin{aligned} &\max\left\{ Sz(P_n(2,3,2,n-7)), f_1(3), f_2(6), f_2\left(\frac{n}{2}\right)\right\} \\ &= \max\left\{ \frac{1}{6}(n^3 - 19n + 198), \frac{1}{6}(n^3 - 25n + 96), \frac{1}{6}(n^3 - 85n + 984), \frac{1}{8}(n^3 + 4n^2 - 4n)\right\} \\ &= \frac{1}{6}(n^3 - 19n + 198) \end{aligned}$$

for $n \geq 12$. Thus $P_{8,3}$ for $n = 8$ and $P_n(2,3,2,n-7)$ for $n \geq 12$ are the unique graphs in $\mathbb{U}_n$ with the second largest Szeged index. It follows that the graphs in $\mathbb{U}_n$ with the third largest Szeged index are just the graphs in $\mathbb{U}_8 \setminus \{P_{8,4}, P_{8,3}\}$ for $n = 8$ and $\mathbb{U}_n \setminus \{P_{n,4}, P_n(2,3,2,n-7)\}$ for $n \geq 12$ with the largest Szeged index, which is equal to $Sz(C_3(P_2, P_3, P_3)) = 66$ for $n = 8$, where one P_3 has a terminal vertex v_2 and the other P_3 has a terminal vertex v_3 in $C_3(P_2, P_3, P_3)$,

$$\begin{aligned} &\max\left\{ Sz(P_n(2,4,2,n-8)), f_1(3), f_2(6), f_2\left(\frac{n}{2}\right)\right\} \\ &= \max\left\{ \frac{1}{6}(n^3 - 25n + 264), \frac{1}{6}(n^3 - 25n + 96), \frac{1}{6}(n^3 - 85n + 984), \frac{1}{8}(n^3 + 4n^2 - 4n)\right\} \\ &= \frac{1}{6}(n^3 - 25n + 264) \end{aligned}$$

for $n = 12, 16$, and

$$\begin{aligned} &\max\left\{ Sz(P'_{n,4}), f_1(3), f_2(6), f_2\left(\frac{n}{2}\right)\right\} \\ &= \max\left\{ \frac{1}{6}(n^3 - 19n + 162), \frac{1}{6}(n^3 - 25n + 96), \frac{1}{6}(n^3 - 85n + 984), \frac{1}{8}(n^3 + 3n^2 + 7n - 27)\right\} \\ &= \frac{1}{6}(n^3 - 19n + 162) \end{aligned}$$

for $n \geq 20$. Thus $C_3(P_2, P_3, P_3)$ with v_2 being a terminal vertex of P_3 and v_3 being a terminal vertex of P_3 for $n = 8$, $P_n(2,4,2,n-8)$ for $n = 12, 16$, and $P'_{n,4}$ for $n \geq 20$ are the unique graphs in $\mathbb{U}_n$ with the third largest Szeged index.

Case 2. $n \geq 9$ and $n \equiv 1 \pmod 4$. The largest Szeged index of graphs in $\mathbb{U}_n \setminus \{P_{n,4}\}$ is equal to $\max\{Sz(S_{9,4}), f_1(3)\} = \max\{120, 100\} = 120$ for $n = 9$ and

$$\max\left\{Sz(P_n(2,3,2,n-7)), f_1(3), f_2(6), f_2\left(\frac{n-1}{2}\right)\right\}$$
$$= \max\left\{\frac{1}{6}(n^3-19n+198), \frac{1}{6}(n^3-25n+96), \frac{1}{6}(n^3-85n+984), \frac{1}{8}(n^3+3n^2+7n-27)\right\}$$
$$= \frac{1}{6}(n^3-19n+198)$$

for $n \geq 13$. Thus $S_{9,4}$ for $n = 9$ and $P_n(2,3,2,n-7)$ for $n \geq 13$ are the unique graphs in $\mathbb{U}_n$ with the second largest Szeged index. Then the graphs in $\mathbb{U}_n$ with the third largest Szeged index are just the graphs in $\mathbb{U}_9 \setminus \{P_{9,4}, S_{9,4}\}$ for $n = 9$ and $\mathbb{U}_n \setminus \{P_{n,4}, P_n(2,3,2,n-7)\}$ for $n \geq 13$ with the largest Szeged index, which is equal to $Sz(P_{9,3}) = 100$ for $n = 9$,

$$\max\{Sz(P_{13}(2,4,2,5)), f_1(3), f_2(6)\} = \max\{356, 328, 346\} = 356$$

for $n = 13$,

$$\max\{Sz(P_{17}(2,4,2,9)), Sz(P'_{17,4}), f_1(3), f_2(6), f_2(8)\}$$
$$= \max\{792, 792, 764, 742, 734\} = 792$$

for $n = 17$, and

$$\max\left\{Sz(P'_{n,4}), f_1(3), f_2(6), f_2\left(\frac{n-1}{2}\right)\right\}$$
$$= \max\left\{\frac{1}{6}(n^3-19n+162), \frac{1}{6}(n^3-25n+96), \frac{1}{6}(n^3-85n+984), \frac{1}{8}(n^3+3n^2+7n-27)\right\}$$
$$= \frac{1}{6}(n^3-19n+162)$$

for $n \geq 21$. Thus $P_{9,3}$ for $n = 9$, $P_{13}(2,4,2,5)$ for $n = 13$, $P_{17}(2,4,2,9)$ and $P'_{17,4}$ for $n = 17$, and $P'_{n,4}$ for $n \geq 21$ are the unique graphs in $\mathbb{U}_n$ with the third largest Szeged index.

Case 3. $n \geq 10$ and $n \equiv 2 \pmod 4$. The largest Szeged index of graphs in $\mathbb{U}_n \setminus \{P_{n,4}\}$ is equal to $\max\{Sz(P_{10}(2,3,2,3)), f_1(3), f_1(5)\} = \max\{168, 141, 125\} = 168$ for $n = 10$, and

$$\max\left\{Sz(P_n(2,3,2,n-7)), f_1(3), f_2(6), f_2\left(\frac{n-2}{2}\right)\right\}$$
$$= \max\left\{\frac{1}{6}(n^3-19n+198), \frac{1}{6}(n^3-25n+96), \frac{1}{6}(n^3-85n+984), \frac{1}{8}(n^3+2n^2+24n-88)\right\}$$
$$= \frac{1}{6}(n^3-19n+198)$$

for $n \geq 14$. Thus for $n \geq 10$, $P_n(2,3,2,n-7)$ is the unique graph in $\mathbb{U}_n$ with the second largest Szeged index. Then the graphs in $\mathbb{U}_n$ with the third largest Szeged index are just the graphs in $\mathbb{U}_n \setminus \{P_{n,4}, P_n(2,3,2,n-7)\}$ with the largest Szeged index, which is equal to $\max\{Sz(P_{10}(2,2,3,3)), f_1(3), f_1(5)\} = \max\{166, 141, 125\} = 166$ for $n = 10$, $\max\{Sz(P_{14}(2,4,2,6)), f_1(3), f_2(6)\} = \max\{443, 415, 423\} = 443$ for $n = 14$, and

$$\begin{aligned}
&\max\left\{Sz(P'_{n,4}), f_1(3), f_2(6), f_2\left(\frac{n-2}{2}\right)\right\} \\
&= \max\left\{\frac{1}{6}(n^3-19n+162), \frac{1}{6}(n^3-25n+96), \frac{1}{6}(n^3-85n+984), \frac{1}{8}(n^3+2n^2+24n-88)\right\} \\
&= \frac{1}{6}(n^3-19n+162)
\end{aligned}$$

for $n \geq 18$. Thus $P_{10}(2,2,3,3)$ for $n = 10$, $P_{14}(2,4,2,6)$ for $n = 14$, and $P'_{n,4}$ for $n \geq 18$ are the unique graphs in $\mathbb{U}_n$ with the third largest Szeged index.

Case 4. $n \geq 11$ and $n \equiv 3 \pmod 4$. The largest Szeged index of graphs in $\mathbb{U}_n \setminus \{P_{n,4}\}$ is equal to

$$\max\{Sz(P_{11}(2,3,2,4)), f_1(3), f_1(5)\} = \max\{220, 214, 164\} = 220$$

for $n = 11$, and

$$\begin{aligned}
&\max\left\{Sz(P_n(2,3,2,n-7)), f_1(3), f_2(6), f_2\left(\frac{n-3}{2}\right)\right\} \\
&= \max\left\{\frac{1}{6}(n^3-19n+198), \frac{1}{6}(n^3-25n+96), \frac{1}{6}(n^3-85n+984), \frac{1}{8}(n^3+n^2+47n-193)\right\} \\
&= \frac{1}{6}(n^3-19n+198)
\end{aligned}$$

for $n \geq 15$. Thus $P_n(2,3,2,n-7)$ for $n \geq 11$ is the unique graph in $\mathbb{U}_n$ with the second largest Szeged index. Then the graphs in $\mathbb{U}_n$ for $n \geq 11$ with the third largest Szeged index are just the graphs in $\mathbb{U}_n \setminus \{P_{n,4}, P_n(2,3,2,n-7)\}$ with the largest Szeged index, which is equal to

$$\max\{Sz(P_{11}(2,2,3,4)), f_1(3), f_1(5)\} = \max\{216, 214, 164\} = 216$$

for $n = 11$, $\max\{Sz(P_{15}(2,4,2,7)), f_1(3), f_2(6)\} = \max\{544, 516, 514\} = 544$ for $n = 15$, and

$$\begin{aligned}
&\max\left\{Sz(P'_{n,4}), f_1(3), f_2(6), f_2\left(\frac{n-3}{2}\right)\right\} \\
&= \max\left\{\frac{1}{6}(n^3-19n+162), \frac{1}{6}(n^3-25n+96), \frac{1}{6}(n^3-85n+984), \frac{1}{8}(n^3+n^2+47n-193)\right\} \\
&= \frac{1}{6}(n^3-19n+162)
\end{aligned}$$

for $n \geq 19$. Thus $P_{11}(2,2,3,4)$ for $n = 11$, $P_{15}(2,4,2,7)$ for $n = 15$, and $P'_{n,4}$ for $n \geq 19$ are the unique graphs in $\mathbb{U}_n$ with the third largest Szeged index.

By combining Cases 1–4, the result follows easily. □

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Author's address

Xuli Qi
School of Mathematics and Statistics, Central China Normal University, Wuhan 430079, P. R. China
E-mail address: qixuli-1212@163.com

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BOUNDING THE CONVEX COMBINATION OF ARITHMETIC AND INTEGRAL MEANS IN TERMS OF ONE-PARAMETER HARMONIC AND GEOMETRIC MEANS

WEI-MAO QIAN, WEN ZHANG, AND YU-MING CHU

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Abstract. In the article, we find the best possible parameters λ_1, μ_1, λ_2 and μ_2 on the interval $[0,1/2]$ such that the double inequalities

$$H(a,b;\lambda_1) < \alpha A(a,b) + (1-\alpha)T(a,b) < H(a,b;\mu_1),$$
$$G(a,b;\lambda_2) < \alpha A(a,b) + (1-\alpha)T(a,b) < G(a,b;\mu_2)$$

hold for all $\alpha \in [0,1]$ and $a,b > 0$ with $a \neq b$, where $A(a,b) = (a+b)/2$, $T(a,b) = 2\int_0^{\pi/2} a^{\cos^2\theta} b^{\sin^2\theta} d\theta/\pi$, $H(a,b;\lambda) = 2[\lambda a + (1-\lambda)b][\lambda b + (1-\lambda)a]/(a+b)$, $G(a,b;\mu) = \sqrt{[\mu a + (1-\mu)b][\mu b + (1-\mu)a]}$ are the arithmetic, integral, one-parameter harmonic and one-parameter geometric means of a and b, respectively.

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1. Introduction

For $\lambda,\mu \in [0,1]$, the arithmetic mean $A(a,b)$, harmonic mean $H(a,b)$, geometric mean $G(a,b)$, integral mean $T(a,b)$ [25], one-parameter harmonic mean $H(a,b;\lambda)$ and one-parameter geometric mean $G(a,b;\mu)$ of two distinct positive real numbers a and b are given by

$$A(a,b) = \frac{a+b}{2}, \quad H(a,b) = \frac{2ab}{a+b}, \tag{1.1}$$

$$G(a,b) = \sqrt{ab}, \quad T(a,b) = \frac{2}{\pi}\int_0^{\pi/2} a^{\cos^2\theta} b^{\sin^2\theta} d\theta, \tag{1.2}$$

$$H(a,b;\lambda) = H[\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)b], \tag{1.3}$$

$$G(a,b;\mu) = G[\mu a + (1-\mu)b, \mu b + (1-\mu)b], \tag{1.4}$$

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respectively. The integral mean $T(a,b)$ has been the subject of intensive research in recent years due to it has been widely applied in pure and applied mathematics, physics and other natural sciences [2, 4–12, 14–16, 19, 21, 22, 24, 26–30, 36–38, 40].

The identity

$$T(a,b) = \sqrt{ab}I_0\left(\log\sqrt{b/a}\right) \tag{1.5}$$

and inequalities

$$L(a,b) < T(a,b) < \frac{A(a,b)+G(a,b)}{2} < \frac{2A(a,b)+G(a,b)}{3} < I(a,b) \tag{1.6}$$

for all $a,b>0$ with $a\neq b$ were established by Qi, Shi, Liu and Yang [18], where

$$I_\nu(t) = \sum_{n=0}^{\infty}\frac{1}{n!\Gamma(n+\nu+1)}\left(\frac{t}{2}\right)^{2n+\nu} \tag{1.7}$$

is the modified Bessel function of the first kind [1], $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$ is the classical gamma function [13,39], and $L(a,b)=(b-a)/(\log b-\log a)$ and $I(a,b)=(b^b/a^a)^{1/(b-a)}/e$ are respectively the logarithmic and inentric means of a and b.

Yang and Chu [31, 32], and Yang, Chu and Song [33] proved that the inequalities

$$\lambda_1\sqrt{L(a,b)A(a,b)} < T(a,b) < \mu_1\sqrt{L(a,b)A(a,b)},$$
$$L^{\lambda_2}(a,b)A^{1-\lambda_2}(a,b) < T(a,b) < \mu_2 L(a,b)+(1-\mu_2)A(a,b),$$
$$T(a,b) > L_p(a,b)$$
$$\lambda_3\sqrt{L(a,b)I(a,b)} < T(a,b) < \mu_3\sqrt{L(a,b)I(a,b)},$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\lambda_1\le\sqrt{2/\pi}$, $\mu_1\ge 1$, $\lambda_2\ge 3/4$, $\mu_2\le 3/4$, $p\le 3/2$, $\lambda_3\le\sqrt{e/\pi}$ and $\mu_3\ge 1$, where $L_p(a,b)=[(b^p-a^p)/(p(\log b-\log a))]^{1/p}$ is the p-order generalized logarithmic mean of a and b.

In [20], the authors proved that $p_1=0, q_1=1/4, p_2=0$ and $q_2=1/2-\sqrt{2}/4$ are the best possible parameters on the interval $[0,1/2]$ such that the double inequalities

$$H(a,b;p_1) < T(a,b) < H(a,b;q_1), \tag{1.8}$$

$$G(a,b;p_2) < T(a,b) < G(a,b;q_2) \tag{1.9}$$

hold for all $a,b>0$ with $a\neq b$.

Let $\alpha\in[0,1]$, $x\in[0,1/2]$, $a,b>0$ with $a\neq b$, $f(x)=H(a,b;x)$, $g(x)=G(a,b;x)$. Then we clearly see that both the functions $f(x)$ and $g(x)$ are strictly increasing on $[0,1/2]$. Inequality (1.6) and the well known inequalities

$$H(a,b) < G(a,b) < L(a,b) < I(a,b) < A(a,b)$$

lead to the conclusion that

$$f(0)=H(a,b) < \alpha A(a,b)+(1-\alpha)T(a,b) < A(a,b)=f(1/2), \tag{1.10}$$

$$g(0)=G(a,b) < \alpha A(a,b)+(1-\alpha)T(a,b) < A(a,b)=g(1/2). \tag{1.11}$$

From inequalities (1.10) and (1.11) together with the monotonicity of the functions $f(x)$ and $g(x)$ on the interval $[0,1/2]$, it is necessary to discover the best possible parameters λ_1, μ_1, λ_2 and μ_2 on the interval $[0,1/2]$ such that the double inequalities

$$H(a,b;\lambda_1) < \alpha A(a,b) + (1-\alpha)T(a,b) < H(a,b;\mu_1),$$
$$G(a,b;\lambda_2) < \alpha A(a,b) + (1-\alpha)T(a,b) < G(a,b;\mu_2)$$

hold for all $\alpha \in [0,1]$ and $a,b > 0$ with $a \neq b$.

2. Lemmas

Lemma 1 (Theorem 2.18 in [3]). *The identity*

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!}{2^{2n}n!}\sqrt{\pi}$$

holds for all $n \in \mathbb{N}$.

Lemma 2 ([17]). *Let* $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *be two real sequences with* $b_n > 0$ *and* $\lim_{n\to\infty} a_n/b_n = s$. *Then the power series* $\sum_{n=0}^{\infty} a_n t^n$ *is convergent for all* $t \in \mathbb{R}$ *and*

$$\lim_{t\to\infty} \frac{\sum_{n=0}^{\infty} a_n t^n}{\sum_{n=0}^{\infty} b_n t^n} = s$$

if the power series $\sum_{n=0}^{\infty} b_n t^n$ *is convergent for all* $t \in \mathbb{R}$.

Lemma 3 (Lemma 2.2 in [35]). *The double inequality*

$$\frac{1}{(x+a)^{1-a}} < \frac{\Gamma(x+a)}{\Gamma(x+1)} < \frac{1}{x^{1-a}}$$

holds for all $x > 0$ *and* $a \in (0,1)$.

Lemma 4 ([34]). *Let* $A(t) = \sum_{k=0}^{\infty} a_k t^k$ *and* $B(t) = \sum_{k=0}^{\infty} b_k t^k$ *be two real power series converging on* $(-r,r)$ $(r > 0)$ *with* $b_k > 0$ *for all* k. *If the non-constant sequence* $\{a_k/b_k\}_{k=0}^{\infty}$ *is increasing (decreasing) for all* k, *then the function* $t \mapsto A(t)/B(t)$ *is strictly increasing (decreasing) on* $(0,r)$.

Lemma 5 ((3.5) in [23]). *The identity*

$$I_\lambda(t)I_\mu(t) = \sum_{n=0}^{\infty} \frac{\Gamma(2n+\lambda+\mu+1)}{n!\Gamma(n+\lambda+\mu+1)\Gamma(n+\lambda+1)\Gamma(n+\mu+1)}\left(\frac{t}{2}\right)^{2n+\lambda+\mu}$$

holds for all $\lambda,\mu > -1$ *and* $t \in \mathbb{R}$.

Lemma 6. *The identity*

$$\cosh(t)I_0(t) = \sum_{n=0}^{\infty} \frac{(4n)!}{2^{2n}[(2n)!]^3} t^{2n}$$

holds for all $t \in \mathbb{R}$, *where* $\cosh(t) = (e^t + e^{-t})/2$ *is the hyperbolic cosine functions.*

Proof. It follows from (1.7) and Lemmas 1 and 5 that

$$I_{-1/2}(t) = \sqrt{\frac{2}{\pi t}} \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} = \sqrt{\frac{2}{\pi t}} \cosh(t),$$

$$\begin{aligned}\cosh(t) I_0(t) &= \sqrt{\frac{\pi t}{2}} I_{-1/2}(t) I_0(t) \\ &= \sqrt{\frac{\pi t}{2}} \sum_{n=0}^{\infty} \frac{\Gamma\left(2n+\frac{1}{2}\right)}{\left[n!\Gamma\left(n+\frac{1}{2}\right)\right]^2} \left(\frac{t}{2}\right)^{2n-1/2} = \sum_{n=0}^{\infty} \frac{(4n)!}{2^{2n}[(2n)!]^3} t^{2n}.\end{aligned}$$

□

Lemma 7. *The function*

$$f(t) = \frac{\cosh^2(t) - \cosh(t) I_0(t)}{\sinh^2(t)} \tag{2.1}$$

is strictly increasing from $(0,\infty)$ *onto* $(1/4,1)$*, where* $\sinh(t) = (e^t - e^{-t})/2$ *is the hyperbolic sine function.*

Proof. Let $n \in \mathbb{N}$, and $\{a_n\}$ and $\{b_n\}$ be defined by

$$a_n = \frac{(4n+4)!}{2^{2n+2}[(2n+2)!]^3}, \quad b_n = \frac{2^{2n+2}}{(2n+2)!}, \tag{2.2}$$

respectively.

Then simple computations lead to

$$\frac{a_0}{b_0} = \frac{3}{8}, \tag{2.3}$$

$$\frac{a_n}{b_n} = \frac{(4n+4)!}{2^{4n+4}[(2n+2)!]^2}, \tag{2.4}$$

$$\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = -\frac{(n+2)(2n+3)(8n+13)(4n+4)!}{2^{4n+5}[(2n+4)!]^2} < 0 \tag{2.5}$$

for all $n \in \mathbb{N}$.

It follows from Lemmas 1, 2, 3 and 6 together with (2.1)-(2.4) that

$$\begin{aligned} f(t) &= 1 - \frac{2[\cosh(t) I_0(t) - 1]}{\cosh(2t) - 1} \\ &= 1 - \frac{2\sum_{n=1}^{\infty} \frac{(4n)!}{2^{2n}[(2n)!]^3} t^{2n}}{\sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} t^{2n}} = 1 - \frac{2\sum_{n=0}^{\infty} a_n t^{2n}}{\sum_{n=0}^{\infty} b_n t^{2n}}, \end{aligned} \tag{2.6}$$

$$f(0^+) = 1 - \frac{2a_0}{b_0} = \frac{1}{4}, \tag{2.7}$$

$$\frac{1}{\sqrt{\pi(2n+5/2)}} < \frac{a_n}{b_n} = \frac{\Gamma(2n+5/2)}{\sqrt{\pi}\Gamma(2n+3)} < \frac{1}{\sqrt{\pi(2n+2)}},$$

$$f(\infty) = 1 - 2\lim_{n\to\infty}\frac{a_n}{b_n} = 1. \tag{2.8}$$

From Lemma 4, (2.5) and (2.6) we clearly see that the function $f(t)$ is strictly increasing on $(0,\infty)$. Therefore, Lemma 7 follows from (2.7) and (2.8) together with the monotonicity of $f(t)$ on the interval $(0,\infty)$. □

Lemma 8. *The function*

$$g(t) = \frac{\cosh^2(t) - I_0^2(t)}{\sinh^2(t)} \tag{2.9}$$

is strictly increasing from $(0,\infty)$ *onto* $(1/2,1)$.

Proof. Let $n \in \mathbb{N}$, and $\{c_n\}$ and $\{d_n\}$ be defined by

$$c_n = \frac{(2n+2)!}{2^{2n+2}[(n+1)!]^4}, \quad d_n = \frac{2^{2n+2}}{(2n+2)!}, \tag{2.10}$$

respectively. Then simple computations lead to

$$\frac{c_0}{d_0} = \frac{1}{4}, \tag{2.11}$$

$$\frac{c_n}{d_n} = \frac{[(2n+2)!]^2}{2^{4n+4}[(n+1)!]^4}, \tag{2.12}$$

$$\frac{c_{n+1}}{d_{n+1}} - \frac{c_n}{d_n} = -\frac{(4n+7)(n+2)^2[(2n+2)!]^2}{2^{4n+6}[(n+1)!]^4} < 0 \tag{2.13}$$

for all $n \in \mathbb{N}$.

From Lemmas 1, 2, 3 and 5 together with (2.9)-(2.12) one has

$$I_0^2(t) = \sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^4}t^{2n},$$

$$g(t) = 1 - \frac{2[I_0^2(t)-1]}{\cosh(2t)-1}$$

$$= 1 - \frac{2\sum_{n=1}^{\infty}\frac{(2n)!}{2^{2n}(n!)^4}t^{2n}}{\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}t^{2n}} = 1 - \frac{2\sum_{n=0}^{\infty}c_n t^{2n}}{\sum_{n=0}^{\infty}d_n t^{2n}}, \tag{2.14}$$

$$g(0^+) = 1 - \frac{2c_0}{d_0} = \frac{1}{2}, \tag{2.15}$$

$$\frac{1}{\pi(n+3/2)} < \frac{c_n}{d_n} = \frac{\Gamma^2(n+3/2)}{\pi\Gamma^2(n+2)} < \frac{1}{\pi(n+1)},$$

$$g(\infty) = 1 - 2\lim_{n\to\infty}\frac{c_n}{d_n} = 1. \tag{2.16}$$

Therefore, Lemma 8 follows easily from Lemma 4 and (2.13)-(2.16). □

3. Main results

Theorem 1. *Let $\lambda_1, \mu_1, \lambda_2, \mu_2 \in [0,1/2]$. Then the double inequalities*

$$H(a,b;\lambda_1) < \alpha A(a,b) + (1-\alpha)T(a,b) < H(a,b;\mu_1), \tag{3.1}$$

$$G(a,b;\lambda_2) < \alpha A(a,b) + (1-\alpha)T(a,b) < G(a,b;\mu_2) \tag{3.2}$$

hold for all $\alpha \in [0,1]$ and $a,b > 0$ with $a \neq b$ if and only if $\lambda_1 \le 1/2 - \sqrt{1-\alpha}/2$, $\mu_1 \ge 1/2 - \sqrt{1-\alpha}/4$, $\lambda_2 \le 1/2 - \sqrt{1-\alpha^2}/2$ and $\mu_2 \ge 1/2 - \sqrt{2(1-\alpha)}/4$.

Proof. Let $p,q \in [0,1/2]$. Without loss of generality, we assume that $b > a > 0$ and $t = \log\sqrt{b/a} > 0$ due to $A(a,b)$, $T(a,b)$, $H(a,b;p)$ and $G(a,b;q)$ are symmetric and homogeneous of degree one with respect to a and b. From (1.1)-(1.5) one has

$$A(a,b) = \sqrt{ab}\cosh(t),$$

$$T(a,b) = \sqrt{ab}I_0(t),$$

$$H(a,b;p) = \sqrt{ab}\cosh(t)\left[1-(1-2p)^2\frac{\sinh^2(t)}{\cosh^2(t)}\right],$$

$$G(a,b;q) = \sqrt{ab}\cosh(t)\sqrt{1-(1-2q)^2\frac{\sinh^2(t)}{\cosh^2(t)}},$$

$$H(a,b;p) - [\alpha A(a,b) + (1-\alpha)T(a,b)] = \frac{\sqrt{ab}\sinh^2(t)}{\cosh(t)}\left[(1-\alpha)f(t) - (1-2p)^2\right], \tag{3.3}$$

$$\begin{aligned} &G(a,b;q) - [\alpha A(a,b) + (1-\alpha)T(a,b)] \\ &= \frac{\sqrt{ab}\left[(1-\alpha)^2(\cosh^2(t) - I_0^2(t)) + 2\alpha(1-\alpha)(\cosh^2(t) - \cosh(t)I_0(t)) - (1-2q)^2)\right]}{\cosh(t)\sqrt{1-(1-2q)^2\frac{\sinh^2(t)}{\cosh^2(t)}} + \alpha\cosh(t) + (1-\alpha)I_0(t)} \\ &= \frac{\sqrt{ab}\sinh^2(t)\left[2\alpha(1-\alpha)f(t) + (1-\alpha)^2 g(t) - (1-2q)^2\right]}{\cosh(t)\sqrt{1-(1-2q)^2\frac{\sinh^2(t)}{\cosh^2(t)}} + \alpha\cosh(t) + (1-\alpha)I_0(t)}, \end{aligned} \tag{3.4}$$

where $f(t)$ and $g(t)$ are defined by Lemmas 7 and 8, respectively.

Therefore, Theorem 1 follows easily from (3.3) and (3.4) together with Lemmas 7 and 8. □

Remark 1. Let $\alpha = 0$. Then inequalities (3.1) and (3.2) reduce to inequalities (1.8) and (1.9), respectively.

Corollary 1. *Let* $\lambda_1 = 1/2 - \sqrt{1-\alpha}/2$ *and* $\mu_1 = 1/2 - \sqrt{1-\alpha}/4$. *Then inequality* (3.1) *leads to*

$$\frac{1}{\cosh(t)} < I_0(t) < \frac{3\cosh(t)}{4} + \frac{1}{4\cosh(t)}$$

for all $t > 0$.

Corollary 2. *Let* $\lambda_2 = 1/2 - \sqrt{1-\alpha^2}/2$ *and* $\mu_2 = 1/2 - \sqrt{2(1-\alpha)}/4$. *Then inequality* (3.2)) *leads to*

$$\frac{\sqrt{1-\alpha^2+\alpha^2\cosh^2(t)} - \alpha\cosh(t)}{1-\alpha} < I_0(t) < \frac{\sqrt{\frac{1-\alpha}{2} + \frac{1+\alpha}{2}\cosh^2(t)} - \alpha\cosh(t)}{1-\alpha}$$

for all $\alpha \in (0,1)$ *and* $t > 0$. *In particular, if* $\alpha = 1/2$, *then one has*

$$\sqrt{3+\cosh^2(t)} - \cosh(t) < I_0(t) < \sqrt{1+3\cosh^2(t)} - \cosh(t)$$

for all $t > 0$.

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Authors' addresses

Wei-Mao Qian

Wei-Mao Qian, School of Continuing Education, Huzhou Vocational & Technical College, Huzhou 313000, Zhejiang, China

E-mail address: qwm661977@126.com

Wen Zhang

Wen Zhang, Friedman Brain Institute, Icahn School of Medicine at Mount Sinai, New York, NY 10029, United States

E-mail address: zhang.wen81@gmail.com

Yu-Ming Chu

Yu-Ming Chu (Corresponding author), Department of Mathematics, Huzhou University, Huzhou 313000, Zhejiang, China

E-mail address: chuyuming@zjhu.edu.cn

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BEST APPROXIMATION AND CHARACTERIZATION OF HILBERT SPACES

SETAREH RAJABI

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Abstract. It is well known that for any nonempty closed convex subset C of a Hilbert space, any best approximation $y \in C$ of the point x satisfies the inequality $\|x-y\|^2 + \|z-y\|^2 \leq \|x-z\|^2$ for all $z \in C$. In this paper, we first introduce and study a new subset of best approximations involving this inequality in general metric spaces. Then, we provide some equivalent conditions which characterize Hilbert spaces.

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1. Introduction and preliminaries

Let (X,d) be a metric space and let C be a nonempty subset of X. For each $x \in X$, the distance between the point x and the set C is denoted by $\operatorname{dist}(x,C)$ and is defined by the following minimum equation

$$\operatorname{dist}(x,C) := \inf_{z \in C} d(x,z).$$

We call $y \in C$ with the property $d(x,y) = \operatorname{dist}(x,C)$ a best approximation of x in C. The set of all best approximation of x in C defines a set valued mapping

$$P_C x := \{y \in C : d(x,y) = \operatorname{dist}(x,C)\}, \tag{1.1}$$

which is called the metric projection operator. It is easily seen that

$$P_C x = \{y \in C : d(x,y) \leq d(x,z), \ \ \forall z \in C\}. \tag{1.2}$$

It is clear that $P_C x$ is a closed subset of C if C is closed. If $P_C x \neq \varnothing$ for every $x \in X$, then C is called proximal. If $P_C x$ is a singleton for every $x \in X$, then C is said to be a Chebyshev set. For more properties of metric projection we refer the reader to [4, 10]. Metric projection operators are widely used in different areas of mathematics such as functional and numerical analysis, theory of optimization and approximation and for problems of optimal control (see e.g., [4]).

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One of the important characterizations of best approximations is the following theorem which is known as Kolmogorov's criterion.

Theorem 1 ([4, Theorem 3.1])**.** *Let H be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, and let C be a nonempty closed convex subset of H. Then for $x \in H$ and $y \in C$ the following conditions are equivalent:*

(i) $y = P_C(x)$;
(ii) $\langle x-y, z-y\rangle \le 0$ *for all* $z \in C$;
(iii) $\|x-y\|^2 + \|z-y\|^2 \le \|x-z\|^2$ *for all* $z \in C$.

The following example shows that the inequality (iii) in Theorem 1 does not hold in general Banach spaces.

Example 1. Let $X = \mathbb{R}^2 = \{x = (u,v) : u, v \in \mathbb{R}\}$ endowed with the norm

$$\|(u,v)\| = |u-v| + \sqrt{u^2+v^2}$$

and $C = \{(u,0) : |u| \le 1\}$. Then C is a Chebyshev set. For $x = (1,1)$, we have $y = P_C x = (0,0)$ and the inequality (iii) in Theorem 1 for $z = (1,0) \in C$ does not hold:

$$\|x-y\|^2 + \|z-y\|^2 = 2+4 > 4 = \|x-z\|^2.$$

A natural question arises here. Does the inequality (iii) in Theorem 1 characterize Hilbert spaces? To answer this question we rewrite the equivalence $(i) \Leftrightarrow (ii)$ as equality of two sets. Let C be a nonempty subset of a metric space X. For each $x \in X$, we define the set $Q_C x$ by

$$Q_C x := \left\{y \in C : d^2(x,y) + d^2(z,y) \le d^2(x,z), \ \ \forall z \in C\right\}. \tag{1.3}$$

It is possible to describe $Q_C x$ by similar way in (1.1). To do this let $y \in C$ and

$$\operatorname{dist}(x,y,C) = \inf\left\{d^2(x,z) - d^2(y,z) : z \in C\right\} \tag{1.4}$$

which may take $-\infty$. Using this notation we have

$$Q_C x = \left\{y \in C : d^2(x,y) = \operatorname{dist}(x,y,C)\right\}. \tag{1.5}$$

Note that if $\operatorname{dist}(x,y,C) < 0$, then $Q_C x = \varnothing$.

Comparison of (1.2) and (1.3) shows that $Q_C x \subseteq P_C x$. By Theorem 1, this turns to equality when C is a nonempty closed convex subset of a Hilbert space.

The main aim of the paper is to study properties of the set $Q_C x$ and to characterize Hilbert spaces by the equality $Q_C x = P_C x$. Some recent characterizations of Hilbert spaces can be found in [2, 8]

2. Main results

We begin with properties of $Q_C x$ in general metric spaces.

Proposition 1. *Let C be a nonempty subset of a metric space (X,d) and $x \in X$. Then the following statements hold:*

(i) $Q_C x \subseteq P_C x$;
(ii) *$Q_C x$ is empty or singleton;*
(iii) *if $Q_C x \neq \varnothing$, then $P_C x$ is singleton;*
(iv) *$Q_C x$ is empty or $Q_C x = P_C x$.*

Proof. The assertion (i) is clear. To prove (ii) let $Q_C x$ be nonempty and $y_1, y_2 \in Q_C x$. Then we have

$$d^2(x,y_1) + d^2(y_1,y_2) \leq d^2(x,y_2),$$

or equivalently

$$d^2(y_1,y_2) \leq d^2(x,y_2) - d^2(x,y_1). \tag{2.1}$$

Interchanging the role of y_1 and y_2 gives us

$$d^2(y_1,y_2) \leq d^2(x,y_1) - d^2(x,y_2). \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\begin{aligned} d^2(y_1,y_2) \leq & \quad \min\left\{d^2(x,y_2) - d^2(x,y_1), -(d^2(x,y_2) - d^2(x,y_1))\right\} \\ \leq & 0. \end{aligned}$$

Therefore $d(y_1,y_2) = 0$ and so $y_1 = y_2$.
Next, we prove (iii). Let $y \in Q_C x$. It follows from (i) that $y \in P_C x$. Now, let $z \in P_C x$. Since $d(x,y) = \operatorname{dist}(x,C) = d(x,z)$, using the inequality

$$d^2(x,y) + d^2(z,y) \leq d^2(x,z),$$

we get $d(y,z) = 0$ and so $y = z$ which implies that $P_C x$ is singleton. Finally, the assertion (iv) follows from (i) and (iii). □

The above proposition clarify the structure of $Q_C x$. Next, we verify that in which metric space (X,d) and for which subset C, the set $Q_C x$ is nonempty. First, we focus on normed linear spaces.

Let $(X, \|\cdot\|)$ be a normed linear space. The subsets $S_X := \{x \in X : \|x\| = 1\}$ and $B_X := \{x \in X : \|x\| \leq 1\}$ are called the unit sphere and the unit ball of X, respectively. Also, the closed ball with center at $x \in X$ and radius r is denoted by $\overline{B}(x,r)$. A subset A of X is said to be admissible [9] if it is the intersection of a family of closed balls. Obviously, any admissible set is a bounded, closed and convex set. Let A be an admissible set of X. A complete family of centers of A is a set S such that

$$A = \cap\left\{\overline{B}(x,r(x)) : x \in S\right\}$$

for a mapping $r : S \to [0,\infty)$. A closed subspace D of X is called a diametral space of A if it contains a complete family of centers of A.

Lemma 1. *Let $(X, \|\cdot\|)$ be a normed linear space and A be an admissible subset of X. If D is a diametral space of A such that $P_D x = Q_D x$ for each $x \in A$, then $P_D(A) \subseteq A$.*

Proof. Since D is a diametral space of the admissible set A, there exists complete family of centers S such that $S \subseteq D$ and

$$A = \cap \left\{ \overline{B}(u, r(u)) : u \in S \right\}.$$

Let $x \in A$. For each $u \in S$, we have $x \in \overline{B}(u, r(u))$ and so

$$\|x - u\| \leq r(u). \tag{2.3}$$

Since $P_D x = Q_D x$, for each $y \in P_D x$ we have

$$\|x - y\|^2 + \|z - y\|^2 \leq \|x - z\|^2, \quad \forall z \in D.$$

This together with the fact that $S \subseteq D$ implies that

$$\|u - y\|^2 \leq \|x - y\|^2 + \|u - y\|^2 \leq \|x - u\|^2, \quad \forall u \in S. \tag{2.4}$$

It follows from (2.3) and (2.4) that for each $u \in S$,

$$\|y - u\| \leq r(u).$$

That is, $y \in A$ and so $P_D(A) \subseteq A$. □

In a normed linear space $(X, \|\cdot\|)$, a vector x is said to be Birkhoff orthogonal to a vector y ($x \perp_B y$) if the inequality $\|x\| \leq \|x + \alpha y\|$ holds for any real number α. It is easy to see that $x \perp_B y$ if and only if $0 \in P_{\overline{0y}} x$ where $\overline{0y}$ is the line $\{\alpha y : \alpha \in \mathbb{R}\}$. Birkhoff orthogonality is said to be symmetric if $x \perp_B y$ implies $y \perp_B x$. Also, Birkhoff orthogonality is said to be homogeneous if $x \perp_B y$ implies $\lambda x \perp_B \mu y$ for any real numbers λ and μ. It is well-known that Birkhoff orthogonality is homogeneous in any normed linear space [1, Theorem 4.5]. The following theorem is the origin of many characterizations of Hilbert spaces. For more details on Birkhoff orthogonality see the survey [1].

Theorem 2 ([5])**.** *Let $(X, \|\cdot\|)$ be a normed linear space, whose dimension is at least three. If Birkhoff orthogonality is symmetric and there is at most one orthogonal from a given line to a point not on that line then X is an inner product space.*

Now, we state and prove the characterization results.

Theorem 3. *Let $(X, \|\cdot\|)$ be a Banach space, whose dimension is at least three. Then the followings are equivalent:*

(i) *X is a Hilbert space;*
(ii) *$Q_C x$ is nonempty for each nonempty closed convex subset C of X and $x \in X \setminus C$;*

(iii) *$P_C x = Q_C x$ for every nonempty closed convex subset C of X and $x \in X \setminus C$;*
(iv) *$P_L x = Q_L x$ for every line L in X and $x \in X \setminus L$;*
(v) *Birkhoff orthogonality is symmetric;*
(vi) *$P_D(A) \subseteq A$ for every admissible set A and every diametral space D of A.*

Proof. $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ follows from Theorem 1. Also, $(ii) \Rightarrow (iii)$ is an immediate consequence of (iv) in Proposition 1. $(iii) \Rightarrow (iv)$ is trivial.
$(iv) \Rightarrow (v)$ Let $P_L x = Q_L x$ for every line L in X. It follows from Proposition 1 that $P_L x$ have at most one point for each line L. Now, let $x \perp_B y$. Then $0 \in P_{\overline{0y}} x$. Since $P_{\overline{0y}} x = Q_{\overline{0y}} x$, we have

$$\|x\|^2 + \|\alpha y\|^2 \leq \|x - \alpha y\|^2, \qquad \forall \alpha \in \mathbb{R},$$

which implies that

$$\left\| \frac{1}{\alpha} x \right\|^2 + \|y\|^2 \leq \left\| \frac{1}{\alpha} x - y \right\|^2, \qquad \forall \alpha \in \mathbb{R} \setminus \{0\}.$$

Hence for all $\beta \in \mathbb{R}$ we get $\|y\| \leq \|y + \beta x\|$ and so $y \perp_B x$.
$(iii) \Rightarrow (vi)$ and $(v) \Rightarrow (i)$ are Lemma 1 and Theorem 2, respectively. Finally, $(vi) \Rightarrow (i)$ follows from [9, Theorem 1]. □

In two dimensional case we have the following theorem.

Theorem 4. *Let $(X, \|\cdot\|)$ be a Banach space, whose dimension is at least two. Then the followings are equivalent:*

(i) *X is a Hilbert space;*
(ii) *$Q_C x$ is nonempty for each nonempty closed convex subset C of X and $x \in X \setminus C$;*
(iii) *$P_C x = Q_C x$ for every nonempty closed convex subset C of X and $x \in X \setminus C$;*
(iv) *$P_L x = Q_L x$ for every line L in X and $x \in X \setminus L$;*
(v) *$x, y \in S_X, \ \ x \perp_B y \ \ \Rightarrow \ \ \|x + y\| \geq \sqrt{2}$.*

Proof. We first prove $(iv) \Rightarrow (v)$. Let $P_L x = Q_L x$ for every line L in X. Now, let $x, y \in S_X$ and $x \perp_B y$. Then, $0 \in P_{\overline{0y}} x$. Since $P_{\overline{0y}} x = Q_{\overline{0y}} x$, we have

$$\|x\|^2 + \|\alpha y\|^2 \leq \|x - \alpha y\|^2, \qquad \forall \alpha \in \mathbb{R},$$

specially for $\alpha = -1$,

$$2 = \|x\|^2 + \|y\|^2 \leq \|x + y\|^2.$$

The equivalence of (i) and (v) follows from $(10.3')$ of [3]. The equivalence of others are the same as Theorem 3. □

As a consequence of Theorems 3 and 4 we would mention that the Hilbert spaces are the only Banach spaces for which $Q_C x$ is nonempty. Of course, there are other metric spaces with this property, namely $CAT(0)$ metric spaces. To see this, we refer

the interested readers to consider [6, Proposition 2.4] and [7, Theorem 2.2], and to follow the same argument as in proof of Theorem 1.

Next, we define a new geometric constant for normed linear spaces.

Definition 1. Let $(X, \|\cdot\|)$ be a normed linear space. Define geometric constant $C(X)$ of X as follows:

$$C(X) = \inf\left\{ \frac{\|x+\alpha y\|^2}{\|x\|^2+\|\alpha y\|^2} : x \perp_B y, \|x\| = \|y\|, x \neq -\alpha y, \alpha \in \mathbb{R} \right\}. \tag{2.5}$$

It is easily seen that for every normed linear space $0 \leq C(X) \leq 1$. If X is an inner product space, then $x \perp_B y$ implies that $\langle x, y\rangle = 0$. Therefore, $\|x+\alpha y\|^2 = \|x\|^2 + \|\alpha y\|^2$ and so $C(X) = 1$. Also, by the homogeneity property of Birkhoff orthogonality we have

$$C(X) = \inf\left\{ \frac{\|x+\alpha y\|^2}{1+\alpha^2} : \ x, y \in S_X, \ x \perp_B y, \ \alpha \in \mathbb{R} \right\}. \tag{2.6}$$

Theorem 5. *Let $(X, \|\cdot\|)$ be a normed linear space and $x, y \in X$. If $C(X) = 1$ and $x \perp_B y$, then*

(i) $0 \in Q_{\overline{0y}}x$;
(ii) $y \perp_B x$;
(iii) *if $x, y \in S_X$, then $\|x+y\| \geq \sqrt{2}$.*

Proof. Let $x, y \in X$ be such that $x \perp_B y$. Without loss of generality we may assume that $y \neq 0$. If x lies on the line $\overline{0y}$, then $x = -\lambda y$ for some $\lambda \in \mathbb{R}$ and it follows from $x \perp_B y$ that $x = 0$ which turns the assertions trivial. Otherwise, for every $\alpha \in \mathbb{R}$ we have $x \neq -\alpha y$. By the homogeneity property of Birkhoff orthogonality we obtain

$$x \perp_B \left(\frac{\|x\|}{\|y\|}\right) y.$$

Since $C(X) = 1$, then for each $\alpha \in \mathbb{R}$ we have

$$1 \leq \frac{\left\|x + \alpha \frac{\|x\|}{\|y\|} y\right\|^2}{\|x\|^2 + \left\|\alpha \frac{\|x\|}{\|y\|} y\right\|^2},$$

or equivalently

$$\|x\|^2 + \left\|\alpha \frac{\|x\|}{\|y\|} y\right\|^2 \leq \left\|x + \alpha \frac{\|x\|}{\|y\|} y\right\|^2. \tag{2.7}$$

Now, for each real umber β, by taking $\alpha = \beta \frac{\|y\|}{\|x\|}$ in (2.7), we have

$$\|x\|^2 + \|\beta y\|^2 \leq \|x + \beta y\|^2, \tag{2.8}$$

which implies that $0 \in Q_{\overline{0y}}x$.

Next, we prove (ii). For each nonzero real umber β, by taking $\alpha = \frac{\|y\|}{\beta\|x\|}$ in (2.7) and then multiplying both sides by $|\beta|$, we have

$$\|y\|^2 + \|\beta x\|^2 \leq \|y + \beta x\|^2,$$

which implies that $y \perp_B x$.

Finally, if $x, y \in S_X$, for $\beta = 1$ in (2.8) we get $\|x + y\| \geq \sqrt{2}$. □

Using Theorems 5, 3 and 4 we have the following corollary.

Corollary 1. *Let $(X, \|\cdot\|)$ be a Banach space. Then X is a Hilbert space if and only if $C(X) = 1$.*

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Author's address

Setareh Rajabi

Institute for Advanced Studies in Basic Sciences (IASBS), Department of Mathematics, Gava Zang, 45137-66731, Zanjan, Iran

E-mail address: setareh.rajaby@yahoo.com

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ON LIE IDEALS AND SYMMETRIC GENERALIZED (α,β)-BIDERIVATION IN PRIME RING

NADEEM UR REHMAN AND SHULIANG HUANG

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Abstract. Let $\mathfrak{R}$ be a prime ring with char$(\mathfrak{R}) \neq 2$. A biadditive symmetric map $\Delta : \mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$ is called symmetric (α,β)-biderivation if, for any fixed $y \in \mathfrak{R}$, the map $x \mapsto \Delta(x,y)$ is a (α,β)-derivation. A symmetric biadditive map $\Gamma : \mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$ is a symmetric generalized (α,β)-biderivation if for any fixed $y \in \mathfrak{R}$, the map $x \mapsto \Gamma(x,y)$ is a generalized (α,β)-derivation of $\mathfrak{R}$ associated with the (α,β)-derivation $\Delta(.,y)$. In the present paper, we investigate the commutativity of a ring having a generalized (α,β)-biderivation satisfying certain algebraic conditions.

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1. Introduction

Throughout this article $\mathfrak{R}$ denotes an associative ring with center Z. An additive subgroup $\mathfrak{L}$ of R is said to be a Lie ideal of $\mathfrak{R}$ if $[l,r] \in \mathfrak{L}$, $\forall\ l \in \mathfrak{L}$ and $r \in \mathfrak{R}$. A Lie ideal $\mathfrak{L}$ is called square closed if $u^2 \in \mathfrak{L}\ \forall\ u \in \mathfrak{L}$. And it is easy to check that $2uv \in \mathfrak{L}\ \forall\ u,v \in \mathfrak{L}$.

A derivation $\mathfrak{d} : \mathfrak{R} \to \mathfrak{R}$ is an additive map such that $\mathfrak{d}(xy) = \mathfrak{d}(x)y + x\mathfrak{d}(y)\ \forall\ x,y \in R$. An additive map $\mathbb{F} : \mathfrak{R} \to \mathfrak{R}$ is a generalized derivation if there exists a derivation $\mathfrak{d} : \mathfrak{R} \to \mathfrak{R}$ such that $\mathbb{F}(xy) = \mathbb{F}(x)y + x\mathfrak{d}(y)$ holds $\forall\ x,y \in \mathfrak{R}$. If $\varphi : \mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$ is a symmetric map ($\varphi(x,y) = \varphi(y,x)\ \forall\ x,y \in \mathfrak{R}$) the map $\tau : \mathfrak{R} \to \mathfrak{R}$ defined by $\tau(x) = \varphi(x,x)$ is the trace of φ. If φ is also biadditive (i.e., additive in both arguments), its trace τ satisfies $\tau(x+y) = \tau(x)+\tau(y)+2\varphi(x,y)$, $\forall\ x,y \in \mathfrak{R}$. A symmetric biadditive map $\varphi : \mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$ is a symmetric biderivation if $\varphi(xy,z) = \varphi(x,z)y + x\varphi(y,z)\ \forall\ x,y,z \in \mathfrak{R}$. The concept of symmetric biderivation was introduced by G. Maksa [9]. A symmetric biadditive map $\zeta : \mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$ is a symmetric left bicentralizer if $\zeta(xy,z) = \zeta(x,z)y$ (and consequently $\zeta(x,yz) = \zeta(x,y)z)\ \forall\ x,y,z \in \mathfrak{R}$.

Symmetric biderivations were proved to be related to the general solution of some functional equations (see [11]). The maps $(u,v) \mapsto \Psi[u,v], \Psi \in \mathcal{C}$, are typical examples of biderivations and they were called inner biderivations. Here $\mathcal{C}$ is the extended centroid of $\mathfrak{R}$, that is, the center of the two-sided Martindale quotient ring $\mathcal{Q}$ (we

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refer the reader to [3] for more details). In [7], it is shown that every biderivation of a noncommutative prime ring $\mathfrak{R}$ is inner. In [5], this result is extended to semiprime. In [14], it is proved that if φ is a nonzero symmetric biderivation, where $\mathfrak{R}$ is a prime ring of $\mathrm{char}(\mathfrak{R}) \neq 2$, with the property:

$$\varphi(x,x)x = x\varphi(x,x), x \in \mathfrak{R}. \tag{1.1}$$

then $\mathfrak{R}$ is commutative. He also proved that if φ_1, φ_2 are nonzero biderivations on $\mathfrak{R}$, $\mathfrak{D}$ is a symmetric biadditive map and $\tau_1(\tau_2(x)) = \mathfrak{d}(x)$ holds $\forall\ x \in \mathfrak{R}$, where τ_1, τ_2, and $\mathfrak{d}$ are the traces of φ_1, φ_2, and $\mathfrak{D}$, respectively, then either $\varphi_1 = 0$ or $\varphi_2 = 0$. Let's mention two results proved in [15]. The first one states that if φ_1 and φ_2 are symmetric biderivations on a prime ring $\mathfrak{R}$, $\mathrm{char}(\mathfrak{R}) \neq 2, 3$, such that $\varphi_1(x,x)\varphi_2(x,x) = 0$ holds $\forall\ x \in \mathfrak{R}$, then either $\varphi_1 = 0$ or $\varphi_2 = 0$. The second result says that if $[[\varphi(x,x),x],x] \in Z\ \forall\ x \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative. In [16] the authors extended the results in [14] assuming condition (1.1) over a nonzero ideal and a nonzero Lie ideal of a prime ring respectively.

The notion of generalized biderivation was introduced in [6]. A biadditive map $\Gamma : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ is a generalized biderivation associated with a biderivation $\Delta : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ if for every $x, y \in \mathfrak{R}$, the maps $y \mapsto \Gamma(x,y)$ and $y \mapsto \Gamma(y,x)$ are generalized derivations of $\mathfrak{R}$ associated with $\Delta(x,.)$ and $\Delta(.,x)$. That is, $\Gamma(xy,z) = \Gamma(x,z)y + x\Delta(y,z)$ and $\Gamma(x,yz) = \Gamma(x,y)z + y\Delta(x,z)$ hold $\forall\ x, y, z \in \mathfrak{R}$. Brešar shown that every generalized biderivation Γ of an ideal I ($\Gamma : I \times I \to \mathfrak{R}$) of a prime ring $\mathfrak{R}$ with $\mathrm{char}(\mathfrak{R}) \neq 2$, is of the form $\Gamma(x,y) = xay + ybx$ for some $a, b \in \mathfrak{Q}$, where $\mathfrak{Q}$ is Martindale quotient ring of $\mathfrak{R}$ (see [8] and [10] for details). In [1] authors extended some results of [14, 15] to generalized biderivations on prime and semiprime rings. Recently, in [2], symmetric generalized (θ,ϕ)-biderivations of a prime ring $\mathfrak{R}$ with $\mathrm{char}(\mathfrak{R}) \neq 2$ have been considered. Notice that a symmetric left bicentralizer is a symmetric generalized biderivation associated with the biderivation $T = 0$.

