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ON THE WEAK SOLUTIONS OF AN OVERDETERMINED SYSTEM OF NONLINEAR FRACTIONAL PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this research work, we employ the critical point theory based on the variational structure to prove the existence of at least three distinct weak solutions for an overdetermined system of nonlinear fractional partial Fredholm-Volterra integro-differential equations.

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1. PRELIMINARIES

The study of partial differential equations (PDE's) started in the 18th century and its area has been growing steadily in the past century. The analysis of physical models has remained to the present day one of the fundamental concerns of the development of PDE's. PDEs can be used for modelling a wide range of physical phenomena, encountered in statistical mechanics, mathematical physics, theoretical neuroscience, fluid dynamics and mathematical finance. On one side PDEs can be used to describe a wide variety of phenomena such as sound, heat, diffusion, electrostatics, electrodynamics, fluid dynamics, elasticity, or quantum mechanics. On the other side PDE's also became an essential tool in other branches of mathematics. Thus the theory of ordinary and partial differential equations is a powerful theory to give a solution to engineering problems (see [22, 23, 25–28]). This theory can be generalized as theory of the overdetermined systems of differential and partial differential equations where it has been developed through some research works, e.g. [6, 10, 24, 30, 31, 34]. On the other hand, to deal with the process having delay, it is not more convenient tool to apply an ordinary or partial differential equation. Instead, a convenient approach to resolve these kind of problems is to employ integro-differential equations. Many

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mathematical models of physical phenomena lead to integro-differential equations [2, 3, 8, 9, 15, 29].

Consider the overdetermined system of nonlinear fractional partial Fredholm-Volterra integro-differential equations as follows:

$$\begin{cases} {}_t D_T^{\alpha_i} (a_i(t) {}_0 D_t^{\alpha_i} u_i(t)) = \lambda F_{u_i}(t, u_1(t), \dots, u_n(t)) \\ \quad + \int_0^T k_{1,i}(t, \tau) h_{1,i}(u_i(\tau)) d\tau \\ \quad + \int_0^t k_{2,i}(t, \tau) h_{2,i}(u_i(\tau)) d\tau, \quad i = 1, \dots, n; \\ u_i(t) = f_i(t) u_i(t) + \int_0^T k_{1,i}(t, \tau) h_{1,i}(u_i(\tau)) d\tau \\ \quad + \int_0^t k_{2,i}(t, \tau) h_{2,i}(u_i(\tau)) d\tau, \quad i = 1, \dots, n; \\ u_i(0) = u_i(T) = 0, \quad i = 1, \dots, n. \end{cases} \quad (1.1)$$

where $t \in (0, T)$, λ is a positive real parameter, $0 < \alpha_i \leq 1$, $a_i(t) \in L^\infty[0, T]$, $\bar{a}_i = \text{essinf}_{[0, T]} a_i(t) > 0$, $f_i : [0, T] \rightarrow \mathbb{R}$ is a continuous function with the maximum norm $\|f_i\|_\infty = \max_{0 \leq t \leq T} |f_i(t)| = M_i^f$, $F : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a function to the extent that $F(t, u_1, \dots, u_n)$ is continuous with respect to t and continuously differentiable with respect to u_i i.e. $F(\cdot, u_1, \dots, u_n) \in C[0, T]$ and $F(t, \cdot, \dots, \cdot) \in C^1(\mathbb{R}^n)$, and further, $k_{1,i}(\cdot, \cdot), k_{2,i}(\cdot, \cdot) \in C([0, T], [0, T])$ hence the kernels $k_{1,i}$ and $k_{2,i}$ are bounded by for example L_i and M_i , respectively. Also, we suppose $h_{1,i}, h_{2,i} : \mathbb{R} \rightarrow \mathbb{R}$ are Lipschitz continuous functions with the Lipschitz constants $L'_i > 0$ and $M'_i > 0$, i.e., $|h_{1,i}(\zeta_1) - h_{1,i}(\zeta_2)| \leq L'_i |\zeta_1 - \zeta_2|$ and $|h_{2,i}(\zeta_1) - h_{2,i}(\zeta_2)| \leq M'_i |\zeta_1 - \zeta_2|$ for every $\zeta_1, \zeta_2 \in \mathbb{R}$ and $i = 1, \dots, n$. Moreover, F_s denotes the partial derivative of F with respect to s and, ${}_t D_T^\gamma$ and ${}_0 D_t^\gamma$ are the right and left Riemann-Liouville type of fractional derivatives of order γ which are defined by [14]

$${}_0 D_t^\gamma u(t) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_0^t \frac{u(s)}{(t-s)^\gamma} ds, \quad (1.2)$$

$${}_t D_T^\gamma u(t) = -\frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_t^T \frac{u(s)}{(s-t)^\gamma} ds. \quad (1.3)$$

The standard and typical approach to study the existence of solutions to the nonlinear fractional boundary value problems is the fixed point theory, see for example [1, 4, 16, 35]. But, another well-developed, successful and recently more attracting method is the calculus of variation to investigate the existence of solutions to the differential equation with type of integer order and very recently fractional order, the readers are referred to [7, 11–13, 17–21, 32, 33, 36] and the references therein to attain more information about this approach.

Here, the well-known three critical points theorem due to Bonanno and Marano [5] is considered. Then the existence of at least three different weak solutions to the overdetermined system (1.1) is proved. In fact, the main steps to apply this theorem is to define two convenient functionals which satisfy all conditions of theorem's assumptions.

2. THREE DISTINCT WEAK SOLUTIONS

In order to prove the existence of at least three weak solutions (1.1), some Theorems and Definitions are recalled in this literature.

Definition 1. Consider $0 < \alpha_i \leq 1, i = 1, \dots, n$, the fractional derivative space $E_0^{\alpha_i}$ is defined by the closure of $C_0^\infty([0, T], \mathbb{R})$ i.e. $E_0^{\alpha_i} = \overline{C_0^\infty([0, T], \mathbb{R})}$ with respect to the following weighted norm

$$\|u\|_{\alpha_i} = \left(\int_0^T a_i(t) |{}_0D_t^{\alpha_i} u(t)|^2 dt + \int_0^T |u(t)|^2 dt \right)^{\frac{1}{2}}, \quad (2.1)$$

for all $u \in E_0^{\alpha_i}, i = 1, \dots, n$.

Lemma 1 ([11]). If $\frac{1}{2} < \alpha_i \leq 1$ then for all $u \in E_0^{\alpha_i}, i = 1, \dots, n$,

(1)

$$\|u\|_{L^2} \leq \frac{T^{\alpha_i}}{\Gamma(\alpha_i + 1)} \|{}_0D_t^{\alpha_i} u\|_{L^2}, \quad (2.2)$$

(2)

$$\|u\|_\infty \leq \frac{T^{\alpha_i - \frac{1}{2}}}{\Gamma(\alpha_i) \sqrt{2\alpha_i - 1}} \|{}_0D_t^{\alpha_i} u\|_{L^2}. \quad (2.3)$$

Remark 1. The following inequalities for $i = 1, \dots, n$, are straightforward from Lemma 1,

(1)

$$\|u\|_{L^2} \leq \frac{T^{\alpha_i}}{\Gamma(\alpha_i + 1) \sqrt{\bar{a}_i}} \left(\int_0^T a_i(t) |{}_0D_t^{\alpha_i} u(t)|^2 dt \right)^{\frac{1}{2}}, \quad (2.4)$$

(2)

$$\|u\|_\infty \leq \frac{T^{\alpha_i - \frac{1}{2}}}{\Gamma(\alpha_i) \sqrt{\bar{a}_i} (2\alpha_i - 1)} \left(\int_0^T a_i(t) |{}_0D_t^{\alpha_i} u(t)|^2 dt \right)^{\frac{1}{2}}. \quad (2.5)$$

Remark 2. The norm which is defined by equation (2.1) is equivalent to the following norm

$$\|u\|_{\alpha_i} = \left(\int_0^T a_i(t) |{}_0D_t^{\alpha_i} u(t)|^2 dt \right)^{\frac{1}{2}}, \text{ for all } u \in E_0^{\alpha_i}, i = 1, \dots, n. \quad (2.6)$$

The norm (2.6) is the norm that we need. Now, define $X = \prod_{i=1}^{i=n} E_0^{\alpha_i}$ augmented to the norm

$$\|U\|_X = \sum_{i=1}^n \|u_i\|_{\alpha_i}, \quad u_i \in E_0^{\alpha_i}, \quad U = (u_1, u_2, \dots, u_n) \in X. \quad (2.7)$$

Definition 2. $U = (u_1, u_2, \dots, u_n) \in X$ is called the weak solution of the system (1.1) if the following equation satisfies

$$\begin{aligned}
0 = & \sum_{i=1}^n \int_0^T a_i(t)_0 D_t^{\alpha_i} u_i(t)_0 D_t^{\alpha_i} v_i(t) dt \\
& - \sum_{i=1}^n \int_0^T \int_0^t k_{1,i}(t, \tau) h_{1,i}(u_i(\tau)) v_i(t) d\tau dt \\
& - \sum_{i=1}^n \int_0^T \int_0^t k_{2,i}(t, \tau) h_{2,i}(u_i(\tau)) v_i(t) d\tau dt \\
& - \lambda \int_0^T \sum_{i=1}^n F_{u_i}(t, u_1(t), \dots, u_n(t)) v_i(t) dt
\end{aligned} \tag{2.8}$$

for every $V = (v_1, v_2, \dots, v_n) \in X$.

Remark 3. Notice that we have used the number of n equations of the system (1.1) to define the weak solution. In fact, we use the remain equations, the number of n integral equations, directly to define an operator, in what follows, so that its Gâteaux derivative is some parts of the equation (2.8).

Definition 3. Define the operator $H_i : E_0^{\alpha_i} \rightarrow E_0^{\alpha_i}$ as

$$\begin{aligned}
H_i(u_i(t)) = & \frac{1}{2} \int_0^T k_{1,i}(t, \tau) h_{1,i}(u_i(\tau)) u_i(t) d\tau \\
& + \frac{1}{2} \int_0^t k_{2,i}(t, \tau) h_{2,i}(u_i(\tau)) u_i(t) d\tau - \frac{1}{2} f_i(t) (u_i(t))^2,
\end{aligned} \tag{2.9}$$

for all $t \in [0, T]$ and for $1 \leq i \leq n$.

From the system (1.1), it is straightforward to see

$$\begin{aligned}
(u_i(t) + \theta v_i(t))^2 = & f_i(t) (u_i(t) + \theta v_i(t))^2 \\
& + \int_0^T k_{1,i}(t, \tau) h_{1,i}(u_i(\tau) + \theta v_i(\tau)) (u_i(t) + \theta v_i(t)) d\tau \\
& + \int_0^t k_{2,i}(t, \tau) h_{2,i}(u_i(\tau) + \theta v_i(\tau)) (u_i(t) + \theta v_i(t)) d\tau.
\end{aligned} \tag{2.10}$$

The Gâteaux derivative of H_i is

$$\begin{aligned}
H'_i(u_i(t))(v_i(t)) &= \frac{d}{d\theta} \left\{ \frac{1}{2} \int_0^T k_{1,i}(t, \tau) h_{1,i}(u_i(\tau) + \theta v_i(\tau))(u_i(t) + \theta v_i(t)) d\tau \right. \\
&\quad + \frac{1}{2} \int_0^t k_{2,i}(t, \tau) h_{2,i}(u_i(\tau) + \theta v_i(\tau))(u_i(t) + \theta v_i(t)) d\tau \\
&\quad \left. - \frac{1}{2} f_i(t)(u_i(t) + \theta v_i(t))^2 \right\}_{\theta=0} \\
&= \frac{d}{d\theta} \left\{ \frac{1}{2} (u_i(t) + \theta v_i(t))^2 - \frac{1}{2} f_i(t)(u_i(t) + \theta v_i(t))^2 \right. \\
&\quad \left. - \frac{1}{2} f_i(t)(u_i(t) + \theta v_i(t))^2 \right\}_{\theta=0} \\
&= u_i(t)v_i(t) - f_i(t)u_i(t)v_i(t) \\
&= (u_i(t) - f_i(t)u_i(t))v_i(t) \\
&= \int_0^T k_{1,i}(t, \tau) h_{1,i}(u_i(\tau))v_i(t) d\tau \\
&\quad + \int_0^t k_{2,i}(t, \tau) h_{2,i}(u_i(\tau))v_i(t) d\tau.
\end{aligned} \tag{2.11}$$

The following theorem is recalled from [5] to obtain at least three distinct critical points.

Theorem 1 ([5, Theorem 3.6]). *Let X be a reflexive real Banach space and $\Phi : X \rightarrow \mathbb{R}$ a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable in which its Gâteaux derivative has a continuous inverse on X^* , moreover, suppose that $\Psi : X \rightarrow \mathbb{R}$ be a sequentially weakly upper semicontinuous and continuously Gâteaux differentiable functional in which its Gâteaux derivative is compact, such that $\Phi(0) = \Psi(0) = 0$. Suppose also there exist $r \in \mathbb{R}$ and $u_1 \in X$ with $0 < r < \Phi(u_1)$, satisfying*

- (1) $\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u) < r \frac{\Psi(u_1)}{\Phi(u_1)}$
- (2) *for all $\lambda \in \Lambda_r := \left] \frac{\Phi(u_1)}{\Psi(u_1)}, \frac{r}{\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)} \right]$, the functional $\Phi - \lambda\Psi$ is coercive.*

Then for any $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ admits at least three distinct critical points in X .

The main result of this paper states the existence of at least different three weak solutions of the system (1.1). For convenience we set the following notations: $\theta_i =$

$$\begin{aligned} & \max \{L_i L'_i, M_i M'_i\}, \\ & \sigma = \min_{1 \leq i \leq n} \{\sigma_i\}, \sigma_i = 1 - \frac{2\theta_i T^{2\alpha_i+1} + M_i^f T^{2\alpha_i}}{(\Gamma(\alpha_i))^2 \bar{a}_i (2\alpha_i - 1)}, \\ & M = \max_{1 \leq i \leq n} \left\{ \frac{T^{2\alpha_i-1}}{(\Gamma(\alpha_i))^2 \bar{a}_i (2\alpha_i - 1) - 2\theta_i T^{2\alpha_i+1} - M_i^f T^{2\alpha_i}} \right\}, \end{aligned} \quad (2.12)$$

$$\mathcal{Y}(c) = \left\{ \mathcal{E} = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n : \frac{1}{2} \sum_{i=1}^n \xi_i^2 \leq c \right\}, \quad c > 0, \quad (2.13)$$

and

$$L = \max_{1 \leq i \leq n} \left\{ \frac{T^{2\alpha_i}}{\sigma (\Gamma(\alpha_i + 1))^2 \bar{a}_i} \right\}. \quad (2.14)$$

The main result is as follows:

Theorem 2. Suppose $F : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a function such that $F(\cdot, u_1, \dots, u_n) \in C[0, T]$, $F(t, \cdot, \dots, \cdot) \in C^1(\mathbb{R}^n)$ and $F(t, 0, \dots, 0) = 0$ for all $t \in [0, T]$. Set

$$\varrho := \frac{\sum_{i=1}^n \left(\|\omega_i\|_{\alpha_i}^2 - 2 \int_0^T H_i(\omega_i(t)) dt \right)}{2 \int_0^T F(t, \omega_1(t), \dots, \omega_n(t)) dt}$$

and

$$\vartheta := \frac{r}{\int_0^T \sup_{(\xi_1, \dots, \xi_n) \in \mathcal{Y}(Mr)} F(t, \xi_1, \dots, \xi_n) dt}.$$

If there exist a constant $r > 0$ and a vector-valued function

$\Omega(t) = (\omega_1(t), \dots, \omega_n(t))$ such that

- (H0) $\frac{1}{2} < \alpha_i \leq 1$;
- (H1) $\frac{2\theta_i T^{2\alpha_i+1} + M_i^f T^{2\alpha_i}}{(\Gamma(\alpha_i))^2 \bar{a}_i (2\alpha_i - 1)} < 1$;
- (H2) $\sum_{i=1}^n \|\omega_i\|_{\alpha_i}^2 \geq 2r + 2 \sum_{i=1}^n \int_0^T H_i(\omega_i(t)) dt$;
- (H3) $\varrho < \vartheta$;
- (H4) $\liminf_{i: |\xi_i| \rightarrow +\infty} \frac{F(t, \xi_1, \dots, \xi_n)}{\sum_{i=1}^n |\xi_i|^2} < \frac{1}{2L\vartheta}$.

Then for any $\lambda \in \Lambda =]\varrho, \vartheta[$, the system (1.1) has at least three different weak solutions in X .

Proof. Consider the space $X = \prod_{i=1}^n E_0^{\alpha_i}$ equipped with the norm $\|U\|_X$ defined by (2.7) which is a reflexive and separable Banach space. For any given $U = (u_1(t), \dots, u_n(t)) \in X$, we present the functionals $\Phi, \Psi : X \rightarrow \mathbb{R}$ as follows:

$$\Phi(U) = \frac{1}{2} \sum_{i=1}^n \|u_i\|_{\alpha_i}^2 - \sum_{i=1}^n \int_0^T H_i(u_i(t)) dt, \quad (2.15)$$

and

$$\Psi(U) = \int_0^T F(t, u_1(t), \dots, u_n(t)) dt. \quad (2.16)$$

These functionals are well-defined Gâteaux differentiable and for all $V = (v_1, v_2, \dots, v_n) \in X$,

$$\begin{aligned} \Phi'(U)(V) &= \sum_{i=1}^n \int_0^T a_i(t) {}_0D_t^{\alpha_i} u_i(t) {}_0D_t^{\alpha_i} v_i(t) dt \\ &\quad - \sum_{i=1}^n \int_0^T \int_0^T k_{1,i}(t, \tau) h_{1,i}(u_i(\tau)) v_i(t) d\tau dt \\ &\quad - \sum_{i=1}^n \int_0^T \int_0^t k_{2,i}(t, \tau) h_{2,i}(u_i(\tau)) v_i(t) d\tau dt, \end{aligned}$$

and

$$\Psi'(U)(V) = \int_0^T \sum_{i=1}^n F_{u_i}(t, u_1(t), \dots, u_n(t)) v_i(t) dt.$$

Notice that $\Phi'(U), \Psi'(U) \in X^*$ in where X^* is dual space of X . It can be easily proved that the functional Φ is sequentially weakly lower semicontinuous and its Gâteaux derivative has a continuous inverse on X^* . Furthermore, since $|k_{1,i}(t, \tau)| \leq L_i$ and $|k_{2,i}(t, \tau)| \leq M_i$ for $0 \leq t, \tau \leq T$ and besides, $h_{1,i}$ and $h_{2,i}$ are Lipschitz continuous functions with the Lipschitz constants $L'_i > 0$ and $M'_i > 0$, hence

$$\begin{aligned} H_i(u_i(t)) &= \frac{1}{2} \int_0^T k_{1,i}(t, \tau) h_{1,i}(u_i(\tau)) u_i(t) d\tau \\ &\quad + \frac{1}{2} \int_0^t k_{2,i}(t, \tau) h_{2,i}(u_i(\tau)) u_i(t) d\tau - \frac{1}{2} f_i(t) (u_i(t))^2 \\ &\leq \frac{1}{2} |u_i(t)| T L_i L'_i \|u_i\|_\infty + \frac{1}{2} |u_i(t)| t M_i M'_i \|u_i\|_\infty + \frac{1}{2} M_i^f \|u_i\|_\infty^2 \\ &\leq \frac{1}{2} T L_i L'_i \|u_i\|_\infty^2 + \frac{1}{2} T M_i M'_i \|u_i\|_\infty^2 + \frac{1}{2} M_i^f \|u_i\|_\infty^2 \\ &\leq \theta_i T \|u_i\|_\infty^2 + \frac{1}{2} M_i^f \|u_i\|_\infty^2 \\ &= (\theta_i T + \frac{1}{2} M_i^f) \|u_i\|_\infty^2. \end{aligned} \quad (2.17)$$

Equations (2.5), (2.6) and (2.15) imply

$$\Phi(U) = \frac{1}{2} \sum_{i=1}^n \|u_i\|_{\alpha_i}^2 - \sum_{i=1}^n \int_0^T H_i(u_i(t)) dt$$

$$\begin{aligned}
&\geq \frac{1}{2} \sum_{i=1}^n \|u_i\|_{\alpha_i}^2 - \sum_{i=1}^n \int_0^T (\theta_i T + \frac{1}{2} M_i^f) \|u_i\|_{\infty}^2 dt \\
&= \frac{1}{2} \sum_{i=1}^n \|u_i\|_{\alpha_i}^2 - \sum_{i=1}^n (\theta_i T^2 + \frac{1}{2} M_i^f T) \|u_i\|_{\infty}^2 \\
&\geq \frac{1}{2} \sum_{i=1}^n \|u_i\|_{\alpha_i}^2 - \sum_{i=1}^n (\theta_i T^2 + \frac{1}{2} M_i^f T) \frac{T^{2\alpha_i-1}}{(\Gamma(\alpha_i))^2 \bar{a}_i (2\alpha_i - 1)} \|u_i\|_{\alpha_i}^2 \\
&= \frac{1}{2} \sum_{i=1}^n \left(1 - \frac{2\theta_i T^{2\alpha_i+1} + M_i^f T^{2\alpha_i}}{(\Gamma(\alpha_i))^2 \bar{a}_i (2\alpha_i - 1)} \right) \|u_i\|_{\alpha_i}^2 \\
&= \frac{1}{2} \sum_{i=1}^n \sigma_i \|u_i\|_{\alpha_i}^2 \\
&\geq \frac{\sigma}{2} \sum_{i=1}^n \|u_i\|_{\alpha_i}^2 \\
&= \frac{\sigma}{2} \|U\|_X,
\end{aligned}$$

we know that σ is positive because each σ_i is positive by assumption (H1), thus $\lim_{\|U\|_X \rightarrow +\infty} \Phi(U) = +\infty$ hence it is coercive. Let us to prove the functional Ψ is definitely sequentially weakly upper semicontinuous and its derivative $\Psi' : X \rightarrow X^*$ is a compact operator. If $\lim_{m \rightarrow +\infty} U_m \rightharpoonup U$ in X where $U_m(t) = (u_{m,1}(t), \dots, u_{m,n}(t))$, then absolutely U_m converges uniformly to U on the interval $[0, T]$, then

$$\begin{aligned}
\limsup_{m \rightarrow +\infty} \Psi(U_m) &\leq \int_0^T \limsup_{m \rightarrow +\infty} F(t, u_{m,1}(t), \dots, u_{m,n}(t)) dt \\
&= \int_0^T F(t, u_1(t), \dots, u_n(t)) dt \\
&= \Psi(U).
\end{aligned}$$

Hence Ψ is sequentially weakly upper semicontinuous. Moreover, since $F(t, \cdot, \dots, \cdot) \in C^1(\mathbb{R}^n)$ then for all $t \in [0, T]$

$$\lim_{m \rightarrow +\infty} F(t, u_{m,1}(t), \dots, u_{m,n}(t)) = F(t, u_1(t), \dots, u_n(t))$$

Now, the Lebesgue control convergence theorem yields $\Psi'(U_m) \rightarrow \Psi'(U)$ strongly, so it results in that Ψ' is strongly continuous on X . Therefore, $\Psi' : X \rightarrow X^*$ is a compact operator.

Suppose that $U_0(t) = (0, \dots, 0)$ and $U_1(t) = \Omega(t)$ then

$$\Phi(U_0(t)) = \Psi(U_0(t)) = 0$$

and the assumption (H2) yields directly

$$0 < r \leq \frac{1}{2} \sum_{i=1}^n \|\omega_i\|_{\alpha_i}^2 - \sum_{i=1}^n \int_0^T H_i(\omega_i(t)) dt = \Phi(U_1).$$

Equations (2.5), (2.6), (2.12) and (2.13), imply

$$\begin{aligned} \Phi^{-1}([-\infty, r]) &= \{U \in X : \Phi(U) \leq r\} \\ &= \left\{ U \in X : \frac{1}{2} \sum_{i=1}^n \|u_i\|_{\alpha_i}^2 - \sum_{i=1}^n \int_0^T H_i(u_i(t)) dt \leq r \right\} \\ &\subseteq \left\{ U \in X : \right. \\ &\quad \left. \frac{1}{2} \sum_{i=1}^n \|u_i\|_{\alpha_i}^2 - \sum_{i=1}^n \int_0^T (\theta_i T + \frac{1}{2} M_i^f) \|u_i\|_{\infty}^2 dt \leq r \right\} \\ &= \left\{ U \in X : \sum_{i=1}^n \frac{(\Gamma(\alpha_i))^2 \bar{a}_i (2\alpha_i - 1)}{2T^{2\alpha_i - 1}} \|u_i\|_{\infty}^2 \right. \\ &\quad \left. - \sum_{i=1}^n (\theta_i T^2 + \frac{1}{2} M_i^f T) \|u_i\|_{\infty}^2 \leq r \right\} \\ &= \left\{ U \in X : \right. \\ &\quad \left. \sum_{i=1}^n \frac{(\Gamma(\alpha_i))^2 \bar{a}_i (2\alpha_i - 1) - 2\theta_i T^{2\alpha_i + 1} - M_i^f T^{2\alpha_i}}{2T^{2\alpha_i - 1}} \|u_i\|_{\infty}^2 \leq r \right\} \\ &\subseteq \left\{ U \in X : \frac{1}{2M} \sum_{i=1}^n \|u_i\|_{\infty}^2 \leq r \right\} \\ &\subseteq \left\{ U \in X : \right. \\ &\quad \left. \frac{1}{2} \sum_{i=1}^n |u_i(t)|^2 \leq Mr, \text{ for all } t \in [0, T] \right\} \\ &\subseteq \mathcal{Y}(Mr), \end{aligned}$$

which leads to the following relation

$$\begin{aligned}
\sup_{U \in \Phi^{-1}([-\infty, r])} \Psi(U) &= \sup_{U \in \Phi^{-1}([-\infty, r])} \int_0^T F(t, u_1(t), \dots, u_n(t)) dt \\
&\leq \sup_{\mathcal{E} \in \mathcal{Y}(Mr)} \int_0^T F(t, \xi_1, \dots, \xi_n) dt \\
&= \int_0^T \sup_{\mathcal{E} \in \mathcal{Y}(Mr)} F(t, \xi_1, \dots, \xi_n) dt.
\end{aligned}$$

By (H3), we have

$$\begin{aligned}
\frac{\sup_{U \in \Phi^{-1}([-\infty, r])} \Psi(U)}{r} &= \frac{\sup_{U \in \Phi^{-1}([-\infty, r])} \int_0^T F(t, u_1(t), \dots, u_n(t)) dt}{r} \\
&\leq \frac{\int_0^T \sup_{\mathcal{E} \in \mathcal{Y}(Mr)} F(t, \xi_1, \dots, \xi_n) dt}{r} \\
&< \frac{2 \int_0^T F(t, \omega_1(t), \dots, \omega_n(t)) dt}{\sum_{i=1}^n \|\omega_i\|_{\alpha_i}^2 - 2 \sum_{i=1}^n \int_0^T H_i(\omega_i(t)) dt} \quad (2.18) \\
&= \frac{\Psi(\Omega(t))}{\Phi(\Omega(t))} \\
&= \frac{\Psi(U_1)}{\Phi(U_1)},
\end{aligned}$$

thus, $\sup_{U \in \Phi^{-1}([-\infty, r])} \Psi(U) < r \frac{\Psi(U_1)}{\Phi(U_1)}$ and hence the hypothesis (1) of Theorem 1 is valid.

By assumption (H4) there exists two constants $\mu, \varepsilon \in \mathbb{R}$ with the properties

$$\frac{\mu}{\sigma} < \frac{\int_0^T \sup_{\mathcal{E} \in \mathcal{Y}(Mr)} F(t, \xi_1, \dots, \xi_n) dt}{r}, \quad (2.19)$$

$$\text{and for all } \mathcal{E} \in \mathbb{R}^n : F(t, \xi_1, \dots, \xi_n) \leq \frac{\mu}{2L\sigma} \sum_{i=1}^n |\xi_i|^2 + \varepsilon,$$

for all $t \in [0, T]$. Then clearly for a fixed $U = (u_1(t), \dots, u_n(t)) \in X$, we have

$$F(t, u_1(t), \dots, u_n(t)) \leq \frac{\mu}{2L\sigma} \sum_{i=1}^n |u_i(t)|^2 + \varepsilon, \quad (2.20)$$

for all $t \in [0, T]$. Now, in the next step we have to show the coercivity of the functional $\Phi(U) - \lambda \Psi(U)$, suppose $\lambda \in \Lambda$, then bringing into accounts (2.4), (2.5), (2.14), (2.19) and (2.20), we have

$$\begin{aligned}
\Phi(U) - \lambda\Psi(U) &= \frac{1}{2} \sum_{i=1}^n \|u_i\|_{\alpha_i}^2 \\
&\quad - \sum_{i=1}^n \int_0^T H_i(u_i(t)) dt - \lambda \int_0^T F(t, u_1(t), \dots, u_n(t)) dt \\
&\geq \frac{1}{2} \sum_{i=1}^n \|u_i\|_{\alpha_i}^2 - \sum_{i=1}^n \int_0^T (\theta_i T + \frac{1}{2} M_i^f) \|u_i\|_{\infty}^2 dt \\
&\quad - \lambda \int_0^T F(t, u_1(t), \dots, u_n(t)) dt \\
&\geq \frac{1}{2} \sum_{i=1}^n \|u_i\|_{\alpha_i}^2 - \sum_{i=1}^n (\theta_i T^2 + \frac{1}{2} M_i^f T) \|u_i\|_{\infty}^2 dt \\
&\quad - \frac{\lambda\mu}{2L\sigma} \int_0^T \left(\sum_{i=1}^n |u_i(t)|^2 \right) dt - \lambda T\varepsilon \\
&\geq \frac{1}{2} \sum_{i=1}^n \|u_i\|_{\alpha_i}^2 - \sum_{i=1}^n (\theta_i T^2 + \frac{1}{2} M_i^f T) \frac{T^{2\alpha_i-1}}{(\Gamma(\alpha_i))^2 \bar{a}_i (2\alpha_i-1)} \|u_i\|_{\alpha_i}^2 \\
&\quad - \frac{\lambda\mu}{2L\sigma} \sum_{i=1}^n \frac{T^{2\alpha_i}}{(\Gamma(\alpha_i+1))^2 \bar{a}_i} \|u_i\|_{\alpha_i}^2 - \lambda T\varepsilon \\
&= \frac{1}{2} \sum_{i=1}^n \sigma_i \|u_i\|_{\alpha_i}^2 - \frac{\lambda\mu}{2L\sigma} \sum_{i=1}^n \frac{T^{2\alpha_i}}{(\Gamma(\alpha_i+1))^2 \bar{a}_i} \|u_i\|_{\alpha_i}^2 - \lambda T\varepsilon \\
&\geq \frac{1}{2} \sum_{i=1}^n \sigma \|u_i\|_{\alpha_i}^2 - \frac{\lambda\mu}{2} \sum_{i=1}^n \|u_i\|_{\alpha_i}^2 - \lambda T\varepsilon \\
&\geq \frac{1}{2} \left(\sigma - \frac{\mu r}{\int_0^T \sup_{\mathcal{E} \in \mathcal{Y}(Mr)} F(t, \xi_1, \dots, \xi_n) dt} \right) \sum_{i=1}^n \|u_i\|_{\alpha_i}^2 - \lambda T\varepsilon,
\end{aligned}$$

since term $\left(\sigma - \frac{\mu r}{\int_0^T \sup_{\mathcal{E} \in \mathcal{Y}(Mr)} F(t, \xi_1, \dots, \xi_n) dt} \right)$ is clearly positive from (2.19), then

$$\lim_{\|U\|_X \rightarrow +\infty} (\Phi(U) - \lambda\Psi(U)) = +\infty. \quad (2.21)$$

Therefore, it is verified that $\Phi - \lambda\Psi$ is coercive and then the hypothesis (2) of Theorem 1 is also established. Applying Theorem 1 and taking into account that the weak solutions of the system (1.1) are exactly constructed through the equation

$\Phi'(U) - \lambda \Psi'(U) = 0$, we conclude that the system (1.1) has at least three distinct weak solutions in X for $\lambda \in \Lambda$ and then the proof is completed. $\square$

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REFERENCES

- [1] R. P. Agarwal, M. Benchohra, and S. Hamani, "A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions." *Acta Appl. Math.*, vol. 109, no. 3, pp. 973–1033, 2010, doi: [10.1007/s10440-008-9356-6](https://doi.org/10.1007/s10440-008-9356-6).
- [2] B. Ahmad and J. J. Nieto, "Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions." *Bound. Value Probl.*, vol. 2011, no. 1, p. 36, 2011, doi: [10.1186/1687-2770-2011-36](https://doi.org/10.1186/1687-2770-2011-36).
- [3] H. Aydi, A. Felhi, E. Karapinar, and S. Sahmim, "A Nadler-type fixed point theorem in dislocated spaces and applications." *Miskolc Math. Notes*, vol. 19, no. 1, pp. 111–124, 2018, doi: [10.18514/MMN.2018.1652](https://doi.org/10.18514/MMN.2018.1652).
- [4] M. Benchohra, S. Hamani, and S. Ntouyas, "Boundary value problems for differential equations with fractional order and nonlocal conditions." *Nonlinear Anal.*, vol. 71, no. 7, pp. 2391–2396, 2009, doi: [10.1016/j.na.2009.01.073](https://doi.org/10.1016/j.na.2009.01.073).
- [5] G. Bonanno and S. A. Marano, "On the structure of the critical set of non-differentiable functions with a weak compactness condition." *Appl. Anal.*, vol. 89, no. 1, pp. 1–10, 2010, doi: [10.1080/00036810903397438](https://doi.org/10.1080/00036810903397438).
- [6] D. Ehsani, M. R. Mokhtarzadeh, and A. Razani, "Expansions of Dirichlet to Neumann operators under perturbations." *Appl. Anal.*, vol. 89, no. 5, pp. 789–799, 2010, doi: [10.1080/00036811003649108](https://doi.org/10.1080/00036811003649108).
- [7] A. Firouzjaji, G. A. Afrouzi, and S. Talebi, "Existence results for Kirchhoff type systems with singular nonlinearity." *Opuscula Math.*, vol. 38, no. 2, pp. 187–199, 2018, doi: [10.7494/OpMath.2018.38.2.187](https://doi.org/10.7494/OpMath.2018.38.2.187).
- [8] S. Gala, Q. Liu, and M. A. Ragusa, "Logarithmically improved regularity criterion for the nematic liquid crystal flows in $\dot{B}_{\infty,\infty}^{-1}$ space." *Comput. Math. Appl.*, vol. 65, no. 11, pp. 1738–1745, 2013, doi: [10.1016/j.camwa.2013.04.003](https://doi.org/10.1016/j.camwa.2013.04.003).
- [9] S. Gala, M. A. Ragusa, Y. Sawano, and H. Tanaka, "Uniqueness criterion of weak solutions for the dissipative quasi-geostrophic equations in Orlicz-Morrey spaces." *Appl. Anal.*, vol. 93, no. 2, pp. 356–368, 2014, doi: [10.1080/00036811.2013.772582](https://doi.org/10.1080/00036811.2013.772582).
- [10] Z. Goodarzi and A. Razani, "A periodic solution of the generalized forced Liénard equation." *Abstr. Appl. Anal.*, vol. 2014, no. 132450, p. 5 pages, 2014, doi: [10.1155/2014/132450](https://doi.org/10.1155/2014/132450).
- [11] F. Jiao and Y. Zhou, "Existence results for fractional boundary value problem via critical point theory." *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, vol. 22, no. 04, 2012, doi: [10.1142/S0218127412500861](https://doi.org/10.1142/S0218127412500861).
- [12] S. M. Khalkhali, S. Heidarkhani, and A. Razani, "Infinitely many solutions for a fourth-order boundary-value problem." *Electron. J. Differential Equations*, vol. 164, pp. 1–14, 2012.
- [13] S. M. Khalkhali and A. Razani, "Multiple solutions for a quasilinear (p, q) -elliptic system." *Electron. J. Differential Equations*, vol. 2013, no. 144, pp. 1–14, 2013.
- [14] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Elsevier Science, Amsterdam, 2006.

- [15] N. Kouhestani and A. Moameni, "Multiplicity results for elliptic problems with super-critical concave and convex nonlinearities." *Calc. Var. Partial Differential Equations*, vol. 57, no. 2, p. 54, 2018, doi: [10.1007/s00526-018-1333-y](https://doi.org/10.1007/s00526-018-1333-y).
- [16] C. Li, R. P. Agarwal, and D.-L. Wu, "Existence and multiplicity of solutions for a class of superlinear elliptic systems." *Adv. Nonlinear Anal.*, vol. 7, no. 2, pp. 183–196, 2018, doi: [10.1515/anona-2016-0045](https://doi.org/10.1515/anona-2016-0045).
- [17] Y.-N. Li, H.-R. Sun, and Q.-G. Zhang, "Existence of solutions to fractional boundary-value problems with a parameter." *Electron. J. Differential Equations*, vol. 2013, no. 141, pp. 1–12, 2013.
- [18] R. Mahdavi Khanghahi and A. Razani, "Solutions for a singular elliptic problem involving the $p(x)$ -Laplacian." *Filomat*, vol. 32, no. 14, 2018.
- [19] M. Makvand Chaharlang and A. Razani, "Infinitely many solutions for a fourth order singular elliptic problem." vol. 32, no. 14, 2018.
- [20] M. Makvand Chaharlang and A. Razani, "Existence of infinitely many solutions for a class of nonlocal problems with Dirichlet boundary condition." *Commun. Korean Math. Soc.*, vol. 34, no. 1, pp. 155–167, 2019, doi: [10.4134/CKMS.c170456](https://doi.org/10.4134/CKMS.c170456).
- [21] M. Makvand Chaharlang and A. Razani, "A fourth order singular elliptic problem involving p -biharmonic operator." *Taiwanese J. Math.*, Accepted, doi: [10.11650/tjm/180906](https://doi.org/10.11650/tjm/180906).
- [22] M. R. Mokhtarzadeh, M. R. Pournaki, and A. Razani, "An existence-uniqueness theorem for a class of boundary value problems." vol. 13, no. 2, pp. 583–592, 2012.
- [23] M. R. Pournaki and A. Razani, "On the existence of periodic solutions for a class of generalized forced Liénard equations." *Appl. Math. Lett.*, vol. 20, pp. 248–254, 2007, doi: [10.1016/j.aml.2006.06.004](https://doi.org/10.1016/j.aml.2006.06.004).
- [24] D. G. Quillen, *Formal properties overdetermined system of linear partial differential equations*. Harvard University, 1964.
- [25] A. Razani, "Shock waves in gas dynamics." *Surv. Math. Appl.*, vol. 2, pp. 59–89, 2007.
- [26] A. Razani, "An existence theorem for ordinary differential equation in Menger probabilistic metric space." *Miskolc Math. Notes*, vol. 15, no. 2, pp. 711–716, 2014, doi: [10.18514/MMN.2014.640](https://doi.org/10.18514/MMN.2014.640).
- [27] A. Razani, "Chapman-Jouguet travelling wave for a two-steps reaction scheme." *Ital. J. Pure Appl. Math.*, vol. 39, pp. 544–553, 2018.
- [28] A. Razani, "Subsonic detonation waves in porous media." *Phys. Scr.*, 2019, doi: [10.1088/1402-4896/ab029b](https://doi.org/10.1088/1402-4896/ab029b).
- [29] A. Samadi, "Applications of measure of noncompactness to coupled fixed points and systems of integral equations." *Miskolc Math. Notes*, vol. 19, no. 1, pp. 537–553, 2018, doi: [10.18514/MMN.2018.2532](https://doi.org/10.18514/MMN.2018.2532).
- [30] E. Shivanian, "Analysis of meshless local radial point interpolation (MLRPI) on a nonlinear partial integro-differential equation arising in population dynamics." *Eng. Anal. Bound. Elem.*, vol. 37, no. 12, pp. 1693–1702, 2013, doi: [10.1016/j.enganabound.2013.10.002](https://doi.org/10.1016/j.enganabound.2013.10.002).
- [31] D. C. Spencer, "Overdetermined systems of linear partial differential equations." *Bull. Amer. Math. Soc. (N.S.)*, vol. 75, no. 2, pp. 179–239, 1969.
- [32] H.-R. Sun and Q.-G. Zhang, "Existence of solutions for a fractional boundary value problem via the Mountain Pass method and an iterative technique." *Comput. Math. Appl.*, vol. 64, no. 10, pp. 3436–3443, 2012, doi: [10.1016/j.camwa.2012.02.023](https://doi.org/10.1016/j.camwa.2012.02.023).
- [33] W. Xie, J. Xiao, and Z. Luo, "Existence of solutions for fractional boundary value problem with nonlinear derivative dependence." in *Abstract and Applied Analysis*, vol. 2014, doi: [10.1155/2014/812910](https://doi.org/10.1155/2014/812910), 2014.
- [34] M. Zaytsev and V. Akkerman, "Reduction of over-determined systems of differential equations." 2012.

- [35] S. Zhang, “Positive solutions to singular boundary value problem for nonlinear fractional differential equation.” *Comput. Math. Appl.*, vol. 59, no. 3, pp. 1300–1309, 2010, doi: [10.1016/j.camwa.2009.06.034](https://doi.org/10.1016/j.camwa.2009.06.034).
- [36] Y. Zhao, H. Chen, and B. Qin, “Multiple solutions for a coupled system of nonlinear fractional differential equations via variational methods.” *Appl. Math. Comput.*, vol. 257, pp. 417–427, 2015, doi: [10.1016/j.amc.2014.12.128](https://doi.org/10.1016/j.amc.2014.12.128).

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THE SECOND REGULARIZED TRACE FORMULA FOR THE STURM-LIOUVILLE OPERATOR

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Abstract. In this paper, the second regularized trace formula for the differential operator with antiperiodic boundary conditions is obtained.

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1. INTRODUCTION

In the Hilbert space $H = L_2[0, \pi]$, we consider operator L generated by the differential expression

$$l(y) = -y'' + q(x)y,$$

with the antiperiodic boundary conditions

$$y(0) = -y(\pi), y'(0) = -y'(\pi),$$

where $q(x) \in C^2[0, \pi]$ is real function and satisfies the condition

$$q(0) = q(\pi). \quad (1.1)$$

It is well known from [13] that the eigenvalues of the operator L form double series

$$\lambda_{n,j} = \left(2n - 1 + \frac{c_0}{2n - 1} + o\left(\frac{1}{n}\right) \right)^2,$$

where $c_0 = \frac{1}{2\pi} \int_0^\pi q(x) dx$, $j = 1, 2, o\left(\frac{1}{n}\right)$ include j and $n = 1, 2, \dots$. Let L_0 denote the operator L with $q(x) \equiv 0$. The eigenvalues of operator L_0 are

$$\mu_n = (2n - 1)^2, \quad n = 1, 2, \dots$$

The orthonormal eigenfunctions corresponding to the eigenvalues μ_n are

$$\varphi_n = \sqrt{\frac{2}{\pi}} \cos(2n - 1)x, \psi_n = \sqrt{\frac{2}{\pi}} \sin(2n - 1)x, n = 1, 2, \dots \quad (1.2)$$

It is natural to define second regularized trace of operator L as

$$\sum_{n=1}^{\infty} '(\lambda_{n,1}^2 + \lambda_{n,2}^2 - 2(2n-1)^4), \quad (1.3)$$

where the symbol "''" means that something in this sum is discarded to provide its convergence. The sum in (1.3) is the main interest of this article.

The regularized trace formula

$$\sum_{n=0}^{\infty} \left(\lambda_n - n^2 - \frac{1}{\pi} \int_0^{\pi} q(x) dx \right) = \frac{q(0) + q(\pi)}{4} - \frac{1}{2\pi} \int_0^{\pi} q(x) dx,$$

of the Sturm-Liouville operator

$$-y'' + q(x)y = \lambda y, y'(0) = 0, y'(\pi) = 0, \quad (1.4)$$

with $q(x) \in C^1[0, \pi]$ was first studied by Gelfand and Levitan ([6]), where the λ_n are the eigenvalues of the operator in (1.4). Afterwards, trace formulas for different differential operators are studied by several mathematicians (see [1–5, 8, 10–12, 14, 15] and references therein).

Note that the first regularized trace formula

$$\sum_{n=0}^{\infty} [\lambda_{n,1} + \lambda_{n,2} - 2(2n-1)^2 - 2c_0] = 0,$$

was obtained for operator L with a real potential $q(x) \in L_2(0, \pi)$, by [10] and with an arbitrary complex $q(x) \in L_2(0, \pi)$, by [15]. The similar formula was obtained by [12] for the operator L with operator function $q(x)$.

Trace formulas are used in inverse problems of spectral analysis of differential equations (see [14]) and for approximate calculation of the first eigenvalues of the related operator [1, 4, 5, 7, 9, 14].

2. SOME FORMULAS ABOUT THE OPERATOR qR_{λ}^0

Let R_{λ}^0 and R_{λ} be the resolvents of the operator L_0 and L , respectively. Then, for any $\lambda \in \rho(L)$ and $\mu \in \rho(L_0)$ where $\rho(\cdot)$ is the resolvent set of an operator, $R_{\lambda} : H \rightarrow H$ and $R_{\lambda}^0 : H \rightarrow H$ are trace class operators, that is, $R_{\lambda} R_{\lambda}^0 \in \sigma_1(H)$. Therefore,

$$tr(R_{\lambda} - R_{\lambda}^0) = \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_{n,1} - \lambda} + \frac{1}{\lambda_{n,2} - \lambda} - \frac{2}{(2n-1)^2 - \lambda} \right).$$

Multiplying both sides of the above equality by $\lambda^2/2\pi i$ then integrating over the circle $|\lambda| = b_p = (2p-1)^2 + 4p$,

$$\frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda^2 tr(R_{\lambda} - R_{\lambda}^0) d\lambda = \sum_{n=1}^p (2(2n-1)^4 - \lambda_{n,1}^2 - \lambda_{n,2}^2) \quad (2.1)$$

is obtained. Taking in the account that $R_\lambda - R_\lambda^0 = -R_\lambda q R_\lambda^0$, from equation (2.1),

$$\sum_{n=1}^p (\lambda_{n,1}^2 + \lambda_{n,2}^2 - 2(2n-1)^4) = \sum_{j=1}^N M_p^j + M_{pN} \quad (2.2)$$

is obtained. Here

$$M_p^j = \frac{(-1)^{j+1}}{2\pi i} \int_{|\lambda|=b_p} \lambda^2 \text{tr} [R_\lambda^0 (q R_\lambda^0)^j] d\lambda,$$

and

$$M_{pN} = \frac{(-1)^N}{2\pi i} \int_{|\lambda|=b_p} \lambda^2 \text{tr} [R_\lambda (q R_\lambda^0)^{N+1}] d\lambda, \quad (2.3)$$

where $q = q(x)$ and N is an integer.

The formula

$$M_p^j = \frac{(-1)^j}{2\pi i j} \int_{|\lambda|=b_p} \lambda \text{tr} (q R_\lambda^0)^j d\lambda, \quad (2.4)$$

can be proved similarly as in Theorem 2 in [12].

From equations (1.2) and (2.4), we write

$$\begin{aligned} M_p^1 &= -\frac{1}{\pi i} \int_{|\lambda|=b_p} \left\{ \lambda \sum_{n=1}^{\infty} [(q R_\lambda^0 \psi_n^1, \psi_n^1) + (q R_\lambda^0 \psi_n^2, \psi_n^2)] \right\} d\lambda \\ &= 2 \sum_{n=1}^{\infty} [(q \psi_n^1, \psi_n^1) + (q \psi_n^2, \psi_n^2)] \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda}{\lambda - \mu_n} d\lambda \\ &= \frac{4}{\pi} \sum_{n=1}^p (2n-1)^2 \int_0^\pi q(x) [\cos^2(2n-1)x + \sin^2(2n-1)x] dx \\ &= \frac{4}{\pi} \sum_{n=1}^p (2n-1)^2 \int_0^\pi q(x) dx. \end{aligned} \quad (2.5)$$

Now, we shall compute M_p^2 . From equation (2.4),

$$\begin{aligned} M_p^2 &= \frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda \left\{ \sum_{n=1}^{\infty} [((q R_\lambda^0)^2 \psi_n^1, \psi_n^1) + ((q R_\lambda^0)^2 \psi_n^2, \psi_n^2)] \right\} d\lambda \\ &= \frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda \left\{ \sum_{n=1}^{\infty} \frac{1}{\mu_n - \lambda} [(q R_\lambda^0 q \psi_n^1, \psi_n^1) + (q R_\lambda^0 q \psi_n^2, \psi_n^2)] \right\} d\lambda \\ &= \frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda \left\{ \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{1}{(\mu_n - \lambda)(\mu_r - \lambda)} [(q \psi_n^1, \psi_r^1) (q \psi_r^1, \psi_n^1) \right. \\ &\quad \left. + (q \psi_n^1, \psi_r^2) (q \psi_r^2, \psi_n^1) + (q \psi_n^2, \psi_r^1) (q \psi_r^1, \psi_n^2) + (q \psi_n^2, \psi_r^2) (q \psi_r^2, \psi_n^2)] \right\} d\lambda. \end{aligned}$$

For convenience, let

$$q_{nr} = |(q\psi_n^1, \psi_r^1)|^2 + |(q\psi_n^1, \psi_r^2)|^2 + |(q\psi_n^2, \psi_r^1)|^2 + |(q\psi_n^2, \psi_r^2)|^2. \quad (2.6)$$

Then,

$$\begin{aligned} M_p^2 &= \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} q_{nr} \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda}{(\lambda - \mu_n)(\lambda - \mu_r)} d\lambda \\ &= \sum_{n=1}^p \sum_{r=1}^p q_{nr} \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda}{(\lambda - \mu_n)(\lambda - \mu_r)} d\lambda \\ &\quad + \sum_{n=1}^p \sum_{r=p+1}^{\infty} q_{nr} \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda}{(\lambda - \mu_n)(\lambda - \mu_r)} d\lambda \\ &\quad + \sum_{n=p+1}^{\infty} \sum_{r=1}^p q_{nr} \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda}{(\lambda - \mu_n)(\lambda - \mu_r)} d\lambda \\ &\quad + \sum_{n=p+1}^{\infty} \sum_{r=p+1}^{\infty} q_{nr} \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda}{(\lambda - \mu_n)(\lambda - \mu_r)} d\lambda \\ &= \sum_{n=1}^p \sum_{r=1}^p q_{nr} + 2 \sum_{n=1}^p \sum_{r=p+1}^{\infty} q_{nr} \frac{\mu_n}{\mu_r - \mu_n} \\ &= \sum_{n=1}^p \sum_{r=1}^{\infty} q_{nr} - \sum_{n=1}^p \sum_{r=p+1}^{\infty} \frac{\mu_r + \mu_n}{\mu_r - \mu_n} q_{nr}. \end{aligned}$$

Thus,

$$M_p^2 = \sum_{n=1}^p \sum_{r=1}^{\infty} q_{nr} - \alpha_p, \quad (2.7)$$

where

$$\alpha_p = \sum_{n=1}^p \sum_{r=p+1}^{\infty} \frac{\mu_r + \mu_n}{\mu_r - \mu_n} q_{nr}.$$

By using equations (1.2) and (2.6), we have

$$\alpha_p = \alpha_{p1} + \alpha_{p2} + \alpha_{p3}, \quad (2.8)$$

where

$$\alpha_{p1} = 2\pi^{-2} \sum_{n=1}^p \sum_{r=1}^{\infty} \frac{(2r-1)^2 + (2n-1)^2}{(2r-1)^2 - (2n-1)^2} \left\{ \left| \int_0^{\pi} q(x) \cos 2(n-r)x dx \right|^2 + \right.$$

$$\begin{aligned}
& + \left| \int_0^\pi q(x) \sin 2(n-r)x dx \right|^2 \Bigg\}, \\
\alpha_{p2} &= 2\pi^{-2} \sum_{n=1}^p \sum_{r=1}^\infty \frac{(2r-1)^2 + (2n-1)^2}{(2r-1)^2 - (2n-1)^2} \left| \int_0^\pi q(x) \cos 2(n+r-1)x dx \right|^2, \\
\alpha_{p3} &= 2\pi^{-2} \sum_{n=1}^p \sum_{r=1}^\infty \frac{(2r-1)^2 + (2n-1)^2}{(2r-1)^2 - (2n-1)^2} \left| \int_0^\pi q(x) \sin 2(n+r-1)x dx \right|^2.
\end{aligned}$$

The formula of α_{p1} can be written as

$$\alpha_{p1} = 2\pi^{-2} \sum_{i=1}^\infty \sum_{\substack{r-n=i \\ n \leq p \\ r > p}} \left(1 + \frac{2(2n-1)^2}{(2r-1)^2 - (2n-1)^2} \right) \quad (2.9)$$

$$\left\{ \left| \int_0^\pi q(x) \cos 2ix dx \right|^2 + \left| \int_0^\pi q(x) \sin 2ix dx \right|^2 \right\}. \quad (2.10)$$

For $i \leq p$

$$\begin{aligned}
\sum_{\substack{cr-n=i \\ n \leq p \\ r > p}} \frac{(2n-1)^2}{(2r-1)^2 - (2n-1)^2} &= \sum_{j=0}^{i-1} \frac{(2(p-j)-1)^2}{(2(p-j+i)-1)^2 - (2(p-j)-1)^2} \\
&= \sum_{j=0}^{i-1} \frac{(2(p-j)-1+i)^2 - 2(2(p-j)-1+i)i + i^2}{4i(2(p-j)-1+i)} \\
&= \frac{2p-1}{4} + \frac{1-2i}{4} + \sum_{j=0}^{i-1} \frac{i}{4(2p-1-2j+i)}.
\end{aligned} \quad (2.11)$$

For any integers p and i , let

$$E = \{(r, n) : r, n \in \mathbb{N}; r-n = i; n \leq p; r > p\}.$$

Then using (2.11), we write

$$\begin{aligned}
\sum_{n, r \in E} \left(1 + \frac{2(2n-1)^2}{(2r-1)^2 - (2n-1)^2} \right) &= i + 2 \sum_{\substack{cr-n=i \\ n \leq p \\ r > p}} \frac{(2n-1)^2}{(2r-1)^2 - (2n-1)^2} \\
&= p + \frac{i}{2} \sum_{j=p-i+1}^p \frac{1}{2j-1+i}, \quad (i \leq p).
\end{aligned} \quad (2.12)$$

It is easy to see that

$$\frac{i}{2} \sum_{j=p-i+1}^p \frac{1}{2j-1+i} < \frac{i^2}{p}.$$

Using this inequality and equation (2.12),

$$\sum_{n,r \in E} \left(1 + \frac{2(2n-1)^2}{(2r-1)^2 - (2n-1)^2} \right) = p + i^2 O(p^{-1}), \quad (i \leq p) \quad (2.13)$$

is obtained.

Here $O(p^{-1})$ depends on p and i , and satisfies the inequality

$$|O(p^{-1})| < \text{const.} \cdot p^{-1}.$$

In a similar form, for $i \geq p$, it can be shown that

$$\sum_{n,r \in E} \left(1 + \frac{2(2n-1)^2}{(2r-1)^2 - (2n-1)^2} \right) = O(p). \quad (2.14)$$

Here $O(p)$ depends on p and i , and satisfies the inequality

$$|O(p)| < \text{const.} \cdot p.$$

From (2.9), (2.13) and (2.14), we obtain

$$\begin{aligned} \alpha_{p1} &= 2\pi^{-2} p \sum_{i=1}^{\infty} \left\{ \left| \int_0^{\pi} q(x) \cos 2ix dx \right|^2 + \left| \int_0^{\pi} q(x) \sin 2ix dx \right|^2 \right\} \\ &\quad + \sum_{i=1}^p i^2 O(p^{-1}) \left\{ \left| \int_0^{\pi} q(x) \cos 2ix dx \right|^2 + \left| \int_0^{\pi} q(x) \sin 2ix dx \right|^2 \right\} \\ &\quad + \sum_{i=p+1}^{\infty} O(p) \left\{ \left| \int_0^{\pi} q(x) \cos 2ix dx \right|^2 + \left| \int_0^{\pi} q(x) \sin 2ix dx \right|^2 \right\} \\ &= \frac{p}{\pi} \int_0^{\pi} |q(x)|^2 dx - \frac{p}{\pi^2} \left| \int_0^{\pi} q(x) dx \right|^2 + \alpha_{p1}^{(1)} + \alpha_{p1}^{(2)}. \end{aligned} \quad (2.15)$$

Here, since $q(0) = q(\pi)$,

$$\begin{aligned} |\alpha_{p1}^{(1)}| &= \left| \sum_{i=1}^p i^2 O(p^{-1}) \left\{ \left| \int_0^{\pi} q(x) \cos 2ix dx \right|^2 + \left| \int_0^{\pi} q(x) \sin 2ix dx \right|^2 \right\} \right| \\ &\leq \text{const.} \cdot p^{-1} \int_0^{\pi} |q'(x)|^2 dx, \end{aligned} \quad (2.16)$$

and

$$|\alpha_{p1}^{(2)}| = \left| \sum_{i=p+1}^{\infty} O(p) \left\{ \left| \int_0^{\pi} q(x) \cos 2ix dx \right|^2 + \left| \int_0^{\pi} q(x) \sin 2ix dx \right|^2 \right\} \right| \quad (2.17)$$

$$\leq \text{const.} p^{-1} \int_0^{\pi} |q'(x)|^2 dx$$

is obtained. From (2.15)-(2.17), we get

$$\alpha_{p1} = \frac{p}{\pi} \int_0^{\pi} q^2(x) dx - \frac{p}{\pi^2} \left| \int_0^{\pi} q(x) dx \right|^2 + O(p^{-1}). \quad (2.18)$$

Since $q(x)$ satisfies the condition in equation (1.1), it can be shown that

$$|\alpha_{pj}| \leq \text{const.}(p^{-1}), \quad (j = 2, 3). \quad (2.19)$$

From equation (2.6), we have

$$\sum_{n=1}^p \sum_{r=1}^{\infty} q_{nr} = \sum_{n=1}^p \{ \|q\psi_n^1\|^2 + \|q\psi_n^2\|^2 \} = \frac{2p}{\pi} \int_0^{\pi} q^2(x) dx. \quad (2.20)$$

From equations (2.7), (2.8) and (2.18)-(2.20),

$$M_p^2 = \frac{p}{\pi} \int_0^{\pi} q(x)^2 dx + \frac{p}{\pi^2} \left(\int_0^{\pi} q(x) dx \right)^2 + O(p^{-1}),$$

is obtained.

3. THE SECOND REGULARIZED TRACE FORMULA

In this section we obtain the second regularized trace formula for the operator L . To do this, we will first show that the formulas

$$\lim_{p \rightarrow \infty} M_p^j = 0, \quad j \geq 3, \quad (3.1)$$

$$\lim_{p \rightarrow \infty} M_p^N = 0, \quad N \geq 6, \quad (3.2)$$

are satisfied.

The inequalities

$$\|qR_{\lambda}^0\|_{\sigma_1(H)} < C, \|R_{\lambda}^0\| < Cp^{-1}, \|R_{\lambda}\| < Cp^{-1}, \quad (|\lambda| = b_p) \quad (3.3)$$

are true (see [12]). Here $C > 0$ is a constant. From (2.4) and (3.3), we have

$$|M_p^j| = \frac{1}{\pi j} \left| \int_{|\lambda|=b_p} \lambda \text{tr}(qR_{\lambda}^0)^j d\lambda \right|$$

$$\leq \frac{b_p}{\pi j} \int_{|\lambda|=b_p} \left\| (qR_{\lambda}^0)^j \right\|_{\sigma_1(H)} |d\lambda|$$

$$\begin{aligned}
&\leq \frac{b_p}{\pi j} \int_{|\lambda|=b_p} \|(qR_\lambda^0)\|_{\sigma_1(H)} \|(qR_\lambda^0)^{j-1}\| |d\lambda| \\
&\leq \frac{Cb_p}{\pi j} \int_{|\lambda|=b_p} \|q\|^{j-1} \|(R_\lambda^0)\|^{j-1} |d\lambda| \\
&< C_1 p^{5-j} (C_1 > 0).
\end{aligned}$$

This implies that

$$\lim_{p \rightarrow \infty} M_p^j = 0, \quad (j \geq 6),$$

but we claim that it is also true for $j = 3, 4, 5$. Now let we prove the formula (3.1) for $j = 3$.

$$\begin{aligned}
M_p^3 &= -\frac{1}{3\pi i} \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \sum_{k=1}^{\infty} \int_{|\lambda|=b_p} \frac{\lambda d\lambda}{(\mu_n - \lambda)(\mu_r - \lambda)(\mu_k - \lambda)} \\
&\quad * \{((q\psi_n^1, \psi_r^1)[(q\psi_r^1, \psi_k^1)\psi_k^1 + (q\psi_r^1, \psi_k^2)\psi_k^2] \\
&\quad + (q\psi_n^1, \psi_r^2)[(q\psi_r^2, \psi_k^1)\psi_k^1 + (q\psi_r^2, \psi_k^2)\psi_k^2], \psi_n^1) \\
&\quad + ((q\psi_n^2, \psi_r^1)[(q\psi_r^1, \psi_k^1)\psi_k^1 + (q\psi_r^1, \psi_k^2)\psi_k^2] \\
&\quad + (q\psi_n^2, \psi_r^2)[(q\psi_r^2, \psi_k^1)\psi_k^1 + (q\psi_r^2, \psi_k^2)\psi_k^2], \psi_n^2)\} \\
&= -\frac{1}{3\pi i} \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \sum_{k=1}^{\infty} \int_{|\lambda|=b_p} \frac{\lambda d\lambda}{(\mu_n - \lambda)(\mu_r - \lambda)(\mu_k - \lambda)} F(n, r, k),
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
F(n, r, k) &= (q\psi_n^1, \psi_r^1)(q\psi_r^1, \psi_k^1)(q\psi_k^1, \psi_n^1) + (q\psi_n^1, \psi_r^1)(q\psi_r^1, \psi_k^2)(q\psi_k^2, \psi_n^1) \\
&\quad + (q\psi_n^1, \psi_r^2)(q\psi_r^2, \psi_k^1)(q\psi_k^1, \psi_n^1) + (q\psi_n^1, \psi_r^2)(q\psi_r^2, \psi_k^2)(q\psi_k^2, \psi_n^1) \\
&\quad + (q\psi_n^2, \psi_r^1)(q\psi_r^1, \psi_k^1)(q\psi_k^1, \psi_n^2) + (q\psi_n^2, \psi_r^1)(q\psi_r^1, \psi_k^2)(q\psi_k^2, \psi_n^2) \\
&\quad + (q\psi_n^2, \psi_r^2)(q\psi_r^2, \psi_k^1)(q\psi_k^1, \psi_n^2) + (q\psi_n^2, \psi_r^2)(q\psi_r^2, \psi_k^2)(q\psi_k^2, \psi_n^2).
\end{aligned}$$

Since $F(n, r, k) = F(r, n, k) = F(k, n, r) = F(k, r, n) = F(n, k, r) = F(r, k, n)$, from equation (3.4)

$$\begin{aligned}
M_p^3 &= \frac{1}{\pi i} \sum_{n=1}^p \sum_{r=p+1}^p \sum_{k=p+1}^{\infty} \int_{|\lambda|=b_p} \frac{\lambda d\lambda}{(\lambda - \mu_n)(\lambda - \mu_r)(\lambda - \mu_k)} F(n, r, k) \\
&\quad + \frac{1}{\pi i} \sum_{n=1}^p \sum_{\substack{r=1 \\ n \neq r}}^p \sum_{k=p+1}^{\infty} \int_{|\lambda|=b_p} \frac{\lambda d\lambda}{(\lambda - \mu_n)(\lambda - \mu_r)(\lambda - \mu_k)} F(n, r, k) \\
&\quad + \frac{1}{\pi i} \sum_{n=1}^p \sum_{k=p+1}^{\infty} \int_{|\lambda|=b_p} \frac{\lambda d\lambda}{(\lambda - \mu_n)(\lambda - \mu_k)} F(n, n, k) \\
&= 2 \sum_{n=1}^p \sum_{r=p+1}^{\infty} \sum_{k=p+1}^{\infty} \frac{\mu_n}{(\mu_r - \mu_n)(\mu_k - \mu_n)} F(n, r, k) \\
&\quad + 4 \sum_{n=1}^p \sum_{\substack{r=1 \\ n \neq r}}^p \sum_{k=p+1}^{\infty} \frac{\mu_n}{(\mu_n - \mu_r)(\mu_n - \mu_k)} F(n, r, k) \\
&\quad - 2 \sum_{n=1}^p \sum_{k=p+1}^{\infty} \frac{\mu_k}{(\mu_k - \mu_n)^2} F(n, n, k)
\end{aligned} \tag{3.5}$$

is obtained. Let

$$F_1(n, r, k) = \pi^{-3} \int_0^{\pi} q(x) \cos 2(n-r)x dx \tag{3.6}$$

$$\begin{aligned}
&\int_0^{\pi} q(x) \cos 2(r-k)x dx \int_0^{\pi} q(x) \cos 2(k-n)x dx, \\
F_2(n, r, k) &= F(n, r, k) - F_1(n, r, k),
\end{aligned} \tag{3.7}$$

$$A_{pi} = \sum_{n=1}^p \sum_{r=p+1}^{\infty} \sum_{k=p+1}^{\infty} \frac{\mu_n}{(\mu_r - \mu_n)(\mu_k - \mu_n)} F_i(n, r, k), \tag{3.8}$$

$$B_{pi} = \sum_{n=1}^p \sum_{\substack{r=1 \\ n \neq r}}^p \sum_{k=p+1}^{\infty} \frac{\mu_n}{(\mu_n - \mu_r)(\mu_n - \mu_k)} F_i(n, r, k), \tag{3.9}$$

$$C_{pi} = \sum_{n=1}^p \sum_{k=p+1}^{\infty} \frac{\mu_k}{(\mu_k - \mu_n)^2} F_i(n, n, k), \quad (i = 1, 2). \tag{3.10}$$

From equation (3.6), we obtain

$$F_1(n, r, k) = F_1(n, k, r), \quad F_1(n, n, k) = F_1(n, k, k).$$

Using these equalities and equation (3.8), we have

$$\begin{aligned} A_{p1} &= \sum_{n=1}^p \sum_{\substack{cr=p+1 \\ r>k}}^{\infty} \sum_{k=p+1}^{\infty} \frac{\mu_n}{(\mu_r - \mu_n)(\mu_k - \mu_n)} F_1(n, r, k) \\ &\quad + \sum_{n=1}^p \sum_{k=p+1}^{\infty} \frac{\mu_n}{(\mu_k - \mu_n)^2} F_1(n, n, k). \end{aligned} \quad (3.11)$$

In similar way, it can be shown that,

$$B_{p1} = - \sum_{n=1}^p \sum_{\substack{cr=1 \\ n<r}}^p \sum_{k=p+1}^{\infty} \frac{\mu_k}{(\mu_r - \mu_n)(\mu_k - \mu_n)} F_1(n, r, k). \quad (3.12)$$

Let

$$A_p^1 = \sum_{n=1}^p \sum_{\substack{cr=p+1 \\ r>k}}^{\infty} \sum_{k=p+1}^{\infty} \frac{\mu_n}{(\mu_r - \mu_n)(\mu_k - \mu_n)} F_1(n, r, k), \quad (3.13)$$

$$B_p^1 = -B_{p1}, \quad (3.14)$$

$$C_p^1 = \sum_{n=1}^p \sum_{k=p+1}^{\infty} \frac{1}{\mu_k - \mu_n} F_1(n, n, k). \quad (3.15)$$

Hence, by using equations (3.5)-(3.15), we obtain

$$M_p^3 = 4A_p^1 - 4B_p^1 - 2C_p^1 + 2A_{p2} + 4B_{p2} - 2C_{p2}. \quad (3.16)$$

Let us find a formula for A_p^1 .

Let

$$E_1 = \{(n, r, k) : n, r, k \in N; r - n = i; k - n = j; n \leq p; r, k > p\},$$

where p, i and j are integers such that $p \geq j, i \geq j$, then

$$\begin{aligned} A_p^1 &= \sum_{n=1}^p \sum_{\substack{r=p+1 \\ r>k}}^{\infty} \sum_{\substack{k=p+1 \\ k-n \leq p}}^{\infty} \frac{\mu_n}{(\mu_r - \mu_n)(\mu_k - \mu_n)} F_1(n, r, k) \\ &\quad + \sum_{n=1}^p \sum_{\substack{r=p+1 \\ r>k}}^{\infty} \sum_{\substack{k=p+1 \\ k-n > p}}^{\infty} \frac{\mu_n}{(\mu_r - \mu_n)(\mu_k - \mu_n)} F_1(n, r, k) \\ &= \pi^{-3} \sum_{\substack{i=2 \\ i>j}}^{\infty} \sum_{j=1}^p \left[\sum_{n, r, k \in E_1} \frac{\mu_n}{(\mu_r - \mu_n)(\mu_k - \mu_n)} \int_0^{\pi} q(x) \cos 2ix dx \right] \end{aligned}$$

$$\begin{aligned} & \times \int_0^\pi q(x) \cos 2(i-j)x dx \int_0^\pi q(x) \cos 2jx dx \Big] \\ & + \sum_{n=1}^p \sum_{\substack{r=p+1 \\ r>k}}^\infty \sum_{\substack{k=p+1 \\ k-n>p}}^\infty \frac{\mu_n}{(\mu_r - \mu_n)(\mu_k - \mu_n)} F_1(n, r, k). \end{aligned}$$

Let

$$\beta_{ij} = \pi^{-3} \int_0^\pi q(x) \cos 2ix dx \int_0^\pi q(x) \cos 2(i-j)x dx \int_0^\pi q(x) \cos 2jx dx, \quad (3.17)$$

and

$$\begin{aligned} A_p^{11} &= \sum_{i=2}^\infty \sum_{\substack{j=1 \\ i>j}}^p \left[\left(\sum_{\substack{n,r,k \in E_1}} \frac{\mu_n}{(\mu_r - \mu_n)(\mu_k - \mu_n)} \right) \beta_{ij} \right], \\ A_p^{12} &= \sum_{n=1}^p \sum_{\substack{r=p+1 \\ r>k}}^\infty \sum_{\substack{k=p+1 \\ k-n>p}}^\infty \frac{\mu_n}{(\mu_r - \mu_n)(\mu_k - \mu_n)} F_1(n, r, k), \end{aligned} \quad (3.18)$$

then we write

$$A_p^1 = A_p^{11} + A_p^{12}. \quad (3.19)$$

By similar proof of (2.14), it can be shown that

$$\begin{aligned} & \sum_{n,r,k \in E_1} \frac{\mu_n}{(\mu_r - \mu_n)(\mu_k - \mu_n)} = \\ &= \sum_{n,r,k \in E_1} \frac{(2n-1)^2}{((2r-1)^2 - (2n-1)^2)((2k-1)^2 - (2n-1)^2)} \\ &= \frac{1}{16i} + \frac{j}{p} O(1), \end{aligned} \quad (3.20)$$

where $O(1)$ satisfies the condition

$$|O(1)| < \text{const.}$$

and depends on p, i and j . Moreover, if $q(x)$ has a continuous derivative of second order at $[0, \pi]$ and satisfies the condition in (1.1), then it can be shown that

$$\beta_{ij} = \frac{\text{const.}}{i^2 j^2}. \quad (3.21)$$

From (3.18), (3.20) and (3.21),

$$A_p^{11} = \sum_{i=2}^\infty \sum_{\substack{j=1 \\ i>j}}^p \left[\frac{\beta_{ij}}{16i} + \frac{O(1)}{pi^2 j} \right],$$

is obtained.

Since

$$\left| \sum_{i=2}^{\infty} \sum_{j=1}^p \frac{O(1)}{pi^2 j} \right| \leq \text{const.} p^{-1} \left(\sum_{i=1}^{\infty} i^{-2} \right) \sum_{j=1}^p j^{-1} < \text{const.} p^{-1} \ln p,$$

we find

$$A_p^{11} = \sum_{i=2}^{\infty} \sum_{\substack{j=1 \\ i>j}}^p \frac{\beta_{ij}}{16i} + o(1). \quad (3.22)$$

Here $o(1)$ is an expression which satisfies the condition

$$\lim_{p \rightarrow \infty} o(1) = 0,$$

and depends on p . From 3.19 and 3.22, we obtain

$$A_p^1 = \sum_{i=2}^{\infty} \sum_{\substack{j=1 \\ i>j}}^p \frac{\beta_{ij}}{16i} + A_p^{12} + o(1). \quad (3.23)$$

Now, to find the formula for B_p^1 , let

$$B_p^{11} = \sum_{n=1}^p \sum_{r=1}^p \sum_{\substack{k=p+1 \\ n< r, k-n \leq p}}^{\infty} \frac{\mu_k}{(\mu_k - \mu_n)(\mu_k - \mu_r)} F_1(n, r, k),$$

and

$$B_p^{12} = \sum_{n=1}^p \sum_{r=1}^p \sum_{\substack{k=p+1 \\ n< r, k-n > p}}^{\infty} \frac{\mu_k}{(\mu_k - \mu_n)(\mu_k - \mu_r)} F_1(n, r, k),$$

then from (3.12) and (3.14), we have

$$B_p^1 = B_p^{11} + B_p^{12}. \quad (3.24)$$

By using equations in (3.6) and (3.7), B_p^{11} can be written as

$$B_p^{11} = \sum_{j=2}^p \sum_{\substack{i=1 \\ j>i}}^{p-1} \left(\sum_{n,r,k \in E_2} \frac{\mu_k}{(\mu_k - \mu_n)(\mu_k - \mu_r)} \right) \beta_{ij}, \quad (3.25)$$

where E_2 is a set defined by

$$E_2 = \{(n, r, k) : n, r, k \in N; r - n = i; k - n = j; n, r \leq p; k > p\}$$

for $i < j \leq p$. Moreover it can be shown that

$$\sum_{n,r,k \in E_2} \frac{\mu_k}{(\mu_k - \mu_n)(\mu_k - \mu_r)} = \frac{1}{16j} + \frac{i}{p}O(1). \quad (3.26)$$

From (3.21), (3.25) and (3.26), we obtain

$$B_p^{11} = \sum_{j=2}^p \sum_{\substack{i=1 \\ j>i}}^{p-1} \left[\frac{\beta_{ij}}{16j} + \frac{O(1)}{pj^2i} \right].$$

and, since $\beta_{ij} = \beta_{ji}$, we write

$$B_p^{11} = \sum_{j=2}^p \sum_{\substack{i=1 \\ j>i}}^{p-1} \frac{\beta_{ij}}{16j} + O(1).$$

By using above equation and 3.24, we have

$$B_p^1 = \sum_{i=2}^p \sum_{\substack{j=1 \\ i>j}}^{p-1} \frac{\beta_{ij}}{16i} + B_p^{12} + o(1). \quad (3.27)$$

From (3.24), we get

$$\left| \sum_{i=2}^p \frac{\beta_{ip}}{i} \right| \leq \sum_{i=2}^p \frac{1}{i^3 p^2} = o(1),$$

$$\left| \sum_{i=p+1}^{\infty} \sum_{j=2}^p \frac{\beta_{ij}}{i} \right| \leq \sum_{i=p+1}^{\infty} \sum_{j=2}^p \frac{1}{i^3 j^2} = \left(\sum_{i=p+1}^{\infty} \frac{1}{i^3} \right) \left(\sum_{j=1}^p \frac{1}{j^2} \right) = o(1).$$

Therefore, by (3.27), we have

$$B_p^1 = \sum_{i=2}^{\infty} \sum_{\substack{j=1 \\ i>j}}^p \frac{\beta_{ij}}{16i} + B_p^{12} + o(1). \quad (3.28)$$

From (3.16), (3.23) and (3.28), we obtain

$$M_p^3 = 4A_p^{12} - 4B_p^{12} - 2C_p^1 + 2A_{p2} + 4B_{p2} - 2C_{p2} + o(1). \quad (3.29)$$

Here, it can be easily seen that,

$$\lim_{p \rightarrow \infty} A_p^{12} = \lim_{p \rightarrow \infty} B_p^{12} = \lim_{p \rightarrow \infty} C_p^1 = \lim_{p \rightarrow \infty} A_{p2} = \lim_{p \rightarrow \infty} B_{p2} = \lim_{p \rightarrow \infty} C_{p2} = 0. \quad (3.30)$$

From (3.29) and (3.30), we obtain

$$\lim_{p \rightarrow \infty} M_p^3 = 0.$$

In a similar way, it can be proved that

$$\lim_{p \rightarrow \infty} M_p^4 = \lim_{p \rightarrow \infty} M_p^5 = 0.$$

Now, let us prove the expression given in (3.2). By employing (2.2) and (3.3), we obtain

$$\begin{aligned} |M_{pN}| &= \frac{1}{2\pi} \left| \int_{|\lambda|=b_p} \lambda^2 \operatorname{tr} \left[R_\lambda (qR_\lambda^0)^{N+1} \right] d\lambda \right| \\ &\leq \frac{b_p^2}{2\pi} \int_{|\lambda|=b_p} \left\| R_\lambda (qR_\lambda^0)^{N+1} \right\|_{\sigma_1(H)} |d\lambda| \\ &\leq b_p^2 \int_{|\lambda|=b_p} \|R_\lambda\| \left\| (qR_\lambda^0)^{N+1} \right\|_{\sigma_1(H)} |d\lambda| \\ &\leq C b_p^2 p^{-1} \int_{|\lambda|=b_p} \|qR_\lambda^0\|^N \left\| (qR_\lambda^0) \right\|_{\sigma_1(H)} |d\lambda| \\ &\leq \operatorname{const.} p^{5-N}. \end{aligned}$$

This shows that

$$\lim_{p \rightarrow \infty} M_{pN} = 0, \quad N \geq 6.$$

By using the equations (2.2), (2.5), (3.1) and (3.2), we find

$$\begin{aligned} \sum_{n=1}^p (\lambda_{n,1}^2 + \lambda_{n,2}^2 - 2(2n-1)^4) &= \frac{4}{\pi} \sum_{n=1}^p (2n-1)^2 \int_0^\pi q(x) dx \\ &\quad + \frac{p}{\pi} \int_0^\pi q^2(x) dx + \frac{p}{\pi^2} \left(\int_0^\pi q(x) dx \right)^2 + o(1). \end{aligned}$$

As a result, we get

$$\begin{aligned} \sum_{n=1}^p \left[\lambda_{n,1}^2 + \lambda_{n,2}^2 - 2(2n-1)^4 - \frac{4}{\pi} (2n-1)^2 \int_0^\pi q(x) dx \right. \\ \left. - \frac{1}{\pi} \int_0^\pi q^2(x) dx - \frac{1}{\pi^2} \left(\int_0^\pi q(x) dx \right)^2 \right] = 0. \end{aligned} \tag{3.31}$$

The left hand-side of this equality is called the second regularized trace of the operator L . Thus we have proven the main result of this article given by the following theorem.

Theorem 1. *If $q(x) \in C^2[0, \pi]$ is a real function which satisfies the condition (1.1), then the equality obtained in (3.31) holds for the second regularized trace of the operator L .*

REFERENCES

- [1] N. M. Aslanova, “ n -th regularized trace of differential operator equation.” *Trans. Natl. Acad. Sci. Azerb., Ser. Phys.-Tech. Math. Sci.*, vol. 26, no. 7, Math. Mech., pp. 27–32, 2006.
- [2] M. Bayramoglu, A. Bayramov, and E. Şen, “A regularized trace formula for a discontinuous Sturm-Liouville operator with delayed argument.” *Electron. J. Differ. Equ.*, vol. 2017, p. 12, 2017.
- [3] A. Bayramov, Z. Oer, and O. Baykal, “On identity for eigenvalues of second order differential operator equation.” *Math. Comput. Modelling*, vol. 49, no. 3-4, pp. 403–412, 2009.
- [4] L. A. Dikii, “A new method for calculation of eigenvalues of sturm-liouville problem,” *Doklady Akademii nauk SSSR*, vol. 156, pp. 12–14, 1957.
- [5] L. A. Dikii, “Trace formulas for sturm-liouville differential operators,” *Uspekhi Matematicheskikh Nauk*, vol. 13, no. 3, pp. 111–143, 1958.
- [6] I. Gel’fand and B. Levitan, “On a simple identity for the characteristic values of a differential operator of the second order differential equations,” *Dokl Akad. Nauk SSSR*, vol. 88, pp. 593–596, 1953.
- [7] M. L. Gorbachuk, *Boundary value problems for operator differential equations*. Springer Science & Business Media, 2012, vol. 48.
- [8] N. J. Guliyev, “The regularized trace formula for the Sturm-Liouville equation with spectral parameter in the boundary conditions.” *Proc. Inst. Math. Mech., Natl. Acad. Sci. Azerb.*, vol. 22, pp. 99–102, 2005.
- [9] F. Hıra and N. Altınışık, “The regularized trace of Sturm-Liouville problem with discontinuities at two points.” *Inverse Probl. Sci. Eng.*, vol. 25, no. 6, pp. 785–794, 2017.
- [10] P. D. Lax, “Trace formulas for the Schrödinger operator.” *Commun. Pure Appl. Math.*, vol. 47, no. 4, pp. 503–512, 1994.
- [11] A. Makin, “Regularized trace of the Sturm-Liouville operator with irregular boundary conditions.” *Electron. J. Differ. Equ.*, vol. 2009, p. 8, 2009.
- [12] F. Maksudov, M. Bayramoglu, and E. Adıgüzelov, “On the regularized trace of the Sturm-Liouville operator with an unbounded operator coefficient on a finite interval.” *Sov. Math., Dokl.*, vol. 30, pp. 169–173, 1984.
- [13] V. Marchenko, *Sturm-Liouville operators and applications*, Revised ed. of the 1986 ed. ed. Providence, RI: AMS Chelsea Publishing, 2011.
- [14] V. A. Sadovnichii and V. E. Podol’skii, “Traces of operators.” *Russ. Math. Surv.*, vol. 61, no. 5, pp. 885–953, 2006.
- [15] J.-J. Sansuc and V. Tkachenko, “Characterization of the periodic and anti-periodic spectra of non-selfadjoint Hill’s operators.” in *New results in operator theory and its applications: the Israel M. Glazman memorial volume*. Basel: Birkhäuser, 1997, pp. 216–224.

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POINCARÉ CHAOS FOR A HYPERBOLIC QUASILINEAR SYSTEM

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Abstract. The existence of unpredictable motions in systems of quasilinear differential equations with hyperbolic linear part is rigorously proved. We make use of the topology of uniform convergence on compact sets and the contraction mapping principle to prove the existence of unpredictable motions. Appropriate examples with simulations that support the theoretical results are provided.

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1. INTRODUCTION AND PRELIMINARIES

A special type of Poisson stable trajectory called an unpredictable trajectory, which leads to Poincaré chaos in the quasi-minimal set, was introduced in the paper [1]. Moreover, the papers [2]-[4] were concerned with the unpredictable solutions of various types of quasilinear differential equations in which the matrix of coefficients admits eigenvalues all with negative linear part. In this study, we deal with the unpredictable solutions of quasilinear differential equations with hyperbolic linear part such that the matrix of coefficients admits eigenvalues both with negative and positive real parts.

Being based on the existence of only one special type of Poisson stable trajectory, the confirmation of the presence of Poincaré chaos is easier compared to the other chaos types [5, 8, 9] since their definitions require the interaction of infinitely many motions such as sensitivity and the existence of a dense set of infinitely many unstable periodic motions embedded in the chaotic attractor. Therefore, it can be accepted that Poincaré chaos is more advantageous from the applications point of view.

Throughout the paper the Euclidean norm for vectors and the norm induced by the Euclidean norm for square matrices will be utilized.

The definition of an unpredictable function is as follows [4].

Definition 1. A uniformly continuous and bounded function $\vartheta(t) : \mathbb{R} \rightarrow \mathbb{R}^P$ is unpredictable if there exist positive numbers ϵ_0, σ and sequences $\{t_n\}, \{u_n\}$ both of

which diverge to infinity such that $\|\vartheta(t + t_n) - \vartheta(t)\| \rightarrow 0$ as $n \rightarrow \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\|\vartheta(t + t_n) - \vartheta(t)\| \geq \epsilon_0$ for each $t \in [u_n - \sigma, u_n + \sigma]$ and $n \in \mathbb{N}$.

For the convenience of our discussions, we will call the convergence of the function's shift on compact subsets of $\mathbb{R}$ in Definition 1 as *Poisson stability* and the existence of the number ϵ_0 as *unpredictability property* of the function. Thus, a function is *unpredictable*, if it is *Poisson stable* and admits the *unpredictability property*.

Compared to the paper [4], we simplified the proof for the unpredictability property in the present study, and this increases the novelty as well as the applicability of the results.

The main object of the present paper is the system of quasilinear differential equations

$$x'(t) = Ax(t) + f(x(t)) + g(t), \quad (1.1)$$

where $x(t) \in \mathbb{R}^p$, p is a fixed natural number, the function $f : \mathbb{R}^p \rightarrow \mathbb{R}^p$ is continuous in all of its arguments, $f(0, 0, \dots, 0) = (0, 0, \dots, 0)$, $g : \mathbb{R} \rightarrow \mathbb{R}^p$ is a uniformly continuous and bounded function, and all eigenvalues of the constant matrix $A \in \mathbb{R}^{p \times p}$ have nonzero real parts. We assume that $\Re(\lambda_i) < 0$, $i = 1, 2, \dots, q$, and $\Re(\lambda_i) > 0$, $i = q + 1, \dots, p$, where $1 \leq q < p$, λ_i , $i = 1, 2, \dots, p$, are the eigenvalues of the matrix A , and $\Re(\lambda_i)$ denotes the real part of the eigenvalue λ_i .

Our purpose is to prove that system (1.1) possesses a unique unpredictable solution, provided that the function $g(t)$ is unpredictable in accordance with Definition 1 and the solution is uniformly globally exponentially stable if all eigenvalues of the matrix A have negative real parts. According to the results of paper [1], the existence of an unpredictable solution implies Poincaré chaos in the dynamics.

The following condition on system (1.1) is required.

- (C1) There exists a positive number L_f such that $\|f(x_1) - f(x_2)\| \leq L_f \|x_1 - x_2\|$ for all $x_1, x_2 \in \mathbb{R}^p$.

It is known that one can find a regular $p \times p$ matrix B such that the transformation $x = By$ reduces system (1.1) to the system

$$y'(t) = Cy(t) + F(y) + G(t), \quad (1.2)$$

where $C = B^{-1}AB$, $F(y) = B^{-1}f(By)$, and $G(t) = B^{-1}g(t)$. In system (1.2), the matrix C is of the form $\text{diag}(C_-, C_+)$, where the eigenvalues of the $q \times q$ matrix C_- and $(p - q) \times (p - q)$ C_+ respectively have negative and positive real parts. Let us denote $F(y) = (F_-(y), F_+(y))$ and $G(t) = (G_-(t), G_+(t))$, where the vector-functions F_- and G_- are of dimension q and the vector-functions F_+ and G_+ are of dimension $p - q$.

It can be verified that $F(0, 0, \dots, 0) = (0, 0, \dots, 0) \in \mathbb{R}^p$ and the function F is Lipschitzian with the Lipschitz constant

$$L_F = \|B\| \|B^{-1}\| L_f,$$

under the condition (C1), that is, $\|F(y_1) - F(y_2)\| \leq L_F \|y_1 - y_2\|$ for all $y_1, y_2 \in \mathbb{R}^p$. Moreover, according to Corollary 5.1 [3], the function $G(t)$ is unpredictable if and only if $g(t)$ is unpredictable.

In the next section, the main result of the paper is formulated for system (1.2) and interpreted for system (1.1).

2. MAIN RESULT

One can confirm that there exist numbers $K \geq 1$ and $\alpha > 0$ such that $\|e^{C-t}\| \leq K e^{-\alpha t}$ for all $t \geq 0$ and $\|e^{C+t}\| \leq K e^{\alpha t}$ for all $t \leq 0$.

The following condition is required.

(C2) $\alpha - 2KL_F > 0$.

According to the theory of differential equations [6,7], a function $\varphi(t) = (\varphi_-(t), \varphi_+(t))$ which is bounded on the whole real axis is a solution of sytem (1.2) if and only if it satisfies the equations

$$\begin{aligned}\varphi_-(t) &= \int_{-\infty}^t e^{C-(t-s)} [F_-(\varphi(s)) + G_-(s)] ds, \\ \varphi_+(t) &= - \int_t^{\infty} e^{C+(t-s)} [F_+(\varphi(s)) + G_+(s)] ds.\end{aligned}\tag{2.1}$$

Let us denote

$$H = \frac{2K \|B^{-1}\| M_g}{\alpha - 2KL_F},\tag{2.2}$$

where $M_g = \sup_{t \in \mathbb{R}} \|g(t)\|$.

The number H defined by (2.2) will be used in the following lemma, which is concerned with the existence and uniqueness of Poisson stable solution of system (1.1).

Lemma 1. *Suppose that the conditions (C1) and (C2) are valid. If the function $g(t)$ is Poisson stable, then system (1.1) possesses a unique Poisson stable solution $\omega(t)$ such that $\sup_{t \in \mathbb{R}} \|\omega(t)\| \leq \|B\| H$.*

Proof. In the proof, we will show that system (1.2) possesses a unique Poisson stable solution by using the contraction mapping principle, and this implies the existence of a unique Poisson stable solution of system (1.1).

Because the function $g(t)$ is Poisson stable, there exists a sequence $\{t_n\}$, $t_n \rightarrow \infty$ as $n \rightarrow \infty$, such that $\|g(t + t_n) - g(t)\| \rightarrow 0$ as $n \rightarrow \infty$ uniformly on each compact subset of $\mathbb{R}$.

Let U be the set of all uniformly continuous and uniformly bounded functions $\psi(t) : \mathbb{R} \rightarrow \mathbb{R}^p$ such that $\|\psi\|_0 \leq H$, where the norm $\|\cdot\|_0$ is defined by $\|\psi\|_0 =$

$\sup_{t \in \mathbb{R}} \|\psi(t)\|$, and $\|\psi(t + t_n) - \psi(t)\| \rightarrow 0$ as $n \rightarrow \infty$ uniformly on each compact subset of $\mathbb{R}$.

Define an operator Π on U through the equations

$$\Pi_- \psi(t) = \int_{-\infty}^t e^{C_-(t-s)} [F_-(\psi(s)) + G_-(s)] ds$$

and

$$\Pi_+ \psi(t) = - \int_t^{\infty} e^{C_+(t-s)} [F_+(\psi(s)) + G_+(s)] ds$$

such that $\Pi \psi(t) = (\Pi_- \psi(t), \Pi_+ \psi(t))$.

First of all, we will show that $\Pi(U) \subseteq U$. Fix an arbitrary function $\psi(t)$ that belongs to U . For all $t \in \mathbb{R}$, we have

$$\begin{aligned} \|\Pi_- \psi(t)\| &\leq \int_{-\infty}^t \|e^{C_-(t-s)}\| \|F_-(\psi(s)) + G_-(s)\| ds \\ &\leq \int_{-\infty}^t K (L_F \|\psi(s)\| + \|B^{-1}\| M_g) e^{-\alpha(t-s)} ds \leq \frac{1}{\alpha} K (L_F H + \|B^{-1}\| M_g) \end{aligned}$$

and

$$\begin{aligned} \|\Pi_+ \psi(t)\| &\leq \int_t^{\infty} \|e^{C_+(t-s)}\| \|F_+(\psi(s)) + G_+(s)\| ds \\ &\leq \int_t^{\infty} K (L_F \|\psi(s)\| + \|B^{-1}\| M_g) e^{\alpha(t-s)} ds \leq \frac{1}{\alpha} K (L_F H + \|B^{-1}\| M_g) \end{aligned}$$

Therefore, $\|\Pi \psi\|_0 \leq \frac{2}{\alpha} K (L_F H + \|B^{-1}\| M_g) = H$.

Now, let us fix an arbitrary positive number ϵ and a compact subset $[a, b]$ of $\mathbb{R}$, where $b > a$. Suppose that a_0 and b_0 are numbers satisfying $a_0 < a$, $b_0 > b$ such that the inequalities

$$\frac{2}{\alpha} K (L_F H + \|B^{-1}\| M_g) e^{-\alpha(a-a_0)} < \frac{\epsilon}{4}, \quad (2.3)$$

$$\frac{2}{\alpha} K (L_F H + \|B^{-1}\| M_g) e^{\alpha(b-b_0)} < \frac{\epsilon}{4} \quad (2.4)$$

are valid, and let γ be a number such that

$$\gamma > \frac{4K(L_F + \|B^{-1}\|)}{\alpha}.$$

There exists a natural number n_0 such that if $n \geq n_0$, then $\|g(t + t_n) - g(t)\| < \frac{\epsilon}{\gamma}$

and $\|\psi(t + t_n) - \psi(t)\| < \frac{\epsilon}{\gamma}$ for all $t \in [a_0, b_0]$.

Therefore, one can verify for $n \geq n_0$ and $t \in [a_0, b_0]$ that

$$\begin{aligned}
& \|\Pi_- \psi(t + t_n) - \Pi_- \psi(t)\| \\
& \leq \int_{-\infty}^{a_0} \|e^{C_-(t-s)}\| (\|F_-(\psi(s + t_n)) - F_-(\psi(s))\| + \|G_-(s + t_n) - G_-(s)\|) ds \\
& \quad + \int_{a_0}^t \|e^{C_-(t-s)}\| (\|F_-(\psi(s + t_n)) - F_-(\psi(s))\| + \|G_-(s + t_n) - G_-(s)\|) ds \\
& \leq \frac{2}{\alpha} K (L_F H + \|B^{-1}\| M_g) e^{-\alpha(t-a_0)} + \frac{\epsilon}{\alpha \gamma} K (L_F + \|B^{-1}\|) (1 - e^{-\alpha(t-a_0)})
\end{aligned}$$

and

$$\begin{aligned}
& \|\Pi_+ \psi(t + t_n) - \Pi_+ \psi(t)\| \\
& \leq \int_{b_0}^{\infty} \|e^{C_+(t-s)}\| (\|F_+(\psi(s + t_n)) - F_+(\psi(s))\| + \|G_+(s + t_n) - G_+(s)\|) ds \\
& \quad + \int_t^{b_0} \|e^{C_+(t-s)}\| (\|F_+(\psi(s + t_n)) - F_+(\psi(s))\| + \|G_+(s + t_n) - G_+(s)\|) ds \\
& \leq \frac{2}{\alpha} K (L_F H + \|B^{-1}\| M_g) e^{\alpha(t-b_0)} + \frac{\epsilon}{\alpha \gamma} K (L_F + \|B^{-1}\|) (1 - e^{\alpha(t-b_0)}).
\end{aligned}$$

Now, using the inequalities (2.3) and (2.4), one can obtain for $t \in [a, b]$ that

$$\|\Pi_- \psi(t + t_n) - \Pi_- \psi(t)\| < \frac{\epsilon}{2}$$

and

$$\|\Pi_+ \psi(t + t_n) - \Pi_+ \psi(t)\| < \frac{\epsilon}{2}$$

provided that $n \geq n_0$. Hence, if $n \geq n_0$, then the inequality $\|\Pi \psi(t + t_n) - \Pi \psi(t)\| < \epsilon$ is valid for all $t \in [a, b]$, and therefore, $\|\Pi \psi(t + t_n) - \Pi \psi(t)\| \rightarrow 0$ uniformly as $n \rightarrow \infty$ on each compact subset of $\mathbb{R}$.

Additionally, it can be shown that $\Pi \psi(t)$ is uniformly continuous since its derivative is bounded. Thus, $\Pi(U) \subseteq U$.

Next, we will show that the operator $\Pi : U \rightarrow U$ is contractive. Let $\psi_1(t)$ and $\psi_2(t)$ be functions in U . Then, we have that

$$\begin{aligned}
& \|\Pi_- \psi_1(t) - \Pi_- \psi_2(t)\| \\
& \leq \int_{-\infty}^t \|e^{C_-(t-s)}\| \|F_-(\psi_1(s)) - F_-(\psi_2(s))\| ds \leq \frac{KL_F}{\alpha} \|\psi_1 - \psi_2\|_0
\end{aligned}$$

and

$$\begin{aligned}
& \|\Pi_+ \psi_1(t) - \Pi_+ \psi_2(t)\| \\
& \leq \int_t^{\infty} \|e^{C_+(t-s)}\| \|F_+(\psi_1(s)) - F_+(\psi_2(s))\| ds \leq \frac{KL_F}{\alpha} \|\psi_1 - \psi_2\|_0.
\end{aligned}$$

Therefore, the inequality $\|\Pi\psi_1 - \Pi\psi_2\|_0 \leq \frac{2KL_F}{\alpha} \|\psi_1 - \psi_2\|_0$ holds, and it implies that the operator $\Pi : U \rightarrow U$ is contractive.

Thus, system (1.2) possesses a unique Poisson stable solution $\xi(t)$, and consequently, $\omega(t) = B\xi(t)$ is the unique Poisson stable solution of system (1.1). Moreover, we have $\sup_{t \in \mathbb{R}} \|\omega(t)\| \leq \|B\|H$, since $\sup_{t \in \mathbb{R}} \|\xi(t)\| \leq H$. $\square$

The next theorem is concerned with the unpredictable solution of system (1.1).

Theorem 1. *Suppose that conditions (C1) and (C2) are valid. If the function $g(t)$ is unpredictable, then system (1.1) possesses a unique unpredictable solution. Moreover, the unpredictable solution is uniformly globally exponentially stable if all eigenvalues of the matrix A have negative real parts.*

Proof. According to Lemma 1, system (1.1) possesses a unique Poisson stable solution $\omega(t)$. Therefore, to prove that system (1.1) possesses a unique unpredictable solution, it remains to show that $\omega(t)$ admits the unpredictability property.

Since $g(t)$ is an unpredictable function, there exist positive numbers ϵ_0 , σ and sequences $\{t_n\}$, $\{u_n\}$ both of which diverge to infinity such that $\|g(t + t_n) - g(t)\| \geq \epsilon_0$ for each $t \in [u_n - \sigma, u_n + \sigma]$ and $n \in \mathbb{N}$.

Assume that $\inf_n \|\omega(u_n + t_n) - \omega(u_n)\| = 0$. This contradicts to the equation

$$\begin{aligned} \omega(t + t_n) - \omega(t) &= \omega(u_n + t_n) - \omega(u_n) + \int_{u_n}^t A(\omega(s + t_n) - \omega(s))ds \\ &\quad + \int_{u_n}^t (f(\omega(s + t_n)) - f(\omega(s)))ds + \int_{u_n}^t (g(s + t_n) - g(s))ds, \end{aligned}$$

since at least one coordinate of $g(s + t_n) - g(s)$ is separated from 0 so that the norm of the last integral is separated from 0, and the norms of the other terms tend to zero as $n \rightarrow \infty$ for each fixed t from the interval $[u_n - \sigma, u_n + \sigma]$. Therefore, we have that

$$\inf_n \|\omega(u_n + t_n) - \omega(u_n)\| = 2\bar{\epsilon}_0$$

for some positive number $\bar{\epsilon}_0$.

Now, fix a positive number η such that

$$\eta(\|A\| \|B\| H + \|B\| HL_f + M_g) \leq \frac{\bar{\epsilon}_0}{2},$$

where H is the number defined by (2.2). Using the inequality

$$\begin{aligned} \|\omega(t + t_n) - \omega(t)\| &\geq \|\omega(u_n + t_n) - \omega(u_n)\| - \left| \int_{u_n}^t \|A\| \|\omega(s + t_n) - \omega(s)\| ds \right| \\ &\quad - \left| \int_{u_n}^t \|f(\omega(s + t_n)) - f(\omega(s))\| ds \right| - \left| \int_{u_n}^t \|g(s + t_n) - g(s)\| ds \right|, \end{aligned}$$

one can obtain for $t \in [u_n - \eta, u_n + \eta]$ that

$$\|\omega(t + t_n) - \omega(t)\| \geq 2\bar{\epsilon}_0 - 2\eta(\|A\| \|B\| H + \|B\| HL_f + M_g) \geq \bar{\epsilon}_0.$$

Hence, the solution $\omega(t)$ is unpredictable.

On the other hand, one can show in a similar way to the proof of Theorem 4.1 [2] that if all eigenvalues of the matrix A have negative real parts, then the unpredictable solution of system (1.1) is uniformly globally exponentially stable under the condition (C2). $\square$

The following two lemmas, which will be utilized in the Section 3, increases the applicability of our results.

Lemma 2. *Suppose that $\phi(t) : \mathbb{R} \rightarrow \mathbb{R}^p$ is an unpredictable function and $\zeta(t) : \mathbb{R} \rightarrow \mathbb{R}^p$ is a continuous and periodic function. Then the function $\phi(t) + \zeta(t)$ is unpredictable.*

Proof. Because the function $\phi(t)$ is unpredictable there exist positive numbers ϵ_0 , σ and sequences $\{t_n\}$, $\{u_n\}$ both of which diverge to infinity such that $\|\phi(t + t_n) - \phi(t)\| \rightarrow 0$ as $n \rightarrow \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\|\phi(t + t_n) - \phi(t)\| \geq \epsilon_0$ for each $t \in [u_n - \sigma, u_n + \sigma]$ and $n \in \mathbb{N}$. Let us denote $h(t) = \phi(t) + \zeta(t)$.

Due to the periodicity of $\zeta(t)$, there exists a subsequence of $\{t_n\}$ (we will assume without loss of generality that it is the sequence $\{t_n\}$ itself) such that $\|\zeta(t + t_n) - \zeta(t)\| \rightarrow 0$ uniformly on the real axis. Hence, $\|h(t + t_n) - h(t)\| \rightarrow 0$ uniformly as $n \rightarrow \infty$ on compact subsets of $\mathbb{R}$. Again, because of the periodicity, one can find a subsequence of $\{u_n\}$, let us say the sequence itself, such that $\|\zeta(t + t_n) - \zeta(t)\| < \epsilon_0/2$ if $t \in [u_n - \sigma, u_n + \sigma]$. Thus, $\|h(t + t_n) - h(t)\| \geq \|\phi(t + t_n) - \phi(t)\| - \|\zeta(t + t_n) - \zeta(t)\| > \epsilon_0 - \epsilon_0/2 = \epsilon_0/2$ for $t \in [u_n - \sigma, u_n + \sigma]$. $\square$

Lemma 3. *Suppose that $\phi(t) : \mathbb{R} \rightarrow \mathbb{R}$ is an unpredictable function. Then the function $\phi^3(t)$ is unpredictable.*

Proof. One can find numbers $\epsilon_0 > 0$, $\sigma > 0$ and sequences $\{t_n\}$, $\{u_n\}$ both of which diverge to infinity such that $\|\phi(t + t_n) - \phi(t)\| \rightarrow 0$ as $n \rightarrow \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\|\phi(t + t_n) - \phi(t)\| \geq \epsilon_0$ for each $t \in [u_n - \sigma, u_n + \sigma]$ and $n \in \mathbb{N}$. The proof for the Poisson stability of $\phi^3(t)$ is trivial, since it follows from the uniform continuity of the cubic function on a compact set.

Fix a natural number n . To prove the unpredictability property, it is sufficient to show for $t \in [u_n - \sigma, u_n + \sigma]$ that $\|\phi(t + t_n) - \phi(t)\| \geq \epsilon_0$ implies $\|\phi^3(t + t_n) - \phi^3(t)\| \geq \epsilon(\epsilon_0)$ for some positive number $\epsilon(\epsilon_0)$. We have that

$$\begin{aligned} |\phi^3(t + t_n) - \phi^3(t)| &= \frac{1}{2} |\phi(t + t_n) - \phi(t)| [\phi^2(t + t_n) + \phi^2(t) \\ &\quad + (\phi(t + t_n) + \phi(t))^2] \geq (\phi^2(t + t_n) + \phi^2(t)) \frac{\epsilon_0}{2}. \end{aligned}$$

Consider the function $F(a, b) = a^2 + b^2$ for $|a - b| \geq \epsilon_0$. The minimum of F occurs at the points (a, b) with $|a| = |b| = \epsilon_0/2$. Therefore, $|\phi^3(t + t_n) - \phi^3(t)| \geq \epsilon_0^3/4$ for $t \in [u_n - \sigma, u_n + \sigma]$. $\square$

3. EXAMPLES

One of the possible ways to confirm the presence of chaos is through simulations. The concept of unpredictable solutions maintain the series of oscillations, but from the other side the chaos accompanies unpredictability. Consequently, we can look for a confirmation of the results for unpredictability observing irregularity in simulations. The approach is effective for asymptotically stable unpredictable solutions, and it is just illustrative for hyperbolic systems with unstable solutions. In the latter case we rely on the fact that any solution becomes unpredictable ultimately.

In the following examples we will utilize the function $\Theta(t) : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\Theta(t) = \int_{-\infty}^t e^{-2(t-s)} \Omega(s) ds, \quad (3.1)$$

which was discussed in paper [4]. In (3.1), the function $\Omega(t)$ is defined by $\Omega(t) = \psi_i$ for $t \in [i, i+1)$, $i \in \mathbb{Z}$, where $\{\psi_i\}$, $i \in \mathbb{Z}$, is an unpredictable solution of the logistic map

$$\lambda_{i+1} = 3.91\lambda_i(1 - \lambda_i) \quad (3.2)$$

inside the unit interval $[0, 1]$. The function $\Theta(t)$ is bounded on the whole real axis such that $\sup_{t \in \mathbb{R}} |\Theta(t)| \leq 1/2$, and it is uniformly continuous since its derivative is bounded. It was proven in paper [4] that $\Theta(t)$ is an unpredictable function.

Example 1. Consider the system

$$\begin{aligned} x_1' &= -3x_1 + 2x_2 - 0.01x_2^3 + 8\Theta(t) - 0.2\sin(3t) \\ x_2' &= x_1 - 2x_2 + 0.05x_1^2 - 5\Theta(t) + 3\cos(3t) \end{aligned} \quad (3.3)$$

where $\Theta(t)$ is the unpredictable function defined by (3.1). The eigenvalues of the matrix of coefficients of system (3.3) are -1 and -3 . One can confirm by means of Lemma 2 that the perturbation function $(8\Theta(t) - 0.2\sin(3t), -5\Theta(t) + 3\cos(3t))$ is unpredictable. According to Theorem 1, there is a unique asymptotically stable unpredictable solution $(\varphi_1(t), \varphi_2(t))$ of system (3.3). Consequently, any solution of the equation behaves irregularly ultimately. This is seen from the simulation of the solution with $x_1(0) = 1, 18$, $x_2(0) = 1, 01$ in Figures 1 and 2.

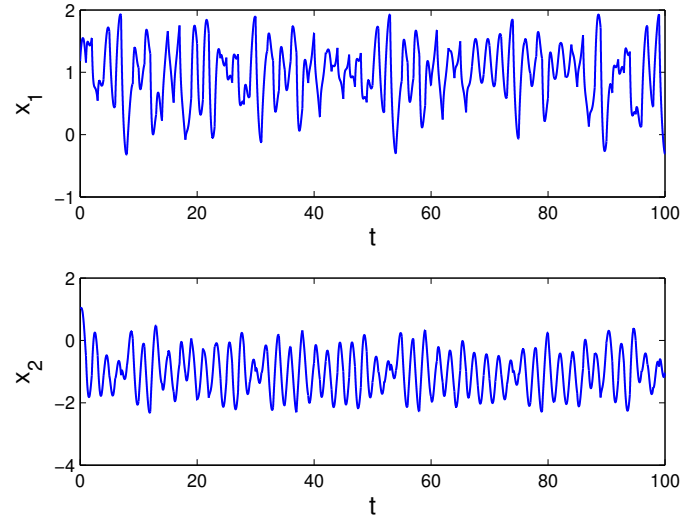


FIGURE 1. The time series of the x_1 and x_2 coordinates of system (3.3) with the initial conditions $x_1(0) = 1,18$, $x_2(0) = 1,01$. The figure manifests the irregular behavior of the system.

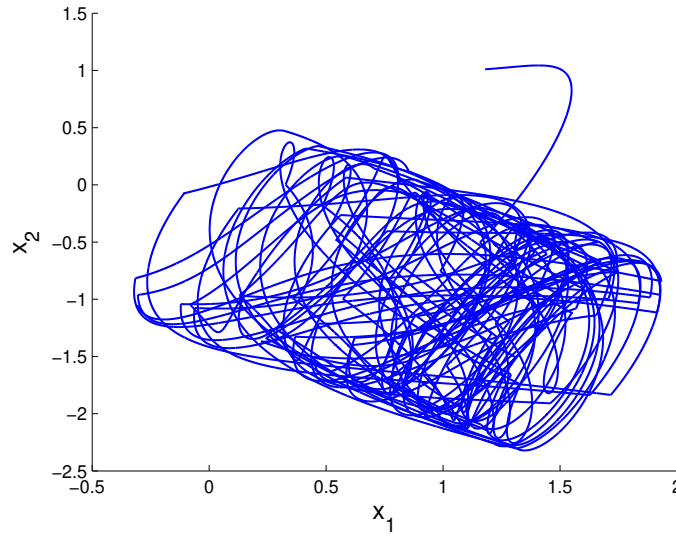


FIGURE 2. The irregular trajectory of system (3.3). The initial data $x_1(0) = 1,18$, $x_2(0) = 1,01$ are utilized in the simulation.

The next example is devoted to a system with hyperbolic linear part such that the matrix of coefficients admit both positive and negative eigenvalues.

Example 2. Let us take into account the system

$$\begin{aligned} y_1' &= 4y_1 - 0.21y_2^3 - 2\varphi_2^3(t) \\ y_2' &= 7y_1 - 6y_2 - 0.14y_1^2 + 3\varphi_1^3(t), \end{aligned} \quad (3.4)$$

where $(\varphi_1(t), \varphi_2(t))$ is the unpredictable solution of system (3.3). One can confirm that the function $(-2\varphi_2^3(t), 3\varphi_1^3(t))$ used as perturbation in system (3.4) is unpredictable in accordance with Lemma 3, and therefore, the system possesses a unique unpredictable solution by Theorem 1. The simulation results for system (3.4) corresponding to the initial conditions $y_1(0) = 0$ and $y_2(0) = 0$ are shown in Figures 3 and 4. Both of the figures reveal the irregular behavior system (3.4).

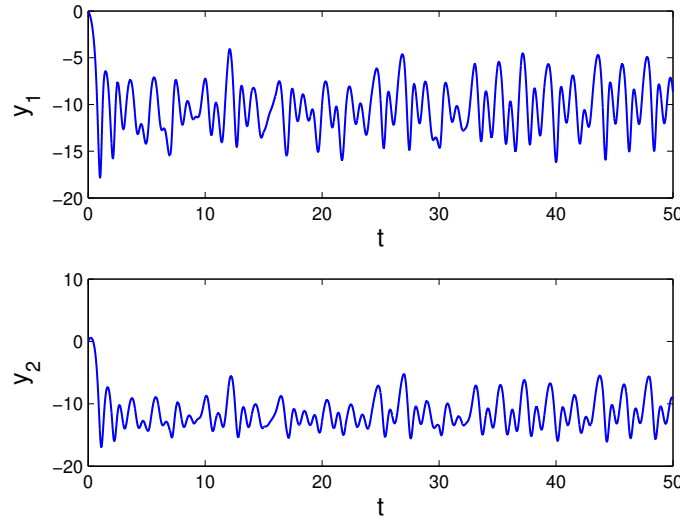


FIGURE 3. The time series for the y_1 and y_2 coordinates of system (3.4) with the initial conditions $y_1(0) = 0$, $y_2(0) = 0$. The irregular behavior of the solution reveals the presence of an unpredictable solution in the dynamics of (3.4).

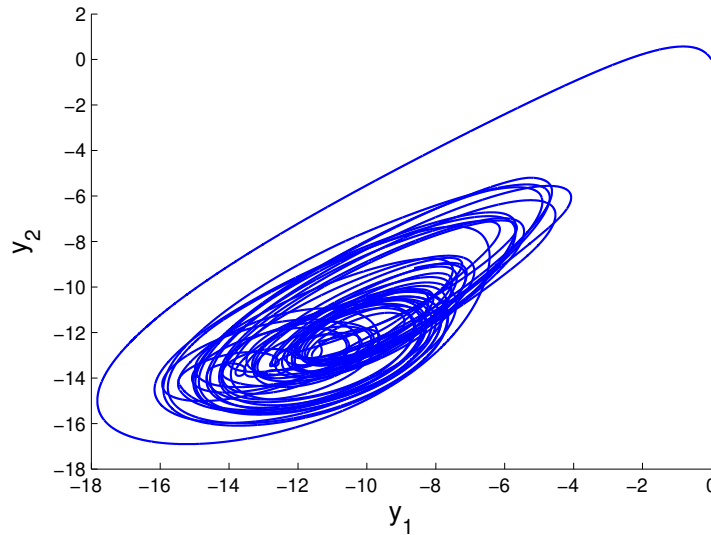


FIGURE 4. The trajectory of the solution of system (3.4). One can observe the irregular behavior of the system in the figure.

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REFERENCES

- [1] M. Akhmet and M. Fen, "Unpredictable points and chaos," *Communications in Nonlinear Science and Numerical Simulation*, vol. 40, pp. 1–5, 2016, doi: [10.1016/j.cnsns.2016.04.007](https://doi.org/10.1016/j.cnsns.2016.04.007).
- [2] M. Akhmet and M. Fen, "Existence of unpredictable solutions and chaos," *Turkish journal of mathematics*, vol. 41, no. 2, pp. 254–256, 2017, doi: [10.3906/mat-1603-51](https://doi.org/10.3906/mat-1603-51).
- [3] M. Akhmet and M. Fen, "Poincaré chaos and unpredictable functions," *Communications in Nonlinear Science and Numerical Simulation*, vol. 48, pp. 85–94, 2017, doi: [10.1016/j.cnsns.2016.12.015](https://doi.org/10.1016/j.cnsns.2016.12.015).
- [4] M. Akhmet and M. Fen, "Non-autonomous equations with unpredictable solutions," *Communications in Nonlinear Science and Numerical Simulation*, vol. 59, pp. 657–670, 2018, doi: [10.1016/j.cnsns.2017.12.011](https://doi.org/10.1016/j.cnsns.2017.12.011).
- [5] R. Devaney, *An Introduction to Chaotic Dynamical Systems*. USA: Addison-Wesley Publishing Company, 1987. doi: [10.1007/978-1-4614-6946-9](https://doi.org/10.1007/978-1-4614-6946-9).
- [6] J. Hale, *Ordinary Differential Equations*. Malabar, FL, USA: Krieger Publishing Company, 1990.
- [7] P. Hartman, *Ordinary Differential Equations*. New York: John Wiley, 1964.

- [8] T. Li and J. Yorke, “Period three implies chaos,” *Journal of Sound and Vibration*, vol. 82, no. 10, pp. 985–992, 1975, doi: [10.2307/2318254](https://doi.org/10.2307/2318254).
- [9] S. Wiggins, *Global Bifurcation and Chaos: Analytical Methods*. New York, Berlin: Springer, 1988. doi: [10.1007/978-1-4612-1042-9](https://doi.org/10.1007/978-1-4612-1042-9).

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SOME PROPERTIES OF COMPLEX QUATERNION AND COMPLEX SPLIT QUATERNION MATRICES

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Abstract. The aim of this study is to investigate some properties of complex quaternion and complex split quaternion matrices. To verify this, we use 2×2 complex matrix representation of these quaternions. Moreover, we present a method to find the determinant of complex quaternion and complex split quaternion matrices. Finally, we research some special matrices for quaternions above.

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1. INTRODUCTION

The real quaternion algebra $\mathbf{H}$ is a four dimensional vector space over the real number field $\mathbb{R}$ and e_0, e_1, e_2, e_3 denote the basis of $\mathbf{H}$ and basis of $\mathbb{R}^4$. The set of real quaternions are a number system that extends the complex numbers field $\mathbb{C}$. Irish mathematician Sir William Rowan Hamilton introduced it in 1843, which is represented as

$$\mathbf{H} = \{a = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

where e_0 acts an identity and $e_1^2 = e_2^2 = e_3^2 = e_1e_2e_3 = -1$. Since $e_2e_3 \neq e_3e_2$ it is obvious that the real quaternions are noncommutative and differ from complex numbers and real numbers. Furthermore any real quaternion can be respresented by a 2×2 complex matrix, [2]. A complex quaternion it is called also biquaternion q can be written as $q = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3$ where $a_0, a_1, a_2, a_3 \in \mathbb{C}$ and its basis elements e_0, e_1, e_2, e_3 satisfy the real quaternion multiplication rules. In [5] and [7] conjugates, 2×2 complex matrices corresponding to basis elements of complex quaternions are expressed.

In 1849, James Cockle introduced the set of real split quaternions which is represented as

$$\mathbf{H}_S = \{p = b_0e_0 + b_1e_1 + b_2e_2 + b_3e_3 : b_0, b_1, b_2, b_3 \in \mathbb{R}\}$$

where $e_1^2 = -1$, $e_2^2 = e_3^2 = 1$ and $e_1e_2e_3 = 1$. Real split quaternions are noncommutative, too, [6]. Also, any real split quaternions can be represented by 2×2 complex matrix, [1]. While coefficients of a real split quaternion are complex numbers, then it is called complex split quaternion. The basis elements of a complex split quaternion have the same rules of a real split quaternion multiplication, [3].

In this study, firstly we associated the results we obtained from the conjugates of the complex quaternion with the real quaternions. Also, we give some properties of matrix representation of complex quaternions and complex split quaternions by expressing these quaternions as 2×2 complex matrices ($M_2(\mathbb{C})$) with using matrices corresponding to the basis of complex quaternions and complex split quaternions. Moreover, we obtain a method to find the determinant for these form of quaternions. Finally, we investigate some special matrices for complex quaternion and complex split quaternion matrices.

2. COMPLEX QUATERNION MATRICES

A real quaternion a is a vector of the form $a = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3$ where a_0, a_1, a_2, a_3 are real numbers. Here $\{e_0, e_1, e_2, e_3\}$ denotes the set of real quaternion basis with the properties

$$e_1^2 = e_2^2 = e_3^2 = e_1e_2e_3 = -1, \quad (2.1)$$

$$e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = e_1, \quad e_3e_1 = -e_1e_3 = e_2. \quad (2.2)$$

A real quaternion a can be written as $a = S_a + V_a$ where $S_a = a_0e_0$ is the scalar part and $V_a = a_1e_1 + a_2e_2 + a_3e_3$ is the vector part of a . For any real quaternion $a = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3$, the conjugate of a is $\bar{a} = a_0e_0 - a_1e_1 - a_2e_2 - a_3e_3$ and the norm of a is $\|a\| = \sqrt{a\bar{a}} = \sqrt{\bar{a}a} = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$. For details, see [2].

A complex quaternion q is of the form $q = A_0e_0 + A_1e_1 + A_2e_2 + A_3e_3$ where A_0, A_1, A_2, A_3 are complex numbers and the elements of $\{e_0, e_1, e_2, e_3\}$ multiply as in real quaternions. Also, a complex quaternion q can be written as $\sum (a_k + ib_k)e_k$ where a_k, b_k are real numbers for $0 \leq k \leq 3$. Here i denotes the complex unit and commutes with e_0, e_1, e_2, e_3 .

For any complex quaternion $q = A_0e_0 + A_1e_1 + A_2e_2 + A_3e_3$, the quaternion conjugate of q is $\bar{q} = A_0e_0 - A_1e_1 - A_2e_2 - A_3e_3$ and $\bar{q}q = q\bar{q} = A_0^2 + A_1^2 + A_2^2 + A_3^2$. The complex conjugate of q is $q^c = \bar{A}_0e_0 + \bar{A}_1e_1 + \bar{A}_2e_2 + \bar{A}_3e_3$ and the Hermitian conjugate of q is $(\bar{q})^c = \bar{A}_0e_0 - \bar{A}_1e_1 - \bar{A}_2e_2 - \bar{A}_3e_3$. For more information of complex quaternions the reader is referred to [5] and [4]. For a complex quaternion $q = (a_0 + ib_0)e_0 + (a_1 + ib_1)e_1 + (a_2 + ib_2)e_2 + (a_3 + ib_3)e_3$, we express the equalities below related to real quaternions and complex quaternions with using the complex conjugate and the Hermitian conjugate of a complex quaternion.

$$q^c q = a^2 + b^2 + 2i (V_a \times V_b) \quad (2.3)$$

$$qq^c = a^2 + b^2 - 2i(V_a \times V_b) \quad (2.4)$$

$$(\bar{q})^c q = \|a\|^2 + \|b\|^2 + 2i(S_a V_b - S_b V_a - V_a \times V_b) \quad (2.5)$$

$$q(\bar{q})^c = \|a\|^2 + \|b\|^2 + 2i(S_a V_b - S_b V_a + V_a \times V_b) \quad (2.6)$$

where $a = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$, $b = b_0 e_0 + b_1 e_1 + b_2 e_2 + b_3 e_3$ and $\times$ denotes the vector product in $\mathbb{R}^3$.

A complex quaternion matrix Q is of the form

$$Q = Q_0 E_0 + Q_1 E_1 + Q_2 E_2 + Q_3 E_3 \quad (2.7)$$

where Q_0, Q_1, Q_2, Q_3 are complex numbers. The complex quaternion matrix basis $\{E_0, E_1, E_2, E_3\}$ satisfying the equalities

$$E_1^2 = E_2^2 = E_3^2 = -E_0, \quad (2.8)$$

$$E_1 E_2 = -E_2 E_1 = E_3, \quad E_2 E_3 = -E_3 E_2 = E_1, \quad E_3 E_1 = -E_1 E_3 = E_2. \quad (2.9)$$

These basis elements are 2×2 matrices, [5]:

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \quad (2.10)$$

The multiplication rules of the 2×2 complex matrices E_0, E_1, E_2, E_3 satisfy the multiplication rules of the complex quaternion basis elements e_0, e_1, e_2, e_3 . Hence, there is an isomorphic relation between the vector form and the matrix form of a complex quaternion.

We denote the algebra of complex quaternion matrices by $\mathbf{H}^{\mathbb{C}}$ and define with the algebra of 2×2 complex matrices:

$$\mathbf{H}^{\mathbb{C}} = \left\{ Q_0 E_0 + Q_1 E_1 + Q_2 E_2 + Q_3 E_3 = \begin{pmatrix} Q_0 + iQ_1 & Q_2 + iQ_3 \\ -Q_2 + iQ_3 & Q_0 - iQ_1 \end{pmatrix} : Q_0, Q_1, Q_2, Q_3 \in \mathbb{C} \right\} \quad (2.11)$$

For any $Q = Q_0 E_0 + Q_1 E_1 + Q_2 E_2 + Q_3 E_3 \in \mathbf{H}^{\mathbb{C}}$, we define $S_Q = Q_0 E_0$, the scalar matrix part of Q ; $\text{Im } Q = Q_1 E_1 + Q_2 E_2 + Q_3 E_3$, the imaginary matrix part of Q . The conjugate, the complex conjugate and the total conjugate of a complex quaternion matrix are denoted by $\bar{Q}, Q^C, (\bar{Q})^C$ respectively these are

$$\bar{Q} = Q_0 E_0 - Q_1 E_1 - Q_2 E_2 - Q_3 E_3, \quad (2.12)$$

$$\begin{aligned} Q^C &= Q_0 \bar{E}_0 + Q_1 \bar{E}_1 + Q_2 \bar{E}_2 + Q_3 \bar{E}_3 \\ &= Q_0 E_0 - Q_1 E_1 + Q_2 E_2 - Q_3 E_3, \end{aligned} \quad (2.13)$$

$$\begin{aligned} (\overline{Q})^C &= \overline{(Q^C)} = Q_0 \overline{E_0} - Q_1 \overline{E_1} - Q_2 \overline{E_2} - Q_3 \overline{E_3} \\ &= Q_0 E_0 + Q_1 E_1 - Q_2 E_2 + Q_3 E_3. \end{aligned} \quad (2.14)$$

In addition, for any $Q = Q_0 E_0 + Q_1 E_1 + Q_2 E_2 + Q_3 E_3 \in \mathbf{H}^{\mathbb{C}}$, we can define transpose and adjoint matrix of Q by Q^t and $Adj Q$ respectively write down as

$$Q^t = Q_0 E_0 + Q_1 E_1 - Q_2 E_2 + Q_3 E_3, \quad (2.15)$$

$$Adj Q = Q_0 E_0 - Q_1 E_1 - Q_2 E_2 - Q_3 E_3. \quad (2.16)$$

So we can get

$$Adj Q = \overline{Q}, \quad (2.17)$$

$$Q^C = (\overline{Q})^t. \quad (2.18)$$

The norm of a complex quaternion matrix

$$Q = Q_0 E_0 + Q_1 E_1 + Q_2 E_2 + Q_3 E_3 = \begin{pmatrix} q_{11} & q_{12} \\ -\overline{q}_{12} & \overline{q}_{11} \end{pmatrix} \quad (2.19)$$

is defined as

$$\|Q\| = \sqrt{|q_{11}|^2 + |q_{12}|^2} \quad (2.20)$$

where $q_{11} = Q_0 + iQ_1$ and $q_{12} = Q_2 + iQ_3$.

Definition 1. A determinant of $Q \in \mathbf{H}^{\mathbb{C}}$ is defined as

$$\det Q = Q_0^2 \det E_0 + Q_1^2 \det E_1 + Q_2^2 \det E_2 + Q_3^2 \det E_3. \quad (2.21)$$

Using the determinant of a complex quaternion matrix basis the above determinant can be written as

$$\det Q = Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2. \quad (2.22)$$

Theorem 1. For any $Q, P \in \mathbf{H}^{\mathbb{C}}$ and $\lambda \in \mathbb{C}$ the following properties are satisfied:

(i) $\det Q = \det(\overline{Q}) = \det(Q^C) = \det(Q^t)$,

(ii) $\det(\lambda Q) = \lambda^2 \det Q$,

(iii) $\det(QP) = \det Q \det P$.

Proof. (i) For $Q = Q_0 E_0 + Q_1 E_1 + Q_2 E_2 + Q_3 E_3 \in \mathbf{H}^{\mathbb{C}}$, from (2.22) it can be found easily that

$$\det Q = \det(\overline{Q}) = \det(Q^C) = \det(Q^t) = Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2.$$

(ii) For any $\lambda \in \mathbb{C}$, we have $\lambda Q = (\lambda Q_0) E_0 + (\lambda Q_1) E_1 + (\lambda Q_2) E_2 + (\lambda Q_3) E_3$. Thus,

$$\begin{aligned} \det(\lambda Q) &= \lambda^2 Q_0^2 + \lambda^2 Q_1^2 + \lambda^2 Q_2^2 + \lambda^2 Q_3^2 \\ &= \lambda^2 (Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2) \end{aligned}$$

$$= \lambda^2 \det Q.$$

(iii) Let $Q = Q_0 E_0 + Q_1 E_1 + Q_2 E_2 + Q_3 E_3$ and $P = P_0 E_0 + P_1 E_1 + P_2 E_2 + P_3 E_3$ be complex quaternion matrices, QP is calculated as

$$\begin{aligned} QP = & (Q_0 P_0 - Q_1 P_1 - Q_2 P_2 - Q_3 P_3) E_0 + (Q_0 P_1 + Q_1 P_0 + Q_2 P_3 - Q_3 P_2) E_1 \\ & + (Q_0 P_2 + Q_2 P_0 - Q_1 P_3 + Q_3 P_1) E_2 + (Q_0 P_3 + Q_3 P_0 + Q_1 P_2 - Q_2 P_1) E_3 \end{aligned}$$

and from (2.22) we have

$$\begin{aligned} \det(QP) = & Q_0^2 P_0^2 + Q_0^2 P_1^2 + Q_0^2 P_2^2 + Q_0^2 P_3^2 + Q_1^2 P_0^2 + Q_1^2 P_1^2 + Q_1^2 P_2^2 + Q_1^2 P_3^2 \\ & + Q_2^2 P_0^2 + Q_2^2 P_1^2 + Q_2^2 P_2^2 + Q_2^2 P_3^2 + Q_3^2 P_0^2 + Q_3^2 P_1^2 + Q_3^2 P_2^2 + Q_3^2 P_3^2. \end{aligned}$$

On the other hand, the determinants of Q and P are $Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2$ and $P_0^2 + P_1^2 + P_2^2 + P_3^2$, respectively, then

$$\begin{aligned} \det Q \det P = & Q_0^2 P_0^2 + Q_0^2 P_1^2 + Q_0^2 P_2^2 + Q_0^2 P_3^2 + Q_1^2 P_0^2 + Q_1^2 P_1^2 + Q_1^2 P_2^2 + Q_1^2 P_3^2 \\ & + Q_2^2 P_0^2 + Q_2^2 P_1^2 + Q_2^2 P_2^2 + Q_2^2 P_3^2 + Q_3^2 P_0^2 + Q_3^2 P_1^2 + Q_3^2 P_2^2 + Q_3^2 P_3^2. \end{aligned}$$

Therefore

$$\det(QP) = \det Q \det P.$$

□

Additionally, using the complex conjugate and the transpose of a complex quaternion matrix we obtain the determinant of a complex quaternion matrix.

$$Q^t Q^C = Q^C Q^t = (Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2) E_0 \quad (2.23)$$

and from (2.22) the determinant of $Q^t Q^C$ is

$$\det(Q^t Q^C) = (Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2)^2 \quad (2.24)$$

so,

$$(\det Q)^2 = \det(Q^t Q^C). \quad (2.25)$$

If $\det Q \neq 0$, the inverse of a complex quaternion matrix is defined as

$$Q^{-1} = \frac{1}{\det Q} \overline{Q}. \quad (2.26)$$

From (2.12), (2.22) and (2.26) the inverse of a complex quaternion matrix can be written as

$$Q^{-1} = \frac{1}{Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2} (Q_0 E_0 - Q_1 E_1 - Q_2 E_2 - Q_3 E_3). \quad (2.27)$$

Example 1. Let $Q = E_0 + iE_2 + E_3$ be a complex quaternion matrix. Then, the complex quaternion matrix Q can be written as $Q = \begin{pmatrix} 1 & 2i \\ 0 & 1 \end{pmatrix}$. From (2.27), the inverse of Q is

$$\begin{aligned} Q^{-1} &= E_0 - iE_2 - E_3 \\ &= \begin{pmatrix} 1 & -2i \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Theorem 2. *Complex quaternion matrices satisfy the following properties for $Q \in \mathbf{H}^{\mathbb{C}}$:*

- (i) $E_1c = cE_1$, $E_2c = cE_2$, $E_3c = cE_3$ for any complex number c ,
- (ii) $Q^2 = S_Q^2 - \det(\operatorname{Im} Q) E_0 + 2S_Q \operatorname{Im} Q$,
- (iii) Every complex quaternion matrix Q is expressed as $Q = Z_1 + Z_2 E_2$ where $Z_1, Z_2 \in M_2(\mathbb{C})$.

Proof. Proofs of (i) and (iii) can be easily shown. Now, we will prove (ii). For $Q = Q_0 E_0 + Q_1 E_1 + Q_2 E_2 + Q_3 E_3 \in \mathbf{H}^{\mathbb{C}}$,

$$Q^2 = (Q_0^2 - Q_1^2 - Q_2^2 - Q_3^2) E_0 + 2Q_0(Q_1 E_1 + Q_2 E_2 + Q_3 E_3)$$

and using the following equalities

$$S_Q = Q_0 E_0, \operatorname{Im} Q = Q_1 E_1 + Q_2 E_2 + Q_3 E_3, \det(\operatorname{Im} Q) = Q_1^2 + Q_2^2 + Q_3^2$$

we get

$$Q^2 = S_Q^2 - \det(\operatorname{Im} Q) E_0 + 2S_Q \operatorname{Im} Q.$$

□

Theorem 3. *For any $Q, P \in \mathbf{H}^{\mathbb{C}}$ the following properties are satisfied:*

- (i) $Q = \left[\overline{(Q^t)^t} \right]^C$,
- (ii) $Q^t = \overline{(Q^C)}$,
- (iii) $(Q^C)^{-1} = \overline{(Q^{-1})^C}$ if Q is invertible,
- (iv) $\overline{(Q)}^{-1} = \overline{(Q^{-1})}$ if Q is invertible,
- (v) $(Q^t)^{-1} = \overline{(Q^{-1})^t}$ if Q is invertible,
- (vi) $(QP)^C = Q^C P^C$,
- (vii) $(QP)^{-1} = P^{-1} Q^{-1}$ if Q and P are invertible.

Proof. Proof of the theorem is easily shown. However, we will prove only (ii), (iii) and (vii).

(ii) For $Q = Q_0 E_0 + Q_1 E_1 + Q_2 E_2 + Q_3 E_3 \in \mathbf{H}^{\mathbb{C}}$, the complex conjugate of a

complex quaternion matrix is $Q^C = Q_0E_0 - Q_1E_1 + Q_2E_2 - Q_3E_3$ and from (2.12) it is obtained that

$$\overline{(Q^C)} = Q_0E_0 + Q_1E_1 - Q_2E_2 + Q_3E_3.$$

The equality (2.15) implies that the transpose of $Q \in \mathbf{H}^C$ is

$$Q^t = Q_0E_0 + Q_1E_1 - Q_2E_2 + Q_3E_3.$$

Thus,

$$Q^t = \overline{(Q^C)}.$$

(iii) Let $Q = Q_0E_0 + Q_1E_1 + Q_2E_2 + Q_3E_3 \in \mathbf{H}^C$ and Q be an invertible complex quaternion matrix. We know $Q^C = Q_0E_0 - Q_1E_1 + Q_2E_2 - Q_3E_3$. From (2.27) we get

$$(Q^C)^{-1} = \frac{1}{Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2} (Q_0E_0 + Q_1E_1 - Q_2E_2 + Q_3E_3).$$

From (2.27) and (2.13), we find

$$(Q^{-1})^C = \frac{1}{Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2} (Q_0E_0 + Q_1E_1 - Q_2E_2 + Q_3E_3).$$

So,

$$(Q^C)^{-1} = (Q^{-1})^C.$$

(vii) Let $Q = Q_0E_0 + Q_1E_1 + Q_2E_2 + Q_3E_3$ and $P = P_0E_0 + P_1E_1 + P_2E_2 + P_3E_3$ be invertible complex quaternion matrices. We denote QP as

$$QP = AE_0 + BE_1 + CE_2 + DE_3,$$

for simplicity, where

$$\begin{aligned} A &= Q_0P_0 - Q_1P_1 - Q_2P_2 - Q_3P_3, \quad B = Q_0P_1 + Q_1P_0 + Q_2P_3 - Q_3P_2, \\ C &= Q_0P_2 - Q_1P_3 + Q_2P_0 + Q_3P_1, \quad D = Q_0P_3 + Q_1P_2 - Q_2P_1 + Q_3P_0. \end{aligned}$$

From (2.22) the determinant of QP is written as

$$\det(QP) = A^2 + B^2 + C^2 + D^2,$$

where

$$\begin{aligned} &A^2 + B^2 + C^2 + D^2 \\ &= Q_0^2P_0^2 + Q_0^2P_1^2 + Q_0^2P_2^2 + Q_0^2P_3^2 + Q_1^2P_0^2 + Q_1^2P_1^2 + Q_1^2P_2^2 + Q_1^2P_3^2 \\ &\quad + Q_2^2P_0^2 + Q_2^2P_1^2 + Q_2^2P_2^2 + Q_2^2P_3^2 + Q_3^2P_0^2 + Q_3^2P_1^2 + Q_3^2P_2^2 + Q_3^2P_3^2 \end{aligned}$$

and from (2.27) can be found

$$(QP)^{-1} = \frac{\overline{(QP)}}{\det(QP)} = \frac{AE_0 - BE_1 - CE_2 - DE_3}{A^2 + B^2 + C^2 + D^2}.$$

On the other hand, from (2.27), the inverses of P and Q can be written as

$$P^{-1} = \frac{P_0E_0 - P_1E_1 - P_2E_2 - P_3E_3}{P_0^2 + P_1^2 + P_2^2 + P_3^2} \text{ and } Q^{-1} = \frac{Q_0E_0 - Q_1E_1 - Q_2E_2 - Q_3E_3}{Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2}$$

and their product is obtained as

$$P^{-1}Q^{-1} = \frac{AE_0 - BE_1 - CE_2 - DE_3}{A^2 + B^2 + C^2 + D^2}.$$

Therefore,

$$(QP)^{-1} = P^{-1}Q^{-1}.$$

□

Example 2. Let $Q = E_0 + E_1$, $P = E_0 + E_2 \in \mathbf{H}^{\mathbb{C}}$. Then,

- (i) $(QP)^C = E_0 - E_1 + E_2 - E_3 \neq E_0 - E_1 + E_2 + E_3 = P^C Q^C$
- (ii) $(QP)^{-1} = \frac{1}{4}(E_0 - E_1 - E_2 - E_3) \neq \frac{1}{4}(E_0 - E_1 - E_2 + E_3) = Q^{-1}P^{-1}$

Example 3. Let $Q = E_0 + E_1 + E_2$. Then,

$$Q(\overline{Q})^C = E_0 + 2E_1 - 2E_3 \neq E_0 + 2E_1 + 2E_3 = (\overline{Q})^C Q.$$

With these examples we get the following Corollary for complex quaternion matrices.

Corollary 1. Let $Q, P \in \mathbf{H}^{\mathbb{C}}$. Then the followings are satisfied:

- (i) $(QP)^C \neq P^C Q^C$ in general;
- (ii) $(QP)^{-1} \neq Q^{-1}P^{-1}$ in general;
- (iii) $Q(\overline{Q})^C \neq (\overline{Q})^C Q$ in general.

Definition 2. For any $Q = Q_0E_0 + Q_1E_1 + Q_2E_2 + Q_3E_3 \in \mathbf{H}^{\mathbb{C}}$,

- (i) if off-diagonal entries of Q are 0 then Q is called a *diagonal matrix* and Q is in form of $Q = Q_0E_0 + Q_1E_1$,
- (ii) if $Q^t = Q$ then Q is called a *symmetric matrix* and Q is in form of $Q = Q_0E_0 + Q_1E_1 + Q_3E_3$,
- (iii) if $Q^t = Q^{-1}$ then Q is called a *orthogonal matrix* and Q is in form of $Q = Q_0E_0 + Q_2E_2$ and $\det Q = 1$,
- (iv) if $(\overline{Q})^t = Q$ then Q is called a *Hermitian matrix* and Q is in form of $Q = Q_0E_0 + Q_2E_2$,
- (v) if $(\overline{Q})^t = Q^{-1}$ then Q is called a *unitary matrix* and Q is in form of $Q = Q_0E_0 + Q_1E_1 + Q_3E_3$ and $\det Q = 1$.

3. COMPLEX SPLIT QUATERNION MATRICES

A complex split quaternion p is a vector of the form $p = b_0e_0 + b_1e_1 + b_2e_2 + b_3e_3$ where b_0, b_1, b_2, b_3 are complex numbers. Here $\{e_0, e_1, e_2, e_3\}$ denotes the complex split quaternion basis with the below properties

$$e_1^2 = -1, \quad e_2^2 = e_3^2 = e_1e_2e_3 = 1, \quad (3.1)$$

$$e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = -e_1, \quad e_3e_1 = -e_1e_3 = e_2. \quad (3.2)$$

For details of complex split quaternions, see [3].

A complex split quaternion matrix P is of the form

$$P = P_0E_0 + P_1E_1 + P_2E_2 + P_3E_3 \quad (3.3)$$

where P_0, P_1, P_2, P_3 are complex numbers. The split quaternion matrix basis $\{E_0, E_1, E_2, E_3\}$ satisfy the equalities

$$E_1^2 = -E_0, \quad E_2^2 = E_3^2 = E_0, \quad (3.4)$$

$$E_1E_2 = -E_2E_1 = E_3, \quad E_2E_3 = -E_3E_2 = -E_1, \quad E_3E_1 = -E_1E_3 = E_2. \quad (3.5)$$

These basis elements are 2×2 matrices, [6]:

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \quad (3.6)$$

The multiplication rules of the complex 2×2 matrices E_0, E_1, E_2, E_3 coincide with the multiplication rules of the complex split quaternion basis elements e_0, e_1, e_2, e_3 . Hence, there is an isomorphic relation between the vector form and the matrix form of a complex split quaternion.

Let us denote the algebra of complex split quaternion matrices by $\mathbf{H}_S^{\mathbb{C}}$. $\mathbf{H}_S^{\mathbb{C}}$ can be defined with the algebra of 2×2 complex matrices:

$$\mathbf{H}_S^{\mathbb{C}} = \left\{ P_0E_0 + P_1E_1 + P_2E_2 + P_3E_3 = \begin{pmatrix} P_0 + iP_1 & P_2 + iP_3 \\ P_2 - iP_3 & P_0 - iP_1 \end{pmatrix} : P_0, P_1, P_2, P_3 \in \mathbb{C} \right\} \quad (3.7)$$

For any $P = P_0E_0 + P_1E_1 + P_2E_2 + P_3E_3 \in \mathbf{H}_S^{\mathbb{C}}$, we define $S_P = P_0E_0$, the scalar matrix part of P ; $\text{Im } P = P_1E_1 + P_2E_2 + P_3E_3$, the imaginary matrix part of P . The conjugate, the complex conjugate and the total conjugate of a complex split quaternion matrix are denoted by $\overline{P}, P^C, (\overline{P})^C$ respectively these are

$$\overline{P} = P_0E_0 - P_1E_1 - P_2E_2 - P_3E_3, \quad (3.8)$$

$$\begin{aligned} P^C &= P_0\overline{E_0} + P_1\overline{E_1} + P_2\overline{E_2} + P_3\overline{E_3} \\ &= P_0E_0 - P_1E_1 + P_2E_2 - P_3E_3, \end{aligned} \quad (3.9)$$

$$(\overline{P})^C = \overline{(P^C)} = P_0\overline{E_0} - P_1\overline{E_1} - P_2\overline{E_2} - P_3\overline{E_3} \quad (3.10)$$

$$= P_0 E_0 + P_1 E_1 - P_2 E_2 + P_3 E_3.$$

Moreover, for any $P = P_0 E_0 + P_1 E_1 + P_2 E_2 + P_3 E_3 \in \mathbf{H}_S^{\mathbb{C}}$, we can define transpose and adjoint matrix of P by P^t and $AdjP$ and these are

$$P^t = P_0 E_0 + P_1 E_1 + P_2 E_2 - P_3 E_3, \quad (3.11)$$

$$AdjP = P_0 E_0 - P_1 E_1 - P_2 E_2 - P_3 E_3, \quad (3.12)$$

$$AdjP = \overline{P}. \quad (3.13)$$

The norm of a complex split quaternion matrix

$$P = P_0 E_0 + P_1 E_1 + P_2 E_2 + P_3 E_3 = \begin{pmatrix} p_{11} & p_{12} \\ \overline{p}_{12} & \overline{p}_{11} \end{pmatrix} \quad (3.14)$$

is defined as

$$\|P\| = \sqrt{|p_{11}|^2 - |p_{12}|^2} \quad (3.15)$$

where $p_{11} = P_0 + iP_1$ and $p_{12} = P_2 + iP_3$.

Definition 3. A determinant of $P \in \mathbf{H}_S^{\mathbb{C}}$ is defined as

$$\det P = P_0^2 \det E_0 + P_1^2 \det E_1 + P_2^2 \det E_2 + P_3^2 \det E_3. \quad (3.16)$$

From the determinant of a complex split quaternion basis can be written as

$$\det P = P_0^2 + P_1^2 - P_2^2 - P_3^2. \quad (3.17)$$

Theorem 4. For any $P, Q \in \mathbf{H}_S^{\mathbb{C}}$ and $\psi \in \mathbb{C}$ the following properties are satisfied:

- (i) $\det P = \det(\overline{P}) = \det(P^C) = \det(P^t)$,
- (ii) $\det(\psi P) = \psi^2 \det P$,
- (iii) $\det(PQ) = \det P \det Q$.

Proof. (i) For $P = P_0 E_0 + P_1 E_1 + P_2 E_2 + P_3 E_3 \in \mathbf{H}_S^{\mathbb{C}}$, from (3.17) we get

$$\det P = \det(\overline{P}) = \det(P^C) = \det(P^t) = P_0^2 + P_1^2 - P_2^2 - P_3^2.$$

(ii) For any $\psi \in \mathbb{C}$ and $P = P_0 E_0 + P_1 E_1 + P_2 E_2 + P_3 E_3 \in \mathbf{H}_S^{\mathbb{C}}$,

$$\psi P = (\psi P_0) E_0 + (\psi P_1) E_1 + (\psi P_2) E_2 + (\psi P_3) E_3.$$

Thus,

$$\begin{aligned} \det(\psi P) &= \psi^2 P_0^2 + \psi^2 P_1^2 - \psi^2 P_2^2 - \psi^2 P_3^2 \\ &= \psi^2 (P_0^2 + P_1^2 - P_2^2 - P_3^2) \\ &= \psi^2 \det P. \end{aligned}$$

(iii) Let $P = P_0 E_0 + P_1 E_1 + P_2 E_2 + P_3 E_3$ and

$Q = Q_0 E_0 + Q_1 E_1 + Q_2 E_2 + Q_3 E_3$ be complex split quaternion matrices, then

PQ

$$= (P_0 Q_0 - P_1 Q_1 + P_2 Q_2 + P_3 Q_3) E_0 + (P_0 Q_1 + P_1 Q_0 - P_2 Q_3 + P_3 Q_2) E_1 \\ + (P_0 Q_2 + P_2 Q_0 - P_1 Q_3 + P_3 Q_1) E_2 + (P_0 Q_3 + P_3 Q_0 + P_1 Q_2 - P_2 Q_1) E_3$$

and from (3.17) the determinant of PQ can be found as in the form of

$$\det(PQ) \\ = P_0^2 Q_0^2 + P_0^2 Q_1^2 + P_1^2 Q_0^2 + P_1^2 Q_1^2 + P_2^2 Q_2^2 + P_2^2 Q_3^2 + P_3^2 Q_2^2 + P_3^2 Q_3^2 \\ - P_0^2 Q_2^2 - P_0^2 Q_3^2 - P_1^2 Q_2^2 - P_1^2 Q_3^2 - P_2^2 Q_0^2 - P_2^2 Q_1^2 - P_3^2 Q_0^2 - P_3^2 Q_1^2.$$

On the other hand, the determinants of P and Q are $P_0^2 + P_1^2 - P_2^2 - P_3^2$ and $Q_0^2 + Q_1^2 - Q_2^2 - Q_3^2$ respectively

$$\det P \det Q \\ = P_0^2 Q_0^2 + P_0^2 Q_1^2 + P_1^2 Q_0^2 + P_1^2 Q_1^2 + P_2^2 Q_2^2 + P_2^2 Q_3^2 + P_3^2 Q_2^2 + P_3^2 Q_3^2 \\ - P_0^2 Q_2^2 - P_0^2 Q_3^2 - P_1^2 Q_2^2 - P_1^2 Q_3^2 - P_2^2 Q_0^2 - P_2^2 Q_1^2 - P_3^2 Q_0^2 - P_3^2 Q_1^2.$$

Thus,

$$\det(PQ) = \det P \det Q.$$

□

Moreover, the determinant of a complex split quaternion matrix can also be found by the complex conjugate and the total conjugate of a complex split quaternion matrix.

$$P^C (\overline{P})^C = (\overline{P})^C P^C = (P_0^2 + P_1^2 - P_2^2 - P_3^2) E_0 \quad (3.18)$$

and from (3.17) the determinant of $P^C (\overline{P})^C$ is found as

$$\det(P^C (\overline{P})^C) = (P_0^2 + P_1^2 - P_2^2 - P_3^2)^2. \quad (3.19)$$

Hence, the determinant of a complex split quaternion matrix can be written as

$$(\det P)^2 = \det(P^C (\overline{P})^C). \quad (3.20)$$

If $\det P \neq 0$, the inverse of a complex split quaternion matrix is defined as

$$P^{-1} = \frac{1}{\det P} \overline{P}. \quad (3.21)$$

From (3.8), (3.17) and (3.21) can be written

$$Q^{-1} = \frac{1}{P_0^2 + P_1^2 - P_2^2 - P_3^2} (P_0 E_0 - P_1 E_1 - P_2 E_2 - P_3 E_3). \quad (3.22)$$

Example 4. Let $P = \left(\frac{1-i}{2}\right)E_0 + \left(\frac{1-i}{2}\right)E_1 - \frac{1}{2}E_2 + \frac{i}{2}E_3$ be a complex split quaternion matrix. Then, P can be written as $P = \begin{pmatrix} 1 & -1 \\ 0 & -i \end{pmatrix}$ and from (3.22), the inverse of P is calculated as

$$\begin{aligned} P^{-1} &= \left(\frac{1+i}{2}\right)E_0 + \left(\frac{-1-i}{2}\right)E_1 + \frac{i}{2}E_2 + \frac{1}{2}E_3 \\ &= \begin{pmatrix} 1 & i \\ 0 & i \end{pmatrix}. \end{aligned}$$

Theorem 5. Complex split quaternion matrices satisfy the following properties for $P \in \mathbf{H}_S^{\mathbb{C}}$.

- (i) $E_1c = cE_1, E_2c = cE_2, E_3c = cE_3$ for any complex number c ,
- (ii) $P^2 = S_P^2 - \det(\operatorname{Im} P)E_0 + 2S_P \operatorname{Im} P$,
- (iii) Every complex split quaternion matrix P can be uniquely expressed as $P = Z_1 + Z_2E_2$, where $Z_1, Z_2 \in M_2(\mathbb{C})$.

Theorem 6. For any $P, Q \in \mathbf{H}_S^{\mathbb{C}}$ the following properties are satisfied:

- (i) $\det P = \|P\|^2$,
- (ii) $(\overline{P})^{-1} = \overline{(P^{-1})}$ if P is invertible,
- (iii) $(P^C)^{-1} = (P^{-1})^C$ if P is invertible,
- (iv) $\left[(\overline{P})^t\right]^{-1} = \left[\overline{(P^{-1})}\right]^t$ if P is invertible,
- (v) $(PQ)^C = P^C Q^C$,
- (vi) $(PQ)^{-1} = Q^{-1}P^{-1}$ if P and Q are invertible.

The proof is analogous to the proof of Theorem 3.

Example 5. Let $P = E_0 + E_2 + E_3$ and $Q = E_0 + E_1$. Then,

- (i) $(PQ)^C = E_0 - E_1 + 2E_2 \neq E_0 - E_1 - 2E_3 = Q^C P^C$,
- (ii) $P(\overline{P})^C = E_0 - 2E_1 + 2E_3 \neq E_0 + 2E_1 + 2E_3 = (\overline{P})^C P$.

With these examples we get the following Corollary for complex split quaternion matrices.

Corollary 2. Let $P, Q \in \mathbf{H}_S^{\mathbb{C}}$. Then the followings are satisfied:

- (i) $(PQ)^C \neq Q^C P^C$ in general;
- (ii) $P(\overline{P})^C \neq (\overline{P})^C P$ in general.

Definition 4. For any $P = P_0E_0 + P_1E_1 + P_2E_2 + P_3E_3 \in \mathbf{H}_S^{\mathbb{C}}$,

- (i) if off-diagonal entries of P are 0 then P is called a *diagonal matrix* and P is in

form of $P_0E_0 + P_1E_1$,

(ii) if $P^t = P$ then P is called a *symmetric matrix* and P is in form of $P = P_0E_0 + P_1E_1 + P_2E_2$,

(iii) if $P^t = P^{-1}$ then P is called an *orthogonal matrix* and P is in form of $P = P_0E_0 + P_3E_3$ and $\det P = 1$,

(iv) if $(\overline{P})^t = P$ then P is called a *Hermitian matrix* and P is in form of $P = P_0E_0 + P_3E_3$,

(v) if $(\overline{P})^t = P^{-1}$ then P is called an *unitary matrix* and P is in form of $P = P_0E_0 + P_1E_1 + P_2E_2$ and $\det P = 1$.

4. CONCLUSION

This paper has investigated the main properties of complex quaternion and complex split quaternion matrices with the use of 2×2 complex matrix representation of them, respectively. Then, the method of computing the determinant of given complex quaternion and complex split quaternion matrices has been proposed. Although, the determinant properties, conjugate products, special matrices of complex quaternion and complex split quaternion matrices were investigated with the similar methods, but different results were obtained due to the difference of the basis elements of these quaternion matrices.

REFERENCES

- [1] Y. Alagöz, K. H. Oral, and S. Yüce, "Split quaternion matrices." *Miskolc Math. Notes*, vol. 13, no. 2, pp. 223–232, 2012, doi: [10.18514/MMN.2012.364](https://doi.org/10.18514/MMN.2012.364).
- [2] H.-D. Ebbinghaus, H. Hermes, F. Hirzebruch, M. Koecher, K. Mainzer, J. Neukirch, A. Prestel, and R. Remmert, *Numbers. With an introduction by Klaus Lamotke. Translated by H. L. S. Orde. Edited by John H. Ewing. Paperback ed., paperback ed. ed.* New York etc.: Springer-Verlag, 1991, vol. 123, doi: [10.1007/978-1-4612-1005-4](https://doi.org/10.1007/978-1-4612-1005-4).
- [3] M. Erdoğan and M. Özdemir, "On complex split quaternion matrices." *Adv. Appl. Clifford Algebr.*, vol. 23, no. 3, pp. 625–638, 2013, doi: [10.1007/s00006-013-0399-z](https://doi.org/10.1007/s00006-013-0399-z).
- [4] K. Gürlebeck, K. Habetha, and W. Sprößig, *Application of holomorphic functions in two and higher dimensions*. Basel: Birkhäuser/Springer, 2016. doi: [10.1007/978-3-0348-0964-1](https://doi.org/10.1007/978-3-0348-0964-1).
- [5] M. L. Mehta, *Random matrices*. Boston, MA: Academic Press, Inc., 1991.
- [6] B. Rosenfeld, *Geometry of Lie groups*. Dordrecht: Kluwer Academic Publishers, 1997.
- [7] J. P. Ward, *Quaternions and Cayley numbers: algebra and applications*. Dordrecht: Kluwer Academic Publishers, 1997. doi: [10.1007/978-94-011-5768-1](https://doi.org/10.1007/978-94-011-5768-1).

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METRICAL FIXED POINT THEOREMS VIA LOCALLY FINITELY T -TRANSITIVE BINARY RELATIONS UNDER CERTAIN CONTROL FUNCTIONS

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Abstract. In this paper, we extend relation-theoretic contraction principle due to Alam and Imdad to a nonlinear contraction using a relatively weaker class of continuous control functions employing a locally finitely T -transitive binary relation, which improves the corresponding fixed point theorems especially due to: Alam and Imdad (J. Fixed Point Theory Appl. 17 (2015) 693–702), Agarwal *et al.* (Applicable Analysis, 87 (1) (2008) 109–116), Berzig and Karapinar (Fixed Point Theory Appl. 2013:205 (2013) 18 pp), Berzig *et al.* (Abstr. Appl. Anal. 2014:259768 (2014) 12 pp) and Turinici (The Sci. World J. 2014:169358 (2014) 10 pp).

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1. INTRODUCTION

The classical Banach contraction principle [6] is a pivotal result of metric fixed point theory. Several extensions of this core result are available in the existing literature of metric fixed point theory. Browder [9] extended Banach contraction principle to a class of nonlinear contractions which was later improved by Boyd and Wong [8], Mukherjea [16] and Jotić [11]. On the other hand, Ran and Reurings [19] and Nieto and Rodríguez-López [17] extended Banach contraction principle to ordered metric spaces. In this continuation, Agarwal *et al.* [1] proved some order-theoretic fixed point results under nonlinear contractions employing comparison as well as continuous control functions, which was latter refined by O'Regan and Petruşel [18].

Recently, Alam and Imdad [2, 3] obtained yet another generalization of classical Banach contraction principle using an amorphous (arbitrary) binary relation. In doing so, the authors introduced the relation-theoretic analogues of certain involved metrical notions such as: contraction, completeness, continuity *etc.* In fact, under the universal relation, all newly defined notions reduce to their corresponding natural analogues.

The aim of this paper, is to extend relation-theoretic contraction principle due to Alam and Imdad [2] to a suitable class of nonlinear contractions. In order to ascertain the existence of fixed points for linear contraction of a self-mapping T , the underlying binary relation is required to be T -closed, whereas for such type nonlinear contraction, transitivity of underlying relation is additionally required. Notice that transitivity requirement is very restrictive. With a view to employ an optimal condition of transitivity, we visited several similar results involving various types of transitive binary relations. Although, all such results are independently proved by their respective authors, yet all such results are in the same mode, which is evident from the fact that our results of this paper improve as well as unify them all.

2. PRELIMINARIES

Recall that for a given nonempty set X , a subset $\mathcal{R}$ of X^2 is called a binary relation on X . For simplicity, we sometimes write $x\mathcal{R}y$ instead of $(x, y) \in \mathcal{R}$. Given subset $E \subseteq X$ and a binary relation $\mathcal{R}$ on X , the restriction of $\mathcal{R}$ to E , denoted by $\mathcal{R}|_E$, is defined as $\mathcal{R}|_E = \mathcal{R} \cap E^2$. Indeed, $\mathcal{R}|_E$ is a relation on E induced by $\mathcal{R}$.

Out of various classes of binary relations in practice, the following ones are relevant to our presentation:

A binary relation $\mathcal{R}$ defined on a nonempty set X is called

- amorphous if it has no specific property at all,
- universal if $\mathcal{R} = X^2$,
- empty if $\mathcal{R} = \emptyset$,
- reflexive if $(x, x) \in \mathcal{R} \forall x \in X$,
- symmetric if whenever $(x, y) \in \mathcal{R}$ implies $(y, x) \in \mathcal{R}$,
- antisymmetric if whenever $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$ imply $x = y$,
- transitive if whenever $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$ imply $(x, z) \in \mathcal{R}$,
- complete if $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R} \forall x, y \in X$,
- partial order if $\mathcal{R}$ is reflexive, antisymmetric and transitive.

Throughout this paper, $\mathcal{R}$ stands for a nonempty binary relation but for the sake of simplicity, we often write ‘binary relation’ instead of ‘nonempty binary relation’. Also, $\mathbb{N}$ stands for the set of natural numbers, while $\mathbb{N}_0$ for the set of whole numbers (*i.e.*, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$).

Definition 1 ([10, 15, 22]). Let X be a nonempty set equipped with a partial order $\preceq$. A self-mapping T defined on X is called increasing (or isotone or order-preserving) if for any $x, y \in X$,

$$x \preceq y \Rightarrow T(x) \preceq T(y).$$

The following notion is formulated by using a suitable property with a view to relax the continuity requirement of the underlying mapping especially in the hypotheses of a fixed point theorem due to Nieto and Rodríguez-López [17].

Definition 2 ([5]). Let (X, d) be a metric space equipped with a partial order $\preceq$. We say that $(X, d, \preceq)$ has *ICU* (increasing-convergence-upper bound) property if every increasing convergent sequence in X is bounded above by its limit (as an upper bound).

Recall that a function $\phi : [0, \infty) \rightarrow [0, \infty)$ is called control function if $\phi(t) < t$ for each $t > 0$. The following order-theoretic fixed point theorem is a consequence of Theorem 2.3 due to Agarwal *et al.* [1] using a continuous control function.

Theorem 1. Let (X, d) be a metric space equipped with a partial order $\preceq$ and T a self-mapping on X . Suppose that the following conditions hold:

- (a) (X, d) is complete,
- (b) T is increasing,
- (c) either T is continuous or $(X, d, \preceq)$ has ICU property,
- (d) there exists a continuous control function ϕ such that
$$d(Tx, Ty) \leq \phi(d(x, y)) \quad \forall x, y \in X \text{ with } x \preceq y.$$

Then T has a fixed point.

In 2014, Berzig and Karapinar [14] introduced the idea of finitely transitive binary relation and proved some fixed point theorems in complete metric spaces endowed with a pair $(\mathcal{R}_1, \mathcal{R}_2)$ of finitely transitive binary relations under the class of $(\alpha\psi, \beta\varphi)$ -contractive conditions.

Definition 3 ([7]). Given $N \in \mathbb{N}_0$, $N \geq 2$, a binary relation $\mathcal{R}$ defined on a nonempty set X is called N -transitive if for any $x_0, x_1, \dots, x_N \in X$,

$$(x_{i-1}, x_i) \in \mathcal{R} \text{ for each } i \ (1 \leq i \leq N) \Rightarrow (x_0, x_N) \in \mathcal{R}.$$

Notice that, 2-transitivity coincides with transitivity. Following Turinici [26], $\mathcal{R}$ is called finitely transitive if it is N -transitive for some $N \geq 2$.

Definition 4 ([7]). Let X be a nonempty set and $\mathcal{R}$ a binary relation on X . We say that a mapping $T : X \rightarrow X$ is an $\mathcal{R}$ -preserving if for all $x, y \in X$,

$$(x, y) \in \mathcal{R} \Rightarrow (Tx, Ty) \in \mathcal{R}.$$

Definition 5 ([7]). Let (X, d) be a metric space and $\mathcal{R}_1$ and $\mathcal{R}_2$ two binary relations on X . We say that (X, d) is $(\mathcal{R}_1, \mathcal{R}_2)$ -regular if for every sequence $\{x_n\} \subseteq X$ such that $x_n \rightarrow x \in X$ as $n \rightarrow \infty$ and

$$(x_n, x_{n+1}) \in \mathcal{R}_1 \text{ and } (x_n, x_{n+1}) \in \mathcal{R}_2 \text{ for all } n \in \mathbb{N},$$

there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$(x_{n_k}, x) \in \mathcal{R}_1 \text{ and } (x_{n_k}, x) \in \mathcal{R}_2 \text{ for all } n \in \mathbb{N}.$$

Definition 6 ([7]). We say that a subset D of X is $(\mathcal{R}_1, \mathcal{R}_2)$ -directed if for all $x, y \in D$, there exists $z \in X$ such that

$$(x, z) \in \mathcal{R}_1, (y, z) \in \mathcal{R}_1, (x, z) \in \mathcal{R}_2, (y, z) \in \mathcal{R}_2.$$

Definition 7 ([7]). We say that the pair of functions (ψ, φ) is a pair of generalized altering distance (where $\psi, \varphi : [0, \infty) \rightarrow [0, \infty)$) if the following hypotheses hold:

- (a₁) ψ is continuous,
- (a₂) ψ is increasing,
- (a₃) $\lim_{n \rightarrow \infty} \varphi(t_n) = 0 \Rightarrow \lim_{n \rightarrow \infty} t_n = 0$.

Definition 8 ([7]). Let (X, d) be a metric space and $T : X \rightarrow X$ a given mapping. We say that T is $(\alpha\psi, \beta\varphi)$ -contractive mapping if there exists a pair of generalized altering distance (ψ, φ) such that

$$d(Tx, Ty) \leq \alpha(x, y)\psi(d(x, y)) - \beta(x, y)\varphi(d(x, y)) \quad \forall x, y \in X,$$

where $\alpha, \beta : X^2 \rightarrow [0, \infty)$.

Theorem 2 ([7]). Let (X, d) be a complete metric space and $T : X \rightarrow X$ an $(\alpha\psi, \beta\varphi)$ -contractive mapping. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two binary relations on X defined by $x, y \in X$: $(x, y) \in \mathcal{R}_1 \Leftrightarrow \alpha(x, y) \leq 1$ and $(x, y) \in \mathcal{R}_2 \Leftrightarrow \beta(x, y) \geq 1$ satisfying the following conditions:

- (i) $\mathcal{R}_1$ and $\mathcal{R}_2$ are finitely transitive,
- (ii) T is $\mathcal{R}_1$ -preserving and $\mathcal{R}_2$ -preserving,
- (iii) either T is continuous or (X, d) is $(\mathcal{R}_1, \mathcal{R}_2)$ -regular,
- (iv) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in \mathcal{R}_1$ and $(x_0, Tx_0) \in \mathcal{R}_2$.

Then T has a fixed point. Moreover, if X is $(\mathcal{R}_1, \mathcal{R}_2)$ -directed, then fixed point remains unique.

In 2014, Theorem 2 is refined and extended by Berzig *et al.* [14] employing a relatively more general $(\alpha\psi, \beta\varphi)$ -contractivity condition. Turinici [26] also extended Theorem 2 using ‘finitely transitivity’ in a local way besides generalizing the contractivity condition due to Meir-Keeler model.

Definition 9 ([26]). A binary relation $\mathcal{R}$ defined on a nonempty set X is called locally finitely transitive if for each (effectively) denumerable subset E of X , there exists $N = N(E) \geq 2$, such that $\mathcal{R}|_E$ is N -transitive.

In a book chapter, Turinici [25] discussed some fixed point results on metric spaces endowed with a locally finitely transitive binary relation. For the sake of brevity, we skip to record the fixed point theorems of Turinici contained in [25, 26] here.

The following notion originated from T -transitive subset of X^2 is essentially due to Roldán-López-de-Hierro *et al.* [20]:

Definition 10 ([3]). Let X be a nonempty set and T a self-mapping on X . A binary relation $\mathcal{R}$ defined on X is called T -transitive if for any $x, y, z \in X$,

$$(Tx, Ty), (Ty, Tz) \in \mathcal{R} \Rightarrow (Tx, Tz) \in \mathcal{R}.$$

Henceforth, the notions of ‘ T -transitivity’ and ‘locally finitely transitivity’ are not only weaker as compared to transitivity but they are also independent of each other. In order to make them compatible, we introduce the following notion of transitivity:

Definition 11. Let X be a nonempty set and T a self-mapping on X . A binary relation $\mathcal{R}$ defined on X is called locally finitely T -transitive if for each (effectively) denumerable subset E of $T(X)$, there exists $N = N(E) \geq 2$, such that $\mathcal{R}|_E$ is N -transitive.

The following result establishes the superiority of the idea of ‘locally finitely T -transitivity’ over other variants of ‘transitivity’:

Proposition 1. Let X be a nonempty set, $\mathcal{R}$ a binary relation on X and T a self-mapping on X . Then

- (i) $\mathcal{R}$ is T -transitive $\Leftrightarrow \mathcal{R}|_{T(X)}$ is transitive,
- (ii) $\mathcal{R}$ is locally finitely T -transitive $\Leftrightarrow \mathcal{R}|_{T(X)}$ is locally finitely transitive,
- (iii) $\mathcal{R}$ is transitive $\Rightarrow \mathcal{R}$ is finitely transitive $\Rightarrow \mathcal{R}$ is locally finitely transitive $\Rightarrow \mathcal{R}$ is locally finitely T -transitive,
- (iv) $\mathcal{R}$ is transitive $\Rightarrow \mathcal{R}$ is T -transitive $\Rightarrow \mathcal{R}$ is locally finitely T -transitive.

3. RELEVANT NOTIONS AND AUXILIARY RESULTS

In this section, for the sake of completeness, we summarize some relevant definitions and basic results for our subsequent discussion:

Definition 12 ([2]). Let $\mathcal{R}$ be a binary relation on a nonempty set X and $x, y \in X$. We say that x and y are $\mathcal{R}$ -comparative if either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$. We denote it by $[x, y] \in \mathcal{R}$.

Definition 13 ([13]). Let X be a nonempty set and $\mathcal{R}$ a binary relation on X .

- (1) The inverse or transpose or dual relation of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$, is defined by $\mathcal{R}^{-1} = \{(x, y) \in X^2 : (y, x) \in \mathcal{R}\}$.
- (2) The symmetric closure of $\mathcal{R}$, denoted by $\mathcal{R}^s$, is defined to be the set $\mathcal{R} \cup \mathcal{R}^{-1}$ (i.e., $\mathcal{R}^s := \mathcal{R} \cup \mathcal{R}^{-1}$). Indeed, $\mathcal{R}^s$ is the smallest symmetric relation on X containing $\mathcal{R}$.

Proposition 2 ([2]). For a binary relation $\mathcal{R}$ defined on a nonempty set X ,

$$(x, y) \in \mathcal{R}^s \iff [x, y] \in \mathcal{R}.$$

Definition 14 ([2]). Let X be a nonempty set and $\mathcal{R}$ a binary relation on X . A sequence $\{x_n\} \subset X$ is called $\mathcal{R}$ -preserving if

$$(x_n, x_{n+1}) \in \mathcal{R} \quad \forall n \in \mathbb{N}_0.$$

Definition 15 ([2]). Let X be a nonempty set and T a self-mapping on X . A binary relation $\mathcal{R}$ defined on X is called T -closed if for any $x, y \in X$,

$$(x, y) \in \mathcal{R} \Rightarrow (Tx, Ty) \in \mathcal{R}.$$

Proposition 3 ([3]). Let X be a nonempty set equipped with a binary relation $\mathcal{R}$ and T a self-mapping on X such that $\mathcal{R}$ is T -closed, then $\mathcal{R}^S$ is also T -closed.

Proposition 4 ([4]). Let X be a nonempty set, $\mathcal{R}$ a binary relation on X and T a self-mapping on X . If $\mathcal{R}$ is T -closed, then for all $n \in \mathbb{N}_0$, $\mathcal{R}$ is also T^n -closed, where T^n denotes n th iterate of T .

Definition 16 ([3]). Let (X, d) be a metric space and $\mathcal{R}$ a binary relation on X . We say that (X, d) is $\mathcal{R}$ -complete if every $\mathcal{R}$ -preserving Cauchy sequence in X converges.

Clearly, every complete metric space is $\mathcal{R}$ -complete with respect to a binary relation $\mathcal{R}$. Particularly, under the universal relation the notion of $\mathcal{R}$ -completeness coincides with usual completeness.

Definition 17 ([3]). Let (X, d) be a metric space and $\mathcal{R}$ a binary relation on X with $x \in X$. A mapping $T : X \rightarrow X$ is called $\mathcal{R}$ -continuous at x if for any $\mathcal{R}$ -preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, we have $T(x_n) \xrightarrow{d} T(x)$. Moreover, T is called $\mathcal{R}$ -continuous if it is $\mathcal{R}$ -continuous at each point of X .

Clearly, every continuous mapping is $\mathcal{R}$ -continuous, for any binary relation $\mathcal{R}$. Particularly, under the universal relation the notion of $\mathcal{R}$ -continuity coincides with usual continuity.

The following notion is a generalization of d -self-closedness of a partial order relation ($\preceq$) contained in Turinici [23, 24]:

Definition 18 ([2]). Let (X, d) be a metric space. A binary relation $\mathcal{R}$ defined on X is called d -self-closed if for any $\mathcal{R}$ -preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x] \in \mathcal{R} \quad \forall k \in \mathbb{N}_0$.

Definition 19 ([21]). Let X be a nonempty set and $\mathcal{R}$ a binary relation on X . A subset E of X is called $\mathcal{R}$ -directed if for each $x, y \in E$, there exists $z \in X$ such that $(x, z) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$.

Definition 20 ([12]). Let X be a nonempty set and $\mathcal{R}$ a binary relation on X . For $x, y \in X$, a path of length k (where k is a natural number) in $\mathcal{R}$ from x to y is a finite sequence $\{z_0, z_1, z_2, \dots, z_k\} \subset X$ satisfying the following:

- (i) $z_0 = x$ and $z_k = y$,
- (ii) $(z_i, z_{i+1}) \in \mathcal{R}$ for each i ($0 \leq i \leq k-1$).

Notice that a path of length k involves $k+1$ elements of X , although they are not necessarily distinct.

Definition 21 ([3]). Let X be a nonempty set and $\mathcal{R}$ a binary relation on X . A subset E of X is called $\mathcal{R}$ -connected if for each pair $x, y \in E$, there exists a path in $\mathcal{R}$ from x to y .

Given a binary relation $\mathcal{R}$ and a self-mapping T on a nonempty set X , we use the following notations:

- (i) $F(T)$ = the set of all fixed points of T ,
- (ii) $X(T, \mathcal{R}) := \{x \in X : (x, Tx) \in \mathcal{R}\}$.

The following result is a relation-theoretic version of Banach contraction principle:

Theorem 3 ([2, 3]). *Let (X, d) be a metric space, $\mathcal{R}$ a binary relation on X and T a self-mapping on X . Suppose that the following conditions hold:*

- (a) (X, d) is $\mathcal{R}$ -complete,
- (b) $\mathcal{R}$ is T -closed,
- (c) either T is $\mathcal{R}$ -continuous or $\mathcal{R}$ is d -self-closed,
- (d) $X(T, \mathcal{R})$ is nonempty,
- (e) there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y) \quad \forall x, y \in X \text{ with } (x, y) \in \mathcal{R}.$$

Then T has a fixed point. Moreover, if X is $\mathcal{R}^S$ -connected, then T has a unique fixed point.

Consider the following family of control functions:

$$\Phi = \{\phi : [0, \infty) \rightarrow [0, \infty) : \phi(t) < t \text{ for each } t > 0 \text{ and } \limsup_{r \rightarrow t} \phi(r) < t \text{ for each } t > 0\}.$$

Proposition 5. *If (X, d) is a metric space, $\mathcal{R}$ is a binary relation on X , T is a self-mapping on X and $\phi \in \Phi$, then the following contractivity conditions are equivalent:*

- (I) $d(Tx, Ty) \leq \phi(d(x, y)) \quad \forall x, y \in X \text{ with } (x, y) \in \mathcal{R}$,
- (II) $d(Tx, Ty) \leq \phi(d(x, y)) \quad \forall x, y \in X \text{ with } [x, y] \in \mathcal{R}$.

We skip the proof of above proposition as it is similar to that of Proposition 2.3 in [2].

Proposition 6 ([5]). *Let $\phi \in \Phi$. If $\{t_n\} \subset (0, \infty)$ is a sequence such that $t_{n+1} \leq \phi(t_n) \quad \forall n \in \mathbb{N}_0$, then $\lim_{n \rightarrow \infty} t_n = 0$.*

Finally, we record the following two known results, which are needed in the proof of our main results:

Lemma 1 ([14]). *Let (X, d) be a metric space and $\{x_n\}$ a sequence in X . If $\{x_n\}$ is not a Cauchy sequence, then there exist $\epsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that*

- (i) $k \leq m_k < n_k \quad \forall k \in \mathbb{N}$,
- (ii) $d(x_{m_k}, x_{n_k}) \geq \epsilon$,
- (iii) $d(x_{m_k}, x_{p_k}) < \epsilon \quad \forall p_k \in \{m_k + 1, m_k + 2, \dots, n_k - 2, n_k - 1\}$.

In addition to this, if $\{x_n\}$ also verifies $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$, then

$$\lim_{k \rightarrow \infty} d(x_{m_k}, x_{n_k+p}) = \epsilon \quad \forall p \in \mathbb{N}_0.$$

Lemma 2 ([26]). *Let X be a nonempty set, $\mathcal{R}$ a binary relation on X and $\{z_n\}$ is an $\mathcal{R}$ -preserving sequence in X . If $\mathcal{R}$ is a N -transitive on $Z := \{z_n : n \in \mathbb{N}_0\}$ for some natural number $N \geq 2$, then*

$$(z_n, z_{n+1+r(N-1)}) \in \mathcal{R} \quad \forall n, r \in \mathbb{N}_0.$$

4. MAIN RESULTS

Firstly, we prove a result on the existence of fixed points for the class Φ (described earlier) of nonlinear contractions employing a locally finitely T -transitive binary relation, which runs as follows:

Theorem 4. *Let (X, d) be a metric space equipped with a binary relation $\mathcal{R}$ and T a self-mapping on X . Suppose that the following conditions hold:*

- (a) (X, d) is $\mathcal{R}$ -complete,
- (b) $\mathcal{R}$ is T -closed and locally finitely T -transitive,
- (c) either T is $\mathcal{R}$ -continuous or $\mathcal{R}$ is d -self-closed,
- (d) $X(T, \mathcal{R})$ is nonempty,
- (e) there exists $\phi \in \Phi$ such that
$$d(Tx, Ty) \leq \phi(d(x, y)) \quad \forall x, y \in X \text{ with } (x, y) \in \mathcal{R}.$$

Then T has a fixed point.

Proof. As $X(T, \mathcal{R})$ is nonempty, one can choose $x_0 \in X(T, \mathcal{R})$. Construct a sequence $\{x_n\}$ of Picard iteration based at the initial point x_0 , i.e.,

$$x_n = T^n(x_0) \quad \forall n \in \mathbb{N}_0. \quad (4.1)$$

As $(x_0, Tx_0) \in \mathcal{R}$, using T -closedness of $\mathcal{R}$ and Proposition 4, we get

$$(T^n x_0, T^{n+1} x_0) \in \mathcal{R}$$

so that

$$(x_n, x_{n+1}) \in \mathcal{R} \quad \forall n \in \mathbb{N}_0. \quad (4.2)$$

Therefore the sequence $\{x_n\}$ is $\mathcal{R}$ -preserving. Now, if $d(x_{n_0+1}, x_{n_0}) = 0$ for some $n_0 \in \mathbb{N}_0$, then in view of (4.1), we have

$$T(x_{n_0}) = x_{n_0}$$

so that x_{n_0} is a fixed point of T and hence we are done.

On the other hand, if $d(x_{n+1}, x_n) > 0 \quad \forall n \in \mathbb{N}_0$, then applying the contractivity condition (e) to (4.2), we deduce, for all $n \in \mathbb{N}_0$ that

$$d(x_{n+2}, x_{n+1}) \leq \phi(d(x_{n+1}, x_n)). \quad (4.3)$$

Using (4.3) and Proposition 6, we have

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \quad (4.4)$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. In fact, suppose that $\{x_n\}$ is not Cauchy. Therefore, owing to Lemma 1, there exist $\epsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that $k \leq m_k < n_k$, $d(x_{m_k}, x_{n_k}) \geq \epsilon$ and $d(x_{m_k}, x_{p_k}) < \epsilon$ where $p_k \in \{m_k + 1, m_k + 2, \dots, n_k - 2, n_k - 1\}$. Further, in view of (4.4) and Lemma 1, we infer

$$\lim_{k \rightarrow \infty} d(x_{m_k}, x_{n_k+p}) = \epsilon \quad \forall p \in \mathbb{N}_0. \quad (4.5)$$

In view of (4.1), $\{x_n\} \subset T(X)$ and hence the range $E := \{x_n : n \in \mathbb{N}_0\}$ (of the sequence $\{x_n\}$) is a denumerable subset of $T(X)$. Hence by locally finitely T -transitivity of $\mathcal{R}$, there exists a natural number $N = N(E) \geq 2$, such that $\mathcal{R}|_E$ is N -transitive.

As $m_k < n_k$ and $N - 1 > 0$, we have by the Division Rule

$$\begin{aligned} n_k - m_k &= (N - 1)(\mu_k - 1) + (N - \eta_k) \\ \mu_k - 1 &\geq 0, \quad 0 \leq N - \eta_k < N - 1 \\ \iff \begin{cases} n_k + \eta_k = m_k + 1 + (N - 1)\mu_k \\ \mu_k \geq 1, \quad 1 < \eta_k \leq N. \end{cases} \end{aligned}$$

Here μ_k and η_k are suitable natural numbers such that η_k can assume any finite positive integral value in interval $(1, N]$. Hence, without loss of generality, we can choose subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ (satisfying (4.5)) such that η_k remains constant, say η , which is independent of k . Write

$$m'_k = n_k + \eta = m_k + 1 + (N - 1)\mu_k \quad (4.6)$$

where η ($1 < \eta \leq N$) is constant.

Owing to (4.5) and (4.6), we obtain

$$\lim_{k \rightarrow \infty} d(x_{m_k}, x_{m'_k}) = \lim_{k \rightarrow \infty} d(x_{m_k}, x_{n_k + \eta}) = \epsilon. \quad (4.7)$$

Using triangular inequality, we have

$$\begin{aligned} d(x_{m_k+1}, x_{m'_k+1}) &\leq d(x_{m_k+1}, x_{m_k}) + d(x_{m_k}, x_{m'_k}) + d(x_{m'_k}, x_{m'_k+1}) \quad \text{and} \\ d(x_{m_k}, x_{m'_k}) &\leq d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{m'_k+1}) + d(x_{m'_k+1}, x_{m'_k}), \end{aligned}$$

therefore, we have

$$\begin{aligned} d(x_{m_k}, x_{m'_k}) - d(x_{m_k}, x_{m_k+1}) - d(x_{m'_k+1}, x_{m'_k}) &\leq d(x_{m_k+1}, x_{m'_k+1}) \\ &\leq d(x_{m_k+1}, x_{m_k}) + d(x_{m_k}, x_{m'_k}) + d(x_{m'_k}, x_{m'_k+1}) \end{aligned}$$

which on letting $k \rightarrow \infty$ and using (4.4) and (4.7), gives rise

$$\lim_{k \rightarrow \infty} d(x_{m_k+1}, x_{m'_k+1}) = \epsilon. \quad (4.8)$$

In view (4.6) and Lemma 2, we have $(x_{m_k}, x_{m'_k}) \in \mathcal{R}$. Denote $\delta_k = d(x_{m_k}, x_{m'_k})$. Now, owing to (4.1) and assumption (e), we have

$$\begin{aligned} d(x_{m_k+1}, x_{m'_k+1}) &= d(Tx_{m_k}, Tx_{m'_k}) \\ &\leq \phi(d(x_{m_k}, x_{m'_k})) \\ &= \phi(\delta_k) \end{aligned}$$

so that

$$d(x_{m_k+1}, x_{m'_k+1}) \leq \phi(\delta_k). \quad (4.9)$$

Utilizing $\delta_k \rightarrow \epsilon$ as $k \rightarrow \infty$ (in view of (4.7)) and the definition of Φ , we have

$$\limsup_{k \rightarrow \infty} \phi(\delta_k) = \limsup_{t \rightarrow \epsilon} \phi(t) < \epsilon. \quad (4.10)$$

On taking limit superior as $k \rightarrow \infty$ in (4.9) besides using (4.8) and (4.10), we obtain

$$\epsilon = \limsup_{k \rightarrow \infty} d(x_{m_k+1}, x_{m'_k+1}) \leq \limsup_{k \rightarrow \infty} \phi(\delta_k) < \epsilon,$$

which is a contradiction so that the sequence $\{x_n\}$ is Cauchy. Hence, $\{x_n\}$ is an $\mathcal{R}$ -preserving Cauchy sequence. By $\mathcal{R}$ -completeness of (X, d) , $\exists x \in X$ such that $x_n \xrightarrow{d} x$.

Finally, we claim that x is a fixed point of T . To substantiate this, suppose that T is $\mathcal{R}$ -continuous. As $\{x_n\}$ is $\mathcal{R}$ -preserving with $x_n \xrightarrow{d} x$, $\mathcal{R}$ -continuity of T implies that $x_{n+1} = T(x_n) \xrightarrow{d} T(x)$. Using the uniqueness of limit, we obtain $T(x) = x$, i.e., x is a fixed point of T .

Alternately, assume that $\mathcal{R}$ is d -self-closed. As $\{x_n\}$ is $\mathcal{R}$ -preserving such that $x_n \xrightarrow{d} x$, the d -self-closedness of $\mathcal{R}$ guarantees the existence of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x] \in \mathcal{R} \quad (\forall k \in \mathbb{N}_0)$. On using assumption (e), Proposition 5 and $[x_{n_k}, x] \in \mathcal{R}$, we have

$$d(x_{n_k+1}, Tx) = d(Tx_{n_k}, Tx) \leq \phi(d(x_{n_k}, x)) \quad \forall k \in \mathbb{N}_0.$$

We assert that

$$d(x_{n_k+1}, Tx) \leq d(x_{n_k}, x) \quad \forall k \in \mathbb{N}.$$

On account of two different possibilities occurring here, we consider a partition of $\mathbb{N}$ i.e., $\mathbb{N}^0 \cup \mathbb{N}^+ = \mathbb{N}$ and $\mathbb{N}^0 \cap \mathbb{N}^+ = \emptyset$ verifying that

- (i) $d(x_{n_k}, x) = 0 \quad \forall k \in \mathbb{N}^0$,
- (ii) $d(x_{n_k}, x) > 0 \quad \forall k \in \mathbb{N}^+$.

In case (i), we have $d(Tx_{n_k}, Tx) = 0 \leq d(x_{n_k}, x) \quad \forall k \in \mathbb{N}^0$. In case (ii), and definition of Φ , we have $d(x_{n_k+1}, Tx) \leq \phi(d(x_{n_k}, x)) < d(x_{n_k}, x)$ for all $k \in \mathbb{N}^+$. Hence in both the cases, we get $d(x_{n_k+1}, Tx) \leq d(x_{n_k}, x) \quad \forall k \in \mathbb{N}$, which by using the fact that $x_{n_k} \xrightarrow{d} x$ as $k \rightarrow \infty$, yields that $x_{n_k+1} \xrightarrow{d} T(x)$. Again, owing to

the uniqueness of limit, we obtain $T(x) = x$ so that x is a fixed point of T . This completes the proof. $\square$

Remark 1. Clearly, $\phi \in \Phi$ utilized in Theorem 4 can be alternatively replaced by the following ones:

- (i) ϕ is a continuous control function,
- (ii) ϕ is an upper semi continuous control function,
- (iii) ϕ is a control function satisfying $\lim_{r \rightarrow t} \phi(r) < t$ for each $t > 0$.

In view of above remark, that Theorem 4 extends Theorem 1 (by taking $\mathcal{R} = \preceq$, partial order).

In view of Proposition 1, we obtain the following consequence of Theorem 4.

Corollary 1. *Theorem 4 remains true if locally finitely T -transitivity of $\mathcal{R}$ (utilized in assumption (b)) is replaced by any one of the following conditions (besides retaining rest of the hypotheses):*

- (i) $\mathcal{R}$ is transitive,
- (ii) $\mathcal{R}$ is T -transitive,
- (iii) $\mathcal{R}$ is finitely transitive,
- (iv) $\mathcal{R}$ is locally finitely transitive.

Now, we establish a uniqueness result corresponding to Theorem 4.

Theorem 5. *In addition to the hypotheses of Theorem 4, suppose that the following condition holds:*

- (u) : $T(X)$ is $\mathcal{R}^s$ -connected.

Then T has a unique fixed point.

Proof. In view of Theorem 4, $F(T) \neq \emptyset$. Take $x, y \in F(T)$, then for all $n \in \mathbb{N}_0$, we have

$$T^n(x) = x \text{ and } T^n(y) = y. \quad (4.11)$$

By assumption (u), there exists a path (say $\{z_0, z_1, z_2, \dots, z_k\}$) of some finite length k in $\mathcal{R}^s$ from x to y so that

$$z_0 = x, z_k = y \text{ and } [z_i, z_{i+1}] \in \mathcal{R} \text{ for each } i \ (0 \leq i \leq k-1). \quad (4.12)$$

As $\mathcal{R}$ is T -closed, using Propositions 3 and 4, we have

$$[T^n z_i, T^n z_{i+1}] \in \mathcal{R} \text{ for each } i \ (0 \leq i \leq k-1) \text{ and for each } n \in \mathbb{N}_0. \quad (4.13)$$

Now, for each $n \in \mathbb{N}_0$ and for each $i \ (0 \leq i \leq k-1)$, write $t_n^i := d(T^n z_i, T^n z_{i+1})$. We assert that

$$\lim_{n \rightarrow \infty} t_n^i = 0. \quad (4.14)$$

With i fixed, we distinguish two cases. Firstly, suppose that $t_{n_0}^i = d(T^{n_0}z_i, T^{n_0}z_{i+1}) = 0$ for some $n_0 \in \mathbb{N}_0$, i.e., $T^{n_0}(z_i) = T^{n_0}(z_{i+1})$, which implies that $T^{n_0+1}(z_i) = T^{n_0+1}(z_{i+1})$. Consequently, we get $t_{n_0+1}^i = d(T^{n_0+1}z_i, T^{n_0+1}z_{i+1}) = 0$. Thus by induction on n , we get $t_n^i = 0 \forall n \geq n_0$, so that $\lim_{n \rightarrow \infty} t_n^i = 0$. Secondly, suppose that $t_n^i > 0 \forall n \in \mathbb{N}_0$. Then on using (4.13), assumption (e) and Proposition 5, we have

$$\begin{aligned} t_{n+1}^i &= d(T^{n+1}z_i, T^{n+1}z_{i+1}) \\ &\leq \phi(d(T^n z_i, T^n z_{i+1})) \\ &= \phi(t_n^i) \end{aligned}$$

so that

$$t_{n+1}^i \leq \phi(t_n^i). \quad (4.15)$$

Using (4.15) and Proposition 6, we have

$$\lim_{n \rightarrow \infty} t_n^i = 0, \text{ for each } i \ (0 \leq i \leq k-1).$$

Hence in both the cases, (4.14) is proved. Making use of (4.11), (4.12), (4.14) and the triangular inequality, we have

$$d(x, y) = d(T^n z_0, T^n z_k) \leq t_n^0 + t_n^1 + \cdots + t_n^{k-1} \rightarrow 0 \text{ as } n \rightarrow \infty$$

so that $x = y$. Hence T has a unique fixed point. $\square$

Corollary 2. *Theorem 5 remains true if we replace condition (u) by one of the following conditions (besides retaining rest of the hypotheses):*

- (u') $\mathcal{R}|_{T(X)}$ is complete,
- (u'') $T(X)$ is $\mathcal{R}^s$ -directed.

A proof of above corollary can be outlined on the lines of the proof of Corollary 3.4 contained in [4].

The following example is adopted to exhibit and substantiate the utility of Theorems 4 and 5 over corresponding earlier known results.

Example 1. Consider $X = [0, \infty)$ equipped with usual metric d . Define a mapping $T : X \rightarrow X$ by $T(x) = \frac{x}{x+1} \forall x \in X$. Let $\mathcal{R} := \{(x, y) \in X^2 : x - y > 0\}$, then $\mathcal{R}$ is locally finitely T -transitive binary relation on X . Clearly X is $\mathcal{R}$ -complete and $\mathcal{R}$ is T -closed. Now, define a control function ϕ by $\phi(t) = \frac{t}{t+1} \forall t \in [0, \infty)$, clearly $\phi \in \Phi$. Now, for all $(x, y) \in \mathcal{R}$, we have

$$\begin{aligned} d(Tx, Ty) &= \left| \frac{x}{x+1} - \frac{y}{y+1} \right| = \left| \frac{x-y}{1+x+y+xy} \right| \\ &\leq \frac{x-y}{1+(x-y)} = \frac{d(x, y)}{1+d(x, y)} = \phi(d(x, y)). \end{aligned}$$

Hence T and ϕ satisfy the assumption (e) of Theorem 5. It is easy to see that, rest of the conditions of Theorem 5 are also satisfied and T has a unique fixed point (namely: $x = 0$). As the relation $\mathcal{R}$ is not partial order and also T is not a linear contraction, Example 1 can not be covered by Theorems 1 and 3, which demonstrate the utility of our newly proved results.

Now, we deduce some special cases, which are well known fixed point theorems of the existing literature.

- (1) Under the universal relation (*i.e.*, $\mathcal{R} = X^2$), Theorem 5 deduces to the classical fixed point theorem under nonlinear contraction.
- (2) Taking $\phi(t) = \alpha t$ (where $\alpha \in [0, 1)$), we obtain Theorem 3. In this case, the requirement of locally finitely T -transitivity on a binary relation is not necessary.
- (3) Putting $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$ (where $\mathcal{R}_1$ and $\mathcal{R}_2$ are finitely transitive) in Theorem 4, we obtain an analogue of Theorem 2 (under the class Φ). Notice that T -closedness of $\mathcal{R}$ is equivalent to $\mathcal{R}$ -preserving property of T and d -self-closedness of $\mathcal{R}$ is slightly weaker than $(\mathcal{R}_1, \mathcal{R}_2)$ -regularity of (X, d) .
- (4) Setting $\mathcal{R}$ to be locally finitely transitive binary relation in Theorem 4, we obtain an analogue of a fixed point theorem of Turinici [26] for the nonlinear class Φ .

In an attempt to extend Theorem 3 from linear contractions to a nonlinear contractions (under the class Φ), we additionally do require locally finitely T -transitivity of $\mathcal{R}$ (optimal condition), which substantiates the utility of the present extension. For possible problems, readers may attempt to prove such results for other type of nonlinear contractions.

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REFERENCES

- [1] R. P. Agarwal, M. A. El-Gebeily, and D. O'Regan, "Generalized contractions in partially ordered metric spaces," *Applicable Analysis*, vol. 87, no. 1, pp. 109–116, 2008.
- [2] A. Alam and M. Imdad, "Relation-theoretic contraction principle." *J. Fixed Point Theory Appl.*, vol. 17, no. 4, pp. 693–702, 2015.
- [3] A. Alam and M. Imdad, "Relation-theoretic metrical coincidence theorems." *Filomat*, vol. 31, no. 14, pp. 4421–4439, 2017.
- [4] A. Alam and M. Imdad, "Nonlinear contractions in metric spaces under locally T-transitive binary relations." *Fixed Point Theory*, vol. 19, no. 1, pp. 13–24, 2018.

- [5] A. Alam, A. R. Khan, and M. Imdad, "Some coincidence theorems for generalized nonlinear contractions in ordered metric spaces with applications." *Fixed Point Theory Appl.*, vol. 2014, 2014, art.no. 216, 30pp.
- [6] S. Banach, "Sur les operations dans les ensembles abstraits et leur application aux equations integrales." *Fund. Math.*, vol. 3, no. 1, pp. 133–181, 1922.
- [7] M. Berzig and E. Karapinar, "Fixed point results for $(\alpha\psi, \beta\varphi)$ - contractive mappings for a generalized altering distance." *Fixed Point Theory Appl.*, vol. 2013, 2013, art. no 205, 18pp.
- [8] D. W. Boyd and J. S. W. Wong, "On nonlinear contractions." *Proc. Amer. Math. Soc.*, vol. 30, no. 27, 1969, 25pp.
- [9] F. E. Browder, "On the convergence of successive approximations for nonlinear functional equations." *Proc. K. Ned. Akad. Wet. Ser. A, Indag. Math.*, vol. 71, pp. 27–35, 1968.
- [10] R. DeMarr, "Common fixed points for isotone mappings." *Colloq. Math.*, vol. 13, no. 1, pp. 45–48, 1964.
- [11] N. Jotić, "Some fixed point theorems in metric spaces." *Indian J. Pure Appl. Math.*, vol. 26, pp. 947–952, 1995.
- [12] B. Kolman, R. C. Busby, and S. Ross, "Discrete mathematical structures." *Third Edition, PHI Pvt. Ltd., New Delhi.*, 2000.
- [13] S. Lipschutz, *Schaum's outlines of theory and problems of set theory and related topics.* New York: McGraw-Hill, New York, 1964.
- [14] M. Berzig, E. Karapinar, and A. Roldán, "Discussion on generalized- $(\alpha\psi, \beta\varphi)$ -contractive mappings via generalized altering distance function and related fixed point theorems." *Abstr. Appl. Anal.*, vol. 2014, 2014, art. no.259768, 12pp.
- [15] E. J. McShane, "Order-preserving maps and integration processes." *Ann. Math. St. 31, Princeton University Press, Princeton.*, vol. 31, 1953.
- [16] A. Mukherjea, "Contractions and completely continuous mappings." *Nonlinear Analysis: Theory, Methods & Applications.*, vol. 1, no. 3, pp. 235–247, 1977.
- [17] J. J. Nieto and R. Rodríguez-López, "Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations." *Order*, vol. 22, no. 3, pp. 223–239, 2005.
- [18] D. O'Regan and A. Petruşel, "Fixed point theorems for generalized contractions in ordered metric spaces." *J. Math. Anal. Appl.*, vol. 341, no. 2, pp. 1241–1252, 2008.
- [19] A. Ran and M. Reurings, "A fixed point theorem in partially ordered sets and some applications to matrix equations." *Proc. Am. Math. Soc.*, vol. 132, no. 5, pp. 1435–1443, 2004.
- [20] A. F. Roldán-López-de-Hierro, E. Karapinar, and M. de-la-Sen, "Coincidence point theorems in quasi-metric spaces without assuming the mixed monotone property and consequences in G-metric spaces." *Fixed Point Theory Appl.*, vol. 2014, 2014, art. no. 184, 29pp.
- [21] B. Samet and M. Turinici, "Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications." *Commun. Math. Anal.*, vol. 13, no. 2, pp. 82–97, 2012.
- [22] A. Tarski, "A lattice-theoretical fixpoint theorem and its applications." *Pacific J. Math.*, vol. 5, no. 2, pp. 285–309, 1955.
- [23] M. Turinici, "Product fixed points in ordered metric spaces." *arXiv:1110.3079v1.*, 2011.
- [24] M. Turinici, "Linear contractions in product ordered metric spaces." *Ann Univ Ferrara.*, vol. 59, no. 1, pp. 187–198, 2013.
- [25] M. Turinici, "Contractive operators in relational metric spaces." *Handbook of Functional Equations, Springer Optimization and its Applications, Springer*, vol. 95, pp. 419–458, 2014.
- [26] M. Turinici, "Contractive maps in locally transitive relational metric spaces." *The Sci. World J.*, vol. 2014, 2014, art. no. 169358, 10pp.

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ON THE VOLTERRA DELAY-INTEGRO-DIFFERENTIAL EQUATION WITH LAYER BEHAVIOR AND ITS NUMERICAL SOLUTION

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Abstract. In this paper, we analyze the convergence of the fitted mesh method applied to singularly perturbed Volterra delay-integro-differential equation. Our mesh comprises a special nonuniform mesh on the first subinterval and uniform mesh on another part. Error estimates are obtained using difference analogue of Gronwall's inequality with delay. A numerical test that confirms the theoretical results is presented.

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1. INTRODUCTION

Volterra delay-integro-differential equations (VDIDEs) have a major influence on the field of science such as ecology, medicine, physics, biology and so on [4, 6, 14]. These equations play a significant role in modelling of some phenomena in engineering and sciences, and hence have led researchers to develop a theory and numerical computation and analysis for VDIDEs.

Here we shall concerned with the development of fitted difference method for singularly perturbed Volterra delay-integro-differential equation (SPVDIDE):

$$Lu := \varepsilon u' + a(t)u + \int_{t-r}^t K(t,s)u(s)ds = f(t), \quad t \in I, \quad (1.1)$$

subject to

$$u(t) = \varphi(t), \quad t \in I_0, \quad (1.2)$$

where $I = (0, T] = \bigcup_{p=1}^m I_p$, $I_p = \{t : r_{p-1} < t \leq r_p\}$, $1 \leq p \leq m$ and $r_s = sr$, for $0 \leq s \leq m$, $\bar{I} = [0, T]$ and $I_0 = [-r, 0]$. $\varepsilon \in (0, 1]$ is the perturbation parameter and r is a constant delay, which is independent of ε . $a(t) \geq \alpha > 0$, $f(t)$ ($t \in \bar{I}$), $\varphi(t)$

$(t \in I_0)$ and $K(t, s)$ $((t, s) \in \bar{I} \times \bar{I})$ are assumed to be sufficiently smooth functions such that the solution, $u(t)$, has initial layer at $t = 0$ for small values of ε .

Singularly perturbed differential equations are typically characterized by a small parameter ε multiplying some or all of the highest order terms in the differential equation. In general, the solutions of such equations exhibit multiscale phenomena. Within certain thin subregions of the domain, the scale of some derivatives is significantly larger than other derivatives. These thin regions of rapid change are called, boundary or interior layers, as appropriate. Such type of equations occur frequently in mathematical problems in the sciences and engineering for example, in fluid flow at high Reynold number, electrical networks, chemical reactions, control theory, the equations governing flow in porous media, the drift-diffusion equations of semi-conductor device physics, and other physical models [5, 16, 17]. It is well-known that standard discretization methods do not work well for these problems as they often produce oscillatory solutions which are inaccurate if the perturbed parameter ε is small. To obtain robust numerical methods it is necessary to fit the coefficients (fitted operator methods) or the mesh (fitted mesh methods) to the behavior of the exact solution [1, 2, 5, 10, 15, 19, 22] (see also references cited in them). For a survey of early results in the theoretical analysis of singularly perturbed Volterra integro-differential equations (VIDEs) and in the numerical analysis and implementation of various techniques for these problems we refer to the book [11]. An analysis of approximate methods when applied to singularly perturbed VIDEs can also be found in [12, 18, 20].

In the last few years, a considerable amount of effort has been devoted to the numerical solution of VIDEs. An overview of the approximate methods for VIDEs may be obtained from [3, 7, 9, 21, 24, 25].

The above mentioned papers, related to VIDEs were only concerned with the regular cases, i.e., in the absence of initial/boundary layers. SPVIDEs also frequently arise in many scientific applications. Wu and Gan [23] investigated error behaviour of linear multistep methods applied to SPVIDEs and derived global error estimates $A(\alpha)$ —stable linear multistep methods with convergent quadrature rule. He and Xu [8] discussed the exponential stability of impulsive SPVIDEs. Amiraliyev and Yilmaz [2] gave an exponentially fitted difference method on a uniform mesh for (1.1)-(1.2) except for a delay term in differential part and shown that the method is first-order convergent uniformly in ε . A useful discussion of uniform convergence on a fitted mesh, for another form of SPVIDEs have been investigated in [13].

In this paper, the main goal is to give a finite difference scheme on a special piecewise uniform mesh on the first subinterval and uniform mesh on the rest of the interval for the numerical solution of the initial-value problem (1.1)-(1.2). The construction of the difference scheme is based on the method of integral identities by using appropriate interpolating quadrature rules with remainder term in integral form. In Section 2, we construct the finite difference discretization and introduce a special piecewise

uniform mesh. In Section 3, the error analysis for the approximate solution is presented and the method is shown theoretically to be uniformly (with respect to the singular perturbation parameter) convergent. Numerical results are given in Section 4 which validate that in practice the method is first order uniformly accurate. The paper ends with a summary of the main conclusions.

Notation: Throughout the paper we use C to denote a generic positive constant independent of the perturbation parameter and mesh.

Assumption: We also will assume that $\varepsilon \leq CN^{-1}$, as is generally the case in practise.

2. THE MESH AND DIFFERENCE SCHEME

In this section, a special piecewise uniform mesh on $[0, T]$ is introduced and the difference scheme consisting of upwind type scheme for differential part and a composite rectangle integration of explicit type for integral part is presented for discretizing (1.1)-(1.2).

Before presenting the numerical method for the solution of (1.1)-(1.2) we need the asymptotic estimates for its differential solution.