2. Preliminaries

Lemma 1 ([12, Lemma 3])**.** *If the prime ring $\mathfrak{R}$ contains a commutative nonzero right ideal I, then $\mathfrak{R}$ is commutative.*

Lemma 2 ([13, Lemma 2.6])**.** *Let $\mathfrak{R}$ be a prime ring with char$(\mathfrak{R}) \neq 2$. If $\mathfrak{L}$ is a commutative Lie ideal of $\mathfrak{R}$, then $\mathfrak{L} \subseteq Z$.*

Lemma 3 ([4, Lemma 4])**.** *Let $\mathfrak{R}$ be a prime ring with char$(\mathfrak{R}) \neq 2$. If $\mathfrak{L} \nsubseteq Z$ is a Lie ideal of $\mathfrak{R}$ and $a\mathfrak{L}b = (0)$, then either $a = 0$ or $b = 0$.*

Lemma 4. *Let $\mathfrak{R}$ be a prime ring, char$(\mathfrak{R}) \neq 2$, $\mathfrak{L}$ a Lie ideal of $\mathfrak{R}$ and α is automorphism. If $T : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ is a symmetric left α-centralizer such that $T(\alpha(x_1), \alpha(x_1)) = 0\ \forall\ x_1 \in \mathfrak{L}$, then either $\mathfrak{L} \subseteq Z$ or $T = 0$.*

Proof. Assume that $\mathfrak{L} \nsubseteq Z$. We have

$$T(x_1, x_1) = 0\ \forall\ x_1 \in \mathfrak{L}. \tag{2.1}$$

Linearizing (2.1), we get

$$T(\alpha(x_1),\alpha(x_2)) = 0 \ \forall \ x_1,x_2 \in \mathfrak{L}. \tag{2.2}$$

Let us replace x_1 by $x_1s - sx_1 \ \forall \ s \in \mathfrak{R}$ in (2.2), to obtain

$$T(\alpha(s),\alpha(x_2))\alpha^2(x_1) = 0 \ \forall x_1,x_2 \in \mathfrak{L}, \ s \in \mathfrak{R}.$$

That is, $\alpha^{-2}(T(\alpha(s),\alpha(x_2)))\mathfrak{L}\alpha^{-2}(T(\alpha(r),\alpha(x_2))) = (0) \ \forall \ x_2 \in \mathfrak{L}, \ s \in \mathfrak{R}$. By Lemma 3, $\alpha^{-2}(T(\alpha(r),\alpha(x_2))) = 0 \ \forall \ x_2 \in \mathfrak{L}, \ r \in \mathfrak{R}$ that is, $T(\alpha(s),\alpha(x_2)) = 0$. Now we replace x_2 by $[x_2,r] \ \forall \ r \in \mathfrak{R}$, we get $T(\alpha(s),\alpha(r))\alpha^2(x_2) = 0$, that is $\alpha^{-2}(T(\alpha(s),\alpha(r)))\mathfrak{L}\alpha^{-2}(T(\alpha(s),\alpha(r))) = (0)$ and hence again by Lemma 3 gives that $\alpha^{-2}(T(\alpha(s),\alpha(r))) = 0 \ \forall \ r,s \in \mathfrak{R}$ i.e., $T(\alpha(s),\alpha(r)) = 0 \ \forall \ r,s \in \mathfrak{R}$. Now, replacing s by $\alpha^{-1}(t_1)$ and r by $\alpha^{-1}(t_2)$, we get $T(t_1,t_2) = 0 \ \forall \ t_1,t_2 \in \mathfrak{R}$, that is $T = 0$. □

Lemma 5. *Let $\mathfrak{R}$ be a ring and α,β automorphisms. If Γ is a symmetric generalized (α,β)-biderivation associated with a symmetric (α,β)-biderivation Δ, then the map $\Gamma - \Delta : \mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$ is a symmetric left α-bicentralizer.*

Proof. The map $\Omega = \Gamma - \Delta$, is clearly biadditive. For all $x_1,x_2,x_3 \in \mathfrak{R}$,

$$\begin{aligned}
\Gamma(x_1x_2,x_3) &= (\Gamma - \Delta)(x_1x_2,x_3) \\
&= \Gamma(x_1x_2,x_3) - \Delta(x_1x_2,x_3) \\
&= \Gamma(x_1,x_3)\alpha(x_2) + \beta(x_1)\Delta(x_2,x_3) \\
&\quad - \Delta(x_1,x_3)\alpha(x_2) - \beta(x_1)\Delta(x_2,x_3) \\
&= \Gamma(x_1,x_3)\alpha(x_2) - \Delta(x_1,x_3)\alpha(x_2) \\
&= \Omega(x_1,x_3)\alpha(x_2).
\end{aligned}$$

Therefore, Ω is a symmetric left α-centralizer of $\mathfrak{R}$. □

3. Main results

Proposition 1. *Let $\mathfrak{R}$ be a prime ring, char$(\mathfrak{R}) \neq 2$, $\mathfrak{L}$ a nonzero Lie ideal of $\mathfrak{R}$, α,β automorphisms, and $\Delta : \mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$ a symmetric (α,β)-biderivation with trace δ. If $\delta(x_1) = 0 \ \forall \ x_1 \in \mathfrak{L}$, then either $\mathfrak{L} \subseteq Z$ or $\Delta = 0$.*

Proof. By our hypothesis

$$\delta(u) = 0 \ \forall \ x_1 \in \mathfrak{L}. \tag{3.1}$$

Linearizing (3.1) and using (3.1), we get

$$2\Delta(x_1,x_2) = 0 \ \forall \ x_1,x_2 \in \mathfrak{L}. \tag{3.2}$$

If we replace x_1 by $x_1r - rx_1$ in (3.2), to get

$$\beta(x_1)\Delta(r,x_2) - \delta(r,x_2)\alpha(x_1) = 0. \tag{3.3}$$

Again replacing r by rx_3 and using (3.2), we find that

$$\beta(x_1)\Delta(r,x_2)\alpha(x_3)-\Delta(r,x_2)\alpha(x_3x_1)=0. \tag{3.4}$$

Multiplying (3.3) from left by x_3, we get

$$\beta(x_1)\Delta(r,x_2)\alpha(x_3)-\Delta(r,x_2)\alpha(x_1x_3)=0. \tag{3.5}$$

From (3.4), (3.5) it follows that $\Delta(r,x_2)\alpha([x_1,x_3])=0\ \forall\ x_1,x_2,x_3\in\mathfrak{L}$ and $r\in\mathfrak{R}$. If we replace r by rs, $s\in\mathfrak{R}$, we obtain $\Delta(r,x_2)\alpha(s)\alpha([x_1,x_3])=0$ and hence $\alpha^{-1}(\Delta(r,x_2))\mathfrak{R}[x_1,x_3]=(0)$ Thus by primeness of $\mathfrak{R}$ it follows that either $[x_1,x_3]=0$ for $x_1,x_3\in\mathfrak{L}$ or $\alpha^{-1}(\Delta(r,x_2))=0$. If $[x_1,x_3]=0\ \forall\ x_1,x_3\in\mathfrak{L}$, then $\mathfrak{L}\subseteq Z$ by Lemma 2. In other case, $\alpha^{-1}(\Delta(r,x_2))=0\ \forall\ x_2\in\mathfrak{L}$, $r\in\mathfrak{R}$, that is, $\Delta(r,x_2)=0$. Replacing x_2 by x_2s-sx_2, we get

$$\beta(x_2)\Delta(r,s)-\Delta(r,s)\alpha(x_2)=0. \tag{3.6}$$

Now, replacing s by sx_3 in (3.6), we get

$$\beta(x_2)\Delta(r,s)\alpha(x_3)-\Delta(r,s)\alpha(x_3x_2)=0. \tag{3.7}$$

If we multiply (3.6) by x_3 to the right, and subtract (3.7) we get

$$\Delta(r,s)\alpha([x_2,x_3])=0\ \forall x_2,x_3\in\mathfrak{L},r,s\in\mathfrak{R}.$$

Replace r by tr, to get $\Delta(t,s)\alpha(t)\alpha([x_2,x_3])=0$, that is $\alpha^{-1}(\Delta(t,s))T[x_2,x_3]=(0)$. So by primeness of $\mathfrak{R}$ either $\Delta=0$ or $\mathfrak{L}$ is commutative. Hence, Lemma 2 gives the required conclusion. □

Theorem 1. *Let $\mathfrak{R}$ be a prime ring, char$(\mathfrak{R})\neq 2$, $\mathfrak{L}$ a nonzero Lie ideal of $\mathfrak{R}$, α,β automorphisms and $\Delta:\mathfrak{R}\times\mathfrak{R}\to\mathfrak{R}$ a symmetric (α,β)-biderivation with trace δ. If $\delta(x_1)\in Z\ \forall\ x_1\in\mathfrak{L}$, then either $\mathfrak{L}\subseteq Z$ or $\Delta=0$.*

Proof. By Assumption we have

$$\delta(x_1)\in Z\ \forall\ x_1\in\mathfrak{L}. \tag{3.8}$$

The linearizing (3.8), we find that

$$2\Delta(x_1,x_2)\in Z\ \forall\ x_1,x_2\in\mathfrak{L}. \tag{3.9}$$

Replace x_1 by $2x_1^2$ in (3.9), to get

$$4\Delta(x_1,x_2)\alpha(x_1)+\beta(x_1)\Delta(x_1,x_2)\in Z. \tag{3.10}$$

Therefore, in particular $\alpha(x_1)\delta(x_1)+\beta(x_1)\delta(x_1)\in Z\ \forall\ x_1\in\mathfrak{L}$. Then we have

$$0=[\alpha(x_1)\delta(x_1)+\beta(x_1)\delta(x_1),r]=[\alpha(x_1)+\beta(x_1),r]\delta(x_1). \tag{3.11}$$

For every $r,s\in\mathfrak{R}$, we have

$$[\alpha(x_1)+\beta(x_1),rs]\delta(x_1)=0=[\alpha(x_1)+\beta(x_1),r]s\delta(x_1)=0.$$

By primeness of $\mathfrak{R}$, given an arbitrary element $x_1\in\mathfrak{L}$, we have either $\delta(x_1)=0$ or $\alpha(x_1)+\beta(x_1)\in Z$. If $(\alpha+\beta)(x_1)\in Z$ then $x_1\in Z$. If $Z\cap\mathfrak{L}=0$, then $\delta(x_1)=0$

$\forall\ x_1 \in \mathfrak{L}$. Assume that $Z \cap \mathfrak{L} \neq 0$. If $\mathfrak{L} \nsubseteq Z$, then there exists $x_2 \in \mathfrak{L} \setminus Z$. Then $\forall\ x_1 \in Z \cap \mathfrak{L}$, the element $x_1 + x_2, x_1 - x_2 \in \mathfrak{L} \setminus Z$. Hence $\delta(x_1 + x_2) = 0$ and $\delta(x_1 - x_2) = 0$ and hence $\delta(x_1) = 0$. In conclusion we prove that $\delta(x_1) = 0\ \forall\ x_1 \in Z \cap \mathfrak{L}$ and above we already know that $\delta(x_1) = 0\ \forall\ x_1 \in \mathfrak{L} \setminus Z$. That is, $\delta(x_1) = 0$ $\forall\ x_1 \in \mathfrak{L}$. □

Theorem 2. *Let $\mathfrak{R}$ be a prime ring, char$(\mathfrak{R}) \neq 2$, $\mathfrak{L}$ a nonzero square closed Lie ideal of $\mathfrak{R}$, α, β automorphisms and $\Gamma_1, \Gamma_2 : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ be two symmetric generalized (α, β)-biderivations associated with symmetric (α, β)-biderivations $\Delta_1, \Delta_2 : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$, respectively. If $\Gamma_1(x_1, x_1)\alpha(x_1) = \beta(x_1)\Gamma_2(x_1, x_1)\ \forall\ x_1 \in \mathfrak{L}$, then either $\mathfrak{L} \subseteq Z$ or $\Delta_2 = 0$.*

Proof. Assume that $\mathfrak{L} \nsubseteq Z$. Suppose that $\gamma_1, \gamma_2, \delta_1, \delta_2$ are traces of $\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$, respectively. We have

$$\gamma_1(x_1)\alpha(x_1) = \beta(x_1)\gamma_2(x_1)\ \forall\ x_1 \in \mathfrak{L}. \tag{3.12}$$

Replacing x_1 by $x_1 + x_2$ in (3.12), we get

$$\begin{aligned}&\gamma_1(x_1)\alpha(x_2) + \gamma_1(x_2)\alpha(x_1) + 2\Gamma_1(x_1, x_2)\alpha(x_1) + 2\Gamma_1(x_1, x_2)\alpha(x_2)\\ &= \beta(x_1)\gamma_2(x_2) + \beta(x_2)\gamma_2(x_1) + 2\beta(x_1)\Gamma_1(x_1, x_2) + 2\beta(x_2)\Gamma_2(x_1, x_2).\end{aligned} \tag{3.13}$$

Substituting x_1 by $-x_1$ in (3.13), we get

$$\begin{aligned}&\gamma_1(x_1)\alpha(x_2) - \gamma_1(x_2)\alpha(x_1) + 2\Gamma_1(x_1, x_2)\alpha(x_1) - 2\Gamma_1(x_1, x_2)\alpha(x_2)\\ &= -\beta(x_1)\gamma_2(x_2) + \beta(x_2)\gamma_2(x_1) + 2\beta(x_1)\Gamma_1(x_1, x_2) - 2\beta(x_2)\Gamma_2(x_1, x_2).\end{aligned} \tag{3.14}$$

Comparing (3.13) and (3.14) and using the fact that char$(\mathfrak{R}) \neq 2$, we obtain

$$\gamma_1(x_1)\alpha(x_2) + 2\Gamma_1(x_1, x_2)\alpha(x_1) = \beta(x_2)\gamma_2(x_1) + 2\beta(x_1)\Gamma_2(x_1, x_2). \tag{3.15}$$

Replacing x_2 by $2x_2x_1$ in (3.15), we have

$$\begin{aligned}&\gamma_1(x_1)\alpha(x_2x_1) + 2\Gamma_1(x_1, x_2)\alpha(x_1^2) + 2\beta(x_2)\delta_1(x_1)\alpha(x_1)\\ &\quad = \beta(x_2x_1)\gamma_2(x_1) + 2\beta(x_1)\Gamma_2(x_1, x_2)\alpha(x_1) + 2\beta(x_1)\beta(x_2)\delta_2(x_1).\end{aligned} \tag{3.16}$$

That is,

$$\begin{aligned}&(\gamma_1(x_1)\alpha(x_2) + 2\Gamma_1(x_1, x_2)\alpha(x_1) - 2\beta(x_1)\Gamma_2(x_1, x_2))\alpha(x_1) + 2\beta(x_2)\delta_1(x_1)\alpha(x_1)\\ &\qquad = \beta(x_2x_1)\gamma_2(x_1) + 2\beta(x_1)\beta(x_2)\delta_2(x_1).\end{aligned} \tag{3.17}$$

Using (3.15) in (3.17), we find that

$$\beta(x_2)\gamma_2(x_1)\alpha(x_1) + 2\beta(x_2)\delta(x_1)\alpha(x_1) = \beta(x_2x_1)\gamma_2(x_1) + 2\beta(x_1x_2)\delta_2(x_1). \tag{3.18}$$

Replacing x_2 by $2x_3x_2$ in (3.18) and using char$(R) \neq 2$, we get

$$\begin{aligned}&\beta(x_2x_3)\gamma_2(x_1)\alpha(x_1) + 2\beta(x_3x_2)\delta(x_1)\alpha(x_1)\\ &\qquad = \beta(x_3x_2x_1)\gamma_2(x_1) + 2\beta(x_1x_3x_2)\delta_2(x_1).\end{aligned} \tag{3.19}$$

Now, subtracting (3.19) from (3.18) multiplied by x_3 to the left, we get

$$\beta([x_3,x_1])\beta(x_2)\delta_2(x_1) \ \forall \ \ x_1,x_2,x_3 \in \mathfrak{L}. \tag{3.20}$$

This implies that $[x_3,x_1]\mathfrak{L}\beta^{-1}(\delta_2(x_1)) = (0)$. By Lemma 3, gives that for an arbitrary element $x_1 \in \mathfrak{L}$ either $x_1 \in Z(\mathfrak{L})$ or $\delta_2(x_1) = 0$. If $Z(\mathfrak{L}) = 0$, then $\delta_2(x_1) = 0$. If $\mathfrak{L} = Z(\mathfrak{L})$, then $\mathfrak{L} \subseteq Z$, by Lemma 2, a contradiction. Let us assume that $\mathfrak{L} \neq Z(\mathfrak{L}) \neq 0$. Then there exists $x_1 \in \mathfrak{L} \setminus Z(\mathfrak{L})$. So $\delta_2(x_1) = 0$ since $\delta_2(x_2) = 0$ $\forall \ x_2 \in \mathfrak{L} \setminus Z(\mathfrak{L})$. Take $0 \neq x_3 \in Z(\mathfrak{L})$. Then $x_1 + x_3, x_1 - x_3 \in \mathfrak{L} \setminus Z(\mathfrak{L})$ and so $\Delta(x_1 + x_3, x_1 + x_3) = 0 = \Delta(x_1 - x_3, x_1 - x_3)$, that is

$$\Delta_2(x_1,x_1) + 2\Delta_2(x_1,x_3) + \Delta_2(x_3,x_3) = 0$$

and

$$\Delta_2(x_1,x_1) - 2\Delta_2(x_1,x_3) + \Delta_2(x_3,x_3) = 0.$$

Adding the above two expression, we find that $2\delta_2(x_3) = 0$. Since $\mathrm{char}(\mathfrak{R}) \neq 2$, we have $\delta_2(x_3) = 0 \ \forall \ x_3 \in \mathfrak{L}$. Using Proposition 1, we get the required result. □

Using the same technique with necessary variation one can prove the following theorem.

Theorem 3. *Let $\mathfrak{R}$ be a prime ring, char$(\mathfrak{R}) \neq 2$, $\mathfrak{L}$ a nonzero square closed Lie ideal of R, α,β automorphisms and $\Gamma_1,\Gamma_2 : \mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$ be two symmetric generalized (α,β)-biderivations associated with symmetric (α,β)-biderivations $\Delta_1,\Delta_2 : \mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$, respectively. If $\Gamma_1(x_1,x_1)\alpha(x_1) + \beta(x_1)\Gamma_2(x_1,x_1) = 0 \ \forall \ x_1 \in \mathfrak{L}$, then either $\mathfrak{L} \subseteq Z$ or $\Delta_2 = 0$.*

It is immediately to get the following corollaries from Theorems 2, 3 and Lemma 1.

Corollary 1. *Let $\mathfrak{R}$ be a prime ring, char$(\mathfrak{R}) \neq 2$, I a nonzero ideal of $\mathfrak{R}$, and $\Gamma_1,\Gamma_2 : \mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$ be two symmetric generalized (α,β)-biderivations associated with symmetric (α,β)-biderivations $\Delta_1,\Delta_2 : \mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$, respectively. If $\Gamma_1(x,x)\alpha(x) = \pm\beta(x)\Gamma_2(x,x) \ \forall \ x \in I$, then $\mathfrak{R}$ is commutative.*

Corollary 2. *Let $\mathfrak{R}$ be a prime ring, char$(\mathfrak{R}) \neq 2$, $\mathfrak{L}$ a nonzero square closed Lie ideal of $\mathfrak{R}$, α,β automorphisms and $\Gamma : \mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$ be a symmetric generalized (α,β)-biderivation associated with a symmetric (α,β)-biderivation $\Delta : \mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$. If $[\Gamma(x,x),x]_{\alpha,\beta} = 0 \ \forall \ x \in I$, then either $\mathfrak{L} \subseteq Z$ or $\Delta = 0$.*

Corollary 3. *Let $\mathfrak{R}$ be a prime ring, char$(\mathfrak{R}) \neq 2$, I a nonzero ideal of $\mathfrak{R}$, α,β automorphisms and $\Gamma : \mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$ be a symmetric generalized (α,β)-biderivation associated with a symmetric (α,β)-biderivation $\Delta : \mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$. If $\Gamma(x,x)\alpha(x) = \pm\beta(x)\Delta(x,x) \ \forall \ x \in I$, then $\mathfrak{R}$ is commutative or Γ is a left α-bicentralizer.*

Theorem 4. *Let R be a prime ring, char$(\mathfrak{R}) \neq 2,3$, $\mathfrak{L}$ a nonzero Lie ideal of $\mathfrak{R}$, $\Gamma : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ a symmetric generalized biderivation associated with a symmetric biderivation Δ. If*

$$\Gamma(\Gamma(x_1,x_1),\Gamma(x_1,x_1)) - \Delta(\Delta(x_1,x_1),\Delta(x_1,x_1)) \\ = \Delta(\Gamma(x_1,x_1),\Gamma(x_1,x_1)) - \Gamma(\Delta(x_1,x_1),\Delta(x_1,x_1))$$

$\forall\ x_1 \in \mathfrak{L}$, *then either $\mathfrak{L} \subseteq Z$ or $\Delta = \Gamma$ or $\Delta = 0$.*

Proof. Let γ, δ be the traces of Γ and Δ, respectively. By our hypothesis we have

$$\gamma^2(x_1) - \delta^2(x_1) = \delta(\gamma(x_1)) - \gamma(\delta(x_1)). \tag{3.21}$$

The substitution of x_1 by $x_1 + x_2$ in (3.21), gives that

$$\begin{aligned} &4\gamma(\Gamma(x_1,x_2)) + 2\Gamma(\gamma(x_1),\gamma(x_2)) + 4\Gamma(\gamma(x_1),\Gamma(x_1,x_2)) \\ &\quad + 4\Gamma(\gamma(x_2),\Gamma(x_1,x_2)) - \delta(\Delta(x_1,x_2)) - 2\Delta(\delta(x_1),\delta(x_2)) \\ &\quad - 4\Delta(\delta(x_1),\Delta(x_1,x_2)) - 4\Delta(\delta(x_2),\Delta(x_1,x_2)) \\ &= 4\delta(\Gamma(x_1,x_2)) + 2\Delta(\gamma(x_1),\gamma(x_2)) + 4\Delta(\gamma(x_1),\Gamma(x_1,x_2)) \\ &\quad + 4\Delta(\gamma(x_2),\Gamma(x_1,x_2)) - 4\gamma(\Delta(x_1,x_2)) - 2\Gamma(\delta(x_1),\delta(x_2)) \\ &\quad - 4\Gamma(\delta(x_1),\Delta(x_1,x_2)) - 4\Gamma(\delta(x_2),\Delta(x_1,x_2)). \end{aligned} \tag{3.22}$$

Now, substituting x_2 by $-x_2$ in the above expression, we find that

$$\begin{aligned} &4\gamma(\Gamma(x_1,x_2)) + 2\Gamma(\gamma(x_1),\gamma(x_2)) - 4\Gamma(\gamma(x_1),\Gamma(x_1,x_2)) \\ &\quad - 4\Gamma(\gamma(x_2),\Gamma(x_1,x_2)) - \delta(\Delta(x_1,x_2)) - 2\delta(\delta(x_1),\delta(x_2)) \\ &\quad + 4\Delta(\delta(x_1),\Delta(x_1,x_2)) + 4\Delta(\delta(x_2),\Delta(x_1,x_2)) \\ &= 4\delta(\Gamma(x_1,x_2)) + 2\Delta(\gamma(x_1),\gamma(x_2)) - 4\Delta(\gamma(x_1),\Gamma(x_1,x_2)) \\ &\quad - 4\Delta(\gamma(x_2),\Gamma(x_1,x_2)) - 4\gamma(\Delta(x_1,x_2)) - 2\Gamma(\delta(x_1),\delta(x_2)) \\ &\quad + 4\Gamma(\delta(x_1),\Delta(x_1,x_2)) + 4\Gamma(\delta(x_2),\Delta(x_1,x_2)). \end{aligned} \tag{3.23}$$

Comparing (3.22) and (3.23) and using the fact that char$(\mathfrak{R}) \neq 2$, we get

$$\begin{aligned} &2\gamma(\Gamma(x_1,x_2)) + \Gamma(\gamma(x_1),\gamma(x_2)) - 2\delta(\Delta(x_1,x_2)) - \Delta(\delta(x_1),\delta(x_2)) \\ &= 2\delta(\Delta(x_1,x_2)) + \Delta(\gamma(x_1),\gamma(x_2)) - 2\gamma(\Delta(x_1,x_2)) - \Gamma(\delta(x_1),\delta(x_2)). \end{aligned} \tag{3.24}$$

Now, substituting x_2 by $x_2 + x_3$ in(3.24), gives that

$$\begin{aligned} &2\Gamma(\Gamma(x_1,x_2),\Gamma(x_1,x_3)) + \Gamma(\gamma(x_1),\Gamma(x_2,x_3)) \\ &\quad - 2\Delta(\Delta(x_1,x_2),\Delta(x_1,x_3)) - \Delta(\delta(x_1),\Delta(x_2,x_3)) \\ &= 2\delta(\Delta(x_1,x_2),\Delta(x_2,x_3)) + \Delta(\gamma(x_1),\Gamma(x_2,x_3)) \\ &\quad - 2\gamma(\Delta(x_1,x_2)) - \Gamma(\delta(x_1),\Delta(x_2,x_3)). \end{aligned} \tag{3.25}$$

Let us take $T = \Gamma - \Delta$, and denote $k = \gamma + \delta$ the trace of $K = \Gamma + \Delta$. By Lemma 5 T is a left bicentralizer of $\mathfrak{R}$. Then (3.24), reduces to

$$2T(\Gamma(x_1,x_2)\Gamma(x_1,x_3))+T(\gamma(x_1),\Gamma(x_2,x_3)) \\ +2T(\Delta(x_1,x_2),\Delta(x_1,x_3))+T(\delta(x_1),\Delta(x_2,x_3))=0. \quad (3.26)$$

Replacing x_3 by $2x_3z$ in (3.26)and using (3.26), we obtain

$$2T(\Gamma(x_1,x_2),x_3)\Delta(x_1,z)+T(\gamma(x_1),x_3)\Delta(x_2,z) \\ +2T(\Delta(x_1,x_2),x_3)\Delta(x_1,z)+T(\delta(x_1),x_3)\Delta(x_2,z)=0. \quad (3.27)$$

That is,

$$2T(K(x_1,x_2),x_3)\Delta(x_1,z)+T(k(x_1),x_3)\Delta(x_2,z)=0. \quad (3.28)$$

Choosing $x_2=x_1$ in (3.28), and using char$(\mathfrak{R})\neq 3$, we get

$$T(k(x_1),x_3)\Delta(x_1,z)=0. \quad (3.29)$$

Choosing $z=x_1$ in (3.28), we obtain

$$2T(K(x_1,x_2),x_3)\delta(x_1)+T(k(x_1),x_3)\Delta(x_2,x_1)=0. \quad (3.30)$$

Comparing (3.29) and (3.30) and using the fact that char$(\mathfrak{R})\neq 2$, gives that

$$T(K(x_1,x_2),x_3)\delta(x_1)=0. \quad (3.31)$$

Replacing x_2 by $2x_2x$ in (3.31), we find that

$$T(K(x_1,x_2),x_3)x\delta(x_1)+2T(x_2,x_3)\Delta(x_1,x)\delta(x_1)=0. \quad (3.32)$$

By replacing x_3 by $2x_3x$ in (3.31), we find that

$$T(K(x_1,x_2),x_3)x\delta(x_1)=0. \quad (3.33)$$

From (3.32) and (3.33) and using the fact that char$(R)\neq 2$, we find that

$$T(x_2,x_3)\Delta(x_1,x)\delta(x_1)=0. \quad (3.34)$$

Replace x_3 by $2x_3w_1$, to get $(T(x_2,x_3)\mathfrak{L}\Delta(x_1,x)\Delta(x_1)=0\ \forall\ x_1,x_2,x,x_3\in\mathfrak{L}$. Lemma 3, gives that either $(T(x_2,x_3)=0$ or $\Delta(x_1,x)\delta(x_1)=0$. In the first case $T(x_2,x_3)=0\ \forall\ x_2,x_3\in\mathfrak{L}$ and hence by Lemma 4, proves that $T=0$. Therefore $\Gamma=\Delta$. On the other hand, if $\Delta(x_1,x)\delta(x_1)=0\ \forall\ x_1,x\in\mathfrak{L}$, then replacing x by $2x_1x$, we have $\delta(x_1)x\delta(x_1)=0$, that is, $\delta(x_1)\mathfrak{L}\delta(x_1)=0$. Again, by Lemma 3, $\delta(x_1)=0$ and hence $\delta(x_1)=0$ for al $x_1\in\mathfrak{L}$. Proposition 1, gives that $\mathfrak{L}\subseteq Z$ or $\Delta=0$. □

Corollary 4. *Let $\mathfrak{R}$ be a prime ring, char$(\mathfrak{R})\neq 2$, I a nonzero ideal of $\mathfrak{R}$, α an automorphism, and $\Gamma:\mathfrak{R}\times\mathfrak{R}\to\mathfrak{R}$ a symmetric generalized (α,α)-biderivation associated with a symmetric (α,α)-biderivation Δ. If*

$$\Gamma(\Gamma(x_1,x_1),\Gamma(x_1,x_1))-\Delta(\Delta(x_1,x_1),\Delta(x_1,x_1)) \\ =\Delta(\Gamma(x_1,x_1),\Gamma(x_1,x_1))-\Gamma(\Delta(x_1,x_1),\Delta(x_1,x_1))$$

$\forall\ x_1\in I$, then either R is commutative or $\Delta=\Gamma$ or Δ is a left α-bicentralizer.

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Authors' addresses

Nadeem ur Rehman
Department of Mathematics, Aligarh Muslim University, 202002, Aligarh, India
E-mail address: nu.rehman.mm@amu.ac.in

Shuliang Huang
Shuliang Huang, Department of Mathematics, Chuzhou University, Chuzhou 239012, China
E-mail address: shulianghuang@sina.com

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ON TWO BIVARIATE KINDS OF (p,q)-BERNOULLI POLYNOMIALS

P. NJIONOU SADJANG AND UGUR DURAN

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Abstract. The main aim of this paper is to introduce and investigate (p,q)-extensions of two bivariate kinds of Bernoulli polynomials and numbers. We firstly examine several (p,q)-analogues of the Taylor expansions of products of some trigonometric functions and determine their coefficients which are also analyzed in detail. Then, we introduce two bivariate kinds of (p,q)-Bernoulli polynomials and acquired multifarious formulas and relations including connection formulas, recurrence formulas, correlations with aforementioned coefficients, partial (p,q)-differential equations and (p,q)-integral representations.

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1. Introduction

The Appell polynomials $A_n(x)$ defined by

$$f(t)\mathrm{e}^{xt} = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}, \tag{1.1}$$

where f is a formal power series in t, have found remarkable applications in different branches of mathematics, theoretical physics and chemistry, see [1, 3–13] and references cited therein. One of the most famous polynomials of the of Appell families is Bernoulli polynomials $B_n(x)$, generated by $f(t) = \dfrac{t}{\mathrm{e}^t - 1}$ in (1.1). Also, Bernoulli numbers, denoted by $B_n := B_n(0)$ are of considerable importance in number theory, combinatorics and numerical analysis (*cf.* [1, 3, 5, 6, 8–10]). Further, they have the following exponential generating function:

$$\frac{t}{\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad (|t| < 2\pi),$$

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and can be generated by the following recurrence relation

$$\sum_{k=0}^{n}\binom{n+1}{k}B_k = 0 \ \text{ for } n \geq 1 \ \text{ and } \ B_0 = 1.$$

Bernoulli numbers are directly related to several combinatorial numbers such as Stirling, Cauchy and harmonic numbers. For example, except B_1 we have

$$B_n = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m m!}{m+1} S_2(n,m), \tag{1.2}$$

where

$$S_2(n,m) = \frac{1}{m!}\sum_{j=0}^{m}(-1)^j \binom{m}{j}(m-j)^n,$$

denote the second kind of Stirling numbers [9] with $S_2(n,m) = 0$ for $n < m$. Very recently in [7], Jamei *et al.* introduced a new kind of bivariate Bernoulli polynomials and studied their main properties. As a valuable application of these extended polynomials, they introduced an extension of the well-known Euler-Maclaurin quadrature formula. Rahmani [8] defined a new family of p-Bernoulli numbers, which are derived from the Gaussian hypergeometric function, and established some basic properties. Based on a three-term recurrence relation, he gave an algorithm for computing Bernoulli numbers and presented a similar algorithm for Bernoulli polynomials.

The Bernoulli polynomials and numbers have found diverse extensions such as poly-Bernoulli numbers, which are somehow connected to multiple zeta values. The q-extension of Bernoulli numbers and polynomials has now found many applications in combinatorics, statistics and various branches of applied mathematics. Mahmudov [5, 6] introduced a class of generalized Bernoulli polynomials, Euler polynomials and Genocchi polynomials based on the q-integers and obtained the q-analogues of well-known formulas including q-analogue of the Srivastava-Pinter addition theorem and correlations with the q-Bernstein polynomials. Duran *et al.* [3] considered the new generating functions of the Bernoulli, the Euler and the Genocchi polynomials under post quantum calculus, denoted by (p,q)-calculus. From those generating functions, they analyzed their various behaviours and derived a relation between the new and old polynomials by making use of the fermionic p-adic integral over the p-adic number fields. Njionou [11] introduced the (p,q)-Appell polynomials which covers generalizations of some famous family of polynomials such as the (p,q)-Bernoulli, the (p,q)-Euler and the (p,q)-Genocchi polynomials. He provided several characterizations of these polynomials.

In this paper, we introduce a (p,q)-extension of the aforesaid bivariate Bernoulli polynomials and establish their properties. Multifarious connections and inversion formulas are stated and proved. In the following section, some preliminaries and

definitions are given and in Section 3, a bivariate kind of (p,q)-Bernoulli polynomials is introduced and some of its fundamental properties are stated and proved.

2. Preliminaries and definitions

Let $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{C}$ denotes the set of all complex numbers. Let us introduce the following notation (see [4, 10, 12])

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}, \quad 0 < |q| < |p| \leq 1$$

for any positive integer.
The twin-basic number is a natural generalization of the q-number, that is

$$\lim_{p\to 1} [n]_{p,q} = [n]_q.$$

The (p,q)-factorial is defined by

$$[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}!, \quad n \geq 1, \quad [0]_{p,q}! = 1.$$

Let us introduce also the so-called (p,q)-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}, \quad 0 \leq k \leq n.$$

Note that as $p \to 1$, the (p,q)-binomial coefficients reduce to the q-binomial coefficients.
It is clear by definition that

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \begin{bmatrix} n \\ n-k \end{bmatrix}_{p,q}.$$

Let us introduce also the so-called falling and raising (p,q)-powers respectively [4, 12]

$$(x \ominus a)^n_{p,q} = (x-a)(px-aq)\cdots(xp^{n-1} - aq^{n-1}),$$
$$(x \oplus a)^n_{p,q} = (x+a)(px+aq)\cdots(xp^{n-1} + aq^{n-1}).$$

These definitions are extended to

$$(a \ominus b)^\infty_{p,q} = \prod_{k=0}^{\infty} (ap^k - q^k b),$$
$$(a \oplus b)^\infty_{p,q} = \prod_{k=0}^{\infty} (ap^k + q^k b),$$

where the convergence is required.

Definition 1 ([4, 12])**.** Let f be an arbitrary function and a be a real number, then the (p,q)-integral of f is defined by

$$\int_0^a f(x)d_{p,q}x = (p-q)a\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}f\left(\frac{q^k}{p^{k+1}}a\right) \quad \text{if } 0<|q|<|p|\leq 1.$$

Let f be a function defined on the set of the complex numbers.

Definition 2 ([4, 12])**.** The (p,q)-derivative of the function f is defined as

$$D_{p,q}f(x) = \frac{f(px)-f(qx)}{(p-q)x}, \quad x\neq 0,$$

and $(D_{p,q}f)(0) = f'(0)$, provided that f is differentiable at 0.

Proposition 1. *The (p,q)-derivative operator fulfills the following product and quotient rules*

$$D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x)+g(qx)D_{p,q}f(x),$$
$$D_{p,q}(f(x)g(x)) = g(px)D_{p,q}f(x)+f(qx)D_{p,q}g(x).$$

$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{g(qx)D_{p,q}f(x)-f(qx)D_{p,q}g(x)}{g(px)g(qx)},$$
$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{g(px)D_{p,q}f(x)-f(px)D_{p,q}g(x)}{g(px)g(qx)}.$$

Proposition 2. *If $F(x)$ is a (p,q)-antiderivative of $f(x)$ and $F(x)$ is continuous at $x=0$, we have*

$$\int_a^b f(x)d_{p,q}x = F(b)-F(a), \quad 0\leq a<b\leq\infty.$$

Corollary 1. *If $f'(x)$ exists in a neighbourhood of $x=0$ and is continuous at $x=0$, where $f'(x)$ denotes the ordinary derivative of $f(x)$, we have*

$$\int_a^b D_{p,q}f(x)d_{p,q}x = f(b)-f(a).$$

Proposition 3. *Suppose that $f(x)$ and $g(x)$ are two functions whose ordinary derivatives exist in a neighbourhood of $x=0$. a and b are two real numbers such that $a<b$, then*

$$\int_a^b f(px)\left(D_{p,q}g(x)\right)d_{p,q}x = f(b)g(b)-f(a)g(a)-\int_a^b g(qx)\left(D_{p,q}f(x)\right)d_{p,q}x. \tag{2.1}$$

As in the q-case, there are many definitions of the (p,q)-exponential function. The following two (p,q)-analogues of the exponential function (see [3,4,10,11]) will be frequently used throughout this paper:

$$e_{p,q}(z) = \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}}}{[n]_{p,q}!} z^n, \tag{2.2}$$

$$E_{p,q}(z) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{[n]_{p,q}!} z^n. \tag{2.3}$$

From the definitions (2.2) and (2.3) of the (p,q)-exponential functions, it is easy to see that [10]

$$e_{p,q}(x) E_{p,q}(-x) = 1. \tag{2.4}$$

From (2.2) we can derive

$$e_{p,q}(iz) = \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}}}{[n]_{p,q}!} (iz)^n = \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n}{2}}}{[2n]_{p,q}!} z^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} z^{2n+1}. \tag{2.5}$$

By (2.5), the (p,q)-cosine and the (p,q)-sine functions are defined (see [4, 10]) as follows:

$$\cos_{p,q}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n}{2}}}{[2n]_{p,q}!} z^{2n}, \tag{2.6}$$

$$\sin_{p,q}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} z^{2n+1}. \tag{2.7}$$

In the following definition, we generalize the notion of q-addition introduced by Jackson and studied later by Ward and Al-Salam, see [1, 2] for more details. Our (p,q)-addition reduces to the q-addition defined by Euler and recalled in [13]

Definition 3. Let x and y be two complex numbers.

(1) The (p,q)-addition of x and y which we denote by $x \oplus_{p,q} y$ is defined by

$$(x \oplus_{p,q} y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{k(k-n)} x^k y^{n-k}. \tag{2.8}$$

(2) The (p,q)-subtration of x and y which we denote by $x \ominus_{p,q} y$ is defined by

$$(x \ominus_{p,q} y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (-1)^{n-k} p^{k(k-n)} x^k y^{n-k}. \tag{2.9}$$

Theorem 1. *The following relation holds true for any $x, y \in \mathbb{R}$.*

$$e_{p,q}(x)e_{p,q}(y) = e_{p,q}(x \oplus_{p,q} y) \tag{2.10}$$

$$e_{p,q}(x)e_{p,q}(-y) = e_{p,q}(x \ominus_{p,q} y). \tag{2.11}$$

Proof. By the definition of the (p,q)-addition and the Cauchy product we can readily see that

$$\begin{aligned} e_{p,q}(x)e_{p,q}(y) &= \sum_{k=0}^{\infty} \frac{p^{\binom{k}{2}} x^k}{[k]_{p,q}!} \sum_{\ell=0}^{\infty} \frac{p^{\binom{\ell}{2}} y^{\ell}}{[\ell]_{p,q}!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{p^{\binom{k}{2}+\binom{n-k}{2}} x^k y^{n-k}}{[k]_{p,q}![n-k]_{p,q}!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{k(k-n)} x^k y^{n-k} \right) \frac{p^{\binom{n}{2}}}{[n]_{p,q}!} = e_{p,q}(x \oplus_{p,q} y). \end{aligned}$$

The second assertion is proved in the same way. □

3. A BIVARIATE KIND OF (p,q)-BERNOULLI POLYNOMIALS

Let $x, y \in \mathbb{R}$. It is well-known that the Taylor expansion of the two functions $e^{xt}\cos yt$ and $e^{xt}\sin yt$ are as follows [7]

$$e^{xt}\cos yt = \sum_{n=0}^{\infty} C_n(x,y)\frac{t^n}{n!}, \tag{3.1}$$

and

$$e^{xt}\sin yt = \sum_{n=0}^{\infty} S_n(x,y)\frac{t^n}{n!}, \tag{3.2}$$

where

$$C_n(x,y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} x^{n-2k} y^{2k}, \tag{3.3}$$

and

$$S_n(x,y) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1}. \tag{3.4}$$

Here we introduce a (p,q)-extension of the two above polynomials $C_n(x,y)$ and $S_n(x,y)$ by the following generating functions:

$$e_{p,q}(xt)\cos_{p,q}(yt) = \sum_{k=0}^{\infty} C_{k,p,q}(x,y)\frac{t^k}{[k]_{p,q}!}, \tag{3.5}$$

and

$$e_{p,q}(xt)\sin_{p,q}(yt)=\sum_{k=0}^{\infty}S_{k,p,q}(x,y)\frac{t^k}{[k]_{p,q}!}, \tag{3.6}$$

Some particular cases are

$$C_{2n,p,q}(0,y)=(-1)^n p^{\binom{2n}{2}}y^{2n}, \quad C_{2n+1,p,q}(0,y)=0$$

and

$$S_{2n,p,q}(0,y)=0, \quad S_{2n+1,p,q}(0,y)=(-1)^n p^{\binom{2n+1}{2}}y^{2n+1}.$$

The following Lemma will be useful in the derivation of several results.

Lemma 1 ([9]). *The following elementary series manipulations hold*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor}A(k,n-2k), \tag{3.7}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor (n-1)/2\rfloor}B(k,n-1-2k). \tag{3.8}$$

Theorem 2. *The following representations hold*

$$C_{n,p,q}(x,y)=p^{\binom{n}{2}}\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^k\begin{bmatrix} n \\ 2k \end{bmatrix}_{p,q}p^{2k(k-n)}x^{n-2k}y^{2k} \tag{3.9}$$

and

$$S_{n,p,q}(x,y)=p^{\binom{n-1}{2}}\sum_{k=0}^{\lfloor \frac{n-1}{2}\rfloor}(-1)^k\begin{bmatrix} n \\ 2k+1 \end{bmatrix}_{p,q}p^{4k^2-2kn}x^{n-2k-1}y^{2k+1} \tag{3.10}$$

Proof. By series manipulation procedure (3.7), we have

$$e_{p,q}(xt)\cos_{p,q}(yt)=\left(\sum_{n=0}^{\infty}\frac{p^{\binom{n}{2}}}{[n]_{p,q}!}(xt)^n\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n p^{\binom{2n}{2}}}{[2n]_{p,q}!}(yt)^{2n}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{p^{\binom{n-2k}{2}}}{[n-2k]_{p,q}!}(xt)^{n-2k}\frac{(-1)^k p^{\binom{2k}{2}}}{[2k]_{p,q}!}(yt)^{2k}$$

$$=\sum_{n=0}^{\infty}\left(p^{\binom{n}{2}}\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^k\begin{bmatrix} n \\ 2k \end{bmatrix}_{p,q}p^{2k(k-n)}x^{n-2k}y^{2k}\right)\frac{t^n}{[n]_{p,q}!},$$

which proves (3.5). The proof of (3.6) is similar by means of the series manipulation method (3.8).

□

Theorem 3. *The following derivative rules are valid*

$$D_{p,q,x}C_{k,p,q}(x,y) = [k]_{p,q}C_{k-1,p,q}(px,y), \tag{3.11}$$
$$D_{p,q,y}C_{k,p,q}(x,y) = -[k]_{p,q}S_{k-1,p,q}(x,py), \tag{3.12}$$
$$D_{p,q,x}S_{k,p,q}(x,y) = [k]_{p,q}S_{k-1,p,q}(px,y), \tag{3.13}$$
$$D_{p,q,y}S_{k,p,q}(x,y) = [k]_{p,q}C_{k-1,p,q}(x,py). \tag{3.14}$$

Proof. Relation (3.5) yields

$$\begin{aligned}\sum_{n=1}^{\infty} D_{p,q,x}C_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!} = te_{p,q}(pxt)\cos_{p,q}(yt) &= \sum_{n=0}^{\infty} C_{n,p,q}(px,y)\frac{t^{n+1}}{[n]_{p,q}!}\\ &= \sum_{n=1}^{\infty} C_{n-1,p,q}(px,y)\frac{t^n}{[n-1]_{p,q}!}\\ &= \sum_{n=0}^{\infty} [n]_{p,q}C_{n-1,p,q}(px,y)\frac{t^n}{[n]_{p,q}!},\end{aligned}$$

proving (3.11). The other equations (3.12), (3.13) and (3.14) can be similarly proved. □

Theorem 4. *The following relations are valid*

$$C_{n,p,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} C_{k,p,q}(0,y)x^{n-k}, \tag{3.15}$$
$$S_{n,p,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} S_{k,p,q}(0,y)x^{n-k}. \tag{3.16}$$

Theorem 5. *The following power representations hold*

$$(-1)^n p^{\binom{2n}{2}} y^{2n} = \sum_{k=0}^{2n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_{p,q} C_{2n-k,p,q}(x,y)x^k, \tag{3.17}$$

and

$$(-1)^n p^{\binom{2n+1}{2}} y^{2n+1} = \sum_{k=0}^{2n+1} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{p,q} S_{2n+1-k,p,q}(x,y)x^k. \tag{3.18}$$

Proof. Multiplying both sides of (3.5) by $E_{p,q}(-xt)$ and using (2.4), it follows that

$$\sum_{n=0}^{\infty} (-1)^n p^{\binom{2n}{2}} y^{2n} \frac{t^{2n}}{[n]_{p,q}!} = \left(\sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{(-x)^n t^n}{[n]_{p,q}!}\right)\left(\sum_{n=0}^{\infty} C_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} C_{n-k,p,q}(x,y)x^k\right)\frac{t^n}{[n]_{p,q}!},$$

which proves (3.17). The proof of (3.18) is similar. □

Theorem 6. *The following connection formulas hold*

$$C_{2n+1,p,q}(x,y) = \sum_{k=0}^{2n}(-1)^k q^{\binom{k+1}{2}}\begin{bmatrix} 2n+1 \\ k+1 \end{bmatrix}_{p,q} C_{2n-k,p,q}(x,y)x^{k+1}, \quad (3.19)$$

and

$$S_{2n,p,q}(x,y) = \sum_{k=0}^{2n-1}(-1)^k q^{\binom{k+1}{2}}\begin{bmatrix} 2n \\ k+1 \end{bmatrix}_{p,q} S_{2n-k-1,p,q}(x,y)x^{k+1}. \quad (3.20)$$

Proof. From the relation

$$\sum_{n=0}^{\infty}(-1)^n p^{\binom{2n}{2}} y^{2n}\frac{t^{2n}}{[n]_{p,q}!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} C_{n-k,p,q}(x,y)x^k\right)\frac{t^n}{[n]_{p,q}!},$$

it follows that

$$\sum_{k=0}^{2n+1}(-1)^k q^{\binom{k}{2}}\begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{p,q} C_{2n+1-k,p,q}(x,y)x^k = 0.$$

Hence (3.19) is proved. We prove (3.20) in the same way.

□

We can now introduce two kinds of bivariate q-Bernoulli polynomials as

$$\frac{te_{p,q}(xt)}{e_{p,q}(t)-1}\cos_{p,q}(yt) = \sum_{n=0}^{\infty} B^{(c)}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}, \quad (3.21)$$

and

$$\frac{te_{p,q}(xt)}{e_{p,q}(t)-1}\sin_{p,q}(yt) = \sum_{n=0}^{\infty} B^{(s)}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}. \quad (3.22)$$

Upon setting $x = y = 0$ for both polynomials in (3.21) and (3.22), we have $B^{(c)}_{n,p,q}(0,0) = B^{(s)}_{n,p,q}(0,0) := B_{n,p,q}$ which are called (p,q)-Bernoulli polynomials defined in [3]. When $y = 0$ in (3.21) and (3.22), we get the usual (p,q)-Bernoulli polynomials, denoted by $B_{n,p,q}(x)$, see [3, 11].

Next, we give some basic properties of these polynomials.

Theorem 7. *$B^{(c)}_{n,p,q}(x,y)$ and $B^{(s)}_{n,p,q}(x,y)$ can be represented in terms of (p,q)-Bernoulli numbers as follows*

$$B^{(c)}_{n,p,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} B_{k,p,q}C_{n-k,p,q}(x,y), \tag{3.23}$$

and

$$B^{(s)}_{n,p,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} B_{k,p,q}S_{n-k,p,q}(x,y), \tag{3.24}$$

Proof. Using the Cauchy product rule, we have

$$\begin{aligned}\sum_{n=0}^{\infty} B^{(c)}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!} &= \frac{t}{e_{p,q}(t)-1}e_{p,q}(xt)\cos_{p,q}(yt)\\ &= \left(\sum_{n=0}^{\infty} B_n\frac{t^n}{[n]_{p,q}!}\right)\left(\sum_{n=0}^{\infty} C_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} B_{k,p,q}C_{n-k,p,q}(x,y)\right)\frac{t^n}{[n]_{p,q}!},\end{aligned}$$

which proves (3.23). The proof of (3.24) is similar. □

We now state the following theorem.

Theorem 8. *The following connection formulas are valid*

$$B^{(c)}_{n,p,q}(x,y)=\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\begin{bmatrix} n \\ 2k \end{bmatrix}_{p,q} B_{n-2k,p,q}(x)p^{\binom{2k}{2}}y^{2k}, \tag{3.25}$$

and

$$B^{(s)}_{n,p,q}(x,y)=\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\begin{bmatrix} n \\ 2k+1 \end{bmatrix}_{p,q} B_{n-1-2k,p,q}(x)p^{\binom{2k+1}{2}}y^{2k+1}. \tag{3.26}$$

Proof. The formula (3.25) follows from (3.7) since

$$\begin{aligned}\sum_{n=0}^{\infty} B^{(c)}_{n,p,q}(x,y)\frac{t^n}{[n]_q} &= \frac{te_{p,q}(xt)}{e_{p,q}(t)-1}\cos_{p,q}(yt)\\ &= \left(\sum_{n=0}^{\infty} B_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n p^{\binom{2n}{2}}}{[2n]_{p,q}!}(yt)^{2n}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\begin{bmatrix} n \\ 2k \end{bmatrix}_{p,q} B_{n-2k,p,q}(x)p^{\binom{2k}{2}}y^{2k}\right)\frac{t^n}{[n]_{p,q}!}.\end{aligned}$$

The proof of (3.26) is similar via (3.8). □

Theorem 9. *The following connection formulas are valid*

$$C_{n,p,q}(x,y)=\sum_{k=0}^{n}\frac{p^{\binom{k+1}{2}}}{[k+1]_{p,q}}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} B^{(c)}_{n,p,q}(x,y), \tag{3.27}$$

$$S_{n,p,q}(x,y)=\sum_{k=0}^{n}\frac{p^{\binom{k+1}{2}}}{[k+1]_{p,q}}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} B^{(s)}_{n,p,q}(x,y). \tag{3.28}$$

Proof. From (3.21), we have

$$\sum_{n=0}^{\infty} B^{(c)}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}=\frac{te_{p,q}(xt)}{e_{p,q}(t)-1}\cos_{p,q}(yt)=\frac{t}{e_{p,q}(t)-1}\sum_{n=0}^{\infty} C_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}.$$

Hence

$$\begin{aligned}\sum_{n=0}^{\infty} C_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}&=\frac{e_{p,q}(t)-1}{t}\sum_{n=0}^{\infty} B^{(c)}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}\\&=\left(\sum_{n=0}^{\infty}\frac{p^{\binom{n+1}{2}}}{[n+1]_{p,q}}\frac{t^n}{[n]_{p,q}!}\right)\left(\sum_{n=0}^{\infty} B^{(c)}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{p^{\binom{k+1}{2}}}{[k+1]_{p,q}}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} B^{(c)}_{n,p,q}(x,y)\right)\frac{t^n}{[n]_{p,q}!}.\end{aligned}$$

Thus (3.27) follows. (3.28) is proved in a similar way. □

Proposition 4. *For every $n\in\mathbb{N}$, the following identities hold*

$$B^{(c)}_{n,p,q}((1\oplus_{p,q}x),y)-B^{(c)}_{n,p,q}(x,y)=[n]_{p,q}C_{n-1,p,q}(x,y), \tag{3.29}$$

$$B^{(s)}_{n,p,q}((1\oplus_{p,q}x),y)-B^{(s)}_{n,p,q}(x,y)=[n]_{p,q}S_{n-1,p,q}(x,y). \tag{3.30}$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} B^{(c)}_{n,p,q}((1\oplus_{p,q}x),y)\frac{t^n}{[n]_{p,q}!}&=\frac{te_{p,q}[(1\oplus_{p,q}x)t]}{e_{p,q}(t)-1}\cos_{p,q}(yt)\\&=\frac{te_{p,q}(xt)[e_{p,q}(t)-1+1]}{e_{p,q}(t)-1}\cos_{p,q}(yt)\\&=te_{p,q}(xt)\cos_{p,q}(yt)+\frac{te_{p,q}(xt)}{e_{p,q}(t)-1}\cos_{p,q}(yt)\\&=\sum_{n=0}^{\infty} C_{n,p,q}(x,y)\frac{t^{n+1}}{[n]_{p,q}!}+\sum_{n=0}^{\infty} B^{(c)}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!},\end{aligned}$$

which proves (3.29). The identity (3.30) is proved similarly. □

Corollary 2. *The following relations hold*

$$B^{(c)}_{2n+1,p,q}(1,y)-B^{(c)}_{2n+1,p,q}(0,y)=[2n+1]_{p,q}(-1)^n p^{\binom{2n}{2}}y^{2n},$$
$$B^{(s)}_{2n,p,q}(1,y)-B^{(s)}_{2n,p,q}(0,y)=[2n]_{p,q}(-1)^{n+1} p^{\binom{2n-1}{2}}y^{2n-1}.$$

Proof. If we replace n by $2n+1$ in (3.29), and x by 0, we obtain

$$B^{(c)}_{2n+1,p,q}(1,y)-B^{(c)}_{2n+1,p,q}(0,y)=[2n+1]_{p,q}C_{2n,p,q}(0,y).$$

The first relation is proved since from (3.9) we have $C_{2n,q}(0,y)=(-1)^n p^{\binom{2n}{2}}y^{2n}$. The second relation is proved similarly. □

Proposition 5. *For every $n\in\mathbb{N}$, the following identities hold*

$$B^{(c)}_{n,p,q}((x\oplus_{p,q}z),y)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}B_{k,p,q}(x)C_{n-k,p,q}(y,z), \tag{3.31}$$

and

$$B^{(s)}_{n,p,q}((x\oplus_{p,q}z),y)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}B_{k,p,q}(x)S_{n-k,p,q}(y,z), \tag{3.32}$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty}B^{(c)}_{n,p,q}((x\oplus_{p,q}z),y)\frac{t^n}{[n]_{p,q}!}&=\frac{te_{p,q}((x\oplus_{p,q}z)t)}{e_{p,q}(t)-1}\cos_{p,q}(yt)\\&=\frac{te_{p,q}(xt)}{e_{p,q}(t)-1}\times e_{p,q}(zt)\cos_{p,q}(yt)\\&=\left(\sum_{n=0}^{\infty}B_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}\right)\left(\sum_{n=0}^{\infty}C_{n,p,q}(y,z)\frac{t^n}{[n]_{p,q}!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}B_{k,p,q}(x)C_{n-k}(y,z)\right)\frac{t^n}{[n]_{q}!},\end{aligned}$$

which proves (3.31). The proof of (3.32) is similar. □

Proposition 6. *For every $n\in\mathbb{N}$, the following identities hold*

$$B^{(c)}_{n,p,q}((x\oplus_{p,q}z),y)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}p^{\binom{n-k}{2}}B^{(c)}_{k,p,q}(x,y)z^{n-k}, \tag{3.33}$$

and

$$B^{(s)}_{n,p,q}((x\oplus_{p,q}z),y)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}p^{\binom{n-k}{2}}B^{(s)}_{k,p,q}(x,y)z^{n-k}. \tag{3.34}$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,p,q}^{(c)}((x\oplus_{p,q} z),y)\frac{t^n}{[n]_{p,q}!} &= \frac{te_{p,q}((x\oplus_{p,q} z)t)}{e_{p,q}(t)-1}\cos_{p,q}(yt)\\ &= \frac{te_{p,q}(xt)}{e_{p,q}(t)-1}\cos_{p,q}(yt)\times e_{p,q}(zt)\\ &= \left(\sum_{n=0}^{\infty} B_{n,p,q}^{(c)}(x,y)\frac{t^n}{[n]_{p,q}!}\right)\left(\sum_{n=0}^{\infty} p^{\binom{n}{2}}z^n\frac{t^n}{[n]_{p,q}!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q} p^{\binom{n-k}{2}}B_{k,p,q}^{(c)}(x,y)z^{n-k}\right)\frac{t^n}{[n]_q!},\end{aligned}$$

which proves (3.33). The proof of (3.34) is similar. □

Proposition 7. *The following equations can be concluded*

$$\sum_{k=0}^{n}\begin{bmatrix} n+1\\ k\end{bmatrix}_{p,q} p^{\binom{n+1-k}{2}}B_{k,p,q}^{(c)}(x,y) = [n+1]_{p,q}C_{n,p,q}(x,y), \tag{3.35}$$

$$\sum_{k=0}^{n}\begin{bmatrix} n+1\\ k\end{bmatrix}_{p,q} p^{\binom{n+1-k}{2}}B_{k,p,q}^{(s)}(x,y) = [n+1]_{p,q}S_{n,p,q}(x,y). \tag{3.36}$$

Proof. From (3.33), we have

$$B_{n+1,p,q}^{(c)}((x\oplus_{p,q} 1),y) - B_{n+1,p,q}^{(c)}(x,y) = \sum_{k=0}^{n}\begin{bmatrix} n+1\\ k\end{bmatrix}_{p,q} p^{\binom{n+1-k}{2}}B_{k,p,q}^{(c)}(x,y).$$

Hence, by using (3.29), relation (3.35) is derived. The proof of (3.36) is concluded in a similar way. □

Corollary 3. *Relations* (3.35) *and* (3.36) *imply that*

$$\sum_{k=0}^{n}\begin{bmatrix} n+1\\ k\end{bmatrix}_{p,q} p^{\binom{n+1-k}{2}}B_{k,p,q}^{(c)}(0,y) = \begin{cases}(-1)^m[2m+1]_{p,q}\,p^{\binom{2m}{2}}y^{2m} & \text{if } n=2m \text{ is odd},\\ 0 & \text{if } n=2m+1 \text{ is even},\end{cases}$$

and

$$\sum_{k=0}^{n}\begin{bmatrix} n+1\\ k\end{bmatrix}_{p,q} p^{\binom{n+1-k}{2}}B_{k,p,q}^{(s)}(0,y) = \begin{cases}0 & \text{if } n=2m \text{ is odd},\\ (-1)^m[2m+2]_{p,q}\,p^{\binom{2m+1}{2}}y^{2m+1} & \text{if } n=2m+1 \text{ is even}.\end{cases}$$

Corollary 4. *For every $n \in \mathbb{N}$, the following partial (p,q)-differential equations hold*

$$D_{p,q,x} B^{(c)}_{n,p,q}(x,y) = [n]_{p,q} B^{(c)}_{n-1,p,q}(px,y),$$
$$D_{p,q,y} B^{(c)}_{n,p,q}(x,y) = -[n]_{p,q} B^{(c)}_{n-1,p,q}(x,py),$$
$$D_{p,q,x} B^{(s)}_{n,p,q}(x,y) = [n]_{p,q} B^{(s)}_{n-1,p,q}(px,y),$$

and

$$D_{p,q,y} B^{(c)}_{n,q}(x,y) = [n]_{p,q} B^{(s)}_{n-1,p,q}(x,py). \tag{3.37}$$

Corollary 5. *The following equations are valid*

$$\int_0^1 B^{(c)}_{2n,p,q}(px,y) d_{p,q}x = (-1)^n p^{\binom{2n}{2}} y^{2n},$$
$$\int_0^1 B^{(s)}_{2n+1,p,q}(px,y) d_{p,q}x = (-1)^{n+1} p^{\binom{2n+1}{2}} y^{2n+1},$$

which are proved by combining Proposition 4 and Corollary 2 using the definition of the (p,q)-integral.

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Authors' addresses

P. Njionou Sadjang
P. Njionou Sadjang, Faculty of Industrial Engineering, University of Douala, Douala, Cameroon
E-mail address: pnjionou@yahoo.fr

Ugur Duran
Ugur Duran, Department of Basic Science of Engineering, Faculty of Engineereing and, Natural Sciences, Iskenderun Technical University, TR-31200, Hatay, Turkey
E-mail address: mtdrnugur@gmail.com

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EMERGENCE OF CONSENSUS OF MULTI-AGENTS SYSTEMS ON TIME SCALES

EWA SCHMEIDEL, URSZULA OSTASZEWSKA, AND MAŁGORZATA ZDANOWICZ

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Abstract. In this paper an emergence of leader-following consensus on arbitrary time scales is investigated. It means that the step size is not necessarily constant but it is a function of time. We propose and prove conditions ensuring a leader-following consensus for discrete time scales. The presented results are illustrated by numerical examples.

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Keywords: time scales, leader-following problem, emergence of consensus, multi-agent systems, networked control systems, graph, Laplacian

1. Introduction

Flocking is a form of collective behaviour of a large number of interacting agents with a common group objective. Such a group is known as a multi-agent system. Examples of these multi-agent systems include crowds, bees, ants, birds, fish and penguins. Often in the group of agents there is one special agent which is called the leader. The leader is an agent, whose motion is independent of all the other agents and is followed by all the other ones. In this paper, we study model of behaviour of multi-agent system with the leader. The emergence of consensus is an important topic in multi-agent systems. The main idea is to drive a team of agents to reach an agreement on a certain issue by negotiating with the leader and with their neighbours. Although, each individual agent has limited processing power, the interconnected system as a whole can perform complex tasks in a coordinated fashion. Agents share their states with their neighbours via a chaotic communication network. The model describes the information transfers between agents particularly between agents and the leader. The model could be applicable for describing negotiation process in a group of number of people in which one of them is more important than the others, as well as for describing movement of group of animals with the leader. The origins of the investigation of the leader-following problem date back to the 1970s.

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In 1974 [9], DeGroot studied explicitly described process leading to the consensus. The general objective of the paper published by French in 1977 [10], was to enhance the understanding of organizational power by constructing a formal theory of power as a process characterizing the relations among organizations. In 2000, Krause [15] proposed the model of a group of experts who have to make a joint assessment of a certain magnitude. Each of the experts has his own opinion but is open to some extent to revise it when being informed about the opinions of all the other experts. Coordination of groups of mobile autonomous agents using nearest neighbour rules was studied by Jadbabaie et al. in [14]. In [3, 4], Blondel et al. investigated Krause's multi-agent consensus model with state-dependent connectivity. Girejko et al. studied Krause's model of opinion dynamics on isolated time scales [11, 13]. In 2007, Cucker and Smale [7, 8] published two papers devoted to an emergent behaviour in flocks. The authors provided the model describing the evolution of a flock for both continuous and discrete time. Cucker-Smale model on isolated time scales is studied by Girejko et al. in [13]. In 2015, Wang et al. [19] studied the leader-following consensus of discrete time linear multi-agent systems with communication noises. Recently, in 2018, Girejko, Machado, Malinowska, and Martins, have published some results for consensus in the Cucker-Smale type model on isolated time scales [12]. In [2], using the time scale theory, Babenko et al. investigated the leader-follower consensus problem for high-order multi-agent systems with inherent non-linear dynamics evolving on an arbitrary time domain. The authors obtained some sufficient conditions to guarantee that the tracking errors exponentially converge to zero using the concept of matrix-valued Lyapunov functions.

Notice that an interaction topology in the multi-agent system is modelled by undirected or directed graph. Our analysis framework is based on tools from matrix theory, algebraic graph theory and time scales theory.

2. Basis of time scales calculus

A time scale is a model of time [1, 5, 6], where the step size is a function of time. From mathematical point of view it is an arbitrary nonempty closed subset $\mathbb{T}$ of the set $\mathbb{R}$ of real numbers.

The mapping $\sigma : \mathbb{T} \to \mathbb{T}$, defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ with $\inf \varnothing = \sup \mathbb{T}$, is called the forward jump operator. Similarly, we define the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ with $\sup \varnothing = \inf \mathbb{T}$. The following classification of points is used within the theory: a point $t \in \mathbb{T}$ is called right-dense, right-scattered, left-dense and left-scattered if $\sigma(t) = t$ (for $t < \sup \mathbb{T}$), $\sigma(t) > t$, $\rho(t) = t$ (for $t > \inf \mathbb{T}$) and $\rho(t) < t$, respectively. We say that t is isolated if $\rho(t) < t < \sigma(t)$, and that t is dense if $\rho(t) = t = \sigma(t)$. The function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$ and called the graininess function. The delta (or Hilger) derivative of

$f:\mathbb{T}\to\mathbb{R}$ at a point $t\in\mathbb{T}^\kappa$, where

$$\mathbb{T}^\kappa := \begin{cases} \mathbb{T}\setminus(\rho(\sup\mathbb{T}),\sup\mathbb{T}] & \text{if } \sup\mathbb{T}<\infty \\ \mathbb{T} & \text{if } \sup\mathbb{T}=\infty \end{cases},$$

is defined in the following way.

Definition 1 ([5])**.** The delta derivative $f^\Delta(t)$ is the number (provided it exists) with the property that given any $\varepsilon>0$, there is a neighbourhood U of t (i.e., $U=(t-\delta,t+\delta)\cap\mathbb{T}$ for some $\delta>0$) such that

$$|(f(\sigma(t))-f(s))-f^\Delta(t)(\sigma(t)-s)|\le\varepsilon|\sigma(t)-s| \quad \text{for all} \quad s\in U.$$

The following definitions will be used in the sequel, too.

Definition 2 ([5])**.** A function $f:\mathbb{T}\to\mathbb{R}$ is called regulated provided its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. A function $f:\mathbb{T}\to\mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-side limits exist (finite) at left-dense points in $\mathbb{T}$.

Definition 3 ([5])**.** Assume $f:\mathbb{T}\to\mathbb{R}$ is a regulated function. We define the indefinite integral of a regulated function f by $\int f(t)\Delta t=F(t)+C$, where C is an arbitrary constant and F is a pre-antiderivative of f. We define the Cauchy integral by $\int_a^b f(t)\Delta t=F(b)-F(a)$ for all $a,b\in\mathbb{T}$.

Definition 4 ([5])**.** We say that a function $p:\mathbb{T}\to\mathbb{R}$ is regressive provided $1+\mu(t)p(t)\neq 0$ holds for all $t\in\mathbb{T}^\kappa$. The set of all regressive and rd-continuous functions $p:\mathbb{T}\to\mathbb{R}$ is denoted by $\mathcal{R}$. The set of all positively regressive elements of $\mathcal{R}$, is defined as $\mathcal{R}^+:=\{p\in\mathcal{R}: 1+\mu(t)p(t)>0 \text{ for all } t\in\mathbb{T}\}$.

An $N\times N$-matrix-valued function P on a time scale $\mathbb{T}$ is called regressive (with respect to $\mathbb{T}$) provided

$$I+\mu(t)P(t) \ \text{ is invertible for all } \ t\in\mathbb{T}^\kappa,$$

where by I we denote the $N\times N$ identity matrix.

Notice that, constant $N\times N$ matrix P is regressive iff the eigenvalues λ_i of P are regressive for all $1\le i\le N$.

The Grönwall inequality is used in the proof of the main result.

Lemma 1 ([5])**.** *Let y be rd-continuous, $p\in\mathcal{R}^+$ and $p(t)\ge 0$ for $t\in\mathbb{T}$ and $C\in\mathbb{R}$. Then*

$$y(t)\le C+\int_{T_0}^t p(\tau)y(\tau)\Delta\tau \textit{ for all } t\in\mathbb{T}$$

implies

$$y(t)\le Ce_p(t,T_0).$$

Here $e_p(t, T_0)$, $T_0 \in \mathbb{T}$, *is a solution of the initial value problem* $y^{\Delta}(t) = p(t)y(t)$, $y(T_0) = 1$ *on* $\mathbb{T}$.

Through this paper, assume that

$$\inf \mathbb{T} = T_0 \text{ and } \sup \mathbb{T} = \infty.$$

It implies that $\mathbb{T}^{\kappa} = \mathbb{T}$.

3. Sufficient conditions for consensus

3.1. *Adjacency matrix*

The multi-agent system can be modelled by directed or undirected graph. A graph is an object that consists of a non-empty set of vertices and another set of edges. In the graph theory an adjacency matrix is a square matrix used to represent a finite graph. We construct an $N \times N$ adjacency matrix A associated to the graph as follows: if there is an edge from node i to node j, then we put 1 as the entry on row i, column j of the matrix A, $i, j = 1, 2, \ldots, N$. Diagonal matrix $D = \mathrm{diag}[d_1, d_2, \ldots, d_N]$ describes communication between the leader and agents. Entries d_i, $i = 1, 2, \ldots, N$ are positive when there exists information exchange between i-th agent and the leader, and $d_i = 0$ otherwise.

Hermitian matrix (or self-adjoint matrix) is a square matrix that is equal to its own conjugate transpose. Obviously, any symmetric real matrix is Hermitian matrix. If a square matrix equals the multiplication of a matrix and its conjugate transpose, then is Hermitian matrix. The matrix norm induced by the Euclidean norm coincides with the spectral norm and it is submultiplicative norm. Throughout this paper the matrix norm means the spectral norm. If P is an $N \times N$ real matrix, then $\|P\| = \sqrt{\rho(PP^T)}$ where $\rho(PP^T)$ is the largest absolute value of eigenvalues of matrix PP^T. Moreover, if P is Hermitian, then $\|P\| = \rho(P)$. Symmetric and Hermitian matrices have the property of being always diagonalizable.

Example 1. We consider multi-agent system consisting of the leader and the five agents. In Figure 1 and 2 the topology of the system is given by undirected and directed graph, respectively. The adjacency matrices are the following.

$$\text{Figure 1:} \quad A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}; \text{ Figure 2:} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

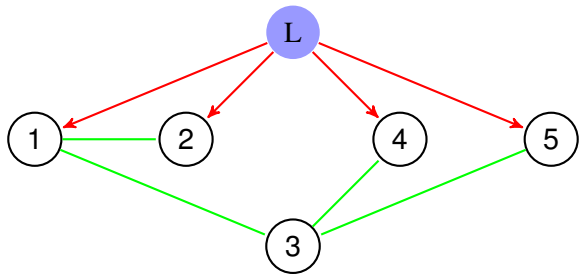

FIGURE 1. The topology of the leader-following multi-agent system under the undirected graph

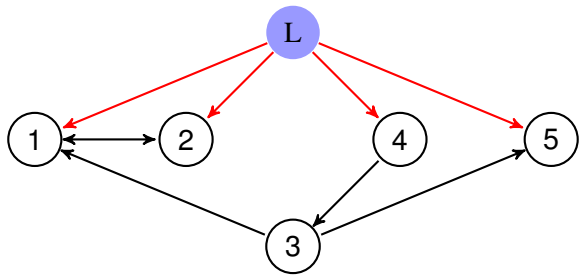

FIGURE 2. The topology of the leader-following multi-agent system under the directed graph

We see that the matrices which represent undirected graph are symmetric. For both graphs we can put for example

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

In this paper, more general adjacency matrix is used which is called weighted adjacency matrix. In weighted adjacency matrix entries are non-negative real numbers. Let a_{ij} $(i, j = 1, \ldots, N)$ denote the entries of the weighted adjacency matrix A associated with fixed graph by which the multi-agent systems are modelled. For an undirected graph, A is symmetric. By L we mean the Laplacian matrix $L = [l_{ij}]$ with $l_{ii} = \sum_{j \neq i} a_{ij}$ and $l_{ij} = -a_{ij}$, $i, j = 1, \ldots, N$, $i \neq j$.

3.2. *Mathematical model of agents dynamics*

We consider a discrete time multi-agent system consisting of N agents and the leader. The dynamics of each agent labelled i, $i = 1,2,\dots N$, is given by the following equation

$$x_i^{\Delta}(t) = f(t,x_i(t)) + \gamma \sum_{j=1}^{N} a_{ij}(x_j(t) - x_i(t)) + \gamma d_i(x_0(t) - x_i(t)),\ t \in \mathbb{T}, \quad (3.1)$$

where $x_i(t)$ represents the state at time t and γ is a feedback control gain. Function $f\colon \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ describes non-linear dynamics. The leader, labelled as $i = 0$, for multi-agent system (3.1) is an isolated agent with trajectory described by

$$x_0^{\Delta}(t) = f(t,x_0(t)),\ t \in \mathbb{T}. \quad (3.2)$$

Notice that the control law $\gamma \sum_{j=1}^{N} a_{ij}(x_j(t) - x_i(t)) + \gamma d_i(x_0(t) - x_i(t))$ for i-th agent used in system (3.1)-(3.2) was studied by many authors including Yu, Jiang and Hu in [20].

Definition 5. The multi-agent system (3.1)–(3.2) is said to be achieved the leader-following consensus if a solution to (3.1)–(3.2) satisfies

$$\lim_{t\to\infty} (x_i(t) - x_0(t)) = 0,\ i = 1,2,\dots,N,\ t \in \mathbb{T},$$

for any initial conditions $x_i(T_0) \in \mathbb{R}$, $i = 0,1,2,\dots,N$, $T_0 \in \mathbb{T}$.

Let us denote by $\varepsilon_i(t) = x_i(t) - x_0(t)$ the distance between the leader and the i-th agent. From (3.1)–(3.2) we obtain

$$\varepsilon_i^{\Delta}(t) = f(t,x_i(t)) - f(t,x_0(t)) + \gamma \sum_{j=1}^{N} a_{ij}(\varepsilon_j(t) - \varepsilon_i(t)) - \gamma d_i \varepsilon_i(t)$$

for $i = 1,2,\dots,N$. Setting

$$\varepsilon(t) = \big(\varepsilon_1(t), \varepsilon_2(t), \dots, \varepsilon_N(t)\big)^T,$$
$$x(t) = \big(x_1(t), x_2(t), \dots, x_N(t)\big)^T$$

and

$$F(t,x(t)) = \big(f(t,x_1(t)), f(t,x_2(t)), \dots, f(t,x_N(t))\big)^T,$$
$$F(t,x_0(t)\mathbb{1}) = \big(f(t,x_0(t)), f(t,x_0(t)), \dots, f(t,x_0(t))\big)^T,$$

system (3.1)–(3.2) takes the following form

$$\varepsilon^{\Delta}(t) = F(t,x(t)) - F(t,x_0(t)\mathbb{1}) - \gamma B \varepsilon(t), \quad (3.3)$$

where $B = L + D$ (for proof see [18]). Here $\mathbb{1}$ is the vector $[1,\dots,1]^T$.

3.3. *Main results*

We assume that function $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ satisfies Lipschitz condition with respect to the second variable in the meaning of following definition.

Definition 6. We say that function $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ fulfills Lipschitz condition if there exists a positive constant $\mathcal{L}$ such that

$$|f(t,x(t)) - f(t,y(t))| \leq \mathcal{L}|x(t) - y(t)|, \ t \in \mathbb{T}. \tag{3.4}$$

Let $P \in \mathcal{R}$. By $e_P(t,T_0)$ we denote the unique solution of initial value problem

$$y^{\Delta}(t) = Py(t), \ y(T_0) = \mathbb{1}.$$

So, for regressive $-\gamma B$, function $e_{-\gamma B}(t,T_0)$ is the solution of initial value problem

$$\varepsilon^{\Delta}(t) = -\gamma B\varepsilon(t), \ \varepsilon(T_0) = \mathbb{1}.$$

Theorem 1. *Assume that* $\mathbb{T} = \mathbb{N}$ *and condition* (3.4) *is satisfied. If*

$$\textit{each eigenvalue of matrix } (-\gamma B) \textit{ is regressive with respect to time scale } \mathbb{N}, \tag{3.5}$$

and

$$\textit{spectral norm of matrix } (I - \gamma B) \textit{ is less than } 1 - \mathcal{L}, \tag{3.6}$$

where $\mathcal{L}$ *is Lipschitz constant, then system* (3.1)–(3.2) *achieves the leader-following consensus.*

Proof. Condition (3.5) implies that matrix $(-\gamma B) \in \mathcal{R}$.

By $\mathcal{M}$ we denote the spectral norm of matrix $(I - \gamma B)$. Thus, by (3.6), we have

$$\mathcal{M} + \mathcal{L} < 1. \tag{3.7}$$

By variation of constants (see [5]), the unique solution of equation (3.3) with initial condition $\varepsilon(T_0)$, for $t \geq T_0$, is given by

$$\varepsilon(t) = e_{-\gamma B}(t,T_0)\varepsilon(T_0) + \int_{T_0}^{t} e_{-\gamma B}(t,\sigma(\tau))\big(F(\tau,x_0(\tau)\mathbb{1}) - F(\tau,x(\tau))\big)\Delta\tau.$$

Using condition (3.4), we obtain

$$\|\varepsilon(t)\| \leq \|\varepsilon(T_0)\| \, \|e_{-\gamma B}(t,T_0)\| + \int_{T_0}^{t} \mathcal{L}\|\varepsilon(\tau)\| \|e_{-\gamma B}(t,\sigma(\tau))\| \Delta\tau. \tag{3.8}$$

On time scale $\mathbb{T} = \mathbb{N}$ function $e_{-\gamma B}(t,T_0) = (I - \gamma B)^{t-T_0}$. This implies

$$\|e_{-\gamma B}(t,T_0)\| = \|(I - \gamma B)^{t-T_0}\| \leq \|I - \gamma B\|^{t-T_0} = \mathcal{M}^{t-T_0}.$$

From the above and (3.8), putting $\sigma(\tau) = \tau + 1$, we get

$$\|\varepsilon(t)\| \leq \|\varepsilon(T_0)\| \mathcal{M}^{t-T_0} + \int_{T_0}^{t} \mathcal{L}\|\varepsilon(\tau)\| \mathcal{M}^{t-\tau-1} \Delta\tau$$

and

$$\mathcal{M}^{-t}\|\varepsilon(t)\| \leq \|\varepsilon(T_0)\| \mathcal{M}^{-T_0} + \int_{T_0}^{t} \mathcal{L}\mathcal{M}^{-1}\|\varepsilon(\tau)\| \mathcal{M}^{-\tau} \Delta\tau.$$

By Lemma 1, we obtain

$$\mathcal{M}^{-t}\|\varepsilon(t)\| \leq \|\varepsilon(T_0)\| \mathcal{M}^{-T_0} e_{\mathcal{L}\mathcal{M}^{-1}}(t, T_0).$$

Thus

$$\|\varepsilon(t)\| \leq \|\varepsilon(T_0)\| \mathcal{M}^{t-T_0}(1 + \mathcal{L}\mathcal{M}^{-1})^{t-T_0} = \|\varepsilon(T_0)\| (\mathcal{M} + \mathcal{L})^{t-T_0}.$$

Therefore, by (3.7)

$$0 \leq \lim_{t\to\infty} \|\varepsilon(t)\| \leq \lim_{t\to\infty} \|\varepsilon(T_0)\| (\mathcal{M} + \mathcal{L})^{t-T_0} = 0.$$

It implies the thesis of Theorem 1. □

Example 2. Let $\mathbb{T} = \mathbb{N}$. We consider a group of 5 followers and the leader with the following initial conditions: $x_0(T_0) = 5$, $x_1(T_0) = 3$, $x_2(T_0) = 14$, $x_3(T_0) = 16$, $x_4(T_0) = 10$ and $x_5(T_0) = 7$, where $T_0 = 1$, and the symmetric adjacency matrix (see Figure 1) and D given in Example 1. Let $f(x, y) = (0.2xy)/(2 + 3x^2)$ and $\gamma = 0.2$. Here $\mathcal{L} = 0.2$, $\mathcal{M} = 0,74$. Therefore, by Theorem 1, the leader-following consensus is achieved. The state trajectories x_i and ε_i, $i = 1,2,3,4,5$ are shown in Figure 3 and Figure 4, respectively. The trajectory of the leader x_0 is drown in red and trajectories of x_i, $i = 1,2,3,4,5$, in green, black, blue, magenta and pink.

Theorem 2. *Assume that $\mathbb{T} = h\mathbb{N}$, $h > 0$ and condition (3.4) is satisfied. If each eigenvalue of matrix $(-\gamma B)$ is regressive with respect to time scale $h\mathbb{N}$, and*

$$\textit{spectral norm of matrix } (I - h\gamma B) \textit{ is less than } 1 - h\mathcal{L}, \tag{3.9}$$

then the leader-following consensus holds for system (3.1)–(3.2).

Proof. Obviously matrix $(-\gamma B) \in \mathcal{R}$. Here, by $\mathcal{M}^*$ we denote spectral norm of matrix $(I - h\gamma B)$. Notice that, if $t, T_0 \in h\mathbb{N}$ then $(t - T_0)/h \in \mathbb{N}$. We have

$$e_{-\gamma B}(t, T_a 0) = (I - h\gamma B)^{\frac{t-T_0}{h}} \text{ and } \|e_{-\gamma B}(t, T_0)\| \leq (\mathcal{M}^*)^{\frac{t-T_0}{h}}.$$

Applying the same arguments as in the proof of Theorem 1 we get the thesis. □

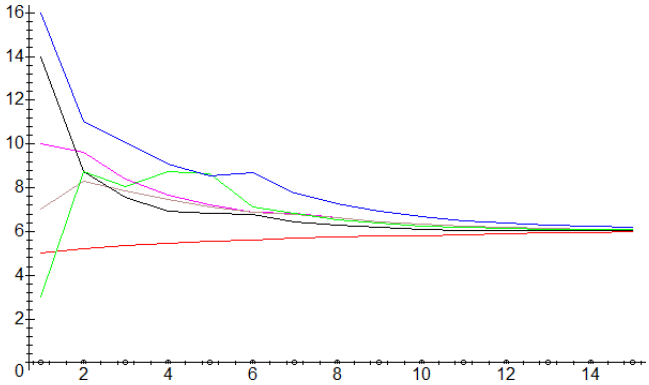

FIGURE 3. Trajectories x_i, $i = 0,1,2,3,4,5$

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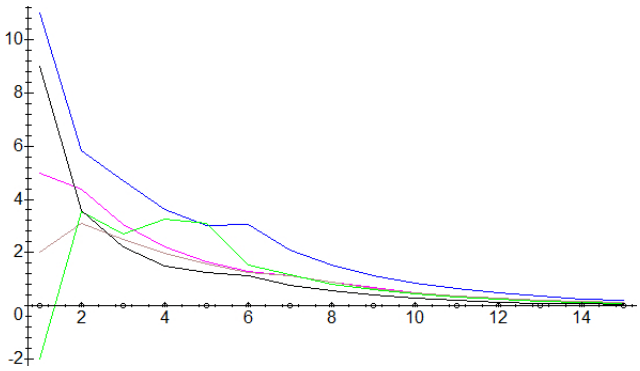

FIGURE 4. Trajectories ε_i, $i = 1,2,3,4,5$

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Example 3. Now let $\mathbb{T} = 0.5\,\mathbb{N}$ and $T_0 = 0.5$. As in Example 2, we consider 5 followers and the leader. We also assume the same initial conditions, the symmetric adjacency matrix, the matrix D, the function $f(x,y) = (0.2xy)/(2+3x^2)$ and $\gamma = 0.2$. Hence, we have $\mathcal{L} = 0.2$ and $\mathcal{M} \leq 0.87 < 1 - 0.5\,\mathcal{L} = 0.9$. According to Theorem 2, the group of agents reaches the consensus. The trajectories of the leader and his followers $(x_i, i = 0,1,2,3,4,5)$ and distances between them $(\varepsilon_i, i = 1,2,3,4,5)$ we draw in Figure 5 and in Figure 6, respectively.

Theorem 3. *Assume that time scale* $\mathbb{T}$ *is discrete and condition* (3.4) *is satisfied. If*

$$(-\gamma B) \text{ is regressive}$$

and

$$\text{spectral norm of matrix } (I - \mu(t)\gamma B) \text{ is less than } 1 - \mu(t)\mathcal{L},$$

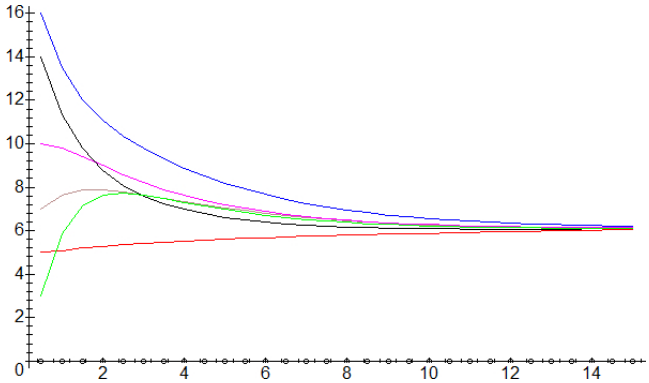

FIGURE 5. Trajectories x_i, $i = 0, 1, 2, 3, 4, 5$

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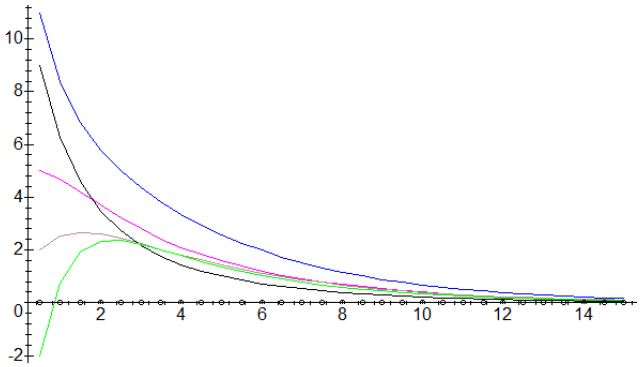

FIGURE 6. Trajectories ε_i, $i = 1, 2, 3, 4, 5$

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then the leader-following consensus holds for system (3.1)–(3.2).

Proof. Let us denote by $\mathcal{M}^{**}$ supremum of spectral norms of matrices $(I - \mu(t)\gamma B)$. Here

$$\begin{aligned}\|e_{-\gamma B}(t, T_0)\| &= \Big\| \prod_{s \in \mathbb{T} \cap [T_0, t)} (I - \mu(s)\gamma B) \Big\| \\ &\le \prod_{s \in \mathbb{T} \cap [T_0, t)} \|(I - \mu(s)\gamma B)\| \le \prod_{s \in \mathbb{T} \cap [T_0, t)} \mathcal{M}^{**}.\end{aligned}$$

For considered here time scales, inequality (3.8) implies

$$\|\varepsilon(t)\| \le \|\varepsilon(T_0)\| \prod_{s\in\mathbb{T}\cap[T_0,t)} \mathcal{M}^{**} + \int_{T_0}^{t} \mathcal{L}\|\varepsilon(\tau)\| \prod_{s\in\mathbb{T}\cap[\sigma(\tau),t)} \mathcal{M}^{**}\Delta\tau. \tag{3.10}$$

Hence, multiplying the both sides of inequality (3.10) by $\prod_{s\in\mathbb{T}\cap(T_0,t)} (\mathcal{M}^{**})^{-1}$, we obtain

$$\prod_{s\in\mathbb{T}\cap(T_0,t)} (\mathcal{M}^{**})^{-1}\|\varepsilon(t)\| \le \|\varepsilon(T_0)\|\mathcal{M}^{**} + \int_{T_0}^{t} \mathcal{L}(\mathcal{M}^{**})^{-1}\|\varepsilon(\tau)\| \prod_{s\in\mathbb{T}\cap(T_0,\tau)} (\mathcal{M}^{**})^{-1}\Delta\tau.$$

By Lemma 1, we obtain

$$\prod_{s\in\mathbb{T}\cap(T_0,t)} (\mathcal{M}^{**})^{-1}\|\varepsilon(t)\| \le \|\varepsilon(T_0)\|\mathcal{M}^{**} \prod_{s\in\mathbb{T}\cap[T_0,t)} \left(1+\mu(s)\mathcal{L}(\mathcal{M}^{**})^{-1}\right)$$

$$\|\varepsilon(t)\| \le \|\varepsilon(T_0)\| \prod_{s\in\mathbb{T}\cap[T_0,t)} \left(\mathcal{M}^{**}+\mu(s)\mathcal{L}\right).$$

By (3.9), we have $\mathcal{M}^{**}+\mu(s)\mathcal{L} < 1$ for any $s \in \mathbb{T}$. Thus

$$\lim_{t\to\infty} \|\varepsilon(t)\| \le \lim_{t\to\infty} \|\varepsilon(T_0)\| \prod_{s\in\mathbb{T}\cap[T_0,t)} \left(\mathcal{M}^{**}+\mu(s)\mathcal{L}\right) = 0.$$

It implies the thesis. □

In the following remark we present the most useful consequence of Theorem 3.

Remark 1. Assume that time scale $\mathbb{T}$ is discrete, graininess function $\mu(t) \in \{\mu_1,\mu_2,\dots,\mu_k\}$ and condition (3.4) is satisfied. If for any $i \in \{1,2,\dots,k\}$ each eigenvalue of matrix $(-\gamma B)$ is regressive with respect to considered time scale, and spectral norm of matrix $(I-\mu_i\gamma B)$ is less than $1-\mu_i\mathcal{L}$, then the leader-following consensus holds for system (3.1)–(3.2).

Proof. By $\mathcal{M}^{\max}$ we denote maximum of spectral norms of $(I-\mu_i\gamma B)$ for $i \in \{1,2,\dots,k\}$. Set $\mu^* = \max_{i\in\{1,2,\dots,k\}} \mu_i(t)$. Here

$$\|e_{-\gamma B}(t,T_0)\| \le \left(\mathcal{M}^{\max}\right)^{\left\lfloor \frac{t-T_0}{\mu^*}\right\rfloor} \text{ for } t \ge T_0.$$

□

Example 4. Consider the time scale

$$\mathbb{T} = \left\{ t : t = \frac{3n+1}{2} \text{ for } n = 2k, t = \frac{3n}{2} \text{ for } n = 2k+1, k \in \mathbb{N} \cup \{0\} \right\},$$

that is $\mathbb{T} = \{0.5, 1.5, 3.5, 4.5, 6.5, 7.5, \dots\}$. In this case we have $T_0 = 0.5$ and $\mu_1 = 1$, $\mu_2 = 2$. Again we observe the behavior of group of 5 followers and the leader with the initial conditions known from the previous examples: $x_0(T_0) = 5$, $x_1(T_0) = 3$, $x_2(T_0) = 14$, $x_3(T_0) = 16$, $x_4(T_0) = 10$ and $x_5(T_0) = 7$. We take the matrix D and the symmetric adjacency matrix given in Example 1. In the numerical computations we use $\gamma = 0.1$, $f(x,y) = (0.1xy)/(2+3x^2)$. Hence $\mathcal{L} = 0.1$ and

$$\|I - \mu_1 \gamma B\| \leq 0.87 < 1 - \mu_1 \mathcal{L} = 0.9,$$

$$\|I - \mu_2 \gamma B\| \leq 0.74 < 1 - \mu_2 \mathcal{L} = 0.8.$$

Finally, by Remark 1, the leader-following consensus is also achieved. Below, in Figure 7 we present trajectories of the leader and 5 agents ($x_i, i = 0,1,2,3,4,5$) and in Figure 8 we draw distances between the leader and 5 agents ($\varepsilon_i, i = 1,2,3,4,5$).

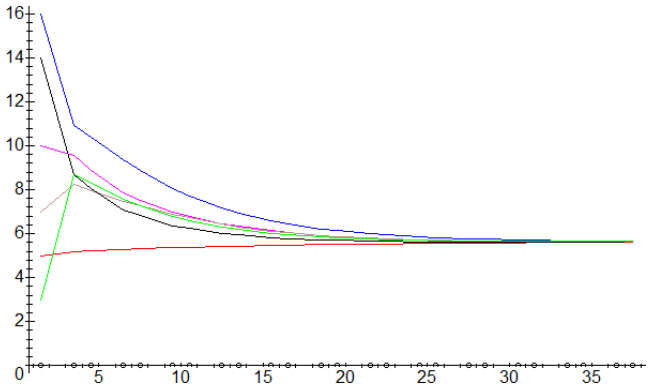

FIGURE 7. Trajectories x_i, $i = 0,1,2,3,4,5$

.

Remark 2. If there exists $T^* \in \mathbb{T}$ such that time scale considered in Theorem 3 is discrete for $t \geq T^*$, then the thesis of this Theorem also holds.

Analogous remarks are valid for Theorems 1 and 2.

The obtained result generalizes the result obtained by Malinowska, Schmeidel and Zdanowicz in [16] and by Ostaszewska, Schmeidel and Zdanowicz in [17]. In [17], the authors study the leader-following problem on discrete time scales with finite codomain of graininess function.

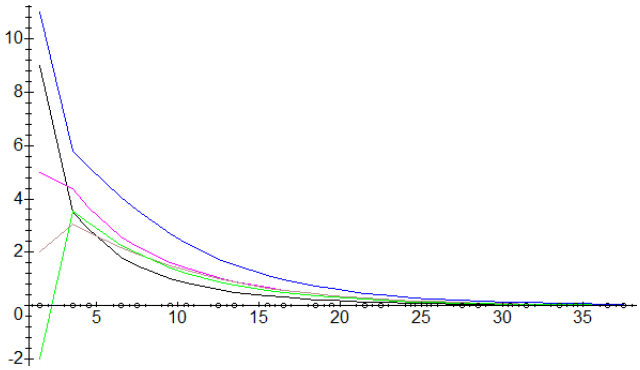

FIGURE 8. Trajectories ε_i, $i = 1,2,3,4,5$

.

CONCLUSION

In this paper, an emergency of the leader-following consensus of multi-agent systems is investigated on time scales. Based on the stability theory of systems on time scales and Grönwall inequality, sufficient conditions ensuring the leader-following consensus of the model are presented for discrete time scale. Finally, examples are given to demonstrate the effectiveness of the proposed method.

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Authors' addresses

Ewa Schmeidel
University of Bialystok, Institute of Mathematics, 1M Ciołkowskiego St., 15245 Białystok, Poland
E-mail address: eschmeidel@math.uwb.edu.pl

Urszula Ostaszewska
University of Bialystok, Institute of Mathematics, 1M Ciołkowskiego St., 15245 Białystok, Poland
E-mail address: uostasze@math.uwb.edu.pl

Małgorzata Zdanowicz
University of Bialystok, Institute of Mathematics, 1M Ciołkowskiego St., 15245 Białystok, Poland
E-mail address: mzdan@math.uwb.edu.pl

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ON THE TRANSLATION HYPERSURFACES WITH GAUSS MAP G SATISFYING $\Delta G = AG$

G. AYDIN ŞEKERCİ, S. SEVİNÇ, AND A.C. ÇÖKEN

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Abstract. It is a known fact that a translation hypersurface is obtained by combination of any three curves in the 4-dimensional Euclidean space. We examine a special situation where the Gauss map of a translation hypersurface satisfies the condition $\Delta G = AG$ where Δ represents the Laplace operator and A is a 4×4-real matrix. Our result is that such a translation hypersurface is one of the following three hypersurfaces: the hypersurface of translation surface and a constant vector along this surface, the hyperplane, the hypersurface $\Sigma \times \mathbb{R}$ where Σ is a translation surface.

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1. Introduction

It was Chen [5], at first, who defined the notation of finite type immersions in order to study the submanifolds in Euclidean and pseudo-Euclidean spaces. Later, the notation was expanded from the finite type immersions to smooth maps. Gauss map is one of the most important smooth map and it is frequently used in the studies of surfaces, submanifolds and etc [7, 8]. The Gauss map G renders each point of a surface to the unit normal vector of surface, and we consider the special case, where the condition $\Delta G = AG$ is satisfied.

Let M be a connected surface in the Euclidean 3-space E^3 and let G be the Gauss map given by $G : M \longrightarrow S^2 \subseteq E^3$ where S^2 is the unit sphere in E^3 which is centered at the origin. The basic connection between Gauss map and Laplacian is arised from surface M with constant mean curvature. For such surfaces, ΔG is equal to $\|dG\|^2 G$ where Δ is the Laplace operator with respect to the induced metric on M and d is a differential operator [9]. This can be considered as surfaces whose Gauss map is an eigenfuction of the Laplacian; that is, $\Delta G = \lambda G, \lambda \in \mathbb{R}$. Generalizing this equation to

$$\Delta G = AG, A \in Mat(3, \mathbb{R}) \tag{1.1}$$

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where $Mat(3,\mathbb{R})$ is the set of 3×3 real matrices, Dillen, Pas and Verstraelen [6] investigated surfaces of revolution in the Euclidean 3-space E^3 which satisfy condition (1.1). Then, they showed that such a surface is part of a plane, a sphere or a circular cylinder. Condition (1.1) is also used in further studies [3, 11]. Thus, the surfaces are classified by considering the relation between the Gauss map and the Laplacian.

Differently from the existing studies, our motivation in this work is to examine the translation hypersurfaces of Euclidean 4-space which satisfies the condition

$$\Delta G = AG, A \in Mat(4,\mathbb{R}). \tag{1.2}$$

We prove that such a translation hypersurface is one of the following three hypersurfaces: a hypersurface of translation surface and a constant vector along this surface, a hyperplane, a hypersurface $\Sigma \times \mathbb{R}$ where Σ is a translation surface. For the proof the Gauss map of translation hypersurface and its Laplacian are obtained firstly. Then, the translation hypersurfaces are classified by using the obtained equations. Our conclusions expand the results of [1, 4].

2. Translation hypersurfaces in 4-dimensional Euclidean space

In this section, by using some basic concepts, we will obtain ΔG for the translation hypersurfaces in Euclidean 4-space.

We assume that M is a hypersurface of a Euclidean 4-space $\mathbb{E}^4$ and $\overline{g}$ is an induced metric tensor to M. It is a translation hypersurface in $\mathbb{E}^4$ if it is given with an immersion

$$x : C \subseteq \mathbb{E}^3 \to \mathbb{E}^4, (u,v,z) \mapsto (u,v,z,\tilde{f}(u)+\tilde{g}(v)+\tilde{h}(z)) \tag{2.1}$$

where $\tilde{f},\tilde{g},\tilde{h}$ are smooth functions [10]. We denote the Gauss map by $G : M \to S^3$ where S^3 is the unit hypersphere of $\mathbb{E}^4$ [2].

The Laplacian Δ with respect to the induced metric tensor $\overline{g}$ is

$$\Delta = \frac{-1}{\sqrt{|\omega|}} \sum_{i,j} \frac{\partial}{\partial x_i}\left(\sqrt{|\omega|}\overline{g}^{ij}\frac{\partial}{\partial x^j}\right) \tag{2.2}$$

where $\omega = \det(\overline{g})$, $\overline{g}^{ij} = (\overline{g})^{-1}$ [11].

Differentiating $x(u,v,z)$ in (2.1) we get

$$x_u = (1,0,0,f),\ x_v = (0,1,0,g),\ x_z = (0,0,1,h),$$

where $f = \frac{d\tilde{f}}{du}$, $g = \frac{d\tilde{g}}{dv}$, $h = \frac{d\tilde{h}}{dz}$. For the induced metric on M we have

$$\begin{aligned}
&\overline{g}_{11} = \langle x_u, x_u\rangle = 1+f^2, && \overline{g}_{12} = \langle x_u, x_v\rangle = fg, && \overline{g}_{13} = \langle x_u, x_z\rangle = fh,\\
&\overline{g}_{21} = \langle x_v, x_u\rangle = fg, && \overline{g}_{22} = \langle x_v, x_v\rangle = 1+g^2, && \overline{g}_{23} = \langle x_v, x_z\rangle = gh,\\
&\overline{g}_{31} = \langle x_z, x_u\rangle = fh, && \overline{g}_{32} = \langle x_z, x_v\rangle = gh, && \overline{g}_{33} = \langle x_z, x_z\rangle = 1+h^2.
\end{aligned}$$

So in a matrix form

$$\overline{g} = \begin{bmatrix} 1+f^2 & fg & fh \\ fg & 1+g^2 & gh \\ fh & gh & 1+h^2 \end{bmatrix},$$

hence

$$\det(\overline{g}) = 1 + f^2 + g^2 + h^2.$$

For later use, we define the smooth function

$$\omega = \|x_u \wedge x_v \wedge x_z\|^2 = \langle (f,g,h,-1), (f,g,h,-1) \rangle,$$

and observe

$$\omega = 1 + f^2 + g^2 + h^2 = \det(\overline{g}) \tag{2.3}$$

Then the Gauss map G of hypersurface is

$$G = (G_1, G_2, G_3, G_4) = \frac{x_u \wedge x_v \wedge x_z}{\|x_u \wedge x_v \wedge x_z\|} = \frac{1}{\sqrt{\omega}}(f,g,h,-1). \tag{2.4}$$

From (2.2), we get

$$\Delta = \frac{-1}{\sqrt{\omega}} \left\{ \begin{array}{l} \frac{\partial}{\partial u}\left(\frac{\omega - f^2}{\sqrt{\omega}} \frac{\partial}{\partial u} - \frac{fg}{\sqrt{\omega}} \frac{\partial}{\partial v} - \frac{fh}{\sqrt{\omega}} \frac{\partial}{\partial z} \right) + \\ + \frac{\partial}{\partial v}\left(\frac{-fg}{\sqrt{\omega}} \frac{\partial}{\partial u} + \frac{\omega - g^2}{\sqrt{\omega}} \frac{\partial}{\partial v} - \frac{gh}{\sqrt{\omega}} \frac{\partial}{\partial z} \right) + \\ + \frac{\partial}{\partial z}\left(\frac{-fh}{\sqrt{\omega}} \frac{\partial}{\partial u} - \frac{gh}{\sqrt{\omega}} \frac{\partial}{\partial v} + \frac{\omega - h^2}{\sqrt{\omega}} \frac{\partial}{\partial z} \right) \end{array} \right\}.$$

By straightforward computations we obtain

$$\Delta = \frac{1}{\omega} \left\{ \begin{array}{l} \left(f^2 - \omega\right) \frac{\partial^2}{\partial u^2} + \left(g^2 - \omega\right) \frac{\partial^2}{\partial v^2} + \left(h^2 - \omega\right) \frac{\partial^2}{\partial z^2} + \\ + 2fg \frac{\partial^2}{\partial u \partial v} + 2fh \frac{\partial^2}{\partial u \partial z} + 2gh \frac{\partial^2}{\partial v \partial z} + \\ + \frac{1}{\omega} \left(\begin{array}{l} \left(\omega - f^2\right) f' + \left(\omega - g^2\right) g' \\ + \left(\omega - h^2\right) h' \end{array} \right) \left(f \frac{\partial}{\partial u} + g \frac{\partial}{\partial v} + h \frac{\partial}{\partial z} \right) \end{array} \right\}. \tag{2.5}$$

Now, we calculate ΔG_1, ΔG_2, ΔG_3 and ΔG_4.

Firstly, by considering (2.5) for $G_1 = \frac{f}{\sqrt{\omega}}$, we obtain

$$\Delta G_1 = \frac{1}{\omega^{\frac{7}{2}}} \left\{ \begin{array}{l} \left(\omega - f^2\right)^2 \left(4ff' - \omega f''\right) + \\ + f\left(g^2 - \omega\right)\left[4g^2\left(g'\right)^2 - \omega\left(\left(g'\right)^2 + gg''\right)\right] + \\ + f\left(h^2 - \omega\right)\left[4h^2\left(h'\right)^2 - \omega\left(\left(h'\right)^2 + hh''\right)\right] + \\ + \left(2f^2 - \omega\right) 4ff'\left(g^2 g' + h^2 h'\right) + \\ + \omega ff'\left(\omega - f^2\right)\left(g' + h'\right) + 6g^2 h^2 h' g' f - \\ - fg'h'\left[\left(\omega - g^2\right)h^2 + \left(\omega - h^2\right)g^2\right] \end{array} \right\}. \tag{2.6}$$

Secondly, by considering (2.5) for $G_2 = \frac{g}{\sqrt{\omega}}$ we get

$$\Delta G_2 = \frac{1}{\omega^{\frac{7}{2}}} \left\{ \begin{array}{l} g\left(f^2 - \omega\right)\left[4\left(f'\right)^2 f^2 - \omega\left(\left(f'\right)^2 + f''f\right)\right] + \\ + \left(\omega^2 - g^2\right)^2\left(4g\left(g'\right)^2 - \omega g''\right) + \\ + g\left(h^2 - \omega\right)\left[4h^2\left(h'\right)^2 - \omega\left(\left(h'\right)^2 + hh''\right) + \right] \\ + \left(2g^2 - \omega\right) 4gg'\left(f'f^2 + h^2 h'\right) + \\ + gg'\omega\left(\omega - g^2\right)\left(f' + h'\right) + 6f^2 h^2 g h' f' - \\ - f'h'g\left(h^2\left(\omega - f^2\right) + f^2\left(\omega - h^2\right)\right) \end{array} \right\}. \tag{2.7}$$

Then, by considering (2.5) for $G_3 = \frac{h}{\sqrt{\omega}}$ we obtain

$$\Delta G_3 = \frac{1}{\omega^{\frac{7}{2}}} \left\{ \begin{array}{l} h\left(f^2 - \omega\right)\left[4\left(f'\right)^2 f^2 - \omega\left(\left(f'\right)^2 + f''f\right)\right] + \\ + h\left(g^2 - \omega\right)\left[4g^2\left(g'\right)^2 - \omega\left(\left(g'\right)^2 + gg''\right)\right] + \\ + \left(\omega - h^2\right)^2\left(4h\left(h'\right)^2 - \omega h''\right) + \\ + 4hh'\left(2h^2 - \omega\right)\left(g'g^2 + f'f^2\right) + \\ + \omega hh'\left(\omega - h^2\right)\left(f' + g'\right) + 6f^2 g^2 h f' g' - \\ - hf'g'\left[\left(\omega - f^2\right)g^2 + \left(\omega - g^2\right)f^2\right] \end{array} \right\}. \tag{2.8}$$

Finally, by considering (2.5) for $G_4 = \frac{-1}{\sqrt{\omega}}$ we get

$$\Delta G_4 = \frac{1}{\omega^{\frac{7}{2}}} \left\{ \begin{array}{l} \left(f^2-\omega\right)\left[\left(f'\right)^2\left(\omega-f^2\right)+ff''\omega-3f^2\left(f'\right)^2-\right. \\ \left.-f'\left(g^2g'+h^2h'\right)\right]+\left(g^2-\omega\right)\left[\left(g'\right)^2\left(\omega-g^2\right)+\right. \\ +gg''\omega-3g^2\left(g'\right)^2-g'\left(f^2f'+h^2h'\right)]+ \\ +\left(h^2-\omega\right)\left[\left(h'\right)^2\left(\omega-h^2\right)+hh''\omega-3h^2\left(h'\right)^2-\right. \\ \left.-h'\left(f^2f'+g^2g'\right)\right]- \\ -6\left(f^2g^2f'g'+f^2h^2f'h'+g^2h^2g'h'\right) \end{array} \right\}. \tag{2.9}$$

So, the left-hand side of condition (1.2), which we are particularly interested in, is obtained for translation hypersurfaces.

3. Main result

Now, let us consider the translation hypersurface M which satisfies the condition $\Delta G = AG$, for a $4\times 4-$real matrix A.

Theorem 1. *A translation hypersurface in Euclidean 4-space whose Gauss map satisfies (1.2) is one of the following hypersurfaces:*

(1) *the hypersurface which consists of translation surface and a constant vector along this surface,*
(2) *the hyperplane,*
(3) *the hypersurface $\Sigma\times\mathbb{R}$, where Σ is a translation surface.*

Proof. By (1.2), we can write ΔG in the matrix form

$$\begin{bmatrix} \Delta G_1 \\ \Delta G_2 \\ \Delta G_3 \\ \Delta G_4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix}. \tag{3.1}$$

When the component G_i of the Gauss map are substituted in (3.1), ΔG_1 is obtained as

$$\Delta G_1 = a_{11}\frac{f}{\sqrt{\omega}} + a_{12}\frac{g}{\sqrt{\omega}} + a_{13}\frac{h}{\sqrt{\omega}} - a_{14}\frac{1}{\sqrt{\omega}}. \tag{3.2}$$

If we substitute (2.6) into (3.2) and then multiply both sides by $\sqrt{\omega}$, we find that

$$\begin{aligned} &\left(\omega-f^2\right)^2\left(4f\left(f'\right)^2-\omega f''\right)+f\left(g^2-\omega\right)\left[4g^2\left(g'\right)^2-\omega\left(\left(g'\right)^2+gg''\right)\right]+ \\ &+f\left(h^2-\omega\right)\left[4h^2\left(h'\right)^2-\omega\left(\left(h'\right)^2+hh''\right)\right]+4ff'\left(2f^2-\omega\right)\left(g^2g'+h^2h'\right)+ \\ &+\omega ff'\left(\omega-f^2\right)\left(g'+h'\right)+6g^2h^2h'g'f-fg'h'\left[\left(\omega-g^2\right)h^2+\left(\omega-h^2\right)g^2\right] \\ &=\omega^3\left(a_{11}f+a_{12}g+a_{13}h-a_{14}\right). \end{aligned} \tag{3.3}$$

Similarly, substitution of (2.7) into (3.1) and then multiplying both sides by $\sqrt{\omega}$, results in

$$\begin{aligned}&\left(\omega-g^2\right)^2\left(4g\left(g'\right)^2-\omega g''\right)+g\left(f^2-\omega\right)\left[4f^2\left(f'\right)^2-\omega\left(\left(f'\right)^2+ff''\right)\right]+\\&+g\left(h^2-\omega\right)\left[4h^2\left(h'\right)^2-\omega\left(\left(h'\right)^2+hh''\right)\right]+4gg'\left(2g^2-\omega\right)\left(f'f^2+h'h^2\right)+\\&+gg'\omega\left(\omega-g^2\right)\left(f'+h'\right)+6f^2h^2gh'f'-f'h'g\left[h^2\left(\omega-f^2\right)+f^2\left(\omega-h^2\right)\right]\\&=\omega^3\left(a_{21}f+a_{22}g+a_{23}h-a_{24}\right).\end{aligned}\tag{3.4}$$

According to the matrix form in (3.1), substitution of (2.8), (2.9) into (3.1) and then multiplying both sides by $\sqrt{\omega}$, we get

$$\begin{aligned}&\left(\omega-h^2\right)^2\left(4h\left(h'\right)^2-\omega h''\right)+h\left(f^2-\omega\right)\left[4f^2\left(f'\right)^2-\omega\left(\left(f'\right)^2+ff''\right)\right]+\\&+h\left(g^2-\omega\right)\left[4g^2\left(g'\right)^2-\omega\left(\left(g'\right)^2+gg''\right)\right]+4hh'\left(2h^2-\omega\right)\left(g'g^2+f'f^2\right)+\\&+\omega hh'\left(\omega-h^2\right)\left(f'+g'\right)+6f^2g^2hf'g'-hf'g'\left[\left(\omega-f^2\right)g^2+\left(\omega-g^2\right)f^2\right]\\&=\omega^3\left(a_{31}f+a_{32}g+a_{33}h-a_{34}\right),\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}&\left(f^2-\omega\right)\left[\left(f'\right)^2\left(\omega-f^2\right)+ff''\omega-3f^2\left(f'\right)^2-f'\left(g'g^2+h'h^2\right)\right]+\\&+\left(g^2-\omega\right)\left[\left(g'\right)^2\left(\omega-g^2\right)+gg''\omega-3g^2\left(g'\right)^2-g'\left(f'f^2+h'h^2\right)\right]+\\&+\left(h^2-\omega\right)\left[\left(h'\right)^2\left(\omega-h^2\right)+hh''\omega-3h^2\left(h'\right)^2-h'\left(f'f^2+g'g^2\right)\right]-\\&-6\left(f^2g^2f'g'+f^2h^2f'h'+g^2h^2h'g'\right)\\&=\omega^3\left(a_{41}f+a_{42}g+a_{43}h-a_{44}\right),\end{aligned}\tag{3.6}$$

respectively.