Lemma 1. For $a, f \in C^1[0, T]$,

$$\left| \frac{\partial}{\partial t} K(t, s) \right| \leq M_0 < \infty,$$

the solution of (1.1)-(1.2) satisfies

$$\begin{aligned} \|u\|_\infty &\leq C_0, \\ |u'(t)| &\leq C \left\{ 1 + \frac{1}{\varepsilon} e^{-\frac{\alpha t}{\varepsilon}} \right\}, \quad 0 \leq t \leq T, \end{aligned}$$

where

$$\begin{aligned} C_0 &= (|\varphi(0)| + \alpha^{-1} \bar{K} \|\varphi\|_{1,0} + \alpha^{-1} \|f\|_\infty) e^{-\alpha^{-1} \bar{K} T}, \\ \bar{K} &= \max_{\bar{I} \times \bar{I}} |K(t, s)|, \\ \|\varphi\|_{1,0} &= \int_{-r}^0 |\varphi(t)| dt. \end{aligned}$$

The proof is being established in the analogous way as in [2, 13].

Our nonuniform mesh

$$\bar{\omega} = \{0 = t_0 < t_1 < \dots < t_{N_0-1} < t_{N_0} = T; h_i = t_i - t_{i-1}\},$$

which consists of m submeshes $\omega_{N,p}$ ($1 \leq p \leq m$), for an even number N , organized as follows

$$\omega = \bigcup_{p=1}^m \omega_{N,p},$$

$$\omega_{N,1} = \left\{ t_i = i h^{(1)}, i = 1, 2, \dots, N/2, h^{(1)} = \frac{2\sigma}{N}, t_i = \sigma + (i - \frac{N}{2})h^{(2)}, \right. \\ \left. i = \frac{N}{2} + 1, \dots, \frac{3N}{2}, h^{(2)} = \frac{r - \sigma}{N} \right\},$$

$$\omega_{N,p} = \left\{ t_i = r_{p-1} + \left(i - (p - \frac{1}{2})N \right) h^{(3)}, i = (p - \frac{1}{2})N + 1, \dots, (p + \frac{1}{2})N, \right. \\ \left. h^{(3)} = \frac{r}{N} \right\}, \quad 2 \leq p \leq m, r_s = sr \quad (0 \leq s \leq m)$$

with transition point $\sigma = \alpha^{-1} \varepsilon \ln N$. Thus the mesh ω contains $\frac{3N}{2}$ points on $\omega_{N,1}$ and N points on each subinterval $\omega_{N,p}$ ($2 \leq p \leq m$) with total number $N_0 = (m + \frac{1}{2})N$.

The problem (1.1)-(1.2) is discretized using the relation

$$h_i^{-1} \int_{t_{i-1}}^{t_i} Lu(t)dt = h_i^{-1} \int_{t_{i-1}}^{t_i} f(t)dt, \quad 1 \leq i \leq N_0 \quad (2.1)$$

followed by the application appropriate quadrature rules.

Namely, applying right side rectangle rule in (2.1), analogous to [13] we get

$$h_i^{-1} \int_{t_{i-1}}^{t_i} [\varepsilon u'(t) + a(t)u(t) - f(t)]dt = \varepsilon u_{i,i} + a_i u_i - f_i + R_i^{(1)}$$

with the truncation error

$$R_i^{(1)} = -h_i^{-1} \int_{t_{i-1}}^{t_i} (t - t_{i-1}) \frac{d}{dt} [a(t)u(t) - f(t)]dt. \quad (2.2)$$

For the remain part in (2.1), after using right side rectangle rule, we have

$$h_i^{-1} \int_{t_{i-1}}^{t_i} \left(\int_{t-r}^t K(t,s)u(s)ds \right) dt = \int_{t_i-r}^{t_i} K(t_i,s)u(s)ds + R_i^{(2)},$$

with

$$R_i^{(2)} = -h_i^{-1} \int_{t_{i-1}}^{t_i} (\xi - t_{i-1}) \left(\frac{d}{d\xi} \int_{\xi-r}^{\xi} K(\xi,s)u(s)ds \right) d\xi. \quad (2.3)$$

Also applying the composite left side rectangle rule, we obtain

$$\int_{t_i-r}^{t_i} K(t_i, s)u(s)ds = \sum_{j=i-\bar{N}+1}^i h_j K(t_i, t_{j-1})u_{j-1} + R_i^{(3)},$$

with

$$\bar{N} = \begin{cases} \frac{3N}{2}, & 1 \leq i \leq \frac{3N}{2} \\ N, & i > \frac{3N}{2} \end{cases}$$

$$R_i^{(3)} = \sum_{j=i-\bar{N}+1}^i \int_{t_{j-1}}^{t_j} (t_j - \xi) \frac{d}{ds} K(t_i, \xi) u(\xi) d\xi, \quad (2.4)$$

for

$$1 \leq i \leq \frac{3N}{2} \text{ and } i > \frac{5N}{2},$$

and

$$R_i^{(3)} = \sum_{j=i-\bar{N}+1}^i \int_{t_{j-1}}^{t_j} (t_j - \xi) \frac{d}{ds} K(t_i, \xi) u(\xi) d\xi - \int_{t_{i-N}}^{t_i-r} K(t_i, s) u(s) ds, \quad (2.5)$$

for

$$\frac{3N}{2} < i \leq \frac{5N}{2}.$$

Therefore we have the exact equality for the $u(t)$:

$$L_N u_i := \varepsilon u_{i,i} + a_i u_i + \sum_{j=i-\bar{N}+1}^i h_j K(t_i, t_{j-1}) u_{j-1} + R_i = f_i, \quad 1 \leq i \leq N_0, \quad (2.6)$$

$$u_i = \varphi_i, \quad -\frac{3N}{2} \leq i \leq 0,$$

with remainder term

$$R_i = R_i^{(1)} + R_i^{(2)} + R_i^{(3)}$$

where $R_i^{(k)}$ ($k = 1, 2, 3$) are given by (2.2), (2.3), (2) and (2.5).

Based on (2.6) we propose the following scheme for approximating (1.1)-(1.2):

$$L_N y_i := \varepsilon y_{i,i} + a_i y_i + \sum_{j=i-\bar{N}+1}^i h_j K_{i,j-1} y_{j-1} = f_i, \quad 1 \leq i \leq N_0, \quad (2.7)$$

$$y_i = \varphi_i, \quad -\frac{3N}{2} \leq i \leq 0. \quad (2.8)$$

3. ERROR ANALYSIS AND CONVERGENCE

Let $z_i = y_i - u_i$. Then for the error of the approximate solution z_i , from (2.6) and (2.7) it follows that

$$L_N z_i = R_i, \quad 1 \leq i \leq N_0, \quad (3.1)$$

$$z_i = 0, \quad -\frac{3N}{2} \leq i \leq 0. \quad (3.2)$$

Lemma 2. *For the solution of (3.1)-(3.2) holds the estimate*

$$\|z\|_{\infty, \omega} \leq C \|R\|_{\infty, \omega}.$$

Proof. Once the inequality

$$\left| \sum_{j=i-\bar{N}+1}^i h_j K_{i,j-1} z_{j-1} \right| \leq \bar{K} \sum_{j=i-\bar{N}+1}^i h_j |z_{j-1}| \leq \bar{K} \sum_{j=1}^i h_j |z_{j-1}|,$$

is satisfied, then by using Lemma 4.1 from [13] we have that

$$|z_i| \leq \alpha^{-1} \|R\|_{\infty, \omega} + \alpha^{-1} \bar{K} \sum_{j=1}^i h_j |z_{j-1}|,$$

which in turn in view of difference analogue of Gronwall's inequality leads to

$$|z_i| \leq \alpha^{-1} \|R\|_{\infty, \omega} \exp \left(\alpha^{-1} \bar{K} \sum_{j=1}^i h_j \right) \leq \alpha^{-1} \exp(\alpha^{-1} \bar{K} T) \|R\|_{\infty, \omega}.$$

Thus the proof is completed. $\square$

Lemma 3. *Under the assumptions of Section 1, the error function R_i of the difference scheme satisfies*

$$\|R\|_{\infty, \omega} \leq C N^{-1} \ln N. \quad (3.3)$$

Proof. We estimate $R_i^{(k)}$ ($k = 1, 2, 3$) separately. For $R_i^{(1)}$, using (2.2) on an arbitrary mesh we have

$$|R_i^{(1)}| \leq C h_i^{-1} \int_{t_{i-1}}^{t_i} (t - t_{i-1}) (1 + |u'(t)|) dt, \quad 1 \leq i \leq N_0.$$

From here by virtue of Lemma 1 we get

$$|R_i^{(1)}| \leq C \left\{ h_i + h_i^{-1} \varepsilon^{-1} \int_{t_{i-1}}^{t_i} (t - t_{i-1}) e^{-\alpha t / \varepsilon} dt \right\}, \quad 1 \leq i \leq N_0.$$

On the layer region $[0, \sigma]$ this reduces to

$$\left| R_i^{(1)} \right| \leq C h^{(1)} (1 + \varepsilon^{-1}) = C 2\alpha^{-1} N^{-1} \ln N (\varepsilon + 1), \quad 1 \leq i \leq \frac{N}{2},$$

therefore

$$\left| R_i^{(1)} \right| \leq C N^{-1} \ln N, \quad 1 \leq i \leq \frac{N}{2}, \quad (3.4)$$

For $i > \frac{N}{2}$, we can write

$$\begin{aligned} \left| R_i^{(1)} \right| &\leq C \left\{ h_i + \varepsilon^{-1} \int_{t_{i-1}}^{t_i} e^{-\alpha t/\varepsilon} dt \right\} = C \left\{ h_i + \alpha^{-1} (e^{-\alpha t_{i-1}/\varepsilon} - e^{-\alpha t_i/\varepsilon}) \right\} \\ &= C \left\{ h_i + \alpha^{-1} e^{-\alpha t_{i-1}/\varepsilon} (1 - e^{-\alpha h_i/\varepsilon}) \right\} \leq C (h_i + \alpha^{-1} e^{-\alpha t_{i-1}/\varepsilon}) \\ &\leq C (h_i + \alpha^{-1} e^{-\alpha \sigma/\varepsilon}) = C (h_i + \alpha^{-1} N^{-1}), \quad \frac{N}{2} + 1 \leq i \leq N_0 \end{aligned}$$

with

$$h_i = h^{(2)}, \quad \frac{N}{2} + 1 \leq i \leq \frac{3N}{2}; \quad h_i = h^{(3)}, \quad \frac{3N}{2} + 1 \leq i \leq N_0.$$

Thereby

$$\left| R_i^{(1)} \right| \leq C N^{-1}, \quad \frac{N}{2} + 1 \leq i \leq N_0. \quad (3.5)$$

Next for $R_i^{(2)}$, according to (2.3) we have

$$\left| R_i^{(2)} \right| \leq \int_{t_{i-1}}^{t_i} \left| \int_{\xi-r}^{\xi} \frac{\partial K(\xi, s)}{\partial \xi} u(s) ds + K(\xi, \xi) u(\xi) - K(\xi, \xi-r) u(\xi-r) \right| d\xi.$$

Taking into consideration the boundness of $\partial K/\partial \xi$, $K(t, s)$ and Lemma 1 we arrive at

$$\left| R_i^{(2)} \right| \leq C r h_i, \quad 1 \leq i \leq N_0,$$

which in turn implies that

$$\left| R_i^{(2)} \right| \leq C N^{-1} \ln N. \quad (3.6)$$

Now estimate $R_i^{(3)}$ for $1 \leq i \leq \frac{3N}{2}$ and $i > \frac{5N}{2}$ we get

$$\left| R_i^{(3)} \right| \leq \sum_{j=i-\overline{N}+1}^i \int_{t_{j-1}}^{t_j} (t_j - \xi) \left| \frac{d}{d\xi} K(t_i, \xi) \right| |u(\xi)| d\xi \leq C \sum_{j=i-\overline{N}+1}^i h_j^2.$$

From here it follows that

$$\left| R_i^{(3)} \right| \leq C \overline{N} \left| h^{(1)} \right|^2 \leq C N^{-1} \ln N, \quad 1 \leq i \leq \frac{N}{2}, \quad (3.7)$$

$$\left| R_i^{(3)} \right| \leq C \overline{N} \left| h^{(2)} \right|^2 \leq C N^{-1}, \quad \frac{N}{2} + 1 \leq i \leq \frac{3N}{2}, \quad (3.8)$$

$$\left| R_i^{(3)} \right| \leq C \overline{N} \left| h^{(3)} \right|^2 \leq C N^{-1}, \quad i > \frac{5N}{2}. \quad (3.9)$$

Finally, we estimate $R_i^{(3)}$ for $\frac{3N}{2} < i \leq \frac{5N}{2}$

$$\begin{aligned} \left| R_i^{(3)} \right| &\leq \sum_{j=i-N+1}^i \int_{t_{j-1}}^{t_j} (t_j - \xi) \left| \frac{d}{d\xi} K(t_i, \xi) \right| |u(\xi)| d\xi + \left| \int_{t_{i-N}}^{t_i-r} K(t_i, s) u(s) ds \right| \\ &\leq C \left(\sum_{j=i-N+1}^i h_j^2 + |t_i - r - t_{i-N}| \right). \end{aligned}$$

Since

$$\begin{aligned} t_i - r - t_{i-N} &= r + (i - \frac{3N}{2})h^{(3)} - r - \sigma - (i - \frac{3N}{2})h^{(2)} \\ &= -\sigma[1 - (i - \frac{3N}{2})N^{-1}] \end{aligned}$$

and

$$\begin{aligned} \sum_{j=i-N+1}^i h_j^2 &= \sum_{j=i-N+1}^{3N/2} h_j^2 + \sum_{j=\frac{3N}{2}+1}^i h_j^2 = (\frac{3N}{2} - i + N) \left| h^{(2)} \right|^2 + (i - \frac{3N}{2}) \left| h^{(3)} \right|^2 \\ &\leq N \left(\left| h^{(2)} \right|^2 + \left| h^{(3)} \right|^2 \right) = (r - \sigma)h^{(2)} + rh^{(3)} = [(r - \sigma)^2 + r^2]N^{-1}, \end{aligned}$$

then

$$\left| R_i^{(3)} \right| \leq C (\sigma + N^{-1}) \leq C N^{-1} \ln N, \quad \frac{3N}{2} < i \leq \frac{5N}{2}. \quad (3.10)$$

Now use (3.4)-(3.10) to get the desired inequality (3.3). $\square$

Lemma 2 and Lemma 3 give the main result of our paper.

Theorem 1. *Let u be solution of (1.1)-(1.2) and y_i its approximation by the difference scheme (2.7)-(2.8). Then*

$$\|u - y\|_{\infty, \bar{\omega}} \leq C N^{-1} \ln N.$$

4. NUMERICAL RESULTS

Example 1. First we study the following test problem

$$\varepsilon u' + 2u - \int_{t-1}^t u(s) ds = -1 + t - \frac{\varepsilon}{2} \left(1 - e^{-\frac{2t}{\varepsilon}} \right), \quad t \in (0, 2],$$

$$u(t) = 1, \quad -1 \leq t \leq 0.$$

The exact solution is given by

$$u(t) = \begin{cases} e^{-\frac{2t}{\varepsilon}}, & 0 \leq t \leq 1 \\ \frac{e^{-\lambda_1(t-1)} - e^{-\lambda_2(t-1)}}{\sqrt{1+\varepsilon}} - 1 + e^{-\frac{2t}{\varepsilon}} + e^{-\frac{2(t-1)}{\varepsilon}}, & 1 \leq t \leq 2 \end{cases}$$

where

$$\lambda_1 = \frac{1 - \sqrt{1+\varepsilon}}{\varepsilon}, \quad \lambda_2 = \frac{1 + \sqrt{1+\varepsilon}}{\varepsilon}.$$

We define the exact error e_ε^N and the computed ε -uniform maximum pointwise error e^N as follows

$$e_\varepsilon^N = \|y - u\|_{\infty, \bar{\omega}_{N_0}},$$

$$e^N = \max_\varepsilon e_\varepsilon^N,$$

where y is the numerical approximation to u for various of N and ε . Parameter-uniform rates of convergence are computed by

$$p^N = \ln(e^N / e^{2N}) / \ln 2.$$

The values of ε and N for which we solve the test problem are $\varepsilon = 2^{-i}$, $i = 0, 6, 12, 18, 24$; $N = 64, 128, 256, 512, 1024$. From Table 1 we observe that the ε -uniform experimental rate of convergence is monotonically increasing towards one, so in agreement with the theoretical rate given by Theorem 1.

TABLE 1. Errors and rates of convergence for Example 1.

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^0	0.008841	0.004804	0.002539	0.001305	0.000666
	0.88	0.92	0.96	0.97	
2^{-6}	0.008783	0.004839	0.002575	0.001333	0.000676
	0.86	0.91	0.95	0.98	
2^{-12}	0.008783	0.004840	0.002575	0.001333	0.000676
	0.86	0.91	0.95	0.98	
2^{-18}	0.008768	0.004831	0.002571	0.001331	0.000675
	0.86	0.91	0.95	0.98	
2^{-24}	0.008768	0.004831	0.002571	0.001331	0.000675
	0.86	0.91	0.95	0.98	
e^N	0.008841	0.004840	0.002575	0.001333	0.000676
p^N	0.86	0.91	0.95	0.98	

Example 2. Consider the initial-value problem

$$\begin{aligned} \varepsilon u' + u + \int_{t-1}^t su(s)ds &= 5t^2 - 1, \quad 0 < t \leq 2 \\ u(t) &= 5 + t, \quad -1 \leq t \leq 0. \end{aligned}$$

For this problem the exact solution is not known. Therefore we use the double-mesh principle to estimate the errors and compute solutions, that is, we compare the computed solution with the solution on a mesh that is twice as fine. The Table 2 shows our numerical results for the second problem. We measure the accuracy in the discrete maximum norm

$$e_\varepsilon^N = \max_i \left| y_i^{\varepsilon, N} - \tilde{y}_i^{\varepsilon, 2N} \right|,$$

where $\tilde{y}_i^{\varepsilon, 2N}$ is the approximate solution of the respective method on the mesh

$$\tilde{\omega}_{2N} = \left\{ x_{\frac{i}{2}} : i = 0, 1, 2, \dots, 2N \right\}$$

with

$$x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2} \quad \text{for } i = 0, 1, 2, \dots, N-1.$$

The rates of convergence are defined as

$$p_\varepsilon^N = \frac{\ln(e_\varepsilon^N / e_\varepsilon^{2N})}{\ln 2}.$$

The ε -uniform errors e^N are estimated from

$$e^N = \max_\varepsilon e_\varepsilon^N.$$

The corresponding ε -uniform the rates of convergence are computed using the formula

$$p^N = \frac{\ln(e^N / e^{2N})}{\ln 2}.$$

TABLE 2. Errors and rates of convergence for Example 2

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^0	0.033904	0.019338	0.010435	0.005515	0.002835
	0.81	0.89	0.92	0.96	
2^{-6}	0.043189	0.024464	0.013293	0.006977	0.003562
	0.82	0.88	0.93	0.97	
2^{-12}	0.059090	0.033471	0.018187	0.009612	0.004907
	0.82	0.88	0.92	0.97	
2^{-18}	0.064587	0.036332	0.019470	0.010290	0.005253
	0.83	0.90	0.92	0.97	
2^{-24}	0.066203	0.037500	0.020096	0.010621	0.005422
	0.82	0.90	0.92	0.97	
e^N	0.066203	0.037500	0.020096	0.010621	0.005422
p^N	0.82	0.90	0.92	0.97	

5. CONCLUSIONS

In this paper, we presented a finite difference scheme on the piecewise uniform mesh to solve singularly perturbed initial value problem for a linear first order Volterra integro-differential equation with delay. The difference scheme is based on the method of integral identities with the use of appropriate interpolating quadrature rules with remainder term in integral form. The emphasis is on the convergence of numerical method. It is shown that the method displays uniform convergence in respect to the perturbation parameter. We have implemented the present method on two particular problems. The numerical results show that the proposed method is first order uniformly accurate and hence can be recommended for singularly perturbed Volterra delay-integro-differential equation. The main lines for the analysis of the uniform convergence carried out here can be extended to more complicated linear differential equations as well as nonlinear differential equations.

REFERENCES

- [1] G. M. Amiraliyev and F. Erdoğan, "Uniform Numerical Method for Singularly Perturbed Delay Differential Equations." *Comput. Math. Appl.*, vol. 53, no. 8, pp. 1251–1259, 2007, doi: [10.1016/j.camwa.2006.07.009](https://doi.org/10.1016/j.camwa.2006.07.009).
- [2] G. M. Amiraliyev and B. Yilmaz, "Fitted Difference Method for a Singularly Perturbed Initial Value Problem." *Int. J. Math. Comput.*, vol. 22, no. 1, pp. 1–10, 2014.
- [3] A. Bellour and M. Bousselsal, "Numerical solution of delay integro-differential equations by using Taylor collocation method." *Math. Meth. Appl. Sci.*, vol. 37, no. 10, pp. 1491–1506, 2014, doi: [10.1002/mma.2910](https://doi.org/10.1002/mma.2910).
- [4] G. A. Bocharov and F. A. Rihan, "Numerical modelling in biosciences using delay differential equations." *J. Comput. Appl. Math.*, vol. 125, no. 1-2, pp. 183–199, 2000, doi: [10.1016/S0377-0427\(00\)00468-4](https://doi.org/10.1016/S0377-0427(00)00468-4).

- [5] P. A. Farrel, A. F. Hegarty, J. J. H. Miller, E. O’Riordan, and G. I. Shishkin, *Robust Computational Techniques for Boundary Layers*. New York: Chapman and Hall/CRC, 2000.
- [6] A. D. Gaetano and O. Arino, “Mathematical modelling of the intravenous glucose tolerance test.” *J. Math. Biol.*, vol. 40, no. 2, pp. 136–168, 2000.
- [7] S. Gan, “Dissipativity of θ -methods for nonlinear Volterra delay-integro-differential equations.” *J. Comput. Appl. Math.*, vol. 206, no. 2, pp. 898–907, 2007, doi: [10.1016/j.cam.2006.08.030](https://doi.org/10.1016/j.cam.2006.08.030).
- [8] D. He and L. Xu, “Integrodifferential Inequality for Stability of Singularly Perturbed Impulsive Delay Integrodifferential Equations.” *J. Inequal. Appl.*, vol. 2009, no. ID 369185, pp. 1–11, 2009, doi: [10.1155/2009/369185](https://doi.org/10.1155/2009/369185).
- [9] C. Huang, “Stability of linear multistep methods for delay integro-differential equations.” *Comput. Math. Appl.*, vol. 55, no. 12, pp. 2830–2838, 2008, doi: [10.1016/j.camwa.2007.09.005](https://doi.org/10.1016/j.camwa.2007.09.005).
- [10] M. K. Kadalbajoo and V. Gupta, “A brief survey on numerical methods for solving singularly perturbed problems.” *Appl. Math. Comput.*, vol. 217, no. 8, pp. 3641–3716, 2010, doi: [10.1016/j.amc.2010.09.059](https://doi.org/10.1016/j.amc.2010.09.059).
- [11] J. P. Kauthen, “A survey on singularly perturbed Volterra equations.” *Appl. Numer. Math.*, vol. 24, no. 2-3, pp. 95–114, 1997, doi: [10.1016/S0168-9274\(97\)00014-7](https://doi.org/10.1016/S0168-9274(97)00014-7).
- [12] A. H. Khater, A. B. Shamardan, D. K. Callebaut, and M. R. A. Sakran, “Numerical solutions of integral and integro-differential equations using Legendre polynomials.” *Numer. Algor.*, vol. 46, no. 3, pp. 195–218, 2007, doi: [10.1007/s11075-007-9130-2](https://doi.org/10.1007/s11075-007-9130-2).
- [13] M. Kudu, I. Amirali, and G. M. Amiraliyev, “A finite-difference method for a singularly perturbed delay integro-differential equation.” *J. Comput. Appl. Math.*, vol. 308, pp. 379–390, 2016, doi: [10.1016/j.cam.2016.06.018](https://doi.org/10.1016/j.cam.2016.06.018).
- [14] S. Marino, E. Beretta, and D. E. Kirschner, “The role of delays in innate and adaptive immunity to intracellular bacterial infection.” *Math. Biosci. Eng.*, vol. 4, no. 2, pp. 261–288, 2007, doi: [10.3934/mbe.2007.4.261](https://doi.org/10.3934/mbe.2007.4.261).
- [15] H. K. Mishra and S. Saini, “Various Numerical Methods for Singularly Perturbed Boundary Value Problems.” *American Journal of Applied Mathematics and Statistics*, vol. 2, no. 3, pp. 129–142, 2014, doi: [10.12691/ajams-2-3-7](https://doi.org/10.12691/ajams-2-3-7).
- [16] A. H. Nayfeh, *Introduction to Perturbation Techniques*. New York: Wiley, 1993.
- [17] R. E. O’Malley, *Singular Perturbation Methods for Ordinary Differential Equations*. New York: Springer-Verlag, 1991.
- [18] J. I. Ramos, “Exponential techniques and implicit Runge-Kutta methods for singularly-perturbed Volterra integro-differential equations.” *Neural Parallel Sci. Comput.*, vol. 16, no. 3, pp. 387–404, 2008.
- [19] H. G. Roos, M. Stynes, and L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations*. Berlin: Springer-Verlag, 1996.
- [20] S. Şevgin, “Numerical solution of a singularly perturbed Volterra integro-differential equation.” *Adv. Difference Equ.*, vol. 2014, no. 171, pp. 1–15, 2014, doi: [10.1186/1687-1847-2014-171](https://doi.org/10.1186/1687-1847-2014-171).
- [21] M. Shakourifar and W. Enright, “Superconvergent interpolants for collocation methods applied to Volterra integro-differential equations with delay.” *Bit Numer. Math.*, vol. 52, no. 3, pp. 725–740, 2012, doi: [10.1007/s10543-012-0373-5](https://doi.org/10.1007/s10543-012-0373-5).
- [22] V. Subburayan and N. Ramanujam, “An Initial Value Technique for Singularly Perturbed Convection-Diffusion Problems with a Negative Shift.” *J. Optim. Theory Appl.*, vol. 158, no. 1, pp. 234–250, 2013, doi: [10.1007/s10957-012-0200-9](https://doi.org/10.1007/s10957-012-0200-9).
- [23] S. Wu and S. Gan, “Errors of linear multistep methods for singularly perturbed Volterra delay-integro-differential equations.” *Math. Comput. Simul.*, vol. 79, no. 10, pp. 3148–3159, 2009, doi: [10.1016/j.matcom.2009.03.006](https://doi.org/10.1016/j.matcom.2009.03.006).

- [24] C. Zhang and S. Vandewalle, “Stability analysis of Volterra delay-integro-differential equations and their backward differentiation time discretization.” *J. Comput. Appl. Math.*, vol. 164-165, pp. 797–814, 2004, doi: [10.1016/j.cam.2003.09.013](https://doi.org/10.1016/j.cam.2003.09.013).
- [25] J. Zhao, Y. Cao, and Y. Xu, “Sinc numerical solution for pantograph Volterra delay-integro-differential equation.” *Int. J. Comput. Math.*, vol. 94, no. 5, pp. 853–865, 2017, doi: [10.1080/00207160.2016.1149577](https://doi.org/10.1080/00207160.2016.1149577).

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GREEN FUNCTION OF TRIHARMONIC HEAT EQUATION USING ADDITION FORMULA OF FUNCTION e^{-x^3}

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Abstract. In this paper, using the Mellin transform of Airy function we present an integral addition formula for the function e^{-x^3} . In deriving this representation, we make use of the Hankel functions of first kind and apply this representation to get the Green function of triharmonic heat equation in terms of the Airy functions.

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1. INTRODUCTION

As we know for the harmonic heat equation

$$\frac{\partial}{\partial t} u(x, y, t) = \Delta u(x, y, t), \quad t > 0, x, y \in \mathbb{R}, \quad u(x, y, 0) = \delta(x)\delta(y), \quad (1.1)$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator, the associated Green function (fundamental solution) is given by [17]

$$u(x, y, t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2+y^2}{4t}}. \quad (1.2)$$

Also, the Green function of biharmonic heat equation [5]

$$\frac{\partial}{\partial t} u(x, y, t) + \Delta^2 u(x, y, t) = 0, \quad t > 0, x, y \in \mathbb{R}, \quad u(x, y, 0) = \delta(x)\delta(y), \quad (1.3)$$

has been recently presented in terms of the quartic Lévy stable function given by [18, 23]

$$L_4(\xi, t) = \frac{1}{\pi} \int_0^\infty e^{-tr^4} \cos(r\xi) dr. \quad (1.4)$$

This function is the Green function of generalized heat equation

$$\frac{\partial}{\partial t} u(x, t) + \frac{\partial^4}{\partial x^4} u(x, t) = 0, \quad t > 0, x \in \mathbb{R}, \quad u(x, 0) = \delta(x), \quad (1.5)$$

related to the following fractional exponential operator

$$u(x, t) = e^{-t \frac{\partial^4}{\partial x^4}} \delta(x) = L_4(x, t). \quad (1.6)$$

The main strategy of obtaining the Green function of biharmonic heat equation is applying the two dimensional fractional exponential operator

$$u(x, y, t) = e^{-t \Delta^2} \delta(x) \delta(y) = e^{-t [\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}]^2} \delta(x) \delta(y), \quad (1.7)$$

which can be extended for the higher order harmonic heat equations. This is our main motivation for showing the Green function of *triharmonic heat equation*

$$\frac{\partial}{\partial t} u(x, y, t) + \Delta^3 u(x, y, t) = 0, \quad t > 0, x, y \in \mathbb{R}, \quad u(x, y, 0) = \delta(x) \delta(y), \quad (1.8)$$

which its solution can be consequently derived as

$$u(x, y, t) = e^{-t \Delta^3} \delta(x) \delta(y). \quad (1.9)$$

For obtaining the above solution which has been much less developed in the literature, see for example [26], first we get an integral addition formula for the function e^{-x^3} . Next, we present the fundamental solution in terms of the Airy functions.

In this sense, we organize the paper as follows. In Section 2, we recall the definitions of the gamma and reciprocal gamma functions along with its triplication formula. We also mention some algebraic and operational properties of the Airy functions of first and second kinds.

In Section 3, we state an integral relation between the function e^{-x^3} and Airy function on the well-known Hankel contour and then we construct an integral addition formula for function e^{-x^3} . The obtained addition formula is given in terms of the imaginary parts of products of Airy functions and can be simplified with respect to the products of Hankel functions of first kind.

In Section 4, using the exponential differential operators we get the Green function of triharmonic heat equation (1.9) in terms of the Airy functions.

2. PRELIMINARIES

2.1. The Gamma functions

The gamma function and the reciprocal gamma function are defined as

$$\Gamma(s) = \int_0^\infty e^{-\tau} \tau^{s-1} d\tau, \quad \Re(s) > 0, \quad (2.1)$$

$$\frac{1}{\Gamma(s)} = \frac{1}{2\pi i} \int_{\text{Ha}} e^\tau \tau^{-s} d\tau, \quad (2.2)$$

where Ha is the Hankel contour in complex plane with a cut along the negative real semi-axis $\arg \tau = \pi$ and consisting of the following three parts

$$(1) \arg \tau = \pi, \quad |\tau| \geq \epsilon,$$

- (2) $-\pi \leq \arg \tau \leq \pi, \quad |\tau| = \epsilon,$
 (3) $\arg \tau = -\pi, \quad |\tau| \geq \epsilon.$

For this function, the well-known triplication formula is given by [1, p. 256, 6.1.19]

$$\Gamma(s) = \frac{1}{2\pi} 3^{s-\frac{1}{2}} \Gamma\left(\frac{s}{3}\right) \Gamma\left(\frac{s+1}{3}\right) \Gamma\left(\frac{s+2}{3}\right), \quad \Re(s) > 0. \quad (2.3)$$

2.2. The Airy functions

The classical Airy functions of first and second kinds $\text{Ai}(x)$ and $\text{Bi}(x)$ are defined as [3, 6, 9, 12, 20, 27]

$$\text{Ai}(x) = \frac{1}{\pi} \int_0^\infty \cos\left(xt + \frac{t^3}{3}\right) dt, \quad (2.4)$$

$$\text{Bi}(x) = \frac{1}{\pi} \int_0^\infty \left(e^{xt - \frac{t^3}{3}} + \sin\left(xt + \frac{t^3}{3}\right) \right) dt, \quad (2.5)$$

and the Mellin transform of function $\text{Ai}(x)$ is given by [27, p. 96]

$$\mathcal{M}\{\text{Ai}(x); s\} = \int_0^\infty x^{s-1} \text{Ai}(x) dx = 3^{-(s+2)/3} \frac{\Gamma(s)}{\Gamma(\frac{s+2}{3})}, \quad \Re(s) > 0. \quad (2.6)$$

Also, for $|\arg(a)| < \frac{\pi}{3}$ and $|\arg(b)| < \frac{\pi}{3}$ some integral addition formula have been derived in the literature, see [12, 25]

$$\text{Ai}(a+b) = \int_0^1 \text{Ai}'\left(\frac{a}{\xi^{\frac{1}{3}}}\right) \text{Ai}'\left(\frac{b}{(1-\xi)^{\frac{1}{3}}}\right) \frac{d\xi}{\xi^{\frac{2}{3}}(1-\xi)^{\frac{2}{3}}}, \quad (2.7)$$

$$\text{Ai}(a+b) = \frac{ab}{a+b} \int_0^1 \text{Ai}\left(\frac{a}{\xi^{\frac{1}{3}}}\right) \text{Ai}\left(\frac{b}{(1-\xi)^{\frac{1}{3}}}\right) \frac{d\xi}{\xi^{\frac{4}{3}}(1-\xi)^{\frac{4}{3}}}. \quad (2.8)$$

Moreover, the following identities hold for the Airy functions of first and second kinds [27, p. 8], [1, p. 447, 10.4.28]

$$\text{Ai}'\left(xe^{-\frac{\pi i}{3}}\right) = \text{Ai}'\left(-xe^{\frac{2\pi i}{3}}\right) = \frac{e^{-\frac{\pi i}{3}}}{2} [\text{Ai}'(-x) - i \text{Bi}'(-x)], \quad (2.9)$$

$$\text{Ai}'(-x) - i \text{Bi}'(-x) = e^{-\frac{\pi i}{6}} \frac{x}{\sqrt{3}} H_{2/3}^{(1)}\left(\frac{2}{3}x^{\frac{3}{2}}\right), \quad (2.10)$$

where the Hankel function of first kind $H_v^{(1)}(x)$ is defined as [19, p. 914, 8.421(1)]

$$H_v^{(1)}(x) = \frac{2e^{-\frac{\nu\pi i}{2}}}{\pi i} \int_{-\infty}^\infty e^{ix \cosh(u) - \nu u} du, \quad -1 < \Re(\nu) < 1, \quad x > 0. \quad (2.11)$$

3. INTEGRAL ADDITION FORMULA

Lemma 1. For $\Re(s) > 0$, the Mellin transform of function e^{-x^3} is given by

$$\begin{aligned}\mathcal{M}\{e^{-x^3}; s\} &= 2\pi \left[3^{-(4s+1)/6} \frac{1}{\Gamma(\frac{s+1}{3})} \right] \left[3^{-(s+2)/3} \frac{\Gamma(s)}{\Gamma(\frac{s+2}{3})} \right], \\ &= 2\pi \left[3^{-(4s+1)/6} \frac{1}{\Gamma(\frac{s+1}{3})} \right] \mathcal{M}\{\text{Ai}(x); s\}.\end{aligned}\quad (3.1)$$

Proof. According to [19, p. 1131], for the Mellin transform of the function e^{-x^3}

$$\mathcal{M}\{e^{-x^3}; s\} = \frac{1}{3} \Gamma\left(\frac{s}{3}\right), \quad (3.2)$$

and taking into account the relations (2.3) and (2.6), we get the result. $\square$

Lemma 2. The following representation holds for the function e^{-x^3} and the Airy function

$$e^{-x^3} = -i 3^{-1/6} \int_{\text{Ha}} e^{\tau} \tau^{-1/3} \text{Ai}\left(3^{2/3} \tau^{1/3} x\right) d\tau, \quad (3.3)$$

where Ha is the Hankel contour.

Proof. Taking into account the relation (2.2) for the reciprocal gamma function $\frac{1}{\Gamma(\frac{s+1}{3})}$ and considering the representation (3.1), we can present a simple comparison between the Mellin transform of the Airy function and e^{-x^3} . After a little algebra and using the following property of Mellin transform

$$\mathcal{M}\{f(ax); s\} = \frac{1}{a^s} F(s), \quad a > 0, \quad (3.4)$$

we obtain the result. $\square$

Theorem 1. For $|\arg(a)| < \frac{\pi}{3}$ and $|\arg(b)| < \frac{\pi}{3}$, the following integral addition formulas hold

$$\begin{aligned}e^{-(a+b)^3} &= \frac{2}{3^{1/6}} \int_0^1 \int_0^\infty \frac{e^{-\tau}}{\sqrt[3]{\tau}} \Im \left[e^{\frac{\pi i}{3}} \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} e^{-\frac{\pi i}{3}} a}{\xi^{1/3}}\right) \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} e^{-\frac{\pi i}{3}} b}{(1-\xi)^{1/3}}\right) \right] \\ &\quad \times \frac{1}{\xi^{\frac{2}{3}} (1-\xi)^{\frac{2}{3}}} d\tau d\xi,\end{aligned}\quad (3.5)$$

$$\begin{aligned}e^{-(a+b)^3} &= 2\sqrt{3} \frac{ab}{a+b} \int_0^1 \int_0^\infty e^{-\tau} \Im \left[\text{Ai}\left(\frac{3^{2/3} \tau^{1/3} e^{-\frac{\pi i}{3}} a}{\xi^{1/3}}\right) \text{Ai}\left(\frac{3^{2/3} \tau^{1/3} e^{-\frac{\pi i}{3}} b}{(1-\xi)^{1/3}}\right) \right] \\ &\quad \times \frac{1}{\xi^{\frac{4}{3}} (1-\xi)^{\frac{4}{3}}} d\tau d\xi.\end{aligned}\quad (3.6)$$

Proof. First, we use the relation (3.3) to present an addition formula for the function e^{-x^3} with respect to the addition formula of Airy function. Next, we compute the remaining integral on the Hankel contour in three parts by applying Titchmarsh theorem for the inverse Laplace transforms of multi-valued functions [8, 10, 21]

$$\begin{aligned}
e^{-(a+b)^3} &= -i 3^{-1/6} \int_0^1 \int_{\text{Ha}} \left[e^{\tau} \tau^{-1/3} \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} a}{\xi^{1/3}}\right) \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} b}{(1-\xi)^{1/3}}\right) \right] \\
&\quad \times \frac{1}{\xi^{2/3} (1-\xi)^{2/3}} d\tau d\xi \\
&= -i 3^{-1/6} \int_0^1 \int_{\infty}^0 \left[e^{\tau} \tau^{-1/3} \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} a}{\xi^{1/3}}\right) \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} b}{(1-\xi)^{1/3}}\right) \right]_{\tau=r e^{i\pi}} \\
&\quad \times \frac{1}{\xi^{2/3} (1-\xi)^{2/3}} d\tau d\xi \\
&\quad - i 3^{-1/6} \int_0^1 \int_{\pi}^{-\pi} \left[e^{\tau} \tau^{-1/3} \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} a}{\xi^{1/3}}\right) \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} b}{(1-\xi)^{1/3}}\right) \right]_{\tau=r e^{i\theta}} \\
&\quad \times \frac{1}{\xi^{2/3} (1-\xi)^{2/3}} d\tau d\xi \\
&\quad - i 3^{-1/6} \int_0^1 \int_0^{\infty} \left[e^{\tau} \tau^{-1/3} \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} a}{\xi^{1/3}}\right) \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} b}{(1-\xi)^{1/3}}\right) \right]_{\tau=r e^{-i\pi}} \\
&\quad \times \frac{1}{\xi^{2/3} (1-\xi)^{2/3}} d\tau d\xi. \tag{3.7}
\end{aligned}$$

Taking $\epsilon \rightarrow 0$, we obtain the addition formula of function e^{-x^3} in terms of the imaginary part of products of Airy functions as mentioned in (3.5). The result (3.6) is obtained by the same procedure. $\square$

Remark 1. For simplicity of the imaginary part of products of Airy functions in (3.5), we employ the relations (2.9) and (2.10), respectively, and present two representations as follows

$$\begin{aligned}
&\Im \left[e^{\frac{\pi i}{3}} \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} e^{-\frac{\pi i}{3}} a}{\xi^{1/3}}\right) \text{Ai}'\left(\frac{3^{2/3} \tau^{1/3} e^{-\frac{\pi i}{3}} b}{(1-\xi)^{1/3}}\right) \right] \\
&= -\frac{1}{4} \left[\text{Ai}'\left(-\frac{3^{2/3} \tau^{1/3} a}{\xi^{1/3}}\right) \text{Bi}'\left(-\frac{3^{2/3} \tau^{1/3} b}{(1-\xi)^{1/3}}\right) + \text{Bi}'\left(-\frac{3^{2/3} \tau^{1/3} a}{\xi^{1/3}}\right) \text{Ai}'\left(-\frac{3^{2/3} \tau^{1/3} b}{(1-\xi)^{1/3}}\right) \right] \\
&\quad - \frac{\sqrt{3}}{4} \left[\text{Ai}'\left(-\frac{3^{2/3} \tau^{1/3} a}{\xi^{1/3}}\right) \text{Ai}'\left(-\frac{3^{2/3} \tau^{1/3} b}{(1-\xi)^{1/3}}\right) - \text{Bi}'\left(-\frac{3^{2/3} \tau^{1/3} a}{\xi^{1/3}}\right) \text{Bi}'\left(-\frac{3^{2/3} \tau^{1/3} b}{(1-\xi)^{1/3}}\right) \right], \tag{3.8}
\end{aligned}$$

$$\begin{aligned} & \Im \left[e^{\frac{\pi i}{3}} \text{Ai}' \left(\frac{3^{2/3} \tau^{1/3} e^{-\frac{\pi i}{3}} a}{\xi^{1/3}} \right) \text{Ai}' \left(\frac{3^{2/3} \tau^{1/3} e^{-\frac{\pi i}{3}} b}{(1-\xi)^{1/3}} \right) \right] = \\ & -\frac{1}{8} \frac{3^{5/6} \tau^{2/3} ab}{\xi^{1/3} (1-\xi)^{1/3}} H_v^{(1)} \left(\frac{2}{3^{1/2}} \frac{\tau^{1/2} a^{3/2}}{\xi^{1/2}} \right) H_v^{(1)} \left(\frac{2}{3^{1/2}} \frac{\tau^{1/2} b^{3/2}}{(1-\xi)^{1/2}} \right). \quad (3.9) \end{aligned}$$

4. THE GREEN FUNCTION OF TRIHARMONIC HEAT EQUATION

In this section, using the exponential differential operators and the obtained addition formula in previous section, we intend to obtain the Green function of triharmonic heat equation in terms of the Airy functions. For knowing more applications of the exponential differential operators in the applied mathematics see [2, 4, 7, 11, 13–16, 22, 24].

At the beginning, we show a fundamental lemma for the Green function of generalized heat equation of third order and state a theorem for the Green function of triharmonic heat equation. Although the obtained solution is very formal, but it presents an analytical form for the structure of Green function.

Lemma 3 ([7, 11]). *The Green function of generalized heat equation*

$$\frac{\partial}{\partial t} u(x, t) = \frac{\partial^3}{\partial x^3} u(x, t), \quad t > 0, x \in \mathbb{R}, \quad u(x, 0) = \delta(x), \quad (4.1)$$

is obtained by applying the following exponential operator on the Dirac delta function

$$u(x, t) = e^{t \frac{\partial^3}{\partial x^3}} \delta(x) = \frac{1}{\sqrt[3]{t}} \text{Ai} \left(-\frac{x}{\sqrt[3]{t}} \right). \quad (4.2)$$

Theorem 2. *The Green function of triharmonic heat equation*

$$\frac{\partial}{\partial t} u(x, y, t) + \Delta^3 u(x, y, t) = 0, \quad t > 0, x, y \in \mathbb{R}, \quad u(x, y, 0) = \delta(x)\delta(y), \quad (4.3)$$

is given by

$$\begin{aligned} u(x, y, t) = & -\frac{xy e^{-\frac{\pi i}{3}}}{2^{4/3} \pi^2} \int_0^1 \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{e^{-\tau} \tau^{-1/3}}{\xi^{2/3} (1-\xi)^{2/3}} \frac{e^{-2(u+v)/3}}{\sqrt[3]{\cosh^2(u) \cosh^2(v)}} \\ & \times \text{Ai} \left(-\frac{x \sqrt[6]{\frac{3\xi}{t\tau}} e^{-\frac{\pi i}{6}}}{\sqrt[3]{2 \cosh(u)}} \right) \text{Ai} \left(-\frac{y \sqrt[6]{\frac{3(1-\xi)}{t\tau}} e^{-\frac{\pi i}{6}}}{\sqrt[3]{2 \cosh(v)}} \right) du dv d\tau d\xi. \end{aligned}$$

Proof. We begin with the following formal solution of triharmonic heat equation

$$u(x, y, t) = e^{-t \Delta^3} \delta(x)\delta(y) = e^{-t[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}]^3} \delta(x)\delta(y), \quad (4.4)$$

and set $a = \sqrt[3]{t} \frac{\partial^2}{\partial x^2}$ and $b = \sqrt[3]{t} \frac{\partial^2}{\partial y^2}$ in relation (3.5) to get the solution as

$$\begin{aligned} e^{-t[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}]} \delta(x)\delta(y) &= -\frac{ab}{4 \times 3^{2/3}} \int_0^1 \int_0^\infty e^{-\tau} \sqrt[3]{\tau} \frac{1}{\xi(1-\xi)} \\ &\quad \times \left[H_{2/3}^{(1)} \left(\frac{2}{3^{1/2}} \frac{t^{1/2} \tau^{1/2} \frac{\partial^3}{\partial x^3}}{\xi^{1/2}} \right) \delta(x) \right] \\ &\quad \times \left[H_{2/3}^{(1)} \left(\frac{2}{3^{1/2}} \frac{t^{1/2} \tau^{1/2} \frac{\partial^3}{\partial y^3}}{(1-\xi)^{1/2}} \right) \delta(y) \right] d\tau d\xi. \end{aligned} \quad (4.5)$$

We now apply the integral representation (2.11) and identity (4.1), and simplify the above solution using the following relations

$$\begin{aligned} \left[H_{2/3}^{(1)} \left(\frac{2}{3^{1/2}} \frac{t^{1/2} \tau^{1/2} \frac{\partial^3}{\partial x^3}}{\xi^{1/2}} \right) \delta(x) \right] &= \frac{2e^{-\frac{\pi i}{3}}}{\pi i} \int_{-\infty}^{\infty} e^{2i \sqrt{\frac{t\tau}{3\xi}} \cosh(u) \frac{\partial^3}{\partial x^3} - 2u/3} \delta(x) du, \\ &= -\frac{2}{\pi} \int_{-\infty}^{\infty} e^{-2u/3} \frac{\sqrt[6]{\frac{3\xi}{t\tau}}}{\sqrt[3]{2 \cosh(u)}} \\ &\quad \times \text{Ai} \left(-\frac{x \sqrt[6]{\frac{3\xi}{t\tau}} e^{-\frac{\pi i}{6}}}{\sqrt[3]{2 \cosh(u)}} \right) du, \end{aligned} \quad (4.6)$$

$$\begin{aligned} \left[H_{2/3}^{(1)} \left(\frac{2}{3^{1/2}} \frac{t^{1/2} \tau^{1/2} \frac{\partial^3}{\partial y^3}}{(1-\xi)^{1/2}} \right) \delta(y) \right] &= \frac{2e^{-\frac{\pi i}{3}}}{\pi i} \int_{-\infty}^{\infty} e^{2i \sqrt{\frac{t\tau}{3(1-\xi)}} \cosh(v) \frac{\partial^3}{\partial y^3} - 2v/3} \delta(y) dv, \\ &= -\frac{2}{\pi} \int_{-\infty}^{\infty} e^{-2v/3} \frac{\sqrt[6]{\frac{3(1-\xi)}{t\tau}}}{\sqrt[3]{2 \cosh(v)}} \\ &\quad \times \text{Ai} \left(-\frac{y \sqrt[6]{\frac{3(1-\xi)}{t\tau}} e^{-\frac{\pi i}{6}}}{\sqrt[3]{2 \cosh(v)}} \right) dv. \end{aligned} \quad (4.7)$$

Finally, by substituting the above relations into (4.5) we obtain the result. $\square$

REFERENCES

- [1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. 9th printing, Dover, New York: John Wiley & Sons., 1972.

- [2] A. Ansari, "Fractional exponential operators and nonlinear partial fractional differential equations in the Weyl fractional derivatives." *Appl. Math. Comput.*, vol. 220, pp. 149–154, 2013, doi: <https://doi.org/10.1016/j.amc.2013.05.066>.
- [3] A. Ansari, "The fractional Airy transform." *Hacet. J. Math. Stat.*, vol. 44, no. 4, pp. 761–766, 2015, doi: [10.15672/HJMS.2015449440](https://doi.org/10.15672/HJMS.2015449440).
- [4] A. Ansari, "Some inverse fractional Legendre transforms of gamma function form." *Kodai Math. J.*, vol. 38, no. 3, pp. 658–671, 2015, doi: [10.2996/kmj/1446210600](https://doi.org/10.2996/kmj/1446210600).
- [5] A. Ansari, "Remarks on the green function of space-fractional biharmonic heat equation using Ramanujan's master theorem." *Kuwait J. Sci.*, vol. 44, no. 4, pp. 1–8, 2017.
- [6] A. Ansari and M. Masomi, "Some results involving integral transforms of Airy functions." *Integral Transforms Spec. Funct.*, vol. 26, no. 7, pp. 539–549, 2015, doi: <https://doi.org/10.1080/10652469.2015.1025072>.
- [7] A. Ansari, M. Ahamadi Darani, and M. Moradi, "On fractional Mittag-Leffler operators." *Rep. Math. Phys.*, vol. 70, no. 1, pp. 119–131, 2012, doi: [https://doi.org/10.1016/S0034-4877\(13\)60017-8](https://doi.org/10.1016/S0034-4877(13)60017-8).
- [8] A. Ansari and A. Ansari, "Solving partial fractional differential equations using the $\mathcal{L}_A$ -transform." *Asian-Eur. J. Math.*, vol. 3, no. 2, pp. 209–220, 2010, doi: [10.1142/S1793557110000143](https://doi.org/10.1142/S1793557110000143).
- [9] A. Ansari and M. R. Masomi, "Some integral identities for products of Airy functions using integral transforms." *Asian-Eur. J. Math.*, vol. 11, no. 1, p. 1850046, 2018, doi: [10.1142/S1793557118500468](https://doi.org/10.1142/S1793557118500468).
- [10] A. Ansari and A. Refahi Sheikhan, "New identities for the Wright and the Mittag-Leffler functions using the Laplace transform." *Asian-Eur. J. Math.*, vol. 7, no. 3, p. 1450038, 2014, doi: [10.1142/S1793557114500387](https://doi.org/10.1142/S1793557114500387).
- [11] A. Ansari, A. Refahi Sheikhan, and S. Kordrostami, "On the generating function $e^{xt+y\phi(t)}$ and its fractional calculus." *Cent. Eur. J. Phys.*, vol. 11, no. 10, pp. 1457–1462, 2013, doi: [10.2478/s11534-013-0195-3](https://doi.org/10.2478/s11534-013-0195-3).
- [12] H. Askari and A. Ansari, "Airy functions and Riesz fractional trigonometric operators." *Integral Transforms Spec. Funct.*, vol. 29, no. 8, pp. 585–604, 2018, doi: <https://doi.org/10.1080/10652469.2018.1472593>.
- [13] G. Dattoli, P. L. Ottaviani, and L. Vazquez, "Evolution operators equations: integration with algebraic and finite difference methods. Applications to physical problems in classical and quantum mechanics and quantum field theory." *Riv. Nuovo Cimento.*, vol. 20, pp. 1–133, 1997, doi: [10.1007/BF02907529](https://doi.org/10.1007/BF02907529).
- [14] G. Dattoli, P. E. Ricci, and D. Sacchetti, "Generalized shift operators and pseudo-polynomials of fractional order." *Appl. Math. Comput.*, vol. 141, pp. 215–224, 2003, doi: [https://doi.org/10.1016/S0096-3003\(02\)00334-X](https://doi.org/10.1016/S0096-3003(02)00334-X).
- [15] G. Dattoli, P. E. Ricci, and I. Khomasuridze, "Operational methods, special polynomial and functions and solution of partial differential equations." *Integral Transforms Spec. Funct.*, vol. 15, pp. 309–321, 2004, doi: <https://doi.org/10.1080/10652460410001673013>.
- [16] G. Dattoli, H. M. Srivastava, and K. Zhukovsky, "Operational methods and differential equations with applications to initial value problems." *Appl. Math. Comput.*, vol. 184, pp. 979–1001, 2007, doi: <https://doi.org/10.1016/j.amc.2006.07.001>.
- [17] L. Debnath and D. Bhatta, *Integral transforms and their applications*. New York: second ed., Chapman & Hall/CRC, Taylor & Francis Group, 2007.
- [18] K. Górska, A. Horzela, K. A. Penson, and G. Dattoli, "The higher-order heat-type equations via signed Lévy stable and generalized Airy functions." *J. Phys. A: Math. Theor.*, vol. 46, p. 16 pages, 2013, doi: [10.1088/1751-8113/46/42/425001](https://doi.org/10.1088/1751-8113/46/42/425001).

- [19] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series and products*. New York: Seventh Edition, Academic Press, 2007.
- [20] L. Moslehi and A. Ansari, "Integral representations of products of Airy functions related to fractional calculus." *J. Class. Anal.*, vol. 7, no. 2, pp. 99–112, 2015, doi: [dx.doi.org/10.7153/jca-07-10](https://doi.org/10.7153/jca-07-10).
- [21] L. Moslehi and A. Ansari, "Some remarks on inverse Laplace transforms involving conjugate branch points with applications." *U. P. B. Sci. Bull. Series A-Appl. Math. Phys.*, vol. 78, no. 3, pp. 107–118, 2016.
- [22] L. Moslehi and A. Ansari, "Squared radial Ornstein-Uhlenbeck processes and inverse Laplace transforms of products of confluent hypergeometric functions." *Hacet. J. Math. Stat.*, 2016, doi: [10.15672/HJMS.20174622761](https://doi.org/10.15672/HJMS.20174622761).
- [23] L. Moslehi and A. Ansari, "Integral representations of products of quartic Lévy stable functions." *Integral Transforms Spec. Funct.*, vol. 28, no. 11, pp. 1–12, 2017, doi: <https://doi.org/10.1080/10652469.2017.1376192>.
- [24] L. Moslehi and A. Ansari, "On M-Wright transforms and time-fractional diffusion equations." *Integral Transforms Spec. Funct.*, vol. 28, no. 2, pp. 113–129, 2017, doi: <https://doi.org/10.1080/10652469.2016.1252763>.
- [25] N. V. Proskurin, "Integral addition formulae for Airy function." *Integral Transforms Spec. Funct.*, vol. 5, no. 1, pp. 143–146, 1997, doi: <https://doi.org/10.1080/10652469708819130>.
- [26] W. Satsanit, "Solution to the triharmonic heat equation." *Electron. J. Differ. Equ.*, no. 4, pp. 1–9, 2011.
- [27] O. Vallee and M. Soares, *Airy functions and applications to physics*. London: Imperial College Press, 2004.

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THE GENERALIZED t -COMTET NUMBERS AND SOME COMBINATORIAL APPLICATIONS

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Abstract. In the present article we use a combinatorial approach to generalize the Comtet numbers. In particular, we establish some combinatorial identities, recurrence relations and generating functions. Additionally, for some particular cases we study their relationship with t -successive associated Stirling numbers and their q -analogue.

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1. INTRODUCTION

It is well-known that the Stirling numbers of the second kind $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ count the number of partitions of a set with n elements into k non-empty blocks. This sequence satisfies the recurrence relation

$$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = k \left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\},$$

with the initial conditions $\left\{ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right\} = 1$ and $\left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} 0 \\ n \end{smallmatrix} \right\} = 0$.

The Stirling numbers $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ can be generalized to the *associated Stirling numbers of the second kind* $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{\geq m}$ (cf. [1, 4, 7, 8, 10, 11, 18, 20]) by means of a restriction on the size of the blocks. In particular, this sequence gives the number of partitions of n elements into k blocks, such that each block contains at least m elements. It is clear that $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{\geq 1} = \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$. This combinatorial sequence has been applied to the study of some special polynomials such as generalized Bernoulli and Cauchy polynomials, (see, e.g., [12–16]).

Recently, Belbachir and Tebtoub [2] considered a variation for the associated Stirling numbers. They introduced the *2-successive associated Stirling numbers of the second kind* $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}^{[2]}$. This new sequence counts the number of partitions of n elements

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into k blocks, with the additional condition that each block contains at least two consecutive elements. Moreover, the last element n must either form a block with its predecessor or belong to another block satisfying the previous conditions. In [2], the authors derived the recurrence

$$\begin{Bmatrix} n \\ k \end{Bmatrix}^{[2]} = k \begin{Bmatrix} n-1 \\ k \end{Bmatrix}^{[2]} + \begin{Bmatrix} n-2 \\ k-1 \end{Bmatrix}^{[2]}, \quad n \geq 2k,$$

with the initial conditions $\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}^{[2]} = 1$, $\begin{Bmatrix} n \\ n-1 \end{Bmatrix}^{[2]} = 0$ and $\begin{Bmatrix} n \\ 0 \end{Bmatrix} = 0$ for $n \geq 1$.

Inspired by these results, in this paper we aim to investigate the sequence $\{a^{[t]}(n, k)\}_{n, k \geq 0}$, defined by the recurrence relation

$$a^{[t]}(n, k) = u_k a^{[t]}(n-1, k) + a^{[t]}(n-t, k-1), \quad n \geq tk, \quad (1.1)$$

with the initial conditions $a^{[t]}(0, 0) = 1$, $a^{[t]}(n, n-\ell) = 0$ for $\ell = 1, 2, \dots, t-1$ and $a^{[t]}(n, 0) = 0$, for $n \geq 1$. Moreover, $\{u_n\}$ is a sequence of real numbers.

We will call the sequence $\{a^{[t]}(n, k)\}_{n, k \geq 0}$ the *generalized t -Comtet numbers*. The reason for this name is that for $t = 1$ we recover the Comtet numbers (see, e.g., [9, 21]). Note that if $u_k = k$, then $a^{[t]}(n, k) = \begin{Bmatrix} n \\ k \end{Bmatrix}^{[t]}$. This sequence is called by Belbachir and Tebtoub [3] as the *t -successive associated Stirling numbers*. If $t = 2$ and $u_k = k$, then $a^{[2]}(n, k) = \begin{Bmatrix} n \\ k \end{Bmatrix}^{[2]}$. If $t = 1$ and $u_k = k$, then $a^{[1]}(n, k) = \begin{Bmatrix} n \\ k \end{Bmatrix}$.

In this paper our goal is to give the recurrence relation, the generating function and some combinatorial identities. For some particular cases, we give combinatorial interpretations.

2. BASIC PROPERTIES

From the recurrence relation (1.1) we obtain the following generating function.

Theorem 1. For $k \geq 1$,

$$A_k^{[t]}(x) := \sum_{n \geq tk} a^{[t]}(n, k) x^n = \frac{x^{tk}}{(1-u_0x)(1-u_1x)(1-u_2x) \cdots (1-u_kx)}, \quad (2.1)$$

with $A_0^{[t]}(x) = \frac{1}{1-u_0x}$.

Proof. Multiplying both sides of (1.1) by x^n and summing over $n \geq tk$, we have

$$\begin{aligned} A_k^{[t]}(x) &= u_k \sum_{n \geq tk} a^{[t]}(n-1, k) x^n + \sum_{n \geq tk} a^{[t]}(n-t, k-1) x^n \\ &= u_k x \sum_{n \geq tk} a^{[t]}(n, k) x^n + \sum_{n \geq tk-t} a^{[t]}(n, k-1) x^{n+t} \\ &= u_k x A_k^{[t]}(x) + x^t A_{k-1}^{[t]}(x). \end{aligned}$$

Then

$$A_k^{[t]}(x) = \frac{x^t A_{k-1}^{[t]}(x)}{1 - u_k x}.$$

Iterating this last recurrence, we obtain (2.1). $\square$

From the above relation, we have the following combinatorial expression.

Corollary 1. *The generalized t -Comtet numbers are given by the explicit identity*

$$a^{[t]}(n, k) = \sum_{i_1 + i_2 + \dots + i_k = n - tk} u_1^{i_1} u_2^{i_2} \dots u_k^{i_k}, \quad (2.2)$$

for $n \geq tk$.

Theorem 2. *The generalized t -Comtet numbers satisfy the following recurrence relation*

$$a^{[t]}(n, k) = \sum_{i=0}^{n-tk} u_k^i a^{[t]}(n-i-t, k-1). \quad (2.3)$$

Proof. For $n \geq tk$,

$$\begin{aligned} a(n, k) &= u_k a(n-1, k) + a(n-t, k-1), \\ u_k a(n-1, k) &= u_k^2 a(n-2, k) + u_k a(n-1-t, k-1), \\ u_k^2 a(n-2, k) &= u_k^3 a(n-3, k) + u_k^2 a(n-2-t, k-1), \\ &\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\ u_k^{n-tk-1} a(tk+1, k) &= u_k^{n-tk} a(tk, k) + u_k^{n-tk-1} a(tk+1-t, k-1), \\ u_k^{n-tk} a(tk, k) &= u_k^{n-tk+1} a(tk-1, k) + u_k^{n-tk} a(tk-1-t, k-1), \end{aligned}$$

by summing, we get the result. $\square$

Theorem 3. *We have the following rational explicit formula*

$$a^{[t]}(n + tk, k) = \sum_{j=0}^k \frac{u_j^{k+n}}{\prod_{i \neq j} (u_j - u_i)}, \quad (2.4)$$

which is independent from t .

Proof. We have

$$A_k^{[t]}(x) = \sum_{n \geq tk} a^{[t]}(n, k) x^n = x^{tk} \sum_{n \geq 0} a^{[t]}(n + tk, k) x^n,$$

then

$$\begin{aligned}
 \sum_{n \geq 0} a^{[t]}(n + tk, k) x^n &= \frac{1}{(1 - u_0 x)(1 - u_1 x) \cdots (1 - u_k x)} \\
 &= \sum_{j=0}^k \frac{\alpha_j}{1 - u_j x} \\
 &= \sum_{j=0}^k \frac{u_j^k}{\prod_{i \neq j} (u_j - u_i)} \sum_{n \geq 0} u_j^n x^n \\
 &= \sum_{n \geq 0} \left(\sum_{j=0}^k \frac{u_j^{k+n}}{\prod_{i \neq j} (u_j - u_i)} \right) x^n,
 \end{aligned}$$

which gives the result. $\square$

Corollary 2. *The dual expression depending on t*

$$a^{[t]}(n, k) = \sum_{j=0}^k \frac{u_j^{n+k(t-1)}}{\prod_{i \neq j} (u_j - u_i)}. \quad (2.5)$$

2.1. Exponential generating function for the t -Comtet numbers

Let $u_1, \dots, u_k$ be a sequence of complex numbers and let $(A_m)_{m=1, \dots, n}$ be the sequence of matrices such that A_m is $m \times m$ -matrix

$$A_m = \begin{bmatrix} u_{k-m} & u_{k-m+1} & \cdots & u_{k-1} \\ u_{k-m+1} & \cdots & & u_k \\ \vdots & \ddots & \ddots & 0 \\ u_{k-1} & u_k & 0 & 0 \end{bmatrix},$$

with the convention that $u_{<0} = 0$.

Consider also

$$\sigma_j = (-1)^j \sum_{1 \leq k_1 < k_2 < \cdots < k_j \leq k} u_{k_1} \cdots u_{k_j},$$

(the alternate sequence of elementary symmetric function associated to $u_1, u_2, \dots, u_k$).

We have $(p - u_1)(p - u_2) \cdots (p - u_k) = p^k + \sigma_1 p^{k-1} + \sigma_2 p^{k-2} + \cdots + \sigma_k$. Now we can state the following lemma which will be used to establish the main result of this subsection.

Lemma 1. *We have the following decomposition*

$$\frac{1}{p^n(p - u_1) \cdots (p - u_k)} = \sum_{i=0}^n \frac{\alpha_{n-i}}{p^i} + \sum_{j=1}^k \frac{\beta_j}{p - u_j}, \quad (2.6)$$

$$\text{with } \alpha_i = \frac{(-1)^{\lfloor (i+1)/2 \rfloor}}{\sigma_k^{i+1}} \det(A_i), \alpha_0 = 1/\sigma_k, \text{ and } \beta_j = \frac{1}{u_j^n \prod_{\substack{i=1 \\ i \neq j}}^k (u_j - u_i)}.$$

Proof. We leave the proof to the reader. $\square$

Let $C_k^{[t]}(x) := \sum_{n \geq tk} a^{[t]}(n, k) \frac{x^n}{n!}$, with $C_0^{[t]}(x) = 1$. We have

$$\frac{\partial^t}{\partial x^t} C_k^{[t]}(x) = \sum_{n \geq t(k-1)} a^{[t]}(n+t, k) \frac{x^n}{n!},$$

which gives using relation (1.1),

$$\frac{\partial^t}{\partial x^t} C_k^{[t]}(x) = u_k \frac{\partial^{t-1}}{\partial x^{t-1}} C_k^{[t]}(x) + C_{k-1}^{[t]}(x). \quad (2.7)$$

To solve the linear recurrence differential equation we use Laplace transform. Using the fact that

$$C_k^{[t]}(0) = \frac{\partial}{\partial x} C_k^{[t]}(y) \Big|_{y=0} = \dots = \frac{\partial^{t-1}}{\partial x^{t-1}} C_k^{[t]}(y) \Big|_{y=0} = 0,$$

we get

$$\prod_{i=1}^k (p^t - u_i p^{t-1}) \mathcal{L}(C_k^{[t]}(y)) = \mathcal{L}(C_{k-1}^{[t]}),$$

where $\mathcal{L}(C_k^{[t]}(y)) = \int_0^\infty C_k^{[t]} e^{py} \partial y$.

Thus by recursion, we get

$$p^{(t-1)k} \prod_{i=1}^k (p - u_i) \mathcal{L}(C_k^{[t]}(y)) = \mathcal{L}(C_0^{[t]}(y)) = \mathcal{L}(u(y)),$$

where $u(t)$ is the Heaviside function. Using Lemma 1, we have

$$\mathcal{L}(C_k^{[t]}(y)) = \mathcal{L}(u(y)) \left[\sum_{i=0}^{(t-1)k} \frac{\alpha_i}{p^i} + \sum_{j=1}^k \frac{\beta_j}{p - u_j} \right].$$

The inverse Laplace transform gives,

$$C_k^{[t]}(y) = \sum_{i=1}^{(t-1)k} \alpha_i \frac{y^{i-1}}{(i-1)!} + \sum_{j=1}^k \beta_j e^{u_j y}. \quad (2.8)$$

Theorem 4. *The exponential generating function of t -Comtet numbers is given by*

$$\sum_{n \geq tk} a^{[t]}(n, k) \frac{x^n}{n!} = \sum_{i=1}^{(t-1)k} \alpha_i \frac{x^{i-1}}{(i-1)!} + \sum_{j=1}^k \beta_j e^{u_j x}. \quad (2.9)$$

3. THE 2-SUCCESSIVE ASSOCIATED r -WHITNEY NUMBERS

In this section, we study the particular case $u_k = km + r$. Let $n, r \geq 0$ be integers. Let $\Pi_r(n, k)$ denote the set of partitions of the set $[n + r] := \{1, \dots, n, n + 1, \dots, n + r\}$ into $k + r$ blocks, such that, the first r elements are in distinct blocks. The elements $\{1, 2, \dots, r\}$ will be called *special elements*. A block of a partition of the above set is called *special* if it contains special element. The cardinality of $\Pi_r(n, k)$ is the r -Stirling numbers of the second kind [5].

The *2-successive associated r -Whitney numbers of the second kind*, denoted $W_{m,r}^{[2]}(n, k)$, count the number of partitions in $\Pi_r(n, k)$, such that:

- the k non-special blocks contain at least two consecutive numbers,
- all the elements but the last one and its predecessor in non-special blocks are coloured with one of m colours independently,
- the elements in the special blocks are not coloured,
- the last element $n + r$ must either form a block with its predecessor or belong to another block (special or not-special) satisfying the previous conditions.

We denote by $\Pi_{r,m}^{[2]}(n, k)$ the set of partitions in $\Pi_r(n, k)$ that satisfying the previous conditions. It is clear that if $r = 0$ and $m = 1$, then $W_{1,0}^{[2]}(n, k) = \{n\}_k^{[2]}$, (see [2]).

For example, $W_{2,3}^{[2]}(5, 2) = 15$ with the partitions being (the $m = 2$ different colours of the elements will be fixed as **red** and **blue**, and the $r = 3$ special elements are $\bar{1}, \bar{2}$ and $\bar{3}$):

$$\begin{aligned} & \{\{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}, \{\mathbf{4}, 5, 6\}, \{7, 8\}\}, \quad \{\{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}, \{\mathbf{4}, 5, 6\}, \{7, 8\}\}, \\ & \{\{\bar{1}, 6\}, \{\bar{2}\}, \{\bar{3}\}, \{4, 5\}, \{7, 8\}\}, \quad \{\{\bar{1}\}, \{\bar{2}, 6\}, \{\bar{3}\}, \{4, 5\}, \{7, 8\}\}, \\ & \{\{\bar{1}\}, \{\bar{2}\}, \{\bar{3}, 6\}, \{4, 5\}, \{7, 8\}\}, \quad \{\{\bar{1}, 4\}, \{\bar{2}\}, \{\bar{3}\}, \{5, 6\}, \{7, 8\}\}, \\ & \{\{\bar{1}\}, \{\bar{2}, 4\}, \{\bar{3}\}, \{5, 6\}, \{7, 8\}\}, \quad \{\{\bar{1}\}, \{\bar{2}\}, \{\bar{3}, 4\}, \{5, 6\}, \{7, 8\}\}, \\ & \{\{\bar{1}, 8\}, \{\bar{2}\}, \{\bar{3}\}, \{4, 5\}, \{6, 7\}\}, \quad \{\{\bar{1}\}, \{\bar{2}, 8\}, \{\bar{3}\}, \{4, 5\}, \{6, 7\}\}, \\ & \{\{\bar{1}\}, \{\bar{2}\}, \{\bar{3}, 8\}, \{4, 5\}, \{6, 7\}\}, \quad \{\{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}, \{\mathbf{4}, 5, 8\}, \{6, 7\}\}, \\ & \{\{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}, \{\mathbf{4}, 5, 8\}, \{6, 7\}\}, \quad \{\{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}, \{4, 5\}, \{\mathbf{6}, 7, 8\}\}, \\ & \quad \{\{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}, \{4, 5\}, \{\mathbf{6}, 7, 8\}\}. \end{aligned}$$

Theorem 5. *For $n \geq 2k$, we have*

$$W_{m,r}^{[2]}(n, k) = (km + r)W_{m,r}^{[2]}(n - 1, k) + W_{m,r}^{[2]}(n - 2, k - 1). \quad (3.1)$$

Proof. For any set partition of $\Pi_{r,m}^{[2]}(n, k)$, there are three options: either $n + r$ form a block with its predecessor $(n + r - 1)$, or $n + r$ is in a special block or $n + r$ is in a non-special block. In the first case, there are $W_{m,r}^{[2]}(n - 2, k - 1)$ possibilities. In the second case, the element $n + r$ can be place into one of the r special blocks and the remaining elements can be chosen in $W_{m,r}^{[2]}(n - 1, k)$. Altogether, we have $rW_{m,r}^{[2]}(n - 1, k)$ possibilities. For the third case, we can follow a similar argument, then we obtain $kmW_{m,r}^{[2]}(n - 1, k)$ possibilities. $\square$

A comparison of (3.1) and (1.1) shows that $a^{[2]}(n, k) = W_{m,r}^{[2]}(n, k)$ for $u_k = km + r$. Therefore, from Theorem 1 and Corollary 1 we get the following corollaries.

Corollary 3. For $k \geq 1$,

$$\begin{aligned} W_k^{[2]}(x) &:= \sum_{n \geq 2k} W_{m,r}^{[2]}(n, k) x^n \\ &= \frac{x^{2k}}{(1 - rx)(1 - (m + r)x)(1 - (2m + r)x) \cdots (1 - (km + r)x)}, \end{aligned} \quad (3.2)$$

with $W_0^{[2]}(x) = \frac{1}{1 - rx}$. Moreover, the 2-successive associated r -Whitney numbers of the second kind are given by the explicit identity

$$W_{m,r}^{[2]}(n, k) = \sum_{i_0 + i_1 + i_2 + \cdots + i_k = n - 2k} r^{i_0} (m + r)^{i_1} \cdots (km + r)^{i_k}, \quad (3.3)$$

for $n \geq 2k$.

In particular, for $m = 1$ and $r = 0$ we obtain the generating function of the 2-successive associated Stirling numbers of the second kind.

Corollary 4. (see [2, Theorem 2.3 and Corollary 2.4] and [3, Theorem 18]) For $k \geq 1$,

$$A_k(x) := \sum_{n \geq 2k} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}^{[2]} x^n = \frac{x^{2k}}{(1 - x)(1 - 2x) \cdots (1 - kx)}, \quad (3.4)$$

with $A_0(x) = 1$. Moreover,

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}^{[2]} = \sum_{i_1 + i_2 + \cdots + i_k = n - 2k} 1^{i_1} 2^{i_2} \cdots k^{i_k}.$$

Our next identity expresses $W_{m,r}^{[2]}(n, k)$ in terms of $\left\{ \begin{matrix} i \\ k \end{matrix} \right\}^{[2]}$ for $i \leq n$.

Theorem 6. Let $n, k \geq 0$,

$$W_{m,r}^{[2]}(n, k) = \sum_{i=2k}^n r^{n-i} \binom{n-k}{n-i} m^{i-2k} \left\{ \begin{matrix} i \\ k \end{matrix} \right\}^{[2]}. \quad (3.5)$$

Proof. From (3.2) we have

$$\begin{aligned} \sum_{n \geq 2k} W_{m,r}^{[2]}(n, k) x^n &= \frac{x^{2k}}{(1-rx)(1-(m+r)x)(1-(2m+r)x) \cdots (1-(km+r)x)} \\ &= \frac{x^{2k}}{(1-rx)^{k+1} \left(1 - \frac{mx}{1-rx}\right) \left(1 - \frac{2mx}{1-rx}\right) \cdots \left(1 - \frac{kmx}{1-rx}\right)} \\ &= \frac{(1-rx)^{k-1}}{m^{2k}} \frac{\left(\frac{mx}{1-rx}\right)^{2k}}{\left(1 - \frac{mx}{1-rx}\right) \left(1 - \frac{2mx}{1-rx}\right) \cdots \left(1 - \frac{kmx}{1-rx}\right)} \\ &= \frac{(1-rx)^{k-1}}{m^{2k}} \frac{y^{2k}}{(1-y)(1-2y) \cdots (1-ky)}, \end{aligned}$$

where $y = \frac{mx}{1-rx}$.

Therefore from (3.4), we have

$$\begin{aligned} \sum_{n \geq 2k} W_{m,r}^{[2]}(n, k) x^n &= \frac{(1-rx)^{k-1}}{m^{2k}} \sum_{i \geq 2k} \left\{ \begin{matrix} i \\ k \end{matrix} \right\}^{[2]} y^i \\ &= \sum_{i \geq 2k} \left\{ \begin{matrix} i \\ k \end{matrix} \right\}^{[2]} \frac{m^{i-2k} x^i}{(1-rx)^{i-k+1}} \\ &= \sum_{i \geq 2k} \sum_{j \geq 0} m^{i-2k} \binom{i-k+j}{i-k} r^j \left\{ \begin{matrix} i \\ k \end{matrix} \right\}^{[2]} x^{i+j}. \end{aligned}$$

Comparing the coefficients of x^n , we obtain (3.5). $\square$

Combinatorial proof: We can construct any set partition of $\Pi_{r,m}^{[2]}(n, k)$ as follows: we put $n-i$ elements in the special blocks. Then there are $\binom{n-k}{i-k} r^{n-i}$ possibilities. Note that we have to subtract k elements of n because in the non-special blocks there are at least two consecutive numbers. The remaining i elements ($i \geq 2k$) can be chosen in $m^{i-2k} \left\{ \begin{matrix} i \\ k \end{matrix} \right\}$ ways. The factor m^{i-2k} accounts for the $i-2k$ non-minimal elements within these blocks that are each to be colored in one of m ways.

From Theorem 5 and by induction on n we obtain the following identity.

Theorem 7. For $n \geq 2k$ we have

$$W_{m,r}^{[2]}(n, k) = \frac{1}{m^k k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (mj + r)^{n-k}.$$

Proof. Let,

$$\begin{aligned} W_{m,r}^{[2]}(n, k) &= \frac{(mk + r)}{m^k k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (mj + r)^{n-1-k} \\ &\quad + \frac{1}{m^{k-1} k!} \sum_{j=0}^k (-1)^{k-1-j} \binom{k-1}{j} (mj + r)^{n-1-k} \\ &= \frac{(mk + r)}{m^{k-1} k!} \sum_{j=1}^k (-1)^{k-j} \binom{k-1}{j-1} (mj + r)^{n-1-k} \\ &\quad + \frac{r}{m^k k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (mj + r)^{n-1-k} \\ &= \frac{1}{m^k k!} \sum_{i=0}^{n-1-k} r^{n-1-k-i} m^i \left[\sum_{j=0}^k (-1)^{k-j} j^{i+1} \binom{k}{j} m \right. \\ &\quad \left. + \sum_{j=0}^k (-1)^{k-j} j^i r \binom{k}{j} \right] \\ &= \frac{1}{m^k k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (mj + r)^{n-k}. \end{aligned}$$

□

3.1. Relations with the r -Whitney numbers

The r -Whitney numbers of the second kind $W_{m,r}(n, k)$ were defined by Mező [17] as the connecting coefficients between some particular polynomials.

For non-negative integers n, k and r with $n \geq k \geq 0$ and for any integer $m > 0$

$$(mx + r)^n = \sum_{k=0}^n m^k W_{m,r}(n, k) x^k, \quad (3.6)$$

where $x^n = x(x-1)\cdots(x-n+1)$ for $n \geq 1$, and $x^0 = 1$.

The r -Whitney numbers of the second kind satisfy the recurrence [17]

$$W_{m,r}(n, k) = W_{m,r}(n-1, k-1) + (km + r)W_{m,r}(n-1, k). \quad (3.7)$$

Comparing (3.7) and (3.1) we have the following relation.

Corollary 5. [2, Theorem 4.1] For $n \geq 2k$,

$$W_{m,r}^{[2]}(n, k) = W_{m,r}(n - k, k). \quad (3.8)$$

Mező and Ramírez [19] studied the r -Whitney matrices of the second and the first kind and they derived several identities for these matrices. In particular, the r -Whitney matrix of the second kind is defined by

$$[W_{m,r}(n, k)]_{n, k \geq 0} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ r & 1 & 0 & 0 & 0 \\ r^2 & m + 2r & 1 & 0 & 0 \\ r^3 & m^2 + 3rm + 3r^2 & 3m + 3r & 1 & 0 \\ r^4 & m^3 + 4rm^2 + 6r^2m + 4r^3 & 7m^2 + 12rm + 6r^2 & 6m + 4r & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Notice that the sequence $(W_{m,r}^{[2]}(n, k))_k$ corresponds with the sequence of elements on rays in direction $(1, 1)$ over the r -Whitney matrix of the second kind.

4. THE t -SUCCESSIVE ASSOCIATED r -WHITNEY NUMBERS

In this section, we consider the rays in direction $(s, 1)$, i.e., we are going to study the sequence $\{W_{m,r}(n - sk, k)\}$. We denote by $W_{m,r}^{[t]}(n, k)$ the number $W_{m,r}(n - sk, k)$, where $t = s + 1$. We call this new sequence *the t -successive associated r -Whitney numbers of the second kind*. It is possible to show that the t -successive associated r -Whitney numbers count the number of partitions in $\Pi_r(n, k)$, such that:

- the k non-special blocks contain at least t consecutive numbers,
- all the elements but the last one and its $t - 1$ predecessors in non-special blocks are coloured with one of m colours independently,
- the elements in the special blocks are not coloured,
- the last element $n + r$ must either form a block with its $t - 1$ -predecessors or belong to another block (special or not-special) satisfying the previous conditions.

Reasoning in a similar manner as in Theorem 5 we obtain the following results.

Theorem 8. For $n \geq tk$, we have

$$W_{m,r}^{[t]}(n, k) = (km + r)W_{m,r}^{[t]}(n - 1, k) + W_{m,r}^{[t]}(n - t, k - 1). \quad (4.1)$$

For $k \geq 1$,

$$\begin{aligned} W_k^{[t]}(x) &:= \sum_{n \geq 0} W_{m,r}^{[t]}(n, k) x^n \\ &= \frac{x^{tk}}{(1 - rx)(1 - (m + r)x)(1 - (2m + r)x) \cdots (1 - (km + r)x)}, \end{aligned} \quad (4.2)$$

with $W_0^{[t]}(x) = \frac{1}{1-rx}$. Moreover, for $n \geq tk$ we have

$$W_{m,r}^{[t]}(n, k) = \frac{1}{m^k k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (mj + r)^{n-(t-1)k}. \quad (4.3)$$

As corollary for $t = 2$ we get [3, Theorem 4, Theorem 6 and Theorem 7]. It is not difficult to generalize the relation given in Theorem 6.

Theorem 9. If $n, k \geq 0$, then

$$W_{m,r}^{[t]}(n, k) = \sum_{i=tk}^n r^{n-i} \binom{n-k}{n-i} m^{i-tk} \left\{ i \atop k \right\}^{[t]}. \quad (4.4)$$

Consequence. From Equation (4.2) we deduce that $W_{m,r}^{[t]}(n + (t-1)k, k)$ are the classical r -Whitney numbers $W_{m,r}(n, k)$.

From the explicit formula given in (4.3) we get the exponential generating function of the t -successive associated r -Whitney numbers.

Theorem 10. The exponential generating function of the t -successive associated r -Whitney numbers is

$$W_k^{[t]}(x) := \sum_{n \geq tk} W_{m,r}^{[t]} \frac{x^n}{n!} = \sum_{j=0}^k \binom{k}{j} \frac{(-1)^{k-j}}{k! m^k} \frac{e^{(jm+r)x}}{(jm+r)^{(t-1)k}}. \quad (4.5)$$

Corollary 6. For the 2-successive associated r -Whitney numbers,

$$W_k^{[2]}(x) = \sum_{j=0}^k \binom{k}{j} \frac{(-1)^{k-j}}{k! m^k} \frac{e^{(jm+r)x}}{(jm+r)^k}. \quad (4.6)$$

These two result are more specified expressions as relation (2.9) of Theorem 4.

Proof. (Theorem 10) We use the derivation $(t-1)k$ times according to x and using the consequence property, we get

$$\begin{aligned} \frac{\partial^{(t-1)k}}{\partial (t-1)k x} W_k(x) &= \sum_{n \geq tk - (t-1)k} W_{m,r}^{[t]}(n + k(t-1), k) \frac{x^n}{n!} \\ &= \frac{1}{k! m^k} e^{rx} (e^{mx} - 1)^k \\ &= \frac{1}{k! m^k} e^{rx} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} e^{jmx} \\ &= \sum_{j=0}^k \binom{k}{j} \frac{(-1)^{k-j}}{k! m^k} e^{(jm+r)x}. \end{aligned}$$

□

Theorem 11. For $n \geq k$, we have

$$W_{m,r}^{[t]}(n + (t-1)k, k) = \sum_{\substack{i_1 + \dots + i_n = n-k \\ i_1, \dots, i_n \in \{0,1\}}} \prod_{j=0}^{n-1} (r + m(j - \sum_{\ell=1}^j i_\ell))^{i_{j+1}}. \quad (4.7)$$

Proof. By induction over n , we suppose that the identity is true until $n-1$

$$\begin{aligned} & \sum_{\substack{i_1 + \dots + i_n = n-k \\ i_1, \dots, i_n \in \{0,1\}}} \prod_{j=0}^{n-1} (r + m(j - \sum_{l=1}^j i_l))^{i_{j+1}} \\ &= \sum_{\substack{i_1 + \dots + i_{n-1} = (n-1) - (k-1) \\ i_1, \dots, i_{n-1} \in \{0,1\}}} \prod_{j=0}^{n-2} (r + m(j - \sum_{l=1}^j i_l))^{i_{j+1}} \\ & \quad + \left(\sum_{\substack{i_1 + \dots + i_{n-1} = (n-1) - k \\ i_1, \dots, i_{n-1} \in \{0,1\}}} \prod_{j=0}^{n-2} (r + m(j - \sum_{l=1}^j i_l))^{i_{j+1}} \right) (r + mk). \end{aligned}$$

We have

$$\begin{aligned} W_{m,r}^{[t]}(n + (t-1)k, k) \\ &= (mk + r)W_{m,r}^{[t]}(n-1 + (t-1)k, k) + W_{m,r}^{[t]}(n-1 + (t-1)(k-1), k), \end{aligned}$$

which gives the desired result. □

Corollary 7. For $n \geq tk$, we have

$$W_{m,r}^{[t]}(n, k) = \sum_{\substack{i_1 + \dots + i_{n-(t-1)k} = n-tk \\ i_1, \dots, i_{n-(t-1)k} \in \{0,1\}}} \prod_{j=0}^{n-(t-1)k-1} (r + m(j - \sum_{\ell=1}^j i_\ell))^{i_{j+1}}, \quad (4.8)$$

with empty sum equal zero.

Example 1. For $k = t = 2$ we have the following formula

$$\begin{aligned} W_{m,r}^{[2]}(n, 2) &= \sum_{i_1 + \dots + i_{n-2} = n-4} r^{i_1} (r + m(1 - i_1))^{i_2} (r + m(2 - i_1 - i_2))^{i_3} \times \\ & \quad \dots \times (r + m(n-3 - i_1 - i_2 - \dots - i_{n-3}))^{i_{n-2}}, \\ W_{m,r}^{[2]}(4, 2) &= \sum_{i+j=0} r^i (r + m(1 - i))^j = 1, \end{aligned}$$

$$\begin{aligned}
W_{m,r}^{[2]}(5,2) &= \sum_{i+j+k=1} r^i (r+m(1-i))^j (r+m(2-i-j))^k = 3(r+m), \\
W_{m,r}^{[2]}(6,2) &= \sum_{i+j+k+\ell=2} r^i (r+m(1-i))^j (r+m(2-i-j))^k (r+m(3-i-j-k))^\ell \\
&= 6r^2 + 12mr + 7m^2.
\end{aligned}$$

Theorem 12. *We have the following explicit formula*

$$W_{m,r}^{[t]}(n+tk, k) = \frac{1}{m^k k!} \sum_{j=0}^{k-j} (-1)^{k-j} \binom{k}{j} (mj+r)^{n+k}, \quad (4.9)$$

and thus

$$W_{m,r}^{[t]}(n, k) = \frac{1}{m^k k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (mj+r)^{n-(t-1)k}. \quad (4.10)$$

Proof. It suffices to set $u_k = mk + r$ then $[u_j - u_i = m(j-i)]$ in Theorem 3. $\square$

For $t = 2$ we get Theorem 7.

Theorem 13. *Expression of t -successive r -Whitney numbers in terms of binomials and Stirling numbers.*

$$W_{m,r}^{[t]}(n+tk, k) = \left(\frac{r}{m}\right)^k \sum_{i=0}^{n+k} \binom{n+k}{i} r^{n-i} m^i \left\{ \begin{matrix} i \\ k \end{matrix} \right\}. \quad (4.11)$$

Proof.

$$\begin{aligned}
W_{m,r}^{[t]}(n+tk, k) &= \frac{1}{m^k k!} \sum_{j=0}^k \sum_{i=0}^{n+k} (-1)^{k-j} \binom{k}{j} \binom{n+k}{i} (mj)^i r^{n+k-i} \\
&= \frac{1}{m^k k!} \sum_{i=0}^{n+k} m^i r^{n+k-i} \binom{n+k}{i} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^i \\
&= \frac{1}{m^k} \sum_{i=0}^{n+k} m^i r^{n+k-i} \binom{n+k}{i} \left\{ \begin{matrix} i \\ k \end{matrix} \right\}.
\end{aligned}$$

Notice that $\sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^i = 0$ for $i < k$. $\square$

5. A q -ANALOGUE OF THE t -SUCCESSIVE ASSOCIATED STIRLING NUMBERS

Finally, we consider a q -analogue of the t -successive associated Stirling numbers of the second kind. For this purpose, we use a similar statistic studied by Carlitz [6], see also [21].

Let $\pi = B_1/B_2/\dots/B_k$ be any block representation of a set partition in $\Pi_{0,1}^{[t]}(n, k) := \Pi^{[t]}(n, k)$, with $\min(B_1) < \min(B_2) < \dots < \min(B_k)$. We define the following statistic on the set $\Pi^{[t]}(n, k)$.

$$w^{[t]}(\pi) := \sum_{i=1}^k (i-1)(|B_i| - t + 1).$$

We now define the q -analogue of the t -successive associated Stirling numbers of the second kind.

Definition 1. Define $\{n\}_q^{[t]}$ as the distribution polynomial for the $w^{[t]}$ statistic on the set $\Pi^{[t]}(n, k)$, that is,

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q^{[t]} = \sum_{\pi \in \Pi^{[t]}(n, k)} q^{w^{[t]}(\pi)}, \quad n, k \geq 0,$$

where q is an indeterminate.

It is clear that $\{n\}_1^{[t]} = \{n\}_1^{[t]}$.

For example, in the set $\Pi^{[2]}(7, 3)$ we have the following partitions:

$$\begin{aligned} & \{\{1, 2\}, \{3, 4, 5\}, \{6, 7\}\}, \quad \{\{1, 2, 3\}, \{4, 5\}, \{6, 7\}\}, \quad \{\{1, 2, 5\}, \{3, 4\}, \{6, 7\}\}, \\ & \{\{1, 2\}, \{3, 4\}, \{5, 6, 7\}\}, \quad \{\{1, 2\}, \{3, 4, 7\}, \{5, 6\}\}, \quad \{\{1, 2, 7\}, \{3, 4\}, \{5, 6\}\}. \end{aligned}$$

Therefore,

$$\left\{ \begin{matrix} 7 \\ 3 \end{matrix} \right\}_q^{[2]} = q^4 + q^3 + q^3 + q^5 + q^4 + q^3 = 3q^3 + 2q^4 + q^5.$$

Let us introduce the following notations.

$$[n]_q = 1 + q + \dots + q^{n-1}, \quad [n]_q! = [1]_q [2]_q \dots [n]_q \quad \text{and} \quad \left[\begin{matrix} n \\ k \end{matrix} \right]_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

The last coefficient is called q -binomial coefficient. If $q = 1$, then $\left[\begin{matrix} n \\ k \end{matrix} \right]_1 = \binom{n}{k}$.

Theorem 14. For $n \geq tk$, we have

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q^{[t]} = [k]_q \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}_q^{[t]} + q^{k-1} \left\{ \begin{matrix} n-t \\ k-1 \end{matrix} \right\}_q^{[t]}. \quad (5.1)$$

Proof. For any set partition of $\Pi^{[t]}(n, k)$, there are two options: either n form a block with its $t-1$ -predecessors, or n is in a block that satisfies the conditions. In the first case, there are $q^{k-1} \{n-t\}_{k-1}^{[t]}$ possibilities. In this case, the size of the last block B_k is t , then this block contributes a factor q^{k-1} . In the second case,

the element n can be place into one of the k blocks and thus contributes a factor $1 + q + q^2 + \cdots + q^{k-1} = [k]_q$. Moreover, the remaining elements can be chosen in $\{n-1\}_q^{[t]}$ ways. Altogether, we have $[k]_q \{n-1\}_q^{[t]}$ possibilities. $\square$

From above theorem, we obtain the following corollaries.

Corollary 8. For $k \geq 1$,

$$\sum_{n \geq tk} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q^{[t]} x^n = \frac{x^{tk} q^{\binom{k}{2}}}{(1-x)(1-[2]_q x)(1-[3]_q x) \cdots (1-[k]_q x)}. \quad (5.2)$$

Moreover, the q -analogue of the t -successive associated r -Stirling numbers are given by the explicit identity

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q^{[t]} = q^{\binom{k}{2}} \sum_{i_1 + i_2 + \cdots + i_k = n - tk} [1]_q^{i_1} [2]_q^{i_2} \cdots [k]_q^{i_k}, \quad (5.3)$$

for $n \geq tk$.

Corollary 9. For $n \geq tk$ we have

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q^{[t]} = \frac{1}{[k]_q!} \sum_{j=1}^k (-1)^{k-j} \left[\begin{matrix} k \\ j \end{matrix} \right]_q q^{\binom{k-j}{2}} ([j]_q)^{n-(t-1)k}.$$

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REFERENCES

- [1] H. Belbachir and I. E. Bousbaa, “Associated Lah numbers and r -Stirling numbers.” *arXiv:1404.5573*, pp. 1–24, 2014.
- [2] H. Belbachir and A. F. Tebtoub, “Les nombres de Stirling associés avec succession d’ordre 2, nombres de Fibonacci-Stirling et unimodalité.” *C. R., Math., Acad. Sci. Paris*, vol. 353, no. 9, pp. 767–771, 2015, doi: doi.org/10.1016/j.crma.2015.06.008.
- [3] H. Belbachir and A. F. Tebtoub, “The t -successive associated Stirling numbers, t -Fibonacci-Stirling numbers, and unimodality.” *Turkish J. Math.*, vol. 41, no. 5, pp. 1279–1291, 2017, doi: doi.org/10.3906/mat-1506-83.
- [4] M. Bóna and I. Mező, “Real zeros and partitions without singleton blocks.” *Eur. J. Comb.*, vol. 51, pp. 500–510, 2016, doi: doi.org/10.1016/j.ejc.2015.07.021.
- [5] A. Z. Broder, “The r -Stirling numbers.” *Discrete Math.*, vol. 49, pp. 241–259, 1984, doi: [doi.org/10.1016/0012-365X\(84\)90161-4](https://doi.org/10.1016/0012-365X(84)90161-4).
- [6] L. Carlitz, *Generalized Stirling numbers*. Combinatorial Analysis Notes, Duke University, 1968.
- [7] J. Y. Choi, L. Long, S.-H. Ng, and J. Smith, “Reciprocity for multirestricted Stirling numbers.” *J. Comb. Theory, Ser. A*, vol. 113, no. 6, pp. 1050–1060, 2006, doi: doi.org/10.1016/j.jcta.2005.10.001.

- [8] J. Y. Choi and J. D. H. Smith, “On the combinatorics of multi-restricted numbers.” *Ars Comb.*, vol. 75, pp. 45–63, 2005.
- [9] L. Comtet, “Nombres de Stirling généraux et fonctions symétriques.” *C. R. Acad. Sci., Paris, Sér. A*, vol. 275, pp. 747–750, 1972.
- [10] L. Comtet, *Advanced combinatorics*. D. Reidel Publishing Co. (Dordrecht, Holland), 1974.
- [11] F. T. Howard, “Associated Stirling numbers.” *Fibonacci Quart.*, vol. 18, pp. 303–315, 1980.
- [12] T. Komatsu, I. Mező, and L. Szalay, “Incomplete Cauchy numbers.” *Acta Math. Hung.*, vol. 149, no. 2, pp. 306–323, 2016, doi: [10.1007/s10474-016-0616-z](https://doi.org/10.1007/s10474-016-0616-z).
- [13] T. Komatsu, “Incomplete poly-Cauchy numbers.” *Monatsh. Math.*, vol. 180, no. 2, pp. 271–288, 2016, doi: [10.1007/s00605-015-0810-z](https://doi.org/10.1007/s00605-015-0810-z).
- [14] T. Komatsu, K. Liptai, and I. Mező, “Incomplete poly-Bernoulli numbers associated with incomplete Stirling numbers.” *Publ. Math.*, vol. 88, no. 3-4, pp. 357–368, 2016.
- [15] T. Komatsu and J. L. Ramírez, “Generalized poly-Cauchy and poly-Bernoulli numbers by using incomplete r -Stirling numbers.” *Aequationes Math.*, vol. 91, no. 6, pp. 1055–1071, 2017, doi: [10.1007/s00010-017-0509-4](https://doi.org/10.1007/s00010-017-0509-4).
- [16] T. Komatsu and J. L. Ramírez, “Incomplete poly-Bernoulli numbers and incomplete poly-Cauchy numbers associated to the q -Hurwitz-lerch zeta function.” *Mediterr. J. Math.*, vol. 14, no. 3, p. 19, 2017, doi: [10.1007/s00009-017-0935-5](https://doi.org/10.1007/s00009-017-0935-5).
- [17] I. Mező, “A new formula for the Bernoulli polynomials.” *Result. Math.*, vol. 58, no. 3-4, pp. 329–335, 2010, doi: [10.1007/s00025-010-0039-z](https://doi.org/10.1007/s00025-010-0039-z).
- [18] I. Mező, “Periodicity of the last digits of some combinatorial sequences.” *J. Integer Seq.*, vol. 17, no. 1, pp. article 14.1.1, 18, 2014.
- [19] I. Mező and J. L. Ramírez, “The linear algebra of the r -Whitney matrices.” *Integral Transforms Spec. Funct.*, vol. 26, no. 3, pp. 213–225, 2015, doi: [10.1080/10652469.2014.984180](https://doi.org/10.1080/10652469.2014.984180).
- [20] V. H. Moll, J. L. Ramírez, and D. Villamizar, “Combinatorial and arithmetical properties of the restricted and associated Bell and factorial numbers.” *J. Comb.*, vol. 9, no. 4, pp. 693–720, 2018, doi: [10.4310/JOC.2018.v9.n4.a7](https://doi.org/10.4310/JOC.2018.v9.n4.a7).
- [21] C. G. Wagner, “Partition statistics and q -Bell numbers ($q = -1$).” *J. Integer Seq.*, vol. 7, no. 1, pp. art. 04.1.1, 12, 2004.

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AN INEQUALITY FOR THE MODULUS OF THE RATIO OF TWO COMPLEX GAMMA FUNCTIONS

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Abstract. The Euler gamma function is closely connected with the theory of zeta-functions. We prove a new inequality for the modulus of the ratio of two complex gamma functions $\Gamma(s)/\Gamma(2-s)$, arising in problems of the size of Selberg zeta-functions at places symmetric with respect to the critical line. This inequality, used together with technics of estimation, allows us in a different way re-prove and extend the result of R. Garunkštis and A. Grigutis for the modified Selberg zeta-function.

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1. INTRODUCTION

The Euler gamma function is closely connected with zeta-functions, and its properties are of great importance to the theory of zeta-functions and applications. E.g., the Riemann zeta-function satisfies the well-known functional equation [3]

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s) \zeta(1-s), \quad (1.1)$$

the Selberg zeta-function associated with the modular group $\mathrm{PSL}(2, \mathbb{Z})$ satisfies the functional equation [4]

$$Z_{\mathrm{PSL}(2, \mathbb{Z})}(s) = Z_{\mathrm{PSL}(2, \mathbb{Z})}(1-s) \frac{\zeta(2s)}{\zeta(2(1-s))} \frac{\Gamma(2s)}{\Gamma(2(1-s))} (2\pi)^{1-2s} \times \\ \times \exp \left(\frac{\pi}{3} \int_0^{s-1/2} v \tan \pi v dv - \frac{\pi}{2} \int_0^{s-1/2} \frac{dv}{\cos \pi v} - \frac{4\pi}{3\sqrt{3}} \int_0^{s-1/2} \frac{\cos \pi v/3}{\cos \pi v} dv \right).$$

It is known [1] that

$$|\zeta(1-s)| > |\zeta(s)|, \quad (1.2)$$

where $s = \sigma + it$, is true for $\sigma > 1/2$ and $t \geq 6.8$ except where $\zeta(s) = 0$ (note that, if the inequality is valid without exceptions, then the Riemann hypothesis is true and vice versa).

In [2] R. Garunkštis and A. Grigutis have proved a similar theorem for the modified Selberg zeta-function

$$W(s) = Z_{\text{PSL}(2, \mathbb{Z})}(s) / \zeta(2s). \quad (1.3)$$

Theorem 1 (R. Garunkštis and A. Grigutis). *If $1/2 < \sigma < 1$ and $t \geq 6.053$, then*

$$|W(1-s)| > |W(s)|. \quad (1.4)$$

In the proof of the theorem R. Garunkštis and A. Grigutis established and used two lemmas for ratios of complex gamma functions [2].

Lemma 1 (R. Garunkštis and A. Grigutis). *For $t \in \mathbb{R}$ the following inequality holds:*

$$\left| \frac{\Gamma(2+it)}{\Gamma(it)} \right| < \left| \frac{\sqrt{2}}{2} + it \right|^2. \quad (1.5)$$

Lemma 2 (R. Garunkštis and A. Grigutis). *For $1/2 < \sigma < 1$ and $t \in \mathbb{R}$ the following inequality holds:*

$$\left| \frac{\Gamma(2s)}{\Gamma(2(1-s))} \right| \leq \left| 2s - 2 - \frac{\sqrt{2}}{2} \right|^{4(\sigma-1/2)}. \quad (1.6)$$

However, in order to obtain more subtle results, the following lemma has to be proved.

Lemma 3. *Let $s = \sigma + it$. For $1 < \sigma < 2$ we have*

$$\left| \frac{\Gamma(s)}{\Gamma(2-s)} \right| \leq |s|^{2(\sigma-1)}. \quad (1.7)$$

We can prove this statement using the theorem of Phragmén and Lindelöf.

Theorem 2 (Phragmén-Lindelöf). *Let $f(z)$ be analytic in the strip*

$$S(\alpha, \beta) = \{z | z = x + iy, \alpha < x < \beta\}. \quad (1.8)$$

Let us assume $|f(z)| \leq 1$ on the boundaries $x = \alpha$ and $x = \beta$, and moreover

$$|f(z)| < C e^{e^{k|y|}}$$

for some $C > 0$ and $0 < k < \frac{\pi}{\beta-\alpha}$. Then $|f(z)| \leq 1$ throughout the strip $S(\alpha, \beta)$.

Proof. See Rademacher [5] for the proof. □

2. PROOF OF LEMMA 3

Proof. Let $s = 2 - w$, $\sigma = 2 - \rho$. Then the statement of the theorem is equivalent to

$$\left| \frac{\Gamma(2-w)}{\Gamma(w)} \right| \leq |2-w|^{2(1-\rho)}, \quad (2.1)$$

here $0 < \rho < 1$. Let us denote

$$f(w) = \frac{\Gamma(2-w)}{\Gamma(w)(2-w)^{2(1-\rho)}}. \quad (2.2)$$

First consider the left boundary $\rho = 0$, $w = -it$. We obtain

$$f(w) = \frac{\Gamma(2+it)}{\Gamma(-it)(2+it)^2}. \quad (2.3)$$

Hence, since $\overline{\Gamma(z)} = \Gamma(\bar{z})$,

$$\begin{aligned} |f(w)| &= \frac{|\Gamma(2+it)|}{|\Gamma(it)||2+it|^2} = \frac{|(1+it)(it)|}{4+t^2} = \\ &= \frac{\sqrt{t^4+t^2}}{t^2+4} = \sqrt{\frac{t^4+t^2}{t^4+8t^2+16}}, \end{aligned} \quad (2.4)$$

yielding us $|f(w)| < 1$.

Now consider the right boundary $\rho = 1$, $w = 1 - it$. We obtain

$$f(w) = \frac{\Gamma(1+it)}{\Gamma(1-it)}. \quad (2.5)$$

Hence,

$$|f(w)| = \frac{|\Gamma(1+it)|}{|\Gamma(1-it)|} = 1. \quad (2.6)$$

Next, let us consider the modulus of the function $f(w)$. Since $|\Gamma(z)| \leq \Gamma(\Re z)$ and $\Gamma(2-\rho) = \Gamma(\sigma) \leq 1$, we obtain

$$|f(w)| = \frac{|\Gamma(2-w)|}{|\Gamma(w)||2-w|^{2(1-\rho)}} \leq \frac{\Gamma(2-\rho)}{((2-\rho)^2+t^2)^{(1-\rho)}} \left| \frac{1}{\Gamma(w)} \right| \leq \left| \frac{1}{\Gamma(w)} \right|. \quad (2.7)$$

It is known [5], that if t is sufficiently large (i.e. $|t| \geq 1$), then the reciprocal gamma function

$$\frac{1}{\Gamma(w)} = O(e^{\frac{\pi}{2}|t|}|t|^{\frac{1}{2}-\rho}). \quad (2.8)$$

Since the reciprocal gamma function is an entire function, it is bounded in every compact subset of the complex plane (in particular, for $0 \leq \rho \leq 1$ and $0 \leq t \leq 1$ the modulus of the reciprocal gamma function $|1/\Gamma(w)| \leq 2$).

Thus, $|f(w)| = O(e^{e^{|t|}})$ for $t \in \mathbb{R}$, which yields us the statement of Lemma 3. $\square$

3. THEOREM FOR THE MODIFIED SELBERG ZETA-FUNCTION

The inequality (1.7) of Lemma 3, used together with technics of estimation, allows us (see Theorem 3) in a different way re-prove and extend the result of R. Garunkštis and A. Grigutis for the modified Selberg zeta-function (cf. Theorem 1).

Theorem 3. *For $1/2 < \sigma < 1$ and $t \in [0, t_1) \cup (t_2, \infty)$ we have*

$$|W(1-s)| > |W(s)|. \quad (3.1)$$

Here $t_1 = 1.740440\dots$ and $t_2 = 6.088036\dots$ are the roots of the function

$$L_1(t) = \log \left(\frac{1+t^2}{\pi \cosh \log(2 \sinh \frac{\pi t}{6})} \right). \quad (3.2)$$

By the definition of the modified Selberg zeta-function (1.3) and Lemma 3 we have

$$\left| \frac{W(s)}{W(1-s)} \right| = \left| \frac{\Gamma(2s)}{\Gamma(2(1-s))} (2\pi)^{1-2s} e^{Q(s)} \right| \leq \left(\frac{|2s|^2}{2\pi} \right)^{2\sigma-1} e^{\Re(Q(s))}, \quad (3.3)$$

here

$$Q(s) = \int_0^{s-1/2} \frac{\pi}{3} v \tan \pi v - \frac{\pi}{2 \cos \pi v} - \frac{4\pi}{3\sqrt{3}} \frac{\cos \pi v/3}{\cos \pi v} dv. \quad (3.4)$$

Integral (3.4) can be evaluated using triangular contour with vertices at $A(0, 0)$, $B(\sigma - 1/2, t)$ and $C(\sigma - 1/2, 0)$: $\int_{AC} + \int_{CB} + \int_{BA} = 0$. Hence,

$$\underbrace{\Re(Q(s))}_{\Re \int_{AB}} = \underbrace{I_1(\sigma)}_{\int_{AC}} + \underbrace{R(\sigma, t)}_{\Re \int_{CB}}. \quad (3.5)$$

Here

$$I_1(\sigma) = \int_0^{\sigma-1/2} \frac{\pi}{3} \theta \tan \pi \theta - \frac{\pi}{2 \cos \pi \theta} - \frac{4\pi}{3\sqrt{3}} \frac{\cos \pi \theta/3}{\cos \pi \theta} d\theta \quad (3.6)$$

and

$$R(\sigma, t) = \Re \left\{ \int_0^t \frac{\frac{\pi i}{3} (\sigma - \frac{1}{2} + i\theta)}{\cot(\pi(\sigma - \frac{1}{2} + i\theta))} - \frac{\frac{\pi i}{2} + \frac{4\pi i}{3\sqrt{3}} \cos(\frac{\pi}{3}(\sigma - \frac{1}{2} + i\theta))}{\cos(\pi(\sigma - \frac{1}{2} + i\theta))} d\theta \right\}. \quad (3.7)$$

Let us denote

$$L(\sigma, t) = (2\sigma - 1) \log \frac{\sigma^2 + t^2}{\pi/2} + I_1(\sigma) + R(\sigma, t). \quad (3.8)$$

Hence (cf. (3.3) and (3.5)),

$$\log \left| \frac{W(s)}{W(1-s)} \right| \leq L(\sigma, t). \quad (3.9)$$

Calculating function $I_1(\sigma)$ (3.6), we obtain

$$\begin{aligned} I_1(\sigma) &= \int_0^{\sigma-1/2} \frac{\pi}{3} \theta \tan \pi \theta - \frac{\pi}{2 \cos \pi \theta} - \frac{4\pi}{3\sqrt{3}} \frac{\cos \pi \theta/3}{\cos \pi \theta} d\theta = \\ &= -\frac{1}{6\pi} \text{Cl}_2(2\pi\sigma) - \frac{2\sigma-1}{6} \log 2 - \frac{2\sigma-1}{6} \log \sin \pi\sigma - \\ &\quad -\frac{1}{2} \log \tan \frac{\pi\sigma}{2} + \frac{2}{3} \log \left(\frac{\sqrt{3}}{2} \cot \frac{\pi\sigma}{3} - \frac{1}{2} \right). \end{aligned} \quad (3.10)$$

Here $\text{Cl}_2(x)$ is the Clausen function of order 2,

$$\text{Cl}_2(x) = -\int_0^x \log \left| 2 \sin \frac{t}{2} \right| dt. \quad (3.11)$$

Noticing that (cf. (3.7))

$$\Re \left\{ \frac{i(\sigma - 1/2 + i\theta)}{\cot(\pi(\sigma - 1/2 + i\theta))} \right\} = \frac{\theta \sin 2\pi\sigma + (1/2 - \sigma) \sinh 2\pi\theta}{\cosh 2\pi\theta - \cos 2\pi\sigma}, \quad (3.12)$$

$$\Re \left\{ \frac{i/2}{\cos(\pi(\sigma - 1/2 + i\theta))} \right\} = \frac{\cos \pi\sigma \sinh \pi\theta}{\cosh 2\pi\theta - \cos 2\pi\sigma} \quad (3.13)$$

and

$$\Re \left\{ \frac{i \cos \left(\frac{\pi}{3} \left(\sigma - \frac{1}{2} + i\theta \right) \right)}{\cos \left(\pi \left(\sigma - \frac{1}{2} + i\theta \right) \right)} \right\} = \frac{\cos \left(\frac{2\pi\sigma}{3} + \frac{\pi}{6} \right) \sinh \frac{4\pi\theta}{3} + \cos \left(\frac{4\pi\sigma}{3} - \frac{\pi}{6} \right) \sinh \frac{2\pi\theta}{3}}{\cosh 2\pi\theta - \cos 2\pi\sigma}. \quad (3.14)$$

we calculate function $R(\sigma, t)$ (3.7), obtaining

$$\begin{aligned} R(\sigma, t) &= \underbrace{\frac{\pi}{3} \sin 2\pi\sigma \int_0^t \frac{\theta}{\cosh 2\pi\theta - \cos 2\pi\sigma} d\theta}_{=I_2(\sigma, t)} + \\ &\quad + \underbrace{\frac{\pi}{3} \left(\frac{1}{2} - \sigma \right) \int_0^t \frac{\sinh 2\pi\theta}{\cosh 2\pi\theta - \cos 2\pi\sigma} d\theta}_{=I_3(\sigma, t)} + \\ &\quad + \underbrace{\pi \cos \pi\sigma \int_0^t \frac{-\sinh \pi\theta}{\cosh 2\pi\theta - \cos 2\pi\sigma} d\theta}_{=I_4(\sigma, t)} + \\ &\quad + \underbrace{\frac{-4\pi}{3\sqrt{3}} \int_0^t \frac{\cos \left(\frac{2\pi\sigma}{3} + \frac{\pi}{6} \right) \sinh \frac{4\pi\theta}{3} + \cos \left(\frac{4\pi\sigma}{3} - \frac{\pi}{6} \right) \sinh \frac{2\pi\theta}{3}}{\cosh 2\pi\theta - \cos 2\pi\sigma} d\theta}_{=I_5(\sigma, t)}. \end{aligned} \quad (3.15)$$

Note that $R(\sigma, t)$ is even function by t , thus, it suffices to consider non-negative t values. Calculating summands of the function $R(\sigma, t)$ (3.15), we obtain

$$I_2(\sigma, t) = \frac{\pi}{3} \sin 2\pi\sigma \int_0^t \frac{\theta}{\cosh 2\pi\theta - \cos 2\pi\sigma} d\theta, \quad (3.16)$$

$$\begin{aligned} I_3(\sigma, t) = & -\frac{2\sigma-1}{12} \log(\cosh 2\pi t - \cos 2\pi\sigma) + \\ & + \frac{2\sigma-1}{6} \log \sin \pi\sigma + \frac{2\sigma-1}{12} \log 2, \end{aligned} \quad (3.17)$$

$$I_4(\sigma, t) = \frac{1}{2} \log \tan \frac{\pi\sigma}{2} - \frac{1}{4} \log \frac{\cosh \pi t - \cos \pi\sigma}{\cosh \pi t + \cos \pi\sigma}, \quad (3.18)$$

$$\begin{aligned} I_5(\sigma, t) = & -\frac{2}{3} \log \left(\frac{\sqrt{3}}{2} \cot \frac{\pi\sigma}{3} - \frac{1}{2} \right) - \\ & - \frac{1}{3} \log \frac{\cosh \frac{2\pi t}{3} - \cos \frac{2\pi\sigma}{3}}{\cosh \frac{2\pi t}{3} - \cos \frac{2\pi(\sigma-1)}{3}}. \end{aligned} \quad (3.19)$$

Expressions (3.16)-(3.19) allow us to calculate function $L(\sigma, t)$ (3.8),

$$\begin{aligned} L(\sigma, t) = & (2\sigma-1) \log \left(\frac{2}{\pi} (\sigma^2 + t^2) \right) - (2\sigma-1) \frac{\log 2}{12} - \\ & - \frac{1}{6\pi} \text{Cl}_2(2\pi\sigma) + \frac{\pi}{3} \sin 2\pi\sigma \int_0^t \frac{\theta}{\cosh 2\pi\theta - \cos 2\pi\sigma} d\theta - \\ & - \frac{2\sigma-1}{12} \log(\cosh 2\pi t - \cos 2\pi\sigma) + \frac{1}{4} \log \frac{\cosh \pi t + \cos \pi\sigma}{\cosh \pi t - \cos \pi\sigma} + \\ & + \frac{1}{3} \log \frac{\cosh \frac{2\pi t}{3} - \cos \frac{2\pi(\sigma-1)}{3}}{\cosh \frac{2\pi t}{3} - \cos \frac{2\pi\sigma}{3}}. \end{aligned} \quad (3.20)$$

Next we will establish several auxiliary lemmas concerning the behaviour of the function $L(\sigma, t)$.

4. THE DERIVATIVES OF THE FUNCTION $L(\sigma, t)$

Lemma 4. *For $1/2 < \sigma < 1$ and fixed $1/2 \leq t < \infty$ the function $L(\sigma, t)$ (3.20) is convex by σ .*

Proof. Let us calculate partial derivatives of the function $L(\sigma, t)$ with respect to the variable σ ,

$$\begin{aligned}
 L'_\sigma(\sigma, t) = & \underbrace{2 \log \frac{\sigma^2 + t^2}{\pi/2}}_{=F_1(\sigma, t)} + \underbrace{\frac{2\sigma(2\sigma-1)}{\sigma^2 + t^2}}_{=F_2(\sigma, t)} + \underbrace{\frac{-\pi t}{3} \frac{\sinh 2\pi t}{\cosh 2\pi t - \cos 2\pi\sigma}}_{=F_3(\sigma, t)} + \\
 & + \underbrace{\frac{\pi}{6}(2\sigma-1) \frac{-\sin 2\pi\sigma}{\cosh 2\pi t - \cos 2\pi\sigma}}_{=F_4(\sigma, t)} + \underbrace{\frac{-\pi \sin \pi\sigma \cosh \pi t}{\cosh 2\pi t - \cos 2\pi\sigma}}_{=F_5(\sigma, t)} + \\
 & + \underbrace{\frac{\pi}{3\sqrt{3}} \frac{1 - 2 \cosh \frac{2\pi t}{3} \cos \left(\frac{2\pi\sigma}{3} - \frac{\pi}{3} \right)}{\left(\cosh \frac{2\pi t}{3} - \cos \frac{2\pi(\sigma-1)}{3} \right) \left(\cosh \frac{2\pi t}{3} - \cos \frac{2\pi\sigma}{3} \right)}}_{=F_6(\sigma, t)}. \tag{4.1}
 \end{aligned}$$

$$\begin{aligned}
 L''_{\sigma\sigma}(\sigma, t) = & \frac{12\sigma-2}{\sigma^2 + t^2} - \frac{4\sigma^2(2\sigma-1)}{(\sigma^2 + t^2)^2} + \\
 & + \underbrace{\frac{-\pi \sin 2\pi\sigma + \pi(2\sigma-1) \cos 2\pi\sigma + 3\pi \cos \pi\sigma \cosh \pi t}{3 \cosh 2\pi t - \cos 2\pi\sigma}}_{=G_1(\sigma, t)} + \\
 & + \underbrace{\frac{\pi^2}{3} \sin 2\pi\sigma \frac{(2\sigma-1) \sin 2\pi\sigma + 6 \sin \pi\sigma \cosh \pi t + 2t \sinh 2\pi t}{(\cosh 2\pi t - \cos 2\pi\sigma)^2}}_{=G_2(\sigma, t)} + \\
 & + \underbrace{\frac{2\pi^2}{9\sqrt{3}} \frac{\sin \left(\frac{2\pi\sigma}{3} - \frac{\pi}{3} \right)}{\left(\cosh \frac{2\pi t}{3} - \cos \frac{2\pi(\sigma-1)}{3} \right)^2 \left(\cosh \frac{2\pi t}{3} - \cos \frac{2\pi\sigma}{3} \right)^2}}_{=G_3(\sigma, t)} \times \\
 & \times \underbrace{\left(2 \cos \left(\frac{2\pi\sigma}{3} - \frac{\pi}{3} \right) - \cosh \frac{2\pi t}{3} \left(\frac{7}{2} + \cos \left(\frac{4\pi\sigma}{3} - \frac{2\pi}{3} \right) \right) + 2 \cosh^3 \frac{2\pi t}{3} \right)}_{=G_4(\sigma, t)}. \tag{4.2}
 \end{aligned}$$

First consider the interval $1/2 \leq t \leq 2$. Let us give a lower bounds of the functions $G_k(\sigma, t)$, which are defined in (4.2). Denote

$$v_1(\sigma, t) = \cosh 2\pi t - \cos 2\pi\sigma,$$

$$u_1(\sigma, t) = \sin 2\pi\sigma + \pi(2\sigma-1) \cos 2\pi\sigma + 3\pi \cos \pi\sigma \cosh \pi t.$$

Note that $v_1(\sigma, t)$ is positive for $t \in \mathbb{R}$ and $1/2 < \sigma < 1$. Next, $u_1(\sigma, t)$ is negative for $t \in \mathbb{R}$ and $1/2 < \sigma \leq 3/4$. For $3/4 < \sigma < 1$, we have $0 < \cos 2\pi\sigma < 1$ and $\cos \pi\sigma < 0$.

Hence,

$$u_1(\sigma, t) \leq \underbrace{\sin 2\pi\sigma + \pi(2\sigma - 1) + 3\pi \cos \pi\sigma}_{=w_1(\sigma)}.$$

The function $w_1(\sigma)$ is convex, because

$$w_1''(\sigma) = -4\pi^2 \underbrace{\sin 2\pi\sigma}_{\leq 0} - 3\pi^3 \underbrace{\cos \pi\sigma}_{\leq 0} > 0,$$

for $3/4 < \sigma < 1$. With $w_1(3/4) < 0$ and $w_1(1) < 0$ it yields us $u_1(\sigma, t) < 0$ for $t \in \mathbb{R}$ and $1/2 < \sigma < 1$. Thus,

$$G_1(\sigma, t) = -\frac{\pi}{3} \frac{u_1(\sigma, t)}{v_1(\sigma, t)} > 0 \quad (4.3)$$

for $t \in \mathbb{R}$ and $1/2 < \sigma < 1$.

The function

$$G_2(\sigma, t) > \underbrace{\frac{2\pi^2}{-3} \frac{(3 + 2t \sinh \pi t) \cosh \pi t}{(\cosh 2\pi t - 1)^2}}_{=u_2(t)-increasing} \geq u_2(1/2) > \varepsilon = -0.8, \quad (4.4)$$

for $t \geq 1/2$ and $1/2 < \sigma < 1$.

The function $G_3(\sigma, t)$ is positive for $1/2 < \sigma < 1$.

Next let us show that $G_4(\sigma, t)$ is increasing by σ and by t . Indeed, consider the derivatives

$$\begin{aligned} (G_4)'_{\sigma} &= -\frac{4\pi}{3} \sin\left(\frac{2\pi\sigma}{3} - \frac{\pi}{3}\right) + \frac{4\pi}{3} \cosh \frac{2\pi t}{3} \sin\left(\frac{4\pi\sigma}{3} - \frac{2\pi}{3}\right) = \\ &= \frac{4\pi}{3} \underbrace{\sin\left(\frac{2\pi\sigma}{3} - \frac{\pi}{3}\right)}_{>0} \left(\underbrace{2\cos\left(\frac{2\pi\sigma}{3} - \frac{\pi}{3}\right)}_{\in(1,2)} \underbrace{\cosh \frac{2\pi t}{3} - 1}_{\geq 1} \right) > 0. \end{aligned}$$

$$(G_4)'_t = \frac{2\pi}{3} \sinh \frac{2\pi t}{3} \left(6 \cosh^2 \frac{2\pi t}{3} - \left(\frac{7}{2} + \cos\left(\frac{4\pi\sigma}{3} - \frac{2\pi}{3}\right) \right) \right) > 0$$

for $t > 0$. Thus $G_4(\sigma, t) \geq G_4(1/2, 1/2) > 0$. Hence,

$$L''_{\sigma\sigma}(\sigma, t) \geq \underbrace{\frac{12\sigma - 2}{\sigma^2 + t^2} - \frac{4\sigma^2(2\sigma - 1)}{(\sigma^2 + t^2)^2}}_{=B(\sigma, t)} - 0.8.$$

Note that the function $B(\sigma, t)$ is decreasing by t . Consider

$$\begin{aligned} B'_t(\sigma, t) &= (12\sigma - 2) \frac{-2t}{(\sigma^2 + t^2)^2} + (8\sigma^3 - 4\sigma^2) \frac{4t}{(\sigma^2 + t^2)^3} = \\ &= \frac{4t}{(\sigma^2 + t^2)^3} \left(\underbrace{(2\sigma - 3)\sigma^2}_{<0} - (6\sigma - 1)t^2 \right) < 0 \end{aligned}$$

for $t > 0$. Thus, $B(\sigma, t) \geq B(\sigma, 2) = B_2(\sigma)$ and

$$B_2(\sigma) = \frac{12\sigma - 2}{\sigma^2 + 4} - \frac{4\sigma^2(2\sigma - 1)}{(\sigma^2 + 4)^2} - 0.8 = 4 \frac{\sigma^3(1 - 0.2\sigma) + (12\sigma - 1.1\sigma^2 - 5.2)}{(\sigma^2 + 4)^2} > 0$$

for $1/2 < \sigma < 1$, yielding us the statement of the lemma for $1/2 \leq t \leq 2$.

Next consider the interval $t \geq C$. Let $C = \sqrt{\frac{1}{2}} + \sqrt{\frac{11}{12}} = 1.20724\dots$. Let us consider the second partial derivative $L''_{\sigma\sigma}$ (4.2). We have shown that the function $G_1(\sigma, t)$ (4.3) and the product $G_3(\sigma, t)G_4(\sigma, t)$ are positive for $t > 0$. Hence, using (4.4), we obtain

$$L''_{\sigma\sigma}(\sigma, t) \geq \underbrace{\frac{12\sigma - 2}{\sigma^2 + t^2} - \frac{4\sigma^2(2\sigma - 1)}{(\sigma^2 + t^2)^2} - \frac{\pi^2}{6} \frac{(3 + 2t \sinh \pi t) \cosh \pi t}{\sinh^4 \pi t}}_{=D(\sigma, t)}.$$

Let us show that $D(\sigma, t)$ is increasing by σ . Indeed, consider the derivative

$$D'_\sigma(\sigma, t) = 4 \frac{P(\sigma, t)}{(\sigma^2 + t^2)^3},$$

here

$$P(\sigma, t) = -\sigma^4 - \sigma^3 - 6t^2\sigma^2 + 3t^2\sigma + 3t^4.$$

For $1/2 < \sigma < 1$ the polynomial $P(\sigma, t)$ is concave by σ , because the second derivative

$$P''_{\sigma\sigma} = -12\sigma^2 - 6\sigma - 12t^2 < 0.$$

The function is nonnegative at the endpoints of the interval,

$$P(1/2, t) = 3t^4 - 3/16 \geq 0$$

for $t \geq 1/2$, and

$$P(1, t) = 3t^4 - 3t^2 - 2 \geq 0$$

for $t \geq C$. Hence, $P(\sigma, t) \geq 0$ and $D'_\sigma(\sigma, t) \geq 0$ for $t \geq C$, yielding us $D(\sigma, t) > D(1/2, t)$. Consider

$$\begin{aligned} D(1/2, t) &= \frac{4}{t^2 + 1/4} - \frac{\pi^2}{6} \frac{(3 + 2t \sinh \pi t) \cosh \pi t}{\sinh^4 \pi t} = \\ &= \frac{4}{t^2} \frac{1}{1 + 1/(4t^2)} - \frac{4\pi^2}{3} t e^{-2\pi t} \frac{(1 + 3e^{-\pi t}/t - e^{-2\pi t})(1 + e^{-2\pi t})}{(1 - e^{-2\pi t})^4} \geq \\ &\geq \frac{4H}{t^2} - 4Ate^{-2\pi t} = \frac{4}{t^2} \underbrace{(H - At^3e^{-2\pi t})}_{=E(t)}. \end{aligned}$$

Here

$$H = \frac{1}{1 + 1/(4C^2)} = 0.854\dots, \quad A = \frac{\pi^2}{3} \frac{(1 + 3e^{-\pi C}/C)(1 + e^{-2\pi C})}{(1 - e^{-2\pi C})^4} = 3.483\dots$$

The function $E(t)$ is increasing for $t \geq C$ and $E(C)$ is positive, yielding us the statement of the lemma. $\square$

Lemma 5. For $1/2 < \sigma < 1$, the derivative $L'_t(\sigma, t)$

- (1) is positive for $t \in (0, 3.53]$,
- (2) is negative for $t \in [3.77, \infty)$.

Proof. Let us calculate the first partial derivative

$$\begin{aligned} L'_t(\sigma, t) &= \underbrace{(2\sigma - 1) \frac{2t}{\sigma^2 + t^2}}_{=N_1(\sigma, t)} + \underbrace{\frac{\pi}{3} \frac{t \sin 2\pi\sigma}{\cosh 2\pi t - \cos 2\pi\sigma}}_{=N_2(\sigma, t)} + \\ &+ \underbrace{\frac{\pi}{6} (2\sigma - 1) \frac{-\sinh 2\pi t}{\cosh 2\pi t - \cos 2\pi\sigma}}_{=N_3(\sigma, t)} + \underbrace{\frac{-\pi \cos \pi\sigma \sinh \pi t}{\cosh 2\pi t - \cos 2\pi\sigma}}_{=N_4(\sigma, t)} + \\ &+ \underbrace{\frac{2\pi}{9} \frac{\sinh \frac{2\pi t}{3} \left(\cos \frac{2\pi(\sigma-1)}{3} - \cos \frac{2\pi\sigma}{3} \right)}{\left(\cosh \frac{2\pi t}{3} - \cos \frac{2\pi(\sigma-1)}{3} \right) \left(\cosh \frac{2\pi t}{3} - \cos \frac{2\pi\sigma}{3} \right)}}_{=N_5(\sigma, t)}. \end{aligned} \quad (4.5)$$

Estimating functions in (4.5) we obtain $N_5(\sigma, t) > 0$, $N_1(\sigma, t) > 0$. Now let us show that $N_2(\sigma, t) + N_3(\sigma, t) + N_4(\sigma, t) > 0$. It is sufficient to prove that

$$\underbrace{\frac{1}{3} t \sin 2\pi\sigma - \frac{2\sigma - 1}{6} \sinh 2\pi t - \cos \pi\sigma \sinh \pi t}_{=N(\sigma, t)} > 0.$$

Consider the derivative

$$\begin{aligned} N'_t(\sigma, t) &= \frac{1}{3} \sin 2\pi\sigma - \frac{2\sigma-1}{3} \pi \cosh 2\pi t - \pi \cos \pi\sigma \cosh \pi t = \\ &= -\frac{1}{3} (2\pi(2\sigma-1) \cosh^2 \pi t + 3\pi \cos \pi\sigma \cosh \pi t - (\sin 2\pi\sigma + (2\sigma-1)\pi)). \end{aligned}$$

The positive root of the quadratic equation

$$r(\sigma) = \frac{-3\pi \cos \pi\sigma + \sqrt{9\pi^2 \cos^2 \pi\sigma + 8\pi(2\sigma-1) \sin 2\pi\sigma + 8\pi^2(2\sigma-1)^2}}{4\pi(2\sigma-1)}.$$

For $1/2 < \sigma < 1$,

$$\frac{3 + \sqrt{17}}{4} < r(\sigma) < \frac{3\pi}{4}.$$

For $0 < t \leq 0.37$

$$1 < \cosh 2\pi t \leq 1.755 < \frac{3 + \sqrt{17}}{4}.$$

Hence, $N'_t(\sigma, t) > 0$ and $N(\sigma, t) > N(\sigma, 0) = 0$, yielding us the statement of the lemma for $t \in (0, 0.37]$.

Consider the first partial derivative in the interval $t \in [0.37, 3.53]$,

$$\begin{aligned} L'_t(\sigma, t) &= (2\sigma-1) \frac{2t}{\sigma^2 + t^2} + \underbrace{\frac{\pi \cos \pi\sigma (2t \sin \pi\sigma - 3 \sinh \pi t)}{3 \cosh 2\pi t - \cos 2\pi\sigma}}_{=H_2(\sigma, t)} + \\ &+ \underbrace{\frac{\pi}{6} (2\sigma-1) \frac{-\sinh 2\pi t}{\cosh 2\pi t - \cos 2\pi\sigma}}_{=H_3(\sigma, t)} + \\ &+ \underbrace{\frac{2\pi}{9} \frac{\sinh \frac{2\pi t}{3} \left(\cos \frac{2\pi(\sigma-1)}{3} - \cos \frac{2\pi\sigma}{3} \right)}{\left(\cosh \frac{2\pi t}{3} - \cos \frac{2\pi(\sigma-1)}{3} \right) \left(\cosh \frac{2\pi t}{3} - \cos \frac{2\pi\sigma}{3} \right)}}_{=H_4(\sigma, t)}. \end{aligned} \tag{4.6}$$

Estimating $H_2(\sigma, t)$ and $H_4(\sigma, t)$ we obtain $H_2(\sigma, t) > 0$ and $H_4(\sigma, t) > 0$ for $1/2 < \sigma < 1$. Thus,

$$\begin{aligned} L'_t(\sigma, t) &> (2\sigma-1) \frac{2t}{\sigma^2 + t^2} + \frac{\pi}{6} (2\sigma-1) \frac{-\sinh 2\pi t}{\cosh 2\pi t - \cos 2\pi\sigma} > \\ &> (2\sigma-1) \left(\frac{2t}{\sigma^2 + t^2} - \frac{\pi}{6} \frac{\sinh 2\pi t}{\cosh 2\pi t - 1} \right) = \\ &= (2\sigma-1) \underbrace{\left(\frac{2t}{\sigma^2 + t^2} - \frac{\pi}{6} \coth \pi t \right)}_{=M(\sigma, t)}. \end{aligned}$$

The function $M(\sigma, t)$ is decreasing by σ , hence

$$M(\sigma, t) \geq \frac{2t}{1+t^2} - \frac{\pi}{6} \coth \pi t \geq \underbrace{\frac{2t}{1+t^2} - \frac{\pi}{6} A_k}_{=M_k(t)}.$$

Here

$$A_k = \begin{cases} \coth 0.37\pi & \text{for } 0.37 \leq t < 2.77, \\ \coth 2.77\pi & \text{for } 2.77 \leq t \leq 3.53. \end{cases}$$

Now

$$M'_k(t) = 2 \frac{1-t^2}{(1+t^2)^2}.$$

For $0.37 \leq t < 2.77$ the function $M_1(t)$ increases in the interval $(0.37, 1)$ and decreases in the interval $(1, 2.77)$. At endpoints the function is positive, $M_1(0.37) > 0$ and $M_1(2.77) > 0$.

For $2.77 \leq t \leq 3.53$ the function $M_2(t)$ decreases. At the endpoint the function is positive, $M_2(3.53) > 0$, yielding us the statement of the lemma for the interval $0.37 \leq t \leq 3.53$.

Estimating functions $N_k(\sigma, t)$ in (4.5) for $t \geq 3.77$ we obtain (note that $N_2(\sigma, t) < 0$)

$$\begin{aligned} L'_t(\sigma, t) &< \frac{2t(2\sigma-1)}{\sigma^2+t^2} - \frac{\pi(2\sigma-1)}{6} - \frac{\pi \cos \pi \sigma}{2 \sinh \pi t} + \frac{2\pi}{3\sqrt{3}} \frac{\coth \frac{\pi t}{3} \sin(\frac{2\pi\sigma}{3} - \frac{\pi}{3})}{\cosh \frac{2\pi t}{3} - \cos \frac{2\pi\sigma}{3}} < \\ &< (2\sigma-1) \underbrace{\left(\frac{2t}{\sigma^2+t^2} + \underbrace{\left(-\frac{\pi}{6} + \frac{\pi^2}{4 \sinh \pi t} + \frac{2\pi^2}{9\sqrt{3}} \frac{\coth \frac{\pi t}{3}}{\cosh \frac{2\pi t}{3} - \frac{1}{2}} \right)}_{=\mu(t)} \right)}_{=K(\sigma, t)}. \end{aligned}$$

Consider the derivative of the function $\mu(t)$

$$\mu'(t) = \underbrace{\frac{\pi^3 \cosh \pi t}{-4 \sinh^2 \pi t}}_{<0} + \underbrace{\frac{-2\pi^3 \sinh^{-2} \frac{\pi t}{3} \left(\cosh \frac{2\pi t}{3} - \frac{1}{2} \right) + 2 \coth \frac{\pi t}{3} \sinh \frac{2\pi t}{3}}_{<0} \frac{1}{\left(\cosh \frac{2\pi t}{3} - \frac{1}{2} \right)^2}.$$

Hence,

$$K'_t(\sigma, t) = (2\sigma-1) \left(\underbrace{\frac{2(\sigma^2-t^2)}{(\sigma^2+t^2)^2}}_{<0} + \underbrace{\mu'(t)}_{<0} \right) < 0$$

the function $K(\sigma, t)$ is decreasing by t and $K(\sigma, t) \leq K(\sigma, 3.77) < 0$, yielding us the statement of the lemma. $\square$

5. AUXILIARY LEMMAS

Let us denote

$$L_1(t) = L(1, t). \quad (5.1)$$

Lemma 6. *The function $L_1(t)$*

- (1) *is negative for $t \in (0, t_1) \cup (t_2, \infty)$,*
- (2) *is positive for (t_1, t_2) ,*
- (3) *has unique maximum point at $t^* \in (t_1, t_2)$.*

Here $t_1 = 1.740440\dots$ and $t_2 = 6.088036\dots$ are the roots of the function.

Proof. By (5.1) and (3.20) we obtain

$$\begin{aligned} L_1(t) &= \log \left(\frac{2}{\pi} (1 + t^2) \right) - \frac{\log 2}{12} - \frac{1}{12} \log(\cosh 2\pi t - 1) + \\ &\quad + \frac{1}{4} \log \frac{\cosh \pi t - 1}{\cosh \pi t + 1} + \frac{1}{3} \log \frac{\cosh \frac{2\pi t}{3} - 1}{\cosh \frac{2\pi t}{3} - \frac{1}{2}} = \\ &= \log \left(\frac{1 + t^2}{\pi} \frac{\sinh \frac{\pi t}{6}}{\sinh^2 \frac{\pi t}{6} + \left(\frac{1}{2}\right)^2} \right) = \log \left(\frac{1 + t^2}{\pi \cosh \log \left(2 \sinh \frac{\pi t}{6} \right)} \right). \end{aligned} \quad (5.2)$$

Note that

$$\lim_{t \rightarrow 0+} L_1(t) = -\infty, \quad \lim_{t \rightarrow +\infty} L_1(t) = -\infty. \quad (5.3)$$

Next, $L_1(t) = 0$ iff

$$\underbrace{\cosh \log \left(2 \sinh \frac{\pi t}{6} \right)}_{\geq 1} = \frac{1 + t^2}{\pi}. \quad (5.4)$$

Hence, there are no zeros in the interval $(0, t_0)$. The function $L_1(t)$ is negative in the interval (cf. (5.3)). Here $t_0 = \sqrt{\pi - 1} = 1.463418\dots$ By (5.2),

$$\begin{aligned} L_1(t) &= \log \left(\frac{2}{\pi} (1 + t^2) e^{-\frac{\pi t}{6}} \frac{1}{1 + \frac{e^{-\frac{2\pi t}{3}}}{1 - e^{-\frac{\pi t}{3}}}} \right) = \\ &= \underbrace{\log \frac{2}{\pi} + \log(1 + t^2) - \frac{\pi t}{6}}_{=\Psi_1(t)} + \underbrace{\log \left(1 - \frac{1}{e^{\frac{2\pi t}{3}} - e^{\frac{\pi t}{3}} + 1} \right)}_{=\Psi_2(t)}. \end{aligned} \quad (5.5)$$

The function $\Psi_1(t)$ is concave for $t > 1$. Indeed, consider the second derivative,

$$\Psi_1''(t) = 2 \frac{1-t^2}{(1+t^2)^2} < 0. \quad (5.6)$$

Next, consider the second derivative of $\Psi_2(t)$,

$$\Psi_2''(t) = \frac{-\pi^2 e^{\frac{2\pi t}{3}}}{9 \left(e^{\frac{2\pi t}{3}} - e^{\frac{\pi t}{3}} + 1 \right)^2 \left(e^{\frac{\pi t}{3}} - 1 \right)^2} \underbrace{\left(4e^{\frac{2\pi t}{3}} - 7e^{\frac{\pi t}{3}} + 4 \right)}_{>0} < 0. \quad (5.7)$$

Combining (5.6) and (5.7) we obtain, that the function $L_1(t)$ is concave for $t > 1$. Thus, the concave function on an open set takes negative, then positive (e.g. $L_1((6/\pi)\log 3) > 0$), then again negative values (cf. (5.3)). Hence, it has unique positive maximum in the interval (t_1, t_2) . Here t_1 and t_2 are the roots of the function $L_1(t)$. The values of the roots we obtain numerically with any sufficient accuracy. $\square$

Next, let us denote

$$L_0(\sigma) = L(\sigma, 0). \quad (5.8)$$

Lemma 7. *For $1/2 < \sigma < 1$, the function $L_0(\sigma)$ is negative.*

Proof. By (3.8) and (3.15) we have

$$L_0(\sigma) = (2\sigma - 1) \log \frac{2}{\pi} \sigma^2 + I_1(\sigma).$$

Calculating the derivative of the function (cf. (3.10)) we obtain

$$\begin{aligned} L_0'(\sigma) &= \underbrace{2 \log \frac{2}{\pi} \sigma^2 + \frac{2\sigma - 2}{\sigma}}_{=Q_1(\sigma)} + \underbrace{\frac{-\pi}{6} (2\sigma - 1) \cot \pi \sigma + \frac{-\pi}{2 \sin \pi \sigma}}_{=Q_2(\sigma)} + \\ &+ \underbrace{\frac{4\pi}{3\sqrt{3}} \frac{1}{1 - 2 \cos \left(\frac{2\pi\sigma}{3} - \frac{\pi}{3} \right)}}_{=Q_3(\sigma)} + 2. \end{aligned} \quad (5.9)$$

Let us estimate $Q_k(\sigma)$ functions from above. For $1/2 < \sigma < 1$ the function $Q_1(\sigma)$ is negative since the derivative

$$Q_1'(\sigma) = \frac{4}{\sigma} + \frac{2}{\sigma^2} > 0$$

and $Q_1(1) < 0$.

For $1/2 < \sigma < 1$ the function

$$Q_2(\sigma) = \underbrace{\frac{-\pi}{6 \sin \pi \sigma}}_{<0} \underbrace{((2\sigma - 1) \cos \pi \sigma + 3)}_{=q_2(\sigma)}.$$

The derivative

$$q_2'(\sigma) = 2 \cos \pi \sigma - (2\sigma - 1) \pi \sin \pi \sigma < 0,$$

while $q_2(1) = 2 > 0$, hence $q_2(\sigma) > 0$, and $Q_2(\sigma)$ is negative.

For $1/2 < \sigma < 1$ the function

$$Q_3(\sigma) = \frac{4\pi}{3\sqrt{3}} \frac{1}{1 - 2\cos\left(\frac{2\pi\sigma}{3} - \frac{\pi}{3}\right)} + 2 < 0.$$

Hence, $L_0'(\sigma) < 0$ with $L_0(1/2) = 0$, yielding us the statement of the lemma. $\square$

6. PROOF OF THE THEOREM FOR THE MODIFIED SELBERG ZETA-FUNCTION

Now we can prove the Theorem 3.

Proof. Consider *max* value of the function $L(\sigma, t)$ in the rectangle $(\sigma, t) \in (1/2, 1) \times (0, t_1)$. By Lemma 5, the function $L(\sigma, t)$ has no stationary points in the interior of the rectangle, so it suffices to investigate the behaviour of the function on vertices of the rectangle. By Lemma 4, the function $L(\sigma, t)$ is convex by σ and the derivative by t is positive, hence we must consider the first zero of the function $L_1(t)$ (cf. Lemma 6). Note that

$$\lim_{\sigma \rightarrow 1/2+} L(\sigma, t) = 0. \quad (6.1)$$

Next let us consider *max* value of the function $L(\sigma, t)$ in the strip $(\sigma, t) \in (1/2, 1) \times (t_2, \infty)$. By Lemma 5, function $L(\sigma, t)$ has no stationary points in the interior of the strip, so it suffices to investigate the behaviour of the function on vertices. By Lemma 4, the function $L(\sigma, t)$ is convex by σ and the derivative by t is negative, hence we must consider the second zero of the function $L_1(t)$ (cf. Lemma 6). By Lemma 7 and (6.1), it gives us $L(\sigma, t) < 0$. Consequently (cf. (3.9))

$$\log \left| \frac{W(s)}{W(1-s)} \right| < 0,$$

yielding us the statement of the theorem. $\square$

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REFERENCES

- [1] R. D. Dixon and L. Schoenfeld, “The size of the Riemann zeta-function at places symmetric with respect to the point $1/2$.” *Duke Math. J.*, vol. 33, no. 2, pp. 291—292, 1966, doi: [10.1215/S0012-7094-66-03333-3](https://doi.org/10.1215/S0012-7094-66-03333-3).
- [2] R. Garunkštis and A. Grigutis, “The size of the Selberg zeta-function at places symmetric with respect to the line $\text{Re}(s) = 1/2$.” *Results. Math.*, vol. 70, no. 1, pp. 271—281, 2016, doi: [10.1007/s00025-015-0486-7](https://doi.org/10.1007/s00025-015-0486-7).

- [3] H. Iwaniec, *Lectures on the Riemann Zeta Function*. American Mathematical Society, 2014.
- [4] M. Minamide, “On zeros of the derivative of the modified Selberg zeta function for the modular group.” *J. Ind. Math. Soc.*, vol. 80, no. 3-4, pp. 275—312, 2013.
- [5] H. Rademacher, *Topics in analytic number theory*. Springer-Verlag Berlin Heidelberg, 1973. doi: [10.1007/978-3-642-80615-5](https://doi.org/10.1007/978-3-642-80615-5).

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PROINOV CONTRACTIONS AND DISCONTINUITY AT FIXED POINT

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Abstract. In this paper, we show that the contractive definition considered by Proinov [Fixed point theorems in metric spaces, *Nonlinear Analysis* 64 (2006) 546 - 557] is strong enough to generate a fixed point but does not force the mapping to be continuous at the fixed point. Thus we provide several answers to the open question posed by B.E. Rhoades in *Contractive definitions and continuity*, *Contemporary Mathematics* 72(1988), 233-245.

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1. INTRODUCTION

For a self-mapping T of a metric space (X, d) we put

$$m(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\}. \quad (1.1)$$

The most general type of contractive condition is either a Ćirić [7] type contractive condition

$$d(Tx, Ty) \leq \alpha \cdot m(x, y), x, y \in X, 0 \leq \alpha < 1,$$

or a φ -contractive condition (see [4, 6, 8, 9]) of the form

$$d(Tx, Ty) \leq \varphi(m(x, y)), x, y \in X,$$

where $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfies different set of conditions (see [1, 18, 23, 24]), or a Meir-Keeler [15] type (ϵ, δ) contractive condition,

given $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that for $x, y \in X$,

$$\epsilon \leq m(x, y) < \epsilon + \delta \implies d(Tx, Ty) < \epsilon.$$

Let us recall that a fixed point of T is said to be contractive [12] if all of the Picard iterates of T converge to this fixed point. The set $O(x; T) = \{T^n x : n = 0, 1, 2, \dots\}$ is called the orbit of the self-mapping T at the point $x \in X$. Mapping T is orbitally continuous at a point $z \in X$ if for any sequence $\{x_n\} \subset O(x; T)$ (for some $x \in X$) $x_n \rightarrow z$ implies $Tx_n \rightarrow Tz$ as $n \rightarrow \infty$. Every continuous self-mapping of a metric

space is orbitally continuous, but the converse need not be true (see Example 2.1 below). Mapping T is asymptotically regular if $\lim_{n \rightarrow \infty} d(T^n x, T^{n+1} x) = 0$ for each $x \in X$ [5].

Let Φ_1 [21] denote the class of all functions $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying; for any $\epsilon > 0$ there exists $\delta > \epsilon$ such that $\epsilon < t < \delta$ implies $\varphi(t) \leq \epsilon$. It is easy to see that $\varphi(t) < t$ for $t > 0$. In 2006, Proinov [21] proved the following very interesting fixed point theorem which subsumes most of the fixed point theorems based on conditions discussed above.

Theorem 1. *Let (X, d) be a complete metric space. Let T be a continuous and asymptotically regular self-mapping on X such that:*

- (i) *There exists $\varphi \in \Phi_1$ such that $d(Tx, Ty) \leq \varphi(D_\gamma(x, y))$ for all $x, y \in X$, where $D_\gamma(x, y) = d(x, y) + \gamma[d(x, Tx) + d(y, Ty)]$, where $\gamma \geq 0$;*
- (ii) *$d(Tx, Ty) < D_\gamma(x, y)$ for all $x, y \in X$ with $x \neq y$.*

Then T has a contractive fixed point. Moreover, in the case $\gamma = 1$ and if φ is continuous and satisfies $\varphi(t) < t$ for all $t > 0$, then the continuity of T can be dropped.

Let us point out (see Theorem 3.2 [21]), that the first part of Theorem 1 is equivalent to the following result.

Theorem 2. *Let (X, d) be a complete metric space. Let T be a continuous and asymptotically regular self-mapping on X such that:*

- (i) *For any $\epsilon > 0$ there exists $\delta > \epsilon$ such that $\epsilon < D_\gamma(x, y) < \delta$ implies $d(Tx, Ty) \leq \epsilon$;*
- (ii) *$d(Tx, Ty) < D_\gamma(x, y)$ for all $x, y \in X$ with $x \neq y$.*

Then T has a contractive fixed point.

The question whether there exists a contractive definition which is strong enough to generate a fixed point but which does not force the mapping to be continuous at the fixed point was investigated by Rhoades in [24] as an existing open problem. The question of the existence of contractive mappings which are discontinuous at their fixed points was settled in the affirmative by Pant [16]. In order to achieve his goal he employed a combination of an $(\epsilon - \delta)$ condition and a ϕ -contractive condition to prove a fixed point in which the fixed point may be a point of discontinuity. Recently in [3] Bisht and Rakočević have studied generalized Meir-Keeler type contractions and discontinuity at fixed point.

In this paper we show that the contractive definition introduced by Proinov [21] need not be continuous at the fixed point. Thus we not only relax the continuity requirement in the results proved by Proinov but also provide more answers to the open question posed in [24].

2. MAIN RESULTS

Our first main result is the following.

Theorem 3. *Let (X, d) be a complete metric space. Let T be an orbitally continuous and asymptotically regular self-mapping on X satisfying (i) and (ii) of Theorem 1. Then T has a contractive fixed point.*

Proof. Let x_0 be any point in X . Define a sequence $\{x_n\}$ in X given by the rule $x_{n+1} = T^n x_0 = T x_n$. From the proof of Theorem 4.1 of [21], we know that $\lim_n d(x_n, x_{n+1}) = 0$ and that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $z \in X$ such that $x_n \rightarrow z$ as $n \rightarrow \infty$. Also $T x_n \rightarrow z$. Orbital continuity of T implies that $\lim_{n \rightarrow \infty} T x_n = T z$. This yields $T z = z$, that is, z is a fixed point of T . Uniqueness of the fixed point follows from (ii). $\square$

As in Theorem 2, the first part of Theorem 3 is equivalent to the following result.

Theorem 4. *Let (X, d) be a complete metric space. Let T be an orbitally continuous and asymptotically regular self-mapping on X such that:*

- (i) *For any $\epsilon > 0$ there exists $\delta > \epsilon$ such that $\epsilon < D_\gamma(x, y) < \delta$ implies $d(Tx, Ty) \leq \epsilon$;*
- (ii) *$d(Tx, Ty) < D_\gamma(x, y)$ for all $x, y \in X$ with $x \neq y$.*

Then T has a contractive fixed point.

The following example [16] illustrates the above theorems.

Example 1. Let $X = [0, 2]$ and d be the usual metric on X . Define $T : X \rightarrow X$ by

$$T(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1; \\ 0, & \text{if } 1 < x \leq 2. \end{cases}$$

Then T satisfies all the conditions of above theorems and has a unique fixed point $x = 1$. The mapping T satisfies condition (i) of Theorems 3 and 4 with

$$\varphi(t) = \begin{cases} 1 & \text{if } t > 1; \\ \frac{t}{2}, & \text{if } t \leq 1, \end{cases}$$

$$\delta(\epsilon) = \begin{cases} 1 & \text{for } \epsilon \geq 1; \\ 1 - \epsilon, & \text{for } \epsilon < 1, \text{ respectively.} \end{cases}$$

It can also be easily seen that $\lim_{x \rightarrow 1} D_\gamma(x, 1) \neq 0$ and T is discontinuous at the fixed point $x = 1$. It can be verified that T neither satisfy Theorem 1 nor Theorem 2. Therefore, Theorems 3 and 4 are effective generalizations of Proinov fixed point theorem [21].

In the next theorem, we shall use the continuity of T^2 and a special case of $D_\gamma(x, y)$, that is $D_1(x, y)$.

Theorem 5. *Let (X, d) be a complete metric space. Let T be an asymptotically regular self-mapping on X such that T^2 is continuous for any $x, y \in X$;*

- (i) There exists $\varphi \in \Phi_1$ such that $d(Tx, Ty) \leq \varphi(D_1(x, y))$ for all $x, y \in X$ and $\varphi(t) < t$ for all $t > 0$;
- (ii) $d(Tx, Ty) < D_1(x, y)$ for all $x, y \in X$ with $x \neq y$.

Then T has a contractive fixed point. say z , and $T^n x \rightarrow z$ for each $x \in X$.

Proof. Let x_0 be any point in X . Define a sequence $\{x_n\}$ in X given by the rule $x_{n+1} = T^n x_0 = Tx_n$. Then from the above theorem (taking $\gamma = 1$) we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $z \in X$ such that $x_n \rightarrow z$ as $n \rightarrow \infty$. Also $Tx_n \rightarrow z$ and $T^2x_n \rightarrow z$. By continuity of T^2 , we have $T^2x_n \rightarrow T^2z$. This implies $T^2z = z$. Using (i) we get

$$d(Tz, T^2x_n) \leq \varphi(D_1(z, Tx_n)) = \varphi(d(z, Tx_n) + d(z, Tz) + d(x_n, T^2x_n)).$$

Taking $n \rightarrow \infty$ and in view of $\varphi(t) < t$ we get $d(z, Tz) = 0$, i.e., z is a fixed point of T . Uniqueness of the fixed point follows from (ii). $\square$

As in Theorem 2 the first part of Theorem 5 is equivalent to the following result.

Theorem 6. Let (X, d) be a complete metric space. Let T be an asymptotically regular self-mapping on X such that T^2 is continuous for any $x, y \in X$;

- (i) For any $\epsilon > 0$ there exists $\delta > \epsilon$ such that $\epsilon < D_1(x, y) < \delta$ implies $d(Tx, Ty) \leq \epsilon$;
- (ii) $d(Tx, Ty) < D_1(x, y)$ for all $x, y \in X$ with $x \neq y$.

Then T has a contractive fixed point. say z , and $T^n x \rightarrow z$ for each $x \in X$.

The following theorems verify that power contraction also allows the possibility of discontinuity at the fixed point.

Theorem 7. Let (X, d) be a complete metric space. Let T be an asymptotically regular self-mapping on X such that:

- (i) There exists $\varphi \in \Phi_1$ such that

$$d(T^n x, T^n y) \leq \varphi(d(x, y) + d(x, T^n x) + d(y, T^n y))$$

for all $x, y \in X$ and $n \in \mathbb{N}$;

- (ii)

$$d(T^n x, T^n y) < d(x, y) + d(x, T^n x) + d(y, T^n y)$$

for all $x, y \in X$ with $x \neq y$.

Then T has a contractive fixed point. say z , and $T^n x \rightarrow z$ for each $x \in X$.

Proof. By Theorem 1, T^n has a unique fixed point $z \in X$; i.e., $T^n(z) = z$. Then $T(z) = T(T^n(z)) = T^n(T(z))$ and so $T(z)$ is a fixed point of T^n . Since the fixed point of T^n is unique, $Tz = z$. To prove the uniqueness, we assume that y is another fixed point of T . Then $Ty = y$ and so $T^n(y) = y$. Again by the uniqueness of the fixed point of T^n , we have $z = y$. Hence z is a fixed point of T . $\square$

As in Theorem 2, the first part of Theorem 7 is equivalent to the following result.

Theorem 8. Let (X, d) be a complete metric space. Let T be an asymptotically regular self-mapping on X such that:

- (i) For any $\epsilon > 0$ there exists $\delta > \epsilon$ such that $\epsilon < d(x, y) + d(x, T^n x) + d(y, T^n y) < \delta$ implies $d(T^n x, T^n y) \leq \epsilon$ and $n \in \mathbb{N}$;
- (ii) $d(T^n x, T^n y) < d(x, y) + d(x, T^n x) + d(y, T^n y)$ for all $x, y \in X$ with $x \neq y$.

Then T has a contractive fixed point. say z , and $T^n x \rightarrow z$ for each $x \in X$.

In view of

$$\begin{aligned} m(x, y) &= \max\{d(x, y), d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/2\} \\ &\leq d(x, y) + d(x, Tx) + d(y, Ty), \end{aligned}$$

we get the following corollaries:

Corollary 1. Let (X, d) be a complete metric space. Let T be an orbitally continuous self-mapping on X such that:

- (i) There exists $\varphi \in \Phi_1$ such that $d(Tx, Ty) \leq \varphi(m(x, y))$ for all $x, y \in X$;
- (ii) $d(Tx, Ty) < m(x, y)$ for all $x, y \in X$ with $x \neq y$.

Then T has a contractive fixed point. say z , and $T^n x \rightarrow z$ for each $x \in X$.

Corollary 2. Let (X, d) be a complete metric space. Let T be an orbitally continuous self-mapping on X such that:

- (i) For any $\epsilon > 0$ there exists $\delta > \epsilon$ such that $\epsilon < m(x, y) < \delta$ implies $d(Tx, Ty) \leq \epsilon$;
- (ii) $d(Tx, Ty) < m(x, y)$ for all $x, y \in X$ with $x \neq y$.

Then T has a contractive fixed point. say z , and $T^n x \rightarrow z$ for each $x \in X$.

Corollary 3. Let (X, d) be a complete metric space. Let T be a self-mapping on X such that T^2 is continuous for any $x, y \in X$;

- (i) There exists $\varphi \in \Phi_1$ such that $d(Tx, Ty) \leq \varphi(m(x, y))$ for all $x, y \in X$;
- (ii) $d(Tx, Ty) < m(x, y)$ for all $x, y \in X$ with $x \neq y$.

Then T has a contractive fixed point. say z , and $T^n x \rightarrow z$ for each $x \in X$.

Corollary 4. Let (X, d) be a complete metric space. Let T be a self-mapping on X such that T^2 is continuous for any $x, y \in X$;

- (i) For any $\epsilon > 0$ there exists $\delta > \epsilon$ such that $\epsilon < m(x, y) < \delta$ implies $d(Tx, Ty) \leq \epsilon$;
- (ii) $d(Tx, Ty) < m(x, y)$ for all $x, y \in X$ with $x \neq y$.

Then T has a contractive fixed point. say z , and $T^n x \rightarrow z$ for each $x \in X$.

Remark 1. The above proved theorems unify and improve the results due to Bisht and Pant [2], Boyd and Wong [4], Ćirić [7], Jachymski [8], Kannan [10], Lim [13], Kuczma et al. [11], Matkowski [14], Pant [16, 17], Park and Bae [19], Park and Rhoades [20], Proinov [21] and Rao and Rao [22].

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REFERENCES

- [1] M. Abtahi, Z. Kadelburg, and S. Radenović, “Fixed points of Ćirić-Matkowski-type contraction in ν -generalized metric spaces.” *RACSAM*, vol. 111, pp. 57–64, 2017, doi: [10.1007/s13398-016-0275-5](https://doi.org/10.1007/s13398-016-0275-5).
- [2] R. K. Bisht and R. P. Pant, “A remark on discontinuity at fixed point.” *J. Math. Anal. Appl.*, vol. 445, pp. 1239–1241, 2017, doi: [10.1016/j.jmaa.2016.02.053](https://doi.org/10.1016/j.jmaa.2016.02.053).
- [3] R. K. Bisht and V. Rakočević, “Generalized Meir-Keeler type contractions and discontinuity at fixed point.” *Fixed Point Theory*, vol. 19, no. 1, pp. 57–64, 2018, doi: [10.24193/fpt-ro.2018.1.06](https://doi.org/10.24193/fpt-ro.2018.1.06).
- [4] D. W. Boyd and J. S. Wong, “On nonlinear contractions.” *Proc. Amer. Math. Soc.*, vol. 20, no. 2, pp. 458–464, 1969, doi: [10.1090/S0002-9939-1969-0239559-9](https://doi.org/10.1090/S0002-9939-1969-0239559-9).
- [5] F. E. Browder and W. V. Petryshyn, “The solution by iteration of nonlinear functional equations in Banach spaces.” *Bull. Am. Math. Soc.*, vol. 72, no. 3, pp. 571–575, 1966, doi: [10.1090/S0002-9904-1966-11544-6](https://doi.org/10.1090/S0002-9904-1966-11544-6).
- [6] K. J. Chung, “On fixed point theorems of Meir and Keeler.” *Math. Japon.*, vol. 23, pp. 381–383, 1978.
- [7] L. B. Ćirić, “Generalized contractions and fixed point theorems.” *Publ. Inst. Math.*, vol. 26, pp. 19–26, 1971.
- [8] J. Jachymski, “Equivalent conditions and Meir-Keeler type theorems.” *J. Math. Anal. Appl.*, vol. 194, pp. 293–303, 1995, doi: [10.1006/jmaa.1995.1299](https://doi.org/10.1006/jmaa.1995.1299).
- [9] G. Jungck, K. Moon, S. Park, and B. Rhoades, “On generalizations of the Meir-Keeler type contraction maps : Corrections.” *J. Math. Anal. Appl.*, vol. 180, pp. 221–222, 1993, doi: [10.1006/jmaa.1993.1397](https://doi.org/10.1006/jmaa.1993.1397).
- [10] R. Kannan, “Some results on fixed points-II.” *Amer. Math. Mon.*, vol. 76, no. 4, pp. 405–408, 1969, doi: [10.1080/00029890.1969.12000228](https://doi.org/10.1080/00029890.1969.12000228).
- [11] M. Kuczma, B. Choczewski, and R. Ger, *Iterative Functional Equations*, “*Encyclopedia of Mathematics and its Applications*.” UK: Cambridge Univ. Press, Cambridge, 1990. doi: [10.1017/CBO9781139086639](https://doi.org/10.1017/CBO9781139086639).
- [12] S. Leader, “Fixed points for operators on metric spaces with conditional uniform equivalence of orbits.” *J. Math. Anal. Appl.*, vol. 61, pp. 466–474, 1977, doi: [10.1016/0022-247X\(77\)90131-7](https://doi.org/10.1016/0022-247X(77)90131-7).
- [13] T. C. Lim, “On Characterizations of Meir-Keeler contractive maps.” *Nonlinear Anal.*, vol. 46, pp. 113–120, 2001, doi: [10.1016/S0362-546X\(99\)00448-4](https://doi.org/10.1016/S0362-546X(99)00448-4).
- [14] J. Matkowski, “Integrable solutions of functional equations.” *Diss. Math.*, vol. 127, pp. 1–68, 1975.
- [15] A. Meir and E. Keeler, “A theorem on contraction mappings.” *J. Math. Anal. Appl.*, vol. 28, pp. 326–329, 1969, doi: [10.1016/0022-247X\(69\)90031-6](https://doi.org/10.1016/0022-247X(69)90031-6).
- [16] R. P. Pant, “Discontinuity and fixed points.” *J. Math. Anal. Appl.*, vol. 240, pp. 284–289, 1999, doi: [10.1006/jmaa.1999.6560](https://doi.org/10.1006/jmaa.1999.6560).
- [17] R. P. Pant, “Noncompatible mappings and common fixed points.” *Soochow J. Math.*, vol. 26, pp. 29–35, 2000.
- [18] R. P. Pant, “A comparison of contractive definitions.” *J. Indian Math. Soc.*, vol. 72, pp. 241–249, 2005.
- [19] S. Park and J. S. Bae, “Extension of a fixed point theorem of Meir-Keeler.” *Ark. Math.*, vol. 19, pp. 223–228, 1981, doi: [10.1007/BF02384479](https://doi.org/10.1007/BF02384479).

- [20] S. Park and B. Rhoades, “Meir-Keeler type contractive conditions.” *Math. Japon.*, vol. 26, pp. 13–20, 1981.
- [21] P. D. Proinov, “Fixed point theorems in metric spaces.” *Nonlinear Anal.*, vol. 64, pp. 546–557, 2006, doi: [10.1016/j.na.2005.04.044](https://doi.org/10.1016/j.na.2005.04.044).
- [22] I. H. N. Rao and K. P. R. Rao, “Generalizations of fixed point theorems of Meir and Keeler type.” *Indian J. Pure Appl. Math.*, vol. 16, pp. 1249–1262, 1985.
- [23] B. E. Rhoades, “A comparison of various definitions of contractive mappings.” *Transactions of Amer. Math. Soc.*, vol. 226, pp. 257–290, 1977, doi: [10.1090/S0002-9947-1977-0433430-4](https://doi.org/10.1090/S0002-9947-1977-0433430-4).
- [24] B. E. Rhoades, “Contractive definitions and continuity.” *Contemporary Mathematics*, vol. 72, pp. 235–245, 1988, doi: [10.1090/conm/072](https://doi.org/10.1090/conm/072).

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BIFURCATION OF SOLUTIONS OF THE SECOND ORDER BOUNDARY VALUE PROBLEMS IN HILBERT SPACES

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Abstract. Bifurcations conditions of solutions of perturbed linear boundary value problems in the Hilbert spaces for the second order evolution equation are obtained.

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1. INTRODUCTION

It is known that differential equations of the second order in Hilbert spaces play important role in the qualitative theory of differential equations, probability theory and stochastic processes. It should be noted some books and papers in this direction [7] - [1].

As a rule such problems were investigated in the regular case when the corresponding linear problem has unique solution. In the presented paper we consider boundary value problem for the second order evolution equation in Hilbert spaces (resonance case). We obtain the necessary and sufficient conditions of the existence of solutions of such problem in the critical (irregular) case when considering linear problem can have set of solutions not for any right hand sides. Bifurcation conditions of solutions are obtained in general case.

2. STATEMENT OF THE PROBLEM

Consider the following boundary value problem (BVP)

$$y''(t, \varepsilon) + A(t)y(t, \varepsilon) = \varepsilon A_1(t)y(t, \varepsilon) + f(t), \quad (2.1)$$

$$l(y(\cdot, \varepsilon), y'(\cdot, \varepsilon))^T = \alpha, \quad (2.2)$$

where $y : J \rightarrow \mathcal{H}$ is a vector-function $y \in C^2(J, \mathcal{H})$ with values in a Hilbert space $\mathcal{H}$, $J \subset \mathbb{R}$, the closed operator-valued function $A(t)$ acts from J into the dense domain $D = D(A(t)) \subset \mathcal{H}$ which is independent from t , l is a linear and bounded operator

which translates solutions of (2.1) into the Hilbert space $\mathcal{H}_1$, $A_1(t)$ is a linear and bounded operator valued function $\|A_1\| = \sup_{t \in J} \|A_1(t)\| < \infty$, $\alpha \in \mathcal{H}_1$. In such a way we consider the case of the so-called abstract hyperbolic equation.

3. LINEAR CASE

At first we find the necessary and sufficient conditions of the existence of solutions of linear unperturbed nonhomogeneous boundary value problem

$$y_0''(t) + A(t)y_0(t) = f(t), \quad l(y_0(\cdot), y_0'(\cdot))^T = \alpha. \quad (3.1)$$

Let $x_1^0(t) = y_0(t)$, $x_2^0(t) = y_0'(t)$, $x_0(t) = (x_1^0(t), x_2^0(t))^T$, then we can rewrite boundary value problem (3.1) in the form of the operator system

$$x_0'(t) = B(t)x_0(t) + g(t), \quad lx_0(\cdot) = \alpha, \quad (3.2)$$

where

$$B(t) = \begin{pmatrix} 0 & I \\ -A(t) & 0 \end{pmatrix}, \quad g(t) = (0, f(t))^T. \quad (3.3)$$

Denote by $U(t)$ the evolution operator of homogeneous equation:

$$U'(t) = B(t)U(t), \quad U(0) = I.$$

Then the set of solutions of (3.2) has the form

$$x_0(t, c) = U(t)c + \int_0^t U(t)U^{-1}(\tau)g(\tau)d\tau.$$

Substituting in boundary condition $lx_0(\cdot) = \alpha$ we obtain the following operator equation

$$Qc = \alpha - l \int_0^\cdot U(\cdot)U^{-1}(\tau)g(\tau)d\tau, \quad Q = lU(\cdot) : \mathcal{H} \rightarrow \mathcal{H}_1, \quad (3.4)$$

or in the form

$$Qc = g_1, \quad c \in \mathcal{H} \quad (3.5)$$

where $Q = lU(\cdot, 0)$, $g_1 = \alpha - l \int_0^\cdot U(\cdot, \tau)f(\tau)d\tau$.

We have three types of solutions of the equation (3.5).

1) *Classical generalized solutions.*

Consider the case when the set of values of Q is closed ($R(Q) = \overline{R(Q)}$). Then operator equation (3.5) is solvable if and only if the element $g_1 \in R(Q)$ or $\mathcal{P}_{N(Q^*)}g_1 = 0$ [6]. Here $\mathcal{P}_{N(Q^*)}$ is the orthoprojector onto the cokernel of the operator Q . The set of solutions of (3.5) has the form

$$c = Q^+g_1 + \mathcal{P}_{N(Q)}\bar{c}, \quad \forall \bar{c} \in \mathcal{H}.$$

2) *Strong generalized solutions.*

Consider the case where $R(Q) \neq \overline{R(Q)}$ and $g_1 \in \overline{R(Q)}$. We show that the operator Q can be extended to the operator $\bar{Q}$ in such a way that $R(\bar{Q})$ is closed. It gives

possibility to define the notion of strong generalized solutions and obtain solvability condition of the corresponding equation in the case $R(Q) \neq \overline{R(Q)}$.

Since the operator Q is linear and bounded, spaces $\mathcal{H}$ and $\mathcal{H}_1$ can be represented in the form of a direct sums of subspaces:

$$\mathcal{H} = N(Q) \oplus X, \quad \mathcal{H}_1 = \overline{R(Q)} \oplus Y,$$

where $X = N(Q)^\perp$, $Y = \overline{R(Q)}^\perp$. Let $\mathcal{H}_2 = \mathcal{H}/N(Q)$ be the quotient space of $\mathcal{H}$ and let $\mathcal{P}_{\overline{R(Q)}}$ and $\mathcal{P}_{N(Q)}$ be the orthoprojectors onto $\overline{R(Q)}$ and $N(Q)$, respectively. Then the operator

$$\mathcal{Q} = \mathcal{P}_{\overline{R(Q)}} Q j^{-1} p : X \rightarrow R(Q) \subset \overline{R(Q)}$$

is linear, continuous, and injective. Here

$$p : X \rightarrow \mathcal{H}_2, j : \mathcal{H} \rightarrow \mathcal{H}_2$$

are continuous bijection and projection, respectively [2]. The triple $(\mathcal{H}, \mathcal{H}_2, j)$ is a locally trivial bundle with a typical fiber $\mathcal{P}_{N(Q)}\mathcal{H}$. In this case, we can define a strong generalized solution of the equation

$$Qc = g_1, c \in X. \quad (3.6)$$

We complete the space X by the norm $\|x\|_{\overline{X}} = \|\mathcal{Q}x\|_F$ [8], where $F = \overline{R(Q)}$. Then the extended operator

$$\overline{\mathcal{Q}} : \overline{X} \rightarrow R(Q) \subset \overline{R(Q)}$$

is a homeomorphism of $\overline{X}$ and $\overline{R(Q)}$. By the construction of a strong generalized solution, the equation

$$\overline{\mathcal{Q}}\overline{c} = g_1,$$

has a unique solution $\overline{\mathcal{Q}}^{-1}g_1$, which is called the strong generalized solution of equation (3.6).

In such a way we obtain an operator $\overline{\mathcal{Q}} = \overline{\mathcal{Q}}\mathcal{P}_{\overline{X}} : \overline{\mathcal{H}} \rightarrow \mathcal{H}_1$ which is normally resolvable ($R(\overline{\mathcal{Q}}) = \overline{R(Q)}$) and has Moore-Penrose pseudoinverse $\overline{\mathcal{Q}}^+$, $\overline{\mathcal{H}} = N(Q) \oplus \overline{X}$. Then the set of strong generalized solutions of the equation (3.5) has the form [6]:

$$c = \overline{\mathcal{Q}}^+ g_1 + \mathcal{P}_{N(Q)} \overline{c}, \forall \overline{c} \in \overline{\mathcal{H}}.$$

3) Strong pseudosolutions.

Consider the case when element $g_1 \notin \overline{R(Q)} \neq R(Q)$. This condition is equivalent to the following $\mathcal{P}_{N(Q^*)}g_1 \neq 0$. In this case, there are elements c of $\overline{\mathcal{H}}$ that minimize the norm $\|\overline{\mathcal{Q}}c - g_1\|_{\overline{\mathcal{H}}}$:

$$c = \overline{\mathcal{Q}}^+ g_1 + \mathcal{P}_{N(Q)} \overline{c}, \quad \overline{c} \in \overline{\mathcal{H}}.$$

These elements are called *strong pseudosolutions* by analogy with [6], [8].

Considered cases give possibility to formulate the following result.

Theorem 1. 1. a) Boundary value problem (3.1) has strong generalized solutions if and only if the following condition holds

$$\mathcal{P}_{N(\overline{Q}^*)}\{\alpha - l \int_0^\cdot U(\cdot)U^{-1}(\tau)g(\tau)d\tau\} = 0; \quad (3.7)$$

if

$$\alpha - l \int_0^\cdot U(\cdot)U^{-1}(\tau)g(\tau)d\tau \in R(Q),$$

then strong generalized solutions are classical;

b) under condition (3.7) the set of solutions has the form

$$x_0(t, c) = U(t)\mathcal{P}_{N(\overline{Q})}c + \overline{(G[g, \alpha])(t)}, \quad \forall c \in \mathcal{H}$$

where $\mathcal{P}_{N(\overline{Q})}, \mathcal{P}_{N(\overline{Q}^*)}$ are the orthoprojectors onto the kernel and cokernel of the operator $\overline{Q}$ respectively,

$$\overline{(G[g, \alpha])(t)} = \int_0^t U(t)U^{-1}(\tau)g(\tau)d\tau + \overline{Q}^+\{\alpha - l \int_0^\cdot U(\cdot)U^{-1}(\tau)g(\tau)d\tau\}$$

is a generalized Green operator of the boundary value problem (3.1);

2. a) Boundary value problem (3.1) has strong pseudosolutions if and only if the following condition holds

$$\mathcal{P}_{N(\overline{Q}^*)}\{\alpha - l \int_0^\cdot U(\cdot)U^{-1}(\tau)g(\tau)d\tau\} \neq 0; \quad (3.8)$$

b) under condition (3.8) the set of strong pseudosolutions has the form

$$x_0(t, c) = U(t)\mathcal{P}_{N(\overline{Q})}c + \overline{(G[g, \alpha])(t)}, \quad \forall c \in \mathcal{H}.$$

4. BIFURCATION CONDITIONS

a) Suppose that condition (3.8) is hold. Our main goal is to obtain conditions for $A_1(t)$ which guarantee strong generalized solvability of perturbed boundary value problem

$$x'(t, \varepsilon) = B(t)x(t, \varepsilon) + g(t) + \varepsilon B_1(t)x(t, \varepsilon), \quad (4.1)$$

$$lx(\cdot, \varepsilon) = \alpha. \quad (4.2)$$

Here is an operator-valued function $B_1(t)$ has the following form:

$$B_1(t) = \begin{pmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{pmatrix}, \quad g(t) = (0, f(t))^T, \quad (4.3)$$

$x(t, \varepsilon) = (x_1(t, \varepsilon), x_2(t, \varepsilon))^T$, $x_1(t, \varepsilon) = y(t, \varepsilon)$, $x_2(t, \varepsilon) = y'(t, \varepsilon)$. We will use the modification of the well-known Vishik-Lyusternik method [9]. A solution of problem

(4.1), (4.2) is sought in the form of a segment of the series in powers of the small parameter ε :

$$x(t, \varepsilon) = \sum_{i=-1}^{+\infty} \varepsilon^i x_i(t) = \frac{x_{-1}(t)}{\varepsilon} + x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \dots \quad (4.4)$$

Substituting series (4.4) into problem (4.1), (4.2) and equating the coefficients of ε^k . For ε^{-1} we obtain the following boundary value problem for finding the coefficient $x_{-1}(t)$ of series (4.4):

$$x'_{-1}(t) = B(t)x_{-1}(t), \quad (4.5)$$

$$lx_{-1}(\cdot) = 0. \quad (4.6)$$

Problem (4.5), (4.6) has a family of solutions:

$$x_{-1}(t, c_{-1}) = U(t)\mathcal{P}_{N(\overline{Q})}c_{-1}, \quad \forall c_{-1} \in \mathcal{H}.$$

An arbitrary element c_{-1} is determined by the condition for the solvability of the following linear inhomogeneous boundary value problem for finding the coefficient $x_0(t)$ in series (4.4):

$$x'_0(t) = B(t)x_0(t) + B_1(t)x_{-1}(t) + g(t), \quad (4.7)$$

$$lx_0(\cdot) = \alpha. \quad (4.8)$$

A necessary and sufficient condition for the solvability of the problem (4.7), (4.8) is given by

$$\mathcal{P}_{N(\overline{Q}^*)}\{\alpha - l \int_0^\cdot U(\cdot)U^{-1}(\tau)(B_1(\tau)x_{-1}(\tau, c_{-1}) + g(\tau))d\tau\} = 0.$$

From this, in view of the form of $x_{-1}(t, c_{-1})$, we obtain an operator equation for $c_{-1} \in \mathcal{H}$:

$$B_0 c_{-1} = \mathcal{P}_{N(\overline{Q}^*)}\{\alpha - l \int_0^\cdot U(\cdot)U^{-1}(\tau)g(\tau)d\tau\}, \quad (4.9)$$

where

$$B_0 = \mathcal{P}_{N(\overline{Q}^*)}l \int_0^\cdot U(\cdot)U^{-1}(\tau)B_1(\tau)U(\tau)d\tau \mathcal{P}_{N(\overline{Q})}.$$

A necessary and sufficient condition of generalized solvability has the following form

$$\mathcal{P}_{N(\overline{B}_0^*)}\mathcal{P}_{N(\overline{Q}^*)}\{\alpha - l \int_0^\cdot U(\cdot)U^{-1}(\tau)g(\tau)d\tau\} = 0. \quad (4.10)$$

Here $\mathcal{P}_{N(\overline{B}_0^*)}, \mathcal{P}_{N(\overline{Q}^*)}$ are orthoprojectors onto cokernels of extended adjoint operators $\overline{B}_0^*, \overline{Q}^*$ respectively. Suppose that $\mathcal{P}_{N(\overline{B}_0^*)}\mathcal{P}_{N(\overline{Q}^*)} = 0$. Then condition (4.10) is hold. The solution set of operator equation for $c_{-1} \in \mathcal{H}$ has the form

$$c_{-1} = \overline{c}_{-1} + \mathcal{P}_{N(\overline{B}_0)}c_\rho, \quad \forall c_\rho \in \mathcal{H},$$

where

$$\bar{c}_{-1} = \bar{B}_0^+ \mathcal{P}_{N(\bar{Q}^*)} \{ \alpha - l \int_0^\cdot U(\cdot) U^{-1}(\tau) g(\tau) d\tau \}.$$

In view of the expression for c_{-1} , the homogeneous boundary value problem (4.5), (4.6) has a ρ -parameter family of solutions

$$x_{-1}(t, c_\rho) = \bar{x}_{-1}(t, \bar{c}_{-1}) + U(t) \mathcal{P}_{N(\bar{Q})} \mathcal{P}_{N(\bar{B}_0)} c_\rho, \quad (4.11)$$

where

$$\bar{x}_{-1}(t, \bar{c}_{-1}) = U(t) \mathcal{P}_{N(\bar{Q})} \bar{c}_{-1}.$$

The general solution of the problem (4.7), (4.8) has the form

$$x_0(t, c_0) = U(t) \mathcal{P}_{N(\bar{Q})} c_0 + F_{-1}(t) + K_{-1}(t) \mathcal{P}_{N(\bar{B}_0)} c_\rho,$$

where

$$F_{-1}(t) = \overline{(G[g + B_1 \bar{x}_{-1}, \alpha])}(t), K_{-1}(t) = \overline{(G[U, 0])}(t) \mathcal{P}_{N(\bar{Q})},$$

c_0 is an element of the space $\mathcal{H}$, which is determined at the next step from the condition for the solvability of the boundary value problem for finding the coefficient $x_1(t)$ in series (4.4). To determine the coefficient $x_1(t)$ of ε^1 in series (4.4), we obtain the following boundary value problem

$$x_1'(t) = B(t)x_1(t) + B_1(t)x_0(t, c_0), \quad (4.12)$$

$$lx_1(\cdot) = 0. \quad (4.13)$$

Under condition of solvability

$$\mathcal{P}_{N(\bar{Q}^*)} l \int_0^\cdot U(\cdot) U^{-1}(\tau) B_1(\tau) x_0(\tau, c_0) d\tau = 0,$$

boundary value problem (4.12), (4.13) has the set of solutions in the form

$$x_1(t, c_1) = U(t) \mathcal{P}_{N(\bar{Q})} c_1 + \overline{(G[B_1 U \mathcal{P}_{N(\bar{Q})} c_0 + F_{-1} + K_{-1}, 0])}(t).$$

The condition for the solvability of the boundary value problem for the element c_0 is

$$B_0 c_0 = -\mathcal{P}_{N(\bar{Q}^*)} l \int_0^\cdot U(\cdot) U^{-1}(\tau) B_1(\tau) F_{-1}(\tau) d\tau - \quad (4.14)$$

$$-\mathcal{P}_{N(\bar{Q}^*)} l \int_0^\cdot U(\cdot) U^{-1}(\tau) B_1(\tau) K_{-1}(\tau) d\tau \mathcal{P}_{N(\bar{Q})} \mathcal{P}_{N(\bar{B}_0)} c_\rho.$$

From the condition $\mathcal{P}_{N(B_0^*)} \mathcal{P}_{N(\bar{Q}^*)} = 0$ follows solvability of the equation (4.14) with the set of solutions in the following form

$$\begin{aligned} c_0 &= -\bar{B}_0^+ \mathcal{P}_{N(\bar{Q}^*)} l \int_0^\cdot U(\cdot) U^{-1}(\tau) B_1(\tau) F_{-1}(\tau) d\tau - \\ &\quad - \bar{B}_0^+ \mathcal{P}_{N(\bar{Q}^*)} l \int_0^\cdot U(\cdot) U^{-1}(\tau) B_1(\tau) K_{-1}(\tau) d\tau \mathcal{P}_{N(\bar{Q})} \mathcal{P}_{N(\bar{B}_0)} c_\rho + \mathcal{P}_{N(\bar{B}_0)} c_\rho, \\ c_0 &= \bar{c}_0 + D_0 \mathcal{P}_{N(\bar{B}_0)} c_\rho, \quad \forall c_\rho \in \mathcal{H}, \end{aligned}$$

where

$$\begin{aligned}\bar{c}_0 &= -\bar{B}_0^+ \mathcal{P}_{N(\bar{Q}^*)} l \int_0^\cdot U(\cdot) U^{-1}(\tau) B_1(\tau) F_{-1}(\tau) d\tau, \\ D_0 &= I - \bar{B}_0^+ \mathcal{P}_{N(\bar{Q}^*)} l \int_0^\cdot U(\cdot) U^{-1}(\tau) B_1(\tau) K_{-1}(\tau) d\tau \mathcal{P}_{N(\bar{Q})}.\end{aligned}$$

Thus, problem (4.7), (4.8) has a ρ -parameter family of solutions:

$$x_0(t, c_0) = \bar{x}_0(t, \bar{c}_0) + \bar{X}_0(t) \mathcal{P}_{N(\bar{B}_0)} c_\rho, \quad \forall c_\rho \in \mathcal{H},$$

where

$$\begin{aligned}\bar{x}_0(t, \bar{c}_0) &= U(t) \mathcal{P}_{N(\bar{Q})} \bar{c}_0 + F_{-1}(t), \\ \bar{X}_0(t) &= U(t) \mathcal{P}_{N(\bar{Q})} D_0 + K_{-1}(t).\end{aligned}$$

Then problem (4.12), (4.13) has a ρ -parameter family of solutions

$$x_1(t, c_1) = U(t) \mathcal{P}_{N(\bar{Q})} c_1 + F_0(t) + K_0(t) \mathcal{P}_{N(B_0)} c_\rho,$$

where

$$\begin{aligned}F_0(t) &= \overline{(G[B_1 U \mathcal{P}_{N(\bar{Q})} \bar{c}_0 + F_{-1} + K_{-1}, 0])}(t), \\ K_0(t) &= \overline{(G[B_1 U \mathcal{P}_{N(\bar{Q})} D_0, 0])}(t),\end{aligned}$$

c_1 is an element of the Hilbert space $\mathcal{H}$, which is determined at the next step from the condition for the solvability of the boundary value problem for finding the coefficient $x_2(t)$ in series (4.4). By induction, we can prove that the coefficients $x_i(t)$ in series (4.4) are determined by solving the boundary value problem

$$x'_i(t) = B(t)x_i(t) + B_1(t)x_{i-1}(t, c_{i-1}), \quad (4.15)$$

$$lx_i(\cdot) = 0, \quad (4.16)$$

which under the same condition of solvability $\mathcal{P}_{N(B_0^*)} \mathcal{P}_{N(\bar{Q}^*)} = 0$ has a ρ -parameter family of solutions

$$x_i(t, c_i) = \bar{x}_i(t, \bar{c}_i) + \bar{X}_i(t) \mathcal{P}_{N(\bar{B}_0)} c_\rho, \quad \forall c_\rho \in \mathcal{H} \quad (4.17)$$

where all the terms are determined by the iterative procedure

$$\bar{x}_i(t, \bar{c}_i) = U(t) \mathcal{P}_{N(\bar{Q})} \bar{c}_i + F_{i-1}(t), \quad (4.18)$$

$$\bar{X}_i(t) = U(t) \mathcal{P}_{N(\bar{Q})} D_i + K_{i-1}(t), \quad (4.19)$$

$$D_i = I - \bar{B}_0^+ \mathcal{P}_{N(\bar{Q}^*)} l \int_0^\cdot U(\cdot) U^{-1}(\tau) B_1(\tau) K_{i-1}(\tau) d\tau \mathcal{P}_{N(\bar{Q})}, \quad (4.20)$$

$$F_{i-1}(t) = \overline{(G[B_1 U \mathcal{P}_{N(\bar{Q})} \bar{c}_{i-1} + F_{i-2} + K_{i-2}, 0])}(t), \quad (4.21)$$

$$K_{i-1}(t) = \overline{(G[B_1 U \mathcal{P}_{N(\bar{Q})} D_{i-1}, 0])}(t). \quad (4.22)$$

The convergence of series (4.4) for sufficiently small fixed $\varepsilon \in (0, \varepsilon^*]$ is proved in the same manner as in [3]. Thus, the following result for perturbed boundary value problem (4.1), (4.2) holds.

Theorem 2. Suppose that for unperturbed boundary value problem (3.2) condition (3.8) is hold. Under condition

$$\mathcal{P}_{N(\overline{B}_0^*)} \mathcal{P}_{N(\overline{Q}^*)} = 0 \quad (4.23)$$

the perturbed boundary value problem (4.1), (4.2) has a ρ -parameter family of strong generalized solutions in the form of the convergent series segment

$$x(t, c_\rho) = \sum_{i=-1}^{+\infty} \varepsilon^i [\overline{x}_i(t, \overline{c}_i) + \overline{X}_i(t) \mathcal{P}_{N(\overline{B}_0)} c_\rho], \quad \forall c_\rho \in \mathcal{H},$$

whose coefficients are given by formulas (4.18)-(4.22).

b) Suppose that condition (3.7) is hold. We obtain the condition on $A_1(t)$ such that the perturbed boundary value problem

$$x'(t, \varepsilon) = B(t)x(t, \varepsilon) + g(t) + \varepsilon B_1(t)x(t, \varepsilon), \quad (4.24)$$

$$lx(\cdot, \varepsilon) = \alpha, \quad (4.25)$$

has strong generalized solutions. As in previous case a solution of problem (4.24), (4.25) is sought in the form of a segment of the series in powers of the small parameter ε :

$$x(t, \varepsilon) = \sum_{i=0}^{+\infty} \varepsilon^i x_i(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \dots \quad (4.26)$$

Substituting series (4.26) into problem (4.24), (4.25) and equating the coefficients of ε^k . For ε^0 , we obtain the following boundary value problem for finding the coefficient $x_0(t)$ of series (4.26):

$$x'_0(t) = B(t)x_0(t) + g(t), \quad (4.27)$$

$$lx_0(\cdot) = \alpha. \quad (4.28)$$

Problem (4.27), (4.28) has a family of strong generalized solutions:

$$x_0(t, c_0) = U(t) \mathcal{P}_{N(\overline{Q})} c_0 + (\overline{G[g, \alpha]})(t).$$

An arbitrary element $c_0 \in \mathcal{H}$ is determined by the condition for the solvability of the following linear inhomogeneous boundary value problem for finding the coefficient $x_1(t)$ in series (4.26):

$$x'_1(t) = B(t)x_1(t) + B_1(t)x_0(t, c_0), \quad (4.29)$$

$$lx_1(\cdot) = 0. \quad (4.30)$$

A necessary and sufficient condition for the solvability of problem (4.29), (4.30) is given by

$$\mathcal{P}_{N(\overline{Q}^*)} \{ l \int_0^\cdot U(\cdot) U^{-1}(\tau) B_1(\tau) x_0(\tau, c_0) d\tau \} = 0.$$

From this, in view of the form of $x_0(t, c_0)$, we obtain an operator equation for $c_0 \in \mathcal{H}$:

$$B_0 c_0 = -\mathcal{P}_{N(\overline{Q}^*)} l \int_0^\cdot U(\cdot) U^{-1}(\tau) B_1(\tau) \overline{(G[g, \alpha])}(\tau) d\tau. \quad (4.31)$$

Under condition $\mathcal{P}_{N(\overline{B}_0^*)} \mathcal{P}_{N(\overline{Q}^*)} = 0$ the equation (4.31) is solvable. The solution set of operator equation for $c_0 \in \mathcal{H}$ has the form

$$c_0 = \bar{c}_0 + \mathcal{P}_{N(\overline{B}_0)} c_\rho, \forall c_\rho \in \mathcal{H},$$

where

$$\bar{c}_0 = -\overline{B_0^+} \mathcal{P}_{N(\overline{Q}^*)} l \int_0^\cdot U(\cdot) U^{-1}(\tau) B_1(\tau) \overline{(G[g, \alpha])}(\tau) d\tau.$$

In view of the expression for c_0 , the homogeneous boundary value problem (4.27), (4.28) has a ρ -parameter family of solutions

$$x_0(t, c_\rho) = \bar{x}_0(t, \bar{c}_0) + U(t) \mathcal{P}_{N(\overline{Q})} \mathcal{P}_{N(\overline{B}_0)} c_\rho, \quad (4.32)$$

where

$$\bar{x}_0(t, \bar{c}_0) = U(t) \mathcal{P}_{N(\overline{Q})} \bar{c}_0 + \overline{(G[g, \alpha])}(t).$$

The general solution of problem (4.29), (4.30) has the form

$$x_1(t, c_1) = U(t) \mathcal{P}_{N(\overline{Q})} c_1 + F_0(t) + K_0(t) \mathcal{P}_{N(\overline{B}_0)} c_\rho,$$

where

$$F_0(t) = \overline{(G[B_1 \bar{x}_0, 0])}(t), K_0(t) = \overline{(G[B_1 U \mathcal{P}_{N(\overline{Q})}, 0])}(t),$$

c_1 is an element of the space $\mathcal{H}$, which is determined at the next step from the condition for the solvability of the boundary value problem for finding the coefficient $x_2(t)$ in series (4.26). By induction, we can prove that the coefficients $x_i(t)$ in series (4.26) are determined by solving the boundary value problem

$$x_i'(t) = B(t)x_i(t) + B_1(t)x_{i-1}(t, c_{i-1}), \quad (4.33)$$

$$l x_i(\cdot) = 0, \quad (4.34)$$

which under condition of solvability has a ρ -parameter family of solutions

$$x_i(t, c_i) = \bar{x}_i(t, \bar{c}_i) + \bar{X}_i(t) \mathcal{P}_{N(\overline{B}_0)} c_\rho, \quad \forall c_\rho \in \mathcal{H} \quad (4.35)$$

where all the terms are determined (as in previous case) by the iterative procedure (4.18)-(4.22).

The convergence of series (4.26) is proved in the same manner as in [3]. Thus, the following result holds.

Theorem 3. Suppose that for unperturbed boundary value problem (3.2) condition (3.7) is hold. Under condition

$$\mathcal{P}_{N(\overline{B}_0^*)} \mathcal{P}_{N(\overline{Q}^*)} = 0 \quad (4.36)$$

the boundary value problem (4.24), (4.25) has a ρ -parameter family of strong generalized solutions in the form of the convergent series segment

$$x(t, c_\rho) = \sum_{i=0}^{+\infty} [\bar{x}_i(t, \bar{c}_i) + \bar{X}_i(t) \mathcal{P}_{N(\bar{B}_0)} c_\rho], \quad \forall c_\rho \in \mathcal{H},$$

whose coefficients are given by formulas (4.18)-(4.22).

Remark 1. In the case when $g_1 \in R(Q) = \overline{R(Q)}$ under the same condition we can obtain classical generalized solutions for the system (4.24), (4.25).

Remark 2. Proposed theory gives possibility to investigate branching of solutions of boundary value problem (3.1) (see [5], [4]).

5. EXAMPLE

Now we find bifurcation conditions of solutions of periodic boundary value problem for hyperbolic equation in a separable (for simplicity) Hilbert space $\mathcal{H}$:

$$y''(t, \varepsilon) + Ty(t, \varepsilon) = \varepsilon A_1(t)y(t, \varepsilon) + f(t), \quad (5.1)$$

$$y(0, \varepsilon) = y(w, \varepsilon), y'(0, \varepsilon) = y'(w, \varepsilon), \quad (5.2)$$

where T is an unbounded operator with compact inverse T^{-1} . Suppose for example that the operator $A_1(t)$ is diagonal: $A_1(t) = \text{diag}\{a_{ii}(t)\}_{i \in N}$. Then there is orthonormal basis $e_i \in \mathcal{H}$ such that $y(t) = \sum_{i=1}^{\infty} c_i(t)e_i$, $Ty(t) = \sum_{i=1}^{\infty} \lambda_i c_i(t)e_i$, $\lambda_i \rightarrow \infty$. In that case boundary value problem (5.1), (5.2) is equivalent to the following countable system of ordinary differential equations:

$$x'_k(t, \varepsilon) = \sqrt{\lambda_k} y_k(t, \varepsilon),$$

$$y'_k(t, \varepsilon) = -\sqrt{\lambda_k} x_k(t, \varepsilon) + \frac{1}{\sqrt{\lambda_k}} (\varepsilon a_{kk}(t) x_k(t, \varepsilon) + f_k(t)), \quad (5.3)$$

$$x_k(0, \varepsilon) = x_k(w, \varepsilon), \quad y_k(0, \varepsilon) = y_k(w, \varepsilon), \quad (5.4)$$

where $c_k(t) = x_k(t)$, $x'_k(t) = y_k(t)$. Consider critical case, when $\lambda_k = \frac{4\pi^2 k^2}{w^2}$. Suppose that $w = 2\pi$. Then $\lambda_k = k^2$. At first we find conditions of solvability of generating ($\varepsilon = 0$) boundary value problem

$$x'_k(t) = \sqrt{\lambda_k} y_k(t),$$

$$y'_k(t) = -\sqrt{\lambda_k} x_k(t) + \frac{1}{\sqrt{\lambda_k}} f_k(t), \quad (5.5)$$

$$x_k(0) = x_k(w), \quad y_k(0) = y_k(w), \quad (5.6)$$

It is easy to see, that generating boundary value problem (5.5), (5.6) is solvable if and only if

$$\int_0^{2\pi} \sin(k\tau) f_k(\tau) d\tau = 0, \quad (5.7)$$

$$\int_0^{2\pi} \cos(k\tau) f_k(\tau) d\tau = 0, k \in N. \quad (5.8)$$

Under condition (5.7), (5.8) generating boundary value problem (5.5), (5.6) has the set of periodic solutions in the form

$$\begin{pmatrix} x_k(t, c_1^k, c_2^k) \\ y_k(t, c_1^k, c_2^k) \end{pmatrix} = \begin{pmatrix} \cos kt & \sin kt \\ -\sin kt & \cos kt \end{pmatrix} \begin{pmatrix} c_1^k \\ c_2^k \end{pmatrix} + \frac{1}{k} \begin{pmatrix} \int_0^t \sin k(t-\tau) f_k(\tau) d\tau \\ \int_0^t \cos k(t-\tau) f_k(\tau) d\tau \end{pmatrix}.$$

Suppose for example that $a_{kk}(t) = a_{kk} \neq 0$ and

$$\int_0^{2\pi} \sin(k\tau) f_k(\tau) d\tau \neq 0, \quad (5.9)$$

$$\int_0^{2\pi} \cos(k\tau) f_k(\tau) d\tau \neq 0, k \in N. \quad (5.10)$$

It means that generating boundary value problem hasn't solutions. Strong generalized solution of the problem (5.3), (5.4) we find in the form of series by the power of small parameter ε :

$$x(t, \varepsilon) = \frac{x_{-1}(t)}{\varepsilon} + x_0(t) + \varepsilon x_1(t) + \dots \quad (5.11)$$

Under ε^{-1} we obtain the following boundary value problem for finding of coefficient $x_{-1}(t) = (x_{-1}^k(t), y_{-1}^k(t))_{k \in N}^T$. For ε^0 we obtain:

$$\frac{dx_{-1}^k(t)}{dt} = ky_{-1}^k(t), \quad x_{-1}^k(0) = x_{-1}^k(2\pi),$$

$$\frac{dy_{-1}^k(t)}{dt} = -kx_{-1}^k(t), \quad y_{-1}^k(0) = y_{-1}^k(2\pi).$$

The set of solutions of such problem has the following form

$$x_{-1}^k(t, c_{-1}) = \cos kt \, c_{-1}^{1k} + \sin kt \, c_{-1}^{2k},$$

$$y_{-1}^k(t, c_{-1}) = -\sin kt \, c_{-1}^{1k} + \cos kt \, c_{-1}^{2k}.$$

Any element $c_{-1} = (c_{-1}^{1k}, c_{-1}^{2k})_{k \in N}^T$ we can find from the condition of solvability of the following boundary value problem for the coefficient $x_0(t) = (x_0^k(t), y_0^k(t))_{k \in N}^T$:

$$\frac{dx_0^k(t)}{dt} = ky_0^k(t), \quad x_0^k(0) = x_0^k(2\pi),$$

$$\frac{dy_0^k(t)}{dt} = -kx_0^k(t) + \frac{1}{k} a_{kk} x_{-1}^k(t, c_{-1}) + \frac{1}{k} f_k(t), \quad y_0^k(0) = y_0^k(2\pi).$$

The set of solutions $x_0^k(t), y_0^k(t)$ of such problem has the following form

$$x_0^k(t, c_0) = \cos kt \, c_0^{1k} + \sin kt \, c_0^{2k} + \frac{1}{k} \int_0^t \sin k(t-\tau) (a_{kk} x_{-1}^k(\tau, c_{-1}) + f_k(\tau)) d\tau,$$

$$y_0^k(t, c_0) = -\sin kt c_0^{1k} + \cos kt c_0^{2k} + \frac{1}{k} \int_0^t \cos k(t-\tau) (a_{kk} x_{-1}^k(\tau, c_{-1}) + f_k(\tau)) d\tau,$$

if and only if the following system is solvable

$$B_0 c_{-1} = g,$$

where B_0 - is a countable invertible matrix in the form of 2×2 block matrices:

$$B_0 = \text{block} \left(\begin{array}{cc} 0 & -\pi a_{kk} \\ \pi a_{kk} & 0 \end{array} \right)_{k \in N},$$

$$g = \left(\int_0^{2\pi} \sin(k\tau) f_k(\tau) d\tau, - \int_0^{2\pi} \cos(k\tau) f_k(\tau) d\tau \right)_{k \in N}^T.$$

It is easy to see that condition (4.23) of the theorem 2 is hold and in this case strong generalized solution is classical solution.

Then constants $\bar{c}_{-1}^{1k}, \bar{c}_{-1}^{2k}$ have the form:

$$\bar{c}_{-1}^{1k} = \frac{-\int_0^{2\pi} \cos k\tau f_k(\tau) d\tau}{\pi a_{kk}}, \quad \bar{c}_{-1}^{2k} = \frac{-\int_0^{2\pi} \sin k\tau f_k(\tau) d\tau}{\pi a_{kk}}.$$

Finally we obtain that

$$x_0^k(t, c_0) = \cos kt c_0^{1k} + \sin kt c_0^{2k} + \frac{1}{k} g_0^{1k}(t),$$

where

$$g_0^{1k}(t) = \int_0^{2\pi} K_1(t, \tau) f_k(\tau) d\tau + \int_0^t \sin k(t-\tau) f_k(\tau) d\tau,$$

$$K_1(t, \tau) = \frac{1}{2\pi k} (kt \sin k(t-\tau) - \sin kt \sin k\tau);$$

$$y_0^k(t, c_0) = -\sin kt c_0^{1k} + \cos kt c_0^{2k} + \frac{1}{k} g_0^{2k}(t),$$

where

$$g_0^{2k}(t) = \int_0^{2\pi} K_2(t, \tau) f_k(\tau) d\tau + \int_0^t \cos k(t-\tau) f_k(\tau) d\tau,$$

$$K_2(t, \tau) = -\frac{1}{2\pi k} (kt \cos k(t-\tau) + \sin kt \cos k\tau).$$

Any element c_{i-1} we can determine from the condition of solvability of boundary value problem for the coefficient $x_i(t)$ under ε^i of the series (5.11):

$$\frac{dx_i^k(t)}{dt} = ky_i^k(t), x_i^k(0) = x_i^k(2\pi), \quad (5.12)$$

$$\frac{dy_i^k(t)}{dt} = -kx_i^k(t) + \frac{1}{k} a_{kk} x_{i-1}^k(t, c_{i-1}), y_i^k(0) = y_i^k(2\pi). \quad (5.13)$$

The set of solutions has the form

$$\begin{aligned}x_i^k(t, c_i) &= \cos kt c_i^{1k} + \sin kt c_i^{2k} + \frac{1}{k} g_i^{1k}(t), \\y_i^k(t, c_i) &= -\sin kt c_i^{1k} + \cos kt c_i^{2k} + \frac{1}{k} g_i^{2k}(t),\end{aligned}$$

where

$$\begin{aligned}g_i^{1k}(t) &= -a_{kk} \int_0^{2\pi} K_1(t, \tau) g_{i-1}^{1k}(\tau) d\tau - a_{kk} \int_0^t \sin k(t-\tau) g_{i-1}^{1k}(\tau) d\tau, \\g_i^{2k}(t) &= a_{kk} \int_0^{2\pi} K_2(t, \tau) g_{i-1}^{1k}(\tau) d\tau + a_{kk} \int_0^t \cos k(t-\tau) g_{i-1}^{1k}(\tau) d\tau, \\ \bar{c}_{i-1}^{1k} &= -\frac{1}{\pi} \int_0^{2\pi} \cos k\tau g_{i-1}^{1k}(\tau) d\tau, \\ \bar{c}_{i-1}^{2k} &= -\frac{1}{\pi} \int_0^{2\pi} \sin k\tau g_{i-1}^{1k}(\tau) d\tau.\end{aligned}$$

REFERENCES

- [1] N. Artamonov, "Estimates of solutions of certain classes of second-order differential equations in a hilbert space," *Sb. Math.*, vol. 194, no. 8, pp. 1113–1123, 2003.
- [2] M. Atiyah, *K-theory*, New-York, Amsterdam, 1967.
- [3] A. Boichuk and O. Pokutnyi, "Bounded solutions of linear perturbed differential equations in a banach space," *Tatra Mountains Math. Publications*, vol. 38, pp. 29–41, 2007.
- [4] A. Boichuk and O. Pokutnyi, "Solutions of the schrodiner equation in a hilbert space. <http://www.boundaryvalueproblems.com/content/2014/1/4>," *Boundary value problems*, 2014, doi: [10.1186/1687-2770-2014-4](https://doi.org/10.1186/1687-2770-2014-4).
- [5] A. Boichuk and O. Pokutnyi, "Perturbation theory of operator equations in the frechet and hilbert spaces," *Ukrainian mathematical journal*, vol. 67, pp. 1327–1335, 2016.
- [6] A. Boichuk and A. Samoilenko, *Generalized Inverse Operators and Fredholm Boundary-Value Problems. 2nd edition.* Berlin/Boston: De Gruyter, 2016.
- [7] G. Da Prato and J. Zabczyk, *Second Order Partial Differential Equations in Hilbert Spaces.* Cambridge University Press, 2002. doi: [10.1112/S0024609303242776](https://doi.org/10.1112/S0024609303242776).
- [8] D. Klyushin, S. Lyashko, D. Nomirowskii, V. V. Semenov, and Y. I. Petunin, *Generalized solutions of operator equations and extreme elements.* Berlin: Springer, 212. doi: [10.1007/978-1-4614-0619-8](https://doi.org/10.1007/978-1-4614-0619-8).
- [9] M. Vishik, "The solution of some perturbation problems for matrices and selfadjoint or non-selfadjoint differential equations i," *Russian Mathematical Surveys*, vol. 15, pp. 1–73, 1960.

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MULTIPLE SOLUTIONS FOR ASYMPTOTICALLY LINEAR $2p$ -ORDER HAMILTONIAN SYSTEMS WITH IMPULSIVE EFFECTS

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Abstract. In this paper, we are concerned with $2p$ -order Hamiltonian systems with impulsive effects. We investigate the variational structure associated to this system. In addition, we obtain some results of multiple solutions for asymptotically linear $2p$ -order Hamiltonian systems via variational methods and critical point theorems. Meanwhile, some examples are presented to illustrate our main results.

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1. INTRODUCTION AND MAIN RESULTS

In recent years, variational methods have been introduced to investigate various impulsive differential equations since papers [10, 13] appeared. As one kind of widely applicable differential equations, Hamiltonian systems with impulsive effects have been also concerned on widely and many new corresponding results have been obtained, see for instance [3, 7–9, 11, 12, 15]. However, in aforementioned papers Hamiltonian systems are second order. To the best of our knowledge, few authors have considered asymptotically linear $2p$ -order Hamiltonian systems with impulsive effects. One difficulty is that the suitable impulsive effects associated to this system have been not found. Another is that the suitable critical point theorems have been not applied. In this paper, we present such impulsive conditions. More precisely, we investigate multiple solutions for

$$\begin{cases} (-1)^{p+1}u^{(2p)} + \nabla V(t, u) = 0, & a.e. \ t \in [0, T], \\ u^{(j)}(0) = u^{(j)}(T), & j = 0, 1, \dots, 2p-1, \\ \Delta(u^{i(2p-j)}(t_k)) = I_{ijk}(u^{i(j-1)}(t_k)), \\ & i = 1, 2, \dots, N, j = 1, 2, \dots, p, k = 1, 2, \dots, q, \end{cases} \quad (1.1)$$

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where p is a positive integer, $u(t) = (u^1(t), u^2(t), \dots, u^N(t))$, $V : [0, T] \times \mathbf{R}^N \rightarrow \mathbf{R}$ is measurable with respect to t , for every $u \in \mathbf{R}^N$, continuously differentiable in u , for almost every $t \in [0, T]$, $0 = t_0 < t_1 < \dots < t_q < t_{q+1} = T$, t_k ($k = 1, 2, \dots, q$) are the instants where the impulses occur, $\Delta(u^{i(2p-j)}(t_k)) = u^{i(2p-j)}(t_k^+) - u^{i(2p-j)}(t_k^-)$ and $I_{ijk} : \mathbf{R} \rightarrow \mathbf{R}$ ($i = 1, 2, \dots, N$, $j = 1, 2, \dots, p$, $k = 1, 2, \dots, q$) are continuous.

From now on, we write $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ as $\{1, 2, \dots, N\}$, $\{1, 2, \dots, p\}$ and $\{1, 2, \dots, q\}$ respectively. In addition, $\mathcal{L}_s(\mathbf{R}^N)$ stands for the space of symmetric matrices of order N and I_N is the unit matrix in $\mathcal{L}_s(\mathbf{R}^N)$. For any $A_1, A_2 \in \mathcal{L}_s(\mathbf{R}^N)$, we denote by $A_1 \leq A_2$ if $A_2 - A_1$ is positively semi-definite, and denote by $A_1 < A_2$ if $A_2 - A_1$ is positively definite. For any $A_1, A_2 \in L^\infty([0, T]; \mathcal{L}_s(\mathbf{R}^N))$, we denote by $A_1 \leq A_2$ if $A_1(t) \leq A_2(t)$ for a.e. $t \in]0, T[$, and denote by $A_1 < A_2$ if $A_1 \leq A_2$ and $A_1(t) < A_2(t)$ on a subset of $]0, T[$ with nonzero measure.

Let us have the space

$$H_1^p := \left\{ u \in H^p([0, T]; \mathbf{R}^N) \mid u^{(j)}(0) = u^{(j)}(T), j = 0, 1, \dots, p-1 \right\}$$

with the inner product

$$\langle u, v \rangle = \int_0^T (u^{(p)}(t), v^{(p)}(t)) + (u(t), v(t)) dt, \quad \forall u, v \in H_1^p,$$

where $(\cdot, \cdot)$ denotes the inner product in $\mathbf{R}^N$. The corresponding norm is defined by

$$\|u\| = \left(\int_0^T |u^{(p)}(t)|^2 + |u(t)|^2 dt \right)^{\frac{1}{2}}, \quad \forall u \in H_1^p.$$

Suppose that I_{ijk} and V satisfy that the following some conditions:

- (I₁) Every I_{ijk} ($i \in \mathbf{A}, j \in \mathbf{B}, k \in \mathbf{C}$) is bounded and $I_{ijk}(0) = 0$.
- (I₂) Every I_{ijk} ($i \in \mathbf{A}, j \in \mathbf{B}, k \in \mathbf{C}$) is odd.
- (V₁) $V(t, u)$ is twice continuously differentiable in u for a.e. $t \in [0, T]$.
- (V₂) $\nabla V(t, 0) \equiv 0$ and set $A_0(t) \equiv D_u^2 V(t, 0)$.
- (V₃) There exist $A_1, A_2 \in L^\infty([0, T]; \mathcal{L}_s(\mathbf{R}^N))$ and $r > 0$ such that

$$A_1(t) \leq D_u^2 V(t, u) \leq A_2(t)$$

for every $u \in \mathbf{R}^N$ with $|u| \geq r$, and a.e. $t \in [0, T]$.

- (V₄) $V(t, -u) = V(t, u)$ for every $u \in \mathbf{R}^N$ and a.e. $t \in [0, T]$.
- (V₅) $V(t, 0) \equiv 0$.

Our main results are the following two theorems.

Theorem 1. Suppose that (I₁), (V₁) – (V₃) with $i_p(A_1) = i_p(A_2) > 0$, $v_p(A_2) = 0$, and $i_p(A_1) \in [i_p(A_0), i_p(A_0) + v_p(A_0)]$ hold. Then, (1.1) has at least one nontrivial weak solution. Further, if $v_p(A_0) = 0$ and $|i_p(A_1) - i_p(A_0)| \geq pN$, then problem (1.1) has two nontrivial weak solutions.

Theorem 2. *Suppose that $(I_1), (I_2), (V_1) - (V_5)$ with $i_p(A_1) = i_p(A_2) > 0$, $v_p(A_2) = 0$, and $v_p(A_0) = 0$ hold. Then, problem (1.1) has at least $|i_p(A_1) - i_p(A_0)|$ distinct pairs of nontrivial weak solutions.*

Remark 1. Here $i_p(A)$ and $v_p(A)$ are called the index and nullity of A respectively. Indeed, for any $A \in L^\infty([0, T]; \mathcal{L}_s(\mathbf{R}^N))$, we define

$$\phi_A(u, v) = \int_0^T (u^{(p)}(t), v^{(p)}(t)) dt - \int_0^T (A(t)u(t), v(t)) dt, \quad \forall u, v \in H_1^p.$$

For any $x, y \in H_1^p$ if $\phi_A(x, y) = 0$, we say that x and y ϕ_A -orthogonal. If H_1, H_2 are the two subsets of H_1^p and for any $x \in H_1$ and $y \in H_2$, $\phi_A(x, y) = 0$, we say that H_1 and H_2 ϕ_A -orthogonal. In addition, H_1^p has a ϕ_A -orthogonal decomposition $H_1^p = H_1^{p+}(A) \oplus H_1^{p0}(A) \oplus H_1^{p-}(A)$ such that ϕ_A is positive definite, zero and negative definite on $H_1^{p+}(A), H_1^{p0}(A)$ and $H_1^{p-}(A)$ respectively. Moreover, $H_1^{p0}(A)$ and $H_1^{p-}(A)$ are finitely dimensional. Hence, for any $A \in L^\infty([0, T]; \mathcal{L}_s(\mathbf{R}^N))$, we define $i_p(A) = \dim H_1^{p-}(A)$, $v_p(A) = \dim H_1^{p0}(A)$. These results are the immediate conclusions of Proposition 2.1.1, Definition 2.1.2 in [5].

This paper is organized as follows. In Section 2, we first recall several critical point theorems. Then, we investigate the variational structure associated to problem (1.1) in H_1^p . Finally, we quote the two lemmas which are crucial in our argument. In Section 3, we verify our main results by applying variational methods and critical point theorems when V satisfies the generalized asymptotically linear conditions. Our results extends some conclusions directly in [4]. Analogously, by new definition of weak solution, one can be dealt with problem (1.1) when V satisfies some other conditions, such as the convex potential condition, the even type potential condition, the Ahmad-Lazer-Paul type coercive condition and its several generalizations, the sublinear potential condition, the superquadratic potential condition, the subquadratic potential condition and the asymptotically quadratic potential condition. In Section 4, we present some examples in order to illustrate our results.

2. PRELIMINARIES

Let X be a Hilbert space. We first recall some critical point theorems in critical point theory. These theorems are due to K. C. Chang.

Theorem 3 ([1, Theorem 4.3.4]). *Let $f \in C^1(X, \mathbf{R})$ be even and $f(0) = 0$. Assume f satisfies PS-condition and*

(i) there is an m -dimensional subspace X_1 and a constant $r > 0$ such that

$$\sup_{x \in X_1 \cap \partial U_r} f(x) < 0,$$

(ii) there is a j -codimensional subspace X_2 such that

$$\inf_{x \in X_2} f(x) > -\infty.$$

Then f has at least $m - j$ distinct pairs of critical points provided $m - j > 0$.

Theorem 4 ([1, Theorem 4.3.6]). Let $f \in C^1(X, \mathbf{R})$ be even and $f(0) = 0$. Assume f satisfies PS-condition and

(i) there is a j -codimensional subspace X_1 and two constants $r, \alpha > 0$ such that

$$f(x) \geq \alpha \quad \text{for any } x \in X_1 \cap \partial U_r,$$

(ii) there is a m -dimensional subspace X_2 and a constant $R > 0$ such that

$$f(x) \leq 0 \quad \text{for any } x \in X_2 \setminus U_R.$$

Then f has at least $m - j$ distinct pairs of critical points provided $m - j > 0$.

The last one is called three solutions theorem and can be verified by Theorem 5.1, Theorem 5.2 and Corollary 5.2 in [2]. One can find its proof in [6].

Theorem 5 ([6, Proposition 5.5.2]). Assume $f \in C^2(X, \mathbf{R})$ and satisfies PS-condition, $f''(x)$ is Fredholm with finite Morse index for each critical point $x \in X$ and $f'(0) = 0$. Suppose there is a positive integer γ such that

$$\gamma \in [m^-(f''(0)), m^0(f''(0)) + m^-(f''(0))]$$

and $H_q(X, f_a; \mathbf{R}) \cong \delta_{q\gamma} \mathbf{R}$ for some regular value $a < I(0)$, where $\delta_{q\gamma} = \begin{cases} 1, & \delta = \gamma, \\ 0, & \delta \neq \gamma. \end{cases}$

Then, f have a critical point $x_0 \neq 0$. Moreover, if 0 is a non-degenerate critical point, and $m^0(f''(x_0)) \leq |\gamma - m^-(f''(0))|$, then f have another critical point $x_1 \neq x_0, 0$.

Remark 2. Here $f_a = \{x \in X \mid f(x) \leq a, a \in \mathbf{R}\}$. The Morse nullity and Morse index of f at $x \in X$ are defined as $\dim(\ker f''(x))$ and the supremum of the dimensions of the vector subsequence of X in which $f''(x)$ is negative definite respectively. Both are denoted by $m^0(f''(x))$ and $m^-(f''(x))$ respectively.

Next, we investigate the variational structure of (1.1). This idea comes from papers [10, 14].

If $u \in H_1^p$, then $u_i^{(2p-j)}(t_k^+)$ and $u_i^{(2p-j)}(t_k^-)$ ($i \in \mathbf{A}, j \in \mathbf{B}, k \in \mathbf{C}$) may not exist. This leads to impulsive effects.

Let us take $y \in H_1^p$, multiply both sides of the equation in (1.1) by y , and integrate between 0 and T , then

$$\int_0^T ((-1)^{p+1} u^{(2p)}, v) dt + \int_0^T (\nabla V(t, u), v) dt = 0. \quad (2.1)$$

Moreover, combining $u^{(2p-1)}(0) = u^{(2p-1)}(T)$, one has

$$\begin{aligned}
 & \int_0^T ((-1)^{p+1} u^{(2p)}, v) dt \\
 &= (-1)^{p+1} \sum_{k=0}^q \int_{t_k}^{t_{k+1}} (u^{(2p)}, v) dt \\
 &= (-1)^{p+1} \sum_{k=0}^q \left((u^{(2p-1)}(t_{k+1}^-), v(t_{k+1}^-)) - (u^{(2p-1)}(t_k^+), v(t_k^+)) \right) \\
 &\quad - (-1)^{p+1} \sum_{k=0}^q \int_{t_k}^{t_{k+1}} (u^{(2p-1)}, \dot{v}) dt \\
 &= (-1)^{p+1} \sum_{i=1}^N \sum_{k=0}^q \left((u_i^{(2p-1)}(t_{k+1}^-), v_i(t_{k+1}^-)) - (u_i^{(2p-1)}(t_k^+), v_i(t_k^+)) \right) \\
 &\quad + (-1)^{p+2} \sum_{k=0}^q \int_{t_k}^{t_{k+1}} (u^{(2p-1)}, \dot{v}) dt \\
 &= (-1)^{p+1} \left((u^{(2p-1)}(T^-), v(T^-)) - (u^{(2p-1)}(0^+), v(0^+)) \right) \\
 &\quad + (-1)^{p+2} \sum_{i=1}^N \sum_{k=1}^q \Delta(u_i^{(2p-1)}(t_k)) v_i(t_k) + (-1)^{p+2} \int_0^T (u^{(2p-1)}, \dot{v}) dt \\
 &= \int_0^T (-1)^{p+2} (u^{(2p-1)}, \dot{v}) dt + (-1)^{p+2} \sum_{i=1}^N \sum_{k=1}^q I_{i1k} (u_i(t_k)) v_i(t_k).
 \end{aligned}$$

Similarly, combining $u^{(2p-j)}(0) = u^{(2p-j)}(T)$, $j \in \mathbf{B} \setminus \{1\}$, one has

$$\begin{aligned}
 & \int_0^T ((-1)^{p+2} u^{(2p-1)}, \dot{v}) dt \\
 &= \int_0^T ((-1)^{p+3} u^{(2p-2)}, \ddot{v}) dt + (-1)^{p+3} \sum_{i=1}^N \sum_{k=1}^q I_{i2k} (\dot{u}_i(t_k)) \dot{v}_i(t_k), \\
 & \quad \dots, \\
 & \int_0^T ((-1)^{2p} u^{(p+1)}, v^{(p-1)}) dt \\
 &= \int_0^T ((-1)^{2p+1} u^{(p)}, v^{(p)}) dt + (-1)^{2p+1} \sum_{i=1}^N \sum_{k=1}^q I_{ipk} (u_i^{(p-1)}(t_k)) v_i^{(p-1)}(t_k).
 \end{aligned}$$

Hence,

$$\begin{aligned} & \int_0^T (u^{(p)}, v^{(p)}) dt - \int_0^T (\nabla V(t, u), v) dt \\ & + \sum_{i=1}^N \sum_{j=1}^p \sum_{k=1}^q (-1)^{j+p} I_{ijk}(u_i^{(j-1)}(t_k)) v_i^{(j-1)}(t_k) = 0. \end{aligned} \quad (2.2)$$

Definition 1. Say that $u \in H_1^p$ is a weak solution for (1.1) if (2.2) holds for any $v \in H_1^p$.

Consider the functional $\varphi : H_1^p \rightarrow \mathbf{R}$ defined by putting

$$\varphi(u) := \frac{1}{2} \int_0^T |u^{(p)}(t)|^2 dt - \int_0^T V(t, u(t)) dt + \sum_{i=1}^N \sum_{j=1}^p \sum_{k=1}^q \int_0^{u_i^{(j-1)}(t_k)} I_{ijk}(s) ds. \quad (2.3)$$

Then, φ is Gâteaux differential at any $u \in H_1^p$ and

$$\begin{aligned} \langle \varphi'(u), v \rangle &= \int_0^T (u^{(p)}, v^{(p)}) dt - \int_0^T (\nabla V(t, u), v) dt \\ &+ \sum_{i=1}^N \sum_{j=1}^p \sum_{k=1}^q (-1)^{j+p} I_{ijk}(u_i^{(j-1)}(t_k)) v_i^{(j-1)}(t_k) \end{aligned} \quad (2.4)$$

for any $\nabla V(t, u) \in H_1^p$. Hence, the following lemma holds by (2.4) and Definition 1.

Lemma 1. If $u \in H_1^p$ is a critical point of φ , then u is a weak solution for (1.1).

Finally, we quote the two important lemmas. The first lemma is identical with Proposition in [6] when $T = 1$ and it's proof is absolutely similar to the proof of Proposition 5.3.1.

Lemma 2. If (V_1) -(V_3) hold, then for any $\varepsilon > 0$, there exists $A : [0, T] \times \mathbf{R}^N \rightarrow \mathcal{L}_s(\mathbf{R}^N)$ and $g : [0, T] \times \mathbf{R}^N \rightarrow \mathbf{R}^N$ such that

$$\nabla v(t, u) = A(t, u)u + g(t, u), \quad (2.5)$$

where

$$A_1 - \varepsilon I_N \leq A(t, u) \leq A_2 + \varepsilon I_N, \text{ for any } u \in \mathbf{R}^N \text{ a.e. } t \in [0, T], \quad (2.6)$$

$$A(\cdot, u(\cdot)) \in L^\infty([0, T], \mathcal{L}_s(\mathbf{R}^N)) \text{ for all } u \in L^2([0, T], \mathbf{R}^N), \quad (2.7)$$

and

$$g(\cdot, u(\cdot)) \in L^\infty([0, T], \mathbf{R}^N) \text{ is bounded for all } u \in L^2([0, T], \mathbf{R}^N). \quad (2.8)$$

Remark 3. Say then V satisfies asymptotically linear conditions if (2.5)-(2.8) hold. Lemma 2 shows that if (V_1) -(V_3) hold, then F satisfies these conditions.

The second lemma is a immediate corollary of Proposition 2.1.3 in [5].

Lemma 3. For any $A \in L^\infty([0, T]; \mathcal{L}_s(\mathbf{R}^N))$, we have

(i) $v_p(A)$ is the dimension of the solution subspace of

$$\begin{cases} (-1)^{p+1}u^{(2p)} + A(t)u = 0, & t \in [0, T], \\ u^{(j)}(0) = u^{(j)}(T), & j = 0, 1, \dots, 2p-1, \end{cases} \quad (2.9)$$

and $v_p(A) \in \{0, 1, \dots, pN\}$.

(ii) $i_p(A) = \sum_{\lambda < 0} v_p(A + \lambda I_N)$.

(iii) If $i_p(A) = 0$, then

$$\int_0^T |u^{(p)}(t)|^2 dt \geq \int_0^T (A(t)u(t), u(t)) dt, \quad \forall u \in H_1^p.$$

And the equality holds if and only if $u \in H_1^{p0}(A)$.

(iv) $v_p(A) = m^0(\phi_A)$, $i_p(A) = m^-(\phi_A)$.

For any $A_1, A_2 \in L^\infty([0, T]; \mathcal{L}_s(\mathbf{R}^N))$, we have

(v) If $A_1 \leq A_2$, then $i_p(A_1) \leq i_p(A_2)$ and $i_p(A_1) + v_p(A_1) \leq i_p(A_2) + v_p(A_2)$; if $A_1 < A_2$, then $i_p(A_1) + v_p(A_1) \leq i_p(A_2)$.

(vi) If $i_p(A_1) = i_p(A_2) > 0$, $v_p(A_2) = 0$, then $H_1^p = H_1^{p-}(A_1) \oplus H_1^{p+}(A_2)$.

3. PROOFS OF MAIN RESULTS

Proof of Theorem 1. By Theorem 5 and Lemma 3(iv), we complete the whole proof by three steps.

Step 1: $\varphi \in C^2(H_1^p, \mathbf{R})$ and 0 is a non-degenerate critical point of φ .

If (I_1) and (V_1) hold, then by the continuity of $I_{ijk}(i \in \mathbf{A}, j \in \mathbf{B}, k \in \mathbf{C})$, for every $u \in H_1^p$, $\varphi''(u)$ is determined by

$$\begin{aligned} \langle \varphi''(u)v, w \rangle &= \int_0^T (w^{(p)}(t), v^{(p)}(t)) dt - \int_0^T (D_u^2 V(t, u(t))w(t), v(t)) dt \\ &\quad + \sum_{i=1}^N \sum_{j=1}^p \sum_{k=1}^q (-1)^{j+p} I_{ijk} (u_i^{(j-1)}(t_k)) w_i^{(j-1)}(t_k) \end{aligned} \quad (3.1)$$

for all $v, w \in H_1^p$. Moreover, we have $\varphi \in C^2(H_1^p, \mathbf{R})$. Meanwhile, (2.4) together with (I_1) and (V_2) implies $\varphi'(0) = 0$. Namely, 0 is a critical point of φ . In addition, by (3.1), we have

$$\langle \varphi''(0)v, w \rangle = \int_0^T (w^{(p)}(t), v^{(p)}(t)) dt - \int_0^T (A_0(t)w(t), v(t)) dt.$$

If $\varphi''(0)v = 0$, then for all $w \in H_1^p$,

$$\int_0^T (w^{(p)}(t), v^{(p)}(t)) dt - \int_0^T (A_0(t)w(t), v(t)) dt = 0.$$

It implies that v is a solution of (2.9) where $A = A_0$. Because of $v_p(A_0) = 0$, we get $v = 0$. The only thing left to do is to prove that for all $w \in H_T^p$, $\varphi''(0)v = w$ has one solution in H_1^p . Indeed, $\varphi''(0) = Id - K$ where $K : H_1^p \rightarrow H_1^p$ defined by

$$\langle Kv, w \rangle = \int_0^T (A_0(t)w(t), v(t)) dt \quad \forall v, w \in H_1^p.$$

Considering that K is compact, we obtain that $R(\varphi''(0)) = H_1^p$ from $\ker \varphi''(0) = \{0\}$. Hence, $\varphi''(0)$ has a bounded inverse. Namely, 0 is a non-degenerate critical point of φ .

Step 2: φ satisfies PS -condition.

If (I_1) and $(V_1) - (V_3)$ with $i(A_1) = i(A_2) > 0$, $v(A_2) = 0$ hold, then φ satisfies PS -condition.

Let $\{\varphi(u_n)\}$ be a bounded sequence such that $\varphi'(u_n) \rightarrow 0$. We first prove that $\{u_n\}$ is bounded in H_T^p . Indeed, by (2.4), we have

$$\begin{aligned} \int_0^T (u_n^{(p)}, v^{(p)}) dt &= \langle \varphi'(u_n), v \rangle + \int_0^T (\nabla V(t, u_n), v) dt \\ &\quad - \sum_{i=1}^N \sum_{j=1}^p \sum_{k=1}^q (-1)^{j+p} I_{ijk} (u_{n_i}^{(j-1)}(t_k)) v_i^{(j-1)}(t_k). \end{aligned} \quad (3.2)$$

Let $F = C([0, T]; \mathbf{R}^N)$ with the norm $\|u\|_\infty = \max_{t \in [0, T]} |u(t)|$. We only need to verify that $\{u_n\}$ will be bounded in F . If not, we can assume $\|u_n\|_\infty \rightarrow +\infty$ and set $v_n = u_n / \|u_n\|_\infty$.

By (3.2) and (2.5), we get

$$\begin{aligned} &\int_0^T (u_n^{(p)}, v^{(p)}) dt \\ &= \|u_n\|_\infty^{-1} \langle \varphi'(u_n), v \rangle + \int_0^T (A(t, u_n) v_n(t), v) dt + \|u_n\|_\infty^{-1} \int_0^T (g(t, u_n), v) dt \\ &\quad - \|u_n\|_\infty^{-1} \sum_{i=1}^N \sum_{j=1}^p \sum_{k=1}^q (-1)^{j+p} I_{ijk} (u_i^{(j-1)}(t_k)) v_i^{(j-1)}(t_k). \end{aligned} \quad (3.3)$$

It implies that $\{v_n\}$ is bounded in H_1^p . Without loss of generality, we assume $v_n \rightharpoonup v_0$ in H_1^p , then $v_n \rightarrow v_0$ in F . By mean of $\|v_n\|_\infty = 1$, we have $\|v_0\|_\infty = 1$. In addition, (3.3) can become

$$\begin{aligned} &\int_0^T \left((u_n^{(p)}, v^{(p)}) - (A(t, u_n) v_n, v) \right) dt \\ &= \|u_n\|_\infty^{-1} \langle \varphi'(u_n), v \rangle + \|u_n\|_\infty^{-1} \int_0^T (g(t, u_n), v) dt \end{aligned}$$

$$-\|u_n\|_\infty^{-1} \sum_{i=1}^N \sum_{j=1}^p \sum_{k=1}^q (-1)^{j+p} I_{ijk}(u_i^{(j-1)}(t_k)) v_i^{(j-1)}(t_k). \quad (3.4)$$

However, by (2.6) and (2.7), there exists $\tilde{A} \in L^\infty([0, T]; \mathcal{L}_s(\mathbf{R}^N))$ such that

$$\int_0^T (A(t, u_n) v(t), w(t)) dt \rightarrow \int_0^T (\tilde{A}(t) v(t), w(t)) dt \quad \forall v, w \in L^2([0, T], \mathbf{R}^N), \quad (3.5)$$

by going to subsequences if necessary, and

$$A_1 - \varepsilon I_N \leq \tilde{A} \leq A_2 + \varepsilon I_N. \quad (3.6)$$

Since $\varphi'(u_n) \rightarrow 0$ and $\|u_n\|_\infty \rightarrow +\infty$, the right side of (3.4) tends to zero. (3.4) and (3.5) imply that

$$\int_0^T \left((v_0^{(p)}(t), v^{(p)}(t)) - (\tilde{A}(t) v_0(t), v(t)) \right) dt = 0, \quad \forall v \in H_1^p. \quad (3.7)$$

Hence, v_0 is the solution of the following problem

$$\begin{cases} (-1)^{p+1} v^{(2p)}(t) + \tilde{A}(t) v(t) = 0, \\ v^{(j)}(0) = v^{(j)}(T), \quad j = 0, 1, \dots, 2p-1. \end{cases} \quad (3.8)$$

By (3.6) and Lemma 3(v), we have $i_p(\tilde{A}) \leq i_p(A_2 + \varepsilon I_N)$ and $i_p(\tilde{A}) + v_p(\tilde{A}) \leq i_p(A_2 + \varepsilon I_N) + v_p(A_2 + \varepsilon I_N)$. But $v_p(A_2 + \varepsilon I_N) = v_p(A_2) = 0$. Hence, $v_p(\tilde{A}) = 0$ and (3.8) has only trivial solution. This contradicts $\|v_0\|_\infty = 1$.

Next, we prove that $\{u_n\}$ contains a convergent subsequence. Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_m}\}$ such that $u_{n_m} \rightharpoonup u_0$ in H_1^p . Then $u_{n_m} \rightarrow u_0$ uniformly in $[0, 1]$ and

$$u_{n_m} \rightarrow u_0 \quad (3.9)$$

in $L^2([0, T], \mathbf{R}^N)$. By (3.2), we have

$$\begin{aligned} \int_0^T (u_{n_m}^{(p)}, v^{(p)}) dt &= \langle \varphi'(u_{n_m}), v \rangle + \int_0^T (\nabla V(t, u_{n_m}), v) dt \\ &\quad - \sum_{i=1}^N \sum_{j=1}^p \sum_{k=1}^q (-1)^{j+p} I_{ijk}(u_{n_m i}^{(j-1)}(t_k)) v_i^{(j-1)}(t_k). \end{aligned}$$

Moreover, $\varphi'(u_{n_m}) \rightarrow 0$ implies

$$\begin{aligned} \int_0^T (u_0^{(p)}, v^{(p)}) dt &= \int_0^T (\nabla V(t, u_0), v) dt \\ &\quad - \sum_{i=1}^N \sum_{j=1}^p \sum_{k=1}^q (-1)^{j+p} I_{ijk}(u_{0i}^{(j-1)}(t_k)) v_i^{(j-1)}(t_k), \quad \forall v \in H_1^p. \end{aligned}$$

Hence,

$$\begin{aligned}
& \int_0^T \left((u_{n_m}^{(p)}(t) - u_0^{(p)}(t)), v^{(p)}(t) \right) dt \\
&= \langle \varphi'(u_{n_m}), v \rangle + \int_0^T ((\nabla V(t, u_{n_m}(t)) - \nabla V(t, u_0(t))), v(t)) dt \\
&\quad - \sum_{i=1}^N \sum_{j=1}^p \sum_{k=1}^q (-1)^{j+p} \left(I_{ijk}(u_{n_m i}^{(j-1)}(t_k)) - I_{ijk}(u_{0i}^{(j-1)}(t_k)) \right) v_i^{(j-1)}(t_k).
\end{aligned}$$

It follows that

$$\begin{aligned}
& \|u_{n_m} - u_0\| \\
&= \|u_{n_m}^{(p)} - u_0^{(p)}\|_{L^2} + \|u_{n_m} - u_0\|_{L^2} \\
&= \sup_{\|v\| \leq 1} \int_0^1 \left((u_{n_m}^{(p)}(t) - u_0^{(p)}(t)), v^{(p)} \right) dt + \|u_{n_m} - u_0\|_{L^2} \\
&= \sup_{\|v\| \leq 1} \left(\varphi'(u_{n_m}), v + \int_0^1 (V'(t, u_{n_m}) - V'(t, u_0)) v dt \right. \\
&\quad \left. - \sum_{i=1}^N \sum_{j=1}^p \sum_{k=1}^q (-1)^{j+p} \left(I_{ijk}(u_{n_m i}^{(j-1)}(t_k)) - I_{ijk}(u_{0i}^{(j-1)}(t_k)) \right) v_i^{(j-1)}(t_k) \right)
\end{aligned}$$

By $\varphi'(u_{n_m}) \rightarrow 0$, (V_1) , the continuity of I_{ijk} ($i \in \mathbf{A}, j \in \mathbf{B}, k \in \mathbf{C}$) and (3.9), we have $\|u_{n_m} - u_0\| \rightarrow 0$. Namely, $u_{n_m} \rightarrow u_0$ in H_1^p .

Step 3: For some regular value $a < \varphi(0)$,

$$H_q(H_1^p, \varphi_a; \mathbf{R}) \cong \delta_{q\gamma} \mathbf{R}, \text{ where } \gamma = i_p(A_1). \quad (3.10)$$

By Lemma 2, we know that

$$\nabla V(t, u) = A(t, u)u + g(t, u), \quad (3.11)$$

where

$$A_1 - \varepsilon I_N \leq A(t, u) \leq A_2 + \varepsilon I_N$$

and $g(t, u)$ is bounded since $(V_1) - (V_3)$ with $i_p(A_1) = i_p(A_2) > 0$ hold. In addition, if let $A_3(t) = A_1(t) - \varepsilon I_N$, $A_4(t) = A_2(t) + \varepsilon I_N$, then

$$A_3(t) \leq A(t, u) \leq A_4(t). \quad (3.12)$$

Moreover, let $H_1 = H_1^{p-}(A_3)$, $H_2 = H_1^{p+}(A_4)$, then $H_1^p = H_1 \oplus H_2$ by Lemma 3(vi). Denote

$$\|u\|_1 := (-\phi_{A_3}(u, u))^{\frac{1}{2}}, \forall u \in H_1, \quad \|u\|_2 := \phi_{A_4}(u, u)^{\frac{1}{2}}, \forall u \in H_2.$$

From Lemma 3(iii), we can verify that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent to $\|\cdot\|$ on H_1 and H_2 respectively. Hence, by (2.4), for any $u = u_1 + u_2$ with $u_1 \in H_1, u_2 \in H_2$, we have

$$\begin{aligned}
& \langle \varphi'(u), (u_2 - u_1) \rangle \\
&= \int_0^T (|u_2^{(p)}(t)|^2 - |u_1^{(p)}(t)|^2) dt + \int_0^T A(t, u(t)) u(t) \cdot (u_2(t) - u_1(t)) dt \\
&\quad - \int_0^T (g(t, u(t)), (u_2(t) - u_1(t))) dt + \sum_{l=1}^M \sum_{k=1}^N I_{kl}(u^k(t_l))(u_2 - u_1)^k(t_l) \\
&= \int_0^T |u_2^{(p)}(t)|^2 dt - \int_0^T (A(t, u(t)) u_2(t), u_2(t)) dt - \int_0^T (g(t, u(t)), u_2(t)) dt \\
&\quad - \int_0^T |u_1^{(p)}(t)|^2 dt + \int_0^T (A(t, u(t)) u_1(t), u_1(t)) dt + \int_0^T (g(t, u(t)), u_1(t)) dt \\
&\quad + \sum_{l=1}^M \sum_{k=1}^N I_{kl}(u^k(t_l)) u_2^k(t_l) - \sum_{l=1}^M \sum_{k=1}^N I_{kl}(u^k(t_l)) u_1^k(t_l) \\
&\geq \int_0^T |u_2^{(p)}(t)|^2 dt - \int_0^T (A_4 u_2(t), u_2(t)) dt - \int_0^T (g(t, u(t)), u_2(t)) dt \\
&\quad - \int_0^T |u_1^{(p)}(t)|^2 dt + \int_0^1 (A_3 u_1(t), u_1(t)) dt + \int_0^T (g(t, u(t)), u_1(t)) dt \\
&\quad + \sum_{l=1}^M \sum_{k=1}^N I_{kl}(u^k(t_l)) u_2^k(t_l) - \sum_{l=1}^M \sum_{k=1}^N I_{kl}(u^k(t_l)) u_1^k(t_l) \\
&= \|u_2\|_2^2 - \int_0^T (g(t, u(t)), u_2(t)) dt + \|u_1\|_1^2 + \int_0^T (g(t, u(t)), u_1(t)) dt \\
&\quad + \sum_{l=1}^M \sum_{k=1}^N I_{kl}(u^k(t_l)) u_2^k(t_l) - \sum_{l=1}^M \sum_{k=1}^N I_{kl}(u^k(t_l)) u_1^k(t_l) \\
&\geq C_1 \|u_2\|^2 + C_2 \|u_1\|^2 - C_3 \|u_2\| - C_4 \|u_1\| + C_5
\end{aligned} \tag{3.13}$$

where $C_i \in \mathbf{R}^+(i = 1, 2)$, $\mathbf{R}^+$ is a set of positive constants, $C_i \in \mathbf{R}(i = 3, 4, 5)$. It follows that there exists $R_0 > 0$ such that

$$\langle \varphi'(u), u_2 - u_1 \rangle > 1, \quad \forall u \in H_1^P$$

with $\|u_2\| > R_0$ or $\|u_1\| > R_0$. Denote by $\mathcal{M} = (H_2 \cap \bar{U}_{R_0}) \oplus H_1$, where $\bar{U}_{R_0}$ is a closed ball with center 0 and radius R_0 . Since φ is decreasing along vector field $V(u) = -u_2 + u_1$ for every $u = u_2 + u_1 \notin \mathcal{M}$, we can define the flow $\sigma(t, u) =$

$e^{-t}u_2 + e^t u_1$, and the time T_u arriving at $\mathcal{M}$ satisfies $e^{-T_u}\|u_2\| = R_0$. Set

$$\eta(t, u_2 + u_1) = \begin{cases} u_2 + u_1, & \text{for } \|u\| \leq R_0, \\ \sigma(T_u t, u), & \text{for } \|u\| > R_0. \end{cases}$$

We can verify that for any $-a > -\varphi(0)$ large enough, $\eta(t, u)$ is a deformation retract from (H_1^P, φ_a) to $(\mathcal{M}, \mathcal{M} \cap \varphi_a)$. Hence,

$$H_q(H_1^P, \varphi_a; \mathbf{R}) \cong H_q(\mathcal{M}, \mathcal{M} \cap \varphi_a; \mathbf{R}). \quad (3.14)$$

On the other hand, by (3.11), we have

$$\begin{aligned} V(t, u) &= \int_0^T (\nabla V(t, su), u) ds + V(t, 0) \\ &= \int_0^T (A(t, su)su, u) ds + \int_0^T (g(t, su), u) ds + V(t, 0). \end{aligned}$$

Thus,

$$\begin{aligned} &\int_0^T V(t, u) dt \\ &= \int_0^T \left(\int_0^T (A(t, su)su, u) ds + \int_0^T (g(t, su), u) ds + V(t, 0) \right) dt \\ &= \int_0^T \int_0^T (A(t, su)su, u) ds dt + \int_0^T \int_0^T (g(t, su), u) ds dt + \int_0^T V(t, 0) dt \\ &\geq \int_0^T \int_0^T (A_3 su, u) ds dt + \int_0^T \int_0^T (g(t, su), u) ds dt + \int_0^T V(t, 0) dt \\ &= \frac{1}{2} \int_0^T (A_3 u, u) dt + \int_0^T \int_0^T (g(t, su), u) ds dt + \int_0^T V(t, 0) dt \end{aligned}$$

because of (3.12). Hence, for any $u = u_1 + u_2 \in \mathcal{M}$, we have

$$\begin{aligned} \varphi(u) &\leq \frac{1}{2} \int_0^T |u^{(p)}|^2 dt - \frac{1}{2} \int_0^T (A_3 u, u) dt - \int_0^T \int_0^T (g(t, su), u) ds dt \\ &\quad - \int_0^T V(t, 0) dt + \sum_{i=1}^N \sum_{j=1}^p \sum_{k=1}^q \int_0^{u_i^{(j-1)}(t_k)} I_{ijk}(s) ds \\ &\leq \frac{1}{2} \int_0^T (|u_1^{(p)}|^2 + |u_2^{(p)}|^2 + 2(u_1^{(p)}, u_2^{(p)})) dt - \frac{1}{2} \int_0^T (A_3 u_1, u_1) dt \\ &\quad - \frac{1}{2} \int_0^T (A_3 u_2, u_2) dt - \int_0^T (A_3 u_1, u_2) dt - \int_0^T \int_0^T (g(t, su), u_1) ds dt \end{aligned}$$

$$\begin{aligned}
 & - \int_0^T \int_0^T (g(t, su), u_2) ds dt + C_6 \\
 & = \frac{1}{2} \int_0^T (|u_1^{(p)}|^2 - (A_3 u_1, u_1)) dt + \frac{1}{2} \int_0^T (|u_2^{(p)}|^2 - (A_3 u_2, u_2)) dt \\
 & \quad + \int_0^T ((u_1^{(p)}, u_2^{(p)}) - (A_3 u_1, u_2)) dt - \int_0^T \int_0^T (g(t, su), u_1) ds dt \\
 & \quad - \int_0^T \int_0^T (g(t, su), u_2) ds dt + C_6 \\
 & = \phi_{A_3}(u_1, u_1) + \phi_{A_3}(u_2, u_2) - \int_0^T \int_0^T (g(t, su), u_1) ds dt \\
 & \quad - \int_0^T \int_0^T (g(t, su), u_2) ds dt + C_6 \\
 & \leq -C_7 \|u_1\|^2 + C_8 \|u_1\| + C_9,
 \end{aligned} \tag{3.15}$$

where $C_7 \in \mathbf{R}^+$, $C_i \in \mathbf{R}$ ($i = 6, 8, 9$).

By (3.12), we get $0 < -\phi_{A_3}(u, u) \leq -\phi_{A_4}(u, u)$. Since $\|\cdot\|_1$ is equivalent to $\|\cdot\|$ on H_1 and H_1 is finitely dimensional, $(-\phi_{A_4}(\cdot, \cdot))^{\frac{1}{2}}$ is also the norm of H_1 . Similar to (3.15), we have

$$\varphi(u) \geq -C_{10} \|u_1\|^2 + C_{11} \|u_1\| + C_{12}, \tag{3.16}$$

where $C_{10} \in \mathbf{R}^+$, $C_i \in \mathbf{R}$ ($i = 11, 12$).

(3.15) and (3.16) show that for any $u = u_1 + u_2 \in \mathcal{M}$,

$$\varphi(u) \rightarrow -\infty \quad \text{if and only if} \quad \|u_1\| \rightarrow +\infty$$

uniformly in $u_2 \in H_2 \cap \bar{U}_{R_0}$. Thus, there exist $T > 0, a_1 < a_2 < -T, R_1 > R_2 > R_0$ such that

$$(H_2 \cap \bar{U}_{R_0}) \oplus (H_1 \setminus U_{R_1}) \subset \varphi_{a_1} \cap \mathcal{M} \subset (H_2 \cap \bar{U}_{R_0}) \oplus (H_1 \setminus U_{R_2}) \subset \varphi_{a_2} \cap \mathcal{M}.$$

Let $\sigma(t, u) = e^{-t} u_2 + e^t u_1$, for every $u \in \mathcal{M} \cap (\varphi_{a_2} \setminus \varphi_{a_1})$, $\varphi(\sigma(t, u))$ is continuous with respect to t , $\varphi(\sigma(0, u)) = \varphi(u) > a_1$ and $\varphi(\sigma(t, u)) \rightarrow -\infty (t \rightarrow +\infty)$, so there exists uniquely $t = T_1(u)$ such that $\sigma(t, u) \in \mathcal{M} \cap \varphi_{a_1}$ and $\varphi(\sigma(t, u)) = a_1$. Define

$$\eta_1(t, u) = \begin{cases} u, & \text{for } u \in \mathcal{M} \cap \varphi_{a_1}, \\ \sigma(T_1(u)t, u), & \text{for } u \in \mathcal{M} \cap (\varphi_{a_2} \setminus \varphi_{a_1}) \end{cases}$$

and

$$\eta_2(t, u) = \begin{cases} u, & \text{for } \|u_1\| \geq R_1, \\ u_2 + t u_1 + (1-t) \frac{R_1}{\|u_1\|} u_1, & \text{for } \|u_1\| < R_1. \end{cases}$$

By the map $\eta(t, u) = \eta_2(t, \eta_1(t, u))$ we can verify that $(H_2 \cap \bar{U}_{R_0} \oplus (H_1 \setminus U_{R_1}); \mathbf{R})$ is a strong deformation retract of $(\mathcal{M} \cap \varphi_{a_2}; \mathbf{R})$. Hence,

$$\begin{aligned} H_q(\mathcal{M}, \mathcal{M} \cap \varphi_{a_2}; \mathbf{R}) &\cong H_q((H_2 \cap \bar{U}_{R_0}) \oplus H_1, (H_2 \cap \bar{U}_{R_0}) \oplus (H_1 \setminus U_{R_1}); \mathbf{R}) \\ &\cong H_q(H_1 \cap \bar{U}_{R_1}, \partial(H_1 \cap U_{R_1}); \mathbf{R}) \\ &\cong \delta_{q\gamma} \mathbf{R}. \end{aligned} \quad (3.17)$$

(3.14) and (3.17) imply that (3.10) holds.

Now all the conditions of Theorem 5 are satisfied, and the corresponding conclusions hold. The proof is complete. $\square$

Proof of Theorem 2. By the proof of Theorem 1, we know that $\varphi \in C^1(H_1^p, \mathbf{R})$, and φ satisfies PS -condition when $(I_1), (V_1) - (V_3)$ with $i_p(A_1) = i_p(A_2) > 0$, $v_p(A_2) = 0$ hold. Moreover, we find that φ is even and $\varphi(0) = 0$ when $(I_1), (I_2), (V_4)$ and (V_5) hold. In fact,

$$\begin{aligned} &\varphi(-u) \\ &= \frac{1}{2} \int_0^T | -u^{(p)} |^2 dt + \sum_{i=1}^N \sum_{j=1}^p \sum_{k=1}^q \int_0^{-u_i^{(j-1)}(t_k)} I_{ijk}(s) ds - \int_0^T V(t, -u) dt. \\ &= \frac{1}{2} \int_0^T | u^{(p)} |^2 dt + \sum_{i=1}^N \sum_{j=1}^p \sum_{k=1}^q \int_0^{u_i^{(j-1)}(t_k)} -I_{ijk}(-s_1) ds_1 - \int_0^T V(t, u) dt. \\ &= \frac{1}{2} \int_0^T | u^{(p)} |^2 dt + \sum_{i=1}^N \sum_{j=1}^p \sum_{k=1}^q \int_0^{u_i^{(j-1)}(t_k)} I_{ijk}(s_1) ds_1 - \int_0^T V(t, u) dt \\ &= \varphi(u). \end{aligned}$$

Now it suffices to consider the two possibilities: $i_p(A_0) > i_p(A_1)$ and $i_p(A_0) < i_p(A_1)$. In fact, we only consider the second case because the first case can be investigated as before by Theorem 3. For small $\varepsilon > 0$ satisfying $v_p(A_0 + \varepsilon I_N) = 0 = v_p(A_1 - \varepsilon I_N)$, $i_p(A_0 + \varepsilon I_N) = i_p(A_0)$ and $i_p(A_1 - \varepsilon I_N) = i_p(A_1)$, similar to (3.13) and (3.15), we can verify that for any $x \in H_1^{p+}(A_0 + \varepsilon I_N)$,

$$\varphi(u) \geq C_{13} \|u\|^2 + C_{14} \|u\| + C_{15},$$

and for any $x \in H_1^{p-}(A_1 - \varepsilon I_N)$,

$$\varphi(u) \leq -C_{16} \|u\|^2 + C_{17} \|u\| + C_{18},$$

where $C_i \in \mathbf{R}^+$ ($i = 13, 16$), $C_i \in \mathbf{R}$ ($i = 14, 15, 17, 18$). Let $j = i_p(A_0)$, $m = i_p(A_1)$. Hence, for these two subspaces $H_1^{p+}(A_0 + \varepsilon I_N)$ and $H_1^{p-}(A_1 - \varepsilon I_N)$ of H_1^p , Theorem 4(i)(ii) hold. The proof is complete. $\square$

4. EXAMPLES

For the sake of simplicity, we only consider (1.1) for the case of $N = p = q = T = 1$. In addition, let $t_1 = \frac{1}{2}$ and $I_{111}(s) = \frac{s}{|s|+1}$, $s \in \mathbf{R}$.

Example 1. Consider the following problem

$$\begin{cases} \ddot{u}(t) + \nabla V(t, u) = 0, \\ u(0) = u(1), \dot{u}(0) = \dot{u}(1), \\ \Delta(\dot{u}(t_1)) = \dot{u}(t_1^+) - \dot{u}(t_1^-) = I_{111}(u(t_1)), \end{cases} \quad (4.1)$$

where $V(t, u) = 4\pi^2 e^{-\frac{t}{10}} u^2 + \pi^2 e^{-\frac{t}{10}} [(u-6) \ln(u^2+1) + 12u \arctan u + 2 \arctan u - 2u]$.

Clearly, $|I_{111}(s)| \leq 1$ and $I_{111}(0) = 0$. $V(t, u)$ is C^2 in u for every $t \in [0, T]$ and $D_u^2 V(t, u) = \pi^2 e^{-\frac{t}{10}} (8 + \frac{2u+12}{u^2+1})$. Then, $7\pi^2 e^{-\frac{t}{10}} < D_u^2 V(t, u) < 9\pi^2 e^{-\frac{t}{10}}$ as $|u| > 5$. In addition, $A_0(t) = 20\pi^2 e^{-\frac{t}{10}}$. By Lemma 3, we obtain that $i_1(7.5\pi^2 e^{-\frac{t}{10}}) = i_1(8.5\pi^2 e^{-\frac{t}{10}}) = 3$, $i_1(A_0) = 5$, $v_1(A_0) = 0$ by simple calculations. Hence, (4.1) has two nontrivial weak solutions by Theorem 1.

Example 2. Consider (4.1), where $V(t, u) = \frac{7}{2}\pi^2 e^{-\frac{t}{10}} u^2 - 8\pi^2 e^{-\frac{t}{10}} [-\frac{1}{2} \ln(u^2+1) + u \arctan u]$. By Example 1, I_{111} satisfies (I_1) . Meanwhile, I_{111} is odd. V satisfies (V_1) – (V_2) and is even in u for every $t \in [0, T]$. In addition, $V(t, 0) \equiv 0$ and $D_u^2 V(t, u) = \pi^2 e^{-\frac{t}{10}} (7 - \frac{8}{u^2+1})$. Then $6.2\pi^2 e^{-\frac{t}{10}} < D_u^2 V(t, u) < 7\pi^2 e^{-\frac{t}{10}}$ as $|u| > 3$. Meanwhile, $A_0(t) = -\pi^2 e^{-\frac{t}{10}}$. $i_1(6.2\pi^2 e^{-\frac{t}{10}}) = i_1(7\pi^2 e^{-\frac{t}{10}}) = 3$, $v_1(7\pi^2 e^{-\frac{t}{10}}) = i_1(A_0) = v_1(A_0) = 0$. Hence, (4.1) has at least three distinct pairs of weak solutions by Theorem 2.

REFERENCES

- [1] K. Chang, *Critical Point Theory and its Applications*. Shanghai: Shanghai Science and Technology Press, 1986.
- [2] K. Chang, *Infinite Dimensional Morse Theory and Multiple Solution Problems*. Basel: Birkhauser, 1993. doi: [10.1007/978-1-4612-0385-8](https://doi.org/10.1007/978-1-4612-0385-8).
- [3] B. Dai and J. Guo, “Solvability of a second-order Hamiltonian system with impulsive effects,” *Boundary Value Prob.*, vol. 2013, no. 1, pp. 1–17, 2013, doi: [10.1186/1687-2770-2013-151](https://doi.org/10.1186/1687-2770-2013-151).
- [4] Y. Dong, “Index theory, nontrivial solutions, and asymptotically linear second-order Hamiltonian systems,” *J. Differ. Equ.*, vol. 214, no. 2, pp. 233–255, 2005, doi: [10.1016/j.jde.2004.10.030](https://doi.org/10.1016/j.jde.2004.10.030).
- [5] Y. Dong, “Index theory for linear selfadjoint operator equations and nontrivial solutions for asymptotically linear operator equations,” *Calc. Var.*, vol. 38, no. 1-2, pp. 75–109, 2010, doi: [10.1007/s00526-009-0279-5](https://doi.org/10.1007/s00526-009-0279-5).
- [6] Y. Dong, *Index Theory for Hamiltonian Systems and Multiple Solution Problems*. Beijing: Science Press, 2015.
- [7] J. Graef, S. Heidarkhani, and L. Kong, “Nontrivial periodic solutions to second-order impulsive Hamiltonian systems,” *Electron J. Differ. Equ.*, vol. 2015, no. 204, pp. 1–17, 2015.

- [8] X. He and P. Chen, “Existence of solutions for a class of second-order sublinear and linear Hamiltonian systems with impulsive effects,” *Elec. J. Quali. Theory Differ. Equ.*, vol. 30, no. 78, pp. 1433–1448, 2011, doi: [10.14232/ejqtde.2011.1.78](https://doi.org/10.14232/ejqtde.2011.1.78).
- [9] S. Heidarkhani, M. Ferrara, and A. Salari, “Infinitely Many Periodic Solutions for a Class of Perturbed Second-Order Differential Equations with Impulses,” *Acta. Appl. Math.*, vol. 139, no. 1, pp. 81–94, 2015, doi: [10.1007/s10440-014-9970-4](https://doi.org/10.1007/s10440-014-9970-4).
- [10] J. Nieto and D. O’Regan, “Variational approach to impulsive differential equations,” *Nonlinear Anal. RWA*, vol. 10, no. 2, pp. 680–690, 2009, doi: [10.1016/j.nonrwa.2007.10.022](https://doi.org/10.1016/j.nonrwa.2007.10.022).
- [11] J. Sun, H. Chen, and J. Nieto, “Infinitely many solutions for second-order Hamiltonian system with impulsive effects,” *Math. Comput. Modelling*, vol. 54, no. 1-2, pp. 544–555, 2011, doi: [10.1016/j.mcm.2011.02.044](https://doi.org/10.1016/j.mcm.2011.02.044).
- [12] J. Sun, H. Chen, J. Nieto, and M. Otero-Novoa, “The multiplicity of solutions for perturbed second-order Hamiltonian systems with impulsive effects,” *Nonlinear Anal.*, vol. 72, no. 12, pp. 4575–4586, 2010, doi: [10.1016/j.na.2010.02.034](https://doi.org/10.1016/j.na.2010.02.034).
- [13] Y. Tian and W. Ge, “Applications of variational methods boundary value problem for impulsive differential equations,” *Proc. Edinb. Math. Soc.*, vol. 51, no. 2, pp. 509–527, 2008, doi: [10.1017/S0013091506001532](https://doi.org/10.1017/S0013091506001532).
- [14] Y. Tian and X. Liu, “Applications of variational methods to Sturm-Liouville boundary-value problem for fourth-order impulsive differential equations,” *Math. Meth. Appl. Sci.*, vol. 37, no. 1, pp. 95–105, 2013, doi: [10.1002/mma.2787](https://doi.org/10.1002/mma.2787).
- [15] J. Zhou and Y. Li, “Existence of solutions for a class of second-order Hamiltonian systems with impulsive effects,” *Nonlinear Anal.*, vol. 72, no. 3-4, pp. 1594–1603, 2010, doi: [10.1016/j.na.2009.08.041](https://doi.org/10.1016/j.na.2009.08.041).

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ON HERMITE-HADAMARD TYPE INEQUALITIES FOR F -CONVEX FUNCTION

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Abstract. In this study, we firstly give some properties the family F and F -convex function which are defined by B. Samet. Then, we obtain some midpoint inequalities for differentiable function. Moreover, we establish some midpoint and trapezoid type inequalities for function whose second derivatives in absolute value are F -convex.

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1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. If f is a convex function then the following double inequality, which is well known in the literature as the Hermite–Hadamard inequality, holds [8]

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}. \quad (1.1)$$

Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f . Both inequalities hold in the reversed direction if f is concave (1.1).

It is well known that the Hermite–Hadamard inequality plays an important role in nonlinear analysis. Over the last decade, this classical inequality has been improved and generalized in a number of ways; there have been a large number of research papers written on this subject, (see, [3, 4, 6, 9, 12, 13]) and the references therein.

Over the years, many type of convexity have been defined, such as quasi-convex [1], pseudo-convex [7], strongly convex [10], ε -convex [2], s -convex [5], h -convex [14] and etc. Recently, Samet [11] have defined a new concept of convexity that depends on a certain function satisfying some axioms, that generalizes different types of convexity, including ε -convex functions, α -convex functions, h -convex functions and many others.

Recall the family $\mathcal{F}$ of mappings $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ satisfying the following axioms:

(A1) If $u_i \in L^1(0, 1)$, $i = 1, 2, 3$, then for every $\lambda \in [0, 1]$, we have

$$\int_0^1 F(u_1(t), u_2(t), u_3(t), \lambda) dt = F\left(\int_0^1 u_1(t) dt, \int_0^1 u_2(t) dt, \int_0^1 u_3(t) dt, \lambda\right).$$

(A2) For every $u \in L^1(0, 1)$, $w \in L^\infty(0, 1)$ and $(z_1, z_2) \in \mathbb{R}^2$, we have

$$\int_0^1 F(w(t)u(t), w(t)z_1, w(t)z_2, t) dt = T_{F,w}\left(\int_0^1 w(t)u(t) dt, z_1, z_2\right),$$

where $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a function that depends on (F, w) , and it is non-decreasing with respect to the first variable.

(A3) For any $(w, u_1, u_2, u_3) \in \mathbb{R}^4$, $u_4 \in [0, 1]$, we have

$$wF(u_1, u_2, u_3, u_4) = F(wu_1, wu_2, wu_3, u_4) + L_w$$

where $L_w \in \mathbb{R}$ is a constant that depends only on w .

Definition 1. Let $f : [a, b] \rightarrow \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $a < b$, be a given function. We say that f is a convex function with respect to some $F \in \mathcal{F}$ (or F -convex function) iff

$$F(f(tx + (1-t)y), f(x), f(y), t) \leq 0, \quad (x, y, t) \in [a, b] \times [a, b] \times [0, 1].$$

Remark 1. 1) Let $\varepsilon \geq 0$, and let $f : [a, b] \rightarrow \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $a < b$, be an ε -convex function, that is (see [2])

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon, \quad (x, y, t) \in [a, b] \times [a, b] \times [0, 1].$$

Define the functions $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ by

$$F(u_1, u_2, u_3, u_4) = u_1 - u_4 u_2 - (1 - u_4) u_3 - \varepsilon \quad (1.2)$$

and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$T_{F,w}(u_1, u_2, u_3) = u_1 - \left(\int_0^1 t w(t) dt\right) u_2 - \left(\int_0^1 (1-t) w(t) dt\right) u_3 - \varepsilon. \quad (1.3)$$

For

$$L_w = (1-w)\varepsilon, \quad (1.4)$$

it is clear that $F \in \mathcal{F}$ and

$F(f(tx + (1-t)y), f(x), f(y), t) = f(tx + (1-t)y) - tf(x) - (1-t)f(y) - \varepsilon \leq 0$, that is f is an F -convex function. Particularly, taking $\varepsilon = 0$, we show that if f is a convex function then f is an F -convex function with respect to F defined above.

2) Let $f : [a, b] \rightarrow \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $a < b$, be an α -convex function, $\alpha \in (0, 1]$, that is

$$f(tx + (1-t)y) \leq t^\alpha f(x) + (1-t^\alpha) f(y), \quad (x, y, t) \in [a, b] \times [a, b] \times [0, 1].$$

Define the functions $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ by

$$F(u_1, u_2, u_3, u_4) = u_1 - u_4^\alpha u_2 - (1 - u_4^\alpha) u_3 \quad (1.5)$$

and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$T_{F,w}(u_1, u_2, u_3) = u_1 - \left(\int_0^1 t^\alpha w(t) dt \right) u_2 - \left(\int_0^1 (1 - t^\alpha) w(t) dt \right) u_3. \quad (1.6)$$

For $L_w = 0$, it is clear that $F \in \mathcal{F}$ and

$$F(f(tx + (1-t)y), f(x), f(y), t) = f(tx + (1-t)y) - t^\alpha f(x) - (1 - t^\alpha) f(y) \leq 0,$$

that is f is an F -convex function.

3) Let $h : J \rightarrow [0, \infty)$ be a given function which is not identical to 0, where J is an interval in $\mathbb{R}$ such that $(0, 1) \subseteq J$. Let $f : [a, b] \rightarrow [0, \infty)$, $(a, b) \in \mathbb{R}^2$, $a < b$, be an h -convex function, that is (see [14])

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y), \quad (x, y, t) \in [a, b] \times [a, b] \times [0, 1].$$

Define the functions $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ by

$$F(u_1, u_2, u_3, u_4) = u_1 - h(u_4)u_2 - h(1 - u_4)u_3 \quad (1.7)$$

and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$T_{F,w}(u_1, u_2, u_3) = u_1 - \left(\int_0^1 h(t)w(t) dt \right) u_2 - \left(\int_0^1 h(1-t)w(t) dt \right) u_3. \quad (1.8)$$

For $L_w = 0$, it is clear that $F \in \mathcal{F}$ and

$$F(f(tx + (1-t)y), f(x), f(y), t) = f(tx + (1-t)y) - h(t)f(x) - h(1-t)f(y) \leq 0,$$

that is f is an F -convex function.

In [11], author established the following Hermite-Hadamard type inequalities using the new convexity concept:

Theorem 1. Let $f : [a, b] \rightarrow \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $a < b$, be an F -convex function, for some $F \in \mathcal{F}$. Suppose that $f \in L_1[a, b]$. Then

$$F\left(f\left(\frac{a+b}{2}\right), \frac{1}{b-a} \int_a^b f(x) dx, \frac{1}{b-a} \int_a^b f(x) dx, \frac{1}{2}\right) \leq 0,$$

$$T_{F,1}\left(\frac{1}{b-a} \int_a^b f(x) dx, f(a), f(b)\right) \leq 0.$$

Theorem 2. Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $(a, b) \in I^\circ \times I^\circ$, $a < b$. Suppose that

(i) $|f'|$ is F -convex on $[a, b]$, for some $F \in \mathcal{F}$

(ii) the function $t \in (0, 1) \rightarrow L_{w(t)}$ belongs to $L^1(0, 1)$, where $w(t) = |1 - 2t|$. Then,

$$T_{F,w} \left(\frac{2}{b-a} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right|, |f'(a)|, |f'(b)| \right) + \int_0^1 L_{w(t)} dt \leq 0.$$

Theorem 3. Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $(a, b) \in I^\circ \times I^\circ$, $a < b$ and let $p > 1$. Suppose that $|f'|^{p/(p-1)}$ is F -convex on $[a, b]$, for some $F \in \mathcal{F}$ and $|f'| \in L^{p/(p-1)}(a, b)$. Then

$$T_{F,1} \left(A(p, f), |f'(a)|^{p/(p-1)}, |f'(b)|^{p/(p-1)} \right) \leq 0$$

where

$$A(p, f) = \left(\frac{2}{b-a} \right)^{\frac{p}{p-1}} (p+1)^{\frac{1}{p-1}} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right|^{\frac{p}{p-1}}.$$

The following lemmas will be useful when we give our main result.

Lemma 1 ([6]). Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $f' \in L^1[a, b]$, then we have

$$\frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) = (b-a) \int_0^1 p(t) f'(ta + (1-t)b) dt,$$

where

$$p(t) = \begin{cases} t, & t \in [0, \frac{1}{2}] \\ t-1, & t \in (\frac{1}{2}, 1]. \end{cases}$$

Lemma 2 ([4]). Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable function on I° , $a, b \in I^\circ$ with $a < b$. If $f'' \in L^1[a, b]$, then

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx = \frac{(b-a)^2}{2} \int_0^1 t(1-t) f''(ta + (1-t)b) dt. \quad (1.9)$$

Lemma 3 ([13]). Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable function on I° , $a, b \in I^\circ$ with $a < b$. If $f'' \in L^1[a, b]$, then

$$\frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \quad (1.10)$$

$$= \frac{(b-a)^2}{4} \int_0^1 m(t) [f''(ta + (1-t)b) + f''(tb + (1-t)a)] dt,$$

where

$$m(t) = \begin{cases} t^2, & t \in [0, \frac{1}{2}] \\ (1-t)^2, & t \in (\frac{1}{2}, 1]. \end{cases}$$

2. MIDPOINT TYPE INEQUALITIES FOR DIFFERENTIABLE FUNCTIONS

In this section, we establish some midpoint type inequalities for functions whose derivatives absolute values are F -convex.

Theorem 4. Let $I \subseteq \mathbb{R}$ be an interval, $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$, $a < b$. Suppose that $|f'|$ is F -convex on $[a, b]$, for some $F \in \mathcal{F}$ and the function $t \in [0, 1] \rightarrow L_{w(t)}$ belongs to $L^1[0, 1]$, where $w(t) = |p(t)|$ ($p(t)$ is defined as in Lemma 1). Then,

$$T_{F,w} \left(\frac{1}{b-a} \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right|, |f'(a)|, |f'(b)| \right) + \int_0^1 L_{w(t)} dt \leq 0.$$

Proof. Since $|f'|$ is F -convex, we have

$$F(|f'(ta + (1-t)b)|, |f'(a)|, |f'(b)|, t) \leq 0, \quad t \in [0, 1].$$

Multiplying this inequality by $w(t)$ and using axiom (A3), we get

$$F(w(t)|f'(ta + (1-t)b)|, w(t)|f'(a)|, w(t)|f'(b)|, t) + L_{w(t)} \leq 0, \quad t \in [0, 1].$$

Integrating over $[0, 1]$ with respect to the variable t and using axiom (A2), we obtain

$$T_{F,w} \left(\int_0^1 w(t) |f'(ta + (1-t)b)| dt, |f'(a)|, |f'(b)| \right) + \int_0^1 L_{w(t)} dt \leq 0, \quad t \in [0, 1].$$

On the other hand using Lemma 1 we have

$$\frac{1}{b-a} \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \int_0^1 |p(t)| |f'(ta + (1-t)b)| dt.$$

Since $T_{F,w}$ is nondecreasing with respect to the first variable, we get

$$T_{F,w} \left(\frac{1}{b-a} \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right|, |f'(a)|, |f'(b)| \right) + \int_0^1 L_{w(t)} dt \leq 0.$$

This completes the proof. $\square$

Corollary 1. *If $|f'|$ is ε -convex on $[a, b]$, $\varepsilon \geq 0$, then we have*

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq (b-a) \left[\frac{|f'(a)| + |f'(b)|}{8} + \frac{\varepsilon}{4} \right].$$

Proof. It is known that an ε -convex is an F -convex. Using (1.4) with $w(t) = |p(t)|$, we obtain

$$\int_0^1 L_{w(t)} dt = \varepsilon \int_0^1 (1 - |p(t)|) dt = \frac{3}{4}\varepsilon.$$

From (1.3) with $w(t) = |p(t)|$, we get

$$\begin{aligned} T_{F,w}(u_1, u_2, u_3) &= u_1 - \left(\int_0^1 t |p(t)| dt \right) u_2 - \left(\int_0^1 (1-t) |p(t)| dt \right) u_3 - \varepsilon \\ &= u_1 - \frac{u_2 + u_3}{8} - \varepsilon \end{aligned}$$

for $u_1, u_2, u_3 \in \mathbb{R}$. Hence,

$$\begin{aligned} &T_{F,w} \left(\frac{1}{b-a} \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right|, |f'(a)|, |f'(b)| \right) \\ &= \frac{1}{b-a} \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| - \frac{|f'(a)| + |f'(b)|}{8} - \varepsilon. \end{aligned}$$

Thus, by Theorem 4, we have

$$\frac{1}{b-a} \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| - \frac{|f'(a)| + |f'(b)|}{8} - \varepsilon + \frac{3}{4}\varepsilon \leq 0,$$

that is

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq (b-a) \left[\frac{|f'(a)| + |f'(b)|}{8} + \frac{\varepsilon}{4} \right].$$

This completes the proof. $\square$

Remark 2. If we take $\varepsilon = 0$ in Corollary 1, then $|f'|$ is convex and we have the inequality

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq (b-a) \left[\frac{|f'(a)| + |f'(b)|}{8} \right]$$

which is given by [6].

Corollary 2. *If $|f'|$ is α -convex on $[a, b]$, $\alpha \in (0, 1]$, then we have*

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq (b-a) \left\{ \frac{1}{(\alpha+1)(\alpha+2)} \left(1 - \frac{1}{2^{\alpha+1}}\right) |f'(a)| \right. \\ \left. + \left[\frac{1}{4} - \frac{1}{(\alpha+1)(\alpha+2)} \left(1 - \frac{1}{2^{\alpha+1}}\right) \right] |f'(b)| \right\}.$$

Proof. It is known that an α -convex is an F -convex. Using (1.6) with $w(t) = |p(t)|$, we obtain

$$\begin{aligned} T_{F,w}(u_1, u_2, u_3) &= u_1 - \left(\int_0^1 t^\alpha |p(t)| dt \right) u_2 - \left(\int_0^1 (1-t^\alpha) |p(t)| dt \right) u_3 \\ &= u_1 - \frac{1}{(\alpha+1)(\alpha+2)} \left(1 - \frac{1}{2^{\alpha+1}}\right) u_2 \\ &\quad - \left[\frac{1}{4} - \frac{1}{(\alpha+1)(\alpha+2)} \left(1 - \frac{1}{2^{\alpha+1}}\right) \right] u_3 \end{aligned}$$

for $u_1, u_2, u_3 \in \mathbb{R}$. Hence, by Theorem 4, we have

$$\begin{aligned} &T_{F,w} \left(\frac{1}{b-a} \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right|, |f'(a)|, |f'(b)| \right) \\ &= \frac{1}{b-a} \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\ &\quad - \frac{1}{(\alpha+1)(\alpha+2)} \left(1 - \frac{1}{2^{\alpha+1}}\right) |f'(a)| \\ &\quad - \left[\frac{1}{4} - \frac{1}{(\alpha+1)(\alpha+2)} \left(1 - \frac{1}{2^{\alpha+1}}\right) \right] |f'(b)| \leq 0, \end{aligned}$$

that is

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq (b-a) \left\{ \frac{1}{(\alpha+1)(\alpha+2)} \left(1 - \frac{1}{2^{\alpha+1}}\right) |f'(a)| \right. \\ \left. + \left[\frac{1}{4} - \frac{1}{(\alpha+1)(\alpha+2)} \left(1 - \frac{1}{2^{\alpha+1}}\right) \right] |f'(b)| \right\}$$

which completes the proof. $\square$

Corollary 3. *If $|f'|$ is h -convex on $[a, b]$, then we have the inequality*

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq (b-a) \left(\int_0^1 h(t) |p(t)| dt \right) [|f'(a)| + |f'(b)|].$$

Proof. It is known that an h -convex is an F -convex. From (1.8) with $w(t) = |p(t)|$, we have

$$\begin{aligned} T_{F,w}(u_1, u_2, u_3) &= u_1 - \left(\int_0^1 h(t) |p(t)| dt \right) u_2 - \left(\int_0^1 h(1-t) |p(t)| dt \right) u_3 \\ &= u_1 - \left(\int_0^1 h(t) |p(t)| dt \right) u_2 - \left(\int_0^1 h(1-t) |p(1-t)| dt \right) u_3 \\ &= u_1 - \left(\int_0^1 h(t) |p(t)| dt \right) (u_2 + u_3) \end{aligned}$$

for $u_1, u_2, u_3 \in \mathbb{R}$. Then,

$$\begin{aligned} &T_{F,w} \left(\frac{1}{b-a} \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right|, |f'(a)|, |f'(b)| \right) \\ &= \frac{1}{b-a} \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| - \left(\int_0^1 h(t) |p(t)| dt \right) [|f'(a)| + |f'(b)|]. \end{aligned}$$

Thus, by Theorem 4, we get

$$\begin{aligned} &\frac{1}{b-a} \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\ &- \left(\int_0^1 h(t) |p(t)| dt \right) [|f'(a)| + |f'(b)|] \leq 0 \end{aligned}$$

which completes the proof. $\square$

Theorem 5. Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$, $a < b$ and $p > 1$. Suppose that $|f'|^{p/(p-1)}$ is F -convex on $[a, b]$, for some $F \in \mathcal{F}$ and $|f'| \in L^{p/(p-1)}(a, b)$. Then we have

$$T_{F,1} \left(A_1(f, p), |f'(a)|^{p/(p-1)}, |f'(b)|^{p/(p-1)} \right) \leq 0$$

where

$$A_1(f, p) = \left(\frac{2}{b-a} \right)^{\frac{p}{p-1}} (p+1)^{\frac{1}{p-1}} \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right|^{\frac{p}{p-1}}.$$

Proof. As $|f'|^{p/(p-1)}$ is F -convex, we have

$$F\left(|f'(ta + (1-t)b)|^{p/(p-1)}, |f'(a)|^{p/(p-1)}, |f'(b)|^{p/(p-1)}, t\right) \leq 0, \quad t \in [0, 1].$$

Using axiom (A2) with $w(t) = 1$, we get

$$T_{F,1}\left(\int_0^1 |f'(ta + (1-t)b)|^{p/(p-1)} dt, |f'(a)|^{p/(p-1)}, |f'(b)|^{p/(p-1)}\right) \leq 0.$$

Using Lemma 1 and Hölder inequality, we obtain

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\ &= (b-a) \int_0^1 |p(t)| |f'(ta + (1-t)b)| dt \\ &\leq (b-a) \left(\int_0^1 |p(t)|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(ta + (1-t)b)|^{p/(p-1)} dt \right)^{\frac{p-1}{p}} \\ &= \frac{b-a}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\int_0^1 |f'(ta + (1-t)b)|^{p/(p-1)} dt \right)^{\frac{p-1}{p}}, \end{aligned}$$

that is

$$A_1(f, p) \leq \int_0^1 |f'(ta + (1-t)b)|^{p/(p-1)} dt.$$

Since $T_{F,1}$ is nondecreasing with respect to the first variable, we can obtain the desired result easily. $\square$

Corollary 4. If $|f'|^{p/(p-1)}$ is ε -convex on $[a, b]$, $\varepsilon \geq 0$, then we have

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\ &\leq \frac{b-a}{2(p+1)^{\frac{1}{p}}} \left[\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2} + \varepsilon \right]^{\frac{p-1}{p}}. \end{aligned}$$

Proof. Using (1.3) with $w(t) = 1$, we have

$$T_{F,1}(u_1, u_2, u_3) = u_1 - \frac{u_2 + u_3}{2} - \varepsilon \quad (2.1)$$

for $u_1, u_2, u_3 \in \mathbb{R}$. Then, by the Theorem 5, we have

$$0 \geq T_{F,1}\left(A_1(f, p), |f'(a)|^{p/(p-1)}, |f'(b)|^{p/(p-1)}\right)$$

$$\begin{aligned}
&= A_1(f, p) - \frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2} - \varepsilon \\
&= \left(\frac{2}{b-a} \right)^{\frac{p}{p-1}} (p+1)^{\frac{1}{p-1}} \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right|^{\frac{p}{p-1}} \\
&\quad - \frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2} - \varepsilon,
\end{aligned}$$

that is,

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\
&\leq \frac{b-a}{2(p+1)^{\frac{1}{p}}} \left[\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2} + \varepsilon \right]^{\frac{p-1}{p}}.
\end{aligned}$$

□

Remark 3. If we choose $\varepsilon = 0$ in Corollary 4, then $|f'|^{p/(p-1)}$ is convex and we have the inequality

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\
&\leq \frac{b-a}{2(p+1)^{\frac{1}{p}}} \left[\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2} \right]^{\frac{p-1}{p}}.
\end{aligned}$$

Corollary 5. If $|f'|^{p/(p-1)}$ is α -convex, $\alpha \in (0, 1]$, then we have the following inequality

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\
&\leq \frac{b-a}{2(p+1)^{\frac{1}{p}}} \left[\frac{|f'(a)|^{p/(p-1)} + \alpha |f'(b)|^{p/(p-1)}}{\alpha + 1} \right]^{\frac{p-1}{p}}.
\end{aligned}$$

Proof. From (1.6) with $w(t) = 1$, we have

$$\begin{aligned}
T_{F,1}(u_1, u_2, u_3) &= u_1 - \left(\int_0^1 t^\alpha dt \right) u_2 - \left(\int_0^1 (1-t^\alpha) dt \right) u_3 \\
&= u_1 - \frac{u_2}{\alpha+1} - \frac{\alpha}{\alpha+1} u_3
\end{aligned} \tag{2.2}$$

for $u_1, u_2, u_3 \in \mathbb{R}$. By the Theorem 5, it follows that

$$\begin{aligned}
 0 &\geq T_{F,1} \left(A_1(f, p), |f'(a)|^{p/(p-1)}, |f'(b)|^{p/(p-1)} \right) \\
 &= A_1(f, p) - \frac{|f'(a)|^{p/(p-1)}}{\alpha + 1} - \frac{\alpha}{\alpha + 1} |f'(b)|^{p/(p-1)} \\
 &= \left(\frac{2}{b-a} \right)^{\frac{p}{p-1}} (p+1)^{\frac{1}{p-1}} \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right|^{\frac{p}{p-1}} \\
 &\quad - \frac{|f'(a)|^{p/(p-1)}}{\alpha + 1} - \frac{\alpha}{\alpha + 1} |f'(b)|^{p/(p-1)},
 \end{aligned}$$

which achieves the proof. $\square$

Corollary 6. *If $|f'|^{p/(p-1)}$ is h -convex, then we have the following inequality*

$$\begin{aligned}
 &\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\
 &\leq \frac{b-a}{2(p+1)^{\frac{1}{p}}} \left(\int_0^1 h(t) dt \right)^{\frac{p-1}{p}} \left[|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)} \right]^{\frac{p-1}{p}}.
 \end{aligned}$$

Proof. Using (1.8) with $w(t) = 1$, we have

$$\begin{aligned}
 T_{F,1}(u_1, u_2, u_3) &= u_1 - \left(\int_0^1 h(t) dt \right) u_2 - \left(\int_0^1 h(1-t) dt \right) u_3 \\
 &= u_1 - \left(\int_0^1 h(t) dt \right) (u_2 + u_3)
 \end{aligned} \tag{2.3}$$

for $u_1, u_2, u_3 \in \mathbb{R}$. By the Theorem 5, we obtain

$$\begin{aligned}
 0 &\geq T_{F,1} \left(A_1(f, p), |f'(a)|^{p/(p-1)}, |f'(b)|^{p/(p-1)} \right) \\
 &= A_1(f, p) - \left(\int_0^1 h(t) dt \right) \left(|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)} \right) \\
 &= \left(\frac{2}{b-a} \right)^{\frac{p}{p-1}} (p+1)^{\frac{1}{p-1}} \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right|^{\frac{p}{p-1}} \\
 &\quad - \left(\int_0^1 h(t) dt \right) \left(|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)} \right)
 \end{aligned}$$

$$-\left(\int_0^1 h(t)dt\right)\left(|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}\right).$$

This completes the proof. $\square$

3. TRAPEZOID TYPE INEQUALITIES FOR TWICE DIFFERENTIABLE FUNCTIONS

In this section, we obtain some trapezoid type inequalities for functions whose second derivatives absolute values are F -convex.

Theorem 6. *Let $I \subseteq \mathbb{R}$ be an interval, $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° , $(a, b) \in I^\circ \times I^\circ$, $a < b$. Suppose that $|f''|$ is F -convex on $[a, b]$, for some $F \in \mathcal{F}$ and the function $t \in [0, 1] \rightarrow L_{w(t)}$ belongs to $L^1[0, 1]$, where $w(t) = t(1-t)$. Then we have the following inequality*

$$T_{F,w} \left(\frac{2}{(b-a)^2} \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right|, |f''(a)|, |f''(b)| \right) + \int_0^1 L_{w(t)} dt \leq 0.$$

Proof. Since $|f''|$ is F -convex, we have

$$F(|f''(ta + (1-t)b)|, |f''(a)|, |f''(b)|, t) \leq 0, \quad t \in [0, 1].$$

Multiplying this inequality by $w(t)$ and using axiom (A3), we get

$$F(w(t)|f''(ta + (1-t)b)|, w(t)|f''(a)|, w(t)|f''(b)|, t) + L_{w(t)} \leq 0, \quad t \in [0, 1].$$

Integrating over $(0, 1)$ with respect to the variable t and using axiom (A2), we obtain

$$T_{F,w} \left(\int_0^1 w(t)|f''(ta + (1-t)b)| dt, |f''(a)|, |f''(b)| \right) + \int_0^1 L_{w(t)} dt \leq 0, \quad t \in [0, 1].$$

On the other hand using Lemma 2, we have

$$\frac{2}{(b-a)^2} \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \int_0^1 t(1-t)|f''(ta + (1-t)b)| dt.$$

Since $T_{F,w}$ is nondecreasing with respect to the first variable, we get

$$T_{F,w} \left(\frac{2}{(b-a)^2} \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right|, |f''(a)|, |f''(b)| \right) + \int_0^1 L_{w(t)} dt \leq 0.$$

This completes the proof. $\square$

Corollary 7. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable function on I° , $a, b \in I^\circ$ with $a < b$. Suppose that the function $|f''|$ is ε -convex on $[a, b]$, $\varepsilon \geq 0$. Then we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{2} \left[\frac{|f''(a)| + |f''(b)|}{12} + \frac{\varepsilon}{6} \right].$$

Proof. We know that an ε -convex is an F -convex. Using (1.4) with $w(t) = t(1-t)$, we obtain

$$\int_0^1 L_{w(t)} dt = \varepsilon \int_0^1 (1-w(t)) dt = \frac{5\varepsilon}{6}.$$

From (1.3) with $w(t) = t(1-t)$, we get

$$\begin{aligned} T_{F,w}(u_1, u_2, u_3) &= u_1 - \left(\int_0^1 t^2(1-t) dt \right) u_2 - \left(\int_0^1 t(1-t)^2 dt \right) u_3 - \varepsilon \\ &= u_1 - \frac{u_2 + u_3}{12} - \varepsilon \end{aligned}$$

for $u_1, u_2, u_3 \in \mathbb{R}$. Hence,

$$\begin{aligned} T_{F,w} \left(\frac{2}{(b-a)^2} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right|, |f''(a)|, |f''(b)| \right) \\ = \frac{2}{(b-a)^2} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| - \frac{|f''(a)| + |f''(b)|}{12} - \varepsilon. \end{aligned}$$

Thus, by Theorem 6, we have

$$\frac{2}{(b-a)^2} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| - \frac{|f''(a)| + |f''(b)|}{12} - \varepsilon + \frac{5\varepsilon}{6} \leq 0.$$

This completes the proof. $\square$

Remark 4. If we taking $\varepsilon = 0$ in Corollary 7, then $|f''|$ is convex and we have the inequality

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{12} \left[\frac{|f''(a)| + |f''(b)|}{2} \right].$$

Corollary 8. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable function on I° , $a, b \in I^\circ$ with $a < b$. Suppose that the function $|f''|$ is α -convex on $[a, b]$, $\alpha \in (0, 1]$. Then we

have

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{2} \left[\frac{|f''(a)|}{(\alpha+2)(\alpha+3)} - \left(\frac{1}{6} - \frac{1}{(\alpha+2)(\alpha+3)} \right) |f''(b)| \right]. \end{aligned}$$

Proof. We know that an α -convex is an F -convex. From (1.6) with $w(t) = t(1-t)$, we have

$$\begin{aligned} T_{F,w}(u_1, u_2, u_3) &= u_1 - \left(\int_0^1 t^{\alpha+1} (1-t) dt \right) u_2 - \left(\int_0^1 (1-t^\alpha) t (1-t) dt \right) u_3 \\ &= u_1 - \frac{u_2}{(\alpha+2)(\alpha+3)} - \left[\frac{1}{6} - \frac{1}{(\alpha+2)(\alpha+3)} \right] u_3 \end{aligned}$$

for $u_1, u_2, u_3 \in \mathbb{R}$. It follows that

$$\begin{aligned} & T_{F,w} \left(\frac{2}{(b-a)^2} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right|, |f''(a)|, |f''(b)| \right) \\ &= \frac{2}{(b-a)^2} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \quad - \frac{|f''(a)|}{(\alpha+2)(\alpha+3)} - \left[\frac{1}{6} - \frac{1}{(\alpha+2)(\alpha+3)} \right] |f''(b)|. \end{aligned}$$

Consequently, by Theorem 6, we obtain

$$\begin{aligned} & \frac{2}{(b-a)^2} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \quad - \frac{|f''(a)|}{(\alpha+2)(\alpha+3)} - \left[\frac{1}{6} - \frac{1}{(\alpha+2)(\alpha+3)} \right] |f''(b)| \leq 0, \end{aligned}$$

which completes the proof. $\square$

Corollary 9. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable function on I° , $a, b \in I^\circ$ with $a < b$. Suppose that the function $|f''|$ is h -convex on $[a, b]$. Then we have

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq (b-a)^2 \left(\int_0^1 h(t) t(1-t) dt \right) \left[\frac{|f''(a)| + |f''(b)|}{2} \right]. \end{aligned}$$

Proof. We know that an h -convex is an F -convex. From (1.8) with $w(t) = t(1-t)$, we have

$$\begin{aligned} T_{F,w}(u_1, u_2, u_3) &= u_1 - \left(\int_0^1 h(t)w(t)dt \right) u_2 - \left(\int_0^1 h(1-t)w(t)dt \right) u_3 \\ &= u_1 - \left(\int_0^1 h(t)t(1-t)dt \right) u_2 - \left(\int_0^1 h(1-t)t(1-t)dt \right) u_3 \\ &= u_1 - \left(\int_0^1 h(t)t(1-t)dt \right) (u_2 + u_3) \end{aligned}$$

for $u_1, u_2, u_3 \in \mathbb{R}$. Then,

$$\begin{aligned} &T_{F,w} \left(\frac{2}{(b-a)^2} \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right|, |f''(a)|, |f''(b)| \right) \\ &= \frac{2}{(b-a)^2} \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ &\quad - \left(\int_0^1 h(t)t(1-t)dt \right) [|f''(a)| + |f''(b)|]. \end{aligned}$$

Thus, by Theorem 6, we get

$$\begin{aligned} &= \frac{2}{(b-a)^2} \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ &\quad - \left(\int_0^1 h(t)t(1-t)dt \right) [|f''(a)| + |f''(b)|] \leq 0. \end{aligned}$$

This finishes the proof. $\square$

Theorem 7. Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° , $a, b \in I^\circ$, $a < b$ and let $p > 1$. Suppose that $|f''|^{p/(p-1)}$ is F -convex on $[a, b]$, for some $F \in \mathcal{F}$ and $|f''| \in L^{p/(p-1)}(a, b)$. Then we have

$$T_{F,1} \left(A_2(f, p), |f''(a)|^{p/(p-1)}, |f''(b)|^{p/(p-1)} \right) \leq 0$$

where

$$A_2(f, p) =$$

$$\left(\frac{2}{(b-a)^2} \right)^{\frac{p}{p-1}} \left(\frac{1}{B(p+1, p+1)} \right)^{\frac{1}{p-1}} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right|^{\frac{p}{p-1}}$$

and $B(x, y)$ is Euler beta function defined by

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

Proof. Since $|f''|^{p/(p-1)}$ is F -convex, we have

$$F\left(|f''(ta + (1-t)b)|^{p/(p-1)}, |f''(a)|^{p/(p-1)}, |f''(b)|^{p/(p-1)}, t\right) \leq 0, \quad t \in [0, 1].$$

Using axiom (A2) with $w(t) = 1$, we get

$$T_{F,1} \left(\int_0^1 |f''(ta + (1-t)b)|^{p/(p-1)} dt, |f''(a)|^{p/(p-1)}, |f''(b)|^{p/(p-1)} \right) \leq 0.$$

Using Lemma 2 and Hölder inequality, we obtain

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ &= \frac{(b-a)^2}{2} \int_0^1 |t(1-t)| |f''(ta + (1-t)b)| dt \\ &\leq \frac{(b-a)^2}{2} \left(\int_0^1 t^p (1-t)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f''(ta + (1-t)b)|^{p/(p-1)} dt \right)^{\frac{p-1}{p}} \\ &= \frac{(b-a)^2}{2} (B(p+1, p+1))^{\frac{1}{p}} \left(\int_0^1 |f''(ta + (1-t)b)|^{p/(p-1)} dt \right)^{\frac{p-1}{p}}, \end{aligned}$$

that is

$$A_2(f, p) \leq \int_0^1 |f''(ta + (1-t)b)|^{p/(p-1)} dt.$$

Since $T_{F,1}$ is nondecreasing with respect to the first variable, we have

$$T_{F,1} \left(A_2(f, p), |f''(a)|^{p/(p-1)}, |f''(b)|^{p/(p-1)} \right) \leq 0.$$

□

Corollary 10. If $|f''|^{p/(p-1)}$ is ε -convex, we have the following inequality

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\leq \frac{(b-a)^2}{2} (B(p+1, p+1))^{\frac{1}{p}} \left[\frac{|f''(a)|^{p/(p-1)} + |f''(b)|^{p/(p-1)}}{2} + \varepsilon \right]^{\frac{p-1}{p}}.$$

Proof. Using (2.1) and by Theorem 7, it can be proved easily. It is omitted. $\square$

Remark 5. If we choose $\varepsilon = 0$ in Corollary 10, then we have

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{2} (B(p+1, p+1))^{\frac{1}{p}} \left[\frac{|f''(a)|^{p/(p-1)} + |f''(b)|^{p/(p-1)}}{2} \right]^{\frac{p-1}{p}}. \end{aligned}$$

Corollary 11. If $|f''|^{p/(p-1)}$ is α -convex, we have the following inequality

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{2} (B(p+1, p+1))^{\frac{1}{p}} \left[\frac{|f'(a)|^{p/(p-1)} + \alpha |f'(b)|^{p/(p-1)}}{\alpha + 1} \right]^{\frac{p-1}{p}}. \end{aligned}$$

Proof. Using (2.2) and by Theorem 7, it can be proved easily. It is omitted. $\square$

Corollary 12. If $|f''|^{p/(p-1)}$ is h -convex, we have the following inequality

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{2} (B(p+1, p+1))^{\frac{1}{p}} \left(\int_0^1 h(t) dt \right)^{\frac{p-1}{p}} \\ & \quad \left[|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)} \right]^{\frac{p-1}{p}}. \end{aligned}$$

Proof. Using (2.3) and by Theorem 7, it can be proved easily. It is omitted. $\square$

4. MIDPOINT TYPE INEQUALITIES FOR TWICE DIFFERENTIABLE FUNCTIONS

In this section, we prove some midpoint type inequalities for functions whose second derivatives absolute values are F -convex.

Theorem 8. Let $I \subseteq \mathbb{R}$ be an interval, $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° , $(a, b) \in I^\circ \times I^\circ$, $a < b$. Suppose that $|f''|$ is F -convex on $[a, b]$,

for some $F \in \mathcal{F}$ the function $t \in [0, 1] \rightarrow L_{m(t)}$ belongs to $L^1[0, 1]$. If F is linear with respect to the first, second and third variables, then

$$T_{F,m} \left(\frac{4}{(b-a)^2} \left| \frac{1}{b-a} \int_a^b f(x) dx - f \left(\frac{a+b}{2} \right) \right|, |f''(a)| + |f''(b)|, |f''(a)| + |f''(b)| \right) + \int_0^1 L_{m(t)} dt \leq 0$$

where

$$m(t) = \begin{cases} t^2, & t \in [0, \frac{1}{2}] \\ (1-t)^2, & t \in (\frac{1}{2}, 1]. \end{cases}$$

Proof. Since $|f''|$ is F -convex, we have

$$F(|f''(ta + (1-t)b)|, |f''(a)|, |f''(b)|, t) \leq 0, \quad t \in [0, 1]$$

and

$$F(|f''(tb + (1-t)a)|, |f''(b)|, |f''(a)|, t) \leq 0, \quad t \in [0, 1].$$

Using the linearity of F , we obtain

$$F(|f''(ta + (1-t)b)| + |f''(tb + (1-t)a)|, |f''(b)| + |f''(a)|, |f''(b)| + |f''(a)|, t) \leq 0,$$

for $t \in [0, 1]$. Multiplying this inequality by $m(t)$ and using axiom (A3), we get

$$F(m(t)[|f''(ta + (1-t)b)| + |f''(tb + (1-t)a)|], m(t)[|f''(b)| + |f''(a)|], m(t)[|f''(b)| + |f''(a)|], t) + L_{m(t)} \leq 0,$$

for $t \in [0, 1]$. Integrating over $[0, 1]$ with respect to the variable t and using axiom (A2), we obtain

$$T_{F,m} \left(\int_0^1 m(t)[|f''(ta + (1-t)b)| + |f''(tb + (1-t)a)|] dt, |f''(b)| + |f''(a)|, |f''(b)| + |f''(a)| \right) + \int_0^1 L_{m(t)} dt \leq 0,$$

for $t \in [0, 1]$. On the other hand, using Lemma 3, we have

$$\begin{aligned} & \frac{4}{(b-a)^2} \left| \frac{1}{b-a} \int_a^b f(x) dx - f \left(\frac{a+b}{2} \right) \right| \\ & \leq \int_0^1 m(t)[|f''(ta + (1-t)b)| + |f''(tb + (1-t)a)|] dt. \end{aligned}$$

Since $T_{F,m}$ is nondecreasing with respect to the first variable, we get

$$T_{F,m} \left(\frac{4}{(b-a)^2} \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right|, \right. \\ \left. |f''(a)| + |f''(b)|, |f''(a)| + |f''(b)| \right) + \int_0^1 L_{m(t)} dt \leq 0.$$

This completes the proof. $\square$

Corollary 13. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable function on I° , $a, b \in I^\circ$ with $a < b$. Suppose that the function $|f''|$ is ε -convex on $[a, b]$, $\varepsilon \geq 0$. Then we have

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{4} \left[\frac{|f''(a)| + |f''(b)|}{12} + \frac{\varepsilon}{12} \right].$$

Proof. Using (1.4) with $w(t) = m(t)$, we obtain

$$\begin{aligned} \int_0^1 L_{m(t)} dt &= \varepsilon \int_0^1 (1-m(t)) dt \\ &= \varepsilon \left(\int_0^{1/2} (1-t^2) dt + \int_{1/2}^1 (1-(1-t)^2) dt \right) \\ &= \frac{11}{12} \varepsilon. \end{aligned}$$

From (1.3) with $w(t) = m(t)$, we get

$$\begin{aligned} T_{F,w}(u_1, u_2, u_3) &= u_1 - \left(\int_0^1 t m(t) dt \right) u_2 - \left(\int_0^1 (1-t) m(t) dt \right) u_3 - \varepsilon \\ &= u_1 - \frac{u_2 + u_3}{24} - \varepsilon \end{aligned}$$

for $u_1, u_2, u_3 \in \mathbb{R}$. Hence,

$$\begin{aligned} T_{F,m} \left(\frac{4}{(b-a)^2} \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right|, \right. \\ \left. |f''(b)| + |f''(a)|, |f''(b)| + |f''(a)| \right) \\ = \frac{4}{(b-a)^2} \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| - \frac{|f''(a)| + |f''(b)|}{12} - \varepsilon. \end{aligned}$$

Thus, by Theorem 8, we have

$$\frac{4}{(b-a)^2} \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| - \frac{|f''(a)| + |f''(b)|}{12} - \varepsilon + \frac{11}{12} \varepsilon \leq 0,$$

that is

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{4} \left[\frac{|f''(a)| + |f''(b)|}{12} + \frac{\varepsilon}{12} \right].$$

This completes the proof. $\square$

Remark 6. If we taking $\varepsilon = 0$ in Corollary 13, then $|f''|$ is convex and we have the inequality

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{24} \left[\frac{|f''(a)| + |f''(b)|}{2} \right]$$

which is given by Sarikaya et al. in [13].

Corollary 14. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable function on I° , $a, b \in I^\circ$ with $a < b$. Suppose that the function $|f''|$ is α -convex on $[a, b]$, $\alpha \in (0, 1]$. Then we have

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{24} \left[\frac{|f''(a)| + |f''(b)|}{2} \right].$$

Proof. From (1.6) with $w(t) = m(t)$, we have

$$\begin{aligned} T_{F,w}(u_1, u_2, u_3) &= u_1 - \left(\int_0^1 t^\alpha m(t) dt \right) u_2 - \left(\int_0^1 (1-t^\alpha) m(t) dt \right) u_3 \\ &= u_1 - \frac{1}{(\alpha+1)(\alpha+2)} \left(\frac{2}{\alpha+3} - \frac{1}{2^{\alpha+1}} \right) u_2 \\ &\quad - \left[\frac{1}{12} - \frac{1}{(\alpha+1)(\alpha+2)} \left(\frac{2}{\alpha+3} - \frac{1}{2^{\alpha+1}} \right) \right] u_3 \end{aligned}$$

for $u_1, u_2, u_3 \in \mathbb{R}$. It follows that

$$\begin{aligned} &T_{F,w} \left(\frac{4}{(b-a)^2} \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right|, \right. \\ &\quad \left. |f''(b)| + |f''(a)|, |f''(b)| + |f''(a)| \right) \\ &= \frac{4}{(b-a)^2} \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{2}{\alpha+3}-\frac{1}{2^{\alpha+1}}\right)[|f''(b)|+|f''(a)|] \\
& -\left[\frac{1}{12}-\frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{2}{\alpha+3}-\frac{1}{2^{\alpha+1}}\right)\right][|f''(b)|+|f''(a)|] \\
& =\frac{4}{(b-a)^2}\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|-\frac{|f''(b)|+|f''(a)|}{12}.
\end{aligned}$$

which completes the proof. $\square$

Corollary 15. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable function on I° , $a, b \in I^\circ$ with $a < b$. Suppose that the function $|f''|$ is h -convex on $[a, b]$. Then we have

$$\begin{aligned}
& \left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
& \leq (b-a)^2\left(\int_0^1 h(t)t(1-t)dt\right)\left[\frac{|f''(a)|+|f''(b)|}{2}\right].
\end{aligned}$$

Proof. From (1.8) with $w(t) = m(t)$, we have

$$\begin{aligned}
T_{F,w}(u_1, u_2, u_3) &= u_1 - \left(\int_0^1 h(t)m(t)dt\right)u_2 - \left(\int_0^1 h(1-t)m(t)dt\right)u_3 \\
&= u_1 - \left(\int_0^1 h(t)m(t)dt\right)u_2 - \left(\int_0^1 h(t)m(1-t)dt\right)u_3 \\
&= u_1 - \left(\int_0^1 h(t)m(t)dt\right)(u_2 + u_3)
\end{aligned}$$

for $u_1, u_2, u_3 \in \mathbb{R}$. Then, by Theorem 8,

$$\begin{aligned}
& T_{F,w}\left(\frac{4}{(b-a)^2}\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|,\right. \\
& \left.|f''(b)|+|f''(a)|, |f''(b)|+|f''(a)|\right) \\
& =\frac{4}{(b-a)^2}\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|
\end{aligned}$$

$$-2 \left(\int_0^1 h(t)m(t)dt \right) [|f''(a)| + |f''(b)|] \leq 0$$

that is,

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq (b-a)^2 \left(\int_0^1 h(t)m(t)dt \right) \left[\frac{|f''(a)| + |f''(b)|}{2} \right]$$

which completes the proof. $\square$

REFERENCES

- [1] B. Defnetti, "Sulla strati cazioni convesse." *Ann. Math. Pura. Appl.*, vol. 30, pp. 173–183, 1949.
- [2] H. D.H. and S. Ulam, "Approximately convex functions," *Proc. Amer. Math. Soc.*, vol. 3, pp. 821–828, 1952.
- [3] S. S. Dragomir and R. P. Agarwal, "Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula," *Applied Mathematics Letters*, vol. 11, no. 5, pp. 91–95, 1998, doi: [10.1016/S0893-9659\(98\)00086-X](https://doi.org/10.1016/S0893-9659(98)00086-X).
- [4] S. S. Dragomir and C. Pearce, "Selected topics on hermite-hadamard inequalities and applications," 2003.
- [5] H. Hudzik and L. Maligranda, "Some remarks ons-convex functions," *Aequationes Mathematicae*, vol. 48, no. 1, pp. 100–111, 1994, doi: [10.1007/BF01837981](https://doi.org/10.1007/BF01837981).
- [6] U. S. Kirmaci, "Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula," *Applied Mathematics and Computation*, vol. 147, no. 1, pp. 137–146, 2004, doi: [10.1016/S0096-3003\(02\)00657-4](https://doi.org/10.1016/S0096-3003(02)00657-4).
- [7] O. L. Mangasarian, "Pseudo-convex functions," in *Stochastic Optimization Models in Finance*. Elsevier, 1975, pp. 23–32, doi: [10.1016/B978-0-12-780850-5.50009-5](https://doi.org/10.1016/B978-0-12-780850-5.50009-5).
- [8] J. E. Peajcariaac and Y. L. Tong, *Convex functions, partial orderings, and statistical applications*. Academic Press, 1992.
- [9] C. E. Pearce and J. Pečarić, "Inequalities for differentiable mappings with application to special means and quadrature formulae," *Applied Mathematics Letters*, vol. 13, no. 2, pp. 51–55, 2000, doi: [10.1016/S0893-9659\(99\)00164-0](https://doi.org/10.1016/S0893-9659(99)00164-0).
- [10] B. T. Polyak, "Existence theorems and convergence of minimizing sequences in extremum problems with restrictions," *Soviet Math. Dokl.*, vol. 7, pp. 72–75, 1966.
- [11] B. Samet, "On an implicit convexity concept and some integral inequalities," *Journal of Inequalities and Applications*, vol. 2016, no. 1, p. 308, 2016, doi: [10.1186/s13660-016-1253-3](https://doi.org/10.1186/s13660-016-1253-3).
- [12] M. Z. Sarikaya, A. Saglam, and H. Yildirim, "On some hadamard-type inequalities for h-convex functions," *J. Math. Inequal*, vol. 2, no. 3, pp. 335–341, 2008, doi: [10.7153/jmi-02-30](https://doi.org/10.7153/jmi-02-30).
- [13] M. Z. Sarikaya, A. Saglam, and H. Yildirim, "New inequalities of hermite-hadamard type for functions whose second derivatives absolute values are convex and quasi-convex," *International Journal of Open Problems in Computer Science and Mathematics*, vol. 238, no. 1390, pp. 1–14, 2012, doi: [10.12816/0006114](https://doi.org/10.12816/0006114).
- [14] S. Varošanec, "On h-convexity," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 303–311, 2007, doi: [10.1016/j.jmaa.2006.02.086](https://doi.org/10.1016/j.jmaa.2006.02.086).

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SIGN-CONSTANCY OF GREEN'S FUNCTIONS FOR TWO-POINT IMPULSIVE BOUNDARY VALUE PROBLEMS

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Abstract. We consider the following second order impulsive differential equation with delays

$$\begin{cases} (Lx)(t) \equiv x''(t) + \sum_{j=1}^p a_j(t)x'(t - \tau_j(t)) + \sum_{j=1}^p b_j(t)x(t - \theta_j(t)) = f(t), \\ t \in [0, \omega], \\ x(t_k) = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = 1, 2, \dots, r. \end{cases}$$

In this paper we obtain sufficient conditions of nonpositivity of Green's functions for impulsive differential equation. All results are formulated in the form of theorems about differential inequalities. It should be noted that the sign-constancy of the coefficients $b_j(t)$ was assumed in all the literature devoted to impulsive functional differential equations. One of the main purposes of this work is to propose a technique allowing us to avoid these assumptions.

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1. INTRODUCTION

Impulsive differential equations have attracted an attention of a number of recognized mathematicians and have applications in many spheres of science from physics, biology, medicine to economical studies. The following well-known books can be noted in this context [14, 17–19]. In the book [3], the concept of the general theory of functional differential equations was presented. On the basis of this concept nonoscillation for the first order functional differential equations was considered in [4], where positivity of the Cauchy and Green's functions of the periodic problem was firstly studied. A concept of nonoscillation for the first order differential equations is also considered in the book [1]. The positivity of Green's function of one- and two-point boundary value problems for functional differential impulsive equations was studied in [2, 5–8, 10–13, 15]. It should be noted that the sign-constancy of the coefficients $b_j(t)$ was assumed in all the results for impulsive functional differential

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equations. One of the main purposes of this work is to propose a technique allowing us to avoid these assumptions.

Let us consider the following impulsive equations:

$$(Lx)(t) \equiv x''(t) + \sum_{j=1}^p a_j(t)x'(t - \tau_j(t)) + \sum_{j=1}^p b_j(t)x(t - \theta_j(t)) = f(t), \quad t \in [0, \omega], \quad (1.1)$$

$$x(t_k) = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = 1, 2, \dots, r, \quad (1.2)$$

$$0 = t_0 < t_1 < t_2 < \dots < t_r < t_{r+1} = \omega,$$

$$x(\zeta) = 0, \quad x'(\zeta) = 0, \quad \zeta < 0, \quad (1.3)$$

where $f, a_j, b_j: [0, \omega] \rightarrow \mathbb{R}$ are summable functions and $\tau_j, \theta_j: [0, \omega] \rightarrow [0, +\infty)$ are measurable functions for $j = 1, 2, \dots, p$, p and r are natural numbers, γ_k and δ_k are real positive numbers.

Let $D(t_1, t_2, \dots, t_r)$ be a space of functions $x: [0, \omega] \rightarrow \mathbb{R}$ such that their derivative $x'(t)$ is absolutely continuous on every interval $t \in [t_i, t_{i+1})$, $i = 0, 1, \dots, r$, $x'' \in L_\infty$, we assume also that there exist the finite limits $x(t_i - 0) = \lim_{t \rightarrow t_i^-} x(t)$ and $x'(t_i - 0) = \lim_{t \rightarrow t_i^-} x'(t)$ and condition (1.2) is satisfied at points t_i ($i = 0, 1, \dots, r$). As a solution x we understand a function $x \in D(t_1, t_2, \dots, t_r)$ satisfying (1.1)-(1.3).

There is not many works on sign-constancy of Green's functions of second-order impulsive boundary value problems. We can note only the results of [5–8, 16] where these problems are considered. In these papers the coefficients $b_j(t)$ were assumed to be nonpositive. Using approaches of these papers, we obtain the results on sign-constancy of Green's functions to impulsive two-point boundary value problems without sign assumption on the coefficients $b_j(t)$. Our results are presented in the form of algebraic inequalities, establishing the smallness of the coefficients $a_j(t)$, $|b_j(t)|$, $j = 1, \dots, p$, $t \in [0, \omega]$.

2. PRELIMINARIES

For equation (1.1)-(1.3) we consider the following variants of boundary conditions:

$$x(0) = 0, \quad x(\omega) = 0, \quad (2.1)$$

$$x'(0) = 0, \quad x(\omega) = 0, \quad (2.2)$$

$$x(0) = 0, \quad x'(\omega) = 0, \quad (2.3)$$

$$x'(0) = 0, \quad x'(\omega) = 0. \quad (2.4)$$

General solution of the equation (1.1)-(1.3) can be represented in the form [4]:

$$x(t) = v_1(t)x(0) + C(t, 0)x'(0) + \int_0^t C(t, s)f(s)ds, \quad (2.5)$$

where

- $v_1(t)$ is a solution of the homogeneous equation

$$(Lx)(t) \equiv x''(t) + \sum_{j=1}^p a_j(t)x'(t - \tau_j(t)) + \sum_{j=1}^p b_j(t)x(t - \theta_j(t)) = 0, \quad t \in [0, \omega], \quad (2.6)$$

$$x(t_k) = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = 1, 2, \dots, r, \quad (2.7)$$

$$0 = t_0 < t_1 < t_2 < \dots < t_r < t_{r+1} = \omega,$$

$$x(\zeta) = 0, \quad x'(\zeta) = 0, \quad \zeta < 0, \quad (2.8)$$

with the initial conditions $x(0) = 1, x'(0) = 0$.

- $C(t, s)$, called the Cauchy function of the equation (2.6)-(2.8), is the solution of the equation

$$(L_s x)(t) \equiv x''(t) + \sum_{j=1}^p a_j(t)x'(t - \tau_j(t)) + \sum_{j=1}^p b_j(t)x(t - \theta_j(t)) = 0, \quad t \in [s, \omega], \quad (2.9)$$

$$x(t_k) = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = m, \dots, r, \quad (2.10)$$

$$0 = t_0 < t_1 < t_2 < \dots < t_r < t_{r+1} = \omega,$$

where m is a number, such that $t_{m-1} < s \leq t_m$,

$$x(\zeta) = 0, \quad x'(\zeta) = 0, \quad \zeta < s, \quad (2.11)$$

satisfying the initial conditions $C(s, s) = 0, C'_t(s, s) = 1$ and $C(t, s) = 0$ for $t < s$.

If the boundary value problem (1.1)-(1.3), (2.i), $i = \overline{1, 4}$ is uniquely solvable, then its solution can be represented as

$$x(t) = \int_0^\omega G_i(t, s) f(s) ds, \quad i = \overline{1, 4}, \quad (2.12)$$

where $G_i(t, s)$ is Green's function of the problem (1.1)-(1.3), (2.i) respectively [5].

Using general representation of the solution (2.5), the following formulas for Green's functions can be obtained:

$$G_1(t, s) = C(t, s) - C(t, 0) \frac{C(\omega, s)}{C(\omega, 0)}, \quad (2.13)$$

$$G_2(t, s) = C(t, s) - C(\omega, s) \frac{v_1(t)}{v_1(\omega)}, \quad (2.14)$$

$$G_3(t, s) = C(t, s) - C(t, 0) \frac{C'_t(\omega, s)}{C'_t(\omega, 0)}, \quad (2.15)$$

$$G_4(t, s) = C(t, s) - C'_t(\omega, s) \frac{v_1(t)}{v'_1(\omega)}. \quad (2.16)$$

Below the following definition will be used.

Definition 1. We call $[0, \omega]$ a semi-nonoscillation interval of (2.6)-(2.8), if every nontrivial solution having zero of derivative does not have zero on this interval.

3. SIGN-CONSTANCY OF GREEN'S FUNCTIONS FOR $b_j(t) \leq 0$

Denote $G^\xi(t, s)$ the Green's function of the problem (1.1)-(1.3) with boundary conditions

$$x(\xi) = 0, \quad x'(\xi) = 0. \quad (3.1)$$

In the paper [5], the following theorem has been proven for the problems (1.1)-(1.3), (2.i).

Lemma 1. Assume that the following conditions are fulfilled:

- (1) $b_j(t) \leq 0$, $j = 1, \dots, p$, $t \in [0, \omega]$.
- (2) The Cauchy function $C_1(t, s)$ of the first order impulsive equation

$$\begin{aligned} y'(t) + \sum_{j=1}^p a_j(t)y(t - \tau_j(t)) &= 0, \quad t \in [0, \omega], \\ y(t_k) &= \delta_k y(t_k - 0), \quad k = 1, 2, \dots, r, \\ y(\zeta) &= 0, \quad \zeta < 0, \end{aligned} \quad (3.2)$$

is positive for $0 \leq s \leq t \leq \omega$.

- (3) Green's function $G^\xi(t, s)$ of the problem (1.1)-(1.3), (3.1) is nonnegative for $t, s \in [0, \xi]$ for every $0 < \xi < \omega$.
- (4) $[0, \omega]$ is a semi-nonoscillation interval of $(Lx)(t) = 0$.

Then Green's functions $G_i(t, s)$, $i = \overline{1, 3}$ are nonpositive for $t, s \in [0, \omega]$ and under the additional condition $\sum_{j=1}^p b_j(t)\chi_{[0, \omega]}(t - \theta_j(t)) \neq 0$, $t \in [0, \omega]$, where

$$\chi_{[0, \omega]}(t) = \begin{cases} 1, & t \in [0, \omega], \\ 0, & t \notin [0, \omega], \end{cases} \quad (3.3)$$

$G_4(t, s) \leq 0$ for $t, s \in [0, \omega]$.

Using the results of [9] (see Theorems 4.1 and 4.2 from [9]), it is easy to see, that if $a_j(t) \geq 0$, $j = 1, \dots, p$, $t \in [0, \omega]$, then the assumption 4) follows from the rest 3 assumptions of Lemma 1, so we can exclude it and rewrite Theorem 1 in the following form.

Theorem 1. Assume that the following conditions are fulfilled:

- (1) $a_j(t) \geq 0$, $b_j(t) \leq 0$, $j = 1, \dots, p$, $t \in [0, \omega]$.
- (2) The Cauchy function $C_1(t, s)$ of the first order equation (3.2) is positive for $0 \leq s \leq t \leq \omega$.
- (3) Green's function $G^\xi(t, s)$ of the problem (1.1)-(1.3), (3.1) is nonnegative for $t, s \in [0, \xi]$ for every $0 < \xi < \omega$.

Then Green's functions $G_i(t, s)$, $i = \overline{1, 3}$ are nonpositive for $t, s \in [0, \omega]$ and under the additional condition $\sum_{j=1}^p b_j(t) \chi_{[0, \omega]}(t - \theta_j(t)) \neq 0$, $t \in [0, \omega]$, Green's function $G_4(t, s) \leq 0$ for $t, s \in [0, \omega]$.

In Theorem 1 we have assumed that the Cauchy function $C_1(t, s)$ of the first order impulsive equation (3.2) is positive. In the lemma below, we will formulate the results of [4] on the conditions of positivity of Cauchy function of the first order impulsive differential equation

$$\begin{aligned} y'(t) + a_1(t) y(t - \tau_1(t)) &= 0, \quad t \in [0, \omega], \\ y(t_k) &= \delta_k y(t_k - 0), \quad k = 1, 2, \dots, r, \\ y(\zeta) &= 0, \quad \zeta < 0. \end{aligned} \quad (3.4)$$

Lemma 2. Let $0 < \delta_j < 1$ for $j = 1, \dots, r$ and the following condition be fulfilled

$$\frac{1 + \ln B(t)}{e} \geq \int_{m(t)}^t a_+(s) ds, \quad (3.5)$$

where $B(t) = \prod_{j \in D_t} \delta_j$, $D_t = \{i : t_i \in [t - \tau_1(t), t]\}$, $a_+(t) = \max\{a_1(t), 0\}$ and $m(t) = \max\{t - \tau_1(t), 0\}$. Then Cauchy function of the first order impulsive differential equation (3.4) is positive.

In the case of $p > 1$ the following sufficient condition proven in [4] can be used.

Lemma 3. Let $a_j(t) \geq 0$ for $j = 1, \dots, p$, $0 < \delta_k \leq 1$ for $k = 1, \dots, r$ and

$$\int_0^\omega \sum_{j=1}^p a_j(s) ds < \prod_{k=1}^r \delta_k, \quad (3.6)$$

then Cauchy function of the first order impulsive differential equation (3.2) is positive.

In the next section we will consider an auxiliary impulsive boundary value problem, which will provide the conditions of nonnegativity of Green's function $G^\xi(t, s)$.

4. AUXILIARY BOUNDARY VALUE PROBLEM $x''(t) = z(t)$

Let us consider the following equation:

$$x''(t) = z(t), \quad t \in [0, \omega], \quad (4.1)$$

$$x(t_k) = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = 1, 2, \dots, r, \quad (4.2)$$

$$x(\zeta) = 0, \quad x'(\zeta) = 0, \quad \zeta < 0. \quad (4.3)$$

Denote by $\bar{C}(t, s)$ the Cauchy function of the equation (4.1)-(4.3). According to [6], Cauchy function for this impulsive equation has the form, represented on the

Figure 1. This corresponds to the following equation:

$$\begin{aligned}
 \bar{C}(t, s) = & \chi_{[0, t]}(s) \left[\sum_{i=0}^r (t-s) \chi_{[t_i, t_{i+1})}(t) \chi_{[t_i, t_{i+1})}(s) \right. \\
 & + \sum_{i=1}^r (\gamma_i(t_i - s) + \delta_i(t - t_i)) \chi_{[t_i, t_{i+1})}(t) \chi_{[t_{i-1}, t_i)}(s) \\
 & + \sum_{i=2}^r \sum_{j=0}^{i-2} \left[\prod_{k=j+1}^i \gamma_k(t_{j+1} - s) + \sum_{l=j+2}^i \prod_{k=l}^i \gamma_k \prod_{k=j+1}^{l-1} \delta_k(t_l - t_{l-1}) \right. \\
 & \left. \left. + \prod_{k=j+1}^i \delta_k(t - t_i) \right] \chi_{[t_i, t_{i+1})}(t) \chi_{[t_j, t_{j+1})}(s) \right].
 \end{aligned} \tag{4.4}$$

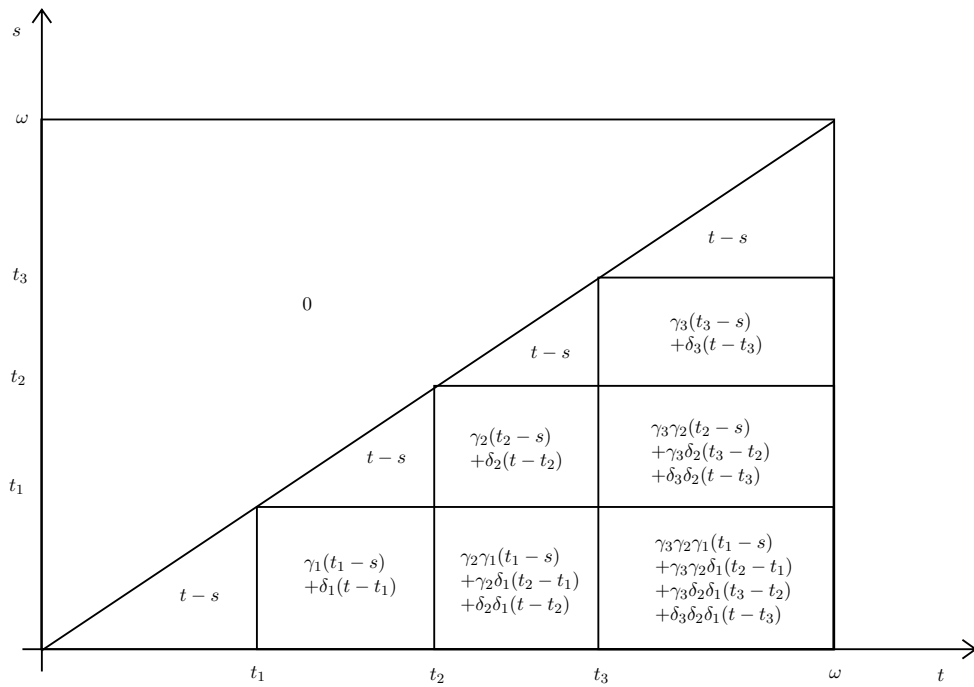


FIGURE 1. The Cauchy function of impulsive equation (4.1) - (4.3).

Let us build now the Green's function $\bar{G}(t, s)$ of the problem (4.1)-(4.3) with boundary conditions

$$x(\omega) = 0, \quad x'(\omega) = 0. \tag{4.5}$$

The solution of this boundary value problem will be of the following form:

$$x(t) = \int_0^\omega \left[\bar{C}(t, s) + \frac{\nu_1(t)}{\nu_1(\omega)} \left[\frac{\bar{C}'_t(\omega, s)}{\bar{C}'_t(\omega, 0)} \bar{C}(\omega, 0) - \bar{C}(\omega, s) \right] - \frac{\bar{C}(t, 0)}{\bar{C}'_t(\omega, 0)} \right] z(s) ds, \quad (4.6)$$

with the corresponding Green's function

$$\bar{G}(t, s) = \bar{C}(t, s) + \frac{\nu_1(t)}{\nu_1(\omega)} \left[\frac{\bar{C}'_t(\omega, s)}{\bar{C}'_t(\omega, 0)} \bar{C}(\omega, 0) - \bar{C}(\omega, s) \right] - \frac{\bar{C}(t, 0)}{\bar{C}'_t(\omega, 0)}, \quad (4.7)$$

where

$$\nu_1(t) = \begin{cases} 1, & t \in [0, t_1), \\ \prod_{k=1}^r \gamma_k, & t \in [t_k, t_{k+1}). \end{cases} \quad (4.8)$$

See the graphical representation of Green's function $\bar{G}(t, s)$ on the Figure 2.

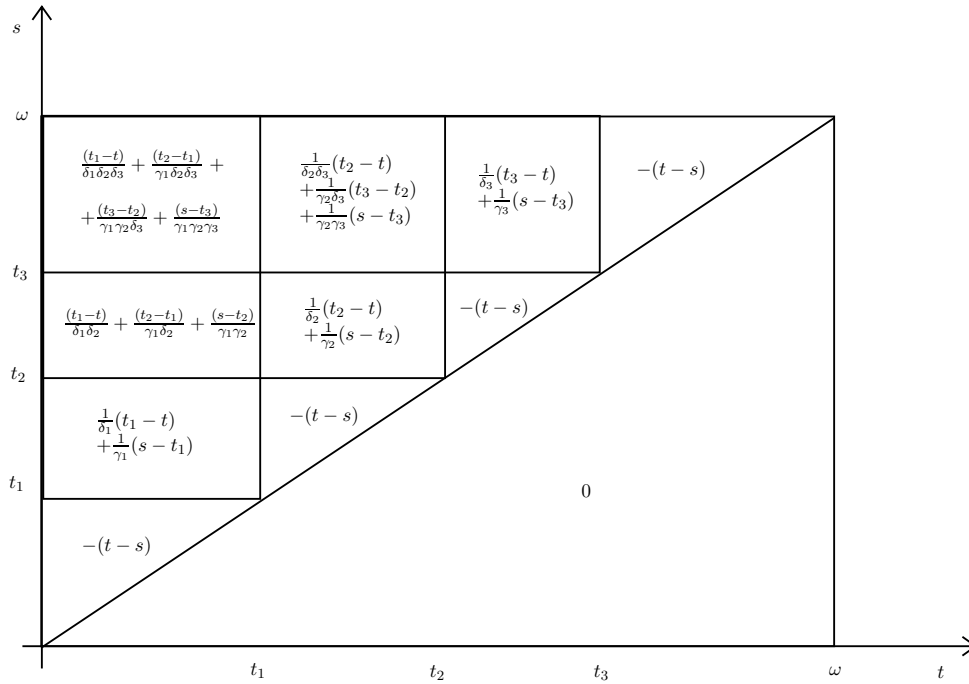


FIGURE 2. The Green's function of impulsive equation (4.1) - (4.3), (4.5).

Theorem 2. *If $a_j(t) \geq 0$, $b_j(t) \leq 0$ for $j = \overline{1, p}$, $0 < \gamma_k \leq 1$, $0 < \delta_k \leq 1$ for $k = \overline{1, r}$, and if the condition*

$$\begin{aligned} & \frac{\omega}{\prod_{k=1}^r \delta_k} \operatorname{ess\,sup}_{t \in [0, \omega]} \sum_{j=1}^p |a_j(t)| \\ & + \omega \left(\sum_{i=1}^r \frac{t_i - t_{i-1}}{\prod_{k=i}^r \delta_k \prod_{k=1}^i \gamma_k} \gamma_i + \frac{\omega - t_r}{\prod_{k=1}^r \gamma_k} \right) \operatorname{ess\,sup}_{t \in [0, \omega]} \sum_{j=1}^p |b_j(t)| < 1 \end{aligned} \quad (4.9)$$

is satisfied, then Green's function $G(t, s)$ of the (1.1)-(1.3), (4.5) is nonnegative.

Proof. We can rewrite equation (1.1)-(1.3) in the form

$$\begin{aligned} z(t) &+ \sum_{j=1}^p a_j(t) \int_0^\omega \bar{G}'_t(t - \tau_j(t), s) \chi_{[0, \omega]}(t - \tau_j(t)) z(s) ds \\ &+ \sum_{j=1}^p b_j(t) \int_0^\omega \bar{G}(t - \theta_j(t), s) \chi_{[0, \omega]}(t - \theta_j(t)) z(s) ds = f(t). \end{aligned} \quad (4.10)$$

Denote the operator $K: L_\infty \rightarrow L_\infty$ as follows:

$$\begin{aligned} (Kz)(s) &= \int_0^\omega \left[- \sum_{j=1}^p a_j(t) \bar{G}'_t(t - \tau_j(t), s) \chi_{[0, \omega]}(t - \tau_j(t)) \right. \\ &\quad \left. - \sum_{j=1}^p b_j(t) \bar{G}(t - \theta_j(t), s) \chi_{[0, \omega]}(t - \theta_j(t)) \right] z(s) ds. \end{aligned} \quad (4.11)$$

For $a_j(t) \geq 0$, $b_j(t) \leq 0$, $j = 1, \dots, p$, the operator $K: L_\infty \rightarrow L_\infty$ is positive. If the spectral radius $\rho(K) < 1$, then the solution of (4.10) can be represented in the form:

$$z = (I - K)^{-1} f = \left[\sum_{j=0}^{\infty} K^j \right] f. \quad (4.12)$$

The solution $x(t)$ of the boundary value problem (1.1)-(1.3), (4.5) can be written in the form:

$$x = \left(\bar{G} \sum_{j=0}^{\infty} K^j \right) f, \quad (4.13)$$

where $\bar{G} \sum_{j=0}^{\infty} K^j$ is Green's operator for (1.1)-(1.3) with boundary conditions (4.5). From nonnegativity of Green's function $\bar{G}(t, s)$ and positivity of operator $K: L_\infty \rightarrow L_\infty$ it follows that Green's function $G(t, s)$ for (1.1)-(1.3), (4.5) is nonnegative.

Now let us find the conditions that the spectral radius of the operator $K: L_\infty \rightarrow L_\infty$ is less than one. It is satisfied, when

$$\begin{aligned} & \operatorname{ess\,sup}_{t \in [0, \omega]} \sum_{j=1}^p |a_j(t)| \operatorname{ess\,sup}_{t \in [0, \omega]} \int_0^\omega |\bar{G}'_t(t, s)| ds \\ & + \operatorname{ess\,sup}_{t \in [0, \omega]} \sum_{j=1}^p |b_j(t)| \operatorname{ess\,sup}_{t \in [0, \omega]} \int_0^\omega |\bar{G}(t, s)| ds < 1. \end{aligned} \quad (4.14)$$

If $0 < \gamma_k \leq 1$, $0 < \delta_k \leq 1$ for $k = \overline{1, r}$, then:

$$\operatorname{ess\,sup}_{t, s \in [0, \omega] \times [0, \omega]} |\bar{G}(t, s)| = \sum_{i=1}^r \frac{t_i - t_{i-1}}{\prod_{k=i}^r \delta_k \prod_{k=1}^i \gamma_k} \gamma_i + \frac{\omega - t_r}{\prod_{k=1}^r \gamma_k}, \quad (4.15)$$

$$\operatorname{ess\,sup}_{t, s \in [0, \omega] \times [0, \omega]} |\bar{G}'_t(t, s)| = \frac{1}{\prod_{k=1}^r \delta_k}. \quad (4.16)$$

Thus, the spectral radius $\rho(K) < 1$, if $0 < \gamma_k \leq 1$, $0 < \delta_k \leq 1$ for $k = \overline{1, r}$ and

$$\begin{aligned} & \frac{\omega}{\prod_{k=1}^r \delta_k} \operatorname{ess\,sup}_{t \in [0, \omega]} \sum_{j=1}^p |a_j(t)| \\ & + \omega \left(\sum_{i=1}^r \frac{t_i - t_{i-1}}{\prod_{k=i}^r \delta_k \prod_{k=1}^i \gamma_k} \gamma_i + \frac{\omega - t_r}{\prod_{k=1}^r \gamma_k} \right) \operatorname{ess\,sup}_{t \in [0, \omega]} \sum_{j=1}^p |b_j(t)| < 1. \end{aligned} \quad (4.17)$$

□

Example 1. Let us consider the following differential equation

$$x''(t) + a_1(t)x'(t - \tau_1(t)) + b_1(t)x(t - \theta_1(t)) = f(t), \quad t \in [0, 3], \quad (4.18)$$

with impulses (1.2), where

$$\begin{aligned} t_1 &= 1, & \gamma_1 &= 0.9, & \delta_1 &= 0.5, \\ t_2 &= 1.5, & \gamma_2 &= 0.6, & \delta_2 &= 0.7, \\ t_3 &= 2.5, & \gamma_3 &= 0.8, & \delta_3 &= 0.4, \end{aligned} \quad (4.19)$$

and $a_1(t) \geq 0$, $b_1(t) \leq 0$.

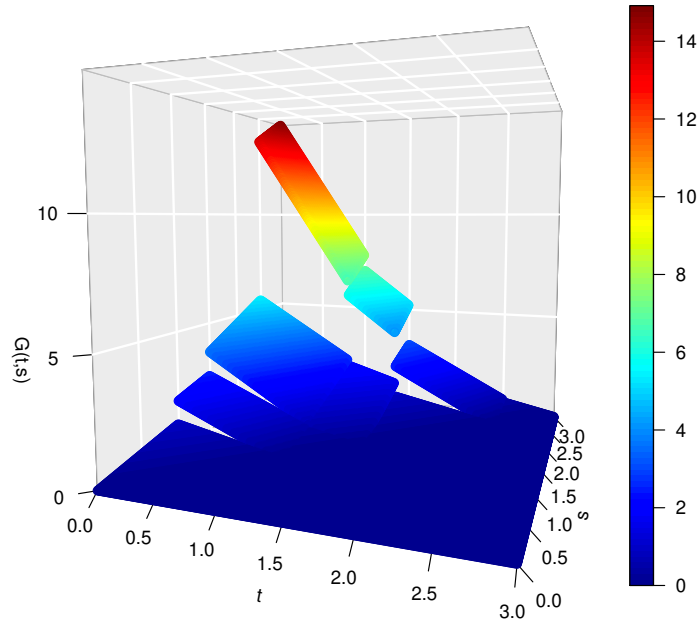
For an auxiliary differential equation

$$x''(t) = z(t), \quad t \in [0, 3], \quad (4.20)$$

with impulses (1.2), (4.19) and boundary conditions

$$x(3) = 0, \quad x'(3) = 0, \quad (4.21)$$

we constructed Green's function $\bar{G}(t, s)$. It is represented on the Figure 3 and its derivative $\bar{G}'_t(t, s)$ - on the Figure 4.

FIGURE 3. $\bar{G}(t,s)$ of (4.20), (4.19), (4.21).

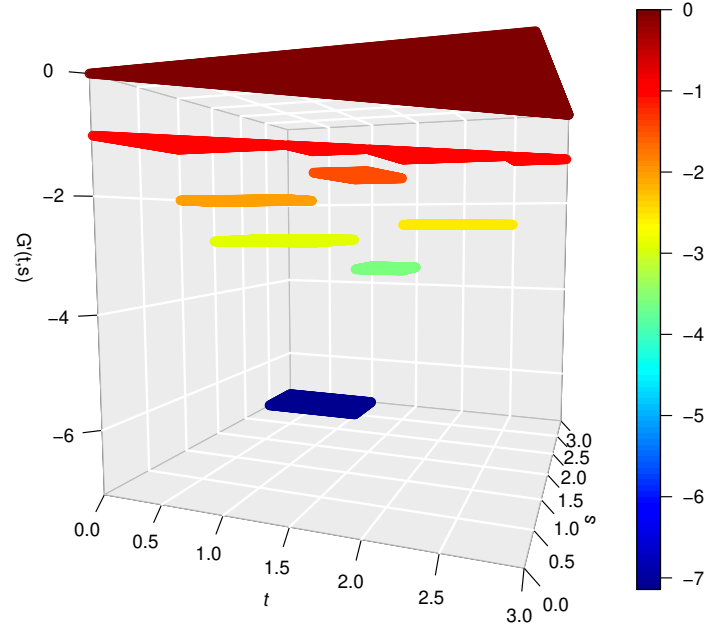
Calculating maximum values of $|\bar{G}(t,s)|$ and $|\bar{G}'_t(t,s)|$ in $t,s \in [0,3] \times [0,3]$, we obtain:

$$\operatorname{ess\,sup}_{t,s \in [0,3] \times [0,3]} |\bar{G}(t,s)| = 14.914, \quad (4.22)$$

$$\operatorname{ess\,sup}_{t,s \in [0,3] \times [0,3]} |\bar{G}'_t(t,s)| = 7.143. \quad (4.23)$$

According to Theorem 2, for our example, we obtain the following condition of positivity of Green's function $G(t,s)$:

$$21.429 \operatorname{ess\,sup}_{t \in [0,3]} |a_1(t)| + 44.742 \operatorname{ess\,sup}_{t \in [0,3]} |b_1(t)| < 1. \quad (4.24)$$

FIGURE 4. $\bar{G}'_t(t, s)$ of (4.20), (4.19), (4.21).

5. SIGN-CONSTANCY OF GREEN'S FUNCTIONS FOR THE CASE WHEN $b_j(t)$ CAN CHANGE SIGN

For the case when $b_j(t)$, $j = 1, \dots, p$, changes sign, there can be considered an auxiliary differential equation:

$$(L^-x)(t) \equiv x''(t) + \sum_{j=1}^p a_j(t)x'(t - \tau_j(t)) + \sum_{j=1}^p b_j^-(t)x(t - \theta_j(t)) = z(t), \quad t \in [0, \omega], \quad (5.1)$$

$$\begin{aligned} x(t_k) &= \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = 1, 2, \dots, r, \\ 0 &= t_0 < t_1 < t_2 < \dots < t_r < t_{r+1} = \omega, \end{aligned} \quad (5.2)$$

$$x(\zeta) = 0, \quad x'(\zeta) = 0, \quad \zeta < 0, \quad (5.3)$$

where

$$b_j^-(t) = \begin{cases} b_j(t), & b_j(t) \leq 0, \\ 0, & b_j(t) > 0. \end{cases} \quad (5.4)$$

The solution for the boundary value problems (5.1)-(5.3), (2.i) can be written in the form

$$x(t) = (G_i^- z)(t) \equiv \int_0^\omega G_i^-(t, s) z(s) ds, \quad i = \overline{1, 4}. \quad (5.5)$$

The given equation (1.1) can be written in the form:

$$(Lx)(t) = (L^- x)(t) + \sum_{j=1}^p b_j^+(t) x(t - \theta_j(t)) = f(t), \quad (5.6)$$

where

$$b_j^+(t) = \begin{cases} b_j(t), & b_j(t) > 0, \\ 0, & b_j(t) \leq 0. \end{cases} \quad (5.7)$$

After the substitution (5.5) into (5.6) we obtain

$$z(t) + \sum_{j=1}^p b_j^+(t) \int_0^\omega G_i^-(t - \theta_j(t), s) \chi_{[0, \omega]}(t - \theta_j(t)) z(s) ds = f(t), \quad i = \overline{1, 4}. \quad (5.8)$$

Define the integral operators $K_i : L_\infty \rightarrow L_\infty$ for each type of boundary conditions ($i = \overline{1, 4}$) by the equality:

$$(K_i z)(t) = - \sum_{j=1}^p b_j^+(t) \int_0^\omega [G_i^-(t - \theta_j(t), s) \chi_{[0, \omega]}(t - \theta_j(t))] z(s) ds. \quad (5.9)$$

Its spectral radius can be denoted by $\rho(K_i)$.

We propose an assertion about nonpositivity of Green's function of (1.1)-(1.3), (2.i) without an assumption $b_j(t) \leq 0$, $j = 1, \dots, p$. For this case we can reformulate Theorem 1 as follows.

Theorem 3. Assume that the following conditions are fulfilled:

- (1) $a_j(t) \geq 0$, $j = 1, \dots, p$, $t \in [0, \omega]$.
- (2) The Cauchy function $C_1(t, s)$ of the first order equation (3.2) is positive for $0 \leq s \leq t \leq \omega$.
- (3) Green's function $G^\xi(t, s)$ of the problem (5.1)-(5.3), (3.1) is nonnegative for $t, s \in [0, \xi]$ for every $0 < \xi < \omega$.
- (4) $\rho(K_i) < 1$.

Then Green's functions $G_i(t, s)$, $i = \overline{1, 3}$ are nonpositive for $t, s \in [0, \omega]$ and under the additional condition $\sum_{j=1}^p b_j^-(t) \chi_{[0, \omega]}(t - \theta_j(t)) \not\equiv 0$, $t \in [0, \omega]$, Green's function $G_4(t, s) \leq 0$ for $t, s \in [0, \omega]$.

Proof. The conditions (1)-(3) of Theorem 3 correspond to all the conditions of Theorem 1, so they guarantee that Green's functions $G_i^-(t, s)$, $i = \overline{1, 4}$, are nonpositive.

We have assumed above, in equation (5.7), that $b_j^+(t) \geq 0$. Together with the fact that $G_i^-(t, s) \leq 0$, according to Corollary 1, it implies that the operators K_i , $i = \overline{1, 4}$, are positive.

If the condition $\rho(K_i) < 1$ holds, then the equation (5.8) can be written as following:

$$z = (I - K_i)^{-1} f = \left[\sum_{j=0}^{\infty} K_i^j \right] f. \quad (5.10)$$

It follows from $G_i^-(t, s) \leq 0$ that all operators K_i^j are positive and, consequently, for this case $\sum_{j=0}^{\infty} K_i^j \geq 0$.

The solution $x(t)$ of the given boundary value problem (1.1)-(1.3), (2.i) can be written in the form:

$$x = \left(G_i^- \sum_{j=0}^{\infty} K_i^j \right) f, \quad (5.11)$$

where Green's operator for the problem (1.1)-(1.3), (2.i) is

$$G_i = G_i^- \sum_{j=0}^{\infty} K_i^j, \quad (5.12)$$

which is nonpositive when the condition $\rho(K_i) < 1$ holds.

□

For the case when $b_j(t)$ can change sign, we can also use the following form of Theorem 1.

Corollary 1. Assume that the following conditions are fulfilled:

- (1) $0 < \gamma_k \leq 1$, $0 < \delta_k \leq 1$ for $k = \overline{1, r}$ and $a_j(t) \geq 0$, $j = 1, \dots, p$, $t \in [0, \omega]$.
- (2) The condition (3.6) is fulfilled.
- (3) The inequality

$$\begin{aligned} & - \sum_{j=1}^p a_j(t) \bar{G}_t'(t - \tau_j(t), s) \chi_{[0, \omega]}(t - \tau_j(t)) \\ & - \sum_{j=1}^p b_j(t) \bar{G}(t - \theta_j(t), s) \chi_{[0, \omega]}(t - \theta_j(t)) \geq 0 \end{aligned} \quad (5.13)$$

is fulfilled for $t \in [0, \omega]$.

- (4) The condition (4.17) is fulfilled.

Then Green's functions $G_i(t, s)$, $i = \overline{1, 3}$ are nonpositive for $t, s \in [0, \omega]$ and under the additional condition $\sum_{j=1}^p b_j(t) \chi_{[0, \omega]}(t - \theta_j(t)) \not\equiv 0$, $t \in [0, \omega]$, Green's function $G_4(t, s) \leq 0$ for $t, s \in [0, \omega]$.

Example 2. Let us consider the same equation as in Example 1, where $a_1(t) = 0.015t + 0.005$, $b_1(t) = 0.005t - 0.005$ (so, $b_1(t)$ changes sign at the point $t = 1$), $\tau_1(t) = 0.5$, $\theta_1(t) = 0.5$.

$$x''(t) + (0.015t + 0.005)x'(t - 0.5) + (0.005t - 0.005)x(t - 0.5) = f(t), \quad t \in [0, 3], \quad (5.14)$$

with impulses (1.2), where

$$\begin{aligned} t_1 &= 1, & \gamma_1 &= 0.9, & \delta_1 &= 0.5, \\ t_2 &= 1.5, & \gamma_2 &= 0.6, & \delta_2 &= 0.7, \\ t_3 &= 2.5, & \gamma_3 &= 0.8, & \delta_3 &= 0.4. \end{aligned} \quad (5.15)$$

Let us verify if the condition (5.13) is satisfied. Denote

$$\begin{aligned} M(t, s) &= -(0.015t + 0.005)\bar{G}'_t(t - 0.5, s)\chi_{[0, 3]}(t - 0.5) \\ &\quad - (0.005t - 0.005)\bar{G}(t - 0.5, s)\chi_{[0, 3]}(t - 0.5). \end{aligned} \quad (5.16)$$

The form of function $M(t, s)$, for our example, is shown on Figure 5. It is easy to see that the function $M(t, s) \geq 0$ for $t, s \in [0, 3] \times [0, 3]$. Thus, the condition 3) of Corollary 1 is fulfilled.

Now let us verify the condition (4.17). It is equivalent to the condition:

$$\operatorname{ess\,sup}_{t, s \in [0, 3] \times [0, 3]} \int_0^3 M(t, s) ds < 1, \quad (5.17)$$

or

$$\operatorname{ess\,sup}_{t, s \in [0, 3] \times [0, 3]} M(t, s) < \frac{1}{3}. \quad (5.18)$$

For our example, we can calculate

$$\operatorname{ess\,sup}_{t, s \in [0, 3] \times [0, 3]} M(t, s) = 0.18. \quad (5.19)$$

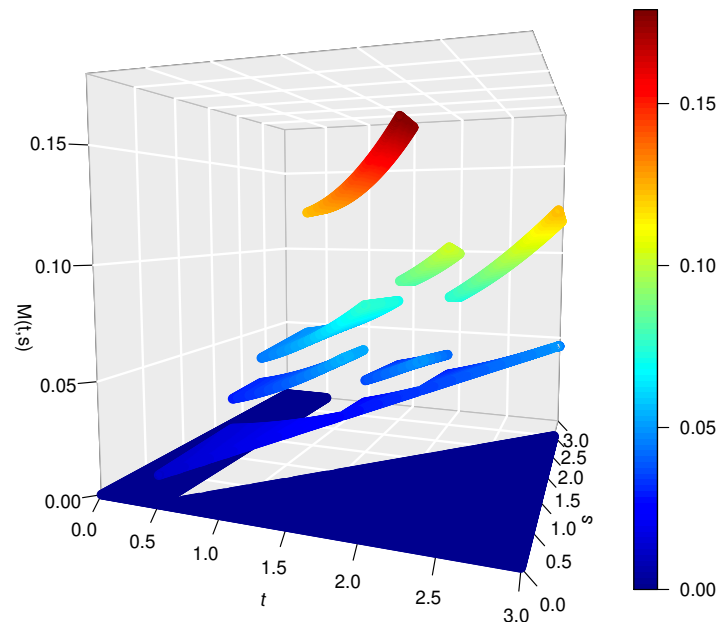
Substituting this into the equation (5.18), we can obtain: $0.18 < \frac{1}{3}$. Thus, the condition 4) of Corollary 1 is fulfilled.

Let us use Lemma 3 to check the condition 2) of Corollary 1. We can calculate:

$$\int_0^3 (0.015t + 0.005) dt = 0.0825,$$

$$\prod_{k=1}^3 \delta_k = 0.14.$$

We obtain $0.0825 < 0.14$, thus, the condition 2) of Corollary 1 holds. So, all the conditions of Corollary 1 are fulfilled. It means that, for our example, Green's functions $G_i(t, s)$, $i = \overline{1, 4}$ are nonpositive.

FIGURE 5. $M(t, s)$.

REFERENCES

- [1] R. P. Agarwal, L. Berezansky, E. Braverman, and A. Domoshnitsky, *Nonoscillation Theory of Functional Differential Equations with Applications*. New York: Springer, 2012. doi: [10.1007/978-1-4614-3455-9](https://doi.org/10.1007/978-1-4614-3455-9).
- [2] R. P. Agarwal and D. O'Regan, "Multiple Nonnegative Solutions for Second Order Impulsive Differential Equations." *Applied Mathematics and Computation*, vol. 114, no. 1, pp. 51–59, 2000, doi: [10.1016/S0096-3003\(99\)00074-0](https://doi.org/10.1016/S0096-3003(99)00074-0).
- [3] N. V. Azbelev, V. P. Maksimov, and L. F. Rakhmatullina, *Introduction to the Theory of Functional Differential Equations: Methods and Applications*. New York: Hindawi, 2007.
- [4] A. Domoshnitsky and M. Drakhlin, "Nonoscillation of First Order Impulsive Differential Equations With Delay." *Journal of Mathematical Analysis and Applications*, vol. 206, no. 1, pp. 254–269, 1997.
- [5] A. Domoshnitsky and G. Landsman, "Semi-nonoscillation Intervals in the Analysis of Sign Constancy of Green's Functions of Dirichlet, Neumann and Focal Impulsive Problems." *Advances in Difference Equations*, vol. 81, no. 1, 2017, doi: [10.1186/s13662-017-1134-1](https://doi.org/10.1186/s13662-017-1134-1).

- [6] A. Domoshnitsky, G. Landsman, and S. Yanetz, "About Sign-constancy of Green's Functions for Impulsive Second Order Delay Equations." *Opuscula Mathematica*, vol. 34, no. 2, pp. 339–362, 2014, doi: [10.7494/OpMath.2014.34.2.339](https://doi.org/10.7494/OpMath.2014.34.2.339).
- [7] A. Domoshnitsky, G. Landsman, and S. Yanetz, "About Sign-constancy of Green's Functions of One-point Problem for Impulsive Second Order Delay Equations." *Functional Differential Equations*, vol. 21, no. 1–2, pp. 3–15, 2014.
- [8] A. Domoshnitsky, G. Landsman, and S. Yanetz, "About Sign-constancy of Green's Function of a Two-point Problem for Impulsive Second Order Delay Equations." *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 9, pp. 1–16, 2016.
- [9] A. Domoshnitsky, I. Mizgireva, and V. Raichik, "Semi-nonoscillation Intervals and Sign-constancy of Green's Functions of Two-point Impulsive Boundary Value Problems." *Submitted to Ukrainian Mathematical Journal*, 2018.
- [10] M. Feng and D. Xie, "Multiple Positive Solutions of Multi-point Boundary Value Problem for Second-order Impulsive Differential Equations." *Journal of Mathematical Analysis and Applications*, vol. 223, no. 1, pp. 438–448, 2009.
- [11] L. Hu, L. Liu, and Y. Wu, "Positive Solutions of Nonlinear Singular Two-point Boundary Value Problems for Second-order Impulsive Differential Equations." *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 11, pp. 3774–3789, 2008, doi: [10.1016/j.na.2007.10.012](https://doi.org/10.1016/j.na.2007.10.012).
- [12] T. Jankowsky, "Positive Solutions to Second Order Four-point Boundary Value Problems for Impulsive Differential Equations." *Applied Mathematics and Computation*, vol. 202, no. 2, pp. 550–561, 2008.
- [13] D. Jiang and X. Lin, "Multiple Positive Solutions of Dirichlet Boundary Value Problems for Second Order Impulsive Differential Equations." *Journal of Mathematical Analysis and Applications*, vol. 321, no. 2, pp. 501–514, 2006.
- [14] V. Lakshmikantham, D. Bainov, and P.S. Simeonov, *Theory of Impulsive Differential Equations*. Singapore: World Scientific, 1989. doi: [10.1142/0906](https://doi.org/10.1142/0906).
- [15] E. Lee and Y. Lee, "Multiple Positive Solutions of Singular Two Point Boundary Value Problems for Second Order Impulsive Differential Equations." *Applied Mathematics and Computation*, vol. 158, no. 3, pp. 745–759, 2004.
- [16] I. Mizgireva, "On Positivity of Green's Functions of Two-point Impulsive Problems." *Submitted to Functional Differential Equations (accepted)*, vol. 24, no. 3–4, 2018.
- [17] S. G. Pandit and S. G. Deo, *Differential Systems Involving Impulses*. Berlin: Springer-Verlag, 1982. doi: [10.1007/BFb0067476](https://doi.org/10.1007/BFb0067476).
- [18] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations*. Singapore: World Scientific, 1995. doi: [10.1142/2892](https://doi.org/10.1142/2892).
- [19] S. G. Zavalishchin and A. N. Seseikin, *Dynamic Impulse Systems: Theory and Applications*. Springer Netherlands, 1997. doi: [10.1007/978-94-015-8893-5](https://doi.org/10.1007/978-94-015-8893-5).

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SPECTRA AND FINE SPECTRA OF THE UPPER TRIANGULAR BAND MATRIX $U(a;0;b)$ OVER THE SEQUENCE SPACE c_0

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Abstract. The aim of this paper is to obtain the spectrum, fine spectrum, approximate point spectrum, defect spectrum and compression spectrum of the operator

$$U(a;0;b) = \begin{bmatrix} a_0 & 0 & b_0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & a_1 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & a_2 & 0 & b_2 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & a_0 & 0 & b_0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & a_1 & 0 & b_1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & a_2 & 0 & b_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \dots \end{bmatrix} \quad (b_0, b_1, b_2 \neq 0)$$

on the sequence space c_0 where the non-zero diagonals are the entries of an oscillatory sequence.

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Keywords: upper triangular band matrix, spectrum, fine spectrum, approximate point spectrum, defect spectrum, compression spectrum

1. INTRODUCTION

We can band matrices which occur finite element or finite difference problems in numerical analysis. We define the relationship between the problem variables helping these matrices. The bandedness is confirmed with variables which are not conjugate in arbitrarily large distances. We can furthermore divide these matrices. For example there are banded matrices with every element in the band is nonzero. We generally encounter to these matrices separating one-dimensional problems.

Also, there are band matrices in problems with higher dimensions. Herein the bands are more thin. For example, the matrix which its bandwidth is the square root of the matrix dimension, correspond to partial differential equation defined in a square domain where the five diagonals are not zero in band. If we apply to this matrix Gaussian elimination, we obtain matrix which has band with many non-zero elements. Therefore the resolvent set of the band operators is important for solving such problems.

Spectral theory is the one of the most useful tools in science. There are many applications in mathematics and physics which contain matrix theory, control theory, function theory, differential and integral equations, complex analysis and quantum physics. For example, atomic energy levels are determined and therefore the frequency of a laser or the spectral signature of a star are obtained by it in quantum mechanics.

1.1. The spectrum

Let $L : X \rightarrow Y$ be a bounded linear operator where X and Y are Banach spaces. Denote the range of L , $R(L)$ and the set of all bounded linear operators on X into itself $B(X)$.

Assume that X be a Banach space and $L \in B(X)$. The adjoint operator $L^* \in B(X^*)$ of L is defined by $(L^* f)(x) = f(Lx)$ for all $f \in X^*$ and $x \in X$ where X^* is the dual space X .

Let X is a complex normed linear space and $D(L) \subset X$ be domain of L where $L : D(L) \rightarrow X$ be a linear operator. For $L \in B(X)$ we determine a complex number λ by the operator $(\lambda I - L)$ denoted by L_λ which has the same domain $D(L)$, such that I is the identity operator. Recall that the resolvent operator of L_λ is $L_\lambda^{-1} := (\lambda I - L)^{-1}$.

Let $\lambda \in \mathbb{C}$. If L_λ^{-1} exists, is bounded and, is defined on a set which is dense in X then λ is called a regular value of L .

The set $\rho(L, X)$ of all regular values of L is called the resolvent set of L . $\sigma(L, X) := \mathbb{C} \setminus \rho(L, X)$ is called the spectrum of L where $\mathbb{C}$ is the complex plane. Hence those values $\lambda \in \mathbb{C}$ for which L_λ is not invertible are contained in the spectrum $\sigma(L, X)$.

The spectrum $\sigma(L, X)$ is union of three disjoint sets as follows: The point spectrum $\sigma_p(L, X)$ is the set such that L_λ^{-1} does not exist. Further $\lambda \in \sigma_p(L, X)$ is called the eigenvalue of L . We say that $\lambda \in \mathbb{C}$ belongs to the continuous spectrum $\sigma_c(L, X)$ of L if the resolvent operator L_λ^{-1} is defined on a dense subspace of X and is unbounded. Furthermore, we say that $\lambda \in \mathbb{C}$ belongs to the residual spectrum $\sigma_r(L, X)$ of L if the resolvent operator L_λ^{-1} exists, but its domain of definition (i.e. the range $R(\lambda I - L)$ of $(\lambda I - L)$ is not dense in X ; in this case L_λ^{-1} may be bounded or unbounded. Together with the point spectrum, these two subspectra form a disjoint subdivision

$$\sigma(L, X) = \sigma_p(L, X) \cup \sigma_c(L, X) \cup \sigma_r(L, X) \quad (1.1)$$

of the spectrum of L .

1.2. Goldberg's classification of spectrum

If $T \in B(X)$, then there are three possibilities for $R(T)$:

(I) $R(T) = X$, (II) $\overline{R(T)} = X$, but $R(T) \neq X$, (III) $\overline{R(T)} \neq X$ and three possibilities for T^{-1} :

(1) T^{-1} exists and continuous, (2) T^{-1} exists but discontinuous, (3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: $I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3$. If an operator is in state III_2 for example, then $\overline{R(T)} \neq X$ and T^{-1} exists but is discontinuous (see [7]).

If λ is a complex number such that $T = \lambda I - L \in I_1$ or $T = \lambda I - L \in II_1$, then $\lambda \in \rho(L, X)$. All scalar values of λ not in $\rho(L, X)$ comprise the spectrum of L . The further classification of $\sigma(L, X)$ gives rise to the fine spectrum of L . That is, $\sigma(L, X)$ can be divided into the subsets $I_2\sigma(L, X) = \emptyset, I_3\sigma(L, X), II_2\sigma(L, X), II_3\sigma(L, X), III_1\sigma(L, X), III_2\sigma(L, X), III_3\sigma(L, X)$. For example, if $T = \lambda I - L$ is in a given state, III_2 (say), then we write $\lambda \in III_2\sigma(L, X)$.

Throughout w denote the space of all real or complex valued sequences. The space of all bounded, convergent, null and bounded variation sequences are denoted by ℓ_∞, c, c_0 and bv , respectively. Also by ℓ_1, ℓ_p, bv_p we denote the spaces of all absolutely summable sequences, p -absolutely summable sequences and p -bounded variation sequences, respectively.

Many researchers have investigated the spectrum and the fine spectrum of linear operators defined by some matrices over certain sequence spaces. There are a lot of studies about spectrum and fine spectrum. For instance, the fine spectrum of the Cesàro operator on the sequence space ℓ_p for $(1 < p < \infty)$ has been examined by Gonzalez [8]. Also, Wenger [17] has studied the fine spectrum of the Hölder summability operator over c , and Rhoades [12] generalized this result to the weighted mean methods. Reade [11] has investigated the spectrum of the Cesàro operator on the sequence space c_0 . The spectrum of the Rhaly operators on the sequence spaces c_0 and c has examined by Yildirim [19]. The spectrum and some subdivisions of the spectrum of discrete generalized Cesàro operators on $\ell_p, (1 < p < \infty)$ has examined by Yildirim and Durna [20]. In [14], Tripathy and Das determined the spectrum and fine spectrum of the upper triangular matrix $U(r, s)$ on the sequence space

$$cs = \left\{ x = (x_n) \in w : \lim_{n \rightarrow \infty} \sum_{i=0}^n x_i \text{ exists} \right\},$$

which is a Banach space with respect to the norm $\|x\|_{cs} = \sup_n \left| \sum_{i=0}^n x_i \right|$. Also they determined the approximate point spectrum, the defect spectrum and the compression spectrum of the operator $U(r, s)$ on the same space. In [16], Tripathy and Saikia determined the norm and spectrum of the Cesàro matrix considered as a bounded operator on $\overline{bv_0} \cap \ell_\infty$. In [15], Tripathy and Paul examined the spectra of the operator $D(r, 0, 0, s)$ on sequence spaces c_0 and c . In [9], Paul and Tripathy investigated the spectrum of the operator $D(r, 0, 0, s)$ over the sequence spaces ℓ_p and bv_p . In [13], Tripathy and Das determined the spectra of the Rhaly operator on the class of bounded statistically null bounded variation sequence space. In [10], Paul and

Tripathy investigated the so-called fine spectrum of the operator $D(r, 0, 0, s)$ over a sequence space bv_0 . In [3], Das and Tripathy determined the spectrum and fine spectrum of the lower triangular matrix $B(r, s, t)$ on the sequence space cs .

2. FINE SPECTRUM

The upper triangular matrix $U(a; 0; b)$ is an infinite matrix with the non-zero diagonals are the entries of an oscillatory sequence of the form

$$U(a; 0; b) = \begin{bmatrix} a_0 & 0 & b_0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & a_1 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & a_2 & 0 & b_2 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & a_0 & 0 & b_0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & a_1 & 0 & b_1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & a_2 & 0 & b_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \dots \end{bmatrix} \quad (2.1)$$

where $b_0, b_1, b_2 \neq 0$.

Lemma 1 (Wilansky [18], Example 8.4.5 A, Page 129). *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c_0)$ from c_0 to itself if and only if*
(i) the rows of A in ℓ_1 and their ℓ_1 norms are bounded,
(ii) the columns of A are in c_0 .

The operator norm of T is the supremum of ℓ_1 norm values of the rows.

Corollary 1. $U(a; 0; b) : c_0 \rightarrow c_0$ is a bounded linear operator and

$$\|U(a; 0; b)\|_{(c_0; c_0)} = \max\{|a_0| + |b_0|, |a_1| + |b_1|, |a_2| + |b_2|\}.$$

Lemma 2 (Golberg [7, p.59]). *T has a dense range if and only if T^* is 1-1.*

Lemma 3 (Golberg [7, p.60]). *T has a bounded inverse if and only if T^* is onto.*

Theorem 1. $\sigma_p(U(a; 0; b), c_0) = \{\alpha \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| < |b_0| |b_1| |b_2|\}$.

Proof. Let λ be an eigenvalue of the operator $U(a; 0; b)$. Then there exists $x \neq \theta = (0, 0, 0, \dots)$ in c_0 such that $U(a; 0; b)x = \lambda x$. Then

$$\begin{aligned} a_0 x_0 + b_0 x_2 &= \lambda x_0 \\ a_1 x_1 + b_1 x_3 &= \lambda x_1 \\ a_2 x_2 + b_2 x_4 &= \lambda x_2 \\ a_0 x_3 + b_0 x_5 &= \lambda x_3 \\ a_1 x_4 + b_1 x_6 &= \lambda x_4 \\ &\vdots \end{aligned}$$

From above, we have

$$\begin{aligned}
 x_{6n} &= q^n x_0, \\
 x_{6n+1} &= q^n x_1, \\
 x_{6n+2} &= \frac{\lambda - a_0}{b_0} q^n x_0, \\
 x_{6n+3} &= \frac{\lambda - a_1}{b_1} q^n x_1, \\
 x_{6n+4} &= \frac{(\lambda - a_0)(\lambda - a_2)}{b_0 b_2} q^n x_0, \\
 x_{6n+5} &= \frac{(\lambda - a_0)(\lambda - a_1)}{b_0 b_1} q^n x_1
 \end{aligned}$$

where $n \geq 0$ and $q = \frac{(\lambda - a_0)(\lambda - a_1)(\lambda - a_2)}{b_0 b_1 b_2}$. Clearly, the subsequences (x_{6n+r}) , $r = 0, 5$ of $x = (x_n)$ are in c_0 if and only if $|\lambda - a_0||\lambda - a_1||\lambda - a_2| < |b_0||b_1||b_2|$ and hence, $x = (x_n) \in c_0$ if and only if $|\lambda - a_0||\lambda - a_1||\lambda - a_2| < |b_0||b_1||b_2|$. Therefore, $\sigma_p(U(a; 0; b), c_0) = \{\alpha \in \mathbb{C} : |\lambda - a_0||\lambda - a_1||\lambda - a_2| < |b_0||b_1||b_2|\}$. $\square$

We will use the following Lemma to find the adjoint of a linear transform on the sequence space c_0 .

Lemma 4 (p.266 [17]). *Let $T : c_0 \rightarrow c_0$ be a linear map and define $T^* : \ell_1 \rightarrow \ell_1$, by $T^*g = g \circ T$, $g \in c_0^* \cong \ell_1$, then T must be given with the matrix A , moreover, T^* must be given with the matrix A^t .*

Theorem 2. $\sigma_p(U(a; 0; b)^*, c_0^* \cong \ell_1) = \emptyset$.

Proof. From Lemma 4, it is clear that the matrix of $U(a; 0; b)^*$ is transpose of the matrix of $U(a; 0; b)$. Let λ be an eigenvalue of the operator $U(a; 0; b)^*$. Then there exists $x \neq \theta = (0, 0, 0, \dots)$ in ℓ_1 such that $U(a; 0; b)^*x = \lambda x$.

Then, we have

$$a_0 x_0 = \lambda x_0 \quad (2.2)$$

$$a_1 x_1 = \lambda x_1 \quad (2.3)$$

$$b_0 x_0 + a_2 x_2 = \lambda x_2 \quad (2.4)$$

$$b_1 x_1 + a_0 x_3 = \lambda x_3 \quad (2.5)$$

$$b_2 x_2 + a_1 x_4 = \lambda x_4 \quad (2.6)$$

$$b_0 x_3 + a_2 x_5 = \lambda x_5 \quad (2.7)$$

$$b_1 x_4 + a_0 x_6 = \lambda x_6 \quad (2.8)$$

$$b_2 x_5 + a_1 x_7 = \lambda x_7 \quad (2.9)$$

$$b_0 x_6 + a_2 x_8 = \lambda x_8 \quad (2.10)$$

$\vdots$

Then we have

$$n = 3k, \quad b_0x_n + a_2x_{n+2} = \lambda x_{n+2} \quad (2.11)$$

$$n = 3k + 1, \quad b_1x_n + a_0x_{n+2} = \lambda x_{n+2} \quad (2.12)$$

$$n = 3k + 2, \quad b_2x_n + a_1x_{n+2} = \lambda x_{n+2} \quad (2.13)$$

Let $x_0 \neq 0$ then we get $\lambda = a_0$ from (2.2), $x_1 = 0$ from (2.5), $x_4 = 0$ from (2.8), $x_2 = 0$ from (2.6) and $x_0 = 0$ from (2.4). But this contradicts with our assumption.

Now let $x_0 = 0$ and $x_1 \neq 0$ then we get $\lambda = a_1$ from (2.3), $x_2 = 0$ from (2.6), $x_5 = 0$ from (2.9), $x_3 = 0$ from (2.7), $x_1 = 0$ from (2.5). But this contradicts with our assumption.

Similarly let $x_0 = 0$, $x_1 = 0$ and $x_2 \neq 0$ then we get $\lambda = a_1$ from (2.4), $x_6 = 0$ from (2.10), $x_4 = 0$ from (2.8), $x_2 = 0$ from (2.6). But this contradicts with our assumption.

Finally, let x_{3k+1} be the first non-zero of the sequence (x_n) . If $n = 3k$, then from (2.11) we have $\lambda = a_2$. Again from (2.11) for $n = 3k + 3$ we have $b_0x_{3k+3} + a_2x_{3k+5} = a_2x_{3k+5}$, then we get $x_{3k+3} = 0$. But from (2.12) for $n = 3k + 1$ we have $b_1x_{3k+1} + a_0x_{3k+3} = a_2x_{3k+3}$, we have $x_{3k+1} = 0$, a contradiction.

Similarly, if x_{3k} or $x_{3k} + 2$ be the first non-zero of the sequence (x_n) we get a contradiction.

Hence, $\sigma_p(U(a; 0; b)^*, c_0^* \cong \ell_1) = \emptyset$. $\square$

Theorem 3. $\sigma_r(U(a; 0; b), c_0) = \emptyset$.

Proof. Since, $\sigma_r(A) = \sigma_p(A^*, \ell_1) \setminus \sigma_p(A, c_0)$, Theorems 1 and 2 give us required result. $\square$

Lemma 5.

$$\sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} a_k b_{nk} \right) = \sum_{k=0}^{\infty} a_k \left(\sum_{n=k+1}^{\infty} b_{nk} \right)$$

where (a_k) and (b_{nk}) are nonnegative real numbers.

Proof.

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} a_k b_{nk} \right) &= \sum_{k=0}^0 a_k b_{1k} + \sum_{k=0}^1 a_k b_{2k} + \sum_{k=0}^2 a_k b_{3k} + \sum_{k=0}^3 a_k b_{4k} + \cdots \\ &= a_0 b_{10} + (a_0 b_{20} + a_1 b_{21}) + (a_0 b_{30} + a_1 b_{31} + a_2 b_{32}) \\ &\quad + (a_0 b_{40} + a_1 b_{41} + a_2 b_{42} + a_3 b_{43}) + \cdots \\ &= a_0 \sum_{n=1}^{\infty} b_{n0} + a_1 \sum_{n=2}^{\infty} b_{n1} + a_2 \sum_{n=3}^{\infty} b_{n2} + \cdots \\ &= \sum_{k=0}^{\infty} a_k \left(\sum_{n=k+1}^{\infty} b_{nk} \right). \end{aligned}$$

□

Theorem 4. $\sigma_c(U(a; 0; b), c_0) = \{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| = |b_0| |b_1| |b_2|\}$ and $\sigma(U(a; 0; b), c_0) = \{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| \leq |b_0| |b_1| |b_2|\}$.

Proof. Let $y = (y_n) \in \ell_1$ be such that $(U(a; 0; b) - \lambda I)^* x = y$ for some $x = (x_n)$. Then we get system of linear equations:

$$\begin{aligned} (a_0 - \lambda)x_0 &= y_0 \\ (a_1 - \lambda)x_1 &= y_1 \\ b_0x_0 + (a_2 - \lambda)x_2 &= y_2 \\ b_1x_1 + (a_0 - \lambda)x_3 &= y_3 \\ b_2x_2 + (a_1 - \lambda)x_4 &= y_4 \\ b_0x_3 + (a_2 - \lambda)x_5 &= y_5 \\ &\vdots \\ b_0x_{3n} + (a_2 - \lambda)x_{3n+2} &= y_{3n+2} \\ b_1x_{3n+1} + (a_0 - \lambda)x_{3n+3} &= y_{3n+3} \\ b_2x_{3n+2} + (a_1 - \lambda)x_{3n+4} &= y_{3n+4} \\ &\vdots \end{aligned}$$

where $n \geq 0$. Solving these equations, we have

$$\begin{aligned} x_0 &= \frac{1}{a_0 - \lambda} y_0 \\ x_1 &= \frac{1}{a_1 - \lambda} y_1 \\ x_2 &= \frac{1}{a_2 - \lambda} y_2 - \frac{b_0}{(a_0 - \lambda)(a_2 - \lambda)} y_0 \\ x_3 &= \frac{1}{a_0 - \lambda} y_3 - \frac{b_1}{(a_0 - \lambda)(a_1 - \lambda)} y_1 \\ x_4 &= \frac{1}{a_1 - \lambda} y_4 - \frac{b_2}{(a_1 - \lambda)(a_2 - \lambda)} y_2 + \frac{b_0b_2}{(a_0 - \lambda)(a_1 - \lambda)(a_2 - \lambda)} y_0 \\ x_5 &= \frac{1}{a_2 - \lambda} y_5 - \frac{b_0}{(a_0 - \lambda)(a_2 - \lambda)} y_3 + \frac{b_0b_1}{(a_0 - \lambda)(a_1 - \lambda)(a_2 - \lambda)} y_1 \\ x_6 &= \frac{1}{a_0 - \lambda} y_6 - \frac{b_1}{(a_0 - \lambda)(a_1 - \lambda)} y_4 + \frac{b_1b_2}{(a_0 - \lambda)(a_1 - \lambda)(a_2 - \lambda)} y_2 \\ &\quad - \frac{b_0b_1b_2}{(a_0 - \lambda)^2(a_1 - \lambda)(a_2 - \lambda)} y_0 \end{aligned}$$

$$\begin{aligned}
x_7 &= \frac{1}{a_1 - \lambda} y_7 - \frac{b_2}{(a_1 - \lambda)(a_2 - \lambda)} y_5 + \frac{b_0 b_2}{(a_0 - \lambda)(a_1 - \lambda)(a_2 - \lambda)} y_3 \\
&\quad - \frac{b_0 b_1 b_2}{(a_0 - \lambda)(a_1 - \lambda)^2(a_2 - \lambda)} y_1 \\
&\quad \vdots
\end{aligned}$$

Thus we get

$$x_{2n+t} = \frac{1}{a_{2n+t} - \lambda} \left[y_{2n+t} + \sum_{k=0}^{n-1} (-1)^{n+k} y_{2k+t} \prod_{v=1}^{n-k} \frac{b_{2n-2v+t}}{a_{2n-2v+t} - \lambda} \right],$$

$t = 0, 1; n = 1, 2, \dots$ Herein $a_x = a_y, b_x = b_y$ for $x \equiv y \pmod{3}$. Therefore we get

$$\begin{aligned}
\sum_{v=0}^{\infty} |x_v| &= |x_0| + |x_1| + |x_2| + |x_3| + \dots \\
&= |x_0| + |x_1| + \sum_{n=1}^{\infty} |x_{2n+t}| \\
&= \left| \frac{y_0}{a_0 - \lambda} \right| + \left| \frac{y_1}{a_1 - \lambda} \right| \\
&\quad + \sum_{n=1}^{\infty} \left| \frac{1}{a_{2n+t} - \lambda} \left[y_{2n+t} + \sum_{k=0}^{n-1} (-1)^{n+k} y_{2k+t} \prod_{v=1}^{n-k} \frac{b_{2n-2v+t}}{a_{2n-2v+t} - \lambda} \right] \right| \\
&\leq \left| \frac{y_0}{a_0 - \lambda} \right| + \left| \frac{y_1}{a_1 - \lambda} \right| \\
&\quad + \frac{1}{|a_{2n+t} - \lambda|} \sum_{n=1}^{\infty} \left[|y_{2n+t}| + \sum_{k=0}^{n-1} |y_{2k+t}| \prod_{v=1}^{n-k} \left| \frac{b_{2n-2v+t}}{a_{2n-2v+t} - \lambda} \right| \right] \\
&= \left| \frac{y_0}{a_0 - \lambda} \right| + \left| \frac{y_1}{a_1 - \lambda} \right| + \frac{1}{|a_{2n+t} - \lambda|} \sum_{n=1}^{\infty} |y_{2n+t}| \\
&\quad + \frac{1}{|a_{2n+t} - \lambda|} \sum_{n=1}^{\infty} \left[\sum_{k=0}^{n-1} |y_{2k+t}| \prod_{v=1}^{n-k} \left| \frac{b_{2n-2v+t}}{a_{2n-2v+t} - \lambda} \right| \right].
\end{aligned}$$

Suppose $t = 0$ and consider the series $\sum_{n=1}^{\infty} \left[\sum_{k=0}^{n-1} |y_{2k}| \prod_{v=1}^{n-k} \left| \frac{b_{2n-2v}}{a_{2n-2v}-\lambda} \right| \right]$. In Lemma 5

if we take $a_k = |y_{2k}|$ and $b_{nk} = \prod_{v=1}^{n-k} \left| \frac{b_{2n-2v}}{a_{2n-2v}-\lambda} \right|$ then we have

$$\begin{aligned} \sum_{n=1}^{\infty} \left[\sum_{k=0}^{n-1} |y_{2k}| \prod_{v=1}^{n-k} \left| \frac{b_{2n-2v}}{a_{2n-2v}-\lambda} \right| \right] &= \sum_{k=0}^{\infty} \left[\sum_{n=k+1}^{\infty} |y_{2k}| \prod_{v=1}^{n-k} \left| \frac{b_{2n-2v}}{a_{2n-2v}-\lambda} \right| \right] \\ &= \sum_{k=0}^{\infty} \left[|y_{2k}| \sum_{n=k+1}^{\infty} \prod_{v=1}^{n-k} \left| \frac{b_{2n-2v}}{a_{2n-2v}-\lambda} \right| \right] \end{aligned}$$

Also since $\prod_{v=1}^{n-k} \left| \frac{b_{2n-2v}}{a_{2n-2v}-\lambda} \right| \sim M \left[\frac{b_2 b_1 b_0}{(a_2-\lambda)(a_1-\lambda)(a_0-\lambda)} \right]^{(n-k-1)/3}$ (M constant) as $n \rightarrow \infty$, the last equation turns into the series

$$\sum_{k=0}^{\infty} \left[|y_{2k}| \sum_{n=0}^{\infty} \left[\frac{b_2 b_1 b_0}{(a_2-\lambda)(a_1-\lambda)(a_0-\lambda)} \right]^{n/3} \right]. \quad (2.14)$$

Since $y = (y_n) \in \ell_1$, the series $\sum_{k=0}^{\infty} |y_{2k}|$ is convergent. Hence the series (2.14) is

convergent if and only if $\left| \frac{b_2 b_1 b_0}{(a_2-\lambda)(a_1-\lambda)(a_0-\lambda)} \right| < 1$. Consequently, if $\lambda \in \mathbb{C}$, $|a_2 - \lambda| |a_1 - \lambda| |a_0 - \lambda| > |b_2| |b_1| |b_0|$, then $(x_n) \in \ell_1$. Therefore, the operator $(U(a; 0; b) - \lambda I)^*$ is onto if $|\lambda - a_0| |\lambda - a_1| |\lambda - a_2| > |b_0| |b_1| |b_2|$. Then by Lemma 3 $U(a; 0; b) - \lambda I$ has a bounded inverse if $|\lambda - a_0| |\lambda - a_1| |\lambda - a_2| > |b_0| |b_1| |b_2|$.

So, $\sigma_c(U(a; 0; b), c_0) \subseteq \{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| \leq |b_0| |b_1| |b_2|\}$.

Since $\sigma(L, c_0)$ is the disjoint union of $\sigma_p(L, c_0)$, $\sigma_r(L, c_0)$ and $\sigma_c(L, c_0)$, therefore

$$\sigma(U(a; 0; b), c_0) \subseteq \{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| \leq |b_0| |b_1| |b_2|\}.$$

By Theorem 1, we get

$$\begin{aligned} \{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| < |b_0| |b_1| |b_2|\} \\ = \sigma_p(U(a; 0; b), c_0) \subset \sigma(U(a; 0; b), c_0) \end{aligned}$$

Since, $\sigma(L, c_0)$ is a compact set, so it is closed and thus,

$$\begin{aligned} \overline{\{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| < |b_0| |b_1| |b_2|\}} &\subset \overline{\sigma(U(a; 0; b), c_0)} \\ &= \sigma(U(a; 0; b), c_0) \end{aligned}$$

and $\{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| \leq |b_0| |b_1| |b_2|\} \subset \sigma(U(a; 0; b), c_0)$.

Hence, $\sigma(U(a; 0; b), c_0) = \{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| \leq |b_0| |b_1| |b_2|\}$ and so $\sigma_c(U(a; 0; b), c_0) = \{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| = |b_0| |b_1| |b_2|\}$. $\square$

Theorem 5. If $|\lambda - a_0| |\lambda - a_1| |\lambda - a_2| < |b_0| |b_1| |b_2|$, then $\lambda \in I_3 \sigma(U(a; 0; b), c_0)$.

Proof. Suppose $|\lambda - a_0||\lambda - a_1||\lambda - a_2| < |b_0||b_1||b_2|$ and so from Theorem 1, $\lambda \in \sigma_p(U(a_0, a_1, a_2; \cdot), c_0)$. Hence, λ satisfies Golberg's condition 3. We shall show that $U(a; 0; b) - \lambda I$ is onto when $|\lambda - a_0||\lambda - a_1||\lambda - a_2| < |b_0||b_1||b_2|$. Let $y = (y_n) \in c_0$ be such that $(U(a; 0; b) - \lambda I)x = y$ for $x = (x_n)$. Then,

$$\begin{aligned}(a_0 - \lambda)x_0 + b_0x_2 &= y_0 \\ (a_1 - \lambda)x_1 + b_1x_3 &= y_1 \\ (a_2 - \lambda)x_2 + b_2x_4 &= y_2 \\ (a_0 - \lambda)x_3 + b_0x_5 &= y_3 \\ (a_1 - \lambda)x_4 + b_1x_6 &= y_4 \\ (a_2 - \lambda)x_5 + b_2x_7 &= y_5 \\ (a_0 - \lambda)x_6 + b_0x_8 &= y_6 \\ &\vdots\end{aligned}$$

Calculating x_k , we get

$$\begin{aligned}x_2 &= \frac{1}{b_0}y_0 + \frac{\lambda - a_0}{b_0}x_0 \\ x_3 &= \frac{1}{b_1}y_1 + \frac{\lambda - a_1}{b_1}x_1 \\ x_4 &= \frac{1}{b_2}y_2 + \frac{\lambda - a_2}{b_0b_2}y_0 + \frac{(\lambda - a_0)(\lambda - a_2)}{b_0b_2}x_0 \\ x_5 &= \frac{1}{b_0}y_3 + \frac{\lambda - a_0}{b_0b_1}y_1 + \frac{(\lambda - a_0)(\lambda - a_1)}{b_0b_1}x_1 \\ x_6 &= \frac{1}{b_1}y_4 + \frac{\lambda - a_1}{b_1b_2}y_2 + \frac{(\lambda - a_1)(\lambda - a_2)}{b_0b_1b_2}y_0 + \frac{(\lambda - a_0)(\lambda - a_1)(\lambda - a_2)}{b_0b_1b_2}x_0 \\ x_7 &= \frac{1}{b_2}y_5 + \frac{\lambda - a_2}{b_0b_2}y_3 + \frac{(\lambda - a_0)(\lambda - a_2)}{b_0b_1b_2}y_1 + \frac{(\lambda - a_0)(\lambda - a_1)(\lambda - a_2)}{b_0b_1b_2}x_1 \\ x_8 &= \frac{1}{b_0}y_6 + \frac{\lambda - a_0}{b_0b_1}y_4 + \frac{(\lambda - a_0)(\lambda - a_1)}{b_0b_1b_2}y_2 + \frac{(\lambda - a_0)(\lambda - a_1)(\lambda - a_2)}{b_0^2b_1b_2}y_0 \\ &\quad + \frac{(\lambda - a_0)^2(\lambda - a_1)(\lambda - a_2)}{b_0^2b_1b_2}x_0 \\ &\vdots\end{aligned}$$

From above, we have

$$x_{2n+t} = \frac{1}{b_{2n+1+t}} \left[y_{2n-2+t} + \sum_{k=0}^{n-2} y_{2k+t} \prod_{v=1}^{n-k-1} \frac{\lambda - a_{2n-2v+t}}{b_{2n-2v+t+1}} \right]$$

$$+ x_t \prod_{v=1}^n \frac{\lambda - a_{2n-2v+t}}{b_{2n-2v+t}},$$

where $t = 0, 1; n = 2, 3, \dots$. Herein $a_x = a_y, b_x = b_y$ for $x \equiv y \pmod{3}$.

Since

$$\prod_{v=1}^n \frac{\lambda - a_{2n-2v+t}}{b_{2n-2v+t}} \sim M_1 \left[\frac{(a_0 - \lambda)(a_1 - \lambda)(a_2 - \lambda)}{b_0 b_1 b_2} \right]^{(n-1)/3} \text{ as } n \rightarrow \infty,$$

where M_1 is a constant, we have

$$\begin{aligned} x_{2n+t} \sim & \frac{1}{b_{2n+1+t}} \left[y_{2n-2+t} + \sum_{k=0}^{n-2} y_{2k+t} \prod_{v=1}^{n-k-1} \frac{\lambda - a_{2n-2v+t}}{b_{2n-2v+t+1}} \right] \\ & + x_t M_2 \left[\frac{(a_0 - \lambda)(a_1 - \lambda)(a_2 - \lambda)}{b_0 b_1 b_2} \right]^{(n-1)/3}, \end{aligned} \quad (2.15)$$

as $n \rightarrow \infty$. Since $y = (y_n) \in c_0$, from (2.15)

$$y = (y_n) \in c_0 \text{ iff } \left| \frac{(\lambda - a_0)(a_1 - \lambda)(a_2 - \lambda)}{b_0 b_1 b_2} \right| < 1.$$

Thus from (2.15), $(x_n) \in c_0$ iff $|\lambda - a_0| |\lambda - a_1| |\lambda - a_2| < |b_0| |b_1| |b_2|$. Therefore, $U(a; 0; b) - \lambda I$ is onto. So, $\lambda \in I$. Hence we get the required result. $\square$

3. SUBDIVISION OF THE SPECTRUM

The spectrum $\sigma(L, X)$ is partitioned into three sets which are not necessarily disjoint as follows:

If there exists a sequence (x_n) in X such that $\|x_n\| = 1$ and $\|Lx_n\| \rightarrow 0$ as $n \rightarrow \infty$ then (x_n) is called Weyl sequence for L .

We call the set

$$\sigma_{ap}(L, X) := \{\lambda \in \mathbb{C} : \text{there exists a Weyl sequence for } \lambda I - L\} \quad (3.1)$$

the approximate point spectrum of L . Moreover, the set

$$\sigma_g(L, X) := \{\lambda \in \sigma(L, X) : \lambda I - L \text{ is not surjective}\} \quad (3.2)$$

is called defect spectrum of L . Finally, the set

$$\sigma_{co}(L, X) = \{\lambda \in \mathbb{C} : \overline{R(\lambda I - L)} \neq X\} \quad (3.3)$$

is called compression spectrum in the literature.

The following Proposition is very useful for calculating the separation of the spectrum of linear operator in Banach spaces.

TABLE 1. Subdivisions of spectrum of a linear operator.

		1	2	3
		L_λ^{-1} exists and is bounded	L_λ^{-1} exists and is unbounded	L_λ^{-1} does not exists
I	$R(\lambda I - L) = X$	$\lambda \in \rho(L, X)$	–	$\lambda \in \sigma_p(L, X)$ $\lambda \in \sigma_{ap}(L, X)$
II	$\overline{R(\lambda I - L)} = X$	$\lambda \in \rho(L, X)$	$\lambda \in \sigma_c(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$	$\lambda \in \sigma_p(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$
III	$\overline{R(\lambda I - L)} \neq X$	$\lambda \in \sigma_r(L, X)$ $\lambda \in \sigma_\delta(L, X)$ $\lambda \in \sigma_{co}(L, X)$	$\lambda \in \sigma_r(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$ $\lambda \in \sigma_{co}(L, X)$	$\lambda \in \sigma_p(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$ $\lambda \in \sigma_{co}(L, X)$

Proposition 1 ([1], Proposition 1.3). *The spectra and subspectra of an operator $L \in B(X)$ and its adjoint $L^* \in B(X^*)$ are related by the following relations:*

- (a) $\sigma(L^*, X^*) = \sigma(L, X)$, (b) $\sigma_c(L^*, X^*) \subseteq \sigma_{ap}(L, X)$,
(c) $\sigma_{ap}(L^*, X^*) = \sigma_\delta(L, X)$, (d) $\sigma_\delta(L^*, X^*) = \sigma_{ap}(L, X)$,
(e) $\sigma_p(L^*, X^*) = \sigma_{co}(L, X)$, (f) $\sigma_{co}(L^*, X^*) \supseteq \sigma_p(L, X)$,
(g) $\sigma(L, X) = \sigma_{ap}(L, X) \cup \sigma_p(L^*, X^*) = \sigma_p(L, X) \cup \sigma_{ap}(L^*, X^*)$.

By the definitions given above, we can write following table

Many authors have examined spectral divisions of generalized difference matrices. For example, Paul and Tripathy, [9] have studied the spectrum of the operator $D(r, 0, 0, s)$ over the sequence spaces ℓ_p and bv_p .

The above-mentioned articles, concerned with the decomposition of spectrum defined by Goldberg. However, in [6] Durna and Yildirim have investigated subdivision of the spectra for factorable matrices on c_0 and in [2] Basar, Durna and Yildirim have investigated subdivisions of the spectra for generalized difference operator on the sequence spaces c_0 and c , in [4] Durna, have studied subdivision of the spectra for the generalized upper triangular double-band matrices Δ^{uv} over the sequence spaces c_0 and c and in [5] Durna, have studied subdivision of the spectra for the generalized difference operator $\Delta_{a,b}$ on the sequence space ℓ_p , ($1 < p < \infty$)

Corollary 2. $III_1\sigma(U(a; 0; b), c_0) = III_2\sigma(U(a; 0; b), c_0) = \emptyset$.

Proof. Since $\sigma_r(L, c_0) = III_1\sigma(L, c_0) \cup III_2\sigma(L, c_0)$ from Table 1, the required result is obtained from Theorem 3. $\square$

Corollary 3. $II_3\sigma(U(a; 0; b), c_0) = III_3\sigma(U(a; 0; b), c_0) = \emptyset$.

Proof. Since $\sigma_p(L, c_0) = I_3\sigma(L, c_0) \cup II I_3\sigma(L, c_0) \cup III I_3\sigma(L, c_0)$ from Table 1, the required result is obtained from Theorem 1 and Theorem 5. $\square$

Theorem 6. *The following statements are hold*

- (a) $\sigma_{ap}(U(a; 0; b), c_0) = \{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| \leq |b_0| |b_1| |b_2|\},$
- (b) $\sigma_\delta(U(a; 0; b), c_0) = \{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| = |b_0| |b_1| |b_2|\},$
- (c) $\sigma_{co}(U(a; 0; b), c_0) = \emptyset.$

Proof. (a) From Table 1, we get

$$\sigma_{ap}(U(a; 0; b), c_0) = \sigma(U(a; 0; b), c_0) \setminus III I_1\sigma(U(a; 0; b), c_0).$$

Hence $\sigma_{ap}(U(a; 0; b), c_0) = \{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| \leq |b_0| |b_1| |b_2|\}$ from Corollary 2.

(b) From Table 1, we have

$$\sigma_\delta(U(a; 0; b), c_0) = \sigma(U(a; 0; b), c_0) \setminus I_3\sigma(U(a; 0; b), c_0).$$

By using Theorem 4 and 5, we get the required result.

(c) By Proposition 1 (e), we have

$$\sigma_p(U^*(a; 0; b), c_0^*) = \sigma_{co}(U(a; 0; b), c_0).$$

From Theorem 2, we get the required result. $\square$

Corollary 4. *The following statements are hold*

- (a) $\sigma_{ap}(U(a; 0; b)^*, c_0^* \cong \ell_1) = \{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| = |b_0| |b_1| |b_2|\}$
- (b) $\sigma_\delta(U(a; 0; b)^*, c_0^* \cong \ell_1) = \{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| \leq |b_0| |b_1| |b_2|\}.$

Proof. Using Proposition 1 (c) and (d), we have

$$\sigma_{ap}(U(a; 0; b)^*, c_0^* \cong \ell_1) = \sigma_\delta(U(a; 0; b), c_0)$$

and

$$\sigma_\delta(U(a; 0; b)^*, c_0^* \cong \ell_1) = \sigma_{ap}(U(a; 0; b), c_0).$$

From Theorem 6 (a) and (b), we get the required results. $\square$

4. RESULTS

We can generalize our operator as follows.

$$U(a_0, a_1, \dots, a_{n-1}; 0; b_0, b_1, \dots, b_{n-1})$$

$$= \begin{bmatrix} a_0 & 0 & b_0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & a_1 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \ddots & 0 & \ddots & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & a_{n-1} & 0 & b_{n-1} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & a_0 & 0 & b_0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & a_1 & 0 & b_1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \dots \end{bmatrix} \quad (4.1)$$

where $b_0, b_1, \dots, b_{n-1} \neq 0$.

One can get parallel all our results obtained in before section as follows.

Theorem 7. *The following statements are provided where*

$S = \left\{ \lambda \in \mathbb{C} : \prod_{k=0}^{n-1} \left| \frac{\lambda - a_k}{b_k} \right| \leq 1 \right\}$, $\overset{\circ}{S}$ be the interior of the set S and ∂S be the boundary of the set S

- (1) $\sigma_p(U(a_0, a_1, \dots, a_{n-1}; 0; b_0, b_1, \dots, b_{n-1}), c_0) = \overset{\circ}{S}$,
- (2) $\sigma_p(U(a_0, a_1, \dots, a_{n-1}; 0; b_0, b_1, \dots, b_{n-1})^*, c_0^* \cong \ell_1) = \emptyset$,
- (3) $\sigma_r(U(a_0, a_1, \dots, a_{n-1}; 0; b_0, b_1, \dots, b_{n-1}), c_0) = \emptyset$,
- (4) $\sigma_c(U(a_0, a_1, \dots, a_{n-1}; 0; b_0, b_1, \dots, b_{n-1}), c_0) = \partial S$,
- (5) $\sigma(U(a_0, a_1, \dots, a_{n-1}; 0; b_0, b_1, \dots, b_{n-1}), c_0) = S$,
- (6) $I_3\sigma(U(a_0, a_1, \dots, a_{n-1}; 0; b_0, b_1, \dots, b_{n-1}), c_0) = \overset{\circ}{S}$,
- (7) $III_1\sigma(U(a_0, a_1, \dots, a_{n-1}; 0; b_0, b_1, \dots, b_{n-1}), c_0) = \emptyset$,
- (8) $III_2\sigma(U(a_0, a_1, \dots, a_{n-1}; 0; b_0, b_1, \dots, b_{n-1}), c_0) = \emptyset$,
- (9) $III_3\sigma(U(a_0, a_1, \dots, a_{n-1}; 0; b_0, b_1, \dots, b_{n-1}), c_0) = \emptyset$,
- (10) $II_3\sigma(U(a_0, a_1, \dots, a_{n-1}; 0; b_0, b_1, \dots, b_{n-1}), c_0) = \emptyset$,
- (11) $\sigma_{ap}(U(a_0, a_1, \dots, a_{n-1}; 0; b_0, b_1, \dots, b_{n-1}), c_0) = S$,
- (12) $\sigma_\delta(U(a_0, a_1, \dots, a_{n-1}; 0; b_0, b_1, \dots, b_{n-1}), c_0) = \partial S$,
- (13) $\sigma_{co}(U(a_0, a_1, \dots, a_{n-1}; 0; b_0, b_1, \dots, b_{n-1}), c_0) = \emptyset$,
- (14) $\sigma_{ap}(U(a_0, a_1, \dots, a_{n-1}; 0; b_0, b_1, \dots, b_{n-1})^*, c_0^* \cong \ell_1) = \partial S$,
- (15) $\sigma_\delta(U(a_0, a_1, \dots, a_{n-1}; 0; b_0, b_1, \dots, b_{n-1})^*, c_0^* \cong \ell_1) = S$.

REFERENCES

- [1] J. Appell, E. De Pascale, and A. Vignoli, *Nonlinear spectral theory*. Berlin: Walter de Gruyter & Co., 2004. doi: [10.1515/9783110199260](https://doi.org/10.1515/9783110199260).
- [2] F. Basar, N. Durna, and M. Yildirim, "Subdivisions of the spectra for generalized difference operator over certain sequence spaces," *Thai J. Math.*, vol. 9, no. 2, pp. 285–295, 2011.
- [3] R. Das and B. C. Tripathy, "Spectrum and fine spectrum of the lower triangular matrix $B(r, s, t)$ over the sequence space cs ," *Songklanakarin J. Sci. Technol.*, vol. 38, no. 3, pp. 265–274, 2016.
- [4] N. Durna, "Subdivision of the spectra for the generalized upper triangular double-band matrices Δ^{uv} over the sequence spaces c_0 and c ," *ADYU Sci.*, vol. 6, no. 1, pp. 31–43, 2016.
- [5] N. Durna, "Subdivision of the spectra for the generalized difference operator $\Delta_{a,b}$ on the sequence space ℓ_p , ($1 < p < \infty$)," *CBU J. Sci.*, vol. 13, no. 2, pp. 359–364, 2017, doi: [10.18466/cbayarfb.319876](https://doi.org/10.18466/cbayarfb.319876).

- [6] N. Durna and M. Yildirim, "Subdivision of the spectra for factorable matrices on c_0 ," *GUJ Sci.*, vol. 24, no. 1, pp. 45–49, 2011.
- [7] S. Goldberg, *Unbounded linear operators: Theory and applications*. McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966.
- [8] M. González, "The fine spectrum of the Cesàro operator in l_p ($1 < p < \infty$)," *Arch. Math. (Basel)*, vol. 44, no. 4, pp. 355–358, 1985, doi: [10.1007/BF01235779](https://doi.org/10.1007/BF01235779).
- [9] A. Paul and B. C. Tripathy, "The spectrum of the operator $D(r, 0, 0, s)$ over the sequence spaces ℓ_p and bv_p ," *Hacet. J. Math. Stat.*, vol. 43, no. 3, pp. 425–434, 2014.
- [10] A. Paul and B. C. Tripathy, "The spectrum of the operator $D(r, 0, 0, s)$ over the sequence space bv_0 ," *Georgian Math. J.*, vol. 22, no. 3, pp. 421–426, 2015.
- [11] J. B. Reade, "On the spectrum of the Cesàro operator," *Bull. London Math. Soc.*, vol. 17, no. 3, pp. 263–267, 1985, doi: [10.1112/blms/17.3.263](https://doi.org/10.1112/blms/17.3.263).
- [12] B. E. Rhoades, "The fine spectra for weighted mean operators," *Pacific J. Math.*, vol. 104, no. 1, pp. 219–230, 1983.
- [13] B. C. Tripathy and R. Das, "Spectra of the Rhaly operator on the sequence space $\overline{bv_0} \cap \ell_\infty$," *Bol. Soc. Parana. Mat. (3)*, vol. 32, no. 1, pp. 265–277, 2014.
- [14] B. C. Tripathy and R. Das, "Spectrum and fine spectrum of the upper triangular matrix $U(r, s)$ over the sequence space," *Proyecciones*, vol. 34, no. 2, pp. 107–125, 2015.
- [15] B. C. Tripathy and A. Paul, "The spectrum of the operator $D(r, 0, 0, s)$ over the sequence spaces c_0 and c ," *Kyungpook Math. J.*, vol. 53, no. 2, pp. 247–256, 2013.
- [16] B. C. Tripathy and P. Saikia, "On the spectrum of the Cesàro operator C_1 on $\overline{bv_0} \cap \ell_\infty$," *Math. Slovaca*, vol. 63, no. 3, pp. 563–572, 2013.
- [17] R. B. Wenger, "The fine spectra of the Hölder summability operators," *Indian J. Pure Appl. Math.*, vol. 6, no. 6, pp. 1–13, 1975.
- [18] A. Wilansky, *Summability through functional analysis*. North-Holland Publishing Co., Amsterdam, 1984.
- [19] M. Yildirim, "On the spectrum of the Rhaly operators on c_0 and c ," *Indian J. Pure Appl. Math.*, vol. 29, no. 12, pp. 1301–1309, 1998.
- [20] M. Yildirim and N. Durna, "The spectrum and some subdivisions of the spectrum of discrete generalized Cesàro operators on l_p ($1 < p < \infty$)," *J. Ineq. Appl.*, vol. 2017, no. 193, pp. 1–13, 2017, doi: [10.1186/s13660-017-1464-2](https://doi.org/10.1186/s13660-017-1464-2).

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ON RELATIVE COMMUTING PROBABILITY OF FINITE RINGS

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Abstract. In this paper we study the probability that the commutator of a randomly chosen pair of elements, one from a subring of a finite ring and other from the ring itself is equal to a given element of the ring.

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1. INTRODUCTION

Let S be a subring of a finite ring R . The relative commuting probability of S in R denoted by $\text{Pr}(S, R)$ is the probability that a randomly chosen pair of elements one from S and the other from R commute. That is

$$\text{Pr}(S, R) = \frac{|\{(x, y) \in S \times R : xy = yx\}|}{|S||R|}.$$

This ratio $\text{Pr}(S, R)$ can also be viewed as the probability that the commutator of a randomly chosen pair of elements, one from the subring S and the other from R , equals the zero of R . We write $[x, y]$ to denote the commutator $xy - yx$ of $x, y \in R$. The study of $\text{Pr}(S, R)$ was initiated in [2]. Note that $\text{Pr}(R, R)$, also denoted by $\text{Pr}(R)$, is the probability that a randomly chosen pair of elements of R commute. The ratio $\text{Pr}(R)$ is called the commuting probability of R and it was introduced by MacHale [6] in the year 1976. It is worth mentioning that the commuting probability of algebraic structures was originated from the works of Erdős and Turán [4] in the year 1968.

In this paper we consider the probability that the commutator of a randomly chosen pair of elements, one from the subring S and the other from R , equals a given element r of R . We write $\text{Pr}_r(S, R)$ to denote this probability. Therefore

$$\text{Pr}_r(S, R) = \frac{|\{(x, y) \in S \times R : [x, y] = r\}|}{|S||R|}. \quad (1.1)$$

Clearly $\text{Pr}_r(S, R) = 0$ if and only if $r \notin K(S, R) := \{[x, y] : x \in S, y \in R\}$. Therefore we consider r to be an element of $K(S, R)$ throughout the paper. Also $\text{Pr}_0(S, R) = \text{Pr}(S, R)$ where 0 is the zero of R . It may be mentioned here that the case when

$S = R$ is already considered in [3] by the authors. Interchanging S and R one may define $\text{Pr}_r(R, S)$ for $r \in R$.

The aim of this paper is to obtain some computing formulas and bounds for $\text{Pr}_r(S, R)$. We also discuss an invariance property of $\text{Pr}_r(S, R)$ under $\mathbb{Z}$ -isoclinism. The motivation of this paper lies in [7] where analogous generalization of commuting probability of finite group is studied.

We write $[S, R]$ and $[x, R]$ for $x \in S$ to denote the additive subgroups of $(R, +)$ generated by the sets $K(S, R)$ and $\{[x, y] : y \in R\}$ respectively. Note that $[x, R] = \{[x, y] : y \in R\}$. Let $Z(S, R) := \{x \in S : xy = yx \forall y \in R\}$. Then $Z(R) := Z(R, R)$ is the center of R . Further, if $r \in R$ then the set $C_S(r) := \{x \in S : xr = rx\}$ is a subring of S and $\bigcap_{r \in R} C_S(r) = Z(S, R)$. We write $\frac{R}{S}$ and $|R : S|$ to denote the additive quotient group and the index of S in R respectively.

2. COMPUTING FORMULA FOR $\text{Pr}_r(S, R)$

In this section, we derive some computing formulas for $\text{Pr}_r(S, R)$. We begin with the following useful lemmas.

Lemma 1 (Lemma 2.1 in [3]). *Let R be a finite ring. Then*

$$|[x, R]| = |R : C_R(x)| \text{ for all } x \in R.$$

Lemma 2. *Let S be a subring of a finite ring R and $T_{x,r}(S, R) = \{y \in R : [x, y] = r\}$ for $x \in S$ and $r \in R$. Then we have the followings*

- (1) $T_{x,r}(S, R) \neq \emptyset$ if and only if $r \in [x, R]$.
- (2) If $T_{x,r}(S, R) \neq \emptyset$ then $T_{x,r}(S, R) = t + C_R(x)$ for some $t \in T_{x,r}(S, R)$.

Proof. Part (1) follows from the fact that $y \in T_{x,r}(S, R)$ if and only if $r \in [x, R]$. Let $t \in T_{x,r}(S, R)$ and $p \in t + C_R(x)$. Then $[x, p] = r$ and so $p \in T_{x,r}(S, R)$. Therefore, $t + C_R(x) \subseteq T_{x,r}(S, R)$. Again, if $y \in T_{x,r}(S, R)$ then $(y - t) \in C_R(x)$ and so $y \in t + C_R(x)$. Therefore, $T_{x,r}(S, R) \subseteq t + C_R(x)$. Hence part (2) follows. $\square$

Now we state and prove the following main result of this section.

Theorem 1. *Let S be a subring of a finite ring R . Then*

$$\text{Pr}_r(S, R) = \frac{1}{|S||R|} \sum_{\substack{x \in S \\ r \in [x, R]}} |C_R(x)| = \frac{1}{|S|} \sum_{\substack{x \in S \\ r \in [x, R]}} \frac{1}{|[x, R]|}.$$

Proof. Note that $\{(x, y) \in S \times R : [x, y] = r\} = \bigcup_{x \in S} (\{x\} \times T_{x,r}(S, R))$. Therefore, by (1.1) and Lemma 2, we have

$$|S||R|\text{Pr}_r(S, R) = \sum_{x \in S} |T_{x,r}(S, R)| = \sum_{\substack{x \in S \\ r \in [x, R]}} |C_R(x)|. \quad (2.1)$$

The second part follows from (2.1) and Lemma 1. $\square$

Proposition 1. *Let S be a subring of a finite ring R and $r \in R$. Then $\Pr_r(S, R) = \Pr_{-r}(R, S)$. However, if $2r = 0$ then $\Pr_r(S, R) = \Pr_r(R, S)$.*

Proof. Let $X = \{(x, y) \in S \times R : [x, y] = r\}$ and $Y = \{(y, x) \in R \times S : [y, x] = -r\}$. It is easy to see that $(x, y) \mapsto (y, x)$ defines a bijective mapping from X to Y . Therefore, $|X| = |Y|$ and the result follows from (1.1).

Second part follows from the fact that $r = -r$ if $2r = 0$. $\square$

Proposition 2. *Let S_1 and S_2 be two subrings of the finite rings R_1 and R_2 respectively. If $(r_1, r_2) \in R_1 \times R_2$ then*

$$\Pr_{(r_1, r_2)}(S_1 \times S_2, R_1 \times R_2) = \Pr_{r_1}(S_1, R_1) \Pr_{r_2}(S_2, R_2).$$

Proof. Let $X_i = \{(x_i, y_i) \in S_i \times R_i : [x_i, y_i] = r_i\}$ for $i = 1, 2$ and

$$Y = \{((x_1, x_2), (y_1, y_2)) \in (S_1 \times S_2) \times (R_1 \times R_2) : [(x_1, x_2), (y_1, y_2)] = (r_1, r_2)\}.$$

Then $((x_1, y_1), (x_2, y_2)) \mapsto ((x_1, x_2), (y_1, y_2))$ defines a bijective map from $X_1 \times X_2$ to Y . Therefore, $|Y| = |X_1| |X_2|$ and hence the result follows from (1.1). $\square$

Using Proposition 1 in Theorem 1, we get the following corollary.

Corollary 1. *Let S be a subring of a finite ring R . Then*

$$\Pr(R, S) = \Pr(S, R) = \frac{1}{|S||R|} \sum_{x \in S} |C_R(x)| = \frac{1}{|S|} \sum_{x \in S} \frac{1}{|[x, R]|}.$$

We conclude this section with the following corollary.

Corollary 2. *Let S be a subring of a finite non-commutative ring R . If $|[S, R]| = p$, a prime, then*

$$\Pr_r(S, R) = \begin{cases} \frac{1}{p} \left(1 + \frac{p-1}{|S:Z(S, R)|} \right), & \text{if } r = 0 \\ \frac{1}{p} \left(1 - \frac{1}{|S:Z(S, R)|} \right), & \text{if } r \neq 0. \end{cases}$$

Proof. For $x \in S \setminus Z(S, R)$, we have $\{0\} \subsetneq [x, R] \subseteq [S, R]$. Since $|[S, R]| = p$, it follows that $[S, R] = [x, R]$ and hence $|[x, R]| = p$ for all $x \in S \setminus Z(S, R)$.