Rearranging (3.6) so that the left-hand side of the equation is $\omega\left(f^2-\omega\right)ff''$, results in

$$\begin{aligned}\omega\left(f^2-\omega\right)ff''=&-\left(f^2-\omega\right)\left(\omega-f^2\right)\left(f'\right)^2+\left(f^2-\omega\right)3f^2\left(f'\right)^2+\\&+\left(f^2-\omega\right)f'\left(g^2g'+h^2h'\right)-\\&-\left(g^2-\omega\right)\left[\left(g'\right)^2\left(\omega-g^2\right)+gg''\omega-3g^2\left(g'\right)^2-\right.\\&\left.-g'\left(f^2f'+h^2h'\right)\right]-\\&-\left(h^2-\omega\right)\left[\left(h'\right)^2\left(\omega-h^2\right)+hh''\omega-3h^2\left(h'\right)^2-\right.\\&\left.-h'\left(f^2f'+g^2g'\right)\right]+\end{aligned}\tag{3.7}$$

$$+6\left(f^2g^2f'g'+f^2h^2f'h'+g^2h^2h'g'\right)+$$
$$+\omega^3\left(a_{41}f+a_{42}g+a_{43}h-a_{44}\right).$$

Then we arrange (3.3) so that the left-hand side of the equation is $-\omega\left(\omega-f^2\right)^2 f''$. After we multiply (3.3) by f and multiply (3.7) by $\left(\omega-f^2\right)$, we get

$$A_1\left(f'\right)^2+B_1f'=\Gamma_1, \tag{3.8}$$

where

$$\begin{aligned}
A_1&=\omega\left(\omega-f^2\right)^2, \qquad (3.9)\\
B_1&=\left(g^2g'+h^2h'\right)\left[\omega\left(3f^2-1\right)+f^2\left(f^2+1\right)\right],\\
\Gamma_1&=\omega\left(\omega-g^2\right)\left[-\left(\left(g'\right)^2\omega+gg''\omega\right)+\left(4g^2\left(g'\right)^2+h^2h'g'\right)\right]+\\
&\quad+\omega\left(\omega-h^2\right)\left[-\left(\left(h'\right)^2\omega+hh''\omega\right)+\left(4h^2\left(h'\right)^2+h'g^2g'\right)\right]-\\
&\quad-6\omega g^2h^2g'h'+f\omega^3\left(a_{11}f+a_{12}g+a_{13}h-a_{14}\right)-\\
&\quad-\left(\omega-f^2\right)\omega^3\left(a_{41}f+a_{42}g+a_{43}h-a_{44}\right).
\end{aligned}$$

Similarly, using equation (3.4) and (3.7) multiplied by $-g$, we get

$$A_2\left(f'\right)^2+B_2f'=\Gamma_2, \tag{3.10}$$

where

$$\begin{aligned}
A_2&=0, \qquad (3.11)\\
B_2&=\omega gg'\left(\omega-3f^2\right),\\
\Gamma_2&=\left(\omega-g^2\right)\left(g''\omega^2-g'\omega h'-3\omega g\left(g'\right)^2\right)-3gg'h^2h'\omega+\\
&\quad+g\omega^3\left(a_{41}f+a_{42}g+a_{43}h-a_{44}\right)+\\
&\quad+\omega^3\left(a_{21}f+a_{22}g+a_{23}h-a_{24}\right).
\end{aligned}$$

Using equation (3.5) and (3.7) multiplied by $-h$, we obtain

$$A_3\left(f'\right)^2+B_3f'=\Gamma_3, \tag{3.12}$$

where

$$\begin{aligned}
A_3&=0, \qquad (3.13)\\
B_3&=\omega hh'\left(4f^2-\omega\right)+h^2f^2\left(h^2-\omega\right)+h^3h'f^2,\\
\Gamma_3&=\left(\omega-h^2\right)\left(3h\left(h'\right)^2\omega-\omega^2h''\right)+\omega hh'g'\left(\omega-3g^2\right)-\\
&\quad-\omega^3\left(a_{31}f+a_{32}g+a_{33}h-a_{34}\right)-\\
&\quad-h\omega^3\left(a_{41}f+a_{42}g+a_{43}h-a_{44}\right).
\end{aligned}$$

So we have the system of equations

$$\begin{aligned} A_1\left(f'\right)^2 + B_1 f' &= \Gamma_1, \\ A_2\left(f'\right)^2 + B_2 f' &= \Gamma_2, \\ A_3\left(f'\right)^2 + B_3 f' &= \Gamma_3, \end{aligned} \tag{3.14}$$

with matrices

$$D = \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \\ A_3 & B_3 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ 0 & B_2 \\ 0 & B_3 \end{bmatrix} \tag{3.15}$$

and

$$E = \begin{bmatrix} A_1 & B_1 & \Gamma_1 \\ A_2 & B_2 & \Gamma_2 \\ A_3 & B_3 & \Gamma_3 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 & \Gamma_1 \\ 0 & B_2 & \Gamma_2 \\ 0 & B_3 & \Gamma_3 \end{bmatrix}. \tag{3.16}$$

For the solution of system (3.14), the rank of E must be equal to the rank of D. So $rankE \neq 3$ hence $B_2\Gamma_3 = \Gamma_2 B_3$. Then, in the case of $rankE = rankD = 1$ or $rankE = rankD = 2$, the solution of (3.14) should be examined.

CASE 1: Let $rankE = rankD = 1$. Then the determinants of $(2\times 2)-$ matrices of E vanish. These determinants are

(1) $\begin{vmatrix} A_1 & B_1 \\ 0 & B_2 \end{vmatrix} = \omega^2 g g' \left(\omega - f^2\right)^2 \left(\omega - 3f^2\right);$

(2) $\begin{vmatrix} A_1 & \Gamma_1 \\ 0 & \Gamma_2 \end{vmatrix} = \omega\left(\omega - f^2\right)^2 \Big\{\left(\omega - g^2\right)\left(g''\omega^2 - g'\omega h' - 3\omega g\left(g'\right)^2\right) -$

$- 3gg'h^2h'\omega + g\omega^3\left(a_{41}f + a_{42}g + a_{43}h - a_{44}\right) +$

$+ \omega^3\left(a_{21}f + a_{22}g + a_{23}h - a_{24}\right)\Big\};$

(3) $\begin{vmatrix} A_1 & B_1 \\ 0 & B_3 \end{vmatrix} = \omega\left(\omega - f^2\right)^2\left[\omega h h'\left(4f^2 - \omega\right) + h^2 f^2\left(h^2 - \omega\right) + h^3 h' f^2\right];$

(4) $\begin{vmatrix} A_1 & \Gamma_1 \\ 0 & \Gamma_3 \end{vmatrix} = \omega\left(\omega - f^2\right)^2 \Big\{\left(\omega - h^2\right)\left(3h\left(h'\right)^2\omega - \omega^2 h''\right) +$

$+ \omega h h' g'\left(\omega - 3g^2\right) - \omega^3\left(a_{31}f + a_{32}g + a_{33}h - a_{34}\right) -$

$- h\omega^3\left(a_{41}f + a_{42}g + a_{43}h - a_{44}\right)\Big\};$

(5) $\begin{vmatrix} B_1 & \Gamma_1 \\ B_2 & \Gamma_2 \end{vmatrix} = \left(g^2 g' + h^2 h'\right)\left[\omega\left(3f^2 - 1\right) + f^2\left(f^2 + 1\right)\right] \times$

$$\times\left\{\left(\omega-g^2\right)\left(g''\omega^2-g'\omega h'-3\omega g\left(g'\right)^2\right)-3gg'h^2h'\omega+\right.$$
$$+g\omega^3\left(a_{41}f+a_{42}g+a_{43}h-a_{44}\right)+$$
$$\left.+\omega^3\left(a_{21}f+a_{22}g+a_{23}h-a_{24}\right)\right\}-$$
$$-\omega gg'\left(\omega-3f^2\right)\left\{\omega\left(\omega-g^2\right)\left[-\left(\left(g'\right)^2\omega+gg''\omega\right)+\right.\right.$$
$$\left.+\left(4g^2\left(g'\right)^2+h^2h'g'\right)\right]+\omega\left(\omega-h^2\right)\left[-\left(\left(h'\right)^2\omega+hh''\omega\right)+\right.$$
$$\left.+\left(4h^2\left(h'\right)^2+h'g^2g'\right)\right]-6\omega g^2h^2g'h'+$$
$$+f\omega^3\left(a_{11}f+a_{12}g+a_{13}h-a_{14}\right)-$$
$$\left.-\left(\omega-f^2\right)\omega^3\left(a_{41}f+a_{42}g+a_{43}h-a_{44}\right)\right\};$$

(6) $$\begin{vmatrix} B_2 & \Gamma_2 \\ B_3 & \Gamma_3 \end{vmatrix}=\omega gg'\left(\omega-3f^2\right)\left\{\left(\omega-h^2\right)\left(3h\left(h'\right)^2\omega-\omega^2h''\right)+\right.$$
$$+\omega hh'g'\left(\omega-3g^2\right)-\omega^3\left(a_{31}f+a_{32}g+a_{33}h-a_{34}\right)-$$
$$\left.-h\omega^3\left(a_{41}f+a_{42}g+a_{43}h-a_{44}\right)\right\}-$$
$$-\left\{\omega hh'\left(4f^2-\omega\right)+h^2f^2\left(h^2-\omega+hh'\right)\right\}\times$$
$$\times\left\{\left(\omega-g^2\right)\left(g''\omega^2-g'\omega h'-3\omega g\left(g'\right)^2\right)-3gg'h^2h'\omega+\right.$$
$$+g\omega^3\left(a_{41}f+a_{42}g+a_{43}h-a_{44}\right)+$$
$$\left.+\omega^3\left(a_{21}f+a_{22}g+a_{23}h-a_{24}\right)\right\}.$$

Also, we have $A_1B_2=0$ and $A_1B_3=0$ for $rankD=1$. Since $rankE=rankD=1$, all determinants of $(2\times 2)-$ matrices of E are zero. When f is isolated by using the results of determinants, it is written by depending on the functions g,g',g'',h,h',h''. Since the functions g and h are independent from the parameter u of the function f, the derivative of f is zero. So, f is a constant.

CASE 2: Let $rankE=rankD=2$. Since $rankD=2$, we have

(1) $$\begin{vmatrix} A_1 & B_1 \\ 0 & B_2 \end{vmatrix}=\omega^2gg'\left(\omega-f^2\right)^2\left(\omega-3f^2\right);$$

(2) $$\begin{vmatrix} A_1 & B_1 \\ 0 & B_3 \end{vmatrix}=\omega\left(\omega-f^2\right)^2\left[\omega hh'\left(4f^2-\omega\right)+h^2f^2\left(h^2-\omega\right)+h^3h'f^2\right].$$

At least one of the products $A_1 B_2$ and $A_1 B_3$ is different from zero. Then, at least one of the determinants of $(2\times 2)-$matrices of E is different from zero and so, $rank E = 2$. For the system

$$\begin{aligned} A_1 \left(f'\right)^2 + B_1 f' &= \Gamma_1, \\ 0 + B_2 f' &= \Gamma_2, \end{aligned}$$

we have

$$f' = \frac{\Gamma_2}{B_2} \text{ and } \left(f'\right)^2 = \frac{\Gamma_1 B_2 - B_1 \Gamma_2}{A_1 B_2},$$

so

$$B_2 \left(\Gamma_1 B_2 - \Gamma_2 B_1\right) = A_1 \Gamma_2^2. \tag{3.17}$$

When we substitute (3.9) and (3.11) into (3.17) and isolate function f, we obtain a polynomial whose coefficients depend on the functions g, g', g'', h, h', h''. Since the functions g and h are independent from the parameter u of function f, the derivative of f is zero. So, f is a constant.

By considering Case 1 and 2, we have two cases: $f = 0$ or $f = c \neq 0$ where c is a constant.

If $f = 0$ then $\tilde{f}$ is a constant. So, this hypersurface has the form

$$x(u,v,z) = \left(u, v, z, c + \tilde{g}(v) + \tilde{h}(z)\right) \tag{3.18}$$

and it consists of a translation surface and a constant vector along it.

If $f = c \neq 0$, c is constant then $\tilde{f} = cu$ and we have

$$x(u,v,z) = \left(u, v, z, cu + \tilde{g}(v) + \tilde{h}(z)\right). \tag{3.19}$$

Thus from [2], we conclude that M is a hypersurface $\sum \times \mathbb{R}$ or a hyperplane. Here $\sum$ is a translation surface.

After that, similar conclusions, which we give for the function f, could be found for the functions g, h. So the desired result is obtained. □

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Authors' addresses

G. Aydın Şekerci
Süleyman Demirel University, Department of Mathematics, 32260 Isparta, Turkey
E-mail address: gulsahaydin@sdu.edu.tr

S. Sevinç
Adnan Menderes University, Department of Mathematics, 09010 Aydın, Turkey
E-mail address: ssevinc@adu.edu.tr

A.C. Çöken
Akdeniz University, Department of Mathematics, 07058 Antalya, Turkey
E-mail address: ceylancoken@akdeniz.edu.tr

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CHEBYSHEV TYPE INEQUALITIES FOR CONFORMABLE FRACTIONAL INTEGRALS

ERHAN SET, AHMET OCAK AKDEMIR, AND İLKER MUMCU

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Abstract. In this article, firstly some necessary definitions and results involving fractional integrals are given. Secondly, a new identity involving conformable fractional integrals is given. Then, by using this identity, we establish new Chebyshev inequalities for the Chebyshev functional via conformable fractional integral.

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1. Introduction

There are many different type integral inequalities in the literature such as classical and analytic inequalities. We will start with the most prominent one of classical inequalities that is called Chebyshev inequality.

This inequality was established by Chebyshev in [3] as following;

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \left\| f' \right\|_\infty \left\| g' \right\|_\infty, \tag{1.1}$$

where $f,g:[a,b]\to\mathbb{R}$ are absolutely continuous functions whose derivatives $f',g'\in L_\infty[a,b]$ and

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right), \tag{1.2}$$

which is called the Čebyšev functional, provided the integrals in (1.2) exist.
We will remind some well-known concepts as followings:
The beta function defined as follows [5, p18]:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt, \quad a,b>0,$$

where $\Gamma(\alpha) = \int_0^\infty e^{-t}u^{\alpha-1}du$ is Gamma function.

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Definition 1. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad x < b$$

respectively. Here $J^{0}_{a+}f(x) = J^{0}_{b-}f(x) = f(x)$.

In [2], Belarbi and Dahmani established following theorems for the Chebyshev inequalities.

Theorem 1. *Let f and g be two synchronous functions on $[0,\infty)$. Then for all $t > 0$, $\alpha > 0$, we have:*

$$J^{\alpha}(fg) \geq \frac{\Gamma(\alpha+1)}{t^{\alpha}} J^{\alpha}f(t)J^{\alpha}g(t). \tag{1.3}$$

Theorem 2. *Let f and g be two synchronous functions on $[0,\infty)$. Then for all $t > 0$, $\alpha > 0$, $\beta > 0$, we have:*

$$\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}(fg)(t) + \frac{t^{\beta}}{\Gamma(\beta+1)}J^{\alpha}(fg)(t) \geq J^{\alpha}f(t)J^{\beta}g(t) + J^{\beta}f(t)J^{\alpha}g(t) \tag{1.4}$$

Theorem 3. *Let $(f_i)_{i=1,\dots,n}$ be n positive inreasing functions on $[0,\infty)$. Then for any $t > 0$, $\alpha > 0$, we have*

$$J^{\alpha}\left(\prod_{i=1}^{n} f_i\right)(t) \geq (J^{\alpha}(1))^{1-n}\prod_{i=1}^{n} J^{\alpha}f_i(t.) \tag{1.5}$$

Theorem 4. *Let f and g be two functions defined on $[0,\infty)$, such that f is increrasing, g is differentiable and there exist a real number $m := inf_{t\geq 0}g^{'}(t)$. Then the inequality*

$$J^{\alpha}(fg)(t) \geq (J^{\alpha}(1))^{-1}J^{\alpha}f(t)J^{\alpha}g(t) - \frac{mt}{\alpha+1}J^{\alpha}f(t) + mJ^{\alpha}(tf(t)) \tag{1.6}$$

is valid for all $t > 0$, $\alpha > 0$.

In [4], Khalil et al. define a new well-behaved simple fractional derivative called conformable fractional derivative depending just on the basic limit definition of the derivative. They also defined the fractional integral of order $0 < \alpha \leq 1$ only.

In [1], Abdeljawad gave the definition of left and right conformable fractional integrals of any order $\alpha > 0$.

Definition 2. Let $\alpha \in (n, n+1]$ and set $\beta = \alpha - n$ then the left conformable fractional integral starting at a if order α is defined by

$$(I_\alpha^a f)(t) = \frac{1}{n!}\int_a^t (t-x)^n (x-a)^{\beta-1} f(x)dx$$

Analogously, the right conformable fractional integral is defined by

$$({}^b I_\alpha f)(t) = \frac{1}{n!}\int_t^b (x-t)^n (b-x)^{\beta-1} f(x)dx.$$

In [7], Set and Mumcu gave following Lemma

Lemma 1. *If $g : [a,b] \to \mathbb{R}$ is integrable and symmetric to $(a+b)/2$ with $a < b$, then*

$$I_\alpha^a g(b) = {}^b I_\alpha g(a) = \frac{1}{2}[{}^b I_\alpha g(a) + I_\alpha^a g(b)]$$

with $\alpha \in (n, n+1]$.

Notice that if $\alpha = n+1$ then $\beta = \alpha - n = n+1-n = 1$ and hence $(I_\alpha^a f)(t) = (J_{n+1}^a f)(t)$.
For some recent results related to conformable fractional integrals, see, e.g., [6,7].
The main purpose of this paper is to establish Chebyshev type inequalities for conformable fractional integral.

2. Chebyshev type inequalities

Lemma 2. *Let $f : [a,b] \to \mathbb{R}_+$ be function, where $0 \le a < b$. If $f \in L[a,b]$ then we get*

$$\int_0^1 t^n (1-t)^{\alpha-n-1} f(ta+(1-t)b)dt + \int_0^1 t^n (1-t)^{\alpha-n-1} f((1-t)a+tb)dt$$
$$= \frac{n!}{(b-a)^\alpha}[{}^b I_\alpha f(a) + I_\alpha^a f(b)].$$

for $\alpha \in (n, n+1]$, $n = 0, 1, 2, \ldots$.

Proof. By integration by parts, we can state

$$\begin{aligned}\int_0^1 t^n (1-t)^{\alpha-n-1} f(ta+(1-t)b)dt &= \\ &= \frac{1}{b-a}\int_a^b \left(\frac{b-x}{b-a}\right)^n \left(\frac{x-a}{b-a}\right)^{\alpha-n-1} f(x)dx \\ &= \frac{1}{(b-a)^\alpha}\int_a^b (b-x)^n (x-a)^{\alpha-n-1} f(x)dx \\ &= \frac{n!}{(b-a)^\alpha}(I_\alpha^a f)(b) \qquad (2.1)\end{aligned}$$

and

$$\begin{aligned}\int_0^1 t^n(1-t)^{\alpha-n-1} f((1-t)a+tb)dt &= \\ &= \frac{1}{b-a}\int_a^b \left(\frac{x-a}{b-a}\right)^n \left(\frac{b-x}{b-a}\right)^{\alpha-n-1} f(x)dx \\ &= \frac{1}{(b-a)^\alpha}\int_a^b (x-a)^n(b-x)^{\alpha-n-1} f(x)dx \\ &= \frac{n!}{(b-a)^\alpha}({}^bI_\alpha f)(a). \end{aligned} \tag{2.2}$$

Adding (2.1) and (2.2), we get desired result. □

Theorem 5. *Let $f,g:[a,b]\to\mathbb{R}$ are two monotonic functions of the same monotonicity, $f,g\in L[a,b]$ and $\alpha\in(n,n+1]$ $(n=0,1,2,...)$, then the following inequality holds:*

$$\begin{aligned}&\frac{\Gamma(\alpha+1)\Gamma(\alpha-n)}{2(b-a)^\alpha})[{}^bI_\alpha(fg)(a)+I_\alpha^a(fg)(b)] \\ &\geq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[{}^bI_\alpha f(a)+I_\alpha^a f(b)]\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[{}^bI_\alpha g(a)+I_\alpha^a g(b)]. \end{aligned} \tag{2.3}$$

Proof. Our proof is based on the property of positivity of the integral. The basic remark is the inequality

$$(f(x)-f(y))(g(x)-g(y))\geq 0$$

for all $x,y\in[a,b]$. For $x=ta+(1-t)b$, $t\in[0,1]$ we obtain

$$(f(ta+(1-t)b)-f(y))(g(ta+(1-t)b)-g(y))\geq 0. \tag{2.4}$$

Similarly, for $x=(1-t)a+tb$, $t\in[0,1]$ we have

$$(f((1-t)a+tb)-f(y))(g((1-t)a+tb)-g(y))\geq 0. \tag{2.5}$$

Adding the inequalities (2.4) and (2.5) , then multiplying both sides with $t^n(1-t)^{\alpha-n-1}$ and integrating with respect to t over $[0,1]$ we have:

$$\begin{aligned}&\int_0^1 t^n(1-t)^{\alpha-n-1}(fg)(ta+(1-t)b)dt \\ &+\int_0^1 t^n(1-t)^{\alpha-n-1}(fg)((1-t)a+tb)dt+2B(n+1,\alpha-n)f(y)g(y) \\ &\geq g(y)\Bigg(\int_0^1 t^n(1-t)^{\alpha-n-1}f(ta+(1-t)b)dt \\ &+\int_0^1 t^n(1-t)^{\alpha-n-1}f((1-t)a+tb)dt\Bigg)\end{aligned}$$

$$+ f(y)\left(\int_0^1 t^n(1-t)^{\alpha-n-1}g(ta+(1-t)b)dt\right.$$
$$\left.+\int_0^1 t^n(1-t)^{\alpha-n-1}g((1-t)a+tb)dt\right). \tag{2.6}$$

We apply Lemma 2 and multiplying both sides by $\frac{1}{n!}$

$$\frac{1}{(b-a)^\alpha}[{}^bI_\alpha(fg)(a)+I_\alpha^a(fg)(b)]+\frac{2}{n!}B(n+1,\alpha-n-1)f(y)g(y)$$
$$\geq \frac{g(y)}{(b-a)^\alpha}[{}^bI_\alpha f(a)+I_\alpha^a f(b)]+\frac{f(y)}{(b-a)^\alpha}[{}^bI_\alpha g(a)+I_\alpha^a g(b)]. \tag{2.7}$$

We put $y=ta+(1-t)b$, $t\in[0,1]$ in (2.7), whence it follows

$$\frac{1}{(b-a)^\alpha}[{}^bI_\alpha(fg)(a)+I_\alpha^a(fg)(b)]+\frac{2}{n!}B(n+1,\alpha-n-1)(fg)(ta+(1-t)b)$$
$$\geq \frac{g(ta+(1-t)b)}{(b-a)^\alpha}[{}^bI_\alpha f(a)+I_\alpha^a f(b)]+\frac{f(ta+(1-t)b)}{(b-a)^\alpha}[{}^bI_\alpha g(a)+I_\alpha^a g(b)]. \tag{2.8}$$

On the other hand, for $y=(1-t)a+tb$, $t\in[0,1]$ in (2.7) we have

$$\frac{1}{(b-a)^\alpha}[{}^bI_\alpha(fg)(a)+I_\alpha^a(fg)(b)]+\frac{2}{n!}B(n+1,\alpha-n-1)(fg)((1-t)a+tb)$$
$$\geq \frac{g((1-t)a+tb)}{(b-a)^\alpha}[{}^bI_\alpha f(a)+I_\alpha^a f(b)]+\frac{f((1-t)a+tb}{(b-a)^\alpha}[{}^bI_\alpha g(a)+I_\alpha^a g(b)]. \tag{2.9}$$

Adding the inequalities (2.8) and (2.9), then the multiplying both sides $t^n(1-t)^{\alpha-n-1}$ and integrating with respect to t over $[0,1]$, one has

$$\frac{2}{(b-a)^\alpha}B(n+1,\alpha-n)[{}^bI_\alpha(fg)(a)+I_\alpha^a(fg)(b)] \tag{2.10}$$
$$+\frac{2}{n!}B(n+1,\alpha-n)\left(\int_0^1 t^n(1-t)^{\alpha-n-1}(fg)(ta+(1-t)b)\right.$$
$$\left.+\int_0^1 t^n(1-t)^{\alpha-n-1}(fg)((1-t)a+tb)\right)$$
$$\geq \frac{1}{(b-a)^\alpha}[{}^bI_\alpha f(a)+I_\alpha^a f(b)]$$
$$\times\left(\int_0^1 t^n(1-t)^{\alpha-n-1}g(ta+(1-t)b)\int_0^1 t^n(1-t)^{\alpha-n-1}g((1-t)a+tb)\right)$$
$$+\frac{1}{(b-a)^\alpha}[{}^bI_\alpha g(a)+I_\alpha^a g(b)]$$

$$\times\left(\int_0^1 t^n(1-t)^{\alpha-n-1}f(ta+(1-t)b)\int_0^1 t^n(1-t)^{\alpha-n-1}f((1-t)a+tb)\right).$$

We apply Lemma 2

$$\frac{4}{(b-a)^\alpha}B(n+1,\alpha-n)[{}^bI_\alpha(fg)(a)+I_\alpha^a(fg)(b)]$$
$$\geq\frac{2n!}{(b-a)^{2\alpha}}[{}^bI_\alpha f(a)+I_\alpha^a f(b)][{}^bI_\alpha g(a)+I_\alpha^a g(b)].$$

Multiplying both sides of the last inequality with $\frac{(\Gamma(\alpha+1))^2}{8}$ and using the properties of the beta functions, we obtain desired inequality. □

Remark 1. If we take $\alpha=n+1$ in above theorem, then the inequality (2.3) becomes:

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha})[J_{a+}^\alpha(fg)(b)+J_{b-}^\alpha(fg)(a)]$$
$$\geq\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)]\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a+}^\alpha g(b)+J_{b-}^\alpha g(a)]. \tag{2.11}$$

Remark 2. In above theorem, if we take $\alpha=n+1$, $b=t$, $a=0$ and f, g is symmetric to $(a+b)/2$, and use Lemma 1 then inequality (2.3) becomes inequality (1.3).

Theorem 6. *Let $f,g:[a,b]\to\mathbb{R}$ are two monotonic functions of the same monotonicity, $f,g\in L[a,b]$, $\alpha\in[n,n+1]$ $(n=0,1,2,...)$ and $\beta\in(k,k+1]$ $(k=0,1,2,...)$, then*

$$\frac{2}{(b-a)^\beta}B(k+1,\beta-k)[{}^bI_\alpha(fg)(a)+I_\alpha^a(fg)(b)]$$
$$+\frac{2}{(b-a)^\alpha}B(n+1,\alpha-n)[{}^bI_\beta(fg)(a)+I_\beta^a(fg)(b)]$$
$$\geq\frac{1}{(b-a)^{\alpha+\beta}}[{}^bI_\alpha f(a)+I_\alpha^a f(b)][{}^bI_\beta g(a)+I_\beta^a g(b)]$$
$$+\frac{1}{(b-a)^{\alpha+\beta}}[{}^bI_\alpha g(a)+I_\alpha^a g(b)][{}^bI_\beta f(a)+I_\beta^a f(b)]. \tag{2.12}$$

Proof. Adding the inequalities (2.8) and (2.9),and then multiplying both sides of obtained inequality with $t^k(1-t)^{\beta-k-1}$ and integrating with respect to t over $[0,1]$, we get

$$\frac{2}{(b-a)^\beta}B(n+1,\beta-n-1)[{}^bI_\alpha(fg)(a)+I_\alpha^a(fg)(b)] \tag{2.13}$$
$$+\frac{2}{n!}B(n+1,\alpha-n-1)\left(\int_0^1 t^k(1-t)^{\beta-k-1}(fg)(ta+(1-t)b)\right.$$

$$
\begin{aligned}
&+\int_0^1 t^k(1-t)^{\beta-k-1}(fg)((1-t)a+tb)\Bigg)\\
&\geq \frac{1}{(b-a)^\alpha}[{}^bI_\alpha f(a)+I_\alpha^a f(b)]\\
&\times\left(\int_0^1 t^k(1-t)^{\beta-k-1}g(ta+(1-t)b)\int_0^1 t^k(1-t)^{\beta-k-1}g((1-t)a+tb)\right)\\
&+\frac{1}{(b-a)^\alpha}[{}^bI_\alpha g(a)+I_\alpha^a g(b)]\\
&\times\left(\int_0^1 t^k(1-t)^{\beta-k-1}f(ta+(1-t)b)\int_0^1 t^k(1-t)^{\beta-k-1}f((1-t)a+tb)\right).
\end{aligned}
$$

So, if we use Lemma 2, we have desired result. □

Remark 3. Applying Theorem 6 for $\alpha=\beta$, we obtain Theorem 5.

Remark 4. If we take $\alpha=n+1$ and $\beta=k+1$ in theorem 6, then inequality (2.12) becomes:

$$
\begin{aligned}
&\frac{2}{(b-a)^{k+1}(k+1)}[J_{a+}^\alpha fg(b)+J_{b-}^\alpha fg(a)]\\
&+\frac{2}{(b-a)^{n+1}(n+1)}[J_{a+}^\beta fg(b)+J_{b-}^\beta fg(a)]\\
&\geq \frac{1}{(b-a)^{n+k+2}}[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)][J_{a+}^\beta g(b)+J_{b-}^\beta g(a)]\\
&+\frac{1}{(b-a)^{n+k+2}}[J_{a+}^\alpha g(b)+J_{b-}^\alpha g(a)][J_{a+}^\beta f(b)+J_{b-}^\beta f(a)].
\end{aligned}
$$

Remark 5. In above theorem, if we take $\alpha=n+1$, $b=t$, $a=0$, and f, g is symmetric to $(a+b)/2$, and use Lemma 1, then inequality (2.12) becomes inequality (1.4).

Theorem 7. *Let $(f_i)_{i=1,\dots,n}$ be a positive increasing functions on [a,b]. For $\alpha\in(n,n+1]$ $n=0,1,2,\dots$, we have*

$$
\begin{aligned}
&\left[{}^bI_\alpha\left(\prod_{i=1}^n f_i\right)(a)+I_\alpha^a\left(\prod_{i=1}^n f_i\right)(b)\right]\\
&\geq\left[\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha\Gamma(\alpha-n)}\right]^{n-1}\prod_{i=1}^n[{}^bI_\alpha(f_i)(a)+I_\alpha^a(f_i)(b)].
\end{aligned}
\tag{2.14}
$$

Proof. We prove this theorem by induction. For $n=1$, we have

$$[{}^bI_\alpha f_1(a)+I_\alpha^a f_1(b)]\geq[{}^bI_\alpha f_1(a)+I_\alpha^a f_1(b)].$$

For $n=2$, applying Theorem 5, we get

$$[{}^bI_\alpha(f_1f_2)(a)+I_\alpha^a(f_1f_2)(b)]$$
$$\geq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha\Gamma(\alpha-n)}[{}^bI_\alpha(f_1)(a)+I_\alpha^a(f_1)(b)][{}^bI_\alpha f_2)(a)+I_\alpha^a(f_2)(b)].$$

Now suppose that

$$\left[{}^bI_\alpha\left(\prod_{i=1}^{n-1}f_i\right)(a)+I_\alpha^a\left(\prod_{i=1}^{n-1}f_i\right)(b)\right]$$
$$\geq\left[\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha\Gamma(\alpha-n)}\right]^{n-2}\prod_{i=1}^{n-1}[{}^bI_\alpha(f_i)(a)+I_\alpha^a(f_i)(b)]. \tag{2.15}$$

Since $(f_i)_{i=1,\dots,n}$ are positive increasing functions, then $\left(\prod_{i=1}^{n-1}f_i\right)(t)$ is an increasing function. So we can apply Theorem 5 to the functions $\prod_{i=1}^{n-1}f_i=g$, $f_n=f$, we obtain

$$\left[{}^bI_\alpha\left(\prod_{i=1}^{n}f_i\right)(a)+I_\alpha^a\left(\prod_{i=1}^{n}f_i\right)(b)\right]$$
$$\geq\left[\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha\Gamma(\alpha-n)}\right]$$
$$\times\left[{}^bI_\alpha\left(\prod_{i=1}^{n-1}f_i\right)(a)+I_\alpha^a\left(\prod_{i=1}^{n-1}f_i\right)(b)\right][{}^bI_\alpha(f_n)(a)+I_\alpha^a(f_n)(b)].$$

Using (2.15), we get desired result. □

Remark 6. If we take $\alpha=n+1$ in Theorem 7, then inequality (2.14) becomes:

$$\left[J_{b-}^\alpha\left(\prod_{i=1}^{n}f_i\right)(a)+J_{a+}^\alpha\left(\prod_{i=1}^{n}f_i\right)(b)\right]$$
$$\geq\left[\frac{(n+1)!}{2(b-a)^\alpha}\right]^{n-1}\prod_{i=1}^{n}[J_{a+}^\alpha f_i(b)+J_{b-}^\alpha f_i(a)]. \tag{2.16}$$

Remark 7. If we take $\alpha=n+1$, $b=t$, $a=0$ and $f_i;i=1,\dots,n$ is symmetric to $(a+b)/2$, then, using Lemma 1, inequality (2.14) becomes inequality (1.5).

Theorem 8. *Let f and g be two functions defined on $[0,\infty)$, such that f is increasing, g is differentiable and there exist a real number $m:=inf_{t\geq 0}g'(t)$. Then the inequality*

$$\Gamma(\alpha-n)[{}^bI_\alpha(fg)(a)+I_\alpha^a(fg)(b)] \tag{2.17}$$

$$\geq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[{}^{b}I_{\alpha}f(a)+I_{\alpha}^{a}f(b)][{}^{b}I_{\alpha}g(a)+I_{\alpha}^{a}g(b)]$$
$$-\frac{\Gamma(n+2)\Gamma(\beta-n)}{2(b-a)^{\alpha}(\alpha+1)}[{}^{b}I_{\alpha}f(a)+I_{\alpha}^{a}f(b)]+m[{}^{b}I_{\alpha}(tf)(a)+I_{\alpha}^{a}(tf)(b)].$$

is valid for all $t>0$, $\alpha\in(n,n+1], n=0,1,2,\ldots$.

Proof. We consider the function $h(t):=g(t)-mt$. It is clear that h is differentiable and it is increasing on $[0,\infty)$. Then by using Theorem 5, we get

$$\Gamma(\alpha-n)[{}^{b}I_{\alpha}((g-mt)f)(a)+I_{\alpha}^{a}((g-mt)f)(b)]$$
$$\geq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[{}^{b}I_{\alpha}f(a)+I_{\alpha}^{a}f(b)][{}^{b}I_{\alpha}g(a)+I_{\alpha}^{a}g(b)-m\,{}^{b}I_{\alpha}t(a)-mI_{\alpha}^{a}t(b)]$$
$$\geq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[{}^{b}I_{\alpha}f(a)+I_{\alpha}^{a}f(b)][{}^{b}I_{\alpha}g(a)+I_{\alpha}^{a}g(b)]$$
$$-mB(n+2,\alpha-n)\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[{}^{b}I_{\alpha}f(a)+I_{\alpha}^{a}f(b)]$$
$$\geq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[{}^{b}I_{\alpha}f(a)+I_{\alpha}^{a}f(b)][{}^{b}I_{\alpha}g(a)+I_{\alpha}^{a}g(b)]$$
$$-\frac{\Gamma(n+2)\Gamma(\beta-n)}{2(b-a)^{\alpha}(\alpha+1)}[{}^{b}I_{\alpha}f(a)+I_{\alpha}^{a}f(b)].$$

So

$$\Gamma(\alpha-n)[{}^{b}I_{\alpha}(fg)(a)+I_{\alpha}^{a}(fg)(b)]$$
$$\geq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[{}^{b}I_{\alpha}f(a)+I_{\alpha}^{a}f(b)][{}^{b}I_{\alpha}g(a)+I_{\alpha}^{a}g(b)]$$
$$-\frac{\Gamma(n+2)\Gamma(\beta-n)}{2(b-a)^{\alpha}(\alpha+1)}[{}^{b}I_{\alpha}f(a)+I_{\alpha}^{a}f(b)]+m[{}^{b}I_{\alpha}(tf)(a)+I_{\alpha}^{a}(tf)(b)].$$

which the proof is completed. □

Remark 8. In Theorem 8, if we take $\alpha=n+1$ and $\beta=k+1$, inequality (2.17) becomes:

$$[J_{a+}^{\beta}(fg)(b)+J_{b-}^{\beta}fg)(a)]$$
$$\geq \frac{(n+2)!}{2(b-a)^{\alpha}}[J_{a+}^{\beta}f(b)+J_{b-}^{\beta}f(a)][J_{a+}^{\beta}g(b)+J_{b-}^{\beta}g(a)] \tag{2.18}$$
$$-\frac{\Gamma(n+2)\Gamma(\beta-n)}{2(b-a)^{\alpha}(\alpha+1)}[J_{a+}^{\beta}f(b)+J_{b-}^{\beta}f(a)]+m[J_{a+}^{\beta}f(b)+J_{b-}^{\beta}g(a)].$$

Remark 9. In Theorem 8, if we take $\alpha=n+1$ and $f,\ g$ is symmetric to $(a+b)/2$, then, using Lemma 1 inequality (2.18) becomes inequality (1.6).

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Authors' addresses

Erhan Set
Ordu University, Department of Mathematics, Faculty of Science and Arts, Ordu, Turkey
E-mail address: erhanset@yahoo.com

Ahmet Ocak Akdemir
Ağrı İbrahim Çeçen University, Department of Mathematics, Faculty of Science and Arts, Ağrı, Turkey
E-mail address: ahmetakdemir@agri.edu.tr

İlker Mumcu
Ordu University, Department of Mathematics, Faculty of Science and Arts, Ordu, Turkey
E-mail address: mumcuilker@msn.com

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A NOTE ON FARTHEST POINT PROBLEM IN BANACH SPACES

SUMIT SOM AND EKREM SAVAS

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Abstract. Farthest point problem states that "Must every uniquely remotal set in a Banach space be singleton?" In this paper we introduce the notion of partial ideal statistical continuity of a function which is way weaker than continuity of a function. We give an example to show that partial ideal statistical continuity is weaker than continuity. In this paper we use Ideal summability to give some answers to FPP problem which improves the result in [13]. We prove that if E is a non-empty, bounded, uniquely remotal subset in a real Banach space X such that E has a Chebyshev center c and the farthest point map $F : X \to E$ restricted to $[c, F(c)]$ is partially ideal statistically continuous at c then E is singleton.

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1. Introduction

Let $\mathbb{X}$ be a real Banach space and E be a nonempty, bounded subset of $\mathbb{X}$. For any $x \in \mathbb{X}$, the farthest distance of x from the set E is denoted by $D(x, E)$ and is defined by

$$D(x, E) = \sup \{\|x - e\| : e \in E\}.$$

The farthest distance from x to E may or may not attained by some elements of E. If the distance is attained then the collection of all such points of E is denoted by $F(x, E)$ i.e

$$F(x, E) = \{e \in E : \|x - e\| = D(x, E)\}.$$

We say that E is remotal if $F(x, E) \neq \phi \ \forall \ x \in \mathbb{X}$ and E is said to be uniquely remotal if $F(x, E)$ is singleton for each $x \in \mathbb{X}$.

The FPP was proposed by Motzkin, Starus and Valentine [11] in context of the Euclidean space E^n. The problem was considered in the setting of Banach spaces by Klee [9] where he proved that every compact uniquely remotal subset of a Banach space is singleton. In [1], Asplund solved the FPP in the affirmation in any finite dimensional Banach space with respect to a norm which is not necessarily symmetric. The study of remotal and uniquely remotal sets has attracted many researchers in the last few decades due to its connection to the geometry of Banach spaces. One can see [2, 12, 15] for further details.

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Chebyshev centers of sets have played a major role in the study of uniquely remotal sets. Recall that a chebyshev center of a subset E of a normed space $\mathbb{X}$ is an element $c \in \mathbb{X}$ such that $D(c,E) = \inf_{x\in\mathbb{X}} D(x,E)$. Astaneh in 1983 proved that in an inner product space $\mathbb{X}$, every closed and bounded set has a Chebyshev center. But it is still unknown that whether every closed and bounded subset of a normed space $\mathbb{X}$ has a center or not.

The idea of convergence of a real sequence has been extended to statistical convergence by Fast [6] and Steinhaus [19] and later on re-introduced by Schoenberg [18] independently and is based on the notion of asymptotic density of the subset of natural numbers. However, the first idea of statistical convergence (by different name) was given by Zygmund [20] in the first edition of his monograph published in Warsaw in 1935. Later on it was further investigated from the sequence space point of view and linked with summability theorem by Fridy [7], Connor [3], Šalát [16], Das et. al. [4, 5], Fridy and Orhan [8]. If $\mathbb{N}$ denotes the set of natural numbers and $K \subset \mathbb{N}$ then $K(m,n)$ denotes the cardinality of the set $K \cap [m,n]$. The upper and lower natural density of the subset K is defined by

$$\bar{d}(K) = \lim_{n\to\infty} \sup \frac{K(1,n)}{n} \quad \text{and} \quad \underline{d}(K) = \lim_{n\to\infty} \inf \frac{K(1,n)}{n}.$$

If $\bar{d}(K) = \underline{d}(K)$, then we say that the natural density of K exists, and it is denoted simply by

$$d(K) = \lim_{n\to\infty} \frac{K(1,n)}{n}.$$

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to a real number x if for each $\varepsilon > 0$, the set $K = \{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\}$ has natural density zero and we write $x_n \xrightarrow{S} x$.

The following definitions and notions will be needed.

Definition 1 ([10])**.** A family $\mathcal{I} \subset 2^{\mathbb{N}}$ of subsets of $\mathbb{N}$ (where $\mathbb{N}$ denotes the set of all non-negative integers) is said to be an ideal in $\mathbb{N}$ provided that the following conditions holds:
(i) $\phi \in \mathcal{I}$,
(ii) $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$,
(iii) $A \in \mathcal{I}, B \subseteq A \Rightarrow B \in \mathcal{I}$.

Definition 2 ([10])**.** An ideal $\mathcal{I}$ is said to be non-trivial if $\mathcal{I} \neq \{\phi\}$ and proper if $\mathbb{N} \notin \mathcal{I}$. A proper ideal $\mathcal{I}$ in $\mathbb{N}$ is said to be admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Throughout this paper $\mathcal{I}$ will denote a non-trivial, proper, admissible ideal.

Definition 3 ([17])**.** Let $\mathcal{I}$ be a non-trivial, proper, admissible ideal. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-statistically convergent to a real number x if

for every $\varepsilon, \delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : |x_k - x| \geq \varepsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

In [13] it was proved that if E is a uniquely remotal subset of a normed space, admitting a center c and if the farthest point map $F : X \to E$ restricted to $[c, F(c)]$ is continuous at c then E is singleton.

In a natural way, in this paper we first introduce the notion of partial ideal statistical continuity of a function via ideal summability and we give an example to show that this notion of partial ideal statistical continuity is much weaker than continuity and also weaker than partial continuity introduced by Sababheh et al. in [14]. We prove that if E is a non-empty, bounded, uniquely remotal subset in a real Banach space $\mathbb{X}$ such that E has a Chebyshev center c and the farthest point map $F : \mathbb{X} \to E$ restricted to $[c, F(c)]$ is partially ideal statistically continuous at c then E is singleton.

2. Results

We first recall the definition of partial continuity of a function from [14] as follows.

Definition 4. Let $\mathbb{X}$ be real Banach space and A be a nonempty subset of $\mathbb{X}$. The function $F : A \to \mathbb{X}$ is said to be partially continuous at $a \in \mathbb{X}$ if there exists a non constant sequence $\{x_n\}_{n\in\mathbb{N}} \subset A$ such that $x_n \to a$ and $F(x_n) \to F(a)$ as $n \to \infty$.

Now we like to introduce the definition of partial ideal statistical continuity of a function as follows.

Definition 5. Let $\mathcal{I}$ be a non-trivial, proper, admissible ideal. Let $\mathbb{X}$ be a real Banach space and A be a nonempty subset of $\mathbb{X}$. The function $F : A \to \mathbb{X}$ is said to be partially ideal statistically continuous at $a \in \mathbb{X}$ if there exists a non constant sequence $\{x_n\}_{n\in\mathbb{N}} \subset A$ such that $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-statistically convergent to a and $\{F(x_n)\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-statistically convergent to $F(a)$.

Now we give an example to show that this notion of partial ideal statistical continuity is much weaker than partial continuity introduced by Sababheh et al. [14].

Example 1. Let $\mathcal{I}$ be a non-trivial, proper, admissible ideal. A function $f : [-1, 0] \to \mathbb{R}$ be defined by $f(x) = [x], x \in [-1, 0]$. It is easy to check that this function is not partially continuous (also not continuous) at the point $x = 0$. Now we show that this function is partially ideal statistically continuous at the point $x = 0$.

Let us define a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[-1, 0]$ by

$$x_n = \begin{cases} 0 \text{ if } n \neq m^2 \ \forall\ m \in \mathbb{N} \\ -\frac{1}{n} \text{ if } n = m^2 \text{ for some } m \in \mathbb{N}. \end{cases}$$

Now the natural density of the set $K = \{m^2 : m \in \mathbb{N}\}$ is 0. Let $\varepsilon > 0$.

As $\left\{n \in \mathbb{N} : |x_n - 0| \geq \varepsilon\right\} \subset \left\{n \in \mathbb{N} : n = m^2 \text{ for some } m \in \mathbb{N}\right\}$, so

$$\lim_{n\to\infty} \frac{1}{n}\left|\{k \leq n : |x_k - x| \geq \varepsilon\}\right| = 0.$$

This shows that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is statistically convergent to 0. Let $\delta > 0$. Then

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : |x_k - x| \geq \varepsilon\}| \geq \delta\right\} \in \mathcal{I}$$

since $\mathcal{I}$ is admissible. So $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-statistically convergent to 0.

Now the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ is defined by

$$f(x_n) = \begin{cases} 0 \text{ if } n \neq m^2 \ \forall\ m \in \mathbb{N} \\ -1 \text{ if } n = m^2 \text{ for some } m \in \mathbb{N}, \end{cases}$$

Now it is easy to check that the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-statistically convergent to $f(0) = 0$. This shows that this function is partially ideal statistically continuous at the point $x = 0$. So the notion of partial ideal statistical continuity is much weaker than partial continuity.

Theorem 1. *Let $\mathcal{I}$ be a non-trivial, proper, admissible ideal. Let E be nonempty, bounded subset of a real Banach space $\mathbb{X}$. If E is uniquely remotal such that E has Chebyshev center c and the farthest point map $F : X \to E$ restricted to $[c, F(c)]$ is partially ideal statistically continuous at c then E is singleton.*

Proof. Since E is uniquely remotal so for each $x \in \mathbb{X}$ there exists unique $e \in E$ such that $\|x - e\| = D(x, E)$ and the farthest point map $F : \mathbb{X} \to E$ defined by $F(x) = F(x, E)$, $\forall\ x \in \mathbb{X}$ is well defined. We assume that E has Chebyshev center at $c = 0$.

Now suppose E is not singleton. So we have $F(0) \neq 0$.

It is given that the farthest point map $F : X \to E$ restricted to $[0, F(0)]$ is partially ideal statistically continuous at 0. So there exist a non constant sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0, F(0)]$ such that $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-statistically convergent to 0 and $\{F(x_n)\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-statistically convergent to $F(0)$.

So we have $x_n = \mu_n F(0)$ with $\mu_n > 0$, $\forall\ n \in \mathbb{N}$ and μ_n is $\mathcal{I}$-statistically convergent to 0. Now for each $n \in \mathbb{N}$ there exists $\psi_n \in \mathbb{X}^*$ such that $\psi_n(F(x_n) - x_n) = \|F(x_n) - x_n\|$ and $\|\psi_n\| = 1$.

$$\begin{aligned} \psi_n(x_n) &= \psi_n(F(x_n)) - \psi_n(F(x_n) - x_n) \\ &\leq \|\psi_n\|\|F(x_n)\| - \|F(x_n) - x_n\| \\ &= \|F(x_n)\| - \|F(x_n) - x_n\| \\ &= \|F(x_n) - 0\| - \|F(x_n) - x_n\| \\ &\leq D(0, E) - D(x_n, E) \end{aligned}$$

$$\leq 0. \text{ (As 0 is the Chebyshev center)}$$

So

$$\begin{aligned} \psi_n(x_n) &\leq 0 \ \forall\ n \in \mathbb{N} \\ \Rightarrow \psi_n(\mu_n F(0)) &\leq 0 \ \forall\ n \in \mathbb{N} \\ \Rightarrow \mu_n \psi_n(F(0)) &\leq 0 \ \forall\ n \in \mathbb{N} \\ \Rightarrow \psi_n(F(0)) &\leq 0 \ \forall\ n \in \mathbb{N} \text{ as } \mu_n > 0. \end{aligned}$$

Now the sequence $\{F(x_n) - x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-statistically convergent to $F(0)$. So for each $\varepsilon, \delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \|F(x_k) - x_k - F(0)\| \geq \varepsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

Now we have

$$\Big|\|F(x_k) - x_k\| - \|F(0)\|\Big| \leq \|F(x_k) - x_k - F(0)\|.$$

Let $\varepsilon > 0$. So we have

$$\left\{k \leq n : \Big|\|F(x_k) - x_k\| - \|F(0)\|\Big| \geq \varepsilon\right\} \subset \left\{k \leq n : \|F(x_k) - x_k - F(0)\| \geq \varepsilon\right\}.$$

This implies that for every $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \Big|\|F(x_k) - x_k\| - \|F(0)\|\Big| \geq \varepsilon\right\}\right| \geq \delta\right\} \subset$$

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \|F(x_k) - x_k - F(0)\| \geq \varepsilon\right\}\right| \geq \delta\right\}.$$

Since the set on the right belongs to the ideal $\mathcal{I}$ so

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \Big|\|F(x_k) - x_k\| - \|F(0)\|\Big| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

So the sequence $\Big\{\|F(x_k) - x_k\|\Big\}_{k\in\mathbb{N}}$ is $\mathcal{I}$-statistically convergent to $\|F(0)\|$.

Now we have

$$\begin{aligned} \psi_n(F(x_n) - x_n) - \psi_n(F(0)) &= \psi_n(F(x_n) - x_n - F(0)) \\ &\leq \|\psi_n\|\|F(x_n) - x_n - F(0)\| \\ &= \|F(x_n) - x_n - F(0)\|. \end{aligned}$$

The sequence in the right is $\mathcal{I}$-statistically convergent to 0. So the sequence $\{\psi_n(F(0))\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-statistically convergent to $\|F(0)\|$. But this is possible only when $F(0) = 0$. This is a contradiction. This proves that the uniquely remotal set E is singleton. □

Corollary 1. *Let $\mathcal{I}$ be a non-trivial, proper, admissible ideal. Let E be nonempty, bounded subset of a real Banach space $\mathbb{X}$. If E is remotal such that E has Chebyshev center c and the extracted farthest point map $F : X \to E$ restricted to $[c, F(c)]$ is partially ideal statistically continuous at c then E is singleton.*

Theorem 2. *Let $\mathcal{I}$ be a non-trivial, proper, admissible ideal in $\mathbb{N}$. Let $\mathbb{X}$ be a real Banach space and E be non-empty, bounded, uniquely remotal set admitting a Chebyshev center c. If E is not singleton then the farthest point map F, restricted to $(c, F(c)]$ is not partially ideal statistically continuous at c.*

Proof. The uniquely remotal set E has a Chebyshev center c. Let $x \in (c, F(c)]$. Then $x = tc + (1-t)F(c)$ for some $t \in [0, 1)$.