If $r = 0$ then by Corollary 1, we have

$$\begin{aligned} \Pr_r(S, R) &= \frac{1}{|S|} \left(|Z(S, R)| + \sum_{x \in S \setminus Z(S, R)} \frac{1}{|[x, R]|} \right) \\ &= \frac{1}{|S|} \left(|Z(S, R)| + \frac{1}{p} (|S| - |Z(S, R)|) \right) \\ &= \frac{1}{p} \left(1 + \frac{p-1}{|S:Z(S, R)|} \right). \end{aligned}$$

If $r \neq 0$ then $r \notin [x, R]$ for all $x \in Z(S, R)$ and $r \in [x, R]$ for all $x \in S \setminus Z(S, R)$. Therefore, by Theorem 1, we have

$$\begin{aligned} \Pr_r(S, R) &= \frac{1}{|S|} \sum_{x \in S \setminus Z(S, R)} \frac{1}{|[x, R]|} = \frac{1}{|S|} \sum_{x \in S \setminus Z(S, R)} \frac{1}{p} \\ &= \frac{1}{p} \left(1 - \frac{1}{|S : Z(S, R)|} \right). \end{aligned}$$

Hence, the result follows. $\square$

3. BOUNDS FOR $\Pr_r(S, R)$

If S is a subring of a finite ring R then it was shown in [2, Theorem 2.16] that

$$\Pr(S, R) \geq \frac{1}{|K(S, R)|} \left(1 + \frac{|K(S, R)| - 1}{|S : Z(S, R)|} \right). \quad (3.1)$$

Also, if p is the smallest prime dividing $|R|$ then by [2, Theorem 2.5] and [2, Corollary 2.6] we have

$$\Pr(S, R) \leq \frac{(p-1)|Z(S, R)| + |S|}{p|S|} \text{ and } \Pr(R) \leq \frac{(p-1)|Z(R)| + |R|}{p|R|}. \quad (3.2)$$

In this section, we obtain several bounds for $\Pr_r(S, R)$ and show that some of our bounds are better than the bounds given in (3.1) and (3.2). We begin with the following upper bound.

Proposition 3. *Let S be a subring of a finite ring R . If p is the smallest prime dividing $|R|$ and $r \neq 0$ then*

$$\Pr_r(S, R) \leq \frac{|S| - |Z(S, R)|}{p|S|} < \frac{1}{p}.$$

Proof. Since $r \neq 0$ we have $S \neq Z(S, R)$. If $x \in Z(S, R)$ then $r \notin [x, R]$. If $x \in S \setminus Z(S, R)$ then $C_R(x) \neq R$. Therefore, by Lemma 1, we have $|[x, R]| = |R : C_R(x)| > 1$. Since p is the smallest prime dividing $|R|$ we have $|[x, R]| \geq p$. Hence the result follows from Theorem 1. $\square$

Proposition 4. *Let S be a subring of a finite ring R . Then $\Pr_r(S, R) \leq \Pr(S, R)$ with equality if and only if $r = 0$.*

Proof. By Theorem 1 and Corollary 1, we have

$$\Pr_r(S, R) = \frac{1}{|S||R|} \sum_{\substack{x \in S \\ r \in [x, R]}} |C_R(x)| \leq \frac{1}{|S||R|} \sum_{x \in S} |C_R(x)| = \Pr(S, R).$$

The equality holds if and only if $r = 0$. $\square$

Proposition 5. *If $S_1 \subseteq S_2$ are two subrings of a finite ring R then*

$$\Pr_r(S_1, R) \leq |S_2 : S_1| \Pr_r(S_2, R).$$

Proof. By Theorem 1, we have

$$\begin{aligned} |S_1| |R| \Pr_r(S_1, R) &= \sum_{\substack{x \in S_1 \\ r \in [x, R]}} |C_R(x)| \\ &\leq \sum_{\substack{x \in S_2 \\ r \in [x, R]}} |C_R(x)| = |S_2| |R| \Pr_r(S_2, R). \end{aligned}$$

Hence the result follows. $\square$

Note that equality holds in Proposition 5 if and only if $r \notin [x, R]$ for all $x \in S_2 \setminus S_1$. If $r = 0$ then the condition of equality reduces to $S_1 = S_2$. Putting $S_1 = S$ and $S_2 = R$ in Proposition 5 we have the following corollary.

Corollary 3. *If S is a subring of a finite ring R then*

$$\Pr_r(S, R) \leq |R : S| \Pr_r(R).$$

For any subring S of R , let $m_S = \min\{|[x, R]| : x \in S \setminus Z(S, R)\}$ and $M_S = \max\{|[x, R]| : x \in S \setminus Z(S, R)\}$. In the following theorem we give bounds for $\Pr(S, R)$ in terms of m_S and M_S .

Theorem 2. *Let S be a subring of a finite ring R . Then*

$$\frac{1}{M_S} \left(1 + \frac{M_S - 1}{|S : Z(S, R)|} \right) \leq \Pr(S, R) \leq \frac{1}{m_S} \left(1 + \frac{m_S - 1}{|S : Z(S, R)|} \right).$$

The equality holds if and only if $m_S = M_S = |[x, R]|$ for all $x \in S \setminus Z(S, R)$.

Proof. Since $m_S \leq |[x, R]|$ and $M_S \geq |[x, R]|$ for all $x \in S \setminus Z(S, R)$, we have

$$\frac{|S| - |Z(S, R)|}{M_S} \leq \sum_{x \in S \setminus Z(S, R)} \frac{1}{|[x, R]|} \leq \frac{|S| - |Z(S, R)|}{m_S}. \quad (3.3)$$

Again, by Corollary 1, we have

$$\Pr(S, R) = \frac{1}{|S|} \left(|Z(S, R)| + \sum_{x \in S \setminus Z(S, R)} \frac{1}{|[x, R]|} \right). \quad (3.4)$$

Hence, the result follows from (3.3) and (3.4). $\square$

Note that for any two integers $m \geq n$, we have

$$\frac{1}{n} \left(1 + \frac{n - 1}{|S : Z(S, R)|} \right) \geq \frac{1}{m} \left(1 + \frac{m - 1}{|S : Z(S, R)|} \right). \quad (3.5)$$

Clearly equality holds in (3.5) if $Z(S, R) = S$. Further, if $Z(S, R) \neq S$ then equality holds if and only if $m = n$. Since $|K(S, R)| \geq M_S$, by (3.5), it follows that

$$\frac{1}{M_S} \left(1 + \frac{M_S - 1}{|S : Z(S, R)|} \right) \geq \frac{1}{|K(S, R)|} \left(1 + \frac{|K(S, R)| - 1}{|S : Z(S, R)|} \right).$$

Therefore, the lower bound obtained in Theorem 2 is better than the lower bound given in (3.1) for $\text{Pr}(S, R)$. Again, if p is the smallest prime divisor of $|R|$ then $p \leq m_S$ and hence, by (3.5), we have

$$\frac{1}{m_S} \left(1 + \frac{m_S - 1}{|S : Z(S, R)|} \right) \leq \frac{(p-1)|Z(S, R)| + |S|}{p|S|}.$$

This shows that the upper bound obtained in Theorem 2 is better than the upper bound given in (3.2) for $\text{Pr}(S, R)$.

Putting $S = R$ in Theorem 2 we have the following corollary.

Corollary 4. *Let R be a finite ring. Then*

$$\frac{1}{M_R} \left(1 + \frac{M_R - 1}{|R : Z(R)|} \right) \leq \text{Pr}(R) \leq \frac{1}{m_R} \left(1 + \frac{m_R - 1}{|R : Z(R)|} \right).$$

The equality holds if and only if $m_R = M_R = |[x, R]|$ for all $x \in R \setminus Z(R)$.

We conclude this section noting that the lower bound obtained in Corollary 4 is better than the lower bound obtained in [2, Corollary 2.18]. Also, if p is the smallest prime divisor of $|R|$ then the upper bound obtained in Corollary 4 is better than the upper bound given in (3.2) for $\text{Pr}(R)$.

4. $\mathbb{Z}$ -ISOCLINISM AND $\text{Pr}_r(S, R)$

The idea of isoclinism of groups was introduced by Hall [5] in 1940. Years after in 2013, Buckley et al. [1] introduced $\mathbb{Z}$ -isoclinism of rings. Recently, Dutta et al. [2] have introduced $\mathbb{Z}$ -isoclinism between two pairs of rings, generalizing the notion of $\mathbb{Z}$ -isoclinism of rings. Let S_1 and S_2 be two subrings of the rings R_1 and R_2 respectively. Recall that a pair of mappings (α, β) is called a $\mathbb{Z}$ -isoclinism between (S_1, R_1) and (S_2, R_2) if $\alpha : \frac{R_1}{Z(S_1, R_1)} \rightarrow \frac{R_2}{Z(S_2, R_2)}$ and $\beta : [S_1, R_1] \rightarrow [S_2, R_2]$ are additive group isomorphisms such that $\alpha \left(\frac{S_1}{Z(S_1, R_1)} \right) = \frac{S_2}{Z(S_2, R_2)}$ and $\beta([x_1, y_1]) = [x_2, y_2]$ whenever $x_i \in S_i, y_i \in R_i$ for $i = 1, 2$; $\alpha(x_1 + Z(S_1, R_1)) = x_2 + Z(S_2, R_2)$ and $\alpha(y_1 + Z(S_1, R_1)) = y_2 + Z(S_2, R_2)$. Two pairs of rings are said to be $\mathbb{Z}$ -isoclinic if there exists a $\mathbb{Z}$ -isoclinism between them.

In [2, Theorem 3.3], Dutta et al. proved that $\text{Pr}(S_1, R_1) = \text{Pr}(S_2, R_2)$ if the rings R_1 and R_2 are finite and the pairs (S_1, R_1) and (S_2, R_2) are $\mathbb{Z}$ -isoclinic. We conclude this paper with the following generalization of [2, Theorem 3.3].

Theorem 3. Let S_1 and S_2 be two subrings of the finite rings R_1 and R_2 respectively. If (α, β) is a $\mathbb{Z}$ -isoclinism between (S_1, R_1) and (S_2, R_2) then

$$\Pr_r(S_1, R_1) = \Pr_{\beta(r)}(S_2, R_2).$$

Proof. By Theorem 1, we have

$$\Pr_r(S_1, R_1) = \frac{|Z(S_1, R_1)|}{|S_1||R_1|} \sum_{\substack{x_1 + Z(S_1, R_1) \in \frac{S_1}{Z(S_1, R_1)} \\ r \in [x_1, R_1]}} |C_{R_1}(x_1)|$$

noting that $r \in [x_1, R_1]$ if and only if $r \in [x_1 + z, R_1]$ and $C_{R_1}(x_1) = C_{R_1}(x_1 + z)$ for all $z \in Z(S_1, R_1)$. Now, by Lemma 1, we have

$$\Pr_r(S_1, R_1) = \frac{|Z(S_1, R_1)|}{|S_1|} \sum_{\substack{x_1 + Z(S_1, R_1) \in \frac{S_1}{Z(S_1, R_1)} \\ r \in [x_1, R_1]}} \frac{1}{|[x_1, R_1]|}. \quad (4.1)$$

Similarly, it can be seen that

$$\Pr_{\beta(r)}(S_2, R_2) = \frac{|Z(S_2, R_2)|}{|S_2|} \sum_{\substack{x_2 + Z(S_2, R_2) \in \frac{S_2}{Z(S_2, R_2)} \\ \beta(r) \in [x_2, R_2]}} \frac{1}{|[x_2, R_2]|}. \quad (4.2)$$

Since (α, β) is a $\mathbb{Z}$ -isoclinism between (S_1, R_1) and (S_2, R_2) we have $\frac{|S_1|}{|Z(S_1, R_1)|} = \frac{|S_2|}{|Z(S_2, R_2)|}$, $|[x_1, R_1]| = |[x_2, R_2]|$ and $r \in [x_1, R_1]$ if and only if $\beta(r) \in [x_2, R_2]$. Hence, the result follows from (4.1) and (4.2). $\square$

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REFERENCES

- [1] S. M. Buckley, D. MacHale, and A. Ní Shé, “Finite rings with many commuting pairs of elements.” *Preprint*.
- [2] J. Dutta, D. K. Basnet, and R. K. Nath, “On commuting probability of finite rings.” *Indag. Math.*, vol. 28, no. 2, pp. 372–382, 2017, doi: [10.1016/j.indag.2016.10.002](https://doi.org/10.1016/j.indag.2016.10.002).
- [3] P. Dutta and R. K. Nath, “On r -commuting probability of finite rings.” *Preprint*.
- [4] P. Erdős and P. Turán, “On some problems of a statistical group-theory. IV.” *Acta. Math. Acad. Sci. Hungar.*, vol. 19, pp. 413–435, 1968, doi: [10.1007/BF01894517](https://doi.org/10.1007/BF01894517).
- [5] P. Hall, “The classification of prime power groups.” *J. Reine Angew. Math.*, vol. 182, pp. 130–141, 1940, doi: [10.1515/crll.1940.182.130](https://doi.org/10.1515/crll.1940.182.130).
- [6] D. MacHale, “Commutativity in Finite Rings.” *Amer. Math. Monthly*, vol. 83, pp. 30–32, 1976, doi: [10.1080/00029890.1976.11994032](https://doi.org/10.1080/00029890.1976.11994032).

- [7] R. K. Nath and M. K. Yadav, “Some results on relative commutativity degree.” *Rend. Circ. Mat. Palermo*, vol. 64, no. 2, pp. 229–239, 2015, doi: [10.1007/s12215-015-0194-x](https://doi.org/10.1007/s12215-015-0194-x).

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SEIBERG–WITTEN–LIKE EQUATIONS ON THE STRICTLY–PSEUDOCONVEX CR 7–MANIFOLDS

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Abstract. In this paper, Seiberg–Witten–like equations are constructed on 7–manifolds endowed with G_2 –structure, lifted by $SU(3)$ –structure. Then a global solution is obtained on the strictly–Pseudoconvex CR 7–manifolds for a given negative and constant scalar curvature.

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1. INTRODUCTION

The exceptional Lie group G_2 is the automorphisms group of the octonion algebra $\mathbb{O}$ which is a subgroup of $SO(7)$. A manifold whose structure group is G_2 is called a G_2 –manifold. G_2 –manifolds have been studied in terms of the covariant derivation of the fundamental 3-form and the parallelism of this form with respect to the Levi–Civita connection [5, 12, 14]. In addition to this, compact G_2 manifolds are currently being studied [4, 11, 17–19].

A 7–manifold M equipped with G_2 –structure is a Riemannian manifold whose structure group is a reduction of the tangent bundle from $Gl(7, \mathbb{R})$ to the subgroup G_2 , which is also a subgroup of $SO(7)$. This implies that the 7–dimensional manifold equipped with G_2 –structure is an orientable Riemannian manifold. Also G_2 –structure on the 7–manifolds determines a non–degenerate global three form Φ on M and G_2 –structure is the stabiliser of Φ . The action of G_2 on the tangent bundle induces an action of G_2 on $\Lambda^2(M)$ and gives the following orthogonal decomposition of $\Lambda^2(M)$:

$$\Lambda^2(M) = \Lambda_7^2(M) \oplus \Lambda_{14}^2(M)$$

where

$$\Lambda_7^2(M) = \{\beta \in \Lambda^2(M) \mid *(\beta \wedge \Phi) = -2\beta\},$$

$$\Lambda_{14}^2(M) = \{\beta \in \Lambda^2(M) \mid *(\beta \wedge \Phi) = \beta\}$$

and $*$ is the Hodge star operator [3]. These two decompositions are used to define self–duality and anti self–duality concept on G_2 –manifolds [9].

On a 6-manifold N equipped with $SU(3)$ -structure, $SU(3)$ -structure acts on the tangent bundle and thus it induces an action of $SU(3)$ -structure on the space of two-forms $\Lambda^2(N)$. According to this $SU(3)$ -structure is the stabiliser in $SO(6)$ of a non-degenerate 2-form ω and a normalized 3-form Ψ . Then, by using ω one can obtain the following decomposition:

$$\Lambda^2(N) = \Lambda_1^2(N) \oplus \Lambda_6^2(N) \oplus \Lambda_8^2(N)$$

where

$$\begin{aligned}\Lambda_1^2(N) &= \{\beta \in \Lambda^2(N) \mid *(\beta \wedge \omega) = 2\beta\}, \\ \Lambda_6^2(N) &= \{\beta \in \Lambda^2(N) \mid *(\beta \wedge \omega) = \beta\}, \\ \Lambda_8^2(N) &= \{\beta \in \Lambda^2(N) \mid *(\beta \wedge \omega) = -\beta\}.\end{aligned}$$

Let N be a subset of M endowed with $SU(3)$ -structure. The relation between $SU(3)$ and G_2 -structure is given by the inclusion $SU(3) \subset G_2$ -structure. This inclusion is characterized by the orthogonal decomposition

$$\mathbb{R}^7 = \mathbb{R}^6 \oplus \alpha \mathbb{R} \quad (1.1)$$

where $\alpha = e^7$ annihilates $\mathbb{R}^6$ at each point.

Then, a non-degenerate 3-form Φ , determined by G_2 -structure on a 7-manifold M , is described by

$$\Phi = \omega \wedge \alpha + \Psi_+ \quad (1.2)$$

where Ψ_+ is the real part of a normalized 3-form Ψ [6, 15, 22]. This implies that $\omega \wedge \alpha$ determines $SU(3)$ -structure on G_2 -manifolds [6]. In the following, Seiberg–Witten equations are briefly reminded.

Seiberg–Witten equations were defined firstly by Witten on any smooth 4-manifold [23]. The solutions of these equations play an important role in the topology of 4-manifolds. Later on, Seiberg–Witten equations have been investigated in higher dimensional manifolds by several authors [7, 9, 16]. In 7-dimension, Seiberg–Witten equations are defined on the manifolds equipped with G_2 -structure by Degirmenci and Ozdemir[9]. In their study they gave a local non-trivial solution to these equations on $\mathbb{R}^7$. In this paper we extend this solution to a global one on the strictly pseudoconvex CR 7-manifolds for a given negative and constant scalar curvature. Since G_2 -structure is lifted by $SU(3)$ -structure, it has a non degenerate 2-form which is the stabilizer of $SU(3)$ -structure. According to this, if wedge product of α is taken by the stabilizer of $SU(3)$ -structure, one gets 3-form which is also stabilizer of $SU(3)$. By using this 3-form, one can decompose the space of 2-form. According to this decomposition, self-duality concept can be defined.

This paper is organized as follows. At first, some basic facts concerning $SU(3)$ -structures contained in G_2 -structure is introduced. In section 2, the space of two-forms $\Omega^2(M)$ is decomposed by considering induced $SU(3)$ -structure. Then the

space of self–dual two-forms is defined. In section 3, Seiberg–Witten–like equations is defined on the 7–manifold endowed with G_2 –structure lifted by an $SU(3)$ –structure. Finally, we give a global solution to these equations on the strictly-Pseudoconvex CR 7–manifolds for a given negative and constant scalar curvature.

2. $SU(3)$ –STRUCTURE ON 7–DIMENSIONAL MANIFOLDS

Let us consider $\mathbb{R}^7$ with a basis $\{e_1, \dots, e_7\}$ and its metric dual $\{e^1, \dots, e^7\}$. An inclusion of $SU(3)$ –structure into G_2 –structure is defined and characterised by the orthogonal decomposition $\mathbb{R}^7 = \mathbb{R}^6 \oplus \alpha\mathbb{R}$ where α annihilates $\mathbb{R}^6$ at each point.

Definition 1. On the 7–manifold M , an $SU(3)$ –structure is a triple $(\alpha, \omega, \Psi) \in \Omega^1(M) \times \Omega^2(M) \times \Omega^3(M, \mathbb{C})$ with model tensor

$$(\alpha, \omega, \Psi) := (e^7, e^{12} + e^{34} + e^{56}, e_{\mathbb{C}}^1 \wedge e_{\mathbb{C}}^2 \wedge e_{\mathbb{C}}^3) \in (\mathbb{R}^7)^* \times \Lambda^2(\mathbb{R}^7)^* \times \Lambda^3(\mathbb{R}^7)^*$$

where $e_{\mathbb{C}}^j := e^{2j-1} - ie^{2j}$ for $j = 1, \dots, 3$ and $e^{ij} = e^i \wedge e^j$.

By setting $\Psi_+ := \operatorname{Re}(\Psi)$ and $\Psi_- := \operatorname{Im}(\Psi)$, the complex–valued $(3, 0)$ form Ψ can be written as $\Psi := \Psi_+ + i\Psi_-$ [6].

On the 6–dimensional manifold N , an $SU(3)$ –structure (α, ω, Ψ) can be lifted to G_2 –structure, which is the holonomy group of the 7–dimensional manifold M , as follows [15]:

$$\Phi = \omega \wedge \alpha + \Psi_+.$$

According to this, there is a natural 6–dimensional distribution $H := TN = \operatorname{Ker} \alpha$ and complementary 1–dimensional distribution $\operatorname{Ker} \omega$. Moreover, the Reeb vector field ξ of (α, ω, Ψ) $SU(3)$ –structure is the section of the vector bundle $H \subset TM$ with $\alpha(\xi) = 1$.

Then we have an almost Hermitian structure (g, J_H) on H with respect to $SU(3)$ –structure. Since $J_H^2 = -I_d$, the following eigenspaces decomposition can be given by:

$$\Lambda_H^1(M) = H \otimes_{\mathbb{R}} \mathbb{C} = \Lambda_H^{1,0}(M) \oplus \Lambda_H^{0,1}(M)$$

where

$$\begin{aligned} \Lambda_H^{1,0}(M) &= \{Z \in H \otimes_{\mathbb{R}} \mathbb{C} \mid J_H Z = iZ\}, \\ \Lambda_H^{0,1}(M) &= \{Z \in H \otimes_{\mathbb{R}} \mathbb{C} \mid J_H Z = -iZ\}. \end{aligned}$$

The complexification of $\Lambda_H^s(M)$ is decomposed as follows

$$\Lambda_H^s(M) = \sum_{q+r=s} \Lambda_H^{q,r}(M),$$

where $\Lambda^{q,r}(M)_H = \operatorname{span}\{u \wedge v \mid u \in \Lambda^q(\Lambda_H^{1,0}(M)), v \in \Lambda^r(\Lambda_H^{0,1}(M))\}$.

The endomorphism map J_H on M induces an endomorphism on $\Lambda_H^s(M)$ and satisfies the identity $J_H^2 = (-1)^r I_d$. The natural action of J_H on 2-form θ is given by

$$J_H \theta(V, W) = \theta(J_H V, J_H W).$$

Then, the following is obtained

$$\begin{aligned}\Lambda_H^{1,1}(M) &= \{\theta \in \Lambda_H^2(M) | J_H \theta = \theta\}, \\ \Lambda_H^{2,0}(M) \oplus \Lambda_H^{0,2}(M) &= \{\theta \in \Lambda_H^2(M) | J_H \theta = -\theta\}.\end{aligned}$$

Since $(H, d\alpha|_H)$ is a symplectic vector bundle equipped with an almost complex structure J_H on M , an almost contact structure can be defined by extending J_H to an endomorphism J of the tangent bundle TM by setting $J\xi = 0$. In that case, an almost contact structure on TM is given as

$$J^2 = -Id + \alpha \otimes \xi.$$

Moreover, A contact manifold (M, α) endowed with an almost contact structure can be endowed by the Riemannian metric g_α on TM such that

$$g_\alpha(V, W) = d\alpha(V, JW) + \alpha(V)\alpha(W)$$

for any $V, W \in \Gamma(TM)$.

After that we denoted contact metric manifold by $(M, g_\alpha, \alpha, J, \xi)$. On the contact metric manifold $(M, g_\alpha, \alpha, J, \xi)$, the generalized Webster-Tanaka connection is given by :

$$\nabla_V^T W = \nabla_V W - (\nabla_V \alpha)(W)\xi - \alpha(V)\nabla_W \xi - \alpha(V)\alpha(W),$$

where ∇ is the Levi-Civita connection and $V, W \in \chi(M)$ [21]. Webster-Tanaka connection satisfies the condition $\nabla^T W \alpha = 0$ and $\nabla^T W g_\alpha = 0$. Also, if $\nabla^T W J = 0$, then $(M, g_\alpha, \alpha, J, \xi)$ is called strictly pseudoconvex CR manifold [20].

3. SELF-DUAL 2-FORMS ON THE CONTACT METRIC MANIFOLDS OF DIMENSION 7

Let $(M, g_\alpha, \alpha, J, \xi)$ be a 7-dimensional contact metric manifold endowed with G_2 -structure which is lifted by $SU(3)$ -structure. Then any 2-form $\eta \in \Omega^2(M)$ splits into $\eta = \eta_H + \eta_\xi$, where $\eta_H = \eta \circ \pi$, $\pi : TM \rightarrow H$ is the canonical projection and $\eta_\xi = \eta \wedge \iota(\xi)\eta$ where ι is the contraction operator. In addition, if $\iota(\xi)\eta = 0$, then η is called a horizontal 2-form. Also, $\Omega^2(M)$ can be decomposed with respect to the bundles of horizontal forms $\Omega_H^2(M)$ and $\Omega_H^1(M)$, as [10]

$$\Omega^2(M) = \Omega_H^2(M) \oplus \alpha \wedge \Omega_H^1(M).$$

Let $\Phi = \omega \wedge \alpha + \Psi_+$ be a fundamental 3-form induced by $SU(3)$ -structure whose stabilizer is G_2 . In an orthonormal basis $\{e_i\} i = 1, \dots, 7$, the fundamental 3-form Φ is described as

$$\Phi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}$$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$.

Φ defines T_Φ duality operator on G_2 –manifold as follows

$$\begin{aligned} T_\Phi : \Omega^2(TM) &\longrightarrow \Omega^2(TM) \\ \beta &\longmapsto T_\Phi(\beta) := *(\Phi \wedge \beta) \end{aligned}$$

with 7 and 14 dimensional eigenspaces corresponding to eigenvalues 2 and -1 , respectively [3].

Let consider the 3–form $\Phi'_\omega = \omega \wedge \alpha$ which is the stabilizer of the $SU(3)$ –structure contained in G_2 –structure. By considering induced $SU(3)$ structure corresponding with 3–form $\Phi'_\omega = \omega \wedge \alpha$ one can obtain another decomposition of $\Omega^2(M)$ [6]. In an orthonormal basis $\{e_i\}$, $i = 1, \dots, 7$, the fundamental 3–form Φ'_ω can be written as:

$$\Phi'_\omega = e^{127} + e^{347} + e^{567}.$$

Also Φ'_ω defines $T_{\Phi'}$ duality operator on 2–forms as

$$\begin{aligned} T_{\Phi'} : \Omega^2(TM) &\longrightarrow \Omega^2(TM) \\ \beta &\longmapsto T_{\Phi'}(\beta) := *(\Phi' \wedge \beta) \end{aligned}$$

with 1, 6, 6 and 8 dimensional eigenspaces corresponding to eigenvalues 2, 1, 0 and -1 , respectively. A basis consisting of the corresponding eigenvalues is given below:

$$\Omega^2(M) = \Omega_H^2(M) \oplus \alpha \wedge \Omega_H^1(M).$$

Eigenvector associated with the eigenvalue 2:

$$\omega = e^1 \wedge e^3 + e^3 \wedge e^4 + e^5 \wedge e^6. \quad (3.1)$$

Eigenvectors associated with the eigenvalue 1:

$$\begin{aligned} a_1 &= -e^1 \wedge e^3 + e^2 \wedge e^4 & a_2 &= e^1 \wedge e^4 + e^2 \wedge e^3 \\ a_3 &= -e^1 \wedge e^5 + e^2 \wedge e^6 & a_4 &= e^1 \wedge e^6 + e^2 \wedge e^5 \\ a_5 &= -e^3 \wedge e^5 + e^4 \wedge e^6 & a_6 &= e^3 \wedge e^6 + e^4 \wedge e^5. \end{aligned}$$

Eigenvectors associated with the eigenvalue -1 :

$$\begin{aligned} b_1 &= -e^1 \wedge e^2 + e^3 \wedge e^4 & b_2 &= -e^1 \wedge e^2 + e^5 \wedge e^6 \\ b_3 &= e^1 \wedge e^3 + e^2 \wedge e^4 & b_4 &= -e^1 \wedge e^4 + e^2 \wedge e^3 \\ b_5 &= e^1 \wedge e^5 + e^2 \wedge e^6 & b_6 &= -e^1 \wedge e^6 + e^2 \wedge e^5 \\ b_7 &= e^3 \wedge e^5 + e^4 \wedge e^6 & b_8 &= -e^3 \wedge e^6 + e^4 \wedge e^5 \end{aligned}$$

Eigenvectors associated with the eigenvalue 0:

$$\begin{aligned} c_1 &= e^1 \wedge e^7 & c_2 &= e^2 \wedge e^7 \\ c_3 &= e^3 \wedge e^7 & c_4 &= e^4 \wedge e^7 \\ c_5 &= e^5 \wedge e^7 & c_6 &= e^6 \wedge e^7 \end{aligned}$$

Considering the natural action of $SU(3)$ -structure on the space of *two*-forms $\Omega_H^2(M)$, the following orthogonal eigenspace decomposition is obtained [2].

$$\Omega_H^2(M) = \Omega_H^{2,1}(M) \oplus \Omega_H^{2,6}(M) \oplus \Omega_H^{2,8}(M)$$

where

$$\begin{aligned} \Omega_H^{2,1}(M) &= \{k\omega : k \in \mathbb{R}\}, \\ \Omega_H^{2,6}(M) &= \{\theta \in \Omega_H^2(M) : J\theta = -\theta\}, \\ \Omega_H^{2,8}(M) &= \{\theta \in \Omega_H^2(M) : J\theta = \theta \text{ and } \theta \wedge \omega \wedge \omega = 0\}. \end{aligned}$$

By complexifying the space of *two*-forms $\Omega_H^2(M)$, we get the following:

$$\Omega_H^2(M) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}\omega \oplus (\Omega_H^{2,6}(M) \otimes_{\mathbb{R}} \mathbb{C}) \oplus \Omega_H^{2,8}(M) \otimes_{\mathbb{R}} \mathbb{C}.$$

The space $\Omega_H^2(M)^+ = \mathbb{C}\omega \oplus (\Omega_H^{2,6}(M) \otimes_{\mathbb{R}} \mathbb{C})$ is called as the space of self-dual *two*-forms. Similarly, the space $\Omega_H^2(M)^-$ is called the space of anti self-dual *two*-forms [24]. Locally, we can express the space of self-dual 2-forms relative to Φ'_ω by $\{\omega, a_1, a_2, a_3, a_4, a_5, a_6\}$.

4. DIRAC OPERATOR ON THE CONTACT METRIC MANIFOLDS

In this section, we talk about the canonical $Spin^c$ -structure of a contact metric manifold and its spinor bundle with associated connection.

Contact metric manifold is defined by a contact distribution and with its complementary. Since contact distribution has an almost-hermitian structure, the results of it can be extended to a contact metric manifold. Since the structure group of any contact metric manifold of dimension $2n + 1$ is $U(n)$, it admits a canonical $Spin^c$ -structure given by:

$$P_{Spin^c(n)} = P_{U(n)} \times_F Spin^c(n)$$

where $F : U(n) \longrightarrow Spin^c(2n)$ is the lifting map [13, 20]. The associated canonical spinor bundle then has the form:

$$\mathbb{S}^{\mathbb{C}} \cong \Omega^{0,*}(M).$$

where $\Omega^{0,*}(M)$ is the direct sum of $\Omega(M)^{0,1} \oplus \Omega(M)^{0,2} \oplus \dots \oplus \Omega(M)^{0,i}$, $i \in \mathbb{N}$. Also, on this spinor bundle, the Clifford multiplication is given by:

$$V \cdot \psi = \sqrt{2} \left((V_H^{0,1})^* \wedge \psi - \iota(V_H^{0,1})\psi \right) + i(-1)^{\deg \psi + 1} \eta(V)\psi. \quad (4.1)$$

where V_H denotes the horizontal part of V . According to these multiplication one can easily obtain $\xi\psi = i(-1)^{\deg\psi+1}\psi$.

As in the almost-Hermitian case, given a metric-connection called Levi-Civita ∇ on TM , there are two ways to include a connection on $\mathbb{S}$:

The first of these is obtained by the extension of the connection to forms and the latter is obtained via $Spin^c$ -structure. In this work, we mainly focused on the canonical $Spin^c$ -structure with the following isomorphism:

$$\mathbb{S}^{\mathbb{C}} \cong \Omega_H^{0,*}(M).$$

On this bundle, we described Dirac operator defined on $\mathbb{S}$ and we give the relation with the Dirac-type operator defined on $\Omega_H^{0,*}(M)$.

In the case of contact metric manifold endowed with a canonical $Spin^c$ structure, there is a spinorial connection ∇^A on the associated spinor bundle $\mathbb{S}^{\mathbb{C}}$ induced by an unitary connection 1-form A on the determinant line bundle L together with the generalized Webster-Tanaka connection ∇^{TW} . Also, on the associated spinor bundle one can describe Dirac operator as follows:

Let $\{e_i\}$ $i = 1, \dots, 2n$ be a local orthonormal frame on H . Then the Kohn-Dirac operator D_H^A is given by:

$$D_H^A = \sum_{i=1}^{2n} e_i \cdot \nabla_{e_i}^A. \quad (4.2)$$

Hence, Dirac operator on the $2n + 1$ dimensional contact metric manifold is [20]:

$$D^A = D_H^A + \xi \cdot \nabla_{\xi}^A. \quad (4.3)$$

Moreover, by considering strictly Pseudoconvex CR manifolds with $\Omega_H^{0,*}(M)$ associated spinor bundle the Dirac type operator is defined as follows

Let

$$\bar{\partial}_H : \Omega_H^{0,r}(M) \longrightarrow \Omega_H^{0,r+1}(M), \quad \bar{\partial}_H^* : \Omega_H^{0,r}(M) \longrightarrow \Omega_H^{0,r-1} \quad (4.4)$$

respectively given by:

$$\bar{\partial}_H = \sum_{i=1}^n \bar{Z}_i^* \wedge \nabla_{\bar{Z}_i}^{TW}, \quad \bar{\partial}_H^* = - \sum_{i=1}^n \iota(\bar{Z}_i)^* \wedge \nabla_{\bar{Z}_i}^{TW}$$

where ∇^{TW} is the extension of the generalized Webster-Tanaka connection to $\Omega_H^{0,*}(M)$ and ι is the contraction operator.

It follows from (4.1) that we have on $\Omega_M^{0,*}(\bar{M})$

$$\mathcal{H} = \sqrt{2} \sum_{r=0}^n (\bar{\partial}_H + \bar{\partial}_H^*) + \sum_{r=0}^n (-1)^{r+1} \sqrt{-1} \cdot \nabla_{\xi}^{TW}. \quad (4.5)$$

Since $\mathbb{S}^{\mathbb{C}} \cong \Omega_H^{0,*}(M)$, (4.3) coincides with (4.5). In this paper we consider the following spinor representation $\kappa : \mathbb{R}^7 \rightarrow \mathbb{C}(8)$:

$$\begin{aligned}\kappa(e_1) &= m_4 \otimes m_1 \otimes m_3, & \kappa(e_3) &= -m_1 \otimes m_3 \otimes m_3, \\ \kappa(e_5) &= -m_3 \otimes m_3 \otimes m_3, & \kappa(e_2) &= I \otimes I \otimes m_2, \\ \kappa(e_4) &= m_4 \otimes m_2 \otimes m_3, & \kappa(e_6) &= -m_2 \otimes m_3 \otimes m_3, \\ \kappa(e_7) &= I \otimes I \otimes m_1,\end{aligned}$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, m_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, m_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, m_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, m_4 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}.$$

In 7–dimension, Seiberg–Witten equations are described by the Dirac equation and Curvature equation. Although, Dirac equation has a common definition on any smooth manifold endowed with $Spin^c$ –structure, the definition of the curvature equation shows some difference with respect to the chosen self–duality concept. In this paper we use the definition given in [9].

An imaginary valued 2–form $\sigma(\psi)$ given by

$$\sigma(\psi)(V, W) = \langle V \cdot W \cdot \psi, \psi \rangle + \langle V, W \rangle |\psi|^2$$

where $V, W \in \Gamma(TM)$ and $\langle \cdot, \cdot \rangle$ is the Hermitian inner product on the spinor bundle $\mathbb{S}^c$. The restriction of $\sigma(\psi)$ to H is denoted by $\sigma_H(\psi) := \sigma(\psi)|_H$.

Definition 2. Let M be the 7–manifold endowed with G_2 –structure, lifted by $SU(3)$ –structure. For any unitary connection 1–form A and spinor field $\psi \in \Gamma(S)$, the Seiberg–Witten equations are defined by:

$$\begin{aligned}D_A \psi &= 0, \\ F_A^+ &= \frac{1}{4} \sigma(\psi)^+\end{aligned}\tag{4.6}$$

where F_A^+ is the self–dual part of the curvature F_A and $\sigma(\psi)^+$ the self–dual part of the 2–form $\sigma(\psi)$ corresponding with the spinor field $\psi \in \Gamma(S)$.

In the following, the method applied by Ş. Bulut in order to give a global solution is used [8].

5. GLOBAL SOLUTION TO THE SEIBERG–WITTEN–LIKE EQUATIONS ON THE STRICTLY–PSEUDOCONVEX CR 7–MANIFOLDS

Let $(M, g_\alpha, \alpha, J, \xi)$ be a strictly–Pseudoconvex CR 7–manifold endowed with a canonical $Spin^c$ –structure and $\{e_1, e_2 = J(e_1), e_3, e_4 = J(e_3), e_5, e_6 = J(e_5), \xi\}$ be a local frame with dual basis $\{e^1, e^2, e^3, e^4, e^5, e^6, \alpha\}$. The spinor bundle $\mathbb{S}^{\mathbb{C}}$ decomposes into eigensubbundles under the action $d\alpha = e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6$,

$$\mathbb{S}^{\mathbb{C}} \cong \Lambda_H^{0,0}(M) \oplus \Lambda_H^{0,1}(M) \oplus \Lambda_H^{0,2}(M) \oplus \Lambda_H^{0,3}(M).$$

Each $\Lambda_H^{0,0}(M)$, $\Lambda_H^{0,1}(M)$, $\Lambda_H^{0,2}(M)$, $\Lambda_H^{0,3}(M)$ is associated with the eigenvalue $-3i, -i, i, 3i$ with dimension 1, 3, 3, 1. Also $\mathbb{S}^{\mathbb{C}}$ can be described as [8]

$$\mathbb{S}^{\mathbb{C}} = \mathbb{S}^{\mathbb{C},+} \oplus \mathbb{S}^{\mathbb{C},-}$$

where

$$\begin{aligned}\mathbb{S}^{\mathbb{C},+} &\cong \Lambda_H^{0,0}(M) \oplus \Lambda_H^{0,2}(M), \\ \mathbb{S}^{\mathbb{C},-} &\cong \Lambda_H^{0,1}(M) \oplus \Lambda_H^{0,3}(M).\end{aligned}$$

This gives the following isomorphisms

$$\begin{aligned}\Lambda_H^{0,0}(M) \oplus \Lambda_H^{0,2}(M) &\cong S_i^{\mathbb{C}} \oplus S_{-3i}^{\mathbb{C}}, \\ \Lambda_H^{0,1}(M) \oplus \Lambda_H^{0,3}(M) &\cong S_{-i}^{\mathbb{C}} \oplus S_{3i}^{\mathbb{C}}.\end{aligned}$$

where $S_i^{\mathbb{C}} = \{\psi \in \Gamma(S), \omega \cdot \psi = i\psi\}$. Let ψ_0 be the spinor in $S_{-3i}^{\mathbb{C}} \cong \Lambda_H^{0,0}(M)$ corresponding to constant function 1, in the chosen coordinates

$$\psi_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

As a result we get $\sigma_H(\psi_0) = -i d\alpha$.

Theorem 1. *Let $(M, g_\alpha, \alpha, J, \xi)$ be a 7-dimensional strictly Pseudoconvex CR manifold. Then, for a given negative and constant scalar curvature s_H , $(A, \psi = \sqrt{-\frac{2}{3}s_H}\psi_0)$ is a solution of the Seiberg–Witten–like equations.*

Proof. Since $\psi = \sqrt{-\frac{2}{3}s_H}\psi_0 \in \Lambda^{0,0}(M)$ and the spinor ψ_0 is a spinor corresponding to the constant function 1, we get $\mathcal{H} + \sum_{q=0}^n (-1)^{q+1} \sqrt{-1} \cdot \nabla \xi^T W = 0$. This means $D_A \psi = 0$. Then, only satisfying the second equation is left. The relation between a curvature of the connection 1-form A and a Ricci form ρ_{ric}^H is given as:

$$F_A = Ric = i\rho_{ric}^H$$

where the unitary connection 1-form A induced by means of $\nabla^T W$ in the line bundle $K = \Omega_H^{0,n}(M)$ [1]. Then, by using the definition of the Ricci form ρ_{ric}^H given by

$$\rho_{ric}^H(V, W) = Ric(V, J_H W) = g(V, J_H \circ Ric W)$$

for any $V, W \in \Gamma(TM)$, one gets

$$\begin{aligned} \rho_{ric}^H = & -R_{11}e_1 \wedge e_2 + R_{14}(e_1 \wedge e_3 + e_2 \wedge e_4) + R_{13}(e_2 \wedge e_3 - e_1 \wedge e_4) \\ & - R_{33}e_3 \wedge e_4 - R_{26}(e_1 \wedge e_5 + e_2 \wedge e_6) + R_{15}(-e_1 \wedge e_6 + e_2 \wedge e_5) \\ & + R_{36}(e_3 \wedge e_5 + e_4 \wedge e_6) + R_{35}(-e_3 \wedge e_6 + e_4 \wedge e_5) - R_{55}e_5 \wedge e_6. \end{aligned} \quad (5.1)$$

Eliminating anti-self dual 2-form in (5.1), one has self-dual part of ρ_{ric}^H as follows

$$\begin{aligned} \rho_{ric}^{H,+} &= \frac{-R_{11} - R_{33} - R_{55}}{3} d\alpha = -\left(\frac{R_{11} + R_{22} + R_{33} + R_{44} + R_{55} + R_{66}}{3} \right) d\alpha \\ &= -\frac{s}{6} d\alpha. \end{aligned}$$

The following is obtained

$$F_A^+ = Ric^+ = i\rho_{ric}^{H,+} = -i\frac{s_H}{6} d\alpha = \frac{1}{4}\sigma_H(\Psi) = \frac{1}{4}\sigma_H^+(\Psi) = \frac{1}{4}\sigma^+(\psi).$$

As a consequence the pair $(A, \psi = \sqrt{-\frac{2}{3}s_H}\psi_0)$ is a solution of the Seiberg–Witten like equations in (4.6). $\square$

REFERENCES

- [1] H. Baum, “Lorentzian twistor spinors and cr-geometry,” *arXiv preprint math/9803090*, 1998.
- [2] L. Bedulli and L. Vezzoni, “The ricci tensor of su (3)-manifolds,” *Journal of Geometry and Physics*, vol. 57, no. 4, pp. 1125–1146, 2007.
- [3] R. L. Bryant, “Metrics with exceptional holonomy,” *Annals of mathematics*, pp. 525–576, 1987.
- [4] R. L. Bryant, “Some remarks on G_2 -structures.” in *Proceedings of the 11th and 12th Gökova geometry-topology conferences, Gökova, Turkey, May 24–28, 2004 and May 30–June 3, 2005. Dedicated to the memory of Raoul Bott*. Cambridge, MA: International Press, 2006, pp. 75–109.
- [5] R. L. Bryant, S. M. Salamon *et al.*, “On the construction of some complete metrics with exceptional holonomy,” *Duke mathematical journal*, vol. 58, no. 3, pp. 829–850, 1989.
- [6] S. Chiossi and S. Salamon, “The intrinsic torsion of su (3) and g 2 structures,” in *Differential Geometry, Valencia 2001*. World Scientific, 2002, pp. 115–133.
- [7] E. Corrigan, C. Devchand, D. B. Fairlie, and J. Nuyts, “First-order equations for gauge fields in spaces of dimension greater than four,” *Nuclear Physics B*, vol. 214, no. 3, pp. 452–464, 1983.
- [8] N. Değirmenci and Senay Bulut, “Seiberg-Witten-like equations on 6-dimensional $SU(3)$ -manifolds.” *Balkan J. Geom. Appl.*, vol. 20, no. 2, pp. 23–31, 2015.
- [9] N. Değirmenci and N. Özdemir, “Seiberg-Witten-like equations on 7-manifolds with G_2 -structure.” *J. Nonlinear Math. Phys.*, vol. 12, no. 4, pp. 457–461, 2005, doi: [10.2991/jnmp.2005.12.4.1](https://doi.org/10.2991/jnmp.2005.12.4.1).
- [10] H. Fan, “Half de Rham complexes and line fields on odd-dimensional manifolds.” *Trans. Am. Math. Soc.*, vol. 348, no. 8, pp. 2947–2982, 1996, doi: [10.1090/S0002-9947-96-01661-3](https://doi.org/10.1090/S0002-9947-96-01661-3).

- [11] M. Fernández *et al.*, “An example of a compact calibrated manifold associated with the exceptional lie group g_2 ,” *Journal of Differential Geometry*, vol. 26, no. 2, pp. 367–370, 1987.
- [12] M. Fernández and A. Gray, “Riemannian manifolds with structure group g_2 ,” *Annali di matematica pura ed applicata*, vol. 132, no. 1, pp. 19–45, 1982.
- [13] T. Friedrich, *Dirac operators in Riemannian geometry*. American Mathematical Soc., 2000, vol. 25.
- [14] A. Gray, “Vector cross products on manifolds,” *Transactions of the American Mathematical Society*, vol. 141, pp. 465–504, 1969.
- [15] N. Hitchin, “Stable forms and special metrics,” *Contemporary Mathematics*, vol. 288, pp. 70–89, 2001.
- [16] A. Hü, T. Dereli, Ş. Koçak *et al.*, “Monopole equations on 8-manifolds with spin (7) holonomy,” *Communications in mathematical physics*, vol. 203, no. 1, pp. 21–30, 1999.
- [17] D. Joyce, “Compact riemannian 7-manifolds with holonomy g_2 . i,” *Journal of differential geometry*, vol. 43, pp. 291–328, 1996.
- [18] D. D. Joyce, *Compact manifolds with special holonomy*. Oxford University Press on Demand, 2000.
- [19] A. Kovalev, “Twisted connected sums and special Riemannian holonomy,” *J. Reine Angew. Math.*, vol. 565, pp. 125–160, 2003, doi: [10.1515/crll.2003.097](https://doi.org/10.1515/crll.2003.097).
- [20] R. Petit, “Spinc-structures and dirac operators on contact manifolds,” vol. 22, no. 2, pp. 229–252, 2005.
- [21] S. Tanno, “Variational problems on contact riemannian manifolds,” *Transactions of the American Mathematical society*, vol. 314, no. 1, pp. 349–379, 1989.
- [22] F. Witt, “Special metric structures and closed forms,” *arXiv preprint math/0502443*, 2005.
- [23] E. Witten, “Monopoles and four-manifolds,” *arXiv preprint hep-th/9411102*, 1994.
- [24] F. Xu, “On instantons on nearly kahler 6-manifolds,” *Asian Journal of Mathematics*, vol. 13, no. 4, pp. 535–568, 2009.

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APPROXIMATION LATTICES DEFINED BY TOLERANCES INDUCED BY IRREDUNDANT COVERINGS

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Abstract. The topic of rough set theory considers a relation to determine the lower and upper approximations of a set X . Originally, this relation was assumed to be an equivalence relation. This research focuses on using tolerance relations instead of equivalences, i.e. we do not assume the transitivity of the relations. More specifically, in this paper we investigate tolerances induced by irredundant coverings. We characterize the interrelation between the lattices of lower and upper approximations of such tolerances R and ρ . The theory of Formal Concept Analysis makes it possible to examine the inclusions of the resulting concepts. We also use quasiorders (denoted by $\preceq(\rho)$ and $\succeq(\rho)$) and an equivalence relation (denoted by $\ker\rho$) for summarizing the connection between tolerances and lattices in a theorem.

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1. INTRODUCTION

The notion of rough sets was introduced by Z. Pawlak [8]. His idea was that our knowledge about the elements of a universe U is given in terms of an information relation $R \subseteq U \times U$ reflecting their indiscernibility. Originally, Pawlak assumed that this binary relation is an equivalence, but later several other types of relations were also examined (see e.g. [4, 12], or [6]). For any binary relation $R \subseteq U \times U$ and any element $u \in U$, denote by $R(u)$ the R -neighbourhood of u , i.e. $R(u) := \{x \in U \mid (u, x) \in R\}$. Now, for any subset $X \subseteq U$ the *lower approximation* of X is defined as

$$X_R := \{x \in U \mid R(x) \subseteq X\},$$

and the *upper approximation* of X is given by

$$X^R := \{x \in U \mid R(x) \cap X \neq \emptyset\}.$$

If R is a reflexive relation then $X_R \subseteq X \subseteq X^R$. The *rough set* of X is the pair (X_R, X^R) , and the *set of all rough sets* is

$$RS(U, R) = \{(X_R, X^R) \mid X \subseteq U\}.$$

The set $RS(U, R)$ may be canonically ordered by the component-wise inclusion:

$$(X_R, X^R) \leq (Y_R, Y^R) \iff X_R \subseteq Y_R \text{ and } X^R \subseteq Y^R,$$

obtaining a partially ordered set $\mathbf{RS}(U, R) := (RS(U, R), \leq)$. If R is an equivalence, then $\mathbf{RS}(U, R)$ is a particular complete distributive lattice.

Ordering the sets $\wp(U)^R = \{X^R \mid X \subseteq U\}$ and $\wp(U)_R = \{X_R \mid X \subseteq U\}$ by the relation $\subseteq$ we obtain dually isomorphic complete lattices $(\wp(U)^R, \subseteq)$ and $(\wp(U)_R, \subseteq)$, called the *lattice of upper approximations*, respectively the *lattice of lower approximations* (see [6]). Let R be a *tolerance*, that is, a reflexive and symmetric relation. In [6] it was shown that $(\wp(U)_R, \subseteq)$ is isomorphic to the concept lattice of the context (U, U, R^c) , where $R^c = (U \times U) \setminus R$ is the complement of the relation R . By using this observation, in [2], we applied FCA methods to describe the sublattices of the lattices of upper (lower) approximations. These lattices play an important role in several applications of rough set theory (see [3, 9–11, 13]).

This paper can be considered as a continuation of [2] where we deduced sufficient conditions which guarantee that for some tolerances $R \subseteq \rho \subseteq U \times U$, the lattice $\wp(U)^\rho$ ($\wp(U)_\rho$) is a complete sublattice of $\wp(U)^R$ (of $\wp(U)_R$). The focus of this paper is on the approximation lattices defined by tolerances induced by irredundant coverings of U . These relations can be considered as a natural generalization of equivalences. If $R \subseteq \rho \subseteq U \times U$ are tolerance relations and R is induced by an irredundant covering of U , we characterize the case when the concept lattice $\mathcal{L}(U, U, \rho^c)$ is a complete sublattice of the concept lattice $\mathcal{L}(U, U, R^c)$. Then this characterization is applied to compare the lattices $(\wp(U)^R, \subseteq)$ and $(\wp(U)_R, \subseteq)$.

2. PRELIMINARIES

First, we note that the above defined approximations for any $X \subseteq U$ and any $\mathcal{H} \subseteq \mathcal{P}(U)$ have the following properties:

- (a) $\left(\bigcup_{X \in \mathcal{H}} X \right)^R = \bigcup_{X \in \mathcal{H}} X^R$ and $\left(\bigcap_{X \in \mathcal{H}} X \right)_R = \bigcap_{X \in \mathcal{H}} X_R$;
- (b) $(X^c)^R = (X_R)^c$, $(X^c)_R = (X^R)^c$.

In view of (a), $\wp(U)_R$ is a *closure system*, being closed under arbitrary intersections and $\wp(U)^R$ is an *interior system*, because it is closed under any union. Therefore, $\wp(U)_R$ and $\wp(U)^R$ are complete lattices with respect to $\subseteq$. If R is a tolerance relation, then for any $X, Y \subseteq U$ we have: $X^R \subseteq Y \iff X \subseteq Y_R$.

Property (b) implies that the lattices $(\wp(U)_R, \subseteq)$ and $(\wp(U)^R, \subseteq)$ are dually isomorphic via the map $H: \wp(U)_R \rightarrow \wp(U)^R$, $H: X \rightarrow X^c$, since $H(X_R) = (X_R)^c = (X^c)^R$. If R is an equivalence, then $\wp(U)_R = \wp(U)^R$ and they form the same Boolean lattice.

A *formal context* is a triple $\mathcal{K} = (G, M, I)$, where G is a set of *objects*, M is a set of *attributes* and $I \subseteq G \times M$ is a relation, called *incidence relation*. The notations

$(g, m) \in I$ and gIm both express that an object g is in relation I with an attribute m . The basics of Formal Concept Analysis (FCA) can be found e.g. in [1]. By defining for all subsets $A \subseteq G$ and $B \subseteq M$

$$A^I = \{m \in M \mid (g, m) \in I, \text{ for all } g \in A\},$$

$$B^I = \{g \in G \mid (g, m) \in I, \text{ for all } m \in B\}$$

we establish a Galois connection between the power-set lattices $\wp(G)$ and $\wp(M)$ and the maps $A \rightarrow A^{II}$, $A \subseteq G$ and $B \rightarrow B^{II}$, $B \subseteq M$ are closure operators on $\wp(G)$, respectively $\wp(M)$.

A *formal concept* of the context $\mathcal{K}$ is a pair $(A, B) \in \wp(G) \times \wp(M)$ with $A^I = B$ and $B^I = A$, where the set A is called the *extent* and B is called the *intent* of the concept (A, B) . It is easy to check that a pair $(A, B) \in \wp(G) \times \wp(M)$ is a concept if and only if $(A, B) = (A^{II}, A^I) = (B^I, B^{II})$. The set of all concepts of the context $\mathcal{K}$ is denoted by $\mathcal{L}(\mathcal{K})$. This set $\mathcal{L}(\mathcal{K})$ is ordered by

$$(A_1, B_1) \leq (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2 \Leftrightarrow B_1 \supseteq B_2.$$

With respect to this order, $\mathcal{L}(\mathcal{K})$ forms a complete lattice, called the *concept lattice of the context* $\mathcal{K} = (G, M, I)$, denoted by $\mathcal{L}(G, M, I)$.

A relation $J \subseteq I$ is called a *closed subrelation* of the context (G, M, I) if every concept of the context (G, M, J) is also a concept of (G, M, I) . In [1] it is proved that this definition is equivalent to the condition that the concept lattice $\mathcal{L}(G, M, J)$ is a complete sublattice of $\mathcal{L}(G, M, I)$.

For a tolerance relation $R \subseteq U \times U$ the relationship between the lattices of approximations and the concept lattice $\mathcal{L}(U, U, R^c)$ was described in [6]. Indeed, let $I = R^c$. Then for any $X \subseteq U$ we have

$$X^I = \{u \in U \mid xR^cu, \text{ for all } x \in X\}$$

$$= \{u \in U \mid (x, u) \notin R, \text{ for all } x \in X\} = U \setminus X^R = (X^R)^c.$$

Thus $X^R = (X^I)^c$, and $X_R = ((X^c)^R)^c = (X^c)^I$, according to (b).

In [6] it is also proved that $(\wp(U)^R, \subseteq) \cong (\wp(U)_R, \supseteq) \cong \mathcal{L}(U, U, R^c)$.

3. COMPLETE SUBLATTICES OF APPROXIMATION LATTICES

Now let ρ, R be two tolerance relations such that $R \subseteq \rho \subseteq U \times U$. Consider the formal contexts $\mathcal{K}_R = (U, U, R^c)$ and $\mathcal{K}_\rho = (U, U, \rho^c)$. Since $J := \rho^c \subseteq R^c := I$, $\mathcal{K}_\rho$ is a subcontext of $\mathcal{K}_R$. We intend to characterize the case when the lattice $\wp(U)^\rho$ ($\wp(U)_\rho$) is isomorphic (dually isomorphic) to a complete sublattice of $\wp(U)^R$ ($\wp(U)_R$, respectively). In [2] we proved that $(\wp(U)^\rho, \subseteq)$ is a complete sublattice of $(\wp(U)^R, \subseteq)$, respectively $(\wp(U)_\rho, \subseteq)$ is a complete sublattice of $(\wp(U)_R, \subseteq)$, whenever $\mathcal{L}(U, U, \rho^c)$ is a complete sublattice of $\mathcal{L}(U, U, R^c)$. Unfortunately, the

converse implication does not necessarily hold. For instance, in [2] we constructed an example where $(\wp(U)^\rho, \subseteq)$ is a complete sublattice of $(\wp(U)^R, \subseteq)$, however $\mathcal{L}(U, U, \rho^c)$ is not even a subset of the lattice $\mathcal{L}(U, U, R^c)$. All we can say is that in general the following conditions are equivalent:

- (1) $(\wp(U)^R, \subseteq)$ is isomorphic to a complete sublattice of $(\wp(U)^\rho, \subseteq)$;
- (2) $(\wp(U)_R, \subseteq)$ is isomorphic to a complete sublattice of $(\wp(U)_\rho, \subseteq)$;
- (3) $\mathcal{L}(U, U, \rho^c)$ is isomorphic to a complete sublattice of $\mathcal{L}(U, U, R^c)$.

Let ρ be a tolerance on U . Let us define

$$\begin{aligned}\leq(\rho) &:= \{(x, y) \in U \times U \mid \rho(x) \subseteq \rho(y)\}, \\ \geq(\rho) &:= \{(x, y) \in U \times U \mid \rho(x) \supseteq \rho(y)\}, \\ \ker \rho &:= \{(x, y) \in U \times U \mid \rho(x) = \rho(y)\}.\end{aligned}$$

Clearly, $\leq(\rho)$ and $\geq(\rho)$ are reflexive and transitive relations, i.e. they are *quasi-orders* and $\geq(\rho)$ is the inverse relation of $\leq(\rho)$. $\ker \rho$ is an equivalence relation, called the kernel of the tolerance ρ . Clearly, $\ker \rho = \leq(\rho) \cap \geq(\rho)$. Let the symbol $\circ$ stand for the relational product, in what follows. It is easy to check that in the case $R \subseteq \rho$ the relations $R \circ \leq(\rho) \subseteq \rho$ and $\geq(\rho) \circ R \subseteq \rho$ always hold. Using these notions, in [2] we proved the following characterization:

Theorem 1. *Let ρ, R be two tolerance relations satisfying $R \subseteq \rho \subseteq U \times U$. Then the following conditions are equivalent:*

- (C): $\mathcal{L}(U, U, \rho^c)$ is a complete sublattice of $\mathcal{L}(U, U, R^c)$;
- (D): For any $(a, b) \in \rho \setminus R$ there exist some elements $c, d \in U$ such that $(b, c), (a, d) \in R$ and $\rho(c) \subseteq \rho(a), \rho(d) \subseteq \rho(b)$;
- (E): $R \circ \leq(\rho) = \rho$;
- (E'): $\geq(\rho) \circ R = \rho$.

Now, the next corollary is immediate:

Corollary 1. *Let R, ρ be two tolerance relations on U such that $R \subseteq \rho$. If R and ρ satisfy one of the equivalent conditions of Theorem 1, then $(\wp(U)^\rho, \subseteq)$ is a complete sublattice of $(\wp(U)^R, \subseteq)$ and $(\wp(U)_\rho, \subseteq)$ is a complete sublattice of $(\wp(U)_R, \subseteq)$.*

In [2] we proved that in the particular case when ρ is an equivalence relation on U such that $R \subseteq \rho$, then condition (D) is satisfied. Hence in such a case $(\wp(U)^\rho, \subseteq)$ is obviously a sublattice of $(\wp(U)^R, \subseteq)$ and $(\wp(U)_\rho, \subseteq)$.

Here we give an algorithm (Algorithm 1) that is checking on a finite set U and two relations $R \subseteq \rho \subseteq U \times U$ whether $\mathcal{L}(U, U, \rho^c)$ is a complete sublattice of $\mathcal{L}(U, U, R^c)$ by using condition (D).

```

CD-EXIST( $R, \rho, a, b$ )
  for  $(b, c) \in R$  do
    for  $(a, d) \in R$  do
      if  $\rho(d) \subseteq \rho(b) \wedge \rho(c) \subseteq \rho(a)$  then
        return TRUE
      end
    end
  end
return FALSE

SATISFIES_D( $R, \rho$ )
  for  $(a, b) \in \rho \setminus R$  do
    if  $\neg \text{CD-EXIST}(R, \rho, a, b)$  then
      return FALSE
    end
  end
return TRUE

```

Algorithm 1: Algorithm for checking condition (D)

4. TOLERANCES INDUCED BY IRREDUNDANT COVERINGS

A collection $\mathcal{C}$ of nonempty subsets of U is called a *covering* of U if $\bigcup \mathcal{C} = U$. The covering $\mathcal{C}$ is called *irredundant* if removing any member X of $\mathcal{C}$, the collection $\mathcal{C} \setminus \{X\}$ remains no longer a covering of U . For instance, the classes of an equivalence relation $E \subseteq U \times U$ provide a simple example of an irredundant covering of U . Each covering $\mathcal{C}$ of U defines a tolerance relation $\rho_{\mathcal{C}} = \bigcup \{X \times X \mid X \in \mathcal{C}\}$, called *the tolerance induced by $\mathcal{C}$* . If $\mathcal{C}$ is an irredundant covering of U , then we say that $\rho_{\mathcal{C}}$ is a *tolerance induced by an irredundant covering*. In [6] the authors proved that the lattices $\wp(U)^{\rho}$, $\wp(U)_{\rho}$ and $\mathbf{RS}(U, \rho)$ are completely distributive if and only if ρ is induced by an irredundant covering of U . It was shown that this condition is also equivalent to the condition that the lattice $\mathcal{L}(U, U, \rho^c)$ is completely distributive. A complete lattice L is called *completely distributive* (see e.g. [1]) if for any doubly indexed family of elements $\{x_{i,j}\}_{i \in I, j \in J}$, ($I, J \neq \emptyset$) we have

$$\bigwedge_{i \in I} \left(\bigvee_{j \in J} x_{i,j} \right) = \bigvee_{f: I \rightarrow J} \left(\bigwedge_{i \in I} x_{i, f(i)} \right).$$

We note that any complete sublattice of a completely distributive lattice is also completely distributive. As immediate consequence of the mentioned results we obtain the following Lemma.

Lemma 1. *Let R, ρ be two tolerance relations on U with $R \subseteq \rho$ and such that $(\wp(U)^R, \subseteq)$ is isomorphic to a complete sublattice of $(\wp(U)^\rho, \subseteq)$. If R is a tolerance induced by an irredundant covering, then ρ is also a tolerance induced by an irredundant covering.*

Proof. Since in view of [6] $(\wp(U)^R, \subseteq)$ is a completely distributive lattice, $(\wp(U)^\rho, \subseteq)$, being a complete sublattice of $(\wp(U)^R, \subseteq)$, is also completely distributive. Therefore, ρ is a tolerance induced by an irredundant covering. $\square$

Since any equivalence relation is a particular tolerance induced by an irredundant covering, the following corollary is immediate.

Corollary 2. *Let R be an equivalence relation and ρ a tolerance relation on U with $R \subseteq \rho$ and such $(\wp(U)^R, \subseteq)$ is a complete sublattice of $(\wp(U)^\rho, \subseteq)$. Then ρ is induced by an irredundant covering.*

Let ρ be a tolerance on U . A nonempty subset X of U is called a *preblock* of ρ if $X \times X \subseteq \rho$. Note that in this case $B \subseteq \rho(x)$ for all $x \in B$. A preblock of ρ that is maximal with respect to the inclusion is called a *block* of ρ .

Remark 1. It is well known that any tolerance relation ρ is determined by its blocks, that is for any $a, b \in U$, $(a, b) \in \rho \Leftrightarrow a, b \in B$, for some block B of ρ . In [6] and in [7] it is shown that if ρ is induced by an irredundant covering $\mathcal{C}$, then B can be chosen as a member of $\mathcal{C}$ having the property $B = \rho(k)$, for some $k \in B$. It is also proved that in this case $\rho = \sup(\rho) \circ \leq(\rho)$ (see [5]).

Theorem 2. *Let $\rho, R \subseteq U \times U$ be two tolerance relations with $R \subseteq \rho$ and assume that R is induced by an irredundant covering. Then condition (E) is equivalent to the condition*

$$(F): \rho = (R \cap \sup(\rho)) \circ \ker \rho \circ (R \cap \leq(\rho)).$$

Proof. First we will show that condition (E) implies condition (F). If (E) holds, then $\mathcal{L}(U, U, \rho^c)$ is a complete sublattice of $\mathcal{L}(U, U, R^c)$, according to Theorem 1. By Corollary 1 this yields that $(\wp(U)^\rho, \subseteq)$ is a complete sublattice of $(\wp(U)^R, \subseteq)$. Then by Lemma 1, ρ is also a tolerance induced by an irredundant covering of U . Then $\rho = \sup(\rho) \circ \leq(\rho)$, according to [5]. It is easy to check that $\sup(\rho) \circ \ker \rho \subseteq \sup(\rho)$. Hence

$$(R \cap \sup(\rho)) \circ \ker \rho \circ (R \cap \leq(\rho)) \subseteq \sup(\rho) \circ \ker \rho \circ \leq(\rho) \subseteq \sup(\rho) \circ \leq(\rho) = \rho.$$

In order to prove the converse inclusion, take any $(a, b) \in \rho$. Then, in view of Remark 1, there exists a $k \in U$ and a block B of ρ such that $a, b \in B = \rho(k)$. Then $B \subseteq \rho(a), \rho(b)$. As $(a, k), (b, k) \in \rho$, now condition (E), i.e. $R \circ \leq(\rho) = \rho$

implies that there exist some elements $c, d \in U$ such that $(a, c) \in R$, $\rho(c) \subseteq \rho(k)$ and $(b, d) \in R$, $\rho(d) \subseteq \rho(k)$. Since $c, d \in \rho(k) = B$, we have $B \subseteq \rho(c), \rho(d)$, whence we get $\rho(c) = \rho(d) = B$, proving $(c, d) \in \ker \rho$. Then $\rho(c) = B \subseteq \rho(a)$ also yields $(a, c) \in \supseteq (\rho)$. Hence $(a, c) \in R \cap \supseteq (\rho)$. Similarly, $\rho(d) = B \subseteq \rho(b)$ yields $(d, b) \in \sqsubseteq (\rho)$. Thus $(d, b) \in R \cap \sqsubseteq (\rho)$. Now, $(a, c) \in R \cap \supseteq (\rho)$, $(c, d) \in \ker \rho$ and $(d, b) \in R \cap \sqsubseteq (\rho)$ together imply $(a, b) \in (R \cap \supseteq (\rho)) \circ \ker \rho \circ (R \cap \sqsubseteq (\rho))$, proving $\rho \subseteq (R \cap \supseteq (\rho)) \circ \ker \rho \circ (R \cap \sqsubseteq (\rho))$.

Conversely, assume that (F) holds, i.e. $\rho = (R \cap \supseteq (\rho)) \circ \ker \rho \circ (R \cap \sqsubseteq (\rho))$. We will prove (E'), which is equivalent to (E) by Theorem 1. Since $\supseteq (\rho) \circ R \subseteq \rho$ is always true, we have to show only the converse inclusion. Indeed, take any $(a, b) \in \supseteq (\rho) \circ R$. Then there exist some elements $d, c \in U$ such that $(a, c) \in R \cap \supseteq (\rho)$, $(c, d) \in \ker \rho$, and $(d, b) \in R \cap \sqsubseteq (\rho)$. Hence $\rho(c) = \rho(d)$. As $(a, c) \in \supseteq (\rho)$ means that $\rho(c) \subseteq \rho(a)$, we get also $\rho(d) \subseteq \rho(a)$, i.e. $(a, d) \in \supseteq (\rho)$. Since $(d, b) \in R$, we obtain $(a, b) \in \supseteq (\rho) \circ R$. Hence $\rho \subseteq \supseteq (\rho) \circ R$, and this means that condition (E') is satisfied. $\square$

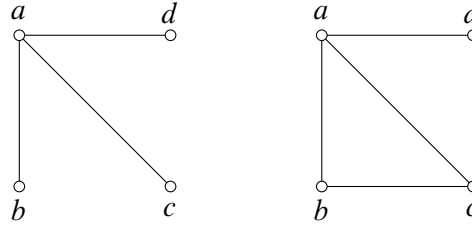


FIGURE 1. Relations R and ρ satisfying condition (F)

TABLE 1. The contexts (U, U, R^c) and (U, U, ρ^c)

R^c	a	b	c	d
a				
b			$\times$	$\times$
c		$\times$		$\times$
d		$\times$	$\times$	

ρ^c	a	b	c	d
a				
b				$\times$
c				$\times$
d		$\times$	$\times$	

Corollary 3. Let R, ρ be two tolerance relations on U such that $R \subseteq \rho$. If R and ρ satisfy condition (F) and R is induced by an irredundant covering of U then $(\wp(U)^\rho, \subseteq)$ is a complete sublattice of $(\wp(U)^R, \subseteq)$ and $(\wp(U)_\rho, \subseteq)$ is a complete sublattice of $(\wp(U)_R, \subseteq)$.

Corollary 4. Let R, ρ be two tolerance relations on U such that $R \subseteq \rho$, condition (F) holds and $R \subseteq \ker \rho$. Then ρ is an equivalence.

Proof. If $R \subseteq \ker \rho$ then $R \cap \supseteq (\rho) \subseteq \ker \rho$ and $R \cap \leq (\rho) \subseteq \ker \rho$. This implies that $(R \cap \supseteq (\rho)) \circ \ker \rho \circ (R \cap \leq (\rho)) \subseteq \ker \rho \circ \ker \rho \circ \ker \rho = \ker \rho$, since $\ker \rho$ is an equivalence. Thus, we get $\rho \subseteq \ker \rho$ using condition (F). Since ρ is a tolerance relation, $\ker \rho \subseteq \rho$ also holds, meaning $\rho = \ker \rho$, therefore ρ is an equivalence. $\square$

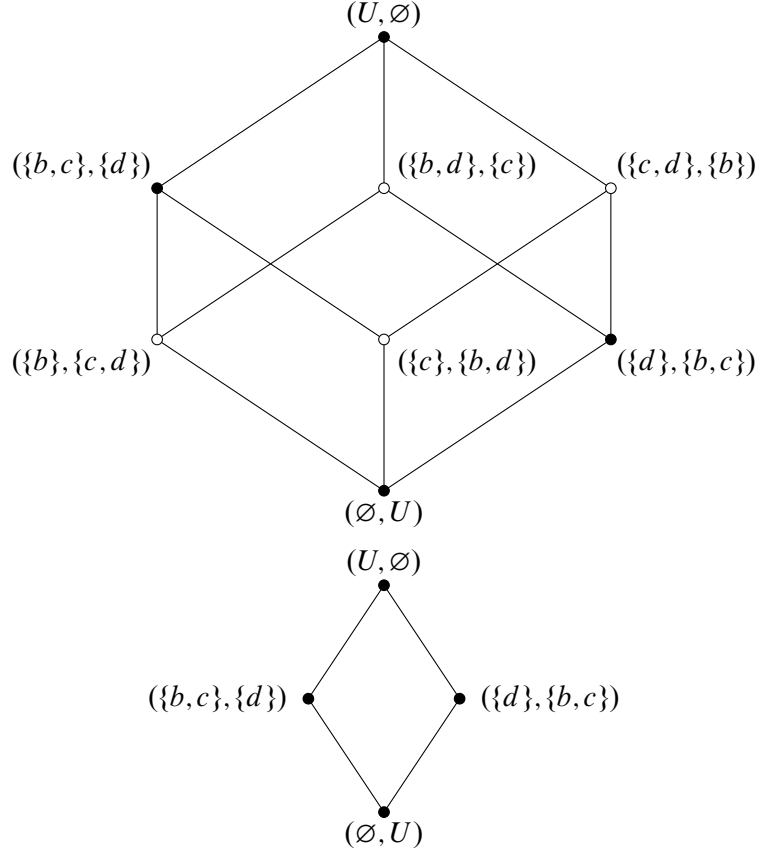


FIGURE 2. The Hasse-diagrams of the concept lattices $\mathcal{L}(U, U, R^c)$ and $\mathcal{L}(U, U, \rho^c)$

Corollary 5. Let R, ρ be two tolerance relations on U such that $R \subseteq \rho$. If ρ is an equivalence, condition (F) automatically holds.

Proof. Since ρ is an equivalence, $\rho = \ker \rho = \supseteq (\rho) = \leq (\rho)$. Therefore $R \cap \supseteq (\rho) = R \cap \leq (\rho) = R \cap \rho = R$. Then it follows that $(R \cap \supseteq (\rho)) \circ \ker \rho \circ (R \cap \leq (\rho)) = R \circ \rho \circ R$. However, $R \circ \rho \circ R \subseteq \rho \circ \rho \circ \rho = \rho$. On the other hand, $R \circ \rho \circ R \supseteq \Delta \circ \rho \circ \Delta = \rho$, where Δ is the identity relation. Combining them yields $R \circ \rho \circ R = \rho$, which proves condition (F). $\square$

5. CONCLUSION

In this paper, we were aiming to extend the characterization found in [2] by further investigating the tolerance relations $R \subseteq \rho$. We deduced condition (F), which is equivalent to the conditions in Theorem 1 whenever R is a tolerance induced by an irredundant covering. Additionally, an algorithm for checking condition (D) was also provided. An example of two relations satisfying condition (F) can be seen in Figure 1. Since tolerance relations are always reflexive, loops are not noted on the figure for simplicity. Figure 2 shows that the concept lattice $\mathcal{L}(U, U, \rho^c)$ is a complete sublattice of $\mathcal{L}(U, U, R^c)$, i.e. condition (C) holds. We also proved some consequences for special cases, e.g. R being a subrelation of $\ker \rho$. As a future work, we propose investigating the results in [2] regarding the so-called compatibility condition in combination with tolerances induced by an irredundant covering.

REFERENCES

- [1] B. Ganter and R. Wille, *Formal concept analysis: mathematical foundations*. Springer Science & Business Media, 2012.
- [2] D. Gégény, I. Piller, S. Radeleczki, and L. Veres, “Approximations induced by tolerance relations,” 2019.
- [3] J. Järvinen, L. Kovács, and S. Radeleczki, “Defining rough sets using tolerances compatible with an equivalence,” 2019.
- [4] J. Järvinen, P. Pagliani, and S. Radeleczki, “Information completeness in Nelson algebras of rough sets induced by quasiorders,” *Studia logica*, vol. 101, no. 5, pp. 1073–1092, 2013, doi: [10.1007/s11225-012-9421-z](https://doi.org/10.1007/s11225-012-9421-z).
- [5] J. Järvinen and S. Radeleczki, “Monteiro spaces and rough sets determined by quasiorder relations: models for Nelson algebras,” *Fundamenta Informaticae*, vol. 131, no. 2, pp. 205–215, 2014.
- [6] J. Järvinen and S. Radeleczki, “Rough sets determined by tolerances,” *International Journal of Approximate Reasoning*, vol. 55, no. 6, pp. 1419–1438, 2014, doi: [10.1016/j.ijar.2013.12.005](https://doi.org/10.1016/j.ijar.2013.12.005).
- [7] J. Järvinen and S. Radeleczki, “Irredundant coverings, tolerances, and related algebras,” in *Algebraic Methods in General Rough Sets*. Springer, 2018, pp. 417–457, doi: [10.1007/978-3-030-01162-8_5](https://doi.org/10.1007/978-3-030-01162-8_5).
- [8] Z. Pawlak, “Rough sets,” *International journal of computer & information sciences*, vol. 11, no. 5, pp. 341–356, 1982, doi: [10.1007/BF01001956](https://doi.org/10.1007/BF01001956).
- [9] Z. Pawlak, “Rough set theory and its applications to data analysis,” *Cybernetics & Systems*, vol. 29, no. 7, pp. 661–688, 1998, doi: [10.1080/019697298125470](https://doi.org/10.1080/019697298125470).
- [10] J. A. Pomykala, “On similarity based approximation of information,” *Demonstratio Mathematica*, vol. 27, no. 3-4, pp. 663–671, 1994, doi: [10.1515/dema-1994-3-412](https://doi.org/10.1515/dema-1994-3-412).
- [11] S. Rissino and G. Lambert-Torres, “Rough set theory-fundamental concepts, principals, data extraction, and applications,” in *Data mining and knowledge discovery in real life applications*. InTech, 2009. doi: [10.5772/6440](https://doi.org/10.5772/6440).
- [12] Y. Yao, “Generalized rough set models,” *Rough sets in knowledge discovery*, vol. 1, pp. 286–318, 1998.
- [13] W. Zakowski, “Approximations in the space (u, π) ,” *Demonstratio mathematica*, vol. 16, no. 3, pp. 761–770, 1983, doi: [10.1515/dema-1983-0319](https://doi.org/10.1515/dema-1983-0319).

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LOGARITHMIC MEANS OF WALSH-FOURIER SERIES

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Abstract. In this paper we discuss some convergence and divergence properties of subsequences of logarithmic means of Walsh-Fourier series. We give necessary and sufficient conditions for the convergence regarding logarithmic variation of numbers.

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1. WALSH FUNCTIONS

We shall denote the set of all non-negative integers by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$ and the set of dyadic rational numbers in the unit interval $\mathbb{I} := [0, 1)$ by $\mathbb{Q}$. In particular, each element of $\mathbb{Q}$ has the form $\frac{p}{2^n}$ for some $p, n \in \mathbb{N}$, $0 \leq p \leq 2^n$.

Denote the dyadic expansion of $n \in \mathbb{N}$ and $x \in \mathbb{I}$ by

$$n = \sum_{j=0}^{\infty} \varepsilon_j(n) 2^j, \varepsilon_j(n) = 0, 1$$

and

$$x = \sum_{j=0}^{\infty} \frac{x_j}{2^{j+1}}, x_j = 0, 1.$$

In the case of $x \in \mathbb{Q}$ chose the expansion which terminates in zeros. Define the dyadic addition $+$ as

$$x + y = \sum_{k=0}^{\infty} |x_k - y_k| 2^{-(k+1)}.$$

The sets $I_n(x) := \{y \in \mathbb{I} : y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}$ for $x \in \mathbb{I}$, $I_n := I_n(0)$ for $0 < n \in \mathbb{N}$ and $I_0(x) := \mathbb{I}$ are the dyadic intervals of $\mathbb{I}$. For $0 < n \in \mathbb{N}$ denote by $|n| := \max \{j \in \mathbb{N} : n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$.

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The Rademacher system is defined by

$$\rho_n(x) := (-1)^{x_n} \quad (x \in \mathbb{I}, n \in \mathbb{N}).$$

The Walsh-Paley system is defined as the sequence of the Walsh-Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (\rho_k(x))^{n_k} = (-1)^{\sum_{k=0}^{|n|} n_k x_k} \quad (x \in \mathbb{I}, n \in \mathbb{N}).$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x) \quad (n \in \mathbb{N}), D_0 = 0.$$

Recall that (see [20])

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n(0) \\ 0, & \text{if } x \in \mathbb{I} \setminus I_n(0) \end{cases}. \quad (1.1)$$

As usual, denote by $L_1(\mathbb{I})$ the set of measurable functions defined on $\mathbb{I}$, for which

$$\|f\|_1 := \int_{\mathbb{I}} |f(t)| dt < \infty.$$

Let $f \in L_1(\mathbb{I})$. The partial sums of the Walsh-Fourier series are defined as follows:

$$S_M(x, f) := \sum_{i=0}^{M-1} \widehat{f}(i) w_i(x),$$

where the number

$$\widehat{f}(i) = \int_{\mathbb{I}} f(t) w_i(t) dt$$

is said to be the i th Walsh-Fourier coefficient of the function f . Set $E_n(x, f) = S_{2^n}(x, f)$. The maximal function is defined by

$$E^*(x, f) = \sup_{n \in \mathbb{N}} E_n(x, |f|).$$

The notation $a \lesssim b$ in the proofs stands for $a < c \cdot b$, where c is an absolute constant.

2. LOGARITHMIC MEANS

In the literature, there is the notion of Riesz's logarithmic means of a Fourier series. The n -th Riesz's logarithmic means of the Fourier series of an integrable function f is defined by

$$R_n(x, f) := \frac{1}{l_n} \sum_{k=1}^n \frac{S_k(x, f)}{k},$$

where $l_n := \sum_{k=1}^n (1/k)$.

Riesz's logarithmic means with respect to the trigonometric system was studied by a lot of authors. This means with respect to the Walsh and Vilenkin systems was discussed by Simon [21], Blahota, Gát [1], Gát [4], Gát, Goginava [8].

Let $\{q_k : k \geq 0\}$ be a sequence of nonnegative numbers. The n -th Nörlund means for the Fourier series of f is defined by

$$\frac{1}{Q_n} \sum_{k=0}^{n-1} q_{n-k} S_k(f),$$

where

$$Q_n := \sum_{k=1}^n q_k.$$

If $q_k = k$, then we get the Nörlund logarithmic means

$$t_n(x, f) := \frac{1}{l_n} \sum_{k=0}^{n-1} \frac{S_k(x, f)}{n-k}.$$

In this paper we call it logarithmic mean although, it is a kind of "reverse" Riesz's logarithmic mean.

It is easy to see that

$$t_n(x, f) = \int_{\mathbb{T}} f(t) F_n(x+t) dt,$$

where by $F_n(t)$ we denote n th logarithmic kernel, i. e.

$$F_n(t) := \frac{1}{l_n} \sum_{k=0}^{n-1} \frac{D_k(t)}{n-k}.$$

and Fejér kernel is defined by

$$K_n(t) := \frac{1}{n} \sum_{k=1}^n D_k(t).$$

3. L_1 - ESTIMATION FOR LOGARITHMIC KERNEL

For $n = \sum_{j=0}^{\infty} \varepsilon_j(n) 2^j$, $\varepsilon_j(n) = 0, 1$ we define

$$n(k) := \sum_{j=0}^k \varepsilon_j(n) 2^j.$$

It is easy to see that $n(|n|) = n$. In this paper for L_1 -norm of logarithmic means we prove the following two sides estimation.

Theorem 1. *Let $n \in \mathbb{N}$. Then*

$$\left\| \frac{1}{l_n} \sum_{j=1}^n \frac{D_{n-j}}{j} \right\|_1 \sim \frac{1}{|n|} \sum_{k=1}^{|n|} |\varepsilon_k(n) - \varepsilon_{k+1}(n)| l_{n(k-1)}.$$

Proof. We can write

$$\begin{aligned} & \sum_{j=1}^n \frac{D_{n-j}(t)}{j} \\ &= \sum_{j=1}^{n(|n|-1)} \frac{D_{n-j}(t)}{j} + \sum_{j=n(|n|-1)+1}^n \frac{D_{n-j}(t)}{j} \\ &= \sum_{j=1}^{n(|n|-1)} \frac{D_{\varepsilon_{|n|}(n)2^{|n|}+n(|n|-1)-j}(t)}{j} + \sum_{j=n(|n|-1)+1}^n \frac{D_{n-j}(t)}{j}. \end{aligned} \quad (3.1)$$

Since

$$D_{\varepsilon_{|n|}(n)2^{|n|}+n(|n|-1)-j}(t) = \varepsilon_{|n|}(n) D_{2^{|n|}}(t) + (w_{2^{|n|}}(t))^{\varepsilon_{|n|}(n)} D_{n(|n|-1)-j}(t),$$

from (3.1) we have

$$\begin{aligned} & \sum_{j=1}^n \frac{D_{n-j}(t)}{j} \\ &= \varepsilon_{|n|}(n) D_{2^{|n|}}(t) l_{n(|n|-1)} \\ & \quad + (w_{2^{|n|}}(t))^{\varepsilon_{|n|}(n)} \sum_{j=1}^{n(|n|-1)} \frac{D_{n(|n|-1)-j}(t)}{j} \\ & \quad + \varepsilon_{|n|}(n) \sum_{j=1}^{2^{|n|}-1} \frac{D_{2^{|n|}-j}(t)}{j + n(|n|-1)}. \end{aligned}$$

Iterating this equality we obtain

$$\begin{aligned}
 & \sum_{j=1}^n \frac{D_{n-j}(t)}{j} \\
 &= \left(\sum_{j=2}^{|n|} \varepsilon_j(n) D_{2^j}(t) l_{n(j-1)} \right) \prod_{k=j+1}^{|n|} (\rho_k(t))^{\varepsilon_k(n)} \\
 &+ \left(\sum_{j=2}^{|n|} \varepsilon_j(n) \sum_{k=1}^{2^j-1} \frac{D_{2^j-k}(t)}{k+n(j-1)} \right) \prod_{s=j+1}^{|n|} (\rho_s(t))^{\varepsilon_s(n)} \\
 &+ \left(\sum_{j=1}^{n(1)} \frac{D_{n(1)-j}(t)}{j} \right) \prod_{k=2}^{|n|} (\rho_k(t))^{\varepsilon_k(n)}.
 \end{aligned} \tag{3.2}$$

Since

$$\varepsilon_j(n) D_{2^j}(t) \prod_{k=0}^j (\rho_k(t))^{\varepsilon_k(n)} = \varepsilon_j(n) D_{2^j}(t) \rho_j(t)$$

we have

$$\begin{aligned}
 & \left(\sum_{j=2}^{|n|} \varepsilon_j(n) D_{2^j}(t) l_{n(j-1)} \right) \prod_{k=j+1}^{|n|} (\rho_k(t))^{\varepsilon_k(n)} \\
 &= w_n(t) \left(\sum_{j=2}^{|n|} \varepsilon_j(n) D_{2^j}(t) l_{n(j-1)} \right) \prod_{k=0}^j (\rho_k(t))^{\varepsilon_k(n)} \\
 &= w_n(t) \left(\sum_{j=2}^{|n|} \varepsilon_j(n) D_{2^j}(t) \rho_j(t) l_{n(j-1)} \right).
 \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3) we conclude that

$$\begin{aligned}
& \sum_{j=1}^n \frac{D_{n-j}(t)}{j} \\
&= w_n(t) \left(\sum_{j=2}^{|n|} \varepsilon_j(n) D_{2^j}(t) \rho_j(t) l_{n(j-1)} \right) \\
&+ \left(\sum_{j=2}^{|n|} \varepsilon_j(n) \sum_{k=1}^{2^j-1} \frac{D_{2^j-k}(t)}{k+n(j-1)} \right) \prod_{s=j+1}^{|n|} (\rho_s(t))^{\varepsilon_s(n)} \\
&+ \left(\sum_{j=1}^{n(1)-1} \frac{D_{n(1)-j}(t)}{j} \right) \prod_{k=2}^{|n|} (\rho_k(t))^{\varepsilon_k(n)} \\
&=: H_n^{(1)}(t) + H_n^{(2)}(t) + H_n^{(3)}(t).
\end{aligned} \tag{3.4}$$

Since (see [9])

$$D_{2^j-k}(t) = D_{2^j}(t) - w_{2^j-1}(t) D_k(t), k = 1, 2, \dots, 2^j - 1$$

for $H_n^{(2)}(t)$ we can write

$$\begin{aligned}
& H_n^{(2)}(t) \\
&= \left(\sum_{j=2}^{|n|} \varepsilon_j(n) D_{2^j}(t) \sum_{k=1}^{2^j-1} \frac{1}{k+n(j-1)} \right) \prod_{s=j+1}^{|n|} (\rho_s(t))^{\varepsilon_s(n)} \\
&- \left(\sum_{j=2}^{|n|} \varepsilon_j(n) w_{2^j-1}(t) \sum_{k=1}^{2^j-1} \frac{D_k(t)}{k+n(j-1)} \right) \prod_{s=j+1}^{|n|} (\rho_s(t))^{\varepsilon_s(n)} \\
&=: H_n^{(21)}(t) + H_n^{(22)}(t).
\end{aligned} \tag{3.5}$$

Since

$$\begin{aligned}
H_n^{(21)}(t) &= \left(\sum_{j=2}^{|n|} \varepsilon_j(n) D_{2^j}(t) (l_{n(j)} - l_{n(j-1)}) \right) \\
&\times \prod_{s=j+1}^{|n|} (\rho_s(t))^{\varepsilon_s(n)}
\end{aligned}$$

from (1.1) we get

$$\left\| H_n^{(21)} \right\|_1 \leq \sum_{j=2}^{|n|} \varepsilon_j(n) (l_{n(j)} - l_{n(j-1)}) \leq c |n|. \tag{3.6}$$

Usin Abel's transformation we obtain

$$\begin{aligned} & \sum_{k=1}^{2^j-1} \frac{D_k(t)}{k+n(j-1)} \\ &= \sum_{k=1}^{2^j-2} \left(\frac{1}{k+n(j-1)} - \frac{1}{k+1+n(j-1)} \right) k K_k(t) \\ & \quad + \frac{2^j-1}{2^j-1+n(j-1)} K_{2^j-1}(t). \end{aligned}$$

Since (see [20]) $\sup_n \|K_n\|_1 < \infty$ for $H_n^{(22)}(t)$ we can write

$$\begin{aligned} & \left\| H_n^{(22)} \right\|_1 \\ & \lesssim \sum_{j=2}^{|n|} \varepsilon_j(n) \sum_{k=1}^{2^j-1} \left(\frac{1}{k+n(j-1)} - \frac{1}{k+1+n(j-1)} \right) k \\ & \quad + \sum_{j=2}^{|n|} \varepsilon_j(n) \frac{2^j-1}{2^j-1+n(j-1)} \\ & \lesssim \sum_{j=2}^{|n|} \varepsilon_j(n) \sum_{k=1}^{2^j-1} \frac{1}{(k+n(j-1))} \\ & \quad + \sum_{j=2}^{|n|} \varepsilon_j(n) \sum_{k=1}^{2^j-1} \frac{n(j-1)}{(k+n(j-1))^2} + \sum_{j=2}^{|n|} \varepsilon_j(n) \frac{2^j-1}{2^j-1+n(j-1)} \\ & \lesssim \sum_{j=2}^{|n|} \varepsilon_j(n) (l_{n(j)} - l_{n(j-1)}) + |n| \lesssim |n|. \end{aligned} \tag{3.7}$$

Combining (3.5)-(3.7) we conclude that

$$\left\| H_n^{(2)} \right\|_1 \leq c |n|. \tag{3.8}$$

It is easy to see that

$$\sup_n \left\| H_n^{(3)} \right\|_1 \leq c. \tag{3.9}$$

First, we find upper estimation for $\left\| H_n^{(1)} \right\|_1$. We can write

$$H_n^{(1)}(t) = w_n(t) \left(\sum_{j=2}^{|n|} \varepsilon_j(n) l_{n(j-1)} (D_{2^{j+1}}(t) - D_{2^j}(t)) \right)$$

$$\begin{aligned}
&= w_n(t) \left(\sum_{j=2}^{|n|-1} (\varepsilon_j(n) l_{n(j-1)} - \varepsilon_{j+1}(n) l_{n(j)}) D_{2^{j+1}}(t) \right) \\
&\quad + w_n(t) l_{n(|n|-1)} D_{2^{|n|+1}} - w_n(t) \varepsilon_2(n) l_{n(1)} D_{2^2}(t).
\end{aligned}$$

Hence, from (1.1) we obtain

$$\begin{aligned}
\|H_n^{(1)}\|_1 &\leq \sum_{j=2}^{|n|-1} |\varepsilon_j(n) l_{n(j-1)} - \varepsilon_{j+1}(n) l_{n(j)}| + c|n| \\
&\leq \sum_{j=2}^{|n|-1} |\varepsilon_j(n) - \varepsilon_{j+1}(n)| l_{n(j-1)} \\
&\quad + \sum_{j=2}^{|n|-1} \varepsilon_{j+1}(n) (l_{n(j)} - l_{n(j-1)}) + c|n| \\
&\leq \sum_{j=2}^{|n|} |\varepsilon_j(n) - \varepsilon_{j+1}(n)| l_{n(j-1)} + c|n|.
\end{aligned}$$

Now, we find lower estimation for $\|H_n^{(1)}\|_1$. Let a_i and $b_i, i = 1, \dots, s$ be strictly increasing sequences, i. e.

$$0 \leq a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_s \leq b_s < a_{s+1} = \infty$$

for which

$$\varepsilon_j(n) = \begin{cases} 1, & a_i \leq j \leq b_i \\ 0, & b_i < j < a_{i+1} \end{cases} \quad (3.10)$$

Then it is evident that

$$b_j + 1 < a_{j+1}. \quad (3.11)$$

Set

$$A_k := \left(\frac{1}{2^{a_k+1}}, \frac{1}{2^{a_k}} \right), B_k := \left(\frac{1}{2^{b_k+2}}, \frac{1}{2^{b_k+1}} \right), k = 1, \dots, s.$$

Let $x \in A_k$. Then we can write

$$\begin{aligned}
|H_n^{(1)}(t)| &= \left| \sum_{j=2}^{|n|} \varepsilon_j(n) (D_{2^{j+1}}(t) - D_{2^j}(t)) l_{n(j-1)} \right| \\
&= \left| \sum_{i=1}^{k-1} \sum_{j=a_i}^{b_i} (D_{2^{j+1}}(t) - D_{2^j}(t)) l_{n(j-1)} \right|
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=a_k}^{b_k} (D_{2^{j+1}}(t) - D_{2^j}(t)) l_{n(j-1)} \Bigg| \\
& = \left| \sum_{i=1}^{k-1} \sum_{j=a_i}^{b_i} 2^j l_{n(j-1)} - 2^{a_k} l_{n(a_k-1)} \right|.
\end{aligned}$$

From (3.11) we can write

$$\begin{aligned}
\sum_{i=1}^{k-1} \sum_{j=a_i}^{b_i} 2^j l_{n(j-1)} & \leq l_{n(b_{k-1}-1)} \sum_{i=1}^{k-1} (2^{b_i+1} - 2^{a_i}) \\
& \leq l_{n(b_{k-1}-1)} \sum_{i=1}^{k-1} (2^{b_i+1} - 2^{b_{i-1}+1}) \\
& \leq 2^{b_{k-1}+1} l_{n(b_{k-1}-1)} \\
& \leq 2^{b_{k-1}+1} l_{n(a_k-1)}.
\end{aligned}$$

Consequently,

$$\left| H_n^{(1)}(t) \right| \geq 2^{a_k} l_{n(a_k-1)} - 2^{b_{k-1}+1} l_{n(a_k-1)} \geq 2^{a_k-1} l_{n(a_k-1)}.$$

Integrating on A_k we get

$$\int_{A_k} \left| H_n^{(1)}(t) \right| dt \geq \int_{A_k} 2^{a_k-1} l_{n(a_k-1)} dt = \frac{l_{n(a_k-1)}}{4}. \quad (3.12)$$

On the interval B_k we have

$$\begin{aligned}
\left| H_n^{(1)}(t) \right| & = \left| \sum_{i=1}^k \sum_{j=a_i}^{b_i} (D_{2^{j+1}}(t) - D_{2^j}(t)) l_{n(j-1)} \right| \\
& = \sum_{i=1}^k \sum_{j=a_i}^{b_i} 2^j l_{n(j-1)} \geq l_{n(b_k-1)} 2^{b_k}.
\end{aligned}$$

Hence,

$$\int_{B_k} \left| H_n^{(1)}(t) \right| dt \geq \int_{B_k} l_{n(b_k-1)} 2^{b_k} dt = \frac{l_{n(b_k-1)}}{4}. \quad (3.13)$$

Since $A_i, B_i, i = 1, \dots, s$ are pairwise disjoint from (3.12) and (3.13) we have

$$\begin{aligned} \int_{\mathbb{I}} |H_n^{(1)}(t)| dt &\geq \sum_{k=1}^s \left(\int_{A_k} |H_n^{(1)}(t)| dt + \int_{B_k} |H_n^{(1)}(t)| dt \right) \\ &\geq \frac{1}{4} \sum_{k=1}^s (l_{n(a_k-1)} + l_{n(b_k-1)}). \end{aligned} \quad (3.14)$$

Since (see (3.10))

$$n(a_k - 1) = n(a_k - 2)$$

and

$$|\varepsilon_j(n) - \varepsilon_{j+1}(n)| = \begin{cases} 1, & j = a_k - 1 \text{ or } b_k, k = 1, 2, \dots, s \\ 0 & \text{otherwise} \end{cases}$$

we conclude that

$$\int_{\mathbb{I}} |H_n^{(1)}(t)| dt \geq \frac{1}{4} \sum_{k=1}^{|n|} |\varepsilon_k(n) - \varepsilon_{k+1}(n)| l_{n(k-1)}.$$

Combining (3.4)-(3.10) and (3.14) we complete the proof of Theorem 1. $\square$

4. ALMOST EVERYWHERE CONVERGENCE OF LOGARITHMIC MEANS

For a non-negative integer n let us denote

$$V_S(n) := \sum_{i=0}^{\infty} |\varepsilon_i(n) - \varepsilon_{i+1}(n)| + \varepsilon_0(n)$$

and

$$V_L(n) := \frac{1}{|n|} \sum_{k=1}^{|n|} |\varepsilon_k(n) - \varepsilon_{k+1}(n)| l_{n(k-1)}.$$

It is known that if $n_j < n_{j+1}$,

$$\sup_j V_S(n_j) < \infty, \quad (4.1)$$

then a. e. $S_{n_j}(f) \rightarrow f$. On the other hand, Konyagin [14] proved that the condition (4.1) is not necessary for a. e. convergence of subsequence of partial sums. Moreover, he gave negative answer to the question of Balashov and proved the validity of the following theorem.

Theorem K (Konyagin). *Suppose $\{n_A\}$ is an increasing sequence of natural numbers, $k_A := [\log_2 n_A] + 1$, and 2^{k_A} is a divider of n_{A+1} for all A . Then $S_{n_A}(f) \rightarrow f$ a. e. for any function $f \in L_1(\mathbb{I})$.*

For instance, a sequence $\{n_A\}, n_A := 2^{A^2} \sum_{i=0}^A 4^i$, such that $\sup_{n_A} V(n_A) = \infty$, satisfies the hypotheses of the theorem.

Almost everywhere convergence of $\{t_{2^A}(f) : A \geq 1\}$ with respect to Walsh-Paley system was studied by author [11]. In particular, the following is proved

Theorem G. *Let $f \in L_1(\mathbb{I})$. Then $t_{2^A}(x, f) \rightarrow f(x)$ as $A \rightarrow \infty$ a. e. $x \in \mathbb{I}$.*

Nagy in [18] established a similar result for the Walsh-Kaczmarz system. However, a divergence on the set with positive measure for the whole sequence $\{t_n(f) : n \geq 1\}$ was proved by Gát and Goginava [7]. Memić [16] improved Theorem G and proved that the following is true.

Theorem M. *Let $f \in L_1(\mathbb{I})$ and*

$$\sup_A \frac{1}{|m_A|} \sum_{k=1}^{|m_A|} \varepsilon_k(m_A) l_{m_A(k-1)} < \infty. \quad (4.2)$$

Then $t_{m_A}(x, f) \rightarrow f(x)$ as $A \rightarrow \infty$ for a. e. $x \in \mathbb{I}$.

In this paper we are going to replace condition (4.2) with more weaker condition

$$\sup_A \frac{1}{|m_A|} \sum_{k=1}^{|m_A|} |\varepsilon_k(m_A) - \varepsilon_{k+1}(m_A)| l_{m_A(k-1)} < \infty. \quad (4.3)$$

It is easy to see that condition (4.2) imply condition (4.3), on other hand, for the sequence $\{2^A - 1 : A \in \mathbb{N}\}$ condition (4.2) does not holds and condition (4.3) holds. So, we prove that the following is valid.

Theorem 2. *Let $f \in L_1(\mathbb{I})$ and condition (4.3) is holds. Then $t_{m_A}(x, f) \rightarrow f(x)$ as $A \rightarrow \infty$ for a. e. $x \in \mathbb{I}$.*

Proof. From (3.4) we have

$$f * (l_{m_A} F_{m_A})(x) = f * H_{m_A}^{(1)}(x) + f * H_{m_A}^{(2)}(x) + f * H_{m_A}^{(3)}(x). \quad (4.4)$$

It is easy to see that

$$\left\| \sup_A \left| f * H_{m_A}^{(3)} \right| \right\|_1 \lesssim \|f\|_1. \quad (4.5)$$

From (3.5) we can write

$$f * H_{m_A}^{(2)}(x) = f * H_{m_A}^{(21)}(x) + f * H_{m_A}^{(22)}(x) + f * H_{m_A}^{(22)}(x). \quad (4.6)$$

Using (1.1) we have

$$\left| f * H_{m_A}^{(21)}(x) \right| \leq \sum_{j=1}^{|m_A|-1} (|f| * D_{2^j})(x) \leq |m_A| E^*(x, f).$$

Hence

$$\sup_A \frac{|f * H_{m_A}^{(21)}(x)|}{|m_A|} \leq E^*(x, f). \quad (4.7)$$

We can write

$$\begin{aligned} |f * H_{m_A}^{(22)}(x)| &\leq \sum_{j=1}^{|m_A|-1} \varepsilon_j(m_A) (l_{m_A(j)} - l_{m_A(j-1)}) (|f| * D_{2^j}(x)) \\ &\lesssim |m_A| E^*(x, f), \\ \sup_A \frac{|f * H_{m_A}^{(22)}(x)|}{|m_A|} &\lesssim E^*(x, f). \end{aligned} \quad (4.8)$$

It is proved in [7]

$$\sup_{\lambda} \lambda \left| \left\{ \sup_A \frac{|f * H_{m_A}^{(23)}|}{|m_A|} > \lambda \right\} \right| \lesssim \|f\|_1. \quad (4.9)$$

Since $\sup_{\lambda} \lambda |\{E^*(f) > \lambda\}| \lesssim \|f\|_1$ from (4.7)- (4.9) we get

$$\sup_{\lambda} \lambda \left| \left\{ \sup_A \frac{|f * H_{m_A}^{(2)}|}{|m_A|} > \lambda \right\} \right| \lesssim \|f\|_1. \quad (4.10)$$

Now, we estimate $|f * H_{m_A}^{(1)}(x)|$. We have

$$\begin{aligned} |f * H_{m_A}^{(1)}(x)| &\lesssim \sum_{j=1}^{|m_A|-1} (\varepsilon_j(m_A) l_{m_A(j-1)} - \varepsilon_{j+1}(m_A) l_{m_A(j)}) (|f| * D_{2^j}(x)) \\ &\quad + l_{(|m_A|-1)} (|f| * D_{2^{|m_A|+1}}(x)) \\ &\lesssim E^*(x, f) \left(l_{(|m_A|-1)} + \sum_{j=1}^{|m_A|-1} (\varepsilon_j(m_A) l_{m_A(j-1)} - \varepsilon_{j+1}(m_A) l_{m_A(j)}) \right) \\ &\lesssim E^*(x, f) \left(l_{(|m_A|-1)} + \sum_{j=1}^{|m_A|-1} |\varepsilon_j(m_A) - \varepsilon_{j+1}(m_A)| l_{m_A(j-1)} \right. \\ &\quad \left. + \sum_{j=1}^{|m_A|-1} \varepsilon_{j+1}(m_A) (l_{m_A(j)} - l_{m_A(j-1)}) \right) \end{aligned}$$

$$\lesssim E^*(x, f) \left(l_{(|m_A|-1)} + \sum_{j=1}^{|m_A|-1} |\varepsilon_j(m_A) - \varepsilon_{j+1}(m_A)| l_{m_A(j-1)} \right).$$

From the condition of the Theorem we can write

$$\sup_A \frac{|f * H_{m_A}^{(1)}(x)|}{l_{m_A}} \lesssim E^*(x, f) V_L(m_A)$$

and

$$\sup_{\lambda} \lambda \left| \left\{ \sup_A \frac{|f * H_{m_A}^{(1)}|}{|m_A|} > \lambda \right\} \right| \lesssim \|f\|_1. \quad (4.11)$$

Combining (3.5), (4.5), (4.10) and (4.11) we conclude that

$$\sup_{\lambda} \lambda \left| \left\{ \sup_A \frac{|f * F_{m_A}|}{|m_A|} > \lambda \right\} \right| \lesssim \|f\|_1.$$

By the well-known density argument we complete the proof of Theorem 2. $\square$

5. UNIFORM AND L -CONVERGENCE OF LOGARITHMIC MEANS

Denote by $C_w(\mathbb{I})$ the space of uniformly continuous functions on $\mathbb{I}$, with the supremum norm

$$\|f\|_{C_w} := \sup_{x \in \mathbb{I}} |f(x)| \quad (f \in C_w(\mathbb{I})).$$

Let $X = X(\mathbb{I})$ be either the space $L_1(\mathbb{I})$, or the space of uniformly continuous functions, that is, $C_w(\mathbb{I})$. The corresponding norm is denoted by $\|\cdot\|_X$.

For Walsh-Fourier series Fine [2] has obtained a sufficient condition for the uniform convergence which is in a complete analogy with the Dini-Lipshitz condition (see also [20]). Similar results are true for the space of integrable functions $L_1(\mathbb{I})$ [19]. Gulicev [13] has estimated the rate of uniform convergence of a Walsh-Fourier series using Lebesgue constant and modulus of continuity. Uniform convergence of Walsh-Fourier series of the functions of classes of generalized bounded variation was investigated by author [10]. This problem has been considered for Vilenkin group by Fridli [3] and Gát [5]. Lukomskii [15] considered uniform and L_1 -convergence of subsequence of partial sums of Walsh-Fourier series. In particular, he proved that the condition $\sup_A V_S(m_A) < \infty$ is necessary and sufficient condition for uniform and

L_1 -convergence of subsequence of partial sums $S_{m_A}(f)$ of Walsh-Fourier series. In Móricz and Siddiqi [17] investigated approximation properties of Nörlund means

$\frac{1}{Q_n} \sum_{k=0}^{n-1} q_{n-k} S_k f$. The case when we have $q_k := 1/k$ differs from the types discussed by Móricz and Siddiqi in [17]. His method is not applicable for logarithmic

means. In [6] it is proved that Theorem of Móricz does not hold for L_1 , C_w and $q_k := 1/k$.

In [12] it is investigated X -norm convergence of subsequence of logarithmic means of Walsh-Fourier series. In particular, the following are proved.

Theorem GT. a) Let $f \in X(\mathbb{I})$ and

$$\sup_A \frac{\log(m_A - 2^{|m_A|} + 1)}{\log m_A} \left\| \frac{1}{l_{m_A - 2^{|m_A|}}} \sum_{j=1}^{m_A - 2^{|m_A|} - 1} \frac{D_{n-j}}{j} \right\|_1 < \infty. \quad (5.1)$$

Then subsequence of Nörlund logarithmic means $t_{m_A}(f)$ converges in the norm of space $X(\mathbb{I})$.

b) If the condition (5.1) does not holds then we can find a function from the space $X(\mathbb{I})$ for which the convergence of logarithmic means $L_{m_A}(f)$ in the norm of space $X(\mathbb{I})$ does not holds.

Since

$$\begin{aligned} & \sup_A \frac{\log(m_A - 2^{|m_A|} + 1)}{\log m_A} \left\| \frac{1}{l_{m_A - 2^{|m_A|}}} \sum_{j=1}^{m_A - 2^{|m_A|} - 1} \frac{D_{n-j}}{j} \right\|_1 \\ & \sim \sup_A \frac{\log(m_A - 2^{|m_A|} + 1)}{\log m_A} \frac{1}{|m_A - 2^{|m_A|}|} \sum_{k=1}^{|m_A - 2^{|m_A|}|} |\varepsilon_k(n) - \varepsilon_{k+1}(n)| l_{n(k-1)} \\ & \sim \sup_A \frac{1}{|m_A|} \sum_{k=1}^{|m_A|} |\varepsilon_k(n) - \varepsilon_{k+1}(n)| l_{n(k-1)}, \end{aligned}$$

from Theorem GT and Theorem 1 we can prove necessary and sufficient condition for norm convergence of subsequence of Nörlund logarithmic means

Theorem 3. Let $f \in X(\mathbb{I})$. Then the condition $\sup_A V_L(m_A) < \infty$ is necessary and sufficient for convergence subsequence of Nörlund logarithmic means of Walsh-Fourier series in norm of space $X(\mathbb{I})$.

REFERENCES

- [1] I. Blahota and G. Gát, "Norm summability of Nörlund logarithmic means on unbounded Vilenkin groups," *Anal. Theory Appl.*, vol. 24, no. 1, pp. 1–17, 2008, doi: [10.1007/s10496-008-0001-z](https://doi.org/10.1007/s10496-008-0001-z). [Online]. Available: <https://doi.org/10.1007/s10496-008-0001-z>
- [2] N. J. Fine, "On the Walsh functions," *Trans. Amer. Math. Soc.*, vol. 65, pp. 372–414, 1949, doi: [10.2307/1990619](https://doi.org/10.2307/1990619). [Online]. Available: <https://doi.org/10.2307/1990619>

- [3] S. Fridli, “Approximation by Vilenkin-Fourier sums,” *Acta Math. Hungar.*, vol. 47, no. 1-2, pp. 33–44, 1986, doi: [10.1007/BF01949122](https://doi.org/10.1007/BF01949122). [Online]. Available: <https://doi.org/10.1007/BF01949122>
- [4] G. Gát, “Investigations of certain operators with respect to the Vilenkin system,” *Acta Math. Hungar.*, vol. 61, no. 1-2, pp. 131–149, 1993, doi: [10.1007/BF01872107](https://doi.org/10.1007/BF01872107). [Online]. Available: <https://doi.org/10.1007/BF01872107>
- [5] G. Gát, “Best approximation by Vilenkin-like systems,” *Acta Math. Acad. Paedagog. Nyházi. (N.S.)*, vol. 17, no. 3, pp. 161–169, 2001.
- [6] G. Gát and U. Goginava, “Uniform and L -convergence of logarithmic means of Walsh-Fourier series,” *Acta Math. Sin. (Engl. Ser.)*, vol. 22, no. 2, pp. 497–506, 2006, doi: [10.1007/s10114-005-0648-8](https://doi.org/10.1007/s10114-005-0648-8). [Online]. Available: <https://doi.org/10.1007/s10114-005-0648-8>
- [7] G. Gát and U. Goginava, “On the divergence of Nörlund logarithmic means of Walsh-Fourier series,” *Acta Math. Sin. (Engl. Ser.)*, vol. 25, no. 6, pp. 903–916, 2009, doi: [10.1007/s10114-009-7013-2](https://doi.org/10.1007/s10114-009-7013-2). [Online]. Available: <https://doi.org/10.1007/s10114-009-7013-2>
- [8] G. Gát and U. Goginava, “Norm convergence of logarithmic means on unbounded Vilenkin groups,” *Banach J. Math. Anal.*, vol. 12, no. 2, pp. 422–438, 2018, doi: [10.1215/17358787-2017-0031](https://doi.org/10.1215/17358787-2017-0031). [Online]. Available: <https://doi.org/10.1215/17358787-2017-0031>
- [9] G. Gát, U. Goginava, and G. Tkebuchava, “Convergence of logarithmic means of multiple Walsh-Fourier series,” *Anal. Theory Appl.*, vol. 21, no. 4, pp. 326–338, 2005, doi: [10.1007/BF02836855](https://doi.org/10.1007/BF02836855). [Online]. Available: <https://doi.org/10.1007/BF02836855>
- [10] U. Goginava, “On the uniform convergence of Walsh-Fourier series,” *Acta Math. Hungar.*, vol. 93, no. 1-2, pp. 59–70, 2001, doi: [10.1023/A:1013865315680](https://doi.org/10.1023/A:1013865315680). [Online]. Available: <https://doi.org/10.1023/A:1013865315680>
- [11] U. Goginava, “Almost everywhere convergence of subsequence of logarithmic means of Walsh-Fourier series,” *Acta Math. Acad. Paedagog. Nyházi. (N.S.)*, vol. 21, no. 2, pp. 169–175, 2005.
- [12] U. Goginava and G. Tkebuchava, “Convergence of subsequences of partial sums and logarithmic means of Walsh-Fourier series,” *Acta Sci. Math. (Szeged)*, vol. 72, no. 1-2, pp. 159–177, 2006.
- [13] N. V. Gulichev, “Approximation to continuous functions by Walsh-Fourier sums,” *Anal. Math.*, vol. 6, no. 4, pp. 269–280, 1980, doi: [10.1007/BF02053633](https://doi.org/10.1007/BF02053633). [Online]. Available: <https://doi.org/10.1007/BF02053633>
- [14] S. V. Konyagin, “On a subsequence of Fourier-Walsh partial sums,” *Mat. Zametki*, vol. 54, no. 4, pp. 69–75, 158, 1993, doi: [10.1007/BF01210421](https://doi.org/10.1007/BF01210421). [Online]. Available: <https://doi.org/10.1007/BF01210421>
- [15] S. F. Lukomskii, “Multiple Walsh series. Convergence in measure and almost everywhere,” *Dokl. Akad. Nauk*, vol. 358, no. 4, pp. 459–460, 1998.
- [16] N. Memić, “Almost everywhere convergence of some subsequences of the Nörlund logarithmic means of Walsh-Fourier series,” *Anal. Math.*, vol. 41, no. 1-2, pp. 45–54, 2015, doi: [10.1007/s10476-015-0104-7](https://doi.org/10.1007/s10476-015-0104-7). [Online]. Available: <https://doi.org/10.1007/s10476-015-0104-7>
- [17] F. Móricz and A. H. Siddiqi, “Approximation by Nörlund means of Walsh-Fourier series,” *J. Approx. Theory*, vol. 70, no. 3, pp. 375–389, 1992, doi: [10.1016/0021-9045\(92\)90067-X](https://doi.org/10.1016/0021-9045(92)90067-X). [Online]. Available: [https://doi.org/10.1016/0021-9045\(92\)90067-X](https://doi.org/10.1016/0021-9045(92)90067-X)
- [18] K. Nagy, “Almost everywhere convergence of a subsequence of the Nörlund logarithmic means of Walsh-Kaczmarz-Fourier series,” *J. Math. Inequal.*, vol. 3, no. 4, pp. 499–510, 2009, doi: [10.7153/jmi-03-49](https://doi.org/10.7153/jmi-03-49). [Online]. Available: <https://doi.org/10.7153/jmi-03-49>
- [19] C. W. Onneweer, “On L_1 -convergence of Walsh-Fourier series,” *Internat. J. Math. Math. Sci.*, vol. 1, no. 1, pp. 47–56, 1978, doi: [10.1155/S016117127800006X](https://doi.org/10.1155/S016117127800006X). [Online]. Available: <https://doi.org/10.1155/S016117127800006X>
- [20] F. Schipp, W. R. Wade, and P. Simon, *Walsh series*. Adam Hilger, Ltd., Bristol, 1990, an introduction to dyadic harmonic analysis, With the collaboration of J. Pál.

- [21] P. Simon, “Strong convergence of certain means with respect to the Walsh-Fourier series,” *Acta Math. Hungar.*, vol. 49, no. 3-4, pp. 425–431, 1987, doi: [10.1007/BF01951006](https://doi.org/10.1007/BF01951006). [Online]. Available: <https://doi.org/10.1007/BF01951006>

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CERTAIN DISCRETE BESSEL CONVOLUTIONS OF THE APPELL POLYNOMIALS

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Abstract. In this article, the Appell polynomials are combined with Bessel functions to introduce a hybrid family namely the Appell-Bessel functions. The infinite sums and Jacobi-Anger expansions for this family are established. The Bernoulli-Bessel and Euler-Bessel functions are introduced as particular cases of the Appell-Bessel functions and their properties are obtained.

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1. INTRODUCTION AND PRELIMINARIES

The importance of the generalized Bessel functions stems from their wide use in applications and from their implications in different fields of pure and applied mathematics. The scattering of free or weakly bounded electrons by intense laser fields is an example where generalized Bessel functions play an important role. The spectral properties of synchrotron radiation by relativistic electrons passing in magnetic undulators, the gain of free electron lasers operating on higher off-axis harmonics are the examples where generalized Bessel functions play a crucial role. The analytical and numerical study of generalized Bessel functions has revealed their interesting properties, which in some sense can be regarded as an extension of the properties of Bessel functions to a two-dimensional domain. In this connection, the relevance of generalized Bessel functions and their multi-variable extension in mathematical physics has been emphasized, since they provide analytical solutions to partial differential equations such as the multi-dimensional diffusion equation, the Schrödinger and Klein-Gordon equations. The algebraic structure underlying generalized Bessel functions can be recognized in full analogy with Bessel functions, thus providing a unifying view to the theory of both Bessel and generalized Bessel functions. Hence the interest for the generalized Bessel functions is justified, for details see [3] and references therein.

We recall that the 2-variable Bessel functions $J_n(x, y)$ are defined by the following generating function [3, p. 332 (2.7)]:

$$\exp\left(\frac{x}{2}\left(t - \frac{1}{t}\right) + \frac{y}{2}\left(t^2 - \frac{1}{t^2}\right)\right) = \sum_{n=-\infty}^{\infty} J_n(x, y)t^n. \quad (1.1)$$

The generalized Bessel functions of the above type are getting more and more importance in physics, mainly in connection with spontaneous or stimulated scattering processes for which the dipole approximation is inadequate.

Sequences of polynomials play a fundamental role in applied mathematics. One of the important classes of polynomial sequences is the class of Appell polynomial sequences [1]. The Appell polynomial sequences arise in numerous problems of applied mathematics, theoretical physics, approximation theory and several other mathematical branches. In the past few decades, there has been a renewed interest in Appell sequences.

Recall that, in 1880, Appell [1] introduced and studied sequences of n -degree polynomials $A_n(x)$, $n = 0, 1, 2, \dots$, satisfying the recurrence relation

$$\frac{d}{dx} A_n(x) = n A_{n-1}(x), \quad n = 1, 2, \dots \quad (1.2)$$

The exponential generating function of Appell polynomials is defined as:

$$A(t)e^{xt} = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!}, \quad (1.3)$$

where $A(t)$ is formal power series of the form

$$A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}, \quad a_0 \neq 0. \quad (1.4)$$

The study of the properties of ordinary and generalized polynomials is, sometimes greatly simplified by the use of concepts and formalisms associated with the “monomiality principle” [2, 7]. It has shown in [4] that the Appell polynomials $A_n(x)$ are *quasi-monomial* with respect to the following “multiplicative” and “derivative” operators:

$$\hat{M}_A = x + \frac{A'(\partial_x)}{A(\partial_x)}, \quad (1.5)$$

and

$$\hat{P}_A = \partial_x, \quad (1.6)$$

respectively, that is $\hat{M}_A$ and $\hat{P}_A$ satisfy the following relations:

$$\hat{M}_A\{A_n(x)\} = A_{n+1}(x), \quad (1.7)$$

$$\hat{P}_A\{A_n(x)\} = n A_{n-1}(x), \quad (1.8)$$

for all $n \in \mathbb{N}$. The operators $\hat{M}_A$ and $\hat{P}_A$ also satisfy the commutation relation

$$[\hat{P}_A, \hat{M}_A] = \hat{P}_A \hat{M}_A - \hat{M}_A \hat{P}_A = \hat{1} \quad (1.9)$$

and thus display the Weyl group structure. In view of the monomiality principle, other properties of the Appell polynomials can be derived from those of the $\hat{M}_A$ and $\hat{P}_A$ operators. In fact:

(i) Combining recurrences (1.7) and (1.8), it follows that

$$\hat{M}_A \hat{P}_A \{A_n(x)\} = n A_n(x), \quad (1.10)$$

which yields the following differential equation satisfied by $A_n(x)$:

$$\left(x \partial_x + \frac{A'(\partial_x)}{A(\partial_x)} \partial_x - n \right) A_n(x) = 0. \quad (1.11)$$

(ii) Since $A_0(x) = 1$, therefore $A_n(x)$ can be explicitly constructed as:

$$A_n(x) = \hat{M}_A^n \{1\}, \quad (1.12)$$

which yields the following series definition of $A_n(x)$:

$$A_n(x) = \sum_{k=0}^n \binom{n}{k} A_k x^{n-k}. \quad (1.13)$$

(iii) Identity (1.12) implies that the exponential generating function of $A_n(x)$ can be given in the form:

$$\exp(t \hat{M}_A) \{1\} = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!}, \quad |t| < \infty, \quad (1.14)$$

which yields generating function (1.3) of $A_n(x)$.

In this paper, the Appell-Bessel functions are introduced by means of generating function and their properties are studied. The corresponding results for the Bernoulli-Bessel and Euler-Bessel functions are also obtained.

2. APPELL-BESSEL FUNCTIONS ${}_A J_n(x, y)$

First, we introduce the Appell-Bessel functions by means of generating function.

Replacing x by the multiplicative operator $\hat{M}_A$ of the Appell polynomials $A_n(x)$ in the l.h.s. of generating function (1.1) and denoting the resultant Appell-Bessel functions in the r.h.s. by ${}_A J_n(x, y)$, we have

$$\exp\left(\frac{\hat{M}_A}{2} \left(t - \frac{1}{t}\right)\right) \exp\left(\frac{y}{2} \left(t^2 - \frac{1}{t^2}\right)\right) = \sum_{n=-\infty}^{\infty} {}_A J_n(x, y) t^n, \quad (2.1)$$

which in view of equation (1.14) with t replaced by $\frac{1}{2} \left(t - \frac{1}{t}\right)$ becomes

$$\exp\left(\frac{y}{2} \left(t^2 - \frac{1}{t^2}\right)\right) \sum_{n=0}^{\infty} A_n(x) \frac{\left(\frac{1}{2} \left(t - \frac{1}{t}\right)\right)^n}{n!} = \sum_{n=-\infty}^{\infty} {}_A J_n(x, y) t^n. \quad (2.2)$$

By virtue of equation (1.3), we find the following generating function for the Appell-Bessel functions ${}_A J_n(x, y)$:

$$A \left(\frac{1}{2} \left(t - \frac{1}{t}\right)\right) \exp\left(\frac{x}{2} \left(t - \frac{1}{t}\right) + \frac{y}{2} \left(t^2 - \frac{1}{t^2}\right)\right) = \sum_{n=-\infty}^{\infty} {}_A J_n(x, y) t^n. \quad (2.3)$$

Note: Since

$$\sum_{n=-\infty}^{\infty} J_n(x, y) = 1, \quad (2.4)$$

therefore, in view of equation (2.3), we have

$$\sum_{n=-\infty}^{\infty} {}_A J_n(x, y) = A(0). \quad (2.5)$$

Next, we establish certain properties of the Appell-Bessel functions ${}_A J_n(x, y)$.

Differentiating generating function (2.3) partially w.r.t. x and y respectively, the following recurrence relations for the Appell-Bessel functions ${}_A J_n(x, y)$ are obtained:

$$\frac{\partial}{\partial x} {}_A J_n(x, y) = \frac{1}{2} \left[{}_A J_{n-1}(x, y) - {}_A J_{n+1}(x, y) \right]. \quad (2.6)$$

$$\frac{\partial}{\partial y} {}_A J_n(x, y) = \frac{1}{2} \left[{}_A J_{n-2}(x, y) - {}_A J_{n+2}(x, y) \right]. \quad (2.7)$$

Consequently, the following relations are obtained:

$$\begin{aligned} \frac{\partial^k}{\partial x^k} {}_A J_n(x, y) &= \frac{1}{2^k} \left[{}_A J_{n-k}(x, y) - \binom{k}{1} {}_A J_{n-k+2}(x, y) \right. \\ &\quad \left. + \binom{k}{2} {}_A J_{n-k+4}(x, y) + \cdots + (-1)^k {}_A J_{n+k}(x, y) \right]. \end{aligned} \quad (2.8)$$

$$\frac{\partial^k}{\partial y^k} {}_A J_n(x, y) = \frac{1}{2^k} \left[{}_A J_{n-2k}(x, y) - \binom{k}{1} {}_A J_{n-2k+4}(x, y) \right]$$

$$+ \binom{k}{2} {}_A J_{n-2k+8}(x, y) + \cdots + (-1)^k {}_A J_{n+2k}(x, y) \Big]. \quad (2.9)$$

Also, we have the following relations involving partial derivatives

$$\left(\frac{1}{x} \frac{\partial}{\partial x} \right)^k \left[x^{\pm n} {}_A J_n(x, y) \right] = x^{\pm n-k} (\pm 1)^k {}_A J_{n-k}(x, y). \quad (2.10)$$

$$\left(\frac{1}{y} \frac{\partial}{\partial y} \right)^k \left[y^{\pm n} {}_A J_n(x, y) \right] = y^{\pm n-k} (\pm 1)^k {}_A J_{n-2k}(x, y). \quad (2.11)$$

As a consequence of the relations (2.8) and (2.9), we also have

$$\frac{\partial^{k+m}}{\partial x^k \partial y^m} {}_A J_n(x, y) = \frac{1}{2^{k+m}} \sum_{i=0}^k (-1)^i \binom{k}{i} \sum_{j=0}^m (-1)^j \binom{m}{j} {}_A J_{n-k-2m+2i+4j}(x, y). \quad (2.12)$$

The Jacobi-Anger expansion is useful in physics (for example, in conversion of plane waves and the cylindrical waves) and in signal processing (to describe frequency modulation signals). The following Jacobi-Anger type expansion involving Appell-Bessel functions ${}_A J_n(x, y)$ is obtained by setting $t = e^{i\phi}$ in generating function (2.3):

$$A(i \sin \phi) (\cos \alpha + i \sin \alpha) = \sum_{n=-\infty}^{\infty} {}_A J_n(x, y) e^{in\phi}, \quad (2.13)$$

where $\alpha := x \sin \phi + y \sin(2\phi)$.

In the next section, the Bernoulli-Bessel and Euler-Bessel functions are considered as members of the Appell-Bessel family.

3. BERNOULLI-BESSEL AND EULER-BESSEL FUNCTIONS

The Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are important members of the Appell family. The Bernoulli polynomials are employed in the integral representation of differentiable periodic functions and play an important role in the approximation of such functions by means of polynomials. They are also used in the remainder term of the composite Euler Maclaurin quadrature formula. The Euler polynomials are strictly connected with the Bernoulli ones and enter in the Taylor expansion in a neighborhood of the origin of the trigonometric and hyperbolic secant functions.

Since for $A(t) = \frac{t}{e^t - 1}$, the Appell polynomials reduce to the Bernoulli polynomials, therefore taking $A\left(\frac{1}{2}\left(t - \frac{1}{t}\right)\right) = \frac{\frac{1}{2}\left(t - \frac{1}{t}\right)}{e^{\frac{1}{2}\left(t - \frac{1}{t}\right)} - 1}$ in the l.h.s. of generating function (2.3) and denoting the resultant Bernoulli-Bessel functions in the r.h.s. by ${}_B J_n(x, y)$,

the following generating function of the Bernoulli-Bessel functions ${}_BJ_n(x, y)$ is obtained:

$$\frac{\frac{1}{2}\left(t - \frac{1}{t}\right)}{e^{\frac{1}{2}\left(t - \frac{1}{t}\right)} - 1} \exp\left(\frac{x}{2}\left(t - \frac{1}{t}\right) + \frac{y}{2}\left(t^2 - \frac{1}{t^2}\right)\right) = \sum_{n=-\infty}^{\infty} {}_BJ_n(x, y)t^n. \quad (3.1)$$

Similarly, for $A(t) = \frac{2}{e^t + 1}$, the Appell polynomials reduce to the Euler polynomials, therefore taking $A\left(\frac{1}{2}\left(t - \frac{1}{t}\right)\right) = \frac{2}{e^{\frac{1}{2}\left(t - \frac{1}{t}\right)} + 1}$ in the l.h.s. of generating function (2.3) and denoting the resultant Euler-Bessel functions in the r.h.s. by ${}_EJ_n(x, y)$, the following generating function of the Euler-Bessel functions ${}_EJ_n(x, y)$ is obtained:

$$\frac{2}{e^{\frac{1}{2}\left(t - \frac{1}{t}\right)} + 1} \exp\left(\frac{x}{2}\left(t - \frac{1}{t}\right) + \frac{y}{2}\left(t^2 - \frac{1}{t^2}\right)\right) = \sum_{n=-\infty}^{\infty} {}_EJ_n(x, y)t^n. \quad (3.2)$$

For the Bernoulli-Bessel functions ${}_BJ_n(x, y)$ and Euler-Bessel functions ${}_EJ_n(x, y)$, we have

$$\lim_{t \rightarrow 1} \sum_{n=-\infty}^{\infty} {}_BJ_n(x, y)t^n = 1 \quad (3.3)$$

and

$$\sum_{n=-\infty}^{\infty} {}_EJ_n(x, y) = 1, \quad (3.4)$$

respectively.

Further, the following Jacobi-Anger type expansions of the Bernoulli-Bessel and Euler-Bessel functions are obtained:

$$\frac{\sin \phi (\sin(\sin \phi - \alpha) + \sin \alpha)}{4 \sin^2\left(\frac{\sin \phi}{2}\right)} = \sum_{n=-\infty}^{\infty} {}_BJ_n(x, y) \cos(n\phi); \quad (3.5)$$

$$\frac{\sin \phi (\cos(\sin \phi - \alpha) - \cos \alpha)}{4 \sin^2\left(\frac{\sin \phi}{2}\right)} = \sum_{n=-\infty}^{\infty} {}_BJ_n(x, y) \sin(n\phi) \quad (3.6)$$

and

$$\frac{\cos(\alpha - \sin \phi) + \cos \alpha}{2 \cos^2\left(\frac{\sin \phi}{2}\right)} = \sum_{n=-\infty}^{\infty} {}_EJ_n(x, y) \cos(n\phi); \quad (3.7)$$

$$\frac{\sin(\alpha - \sin \phi) + \sin \alpha}{2 \cos^2\left(\frac{\sin \phi}{2}\right)} = \sum_{n=-\infty}^{\infty} {}_EJ_n(x, y) \sin(n\phi), \quad (3.8)$$

where $\alpha := x \sin \phi + y \sin(2\phi)$.

4. CONCLUDING REMARKS

The Appell family generated by (1.3) is obviously rather restrictive; it does not allow the treatment of some other polynomial sets on the Laguerre or the Bessel polynomials within the context of the operational formalism. Recently, Dattoli *et al.* [5] have shown that the extension of Appell family to Sheffer family [6] allows such a possibility. To give an illustration of this approach, we extend the Appell-Bessel functions to the Sheffer-Bessel functions.

A polynomial sequence $s_n(x)$ ($n = 0, 1, 2, \dots$), is called Sheffer polynomials [6], if $s_n(x)$ possesses the following exponential generating function:

$$A(t) \exp(xH(t)) = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \quad (4.1)$$

where $A(t)$ has expansion (1.4) and $H(t)$ is a power series such that

$$H(t) = \sum_{n=0}^{\infty} h_n \frac{t^n}{n!}, \quad h_1 \neq 0. \quad (4.2)$$

Clearly the Appell polynomials belong to the Sheffer family. If $s_n(x)$ are of Sheffer type, then it is always possible to find the explicit representations of the multiplicative and derivative operators $\hat{M}$ and $\hat{P}$. Among the polynomials encountered in quantum mechanics, Hermite and Laguerre polynomials are of Sheffer type, whereas Legendre, Jacobi and Gegenbauer polynomials are not.

Proceeding on the same lines as in the case of Appell-Bessel functions, following generating function for the Sheffer-Bessel functions ${}_s J_n(x, y)$ is obtained:

$$A\left(\frac{1}{2}\left(t - \frac{1}{t}\right)\right) \exp\left(xH\left(\frac{1}{2}\left(t - \frac{1}{t}\right)\right) + \frac{y}{2}\left(t^2 - \frac{1}{t^2}\right)\right) = \sum_{n=-\infty}^{\infty} {}_s J_n(x, y) t^n. \quad (4.3)$$

We note that for $A(t) = \exp(-t^2)$ and $H(t) = 2t$, the Sheffer polynomials reduce to the Hermite polynomials $H_n(x)$ [6]. Hence from equation (4.3), the following generating function for the Hermite-Bessel functions ${}_H J_n(x, y)$ is obtained:

$$\exp\left(x\left(t - \frac{1}{t}\right) + \frac{y}{2}\left(t^2 - \frac{1}{t^2}\right) - \frac{1}{4}\left(t - \frac{1}{t}\right)^2\right) = \sum_{n=-\infty}^{\infty} {}_H J_n(x, y) t^n. \quad (4.4)$$

The following recurrence relations for the Hermite-Bessel functions ${}_H J_n(x, y)$ are obtained:

$$\frac{\partial}{\partial x} {}_H J_n(x, y) = ({}_H J_{n-1}(x, y) - {}_H J_{n+1}(x, y)); \quad (4.5)$$

$$\frac{\partial}{\partial y} {}_H J_n(x, y) = \frac{1}{2} ({}_H J_{n-2}(x, y) - {}_H J_{n+2}(x, y)). \quad (4.6)$$

On taking $t = e^{i\phi}$ in equation (4.4), the following Jacobi-Anger type expansions of the Hermite-Bessel functions ${}_H J_n(x, y)$ are obtained:

$$\cos \beta \exp(\sin^2 \phi) = \sum_{n=-\infty}^{\infty} {}_H J_n(x, y) \cos(n\phi) \quad (4.7)$$

and

$$\sin \beta \exp(\sin^2 \phi) = \sum_{n=-\infty}^{\infty} {}_H J_n(x, y) \sin(n\phi), \quad (4.8)$$

where $\beta := 2x \sin \phi + y \sin(2\phi)$.

From equation (4.4), the following infinite sums for the Hermite-Bessel functions of even and odd indices are obtained:

$$\cosh\left(x\left(t - \frac{1}{t}\right)\right) \exp\left(\frac{y}{2}\left(t^2 - \frac{1}{t^2}\right) - \frac{1}{4}\left(t - \frac{1}{t}\right)^2\right) = \sum_{n=-\infty}^{\infty} {}_H J_{2n}(x, y) t^{2n} \quad (4.9)$$

and

$$\sinh\left(x\left(t - \frac{1}{t}\right)\right) \exp\left(\frac{y}{2}\left(t^2 - \frac{1}{t^2}\right) - \frac{1}{4}\left(t - \frac{1}{t}\right)^2\right) = \sum_{n=-\infty}^{\infty} {}_H J_{2n+1}(x, y) t^{2n+1}. \quad (4.10)$$

Setting $t = e^{i\phi}$ in equations (4.9) and (4.10), the following relations are deduced:

$$\cos(2x \sin \phi) \cos(y \sin(2\phi)) \exp(\sin^2 \phi) = \sum_{n=-\infty}^{\infty} {}_H J_{2n}(x, y) \cos(2n\phi); \quad (4.11)$$

$$\cos(2x \sin \phi) \sin(y \sin(2\phi)) \exp(\sin^2 \phi) = \sum_{n=-\infty}^{\infty} {}_H J_{2n}(x, y) \sin(2n\phi); \quad (4.12)$$

$$-\sin(2x \sin \phi) \sin(y \sin(2\phi)) \exp(\sin^2 \phi) = \sum_{n=-\infty}^{\infty} {}_H J_{2n+1}(x, y) \cos((2n+1)\phi); \quad (4.13)$$

$$\sin(2x \sin \phi) \cos(y \sin(2\phi)) \exp(\sin^2 \phi) = \sum_{n=-\infty}^{\infty} {}_H J_{2n+1}(x, y) \sin((2n+1)\phi). \quad (4.14)$$

Also from equation (4.4), we find the following integral representation for the Hermite-Bessel functions ${}_H J_n(x, y)$ as

$${}_H J_n(x, y) = \frac{1}{\pi} \int_0^\pi e^{\sin^2 \phi} \cos(2x \sin \phi + y \sin(2\phi) - n\phi) d\phi. \quad (4.15)$$

The Sheffer polynomials, which include Appell polynomials as a special case along with the underlying operational formalism, offer a powerful tool for investigation of the properties of a wide class of special functions. Here, we have considered Hermite-Bessel functions as one of the members of the Sheffer-Bessel family. A general approach to generate Sheffer-Bessel functions will be discussed in a forthcoming investigation.

REFERENCES

- [1] P. Appell, "Sur une classe de polynômes." *Ann. Sci. École. Norm. Sup.*, vol. 9, no. 2, pp. 119–144, 1880.
- [2] G. Dattoli, "Hermite-Bessel and Laguerre-Bessel functions: A by-product of the monomiality principle." *Proc. Melfi Sch. Adv. Top. Math. Phys.*, vol. 1, pp. 147–164, 2000.
- [3] G. Dattoli, L. Gainnnessi, L. Mezi, and A. Torre, "Theory of Generalized Bessel functions." *Nuovo Cimento*, vol. 105, no. 3, pp. 327–347, 1990.
- [4] G. Dattoli, B. Germano, M. R. Martinelli, and P. E. Ricci, "Monomiality and partial differential equations." *Math. Comput. Modelling*, vol. 50, pp. 1332–1337, 2009, doi: <https://doi.org/10.1016/j.mcm.2009.06.013>.
- [5] G. Dattoli, M. Migliorati, and H. M. Srivastava, "Sheffer polynomials, monomiality principle, algebraic methods and the theory of classical polynomials." *Math. Comput. Modelling*, vol. 45, no. 9–10, pp. 1033–1041, 2007, doi: <https://doi.org/10.1016/j.mcm.2006.08.010>.
- [6] E. D. Rainville, *Special Functions*. New York: Chelsea Publishing Co., 1971.
- [7] J. F. Steffensen, "The poweroid, an extension of the mathematical notion of power." *Acta. Math.*, vol. 73, pp. 333–366, 1941.

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APPROXIMATING SOLUTIONS OF EQUILIBRIUM PROBLEMS IN HADAMARD SPACES

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Abstract. In this paper, we study Δ -convergence and strong convergence of the sequence generated by the extragradient method for pseudo-monotone equilibrium problems in Hadamard spaces. We first show Δ -convergence of the generated sequence to a solution of the equilibrium problem, then the strong convergence of Halpern regularization method is proved. Finally we give some examples where the main results can be applied.

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1. PRELIMINARIES

Let (X, d) be a metric space. For $x, y \in X$, a mapping $c : [0, l] \rightarrow X$, where $l \geq 0$, is called a geodesic with endpoints x, y , if $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. If, for every $x, y \in X$, a geodesic with endpoints x, y exists, then we call (X, d) a geodesic metric space. Furthermore, if there exists a unique geodesic for each $x, y \in X$, then (X, d) is said to be uniquely geodesic.

A subset K of a uniquely geodesic space X is said to be convex when for any two points $x, y \in K$, the geodesic joining x and y is contained in K . For each $x, y \in X$, the image of a geodesic c with endpoints x, y is called a geodesic segment joining x and y and is denoted by $[x, y]$.

Let X be a uniquely geodesic metric space. For each $x, y \in X$ and for each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that $d(x, z) = td(x, y)$ and $d(y, z) = (1 - t)d(x, y)$. We will use the notation $(1 - t)x \oplus ty$ for denoting the unique point z satisfying the above statement.

Definition 1 ([7]). A geodesic space X is called CAT(0) space if for all $x, y, z \in X$ and $t \in [0, 1]$ it holds that

$$d^2(tx \oplus (1 - t)y, z) \leq td^2(x, z) + (1 - t)d^2(y, z) - t(1 - t)d^2(x, y).$$

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A complete CAT(0) space is called a Hadamard space.

Let (X, d) be a Hadamard space and $\{x_n\}$ be a bounded sequence in X . Take $x \in X$. Let $r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x, x_n)$. The asymptotic radius of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) | x \in X\}$$

and the asymptotic center of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X | r(x, \{x_n\}) = r(\{x_n\})\}$. It is known that in a Hadamard space, $A(\{x_n\})$ consists exactly one point.

Definition 2 (see [23], p. 3690). A sequence $\{x_n\}$ in a Hadamard space (X, d) Δ -converges to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$, for each subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

We denote Δ -convergence in X by $\xrightarrow{\Delta}$ and the metric convergence by $\rightarrow$.

We present next two known results related to the notion of Δ -convergence.

Lemma 1 ([23], Proposition 3.6). *Let X be a Hadamard space. Then, every bounded, closed and convex subset of X is Δ -compact; i.e. every bounded sequence in it, has a Δ -convergent subsequence.*

Lemma 2 ([7]). *Let (X, d) be a CAT(0) space. Then, for all $x, y, z \in X$ and $t \in [0, 1]$:*

$$d(tx \oplus (1-t)y, z) \leq td(x, z) + (1-t)d(y, z).$$

Let $C \subset X$ be nonempty, closed and convex. It is well known for any $x \in X$ there exists a unique $u \in C$ such that

$$d(u, x) = \inf\{d(z, x) : z \in C\}. \quad (1.1)$$

We define the *projection on C* , $P_C : X \rightarrow C$, by taking as $P_C(x)$ the unique $u \in C$ which satisfies (1.1).

A function $h : X \rightarrow (-\infty, +\infty]$ is called:

i) convex iff

$$h(tx \oplus (1-t)y) \leq th(x) + (1-t)h(y), \quad \forall x, y \in X \text{ and } t \in (0, 1)$$

ii) strictly convex iff

$$h(tx \oplus (1-t)y) < th(x) + (1-t)h(y), \quad \forall x, y \in X, x \neq y \text{ and } t \in (0, 1).$$

It is easy to see that each strictly convex function has at most one minimizer on X .

Take a closed and convex set $K \subset X$ and $f : X \times X \rightarrow \mathbb{R}$. The equilibrium problem $EP(f, K)$ consists of finding $x^* \in K$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in K.$$

The set of solutions of $EP(f, K)$ will be denoted as $S(f, K)$.

The equilibrium problem encompasses, among its particular cases, convex optimization problems, variational inequalities (monotone or otherwise), Nash equilibrium problems, and other problems of interest in many applications.

Equilibrium problems with monotone and pseudo-monotone bifunctions have been studied extensively in Hilbert, Banach as well as in topological vector spaces by many authors (e.g. [3, 4, 6, 12, 16]). Recently some authors have studied equilibrium problems on Hadamard manifolds (see [5, 28]). The first author and Ranjbar [21] studied a variational inequality for a nonexpansive mapping in Hadamard spaces using the notion of quasi-linearization. This variational inequality is convertible to an equilibrium problem on Hadamard spaces. The authors have studied optimization and equilibrium problems in Hilbert and Hadamard spaces (see [18–20]). In [19] the authors extended some results of pseudo-monotone equilibrium problems and the proximal point algorithm to Hadamard spaces that contain Hilbert spaces and Hadamard manifolds. In this article we study pseudo-monotone equilibrium problems in Hadamard spaces by the extragradient method.

We will deal in this paper with the extragradient (or Korpelevich's) method for equilibrium problems in Hadamard spaces, since it plays an important role in the sequel, we describe it now in some detail. Note that the prototypical example of an equilibrium problem is a variational inequality problem. Therefore we start with an introduction to its well known finite dimensional formulation when applied to variational inequalities, i.e., we assume that $X = \mathbb{R}^n$ and $f(x, y) = \langle T(x), y - x \rangle$ with $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Then $EP(f, K)$ is equivalent to the variational inequality problem $VIP(T, K)$, consisting of finding a point $x^* \in K$ such that $\langle T(x^*), y - x^* \rangle \geq 0$ for all $y \in K$. In this setting, there are several iterative methods for solving $VIP(T, K)$. In order to solve this problem, Korpelevich suggested in [26] an algorithm of the form:

$$y_n = P_K(x_n - \beta_n T(x_n)), \quad (1.2)$$

$$x_{n+1} = P_K(x_n - \beta_n T(y_n)), \quad (1.3)$$

where P_K is the projection mapping on K and β_n is a positive sequence. If T is Lipschitz continuous with constant L and $VIP(T, K)$ has solutions, then the sequence generated by (1.2)–(1.3) converges to a solution of $VIP(T, K)$ provided that $\beta_n = \beta \in (0, 1/L)$ (see [26]). Other variants of Korpelevich's method can be found in [8, 22, 27].

Extensions of Korpelevich's method to the point-to-set setting (in which case Lipschitz continuity assumption must be carefully reworked, see e.g. [30]), can be found in [1, 13, 24, 25]. All these references deal with finite dimensional spaces.

Let (X, d) be a Hadamard space and take $f : X \times X \rightarrow \mathbb{R}$. The following condition on the bifunction f is essential and we will need it in the sequel:

B_1 : $f(x, \cdot) : X \rightarrow \mathbb{R}$ is convex and lower semicontinuous (shortly, lsc) for all $x \in X$.

An extragradient method for equilibrium problems in a Hilbert space H has been studied in [32]. It has the following form:

$$y_n \in \operatorname{Argmin}_{y \in K} \left\{ \beta_n f(x_n, y) + \frac{1}{2} \|y - x_n\|^2 \right\}, \quad (1.4)$$

$$x_{n+1} \in \operatorname{Argmin}_{y \in K} \left\{ \beta_n f(y_n, y) + \frac{1}{2} \|y - x_n\|^2 \right\}, \quad (1.5)$$

where β_n is a positive sequence. Weak convergence of the sequence generated by (1.4)–(1.5) to a solution of the equilibrium problem was established in [32]. In [10], this algorithm was upgraded by adding, in each iteration, the orthogonal projection of the initial iterate x_0 onto the intersection of two halfspaces separating the solution set from the point x_{n+1} given by (1.5). With this upgrade, it was proved in [10] that the sequence of such projections is strongly convergent to a solution of the problem.

It is easy to check that for the variational inequality case, i.e. when $f(x, y) = \langle T(x), y - x \rangle$ for some $T : H \rightarrow H$, (1.4)–(1.5) reduce precisely to (1.2)–(1.3); indeed the first order optimality condition for the minimization problem in (1.4) (sufficient under convexity of $f(x_n, \cdot)$ and K) is $\langle \beta_n T(x_n) - x_n + y, u - y \rangle \geq 0$ for all $u \in K$, in which case, by an elementary property of orthogonal projections, $y = P_K(x_n - \beta_n T(x_n))$. Since y_n satisfies such condition, we have $y_n = P_K(x_n - \beta_n T(x_n))$, i.e. (1.2). In the same way, (1.5) reduces to (1.3) in the variational inequality case. Thus, it seems reasonable to view (1.4)–(1.5) as an extragradient method for equilibrium problems.

An extragradient method for solving variational inequalities has been studied in [15] and [11]. Then the extragradient method for nonsmooth equilibrium problems in Banach spaces has been recently studied in [14].

In the sequel, let $\varphi : X \rightarrow (-\infty, +\infty]$ be a convex, proper and lsc function where X is a Hadamard space. The resolvent of φ of order $\lambda > 0$ is defined at each point $x \in X$ as follows

$$J_\lambda^\varphi x := \operatorname{Argmin}_{y \in X} \left\{ \varphi(y) + \frac{1}{2\lambda} d^2(y, x) \right\}.$$

By Lemma 3.1.2 of [17] (see also Lemma 2.2.19 of [2]) for each $x \in X$, $J_\lambda^\varphi x$ exists. Therefore we easily conclude the existence of the sequence generated by the extragradient method.

For convergence of the extragradient method, some monotonicity assumptions on the bifunction f are needed. We define next two such properties for future reference: The bifunction f is said to be *monotone* if $f(x, y) + f(y, x) \leq 0$ for all $x, y \in X$, and *pseudo-monotone* if for any pair $x, y \in X$, $f(x, y) \geq 0$ implies $f(y, x) \leq 0$.

We will add now some additional conditions on the bifunction f which will be needed in the convergence analysis:

B_2 : $f(\cdot, y)$ is Δ -upper semicontinuous for all $y \in X$.

B_3 : f is Lipschitz-type continuous, i.e. there exist two positive constants c_1 and c_2 such that

$$f(x, y) + f(y, z) \geq f(x, z) - c_1 d^2(x, y) - c_2 d^2(y, z), \quad \forall x, y, z \in X.$$

B_4 : f is pseudo-monotone.

It is well known that a concave and upper semicontinuous function is always Δ -upper semicontinuous.

In this paper we will consider an extragradient method which improves upon (1.4)-(1.5) in two senses:

- i) We will deal with a rather general class of Hadamard spaces, while [10] only considers Hilbert spaces.
- ii) The convergence analysis of the method in [10] requires weak continuity of $f(\cdot, \cdot)$, which seldom holds in infinite dimensional spaces, beyond the case of affine functions. Our continuity assumptions (lower semicontinuity of $f(x, \cdot)$ for all $x \in X$ and Δ -upper semicontinuity of $f(\cdot, y)$ for all $y \in X$) are much less demanding, and covers the important concave-convex case.

We also mention another difference between our results and those of [10], regarding the additional step required for getting strong convergence, rather than weak. In [10], it consists of a projection of the initial iterate onto the intersection of two halfspaces. Our method, instead, takes a geodesic convex combination in X of the current iterate with a given point in X (see (3.3)).

The paper is organized as follows. In Section 2, we will present an extragradient method for equilibrium problems in Hadamard spaces. We prove Δ -convergence of the generated sequence to a solution of the equilibrium problem, assuming pseudo-monotonicity of the bifunction. In Section 3, we propose a variant of the extragradient method for which the generated sequence can be shown to be strongly convergent to an equilibrium point, when the bifunction is pseudo-monotone. In Section 4, we give some examples where the main results can be applied.

2. EXTRAGRADIENT METHOD AND Δ -CONVERGENCE

In this section, we study Δ -convergence of the sequence generated by the extragradient method to an equilibrium point of $EP(f, K)$. The method considered in this section is a particular case of the one studied in [32] in finite dimensional Euclidean spaces. We start with a Hadamard space X , a closed and convex set $K \subset X$ and a bifunction $f : X \times X \rightarrow \mathbb{R}$. We assume that the bifunction f satisfies B_1, B_2, B_3, B_4 and $S(f, K) \neq \emptyset$, and propose the following **Extragradient Method (EM)** for solving this problem. **Initialize:** $x_0 \in X$, $n := 0$, $0 < \alpha \leq \lambda_k \leq \beta < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$

and $k = 0, 1, 2, \dots$

Step 1: Solve the following minimization problem and let y_n be the solution of it, i.e.

$$y_n \in \operatorname{Argmin}_{y \in K} \left\{ f(x_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \right\}. \quad (2.1)$$

Step 2: Solve the following minimization problem and let x_{n+1} be the solution of it, i.e.

$$x_{n+1} \in \operatorname{Argmin}_{y \in K} \left\{ f(y_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \right\}. \quad (2.2)$$

Step 3: $n := n + 1$ and go back Step 1.

In order to prove Δ -convergence of the sequences generated by Algorithm EM to an equilibrium point, we need the following lemma.

Lemma 3. Assume that $\{x_n\}$ and $\{y_n\}$ are generated by Algorithm EM and $x^* \in S(f, K)$, then

$$d^2(x_{n+1}, x^*) \leq d^2(x_n, x^*) - (1 - 2c_1\lambda_n)d^2(x_n, y_n) - (1 - 2c_2\lambda_n)d^2(y_n, x_{n+1}) \quad (2.3)$$

Proof. Take $x^* \in S(f, K)$. Note that x_{n+1} solves the minimization problem in (2.2). Now, letting $y = tx_{n+1} \oplus (1-t)x^*$ such that $t \in [0, 1)$, we have

$$\begin{aligned} f(y_n, x_{n+1}) + \frac{1}{2\lambda_n} d^2(x_n, x_{n+1}) &\leq f(y_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \\ &= f(y_n, tx_{n+1} \oplus (1-t)x^*) + \frac{1}{2\lambda_n} d^2(x_n, tx_{n+1} \oplus (1-t)x^*) \\ &\leq tf(y_n, x_{n+1}) + (1-t)f(y_n, x^*) \\ &\quad + \frac{1}{2\lambda_n} \{td^2(x_n, x_{n+1}) + (1-t)d^2(x_n, x^*) - t(1-t)d^2(x_{n+1}, x^*)\}. \end{aligned}$$

Since $f(x^*, y_n) \geq 0$, pseudo-monotonicity of f implies that $f(y_n, x^*) \leq 0$. Hence we can write the above inequality as

$$f(y_n, x_{n+1}) \leq \frac{1}{2\lambda_n} \{d^2(x_n, x^*) - d^2(x_n, x_{n+1}) - td^2(x_{n+1}, x^*)\}.$$

By letting $t \rightarrow 1^-$ we get

$$f(y_n, x_{n+1}) \leq \frac{1}{2\lambda_n} \{d^2(x_n, x^*) - d^2(x_n, x_{n+1}) - d^2(x_{n+1}, x^*)\}. \quad (2.4)$$

On the other hand, we have

$$y_n \in \operatorname{Argmin}\{f(x_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y), y \in K\}.$$

Therefore, letting $y = ty_n \oplus (1-t)x_{n+1}$ such that $t \in [0, 1)$, we obtain

$$\begin{aligned} f(x_n, y_n) + \frac{1}{2\lambda_n} d^2(x_n, y_n) &\leq f(x_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \\ &= f(x_n, ty_n \oplus (1-t)x_{n+1}) + \frac{1}{2\lambda_n} d^2(x_n, ty_n \oplus (1-t)x_{n+1}) \\ &\leq tf(x_n, y_n) + (1-t)f(x_n, x_{n+1}) \end{aligned}$$

$$+ \frac{1}{2\lambda_n} \{t d^2(x_n, y_n) + (1-t) d^2(x_n, x_{n+1}) - t(1-t) d^2(y_n, x_{n+1})\}.$$

Then we receive to

$$f(x_n, y_n) - f(x_n, x_{n+1}) \leq \frac{1}{2\lambda_n} \{d^2(x_n, x_{n+1}) - d^2(x_n, y_n) - t d^2(y_n, x_{n+1})\}.$$

Now, if $t \rightarrow 1^-$ we get

$$f(x_n, y_n) - f(x_n, x_{n+1}) \leq \frac{1}{2\lambda_n} \{d^2(x_n, x_{n+1}) - d^2(x_n, y_n) - d^2(y_n, x_{n+1})\}. \quad (2.5)$$

Also, by B_3 , f is Lipschitz-type continuous with constants c_1 and c_2 , hence we have

$$-c_1 d^2(x_n, y_n) - c_2 d^2(y_n, x_{n+1}) + f(x_n, x_{n+1}) - f(x_n, y_n) \leq f(y_n, x_{n+1}). \quad (2.6)$$

Note that by (2.5) and (2.6), we obtain

$$\left(\frac{1}{2\lambda_n} - c_1\right) d^2(x_n, y_n) + \left(\frac{1}{2\lambda_n} - c_2\right) d^2(y_n, x_{n+1}) - \frac{1}{2\lambda_n} d^2(x_n, x_{n+1}) \leq f(y_n, x_{n+1}). \quad (2.7)$$

In the sequel by (2.4) and (2.7), we have

$$(1 - 2c_1\lambda_n) d^2(x_n, y_n) + (1 - 2c_2\lambda_n) d^2(y_n, x_{n+1}) \leq d^2(x_n, x^*) - d^2(x_{n+1}, x^*).$$

□

Remark 1. In Lemma 3, it is obvious that $\lim_{n \rightarrow \infty} d(x_n, x^*)$ exists and hence $\{x_n\}$ is bounded. Note that $\liminf_{n \rightarrow \infty} (1 - 2c_i\lambda_n) > 0$ for $i = 1, 2$. Thus we conclude from Lemma 3 that

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = \lim_{n \rightarrow \infty} d(x_{n+1}, y_n) = \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0, \quad (2.8)$$

and in the sequel, by (2.4), (2.7) and taking limit, we have $\lim_{n \rightarrow \infty} f(y_n, x_{n+1}) = 0$.

Theorem 1. Assume that the bifunction f satisfies B_1, B_2, B_3 and B_4 . In addition the solution set $S(f, K)$ is nonempty. Then the sequence $\{x_n\}$ generated by Algorithm EM, Δ -converges to a point of $S(f, K)$.

Proof. Note that x_{n+1} solves the minimization problem in (2.2). By letting $z = tx_{n+1} \oplus (1-t)y$ such that $t \in [0, 1)$ and $y \in K$, we get

$$\begin{aligned} f(y_n, x_{n+1}) + \frac{1}{2\lambda_n} d^2(x_n, x_{n+1}) &\leq f(y_n, z) + \frac{1}{2\lambda_n} d^2(x_n, z) \\ &= f(y_n, tx_{n+1} \oplus (1-t)y) + \frac{1}{2\lambda_n} d^2(x_n, tx_{n+1} \oplus (1-t)y) \\ &\leq t f(y_n, x_{n+1}) + (1-t) f(y_n, y) \\ &\quad + \frac{1}{2\lambda_n} \{t d^2(x_n, x_{n+1}) + (1-t) d^2(x_n, y) - t(1-t) d^2(x_{n+1}, y)\}. \end{aligned}$$

Using the above inequality, a straightforward calculation leads to

$$f(y_n, x_{n+1}) - f(y_n, y) \leq \frac{1}{2\lambda_n} \{d^2(x_n, y) - d^2(x_n, x_{n+1}) - t d^2(x_{n+1}, y)\}.$$

Now, if $t \rightarrow 1^-$ we obtain

$$\frac{1}{2\lambda_n} \{d^2(x_n, x_{n+1}) + d^2(x_{n+1}, y) - d^2(x_n, y)\} \leq f(y_n, y) - f(y_n, x_{n+1}), \quad (2.9)$$

also, it is easy to see that

$$-\frac{1}{2\lambda_n} d(x_n, x_{n+1}) \{d(x_{n+1}, y) + d(x_n, y)\} \leq f(y_n, y) - f(y_n, x_{n+1}). \quad (2.10)$$

Remark 1 shows that $\{x_n\}$ is bounded, therefore there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $p \in K$ such that $x_{n_k} \xrightarrow{\Delta} p$ and hence by (2.8), $y_{n_k} \xrightarrow{\Delta} p$. Replacing n by n_k in (2.10), taking limsup and using Remark 1, since $f(\cdot, y)$ is Δ -upper semicontinuous, we have

$$0 \leq \limsup_{k \rightarrow \infty} f(y_{n_k}, y) \leq f(p, y), \quad \forall y \in K.$$

Therefore, $p \in S(f, K)$. Finally Opial's Lemma in Hadamard spaces (see Lemma 2.1 in [29]) implies that $\{x_n\}$ Δ -converges to a point of $S(f, K)$. $\square$

3. HALPERN'S REGULARIZATION OF THE EM METHOD

In this section, we perform a minor modification on the EM algorithm which ensures the strong convergence of the generated sequence to a solution of $EP(f, K)$. In Hilbert spaces, this procedure, called Halpern's regularization (see, e.g., [9]), consists of taking a convex combination of a given EM iterate with a fixed point $u \in X$, where the weight given to u decreases to 0 (see (3.3)). The strong limit of the generated sequence is the projection of u onto the solution set. The modified method will be called HEM. We will assume in the sequel that X is a Hadamard space, $K \subset X$ is closed and convex, and $f : X \times X \rightarrow \mathbb{R}$ is a bifunction which satisfies B_1, B_2, B_3, B_4 and $S(f, K) \neq \emptyset$, and propose the following **Halpern's regularization of the Extragradient Method (HEM)** for solving this problem.

Initialize: $x_0, u \in X, n := 0, 0 < \alpha \leq \lambda_k \leq \beta < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$ and $k = 0, 1, 2, \dots$. Take $\{\alpha_k\} \subset (0, 1)$ such that $\lim_{k \rightarrow \infty} \alpha_k = 0$ and $\sum_{k=0}^{\infty} \alpha_k = \infty$.

Step 1: Solve the following minimization problem and let y_n be the solution of it, i.e.

$$y_n \in \operatorname{Argmin}_{y \in K} \left\{ f(x_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \right\}. \quad (3.1)$$

Step 2: Solve the following minimization problem and let z_n be the solution of it, i.e.

$$z_n \in \operatorname{Argmin}_{y \in K} \left\{ f(y_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \right\}. \quad (3.2)$$

Step 3: Determine the next approximation x_{n+1} as

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) z_n. \quad (3.3)$$

Step 4: $n := n + 1$ and go back Step 1.

In order to prove the strong convergence result by Algorithm HEM, we need the following lemmas. Although the proof of Lemma 4 is exactly similar to the proof of Lemma 3, but we rewrite it, because we need relations (3.5) and (3.8) to prove our main result in this section.

Lemma 4. Assume that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are generated by Algorithm HEM and $x^* \in S(f, K)$, then

$$d^2(z_n, x^*) \leq d^2(x_n, x^*) - (1 - 2c_1\lambda_n)d^2(x_n, y_n) - (1 - 2c_2\lambda_n)d^2(y_n, z_n) \quad (3.4)$$

Proof. Take $x^* \in S(f, K)$. Since z_n solves the minimization problem in (3.2) by letting $y = tz_n \oplus (1 - t)x^*$ such that $t \in [0, 1)$, we have

$$\begin{aligned} f(y_n, z_n) + \frac{1}{2\lambda_n} d^2(x_n, z_n) &\leq f(y_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \\ &= f(y_n, tz_n \oplus (1 - t)x^*) + \frac{1}{2\lambda_n} d^2(x_n, tz_n \oplus (1 - t)x^*) \\ &\leq tf(y_n, z_n) + (1 - t)f(y_n, x^*) + \frac{1}{2\lambda_n} \{td^2(x_n, z_n) \\ &\quad + (1 - t)d^2(x_n, x^*) - t(1 - t)d^2(z_n, x^*)\}. \end{aligned}$$

Since $f(x^*, y_n) \geq 0$, pseudo-monotonicity of f implies that $f(y_n, x^*) \leq 0$. Hence we can write the above inequality as

$$f(y_n, z_n) \leq \frac{1}{2\lambda_n} \{d^2(x_n, x^*) - d^2(x_n, z_n) - td^2(z_n, x^*)\}.$$

Now, if $t \rightarrow 1^-$, we get

$$f(y_n, z_n) \leq \frac{1}{2\lambda_n} \{d^2(x_n, x^*) - d^2(x_n, z_n) - d^2(z_n, x^*)\}. \quad (3.5)$$

On the other hand, since y_n solves the minimization problem in (3.1), by letting $y = ty_n \oplus (1 - t)z_n$ such that $t \in [0, 1)$, we have

$$\begin{aligned} f(x_n, y_n) + \frac{1}{2\lambda_n} d^2(x_n, y_n) &\leq f(x_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \\ &= f(x_n, ty_n \oplus (1 - t)z_n) + \frac{1}{2\lambda_n} d^2(x_n, ty_n \oplus (1 - t)z_n) \end{aligned}$$

$$\begin{aligned} &\leq tf(x_n, y_n) + (1-t)f(x_n, z_n) + \frac{1}{2\lambda_n}\{td^2(x_n, y_n) \\ &\quad + (1-t)d^2(x_n, z_n) - t(1-t)d^2(y_n, z_n)\}, \end{aligned}$$

which implies that

$$f(x_n, y_n) - f(x_n, z_n) \leq \frac{1}{2\lambda_n}\{d^2(x_n, z_n) - d^2(x_n, y_n) - td^2(y_n, z_n)\}.$$

Now, if $t \rightarrow 1^-$, we get

$$f(x_n, y_n) - f(x_n, z_n) \leq \frac{1}{2\lambda_n}\{d^2(x_n, z_n) - d^2(x_n, y_n) - d^2(y_n, z_n)\}. \quad (3.6)$$

Also, by B_3 , f is Lipschitz-type continuous with constants c_1 and c_2 , hence we have

$$-c_1d^2(x_n, y_n) - c_2d^2(y_n, z_n) + f(x_n, z_n) - f(x_n, y_n) \leq f(y_n, z_n). \quad (3.7)$$

Note that by (3.6) and (3.7), we obtain

$$\left(\frac{1}{2\lambda_n} - c_1\right)d^2(x_n, y_n) + \left(\frac{1}{2\lambda_n} - c_2\right)d^2(y_n, z_n) - \frac{1}{2\lambda_n}d^2(x_n, z_n) \leq f(y_n, z_n). \quad (3.8)$$

In the sequel from (3.5) and (3.8), we conclude that

$$(1 - 2c_1\lambda_n)d^2(x_n, y_n) + (1 - 2c_2\lambda_n)d^2(y_n, z_n) \leq d^2(x_n, x^*) - d^2(z_n, x^*).$$

□

Lemma 5 ([31]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ with $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\{t_n\}$ be a sequence of real numbers. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n t_n, \quad \forall n \geq 0.$$

If $\limsup_{k \rightarrow \infty} t_{n_k} \leq 0$ for every subsequence $\{s_{n_k}\}$ of $\{s_n\}$ satisfying

$$\liminf_{k \rightarrow \infty} (s_{n_k+1} - s_{n_k}) \geq 0,$$

then $\lim_{n \rightarrow \infty} s_n = 0$.

Theorem 2. *Assume that the bifunction f satisfies B_1, B_2, B_3 and B_4 . In addition the solution set $S(f, K)$ is nonempty. Then the sequence $\{x_n\}$ generated by Algorithm HEM converges strongly to $P_{S(f, K)}u$.*

Proof. Let $x^* = P_{S(f, K)}u$. Lemma 4 shows that

$$d(z_n, x^*) \leq d(x_n, x^*). \quad (3.9)$$

By (3.3) and (3.9), we obtain

$$\begin{aligned} d(x_{n+1}, x^*) &\leq \alpha_n d(u, x^*) + (1 - \alpha_n)d(z_n, x^*) \\ &\leq \alpha_n d(u, x^*) + (1 - \alpha_n)d(x_n, x^*) \end{aligned}$$

$$\begin{aligned} &\leq \max\{d(u, x^*), d(x_n, x^*)\} \\ &\leq \cdots \leq \max\{d(u, x^*), d(x_0, x^*)\}, \end{aligned}$$

which implies that $\{x_n\}$ is bounded. Thus, by (3.9), $\{z_n\}$ is also bounded. On the other hand, (3.3) and (3.9) imply

$$\begin{aligned} d^2(x_{n+1}, x^*) &\leq (1 - \alpha_n)d^2(z_n, x^*) + \alpha_n d^2(u, x^*) - \alpha_n(1 - \alpha_n)d^2(u, z_n) \\ &\leq (1 - \alpha_n)d^2(x_n, x^*) + \alpha_n d^2(u, x^*) - \alpha_n(1 - \alpha_n)d^2(u, z_n). \end{aligned} \quad (3.10)$$

We are going to prove $d^2(x_n, x^*) \rightarrow 0$. By Lemma 5, it suffices to show that

$$\limsup_{k \rightarrow \infty} (d^2(u, x^*) - (1 - \alpha_{n_k})d^2(u, z_{n_k})) \leq 0$$

for every subsequence $\{d^2(x_{n_k}, x^*)\}$ of $\{d^2(x_n, x^*)\}$ satisfying

$$\liminf_{k \rightarrow \infty} (d^2(x_{n_k+1}, x^*) - d^2(x_{n_k}, x^*)) \geq 0.$$

Consider such a sequence. We have

$$\begin{aligned} 0 &\leq \liminf_{k \rightarrow \infty} (d^2(x_{n_k+1}, x^*) - d^2(x_{n_k}, x^*)) \\ &\leq \liminf_{k \rightarrow \infty} (\alpha_{n_k} d^2(u, x^*) + (1 - \alpha_{n_k})d^2(z_{n_k}, x^*) - d^2(x_{n_k}, x^*)) \\ &= \liminf_{k \rightarrow \infty} (\alpha_{n_k} (d^2(u, x^*) - d^2(z_{n_k}, x^*)) + d^2(z_{n_k}, x^*) - d^2(x_{n_k}, x^*)) \\ &\leq \limsup_{k \rightarrow \infty} \alpha_{n_k} (d^2(u, x^*) - d^2(z_{n_k}, x^*)) + \liminf_{k \rightarrow \infty} (d^2(z_{n_k}, x^*) - d^2(x_{n_k}, x^*)) \\ &= \liminf_{k \rightarrow \infty} (d^2(z_{n_k}, x^*) - d^2(x_{n_k}, x^*)) \\ &\leq \limsup_{k \rightarrow \infty} (d^2(z_{n_k}, x^*) - d^2(x_{n_k}, x^*)) \leq 0. \end{aligned}$$

This shows that $\lim_{k \rightarrow \infty} (d^2(z_{n_k}, x^*) - d^2(x_{n_k}, x^*)) = 0$, i.e.

$$\lim_{k \rightarrow \infty} d^2(x_{n_k}, x^*) = \lim_{k \rightarrow \infty} d^2(z_{n_k}, x^*). \quad (3.11)$$

Replacing n by n_k in Lemma 4, since $\liminf_{n \rightarrow \infty} (1 - 2c_i \lambda_n) > 0$ for $i = 1, 2$, we conclude that

$$\lim_{k \rightarrow \infty} d^2(x_{n_k}, y_{n_k}) = \lim_{k \rightarrow \infty} d^2(y_{n_k}, z_{n_k}) = \lim_{k \rightarrow \infty} d^2(x_{n_k}, z_{n_k}) = 0. \quad (3.12)$$

Now, replacing n by n_k in (3.5) and (3.8), and taking limit, we get

$$\lim_{k \rightarrow \infty} f(y_{n_k}, z_{n_k}) = 0. \quad (3.13)$$

On the other hand, there exists a subsequence $\{z_{n_{k_t}}\}$ of $\{z_{n_k}\}$ and $p \in K$ such that $z_{n_{k_t}} \xrightarrow{\Delta} p$ and

$$\limsup_{k \rightarrow \infty} (d^2(u, x^*) - (1 - \alpha_{n_k})d^2(u, z_{n_k})) = \lim_{t \rightarrow \infty} (d^2(u, x^*) - (1 - \alpha_{n_{k_t}})d^2(u, z_{n_{k_t}})).$$

By Δ -lower semicontinuity of $d^2(u, \cdot)$, we obtain

$$\begin{aligned} & \limsup_{k \rightarrow \infty} (d^2(u, x^*) - (1 - \alpha_{n_k})d^2(u, z_{n_k})) \\ &= \lim_{t \rightarrow \infty} (d^2(u, x^*) - (1 - \alpha_{n_{k_t}})d^2(u, z_{n_{k_t}})) \\ &\leq d^2(u, x^*) - d^2(u, p). \end{aligned} \quad (3.14)$$

Now, note that since z_n solves the minimization problem in (3.2) by letting $z = tz_n \oplus (1-t)y$ such that $t \in [0, 1]$ and $y \in K$, we have

$$\begin{aligned} f(y_n, z_n) + \frac{1}{2\lambda_n} d^2(x_n, z_n) &\leq f(y_n, z) + \frac{1}{2\lambda_n} d^2(x_n, z) \\ &= f(y_n, tz_n \oplus (1-t)y) + \frac{1}{2\lambda_n} d^2(x_n, tz_n \oplus (1-t)y) \\ &\leq tf(y_n, z_n) + (1-t)f(y_n, y) + \frac{1}{2\lambda_n} \{td^2(x_n, z_n) \\ &\quad + (1-t)d^2(x_n, y) - t(1-t)d^2(z_n, y)\}. \end{aligned}$$

A straightforward calculation leads to

$$f(y_n, z_n) - f(y_n, y) \leq \frac{1}{2\lambda_n} \{d^2(x_n, y) - d^2(x_n, z_n) - td^2(z_n, y)\}.$$

Now, if $t \rightarrow 1^-$ we obtain

$$\frac{1}{2\lambda_n} \{d^2(x_n, z_n) + d^2(z_n, y) - d^2(x_n, y)\} \leq f(y_n, y) - f(y_n, z_n). \quad (3.15)$$

It is easy to see that

$$\frac{-1}{2\lambda_n} d(x_n, z_n) \{d(z_n, y) + d(x_n, y)\} \leq f(y_n, y) - f(y_n, z_n). \quad (3.16)$$

Since $y_{n_{k_t}} \xrightarrow{\Delta} p$, replacing n by n_{k_t} in (3.16) and then taking limsup and using (3.12) and (3.13) we get

$$0 \leq \limsup_{t \rightarrow \infty} f(y_{n_{k_t}}, y), \quad \forall y \in K.$$

Now, since $f(\cdot, y)$ is Δ -upper semicontinuous, we get

$$f(p, y) \geq 0, \quad \forall y \in K$$

i.e. $p \in S(f, K)$. Therefore we have $d(u, x^*) \leq d(u, p)$ by the definition of x^* , thus (3.14) implies that

$$\limsup_{k \rightarrow \infty} (d^2(u, x^*) - (1 - \alpha_{n_k})d^2(u, z_{n_k})) \leq 0. \quad (3.17)$$

Hence

$$d^2(x_n, x^*) \rightarrow 0,$$

i.e. $x_n \rightarrow x^* = P_{S(f, K)}u$. $\square$

4. SOME EXAMPLES AND APPLICATIONS

In this section, in order to show some applications of our main results, we give some examples of equilibrium problems in Hadamard spaces. Hilbert spaces and $\mathbb{R}$ -trees are two basic examples of Hadamard spaces, which in some sense represent the most extreme cases; curvature 0 and curvature $-\infty$. The most illuminating instances of Hadamard spaces are Hadamard manifolds. A Hadamard manifold is a complete simply connected Riemannian manifold of nonpositive sectional curvature. The class of Hadamard manifolds includes hyperbolic spaces, manifolds of positive definite matrices, the complex Hilbert ball with the hyperbolic metric and etc. The details are described in [2].

In the sequel, we first recall hyperbolic spaces. We equip $\mathbb{R}^{n+1}$ with the inner product

$$\langle x, y \rangle = -x_0 y_0 + \sum_{i=1}^n x_i y_i,$$

for $x = (x_0, x_1, \dots, x_n)$ and $y = (y_0, y_1, \dots, y_n)$. Define

$$\mathbb{H}^n := \{x = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : \langle x, x \rangle = -1, x_0 > 0\}.$$

Then $\langle \cdot, \cdot \rangle$ induces the Riemannian metric d on the tangent spaces $T_p \mathbb{H}^n \subset T_p \mathbb{R}^{n+1}$ as

$$d(x, y) = \operatorname{arccosh}(-\langle x, y \rangle), \quad \forall x, y \in \mathbb{H}^n$$

for $p \in \mathbb{H}^n$. Then $(\mathbb{H}^n, d)$ is Hadamard manifold with sectional curvature -1 at every point (see [2]).

In the following example, we give an equilibrium problem in the hyperbolic space.

Example 1. Let $X = \mathbb{H}^n$ be the hyperbolic space and $T : X \rightarrow X$ is the nonexpansive mapping defined by

$$T(x) = (x_0, -x_1, -x_2, \dots, -x_n).$$

Now, we define $f : X \times X \rightarrow \mathbb{R}$ as $f(x, y) = \langle x - T(x), y - x \rangle$. If x is an equilibrium point of $EP(f, X)$, then $f(x, y) \geq 0$ for all $y \in X$. Taking $y = T(x)$, we get

$$\langle x - T(x), T(x) - x \rangle \geq 0,$$

which implies that $-4x_1^2 - 4x_2^2 - \dots - 4x_n^2 \geq 0$, i.e. $x_1 = x_2 = \dots = x_n = 0$. On the other hand, we have $\langle x, x \rangle = -1$. Therefore we conclude that $x_0 = 1$. Hence $(1, 0, 0, \dots, 0)$ is an equilibrium point of $EP(f, X)$ and it is also a fixed point for T .

The space $P(n, \mathbb{R})$ of symmetric positive definite matrices $n \times n$ with real entries is a Hadamard manifold if it is equipped with the Riemannian metric

$$\langle X, Y \rangle_A := \text{Tr}(A^{-1}XA^{-1}Y), \quad \forall X, Y \in T_A(P(n, \mathbb{R}))$$

for every $A \in P(n, \mathbb{R})$.

Example 2. Consider $P(n, \mathbb{R})$ with $n = 1$, i.e. the positive real numbers $\mathbb{R}^+$ along with the inner product $\langle x, y \rangle_a = \frac{xy}{a^2}$ for $a \in \mathbb{R}^+$ and $x, y \in T_a\mathbb{R}^+ = \mathbb{R}$. The metric on $\mathbb{R}^+$ is defined as $d(a, b) = |\ln a - \ln b|$ and $\gamma(t) = a(\frac{b}{a})^t$ is the geodesic between $a, b \in \mathbb{R}^+$. Therefore $X = \mathbb{R}^+$ with this metric is a CAT(0) space and the inequality in Definition 1 becomes equality. If $\gamma(t) = a(\frac{b}{a})^t$ is the geodesic between $a, b \in \mathbb{R}^+$, we have

$$\ln \gamma(t) = \ln a \left(\frac{b}{a}\right)^t = \ln a + t(\ln b - \ln a) = (1-t)\ln a + t\ln b.$$

Now, we define $f : X \times X \rightarrow \mathbb{R}$ as $f(x, y) = (\ln x)(\ln y) - (\ln x)^2$. Note that

$$f(x, \gamma(t)) = (\ln x)(\ln \gamma(t)) - (\ln x)^2 = (1-t)f(x, a) + tf(x, b).$$

It is obvious that f satisfies B_2 . Now we show that f satisfies B_3 with Lipschitz constants $c_1 = c_2 = \frac{1}{2}$. Take $x, y, z \in X$, then we have

$$\begin{aligned} f(x, y) + f(y, z) - f(x, z) &= (\ln x)(\ln y) + (\ln y)(\ln z) - (\ln y)^2 - (\ln x)(\ln z) \\ &= (\ln x)(\ln y - \ln z) - (\ln y)(\ln y - \ln z) \\ &= (\ln x - \ln y)(\ln y - \ln z) \\ &\geq -d(x, y)d(y, z) \\ &\geq -\frac{1}{2}d^2(x, y) - \frac{1}{2}d^2(y, z). \end{aligned}$$

Also, the following statement shows that f is monotone

$$\begin{aligned} f(x, y) + f(y, x) &= (\ln x)(\ln y) - (\ln x)^2 + (\ln y)(\ln x) - (\ln y)^2 \\ &= -(\ln x - \ln y)^2 \leq 0. \end{aligned}$$

Finally, if we consider $K = [2, +\infty)$, then $S(f, K) \neq \emptyset$.

In the following example, we give a class of pseudo-monotone bifunctions.

Example 3. Suppose that (X, d) is a Hadamard space. Let $\varphi : X \rightarrow \mathbb{R}^+$ and $\psi : X \rightarrow \mathbb{R}$ be two functions. We define $f : X \times X \rightarrow \mathbb{R}$ as $f(x, y) = \varphi(x)(\psi(y) -$

$\psi(x)$). It is clear that f is pseudo-monotone bifunction. Now, if there exist $c_1 > 0$ and $c_2 > 0$ such that

$$(\varphi(x) - \varphi(y))(\psi(y) - \psi(z)) \geq -c_1 d^2(x, y) - c_2 d^2(y, z),$$

then f satisfies B_3 .

In the last example, we first introduce the geometric median and the Fréchet mean for a finite set of points in Hadamard spaces. Then we define two bifunctions f_j for $j = 1, 2$, which satisfy B_1, B_2, B_3, B_4 and their equilibrium points are the geometric median or the Fréchet mean of the finite set of points. This result can be used for finding median and mean in a Hadamard space.

Example 4. Let $w_1, \dots, w_n$ be positive weights satisfying $\sum_{i=1}^n w_i = 1$ and $z_1, \dots, z_n$ be in a Hadamard space (X, d) . We define the geometric median for $z_1, \dots, z_n$ as

$$\operatorname{Argmin} \left\{ \sum_{i=1}^n w_i d(x, z_i) : x \in X \right\}$$

and the Fréchet mean as

$$\operatorname{Argmin} \left\{ \sum_{i=1}^n w_i d^2(x, z_i) : x \in X \right\}.$$

Now, we define $\phi_j : X \rightarrow \mathbb{R}$ as $\phi_j(x) = \sum_{i=1}^n w_i d^j(x, z_i)$ and then we consider $f_j : X \times X \rightarrow \mathbb{R}$ as $f_j(x, y) = \phi_j(y) - \phi_j(x)$ for $j = 1, 2$. It is obvious that f_j satisfies B_1, B_2, B_3, B_4 and $S(f_j, X) \neq \emptyset$, and any equilibrium point of $EP(f_j, X)$ is the minimum point of ϕ_j for $j = 1, 2$.

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REFERENCES

- [1] T. Q. Bao and P. Q. Khanh, *A projection-type algorithm for pseudomonotone nonlipschitzian multi-valued variational inequalities*. New York, NY: Springer, 2005. doi: [10.1007/0-387-23639-2-6](https://doi.org/10.1007/0-387-23639-2-6).
- [2] M. Bačák, *Convex analysis and optimization in Hadamard spaces*. Berlin: De Gruyter, 2014, vol. 22, doi: [10.1515/9783110361629](https://doi.org/10.1515/9783110361629).
- [3] M. Bianchi and S. Schaible, "Generalized monotone bifunctions and equilibrium problems." *J. Optim. Theory Appl.*, vol. 90, no. 1, pp. 31–43, 1996, doi: [10.1007/BF02192244](https://doi.org/10.1007/BF02192244).
- [4] O. Chadli, Z. Chbani, and H. Riahi, "Equilibrium problems with generalized monotone bifunctions and applications to variational inequalities." *J. Optim. Theory Appl.*, vol. 105, no. 2, pp. 299–323, 2000.
- [5] V. Colao, G. López, G. Marino, and V. Martín-Márquez, "Equilibrium problems in Hadamard manifolds," *J. Math. Anal. Appl.*, vol. 388, no. 1, pp. 61–77, 2012, doi: [10.1016/j.jmaa.2011.11.001](https://doi.org/10.1016/j.jmaa.2011.11.001).

- [6] P. L. Combettes and S. A. Hirstoaga, "Equilibrium programming in Hilbert spaces," *J. Nonlinear Convex Anal.*, vol. 6, no. 1, pp. 117–136, 2005.
- [7] S. Dhompongsa and B. Panyanak, "On Δ -convergence theorems in CAT(0) spaces," *Comput. Math. Appl.*, vol. 56, no. 10, pp. 2572–2579, 2008, doi: [10.1016/j.camwa.2008.05.036](https://doi.org/10.1016/j.camwa.2008.05.036).
- [8] E. G. Golshtein and N. V. Tretyakov, *Modified Lagrangians and monotone maps in optimization. Transl. from the Russian by N. V. Tretyakov*. New York, NY: Wiley, 1996.
- [9] B. Halpern, "Fixed points of nonexpanding maps." *Bull. Am. Math. Soc.*, vol. 73, pp. 957–961, 1967, doi: [10.1090/S0002-9904-1967-11864-0](https://doi.org/10.1090/S0002-9904-1967-11864-0).
- [10] D. V. Hieu, "Common solutions to pseudomonotone equilibrium problems." *Bull. Iran. Math. Soc.*, vol. 42, no. 5, pp. 1207–1219, 2016.
- [11] A. N. Iusem and B. F. Svaiter, "A variant of Korpelevich's method for variational inequalities with a new search strategy." *Optimization*, vol. 42, no. 4, pp. 309–321, 1997, doi: [10.1080/02331939708844365](https://doi.org/10.1080/02331939708844365).
- [12] A. N. Iusem, G. Kassay, and W. Sosa, "On certain conditions for the existence of solutions of equilibrium problems." *Math. Program.*, vol. 116, no. 1-2 (B), pp. 259–273, 2009, doi: [10.1007/s10107-007-0125-5](https://doi.org/10.1007/s10107-007-0125-5).
- [13] A. N. Iusem and L. R. Lucambio Pérez, "An extragradient-type algorithm for non-smooth variational inequalities." *Optimization*, vol. 48, no. 3, pp. 309–332, 2000, doi: [10.1080/02331930008844508](https://doi.org/10.1080/02331930008844508).
- [14] A. N. Iusem and V. Mohebbi, "Extragradient method for nonsmooth equilibrium problems in Banach spaces," *Optimization*, 2018, doi: [10.1080/02331934.2018.1462808](https://doi.org/10.1080/02331934.2018.1462808).
- [15] A. N. Iusem and M. Nasri, "Korpelevich's method for variational inequality problems in Banach spaces," *J. Global Optim.*, vol. 50, no. 1, pp. 59–76, 2011, doi: [10.1007/s10898-010-9613-x](https://doi.org/10.1007/s10898-010-9613-x).
- [16] A. N. Iusem and W. Sosa, "On the proximal point method for equilibrium problems in Hilbert spaces." *Optimization*, vol. 59, no. 7-8, pp. 1259–1274, 2010, doi: [10.1080/02331931003603133](https://doi.org/10.1080/02331931003603133).
- [17] J. Jost, *Nonpositive curvature: geometric and analytic aspects*. Basel: Birkhäuser, 1997. doi: [10.1007/978-3-0348-8918-6](https://doi.org/10.1007/978-3-0348-8918-6).
- [18] H. Khatibzadeh and V. Mohebbi, "Proximal point algorithm for infinite pseudo-monotone bifunctions." *Optimization*, vol. 65, no. 8, pp. 1629–1639, 2016, doi: [10.1080/02331934.2016.1153639](https://doi.org/10.1080/02331934.2016.1153639).
- [19] H. Khatibzadeh and V. Mohebbi, "Monotone and pseudo-monotone equilibrium problems in Hadamard spaces." *J. Aust. Math. Soc.*, 2019, doi: [10.1017/S1446788719000041](https://doi.org/10.1017/S1446788719000041).
- [20] H. Khatibzadeh, V. Mohebbi, and S. Ranjbar, "New results on the proximal point algorithm in non-positive curvature metric spaces." *Optimization*, vol. 66, no. 7, pp. 1191–1199, 2017, doi: [10.1080/02331934.2017.1326044](https://doi.org/10.1080/02331934.2017.1326044).
- [21] H. Khatibzadeh and S. Ranjbar, "A variational inequality in complete CAT(0) spaces." *J. Fixed Point Theory Appl.*, vol. 17, no. 3, pp. 557–574, 2015, doi: [10.1007/s11784-015-0245-0](https://doi.org/10.1007/s11784-015-0245-0).
- [22] E. N. Khobotov, "A modification of the extragradient method for solving variational inequalities and some optimization problems," *Zh. Vychisl. Mat. i Mat. Fiz.*, vol. 27, no. 10, pp. 1462–1473, 1597, 1987.
- [23] W. A. Kirk and B. Panyanak, "A concept of convergence in geodesic spaces." *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 68, no. 12, pp. 3689–3696, 2008, doi: [10.1016/j.na.2007.04.011](https://doi.org/10.1016/j.na.2007.04.011).
- [24] I. V. Konnov, "On combined relaxation methods' convergence rates." *Russ. Math.*, vol. 37, no. 12, pp. 89–92, 1993.
- [25] I. Konnov, *Combined relaxation methods for variational inequalities*. Berlin: Springer, 2001, vol. 495, doi: [10.1007/978-3-642-56886-2](https://doi.org/10.1007/978-3-642-56886-2).
- [26] G. M. Korpelevich, "An extragradient method for finding saddle points and for other problems." *Mat. Metody Resheniya Èkon. Zadach*, vol. 12, pp. 747–756, 1976.

- [27] P. Marcotte, “Application of Khobotov’s algorithm to variational inequalities and network equilibrium problems.” *INFOR*, vol. 29, no. 4, pp. 258–270, 1991.
- [28] M. A. Noor and K. I. Noor, “Some algorithms for equilibrium problems on Hadamard manifolds.” *J. Inequal. Appl.*, vol. 2012, p. 8, 2012, doi: [10.1186/1029-242X-2012-230](https://doi.org/10.1186/1029-242X-2012-230).
- [29] S. Ranjbar and H. Khatibzadeh, “ Δ -convergence and w -convergence of the modified Mann iteration for a family of asymptotically nonexpansive type mappings in complete CAT(0) spaces.” *Fixed Point Theory*, vol. 17, no. 1, pp. 151–158, 2016.
- [30] S. M. Robinson and S. Lu, “Solution continuity in variational conditions.” *J. Glob. Optim.*, vol. 40, no. 1-3, pp. 405–415, 2008, doi: [10.1007/s10898-007-9192-7](https://doi.org/10.1007/s10898-007-9192-7).
- [31] S. Saejung and P. Yotkaew, “Approximation of zeros of inverse strongly monotone operators in Banach spaces.” *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 75, no. 2, pp. 742–750, 2012, doi: [10.1016/j.na.2011.09.005](https://doi.org/10.1016/j.na.2011.09.005).
- [32] D. Q. Tran, M. L. Dung, and V. H. Nguyen, “Extragradient algorithms extended to equilibrium problems.” *Optimization*, vol. 57, no. 6, pp. 749–776, 2008, doi: [10.1080/02331930601122876](https://doi.org/10.1080/02331930601122876).

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PARTIAL SUMS OF THE GAUSSIAN q -BINOMIAL COEFFICIENTS, THEIR RECIPROCAL, SQUARE AND SQUARED RECIPROCAL WITH APPLICATIONS

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Abstract. In this paper, we shall derive formulae for *partial* sums of the Gaussian q -binomial coefficients, their reciprocals, squares and squared reciprocals. To prove the claimed results, we use q -calculus. As applications of our results, we give some interesting generalized Fibonomial sums formulae.

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1. INTRODUCTION

For $n > 1$, define the second order linear sequences $\{U_n\}$ and $\{V_n\}$ by

$$\begin{aligned}U_n &= pU_{n-1} + U_{n-2}, \quad U_0 = 0, \quad U_1 = 1, \\V_n &= pV_{n-1} + V_{n-2}, \quad V_0 = 2, \quad V_1 = p.\end{aligned}$$

When $p = 1$, $U_n = F_n$ (n th Fibonacci number) and $V_n = L_n$ (n th Lucas number), resp. Falcon and Plaza named the previous sequences as k -Fibonacci and k -Lucas numbers, see [5, 6].

For $n \geq k \geq 1$, define the generalized Fibonomial coefficients by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_U := \frac{U_1 U_2 \dots U_n}{(U_1 U_2 \dots U_k)(U_1 U_2 \dots U_{n-k})}$$

with $\left\{ \begin{matrix} n \\ 0 \end{matrix} \right\}_U = \left\{ \begin{matrix} n \\ n \end{matrix} \right\}_U = 1$. When $p = 1$, we obtain the usual Fibonomial coefficients, denoted by $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_F$. For more details about the Fibonomial and generalized Fibonomial coefficients, see [7, 9, 22].

The Binet forms are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

where $\alpha, \beta = \left(p \pm \sqrt{p^2 + 4} \right) / 2$.

Throughout this paper we will use the following notations: the q -Pochhammer symbol $(x; q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})$ and the Gaussian q -binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$

The link between the generalized Fibonomial and Gaussian q -binomial coefficients is

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_U = \alpha^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q \quad \text{with } q = -\alpha^{-2}.$$

By taking $q = \beta/\alpha$, the Binet formulæ are reduced to the following forms:

$$U_n = \alpha^{n-1} \frac{1-q^n}{1-q} \quad \text{and} \quad V_n = \alpha^n (1+q^n),$$

where $\mathbf{i} = \sqrt{-1} = \alpha\sqrt{q}$. For later use note that q -form of the coefficient p in the recurrence relations of $\{U_n\}$ and $\{V_n\}$ is $(1+q)(-q)^{-1/2}$.

The Fibonomial coefficients surprisingly appear in several places in the literature (for more details, we refer to [4, 10, 11, 17, 18]). Nowadays interesting sums including the Fibonomial coefficients with certain factors or sign functions have been introduced and computed by several authors (see [12–16, 19–21, 23]).

Marques and Trojovský [19] presented some Fibonomial sums formulæ with the Fibonacci and Lucas numbers as coefficients. For example, for positive integers m and n , they showed that

$$\sum_{j=0}^{4m+2} (-1)^{\frac{j(j-1)}{2}} \left\{ \begin{matrix} 4m \\ j \end{matrix} \right\}_F F_{n+4m-j} = \frac{1}{2} F_{2m+n} \sum_{j=0}^{4m} (-1)^{\frac{j(j-1)}{2}} \left\{ \begin{matrix} 4m \\ j \end{matrix} \right\}_F L_{2m-j}.$$

Kiliç and Prodinger [14] gave a systematic approach to compute certain sums of squares of Fibonomial coefficients with finite products of generalized Fibonacci and Lucas numbers as coefficients. For example, if n is nonnegative integer, then they proved the following Gaussian q -binomial sums identity

$$\begin{aligned} \sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 (-1)^k q^{k^2-2kn-3k} (1-q^{2k})^2 \\ = 2(-1)^{n+1} q^{-n^2-2n-2} \frac{(1+q)(1-q^{2n+1})(1-q^{2n+1})}{(1+q^{2n})} \begin{bmatrix} 2n+1 \\ n \end{bmatrix}_{q^2}. \end{aligned}$$

Recently Marques and Trojovský [20] derived various interesting Fibonomial sums formulæ with certain weight functions. For example, they gave that for any nonnegative integers l and n ,

$$\sum_{j=0}^{4l+3} \operatorname{sgn}(2l+1-j) \left\{ \begin{matrix} 4l+3 \\ j \end{matrix} \right\}_F F_{n-j} = \frac{F_{2l}}{F_{4l+3}} \left\{ \begin{matrix} 4l+3 \\ 2l+1 \end{matrix} \right\}_F F_{n-4l-3} \quad (1.1)$$

and

$$\sum_{j=0}^{4l+1} \operatorname{sgn}(2l-j) \left\{ \begin{matrix} 4l+1 \\ j \end{matrix} \right\}_F F_{n-j} = -\frac{F_{2l-1}}{F_{4l+1}} \left\{ \begin{matrix} 4l+1 \\ 2l \end{matrix} \right\}_F F_{n-4l-1}, \quad (1.2)$$

where $\operatorname{sgn}(x)$ denotes the sign function of x , defined by

$$\operatorname{sgn}(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Much recently Kılıç and Akkuş [1] generalized all the results of [20] through the Gaussian q -binomial coefficients instead of the Fibonomial coefficients with additional parameters. They also gave analogues of all the sums formulæ whose upper bounds are even integers. The authors proved the claimed results by mainly and analytically q -calculus, and the celebrated Zeilberger algorithm for some steps of their proofs. For convenience of the readers, we recall two sums formulæ from [1]:

$$\begin{aligned} & \sum_{j=0}^{4l+3} \operatorname{sgn}(2l+1-j) \left[\begin{matrix} 4l+3 \\ j \end{matrix} \right]_q (-1)^{\frac{1}{2}j(j-2)} q^{-\frac{1}{2}j(4l-j+2)} (1-q^{n-j}) z^{[2 \uparrow j]} \\ &= q^{2-2l^2} \frac{1-q^{2l}}{1-q^{4l+3}} \left[\begin{matrix} 4l+3 \\ 2l+1 \end{matrix} \right]_q (1-q^{n-4l-3}), \end{aligned}$$

and

$$\begin{aligned} & \sum_{j=0}^{4l+1} \operatorname{sgn}(2l-j) \left[\begin{matrix} 4l+1 \\ j \end{matrix} \right]_q (-1)^{\frac{1}{2}j^2} q^{\frac{1}{2}j(j-4l)} (1-q^{n-j}) z^{[2 \downarrow j]} \\ &= -(-q)^{\frac{4l+3-4l^2}{2}} \frac{1-q^{2l-1}}{1-q^{4l+1}} \left[\begin{matrix} 4l+1 \\ 2l \end{matrix} \right]_q (1-q^{n-4l-1}), \end{aligned}$$

where $z = (1+q)(-q)^{-1/2}$, resp. If one take $q = \left(p - \sqrt{p^2 + 4} \right) / \left(p + \sqrt{p^2 + 4} \right)$, then these sums are reduced to the following generalized Fibonomial sums formulæ: For nonnegative integers l and n

$$\sum_{j=0}^{4l+3} \operatorname{sgn}(2l+1-j) p^{[2 \uparrow j]} \left\{ \begin{matrix} 4l+3 \\ j \end{matrix} \right\}_U U_{n-j} = \frac{U_{2l}}{U_{4l+3}} \left\{ \begin{matrix} 4l+3 \\ 2l+1 \end{matrix} \right\}_U U_{n-4l-3},$$

and

$$\sum_{j=0}^{4l+1} \text{sgn}(2l-j) p^{[2|j]} \begin{Bmatrix} 4l+1 \\ j \end{Bmatrix}_U U_{n-j} = -\frac{U_{2l-1}}{U_{4l+1}} \begin{Bmatrix} 4l+1 \\ 2l \end{Bmatrix}_U U_{n-4l-1},$$

where $[]$ stands for the Iverson notation (see [8]). We would like to take attention of the readers to factors $p^{[2|j]}$ and $p^{[2\vdash j]}$ in these generalizations just above. These are not easily seen while deriving the generalized sums formulæ. Indeed, when $p = 1$, these generalized Fibonomial sums formulæ are reduced to the sums formulæ (1.1) and (1.2) given in [20]. Similarly when $q = (1 + \sqrt{5}) / (1 - \sqrt{5})$ or equivalently $p = 1$, some of the results of [1] cover the results of [20].

Othsuka conjectured the following two advanced problems, H-764 and H-768 (see [2, 3]). Here we recall these problems:

Advanced Problem H-764: For $n \geq 1$, prove that

$$(i) \sum_{k=0}^n F_{2(n-k)} \begin{Bmatrix} 2n \\ k \end{Bmatrix}_F = \frac{F_n F_{n+1}}{F_{2n-1}} \begin{Bmatrix} 2n \\ n \end{Bmatrix}_F,$$

$$(ii) \sum_{k=0}^n F_{2(n-k)} \begin{Bmatrix} 2n \\ k \end{Bmatrix}_F^2 = \frac{F_n}{L_n} \begin{Bmatrix} 2n \\ n \end{Bmatrix}_F^2.$$

Advanced Problem H-768: For $n \geq 1$, prove that

$$(i) \sum_{k=0}^n F_{2(n-k)} \begin{Bmatrix} 2n \\ k \end{Bmatrix}_F^{-1} = \frac{F_{2n+1}(F_{2n+2} + 1)}{F_{2n+3}} - \frac{F_{n+1}F_{n+3}}{F_{2n+3}} \begin{Bmatrix} 2n \\ n \end{Bmatrix}_F^{-1},$$

$$(ii) \sum_{k=0}^n F_{2(n-k)} \begin{Bmatrix} 2n \\ k \end{Bmatrix}_F^{-2} = \frac{F_{2n+1}^2}{F_{2n+2}} - \frac{F_{n+1}}{L_{n+1}} \begin{Bmatrix} 2n \\ n \end{Bmatrix}_F^2.$$

In this paper, inspired by the results of [1] and earlier partial q -binomial sums formulæ, we shall derive new kinds interesting *partial* sums formulæ including the Gaussian q -binomial coefficients which are completely different from the sums formulæ given in [1]. We summarize what we present in this paper below.

- *Sums of half* of the Gaussian q -binomial coefficients.
- *Partial sums of square* of the Gaussian q -binomial coefficients.
- *Partial sums of reciprocals* of the Gaussian q -binomial coefficients.
- *Partial sums of squared reciprocals* of the Gaussian q -binomial coefficients.

All above sums will be computed with certain weight functions. Further we notice that special cases of our results give us solutions for Advanced Problems H-764 and H-768 in [2, 3]. All the identities and formulæ we will obtain hold for general q , and results about Fibonomial and Fibonacci numbers come out as corollaries for the

special choice of q . One could derive many special corollaries by choosing special q values.

2. UPPER BOUND CASES

Now we present our results. Before this, we give an auxiliary lemma and then give one of our main results.

Lemma 1. *For nonnegative integer n , any nonzero constant c and any function f ,*
(i)

$$\sum_{j=0}^n c^{[2 \nmid j]} f(j) = \sum_{j=0}^n \left(\frac{c+1-(c-1)(-1)^j}{2} \right) f(j),$$

(ii)

$$\sum_{j=0}^n c^{[2 | j]} f(j) = \sum_{j=0}^n \left(\frac{c+1+(c-1)(-1)^j}{2} \right) f(j),$$

where $[]$ stands for the Iverson notation.

Proof. For any integer j , since $[2 | j] = (1 + (-1)^j)/2$ and

$$\begin{aligned} c^{[2 \nmid j]} f(j) &= [2 | j] f(j) + c [2 \nmid j] f(j) = ([2 | j] + c(1 - [2 | j])) f(j) \\ &= (c - (c-1)[2 | j]) f(j) = \left(\frac{c+1-(c-1)(-1)^j}{2} \right) f(j), \end{aligned}$$

the first claim (i) follows. The latter is similarly proven. $\square$

Theorem 1. (i) For even n ,

$$\begin{aligned} \sum_{j=0}^n \begin{bmatrix} 2n \\ j \end{bmatrix}_q \mathbf{i}^{j^2} (-1)^{j(n-1)} q^{\frac{j(j-2n+2)}{2}} (1 - q^{2n-2j}) z^{[2 \nmid j]} \\ = (-1)^{n+1} \mathbf{i}^{-n^2} q^{-\frac{1}{2}n^2+n-1} \frac{(1-q^n)(1-q^{n+1})}{1-q^{2n-1}} \begin{bmatrix} 2n \\ n \end{bmatrix}_q, \end{aligned}$$

(ii) For odd n ,

$$\begin{aligned} \sum_{j=0}^n \begin{bmatrix} 2n \\ j \end{bmatrix}_q \mathbf{i}^{j^2} (-1)^{j(n-1)} (1 - q^{2n-2j}) z^{[2 | j]} \\ = (-1)^{n+1} \mathbf{i}^{-n^2} q^{-\frac{1}{2}n^2+n-1} \frac{(1-q^n)(1-q^{n+1})}{1-q^{2n-1}} \begin{bmatrix} 2n \\ n \end{bmatrix}_q, \end{aligned}$$

where $z = -\mathbf{i}q^{-1/2}(1+q)$.

Proof. We prove the claim (i). The latter is similar. Since n is even, say $n = 2k$, then we have to prove that

$$\begin{aligned} \sum_{j=0}^{2k} \begin{bmatrix} 4k \\ j \end{bmatrix}_q (-q)^{\frac{j(j-4k+2)}{2}} (1-q^{4k-2j}) \left(\frac{z+1-(z-1)(-1)^j}{2} \right) \\ = (-q)^{-2k^2+2k-1} \frac{(1-q^{2k})(1-q^{2k+1})}{(1-q^{4k-1})} \begin{bmatrix} 4k \\ 2k \end{bmatrix}_q. \end{aligned}$$

Note that for any functions $F(j)$ and $G(j)$ of j , the following equality holds

$$\sum_{j=0}^{2n} [F(j) - G(j)] = \sum_{j=0}^n [F(2j) - G(2j)] + \sum_{j=0}^{n-1} [F(2j+1) - G(2j+1)].$$

By this fact, we rewrite the LHS of the claim mentioned above as

$$\begin{aligned} \sum_{j=0}^{2k} \begin{bmatrix} 4k \\ j \end{bmatrix}_q (-q)^{\frac{j(j-4k+2)}{2}} (1-q^{4k-2j}) \left(\frac{z+1-(z-1)(-1)^j}{2} \right) \\ = \frac{1}{2} \sum_{j=0}^{2k} \begin{bmatrix} 4k \\ j \end{bmatrix}_q (-q)^{\frac{j(j-4k+2)}{2}} (1-q^{4k-2j}) (z+1-(z-1)(-1)^j) \\ = \frac{(z+1)}{2} \sum_{j=0}^k \begin{bmatrix} 4k \\ 2j \end{bmatrix}_q q^{2j(j-2k+1)} (1-q^{4k-4j}) \\ - \frac{(z-1)}{2} \sum_{j=0}^k \begin{bmatrix} 4k \\ 2j \end{bmatrix}_q q^{2j(j-2k+1)} (1-q^{4k-4j}) \\ + \frac{1}{2} \sum_{j=0}^{k-1} \begin{bmatrix} 4k \\ 2j+1 \end{bmatrix}_q (-q)^{\frac{(2j+1)(2j-4k+3)}{2}} (1-q^{4k-4j-2}) (z+1) \\ + \frac{1}{2} \sum_{j=0}^{k-1} \begin{bmatrix} 4k \\ 2j+1 \end{bmatrix}_q (-q)^{\frac{(2j+1)(2j-4k+3)}{2}} (1-q^{4k-4j-2}) (z-1) \\ = \sum_{j=0}^k \begin{bmatrix} 4k \\ 2j \end{bmatrix}_q q^{2j(j-2k+1)} (1-q^{4k-4j}) \\ + z \sum_{j=0}^{k-1} \begin{bmatrix} 4k \\ 2j+1 \end{bmatrix}_q (-q)^{\frac{(2j+1)(2j-4k+3)}{2}} (1-q^{4k-4j-2}), \end{aligned}$$

which, by the identity

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q = \frac{1-q^{n-k+1}}{1-q^{k+1}} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q,$$

equals

$$\begin{aligned} & \sum_{j=0}^k \begin{bmatrix} 4k \\ 2j \end{bmatrix}_q q^{2j(j-2k+1)} (1-q^{4k-4j}) \\ & + \sum_{j=0}^{k-1} \begin{bmatrix} 4k \\ 2j \end{bmatrix}_q (-q)^{\frac{(2j+1)(2j-4k+3)}{2}} (1-q^{4k-4j-2}) \frac{(1-q^{4k-2j})}{1-q^{2j+1}} z \\ & = \sum_{j=0}^{k-1} \begin{bmatrix} 4k \\ 2j \end{bmatrix}_q q^{2j(j-2k+1)} (1-q^{4k-4j}) \\ & + \sum_{j=0}^{k-1} \begin{bmatrix} 4k \\ 2j \end{bmatrix}_q (-q)^{\frac{(2j+1)(2j-4k+3)}{2}} (1-q^{4k-4j-2}) \frac{(1-q^{4k-2j})}{1-q^{2j+1}} z \\ & = \frac{-q^{1+2k-2k^2} (q; q)_{4k}}{(-q+q^{4k})(q; q)_{2k}(q; q)_{2k-2}} + \frac{q^{2k-2k^2} (1+q)(q; q)_{4k}}{(-q+q^{4k})(q; q)_{2k-1}^2} \end{aligned}$$

which, by the definition of the Gaussian q -binomial coefficients, equals

$$\begin{aligned} & q^{2k-2k^2} \frac{(1-q^{2k-1})(1-q^{2k})}{1-q^{4k-1}} \begin{bmatrix} 4k \\ 2k \end{bmatrix}_q \\ & - q^{2k-2k^2-1} (1+q) \frac{(1-q^{2k})(1-q^{2k+1})}{1-q^{4k-1}} \begin{bmatrix} 4k \\ 2k-1 \end{bmatrix}_q \\ & = q^{2k-2k^2} \frac{(1-q^{2k-1})(1-q^{2k})}{1-q^{4k-1}} \begin{bmatrix} 4k \\ 2k \end{bmatrix}_q - q^{2k-2k^2-1} (1+q) \frac{(1-q^{2k})^2}{1-q^{4k-1}} \begin{bmatrix} 4k \\ 2k \end{bmatrix}_q \\ & = q^{2k-2k^2-1} \frac{1-q^{2k}}{1-q^{4k-1}} \begin{bmatrix} 4k \\ 2k \end{bmatrix}_q \left[q(1-q^{2k-1}) - (1+q)(1-q^{2k}) \right] \\ & = -q^{2k-2k^2-1} \frac{(1-q^{2k})(1-q^{2k+1})}{1-q^{4k-1}} \begin{bmatrix} 4k \\ 2k \end{bmatrix}_q, \end{aligned}$$

as claimed for even n such that $n = 2k$. The other claim is similarly proven. $\square$

By taking $q = (p - \sqrt{p^2 + 4}) / (p + \sqrt{p^2 + 4})$ in Theorem 1, we have the following result.

Corollary 1. (i)

For odd n ,

$$\sum_{j=0}^n \left\{ \begin{matrix} 2n \\ j \end{matrix} \right\}_U p^{[2|j]} U_{2n-2j} = \frac{U_n U_{n+1}}{U_{2n-1}} \left\{ \begin{matrix} 2n \\ n \end{matrix} \right\}_U.$$

(ii)

For even n ,

$$\sum_{j=0}^n \left\{ \begin{matrix} 2n \\ j \end{matrix} \right\}_U p^{[2 \nmid j]} U_{2n-2j} = \frac{U_n U_{n+1}}{U_{2n-1}} \left\{ \begin{matrix} 2n \\ n \end{matrix} \right\}_U.$$

We notice that when $p = 1$, the results of Corollary 1 cover solutions for the advanced problem 764(i) [2].

Theorem 2. For nonnegative integers n and m ,

$$\sum_{j=0}^m \left[\begin{matrix} n \\ j \end{matrix} \right]_q^2 q^{j(j-n+1)} (1 - q^{n-2j}) = q^{m(m-n+1)} (1 - q^{n-m}) \left[\begin{matrix} n \\ m \end{matrix} \right]_q \left[\begin{matrix} n-1 \\ m \end{matrix} \right]_q.$$

Proof. Denote the LHS of the claim by $F(n, m)$, that is,

$$F(n, m) = \sum_{j=0}^m \left[\begin{matrix} n \\ j \end{matrix} \right]_q^2 q^{j(j-n+1)} (1 - q^{n-2j}).$$

For $m \geq n$, $F(n, m)$ is a whole sum and equals 0. To see this fact, consider

$$F(n, m) = \sum_{j \geq 0} \left[\begin{matrix} n \\ j \end{matrix} \right]_q^2 q^{j(j-n+1)} (1 - q^{n-2j}),$$

which, by taking $n - j$ instead of j , gives us

$$F(n, m) = \sum_{j \geq 0} \left[\begin{matrix} n \\ j \end{matrix} \right]_q^2 q^{(j-n)(j-1)} (1 - q^{2j-n}) = -F(n, m),$$

which gives us $F(n, m) = 0$, as claimed.

Define

$$G(n, m) = q^{m(m-n+1)} (1 - q^{n-m}) \left[\begin{matrix} n \\ m \end{matrix} \right]_q \left[\begin{matrix} n-1 \\ m \end{matrix} \right]_q.$$

Then we have

$$G(n, m) = F(n, m),$$

which follows from

$$\begin{aligned}
& G(n, j) - G(n, j-1) \\
&= q^{j(j-n+1)}(1-q^{n-j}) \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} n-1 \\ j \end{bmatrix}_q - q^{(j-1)(j-n)}(1-q^{n-j+1}) \begin{bmatrix} n \\ j-1 \end{bmatrix}_q \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q \\
&= q^{j(j-n+1)} \begin{bmatrix} n \\ j \end{bmatrix}_q \left((1-q^{n-j}) \frac{(q; q)_{n-1}}{(q; q)_j (q; q)_{n-j-1}} - \frac{q^{n-2j} (1-q^j)^2 (q; q)_{n-1}}{(1-q^{n-j}) (q; q)_j (q; q)_{n-j-1}} \right) \\
&= q^{j(j-n+1)} \begin{bmatrix} n \\ j \end{bmatrix}_q \left(\frac{(1-q^{n-j})^2}{1-q^{cn}} \begin{bmatrix} n \\ j \end{bmatrix}_q - \frac{q^{n-2j} (1-q^j)^2}{1-q^n} \begin{bmatrix} n \\ j \end{bmatrix}_q \right) \\
&= q^{j(j-n+1)} \begin{bmatrix} n \\ j \end{bmatrix}_q^2 \left(\frac{(1-q^{n-j})^2 - q^{n-2j} (1-q^j)^2}{1-q^n} \right) \\
&= q^{j(j-n+1)} \begin{bmatrix} n \\ j \end{bmatrix}_q^2 \left(\frac{(1-q^{n-2j})(1-q^n)}{1-q^n} \right) \\
&= q^{j(j-n+1)}(1-q^{n-2j}) \begin{bmatrix} n \\ j \end{bmatrix}_q^2,
\end{aligned}$$

as claimed. $\square$

As special cases of Theorem 2 with $q = (p - \sqrt{p^2 + 4}) / (p + \sqrt{p^2 + 4})$, " $m \rightarrow n, n \rightarrow 2n$ " and " $m \rightarrow n, n \rightarrow 2n$ ", we have the following result, resp.

Corollary 2. For nonnegative integer n ,

$$\sum_{j=0}^n \left\{ \begin{matrix} 2n \\ j \end{matrix} \right\}_U^2 U_{2n-2j} = \frac{U_n}{V_n} \left\{ \begin{matrix} 2n \\ n \end{matrix} \right\}_U^2 \quad \text{and} \quad \sum_{j=0}^{2n} U_{2n-2j} \left\{ \begin{matrix} 2n \\ j \end{matrix} \right\}_U^2 = 0.$$

When $p = 1$, the first result of Corollary 2 gives us a solution for the advanced problem 764(ii) [2].

Now we present a sum formula for the squared reciprocals of the Gaussian q -binomial coefficient without proof.

Theorem 3. For nonnegative integers n and m ,

$$\sum_{k=0}^m \begin{bmatrix} n \\ k \end{bmatrix}_q^{-2} q^{k(n+1-k)} (1-q^{n-2k}) = \frac{(1-q^{n+1})^2}{1-q^{n+2}} - q^{(m+1)(n-m)} \frac{(1-q^{m+1})^2}{(1-q^{n+2})} \begin{bmatrix} n \\ m \end{bmatrix}_q^{-2}.$$

As special cases of Theorem 3 with $q = (p - \sqrt{p^2 + 4}) / (p + \sqrt{p^2 + 4})$, " $m \rightarrow n, n \rightarrow 2n$ " and " $m \rightarrow n, n \rightarrow 2n$ ", we have the following result, resp.

Corollary 3. For nonnegative integer n ,

$$\sum_{j=0}^n U_{2n-2j} \left\{ \begin{matrix} 2n \\ j \end{matrix} \right\}_U^{-2} = \frac{U_{2n+1}^2}{U_{2n+2}} - \frac{U_{n+1}}{V_{n+1}} \left\{ \begin{matrix} 2n \\ n \end{matrix} \right\}_U^{-2} \quad \text{and} \quad \sum_{j=0}^{2n} U_{2n-2j} \left\{ \begin{matrix} 2n \\ j \end{matrix} \right\}_U^{-2} = 0.$$

When $p = 1$, the first result of Corollary 3 gives a solution for the advanced problem 768(ii) [3].

3. ADDITIONAL SUMS FORMULÆ

In this Section, we will give new partial sums formulæ including the Gaussian q -binomial coefficients and their reciprocals.

Theorem 4. For nonnegative integers n and m ,

(i)

$$\sum_{j=0}^m \left[\begin{matrix} n \\ j \end{matrix} \right]_q q^{j(j+1-n)} (1 - q^{n-2j}) = q^{m(m-n+1)} (1 - q^{n-m}) \left[\begin{matrix} n \\ m \end{matrix} \right]_q.$$

(ii)

$$\sum_{j=0}^m \left[\begin{matrix} n \\ j \end{matrix} \right]_q^{-1} q^{j(n+1-j)} (1 - q^{n-2j}) = (1 - q^{n+1}) - q^{(m+1)(n-m)} (1 - q^{m+1}) \left[\begin{matrix} n \\ m \end{matrix} \right]_q^{-1}.$$

Proof. As a showcase, we only prove the second formula. Denote

$$F(n, m) = \sum_{j=0}^m \left[\begin{matrix} n \\ j \end{matrix} \right]_q^{-1} q^{j(n+1-j)} (1 - q^{n-2j}).$$

For $m \geq n$, $F(n, m)$ is a whole sum and equals 0. To see this, consider by taking $n - j$ instead of j ,

$$\begin{aligned} F(n, m) &= \sum_{j \geq 0} \left[\begin{matrix} n \\ j \end{matrix} \right]_q^{-1} q^{j(n+1-j)} (1 - q^{n-2j}) \\ &= \sum_{j \geq 0} \left[\begin{matrix} n \\ j \end{matrix} \right]_q^{-1} q^{(n-j)(j+1)} (1 - q^{2j-n}) \\ &= -F(n, m). \end{aligned}$$

So we get $F(n, m) = 0$.

Define

$$G(n, m) = (1 - q^{n+1}) - q^{(m+1)(n-m)} (1 - q^{m+1}) \left[\begin{matrix} n \\ m \end{matrix} \right]_q^{-1}.$$

Then we have

$$G(n, m) = F(n, m),$$

which follows from

$$\begin{aligned} & G(n, j) - G(n, j-1) \\ &= (1 - q^{n+1}) - q^{(j+1)(n-j)} (1 - q^{j+1}) \begin{bmatrix} n \\ j \end{bmatrix}_q^{-1} \\ &\quad - \left((1 - q^{n+1}) - q^{j(n-j+1)} (1 - q^j) \begin{bmatrix} n \\ j-1 \end{bmatrix}_q^{-1} \right) \\ &= -q^{(j+1)(n-j)} (1 - q^{j+1}) \begin{bmatrix} n \\ j \end{bmatrix}_q^{-1} + q^{j(n-j+1)} (1 - q^j) \begin{bmatrix} n \\ j-1 \end{bmatrix}_q^{-1} \\ &= -q^{(j+1)(n-j)} (1 - q^{j+1}) \begin{bmatrix} n \\ j \end{bmatrix}_q^{-1} + q^{j(n-j+1)} (1 - q^j) \frac{(1 - q^{n+1-j})}{(1 - q^j)} \begin{bmatrix} n \\ j \end{bmatrix}_q^{-1} \\ &= q^{j-j^2+jn} \left[q^{-2j+n} (-1 + q^{j+1}) + (1 - q^{n+1-j}) \right] \begin{bmatrix} n \\ j \end{bmatrix}_q^{-1} \\ &= q^{j(n+1-j)} (1 - q^{n-2j}) \begin{bmatrix} n \\ j \end{bmatrix}_q^{-1}, \end{aligned}$$

as claimed. $\square$

REFERENCES

- [1] E. Kılıç and I. Akkus, “On Fibonomial sums identities with special sign functions: analytically q -calculus approach.” *Mathematica Slovaca*, vol. 68, no. 3, pp. 501–518, 2018.
- [2] H. Ohtsuka, “Advanced Problem H-764.” *Fibonacci Q.*, vol. 52, no. 4, p. 375, 2014.
- [3] H. Ohtsuka, “Advanced Problem H-768.” *Fibonacci Q.*, vol. 53, no. 1, p. 89, 2015.
- [4] L. Carlitz, “The characteristic polynomial of a certain matrix of binomial coefficients.” *Fibonacci Q.*, vol. 3, pp. 81–89, 1965.
- [5] S. Falcon, “On the k -Lucas numbers.” *Int. J. Contemp. Math. Sci.*, vol. 6, no. 21-24, pp. 1039–1050, 2011.
- [6] S. Falcón and A. Plaza, “On the Fibonacci k -numbers.” *Chaos Solitons Fractals*, vol. 32, no. 5, pp. 1615–1624, 2007, doi: [10.1016/j.chaos.2006.09.022](https://doi.org/10.1016/j.chaos.2006.09.022).
- [7] H. Gould, “The bracket function and Fontené-Ward generalized binomial coefficients with application to Fibonomial coefficients.” *Fibonacci Q.*, vol. 7, pp. 23–40, 1969.
- [8] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete mathematics: a foundation for computer science*. 2nd ed., 2nd ed. Amsterdam: Addison-Wesley Publishing Group, 1994.
- [9] V. E. Hoggatt, “Fibonacci numbers and generalized binomial coefficients.” *Fibonacci Q.*, vol. 5, pp. 383–400, 1967.
- [10] D. Jarden, “Recurring sequences.” *Math. Stud.*, vol. 16, pp. 28–30, 1949.

- [11] E. Kılıç, "The generalized Fibonomial matrix." *Eur. J. Comb.*, vol. 31, no. 1, pp. 193–209, 2010, doi: [10.1016/j.ejc.2009.03.041](https://doi.org/10.1016/j.ejc.2009.03.041).
- [12] E. Kılıç, "Evaluation of sums containing triple aerated generalized Fibonomial coefficients." *Math. Slovaca*, vol. 67, no. 2, pp. 355–370, 2017, doi: [10.1515/ms-2016-0272](https://doi.org/10.1515/ms-2016-0272).
- [13] E. Kılıç, I. Akkus, and H. Ohtsuka, "Some generalized Fibonomial sums related with the Gaussian q -binomial sums." *Bull. Math. Soc. Sci. Math. Roum., Nouv. Sér.*, vol. 55, no. 1, pp. 51–61, 2012.
- [14] E. Kılıç and H. Prodinger, "Closed form evaluation of sums containing squares of Fibonomial coefficients." *Math. Slovaca*, vol. 66, no. 3, pp. 757–765, 2016, doi: [10.1515/ms-2015-0177](https://doi.org/10.1515/ms-2015-0177).
- [15] E. Kılıç and H. Prodinger, "Evaluation of sums involving Gaussian q -binomial coefficients with rational weight functions." *Int. J. Number Theory*, vol. 12, no. 2, pp. 495–504, 2016, doi: [10.1142/S1793042116500305](https://doi.org/10.1142/S1793042116500305).
- [16] E. Kılıç, H. Prodinger, I. Akkus, and H. Ohtsuka, "Formulas for fibonomial sums with generalized Fibonacci and Lucas coefficients." *Fibonacci Q.*, vol. 49, no. 4, pp. 320–329, 2011.
- [17] E. Kılıç, G. N. Stănică, and P. Stănică, "Spectral properties of some combinatorial matrices." *Congr. Numerantium*, vol. 201, pp. 223–235, 2010.
- [18] E. Kılıç and P. Stănică, "Generating matrices of C -nomial coefficients and their spectra." in *Proceedings of the 14th international conference on Fibonacci numbers and their applications, Morelia, Mexico, July 5–9, 2010*. México: Sociedad Matemática Mexicana, 2011, pp. 139–154.
- [19] D. Marques and P. Trojovský, "On some new sums of Fibonomial coefficients." *Fibonacci Q.*, vol. 50, no. 2, pp. 155–162, 2012.
- [20] D. Marques and P. Trojovský, "On some new identities for the Fibonomial coefficients." *Math. Slovaca*, vol. 64, no. 4, pp. 809–818, 2014, doi: [10.2478/s12175-014-0241-7](https://doi.org/10.2478/s12175-014-0241-7).
- [21] J. Seibert and P. Trojovský, "On some identities for the Fibonomial coefficients." *Math. Slovaca*, vol. 55, no. 1, pp. 9–19, 2005.
- [22] R. Torretto and J. Fuchs, "Generalized binomial coefficients." *Fibonacci Q.*, vol. 2, pp. 296–302, 1964.
- [23] P. Trojovský, "On some identities for the Fibonomial coefficients via generating function." *Discrete Appl. Math.*, vol. 155, no. 15, pp. 2017–2024, 2007, doi: [10.1016/j.dam.2007.05.003](https://doi.org/10.1016/j.dam.2007.05.003).

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CERTAIN COMBINATORIC CONVOLUTION SUMS ARISING FROM BERNOULLI AND EULER POLYNOMIALS

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Abstract. In this study, we introduce the absolute Möbius divisor function $U(n)$. According to some numerical computational evidence, we consider integer pairs $(n, n+1)$ satisfying; $\varphi(n) = \varphi(n+1) = U(n) = U(n+1)$. Furthermore, we give some examples and proofs for our results.

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1. INTRODUCTION

Let $\mathbb{N}$ denote the set of positive integers. A positive integer n has a unique prime factorization $n = \prod_{i=1}^{\omega(n)} p_i^{\alpha_i}$, where $\omega(n)$ is the number of distinct prime factors of n and each prime factor being counted only once. The divisor function $\sigma(n) = \sum_{d|n} d$ and the Euler totient function $\varphi(n)$ are widely studied in the field of elementary number theory. It is well-known [12, p. 22–23] that

$$n = \sum_{d|n} d \text{ and } \varphi(n) = \sum_{d|n} \mu(d) \cdot \frac{n}{d}.$$

Here, $\mu(d)$ is the Möbius μ -function. We investigate the absolute Möbius divisor function U , given by $U(n) = \left| \sum_{d|n} \mu(d) d \right|$. If n is a square-free (resp., not square-free) integer then $U(n) = \varphi(n)$ (resp., $U(n) \neq \varphi(n)$). For $n \in \mathbb{N}$, let us define the function $\omega_0(n)$ is the number of odd primes factors of n . That is,

$$\omega_0(n) = \begin{cases} \omega(n), & \text{if } n \text{ is odd,} \\ \omega(n) - 1, & \text{if } n \text{ is even.} \end{cases}$$

Ratat [11], who asked for which values of n the equation $\varphi(n) = \varphi(n+1)$ holds and gave $n = 1, 3, 15, 104$ for examples. In 1918, answering to Ratat's question, Goormaghtigh [6] gave $n = 164, 194, 255, 495$.

Klee [8], Moser [10], Lal and Gillard [9], Ballew, Case and Higgins [3], Baillie [1, 2] and Erdős, Pomerance and Sarkozy [5] studied for solutions to $\varphi(n) = \varphi(n+k)$. On the other hand, there is an unsolved problem on the divisor function, which asks that if $\sigma(n) = \sigma(n+1)$ infinitely often (see [7, p. 103], [13, p. 166]).

It should be mentioned that the Möbius conjecture associated with the Möbius function have been studied by Bayad and Goubi [4].

Throughout the paper, $p, q_1, \dots, q_6$ are distinct odd primes with $q_1 < \dots < q_6$ unless otherwise specified hereafter.

The aim of this article is to study an equation

$$\varphi(n) = \varphi(n+1) = U(n) = U(n+1). \quad (1.1)$$

More precisely, we prove the following theorem.

Theorem 1. $(\omega_0(n), \omega_0(n+1)) = (1, t)$ or $(t, 1)$ with $t \leq 6$

Let

$$\begin{cases} \text{Case 1)} & n = p, & n+1 = 2 \prod_{i=1}^t q_i & \text{or} \\ \text{Case 2)} & n = 2 \prod_{i=1}^t q_i, & n+1 = p & \text{or} \\ \text{Case 3)} & n = 2p, & n+1 = \prod_{i=1}^t q_i & \text{or} \\ \text{Case 4)} & n = \prod_{i=1}^t q_i, & n+1 = 2p. \end{cases} \quad (1.2)$$

Then there exist two pairs of $(n, n+1) = (194, 195)$ and $(n, n+1) = (5186, 5187)$ satisfying Eq. (1.1) and (1.2).

Remark 1. Let $p, q, p_1, \dots, p_t$ be distinct odd primes. Euler's totient function $\varphi(n)$ plays a key role in the RSA encryption. Given $N = pq$ with p and q distinct odd primes. If $n = 2p$ (resp., $p_1 \cdots p_t$) and $n+1 = p_1 \cdots p_t$ (resp., $2p$) then the computation time of $2(p+1)q$ is more shorter than it of pq . Furthermore, we find $N = pq$ derived from $2(p+1)q$.

Remark 2. Table 18 and Table 19 in Appendix give us examples of values of $U(n) = U(n+1)$ and $\varphi(n) = \varphi(n+1) = U(n) = U(n+1)$. We conjectured that $12 \mid U(n) = U(n+1)$ except $n = 1$. In fact, if $n \neq 1$, we note that $12 \mid \varphi(n) = \varphi(n+1) = U(n) = U(n+1)$ and $(\omega(n), \omega(n+1)) = (1, t)$ or $(t, 1)$ with $t \leq 6$ by Theorem 1.

2. LEMMAS FOR THEOREM 1 $((1, t)$ OR $(t, 1)$ WITH $t \leq 6$)

To prove Theorem 1, we need following lemmas.

Lemma 1. $((\omega_0(n), \omega_0(n+1)) = (1, 1))$ Let p and q_1 be distinct odd primes and let

$$\begin{cases} n = p, & n+1 = 2q_1 & \text{or} \\ n = 2q_1, & n+1 = p. \end{cases} \quad (2.1)$$

Then there does not exist a pair of positive integers $(n, n+1)$ satisfying Eq. (1.1) and (2.1).

Proof. Assume that $U(n) = U(n+1)$. It is easily check that $p = q_1$. This is a contradiction for $p \neq q_1$. This completes the proof of Lemma 1. $\square$

Remark 3. If the both p and $2p+1$ are primes, then p is a Sophie Germain prime (or safe prime). Then there does not exist a Sophie Germain prime p satisfying $U(2p) = U(q_1) = \varphi(2p) = \varphi(q_1)$ by Lemma 1.

Lemma 2. Let $q_1, q_2, \dots, q_r$ be distinct odd primes with $q_1 < q_2 < \dots < q_r$ and $r \geq 2$. If $S = \{1, 2, \dots, r\}$ and $S'' \subset S' \subset S$ then

$$\prod_{j \in S'} (1 - q_j^{-1}) < \prod_{j \in S''} (1 - q_j^{-1}) \quad (2.2)$$

and

$$\prod_{i=1}^t q_i^{-1} > \prod_{j=1}^{t+j} q_j^{-1} \quad (j \geq 1). \quad (2.3)$$

Furthermore, if $i \leq i'$ with $i, i' \in S$, then

$$\prod_{i=1}^t (1 - q_i^{-1}) > \prod_{i'=1}^t (1 - q_{i'}^{-1}) \quad (2.4)$$

and

$$\prod_{i=1}^t q_i^{-1} \geq \prod_{i'=1}^{t+j} q_{i'}^{-1}. \quad (2.5)$$

Proof. It is trivial. $\square$

The following lemma makes to reduce conditions of (1.2).

Lemma 3. Let $p+1 = 2q_1 \cdots q_s$ or $p-1 = 2q_1 \cdots q_s$ be positive integers, where $p, q_1, \dots, q_s$ are distinct odd prime integers with $s \geq 2$. Then there does not exist $n = p$ (resp., $n = 2q_1 \cdots q_s$) and $n+1 = 2q_1 \cdots q_s$ (resp., $n+1 = p$) satisfying $\varphi(n) = \varphi(n+1) = U(n) = U(n+1)$.

Proof. Assume $U(n) = U(n+1)$. Then, we get $p-1 = (q_1-1) \cdots (q_s-1)$ and

$$\begin{cases} \frac{p-1}{p+1} = \frac{1}{2} \left(1 - \frac{1}{q_1}\right) \cdots \left(1 - \frac{1}{q_s}\right) & \text{or} \\ 1 = \frac{1}{2} \left(1 - \frac{1}{q_1}\right) \cdots \left(1 - \frac{1}{q_s}\right). \end{cases} \quad (2.6)$$

We note that

$$\frac{1}{2} \left(1 - \frac{1}{q_1}\right) \cdots \left(1 - \frac{1}{q_s}\right) < \frac{1}{2} \quad (2.7)$$

and

$$1 > \frac{p-1}{p+1} = 1 - \frac{2}{p+1} = 1 - \frac{2}{q_1 \cdots q_s} \geq 1 - \frac{2}{3 \cdot 5} > \frac{1}{2} \quad \text{by (2.5)}. \quad (2.8)$$

Thus the proof is completed by (2.6), (2.7) and (2.8). $\square$

Lemma 4. $((\omega_0(n), \omega_0(n+1)) = (1, 2) \text{ or } (2, 1))$ Let p, q_1, q_2 be distinct odd prime integers with $q_1 < q_2$ and let

$$\begin{cases} \text{Case 1)} & n = p, & n+1 = 2q_1q_2 & \text{or} \\ \text{Case 2)} & n = 2q_1q_2, & n+1 = p & \text{or} \\ \text{Case 3)} & n = 2p & n+1 = q_1q_2 & \text{or} \\ \text{Case 4)} & n = q_1q_2, & n+1 = 2p. \end{cases} \quad (2.9)$$

Then we can not find positive integers n and $n+1$ satisfying (1.1) and (2.9).

Proof. Eq. (1.1) has no positive integers of n and $n+1$ in Case 1) and Case 2) by Lemma 3. Thus, we consider Cases 3) and Case 4).

Case 3) Put $n = 2p$ and $n+1 = q_1q_2$ in (1.1). This yields that

$$p-1 = U(2p) = U(q_1q_2) = (q_1-1)(q_2-1) \text{ and } (q_1-2)(q_2-2) = -1.$$

This is completed the Case 3).

Case 4) Finally, we consider Eq. (1.1) with $n = q_1q_2$ and $n+1 = 2p$. Similarly, we get $(q_1-2)(q_2-2) = -1$ and $q_1 = q_2 = 3$. Thus, for this case, we cannot find distinct odd prime integers q_1 and q_2 satisfying Eq. (1.1). Therefore, we prove Lemma 4. $\square$

Remark 4. Using Mathematica 11.0, we find odd primes p, q_1, q_2 satisfying $(q_1-1)(q_2-1) = (p-1)$ with $p < 200$.

But these odd primes p, q_1 and q_2 in the Table 1 do not satisfy $n = 2p$ and $n+1 = q_1q_2$.

Lemma 5. $((\omega_0(n), \omega_0(n+1)) = (1, 3) \text{ or } (3, 1))$ Let p, q_1, q_2 and q_3 be distinct odd prime integers with $q_1 < q_2 < q_3$ and let

$$\begin{cases} \text{Case 1)} & n = p, & n+1 = 2q_1q_2q_3 & \text{or} \\ \text{Case 2)} & n = 2q_1q_2q_3, & n+1 = p & \text{or} \\ \text{Case 3)} & n = 2p, & n+1 = q_1q_2q_3 & \text{or} \\ \text{Case 4)} & n = q_1q_2q_3, & n+1 = 2p. \end{cases} \quad (2.10)$$

Then there exists a unique pair of $(n, n+1) = (194, 195)$ satisfying (1.1) and (2.10).

Proof. To prove Lemma 5, we need to check Case 3) and Case 4) only by Lemma 3.

Recall the identity

$$(p-1) = (q_1-1)(q_2-1)(q_3-1). \quad (2.11)$$

We consider two cases step by step.

Case 3) Let $n = 2p$ and $n+1 = q_1q_2q_3$. Using (2.11), we get

$$\left(1 - \frac{3}{q_1q_2q_3}\right) = 2\left(1 - \frac{1}{q_1}\right)\left(1 - \frac{1}{q_2}\right)\left(1 - \frac{1}{q_3}\right). \quad (2.12)$$

If $q_1 \geq 5$, then R.H.S.(the right hand side) of (2.12) is

$$2\left(1 - \frac{1}{q_1}\right)\left(1 - \frac{1}{q_2}\right)\left(1 - \frac{1}{q_3}\right) \geq 2\left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{7}\right)\left(1 - \frac{1}{11}\right) = \frac{96}{77} > 1$$

and L.H.S.(the left hand side) of (2.12) is $\left(1 - \frac{3}{q_1 q_2 q_3}\right) < 1$. Thus we cannot find distinct odd primes q_1, q_2, q_3 satisfying Eq. (2.12) with $q_1 \geq 5$.

Assume $q_1 = 3$. Then we have

$$p - 1 = 2(q_2 - 1)(q_3 - 1) \text{ and } 2p + 1 = 3q_2 q_3. \quad (2.13)$$

By (2.13), it is easy to see that

$\frac{q_2 - 4}{1}$	$\frac{q_3 - 4}{9}$	$\frac{(q_2, q_3)}{(5, 13)}$	$\frac{2p + 1}{195}$	$\frac{p}{97}$	prime O
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Therefore, an unique solution of this case is $(n, n + 1) = (194, 195)$.

Case 4) Let $n = q_1 q_2 q_3$ and $n + 1 = 2p$. From (2.11), we deduce

$$\left(1 - \frac{1}{q_1 q_2 q_3}\right) = 2\left(1 - \frac{1}{q_1}\right)\left(1 - \frac{1}{q_2}\right)\left(1 - \frac{1}{q_3}\right). \quad (2.14)$$

Similarly, the same method of (2.13) in Case 3), we have two inequalities

$2\left(1 - \frac{1}{q_1}\right)\left(1 - \frac{1}{q_2}\right)\left(1 - \frac{1}{q_3}\right) > 1$ and $1 - \frac{1}{q_1 q_2 q_3} < 1$ with $q_1 \geq 5$. Thus, we consider the only prime $q_1 = 3$. An equation (2.14) yields $(q_2 - 4)(q_3 - 4) = 11$ and $(q_2, q_3) = (5, 15)$. But q_3 is not a prime integer.

This proves the lemma. □

Remark 5. Using Mathematica 11.0, we find the number of odd prime pairs (p, q_1, q_2, q_3) satisfying $U(2p) = p - 1 = (q_1 - 1)(q_2 - 1)(q_3 - 1) = U(q_1 q_2 q_3)$ with $p < 400$ as follows in the Table 2.

Lemma 6. $((\omega_0(n), \omega_0(n + 1)) = (1, 4) \text{ or } (4, 1))$ Let p, q_1, q_2, q_3, q_4 be distinct odd prime integers with $q_1 < q_2 < q_3 < q_4$ and let

$$\begin{cases} \text{Case 3)} & n = 2p & n + 1 = q_1 q_2 q_3 q_4 & \text{or} \\ \text{Case 4)} & n = q_1 q_2 q_3 q_4, & n + 1 = 2p. \end{cases} \quad (2.15)$$

Then there exists an unique pairs of $(n, n + 1) = (5186, 5197)$ satisfying (1.1) and (2.15).

Proof. The proof is similar to Lemma 5. □

Remark 6. Let $n = 2p$ and $n + 1 = \prod_{i=1}^l p_i$ (or $n - 1 = \prod_{i=1}^l p_i$), where p_i are distinct odd prime integers with $p_1 < p_2 < \dots < p_l$. Assume $\varphi(n) = \varphi(n + 1) =$

$U(n) = U(n+1)$. We note that

$$2p+1 = \prod_{i=1}^l p_i \quad (\text{or } 2p-1 = \prod_{i=1}^l p_i) \quad \text{and} \quad 2(p-1) = 2 \prod_{i=1}^l (p_i-1). \quad (2.16)$$

By Eq. (2.16) we get

$$\left(1 - \frac{3}{\prod_{i=1}^l p_i}\right) = 2 \prod_{i=1}^l \left(1 - \frac{1}{p_i}\right) \quad \text{or} \quad \left(1 - \frac{1}{\prod_{i=1}^l p_i}\right) = 2 \prod_{i=1}^l \left(1 - \frac{1}{p_i}\right)$$

and

$$2 \prod_{i=1}^l \left(1 - \frac{1}{p_i}\right) + \frac{3}{\prod_{i=1}^l p_i} \geq 2 \prod_{i=1}^l \left(1 - \frac{1}{p[i]}\right) + \frac{3}{\prod_{i=1}^l p[i]}. \quad (2.17)$$

Here, $p[i]$ is the i -th prime integer. Inequality (2.17) yields on the Table 3.

For example, if $q_1 = 37$ then $n+1$ (or $n-1$) $\geq 75347738233715510682119691$
 21548009585444856281368482589916445090657521119430259527166840110198
 38699393817326126389986232066116624296065771602290209012057820739178
 72287545170087475416051810829082770128252239632108848112410020471053
 53707182094449546678344114135023520667353779163828815640309227081776
 53264817868019846139296555645562400572533870903274086461776912104807
 86948103160891681120747752822120795047461510487499681318260518950750
 72046046180977008378417515269482748132721747320752111324046327166749
 59534769799425231633065885082282111753064922202067920112201792625821
 255993169428654728093503918276240568080769187150768754091 and the digit
 of $n+1$ (or $n-1$) is greater than 627 with $l \geq 229$.

Lemma 7. Let $p, q_1, q_2, q_3, q_4, q_5$ be distinct odd prime integers with $q_1 < q_2 < q_3 < q_4 < q_5$ and let

$$\begin{cases} \text{Case 3)} & n = 2p, & n+1 = q_1 q_2 q_3 q_4 q_5 & \text{or} \\ \text{Case 4)} & n = q_1 q_2 q_3 q_4 q_5 & n+1 = 2p. \end{cases} \quad (2.18)$$

Then there does not exist n and $n+1$ satisfying (1.1) and (2.18).

Proof. The proof is similar to Lemma 4. □

As a result, from Lemma 1, Lemma 4, Lemma 5, Lemma 6 and Lemma 7, there exist two pairs of $(n, n+1) = (194, 195)$ and $(n, n+1) = (5186, 5187)$ satisfying Eq. (1.1) and (1.2).

3. LEMMAS FOR THEOREM 1 ((1, 6) OR (6, 1))

To prove Theorem 1, we need following lemmas.

Lemma 8. Let $q_2, \dots, q_6$ be distinct odd primes integers greater than 3 and let $S = \{q_2, q_3, q_4, q_5, q_6\}$, $S' = \{q_i \in S \mid q_i \equiv -1 \pmod{3}\}$ and $S'' = S - S'$. If $\#S' \equiv 1 \pmod{2}$ then

$$3 \left(\prod_{i=2}^6 q_i - 1 \right) \not\equiv 4 \prod_{i=2}^6 (q_i - 1). \quad (3.1)$$

Proof. If $\#S' = 1$ or 3 then

$$\begin{cases} \prod_{i=2}^6 q_i - 1 \equiv -1 - 1 \equiv -2 \pmod{3}, \\ \frac{4}{3} \prod_{i=2}^6 (q_i - 1) \equiv 0 \pmod{3}. \end{cases} \quad (3.2)$$

If $\#S' = 5$ then

$$\begin{cases} 3 \left(\prod_{i=2}^6 (q_i - 1) \right) \equiv 0 \pmod{3}, \\ 4 \prod_{i=2}^6 (q_i - 1) \not\equiv 0 \pmod{3}. \end{cases} \quad (3.3)$$

From Eq. (3.2) and Eq. (3.3), our claim follows. $\square$

Remark 7. Let $ayz - b(y + z) + d = 0$ with $a \in \mathbb{N}$ and $b, d \in \mathbb{Z}$. Here, $\mathbb{Z}$ denotes the set of ring of integers. More precisely, we can write it as

$$(ay - b)(az - b) = -ad + b^2. \quad (3.4)$$

If Eq. (3.4) have (at least one) positive integer solutions (y, z) then there exist (at least one) positive integers d_i satisfying

$$\begin{cases} d_i \mid -ad + b^2, \\ d_i + b \equiv 0 \pmod{a}, \\ \frac{-ad + b^2}{d_i} \equiv 0 \pmod{a}. \end{cases} \quad (3.5)$$

We can focus our attention to identity, namely, Eq. (1.1) with $\omega_0(n) = 1$ (resp., $\omega_0(n) = 6$) and $\omega_0(n + 1) = 6$ (resp., $\omega_0(n + 1) = 1$).

Lemma 9. Let $p, q_1, \dots, q_6$ be distinct odd primes and let

$$\begin{cases} \text{Case 3) } & n = 2p, & n + 1 = q_1 \dots q_6 \text{ or} \\ \text{Case 4) } & n = q_1 \dots q_6, & n + 1 = 2p \end{cases} \quad (3.6)$$

with $q_1 < \dots < q_6$. Then, we cannot find 7-tuples $(p, q_1, \dots, q_6)$ primes satisfying Eq. (3.6).

Proof. Since the proof is very similar, we consider only Case 3) and Case 4) by Lemma 3. By the Table 3, we can obtain $q_1 = 3$.

First, we consider the Case 3), that is,

$$\begin{cases} n = 2p, \\ n + 1 = 2p + 1 = 3q_2 \dots q_6, \end{cases} \quad (3.7)$$

where $q_1, \dots, q_6$ are distinct odd primes with $3 < q_2 < \dots < q_6$.

From Eq. (1.1) we deduce that

$$3 \left(\prod_{i=2}^6 q_i - 1 \right) = 4 \prod_{i=2}^6 (q_i - 1) \quad (3.8)$$

and

$$1 = \frac{4}{3} \prod_{i=2}^6 \left(1 - \frac{1}{q_i} \right) + \prod_{i=2}^6 \frac{1}{(q_i - 1)}. \quad (3.9)$$

Let $f(p[i]) = \frac{4}{3} \prod_{j=i}^{i+4} \left(1 - \frac{1}{p[j]} \right) + \prod_{j=i}^{i+4} \frac{1}{p[j]} - 1$. Here, $p[i]$ is the i -th prime integer.

That is, $p[1] = 2, p[2] = 3, \dots$

Since $f(p[5]) = -\frac{2858}{62491} < 0$ and $f(p[6]) = \frac{37796}{2800733} > 0$, the set of possible primes satisfying (3.8) are $\{5, 7, 11\}$. We note that we can write Eq. (3.8) as

$$15 \prod_{i=3}^6 q_i - 3 = 16 \prod_{i=3}^6 (q_i - 1), \quad (3.10)$$

$$7 \prod_{i=3}^6 q_i - 1 = 8 \prod_{i=3}^6 (q_i - 1), \quad (3.11)$$

$$33 \prod_{i=3}^6 q_i - 3 = 40 \prod_{i=3}^6 (q_i - 1). \quad (3.12)$$

In Eq. (3.10), if $q_3 \equiv 1 \pmod{5}$, then $L.H.S \not\equiv R.H.S \pmod{5}$ and so,

$$q_3 \not\equiv 1 \pmod{5}. \quad (3.13)$$

Furthermore, we get

q_3	$3 \cdot 5 \cdot q_3$	$16(q_3 - 1)$
13	195	192
17	255	256

(3.14)

Let $f_1(p[i]) = \frac{16}{15} \prod_{j=i}^{i+3} \left(1 - \frac{1}{p[j]} \right) + \prod_{j=i}^{i+3} \frac{1}{p[j]} - 1$. Then, we observe that

$$\begin{cases} f_1(p[16]) = -\frac{202324}{6390045} < 0, \\ f_1(p[17]) = \frac{144986}{85602215} > 0. \end{cases} \quad (3.15)$$

Possible prime integers for q_3 are 17, 19, 23, 29, 31, 37, 41, 43, 47, 53 by (3.13), (3.14) and (3.15).

Case 3-a) $(q_1, q_2, q_3) = (3, 5, 17)$. Similarly, we consider

$$3 \times 5 \times 17 \times q_4 q_5 q_6 - 3 = 16^2 (q_4 - 1)(q_5 - 1)(q_6 - 1). \quad (3.16)$$

Let $f_2(p[i]) = \frac{256}{255} \prod_{j=i}^{i+2} \left(1 - \frac{1}{p[j]}\right) + \frac{1}{85} \prod_{j=i}^{i+2} \frac{1}{p[j]} - 1$. Then, we observe that

$$f_2(p[134]) < 0 \text{ and } f_2(p[135]) > 0. \quad (3.17)$$

It is easily checked that

$$255q_4 < 256(q_4 - 1) \text{ and } q_4 > 256. \quad (3.18)$$

On the other hand, if $q_4 \equiv 1 \pmod{5}$ or $q_4 \equiv 1 \pmod{17}$, then Eq. (3.16) has no solution. Thus, the set of possible prime integers for q_4 is $P_1 := \{257, 263, 269, 277, 283, 293, 313, 317, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, 419, 433, 439, 449, 457, 463, 467, 479, 487, 499, 503, 509, 523, 547, 557, 563, 569, 577, 587, 593, 599, 607, 617, 619, 643, 653, 659, 673, 677, 683, 709, 719, 727, 733, 739, 743, 757\}$.

Let $f_3(x) := -255xyz + 3 + 256(x-1)(y-1)(z-1)$. Inserting $x \in P_1$ into $f_3(x)$, we have Table 4. Eight Diophantine equations $65539 - 65536(y+z) + yz = 0$, $67075 - 67072(y+z) + 7yz = 0$, $70659 - 70656y - 70656z + 21yz = 0$, $90115 - 90112(y+z) + 97yz = 0$, $101379 - 101376y - 101376z + 141yz = 0$, $119299 - 119296(y+z) + 211yz = 0$, $139779 - 139776y - 139776z + 291yz = 0$, $183811 - 183808(y+z) + 463yz = 0$ have integer solutions (y, z) using (3.5). But solutions (y, z) of these Diophantine equations have not pairs of prime integer solutions.

Therefore, Eq. (3.16) has no solution.

Case 3-b) $(q_1, q_2, q_3) = (3, 5, 19)$. Consider

$$3 \times 5 \times 19 q_4 q_5 q_6 - 3 = 16 \times 18 (q_4 - 1)(q_5 - 1)(q_6 - 1)$$

and

$$95q_4q_5q_6 - 1 = 96(q_4 - 1)(q_5 - 1)(q_6 - 1). \quad (3.19)$$

An inequality $95q_4 < 96(q_4 - 1)$ deduces the lower bound for q_4 , that is,

$$q_4 > 96. \quad (3.20)$$

Let $f_4(p[i]) := \frac{96}{95} \prod_{j=i}^{i+2} \left(1 - \frac{1}{p[j]}\right) + \frac{1}{95} \prod_{j=i}^{i+2} \frac{1}{p[j]} - 1$.

Inequalities $f_4(p[60]) < 0$ and $f_4(p[61]) > 0$ deduce the upper bound for q_4 , that is,

$$q_4 \leq p[60] = 281. \quad (3.21)$$

If $q_4 \equiv 1 \pmod{5}$ or $q_4 \equiv 1 \pmod{19}$, then Eq. (3.19) has no solution. Thus, the set of possible prime integers for q_4 is $P_2 := \{97, 103, 107, 109, 113, 127, 137, 139, 149, 157, 163, 167, 173, 179, 193, 197, 199, 223, 227, 233, 239, 257, 263, 269, 277\}$.

Let $f_5(x) := -95xyz + 3 + 96(x-1)(y-1)(z-1)$. We derive Table 5. Four Diophantine equations $99217 - 9216y - 9216z + yz = 0$, $9793 - 9792y - 9792z + 7yz = 0$, $10369 - 10368y - 10368z + 13yz = 0$, $13249 - 13248y - 13248z + 43yz = 0$ have integer solutions (y, z) by (3.5), but these solutions are not pairs of prime integers. Therefore, Eq. (3.19) has no solution.

Case 3-c) $(q_1, q_2, q_3) = (3, 5, 23)$. Consider

$$3 \times 5 \times 23q_4q_5q_6 - 3 = 16 \times 22(q_4 - 1)(q_5 - 1)(q_6 - 1). \quad (3.22)$$

Since $345q_4 < 352(q_4 - 1)$ the lower bound of possible prime q_4 is 53.

$$\text{Let } f_6(p[i]) := \frac{352}{345} \prod_{j=i}^{i+2} \left(1 - \frac{1}{p[j]}\right) + \frac{1}{115} \prod_{j=i}^{i+2} \frac{1}{p[j]} - 1.$$

The upper bound of possible prime q_4 is 139 since $f_6(p[34]) < 0$ and $f_6(p[35]) > 0$. Similarly, if $q_4 \equiv 1 \pmod{5}$ or $q_4 \equiv 1 \pmod{23}$ has no solution. Thus, the set of possible primes for q_4 is $\{53, 59, 67, 73, 79, 83, 89, 97, 103, 107, 109, 113, 127, 137\}$.

$$\text{Let } f_7(x) := -345xyz + 352(x-1)(y-1)(z-1) + 3.$$

We derive Table 6. According to the Table 6, we have no pairs of prime integer solutions (y, z) . Therefore, Eq. (3.22) has no solution.

Case 3-d) $(q_1, q_2, q_3) = (3, 5, 29)$. Consider

$$435q_4q_5q_6 - 3 = 448(q_4 - 1)(q_5 - 1)(q_6 - 1). \quad (3.23)$$

Since $435q_4 < 448(q_4 - 1)$ the lower bound of possible prime q_4 is 37.

$$\text{Let } f_8(p[i]) := \frac{448}{435} \prod_{j=i}^{i+2} \left(1 - \frac{1}{p[j]}\right) + \frac{1}{145} \prod_{j=i}^{i+2} \frac{1}{p[j]} - 1.$$

The upper bound of possible prime q_4 is 97 since $f_8(p[25]) < 0$ and $f_8(p[26]) > 0$. If $q_4 \equiv 1 \pmod{5}$ and $q_4 \equiv 1 \pmod{29}$ then Eq. (3.23) has no solution.

$$\text{Let } f_9(x) := -435xyz + 448(x-1)(y-1)(z-1) + 3.$$

Similarly, we get Table 7. According to the Table 7, there are no pairs of integer solutions $(y, z) \in \mathbb{Z} \times \mathbb{Z}$.

Case 3-e) $(q_1, q_2, q_3) = (3, 5, 37)$. In (3.8), put $(q_1, q_2, q_3) = (3, 5, 37)$. Then

$$185q_4q_5q_6 - 1 = 192(q_4 - 1)(q_5 - 1)(q_6 - 1). \quad (3.24)$$

Here,

$$q_4 > 37, \quad q_4 \not\equiv 1 \pmod{5} \quad \text{and} \quad q_4 \not\equiv 1 \pmod{37}. \quad (3.25)$$

$$\text{Let } f_{10}(p[i]) := \frac{192}{185} \prod_{j=i}^{i+2} \left(1 - \frac{1}{p[j]}\right) + \frac{1}{185} \prod_{j=i}^{i+2} \frac{1}{p[j]} - 1. \text{ Clearly,}$$

$$f_{10}(p[21]) < 0 \quad \text{and} \quad f_{10}(p[22]) > 0. \quad (3.26)$$

Let $f_{11}(x) := -185xyz + 192(x-1)(y-1)(z-1) + 1$. Combining (3.24), (3.25) and (3.26), we have Table 8 for the set of possible primes q_4 and $f_{11}(q_4)$.

Similarly, we get Table 8. Using the methods of Case 3-d) these Diophantine equations have no $(y, z) \in \mathbb{Z} \times \mathbb{Z}$.

Case 3-f) $(q_1, q_2, q_3) = (3, 5, 43)$ or $(3, 5, 47)$ or $(3, 5, 53)$. In (3.8), put $q_3 = 43$ or 47 or 53. Then

$$\begin{aligned} 215q_4q_5q_6 - 1 &= 224(q_4 - 1)(q_5 - 1)(q_6 - 1) \text{ or} \\ 705q_4q_5q_6 - 3 &= 736(q_4 - 1)(q_5 - 1)(q_6 - 1) \text{ or} \\ 795q_4q_5q_6 - 3 &= 832(q_4 - 1)(q_5 - 1)(q_6 - 1). \end{aligned}$$

Let $f_{12}(x) := -215xyz + 224(x-1)(y-1)(z-1) + 1$, $f_{13}(x) := -705xyz + 736(x-1)(y-1)(z-1) + 3$ and $f_{14}(x) := -795xyz + 832(x-1)(y-1)(z-1) + 3$. Similarly, using the same method of Case 3-d), we get Table 9. It is easily checked that seven Diophantine equations have no integer solutions.

Next, we consider Eq. (3.11) with $q_1 = 3$ and $q_2 = 7$.

$$\text{Let } f_{15}(p[i]) := \frac{8}{7} \prod_{j=i}^{i+3} \left(1 - \frac{1}{p[j]}\right) + \frac{1}{7} \prod_{j=i}^{i+3} \frac{1}{p[j]} - 1.$$

Since $f_{15}(p[9]) = -\frac{33102}{5355343} < 0$ and $f_{15}(p[10]) = \frac{130320}{9546481} > 0$, the set of possible primes for q_3 is $\{11, 13, 17, 19, 23\}$. Next, we consider (3.11) with $q_3 = 11, 13, 17, 19, 23$ step by step.

Case 3-g) $(q_1, q_2, q_3) = (3, 7, 11)$. In (3.11), put $(q_1, q_2, q_3) = (3, 7, 11)$. That is, we consider

$$77q_4q_5q_6 - 1 = 80(q_4 - 1)(q_5 - 1)(q_6 - 1). \quad (3.27)$$

Similarly, we obtain the lower bound of possible prime integers q_4 is 29 and the upper bound of possible prime integers q_4 is 73. If $q_4 \equiv 1 \pmod{7}$ or $q_4 \equiv 1 \pmod{11}$ then Eq. (3.27) has no solution. Thus, the set of possible primes of q_4 is $\{31, 37, 41, 47, 53, 61, 73\}$. Let $f_{16}(x) := -77xyz + 80(x-1)(y-1)(z-1) + 1$.

Similarly, using the same method of Case 3-d), we get Table 10. Two Diophantine equations $2881 - 2880(y+z) + 31yz = 0$ and $3201 - 3200(y+z) + 43yz = 0$ have integer solutions (y, z) but not pairs of prime integers. Therefore, it is easily checked that eight Diophantine equations have no pairs of prime integer solutions (y, z) .

Case 3-h) $(q_1, q_2, q_3) = (3, 7, 13)$. In (3.11), put $(q_1, q_2, q_3) = (3, 7, 13)$. Then

$$91q_4q_5q_6 - 1 = 96(q_4 - 1)(q_5 - 1)(q_6 - 1). \quad (3.28)$$

Let $f_{17}(x) := -91xyz + 96(x-1)(y-1)(z-1) + 1$.

Similarly, we have Table 11. If we use (3.5), then five Diophantine equations have no $(y, z) \in \mathbb{Z} \times \mathbb{Z}$. Thus, Eq. (3.28) has no solution.

Case 3-i) $(q_1, q_2, q_3) = (3, 7, 17)$ or $(3, 7, 19)$. Given two equations

$$\begin{aligned} 119q_4q_5q_6 - 1 &= 128(q_4 - 1)(q_5 - 1)(q_6 - 1) \text{ and} \\ 133q_4q_5q_6 - 1 &= 144(q_4 - 1)(q_5 - 1)(q_6 - 1), \end{aligned}$$

we consider two Diophantine equations $f_{18}(x) := -119xyz + 128(x-1)(y-1)(z-1) + 1$ and $f_{19}(x) := -133xyz + 144(x-1)(y-1)(z-1) + 1$.

Similarly, we have Table 12. Six Diophantine equations have no $(y, z) \in \mathbb{Z} \times \mathbb{Z}$.

Case 3-j) $(q_1, q_2, q_3) = (3, 7, 23)$. In (3.11), put $(q_1, q_2, q_3) = (3, 7, 23)$. Then

$$161q_4q_5q_6 - 1 = 176(q_4 - 1)(q_5 - 1)(q_6 - 1). \quad (3.29)$$

The lower bound and upper bound of possible primes q_4 is 29. But, $29 \equiv 1 \pmod{7}$. So, we cannot find pairs of prime integers of (3.29).

Next, we consider Eq. (3.12) with $q_1 = 3$ and $q_2 = 11$, that is,

$$33q_3q_4q_5q_6 - 3 = 40(q_3 - 1)(q_4 - 1)(q_5 - 1)(q_6 - 1). \quad (3.30)$$

$$\text{Let } f_{20}(p[i]) := \frac{40}{33} \prod_{j=i}^{i+3} \left(1 - \frac{1}{p[j]}\right) + \frac{1}{11} \prod_{j=i}^{i+3} \frac{1}{p[j]} - 1.$$

Since $f_{20}(p[7]) < 0$ and $f_{20}(p[8]) > 0$, the set of possible primes q_3 is $\{13, 17\}$. Thus, we solve the equation (3.12) one by one.

Case 3-k) $(q_1, q_2, q_3) = (3, 11, 13)$ and $(3, 11, 17)$. Let $f_{21}(x) := -143xyz + 160(x-1)(y-1)(z-1) + 1$ and $f_{22}(x) := -561xyz + 640(x-1)(y-1)(z-1) + 3$.

Then we get Table 13. Three Diophantine equations have no pairs of integer solutions.

Therefore, we cannot find 7-tuples of prime integers $(p, q_1, \dots, q_6)$ for Case 3) by Case 3-a)~Case 3-k).

Second, we consider the Case 4), that is, $n = 3q_2 \cdots q_6$ and $n + 1 = 2p$.

From (1.1), we observe that

$$3 \prod_{i=2}^6 q_i - 1 = 4 \prod_{i=2}^6 (q_i - 1). \quad (3.31)$$

It is easy to see that

$$3 \prod_{i=2}^6 q_i - 1 \not\equiv 4 \prod_{i=2}^6 (q_i - 1) \pmod{3} \text{ with } q_2 \equiv 1 \pmod{3}. \quad (3.32)$$

Let $f_{23}(p[i]) := \frac{4}{3} \prod_{j=i}^{i+4} \left(1 - \frac{1}{p[j]}\right) + \frac{1}{3} \prod_{j=i}^{i+4} \frac{1}{p[j]} - 1$. Since $f_{23}(p[5]) = -\frac{145760}{3187041}$ and $f_{23}(p[6]) = \frac{8722}{646323}$, we have the set of possible primes q_2 satisfying (3.31) is $\{5, 11\}$.

Let $q_1 = 3$ and $q_2 = 5$. Then (3.31) becomes

$$15 \prod_{i=3}^6 q_i - 1 = 16 \prod_{i=3}^6 (q_i - 1). \quad (3.33)$$

Similar to (3.14) and (3.15), we derive that the lower bound and upper bound of possible prime integers q_3 are 17 and 53. The equation (3.33) deduces $q_3 \not\equiv 1 \pmod{3}$ and $q_3 \not\equiv 1 \pmod{5}$. Therefore, the set of possible prime integers q_3 satisfying (3.33) is $\{17, 23, 29, 47, 53\}$.

Case 4-a) $(q_1, q_2, q_3) = (3, 5, 17)$. In (3.31), put $(q_1, q_2, q_3) = (3, 5, 17)$. Then

$$255q_4q_5q_6 - 1 = 256(q_4 - 1)(q_5 - 1)(q_6 - 1). \quad (3.34)$$

Using the same method in Case 3-a), we have the same lower bound and upper bound of possible primes q_4 in Case 3-a). Furthermore, we get $q_4 \not\equiv 1 \pmod{3}$, $q_4 \not\equiv 1 \pmod{5}$ and $q_4 \not\equiv 1 \pmod{17}$ by (3.34).

Let $f_{24}(x) = f_3(x) - 2 = -255q_4q_5q_6 + 1 + 256(q_4 - 1)(q_5 - 1)(q_6 - 1)$. We have the following Table 14. Two Diophantine equations $65537 - 65536(y + z) + yz = 0$ and $90113 - 90112(y + z) + 97yz = 0$ have pairs of integer solutions (y, z) satisfying $f_{24}(x) = 0$ but these are not pairs of primes.

Therefore, Eq. (3.34) has no solution.

Case 4-b) $(q_1, q_2, q_3) = (3, 5, 23)$. In (3.31), put $(q_1, q_2, q_3) = (3, 5, 23)$. Then

$$345q_4q_5q_6 - 1 = 352(q_4 - 1)(q_5 - 1)(q_6 - 1). \quad (3.35)$$

It is easy to see that the lower bound and upper bound of possible primes q_4 are same numbers in Case 3-c). Furthermore, we get $q_4 \not\equiv 1 \pmod{3}$, $q_4 \not\equiv 1 \pmod{5}$ and $q_4 \not\equiv 1 \pmod{23}$ by (3.35).

Thus, the set of possible primes for q_4 is $\{53, 59, 83, 89, 107, 113, 137\}$.

Let $f_{25}(x) = f_7(x) - 2 := -345xyz + 352(x - 1)(y - 1)(z - 1) + 1$.

We derive Table 15. Diophantine equations have no pairs of integer solutions (y, z) .

Case 4-c) $(q_1, q_2, q_3) = (3, 5, 29)$. In (3.31), put $(q_1, q_2, q_3) = (3, 5, 29)$. Then

$$435q_4q_5q_6 - 1 = 448(q_4 - 1)(q_5 - 1)(q_6 - 1). \quad (3.36)$$

It is easy to see that the lower bound and upper bound of possible primes q_4 are same numbers in Case 3-d). Furthermore, we get $q_4 \not\equiv 1 \pmod{3}$, $q_4 \not\equiv 1 \pmod{5}$ and $q_4 \not\equiv 1 \pmod{29}$ by (3.36).

Let $f_{26}(x) = f_9(x) - 2 := -435xyz + 448(x - 1)(y - 1)(z - 1) + 1$. Similarly, we get Table 16. There is no integer solution $(y, z) \in \mathbb{Z} \times \mathbb{Z}$.

Case 4-d) $(q_1, q_2, q_3) = (3, 5, 47)$ and $(3, 5, 53)$. Let $f_{27}(x) := -705xyz + 736(x - 1)(y - 1)(z - 1) + 1$ and $f_{28}(x) := -795xyz + 832(x - 1)(y - 1)(z - 1) + 1$.

Then we get Table 17. Three Diophantine equations have no pairs of integer solutions. Next, we consider the equation (3.12) with $q_1 = 3$ and $q_2 = 11$. It is easily checked that $q_3 \not\equiv 1 \pmod{3}$ and $q_3 \not\equiv 1 \pmod{11}$ by (3.12). So, we only consider a prime $q_3 = 17$.

Case 4-e) $(q_1, q_2, q_3) = (3, 11, 17)$. Consider

$$561q_4q_5q_6 - 1 = 640(q_4 - 1)(q_5 - 1)(q_6 - 1). \quad (3.37)$$

Similarly, the possible prime q_4 satisfying (3.37) is 19. Let $f_{29}(x) := -561q_4q_5q_6 + 640(q_4 - 1)(q_5 - 1)(q_6 - 1) + 1$.

Then the Diophantine equation $f_{29}(19) = 11523 - 11520y - 11520z + 861yz = 0$ has no pairs of integer solutions (y, z) .

Therefore, we complete the proof of Lemma 9. $\square$

As a result, from Lemma 9, there does not exist satisfying Eq. (1.1) and (1.2).

Proof of Theorem 1. From Lemma 1, Lemma 4, Lemma 5, Lemma 6, Lemma 7 and Lemma 9, we complete the proof of Theorem 1. $\square$

APPENDIX

TABLE 1. Primes p , q and r

p	(p, q_1, q_2)	$(2p+1, q_1, q_2)$	p	(p, q_1, r_1)	$(2p+1, q_1, q_2)$
13	(13, 3, 7)	(27, 21)	97	(97, 7, 17)	(195, 119)
37	(37, 3, 19)	(75, 57)	109	(109, 7, 19)	(219, 133)
41	(41, 5, 11)	(83, 55)	113	(113, 5, 29)	(227, 145)
61	(61, 3, 31)	(123, 93)	157	(157, 3, 79)	(315, 237)
61	(61, 7, 11)	(123, 77)	181	(181, 7, 31)	(363, 217)
73	(73, 3, 37)	(147, 111)	181	(181, 11, 19)	(363, 209)
73	(73, 5, 19)	(147, 95)	193	(193, 3, 97)	(387, 291)
73	(73, 7, 13)	(147, 91)	193	(193, 13, 17)	(387, 221)
89	(89, 5, 23)	(179, 115)			

TABLE 2. Primes p , q_1 , q_2 and q_3

p	(q_1, q_2, q_3)	$2p$	$q_1 q_2 q_3 - 1$	$q_1 q_2 q_3 + 1$	$U(n) = U(n+1)$ $= \varphi(n) = \varphi(n+1)$
97	(3, 5, 13)	194	194	196	O
193	(3, 7, 17)	386	356	358	×
241	(3, 5, 31)	482	464	466	×
241	(3, 11, 13)	482	428	430	×
241	(5, 7, 11)	482	384	386	×
337	(3, 5, 43)	674	644	646	×
337	(3, 7, 29)	674	608	610	×

TABLE 3. Bound of prime p_1 with respect to l

l	possible prime p_1
$l \geq 3$	$p_1 = 3$
$l \geq 7$	$p_1 = 3, 5$
$l \geq 15$	$p_1 = 3, 5, 7$
$l \geq 27$	$p_1 = 3, 5, 7, 11$
$l \geq 41$	$p_1 = 3, 5, 7, 11, 13$
$l \geq 62$	$p_1 = 3, 5, 7, 11, 13, 17$
$l \geq 85$	$p_1 = 3, 5, 7, 11, 13, 17, 19$
$l \geq 115$	$p_1 = 3, 5, 7, 11, 13, 17, 19, 23$
$l \geq 150$	$p_1 = 3, 5, 7, 11, 13, 17, 19, 23, 29$
$l \geq 186$	$p_1 = 3, 5, 7, 11, 13, 17, 19, 23, 29, 31$
$l \geq 229$	$p_1 = 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37$

TABLE 4. Diophantine equations for Lemma 9 Case 3-a

x	$f_3(x)$	x	$f_3(x)$
257	$65539 - 65536(y+z) + yz$	263	$67075 - 67072(y+z) + 7yz$
269	$68611 - 68608(y+z) + 13yz$	277	$70659 - 70656(y+z) + 21yz$
283	$72195 - 72192(y+z) + 27yz$	293	$74755 - 74752(y+z) + 37yz$
313	$79875 - 79872(y+z) + 57yz$	317	$80899 - 80896(y+z) + 61yz$
337	$86019 - 86016(y+z) + 81yz$	347	$88579 - 88576(y+z) + 91yz$
349	$89091 - 89088(y+z) + 93yz$	353	$90115 - 90112(y+z) + 97yz$
359	$91651 - 91648(y+z) + 103yz$	367	$93699 - 93696(y+z) + 111yz$
373	$95235 - 95232(y+z) + 117yz$	379	$96771 - 96768(y+z) + 123yz$
383	$97795 - 97792(y+z) + 127yz$	389	$99331 - 99328(y+z) + 133yz$
397	$101379 - 101376(y+z) + 141yz$	419	$107011 - 107008(y+z) + 163yz$
433	$110595 - 110592(y+z) + 177yz$	439	$112131 - 112128(y+z) + 183yz$
449	$114691 - 114688(y+z) + 193yz$	457	$116739 - 116736(y+z) + 201yz$
463	$118275 - 118272(y+z) + 207yz$	467	$119299 - 119296(y+z) + 211yz$
479	$122371 - 122368(y+z) + 223yz$	487	$124419 - 124416(y+z) + 231yz$
499	$127491 - 127488(y+z) + 243yz$	503	$128515 - 128512(y+z) + 247yz$
509	$130051 - 130048(y+z) + 253yz$	523	$133635 - 133632(y+z) + 267yz$
547	$139779 - 139776(y+z) + 291yz$	557	$142339 - 142336(y+z) + 301yz$
563	$143875 - 143872(y+z) + 307yz$	569	$145411 - 145408(y+z) + 313yz$
577	$147459 - 147456(y+z) + 321yz$	587	$150019 - 150016(y+z) + 331yz$
593	$151555 - 151552(y+z) + 337yz$	599	$153091 - 153088(y+z) + 343yz$
607	$155139 - 155136(y+z) + 351yz$	617	$157699 - 157696(y+z) + 361yz$
619	$158211 - 158208(y+z) + 363yz$	643	$164355 - 164352(y+z) + 387yz$
653	$166915 - 166912(y+z) + 397yz$	659	$168451 - 168448(y+z) + 403yz$
673	$172035 - 172032(y+z) + 417yz$	677	$173059 - 173056(y+z) + 421yz$
683	$174595 - 174592(y+z) + 427yz$	709	$181251 - 181248(y+z) + 453yz$
719	$183811 - 183808(y+z) + 463yz$	727	$185859 - 185856(y+z) + 471yz$
733	$187395 - 187392(y+z) + 477yz$	739	$188931 - 188928(y+z) + 483yz$
743	$189955 - 189952(y+z) + 487yz$	757	$193539 - 193536(y+z) + 501yz$

TABLE 5. Diophantine equations for Lemma 9 Case 3-b

x	$f_5(x)$	x	$f_5(x)$
97	$9217 - 9216(y+z) + yz$	103	$9793 - 9792(y+z) + 7yz$
107	$10177 - 10176(y+z) + 11yz$	109	$10369 - 10368(y+z) + 13yz$
113	$10753 - 10752(y+z) + 17yz$	127	$12097 - 12096(y+z) + 31yz$
137	$13057 - 13056(y+z) + 41yz$	139	$13249 - 13248(y+z) + 43yz$
149	$14209 - 14208(y+z) + 53yz$	157	$14977 - 14976(y+z) + 61yz$
163	$15553 - 15552(y+z) + 67yz$	167	$15937 - 15936(y+z) + 71yz$
173	$16513 - 16512(y+z) + 77yz$	179	$17089 - 17088(y+z) + 83yz$
193	$18433 - 18432(y+z) + 97yz$	197	$18817 - 18816(y+z) + 101yz$
199	$19009 - 19008(y+z) + 103yz$	223	$21313 - 21312(y+z) + 127yz$
227	$21697 - 21696(y+z) + 131yz$	233	$22273 - 22272(y+z) + 137yz$
239	$22849 - 22848(y+z) + 143yz$	257	$24577 - 24576(y+z)z + 161yz$
263	$25153 - 25152(y+z) + 167yz$	269	$25729 - 25728(y+z) + 173yz$
277	$26497 - 26496(y+z) + 181yz$		

TABLE 6. Diophantine equations for Lemma 9 Case 3-c

x	$f_7(x)$	x	$f_7(x)$
53	$18307 - 18304(y+z) + 19yz$	59	$20419 - 20416(y+z) + 61yz$
67	$23235 - 23232(y+z) + 117yz$	73	$25347 - 25344(y+z) + 159yz$
79	$27459 - 27456(y+z) + 201yz$	83	$28867 - 28864(y+z) + 229yz$
89	$30979 - 30976(y+z) + 271yz$	97	$33795 - 33792(y+z) + 327yz$
109	$38019 - 38016(y+z) + 411yz$	113	$39427 - 39424(y+z) + 439yz$
127	$44355 - 44352(y+z) + 537yz$	137	$47875 - 47872(y+z) + 607yz$

TABLE 7. Diophantine equations for Lemma 9 Case 3-d

x	$f_9(x)$	x	$f_9(x)$
37	$16131 - 16128(y+z) + 33yz$	43	$18819 - 18816(y+z) + 111yz$
47	$20611 - 20608(y+z) + 163yz$	53	$23299 - 23296(y+z) + 241yz$
67	$29571 - 29568(y+z) + 423yz$	73	$32259 - 32256(y+z) + 501yz$
79	$34947 - 34944(y+z) + 579yz$	83	$36739 - 36736(y+z) + 631yz$
89	$39427 - 39424(y+z) + 709yz$	97	$43011 - 43008(y+z) + 813yz$

TABLE 8. Diophantine equations for Lemma 9 Case 3-e

x	$f_{11}(x)$	x	$f_{11}(x)$
43	$8065 - 8064(y+z) + 109yz$	47	$8833 - 8832(y+z) + 137yz$
53	$9985 - 9984(y+z) + 179yz$	59	$11137 - 11136(y+z) + 221yz$
67	$12673 - 12672(y+z) + 277yz$	73	$13825 - 13824(y+z) + 319yz$

TABLE 9. Diophantine equations for Lemma 9 Case 3-f

x	$f_{12}(x) = 0$	x	$f_{12}(x) = 0$
47	$10305 = 10304(y+z) - 199yz$	53	$11649 = 11648(y+z) - 253yz$
59	$12993 = 12992(y+z) - 307yz$	67	$14785 = 14784(y+z) - 379yz$
x	$f_{13}(x) = 0$	x	$f_{13}(x) = 0$
53	$38275 = 38272(y+z) - 907yz$	59	$42691 = 42688(y+z) - 1093yz$
x	$f_{14}(x) = 0$		
59	$48259 = 48256(y+z) - 1351yz$		

TABLE 10. Diophantine equations for Lemma 9 Case 3-g

x	$f_{16}(x)$	x	$f_{16}(x)$
31	$2401 - 2400(y+z) + 13yz$	37	$2881 - 2880(y+z) + 31yz$
41	$3201 - 3200(y+z) + 43yz$	47	$3681 - 3680(y+z) + 61yz$
53	$4161 - 4160(y+z) + 79yz$	59	$4641 - 4640(y+z) + 97yz$
61	$4801 - 4800(y+z) + 103yz$	73	$5761 - 5760(y+z) + 139yz$

TABLE 11. Diophantine equations for Lemma 9 Case 3-h

x	$f_{17}(x) = 0$	x	$f_{17}(x) = 0$
23	$2113 = 2112(y+z) - 19yz$	31	$2881 = 2880(y+z) - 59yz$
37	$3457 = 3456(y+z) - 89yz$	41	$3841 = 3840(y+z) - 109yz$
47	$4417 = 4416(y+z) - 139yz$		

TABLE 12. Diophantine equations for Lemma 9 Case 3-i

x	$f_{18}(x) = 0$	x	$f_{18}(x) = 0$
19	$2305 = 2304(y+z) - 43yz$	23	$2817 = 2816(y+z) - 79yz$
31	$3841 = 3840(y+z) - 151yz$	37	$4609 = 4608(y+z) - 205yz$
x	$f_{19}(x) = 0$	x	$f_{19}(x) = 0$
23	$3169 = 3168(y+z) - 109yz$	31	$4321 = 4320(y+z) - 197yz$

TABLE 13. Diophantine equations for Lemma 9 Case 3-k

x	$f_{21}(x) = 0$	x	$f_{22}(x) = 0$
13	$1921 = 1920(y+z) - 61yz$	19	$11523 = 11520(y+z) - 861yz$
17	$2561 = 2560(y+z) - 129yz$		

TABLE 14. Diophantine equations for Lemma 9 Case 4-a

x	$f_{24}(x)$	x	$f_{24}(x)$
257	$65537 - 65536(y+z) + yz$	263	$67073 - 67072(y+z) + 7yz$
269	$68609 - 68608(y+z) + 13yz$	293	$74753 - 74752(y+z) + 37yz$
317	$80897 - 80896(y+z) + 61yz$	347	$88577 - 88576(y+z) + 91yz$
353	$90113 - 90112(y+z) + 97yz$	359	$91649 - 91648(y+z) + 103yz$
383	$97793 - 97792(y+z) + 127yz$	389	$99329 - 99328(y+z) + 133yz$
419	$107009 - 107008(y+z) + 163yz$	449	$114689 - 114688(y+z) + 193yz$
467	$119297 - 119296(y+z) + 211yz$	479	$122369 - 122368(y+z) + 223yz$
503	$128513 - 128512(y+z) + 247yz$	509	$130049 - 130048(y+z) + 253yz$
557	$142337 - 142336(y+z) + 301yz$	563	$143873 - 143872(y+z) + 307yz$
569	$145409 - 145408(y+z) + 313yz$	587	$150017 - 150016(y+z) + 331yz$
593	$151553 - 151552(y+z) + 337yz$	599	$153089 - 153088(y+z) + 343yz$
617	$157697 - 157696(y+z) + 361yz$	653	$166913 - 166912(y+z) + 397yz$
659	$168449 - 168448(y+z) + 403yz$	677	$173057 - 173056(y+z) + 421yz$
683	$174593 - 174592(y+z) + 427yz$	719	$183809 - 183808(y+z) + 463yz$
743	$189953 - 189952(y+z) + 487yz$		

TABLE 15. Diophantine equations for Lemma 9 Case 4-b

x	$f_{25}(x)$	x	$f_{25}(x)$
53	$18305 - 18304(y+z) + 19yz$	59	$20417 - 20416(y+z) + 61yz$
83	$28865 - 28864(y+z) + 229yz$	89	$30977 - 30976(y+z) + 271yz$
107	$37313 - 37312(y+z) + 397yz$	113	$39425 - 39424(y+z) + 439yz$
137	$47873 - 47872(y+z) + 607yz$		

TABLE 16. Diophantine equations for Lemma 9 Case 4-c

x	$f_{26}(x)$	x	$f_{26}(x)$
47	$20609 - 20608(y+z) + 163yz$	53	$23297 - 23296(y+z) + 241yz$
83	$36737 - 36736(y+z) + 631yz$	89	$39425 - 39424(y+z) + 709yz$

TABLE 17. Diophantine equations for Lemma 9 Case 4-d

x	$f_{28}(x) = 0$	x	$f_{29}(x) = 0$
53	$38273 = 38272(y+z) - 907yz$	59	$48257 = 48256(y+z) - 1351yz$
59	$42689 = 42688(y+z) - 1093yz$		

TABLE 18. Values of $U(n) = U(n+1)$ for $(1 \leq n \leq 10^5)$. Except $1, 12|U(n) = U(n+1)$

n	$n+1$	<i>Factor of n</i>	<i>Factor of $n+1$</i>	$U(n) = U(n+1)$
168	169	$2^3 \times 3 \times 7$	13^2	12
194	195	2×97	$3 \times 5 \times 13$	96
350	351	$2 \times 5^2 \times 7$	$3^3 \times 13$	24
1368	1369	$2^3 \times 3^2 \times 19$	37^2	36
1628	1629	$2^2 \times 11 \times 37$	$3^2 \times 181$	360
3705	3706	$3 \times 5 \times 13 \times 19$	$2 \times 17 \times 109$	1728
5186	5187	2×2593	$3 \times 7 \times 13 \times 19$	2592
6929	6930	$13^2 \times 41$	$2 \times 3^2 \times 5 \times 7 \times 11$	480
7475	7476	$5^2 \times 13 \times 23$	$2^2 \times 3 \times 7 \times 89$	1056
25545	25546	$3 \times 5 \times 13 \times 131$	$2 \times 53 \times 241$	12480
26047	26048	7×61^2	$2^6 \times 11 \times 37$	360
26864	26865	$2^4 \times 23 \times 73$	$3^3 \times 5 \times 199$	1584
28251	28252	$3^2 \times 43 \times 73$	$2^2 \times 7 \times 1009$	6048
34936	34937	$2^3 \times 11 \times 397$	$7^2 \times 23 \times 31$	3960
37248	37249	$2^7 \times 3 \times 97$	193^2	192
56574	56575	$2^2 \times 3 \times 29 \times 163$	$5^2 \times 2269$	9072
65575	65576	$5^2 \times 37 \times 71$	$2^2 \times 3 \times 13 \times 421$	10080
81732	81733	$2^2 \times 3 \times 7^2 \times 139$	37×47^2	1656
82368	82369	$2^6 \times 3^2 \times 11 \times 13$	$7^2 \times 41^2$	240
87308	87309	$2^2 \times 13 \times 23 \times 73$	$3^2 \times 89 \times 109$	19008
87367	87368	$7^2 \times 1783$	$2^3 \times 67 \times 163$	10692
88450	88451	$2 \times 5^2 \times 29 \times 61$	$11^2 \times 17 \times 43$	6720
91539	91539	$3^2 \times 7 \times 1453$	$2^2 \times 5 \times 23 \times 199$	17424

TABLE 19. Values of $\varphi(n) = \varphi(n+1) = U(n) = U(n+1)$ for $(1 \leq n \leq 10^6)$

n	$n+1$	<i>Factor of n</i>	<i>Factor of $n+1$</i>	$U(n)$	<i>Factor of $U(n)$</i>
1	2	•	•	1	•
194	195	2×97	$3 \times 5 \times 13$	96	$2^5 \times 3$
3705	3706	$3 \times 5 \times 13 \times 19$	$2 \times 17 \times 109$	1728	$2^6 \times 3^3$
5186	5187	2×2593	$3 \times 7 \times 13 \times 19$	2592	$2^5 \times 3^4$
25545	25546	$3 \times 5 \times 13 \times 131$	$2 \times 53 \times 141$	12480	$2^6 \times 3 \times 5 \times 13$
388245	388246	$3 \times 5 \times 11 \times 13 \times 181$	$2 \times 17 \times 19 \times 601$	172800	$2^8 \times 3^3 \times 5^2$

REFERENCES

- [1] R. Baillie, “Table of $\varphi(n) = \varphi(n+1)$,” *Math. Comp.*, vol. 30, pp. 189–190, 1976, doi: [10.2307/2005449](https://doi.org/10.2307/2005449).
- [2] R. Baillie, “Solutions of $\varphi(n) = \varphi(n+1)$ for euler’s function,” *Math. Comp.*, vol. 32, no. 144, p. 1326, 1978, doi: [10.2307/2006368](https://doi.org/10.2307/2006368).
- [3] D. Ballew, J. Case, and R. N. Higgins, “Table of $\varphi(n) = \varphi(n+1)$,” *Math. Comp.*, vol. 29, no. 129, pp. 329–330, 1975, doi: [10.2307/2005486](https://doi.org/10.2307/2005486).
- [4] A. Bayad and M. Goubi, “Proof of the möbius conjecture revisited,” *Proc. Jangjeon Math. Soc.*, vol. 16(2), pp. 237–243, 2013.
- [5] P. Erdős, C. Pomerance, and A. Sárközy, “On locally repeated values of certain arithmetic functions,” *II, Acta Math. Hungarica*, vol. 49, pp. 251–259, 1987, doi: [10.1007/bf01956329](https://doi.org/10.1007/bf01956329).
- [6] R. Goormaghtigh, “L’intermédiaire des math.” vol. 25, pp. 42–44, 1918.
- [7] R. K. Guy, “Unsolved problems in number theory,” *Springer*, 2004, doi: [10.1007/978-0-387-26677-0](https://doi.org/10.1007/978-0-387-26677-0).
- [8] V. L. Klee, “Some remarks on euler’s totient,” *Amer.Math.Monthly*, vol. 54 and 55, p. 332 and 360, 1947 and 1948, doi: [10.2307/2304964](https://doi.org/10.2307/2304964).
- [9] M. Lal and P. Gillard, “On the equation $\varphi(n) = \varphi(n+k)$,” *Math. Comp.*, vol. 26, pp. 579–583, 1972, doi: [10.1090/s0025-5718-1972-0319391-6](https://doi.org/10.1090/s0025-5718-1972-0319391-6).
- [10] L. Moser, “Some equations involving euler’s totient function,” *Amer. Math. Monthly*, vol. 56, pp. 22–23, 1949, doi: [10.2307/2305815](https://doi.org/10.2307/2305815).
- [11] R. Ratat, “L’intermédiaire des math.” vol. 24, pp. 101–102, 1917.
- [12] H. E. Rose, “A course in number theory,” *Clarendon Press*, 1987.
- [13] W. Sierpiński, “Elementary theory of numbers,” *Polska Akademia Nauk*, 1964.

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AN ACCURACY-TYPE ORDER ON ROUGH SETS

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Abstract. We introduce a new order on rough sets inspired by the notion of accuracy measure. Rough sets ordered by this accuracy-type order form a complete lattice, which is isomorphic to the lattice where rough sets are ordered traditionally by the inclusion relation. Although the lattices are order-isomorphic, the elements can be totally differently located in them. The isomorphism of the lattices is verified by introducing the so-called boundary representation of rough sets. Some basic properties of the introduced order are also presented and an illustrative example demonstrates the relationship between the different structures.

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1. INTRODUCTION

In this paper we examine different orders which can be defined on rough sets. The notion of rough sets were introduced by Pawlak in [8] aiming an extension of standard set theory. In his original model, Pawlak assumed that our knowledge about the objects of a universe U is given in terms of an equivalence relation $E \subseteq U \times U$ reflecting the indiscernibility of the objects. Two elements of U are E -equivalent if we cannot distinguish them by their attributes known by us. Using the classes of this equivalence for any subset of the object set a pair of sets called *lower and upper approximations* is defined. These pairs may be identified with rough sets and ordering them by the component-wise inclusion, we obtain a well-known complete lattice, *the rough set lattice*, whose properties were investigated in details. Forming the ratio of the cardinalities of the lower and upper approximations we obtain the so-called accuracy measure which is a basic numerical characterization of rough sets. This measure and its improved versions play an important role in granular computing and were studied in different papers [10–14]. Inspired by the accuracy measure we propose a new order on rough sets. Rough sets ordered by this accuracy-type order form a complete lattice which is isomorphic to the lattice obtained by the traditional inclusion order, although the sets can be completely differently positioned in this new lattice. Our work was also motivated by the papers [1, 6]. In [6] an extension

of the classical set algebra is presented, which can be considered as a counterpart of Belnap's four-valued logic. It is known that Belnap's four-valued logic deals with two truth values [1]: one of them describes the incompleteness of the information, and the other expresses its consistency.

2. PRELIMINARIES

For any equivalence relation $E \subseteq U \times U$ and any element $u \in U$, let $E(u)$ denote the E -equivalence class of u , i.e.

$$E(u) := \{x \in U \mid (u, x) \in E\}.$$

Then for any subset $X \subseteq U$ the *lower approximation* of X with respect to E is defined as

$$X_E := \{x \in U \mid E(x) \subseteq X\},$$

and the *upper approximation* of X is given by

$$X^E := \{x \in U \mid E(x) \cap X \neq \emptyset\}.$$

The set $\text{BN}_E(X) := X^E \setminus X_E$ will be referred to as the E -boundary region of X . Since $x \in E(x)$ for each $x \in U$, we get that $X_E \subseteq X \subseteq X^E$ and hence we can partition the elements of U into three disjoint classes with respect to the set X (see [5]):

- (1) The elements which are *certainly in* X . These elements are interpreted as the elements of X_E , because if $x \in X_E$, then all the elements of the equivalence class $E(x)$ of x are in X .
- (2) The elements which are *certainly not in* X . The elements x of U such that all the elements of the class $E(x)$ of x are not in X : $E(x) \cap X = \emptyset$.
- (3) The elements which are *possibly in* X , i.e. $x \in U$ which are E -related at least with one element of X and also with at least one element from its complement $X^c = U \setminus X$. In other words, now $E(x) \cap X \neq \emptyset$ and $E(x) \not\subseteq X$, i.e. $x \in X^E \setminus X_E = \text{BN}_E(X)$.

If $X_E = X^E$, i.e. the boundary region of X is the empty set $\emptyset$; then the set X is called *exact* with respect to E . Now let $X, Y \subseteq U$. The following properties will be used in our paper:

- (a) $\emptyset^E = \emptyset_E = \emptyset$, $U^E = U_E = U$;
- (b) $(X_E)^E = (X_E)_E = X_E$, $(X^E)^E = (X^E)_E = X^E$;
- (c) If $X \subseteq Y$ then $X_E \subseteq Y_E$ and $X^E \subseteq Y^E$.

By definition, for any set $X \subseteq U$, X_E , X^E and also $\text{BN}_E(X)$ are unions of some classes of the equivalence E . In what follows, unions of E -equivalence classes will be called *definable sets* and their set will be denoted by $\text{DF}(U, E)$. Thus

$$\text{DF}(U, E) = \left\{ X \subseteq U \mid X = \bigcup \{E(x) \mid x \in X\} \right\}.$$

As for each $X \in \text{DF}(U, E)$, $X^E = X_E = X$, according to (b), we deduce that $\text{DF}(U, E)$, $\{X^E \mid X \subseteq U\}$ and $\{X_E \mid X \subseteq U\}$ are the same systems of sets, and the definable sets are just the exact sets with respect to E . Clearly, with respect to $\subseteq$ they form an ordered set $(\text{DF}(U, E), \subseteq)$. It is known that $(\text{DF}(U, E), \subseteq)$ is a complete Boolean lattice, where the lattice operations $\vee$ and $\wedge$ are just the set-theoretical union and intersection of the definable sets. Since the set-theoretical complement X^c of a set $X \in \text{DF}(U, E)$ is also definable, the complement of X in the lattice $(\text{DF}(U, E), \subseteq)$ is just X^c .

The *rough set* of X can be identified with the pair (X_E, X^E) and the set of all rough sets of E is usually given as

$$RS(U, E) := \{(X_E, X^E) \mid X \subseteq U\}.$$

In other words, with every rough set we associate two exact sets, i.e. the lower approximation X_E and the upper approximation X^E of the target set X . X_E is also called the *positive region* of X , while $(X^E)^c$ the *negative region* of X . The set $RS(U, E)$ may be canonically ordered by the component-wise inclusion (called *inclusion order*):

$$(X_E, X^E) \leq (Y_E, Y^E) \Leftrightarrow X_E \subseteq Y_E \text{ and } X^E \subseteq Y^E,$$

obtaining a partially ordered set $\mathbb{RS}(U, E) := (RS(U, E), \leq)$. It is well-known that $\mathbb{RS}(U, E)$ is a completely distributive Stone lattice (see e.g. [9]) with the least element $(\emptyset, \emptyset)$ and the greatest element (U, U) . (Later this result was improved by S. D. Comer [2] by showing that $\mathbb{RS}(U, E)$ is a so-called regular double Stone algebra). Of course, a rough set can be characterized not only by the fact that how "wide" are the sets X_E and X^E assigned to it, but also by the fact that how "vague" is the rough set itself, i.e. how wide is its boundary with respect to its lower or upper approximation. For instance, a (finite) rough set different from $(\emptyset, \emptyset)$ can also be characterized numerically by the coefficient:

$$\alpha_E(X) = \frac{|X_E|}{|X^E|} = 1 - \frac{|\text{BN}_E(X)|}{|X_E| + |\text{BN}_E(X)|}$$

called the *accuracy of approximation*, where $|X|$ denotes the cardinality of the set $X \neq \emptyset$. For $\emptyset$ we define $\alpha_E(\emptyset) = 1$. Inspired by this notion introduced by Z. Pawlak in [8], we define a new partial order $\sqsubseteq$ on the set $RS(U, E)$ as follows:

$$(X_E, X^E) \sqsubseteq (Y_E, Y^E) \Leftrightarrow X_E \subseteq Y_E \text{ and } \text{BN}_E(Y) \subseteq \text{BN}_E(X).$$

Clearly, $\sqsubseteq$ is a reflexive and transitive relation. Let $(X_E, X^E) \sqsubseteq (Y_E, Y^E)$ and $(Y_E, Y^E) \sqsubseteq (X_E, X^E)$. Then $X_E = Y_E$ and $\text{BN}_E(Y) = \text{BN}_E(X)$ together imply

$$X^E = X_E \cup \text{BN}_E(X) = Y_E \cup \text{BN}_E(Y) = Y^E,$$

whence we get

$$(X_E, X^E) = (Y_E, Y^E).$$

This proves that $\sqsubseteq$ is also antisymmetric, thus it is a partial order.

On the other hand, if (X_E, X^E) and (Y_E, Y^E) are rough sets different from $(\emptyset, \emptyset)$ and $(X_E, X^E) \sqsubseteq (Y_E, Y^E)$, then $|X_E| \leq |Y_E|$ and $|\text{BN}_E(X)| \geq |\text{BN}_E(Y)|$ imply

$$\frac{|X_E|}{|\text{BN}_E(X)|} + 1 \leq \frac{|Y_E|}{|\text{BN}_E(Y)|} + 1,$$

and hence

$$\alpha_E(X) = 1 - \frac{1}{\frac{|X_E|}{|\text{BN}_E(X)|} + 1} \leq 1 - \frac{1}{\frac{|Y_E|}{|\text{BN}_E(Y)|} + 1} = \alpha_E(Y).$$

We also note that the accuracy measure can be generated using the Marczewski-Steinhaus metric defined on the subsets of a universe (see [12, 13]).

In what follows, we will study the partially ordered set $(RS(U, E), \sqsubseteq)$. Comparing two rough sets (A_E, A^E) and (B_E, B^E) by the relation $\sqsubseteq$, the inclusion $A_E \subseteq B_E$ expresses that the positive region of the first rough set is less than the positive region of the second one, i.e. the truth-value represented by the second one is bigger than the truth-value of the first one. The relation $\text{BN}_E(B) \subseteq \text{BN}_E(A)$ expresses that the second rough set is closer to an exact set than the first one, i.e. the information in the second one is more consistent than in the first one. Surprisingly, although $\sqsubseteq$ is significantly different from the inclusion order, $(RS(U, E), \sqsubseteq)$ is isomorphic to the lattice $(RS(U, E), \leq)$. To understand the properties of the lattice $(RS(U, E), \sqsubseteq)$ we need some more notions.

Lattice theoretical notions

A complete lattice L is called *completely distributive* (see e.g. [3]) if for any doubly indexed family of elements $\{x_{i,j}\}_{i \in I, j \in J}$, $(I, J \neq \emptyset)$ we have

$$\bigwedge_{i \in I} \left(\bigvee_{j \in J} x_{i,j} \right) = \bigvee_{f: I \rightarrow J} \left(\bigwedge_{i \in I} x_{i, f(i)} \right).$$

For instance, any complete Boolean lattice and finite distributive lattice is completely distributive. A bounded lattice L is called *pseudocomplemented*, if for any $x \in L$ there exists an element $x^* \in L$ such that $x \wedge y = 0$, for some $y \in L \Leftrightarrow y \leq x^*$. A bounded distributive lattice L is called a *Stone lattice*, whenever the equality

$$x^* \vee x^{**} = 1$$

holds for all $x \in L$ (see [4]).

A bijective function $\phi: M \rightarrow N$ between two ordered sets $(M_1, \leq_1)$ and $(M_2, \leq_2)$ is called an *order isomorphism* if for every x and y in M_1

$$x \leq_1 y \Leftrightarrow \phi(x) \leq_2 \phi(y).$$

Specially, if ϕ is an order isomorphism from M_1 to M_2 where $(M_1, \leq_1)$ and $(M_2, \leq_2)$ are complete lattices, then ϕ preserves the lattice operations.

A *singleton* of a binary relation $E \subseteq U \times U$ is an element $s \in U$ such that $E(s) = \{s\}$. The set of singletons will be denoted by $S(E)$, or simply by S , when there is no danger of confusion. Let $E \subseteq U \times U$ be an equivalence. Then clearly, for any $s \in S$ we have $\{s\}_E = \{s\}^E = \{s\}$, whence $S_E = S^E = S$. Thus $S \in \text{DF}(U, E)$. It was proved in [7] that for any $X, Y \in \text{DF}(U, E)$ with $X \subseteq Y$, the pair (X, Y) is a rough set defined by E if and only if $Y \setminus X$ does not contain any singleton of E . Since Y is a disjoint union of X and $Y \setminus X$, the latter condition is equivalent to $X \cap S = Y \cap S$. In other words,

$$(X, Y) \in \text{RS}(U, E) \Leftrightarrow X, Y \in \text{DF}(U, E), X \subseteq Y \text{ and } X \cap S = Y \cap S. \quad (2.1)$$

Therefore, the rough sets of E can be identified with the set

$$\text{RS}(U, E) = \{(X, Y) \in \text{DF}(U, E)^2 \mid X \subseteq Y \text{ and } X \cap S = Y \cap S\}. \quad (2.2)$$

Since the operations in the lattice $\mathbb{RS}(U, E) = (\text{RS}(U, E), \leq)$ are defined as the component-wise union and intersection of the pairs of definable sets, $\mathbb{RS}(U, E)$ constitute a complete sublattice of the direct square $(\text{DF}(U, E), \subseteq) \times (\text{DF}(U, E), \subseteq)$. The representation given in (2.2) is usually called *increasing representation of rough sets induced by E* .

Disjoint representations of rough sets were introduced by P. Pagliani in [5]. Each rough set (X_E, X^E) may be represented as a pair $(X_E, (X^E)^c)$, called *the disjoint rough set of X* . Clearly, $(X_E, (X^E)^c)$ belongs to $\text{DF}(U, E)^2$, and now $(X^E)^c$ can be interpreted as the set of elements that certainly are outside of X . (This is the negative region of X .) We denote

$$d\text{RS}(U, E) := \{(X_E, (X^E)^c) \mid X \subseteq U\}.$$

It was proved in [5] that

$$d\text{RS}(U, E) = \{(A, B) \in \text{DF}(U, E)^2 \mid A \cap B = \emptyset, S \subseteq A \cup B\}. \quad (3)$$

The pairs $(A, B), (C, D) \in d\text{RS}(U, E)$ are ordered as follows:

$$(A, B) \lesssim (C, D) \Leftrightarrow (A \subseteq C \text{ and } B \supseteq D)$$

In [5] it was shown that $\sigma: (A, B) \rightarrow (A, B^c)$ is an order-isomorphism between $(\text{RS}(U, E), \leq)$ and $(d\text{RS}(U, E), \lesssim)$, which allows us to call (3) *the disjoint representation of rough sets induced by E* .

3. BOUNDARY REPRESENTATION OF ROUGH SETS

Let us observe first that the rough set corresponding to a reference set $X \subseteq U$ can be determined not only by its lower and upper approximations X_E and X^E , but also by X_E and its boundary $\text{BN}_E(X) = X^E \setminus X_E \in \text{DF}(U, E)$. Indeed, we have $X^E = X_E \cup \text{BN}_E(X)$ and $X_E \cap \text{BN}_E(X) = \emptyset$, $S \cap \text{BN}_E(X) = \emptyset$. In fact, we have the following

Lemma 1. *Let $E \subseteq U \times U$ be an equivalence relation and S be the set of its singletons. Then for any $A, D \in \text{DF}(U, E)$, the pair $(A, A \cup D) \in \text{RS}(U, E)$ if and only if $(A \cup S) \cap D = \emptyset$.*

Proof. If $(A, A \cup D) \in \text{RS}(U, E)$, then $(A, A \cup D) = (X_E, X^E)$, for some $X \subseteq U$ and hence

$$D = (A \cup D) \setminus A = X^E \setminus X_E = \text{BN}_E(X).$$

Then $A \cap D = \emptyset$ and

$$S \cap D = S \cap \text{BN}_E(X) = \emptyset$$

imply $(A \cup S) \cap D = \emptyset$.

Conversely, assume that $(A \cup S) \cap D = \emptyset$. Then $S \cap D = \emptyset$. Since $A, A \cup D \in \text{DF}(U, E)$, $A \subseteq A \cup D$ and

$$(A \cup D) \cap S = (A \cap S) \cap (D \cap S) = A \cap S,$$

in view of (2.1) we obtain that $(A, A \cup D) \in \text{RS}(U, E)$. $\square$

As an immediate corollary we obtain:

Corollary 1. *For any equivalence $E \subseteq U \times U$,*

$$\text{RS}(U, E) = \{(A, A \cup D) \mid A, D \in \text{DF}(U, E), \text{ and } (A \cup S) \cap D = \emptyset\}.$$

Now, let us consider the following set

$$\text{BRS}(U, E) = \{(A, D) \mid A, D \in \text{DF}(U, E), \text{ and } (A \cup S) \cap D = \emptyset\}.$$

Defining on $\text{BRS}(U, E)$ a partial order as follows:

$$(A_1, D_1) \trianglelefteq (A_2, D_2) \iff A_1 \subseteq A_2 \text{ and } D_2 \subseteq D_1,$$

we obtain a partially ordered set $\mathbb{BRS}(U, E) := (\text{BRS}(U, E), \trianglelefteq)$. Since we will prove that $\mathbb{BRS}(U, E)$ is isomorphic to the lattice $\mathbb{RS}(U, E)$, we can call $\mathbb{BRS}(U, E)$ *boundary representation of the rough sets of $E \subseteq U \times U$* . We can see that this is closely related to the disjoint representation of rough sets.

Theorem 1. *Let E be an arbitrary equivalence on the universe U . Then $\mathbb{BRS}(U, E)$ and $\mathbb{RS}(U, E)$ are order-isomorphic.*

Proof. Let us consider the maps

$$\varphi: \text{RS}(U, E) \rightarrow \text{BRS}(U, E), \quad \varphi(A, B) = (A, (B \cup S)^c),$$

for all $(A, B) \in \text{RS}(U, E)$ and

$$\psi: \text{BRS}(U, E) \rightarrow \text{RS}(U, E), \quad \psi(A, D) = (A, A \cup (D \cup S)^c),$$

for all $(A, D) \in \text{BRS}(U, E)$.

First, check that φ and ψ are well-defined: Indeed, for any $(A, B) \in \text{RS}(U, E)$ we have $A, B \cup S \in \text{DF}(U, E)$, whence $(B \cup S)^c \in \text{DF}(U, E)$ and

$$(A \cup S) \cap (B \cup S)^c = (A \cup S) \cap B^c \cap S^c = (A \cap B^c \cap S^c) \cup (S \cap B^c \cap S^c) = \emptyset,$$

because $S \cap S^c = \emptyset$, and $A \subseteq B$ implies $A \cap B^c = \emptyset$. Hence $(A, (B \cup S)^c) \in BRS(U, E)$, according to (2). Further, for each $(A, D) \in BRS(U, E)$, $A, D, S \in DF(U, E)$ yields $A \cup (D \cup S)^c \in DF(U, E)$. Clearly, $A \subseteq A \cup (D \cup S)^c$ and

$$\begin{aligned} (A \cup (D \cup S)^c) \cap S &= (A \cup (D^c \cap S^c)) \cap S = (A \cap S) \cup (D^c \cap S^c \cap S) \\ &= (A \cap S) \cup \emptyset = A \cap S. \end{aligned}$$

Hence $(A, A \cup (D \cup S)^c) \in RS(U, E)$.

It is easy to see that both φ and ψ are order-preserving. Indeed, for some $(A_1, B_1), (A_2, B_2) \in RS(U, E)$ let $(A_1, B_1) \leq (A_2, B_2)$. Then $A_1 \subseteq A_2$ and $B_1 \cup S \subseteq B_2 \cup S$, and hence $(B_2 \cup S)^c \subseteq (B_1 \cup S)^c$. This implies

$$\varphi(A_1, B_1) = (A_1, (B_1 \cup S)^c) \leq (A_2, (B_2 \cup S)^c) = \varphi(A_2, B_2).$$

Similarly, let $(A_1, D_1) \leq (A_2, D_2)$ for some $(A_1, D_1), (A_2, D_2) \in BRS(U, E)$. Then $A_1 \subseteq A_2$ and $D_2 \subseteq D_1$ yields $(D_1 \cup S)^c \subseteq (D_2 \cup S)^c$, and hence we get

$$\psi(A_1, D_1) = (A_1, (D_1 \cup S)^c) \leq (A_2, (D_2 \cup S)^c) = \psi(A_2, D_2).$$

Finally, observe that the mappings φ and ψ are inverses each of other. Indeed, for any $(A, D) \in BRS(U, E)$ we have

$$\begin{aligned} \varphi(\psi(A, D)) &= \varphi(A, A \cup (D \cup S)^c) = (A, (A \cup (D \cup S)^c \cup S)^c) \\ &= (A, A^c \cap (D \cup S) \cap S^c) = (A, (A^c \cap D \cap S^c) \cup (A^c \cap S \cap S^c)) \\ &= (A, (A \cup S)^c \cap D) = (A, D), \end{aligned}$$

because $(A \cup S) \cap D = \emptyset$ implies $D \subseteq (A \cup S)^c$, i.e. $(A \cup S)^c \cap D = D$.

On the other hand, let $(A, B) \in RS(U, E)$. Then

$$\begin{aligned} \psi(\varphi(A, B)) &= \psi(A, (B \cup S)^c) = (A, A \cup [(B \cup S)^c \cup S]^c) \\ &= (A, A \cup [(B \cup S) \cap S^c]) = (A, A \cup [(B \cap S^c) \cup (S \cap S^c)]) \\ &= (A, A \cup (B \cap S^c)) = (A, B). \end{aligned}$$

The last equality holds because

$$B = B \cap (S \cup S^c) = (B \cap S) \cup (B \cap S^c) = (A \cap S) \cup (B \cap S^c)$$

implies

$$A \cup (B \cap S^c) = A \cup (A \cap S) \cup (B \cap S^c) = A \cup B = B.$$

The above results show that φ and ψ are order-isomorphism, hence the ordered sets $\mathbb{BRS}(U, E)$ and $\mathbb{RS}(U, E)$ are isomorphic. $\square$

Now let us consider the partially ordered set $(RS(U, E), \sqsubseteq)$ defined by the order $\sqsubseteq$. Now we are able to prove

Proposition 1. *Let E be an equivalence on the universe U . Then the partially ordered sets $(RS(U, E), \sqsubseteq)$ and $(RS(U, E), \leq)$ are order-isomorphic.*

Proof. Observe that it is enough to prove that $(RS(U, E), \sqsubseteq)$ and $(BRS(U, E), \leq)$ are order-isomorphic. Then Theorem 1 will imply that $(RS(U, E), \sqsubseteq) \cong (RS(U, E), \leq)$.

Let us consider the maps for all $(A_E, A^E) \in RS(U, E)$

$$f: RS(U, E) \rightarrow BRS(U, E), \quad f(A_E, A^E) = (A_E, A^E \setminus A_E),$$

and for all $(A, D) \in BRS(U, E)$

$$g: BRS(U, E) \rightarrow RS(U, E), \quad g(A, D) = (A, A \cup D).$$

Since $A_E, A^E \setminus A_E \in DF(U, E)$ and

$$(A_E \cup S) \cap (A^E \setminus A_E) = (A_E \cup S) \cap BN_E(A) = \emptyset,$$

we have $(A_E, A^E \setminus A_E) \in BRS(U, E)$, and hence f is well-defined. Observe also that for any $(A, D) \in BRS(U, E)$, we have $(A, A \cup D) \in RS(U, E)$. Indeed,

$$(A \cup S) \cap D = \emptyset$$

by definition of $BRS(U, E)$, and hence Lemma 1 yields $(A, A \cup D) \in RS(U, E)$. Thus the map g is also well-defined. Since

$$f(g(A, D)) = f(A, A \cup D) = (A, (A \cup D) \setminus A) = (A, D),$$

and

$$g(f(A_E, A^E)) = g(A_E, A^E \setminus A_E) = (A_E, A^E),$$

for all $(A, D) \in BRS(U, E)$, and $(A_E, A^E) \in RS(U, E)$, the maps f and g are inverses each of other. Therefore f and g are bijective mappings.

Now let $(X_E, X^E), (Y_E, Y^E) \in RS(U, E)$. Then we can write:

$$\begin{aligned} (X_E, X^E) \sqsubseteq (Y_E, Y^E) &\Leftrightarrow X_E \subseteq Y_E \text{ and } Y^E \setminus Y^E \subseteq X^E \setminus X_E \\ &\Leftrightarrow (X_E, X^E \setminus X_E) \leq (Y_E, Y^E \setminus Y_E) \\ &\Leftrightarrow f(X_E, X^E) \leq f(Y_E, Y^E). \end{aligned}$$

This proves that f (and also its inverse g) is an order-isomorphism. $\square$

Corollary 2. *Let $E \subseteq U \times U$ be an equivalence relation. Then $(BRS(U, E), \leq)$ and $(RS(U, E), \sqsubseteq)$ are completely distributive Stone lattices.*

It is not hard to determine the form of the join and meet operation in the new lattice $(RS(U, E), \sqsubseteq)$.

Proposition 2. *Let $E \subseteq U \times U$ be an equivalence relation. Then for any system $\mathcal{H} \subseteq \mathcal{P}(U)$ and $(X_E, X^E) \in RS(U, E)$, $X \in \mathcal{H}$, the join and meet in the complete lattice $(RS(U, E), \sqsubseteq)$ are given by the formulas:*

$$\bigsqcup_{X \in \mathcal{H}} (X_E, X^E) = \left(\bigcup_{X \in \mathcal{H}} X_E, \left(\bigcup_{X \in \mathcal{H}} X_E \right) \cup \left(\bigcap_{X \in \mathcal{H}} BN_E(X) \right) \right),$$

$$\bigcap_{X \in \mathcal{H}} (X_E, X^E) = \left(\bigcap_{X \in \mathcal{H}} X_E, \left(\bigcap_{X \in \mathcal{H}} X_E \right) \cup \left(\bigcup_{X \in \mathcal{H}} BN_E(X) \right) \right).$$

Proof. Since $X_E \subseteq \bigcup_{X \in \mathcal{H}} X_E$ and $\bigcap_{X \in \mathcal{H}} BN_E(X) \subseteq BN_E(X)$, for each $X \in \mathcal{H}$, we have $(X_E, X^E) \sqsubseteq \left(\bigcup_{X \in \mathcal{H}} X_E, \left(\bigcup_{X \in \mathcal{H}} X_E \right) \cup \left(\bigcap_{X \in \mathcal{H}} BN_E(X) \right) \right)$ for all $X \in \mathcal{H}$, i.e. the right side is an upperbound for the system $\{(X_E, X^E) \mid X \in \mathcal{H}\}$. Now, suppose that (Y_E, Y^E) is also an upper bound for $\{(X_E, X^E) \mid X \in \mathcal{H}\}$.

Then the relations $X_E \subseteq Y_E$ and $BN_E(Y) \subseteq BN_E(X)$, for all $X \in \mathcal{H}$ imply $\bigcup_{X \in \mathcal{H}} X_E \subseteq Y_E$ and $BN_E(Y) \subseteq \bigcap_{X \in \mathcal{H}} BN_E(X)$, proving that

$$\left(\bigcup_{X \in \mathcal{H}} X_E, \left(\bigcup_{X \in \mathcal{H}} X_E \right) \cup \left(\bigcap_{X \in \mathcal{H}} BN_E(X) \right) \right) \sqsubseteq (Y_E, Y^E).$$

This means that

$$\left(\bigcup_{X \in \mathcal{H}} X_E, \left(\bigcup_{X \in \mathcal{H}} X_E \right) \cup \left(\bigcap_{X \in \mathcal{H}} BN_E(X) \right) \right)$$

equals to the supremum $\bigsqcup_{X \in \mathcal{H}} (X_E, X^E)$ of $\{(X_E, X^E) \mid X \in \mathcal{H}\}$. The second equality is proved dually. $\square$

Let L be a complete lattice and an element $p \in L$. Then the set

$$[p) = \{x \in L \mid x \geq p\}$$

is called the *principal filter* belonging to p . In addition, we use the notion of an *exact element* of $RS(U, E)$ as follows:

$$(X_E, X^E) \in RS(U, E) \text{ is } \textit{exact}, \text{ if } X_E = X = X^E.$$

Corollary 3. *The principal filter*

$$[(\emptyset, \emptyset)) = \{(X, Y) \in RS(U, E) \mid (\emptyset, \emptyset) \sqsubseteq (X, Y)\}$$

is equal to the set of the exact elements of $RS(U, E)$.

Proof. Let (X, X) be an exact element of the lattice $RS(U, E)$. Then clearly $(\emptyset, \emptyset) \sqsubseteq (X, X)$. Conversely, for any $(X, Y) \in [(\emptyset, \emptyset))$ the relation $(\emptyset, \emptyset) \sqsubseteq (X, Y)$ implies $Y \setminus X \subseteq \emptyset$, i.e. $X = Y$. This means that (X, Y) is an exact element of $RS(U, E)$. $\square$

We give an example to illustrate the discussed structures.

TABLE 1.

X	X_E	X^E	$BN_E(X)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\{a\}$	$\{a\}$	$\emptyset$
$\{b\}$	$\{b\}$	$\{b\}$	$\emptyset$
$\{c\}$	$\emptyset$	$\{c;d\}$	$\{c;d\}$
$\{d\}$	$\emptyset$	$\{c;d\}$	$\{c;d\}$
$\{a;b\}$	$\{a;b\}$	$\{a;b\}$	$\emptyset$
$\{a;c\}$	$\{a\}$	$\{a;c;d\}$	$\{c;d\}$
$\{a;d\}$	$\{a\}$	$\{a;c;d\}$	$\{c;d\}$
$\{b;c\}$	$\{b\}$	$\{b;c;d\}$	$\{c;d\}$
$\{b;d\}$	$\{b\}$	$\{b;c;d\}$	$\{c;d\}$
$\{c;d\}$	$\{c;d\}$	$\{c;d\}$	$\emptyset$
$\{a;b;c\}$	$\{a;b\}$	U	$\{c;d\}$
$\{a;b;d\}$	$\{a;b\}$	U	$\{c;d\}$
$\{a;c;d\}$	$\{a;c;d\}$	$\{a;c;d\}$	$\emptyset$
$\{b;c;d\}$	$\{b;c;d\}$	$\{b;c;d\}$	$\emptyset$
U	U	U	$\emptyset$

Example 1. Consider a universe consisting of four elements $U = \{a, b, c, d\}$. Let E be an equivalence relation on U yielding the partitions

$$U/E = \{\{a\}, \{b\}, \{c;d\}\}.$$

Table 1 contains the lower and upper approximations and the boundary regions for all subsets of U .

Based on Table 1 twelve different rough sets are detected and the rough set lattice with the inclusion order is presented in Figure 1. Using the boundary regions we construct the lattices $(BRS(U, E), \leq)$ and $(RS(U, E), \sqsubseteq)$ and plot their Hasse diagrams in the next Figure 2. Comparing the diagrams of the lattices $(RS(U, E), \leq)$ (see Figure 1) and $(RS(U, E), \sqsubseteq)$ (see Figure 2) their order-isomorphism is obvious, but the rough sets are located in different way in them.

As an illustration to Corollary 3 we list the exact rough sets from Table 1, which form the principal filter belonging to $(\emptyset, \emptyset)$ as follows:

$$\begin{aligned} [(\emptyset, \emptyset)] = & \{(\emptyset, \emptyset); (\{a\}, \{a\}); (\{b\}, \{b\}); (\{a;b\}, \{a;b\}); (\{c;d\}, \{c;d\}); \\ & (\{a;c;d\}, \{a;c;d\}); (\{b;c;d\}, \{b;c;d\}); (U, U)\}. \end{aligned}$$

The elements of the filter are the edges of the upper "cube" in the lattice $(RS(U, E), \sqsubseteq)$ as it is visible in Figure 2.

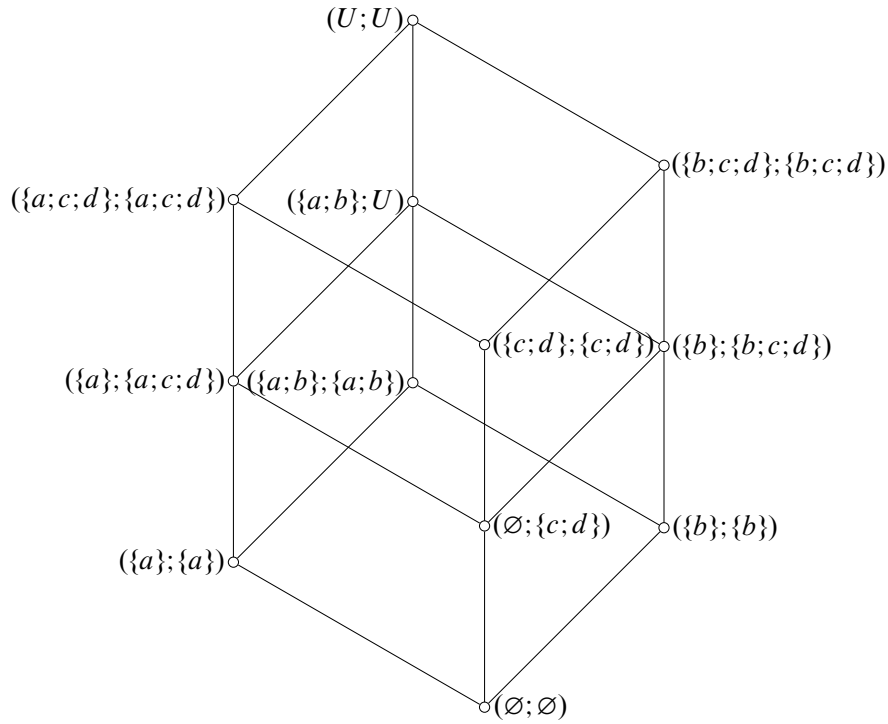


FIGURE 1. Forming rough sets and the rough set lattice $(RS(U, E), \leq)$ to the equivalence relation given in Example 1

4. CONCLUSIONS

Except for the properties of the well-known rough set lattice the order-theoretical consequences of rough set theory are less investigated. In this paper, starting from the traditional accuracy measure we propose a new order on the set of rough sets. This order induces a complete lattice which is isomorphic to the rough set lattice ordered by the inclusion relation. In order to prove the isomorphism of the structures we used a new type of representation of rough sets called boundary representation. For the sake of completeness the join and meet operators are also derived to this new lattice. We prove that the exact rough sets (for which the lower and upper approximations coincide) are specially located in our construction. To illustrate our examination a simple example is studied and the corresponding lattices are presented. As a future work we plan to explore further properties of this new order and study possible applications and examples.

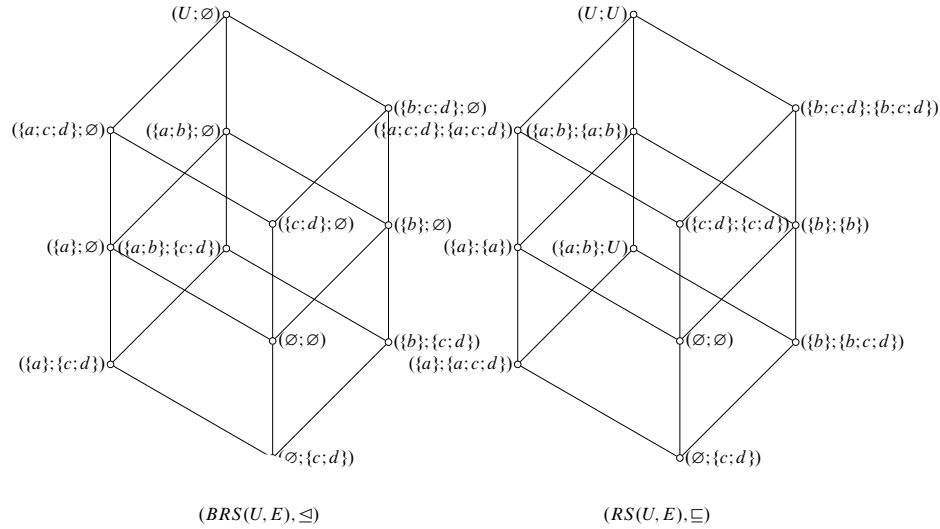


FIGURE 2. The Hasse diagram of the lattices $(BRS(U, E), \leq)$ and $(RS(U, E), \subseteq)$

REFERENCES

- [1] Belnap N.D., “A useful four-valued logic,” in *Modern Uses of Multiple-Valued Logic*, J.M. Dunn and G. Epstein, Ed. D.Reidel, 1977, pp. 8–37.
- [2] Comer, S. D., “On connections between information systems, rough sets and algebraic logic,” *Algebraic Methods in Logic and Computer Science, Banach Center Publications*, vol. 28, pp. 117–124, 1993, doi: [10.4064/-28-1-117-124](https://doi.org/10.4064/-28-1-117-124).
- [3] Ganter, B. and Wille, R., *Formal Concept Analysis, Mathematical Foundations*. Berlin: Springer-Verlag, 1999. doi: [10.1007/978-3-642-59830-2](https://doi.org/10.1007/978-3-642-59830-2).
- [4] Grätzer, G., *General Lattice Theory, 2nd ed.* Basel: Birkhäuser, 1998. doi: [10.1007/978-3-0348-7633-9](https://doi.org/10.1007/978-3-0348-7633-9).
- [5] Järvinen, J., “Approximations and rough sets based on tolerances,” in *Rough Sets and Current Trends in Computing 2000*, Ziarko, W. and Yao, Y. Y., Ed. Berlin, Heidelberg: Springer, 2001, pp. 182–189, doi: [10.1007/3-540-45554-X-21](https://doi.org/10.1007/3-540-45554-X-21).
- [6] Mousavi, A., Jabedar-Maralani, P., “Relative sets and rough sets,” *International Journal of Applied Mathematics and Computer Science*, vol. 11, pp. 637–653, 2001.
- [7] Pagliani, P., “Rough set systems and logico-algebraic structures,” in *Incomplete Information: Rough Set Analysis*, Orlowska, E., Ed. Heidelberg: Physica, 1997, pp. 109–190, doi: [10.1007/978-3-7908-1888-8-6](https://doi.org/10.1007/978-3-7908-1888-8-6).
- [8] Pawlak, Z., “Rough sets,” *Int. J. Comput. Inf.*, vol. 11, pp. 341–356, 1982, doi: [10.1007/BF01001956](https://doi.org/10.1007/BF01001956).
- [9] Pomykala, J. and Pomykala, J. A., “The Stone algebra of rough sets,” *Bull. Polish Acad. Sci. Math.*, vol. 36, no. 7-8, pp. 495–508, 1988.

- [10] Wang, J., Liang, J., Qian, Y. and Dang, C., “Uncertainty measure of rough sets based on a knowledge granulation for nincomplete information systems,” *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 16, no. 02, pp. 233–244, 2008, doi: [10.1142/S0218488508005157](https://doi.org/10.1142/S0218488508005157).
- [11] Xu, B., Zhou, Y and Lu, H., “An improved accuracy measure for rough sets,” *Journal of Computer and System Sciences*, vol. 71, no. 2, pp. 163–173, 2005, doi: [10.1016/j.jcss.2005.02.002](https://doi.org/10.1016/j.jcss.2005.02.002).
- [12] Yao, Y. Y., “Information granulation and rough set approximation,” *International Journal of Intelligent Systems*, vol. 16, no. 1, pp. 87–104, 2001, doi: [10.1002/1098-111X\(200101\)](https://doi.org/10.1002/1098-111X(200101)).
- [13] Yao, Y. Y., “Notes on rough set approximations and associated measures,” *Journal of Zhejiang Ocean University (Natural Science)*, vol. 29, no. 5, pp. 399–410, 2010.
- [14] Yao Y.Y., Wong S.K.M., Lin T.Y., “A Review of Rough Set Models,” in *Rough Sets and Data Mining*. Boston, MA: Springer, 1997, pp. 47–75, doi: [10.1007/978-1-4613-1461-5-3](https://doi.org/10.1007/978-1-4613-1461-5-3).

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A GENERALIZATION OF g -SUPPLEMENTED MODULES

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Abstract. In this work g -radical supplemented modules are defined which generalize g -supplemented modules. Some properties of g -radical supplemented modules are investigated. It is proved that the finite sum of g -radical supplemented modules is g -radical supplemented. It is also proved that every factor module and every homomorphic image of a g -radical supplemented module is g -radical supplemented. Let R be a ring. Then ${}_R R$ is g -radical supplemented if and only if every finitely generated R -module is g -radical supplemented. In the end of this work, it is given two examples for g -radical supplemented modules separating with g -supplemented modules.

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1. INTRODUCTION

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an R -module. We will denote a submodule N of M by $N \leq M$. Let M be an R -module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small submodule* of M and denoted by $N \ll M$. Let M be an R -module and $N \leq M$. If there exists a submodule K of M such that $M = N + K$ and $N \cap K = 0$, then N is called a *direct summand* of M and it is denoted by $M = N \oplus K$. For any module M , we have $M = M \oplus 0$. $Rad M$ indicates the radical of M . A submodule N of an R -module M is called an *essential submodule* of M , denoted by $N \leq_e M$, in case $K \cap N \neq 0$ for every submodule $K \neq 0$. Let M be an R -module and K be a submodule of M . K is called a *generalized small* (briefly, *g -small*) *submodule* of M if for every $T \leq M$ with $M = K + T$ implies that $T = M$, this is written by $K \ll_g M$ (in [6], it is called an *e -small submodule* of M and denoted by $K \ll_e M$). It is clear that every small submodule is a generalized small submodule but the converse is not true generally. Let M be an R -module. M is called an *hollow module* if every proper submodule of M is small in M . M is called a *local module* if M has the largest submodule, i.e.

a proper submodule which contains all other proper submodules. Let U and V be submodules of M . If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M . M is called a *supplemented module* if every submodule of M has a supplement in M . Let M be an R -module and $U, V \leq M$. If $M = U + V$ and $M = U + T$ with $T \leq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g M$, then V is called a *g-supplement* of U in M . M is called *g-supplemented* if every submodule of M has a g-supplement in M . The intersection of maximal essential submodules of an R -module M is called a *generalized radical* of M and denoted by $\text{Rad}_g M$ (in [6], it is denoted by $\text{Rad}_e M$). If M have no maximal essential submodules, then we denote $\text{Rad}_g M = M$.

Lemma 1 ([2, 4, 6]). *Let M be an R -module and $K, L, N, T \leq M$. Then the followings are hold.*

(1) *If $K \leq N$ and N is generalized small submodule of M , then K is a generalized small submodule of M .*

(2) *If K is contained in N and a generalized small submodule of N , then K is a generalized small submodule in submodules of M which contains submodule N .*

(3) *Let S be an R -module and $f : M \rightarrow S$ be an R -module homomorphism. If $K \ll_g M$, then $f(K) \ll_g S$.*

(4) *If $K \ll_g L$ and $N \ll_g T$, then $K + N \ll_g L + T$.*

Corollary 1. *Let $M_1, M_2, \dots, M_n \leq M$, $K_1 \ll_g M_1$, $K_2 \ll_g M_2$, ..., $K_n \ll_g M_n$. Then $K_1 + K_2 + \dots + K_n \ll_g M_1 + M_2 + \dots + M_n$.*

Corollary 2. *Let M be an R -module and $K \leq N \leq M$. If $N \ll_g M$, then $N/K \ll_g M/K$.*

Corollary 3. *Let M be an R -module, $K \ll_g M$ and $L \leq M$. Then $(K + L)/L \ll_g M/L$.*

Lemma 2. *Let M be an R -module. Then $\text{Rad}_g M = \sum_{L \ll_g M} L$.*

Proof. See [2]. □

Lemma 3. *The following assertions are hold.*

(1) *If M is an R -module, then $Rm \ll_g M$ for every $m \in \text{Rad}_g M$.*

(2) *If $N \leq M$, then $\text{Rad}_g N \leq \text{Rad}_g M$.*

(3) *If $K, L \leq M$, then $\text{Rad}_g K + \text{Rad}_g L \leq \text{Rad}_g (K + L)$.*

(4) *If $f : M \rightarrow N$ is an R -module homomorphism, then $f(\text{Rad}_g M) \leq \text{Rad}_g N$.*

(5) *If $K, L \leq M$, then $\frac{\text{Rad}_g K + L}{L} \leq \text{Rad}_g \frac{K + L}{L}$.*

Proof. Clear from Lemma 1 and Lemma 2. □

Lemma 4. *Let $M = \oplus_{i \in I} M_i$. Then $\text{Rad}_g M = \oplus_{i \in I} \text{Rad}_g M_i$.*

Proof. Since $M_i \leq M$, then by Lemma 3(2), $Rad_g M_i \leq Rad_g M$ and $\bigoplus_{i \in I} Rad_g M_i \leq Rad_g M$. Let $x \in Rad_g M$. Then by Lemma 3(1), $Rx \ll_g M$. Since $x \in M = \bigoplus_{i \in I} M_i$, there exist $i_1, i_2, \dots, i_k \in I$ and $x_{i_1} \in M_{i_1}, x_{i_2} \in M_{i_2}, \dots, x_{i_k} \in M_{i_k}$ such that $x = x_{i_1} + x_{i_2} + \dots + x_{i_k}$. Since $Rx \ll_g M$, then by Lemma 1(4), under the canonical epimorphism π_{i_t} ($t = 1, 2, \dots, k$) $Rx_{i_t} = \pi_{i_t}(Rx) \ll_g Rx_{i_t}$. Then $x_{i_t} \in Rad_g M_{i_t}$ ($t = 1, 2, \dots, k$) and $x = x_{i_1} + x_{i_2} + \dots + x_{i_k} \in \bigoplus_{i \in I} Rad_g M_i$. Hence $Rad_g M \leq \bigoplus_{i \in I} Rad_g M_i$ and since $\bigoplus_{i \in I} Rad_g M_i \leq Rad_g M$, $Rad_g M = \bigoplus_{i \in I} Rad_g M_i$. $\square$

2. G-RADICAL SUPPLEMENTED MODULES

Definition 1. Let M be an R -module and $U, V \leq M$. If $M = U + V$ and $U \cap V \leq Rad_g V$, then V is called a generalized radical supplement (briefly, g-radical supplement) of U in M . If every submodule of M has a generalized radical supplement in M , then M is called a generalized radical supplemented (briefly, g-radical supplemented) module.

Clearly we see that every g-supplemented module is g-radical supplemented. But the converse is not true in general. (See Example 1 and 2.)

Lemma 5. Let M be an R -module and $U, V \leq M$. Then V is a g-radical supplement of U in M if and only if $M = U + V$ and $Rm \ll_g V$ for every $m \in U \cap V$.

Proof. ($\Rightarrow$) Since V is a g-radical supplement of U in M , $M = U + V$ and $U \cap V \leq Rad_g V$. Let $m \in U \cap V$. Since $U \cap V \leq Rad_g V$, $m \in Rad_g V$. Hence by Lemma 3(1), $Rm \ll_g V$.

($\Leftarrow$) Since $Rm \ll_g V$ for every $m \in U \cap V$, then by Lemma 2, $U \cap V \leq Rad_g V$ and hence V is a g-radical supplement of U in M . $\square$

Lemma 6. Let M be an R -module, $M_1, U, X \leq M$ and $Y \leq M_1$. If X is a g-radical supplement of $M_1 + U$ in M and Y is a g-radical supplement of $(U + X) \cap M_1$ in M_1 , then $X + Y$ is a g-radical supplement of U in M .

Proof. Since X is a g-radical supplement of $M_1 + U$ in M , $M = M_1 + U + X$ and $(M_1 + U) \cap X \leq Rad_g X$. Since Y is a g-radical supplement of $(U + X) \cap M_1$ in M_1 , $M_1 = (U + X) \cap M_1 + Y$ and $(U + X) \cap Y = (U + X) \cap M_1 \cap Y \leq Rad_g Y$. Then $M = M_1 + U + X = (U + X) \cap M_1 + Y + U + X = U + X + Y$ and, by Lemma 3(3), $U \cap (X + Y) \leq (U + X) \cap Y + (U + Y) \cap X \leq Rad_g Y + (M_1 + U) \cap X \leq Rad_g Y + Rad_g X \leq Rad_g (X + Y)$. Hence $X + Y$ is a g-radical supplement of U in M . $\square$

Lemma 7. Let $M = M_1 + M_2$. If M_1 and M_2 are g-radical supplemented, then M is also g-radical supplemented.

Proof. Let $U \leq M$. Then 0 is a g-radical supplement of $M_1 + M_2 + U$ in M . Since M_1 is g-radical supplemented, there exists a g-radical supplement X of

$(M_2 + U) \cap M_1 = (M_2 + U + 0) \cap M_1$ in M_1 . Then by Lemma 6, $X + 0 = X$ is a g-radical supplement of $M_2 + U$ in M . Since M_2 is g-radical supplemented, there exists a g-radical supplement Y of $(U + X) \cap M_2$ in M_2 . Then by Lemma 6, $X + Y$ is a g-radical supplement of U in M . $\square$

Corollary 4. *Let $M = M_1 + M_2 + \dots + M_k$. If M_i is g-radical supplemented for every $i = 1, 2, \dots, k$, then M is also g-radical supplemented.*

Proof. Clear from Lemma 7. $\square$

Lemma 8. *Let M be an R -module, $U, V \leq M$ and $K \leq U$. If V is a g-radical supplement of U in M , then $(V + K)/K$ is a g-radical supplement of U/K in M/K .*

Proof. Since V is a g-radical supplement of U in M , $M = U + V$ and $U \cap V \leq \text{Rad}_g V$. Then $M/K = U/K + (V + K)/K$ and by Lemma 3(5), $(U/K) \cap ((V + K)/K) = (U \cap V + K)/K \leq (\text{Rad}_g V + K)/K \leq \text{Rad}_g [(V + K)/K]$. Hence $(V + K)/K$ is a g-radical supplement of U/K in M/K . $\square$

Lemma 9. *Every factor module of a g-radical supplemented module is g-radical supplemented.*

Proof. Clear from Lemma 8. $\square$

Corollary 5. *The homomorphic image of a g-radical supplemented module is g-radical supplemented.*

Proof. Clear from Lemma 9. $\square$

Lemma 10. *Let M be a g-radical supplemented module. Then every finitely M -generated module is g-radical supplemented.*

Proof. Clear from Corollary 4 and Corollary 5. $\square$

Corollary 6. *Let R be a ring. Then ${}_R R$ is g-radical supplemented if and only if every finitely generated R -module is g-radical supplemented.*

Proof. Clear from Lemma 10. $\square$

Theorem 1. *Let M be an R -module. If M is g-radical supplemented, then $M/\text{Rad}_g M$ is semisimple.*

Proof. Let $U/\text{Rad}_g M \leq M/\text{Rad}_g M$. Since M is g-radical supplemented, there exists a g-radical supplement V of U in M . Then $M = U + V$ and $U \cap V \leq \text{Rad}_g V$. Thus $M/\text{Rad}_g M = U/\text{Rad}_g M + (V + \text{Rad}_g M)/\text{Rad}_g M$ and

$$\begin{aligned} (U/\text{Rad}_g M) \cap ((V + \text{Rad}_g M)/\text{Rad}_g M) &= (U \cap V + \text{Rad}_g M)/\text{Rad}_g M \\ &\leq (\text{Rad}_g V + \text{Rad}_g M)/\text{Rad}_g M \\ &= \text{Rad}_g M/\text{Rad}_g M = 0. \end{aligned}$$

Hence $M/\text{Rad}_g M = U/\text{Rad}_g M \oplus (V + \text{Rad}_g M)/\text{Rad}_g M$ and $U/\text{Rad}_g M$ is a direct summand of M . $\square$

Lemma 11. *Let M be a g-radical supplemented module and $L \leq M$ with $L \cap \text{Rad}_g M = 0$. Then L is semisimple. In particular, a g-radical supplemented module M with $\text{Rad}_g M = 0$ is semisimple.*

Proof. Let $X \leq L$. Since M is g-radical supplemented, there exists a g-radical supplement T of X in M . Hence $M = X + T$ and $X \cap T \leq \text{Rad}_g T \leq \text{Rad}_g M$. Since $M = X + T$ and $X \leq L$, by Modular Law, $L = L \cap M = L \cap (X + T) = X + L \cap T$. Since $X \cap T \leq \text{Rad}_g M$ and $L \cap \text{Rad}_g M = 0$, $X \cap L \cap T = L \cap X \cap T \leq L \cap \text{Rad}_g M = 0$. Hence $L = X \oplus L \cap T$ and X is a direct summand of L . $\square$

Proposition 1. *Let M be a g-radical supplemented module. Then $M = K \oplus L$ for some semisimple module K and some module L with essential generalized radical.*

Proof. Let K be a complement of $\text{Rad}_g M$ in M . Then by [5, 17.6], $K \oplus \text{Rad}_g M \leq M$. Since $K \cap \text{Rad}_g M = 0$, then by Lemma 11, K is semisimple. Since M is g-radical supplemented, there exists a g-radical supplement L of K in M . Hence $M = K + L$ and $K \cap L \leq \text{Rad}_g L \leq \text{Rad}_g M$. Then by $K \cap \text{Rad}_g M = 0$, $K \cap L = 0$. Hence $M = K \oplus L$. Since $M = K \oplus L$, then by Lemma 4, $\text{Rad}_g M = \text{Rad}_g K \oplus \text{Rad}_g L$. Hence $K \oplus \text{Rad}_g M = K \oplus \text{Rad}_g L$. Since $K \oplus \text{Rad}_g L = K \oplus \text{Rad}_g M \leq M = K \oplus L$, then by [1, Proposition 5.20], $\text{Rad}_g L \leq L$. $\square$

Proposition 2. *Let M be an R -module and $U \leq M$. The following statements are equivalent.*

- (1) *There is a decomposition $M = X \oplus Y$ with $X \leq U$ and $U \cap Y \leq \text{Rad}_g Y$.*
- (2) *There exists an idempotent $e \in \text{End}(M)$ with $e(M) \leq U$ and $(1-e)(U) \leq \text{Rad}_g(1-e)(M)$.*
- (3) *There exists a direct summand X of M with $X \leq U$ and $U/X \leq \text{Rad}_g(M/X)$.*
- (4) *U has a g-radical supplement Y such that $U \cap Y$ is a direct summand of U .*

Proof. (1) $\Rightarrow$ (2) For a decomposition $M = X \oplus Y$, there exists an idempotent $e \in \text{End}(M)$ with $X = e(M)$ and $Y = (1-e)(M)$. Since $e(M) = X \leq U$, we easily see that $(1-e)(U) = U \cap (1-e)(M)$. Then by $Y = (1-e)(M)$ and $U \cap Y \leq \text{Rad}_g Y$, $(1-e)(U) = U \cap (1-e)(M) = U \cap Y \leq \text{Rad}_g Y = \text{Rad}_g(1-e)(M)$.

(2) $\Rightarrow$ (3) Let $X = e(M)$ and $Y = (1-e)(M)$. Since $e \in \text{End}(M)$ is idempotent, we easily see that $M = X \oplus Y$. Then $M = U + Y$. Since $e(M) = X \leq U$, we easily see that $(1-e)(U) = U \cap (1-e)(M)$. Since $M = U + Y$ and $U \cap Y = U \cap (1-e)(M) = (1-e)(U) \leq \text{Rad}_g(1-e)(M) = \text{Rad}_g Y$, Y is a g-radical supplement of U in M . Then by Lemma 8, $M/X = (Y + X)/X$ is a g-radical supplement of U/X in M/X . Hence $U/X = (U/X) \cap (M/X) \leq \text{Rad}_g(M/X)$.

(3) $\Rightarrow$ (4) Let $M = X \oplus Y$. Since $X \leq U$, $M = U + Y$. Let $t \in U \cap Y$ and $Rt + T = Y$ for an essential submodule T of Y . Let $((T + X)/X) \cap (L/X) = 0$ for a submodule L/X of M/X . Then $(L \cap T + X)/X = ((T + X)/X) \cap (L/X) = 0$ and $L \cap T + X = X$. Hence $L \cap T \leq X$ and since $X \cap Y = 0$, $L \cap T \cap Y \leq X \cap Y = 0$. Since $L \cap Y \cap T = L \cap T \cap Y = 0$ and $T \leq Y$, $L \cap Y = 0$. Since $X \leq L$ and

$M = X + Y$, by Modular Law, $L = L \cap M = L \cap (X + Y) = X + L \cap Y = X + 0 = X$. Hence $L/X = 0$ and $(T + X)/X \leq M/X$. Since $Rt + T = Y$, $R(t + X) + (T + X)/X = (Rt + X)/X + (T + X)/X = (Rt + T + X)/X = (Y + X)/X = M/X$. Since $t \in U$, $t + X \in U/X \leq \text{Rad}_g(M/X)$ and hence $R(t + X) \ll_g M/X$. Then by $R(t + X) + (T + X)/X = M/X$ and $(T + X)/X \leq M/X$, $(T + X)/X = M/X$ and then $X + T = M$. Since $X + T = M$ and $T \leq Y$, by Modular Law, $Y = Y \cap M = Y \cap (X + T) = X \cap Y + T = 0 + T = T$. Hence $Rt \ll_g Y$ and by Lemma 5, Y is a g -radical supplement of U in M . Since $M = X \oplus Y$ and $X \leq U$, by Modular Law, $U = U \cap M = U \cap (X \oplus Y) = X \oplus U \cap Y$. Hence $U \cap Y$ is a direct summand of U .

(4) $\Rightarrow$ (1) Let $U = X \oplus U \cap Y$ for a submodule X of U . Since Y is a g -radical supplement of U in M , $M = U + Y$ and $U \cap Y \ll_g Y$. Hence $M = U + Y = (X \oplus U \cap Y) + Y = X \oplus Y$. $\square$

Lemma 12. *Let V be a g -radical supplement of U in M . If U is a generalized maximal submodule of M , then $U \cap V$ is a unique generalized maximal submodule of V .*

Proof. Since U is a generalized maximal submodule of M and $V/(U \cap V) \simeq (V + U)/U = M/U$, $U \cap V$ is a generalized maximal submodule of V . Hence $\text{Rad}_g V \leq U \cap V$ and since $U \cap V \leq \text{Rad}_g V$, $\text{Rad}_g V = U \cap V$. Thus $U \cap V$ is a unique generalized maximal submodule of V . $\square$

Definition 2. Let M be an R -module. If every proper essential submodule of M is generalized small in M or M has no proper essential submodules, then M is called a generalized hollow module.

Clearly we see that every hollow module is generalized hollow.

Definition 3. Let M be an R -module. If M has a large proper essential submodule which contain all essential submodules of M or M has no proper essential submodules, then M is called a generalized local module.

Clearly we see that every local module is generalized local.

Proposition 3. *Let M be an R -module and $\text{Rad}_g M \neq M$. Then M is generalized hollow if and only if M is generalized local.*

Proof. ($\Rightarrow$) Let M be generalized hollow and let L be a proper essential submodule of M . Then $L \ll_g M$ and by Lemma 2, $L \leq \text{Rad}_g M$. Thus $\text{Rad}_g M$ is a proper essential submodule of M which contain all proper essential submodules of M .

($\Leftarrow$) Let M be a generalized local module, T be the largest proper essential submodule of M and L be a proper essential submodule of M . Let $L + S = M$ with $S \leq M$. If $S \neq M$, then $L + S \leq T \neq M$. Thus $S = M$ and $L \ll_g M$. $\square$

Definition 4. Let M be an R -module and $U, V \leq M$. If $M = U + V$ and $U \cap V \ll_g M$, then V is called a weak g -supplement of U in M . If every submodule of M has a weak g -supplement in M , then M is called a weakly g -supplemented module. (See [3]).

Clearly we can see that if M is a weakly g -supplemented module, then M is g -semilocal ($M/\text{Rad}_g M$ is semisimple, see [3]).

Proposition 4. Generalized hollow and generalized local modules are weakly g -supplemented, so are g -semilocal.

Proof. Clear from definitions. $\square$

Proposition 5. Let M be a g -radical supplemented module with $\text{Rad}_g M \ll_g M$. Then M is weakly g -supplemented.

Proof. Clear from definitions. $\square$

Example 1. Consider the $\mathbb{Z}$ -module $\mathbb{Q}$. Since $\text{Rad}_g \mathbb{Q} = \text{Rad} \mathbb{Q} = \mathbb{Q}$, $\mathbb{Z}\mathbb{Q}$ is g -radical supplemented. But, since $\mathbb{Z}\mathbb{Q}$ is not supplemented and every nonzero submodule of $\mathbb{Z}\mathbb{Q}$ is essential in $\mathbb{Z}\mathbb{Q}$, $\mathbb{Z}\mathbb{Q}$ is not g -supplemented.

Example 2. Consider the $\mathbb{Z}$ -module $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ for a prime p . It is easy to check that $\text{Rad}_g \mathbb{Z}_{p^2} \neq \mathbb{Z}_{p^2}$. By Lemma 4, $\text{Rad}_g (\mathbb{Q} \oplus \mathbb{Z}_{p^2}) = \text{Rad}_g \mathbb{Q} \oplus \text{Rad}_g \mathbb{Z}_{p^2} \neq \mathbb{Q} \oplus \mathbb{Z}_{p^2}$. Since $\mathbb{Q}$ and $\mathbb{Z}_{p^2}$ are g -radical supplemented, by Lemma 7, $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ is g -radical supplemented. But $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ is not g -supplemented.

REFERENCES

- [1] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules (Graduate Texts in Mathematics)*. New York: Springer, 1998.
- [2] B. Koşar, C. Nebiyev, and N. Sökmez, “ G -supplemented modules,” *Ukrainian Mathematical Journal*, vol. 67, no. 6, pp. 861–864, 2015, doi: [10.1007/s11253-015-1127-8](https://doi.org/10.1007/s11253-015-1127-8).
- [3] C. Nebiyev and H. H. Ökten, “Weakly g -supplemented modules,” *European Journal of Pure and Applied Mathematics*, vol. 10, no. 3, pp. 521–528, 2017.
- [4] N. Sökmez, B. Koşar, and C. Nebiyev, “Genelleştirilmiş küçük alt modüller,” in *XIII. Ulusal Matematik Sempozyumu*. Kayseri: Erciyes Üniversitesi, 2010.
- [5] R. Wisbauer, *Foundations of Module and Ring Theory*. Philadelphia: Gordon and Breach, 1991.
- [6] D. X. Zhou and X. R. Zhang, “Small-essential submodules and morita duality,” *Southeast Asian Bulletin of Mathematics*, vol. 35, pp. 1051–1062, 2011.

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ON THE QUALITATIVE BEHAVIOUR OF PARAMETER DEPENDENT SYSTEMS

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Abstract. In the first part of this work we present some basics about the connection of exponential dichotomy and Floquet-theory. Then we study some stability notions of linear differential equations based on the concept of the exponential and ordinary dichotomy. More precisely we shall show that the definition of the asymptotical (Lyapunov-) stability can be formulated by using the exponential dichotomy. In the main part of the paper we consider parameter dependent non-autonomous linear and autonomous nonlinear systems which have at a certain value of the parameter an asymptotically stable solution. Then we examine the question how much the parameter can be changed such that the solution remains asymptotically stable. Finally, we study the orbital stability of periodic solutions of autonomous systems applying the previous results.

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1. INTRODUCTION

In many fields of the applied mathematics it may often happen that when we would like to study a phenomenon when we don't know the exact mathematical model or the system of the differential equations which describe the given phenomenon. For example it may happen that there are some parameters in the mathematical model, which depend on the variables of the system or on environmental factors etc. In many cases the exact values of these parameters are not known because we can just measure or estimate them, or even although if we can calculate somehow the exact values of the parameters but we can not give their exact value numerically. In these cases we can study the qualitative behaviour of the model with the estimated values of the parameters. Therefore, it is a legitimate expectation that the qualitative behaviour of the model with the estimated parameters must be the same as the behaviour of the exact model. However, often we couldn't verify the correspondence of the qualitative properties of these two models. But we can get bounds and intervals for the parameters such that we can prove theoretically that if the parameters are within these

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bounds or intervals, then the behaviour of the unknown functions in the models do not change, which means that the estimated model reflects faithfully the exact model, and consequently, the studied phenomenon, too.

The objective of this paper is to study the Lyapunov stability of the solutions of system of differential equations in the form

$$\dot{x} = f(id, x, \mu), \quad (1.1)$$

where μ is a parameter. More precisely, we would like to know that if with a certain value of the parameter μ_0 a solution of system (1.1) is asymptotically stable, then under what conditions we can prove that the solution remains asymptotically stable, if the parameter changes are sufficiently small.

In this paper we are going to use the notion of the exponential dichotomy and two results of the theory of the exponential dichotomy: the relation between the exponential dichotomy and the stability of linear systems, and a theorem about the roughness of the exponential dichotomy.

It is well known that the qualitative properties of the solutions of systems of ordinary differential equations depend continuously on the parameters of the right hand side. In this paper we use the robustness of the exponential dichotomy in order to show that the stability properties of the system change continuously.

The paper is organized as follows: in Section 2 we present the basic notions and theorems with respect to the exponential dichotomy and the Floquet theory, which we will use in the next section, then in Section 3 we are going to study the stability of different types of solutions of system (1.1): firstly the stability of (the trivial solution) of linear systems, then the stability of the equilibrium point of autonomous (but not necessarily linear) systems, and finally the stability of periodic solutions of autonomous systems.

2. MATHEMATICAL BACKGROUND: THE USED FRAMEWORK AND TOOLS

2.1. Exponential dichotomy

In the theory of the exponential dichotomy we study systems of linear differential equations. There are a lot of papers where the authors study linear systems in general cases, for example with arbitrary finite dimensional phase space or infinite phase space (cf. [5]). In this paper we use simply the usual phase space $(\mathbb{R}^n, \|\cdot\|)$, where $\|\cdot\|$ denotes an arbitrary vector norm. Let us consider the linear system

$$\dot{x} = Ax, \quad (2.1)$$

where $A \in \mathcal{C}(J, \mathbb{R}^{n \times n})$ with a fixed dimension $n \in \mathbb{N}$ and interval $J := [\tau, +\infty)$, where $\tau \in \mathbb{R}$. Let us denote the fundamental matrix of system (2.1) by Φ , i.e. let Φ be a regular matrix solution of (2.1). Thus, the entire solution φ of system (2.1) with an initial condition $x(\tau) = \xi$ has the form

$$\varphi(t) = \Lambda(t, \tau)\xi := \Phi(t)\Phi^{-1}(\tau)\xi \quad (t \in J)$$

where $\Lambda(\cdot, \tau)$ denotes the Cauchy matrix of (2.1).

In this paper we will use the definition of the exponential dichotomy in the sense discussed in the book of Coppel (cf. [1]).

Definition 1 (cf. [1]). The equation (2.1) is said to possess an exponential dichotomy if there exist a projection P and positive constants $K_1, K_2, \alpha_1, \alpha_2$ such that

$$\|\Phi(t)P\Phi^{-1}(s)\| \leq K_1 e^{-\alpha_1(t-s)} \quad (t \geq s), \quad (2.2)$$

$$\|\Phi(t)(I - P)\Phi^{-1}(s)\| \leq K_2 e^{-\alpha_2(s-t)} \quad (s \geq t).$$

It is said to possess an ordinary dichotomy if the inequalities (2.2) hold with $\alpha_1 = \alpha_2 = 0$.

In the autonomous case, i. e. when A is a constant matrix, we can easily determine the existence of the exponential dichotomy by looking for the eigenvalues of the coefficient matrix.

Proposition 1 (cf. [1], [6]). Let $A \in \mathbb{R}^{n \times n}$ be a matrix and let us consider the autonomous system

$$\dot{x} = Ax. \quad (2.3)$$

It has an exponential dichotomy on $J := [0, +\infty)$ if and only if no eigenvalue of the constant matrix A has zero real part. It has an ordinary dichotomy, if and only if all eigenvalues of A with zero real part are semisimple (which means that these eigenvalues are simple roots of the minimal polynomial of A). In each case we can take the projection P as the spectral projection defined by

$$P := \frac{1}{2\pi i} \int_{\gamma} (zI - A) dz. \quad (2.4)$$

To prove the main results of this paper we are going to use the following theorems: the first states the roughness of the exponential dichotomy, the second gives the relation between the exponential dichotomy and the Lyapunov-stability of linear systems.

Theorem 1 (cf. [4]). Let us assume that the linear system with continuous coefficients

$$\dot{x} = Ax \quad (2.5)$$

possesses an exponential dichotomy on $J := [0, +\infty)$ with some constants $K_1, K_2, \alpha_1, \alpha_2 > 0$ and projection P . Let B a matrix valued function with continuous coefficients defined on the interval J . If for the function B the inequality

$$\limsup_{s \rightarrow +\infty} \|B(s)\| < \left(\frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2} \right)^{-1} \quad (2.6)$$

holds, then system

$$\dot{x} = (A + B)x. \quad (2.7)$$

possesses an exponential dichotomy on the interval J with projection Q , where Q is similar to P .

In Definition 1 we can assume that $K_1 = K_2 =: K$ and $\alpha_1 = \alpha_2 =: \alpha$, and if we use these constants, the upper bound at condition (2.6) is the following:

$$\limsup_{s \rightarrow +\infty} \|B(s)\| < \frac{\alpha}{2K}. \quad (2.8)$$

In Section 3 the other key theorem will be the following result, which connects the notion of the exponential dichotomy and the stability of linear systems.

Theorem 2 (cf. [2]). *The linear system (2.1) is*

- (1) *asymptotically stable if and only if it admits an exponential dichotomy with projection $P = I$, furthermore*
- (2) *stable if and only if it admits an ordinary dichotomy with projection $P = I$.*

2.2. Some basics from the Floquet-theory

In the last part of Section 2 we are going to study the stability of periodic orbits (of parameter dependent systems) using some tools from the Floquet theory, therefore in this subsection we summarize the necessary results about the stability of periodic orbits and some relevant fields.

Let $A \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{n \times n})$ be a matrix-valued function, for which $A(t + T) = A(t)$ is fulfilled for each $t \in \mathbb{R}$ with some period $T > 0$. Let us consider the linear system with periodic coefficient matrix

$$\dot{x} = Ax. \quad (2.9)$$

Let us denote by C the principal or monodromy matrix of (2.9):

$$C := \Lambda(T, 0).$$

Following from the well-known Floquet Theorem, we can write the Cauchy matrix of system (2.9) in a special form.

Theorem 3 (Floquet). *The fundamental matrix of (2.9) can be written in the form*

$$\Lambda(t, 0) = P(t)e^{Bt} \quad (t \geq 0),$$

where $P \in \mathcal{C}^1([0, +\infty), \mathbb{R}^{n \times n})$ is a regular T -periodic matrix, $P(0) = I$ and $B \in \mathbb{R}^{n \times n}$ is a constant matrix.

The eigenvalues of these two matrices C and B play an important role at the investigation of the stability of system (2.9). The eigenvalues of the matrix C are the characteristic multipliers of the system (2.9), and the eigenvalues of the matrix B are the characteristic exponents of the system (2.9). Following from the identity

$$e^{BT} = I \cdot e^{BT} = P(T) \cdot e^{BT} = \Lambda(T, 0) = C,$$

if ν is a characteristic exponent, then $\lambda = e^{\nu T}$ is a characteristic multiplier and vice versa. Hence, by $T > 0$

$$|\lambda| < 1 \iff \Re(\nu) < 0. \quad (2.10)$$

Let us assume that 1 is a simple characteristic multiplier of system (2.9) and there is no other characteristic multiplier on the unit circle. Then, the fundamental matrix of the system has the following form

$$\Phi(t) = P(t) \begin{bmatrix} 1 & 0 \\ 0 & \exp(B_1 t) \end{bmatrix} \quad (t \in \mathbb{R}),$$

where P is regular, continuously differentiable, periodic matrix valued function with period T , and $B_1 \in \mathbb{R}^{(n-1) \times (n-1)}$ is a hyperbolic matrix.

Let $\Omega \subset \mathbb{R}^n$ be an open, connected set, $f \in \mathcal{C}^1(\Omega)$ and let us consider the system

$$\dot{x} = f(x). \quad (2.11)$$

Let us assume that $\varphi : \mathbb{R} \rightarrow \mathbb{R}^n$ is a nontrivial, T -periodic solution of the system. We know that the variational system

$$\dot{x} = (\partial f \circ \varphi)x \quad (2.12)$$

has a T -periodic solution if and only if 1 is a characteristic multiplier of the system. The theorem of Andronov and Witt gives a sufficient condition for the asymptotical stability of the periodic solution φ .

Theorem 4 (Andronov-Witt, see [3]). *If 1 is a simple characteristic multiplier of the system (2.12) and the remaining $n - 1$ characteristic multipliers are in modulus less than 1, then the periodic solution φ is orbitally asymptotically stable.*

The Andronov-Witt Theorem has a version which states the instability of the periodic solutions.

Theorem 5 (cf. [3]). *If at least one of the characteristic multipliers of the variational system is in modulus greater than 1, then the periodic solution φ is orbitally unstable.*

Let us assume that φ is orbitally asymptotically stable, 1 is a simple characteristic multiplier and there is no other characteristic multiplier on the complex unit circle, i.e. if $M \subset \mathbb{C}$ denotes the set of the characteristic multipliers, then

$$M = \{1\} \cup \{\lambda \in \mathbb{C} : \lambda \text{ is a characteristic multiplier, } |\lambda| \neq 1\}.$$

If there would be a characteristic multiplier $\lambda \in M$ such that $|\lambda| > 1$, by Theorem 4 it would contradict the fact that φ is asymptotically stable. Hence, because we assumed that there are no characteristic multiplier on the unit circle except 1, we proved that the remaining $n - 1$ characteristic multipliers are in modulus less than 1. This means that in this case the conversion of the Andronov-Witt Theorem 4 is also true.

Let us see how the variational system looks like, in the case when the periodic solution φ is orbitally asymptotically stable, and hence 1 is a characteristic multiplier, and there are no other characteristic multipliers on the complex unit circle. Then, by Floquet Theorem 3, with a suitable invertible matrix $S \in \mathbb{R}^{n \times n}$ we have

$$S^{-1}CS = \begin{bmatrix} 1 & 0 \\ 0 & C_1 \end{bmatrix},$$

where $C_1 \in \mathbb{R}^{(n-1) \times (n-1)}$ is such that $\exp(B_1 T) = C_1$ is fulfilled with some matrix B_1 . Let us introduce one more notation, namely

$$B := \begin{bmatrix} 0 & 0 \\ 0 & B_1 \end{bmatrix}. \quad (2.13)$$

With the above matrix B the equality

$$\exp(Bt) = \begin{bmatrix} 1 & 0 \\ 0 & \exp(B_1 t) \end{bmatrix}$$

is satisfied, and hence

$$S^{-1}CS = e^{BT}$$

is also true.

Let us study a little bit more the second assumption of the Andronov-Witt Theorem (the remaining characteristic multipliers are in modulus less than 1). By (2.10) this assumption means that the eigenvalues of the matrix B_1 have negative real part, or in other words, the $n - 1$ dimensional autonomous linear system

$$\dot{y} = B_1 y \quad (2.14)$$

is asymptotically stable, i.e. this system possesses an exponential dichotomy with projection $P = I \in \mathbb{R}^{(n-1) \times (n-1)}$.

In conclusion, if φ is a periodic solution of system (2.11), 1 is a simple characteristic multiplier and the other characteristic multipliers are in modulus less or greater than 1, then φ is asymptotically stable if and only if system (2.14) possesses an exponential dichotomy with projection $P = I$.

3. MAIN RESULTS

3.1. Nonautonomous linear systems

In the first application we are going to consider the stability of the system

$$\dot{x} = A(\cdot; \mu)x, \quad (3.1)$$

where $A \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^p, \mathbb{R}^{n \times n})$, $\mu \in \mathbb{R}^p$ represents the parameter of the system, $p \in \mathbb{N}$. We are going to investigate the following problem: if for a certain value of the parameter μ_0 system (3.1) is asymptotically stable, then how much we can change the value of the parameter such that the stability of the system does not change. For this we are going to use Theorem 1 on exponential dichotomy.

Theorem 6. *Let us assume that the coefficient matrix A of (3.1) is Lipschitz continuous w.r.t. the second variable on the positive half line, i.e. there exists a constant $L > 0$ such that for each μ_1 and μ_2*

$$\|A(t, \mu_1) - A(t, \mu_2)\| \leq L\|\mu_1 - \mu_2\|$$

is satisfied with arbitrary $t \geq 0$. We assume that for $\mu_0 \in \mathbb{R}^p$ system (3.1) is asymptotically stable or equivalently, it has an exponential dichotomy on $J = [0, +\infty)$ with constants $K, \alpha > 0$ and projection $P = I$. Then for every parameter $\lambda \in \mathbb{R}^p$ for which

$$\|\mu_0 - \lambda\| \leq \frac{1}{L} \cdot \frac{\alpha}{2K} \quad (3.2)$$

holds, system (3.1) is asymptotically stable with λ .

Proof. Let us assume that with the parameter value μ_0 the system

$$\dot{x} = A_0 x$$

is asymptotically stable, where $A_0 := A(\cdot, \mu_0)$. Then we know from Theorem 2 that this system possesses an exponential dichotomy with projection $P = I$ and with constants $K, \alpha > 0$. Let λ be a parameter value such that the condition (3.2) of Theorem 6 is fulfilled, i. e.

$$\|\mu_0 - \lambda\| \leq \frac{1}{L} \cdot \frac{\alpha}{2K}$$

holds. Then let us write the matrix $A(t, \lambda)$ in the following form:

$$A(t, \lambda) = A(t, \mu_0) + (A(t, \lambda) - A(t, \mu_0)),$$

and define the function B as

$$B(t) := A(t, \lambda) - A(t, \mu_0) \quad (t \geq 0).$$

Hence, by using the perturbation Theorem 1, the system

$$\dot{x}(t) = (A(t, \mu_0) + B(t))x(t) \quad (t \geq 0)$$

has an exponential dichotomy, too, with projection Q , because for each $t \geq 0$ we know from the assumptions that

$$\|B(t)\| = \|A(t, \lambda) - A(t, \mu_0)\| \leq L\|\lambda - \mu_0\| \leq \frac{\alpha}{2K}.$$

So we proved that the system

$$\dot{x}(t) = (A(t, \mu_0) + A(t, \lambda) - A(t, \mu_0))x(t) = A(t, \lambda)x(t) \quad (t \geq 0) \quad (3.3)$$

admits an exponential dichotomy with projection Q . By the perturbation Theorem 1 we know that the two projections $P = I$ and Q are similar, which means that $Q = c \cdot I$, where $c \in \mathbb{R}$ is a constant. Then we get that the perturbed system (3.3) has an exponential dichotomy, too, with projection $\tilde{Q} = I$, and as a consequence we get that system (3.3) is asymptotically stable. $\square$

Assume now that system (3.1) is autonomous, i.e. for the coefficient function $A \in \mathcal{C}^1(\mathbb{R}^p, \mathbb{R}^{n \times n})$ holds and with a certain value μ_0 of the parameter the autonomous system

$$\dot{x} = A(\mu_0)x$$

is asymptotically stable. Then we know that all eigenvalues of the coefficient matrix $A(\mu_0) \in \mathbb{R}^{n \times n}$ have negative real part or equivalently (cf. Proposition 1 or Theorem 2) the system possesses an exponential dichotomy with projection $P = I$ (and with constants $K, \alpha > 0$). Applying Theorem 6, we obtain that system

$$\dot{x} = A(\lambda)x$$

is asymptotically stable, too, if condition (3.2) is fulfilled for λ . It follows that with these λ values of the parameters all eigenvalues of matrix $A(\lambda)$ have negative real parts. It means that this way we get a condition under which the eigenvalues of a matrix (which depends on a parameter) remain in the left half plane. (It is easy to show that the real part of the eigenvalues of $A(\lambda)$ remain negative with small changes of the parameter λ).

3.2. The stability of an equilibrium point of a system of nonlinear autonomous equations

Let us consider the autonomous system

$$\dot{x} = f(x; \mu) \tag{3.4}$$

where $f \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^p, \mathbb{R}^n)$ and $\mu \in \mathbb{R}^p$ is a parameter. Let us denote by $x_0(\mu)$ the equilibrium point of the system. Similarly to the previous subsection, we are going to study the following problem: if the equilibrium point is asymptotically stable then how much we can change the value of the parameter such that the equilibrium point remains asymptotically stable.

Theorem 7. *Let us assume that $x_0(\mu)$ is an equilibrium of system (3.4) for each $\mu \in \mathbb{R}^p$, and there exists a $\mu_0 \in \mathbb{R}^p$ such that $x_0(\mu_0)$ is an asymptotically stable equilibrium point. If the function F*

$$F(\mu) := \partial_1 f(x_0(\mu); \mu) \quad (\mu \in \mathbb{R}^p)$$

is Lipschitz continuous, i.e. there exists a constant $L > 0$ such that for all parameters $\mu_1, \mu_2 \in \mathbb{R}^p$

$$\|F(\mu_1) - F(\mu_2)\| \leq L\|\mu_1 - \mu_2\|$$

is satisfied, then there is a constant $R > 0$ such that for every

$$\lambda \in B_R(\mu_0) := \{\mu \in \mathbb{R}^p : \|\mu - \mu_0\| < R\},$$

$x_0(\lambda) \in \mathbb{R}^n$ is an asymptotically stable equilibrium point of (3.4).

Proof. Let us assume that with the parameter value μ_0 the equilibrium point $x_0(\mu_0)$ is asymptotically stable. This is equivalent to the fact that the linear autonomous system

$$\dot{y} = \partial_1 f(x_0(\mu_0); \mu_0) y$$

is asymptotically stable, where $\partial_1 f(\cdot, \mu)$ denotes the Jacobian matrix of the function f w.r.t. the first variable. By Theorem 2 this is equivalent to the fact that the linear system possesses an exponential dichotomy with projection $P = I$. Hence we can apply the previous theorem, which says that there exists a constant $R \in \mathbb{R}$ such that if $\|\lambda - \mu_0\| < R$, then the system

$$\dot{y} = \partial_1 f(x_0(\lambda); \lambda) y$$

is asymptotically stable, which means that the equilibrium point $x_0(\lambda)$ is also asymptotically stable. $\square$

Now we show an example: how the previous theorem works at a concrete system of differential equations.

Example 1. Let us consider the following system of differential equations of Lotka-Volterra type:

$$\begin{cases} \dot{x} = x(a - x - y), \\ \dot{y} = y(b - 2x - y), \end{cases} \quad (3.5)$$

where $(a, b) \in \mathbb{R}^2$ is a parameter. (Assume that we have two populations, $x(t)$ and $y(t)$ denote the number of the individuals of the populations, further a and b are unknown rates, which depend on a lot of things, for example on environmental factors.) Let us study the stability of the positive equilibrium point

$$(x_0, y_0) := (b - a, 2a - b).$$

Let us denote by f the function of the right hand side of the system:

$$f(x, y) := \begin{bmatrix} x(a - x - y) \\ y(b - 2x - y) \end{bmatrix} \quad (x, y \in \mathbb{R}). \quad (3.6)$$

The Jacobian of f at (x_0, y_0) is

$$\partial f(x_0, y_0) = \begin{bmatrix} a - b & a - b \\ 2b - 4a & b - 2a \end{bmatrix},$$

hence the function F in Theorem 7 has the following form:

$$F(a, b) := \begin{bmatrix} a - b & a - b \\ 2b - 4a & b - 2a \end{bmatrix} \quad (a, b \in \mathbb{R}).$$

We can easily prove that this function is Lipschitz continuous: for each $(a, b), (a_1, b_1) \in \mathbb{R}^2$ we have

$$\|F(a, b) - F(a_1, b_1)\|_1 = \left\| \begin{bmatrix} a - a_1 - (b - b_1) & a - a_1 - (b - b_1) \\ 2(b - b_1) - 4(a - a_1) & b - b_1 - 2(a - a_1) \end{bmatrix} \right\|_1$$

$$\begin{aligned}
&= 2|b - b_1 - 2(a - a_1)| + |b - b_1 - (a - a_1)| \\
&\leq 2(|b - b_1| + 2|a - a_1|) + |b - b_1| + |a - a_1| \\
&= 3|b - b_1| + 5|a - a_1| \\
&\leq 5\|(a, b) - (a_1, b_1)\|_1.
\end{aligned}$$

Let us assume that $a = 1/2$ and $b = 1/3$ (for example let us imagine that we have two concrete populations and we get these estimations for the parameters from an experiment.) In this case

$$\partial f(x_0, y_0) = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} \\ -\frac{4}{3} & -\frac{2}{3} \end{bmatrix},$$

and the eigenvalues are

$$\lambda_{1,2} = -\frac{1}{4} \pm \frac{\sqrt{7}}{12}i,$$

which means that the equilibrium point is asymptotically stable. By Theorem 2 the linear system

$$\dot{y} = \partial f(x_0, y_0)y$$

possesses an exponential dichotomy with $P = I$. It is easy to calculate that the constants of the definition of the exponential dichotomy are $K = 2$ and $\alpha = 1/4$ at this system. Hence the previous Theorem 7 says that the equilibrium point is asymptotically stable with some other parameters $(\tilde{a}, \tilde{b}) \in \mathbb{R}^2$, if for the parameters $(\tilde{a}, \tilde{b})$

$$\|(1/2, 1/3) - (\tilde{a}, \tilde{b})\|_1 \leq \frac{1}{L} \cdot \left(\frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2} \right)^{-1} = 0.0125.$$

3.3. The stability of periodic solutions

In the last part of this section let us consider the system (3.4) again:

$$\dot{x} = f(x, \mu) \tag{3.7}$$

where $\mu \in \mathbb{R}^p$ and assume that φ_μ is a periodic solution of (3.7). Then we know that 1 is a characteristic multiplier of the variational system (2.12), furthermore we assume that there are no other characteristic multipliers on the unit circle. We have seen in Section 2.2 that in this case φ_μ is asymptotically stable if and only if the linear system

$$\dot{y} = B_{1_\mu} y$$

possesses an exponential dichotomy with projection $P = I$, where the matrix $B_{1_\mu} \in \mathbb{R}^{(n-1) \times (n-1)}$ comes from (2.13) and is related to the monodromy matrix of the variational system of φ_μ . With this observation we can apply the perturbation Theorem 1 referring to the stability of the exponential dichotomy, to study the stability of periodic orbits, similarly to the previous two subsections.

Theorem 8. *Let us denote by φ_μ a periodic solution of system (3.4) (with parameter value μ in a neighbourhood of some μ_0). Let us assume that with a certain value of the parameter μ_0 , φ_{μ_0} is orbitally asymptotically stable, and $\lambda = 1$ is a simple characteristic multiplier, and except for this, there is no other characteristic multiplier on the unit circle, consequently, the system*

$$\dot{y} = B_{1_{\mu_0}} y$$

possesses an exponential dichotomy with some constants $K_1, K_2, \alpha_1, \alpha_2$ and projection $P = I$. Then, for each parameter value $\lambda \in \mathbb{R}^p$, for which

$$\|B_{1_{\mu_0}} - B_{1_\lambda}\| < \left(\frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2} \right)^{-1} \quad (3.8)$$

holds, the periodic solution φ_λ is orbitally asymptotically stable.

Proof. Since φ_λ is a periodic solution (if λ is in a neighbourhood of μ_0), we know that 1 is a characteristic multiplier, furthermore we assumed that the other characteristic multipliers are in modulus strictly less or greater than 1. Hence, if condition (3.8) holds, then the following system

$$\dot{y} = B_{1_\lambda} y$$

has an exponential dichotomy with projection $P = I$, which means that the matrix B_{1_λ} has eigenvalues with negative real parts. Thus we know that with the parameter value λ the other characteristic multipliers are in modulus less than 1. Hence, by Theorem 4 the periodic solution φ_λ is orbitally asymptotically stable. $\square$

REFERENCES

- [1] W. A. Coppel, *Dichotomies in stability theory*. Berlin Heidelberg New York: Springer Verlag, 1978.
- [2] S. Császár and S. Kovács, “Exponential dichotomy and the stability of linear systems.” *Annales Univ. Sci. Budapest., Sec. Comp.*, vol. 47, pp. 211–226, 2018.
- [3] M. Farkas, *Periodic Motions*. New York: Springer-Verlag, 1994.
- [4] N. Ju and S. Wiggins, “On Roughness of Exponential Dichotomy.” *J. Appl. Math. Anal. Appl.*, vol. 262, pp. 39–49, 2001, doi: [doi:10.1006/jmaa.2001.7496](https://doi.org/10.1006/jmaa.2001.7496).
- [5] N. Lupa and M. Megan, “Exponential dichotomies of evolution operators in Banach spaces.” *Monats. Math.*, vol. 174, pp. 265–284, 2014, doi: [DOI 10.1007/s00605-013-0517-y](https://doi.org/10.1007/s00605-013-0517-y).
- [6] C. Pötzsche, “Bifurcations in nonautonomous dynamical systems: Results and tools in discrete time.” *Proceedings of the International Workshop Future Directions in Difference Equations*, pp. 163–212, 2011, June 13–17, Vigo, Spain, doi: [978-84-8158-541-4/pbk](https://doi.org/10.1007/978-84-8158-541-4/pbk).

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SOME PROPERTIES OF GREEN'S FUNCTIONS OF SHILOV-TYPE PARABOLIC SYSTEMS

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Abstract. For Shilov-type parabolic systems with nonnegative genus and coefficients of bounded smoothness, the properties of differentiability of Green's function with respect to spatial variables are studied.

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1. INTRODUCTION

Developing the idea advanced by Ya.I. Zhitomirskii in [10] concerning the description of the systems of partial differential equations parabolically steady to a change in coefficients, the following wide class of parabolic systems with variable coefficients was defined in [7]:

$$\partial_t u(t; x) = \{P_0(t; i \partial_x) + P_1(t, x; i \partial_x)\} u(t; x), \quad (t; x) \in \Pi_{(0; T]}. \quad (1.1)$$

Here, $u := \text{col}(u_1, \dots, u_m)$, $\Pi_{(0; T]} := \{(t; x) : t \in (0; T], x \in \mathbb{R}^n\}$, and $P_0(t; i \partial_x)$ and $P_1(t, x; i \partial_x)$ are matrix differentiable expressions of the orders, respectively, p and p_1 , $p > p_1 \geq 0$, with coefficients depending on the time variable t . In this case, the coefficients of the expression P_1 can depend also on the spatial variable x . It is also assumed that the corresponding system

$$\partial_t u(t; x) = P_0(t; i \partial_x) u(t; x), \quad (t; x) \in \Pi_{(0; T]}, \quad (1.2)$$

is uniformly parabolic by Shilov with parabolicity index h , $0 < h \leq p$, nonnegative genus μ , and reduced order p_0 [2]. In other words, there exist constants $\delta_0 > 0$ and $\delta_1 \geq 0$ such that, for all $x \in \mathbb{R}^n$ and $t \in [0; T]$, the following inequality holds:

$$\Lambda(t; x) := \max_{j \in \mathbb{N}_m} \text{Re} \lambda_j(t; x) \leq -\delta_0 \|x\|^h + \delta_1.$$

Here, λ_j are roots of the equation $\det(P_0(t; \zeta) - \lambda E) = 0$, $\zeta \in \mathbb{C}^n$, E is the identity matrix of the order m , $\|\cdot\|$ is the Euclid norm in $\mathbb{R}^n$, and $\mathbb{N}_m := \{1, 2, \dots, m\}$.

We recall also that the reduced order p_0 of a system is called the exact power order of growth of a function $\Lambda(\cdot; \cdot)$ in the complex space $\mathbb{C}^n$ (for a parabolic system,

we have always $p_0 > 1$). In addition, the genus of a system parabolic by Shilov is called the largest index μ such that the function $\Lambda(\cdot; \cdot)$ in the domain $G^\mu(K) := [0; T] \times \{x + iy \in \mathbb{C}^n : \|y\| \leq K(1 + \|x\|)^\mu\}$ with some $K > 0$ satisfies the estimate

$$\Lambda(t; x + iy) \leq -\delta_* \|x\|^h + \delta_1, \delta_* > 0, \delta_1 \geq 0.$$

It is known that $1 - (p_0 - h) \leq \mu \leq 1$.

The following condition must be satisfied for (1.1):

$$(A) : 0 \leq p_1 < h - n(1 - h\mu/p_0) - (m - 1)(p - h).$$

It is obvious that the class of systems (1.1) envelops completely the class of Shilov-type parabolic systems with nonnegative genus. In this connection, it was proposed in [10] to call such systems Shilov-type parabolic systems with nonnegative genus and variable coefficients. To a certain extent, these systems solve the problem of proper extension of the notion of parabolicity by Shilov to the linear systems of equations with variable coefficients, since the Shilov class of systems is not parabolically stable to a change in its coefficients [4].

Let us consider systems (1.1) under the condition that their coefficients are the complex-valued functions that are continuous in the variable t , infinitely differentiable with respect to the variable x , and bounded together with their derivatives in a ball $\Pi_{[0; T]}$. For these systems, Green's function $Z(t, x; \tau, \xi)$, $0 \leq \tau < t \leq T$, $\{x, \xi\} \subset \mathbb{R}^n$, was constructed, and its main properties such as the smoothness and the behavior in vicinities of infinitely remote spatial points were studied [7]. On the basis of these results, the theory of the Cauchy problem was developed in [5, 8, 9] for such systems in spaces of the type S by Gelfand and Shilov [3]. In particular, the proper solvability of the Cauchy problem with generalized initial data like the Gevrey ultradistributions was established, the form of a representation of classical solutions with generalized limiting values on the input hyperplane was obtained, their qualitative properties were studied, and the sets of generalized input functions, for which the corresponding solutions are elements of the Schwartz space S or some space of the type S , were described.

In order to get similar results under weaker conditions imposed on the coefficients of system (1.1), it is necessary firstly to clarify the properties of corresponding Green's function Z . In the present work, we will study properties of the function $Z(t, x; \tau, \xi)$ with respect to spatial variables x and ξ under the minimum conditions of smoothness imposed on the coefficients of this system.

2. AUXILIARY INFORMATION

Let the differential expressions P_0 and P_1 of system (1.1) have the structure

$$P_0(t; i\partial_x) = \sum_{|k|_+ \leq p} A_{0,k}(t) \partial_x^k, \quad P_1(t, x; i\partial_x) = \sum_{|k|_+ \leq p_1} A_{1,k}(t; x) \partial_x^k,$$

where $A_{0,k}(t) := i^{|k|+} \left(a_{0,k}^{lj}(t) \right)_{l,j=1}^m$, $A_{1,k}(t; x) := i^{|k|+} \left(a_{1,k}^{lj}(t; x) \right)_{l,j=1}^m$ are matrix coefficients, i is the imaginary unity, and $|k|_+ := k_1 + \dots + k_n$, $k \in \mathbb{Z}_+^n$.

By $G(t, \tau; \cdot)$, $0 \leq \tau < t \leq T$, we denote Green's function of system (1.2). It is known that $G(t, \tau; \cdot) = F[\Theta_\tau^t(\xi)](t, \tau; \cdot)$, where $F[\cdot]$ is the Fourier transformation operator, and $\Theta_\tau^t(\cdot)$ is a matriciant of the corresponding Fourier duality of the system. The following proposition is proper [5, 6]:

$$\forall T > 0 \exists \delta > 0 \forall k \in \mathbb{Z}_+^n \exists c > 0 \forall t \in (\tau; T] \forall \tau \in [0; T] \forall \{x, \xi\} \subset \mathbb{R}^n :$$

$$|\partial_x^k G(t, \tau; x - \xi)| \leq c(t - \tau)^{-\frac{n+|k|_++\gamma}{h}} e^{-\delta \left(\frac{\|x-\xi\|}{(t-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}}}, \quad (2.1)$$

where $\gamma := (m-1)(p-h)$ and $\alpha := \mu/p_0$, $|(a_{lj})_{l,j=1}^m| := \max_{\{l,j\} \subset \mathbb{N}_m} |a_{lj}|$.

Here, we consider systems (1.1), which satisfy, in addition to condition (A), the following condition:

- (B): the coefficients $a_{0,k}^{lj}(t)$, $a_{1,k}^{lj}(t; x)$ are continuous in the variable t uniformly with respect to x , differentiable with respect to the variable x up to the order α_* inclusively, and bounded together with their derivatives by complex-valued functions in a ball $\Pi_{[0;T]}$.

In [7], Green's function of system (1.1) was constructed in the form

$$Z(t, x; \tau, \xi) = G(t, \tau; x - \xi) + W(t, x; \tau, \xi), \quad (t, x; \tau, \xi) \in \Pi_T^2, \quad (2.2)$$

where $\Pi_T^2 := \{(t, x; \tau, \xi) | 0 \leq \tau < t \leq T, \{x, \xi\} \subset \mathbb{R}^n\}$,

$$W(t, x; \tau, \xi) := \int_{\tau}^t d\beta \int_{\mathbb{R}^n} G(t, \beta; x - y) \Phi(\beta, y; \tau, \xi) dy, \quad (2.3)$$

and

$$\Phi(t, x; \tau, \xi) = \sum_{l=1}^{\infty} K_l(t, x; \tau, \xi), \quad (2.4)$$

where

$$K_1(t, x; \tau, \xi) := P_1(t, x; i \partial_x) G(\tau, \xi; x - \xi),$$

$$K_l(t, x; \tau, \xi) := \int_{\tau}^t d\beta \int_{\mathbb{R}^n} K_1(t, x; \beta, y) K_{l-1}(\beta, y; \tau, \xi) dy, \quad l > 1. \quad (2.5)$$

In this case, it was established that condition (A) and the boundedness of the coefficients of system (1.1) ensure the absolute uniform convergence of the functional series (2.4) for all $\{x, \xi\} \subset \mathbb{R}^n$, $t \in (\tau; T]$, and $\tau \in [0, T)$. Moreover, its sum Φ and the iterated kernels K_l satisfy the estimates

$$|\Phi(t, x; \tau, \xi)| \leq c_1(t - \tau)^{\alpha_0 - (1 + \alpha n)} e^{-\delta_1 \left(\frac{\|x-\xi\|}{(t-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}}}, \quad (2.6)$$

$$|K_l(t, x; \tau, \xi)| \leq c_0^l \left(\prod_{j=1}^{l-1} c_{(j\varepsilon)} B(\alpha_0, j\alpha_0) \right) \times (t - \tau)^{l\alpha_0 - (1 + \alpha n)} e^{-\delta(1 - (l-1)\varepsilon) \left(\frac{\|x - \xi\|}{(t - \tau)^\alpha} \right)^{\frac{1}{1-\alpha}}}, \quad \varepsilon \in (0; 1), \quad (2.7)$$

with the estimating constants independent of t, τ, x , and ξ . Here $\alpha_0 := 1 + \alpha n - (n + p_1 + \gamma)/h > 0$ and $B(\cdot, \cdot)$ is the Euler beta-function.

We note that estimates (2.1) and (2.6) for $\{x, \xi\} \subset \mathbb{R}^n$ and $0 \leq \tau < t \leq T$ guarantee the absolute convergence of the integral, by which the potential W is determined. Thus, the matrix function $Z(t, x; \tau, \xi)$ is properly determined by formula (2.2) on the whole set Π_T^2 .

Completing this item, we present the following estimates from [1], which will be of importance in what follows:

$$e^{-\delta \left\{ \left(\frac{\|x - y\|}{(t - \beta)^\alpha} \right)^{\frac{1}{1-\alpha}} + \left(\frac{\|y - \xi\|}{(\beta - \tau)^\alpha} \right)^{\frac{1}{1-\alpha}} \right\}} \leq e^{-\delta \left(\frac{\|x - \xi\|}{(t - \tau)^\alpha} \right)^{\frac{1}{1-\alpha}}}; \quad (2.8)$$

$$\begin{aligned} & \int_{\mathbb{R}^n} e^{-\delta \left\{ \left(\frac{\|x - y\|}{(t - \beta)^\alpha} \right)^{\frac{1}{1-\alpha}} + \left(\frac{\|y - \xi\|}{(\beta - \tau)^\alpha} \right)^{\frac{1}{1-\alpha}} \right\}} \frac{dy}{((t - \beta)(\beta - \tau))^{\alpha n}} \\ & \leq \frac{c_\varepsilon e^{-\delta(1 - \varepsilon) \left(\frac{\|x - \xi\|}{(t - \tau)^\alpha} \right)^{\frac{1}{1-\alpha}}}}{(t - \tau)^{\alpha n}}, \quad \delta > 0 \end{aligned} \quad (2.9)$$

(here, $\{x, y, \xi\} \subset \mathbb{R}^n$, $\beta \in (\tau; t)$, $0 \leq \tau < t \leq T$, $\varepsilon \in (0; 1)$, and $\delta > 0$, and the quantity c_ε depends only on ε).

3. PROPERTIES OF GREEN'S FUNCTION

First, we estimate the derivatives of the iterated kernels K_l .

According to representation (2.5), the smoothness of the kernel $K_1(t, x; \tau, \xi)$ in the spatial variables x and ξ is determined, respectively, by the smoothness of the coefficients of system (1.1) and the function $G(t, \tau; x - \xi)$. Therefore, there exist the derivatives $\partial_\xi^r \partial_x^q K_1$ for $\{r, q\} \subset \mathbb{Z}_+^n$, $|q|_+ \leq \alpha_*$, and the following equality holds:

$$\partial_\xi^r \partial_x^q K_1(t, x; \tau, \xi) = \sum_{|k|_+ \leq p_1} \sum_{l=0}^q C_q^l \left(\partial_x^l A_{1,k}(t; x) \right) \left(\partial_{(x - \xi)}^{k+r+q-l} G(t, \tau; x - \xi) \right),$$

where C_q^l is a binomial coefficient. From whence, with regard for condition (B) and estimate (2.1) for $\{r, q\} \subset \mathbb{Z}_+^n$, $|q|_+ \leq \alpha_*$, $(t, x; \tau, \xi) \in \Pi_T^2$, we get

$$|\partial_\xi^r \partial_x^q K_1(t, x; \tau, \xi)| \leq c_{r,q} (t - \tau)^{-\frac{n + p_1 + \gamma + |r + q|_+}{h}} e^{-\delta \left(\frac{\|x - \xi\|}{(t - \tau)^\alpha} \right)^{\frac{1}{1-\alpha}}} \quad (3.1)$$

(here, the estimating constants are independent of t, τ, x , and ξ).

For $l > 1$, we will use the representation

$$\begin{aligned} K_l(t, x; \tau, \xi) &= \int_{\tau}^{t_1} d\beta \int_{\mathbb{R}^n} K_1(t, x; \beta, \eta + \xi) K_{l-1}(\beta, \eta + \xi; \tau, \xi) d\eta \\ &\quad + \int_{t_1}^t d\beta \int_{\mathbb{R}^n} K_1(t, x; \beta, x - z) K_{l-1}(\beta, x - z; \tau, \xi) dz, \quad t_1 := \frac{t + \tau}{2}. \end{aligned}$$

According to it,

$$\begin{aligned} \partial_{\xi}^r \partial_x^q K_l(t, x; \tau, \xi) &= \sum_{|r|_+ \leq |r|_+} C_r^{r_1} \int_{\tau}^{t_1} d\beta \int_{\mathbb{R}^n} \left(\partial_{\xi}^{r_1} \partial_x^q K_1(t, x; \beta, \eta + \xi) \right) \\ &\quad \times \left(\partial_{\xi}^{r-r_1} K_{l-1}(\beta, \eta + \xi; \tau, \xi) \right) d\eta + \sum_{|q|_+ \leq |q|_+} C_q^{q_1} \int_{t_1}^t d\beta \int_{\mathbb{R}^n} \left(\partial_x^{q_1} K_1(t, x; \beta, x - z) \right) \\ &\quad \times \left(\partial_{\xi}^r \partial_x^{q-q_1} K_{l-1}(\beta, x - z; \tau, \xi) \right) dz, \quad |q|_+ \leq \alpha_*, (t, x; \tau, \xi) \in \Pi_T^2. \end{aligned} \quad (3.2)$$

Hence, the estimation of $|\partial_{\xi}^r \partial_x^q K_l(t, x; \tau, \xi)|$ is reduced to that of the expressions

$$\begin{aligned} &|\partial_{\xi}^r \partial_x^q K_1(t, x; \tau, \eta + \xi)|, \quad |\partial_x^q K_1(t, x; \tau, x - z)|, \\ &|\partial_{\xi}^r K_{l-1}(t, \eta + \xi; \tau, \xi)|, \quad |\partial_{\xi}^r \partial_x^q K_{l-1}(t, x - z; \tau, \xi)|. \end{aligned}$$

In view of the boundedness of $\partial_x^q a_{1,k}^{lj}(t; x)$, $|q|_+ \leq \alpha_*$, and estimate (2.1), for all $\{q, r\} \in \mathbb{Z}_+^n$, $|q|_+ \leq \alpha_*$, $\{x, \eta, \xi\} \in \mathbb{R}^n$, $t \in (\tau; T]$, and $\tau \in [0; T)$, we have

$$\begin{aligned} |\partial_{\xi}^r \partial_x^q K_1(t, x; \tau, \eta + \xi)| &\leq m \sum_{|k|_+ \leq p_1} \sum_{|q|_+ \leq |q|_+} C_q^{q_1} |\partial_x^{q_1} A_{1,k}(t; x)| \\ &\quad \times \left| \partial_{(x-\eta-\xi)}^{k+r+q-q_1} G(t, \tau; x - \eta - \xi) \right| \leq c_{r,q} (t - \tau)^{-\frac{n+p_1+\gamma+|r+q|_+}{h}} e^{-\delta \left(\frac{\|x-\eta-\xi\|}{(t-\tau)^{\alpha}} \right)^{\frac{1}{1-\alpha}}}; \end{aligned} \quad (3.3)$$

$$\begin{aligned} |\partial_x^q K_1(t, x; \tau, x - \xi)| &= \left| \partial_x^q \left(\sum_{|k|_+ \leq p_1} A_{1,k}(t; x) \partial_x^k G(t, \tau; \xi) \right) \right| \leq m \left| \partial_x^q A_{1,0}(t; x) \right| \\ &\quad \times \left| G(t, \tau; \xi) \right| \leq \widehat{c}_q (t - \tau)^{-\frac{n+\gamma}{h}} e^{-\delta \left(\frac{\|\xi\|}{(t-\tau)^{\alpha}} \right)^{\frac{1}{1-\alpha}}} \leq c_q (t - \tau)^{-\frac{n+p_1+\gamma}{h}} e^{-\delta \left(\frac{\|\xi\|}{(t-\tau)^{\alpha}} \right)^{\frac{1}{1-\alpha}}}. \end{aligned} \quad (3.4)$$

We now estimate the expression $|\partial_\xi^r K_l(t, \eta + \xi; \tau, \xi)|$. Since

$$\partial_\xi^r K_1(t, \eta + \xi; \tau, \xi) = \sum_{|k|_+ \leq p_1} \partial_\xi^r A_{1,k}(t; \eta + \xi) \partial_\eta^k G(t, \tau; \eta), \quad (t, x; \tau, \xi) \in \Pi_T^2, \quad (3.5)$$

we have, according to condition (A), that the iterated kernels $K_l(t, \eta + \xi; \tau, \xi)$ are differentiable with respect to the variable ξ only to the order α_* . This fact and (3.2) imply that $\partial_x^q K_l(t, x; \tau, \xi)$, $|q|_* \leq \alpha_*$, is also a function differentiable with respect to ξ only to this order α_* .

Representation (3.5) and estimate (2.1) yield

$$|\partial_\xi^r K_1(t, \eta + \xi; \tau, \xi)| \leq c_{1,r} (t - \tau)^{-\frac{n+p_1+\gamma}{h}} e^{-\delta \left(\frac{\|\eta\|}{(t-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}}}. \quad (3.6)$$

We note that

$$\partial_\xi^r K_2(t, \eta + \xi; \tau, \xi) = \partial_\xi^r \left(\int_\tau^t d\beta \int_{\mathbb{R}^n} K_1(t, \eta + \xi; \beta, y) K_1(\beta, y; \tau, \xi) dy \right).$$

Let us change the variable of integration in the last integral by the formula $y = z + \xi$. In view of estimates (3.6) and (2.9) and the equalities

$$\int_\tau^t ((t - \beta)(\beta - \tau))^{\alpha_0 - 1} d\beta = (t - \tau)^{2\alpha_0 - 1} B(\alpha_0, \alpha_0) \quad (3.7)$$

and

$$\partial_\xi^r K_1(t, \eta + \xi; \tau, z + \xi) = \partial_\xi^r K_1(t, (\eta - z) + \xi; \tau, \xi) \Big|_{\xi=z+\xi},$$

we get

$$\begin{aligned} |\partial_\xi^r K_2(t, \eta + \xi; \tau, \xi)| &\leq m \sum_{|r_1|_+ \leq |r|_+} C_r^{r_1} \int_\tau^t d\beta \int_{\mathbb{R}^n} |\partial_\xi^{r_1} K_1(t, \eta + \xi; \beta, z + \xi)| \\ &\quad \times |\partial_\xi^{r-r_1} K_1(\beta, z + \xi; \tau, \xi)| dz \leq m \sum_{|r_1|_+ \leq |r|_+} C_r^{r_1} c_{1,r_1} c_{1,(r-r_1)} \\ &\quad \times \int_\tau^t ((t - \beta)(\beta - \tau))^{-\frac{n+p_1+\gamma}{h}} \int_{\mathbb{R}^n} e^{-\delta \left(\left(\frac{\|\eta-z\|}{(t-\beta)^\alpha} \right)^{\frac{1}{1-\alpha}} + \left(\frac{\|z\|}{(\beta-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}} \right)} dz d\beta \\ &\leq c_{2,r}(\varepsilon) B(\alpha_0, \alpha_0) (t - \tau)^{\alpha_0 - \frac{n+p_1+\gamma}{h}} e^{-\delta(1-\varepsilon) \left(\frac{\|\eta\|}{(t-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}}}, \quad \varepsilon \in (0; 1). \end{aligned}$$

By reasoning analogously step by step, we arrive at the inequality

$$|\partial_\xi^r K_l(t, \eta + \xi; \tau, \xi)| \leq c_{l,r}(\varepsilon) \left(\prod_{j=1}^{l-1} B(\alpha_0, j\alpha_0) \right) (t - \tau)^{(l-1)\alpha_0 - \frac{n+p_1+\gamma}{h}}$$

$$\times e^{-\delta(1-(l-1)\varepsilon)\left(\frac{\|\eta\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}, \quad (3.8)$$

which is satisfied for all $\{\eta, \xi\} \subset \mathbb{R}^n$, $|r|_+ \leq \alpha_*$, $0 \leq \tau < t \leq T$, $\varepsilon \in (0; 1)$, and $l \in \mathbb{N} \setminus \{1\}$ and, hence, until the existence of such number l_* , for which

$$|\partial_\xi^r K_{l_*}(t, \eta + \xi; \tau, \xi)| \leq c_{l_*, r}(\varepsilon) \left(\prod_{j=1}^{l_*-1} B(\alpha_0, j\alpha_0) \right) e^{-\delta(1-(l_*-1)\varepsilon)\left(\frac{\|\eta\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}} \quad (3.9)$$

(here, the quantities $c_{l, r}(\varepsilon) > 0$ do not depend on the variables t, τ, η , and ξ , which vary in the above-indicated way).

Since

$$\begin{aligned} \partial_\xi^r \partial_x^q K_l(t, x; \tau, \eta + \xi) &= \partial_\xi^r \partial_x^q K_l(t, x; \tau, \xi) \Big|_{\xi=\eta+\xi}, \\ \partial_\xi^r \partial_x^q K_l(t, x - z; \tau, \xi) &= \partial_\xi^r \partial_y^q K_l(t, y; \tau, \xi) \Big|_{y=x-z}, \end{aligned}$$

the expressions $\partial_\xi^r \partial_x^q K_l(t, x; \tau, \eta + \xi)$, $\partial_\xi^r \partial_x^q K_l(t, x - z; \tau, \xi)$ and $\partial_\xi^r \partial_x^q K_l(t, x; \tau, \xi)$ are of the same type. Therefore, with regard for representation (3.2) and estimates (3.3), (3.4), (3.8), and (2.9), we have

$$\begin{aligned} |\partial_\xi^r \partial_x^q K_2(t, x; \tau, \xi)| &\leq m 2^{|r+q|_+} \left(\sum_{|r_1|_+ \leq |r|_+} c_{r_1, q} c_{1, (r-r_1)} \int_{\tau}^{t_1} (t-\beta)^{-\frac{n+p_1+\gamma+|r_1+q|_+}{h}} \right. \\ &\times (\beta-\tau)^{-\frac{n+p_1+\gamma}{h}} \int_{\mathbb{R}^n} e^{-\delta\left(\left(\frac{\|x-\eta-\xi\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\|\eta\|}{(\beta-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}\right)} d\eta d\beta \\ &+ \sum_{|q_1|_+ \leq |q|_+} c_{q_1} c_{r, (q-q_1)} \times \int_{t_1}^t (\beta-\tau)^{-\frac{n+p_1+\gamma+|r+q-q_1|_+}{h}} (t-\beta)^{-\frac{n+p_1+\gamma}{h}} \\ &\left. \int_{\mathbb{R}^n} e^{-\delta\left(\left(\frac{\|z\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\|x-z-\xi\|}{(\beta-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}\right)} dz d\beta \right) \\ &\leq m 2^{|r+q|_+} c_\varepsilon e^{-\delta(1-\varepsilon)\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}} (t-\tau)^{-\alpha n} \left(\sum_{|r_1|_+ \leq |r|_+} c_{r_1, q} c_{1, (r-r_1)} \right. \\ &\times \int_{\tau}^{t_1} (t-\beta)^{\alpha n - \frac{n+p_1+\gamma+|r_1+q|_+}{h}} (\beta-\tau)^{\alpha_0-1} d\beta + \sum_{|q_1|_+ \leq |q|_+} c_{q_1} c_{r, (q-q_1)} \\ &\left. \times \int_{t_1}^t (t-\beta)^{\alpha_0-1} (\beta-\tau)^{\alpha n - \frac{n+p_1+\gamma+|r+q-q_1|_+}{h}} d\beta \right), |r|_+ \leq \alpha_*, |q|_+ \leq \alpha_*, \varepsilon \in (0; 1). \end{aligned}$$

In view of the estimates

$$\begin{aligned} \int_{\tau}^{t_1} (t-\beta)^{\alpha n - \frac{n+p_1+\gamma+|r_1+q|_+}{h}} (\beta-\tau)^{\alpha_0-1} d\beta \\ \leq 2^{\frac{|r_1+q|_+}{h}} (t-\tau)^{2\alpha_0 - \left(1 + \frac{|r_1+q|_+}{h}\right)} B(\alpha_0, \alpha_0) \end{aligned}$$

and

$$\begin{aligned} \int_{t_1}^t (t-\beta)^{\alpha_0-1} (\beta-\tau)^{\alpha n - \frac{n+p_1+\gamma+|r+q-q_1|_+}{h}} d\beta \\ \leq 2^{\frac{|r+q-q_1|_+}{h}} (t-\tau)^{2\alpha_0 - \left(1 + \frac{|r+q-q_1|_+}{h}\right)} B(\alpha_0, \alpha_0), \end{aligned}$$

we get the inequality

$$|\partial_{\xi}^r \partial_x^q K_2(t, x; \tau, \xi)| \leq c_{2, \xi}^{r, q} (t-\tau)^{2\alpha_0 - \left(1 + \alpha n + \frac{|r+q|_+}{h}\right)} B(\alpha_0, \alpha_0) e^{-\delta(1-\varepsilon) \left(\frac{\|x-\xi\|}{(t-\tau)^{\alpha}}\right)^{\frac{1}{1-\alpha}}}.$$

By continuing stepwise the process of estimation, we obtain

$$\begin{aligned} |\partial_{\xi}^r \partial_x^q K_l(t, x; \tau, \xi)| &\leq c_{l, \varepsilon}^{r, q} (t-\tau)^{l\alpha_0 - \left(1 + \alpha n + \frac{|r+q|_+}{h}\right)} \\ &\leq e^{-\delta(1-(l-1)\varepsilon) \left(\frac{\|x-\xi\|}{(t-\tau)^{\alpha}}\right)^{\frac{1}{1-\alpha}}} \left(\prod_{j=1}^{l-1} B(\alpha_0, j\alpha_0) \right), \end{aligned} \quad (3.10)$$

for all $|r|_+ \leq \alpha_*$, $|q|_+ \leq \alpha_*$, $\{x, \xi\} \subset \mathbb{R}^n$, $0 \leq \tau < t \leq T$, $\varepsilon \in (0; 1)$, and $l \in \mathbb{N} \setminus \{1\}$.