Now

$$\begin{aligned}
\|x - F(x)\| &= \|tc + (1-t)F(c) - F(x)\| \\
&= \|tc - tF(x) + (1-t)F(c) - (1-t)F(x)\| \\
&\le t\|c - F(x)\| + (1-t)\|F(c) - F(x)\| \\
&\le t\|c - F(c)\| + (1-t)\|F(c) - F(x)\| \\
&\le t\|x - F(x)\| + (1-t)\|F(c) - F(x)\| \\
\Rightarrow \|F(x) - F(c)\| &\ge \|x - F(x)\| \ge \|c - F(c)\| = r.
\end{aligned}$$

If we choose $||c - F(c)|| = r > 0$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $(c, F(c)]$, $\mathcal{I}$-statistically convergent to c but we have

$$\frac{1}{n}\Big|\{k \le n : \|F(x_k) - F(c)\| \ge r\}\Big| = 1 \text{ for all } n \in \mathbb{N}.$$

Let $0 < \delta < 1$. Then we have

$$\{n \in \mathbb{N} : \frac{1}{n}|\{k \le n : \|F(x_k) - F(c)\| \ge r\}| \ge \delta\} = \mathbb{N} \notin \mathcal{I}.$$

This shows that The farthest point map F is not partially ideal statistically continuous at c. This completes the proof. □

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Authors' addresses

Sumit Som
Research Associate, National Institute of Technology Durgapur, Department of Mathematics, West Bengal, Pin- 713209, Durgapur, India.
E-mail address: somkakdwip@gmail.com

Ekrem Savas
Usak University, Department of Mathematics, Usak, Turkey
E-mail address: ekremsavas@yahoo.com

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SOME PROPERTIES OF ANALYTIC FUNCTIONS ASSOCIATED WITH FRACTIONAL q-CALCULUS OPERATORS

H. M. SRIVASTAVA, M. K. AOUF, AND A. O. MOSTAFA

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Abstract. By applying a fractional q-calculus operator, we define the subclasses $\mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$ and $\mathcal{G}_n^{\alpha}(\lambda,\beta,b,q)$ of normalized analytic functions with complex order and negative coefficients. Among the results investigated for each of these function classes, we derive their associated coefficient estimates, radii of close-to-convexity, starlikeness and convexity, extreme points, and growth and distortion theorems.

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Keywords: analytic functions, fractional q-calculus operators, q-gamma functions, starlike functions of complex order, convex functions of complex order, close-to-convex functions of complex order, coefficient estimates, radii of close-to-convexity, starlikeness and convexity, extreme points, growth and distortion theorems

1. Introduction and definitions

Here, in this paper, we denote by $\mathcal{A}(n)$ the class of functions of the following normalized form:

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \qquad (n \in \mathbb{N};\ \mathbb{N} := \{1,2,3,\cdots\}), \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}$ centered at the origin ($z = 0$) in the complex z-plane. We write $\mathcal{A}(1) = \mathcal{A}$. We also denote by $\mathcal{T}(n)$ the subclass of $\mathcal{A}(n)$ consisting of functions of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \qquad (a_k \geqq 0;\ k \geqq n+1;\ n \in \mathbb{N}). \tag{1.2}$$

In our investigation, we make use of various operators of q-calculus and fractional q-calculus. For this purpose, we refer the reader to the various definitions, notations and conventions, which are considerably detailed in our earlier paper (see, for details, [22]; see also [8]).

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For a fixed $\mu \in \mathbb{C}$, a set $\mathbb{D}$ is called a μ-geometric set if and only if both $z \in \mathbb{D}$ and $\mu z \in \mathbb{D}$. For a function f defined on a q-geometric set, we make use of Jackson's q-derivative and q-integral $(0 < q < 1)$ of a function on a subset of $\mathbb{C}$, which are already introduced in several earlier investigations (see, for example, [2], [4], [6], [8], [9], [10], [14], [15], [16], [17], [21], [22] and [25]).

Now, for a complex-valued function $f(z)$, we introduce the fractional q-calculus operators as follows (see, for example, [12] and [13]; see also [1]).

Definition 1 (Fractional q-integral operator)**.** The fractional q-integral operator $I_{q,z}^{\lambda}$ of order λ is defined, for a function $f(z)$, by

$$I_{q,z}^{\lambda} f(z) = D_{q,z}^{-\lambda} f(z) = \frac{1}{\Gamma_q(\lambda)} \int_0^z (z-tq)_{\lambda-1} f(t) d_q t \qquad (\lambda > 0), \tag{1.3}$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z-plane containing the origin. Here, and elsewhere in this paper, the q-binomial $(z-tq)_{\lambda-1}$ is given by

$$\begin{aligned}(z-tq)_{\lambda-1} &= z^{\lambda-1} \prod_{k=0}^{\infty} \left[\frac{1-(tqz^{-1})q^k}{1-(tqz^{-1})q^{\lambda+k-1}} \right] \\ &= z^{\lambda}\, {}_1\Phi_0(q^{1-\lambda}; \text{—}\, ; q, tq^{\lambda} z^{-1}).\end{aligned} \tag{1.4}$$

Remark 1. The q-hypergeometric series ${}_1\Phi_0(\lambda; \text{—}; q, z)$ is known to be single-valued when $|\arg(z)| < \pi$ (see, for example, [8]). Therefore, the q-binomial $(z-tq)_{\lambda-1}$ in (1.4) is single-valued when

$$\left| \arg\left(-tq^{\lambda} z^{-1}\right) \right| < \pi, \qquad \left| \frac{tq^{\lambda}}{z} \right| < 1 \text{ and } |\arg(z)| < \pi.$$

Definition 2 (Fractional q-derivative operator)**.** The fractional q-derivative operator $D_{q,z}^{\lambda}$ of order λ $(0 \leqq \lambda < 1)$ is defined, for a function $f(z)$, by

$$D_{q,z}^{\lambda} f(z) = D_{q,z} I_{q,z}^{1-\lambda} f(z) = \frac{1}{\Gamma_q(1-\lambda)} D_q \int_0^z (z-tq)_{-\lambda} f(t) d_q t, \tag{1.5}$$

where $f(z)$ is suitably constrained and the multiplicity of $(z-tq)_{-\lambda}$ is removed as in Definition 1.

Definition 3 (Extended fractional q-derivative operator)**.** Under the hypotheses of Definition 2, for a function $f(z)$, the fractional q-derivative of order λ is defined by

$$D_{q,z}^{\lambda} f(z) = D_{q,z}^{m} I_{q,z}^{m-\lambda} f(z) \qquad (m-1 \leqq \lambda < 1;\ m \in \mathbb{N}). \tag{1.6}$$

Clearly, we have

$$D_{q,z}^{\lambda} z^n = \frac{\Gamma_q(n+1)}{\Gamma_q(n+1-\lambda)} z^{n-\lambda} \qquad (\lambda \geqq 0;\ n > -1).$$

Now, by using the operator $D_{q,z}^{\lambda}$, we define (for $-\infty < \lambda < 2$, $0 < q < 1$ and $z \in \mathbb{U}$,) a q-differintegral operator $\Omega_{q,z}^{\lambda} : \mathcal{T}(n) \to \mathcal{T}(n)$ as follows (see [12] and [13]):

$$\Omega_{q,z}^{\lambda} f(z) = \frac{\Gamma_q(2-\lambda)}{\Gamma_q(\lambda)} z^{\lambda} D_{q,z}^{\lambda} f(z) = z - \sum_{k=n+1}^{\infty} A_q(\lambda,k) a_k z^k \tag{1.7}$$

where

$$A_q(\lambda,k) = \frac{\Gamma_q(k+1)\Gamma_q(2-\lambda)}{\Gamma_q(2)\Gamma_q(k+1-\lambda)} \tag{1.8}$$

and $D_{q,z}^{\lambda} f(z)$ in (1.7) represents, respectively, the fractional q-integral of $f(z)$ of order λ $(-\infty < \lambda < 0)$ and the fractional q-derivative of $f(z)$ of order λ $(0 \leqq \lambda < 2)$ (see, for details, [7, 18–20]). We note that some interesting special and limit cases of (1.7) were investigated in the earlier works by Owa and Srivastava [11] and by Srivastava and Owa (see [23] and [24]).

Remark 2. From (1.3), (1.7) and (1.8), we find that

$$\begin{aligned}\Omega_{q,z}^{-\lambda} f(z) &= \frac{\Gamma_q(2+\lambda)}{\Gamma_q(2)} z^{-\lambda} D_{q,z}^{-\lambda} f(z) = \frac{\Gamma_q(2+\lambda)}{\Gamma_q(2)} z^{-\lambda} I_{q,z}^{\lambda} f(z) \\ &= z - \sum_{k=n+1}^{\infty} A_q(-\lambda,k) a_k z^k,\end{aligned} \tag{1.9}$$

where

$$A_q(-\lambda,k) = \frac{\Gamma_q(k+1)\Gamma_q(2+\lambda)}{\Gamma_q(2)\Gamma_q(k+1+\lambda)} \qquad (\lambda > 0;\ 0 < q < 1). \tag{1.10}$$

Definition 4. A function $f(z) \in \mathcal{T}(n)$ is said to be in the function class:

$$\mathcal{S}_n^{\alpha}(\lambda,\beta,b,q) \qquad (\lambda < 2;\ 0 \leqq \alpha \leqq 1;\ 0 < q < 1;\ \beta > 0;\ b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\})$$

if it satisfies the following condition:

$$\left| \frac{1}{b} \left(\frac{(1-\alpha) z D_q\left(\Omega_{q,z}^{\lambda} f(z)\right) + \alpha z D_q\left(z D_q\left(\Omega_{q,z}^{\lambda} f(z)\right)\right)}{(1-\alpha)\Omega_{q,z}^{\lambda} f(z) + \alpha z D_q\left(\Omega_{q,z}^{\lambda} f(z)\right)} - 1 \right) \right| < \beta. \tag{1.11}$$

Some of the interesting particular cases of the function class $\mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$ are being recorded below:

(i) $\mathcal{S}_n^{\alpha}(\lambda,1,b,q) = \mathcal{S}_n^{\alpha}(\lambda,b,q)$ (see [12]);
(ii) $\mathcal{S}_n^{\alpha}(0,\beta,b,q) = \mathcal{S}_n^{\alpha}(\beta,b,q)$, where

$$\mathcal{S}_n^{\alpha}(\beta,b,q) := \left\{ f : f \in T(n) \qquad \text{and} \right.$$

$$\left|\frac{1}{b}\left(\frac{(1-\alpha)zD_q f(z)+\alpha z^2 D_q^2 f(z)}{(1-\alpha)f(z)+\alpha z D_q f(z)}-1\right)\right|<\beta\Bigg\}.$$

(iii) $\lim\limits_{q\to 1-}\mathcal{S}_n^{\alpha}(\beta,b,q)=\mathcal{S}_n(b,\alpha,\beta)$ (see [3]);

(iv) $\mathcal{S}_n^{0}(\lambda,\beta,b,q)=\mathcal{S}_n^{*}(\lambda,\beta,b,q)$, where

$$\mathcal{S}_n(b,\alpha,\beta):=\left\{f:f\in\mathcal{T}(n)\quad\text{and}\quad\left|\frac{1}{b}\left(\frac{zD_q\big(\Omega_{q,z}^{\lambda}f(z)\big)}{\Omega_{q,z}^{\lambda}f(z)}-1\right)\right|<\beta\right\};$$

(v) $\lim\limits_{q\to 1-}\mathcal{S}_n^{*}(\lambda,\beta,b,q)=\mathcal{K}_n(\lambda,b,\beta)$ (see [5] *with* $p=1$);

(vi) $\mathcal{S}_n^{1}(\lambda,\beta,b,q)=\mathcal{C}_n(\lambda,\beta,b,q)$, where

$$\mathcal{C}_n(\lambda,\beta,b,q):=\left\{f:f\in\mathcal{T}(n)\quad\text{and}\quad\left|\frac{1}{b}\left(\frac{zD_q^2\big(\Omega_{q,z}^{\lambda}f(z)\big)}{D_q\big(\Omega_{q,z}^{\lambda}f(z)\big)}-1\right)\right|<\beta\right\}.$$

Definition 5. A function $f(z)\in\mathcal{T}(n)$ is in the function class

$$\mathcal{G}_n^{\alpha}(\lambda,\beta,b,q)\ \ (\lambda<2;\ 0\leqq\alpha\leqq 1;\ 0<q<1;\ b\in\mathbb{C}^*;\ \beta>0)$$

if it satisfies the following condition:

$$\left|\frac{1}{b}\left(D_q\big(\Omega_{q,z}^{\lambda}f(z)\big)+\alpha zD_q^2\big(\Omega_{q,z}^{\lambda}f(z)\big)-1\right)\right|<\beta. \tag{1.12}$$

We choose to note the following special case of the function class $\mathcal{G}_n^{\alpha}(\lambda,\beta,b,q)$:

(i) $\mathcal{G}_n^{\alpha}(0,\beta,b,q)=\mathcal{G}_n^{\alpha}(\beta,b,q)$, where

$$\mathcal{G}_n^{\alpha}(\beta,b,q)=\left\{f:f\in\mathcal{T}(n)\quad\text{and}\quad\left|\frac{1}{b}\left(D_q f(z)+\alpha zD_q^2 f(z)-1\right)\right|<\beta\right\};$$

(ii) $\mathcal{G}_n^{\alpha}(\lambda,1,b,q)=\mathcal{R}_n^{\alpha}(\lambda,b,q)$ (see [13]);

(iii) $\mathcal{G}_n^{\alpha}(0,\beta,b,q)=\mathcal{R}_n(\alpha,\beta,b,q)$ (see [13]);

(iv) $\lim\limits_{q\to 1-}\mathcal{G}_n^{\alpha}(0,\beta,b,q)=\mathcal{R}_n(\alpha,\beta,b)$ (see [3]).

For each of the above-defined general function classes $\mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$ and $\mathcal{G}_n^{\alpha}(\lambda,\beta,b,q)$ of analytic functions with complex order and negative coefficients, we propose here to investigate the associated coefficient estimates, radii of close-to-convexity, starlikeness and convexity, extreme points, and growth and distortion theorems.

2. Properties of the function classes $\mathcal{S}_n^\alpha(\lambda,\beta,b,q)$ and $\mathcal{G}_n^\alpha(\lambda,\beta,b,q)$

Henceforth in this paper, unless otherwise mentioned, we assume that $\lambda < 2$, $0 \leqq \alpha \leqq 1$, $0 < q < 1$, $b \in \mathbb{C}^*$, $\beta > 0$, $[\lambda]_q$ denotes the basic (or q-) number defined by

$$[\lambda]_q = \frac{1-q^\lambda}{1-q} \qquad (|q| < 1), \tag{2.1}$$

which readily yields

$$[\lambda]_q = \frac{1-q^\lambda}{1-q} \to \lambda \qquad (q \to 1-),$$

$A_q(\lambda,k)$ is given by (1.8), $f(z)$ is in the form (1.2) and $z \in \mathbb{U}$.

Theorem 1. *The function $f(z) \in \mathcal{S}_n^\alpha(\lambda,\beta,b,q)$ if and only if*

$$\sum_{k=n+1}^{\infty} \big([k]_q + \beta|b| - 1\big)[1+\alpha([k]_q-1)]A_q(\lambda,k)a_k \leqq \beta|b|. \tag{2.2}$$

Proof. Let $f(z) \in \mathcal{S}_n^\alpha(\lambda,\beta,b,q)$. Then, in view of (1.11) and (1.7), we readily find that

$$\Re\left(\frac{-\sum\limits_{k=n+1}^{\infty}[1+\alpha([k]_q-1)]([k]_q-1)A_q(\lambda,k)a_k z^{k-1}}{1-\sum\limits_{k=n+1}^{\infty}[1+\alpha([k]_q-1)]A_q(\lambda,k)a_k z^{k-1}}\right) > -\beta|b|. \tag{2.3}$$

Setting $z = r$ $(0 \leqq r < 1)$ in (2.3), we observe that the expression in the denominator of the left-hand side of (2.3) is positive for $r = 0$ and also for $0 < r < 1$. Thus, if we let $r \to 1-$ through real values, (2.3) would lead us to (2.2).

Conversely, let (2.2) hold true and $|z| = 1$. We then find that

$$\left|\frac{(1-\alpha)zD_q\big(\Omega_{q,z}^\lambda f(z)\big) + \alpha zD_q\Big(zD_q\big(\Omega_{q,z}^\lambda f(z)\big)\Big)}{(1-\alpha)\Omega_{q,z}^\lambda f(z) + \alpha zD_q\big(\Omega_{q,z}^\lambda f(z)\big)} - 1\right|$$

$$\leqq \frac{\beta|b|\{1-\sum\limits_{k=n+1}^{\infty}[1+\alpha([k]_q-1)]A_q(\lambda,k)a_k\}}{1-\sum\limits_{k=n+1}^{\infty}[1+\alpha([k]_q-1)]A_q(\lambda,k)a_k} = \beta|b|.$$

Hence, by the *Maximum Modulus Theorem*, we conclude that $f(z) \in \mathcal{S}_n^\alpha(\lambda,\beta,b,q)$, which completes the proof of Theorem 1. □

The following corollary follows easily from Theorem 1.

Corollary 1. *Let $f(z) \in \mathcal{S}_n^\alpha(\lambda,\beta,b,q)$. Then*

$$a_k \leqq \frac{\beta\,|b|}{([k]_q+\beta\,|b|-1)[1+\alpha([k]_q-1)]A_q(\lambda,k)} \qquad (k \geqq n+1). \tag{2.4}$$

The result is sharp for the function $f(z)$ given (for $(k \geqq n+1)$ by

$$f(z) = z - \frac{\beta\,|b|}{([k]_q+\beta\,|b|-1)[1+\alpha([k]_q-1)]A_q(\lambda,k)} z^k. \tag{2.5}$$

Putting $\beta = 1$ in Theorem 1, we have Corollary 2 below.

Corollary 2. *Let $f(z) \in \mathcal{S}_n^\alpha(\lambda,b,q)$. Then*

$$\sum_{k=n+1}^{\infty} ([k]_q+|b|-1)[1+\alpha([k]_q-1)]A_q(\lambda,k)a_k \leqq |b|.$$

Corollary 3. *Let $f(z) \in \mathcal{S}_n^\alpha(\lambda,b,q)$. Then*

$$a_k \leqq \frac{|b|}{([k]_q+|b|-1)[1+\alpha([k]_q-1)]A_q(\lambda,k)} \qquad (k \geqq n+1).$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{|b|}{([k]_q+|b|-1)[1+\alpha([k]_q-1)]A_q(\lambda,k)} z^k \qquad (k \geqq n+1).$$

It is not difficult to prove the following results. The details involved are being left as an exercise for the interested reader.

Theorem 2. *The function $f(z) \in \mathcal{G}_n^\alpha(\lambda,\beta,b,q)$ if and only if*

$$\sum_{k=n+1}^{\infty} [k]_q\big[1+\alpha([k]_q-1)\big]A_q(\lambda,k)a_k \leqq \beta\,|b|. \tag{2.6}$$

Corollary 4. *Let $f(z) \in \mathcal{G}_n^\alpha(\lambda,\beta,b,q)$. Then*

$$a_k \leqq \frac{\beta\,|b|}{[k]_q\big[1+\alpha([k]_q-1)\big]A_q(\lambda,k)}. \tag{2.7}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{\beta\,|b|}{[k]_q\big[1+\alpha[k]_q-1\big]A_q(\lambda,k)} z^k \qquad (k \geqq n+1). \tag{2.8}$$

We now state (without proof) Theorem 3 below.

Theorem 3. *If $b_1, b_2 \in \mathbb{C}^*$ and $|b_1| < |b_2|$, then*

$$\mathcal{S}_n^\alpha(\lambda,\beta,b_1,q) \subset \mathcal{S}_n^\alpha(\lambda,\beta,b_2,q).$$

The following result can indeed be proven along the lines which we have already indicated above.

Theorem 4. *If $b_1, b_2 \in \mathbb{C}^*$ and $|b_1| < |b_2|$, then*

$$\mathcal{G}_n^\alpha(\lambda,\beta,b_1,q) \subset \mathcal{G}_n^\alpha(\lambda,\beta,b_2,q). \tag{2.9}$$

3. Extreme points for the function classes $\mathcal{S}_n^\alpha(\lambda,\beta,b,q)$ and $\mathcal{G}_n^\alpha(\lambda,\beta,b,q)$

In this section, we first prove the following result.

Theorem 5. *Let $f_n(z) = z$ and*

$$f_k(z) = z - \frac{\beta\,|b|}{([k]_q + \beta\,|b| - 1)[1+\alpha([k]_q - 1)]A_q(\lambda,k)} z^k \tag{3.1}$$

$$(k \geqq n+1).$$

Then the function $f(z)$ is in the class $\mathcal{S}_n^\alpha(\lambda,\beta,b,q)$ if and only if it can be expressed in the following form:

$$f(z) = \sum_{k=n}^{\infty} \mu_k f_k(z), \tag{3.2}$$

where

$$\sum_{k=n}^{\infty} \mu_k = 1 \qquad \text{and} \qquad \mu_k \geqq 0.$$

Proof. By assuming (3.2) to hold true, if we appropriately apply Theorem 1, it is not difficult to conclude that $f(z) \in \mathcal{S}_n^\alpha(\lambda,\beta,b,q)$.

Conversely, upon leting $f(z) \in \mathcal{S}_n^\alpha(\lambda,\beta,b,q)$, if we set

$$\mu_k = \frac{([k]_q + \beta\,|b| - 1)[1+\alpha([k]_q - 1)]A_q(\lambda,k)}{\beta\,|b|} a_k \qquad (k \geqq n+1)$$

and

$$\mu_n = 1 - \sum_{k=n+1}^{\infty} \mu_k,$$

we can easily see that $f(z)$ can be expressed in the form (3.2). This completes the proof of Theorem 5. □

Corollary 5. *The extreme points of the function class $\mathcal{S}_n^\alpha(\lambda,\beta,b,q)$ are the functions $f_n(z) = z$ and $f_k(z)$ $(k \geqq n+1)$ given by (3.1).*

Similarly, we can prove the following theorem.

Theorem 6. *Let $f_n(z) = z$ and*

$$f_k(z) = z - \frac{\beta\,|b|}{[k]_q[1+\alpha([k]_q - 1)]A_q(\lambda,k)} z^k \qquad (k \geqq n+1). \tag{3.3}$$

Then the function $f(z)$ *is in the class* $\mathcal{S}_n^\alpha(\lambda,\beta,b,q)$ *if and only if it can be expressed in the form given by*

$$f(z)=\sum_{k=n}^{\infty}\mu_k f_k(z), \tag{3.4}$$

where

$$\sum_{k=n}^{\infty}\mu_k=1 \qquad \text{and} \qquad \mu_k \geqq 0. \tag{3.5}$$

Corollary 6. *The extreme points of the function class* $\mathcal{G}_n^\alpha(\lambda,\beta,b,q)$ *are the functions* $f_n(z)=z$ *and* $f_k(z)$ $(k \geqq n+1)$ *given by* (3.3).

4. Radii of close-to-convexity, starlikeness and convexity of the function class $\mathcal{S}_n^\alpha(\lambda,\beta,b,q)$

Theorem 7. *Let* $f(z)\in\mathcal{S}_n^\alpha(\lambda,\beta,b,q)$. *Then* $f(z)$ *is close-to-convex of order* ρ $(0\leqq\rho<1)$ *in* $|z|<r_1$, *where*

$$r_1:=\inf_{k\geqq n+1}\left\{\frac{(1-\rho)([k]_q+\beta|b|-1)[1+\alpha([k]_q-1)]A_q(\lambda,k)}{k\beta|b|}\right\}^{\frac{1}{k-1}}. \tag{4.1}$$

The sharpness of this result is attained for the function $f(z)$ *given by* (2.5).

Proof. By showing that

$$\left|f'(z)-1\right|\leqq 1-\rho \quad \text{for} \quad |z|<r_1,$$

where r_1 is given by (4.1), we readily find that

$$\left|f'(z)-1\right|\leqq 1-\rho,$$

if

$$\sum_{k=n+1}^{\infty}\frac{k}{1-\rho}a_k|z|^{k-1}\leqq 1. \tag{4.2}$$

But, by Theorem 1, it is seen that (4.2) will hold true if (for $k\geqq n+1$)

$$|z|\leqq\left(\frac{(1-\rho)([k]_q+\beta|b|-1)[1+\alpha([k]_q-1)]A_q(\lambda,k)}{k\beta|b|}\right)^{\frac{1}{k-1}}.$$

This completes the proof of Theorem 7. □

By using arguments and analysis similar to those in the proof of Theorem 7, we can analogously derive Theorem 8 and Corollary 7 below.

Theorem 8. *Let $f(z) \in \mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$. Then the function $f(z)$ is starlike of order ρ $(0 \leqq \rho < 1)$ in $|z| < r_2$, where*

$$r_2 := \inf_{k \geqq n+1} \left\{ \frac{(1-\rho)([k]_q + \beta\,|b| - 1)[1+\alpha([k]_q - 1)]A_q(\lambda,k)}{(k-\rho)\beta\,|b|} \right\}^{\frac{1}{k-1}}. \tag{4.3}$$

The sharpness of this result is attained for the function $f(z)$ given by (2.5).

Corollary 7. *Let $f(z) \in \mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$. Then the function $f(z)$ is convex of order ρ $(0 \leqq \rho < 1)$ in $|z| < r_3$, where*

$$r_3 := \inf_{k \geqq n+1} \left\{ \frac{(1-\rho)([k]_q + \beta\,|b| - 1)[1+\alpha([k]_q - 1)]A_q(\lambda,k)}{k(k-\rho)\beta\,|b|} \right\}^{\frac{1}{k-1}}.$$

The sharpness of the result is attained for the function $f(z)$ given by (2.5).

5. Growth and distortion theorems

For convenience in this section, for $k \geqq n+1$, we shall henceforth use the following notations:

$$\sigma_{k,\alpha}(\lambda,\beta,b,q) := ([k]_q + \beta\,|b| - 1)[1+\alpha([k]_q - 1)]A_q(\lambda,k) \tag{5.1}$$

and

$$\phi_{k,\alpha}(\lambda,\beta,b,q) := [k]_q[1+\alpha([k]_q - 1)]A_q(\lambda,k). \tag{5.2}$$

We now prove the following which will be needed in our further investigation in this section.

Lemma 1. *The sequence $\{A_q(\lambda,k)\}_{k=n+1}^{\infty}$ is a decreasing sequence in k $(k \geqq n+1)$ for $\lambda < 2$ and $0 < q < 1$.*

Proof. It follows from (1.8) and the recurrence relation:

$$\Gamma_q(z+1) = [z]_q\,\Gamma_q(z)$$

that

$$\begin{aligned}\frac{A_q(\lambda,k+1)}{A_q(\lambda,k)} &= \frac{\Gamma_q(k+2)\Gamma_q(k-\lambda+1)}{\Gamma_q(k+1)\Gamma_q(k-\lambda+2)} \\ &= \frac{[k+1]_q\Gamma_q(k+1)\Gamma_q(k-\lambda+1)}{\Gamma_q(k+1)[k-\lambda+1]_q\Gamma_q(k-\lambda+2)} = \frac{[k+1]_q}{[k-\lambda+1]_q}.\end{aligned}$$

It is sufficient to consider the value $k = n+1$. By using the definition (2.1) of the basic (or q-) number $[\lambda]_q$ again, we thus find that

$$\frac{A_q(\lambda,k+1)}{A_q(\lambda,k)} = \frac{[n+2]_q}{[n-\lambda+2]_q} = \frac{1-q^{n+2}}{1-q^{n-\lambda+2}} \qquad (0 < q < 1;\ -\infty < \lambda < 2).$$

The sequence $\{A_q(\lambda,k)\}_{k=n+1}^{\infty}$ is a decreasing sequence in k if

$$\frac{A_q(\lambda,k+1)}{A_q(\lambda,k)} < 1 \qquad (k \geqq n+1),$$

that is, if

$$\frac{1-q^{n+2}}{1-q^{n-\lambda+2}} < 1 \qquad (0<q<1;\ -\infty<\lambda<2), \tag{5.3}$$

which implies that $\lambda < 0$. Thus $\{A_q(\lambda,k)\}_{k=n+1}^{\infty}$ is a decreasing sequence in k $(k \geqq n+1)$ for $-\infty<\lambda<2$ and $0<q<1$. □

Theorem 9. *Let $f(z) \in \mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$. Then*

$$|z| - \frac{\beta\,|b|}{\sigma_{n+1,\alpha}(\lambda,\beta,b,q)}\,|z|^{n+1} \leqq |f(z)| \leqq |z| + \frac{\beta\,|b|}{\sigma_{n+1,\alpha}(\lambda,\beta,b,q)}\,|z|^{n+1}. \tag{5.4}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{\beta\,|b|}{\sigma_{n+1,\alpha}(\lambda,\beta,b,q)}\,z^{n+1}. \tag{5.5}$$

Proof. Since $f(z) \in \mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$, in view of Theorem 1, we have

$$\sigma_{n+1,\alpha}(\lambda,\beta,b,q)\sum_{k=n+1}^{\infty} a_k \leqq \sum_{k=n+1}^{\infty} \sigma_{k,\alpha}(\lambda,\beta,b,q)a_k \leqq \beta\,|b|,$$

that is,

$$\sum_{k=n+1}^{\infty} a_k \leqq \frac{\beta\,|b|}{\sigma_{n+1,\alpha}(\lambda,\beta,b,q)}. \tag{5.6}$$

We thus obtain

$$\begin{aligned} |f(z)| &\geqq |z| - \sum_{k=n+1}^{\infty} a_k\,|z|^k \geqq |z| - |z|^{n+1}\sum_{k=n+1}^{\infty} a_k \\ &\geqq |z| - \frac{\beta\,|b|}{\sigma_{n+1,\alpha}(\lambda,\beta,b,q)}\,|z|^{n+1} \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} |f(z)| &\leqq |z| + \sum_{k=n+1}^{\infty} a_k\,|z|^k \leqq |z| + |z|^{n+1}\sum_{k=n+1}^{\infty} a_k \\ &\leqq |z| + \frac{\beta\,|b|}{\sigma_{n+1,\alpha}(\lambda,\beta,b,q)}\,|z|^{n+1}. \end{aligned} \tag{5.8}$$

These last inequalities (5.7) and (5.8) complete the proof of Theorem 9. □

Corollary 8. *Under the hypothesis of Theorem 9, the function $f(z)$ is included in a disk with center at the origin and radius r given by*

$$r = 1 + \frac{\beta\,|b|}{\sigma_{n+1,\alpha}(\lambda,\beta,b,q)}.$$

Similarly, we can prove the following distortion theorem for $f(z) \in \mathcal{G}_n^{\alpha}(\lambda,\beta,b,q)$.

Theorem 10. *Let $f(z) \in \mathcal{G}_n^{\alpha}(\lambda,\beta,b,q)$ and let $\phi_{k,\alpha}(\lambda,\beta,b,q)$ be given by (5.2). Then*

$$|z| - \frac{\beta\,|b|}{\phi_{n+1,\alpha}(\lambda,\beta,b,q)}\,|z|^{n+1} \leqq |f(z)| \leqq |z| + \frac{\beta\,|b|}{\phi_{n+1,\alpha}(\lambda,\beta,b,q)}\,|z|^{n+1}. \tag{5.9}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{\beta\,|b|}{\phi_{n+1,\alpha}(\lambda,\beta,b,q)}\,z^{n+1}. \tag{5.10}$$

Corollary 9. *Under the hypothesis of Theorem 10, the function $f(z)$ is included in a disk with its center at the origin and its radius r given by*

$$r = 1 + \frac{\beta\,|b|}{\phi_{n+1,\alpha}(\lambda,\beta,b,q)}.$$

A further distortion theorem involving the generalized fractional q-differintegral operator $\Omega_{q,z}^{\lambda}$ defined by (1.7) is given by the following theorem.

Theorem 11. *Let $f(z) \in \mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$. Then*

$$\begin{aligned} &|z| - \frac{\beta\,|b|}{([n+1]_q + \beta\,|b| - 1)[1+\alpha([n+1]_q - 1)]}\,|z|^{n+1} \\ &\quad \leqq \left|\Omega_{q,z}^{\lambda} f(z)\right| \\ &\quad \leqq |z| + \frac{\beta\,|b|}{([n+1]_q + \beta\,|b| - 1)[1+\alpha([n+1]_q - 1)]}\,|z|^{n+1}. \end{aligned} \tag{5.11}$$

The result is sharp.

Proof. From the above Lemma 1, in conjunction with the equations (5.6) and (1.7), we have

$$\begin{aligned} \left|\Omega_{q,z}^{\lambda} f(z)\right| &\geqq |z| - A_q(\lambda,n+1)\,|z|^{n+1} \sum_{k=n+1}^{\infty} a_k \\ &\geqq |z| - \frac{\beta\,|b|}{([n+1]_q + \beta\,|b| - 1)[1+\alpha([n+1]_q - 1)]}\,|z|^{n+1} \end{aligned} \tag{5.12}$$

and

$$\left|\Omega_{q,z}^{\lambda} f(z)\right| \leqq |z| + A_q(\lambda,n+1)\,|z|^{n+1} \sum_{k=n+1}^{\infty} a_k$$

$$\leqq |z| + \frac{\beta\,|b|}{([n+1]_q + \beta\,|b| - 1)[1+\alpha([n+1]_q - 1)]}\,|z|^{n+1}. \tag{5.13}$$

The equalities in (5.11) are attained for the function $f(z)$ given by

$$D_{q,z}^{\lambda} f(z) = \frac{\Gamma_q(z) z^{1-\lambda}}{\Gamma_q(2-\lambda)} \cdot \left(1 - \frac{\beta\,|b|}{([n+1]_q + \beta\,|b| - 1)[1+\alpha([n+1]_q - 1)]}\,|z|^{n}\right) \tag{5.14}$$

or by the function $f(z)$ given by (5.5). We have thus completed our demonstration of Theorem 11. □

From Theorem 10 and (1.7), we have the following distortion inequality involving the fractional q-derivative operator $D_{q,z}^{\lambda}$.

Corollary 10. *Let $f(z) \in \mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$. Then*

$$\frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)}\,|z|^{1-\lambda}\left(1 - \frac{\beta\,|b|}{([n+1]_q + \beta\,|b| - 1)[1+\alpha([n+1]_q - 1)]}\,|z|^{n}\right) \leqq \left|D_{q,z}^{\lambda} f(z)\right| \leqq \frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)}\,|z|^{1-\lambda} \cdot \left(1 + \frac{\beta\,|b|}{([n+1]_q + \beta\,|b| - 1)[1+\alpha([n+1]_q - 1)]}\,|z|^{n}\right). \tag{5.15}$$

The result is sharp for the function $f(z)$ given by (5.5).

Upon setting $\beta = 1$ in Corollary 10, we get the following corollary which provided the *corrected* version of a result obtained by Purohit and Raina [12, Corollary 1].

Corollary 11. *Let $f(z) \in \mathcal{S}_n^{\alpha}(\lambda,b,q)$. Then*

$$\frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)}\,|z|^{1-\lambda}\left(1 - \frac{|b|}{([n+1]_q + |b| - 1)[1+\alpha([n+1]_q - 1)]}\,|z|^{n}\right) \leqq \left|D_{q,z}^{\lambda} f(z)\right| \leqq \frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)}\,|z|^{1-\lambda} \cdot \left(1 + \frac{|b|}{([n+1]_q + |b| - 1)[1+\alpha([n+1]_q - 1)]}\,|z|^{n}\right). \tag{5.16}$$

The result is sharp for the function $f(z)$ given by (5.5) with $\beta = 1$.

Also, in view of (1.9) or by virtue of (1.3), Theorem 10 gives the following distortion inequality involving the fractional q-integral operator $I_{q,z}^{\lambda}$.

Corollary 12. *Let $f(z) \in \mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$. Then*

$$\frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)}\,|z|^{1+\lambda}\left(1 - \frac{\beta\,|b|}{([n+1]_q + \beta\,|b| - 1)[1+\alpha([n+1]_q - 1)]}\,|z|^{n}\right)$$

$$\leqq \left|I_{q,z}^{\lambda} f(z)\right| \frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)} |z|^{1+\lambda} \cdot \left(1 + \frac{\beta |b|}{([n+1]_q + \beta |b| - 1)[1+\alpha([n+1]_q - 1)]} |z|^n\right). \tag{5.17}$$

The result is sharp for the function $f(z)$ given by (5.5).

Putting $\beta = 1$ in Corollary 12, we have the following result.

Corollary 13. *Let $f(z) \in \mathcal{S}_n^{\alpha}(\lambda, b, q)$. Then*

$$\frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)} |z|^{1+\lambda} \left(1 - \frac{|b|}{([n+1]_q + |b| - 1)[1+\alpha([n+1]_q - 1)]} |z|^n\right) \leqq \left|I_{q,z}^{\lambda} f(z)\right| \leqq \frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)} |z|^{1+\lambda} \cdot \left(1 + \frac{|b|}{([n+1]_q + |b| - 1)[1+\alpha([n+1]_q - 1)]} |z|^n\right). \tag{5.18}$$

The result is sharp for the function $f(z)$ given by (5.5) with $\beta = 1$ and λ replaced by $-\lambda$.

Theorem 12. *Let $f(z) \in \mathcal{G}_n^{\alpha}(\lambda, \beta, b, q)$. Then*

$$|z| - \frac{\beta |b|}{[n+1]_q [1+\alpha([n+1]_q - 1)]} |z|^{n+1} \leqq \left|\Omega_{q,z}^{\lambda} f(z)\right| \leqq |z| + \frac{\beta |b|}{[n+1]_q [1+\alpha([n+1]_q - 1)]} |z|^{n+1}. \tag{5.19}$$

The result is sharp for the function $f(z)$ given by

$$D_{q,z}^{\lambda} f(z) = \frac{\Gamma_q(z) z^{1-\lambda}}{\Gamma_q(2-\lambda)} \left(1 - \frac{\beta |b|}{[n+1]_q [1+\alpha([n+1]_q - 1)]} |z|^n\right) \tag{5.20}$$

or by the function $f(z)$ given by (5.10).

Similarly, we can prove the following distortion inequalities for $f(z) \in \mathcal{G}_n^{\alpha}(\lambda, \beta, b, q)$ involving the fractional q-derivative operator $D_{q,z}^{\lambda}$ and the fractional q-integral operator $I_{q,z}^{\lambda}$.

Corollary 14. *Let $f(z) \in G_n^{\alpha}(\lambda, \beta, b, q)$. Then*

$$\frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)} |z|^{1-\lambda} \left(1 - \frac{\beta |b|}{[n+1]_q [1+\alpha([n+1]_q - 1)]} |z|^n\right) \leqq \left|D_{q,z}^{\lambda} f(z)\right|$$

$$\leqq \frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)}|z|^{1-\lambda}\left(1+\frac{\beta\,|b|}{[n+1]_q[1+\alpha([n+1]_q-1)]}|z|^n\right). \tag{5.21}$$

The result is sharp for the function $f(z)$ *given by* (5.10).

Corollary 15. *Let* $f(z)\in\mathcal{G}_n^\alpha(\lambda,\beta,b,q)$. *Then*

$$\begin{aligned}&\frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)}|z|^{1+\lambda}\left(1-\frac{\beta\,|b|}{[n+1]_q[1+\alpha([n+1]_q-1)]}|z|^n\right)\\&\quad\leqq\left|I_{q,z}^\lambda f(z)\right|\\&\quad\leqq\frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)}|z|^{1+\lambda}\left(1+\frac{\beta\,|b|}{[n+1]_q[1+\alpha([n+1]_q-1)]}|z|^n\right).\end{aligned} \tag{5.22}$$

The result is sharp for the function $f(z)$ *given by* (5.10).

Putting $\beta=1$ in Corollaries 14 and 15, respectively, we have the following corollaries.

Corollary 16. *Let* $f(z)\in\mathcal{G}_n^\alpha(\lambda,b,q)$. *Then*

$$\begin{aligned}&\frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)}|z|^{1-\lambda}\left(1-\frac{|b|}{[n+1]_q[1+\alpha([n+1]_q-1)]}|z|^n\right)\\&\quad\leqq\left|D_{q,z}^\lambda f(z)\right|\\&\quad\leqq\frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)}|z|^{1-\lambda}\left(1+\frac{|b|}{[n+1]_q[1+\alpha([n+1]_q-1)]}|z|^n\right).\end{aligned} \tag{5.23}$$

The result is sharp for the function $f(z)$ *given by* (5.10) *with* $\beta=1$.

Corollary 17. *Let* $f(z)\in\mathcal{G}_n^\alpha(\lambda,\beta,b,q)$. *Then*

$$\begin{aligned}&\frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)}|z|^{1+\lambda}\left(1-\frac{|b|}{[n+1]_q[1+\alpha([n+1]_q-1)]}|z|^n\right)\\&\quad\leqq\left|I_{q,z}^\lambda f(z)\right|\\&\quad\leqq\frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)}|z|^{1+\lambda}\left(1+\frac{|b|}{[n+1]_q[1+\alpha([n+1]_q-1)]}|z|^n\right).\end{aligned} \tag{5.24}$$

The result is sharp for the function $f(z)$ *given by* (5.10) *with* $\beta=1$.

Remark 3. The results asserted by Corollaries 15 and 16 provide, respectively, the *corrected* versions of the results obtained by Purohit and Raina [12, Corollaries 3 and 4].

Remark 4. Putting $\lambda=0$ in our results, we obtain a number of new results for the function classes $\mathcal{S}_n^\alpha(\beta,b,q)$ and $\mathcal{G}_n^\alpha(\beta,b,q)$.

6. Conclusion

In our present investigation, we applied various operators of q-calculus and fractional q-calculus in the study of two general subclasses $\mathcal{S}_n^\alpha(\lambda,\beta,b,q)$ and $\mathcal{G}_n^\alpha(\lambda,\beta,b,q)$ of normalized analytic functions with complex order and negative coefficients. For each of these function classes, we have derived their associated coefficient estimates, radii of close-to-convexity, starlikeness and convexity, extreme points, and growth and distortion theorems. Our main results and their new or known consequences are stated and proved as theorems and corollaries.

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Authors' addresses

H. M. Srivastava

Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada and Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan, Republic of China and Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, AZ1007 Baku, Azerbaijan

E-mail address: harimsri@math.uvic.ca

M. K. Aouf

Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

E-mail address: mkaouf127@yahoo.com

A. O. Mostafa

Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

E-mail address: adelaeg254@yahoo.com

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MULTIPLE SOLUTIONS OF A DIRICHLET PROBLEM IN ONE-DIMENSIONAL BILLIARD SPACE

JAN TOMEČEK

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Abstract. The paper gives multiplicity results for the impulsive boundary value problem

$$x'' = f(t,x), \qquad \text{for a.e. } t \in [0,T], \text{ such that } x(t) \in \operatorname{int} K,$$
$$x'(s+) = -x'(s-), \quad \text{if } s \in (0,T),\ x(s) \in \partial K,$$
$$x(0) = A, \quad x(T) = B,$$

where $K \subset \mathbb{R}$ is a compact interal, f is Carathéodory function on $[0,T] \times K$ and $A, B \in \operatorname{int} K$. This problem can be understood as a problem in one-dimensional billiard space and it is also a generalization of oscillator with obstacles from below and from above and absolutely elastic impacts. A simple condition for the existence of solution with exact number of impacts is given, as well as the multiplicity result. The results are obtained by a transformation into problem without impulses (without impacts) and using Schauder fixed point theorem.

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Keywords: state-dependent impulses, system with impact, billiard space, Dirichlet problem, existence and multiplicity result

1. Introduction

The theory of impulsive differential equations is quite well developed theory. It received a lot of attention during the last decades, see e.g. monographs [1, 3, 5, 6, 8, 10, 11]. The theory enables to model phenomena described by differential equations where abrupt changes on short time intervals (often caused by external forces) occur. It is more advantageous to model these changes as instantaneous (so called impulses).

Systems with impact draw attention of many mathematicians and physicists in the past several centuries. Let us mention a monograph [7] and some papers investigating periodic solutions of linear impact oscillator, see e.g. [2, 9] and references therein. Note, that the dynamics with impacts can be understood by means of impulsive differential equations. Here the impulse (discontinuity) occurs at the time of impact and the velocity (derivative) undergoes discontinuous change (the mass bounces off of an obstacle), e.g. billiard ball hits the boundary of the table, mass on spring hits some obstacle, etc.

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Recently, in [4], Dirichlet problem in billiard spaces with absolutely elastic impact

$$\begin{aligned} &x'' = f(t,x) \quad \text{for a.e. } t \in [0,T], x(t) \in \operatorname{int} K, \\ &\triangle x'(s) = I(x(s), x'(s)), \quad x(s) \in \partial K, \\ &x(0) = x(T) = 0, \end{aligned}$$

was investigated for two cases: (i) one-dimensional case, i.e. $K \subset \mathbb{R}$ is a compact interval, (ii) multidimensional case, for $K \subset \mathbb{R}^{2k}$, $k \in \mathbb{N}$, certain appropriate compact set. In the one-dimensional case, there were reached multiplicity results under conditions that f is Carathéodory condition and is Lipschitz continuous in the second variable. The existence of infinitely solutions are then guaranteed. The purpose of this paper was to give more general and more detailed information than in [4] for the one-dimensional case. Here, the Lipschitz condition is not needed. Moreover, the exact number of impacts is given (see Theorem 1). The reason of the weakened assumptions lies in the fact that paper [4] relies on the shooting method and continuous dependence on initial values of solution. On the other hand, the idea in [4] can be quite easily generalized for higher dimensions. Here, the idea is to investigate an auxiliary problem without impulses.

In this paper the impulsive boundary value problem

$$x'' = f(t,x), \qquad \text{for a.e. } t \in [0,T], \text{ such that } x(t) \in \operatorname{int} K, \tag{1.1}$$

$$x'(s+) = -x'(s-), \quad \text{if } s \in (0,T),\ x(s) \in \partial K, \tag{1.2}$$

$$x(0) = A, \quad x(T) = B, \tag{1.3}$$

where $K \subset \mathbb{R}$ is some compact interval, $f : [0,T] \times K \to \mathbb{R}$ is a Carathéodory function, A, $B \in \operatorname{int} K$, is investigated. The impulsive condition (1.2) represents absolutely elastic impact on the boundary of K.

First let us give the definition of a solution to this problem.

Definition 1. The function $x \in \mathrm{C}([0,T])$ is called a solution of the problem (1.1),(1.2), if and only if

- there exists a finite set $\{t_1, \dots, t_p\} \subset (0,T)$, $t_1 < t_2 < \dots < t_p$ such that $x|_{[t_i, t_{i+1}]} \in \mathrm{AC}^1([t_i, t_{i+1}])$ for $i = 0, \dots, p$, where we put $t_0 = 0$, $t_{p+1} = T$,
- $x(t) \in K$ for each $t \in [0,T]$ and $x(t) \in \partial K$ iff $t = t_i$ for some $i = 1, \dots, p$,
- x satisfies the differential equation (1.1) and impulsive conditions (1.2).

The number p is called a number of impacts of a solution x. A solution x of (1.1),(1.2) is called a solution of (1.1)–(1.3), if it satisfies (1.3).

The main result of this paper is the following.

Theorem 1. *Let $K = [a,b]$ be a compact interval, $f \in \mathrm{Car}([0,T] \times K)$ and $A, B \in \operatorname{int} K$. Then for each $p \in \mathbb{N}$ satisfying*

$$p > \frac{T}{b-a} \int_0^T |m(t)| \, \mathrm{d}t + 1,$$

where

$$m(t) = \max_{x \in K} |f(t,x)|, \quad a.e.\ t \in [0,T],$$

the problem (1.1)–(1.3) *has at least two solutions with exactly* p *impacts.*

The paper is organized as follows. In Section 2 necessary notation and some trivial results are presented. In Section 3, auxiliary singular nonimpulsive problem is investigated and its relation to the original problem is shown. Finally, Section 4 contains the main results of the paper, incl. the proof of Theorem 1.

2. Notation and preliminaries

By $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ we denote the set of all positive integers, integers and reals, respectively. By $\mathrm{AC}(J)$, $(\mathrm{AC}^1(J))$ we denote the set of absolutely continuous functions on the interval J (having absolutely continuous derivatives on J) and by $\mathrm{C}([0,T])$, $\mathrm{C}^1([0,T])$ the space of continuous functions on the interval $[0,T]$ (having continuous derivatives) equipped with standard norms. Moreover, $\mathrm{L}^1([0,T])$ is the space of all Lebesgue integrable functions on $[0,T]$ with the norm $\|x\|_{\mathrm{L}^1} = \int_0^T |x(t)|\,\mathrm{d}t$, $x \in \mathrm{L}^1([0,T])$. By $\mathrm{Car}([0,T]\times\Omega)$, where $\Omega \subset \mathbb{R}$ we mean the set of all functions $f : [0,T]\times\Omega \to \mathbb{R}$ satisfying Carathéodory conditions:

- for each $x \in \Omega$ the function $f(\cdot,x)$ is Lebesgue measurable on $[0,T]$,
- for a.e. $t \in [0,T]$ the function $f(t,\cdot)$ is continuous on Ω,
- for each compact set $K \subset \Omega$, there exists $m \in \mathrm{L}^1([0,T])$ such that $|f(t,x)| \le m(t)$ for a.e. $t \in [0,T]$.

Also, let us remind some facts about the fractional part of a real number: For $x, a \in \mathbb{R}$, $a > 0$ we denote

$$x \bmod a := x - a\left\lfloor \frac{x}{a} \right\rfloor,$$

where $\lfloor x \rfloor$ stands for the integer part of x. Let us note that the function

$$\mathbb{R} \ni t \mapsto t \bmod a$$

is a-periodic function.

Remark 1. Let $x : J \to \mathbb{R}$, $J \subset \mathbb{R}$ be an interval, $a \in \mathbb{R}$, $a > 0$, $x \bmod a \neq 0$ on J, $i \in \mathbb{N}$. Then

- if x is continuous on J, then is $x \bmod a$;
- if x has an i-th derivative at $t \in \operatorname{int} J$, then $(x \bmod a)^{(i)}(t) = x^{(i)}(t)$.

Lemma 1. *Let* $R \in \mathbb{R}$, $R > 0$ *and* $\varphi : (0,R) \to \mathbb{R}$. *Then the function* $\varphi^* : \mathbb{R} \to \mathbb{R}$ *defined by*

$$\varphi^*(x) = \begin{cases} \varphi(x \bmod 2R) & \text{if } x \bmod 2R \in (0,R), \\ -\varphi(2R - (x \bmod 2R)) & \text{if } x \bmod 2R \in (R,2R), \\ 0 & \text{if } x \bmod R = 0 \end{cases}$$

is $2R$-periodic and odd function.

Proof. The periodicity follows directly from the definition of φ^* and periodicity of the mapping $x \mapsto x \bmod 2R$. In order to prove that the function is odd we consider three cases:
CASE A. Let $x \bmod R = 0$. Then also $(-x) \bmod R = 0$ and therefore $\varphi^*(-x) = 0 = -\varphi^*(x)$.
CASE B. Let $x \bmod 2R \in (0, R)$. Then there exists an integer k such that $x = 2Rk + x \bmod 2R$. Then

$$-x = 2R(-k) - x \bmod 2R = 2R(-k-1) + 2R - x \bmod 2R,$$

where $2R - x \bmod 2R \in (R, 2R)$ and therefore $(-x) \bmod 2R = 2R - x \bmod 2R \in (R, 2R)$. Then

$$\begin{aligned}\varphi^*(-x) &= -\varphi(2R - (-x) \bmod 2R) = -\varphi(2R - (2R - x \bmod 2R)) \\ &= -\varphi(x \bmod 2R) = -\varphi^*(x).\end{aligned}$$

CASE C. The case $x \bmod 2R \in (R, 2R)$ can be proved similarly to the previous one. □

Remark 2. If φ is continuous on $(0, R)$, then from the properties of φ^* (from Lemma 1) it follows only that φ^* is continuous on each interval $(kR, (k+1)R)$, $k \in \mathbb{Z}$ and

$$\varphi^*(kR+) = -\varphi^*(kR-) = \begin{cases} \varphi(0+) & \text{if } k \text{ is even,} \\ -\varphi(R-) & \text{if } k \text{ is odd.} \end{cases}$$

Since $\varphi^*(kR) = 0$ for $k \in \mathbb{Z}$, the function φ^* has discontinuities at $x = kR$, $k \in \mathbb{Z}$ unless $\varphi(0+) = \varphi(R-) = 0$.

Remark 3. In the next section will use function

$$\Delta(t) = \begin{cases} t \bmod 2R & \text{if } t \bmod 2R \in [0, R), \\ 2R - t \bmod 2R & \text{if } t \bmod 2R \in [R, 2R). \end{cases}$$

Let us note that inclusion $t \bmod 2R \in [0, R)$ is equivalent to $t \in [2kR, (2k+1)R)$ for some $k \in \mathbb{Z}$ and the inclusion $t \bmod 2R \in [R, 2R)$ is equivalent to $t \in [(2k+1)R, 2(k+1)R)$ for some $k \in \mathbb{Z}$. Then, we can see that $\Delta : \mathbb{R} \to \mathbb{R}$, $\Delta(t) = |t|$ for $t \in [-R, R]$ and Δ is $2R$-periodic. Therefore, range of Δ is $[0, R]$, it is a piecewise linear function, it is Lipschitz continuous on $\mathbb{R}$. Moreover,

$$\Delta'(t) = (-1)^k \quad \text{if } t \in (kR, (k+1)R),\ k \in \mathbb{Z}.$$

3. Singular auxiliary problem

In this section we assume that

$$R \in \mathbb{R},\ R > 0,\ K = [0,R],\ T > 0,\ f \in \mathrm{Car}([0,T] \times K), \tag{3.1}$$

and let us denote

$$m(t) := \sup_{x \in K} |f(t,x)| \quad \text{for a.e. } t \in [0,T], \quad \overline{m} := \|m\|_{\mathrm{L}^1}. \tag{3.2}$$

Unlike the paper [4], here the existence result is obtained by using an appropriate "extension" of the right-hand side of the equation (1.1) onto the set $[0,T] \times \mathbb{R}$. Note that this extension is not neccessarily a Carathéodory function on $[0,T] \times \mathbb{R}$, but it has possible discontinuities in the state variable.

We define an auxiliary function $f^* : [0,T] \times \mathbb{R} \to \mathbb{R}$ by

$$f^*(t,x) = \begin{cases} f(t, x \bmod 2R) & \text{if } x \bmod 2R \in (0,R), \\ -f(t, 2R - (x \bmod 2R)) & \text{if } x \bmod 2R \in (R, 2R), \\ 0, & \text{if } x \bmod R = 0, \end{cases} \quad \text{for a.e. } t \in [0,T]. \tag{3.3}$$

As we can see from Lemma 1 the function f^* is odd and $2R$-periodic in the second variable. We consider an auxiliary differential equation

$$y'' = f^*(t,y). \tag{3.4}$$

Its purpose is cleared out in Lemma 2. By *a solution of* (3.4) we understand a function $y \in \mathrm{AC}^1([0,T])$ such that (3.4) is satisfied almost everywhere on $[0,T]$.