Let us pass to the estimation of the expression $|\partial_{\xi}^r \partial_x^q K_l(t, x; \tau, \xi)|$, which will be suitable for the establishment of the differentiability of the matrix function Φ with respect to the spatial variables. Directly from (3.10), we arrive at the existence of a number l^* such that

$$|\partial_{\xi}^r \partial_x^q K_{l^*}(t, x; \tau, \xi)| \leq c_{l^*, \varepsilon}^{r, q} e^{-\delta(1-(l^*-1)\varepsilon) \left(\frac{\|x-\xi\|}{(t-\tau)^{\alpha}}\right)^{\frac{1}{1-\alpha}}} \left(\prod_{j=1}^{l^*-1} B(\alpha_0, j\alpha_0) \right).$$

Let us set $l_+ := \max\{l_*, l^*\}$, $l_- := \min\{l_*, l^*\}$, where l_* is the corresponding number from (3.9), $\varepsilon := \frac{1}{r_* l_+}$, $\delta_* := \delta \left(1 - \frac{1}{r_*}\right)$, $r_* > 2$, $T_0 := \max\{1, T\}$, and

$$c_*^0 := \max_{l \in \mathbb{N}_{l_+} \setminus \{1\}} \left\{ c_{1, r}, c_{l, r}(\varepsilon) \left(\prod_{j=1}^{l-1} B(\alpha_0, j\alpha_0) \right), c_{r, q}, c_{l, \varepsilon}^{r, q} \left(\prod_{j=1}^{l-1} B(\alpha_0, j\alpha_0) \right) \right\},$$

$c_* := c_*^0(T_0)^{l_+ - l_-}$. Then (3.8) and (3.10) imply that, for all $\{x, \xi, \eta\} \subset \mathbb{R}^n$, $0 \leq \tau < t \leq T$, $|r|_+ \leq \alpha_*$, and $|q|_+ \leq \alpha_*$,

$$|\partial_\xi^r \partial_x^q K_{l_+}(t, x; \tau, \xi)| \leq c_* e^{-\delta_* \left(\frac{\|x - \xi\|}{(t - \tau)^\alpha} \right)^{\frac{1}{1 - \alpha}}},$$

$$|\partial_\xi^r K_{l_+}(t, \eta + \xi; \tau, \xi)| \leq c_* e^{-\delta_* \left(\frac{\|\eta\|}{(t - \tau)^\alpha} \right)^{\frac{1}{1 - \alpha}}}.$$

In view of this result, estimate (2.8), the equality

$$\int_{\mathbb{R}^n} e^{-\delta_0 \left(\frac{\|x - y\|}{(t - \beta)^\alpha} \right)^{\frac{1}{1 - \alpha}}} \frac{dy}{(t - \beta)^{\alpha n}} = \int_{\mathbb{R}^n} e^{-\delta_0 \|z\|^{\frac{1}{1 - \alpha}}} dz =: \widehat{E} < +\infty,$$

representation (3.2), and inequalities (3.3) and (3.4), we obtain

$$\begin{aligned} |\partial_\xi^r K_{l_+ + 1}(t, \eta + \xi; \tau, \xi)| &\leq \sum_{|r_1|_+ \leq |r|_+} C_r^{r_1} \int_\tau^t d\beta \int_{\mathbb{R}^n} |\partial_\xi^{r_1} K_1(t, \eta + \xi; \beta, z + \xi) \partial_\xi^{r - r_1} \\ &\times K_{l_+}(\beta, z + \xi; \tau, \xi)| dz \leq m c_*^2 \left(\sum_{|r_1|_+ \leq |r|_+} C_r^{r_1} \right) \int_\tau^t (t - \beta)^{\alpha_0 - 1} \\ &\times \int_{\mathbb{R}^n} e^{-\delta_* \left(\left(\frac{\|\eta - z\|}{(t - \beta)^\alpha} \right)^{\frac{1}{1 - \alpha}} + \left(\frac{\|z\|}{(\beta - \tau)^\alpha} \right)^{\frac{1}{1 - \alpha}} \right)} e^{-\frac{\delta}{r_*} \left(\frac{\|\eta - z\|}{(t - \beta)^\alpha} \right)^{\frac{1}{1 - \alpha}}} \frac{dz}{(t - \beta)^{n\alpha}} d\beta \\ &\leq m c_r^0 \widehat{E} c_*^2 B(\alpha_0, 1) (t - \tau)^{\alpha_0} e^{-\delta_* \left(\frac{\|\eta\|}{(t - \tau)^\alpha} \right)^{\frac{1}{1 - \alpha}}}, \quad c_r^0 := \sum_{|r_1|_+ \leq |r|_+} C_r^{r_1}; \\ |\partial_\xi^r \partial_x^q K_{l_+ + 1}(t, x; \tau, \xi)| &\leq \sum_{|r_1|_+ \leq |r|_+} C_r^{r_1} \int_\tau^{t_1} d\beta \int_{\mathbb{R}^n} |\partial_\xi^{r_1} \partial_x^q K_1(t, x; \beta, \eta + \xi) \\ &\times \partial_\xi^{r - r_1} K_{l_+}(\beta, \eta + \xi; \tau, \xi)| d\eta + \sum_{|q_1|_+ \leq |q|_+} C_q^{q_1} \int_{t_1}^t d\beta \int_{\mathbb{R}^n} |\partial_x^{q_1} K_1(t, x; \beta, x - z) \\ &\times \partial_\xi^r \partial_x^{q - q_1} K_{l_+}(\beta, x - z; \tau, \xi)| dz \leq m c_*^2 \left(\sum_{|r_1|_+ \leq |r|_+} C_r^{r_1} \int_\tau^{t_1} (t - \beta)^{\alpha_0 - (1 + \frac{|r_1|_+ + |q|_+}{h})} \right. \\ &\times \int_{\mathbb{R}^n} e^{-\delta_* \left(\left(\frac{\|x - \eta - \xi\|}{(t - \beta)^\alpha} \right)^{\frac{1}{1 - \alpha}} + \left(\frac{\|\eta\|}{(\beta - \tau)^\alpha} \right)^{\frac{1}{1 - \alpha}} \right)} e^{-\frac{\delta}{r_*} \left(\frac{\|x - \eta - \xi\|}{(t - \beta)^\alpha} \right)^{\frac{1}{1 - \alpha}}} \frac{d\eta}{(t - \beta)^{n\alpha}} d\beta \end{aligned}$$

$$\begin{aligned}
& + \sum_{|q|_+ \leq |q|_+} C_q^{q_1} \int_{t_1}^t (t-\beta)^{\alpha_0-1} \int_{\mathbb{R}^n} e^{-\delta_* \left(\left(\frac{\|z\|}{(t-\beta)^\alpha} \right)^{\frac{1}{1-\alpha}} + \left(\frac{\|x-z-\xi\|}{(\beta-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}} \right)} \\
& \times e^{-\frac{\delta}{r_*} \left(\frac{\|z\|}{(t-\beta)^\alpha} \right)^{\frac{1}{1-\alpha}}} \frac{dz}{(t-\beta)^{\alpha n}} d\beta \Bigg) \leq mc_*^2 \widehat{E} e^{-\delta_* \left(\frac{\|x-\xi\|}{(t-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}}} \int_{\tau}^t (t-\beta)^{\alpha_0-1} d\beta \\
& \times \left(\left(\sum_{|r|_+ \leq |r|_+} C_r^{r_1} (t-t_1)^{-\frac{|r|_++q|_+}{h}} \right) + c_q \right) \leq mc_*^2 \widehat{E} e^{-\delta_* \left(\frac{\|x-\xi\|}{(t-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}}} (t-\tau)^{\alpha_0} \\
& \times B(\alpha_0, 1) \left(\left(2^{\frac{|r|_++q|_+}{h}} \sum_{|r|_+ \leq |r|_+} C_r^{r_1} (t-\tau)^{-\frac{|r|_++q|_+}{h}} \right) + c_q \right) \\
& \leq mc_{r,q}^0 c_*^2 \widehat{E} (2T_0)^{\frac{|r|_++q|_+}{h}} B(\alpha_0, 1) (t-\tau)^{\alpha_0 - \frac{|r|_++q|_+}{h}} e^{-\delta_* \left(\frac{\|x-\xi\|}{(t-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}}}, \quad c_{r,q}^0 := c_r + c_q.
\end{aligned}$$

Applying the method of induction, we can verify firstly the validity of the estimate

$$\begin{aligned}
& \left| \partial_\xi^r K_{l_++l}(t, \eta + \xi; \tau, \xi) \right| \\
& \leq c_* (mc_r^0 c_* \widehat{E} (t-\tau)^{\alpha_0})^l e^{-\delta_* \left(\frac{\|\eta\|}{(t-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}}} \left(\prod_{j=1}^{l-1} B(\alpha_0, 1 + j\alpha_0) \right), \quad (3.11)
\end{aligned}$$

and, hence, the estimate

$$\begin{aligned}
& \left| \partial_\xi^r \partial_x^q K_{l_++l}(t, x; \tau, \xi) \right| \leq c_* \left(mc_{r,q}^0 c_* \widehat{E} (2T_0)^{\frac{|r|_++q|_+}{h}} \right)^l (t-\tau)^{l\alpha_0 - \frac{|r|_++q|_+}{h}} \\
& \times e^{-\delta_* \left(\frac{\|x-\xi\|}{(t-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}}} \left(\prod_{j=1}^{l-1} B(\alpha_0, 1 + j\alpha_0) \right), \quad (3.12)
\end{aligned}$$

for $|r|_+ \leq \alpha_*$, $|q|_+ \leq \alpha_*$, $(t, x; \tau, \xi) \in \Pi_T^2$ and $l \in \mathbb{N} \setminus \{1\}$.

The following propositions hold true.

Lemma 1. *The matrix function $\Phi(t, x; \tau, \xi)$ on the set Π_T^2 is a function differentiable with respect to each of the spatial variables x and ξ to the order α_* inclusively, and their derivatives satisfy the following estimates:*

$$\left| \partial_\xi^r \partial_x^q \Phi(t, x; \tau, \xi) \right| \leq c_1 (t-\tau)^{\alpha_0 - (1+\alpha n + \frac{|r|_++q|_+}{h})} e^{-\delta_* \left(\frac{\|x-\xi\|}{(t-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}}}, \quad (3.13)$$

$$\left| \partial_\xi^r \Phi(t, \eta + \xi; \tau, \xi) \right| \leq c_2 (t-\tau)^{\alpha_0 - (1+\alpha n)} e^{-\delta_* \left(\frac{\|\eta\|}{(t-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}}}, \quad \{\eta, \xi\} \subset \mathbb{R}^n \quad (3.14)$$

(here, the estimating constants c_1, c_2 , and δ_* are independent of t, τ, x, ξ, η).

Proof. In any way, let us fix a point $(x_0; \xi_0)$ from $\mathbb{R}^{2n}$, and let us consider a ball $\mathbb{K}_{(x_0; \xi_0)}^\delta$ with radius $\delta > 0$, which is centered at the point $(x_0; \xi_0)$, in this space. Then, in view of structure (2.4) of the function Φ and the differentiability of the iterated kernels K_l with respect to spatial variables on $\mathbb{R}^{2n}$ to the order α_* inclusively, we can conclude that, in order to prove the differentiability of the matrix function Φ at the point $(x_0; \xi_0)$ to the indicated order, it is necessary only to prove the uniform convergence of the formally differentiated series (2.4) in the variables x and ξ on the set $\mathbb{K}_{(x_0; \xi_0)}^\delta$, $\delta > 0$ (at every fixed t and τ , $0 \leq \tau < t \leq T$):

$$\sum_{l=1}^{\infty} \partial_{\xi}^r \partial_x^q K_l(t, x; \tau, \xi), \quad |r|_+ \leq \alpha_*, |q|_+ \leq \alpha_*. \quad (3.15)$$

Directly from estimates (3.10) and (3.12) and the equality

$$\prod_{j=0}^{l-1} B(\alpha_0, 1 + j\alpha_0) = \frac{(\Gamma(\alpha_0))^l}{\Gamma(1 + l\alpha_0)},$$

where $\Gamma(\cdot)$ is the Euler gamma-function, for $\{r, q\} \subset \mathbb{Z}_+^n$, $|r|_+ \leq \alpha_*$, $|q|_+ \leq \alpha_*$, and $(t, x; \tau, \xi) \in \Pi_T^2$, we have

$$\begin{aligned} \left| \sum_{l=1}^{\infty} \partial_{\xi}^r \partial_x^q K_l(t, x; \tau, \xi) \right| &\leq \sum_{l=1}^{l_+} \left| \partial_{\xi}^r \partial_x^q K_l(t, x; \tau, \xi) \right| + \sum_{l=l_++1}^{\infty} \left| \partial_{\xi}^r \partial_x^q K_l(t, x; \tau, \xi) \right| \\ &\leq c_* \left(\sum_{l=1}^{l_+} (t - \tau)^{l\alpha_0 - (1 + \alpha n + \frac{|r+q|_+}{h})} \right. \\ &\quad \left. + \sum_{l=1}^{\infty} (mc_{r,q}^0 c_* \widehat{E}(2T_0)^{\frac{|r+q|_+}{h}})^l (t - \tau)^{l\alpha_0 - \frac{|r+q|_+}{h}} \right. \\ &\quad \left. \times \left(\prod_{j=1}^{l-1} B(\alpha_0, 1 + j\alpha_0) \right) \right) e^{-\delta_* \left(\frac{\|x - \xi\|}{(t - \tau)^{\alpha}} \right)^{\frac{1}{1-\alpha}}} \leq c_1 (t - \tau)^{\alpha_0 - (1 + \alpha n + \frac{|r+q|_+}{h})} \\ &\quad \times e^{-\delta_* \left(\frac{\|x - \xi\|}{(t - \tau)^{\alpha}} \right)^{\frac{1}{1-\alpha}}}. \end{aligned}$$

From whence, we get the uniform convergence of series (3.15) in x and ξ and, hence, the validity of estimates (3.13).

Due to the corresponding estimates (3.8) and (3.11), we can verify analogously the validity of estimate (3.14).

The lemma is proved. $\square$

Lemma 2. *The volumetric potential $W(t, x; \tau, \xi)$ on the set Π_T^2 is a function differentiable with respect to each of the spatial variables x and ξ to, respectively, the*

orders $\alpha_* + p_1$ and α_* inclusively. In this case,

$$\begin{aligned} \partial_\xi^r \partial_x^q W(t, x; \tau, \xi) &= \sum_{l=0}^r C_r^l \int_{\tau}^{t_1} d\beta \int_{\mathbb{R}^n} \partial_\xi^l \partial_x^q G(t, \beta; x - y - \xi) \partial_\xi^{r-l} \Phi(\beta, y + \xi; \tau, \xi) dy \\ &+ \int_{t_1}^t d\beta \int_{\mathbb{R}^n} \partial_x^q G(t, \beta; x - y) \partial_\xi^r \Phi(\beta, y; \tau, \xi) dy, \quad |q|_+ \leq p_1, |r|_+ \leq \alpha_*, \quad (3.16) \end{aligned}$$

$$\begin{aligned} \partial_\xi^r \partial_x^q W(t, x; \tau, \xi) &= \sum_{l=0}^r C_r^l \int_{\tau}^{t_1} d\beta \int_{\mathbb{R}^n} \partial_\xi^l \partial_x^q G(t, \beta; x - y - \xi) \partial_\xi^{r-l} \Phi(\beta, y + \xi; \tau, \xi) dy \\ &+ \int_{t_1}^t d\beta \int_{\mathbb{R}^n} \partial_\eta^k G(t, \beta; \eta) \partial_\xi^r \partial_x^{q-k} \Phi(\beta, x - \eta; \tau, \xi) d\eta, \quad |r|_+ \leq \alpha_*, \quad (3.17) \end{aligned}$$

$$|k|_+ = p_1, \quad p_1 < |q|_+ \leq \alpha_* + p_1.$$

Proof. For $|q|_+ \leq p_1$ and $|r|_+ \leq \alpha_*$, we use the representation

$$\begin{aligned} W(t, x; \tau, \xi) &= \int_{\tau}^{t_1} d\beta \int_{\mathbb{R}^n} G(t, \beta; x - y - \xi) \Phi(\beta, y + \xi; \tau, \xi) dy \\ &+ \int_{t_1}^t d\beta \int_{\mathbb{R}^n} G(t, \beta; x - y) \Phi(\beta, y; \tau, \xi) dy. \end{aligned}$$

From whence, by the formal differentiation under the sign of integral, we obtain equality (3.16). Hence, in order to substantiate the validity of equality (3.16), it is sufficient to prove the uniform convergence of the following integrals in the variables x and ξ on $\mathbb{R}^{2n}$:

$$\begin{aligned} \mathcal{J}_1^{r,l,q}(t_1, x; \tau, \xi) &:= \int_{\tau}^{t_1} d\beta \int_{\mathbb{R}^n} |\partial_\xi^l \partial_x^q G(t, \beta; x - y - \xi)| |\partial_\xi^{r-l} \Phi(\beta, y + \xi; \tau, \xi)| dy, \\ |l|_+ \leq |r|_+; \mathcal{J}_2^{r,q}(t, x; t_1, \xi) &:= \int_{t_1}^t d\beta \int_{\mathbb{R}^n} |\partial_x^q G(t, \beta; x - y)| |\partial_\xi^r \Phi(\beta, y; \tau, \xi)| dy. \quad (3.18) \end{aligned}$$

This convergence becomes obvious, if we take condition (A) and the following estimates into account for $\{x, \xi\} \subset \mathbb{R}^n$ and $0 \leq \tau < t \leq T$:

$$\begin{aligned} \mathcal{J}_1^{r,l,q}(t_1, x; \tau, \xi) &\leq c c_2 \widehat{E} e^{-\delta_* \left(\frac{\|x-\xi\|}{(t-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}}} (t-t_1)^{-\frac{n+\gamma+|l+q|_+}{h}} \\ &\quad \times \int_{\tau}^{t_1} (\beta-\tau)^{\alpha_0-1} d\beta, \quad |l|_+ \leq |r|_+; \end{aligned} \quad (3.19)$$

$$\begin{aligned} \mathcal{J}_2^{r,q}(t, x; t_1, \xi) \\ \leq c c_1 \widehat{E} e^{-\delta_* \left(\frac{\|x-\xi\|}{(t-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}}} (t_1-\tau)^{-\frac{n+p_1+\gamma+|r|_+}{h}} \int_{t_1}^t (t-\beta)^{\alpha_0-1+\frac{p_1-|q|_+}{h}} d\beta. \end{aligned} \quad (3.20)$$

These estimates follow directly from (2.1), (3.13), and (3.14).

We now prove the validity of formula (3.17). For this purpose, we fix any $k \in \mathbb{Z}_+^n$ such that $|k|_+ = p_1$. Then, according to (3.16) for $p_1 < |q|_+ \leq \alpha_* + p_1$ and $|r|_+ \leq \alpha_*$, we have

$$\begin{aligned} \partial_\xi^r \partial_x^q W(t, x; \tau, \xi) &= \sum_{l=0}^r C_r^l \partial_x^{q-k} \int_{\tau}^{t_1} d\beta \int_{\mathbb{R}^n} \partial_\xi^l \partial_x^k G(t, \beta; x-y-\xi) \partial_\xi^{r-l} \Phi(\beta, y+\xi; \tau, \xi) \\ &\quad \times dy + \partial_x^{q-k} \int_{t_1}^t d\beta \int_{\mathbb{R}^n} \partial_\eta^k G(t, \beta; \eta) \partial_\xi^r \Phi(\beta, x-\eta; \tau, \xi) d\eta, \quad (t, x; \tau, \xi) \in \Pi_T^2. \end{aligned}$$

Hence, it remains to substantiate the possibility to introduce the operation ∂_x^{q-k} under the signs of the corresponding integrals. In other words, we should prove the uniform convergence in x and ξ of the following integrals on $\mathbb{R}^{2n}$ for $0 \leq \tau < t \leq T$:

$$\begin{aligned} \int_{\tau}^{t_1} d\beta \int_{\mathbb{R}^n} \partial_\xi^l \partial_x^q G(t, \beta; x-y-\xi) \Phi(\beta, y+\xi; \tau, \xi) dy, \\ \int_{t_1}^t d\beta \int_{\mathbb{R}^n} \partial_\eta^k G(t, \beta; \eta) \partial_\xi^r \partial_x^{q-k} \Phi(\beta, x-\eta; \tau, \xi) d\eta. \end{aligned}$$

By reasoning similarly to the case of integrals (3.18) and using estimates (2.1), (3.13), and (3.14), we get the necessary convergence of the indicated integrals.

The lemma is proved. $\square$

The main result can be formulated as the following proposition.

Theorem 1. *Let system (1.1) satisfy conditions (A) and (B). Then the corresponding function $Z(t, x; \tau, \xi)$ defined by equality (2.2) is a function differentiable with respect to each of the spatial variables x and ξ on the set Π_T^2 to, respectively, the orders $\alpha_* + p_1$ and α_* inclusively, and*

$$\exists \delta > 0 \forall \{r, q\} \subset \mathbb{Z}_+^n, |q|_+ \leq \alpha_* + p_1, |r|_+ \leq \alpha_*, \exists c > 0 \forall (t, x; \tau, \xi) \in \Pi_T^2 :$$

$$|\partial_\xi^r \partial_x^q Z(t, x; \tau, \xi)| \leq c(t - \tau)^{-\frac{n+|r+q|_++\gamma}{h}} e^{-\delta \left(\frac{\|x-\xi\|}{(t-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}}} ; \quad (3.21)$$

$$|\partial_\xi^k Z(t, x + \xi; \tau, \xi)| \leq c_k(t - \tau)^{\beta_k - \frac{n+\gamma}{h}} e^{-\delta_1 \left(\frac{\|x\|}{(t-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}}} , \quad (3.22)$$

where $|k|_+ \leq \alpha_*$, $0 \leq \tau < t \leq T$, $\{x, \xi\} \subset \mathbb{R}^n$, $\beta_k := \begin{cases} 0, & k = 0, \\ \alpha_0, & k \neq 0 \end{cases}$ (here, the estimating constants are independent of t , τ , x , and ξ).

Proof. With regard for structure (2.2) and the infinite differentiability of the function $G(t, \tau; \xi)$ with respect to the variable ξ , the smoothness of the function $Z(t, x; \tau, \xi)$ in the variables x and ξ becomes obvious directly from the assertion of Lemma 2.

Let $|q|_+ \leq p_1$ and $|r|_+ \leq \alpha_*$. Then, according to (3.16), we get

$$|\partial_\xi^r \partial_x^q Z(t, x; \tau, \xi)| \leq |\partial_{x-\xi}^{r+q} G(t, \tau; x - \xi)| + \sum_{l=0}^r C_r^l \mathcal{J}_1^{r,l,q}(t_1, x; \tau, \xi) + \mathcal{J}_2^{r,q}(t, x; t_1, \xi).$$

From whence, by using estimates (2.1), (3.19), and (3.20), we obtain assertion (3.21).

In a similar way, by using formula (3.17), we verify the validity of assertion (3.21) also for $p_1 < |q|_+ \leq \alpha_*$ and $|r|_+ \leq \alpha_*$.

Then, according to estimates (2.1) and (3.14), we have

$$\begin{aligned} Y_k(t, x; \tau, \xi) &:= \left| \int_\tau^t d\beta \int_{\mathbb{R}^n} G(t, \beta; x - \zeta) \partial_\xi^k \Phi(\beta, \zeta + \xi; \tau, \xi) d\zeta \right| \\ &\leq c c_2 \int_\tau^t (t - \beta)^{\alpha_0 + \frac{p_1}{h} - 1} (\beta - \tau)^{\alpha_0 - 1} \int_{\mathbb{R}^n} \exp \left\{ -\delta_0 \left\{ \left(\frac{\|x - \zeta\|}{(t - \beta)^\alpha} \right)^{\frac{1}{1-\alpha}} \right. \right. \\ &\quad \left. \left. + \left(\frac{\|\zeta\|}{(\beta - \tau)^\alpha} \right)^{\frac{1}{1-\alpha}} \right\} \right\} \frac{dy d\beta}{((t - \beta)(\beta - \tau))^{\alpha n}}, \delta_0 := \min\{\delta, \delta_*\}, |k|_+ \leq \alpha_*. \end{aligned}$$

Using estimate (2.9) and equality (3.7), we get

$$Y_k(t, x; \tau, \xi) \leq c_\varepsilon(t - \tau)^{\alpha_0 - \frac{n+\gamma}{h}} e^{-\delta_0(1-\varepsilon) \left(\frac{\|x\|}{(t-\tau)^\alpha} \right)^{\frac{1}{1-\alpha}}} , \quad \varepsilon \in (0; 1),$$

where $|k|_+ \leq \alpha_*$, $0 \leq \tau < t \leq T$ and $\{x, \xi\} \subset \mathbb{R}^n$. From whence, with regard for inequality (2.1) and the representation

$$Z(t, x + \xi; \tau, \xi) = G(t, \tau; x) + \int_{\tau}^t d\beta \int_{\mathbb{R}^n} G(t, \beta; x - \zeta) \Phi(\beta, \zeta + \xi; \beta, \xi) d\zeta,$$

we arrive at estimate (3.22).

The theorem is proved. $\square$

REFERENCES

- [1] A. Friedman, *Partial Differential Equations of Parabolic Type*. Englewood Cliffs: Prentice-Hall, 1964.
- [2] I. Gelfand and G. Shilov, *Generalized Functions. Vol. 3. Theory of Differential Equations*. Boston, MA: Academic Press, 1967.
- [3] I. Gelfand and G. Shilov, *Generalized Functions. Vol. 2. Spaces of Fundamental and Generalized Functions*. Boston, MA: Academic Press, 1968.
- [4] U. Hou-sin, "On the definition of parabolicity of systems of equations with partial derivatives." *Uspekhi Mat. Nauk*, vol. 15, no. 6, pp. 157–161, 1960.
- [5] I.M.Dovzhytska, "The Cauchy problem for Shilov-type parabolic systems with variable coefficients and nonnegative genus." Author's abstract of the Candidate thesis, Yu. Fed'kovych Chernivtsi National University, 2, Kotsubins'ky Str., 58012 Chernivtsi, Ukraine, 30 January 2015.
- [6] V. Litovchenko, "The Cauchy problem for parabolic equations by Shilov." *Siber. Mat. Zh.*, vol. 45, no. 4, pp. 809–821, 2004, doi: [10.1023/B:SIMJ.0000035831.63036.bb](https://doi.org/10.1023/B:SIMJ.0000035831.63036.bb).
- [7] V. Litovchenko and I. Dovzhytska, "The fundamental matrix of solutions of the Cauchy problem for a class of parabolic system of the Shilov type with variable coefficients." *J. of Math. Sci.*, vol. 175, no. 4, pp. 450–476, 2011, doi: [10.1007/s10958-011-0356-0](https://doi.org/10.1007/s10958-011-0356-0).
- [8] V. Litovchenko and I. Dovzhytska, "Cauchy problem for a class of parabolic systems of Shilov type with variable coefficients." *Cent. Eur. J. Math.*, vol. 10, no. 3, pp. 1084–1102, 2012, doi: [10.2478/s11533-012-0025-7](https://doi.org/10.2478/s11533-012-0025-7).
- [9] V. Litovchenko and I. Dovzhytska, "Stabilization of solutions of Shilov-type parabolic systems with nonnegative genus." *Siber. Mat. Zh.*, vol. 55, no. 2, pp. 341–349, 2014, doi: [10.1134/S0037446614020104](https://doi.org/10.1134/S0037446614020104).
- [10] Y. Zhitomirskii, "Cauchy problem for some types of systems of linear equations, which are parabolic by G.E. Shilov, with partial derivatives with continuous coefficients." *Izv. AN SSSR. Ser. Mat.*, vol. 23, pp. 925–932, 1959.

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ASYMPTOTIC STABILITY OF SOLUTIONS FOR A KIND OF THIRD-ORDER STOCHASTIC DIFFERENTIAL EQUATIONS WITH DELAYS

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Abstract. This work is devoted to investigate the stochastic asymptotically stability of the zero solution for a kind of third-order stochastic differentials equation with variable and constant delays by a suitable Lyapunov functional. Our results improve and form a complement to some results that can be found in the literature. In the last section, we give an example to illustrate our main result.

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1. INTRODUCTION

Stochastic delay differential equations (SDDEs) are natural generalizations of stochastic ordinary differential equations (SODEs) by allowing the coefficients to depend on the past values.

Recently, the studies of stochastic differential equations (SDEs) have attracted the considerable attentions of many scholars in the last forty years.

SDEs play an important role in many branches of science and engineering, and there are a large number of books, which provide full details for the background of probability theory and stochastic calculus, see for example, [6, 7, 9, 10, 13–16] and the references therein.

Systems of SDDEs occupy now a place of central importance in many areas of science including medicine, engineering, biology and physics.

Stability theory is one of the main components of SDDEs. The Lyapunov's direct method has been successfully used to investigate stability problems in deterministic SDDEs for more than one hundred years, when there is no analytical expression for solutions.

An apparent advantage of this method is the stability in the large can be obtained without any prior knowledge of solutions. Therefore the method yields stability information directly without solving differential equations.

However, there are many difficulties encountered in the study of stability by means of Lyapunov's direct method. Therefore, in the relevant literature, some authors investigated the stability of solutions for DDEs and SDDEs by using different approaches such as the fixed point method, the inequalities techniques, the perturbation methods, the second method of Lyapunov and so on (see e.g. [19, 22] and reference therein).

In this direction, many authors have proposed different approaches to investigate the stability of solutions of third-order DDEs. We can mention the papers of Ademola et al. [4, 5], Graef and Tunç [8], Mahmoud [11], Omeike [17], Oudjedi et al. [18], Remili et al. [20, 21], Sadek [23], Shekhar et al. [24], Tunç [25–29] and the references cited therein.

Meanwhile, the scarcity of works on stability and boundedness of solutions for third-order SDEs with or without delay were studied very rarely, interesting results are contained, for instance, in [1], [2], [3].

In 2015, Abou-El-Ela et al. [1] considered the stochastic asymptotic stability of the zero solution and the uniform stochastic boundedness of all solutions for the third-order SDE of the form

$$\ddot{x}(t) + a\ddot{x}(t) + b\dot{x}(t) + cx(t) + \sigma x(t)\dot{\omega}(t) = p(t, x(t), \dot{x}(t), \ddot{x}(t)),$$

where a, b, c and σ are positive constants; $\omega(t) \in \mathbb{R}^m$ is a standard Wiener process, p is a continuous function.

In 2015, Abou-El-Ela et al. [2] investigated the asymptotic stability of the zero solution for the third-order SDDEs given by

$$\begin{aligned}\ddot{x}(t) + a_1\ddot{x}(t) + g_1(\dot{x}(t - r_1(t))) + f_1(x(t)) + \sigma_1 x(t)\dot{\omega}(t) &= 0, \\ \ddot{x}(t) + a_2\ddot{x}(t) + f_2(x(t))\dot{x}(t) + f_3(x(t - r_2(t))) + \sigma_2 x(t - h(t))\dot{\omega}(t) &= 0,\end{aligned}$$

where a_1, a_2, σ_1 and σ_2 are positive constants; $0 \leq r_1(t) \leq \gamma_1$, $0 \leq r_2(t) \leq \gamma_2$, γ_1 and γ_2 are two positive constants which will be determined later. $0 \leq h(t)$, $\sup h(t) = H$; $\omega(t) \in \mathbb{R}^m$ is a standard Wiener process; g_1, f_1, f_2 and f_3 are continuous functions with $g_1(0) = f_1(0) = f_3(0) = 0$.

In 2017, Ademola [3] studied the problems of stability, boundedness and uniqueness of solutions of a certain third-order SDDE as the following form

$$\ddot{x}(t) + a\ddot{x}(t) + b\dot{x}(t) + h(x(t - \tau)) + \sigma x(t)\dot{\omega}(t) = p(t, x(t), \dot{x}(t), \ddot{x}(t)),$$

where a, b and σ are positives constants, h, p are nonlinear continuous functions in their respective arguments with $h(0) = 0$, $\tau > 0$ is a constant delay.

The main purpose of this work is to establish new criteria for the stochastic asymptotic stability of the zero solution for a kind of third-order nonlinear SDE with variable and constant delays as the following form

$$\ddot{x}(t) + a\ddot{x}(t) + \phi(\dot{x}(t - r(t))) + \psi(x(t - r(t))) + \sigma x(t - h)\dot{\omega}(t) = 0, \quad (1.1)$$

where a, σ and h are positive constants, $r(t)$ is a continuously differentiable function with $0 \leq r(t) \leq \gamma_1$, γ_1 is a positive constant which will be determined later, ϕ, ψ are two nonlinear continuous functions in their respective arguments with $\phi(0) = \psi(0) = 0$, $\omega(t) = (\omega_1(t), \omega_2(t), \dots, \omega_n(t)) \in \mathbb{R}^m$ is m -dimensional standard Brownian motion, defined on the probability space. The functions ϕ and ψ are also differentiable throughout this work.

In this paper, by constructing a suitable Lyapunov functional, sufficient conditions for the stochastically asymptotically stability of the zero solution of (1.1) are established. Our result includes and improves the former results that can be found in the literature.

The remainder of this work is organized as follows. In section 2, we give a theorem, which deals with stochastic asymptotically stability of the zero solution for (1.1). In section 3, we introduced the proof of the main theorem. In the last section, we gave an example to verify the analysis made in this work.

2. STABILITY RESULT

Let $\omega(t) = (\omega_1(t), \dots, \omega_m(t))$ be an m -dimensional Brownian motion defined on the probability space. Consider an n -dimensional SDE

$$dx(t) = f(t, x(t))dt + g(t, x(t))dB(t) \quad \text{on } t \geq 0, \quad (2.1)$$

with initial value $x(0) = x_0 \in \mathbb{R}^n$. As a standing condition, we assume that $f : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ satisfy the local Lipschitzian condition and the linear growth condition for the existence and uniqueness of solutions for equation (2.1) (see for example, [12, 30]). It is therefore known that equation (2.1) has a unique continuous solution on $t \geq 0$, which is denoted by $x(t; x_0)$ in this work. Assume furthermore that $f(t, 0) = 0$ and $g(t, 0) = 0$, for all $t \geq 0$. Hence the stochastic differential equation admits the zero solution $x(t; 0) \equiv 0$.

Let $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ denote the family of non-negative functions $V(t, x)$ defined on $\mathbb{R}^+ \times \mathbb{R}^n$, which are once continuously differentiable in t and twice continuously differentiable in x .

Define the differential operator $\mathcal{L}$ associated with equation (2.1) by

$$\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^n f_i(t, x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n [g(t, x) g^T(t, x)]_{ij} \frac{\partial^2}{\partial x_i \partial x_j},$$

If $\mathcal{L}$ acts on a function $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$, then

$$\mathcal{L}V(t, x) = V_t(t, x) + V_x(t, x) \cdot f(t, x) + \frac{1}{2} \text{trace}[g^T(t, x) V_{xx}(t, x) g(t, x)], \quad (2.2)$$

where $V_t = \frac{\partial V}{\partial t}$, $V_x = (\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n})$ and

$$V_{xx} = (\frac{\partial^2 V}{\partial x_i \partial x_j})_{n \times n} = \begin{pmatrix} \frac{\partial^2 V}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 V}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_n \partial x_n} \end{pmatrix}.$$

Moreover, let $\mathcal{K}$ denote the family of all continuous nondecreasing functions $\theta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\theta(0) = 0$ and $\theta(r) > 0$, if $r > 0$.

Lemma 1 ([13]). Assume that there exist $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ and $\theta \in \mathcal{K}$ such that

$$V(t, 0) = 0, \quad \theta(|x|) \leq V(t, x), \quad \text{and} \\ \mathcal{L}V(t, x) \leq 0, \quad \text{for all } (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n.$$

Then the zero solution of the stochastic differential equation (2.1) is stochastically stable.

Lemma 2 ([13]). Assume that there exist $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ and $\theta_1, \theta_2, \theta_3 \in \mathcal{K}$ such that

$$\theta_1(|x|) \leq V(t, x) \leq \theta_2(|x|), \quad \text{and} \\ \mathcal{L}V(t, x) \leq -\theta_3(|x|), \quad \text{for all } (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n.$$

Then the zero solution of the stochastic differential equation (2.1) is stochastically asymptotically stable.

Now we present the main stability result of (1.1).

Theorem 1. In additions to the basic assumptions imposed on the functions ϕ and ψ appearing in (1.1), suppose that there exists positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, c_1, L$ and M such that:

- (i) $\alpha_1 \leq \frac{\psi(x)}{x} \leq \alpha_2$ and $\psi(x) \operatorname{sgn} x > 0$, for all $x \neq 0$.
- (ii) $\sup\{\psi'(x)\} = \frac{c_1}{2}$ and $|\psi'(x)| \leq L$, for all x .
- (iii) $\beta_1 \leq \frac{\phi(y)}{y} \leq \beta_2$, for all $y \neq 0$ and $|\phi'(y)| \leq M$, for all y .
- (iv) $0 \leq r(t) \leq \gamma_1$ and $r'(t) \leq \gamma_2$, such that $0 < \gamma_2 < 1$.
- (v) $a\beta_1 - c_1 > 2\beta_1 + 6$.
- (vi) $\sigma^2 < 2\alpha_1 - a - \beta_1 - 2$.

Then the zero solution of (1.1) is stochastically asymptotically stable, provided that

$$\gamma_1 < \min \left\{ \frac{2\alpha_1 - \sigma^2 - a - \beta_1 - 2}{2(L + M)}, \frac{(a\beta_1 - c_1 - 2\beta_1 - 6)(1 - \gamma_2)}{4\lambda(L + M)(1 - \gamma_2) + 4L(\lambda + 2)}, \right. \\ \left. \frac{(a\beta_1 - c_1 - 2\beta_1)(1 - \gamma_2)}{4\beta_1(L + M)(1 - \gamma_2) + 4\beta_1(\lambda + 2)M} \right\},$$

with $\lambda = \frac{a\beta_1 + c_1}{4\beta_1}$.

3. PROOF OF THEOREM 1

The equation (1.1) can be written in the following equivalent system:

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -az - \phi(y) - \psi(x) + \int_{t-r(t)}^t \phi'(y(s))z(s)ds + \int_{t-r(t)}^t \psi'(x(s))y(s)ds \\ &\quad - \sigma x(t-h)\dot{\omega}(t).\end{aligned}\quad (3.1)$$

Define the Lyapunov functional $V(t, X_t)$, where $X_t = (x_t, y_t, z_t)$, as the following

$$\begin{aligned}V(t, X_t) &= \lambda \int_0^x \psi(\xi)d\xi + \psi(x)y + \frac{1}{2}\lambda ay^2 + \int_0^y \phi(\eta)d\eta + \lambda yz + \frac{1}{2}z^2 \\ &\quad + xz + x^2 + \mu \int_{-r(t)}^0 \int_{t+s}^t y^2(\vartheta)d\vartheta ds \\ &\quad + \nu \int_{-r(t)}^0 \int_{t+s}^t z^2(\vartheta)d\vartheta ds + \frac{1}{2}\sigma^2 \int_{t-h}^t x^2(s)ds,\end{aligned}\quad (3.2)$$

where μ and ν are two positive constants, which will be determined later.

Our target here is to show that the Lyapunov functional $V(t, X_t)$ satisfies the conditions of Lemma 2.

Thus from (3.2), (3.1) and by using Itô formula (2.2), we get

$$\begin{aligned}\mathcal{L}V(t, X_t) &= \psi'(x)y^2 + \lambda z^2 - \lambda \phi(y)y - az^2 + yz - axz - x\phi(y) - x\psi(x) + 2xy \\ &\quad + (x + \lambda y + z) \left(\int_{t-r(t)}^t \phi'(y(s))z(s)ds + \int_{t-r(t)}^t \psi'(x(s))y(s)ds \right) \\ &\quad + \mu r(t)y^2 - \mu(1-r'(t)) \int_{t-r(t)}^t y^2(\vartheta)d\vartheta \\ &\quad + \nu r(t)z^2 - \nu(1-r'(t)) \int_{t-r(t)}^t z^2(\vartheta)d\vartheta + \frac{1}{2}\sigma^2 x^2.\end{aligned}$$

In view the assumptions (i) – (iv) of Theorem 1, we obtain

$$\begin{aligned}\mathcal{L}V(t, X_t) &\leq \frac{c_1}{2}y^2 + \lambda z^2 - \lambda \beta_1 y^2 - az^2 + yz - axz - \beta_1 xy - \alpha_1 x^2 + 2xy + \frac{1}{2}\sigma^2 x^2 \\ &\quad + (x + \lambda y + z) \left(M \int_{t-r(t)}^t z(s)ds + L \int_{t-r(t)}^t y(s)ds \right) \\ &\quad + \mu \gamma_1 y^2 + \nu \gamma_1 z^2 - (1-\gamma_2) \left(\mu \int_{t-r(t)}^t y^2(\vartheta)d\vartheta + \nu \int_{t-r(t)}^t z^2(\vartheta)d\vartheta \right).\end{aligned}$$

Then by using the inequality $2uv \leq u^2 + v^2$, with the condition $r(t) \leq \gamma_1$ in Theorem 1, we have

$$\begin{aligned} \mathcal{L}V(t, X_t) \leq & - \left\{ \alpha_1 - \frac{1}{2}(\sigma^2 + a + \beta_1 + 2) - \frac{1}{2}(L + M)\gamma_1 \right\} x^2 \\ & - \left\{ \lambda\beta_1 - \frac{1}{2}(c_1 + \beta_1 + 3) - \frac{1}{2}\lambda(L + M)\gamma_1 - \mu\gamma_1 \right\} y^2 \\ & - \left\{ \frac{1}{2}(a - 1) - \frac{1}{2}(L + M)\gamma_1 - \lambda - v\gamma_1 \right\} z^2 \\ & + \left\{ L + \frac{\lambda L}{2} - \mu(1 - \gamma_2) \right\} \int_{t-r(t)}^t y^2(\vartheta) d\vartheta \\ & + \left\{ M + \frac{\lambda M}{2} - v(1 - \gamma_2) \right\} \int_{t-r(t)}^t z^2(\vartheta) d\vartheta. \end{aligned}$$

If we take

$$\mu = \frac{L(\lambda + 2)}{2(1 - \gamma_2)} > 0 \text{ and } v = \frac{M(\lambda + 2)}{2(1 - \gamma_2)} > 0,$$

it follows that

$$\begin{aligned} \mathcal{L}V(t, X_t) \leq & - \left\{ \alpha_1 - \frac{1}{2}(\sigma^2 + a + \beta_1 + 2) - \frac{1}{2}(L + M)\gamma_1 \right\} x^2 \\ & - \left\{ \lambda\beta_1 - \frac{1}{2}(c_1 + \beta_1 + 3) - \frac{1}{2}\lambda(L + M)\gamma_1 - \frac{L(\lambda + 2)}{2(1 - \gamma_2)}\gamma_1 \right\} y^2 \\ & - \left\{ \frac{1}{2}(a - 1) - \frac{1}{2}(L + M)\gamma_1 - \lambda - \frac{M(\lambda + 2)}{2(1 - \gamma_2)}\gamma_1 \right\} z^2. \end{aligned}$$

In view of

$$\lambda\beta_1 - \frac{1}{2}c_1 = \frac{a\beta_1 - c_1}{4} > 0 \text{ and } \frac{a}{2} - \lambda = \frac{a\beta_1 - c_1}{4\beta_1} > 0,$$

We have

$$\begin{aligned} \mathcal{L}V(t, X_t) \leq & - \left\{ \alpha_1 - \frac{1}{2}(\sigma^2 + a + \beta_1 + 2) - \frac{L + M}{2}\gamma_1 \right\} x^2 \\ & - \left\{ \frac{1}{4}(a\beta_1 - c_1 - 2\beta_1 - 6) - \frac{\lambda(L + M)(1 - \gamma_2) + L(\lambda + 2)}{2(1 - \gamma_2)}\gamma_1 \right\} y^2 \\ & - \left\{ \frac{a\beta_1 - c_1 - 2\beta_1}{4\beta_1} - \frac{(L + M)(1 - \gamma_2) + M(\lambda + 2)}{2(1 - \gamma_2)}\gamma_1 \right\} z^2. \end{aligned} \tag{3.3}$$

Thus, in view of (3.3), one can conclude that $\mathcal{L}V(t, X_t)$ satisfies the condition (ii) of Lemma 2 as:

$$\mathcal{L}V(t, X_t) \leq -D_1(x^2 + y^2 + z^2), \text{ for some } D_1 > 0, \tag{3.4}$$

provided that

$$\gamma_1 < \min \left\{ \frac{2\alpha_1 - \sigma^2 - a - \beta_1 - 2}{2(L + M)}, \frac{(a\beta_1 - c_1 - 2\beta_1 - 6)(1 - \gamma_2)}{4\lambda(L + M)(1 - \gamma_2) + 4L(\lambda + 2)}, \frac{(a\beta_1 - c_1 - 2\beta_1)(1 - \gamma_2)}{4\beta_1(L + M)(1 - \gamma_2) + 4\beta_1(\lambda + 2)M} \right\}.$$

Next, we shall show that the assumption (i) of Lemma 2 is satisfied.

Since $\int_{-r(t)}^0 \int_{t+s}^t y^2(\vartheta) d\vartheta ds$ and $\int_{-r(t)}^0 \int_{t+s}^t z^2(\vartheta) d\vartheta ds$ are non-negative and by using the assumption (iii) of Theorem 1, we obtain

$$\begin{aligned} V(t, X_t) &\geq \lambda \int_0^x \psi(\xi) d\xi + \psi(x)y + \frac{1}{2}\lambda ay^2 + \frac{1}{2}\beta_1 y^2 + \lambda yz + \frac{1}{2}z^2 + xz + x^2 \\ &= \frac{1}{2\beta_1}(\beta_1 y + \psi(x))^2 + \left(\lambda y + \frac{z}{2}\right)^2 + \left(x + \frac{z}{2}\right)^2 + \frac{1}{2}\lambda(a - 2\lambda)y^2 \\ &\quad + \frac{2}{\beta_1 y^2} \int_0^x \psi(\xi) \left\{ \int_0^y (\lambda\beta_1 - \psi'(\xi))\eta d\eta \right\} d\xi; \quad y \neq 0. \end{aligned} \tag{3.5}$$

Now we recall that:

$$a - 2\lambda = \frac{a\beta_1 - c_1}{2\beta_1} > 0,$$

and

$$\lambda\beta_1 - \psi'(\xi) \geq \frac{a\beta_1 + c_1}{4} - \frac{c_1}{2} = \frac{a\beta_1 - c_1}{4} > 0; \quad \text{by condition (ii) of Theorem 1.}$$

Then, we get

$$\begin{aligned} &\frac{2}{\beta_1 y^2} \int_0^x \psi(\xi) \left\{ \int_0^y (\lambda\beta_1 - \psi'(\xi))\eta d\eta \right\} d\xi \\ &\geq \frac{a\beta_1 - c_1}{4\beta_1} \int_0^y \psi(\xi) d\xi, \end{aligned}$$

which together with (3.5), implies the following inequality

$$\begin{aligned} V(t, X_t) &\geq \frac{1}{2\beta_1}(\beta_1 y + \psi(x))^2 + \left(x + \frac{z}{2}\right)^2 + \frac{1}{2}\lambda \left(\frac{a\beta_1 - c_1}{2\beta_1}\right)y^2 \\ &\quad + \left(\lambda y + \frac{z}{2}\right)^2 + \frac{a\beta_1 - c_1}{4\beta_1} \int_0^y \psi(\xi) d\xi. \end{aligned}$$

Hence, we can see that

$$V(t, X_t) \geq D_2(x^2 + y^2 + z^2), \quad \text{for some } D_2 > 0. \tag{3.6}$$

In view of the assumptions $\psi(x) \leq \alpha_2 x$, $\psi(y) \leq \beta_2 y$ from the conditions (i) and (iii) of Theorem 1 respectively; and the inequality $uv \leq \frac{1}{2}(u^2 + v^2)$, then we can

write from (3.2) that

$$\begin{aligned} V(t, X_t) \leq & \lambda \int_0^x \alpha_1 \xi d\xi + \frac{\alpha_2}{2}(x^2 + y^2) + \frac{1}{2}\lambda a y^2 + \int_0^y \beta_2 \eta d\eta + \frac{\lambda}{2}(y^2 + z^2) \\ & + \frac{1}{2}z^2 + \frac{1}{2}(x^2 + z^2) + x^2 + \mu \int_{t-r(t)}^t \left(\vartheta - t + r(t) \right) y^2(\vartheta) d\vartheta \\ & + \nu \int_{t-r(t)}^t \left(\vartheta - t + r(t) \right) z^2(\vartheta) d\vartheta + \frac{1}{2}\sigma^2 \int_{t-h}^t x^2(s) ds. \end{aligned}$$

Since $r(t) \leq \gamma_1$, then it follows that

$$\begin{aligned} V(t, X_t) \leq & \frac{1}{2} \left\{ (\lambda + 1)\alpha_2 + 3 + \sigma^2 h \right\} \|x\|^2 + \frac{1}{2} \left\{ \alpha_2 + \beta_2 + \lambda(a + 1) + \mu\gamma_1^2 \right\} \|y\|^2 \\ & + \frac{1}{2} \left\{ \lambda + 2 + \nu\gamma_1^2 \right\} \|z\|^2. \end{aligned}$$

Then there exists a positive constant D_3 such that

$$V(t, X_t) \leq D_3(x^2 + y^2 + z^2), \quad D_3 > 0. \quad (3.7)$$

Therefore from (3.6) and (3.7), we note that $V(t, X_t)$ satisfies condition (i) of Lemma 2.

Thus all the assumptions of Lemma 2 are satisfied, so the zero solution of (1.1) is stochastically asymptotically stable.

This completes the proof of Theorem 1.

4. EXAMPLE

In this section, we give an example to show the applicability of the result obtained and for illustrations.

As an application of Theorem 1, we consider the third-order stochastic delay differential equation as the following form:

$$\begin{aligned} \ddot{x}(t) + 12\ddot{x}(t) + 8\dot{x}(t - r(t)) + \sin(\dot{x}(t - r(t))) + 24x(t - r(t)) \\ + \frac{x(t - r(t))}{1 + x^2(t - r(t))} + 3x(t)\dot{\omega}(t) = 0. \end{aligned} \quad (4.1)$$

Its equivalent system is given as:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -12z - (8y + \sin y) - \left(24x + \frac{x}{1 + x^2} \right) + \int_{t-r(t)}^t \{8 + \cos y(s)\} z(s) ds \\ &+ \int_{t-r(t)}^t \left\{ 24 + \frac{1 - x^2(s)}{(1 + x^2(s))^2} \right\} y(s) ds - 2x(t)\dot{\omega}(t). \end{aligned}$$

By comparing the above differential system (3.1) and taking into account the assumptions of Theorem 1.

The path of the function $\sin y$ is shown in Figure 1. It follows that

$$a = 12, \quad \phi(y) = 8y + \sin y, \quad \frac{\phi(y)}{y} - 8 = \frac{\sin y}{y},$$

then we find

$$-1 \leq \frac{\phi(y)}{y} - 8 \leq 1.$$

Hence, we have

$$\beta_1 = 7, \quad \beta_2 = 9, \quad \text{also } |\phi'(y)| = |8 + \cos y| \leq 9 = M.$$

The behaviour of $\frac{x}{1+x^2}$ is shown in Figure 2. Therefore we obtain

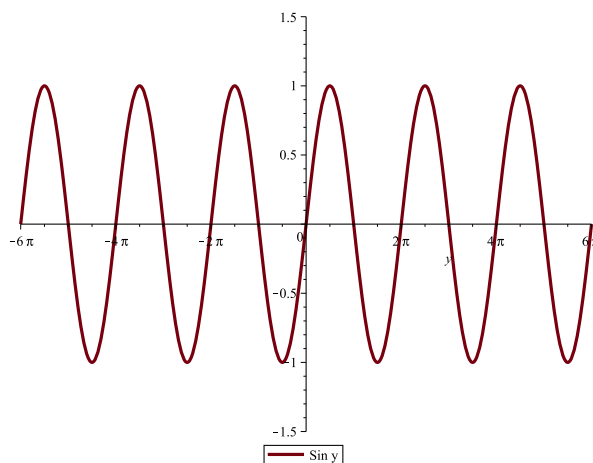


FIGURE 1. The behaviour of the function $\sin y$

$$\psi(x) = 24x + \frac{x}{1+x^2}, \quad \frac{\psi(x)}{x} - 24 = \frac{1}{1+x^2}, \quad \text{then } 0 \leq \frac{\psi(x)}{x} - 24 \leq 1.$$

It tends to

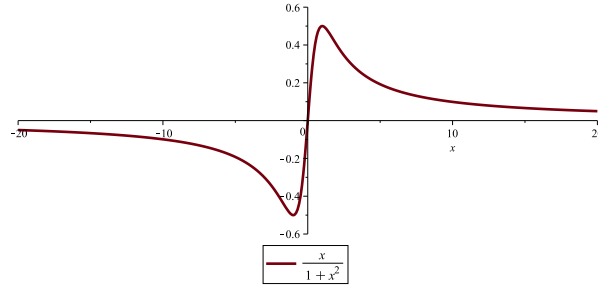
$$\alpha_1 = 24, \quad \alpha_2 = 25, \quad \psi'(x) = 24 + \frac{1-x^2}{(1+x^2)^2}, \quad \text{since}$$

$$\left| \frac{1-x^2}{(1+x^2)^2} \right| \leq 1, \quad \text{then } |\psi'(x)| \leq 25 = L.$$

Therefore $\sup\{|\psi'(x)|\} = 25$, then we obtain $c_1 = 50$, and $\lambda = \frac{a\beta_1 + c_1}{4\beta_1} = \frac{67}{14}$.

It is obvious that

$$a\beta_1 - c_1 = 34 \quad \text{and} \quad 2\beta_1 + 6 = 20.$$

FIGURE 2. The behaviour of the function $\frac{x}{1+x^2}$

Therefore

$$a\beta_1 - c_1 > 2\beta_1 + 6, \text{ and } 2\alpha_1 - a - \beta_1 - 2 = 27 > \sigma^2 = 9 \text{ (since } \sigma = 3\text{)}.$$

Thus, the above estimates show all the assumptions of Theorem 1 hold, so we can prove that

$$\begin{aligned} \mathcal{L}V(t, X_t) &\leq (9 - 17\gamma_1)x^2 - \left\{ \frac{14}{4} - \frac{1}{2}(163 + 2\mu)\gamma_1 \right\} y^2 \\ &\quad - \left\{ \frac{10}{14} - (17 + \nu)\gamma_1 \right\} z^2 + \left\{ 85 - \mu(1 - \gamma_2) \right\} \int_{t-r(t)}^t y^2(\vartheta) d\vartheta \\ &\quad + \left\{ 30.5 - \nu(1 - \gamma_2) \right\} \int_{t-r(t)}^t z^2(\vartheta) d\vartheta. \end{aligned}$$

Let us choose

$$\mu = \frac{85}{1 - \gamma_2} > 0 \text{ and } \nu = \frac{30.5}{1 - \gamma_2} > 0,$$

where $0 < \gamma_2 < 1$.

Consequently, it follows for a positive constant ρ_1 that

$$\mathcal{L}V(t, X_t) \leq -\rho_1(x^2 + y^2 + z^2), \quad (4.2)$$

provided that

$$\gamma_1 < \min \left\{ \frac{9}{34}, \frac{7(1 - \gamma_2)}{326(1 - \gamma_2) + 340}, \frac{5(1 - \gamma_2)}{14\{17(1 - \gamma_2) + 30.5\}} \right\}.$$

Also, we can see that

$$\begin{aligned} V(t, X_t) &\geq \frac{1}{14} \left(7y + \psi(x) \right)^2 + \left(\frac{67}{14}y + \frac{z}{2} \right)^2 + \left(x + \frac{z}{2} \right)^2 + 5.8 y^2 \\ &\quad + \frac{17}{14} \int_0^y \psi(\xi) d\xi. \end{aligned}$$

Then, there exists a positive constant ρ_2 such that

$$V(t, X_t) \geq \rho_2(x^2 + y^2 + z^2). \quad (4.3)$$

As well it can be shown that:

$$V(t, X_t) \leq \frac{157}{2} \|x\|^2 + \frac{1}{2} \left\{ 96 + \frac{25(\lambda + 2)}{2(1 - \gamma_2)} \gamma_1^2 \right\} \|y\|^2 + \frac{1}{2} \left\{ \frac{95}{14} + \frac{9(\lambda + 2)}{2(1 - \gamma_2)} \gamma_1^2 \right\} \|z\|^2.$$

Hence there exists a positive constant ρ_3 satisfying

$$V(t, X_t) \leq \rho_3(x^2 + y^2 + z^2). \quad (4.4)$$

Now from the results (4.2), (4.3) and (4.4), we note that all the conditions of Lemma 2 are satisfied, then the zero solution of (4.1) is stochastically asymptotically stable.

5. ACKNOWLEDGEMENT

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REFERENCES

- [1] A. M. A. Abou-El-Ela, A. I. Sadek, A. M. Mahmoud, and E. S. Farghaly, "New stability and boundedness results for solutions of a certain third-order nonlinear stochastic differential equation." *Asian Journal of Mathematics and Computer Research*, vol. 5, no. 1, pp. 60–70, 2015.
- [2] A. M. A. Abou-El-Ela, A. I. Sadek, A. M. Mahmoud, and E. S. Farghaly, "Stability of solutions for certain third-order nonlinear stochastic delay differential equation." *Ann. of Appl. Math.*, vol. 31, no. 3, pp. 253–261, 2015.
- [3] A. T. Ademola, "Stability, boundedness and uniqueness of solutions to certain third-order stochastic delay differential equations." *Differ. Equ. Control Processes*, vol. 2, pp. 24–50, 2017.
- [4] A. T. Ademola and P. O. Arawomo, "Uniform stability and boundedness of solutions of nonlinear delay differential equations of the third-order." *Mathematical Journal Okayama University*, vol. 55, pp. 157–166, 2013.
- [5] A. T. Ademola, B. S. Ogundare, M. O. Ogundiran, and O. Adesina, "Stability; boundedness and existence of periodic solutions to certain third-order delay differential equations with multiple deviating arguments." *International Journal of Differential Equations*, vol. 2015, pp. 1–12, 2015, doi: [10.1155/2015/213935](https://doi.org/10.1155/2015/213935).
- [6] A. Friedman, *Stochastic Differential Equations and Applications*. New York: Academic Press, 1976.
- [7] I. I. Gikhman and A. V. Skorokhod, *Stochastic Differential Equations*. Berlin, Heidelberg, New York: Springer-Verlag, 1972.
- [8] J. R. Graef and C. Tunç, "Global asymptotic stability and boundedness of certain multi-delay functional differential equations of third-order." *Math. Meth. Appl. Sci.*, vol. 38, no. 17, pp. 3747–3752, 2015, doi: [10.1002/mma.3314](https://doi.org/10.1002/mma.3314).
- [9] R. Z. Khasminskii, *Stochastic Stability of Differential Equations*. Germantown: Sijthoff and Noordhoff, 1980.
- [10] G. J. Kushner, *Stochastic Stability and Control*. Moscow: World Publ. Co., 1969.

- [11] A. M. Mahmoud, "On the asymptotic stability of solutions for a certain non-autonomous third-order delay differential equation." *British Journal of Mathematics and Computer Science*, vol. 16, no. 3, pp. 1–12, 2016, doi: [10.9734/BJMCS/2016/23936](https://doi.org/10.9734/BJMCS/2016/23936).
- [12] X. Mao, "Existence and uniqueness of solutions of stochastic integral equations." *J. Fuzhou Univ.*, vol. 4, pp. 41–50, 1983.
- [13] X. Mao, *Stochastic Differential Equations and Their Applications*. Chichester: Horwood Publishing, 1997.
- [14] X. Mao, *Stochastic Differential Equations and Applications (2nd Ed.)*. United Kingdom: Woodhead Publishing Limited, 2007.
- [15] S. E. A. Mohammed, *Stochastic Functional Differential Equations*. Boston, London, Melbourne: Pitman Advanced Publishing Program, 1984.
- [16] B. Oksendal, *Stochastic Differential Equations, An Introduction with Applications*. New York: Springer-Verlag, 2000.
- [17] M. O. Omeike, "New results on the stability of solution of some non-autonomous delay differential equations of the third-order." *Differ. Equ. Control Processes*, vol. 1, pp. 18–29, 2010.
- [18] L. Oudjedi, D. Beldjerd, and M. Remili, "On the stability of solutions for non-autonomous delay differential equations of third-order." *Differ. Equ. Control Processes*, vol. 1, pp. 22–34, 2014.
- [19] S. Polidoro and M. A. Ragusa, "Hölder regularity for solutions of ultraparabolic equations in divergence form." *Potential Analysis*, vol. 14, no. 4, pp. 341–350, 2001, doi: [10.1023/A:1011261019736](https://doi.org/10.1023/A:1011261019736).
- [20] M. Remili and D. Beldjerd, "Boundedness and stability results for a kind of third-order delay differential equations." *Applications and Applied Mathematics: An International Journal*, vol. 10, no. 2, pp. 772–782, 2015.
- [21] M. Remili and L. Oudjedi, "Stability and boundedness of the solutions of nonautonomous third-order differential equations with delay." *Acta Univ. Palacki. Olomuc., Far. rer. nat., Mathematica*, vol. 35, no. 2, pp. 139–147, 2014.
- [22] A. Rontó, M. Rontó, and I. Varga, "Partially solved differential system with two point nonlinear boundary conditions." *Miskolc Mathematical Notes*, vol. 18, no. 2, pp. 1001–1014, 2017, doi: [10.18514/MMN.2017.2491](https://doi.org/10.18514/MMN.2017.2491).
- [23] A. I. Sadek, "On the stability of solutions of some non-autonomous delay differential equations of the third-order." *Asymptotic Anal.*, vol. 43, pp. 1–7, 2005.
- [24] P. Shekhar, V. Dharmiah, and G. Mahadevi, "Stability and boundedness of solutions of delay differential equations of third-order." *IOSR J. Math.*, vol. 5, pp. 9–13, 2013.
- [25] C. Tunç, "On the stability of solutions for non-autonomous delay differential equations of third-order." *Iran J. Sci. Technol. A*, vol. 32, pp. 261–273, 2008.
- [26] C. Tunç, "On the stability and boundedness of solutions to third-order nonlinear differential equations with retarded argument." *Nonlinear Dynam.*, vol. 57, no. 1-2, pp. 97–106, 2009, doi: [10.1007/s11071-008-9423-6](https://doi.org/10.1007/s11071-008-9423-6).
- [27] C. Tunç, "Bound of solutions to third-order nonlinear differential equations with bounded delay." *J. Franklin Ins.*, vol. 347, no. 2, pp. 415–425, 2010, doi: [10.1016/j.jfranklin.2009.05.001](https://doi.org/10.1016/j.jfranklin.2009.05.001).
- [28] C. Tunç, "Stability and bounded of solutions to non-autonomous delay differential equations of third-order." *Nonlinear Dynam.*, vol. 62, no. 4, pp. 945–953, 2010, doi: [10.1007/s11071-010-9776-5](https://doi.org/10.1007/s11071-010-9776-5).
- [29] C. Tunç, "On the stability and boundedness of certain third-order nonautonomous differential equations of retarded type." *Proyecciones*, vol. 34, no. 2, pp. 147–159, 2015, doi: [10.4067/S0716-09172015000200004](https://doi.org/10.4067/S0716-09172015000200004).
- [30] R. Wu and X. Mao, "Existence and uniqueness of solutions of stochastic differential equations." *Stochastics*, vol. 11, pp. 19–32, 1983.

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IDENTITIES AND CONGRUENCES INVOLVING THE GEOMETRIC POLYNOMIALS

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Abstract. In this paper, we investigate the umbral representation of the geometric polynomials $\mathbf{w}_x^n := w_n(x)$ to derive some properties involving these polynomials. Furthermore, for any prime number p and any polynomial f with integer coefficients, we show $(f(\mathbf{w}_x))^p \equiv f(\mathbf{w}_x) \pmod{p}$ and we give other curious congruences.

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1. INTRODUCTION

The geometric numbers are quantities arising from enumerative combinatorics and have nice number-theoretic properties. In combinatorics, the n -th geometric number (named also the n -th ordered Bell number) counts the number of ways to partition the set $[n] := \{1, \dots, n\}$ into ordered subsets [2, 3, 6]. The geometric polynomials are defined by $w_n(x) = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} k! x^k$ and satisfy the recurrence relation $(x+1)w_n(x) = x \sum_{j=0}^n \binom{n}{j} w_j(x)$, $n \geq 1$, [9], where $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ is the (n, k) -th Stirling number of the second kind [2, 26]. These polynomials have attracted attention from many researchers, see for instance [9, 10, 15–17]. For $x = 1$ we obtain the geometric numbers $w_n := w_n(1) = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} k!$, for more information about these numbers, see [6–8, 11, 12, 14, 28, 29]. More generally, let $w_n(x; r, s)$ be the n -th (r, s) -geometric polynomial defined by

$$w_n(x; r, s) = \sum_{k=0}^n \left\{ \begin{smallmatrix} n+r \\ k+r \end{smallmatrix} \right\}_r (k+s)! x^k.$$

This polynomial generalizes the geometric polynomial $w_n(x) = w_n(x; 0, 0)$ and the polynomial $w_n(x; r, r)$ introduced by Mező [18]. Here, $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_r$ denotes the (n, k) -th r -Stirling number of the second kind [4]. One can see easily that

$$\begin{aligned} w_0(x; r, s) &= s!, \\ w_1(x; r, s) &= s!(r + (s+1)x), \end{aligned}$$

$$w_2(x, r, s) = s!(r^2 + (2r + 1)(s + 1)x + (s + 1)(s + 2)x^2).$$

We note that this generalization can be viewed as a particular case of that defined by Kargin et al. [16]. As it shown below, these polynomials are also linked to the absolute r -Stirling numbers of first kind denoted by $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_r$.

Recall that the r -Stirling numbers can be defined by [4, 26]

$$(x)_n = \sum_{k=0}^n (-1)^{n-k} \left[\begin{smallmatrix} n+r \\ k+r \end{smallmatrix} \right]_r (x+r)^k \text{ and } (x+r)^n = \sum_{k=0}^n \left\{ \begin{smallmatrix} n+r \\ k+r \end{smallmatrix} \right\}_r (x)_k,$$

where $(\alpha)_n = \alpha \cdots (\alpha - n + 1)$ if $n \geq 1$, $(\alpha)_0 = 1$.

This work is motivated by application of the umbral calculus method to determine identities and congruences involving Bell numbers and polynomials in the works of Gessel [13], Sun et al. [27], Mező et al. [19] and Benyattou et al. [1]. In this paper, we will talk about identities and congruences involving the (r, s) -geometric polynomials based on the geometric umbra defined by $\mathbf{w}_x^n := w_n(x)$. For more information about umbral calculus, see [5, 13, 22–25].

2. IDENTITIES INVOLVING THE (r, s) -GEOMETRIC POLYNOMIALS

The above recurrence relation is equivalent to $(x+1)\mathbf{w}_x^n = x(\mathbf{w}_x + 1)^n, n \geq 1$. Furthermore, we have

Proposition 1. *Let f be a polynomial and r, s be non-negative integers. Then*

$$\begin{aligned} (x+1)f(\mathbf{w}_x + r) &= xf(\mathbf{w}_x + r + 1) + f(r), \\ (\mathbf{w}_x + r)_{n+r} &= (n+r)!x^n(x+1)^r, \\ (\mathbf{w}_x + r - s)^n(\mathbf{w}_x)_s &= x^s w_n(x; r, s), \\ (\mathbf{w}_x + r)^n(\mathbf{w}_x + s)_s &= (x+1)^s w_n(x; r, s). \end{aligned}$$

Proof. It suffices to show the first identity for $f(x) = x^n$. For $r = 0$ we have $(x+1)\mathbf{w}_x^n - x(\mathbf{w}_x + 1)^n = \delta_{(n=0)}$. Assume it is true for $r - 1$, then if we set

$$h_n(r) := (x+1)(\mathbf{w}_x + r)^n - x(\mathbf{w}_x + r + 1)^n$$

we obtain $h_n(r) = \sum_{j=0}^n \binom{n}{j} h_j(r-1) = \sum_{j=0}^n \binom{n}{j} (r-1)^j = r^n$, which concludes the induction step. For the other identities, since $(x)_n = \sum_{k=0}^n (-1)^{n-k} \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] x^k$ and $(x)_n$ is a sequence of binomial type [20, 23], we obtain

$$(\mathbf{w}_x + r)_{n+r} = \sum_{j=0}^{n+r} \binom{n+r}{j} (r)_j (\mathbf{w}_x)_{n+r-j} = (n+r)!x^n(x+1)^r.$$

So, the polynomials $x^s w_n(x; r, s)$ and $(x+1)^s w_n(x, r, s)$ must be, respectively,

$$\sum_{j=0}^n \left\{ \begin{smallmatrix} n+r \\ j+r \end{smallmatrix} \right\}_r (\mathbf{w}_x)_{j+s} = \sum_{j=0}^n \left\{ \begin{smallmatrix} n+r \\ j+r \end{smallmatrix} \right\}_r (\mathbf{w}_x - s)_j (\mathbf{w}_x)_s = (\mathbf{w}_x + r - s)^n (\mathbf{w}_x)_s,$$

$$\sum_{j=0}^n \left\{ \begin{matrix} n+r \\ j+r \end{matrix} \right\}_r (\mathbf{w}_x + s)_{j+s} = \sum_{j=0}^n \left\{ \begin{matrix} n+r \\ j+r \end{matrix} \right\}_r (\mathbf{w}_x)_j (\mathbf{w}_x + s)_s = (\mathbf{w}_x + r)^n (\mathbf{w}_x + s)_s.$$

□

The last two identities of Proposition 1 lead to:

Corollary 1. *Let r, s be non-negative integers and f be a polynomial. Then*

$$(x+1)^s f(\mathbf{w}_x + r - s)(\mathbf{w}_x)_s = x^s f(\mathbf{w}_x + r)(\mathbf{w}_x + s)_s.$$

Proposition 2. *Let $\mathcal{P}_n$ and $\mathcal{T}_n$ be the polynomials*

$$\mathcal{P}_n(x; r) = \sum_{j=0}^n (-1)^j \binom{j+r}{r} x^{n-j} \quad \text{and} \quad \mathcal{T}_n(x; r) = \sum_{j=0}^n \binom{n+r}{j+r} x^j.$$

Then $(\mathbf{w}_x - r - 1)_n = n! \mathcal{P}_n(x; r)$ and $(\mathbf{w}_x + n + r)_n = n! \mathcal{T}_n(x; r)$.

Proof. It suffices to observe that

$$\begin{aligned} (\mathbf{w}_x - r - 1)_n &= \sum_{j=0}^n \binom{n}{j} (-r-1)_j (\mathbf{w}_x)_{n-j} = n! \sum_{j=0}^n (-1)^j \binom{j+r}{r} x^{n-j}, \\ (\mathbf{w}_x + n + r)_n &= \sum_{j=0}^n \binom{n}{j} (n+r)_{n-j} (\mathbf{w}_x)_j = n! \sum_{j=0}^n \binom{n+r}{j+r} x^j. \end{aligned}$$

□

The following theorem can be served to derive several identities and congruences for the (r, s) -geometric polynomials.

Theorem 1. *Let m, s be non-negative integers and f be a polynomial. Then*

$$(x+1)^m f(\mathbf{w}_x) - x^m f(\mathbf{w}_x + m) = \sum_{k=0}^{m-1} f(k) (x+1)^{m-1-k} x^k, \quad m \geq 1.$$

Proof. Set $f(x) = \sum_{k=0}^n a_k x^k$ and use Proposition 1 to obtain

$$(x+1)f(\mathbf{w}_x) - xf(\mathbf{w}_x + 1) = f(0) + \sum_{k=0}^n a_k \left((x+1)\mathbf{w}_x^k - x(\mathbf{w}_x + 1)^k \right) = f(0).$$

So, the identity is true for $m = 1$. Assume it is true for m . Then

$$\begin{aligned} (x+1)^{m+1} f(\mathbf{w}_x) &= (x+1) \left(\sum_{k=0}^{m-1} (x+1)^{m-1-k} x^k f(k) + x^m f(\mathbf{w}_x + m) \right) \\ &= \sum_{k=0}^{m-1} (x+1)^{m-k} x^k f(k) + x^m (x+1) f(\mathbf{w}_x + m) \end{aligned}$$

and since $(x+1)f(\mathbf{w}_x+m) - xf(\mathbf{w}_x+m+1) = f(m)$, we can write

$$\begin{aligned} (x+1)^{m+1}f(\mathbf{w}_x) &= \sum_{k=0}^{m-1} (x+1)^{m-k} x^k f(k) + x^m \left(xf(\mathbf{w}_x+m+1) + f(m) \right) \\ &= \sum_{k=0}^{m-1} (x+1)^{m-k} x^k f(k) + x^m f(m) + x^{m+1} f(\mathbf{w}_x+m+1) \\ &= \sum_{k=0}^m (x+1)^{m-k} x^k f(k) + x^{m+1} f(\mathbf{w}_x+m+1) \end{aligned}$$

which concludes the induction step. $\square$

We note that for $f(x) = x^n$ and $x = 1$ in Theorem 1 we obtain Proposition 3.3 given in [8].

Corollary 2. *For any polynomial f there holds*

$$f(\mathbf{w}_x) = \frac{1}{1+x} \sum_{k \geq 0} f(k) \left(\frac{x}{1+x} \right)^k, \quad x > -\frac{1}{2}.$$

Proof. For $m = 1$ in Theorem 1, when we replace $f(x)$ by $f(x+r)$ we get the identity $f(r) = (x+1)f(\mathbf{w}_x+r) - xf(\mathbf{w}_x+r+1)$. Then

$$\begin{aligned} RHS &= \lim_{n \rightarrow \infty} \frac{1}{1+x} \sum_{k=0}^n \left(\frac{x}{1+x} \right)^k \left((x+1)f(\mathbf{w}_x+k) - xf(\mathbf{w}_x+k+1) \right) \\ &= \lim_{n \rightarrow \infty} (f(\mathbf{w}_x) - \left(\frac{x}{1+x} \right)^{n+1} f(\mathbf{w}_x+n+1)) = f(\mathbf{w}_x) \end{aligned}$$

which completes the proof. $\square$

Corollary 3. *Let n, r, s be non-negative integers.*

For $f(x) = (x+r)^n(x+s)_s$ or $(x+r-s)^n(x)_s$ in Corollary 2 we obtain

$$w_n(x; r, s) = \frac{s!}{(1+x)^{s+1}} \sum_{k \geq 0} \binom{k+s}{s} (k+r)^n \left(\frac{x}{1+x} \right)^k, \quad x > -\frac{1}{2}.$$

Corollary 4. *For any integers $r \geq 0$, $s \geq 0$ and $n \geq 1$ the polynomial $w_n(x, r, s+r)$ has only real non-positive zeros.*

Proof. From Corollary 3 we may state

$$x^r(x+1)^s w_{n+1}(x; r, s+r) = x \frac{d}{dx} \left(x^r(x+1)^{s+1} w_n(x; r, s+r) \right)$$

and using the recurrence relation of r -Stirling numbers we conclude that this identity remains true for all real number x . So, by induction on n , it follows that $w_n(x; r, s + r)$, $n \geq 1$, has only real non-positive zeros. $\square$

Lemma 1. *For any non-negative integers $n \geq 2$ there holds*

$$(1+x)w_{n-1}(x) = \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (k-1)!x^k.$$

Proof. From the definition of geometric polynomials, we have

$$\begin{aligned} (1+x)w_{n-1}(x) &= \sum_{k=1}^{n-1} \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} k!x^k + \sum_{k=1}^{n-1} \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} k!x^{k+1} \\ &= \sum_{k=1}^n \left(\left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} \right) (k-1)!x^k \\ &= \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (k-1)!x^k. \end{aligned}$$

$\square$

For more explicit formulae for geometric polynomials, see for example [15].

Proposition 3. *Let n, r, s be non-negative integers. Then*

$$\log \left(1 + \sum_{n \geq 1} \frac{w_n(x; r, s)}{s!} \frac{t^n}{n!} \right) = (r + (s+1)x)t + (s+1)(x+1) \sum_{n \geq 2} w_{n-1}(x) \frac{t^n}{n!}.$$

In particular, for $r = s = 0$ we get

$$\log \left(1 + \sum_{n \geq 1} w_n(x) \frac{t^n}{n!} \right) = xt + (x+1) \sum_{n \geq 2} w_{n-1}(x) \frac{t^n}{n!}.$$

Proof. One can verify easily that the exponential generating function of the polynomials $w_n(x; r, s)$ is to be $s! \exp(rt)(1 - x(\exp(t) - 1))^{-s-1}$. Then, upon using this generating function and the last Lemma, we can write

$$\begin{aligned} LHS &= rt - (s+1) \ln(1 - x(\exp(t) - 1)) \\ &= rt + (s+1) \sum_{k \geq 1} \frac{x^k}{k} (\exp(t) - 1)^k \\ &= rt + (s+1) \sum_{k \geq 1} (k-1)!x^k \sum_{n \geq k} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{t^n}{n!} \end{aligned}$$

$$\begin{aligned}
&= rt + (s+1)xt + (s+1) \sum_{n \geq 2} \frac{t^n}{n!} \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (k-1)! x^k \\
&= (r + (s+1)x)t + (s+1)(x+1) \sum_{n \geq 2} w_{n-1}(x) \frac{t^n}{n!}.
\end{aligned}$$

□

3. CONGRUENCES INVOLVING THE (R,S)-GEOMETRIC POLYNOMIALS

In this section, we give some congruences involving the (r, s) -geometric polynomials. Let $\mathbb{Z}_p$ be the ring of p -adic integers and for two polynomials $f(x), g(x) \in \mathbb{Z}_p[x]$, the congruence $f(x) \equiv g(x) \pmod{p\mathbb{Z}_p[x]}$ means that the corresponding coefficients of $f(x)$ and $g(x)$ are congruent modulo p . This congruence will be used later as $f(x) \equiv g(x)$ and we will use $a \equiv b$ instead $a \equiv b \pmod{p}$.

Proposition 4. *Let n, r, s be non-negative integers and p be a prime number. Then, for any polynomial f with integer coefficients there holds*

$$\sum_{k=0}^{p-1} f(k)(x+1)^{p-1-k} x^k \equiv f(\mathbf{w}_x).$$

In particular, for $f(x) = (x+r-s)^n(x)_s$ or $(x+r)^n(x+s)_s$ we get, respectively,

$$\begin{aligned}
\sum_{k=0}^{p-1} (r-s+k)^n(k)_s(x+1)^{p-1-k} x^k &\equiv x^s w_n(x; r, s), \\
\sum_{k=0}^{p-1} (r+k)^n(s+k)_s(x+1)^{p-1-k} x^k &\equiv (x+1)^s w_n(x; r, s).
\end{aligned}$$

Proof. For $m = p$ be a prime number, Theorem 1 implies

$$LHS = (x+1)^p f(\mathbf{w}_x) - x^p f(\mathbf{w}_x + p) \equiv (x^p + 1)f(\mathbf{w}_x) - x^p f(\mathbf{w}_x) = f(\mathbf{w}_x).$$

For the particular cases, use Proposition 1. □

Corollary 5. *Let n, r, s, m, q be non-negative integers and p be a prime number. Then, for any polynomials f and g with integer coefficients there holds*

$$(f(\mathbf{w}_x))^p g(\mathbf{w}_x) \equiv f(\mathbf{w}_x)g(\mathbf{w}_x).$$

In particular, we have $w_{mp+q}(x; r, s) \equiv w_{m+q}(x; r, s)$.

Proof. By Fermat's little theorem and by twice application of Proposition 4 we may state

$$LHS \equiv \sum_{k=0}^{p-1} (f(k))^p g(k)(x+1)^{p-1-k} x^k \equiv \sum_{k=0}^{p-1} f(k)g(k)(x+1)^{p-1-k} x^k = RHS.$$

□

We note that, for $f(x) = x^m$, $g(x) = x^q$ and $x = 1$, Corollary 5 may be seen as a particular case of Theorem 3.1 given in [8].

Corollary 6. *For any non-negative integers $m \geq 1, n, r, s$ and any prime number p , there hold*

$$\begin{aligned}(x+1)^{s+1}(w_{m(p-1)}(x; r, s) - s!) &\equiv -(s-r')_s(x+1)^{r'}x^{p-r'}, \quad r' \neq 0, \\ (x+1)^{s+1}(w_{m(p-1)}(x; r, s) - s!) &\equiv -s!(x^p+1), \quad r' = 0,\end{aligned}$$

where $r' \equiv r$ and $r' \in \{0, 1, \dots, p-1\}$.

Proof. Set $n = m(p-1)$ in Proposition 4. If $r' \neq 0$ we get

$$\begin{aligned}(x+1)^s w_{m(p-1)}(x; r, s) &\equiv \sum_{k=0}^{p-1} (r'+k)^{m(p-1)} (s+k)_s (x+1)^{p-1-k} x^k \\ &\equiv \sum_{k=0, r'+k \neq p}^{p-1} (s+k)_s (x+1)^{p-1-k} x^k \\ &= \sum_{k=0}^{p-1} (s+k)_s (x+1)^{p-1-k} x^k \\ &\quad - (s-r'+p)_s (x+1)^{r'-1} x^{p-r'} \\ &\equiv (x+1)^s w_0(x; 0, s) - (s-r')_s (x+1)^{r'-1} x^{p-r'} \\ &\equiv s!(x+1)^s - (s-r')_s (x+1)^{r'-1} x^{p-r'}\end{aligned}$$

and if $r' = 0$ we get

$$\begin{aligned}(x+1)^{s+1} w_{m(p-1)}(x; r, s) &\equiv \sum_{k=1}^{p-1} (s+k)_s (x+1)^{p-k} x^k \\ &= \sum_{k=0}^{p-1} (s+k)_s (x+1)^{p-k} x^k - s!(x+1)^p \\ &= (x+1)^{s+1} w_0(x; 0, s) - s!(x+1)^p \\ &= s!(x+1)^{s+1} - s!(x^p+1).\end{aligned}$$

which complete the proof. □

Remark 1. For $r = s = m-1 = 0$ in Corollary 6 or $n = p$ in Lemma 1 we obtain $(x+1)w_{p-1}(x) \equiv x - x^p$ which gives for $x = 1$ the known congruence $w_{p-1} \equiv 0$, see [8].

Now, we give some curious congruences on (r, s) -geometric polynomials and on $(r_1, \dots, r_q)$ -geometric polynomials defined below.

Theorem 2. *For any integers $n, m, r, s \geq 0$ and any prime number $p \nmid m$, there holds*

$$\sum_{k=1}^{p-1} \frac{w_{n+k}(x; r, s)}{(-m)^k} \equiv (-m)^n (w_{p-1}(x; r+m, s) - s!).$$

Proof. Upon using the identity $x^s w_n(x; r, s) = (\mathbf{w}_x + r - s)^n (\mathbf{w}_x)_s$ and the known congruence $(-m)^{-k} \equiv \binom{p-1}{k} m^{p-1-k}$ we obtain

$$\begin{aligned} x^s LHS &\equiv \sum_{k=0}^{p-1} \binom{p-1}{k} m^{p-1-k} (\mathbf{w}_x + r - s)^{n+k} (\mathbf{w}_x)_s \\ &= (\mathbf{w}_x + r - s)^n (\mathbf{w}_x + r + m - s)^{p-1} (\mathbf{w}_x)_s \\ &= \sum_{j=0}^n \binom{n}{j} (-m)^{n-j} (\mathbf{w}_x + r + m - s)^{j+p-1} (\mathbf{w}_x)_s \\ &= (-m)^n (\mathbf{w}_x + r + m - s)^{p-1} (\mathbf{w}_x)_s \\ &\quad + \delta_{(n \geq 1)} \sum_{j=1}^n \binom{n}{j} (-m)^{n-j} (\mathbf{w}_x + r + m - s)^{j+p-1} (\mathbf{w}_x)_s \\ &= x^s (-m)^n w_{p-1}(x; r+m, s) \\ &\quad + \delta_{(n \geq 1)} x^s \sum_{j=1}^n \binom{n}{j} (-m)^{n-j} w_{p+j-1}(x; r+m, s) \\ &\equiv x^s (-m)^n w_{p-1}(x; r+m, s) \\ &\quad + \delta_{(n \geq 1)} x^s \sum_{j=1}^n \binom{n}{j} (-m)^{n-j} w_j(x; r+m, s) \\ &= x^s (-m)^n w_{p-1}(x; r+m, s) + \delta_{(n \geq 1)} x^s (w_n(x; r, s) - (-m)^n s!) \\ &= x^s [(-m)^n w_{p-1}(x; r+m, s) + w_n(x; r, s) - (-m)^n s!], \end{aligned}$$

where δ is the Kronecker's symbol, i.e. $\delta_{(n \geq 1)} = 1$ if $n \geq 1$ and 0 otherwise. $\square$

Let $\mathbf{r}_q = (r_1, \dots, r_q)$ be a vector of non-negative integers and let

$$w_n(x; \mathbf{r}_q) = \sum_{j=0}^{n+|\mathbf{r}_q|-1} \left\{ \begin{matrix} n+|\mathbf{r}_q| \\ j+r_q \end{matrix} \right\}_{\mathbf{r}_q} (j+r_q)! x^j, \quad 0 \leq r_1 \leq \dots \leq r_q,$$

where $\left\{ \begin{smallmatrix} n+|\mathbf{r}_q| \\ j+r_q \end{smallmatrix} \right\}_{\mathbf{r}_q}$ are the $(r_1, \dots, r_q)$ -Stirling numbers defined by Mihoubi et al. [21]. This polynomial is a generalization of the r -geometric polynomials $w_n(x; r) := w_n(x; r, r)$.

Proposition 5. *For any non-negative integers n, m and any prime $p \nmid m$, there holds*

$$x^{r_q} \sum_{k=1}^{p-1} \frac{w_{n+k}(x; \mathbf{r}_q)}{(-m)^k} \equiv (-m)^n (-m)_{r_1} \cdots (-m)_{r_q} (w_{p-1}(x; m, 0) - 1).$$

In particular, for $q = 1$ and $r_q = r$ we obtain

$$x^r \sum_{k=1}^{p-1} \frac{w_{n+k}(x; r, r)}{(-m)^k} \equiv (-m)^n (-m)_r (w_{p-1}(x; m, 0) - 1).$$

Proof. By the identity $(\mathbf{w}_\mathbf{x})_n = n!x^n$ and by [21, Th. 10] we have

$$\begin{aligned} x^{r_q} w_n(x; \mathbf{r}_q) &= \sum_{j=0}^{n+|\mathbf{r}_q|-1} \left\{ \begin{smallmatrix} n+|\mathbf{r}_q| \\ j+r_q \end{smallmatrix} \right\}_{\mathbf{r}_q} (\mathbf{w}_\mathbf{x})_{j+r_q} \\ &= \sum_{j=0}^{n+|\mathbf{r}_q|-1} \left\{ \begin{smallmatrix} n+|\mathbf{r}_q| \\ j+r_q \end{smallmatrix} \right\}_{\mathbf{r}_q} (\mathbf{w}_\mathbf{x} - r_q)_j (\mathbf{w}_\mathbf{x})_{r_q} \\ &= \mathbf{w}_\mathbf{x}^n (\mathbf{w}_\mathbf{x})_{r_1} \cdots (\mathbf{w}_\mathbf{x})_{r_q} \\ &= \sum_{k=0}^{|\mathbf{r}_q|} a_k(\mathbf{r}_q) \mathbf{w}_\mathbf{x}^{n+k} \\ &= \sum_{j=0}^{|\mathbf{r}_q|} a_j(\mathbf{r}_q) w_{n+j}(x), \end{aligned}$$

where $\sum_{k=0}^{|\mathbf{r}_q|} a_k(\mathbf{r}_q) u^k = (u)_{r_1} \cdots (u)_{r_q}$. So, by application of Theorem 2 we get

$$\begin{aligned} x^{r_q} \sum_{k=1}^{p-1} \frac{w_{n+k}(x; \mathbf{r}_q)}{(-m)^k} &= \sum_{j=0}^{|\mathbf{r}_q|} a_j(\mathbf{r}_q) \sum_{k=1}^{p-1} \frac{w_{n+j+k}(x; 0, 0)}{(-m)^k} \\ &\equiv \sum_{j=0}^{|\mathbf{r}_q|} a_j(\mathbf{r}_q) (-m)^{n+j} (w_{p-1}(x; m, 0) - 1) \\ &= (-m)^n (-m)_{r_1} \cdots (-m)_{r_q} (w_{p-1}(x; m, 0) - 1). \end{aligned}$$

□

Remark 2. Since $x^{r_q} w_n(x; \mathbf{r}_q) = \mathbf{w}_x^n(\mathbf{w}_x)_{r_1} \cdots (\mathbf{w}_x)_{r_q}$, then, for $g(x) = x^q(x)_{r_1} \cdots (x)_{r_q}$ and $f(x) = x^m$ in Corollary 5 we obtain

$$w_{mp+q}(x; \mathbf{r}_q) \equiv w_{m+q}(x; \mathbf{r}_q),$$

$$w_{m(p-1)}(x; \mathbf{r}_q) \equiv w_0(x; \mathbf{r}_q), \quad r_1 \cdots r_q \neq 0, \quad m \geq 0.$$

Corollary 7. Let $a_0(x), \dots, a_t(x)$ be polynomials with integer coefficients,

$$\mathcal{R}_{n,t}(x; r, s) = \sum_{i=0}^t a_i(x) w_{n+i}(x; r, s) \quad \text{and} \quad \mathcal{L}_t(x, y) = \sum_{i=0}^t a_i(x) y^i.$$

Then, for any non-negative integers n, m, r, s and any prime $p \nmid m$, there hold

$$\sum_{k=1}^{p-1} \frac{\mathcal{R}_{n+k,t}(x; r, s)}{(-m)^k} \equiv (-m)^n \mathcal{L}_t(x, -m) (w_{p-1}(x; r+m, s) - s!).$$

Proof. Theorem 2 implies

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{\mathcal{R}_{n+k,t}(x; r, s)}{(-m)^k} &= \sum_{j=0}^t a_j(x) \sum_{k=1}^{p-1} \frac{w_{n+k+j}(x; r, s)}{(-m)^k} \\ &\equiv \sum_{j=0}^t a_j(x) (-m)^{n+j} (w_{p-1}(x; r+m, s) - s!) \\ &= (-m)^n \mathcal{L}_t(x, -m) (w_{p-1}(x; r+m, s) - s!). \end{aligned}$$

□

4. CONGRUENCES INVOLVING $w_n(x; r, s)$, $\mathcal{P}_n(x, r)$ AND $\mathcal{T}_n(x, r)$

The following theorem gives connection in congruences between the polynomials w_n and $\mathcal{P}_n$.

Theorem 3. Let n, r be non-negative integers and p be a prime number. Then, for $m \in \{0, \dots, p-1\}$ there holds

$$\sum_{k=m}^{p-1} (-x)^k \frac{w_n(x; r+k, k)}{(k-m)!} \equiv (-1)^m m! (r+m)^n \mathcal{P}_{p-1}(x, m).$$

In particular, for $m = 0$, we get

$$\sum_{k=0}^{p-1} (-x)^k \frac{w_n(x; r+k, k)}{k!} \equiv r^n (1+x+\cdots+x^{p-1}).$$

Proof. For $k < m$ we get $\langle m+1 \rangle_{p-1-k} = 0$ and for $m \leq k \leq p-1$ we have

$$\langle m+1 \rangle_{p-1-k} = \frac{(m+p-k-1)!}{m!} = \frac{(p-1-(k-m))!}{m!} \equiv -\frac{1}{m!} \frac{(-1)^{k-m}}{(k-m)!}.$$

where $\langle x \rangle_n = x(x+1)\cdots(x+n-1)$ if $n \geq 1$ and $\langle x \rangle_0 = 1$. Then

$$\begin{aligned} LHS &\equiv -(-1)^m m! \sum_{k=0}^{p-1} \langle m+1 \rangle_{p-1-k} x^k w_n(x; r+k, k) \\ &\equiv -(-1)^m m! \sum_{k=0}^{p-1} \langle m-p+1 \rangle_{p-1-k} (\mathbf{w}_x + r)^n (\mathbf{w}_x)_k \\ &\equiv -(-1)^m m! \sum_{k=0}^{p-1} \binom{p-1}{k} \langle m-p+1 \rangle_{p-1-k} (\mathbf{w}_x + r)^n \langle -\mathbf{w}_x \rangle_k \\ &= -(-1)^m m! \langle m-p+1-\mathbf{w}_x \rangle_{p-1} (\mathbf{w}_x + r)^n \\ &= -(-1)^m m! (\mathbf{w}_x - m + p - 1)_{p-1} (\mathbf{w}_x + r)^n \\ &= -(-1)^m m! (\mathbf{w}_x - m + r + m)^n (\mathbf{w}_x - m + p - 1)_{p-1} \\ &= -(-1)^m m! \sum_{j=0}^n \left\{ \begin{matrix} n+r+m \\ j+r+m \end{matrix} \right\}_{r+m} (\mathbf{w}_x - m)_j (\mathbf{w}_x - m + p - 1)_{p-1}. \end{aligned}$$

But for $j \geq 1$ we have

$$\begin{aligned} (\mathbf{w}_x - m)_j (\mathbf{w}_x - m + p - 1)_{p-1} &= (\mathbf{w}_x - m + p - 1)_{j+p-1} \\ &\equiv (\mathbf{w}_x - m - 1)_{j+p-1} = (j+p-1)! \mathcal{P}_{j+p-1}(x, m+1) \\ &\equiv -\delta_{(j=0)} \mathcal{P}_{p-1}(x, m+1), \end{aligned}$$

hence, it follows $LHS \equiv (-1)^m m! (r+m)^n \mathcal{P}_{p-1}(x, m)$. $\square$

A connection in congruences between the polynomials w_n and $\mathcal{T}_n$ is to be:

Theorem 4. For any integers $n, m, r \geq 0$ and any prime p , there holds

$$\sum_{k=0}^{p-1} (-m)_{p-1-k} (x+1)^k w_n(x; r+m, k) \equiv -r^n \mathcal{T}_{p-1}(x; m).$$

Proof. Upon using the identity $(x+1)^s w_n(x; r, s) = (\mathbf{w}_x + r)^n (\mathbf{w}_x + s)_s$ and the known congruence $(m)_{p-1-k} \equiv \binom{p-1}{k} \langle -m \rangle_{p-1-k}$ we obtain

$$LHS \equiv \sum_{k=0}^{p-1} \binom{p-1}{k} \langle m \rangle_{p-1-k} (\mathbf{w}_x + r + m)^n (\mathbf{w}_x + k)_k$$

$$\begin{aligned}
&\equiv \sum_{k=0}^{p-1} \binom{p-1}{k} \langle m \rangle_{p-1-k} (\mathbf{w}_x + r + m)^n \langle \mathbf{w}_x + 1 \rangle_k \\
&= (\mathbf{w}_x + r + m)^n \langle \mathbf{w}_x + m + 1 \rangle_{p-1} \\
&\equiv (\mathbf{w}_x + m + r)^n (\mathbf{w}_x + m + p - 1)_{p-1} \\
&= \sum_{j=0}^n \left\{ \begin{matrix} n+r \\ j+r \end{matrix} \right\}_r (\mathbf{w}_x + m)_j (\mathbf{w}_x + m + p - 1)_{p-1} \\
&= \sum_{j=0}^n \left\{ \begin{matrix} n+r \\ j+r \end{matrix} \right\}_r (\mathbf{w}_x + m + p - 1)_{j+p-1} \\
&= \sum_{j=0}^n \left\{ \begin{matrix} n+r \\ j+r \end{matrix} \right\}_r (j + p - 1)! \mathcal{T}_{j+p-1}(x; m - j) \\
&= (p-1)! \mathcal{T}_{p-1}(x; m) + \sum_{j=1}^n \left\{ \begin{matrix} n+r \\ j+r \end{matrix} \right\}_r (j + p - 1)! \mathcal{T}_{j+p-1}(x; m - j) \\
&\equiv -r^n \mathcal{T}_{p-1}(x; m).
\end{aligned}$$

□

Corollary 8. Let $\mathcal{R}_{n,t}(x; r, s)$ be as in Corollary 7. Then, for any non-negative integers n, m, r, s and any prime $p \nmid m$, there holds

$$\sum_{k=m}^{p-1} (-x)^k \binom{k}{m} \frac{\mathcal{R}_{n,t}(x; r + k, k)}{k!} \equiv (-1)^m (r + m)^n \mathcal{L}_t(x, r + m) \mathcal{P}_{p-1}(x, m).$$

Proof. Theorem 3 implies

$$\begin{aligned}
LHS &= \sum_{j=0}^t a_j(x) \sum_{k=m}^{p-1} (-x)^k \binom{k}{m} \frac{w_{n+j}(x; r + k, k)}{k!} \\
&\equiv \sum_{j=0}^t a_j(x) (-1)^m (r + m)^{n+j} \mathcal{P}_{p-1}(x, m) \\
&\equiv (-1)^m (r + m)^n \mathcal{L}_t(x, r + m) \mathcal{P}_{p-1}(x, m).
\end{aligned}$$

□

REFERENCES

- [1] A. Benyattou and M. Mihoubi, “Curious congruences related to the Bell polynomials,” *Quaest Math.*, vol. 40, pp. 1–12, 2017, doi: [10.2989/16073606.2017.1391349](https://doi.org/10.2989/16073606.2017.1391349).
- [2] K. N. Boyadzhiev, “A series transformation formula and related polynomials,” *Int. J. Math. Math. Sci.*, vol. 2005, no. 23, pp. 3849–3866, 2005, doi: [10.1155/IJMMS.2005.3849](https://doi.org/10.1155/IJMMS.2005.3849).

- [3] K. N. Boyadzhiev and A. Dil, "Geometric polynomials: properties and applications to series with zeta values," *Analysis Math.*, vol. 42, no. 3, pp. 203–224, 2016, doi: [10.1007/s10476-016-0302-y](https://doi.org/10.1007/s10476-016-0302-y).
- [4] A. Z. Broder, "The r-Stirling numbers," *Discrete Math.*, vol. 49, no. 3, pp. 241–259, 1984, doi: [10.1016/0012-365X\(84\)90161-4](https://doi.org/10.1016/0012-365X(84)90161-4).
- [5] A. D. Bucchianico and D. Loeb, "A selected survey of umbral calculus," *Electron. J. Combin.*, vol. 2, pp. 1–34, 2000.
- [6] M. B. Can and M. Joyce, "Ordered Bell numbers, Hermite polynomials, skew Young tableaux, and Borel orbits," *J. Comb. Theory Ser. A*, vol. 119, no. 8, pp. 1798–1810, 2012, doi: [10.1016/j.jcta.2012.06.002](https://doi.org/10.1016/j.jcta.2012.06.002).
- [7] M. E. Dasef and S. M. Kautz, "Some sums of some importance," *College Math. J.*, vol. 28, pp. 52–55, 1997.
- [8] T. Diagana and H. Maïga, "Some new identities and congruences for Fubini numbers," *J. Number Theory*, vol. 173, pp. 547–569, 2017.
- [9] A. Dil and V. Kurt, "Investigating geometric and exponential polynomials with Euler-Seidel matrices," *J. Integer Seq.*, vol. 14, no. 4, 2011.
- [10] A. Dil and V. Kurt, "Polynomials related to harmonic numbers and evaluation of harmonic number series II," *Appl. Anal. Discrete Math.*, vol. 5, pp. 212–229, 2011.
- [11] D. Dumont, "Matrices d'Euler-Siedel," *Sémin. Lothar. Comb.*, vol. 5, 1981.
- [12] P. Flajolet and R. Sedgewick, *Analytic combinatorics*. Cambridge university press, 2009.
- [13] I. M. Gessel, "Applications of the classical umbral calculus," *Algebra Universalis*, vol. 49, no. 4, pp. 397–434, 2003, doi: [10.1007/s00012-003-1813-5](https://doi.org/10.1007/s00012-003-1813-5).
- [14] R. D. James, "The factors of a square-free integer," *Canad. Math. Bull.*, vol. 11, pp. 733–735, 1968, doi: [10.4153/CMB-1968-089-7](https://doi.org/10.4153/CMB-1968-089-7).
- [15] L. Kargin, "Some formulae for products of geometric polynomials with applications," *J. Integer Seq.*, vol. 20, no. 2, 2017.
- [16] L. Kargin and B. Çekim, "Higher order generalized geometric polynomials," *Turk. J. Math.*, vol. 42, pp. 887–903, 2018.
- [17] L. Kargin and R. B. Corcino, "Generalization of Mellin derivative and its applications," *Integral Transforms Spec. Funct.*, vol. 27, pp. 620–631, 2016, doi: [10.1080/10652469.2016.1174701](https://doi.org/10.1080/10652469.2016.1174701).
- [18] I. Mezö, "Periodicity of the last digits of some combinatorial sequences," *J. Integer Seq.*, vol. 17, no. 1, 2014.
- [19] I. Mezö and J. L. Ramírez, "Divisibility properties of the r-Bell numbers and polynomials," *J. Number Theory*, vol. 177, pp. 136–152, 2017, doi: [10.1016/j.jnt.2017.01.022](https://doi.org/10.1016/j.jnt.2017.01.022).
- [20] M. Mihoubi, "Bell polynomials and binomial type sequences," *Discrete Math.*, vol. 308, no. 12, pp. 2450–2459, 2008, doi: [10.1016/j.disc.2007.05.010](https://doi.org/10.1016/j.disc.2007.05.010).
- [21] M. Mihoubi and M. S. Maamra, "The $(r_1, \dots, r_p)$ -Stirling numbers of the second kind," *Integers*, vol. 12, no. 5, pp. 1047–1059, 2012, doi: [10.1515/integers-2012-0022](https://doi.org/10.1515/integers-2012-0022).
- [22] T. J. Robinson, "Formal calculus and umbral calculus," *Electron. J. Combin.*, vol. 17, no. 1, p. R95, 2010.
- [23] S. Roman, *The Umbral Calculus*. Courier Corporation, 2013.
- [24] S. M. Roman and G.-C. Rota, "The umbral calculus," *Adv. Math.*, vol. 27, no. 2, pp. 95–188, 1978, doi: [10.1016/0001-8708\(78\)90087-7](https://doi.org/10.1016/0001-8708(78)90087-7).
- [25] G. C. Rota and B. D. Taylor, "The classical umbral calculus," *SIAM J. Math. Anal.*, vol. 25, no. 2, pp. 694–711, 1994, doi: [10.1137/S0036141093245616](https://doi.org/10.1137/S0036141093245616).
- [26] R. P. Stanley, *Enumerative Combinatorics I*. Cambridge University press, 1997.
- [27] Y. Sun, X. Wu, and J. Zhuang, "Congruences on the Bell polynomials and the derangement polynomials," *J. Number Theory*, vol. 133, no. 5, pp. 1564–1571, 2013, doi: [10.1016/j.jnt.2012.08.031](https://doi.org/10.1016/j.jnt.2012.08.031).
- [28] S. M. Tanny, "On some numbers related to the Bell numbers," *Canad. Math. Bull.*, vol. 17, no. 5, p. 733, 1975, doi: [10.4153/CMB-1974-132-8](https://doi.org/10.4153/CMB-1974-132-8).

- [29] D. J. Velleman and G. S. Call, “Permutations and combination locks,” *Math. Mag.*, vol. 68, no. 4, pp. 243–253, 1995, doi: [10.2307/2690567](https://doi.org/10.2307/2690567).

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HERMITE-HADAMARD TYPE INEQUALITIES FOR HARMONICALLY CONVEX FUNCTIONS VIA KATUGAMPOLA FRACTIONAL INTEGRALS

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Abstract. In this work, firstly, we established Hermite-Hadamard's inequalities for harmonically convex functions via Katugampola fractional integrals. Then we give some Hermite-Hadamard type inequalities of these classes functions.

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1. INTRODUCTION AND PRELIMINARIES

We will start with a definition of mathematical analysis that has a high degree precedence for the inequality theory.

A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if the inequality

$$f(\lambda u + (1 - \lambda)v) \leq \lambda f(u) + (1 - \lambda)f(v)$$

holds for all $u, v \in I$ and $\lambda \in [0, 1]$.

This definition has been used in the celebrated Hermite-Hadamard inequality;

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}. \quad (1.1)$$

In addition to giving upper and lower bounds for the mean value of a convex function, this double inequality has many applications.

Convexity plays an important role in different fields of pure and applied sciences. In recent years we have noticed that theory of convexity developed rapidly. Consequently several new generalizations of convex functions have been proposed in the literature. Recently İşcan [4] introduced the notion of harmonic convex function.

Definition 1. Let $I \subset \mathbb{R}/\{0\}$ be a real interval. A function $f : I \rightarrow \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x) \quad (1.2)$$

for all $x, y \in I$ and $t \in [0, 1]$.

The following theorem involve a different variant of Hadamard's inequality for harmonically convex functions.

Theorem 1 ([4]). Let $I \subset \mathbb{R}/\{0\} \rightarrow \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$ then the following inequalities hold.

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2}.$$

To prove our results, we will use the following concepts and definitions. The Beta function [11, p.18]:

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1} (1-t)^{b-1} dt, \quad a, b > 0,$$

where $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$ is Gamma function.

The hypergeometric function [7]:

$${}_2F_1(a, b; c; z) = \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt, \quad c > b > 0, \quad z < 1.$$

Lemma 1 ([10]). For $0 < \alpha \leq 1$ and $0 \leq a < b$, we have

$$|a^\alpha - b^\alpha| \leq (b-a)^\alpha.$$

Definition 2. Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a+}^\mu f$ and $J_{b-}^\mu f$ of order $\alpha > 0$ are defined by

$$J_{a+}^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (t-x)^{\mu-1} f(t) dt, \quad x < b$$

respectively where $\Gamma(\mu) = \int_0^\infty e^{-t} t^{\mu-1} dt$. Here $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$

In the case of $\mu = 1$, the fractional integral reduces to classical integral. The great impact of fractional calculus in pure and applied sciences can not be denied. Resultantly many researchers used the techniques of fractional calculus intensively to get the new refinements of the previously known results. For example, we refer the reader to [1-3] and references cited therein. In [12], Sarıkaya et. al. proved a new

version of Hermite-Hadamard's inequalities in Riemann-Liouville fractional integral form as follows:

Theorem 2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \leq \frac{f(a) + f(b)}{2} \quad (1.3)$$

with $\alpha > 0$.

For further results related to Hermite-Hadamard type inequalities involving fractional integrals on can see [8, 9, 12–19].

In [5], Iscan et al. gave a generalization of (1.3) for harmonically convex functions as follows:

Theorem 3. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a function such that $f \in L[a, b]$, where $a, b \in I$ with $a < b$. If f is a harmonically convex function on $[a, b]$, then the following inequalities for fractional integrals hold:

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq \frac{\Gamma(\alpha+1)}{2} \left(\frac{ab}{b-a}\right)^\alpha \left\{ J_{1/a-}^\alpha (f \circ g)(1/b) + J_{1/b+}^\alpha (f \circ g)(1/a) \right\} \\ &\leq \frac{f(a) + f(b)}{2} \end{aligned} \quad (1.4)$$

where $g(x) = 1/x$.

Katugampola gave a new fractional integral that generalizes the Riemann-Liouville and the Hadamard fractional integrals into a single form.

Definition 3 ([6]). Let $[a, b] \subset \mathbb{R}$ be a finite interval. Then, the left- and right-side Katugampola fractional integrals of order $(\alpha > 0)$ of $f \in X_c^p(a, b)$ are defined:

$${}^\rho J_{a+}^\alpha f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} f(t) dt$$

and

$${}^\rho J_{b-}^\alpha f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}} f(t) dt$$

with $a < x < b$ and $\rho > 0$, if the integral exist.

Theorem 4 ([6]). Let $\alpha > 0$ and $\rho > 0$. Then for $x > a$,

1. $\lim_{\rho \rightarrow 1} {}^\rho J_{a+}^\alpha f(x) = J_{a+}^\alpha f(x)$,
2. $\lim_{\rho \rightarrow 0+} {}^\rho J_{a+}^\alpha f(x) = H_{a+}^\alpha f(x)$.

Similar results also hold for right-sided operators.

The main purpose of this paper is to establish Hermite-Hadamard's inequalities for harmonically convex functions via Katugampola fractional integral. We also obtain Hermite-Hadamard type inequalities of these classes functions.

2. HERMITE-HADAMARD INEQUALITIES FOR HARMONICALLY CONVEXITY VIA KATUGAMPOLA FRACTIONAL INTEGRALS

Consider the space $X_c^p(a, b)$ ($c \in \mathbb{R}$, $1 \leq p \leq \infty$) consist of those complex-valued Lebesgue measurable functions φ on (a, b) for which $\|\varphi\|_{X_c^p} < \infty$, with

$$\|\varphi\|_{X_c^p} = \left(\int_a^b |x^c \varphi(x)|^p \frac{dx}{x} \right)^{1/p} \quad (1 \leq p < \infty)$$

and

$$\|\varphi\|_{X_c^p} = \text{esssup}_{x \in (a, b)} [x^c |\varphi(x)|].$$

Hermite-Hadamard's inequalities for harmonically convex functions can be represented in Katugampola fractional integral forms as follows:

Theorem 5. Let $\alpha > 0$ and $\rho > 0$. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a function such that $f \in X_c^p(a^\rho, b^\rho)$, where $a^\rho, b^\rho \in I$ with $a < b$. If f is a harmonically convex function on $[a, b]$, then the following inequalities hold:

$$\begin{aligned} & f\left(\frac{2a^\rho b^\rho}{a^\rho + b^\rho}\right) \\ & \leq \frac{\rho^\alpha \Gamma(\alpha + 1)}{2} \left(\frac{a^\rho b^\rho}{b^\rho - a^\rho}\right)^\alpha \left\{ {}^\rho J_{1/a-}^\alpha (f \circ g)(1/b) + {}^\rho J_{1/b+}^\alpha (f \circ g)(1/a) \right\} \quad (2.1) \\ & \leq \frac{f(a^\rho) + f(b^\rho)}{2}. \end{aligned}$$

where $g(x) = 1/x^\rho$.

Proof. Let $t \in [0, 1]$. Consider $x, y \in [a, b]$, $a \geq 0$, choosing $x^\rho = \frac{a^\rho b^\rho}{t^\rho b^\rho + (1-t^\rho)a^\rho}$, $y^\rho = \frac{a^\rho b^\rho}{t^\rho a^\rho + (1-t^\rho)b^\rho}$. Since f is harmonically convex function on $[a, b]$, and from definition, we can write

$$f\left(\frac{2x^\rho y^\rho}{x^\rho + y^\rho}\right) \leq \frac{f(x^\rho) + f(y^\rho)}{2}$$

Then we have

$$f\left(\frac{2a^\rho b^\rho}{a^\rho + b^\rho}\right) \leq \frac{f\left(\frac{a^\rho b^\rho}{t^\rho b^\rho + (1-t^\rho)a^\rho}\right) + f\left(\frac{a^\rho b^\rho}{t^\rho a^\rho + (1-t^\rho)b^\rho}\right)}{2} \quad (2.2)$$

Multiplying both sides of (2.2) by $t^{\rho\alpha-1}$, then integrating the resulting inequality with respect to t over $[0, 1]$, we obtain

$$f\left(\frac{2a^\rho b^\rho}{a^\rho + b^\rho}\right)$$

$$\begin{aligned}
&\leq \frac{\rho\alpha}{2} \left\{ \int_0^1 t^{\rho\alpha-1} f\left(\frac{a^\rho b^\rho}{t^\rho b^\rho + (1-t^\rho)a^\rho}\right) dt + \int_0^1 t^{\rho\alpha-1} f\left(\frac{a^\rho a^\rho}{t^\rho b^\rho + (1-t^\rho)b^\rho}\right) dt \right\} \\
&= \frac{\rho\alpha}{2} \left(\frac{a^\rho b^\rho}{b^\rho - a^\rho}\right)^\alpha \left\{ \int_{1/b}^{1/a} \frac{x^{\rho-1}}{(x^\rho - \frac{1}{b^\rho})^{1-\alpha}} f\left(\frac{1}{x^\rho}\right) dx \right. \\
&\quad \left. + \int_{1/a}^{1/b} \frac{x^{\rho-1}}{(\frac{1}{a^\rho} - x^\rho)^{1-\alpha}} f\left(\frac{1}{x^\rho}\right) dx \right\} \\
&= \frac{\rho^\alpha \Gamma(\alpha+1)}{2} \left(\frac{a^\rho b^\rho}{b^\rho - a^\rho}\right)^\alpha \left\{ {}^\rho J_{1/a-}^\alpha (f \circ g)(1/b) + {}^\rho J_{1/b+}^\alpha (f \circ g)(1/a) \right\}
\end{aligned}$$

where $g(x) = 1/x^\rho$. So the first inequality is proved.

For the proof of the second inequality in (2.1), we first note that for a harmonically convex function f , we have

$$f\left(\frac{a^\rho b^\rho}{t^\rho b^\rho + (1-t^\rho)a^\rho}\right) \leq t^\rho f(a^\rho) + (1-t^\rho)b^\rho$$

and

$$f\left(\frac{a^\rho b^\rho}{t^\rho a^\rho + (1-t^\rho)b^\rho}\right) \leq t^\rho f(b^\rho) + (1-t^\rho)a^\rho.$$

By adding these inequalities, we have

$$f\left(\frac{a^\rho b^\rho}{t^\rho b^\rho + (1-t^\rho)a^\rho}\right) + f\left(\frac{a^\rho b^\rho}{t^\rho a^\rho + (1-t^\rho)b^\rho}\right) \leq f(a^\rho) + f(b^\rho). \quad (2.3)$$

Then multiplying both sides of (2.3) by $t^{\rho\alpha-1}$, and integrating the resulting inequality with respect to t over $[0, 1]$, we get

$$\begin{aligned}
&\int_0^1 f\left(\frac{a^\rho b^\rho}{t^\rho b^\rho + (1-t^\rho)a^\rho}\right) t^{\rho\alpha-1} dt + \int_0^1 f\left(\frac{a^\rho b^\rho}{t^\rho a^\rho + (1-t^\rho)b^\rho}\right) t^{\rho\alpha-1} dt \\
&\leq [f(a) + f(b)] \int_0^1 t^{\rho\alpha-1} dt
\end{aligned}$$

i.e.

$$\begin{aligned}
&\frac{\rho^\alpha \Gamma(\alpha+1)}{2} \left(\frac{a^\rho b^\rho}{b^\rho - a^\rho}\right)^\alpha \left\{ {}^\rho J_{1/a-}^\alpha (f \circ g)(1/b) + {}^\rho J_{1/b+}^\alpha (f \circ g)(1/a) \right\} \\
&\leq \frac{f(a^\rho) + f(b^\rho)}{2}.
\end{aligned}$$

The proof is completed. $\square$

Remark 1. In Theorem 5, taking limit $\rho \rightarrow 1$ we obtain inequality of (1.4).

3. HERMITE-HADAMARD TYPE INEQUALITIES FOR KATUGAMPOLA FRACTIONAL INTEGRALS

Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° , the interior of I , throughout this section we will take

$$I_f(g; \alpha, a, b) = \frac{f(a^\rho) + b^\rho}{2} - \frac{\rho^\alpha \Gamma(\alpha + 1)}{2} \left(\frac{a^\rho b^\rho}{b^\rho - a^\rho} \right)^\alpha \left\{ {}^\rho J_{1/a-}^\alpha (f \circ g)(1/b) + {}^\rho J_{1/b+}^\alpha (f \circ g)(1/a) \right\},$$

where $a^\rho, b^\rho \in I$ with $a < b$. $g(x) = 1/x^\rho$ and Γ is Euler Gamma function.

Lemma 2. Let $\alpha > 0$ and $\rho > 0$. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function such that $f \in X_c^P(a^\rho, b^\rho)$, where $a^\rho, b^\rho \in I$ with $a < b$. Then the following equality holds:

$$I_f(g; \alpha, a, b) = \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} \int_0^1 \frac{[t^{\rho\alpha} - (1-t^\rho)^\alpha] t^{\rho-1}}{[t^\rho a^\rho + (1-t^\rho) b^\rho]^2} f' \left(\frac{a^\rho b^\rho}{t^\rho a^\rho + (1-t^\rho) b^\rho} \right) dt. \quad (3.1)$$

Proof. Let $A_t = t^\rho a^\rho + (1-t^\rho) b^\rho$ and $B_t = t^\rho b^\rho + (1-t^\rho) a^\rho$. It suffices to note that

$$\begin{aligned} I_f(g; \alpha, a, b) &= \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} \int_0^1 \frac{[t^{\rho\alpha} - (1-t^\rho)^\alpha] t^{\rho-1}}{[t^\rho a^\rho + (1-t^\rho) b^\rho]^2} f' \left(\frac{a^\rho b^\rho}{t^\rho a^\rho + (1-t^\rho) b^\rho} \right) dt \\ &= \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} \int_0^1 \frac{t^{\rho\alpha} t^{\rho-1}}{A_t^2} f' \left(\frac{a^\rho b^\rho}{A_t} \right) dt \\ &\quad - \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} \int_0^1 \frac{(1-t^\rho)^\alpha t^{\rho-1}}{A_t^2} f' \left(\frac{a^\rho b^\rho}{A_t} \right) dt \\ &= I_1 + I_2. \end{aligned} \quad (3.2)$$

By integrating by part, we get

$$\begin{aligned} I_1 &= \frac{1}{2} \left[t^{\rho\alpha} f \left(\frac{a^\rho b^\rho}{A_t} \right) \Big|_0^1 - \rho\alpha \int_0^1 t^{\rho\alpha-1} f \left(\frac{a^\rho b^\rho}{A_t} \right) dt \right] \\ &= \frac{1}{2} \left[f(b^\rho) - \rho\alpha \left(\frac{a^\rho b^\rho}{b^\rho - a^\rho} \right)^\alpha \int_{1/b}^{1/a} \frac{x^{\rho-1}}{(x^\rho - \frac{1}{b^\rho})^{1-\alpha}} f \left(\frac{1}{x^\rho} \right) dx \right] \\ &= \frac{1}{2} \left[f(b^\rho) - \rho^\alpha \Gamma(\alpha + 1) \left(\frac{a^\rho b^\rho}{b^\rho - a^\rho} \right)^\alpha {}^\rho J_{1/a-}^\alpha (f \circ g)(1/b) \right] \end{aligned} \quad (3.3)$$

and similarly we get

$$\begin{aligned}
 I_2 &= -\frac{1}{2} \left[(1-t^\rho)^\alpha f\left(\frac{a^\rho b^\rho}{A_t}\right) \Big|_0^1 + \rho\alpha \int_0^1 (1-t^\rho)^{\alpha-1} t^{\rho-1} f\left(\frac{a^\rho b^\rho}{A_t}\right) dt \right] \\
 &= \frac{1}{2} \left[f(a^\rho) - \rho\alpha \int_0^1 u^{\rho\alpha-1} f\left(\frac{a^\rho b^\rho}{B_t}\right) du \right] \\
 &= \frac{1}{2} \left[f(a^\rho) - \rho\alpha \left(\frac{a^\rho b^\rho}{b^\rho - a^\rho}\right)^\alpha \int_{1/a}^{1/b} \frac{x^{\rho-1}}{\left(\frac{1}{a^\rho} - x^\rho\right)^{1-\alpha}} f\left(\frac{1}{x^\rho}\right) dx \right] \\
 &= \frac{1}{2} \left[f(a^\rho) - \rho^\alpha \Gamma(\alpha+1) \left(\frac{a^\rho b^\rho}{b^\rho - a^\rho}\right)^\alpha {}^\rho J_{1/b+}^\alpha (f \circ g)(1/a) \right] \quad (3.4)
 \end{aligned}$$

Using (3.3) and (3.4) in (3.2), we get equality (3.1). $\square$

Remark 2. In Lemma 2, taking limit $\rho \rightarrow 1$ we obtain inequality Lemma 3 in [5].

Theorem 6. Let $\alpha > 0$ and $\rho > 0$. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function such that $f \in X_C^\rho(a^\rho, b^\rho)$, where $a^\rho, b^\rho \in I$ with $a < b$. If $|f|^q$ is a harmonically convex function on $[a, b]$ for some fixed $q \geq 1$, then the following inequalities holds:

$$\begin{aligned}
 |I_f(g; \alpha, a, b)| &\leq \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} \Lambda_1^{1-1/q}(\alpha; a, b) \\
 &\quad \left(\Lambda_2(\alpha; a, b) |f'(b)|^q + \Lambda_3(\alpha; a, b) |f'(a)|^q \right)^{1/q} \quad (3.5)
 \end{aligned}$$

where

$$\begin{aligned}
 \Lambda_1(\alpha; a, b) &= \frac{b^{-2\rho}}{\rho(\alpha+1)} \left[{}_2F_1\left(2, \alpha+1; \alpha+2; 1 - \frac{a^\rho}{b^\rho}\right) \right. \\
 &\quad \left. + {}_2F_1\left(2, 1; \alpha+2; 1 - \frac{a^\rho}{b^\rho}\right) \right], \\
 \Lambda_2(\alpha; a, b) &= \frac{b^{-2\rho}}{\rho(\alpha+2)} \left[{}_2F_1\left(2, \alpha+2; \alpha+3; 1 - \frac{a^\rho}{b^\rho}\right) \right. \\
 &\quad \left. + \frac{1}{\alpha+1} {}_2F_1\left(2, 2; \alpha+3; 1 - \frac{a^\rho}{b^\rho}\right) \right], \\
 \Lambda_3(\alpha; a, b) &= \frac{b^{-2\rho}}{\rho(\alpha+2)} \left[\frac{1}{\alpha+1} {}_2F_1\left(2, \alpha+1; \alpha+3; 1 - \frac{a^\rho}{b^\rho}\right) \right. \\
 &\quad \left. + {}_2F_1\left(2, 1; \alpha+3; 1 - \frac{a^\rho}{b^\rho}\right) \right].
 \end{aligned}$$

Proof. Let $A_t = t^\rho a^\rho + (1-t^\rho)b^\rho$. From Lemma 2, using the property of the modulus, the power mean inequality and the harmonically convexity of $|f|^q$, we get

$$|I_f(g; \alpha, a, b)|$$

$$\begin{aligned}
&\leq \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} \int_0^1 \frac{|t^{\rho\alpha} - (1-t^\rho)^\alpha| |t^{\rho-1}|}{A_t^2} \left| f' \left(\frac{a^\rho b^\rho}{A_t} \right) \right| dt \\
&\leq \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} \left(\int_0^1 \frac{|t^{\rho\alpha} - (1-t^\rho)^\alpha| |t^{\rho-1}|}{A_t^2} dt \right)^{1-1/q} \\
&\quad \times \left(\int_0^1 \frac{|t^{\rho\alpha} - (1-t^\rho)^\alpha| |t^{\rho-1}|}{A_t^2} \left| f' \left(\frac{a^\rho b^\rho}{A_t} \right) \right| dt \right)^{1/q} \\
&\leq \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} \left(\int_0^1 \frac{|t^{\rho\alpha} + (1-t^\rho)^\alpha| |t^{\rho-1}|}{A_t^2} dt \right)^{1-1/q} \\
&\quad \times \left(\int_0^1 \frac{|t^{\rho\alpha} + (1-t^\rho)^\alpha| |t^{\rho-1}|}{A_t^2} \left(t^\rho |f'(b^\rho)|^q + (1-t^\rho) |f'(a^\rho)|^q \right) dt \right)^{1/q} \\
&\leq \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} \Lambda_1^{1-1/q}(\alpha; a, b) \left(\Lambda_2(\alpha; a, b) |f'(b)|^q + \Lambda_3(\alpha; a, b) |f'(a)|^q \right)^{1/q}.
\end{aligned} \tag{3.6}$$

Calculating $\Lambda_1(\alpha; a, b)$, $\Lambda_2(\alpha; a, b)$ and $\Lambda_3(\alpha; a, b)$, we have

$$\begin{aligned}
&\Lambda_1(\alpha; a, b) \\
&= \int_0^1 \frac{[t^{\rho\alpha} + (1-t^\rho)^\alpha] t^{\rho-1}}{A_t^2} dt \\
&= b^{-2\rho} \int_0^1 (u^\alpha + (1-u)^\alpha) \left(1 - \left(1 - \frac{a^\rho}{b^\rho} \right) u \right)^{-2} dt \\
&= \frac{b^{-2\rho}}{\rho(\alpha+1)} \left[{}_2F_1 \left(2, \alpha+1; \alpha+2; 1 - \frac{a^\rho}{b^\rho} \right) + {}_2F_1 \left(2, 1; \alpha+2; 1 - \frac{a^\rho}{b^\rho} \right) \right].
\end{aligned} \tag{3.7}$$

Similarly, we get

$$\begin{aligned}
&\Lambda_2(\alpha; a, b) \\
&= \int_0^1 \frac{[t^{\rho\alpha} + (1-t^\rho)^\alpha] t^{\rho-1}}{A_t^2} t^\rho dt \\
&= \frac{b^{-2\rho}}{\rho(\alpha+2)} \left[{}_2F_1 \left(2, \alpha+2; \alpha+3; 1 - \frac{a^\rho}{b^\rho} \right) + \frac{1}{\alpha+1} {}_2F_1 \left(2, 2; \alpha+3; 1 - \frac{a^\rho}{b^\rho} \right) \right]
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
&\Lambda_3(\alpha; a, b) \\
&= \int_0^1 \frac{[t^{\rho\alpha} + (1-t^\rho)^\alpha] t^{\rho-1}}{A_t^2} t^\rho dt
\end{aligned} \tag{3.9}$$

$$= \frac{b^{-2\rho}}{\rho(\alpha+2)} \left[\frac{1}{\alpha+1} {}_2F_1 \left(2, \alpha+1; \alpha+3; 1 - \frac{a^\rho}{b^\rho} \right) + {}_2F_1 \left(2, 1; \alpha+3; 1 - \frac{a^\rho}{b^\rho} \right) \right]$$

So, if we use (3.7)-(3.9) in (3.6), we obtain the inequality of (3.5). This completes the proof. $\square$

Remark 3. In Theorem 6, taking limit $\rho \rightarrow 1$ we obtain Theorem 5 in [5].

Theorem 7. Let $\alpha > 0$ and $\rho > 0$. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function such that $f \in X_c^p(a^\rho, b^\rho)$, where $a^\rho, b^\rho \in I$ with $a < b$. If $|f|^l$ is a harmonically convex function on $[a, b]$ for some fixed $l \geq 1$, then the following inequalities holds:

$$|I_f(g; \alpha, a, b)| \leq \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} \Lambda_4^{1-1/q}(\alpha; a, b) \left(\Lambda_5(\alpha; a, b) |f'(b)|^q + \Lambda_6(\alpha; a, b) |f'(a)|^q \right)^{1/q} \quad (3.10)$$

where

$$\begin{aligned} \Lambda_4 &= \frac{b^{-2\rho}}{\rho(\alpha+1)} \left[{}_2F_1 \left(2, \alpha+1; \alpha+2; 1 - \frac{a^\rho}{b^\rho} \right) - {}_2F_1 \left(2, 1; \alpha+2; 1 - \frac{a^\rho}{b^\rho} \right) \right. \\ &\quad \left. + {}_2F_1 \left(2, 1; \alpha+2; \frac{1}{2} \left(1 - \frac{a^\rho}{b^\rho} \right) \right) \right] \\ \Lambda_5 &= \frac{b^{-2\rho}}{\rho(\alpha+2)} \left[{}_2F_1 \left(2, \alpha+2; \alpha+3; 1 - \frac{a^\rho}{b^\rho} \right) - \frac{1}{\alpha+1} {}_2F_1 \left(2, 2; \alpha+3; 1 - \frac{a^\rho}{b^\rho} \right) \right. \\ &\quad \left. + \frac{1}{2(\alpha+1)} {}_2F_1 \left(2, 2; \alpha+3; \frac{1}{2} \left(1 - \frac{a^\rho}{b^\rho} \right) \right) \right] \\ \Lambda_6 &= \frac{b^{-2\rho}}{\rho(\alpha+2)} \left[\frac{1}{\alpha+1} {}_2F_1 \left(2, \alpha+1; \alpha+3; 1 - \frac{a^\rho}{b^\rho} \right) - {}_2F_1 \left(2, 1; \alpha+3; 1 - \frac{a^\rho}{b^\rho} \right) \right. \\ &\quad \left. + {}_2F_1 \left(2, 1; \alpha+3; \frac{1}{2} \left(1 - \frac{a^\rho}{b^\rho} \right) \right) \right]. \end{aligned}$$

Proof. Let $A_t = t^\rho a^\rho + (1-t^\rho)b^\rho$. From Lemma 2, using the property of the modulus, the power mean inequality and the harmonically convexity of $|f|^q$, we have

$$\begin{aligned} &|I_f(g; \alpha, a, b)| \\ &\leq \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} \int_0^1 \frac{|t^{\rho\alpha} - (1-t^\rho)^\alpha| |t^{\rho-1}|}{A_t^2} \left| f' \left(\frac{a^\rho b^\rho}{A_t} \right) \right| dt \\ &\leq \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} \left(\int_0^1 \frac{|t^{\rho\alpha} - (1-t^\rho)^\alpha| |t^{\rho-1}|}{A_t^2} dt \right)^{1-1/q} \end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^1 \frac{|t^{\rho\alpha} - (1-t^\rho)^\alpha| |t^{\rho-1}|}{A_t^2} \left| f' \left(\frac{a^\rho b^\rho}{A_t} \right) \right| dt \right)^{1/q} \\
& \leq \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} \left(\int_0^1 \frac{|t^{\rho\alpha} - (1-t^\rho)^\alpha| |t^{\rho-1}|}{A_t^2} dt \right)^{1-1/q} \\
& \quad \times \left(\int_0^1 \frac{|t^{\rho\alpha} - (1-t^\rho)^\alpha| |t^{\rho-1}|}{A_t^2} \left(t^\rho |f'(b^\rho)|^q + (1-t^\rho) |f'(a^\rho)|^q \right) dt \right)^{1/q} \\
& \leq \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} K_1^{1-1/q}(\alpha; a, b) \left(K_2(\alpha; a, b) |f'(b)|^q + K_3(\alpha; a, b) |f'(a)|^q \right)^{1/q}.
\end{aligned} \tag{3.11}$$

Calculating K_1 , K_2 and K_3 , by Lemma 1, we get

$$\begin{aligned}
K_1 &= \int_0^1 \frac{|t^{\rho\alpha} - (1-t^\rho)^\alpha| |t^{\rho-1}|}{A_t^2} dt \\
&= \int_0^{1/2} \frac{((1-t^\rho)^\alpha - t^{\rho\alpha}) t^{\rho-1}}{A_t^2} dt + \int_{1/2}^1 \frac{(t^{\rho\alpha} - (1-t^\rho)^\alpha) t^{\rho-1}}{A_t^2} dt \\
&= \int_0^1 \frac{(t^{\rho\alpha} - (1-t^\rho)^\alpha) t^{\rho-1}}{A_t^2} dt + 2 \int_0^{1/2} \frac{((1-t^\rho)^\alpha - t^{\rho\alpha}) t^{\rho-1}}{A_t^2} dt \\
&\leq \int_0^1 u^\alpha A_u^{-2} du - \int_0^1 (1-u)^\alpha A_u^{-2} du + 2 \int_0^{1/2} (1-2u)^\alpha A_u^{-2} du \\
&= \frac{b^{-2\rho}}{\rho(\alpha+1)} \left[{}_2F_1 \left(2, \alpha+1; \alpha+2; 1 - \frac{a^\rho}{b^\rho} \right) - {}_2F_1 \left(2, 1; \alpha+2; 1 - \frac{a^\rho}{b^\rho} \right) \right. \\
&\quad \left. + {}_2F_1 \left(2, 1; \alpha+2; \frac{1}{2} \left(1 - \frac{a^\rho}{b^\rho} \right) \right) \right].
\end{aligned} \tag{3.12}$$

and similarly we obtain

$$\begin{aligned}
K_2 &= \int_0^1 \frac{|t^{\rho\alpha} - (1-t^\rho)^\alpha| |t^{\rho-1}|}{A_t^2} t^\rho dt \\
&\leq \int_0^1 u^{\alpha+1} A_u^{-2} du - \int_0^1 (1-u)^\alpha u A_u^{-2} du + 2 \int_0^{1/2} (1-2u)^\alpha u A_u^{-2} du \\
&= \frac{b^{-2\rho}}{\rho(\alpha+2)} \left[{}_2F_1 \left(2, \alpha+2; \alpha+3; 1 - \frac{a^\rho}{b^\rho} \right) - \frac{1}{\alpha+1} {}_2F_1 \left(2, 2; \alpha+3; 1 - \frac{a^\rho}{b^\rho} \right) \right. \\
&\quad \left. + \frac{1}{2(\alpha+1)} {}_2F_1 \left(2, 2; \alpha+3; \frac{1}{2} \left(1 - \frac{a^\rho}{b^\rho} \right) \right) \right],
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
 K_3 &= \int_0^1 \frac{|t^{\rho\alpha} - (1-t^\rho)^\alpha| |t^{\rho-1}|}{A_t^2} (1-t)^\rho dt \\
 &\leq \int_0^1 t^\alpha (1-u) A_u^{-2} du - \int_0^1 (1-u)^{\alpha+1} A_u^{-2} du + 2 \int_0^{1/2} (1-2u)^\alpha (1-u) A_u^{-2} du \\
 &= \frac{b^{-2\rho}}{\rho(\alpha+2)} \left[\frac{1}{\alpha+1} {}_2F_1 \left(2, \alpha+1; \alpha+3; 1 - \frac{a^\rho}{b^\rho} \right) - {}_2F_1 \left(2, 1; \alpha+3; 1 - \frac{a^\rho}{b^\rho} \right) \right. \\
 &\quad \left. + {}_2F_1 \left(2, 1; \alpha+3; \frac{1}{2} \left(1 - \frac{a^\rho}{b^\rho} \right) \right) \right]. \tag{3.14}
 \end{aligned}$$

So, if we use (3.12)-(3.14) in (3.11), we get the inequality (3.10). $\square$

Remark 4. In Theorem 7, taking limit $\rho \rightarrow 1$ we obtain Theorem 6 in [5].

Theorem 8. Let $\alpha > 0$ and $\rho > 0$. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function such that $f \in X_c^\rho(a^\rho, b^\rho)$, where $a^\rho, b^\rho \in I$ with $a < b$. If $|f|^l$ is a harmonically convex function on $[a, b]$ for some fixed $l > 1, 1/k + 1/l = 1$, then the following inequalities holds:

$$|I_f(g; \alpha, a, b)| = \frac{a^\rho(b^\rho - a^\rho)}{2b^\rho} \left(\Lambda_7^{1/k} + \Lambda_8^{1/k} \right) \left(\frac{|f'(a^\rho)|^l + |f'(b^\rho)|^l}{\rho+1} \right)^{1/l} \tag{3.15}$$

where

$$\begin{aligned}
 \Lambda_7 &= B \left(\frac{\rho k - k + 1}{\rho}, \alpha k + 1 \right) {}_2F_1 \left(2k, \frac{\rho k - k + 1}{\rho}; \alpha k + k + 1 + \frac{1-k}{\rho}; 1 - \frac{a^\rho}{b^\rho} \right) \\
 \Lambda_8 &= \frac{1}{\left(\alpha k + k + \frac{1-k}{\rho} \right)} {}_2F_1 \left(2k, \alpha k + k + \frac{1-k}{\rho}; \alpha k + k + 1 + \frac{1-k}{\rho}; 1 - \frac{a^\rho}{b^\rho} \right)
 \end{aligned}$$

Proof. Let $A_t = t^\rho a^\rho + (1-t^\rho)b^\rho$. From Lemma 2, Hölder inequality and the harmonically convexity of $|f|^q$, we have

$$\begin{aligned}
 &|I_f(g; \alpha, a, b)| \\
 &\leq \frac{a^\rho b^\rho (b^\rho - a^\rho)}{2} \left[\int_0^1 \frac{t^{\rho\alpha} t^{\rho-1}}{A_t^2} \left| f' \left(\frac{a^\rho b^\rho}{A_t} \right) \right| dt + \int_0^1 \frac{(1-t^\rho)^\alpha t^{\rho-1}}{A_t^2} \left| f' \left(\frac{a^\rho b^\rho}{A_t} \right) \right| dt \right] \\
 &\leq \frac{a^\rho b^\rho (b^\rho - a^\rho)}{2} \left\{ \left(\int_0^1 \frac{t^{\rho\alpha} t^{k(\rho-1)}}{A_t^{2k}} dt \right)^{1/k} \left(\int_0^1 \left| f' \left(\frac{a^\rho b^\rho}{A_t^{2k}} \right) \right|^l dt \right)^{1/l} \right. \\
 &\quad \left. + \left(\int_0^1 \frac{(1-t^\rho)^\alpha t^{k(\rho-1)}}{A_t^{2k}} dt \right)^{1/k} \left(\int_0^1 \left| f' \left(\frac{a^\rho b^\rho}{A_t^{2k}} \right) \right|^l dt \right)^{1/l} \right\}
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{a^\rho b^\rho (b^\rho - a^\rho)}{2} \left(K_4^{1/k} + K_5^{1/k} \right) \left(\int_0^1 \left[t^\rho |f'(b^\rho)|^l + (1-t^\rho) |f'(a^\rho)|^l \right] dt \right)^{1/l} \\
&\leq \frac{a^\rho b^\rho (b^\rho - a^\rho)}{2} \left(K_4^{1/k} + K_5^{1/k} \right) \left(\frac{|f'(a^\rho)|^l + |f'(b^\rho)|^l}{\rho + 1} \right)^{1/l}. \quad (3.16)
\end{aligned}$$

Calculating K_4 and K_5 , we obtain

$$\begin{aligned}
K_4 &= \int_0^1 \frac{(1-t^\rho)^{\alpha k} t^{k(\rho-1)}}{A_t^{2k}} dt \\
&= \frac{b^{-2\rho k}}{B(\frac{\rho k - k + 1}{\rho}, \alpha k + 1)} {}_2F_1 \left(2k, \frac{\rho k - k + 1}{\rho}; \alpha k + k + 1 + \frac{1-k}{\rho}; 1 - \frac{a^\rho}{b^\rho} \right) \quad (3.17)
\end{aligned}$$

$$\begin{aligned}
K_5 &= \int_0^1 \frac{t^{\rho \alpha k} t^{k(\rho-1)}}{A_t^{2k}} dt \\
&= \left(\alpha k + k + \frac{1-k}{\rho} \right) b^{-2\rho k} {}_2F_1 \left(2k, \alpha k + k + \frac{1-k}{\rho}; \alpha k + k + 1 + \frac{1-k}{\rho}; 1 - \frac{a^\rho}{b^\rho} \right) \quad (3.18)
\end{aligned}$$

So, if we use (3.17) and (3.18) in (3.16), we get the inequality of (3.15). This completes the proof. $\square$

Remark 5. In Theorem 8, taking limit $\rho \rightarrow 1$ we obtain Theorem 7 in [5].

Theorem 9. Let $\alpha > 0$ and $\rho > 0$. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function such that $f \in X_c^\rho(a^\rho, b^\rho)$, where $a^\rho, b^\rho \in I$ with $a < b$. If $|f|^l$ is a harmonically convex function on $[a, b]$ for some fixed $l > 1$, $1/k + 1/l = 1$, then the following inequalities holds:

$$|I_f(g; \alpha, a, b)| \leq \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} \Lambda_9^{1/k} \left(\Lambda_{10} |f'(b^\rho)|^l + \Lambda_{11} |f'(a^\rho)|^l \right)^{1/l} \quad (3.19)$$

where

$$\Lambda_9 = b^{-2\rho k} {}_2F_1 \left(2k, \frac{1}{\rho}; \frac{\rho+1}{\rho}; 1 - \frac{a^\rho}{b^\rho} \right) \quad (3.20)$$

$$\Lambda_{10} = \frac{1}{\rho 2^{\frac{\rho+1}{\rho}}} B \left(\frac{\rho+1}{\rho}, \alpha l + 1 \right) + \frac{\alpha l + 1}{2\rho} {}_2F_1 \left(\frac{-1}{\rho}, 1; \alpha l + 2; \frac{1}{2} \right) \quad (3.21)$$

$$\begin{aligned}
\Lambda_{11} &= \frac{1}{\rho 2^{\frac{1}{\rho}}} B \left(\frac{1}{\rho}, \alpha l + 1 \right) {}_2F_1 \left(-1, \frac{1}{\rho}; \alpha l + \frac{1}{\rho} + 1; \frac{1}{2} \right) \\
&\quad + \frac{(\alpha l + 1)(\alpha l + 2)}{\rho 2^{\frac{2-\rho}{\rho}}} {}_2F_1 \left(\frac{1-\rho}{\rho}, 2; \alpha l + 3; \frac{1}{2} \right). \quad (3.22)
\end{aligned}$$

Proof. Let $A_t = t^\rho a^\rho + (1-t^\rho)b^\rho$. From Lemma 1, Lemma 2, Hölder inequality and the harmonically convexity of $|f|^q$, we have

$$\begin{aligned}
 |I_f(g; \alpha, a, b)| &\leq \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} \int_0^1 \frac{|t^{\rho\alpha} - (1-t^\rho)^\alpha| |t^{\rho-1}|}{A_t^{2k}} \left| f' \left(\frac{a^\rho b^\rho}{A_t} \right) \right| dt \\
 &\leq \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} \left(\int_0^1 \frac{1}{A_t^{2k}} dt \right)^{1/k} \\
 &\quad \times \left(\int_0^1 |t^{\rho\alpha} - (1-t^\rho)^\alpha|^l |t^{\rho-1}|^l \left| f' \left(\frac{a^\rho b^\rho}{A_t} \right) \right|^l dt \right)^{1/l} \\
 &\leq \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} \left(\int_0^1 \frac{1}{A_t^{2k}} dt \right)^{1/k} \\
 &\quad \times \left(\int_0^1 |1-2t^\rho|^{\alpha l} \left[t^\rho |f'(b^\rho)|^l + (1-t^\rho) |f'(a^\rho)|^l \right] dt \right)^{1/l} \\
 &\leq \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} K_6^{1/k} \left(K_7 |f'(b^\rho)|^l + K_8 |f'(a^\rho)|^l \right)^{1/l}. \quad (3.23)
 \end{aligned}$$

where

$$\Lambda_9 = \int_0^1 \frac{1}{A_t^{2k}} dt = b^{-2\rho k} {}_2F_1 \left(2k, \frac{1}{\rho}; \frac{\rho+1}{\rho}; 1 - \frac{a^\rho}{b^\rho} \right) \quad (3.24)$$

$$\begin{aligned}
 \Lambda_{10} &= \int_0^1 |1-2t^\rho|^{\alpha l} t^\rho dt \\
 &= \int_0^{1/2^{1/\rho}} (1-2t^\rho)^{\alpha l} t^\rho dt + \int_{1/2^{1/\rho}}^1 (2t^\rho-1)^{\alpha l} t^\rho dt \\
 &= \frac{1}{\rho 2^{\frac{\rho+1}{\rho}}} B \left(\frac{\rho+1}{\rho}, \alpha l + 1 \right) + \frac{\alpha l + 1}{2\rho} {}_2F_1 \left(\frac{-1}{\rho}, 1; \alpha l + 2; \frac{1}{2} \right) \quad (3.25)
 \end{aligned}$$

$$\begin{aligned}
 \Lambda_{11} &= \int_0^1 |1-2t^\rho|^{\alpha l} (1-t^\rho) dt \\
 &= \int_0^{1/2^{1/\rho}} (1-2t^\rho)^{\alpha l} (1-t^\rho) dt + \int_{1/2^{1/\rho}}^1 (2t^\rho-1)^{\alpha l} (1-t^\rho) dt \\
 &= \frac{1}{\rho 2^{\frac{1}{\rho}}} B \left(\frac{1}{\rho}, \alpha l + 1 \right) {}_2F_1 \left(-1, \frac{1}{\rho}; \alpha l + \frac{1}{\rho} + 1; \frac{1}{2} \right) \\
 &\quad + \frac{(\alpha l + 1)(\alpha l + 2)}{\rho 2^{\frac{2-\rho}{\rho}}} {}_2F_1 \left(\frac{1-\rho}{\rho}, 2; \alpha l + 3; \frac{1}{2} \right). \quad (3.26)
 \end{aligned}$$

So, if we use (3.24)-(3.26) in (3.23), we get desired result. $\square$

Remark 6. In Theorem 9, taking limit $\rho \rightarrow 1$ we obtain Theorem 8 in [5].

Theorem 10. Let $\alpha > 0$ and $\rho > 0$. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function such that $f \in X_c^p(a^\rho, b^\rho)$, where $a^\rho, b^\rho \in I$ with $a < b$. If $|f|^q$ is a harmonically convex function on $[a, b]$ for some fixed $q > 1$, $1/k + 1/l = 1$, then the following inequalities holds:

$$|I_f(g; \alpha, a, b)| = \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} \Lambda_{12}^{1/k} \left(\Lambda_{13} |f'(b^\rho)|^l + \Lambda_{14} |f'(a^\rho)|^l \right)^{1/l} \quad (3.27)$$

where

$$\begin{aligned} \Lambda_{12} &= \frac{1}{\rho 2^{(k\rho-k+1)/\rho}} B\left(\frac{k\rho-k+1}{\rho}, \alpha k + 1\right) \\ &\quad + \frac{1}{\rho 2^\rho} {}_2F_1\left(\frac{k+\rho-k\rho-1}{\rho}, 1; \alpha k + 2; \frac{1}{2}\right) \\ \Lambda_{13} &= \frac{1}{(\rho+1)b^{2\rho l}} {}_2F_1\left(2l, \frac{\rho+1}{\rho}; \frac{2\rho+1}{\rho}; 1 - \frac{a^\rho}{b^\rho}\right) \\ \Lambda_{14} &= \frac{\rho}{(\rho+1)b^{2\rho l}} {}_2F_1\left(2l, \frac{1}{\rho}; \frac{2\rho+1}{\rho}; 1 - \frac{a^\rho}{b^\rho}\right). \end{aligned}$$

Proof. Let $A_t = t^\rho a^\rho + (1-t^\rho)b^\rho$. From Lemma 1, Lemma 2, Hölder inequality and the harmonically convexity of $|f|^q$, we have

$$\begin{aligned} |I_f(g; \alpha, a, b)| &\leq \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} \int_0^1 \frac{|t^{\rho\alpha} - (1-t^\rho)^\alpha| |t^{\rho-1}|}{A_t^2} \left| f' \left(\frac{a^\rho b^\rho}{A_t} \right) \right| dt \\ &\leq \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} \left(\int_0^1 |t^{\rho\alpha} - (1-t^\rho)^\alpha|^k |t^{\rho-1}|^k dt \right)^{1/k} \\ &\quad \times \left(\int_0^1 \frac{1}{A_t^{2l}} \left| f' \left(\frac{a^\rho b^\rho}{A_t} \right) \right|^l dt \right)^{1/l} \\ &\leq \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} \left(\int_0^1 |(2t^\rho - 1)|^{\alpha k} t^{k(\rho-1)} dt \right)^{1/k} \\ &\quad \times \left(\int_0^1 \frac{1}{A_t^{2l}} \left[t^\rho |f'(b^\rho)|^l + (1-t^\rho) |f'(a^\rho)|^l \right] dt \right)^{1/l} \\ &\leq \frac{\rho a^\rho b^\rho (b^\rho - a^\rho)}{2} K_6^{1/k} \left(K_7 |f'(b^\rho)|^l + K_8 |f'(a^\rho)|^l \right)^{1/l}. \quad (3.28) \end{aligned}$$

where

$$\Lambda_{12} = \int_0^1 |(2t^\rho - 1)|^{\alpha k} t^{k(\rho-1)} dt$$

$$\begin{aligned}
&= \int_0^{1/2^{1/\rho}} (1-2t^\rho)^{\alpha k} t^{k(\rho-1)} dt + \int_{1/2^{1/\rho}}^1 (2t^\rho-1)^{\alpha k} t^{k(\rho-1)} dt \\
&= \frac{1}{\rho 2^{(k\rho-k+1)/\rho}} B\left(\frac{k\rho-k+1}{\rho}, \alpha k+1\right)
\end{aligned} \tag{3.29}$$

$$+ \frac{1}{\rho 2^\rho} {}_2F_1\left(\frac{k+\rho-k\rho-1}{\rho}, 1; \alpha k+2; \frac{1}{2}\right), \tag{3.30}$$

$$\begin{aligned}
\Lambda_{13} &= \int_0^1 t^\rho A_t^{-2l} dt \\
&= \frac{1}{(\rho+1)b^{2\rho l}} {}_2F_1\left(2l, \frac{\rho+1}{\rho}; \frac{2\rho+1}{\rho}; 1-\frac{a^\rho}{b^\rho}\right)
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
\Lambda_{14} &= \int_0^1 (1-t^\rho) A_t^{-2l} dt \\
&= \frac{\rho}{(\rho+1)b^{2\rho l}} {}_2F_1\left(2l, \frac{1}{\rho}; \frac{2\rho+1}{\rho}; 1-\frac{a^\rho}{b^\rho}\right)
\end{aligned} \tag{3.32}$$

So, if we use (3.30)-(3.32) in (3.28), we obtain desired result. $\square$

Remark 7. In Theorem 10, taking limit $\rho \rightarrow 1$ we obtain Theorem 9 in [5].

REFERENCES

- [1] S. Belarbi and Z. Dahmani, "On some new fractional integral inequalities," *J. Inequal. Pure Appl. Math.*, vol. 10, no. 3, p. 5, 2009.
- [2] Z. Dahmani, "New inequalities in fractional integrals," *IJNS, International Journal of Nonlinear Sciences*, vol. 9, no. 4, pp. 493–497, 2010.
- [3] Z. Dahmani, L. Tabharit, and S. Taf, "New generalisations of gruss inequality using riemann-liouville fractional integrals," *Bull. Math. Anal. Appl.*, vol. 2, no. 3, pp. 93–99, 2010.
- [4] I. Iscan, "Hermite-hadamard type inequalities for harmonically convex functions," *Hacet. J. Math. Stat.*, 2014.
- [5] I. Iscan and S. Wu, "Hermite-hadamard type inequalities for harmonically convex functions via fractional integrals," *Applied Mathematics and Computation*, vol. 238, pp. 237–244, 2014, doi: [10.1016/j.amc.2014.04.020](https://doi.org/10.1016/j.amc.2014.04.020).
- [6] U. Katugampola, "New approach to generalized fractional derivatives," *Bull. Math. Anal. Appl.*, vol. 6, no. 4, pp. 1–15, 2014.
- [7] A. Kilbas, H. Srivastava, and J. Trujillo, *Theory and applications of fractional differential equations*. Amsterdam: Elsevier, 2006.
- [8] M. A. Noor, M. U. Awan, and K. I. Noor, "On some inequalities for relative semi-convex functions," *Journal of Inequalities and Applications*, vol. 332, 2013.
- [9] M. A. Noor, K. I. Noor, and M. U. Awan, "Generalized convexity and integral inequalities," *Applied Mathematics and Information Sciences*, vol. 9, no. 1, pp. 233–243, 2015, doi: [10.12785/amis/090129](https://doi.org/10.12785/amis/090129).
- [10] A. Prudnikov, Y. Brychkov, and O. Marichev, *Integral and series In: Elementary Functions*. vol. 1. Nauka, Moscow, 1981.
- [11] E. Rainville, *Special Functions*. The Mcmillan Company, New York, 1960.

- [12] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, “Hermite-hadamard’s inequalities for fractional integrals and related fractional inequalities,” *Math. Comput. Model.*, vol. 57, pp. 2403–2407, 2013, doi: [10.1016/j.mcm.2011.12.048](https://doi.org/10.1016/j.mcm.2011.12.048).
- [13] E. Set, A. Akdemir, and B. Çelik, “On generalization of fejér type inequalities via fractional integral operator,” *Filomat*, vol. 32, no. 16, pp. 5537–5547, 2018, doi: [10.2298/FIL1816537S](https://doi.org/10.2298/FIL1816537S).
- [14] E. Set, A. Akdemir, and I. Mumcu, “Ostrowski type inequalities involving special functions via conformable fractional integrals,” *J. Adv. Math. Stud.*, vol. 10, no. 3, pp. 386–395, 2017.
- [15] E. Set, A. Akdemir, and I. Mumcu, “Hadamard’s inequality and its extensions for conformable fractional integrals of any order $\alpha > 0$,” *Creat. Math. Inform.*, vol. 27, no. 2, pp. 197–206, 2018.
- [16] E. Set and B. Çelik, “Fractional hermite-hadamard type inequalities for quasi-convex functions,” *Ordu Univ. J. Sci. Tech.*, vol. 6, no. 1, pp. 137–149, 2016.
- [17] E. Set, I. Iscan, and F. Zehir, “On some new inequalities of hermite-hadamard type involving harmonically convex functions via fractional integrals,” *Konuralp J. Math.*, vol. 3, no. 1, pp. 42–55, 2015.
- [18] E. Set, M. Noor, M. Awan, and A. Gözpinar, “Generalized hermite-hadamard type inequalities involving fractional integral operators,” *Journal of Inequalities and Applications*, vol. 2107, no. 169, pp. 1–10, 2017, doi: [10.1186/s13660-017-1444-6](https://doi.org/10.1186/s13660-017-1444-6).
- [19] E. Set, M. Z. Sarikaya, M. E. Ozdemir, and H. Yildirim, “The hermite-hadamard’s inequality for some convex functions via fractional integrals and related results,” *JAMSI*, vol. 10, no. 2, pp. 69–83, 2014, doi: [10.2478/jamsi-2014-0014](https://doi.org/10.2478/jamsi-2014-0014).

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ON THE REDUCTION OF THE IDENTIFICATION OF THE PARAMETERS OF A DIFFERENTIAL EQUATION IN A HILBERT SPACE TO A BOUNDARY VALUE PROBLEM

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Abstract. The work considers the problem of differential equation parameter identification given in Hilbert space. Conditions of existence of problem solutions are established. They correspond to conditions of continuous dependence of solutions of differential equations on parameters. In case of linear model and space L_2 condition means convergence in mean quadratic sense on aprioric set.

A constructive algorithm of identification problem solution in Hilbert space is developed. It is based on the considering of the estimation of differential equation solution and it results in the solving corresponding boundary value problem. The way of its reduction to initial value problems is offered. One partial case allowing problem solution not only in operator form is considered.

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1. INTRODUCTION

When modeling physical and biological processes of real world [9], [10], [11], [12], [14], [15] problems of search of system parameter estimates which can be elements of some functional spaces arise. Such models include differential equations with delay, integro-differential equations. Their theory is presented in works [5], [6]. Some results on parameter identification of these systems were presented in earlier monographs [2], [13]. Infinite-dimensional minimax problems were displayed in [8], [17].

Last time there appeared interest on such problems dealing with investigation of the "solution semigroup and the structural operators in the initial data space" [3], [7]. As a rule the results are presented "in terms of the initial values and eigenvectors of adjoint operator" [7]. Practical applications require more constructive presentations of the methods in the form of initial or boundary value problems.

Hence, the purpose of this research is to establish conditions of solution existence for parameter identification problems given in abstract Hilbert spaces and development of constructive algorithms resulting in corresponding boundary value problems.

2. CONDITIONS FOR EXISTENCE OF SOLUTIONS OF IDENTIFICATION PROBLEMS IN HILBERT SPACE

In order to solve identification problems in Hilbert space we need to establish some conditions of existence of solutions of these problems. The conditions that will be presented are based on the following theoretical background.

Let $\mathbb{H}$ be abstract Hilbert space of parameters α which should be identified. A priori set of parameters G is often described as

$$G = \{\alpha \in \mathbb{H} : \mathcal{J}(\alpha) \leq 1\}$$

where $\mathcal{J}(\alpha), \alpha \in \mathbb{H}$ is some functional. It is natural to choose as aposterioric estimate for $\alpha \in \mathbb{H}$ (denote it as $\hat{\alpha}$) minimum point of functional $\mathcal{J}(\alpha)$, that is

$$\hat{\alpha} = \arg \min_{\alpha \in G} \mathcal{J}(\alpha).$$

In case of quadratic functional $\mathcal{J}(\alpha)$ such an estimate corresponds to Chebyshev's center.

Thus, firstly let's establish existence conditions for minimum point of functional $\mathcal{J}(\alpha)$.

Definition 1. Functional $\mathcal{J}(\alpha), \alpha \in \mathbb{H}$ is said to be weakly lower semicontinuous at point α_0 if for arbitrary sequence $\{\alpha_n\}_{n=1, \infty}$ which converges weakly to α_0 , that is $\forall \beta \in \mathbb{H} \quad (\alpha_n, \beta)_{\mathbb{H}} \xrightarrow{n \rightarrow \infty} (\alpha_0, \beta)_{\mathbb{H}}$ holds

$$\liminf_{n \rightarrow \infty} \mathcal{J}(\alpha_n) \geq \mathcal{J}(\alpha_0) \quad (2.1)$$

The following result follows from the generalized Weierstrass Theorem ([1], Theorem 7.3.1)

Theorem 1. *Let*

- (i) $G \subset \mathbb{H}$ be bounded and weakly closed set;
 - (ii) $\mathcal{J} : \mathbb{H} \rightarrow R$ is weakly lower semicontinuous.
- Then there is point $\alpha_0 \in G$ such that*

$$\min_{\alpha \in G} \mathcal{J}(\alpha) = \mathcal{J}(\alpha_0)$$

Solving identification problems in Hilbert spaces they consider restrictions given by functionals of the following form

$$\mathcal{J}(\alpha) = \int_0^T (Q(y(t) - Hx(t)), (y(t) - Hx(t))) dt = \mathcal{J}_1(x) \quad (2.2)$$

where

$$Q > 0 \quad (2.3)$$

is some given positive definite matrix, $x(t) \in \mathbb{R}^n$ is system state dependent on parameter $\alpha \in \mathbb{H}$. Note, that as it follows from (2.3) functional (2.2) is lower continuous with respect to $x \in \mathbb{R}^n$. It is left to show that functional (2.2) is weakly lower semi-continuous with respect to $\alpha \in G$.

So, in order to prove existence of minimum point of functional $\mathcal{J}(\alpha)$ of the form (2.2) (in other words, existence of aposterioric estimate) it is necessary to establish conditions of continuous dependence $x(t)$ with respect to α .

Consider the following system

$$\begin{cases} \frac{dx(t)}{dt} = \mathcal{A}(t, \alpha) x(t) \\ x(t_0) = x_0 \end{cases} \quad (2.4)$$

where $x(t) \in \mathbb{C}^1([0, T], \mathbb{R}^n)$, $\mathcal{A}(t, \alpha) : [0, T] \times G \times \mathbb{C}^1([0, T], \mathbb{R}^n) \rightarrow \mathbb{C}([0, T], \mathbb{R}^n)$ is linear operator with respect to $x(t)$.

Lemma 1. Let operator $\mathcal{A}(\bullet, \bullet)$ be linear with respect to $x(t)$ and is such that

- (i) $\|\mathcal{A}(s, \alpha)\| \leq k(s)$ at $\alpha \in G, s \in [t_0, T]$;
- (ii) for arbitrary $x(s) \in \mathbb{C}^1([0, T], \mathbb{R}^n)$, arbitrary $\alpha_0 \in G$ and $\alpha_n \xrightarrow{n \rightarrow \infty} \alpha_0$ it holds

$$\lim_{\alpha_n \rightarrow \alpha_0} \int_{t'}^{t''} [\mathcal{A}(s, \alpha_n) x(s) - \mathcal{A}(s, \alpha_0) x(s)] ds = 0 \quad (2.5)$$

Then solution (2.4) $x(t)$ depends on $\alpha \in G$ continuously.

Proof. Let's denote solution (2.4) corresponding to parameter $\alpha \in \mathbb{H}$ as $x(t, \alpha)$. Further we estimate

$$\begin{aligned} & \|x(t, \alpha_n) - x(t, \alpha_0)\| = \\ & = \left\| x(t_0, \alpha_n) + \int_{t_0}^t \mathcal{A}(s, \alpha_n) x(s, \alpha_n) ds - x(t_0, \alpha_0) - \int_{t_0}^t \mathcal{A}(s, \alpha_0) x(s, \alpha_0) ds \right\| \leq \\ & \leq \|x(t_0, \alpha_n) - x(t_0, \alpha_0)\| + \left\| \int_{t_0}^t \mathcal{A}(s, \alpha_n) x(s, \alpha_n) ds - \int_{t_0}^t \mathcal{A}(s, \alpha_0) x(s, \alpha_0) ds \right\| \leq \\ & \leq \|x(t_0, \alpha_n) - x(t_0, \alpha_0)\| + \left\| \int_{t_0}^t \mathcal{A}(s, \alpha_n) x(s, \alpha_0) ds - \int_{t_0}^t \mathcal{A}(s, \alpha_0) x(s, \alpha_0) ds \right\| + \\ & + \left\| \int_{t_0}^t \mathcal{A}(s, \alpha_n) x(s, \alpha_n) ds - \int_{t_0}^t \mathcal{A}(s, \alpha_n) x(s, \alpha_0) ds \right\| \leq \\ & \leq \|x(t_0, \alpha_n) - x(t_0, \alpha_0)\| + \left\| \int_{t_0}^t \mathcal{A}(s, \alpha_n) x(s, \alpha_0) ds - \int_{t_0}^t \mathcal{A}(s, \alpha_0) x(s, \alpha_0) ds \right\| + \\ & + \int_{t_0}^t k(s) \|x(s, \alpha_n) - x(s, \alpha_0)\| ds \end{aligned}$$

Due to Gronwall lemma

$$\begin{aligned} & \|x(t, \alpha_n) - x(t, \alpha_0)\| \leq \\ & \leq \left(\|x(t_0, \alpha_n) - x(t_0, \alpha_0)\| + \left\| \int_{t_0}^t \mathcal{A}(s, \alpha_n) x(s, \alpha_0) ds - \int_{t_0}^t \mathcal{A}(s, \alpha_0) x(s, \alpha_0) ds \right\| \right) \\ & e^{\left\| \int_{t_0}^t k(s) ds \right\|} \end{aligned} \quad (2.6)$$

Taking into account (2.5) right side of inequality (2.6) tends to zero as $\alpha_n \xrightarrow{n \rightarrow \infty} \alpha_0$. $\square$

Corollary 1. *If operator $\mathcal{A}(s, \alpha)$ is linear with respect to α then condition (2.5) can be rewritten as*

$$\lim_{\alpha_n \rightarrow \alpha_0} \int_{t'}^{t''} \|\mathcal{A}(s, \alpha_n) - \mathcal{A}(s, \alpha_0)\| ds = 0 \quad (2.7)$$

Corollary 2. *If operator $\mathcal{A}(s, \alpha)$ is linear with respect to α and doesn't depend on t , that is $\mathcal{A}(s, \alpha) = \mathcal{A}(\alpha)$ then condition (2.5) can be rewritten as*

$$\lim_{\alpha_n \rightarrow \alpha_0} \|\mathcal{A}(\alpha_n) - \mathcal{A}(\alpha_0)\| = 0 \quad (2.8)$$

3. LINEAR SYSTEM WITH INTEGRAL KERNEL

Consider the following system which is partial case of (2.4)

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + \int_0^t K(t-s)x(s)ds \\ x(0) = x_0 \end{cases} \quad (3.1)$$

where $A, K(s) \in \mathbb{R}^{n \times n}$ are matrix and matrix-valued function correspondingly.

Lemma 2. *Let matrix-valued function $K(s)$ belong to set G which satisfies to (i) for any $K_0(s) \in G$ and $K_n \xrightarrow{n \rightarrow \infty} K_0$ we have*

$$\lim_{n \rightarrow \infty} \int_0^T \|K_n(\tau) - K_0(\tau)\|^2 d\tau = 0 \quad (3.2)$$

That is we require convergence in strong-quadratic sense.

Then the solution (2.4) $x(t)$ depends on $K(\bullet) \in G$ continuously.

Proof. Denote solution of (3.1) corresponding to some integral kernel $K(\bullet)$ as $x(t, K)$. Let's estimate

$$\begin{aligned}
& \|x(t, K_n) - x(t, K_0)\| = \\
& = \left\| \begin{aligned} & x(0, K_n) + \int_0^t \int_0^t K_n(t_1 - s) x(s, K_n) ds dt_1 - x(0, K_0) \\ & - \int_0^t \int_0^t K_0(t_1 - s) x(s, K_0) ds dt_1 + \int_0^t A(x(t, K_n) - x(t, K_0)) dt \end{aligned} \right\| \leq \\
& \leq \|x(0, K_n) - x(0, K_0)\| + \|A\| \int_0^t \|x(t, K_n) - x(t, K_0)\| dt + \\
& + \left\| \int_0^t \int_0^t K_n(t_1 - s) x(s, K_n) ds dt_1 - \int_0^t \int_0^t K_n(t_1 - s) x(s, K_0) ds dt_1 \right\| + \\
& + \left\| \int_0^t \int_0^t K_n(t_1 - s) x(s, K_0) ds dt_1 - \int_0^t \int_0^t K_0(t_1 - s) x(s, K_0) ds dt_1 \right\| \leq \\
& \leq \|x(0, K_n) - x(0, K_0)\| + \|A\| \int_0^t \|x(t, K_n) - x(t, K_0)\| dt + \\
& + \left\| \int_0^t \int_0^t K_n(t_1 - s) [x(s, K_n) - x(s, K_0)] ds dt_1 \right\| + \\
& + \left\| \int_0^t \int_0^t [K_n(t_1 - s) - K_0(t_1 - s)] x(s, K_0) ds dt_1 \right\| = \\
& = \|x(0, K_n) - x(0, K_0)\| + \|A\| \int_0^t \|x(t, K_n) - x(t, K_0)\| dt + \\
& + \left\| \int_0^t \int_s^t K_n(t_1 - s) dt_1 [x(s, K_n) - x(s, K_0)] ds \right\| + \\
& + \left\| \int_0^t \int_s^t [K_n(t_1 - s) - K_0(t_1 - s)] dt_1 x(s, K_0) ds \right\|
\end{aligned} \tag{3.3}$$

Further we use the following estimates

(a)

$$\begin{aligned}
& \left\| \int_0^t \int_s^t K_n(t_1 - s) dt_1 [x(s, K_n) - x(s, K_0)] ds \right\| \leq \\
& \leq \left\| \int_0^t \int_s^t |K_n(t_1 - s)| dt_1 [x(s, K_n) - x(s, K_0)] ds \right\| \leq \\
& \leq \left\| \int_0^T |K_n(\tau)| d\tau \right\| \int_0^t \|x(s, K_n) - x(s, K_0)\| ds
\end{aligned} \tag{3.4}$$

Due to Cauchy-Bunyakovsky inequality

$$\left\| \int_0^T |K_n(\tau)| d\tau \right\| \leq T \int_0^T \|K_n(\tau)\|^2 d\tau = TM \tag{3.5}$$

where $\int_0^T \|K_n(\tau)\|^2 d\tau = M < \infty$ due to condition (3.2), we can continue (3.4) as

$$\begin{aligned} & \left\| \int_0^t \int_s^t K_n(t_1 - s) dt_1 [x(s, K_n) - x(s, K_0)] ds \right\| \leq \\ & \leq TM \int_0^t \|x(s, K_n) - x(s, K_0)\| ds \end{aligned} \quad (3.6)$$

(b)

$$\begin{aligned} & \left\| \int_0^t \int_s^t [K_n(t_1 - s) - K_0(t_1 - s)] dt_1 x(s, K_0) ds \right\| \leq \\ & \leq \int_0^t \left\| \int_s^t [K_n(t_1 - s) - K_0(t_1 - s)] dt_1 \right\| \|x(s, K_0)\| ds \leq \\ & \leq \int_0^t \int_s^t \|K_n(t_1 - s) - K_0(t_1 - s)\|^2 dt_1 (t_1 - s) \|x(s, K_0)\| ds \leq \\ & \leq \int_0^T \int_0^T \|K_n(t_1 - s) - K_0(t_1 - s)\|^2 dt_1 T \|x(s, K_0)\| ds \end{aligned} \quad (3.7)$$

Applying (3.6) and (3.7) in (3.3) we get

$$\begin{aligned} & \|x(t, K_n) - x(t, K_0)\| \leq \\ & \leq \|x(0, K_n) - x(0, K_0)\| + \|A\| \int_0^t \|x(t, K_n) - x(t, K_0)\| dt + \\ & + TM \int_0^t \|x(s, K_n) - x(s, K_0)\| ds + \\ & + \int_0^T \int_0^T \|K_n(t_1 - s) - K_0(t_1 - s)\|^2 dt_1 T \|x(s, K_0)\| ds \end{aligned} \quad (3.8)$$

Using in (3.8) Gronwall lemma we get

$$\begin{aligned} & \|x(t, K_n) - x(t, K_0)\| \leq \\ & \leq (\|x(0, K_n) - x(0, K_0)\| + \\ & + T \int_0^T \int_0^T \|K_n(t_1 - s) - K_0(t_1 - s)\|^2 dt_1 \|x(s, K_0)\| ds) e^{(\|A\| + TM)t} \end{aligned} \quad (3.9)$$

By virtue to (3.2) right side of (3.9) tends to zero as $n \rightarrow \infty$. $\square$

4. ESTIMATION PROBLEMS IN HILBERT SPACE

Consider identification problem of system

$$\frac{dx}{dt} = f(t, x, \alpha) + \int_{t_0}^t g(t, s, x(s), \beta) ds \quad (4.1)$$

$$x(t_0) = x_0 \quad (4.2)$$

where $x(t) \in \mathbb{R}^n$, $\alpha \in \mathbb{H}_1$, $\beta \in \mathbb{H}_2$. Here $\mathbb{H}_1, \mathbb{H}_2$ are abstract Hilbert spaces. Introduce denotations

$$f_1(t, x, \theta) = f(t, x, \alpha) + \int_{t_0}^t g(t, s, x(s), \beta) ds, \quad \theta = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Assume that values of parameters α and β are unknown and their aprioric sets (G_α and G_β respectively) are only given. Consider the following problems.

Problem 1. Knowing values of function $x(s)$ and $\frac{dx(s)}{ds}$, $s \leq T$ with some unknown error, find estimates of parameters α and β .

Problem 2. For given function $y(t)$ such that

$$y(t) = h(t, x(t)) + \eta(t), t_0 \leq t \leq T, \quad \text{where } y \in \mathbb{R}^m,$$

h is known vector-valued function, $\eta(t)$ is some unknown function belonging to set G_η from space $L_2(t_0, T)$ find estimates of parameters α and β , and estimate of function $x(s)$, $s \geq T$.

Firstly note that these problems are covered by problems of estimation of differential equation solutions in Hilbert spaces.

Indeed if we introduce function $\theta(t)$ with values in $\mathbb{H} = \mathbb{H}_1 \times \mathbb{H}_2$ as solution of equation

$$\frac{d\theta}{dt} = 0, \theta(t_0) = \theta,$$

then we get system of equations

$$\begin{aligned} \frac{dx(t)}{dt} &= f_1(t, x(t), \theta(t)), \\ x(t_0) &= x_0, \\ \frac{d\theta}{dt} &= 0, \theta(t_0) = \theta. \end{aligned} \quad (4.3)$$

Thus in case of Problem 1 we should find an estimate $\theta(t)$ as solution of differential equation observing $x(t)$, $t_0 < t < T$ satisfying equation (4.1).

In case of the Problem 2 the problem of estimation of θ and $x(s)$ is reduced to estimate of differential equation solution in Hilbert space $\mathbb{R}^n \times \mathbb{H}$.

Hence it is reasonable to investigate problems on estimation of differential equation solution parameters in Hilbert space firstly.

Firstly consider case of linear differential equation on $\mathbb{H}$

$$\begin{aligned} \frac{dx}{dt} &= \mathcal{A}(t)x(t) + \mathcal{B}(t)f_1(t) \\ x(t_0) &= x_0, \end{aligned} \quad (4.4)$$

where $\mathcal{A}(t) \in \mathcal{L}(\mathbb{H}, \mathbb{H})$, $f_1(t)$ is unknown function from $L_2((t_0, T), \mathbb{F}_1)$, where $\mathbb{F}_1$ is some Hilbert space, $\mathcal{B}(t) \in \mathcal{L}(\mathbb{F}_1, \mathbb{H})$, norms $\|\mathcal{A}(t)\|$ and $\|\mathcal{B}(t)\|$ are continuous functions, x_0 is unknown vector from $\mathbb{H}$.

Note we mean solution of the following integral equation as generalized solution of equation (4.4)

$$x(t) = \int_{t_0}^t \mathcal{A}(s)x(s)ds + \int_{t_0}^t \mathcal{B}(s)f_1(s)ds$$

Such solution exists and is unique (it was proved in [16]). Besides that, let observations are given

$$y(t) = \mathcal{H}(t, y(\bullet), x(\bullet)) + \mathcal{D}(t)f_2(t)$$

where $\mathbb{F}_2, \mathbb{Y}$ are Hilbert spaces, $\|\mathcal{D}(t)\|$ is continuous with respect to t , f_2 is unknown function from space $\mathbb{F}_2$, $\mathcal{H}(t, y(\bullet), x(\bullet))$ depends on observations $y(s)$, $s < t$ and at fixed t and y is mapping the space $\mathbb{H}$ into $\mathbb{Y}$.

Also let triple $(x_0, f_1(\bullet), f_2(\bullet))$ belong to some set G of Hilbert space $\mathbb{H} \times L_2((t_0, T), \mathbb{F}_1) \times L_2((t_0, T), \mathbb{F}_2)$.

Definition 2. Aposterioric estimate of vector $\mathcal{S}x(T)$, where $\mathcal{S} \in \mathcal{L}(\mathbb{H}, \mathbb{F}_3)$, $\mathbb{F}_3$ is Hilbert space is vector $\mathcal{S}\hat{x}(T)$, where $\hat{x}(T)$ is solution of equation

$$\begin{aligned} \frac{d\hat{x}}{dt} &= \mathcal{A}(t)\hat{x}(t) + \mathcal{B}(t)\hat{f}_1(t), \\ \hat{x}(t_0) &= \hat{x}_0, \end{aligned}$$

and pair $(\hat{x}_0, \hat{f}_1)$ belongs to aposterioric set G_y determined by the following way

$$G_y = \left\{ (x_0, f_1) : (x_0, f_1, f_2) \in G_y^{(1)} \right\},$$

where

$$G_y^{(1)} = \{(x_0, f_1, f_2) : (x_0, f_1, f_2) \in G, y(t) = \mathcal{H}(t, y(\bullet), x(\bullet)) + \mathcal{D}(t)f_2(t), t_0 \leq t \leq T\}$$

It is obvious that if $\mathcal{H}(t, y(\bullet), x(\bullet))$ is continuous mapping with respect to x and set G is bounded then aposterioric set is also bounded.

Remark 1. If $\hat{x}_0, \hat{f}_1$ are aposteriric estimates of vectors x_0 and f_1 correspondingly then inequality holds

$$\left\{ \|\hat{x}_0 - x_0\|^2 + \|\hat{f}_1 - f_1\|^2 \right\}^{1/2} \leq \sup_{(x_0, f_1) \in G_y} \left[\|\hat{x}_0 - x_0\|^2 + \|\hat{f}_1 - f_1\|^2 \right]^{1/2} = \sigma_a^{(1)}$$

On the other hand for minimax aposterioric estimates $\hat{x}_0, \hat{f}_1$ determined by equality

$$\begin{aligned} \inf_{(\hat{x}_0, \hat{f}_1) \in G_y} \sup_{(x_0, f_1) \in G_y} \left\{ \|\hat{x}_0 - x_0\|^2 + \|\hat{f}_1 - f_1\|^2 \right\}^{1/2} &\leq \\ &\leq \sup_{(x_0, f_1) \in G_y} \left[\|\hat{x}_0 - x_0\|^2 + \|\hat{f}_1 - f_1\|^2 \right]^{1/2} = \sigma_a \end{aligned}$$

the following inequality holds

$$\sigma_a \leq \sigma_a^{(1)}.$$

Remark 2. Let G be bounded closed set. Then

$$\sup_{(x_0, f_1) \in G_y} \left[\|\hat{x}_0 - x_0\|^2 + \|\hat{f}_1 - f_1\|^2 \right]^{1/2} = \sup_{(l_1, l_1) + (l_2, l_2) \leq 1} \left[\sigma_l + |L(\hat{f}) - \hat{L}(f)| \right]^{1/2}$$

where

$$\begin{aligned} \sigma_l &= \frac{1}{2} \left[\sup_{G_y} L(f) - \inf_{G_y} L(f) \right] \\ \hat{L}(f) &= \frac{1}{2} \left[\sup_{G_y} L(f) + \inf_{G_y} L(f) \right] \\ L(f) &= (l_1, x_0) + (l_2, f_1) \\ L(\hat{f}) &= (l_1, \hat{x}_0) + (l_2, \hat{f}_1) \end{aligned}$$

Proof. Since

$$\|\hat{x}_0 - x_0\|^2 + \|\hat{f}_1 - f_1\|^2 = \sup_{(l_1, l_1) + (l_2, l_2) \leq 1} \left[L(\hat{f}) - L(f) \right]^2$$

then

$$\sigma_a^{(1)} = \sup_{(l_1, l_1) + (l_2, l_2) \leq 1} \sup_{f \in G_y} |L(\hat{f}) - L(f)|$$

Note that $L(f)$ is changing within

$$\inf_{G_y} L(f) \leq L(f) \leq \sup_{G_y} L(f)$$

or

$$|L(f) - \hat{L}(f)| \leq \sigma_l$$

□

Remark 3. Let set G_y be bounded and closed, and also centrally symmetric with respect to vector $\bar{x}_0, \bar{f}_1$ (that is if $(x_0 - \bar{x}_0, f_1 - \bar{f}_1) \in G_y$ then $-(x_0 - \bar{x}_0, f_1 - \bar{f}_1) \in G_y$ also). Then it is easy seen that

$$\sup_{G_y} L(f) = \sup_{\tilde{G}_y} L(f) + L(\bar{f})$$

$$\inf_{G_y} L(f) = -\sup_{\tilde{G}_y} L(f) + L(\bar{f})$$

where $\tilde{G}_y = G_y - (\bar{x}_0, \bar{f}_0)$ and is centrally symmetric with respect to zero and it implies

$$\sigma_a^{(1)} = \sup_{(l_1, l_1) + (l_2, l_2) \leq 1} \left[\sup_{\tilde{G}_y} L(f) + \left| L(\hat{f}) - L(\bar{f}) \right| \right] \geq \sup_{\tilde{G}_y} [(x_0, x_0) + (f_1, f_1)]^{1/2}$$

Equality sign is reached as $\hat{f} = \bar{f}$. That is $\bar{f}$ is minimax aposterioric estimate with estimation error

$$\sigma_a = \sup_{\tilde{G}_y} [(x_0, x_0) + (f_1, f_1)]^{1/2}$$

Further we consider case when $\mathcal{H}(t, y(\bullet), x(\bullet))$ depends on $x(\bullet)$ linearly. Moreover,

$$\mathcal{H}(t, y(\bullet), x(\bullet)) = \mathcal{H}(t, y(\bullet))x(t) \quad (4.5)$$

where $\mathcal{H}(t, y(\bullet)) \in \mathcal{L}(H, Y)$.

Introduce functional

$$\begin{aligned} \mathcal{J}(x_0, f_1, f_2) = & (\mathcal{Q}_0 x_0, x_0) + \int_{t_0}^T (\mathcal{Q}_1 f_1, f_1) dt + \int_{t_0}^T (\mathcal{Q}_2 f_2, f_2) dt + \\ & + \int_{t_0}^T (\lambda(t), y(t) - \mathcal{H}(t, y(\bullet))x(t) - \mathcal{D}(t)f_2(t)) dt \end{aligned}$$

and consider the following aposterioric set

$$G_y = \{(x_0, f_1) : \mathcal{J}(x_0, f_1, f_2) \leq 1\} \quad (4.6)$$

Lemma 3. *Aposterioric estimates of the problem (4.4)-(4.6) can be found as a result of solution of the following system of equations with respect to $p(t), \hat{x}(t), \hat{\lambda}(t)$*

$$\begin{cases} -p'(t) - \mathcal{A}^*(t)p(t) = \mathcal{H}^*(t, y(\bullet))\hat{\lambda}(t), \\ p(T) = 0, \end{cases} \quad (4.7)$$

$$\begin{cases} \frac{d\hat{x}(t)}{dt} = \mathcal{A}(t)\hat{x}(t) + \mathcal{B}(t)\mathcal{Q}_1^{-1}\mathcal{B}^*(t)p(t), \\ \hat{x}(t_0) = \mathcal{Q}_0^{-1}p(t_0), \end{cases} \quad (4.8)$$

$$y(t) = \mathcal{H}^*(t, y(\bullet))\hat{x}(t) + \frac{1}{2}\mathcal{D}(t)\mathcal{Q}_2^{-1}\mathcal{D}^*(t)\hat{\lambda}(t) \quad (4.9)$$

and have the following form

$$\hat{f}_1(t) = \mathcal{Q}_1^{-1} \mathcal{B}^*(t) p(t), \hat{f}_2(t) = \frac{1}{2} \mathcal{Q}_2^{-1} \mathcal{D}^*(t) \hat{\lambda}(t) \quad (4.10)$$

Proof. Calculate

$$\begin{aligned} \left. \frac{d}{d\tau} \mathcal{J}(x_0 + \tau w, f_1 + \tau g_1, f_2 + \tau g_2) \right|_{\tau=0} &= 2(\mathcal{Q}_0 x_0, w) + 2 \int_{t_0}^T (\mathcal{Q}_1 f_1, g_1) dt + \\ &+ 2 \int_{t_0}^T (\mathcal{Q}_2 f_2, g_2) dt - \int_{t_0}^T (\lambda(t), \mathcal{H}(t, y(\bullet)) x(g_1) - \mathcal{D}(t) g_2(t)) dt = \\ &= 2(\mathcal{Q}_0 x_0, w) + 2 \int_{t_0}^T (\mathcal{Q}_1 f_1, g_1) dt + 2 \int_{t_0}^T (\mathcal{Q}_2 f_2, g_2) dt - \int_{t_0}^T (\mathcal{D}^*(t) \lambda(t), g_2) dt - \\ &- \int_{t_0}^T (\lambda(t), \mathcal{H}(t, y(\bullet)) x(g_1)) dt \end{aligned}$$

Introduce adjoined function $p(t)$ as solution of equation

$$-p'(t) - \mathcal{A}^*(t) p(t) = \mathcal{H}^*(t, y(\bullet)) \hat{\lambda}(t)$$

Hence

$$\begin{aligned} \left. \frac{d}{d\tau} \mathcal{J}(x_0 + \tau w, f_1 + \tau g_1, f_2 + \tau g_2) \right|_{\tau=0} &= 2(\mathcal{Q}_0 x_0, w) + 2 \int_{t_0}^T (\mathcal{Q}_1 f_1, g_1) dt + \\ &+ 2 \int_{t_0}^T (\mathcal{Q}_2 f_2, g_2) dt - \int_{t_0}^T (\mathcal{D}^*(t) \lambda(t), g_2) dt + (p(T), x(T)) - (p(t_0), w) - \\ &- \int_{t_0}^T (p(t), \mathcal{A}(t) x(t)) dt - \int_{t_0}^T (p(t), \mathcal{B}(t) g_1(t)) dt + \int_{t_0}^T (\mathcal{A}^*(t) p(t), x(t)) dt = 0 \end{aligned}$$

□

Lemma 4. Let there exists $\mathcal{G} = [\mathcal{D}(t) \mathcal{Q}_2^{-1} \mathcal{D}^*(t)]^{-1}$. Then solution of system (4.7)-(4.9) can be found in form

$$\hat{x}(t) = \mathcal{P}(t) p(t) + q(t) \quad (4.11)$$

where $q(t) \in \mathbb{H}$ is solution of problem

$$\begin{cases} \frac{dq(t)}{dt} = \mathcal{A}(t) q(t) + 2\mathcal{P}(t) \mathcal{H}^*(t, y(\bullet)) \mathcal{G} [y(t) - \mathcal{H}(t, y(\bullet)) q(t)] \\ q(t_0) = \mathcal{O}, \quad \mathcal{O} \text{ is null-vector,} \end{cases} \quad (4.12)$$

operator $\mathcal{P}(t) \in \mathcal{L}(\mathbb{H}, \mathbb{H})$ is solution of the following Riccati equation

$$\begin{cases} \frac{d\mathcal{P}(t)}{dt} = \mathcal{A}(t) \mathcal{P}(t) + \mathcal{P}(t) \mathcal{A}^*(t) - 2\mathcal{P}(t) \mathcal{H}^*(t, y(\bullet)) \mathcal{G} \mathcal{H}(t, y(\bullet)) \mathcal{P}(t) + \\ \quad + \mathcal{B}(t) \mathcal{Q}_1^{-1} \mathcal{B}^*(t) \\ \mathcal{P}(t_0) = \mathcal{Q}_0^{-1}. \end{cases} \quad (4.13)$$

Proof. From (4.11) it follows that

$$\frac{d\hat{x}(t)}{dt} = \frac{d\mathcal{P}(t)}{dt} p(t) + \mathcal{P}(t) p'(t) + q'(t) \quad (4.14)$$

Using in (4.14) equations (4.7), (4.8) we have

$$\begin{aligned} \mathcal{A}(t)\hat{x}(t) + \mathcal{B}(t)\mathcal{Q}_1^{-1}\mathcal{B}^*(t)p(t) &= \frac{d\mathcal{P}(t)}{dt}p(t) + \\ + \mathcal{P}(t) \left[-\mathcal{A}^*(t)p(t) - \mathcal{H}^*(t, y(\bullet))\hat{\lambda}(t) \right] &+ q'(t) \end{aligned} \quad (4.15)$$

Due to (4.9) we have

$$\hat{\lambda} = 2\mathcal{G}[y(t) - \mathcal{H}(t, y(\bullet))\hat{x}(t)]$$

Since (4.15) it follows that

$$\begin{aligned} &\left\{ \mathcal{A}(t)\mathcal{P}(t) + \mathcal{B}(t)\mathcal{Q}_1^{-1}\mathcal{B}^*(t) - \frac{d\mathcal{P}(t)}{dt} + \right. \\ &\quad \left. + \mathcal{P}(t)\mathcal{A}^*(t) - 2\mathcal{P}(t)\mathcal{H}^*(t, y(\bullet))\mathcal{G}\mathcal{H}(t, y(\bullet))\mathcal{P}(t) \right\} = \\ &= \left\{ -\mathcal{A}(t)q(t) - 2\mathcal{P}(t)\mathcal{H}^*(t, y(\bullet))\mathcal{G}y(t) + \right. \\ &\quad \left. + 2\mathcal{P}(t)\mathcal{H}^*(t, y(\bullet))\mathcal{G}\mathcal{H}(t, y(\bullet))q(t) + q'(t) \right\} \end{aligned}$$

Thus we obtained problem (4.12) for $q(t)$ and (4.13) for $\mathcal{P}(t)$. $\square$

At last we consider case of differential equation (4.4) over space $\mathbb{R}^n \times \mathbb{H}$, where $\mathbb{H}$ is abstract Hilbert space and operator $\mathcal{H}(t, y(\bullet))$ has the form $\mathcal{H}(t, y(\bullet)) = (H_1(t), \mathcal{O})$, where $H_1(t) \in \mathbb{R}^{m \times n}$, $\mathcal{O}$ is null-vector. Let vector $x(t) \in \mathbb{R}^n \times \mathbb{H}$ has presentation $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, where $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{H}$. Let we consider space $\mathbb{R}^m$ as space of observations Y and $Q_2 \in \mathbb{R}^{m \times m}$. Then

$$y(t) = H_1(t)x_1(t) + \mathcal{D}(t)f_2(t),$$

where $\mathcal{D}(t)f_2(t) \in \mathbb{R}^m$.

Further applying result [4] on presentation of linear operator $A \in \mathcal{L}(\mathbb{R}^n \times \mathbb{H}, \mathbb{R}^n \times \mathbb{H})$, where $\mathbb{H}$ is abstract Hilbert space in the form

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{pmatrix},$$

where $\mathcal{A}_{11} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $\mathcal{A}_{12} \in \mathcal{L}(\mathbb{H}, \mathbb{R}^n)$, $\mathcal{A}_{21} \in \mathcal{L}(\mathbb{R}^n, \mathbb{H})$, $\mathcal{A}_{22} \in \mathcal{L}(\mathbb{H}, \mathbb{H})$, we lead to the following presentations of operators

$$\begin{aligned} \mathcal{A}(t) &= \begin{pmatrix} \mathcal{A}_{11}(t) & \mathcal{A}_{12}(t) \\ \mathcal{A}_{21}(t) & \mathcal{A}_{22}(t) \end{pmatrix}, \\ \mathcal{P}(t) &= \begin{pmatrix} \mathcal{P}_{11}(t) & \mathcal{P}_{21}(t) \\ \mathcal{P}_{21}(t) & \mathcal{P}_{22}(t) \end{pmatrix}, \end{aligned}$$

$$\mathcal{Q}_1^{-1} = \begin{pmatrix} \mathcal{Q}_1^{11} & \mathcal{Q}_1^{12} \\ \mathcal{Q}_1^{21} & \mathcal{Q}_1^{22} \end{pmatrix},$$

$$\mathcal{B}(t) = \begin{pmatrix} \mathcal{B}_{11}(t) & \mathcal{B}_{12}(t) \\ \mathcal{B}_{21}(t) & \mathcal{B}_{22}(t) \end{pmatrix},$$

where elements of operators belong to corresponding spaces.

In this case the problem (4.12) for $q(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}$ results in problems

$$\begin{cases} \frac{dq_1(t)}{dt} = \mathcal{A}_{11}(t)q_1(t) + \mathcal{A}_{12}(t)q_2(t) + 2\mathcal{P}_{11}(t)H_1^*(t)\mathcal{G}[y(t) - H_1(t)q_1(t)] \\ q_1(t_0) = 0, \end{cases}$$

$$\begin{cases} \frac{dq_2(t)}{dt} = \mathcal{A}_{21}(t)q_1(t) + \mathcal{A}_{22}(t)q_2(t) + 2\mathcal{P}_{21}(t)H_1^*(t)\mathcal{G}[y(t) - H_1(t)q_1(t)] \\ q_2(t_0) = 0, \end{cases}$$

and four problems for components of operator $\mathcal{P}(t)$.

5. EXAMPLE

Let along interval of time $(0, t_1)$ there are observed the function $x(t) \in \mathbb{R}^1$ and its derivative $\frac{dx(t)}{dt}$ with error $v(t)$. The function $x(t)$ is solution of the equation

$$\frac{dx(t)}{dt} = a(t)x(t) + \int_0^1 \theta(t)x(t-\tau)d\tau.$$

Function $a(t)$ is known and is continuous on $[0, t_1]$, $\theta(t)$ is unknown function from space $L_2(0, 1)$.

Assume, there are known restrictions for $\theta(t)$ and measurements errors $v(t)$. We determine the following set in the form

$$G = \left\{ (\theta, v) : q_0^2 \int_0^1 \theta^2(t)dt + q_2^2 \int_0^{t_1} v^2(t)dt \leq \gamma^2 \right\}$$

Then aposterioric set will have the form

$$G_y = \{ \theta : \mathcal{J}_y \leq \gamma^2 \},$$

where $\mathcal{J}_y(\theta) = q_0^2 \int_0^1 \theta^2(\tau)d\tau + \int_0^{t_1} (y(t) - \int_0^1 \theta(\tau)x(t-\tau)ds)^2 q_2^2$, where $y(t) = \frac{dx(t)}{dt} - a(t)x(t)$.

Rewrite the problem in vector-like form. The observations are

$$y(t) = (h(t), \theta)_{L_2(0,1)} + v(t),$$

Hereinafter $(h(t), \theta) = (h(t), \theta)_{L_2(0,1)} := \int_0^1 \theta(\tau)x(t-\tau)d\tau$.

We write the equation for the parameter θ

$$\frac{d\theta}{dt} = 0, \quad \theta(0) = \theta$$

For aposterioric estimate of the parameter $\theta(t) = \theta$ which is based on observations till the instant t_1 we get the following boundary value problem

$$\begin{aligned} -\frac{d\hat{p}}{dt} &= h(t)q_2^2(y(t) - (h(t), \hat{\theta}(t))), \\ \hat{p}(t_1) &= 0, \\ \frac{d\hat{\theta}(t)}{dt} &= 0, \\ \hat{\theta}(0) &= q_0^{-2}\hat{p}(0) \end{aligned}$$

Due to these system of equations we can get integral equation for determining the estimate of the function $\hat{\theta}(\tau)$

$$\hat{\theta}(\tau) + \int_0^1 R(\tau, \tau_1)\hat{\theta}(\tau_1)d\tau_1 = \int_0^1 x(t-\tau)q_0^{-2}q_2^2y(t)dt,$$

where $R(\tau, \tau_1) = \int_0^{t_1} x(t-\tau)x(t-\tau_1)dtq_0^{-2}q_2^2$.

Based on the results which were obtained above it follows that the estimate of parameter $\theta(t_1)$ using the observations till the instant t_1 satisfies to the differential equation

$$\begin{aligned} \frac{d\hat{\theta}(t)}{dt} &= \mathcal{P}^2(t)q_2^2h(t)(y(t) - (h(t), \hat{\theta}(t))), \\ \hat{\theta}(0) &= 0. \end{aligned}$$

Here operator-function $\mathcal{P}(t)$ is solution of the equation

$$\begin{aligned} \frac{d\mathcal{P}(t)}{dt} &= -\mathcal{P}(t)h(t) \otimes h(t)q_2^2\mathcal{P}(t), \\ \mathcal{P}(0) &= q_0^{-2}\mathcal{E}, \end{aligned}$$

which has the solution

$$\mathcal{P}(t) = \left(q_0^{-2}\mathcal{E} + \int_0^t h(\tau) \otimes h(\tau)d\tau q_2^2 \right)^{-1},$$

where $h(\tau) \otimes h(\tau)$ is linear continuous operator mapping in the following way

$$h(\tau) \otimes h(\tau)x = h(\tau)(h(\tau), x)$$

Aposterioric error of such estimate has the form

$$\sigma_a = \sup_{\theta \in G_y} |\hat{\theta}(t_1) - \theta| = \left(\gamma^2 - \mathcal{J}_y(\hat{\theta}) \right)^{1/2} \left(q_0^{-2} + q_2^2 \lambda_{\min}(\mathcal{H}) \right)^{-1/2},$$

where $\lambda_{\min}(\mathcal{H})$ is minimal eigenvalue of operator $\mathcal{H}$, where $\mathcal{H} := \int_0^{t_1} h(\tau) \otimes h(\tau)d\tau$.

Note that

$$\sigma_a \leq \gamma \left(q_0^{-2} + q_2^2 \lambda_{\min}(\mathcal{H}) \right)^{-1/2},$$

6. CONCLUSIONS

Thus the work considers the problem of differential equation parameter identification given in Hilbert space. Conditions of existence of problem solutions are established. They correspond to conditions of continuous dependence of solutions of differential equations on parameters. In case of linear model and space L_2 condition means convergence in mean quadratic sense on aprioric set.

A constructive algorithm of identification problem solution in Hilbert space is developed. It results in solution of corresponding boundary value problem. The way of its reduction to Cauchy problems is offered, one partial case allowing problem solution not only in operator form is considered.

In the next works solution of Problems 1 and 2 should be considered in case of models given by integro-differential equations.

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REFERENCES

- [1] “Lower semicontinuous functionals,” in *Systems & Control: Foundations & Applications*. Birkhäuser-Verlag, 2005, pp. 205–219. [Online]. Available: https://doi.org/10.1007/3-7643-7357-1_7. doi: [10.1007/3-7643-7357-1_7](https://doi.org/10.1007/3-7643-7357-1_7)
- [2] B. Bublik, V. Y. Danilov, and A. Nakonechny, *Some problems of observation and control in linear systems*. Kiev: UMK PO (in Russian), 1988.
- [3] A. Favini and H. Tanabe, “Identification Problems for Integrodifferential Equations with Delay: An Improvement of the Results by G. Di Blasio and A. Lorenzi,” *Funkcialaj Ekvacioj-Serio Internacia*, vol. 60, no. 2, pp. 133–170, AUG 2017, doi: [10.1619/fesi.60.133](https://doi.org/10.1619/fesi.60.133).
- [4] I. Gohberg and M. G. Krein, *Theory and applications of Volterra operators in Hilbert space*. American Mathematical Soc., 1970, vol. 24.
- [5] K. Gopalsamy, *Stability and oscillations in delay differential equations of population dynamics*. Springer Science & Business Media, 2013, vol. 74.
- [6] J. K. Hale and S. M. V. Lunel, *Introduction to functional differential equations*. Springer Science & Business Media, 2013, vol. 99.
- [7] J.-M. Jeong and S.-H. Cho, “Identification problems of retarded differential systems in hilbert spaces,” *Evolution Equations and Control Theory*, vol. 6, no. 1, pp. 77–91, MAR 2017, doi: [10.3934/eect.2017005](https://doi.org/10.3934/eect.2017005).
- [8] E. A. Kapustyan and A. G. Nakonechnyi, “The minimax problems of pointwise observation for a parabolic boundary-value problem,” *Journal of Automation and Information Sciences*, vol. 34, no. 5, 2002.
- [9] I. M. Lyashenko and A. Mukoyed, *Modeling biological and ecological processes*. Kyiv: Kyiv University, 2002.

- [10] V. Martsenyuk, “Integro-differential models with memory in population dynamics problems,” *Journal of Automation and Information Sciences*, vol. 36, no. 10, pp. 56–65, 2004, doi: [10.1615/JAutomatInfScien.v36.i10.70](https://doi.org/10.1615/JAutomatInfScien.v36.i10.70).
- [11] V. Martsenyuk and O. Nakonechny, “Models and methods of population dynamics for software environment supporting medical system researches,” vol. 2007-January, 2007, pp. 195–202. [Online]. Available: <https://www.scopus.com/inward/record.uri?eid=2-s2.0-84958048021&partnerID=40&md5=c5f7c7f04689f85e11ffac94e6c2726a>
- [12] V. Marzeniuk and A. Nakonechny, *System analysis methods of medical and biological processes*. Ternopil: Ukrmedknuha, 2003.
- [13] A. G. Nakonechny, *Minimax estimation of functionals of variational equation solutions in Hilbert spaces*. Kiev: KGU (in Russian), 1985.
- [14] A. Nakonechny and V. Marzeniuk, “Uncertainties in medical processes control,” *Lecture Notes in Economics and Mathematical Systems*, vol. 581, pp. 185–192, 2006, doi: [10.1007/3-540-35262-7_11](https://doi.org/10.1007/3-540-35262-7_11).
- [15] O. G. Nakonechnyi, O. Trofymchuk, I. Trofimova, and D. Cherniy, *Modeling and analysis of global biospherical processes*. Kyiv: Kyiv University, 2002.
- [16] F. P. Vasiliev, “Numerical methods for solution of extreme problems,” p. 518, 1980.
- [17] S. Zhuk and O. Nakonechnii, “Minimax state estimates for abstract neumann problems,” *Minimax Theory and its Applications*, vol. 3, no. 1, pp. 1–21, 2018.

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ON SUPPLEMENT ELEMENTS IN LATTICES

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Abstract. In this work, some properties of supplement elements in lattices are investigated. Some relation between lying above and (weak) supplement elements also studied. Some properties of supplement submodules in modules which given in [8] are generalized to lattices. Let a be a supplement of b in a lattice L . If $a/0$ has at least one maximal ($\neq a$) element, then it is possible to define a bijective map between the maximal elements ($\neq a$) of $a/0$ and the maximal elements ($\neq 1$) of $1/b$. Let a be a supplement element in a lattice L . If L is amply supplemented, then $a/0$ is also amply supplemented. If L is weakly supplemented, then $a/0$ is also weakly supplemented.

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1. INTRODUCTION

Throughout this paper, all lattices are complete modular lattices with the smallest element 0 and the greatest element 1. Let L be a lattice, $a, b \in L$ and $a \leq b$. A sublattice $\{x \in L \mid a \leq x \leq b\}$ is called a *quotient sublattice*, denoted by b/a . Let L be a lattice, $a, b \in L$ and $a < b$. If there is no element c of L such that $a < c < b$, we say that a is *covered* by b (or b *covers* a) and denoted by $a < b$. An element a of a lattice L is called an *atom* if $0 < a$. An element a' of a lattice L is called a *complement* of a if $a \wedge a' = 0$ and $a \vee a' = 1$, this case we denote $1 = a \oplus a'$ (a and a' also is called *direct summands* of L). L is called a *complemented lattice* if each element has at least one complement in L . An element a of L is said to be *small* or *superfluous* in L and denoted by $a \ll L$ if $b = 1$ for every element b of L such that $a \vee b = 1$. The meet of all maximal elements ($\neq 1$) of a lattice L is called the *radical* of L and denoted by $r(L)$. An element c of L is called a *supplement* of b in L if it is minimal for $b \vee c = 1$. a is a supplement of b in a lattice L if and only if $a \vee b = 1$ and $a \wedge b \ll a/0$. A lattice L is said to be *supplemented* if every element of L has a supplement in L . We say that an element b of L *lies above* an element a of L if $a \leq b$ and $b \ll 1/a$. An element a of L is called a *weak supplement* of b in L if $a \vee b = 1$ and $a \wedge b \ll L$. A lattice L is said to be *weakly supplemented*, if every element of L has a weak supplement in L . An element $a \in L$ has *ample supplements*

in L if for every $b \in L$ with $a \vee b = 1$, a has a supplement b' in L with $b' \leq b$. L is called an *amply supplemented lattice*, if every element of L has ample supplements in L . It is clear that every supplemented lattice is weakly supplemented and every amply supplemented lattice is supplemented. An element c of a lattice L is called a *generalized supplement* or *Rad-supplement* of b in L if $b \vee c = 1$ and $b \wedge c \leq r(c/0)$. L is said to be *generalized supplemented* (or *Rad-supplemented*) if every element of L has a generalized supplement in L . It is clear that every supplemented lattice is generalized supplemented, but the converse is not true in general (See [3, Example 1]). Let L be a lattice. It is defined β_* relation on the elements of L by $a\beta_*b$ with $a, b \in L$ if and only if for each $t \in L$ such that $a \vee t = 1$ then $b \vee t = 1$ and for each $k \in L$ such that $b \vee k = 1$ then $a \vee k = 1$.

More details about (amply) supplemented lattices are in [1, 2, 6]. More results about (amply) supplemented modules are in [5, 9]. Some important properties of supplement submodules are in [8]. The definition of generalized supplemented lattices and some properties of them are in [3]. The definition of β_* relation on lattices and some properties of this relation are in [7]. The definition of β^* relation on modules and some properties of this relation are in [4].

In this paper, it is generalized some properties of supplement submodules and lying above to lattices. It is studied some relationships between (weak) supplement elements and lying above by Proposition 6, Proposition 7, Lemma 14, Lemma 15 and Corollary 6. If a is a supplement of b in a lattice L , it is defined a bijective map between maximal ($\neq a$) elements of $a/0$ and maximal ($\neq 1$) elements of $1/b$ by Proposition 3. It is proved that a lattice L is amply supplemented if and only if every element of L lies above a supplement element in L by Lemma 16.

Lemma 1 (See [6, Lemma 12.3]). *In any modular lattice $[(c \vee d) \wedge b] \leq [c \wedge (b \vee d)] \vee [d \wedge (b \vee c)]$ holds for every $b, c, d \in L$.*

Lemma 2 (See [6, Lemma 7.4]). *Let L be a lattice, $a, b \in L$ and $a \leq b$. If $a \ll b/0$ then $a \ll L$.*

Lemma 3 (See [6, Lemma 7.5]). *In a lattice L let $c' \ll c/0$ and $d' \ll d/0$. Then $c' \vee d' \ll (c \vee d)/0$.*

Lemma 4 (See [6, Lemma 7.6]). *If $a \ll L$, then $a \leq r(L)$.*

Lemma 5. *Let L be a lattice, $a, b \in L$ and $a \leq b$. Then b lies above a if and only if $a\beta_*b$.*

Proof. ($\Rightarrow$) See [7, Theorem 3].

($\Leftarrow$) Let $b \vee t = 1$ with $t \in 1/a$. Since $a\beta_*b$, $a \vee t = 1$ and since $a \leq t$, $t = 1$. Hence $b \ll 1/a$ and b lies above a . $\square$

Lemma 6 (See [7, Lemma 2]). *Let L be a lattice and $a, b, c \in L$. If $a \vee b = 1$ and $(a \wedge b) \vee c = 1$, then $a \vee (b \wedge c) = b \vee (a \wedge c) = 1$.*

Lemma 7. *Let L be a lattice, $a, b, k \in L$ and $1 = a \vee b$. If $a < k < 1$, then $0 < k \wedge b < b$.*

Proof. Since $a < k$ and $1 = a \vee b$, $k = k \wedge 1 = k \wedge (a \vee b) = a \vee (k \wedge b)$ and since $k \neq a$, we get $k \wedge b \neq 0$. Since $a \vee (k \wedge b) = k \neq 1$ and $1 = a \vee b$, we get $k \wedge b \neq b$. Hence $0 < k \wedge b < b$ holds. $\square$

Lemma 8 (See [6, Proposition 12.2]). *Let a be a supplement of b in L . If b is a maximal ($\neq 1$) element of L , then $a \wedge b$ is the unique maximal ($\neq a$) element of $a/0$ and $r(a/0) = a \wedge b$ holds.*

Lemma 9. *Let L be a lattice and $a, b \in L - \{1\}$. If $1 = a \vee b$ and b is an atom, then a is a maximal ($\neq 1$) element of L .*

Proof. If $k \in L$ and $a < k < 1$, then by Lemma 7, $0 < k \wedge b < b$. Since b is an atom, this is contradiction. Hence a is a maximal ($\neq 1$) element of L . $\square$

Corollary 1. *Let a be a supplement of b in L . Then b is maximal ($\neq 1$) if and only if $0 < a$ or $a \wedge b < a$.*

Proof. Clear from Lemma 8 and Lemma 9. $\square$

2. SOME PROPERTIES OF SUPPLEMENT ELEMENTS

Lemma 10. *Let a be a supplement of b in L and $c \in a/0$. Then $c \ll L$ if and only if $c \ll a/0$.*

Proof. ($\Rightarrow$) Let $c \vee t = a$ with $t \in a/0$. Then $b \vee c \vee t = b \vee a = 1$ and since $c \ll L$, $b \vee t = 1$. By minimality of a , we get $t = a$. Hence $c \ll a/0$.

($\Leftarrow$) Clear from Lemma 2. $\square$

Proposition 1. *Let a be a supplement element in L . If L is weakly supplemented, then $a/0$ is also weakly supplemented.*

Proof. Let $x \in a/0$. Since L is weakly supplemented, there exists an element y of L such that $x \vee y = 1$ and $x \wedge y \ll L$. Here $a = a \wedge 1 = a \wedge (x \vee y) = x \vee (a \wedge y)$ holds. Since $x \wedge a \wedge y = x \wedge y \ll L$, by Lemma 10, $x \wedge a \wedge y \ll a/0$. Hence $a/0$ is weakly supplemented. $\square$

Lemma 11. *Let a be a supplement of b in L and $x, y \in a/0$. Then y is a supplement of x in $a/0$ if and only if y is a supplement of $b \vee x$ in L .*

Proof. ($\Rightarrow$) Let y be a supplement of x in $a/0$ and $b \vee x \vee z = 1$ with $z \leq y$. Because of $x, y \in a/0$ and $z \leq y$, $x \vee z \leq a$. Since a is a supplement of b in L , $a = x \vee z$. Since y is a supplement of x in $a/0$, $z = y$. Hence y is a supplement of $b \vee x$ in L .

($\Leftarrow$) Let y be a supplement of $b \vee x$ in L . So, $b \vee x \vee y = 1$ and $(b \vee x) \wedge y \ll y/0$. Since $x \vee y \leq a$ and a is a supplement of b in L , $x \vee y = a$ and $x \wedge y \leq (b \vee x) \wedge y \ll y/0$. Hence y is a supplement of x in $a/0$. $\square$

Corollary 2. *Let a and b be mutual supplements in L , x be a supplement of y in $a/0$ and z be a supplement of t in $b/0$. Then $x \vee z$ is a supplement of $y \vee t$ in L .*

Proof. By Lemma 11, x is a supplement of $b \vee y$ and z is a supplement of $a \vee t$ in L . Hence $x \wedge (b \vee y) \ll x/0$ and $z \wedge (a \vee t) \ll z/0$. By Lemma 1 and Lemma 3, $(x \vee z) \wedge (y \vee t) \leq [x \wedge (y \vee t \vee z)] \vee [z \wedge (x \vee y \vee t)] = [x \wedge (y \vee b)] \vee [z \wedge (a \vee t)] \ll (x \vee z)/0$ and since $1 = a \vee b = x \vee y \vee z \vee t = x \vee z \vee y \vee t$, $x \vee z$ is a supplement of $y \vee t$ in L . $\square$

Corollary 3. *Let $1 = a \oplus b$ in L , x be a supplement of y in $a/0$ and z be a supplement of t in $b/0$. Then $x \vee z$ is a supplement of $y \vee t$ in L .*

Proof. Clear from Corollary 2. $\square$

Proposition 2. *Let a be a supplement element in L . If L is amply supplemented, then $a/0$ is also amply supplemented.*

Proof. Let a be a supplement of b in L and $a = x \vee y$ with $x, y \in a/0$. Since $1 = a \vee b = b \vee x \vee y$ and L is amply supplemented, there exists a supplement y' of $b \vee x$ in L with $y' \leq y$. By Lemma 11, y' is a supplement of x in $a/0$. Hence $a/0$ is amply supplemented. $\square$

Lemma 12. *Let a be a supplement of b in L and x be a maximal ($\neq a$) element of $a/0$. Then $b \vee x$ is a maximal ($\neq 1$) element of L and $x = a \wedge (b \vee x)$.*

Proof. Since a is a supplement of b in L , $1 = a \vee b$ and $a \wedge b \ll a/0$. By Lemma 4, we get $a \wedge b \leq r(a/0) \leq x$. By modularity, $a \wedge (b \vee x) = (a \wedge b) \vee x = x$ holds. Since x is a maximal ($\neq a$) element of $a/0$ and $\frac{1}{b \vee x} = \frac{a \vee b}{b \vee x} \cong \frac{a}{a \wedge (b \vee x)} = \frac{a}{x}$, $b \vee x$ is a maximal ($\neq 1$) element of L . $\square$

Lemma 13. *Let L be a lattice and $a \in L$. If k is a maximal ($\neq 1$) element of L and $a \not\leq k$, then $a \wedge k$ is a maximal ($\neq a$) element of $a/0$.*

Proof. Since $a \not\leq k$, we get $a \wedge k \neq a$. Let $a \wedge k < t \leq a$ with $t \in a/0$. Then $t \not\leq k$ and by maximality of k , $k \vee t = 1$. Here $a = a \wedge 1 = a \wedge (k \vee t) = (a \wedge k) \vee t = t$. Hence $a \wedge k$ is a maximal ($\neq a$) element of $a/0$. $\square$

Proposition 3. *Let a be a supplement of b in a lattice L . If $a/0$ has at least one maximal ($\neq a$) element, then it is possible to define a bijective map between the maximal ($\neq a$) elements of $a/0$ and maximal ($\neq 1$) elements of $1/b$.*

Proof. Let $\Gamma = \{k \in a/0 \mid k \text{ is maximal } (\neq a) \text{ in } a/0\}$ and $\Lambda = \{t \in L \mid b \leq t \text{ and } t \text{ is maximal } (\neq 1) \text{ in } 1/b\}$. We can define a map

$$f: \Gamma \longrightarrow \Lambda, \quad k \longmapsto f(k) = b \vee k.$$

By Lemma 12, $b \vee k$ is maximal ($\neq 1$) in $1/b$. Hence f is a function. Let $f(k_1) = f(k_2)$ for $k_1, k_2 \in \Gamma$. Then $b \vee k_1 = b \vee k_2$ and $a \wedge (b \vee k_1) = a \wedge (b \vee k_2)$. By

Lemma 12, $k_1 = a \wedge (b \vee k_1)$ and $k_2 = a \wedge (b \vee k_2)$. Hence $k_1 = k_2$ and f is injective.

Let $t \in \Lambda$. Then t is a maximal ($\neq 1$) element of L and $b \leq t$. Since $1 = a \vee b$ and $b \leq t$, by maximality of t , $a \not\leq t$. Then by Lemma 13, $a \wedge t$ is a maximal ($\neq a$) element of $a/0$ and $a \wedge t \in \Gamma$. By definition of f and the modularity of L , $f(a \wedge t) = b \vee (a \wedge t) = (b \vee a) \wedge t = 1 \wedge t = t$ and f is surjective.

Hence f is bijective. $\square$

The following Proposition can be given as a consequence of [3, Theorem 2]. It is proved by different way as follows.

Proposition 4. *Let a be a supplement of b in L and $b \neq 1$. If $r(L) \ll L$, then there exists a maximal ($\neq 1$) element t of L such that $b \leq t$.*

Proof. If $a \leq r(L)$, then $a \ll L$ and since $1 = a \vee b$, $b = 1$. This is contradiction. Hence $a \not\leq r(L)$ and there exists a maximal ($\neq 1$) element k of L such that $a \not\leq k$. By Lemma 13 $a \wedge k$ is a maximal ($\neq a$) element of $a/0$ and by Lemma 12, $t = b \vee (a \wedge k)$ is a maximal ($\neq 1$) element of L with $b \leq t$. $\square$

The following Proposition can be given as a consequence of [3, Lemma 2]. It is proved by different way as follows.

Proposition 5. *Let a be a supplement of b in L . Then $r(a/0) = a \wedge r(L)$.*

Proof. Assume that $a \leq r(L)$. If k is a maximal ($\neq a$) element of $a/0$, by Lemma 12, $b \vee k$ is a maximal ($\neq 1$) element of L and $k = a \wedge (b \vee k)$. Then by $a \leq r(L) \leq b \vee k$, $a = a \wedge (b \vee k) = k$. This is contradiction. Hence $a/0$ have no maximal ($\neq a$) elements and since $a \leq r(L)$, $r(a/0) = a = a \wedge r(L)$.

Now assume that $a \not\leq r(L)$. Then there exists a maximal ($\neq 1$) element t of L such that $a \not\leq t$. By Lemma 13, $a \wedge t$ is a maximal ($\neq a$) element of $a/0$ and hence $a/0$ has at least one maximal ($\neq a$) element. By using Lemma 12, we can see that $r(a/0) = \bigwedge \{x \mid x \text{ is maximal } (\neq a) \text{ in } a/0\} = \bigwedge \{a \wedge (b \vee x) \mid x \text{ is maximal } (\neq a) \text{ in } a/0\} = a \wedge [\bigwedge \{b \vee x \mid x \text{ is maximal } (\neq a) \text{ in } a/0\}]$.

Since $r(L) \leq b \vee x$ for every maximal ($\neq a$) element x of $a/0$, $r(L) \leq \bigwedge \{b \vee x \mid x \text{ is maximal } (\neq a) \text{ in } a/0\}$ and $a \wedge r(L) \leq a \wedge [\bigwedge \{b \vee x \mid x \text{ is maximal } (\neq a) \text{ in } a/0\}] = r(a/0)$. By Lemma 13, for every maximal ($\neq 1$) element t of L , $a \wedge t = a$ or $a \wedge t$ is a maximal ($\neq a$) element of $a/0$. Then by using the definition of $r(L)$ we can see that $r(a/0) \leq \bigwedge \{a \wedge t \mid t \text{ is maximal } (\neq 1) \text{ in } L\} = a \wedge [\bigwedge \{t \mid t \text{ is maximal } (\neq 1) \text{ in } L\}] = a \wedge r(L)$.

Hence $r(a/0) = a \wedge r(L)$. $\square$

Proposition 6. *Let L be a lattice, $a, b \in L$, $a \leq b$ and b lies above a in L . If a and b have weak supplements in L , then they have the same weak supplements in L .*

Proof. Since b lies above a , by Lemma 5, $a\beta_*b$. Then by [7, Theorem 4], the desired is obtained. $\square$

Proposition 7. *Let L be a lattice, $a, b \in L$, $a \leq b$ and b lies above a . If a and b have supplements in L , then they have the same supplements in L .*

Proof. Since b lies above a , by Lemma 5, $a\beta_*b$. Then by [7, Theorem 4], the desired is obtained. $\square$

Lemma 14. *Let a be a weak supplement of b in L and $k \leq a$. Then k is a weak supplement of b in L if and only if a lies above k in L .*

Proof. $(\Rightarrow)$ Since k is a weak supplement of b in L , $1 = k \vee b$ and $k \wedge b \ll L$. By modularity, $a = a \wedge 1 = a \wedge (k \vee b) = k \vee (a \wedge b)$. Let $a \vee t = 1$ with $t \in 1/k$. Then $1 = a \vee t = k \vee (a \wedge b) \vee t = (a \wedge b) \vee t$ and since $a \wedge b \ll L$, $t = 1$. Hence $a \ll 1/k$ and a lies above k in L .

$(\Leftarrow)$ By Lemma 5, $k\beta_*a$ and since $a \vee b = 1$, $k \vee b = 1$. In addition to this, $k \wedge b \leq a \wedge b \ll L$. Hence k is a weak supplement of b in L . $\square$

Lemma 15. *Let a be a weak supplement of b in L and $b \leq c$. Then a is a weak supplement of c in L if and only if c lies above b in L .*

Proof. $(\Rightarrow)$ Since a is a weak supplement of c in L , c is a weak supplement of a in L . Then by Lemma 14, c lies above b in L .

$(\Leftarrow)$ Since c lies above b in L , by Lemma 5, $b\beta_*c$. Since $a \vee b = 1$ and $b\beta_*c$, $a \vee c = 1$. Let $(a \wedge c) \vee t = 1$ with $t \in L$. By Lemma 6, $c \vee (a \wedge t) = 1$ and since $b\beta_*c$, $b \vee (a \wedge t) = 1$. By using Lemma 6 again, we can see that $(a \wedge b) \vee t = 1$ and since $a \wedge b \ll L$, $t = 1$. Hence a is a weak supplement of c in L . $\square$

Corollary 4. *Let L be a weakly supplemented lattice, $a, b \in L$ and $a \leq b$. Then a and b have the same weak supplements in L if and only if b lies above a in L .*

Proof. Clear from Proposition 6 and Lemma 15. $\square$

Corollary 5. *Let a be a supplement of b in L and $k \leq b$. Then a is a supplement of k in L if and only if b lies above k in L .*

Proof. Clear from Lemma 14. $\square$

Corollary 6. *Let a be a weak supplement of b in L . Then a is a supplement of b in L if and only if a does not lie above any element c with $c < a$.*

Proof. Clear from Lemma 14. $\square$

Definition 1. Let L be a lattice. L is said to be $\oplus$ -supplemented if every element of L has a supplement that is a direct summand of L .

Proposition 8. *Let L be a lattice. If every element of L has a weak supplement that is a direct summand in L , then L is $\oplus$ -supplemented.*

Proof. Assume that a has a weak supplement x that is a direct summand in L and $1 = x \oplus y$ with $y \in L$. Then x is a supplement of y in L and by Corollary 6, x does not lie above any element c with $c < x$. Also by Corollary 6, x is a supplement of a in L . This means that L is $\oplus$ -supplemented. $\square$

Proposition 9. *A lattice L is weakly supplemented if and only if every element of L lies above a weak supplement in L .*

Proof. ($\Rightarrow$) Since L is weakly supplemented, every element of L is a weak supplement in L . Since every element of L lies above itself, every element of L lies above a weak supplement in L .

($\Leftarrow$) Let $a \in L$. By hypothesis, a lies above a weak supplement b in L . Let b be a weak supplement of c in L . Then c is a weak supplement of b in L and since a lies above b in L , by Lemma 15, c is a weak supplement of a in L . Hence L is weakly supplemented. $\square$

Lemma 16. *A lattice L is amply supplemented if and only if every element of L lies above a supplement element in L .*

Proof. ($\Rightarrow$) Since L is amply supplemented, then L is supplemented. Let $a \in L$ and b is a supplement of a in L . Since $1 = a \vee b$ and L is amply supplemented, b has a supplement x in L with $x \leq a$. Since b is a supplement of a in L , a is a weak supplement of b in L . Since x is a supplement of b in L , x is a weak supplement of b in L . Then by Lemma 14, a lies above x in L . Hence every element of L lies above a supplement element in L .

($\Leftarrow$) Let $a, b \in L$ and $a \vee b = 1$. Since every element of L lies above a supplement element in L , $a \wedge b$ lies above a supplement element x in L . Let x be a supplement of y in L . Then y is a weak supplement of x in L and since $a \wedge b$ lies above x in L , by Lemma 15, y is a weak supplement of $a \wedge b$ in L . Here $1 = (a \wedge b) \vee y$ and $a \wedge b \wedge y \ll L$. By modularity, $b = b \wedge 1 = b \wedge ((a \wedge b) \vee y) = (a \wedge b) \vee (b \wedge y)$. Then $1 = a \vee b = a \vee (a \wedge b) \vee (b \wedge y) = a \vee (b \wedge y)$ and since $a \wedge b \wedge y \ll L$, $b \wedge y$ is a weak supplement of a in L . By hypothesis, there exists a supplement element b' in L such that $b \wedge y$ lies above b' in L . By Lemma 14, b' is a weak supplement of a in L . Here $1 = a \vee b'$ and $a \wedge b' \ll L$. Since b' is a supplement in L and $a \wedge b' \ll L$, by Lemma 10, $a \wedge b' \ll b'/0$. Thus b' is a supplement of a in L . Note that $b' \leq b$. Hence L is amply supplemented. $\square$

Proposition 10. *Let L be a lattice. The following statements are equivalent.*

- (i) *Every element of L lies above a direct summand in L .*
- (ii) *L is amply supplemented and every supplement element in L is a direct summand of L .*
- (iii) *For every elements a and b of L with $1 = a \vee b$, there exists a supplement x of a in L such that $x \leq b$ and x is a direct summand of L .*

Proof. (i) $\Rightarrow$ (ii) Since every direct summand of L is a supplement in L , by Lemma 16, L is amply supplemented. Let a be a supplement element in L . By hypothesis, a lies above a direct summand x of L . Since a is a supplement element in L , by Corollary 6, $a = x$. Hence every supplement element in L is a direct summand of L .

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (i) Clearly we can see that L is supplemented. Let $a \in L$ and b is a supplement of a in L . By hypothesis, there exists a supplement x of b in L such that $x \leq a$ and x is a direct summand of L . Since b is a supplement of a in L , a is a weak supplement of b in L . Since x is a supplement of b in L , x is a weak supplement of b in L . Then by Lemma 14, a lies above x in L . Hence every element of L lies above a direct summand in L . $\square$

Lemma 17. *Let L be a lattice. Then the following statements are equivalent.*

(i) L is complemented.

(ii) Every element of L is a direct summand of L .

(iii) Every element of L is a supplement in L .

Proof. (i) $\Leftrightarrow$ (ii) Clear.

(ii) $\Rightarrow$ (iii) Clear, since every direct summand of L is a supplement in L .

(iii) $\Rightarrow$ (ii) Let a be any element of L . By hypothesis a is a supplement in L and there exists an element b of L such that a is a supplement of b in L . Here $1 = a \vee b$ and $a \wedge b \ll b/0$. By hypothesis, $a \wedge b$ is a supplement in L . Let $a \wedge b$ be a supplement of k in L . Then $(a \wedge b) \vee k = 1$ and since $a \wedge b \ll L$, $k = 1$. Therefore $a \wedge b$ is a supplement of 1 in L . Thus $a \wedge b = 0$ and $1 = a \oplus b$. Hence every element of L is a direct summand of L . $\square$

Proposition 11. *Let L be a weakly supplemented lattice. Then every weak supplement in L is a supplement in L if and only if L is complemented.*

Proof. ($\Rightarrow$) Let $a \in L$. By hypothesis, a has a weak supplement b in L . Then a is a weak supplement of b in L . By hypothesis, a is a supplement element in L . Hence every element of L is a supplement in L and by Lemma 17, L is complemented.

($\Leftarrow$) Since L is complemented, by Lemma 17, every element of L is a supplement in L . Hence every weak supplement in L is a supplement in L . $\square$

REFERENCES

- [1] R. Alizade and E. Toksoy, "Cofinitely weak supplemented lattices," *Indian Journal of Pure and Applied Mathematics*, vol. 40:5, pp. 337–346, 2009.
- [2] R. Alizade and S. E. Toksoy, "Cofinitely supplemented modular lattices," *Arabian Journal for Science and Engineering*, vol. 36, no. 6, p. 919, 2011, doi: [10.1007/s13369-011-0095-z](https://doi.org/10.1007/s13369-011-0095-z). [Online]. Available: <https://doi.org/10.1007/s13369-011-0095-z>
- [3] Ç. Biçer, C. Nebiyev, and A. Pancar, "Generalized supplemented lattices," *Miskolc Mathematical Notes*, vol. 19, no. 1, pp. 141–147, 2018, doi: [10.18514/MMN.2018.1974](https://doi.org/10.18514/MMN.2018.1974). [Online]. Available: <https://doi.org/10.18514/MMN.2018.1974>
- [4] G. F. Birkenmeier, F. T. Mutlu, C. Nebiyev, N. Sökmez, and A. Tercan, "Goldie*-supplemented modules," *Glasgow Mathematical Journal*, vol. 52A, pp. 41–52, 2010, doi: [10.1017/S0017089510000212](https://doi.org/10.1017/S0017089510000212). [Online]. Available: <https://doi.org/10.1017/S0017089510000212>
- [5] J. Clark, C. Lomp, N. Vanaja, and R. Wisbauer, *Lifting Modules: Supplements and Projectivity in Module Theory (Frontiers in Mathematics)*, 2006th ed. Basel: Birkhäuser, 8 2006.
- [6] G. Călugăreanu, *Lattice Concepts of Module Theory*. Kluwer Academic Publisher, 2000.

- [7] C. Nebiyev and H. H. Ökten, “ β^* relation on lattices,” *Miskolc Mathematical Notes*, vol. 18, no. 2, 2017, doi: [10.18514/MMN.2017.1782](https://doi.org/10.18514/MMN.2017.1782). [Online]. Available: <https://doi.org/10.18514/MMN.2017.1782>
- [8] C. Nebiyev and A. Pancar, “On supplement submodules,” *Ukrainian Mathematical Journal*, vol. 65, no. 7, pp. 1071–1078, 2013, doi: [10.1007/s11253-013-0842-2](https://doi.org/10.1007/s11253-013-0842-2). [Online]. Available: <https://doi.org/10.1007/s11253-013-0842-2>
- [9] R. Wisbauer, *Foundations of Module and Ring Theory*. Philadelphia: Gordon and Breach, 1991.

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ON (p, q) -EXTENSION OF FURTHER MEMBERS OF BESSEL-STRUVE FUNCTIONS CLASS

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Abstract. In [10] (p, q) -extensions of the modified Bessel and the modified Struve functions of the first kind are presented. This article companion to [10] contains the (p, q) -extension of modified Struve function of the second kind $\mathbf{M}_{\nu, p, q}$ and the Bessel-Struve kernel function $S_{\nu, p, q}$. Systematic investigation of its properties, among integral representation, Mellin transform, Laguerre polynomial representation for both introduced special functions, while additional differential-difference equation, log-convexity property and Turán-type inequalities are realized for the latter.

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1. INTRODUCTION AND PRELIMINARIES

The theory of special functions has been one of the most rapidly growing research subjects in Mathematical Analysis due to their diverse applications in many different areas of mathematical, physical, statistical, and engineering sciences. Further, extensions of the Eulerian Gamma function enable to define important generalizations not only of the higher transcendental hypergeometric functions, but the Bessel and alike functions class members. So, bearing in mind that

$$\frac{4^{-n}}{n! \Gamma(n + \nu + 1)} = \frac{B\left(n + \frac{1}{2}, \nu + \frac{1}{2}\right)}{\sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right) (2n)!}, \quad 2\nu > -1, n \in \mathbb{N}_0,$$

the Bessel function of the first kind can be rewritten into

$$J_{\nu}(z) = \sum_{n \geq 0} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu}}{n! \Gamma(n + \nu + 1)} = \frac{\left(\frac{z}{2}\right)^{\nu}}{\sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right)} \sum_{n \geq 0} B\left(n + \frac{1}{2}, \nu + \frac{1}{2}\right) \frac{(-z^2)^n}{(2n)!}.$$

Then making use of the (p, q) -extended Beta function [7] (see also, [18])

$$B(x, y; p, q) = \int_0^1 t^{x-1} (1-t)^{y-1} e^{-\frac{p}{t} - \frac{q}{1-t}} dt,$$

when $\min\{\Re(x), \Re(y)\} > 0$; $\min\{\Re(p), \Re(q)\} \geq 0$ to $J_\nu(z)$ several extensions, generalizations and unifications of various special functions of (p, q) -variant, and in turn the (p, p) -, that is the p -variant have been studied widely together with the set of related higher transcendental hypergeometric type special functions by several authors, consult for instance [3–6, 12, 15, 18]). In particular, Maširević *et al.* [10] applied the (p, q) -extended Beta in introducing the (p, q) -Bessel function of the first kind

$$J_{\nu, p, q}(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \sum_{n \geq 0} B\left(n + \frac{1}{2}, \nu + \frac{1}{2}; p, q\right) \frac{(-z^2)^n}{(2n)!}. \quad (1.1)$$

These publication also contains the newly defined the (p, q) -extended modified Bessel $I_{\nu, p, q}$ together with the modified Struve $\mathbf{L}_{\nu, p, q}$ functions of the first kind of order ν in the form:

$$I_{\nu, p, q}(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \sum_{n \geq 0} B\left(n + \frac{1}{2}, \nu + \frac{1}{2}; p, q\right) \frac{z^{2n}}{(2n)!}$$

$$\mathbf{L}_{\nu, p, q}(z) = \frac{2\left(\frac{z}{2}\right)^{\nu+1}}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \sum_{n \geq 0} B\left(n + 1, \nu + \frac{1}{2}; p, q\right) \frac{z^{2n}}{(2n+1)!},$$

where $\min\{p, q\} \geq 0$ while for $p = 0 = q$ necessarily $\Re(\nu) > -\frac{1}{2}$. Among other findings they reported about the integral representations [10]

$$I_{\nu, p, q}(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \cosh(zt) e^{-\frac{p}{t^2} - \frac{q}{1-t^2}} dt, \quad (1.2)$$

$$\mathbf{L}_{\nu, p, q}(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \sinh(zt) e^{-\frac{p}{t^2} - \frac{q}{1-t^2}} dt. \quad (1.3)$$

Clearly, for $p = 0 = q$ all these extensions reduce to their classical ancestors.

Here our aim is to introduce and investigate, in a rather systematic manner, (p, q) -extended modified Struve function $\mathbf{M}_{\nu, p, q}(x)$ and the so-called (p, q) -extended Bessel–Struve kernel function $S_{\nu, p, q}(x)$ presenting their various analytical properties.

2. THE (p, q) -EXTENDED MODIFIED STRUVE FUNCTION $\mathbf{M}_{\nu, p, q}(x)$

The series definition of the Fox-Wright function ${}_p\Psi_q(z)$ [17] reads

$${}_p\Psi_q \left[\begin{matrix} (\alpha_1, A_1), \dots, (\alpha_p, A_p) \\ (\beta_1, B_1), \dots, (\beta_q, B_q) \end{matrix} \middle| z \right] = \sum_{n \geq 0} \frac{\prod_{j=1}^p \Gamma(\alpha_j + A_j n)}{\prod_{k=1}^q \Gamma(\beta_k + B_k n)} \frac{z^n}{n!}, \quad (2.1)$$

where $A_j, B_k > 0$ $j = 1, \dots, p; k = 1, \dots, q$ and the series converges for all $z \in \mathbb{C}$ when $\Delta = 1 + \sum_{k=1}^q B_k - \sum_{j=1}^p A_j > 0$; in the case $\Delta = 0$ the convergence occurs inside the open disc

$$|z| < \nabla = \left(\prod_{j=1}^p A_j^{-A_j} \right) \left(\prod_{j=1}^q B_j^{B_j} \right).$$

The modified Struve functions of the second kind [2, p. 1388, Eq. (2)], [13, Eq. 11.2.6]

$$\mathbf{M}_\nu(x) = \mathbf{L}_\nu(x) - I_\nu(x),$$

having integral form [13, Eq. 11.5.4]

$$\mathbf{M}_\nu(x) = -\frac{2\left(\frac{x}{2}\right)^\nu}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} e^{-xt} dt, \quad 2\nu + 1 > 0, x > 0.$$

By routine calculation we infer the related power series representation:

Theorem 1. For all $2\nu + 1 > 0, x > 0$ we have

$$\mathbf{M}_\nu(x) = -\frac{2^{-\nu} x^\nu}{\Gamma(\nu + \frac{1}{2})} {}_1\Psi_1 \left[\begin{matrix} (\frac{1}{2}, \frac{1}{2}) \\ (\nu + 1, \frac{1}{2}) \end{matrix} \middle| -x \right].$$

The proof is straightforward mentioning only that being $\Delta = 1$, the series converges for all $x > 0$.

We now introduce the (p, q) -extended modified Struve function of the second kind as

$$\mathbf{M}_{\nu,p,q}(x) = \mathbf{L}_{\nu,p,q}(x) - I_{\nu,p,q}(x).$$

Applying this definition by taking (1.2) and (1.3), we obtain the related integral representation

$$\mathbf{M}_{\nu,p,q}(x) = -\frac{2\left(\frac{x}{2}\right)^\nu}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} e^{-xt - \frac{p}{t^2} - \frac{q}{1-t^2}} dt, \quad (2.2)$$

with $\min\{p, q\} \geq 0$ and $\Re(\nu) > -\frac{1}{2}$ when $p = q = 0$. Clearly, the latter case coincides with the original definition of modified Struve function $\mathbf{M}_\nu(x)$ of second kind.

The double Mellin transforms [14, p. 293, Eq. (7.1.6)] of a suitable class of $L_1(\mathbb{R}_+^2)$ -integrable function $f(x, y)$ with respect to the indices r and s is defined by

$$\mathcal{M}\{f(x, y)\}(r, s) = \int_0^\infty \int_0^\infty x^{r-1} y^{s-1} f(x, y) dx dy, \quad (2.3)$$

provided that the improper integral in (2.3) exists. Now, we examine the existence and the establish the shape of the double Mellin transform of $\mathbf{M}_{\nu,p,q}(x)$ when the arguments are the extension parameters p, q .

Theorem 2. For all $\min\{\Re(r), \Re(s), 2\Re(v) + 1\} > 0$ and $x > 0$ the Mellin transform of $\mathbf{M}_{v,p,q}(x)$ with respect to $p, q \geq 0$ reads as follows

$$\begin{aligned} \mathcal{M}\{\mathbf{M}_{v,p,q}(x)\}(r,s) &= -\frac{\left(\frac{x}{2}\right)^v \Gamma(r)\Gamma(s)\Gamma(v+s+\frac{1}{2})}{\sqrt{\pi} \Gamma(v+\frac{1}{2})} \\ &\quad \times {}_1\Psi_1\left[\begin{matrix} (\frac{1}{2}+r, \frac{1}{2}) \\ (v+r+s+1, \frac{1}{2}) \end{matrix} \middle| -x\right]. \end{aligned}$$

Proof. By the definition of Mellin transform we find from (2.2) that

$$\begin{aligned} \mathcal{M}\{\mathbf{M}_{v,p,q}(x)\}(r,s) &= \int_0^\infty \int_0^\infty p^{r-1} q^{s-1} \mathbf{M}_{v,p,q}(x) dp dq \\ &= -\frac{2\left(\frac{x}{2}\right)^v}{\sqrt{\pi} \Gamma(v+\frac{1}{2})} \int_0^1 (1-t^2)^{v-\frac{1}{2}} e^{-xt} \\ &\quad \times \left(\int_0^\infty p^{r-1} e^{-\frac{p}{t^2}} dp\right) \left(\int_0^\infty q^{s-1} e^{-\frac{q}{1-t^2}} dq\right) dt, \end{aligned}$$

where we have interchanged the integration order. Applying the well-known Gamma function formula [13, p.136, Eq. (5.2.1)]

$$\Gamma(\eta) \xi^{-\eta} = \int_0^\infty e^{-\xi t} t^{\eta-1} dt, \quad \Re(\xi) > 0, \Re(\eta) > 0$$

to the inner p - and q -integrals we get

$$\begin{aligned} \mathcal{M}\{\mathbf{M}_{v,p,q}(x)\}(r,s) &= -\frac{2\left(\frac{x}{2}\right)^v \Gamma(r)\Gamma(s)}{\sqrt{\pi} \Gamma(v+\frac{1}{2})} \int_0^1 t^{2r} (1-t^2)^{v+s-\frac{1}{2}} e^{-xt} dt \\ &= -\frac{2\left(\frac{x}{2}\right)^v \Gamma(r)\Gamma(s)}{\sqrt{\pi} \Gamma(v+\frac{1}{2})} \sum_{n \geq 0} \frac{(-x)^n}{n!} \int_0^1 t^{2r+n} (1-t^2)^{v+s-\frac{1}{2}} dt \\ &= -\frac{\left(\frac{x}{2}\right)^v \Gamma(r)\Gamma(s)}{\sqrt{\pi} \Gamma(v+\frac{1}{2})} \sum_{n \geq 0} \frac{(-x)^n}{n!} \int_0^1 u^{r+\frac{n}{2}-\frac{1}{2}} (1-u)^{v+s-\frac{1}{2}} du \\ &= -\frac{\left(\frac{x}{2}\right)^v \Gamma(r)\Gamma(s)}{\sqrt{\pi} \Gamma(v+\frac{1}{2})} \sum_{n \geq 0} \frac{\Gamma(r+\frac{n}{2}+\frac{1}{2}) \Gamma(v+s+\frac{1}{2})}{\Gamma(v+r+s+1+\frac{n}{2})} \frac{(-x)^n}{n!}, \end{aligned}$$

taking into account the series e^{-xt} . Thus, (2.1) finishes the proof. $\square$

Remark 1. If we set $r = 1 = s$ in Theorem 2 we get

$$\int_0^\infty \int_0^\infty \mathbf{M}_{v,p,q}(x) dp dq = -\frac{\left(\frac{x}{2}\right)^v (2v+1)}{2\sqrt{\pi}} {}_1\Psi_1\left[\begin{matrix} (\frac{3}{2}, \frac{1}{2}) \\ (v+3, \frac{1}{2}) \end{matrix} \middle| -x\right], \quad x > 0.$$

Next, we report on the Laguerre polynomial representation for $\mathbf{M}_{v,p,q}(x)$. The Laguerre polynomial $L_n^{(\alpha)}(p)$ of the degree $n \in \mathbb{N}_0$ is given by the generating function [16, p. 202]

$$\frac{1}{(1-t)^{\alpha+1}} \exp\left(-\frac{pt}{1-t}\right) = \sum_{n \geq 0} L_n^{(\alpha)}(p) t^n, \quad \alpha \in \mathbb{C}; |t| < 1.$$

Theorem 3. For all $\min\{\Re(p), \Re(q)\} > 0$, $\Re(v) > -\frac{1}{2}$ and $\alpha \in \mathbb{C}$, for which $v + \alpha > -\frac{3}{2}$, the following Laguerre polynomial representation holds true

$$\begin{aligned} \mathbf{M}_{v,p,q}(x) = & -\frac{(v + \frac{1}{2})_{\alpha+1} x^v}{\sqrt{\pi} 2^v e^{p+q}} \sum_{m,n \geq 0} (v + \alpha + \frac{3}{2})_n L_m^\alpha(p) L_n^\alpha(q) \\ & \times {}_1\Psi_1 \left[\begin{matrix} (\alpha + n + \frac{3}{2}, \frac{1}{2}) \\ (v + 2\alpha + m + n + 3, \frac{1}{2}) \end{matrix} \middle| -x \right]. \end{aligned}$$

Proof. Simple transformation gives

$$\begin{aligned} \exp\left(-\frac{p}{t^2}\right) &= \frac{t^{2\alpha+2}}{e^p} \sum_{m \geq 0} L_m^{(\alpha)}(p) (1-t^2)^m; \\ \exp\left(-\frac{q}{1-t^2}\right) &= \frac{(1-t^2)^{\alpha+1}}{e^q} \sum_{n \geq 0} L_n^{(\alpha)}(q) t^{2n}, \end{aligned}$$

that is

$$e^{-\frac{p}{t^2} - \frac{q}{1-t^2}} = e^{-p-q} t^2 (1-t^2)^{\alpha+1} \sum_{m,n \geq 0} L_m^{(\alpha)}(p) L_n^{(\alpha)}(q) t^{2n} (1-t^2)^m. \quad (2.4)$$

By inserting (2.4) into (2.2) and changing the order of integration and summation, we conclude

$$\begin{aligned} \mathbf{M}_{v,p,q}(x) = & -\frac{2 \left(\frac{x}{2}\right)^v e^{-p-q}}{\sqrt{\pi} \Gamma(v + \frac{1}{2})} \sum_{m,n \geq 0} L_m^\alpha(p) L_n^\alpha(q) \\ & \times \int_0^1 t^{2(\alpha+n+1)} (1-t^2)^{v+\alpha+m+\frac{1}{2}} e^{-xt} dt, \end{aligned}$$

and then series expansion of e^{-xt} is used. The rest is obvious. $\square$

We close this section with a certain differential-difference equation for real argument $\mathbf{M}_{v,p,q}(x)$.

Theorem 4. For all $\min\{\Re(p), \Re(q)\} > 0$ and for $\Re(v) > -\frac{1}{2}$ when $p = q = 0$, we have

$$x^2 w_v'' - 2v x w_v' - [x^2 - v(v+1)] w_v = -(2v+1)x w_{v+1}, \quad x > 0, \quad (2.5)$$

where $w_v = w_v(x) \equiv \mathbf{M}_{v,p,q}(x)$.

Proof. Upon differentiating both sides of (2.2) with respect to x , we find that

$$\begin{aligned} \left(\frac{d}{dx}\right)^2 (x^{-\nu} \mathbf{M}_{\nu,p,q}(x)) &= -\frac{2^{1-\nu}}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_0^1 t^2 (1-t^2)^{\nu-\frac{1}{2}} e^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}} dt \\ &= -\frac{2^{1-\nu}}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_0^1 \{1 - (1-t^2)\} (1-t^2)^{\nu-\frac{1}{2}} e^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}} dt, \end{aligned}$$

which can be rewritten as follows:

$$\begin{aligned} \left(\frac{d}{dx}\right)^2 (x^{-\nu} \mathbf{M}_{\nu,p,q}(x)) &= -\frac{2^{1-\nu}}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} e^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}} dt \\ &\quad + \frac{2^{1-\nu}}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_0^1 (1-t^2)^{\nu+1} e^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}} dt \\ &= \frac{1}{x^\nu} \mathbf{M}_{\nu,p,q}(x) - \frac{2\nu+1}{x^{\nu+1}} \mathbf{M}_{\nu+1,p,q}(x), \end{aligned}$$

where we use the definition (2.2). Expanding the left-hand-side second derivative and reducing the material we arrive at (2.5). $\square$

3. THE (p, q) -BESSEL-STRUVE KERNEL FUNCTION

The so-called Bessel-Struve kernel function S_ν is defined by the series [1, p. 1845]

$$S_\nu(x) = \frac{\Gamma(\nu+1)}{\sqrt{\pi}} \sum_{n \geq 0} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2} + \nu + 1)} \frac{x^n}{n!}, \quad \nu > -1, x \in \mathbb{R}.$$

Consider the initial value problem (see [8, 9, 11]):

$$\mathcal{L}_\nu u(x) = \lambda^2 u(x), \quad u(0) = 1, \quad u'(0) = \frac{\lambda \Gamma(\nu+1)}{\sqrt{\pi} \Gamma(\nu + \frac{3}{2})},$$

where for $\nu > -\frac{1}{2}$, $\mathcal{L}_\nu$ stands for the Bessel-Struve operator defined by

$$\mathcal{L}_\nu u(x) = \frac{d^2 u}{dx^2}(x) + \frac{2\nu+1}{x} \left(\frac{du}{dx}(x) - \frac{du}{dx}(0) \right)$$

with some $u \in C^\infty(\mathbb{R})$. The Bessel-Struve kernel function becomes the particular case of the unique solution $S_\nu(\lambda x)$ of the above initial value problem.

The function S_ν is expressible in terms of the Beta function viz.

$$S_\nu(x) = \frac{\Gamma(\nu+1)}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \sum_{n \geq 0} \mathbf{B}\left(\frac{n}{2} + \frac{1}{2}, \nu + \frac{1}{2}\right) \frac{x^n}{n!},$$

and also as a specific Fox-Wright function

$$S_\nu(x) = {}_1\Psi_1 \left[\begin{matrix} (\frac{1}{2}, \frac{1}{2}) \\ (\nu+1, \frac{1}{2}) \end{matrix} \middle| x \right], \quad x \in \mathbb{R},$$

mentioning that $S_\nu(0) = 1$ and $\Delta = 1$ which ensures the convergence of this series on the whole real x axis.

Here, our aim is introducing and studying the novel (p, q) -extended Bessel-Struve kernel function $S_{\nu, p, q}(x)$ by presenting functional bound, integral representation, Mellin transform and other properties.

Analogously to (1.1), we introduce the (p, q) -extended Bessel-Struve kernel function $S_{\nu, p, q}(x)$ in the form

$$S_{\nu, p, q}(x) = \frac{\Gamma(\nu + 1)}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \sum_{n \geq 0} B\left(\frac{n}{2} + \frac{1}{2}, \nu + \frac{1}{2}; p, q\right) \frac{x^n}{n!}. \quad (3.1)$$

with $\min\{p, q\} \geq 0$ and $\Re(\nu) > -\frac{1}{2}$ when $p = q = 0$.

Firstly, we derive an integral representation and a functional upper bound for $S_{\nu, p, q}(x)$.

Theorem 5. For all $\min\{\Re(p), \Re(q)\} > 0$ and for $\Re(\nu) > -\frac{1}{2}$ when $p = q = 0$, we have

$$S_{\nu, p, q}(x) = \frac{2 \Gamma(\nu + 1)}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} e^{xt - \frac{p}{t^2} - \frac{q}{1-t^2}} dt, \quad x \geq 0. \quad (3.2)$$

Proof. By virtue of the integral expression [10]

$$B(x, y; p, q) = 2 \int_0^1 t^{2x-1} (1-t^2)^{y-1} e^{-\frac{p}{t^2} - \frac{q}{1-t^2}} dt,$$

valid for all $\min\{\Re(x), \Re(y)\} > 0; \min\{\Re(p), \Re(q)\} \geq 0$, from (3.1) one yields

$$\begin{aligned} S_{\nu, p, q}(x) &= \frac{2 \Gamma(\nu + 1)}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \sum_{n \geq 0} \frac{x^n}{n!} \int_0^1 t^n (1-t^2)^{\nu-\frac{1}{2}} e^{-\frac{p}{t^2} - \frac{q}{1-t^2}} dt \\ &= \frac{2 \Gamma(\nu + 1)}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} e^{-\frac{p}{t^2} - \frac{q}{1-t^2}} \sum_{n \geq 0} \frac{(xt)^n}{n!} dt, \end{aligned}$$

with interchanged order of summation and integration. This is equivalent to the assertion. $\square$

Now, making use of the estimate [15, Lemma 2]

$$\sup_{0 < t < 1} e^{-\frac{p}{t^2} - \frac{q}{1-t^2}} = e^{-(\sqrt{p} + \sqrt{q})^2}, \quad \min\{p, q\} \geq 0,$$

in (3.2) we deduce

Corollary 1. For all $\min\{p, q\} > 0$ and for $\nu > -\frac{1}{2}$ if $p = q = 0$ we have

$$S_{\nu, p, q}(x) \leq e^{-(\sqrt{p} + \sqrt{q})^2} S_\nu(x), \quad x \geq 0.$$

Theorem 6. For all $\min\{\Re(p), \Re(q)\} > 0$, for $\Re(v) > -\frac{1}{2}$ if $p = q = 0$ and for all $x \in \mathbb{R} \setminus \{0\}$ we have

$$S_{v,p,q}(x) = \Gamma(v+1) \left(\frac{2}{x}\right)^v (J_{v,p,q}(ix) - i\mathbf{H}_{v,p,q}(ix)),$$

$$S_{v,p,q}(x) = \Gamma(v+1) \left(\frac{2}{x}\right)^v (I_{v,p,q}(x) + \mathbf{L}_{v,p,q}(x)).$$

Proof. Using the known integral representations [10]

$$J_{v,p,q}(x) = \frac{2 \left(\frac{x}{2}\right)^v}{\sqrt{\pi} \Gamma(v + \frac{1}{2})} \int_0^1 (1-t^2)^{v-\frac{1}{2}} \cos(xt) e^{-\frac{p}{t^2} - \frac{q}{1-t^2}} dt,$$

$$\mathbf{H}_{v,p,q}(x) = \frac{2 \left(\frac{x}{2}\right)^v}{\sqrt{\pi} \Gamma(v + \frac{1}{2})} \int_0^1 (1-t^2)^{v-\frac{1}{2}} \sin(xt) e^{-\frac{p}{t^2} - \frac{q}{1-t^2}} dt,$$

in right-hand side of the first stated relation we recover $S_{v,p,q}(x)$. Similarly, with the aid of integral forms [10]

$$I_{v,p,q}(x) = \frac{2 \left(\frac{x}{2}\right)^v}{\sqrt{\pi} \Gamma(v + \frac{1}{2})} \int_0^1 (1-t^2)^{v-\frac{1}{2}} \cosh(xt) e^{-\frac{p}{t^2} - \frac{q}{1-t^2}} dt,$$

$$\mathbf{L}_{v,p,q}(x) = \frac{2 \left(\frac{x}{2}\right)^v}{\sqrt{\pi} \Gamma(v + \frac{1}{2})} \int_0^1 (1-t^2)^{v-\frac{1}{2}} \sinh(xt) e^{-\frac{p}{t^2} - \frac{q}{1-t^2}} dt,$$

we finish the proof of the second formula. $\square$

Our next goal is the Mellin transform result.

Theorem 7. For all $\min\{\Re(r), \Re(s)\} > 0$ and $\Re(v) > -\frac{1}{2}$ the Mellin transforms of $S_{v,p,q}$ with respect to $p, q \geq 0$, read as follows:

$$\mathcal{M}\{S_{v,p,q}(x)\}(r,s) = \frac{\Gamma(v+1)\Gamma(r)\Gamma(s)\Gamma(v+s+\frac{1}{2})}{\sqrt{\pi}\Gamma(v+\frac{1}{2})} {}_1\Psi_1\left[\begin{matrix} (\frac{1}{2}+r, \frac{1}{2}) \\ (v+r+s+1, \frac{1}{2}) \end{matrix} \middle| x\right].$$

Proof. We find from (3.2) ad definitionem

$$\begin{aligned} \mathcal{M}\{S_{v,p,q}(x)\}(r,s) &= \int_0^\infty \int_0^\infty p^{r-1} q^{s-1} S_{v,p,q}(x) dp dq \\ &= \int_0^\infty \int_0^\infty \left(\frac{\Gamma(v+1) p^{r-1} q^{s-1}}{\sqrt{\pi} \Gamma(v+\frac{1}{2})} \sum_{n \geq 0} \frac{x^n \mathbf{B}(\frac{n}{2} + \frac{1}{2}, v + \frac{1}{2}; p, q)}{n!} \right) dp dq \\ &= \frac{\Gamma(v+1)}{\sqrt{\pi} \Gamma(v+\frac{1}{2})} \sum_{n \geq 0} \frac{x^n}{n!} \int_0^\infty \int_0^\infty p^{r-1} q^{s-1} \mathbf{B}(\frac{n}{2} + \frac{1}{2}, v + \frac{1}{2}; p, q) dp dq. \end{aligned}$$

By virtue of the formula [7, p.342, Eq. (2.1)]

$$\int_0^\infty \int_0^\infty p^{r-1} q^{s-1} \mathbf{B}(x, y; p, q) \, dp \, dq = \Gamma(r) \Gamma(s) \mathbf{B}(x+r, y+s),$$

valid for all $\min\{\Re(r), \Re(s)\} > 0$, the double integral becomes

$$\mathcal{M}\{S_{v,p,q}(x)\}(r, s) = \frac{\Gamma(v+1)\Gamma(r)\Gamma(s)}{\sqrt{\pi}\Gamma(v+\frac{1}{2})} \sum_{n \geq 0} \frac{\mathbf{B}(\frac{n}{2} + \frac{1}{2} + r, v + \frac{1}{2} + s)}{n!} x^n.$$

After simplification we infer the asserted formula. $\square$

Remark 2. For $r = 1 = s$ in Theorem 7 we obtain the integration formula

$$\int_0^\infty \int_0^\infty S_{v,p,q}(x) \, dp \, dq = \frac{\Gamma(v+1)(v+\frac{1}{2})}{\sqrt{\pi}} {}_1\Psi_1\left[\begin{matrix} (\frac{3}{2}, \frac{1}{2}) \\ (v+3, \frac{1}{2}) \end{matrix} \middle| x\right].$$

The following Laguerre polynomial representation for $S_{v,p,q}(x)$ has been established by the lines of the same fashion result for $\mathbf{M}_{v,p,q}(x)$ in Theorem 3, therefore we omit the proof.

Theorem 8. For all $\min\{\Re(p), \Re(q)\} > 0$ and $\Re(v) > -\frac{1}{2}$ we have

$$S_{v,p,q}(x) = \frac{\Gamma(v+1)(v+\frac{1}{2})_{\alpha+1}}{\sqrt{\pi} e^{p+q}} \sum_{m,n \geq 0} L_m^\alpha(p) L_n^\alpha(q) (v+\alpha+\frac{3}{2})_m \\ \times {}_1\Psi_1\left[\begin{matrix} (\alpha+n+\frac{3}{2}, \frac{1}{2}) \\ (v+2\alpha+m+n+3, \frac{1}{2}) \end{matrix} \middle| x\right],$$

where the Pochhammer symbol notation $(a)_b = \Gamma(a+b)/\Gamma(a)$ has been employed.

To end the exposition we obtain a derivation formula for $S_{v,p,q}(x)$.

Theorem 9. For all $\min\{\Re(p), \Re(q)\} > 0$ and for $\Re(v) > -\frac{1}{2}$ when $p = q = 0$, we have

$$\left(\frac{d}{dx}\right)^2 (S_{v,p,q}(x)) = S_{v,p,q}(x) - \frac{(2v+1)}{2(v+1)} S_{v+1,p,q}(x).$$

Proof. Recall the series representation (3.1) of the Bessel–Struve kernel

$$S_{v,p,q}(x) = \frac{\Gamma(v+1)}{\sqrt{\pi}\Gamma(v+\frac{1}{2})} \sum_{n \geq 0} \mathbf{B}\left(\frac{n}{2} + \frac{1}{2}, v + \frac{1}{2}; p, q\right) \frac{x^n}{n!}.$$

With the help of the contiguous relation [7, p. 362, Theorem 3]

$$\mathbf{B}(x+1, y; p, q) + \mathbf{B}(x, y+1; p, q) = \mathbf{B}(x, y; p, q),$$

a direct calculation gives

$$\left(\frac{d}{dx}\right)^2 (S_{v,p,q}(x)) = \frac{\Gamma(v+1)}{\sqrt{\pi}\Gamma(v+\frac{1}{2})} \sum_{n \geq 2} \left(\mathbf{B}\left(\frac{n}{2} - \frac{1}{2}, v + \frac{1}{2}; p, q\right)\right)$$

$$\begin{aligned}
& -B\left(\frac{n}{2}-\frac{1}{2}, \nu+\frac{3}{2}; p, q\right) \frac{x^{n-2}}{n!} \\
& = \frac{\Gamma(\nu+1)}{\sqrt{\pi} \Gamma\left(\nu+\frac{1}{2}\right)} \sum_{k \geq 0} B\left(\frac{k}{2}+\frac{1}{2}, \nu+\frac{1}{2}; p, q\right) \frac{x^k}{k!} \\
& \quad + \frac{\Gamma(\nu+1)}{\sqrt{\pi} \Gamma\left(\nu+\frac{1}{2}\right)} \sum_{k \geq 0} B\left(\frac{k}{2}+\frac{1}{2}, \nu+\frac{3}{2}; p, q\right) \frac{x^k}{k!}.
\end{aligned}$$

The rest is obvious. $\square$

4. LOG-CONVEXITY AND TURÁN TYPE INEQUALITIES

In this section, first we prove the log-convexity properties and Turán type inequalities for the normalized variant of modified Struve function of second kind $\mathbf{M}_{\nu,p,q}(x)$. Let us consider the normalized variant $\mathcal{M}_{\nu,p,q}: \mathbb{R}_+ \mapsto \mathbb{R}$, defined by

$$\mathcal{M}_{\nu,p,q}(x) = -\Gamma\left(\nu+\frac{1}{2}\right) \left(\frac{x}{2}\right)^{-\nu} \mathbf{M}_{\nu,p,q}(x) \quad (4.1)$$

$$= \frac{2}{\sqrt{\pi}} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} e^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}} dt. \quad (4.2)$$

Theorem 10. *Let $\min\{p, q\} \geq 0$. Then the following assertions are true:*

- The function $\nu \mapsto \mathcal{M}_{\nu,p,q}(x)$ is log-convex on $(-\frac{1}{2}, \infty)$ for all $x > 0$.
- The function $x \mapsto \mathcal{M}_{\nu,p,q}(x)$ is log-convex on $(0, \infty)$ for all $\nu > -\frac{1}{2}$.
- The function $(p, q) \mapsto \mathcal{M}_{\nu,p,q}(x)$ is log-convex on $(0, \infty)$ for all $x > 0$ and $\nu > -\frac{1}{2}$.

Moreover, for the same parameter range there holds the Turán inequality

$$\mathcal{M}_{\nu,p,q}^2(x) - \mathcal{M}_{\nu-1,p,q}(x) \mathcal{M}_{\nu+1,p,q}(x) \leq 0, \quad \nu > -\frac{1}{2}, \quad (4.3)$$

which is equivalent with the Turán-type inequality

$$\mathbf{M}_{\nu,p,q}^2(x) - \mathbf{M}_{\nu-1,p,q}(x) \mathbf{M}_{\nu+1,p,q}(x) \leq \frac{1}{\nu+\frac{1}{2}} \mathbf{M}_{\nu,p,q}^2(x), \quad \nu > -\frac{1}{2}. \quad (4.4)$$

Furthermore, for the same parameter range there holds the Turán inequality

$$\mathcal{M}_{\nu,p,q}^2(x) - \mathcal{M}_{\nu,p-1,q-1}(x) \mathcal{M}_{\nu,p+1,q+1}(x) \leq 0, \quad \nu > -\frac{1}{2}. \quad (4.5)$$

Proof. Using the integral representation (4.2) by aid of classical Hölder–Rogers inequality for integrals, we have

$$\mathcal{M}_{\lambda\nu_1+(1-\lambda)\nu_2,p,q}(x) = \frac{2}{\sqrt{\pi}} \int_0^1 (1-t^2)^{\lambda\nu_1+(1-\lambda)\nu_2-\frac{1}{2}} e^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}} dt$$

$$\begin{aligned}
 &= \frac{2}{\sqrt{\pi}} \int_0^1 (1-t^2)^{\lambda(v_1-\frac{1}{2})+(1-\lambda)(v_2-\frac{1}{2})} e^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}} dt \\
 &= \frac{2}{\sqrt{\pi}} \int_0^1 \left\{ (1-t^2)^{v_1-\frac{1}{2}} e^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}} \right\}^{\lambda} \\
 &\quad \times \left\{ (1-t^2)^{v_1-\frac{1}{2}} e^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}} \right\}^{1-\lambda} dt \\
 &\leq \left\{ \frac{2}{\sqrt{\pi}} \int_0^1 (1-t^2)^{v_1-\frac{1}{2}} e^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}} dt \right\}^{\lambda} \\
 &\quad \times \left\{ \frac{2}{\sqrt{\pi}} \int_0^1 (1-t^2)^{v_2-\frac{1}{2}} e^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}} dt \right\}^{1-\lambda}.
 \end{aligned}$$

This is equivalent to

$$\mathcal{M}_{\lambda v_1+(1-\lambda)v_2,p,q}(x) \leq [\mathcal{M}_{v_1,p,q}(x)]^{\lambda} [\mathcal{M}_{v_2,p,q}(x)]^{1-\lambda}, \quad (4.6)$$

which proves the first assertion for all $v_1, v_2 > -\frac{1}{2}$, $\lambda \in [0, 1]$ and $x > 0$.

In a similar manner, we can prove the next two assertions that $x \mapsto \mathcal{M}_{v,p,q}(x)$ is log-convex for all $v > -\frac{1}{2}$, $\lambda \in [0, 1]$ and $x, y > 0$, that is

$$\mathcal{M}_{v,p,q}(\lambda x + (1-\lambda)y) \leq [\mathcal{M}_{v,p,q}(x)]^{\lambda} [\mathcal{M}_{v,p,q}(y)]^{1-\lambda},$$

while the function $(p, q) \mapsto \mathcal{M}_{v,p,q}(x)$ is log-convex for all $p_1, p_2, q_1, q_2 > 0$, $v > -\frac{1}{2}$, $\lambda \in [0, 1]$ and $x > 0$:

$$\mathcal{M}_{v,\lambda p_1+(1-\lambda)p_2,\lambda q_1+(1-\lambda)q_2}(x) \leq [\mathcal{M}_{v,p_1,q_1}(x)]^{\lambda} [\mathcal{M}_{v,p_2,q_2}(x)]^{1-\lambda}. \quad (4.7)$$

Next, choosing $v_1 = v - 1$, $v_2 = v + 1$ and $\lambda = \frac{1}{2}$ in (4.6) we conclude the Turán inequality (4.3).

As to the Turán-type inequality (4.4), we replace (4.1) into (4.3) and transform the result by

$$\frac{\Gamma(v+\frac{3}{2})\Gamma(v-\frac{1}{2})}{\Gamma^2(v+\frac{1}{2})} = \frac{2v+1}{2v-1}, \quad v > \frac{1}{2}.$$

Again, specifying $p_1 = p - 1$, $q_1 = q - 1$, $p_2 = p + 1$, $q_2 = q + 1$ and $\lambda = \frac{1}{2}$ in (4.7), we deduce the Turán inequality (4.5). $\square$

Also one proves the log-convexity of $S_{v,p,q}(x)$ for $x > 0$.

Theorem 11. *Let $\min\{p, q\} \geq 0$. Then the following assertions are true:*

- *The function $x \mapsto S_{v,p,q}(x)$ is log-convex on positive real x half-axis for all $v > -\frac{1}{2}$.*

- The function $(p, q) \mapsto S_{v,p,q}(x)$ is log-convex on $(0, \infty)$ for all $x > 0$ and $v > -\frac{1}{2}$.

Moreover, for the same parameter range there holds the Turán inequality

$$S_{v,p,q}^2(x) - S_{v,p-1,q-1}(x)S_{v,p+1,q+1}(x) \leq 0, \quad v > -\frac{1}{2}, \quad (4.8)$$

Proof. Considering the integral representation (3.2) assuming $x, y > 0$, $\lambda \in [0, 1]$ we have

$$S_{v,p,q}(\lambda x + (1-\lambda)y) \leq \{S_{v,p,q}(x)\}^\lambda \{S_{v,p,q}(y)\}^{1-\lambda}.$$

Indeed, this follows with the aid of the Hölder–Rogers inequality for integrals, viz.

$$\begin{aligned} & \int_0^1 \left\{ (1-t^2)^{v-\frac{1}{2}} e^{xt-\frac{p}{t^2}-\frac{q}{1-t^2}} \right\}^\lambda \left\{ (1-t^2)^{v-\frac{1}{2}} e^{yt-\frac{p}{t^2}-\frac{q}{1-t^2}} \right\}^{1-\lambda} dt \\ & \leq \left\{ \int_0^1 (1-t^2)^{v-\frac{1}{2}} e^{xt-\frac{p}{t^2}-\frac{q}{1-t^2}} dt \right\}^\lambda \left\{ \int_0^1 (1-t^2)^{v-\frac{1}{2}} e^{yt-\frac{p}{t^2}-\frac{q}{1-t^2}} dt \right\}^{1-\lambda}. \end{aligned}$$

completing the proof in the prescribed form.

Letting $p_1, p_2, q_1, q_2 > 0$, $v > -\frac{1}{2}$, $\lambda \in [0, 1]$ and $x > 0$, we obtain

$$S_{v,\lambda p_1+(1-\lambda)p_2,\lambda q_1+(1-\lambda)q_2}(x) \leq [S_{v,p_1,q_1}(x)]^\lambda [S_{v,p_2,q_2}(x)]^{1-\lambda}.$$

Specifying here $p_1 = p-1$, $q_1 = q-1$, $p_2 = p+1$, $q_2 = q+1$ and $\lambda = \frac{1}{2}$ we conclude the Turán inequality (4.8). $\square$

REFERENCES

- [1] Á. Baricz, S. R. Mondal, and A. Swaminathan, “Monotonicity properties of the Bessel–Struve kernel.” *Bull. Korean Math. Soc.*, vol. 53, no. 6, pp. 1845–1856, 2016, doi: [10.4134/BKMS.b151021](#).
- [2] Á. Baricz and T. K. Pogány, “Functional inequalities for modified Struve functions II.” *Math. Inequal. Appl.*, vol. 17, no. 4, pp. 1387–1398, 2014, doi: [10.7153/mia-17-102](#).
- [3] M. A. Chaudhry, A. Qadir, M. Rafique, and S. M. Zubair, “Extension of Euler’s Beta function.” *J. Comput. Appl. Math.*, vol. 78, no. 1, pp. 19–32, 1997, doi: [10.1016/S0377-0427\(96\)00102-1](#).
- [4] M. A. Chaudhry, A. Qadir, H. M. Srivastava, and R. B. Paris, “Extended hypergeometric and confluent hypergeometric functions.” *Appl. Math. Comput.*, vol. 159, no. 2, pp. 589–602, 2004, doi: [10.1016/j.amc.2003.09.017](#).
- [5] M. A. Chaudhry and S. M. Zubair, *On a Class of Incomplete Gamma Functions with Applications*. Boca Raton, FL: CRC Press (Chapman and Hall), 2002. doi: [1-58488-143-7/hbk](#).
- [6] J. Choi, R. K. Parmar, and T. K. Pogány, “Mathieu-type series built by (p, q) -extended Gaussian hypergeometric function.” *Bull. Korean Math. Soc.*, vol. 54, no. 3, pp. 789–797, 2017, doi: [10.4134/BKMS.b160313](#).
- [7] J. Choi, A. K. Rathie, and R. K. Parmar, “Extension of extended beta, hypergeometric and confluent hypergeometric functions.” *Honam Math. J.*, vol. 36, no. 2, pp. 339–367, 2014, doi: [HMJ.2014.36.2.357](#).
- [8] A. Gasmi and M. Sifi, “The Bessel–Struve intertwining operator on $\mathbb{C}$ and mean-periodic functions.” *Int. J. Math. Math. Sci.*, vol. 2004, no. 57–60, pp. 3171–3185, 2004, doi: [10.1155/S0161171204309178](#).

- [9] A. Gasmin and F. Soltani, "Fock spaces for the Bessel–Struve kernel." *J. Anal. Appl.*, vol. 3, no. 2, pp. 91–106, 2005.
- [10] D. Jankov Maširević, R. K. Parmar, and T. K. Pogány, " (p, q) -extended Bessel and modified Bessel functions of the first kind." *Results Math.*, vol. 72, no. 1-2, pp. 617–632, 2017, doi: [10.1007/s00025-016-0649-1](https://doi.org/10.1007/s00025-016-0649-1).
- [11] L. Kamoun and M. Sifi, "Bessel–Struve intertwining operator and generalized Taylor series on the real line." *Integral Transforms Spec. Funct.*, vol. 16, no. 1, pp. 39–55, 2005, doi: [10.1080/1065246042000272063](https://doi.org/10.1080/1065246042000272063).
- [12] M. J. Luo, R. K. Parmar, and R. K. Raina, "On extended Hurwitz–Lerch zeta function." *J. Math. Anal. Appl.*, vol. 448, no. 2, pp. 1281–1304, 2017, doi: [10.1016/j.jmaa.2016.11.046](https://doi.org/10.1016/j.jmaa.2016.11.046).
- [13] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. C. (eds.), *NIST Handbook of Mathematical Functions*. Cambridge: Cambridge University Press, 2010. doi: [10.1111/j.1751-5823.2011.00134.18.x](https://doi.org/10.1111/j.1751-5823.2011.00134.18.x).
- [14] R. B. Paris and D. Kaminski, *Asymptotics and Mellin-Barnes Integrals*. Cambridge: Cambridge University Press, 2001. doi: [10.1017/CBO9780511546662](https://doi.org/10.1017/CBO9780511546662).
- [15] R. K. Parmar and T. K. Pogány, "Extended Srivastava's triple hypergeometric $H_{A,p,q}$ function and related bounding inequalities." *J. Contemp. Math. Anal.*, vol. 52, no. 6, pp. 261–272, 2017, doi: [10.3103/S1068362317060036](https://doi.org/10.3103/S1068362317060036).
- [16] E. D. Rainville, *Special Functions*. Bronx, New York: Chelsea Publishing Company, 1971.
- [17] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*. New York, Chichester, Brisbane and Toronto: Ellis Horwood Limited, Chichester; Halsted Press, a division of John Wiley and Sons, 1985.
- [18] H. M. Srivastava, R. K. Parmar, and P. Chopra, "A class of extended fractional derivative operators and associated generating relations involving hypergeometric functions." *Axioms*, vol. 1, no. 3, pp. 238–258, 2012, doi: [10.3390/axioms1030238](https://doi.org/10.3390/axioms1030238).

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NOTES ON TWO KINDS OF SPECIAL VALUES FOR THE BELL POLYNOMIALS OF THE SECOND KIND

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Abstract. In the paper, by methods and techniques in combinatorial analysis and the theory of special functions, the authors discuss two kinds of special values for the Bell polynomials of the second kind for two special sequences, find a relation between these two kinds of special values for the Bell polynomials of the second kind, and derive an identity involving the combinatorial numbers.

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1. MOTIVATION

In [1, Definition 11.2] and [2, p. 134, Theorem A], the Bell polynomials of the second kind, denoted by $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ for $n \geq k \geq 0$, are defined by

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \leq i \leq n, \ell_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^n i \ell_i = n \\ \sum_{i=1}^n \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i}.$$

For more information on the Bell polynomials of the second kind $B_{n,k}$, please refer to the monographs and handbooks [1–3] and closely related references therein.

In [1, p. 451], the formulas

$$\begin{aligned} B_{2r,k}(0, 2!, \dots, 0, (2r)!) &= \frac{(2r)!}{k!} \binom{r-1}{k-1}, \\ B_{2r-1,k}(0, 2!, \dots, (2r-2)!, 0) &= 0, \\ B_{2r,2s}(1!, 0, \dots, (2r-1)!, 0) &= \frac{(2r)!}{(2s)!} \binom{r+s-1}{2s-1}, \\ B_{2r,2s-1}(1!, 0, \dots, (2r-1)!, 0) &= 0, \end{aligned}$$

$$B_{2r-1,2s-1}(1!, 0, \dots, (2r-1)!, 0) = \frac{(2r-1)!}{(2s-1)!} \binom{r+s-2}{2s-2},$$

$$B_{2r-1,2s}(1!, 0, \dots, 0, (2r-1)!) = 0$$

were stated, but no proof was supplied for them there.

For simplicity, we denote

$$\lambda(n, k) = B_{n,k} \left(0, 2!, 0, 4!, \dots, (n-k+1)! \frac{1 + (-1)^{n-k+1}}{2} \right)$$

and

$$\mu(n, k) = B_{n,k} \left(1!, 0, 3!, 0, \dots, (n-k+1)! \frac{1 - (-1)^{n-k+1}}{2} \right).$$

In terms of these notations, the above claims in [1, p. 451] can be restated as

$$\begin{aligned} \lambda(2r, k) &= \frac{(2r)!}{k!} \binom{r-1}{k-1}, & \lambda(2r-1, k) &= 0, \\ \mu(2r, 2s) &= \frac{(2r)!}{(2s)!} \binom{r+s-1}{2s-1}, & \mu(2r, 2s-1) &= 0, \\ \mu(2r-1, 2s-1) &= \frac{(2r-1)!}{(2s-1)!} \binom{r+s-2}{2s-2}, & \mu(2r-1, 2s) &= 0. \end{aligned}$$

In this paper, by methods and techniques in combinatorial analysis and the theory of special functions, we will provide alternative proofs for the above six formulas, find a relation between them, and derive an identity involving combinatorial numbers.

2. MAIN RESULTS

We first derive an identity involving combinatorial numbers, which will be useful in next proofs of our main results.

Theorem 1. For $k \geq 1$ and $n \geq 0$, we have

$$\sum_{q=0}^n \frac{(-1)^q}{k+q} \binom{n}{q} = \frac{1}{k \binom{k+n}{k}}. \quad (2.1)$$

Proof. Let

$$f(x) = \sum_{q=0}^n \frac{(-1)^q}{k+q} \binom{n}{q} x^{k+q}, \quad x \in [-1, 1].$$

Then $f(0) = 0$ and

$$f'(x) = \sum_{q=0}^n (-1)^q \binom{n}{q} x^{k+q-1} = x^{k-1} \sum_{q=0}^n (-1)^q \binom{n}{q} x^q = x^{k-1} (1-x)^n.$$

Integrating from 0 to $t \in (0, 1]$ on both sides of the above equality yields

$$f(t) = \int_0^t x^{k-1} (1-x)^n dx = B(t; k, n+1),$$

where $B(z; a, b)$ denotes the incomplete beta function, see [3, p. 183]. Therefore, we obtain

$$\begin{aligned} f(1) &= B(1; k, n+1) = B(k, n+1) \\ &= \frac{\Gamma(k)\Gamma(n+1)}{\Gamma(k+n+1)} = \frac{(k-1)!n!}{(k+n)!} = \frac{1}{k} \frac{k!n!}{(k+n)!} = \frac{1}{k \binom{n+k}{k}}, \end{aligned}$$

where $B(a, b)$ and $\Gamma(z)$ denote the beta function and the classical Euler gamma function respectively, see [3, p. 142] and [3, Chapter 5]. The formula (2.1) is thus proved. The proof of Theorem 1 is complete. $\square$

We are now in a position to state and prove our main results.

Theorem 2. For $n \geq k \geq 0$, we have the relation

$$\frac{\mu(n, k)}{n!} = \frac{\lambda(n+k, k)}{(n+k)!} \quad (2.2)$$

and two explicit formulas

$$\lambda(n, k) = \frac{1 + (-1)^n}{2} \frac{n!}{k!} \binom{\frac{n}{2} - 1}{k-1} \quad (2.3)$$

and

$$\mu(n, k) = \frac{1 + (-1)^{n+k}}{2} \frac{n!}{k!} \binom{\frac{n+k}{2} - 1}{k-1}. \quad (2.4)$$

Proof. It is known [2, 10] that the quantities

$$\langle x \rangle_n = \prod_{\ell=0}^{n-1} (x - \ell) = \begin{cases} x(x-1) \cdots (x-n+1), & n \geq 1 \\ 1, & n = 0 \end{cases}$$

and

$$(x)_n = \prod_{\ell=0}^{n-1} (x + \ell) = \begin{cases} x(x+1) \cdots (x+n-1), & n \geq 1 \\ 1, & n = 0 \end{cases}$$

are called the falling and rising factorials respectively. In [2, p. 133], it was listed that

$$\frac{1}{k!} \left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!} \right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \frac{t^n}{n!} \quad (2.5)$$

for $k \geq 0$. Hence, we have

$$\sum_{n=k}^{\infty} \lambda(n, k) \frac{t^n}{n!} = \frac{1}{k!} \left[\sum_{m=1}^{\infty} m! \frac{1 + (-1)^m}{2} \frac{t^m}{m!} \right]^k = \frac{1}{k!} \left(\frac{t^2}{1-t^2} \right)^k$$

which is equivalent to

$$\sum_{n=0}^{\infty} \lambda(n+k, k) \frac{1}{\binom{n+k}{n}} \frac{t^n}{n!} = \left(\frac{t}{1-t^2} \right)^k.$$

Since the function $\frac{t}{1-t^2}$ is odd, we derive $\lambda(2r-1, k) = 0$. Further computation yields

$$\begin{aligned} \lambda(n+k, k) &= \binom{n+k}{n} \lim_{t \rightarrow 0} \frac{d^n}{dt^n} \left(\frac{t}{1-t^2} \right)^k \\ &= \binom{n+k}{n} \frac{1}{2^k} \lim_{t \rightarrow 0} \left[\left(\frac{1}{1-t} - \frac{1}{1+t} \right)^k \right]^{(n)} \\ &= \binom{n+k}{n} \frac{1}{2^k} \lim_{t \rightarrow 0} \left[\sum_{\ell=0}^k \binom{k}{\ell} \left(\frac{1}{1-t} \right)^{\ell} (-1)^{k-\ell} \left(\frac{1}{1+t} \right)^{k-\ell} \right]^{(n)} \\ &= \binom{n+k}{n} \frac{1}{2^k} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \lim_{t \rightarrow 0} \sum_{q=0}^n \binom{n}{q} \left[\left(\frac{1}{1-t} \right)^{\ell} \right]^{(q)} \left[\left(\frac{1}{1+t} \right)^{k-\ell} \right]^{(n-q)} \\ &= \binom{n+k}{n} \frac{1}{2^k} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \lim_{t \rightarrow 0} \sum_{q=0}^n \binom{n}{q} \\ &\quad \times \langle -\ell \rangle_q (-1)^q \left(\frac{1}{1-t} \right)^{\ell+q} \langle -(k-\ell) \rangle_{n-q} \left(\frac{1}{1+t} \right)^{k-\ell+(n-q)} \\ &= \frac{(-1)^k}{2^k} \binom{n+k}{n} \sum_{\ell=0}^k (-1)^{\ell} \binom{k}{\ell} \sum_{q=0}^n (-1)^q \binom{n}{q} \langle -\ell \rangle_q \langle \ell-k \rangle_{n-q} \\ &= \frac{(-1)^{n+k}}{2^k} \binom{n+k}{n} \sum_{\ell=0}^k (-1)^{\ell} \binom{k}{\ell} \sum_{q=0}^n (-1)^q \binom{n}{q} (\ell)_q (k-\ell)_{n-q}. \end{aligned}$$

In summary, we obtain

$$\lambda(n, k) = \frac{(-1)^k}{2^k} \binom{n}{k} \sum_{\ell=0}^k (-1)^{\ell} \binom{k}{\ell} \sum_{q=0}^{n-k} (-1)^q \binom{n-k}{q} \langle -\ell \rangle_q \langle \ell-k \rangle_{n-k-q}$$

$$= \frac{(-1)^n}{2^k} \binom{n}{k} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \sum_{q=0}^{n-k} (-1)^q \binom{n-k}{q} (\ell)_q (k-\ell)_{n-k-q}. \quad (2.6)$$

By similar argument as above, it follows that

$$\sum_{n=k}^{\infty} \mu(n, k) \frac{t^n}{n!} = \frac{1}{k!} \left[\sum_{m=1}^{\infty} m! \frac{1 - (-1)^m}{2} \frac{t^m}{m!} \right]^k = \frac{(-1)^k}{k! 2^k} \left(\frac{1}{t-1} + \frac{1}{t+1} \right)^k$$

which is equivalent to

$$\sum_{n=0}^{\infty} \mu(n+k, k) \frac{1}{\binom{n+k}{n}} \frac{t^n}{n!} = \frac{(-1)^k}{2^k} \frac{1}{t^k} \left(\frac{1}{t-1} + \frac{1}{t+1} \right)^k = \left(\frac{1}{1-t^2} \right)^k. \quad (2.7)$$

Since $\frac{1}{1-t^2}$ is even, we deduce immediately that $\mu(2r-1, 2s) = 0$ and $\mu(2r, 2s-1) = 0$. By the L'Hôpital rule and the identity (2.1) in Theorem 1, it follows that

$$\begin{aligned} \mu(n+k, k) &= \binom{n+k}{n} \frac{(-1)^k}{2^k} \lim_{t \rightarrow 0} \frac{d^n}{dt^n} \left[\frac{1}{t^k} \left(\frac{1}{t-1} + \frac{1}{t+1} \right)^k \right] \\ &= \binom{n+k}{n} \frac{(-1)^k}{2^k} \lim_{t \rightarrow 0} \sum_{q=0}^n \binom{n}{q} \frac{\langle -k \rangle_q}{t^{k+q}} \left[\left(\frac{1}{t-1} + \frac{1}{t+1} \right)^k \right]^{(n-q)} \\ &= \binom{n+k}{n} \frac{(-1)^k}{2^k} \sum_{q=0}^n \binom{n}{q} \frac{\langle -k \rangle_q}{(k+q)!} \lim_{t \rightarrow 0} \left[\left(\frac{1}{t-1} + \frac{1}{t+1} \right)^k \right]^{(n+k)} \\ &= \binom{n+k}{n} \frac{(-1)^k}{2^k} \sum_{q=0}^n \binom{n}{q} \frac{(-1)^q (k+q-1)!}{(k-1)!(k+q)!} \\ &\quad \times \lim_{t \rightarrow 0} \left[\sum_{\ell=0}^k \binom{k}{\ell} \left(\frac{1}{t-1} \right)^\ell \left(\frac{1}{t+1} \right)^{k-\ell} \right]^{(n+k)} \\ &= \binom{n+k}{n} \frac{(-1)^k}{2^k (k-1)!} \sum_{q=0}^n \frac{(-1)^q}{k+q} \binom{n}{q} \sum_{\ell=0}^k \binom{k}{\ell} \\ &\quad \times \lim_{t \rightarrow 0} \sum_{m=0}^{n+k} \binom{n+k}{m} \left[\left(\frac{1}{t-1} \right)^\ell \right]^{(m)} \left[\left(\frac{1}{t+1} \right)^{k-\ell} \right]^{(n+k-m)} \\ &= \frac{(-1)^k}{2^k k!} \sum_{\ell=0}^k \binom{k}{\ell} \lim_{t \rightarrow 0} \sum_{m=0}^{n+k} \binom{n+k}{m} \\ &\quad \times \langle -\ell \rangle_m \left(\frac{1}{t-1} \right)^{\ell+m} \langle \ell-k \rangle_{n+k-m} \left(\frac{1}{t+1} \right)^{(k-\ell)+(n+k-m)} \end{aligned}$$

$$= \frac{(-1)^k}{2^k k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \sum_{m=0}^{n+k} (-1)^m \binom{n+k}{m} \langle -\ell \rangle_m \langle \ell - k \rangle_{n+k-m}.$$

In short, we obtain

$$\begin{aligned} \mu(n, k) &= \frac{(-1)^k}{2^k k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \sum_{m=0}^n (-1)^m \binom{n}{m} \langle -\ell \rangle_m \langle \ell - k \rangle_{n-m} \\ &= \frac{(-1)^{n+k}}{2^k k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \sum_{m=0}^n (-1)^m \binom{n}{m} (\ell)_m (k - \ell)_{n-m}. \end{aligned} \quad (2.8)$$

Comparing between (2.6) and (2.8) reveals

$$\mu(n, k) = \frac{(-1)^{n+k}}{2^k k!} \frac{2^k}{(-1)^{n+k}} \frac{1}{\binom{n+k}{k}} \lambda(n+k, k)$$

which can be rearranged as (2.2).

The Faà di Bruno formula, see [1, Theorem 11.4] and [2, p. 139, Theorem C], can be described in terms of $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ by

$$\frac{d^n}{dx^n} f \circ h(x) = \sum_{k=0}^n f^{(k)}(h(x)) B_{n,k}(h'(x), h''(x), \dots, h^{(n-k+1)}(x)). \quad (2.9)$$

In [1, p. 412] and [2, p. 135], one can find the identity

$$\begin{aligned} B_{n,k}(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}) \\ = a^k b^n B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \end{aligned} \quad (2.10)$$

for $n \geq k \geq 0$ and $a, b \in \mathbb{C}$. In [8, Theorem 4.1] and [12, Section 3], it was set up little by little that

$$B_{n,k}(x, 1, 0, \dots, 0) = \frac{1}{2^{n-k}} \frac{n!}{k!} \binom{k}{n-k} x^{2k-n}, \quad (2.11)$$

where $\binom{0}{0} = 1$ and $\binom{p}{q} = 0$ for $q > p \geq 0$. For detailed information on applications of the formula (2.11), please refer to the papers [4–9, 11–14] and closely related references therein. Then it follows from (2.7), (2.9), (2.10), and (2.11) that, when denoting $u = u(t) = 1 - t^2$,

$$\begin{aligned} \mu(n+k, k) &= \binom{n+k}{n} \lim_{t \rightarrow 0} \frac{d^n}{dt^n} \left(\frac{1}{1-t^2} \right)^k \\ &= \binom{n+k}{n} \lim_{t \rightarrow 0} \sum_{\ell=0}^n \left(\frac{1}{u^k} \right)^{(\ell)} B_{n,\ell}(-2t, -2, 0, \dots, 0) \end{aligned}$$

$$\begin{aligned}
 &= \binom{n+k}{n} \lim_{t \rightarrow 0} \sum_{\ell=0}^n \langle -k \rangle_{\ell} \frac{1}{u^{k+\ell}} (-2)^{\ell} B_{n,\ell}(t, 1, 0, \dots, 0) \\
 &= \binom{n+k}{n} \lim_{t \rightarrow 0} \sum_{\ell=0}^n \langle -k \rangle_{\ell} \frac{1}{(1-t^2)^{k+\ell}} (-2)^{\ell} \frac{1}{2^{n-\ell}} \frac{n!}{\ell!} \binom{\ell}{n-\ell} t^{2\ell-n} \\
 &= \binom{n+k}{n} \lim_{t \rightarrow 0} \sum_{\ell=0}^n \langle k \rangle_{\ell} \frac{n!}{\ell!} \binom{\ell}{n-\ell} \frac{(2t)^{2\ell-n}}{(1-t^2)^{k+\ell}} \\
 &= \begin{cases} 0, & n = 2m-1 \\ \binom{2m+k}{2m} (k)_m \frac{(2m)!}{m!} \binom{m}{2m-m}, & n = 2m \end{cases} \\
 &= \begin{cases} 0, & n = 2m-1 \\ \frac{(2m+k)!}{k!} \binom{m+k-1}{k-1}, & n = 2m \end{cases} \\
 &= \begin{cases} 0, & n = 2m-1 \\ \frac{(n+k)!}{k!} \binom{\frac{n+2k}{2}-1}{k-1}, & n = 2m \end{cases}
 \end{aligned}$$

for $m \in \mathbb{N}$. The formula in (2.4) is thus proved.

Substituting the formula in (2.4) into (2.2) and simplifying lead readily to the formula in (2.3). The proof of Theorem 2 is complete. $\square$

3. REMARKS

In this section, we state several remarks on something related.

Remark 1. In [8, Theorem 2.1], it was proved that

$$B_{n,k} \left(1, 0, 1, \dots, \frac{1 - (-1)^{n-k+1}}{2} \right) = \frac{1}{2^k k!} \sum_{\ell=0}^k (-1)^{\ell} \binom{k}{\ell} (k-2\ell)^n$$

and

$$B_{n,k} \left(0, 1, 0, \dots, \frac{1 + (-1)^{n-k+1}}{2} \right) = \frac{1}{2^k k!} \sum_{\ell=0}^{2k} (-1)^{\ell} \binom{2k}{\ell} (k-\ell)^n$$

for $n \geq k \geq 0$, where 0^0 is regarded as 1. In [8, Section 3], basing on numerical calculation, the authors guessed that

$$B_{2\ell-1,k} \left(0, 1, 0, \dots, \frac{1 + (-1)^{(2\ell-1)-k+1}}{2} \right) = 0, \quad 2\ell-1 \geq k \geq 0, \quad (3.1)$$

$$B_{2\ell,k}\left(0, 1, 0, \dots, \frac{1 - (-1)^{2\ell-k+1}}{2}\right) = 0, \quad 2\ell > k > \ell, \quad (3.2)$$

$$B_{2\ell,k}\left(0, 1, 0, \dots, \frac{1 - (-1)^{2\ell-k+1}}{2}\right) \neq 0, \quad \ell \geq k \geq 1, \quad (3.3)$$

$$B_{k+2\ell,k}(1, 0, 1, \dots, 1, 0, 1) \neq 0, \quad k, \ell \in \mathbb{N}, \quad (3.4)$$

and

$$B_{k+2\ell-1,k}(1, 0, 1, \dots, 0, 1, 0) = 0, \quad k, \ell \in \mathbb{N}. \quad (3.5)$$

In [8, Theorem 3.1], the identity (3.5) was proved to be true. In fact, the identities (3.1) and (3.5) can be concluded readily from the proof of [8, Theorem 2.1] as follows. From the formula (2.5), it followed that

$$\sum_{n=k}^{\infty} B_{n,k}\left(0, 1, 0, \dots, \frac{1 + (-1)^{n-k+1}}{2}\right) \frac{t^n}{n!} = \frac{(\cosh t - 1)^k}{k!} \quad (3.6)$$

and

$$\sum_{n=k}^{\infty} B_{n,k}\left(1, 0, 1, \dots, \frac{1 - (-1)^{n-k+1}}{2}\right) \frac{t^n}{n!} = \frac{\sinh^k t}{k!}. \quad (3.7)$$

In (3.6), the function $\cosh t - 1$ is even, so the identity (3.1) is clearly valid. Since the function $\sinh t$ in (3.7) is odd, then

$$B_{2n,2k-1}(1, 0, 1, \dots, 1, 0) = 0 \quad \text{and} \quad B_{2n-1,2k}(1, 0, 1, \dots, 1, 0) = 0$$

which are equivalent to the identity (3.5). However, the validity of the identities (3.2), (3.3), and (3.4) has not been verified yet.

Remark 2. The formula (2.8) can be rewritten as

$$\begin{aligned} \mu(n, k) &= \frac{(-1)^{n+k} n!}{2^k k!} \\ &\times \sum_{\ell=0}^k \sum_{m=0}^n (-1)^{\ell+m} \binom{k}{\ell} \binom{\ell+m-1}{\ell-1} \binom{(k-\ell)+(n-m)-1}{k-\ell-1}. \end{aligned} \quad (3.8)$$

Then combining the formulas (2.8) and (3.8) with the formula in (2.4) and rearranging arrive at

$$\begin{aligned} \sum_{\ell=0}^k (-1)^{\ell} \binom{k}{\ell} \sum_{m=0}^n (-1)^m \binom{n}{m} (\ell)_m (k-\ell)_{n-m} \\ = [1 + (-1)^{n+k}] 2^{k-1} n! \binom{\frac{n+k}{2} - 1}{k-1} \end{aligned}$$

and

$$\sum_{m=0}^n \sum_{\ell=0}^k (-1)^{\ell+m} \binom{k}{\ell} \binom{\ell+m-1}{\ell-1} \binom{(k-\ell)+(n-m)-1}{k-\ell-1} \\ = [1 + (-1)^{n+k}] 2^{k-1} \binom{\frac{n+k}{2}-1}{k-1}.$$

Comparing these identities with the Vandermonde convolution formula

$$\sum_{k=0}^n \binom{n}{k} \langle x \rangle_k \langle y \rangle_{n-k} = \langle x+y \rangle_n$$

in [1, Theorem 3.1] and [2, p. 44] motivates us to ask a question: is the quantity

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \langle x \rangle_k \langle y \rangle_{n-k}$$

summable?

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REFERENCES

- [1] C. A. Charalambides, *Enumerative combinatorics*, ser. CRC Press Series on Discrete Mathematics and its Applications. Chapman & Hall/CRC, Boca Raton, FL, 2002.
- [2] L. Comtet, *Advanced combinatorics*, enlarged ed. D. Reidel Publishing Co., Dordrecht, 1974, the art of finite and infinite expansions.
- [3] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, Eds., *NIST handbook of mathematical functions*. U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010, with 1 CD-ROM (Windows, Macintosh and UNIX).
- [4] F. Qi, "Integral representations for multivariate logarithmic polynomials," *J. Comput. Appl. Math.*, vol. 336, pp. 54–62, 2018, doi: [10.1016/j.cam.2017.11.047](https://doi.org/10.1016/j.cam.2017.11.047).
- [5] F. Qi and V. Cernanova, "Some discussions on a kind of improper integrals," *Int. J. Anal. Appl.*, vol. 11, pp. 101–109, 2016.
- [6] F. Qi, V. Cernanova, X.-T. Shi, and B.-N. Guo, "Some properties of central Delannoy numbers," *J. Comput. Appl. Math.*, vol. 328, pp. 101–115, 2018, doi: [10.1016/j.cam.2017.07.013](https://doi.org/10.1016/j.cam.2017.07.013).
- [7] F. Qi and B.-N. Guo, "Explicit and recursive formulas, integral representations, and properties of the large Schröder numbers," *Kragujevac J. Math.*, vol. 41, no. 1, pp. 121–141, 2017, doi: [10.5937/KgJMath1701121F](https://doi.org/10.5937/KgJMath1701121F).
- [8] F. Qi and B.-N. Guo, "Explicit formulas for special values of the Bell polynomials of the second kind and for the Euler numbers and polynomials," *Mediterr. J. Math.*, vol. 14, no. 3, pp. Art. 140, 14, 2017, doi: [10.1007/s00009-017-0939-1](https://doi.org/10.1007/s00009-017-0939-1).
- [9] F. Qi, X.-T. Shi, and B.-N. Guo, "Two explicit formulas of the Schröder numbers," *Integers*, vol. 16, pp. Paper No. A23, 15, 2016.

- [10] F. Qi, X.-T. Shi, and F.-F. Liu, “Several identities involving the falling and rising factorials and the Cauchy, Lah, and Stirling numbers,” *Acta Univ. Sapientiae Math.*, vol. 8, no. 2, pp. 282–297, 2016, doi: [10.1515/ausm-2016-0019](https://doi.org/10.1515/ausm-2016-0019).
- [11] F. Qi, X.-T. Shi, F.-F. Liu, and D. V. Kruchinin, “Several formulas for special values of the Bell polynomials of the second kind and applications,” *J. Appl. Anal. Comput.*, vol. 7, no. 3, pp. 857–871, 2017, doi: [10.11948/2017054](https://doi.org/10.11948/2017054).
- [12] F. Qi and M.-M. Zheng, “Explicit expressions for a family of the Bell polynomials and applications,” *Appl. Math. Comput.*, vol. 258, pp. 597–607, 2015, doi: [10.1016/j.amc.2015.02.027](https://doi.org/10.1016/j.amc.2015.02.027).
- [13] C.-F. Wei and F. Qi, “Several closed expressions for the Euler numbers,” *J. Inequal. Appl.*, pp. 2015:219, 8, 2015, doi: [10.1186/s13660-015-0738-9](https://doi.org/10.1186/s13660-015-0738-9).
- [14] J.-L. Zhao and F. Qi, “Two explicit formulas for the generalized Motzkin numbers,” *J. Inequal. Appl.*, pp. Paper No. 44, 8, 2017, doi: [10.1186/s13660-017-1313-3](https://doi.org/10.1186/s13660-017-1313-3).

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HERMITE-HADAMARD-FEJÉR TYPE INEQUALITIES FOR CONFORMABLE FRACTIONAL INTEGRALS

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Abstract. In this work, firstly we have established Hermite-Hadamard-Fejér inequality for conformable fractional integrals. Secondly, we give a new lemma and obtain Hermite-Hadamard-Fejér type integral inequalities for conformable fractional integrals by using this lemma.

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1. INTRODUCTION

A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if the inequality

$$f(\lambda u + (1 - \lambda)v) \leq \lambda f(u) + (1 - \lambda)f(v)$$

holds for all $u, v \in I$ and $\lambda \in [0, 1]$.

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2} \quad (1.1)$$

is known as the Hermite-Hadamard inequality.

Fejér gave a generalization of the inequalities of (1.1) as the following: If $f : [a, b] \rightarrow \mathbb{R}$ is a convex function, $g : [a, b] \rightarrow \mathbb{R}$ is non-negative, integrable and symmetric to $(a+b)/2$, then

$$f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx \leq \frac{1}{b-a} \int_a^b f(x) g(x) dx \leq \frac{f(a) + f(b)}{2} \int_a^b g(x) dx \quad (1.2)$$

Definition 1. Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a+}^\mu f$ and $J_{b-}^\mu f$ of order $\mu > 0$ are defined by

$$J_{a+}^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (t-x)^{\mu-1} f(t) dt, \quad x < b$$

respectively where $\Gamma(\mu) = \int_0^{\infty} e^{-t} t^{\mu-1} dt$. Here $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

In the case of $\mu = 1$, the fractional integral reduces to the classical integral. We define the Beta function [4, p18]:

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1} (1-t)^{b-1} dt, \quad a, b > 0,$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$ is the Gamma function.

The incomplete Beta function is defined by

$$B_x(a, b) = \int_0^x t^{a-1} (1-t)^{b-1} dt, \quad a, b > 0, 0 \leq x \leq 1$$

Incomplete Beta function satisfies the following identity

$$B_t(a, b) + B_{1-t}(b, a) = B(a, b) \quad (1.3)$$

We use in this paper the Newton-Leibniz formula

$$\frac{d}{dt} \left(\int_{a(t)}^{b(t)} f(x, t) dx \right) = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t} dx + f(b(t), t) b'(t) + f(a(t), t) a'(t) \quad (1.4)$$

where $f(x, t)$ be a function such that the partial derivative of f with respect to t exists, and is continuous.

In spite of its valuable contributions to mathematical analysis, the Riemann-Liouville fractional integrals have deficiencies. For example the solution of the differential equation that is given as

$$y^{(\frac{1}{2})} + y = x^{(\frac{1}{2})} + \frac{2}{\Gamma(2.5)} x^{(\frac{3}{2})}, \quad y(0) = 0$$

where $y^{(\frac{1}{2})}$ is the fractional derivative of y of order $\frac{1}{2}$.

The solution of the above differential equation have caused to imagine on a new and simple representation of the definition of fractional derivative. In [3], Khalil et al. gave a new definition that is called "Conformable fractional derivative". They not only proved further properties of this definitions but also gave the differences with the other fractional derivatives. Besides, another considerable study have presented by Abdeljawad to discuss the basic concepts of fractional calculus. In [1], Abdeljawad gave the following definitions of right-left fractional integrals:

Definition 2. Let $\alpha \in (n, n+1]$, $n = 0, 1, 2, \dots$ and set $\beta = \alpha - n$ then the left conformable fractional integral starting at a if order α is defined by

$$(I_a^{\alpha} f)(t) = \frac{1}{n!} \int_a^t (t-x)^n (x-a)^{\beta-1} f(x) dx$$

Analogously, the right conformable fractional integral is defined by

$$({}^b I_\alpha f)(t) = \frac{1}{n!} \int_t^b (x-t)^n (b-x)^{\beta-1} f(x) dx.$$

Notice that if $\alpha = n + 1$ then $\beta = \alpha - n = n + 1 - n = 1$ and hence $(I_\alpha^a f)(t) = (J_{n+1}^a f)(t)$.

The main purpose of this paper is to establish Hermite-Hadamard-Fejér inequalities for convex functions via conformable fractional integral. We also obtain Hermite-Hadamard type inequalities of these classes of functions.

2. HERMITE-HADAMARD-FEJÉR INEQUALITIES FOR CONFORMABLE FRACTIONAL INEQUALITIES

Throughout this section, let $\|g\|_\infty = \sup_{t \in [a,b]} |g(x)|$, for the continuous function $g : [a, b] \rightarrow \mathbb{R}$.

Lemma 1. *If $g : [a, b] \rightarrow \mathbb{R}$ is integrable and symmetric to $(a+b)/2$ with $a < b$, then*

$$I_\alpha^a g(b) = {}^b I_\alpha g(a) = \frac{1}{2} [I_\alpha^a g(b) + {}^b I_\alpha g(a)]$$

with $\alpha \in (n, n+1]$, $n \in \mathbb{N}$.

Proof. Since g is symmetric to $(a+b)/2$, we have $g(a+b-x) = g(x)$, for all $g(x)$, for all $x \in [a, b]$. Hence, in the following integral setting $x = a+b-t$ and $dx = -dt$ gives

$$\begin{aligned} I_\alpha^a g(b) &= \frac{1}{n!} \int_a^b (b-x)^n (x-a)^{\alpha-n-1} g(x) dx \\ &= \frac{1}{n!} \int_a^b (t-a)^n (b-t)^{\alpha-n-1} g(a+b-t) dt \\ &= \frac{1}{n!} \int_a^b (t-a)^n (b-t)^{\alpha-n-1} g(t) dt \\ &= {}^b I_\alpha g(a). \end{aligned}$$

This completes the proof. $\square$

Theorem 1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be convex function with $a < b$ and $f \in L[a, b]$. If $g : [a, b] \rightarrow \mathbb{R}$ is non-negative, integrable and symmetric to $(a+b)/2$, then the following inequalities holds for conformable fractional integrals*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) [I_\alpha^a g(b) + {}^b I_\alpha g(a)] &\leq [I_\alpha^a (fg)(b) + {}^b I_\alpha (fg)(a)] \\ &\leq \left(\frac{f(a) + f(b)}{2}\right) [I_\alpha^a g(b) + {}^b I_\alpha g(a)] \quad (2.1) \end{aligned}$$

$\alpha \in (n, n+1]$.

Proof. Since f is a convex function on $[a, b]$, we have for all $t \in [0, 1]$

$$\begin{aligned} 2f\left(\frac{a+b}{2}\right) &= 2f\left(\frac{ta + (1-t)b + tb + (1-t)a}{2}\right) \\ &\leq \frac{f(ta + (1-t)b) + f(tb + (1-t)a)}{2} \end{aligned} \quad (2.2)$$

Multiplying both sides of (2.2) by $2t^\alpha(1-t)^{\alpha-n-1}g(tb + (1-t)a)$ then integrating the resulting inequality with respect to t over $[0, 1]$, we obtain

$$\begin{aligned} &2f\left(\frac{a+b}{2}\right) \int_0^1 t^n(1-t)^{\alpha-n-1}g(tb + (1-t)a)dt \\ &\leq \int_0^1 t^n(1-t)^{\alpha-n-1}[f(ta + (1-t)b) + f(tb + (1-t)a)]g(tb + (1-t)a)dt \\ &= \int_0^1 t^n(1-t)^{\alpha-n-1}f(ta + (1-t)b)g(tb + (1-t)a)dt \\ &\quad + \int_0^1 t^n(1-t)^{\alpha-n-1}f(tb + (1-t)a)g(tb + (1-t)a)dt. \end{aligned}$$

Setting $x = tb + (1-t)a$, we get

$$\begin{aligned} &\frac{2}{(b-a)^\alpha} f\left(\frac{a+b}{2}\right) \int_a^b (x-a)^n(b-x)^{\alpha-n-1}g(x)dx \\ &\leq \frac{1}{(b-a)^\alpha} \left\{ \int_a^b (x-a)^n(b-x)^{\alpha-n-1}f(a+b-x)g(x)dx \right. \\ &\quad \left. + \int_a^b (x-a)^n(b-x)^{\alpha-n-1}f(x)g(x)dx \right\} \\ &= \frac{1}{(b-a)^\alpha} \left\{ \int_a^b (b-x)^n(x-a)^{\alpha-n-1}f(x)g(a+b-x)dx \right. \\ &\quad \left. + \int_a^b (x-a)^n(b-x)^{\alpha-n-1}f(x)g(x)dx \right\} \\ &= \frac{1}{(b-a)^\alpha} \left\{ \int_a^b (b-x)^n(x-a)^{\alpha-n-1}f(x)g(x)dx \right. \\ &\quad \left. + \int_a^b (x-a)^n(b-x)^{\alpha-n-1}f(x)g(x)dx \right\} \end{aligned}$$

Therefore, by Lemma 1 we have

$$\frac{n!}{(b-a)^\alpha} f\left(\frac{a+b}{2}\right) [I_\alpha^a g(b) + {}^b I_\alpha g(a)] \leq \frac{n!}{(b-a)^\alpha} [I_\alpha^a (fg)(b) + {}^b I_\alpha (fg)(a)]$$

and the first inequality is proved. For the proof of the second inequality in (2.1) we first note that if f is a convex function, then, for all $t \in [0, 1]$, it yields

$$f(ta + (1-t)b) + f(tb + (1-t)a) \leq f(a) + f(b). \quad (2.3)$$

Then multiplying both sides of (3.3) by $2t^\alpha(1-t)^{\alpha-n-1}g(tb + (1-t)a)$ and integrating the resulting inequality with respect to t over $[0, 1]$, we obtain

$$\begin{aligned} & \int_0^1 t^n(1-t)^{\alpha-n-1} f(ta + (1-t)b)g(tb + (1-t)a)dt \\ & + \int_0^1 t^n(1-t)^{\alpha-n-1} f(tb + (1-t)a)g(tb + (1-t)a)dt \\ & \leq [f(a) + f(b)] \int_0^1 t^n(1-t)^{\alpha-n-1} g(tb + (1-t)a)dt \end{aligned}$$

i.e.

$$\frac{n!}{(b-a)^\alpha} [I_\alpha^a (fg)(b) + {}^b I_\alpha (fg)(a)] \leq \frac{n!}{(b-a)^\alpha} \left(\frac{f(a) + f(b)}{2} \right) [I_\alpha^a g(b) + {}^b I_\alpha g(a)]$$

The proof is completed. $\square$

Remark 1. If we take $\alpha = n + 1$ in Theorem 2.1, then we obtain Theorem 2.2 in [2].

3. HERMITE-HADAMARD-FEJÉR TYPE INEQUALITIES FOR CONFORMABLE FRACTIONAL INEQUALITIES

Lemma 2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$ and $f' \in L[a, b]$. If $g : [a, b] \rightarrow \mathbb{R}$ is integrable and symmetric to $(a+b)/2$ then the following equality holds for conformable fractional integral

$$\begin{aligned} & \left(\frac{f(a) + f(b)}{2} \right) [I_\alpha^a g(b) + {}^b I_\alpha g(a)] - [I_\alpha^a (fg)(b) + {}^b I_\alpha (fg)(a)] \\ & = \frac{(b-a)^{\alpha+1}}{n!} \int_0^1 k(t) f'((1-t)a + tb) dt \end{aligned} \quad (3.1)$$

where

$$k(t) = \left[\int_0^t s^n(1-s)^{\alpha-n-1} g((1-s)a + sb) ds + \int_1^t (1-s)^n s^{\alpha-n-1} g((1-s)a + sb) ds \right]$$

with $\alpha \in (n, n+1]$.

Proof. It suffices to note that

$$\begin{aligned}
 I &= \int_0^1 k(t) f'(t) dt \\
 &= \int_0^1 \left[\int_0^t s^n (1-s)^{\alpha-n-1} g((1-s)a + sb) ds \right. \\
 &\quad \left. + \int_1^t (1-s)^n s^{\alpha-n-1} g((1-s)a + sb) ds \right] f'((1-t)a + tb) dt \\
 &= \int_0^1 \left(\int_0^t s^n (1-s)^{\alpha-n-1} g((1-s)a + sb) ds \right) f'((1-t)a + tb) dt \\
 &\quad + \int_0^1 \left(\int_1^t (1-s)^n s^{\alpha-n-1} g((1-s)a + sb) ds \right) f'((1-t)a + tb) dt \\
 &= I_1 + I_2.
 \end{aligned}$$

By integrating by parts and Lemma 1 we get

$$\begin{aligned}
 I_1 &= \left(\int_0^t s^n (1-s)^{\alpha-n-1} g((1-s)a + sb) ds \right) \frac{f((1-t)a + tb)}{b-a} \Big|_0^1 \\
 &\quad - \int_0^1 t^n (1-t)^{\alpha-n-1} g((1-t)a + tb) f((1-t)a + tb) dt \\
 &= \left(\int_0^1 s^n (1-s)^{\alpha-n-1} g((1-s)a + sb) ds \right) \frac{f(b)}{b-a} \\
 &\quad - \int_0^1 t^n (1-t)^{\alpha-n-1} (fg)((1-t)a + tb) dt \\
 &= \left(\int_a^b (x-a)^n (b-x)^{\alpha-n-1} g(x) dx \right) \frac{f(b)}{(b-a)^{\alpha+1}} \\
 &\quad - \frac{1}{(b-a)^{\alpha+1}} \int_a^b (x-a)^n (b-x)^{\alpha-n-1} (fg)(x) dx \\
 &= \frac{n!}{(b-a)^{\alpha+1}} [I_\alpha^a g(b) + {}^b I_\alpha g(a)] \frac{f(b)}{2} - \frac{n!}{(b-a)^{\alpha+1}} {}^b I_\alpha (fg)(a)
 \end{aligned}$$

and similarly

$$\begin{aligned}
 I_2 &= \left(\int_1^t (1-s)^n s^{\alpha-n-1} g((1-s)a + sb) ds \right) \frac{f((1-t)a + tb)}{b-a} \Big|_0^1 \\
 &\quad - \int_0^1 (1-t)^n t^{\alpha-n-1} g((1-t)a + tb) f((1-t)a + tb) dt \\
 &= \left(\int_0^1 (1-s)^n s^{\alpha-n-1} g((1-s)a + sb) ds \right) \frac{f(a)}{b-a} \\
 &\quad - \int_0^1 (1-t)^n t^{\alpha-n-1} g((1-t)a + tb) f((1-t)a + tb) dt
 \end{aligned}$$

$$\begin{aligned}
& - \int_0^1 (1-t)^n t^{\alpha-n-1} (fg)((1-t)a + tb) dt \\
& = \left(\int_a^b (b-x)^n (x-a)^{\alpha-n-1} g(x) dx \right) \frac{f(a)}{(b-a)^{\alpha+1}} \\
& \quad - \frac{1}{(b-a)^{\alpha+1}} \int_a^b (b-x)^n (x-a)^{\alpha-n-1} (fg)(x) dx \\
& = \frac{n!}{(b-a)^{\alpha+1}} [I_\alpha^a g(b) + {}^b I_\alpha g(a)] \frac{f(a)}{2} - \frac{n!}{(b-a)^{\alpha+1}} I_\alpha^a (fg)(b).
\end{aligned}$$

Thus, we can write

$$\begin{aligned}
I &= I_1 + I_2 \\
&= \frac{n!}{(b-a)^{\alpha+1}} \\
&\quad \times \left\{ \left(\frac{f(a) + f(b)}{2} \right) [I_\alpha^a g(b) + {}^b I_\alpha g(a)] - [I_\alpha^a (fg)(b) + {}^b I_\alpha (fg)(a)] \right\}.
\end{aligned}$$

Multiplying both sides by $\frac{(b-a)^{\alpha+1}}{n!}$ we obtain (3.1) which completes the proof. $\square$

Remark 2. If we take $\alpha = n + 1$ in Lemma 2, then we obtain Lemma 2.4 in [2].

Theorem 2. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a, b]$ with $a < b$. If $|f'|$ is convex on $[a, b]$ and $g : [a, b] \rightarrow \mathbb{R}$ is continuous and symmetric to $(a+b)/2$, then the following inequality for conformable fractional integrals holds

$$\begin{aligned}
& \left| \left(\frac{f(a) + f(b)}{2} \right) [I_\alpha^a g(b) + {}^b I_\alpha g(a)] - [I_\alpha^a (fg)(b) + {}^b I_\alpha (fg)(a)] \right| \quad (3.2) \\
& \leq \frac{(b-a)^{\alpha+1} \|g\|_\infty}{n!} (|f'(a)| + |f'(b)|) \\
& \quad \times \left[\frac{1}{2} B(n+1, \alpha-n) + B_{1/2}(\alpha-n+1, n+1) + B_{1/2}(n+2, \alpha-n) \right]
\end{aligned}$$

with $\alpha \in (n, n+1]$.

Proof. From Lemma 2 and the convexity of $|f'|$, we have

$$\begin{aligned}
& \left| \left(\frac{f(a) + f(b)}{2} \right) [I_\alpha^a g(b) + {}^b I_\alpha g(a)] - [I_\alpha^a (fg)(b) + {}^b I_\alpha (fg)(a)] \right| \\
& \leq \frac{(b-a)^{\alpha+1}}{n!} \int_0^1 |k(t)| |f'((1-t)a + tb)| dt \\
& \leq \frac{(b-a)^{\alpha+1}}{n!} \int_0^1 |k(t)| [(1-t)|f'(a)| + t|f'(b)|] dt. \quad (3.3)
\end{aligned}$$

Since $g : [a, b] \rightarrow \mathbb{R}$ is symmetric to $(a + b)/2$ we write

$$\begin{aligned} & \int_1^t (1-s)^n s^{\alpha-n-1} g((1-s)a + sb) ds \\ &= \int_0^{1-t} s^n (1-s)^{\alpha-n-1} g(sa + (1-s)b) ds \\ &= \int_0^{1-t} s^n (1-s)^{\alpha-n-1} g((1-s)a + sb) ds \end{aligned}$$

then we have

$$\begin{aligned} |k(t)| &= \left| \int_0^t s^n (1-s)^{\alpha-n-1} g((1-s)a + sb) ds \right. \\ &\quad \left. + \int_1^t (1-s)^n s^{\alpha-n-1} g((1-s)a + sb) ds \right| \\ &= \left| \int_{1-t}^t s^n (1-s)^{\alpha-n-1} g((1-s)a + sb) ds \right| \end{aligned}$$

so we get

$$|k(t)| \leq \begin{cases} \int_t^{1-t} s^n (1-s)^{\alpha-n-1} ds, & t \in [0, \frac{1}{2}] \\ \int_{1-t}^t s^n (1-s)^{\alpha-n-1} ds, & t \in [\frac{1}{2}, 1] \end{cases}. \quad (3.4)$$

Combining (3.3) and (3.4), we get

$$\begin{aligned} & \left| \left(\frac{f(a) + f(b)}{2} \right) [I_\alpha^a g(b) + {}^b I_\alpha g(a)] - [I_\alpha^a (fg)(b) + {}^b I_\alpha (fg)(a)] \right| \\ & \leq \frac{(b-a)^{\alpha+1}}{n!} \int_0^{1/2} \left(\int_t^{1-t} |s^n (1-s)^{\alpha-n-1} g((1-s)a + sb)| ds \right) \\ & \quad \times \left((1-t)|f'(a)| + t|f'(b)| \right) dt \\ & \quad + \frac{(b-a)^{\alpha+1}}{n!} \int_{1/2}^1 \left(\int_{1-t}^t |s^n (1-s)^{\alpha-n-1} g((1-s)a + sb)| ds \right) \\ & \quad \times \left((1-t)|f'(a)| + t|f'(b)| \right) dt \\ & \leq \frac{(b-a)^{\alpha+1} \|g\|_\infty |f'(a)|}{n!} \left[\int_0^{1/2} \left(\int_t^{1-t} s^n (1-s)^{\alpha-n-1} ds \right) (1-t) dt \right. \\ & \quad \left. + \int_{1/2}^1 \left(\int_{1-t}^t s^n (1-s)^{\alpha-n-1} ds \right) (1-t) dt \right] \\ & \quad + \frac{(b-a)^{\alpha+1} \|g\|_\infty |f'(b)|}{n!} \left[\int_0^{1/2} \left(\int_t^{1-t} s^n (1-s)^{\alpha-n-1} ds \right) t dt \right. \\ & \quad \left. + \int_{1/2}^1 \left(\int_{1-t}^t s^n (1-s)^{\alpha-n-1} ds \right) t dt \right] \end{aligned}$$

$$+ \int_{1/2}^1 \left(\int_{1-t}^t s^n (1-s)^{\alpha-n-1} ds \right) t dt \Big]. \quad (3.5)$$

Using integration by parts, from equality (1.3) and (1.4)

$$\begin{aligned} & \int_0^{1/2} \left(\int_t^{1-t} s^n (1-s)^{\alpha-n-1} ds \right) (1-t) dt \\ &= \left(\int_t^{1-t} s^n (1-s)^{\alpha-n-1} ds \right) \left(t - \frac{t^2}{2} \right) \Big|_0^{1/2} \\ &\quad - \int_0^{1/2} \left[-(1-t)^n t^{\alpha-n-1} - t^n (1-t)^{\alpha-n-1} \right] \left(t - \frac{t^2}{2} \right) dt \\ &= B_{1/2}(\alpha-n+1, n+1) + \frac{1}{2} B_{1/2}(\alpha-n+2, n+1) \\ &\quad + B_{1/2}(n+2, \alpha-n) + \frac{1}{2} B_{1/2}(n+3, \alpha-n), \end{aligned} \quad (3.6)$$

$$\begin{aligned} & \int_0^{1/2} \left(\int_t^{1-t} s^n (1-s)^{\alpha-n-1} ds \right) t dt \\ &= \left(\int_t^{1-t} s^n (1-s)^{\alpha-n-1} ds \right) \frac{t^2}{2} \Big|_0^{1/2} \\ &\quad - \int_0^{1/2} \left[-(1-t)^n t^{\alpha-n-1} - t^n (1-t)^{\alpha-n-1} \right] \frac{t^2}{2} dt \\ &= \frac{1}{2} B_{1/2}(\alpha-n+2, n+1) + \frac{1}{2} B_{1/2}(n+3, \alpha-n), \end{aligned} \quad (3.7)$$

$$\begin{aligned} & \int_{1/2}^1 \left(\int_{1-t}^t s^n (1-s)^{\alpha-n-1} ds \right) (1-t) dt \\ &= \left(\int_{1-t}^t s^n (1-s)^{\alpha-n-1} ds \right) \left(t - \frac{t^2}{2} \right) \Big|_{1/2}^1 \\ &\quad - \int_{1/2}^1 \left[t^n (1-t)^{\alpha-n-1} + (1-t)^n t^{\alpha-n-1} \right] \left(t - \frac{t^2}{2} \right) dt \\ &= \left(\int_{1-t}^t s^n (1-s)^{\alpha-n-1} ds \right) \left(t - \frac{t^2}{2} \right) \Big|_{1/2}^1 \\ &\quad - \int_0^{1/2} \left[t^n (1-t)^{\alpha-n-1} + (1-t)^n t^{\alpha-n-1} \right] \left(\frac{1}{2} - \frac{t^2}{2} \right) dt \\ &= B(n+1, \alpha-n) - \frac{1}{2} B(n+1, \alpha-n) - B_{1/2}(\alpha-n, n+1) \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}B_{1/2}(n+3, \alpha-n) - \frac{1}{2}B_{1/2}(\alpha-n+2, n+1) \\
& = \frac{1}{2}B(n+1, \alpha-n) - \frac{1}{2}B_{1/2}(n+3, \alpha-n) - \frac{1}{2}B_{1/2}(\alpha-n+2, n+1)
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
& \int_{1/2}^1 \left(\int_{1-t}^t s^n (1-s)^{\alpha-n-1} ds \right) t dt \\
& = \left(\int_{1-t}^t s^n (1-s)^{\alpha-n-1} ds \right) \frac{t^2}{2} \Big|_{1/2}^1 \\
& \quad - \int_{1/2}^1 [t^n (1-t)^{\alpha-n-1} + (1-t)^n t^{\alpha-n-1}] \frac{t^2}{2} dt \\
& = \left(\int_{1-t}^t s^n (1-s)^{\alpha-n-1} ds \right) \frac{t^2}{2} \Big|_{1/2}^1 \\
& \quad - \int_0^{1/2} [t^n (1-t)^{\alpha-n-1} + (1-t)^n t^{\alpha-n-1}] \frac{(1-t)^2}{2} dt \\
& = \left(\int_{1-t}^t s^n (1-s)^{\alpha-n-1} ds \right) \frac{t^2}{2} \Big|_{1/2}^1 \\
& \quad - \int_0^{1/2} [t^n (1-t)^{\alpha-n-1} + (1-t)^n t^{\alpha-n-1}] \left(\frac{1}{2} - t + \frac{t^2}{2} \right) dt \\
& = B(n+1, \alpha-n) - \frac{1}{2}B_{1/2}(n+1, \alpha-n) - \frac{1}{2}B_{1/2}(\alpha-n, n+1) \\
& \quad + B_{1/2}(\alpha-n+1, n+1) + B_{1/2}(n+2, \alpha-n) \\
& \quad - \frac{1}{2}B_{1/2}(n+3, \alpha-n) - \frac{1}{2}B_{1/2}(\alpha-n+2, n+1) \\
& = \frac{1}{2}B(n+1, \alpha-n) + B_{1/2}(n+2, \alpha-n) + B_{1/2}(\alpha-n+1, n+1) \\
& \quad - \frac{1}{2}B_{1/2}(n+3, \alpha-n) - \frac{1}{2}B_{1/2}(\alpha-n+2, n+1).
\end{aligned} \tag{3.9}$$

Using (3.6), (3.7), (3.8) and (3.9) in (3.5), we get

$$\begin{aligned}
& \left| \left(\frac{f(a) + f(b)}{2} \right) [I_\alpha^a g(b) + {}^b I_\alpha g(a)] - [I_\alpha^a (fg)(b) + {}^b I_\alpha (fg)(a)] \right| \\
& \leq \frac{(b-a)^{\alpha+1} \|g\|_\infty |f'(a)|}{n!} [B_{1/2}(\alpha-n+1, n+1) + \frac{1}{2}B_{1/2}(\alpha-n+2, n+1) \\
& \quad + B_{1/2}(n+2, \alpha-n) + \frac{1}{2}B_{1/2}(n+3, \alpha-n)]
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}B_{1/2}(n+3, \alpha-n) - \frac{1}{2}B_{1/2}(\alpha-n+2, n+1)] \\
& + \frac{(b-a)^{\alpha+1}\|g\|_{\infty}|f'(b)|}{n!} [B_{1/2}(\alpha-n+2, n+1) + B_{1/2}(n+3, \alpha-n) \\
& - \frac{1}{2}B_{1/2}(n+3, \alpha-n) + B_{1/2}(n+2, \alpha-n) + B_{1/2}(\alpha-n+1, n+1) \\
& - \frac{1}{2}B_{1/2}(n+3, \alpha-n) - \frac{1}{2}B_{1/2}(\alpha-n+2, n+1)] \\
& = \frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{n!} (|f'(a)| + |f'(b)|) \\
& \quad \times [\frac{1}{2}B(n+1, \alpha-n) + B_{1/2}(\alpha-n+1, n+1) + B_{1/2}(n+2, \alpha-n)]
\end{aligned}$$

which completes the proof. $\square$

Remark 3. If we take $\alpha = n+1$ in Theorem 2, then we obtain Theorem 2.6 in [2].

Theorem 3. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a, b]$ with $a < b$. If $|f'|^q$, $q > 1$, is convex on $[a, b]$ and $g : [a, b] \rightarrow \mathbb{R}$ is continuous and symmetric to $(a+b)/2$, then the following inequality for fractional integrals holds

$$\begin{aligned}
& \left| \left(\frac{f(a) + f(b)}{2} \right) [I_{\alpha}^a g(b) + {}^b I_{\alpha} g(a)] - [I_{\alpha}^a (fg)(b) + {}^b I_{\alpha} (fg)(a)] \right| \\
& \leq \frac{(b-a)^{\alpha+1}\|g\|_{\infty} 2^{1/p}}{n!} \\
& \quad \times \left(\int_0^{1/2} [B_t(n+1, \alpha-n)^p - B_{1-t}(n+1, \alpha-n)^p] dt \right)^{1/p} \\
& \quad \times \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q} \tag{3.10}
\end{aligned}$$

for $\alpha \in (n, n+1]$, where $1/p + 1/q = 1$

Proof. Using Lemma 2, Holder's inequality, (3.4) and the convexity of $|f'|^q$, it follows that

$$\begin{aligned}
& \left| \left(\frac{f(a) + f(b)}{2} \right) [I_{\alpha}^a g(b) + {}^b I_{\alpha} g(a)] - [I_{\alpha}^a (fg)(b) + {}^b I_{\alpha} (fg)(a)] \right| \\
& \leq \frac{(b-a)^{\alpha+1}}{n!} \left(\int_0^{1/2} \left(\int_t^{1-t} |s^n(1-s)^{\alpha-n-1} g((1-s)a + sb)| ds \right)^p dt \right)^{1/p} \\
& \quad \times \left(\int_0^{1/2} |f'(1-t)a + tb|^q dt \right)^{1/q}
\end{aligned}$$

$$\begin{aligned}
& + \frac{(b-a)^{\alpha+1}}{n!} \left(\int_{1/2}^1 \left(\int_{1-t}^t |s^n(1-s)^{\alpha-n-1} g((1-s)a+sb)| ds \right)^p dt \right)^{1/p} \\
& \times \left(\int_{1/2}^1 |f'(1-t)a+tb|^q dt \right)^{1/q} \\
& \leq \frac{(b-a)^{\alpha+1} \|g\|}{n!} \left(\int_0^{1/2} \left(\int_t^{1-t} s^n(1-s)^{\alpha-n-1} ds \right)^p dt \right)^{1/p} \\
& \times \left(\int_0^{1/2} [(1-t)|f'(a)|^q + t|f'(a)|^q] dt \right)^{1/q} \\
& + \frac{(b-a)^{\alpha+1} \|g\|}{n!} \left(\int_0^{1/2} \left(\int_{1-t}^t s^n(1-s)^{\alpha-n-1} ds \right)^p dt \right)^{1/p} \\
& \times \left(\int_{1/2}^1 [(1-t)|f'(a)|^q + t|f'(a)|^q] dt \right)^{1/q} \\
& = \frac{(b-a)^{\alpha+1} \|g\|}{n!} \left[\left(\int_0^{1/2} [B_{1-t}(n+1, \alpha-n)^p - B_t(n+1, \alpha-n)^p] dt \right)^{1/p} \right. \\
& \quad \left. + \left(\int_{1/2}^1 [B_t(n+1, \alpha-n)^p - B_{1-t}(n+1, \alpha-n)^p] dt \right)^{1/p} \right] \\
& \times \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q} \\
& = \frac{(b-a)^{\alpha+1} \|g\| 2^{1/p}}{n!} \left(\int_0^{1/2} [B_t(n+1, \alpha-n)^p - B_{1-t}(n+1, \alpha-n)^p] dt \right)^{1/p} \\
& \times \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q}.
\end{aligned}$$

Here we use

$$(A-B)^q \leq A^q - B^q$$

for any $A \geq B \geq 0$ and $q \geq 1$. The proof is completed. $\square$

Remark 4. If we take $\alpha = n + 1$ in Theorem 3, then we obtain Theorem 2.9 (i) in [2].

Theorem 4. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a, b]$ with $a < b$. If $|f'|^q$, $q > 1$, is convex on $[a, b]$ and $g : [a, b] \rightarrow \mathbb{R}$ is continuous and

symmetric to $(a+b)/2$, then the following inequality for fractional integrals holds

$$\begin{aligned} & \left| \left(\frac{f(a)+f(b)}{2} \right) [I_{\alpha}^a g(b) + {}^b I_{\alpha} g(a)] - [I_{\alpha}^a (fg)(b) + {}^b I_{\alpha} (fg)(a)] \right| \\ & \leq \frac{2(b-a)^{\alpha+1} \|g\|_{\infty}}{n!} \left(\int_0^1 \left| B_t(n+1, \alpha-n) - B_{1-t}(n+1, \alpha-n) \right| dt \right)^{1-1/q} \\ & \quad \times \left(\left[\frac{1}{2} B(n+1, \alpha-n) + B_{1/2}(\alpha-n+1, n+1) + B_{1/2}(n+2, \alpha-n) \right] \right)^{1/q} \\ & \quad \times \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q} \end{aligned}$$

for $\alpha \in (n, n+1]$, where $1/p + 1/q = 1$.

Proof. Using Lemma 2, Power-mean inequality, (3.4) and the convexity of $|f'|^q$, it follows that

$$\begin{aligned} & \left| \left(\frac{f(a)+f(b)}{2} \right) [I_{\alpha}^a g(b) + {}^b I_{\alpha} g(a)] - [I_{\alpha}^a (fg)(b) + {}^b I_{\alpha} (fg)(a)] \right| \\ & \leq \frac{1}{n!} \left(\int_0^1 \left(\left| \int_{1-t}^t s^n (1-s)^{\alpha-n-1} g((1-s)a + sb) ds \right| \right) dt \right)^{1-1/q} \\ & \quad \times \left((b-a)^{\alpha+1} \int_0^1 \left(\left| \int_{1-t}^t s^n (1-s)^{\alpha-n-1} g((1-s)a + sb) ds \right| \right) \right. \\ & \quad \times \left. |f'((1-t)a + tb)|^q dt \right)^{1/q} \\ & \leq \frac{(b-a)^{\alpha+1} \|g\|_{\infty}}{(b-a)^{1/q} n!} \left(\int_0^1 \left(\left| \int_{1-t}^t s^n (1-s)^{\alpha-n-1} ds \right| \right) dt \right)^{1-1/q} \\ & \quad \times \left(\int_0^1 \left(\left| \int_{1-t}^t s^n (1-s)^{\alpha-n-1} ds \right| \right) [(1-t)|f'(a)|^q + t|f'(b)|^q] dt \right)^{1/q}. \end{aligned} \quad (3.11)$$

We can write

$$\begin{aligned} & \int_0^1 \left(\left| \int_{1-t}^t s^n (1-s)^{\alpha-n-1} ds \right| \right) dt \\ & = \int_0^1 \left(\left| \int_0^t s^n (1-s)^{\alpha-n-1} ds - \int_0^{1-t} s^n (1-s)^{\alpha-n-1} ds \right| \right) dt \end{aligned} \quad (3.12)$$

$$= \int_0^1 \left(\left| B_t(n+1, \alpha-n) - B_{1-t}(n+1, \alpha-n) \right| \right) dt \quad (3.13)$$

and similarly to the proof of Theorem 2

$$\begin{aligned} & \int_0^1 \left(\left| \int_{1-t}^t s^n (1-s)^{\alpha-n-1} ds \right| \right) [(1-t)|f'(a)|^q + t|f'(b)|^q] dt \\ & \leq \left(|f'(a)|^q + |f'(b)|^q \right) \left[\frac{1}{2} B(n+1, \alpha-n) \right. \\ & \quad \left. + B_{1/2}(\alpha-n+1, n+1) + B_{1/2}(n+2, \alpha-n) \right] \end{aligned} \quad (3.14)$$

We write (3.13) and (3.14) in (3.11), which completes the proof. $\square$

Remark 5. If we take $\alpha = n + 1$ in Theorem 4, then we obtain Theorem 2.8 in [2].

REFERENCES

- [1] T. Abdeljawad, "On conformable fractional calculus," *J. Comput. Appl. Math.*, vol. 279, pp. 57–66, 2015, doi: [10.1016/j.cam.2014.10.016](https://doi.org/10.1016/j.cam.2014.10.016).
- [2] I. Iscan, "Hermite-hadamard-fejer type inequalities for convex functions via fractional integrals," *Stud. Univ. Babeş-Bolyai Math.*, vol. 60, no. 3, pp. 355–366, 2015.
- [3] R. Khalil, M. A. Horani, A. Yousef, and M. Sababheh, "A new definition of fractional derivative," *J. Comput. Appl. Math.*, vol. 264, pp. 65–70, 2014, doi: [10.1016/j.cam.2014.01.002](https://doi.org/10.1016/j.cam.2014.01.002).
- [4] E. Rainville, *Special Functions*. The Mcmillan Company, New York, 1960.

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BASIC AND FRACTIONAL q -CALCULUS AND ASSOCIATED FEKETE-SZEGŐ PROBLEM FOR p -VALENTLY q -STARLIKE FUNCTIONS AND p -VALENTLY q -CONVEX FUNCTIONS OF COMPLEX ORDER

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Abstract. In this paper, we introduce and study some subclasses of p -valently analytic functions in the open unit disk $\mathbb{U}$ by applying the q -derivative operator and the fractional q -derivative operator in conjunction with the principle of subordination between analytic functions. For the Taylor-Maclaurin coefficients $\{a_k\}_{k=p+1}^{\infty}$ of each of these subclasses of p -valently analytic functions, we derive sharp bounds for the Fekete-Szegő functional given by

$$\left| a_{p+2} - \mu a_{p+1}^2 \right|.$$

Relevant connections of the results presented in this paper with those derived in earlier works are also considered.

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Keywords: analytic functions, univalent functions, p -valent functions, q -derivative operator, fractional q -derivative operator, q -starlike and fractional q -starlike functions, q -convex and fractional q -convex functions, Fekete-Szegő problem and Fekete-Szegő functional, principle of subordination between analytic functions

1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

The theory of the basic and the fractional quantum calculus (that is, the basic q -calculus and the fractional q -calculus) plays important roles in many diverse areas of the mathematical, physical and engineering sciences (see, for example, [5, 9, 16] and [21]). Our main objective in this paper is to introduce and study some subclasses of p -valently analytic functions in the open unit disk $\mathbb{U}$ by applying the q -derivative operator and the fractional q -derivative operator in conjunction with the principle of subordination between analytic functions (see, for details, [12]).

We begin by denoting by $\mathcal{A}(p)$ the class of functions $f(z)$ of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}), \quad (1.1)$$

which are analytic and p -valent in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

In particular, we write

$$\mathcal{A}(1) = \mathcal{A}.$$

A function $f(z) \in \mathcal{A}(p)$ is said to be in the class $\mathcal{S}_p^*(\alpha)$ of p -valently starlike of order α in $\mathbb{U}$ if and only if

$$\Re \left(\frac{zf'(z)}{f(z)} \right) > \alpha \quad (0 \leq \alpha < p; z \in \mathbb{U}). \quad (1.2)$$

A function $f(z) \in \mathcal{A}(p)$ is said to be in the class $\mathcal{C}_p(\alpha)$ of p -valently convex of order α in $\mathbb{U}$ if and only if

$$\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (0 \leq \alpha < p; z \in \mathbb{U}). \quad (1.3)$$

The p -valent function classes $\mathcal{S}_p^*(\alpha)$ and $\mathcal{C}_p(\alpha)$ were studied by Owa [13]. From (1.2) and (1.3), it follows that

$$f(z) \in \mathcal{C}_p(\alpha) \iff \frac{zf'(z)}{p} \in \mathcal{S}_p^*(\alpha). \quad (1.4)$$

We now recall some basic definitions and concept details of the q -calculus which are used in this paper (see, for details, [7] and [8]; see also [5] and [21]).

Definition 1. Let $q \in (0, 1)$ and define the q -number $[\lambda]_q$ and the q -factorial $[n]_q!$ by

$$[\lambda]_q = \begin{cases} \frac{1 - q^\lambda}{1 - q} & (\lambda \in \mathbb{C}) \\ \sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\lambda = n \in \mathbb{N}) \end{cases} \quad (1.5)$$

and

$$[n]_q! = \begin{cases} 1 & (n = 0) \\ \prod_{k=1}^n [k]_q & (n \in \mathbb{N}), \end{cases} \quad (1.6)$$

respectively.

Definition 2. The q -derivative (or the q -difference) $D_q f(z)$ of a function $f(z)$ is defined in a given subset of $\mathbb{C}$ by

$$D_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z} & (z \neq 0) \\ f'(0) & (z = 0), \end{cases} \quad (1.7)$$

provided that $f'(0)$ exists.

We note from Definition 2 that

$$\lim_{q \rightarrow 1^-} D_q f(z) = \lim_{q \rightarrow 1^-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z) \quad (1.8)$$

for a function f which is differentiable in a given subset of $\mathbb{C}$. It is readily deduced from (1.1) and (1.7) that

$$D_q f(z) = [p]_q z^{p-1} + \sum_{k=p+1}^{\infty} [k]_q a_k z^{k-1} \quad (z \neq 0), \quad (1.9)$$

where $[\lambda]_q$ is given by (1.5) and the function $f(z) \in \mathcal{A}(p)$ is given by (1.1).

Making use of the q -derivative operator D_q given by (1.7), we introduce the subclass $\mathcal{S}_q^*(p, \alpha)$ of p -valently q -starlike functions of order α in $\mathbb{U}$ and the subclass $\mathcal{C}_q(p, \alpha)$ of p -valently q -convex functions of order α in $\mathbb{U}$ as follows:

$$f(z) \in \mathcal{S}_q^*(p, \alpha) \iff \Re \left(\frac{1}{[p]_q} \frac{z D_q f(z)}{f(z)} \right) > \alpha \quad (1.10)$$

($0 < q < 1$; $0 \leq \alpha < 1$; $z \in \mathbb{U}$)

and

$$f(z) \in \mathcal{C}_q(p, \alpha) \iff \Re \left(\frac{1}{[p]_q} \frac{D_q(z D_q f(z))}{D_q f(z)} \right) > \alpha \quad (1.11)$$

($0 < q < 1$; $0 \leq \alpha < 1$; $z \in \mathbb{U}$),

respectively. From (1.10) and (1.11), it follows that

$$f(z) \in \mathcal{C}_q(p, \alpha) \iff \frac{z D_q f(z)}{[p]_q} \in \mathcal{S}_q^*(p, \alpha). \quad (1.12)$$

We note also that

$$\lim_{q \rightarrow 1^-} \mathcal{S}_q^*(p, \alpha) = \mathcal{S}_p^*(\alpha) \quad \text{and} \quad \lim_{q \rightarrow 1^-} \mathcal{C}_q(p, \alpha) = \mathcal{C}_p(\alpha).$$

We next introduce the familiar principle of subordination between analytic functions. Indeed, for two given functions $f(z)$ and $g(z)$ which are analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to the function $g(z)$ and write

$$f(z) \prec g(z),$$

if there exists a Schwarz function $w(z)$, which is analytic in $\mathbb{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1$$

such that

$$f(z) = g(w(z)).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then it follows that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Throughout our present investigation, let the function $\varphi(z)$ be analytic with positive real part in $\mathbb{U}$ and satisfy the following conditions:

$$\varphi(0) = 1 \quad \text{and} \quad \varphi'(0) > 0,$$

which maps $\mathbb{U}$ onto a region which is starlike with respect to 1 and symmetric with respect to the real axis. Suppose now that $\mathcal{S}_{b,p}^*(\varphi)$ denotes the class of functions $f(z) \in \mathcal{A}(p)$ for which

$$1 + \frac{1}{b} \left(\frac{1}{p} \frac{zf'(z)}{f(z)} - 1 \right) \prec \varphi(z) \quad (b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}; z \in \mathbb{U}). \quad (1.13)$$

Also let $\mathcal{C}_{b,p}(\varphi)$ be the class of functions $f(z) \in \mathcal{A}(p)$ for which

$$1 - \frac{1}{b} + \frac{1}{bp} \left(1 + \frac{zf''(z)}{f'(z)} \right) \prec \varphi(z) \quad (b \in \mathbb{C}^*; z \in \mathbb{U}). \quad (1.14)$$

The p -valent function classes $\mathcal{S}_{b,p}^*(\varphi)$ and $\mathcal{C}_{b,p}(\varphi)$ were introduced and studied by Ali *et al.* [2]. We note that each of the following function classes:

$$\mathcal{S}_{1,1}^*(\varphi) =: \mathcal{S}^*(\varphi) \quad \text{and} \quad \mathcal{C}_{1,1}(\varphi) =: \mathcal{C}(\varphi)$$

was introduced and studied by Ma and Minda [11]. In fact, the widely-investigated function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ are the special cases of $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$, respectively, when

$$\varphi(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (0 \leq \alpha < 1; z \in \mathbb{U}).$$

Finally, for $b \in \mathbb{C}^*$, $0 < q < 1$, $0 \leq \lambda \leq 1$ and $p \in \mathbb{N}$, we define the subclass $\mathcal{S}_{\lambda,q,b}(p, \varphi)$ of the p -valently analytic function class $\mathcal{A}(p)$ consisting of functions $f(z)$ of the form (1.1) and satisfying the following subordination condition:

$$1 + \frac{1}{b} \left(\frac{1}{[p]_q} \frac{z D_q f(z) + \lambda q z^2 D_q (D_q f(z))}{(1 - \lambda) f(z) + \lambda z D_q f(z)} - 1 \right) \prec \varphi(z), \quad (1.15)$$

where D_q denotes the q -derivative operator given by Definition 2.

The following function classes are included in the class $\mathcal{S}_{\lambda,q,b}(p, \varphi)$ of p -valently q -starlike functions of complex order b in $\mathbb{U}$ (see also [1]):

(i) $\lim_{q \rightarrow 1^-} \mathfrak{S}_{0,q,b}(p, \varphi) =: \mathfrak{S}_{b,p}^*(\varphi)$ and $\lim_{q \rightarrow 1^-} \mathfrak{S}_{1,q,b}(p, \varphi) =: \mathfrak{C}_{b,p}(\varphi)$
(see Ali *et al.* [2]);

(ii) $\lim_{q \rightarrow 1^-} \mathfrak{S}_{\lambda,q,b}(p, \varphi) =: \mathfrak{S}_{\lambda,b,p}(\varphi)$
(see Aouf *et al.* [3]) with

$$g(z) = \frac{z^p}{1-z};$$

(iii) $\mathfrak{S}_{0,q,b}(1, \varphi) =: \mathfrak{S}_{q,b}(\varphi)$ and $\mathfrak{S}_{1,q,b}(1, \varphi) =: \mathfrak{C}_{q,b}(\varphi)$
(see Seoudy and Aouf [19]);

(iv) $\lim_{q \rightarrow 1^-} \mathfrak{S}_{0,q,b}(1, \varphi) =: \mathfrak{S}_b^*(\varphi)$ and $\lim_{q \rightarrow 1^-} \mathfrak{S}_{1,q,b}(1, \varphi) =: \mathfrak{C}_b(\varphi)$
(see Ravichandran *et al.* [17]).

We also note here that

$$\begin{aligned} \text{(i)} \quad \mathfrak{S}_{\lambda,q,\left(1-\frac{\gamma}{[p]_q}\right)e^{-i\alpha \cos \alpha}}(p, \varphi) &= \mathfrak{S}_{q,p,\lambda}^{\alpha,\gamma}(\varphi) \\ &= \left\{ f(z) \in \mathcal{A}(p) : \frac{e^{i\alpha} \left(\frac{z D_q f(z) + \lambda q z^2 D_q(D_q f(z))}{(1-\lambda)f(z) + \lambda z D_q f(z)} \right) - \gamma \cos \alpha - i [p]_q \sin \alpha}{([p]_q - \gamma) \cos \alpha} \prec \varphi(z) \right. \\ &\quad \left. \left(|\alpha| < \frac{\pi}{2}; 0 \leq \gamma < [p]_q; 0 \leq \lambda \leq 1 \right) \right\}; \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \mathfrak{S}_{0,q,\left(1-\frac{\gamma}{[p]_q}\right)e^{-i\alpha \cos \alpha}}(p, \varphi) &= \mathfrak{S}_{q,p,\gamma}^{\alpha}(\varphi) \\ &= \left\{ f(z) \in \mathcal{A}(p) : \frac{e^{i\alpha} \left(\frac{z D_q f(z)}{f(z)} \right) - \gamma \cos \alpha - i [p]_q \sin \alpha}{([p]_q - \gamma) \cos \alpha} \prec \varphi(z) \right. \\ &\quad \left. \left(|\alpha| < \frac{\pi}{2}; 0 \leq \gamma < [p]_q \right) \right\}; \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad \mathfrak{S}_{0,q,\left(1-\frac{\gamma}{[p]_q}\right)e^{-i\alpha \cos \alpha}}(p, \varphi) &= \mathfrak{C}_{q,p,\gamma}^{\alpha}(\varphi) \\ &= \left\{ f(z) \in \mathcal{A}(p) : \frac{e^{i\alpha} \left(\frac{D_q(z D_q f(z))}{D_q f(z)} \right) - \gamma \cos \alpha - i [p]_q \sin \alpha}{([p]_q - \gamma) \cos \alpha} \prec \varphi(z) \right. \\ &\quad \left. \left(|\alpha| < \frac{\pi}{2}; 0 \leq \gamma < [p]_q \right) \right\}. \end{aligned}$$

In Geometric Function Theory, various subclasses of the normalized analytic function class $\mathcal{A}$ as well as the normalized p -valently analytic function class $\mathcal{A}(p)$ have been studied from different viewpoints. The above-defined q -calculus provides important tools that have been used in order to investigate various subclasses of $\mathcal{A}$ and $\mathcal{A}(p)$. Historically speaking, even though the q -derivative operator D_q was first applied by Ismail *et al.* [6] to study a q -extension of the class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$, a firm footing of the usage of the q -calculus in the context of Geometric Function Theory was actually provided and the basic (or q -) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [20, pp. 347 *et seq.*]).

2. FEKETE-SZEGŐ PROBLEM FOR THE FUNCTION CLASS $\mathcal{S}_{\lambda,q,b}(p,\varphi)$

Let Ω be the class of functions $w(z)$ of the form:

$$w(z) = w_1z + w_2z^2 + w_3z^3 + \cdots \quad (z \in \mathbb{U}), \quad (2.1)$$

which satisfy the following inequality:

$$|w(z)| < 1 \quad (z \in \mathbb{U}).$$

Each of the following lemmas will be needed in our present investigation of the Fekete-Szegő problem for the function class $\mathcal{S}_{\lambda,q,b}(p,\varphi)$ which we have introduced by using the subordination condition (1.15) (see, for example, [4]; see also [22] and [24]).

Lemma 1 ([10]). *Let the function $w(z) \in \Omega$ be given by (2.1). Then*

$$|w_2 - \tau w_1^2| \leq \max\{1, |\tau|\} \quad (\tau \in \mathbb{C}).$$

The result is sharp for the function given by

$$w(z) = z \quad \text{or} \quad w(z) = z^2 \quad (z \in \mathbb{U}).$$

Lemma 2 ([2], [11]). *Let the function $w(z) \in \Omega$ be given by (2.1). Then*

$$|w_2 - \kappa w_1^2| \leq \begin{cases} -\kappa & (\kappa \leq -1) \\ 1 & (-1 \leq \kappa \leq 1) \\ \kappa & (\kappa \geq 1). \end{cases} \quad (2.2)$$

For $\kappa < -1$ or $\kappa > 1$, the equality holds true in (2.2) if and only if $w(z) = z$ or one of its rotations. If $-1 < \kappa < 1$, then the equality holds true in (2.2) if and only if $w(z) = z^2$ or one of its rotations. If $\kappa = -1$, then the equality holds true in (2.2) if and only if

$$w(z) = \frac{z(z+\eta)}{1+\eta z} \quad (0 \leq \eta \leq 1)$$

or one of its rotations. If $\kappa = 1$, then the equality holds true in (2.2) if and only if

$$w(z) = -\frac{z(z+\eta)}{1+\eta z} \quad (0 \leq \eta \leq 1)$$

or one of its rotations. The upper bound in (2.2) is sharp and it can be improved as follows when $-1 < \kappa < 1$:

$$|w_2 - \kappa w_1^2| + (\kappa + 1)|w_1|^2 \leq 1 \quad (-1 < \kappa \leq 0)$$

and

$$|w_2 - \kappa w_1^2| + (1 - \kappa)|w_1|^2 \leq 1 \quad (0 < \kappa < 1).$$

Lemma 3 ([14]). Let the function $w(z) \in \Omega$ be given by (2.1). Then, for any real numbers q_1 and q_2 , the following sharp estimates hold true:

$$|w_3 + q_1 w_1 w_2 + q_2 w_1^3| \leq H(q_1, q_2), \quad (2.3)$$

where

$$H(q_1, q_2) = \begin{cases} 1 & ((q_1, q_2) \in D_1 \cup D_2) \\ |q_2| & ((q_1, q_2) \in \bigcup_{k=3}^7 D_k) \\ \frac{2}{3}(|q_1| + 1) \left(\frac{|q_1| + 1}{3(|q_1| + 1 + q_2)} \right)^{\frac{1}{2}} & ((q_1, q_2) \in D_8 \cup D_9) \\ \frac{q_2}{3} \left(\frac{q_1^2 - 4}{q_1^2 - 4q_2} \right) \left(\frac{q_1^2 - 4}{3(q_2 - 1)} \right)^{\frac{1}{2}} & ((q_1, q_2) \in D_{10} \cup D_{11} \setminus \{\pm 2, 1\}) \\ \frac{2}{3}(|q_1| - 1) \left(\frac{|q_1| - 1}{3(|q_1| - 1 - q_2)} \right)^{\frac{1}{2}} & ((q_1, q_2) \in D_{12}). \end{cases}$$

The extremal functions, up to rotations, are of the form given by

$$w(z) = z^3, \quad w(z) = z, \quad w(z) = w_0(z) = \frac{z[(1-\lambda)\varepsilon_2 + \lambda\varepsilon_1]z - \varepsilon_1\varepsilon_2 z}{1 - [(1-\lambda)\varepsilon_1 + \lambda\varepsilon_2]z},$$

$$w(z) = w_1(z) = \frac{z(t_1 - z)}{1 - t_1 z}, \quad w(z) = w_2(z) = \frac{z(t_2 + z)}{1 + t_2 z}, \quad |\varepsilon_1| = |\varepsilon_2| = 1,$$

$$\varepsilon_1 = t_0 - e^{-i\left(\frac{\theta_0}{2}\right)} (a \mp b), \quad \varepsilon_2 = -e^{-i\left(\frac{\theta_0}{2}\right)} (ia \pm b),$$

$$a = t_0 \cos\left(\frac{\theta_0}{2}\right), \quad b = \sqrt{1 - t_0^2 \sin^2\left(\frac{\theta_0}{2}\right)}, \quad \lambda = \frac{b \pm a}{2b},$$

$$t_0 = \left(\frac{2q_2(q_1^2 + 2) - 3q_1^2}{3(q_2 - 1)(q_1^2 - 4q_2)} \right)^{\frac{1}{2}}, \quad t_1 = \left(\frac{|q_1| + 1}{3(|q_1| + 1 + q_2)} \right)^{\frac{1}{2}},$$

$$t_2 = \left(\frac{|q_1| - 1}{3(|q_1| - 1 - q_2)} \right)^{\frac{1}{2}} \text{ and } \cos\left(\frac{\theta_0}{2}\right) = \frac{q_1}{2} \left(\frac{q_2(q_1^2 + 8) - 2(q_1^2 + 2)}{2q_2(q_1^2 + 2) - 3q_1^2} \right).$$

The sets D_k ($k = 1, \dots, 12$) are defined as follows:

$$D_1 := \left\{ (q_1, q_2) : |q_1| \leq \frac{1}{2} \text{ and } |q_2| \leq 1 \right\},$$

$$D_2 := \left\{ (q_1, q_2) : \frac{1}{2} \leq |q_1| \leq 2 \text{ and } \frac{4}{27}(|q_1| + 1)^3 - (|q_1| + 1) \leq q_2 \leq 1 \right\},$$

$$D_3 := \left\{ (q_1, q_2) : |q_1| \leq \frac{1}{2} \text{ and } q_2 \leq -1 \right\},$$

$$D_4 := \left\{ (q_1, q_2) : |q_1| \geq \frac{1}{2} \text{ and } q_2 \leq -\frac{2}{3}(|q_1| + 1) \right\},$$

$$D_5 := \{(q_1, q_2) : |q_1| \leq 2 \text{ and } q_2 \geq 1\},$$

$$D_6 := \left\{ (q_1, q_2) : 2 \leq |q_1| \leq 4 \text{ and } q_2 \geq \frac{1}{12}(q_1^2 + 8) \right\},$$

$$D_7 := \left\{ (q_1, q_2) : |q_1| \geq 4 \text{ and } q_2 \geq \frac{2}{3}(|q_1| - 1) \right\},$$

$$D_8 := \left\{ (q_1, q_2) : \frac{1}{2} \leq |q_1| \leq 2 \text{ and } -\frac{2}{3}(|q_1| + 1) \leq q_2 \leq \frac{4}{27}(|q_1| + 1)^3 - (|q_1| + 1) \right\},$$

$$D_9 := \left\{ (q_1, q_2) : |q_1| \geq 2 \text{ and } -\frac{2}{3}(|q_1| + 1) \leq q_2 \leq \frac{2|q_1|(|q_1| + 1)}{q_1^2 + 2|q_1| + 4} \right\},$$

$$D_{10} := \left\{ (q_1, q_2) : 2 \leq |q_1| \leq 4 \text{ and } \frac{2|q_1|(|q_1| + 1)}{q_1^2 + 2|q_1| + 4} \leq q_2 \leq \frac{1}{12}(q_1^2 + 8) \right\},$$

$$D_{11} := \left\{ (q_1, q_2) : |q_1| \geq 4 \text{ and } \frac{2|q_1|(|q_1| + 1)}{q_1^2 + 2|q_1| + 4} \leq q_2 \leq \frac{2|q_1|(|q_1| - 1)}{q_1^2 - 2|q_1| + 4} \right\}$$

and

$$D_{12} := \left\{ (q_1, q_2) : |q_1| \geq 4 \text{ and } \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4} \leq q_2 \leq \frac{2}{3}(|q_1|-1) \right\}.$$

Remark 1. Unless otherwise mentioned, we assume throughout this paper that

$$b \in \mathbb{C}^*, \quad 0 \leq \lambda \leq 1, \quad 0 < q < 1 \quad \text{and} \quad p \in \mathbb{N}.$$

Theorem 1. Let the function $\varphi(z)$ be given by

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots (B_1 > 0).$$

If the function $f(z)$ given by (1.1) belongs to the class $\mathcal{S}_{\lambda,q,b}(p, \varphi)$ and $\mu \in \mathbb{C}$, then

$$\begin{aligned} |a_{p+2} - \mu a_{p+1}^2| &\leq \frac{[p]_q B_1 |b|}{[p+2]_q - [p]_q} \left(\frac{1 + \lambda([p]_q - 1)}{1 + \lambda([p+2]_q - 1)} \right) \\ &\cdot \max \left\{ 1, \left| \frac{B_2}{B_1} + \frac{[p]_q B_1 b}{[p+1]_q - [p]_q} \right| \right. \\ &\cdot \left. \left(1 - \mu \frac{[p+2]_q - [p]_q}{[p+1]_q - [p]_q} \frac{[1 + \lambda([p+2]_q - 1)][1 + \lambda([p]_q - 1)]}{[1 + \lambda([p+1]_q - 1)]^2} \right) \right\} \end{aligned} \quad (2.4)$$

and

$$|a_{p+3}| \leq \frac{[p]_q B_1 |b|}{[p+3]_q - [p]_q} \left(\frac{1 + \lambda([p]_q - 1)}{1 + \lambda([p+3]_q - 1)} \right) H(q_1, q_2), \quad (2.5)$$

where

$$q_1 = \frac{2B_2}{B_1} - \frac{[p]_q (2[p]_q - [p+1]_q - [p+2]_q) B_1 b}{([p+1]_q - [p]_q)([p+2]_q - [p]_q)} \quad (2.6)$$

and

$$\begin{aligned} q_2 &= \frac{B_3}{B_1} - \left(\frac{[p]_q B_1 b}{[p+1]_q - [p]_q} \right)^2 - \frac{[p]_q (2[p]_q - [p+1]_q - [p+2]_q) B_1 b}{([p+1]_q - [p]_q)([p+2]_q - [p]_q)} \\ &\cdot \left(\frac{B_2}{B_1} + \frac{[p]_q B_1 b}{[p+1]_q - [p]_q} \right). \end{aligned} \quad (2.7)$$

The result is sharp.

Proof. If $f(z) \in \mathcal{S}_{\lambda,q,b}(p, \varphi)$, then there is a Schwarz function $w(z) \in \Omega$ given by

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots$$

such that

$$1 + \frac{1}{b} \left(\frac{1}{[p]_q} \frac{z D_q f(z) + \lambda q z^2 D_q(D_q f(z))}{(1-\lambda)f(z) + \lambda z D_q f(z)} - 1 \right) = \varphi(w(z)).$$

Since

$$\begin{aligned} & \frac{1}{[p]_q} \frac{z D_q f(z) + \lambda q z^2 D_q(D_q f(z))}{(1-\lambda)f(z) + \lambda z D_q f(z)} \\ &= 1 + \frac{1 + \lambda([p+1]_q - 1)}{1 + \lambda([p]_q - 1)} \left(\frac{[p+1]_q}{[p]_q} - 1 \right) a_{p+1} z \\ &+ \left[\frac{1 + \lambda([p+2]_q - 1)}{1 + \lambda([p]_q - 1)} \left(\frac{[p+2]_q}{[p]_q} - 1 \right) a_{p+2} - \left(\frac{1 + \lambda([p+1]_q - 1)}{1 + \lambda([p]_q - 1)} \right)^2 \right. \\ &\cdot \left. \left(\frac{[p+1]_q}{[p]_q} - 1 \right) a_{p+1}^2 \right] z^2 + \left[\frac{1 + \lambda([p+3]_q - 1)}{1 + \lambda([p]_q - 1)} \left(\frac{[p+3]_q}{[p]_q} - 1 \right) a_{p+3} \right. \\ &+ \left. \left(\frac{1 + \lambda([p+1]_q - 1)}{1 + \lambda([p]_q - 1)} \right)^3 \left(\frac{[p+1]_q}{[p]_q} - 1 \right) a_{p+1}^3 + \frac{1 + \lambda([p+1]_q - 1)}{1 + \lambda([p]_q - 1)} \right. \\ &\cdot \left. \frac{1 + \lambda([p+2]_q - 1)}{1 + \lambda([p]_q - 1)} \left(2 - \frac{[p+1]_q}{[p]_q} - \frac{[p+2]_q}{[p]_q} \right) a_{p+1} a_{p+2} \right] z^3 + \dots \end{aligned}$$

and

$$\varphi(w(z)) = 1 + w_1 B_1 z + (w_1^2 B_2 + w_2 B_1) z^2 + (w_3 B_1 + w_1^3 B_3 + 2w_1 w_2 B_2) z^3 + \dots,$$

we observe that

$$a_{p+1} = \frac{[p]_q B_1 b w_1}{[p+1]_q - [p]_q} \left(\frac{1 + \lambda([p]_q - 1)}{1 + \lambda([p+1]_q - 1)} \right), \quad (2.8)$$

$$\begin{aligned} a_{p+2} &= \frac{[p]_q B_1 b}{[p+2]_q - [p]_q} \left(\frac{1 + \lambda([p]_q - 1)}{1 + \lambda([p+2]_q - 1)} \right) \\ &\cdot \left[w_2 + w_1^2 \left(\frac{B_2}{B_1} + \frac{[p]_q B_1 b}{[p+1]_q - [p]_q} \right) \right] \end{aligned} \quad (2.9)$$

and

$$a_{p+3} = \frac{1 + \lambda([p]_q - 1)}{1 + \lambda([p+3]_q - 1)} \frac{[p]_q B_1 b}{[p+3]_q - [p]_q} \left\{ w_3 + \left[\frac{2B_2}{B_1} \right. \right.$$

$$\begin{aligned}
& - \frac{[p]_q (2[p]_q - [p+1]_q - [p+2]_q) B_1 b}{([p+1]_q - [p]_q) ([p+2]_q - [p]_q)} w_1 w_2 + \left[\frac{B_3}{B_1} - \left(\frac{[p]_q B_1 b}{[p+1]_q - [p]_q} \right)^2 \right. \\
& \left. - \frac{[p]_q (2[p]_q - [p+1]_q - [p+2]_q) B_1 b}{([p+1]_q - [p]_q) ([p+2]_q - [p]_q)} \left(\frac{B_2}{B_1} + \frac{[p]_q B_1 b}{[p+1]_q - [p]_q} \right) w_1^3 \right] w_1^3 \Big\}. \quad (2.10)
\end{aligned}$$

Therefore, we have

$$a_{p+2} - \mu a_{p+1}^2 = \frac{[p]_q B_1 b}{[p+2]_q - [p]_q} \frac{1 + \lambda ([p]_q - 1)}{1 + \lambda ([p+2]_q - 1)} [w_2 - \nu w_1^2],$$

where

$$\begin{aligned}
\nu &= \frac{[p]_q B_1 |b|}{[p+1]_q - [p]_q} \\
&\cdot \left(\mu \frac{[p+2]_q - [p]_q}{[p+1]_q - [p]_q} \frac{[1 + \lambda ([p+2]_q - 1)][1 + \lambda ([p]_q - 1)]}{[1 + \lambda ([p+1]_q - 1)]^2} - 1 \right) - \frac{B_2}{B_1}. \quad (2.11)
\end{aligned}$$

The result (2.4) follows by an application of Lemma 1 and the result (2.5) follows by an application of Lemma 3. These results are sharp for the functions given by

$$1 + \frac{1}{b} \left(\frac{1}{[p]_q} \frac{z D_q f(z) + \lambda q z^2 D_q (D_q f(z))}{(1 - \lambda) f(z) + \lambda z D_q f(z)} - 1 \right) = \varphi(z^2)$$

and

$$1 + \frac{1}{b} \left(\frac{1}{[p]_q} \frac{z D_q f(z) + \lambda q z^2 D_q (D_q f(z))}{(1 - \lambda) f(z) + \lambda z D_q f(z)} - 1 \right) = \varphi(z),$$

respectively. This completes the proof of Theorem 1. $\square$

By using Lemma 2, we can obtain the following theorem.

Theorem 2. Let $b > 0$ and let the function $\varphi(z)$ be given by

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots (B_k > 0; k \in \{1, 2\}).$$

If the function $f(z)$ given by (1.1) belongs to the class $\mathcal{S}_{\lambda, q, b}(p, \varphi)$ and $\mu \in \mathbb{R}$, then

$$|a_{p+2} - \mu a_{p+1}^2| \leq$$

$$\left\{ \begin{array}{l} \frac{[p]_q b}{[p+2]_q - [p]_q} \left(\frac{1 + \lambda([p]_q - 1)}{1 + \lambda([p+2]_q - 1)} \right) \\ \cdot \left\{ B_2 + \frac{[p]_q B_1^2 b}{[p+1]_q - [p]_q} \left(1 - \mu \frac{[p+2]_q - [p]_q}{[p+1]_q - [p]_q} \frac{[1 + \lambda([p+2]_q - 1)][1 + \lambda([p]_q - 1)]}{[1 + \lambda([p+1]_q - 1)]^2} \right) \right\} \\ (\mu \leq \sigma_1) \\ \frac{[p]_q B_1 b}{[p+2]_q - [p]_q} \left(\frac{1 + \lambda([p]_q - 1)}{1 + \lambda([p+2]_q - 1)} \right) \\ (\sigma_1 \leq \mu \leq \sigma_2) \\ \frac{[p]_q b}{[p+2]_q - [p]_q} \left(\frac{1 + \lambda([p]_q - 1)}{1 + \lambda([p+2]_q - 1)} \right) \cdot \\ \cdot \left\{ -B_2 + \frac{[p]_q B_1^2 b}{[p+1]_q - [p]_q} \left(\mu \frac{[p+2]_q - [p]_q}{[p+1]_q - [p]_q} \frac{[1 + \lambda([p+2]_q - 1)][1 + \lambda([p]_q - 1)]}{[1 + \lambda([p+1]_q - 1)]^2} - 1 \right) \right\} \\ (\mu \geq \sigma_2), \end{array} \right.$$

where

$$\sigma_1 = \frac{([p+1]_q - [p]_q) [(B_2 - B_1) ([p+1]_q - [p]_q) + [p]_q B_1^2 b] [1 + \lambda([p+1]_q - 1)]^2}{[p]_q ([p+2]_q - [p]_q) [1 + \lambda([p+2]_q - 1)] [1 + \lambda([p]_q - 1)] B_1^2 b}$$

and

$$\sigma_2 = \frac{([p+1]_q - [p]_q) [(B_2 + B_1) ([p+1]_q - [p]_q) + [p]_q B_1^2 b] [1 + \lambda([p+1]_q - 1)]^2}{[p]_q ([p+2]_q - [p]_q) [1 + \lambda([p+2]_q - 1)] [1 + \lambda([p]_q - 1)] B_1^2 b}.$$

The result is sharp.

Further, if we set

$$\sigma_3 = \frac{([p+1]_q - [p]_q) [([p+1]_q - [p]_q) B_2 + [p]_q B_1^2 b] [1 + \lambda([p+1]_q - 1)]^2}{[p]_q ([p+2]_q - [p]_q) [1 + \lambda([p+2]_q - 1)] [1 + \lambda([p]_q - 1)] B_1^2 b},$$

then each of the following assertions holds true:

(i) If $\sigma_1 \leq \mu \leq \sigma_3$, then

$$\begin{aligned} |a_{p+2} - \mu a_{p+1}^2| + \frac{([p+1]_q - [p]_q)^2}{[p]_q ([p+2]_q - [p]_q) B_1^2 b} \frac{[1 + \lambda([p+1]_q - 1)]^2}{[1 + \lambda([p+2]_q - 1)] [1 + \lambda([p]_q - 1)]} \\ \cdot \left\{ (B_1 - B_2) - \frac{[p]_q B_1^2 b}{[p+1]_q - [p]_q} \left[1 - \mu \frac{[1 + \lambda([p+2]_q - 1)][1 + \lambda([p]_q - 1)]}{[1 + \lambda([p+1]_q - 1)]^2} \right] \right. \\ \left. \cdot \frac{[p+2]_q - [p]_q}{[p+1]_q - [p]_q} \right\} |a_{p+1}|^2 \leq \frac{[p]_q B_1 b}{[p+2]_q - [p]_q} \left(\frac{1 + \lambda([p]_q - 1)}{1 + \lambda([p+2]_q - 1)} \right). \end{aligned}$$

(ii) If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|a_{p+2} - \mu a_{p+1}^2| + \frac{([p+1]_q - [p]_q)^2}{[p]_q ([p+2]_q - [p]_q) B_1^2 b} \frac{[1 + \lambda ([p+1]_q - 1)]^2}{[1 + \lambda ([p+2]_q - 1)][1 + \lambda ([p]_q - 1)]} \\ \cdot \left\{ (B_1 - B_2) - \frac{[p]_q B_1^2 b}{[p+1]_q - [p]_q} \left[\mu \frac{[1 + \lambda ([p+2]_q - 1)][1 + \lambda ([p]_q - 1)]}{[1 + \lambda ([p+1]_q - 1)]^2} \right. \right. \\ \left. \left. \cdot \frac{[p+2]_q - [p]_q}{[p+1]_q - [p]_q} - 1 \right] \right\} |a_{p+1}|^2 \leq \frac{[p]_q B_1 b}{[p+2]_q - [p]_q} \left(\frac{1 + \lambda [p]_q - 1}{1 + \lambda ([p+2]_q - 1)} \right).$$

Proof. By applying Lemma 2 to (2.8), (2.9) and (2.11), we obtain the required results asserted by Theorem 2. In order to show that the bounds are sharp, we define the functions $K_{\varphi n}$ ($n \in \mathbb{N} \setminus \{1\}$) by

$$1 + \frac{1}{b} \left(\frac{1}{[p]_q} \frac{z D_q K_{\varphi n}(z) + \lambda q z^2 D_q (D_q K_{\varphi n}(z))}{(1 - \lambda) K_{\varphi n}(z) + \lambda z D_q K_{\varphi n}(z)} - 1 \right) = \varphi(z^{n-1}) \\ (z^{1-p} K_{\varphi n}(z)|_{z=0} = 0 = z^{1-p} K'_{\varphi n}(z)|_{z=0} - p)$$

and the functions F_β and G_β ($0 \leq \beta \leq 1$) by

$$1 + \frac{1}{b} \left(\frac{1}{[p]_q} \frac{z D_q F_\beta(z) + \lambda q z^2 D_q (D_q F_\beta(z))}{(1 - \lambda) F_\beta(z) + \lambda z D_q F_\beta(z)} - 1 \right) = \varphi \left(\frac{z(z + \beta)}{1 + \beta z} \right) \\ (z^{1-p} F_\beta(z)|_{z=0} = 0 = z^{1-p} F'_\beta(z)|_{z=0} - p)$$

and

$$1 + \frac{1}{b} \left(\frac{1}{[p]_q} \frac{z D_q G_\beta(z) + \lambda q z^2 D_q (D_q G_\beta(z))}{(1 - \lambda) G_\beta(z) + \lambda z D_q G_\beta(z)} - 1 \right) = \varphi \left(-\frac{z(z + \beta)}{1 + \beta z} \right). \\ (z^{1-p} G_\beta(z)|_{z=0} = 0 = z^{1-p} G'_\beta(z)|_{z=0} - p).$$

Clearly, the functions $K_{\varphi n}$, F_β and G_β are in the class $\mathcal{S}_{\lambda, q, b}(p, \varphi)$. We also write

$$K_\varphi = K_{\varphi 2}.$$

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds true if and only if the function f is K_φ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds true if the function f is $K_{\varphi 3}$ or one of its rotations. If $\mu = \sigma_1$, then the equality holds true if and only if the function f is F_β or one of its rotations. If $\mu = \sigma_2$, then the equality holds true if and only if the function f is G_β or one of its rotations. This completes the proof of Theorem 2. $\square$

Remark 2. For different choices of the parameters q , b and λ in Theorem 1 and Theorem 2, we can deduce the corresponding results derived earlier by Ali *et al.* [2], Aouf *et al.* [3] and Seoudy and Aouf [19].

3. APPLICATIONS TO FUNCTIONS DEFINED BY THE FRACTIONAL q -DERIVATIVE OPERATOR

We first recall some definitions of the *fractional* q -calculus which we will be used in this section.

First of all, for $0 < q < 1$ and $\lambda, \mu \in \mathbb{C}$, the basic (or q -) shifted factorial $(\lambda; q)_\mu$ is defined by (see, for example, [5, 21] and [23])

$$(\lambda; q)_\mu = \prod_{j=0}^{\infty} \left(\frac{1 - \lambda q^j}{1 - \lambda q^{\mu+j}} \right) \quad (0 < q < 1; \lambda, \mu \in \mathbb{C}),$$

so that

$$(\lambda; q)_n := \begin{cases} 1 & (n = 0) \\ \prod_{j=0}^{n-1} (1 - \lambda q^j) & (n \in \mathbb{N}) \end{cases} \quad (3.1)$$

and

$$(\lambda; q)_\infty := \prod_{j=0}^{\infty} (1 - \lambda q^j) \quad (0 < q < 1; \lambda \in \mathbb{C}), \quad (3.2)$$

Furthermore, in terms of the basic (or q -) Gamma function $\Gamma_q(z)$ defined by

$$\Gamma_q(z) := \frac{(q; q)_\infty}{(q^z; q)_\infty} (1 - q)^{1-z} \quad (0 < q < 1; z \in \mathbb{C}), \quad (3.3)$$

so that

$$\lim_{q \rightarrow 1-} \{\Gamma_q(z)\} = \Gamma(z)$$

for the familiar (Euler's) Gamma function $\Gamma(z)$, we find from (3.1) that

$$(q^\alpha; q)_n = \frac{\Gamma_q(\alpha + n)}{\Gamma_q(\alpha)} (1 - q)^n \quad (n \in \mathbb{N}; \alpha \in \mathbb{C}).$$

For $0 < q < 1$, the (Jackson's) q -integral is defined (in general) by

$$\int_0^x f(t) d_q t = (1 - q)x \sum_{k=0}^{\infty} f(xq^k) q^k \quad (x > 0), \quad (3.4)$$

provided that the series on the right-hand side converges absolutely. In the limit case when $q \rightarrow 1-$, the q -integral in (3.4) reduces to

$$\int_0^x f(t) dt.$$

In general, for any closed interval $[a, b]$ ($0 \leq a < b$), we write

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t. \quad (3.5)$$

In order to introduce the subclasses

$$\mathcal{S}_{q,b,\delta}^*(p, \varphi) \quad \text{and} \quad \mathcal{C}_{q,b,\delta}(p, \varphi),$$

we need the following definitions.

Definition 3 ([15], [16] and see also the references cited therein). The *fractional* q -integral operator $\mathfrak{I}_{q,z}^\delta$ of order δ is defined, for a function $f(z)$, by

$$\mathfrak{I}_{q,z}^\delta f(z) = D_{q,z}^{-\delta} f(z) = \frac{1}{\Gamma_q(\delta)} \int_0^z (z - tq)_q^{\delta-1} f(t) d_q t \quad (\delta > 0), \quad (3.6)$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z -plane containing the origin and the q -binomial $(\kappa + z)_q^n$ is defined by (see, for example, [21, p. 486])

$$(\kappa + z)_q^\mu = \sum_{k=0}^{\infty} \begin{bmatrix} \mu \\ k \end{bmatrix}_q q^{\binom{k}{2}} \kappa^{\mu-k} z^k = \kappa^\mu {}_1\Phi_0 \left[\begin{matrix} q^{-\mu}; \\ \text{---}; \end{matrix} q, -\frac{zq^\mu}{\kappa} \right] \quad (3.7)$$

in which the generalized basic (or q -) binomial coefficient $\begin{bmatrix} \lambda \\ n \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} \mu \\ n \end{bmatrix}_q = \frac{(q^{-\mu}; q)_n}{(q; q)_n} (-q^\mu)^n q^{-\binom{n}{2}} \quad (0 < q < 1; \mu \in \mathbb{C}; n \in \mathbb{N}_0) \quad (3.8)$$

and ${}_r\Phi_s$ denotes the generalized q -hypergeometric function with r numerator and s denominator parameters (see, for example, [23, p. 347, Eq. 9.4 (272)]).

Remark 3. It follows from Definition 3 that

$$(z - tq)_q^{\delta-1} = z^{\delta-1} {}_1\Phi_0 \left[\begin{matrix} q^{1-\delta}; \\ \text{---}; \end{matrix} q, \frac{tq^\delta}{z} \right]. \quad (3.9)$$

Since the series

$${}_1\Phi_0 \left[\begin{matrix} \lambda; \\ \text{---}; \end{matrix} q, z \right]$$

is single-valued when (see, for details, [5])

$$|\arg(-z)| < \pi \quad \text{and} \quad |z| < 1,$$

the q -binomial $(z - tq)_q^{\delta-1}$ in (3.9) is single-valued when

$$\left| \arg \left(-\frac{tq^\delta}{z} \right) \right| < \pi, \quad \left| \frac{tq^\delta}{z} \right| < 1 \quad \text{and} \quad |\arg(z)| < \pi.$$

Definition 4 ([15], [16] and see also the references cited therein). The *fractional* q -derivative operator $\mathfrak{D}_{q,z}^\delta$ of order δ is defined, for a function $f(z)$, by

$$\mathfrak{D}_{q,z}^\delta f(z) = \mathfrak{D}_{q,z}^\delta \mathfrak{I}_{q,z}^{1-\delta} f(z) = \frac{1}{\Gamma_q(1-\delta)} \mathfrak{D}_{q,z} \int_0^z (z-tq)_q^{-\delta} f(t) \, d_q t \quad (3.10)$$

$$(0 \leq \delta < 1),$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z -plane containing the origin and the multiplicity of the q -binomial $(z-tq)_q^{-\delta}$ is removed as in Definition 3.

Definition 5 ([15], [16] and see also the references cited therein). Under the hypotheses of Definition 4, the *fractional* q -derivative operator $\mathfrak{D}_{q,z}^\delta$ of order δ is defined, for a function $f(z)$, by

$$\mathfrak{D}_{q,z}^\delta f(z) = \mathfrak{D}_{q,z}^m \mathfrak{I}_{q,z}^{m-\delta} f(z) \quad (m-1 \leq \delta < 1; m \in \mathbb{N}_0 = \mathbb{N} \setminus \{0\}). \quad (3.11)$$

Definition 6 ([18]). In terms of the *fractional* q -derivative operator $\mathfrak{D}_{q,z}^\delta$ given by Definition 3.13, the *fractional* q -derivative operator

$$\Omega_{q,p}^\delta : \mathcal{A}(p) \rightarrow \mathcal{A}(p)$$

is defined as follows:

$$\begin{aligned} \Omega_{q,p}^\delta f(z) &= \frac{\Gamma_q(p+1-\delta)}{\Gamma_q(p+1)} z^\delta \mathfrak{D}_{q,z}^\delta f(z) \\ &= z^p + \sum_{k=p+1}^{\infty} \frac{\Gamma_q(k+1)\Gamma_q(p+1-\delta)}{\Gamma_q(p+1)\Gamma_q(k+1-\delta)} a_k z^k, \end{aligned} \quad (3.12)$$

where the function $f(z) \in \mathcal{A}(p)$ is given by (1.1).

Thus, for $b \in \mathbb{C}^*$, $0 < q < 1$, $0 \leq \lambda \leq 1$, $0 \leq \delta < 1$ and $p \in \mathbb{N}$, we now let $\mathcal{S}_{\lambda,q,b,\delta}(p, \varphi)$ be the subclass of the normalized p -valently analytic function class $\mathcal{A}(p)$ consisting of functions $f(z)$ of the form (1.1) and satisfying the following subordination condition:

$$1 + \frac{1}{b} \left(\frac{1}{[p]_q} \frac{z D_q \left(\Omega_{q,p}^\delta f(z) \right) + \lambda q z^2 D_q^2 \left(\Omega_{q,p}^\delta f(z) \right)}{(1-\lambda) \Omega_{q,p}^\delta f(z) + \lambda z D_q \left(\Omega_{q,p}^\delta f(z) \right)} - 1 \right) \prec \varphi(z). \quad (3.13)$$

Remark 4. By using arguments and analysis to those in the proofs of Theorem 1 and Theorem 2, we can analogously derive Theorem 3 and Theorem 4 below. The details involved are being left as an exercise for the interested reader.

Theorem 3. Let the function $\varphi(z)$ be given by

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots (B_1 > 0).$$

If the function $f(z)$ given by (1.1) belongs to the class $\mathcal{S}_{\lambda,q,b,\delta}(p,\varphi)$ and $\mu \in \mathbb{C}$, then

$$\begin{aligned} |a_{p+2} - \mu a_{p+1}^2| &\leq \frac{[p]_q B_1 |b|}{[p+2]_q - [p]_q} \frac{(1-q^{p-\delta+1})(1-q^{p-\delta+2})}{(1-q^{p+1})(1-q^{p+2})} \left(\frac{1+\lambda([p]_q-1)}{1+\lambda([p+2]_q-1)} \right) \\ &\cdot \max \left\{ 1, \left| \frac{B_2}{B_1} + \frac{[p]_q B_1 b}{[p+1]_q - [p]_q} \left(1 - \mu \frac{[p+2]_q - [p]_q}{[p+1]_q - [p]_q} \right. \right. \right. \\ &\cdot \left. \left. \frac{(1-q^{p+2})(1-q^{p-\delta+1})}{(1-q^{p+1})(1-q^{p-\delta+2})} \frac{[1+\lambda([p+2]_q-1)][1+\lambda([p]_q-1)]}{[1+\lambda([p+1]_q-1)]^2} \right) \right| \Bigg\} \end{aligned}$$

and

$$\begin{aligned} |a_{p+3}| &\leq \frac{[p]_q B_1 |b|}{[p+3]_q - [p]_q} \frac{(1-q^{p-\delta+1})(1-q^{p-\delta+2})(1-q^{p-\delta+3})}{(1-q^{p+1})(1-q^{p+2})(1-q^{p+3})} \\ &\cdot \left(\frac{1+\lambda([p]_q-1)}{1+\lambda([p+3]_q-1)} \right) H(q_1, q_2), \end{aligned}$$

where q_1 and q_2 are defined by (2.6) and (2.7). The result is sharp.

Theorem 4. Let the function $\varphi(z)$ be given by

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots \quad (B_k > 0; k \in \{1, 2\}).$$

If the function $f(z)$ given by (1.1) belongs to the class $\mathcal{S}_{\lambda,q,b,\delta}(p,\varphi)$ and $\mu \in \mathbb{R}$, then

$$|a_{p+2} - \mu a_{p+1}^2| \leq$$

$$\left\{ \begin{array}{l} \frac{[p]_q b}{[p+2]_q - [p]_q} \frac{(1-q^{p-\delta+1})(1-q^{p-\delta+2})}{(1-q^{p+1})(1-q^{p+2})} \frac{1+\lambda([p]_q-1)}{1+\lambda([p+2]_q-1)} \\ \cdot \left\{ B_2 + \frac{[p]_q B_1^2 b}{[p+1]_q - [p]_q} \left(1 - \mu \frac{[p+2]_q - [p]_q}{[p+1]_q - [p]_q} \frac{(1-q^{p+2})(1-q^{p-\delta+1})}{(1-q^{p+1})(1-q^{p-\delta+2})} \right. \right. \\ \left. \left. \cdot \frac{[1+\lambda([p+2]_q-1)][1+\lambda([p]_q-1)]}{[1+\lambda([p+1]_q-1)]^2} \right) \right\} \\ (\mu \leq \sigma_1^*) \\ \frac{[p]_q B_1 b}{[p+2]_q - [p]_q} \frac{(1-q^{p-\delta+1})(1-q^{p-\delta+2})}{(1-q^{p+1})(1-q^{p+2})} \frac{1+\lambda([p]_q-1)}{1+\lambda([p+2]_q-1)} \\ (\sigma_1^* \leq \mu \leq \sigma_2^*) \\ \frac{[p]_q b}{[p+2]_q - [p]_q} \frac{(1-q^{p-\delta+1})(1-q^{p-\delta+2})}{(1-q^{p+1})(1-q^{p+2})} \frac{1+\lambda([p]_q-1)}{1+\lambda([p+2]_q-1)} \\ \cdot \left\{ B_2 + \frac{[p]_q B_1^2 b}{[p+1]_q - [p]_q} \left(\mu \frac{[p+2]_q - [p]_q}{[p+1]_q - [p]_q} \frac{(1-q^{p+2})(1-q^{p-\delta+1})}{(1-q^{p+1})(1-q^{p-\delta+2})} \right. \right. \\ \left. \left. \cdot \frac{[1+\lambda([p+2]_q-1)][1+\lambda([p]_q-1)]}{[1+\lambda([p+1]_q-1)]^2} - 1 \right) \right\} \\ (\mu \leq \sigma_2^*), \end{array} \right.$$

where

$$\sigma_1^* = \frac{([p+1]_q - [p]_q) [(B_2 - B_1) ([p+1]_q - [p]_q) + [p]_q B_1^2 b] [1+\lambda([p+1]_q-1)]^2}{[p]_q ([p+2]_q - [p]_q) [1+\lambda([p+2]_q-1)] [1+\lambda([p]_q-1)] B_1^2 b} \cdot \frac{(1-q^{p+1})(1-q^{p-\delta+2})}{(1-q^{p+2})(1-q^{p-\delta+1})}$$

and

$$\sigma_2^* = \frac{([p+1]_q - [p]_q) [(B_2 + B_1) ([p+1]_q - [p]_q) + [p]_q B_1^2 b] [1+\lambda([p+1]_q-1)]^2}{[p]_q ([p+2]_q - [p]_q) [1+\lambda([p+2]_q-1)] [1+\lambda([p]_q-1)] B_1^2 b} \cdot \frac{(1-q^{p+1})(1-q^{p-\delta+2})}{(1-q^{p+2})(1-q^{p-\delta+1})}.$$

The result is sharp.

Further, if we set

$$\sigma_3^* = \frac{([p+1]_q - [p]_q) [([p+1]_q - [p]_q) B_2 + [p]_q B_1^2 b] [1+\lambda([p+1]_q-1)]^2}{[p]_q ([p+2]_q - [p]_q) [1+\lambda([p+2]_q-1)] [1+\lambda([p]_q-1)] B_1^2 b} \cdot \frac{(1-q^{p+1})(1-q^{p-\delta+2})}{(1-q^{p+2})(1-q^{p-\delta+1})}.$$

then each of the following assertions holds true:

(i) If $\sigma_1^* \leq \mu \leq \sigma_3^*$, then

$$\begin{aligned} & |a_{p+2} - \mu a_{p+1}^2| + \frac{([p+1]_q - [p]_q)^2}{[p]_q ([p+2]_q - [p]_q) B_1^2 b} \frac{(1-q^{p+1})(1-q^{p-\delta+2})}{(1-q^{p+2})(1-q^{p-\delta+1})} \\ & \cdot \frac{[1+\lambda([p+1]_q-1)]^2}{[1+\lambda([p+2]_q-1)][1+\lambda([p]_q-1)]} \left\{ (B_1 - B_2) - \frac{[p]_q B_1^2 b}{[p+1]_q - [p]_q} \right. \\ & \cdot \left(1 - \mu \frac{[p+2]_q - [p]_q}{[p+1]_q - [p]_q} \frac{(1-q^{p+2})(1-q^{p-\delta+1})}{(1-q^{p+1})(1-q^{p-\delta+2})} \right. \\ & \cdot \left. \left. \frac{[1+\lambda([p+2]_q-1)][1+\lambda([p]_q-1)]}{[1+\lambda([p+1]_q-1)]^2} \right) \right\} |a_{p+1}|^2 \\ & \leq \frac{[p]_q B_1 b}{[p+2]_q - [p]_q} \frac{(1-q^{p-\delta+1})(1-q^{p-\delta+2})}{(1-q^{p+1})(1-q^{p+2})} \frac{1+\lambda([p]_q-1)}{1+\lambda([p+2]_q-1)} \end{aligned}$$

(ii) If $\sigma_3^* \leq \mu \leq \sigma_2^*$, then

$$\begin{aligned} & |a_{p+2} - \mu a_{p+1}^2| + \frac{([p+1]_q - [p]_q)^2}{[p]_q ([p+2]_q - [p]_q) B_1^2 b} \frac{(1-q^{p+1})(1-q^{p-\delta+2})}{(1-q^{p+2})(1-q^{p-\delta+1})} \\ & \cdot \frac{[1+\lambda([p+1]_q-1)]^2}{[1+\lambda([p+2]_q-1)][1+\lambda([p]_q-1)]} \left\{ (B_1 - B_2) - \frac{[p]_q B_1^2 b}{[p+1]_q - [p]_q} \right. \\ & \cdot \left(\mu \frac{[p+2]_q - [p]_q}{[p+1]_q - [p]_q} \frac{(1-q^{p+2})(1-q^{p-\delta+1})}{(1-q^{p+1})(1-q^{p-\delta+2})} \right. \\ & \cdot \left. \left. \frac{[1+\lambda([p+2]_q-1)][1+\lambda([p]_q-1)]}{[1+\lambda([p+1]_q-1)]^2} - 1 \right) \right\} |a_{p+1}|^2 \\ & \leq \frac{[p]_q B_1 b}{[p+2]_q - [p]_q} \frac{(1-q^{p-\delta+1})(1-q^{p-\delta+2})}{(1-q^{p+1})(1-q^{p+2})} \frac{1+\lambda([p]_q-1)}{1+\lambda([p+2]_q-1)}. \end{aligned}$$

Remark 5. For different choices of the parameters p , q and λ in Theorem 3 and Theorem 4, we can obtain new results for each of the following p -valently analytic function classes:

$$\mathcal{S}_{b,p}^*(\varphi), \quad \mathcal{C}_{b,p}(\varphi), \quad \mathcal{S}_{\lambda,b,p}(\varphi), \quad \mathcal{S}_{q,b}(\varphi), \quad \mathcal{C}_{q,b}(\varphi), \quad \mathcal{S}_b^*(\varphi) \quad \text{and} \quad \mathcal{C}_b(\varphi),$$

which are defined in Section 1.

Remark 6. For different choices of the parameters b and λ in Theorem 3 and Theorem 4, we can deduce new results for each of the following p -valently analytic

function classes:

$$\mathcal{S}_{q,p,\lambda}^{\alpha,\gamma}(\varphi), \quad \mathcal{S}_{q,p,\lambda}^{\alpha}(\varphi) \quad \text{and} \quad \mathcal{C}_{q,p,\lambda}^{\alpha}(\varphi),$$

which are defined in Section 1.

REFERENCES

- [1] S. Agrawal and S. K. Sahoo, "A generalization of starlike functions of order α ." *Hokkaido Math. J.*, vol. 46, pp. 15–27, 2017.
- [2] R. M. Ali, V. Ravichandran, and N. Seenivasagan, "Coefficient bounds for p-valent functions." *Appl. Math. Comput.*, vol. 187, pp. 35–46, 2007, doi: [10.1016/j.amc.2006.08.100](https://doi.org/10.1016/j.amc.2006.08.100).
- [3] M. K. Aouf, R. M. El-Ashwah, and H. M. Zayed, "Fekete-Szegő inequalities for p-valent starlike and convex functions of complex order." *J. Egyptian Math. Soc.*, vol. 22, pp. 190–196, 2014, doi: [10.1016/j.joems.2013.06.012](https://doi.org/10.1016/j.joems.2013.06.012).
- [4] M. Fekete and G. Szegő, "Eine Bemerkung Über Ungerade Schlichte Funktionen." *J. London Math. Soc.*, vol. [s1-8 (2)], pp. 85–89, 1933, doi: [10.1112/jlms/s1-8.2.85](https://doi.org/10.1112/jlms/s1-8.2.85).
- [5] G. Gasper and M. Rahman, *Basic Hypergeometric Series*. Cambridge, London and New York: (with a Foreword by Richard Askey), Encyclopedia of Mathematics and Its Applications, Vol. 35, Cambridge University Press, Cambridge, New York, Port Chester, Melbourne and Sydney, 1990; Second edition, Encyclopedia of Mathematics and Its Applications, Vol. 96; Cambridge University Press, 2004.
- [6] M. E. H. Ismail, E. Merkes, and D. Styer, "A generalization of starlike functions." *Complex Variables Theory Appl.*, vol. 14, pp. 77–84, 1990, doi: [10.1080/17476939008814407](https://doi.org/10.1080/17476939008814407).
- [7] F. H. Jackson, "On q-definite integrals." *Quart. J. Pure Appl. Math.*, vol. 41, pp. 193–203, 1910.
- [8] F. H. Jackson, "q-difference equations." *Amer. J. Math.*, vol. 32, pp. 305–314, 1910.
- [9] V. G. Kac and P. Cheung, *Quantum Calculus*. New York: Universitext, Springer-Verlag, 2002. doi: [10.1007/978-1-4613-0071-7](https://doi.org/10.1007/978-1-4613-0071-7).
- [10] F. R. Keogh and E. P. Merkes, "A coefficient inequality for certain classes of analytic functions." *Proc. Amer. Math. Soc.*, vol. 20, pp. 8–12, 1969, doi: [10.2307/2035949](https://doi.org/10.2307/2035949).
- [11] W. Ma and D. Minda, "A unified treatment of some special classes of univalent functions. Proceedings of the Conference on Complex Analysis, Z. Li, F. Ren, L. Lang and S. Zhang (Editors), International Press," pp. 157–169, 1994.
- [12] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*. New York: Series on Monographs and Textbooks in Pure and Appl. Math. No. 255; Marcel Dekker, 2000.
- [13] S. Owa, *The quasi-Hadamard products of certain analytic functions*. Singapore, New Jersey, London and Hong Kong: In: *Current Topics in Analytic Function Theory* (H. M. Srivastava and S. Owa, Editors), pp. 234–251; World Scientific Publishing Company, 1992. doi: [10.1142/9789814355896_0019](https://doi.org/10.1142/9789814355896_0019).
- [14] D. V. Prokhorov and J. Szynal, "Inverse coefficients for (α, β) -convex functions." *Ann. Univ. Mariae Curie-Skłodowska Sect. A*, vol. 35, no. 234-261, pp. 125–143, 1981.
- [15] S. D. Purohit and R. K. Raina, "Certain subclasses of analytic functions associated with fractional q-calculus operators." *Math. Scand.*, vol. 109, pp. 55–70, 2011, doi: [10.7146/math.scand.a-15177](https://doi.org/10.7146/math.scand.a-15177).
- [16] P. M. Rajković, S. D. Marinković, and M. S. Stanković, "Fractional integrals and derivatives in q-calculus." *Appl. Anal. Discrete Math.*, vol. 1, pp. 311–323, 2007, doi: [10.2298/AADM0701311R](https://doi.org/10.2298/AADM0701311R).
- [17] V. Ravichandran, Y. Polatoğlu, M. Bolcal, and A. Sen, "Certain subclasses of starlike and convex functions of complex order." *Hacettepe J. Math. Stat.*, vol. 34, pp. 9–15, 2005.
- [18] K. A. Selvakumaran, S. D. Purohit, A. Secer, and M. Bayram, "Convexity of certain q-integral operators of p-valent functions." *Abstr. Appl. Anal.*, vol. 2014, pp. 1–7, 2014, doi: [10.1155/2014/925902](https://doi.org/10.1155/2014/925902).

- [19] T. M. Seoudy and M. K. Aouf, "Coefficient estimates of new classes of q -starlike and q -convex functions of complex order." *J. Math. Inequal.*, vol. 10, pp. 135–145, 2016, doi: [10.7153/jmi-10-11](https://doi.org/10.7153/jmi-10-11).
- [20] H. M. Srivastava, *Univalent functions, fractional calculus, and associated generalized hypergeometric functions*. New York, Chichester, Brisbane and Toronto: In: *Univalent Functions, Fractional Calculus, and Their Applications* (H. M. Srivastava and S. Owa, Editors), Halsted Press (Ellis Horwood Limited, Chichester), pp. 329–354; John Wiley and Sons, 1989.
- [21] H. M. Srivastava and J. Choi, *Zeta and q -Zeta Functions and Associated Series and Integrals*. Amsterdam, London and New York: Elsevier Science Publishers, 2012.
- [22] H. M. Srivastava, A. K. Mishra, and M. K. Das, "The Fekete-Szegő problem for a subclass of close-to-convex functions." *Complex Variables Theory Appl.*, vol. 44, pp. 145–163, 2001, doi: [10.1080/17476930108815351](https://doi.org/10.1080/17476930108815351).
- [23] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*. New York, Chichester, Brisbane and Toronto: Halsted Press (Ellis Horwood Limited, Chichester); John Wiley and Sons, 1985.
- [24] H. Tang, H. M. Srivastava, S. Sivasubramanian, and P. Gurusamy, "The Fekete-Szegő functional problems for some classes of m -fold symmetric bi-univalent functions." *J. Math. Inequal.*, vol. 10, pp. 1063–1092, 2016, doi: [10.7153/jmi-10-85](https://doi.org/10.7153/jmi-10-85).

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SOME RESULTS ON THE q -ANALOGUES OF THE INCOMPLETE FIBONACCI AND LUCAS POLYNOMIALS

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Abstract. In the present paper, we introduce new families of the q -Fibonacci and q -Lucas polynomials, which are represented here as the incomplete q -Fibonacci polynomials $F_n^k(x, s, q)$ and the incomplete q -Lucas polynomials $L_n^k(x, s, q)$, respectively. These polynomials provide the q -analogues of the incomplete Fibonacci and Lucas numbers. We give several properties and generating functions of each of these families q -polynomials. We also point out the fact that the results for the q -analogues which we consider in this article for $0 < q < 1$ can easily be translated into the corresponding results for the (p, q) -analogues (with $0 < q < p \leq 1$) by applying some obvious parametric variations, the additional parameter p being redundant.

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1. INTRODUCTION

The Fibonacci numbers are defined by the following recurrence relation:

$$F_{n+1} = F_n + F_{n-1} \quad (n \in \{0, 1, 2, \dots\})$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$ and the Lucas numbers are defined by the same recurrence relation with the different initial conditions $L_0 = 2$ and $L_1 = 1$.

In existing literature, there are many extensions and generalizations of the Fibonacci numbers. For instance, Filipponi [13] defined the incomplete Fibonacci and Lucas numbers as follows:

$$F_n(k) = \sum_{j=0}^k \binom{n-1-j}{j} \quad (0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor) \quad (1.1)$$

and

$$L_n(k) = \sum_{j=0}^k \frac{n}{n-j} \binom{n-j}{j} \quad (0 \leq k \leq \lfloor \frac{n}{2} \rfloor), \quad (1.2)$$

where $n = 1, 2, 3, \dots$. We note that

$$F_n(\lfloor \frac{n-1}{2} \rfloor) = F_n$$

and

$$L_n(\lfloor \frac{n}{2} \rfloor) = L_n.$$

The generating functions of these numbers were studied by Pintér and Srivastava [17]. Many other authors have also studied this topic (see, for example, [8–12, 18, 19, 21, 22]).

For $0 < q < 1$, the q -integer is defined by

$$[n] := [n]_q = \frac{1 - q^n}{1 - q} \quad (1.3)$$

and the q -factorial is defined by

$$[n]! := \begin{cases} [n] \cdot [n-1] \cdots [1] & (n = 1, 2, 3, \dots) \\ 1 & (n = 0). \end{cases} \quad (1.4)$$

The q -binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[n-k]! [k]!} \quad (0 \leq k \leq n) \quad (1.5)$$

with (see [2] and [20])

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = 1 \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix} = 0 \quad (n < k).$$

The Heine's binomial formula is recalled here as follows (see [2, p. 2]):

$$\frac{1}{(1-x)_q^n} = 1 + \sum_{j=1}^{\infty} \frac{[n]_q [n+1]_q \cdots [n+j-1]_q}{[j]_q!} x^j.$$

The q -difference operator D_q is defined as follows:

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}$$

if $x \neq 0$.

Cigler [6] introduced the q -Fibonacci polynomials which are defined below:

$$F_n(x, s, q) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} q^{\binom{j+1}{2}} \begin{bmatrix} n-j-1 \\ j \end{bmatrix} s^j x^{n-1-2j} \quad (n \in \{0, 1, 2, \dots\}). \quad (1.6)$$

Also, in [4], we have the following explicit formula for the q -Lucas polynomials:

$$L_n(x, s, q) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j} \quad (n \in \{0, 1, 2, \dots\}). \quad (1.7)$$

A recurrence relation for the q -Fibonacci polynomials is given by

$$F_n(x, s, q) = xF_{n-1}(x, s, q) + (q-1)sD_q F_{n-1}(x, s, q) + sF_{n-2}(x, s, q),$$

with the initial values $F_0(x, s, q) = 0$ and $F_1(x, s, q) = 1$. The q -Lucas polynomials satisfy the same recurrence relation as above:

$$L_n(x, s, q) = xL_{n-1}(x, s, q) + (q-1)sD_q L_{n-1}(x, s, q) + sL_{n-2}(x, s, q),$$

but with the initial values given by (see, for details, [4])

$$L_0(x, s, q) = 2 \quad \text{and} \quad L_1(x, s, q) = x.$$

The following formulas provide relationship between these polynomials:

$$L_n(x, s, q) = F_{n+1}(x, s, q) + sF_{n-1}(x, s, q)$$

and

$$L_n(x, qs, q) = F_{n+1}(x, s, q) + q^n s F_{n-1}(x, s, q).$$

For more details about the q -analogues of the Fibonacci polynomials, see [1, 3–5, 7, 14]. There are only a few studies for the q -Fibonacci and q -Lucas polynomials and for the extensions and generalizations of these polynomials.

We choose to remark in passing that several authors (see, for example, [23] and [24]) studied the so-called (p, q) -Fibonacci and (p, q) -Lucas polynomials by introducing a seemingly redundant parameter p , since the so-called (p, q) -number $[n]_{p,q}$ is given (for $0 < q < p \leq 1$) by

$$\begin{aligned} [n]_{p,q} &:= \begin{cases} \frac{p^n - q^n}{p - q} & (n \in \{1, 2, 3, \dots\}) \\ 0 & (n = 0) \end{cases} \\ &= p^{n-1} \left(\frac{q}{p} \right)_n \\ &=: p^{n-1} [n]_{\frac{q}{p}} \end{aligned} \quad (1.8)$$

and

$$\begin{aligned} [n]_q &:= \frac{1 - q^n}{1 - q} \\ &= p^{1-n} \left(\frac{p^n - (pq)^n}{p - (pq)} \right) \end{aligned}$$

$$= p^{1-n} [n]_{p,pq}, \quad (1.9)$$

which do exhibits the fact that the results for the q -analogues which we consider in this article for $0 < q < 1$ can easily be translated into the corresponding results for the (p, q) -analogues (with $0 < q < p \leq 1$) by applying some obvious parametric variations, the additional parameter p being redundant.

The aim of this paper is to introduce and study q -analogues of (1.1) and (1.2). We thus investigate the incomplete q -Fibonacci and q -Lucas polynomials and derive some of their properties including their generating functions.

2. MAIN RESULTS

We define the incomplete q -Fibonacci and q -Lucas polynomials by using the explicit formulas as follows.

Definition 1. The incomplete q -Fibonacci polynomials are defined by

$$\mathcal{F}_n^k := F_n^k(x, s, q) = \sum_{j=0}^k q^{\binom{j+1}{2}} \begin{bmatrix} n-j-1 \\ j \end{bmatrix} s^j x^{n-1-2j} \quad (2.1)$$

for $0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$.

If we take $q \rightarrow 1-$ in (2.1), we get incomplete bivariate Fibonacci numbers in [22]. On the other hand, if we take $q \rightarrow 1-$ and $s = x = 1$ in (2.1), we get incomplete Fibonacci numbers studied in [13].

TABLE 1. Incomplete q -Fibonacci polynomials for $n = 1, \dots, 8$ and $k = 0, 1, 2, 3, 4$.

n/k	0	1	2	$F_n^k(x, s, q) = \mathcal{F}_n^k$		
1	1					
2	x					
3	x^2	$x^2 + qs$				
4	x^3	$x^3 + [2] qsx$				
5	x^4	$x^4 + [3] qsx^2$	$x^4 + [3] qsx^2 + q^3 s^2$			
6	x^5	$x^5 + [4] qsx^3$	$x^5 + [4] qsx^3 + [3] q^3 s^2 x$			
7	x^6	$x^6 + [5] qsx^4$	$x^6 + [5] qsx^4 + \frac{[3][4]}{[2]} q^3 s^2 x^4$	$x^6 + [5] qsx^4 + \frac{[3][4]}{[2]} q^3 s^2 x^4 + q^6 s^3$		
8	x^7	$x^7 + [6] qsx^5$	$x^7 + [6] qsx^5 + \frac{[4][5]}{[2]} q^3 s^2 x^3$	$x^7 + [6] qsx^5 + \frac{[4][5]}{[2]} q^3 s^2 x^3 + [4] q^6 s^3 x$		

We now define incomplete q -Lucas polynomials.

Definition 2. The incomplete q -Lucas polynomials are defined by

$$\mathcal{L}_n^k := L_n^k(x, s, q) = \sum_{j=0}^k q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j} \quad (2.2)$$

for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

TABLE 2. Incomplete q -Lucas polynomials for $n = 1, \dots, 7$ and $k = 0, 1, 2, 3$.

n/k	0	1	2	$U_n^k(x, s, q) = \mathcal{L}_n^k$	3
1	x				
2	x^2	$x^2 + [2]s$			
3	x^3	$x^3 + [3]sx$			
4	x^4	$x^4 + [4]sx^2$	$x^4 + [4]sx^2 + \frac{[4]}{[2]}qs^2$		
5	x^5	$x^5 + [5]sx^3$	$x^5 + [5]sx^3 + \frac{[5]}{[2]}qs^2x$		
6	x^6	$x^6 + [6]sx^4$	$x^6 + [6]sx^4 + \frac{[3][6]}{[2]}qs^2x^2$	$x^6 + [6]sx^4 + \frac{[3][6]}{[2]}qs^2x^2 + [6]q^3s$	
7	x^7	$x^7 + [7]sx^5$	$x^7 + [7]sx^5 + \frac{[4][7]}{[2]}qs^2x^3$	$x^7 + [7]sx^5 + \frac{[4][7]}{[2]}qs^2x^3$	$x^7 + [7]sx^5 + \frac{[4][7]}{[2]}qs^2x^3 + [7]q^3s^3x$

In particular, if we take $q \rightarrow 1-$ in (2.2), then we get incomplete bivariate Lucas numbers studied in [22]. Moreover, if we take $q \rightarrow 1-$ and $s = x = 1$ in (2.2), then we get incomplete Lucas numbers studied in [13].

2.1. Recurrence relations

Theorem 1. The recurrence relation of the incomplete q -Fibonacci polynomials is given by

$$\mathcal{F}_{n+2}^{k+1} = x\mathcal{F}_{n+1}^{k+1} + (q-1)sD_q\mathcal{F}_{n+1}^k + s\mathcal{F}_n^k \quad (2.3)$$

for $0 \leq k \leq \frac{n-2}{2}$.

Proof. We find from (2.1) that

$$\begin{aligned}
 & x\mathcal{F}_{n+1}^{k+1} + (q-1)sD_q\mathcal{F}_{n+1}^k + s\mathcal{F}_n^k \\
 &= \sum_{j=0}^{k+1} q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n+1-2j} \\
 &+ (q-1) \sum_{j=0}^k q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} [n-2j] s^{j+1} x^{n-1-2j} \\
 &+ \sum_{j=0}^k q^{\binom{j+1}{2}} \begin{bmatrix} n-j-1 \\ j \end{bmatrix} s^{j+1} x^{n-1-2j} \\
 &= \sum_{j=0}^{k+1} \left\{ q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} + (q-1)q^{\binom{j}{2}} \begin{bmatrix} n-j+1 \\ j-1 \end{bmatrix} [n-2j+2] \right. \\
 &\quad \left. + q^{\binom{j}{2}} \begin{bmatrix} n-j \\ j-1 \end{bmatrix} \right\} s^j x^{n+1-2j}.
 \end{aligned}$$

It is known that (see [5])

$$q^{\binom{j+1}{2}} \begin{bmatrix} n-j+1 \\ j \end{bmatrix} =$$

$$= q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} + (q-1)q^{\binom{j}{2}} \begin{bmatrix} n-j+1 \\ j-1 \end{bmatrix} [n-2j+2] + q^{\binom{j}{2}} \begin{bmatrix} n-j \\ j-1 \end{bmatrix}.$$

Therefore, the recurrence relation (2.3) is seen to hold true. $\square$

In Theorem 1, by taking $q \rightarrow 1-$ and $x = s = 1$, we get the following result:

$$F_{n+2}(k+1) = F_{n+1}(k+1) + F_n(k) \quad (0 \leq k \leq \frac{n-2}{2}),$$

where $F_n(k)$ are the incomplete Fibonacci numbers studied in [13].

The following theorem results, in part, from the recurrence relation (2.3).