Lemma 2. *If y is a strictly monotone solution of equation* (3.4) *such that $y(0) \bmod R \neq 0$, $y(T) \bmod R \neq 0$, then the function*

$$x(t) = (\Delta \circ y)(t) = \begin{cases} y(t) \bmod 2R & \text{if } y(t) \bmod 2R \in [0,R), \\ 2R - (y(t) \bmod 2R) & \text{if } y(t) \bmod 2R \in [R, 2R), \end{cases} \quad t \in [0,T] \tag{3.5}$$

is a solution of impulsive equation (1.1),(1.2) *having exactly*

$$\left| \left\lfloor \frac{y(0)}{R} \right\rfloor - \left\lfloor \frac{y(T)}{R} \right\rfloor \right|$$

impacts.

Proof. Let y be a strictly increasing solution of (3.4). From Remark 3 it follows that x is continuous on $[0,T]$ and $x(t) \in K$ for each $t \in [0,T]$. Let us denote

$$k := \left\lfloor \frac{y(0)}{R} \right\rfloor, \quad l := \left\lfloor \frac{y(T)}{R} \right\rfloor, \quad p := \left| \left\lfloor \frac{y(0)}{R} \right\rfloor - \left\lfloor \frac{y(T)}{R} \right\rfloor \right|.$$

Then $p = |k - l| = l - k$, $kR < y(0) < (k+1)R$, $lR < y(T) < (l+1)R$. From the continuity and monotonicity it follows that there exist $0 < t_1 < t_2 < \ldots < t_p < T$ such

that

$$y(t_i) = (k+i)R, \quad i = 1, \dots, p.$$

Then for every $t \in [0,T]$ the equation $y(t) \bmod R = 0$ is satisfied iff $t = t_i$ for some $i \in \{1, \dots, p\}$, i.e. $x(t) \in \{0, R\} = \partial K$ iff $t = t_i$. This means that x has exactly p impacts.

Let us put $t_0 = 0$, $t_{p+1} = T$. For $i = 0, \dots, p$, $t \in (t_i, t_{i+1})$ we have $y(t) \in ((k+i)R, (k+i+1)R)$ and according to Remark 3 we see that

$$x'(t) = \Delta'(y(t))y'(t) = (-1)^{k+i} y'(t) \quad t \in (t_i, t_{i+1}),$$

i.e. x' is absolutely continuous on (t_i, t_{i+1}). Moreover

$$x'(t_i+) = \lim_{t \to t_i+} x'(t) = \lim_{t \to t_i+} (-1)^{k+i} y'(t) = (-1)^{k+i} y'(t_i), \quad i = 0, \dots, p$$

and

$$x'(t_i-) = \lim_{t \to t_i-} x'(t) = \lim_{t \to t_i-} (-1)^{k+i-1} y'(t) = (-1)^{k+i-1} y'(t_i), \quad i = 1, \dots, p+1.$$

Therefore $x|_{[t_i, t_{i+1}]}$ has absolutely continuous derivative on $[t_i, t_{i+1}]$, $i = 0, \dots, p$ and x satisfies (1.2).

Let $i \in \{0, \dots, p\}$. If $y(t) \bmod 2R \in (0, R)$ for each $t \in (t_i, t_{i+1})$, then by (3.3), (3.5) we have

$$x''(t) = (y(t) \bmod 2R)'' = y''(t) = f^*(t, y(t)) = f(t, y(t) \bmod 2R) = f(t, x(t))$$

for a.e. $t \in (t_i, t_{i+1})$. If $y(t) \bmod 2R \in (R, 2R)$ for $t \in (t_i, t_{i+1})$, then

$$\begin{aligned} x''(t) &= (2R - (y(t) \bmod 2R))'' = -y''(t) = -f^*(t, y(t)) \\ &= -(-f(t, 2R - (y(t) \bmod 2R))) = f(t, x(t)) \end{aligned}$$

for a.e. $t \in (t_i, t_{i+1})$. Therefore x satisfies Eq. (1.1). The proof is similar for the case when y is strictly decreasing. □

Note that the function f^* is not Carathéodory in general, since

$$\lim_{x \to Rk+} f^*(t,x) = -\lim_{x \to Rk-} f^*(t,x) \in \mathbb{R}.$$

for each $k \in \mathbb{Z}$ and a.e. $t \in [0,T]$, see Remark 2. Therefore we use regularization technique to obtain a solution of problem (3.4),(1.3), with $A, B \in \mathbb{R}$, satisfying $A \bmod R \neq 0 \neq B \bmod R$.

For $n \in \mathbb{N}$, we put

$$\eta_n(x) = \begin{cases} \frac{2n}{R} x & \text{if } x \in \left[0, \frac{R}{2n}\right), \\ 1 & \text{if } x \in \left[\frac{R}{2n}, R\left(1 - \frac{1}{2n}\right)\right], \\ \frac{2n}{R}(R - x) & \text{if } x \in \left(R\left(1 - \frac{1}{2n}\right), R\right]. \end{cases}$$

Let us emphasize that for every $n \in \mathbb{N}$ the function η_n has the following properties:

- $\eta_n : [0, R] \to \mathbb{R}$ is continuous,

- $\eta_n(0) = \eta_n(R) = 0$,
- $0 \le \eta_n(x) \le 1$ for $x \in [0, R]$ and
- $\eta_n(x) = 1$ for $x \in \left[\frac{R}{2n}, R\left(1 - \frac{1}{2n}\right)\right]$.

Let us consider truncated functions

$$f_n^*(t,x) = \eta_n(x \bmod R) f^*(t,x), \tag{3.6}$$

for a.e. $t \in [0,T]$, all $x \in \mathbb{R}$, $n \in \mathbb{N}$.

Remark 4. Let us note that from the assumption (3.1), properties of η_n and Remark 2, it follows that $f_n^* \in \mathrm{Car}([0,T] \times \mathbb{R})$, and moreover $|f^*(t,x)| \le m(t)$ as well as $|f_n^*(t,x)| \le m(t)$ for a.e. $t \in [0,T]$, all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

To obtain the existence result for boundary value problem (3.4), (1.3) we consider auxiliary equation

$$y'' = f_n^*(t,y) \qquad \text{for a.e. } t \in [0,T]. \tag{3.7}$$

for $n \in \mathbb{N}$.

Lemma 3. *Let us assume* (3.1), (3.2), $A, B \in \mathbb{R}$, $A \bmod R \neq 0 \neq B \bmod R$, $n \in \mathbb{N}$. *Then there exists at least one solution* $y_n \in \mathrm{AC}^1([0,T])$ *to the problem* (3.7),(1.3) *satisfying*

$$\|y_n\|_\infty \le K, \quad \left|y_n'(t) - \frac{B-A}{T}\right| \le \overline{m}, \ t \in [0,T], \tag{3.8}$$

where

$$K := |A| + |B| + T\overline{m}. \tag{3.9}$$

Proof. Let us put

$$G(t,s) = \begin{cases} \frac{t(s-T)}{T} & t \le s, \\ \frac{s(t-T)}{T} & t > s, \end{cases}$$

which is the Green's function of the boundary value problem

$$y'' = 0 \quad \text{on } [0,T], \qquad y(0) = 0, \quad y(T) = 0.$$

Let $T_n : \mathrm{C}^1([0,T]) \to \mathrm{C}^1([0,T])$ be an operator defined by

$$T_n y(t) = \frac{t}{T} B + \frac{T-t}{T} A + \int_0^T G(t,s) f_n^*(s, y(s))\, \mathrm{d}s, \quad y \in \mathrm{C}^1([0,T]),$$

and a set

$$\Omega = \{y \in \mathrm{C}^1([0,T]) \quad : \quad \|y\|_\infty \le K, \quad \|y'\|_\infty \le K_1\}$$

with (3.9) and $K_1 := \frac{|B-A|}{T} + \overline{m}$. According to Remark 4 and definition of T_n we see that $T_n(\mathrm{C}^1([0,T])) \subset \Omega$. It is a standard argument to prove that T_n are totally continuous on Ω – using Arzelà–Ascoli theorem. Therefore according to the Schauder fixed point theorem there exists a fixed point $y_n \in \Omega$ of T_n. Again, it is a standard procedure to prove that y_n is a solution of (3.7),(1.3). □

Let us emphasize the fact that the upper estimates K and $\overline{m}$ in (3.8) are independent of n.

Lemma 4. *Let us assume* (3.1), (3.2), $A, B \in \mathbb{R}$, $A \bmod R \neq 0 \neq B \bmod R$. *If*

$$|B - A| > T\overline{m}, \tag{3.10}$$

then there exists a strictly monotone solution of problem (3.4), (1.3).

Proof. According to Lemma 3 for every $n \in \mathbb{N}$ there exists a solution $y_n \in \mathrm{AC}^1([0,T])$ of problem (3.7), (1.3) satisfying (3.8), (3.9). Obviously, the sequences $\{y_n\}_{n=1}^{\infty}$ and $\{y_n'\}_{n=1}^{\infty}$ are uniformly bounded. Moreover, due to (3.7) and Remark 4, for every $n \in \mathbb{N}$ and $t, s \in [0,T]$ we have

$$|y_n'(t) - y_n'(s)| = \left| \int_s^t y_n''(\xi)\,\mathrm{d}\xi \right| \leq \left| \int_s^t |f_n^*(\xi, y_n(\xi))|\,\mathrm{d}\xi \right| \leq \left| \int_s^t m(\xi)\,\mathrm{d}\xi \right|.$$

Since $m \in \mathrm{L}^1([0,T])$, the sequence $\{y_n'\}_{n=1}^{\infty}$ is equicontinuous. According to Arzelà–Ascoli theorem there exists a convergent (in $\mathrm{C}^1([0,T])$) subsequence $\{y_{k_n}\}_{n=1}^{\infty}$ and its limit $y \in \mathrm{C}^1([0,T])$. From (3.8) by taking limit $n \to \infty$ we have

$$\frac{B-A}{T} - \overline{m} \leq y'(t) \leq \frac{B-A}{T} + \overline{m}, \quad t \in [0,T]. \tag{3.11}$$

From assumption (3.10) we see that $A \neq B$. There are two cases:
If $A < B$, then from (3.10) we get

$$\frac{B-A}{T} > \overline{m}.$$

Consequently from the first inequality in (3.11) we see that $y' > 0$ on $[0,T]$, therefore y is strictly increasing.
If $A > B$, then from (3.10) we have

$$-\frac{B-A}{T} > \overline{m}.$$

Consequently from the second inequality in (3.11) we see that $y' < 0$ on $[0,T]$, therefore y is strictly decreasing.
In both cases y is a strictly monotone function. Since y is strictly monotone and continuous, it follows that there exists a finite set T_y such that for $t \in [0,T]$ the equality $y(t) \bmod R = 0$ holds iff $t \in T_y$. If $T_y = \varnothing$, then we put $t_0 = 0$, $t_1 = T$ and $p = 0$. If $T_y \neq \varnothing$, we denote $T_y = \{t_1, t_2, \ldots, t_p\}$, where

$$0 =: t_0 < t_1 < t_2 < \ldots < t_p < t_{p+1} := T.$$

Let $i \in \{0, \ldots, p\}$. Then for each $s_1, s_2 \in (t_i, t_{i+1})$, $s_1 < s_2$ we have

$$\begin{aligned} y'(s_2) - y'(s_1) &= \lim_{n\to\infty} (y_{k_n}'(s_2) - y_{k_n}'(s_1)) \\ &= \lim_{n\to\infty} \int_{s_1}^{s_2} y_{k_n}''(s)\,\mathrm{d}s = \lim_{n\to\infty} \int_{s_1}^{s_2} f_{k_n}^*(s, y_{k_n}(s))\,\mathrm{d}s. \end{aligned}$$

Since $\{y_{k_n}\}_{n=1}^{\infty}$ uniformly converges to y on (t_i, t_{i+1}) and y is strictly monotone and continuous on this interval, it follows that there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we have

$$y_{k_n}(t) \bmod R \in \left[\frac{R}{2k_n}, R\left(1 - \frac{1}{2k_n}\right)\right] \subset (0, R) \quad \text{for each } t \in (s_1, s_2).$$

Therefore by (3.6) and the last property of η_n we get

$$\lim_{n\to\infty} \int_{s_1}^{s_2} f_{k_n}^*(s, y_{k_n}(s))\,\mathrm{d}s = \lim_{n\to\infty} \int_{s_1}^{s_2} f^*(s, y_{k_n}(s))\,\mathrm{d}s = \int_{s_1}^{s_2} f^*(s, y(s))\,\mathrm{d}s.$$

Summarizing, we obtain

$$y'(s_2) - y'(s_1) = \int_{s_1}^{s_2} f^*(s, y(s))\,\mathrm{d}s$$

for each $s_1, s_2 \in (t_i, t_{i+1})$. According to Remark 4 we see that $y \in \mathrm{AC}^1([t_i, t_{i+1}])$ and therefore $y \in \mathrm{AC}^1([0, T])$. Moreover y satisfies differential equation (3.4) for a.e. $t \in [0, T]$. Obviously, y satisfies (1.3), as well. □

4. Main results

Now we are able to formulate and prove the existence theorems.

Theorem 2. *Let* (3.1) *be satisfied,* $A, B \in (0, R)$ *and* $p \in \mathbb{N}$ *be such that*

$$p > \frac{T\overline{m}}{R} + 1, \tag{4.1}$$

where $\overline{m}$ *is defined in* (3.2). *Then there exist at least two solutions of* (1.1)–(1.3) *having exactly* p*-impacts.*

Proof. Step 1. (Existence of solutions of auxiliary problems) For $i \in \mathbb{Z}$ satisfying $|i| = p$ we consider the differential equation (3.4) together with boundary conditions

$$y(0) = A, \quad y(T) = iR + \theta_i, \tag{4.2}$$

where

$$\theta_i = \begin{cases} B & \text{if } i \text{ is even,} \\ R - B & \text{if } i \text{ is odd.} \end{cases}$$

Let us note that if y satisfies (4.2), then

$$\left|\left\lfloor \frac{y(0)}{R} \right\rfloor - \left\lfloor \frac{y(T)}{R} \right\rfloor\right| = \left|\left\lfloor \frac{A}{R} \right\rfloor - \left\lfloor \frac{iR + \theta_i}{R} \right\rfloor\right| = \left|\left\lfloor \frac{iR + \theta_i}{R} \right\rfloor\right| = \left|\left\lfloor i + \frac{\theta_i}{R} \right\rfloor\right| = |i|. \tag{4.3}$$

According to (4.1), we have

$$|iR + \theta_i - A| \geq |iR| - |\theta_i - A| \geq |i|R - R = (|i| - 1)R = (p - 1)R > T\overline{m},$$

which is the assumption (3.10) for $B = iR + \theta_i$. Therefore we conclude that by Lemma 4 there exists at least one increasing solution y_1 of (3.4) with boundary condition

$$y(0) = A, \quad y(T) = pR + \theta_p \ ,$$

and at least one decreasing solution y_2 of (3.4) with boundary condition

$$y(0) = A, \quad y(T) = -pR + \theta_{-p} \ .$$

From (4.3) we get

$$\left|\left\lfloor \frac{y_1(0)}{R} \right\rfloor - \left\lfloor \frac{y_1(T)}{R} \right\rfloor\right| = \left|\left\lfloor \frac{y_2(0)}{R} \right\rfloor - \left\lfloor \frac{y_2(T)}{R} \right\rfloor\right| = p \ . \tag{4.4}$$

STEP 2. (Solutions of (1.1)–(1.3)) Let us define x_1 by (3.5) with x_1 and y_1 in place of x and y, respectively. Then according to Lemma 2 and (4.4), the function x_1 is a solution of (1.1), (1.2) and it has exactly p impacts. Moreover, $x_1(0) = y_1(0) = A$. It remains to prove that $y_1(T) = B$. If p is even, then $y_1(T) = pR + B$. Obviously $y_1(T) \bmod 2R = B \in (0, R)$ and then

$$x_1(T) = y_1(T) \bmod 2R = B.$$

If p is odd, then $y_1(T) = (p+1)R - B = (p-1)R + 2R - B$. Obviously $y_1(T) \bmod 2R = 2R - B$ and then

$$x_1(T) = 2R - (y_1(T) \bmod 2R) = 2R - (2R - B) = B.$$

So the function x_1 is a solution of (1.1)–(1.3) having exactly p impacts. Similarly, we define x_2 by (3.5) with x_2 and y_2 in place of x and y, respectively, and prove that x_2 is solution of (1.1)–(1.3) having exactly p impacts, in the same way. From the properties of y_i we can see that $x_1 \neq x_2$. □

Corollary 1. *Let* (3.1) *be satisfied,* $A, B \in (0, R)$. *Then there exist infinitely many solutions of* (1.1)–(1.3).

Remark 5. Theorem 2 states that for p satisfying (4.1) there are two solutions x_1, x_2 of (1.1)–(1.3). But from the proof we can say much more about these solutions. Let $t_1^i, \dots, t_p^i$ be times of impacts of x_i, $t_0^i = 0$, $t_{p+1}^i = T$, $i = 1, 2$. Then

- $x_i'(t)(-1)^{i+j} > 0$ for $t \in (t_{j-1}^i, t_j^i)$, $j = 1, \dots, p+1$, $i = 1, 2$,
- $x_1(t_j^1) = R$ and $x_2(t_j^2) = 0$ for odd $j \in \{1, \dots, p\}$,
- $x_1(t_j^1) = 0$ and $x_2(t_j^2) = R$ for even $j \in \{1, \dots, p\}$.

For instance, this means that x_1 is increasing on the interval $[0, t_1^1]$, it hits $x = R$ at $t = t_1^1$, then it is decreasing until it is positive and so on.

The proof of Theorem 1 is almost a consequence of Theorem 2.

Proof. (Proof of Theorem 1) Let us put $\widetilde{R} := b - a$, $\widetilde{K} = [0, \widetilde{R}]$ and

$$\widetilde{f}(t, x) := f(t, x + a) \quad \text{a.e. } t \in [0, T], \text{ all } x \in [0, \widetilde{R}] = \widetilde{K},$$

$\widetilde{A} := A - a$, $\widetilde{B} := B - a$. Then

$$|\widetilde{f}(t,x)| \le m(t) \quad \text{for a.e. } t \in [0,T], \text{ all } x \in \widetilde{K}.$$

According to Theorem 2 there exist two solutions $\widetilde{x}_1, \widetilde{x}_2$ of (1.1)–(1.3) with $f = \widetilde{f}$, $R = \widetilde{R}$, $K = \widetilde{K}$, $A = \widetilde{A}$, $B = \widetilde{B}$ having exactly p impacts. It is not difficult to check that

$$x_1(t) := \widetilde{x}_1(t) + a, \quad x_2(t) := \widetilde{x}_2(t) + a, \quad t \in [0,T]$$

are solutions to problem (1.1)–(1.3) for $K = [a,b]$. □

Example 1. Let us consider a one-dimensional billiard problem

$$y'' = t^\alpha + \sqrt{|y|}, \qquad \text{for a.e. } t \in [0,T],\ y(t) \in (-r,r),$$
$$y'(s+) = -y'(s-), \quad s \in (0,T),\ |y(s)| = r,$$
$$y(0) = y(T) = 0,$$

where $\alpha > 0$. According to Theorem 1 putting $a = -r$, $b = r$, $A = 0$, $B = 0$, then for each $p \in \mathbb{N}$,

$$p > \frac{T^2}{2r}\left(\frac{T^\alpha}{\alpha+1} + \sqrt{r}\right) + 1,$$

there exist at least two solutions of the Dirichlet problem having exactly p impacts. Note that a similar result in [4] (Theorem 3.6) cannot be applied in this case, because the right-hand side of the differential equation is not Lipschitz–continuous in the second variable. Moreover, Theorem 1 gives more detailed multiplicity results than the result in [4].

Example 2. Let us consider oscillator

$$y'' = -\lambda y + q(t),$$

with $\lambda > 0$, p the Lebesgue integrable function on $[0,T]$ with obstacles from above and from below, i.e. the solution has values in a compact interval $K = [-r,r]$. If the value reaches the boundary of K, then the impact occurs and the oscillator bounces back with absolutely elastic impact, which can be expressed by condition

$$y'(s+) = -y'(s-), \quad \text{if } |y(s)| = r\ .$$

Let us look for solutions of this equation with impact satisfying Dirichlet conditions $y(0) = A$, $y(T) = B$, where $|A|, |B| < r$. Since

$$\max_{x \in [-r,r]} |-\lambda x + q(t)| \le \lambda r + |q(t)|, \qquad \text{for a.e. } t \in [0,T]$$

we can conclude from Theorem 1 that for each integer p greater than

$$\frac{T}{2r}(\lambda r T + \|q\|_{L^1}) + 1\ ,$$

there exist at least two solutions of this boundary value problem with exactly p-impacts.

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Author's address

Jan Tomeček

Department of Mathematical Analysis, and Applications of Mathematics,, Faculty of Science, Palacký University,, 17. listopadu 12, 771 46 Olomouc, Czechia

E-mail address: jan.tomecek@upol.cz

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GENERALIZED ELASTICA IN $SO(3)$

GÖZDE ÖZKAN TÜKEL, TUNAHAN TURHAN, AND AHMET YÜCESAN

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Abstract. In a Lie group G equipped with bi-invariant Riemannian metric, we characterize the generalized elastica by an Euler-Lagrange equation in terms of the Lie reduction V of a curve γ in G. We define a generalized elastic Lie quadratic in the Lie algebra of G. For a generalized elastic Lie quadratic, we construct the Lax equation that is crucial to the solution of a generalized elastica with regard to its generalized elastic Lie quadratic. Then we solve this equation for a null generalized elastic Lie quadratic with $\|\dot{V}(t)\|$ =constant when G Lie group is $SO(3)$.

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1. Introduction

Elastica (or elastic curve) proposed by Bernoulli to Euler is an extremal of the bending energy functional $\int_\gamma \left(\kappa^2+\lambda\right) ds$, where γ is a curve, κ is the curvature of γ and λ is a Lagrange multiplier depending on the length [15]. The history of elastica is quite old. So far it has been studied and developed by a lot of authors under various point of view including a generalization of elastica to Riemannian manifold [8–12]. The generalized elastica is defined as critical point of the functional $\mathcal{F}(\gamma) = \int_\gamma P(\kappa)\, ds$, under some boundary conditions, where $P(\kappa)$ is a differentiable function of κ. Existence, classification or stability of this variational problem have been investigated in an Euclidean space, a Riemannian manifold, etc. [1–4,6,7]. Extremals of the functional $\mathcal{F}$ correspond to geodesics (when $P(\kappa) = \kappa^r$, $r = 0$), classical elastica (when $P(\kappa) = \kappa^2+\lambda$), free elastica (when $P(\kappa) = \kappa^2$), elastica circular at rest (when $P(\kappa) = (\kappa+\lambda)^2$), $r-$elastica (or free hyperelastic curves, when $P(\kappa) = \kappa^r$, $r > 2$), etc. [1,4].

Lie groups which lie at the intersection of algebra and geometry play an important role both of them. While algebraic properties of Lie groups come from the group axioms, their geometric properties come from the identification of group operations with points in a topological space. The Lie group structures derive from combining the algebraic and topological properties via differentiability requirements. So, the

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elements of this group are the points in a manifold that are parametrized by continuous real variables [5]. In this paper, we study the problem of generalized elastica in a manifold which is a Lie group equipped with bi-invariant Riemannian metric. A curve defined in the Lie group corresponds to the Lie reduction in its Lie algebra. Popiel and Noakes (2007) gave a characterization of elastic curves in Lie groups with regard to corresponding Lie reduction. They define "elastic Lie quadratics" as solutions of the Euler-Lagrange equation in the Lie algebra [14]. They solve this variational problem in $SO(3)$ by quadratures. Motivated by [14], we survey the theory of finding extremal of the generalized curvature energy functional in Lie groups equipped with a bi-invariant Riemannian metric.

Now we remind the characterization of a generalized elastica in a Riemannian manifold. Let M be a $n-$dimensional Riemannian manifold with Riemannian metric $<,>$, Riemannian norm $\|.\|$, Levi-Civita connection ∇ and Riemannian curvature tensor R. Let Ω be the space of C^∞ curves $\gamma : [0,\ell] \to M$ satisfying

$$\|\dot{\gamma}\| = \left\| \frac{d\gamma}{dt} \right\| = 1$$

$$\gamma(i\ell) = p_i, \dot{\gamma}(i\ell) = v_i$$

for $p_i \in M$ and $v_i \in T_{p_i} M$, $i = 0, 1$.

A generalized elastic curve (or P-elastica) is an extremal of the generalized Euler-Bernoulli energy functional

$$\begin{array}{rccl} \mathcal{F}: & \Omega & \to & \mathbb{R} \\ & \gamma & \to & \mathcal{F}(\gamma) = \int\limits_0^\ell P(\kappa)dt, \end{array} \tag{1}$$

where $P(\kappa)$ is a C^∞ function and $\kappa = \left\| \nabla_{\frac{d}{dt}} \dot{\gamma} \right\|$ is the geodesic curvature of γ, acting on space curves in a Riemannian manifold satisfying given boundary conditions. Any critical point of the functional (1) satisfies the following Euler-Lagrange equation

$$\nabla^2_{\frac{d}{dt}} \left(\frac{P'(\kappa)}{\kappa} \nabla_{\frac{d}{dt}} \dot{\gamma} \right) + \frac{P'(\kappa)}{\kappa} R(\nabla_{\frac{d}{dt}} \dot{\gamma}, \dot{\gamma})\dot{\gamma} + \nabla_{\frac{d}{dt}} \left[\left(2\kappa P'(\kappa) - P(\kappa) \right) \dot{\gamma} \right] = 0, \tag{2}$$

where $P'(\kappa) = \frac{dP}{d\kappa}$, [3,4,6].

In the following, we present a theorem which characterizes generalized elastica as a differential equation with boundary conditions of a special form.

Theorem 1. *Any C^∞ curve $\gamma : I \subset \mathbb{R} \to M$ is a generalized elastica iff γ satisfies the Euler Lagrange equation* (2) *for all $t \in I$ and following equalities*

$$1 = \|\dot{\gamma}(t_0)\|, \tag{3}$$

$$0 = < \nabla_{\frac{d}{dt}} \dot{\gamma} \Big|_{t=t_0}, \dot{\gamma}(t_0) >, \tag{4}$$

$$0 =< \nabla^2_{\frac{d}{dt}} \dot{\gamma}\Big|_{t=t_0}, \dot{\gamma}(t_0) > + \left\| \nabla_{\frac{d}{dt}} \dot{\gamma}\Big|_{t=t_0} \right\|^2, \tag{5}$$

for some $t_0 \in I$.

Proof. Let $\gamma : I \to M$ be a generalized elastica. Then $\dot{\gamma}$ satisfies the Euler-Lagrange equation (2) and

$$\|\dot{\gamma}(t)\|^2 = 1 \tag{6}$$

for any $t \in I$. By taking the first and second derivative of (6), we obtain

$$< \nabla_{\frac{d}{dt}} \dot{\gamma}\Big|_t, \dot{\gamma}(t) >= 0,$$

$$< \nabla^2_{\frac{d}{dt}} \dot{\gamma}\Big|_t, \dot{\gamma}(t) > + \left\| \nabla_{\frac{d}{dt}} \dot{\gamma}\Big|_t \right\|^2 = 0.$$

In particular, (3), (4) and (5) holds for any $t_0 \in I$.

Now we need to suppose that γ satisfies Equations $(2-5)$. Then we show that $\|\dot{\gamma}(t)\|^2 = 1$ for all $t \in I$. If we consider $I = (s_1, s_2)$, we show that $\|\dot{\gamma}(t)\|^2 = 1$ for all $t \in [t_0, s_2)$ and $t \in (s_1, t_0]$. We write $\mathcal{L} = \{t \in [t_0, s_2) : \|\dot{\gamma}(t)\| = 1\}$ and $\aleph = \sup(\mathcal{L})$. We show that in fact $\|\dot{\gamma}(t)\|^2 = 1$ on some open interval containing $\aleph$; this contradicts $\aleph = \sup(\mathcal{L})$, so we get $\|\dot{\gamma}(t)\|^2 = 1$ on $[t_0, s_2)$, (the proof for $(s_1, t_0]$ is similar). Write $\gamma_1 = \gamma, \gamma_2 = \dot{\gamma}, \gamma_3 = \nabla_{\frac{d}{dt}} \dot{\gamma}$ and $\gamma_4 = \nabla^2_{\frac{d}{dt}} \dot{\gamma}$. Then (2) can be written as the following system:

$$\dot{\gamma}_1 = \gamma_2, \ \nabla_{\frac{d}{dt}} \gamma_2 = \gamma_3, \ \nabla_{\frac{d}{dt}} \gamma_3 = \gamma_4$$

$$\begin{aligned}
\nabla_{\frac{d}{dt}} \gamma_4 = &-\frac{\|\gamma_3\|}{P'(\|\gamma_3\|)} \nabla^2_{\frac{d}{dt}} \left(\frac{P'(\gamma_3)}{\|\gamma_3\|} \right) \gamma_3 - R(\gamma_3, \gamma_2)\gamma_2 \\
&- \frac{\|\gamma_3\|}{P'(\|\gamma_3\|)} \left(\left(\nabla_{\frac{d}{dt}} \left(2\|\gamma_3\| P'(\|\gamma_3\|) - P(\|\gamma_3\|) \right) \right) \gamma_2 \right. \\
&\left. + \left(2\|\gamma_3\| P'(\|\gamma_3\|) - P(\|\gamma_3\|) \right) \gamma_3 \right).
\end{aligned}$$

The rest of the proof can be seen by the similar methodology in the proof of Theorem 1.2 in [14]. □

After we give Theorem 1 which characterizes generalized elastica in M, we organize the next part of the manuscript as follows. In Section 2 we study the $n-$dimensional manifold M to be a Lie group equipped with bi-invariant Riemannian metric. We briefly talk about the Lie group and the corresponding Lie algebraic structure and remind to the reader basic structures to be needed throughout the paper. Then we give the transitions between some covariant derivatives of γ and the Lie reduction of γ. We derive an Euler-Lagrange equation which characterizes the generalized elastica with regard to the Lie reduction of a curve γ in Lie group G. In

Section 3, we assume the Lie group G is $SO(3)$ which is known as the group of rotations in the Euclidean 3−space. We solve this equation for a null generalized elastic Lie quadratic in $\mathfrak{so}(3)$ which is the set of skew-symmetric 3×3 matrices.

2. Generalized elastica in Lie groups

A Lie group G is a C^{∞} manifold that is also a group with smooth group operations [5]. The identity element of G is denoted by e. The left and right multiplications by $g \in G$ are the maps L_x, $R_x : G \to G$ defined by $L_x(y) := xy$ and $R_x(y) := yx$, respectively [13]. If a Riemannian metric $<,>$ satisfies for all $x, y \in G$ and $u, v \in T_yG$

$$< u, v >_y = < d\,(L_x)_y\,(u)\,, d\,(L_x)_y\,(v) >_{L_x(y)}, \tag{7}$$

then the metric $<,>$ is called left-invariant. A Riemannian metric $<,>$ is known bi-invariant if it is invariant both left and right invariance. Throughout this paper, we consider that the manifold M is a Lie group G equipped with bi-invariant Riemannian metric $<,>$. Bi-invariance of a left-invariant metric $<,>$ for all $X, Y, Z \in \mathfrak{g}$ is equivalent

$$< [X, Y], Z > = < [Z, X], Y > \tag{8}$$

and the following properties hold:

$$\nabla_X Y = \frac{1}{2}[X, Y],\ R(X, Y)Z = -\frac{1}{4}[[X, Y], Z]$$

where $[,]$ is the Lie bracket [5, 12].

Now, we suppose that $\gamma : I \subset \mathbb{R} \to G$ be a curve on G. Then we define $V : I \to \mathfrak{g}$ by

$$V(t) = \left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} \dot{\gamma}\,(t)\,. \tag{9}$$

The curve V is known the Lie reduction corresponding to γ. (9) is equal to the differential equation

$$\dot{\gamma}\,(t) = \left(dL_{\gamma(t)}\right)_e V\,(t)$$

[10]. For all $t \in I$, the Riemannian curvature tensor is given by

$$\left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} R(\nabla_{\frac{d}{dt}} \dot{\gamma}(t), \dot{\gamma}(t))\dot{\gamma}(t) = -\frac{1}{4}[V(t), [V(t), \dot{V}(t)]]$$

in the Lie algebra $\mathfrak{g}$ [14].

Lemma 1. *Let $\gamma : I \to G$ be a differentiable curve. Suppose that the Lie reduction of γ is given by $V : I \to \mathfrak{g}$. Then we have for all $t \in I$ in the following equations:*

$$i)\left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} \nabla_{\frac{d}{dt}} \dot{\gamma}(t) = \dot{V}(t),$$

$$ii)\left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} \nabla^2_{\frac{d}{dt}} \left(\frac{P'(\kappa)}{\kappa} \nabla_{\frac{d}{dt}} \dot{\gamma}(t)\right) = \frac{d^2}{dt^2}\left(\frac{P'(\left\|\dot{V}(t)\right\|)}{\left\|\dot{V}(t)\right\|}\right)\dot{V}(t)$$
$$+2\frac{d}{dt}\left(\frac{P'(\left\|\dot{V}(t)\right\|)}{\left\|\dot{V}(t)\right\|}\right)\left(\ddot{V}(t)+\frac{1}{2}[V(t),\dot{V}(t)]\right)$$
$$+\frac{P'(\left\|\dot{V}(t)\right\|)}{\left\|\dot{V}(t)\right\|}\left(\frac{d^3V(t)}{dt^3}+[V(t),\ddot{V}(t)]+\frac{1}{4}[V(t),[V(t),\dot{V}(t)]]\right),$$

$$iii)\left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} \frac{P'(\kappa)}{\kappa} R(\nabla_{\frac{d}{dt}} \dot{\gamma}(t),\dot{\gamma}(t))\dot{\gamma}(t) = -\frac{1}{4}\frac{P'(\left\|\dot{V}(t)\right\|)}{\left\|\dot{V}(t)\right\|}[V(t),[V(t),\dot{V}(t)]],$$

$$iv)\left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} \nabla_{\frac{d}{dt}} [(2\kappa P'(\kappa)-P(\kappa))\dot{\gamma}] = \frac{d}{dt}\left(2\left\|\dot{V}(t)\right\| P'(\left\|\dot{V}(t)\right\|)\right.$$
$$\left.-P(\left\|\dot{V}(t)\right\|)\right)V(t)+\left(2\left\|\dot{V}(t)\right\| P'(\left\|\dot{V}(t)\right\|)-P(\left\|\dot{V}(t)\right\|)\right)\dot{V}(t)$$

Proof. Let $\{E_1(t), E_2(t),...,E_n(t)\}$ be an orthonormal frame of the Lie algebra $\mathfrak{g}$. Thus we have an orthonormal frame $\{\bar{E}_1(\gamma(t)), \bar{E}_2(\gamma(t)),...,\bar{E}_n(\gamma(t))\}$ for $T_{\gamma(t)}G$, [11, 12]. By using the fact that

$$\nabla_{\bar{E}_i(\gamma(t))} \bar{E}_j(\gamma(t)) = \frac{1}{2}[\bar{E}_i(\gamma(t)), \bar{E}_j(\gamma(t))]$$

and the left invariance for vector fields of G, we can write

$$\dot{\gamma}(t) = \sum_i v_i \bar{E}_i(\gamma(t)).$$

One can found the proof of (i) of Lemma 1 in [11].

By using (7) and (i) of Lemma 1, we get

$$\kappa = \left\|\nabla_{\frac{d}{dt}} \dot{\gamma}(t)\right\| = \left\|\left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} \nabla_{\frac{d}{dt}} \dot{\gamma}(t)\right\| = \left\|\dot{V}(t)\right\|.$$

So we can make the following calculations:

$$\nabla_{\frac{d}{dt}}\left(\frac{P'(\kappa)}{\kappa}\nabla_{\frac{d}{dt}}\dot{\gamma}(t)\right)=\tfrac{d}{dt}\left(\frac{P'(\kappa)}{\kappa}\right)\left(dL_{\gamma(t)}\right)_e\sum_i\frac{dv_i}{dt}E_i(t)$$
$$+\frac{P'(\kappa)}{\kappa}\left(dL_{\gamma(t)}\right)_e\left[\sum_i\frac{d^2v_i}{dt^2}E_i(t)+\frac{dv_i}{dt}\left(v_j\nabla_{\bar{E}_i(\gamma(t))}\bar{E}_j(\gamma(t))\right)\right]$$
$$=\left(dL_{\gamma(t)}\right)_e\left[\frac{d}{dt}\left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|}\right)\dot{V}(t)+\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|}(\ddot{V}(t)\frac{1}{2}[V(t),\dot{V}(t)])\right]$$

and

$$\nabla^2_{\frac{d}{dt}}\left(\frac{P'(\kappa)}{\kappa}\nabla_{\frac{d}{dt}}\dot{\gamma}(t)\right)=\left(dL_{\gamma(t)}\right)_e\left[\frac{d^2}{dt^2}\left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|}\right)\dot{V}(t)\right.$$
$$+2\frac{d}{dt}\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|}(\ddot{V}(t)\frac{1}{2}[V(t),\dot{V}(t)])$$
$$\left.+\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|}\left(\frac{d^3V}{dt^3}+[V(t),\ddot{V}(t)]+\frac{1}{4}[V(t),[V(t),\ddot{V}(t)]]\right)\right]$$

$ii)$

$$\begin{aligned}\frac{P'(\kappa)}{\kappa}R(\nabla_{\frac{d}{dt}}\dot{\gamma}(t),\dot{\gamma}(t))\dot{\gamma}(t) &= \frac{P'(\kappa)}{\kappa}\left(dL_{\gamma(t)}\right)_e\left[-\frac{1}{4}[V(t),[V(t),\dot{V}(t)]]\right]\\ &= \left(dL_{\gamma(t)}\right)_e\left[-\frac{1}{4}\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|}[V(t),[V(t),\dot{V}(t)]]\right].\end{aligned}$$

The proof of (iv) of Lemma 1 is a result of (i) and (ii) of Lemma 1. □

The following theorem gives the characterization of a generalized elastica in G.

Theorem 2. *Any differentiable curve $\gamma : I \to G$ in the Lie group G is a generalized elastica iff the curve $V : I \to \mathfrak{g}$ defined by* (9) *satisfies*

$$\|V(t)\|^2 = 1, \tag{10}$$

$$\begin{aligned}\tfrac{d}{dt}\left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|}\dot{V}(t)\right)+\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|}[V(t),\dot{V}(t)]\\ +\left(P(\|\dot{V}(t)\|)-2<C,V(t)>\right)V(t)+C=0\end{aligned} \tag{11}$$

for some constant $C \in \mathfrak{g}$ and all $t \in I$.

Proof. Assume that $\gamma : I \to G$ is a generalized elastica in G. Then we have from the left invariance of Riemannian metric and Eq. (9), we get

$$1 = \left\| \dot{\gamma}(t) \right\|^2 = \left\| \left(dL_{\gamma(t)^{-1}} \right)_{\gamma(t)} \dot{\gamma}(t) \right\|^2 = \| V(t) \|^2 . \tag{12}$$

If γ is a generalized elastica, then γ satisfies the Euler-Lagrange equation (2). Applying $\left(dL_{\gamma(t)^{-1}} \right)_{\gamma(t)}$ to (2) and using Lemma 1, we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\frac{d}{dt}\left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|} \right) \dot{V}(t) \right) + \frac{d}{dt}\left(\left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|} \right) [V(t), \dot{V}(t)] \right) \\ &+ \frac{d}{dt}\left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|} \ddot{V}(t) \right) + \frac{d}{dt}\left(\left(2 \left\| \dot{V}(t) \right\| P'(\left\| \dot{V}(t) \right\|) - P(\left\| \dot{V}(t) \right\|) \right) V(t) \right) = 0. \end{aligned}$$

Integrating once, we have,

$$\begin{aligned} &\frac{d}{dt}\left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|} \dot{V}(t) \right) + \left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|} \right) [V(t), \dot{V}(t)] \\ &+ \left(2 \left\| \dot{V}(t) \right\| P'(\left\| \dot{V}(t) \right\|) - P(\left\| \dot{V}(t) \right\|) \right) V(t) + C = 0 \end{aligned} \tag{13}$$

where $C \in \mathfrak{g}$ is a constant. The first and second derivative of (12) are found as follows

$$< \dot{V}(t), V(t) >= 0, \tag{14}$$

$$< \ddot{V}(t), V(t) > + \left\| \dot{V}(t) \right\|^2 = 0. \tag{15}$$

Taking inner product of (13) with $\dot{V}(t)$ and applying (14), we have

$$\begin{aligned} &\frac{d}{dt}\left(\frac{P'(\left\| \dot{V}(t) \right\|)}{\left\| \dot{V}(t) \right\|} < \dot{V}(t), \dot{V}(t) > \right) \\ &- \frac{P'(\left\| \dot{V}(t) \right\|)}{\left\| \dot{V}(t) \right\|} < \ddot{V}(t), \dot{V}(t) > + < C, \dot{V}(t) >= 0. \end{aligned} \tag{16}$$

Integrating (16), we obtain

$$P(\left\| \dot{V}(t) \right\|) = P'(\left\| \dot{V}(t) \right\|) \left\| \dot{V}(t) \right\| + < C, V(t) > + b. \tag{17}$$

for some constant $b \in \mathbb{R}$. If we take inner product of (13) with $V(t)$ and using (14) and (15), we have

$$P(\left\| \dot{V}(t) \right\|) = \left\| \dot{V}(t) \right\| P'(\left\| \dot{V}(t) \right\|) + < V(t), C > . \tag{18}$$

Combining (17) and (18), we obtain $b = 0$. Substituting (18) into (13), we have (11).

Conversely, let $V : I \to \mathfrak{g}$ corresponds the Lie reduction of a curve $\gamma : I \to G$. Suppose (10) and (11) are satisfied. From the left invariance of Riemannian metric, we have

$$\|V(t)\|^2 = \left\| \left(dL_{\gamma(t)^{-1}} \right)_{\gamma(t)} \dot{\gamma}(t) \right\|^2 = \left\| \dot{\gamma}(t) \right\|^2 = 1.$$

Then it remains the show that γ satisfies (2). Writing

$$< C, V(t) >= P(\|\dot{V}(t)\|) - \|\dot{V}(t)\| \, P'(\|\dot{V}(t)\|)$$

and differentiating (11), we get

$$\begin{aligned} &\nabla^2_{\frac{d}{dt}} \left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|} \dot{V}(t) \right) + \nabla_{\frac{d}{dt}} \left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|} [V(t), \dot{V}(t)] \right) \\ &+ \nabla_{\frac{d}{dt}} (2 \|\dot{V}(t)\| \, P'(\|\dot{V}(t)\|) - P(\|\dot{V}(t)\|) V(t)) = 0 \end{aligned}$$

Applying (9) and using Lemma 1, we obtain

$$\begin{aligned} &\left(dL_{\gamma(t)^{-1}} \right)_{\gamma(t)} (\nabla^2_{\frac{d}{dt}} \left(\frac{P'(\left\| \nabla_{\frac{d}{dt}} \dot{\gamma}(t) \right\|)}{\left\| \nabla_{\frac{d}{dt}} \dot{\gamma}(t) \right\|} \nabla_{\frac{d}{dt}} \dot{\gamma}(t) \right) + \frac{P'(\left\| \nabla_{\frac{d}{dt}} \dot{\gamma}(t) \right\|)}{\left\| \nabla_{\frac{d}{dt}} \dot{\gamma} \right\|} R(\nabla_{\frac{d}{dt}} \dot{\gamma}(t), \dot{\gamma}(t)) \dot{\gamma}(t) \\ &+ \nabla_{\frac{d}{dt}} (2 \left\| \nabla_{\frac{d}{dt}} \dot{\gamma}(t) \right\| P'(\left\| \nabla_{\frac{d}{dt}} \dot{\gamma}(t) \right\|) - P(\left\| \nabla_{\frac{d}{dt}} \dot{\gamma}(t) \right\|) \dot{\gamma}(t))) = 0. \end{aligned}$$

for $\forall t \in I$. Since $\left(dL_{\gamma(t)^{-1}} \right)_{\gamma(t)}$ is an isomorphism, γ satisfies (2). □

Definition 1. Any curve $V : I \to \mathfrak{g}$ satisfying (10) and (11) for some $C \in \mathfrak{g}$ and $\forall t \in I$ is called a generalized elastic Lie quadratic with constant C. Also, V defined by (9) is called a generalized elastic Lie quadratic associated with γ, if $\gamma : I \to G$ is a generalized elastica.

Corollary 1. *Let $V : I \to \mathfrak{g}$ be a generalized elastic Lie quadratic. We define $W : I \to \mathfrak{g}$ by*

$$W(t) = \frac{d}{dt} \left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|} \dot{V}(t) \right) + (P(\|\dot{V}(t)\|) - 2 < C, V(t) >) V(t). \tag{19}$$

Then we have

$$\dot{W}(t) = [W(t), V(t)] \tag{20}$$

for all $t \in I$, and $\|W(t)\|$ is a constant.

Proof. Substituting (19) in (11), we have

$$W(t) = \frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|} [\dot{V}(t), V(t)] - C. \tag{21}$$

Differentiating (21), we obtain

$$\dot{W}(t) = [\frac{d}{dt}\left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|}\dot{V}(t)\right), V(t)]. \tag{22}$$

Combining (19) and (22), we obtain desired Eq. (20). On the other hand from (20), we have

$$\begin{aligned}\frac{d}{dt}\|W(t)\|^2 &= \frac{d}{dt} < W(t), W(t) >= 2 < \dot{W}(t), W(t) > \\ &= 2 < [W(t), V(t)], W(t) >= 0.\end{aligned}$$

Therefore $\|W(t)\|$ is found a constant. □

The differential equation (20) known as Lax equation is an extremal to solution of (9) or equivalently $\dot{\gamma}(t) = \left(dL_{\gamma(t)}\right)_e V(t)$ for a generalized elastic Lie quadratic V. Popiel and Noakes prove that the differential equation that gives the elastic curve can expand the whole real axis by Picard's theorem and Lax equations[14]. Then by Theorem 3.1 in [14] and Theorem 1, all generalized elastica in G extend uniquely to $\mathbb{R}$ when G is compact.

3. Generalized elastica in SO(3)

In this section we suppose $G = SO(3)$ which is the group of rotations of Euclidean 3−space. Then the Lie algebra of G is $\mathfrak{g} = \mathfrak{so}(3)$ which is the set off all skew-symmetric real 3×3 matrices. Recall that $\mathfrak{so}(3)$ is a Lie algebra with the Lie bracket $[A, B] = AB - BA$, for $A, B \in \mathfrak{so}(3)$ and E^3 is a Lie algebra with the Lie bracket the cross product $\times$. The Euclidean inner product and norm associated with the inner product are denoted by $<,>$ and $\|.\|$. $B : E^3 \to \mathfrak{so}(3)$ is a Lie algebra isomorphism given by

$$B(v)w = v \times w.$$

The unique dot product on E^3 satisfying (8) because dot product is up to a positive multiple. We may assume that B is an isometry without loss of generality.

Now we consider $\gamma : \mathbb{R} \to SO(3)$ is a generalized elastica and $\tilde{V} : \mathbb{R} \to \mathfrak{so}(3)$ is the associated generalized elastic Lie quadratic with the constant $\tilde{C}$. The inverse function is defined as follows:

$$V = B^{-1}(\tilde{V}) : \mathbb{R} \to E^3 \tag{23}$$

and $C = B^{-1}(\tilde{C})$ for convenience. V satisfies for all $t \in I$

$$\|V(t)\|^2 = \left\|B^{-1}(\tilde{V})\right\|^2 = \left\|\tilde{V}\right\|^2 = 1 \tag{24}$$

because B is a Lie algebra isomorphism and isometry. By using Eq. (11)

$$\begin{aligned}\frac{d}{dt}\left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|}\dot{V}(t)\right) + \frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|}V(t) \times \dot{V}(t) + (P(\left\|\dot{V}(t)\right\|) \\ -2 < C, V(t) >)V(t) + C = 0.\end{aligned} \tag{25}$$

This implies V is a generalized elastic Lie quadratic with constant C in the Lie algebra $(E^3, \times)$. We study with V rather than $\tilde{V}$, solving (25) with (24). So, we can say that for any $A \in SO(3)$ and $t_0 \in \mathbb{R}$, $t \to A(V(t))$ is a generalized elastic Lie quadratic in E^3 with constant $A(C)$, and $t \to V(t-t_0)$ is a generalized elastic Lie quadratic in E^3 with constant C by local uniqueness in Picard theorem.

Now, we may suppose without loss of generality that

$$C = \begin{bmatrix} 0 & 0 & c \end{bmatrix}^T \text{ for some } c \in \mathbb{R},\ V_1(0) = 0 \text{ for } V(t) = \begin{bmatrix} V_1(t) & V_2(t) & V_3(t) \end{bmatrix}^T. \tag{26}$$

If V is a generalized elastic Lie quadratic in E^3 with constant $C = 0$, then we call that V is a null generalized elastic Lie quadratic (see [10] and [14]). Then Eq. (25) reduces to

$$\frac{d}{dt}\left(\frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|}\dot{V}(t)\right) + \frac{P'(\|\dot{V}(t)\|)}{\|\dot{V}(t)\|}V(t)\times\dot{V}(t) + P(\|\dot{V}(t)\|)V(t) = 0. \tag{27}$$

Now, we suppose that $\|\dot{V}(t)\| = const.$ in the next part of the paper. Then we have from the first derivative of $\|\dot{V}(t)\|$

$$< \ddot{V}(t), \dot{V}(t) >= 0.$$

This implies that

$$\frac{d}{dt}\left(P\left(\|\dot{V}(t)\|\right)\right) = \frac{P'\left(\|\dot{V}(t)\|\right)}{\|\dot{V}(t)\|} < \ddot{V}(t), \dot{V}(t) >= 0. \tag{28}$$

From (18) and (28), (27) reduces to

$$\ddot{V}(t) = \dot{V}(t)\times V(t) - \|\dot{V}(t)\|^2 V(t).$$

Then we can give the following proposition;

Proposition 1. If V is a null generalized elastic Lie quadratic with $\|\dot{V}(t)\| = const.$ and satisfies (26), then we have

$$V(t) = \begin{bmatrix} a\sin(wt) & a\cos(wt) & \sqrt{1-a^2} \end{bmatrix}^T$$

for all $t \in \mathbb{R}$ and $w = 0$ or $w = 1/\sqrt{1-a^2}$, $a \in (-1, 1)$.

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Authors' addresses

Gözde Özkan Tükel
Isparta University of Applied Sciences, 32200 Isparta, Turkey
E-mail address: gozdetukel@isparta.edu.tr

Tunahan Turhan
Isparta University of Applied Sciences, 32200 Isparta, Turkey
E-mail address: tunahanturhan@isparta.edu.tr

Ahmet Yücesan
Süleyman Demirel University, Department of Mathematics, 32200 Isparta, Turkey
E-mail address: ahmetyucesan@sdu.edu.tr

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SOME CONVERGENCE THEOREMS FOR NEW ITERATION SCHEME IN CAT(0) SPACES

IZHAR UDDIN, JAVID ALI, AND VLADIMIR RAKOČEVIĆ

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Abstract. In this paper, we construct an iteration scheme involving a hybrid pair of nonexpansive mappings. For this scheme, we prove some convergence theorems in CAT(0) spaces. In process, we remove a restricted condition (also called end-point condition) in previous several existing results. Thus, several relevant results cited in the literature generalize and improve.

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1. Introduction

A metric space (X,d) is a CAT(0) space if it is geodesically connected and every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane. A complete CAT(0) space is also called Hadamard space. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples are the classes of Pre-Hilbert spaces, $\mathbb{R}$-tress and some others. For more details on these spaces, one can consult [3].

Fixed point theory in CAT(0) spaces was initiated by Kirk [13] wherein he proved that every single-valued nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space has a fixed point. Since then the fixed point theory for single-valued as well as multi-valued mappings is rapidly developing in complete CAT(0) space (e.g., [4–9]). Here it is worth mentioning that the results in complete CAT(0) space can be applied to any CAT(k) space with $k \leq 0$ as any CAT(k) space is a CAT(k') space for every $k' \geq k$.

In recent years, different iterative schemes have been used to approximate the fixed points of multi-valued nonexpansive mappings in Banach spaces. Among these iterative schemes, iteration schemes due to Sastry and Babu [17], Panyanak [16] and Song and Wang [23] are notable generalizations of Mann and Ishikawa iteration schemes especially in the case of multi-valued mappings. By now, there exists an extensive

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literature on the iterative fixed points for various classes of mappings. For an up to date account of literature on this theme, we refer the readers to Berinde [2].

In 2010, Sokhuma and Kaewkhao [21] introduced an iteration scheme for a pair of single valued and multivalued mapping and same has been utilized to prove some convergence theorems in Banach spaces. This scheme has also been studied by several authors [1, 19, 20, 25] with respect to different class of mappings in different spaces. All the authors proved their results under a very strong condition, i.e. end point condition $Tw = \{w\}$ for all $w \in F(T)$, where T is a multi-valued mapping. With a motivation to remove this strong condition, Uddin and Imdad [24] introduced a new iteration scheme for a pair of hybrid mappings in Banach space.

In this paper, we study newly defined iteration scheme due to Uddin and Imdad [24] in complete CAT(0) space and prove some convergence theorems. In process, several relevant results in Sokhuma and Kaewkhao [21], Akkasriworn et al. [1], Uddin et al. [25], Sokhuma [19], Sokhuma [20] and Uddin and Imdad [24] are generalized and improved.

2. Preliminaries

With a view to make, our presentation self contained, we collect some basic definitions and needed results which will be used frequently in the text later.
Let X be a Banach space and K be a nonempty subset of X. Let $CB(K)$ be the family of nonempty closed bounded subsets of K while $KC(K)$ be the family of nonempty compact convex subsets of K. A subset K of X is called proximinal if for each $x \in X$, there exists an element $k \in K$ such that

$$d(x,k) = d(x,K) = \inf\{\|x-y\| : y \in K\}.$$

It is well known that every closed convex subset of a uniformly convex Banach space is proximinal. We shall denote by $PB(K)$, the family of nonempty bounded proximinal subsets of K. The Hausdorff metric H on $CB(K)$ is defined as

$$H(A,B) = max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\} \text{ for } A,\ B \in CB(K).$$

A multi-valued mapping $T : K \to CB(K)$ is said to be nonexpansive if

$$H(Tx,Ty) \leq \|x-y\|, \text{ for all } x,\ y \in K.$$

We use the notation $F(T)$ for the set of fixed points of the mapping T while $F(t,T)$ denotes the set of common fixed points of t and T, i.e. a point x is said to be a common fixed point of t and T if $x = tx \in Tx$.

Now, we recall some basic geometric properties which are instrumental throughout the discussions. Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, write:

$$r(x, (\{x_n\})) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ is given by

$$r(\{x_n\}) = \inf\{r(x, x_n) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined as:

$$A(\{x_n\}) = \{x \in X : r(x, x_n) = r(\{x_n\})\}.$$

It is well known that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point (see Proposition 5 of [7]).

In 2008, Kirk and Panyanak [14] gave a concept of convergence in CAT(0) spaces which is an analogue of weak convergence in Banach spaces and restriction of Lim's concept of convergence [15] to CAT(0) space.

Definition 1 ([14])**.** A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_n x_n = x$ and read as x is the Δ-limit of $\{x_n\}$.

Notice that for a given $\{x_n\} \subset X$ which Δ-converges to x and for any $y \in X$ with $y \neq x$ (owing to uniqueness of asymptotic center), we have

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

Thus every CAT(0) space satisfies the Opial property.

Now, we state some basic facts about CAT(0) spaces which will be frequently used throughout the text.

Lemma 1 ([14])**.** *Every bounded sequence in a complete CAT(0) space admits a Δ-convergent subsequence.*

Lemma 2 ([8])**.** *If K is closed convex subset of a complete CAT(0) space X and if $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K.*

Kirk and Panyanak [14] also proved analogoue of famous demiclosedness principle for nonexpansive mappings in CAT(0) spaces.

Lemma 3. *Let K be a closed convex subset of X and $T : K \to X$ a nonexpansive mapping. If $\{x_n\}$ is a sequence in X which Δ-converges to x and $d(x_n, Tx_n) \to 0$, then $x \in K$ and $Tx = x$.*

The following theorem is a consequence of Theorem 3.2 of Dhompongsa et al. [6].

Lemma 4 ([6]). *Let K be a nonempty closed convex subset of complete CAT(0) space X and $T : K \to C(K)$ be a multivalued nonexpansive mapping. If $\Delta - \lim_n x_n = x$ and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$, then x is a fixed point of T.*

Lemma 5 ([9]). *Let (X, d) be a CAT(0) space. For $x, y \in X$ and $\alpha \in [0,1]$, there exists a unique $z \in [x, y]$ such that*

$$d(x,z) = \alpha d(x,y) \quad \text{and} \quad d(y,z) = (1-\alpha)d(x,y).$$

Notice that we use the notation $(1-\alpha)x \oplus \alpha y$ for the unique point z in the case of preceding lemma.

Lemma 6 ([9]). *For $x, y, z \in X$ and $\alpha \in [0,1]$ we have*

$$d((1-\alpha)x \oplus \alpha y, z) \le (1-\alpha)d(x,z) + \alpha d(y,z).$$

The following lemma is very important to prove our main results, which is an analogue of Proposition 2 of [11] in CAT(0) spaces.

Lemma 7. *Let X be a CAT(0) space and let $x \in X$. Suppose $\{t_n\}$ is a sequence in $[b,c]$ for some $b, c \in (0,1)$ and$\{x_n\}$, $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \le a$, $\limsup_{n\to\infty} d(y_n, x) \le a$, and $\lim_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) = a$ for some $a \ge 0$. Then*

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Lemma 8. *Let X be a Banach space, and let K be a nonempty closed convex subset of X. Then,*

$$d(y, Ty) \le \|y - x\| + d(x, Tx) + H(Tx, Ty),$$

where $x, y \in K$ and T is a multi-valued nonexpansive mapping from K into $CB(K)$.

The following result due to Song and Cho [22] is very useful.

Lemma 9. *Let $T : K \to P(K)$ be a multi-valued mapping and $P_T(x) = \{y \in Tx : \|x - y\| = d(x, Tx)\}$. Then the following are equivalent.*

(1) $x \in F(T)$,
(2) $P_T(x) = \{x\}$,
(3) $x \in F(P_T)$. *Moreover,* $F(T) = F(P_T)$.

3. Main results

In this section, we opt the following iteration scheme in CAT(0) spaces.
Let K be a nonempty convex subset of CAT(0) space X, let $t : K \to K$ be a single-valued nonexpansive mapping and $T : K \to PB(K)$ be a multi-valued nonexpansive mapping. The sequence $\{x_n\}$ of the modified Ishikawa iteration is defined by

$$\begin{cases} y_n = \alpha_n z_n \oplus (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n t y_n \oplus (1-\beta_n)x_n, \end{cases} \tag{3.1}$$

where $x_0 \in K$, $z_n \in P_T x_n$ and $0 < a \leq \alpha_n,\ \beta_n \leq b < 1$.

We begin with following lemma.

Lemma 10. *Let K be a non-empty bounded closed convex subset of a complete CAT(0) space X. Let $t : K \to K$ be a single-valued nonexpansive mapping and $T : K \to PB(K)$ be a multi-valued mapping whose P_T is a nonexpansive mapping such that $F(t,T) \neq \emptyset$. If $\{x_n\}$ is the sequence of the modified Ishikawa iteration defined by (3.1) with $0 < a \leq \alpha_n,\ \beta_n \leq b < 1$, then $\lim_{n\to\infty} d(x_n, w)$ exists for all $w \in F(t,T)$.*

Proof. Let $w \in F(t,T)$ and $\{x_n\}$ be the sequence described by (3.1). Then in view Lemma 9

$$w \in P_T(w) = \{w\}.$$

Now, consider

$$\begin{aligned} d(x_{n+1}, w) &= d((1-\beta_n)x_n \oplus \beta_n t y_n, w) \\ &\leq (1-\beta_n)d(x_n, w) \oplus \beta_n d(t y_n, t w) \\ &\leq (1-\beta_n)d(x_n, w) \oplus \beta_n d(y_n, w). \end{aligned} \tag{3.2}$$

But

$$\begin{aligned} d(y_n, w) &= d((1-\alpha_n)x_n \oplus \alpha_n z_n, w) \\ &\leq (1-\alpha_n)d(x_n, w) + \alpha_n d(z_n, w) \\ &= (1-\alpha_n)d(x_n, w) + \alpha_n d(z_n, P_T w) \\ &\leq (1-\alpha_n)d(x_n, w) + \alpha_n H(P_T x_n, P_T w) \\ &\leq (1-\alpha_n)d(x_n, w) + \alpha_n d(x_n, w) \\ &= d(x_n, w). \end{aligned} \tag{3.3}$$

In view of (3.2) and (3.3), we have

$$d(x_{n+1}, w) \leq d(x_n, w).$$

Which shows that $\{d(x_n, w)\}$ is a decreasing sequence of non-negative reals. Thus in all, sequence $\{d(x_n, w\}$ is bounded below and decreasing, therefore remains convergent. □

Lemma 11. *Let K be a non-empty bounded closed convex subset of a complete CAT(0) space X. Let $t : K \to K$ be a single-valued nonexpansive mapping and $T : K \to PB(K)$ be a multi-valued mapping whose P_T is a nonexpansive mapping such that $F(t,T) \neq \emptyset$. If $\{x_n\}$ is the sequence of the modified Ishikawa iteration defined by (3.1) with $0 < a \leq \alpha_n,\ \beta_n \leq b < 1$, then $\lim_{n\to\infty} d(t y_n, x_n) = 0$.*

Proof. In view of Lemma 10, $\lim_{n\to\infty} d(x_n, w)$ exists for all $w \in F(t,T)$. Write

$$\lim_{n\to\infty} d(x_n, w) = c. \tag{3.4}$$

Now, consider

$$\begin{aligned} d(ty_n, w) &\le d(y_n, w) \\ &\le d((1-\alpha_n)x_n \oplus \alpha_n z_n, w) \\ &\le (1-\alpha_n)d(x_n, w) + \alpha_n d(z_n, w) \\ &= (1-\alpha_n)d(x_n, w) + \alpha_n d(z_n, P_T w) \\ &\le (1-\alpha_n)d(x_n, w) + \alpha_n H(P_T x_n, P_T w) \\ &\le (1-\alpha_n)d(x_n, w) + \alpha_n d(x_n, w) \\ &= d(x_n, w). \end{aligned} \tag{3.5}$$

On taking lim sup of both the sides, we obtain

$$\limsup_{n\to\infty} d(ty_n, w) \le c. \tag{3.6}$$

Also,

$$\begin{aligned} c &= \lim_{n\to\infty} d(x_{n+1}, w) \\ &= \lim_{n\to\infty} d((1-\beta_n)x_n \oplus \beta_n t y_n, w). \end{aligned} \tag{3.7}$$

In view of (3.5), (3.6), (3.7) and Lemma 7, we get

$$\lim_{n\to\infty} d(ty_n, x_n) = 0.$$

□

Lemma 12. *Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X. Let $t : K \to K$ be a single-valued nonexpansive mapping and $T : K \to PB(K)$ be a multi-valued mapping whose P_T is a nonexpansive mapping such that $F(t,T) \neq \varnothing$. If $\{x_n\}$ is the sequence of the modified Ishikawa iteration defined by (3.1) with $0 < a \le \alpha_n,\ \beta_n \le b < 1$, then $\lim_{n\to\infty} d(z_n, x_n) = 0$.*

Proof. Let $w \in F(t,T)$ and $\{x_n\}$ be the sequence described by (3.1). Since P_T is nonexpansive, so in view of Lemma 9, we have

$$w \in P_T(w) = \{w\}.$$

Now, consider

$$\begin{aligned} d(x_{n+1}, w) &= d((1-\beta_n)x_n \oplus \beta_n t y_n, w) \\ &\le (1-\beta_n)d(x_n, w) + \beta_n d(ty_n, tw) \\ &\le (1-\beta_n)d(x_n, w) + \beta_n d(y_n, w). \end{aligned} \tag{3.8}$$

and therefore

$$d(x_{n+1},w)-d(x_n,w)\le\beta_n(d(y_n,w)-d(x_n,w)),$$
$$\frac{d(x_{n+1},w)-d(x_n,w)}{\beta_n}\le d(y_n,w)-d(x_n,w).$$

Since $0<a\le\beta_n\le b<1$, we have

$$\liminf_{n\to\infty}\left\{\left(\frac{d(x_{n+1},w)-d(x_n,w)}{\beta_n}\right)+d(x_n,w)\right\}\le\liminf_{n\to\infty}d(y_n,w).$$

It follows that

$$c\le\liminf_{n\to\infty}d(y_n,w).$$

Since, from (3.3) $\limsup_{n\to\infty}d(y_n,w)\le c$, hence we have

$$\begin{aligned}c&=\lim_{n\to\infty}d(y_n,w)\\&=\lim_{n\to\infty}d((1-\alpha_n)x_n\oplus\alpha_n z_n,w).\end{aligned}\tag{3.9}$$

Recall that $d(z_n,w)=d(z_n,P_Tw)\le H(P_Tx_n,P_Tw)\le d(x_n,w)$. Thus, we have

$$\limsup_{n\to\infty}d(z_n,w)\le\limsup_{n\to\infty}d(x_n,w)=c.\tag{3.10}$$

Owing to (3.9), (3.10) and Lemma 7, we obtain $\lim_{n\to\infty}d(x_n,z_n)=0$. □

Lemma 13. *Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X. Let $t:K\to K$ be a single-valued nonexpansive mapping and $T:K\to PB(K)$ be a multi-valued mapping whose P_T is a nonexpansive mapping such that $F(t,T)\neq\varnothing$. If $\{x_n\}$ is the sequence of the modified Ishikawa iteration defined by (3.1) with $0<a\le\alpha_n,\ \beta_n\le b<1$, then $\lim_{n\to\infty}d(tx_n,x_n)=0$.*

Proof.

$$\begin{aligned}d(tx_n,x_n)&\le d(tx_n,ty_n)+d(ty_n,x_n)\\&\le d(x_n,y_n)+d(ty_n,x_n)\\&\le d(x_n,(1-\alpha_n)x_n\oplus\alpha_n z_n)+d(ty_n,x_n)\\&=\alpha_n d(x_n,z_n)+d(ty_n,x_n)\end{aligned}$$

therefore,

$$\lim_{n\to\infty}d(tx_n,x_n)\le\lim_{n\to\infty}\alpha_n d(x_n,z_n)+\lim_{n\to\infty}d(ty_n,x_n).$$

Thus by Lemma 11 and Lemma 12, we have

$$\lim_{n\to\infty}d(tx_n,x_n)=0.$$

□

Now, we prove the following Δ-convergence theorem.

Theorem 1. *Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X. Let $t : K \to K$ be a single-valued nonexpansive mapping and $T : K \to P(K)$ be a multi-valued mapping such that $F(t,T) \neq \emptyset$ with P_T is a nonexpansive mapping. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (3.1) with $0 < a \leq \alpha_n,\ \beta_n \leq b < 1$, then $\{x_n\}$ Δ-converge to $y \in F(t,T)$.*

Proof. From Lemma 10, we have $\lim_{n\to\infty} d(x_n, w)$ exists for each $w \in F(t,T)$ so that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, tx_n) = 0$.
Let $W_\omega(\{x_n\}) := \cup A(\{u_n\})$, where union is taken over all subsequences $\{u_n\}$ over $\{x_n\}$. In order to show that the Δ-convergence of $\{x_n\}$ to a common fixed point of t and T, firstly we will prove $W_\omega(\{x_n\}) \subset F(t,T)$ and thereafter argue that $W_\omega(\{x_n\})$ is a singleton set. To show $W_\omega(\{x_n\}) \subset F(t,T)$, let $y \in W_\omega(\{x_n\})$. Then, there exists a subsequence $\{y_n\}$ of $\{x_n\}$ such that $A(\{y_n\}) = y$. By Lemmas 1 and 2, there exists a subsequence $\{w_n\}$ of $\{y_n\}$ such that $\Delta - \lim_n w_n = w$ and $w \in K$. Since, $\lim_{n\to\infty} d(tw_n, w_n) = 0$ so that in view of Lemma 4, $w \in F(t)$. Also, $\lim_{n\to\infty} d(x_n, P_T x_n) \leq d(x_n, z_n)$. In view of Lemma 12, we have $\lim_{n\to\infty} d(x_n, P_T x_n) = 0$ and so is $\lim_{n\to\infty} d(w_n, P_T w_n) = 0$. Owing to Lemma 4, $w \in F(P_T)$ and hence $w \in F(t,T)$. Now, we claim that $w = y$. Let on contrary that $w \neq y$, then we have

$$\begin{aligned}
\limsup_{n\to\infty} d(w_n, w) &< \limsup_{n\to\infty} d(w_n, y) \\
&\leq \limsup_{n\to\infty} d(y_n, y) \\
&< \limsup_{n\to\infty} d(y_n, w) \\
&= \limsup_{n\to\infty} d(x_n, w) \\
&= \limsup_{n\to\infty} d(w_n, w)
\end{aligned}$$

which is a contradiction and hence $w = y \in F$. To show that $W_\omega(\{(x_n\})$ is a singleton, let $\{y_n\}$ be a subsequence of $\{x_n\}$. In view of Lemmas 1 and 2, there exists a subsequence $\{w_n\}$ of $\{y_n\}$ such that $\Delta - \lim_n w_n = w$. Let $A(\{y_n\}) = y$ and $A(\{x_n\}) = x$. Earlier, we have shown that $y = w$, therefore it is enough to show $w = x$. If $w \neq x$, so by Lemma 10 $\{d(x_n, w)\}$ is convergent. By uniqueness of asymptotic center

$$\begin{aligned}
\limsup_{n\to\infty} d(w_n, w) &< \limsup_{n\to\infty} d(w_n, x) \\
&\leq \limsup_{n\to\infty} d(x_n, x) \\
&< \limsup_{n\to\infty} d(x_n, w) \\
&= \limsup_{n\to\infty} d(w_n, w)
\end{aligned}$$

which is a contradiction so that the conclusion follows. □

Theorem 2. *Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X. Let $t : K \to K$ be a single-valued nonexpansive mapping and $T : K \to P(K)$ be a multi-valued mapping whose P_T is a nonexpansive mapping such that $F(t,T) \neq \emptyset$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (3.1) with $0 < a \leq \alpha_n,\ \beta_n \leq b < 1$, then $\{x_{n_i}\} \to y$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ implies $y \in F(t,T)$.*

Proof. Assume that $\lim_{i\to\infty} d(x_{n_i}, y) = 0$. By Lemma 13, we obtain

$$\lim_{i\to\infty} d(tx_{n_i}, x_{n_i}) = 0$$

Now, we have

$$\begin{aligned} d(x_{n_i}, ty) &\leq d(x_{n_i}, tx_{n_i}) + d(tx_{n_i}, ty) \\ &\leq d(x_{n_i}, tx_{n_i}) + d(x_{n_i}, y). \end{aligned}$$

On taking limit of both the sides, we get

$$\lim_{i\to\infty} d(x_{n_i}, ty) = 0.$$

Hence by the uniqueness of limit, we obtain $y = ty$, that is, $y \in F(t)$. By Lemma 8, we have

$$\begin{aligned} d(y, P_T y) &\leq d(y, x_{n_i}) + d(x_{n_i}, P_T x_{n_i}) + H(P_T x_{n_i}, P_T y) \\ &\leq d(y, x_{n_i}) + d(x_{n_i}, z_{n_i}) + d(x_{n_i}, y) \ \to 0 \end{aligned}$$

as $i \to \infty$. It follows that $y \in F(P_T) = F(T)$. Thus $y \in F(t,T)$. □

Theorem 3. *Let K be a nonempty compact convex subset of a complete CAT(0) space X. Let $t : K \to K$ be a single-valued nonexpansive mapping and $T : K \to PB(K)$ be a multi-valued mapping whose P_T is a nonexpansive mapping such that $F(t,T) \neq \emptyset$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (3.1) with $0 < a \leq \alpha_n,\ \beta_n \leq b < 1$, then $\{x_n\}$ converges strongly to a common fixed point of t and T.*

Proof. Since $\{x_n\}$ is contained in K which is compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $y \in K$, that is, $\lim_{i\to\infty} d(x_{n_i}, y) = 0$. By Theorem 2, we get $y \in F(t,T)$ and by Lemma 10, we have that $\lim_{n\to\infty} d(x_n, y)$ exists. It must be the case in which $\lim_{n\to\infty} d(x_n, y) = \lim_{i\to\infty} d(x_{n_i}, y) = 0$. Thus, $\{x_n\}$ converges strongly to $y \in F(t,T)$. □

Khan and Fukhar-ud-din [12] introduced the so-called condition (A') for two mappings and gave an improved version in [10] of condition (I) of Senter and Dotson [18]. A hybrid version of condition (A') for a pair of single valued and multivalued mapping which is weaker than compactness of the domain, is given as follows:

A pair of single-valued mapping $t : K \to K$ and a multi-valued mapping $T : K \to CB(K)$ is said to satisfy condition (A') if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that either $d(x,tx) \geq f(d(x,F))$ or $d(x,Tx) \geq f(d(x,F))$ for all $x \in K$.

Theorem 4. *Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X. Let $t : K \to K$ be a single-valued nonexpansive mapping and $T : K \to PB(K)$ be a multi-valued mapping whose P_T is a nonexpansive mapping such that $F(t,T) \neq \emptyset$. Moreover pair (t, P_T) satisfies condition (A'). If $\{x_n\}$ is sequence of the modified Ishikawa iteration defined by (3.1) with $0 < a \leq \alpha_n,\ \beta_n \leq b < 1$, then $\{x_n\}$ converges strongly to a common fixed point of t and T.*

Proof. First, we show that $F(t,T)$ is closed. Let $\{x_n\}$ be a sequence in $F(t,T)$ converging to some point $z \in K$. Since

$$\begin{aligned} d(x_n, tz) &= d(tx_n, tz) \\ &\leq d(x_n, z), \end{aligned}$$

we have

$$\limsup_n d(x_n, tz) \leq \limsup d(x_n, z) = 0.$$

By uniqueness of the limit, we have $tz = z$. Also,

$$\begin{aligned} d(x_n, P_T z) &\leq H(P_T x_n, P_T z) \\ &\leq d(x_n, z) \to 0 \ as\ n \to \infty. \end{aligned}$$

This implies that $\{x_n\}$ converges to some point of $P_T z$ and hence $z \in F(P_T) = F(T)$.

By Lemma 10, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(t,T)$ and let us take to be c. If $c = 0$, then there is nothing to prove. If $c > 0$, then in view of Equation (3.4) for all $p \in F(t,T)$, we have

$$d(x_{n+1}, p) \leq d(x_n, p),$$

so that

$$\inf_{p\in F(t,T)} d(x_{n+1}, p) \leq \inf_{p\in F(t,T)} d(x_n, p),$$

which amounts to say that

$$d(x_{n+1}, F(t,T)) \leq d(x_n, F(t,T))$$

and hence $\lim_{n\to\infty} d(x_n, F(t,T))$ exists. Owing to condition (A') there exists a nondecreasing function f such that

$$\lim_{n\to\infty} f(d(x_n, F(t,T))) \leq \lim_{n\to\infty} d(x_n, tx_n) = 0$$

or,

$$\lim_{n\to\infty} f(d(x_n, F(t,T))) \leq \lim_{n\to\infty} d(x_n, P_T x_n) \leq \lim_{n\to\infty} d(x_n, z_n) = 0$$

so that in both the cases $\lim_{n\to\infty} f(d(x_n, F(t,T))) = 0$. Since, f is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n\to\infty} d(x_n, F(t,T)) = 0$.
This implies that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d(x_{n_k}, p_k) \leq \frac{1}{2^k} \text{ for all k} \geq 1$$

where $\{p_k\}$ is in $F(t,T)$. By Lemma 10, we have

$$d(x_{n_{k+1}}, p_k) \leq d(x_{n_k}, p_k) \leq \frac{1}{2^k},$$

so that

$$\begin{aligned} d(p_{k+1}, p_k) &\leq d(p_{k+1}, x_{n_{k+1}}) + d(x_{n_{k+1}}, p_k) \\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}, \end{aligned}$$

which implies that $\{p_k\}$ is a Cauchy sequence. Since $F(t,T)$ is closed, therefore $\{p_k\}$ is a convergent sequence. Write $\lim_{k\to\infty} p_k = p$. Now, in order to show that $\{x_n\}$ converges to p, lets proceed as follows:

$$d(x_{n_k}, p) \leq d(x_{n_k}, p_k) + d(p_k, p) \to 0 \; as \; k \to \infty,$$

so that $\lim_{k\to\infty} d(x_{n_k}, p) = 0$. Since $\lim_{n\to\infty} d(x_n, p)$ exists, therefore $x_n \to p$. □

Remark 1. The condition of nonexpansiveness on P_T is necessary. By the following example we can illustrate that P_T need not be necessarily nonexpansive even T is nonexpansive.

Example 1. Let X be a rectangle with vertices $(0,0), (2,0), (2,1)$ and $(0,1)$. Define $T((u,v)) = T((u,0)) = [(\frac{u}{\sqrt{2}}, \frac{u}{\sqrt{2}}), (1+\frac{u}{\sqrt{2}}, \frac{u}{\sqrt{2}})]$ for $u \leq \sqrt{2}$ and $T(u,v) = T(u,0) = [(1,1),(2,1)]$ for $u > \sqrt{2}$. Then it can be verified that

$$H(Tu_1, Tu_2) = |u_1 - u_2|$$

for $u_1, u_2 \leq \sqrt{2}$ and less than other case so that T is nonexpansive. Also, if $x = (0,1)$ then $P_T(x) = (0,0)$ while for $p = (\sqrt{2}, 1)$, $P_T(p) = (\sqrt{2}, 1)$. Now, $||P_T(p) - P_T(x)|| = \sqrt{3}$ while $||x - p|| = \sqrt{2}$ that is $||P_T(p) - P_T(x)|| > ||x - p||$ which assures that P_T is not a nonexpansive mapping.

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Authors' addresses

Izhar Uddin
Department of Mathematics, Jamia Millia Islamia, New Delhi, India
E-mail address: izharuddin1@jmi.ac.in

Javid Ali
Department of Mathematics, Aligarh Muslim University, 202002 Aligarh, India
E-mail address: javid.mm@amu.ac.in

Vladimir Rakočević
University of Niš, Faculty of Sciences and Mathematics, Višegradska 33, 18000 Niš, Serbia
E-mail address: vrakoc@sbb.rs

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PERIODIC SOLUTIONS AND STABILITY OF LINEAR EVOLUTION EQUATIONS WITH NONINSTANTANEOUS IMPULSES

JINRONG WANG AND MICHAL FEČKAN

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Abstract. This paper concerns existence and stability of solutions to periodic linear evolution equations with noninstantaneous impulses via the theory of operator semigroup. A series of fundamental results including compactness, semigroup property, exponential estimate and periodicity are established for a new introduced impulsive evolution operator. Moreover, triple sufficient conditions are given to guarantee this impulsive evolution operator is exponentially stable. In addition, a relationship between existence of periodic solutions and fixed point of impulsive evolution operator is determined, and the alternative results on periodic solutions and their asymptotical stability are obtained by using the well known Fredholm alternative theorem. Finally, an example of periodic impulsive parabolic linear partial differential equation is given for illustration of the theoretically results.

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Keywords: linear evolution equations, noninstantaneous impulses, periodic solutions, existence, stability

1. Introduction

Differential equations with instantaneous impulses [5,6,12] have been widely used to characterize the mathematical modeling in the real world life that undergo rapid changes in their state. The duration of these rapid changes is relatively short compared to the overall duration of the whole process. However, there exists another impulsive action in many real world processes, i.e., the evolution process involving abrupt state changes as well as keeping active on a finite time interval.

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Differential equations with non-instantaneous impulses can be used to describe the dynamic of periodic evolution processes in hemodynamic equilibrium of a person. This new class impulsive equation was introduced in [7], which is an extension of standard impulsive differential equations. Owing to the widely application of this new type impulsive equations, there are some generalized, improved, and extended work on this fields, one can see [1–4, 8, 10, 13–17] and the reference therein. In fact, Wang and Fečkan [14, p.917] mentioned that the algebraic equation $x = g(t,x)$ in [7] has the only solution depending on t, if g satisfied a contraction condition. To overcome this shortcoming, a general class of impulsive evolution equations [14, (1.6)] was established. These systems can be related to differential algebraic equations [11].

In this paper, we consider the following periodic linear non-instantaneous impulsive evolution equations

$$\begin{cases} u'(t) = Au(t),\ t \in [s_i, t_{i+1}],\ i \in \mathbb{N}_0 := \{0,1,2,\cdots\}, \\ u(t_i^+) = (\mathbb{E} + B_i)u(t_i^-) + c_i,\ i \in \mathbb{N} := \{1,2,\cdots\}, \\ u(t) = (\mathbb{E} + B_i)u(t_i^-) + c_i,\ t \in (t_i, s_i],\ i \in \mathbb{N}, \\ u(s_i^+) = u(s_i^-),\ i \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $A : D(A) \subseteq X \to X$ is the generator of a C_0-semigroup $\{T(t) : t \geq 0\}$ on a Banach space X with a norm $\|\cdot\|$, $B_i : X \to X, i \in \mathbb{N}$ are bounded linear operators with $B_{i+p} = B_i$ and $c_i \in X$ with $c_{i+p} = c_i$, the ω-periodic time sequences $\{t_i\}_{i\in\mathbb{N}_0}$ and $\{s_i\}_{i\in\mathbb{N}_0}$ satisfying $t_i < s_i < t_{i+1}, i \in \mathbb{N}$ with $t_{i+p} = t_i + \omega$ and $s_{i+p} = s_i + \omega$ where $p \in \mathbb{N}$ denotes the number of impulsive points and connection points of a periodic interval $[0,\omega]$ and set $s_0 = 0$, so $s_p = \omega$. Moreover, $\mathbb{E}$ denotes the standard identity operator. We remark that the second equation in (1.1) follows from the third one, but we write it for emphasizing impulses.

Denote $r(t,0)$ by the number of impulsive points existing in $(0,t)$ and $z_+ = \max\{0,z\}$ for $z \in \mathbb{R}$. Throughout this paper, we suppose that $T(t)B_k = B_kT(t)$ for $t > 0$.

Consider $U_T(\cdot,\cdot) : [0,\infty) \times [0,\infty) \to X$ given by

$$\begin{aligned} &U_T(t,s) \\ &= T\left((t - s_{r(t,0)})_+ - (s - s_{r(s,0)})_+ + \sum_{k=r(s,0)}^{r(t,0)-1} (t_{k+1} - s_k)\right) \prod_{k=r(s,0)}^{r(t,0)} (\mathbb{E} + B_k), \end{aligned} \tag{1.2}$$

where we set $\prod_{k=r(s,0)}^{r(t,0)-1} (\mathbb{E} + B_k) = \mathbb{E}$ if $r(s,0) = r(t,0)$. Obviously,

$$\begin{aligned} &U_T(t,0) \\ &= T\left((t - s_{r(t,0)})_+ - (0 - s_{r(0,0)})_+ + \sum_{k=r(0,0)}^{r(t,0)-1} (t_{k+1} - s_k)\right) \prod_{k=r(0,0)}^{r(t,0)} (\mathbb{E} + B_k) \end{aligned}$$

$$= T\left((t-s_{r(t,0)})_+ + \sum_{k=0}^{r(t,0)-1}(t_{k+1}-s_k)\right)\prod_{k=0}^{r(t,0)}(\mathbb{E}+B_k),$$

and

$$\begin{aligned}
&U_T(\omega,0)\\
&= T\left((\omega-s_{r(\omega,0)})_+ - (0-s_{r(0,0)})_+ + \sum_{k=r(0,0)}^{r(\omega,0)-1}(t_{k+1}-s_k)\right)\prod_{k=r(0,0)}^{r(\omega,0)}(\mathbb{E}+B_k)\\
&= T\left((\omega-s_p)_+ - (0-s_0)_+ + \sum_{k=0}^{r(\omega,0)-1}(t_{k+1}-s_k)\right)\prod_{k=0}^{p}(\mathbb{E}+B_k)\\
&= T\left(\sum_{k=0}^{p-1}(t_{k+1}-s_k)\right)\prod_{k=0}^{p}(\mathbb{E}+B_k).
\end{aligned}$$

Introducing

$$PC_\omega([0,+\infty),X) = \{u \in PC([0,\infty),X) : u(t) = u(t+\omega), t \geq 0\},$$

where

$$\begin{aligned}
PC([0,\infty),X) = \{u : [0,\infty) \to X : u_{|J_i} \in C(J_i,X), J_i = (t_i,t_{i+1}], i \in \mathbb{N}_0\\
\text{and } u(t_i^+) \text{ and } u(t_i^-) \text{ exist for each } i \in \mathbb{N}\},
\end{aligned}$$

where $u_{|J_i}$ denotes the domain of u restricted to the subinterval $J_i \subset [0,\infty)$.

The following result is clear.

Theorem 1. *The mild solution $u(\cdot,u_0) \in PC([0,\infty),X)$ of (1.1) with the initial condition $u(0) = u_0$ has the form*

$$u(t,u_0) = U_T(t,0)u_0 + \sum_{k=1}^{r(t,0)} U_T(t,s_k)c_k, \quad \forall\, t \geq 0. \tag{1.3}$$

Definition 1. We say the mild solution $u(\cdot,u_0) \in PC([0,\infty),X)$ of (1.1) is ω-periodic if $u(t,u_0) = u(t+\omega,u_0), t \geq 0$.

Definition 2. We say the mild solution $u(\cdot,u_0) \in PC_\omega([0,+\infty),X)$ of (1.1) is locally asymptotically stable if there exists a $\delta > 0$ such that for any $y_0 \in X$ with $\|u_0 - y_0\| < \delta$ such that

$$\lim_{t\to\infty}\|u(t,u_0) - u(t,y_0)\| = 0.$$

If δ can be arbitrary then $u(\cdot,u_0) \in PC_\omega([0,+\infty),X)$ is globally asymptotically stable.

Clearly, local asymptotic stability for ω-periodic mild solution of (1.1) coincides with global asymptotic stability.

This paper is organized as follows. Section 2 presents the fundamental properties for the new introduced impulsive evolution operator $U_T(\cdot,\cdot)$. The compactness, semigroup property, exponential estimate and periodicity of $U_T(\cdot,\cdot)$ as well as exponential stability are obtained for general impulses. Section 3 establishes a relationship between periodic mild solutions of (1.1) and fixed point of $U_T(\cdot,\cdot)$ and presents alternative results on periodic solutions of (1.1) and their asymptotical stability. An example is given for illustration of the theoretically results in final section.

2. Basic properties for $U_T(\cdot,\cdot)$

In this section, we study basic properties including compactness, periodicity and exponential stability for (1.2).

We impose the following assumptions:

- $(H0)$ Let A be the infinitesimal generator of a C_0-semigroup $\{T(t) : t \geq 0\}$ in X.
- $(H0^+)$ $\{T(t) : t \geq 0\}$ is compact.
- $(H1)$ For each $i \in \mathbb{N}$, $t_{i+p} = t_i + \omega$ and $s_{i+p} = s_i + \omega$.
- $(H2)$ For each $i \in \mathbb{N}$, $B_i : X \to X$ are bounded linear operators with $B_{i+p} = B_i$ and $c_{i+p} = c_i$.

By $(H1)$ and $(H2)$, for each $N \in \mathbb{N}$,

$$t_{i+Np} = t_i + N\omega,\ s_{i+Np} = s_i + N\omega,\ B_{i+Np} = B_i,\ c_{i+Np} = c_i.$$

Theorem 2. *Assumptions $(H0)$, $(H0^+)$ hold and suppose $B_i : X \to X$ be bounded linear operators for each $i \in \mathbb{N}$. Then, $U_T(t,s)$ is linear and compact for any $\omega \geq t > s \geq 0$.*

Proof. The proof directly follows from (1.2), $(H0)$, $(H0^+)$ and that B_i are bounded linear operators. □

Theorem 3. *The property of semigroup holds for $U_T(\cdot,\cdot)$ in $[0,\omega]$, i.e.,*

$$U_T(t,s) = U_T(t,\tau)U_T(\tau,s),\ 0 \leq s < \tau < t \leq \omega$$

Proof. Note that the form of (1.2) and the semigroup property of $T(t)$, one can derive that

$$\begin{aligned}
&U_T(t,\tau)U_T(\tau,s)\\
&= T\left((t-s_{r(t,0)})_+ - (\tau - s_{r(\tau,0)})_+ + \sum_{k=r(\tau,0)}^{r(t,0)-1}(t_{k+1}-s_k)\right)\prod_{k=r(\tau,0)}^{r(t,0)}(\mathbb{E}+B_k)\\
&\quad\times\prod_{k=r(s,0)}^{r(\tau,0)}(\mathbb{E}+B_k)T\left((\tau-s_{r(\tau,0)})_+ - (s-s_{r(s,0)})_+ + \sum_{k=r(s,0)}^{r(\tau,0)-1}(t_{k+1}-s_k)\right)
\end{aligned}$$

$$= U_T(t,s).$$

The proof is completed. □

Theorem 4. *Assumptions* $(H0)$, $(H1)$ *and* $(H2)$ *hold. Then,* $U_T(\cdot,\cdot)$ *is* ω*-periodic, i.e.,*

$$U_T(t+N\omega, s+N\omega) = U_T(t,s),\ N \in \mathbb{N}.$$

Proof. Note that the form of (1.2) and the facts $r(v+N\omega,0) = r(v,0)+Np$, one can derive that

$$\begin{aligned}
U_T(t+N\omega, s+N\omega) &= T\Bigg((t+N\omega - s_{r(t+N\omega,0)})_+ - (s+N\omega - s_{r(s+N\omega,0)})_+ \\
&\quad + \sum_{k=r(s+N\omega,0)}^{r(t+N\omega,0)-1} (t_{k+Np+1} - s_{k+Np})\Bigg) \prod_{k=r(s+N\omega,0)}^{r(t+N\omega,0)} (\mathbb{E} + B_{k+Np}) \\
&= T\Bigg((t+N\omega - (s_{r(t,0)} + N\omega))_+ - (s+N\omega - (s_{r(s,0)} + N\omega))_+ \\
&\quad + \sum_{k=r(s,0)+Np}^{r(t,0)+Np-1} (t_{k+Np+1} - s_{k+Np})\Bigg) \prod_{k=r(s,0)+Np}^{r(t,0)+Np} (\mathbb{E} + B_{k+Np}) = U_T(t,s).
\end{aligned}$$

The proof is completed. □

Theorem 5. *Assumptions* $(H0)$, $(H1)$ *and* $(H2)$ *hold. Then, we have*

$$U_T(t+N\omega,0) = U_T(t,0)[U_T(\omega,0)]^N,\ N \in \mathbb{N}.$$

Proof. We give two proofs as follows:

Method 1. Note that the form of (1.2) and the facts $r(v+N\omega,0) = r(v,0)+Np$, one can derive that

$$\begin{aligned}
U_T(t,0)[U_T(\omega,0)]^N &= T\left((t - s_{r(t,0)})_+ + \sum_{k=0}^{r(t,0)-1} (t_{k+1} - s_k)\right) \prod_{k=0}^{r(t,0)} (\mathbb{E} + B_k) \\
&\quad \times \left[T\left((\omega - s_p)_+ + \sum_{k=0}^{r(\omega,0)-1} (t_{k+1} - s_k)\right) \prod_{k=0}^{p} (\mathbb{E} + B_k)\right]^N \\
&= T\left((t+N\omega - (s_{r(t,0)} + N\omega))_+ + \sum_{k=0}^{r(t,0)+Np-1} (t_{k+Np+1} - s_{k+Np})\right) \\
&\quad \times \prod_{k=0}^{r(t,0)+Np} (\mathbb{E} + B_{k+Np}) = U_T(t+N\omega,0).
\end{aligned}$$

Method 2. By Theorem 3 and Theorem 4,

$$\begin{aligned} U_T(t+N\omega,0) &= U_T(t+N\omega,N\omega)U_T(N\omega,0) \\ &= U_T(t,0)U_T(N\omega,(N-1)\omega)U_T((N-1)\omega,0) \\ &= U_T(t,0)U_T(\omega,0)U_T((N-1)\omega,0) \\ &\vdots \\ &= U_T(t,0)[U_T(\omega,0)]^N,\ N\in\mathbb{N}. \end{aligned}$$

The proof is completed. □

By $(H0)$, $\exists\ \nu\in\mathbb{R}$ and $L\geq 1$ such that

$$\|T(t)\|\leq Le^{\nu t},\ \forall\ t\geq 0.$$

Theorem 6. *For any* $0\leq s<t$,

$$\begin{aligned} &\|U_T(t,s)\| \\ &\leq L\exp\left\{\nu\left((t-s_{r(t,0)})_+-(s-s_{r(s,0)})_+ + \sum_{k=r(s,0)}^{r(t,0)-1}(t_{k+1}-s_k)\right)\right\} \\ &\quad\times\exp\left\{\sum_{k=r(s,0)}^{r(t,0)}\ln\|\mathbb{E}+B_k\|\right\}. \end{aligned} \tag{2.1}$$

Proof. Taking the norm for (1.2), one can obtain the desired result after some simple calculation. □

In the sequel, for general impulses, we suppose

$$\rho=\sup_{k\geq 1}\|\mathbb{E}+B_k\|<\infty,\quad \mu_1=\inf_{k\geq 1}(t_{k+1}-s_k)>0,\quad \mu_2=\sup_{k\geq 1}(t_{k+1}-s_k)<\infty$$

and set

$$\mu=\begin{cases}\mu_1,\ \nu<0,\\ \mu_2,\ \nu\geq 0.\end{cases} \tag{2.2}$$

By (2.1), one can derive the following result.

Theorem 7. *Assumptions* $(H0)$ *and* $(H1)$ *hold. Then it holds*

$$\|U_T(t,s)\|\leq L\rho\exp\{r(t,s)(\nu\mu+\ln\rho)\}. \tag{2.3}$$

Proof. The proof is straightway, so we omit it here. □

$(H3)$ Suppose that

$$\lim_{t-s\to\infty}\frac{r(t,s)}{t-s}=\sigma<\infty.$$

Theorem 8. *Assumptions* $(H0)$, $(H1)$ *and* $(H3)$ *hold. Then,* $U_T(t,s)$ *is exponentially stable provided that* $\nu\mu + \ln\rho < 0$.

Proof. By $(H3)$, for any $1 > \epsilon > 0$ there exits a $t_\epsilon > 0$ such that

$$\left|\frac{r(t,s)}{t-s} - \sigma\right| < \sigma\epsilon, \quad t-s \geq t_\epsilon,$$

that is,

$$(1-\epsilon)(t-s) < \frac{r(t,s)}{\sigma} < (1+\epsilon)(t-s), \quad t-s \geq t_\epsilon \tag{2.4}$$

Linking (2.3) and (2.4), we derive

$$\|U_T(t,s)\| \leq L\rho\exp\{\sigma(1-\epsilon)(\nu\mu + \ln\rho)(t-s)\}$$

for $t-s \geq t_\epsilon$. This completes the proof. □

$(H4)$ There exists an $\alpha > 0$ such that

$$\nu + \frac{1}{\mu}\ln\rho \leq -\alpha < 0,$$

where $\mu = \begin{cases} \mu_1, \alpha+\nu < 0, \\ \mu_2, \alpha+\nu \geq 0. \end{cases}$

Theorem 9. *Assumptions* $(H0)$, $(H1)$ *and* $(H4)$ *hold. Then,* $U_T(t,s)$ *satisfies the following estimate:*

$$\|U_T(t,s)\| \leq L\exp\{\mu|\alpha+\nu| + \mu_1\alpha\}\exp\{-\alpha\mu_1 r(s,t)\}. \tag{2.5}$$

Moreover, $U_T(t,s)$ *is exponentially stable when* $(H3)$ *holds.*

Proof. Obviously,

$$(t-s_{r(t,0)})_+ - (s-s_{r(s,0)})_+ + \sum_{k=r(s,0)}^{r(t,0)-1}(t_{k+1}-s_k) \geq (r(t,s)-1)\mu_1, \tag{2.6}$$

$$(t-s_{r(t,0)})_+ - (s-s_{r(s,0)})_+ + \sum_{k=r(s,0)}^{r(t,0)-1}(t_{k+1}-s_k) \leq (r(t,s)+1)\mu_2. \tag{2.7}$$

Linking (2.6) and (2.7), we obtain

$$\begin{aligned}\mu_1(r(t,s)-1) &\leq (t-s_{r(t,0)})_+ - (s-s_{r(s,0)})_+ + \sum_{k=r(s,0)}^{r(t,0)-1}(t_{k+1}-s_k) \\ &\leq \mu_2(r(t,s)+1),\end{aligned}$$

which implies that

$$\frac{1}{\mu_2}((t-s_{r(t,0)})_+-(s-s_{r(s,0)})_+ + \sum_{k=r(s,0)}^{r(t,0)-1}(t_{k+1}-s_k))-1 \tag{2.8}$$
$$\le r(t,s) \le \frac{1}{\mu_1}\left((t-s_{r(t,0)})_+-(s-s_{r(s,0)})_+ + \sum_{k=r(s,0)}^{r(t,0)-1}(t_{k+1}-s_k)\right)+1.$$

Then

$$-\mu(\alpha+\nu)r(t,s) \tag{2.9}$$
$$\le -(\alpha+\nu)\left((t-s_{r(t,0)})_+-(s-s_{r(s,0)})_+ + \sum_{k=r(s,0)}^{r(t,0)-1}(t_{k+1}-s_k)\right)+\mu|\alpha+\nu|.$$

Note that $(H4)$ implies that

$$-\mu(\alpha+\nu)r(t,s) = -\sum_{k=r(s,0)}^{r(t,0)}\mu(\alpha+\nu) \ge \sum_{k=r(s,0)}^{r(t,0)}\ln\rho \ge \sum_{k=r(s,0)}^{r(t,0)}\ln\|\mathbb{E}+B_k\|. \tag{2.10}$$

Linking (2.8), (2.9) and (2.10), we obtain

$$\nu\left((t-s_{r(t,0)})_+-(s-s_{r(s,0)})_+ + \sum_{k=r(s,0)}^{r(t,0)-1}(t_{k+1}-s_k)\right)+\sum_{k=r(s,0)}^{r(t,0)}\ln\|\mathbb{E}+B_k\|$$
$$\le -\alpha\left((t-s_{r(t,0)})_+-(s-s_{r(s,0)})_+ + \sum_{k=r(s,0)}^{r(t,0)-1}(t_{k+1}-s_k)\right)+\mu|\alpha+\nu| \tag{2.11}$$
$$\le -\alpha\mu_1 r(t,s)+\mu_1\alpha+\mu|\alpha+\nu|.$$

Finally, substituting (2.11) to (2.1), one can derive (2.5).

Linking (2.4) and (2.5), one can verify the exponential stability of $U_T(t,s)$ given by

$$\|U_T(t,s)\| \le L\exp\{\mu|\alpha+\nu|+\mu_1\alpha\}\exp\{-\alpha\sigma(1-\epsilon)\mu_1(t-s)\}$$

for $t-s \ge t_\epsilon$. The proof is completed. □

Theorem 10. *Assume $(H0)$ and $(H3)$ hold. Suppose that there is a $p \in [1,\infty)$ such that*

$$\int_0^\infty \|T(t)x\|^p\,dt < \infty,\ \forall x \in X, \tag{2.12}$$

and

$$\prod_{k=0}^{\infty}\|\mathbb{E}+B_k\| := M_B < \infty. \tag{2.13}$$

Then, $U_T(t,s)$ is exponentially stable.

Proof. By (2.12) and [9, Theorem 4.1], there are constants $M_T \geq 1$ and $\upsilon > 0$ such that $\|T(t)\| \leq M_T e^{-\upsilon t}$. Taking the norm for both sides of (1.2) and noticing (2.13), one has

$$\begin{aligned}
&\|U_T(t,s)\| \\
&\leq M_T M_B \exp\left\{-\upsilon\left[(t-s_{r(t,0)})_+ - (s-s_{r(s,0)})_+ + \sum_{k=r(s,0)}^{r(t,0)-1}(t_{k+1}-s_k)\right]\right\} \\
&\leq M_T M_B \exp\{\mu_1\alpha\}\exp\{-\alpha\sigma(1-\epsilon)\mu_1(t-s)\}
\end{aligned}$$

for $t-s \geq t_\epsilon$, which is exponentially stable. □

3. Periodic solutions and asymptotical stability

In this section, we give a sufficient and necessary condition to guarantee that (1.1) has an an ω-periodic mild solution.

Theorem 11. *Assumptions $(H0)$, $(H1)$ and $(H2)$ hold. Then (1.3) is an ω-periodic mild solution of (1.1) if and only if u_0 satisfies*

$$(\mathbb{E} - U_T(\omega,0))u_0 = \sum_{k=1}^{p} U_T(\omega,s_k)c_k. \tag{3.1}$$

Proof. Suppose (1.3) is an ω-periodic mild solution of (1.1), thus, $u(t+\omega,u_0) = u(t,u_0)$, $t \geq 0$ due to Definition 1. Of course, $u_0 = u(\omega,u_0)$, which implies (3.1), since $r(\omega,0) = p$. Now, suppose $u_0 \in X$ solves (3.1). Then (1.3) implies

$$\begin{aligned}
u(t+\omega,u_0) &= U_T(t+\omega,0)u_0 + \sum_{k=1}^{r(t+\omega,0)} U_T(t+\omega,s_k)c_k \\
&= U_T(t+\omega,\omega)\left(U_T(\omega,0)u_0 + \sum_{k=1}^{p} U_T(\omega,s_k)c_k\right) + \sum_{k=p+1}^{r(t,0)+p} U_T(t+\omega,s_k)c_k \\
&= U_T(t,0)u_0 + \sum_{k=1}^{r(t,0)} U_T(t+\omega,s_{k+p})c_{k+p} \\
&= U_T(t,0)u_0 + \sum_{k=1}^{r(t,0)} U_T(t+\omega,s_k+\omega)c_k \\
&= U_T(t,0)u_0 + \sum_{k=1}^{r(t,0)} U_T(t,s_k)c_k = u(t,u_0),
\end{aligned}$$

where we use Theorems 3, 4 and properties $r(t+\omega,0)=r(t,0)+p$, $s_{k+p}=s_k+\omega$, $c_{k+p}=c_k$. The proof is completed. □

Next, we give an alternative result.

Theorem 12. *Let $(H0)$, $(H0^+)$, $(H1)$ and $(H2)$ be satisfied. Then one of the following facts is fulfilled:*

(i) *For any $c_i \in X$ with $c_{i+p}=c_i$, (1.1) has the unique ω-periodic mild solution. Then $[\mathbb{E}-U_T(\omega,0)]^{-1}$ exists and it is bounded.*
(ii) *For any $c_i \in X$ with $c_{i+p}=c_i$, either (1.1) has no ω-periodic mild solution or it has a finite dimensional set of ω-periodic mild solutions. Then $[\mathbb{E}-U_T(\omega,0)]^{-1}$ does not exist.*

Proof. Under $(H0)$ and $(H0^+)$, one can see $U_T(\omega,0)$ is linear and compact operator via Theorem 2. The result follows from Theorem 11 by Fredholm Alternative Theorem. □

To end this section, we show that ω-periodic mild solution is asymptotically stable.

Theorem 13. *Assumptions $(H0)$, $(H1)$, $(H2)$ and $(H4)$ hold. Then for any $c_i \in X$ with $c_{i+p}=c_i$, (1.1) has the unique ω-periodic mild solution which is also asymptotically stable.*

Proof. Theorem 9 implies that $U_T(t,0)$ is exponentially stable. This gives that the spectral radius of $U_T(\omega,0)$ is less than 1. Then Neumann lemma leads that $[\mathbb{E}-U_T(\omega,0)]^{-1}$ exists and it is bounded. Thus Theorem 11 gives the existence and uniqueness result. Next, by Definition 2, it remains to verify that

$$\lim_{t\to\infty}\|u(t,u_0)-u(t,y_0)\| = \lim_{t\to\infty}\|U_T(t,0)\|\|u_0-y_0\| = 0,$$

which is satisfied again via Theorem 9. The proof is completed. □

4. Application to linear heat equations

In this section, we use an example of periodic impulsive parabolic linear partial differential equation to illustrate the above theoretical results.

Set $X=L^2(0,1)$ and $s_0=0$, $t_i=(2i-1)\pi$, $s_i=2i\pi$, $i\in\mathbb{N}$, $p=1$ and $\omega=2\pi$. Obviously, $t_{i+p}=t_i+\omega$ and $s_{i+p}=s_i+\omega$.

Consider a parabolic type periodic linear heat equations with non-instantaneous impulses:

$$\begin{cases} \frac{\partial}{\partial t}u(t,y) = \kappa^2 \frac{\partial^2}{\partial y^2}u(t,y), \\ y \in (0,1),\ t \in [0,\pi] \cup [2i\pi,(2i+1)\pi],\ i \in \mathbb{N},\ \kappa > 0, \\ u(t,0) = u(t,1) = 0,\ t > 0, \\ u((2i-1)\pi^+,y) = u((2i-1)\pi^-,y) + b_i u((2i-1)\pi^-,y) + c_i(y),\ i \in \mathbb{N}, \\ u(t,y) = u((2i-1)\pi^-,y) + b_i u((2i-1)\pi^-,y) + c_i(y), \\ t \in ((2i-1)\pi,2i\pi],\ i \in \mathbb{N}, \\ u(2i^+,y) = u(2i^-,y),\ i \in \mathbb{N}, \end{cases} \tag{4.1}$$

for $b_i \in \mathbb{R} \setminus \{0\}$ and $c_i \in X$. Define $Au = \frac{\partial^2}{\partial y^2}u$ for $u \in D(A)$ with

$$D(A) = \left\{ u \in X : \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial y^2} \in X,\ u(0) = u(1) = 0 \right\}.$$

Then, A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ in X.

Let $u_n(y) = \sqrt{2}\sin(\pi n y)$, $n = 1,2,\cdots$ be the orthonormal set of eigenfunctions of A. For any $u \in D(A)$, $Au = \sum_{n=1}^{\infty}(-\kappa^2 n^2)\langle u,u_n\rangle u_n$. Thus,

$$T(t)u := \sum_{n=1}^{\infty} \exp\{-\kappa^2 n^2 t\}\langle u,u_n\rangle u_n,$$

with $T(\cdot)$ is dissipative and compact with $\|T(t)\| \leq e^{-\kappa^2 t}$ for all $t \geq 0$. Obviously, $L = 1$ and $\nu = -\kappa^2$. Let $B_i = b_i \mathbb{E}$ and $c_i = c_i(y)$. Then,

$$T(\cdot)B_i = B_i T(\cdot) = \sum_{n=1}^{\infty} b_i e^{-\kappa^2 n^2 \cdot}\langle u,u_n\rangle u_n.$$

In addition, $(H0)$, $(H0^+)$ and $(H1)$ are satisfied. Denoting $u(\cdot)(y) = u(\cdot,y)$, (4.1) can be abstracted into (1.1). Clearly $(H3)$ holds with $\sigma = 1$ and $\rho = \sup_{i\in\mathbb{N}}|1+b_i|$, $\mu = \pi$ in (2.2). By (1.2),

$$\begin{aligned} U_T(t,s) = &\prod_{k=r(s,0)}^{r(t,0)} (1+b_k) \\ &\times \sum_{n=1}^{\infty} \exp\left\{-\kappa^2 n^2 \left((t-s_{r(t,0)})_+ - (s-s_{r(s,0)})_+ + (r(t,0)-r(s,0))\pi\right)\right\}, \end{aligned} \tag{4.2}$$

is exponentially stable provided that

$$0 > \nu\mu + \ln\rho \Longleftrightarrow \sup_{i\in\mathbb{N}_0} |1+b_i| < e^{\pi\kappa^2} \tag{4.3}$$

via Theorem 8. Next, (2.13) holds either if

$$\sum_{i=1}^{\infty}|b_i|<\infty,\ \text{for example, } b_i=\frac{1}{i^2}$$

or

$$\sum_{i=1}^{\infty}\ln|1+b_i|<\infty,\ \text{for example, } b_i=\pm e^{\frac{1}{i^2}}-1.$$

Then by Theorem 10, (4.2) is exponentially stable.

Now we consider the simplest periodic case, so $b_1=b$ and $c_1=c$ for any $i\in\mathbb{N}$. Then obviously,

$$U_T(\omega,0)u=(1+b)\sum_{n=1}^{\infty}e^{-\kappa^2n^2\pi}\langle u,u_n\rangle u_n$$

and

$$u(\omega,u_0)=(1+b)\sum_{n=1}^{\infty}e^{-\kappa^2n^2\pi}\langle u_0,u_n\rangle u_n+c(y).$$

So (3.1) becomes

$$(1-(1+b)e^{-\kappa^2n^2\pi})\bar{u}_n=\bar{c}_n,\quad n\in\mathbb{N}, \tag{4.4}$$

where $u_0=\sum_{n=1}^{\infty}\bar{u}_nu_n$ and $c=\sum_{n=1}^{\infty}\bar{c}_nu_n$. From (4.4), we deduce: If

$$e^{\kappa^2n^2\pi}\neq 1+b,\quad \forall n\in\mathbb{N}, \tag{4.5}$$

then the alternative (i) of Theorem 12 holds for (4.1): There is a unique 2π-periodic mild solution of (4.1) given by

$$u(t,u_0)=\begin{cases}\sum_{n=1}^{\infty}\frac{e^{-\kappa^2n^2t}\bar{c}_n}{1-(1+b)e^{-\kappa^2n^2\pi}}u_n & t\in[0,\pi]\\ \sum_{n=1}^{\infty}\left(\frac{(1+b)e^{-\kappa^2n^2\pi}}{1-(1+b)e^{-\kappa^2n^2\pi}}+1\right)\bar{c}_nu_n & t\in(\pi,2\pi].\end{cases}$$

If

$$e^{\kappa^2\tilde{n}^2\pi}=1+b$$

for some $\tilde{n}\in\mathbb{N}$, then the alternative (ii) of Theorem 12 holds for (4.1): If $\bar{c}_{\tilde{n}}\neq 0$ then there is no a 2π-periodic mild solution of (4.1), but if $\bar{c}_{\tilde{n}}=0$ then there is a 1-dimensional space $[u_{\tilde{n}}]$ of 2π-periodic mild solutions of (4.1).

Next, (4.3) becomes

$$|1+b|<e^{\pi\kappa^2},$$

which implies (4.5). For example, we choose $b=0.05$ satisfying $|1+0.05|<e^{0.02\pi}\approx 1.0648$.

Hence then there is a unique 2π-periodic mild solution of (4.1), which is in addition asymptotically stable (see Figures 1-3).

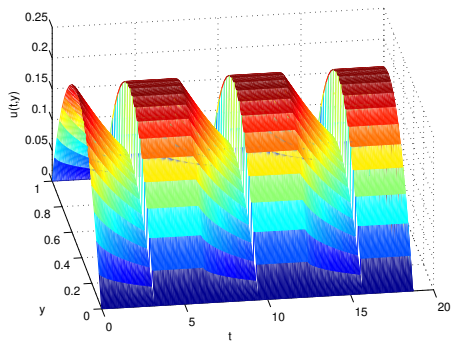

FIGURE 1. The periodic solution of (4.1) with $\kappa^2 = 0.02$, $u_0(y) = y(1-y)$, $y \in [0,1]$, $b_i = 0.05$.

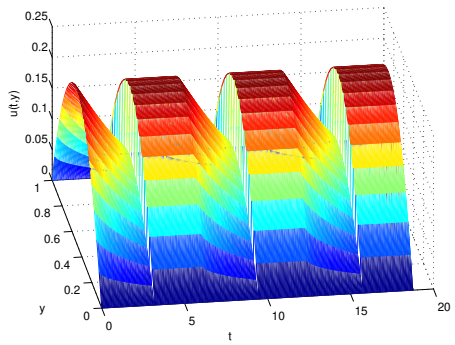

FIGURE 2. The periodic solution of (4.1) with $\kappa^2 = 0.02$, $u_0(y) = 1.01y(1-y)$, $y \in [0,1]$, $b_i = 0.05$.

Note that for $b_i = b$, (4.4) gives

$$c_i(y) = c(y) = \sum_{n=1}^{\infty} (1-(1+b)e^{-\kappa^2 n^2 \pi})\bar{u}_n u_n(y)$$

knowing $u_0(y)$. For instance, if $u_0(y) = y(1-y)$, then using the Fourier series of the function $y(1-y)$ with respect to the orthogonal basis $\sin \pi n y$, we have

$$u_0(y) = \frac{8}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin(2n+1)\pi y}{(2n+1)^3} = \sum_{n=1}^{\infty} \frac{4\sqrt{2}}{(2n+1)^3 \pi^3} u_{2n+1}(y),$$

so

$$\bar{u}_{2n+1} = \frac{4\sqrt{2}}{(2n+1)^3 \pi^3}, \quad \bar{u}_{2n} = 0,$$

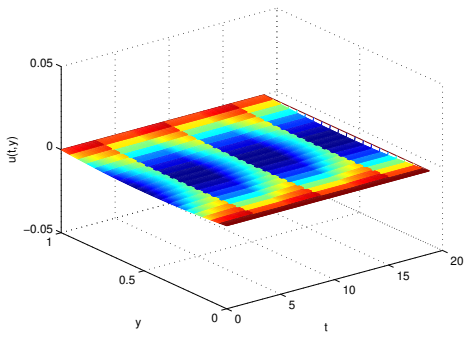

FIGURE 3. The difference between the Figures 1 and 2.

and

$$c(y) = \sum_{n=1}^{\infty} \frac{4\sqrt{2}(1-(1+b)e^{-\kappa^2 n^2 \pi})}{(2n+1)^3 \pi^3} u_{2n+1}(y)$$
$$= \frac{8}{\pi^3} \sum_{n=1}^{\infty} \frac{(1-(1+b)e^{-\kappa^2 n^2 \pi}) \sin(2n+1)\pi y}{(2n+1)^3}.$$

For numerical computation, it is useful to use also another formula

$$c = u_0 - (1+b)u(\pi, u_0).$$

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Authors' addresses

JinRong Wang

JinRong Wang, Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, P.R. China, and Qufu Normal University, School of Mathematical Sciences, Qufu, Shandong 273165, P.R. China

E-mail address: jrwang@gzu.edu.cn

Michal Fečkan

Michal Fečkan, Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University, Mlynská dolina, 842 48 Bratislava, Slovakia, and Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia

E-mail address: Michal.Feckan@fmph.uniba.sk

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BICOMPLEX GENERALIZED $k-$HORADAM QUATERNIONS

YASIN YAZLIK, SURE KÖME, AND CAHIT KÖME

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Abstract. This study provides a broad overview of the generalization of the various quaternions, especially in the context of its enhancing importance in the disciplines of mathematics and physics. By the help of bicomplex numbers, in this paper, we define the bicomplex generalized $k-$Horadam quaternions. Fundamental properties and mathematical preliminaries of these quaternions are outlined. Finally, we give some basic conjucation identities, generating function, the Binet formula, summation formula, matrix representation and a generalized identity, which is generalization of the well-known identities such as Catalan's identity, Cassini's identity and d'Ocagne's identity, of the bicomplex generalized $k-$Horadam quaternions in detail.

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Keywords: bicomplex generalized $k-$Horadam quaternions, recurrence relations, quaternions

1. Introduction

Bicomplex numbers emerge in various scientific areas such as quantum mechanics, digital signal processing, electromagnetic waves and curved structures, determination of antenna patterns, fractal structures and many related fields. Recently, several remarkable studies have been conducted related with bicomplex numbers (see [6, 11, 13, 19, 22, 24, 25, 27, 29, 30, 35]). For instance, Rochon and Tremblay, in [29], studied the bicomplex Schrödinger equations. They also mentioned that the bicomplex quantum mechanics are the generalization of both the classical and hyperbolic quantum mechanics. Kabadayi and Yayli, in [19], represented a curve by means of bicomplex numbers in a hypersurface in E^4 and then they defined the homothetic motion of this curve. Lavoie et al., in [21], determined the eigenkets and eigenvalues of the bicomplex quantum harmonic oscillator Hamiltonian. They asserted that these eigenvalues and eigenkets, first in the literature, were derived with a number system larger than $\mathbb{C}$. The bicomplex number q, which extends the complex numbers, can be defined as

$$\mathbb{C}_2 = \{q = q_1 + \mathbf{i}q_2 + \mathbf{j}q_3 + \mathbf{ij}q_4 \quad | \quad q_1, q_2, q_3, q_4 \in \mathbb{R}\}, \tag{1.1}$$

where $\mathbf{i}, \mathbf{j}$ and $\mathbf{ij}$ satisfy the multiplication rules

$$\mathbf{i}^2 = \mathbf{j}^2 = -1, \quad \mathbf{ij} = \mathbf{ji}. \tag{1.2}$$

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The conjugations of the bicomplex numbers are defined in [28] as:

$$q_i^\star = q_1 - \mathbf{i}q_2 + \mathbf{j}q_3 - \mathbf{ij}q_4, \quad q_j^\star = q_1 + \mathbf{i}q_2 - \mathbf{j}q_3 - \mathbf{ij}q_4$$

and

$$q_{ij}^\star = q_1 - \mathbf{i}q_2 - \mathbf{j}q_3 + \mathbf{ij}q_4$$

and the basic properties of the conjugations are as follows:

$$\left(q^\star\right)^\star = q, \quad (q_1q_2)^\star = q_2^\star q_1^\star, \quad (q_1+q_2)^\star = q_1^\star + q_2^\star, \quad (\lambda q_1)^\star = \lambda q_1^\star$$

and

$$(\lambda q_1 \pm \mu q_2)^\star = \lambda q_1^\star + \mu q_2^\star,$$

where $q_1, q_2 \in \mathbb{C}_2$ and $\lambda, \mu \in \mathbb{R}$. Furthermore, three different norms for the bicomplex numbers are given by

$$N_{q_i} = ||q \times q_i|| = \sqrt{|q_1^2 + q_2^2 - q_3^2 - q_4^2 + 2j(q_1q_3 + q_2q_4)|},$$
$$N_{q_j} = ||q \times q_j|| = \sqrt{|q_1^2 - q_2^2 + q_3^2 - q_4^2 + 2i(q_1q_2 + q_3q_4)|},$$
$$N_{q_{ij}} = ||q \times q_{ij}|| = \sqrt{|q_1^2 + q_2^2 + q_3^2 + q_4^2 + 2ij(q_1q_4 - q_2q_3)|}.$$

Quaternions, which are a number system that extends the complex numbers, arise in quantum mechanics, physics, mathematics, computer science and related areas (see [1, 2, 5, 9, 10, 12, 14, 15, 26, 31–33]). They were first introduced by William Rowan Hamilton in 1843 [14]. In general, a quaternion q, which is member of a noncommutative division algebra, is defined by

$$\mathbb{H} = \{q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \quad | \quad q_0, q_1, q_2, q_3 \in \mathbb{R}\}, \tag{1.3}$$

where $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ satisfy the multiplication rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \tag{1.4}$$

Note that, although quaternions are noncommutative, the bicomplex numbers and bicomplex quaternions are commutative. The conjugate of a quaternion $\bar{q}$ is defined by

$$\overline{q} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3, \tag{1.5}$$

where $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ satisfy the rules (1.4).

The quaternions have been studied by several authors in the recent years (see [1, 2, 4, 5, 9, 10, 12, 15, 26, 31–33]). For example, Horadam, in [15], defined the Fibonacci quaternions. Motivated by Horadam's study, Halici, in [9], examine some basic properties of Fibonacci and Lucas quaternions. She also gave the generating functions, the Binet formulas and derived some sums formulas for these quaternions. Liana and Wloch [31] introduced the Jacobsthal and Jacobsthal-Lucas quaternions and they gave some properties and matrix representations of these quaternions. Tan proposed the biperiodic Fibonacci quaternions whose coefficients are the biperiodic Fibonacci numbers in [33]. Later, Tan et al. described the biperiodic

Lucas quaternions and gave the generating functions, the Binet formulas and Cassini and Catalan like identities [32]. Later, by using the bicomplex numbers, Aydın, in [1], defined the bicomplex Fibonacci and Lucas quaternions as:

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + ijF_{n+3}, \tag{1.6}$$

where F_n is the nth Fibonacci number. She also studied addition, subtraction, multiplication of the bicomplex Fibonacci quaternions and then gave several properties of this quaternion. Although, she mentioned the bicomplex Lucas quaternions in Theorem 2.5 in [1], she didn't give any definition of the bicomplex Lucas quaternions.

For $n \in \mathbb{N}_0$, the Fibonacci and Lucas numbers are defined by the recurrence relations

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \tag{1.7}$$

and

$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1, \tag{1.8}$$

respectively. Recently, many researchers have studied several applications and generalizations for the number sequences(see [7, 8, 16–18, 20, 36]). For further information, we specially refer to book in [20]. For example, Yazlik and Taskara, in [36], defined the generalized $k-$Horadam sequence, which is generalization of many number sequences in the literature. For $n \in \mathbb{N}_0$ and $f(k)^2 + 4g(k) > 0$, the generalized $k-$Horadam sequence defined by

$$H_{k,n+2} = f(k)H_{k,n+1} + g(k)H_{k,n}, \quad H_{k,0} = a, \quad H_{k,1} = b. \tag{1.9}$$

Note that, the Binet formula of the generalized $k-$Horadam sequence is given by, for $n \in \mathbb{N}_0$,

$$H_{k,n} = \frac{Xr_1^n - Yr_2^n}{r_1 - r_2}, \tag{1.10}$$

where $X = b - ar_2$ and $Y = b - ar_1$.

In this paper, by analogy to generalizations of Fibonacci and Lucas quaternions explained for example in [12, 26, 32, 33], we generalize families of the Fibonacci and Lucas quaternions. Hence, the next section describes the bicomplex generalized $k-$Horadam quaternions which are both generalization of the results in [1] and they include several bicomplex quaternions which are not defined before.

2. Bicomplex generalized $k-$Horadam quaternions

Definition 1. For $n \in \mathbb{N}_0$, the bicomplex generalized $k-$Horadam quaternion is defined by

$$\mathcal{H}_{k,n}^{Q} = H_{k,n} + \mathbf{i}H_{k,n+1} + \mathbf{j}H_{k,n+2} + \mathbf{ij}H_{k,n+3}, \tag{2.1}$$

where $H_{k,n}$ is the generalized $k-$Horadam numbers which is defined in (1.9).

TABLE 1. The bicomplex generalized $k-$Horadam quaternions

$f(k)$	$g(k)$	a	b	Bicomplex generalized $k-$Horadam quaternions $\mathcal{H}_{k,n}^{Q} = H_{k,n} + \mathbf{i}H_{k,n+1} + \mathbf{j}H_{k,n+2} + \mathbf{ij}H_{k,n+3}$ $H_{k,n} = f(k)H_{k,n-1} + g(k)H_{k,n-2}$, $H_{k,0} = a$ and $H_{k,1} = b$
1	1	0	1	Bicomplex Fibonacci quaternions [1] $Q_n = F_n + \mathbf{i}F_{n+1} + \mathbf{j}F_{n+2} + \mathbf{ij}F_{n+3}$ $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$ and $F_1 = 1$
1	1	2	1	Bicomplex Lucas quaternions $\mathcal{L}_n^{Q} = L_n + \mathbf{i}L_{n+1} + \mathbf{j}L_{n+2} + \mathbf{ij}L_{n+3}$ $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$ and $L_1 = 1$
2	k	0	1	Bicomplex $k-$Pell quaternions [3] $BC_{k,n}^{P} = P_{k,n} + \mathbf{i}P_{k,n+1} + \mathbf{j}P_{k,n+2} + \mathbf{ij}P_{k,n+3}$ $P_{k,n} = 2P_{k,n-1} + kP_{k,n-2}$, $P_{k,0} = 0$ and $P_{k,1} = 1$
2	1	0	1	Bicomplex Pell quaternions [3] $\mathcal{P}_n^{Q} = P_n + \mathbf{i}P_{n+1} + \mathbf{j}P_{n+2} + \mathbf{ij}P_{n+3}$ $P_n = 2P_{n-1} + P_{n-2}$, $P_0 = 0$ and $P_1 = 1$
2	1	2	2	Bicomplex Pell-Lucas quaternions $\mathcal{PL}_n^{Q} = Q_n + \mathbf{i}Q_{n+1} + \mathbf{j}Q_{n+2} + \mathbf{ij}Q_{n+3}$ $Q_n = 2Q_{n-1} + Q_{n-2}$, $Q_0 = 2$ and $Q_1 = 2$
1	2	0	1	Bicomplex Jacobsthal quaternions $\mathcal{J}_n^{Q} = J_n + \mathbf{i}J_{n+1} + \mathbf{j}J_{n+2} + \mathbf{ij}J_{n+3}$ $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = 0$ and $J_1 = 1$
1	2	2	1	Bicomplex Jacobsthal-Lucas quaternions $\mathcal{JL}_n^{Q} = j_n + \mathbf{i}j_{n+1} + \mathbf{j}j_{n+2} + \mathbf{ij}j_{n+3}$ $j_n = j_{n-1} + 2j_{n-2}$, $j_0 = 2$ and $j_1 = 1$

It is not difficult to see from the following table that the bicomplex generalized $k-$Horadam quaternions can be reduced into several quaternions for the special cases of $f(k), g(k), a$ and b.

For $n, m \in \mathbb{N}_0$, $\mathcal{H}_{k,n}^{Q}$ and $\mathcal{H}_{k,m}^{Q}$ be two bicomplex generalized $k-$Horadam quaternions. Thus, the addition and subtraction of these two quaternions can be given as:

$$\begin{aligned}
\mathcal{H}_{k,n}^{Q} \pm \mathcal{H}_{k,m}^{Q} &= \left(H_{k,n} + \mathbf{i}H_{k,n+1} + \mathbf{j}H_{k,n+2} + \mathbf{ij}H_{k,n+3}\right) \\
&\quad \pm \left(H_{k,m} + \mathbf{i}H_{k,m+1} + \mathbf{j}H_{k,m+2} + \mathbf{ij}H_{k,m+3}\right) \\
&= \left(H_{k,n} \pm H_{k,m}\right) + \mathbf{i}\left(H_{k,n+1} \pm H_{k,m+1}\right) \\
&\quad + \mathbf{j}\left(H_{k,n+2} \pm H_{k,m+2}\right) + \mathbf{ij}\left(H_{k,n+3} \pm H_{k,m+3}\right).
\end{aligned}$$

TABLE 2. The bicomplex generalized $k-$Horadam quaternions (continued from previous page)

k	1	0	1	Bicomplex $k-$Fibonacci quaternions [34] $Q_{F_{k,n}} = F_{k,n} + \mathbf{i}F_{k,n+1} + \mathbf{j}F_{k,n+2} + \mathbf{ij}F_{k,n+3}$ $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$, $F_{k,0} = 0$ and $F_{k,1} = 1$
k	1	2	k	Bicomplex $k-$Lucas quaternions $\mathcal{L}^{Q}_{k,n} = L_{k,n} + \mathbf{i}L_{k,n+1} + \mathbf{j}L_{k,n+2} + \mathbf{ij}L_{k,n+3}$ $L_{k,n} = kL_{k,n-1} + L_{k,n-2}$, $L_{k,0} = 2$ and $L_{k,1} = k$
p	q	a	b	Bicomplex Horadam quaternions $\mathcal{H}^{Q}_{n} = H_n + \mathbf{i}H_{n+1} + \mathbf{j}H_{n+2} + \mathbf{ij}H_{n+3}$ $H_n = pH_{n-1} + qH_{n-2}$, $H_0 = a$ and $H_1 = b$

On the other hand, the multiplication of these two quaternions can be computed as:

$$\begin{aligned}
&\mathcal{H}^{Q}_{k,n} \times \mathcal{H}^{Q}_{k,m} \\
&\quad = \left(H_{k,n} + \mathbf{i}H_{k,n+1} + \mathbf{j}H_{k,n+2} + \mathbf{ij}H_{k,n+3}\right) \\
&\qquad \times \left(H_{k,m} + \mathbf{i}H_{k,m+1} + \mathbf{j}H_{k,m+2} + \mathbf{ij}H_{k,m+3}\right) \\
&\quad = \left(H_{k,n}H_{k,m} - H_{k,n+1}H_{k,m+1} - H_{k,n+2}H_{k,m+2} + H_{k,n+3}H_{k,m+3}\right) \\
&\qquad + \mathbf{i}\left(H_{k,n}H_{k,m+1} + H_{k,n+1}H_{k,m} - H_{k,n+2}H_{k,m+3} - H_{k,n+3}H_{k,m+2}\right) \\
&\qquad + \mathbf{j}\left(H_{k,n}H_{k,m+2} + H_{k,n+2}H_{k,m} - H_{k,n+1}H_{k,m+3} - H_{k,n+3}H_{k,m+1}\right) \\
&\qquad + \mathbf{ij}\left(H_{k,n}H_{k,m+3} + H_{k,n+3}H_{k,m} + H_{k,n+1}H_{k,m+2} + H_{k,n+2}H_{k,m+1}\right) \\
&\quad = \mathcal{H}^{Q}_{k,m} \times \mathcal{H}^{Q}_{k,n}.
\end{aligned}$$

In addition, the conjugations of the bicomplex generalized $k-$Horadam quaternions are defined as:

$$\left(\mathcal{H}^{Q}_{k,n}\right)^{\star}_{i} = H_{k,n} - \mathbf{i}H_{k,n+1} + \mathbf{j}H_{k,n+2} - \mathbf{ij}H_{k,n+3} \tag{2.2}$$

$$\left(\mathcal{H}^{Q}_{k,n}\right)^{\star}_{j} = H_{k,n} + \mathbf{i}H_{k,n+1} - \mathbf{j}H_{k,n+2} - \mathbf{ij}H_{k,n+3} \tag{2.3}$$

$$\left(\mathcal{H}^{Q}_{k,n}\right)^{\star}_{ij} = H_{k,n} - \mathbf{i}H_{k,n+1} - \mathbf{j}H_{k,n+2} + \mathbf{ij}H_{k,n+3}. \tag{2.4}$$

Theorem 1. *For any given two bicomplex generalized $k-$Horadam quaternions $\mathcal{H}^{Q}_{k,n}$ and $\mathcal{H}^{Q}_{k,m}$, where $n, m \in \mathbb{N}_0$, the following conjugation identities hold.*

$$\left(\mathcal{H}^{Q}_{k,n}\mathcal{H}^{Q}_{k,m}\right)^{\star}_{i} = \left(\mathcal{H}^{Q}_{k,m}\right)^{\star}_{i}\left(\mathcal{H}^{Q}_{k,n}\right)^{\star}_{i} = \left(\mathcal{H}^{Q}_{k,n}\right)^{\star}_{i}\left(\mathcal{H}^{Q}_{k,m}\right)^{\star}_{i}, \tag{2.5}$$

$$\left(\mathcal{H}_{k,n}^{Q}\mathcal{H}_{k,m}^{Q}\right)_{j}^{\star}=\left(\mathcal{H}_{k,m}^{Q}\right)_{j}^{\star}\left(\mathcal{H}_{k,n}^{Q}\right)_{j}^{\star}=\left(\mathcal{H}_{k,n}^{Q}\right)_{j}^{\star}\left(\mathcal{H}_{k,m}^{Q}\right)_{j}^{\star}, \tag{2.6}$$

$$\left(\mathcal{H}_{k,n}^{Q}\mathcal{H}_{k,m}^{Q}\right)_{ij}^{\star}=\left(\mathcal{H}_{k,m}^{Q}\right)_{ij}^{\star}\left(\mathcal{H}_{k,n}^{Q}\right)_{ij}^{\star}=\left(\mathcal{H}_{k,n}^{Q}\right)_{ij}^{\star}\left(\mathcal{H}_{k,m}^{Q}\right)_{ij}^{\star}. \tag{2.7}$$

Proof. By considering the Eqs. (2.2), (2.3) and (2.4), the theorem can be proved easily. □

Theorem 2. *Let* $\left(\mathcal{H}_{k,n}^{Q}\right)_{i}^{\star}$, $\left(\mathcal{H}_{k,n}^{Q}\right)_{j}^{\star}$ *and* $\left(\mathcal{H}_{k,n}^{Q}\right)_{ij}^{\star}$ *be three conjugations of the generalized* $k-$*Horadam quaternion. Then we obtain the following relations*

$$\begin{aligned}\mathcal{H}_{k,n}^{Q}\left(\mathcal{H}_{k,n}^{Q}\right)_{i}^{\star}&=H_{k,n}^{2}+H_{k,n+1}^{2}-H_{k,n+2}^{2}-H_{k,n+3}^{2}\\&\quad+2j\left(H_{k,n}H_{k,n+2}+H_{k,n+1}H_{k,n+3}\right)\end{aligned} \tag{2.8}$$

$$\begin{aligned}\mathcal{H}_{k,n}^{Q}\left(\mathcal{H}_{k,n}^{Q}\right)_{j}^{\star}&=H_{k,n}^{2}-H_{k,n+1}^{2}+H_{k,n+2}^{2}-H_{k,n+3}^{2}\\&\quad+2i\left(H_{k,n}H_{k,n+1}+H_{k,n+2}H_{k,n+3}\right)\end{aligned} \tag{2.9}$$

$$\begin{aligned}\mathcal{H}_{k,n}^{Q}\left(\mathcal{H}_{k,n}^{Q}\right)_{ij}^{\star}&=H_{k,n}^{2}+H_{k,n+1}^{2}+H_{k,n+2}^{2}+H_{k,n+3}^{2}\\&\quad+2ij\left(H_{k,n}H_{k,n+3}-H_{k,n+1}H_{k,n+2}\right).\end{aligned} \tag{2.10}$$

Proof. The theorem can be proved easily by using the Eqs. (2.1), (2.2), (2.3) and (2.4). Hence we omit the proof. □

Theorem 3. *For* $n\in\mathbb{N}_0$, *the generalized* $k-$*Horadam quaternion satisfies the recurrence relation*

$$\mathcal{H}_{k,n+2}^{Q}=f(k)\mathcal{H}_{k,n+1}^{Q}+g(k)\mathcal{H}_{k,n}^{Q}. \tag{2.11}$$

Proof. By considering the right hand side of the Eq. (2.11), we get

$$\begin{aligned}&f(k)\mathcal{H}_{k,n+1}^{Q}+g(k)\mathcal{H}_{k,n}^{Q}\\&\quad=f(k)\left(H_{k,n+1}+\mathbf{i}H_{k,n+2}+\mathbf{j}H_{k,n+3}+\mathbf{ij}H_{k,n+4}\right)\\&\qquad+g(k)\left(H_{k,n}+\mathbf{i}H_{k,n+1}+\mathbf{j}H_{k,n+2}+\mathbf{ij}H_{k,n+3}\right)\\&\quad=\left(f(k)H_{k,n+1}+g(k)H_{k,n}\right)+\mathbf{i}\left(f(k)H_{k,n+2}+g(k)H_{k,n+1}\right)\\&\qquad+\mathbf{j}\left(f(k)H_{k,n+3}+g(k)H_{k,n+2}\right)+\mathbf{ij}\left(f(k)H_{k,n+4}+g(k)H_{k,n+3}\right)\\&\quad=H_{k,n+2}+\mathbf{i}H_{k,n+3}+\mathbf{j}H_{k,n+4}+\mathbf{ij}H_{k,n+5}\\&\quad=\mathcal{H}_{k,n+2}^{Q},\end{aligned}$$

which is the desired result. □

The following theorem explains the generating function of the bicomplex generalized $k-$Horadam quaternions.

Theorem 4. *For $n \in \mathbb{N}_0$, the generating function for the generalized $k-$Horadam quaternion is*

$$H(t) = \frac{\mathcal{H}_{k,0}^{Q} + \left(\mathcal{H}_{k,1}^{Q} - f(k)\mathcal{H}_{k,0}^{Q}\right)t}{1 - f(k)t - g(k)t^2}. \tag{2.12}$$

Proof. We use the formal power series to find the generating function of $\mathcal{H}_{k,n}^{Q}$. Now, we define

$$H(t) = \sum_{n=0}^{\infty} \mathcal{H}_{k,n}^{Q} t^n = \mathcal{H}_{k,0}^{Q} + \mathcal{H}_{k,1}^{Q} t + \sum_{n=2}^{\infty} \mathcal{H}_{k,n}^{Q} t^n. \tag{2.13}$$

Multiplying the Eq. (2.13) both $f(k)t$ and $g(k)t^2$, we get

$$f(k)tH(t) = \sum_{n=0}^{\infty} f(k)\mathcal{H}_{k,n}^{Q} t^{n+1} = f(k)\mathcal{H}_{k,0}^{Q} t + \sum_{n=2}^{\infty} f(k)\mathcal{H}_{k,n-1}^{Q} t^n \tag{2.14}$$

and

$$g(k)t^2 H(t) = \sum_{n=2}^{\infty} g(k)\mathcal{H}_{k,n-2}^{Q} t^n. \tag{2.15}$$

By considering the above equations and doing some basic operations, we get

$$\begin{aligned}\left(1 - f(k)t - g(k)t^2\right) H(t) &= \mathcal{H}_{k,0}^{Q} + \mathcal{H}_{k,1}^{Q} t - f(k)\mathcal{H}_{k,0}^{Q} t \\ &\quad + \sum_{n=2}^{\infty} \left(\underbrace{\mathcal{H}_{k,n}^{Q} - f(k)\mathcal{H}_{k,n-1}^{Q} - g(k)\mathcal{H}_{k,n-2}^{Q}}_{0} \right) t^n.\end{aligned}$$

Therefore, we obtain

$$H(t) = \frac{\mathcal{H}_{k,0}^{Q} + \left(\mathcal{H}_{k,1}^{Q} - f(k)\mathcal{H}_{k,0}^{Q}\right)t}{1 - f(k)t - g(k)t^2}, \tag{2.16}$$

which is the desired result. □

Now, we give the Binet formula for the bicomplex generalized $k-$Horadam quaternions by means of the Binet formula of the generalized $k-$Horadam numbers which is defined in the Eq. (1.10).

Theorem 5. *The Binet formula for the generalized $k-$Horadam quaternion is*

$$\mathcal{H}_{k,n}^{Q} = \frac{\alpha^{\star} X r_1^n - \beta^{\star} Y r_2^n}{r_1 - r_2}, \tag{2.17}$$

where $\alpha^\star = 1+\boldsymbol{i}r_1+\boldsymbol{j}r_1^2+\boldsymbol{ij}r_1^3$, $\beta^\star = 1+\boldsymbol{i}r_2+\boldsymbol{j}r_2^2+\boldsymbol{ij}r_2^3$, $X = b-ar_2$, $Y = b-ar_1$, $r_1 = \frac{f(k)+\sqrt{f(k)^2+4g(k)}}{2}$ *and* $r_2 = \frac{f(k)-\sqrt{f(k)^2+4g(k)}}{2}$.

Proof. By considering the Binet formula of the generalized $k-$Horadam numbers, we get

$$\begin{aligned}
\mathcal{H}_{k,n}^{Q} &= H_{k,n} + \mathbf{i}H_{k,n+1} + \mathbf{j}H_{k,n+2} + \mathbf{ij}H_{k,n+3} \\
&= \frac{Xr_1^n - Yr_2^n}{r_1-r_2} + \mathbf{i}\frac{Xr_1^{n+1} - Yr_2^{n+1}}{r_1-r_2} + \mathbf{j}\frac{Xr_1^{n+2} - Yr_2^{n+2}}{r_1-r_2} \\
&\quad + \mathbf{ij}\frac{Xr_1^{n+3} - Yr_2^{n+3}}{r_1-r_2} \\
&= \frac{Xr_1^n}{r_1-r_2}\left(1+\mathbf{i}r_1+\mathbf{j}r_1^2+\mathbf{ij}r_1^3\right) - \frac{Yr_2^n}{r_1-r_2}\left(1+\mathbf{i}r_2+\mathbf{j}r_2^2+\mathbf{ij}r_2^3\right) \\
&= \frac{1}{r_1-r_2}\left(\alpha^\star Xr_1^n - \beta^\star Yr_2^n\right),
\end{aligned}$$

where $\alpha^\star = 1+\mathbf{i}r_1+\mathbf{j}r_1^2+\mathbf{ij}r_1^3$, $\beta^\star = 1+\mathbf{i}r_2+\mathbf{j}r_2^2+\mathbf{ij}r_2^3$, $X = b-ar_2$, $Y = b-ar_1$, $r_1 = \frac{f(k)+\sqrt{f(k)^2+4g(k)}}{2}$ and $r_2 = \frac{f(k)-\sqrt{f(k)^2+4g(k)}}{2}$. Therefore the proof is completed. □

The matrix representation of the bicomplex generalized $k-$Horadam quaternions can be given in the following theorem.

Theorem 6. *Let* $n \geq 1$ *be integer. Then*

$$\begin{pmatrix} \mathcal{H}_{k,2n+l}^{Q} & \mathcal{H}_{k,2(n-1)+l}^{Q} \\ \mathcal{H}_{k,2(n+1)+l}^{Q} & \mathcal{H}_{k,2n+l}^{Q} \end{pmatrix} = \begin{pmatrix} \mathcal{H}_{k,2+l}^{Q} & \mathcal{H}_{k,l}^{Q} \\ \mathcal{H}_{k,4+l}^{Q} & \mathcal{H}_{k,2+l}^{Q} \end{pmatrix} \begin{pmatrix} f(k)^2+2g(k) & 1 \\ -g(k)^2 & 0 \end{pmatrix}^{n-1}, \tag{2.18}$$

where $l \in \{0,1\}$.

Proof. We prove the theorem by induction on n. If $n = 1$ then the result is clear. Now we assume that, for any integer m such as $1 \leq m \leq n$,

$$\begin{pmatrix} \mathcal{H}_{k,2m+l}^{Q} & \mathcal{H}_{k,2(m-1)+l}^{Q} \\ \mathcal{H}_{k,2(m+1)+l}^{Q} & \mathcal{H}_{k,2m+l}^{Q} \end{pmatrix} = \begin{pmatrix} \mathcal{H}_{k,2+l}^{Q} & \mathcal{H}_{k,l}^{Q} \\ \mathcal{H}_{k,4+l}^{Q} & \mathcal{H}_{k,2+l}^{Q} \end{pmatrix} \begin{pmatrix} f(k)^2+2g(k) & 1 \\ -g(k)^2 & 0 \end{pmatrix}^{m-1}.$$

Then, for $n = m+1$, we get

$$\begin{pmatrix} \mathcal{H}_{k,2+l}^{Q} & \mathcal{H}_{k,l}^{Q} \\ \mathcal{H}_{k,4+l}^{Q} & \mathcal{H}_{k,2+l}^{Q} \end{pmatrix} \begin{pmatrix} f(k)^2+2g(k) & 1 \\ -g(k)^2 & 0 \end{pmatrix}^{m}$$

$$
\begin{aligned}
&= \begin{pmatrix} \mathcal{H}^{Q}_{k,2+l} & \mathcal{H}^{Q}_{k,l} \\ \mathcal{H}^{Q}_{k,4+l} & \mathcal{H}^{Q}_{k,2+l} \end{pmatrix} \begin{pmatrix} f(k)^2+2g(k) & 1 \\ -g(k)^2 & 0 \end{pmatrix}^{m-1} \begin{pmatrix} f(k)^2+2g(k) & 1 \\ -g(k)^2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{H}^{Q}_{k,2m+l} & \mathcal{H}^{Q}_{k,2(m-1)+l} \\ \mathcal{H}^{Q}_{k,2(m+1)+l} & \mathcal{H}^{Q}_{k,2m+l} \end{pmatrix} \begin{pmatrix} f(k)^2+2g(k) & 1 \\ -g(k)^2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{H}^{Q}_{k,2(m+1)+l} & \mathcal{H}^{Q}_{k,2m+l} \\ \mathcal{H}^{Q}_{k,2(m+2)+l} & \mathcal{H}^{Q}_{k,2(m+1)+l} \end{pmatrix},
\end{aligned}
$$

where $l \in \{0,1\}$. Therefore, the proof is completed. □

Theorem 7. *For $n \in \mathbb{N}_0$, the summation formula for the generalized $k-$Horadam quaternion is*

$$
\sum_{s=1}^{n} \mathcal{H}^{Q}_{k,s} = \begin{cases} \frac{\mathcal{H}^{Q}_{k,n+1}+g(k)\mathcal{H}^{Q}_{k,n}-\mathcal{H}^{Q}_{k,1}-g(k)\mathcal{H}^{Q}_{k,0}}{f(k)+g(k)-1}, & \text{if } f(k)+g(k)-1 \neq 0 \\ \frac{g(k)\mathcal{H}^{Q}_{k,n}+\mathcal{H}^{Q}_{k,1}+(n-1)[g(k)a+b](1+i+j+ij)}{1+g(k)}, & \text{if } f(k)+g(k)-1 = 0 \end{cases} \tag{2.19}
$$

Proof. We prove the theorem with two cases. First we assume that $f(k)+g(k)-1 \neq 0$. By using the definition of the generalized $k-$Horadam quaternion, we have

$$
\sum_{s=1}^{n} \mathcal{H}^{Q}_{k,s} = \sum_{s=1}^{n} H_{k,s} + \mathbf{i}\sum_{s=1}^{n} H_{k,s+1} + \mathbf{j}\sum_{s=1}^{n} H_{k,s+2} + \mathbf{ij}\sum_{s=1}^{n} H_{k,s+3}. \tag{2.20}
$$

Now, we compute each term on the right hand side of the Eq. (2.20). By using the Eq. (12) in [23], we get

$$
\sum_{s=1}^{n} H_{k,s} = \frac{H_{k,n+1}+g(k)H_{k,n}-H_{k,1}-g(k)H_{k,0}}{f(k)+g(k)-1}. \tag{2.21}
$$

Moreover,

$$
\begin{aligned}
&\sum_{s=1}^{n} H_{k,s+1} \\
&= f(k)\sum_{s=1}^{n} H_{k,s} + g(k)\sum_{s=1}^{n} H_{k,s-1} \\
&= f(k)\left(\sum_{s=1}^{n} H_{k,s+1} - H_{k,n+1} + H_{k,1}\right) \\
&\quad + g(k)\left(\sum_{s=1}^{n} H_{k,s+1} - H_{k,n+1} - H_{k,n} + H_{k,0} + H_{k,1}\right)
\end{aligned}
$$

$$= \frac{g(k)H_{k,n+1} + g(k)H_{k,n} - g(k)H_{k,0} - g(k)H_{k,1} + f(k)H_{k,n+1} - f(k)H_{k,1}}{(f(k)+g(k)-1)}$$
$$= \frac{H_{k,n+2} + g(k)H_{k,n+1} - H_{k,2} - g(k)H_{k,1}}{f(k)+g(k)-1}. \tag{2.22}$$

By doing the similar operations as in the Eq. (2.22), we obtain

$$\sum_{s=1}^{n} H_{k,s+2} = \frac{H_{k,n+3} + g(k)H_{k,n+2} - H_{k,3} - g(k)H_{k,2}}{f(k)+g(k)-1} \tag{2.23}$$

and

$$\sum_{s=1}^{n} H_{k,s+3} = \frac{H_{k,n+4} + g(k)H_{k,n+3} - H_{k,4} - g(k)H_{k,3}}{f(k)+g(k)-1}. \tag{2.24}$$

By substituting the Equations (2.21), (2.22), (2.23) and (2.24) in the Eq. (2.20), one can obtain that

$$\sum_{s=1}^{n} \mathcal{H}_{k,s}^{Q} = \frac{\mathcal{H}_{k,n+1}^{Q} + g(k)\mathcal{H}_{k,n}^{Q} - \mathcal{H}_{k,1}^{Q} - g(k)\mathcal{H}_{k,0}^{Q}}{f(k)+g(k)-1}. \tag{2.25}$$

Next we assume that $f(k)+g(k)-1=0$. By using the Eq. (13) in [23], we get

$$\sum_{s=1}^{n} H_{k,s} = \frac{g(k)H_{k,n} + (n-1)[g(k)a+b] + H_{k,1}}{1+g(k)}. \tag{2.26}$$

Furthermore, we have

$$\sum_{s=1}^{n} H_{k,s+2} + g(k)\sum_{s=1}^{n} H_{k,s+1} = n[g(k)a+b]$$
$$\sum_{s=1}^{n} H_{k,s+1} + H_{k,n+2} - H_{k,2} + g(k)\sum_{s=1}^{n} H_{k,s+1} = n[g(k)a+b]$$
$$(1+g(k))\sum_{s=1}^{n} H_{k,s+1} = (n-1)[g(k)a+b] + (g(k)a+b) - H_{k,n+2} + H_{k,2}$$
$$\sum_{s=1}^{n} H_{k,s+1} = \frac{g(k)H_{k,n+1} + (n-1)[g(k)a+b] + H_{k,2}}{1+g(k)}. \tag{2.27}$$

By doing the similar operations as in the Eq. (2.27), we obtain

$$\sum_{s=1}^{n} H_{k,s+2} = \frac{g(k)H_{k,n+2} + (n-1)[g(k)a+b] + H_{k,3}}{1+g(k)} \tag{2.28}$$

and

$$\sum_{s=1}^{n} H_{k,s+3} = \frac{g(k)H_{k,n+3} + (n-1)[g(k)a+b] + H_{k,4}}{1+g(k)}. \tag{2.29}$$

Therefore, by substituting the Equations (2.26), (2.27), (2.28) and (2.29) in the Eq. (2.20), one can obtain that

$$\sum_{s=1}^{n} \mathcal{H}_{k,s}^{Q} = \frac{g(k)\mathcal{H}_{k,n}^{Q} + \mathcal{H}_{k,1}^{Q} + (n-1)[g(k)a+b](1+\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{ij})}{1+g(k)}. \tag{2.30}$$

□

Now, we give a theorem which generalizes the well-known identities such as Catalan's identity, Cassini's identity and d'Ocagne's identity.

Theorem 8. *The generalized $k-$Horadam quaternions satisfy the identity that is*

$$\begin{aligned}
&\mathcal{H}_{k,n}^{Q}\mathcal{H}_{k,n-r+s}^{Q} - \mathcal{H}_{k,n+s}^{Q}\mathcal{H}_{k,n-r}^{Q} \\
&= \frac{(-g(k))^{n-r}\left(bH_{k,r} - aH_{k,r+1}\right)\left(bH_{k,s} - aH_{k,s+1}\right)}{b^2 - a^2 g(k) - abf(k)} \\
&\quad \times \Big[1 + g(k) - g(k)^2 - g(k)^3 + \boldsymbol{i}\left(1 - g(k)^2\right) f(k) \\
&\qquad + \boldsymbol{j}(1+g(k))\left(f(k)^2 + 2g(k)\right) + \boldsymbol{ij} f(k)\left(f(k)^2 + 2g(k)\right)\Big].
\end{aligned} \tag{2.31}$$

Proof. By using the Theorem 7 in [36] and definition of the generalized $k-$Horadam numbers, we get

$$\begin{aligned}
&\mathcal{H}_{k,n}^{Q}\mathcal{H}_{k,n-r+s}^{Q} - \mathcal{H}_{k,n+s}^{Q}\mathcal{H}_{k,n-r}^{Q} \\
&= \left(H_{k,n} + \mathbf{i}H_{k,n+1} + \mathbf{j}H_{k,n+2} + \mathbf{ij}H_{k,n+3}\right) \\
&\quad \times \left(H_{k,n-r+s} + \mathbf{i}H_{k,n-r+s+1} + \mathbf{j}H_{k,n-r+s+2} + \mathbf{ij}H_{k,n-r+s+3}\right) \\
&\quad - \left(H_{k,n+s} + \mathbf{i}H_{k,n+s+1} + \mathbf{j}H_{k,n+s+2} + \mathbf{ij}H_{k,n+s+3}\right) \\
&\quad \times \left(H_{k,n-r} + \mathbf{i}H_{k,n-r+1} + \mathbf{j}H_{k,n-r+2} + \mathbf{ij}H_{k,n-r+3}\right) \\
&= H_{k,n}H_{k,n-r+s} - H_{k,n+1}H_{k,n-r+s+1} - H_{k,n+2}H_{k,n-r+s+2} \\
&\quad + H_{k,n+3}H_{k,n-r+s+3} - H_{k,n+s}H_{k,n-r} + H_{k,n+s+1}H_{k,n-r+1} \\
&\quad + H_{k,n+s+2}H_{k,n-r+2} - H_{k,n+s+3}H_{k,n-r+3} \\
&\quad + \mathbf{i}\Big(H_{k,n+1}H_{k,n-r+s} + H_{k,n}H_{k,n-r+s+1}
\end{aligned}$$

$$
\begin{aligned}
&- H_{k,n+2}H_{k,n-r+s+3} - H_{k,n+3}H_{k,n-r+s+2} - H_{k,n+s}H_{k,n-r+1} \\
&- H_{k,n+s+1}H_{k,n-r} + H_{k,n+s+2}H_{k,n-r+3} + H_{k,n+s+3}H_{k,n-r+2}\Big) \\
&+\mathbf{j}\Big(H_{k,n+2}H_{k,n-r+s} + H_{k,n}H_{k,n-r+s+2} - H_{k,n+1}H_{k,n-r+s+3} \\
&- H_{k,n+3}H_{k,n-r+s+1} - H_{k,n+s+2}H_{k,n-r} - H_{k,n+s}H_{k,n-r+2} \\
&+ H_{k,n+s+1}H_{k,n-r+3} + H_{k,n+s+3}H_{k,n-r+1}\Big) \\
&+\mathbf{ij}\Big(H_{k,n}H_{k,n-r+s+3} + H_{k,n+3}H_{k,n-r+s} + H_{k,n+1}H_{k,n-r+s+2} \\
&+ H_{k,n+2}H_{k,n-r+s+1} - H_{k,n+s}H_{k,n-r+3} - H_{k,n+s+3}H_{k,n-r} \\
&- H_{k,n+s+1}H_{k,n-r+2} - H_{k,n+s+2}H_{k,n-r+1}\Big) \\
&= \frac{(-g(k))^{n-r}\left(bH_{k,r} - aH_{k,r+1}\right)\left(bH_{k,s} - aH_{k,s+1}\right)}{b^2 - a^2 g(k) - abf(k)} \\
&\times\left[1 + g(k) - g(k)^2 - g(k)^3\right] \\
&+\mathbf{i}\frac{(-g(k))^{n-r}\left(bH_{k,r} - aH_{k,r+1}\right)\left(bH_{k,s} - aH_{k,s+1}\right)}{b^2 - a^2 g(k) - abf(k)} \\
&\times\left[\left(1 - g(k)^2\right) f(k)\right] \\
&+\mathbf{j}\frac{(-g(k))^{n-r}\left(bH_{k,r} - aH_{k,r+1}\right)\left(bH_{k,s} - aH_{k,s+1}\right)}{b^2 - a^2 g(k) - abf(k)} \\
&\times\left[(1 + g(k))\left(f(k)^2 + 2g(k)\right)\right] \\
&+\mathbf{ij}\frac{(-g(k))^{n-r}\left(bH_{k,r} - aH_{k,r+1}\right)\left(bH_{k,s} - aH_{k,s+1}\right)}{b^2 - a^2 g(k) - abf(k)} \\
&\times\left[f(k)\left(f(k)^2 + 2g(k)\right)\right] \\
&= \frac{(-g(k))^{n-r}\left(bH_{k,r} - aH_{k,r+1}\right)\left(bH_{k,s} - aH_{k,s+1}\right)}{b^2 - a^2 g(k) - abf(k)} \\
&\times\Big[1 + g(k) - g(k)^2 - g(k)^3 + \mathbf{i}\left(1 - g(k)^2\right) f(k)
\end{aligned}
$$

$$+\mathbf{j}(1+g(k))\left(f(k)^2+2g(k)\right)+\mathbf{ij}f(k)\left(f(k)^2+2g(k)\right)\Bigg].$$

□

Corollary 1. *By taking $s=m-n+r$ in the Theorem 8, we obtain the following identity:*

$$\begin{aligned}
&\mathcal{H}^{Q}_{k,n}\mathcal{H}^{Q}_{k,m}-\mathcal{H}^{Q}_{k,m+r}\mathcal{H}^{Q}_{k,n-r}\\
&\quad=\frac{(-g(k))^{n-r}\left(bH_{k,r}-aH_{k,r+1}\right)\left(bH_{k,m-n+r}-aH_{k,m-n+r+1}\right)}{b^2-a^2g(k)-abf(k)}\\
&\qquad\times\Bigg[1+g(k)-g(k)^2-g(k)^3+\boldsymbol{i}\left(1-g(k)^2\right)f(k)\\
&\qquad\quad+\boldsymbol{j}(1+g(k))\left(f(k)^2+2g(k)\right)+\boldsymbol{ij}f(k)\left(f(k)^2+2g(k)\right)\Bigg].
\end{aligned}$$

Corollary 2. *(Catalan's identity) By taking $m=n$ in Corollary 1, we obtain Catalan's identity for the generalized $k-$Horadam quaternions as:*

$$\begin{aligned}
&\left(\mathcal{H}^{Q}_{k,n}\right)^2-\mathcal{H}^{Q}_{k,n+r}\mathcal{H}^{Q}_{k,n-r}\\
&\quad=\frac{(-g(k))^{n-r}\left(bH_{k,r}-aH_{k,r+1}\right)^2}{b^2-a^2g(k)-abf(k)}\\
&\qquad\times\Bigg[1+g(k)-g(k)^2-g(k)^3+\boldsymbol{i}\left(1-g(k)^2\right)f(k)\\
&\qquad\quad+\boldsymbol{j}(1+g(k))\left(f(k)^2+2g(k)\right)+\boldsymbol{ij}f(k)\left(f(k)^2+2g(k)\right)\Bigg].
\end{aligned}$$

Corollary 3. *(Cassini's identity) By taking $m=n$ and $r=1$ in Corollary 1, we obtain Cassini's identity for the generalized $k-$Horadam quaternions as:*

$$\begin{aligned}
&\left(\mathcal{H}^{Q}_{k,n}\right)^2-\mathcal{H}^{Q}_{k,n+1}\mathcal{H}^{Q}_{k,n-1}\\
&\quad=(-g(k))^{n-1}\left(b^2-a^2g(k)-abf(k)\right)\\
&\qquad\times\Bigg[1+g(k)-g(k)^2-g(k)^3+\boldsymbol{i}\left(1-g(k)^2\right)f(k)\\
&\qquad\quad+\boldsymbol{j}(1+g(k))\left(f(k)^2+2g(k)\right)+\boldsymbol{ij}f(k)\left(f(k)^2+2g(k)\right)\Bigg].
\end{aligned}$$

Corollary 4. *(d'Ocagne's identity) By taking $n = n+1$ and $r = 1$ in Corollary 1, we obtain d'Ocagne's identity for the generalized $k-$Horadam quaternions as:*

$$\begin{aligned}
&\mathcal{H}^{Q}_{k,n+1}\mathcal{H}^{Q}_{k,m} - \mathcal{H}^{Q}_{k,m+1}\mathcal{H}^{Q}_{k,n} \\
&\quad = (-g(k))^n \left(bH_{k,m-n} - aH_{k,m-n+1}\right) \\
&\qquad \times \Big[1 + g(k) - g(k)^2 - g(k)^3 + \boldsymbol{i}\left(1 - g(k)^2\right) f(k) \\
&\qquad\quad + \boldsymbol{j}(1+g(k))\left(f(k)^2 + 2g(k)\right) + \boldsymbol{ij} f(k)\left(f(k)^2 + 2g(k)\right) \Big].
\end{aligned}$$

3. Conclusion

This study presents the bicomplex generalized $k-$Horadam quaternions which are generalization of the results in [1, 3, 34]. Moreover, for the special cases of $f(k), g(k), a$ and b, we obtain several new quaternions(see Table 1) which are not defined before in the literature. We derive the Binet formula, generating function, matrix representation and the summation formula for this quaternion. We also give Theorem 8 which is the generalization of the Catalan's identity, Cassini's identity and d'Ocagne's identity. Since this study includes some new generalized results for the bicomplex quaternions, it contributes to the literature by providing essential information on the generalization of the bicomplex quaternions.

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Authors' addresses

Yasin Yazlik
Nevşehir Hacı Bektaş Veli University, Department of Mathematics, 50300 Nevşehir, Turkey
E-mail address: yyazlik@nevsehir.edu.tr

Sure Köme
Nevşehir Hacı Bektaş Veli University, Department of Mathematics, 50300, Turkey
E-mail address: surekome@gmail.com

Cahit Köme
Nevşehir Hacı Bektaş Veli University, Department of Mathematics, 50300 Nevşehir, Turkey
E-mail address: cahitkome@gmail.com

EDITOR-IN-CHIEF

Miklós Rontó, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matronto@uni-miskolc.hu

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Michal Fečkan, Comenius University, Department of Mathematical Analysis; Mlyn-ská dolina, 842 15 Bratislava, Slovakia; Michal.Feckan@fmph.uniba.sk

Sándor Fegyverneki, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matfs@uni-miskolc.hu

Stephan Foldes, University of Miskolc, Institute of Information Technology; H-3515 Miskolc-Egyetemváros, Hungary; mmn.foldes@gmail.com

Aurél Galántai, Óbuda University, John von Neumann Faculty of Informatics; H-1034 Budapest, Bcsi u. 96/b, Hungary; galantai.aurel@nik.uni-obuda.hu

István Győri, Department of Mathematics and Computing, University of Veszprém; P. O. Box 158, H-8201 Veszprém, Hungary; gyori@almos.vein.hu

László Hatvani, Bolyai Institute, University of Szeged; 1 Aradi vértanúk tere, H-6720 Szeged, Hungary; hatvani@math.u-szeged.hu

Attila Házy, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matha@uni-miskolc.hu

Imre Juhász, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; agtji@uni-miskolc.hu

Judit Makó, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matjudit@uni-miskolc.hu

Nino Partsvania, Andrea Razmadze Mathematical Institute; 2 University St., Tbilisi 0186 Georgia; ninopa@rmi.ge

Nikolai Perestyuk, National Shevchenko University of Kiev, Faculty of Mechanics and Mathematics; 64 Vladimirskaya St., 01006 Kiev, Ukraine; pmo@univ.kiev.ua

Adrian Petrusel, Department of Mathematics, Babes-Bolyai University of Cluj-Napoca; No. 1, 400084, Cluj-Napoca, Romania; petrusel@math.ubbcluj.ro

Sándor Radeleczki, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matradi@uni-miskolc.hu

Maria Alessandra Ragusa, University of Catania, Italy Viale Andrea Doria; 6-95125, Catania, Italy; maragusa@dmi.unict.it

Csaba Rakaczki, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matrcs@uni-miskolc.hu

András Rontó, Institute of Mathematics, Academy of Sciences of Czech Republic; Žižkova 22, 616 62 Brno, Czech Republic; ronto@math.cas.cz

Nikolai Rozov, Moscow State University, Faculty of Mechanics and Mathematics; The Main Building, Vorobiovy Gory, 119899 Moscow, Russia; rozov@rozov.mccme.ru

Anatoly Samoilenko, Institute of Mathematics, National Academy of Sciences of Ukraine; 3 Tereshenkovskaya St., 01601 Kiev, Ukraine; sam@imath.kiev.ua

Jenő Szigeti, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matjeno@uni-miskolc.hu

Jiří Šremr, Institute of Mathematics, Academy of Sciences of Czech Republic; Žižkova 22, 616 62 Brno, Czech Republic; sremr@fme.vutbr.cz

Hari Mohan Srivastava, Department of Mathematics and Statistics, University of Victoria; Victoria, British Columbia V8W 3R4, Canada; harimsri@math.uvic.ca

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TECHNICAL EDITORS

Krisztián Hriczó, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; mathk@uni-miskolc.hu

Attila Körei, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matka@uni-miskolc.hu

Péter Olajos, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matolaj@uni-miskolc.hu

Péter Varga, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matvarga@uni-miskolc.hu

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