Theorem 2. *The following non-homogeneous recurrence relation of the incomplete q -Fibonacci polynomials holds true:*

$$\mathcal{F}_{n+2}^k = (x + (q-1)sD_q)\mathcal{F}_{n+1}^k + s\mathcal{F}_n^k - q^{\binom{k}{2}}q^n \begin{bmatrix} n-k-1 \\ k \end{bmatrix} s^{k+1}x^{n-2k-1}. \quad (2.4)$$

Proof. The proof of the non-homogeneous recurrence relation (2.4) follows from Definition 1 and the equation (2.3). $\square$

Theorem 3. *It is asserted that*

$$\mathcal{L}_n^k = \mathcal{F}_{n+1}^k + s\mathcal{F}_{n-1}^{k-1} \quad (0 \leq k \leq \frac{n}{2}). \quad (2.5)$$

Proof. Using the equation (2.1), we have

$$\begin{aligned} \mathcal{F}_{n+1}^k + s\mathcal{F}_{n-1}^{k-1} &= \sum_{j=0}^k q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j} + s \sum_{j=0}^{k-1} q^{\binom{j+1}{2}} \begin{bmatrix} n-j-2 \\ j \end{bmatrix} s^j x^{n-2-2j} \\ &= \sum_{j=0}^k q^{\binom{j}{2}} \left\{ q^{-\binom{j}{2}} q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} + \begin{bmatrix} n-j-1 \\ j-1 \end{bmatrix} \right\} s^j x^{n-2j} \\ &= \sum_{j=0}^k q^{\binom{j}{2}} \left\{ q^j \begin{bmatrix} n-j \\ j \end{bmatrix} + \begin{bmatrix} n-j-1 \\ j-1 \end{bmatrix} \right\} s^j x^{n-2j} \\ &= \sum_{j=0}^k q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j} \\ &= \mathcal{L}_n^k \end{aligned}$$

for $0 \leq k \leq \frac{n}{2}$. $\square$

Theorem 4. *The following recurrence relation holds true:*

$$L_n^k(x, qs, q) = \mathcal{F}_{n+1}^k + q^n s \mathcal{F}_{n-1}^{k-1} \quad (0 \leq k \leq \frac{n}{2}). \quad (2.6)$$

Proof. We consider

$$\begin{aligned}
 \mathcal{F}_{n+1}^k + q^n s \mathcal{F}_{n-1}^{k-1} &= \sum_{j=0}^k q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j} \\
 &\quad + q^n s \sum_{j=0}^{k-1} q^{\binom{j+1}{2}} \begin{bmatrix} n-j-2 \\ j \end{bmatrix} j s^j x^{n-2-2j} \\
 &= \sum_{j=0}^k q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j} \\
 &\quad + q^n \sum_{j=1}^k q^{\binom{j}{2}} \begin{bmatrix} n-j-1 \\ j-1 \end{bmatrix} s^j x^{n-2j} \\
 &= \sum_{j=1}^k q^{\binom{j}{2}} \left\{ q^j \begin{bmatrix} n-j \\ j \end{bmatrix} + q^n \begin{bmatrix} n-j-1 \\ j-1 \end{bmatrix} \right\} s^j x^{n-2j} \\
 &= \sum_{j=1}^k q^{\binom{j}{2}} q^j \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j} \\
 &= L_n(x, qs, q)
 \end{aligned}$$

for $0 \leq k \leq \frac{n}{2}$. $\square$

Theorem 5. The recurrence relation of incomplete q -Lucas polynomials is given by

$$\mathcal{L}_{n+2}^{k+1} = x \mathcal{L}_{n+1}^{k+1} + (q-1) s D_q \mathcal{L}_{n+1}^k + s \mathcal{L}_n^k \quad \left(0 \leq k \leq \frac{n-1}{2} \right). \quad (2.7)$$

Proof. In view of (2.3) and (2.5), we find that

$$\begin{aligned}
 \mathcal{L}_{n+2}^{k+1} &= \mathcal{F}_{n+3}^{k+1} + s \mathcal{F}_{n+1}^k \\
 &= \left(x \mathcal{F}_{n+2}^{k+1} + (q-1) s D_q \mathcal{F}_{n+2}^k + s \mathcal{F}_{n+1}^k \right) \\
 &\quad + s \left(x \mathcal{F}_n^k + (q-1) s D_q \mathcal{F}_n^{k-1} + s \mathcal{F}_n^{k-1} \right) \\
 &= x \left(\mathcal{F}_{n+2}^{k+1} + s \mathcal{F}_n^k \right) + (q-1) s D_q \left(\mathcal{F}_{n+2}^k + \mathcal{F}_n^{k-1} \right) + s \left(\mathcal{F}_{n+1}^k + s \mathcal{F}_{n-1}^{k-1} \right) \\
 &= x \mathcal{L}_{n+1}^{k+1} + (q-1) s D_q \mathcal{L}_{n+1}^k + s \mathcal{L}_n^k
 \end{aligned}$$

for $0 \leq k \leq \frac{n-1}{2}$. $\square$

Theorem 6. *The following non-homogeneous recurrence relation of the incomplete q -Lucas polynomials holds true:*

$$\begin{aligned} \mathcal{L}_{n+2}^k &= (x + (q-1)sD_q) \mathcal{L}_{n+1}^k \\ &+ s\mathcal{L}_n^k - \frac{q^{\binom{k}{2}} q^n (q^{-k}[n+1] - 1)}{[n-k]} \begin{bmatrix} n-k \\ k \end{bmatrix} s^{k+1} x^{n-2k}. \end{aligned} \quad (2.8)$$

Proof. It is easy to derive non-homogeneous recurrence relation (2.8) by using (2.2) and (2.7). $\square$

2.2. Summation formulas

Theorem 7. *The following summation formula for the incomplete q -Fibonacci polynomials holds true:*

$$\sum_{j=0}^{h-1} \frac{1}{x^j} \left(s\mathcal{F}_{n+j}^k + (q-1)sD_q \mathcal{F}_{n+1+j}^k \right) = \frac{1}{x^{h-1}} \mathcal{F}_{n+1+h}^{k+1} - x \mathcal{F}_{n+1}^{k+1}, \quad (2.9)$$

where $x \neq 0$ and $n \geq 2k+2$.

Proof. Our proof uses the principle of mathematical induction on h . For $h=1$, the equation (2.9) holds true. Suppose that the equation (2.9) holds true for some integer $h > 1$. Then, by using the equation (2.1), we get

$$\begin{aligned} &\sum_{j=0}^h \frac{1}{x^j} \left(s\mathcal{F}_{n+j}^k + (q-1)sD_q \mathcal{F}_{n+1+j}^k \right) \\ &= \sum_{j=0}^{h-1} \frac{1}{x^j} \left(s\mathcal{F}_{n+j}^k + (q-1)sD_q \mathcal{F}_{n+1+j}^k \right) \\ &\quad + \frac{1}{x^h} \left(s\mathcal{F}_{n+h}^k + (q-1)sD_q \mathcal{F}_{n+1+h}^k \right) \\ &= \frac{1}{x^{h-1}} \mathcal{F}_{n+1+h}^{k+1} - x \mathcal{F}_{n+1}^{k+1} + \frac{1}{x^h} \left(s\mathcal{F}_{n+h}^k + (q-1)sD_q \mathcal{F}_{n+1+h}^k \right) \\ &= \frac{1}{x^h} \left(x \mathcal{F}_{n+1+h}^{k+1} + s\mathcal{F}_{n+h}^k + (q-1)sD_q \mathcal{F}_{n+1+h}^k \right) - x \mathcal{F}_{n+1}^{k+1} \\ &= \frac{1}{x^h} \mathcal{F}_{n+h+2}^{k+1} - x \mathcal{F}_{n+1}^{k+1}, \end{aligned}$$

which completes the proof of the assertion (2.9) by the principle of mathematical induction on h . $\square$

Theorem 8. A summation formula for the incomplete q -Lucas polynomials is given by

$$\sum_{j=0}^{h-1} \frac{1}{x^j} \left(s\mathcal{L}_{n+j}^k + (q-1)sD_q\mathcal{L}_{n+1+j}^k \right) = \frac{1}{x^{h-1}} \mathcal{L}_{n+1+h}^{k+1} - x\mathcal{L}_{n+1}^{k+1}, \quad (2.10)$$

where $x \neq 0$ and $n \geq 2k+1$.

Proof. For using the principle of mathematical induction on h , we suppose that the assertion (2.10) is true for some integer $h > 1$. We thus find from the equation (2.7) that

$$\begin{aligned} & \sum_{j=0}^h \frac{1}{x^j} \left(s\mathcal{L}_{n+j}^k + (q-1)sD_q\mathcal{L}_{n+1+j}^k \right) \\ &= \sum_{j=0}^{h-1} \frac{1}{x^j} \left(s\mathcal{L}_{n+j}^k + (q-1)sD_q\mathcal{L}_{n+1+j}^k \right) \\ & \quad + \frac{1}{x^h} \left(s\mathcal{L}_{n+h}^k + (q-1)sD_q\mathcal{L}_{n+1+h}^k \right) \\ &= \frac{1}{x^{h-1}} \mathcal{L}_{n+1+h}^{k+1} - x\mathcal{L}_n^{k+1} + \frac{1}{x^h} \left(s\mathcal{L}_{n+h}^k + (q-1)sD_q\mathcal{L}_{n+1+h}^k \right) \\ &= \frac{1}{x^h} \left(x\mathcal{L}_{n+1+h}^{k+1} + s\mathcal{L}_{n+h}^k + (q-1)sD_q\mathcal{L}_{n+1+h}^k \right) - x\mathcal{L}_{n+1}^{k+1} \\ &= \frac{1}{x^h} \mathcal{L}_{n+h+2}^{k+1} - x\mathcal{L}_{n+1}^{k+1}, \end{aligned}$$

which completes the proof of the assertion (2.10) by the principle of mathematical induction on h . $\square$

We now derive some summation formulas for the incomplete q -Lucas polynomials.

Lemma 1. It is asserted that

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} j s^j x^{n-2j} = \frac{1}{2} \left(n\mathcal{L}_n - x \frac{d}{dx} \mathcal{L}_n \right). \quad (2.11)$$

Proof. We observe that

$$\frac{d}{dx} \mathcal{L}_n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j (n-2j) x^{n-2j-1}$$

$$\begin{aligned}
&= n \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j-1} \\
&\quad - 2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} j s^j x^{n-2j-1} \\
&= n x^{-1} \mathcal{L}_n - 2 x^{-1} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} j s^j x^{n-2j},
\end{aligned}$$

which proves the assertion (2.11) of Lemma 1. $\square$

The following theorem asserts the summation formula of the incomplete q -Lucas polynomials.

Theorem 9. *Let $\mathcal{L}_n^k$ be the n th incomplete q -Lucas polynomial. Then*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{L}_n^k = \left(\lfloor \frac{n}{2} \rfloor - \frac{n}{2} + 1 \right) \mathcal{L}_n + \frac{x}{2} \frac{d}{dx} \mathcal{L}_n. \quad (2.12)$$

Proof. By using Definition 2 of the incomplete q -Lucas polynomials, we get

$$\begin{aligned}
&\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{L}_n^k = \left\{ q^{\binom{0}{2}} \begin{bmatrix} n \\ 0 \end{bmatrix} s^0 x^n \right\} + \left\{ q^{\binom{0}{2}} \begin{bmatrix} n \\ 0 \end{bmatrix} s^0 x^n + q^{\binom{1}{2}} \frac{[n]}{[n-1]} \begin{bmatrix} n-1 \\ 1 \end{bmatrix} s x^{n-2} \right\} \\
&\quad + \left\{ q^{\binom{0}{2}} \begin{bmatrix} n \\ 0 \end{bmatrix} s^0 x^n + q^{\binom{1}{2}} \frac{[n]}{[n-1]} \begin{bmatrix} n-1 \\ 1 \end{bmatrix} s x^{n-2} + q^{\binom{2}{2}} \frac{[n]}{[n-2]} \begin{bmatrix} n-2 \\ 2 \end{bmatrix} s^2 x^{n-4} \right\} \\
&\quad + \cdots + \left\{ q^{\binom{0}{2}} \begin{bmatrix} n \\ 0 \end{bmatrix} s^0 x^n + \cdots + q^{\binom{\lfloor \frac{n}{2} \rfloor}{2}} \frac{[n]}{[n-\lfloor \frac{n}{2} \rfloor]} \begin{bmatrix} n-\lfloor \frac{n}{2} \rfloor \\ \lfloor \frac{n}{2} \rfloor \end{bmatrix} s^{\lfloor \frac{n}{2} \rfloor} x^{n-2\lfloor \frac{n}{2} \rfloor} \right\} \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\lfloor \frac{n}{2} \rfloor + 1 - j \right) q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j} \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\lfloor \frac{n}{2} \rfloor + 1 \right) q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j} - \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{j}{2}} j \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j},
\end{aligned}$$

which, in view of Lemma 1, yields

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{L}_n^k = \left(\lfloor \frac{n}{2} \rfloor - \frac{n}{2} + 1 \right) \mathcal{L}_n + \frac{x}{2} \frac{d}{dx} \mathcal{L}_n.$$

□

3. GENERATING FUNCTIONS

In this section, we obtain the generating functions of the incomplete q -Fibonacci and the q -Lucas polynomials.

Lemma 2 (see [17]). *Let $\{s_n\}_{n=0}^\infty$ be a complex sequence satisfying the following non-homogeneous recurrence relation:*

$$s_n = xs_{n-1} + ys_{n-2} + r_n \quad (n > p), \quad (3.1)$$

where $\{r_n\}$ is a given complex sequence. Then the generating function $S^k(x, y; t)$ of the sequence $\{s_n\}$ is given by

$$S^k(x, y; t) = (s_0 - r_0 + (s_1 - xs_0 - r_0)t + G(t))(1 - xt - yt^2)^{-1}, \quad (3.2)$$

where $G(t)$ denotes the generating function of the given sequence $\{r_n\}$.

Theorem 10. *The generating function of the incomplete q -Fibonacci polynomials is given by*

$$\begin{aligned} U^k(x, s, q; t) = t^{2k+1} & \left(\mathcal{F}_{2k+1} + (\mathcal{F}_{2k+2} - (x + (q-1)sD_q)\mathcal{F}_{2k+1})t \right. \\ & \left. + q^{\binom{k+2}{2}} s^{k+1} t^2 \frac{1}{(1 - xtq)_q^{k+1}} \right) (1 - (x + (q-1)sD_q)t - st^2)^{-1}, \end{aligned} \quad (3.3)$$

where $\mathcal{F}_k = F_k(x, s, q)$ are the q -Fibonacci polynomials.

Proof. From (2.1) and (2.4), we get

$$\mathcal{F}_n^k = 0 \quad (0 \leq n \leq 2k+1)$$

and

$$\mathcal{F}_{2k+1}^k = \mathcal{F}_{2k+1} \quad \text{and} \quad \mathcal{F}_{2k+2}^k = \mathcal{F}_{2k+2}.$$

Also, for $n \geq 2k+3$, we have

$$\begin{aligned} \mathcal{F}_{n+2k+1}^k &= (x + (q-1)sD_q)\mathcal{F}_{n+2k}^k \\ &+ s\mathcal{F}_{n+2k-1}^k - q^{\binom{k}{2}} q^{n+2k-1} \begin{bmatrix} n+k-2 \\ k \end{bmatrix} s^{k+1} x^{n-2}. \end{aligned}$$

Let

$$s_n = \mathcal{F}_{n+2k+1}^k$$

and

$$r_0 = r_1 = 0 \quad \text{and} \quad r_n = q^{\binom{k}{2}} q^{n+2k-1} \begin{bmatrix} n+k-2 \\ k \end{bmatrix} s^{k+1} x^{n-2}.$$

We then find that

$$G(t) = q^{\binom{k}{2}} q^{2k+1} s^{k+1} t^2 \frac{1}{(1-xtq)_q^{k+1}}$$

and

$$\begin{aligned} S^k(x, s, q; t) &= \left(\mathcal{F}_{2k+1}^k + \left(\mathcal{F}_{2k+2}^k - (x + (q-1)sD_q) \mathcal{F}_{2k+1}^k \right) t \right. \\ &\quad \left. + q^{\binom{k}{2}} q^{2k+1} s^{k+1} t^2 \frac{1}{(1-xtq)_q^{k+1}} \right) \\ &\quad \cdot (1 - (x + (q-1)sD_q)t - st^2)^{-1}. \end{aligned}$$

Therefore, we have

$$U^k(x, s, q; t) = t^{2k+1} S^k(x, s, q; t).$$

□

When $q \rightarrow 1-$ in Theorem 10, then we obtain the generating function of the incomplete bivariate Fibonacci polynomials (see [22]).

Theorem 11. *The generating function of the incomplete q -Lucas polynomials is given by*

$$\begin{aligned} V^k(x, s, q; t) &= t^{2k} \left(\mathcal{L}_{2k} + (\mathcal{L}_{2k+1} - (x + (q-1)sD_q) \mathcal{L}_{2k}) t \right. \\ &\quad \left. + q^{\binom{k+1}{2}} s^{k+1} t^2 (q^{k+1} (1-xt) + 1) \frac{1}{(1-xtq)_q^{k+1}} \right) \\ &\quad \cdot (1 - (x + (q-1)sD_q)t - st^2)^{-1}, \end{aligned} \quad (3.4)$$

where $\mathcal{L}_k = L_k(x, s, q)$ are the q -Lucas polynomials.

Proof. Considering the relation between the incomplete q -Fibonacci and q -Lucas polynomials, we get

$$V^k(x, s, q; t) = \sum_{n=0}^{\infty} \mathcal{L}_n^k t^n$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \mathcal{F}_{n+1}^k t^n + s \sum_{n=0}^{\infty} \mathcal{F}_{n+1}^{k-1} t^n \\
&= t^{-1} U^k(x, s, q; t) + st U^{k-1}(x, s, q; t).
\end{aligned}$$

□

4. CONCLUSION

In the present study, we have introduced the q -analogues of the incomplete Fibonacci and Lucas polynomials which satisfy essentially analogous recursion formulas and recurrence relations with the familiar q -Fibonacci and q -Lucas polynomials. Applications and some generalizations of the q -Fibonacci polynomials are given earlier in [15, 16], which contain some nice results for the q -Fibonacci polynomials. Also, Erkuş-Duman and Tuglu [12] studied various families of multilinear and multilateral generating functions for the generalized bivariate Fibonacci and Lucas polynomials. These works motivate the derivations of similar results for the incomplete q -Fibonacci and q -Lucas polynomials which we have investigated in this paper. By means of the relationships (1.8) and (1.9), we have exhibited the fact that the results for the q -analogues which we consider in this article for $0 < q < 1$ and the corresponding results for the (p, q) -analogues (with $0 < q < p \leq 1$) are essentially equivalent, requiring only some obvious parametric variations, the additional parameter p being redundant.

REFERENCES

- [1] G. E. Andrews, "Fibonacci numbers and the Rogers-Ramanujan identities." *Fibonacci Quart.*, vol. 42, no. 1, pp. 3–19, 2004.
- [2] A. Aral, V. Gupta, and R. P. Agarwal, *Applications of q -Calculus in Operator Theory*. New York: Springer, 2013. doi: [10.1007/978-1-4614-6946-9](https://doi.org/10.1007/978-1-4614-6946-9).
- [3] L. Carlitz, "Fibonacci notes. IV: q -Fibonacci polynomials." *Fibonacci Q.*, vol. 13, no. 2, pp. 97–102, 1975.
- [4] J. Cigler, "Einige q -Analoge der Lucas-und Fibonacci-Polynome." *Sitzungsber., Abt. II, Österr. Akad. Wiss. Math.-Naturwiss. Kl.*, vol. 211, pp. 3–20, 2002.
- [5] J. Cigler, "A new class of q -Fibonacci polynomials." *Electron. J. Combin.*, vol. 10, no. 3, pp. 1–15, 2003.
- [6] J. Cigler, " q -Fibonacci polynomials." *Fibonacci Quart.*, vol. 41, no. 1, pp. 31–40, 2003.
- [7] J. Cigler, " q -Fibonacci polynomials and the Rogers-Ramanujan identities." *Ann. Combin.*, vol. 8, no. 3, pp. 269–285, 2004, doi: [10.1007/s00026-004-0220-8](https://doi.org/10.1007/s00026-004-0220-8).
- [8] G. B. Djordjević, "Generating functions of the incomplete generalized Fibonacci and generalized Lucas numbers." *Fibonacci Quart.*, vol. 42, no. 2, pp. 106–113, 2004.
- [9] G. B. Djordjević and H. M. Srivastava, "Incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers." *Math. Comput. Model.*, vol. 42, no. 9–10, pp. 1049–1056, 2005, doi: [10.1016/j.mcm.2004.10.026](https://doi.org/10.1016/j.mcm.2004.10.026).
- [10] G. B. Djordjević and H. M. Srivastava, "Some generalizations of the incomplete Fibonacci and the incomplete Lucas polynomials." *Adv. Stud. Contemp. Math. (Kyungshang)*, vol. 11, no. 1, pp. 11–32, 2005.

- [11] G. B. Djordjević and H. M. Srivastava, "Some generalizations of certain sequences associated with the Fibonacci numbers." *J. Indones. Math. Soc.*, vol. 12, no. 1, pp. 99–112, 2006.
- [12] E. Erkuş-Duman and N. Tuglu, "Generating functions for the generalized bivariate Fibonacci and Lucas polynomials." *J. Comput. Anal. Appl.*, vol. 18, no. 5, pp. 815–821, 2005.
- [13] P. Filippini, "Incomplete Fibonacci and Lucas numbers." *Rend. Circ. Mat. Palermo* (2), vol. 45, no. 1, pp. 37–56, 1996, doi: [10.1007/BF02845088](https://doi.org/10.1007/BF02845088).
- [14] C. Jia, H. M. Liu, and T. M. Wang, " q -Analogues of generalized Fibonacci and Lucas polynomials." *Fibonacci Quart.*, vol. 45, no. 1, pp. 26–34, 2007.
- [15] T. Mansour and M. Shattuck, "Free rises, restricted partitions, and q -Fibonacci polynomials." *Afrika Mat.*, vol. 24, no. 3, pp. 305–320, 2013, doi: [10.1007/s13370-011-0060-8](https://doi.org/10.1007/s13370-011-0060-8).
- [16] E. Munarini, "Generalized q -Fibonacci numbers." *Fibonacci Quart.*, vol. 43, no. 3, pp. 234–242, 2005.
- [17] A. Pintér and H. M. Srivastava, "Generating functions of the incomplete Fibonacci and Lucas numbers." *Rend. Circ. Mat. Palermo* (2), vol. 48, no. 3, pp. 591–596, 1999, doi: [10.1007/BF02844348](https://doi.org/10.1007/BF02844348).
- [18] R. K. Raina and H. M. Srivastava, "A class of numbers associated with the Lucas numbers." *Math. Comput. Model.*, vol. 25, no. 7, pp. 15–22, 1997, doi: [10.1016/S0895-7177\(97\)00045-9](https://doi.org/10.1016/S0895-7177(97)00045-9).
- [19] J. L. Ramirez, "Incomplete generalized Fibonacci and Lucas polynomials." *Hacet. J. Math. Statist.*, vol. 44, no. 2, pp. 363–373, 2015.
- [20] H. M. Srivastava and J. Choi, *Zeta and q -Zeta Functions and Associated Series and Integrals*. Amsterdam: Elsevier, 2012.
- [21] D. Tasci and M. Cetin-Firengiz, "Incomplete Fibonacci and Lucas p -numbers." *Math. Comput. Model.*, vol. 52, no. 9-10, pp. 1763–1770, 2010, doi: [10.1016/j.mcm.2010.07.003](https://doi.org/10.1016/j.mcm.2010.07.003).
- [22] D. Tasci, M. Cetin-Firengiz, and N. Tuglu, "Incomplete bivariate Fibonacci and Lucas p -numbers." *Discrete Dyn. Nature Soc.*, vol. 2012, Article ID 840345, pp. 1–11, 2012, doi: [10.1155/2012/840345](https://doi.org/10.1155/2012/840345).
- [23] J.-Z. Wang, "Some new results for the (p, q) -Fibonacci and Lucas polynomials." *Adv. Difference Equ.*, vol. 2014, Article ID 64, pp. 1–15, 2014, doi: [10.1186/1687-1847-2014-64](https://doi.org/10.1186/1687-1847-2014-64).
- [24] W.-P. Wang and H. Wang, "Some results on convolved (p, q) -Fibonacci polynomials." *Integral Transforms Spec. Funct.*, vol. 26, no. 5, pp. 340–356, 2015, doi: [10.1080/10652469.2015.1007502](https://doi.org/10.1080/10652469.2015.1007502).

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INFINITELY MANY WEAK SOLUTIONS FOR A FOURTH-ORDER EQUATION WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. Existence results of infinitely many solutions for a fourth-order differential equation are established. This equation depends on two real parameters. The approach is based on an infinitely many critical points theorem.

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1. INTRODUCTION

The aim of the present paper is to investigate the existence of infinitely many solutions for the following fourth-order problem

$$\begin{cases} u^{(iv)}(x) = \lambda \alpha(x) f(x, u(x)) + h(x, u(x)), & x \in [0, 1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = \mu g(u(1)) \end{cases} \quad (1.1)$$

where λ and μ are positive parameters, $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is L^1 -Carathéodory function, $g : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, $\alpha \in L^\infty([0, 1])$, $\alpha(x) \geq 0$, for a.e. $x \in \mathbb{R}$, $\alpha \not\equiv 0$ and $h : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function, there exists $0 < L < 1$, such that $h(x, t) \leq L|t|$ for each $x \in [0, 1]$ and $t \in \mathbb{R}$.

The problem (1.1) is related to the deflections of elastic beams based on nonlinear elasticity. In relation with the problem (1.1), there is an interesting physical description.

Suppose an elastic beam of length $d = 1$, which is clamped at its left side $x = 0$, and resting on a kind of elastic bearing at its right side $x = 1$ is given by μg . Along its length, a load $\lambda \alpha f + h$, is added to cause deformations. If $u = u(x)$ denotes the configuration of the deformed beam, then since $u'''(1)$ represents the shear force at $x = 1$, the condition $u'''(1) = \mu g(u(1))$ means that the vertical force is equal to $\mu g(u(1))$, which denotes a relation, possibly nonlinear, between the vertical force and the displacement $u(1)$. In addition, since $u''(1) = 0$ indicates that there is no

bending moment at $x = 1$, the beam is resting on the bearing μg .

Different models and their applications for problems such as (1.1) can be derived from [4]. We refer the reader to the references [14] and [10] for a physical justification of this model. There is increasing interest in studying fourth-order boundary value problems. Because the change of the static form beam or the support of rigid body can be described by a fourth-order equation. Also a model to study travelling waves in suspension bridges can be furnished by nonlinear fourth-order equations (for instance, see [5]).

When the boundary conditions are nonzero or nonlinear, fourth-order equations can model beams resting on elastic bearings located in their extremities (see for instance [1, 3, 6–8, 11–13] and the references therein).

For example, using a variational methods, the existence of three solutions for special cases of problem (1.1) has been established in [13] and [12]. In [7] the author obtained the existence of at least two positive solutions for the problem

$$\begin{cases} u^{(iv)}(x) = f(x, u(x)), & x \in [0, 1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = g(u(1)), \end{cases} \quad (1.2)$$

based on variational methods and maximum principle. It should be noted that the function f is assumed to be continuous. By assuming appropriate conditions on f and g , the author guarantees positive solutions to problem (1.2). Also existence and multiplicity results for this kind of problems were considered in [1, 3, 6]. In all these works the critical point theory is applied.

Moreover in [8] authors, considered numerical solutions for problem (1.2) with nonlinear boundary conditions.

In particular, using a variational methods the existence of non-zero solutions for problem

$$\begin{cases} u^{(iv)}(x) = f(x, u(x)), & x \in [0, 1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = \mu g(u(1)), \end{cases} \quad (1.3)$$

has been established in [1].

In the present paper, using one kind of infinitely many critical points theorem obtained in [2], we establish the existence of infinitely many weak solutions for the problem (1.1). The paper is organized as follows.

In section 2 we establish all the preliminary results that we need, and in section 3 we present our main results.

2. PRELIMINARIES

The following theorem is a smooth version of Theorem 2.1 of [2] which is a more precise version of Ricceri's Variational Principle [9] and it is the main tool of the next section.

Theorem 1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \rightarrow \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u \in \Phi^{-1}([-\infty, r])} \frac{\sup_{v \in \Phi^{-1}([-\infty, r])} \Psi(v) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r \rightarrow +\infty} \varphi(r), \quad \delta := \liminf_{r \rightarrow (\inf_X \Phi)^+} \varphi(r).$$

Then, one has

(a) *for every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}([-\infty, r])$ admits a global minimum, which is a critical point (local minimum) of I_λ in X .*

(b) *If $\gamma < +\infty$ then, for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternative holds:*

either

(b₁) I_λ possesses a global minimum,

or

(b₂) there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that

$$\lim_{n \rightarrow +\infty} \Phi(u_n) = +\infty.$$

(c) *If $\delta < +\infty$ then, for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternative holds:*

either

(c₁) there is a global minimum of Φ which is a local minimum of I_λ ,

or

(c₂) there is a sequence of pairwise distinct critical points (local minima) of I_λ which weakly converges to a global minimum of Φ .

Now, we recall some basic facts and introduce the needed notations.

Definition 1. A function $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is said to be a Carathéodory function if

(C₁) the function $x \rightarrow f(x, t)$ is measurable for every $t \in \mathbb{R}$;

(C₂) the function $t \rightarrow f(x, t)$ is continuous for a.e. $x \in [0, 1]$.

And $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is said to be a L^1 -Carathéodory function if, in addition to conditions (C₁) and (C₂), the following condition is also satisfied :

(C₃) for every $\rho > 0$ there is function $l_\rho \in L^1([0, 1])$ such that $\sup_{|t| \leq \rho} |f(x, t)| \leq l_\rho(x)$ for almost every $x \in [0, 1]$.

Denote

$$X := \{u \in H^2([0, 1]) | u(0) = u'(0) = 0\},$$

where $H^2([0, 1])$ is the Sobolev space of all functions $u : [0, 1] \rightarrow \mathbb{R}$ such that u and its distributional derivative u' are absolutely continuous and u'' belongs to $L^2([0, 1])$. X is a Hilbert space with the usual norm

$$\|u\|_X = \left(\int_0^1 (|u''(x)|^2 + |u'(x)|^2 + |u(x)|^2) dx \right)^{1/2}$$

which is equivalent to the norm

$$\|u\| = \left(\int_0^1 |u''(x)|^2 dx \right)^{1/2}.$$

Also the embedding $X \hookrightarrow C^1([0, 1])$ is compact and we have

$$\|u\|_{C^1([0,1])} = \max\{\|u\|_\infty, \|u'\|_\infty\} \leq \|u\| \quad (2.1)$$

for each $u \in X$ (see [12]).

Put

$$F(x, t) = \int_0^t f(x, \xi) d\xi \quad \text{for all } (x, t) \in [0, 1] \times \mathbb{R},$$

$$G(t) = \int_0^t g(\xi) d\xi \quad \text{for all } t \in \mathbb{R},$$

and

$$H(x, t) = \int_0^t h(x, \xi) d\xi \quad \text{for all } (x, t) \in [0, 1] \times \mathbb{R}.$$

Let $\Phi, \Psi : X \rightarrow \mathbb{R}$ be defined by

$$\Phi(u) = \frac{1}{2} \int_0^1 |u''(x)|^2 dx - \int_0^1 H(x, u(x)) dx = \frac{1}{2} \|u\|^2 - \int_0^1 H(x, u(x)) dx \quad (2.2)$$

and

$$\Psi(u) = \int_0^1 \alpha(x) F(x, u(x)) dx - \frac{\mu}{\lambda} G(u(1)) \quad (2.3)$$

for every $u \in X$.

Now according to (2.1) we observe that

$$\frac{(1-L)}{2} \|u\|^2 \leq \Phi(u) \leq \frac{(1+L)}{2} \|u\|^2 \quad (2.4)$$

for every $u \in X$. Similar to [1]-page 3, Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_0^1 \alpha(x) f(x, u(x)) v(x) dx - \frac{\mu}{\lambda} g(u(1)) v(1),$$

and, Φ is continuously Gâteaux differentiable functional whose differential at the point $u \in X$ is

$$\Phi'(u)(v) = \int_0^1 u''(x)v''(x)dx - \int_0^1 h(x, u(x))v(x)dx$$

for every $v \in X$.

Definition 2. A function $u \in X$ is a weak solution to the problem (1.1) if

$$\begin{aligned} \int_0^1 u''(x)v''(x)dx - \lambda \int_0^1 \alpha(x)f(x, u(x))v(x)dx + \mu g(u(1))v(1) \\ - \int_0^1 h(x, u(x))v(x)dx = 0 \end{aligned} \quad (2.5)$$

for every $v \in X$.

3. MAIN RESULTS

Let

$$\begin{aligned} G_\eta = \min_{|t| \leq \eta} G(t) = \inf_{|t| \leq \eta} G(t) \quad \text{for all } \eta > 0, \\ \tau := \frac{1-L}{8\pi^4(\frac{2}{3})^3(1+L)}, \end{aligned} \quad (3.1)$$

$$C := \liminf_{\xi \rightarrow +\infty} \frac{\int_0^1 \sup_{|t| \leq \xi} \alpha(x)F(x, t)dx}{\xi^2} \quad (3.2)$$

and

$$D := \limsup_{\xi \rightarrow +\infty} \frac{\int_{\frac{3}{4}}^1 \alpha(x)F(x, \xi)dx}{\xi^2}. \quad (3.3)$$

Now we formulate our main result as follows.

Theorem 2. Assume that

$$(A_1) \int_0^{\frac{3}{4}} \alpha(x)F(x, t)dx \geq 0 \quad \text{for all } t \geq 0.$$

$$(A_2) C < \tau D, \text{ where } \tau, C \text{ and } D \text{ are given by (3.1), (3.2) and (3.3).}$$

Then, setting

$$\lambda_1 := \frac{4\pi^4(\frac{2}{3})^3(1+L)}{D}, \quad \lambda_2 := \frac{1-L}{2C},$$

for each $\lambda \in (\lambda_1, \lambda_2)$, for every arbitrary continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ whose potential

$$G(t) = \int_0^t g(\xi) d\xi \quad \text{for all } t \in \mathbb{R},$$

is a non-positive function satisfying the condition

$$G^* := -\frac{2}{1-L} \lim_{\xi \rightarrow +\infty} \frac{G_\xi}{\xi^2} < +\infty \quad (3.4)$$

and for every $\mu \in (0, \mu_{g,\lambda})$ where $\mu_{g,\lambda} = \frac{1}{G^*} \left(1 - \frac{\lambda}{\lambda_2}\right)$, the problem (1.1) has unbounded sequence of weak solution in X .

Proof. From (A_2) we see that $\lambda_1 < \lambda_2$. Our aim is to apply Theorem 1, to problem (1.1). Fix $\bar{\lambda} \in (\lambda_1, \lambda_2)$ and let g be a function satisfies the condition (3.4). Since $\bar{\lambda} < \lambda_2$, we have $\mu_{g,\bar{\lambda}} = \frac{1}{G^*} \left(1 - \frac{\bar{\lambda}}{\lambda_2}\right) > 0$. Now fix $\bar{\mu} \in (0, \mu_{g,\bar{\lambda}})$ and put $z_1 = \lambda_1$ and $z_2 = \frac{\lambda_2}{1 + \frac{\bar{\mu}}{\bar{\lambda}} \lambda_2 G^*}$. If $G^* = 0$, clearly, $z_1 = \lambda_1$, $z_2 = \lambda_2$ and $\bar{\lambda} \in (z_1, z_2)$. If $G^* \neq 0$, since $\bar{\mu} < \mu_{g,\bar{\lambda}}$, we obtain $\frac{\bar{\lambda}}{\lambda_2} + \bar{\mu} G^* < 1$, and so $\frac{\lambda_2}{1 + \frac{\bar{\mu}}{\bar{\lambda}} \lambda_2 G^*} > \bar{\lambda}$, namely, $\bar{\lambda} < z_2$. Hence, since $\bar{\lambda} > \lambda_1 = z_1$, one has $\bar{\lambda} \in (z_1, z_2)$.

Take X , Φ and Ψ as in the previous section.

For each $u \in X$, we let the functional $I_{\bar{\lambda}} : X \rightarrow \mathbb{R}$ be defined

$$I_{\bar{\lambda}}(u) = \Phi(u) - \bar{\lambda} \Psi(u).$$

Φ is a sequentially weakly lower semicontinuous on X . Indeed, consider an arbitrary $u \in X$ and $\{u_n\}_{n=1}^\infty \subset X$ such that $u_n \rightharpoonup u$ in X . Due to the compact embedding X into $C([0, 1])$, we have that $u_n \rightarrow u$ in $C([0, 1])$. This implies

$$\int_0^1 H(x, u_n(x)) dx \rightarrow \int_0^1 H(x, u(x)) dx, \quad (3.5)$$

and the weakly sequentially lower semicontinuous property of the $\|\cdot\|$ implies

$$\liminf_{n \rightarrow +\infty} \|u_n\|^2 \geq \|u\|^2. \quad (3.6)$$

From (3.5)-(3.6) we have

$$\liminf_{n \rightarrow +\infty} \Phi(u_n) \geq \Phi(u).$$

Since

$$\Phi(u) \geq \left(\frac{1-L}{2}\right) \|u\|^2,$$

taking the condition $0 < L < 1$ into account we observe Φ is coercive. Moreover, Φ is strongly continuous. On the other hand, the compact embedding X into $C[0, 1]$

implies that the functional Ψ is continuously differentiable and with compact derivative. Hence Ψ is sequentially weakly upper semicontinuous (see [15, Corollary 41.9]). Therefore we observe that the regularity assumptions of Theorem 1 on Φ and Ψ are satisfied. Note that the weak solution of problem (1.1) are exactly the critical points of $I_{\bar{\lambda}}$ (in particular, see [12, Lemma 2.1]).

Let $\{\xi_n\}$ be a sequence of positive numbers such that $\xi_n \rightarrow +\infty$ as $n \rightarrow +\infty$ and

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \left(\frac{\int_0^1 \sup_{|t| \leq \xi_n} \alpha(x) F(x, t) dx}{\xi_n^2} - \frac{\bar{\mu}}{\bar{\lambda}} \frac{G_{\xi_n}}{\xi_n^2} \right) \\ &= \liminf_{\xi \rightarrow +\infty} \left(\frac{\int_0^1 \sup_{|t| \leq \xi} \alpha(x) F(x, t) dx}{\xi^2} - \frac{\bar{\mu}}{\bar{\lambda}} \frac{G_{\xi}}{\xi^2} \right). \end{aligned} \quad (3.7)$$

According to (A_2) and (3.4), it is obvious that the limit on the right-hand side in (3) is finite and therefore we have

$$\lim_{n \rightarrow +\infty} \left(\frac{\int_0^1 \sup_{|t| \leq \xi_n} \alpha(x) F(x, t) dx}{\xi_n^2} - \frac{\bar{\mu}}{\bar{\lambda}} \frac{G_{\xi_n}}{\xi_n^2} \right) < +\infty. \quad (3.8)$$

Now for all $n \in \mathbb{N}$, put $r_n = \left(\frac{1-L}{2}\right)\xi_n^2$. Since $\left(\frac{1-L}{2}\right)\|u\|^2 \leq \Phi(u)$, for each $u \in X$ and bearing (2.1) in mind, we see that

$$\begin{aligned} \Phi^{-1}([-\infty, r_n]) &= \{u \in X; \Phi(u) < r_n\} \\ &\subseteq \left\{u \in X; \left(\frac{1-L}{2}\right)\|u\|^2 < r_n\right\} \\ &\subseteq \{u \in X; |u(x)| \leq \xi_n \text{ for each } x \in [0, 1]\}. \end{aligned}$$

Note that $\Phi(0) = \Psi(0) = 0$. Hence, for all $n \in \mathbb{N}$, one has

$$\begin{aligned} \varphi(r_n) &= \inf_{u \in \Phi^{-1}([-\infty, r_n])} \frac{\sup_{v \in \Phi^{-1}([-\infty, r_n])} \Psi(v) - \Psi(u)}{r_n - \Phi(u)} \\ &\leq \frac{\sup_{v \in \Phi^{-1}([-\infty, r_n])} \Psi(v)}{r_n} \\ &= \frac{\sup_{v \in \Phi^{-1}([-\infty, r_n])} \left(\int_0^1 \alpha(x) F(x, v(x)) dx - \frac{\bar{\mu}}{\bar{\lambda}} G(v(1)) \right)}{r_n} \leq \end{aligned}$$

$$\begin{aligned}
& \frac{\left(\sup_{v \in \Phi^{-1}([-\infty, r_n])} \int_0^1 \alpha(x) F(x, v(x)) dx \right) - \frac{\bar{\mu}}{\bar{\lambda}} \left(\inf_{v \in \Phi^{-1}([-\infty, r_n])} G(v(1)) \right)}{r_n} \leq \\
& \frac{\left(\int_0^1 \sup_{|t| \leq \xi_n} \alpha(x) F(x, t) dx \right) - \frac{\bar{\mu}}{\bar{\lambda}} \left(\inf_{|t| \leq \xi_n} G(t) \right)}{r_n} = \\
& \frac{2}{1-L} \left(\frac{\int_0^1 \sup_{|t| \leq \xi_n} \alpha(x) F(x, t) dx}{\xi_n^2} - \frac{\bar{\mu}}{\bar{\lambda}} \frac{G_{\xi_n}}{\xi_n^2} \right).
\end{aligned}$$

Therefore, from (3.8) we obtain

$$\gamma \leq \liminf_{n \rightarrow +\infty} \varphi(r_n) \leq \lim_{n \rightarrow +\infty} \frac{2}{1-L} \left(\frac{\int_0^1 \sup_{|t| \leq \xi_n} \alpha(x) F(x, t) dx}{\xi_n^2} - \frac{\bar{\mu}}{\bar{\lambda}} \frac{G_{\xi_n}}{\xi_n^2} \right) < +\infty. \quad (3.9)$$

Now we show that $\bar{\lambda} < \frac{1}{\gamma}$. Since $\bar{\mu} \in (0, \mu_{g, \bar{\lambda}})$ then $\bar{\mu} < \frac{1}{G^*} (1 - \frac{\bar{\lambda}}{\lambda_2}) = \frac{1}{G^*} (1 - \frac{2C\bar{\lambda}}{1-L})$. Hence from (3) and (3.9) we have

$$\begin{aligned}
\gamma \leq \liminf_{n \rightarrow +\infty} \varphi(r_n) & \leq \lim_{n \rightarrow +\infty} \frac{2}{1-L} \left(\frac{\int_0^1 \sup_{|t| \leq \xi_n} \alpha(x) F(x, t) dx}{\xi_n^2} - \frac{\bar{\mu}}{\bar{\lambda}} \frac{G_{\xi_n}}{\xi_n^2} \right) = \\
& \liminf_{\xi \rightarrow +\infty} \frac{2 \left[\int_0^1 \sup_{|t| \leq \xi} \alpha(x) F(x, t) dx - \frac{\bar{\mu}}{\bar{\lambda}} G_{\xi} \right]}{(1-L)\xi^2} < \\
& \frac{2C}{1-L} - \lim_{\xi \rightarrow +\infty} \frac{2(1 - \frac{2C\bar{\lambda}}{1-L})G_{\xi}}{\bar{\lambda}(1-L)G^*\xi^2} = \frac{2C}{1-L} + \frac{1 - \frac{2C\bar{\lambda}}{1-L}}{\bar{\lambda}} = \frac{1}{\bar{\lambda}}.
\end{aligned}$$

This implies that $\bar{\lambda} < \frac{1}{\gamma}$. So we proved that

$$\bar{\lambda} \in (z_1, z_2) \subseteq (\lambda_1, \lambda_2) \subseteq \left(0, \frac{1}{\gamma}\right).$$

For the fixed $\bar{\lambda}$, the inequality (3.9) concludes that the condition (b) of Theorem 1 can be applied and either $I_{\bar{\lambda}}$ has a global minimum or there exists a sequence $\{u_n\}$ of weak solutions of the problem (1.1) such that $\lim_{n \rightarrow +\infty} \|u_n\| = +\infty$.

The other step is to prove that for the fixed $\bar{\lambda}$ the functional $I_{\bar{\lambda}}$ has no global minimum. Let us show that the functional $I_{\bar{\lambda}}$ is unbounded from below. Since $\bar{\lambda} > \lambda_1$ then we have

$$\frac{1}{\bar{\lambda}} < \frac{1}{4\pi^4(\frac{2}{3})^3(1+L)} \limsup_{\xi \rightarrow +\infty} \frac{\int_{\frac{3}{4}}^1 \alpha(x) F(x, \xi) dx}{\xi^2}$$

and so there exists a sequence $\{\eta_n\}$ of positive numbers and a constant θ such that $\eta_n \rightarrow \infty$ as $n \rightarrow \infty$ and

$$\frac{1}{\bar{\lambda}} < \theta < \frac{1}{4\pi^4(\frac{2}{3})^3(1+L)} \frac{\int_{\frac{3}{4}}^1 \alpha(x) F(x, \eta_n) dx}{\eta_n^2} \quad (3.10)$$

for each $n \in \mathbb{N}$ large enough. For all $n \in \mathbb{N}$ define

$$w_n(x) := \begin{cases} 0 & \text{if } x \in [0, \frac{3}{8}] \\ \eta_n \cos^2(\frac{4\pi x}{3}) & \text{if } x \in]\frac{3}{8}, \frac{3}{4}] \\ \eta_n & \text{if } x \in]\frac{3}{4}, 1]. \end{cases}$$

We observe that $w_n \in X$ and $\|w_n\|^2 = 8\pi^4(\frac{2}{3})^3\eta_n^2$ and so from (2.4) we see that

$$\Phi(w_n) \leq \frac{(1+L)}{2} \|w_n\|^2 = 4\pi^4(\frac{2}{3})^3(1+L)\eta_n^2. \quad (3.11)$$

On the other hand, bearing (A_1) in mind and since G is non-positive, we have

$$\Psi(w_n) \geq \int_{\frac{3}{4}}^1 \alpha(x) F(x, \eta_n) dx. \quad (3.12)$$

It follows from (3.10)-(3.12) that

$$\begin{aligned} I_{\bar{\lambda}}(w_n) &= \Phi(w_n) - \bar{\lambda} \Psi(w_n) \\ &\leq 4(1+L)\pi^4(\frac{2}{3})^3\eta_n^2 - \bar{\lambda} \int_{\frac{3}{4}}^1 \alpha(x) F(x, \eta_n) dx \\ &< 4\pi^4(\frac{2}{3})^3(1+L)\eta_n^2 - 4\pi^4(\frac{2}{3})^3(1+L)\eta_n^2 \bar{\lambda} \theta = \\ &4\pi^4(\frac{2}{3})^3(1+L)\eta_n^2(1 - \bar{\lambda}\theta) \end{aligned}$$

for every $n \in \mathbb{N}$ large enough. Since $\bar{\lambda}\theta > 1$ and $\eta_n \rightarrow +\infty$ as $n \rightarrow +\infty$, we have

$$\lim_{n \rightarrow +\infty} I_{\bar{\lambda}}(w_n) = -\infty$$

and it follows that $I_{\bar{\lambda}}$ has no global minimum. Therefore, taking the fact

$$\Phi(u) \leq \frac{1+L}{2} \|u\|^2$$

into account, by (b) from Theorem 1, there exist a sequence $\{u_n\}$ of critical points of $I_{\bar{\lambda}}$ such that $\lim_{n \rightarrow \infty} \|u_n\| = +\infty$, and the conclusion is achieved. $\square$

Now we present the following example to illustrate the result.

Example 1. Let $F : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be the function defined as

$$F(x, t) := \begin{cases} e^{x^2} t^3 (1 - \cos(\ln(|t|))) & \text{if } (x, t) \in [0, 1] \times (\mathbb{R} - \{0\}) \\ 0 & \text{if } (x, t) \in [0, 1] \times \{0\} \end{cases}$$

and

$$\alpha(x) := \begin{cases} \frac{1}{4} & \text{if } x \in [0, \frac{1}{4}] \\ 0 & \text{if } x \in (\frac{1}{4}, 1]. \end{cases}$$

Consider the problem

$$\begin{cases} u^{(iv)} = \lambda \alpha(x) e^{x^2} \left(3u^2 (1 - \cos(\ln(|u|))) + u^2 \sin(\ln(|u|)) \right) + \frac{1}{2} \sin u, \\ \quad \quad \quad x \in (0, 1), \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = \mu g(u(1)). \end{cases} \quad (3.13)$$

Let $f(x, u) = e^{x^2} \left(3u^2 (1 - \cos(\ln(|u|))) + u^2 \sin(\ln(|u|)) \right)$, $h(x, u) = \frac{1}{2} \sin u$ with $L = \frac{1}{2}$, $g(u) = -u$.

We observe that

$$C := \liminf_{\xi \rightarrow +\infty} \frac{\int_0^1 \sup_{|t| \leq \xi} \alpha(x) F(x, t) dx}{\xi^2} = 0$$

$$D := \limsup_{\xi \rightarrow +\infty} \frac{\int_{\frac{3}{4}}^1 \alpha(x) F(x, \xi) dx}{\xi^2} = +\infty$$

and

$$G^* := -\frac{2}{1-L} \lim_{\xi \rightarrow +\infty} \frac{G_\xi}{\xi^2} = 2.$$

Hence, by Theorem 2, for every $\lambda \in (0, +\infty)$ and $\mu \in (0, \frac{1}{2})$ the problem (3.13) has a sequence of generalized solutions which is unbounded in X .

Theorem 3. Assume that assumption (A_1) in Theorem 2 holds and

$$D > 4\pi^4 \left(\frac{2}{3}\right)^3 (1+L) \quad \text{and} \quad C < \frac{1-L}{2}$$

and for every arbitrary continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ whose potential

$G(t) = \int_0^t g(\xi) d\xi$ for all $t \in \mathbb{R}$, is a non-positive function, satisfying the condition $\lim_{\xi \rightarrow +\infty} \frac{G_\xi}{\xi^2} = 0$. Then, the problem

$$\begin{cases} u^{iv} = f(x, u) + h(x, u), & x \in (0, 1), \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = \mu g(u(1)) \end{cases} \quad (3.14)$$

for every $\mu \in (0, +\infty)$ has an unbounded sequence of weak solution in X .

Proof. Theorem is an immediately consequence of Theorem 2 when $\lambda = 1$. $\square$

Remark 1. In Theorem 2, if we assume that the function f is non-negative, the assumption (A_2) can be written as

$$\liminf_{\xi \rightarrow +\infty} \frac{\int_0^1 \alpha(x) F(x, \xi) dx}{\xi^2} < \tau \limsup_{\xi \rightarrow +\infty} \frac{\int_{\frac{3}{4}}^1 \alpha(x) F(x, \xi) dx}{\xi^2}$$

as well as $\mu_{g,\lambda} = \frac{1}{G^*} \left(1 - \frac{2\lambda}{1-L} \liminf_{\xi \rightarrow +\infty} \frac{\int_0^1 \alpha(x) F(x, \xi) dx}{\xi^2} \right)$. Moreover, in the autonomous case, putting $F(t) = \int_0^t f(\xi) d\xi$ for all $t \in \mathbb{R}$, and $\alpha(x) = 1$ for a.e. $x \in \mathbb{R}$, the assumption (A_2) assumes the form

$$\liminf_{\xi \rightarrow +\infty} \frac{F(\xi)}{\xi^2} < \frac{\tau}{4} \limsup_{\xi \rightarrow +\infty} \frac{F(\xi)}{\xi^2},$$

and in this case, we have

$$\lambda_1 = \frac{16\pi^4 \left(\frac{2}{3}\right)^3 (1+L)}{\limsup_{\xi \rightarrow +\infty} \frac{F(\xi)}{\xi^2}} \quad \text{and} \quad \lambda_2 = \frac{1-L}{2 \liminf_{\xi \rightarrow +\infty} \frac{F(\xi)}{\xi^2}}$$

and $\mu_{g,\lambda} = \frac{1}{G^*} \left(1 - \frac{2\lambda}{1-L} \liminf_{\xi \rightarrow +\infty} \frac{F(\xi)}{\xi^2} \right)$.

Remark 2. Replacing $\xi \rightarrow +\infty$ with $\xi \rightarrow 0^+$ and also put

$$C := \liminf_{\xi \rightarrow 0^+} \frac{\int_0^1 \sup_{|t| \leq \xi} \alpha(x) F(x, t) dx}{\xi^2} \quad (3.15)$$

and

$$D := \limsup_{\xi \rightarrow 0^+} \frac{\int_{\frac{3}{4}}^1 \alpha(x) F(x, \xi) dx}{\xi^2} \quad (3.16)$$

and

$$G^* := -\frac{2}{1-L} \lim_{\xi \rightarrow 0^+} \frac{G_\xi}{\xi^2} < +\infty \quad (3.17)$$

in Theorem 2, by the same way as in the proof of Theorem 2 but using conclusion (c) of Theorem 2 instead of (b), we can obtain a sequence of pairwise distinct weak solutions to the problem (1.1) which converges uniformly to zero.

We present the following example to illustrate Remark 2.

Example 2. Let $\beta > 1847$ be a real number and $F : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by putting

$$F(t) := \begin{cases} t^2 \left(1 + \beta \sin^2\left(\frac{1}{t}\right) \right) & \text{if } t \in]0, +\infty[\\ 0 & \text{if } t \in]-\infty, 0]. \end{cases}$$

We see that $F(t) = \int_0^t f(\xi) d\xi$, where $f(t) = 2t + 2\beta t \sin^2\left(\frac{1}{t}\right) - \beta \sin\left(\frac{2}{t}\right)$.

Put $g(t) = -t^3$, $\alpha(x) = 1$ and $h(x, t) = \frac{1}{3} x t \sin t$ for every $x \in (0, 1)$ and $t \in \mathbb{R}$. We can consider $L = \frac{1}{3}$ and in this case $\tau = \frac{27}{128\pi^4}$.

Let $a_n = \frac{1}{n\pi}$, $b_n = \frac{1}{\frac{2n+1}{2}\pi}$. For every $n \in \mathbb{N}$, we have

$$C := \liminf_{\xi \rightarrow 0^+} \frac{F(\xi)}{\xi^2} \leq \lim_{n \rightarrow \infty} \frac{F(a_n)}{a_n^2} = 1, \quad (3.18)$$

and

$$D := \limsup_{\xi \rightarrow 0^+} \frac{F(\xi)}{\xi^2} \geq \lim_{n \rightarrow \infty} \frac{F(b_n)}{b_n^2} = \beta + 1, \quad (3.19)$$

and by (3.18) and (3.19) one has

$$\liminf_{\xi \rightarrow 0^+} \frac{F(\xi)}{\xi^2} < \frac{\tau}{4} \limsup_{\xi \rightarrow 0^+} \frac{F(\xi)}{\xi^2},$$

as well as

$$\left(\frac{512\pi^4}{81(\alpha+1)}, \frac{1}{3} \right) \subset \left(\frac{16\pi^4 \left(\frac{2}{3}\right)^3 (1+L)}{\limsup_{\xi \rightarrow 0^+} \frac{F(\xi)}{\xi^2}}, \frac{1-L}{2 \liminf_{\xi \rightarrow 0^+} \frac{F(\xi)}{\xi^2}} \right).$$

Therefore, by applying Remark 1 and Remark 2, for every $\beta > 1847$, and every $\lambda \in \left(\frac{512\pi^4}{81(\alpha+1)}, \frac{1}{3}\right)$ and $\mu \in (0, +\infty)$ the problem

$$\begin{cases} u^{iv} = \lambda \left(2u + 2\beta u \sin^2\left(\frac{1}{u}\right) - \beta \sin\left(\frac{2}{u}\right) \right) + \frac{1}{3}x u \sin u, & x \in (0, 1), \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = \mu g(u(1)) \end{cases} \quad (3.20)$$

has a sequence of pairwise distinct weak solutions which converges uniformly to zero.

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REFERENCES

- [1] G. Bonanno, A. Chinnì, and S. Tersian, “Existence results for a two point boundary value problem involving a fourth-order equation.” *Electron. j. Qual. Theory Differ. Equ.*, vol. 2015, no. 33, pp. 1–9, 2015, doi: [10.14232/ejqtde.2015.1.33](https://doi.org/10.14232/ejqtde.2015.1.33).
- [2] G. Bonanno and G. Molica Bisci, “Infinitely many solutions for a boundary value problem with discontinuous nonlinearities.” *Bound. Value Probl.*, vol. 2009, pp. 1–20, 2009, doi: [10.1155/2009/670675](https://doi.org/10.1155/2009/670675).
- [3] M. Grossinho and S. A. Tersian, “The dual variational principle and equilibria for a beam resting on a discontinuous nonlinear elastic foundation.” *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 41, no. 3-4, pp. 417–431, 2000, doi: [10.1016/S0362-546X\(98\)00285-5](https://doi.org/10.1016/S0362-546X(98)00285-5).
- [4] W. W. Jr., S. Timoshenko, and D. Young, *Vibrations Problems in Engineering*. New York: Wiley, 1990.
- [5] A. Lazer and P. J. McKenna, “Large- amplitude periodic oscillations in suspension bridges: Some new connections with nonlinear analysis.” *SIAM Rev.*, vol. 32, no. 4, pp. 537–578, 1990, doi: [10.1137/1032120](https://doi.org/10.1137/1032120).
- [6] T. F. Ma, “Existence results and numerical solutions for a beam equation with nonlinear boundary conditions.” *Appl. Numer. Math.*, vol. 47, no. 2, pp. 189–196, 2003, doi: [10.1016/S0168-9274\(03\)00065-5](https://doi.org/10.1016/S0168-9274(03)00065-5).
- [7] T. F. Ma, “Positive solutions for a beam equation on a nonlinear elastic foundation.” *Math. Comput. Modelling.*, vol. 39, no. 11-12, pp. 1195–1201, 2004, doi: [10.1016/j.mcm.2004.06.001](https://doi.org/10.1016/j.mcm.2004.06.001).
- [8] T. F. Ma and J. da Silva, “Iterative solutions for a beam equation with nonlinear boundary conditions of third order.” *Appl. Math. Comput.*, vol. 159, no. 1, pp. 11–18, 2004, doi: [10.1016/j.amc.2003.08.088](https://doi.org/10.1016/j.amc.2003.08.088).
- [9] B. Ricceri, “A general variational principle and some of its applications.” *J. Comput. Appl. Math.*, vol. 113, no. 1-2, pp. 401–410, 2000, doi: [10.1016/S0377-0427\(99\)00269-1](https://doi.org/10.1016/S0377-0427(99)00269-1).
- [10] D. Russel, *Mathematical models for the elastic beam and their control- theoretic implications*. Longman, Essex, 1986, pp. 177–216.
- [11] X. Wang, “Infinitely many solutions for a fourth-order differential equation on a nonlinear elastic foundation.” *Boundl. Value Probl.*, vol. 2013, no. 9, pp. 1–10, 2013, doi: [10.1186/1687-2770-2013-258](https://doi.org/10.1186/1687-2770-2013-258).

- [12] L. Yang, H. Chen, and X. Yang, “The multiplicity of solutions for fourth-order equations generated from a boundary condition.” *Appl. Math. Lett.*, vol. 24, no. 9, pp. 1599–1603, 2011, doi: [10.1016/j.aml.2011.04.008](https://doi.org/10.1016/j.aml.2011.04.008).
- [13] A. Cabada and S. Tersian, “Multiplicity of solutions of a two point boundary value problem for a fourth-order equation.” *Appl. Math. Comput.*, vol. 219, no. 10, pp. 5261–5267, 2011, doi: [10.1016/j.amc.2012.11.066](https://doi.org/10.1016/j.amc.2012.11.066).
- [14] E. Feireisl, “Non-zero time periodic solutions to an equation of Petrovsky type with nonlinear boundary conditions: slow oscillations of beams on elastic bearings.” *Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser.*, vol. 20, no. 1, pp. 133–146, 1993.
- [15] E. Zeidler, *Nonlinear functional analysis and its applications*. New York: Springer, 1989.

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EXISTENCE OF WEAK SOLUTIONS FOR THREE-POINT BOUNDARY VALUE PROBLEMS OF KIRCHHOFF-TYPE

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Abstract. In this paper, we study the existence and multiplicity of solutions for a perturbed three-point boundary value problem of Kirchhoff-type. The approach is based on variational methods and critical point theory. As applications of these methods, we get several multiplicity results for the problems under consideration.

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1. INTRODUCTION

In this paper, we consider the following perturbed second-order Kirchhoff-type problem

$$\begin{cases} -K\left(\int_a^b |u'(x)|^2 dx\right) u''(x) = \lambda \alpha(x) f(x, u(x)), & x \in (a, b), \\ u(a) = 0, u(b) = \beta u(\gamma) \end{cases} \quad (1.1)$$

where λ is a positive parameter, $K : [0, +\infty[\rightarrow \mathbb{R}$ is a continuous function such that there exist positive numbers m_0 and m_1 with $m_0 \leq K(x) \leq m_1$ for all $x \geq 0$, $a, b \in \mathbb{R}$ with $a < b$, $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is a non-negative L^1 -Carathéodory function, $\alpha \in L^\infty([a, b])$, $\alpha(x) \geq 0$, for a.e. $x \in [a, b]$, $\alpha \not\equiv 0$, $\beta \in \mathbb{R}$ and $\gamma \in (a, b)$.

In recent years, existence and multiplicity results for this kinds of problems has been extensively studied (see [1, 2, 4, 9, 15] and the references therein).

For example, Gupta [15] under natural conditions on f using degree-theoretic arguments, obtained existence and uniqueness theorems to the three-point nonlinear second-order boundary value problem

$$\begin{cases} u'' = f(x, u(x), u'(x)) - e(x), & 0 < x < 1, \\ u(0) = 0, u(\eta) = u(1) \end{cases}$$

where $f : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a function satisfying Carathéodorys conditions, $e : [0, 1] \rightarrow \mathbb{R}$ is a function in $L^1(0, 1)$ and $\eta \in (0, 1)$.

Authors in [4] based upon Leggett-Williams fixed-point theorem, provided conditions for the existence of three positive solutions to the nonlinear boundary value problem

$$\begin{cases} u'' + f(t, u) = 0, & t \in (0, 1), \\ u(0) = 0, \quad au(\eta) = u(1) \end{cases} \quad (1.2)$$

where $0 < \eta < 1$, $0 < a$ and $a\eta < 1$. Also $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is a continuous function and $f(t, \cdot)$ does not vanish identically on any subset of $[0, 1]$ with positive measure. Recently, Lin in [9] by using variational methods and a three-critical-point theorem, considered the existence of at least three solutions for problem (1.2). In [2] authors, based on variational methods and a critical point theorem established the existence of at least one weak solution for problem

$$\begin{cases} -K \left(\int_a^b |u'(t)|^2 dt \right) u''(t) = f(t, u(t)) + h(u(t)), & t \in (a, b), \\ u(a) = 0, \quad u(b) = \alpha u(\eta) \end{cases}$$

where $K : [0, +\infty[\rightarrow \mathbb{R}$ is a continuous function such that there exist positive numbers m and M with $m \leq K(x) \leq M$ for all $x \geq 0$, $a, b \in \mathbb{R}$ with $a < b$, $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is an L^1 -Carathéodory function, $h : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, $\alpha \in \mathbb{R}$ and $\eta \in (a, b)$. Different types of Kirchhoff equation are expressed in [16]. It was proposed as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. The Kirchhoff's model takes into account the length changes of the string produced by transverse vibrations. Some interesting results can be found, for example in [3, 8, 11, 16]. Many researchers have studied the problems of Kirchhoff-type. We refer the reader to the papers [5–7, 10, 17] and the references therein.

In this paper, using two kinds of critical points theorems obtained in [12], we establish the existence of weak solutions for the problem (1.1). The paper is organized as follows.

In section 2 we establish all the preliminary results that we need, and in section 3 we present our main results.

2. PRELIMINARIES

Let us recall some basic concepts.

Definition 1. A function $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is said to be a Carathéodory function if

- (C₁) the function $x \rightarrow f(x, t)$ is measurable for every $t \in \mathbb{R}$;
- (C₂) the function $t \rightarrow f(x, t)$ is continuous for a.e. $x \in [a, b]$.

And $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is said to be a L^1 -Carathéodory function if, in addition to conditions (C₁) and (C₂), the following condition is also satisfied :

- (C₃) for every $\rho > 0$ there is function $l_\rho \in L^1([a, b])$ such that $\sup_{|t| \leq \rho} |f(x, t)| \leq l_\rho(x)$ for almost every $x \in [a, b]$.

Denote

$$X := \{u \in W^{1,2}(a,b) | u(a) = 0, u(b) = \beta u(\gamma)\},$$

endowed with the norm

$$\|u\| := \left(\int_a^b |u'(x)|^2 dx \right)^{1/2}.$$

Theorem 1 ([9], Theorem 3.2). *X is a separable and reflexive real Banach space.*

The following lemma is useful for proving our main result.

Lemma 1 ([2], Lemma 2.4). *For all $u \in X$, we have*

$$\max_{x \in [a,b]} |u(x)| \leq \frac{(1 + |\beta|)\sqrt{b-a}}{2} \|u\|. \quad (2.1)$$

Remark 1. If $\{u_n\}_{n \in \mathbb{N}}$ is a bounded sequence in X , then from the compact embedding $X \hookrightarrow C([a, b])$ it has a subsequence that pointwise converges to some $u \in X$ (it comes from the definition of compact embedding). Also since X is reflexive space then there exist a subsequence that weakly converges in X (see [14]-Theorem 3.18) and so according to continuous embedding $X \rightarrow L^\infty([a, b])$ weakly converges in $L^\infty([a, b])$.

Corresponding to the functions f and K we introduce the functions $F : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ and $\widetilde{K} : [0, \infty) \rightarrow \mathbb{R}$ respectively, as follows

$$F(x, t) = \int_0^t f(x, \xi) d\xi \quad \text{for all } (x, t) \in [a, b] \times \mathbb{R},$$

and

$$\widetilde{K}(t) = \int_0^t K(\xi) d\xi \quad \text{for all } t \geq 0.$$

Moreover, set

$$F^\theta = \int_a^b \sup_{|\xi| \leq \theta} F(x, \xi) dx \quad \text{for all } \theta > 0.$$

Let $\Phi, \Psi : X \rightarrow \mathbb{R}$ be defined by

$$\Phi(u) = \frac{1}{2} \widetilde{K}(\|u\|^2) \quad (2.2)$$

and

$$\Psi(u) = \int_a^b \alpha(x) F(x, u(x)) dx \quad (2.3)$$

for every $u \in X$.

We observe that

$$\frac{m_0}{2} \|u\|^2 \leq \Phi(u) \leq \frac{m_1}{2} \|u\|^2 \quad (2.4)$$

for every $u \in X$. Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_a^b \alpha(x) f(x, u(x)) v(x) dx,$$

and, Φ is continuously Gâteaux differentiable functional whose differential at the point $u \in X$ is

$$\Phi'(u)(v) = K \left(\int_a^b |u'(x)|^2 \right) \left(\int_a^b u'(x) v'(x) dx \right)$$

for every $v \in X$.

Definition 2. Let Φ and Ψ be defined as above and put $I_\lambda = \Phi - \lambda\Psi$. We say that $u \in X$ is a critical point of I_λ when $I'_\lambda(u) = 0_{\{X^*\}}$, that is, $I'_\lambda(u)(v) = \Phi'(u)(v) - \lambda\Psi'(u)(v) = 0$ for all $v \in X$.

Definition 3. A function $u \in X$ is a weak solution to the problem (1.1) if

$$K \left(\int_a^b |u'(x)|^2 \right) \left(\int_a^b u'(x) v'(x) dx \right) - \lambda \int_a^b \alpha(x) f(x, u(x)) v(x) dx = 0 \quad (2.5)$$

for every $v \in X$.

Remark 2. We observe that the weak solutions of the problem (1.1) are exactly the solutions of the equation $I'_\lambda(u)(v) = 0$. Also if α and f are, in addition, continuous functions, then each weak solution of (1.1) is a classical solution.

Lemma 2. If $u_0 \not\equiv 0$ is a weak solution for problem (1.1) then u_0 is non-negative.

Proof. From remark 2 one has, $I'_\lambda(u_0)(v) = 0$ for all $v \in X$. Choosing $v(x) = \max\{-u_0(x), 0\}$ and let $E = \{x \in [a, b] : u_0(x) < 0\}$. Then we have

$$\begin{aligned} & K \left(\int_{E \cup E^c} |u'_0(x)|^2 dx \right) \left(\int_{E \cup E^c} u'_0(x) v'(x) dx \right) \\ &= \lambda \int_a^b \alpha(x) f(x, u_0(x)) v(x) dx \geq 0 \end{aligned}$$

that is

$$-m_1 \int_E v'(x) v'(x) dx \geq 0$$

which means that $-m_1 \|v\|^2 \geq 0$ and one has $\|v\| = 0$ and therefore $v = 0$. Hence $-u_0 \leq 0$, that is, $u_0 \geq 0$ and the proof is complete. $\square$

Definition 4 (see [13]). Fix $r \in]-\infty, +\infty]$. A Gâteaux differentiable function I satisfies the Palais-Smale condition cut off upper at r (in short $(PS)^{[r]}$ -condition) if any sequence $\{u_n\}$ such that

- (a) $\{I(u_n)\}$ is bounded,
 - (b) $\lim_{n \rightarrow +\infty} \|I'(u_n)\|_{X^*} = 0$,
 - (c) $\Phi(u_n) < r \quad \forall n \in \mathbb{N}$,
- has a convergent subsequence.

Definition 5. A Gâteaux differentiable function I satisfies the Palais-Smale condition (in short (PS) -condition) if any sequence $\{u_n\}$ such that

- (a) $\{I(u_n)\}$ is bounded,
 - (b) $\lim_{n \rightarrow +\infty} \|I'(u_n)\|_{X^*} = 0, \quad \forall n \in \mathbb{N}$,
- has a convergent subsequence.

Our main tools are the following critical point theorems.

Theorem 2 ([12], Theorem 2.3). Let X be a real Banach space, and let $\Phi, \Psi : X \rightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$.

Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that

$$\frac{\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}, \quad (2.6)$$

and, for each $\lambda \in \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)} \right]$ the functional $\Phi - \lambda \Psi$ satisfies the $(PS)^{[r]}$ -condition. Then, for each $\lambda \in \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)} \right]$ there is $u_\lambda \in \Phi^{-1}([0, r])$ (hence $u_\lambda \neq 0$) such that $I_\lambda(u_\lambda) < I_\lambda(u)$ for all $u \in \Phi^{-1}([0, r])$ and $I'_\lambda(u_\lambda) = 0$.

Theorem 3 ([12], Theorem 3.2). Let X be a real Banach space, and let $\Phi, \Psi : X \rightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and

$\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that $\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u) < +\infty$ and assume

that for each $\lambda \in \left] 0, \frac{r}{\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)} \right]$ the functional $\Phi - \lambda \Psi$ satisfies the (PS) -condition and it is unbounded from below. Then, for each

$$\lambda \in \left] 0, \frac{r}{\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)} \right]$$

the functional I_λ admits two distinct critical points.

Now we present two propositions that will be needed to prove the main theorems of this paper.

Proposition 1. Take $I_\lambda = \Phi - \lambda\Psi$ as in the definition 2. Then I_λ satisfies the $(PS)^{[r]}$ -condition for any $r > 0$.

Proof. Consider sequence $\{u_n\} \subseteq X$ such that $\{I_\lambda(u_n)\}$ is bounded, $\lim_{n \rightarrow +\infty} \|I'_\lambda(u_n)\|_{X^*} = 0$ and $\Phi(u_n) < r, \forall n \in \mathbb{N}$. Since $\Phi(u_n) < r$, then from (2.4) we see that, $\{u_n\}$ is bounded in X . Therefore passing to a subsequence if necessary we can assume that $u_n(x) \rightarrow u(x)$, and there is $s > 0$ such that $|u_n(x)| \leq s$ for all $x \in [a, b]$ and for all $n \in \mathbb{N}$ and also $\{u_n\}$ weakly converges to u in $L^\infty([a, b])$ (see Remark 1). Now according to Hölder's inequality and Lebesgue's Dominated Convergence Theorem, Since $\alpha(x)f(x, u_n(x)) \leq \alpha(x) \cdot \max_{|\xi| \leq s} f(x, \xi) \in L^1([a, b])$ for all $n \in \mathbb{N}$ and $f(x, u_n(x)) \rightarrow f(x, u(x))$ for a.e. $x \in [a, b]$ (f is L^1 -Carathéodory function), one has $\alpha f(x, u_n)$ is strongly converging to $\alpha f(x, u)$ in $L^1([a, b])$. Now since $u_n \rightharpoonup u$ in $L^\infty([a, b])$ and $\alpha f(x, u_n) \rightarrow \alpha f(x, u)$ in $L^1([a, b]) \subseteq (L^\infty([a, b]))^*$ then from [[14], proposition 3.5(iv)], one has

$$\lim_{n \rightarrow +\infty} \int_a^b \alpha(x) f(x, u_n(x)) (u_n(x) - u(x)) dx = 0. \quad (2.7)$$

From $\lim_{n \rightarrow +\infty} \|I'_\lambda(u_n)\|_X = 0$, there exists a sequence $\{\varepsilon_n\}$, with $\varepsilon_n \rightarrow 0^+$, such that

$$\left| K \left(\int_a^b |u'_n(x)|^2 \right) \left(\int_a^b u'_n(x) v'(x) dx \right) - \lambda \int_a^b \alpha(x) f(x, u_n(x)) v(x) dx \right| \leq \varepsilon_n, \quad (2.8)$$

, for all $n \in \mathbb{N}$ and for all $v \in X$ with $\|v\| \leq 1$. Taking into account $v(x) = \frac{u_n(x) - u(x)}{\|u_n - u\|}$, from (2.8) one has

$$\left| K \left(\int_a^b |u'_n(x)|^2 \right) \int_a^b u'_n(x) (u'_n(x) - u'(x)) dx - \lambda \int_a^b \alpha(x) f(x, u_n(x)) (u_n(x) - u(x)) dx \right| \leq \varepsilon_n \|u_n - u\| \quad (2.9)$$

for all $n \in \mathbb{N}$. Now according to inequality $|a||b| \leq \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2$ one has

$$\begin{aligned} & K \left(\int_a^b |u'_n(x)|^2 \right) \int_a^b u'_n(x) (u'_n(x) - u'(x)) dx = \\ & K \left(\int_a^b |u'_n(x)|^2 \right) \left(\int_a^b |u'_n(x)|^2 dx - \int_a^b u'_n(x) u'(x) dx \right) \geq \end{aligned}$$

$$m_0 \left(\|u_n\|^2 - \int_a^b \left(\frac{1}{2} |u'_n(x)|^2 + \frac{1}{2} |u'(x)|^2 \right) dx \right) =$$

$$m_0 \left(\|u_n\|^2 - \frac{1}{2} \|u_n\|^2 - \frac{1}{2} \|u\|^2 \right) = m_0 \left(\frac{1}{2} \|u_n\|^2 - \frac{1}{2} \|u\|^2 \right).$$

Hence from (2.9), we have

$$m_0 \left(\frac{1}{2} \|u_n\|^2 - \frac{1}{2} \|u\|^2 \right) \leq \lambda \int_a^b \alpha(x) f(x, u_n(x)) (u_n(x) - u(x)) dx$$

$$+ \varepsilon_n \|u_n - u\|$$

that is,

$$\frac{m_0}{2} \|u_n\|^2 \leq \frac{m_0}{2} \|u\|^2$$

$$+ \lambda \int_a^b \alpha(x) f(x, u_n(x)) (u_n(x) - u(x)) dx + \varepsilon_n \|u_n - u\|. \quad (2.10)$$

Now, according to (2.7) and (2.10) when $\varepsilon_n \rightarrow 0^+$, we have,

$$\limsup_{n \rightarrow +\infty} \|u_n\| \leq \|u\|.$$

Thus [14], Proposition 3.32 ensures that $u_n \rightarrow u$, strongly in X and the proof is complete. $\square$

Proposition 2. Assume that there are $M > 0$ and θ with $\theta m_0 > 2m_1$ such that

$$0 < \theta F(x, \xi) \leq \xi f(x, \xi) \quad (2.11)$$

for all $\xi \geq M$ and $x \in [a, b]$. Then $I_\lambda = \Phi - \lambda \Psi$ satisfies the (PS)-condition and it is unbounded from below.

Proof. First we prove that I_λ satisfies (PS)-condition for every $\lambda > 0$. For this purpose we will prove that for arbitrary sequence $\{u_n\} \subset X$ satisfying

$$|I_\lambda(u_n)| \leq D \text{ for some } D > 0 \text{ and for all } n \in \mathbb{N}, \quad (2.12)$$

$$I'_\lambda(u_n) \rightarrow 0 \text{ in } X^* \text{ as } n \rightarrow \infty, \quad (2.13)$$

contains a convergent subsequence. Let $\bar{u}_n(x) = \max\{0, -u_n(x)\}$. From (2.13) we have

$$|I'_\lambda(u_n)(v)| \leq \varepsilon_n \|v\| \quad (2.14)$$

for all $v \in X$ with $\varepsilon_n \rightarrow 0^+$. Hence $|I'_\lambda(u_n)(\bar{u}_n)| \leq \varepsilon_n \|\bar{u}_n\|$, that is,

$$\left| K \left(\int_a^b |u'_n(x)|^2 \right) \left(\int_a^b u'_n(x) \bar{u}'_n(x) dx \right) \right.$$

$$\left. - \lambda \int_a^b \alpha(x) f(x, u_n(x)) \bar{u}_n(x) dx \right| \leq \varepsilon_n \|\bar{u}_n\|.$$

Therefore

$$m_0 \| \bar{u}_n \|^2 + \lambda \int_a^b \alpha(x) f(x, u_n(x)) \bar{u}_n(x) dx \leq \varepsilon_n \| \bar{u}_n \|.$$

Hence $m_0 \| \bar{u}_n \|^2 \leq \varepsilon_n \| \bar{u}_n \|$ and so $\| \bar{u}_n \| \leq \frac{\varepsilon_n}{m_0}$. Therefore $\{\bar{u}_n\}$ is strongly converges to 0 in X and so it is bounded in X . Thus according to compact embedding $X \rightarrow C([a, b])$ it is bounded in $C([a, b])$ and hence $0 \leq \bar{u}_n(x) \leq L$ for some $L \geq 0$ and for all $x \in [a, b]$ and one has,

$$u_n(x) \geq -L \quad (2.15)$$

for all $x \in [a, b]$ and for all $n \in \mathbb{N}$. Now we prove that $\{u_n\}$ is bounded in X . For this end from (2.14), we have

$$-I'_\lambda(u_n)(u_n) \leq \varepsilon_n \| u_n \| \quad (2.16)$$

for all $n \in \mathbb{N}$ with $\varepsilon_n \rightarrow 0^+$. Now we have

$$\begin{aligned} I_\lambda(u_n) - \frac{1}{\theta} I'_\lambda(u_n)(u_n) &= \left(\frac{m_0}{2} - \frac{m_1}{\theta} \right) \| u_n \|^2 \\ &+ \frac{\lambda}{\theta} \int_a^b \alpha(x) \left[f(x, u_n(x)) u_n(x) - \theta F(x, u_n(x)) \right] dx. \end{aligned} \quad (2.17)$$

From (2.11) one has

$$\int_{u_n(x) \geq M} \alpha(x) \left[f(x, u_n(x)) u_n(x) - \theta F(x, u_n(x)) \right] dx \geq 0.$$

On the other hand, in view of (2.15) we have

$$\begin{aligned} &\left| \int_{-L \leq u_n(x) < M} \alpha(x) \left[f(x, u_n(x)) u_n(x) - \theta F(x, u_n(x)) \right] dx \right| \\ &\leq \int_{-L \leq u_n(x) < M} \alpha(x) \sup_{\xi \in [-L, M]} \left[f(x, \xi) \xi - \theta F(x, \xi) \right] dx. \end{aligned}$$

Hence from (2.17) we have

$$\begin{aligned} I_\lambda(u_n) - \frac{1}{\theta} I'_\lambda(u_n)(u_n) &\geq \left(\frac{m_0}{2} - \frac{m_1}{\theta} \right) \| u_n \|^2 \\ &- \frac{\lambda}{\theta} \int_{-L \leq u_n(x) < M} \alpha(x) \sup_{\xi \in [-L, M]} \left[f(x, \xi) \xi - \theta F(x, \xi) \right] dx. \end{aligned} \quad (2.18)$$

Taking into account (2.12), (2.16), from (2.18) one has

$$\left(\frac{m_0}{2} - \frac{m_1}{\theta} \right) \| u_n \|^2 - \frac{\lambda}{\theta} \int_{-L \leq u_n(x) < M} \alpha(x) \sup_{\xi \in [-L, M]} \left[f(x, \xi) \xi - \theta F(x, \xi) \right] dx$$

$$\leq D + \frac{\varepsilon_n}{\theta} \|u_n\|$$

that is,

$$\left(\frac{m_0}{2} - \frac{m_1}{\theta}\right) \|u_n\|^2 \leq D + \frac{\varepsilon_n}{\theta} \|u_n\| + \frac{\lambda}{\theta} \int_{-L \leq u_n(x) < M} \alpha(x) \sup_{\xi \in [-L, M]} [f(x, \xi)\xi - \theta F(x, \xi)] dx \quad (2.19)$$

and thus, (2.19) ensures that $\{u_n\}$ is bounded in X . Finally, as the argument used to prove Proposition 1, $\{u_n\}$ admits a convergent subsequence and so I_λ satisfies (PS) -condition.

Now, from (2.11) we will show that there are positive constants c_1 and c_2 such that

$$F(x, \xi) \geq c_1 \xi^\theta - c_2 \quad (2.20)$$

for all $\xi \geq 0$ and $x \in (a, b)$. Put $\phi_\xi(s) := F(x, s\xi)$, $\forall s > 0$. From (2.11), for all $\xi > M$ and $x \in (a, b)$ we have

$$0 < \theta \phi_\xi(s) = \theta F(x, s\xi) \leq s\xi \cdot f(x, s\xi) = s\phi'_\xi(s), \quad \forall s > \frac{M}{\xi}. \quad (2.21)$$

Hence,

$$\int_{\frac{M}{\xi}}^1 \frac{\phi'_\xi(s)}{\phi_\xi(s)} ds \geq \int_{\frac{M}{\xi}}^1 \frac{\theta}{s} ds.$$

Therefore, we have

$$\phi_\xi(1) \geq \phi_\xi\left(\frac{M}{\xi}\right) \left(\frac{\xi}{M}\right)^\theta.$$

Now, in view of (2.21) one has

$$F(x, \xi) \geq \frac{A}{M^\theta} \xi^\theta \quad (2.22)$$

where $A = \inf_{x \in (a, b)} F(x, M)$. On the other hand for every $0 \leq \xi \leq M$ we have $F(x, \xi) \leq b(x)$, where $b(x) = \sup_{0 \leq \xi \leq M} F(x, \xi)$. Therefore for every $0 \leq \xi \leq M$ we have

$$F(x, \xi) \geq -b(x). \quad (2.23)$$

Thus, (2.20) is proved according to (2.22) and (2.23).

Fixed $u_0 \in X - \{0\}$, and suppose that $u_0 \geq 0$. For each $t > 1$, we have

$$I_\lambda(tu_0) = \frac{1}{2} \widetilde{K}(\|tu_0\|^2) - \lambda \int_a^b \alpha(x) \cdot F(x, tu_0) dx.$$

Taking into account (2.20), one has

$$I_\lambda(tu_0) \leq \frac{m_1 t^2}{2} \|u_0\|^2 - \lambda c_1 t^\theta \int_a^b \alpha(x) \cdot u_0^\theta dx + \lambda c_2 \int_a^b \alpha(x) dx$$

and since $\theta > 2$ (it comes from condition $\theta m_0 > 2m_1$), this condition guarantees that I_λ is unbounded from below. $\square$

3. MAIN RESULTS

In this section we formulate our main results. Put

$$Z := \frac{1}{\gamma - a} + \frac{2(\beta - 1)^2}{b - \gamma}, \quad (3.1)$$

$$\tau := \sqrt{\frac{m_0}{m_1}} \quad (3.2)$$

and

$$\Theta := \frac{2m_0}{(1 + |\beta|)^2(b - a)}. \quad (3.3)$$

Now we express the main results.

Theorem 4. Assume that there exist two positive constants δ and θ with

$$\delta < \frac{2\tau\theta}{(1 + |\beta|)\sqrt{Z(b - a)}} \quad (3.4)$$

such that

$$(i) \quad \frac{F^\theta}{\theta^2} < \frac{2\Theta \int_\gamma^{\frac{b+\gamma}{2}} \alpha(x)F(x, \delta)dx}{\|\alpha\|_\infty m_1 Z \delta^2},$$

where $\|\alpha\|_\infty = \|\alpha\|_{L^\infty}$. Then, for each

$$\lambda \in \Lambda := \left[\frac{m_1 Z \delta^2}{2 \int_\gamma^{\frac{b+\gamma}{2}} \alpha(x)F(x, \delta)dx}, \frac{\Theta \theta^2}{\|\alpha\|_\infty F^\theta} \right], \quad (3.5)$$

the problem (1.1) admits at least one non-trivial and non-negative weak solution $u_\lambda \in X$ such that $|u_\lambda(x)| < \theta$ for all $x \in [a, b]$.

Proof. Our aim is to apply Theorem 2, to problem (1.1). Fix λ , as in the conclusion. Take X , Φ and Ψ as in the previous section. We observe that the regularity assumptions of Theorem 2 on Φ and Ψ are satisfied. Also according to Proposition 1, the functional I_λ satisfies the $(PS)^{[r]}$ -condition for all $r > 0$.

Put

$$r := \Theta \theta^2 \quad (3.6)$$

and

$$w(x) := \begin{cases} \frac{\delta}{\gamma-a}(x-a) & \text{if } x \in [a, \gamma) \\ \delta & \text{if } x \in [\gamma, \frac{b+\gamma}{2}] \\ \delta(\frac{2(\beta-1)}{b-\gamma}x - \frac{\beta(b+\gamma)-2b}{b-\gamma}) & \text{if } x \in (\frac{b+\gamma}{2}, b]. \end{cases} \quad (3.7)$$

We observe that $w \in X$ and

$$\|w\|^2 = \int_a^b |w'(x)|^2 dx = \left(\frac{1}{\gamma-a} + \frac{2(\beta-1)^2}{b-\gamma} \right) \delta^2 = Z\delta^2.$$

In particular, from (2.4) one has

$$\frac{m_0}{2} Z\delta^2 \leq \Phi(w) \leq \frac{m_1}{2} Z\delta^2. \quad (3.8)$$

Therefore, from (3.4) one has $0 < \Phi(w) < r$. Now for each $u \in X$ and bearing (2.1) in mind, we see that

$$\begin{aligned} \Phi^{-1}([-\infty, r]) &= \{u \in X; \Phi(u) < r\} \\ &\subseteq \left\{u \in X; \frac{m_0}{2} \|u\|^2 < r\right\} \\ &\subseteq \{u \in X; |u(x)| < \theta \text{ for each } x \in [a, b]\}, \end{aligned}$$

and it follows that

$$\begin{aligned} \sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u) &= \sup_{u \in \Phi^{-1}([-\infty, r])} \left(\int_a^b \alpha(x) F(x, u(x)) dx \right) \\ &\leq \int_a^b \alpha(x) \sup_{|\xi| \leq \theta} F(x, \xi) dx \leq \|\alpha\|_\infty F^\theta. \end{aligned}$$

Hence, we have

$$\frac{\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)}{r} \leq \frac{\|\alpha\|_\infty F^\theta}{\Theta \theta^2} < \frac{1}{\lambda}. \quad (3.9)$$

On the other hand, from (3.8) and since $\lambda \in \Lambda$, one has,

$$\frac{\Psi(w)}{\Phi(w)} \geq \frac{\int_a^b \alpha(x) F(x, w(x)) dx}{\frac{m_1}{2} Z\delta^2} \geq \frac{\int_\gamma^{\frac{b+\gamma}{2}} \alpha(x) F(x, \delta) dx}{\frac{m_1}{2} Z\delta^2} > \frac{1}{\lambda}. \quad (3.10)$$

Now from (3.9) and (3.10) we have,

$$\frac{\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)}{r} < \frac{\Psi(w)}{\Phi(w)}.$$

Finally for every

$$\lambda \in \Lambda \subseteq \left[\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{u \in \Phi^{-1}(-\infty, r]} \Psi(u)} \right],$$

since the weak solutions of the problem (1.1) are exactly the solutions of the equation $I'_\lambda(u) = 0$, then Theorem 2 (with $\tilde{u} = w$) and lemma 2 will be guaranteed the conclusion. $\square$

Remark 3. According to $F(x, t) = \int_0^t f(x, \xi) d\xi$ for all $(x, t) \in [a, b] \times \mathbb{R}$, we can consider $F(t) = \int_0^t f(\xi) d\xi$ for all $t \in \mathbb{R}$. When f is a continuous and non-negative function then $F \in C^1(\mathbb{R})$ and since $F'(t) = f(t) \geq 0$ for all $t \in \mathbb{R}$ then $F(t)$ is non-decreasing and so

$$F^\theta = \int_a^b \sup_{|\xi| \leq \theta} F(\xi) dx = (b-a) \sup_{|\xi| \leq \theta} F(\xi) = (b-a) F(\theta) \quad \text{for all } \theta > 0.$$

Hence when f does not depend on x , hypothesis (i) in Theorem 4 becomes the following form:

$$(I) \quad \frac{F(\theta)}{\theta^2} < \frac{2\Theta F(\delta) \int_\gamma^{\frac{b+\gamma}{2}} \alpha(x) dx}{(b-a) \|\alpha\|_\infty m_1 Z \delta^2},$$

and the interval becomes

$$\left[\frac{m_1 Z \delta^2}{2F(\delta) \int_\gamma^{\frac{b+\gamma}{2}} \alpha(x) dx}, \frac{\Theta \theta^2}{(b-a) \|\alpha\|_\infty F(\theta)} \right].$$

Now, we present the following example to illustrate Theorem 4 and Remark 3.

Example 1. Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and non-negative function and

$$\alpha(x) := \begin{cases} 4 & \text{if } x \in [\frac{1}{2}, \frac{3}{4}], \\ 0 & \text{if } x \in [0, 1] - [\frac{1}{2}, \frac{3}{4}]. \end{cases}$$

Also put $K(t) = 1 + \tanh \sqrt{t}$ for all $t \geq 0$ and therefore we can consider $m_0 = 1$ and $m_1 = 2$. Let $a = 0$, $b = 1$, $\beta = 2$, $\gamma = \frac{1}{2}$ and hence we have $Z = 6$ and $\Theta = \frac{2}{9}$. Now, if we assume that $\delta = 0.1$, $\theta = 10$ and

$$\frac{1}{100} \int_0^{10} f(\xi) d\xi < \frac{25}{27} \int_0^{0.1} f(\xi) d\xi,$$

then according to Theorem 4 and Remark 3 for each

$$\lambda \in \left] \frac{3}{50 \int_0^{0.1} f(\xi) d\xi}, \frac{50}{9 \int_0^{10} f(\xi) d\xi} \right],$$

problem

$$\begin{cases} -\left(1 + \tanh(\sqrt{\int_0^1 |u'(x)|^2 dx})\right) u''(x) = \lambda \alpha(x) f(u(x)), & x \in (0, 1), \\ u(0) = 0, u(1) = 2u(\frac{1}{2}) \end{cases} \quad (3.11)$$

has at least one non-trivial and non-negative weak solution u_0 such that $|u_0(x)| < 10$ for every $x \in [0, 1]$.

Remark 4. For example in problem (3.11) we can consider ,

$$f(t) := \begin{cases} 100t^2 & \text{if } t \leq 0.1 \\ -100t + 11 & \text{if } 0.1 < t \leq 0.11 \\ 0 & \text{if } t > 0.11 \end{cases}$$

Now we will see that under the appropriate conditions on the function f , existence a non-trivial weak solution will be guaranteed for problem

$$\begin{cases} -K \left(\int_a^b |u'(x)|^2 dx \right) u''(x) = \lambda \alpha(x) f(u(x)), & x \in (a, b), \\ u(a) = 0, u(b) = \beta u(\gamma) \end{cases} \quad (3.12)$$

for any $\lambda \in \mathbb{R}^+$. In this case the following proposition is a consequence of Theorem 4.

Corollary 1. Assume that $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and non-negative function such that

$$\lim_{\xi \rightarrow 0^+} \frac{f(\xi)}{\xi} = +\infty \text{ and } \lim_{\xi \rightarrow +\infty} \frac{f(\xi)}{\xi} = 0.$$

Then, for each $\lambda \in \mathbb{R}^+$, problem (3.12) admits at least one non-trivial and non-negative solution.

Proof. Suppose that $\lambda > 0$ is fixed. Then from $\lim_{\xi \rightarrow 0^+} \frac{f(\xi)}{\xi} = +\infty$ we have

$\lim_{\xi \rightarrow 0^+} \frac{F(\xi)}{\xi^2} = +\infty$ and so $\frac{1}{\lambda} < \frac{2 \int_{\gamma}^{\frac{b+\gamma}{2}} \alpha(x) dx}{m_1 Z} \frac{F(\delta)}{\delta^2}$, for some $\delta > 0$ such that sufficiently small.

On the other hand from $\lim_{\xi \rightarrow +\infty} \frac{f(\xi)}{\xi} = 0$ we have

$\lim_{\xi \rightarrow +\infty} \frac{F(\xi)}{\xi^2} = 0$ and so $\frac{(b-a)\|a\|_\infty}{\Theta} \frac{F(\theta)}{\theta^2} < \frac{1}{\lambda}$, for some $\theta > 0$ with $\delta < \frac{2\tau\theta}{(1+|\beta|)\sqrt{Z(b-a)}}$ such that sufficiently large. Now we can apply Theorem 4 and the conclusion follows. $\square$

Now, we point out the following existence results, as consequences of Theorem 3.

Theorem 5. Suppose that there are $M > 0$ and θ with $\theta m_0 > 2m_1$ such that

$$0 < \theta F(x, \xi) \leq \xi f(x, \xi) \quad (3.13)$$

for all $\xi \geq M$ and $x \in [a, b]$. Then for each $\lambda \in \Lambda' =]0, \frac{1}{\|\alpha\|_\infty F^s}[$ where $s = \frac{(1+|\beta|)\sqrt{\frac{2(b-a)}{m_1}}}{2}$, problem (1.1) admits at least two distinct non-negative weak solutions.

Proof. Our aim is to apply Theorem 3, to problem (1.1). Put $r = 1$ and fixed $\lambda \in \Lambda'$. Let X , Φ and Ψ be as given in the proof of Theorem 4. We observe that the regularity assumptions of Theorem 3 on Φ and Ψ are satisfied and also according to proposition 2, the functional I_λ satisfies the (PS) -condition and it is unbounded from below. Since for each $u \in X$ such that $u \in \Phi^{-1}(]-\infty, 1])$ we have $\|u\| < \sqrt{\frac{2}{m_1}}$, hence from (2.1) one has

$$\begin{aligned} \frac{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} &= \sup_{u \in \Phi^{-1}(]-\infty, 1])} \Psi(u) \\ &= \sup_{u \in \Phi^{-1}(]-\infty, 1])} \int_a^b \alpha(x) F(x, u(x)) dx \\ &\leq \int_a^b \alpha(x) \sup_{|\xi| \leq \frac{(1+|\beta|)\sqrt{\frac{2(b-a)}{m_1}}}{2}} F(x, \xi) dx \leq \|\alpha\|_\infty F^s < \frac{1}{\lambda}. \end{aligned} \quad (3.14)$$

From (3.14) we have

$$\lambda \in \Lambda' \subseteq \left] 0, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right[.$$

So all hypotheses of Theorem 3 are verified. Therefore, for each $\lambda \in \Lambda'$, the functional I_λ admits at least two distinct critical points which are, non-negative solutions of problem (1.1) and the proof is complete. $\square$

To illustrate Theorem 5, we give the following example.

Example 2. Let $f : [1, 2] \times \mathbb{R} \rightarrow \mathbb{R}$ be the function defined as $f(x, t) = x e^{2t}$ and $K(t) = (\cos t)^2 + 1$ for all $t \geq 0$. In this case we can consider $m_0 = 1$ and $m_1 = 2$. Now we have

$$F(x, t) = \int_0^t f(x, \xi) d\xi = \frac{1}{2} x (e^{2t} - 1).$$

Therefore if for example we consider $\theta = 5$ then (3.13) is satisfied for $t \geq \frac{5}{2}$ and so according to Theorem 5 for any $\lambda \in]0, \frac{1}{\|a\|_{\infty} F^1}[$ where

$$F^1 = \int_1^2 \sup_{|\xi| \leq 1} F(x, \xi) dx = \frac{3(e^2 - 1)}{4}$$

the problem

$$\begin{cases} -\left(\cos^2\left(\int_1^2 |u'(x)|^2 dx\right) + 1\right) u''(x) = \lambda \alpha(x) x e^{2u(x)}, x \in (1, 2), \\ u(1) = 0, u(2) = u\left(\frac{3}{2}\right) \end{cases} \quad (3.15)$$

has at least two distinct non-negative weak solutions.

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REFERENCES

- [1] G. A. Afrouzi, G. Carisiti, D. Barilla, and S. Moradi, "A variational approach to perturbed three-point boundary value problems of Kirchhoff-type." *Complex Var. Elliptic Equ.*, vol. 62, no. 3, pp. 397–412, 2017, doi: [10.1080/17476933.2016.1221948](https://doi.org/10.1080/17476933.2016.1221948).
- [2] G. A. Afrouzi, S. Heidarkhani, and S. Moradi, "Existence of weak solutions for three-point boundary value problems of Kirchhoff-type." *Electron. J. Differ. Equ.*, vol. 2016, no. 234, pp. 1–13, 2016.
- [3] M. Chipot and B. Lovat, "Some remarks on non local elliptic and parabolic problems." *Nonlinear Anal., Theory Methods Appl.*, vol. 30, no. 7, pp. 4619–4627, 1997, doi: [10.1016/S0362-546X\(97\)00169-7](https://doi.org/10.1016/S0362-546X(97)00169-7).
- [4] X. He and W. Ge, "Triple solutions for second-order three-point boundary value problems." *J. Math. Anal. Appl.*, vol. 268, no. 1, pp. 256–265, 2002, doi: [10.1006/jmaa.2001.7824](https://doi.org/10.1006/jmaa.2001.7824).
- [5] X. He and W. Zou, "Infinitely many positive solutions for Kirchhoff-type problems." *Nonlinear Anal., Theory Methods Appl.*, vol. 70, no. 3, pp. 1407–1414, 2009, doi: [10.1016/j.na.2008.02.021](https://doi.org/10.1016/j.na.2008.02.021).
- [6] S. Heidarkhani, G. A. Afrouzi, and D. O'Regan, "Existence of three solutions for a Kirchhoff-type boundary value problem." *Electron. J. Differ. Equ.*, vol. 2011, no. 3, pp. 1–11, 2011.
- [7] K. Perera and Z. Zhang, "Nontrivial solutions of Kirchhoff-type problems via the Yang index." *J. Differ. Equations.*, vol. 221, no. 1, pp. 246–255, 2006, doi: [10.1016/j.jde.2005.03.006](https://doi.org/10.1016/j.jde.2005.03.006).
- [8] S. Shahrzad and S. Parasurama, "Suppression of vibration in the axially moving Kirchhoff string by boundary control." *J. Sound Vib.*, vol. 214, no. 3, pp. 567–575, 1998, doi: [10.1006/jsvi.1998.1506](https://doi.org/10.1006/jsvi.1998.1506).
- [9] Xiaojie Lin, "Existence of three solutions for a three-point boundary value problem via a three-critical-point theorem." *Carpath. J. Math.*, vol. 31, no. 2, pp. 213–220, 2015.

- [10] C. O. Alves and G. M. Figueiredo, “Multi-bump solutions for a Kirchhoff-type problem.” *Adv. Nonlinear Anal.*, vol. 5, no. 1, pp. 1–26, 2016, doi: [10.1515/anona-2015-0101](https://doi.org/10.1515/anona-2015-0101).
- [11] A. Arosio and S. Panizzi, “On the well-posedness of the Kirchhoff string.” *Trans. Am. Math. Soc.*, vol. 348, no. 1, pp. 305–330, 1996, doi: [10.1090/S0002-9947-96-01532-2](https://doi.org/10.1090/S0002-9947-96-01532-2).
- [12] G. Bonanno, “Relations between the mountain pass theorem and local minima.” *Adv. Nonlinear Anal.*, vol. 1, no. 3, pp. 205–220, 2012, doi: [10.1155/anona-2012-0003](https://doi.org/10.1155/anona-2012-0003).
- [13] G. Bonanno, “A critical point theorem via the Ekeland variational principle.” *Nonlinear Anal., Theory Methods Appl.*, vol. 75, no. 5, pp. 2992–3007, 2012, doi: [10.1016/j.na.2011.003](https://doi.org/10.1016/j.na.2011.003).
- [14] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. New York: Springer, 2011. doi: [10.1007/978-0-387-70914-7](https://doi.org/10.1007/978-0-387-70914-7).
- [15] C. P. Gupta, “Solvability of a three-point boundary value problems for a second order ordinary differential equation.” *J. Math. Anal. Appl.*, vol. 168, no. 2, pp. 540–551, 1992, doi: [10.1016/0022-247X\(92\)90179-H](https://doi.org/10.1016/0022-247X(92)90179-H).
- [16] G. Kirchhoff, “Vorlesungen über mathematische Physik: Mechanik. 2^{te} Aufl.” Leipzig: Teubner (1877), 1877.
- [17] B. Ricceri, “On an elliptic Kirchhoff-type problem depending on two parameters.” *J. Glob. Optim.*, vol. 46, no. 4, pp. 543–549, 2010, doi: [10.1007/s10898-009-9438-7](https://doi.org/10.1007/s10898-009-9438-7).

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GENERALIZED WEYL CONFORMAL CURVATURE TENSOR OF GENERALIZED RIEMANNIAN SPACE

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Abstract. In this paper certain invariants of conformal mappings of the generalized Riemannian space are obtained. Here are the general results. These invariants are the analogues of the Thomas projective parameter and generalized Weyl projective tensor.

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1. INTRODUCTION AND MOTIVATION

A differentiable N -dimensional manifold on which the non-symmetric basic tensor G_{ij} is defined is a generalized Riemannian space. Because of the non-symmetry $G_{ij} \neq G_{ji}$, it is possible to determine the symmetric and antisymmetric part of the metric tensor G_{ij} :

$$g_{ij} = \frac{1}{2}(G_{ij} + G_{ji}) \quad \text{and} \quad F_{ij} = \frac{1}{2}(G_{ij} - G_{ji}). \quad (1.1)$$

Metric g is non-degenerate, i.e. the matrix $(g_{ij})_{N \times N}$ is non-singular. The coordinates of contravariant metric tensor g of associated space $\mathbb{R}_N$ are the corresponding inverse matrix elements $(g^{ij}) = (g_{ij})^{-1}$. It also holds that: $g^{i\alpha} g_{j\alpha} = \delta_j^i$.

The use of a non-symmetric basic tensor and a non-symmetric affine connection is motivated by Einstein's and Eisenhart's works [5–7]. In Einstein's Unified Field Theory, the symmetric part g_{ij} of the basic tensor G_{ij} is related to gravitation, and the anti-symmetric part F_{ij} is related to electromagnetism. Some mathematicians have studied and developed the theory of non-symmetric affine connection spaces and the theory of generalized Riemannian spaces, for example, L. P. Eisenhart [7], S. M. Minčić [14, 15], M. Prvanović [17].

Affine connection coefficients of the space $\mathbb{G}\mathbb{R}_N$ are (generalized) Christoffel symbols of the second kind

$$\Gamma_{jk}^i = \frac{1}{2} g^{i\alpha} (G_{j\alpha,k} - G_{jk,\alpha} + G_{\alpha k,j}), \quad (1.2)$$

where the partial derivative $\partial/\partial x^i$ is marked with a comma. In general, $\Gamma_{jk}^i \neq \Gamma_{kj}^i$ is valid. This coefficient is decomposed in two parts, the symmetric part $\overset{0}{\Gamma}_{jk}^i$ and the anti-symmetric part T_{jk}^i , where

$$\overset{0}{\Gamma}_{jk}^i = \frac{1}{2} (\Gamma_{jk}^i + \Gamma_{kj}^i) \quad \text{and} \quad T_{jk}^i = \frac{1}{2} (\Gamma_{jk}^i - \Gamma_{kj}^i). \quad (1.3)$$

The symmetric part $\overset{0}{\Gamma}_{jk}^i$ is the Levi-Civita affine connection of the symmetric metric g and the anti-symmetric part T_{jk}^i is called the torsion tensor.

On the basis of a non-symmetric affine connection, it is possible to consider four types of covariant derivatives with regard to the affine connection of the space $\mathbb{G}\mathbb{R}_N$ (see [14, 15]). Afterwards, a family of $\mathbb{G}\mathbb{R}_N$ curvature tensors was obtained

$$R_{jmn}^i = \overset{0}{R}_{jmn}^i + u T_{jm;n}^i + u' T_{jn;m}^i + v T_{jm}^\alpha T_{\alpha n}^i + v' T_{jn}^\alpha T_{\alpha m}^i + w T_{mn}^\alpha T_{\alpha j}^i, \quad (1.4)$$

where u, u', v, v', w are real constants, a covariant derivative based on an affine connection $\overset{0}{\Gamma}$ is denoted by a semicolon and a curvature tensor

$$\overset{0}{R}_{jmn}^i = \overset{0}{\Gamma}_{jm,n}^i - \overset{0}{\Gamma}_{jn,m}^i + \overset{0}{\Gamma}_{jm}^\alpha \overset{0}{\Gamma}_{\alpha n}^i - \overset{0}{\Gamma}_{jn}^\alpha \overset{0}{\Gamma}_{\alpha m}^i. \quad (1.5)$$

Geometrical objects that are invariant with respect to conformal mappings play an important role in the theory of gravity [9, 22]. The aim of our paper is to generalize invariants of conformal mappings from [2, 19]. These mappings generalize the concept of conformal mappings defined by N. S. Sinyukov [18], J. Mikeš [11–13], S. E. Stepanov [20, 21].

2. CONFORMAL MAPPINGS OF GENERALIZED RIEMANNIAN SPACE

Suppose

$$f : \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\overline{\mathbb{R}}_N$$

is a diffeomorphism. We can consider the manifolds $\mathbb{G}\mathbb{R}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ in the common system of local coordinates with respect to this mapping (see Figure 1). Namely, if f maps a point $M \in \mathbb{G}\mathbb{R}_N$ to point $\overline{M} \in \mathbb{G}\overline{\mathbb{R}}_N$ and if $(\mathcal{U}, \varphi)$ is a local chart around the point M it will be $\varphi(M) = x = (x^1, \dots, x^N) \in \mathbb{E}^N$ (Euclidean N -space). In this case, we define (for the coordinate mapping in the $\mathbb{G}\overline{\mathbb{R}}_N$) the mapping $\overline{\varphi} = \varphi \circ f^{-1}$, and then

$$\overline{\varphi}(\overline{M}) = (\varphi \circ f^{-1})(f(M)) = \varphi(M) = x = (x^1, \dots, x^N) \in \mathbb{E}^N. \quad (2.1)$$

Therefore the points M and $\bar{M} = f(M)$ have the same local coordinates.

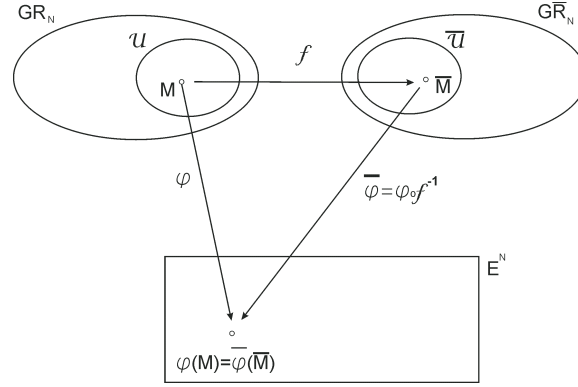


FIGURE 1. Common system of coordinates

Based on Eisenhart's results [7], many authors started the researches on conformal mappings between Riemannian and generalized Riemannian spaces as well as invariants of these mappings (see [1–4, 8, 10–13, 16–21, 23]).

One says that diffeomorphism $f : \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\bar{\mathbb{R}}_N$ is **conformal mapping** [19] if the metric tensors G_{ij} and $\bar{G}_{ij}$ of the spaces $\mathbb{G}\mathbb{R}_N$ and $\mathbb{G}\bar{\mathbb{R}}_N$ satisfy the condition

$$\bar{G}_{ij} = e^{2\psi} G_{ij}, \quad (2.2)$$

where ψ is a scalar function. The basic equation of conformal mapping f is

$$\bar{\Gamma}_{jk}^i = \Gamma_{jk}^i + \psi_j \delta_k^i + \psi_k \delta_j^i - \psi^i g_{jk} + \xi_{jk}^i, \quad (2.3)$$

for $\psi_i = \partial\psi/\partial x^i$, $\psi^i = g^{i\alpha}\psi_\alpha$, and tensor ξ_{jk}^i anti-symmetric in the indices j and k . After symmetrizing the equation (2.3) by indices j and k and contracting the symmetrized equation by indices i and k , we obtain that is

$$\psi_j = \frac{1}{N} (\bar{\Gamma}_{j\alpha}^0 - \Gamma_{j\alpha}^0). \quad (2.4)$$

This result proves that it holds

$$\begin{aligned} \bar{\Gamma}_{jk}^0 - \Gamma_{jk}^0 &= \frac{1}{N} (\bar{\Gamma}_{j\alpha}^\alpha \delta_k^i + \bar{\Gamma}_{k\alpha}^\alpha \delta_j^i - \bar{\Gamma}_{\alpha\beta}^\beta g^{i\alpha} g_{jk}) \\ &\quad - \frac{1}{N} (\Gamma_{j\alpha}^\alpha \delta_k^i + \Gamma_{k\alpha}^\alpha \delta_j^i - \Gamma_{\alpha\beta}^\beta g^{i\alpha} g_{jk}). \end{aligned} \quad (2.5)$$

Weyl conformal curvature tensor

$$\begin{aligned} \overset{0}{C}_{jmn}^i &= \overset{0}{R}_{jmn}^i + \frac{1}{N-2} (\delta_n^i \overset{0}{R}_{jm} - \delta_m^i \overset{0}{R}_{jn} + \overset{0}{R}_n^i g_{jm} - \overset{0}{R}_m^i g_{jn}) \\ &\quad + \frac{\overset{0}{R}}{(N-1)(N-2)} (\delta_m^i g_{jn} - \delta_n^i g_{jm}), \end{aligned} \quad (2.6)$$

is an invariant of the mapping f .

After anti-symmetrizing the equation (2.3) by indices j and k , we obtain that is $\xi_{jk}^i = \bar{T}_{jk}^i - T_{jk}^i$. If one substitutes this equality in the equation (2.3), the non-symmetric affine connection of the space $\mathbb{G}\mathbb{R}_N$ will be reduced to the symmetric affine connection $\overset{0}{\Gamma}$. For this reason, different authors have studied equitorsion conformal mappings (the case of $\xi_{jk}^i = 0$) or conformal mappings of special generalized Riemannian spaces [16, 19, 23]. We will generalize the existing invariants of conformal mappings in this paper.

3. GENERALIZED INVARIANTS OF CONFORMAL MAPPINGS

From the equation (2.2), we get

$$\bar{g}_{ij} = e^{2\psi} g_{ij}, \quad \bar{F}_{ij} = e^{2\psi} F_{ij}, \quad \bar{g}^{ij} = e^{-2\psi} g^{ij},$$

which proves that is

$$\bar{g}^{ij} \bar{g}_{mn} = g^{ij} g_{mn} \quad \text{and} \quad \bar{g}^{ij} \bar{F}_{mn} = g^{ij} F_{mn}. \quad (3.1)$$

With regard to the equation (2.5), we obtain that it holds

$$\overset{0}{\bar{\Gamma}}_{jk}^i - \overset{0}{\Gamma}_{jk}^i = \bar{\zeta}_{jk}^i - \zeta_{jk}^i, \quad (3.2)$$

$p = 1, 2$, for

$$\zeta_{jk}^i = \overset{0}{\Gamma}_{jk}^i \quad \text{and} \quad \zeta_{jk}^i = \frac{1}{N} (\overset{0}{\Gamma}_{j\alpha}^\alpha \delta_k^i + \overset{0}{\Gamma}_{k\alpha}^\alpha \delta_j^i - \overset{0}{\Gamma}_{\alpha\beta}^\beta g^{i\alpha} g_{jk}), \quad (3.3)$$

and the corresponding $\bar{\zeta}_{jk}^i$.

From the equations (1.2, 1.3) and the equality $g_{;k}^{ij} = 0$, one obtains that the torsion tensor T_{jk}^i may be expressed as

$$\begin{aligned} T_{jk}^i &= \frac{1}{2} g^{i\alpha} (F_{j\alpha,k} - F_{jk,\alpha} + F_{\alpha k,j}) \\ &= \frac{1}{2} \left((g^{i\alpha} F_{j\alpha})_{;k} - (g^{i\alpha} F_{k\alpha})_{;j} - (g^{i\alpha} F_{jk})_{;\alpha} \right). \end{aligned} \quad (3.4)$$

Based on this equation and the second of the equalities (3.1), one establishes that it holds

$$\bar{T}_{jk}^i - T_{jk}^i = \bar{\tau}_{jk}^i - \tau_{jk}^i \quad (3.5)$$

$r = (r_1, r_2, r_3, r_4, r_5), r_1, \dots, r_5 \in \{1, 2\}$, for

$$\tau_{rjk}^i = \frac{1}{2} g^{\alpha\beta} (\zeta_{\beta k}^i F_{j\alpha} - \zeta_{\beta j}^i F_{k\alpha} - \zeta_{\alpha\beta}^i F_{jk} + \delta_{\beta}^i \zeta_{j\alpha}^{\gamma} F_{\gamma k} - \delta_{\beta}^i \zeta_{k\alpha}^{\gamma} F_{j\gamma})$$

and the corresponding $\bar{\tau}_{rjk}^i$.

Plugging the equations (2.4, 3.5) into the basic equation (2.3), we get

$$\bar{\mathcal{T}}_{pjk}^i = \mathcal{T}_{pjk}^i,$$

for

$$\mathcal{T}_{pjk}^i = \Gamma_{jk}^i - \zeta_{jk}^i - \tau_{rjk}^i, \quad (3.6)$$

the above defined $\zeta_{pjk}^i, \tau_{rjk}^i$ and the corresponding $\bar{\mathcal{T}}_{pjk}^i$.

It holds the following lemma:

Lemma 1. *Let $f : \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\bar{\mathbb{R}}_N$ be a conformal mapping. The set of geometrical object given by the equation (3.6) is the set of invariants of the mapping f . $\square$*

Corollary 1. *Let $f : \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\bar{\mathbb{R}}_N$ be an equitorsion conformal mapping. The geometrical object*

$$\bar{\mathcal{T}}_{jk}^i = \bar{\Gamma}_{jk}^i - \bar{\zeta}_{jk}^i \quad (3.7)$$

is an invariant of the mapping f . $\square$

The families R_{jmn}^i and $\bar{R}_{jmn}^i$ of curvature tensors of the spaces $\mathbb{G}\mathbb{R}_N$ and $\mathbb{G}\bar{\mathbb{R}}_N$ satisfy the equation

$$\begin{aligned} \bar{R}_{jmn}^i - R_{jmn}^i &= \bar{R}_{jmn}^i - R_{jmn}^i + u(\bar{T}_{jm;n}^i - T_{jm;n}^i) \\ &+ u'(\bar{T}_{jn;m}^i - T_{jn;m}^i) + v(\bar{T}_{jm}^{\alpha} \bar{T}_{\alpha n}^i - T_{jm}^{\alpha} T_{\alpha n}^i) \\ &+ v'(\bar{T}_{jn}^{\alpha} \bar{T}_{\alpha m}^i - T_{jn}^{\alpha} T_{\alpha m}^i) + w(\bar{T}_{mn}^{\alpha} \bar{T}_{\alpha j}^i - T_{mn}^{\alpha} T_{\alpha j}^i). \end{aligned} \quad (3.8)$$

With regard to the invariance $\overset{0}{C}_{jmn}^i = \overset{0}{C}_{jmn}^i$, we obtain that is

$$\begin{aligned} \overset{0}{R}_{jmn}^i - \overset{0}{R}_{jmn}^i &= \frac{1}{N-2} (\delta_n^i \overset{0}{R}_{jm} - \delta_m^i \overset{0}{R}_{jn} + \overset{0}{R}_n^i g_{jm} - \overset{0}{R}_m^i g_{jn}) \\ &\quad - \frac{1}{N-2} (\delta_n^i \overset{0}{R}_{jm} - \delta_m^i \overset{0}{R}_{jn} + \overset{0}{R}_n^i \bar{g}_{jm} - \overset{0}{R}_m^i \bar{g}_{jn}) \\ &\quad + \frac{1}{(N-1)(N-2)} (\overset{0}{R} (\delta_m^i g_{jn} - \delta_n^i g_{jm}) - \overset{0}{R} (\delta_m^i \bar{g}_{jn} - \delta_n^i \bar{g}_{jm})). \end{aligned} \quad (3.9)$$

From the equations (3.2, 3.5), we establish that

$$\bar{T}_{jm;n}^i - T_{jm;n}^i = \bar{\sigma}_{s;jmn}^i - \sigma_{s;jmn}^i \quad (3.10)$$

$$\bar{T}_{jm}^\alpha \bar{T}_{\alpha n}^i - T_{jm}^\alpha T_{\alpha n}^i = \bar{\Theta}_{r^1;jmn}^i - \Theta_{r^1;jmn}^i, \quad (3.11)$$

for $s = (s_1, s_2, s_3)$, $r = (r_1, \dots, r_5)$, $r^u = (r_1^u, \dots, r_5^u)$, $r_v, r_v^u, s_1, s_2, s_3 \in \{1, 2\}$,

$$\sigma_{s;jmn}^i = \tau_{r;jm;n}^i + \zeta_{s_1}^i (T - \tau_r)_{jm}^\alpha - \zeta_{s_2}^\alpha (T - \tau_r)_{jm}^i - \zeta_{s_3}^\alpha (T - \tau_r)_{j\alpha}^i \quad (3.12)$$

$$\Theta_{r^1;jmn}^i = T_{jm}^\alpha \tau_{r^2;\alpha n}^i + T_{\alpha n}^i \tau_{r^1;jm}^\alpha - \tau_{r^1;jm}^\alpha \tau_{r^2;\alpha n}^i, \quad (3.13)$$

and the corresponding $\bar{\sigma}_{s;jmn}^i$ and $\bar{\Theta}_{r^1;jmn}^i$.

From equation (1.5) it turns out that:

$$\overset{0}{R}_{jmn}^i = R_{jmn}^i - u T_{jm;n}^i - u' T_{jn;m}^i - v T_{jm}^\alpha T_{\alpha n}^i - v' T_{jn}^\alpha T_{\alpha m}^i - w T_{mn}^\alpha T_{\alpha j}^i, \quad (3.14)$$

$$\overset{0}{R}_{ij} = R_{ij} - u T_{ij;\alpha}^\alpha - (v' + w) T_{i\beta}^\alpha T_{\alpha j}^\beta, \quad (3.15)$$

$$\overset{0}{R}_j^i = R_j^i - u g^{i\alpha} T_{\alpha j;\beta}^\beta - (v' + w) g^{i\alpha} T_{\alpha\gamma}^\beta T_{\beta j}^\gamma, \quad (3.16)$$

$$\overset{0}{R} = R - (v' + w) T_{\gamma\beta}^\alpha T_{\alpha\delta}^\beta g^{\gamma\delta}, \quad (3.17)$$

for Ricci-tensor $R_{ij} = R_{i\alpha}^\alpha$, mixed Ricci-curvature tensor $R_j^i = g^{i\alpha} R_{j\alpha}$ and the scalar curvature $R = R_\alpha^\alpha$.

After introducing the equations (3.9–3.17) into the equation (3.8), one obtains that

$$\bar{C}_{\rho;jmn}^i = C_{\rho;jmn}^i,$$

for

$$\begin{aligned}
C_{\rho}^i{}_{jmn} = & R_{jmn}^i + \frac{1}{N-2}(\delta_m^i R_{jn} - \delta_n^i R_{jm} + R_m^i g_{jn} - R_n^i g_{jm}) \\
& + \frac{R}{(N-1)(N-2)}(\delta_m^i g_{jn} - \delta_n^i g_{jm}) \\
& - \frac{u}{N-2}(\delta_m^i T_{jn;\alpha}^\alpha - \delta_n^i T_{jm;\alpha}^\alpha + g^{i\alpha} T_{\alpha m;\beta}^\beta g_{jn} - g^{i\alpha} T_{\alpha n;\beta}^\beta g_{jm}) \\
& - \frac{v'+w}{N-2}(\delta_m^i T_{j\beta}^\alpha T_{\alpha n}^\beta - \delta_n^i T_{j\beta}^\alpha T_{\alpha m}^\beta + g^{i\alpha} T_{\alpha\gamma}^\beta (T_{\beta m}^\gamma g_{jn} - T_{\beta n}^\gamma g_{jm})) \\
& - \frac{v'+w}{(N-1)(N-2)} T_{\gamma\beta}^\alpha T_{\alpha\delta}^\beta g^{\gamma\delta} (\delta_m^i g_{jn} - \delta_n^i g_{jm}) \\
& - \underbrace{\sigma_{s^1}^i}_{r^1} jmn - \underbrace{u' \sigma_{s^2}^i}_{r^2} jnm - \underbrace{v}_{r^4} \Theta_{jmn}^i - \underbrace{v' \Theta_{jnm}^i}_{r^6} - \underbrace{w \Theta_{mnj}^i}_{r^8},
\end{aligned} \tag{3.18}$$

$\rho = (s^1, s^2, r^1, \dots, r^8)$, and the corresponding $\overline{C}_{\rho}^i{}_{jmn}$.

The following theorem holds:

Theorem 1. *Let $f : \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\overline{\mathbb{R}}_N$ be a conformal mapping of generalized Riemannian space $\mathbb{G}\mathbb{R}_N$. The families $C_{\rho}^i{}_{jmn}$ of geometrical objects given by the equation (3.18) are families of invariants of the conformal mapping f . $\square$*

Corollary 2. *Let $f : \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\overline{\mathbb{R}}_N$ be an equitorsion conformal mapping. The family of geometrical objects*

$$\begin{aligned}
\widetilde{C}_s^i{}_{jmn} = & R_{jmn}^i + \frac{1}{N-2}(\delta_m^i R_{jn} - \delta_n^i R_{jm} + R_m^i g_{jn} - R_n^i g_{jm}) \\
& + \frac{R}{(N-1)(N-2)}(\delta_m^i g_{jn} - \delta_n^i g_{jm}) - u \widetilde{\sigma}_{s^1}^i{}_{jmn} - u' \widetilde{\sigma}_{s^2}^i{}_{jnm} \\
& - \frac{u}{N-2}(\delta_m^i \widetilde{\sigma}_{s^3}^\alpha{}_{jn\alpha} - \delta_n^i \widetilde{\sigma}_{s^4}^\alpha{}_{jma} + g^{i\alpha} \widetilde{\sigma}_{s^5}^\beta{}_{\alpha m\beta} g_{jn} - g^{i\alpha} \widetilde{\sigma}_{s^6}^\beta{}_{\alpha n\beta} g_{jm}),
\end{aligned} \tag{3.19}$$

for

$$\widetilde{\sigma}_{s^k}^i{}_{jmn} = \zeta_{s_1^k}^i{}_{\alpha n} T_{jm}^\alpha - \zeta_{s_2^k}^\alpha{}_{jn} T_{\alpha m}^i - \zeta_{s_3^k}^\alpha{}_{mn} T_{j\alpha}^i,$$

$k = 1, \dots, 6$, is the family of invariants of the mapping f . $\square$

4. CONCLUSION

Weyl conformal curvature tensor was generalized in this paper. We also obtained invariants of conformal mappings analogous to the Thomas projective parameter.

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REFERENCES

- [1] H. M. Abood, "Almost Hermitian Manifold with Flat Bochner Tensor." *Eur. J. Pure Appl. Math.*, vol. 3, no. 4, pp. 730–736, 2010.
- [2] S. Bochner, "Curvature and Betti Numbers. II." *Ann. Math. (2)*, vol. 50, no. 1, pp. 77–93, 1949, doi: [10.2307/1969353](https://doi.org/10.2307/1969353).
- [3] H. Chuda and J. Mikeš, "Conformally geodesic mappings satisfying a certain initial condition," *Arch. Math., Brno*, vol. 47, no. 5, pp. 389–394, 2011.
- [4] H. Chuda and M. Shiha, "Conformal holomorphically projective mappings satisfying a certain initial condition," *Miskolc Math. Notes*, vol. 14, no. 2, pp. 569–574, 2013.
- [5] A. Einstein, "A generalization of the relativistic theory of gravitation." *Ann. Math. (2)*, vol. 46, no. 4, pp. 578–584, 1945, doi: [10.2307/1969197](https://doi.org/10.2307/1969197).
- [6] A. Einstein, "Bianchi Identities in the Generalized Theory of Gravitation." *Can. J. Math.*, vol. 2, pp. 120–128, 1950.
- [7] L. P. Eisenhart, *Riemannian Geometry*. New York: Princeton University Press, 1967.
- [8] L. E. Evtushik, I. Hinterleitner, N. I. Guseva, and J. Mikeš, "Conformal mappings onto Einstein spaces," *Russ Math.*, vol. 60, no. 10, pp. 5–9, 2016, doi: [10.3103/S1066369X16100029](https://doi.org/10.3103/S1066369X16100029).
- [9] E. Goulart and M. Novello, "Cosmological Stability of Weyl Conformal Tensor," *Gravit. Cosmol.*, vol. 14, no. 4, pp. 321–326, 2008.
- [10] S. B. Mathur, "Conformal spaces and curvature tensors," *Indian J. Pure Appl. Math.*, vol. 2, pp. 29–31, 1971.
- [11] J. Mikeš, V. Kiosak, and A. Vanžurová, *Geodesic mappings of manifolds with affine connection*. Olomouc: Palacky University, 2008.
- [12] J. Mikeš, E. Stepanova, A. Vanžurová, and et al., *Differential Geometry of Special Mappings*. Olomouc: Palacky University, 2015.
- [13] J. Mikeš, A. Vanžurová, and I. Hinterleitner, *Geodesic Mappings and Some Generalizations*. Olomouc: Palacky University, 2009.
- [14] S. M. Minčič, "Ricci identities in the space of non-symmetric affine connection." *Mat. Vesnik*, vol. 10 (25), no. 2, pp. 161–172, 1973.
- [15] S. M. Minčič, "Independent curvature tensors and pseudotensors of spaces with non-symmetric affine connexion." *Colloq. Math. Soc. Janos Bolyai*, vol. 31, pp. 445–460, 1979.
- [16] M. S. Najdanović, M. L. Zlatanović, and I. Hinterleitner, "Conformal and Geodesic Mappings of Generalized Equidistant Spaces." *Publ. Inst. Math., Nouv. Ser.*, vol. 98, no. 112, pp. 71–84, 2015, doi: [10.2298/PIM1512071N](https://doi.org/10.2298/PIM1512071N).
- [17] M. Prvanović, "Projective and conformal transformations in recurrent and Ricci-recurrent Riemannian space." *Tensor, New Ser.*, vol. 12, no. 3, pp. 219–226, 1962.
- [18] N. S. Sinyukov, *Geodesic Mappings of Riemannian Spaces, (in Russian)*. Moskva: Nauka, 1979.
- [19] M. S. Stanković, L. S. Velimirović, S. M. Minčič, and M. L. Zlatanović, "Equitortion Conform Mappings of Generalized Riemannian Spaces." *Mat. Vesnik*, vol. 61, no. 2, pp. 119–129, 2009.
- [20] S. E. Stepanov, I. A. Alexandrova, I. I. Tsyganok, and J. Mikeš, "Conformal Killing forms on totally umbilical submanifolds." *J. Math. Sci., New York*, vol. 217, no. 5, pp. 525–539, 2016, doi: [10.1007/s10958-016-2989-5](https://doi.org/10.1007/s10958-016-2989-5).

- [21] S. E. Stepanov and I. I. Tsyganok, "Theorems of existence and non-existence of conformal Killing forms." *zv. Vyssh. Uchebn. Zaved. Mat.*, no. 10, pp. 54–61, 2014.
- [22] H. Zhang, Y. Zhang, and X.-Z. Li, "Dynamical spacetimes in conformal gravity." *Nucl. Phys., B*, vol. 921, pp. 522–537, 2017.
- [23] M. Zlatanović, I. Hinterleitner, and M. Najdanović, "On equitorsion concircular tensors of generalized Riemannian spaces." *Filomat*, vol. 28, no. 3, pp. 463–471, 2014, doi: [10.2298/FIL1403463Z](https://doi.org/10.2298/FIL1403463Z).

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GENERALIZED CORE INVERSES OF MATRICES

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Abstract. In this paper, we introduce two new generalized inverses of matrices, namely, the $\langle i, m \rangle$ -core inverse and the $\langle j, m \rangle$ -core inverse. The $\langle i, m \rangle$ -core inverse of a complex matrix extends the notions of the core inverse defined by Baksalary and Trenkler [1] and the core-EP inverse defined by Manjunatha Prasad and Mohana [10]. The $\langle j, m \rangle$ -core inverse of a complex matrix extends the notions of the core inverse and the DMP-inverse defined by Malik and Thome [9]. Moreover, the formulae and properties of these two new concepts are investigated by using matrix decompositions and matrix powers.

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1. INTRODUCTION

Let $\mathbb{C}^{m \times n}$ denote the set of all $m \times n$ complex matrices. Let A^* , $\mathcal{R}(A)$ and $\text{rk}(A)$ denote the conjugate transpose, column space, and rank of $A \in \mathbb{C}^{m \times n}$, respectively. For $A \in \mathbb{C}^{m \times n}$, if $X \in \mathbb{C}^{n \times m}$ satisfies $AXA = A$, $XAX = X$, $(AX)^* = AX$, and $(XA)^* = XA$, then X is called a *Moore-Penrose inverse* of A . This matrix X is unique and denoted by $A^\dagger$. A matrix $X \in \mathbb{C}^{n \times m}$ is called an *outer inverse* of A if it satisfies $XAX = X$; is called a $\{2, 3\}$ -inverse of A if it satisfies $XAX = X$ and $(AX)^* = AX$; is called a $\{1, 3\}$ -inverse of A if it satisfies $AXA = A$ and $(AX)^* = AX$; is called a $\{1, 2, 3\}$ -inverse of A if it satisfies $AXA = A$, $XAX = X$ and $(AX)^* = AX$.

The core inverse of a complex matrix was introduced by Baksalary and Trenkler [1]. Let $A \in \mathbb{C}^{n \times n}$. A matrix $X \in \mathbb{C}^{n \times n}$ is called a *core inverse* of A , if it satisfies $AX = P_A$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A)$, here P_A denotes the orthogonal projector onto $\mathcal{R}(A)$. If such a matrix exists, then it is unique and denoted by $A^\oplus$. For a square complex matrix A , one has that A is core invertible, A is group invertible, and $\text{rk}(A) = \text{rk}(A^2)$ are three equivalent conditions (see [2]). We denote $\mathbb{C}_n^{CM} = \{A \in \mathbb{C}^{n \times n} \mid \text{rk}(A) = \text{rk}(A^2)\}$.

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Let $A \in \mathbb{C}^{n \times n}$. A matrix $X \in \mathbb{C}^{n \times n}$ such that $XA^{k+1} = A^k$, $XAX = X$ and $AX = XA$ is called the *Drazin inverse* of A and denoted by A^D . The Drazin inverse of a square matrix always exists and it is unique. Such integer k is called the Drazin index of A , denoted by $\text{ind}(A)$. If $\text{ind}(A) \leq 1$, then the Drazin inverse of A is called the *group inverse* and denoted by $A^\#$.

The DMP-inverse for a complex matrix was introduced by Malik and Thome [9]. Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$. A matrix $X \in \mathbb{C}^{n \times n}$ is called a *DMP-inverse* of A , if it satisfies $XAX = X$, $XA = A^D A$ and $A^k X = A^k A^\dagger$. It is unique and denoted by $A^{D,\dagger}$. Malik and Thome gave several characterizations of the DMP-inverse by using the decomposition of Hartwig and Spindelböck [8].

The notion of the core-EP inverse for a complex matrix was introduced by Manjunatha Prasad and Mohana [10]. A matrix $X \in \mathbb{C}^{n \times n}$ is a *core-EP inverse* of $A \in \mathbb{C}^{n \times n}$ if X is an outer inverse of A satisfying $\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$, where k is the index of A . The core-EP inverse is unique and denoted by $A^\oplus$.

In addition, $\mathbf{1}_n$ and $\mathbf{0}_n$ will denote the $n \times 1$ column vectors all of whose components are 1 and 0, respectively. $0_{m \times n}$ (abbr. 0) denotes the zero matrix of size $m \times n$. If $\mathcal{S}$ is a subspace of $\mathbb{C}^n$, then $P_{\mathcal{S}}$ stands for the *orthogonal projector* onto the subspace $\mathcal{S}$. A matrix $A \in \mathbb{C}^{n \times n}$ is called an *EP matrix* if $\mathcal{R}(A) = \mathcal{R}(A^*)$, A is called *Hermitian* if $A^* = A$ and A is *unitary* if $AA^* = I_n$, where I_n denote the *identity matrix* of size n . Let $\mathbb{N}$ denote the set of positive integers.

2. PRELIMINARIES

A related decomposition of the matrix decomposition of Hartwig and Spindelböck [8] was given in [2, Theorem 2.1] by Benítez, in [3] it can be found a simpler proof of this decomposition. Let us start this section with the concept of principal angles.

Definition 1 ([12]). Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two nontrivial subspaces of $\mathbb{C}^n$. We define the *principal angles* $\theta_1, \dots, \theta_r \in [0, \pi/2]$ between $\mathcal{S}_1$ and $\mathcal{S}_2$ by

$$\cos \theta_i = \sigma_i(P_{\mathcal{S}_1} P_{\mathcal{S}_2}),$$

for $i = 1, \dots, r$, where $r = \min\{\dim \mathcal{S}_1, \dim \mathcal{S}_2\}$. The real numbers $\sigma_i(P_{\mathcal{S}_1} P_{\mathcal{S}_2}) \geq 0$ are the singular values of $P_{\mathcal{S}_1} P_{\mathcal{S}_2}$.

The following theorem can be found in [2, Theorem 2.1].

Theorem 1. Let $A \in \mathbb{C}^{n \times n}$, $r = \text{rk}(A)$, and let $\theta_1, \dots, \theta_p$ be the principal angles between $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$ belonging to $]0, \pi/2[$. Denote by x and y the multiplicities of the angles 0 and $\pi/2$ as a canonical angle between $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$, respectively. There exists a unitary matrix $U \in \mathbb{C}^{n \times n}$ such that

$$A = U \begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix} U^*, \quad (2.1)$$

where $M \in \mathbb{C}^{r \times r}$ is nonsingular,

$$C = \text{diag}(\mathbf{0}_y, \cos \theta_1, \dots, \cos \theta_p, \mathbf{1}_x),$$

$$S = \begin{bmatrix} \text{diag}(\mathbf{1}_y, \sin \theta_1, \dots, \sin \theta_p) & 0_{p+y, n-(r+p+y)} \\ 0_{x, p+y} & 0_{x, n-(r+p+y)} \end{bmatrix},$$

and $r = y + p + x$. Furthermore, x and $y + n - r$ are the multiplicities of the singular values 1 and 0 in $P_{\mathcal{R}(A)} P_{\mathcal{R}(A^*)}$, respectively.

In this decomposition, one has $C^2 + SS^* = I_r$. Recall that $A^\dagger$ always exists. We have that $A^\#$ exists if and only if C is nonsingular in view of [2, Theorem 3.7]. The following equalities hold

$$A^\dagger = U \begin{bmatrix} CM^{-1} & 0 \\ S^*M^{-1} & 0 \end{bmatrix} U^*, \quad A^\# = U \begin{bmatrix} C^{-1}M^{-1} & C^{-1}M^{-1}C^{-1}S \\ 0 & 0 \end{bmatrix} U^*.$$

By [3, Theorem 2], we have that

$$A^D = U \begin{bmatrix} (MC)^D & [(MC)^D]^2MS \\ 0 & 0 \end{bmatrix} U^*. \quad (2.2)$$

We also have

$$AA^\dagger = U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^*, \quad (2.3)$$

$$A^\oplus = A^\# AA^\dagger = U \begin{bmatrix} C^{-1}M^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*. \quad (2.4)$$

Lemma 1 ([13, Theorem 3.1]). *Let $A \in \mathbb{C}^{n \times n}$. Then A is core invertible if and only if there exists $X \in \mathbb{C}^{n \times n}$ such that $(AX)^* = AX$, $XA^2 = A$ and $AX^2 = X$. In this situation, we have $A^\oplus = X$.*

Lemma 2. *Let $A \in \mathbb{C}^{n \times n}$. If there exists $X \in \mathbb{C}^{n \times n}$ such that $AX^{k+1} = X^k$ and $XA^{k+1} = A^k$ for some $k \in \mathbb{N}$, then for $m \in \mathbb{N}$ we have*

- (1) $A^k = X^m A^{k+m}$;
- (2) $X^k = A^m X^{k+m}$;
- (3) $A^k X^k = A^{k+m} X^{k+m}$;
- (4) $X^k A^k = X^{k+m} A^{k+m}$;
- (5) $A^k = A^m X^m A^k$;
- (6) $X^k = X^m A^m X^k$.

Proof. (1). For $m = 1$, it is clear by the hypotheses. If the formula is true for $m \in \mathbb{N}$, then $X^{m+1} A^{k+m+1} = XX^m A^{k+m} A = XA^k A = XA^{k+1} = A^k$.

(3). It is easy to check that $A^k X^k = A^{k+1} X^{k+1}$ by $AX^{k+1} = X^k$. It is not difficult to check the equality $A^k X^k = A^{k+m} X^{k+m}$ by induction.

(5). From (1) we have $A^k = X^k A^{2k}$. Thus by $AX^{k+1} = X^k$, we have $A^k = X^k A^{2k} = AX^{k+1} A^{2k} = AX^k XA^{2k} = A(AX^{k+1})XA^{2k} = A^2 X^{k+2} A^{2k} = A^2 X^2 X^k A^{2k} = \dots = A^m X^m X^k A^{2k} = A^m X^m A^k$.

The proofs of (2), (4), and (6) are similar to the proofs of (1), (3), (5), respectively. $\square$

Lemma 3. *Let $A \in \mathbb{C}^{n \times n}$. If there exists $X \in \mathbb{C}^{n \times n}$ such that $AX^{k+1} = X^k$ and $XA^{k+1} = A^k$ for some $k \in \mathbb{N}$, then $A^D = X^{k+1}A^k$.*

Proof. Since A is Drazin invertible. We will check that $A^D = X^{k+1}A^k$. Have in mind, $AX^{k+1} = X^k$ and $XA^{k+1} = A^k$, thus

$$A(X^{k+1}A^k) = X^kA^k = X^k(XA^{k+1}) = X^{k+1}A^kA. \quad (2.5)$$

That is, $X^{k+1}A^k$ and A commute. Then by (1) and (4) in Lemma 2, we have that

$$\begin{aligned} (X^{k+1}A^k)A(X^{k+1}A^k) &= X^{k+1}A^{k+1}X^{k+1}A^k = X^kA^k(X^{k+1}A^k) \\ &= X^kX^{k+1}A^kA^k = X^{k+1}X^kA^{2k} = X^{k+1}A^k. \end{aligned} \quad (2.6)$$

From (1) in Lemma 2, we have that

$$(X^{k+1}A^k)A^{k+1} = X(X^kA^{2k})A = XA^{k+1} = A^k. \quad (2.7)$$

Thus we have $A^D = X^{k+1}A^k$ by the definition of the Drazin inverse and in view of (2.5), (2.6), and (2.7). $\square$

Remark 1. From the proofs of Lemma 2 and Lemma 3, it is obvious that Lemma 2 and Lemma 3 are valid for rings. Moreover, we can get that for an element $a \in R$, a is Drazin invertible if and only if there exist $x \in R$ and $k \in \mathbb{N}$ such that $ax^{k+1} = x^k$ and $xa^{k+1} = a^k$, where R is a ring.

The following lemma is similar to [9, Theorem 2.5].

Lemma 4. *Let $A \in \mathbb{C}^{n \times n}$ be the form (2.1). Then*

$$A^{D,\dagger} = U \begin{bmatrix} (MC)^D & 0 \\ 0 & 0 \end{bmatrix} U^*. \quad (2.8)$$

Proof. By (2.2), (2.3) and the definition of DMP-inverse we have

$$\begin{aligned} A^{D,\dagger} &= A^D A A^\dagger \\ &= U \begin{bmatrix} (MC)^D & [(MC)^D]^2 MS \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} (MC)^D & 0 \\ 0 & 0 \end{bmatrix} U^*. \end{aligned}$$

$\square$

Lemma 5 ([11, Corollary 3.3]). *Let $A \in \mathbb{C}^{n \times n}$ be a matrix of index k . Then $AA^\oplus = A^k(A^k)^\dagger$.*

3. $\langle i, m \rangle$ -CORE INVERSE

Let us start this section by introducing the definition of the $\langle i, m \rangle$ -core inverse.

Definition 2. Let $A \in \mathbb{C}^{n \times n}$ and $m, i \in \mathbb{N}$. A matrix $X \in \mathbb{C}^{n \times n}$ is called an $\langle i, m \rangle$ -core inverse of A , if it satisfies

$$X = A^D A X \quad \text{and} \quad A^m X = A^i (A^i)^\dagger. \quad (3.1)$$

It will be proved that if X exists, then it is unique and denoted by $A_{i,m}^\oplus$.

Theorem 2. Let $A \in \mathbb{C}^{n \times n}$. If exists $X \in \mathbb{C}^{n \times n}$ such that (3.1) holds, then X is unique.

Proof. Assume that X satisfies the system in (3.1), that is $X = A^D A X$ and $A^m X = A^i (A^i)^\dagger$. Thus $X = A^D A X = (A^D)^m A^m X = (A^D)^m A^i (A^i)^\dagger$. Therefore, X is unique by the uniqueness of A^D and $A^i (A^i)^\dagger$. $\square$

Theorem 3. The system in (3.1) is consistent if and only if $i \geq \text{ind}(A)$. In this case, the solution of (3.1) is $X = (A^D)^m A^i (A^i)^\dagger$.

Proof. Assume that $i \geq \text{ind}(A)$. Let $X = (A^D)^m A^i (A^i)^\dagger$. We have

$$\begin{aligned} A^D A X &= A^D A (A^D)^m A^i (A^i)^\dagger = (A^D)^m A^D A A^i (A^i)^\dagger = (A^D)^m A^i (A^i)^\dagger = X \\ A^m X &= A^m (A^D)^m A^i (A^i)^\dagger = A^D A A^i (A^i)^\dagger = A^i (A^i)^\dagger. \end{aligned}$$

Thus, the system in (3.1) is consistent and the solution of (3.1) is $X = (A^D)^m A^i (A^i)^\dagger$.

If the system in (3.1) is consistent, then exists X_0 such that $X_0 = A^D A X_0$ and $A^m X_0 = A^i (A^i)^\dagger$. Then $X_0 = A^D A X_0 = (A^D)^m A^m X_0 = (A^D)^m A^i (A^i)^\dagger$ and $A^i (A^i)^\dagger = A^m X_0 = A^m (A^D)^m A^i (A^i)^\dagger = A A^D A^i (A^i)^\dagger$. Hence $A^i = A^i (A^i)^\dagger A^i = A A^D A^i (A^i)^\dagger A^i = A A^D A^i$, that is $i \geq \text{ind}(A)$. $\square$

Example 1. We will give an example that shows if $i < \text{ind}(A)$, then the system in (3.1) is not consistent. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. It is easy to get $\text{ind}(A) = 2$ and $A^D = 0$. Let $i = 1$ and suppose that X is the solution of system in (3.1), then $X = A^D A X = 0$, which gives $A A^\dagger = A^m X = 0$, thus $A = A A^\dagger A = 0$, this is a contradiction.

Remark 2. If $i \geq \text{ind}(A)$, then $A_{i,m+1}^\oplus = A^D A_{i,m}^\oplus$.

Remark 3. The $\langle i, m \rangle$ -core inverse is a generalization of the core inverse and the core-EP inverse. More precisely, we have the following statements:

- (1) If $m = i = \text{ind}(A) = 1$, then the $\langle 1, 1 \rangle$ -core inverse coincides with the core inverse;
- (2) If $m = 1$ and $i = \text{ind}(A)$, then the $\langle i, 1 \rangle$ -core inverse coincides with the core-EP inverse.

For the convenience of the readers, in the following, we give some notes of (1) and (2) in Remark 3.

- (1) If $m = i = \text{ind}(A) = 1$, then A is group invertible and $A^D = A^\#$ and (3.1) is equivalent to $X = A^\#AX$ and $AX = AA^\dagger$. Thus $X = A^\#AX = A^\#AA^\dagger$, $(AX)^* = (AA^\dagger)^* = AA^\dagger = AX$, $AX^2 = AA^\#AA^\dagger A^\#AA^\dagger = AA^\#AA^\dagger AA^\#A^\dagger = AA^\#A^\dagger = X$ and $XA^2 = A^\#AA^\dagger A^2 = A^\#A^2 = A$. Hence, $\langle 1, 1 \rangle$ -core inverse coincides with the core inverse by Lemma 1. Note that if A is group invertible, then we have that X is the core inverse of A if and only if $X = A^\#AX$ and $AX = AA^\dagger$.
- (2) If $m = 1$ and $i = \text{ind}(A)$, then by Theorem 3.3, $A_{i,1}^\oplus$ exists and $A_{i,1}^\oplus = A^D A^i (A^i)^\dagger$. Let us denote $X = A_{i,1}^\oplus = A^D A^i (A^i)^\dagger$. Observe that $AX = A^i (A^i)^\dagger$ is Hermitian. Now,

$$XAX = A^D A^i (A^i)^\dagger A^i (A^i)^\dagger = A^D A^i (A^i)^\dagger = X,$$

that is X is an outer inverse of A . From

$$A^i = A^D A^{i+1} = A^D A^i (A^i)^\dagger A^i A = XA^{i+1}$$

we get $\mathcal{R}(A^i) \subseteq \mathcal{R}(X)$. Also,

$$\begin{aligned} AX^2 &= (AX)X = A^i (A^i)^\dagger A^D A^i (A^i)^\dagger \\ &= A^i (A^i)^\dagger A^i A^D (A^i)^\dagger = A^D A^i (A^i)^\dagger = X, \end{aligned}$$

which implies $X = (AX)^*X \in \mathcal{R}(X^*)$, therefore, $\mathcal{R}(X) \subseteq \mathcal{R}(X^*)$. Finally, $X^* = [A^D A^i (A^i)^\dagger]^* = A^i (A^i)^\dagger (A^D)^*$ implies $\mathcal{R}(X^*) \subseteq \mathcal{R}(A^i)$. Hence $\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^i)$. Therefore, the $\langle i, 1 \rangle$ -core inverse coincides with the core-EP inverse by the definition of the core-EP inverse.

From the above statement, we have the following theorem.

Theorem 4. *Let $A \in \mathbb{C}^{n \times n}$ with $i = \text{ind}(A)$. Then X is the core-EP inverse of A if and only if $X = A^D AX$ and $AX = A^i (A^i)^\dagger$.*

Corollary 1. *Let $A \in \mathbb{C}^{n \times n}$ with $1 = \text{ind}(A)$. Then X is the core inverse of A if and only if $X = A^\#AX$ and $AX = AA^\dagger$.*

For any $A \in \mathbb{C}^{n \times n}$, either $A^l = 0$ for some $l \in \mathbb{N}$, or $A^l \neq 0$ for all positive integers. Moreover, if $\text{ind}(A) = k$, then $G_k B_k$ is nonsingular (see [5–7]), where $A = B_1 G_1$ is a full rank factorization of A and $G_l B_l = B_{l+1} G_{l+1}$ is a full rank factorization of $G_l B_l$, $l = 1, \dots, k-1$. When $A^k \neq 0$, then it can be written as

$$A^k = \prod_{l=1}^k B_l \prod_{l=1}^k G_{k+1-l}. \quad (3.2)$$

We have the following results, (see [5, Theorem 4] or [4, Theorem 7.8.2]):

$$\text{ind}(A) = \begin{cases} k & \text{when } G_k B_k \text{ is nonsingular,} \\ k+1 & \text{when } G_k B_k = 0. \end{cases}$$

and

$$A^D = \begin{cases} \prod_{l=1}^k B_l (G_k B_k)^{-k-1} \prod_{l=1}^k G_{k+1-l} & \text{when } G_k B_k \text{ is nonsingular,} \\ 0 & \text{when } G_k B_k = 0. \end{cases} \quad (3.3)$$

In the sequel, we always assume that $A^k \neq 0$.

It is well-known that if $A = EF$ is a full rank factorization of A , where $r = \text{rk}(A)$, $E \in \mathbb{C}^{n \times r}$ and $F \in \mathbb{C}^{r \times n}$, then (see [4, Theorem 1.3.2])

$$A^\dagger = F^* (FF^*)^{-1} (E^* E)^{-1} E^*. \quad (3.4)$$

Remark 4. The notations and results in above paragraph will be used many times in the sequel.

We will investigate the $\langle i, m \rangle$ -core inverse of a matrix $A \in \mathbb{C}^{n \times n}$ by using Remark 4.

Theorem 5. Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$. If $i \geq k$, then $A_{i,m}^\oplus = A_{k,m}^\oplus$.

Proof. Since $\text{ind}(A) = k$, we have $\mathcal{R}(A^k) = \mathcal{R}(A^i)$ for any $i \geq k$, and therefore, $A^k (A^k)^\dagger = A^i (A^i)^\dagger$. Now, the conclusion follows from Theorem 3. $\square$

Remark 5. The proof of Theorem 5 also can be proved as follows. Since the proof in this remark will be used several times in the sequel, we write this proof here.

Proof. If A is nilpotent, then $A^D = 0$, hence by Theorem 3, one has $A_{i,m}^\oplus = A_{k,m}^\oplus = 0$. Therefore, we can assume that $A^k \neq 0$. By equality (3.2), we have

$$A^k = \prod_{l=1}^k B_l \prod_{l=1}^k G_{k+1-l}. \quad (3.5)$$

where $A = B_1 G_1$ is a full rank factorization of A and $G_l B_l = B_{l+1} G_{l+1}$ is a full rank factorization of $G_l B_l$, $l = 1, \dots, k-1$. Let $M = \prod_{l=1}^k B_l$, $N = \prod_{l=1}^k G_{k+1-l}$ and $L = G_k B_k$. Now, we will show that

$$A^i = \prod_{l=1}^k B_l (G_k B_k)^{i-k} \prod_{l=1}^k G_{k+1-l} = M L^{i-k} N.$$

In fact,

$$\begin{aligned}
A^i &= \prod_{l=1}^i B_l \prod_{l=1}^i G_{k+1-l} \\
&= B_1 \cdots B_i G_i \cdots G_1 \\
&= B_1 \cdots B_{i-1} (B_i G_i) G_{i-1} \cdots G_1 \\
&= B_1 \cdots B_{i-1} (G_{i-1} B_{i-1}) G_{i-1} \cdots G_1 \\
&= B_1 \cdots B_{i-2} (G_{i-2} B_{i-2})^2 G_{i-2} \cdots G_1 \\
&= \cdots \\
&= B_1 \cdots B_k (G_k B_k)^{i-k} G_k \cdots G_1 = ML^{i-k} N.
\end{aligned} \tag{3.6}$$

If we let $M_1 = ML^{i-k}$, then $A^i = ML^{i-k} N = M_1 N$ is a full rank factorization of A^i (see [7, p.183]). Thus

$$(A^i)^\dagger = N^* (NN^*)^{-1} (M_1^* M_1)^{-1} M_1^*. \tag{3.7}$$

Note that $NM = \prod_{l=1}^k G_{k+1-l} \prod_{l=1}^k B_l = L^k$. By Theorem 3, (3.3) and (3.7) we have

$$\begin{aligned}
A_{i,1}^\oplus &= A^D A^i (A^i)^\dagger \\
&= ML^{-k-1} NML^{i-k} N (A^i)^\dagger \\
&= ML^{-k-1} NML^{i-k} NN^* (NN^*)^{-1} (M_1^* M_1)^{-1} M_1^* \\
&= ML^{i-k-1} NN^* (NN^*)^{-1} (M_1^* M_1)^{-1} M_1^* \\
&= ML^{i-k-1} (M_1^* M_1)^{-1} M_1^* \\
&= ML^{i-k-1} [(L^{i-k})^* M^* ML^{i-k}]^{-1} (L^{i-k})^* M^* \\
&= ML^{i-k-1} L^{k-i} (M^* M)^{-1} [(L^{i-k})^*]^{-1} (L^{i-k})^* M^* \\
&= ML^{-1} (M^* M)^{-1} M^*.
\end{aligned} \tag{3.8}$$

The last expression does not depend on i , then $A_{i,1}^\oplus = A_{k,1}^\oplus$. Thus, by Remark 2, we have $A_{i,m}^\oplus = A^D A_{i,m-1}^\oplus = A^D (A^D A_{i,m-2}^\oplus) = (A^D)^2 A_{i,m-2}^\oplus = \cdots = (A^D)^{m-1} A_{i,1}^\oplus = (A^D)^{m-1} A_{k,1}^\oplus = A_{k,m}^\oplus$. $\square$

Remark 6. By Theorem 5, it is enough to investigate the $i = \text{ind}(A) = k$ case, when we discuss the $\langle i, m \rangle$ -core inverse of a matrix $A \in \mathbb{C}^{n \times n}$. That is, the Theorem 5 is a key theorem.

Theorem 6. Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$ and $k, m \in \mathbb{N}$. If $A = B_1 G_1$ is a full rank factorization of A and $G_l B_l = B_{l+1} G_{l+1}$ is a full rank factorization of $G_l B_l$,

$l = 1, \dots, k-1$, then $A_{k,m}^\oplus = ML^{-m}M^\dagger$, where $M = \prod_{l=1}^k B_l$, $N = \prod_{l=1}^k G_{k+1-l}$ and $L = G_k B_k$.

Proof. By the proof of Remark 5, we have $A_{k,1}^\oplus = ML^{-1}(M^*M)^{-1}M^*$ and $NM = L^k$. Now, we will prove $(A^D)^s A_{k,1}^\oplus = ML^{-s-1}(M^*M)^{-1}M^*$ for any $s \in \mathbb{N}$. By (3.3) we have $A^D = \prod_{l=1}^k B_l (G_k B_k)^{-k-1} \prod_{l=1}^k G_{k+1-l} = ML^{-k-1}N$. When $s = 1$, we have

$$\begin{aligned} A^D A_{k,1}^\oplus &= ML^{-k-1}NML^{-1}(M^*M)^{-1}M^* = ML^{-k-1}(NM)L^{-1}(M^*M)^{-1}M^* \\ &= ML^{-k-1}L^k L^{-1}(M^*M)^{-1}M^* = ML^{-2}(M^*M)^{-1}M^*. \end{aligned}$$

Assume that $(A^D)^{s-1} A_{k,1}^\oplus = ML^{-s}(M^*M)^{-1}M^*$. Then

$$\begin{aligned} (A^D)^s A_{k,1}^\oplus &= A^D (A^D)^{s-1} A_{k,1}^\oplus = A^D ML^{-s}(M^*M)^{-1}M^* \\ &= ML^{-k-1}NML^{-s}(M^*M)^{-1}M^* \\ &= ML^{-k-1}L^k L^{-s}(M^*M)^{-1}M^* \\ &= ML^{-s-1}(M^*M)^{-1}M^*. \end{aligned}$$

Thus by Remark 2, we have

$$A_{k,m}^\oplus = (A^D)^{m-1} A_{k,1}^\oplus = ML^{-m}(M^*M)^{-1}M^* = ML^{-m}M^\dagger.$$

□

In the following theorem, we will give a canonical form for the $\langle k, m \rangle$ -core inverse of a matrix $A \in \mathbb{C}^{n \times n}$ by using the matrix decomposition in Theorem 1. We will also use the following simple fact: Let $X \in \mathbb{C}^{n \times m}$ and $\mathbf{b} \in \mathbb{C}^n$. If $\mathbf{y} \in \mathbb{C}^m$ satisfies $X^*X\mathbf{y} = X^*\mathbf{b}$, then $XX^\dagger\mathbf{b} = X\mathbf{y}$.

Theorem 7. Let $A \in \mathbb{C}^{n \times n}$ have the form (2.1) with $\text{ind}(A) = k$ and $m \in \mathbb{N}$. Then

$$A_{k,m}^\oplus = U \begin{bmatrix} (MC)_{k-1,m}^\oplus & 0 \\ 0 & 0 \end{bmatrix} U^*. \quad (3.9)$$

Proof. Let r be the rank of A . By Theorem 3 we have

$$A_{k,m}^\oplus = (A^D)^m A^k (A^k)^\dagger. \quad (3.10)$$

Since A has the form given in Theorem 1 we have

$$A^k = U \begin{bmatrix} (MC)^k & (MC)^{k-1}MS \\ 0 & 0 \end{bmatrix} U^*. \quad (3.11)$$

Let $\mathbf{b} \in \mathbb{C}^n$ be arbitrary and let us decompose $\mathbf{b} = U \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$, where $\mathbf{b}_1 \in \mathbb{C}^r$. Let $\mathbf{x}_0 \in \mathbb{C}^n$ satisfy $(A^k)^* A^k \mathbf{x}_0 = (A^k)^* \mathbf{b}$ [this $\mathbf{x}_0$ always exists because the normal

equations always have a solution]. We can decompose $\mathbf{x}_0$ by writing $\mathbf{x}_0 = U \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$, where $\mathbf{x}_1 \in \mathbb{C}^r$. Let us denote $N = (MC)^{k-1}M$. Using (3.11),

$$U \begin{bmatrix} CN^* & 0 \\ S^*N^* & 0 \end{bmatrix} \begin{bmatrix} NC & NS \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = U \begin{bmatrix} CN^* & 0 \\ S^*N^* & 0 \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}.$$

Therefore,

$$CN^*N(C\mathbf{x}_1 + S\mathbf{x}_2) = CN^*\mathbf{b}_1 \quad \text{and} \quad S^*N^*N(C\mathbf{x}_1 + S\mathbf{x}_2) = S^*N^*\mathbf{b}_1.$$

Premultiplying the first equality by C and the second equality by S and after, adding them, we get $N^*N(C\mathbf{x}_1 + S\mathbf{x}_2) = N^*\mathbf{b}_1$, and hence, $N(C\mathbf{x}_1 + S\mathbf{x}_2) = NN^\dagger\mathbf{b}_1$. Now,

$$\begin{aligned} A^k(A^k)^\dagger\mathbf{b} &= A^k\mathbf{x}_0 = U \begin{bmatrix} NC & NS \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \\ &= U \begin{bmatrix} NC\mathbf{x}_1 + NS\mathbf{x}_2 \\ \mathbf{0} \end{bmatrix} = U \begin{bmatrix} NN^\dagger\mathbf{b}_1 \\ \mathbf{0} \end{bmatrix} = U \begin{bmatrix} NN^\dagger & 0 \\ 0 & 0 \end{bmatrix} U^*\mathbf{b}. \end{aligned}$$

Since $\mathbf{b}$ is arbitrary,

$$A^k(A^k)^\dagger = U \begin{bmatrix} NN^\dagger & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

Now we will prove $NN^\dagger = (MC)^{k-1}[(MC)^{k-1}]^\dagger$. Recall that we have $N = (MC)^{k-1}M$, and so, $\mathcal{R}(N) \subseteq \mathcal{R}((MC)^{k-1})$. Since M is nonsingular, $\text{rk}(N) = \text{rk}((MC)^{k-1})$, and thus, $\mathcal{R}(N) = \mathcal{R}((MC)^{k-1})$. Since $(MC)^{k-1}[(MC)^{k-1}]^\dagger$ and $NN^\dagger$ are the orthogonal projectors onto $\mathcal{R}((MC)^{k-1})$ and $\mathcal{R}(N)$, respectively, we get $NN^\dagger = (MC)^{k-1}[(MC)^{k-1}]^\dagger$.

By (2.2) we have

$$A^D = U \begin{bmatrix} (MC)^D & [(MC)^D]^2MS \\ 0 & 0 \end{bmatrix} U^*. \quad (3.12)$$

Thus, we have

$$(A^D)^m = U \begin{bmatrix} [(MC)^D]^m & [(MC)^D]^{m+1}MS \\ 0 & 0 \end{bmatrix} U^*.$$

Since $\text{ind}(A) = k$, we have $A^D A^{k+1} = A^k$. By using the above representations of A^D and A^k given in (3.11) and (3.12), respectively,

$$\begin{aligned} \begin{bmatrix} (MC)^D & [(MC)^D]^2MS \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (MC)^{k+1} & (MC)^kMS \\ 0 & 0 \end{bmatrix} \\ = \begin{bmatrix} (MC)^k & (MC)^{k-1}MS \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Therefore,

$$(MC)^D(MC)^kM[C \mid S] = (MC)^{k-1}M[C \mid S]. \quad (3.13)$$

Have in mind that we have $C^2 + SS^* = I_r$. Thus, postmultiplying (3.13) by $\begin{bmatrix} C \\ S^* \end{bmatrix}$ gives us $(MC)^D(MC)^k M = (MC)^{k-1} M$ and from the nonsingularity of M we obtain $(MC)^D(MC)^k = (MC)^{k-1}$, and so, $\text{ind}(MC) \leq k-1$. Therefore we have

$$\begin{aligned} A_{k,m}^\oplus &= (A^D)^m A^k (A^k)^\dagger \\ &= U \begin{bmatrix} [(MC)^D]^m & [(MC)^D]^{m+1} MS \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (MC)^{k-1} ((MC)^{k-1})^\dagger & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} [(MC)^D]^m (MC)^{k-1} ((MC)^{k-1})^\dagger & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} (MC)_{k-1,m}^\oplus & 0 \\ 0 & 0 \end{bmatrix} U^*. \end{aligned}$$

□

Remark 7. If we use the decomposition of Hartwig and Spindelböck in [8, Corollary 6], then an expression of the $\langle k, m \rangle$ -core inverse of A is

$$A_{k,m}^\oplus = U \begin{bmatrix} (\Sigma K)_{k-1,m}^\oplus & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

which is similar to the expression of $A_{k,m}^\oplus$ in Theorem 7. Since the proof of this result can be proved as the proof of Theorem 7, we omit this proof.

Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$. The Jordan Canonical form of A is $P^{-1}AP = J$, where $P \in \mathbb{C}^{n \times n}$ is nonsingular and $J \in \mathbb{C}^{n \times n}$ is a block diagonal matrix composed of Jordan blocks. In the following theorem, we will compute the $\langle k, m \rangle$ -core inverse by using the Jordan Canonical form of A .

Theorem 8. Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$, then $A_{k,m}^\oplus = P_1 D^{-m} P_1^\dagger$, where $A = P \begin{bmatrix} D & 0 \\ 0 & N \end{bmatrix} P^{-1}$ with $D \in \mathbb{C}^{r \times r}$ is nonsingular, N is nilpotent and $P = [P_1 \mid P_2]$ with $P_1 \in \mathbb{C}^{n \times r}$.

Proof. The Jordan Canonical form of A is $P^{-1}AP = J$, where $P \in \mathbb{C}^{n \times n}$ is nonsingular and $J \in \mathbb{C}^{n \times n}$ is a block diagonal matrix. Rearrange the elements of J such that $A = P \begin{bmatrix} D & 0 \\ 0 & N \end{bmatrix} P^{-1}$, where D is nonsingular and N is nilpotent. It is well-known that $A^D = P \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$ and $A^k = P \begin{bmatrix} D^k & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$. If we let $P = [P_1 \mid P_2]$ and $P^{-1} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$, then

$$(A^D)^m A^k = [P_1 \mid P_2] \begin{bmatrix} (D^{-1})^m & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D^k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = P_1 D^{k-m} Q_1.$$

Observe that $A^k = (P_1 D^k) Q_1$ is a full rank factorization of A^k . Hence by (3.4) we have

$$\begin{aligned}
 (A^k)^\dagger &= (P_1 D^k Q_1)^\dagger \\
 &= Q_1^* (Q_1 Q_1^*)^{-1} [(P_1 D^k)^* P_1 D^k]^{-1} (P_1 D^k)^* \\
 &= Q_1^* (Q_1 Q_1^*)^{-1} D^{-k} (P_1^* P_1)^{-1} [(D^k)^*]^{-1} (D^k)^* P_1^* \\
 &= Q_1^* (Q_1 Q_1^*)^{-1} D^{-k} (P_1^* P_1)^{-1} P_1^* \\
 &= Q_1^\dagger D^{-k} P_1^\dagger.
 \end{aligned}$$

By Theorem 3, we have $A_{k,m}^\oplus = (A^D)^m A^k (A^k)^\dagger$. Thus we have

$$\begin{aligned}
 A_{k,m}^\oplus &= (A^D)^m A^k (A^k)^\dagger = P_1 D^{k-m} Q_1 Q_1^\dagger D^{-k} P_1^\dagger \\
 &= P_1 D^{k-m} Q_1 Q_1^* (Q_1 Q_1^*)^{-1} D^{-k} P_1^\dagger = P_1 D^{-m} D^k D^{-k} P_1^\dagger = P_1 D^{-m} P_1^\dagger.
 \end{aligned}$$

□

Proposition 1. *Let $A \in \mathbb{C}^{n \times n}$. If $i \geq \text{ind}(A)$, then $A^m A_{i,m}^\oplus$ is the projector onto $\mathcal{R}(A^i)$ along $\mathcal{R}(A^i)^\perp$.*

Proof. It is trivial. □

In the following proposition, we will investigate some properties of the $\langle i, m \rangle$ -core inverse.

Proposition 2. *Let $A \in \mathbb{C}^{n \times n}$, $m, i \in \mathbb{N}$. If $i \geq \text{ind}(A)$, then*

- (1) $A_{i,m}^\oplus$ is a $\{2, 3\}$ -inverse of A^m ;
- (2) $A_{i,m}^\oplus = (A^D)^m P_{A^i}$;
- (3) $(A_{i,m}^\oplus)^n = (A^D)^{m(n-1)} A_{i,m}^\oplus = (A^D)^{mn} P_{A^i}$;
- (4) $A^i A_{i,m}^\oplus = A_{i,m}^\oplus A^i$ if and only if $\mathcal{R}(A^i)^\perp \subseteq \mathcal{N}(A^i)$;
- (5) $A_{i,m}^\oplus = A$ implies that A is EP.

Proof. (1). By Theorem 3 we have $A_{i,m}^\oplus = (A^D)^m A^i (A^i)^\dagger$, thus

$$\begin{aligned}
 A_{i,m}^\oplus A^m A_{i,m}^\oplus &= (A^D)^m A^i (A^i)^\dagger A^m (A^D)^m A^i (A^i)^\dagger \\
 &= (A^D)^m A^i (A^i)^\dagger A^i A^m (A^D)^m (A^i)^\dagger \\
 &= (A^D)^m A^i A^m (A^D)^m (A^i)^\dagger = (A^D)^m A^m (A^D)^m A^i (A^i)^\dagger \\
 &= A^D A (A^D)^m A^i (A^i)^\dagger = (A^D)^m A^i (A^i)^\dagger = A_{i,m}^\oplus.
 \end{aligned}$$

Thus $A_{i,m}^\oplus$ is a $\{2, 3\}$ -inverse of A^m in view of $A^m A_{i,m}^\oplus = A^i (A^i)^\dagger$.

(2) is trivial.

(3). By

$$\begin{aligned} (A_{i,m}^\oplus)^n &= (A^D)^m A^i (A^i)^\dagger (A^D)^m A^i (A^i)^\dagger (A_{i,m}^\oplus)^{n-2} \\ &= (A^D)^m (A^D)^m A^i (A^i)^\dagger (A_{i,m}^\oplus)^{n-2} \\ &= (A^D)^m A_{i,m}^\oplus (A_{i,m}^\oplus)^{n-2} = (A^D)^m (A_{i,m}^\oplus)^{n-1}, \end{aligned}$$

it is easy to check (3).

(4). By $\mathcal{R}[(I_n - A^i (A^i)^\dagger) = \mathcal{N}[(A^i)^\dagger]$, we have

$$\begin{aligned} A^i A_{i,m}^\oplus &= A_{i,m}^\oplus A^i \Leftrightarrow A^i (A^D)^m A^i (A^i)^\dagger = (A^D)^m A^i (A^i)^\dagger A^i \\ &\Leftrightarrow A^i (A^D)^m A^i (A^i)^\dagger = (A^D)^m A^i \\ &\Leftrightarrow A^i (A^D)^m (I_n - A^i (A^i)^\dagger) = 0 \\ &\Leftrightarrow \mathcal{R}[I_n - A^i (A^i)^\dagger] \subseteq \mathcal{N}[A^i (A^D)^m] \\ &\Leftrightarrow \mathcal{N}[(A^i)^\dagger] \subseteq \mathcal{N}[(A^D)^m] \\ &\Leftrightarrow \mathcal{N}[(A^i)^*] \subseteq \mathcal{N}[(A^D)^m] \\ &\Leftrightarrow \mathcal{R}(A^i)^\perp \subseteq \mathcal{N}[(A^D)^m] \\ &\Leftrightarrow \mathcal{R}(A^i)^\perp \subseteq \mathcal{N}[A^i]. \end{aligned}$$

(5). Let A be written in the form (2.1). We have $A_{i,m}^\oplus = U \begin{bmatrix} (MC)_{i-1,m}^\oplus & 0 \\ 0 & 0 \end{bmatrix} U^*$ by Theorem 7. Thus, $A_{i,m}^\oplus = A$ implies $MS = 0$. From the nonsingularity of M , we have $S = 0$, which is equivalent to say that A is EP in view of [2, Theorem 3.7]. $\square$

4. (j, m) -CORE INVERSE

Let us start this section by introducing the definition of the (j, m) -core inverse.

Definition 3. Let $A \in \mathbb{C}^{n \times n}$ and $m, j \in \mathbb{N}$. A matrix $X \in \mathbb{C}^{n \times n}$ is called a (j, m) -core inverse of A , if it satisfies

$$X = A^D A X \quad \text{and} \quad A^m X = A^m (A^j)^\dagger. \quad (4.1)$$

Theorem 9. Let $A \in \mathbb{C}^{n \times n}$. If the system in (4.1) is consistent, then the solution is unique.

Proof. Assume that X satisfies (4.1), that is $X = A^D A X$ and $A^m X = A^m (A^j)^\dagger$. Then $X = A^D A X = (A^D)^m A^m X = (A^D)^m A^m (A^j)^\dagger = A^D A (A^j)^\dagger$. Thus X is unique. $\square$

By Theorem 9 if X exists, then it is unique and denoted by $A_{j,m}^\ominus$.

Theorem 10. Let $A \in \mathbb{C}^{n \times n}$ and $m, j \in \mathbb{N}$.

- (1) If $m \geq \text{ind}(A)$, then the system in (4.1) is consistent and the solution is $X = A^D A (A^j)^\dagger$.

(2) If the system in (4.1) is consistent, then $\text{ind}(A) \leq \max\{j, m\}$.

Proof. (1). Let $X = A^D A(A^j)^\dagger$. We have $A^D AX = A^D AA^D A(A^j)^\dagger = A^D A(A^j)^\dagger = X$ and $A^m X = A^m A^D A(A^j)^\dagger = A^D AA^m(A^j)^\dagger = A^m(A^j)^\dagger$.

(2). If the system in (4.1) is consistent, then exists $X_0 \in \mathbb{C}^{n \times n}$ such that $X_0 = A^D AX_0 = (A^D)^m A^m X_0 = (A^D)^m A^m(A^j)^\dagger = A^D A(A^j)^\dagger$ and $A^m(A^j)^\dagger = A^m X_0 = A^m A^D A(A^j)^\dagger = A^m(A^D)^j A^j(A^j)^\dagger$. Thus

$$A^m(A^j)^\dagger A^j = A^m(A^D)^j A^j(A^j)^\dagger A^j = A^m(A^D)^j A^j = A^m A^D A.$$

If $m \geq j$, then $A^m A^D A = A^m(A^j)^\dagger A^j = A^{m-j} A^j(A^j)^\dagger A^j = A^{m-j} A^j = A^m$. That is, $\text{ind}(A) \leq m$. If $j > m$, then $A^j = A^j(A^j)^\dagger A^j = A^{j-m} A^m(A^j)^\dagger A^j = A^{j-m} A^m A^D A = A^j A^D A$. That is, $\text{ind}(A) \leq j$. Thus, $\text{ind}(A) \leq \max\{j, m\}$. $\square$

Example 2. We will give an example that shows if $m < \text{ind}(A)$, then the system in (4.1) is not consistent. Let A be the same matrix in Example 1. It is easy to get $\text{ind}(A) = 2$ and $A^D = 0$. Let $m = j = 1$ and suppose that X is the solution of system in (4.1), then $X = A^D AX = 0$, which gives $AA^\dagger = AX = 0$, thus $A = AA^\dagger A = 0$, this is a contradiction.

Example 3. The converse of Theorem 10 (1) is not true. Let $m = 1$ and $j = 3$. If we let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

then $\text{ind}(A) = 3$ and $A^3 = 0$. Hence $X = 0$ is a solution of (4.1), but $m < \text{ind}(A)$.

Example 4. If $\text{ind}(A) \leq \max\{j, m\}$, then the system in (4.1) may be not consistent. If we let

$$A = \begin{bmatrix} 2 & 2 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then

$$A^3 = A^2 = \begin{bmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

$A^D = A^2$ and $\text{ind}(A) = 2$. Let $m = 1$ and $j = 2$, then $\text{ind}(A) \leq \max\{j, m\}$. It is easy to check that

$$(A^2)^\dagger = \frac{1}{15} \begin{bmatrix} 2 & -1 & 0 \\ 2 & -1 & 0 \\ 2 & -1 & 0 \end{bmatrix}.$$

If the system in (4.1) has a solution X_0 , then $X_0 = A^D A X_0 = A^D A(A^2)^\dagger$ and $A(A^2)^\dagger = A X_0 = A A^D A(A^2)^\dagger = A^4(A^2)^\dagger = A^2(A^2)^\dagger$ would hold. But

$$A(A^2)^\dagger = \frac{1}{15} \begin{bmatrix} 10 & -5 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \frac{1}{15} \begin{bmatrix} 12 & -6 & 0 \\ -6 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A^2(A^2)^\dagger.$$

Thus, the system in (4.1) is not consistent.

Remark 8. If $m \geq \text{ind}(A) = k$, it is not difficult to see that $A_{j,m}^\ominus = A_{j,m+1}^\ominus$. That is to say, the (j, m) -core inverse of A coincides with the $(j, m+1)$ -core inverse of A . Thus, in the sequel, we only discuss the $m = \text{ind}(A)$ case.

Theorem 11. Let $A, X \in \mathbb{C}^{n \times n}$, $k, j \in \mathbb{N}$. If $\text{ind}(A) = k$ and X is the (j, k) -core inverse of A , then we have $X^j A^j X^j = (A^D)^{j(j-1)} X^j$ and $X A^j = A^D A$.

Proof. By the definition of the (j, k) -core inverse, we have $X = A^D A X$ and $A^k X = A^k (A^j)^\dagger$. By $X = A^D A (A^j)^\dagger$, it is easy to check that $X^{n+1} = (A^D)^j X^n$ for arbitrary $n \in \mathbb{N}$, which gives that $X^j = (A^D)^{j(j-1)} X$.

$$\begin{aligned} X A^j &= A^D A (A^j)^\dagger A^j = (A^D)^j A^j (A^j)^\dagger A^j = (A^D)^j A^j = A^D A; \\ X^j A^j X^j &= (A^D)^{j(j-1)} X A^j X^j = (A^D)^{j(j-1)} A^D A X^j = (A^D)^{j(j-1)} X^j. \end{aligned}$$

□

Corollary 2. Let $A, X \in \mathbb{C}^{n \times n}$ and $\text{ind}(A) = k$. If X is the $(1, k)$ -core inverse of A , then we have $X A X = X$ and $X A = A^D A$.

The (j, m) -core inverse is a generalization of the core inverse and the DMP-inverse in view of Theorem 11.

Remark 9. When $j = m = 1 = \text{ind}(A)$, the equations in (4.1) are equivalent to $X A X = X$, $X A = A^\# A$, and $A X = A A^\dagger$. Thus $A X = A A^\dagger$ implies that $(A X)^* = A X$; $X A = A^\# A$ gives that $X A^2 = A$ and $A X A = A$; and $X = X A X = A^\# A X = A A^\# X$, which means that $\mathcal{R}(X) \subseteq \mathcal{R}(A)$, then $X = A Y$ for some $Y \in \mathbb{C}^{n \times n}$, thus $X = A Y = A X A Y = A X^2$. Therefore, we have $A^\oplus = X$ by Lemma 1. In a word, the $(1, 1)$ -core inverse coincides with the usual core inverse.

Remark 10. If we let $j = 1$ and $m = \text{ind}(A)$, then the equations in (4.1) are equivalent to $X A X = X$, $X A = A^D A$, and $A^k X = A^k A^\dagger$ by Theorem 11. Thus $(1, k)$ -core inverse coincides with the DMP-inverse.

From Remark 10, Theorem 11 and the definition of the (j, k) -core inverse, we have the following theorem, which says that the conditions $X A X = X$, and $X A = A^D A$ in the definition of the DMP-inverse can be replaced by $X = A^D A X$.

Theorem 12. Let $A \in \mathbb{C}^{n \times n}$ with $k = \text{ind}(A)$. Then $X \in \mathbb{C}^{n \times n}$ is the DMP-inverse of A if and only if $X = A^D A X$ and $A^k X = A^k A^\dagger$.

In the following theorem, we will give a canonical form for the (j, k) -core inverse of a matrix $A \in \mathbb{C}^{n \times n}$ by using the matrix decomposition in Theorem 1.

Theorem 13. *Let $A \in \mathbb{C}^{n \times n}$ have the form (2.1) with $\text{ind}(A) = k$ and $j \in \mathbb{N}$. Then*

$$A_{j,k}^{\ominus} = U \begin{bmatrix} (MC)^D (MC)_{j-1,k}^{\ominus} & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

Proof. By Theorem 10 and the idempotency of $A^D A$ we have

$$A_{j,k}^{\ominus} = A^D A (A^j)^{\dagger} = (A^D)^j A^j (A^j)^{\dagger}. \quad (4.2)$$

From the proof of Theorem 7, we have

$$A^j (A^j)^{\dagger} = U \begin{bmatrix} (MC)^{j-1} ((MC)^{j-1})^{\dagger} & 0 \\ 0 & 0 \end{bmatrix} U^*. \quad (4.3)$$

By (2.2) we have

$$(A^D)^j = U \begin{bmatrix} [(MC)^D]^j & [(MC)^D]^{j+1} MS \\ 0 & 0 \end{bmatrix} U^*. \quad (4.4)$$

By the proof of Theorem 7, we have $\text{ind}(MC) \leq k - 1 < k$. From (4.2), (4.3) and (4.4), we have

$$\begin{aligned} A_{j,k}^{\ominus} &= (A^D)^j A^j (A^j)^{\dagger} \\ &= U \begin{bmatrix} [(MC)^D]^j & [(MC)^D]^{j+1} MS \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (MC)^{j-1} ((MC)^{j-1})^{\dagger} & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} [(MC)^D]^j (MC)^{j-1} ((MC)^{j-1})^{\dagger} & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} (MC)^D [(MC)^D]^{j-1} (MC)^{j-1} ((MC)^{j-1})^{\dagger} & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} (MC)^D (MC)^D MC ((MC)^{j-1})^{\dagger} & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} (MC)^D (MC)_{j-1,k}^{\ominus} & 0 \\ 0 & 0 \end{bmatrix} U^*. \end{aligned}$$

□

Remark 11. If we use the decomposition of Hartwig and Spindelböck in [8, Corollary 6], then an expression of the (j, k) -core inverse of A is

$$A_{j,k}^{\ominus} = U \begin{bmatrix} (\Sigma K)^D (\Sigma K)_{j-1,k}^{\ominus} & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

which is similar to the expression of $A_{j,k}^{\ominus}$ in Theorem 13. Since the proof of this result can be proved like the proof of Theorem 13, we omit this proof.

Theorem 14. Let $A \in \mathbb{C}^{n \times n}$ and $\text{ind}(A) = k$. If $(A^k X^k)^* = A^k X^k$, $AX^{k+1} = X^k$ and $XA^{k+1} = A^k$, then A is (k, k) -core invertible and $A_{k,k}^\ominus = X^k$.

Proof. By Lemma 2 and Lemma 3, we have $A^k X^k A^k = A^k$, $X^k A^k X^k = X^k$, $A^k = X^k A^{2k}$, and $A^D = X^{k+1} A^k$. Equalities $(A^k X^k)^* = A^k X^k$ and $A^k X^k A^k = A^k$ imply that X^k is a $\{1, 3\}$ -inverse of A^k . From $A^D = X^{k+1} A^k$, we can obtain $(A^D)^k = X^{k-1} A^D$ by induction. Thus

$$\begin{aligned} A_{k,k}^\ominus &= A^D A(A^k)^\dagger = (A^D)^k A^k (A^k)^\dagger = (A^D)^k A^k (A^k)^{(1,3)} \\ &= (A^D)^k A^k X^k = (X^{k+1} A^k)^k A^k X^k = X^{k-1} X^{k+1} A^k A^k X^k \\ &= X^{2k} A^{2k} X^k = X^k (X^k A^{2k}) X^k = X^k A^k X^k = X^k. \end{aligned}$$

□

Proposition 3. Let $A \in \mathbb{C}^{n \times n}$ be a matrix with $j \geq \text{ind}(A) = k$. If A is (j, k) -core invertible, then $A^j A_{j,k}^\ominus$ is the projector onto $\mathcal{R}(A^j)$ along $\mathcal{R}(A^j)^\perp$.

Proof. It is trivial. □

In the following proposition, we will investigate some properties of the (j, k) -core inverse.

Proposition 4. Let $A \in \mathbb{C}^{n \times n}$ with $j \geq \text{ind}(A) = k$. If A is (j, k) -core invertible, then

- (1) $A_{j,k}^\ominus$ is a $\{1, 2, 3\}$ -inverse of A^j ;
- (2) $A_{j,k}^\ominus = (A^D)^j P_{A^j}$;
- (3) $(A_{j,k}^\ominus)^n = \begin{cases} [(A^D)^j (A^j)^\dagger]^{n/2} & \text{if } n \text{ is even,} \\ A^j [(A^D)^j (A^j)^\dagger]^{(n+1)/2} & \text{if } n \text{ is odd.} \end{cases}$
- (4) $A_{j,k}^\ominus A^D = (A^D)^{j+1}$;
- (5) $A^j A_{j,k}^\ominus = A_{j,k}^\ominus A^j$ if and only if $\mathcal{R}(A^j)^\perp \subseteq \mathcal{N}(A^j)$;
- (6) $A_{j,k}^\ominus = A$ implies that A is EP.

Proof. (1). By Theorem 10 we have $A_{j,k}^\ominus = A^D A(A^j)^\dagger = (A^D)^j A^j (A^j)^\dagger$, thus

$$\begin{aligned} A^j A_{j,k}^\ominus A^j &= A^j (A^D)^j A^j (A^j)^\dagger A^j = A^j (A^D)^j A^j = A^j A^D A = A^j; \\ A_{j,k}^\ominus A^j A_{j,k}^\ominus &= (A^D)^j A^j (A^j)^\dagger A^j A_{j,k}^\ominus = A^D A A_{j,k}^\ominus \\ &= A^D A A^D A(A^j)^\dagger = A^D A(A^j)^\dagger = A_{j,k}^\ominus; \\ A^j A_{j,k}^\ominus &= A^j (A^D)^j A^j (A^j)^\dagger = A^j (A^j)^\dagger. \end{aligned}$$

(2) is trivial.

(3). By $(A_{j,k}^\ominus)^2 = (A^D)^j A^j (A^j)^\dagger (A^D)^j A^j (A^j)^\dagger = (A^D)^j (A^j)^\dagger$ and induction it is easy to check (3).

(4). $A_{j,k}^\ominus A^D = (A^D)^j A^j (A^j)^\dagger A^D = (A^D)^j A^j (A^j)^\dagger (A^D)^j A^j A^D = (A^D)^{j+1}$.

(5). By $\mathcal{R}[I_n - A^j (A^j)^\dagger] = \mathcal{N}[(A^j)^\dagger]$ and $\mathcal{N}(A^D A) = \mathcal{N}(A^D)$, we have

$$\begin{aligned} A^j A_{j,k}^\ominus &= A_{j,k}^\ominus A^j \Leftrightarrow A^j (A^D)^j A^j (A^j)^\dagger = (A^D)^j A^j (A^j)^\dagger A^j \\ &\Leftrightarrow A^j (A^D)^j A^j (A^j)^\dagger = (A^D)^j A^j \\ &\Leftrightarrow A^j (A^D)^j [I_n - A^j (A^j)^\dagger] = 0 \\ &\Leftrightarrow \mathcal{R}[I_n - A^j (A^j)^\dagger] \subseteq \mathcal{N}(A^D A) \\ &\Leftrightarrow \mathcal{N}[(A^j)^\dagger] \subseteq \mathcal{N}(A^D A) \\ &\Leftrightarrow \mathcal{N}[(A^j)^*] \subseteq \mathcal{N}(A^D) \\ &\Leftrightarrow \mathcal{R}(A^j)^\perp \subseteq \mathcal{N}(A^j). \end{aligned}$$

(6). Let A be written in the form (2.1). We have

$$A_{j,k}^\ominus = U \begin{bmatrix} (MC)^D (MC)_{j-1,k}^\ominus & 0 \\ 0 & 0 \end{bmatrix} U^*$$

by Theorem 13. Thus, $A_{j,k}^\ominus = A$ implies $MS = 0$. From the nonsingularity of M , we have $S = 0$, which is equivalent to say that A is EP in view of [2, Theorem 3.7]. $\square$

In the following proposition, we shall give the relationship between the (j, k) -core inverse and DMP-inverse and core-EP inverse.

Proposition 5. *Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$. Then*

$$A_{k,k}^\ominus = A^{D,\dagger} (A^D)^{k-1} A A^\oplus.$$

Proof. We have that $A^k (A^k)^\dagger = A A^\oplus$ by Lemma 5 and $A^{D,\dagger} = A^D A A^\dagger$. Thus

$$\begin{aligned} A_{k,k}^\ominus &= A^D A (A^k)^\dagger = (A^D)^k A^k (A^k)^\dagger = A^D A^k (A^D)^{k-1} (A^k)^\dagger \\ &= A^D A A^\dagger A^k (A^D)^{k-1} (A^k)^\dagger = A^{D,\dagger} (A^D)^{k-1} A^k (A^k)^\dagger \\ &= A^{D,\dagger} (A^D)^{k-1} A A^\oplus. \end{aligned}$$

$\square$

In the following theorem, we will give a relationship between the $\langle i, m \rangle$ -core inverse and $\langle j, m \rangle$ -core inverse.

Theorem 15. *Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$. Then $A_{k,m}^\oplus = A_{m,k}^\ominus$ for any $m \geq k$.*

Proof. By Theorem 10, we have $A_{m,k}^{\ominus} = A^D A(A^m)^{\dagger} = (A^D)^k A^k (A^m)^{\dagger}$. By the proof of Remark 5, we have $A^k = MN$ and $NM = L^k$, where $M = \prod_{l=1}^k B_l$, $N = \prod_{l=1}^k G_{k+1-l}$ and $L = G_k B_k$. It is easy to see that $(A^D)^s = ML^{-k-s}N$ for any $s \in \mathbb{N}$ by $NM = L^k$. Thus $(A^D)^k = ML^{-2k}N$ and

$$(A^D)^k A^k = ML^{-2k}NMN = ML^{-2k}L^kN = ML^{-k}N.$$

By the proof of Remark 5, we have $A^m = ML^{m-k}N = M_1N$ is a full rank factorization of A^m , where $M_1 = ML^{m-k}$ and

$$(A^m)^{\dagger} = N^*(NN^*)^{-1}(M_1^*M_1)^{-1}(M_1)^*.$$

By Theorem 6, we have $A_{k,m}^{\oplus} = ML^{-m}M^{\dagger}$. In the following steps, we will show that $A_{m,k}^{\ominus} = ML^{-m}M^{\dagger}$. From $A_{m,k}^{\ominus} = (A^D)^k A^k (A^m)^{\dagger}$, we have

$$\begin{aligned} A_{k,m}^{\ominus} &= (A^D)^k A^k (A^m)^{\dagger} = ML^{-k}NN^*(NN^*)^{-1}(M_1^*M_1)^{-1}(M_1)^* \\ &= ML^{-k}(M_1^*M_1)^{-1}(M_1)^* = ML^{-k}[(L^{m-k})^*M^*ML^{m-k}]^{-1}(L^{m-k})^*M^* \\ &= ML^{-k}L^{k-m}(M^*M)^{-1}[(L^{m-k})^*]^{-1}(L^{m-k})^*M^* \\ &= ML^{-m}(M^*M)^{-1}M^* = ML^{-m}M^{\dagger}. \end{aligned}$$

□

Theorem 16. Let $A \in \mathbb{C}^{n \times n}$ with $i \geq \text{ind}(A) = k$, then $A_{i,k}^{\ominus} = P_1 D^{-i} P_1^{\dagger}$, where $A = P \begin{bmatrix} D & 0 \\ 0 & N \end{bmatrix} P^{-1}$ with $D \in \mathbb{C}^{r \times r}$ is nonsingular, N is nilpotent and $P = [P_1 \mid P_2]$ with $P_1 \in \mathbb{C}^{n \times r}$.

Proof. It is easy to see that by Theorem 8 and Theorem 15. □

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REFERENCES

- [1] O. M. Baksalary and G. Trenkler, "Core inverse of matrices," *Linear Multilinear Algebra*, vol. 58, no. 6, pp. 681–697, 2010, doi: [10.1080/03081080902778222](https://doi.org/10.1080/03081080902778222).
- [2] J. Benítez, "A new decomposition for square matrices," *Electron. J. Linear Algebra*, vol. 20, pp. 207–225, 2010, doi: [10.13001/1081-3810.1369](https://doi.org/10.13001/1081-3810.1369).
- [3] J. Benítez and X. Liu, "A short proof of a matrix decomposition with applications," *Linear Algebra Appl.*, vol. 438, no. 3, pp. 1398–1414, 2013, doi: [10.1016/j.laa.2012.10.002](https://doi.org/10.1016/j.laa.2012.10.002).
- [4] S. L. Campbell and C. D. Meyer, *Generalized Inverses of Linear Transformations*. Philadelphia, SIAM, 2009.

- [5] J. Chen, “Group inverses and drazin inverses of matrices over rings, (chinese),” *J. Xinjiang Univ. Natur. Sci.*, vol. 9, no. 1, pp. 44–49, 1992.
- [6] R. E. Cline, “An application of representations for the generalized inverse of a matrix,” Mathematics Research Center, University of Wisconsin, Madison, Tech. Rep., 1965.
- [7] R. E. Cline, “Inverses of rank invariant powers of a matrix,” *SIAM J. Numer. Anal.*, vol. 5, no. 1, pp. 182–197, 1968, doi: [10.1137/0705015](https://doi.org/10.1137/0705015).
- [8] R. E. Hartwig and K. Spindelböck, “Matrices for which a^* and $a^\dagger$ commute,” *Linear Multilinear Algebra*, vol. 14, no. 3, pp. 241–256, 1983, doi: [10.1080/03081088308817561](https://doi.org/10.1080/03081088308817561).
- [9] S. B. Malik and N. Thome, “On a new generalized inverse for matrices of an arbitrary index,” *Appl. Math. Comput.*, vol. 226, pp. 575–580, 2014, doi: doi.org/10.1016/j.amc.2013.10.060.
- [10] K. Manjunatha Prasad and K. S. Mohana, “Core-ep inverse,” *Linear Multilinear Algebra*, vol. 62, no. 6, pp. 792–802, 2014, doi: [10.1080/03081087.2013.791690](https://doi.org/10.1080/03081087.2013.791690).
- [11] H. Wang, “Core-ep decomposition and its applications,” *Linear Algebra Appl.*, vol. 508, pp. 289–300, 2016, doi: [10.1016/j.laa.2016.08.008](https://doi.org/10.1016/j.laa.2016.08.008).
- [12] H. K. Wimmer, “Canonical angles of unitary spaces and perturbations of direct complements,” *Linear Algebra Appl.*, vol. 287, pp. 373–379, 1999, doi: [10.1016/S0024-3795\(98\)10017-4](https://doi.org/10.1016/S0024-3795(98)10017-4).
- [13] S. Xu, J. Chen, and X. Zhang, “New characterizations for core inverses in rings with involution,” *Front. Math. China*, vol. 12, no. 1, pp. 231–246, 2017, doi: [10.1007/s11464-016-0591-2](https://doi.org/10.1007/s11464-016-0591-2).

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STABILITY AND BIFURCATION ANALYSIS FOR A DELAYED SHIMIZU-MORIOKA MODEL

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Abstract. This paper is concerned with a Shimizu-Morioka model with constant delays. Its stability of the equilibrium is investigated and the existence of Hopf bifurcations is demonstrated by analyzing the associated characteristic equation. Furthermore, the explicit formulae determining the stability and the direction of periodic solutions bifurcating from Hopf bifurcations are obtained by applying the center manifold theory and the normal form method. Finally, special attention is paid to numerical simulations in order to verify the theoretical predictions.

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Keywords: Shimizu-Morioka model, time delays, stability, local Hopf bifurcation

1. INTRODUCTION

The original Shimizu-Morioka model [16] is described by the following equation:

$$\begin{cases} \dot{x} \equiv \frac{dx}{dt} = y, \\ \dot{y} \equiv \frac{dy}{dt} = x - \beta y - xz, \\ \dot{z} \equiv \frac{dz}{dt} = -\alpha z + x^2, \end{cases} \quad (1.1)$$

where $(x, y, z) \in \mathbb{R}^3$ are the state variables and α, β are real positive parameters. This model has been proposed as a simplified and an alternative model for studying the dynamics of the well-known Lorenz system for large Rayleigh numbers (Ra), in which complex behavior of the trajectories has been discovered by means of computer simulation. As in the Lorenz model, the Shimizu-Morioka model is invariant with respect to the substitution $(x, y, z) \rightarrow (-x, -y, z)$. The model received much attention due to its stability to describe bifurcation of the associated Lorenz like strange attractors [15], e.g., taking $\alpha = 0.45$ and $\beta = 0.75$ (see Figure 1). In case that from the dynamical system of equation (1.1) follows that

$$\nabla V = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = -(\alpha + \beta) < 0, \quad (1.2)$$

then the system is known to be a dissipative one. According to the definition of the dissipative system [19], a volume element V_0 is contracted by the flow into a volume

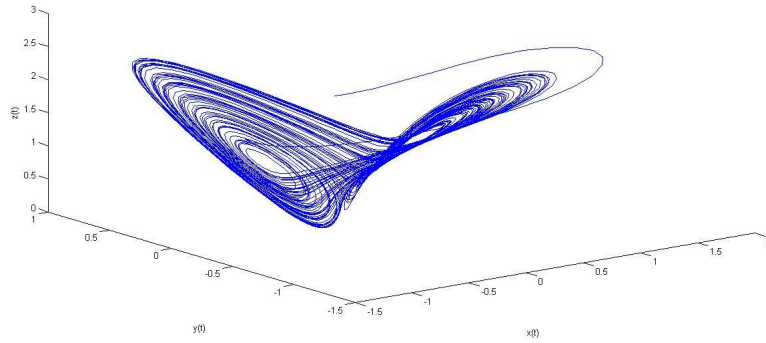


FIGURE 1. Phase portraits of Shimizu-Morioka model in the $x - y - z$ space.

element $V_0 e^{\nabla V_t}$ in time t . Hence, each volume containing the system orbit shrinks to zero as $t \rightarrow \infty$ at an exponential rate.

During the past decades, there are many researchers focus on the study of dynamical behaviour analysis of Shimizu-Morioka model. In particular, they use feedback control laws and delay feedback control method to study the local and global stabilization and bifurcation of Shimizu-Morioka model [9] [3] [18].

Delay differential equations (DDEs) are well-known as differential equations with time delay term, which exhibit considerably more complex dynamical behavior than ordinary differential equations (ODEs) since the delays could cause a stable equilibrium to become unstable and fluctuate. The research of the dynamical behaviors for DDEs has received much attention in interdisciplinary subjects including natural science [11] [2] [7], engineering [1], life sciences [17] and others [20] [21]. So, there's great significance to study the delayed Shimizu-Morioka model as follows:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = x(t - \tau) - \beta y(t) - x(t - \tau)z(t), \\ \dot{z}(t) = -\alpha z(t) + x^2(t - \tau), \end{cases} \quad (1.3)$$

where τ is a non-negative constant delay term.

We define $\mathbb{R}_+ \equiv \{x \in \mathbb{R} : x \geq 0\}$, $\text{int } \mathbb{R}_+ \equiv \{x \in \mathbb{R} : x > 0\}$. Denoted by $C([-\tau, 0], \mathbb{R}_+)$, the infinite dimensional Banach space of continuous functions from the interval $[-\tau, 0]$ into $\mathbb{R}_+$, equipped with the uniform norm. We assume initial data for model (1.3) is taken from $X = C([-\tau, 0], \mathbb{R}_+) \times C([-\tau, 0], \mathbb{R}_+) \times C([-\tau, 0], \mathbb{R}_+)$. The variables $x(t)$, $y(t)$ and $z(t)$ in model (1.3) belong to X for $t \in [-\tau, 0]$.

In this paper, we will take the delay τ as bifurcation parameter and show that when τ passes through the critical values, the equilibrium loses its stability and the Hopf bifurcation occurs. The rest of this paper is organized as follows. In Sect.2, the stability and the existence of Hopf bifurcation parameter are determined. In Sect.3, we give details of the bifurcation values that describe the direction of the Hopf bifurcation and the stability for the bifurcated periodic solution using the center manifold theory and the normal form method introduced by Hassard *et al.* [5]. Finally, some numerical simulations and conclusions are given to illustrate theoretical predictions in Sect.4.

2. THE EXISTENCE OF LOCAL HOPF BIFURCATION

In this section, we firstly give out the existence and stability of the equilibrium for system (1.3). It has three equilibrium points $E(0,0,0)$, $E_+(\sqrt{\alpha},0,1)$ and $E_-(-\sqrt{\alpha},0,1)$. Due to the symmetry of the system, it is sufficient to study the existence of local Hopf bifurcations occurring at E_+ .

Let $x_1(t) = x(t) - \sqrt{\alpha}$, $x_2(t) = y(t)$, $x_3(t) = z(t) - 1$, then system (1.3) becomes

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -\beta x_2(t) - \sqrt{\alpha} x_3(t) - x_1(t - \tau) x_3(t), \\ \dot{x}_3(t) = 2\sqrt{\alpha} x_1(t - \tau) - \alpha x_3(t) + x_1^2(t - \tau), \end{cases} \quad (2.1)$$

the linearization of system (2.1) is

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -\beta x_2(t) - \sqrt{\alpha} x_3(t), \\ \dot{x}_3(t) = 2\sqrt{\alpha} x_1(t - \tau) - \alpha x_3(t). \end{cases} \quad (2.2)$$

The associated characteristic equation of (2.2) is

$$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda + \beta & \sqrt{\alpha} \\ -2\sqrt{\alpha}e^{-\lambda\tau} & 0 & \lambda + \alpha \end{vmatrix} = 0,$$

i.e.

$$\lambda^3 + (\alpha + \beta)\lambda^2 + \alpha\beta\lambda + 2\alpha e^{-\lambda\tau} = 0. \quad (2.3)$$

When $\tau = 0$, Eq.(2.3) becomes

$$\lambda^3 + (\alpha + \beta)\lambda^2 + \alpha\beta\lambda + 2\alpha = 0. \quad (2.4)$$

We give out the assumption (H_1) $(\alpha + \beta)\beta > 2$.

By Routh-Hurwitz stability criterion, we obtain

Lemma 1. Assume that (H_1) holds, then when $\tau = 0$, the equilibrium E_+ of system (2.1) is asymptotically stable.

If $i\omega$ ($\omega > 0$) is a root of Eq.(2.3), then

$$-i\omega^3 - (\alpha + \beta)\omega^2 + i\omega\alpha\beta + 2\alpha(\cos\omega\tau - i\sin\omega\tau) = 0. \quad (2.5)$$

Separating the real and imaginary parts, we obtain

$$\begin{cases} -(\alpha + \beta)\omega^2 + 2\alpha\cos\omega\tau = 0, \\ \alpha\beta\omega - \omega^3 - 2\alpha\sin\omega\tau, \end{cases} \quad (2.6)$$

it follows from (2.6) that

$$\begin{cases} \sin(\omega\tau) = \frac{\omega(\alpha\beta - \omega^2)}{2\alpha}, \\ \cos(\omega\tau) = \frac{(\alpha + \beta)\omega^2}{2\alpha}. \end{cases} \quad (2.7)$$

By $\sin^2(\omega\tau) + \cos^2(\omega\tau) = 1$, then (2.3) is equivalent to

$$\omega^6 + (\alpha^2 + \beta^2)\omega^4 + \alpha^2\beta^2\omega^2 - 4\alpha^2 = 0. \quad (2.8)$$

Let $z = \omega^2$, Eq.(2.8) becomes

$$z^3 + (\alpha^2 + \beta^2)z^2 + \alpha^2\beta^2z - 4\alpha^2 = 0. \quad (2.9)$$

From (2.8), we have $h(z) = z^3 + (\alpha^2 + \beta^2)z^2 + \alpha^2\beta^2z - 4\alpha^2$. Then, $h'(z) = 3z^2 + 2(\alpha^2 + \beta^2)z + \alpha^2\beta^2$. Denote $\Delta = (\alpha^2 + \beta^2)^2 - 3\alpha^2\beta^2$, obviously $\Delta \geq \alpha^2\beta^2 > 0$, the equation $3z^2 + 2(\alpha^2 + \beta^2)z + \alpha^2\beta^2 = 0$ has two real negative roots:

$$z_1^* = \frac{-(\alpha^2 + \beta^2) + \sqrt{\Delta}}{3}, \quad z_2^* = \frac{-(\alpha^2 + \beta^2) - \sqrt{\Delta}}{3}. \quad (2.10)$$

Noticing that $h(0) < 0$, $\lim_{z \rightarrow +\infty} h(z) = +\infty$. Eq.(2.9) has one positive real roots z , and

Eq.(2.8) has two real root ω_k ($k = 1, 2$), suppose $\omega_1 = -\sqrt{z}$, $\omega_2 = \sqrt{z}$. Substituting ω_k ($k = 1, 2$) into (2.7), we have

$$\tau_k^j = \frac{1}{\omega_k} \left\{ \arccos \left(\frac{(\alpha + \beta)\omega_k^2}{2\alpha} \right) + 2j\pi \right\}, \quad (2.11)$$

where $k = 1, 2$ and $j = 0, 1, 2, \dots$, then $i\omega_k$ ($k = 1, 2$) is a pair of pure imaginary roots of Eq.(2.3) with τ_k^j . Define

$$\tau_0 = \tau_k^{(0)} = \min_{k \in \{1, 2\}} \{\tau_k^{(0)}\}. \quad (2.12)$$

Lemma 2. When $\tau = \tau_k^j$ ($k = 1, 2$ and $j = 0, 1, 2, \dots$), then (2.3) has a pair of pure imaginary roots $i\omega_k$, and all other roots of (2.3) have nonzero real parts.

Let $\lambda(\tau) = \alpha(\tau) \pm i\omega(\tau)$ be the roots of (2.3) near $\tau = \tau^j$ satisfying $\alpha(\tau^j) = 0$, $\omega(\tau^j) = \omega_k$ ($j = 0, 1, 2, \dots$). By the theory of functional differential equation [5], for $\forall \tau^j$, $\exists \varepsilon > 0$ s.t. $\lambda(\tau)$ in $|\tau - \tau^j| < \varepsilon$ about τ is continuous and differentiable.

Lemma 3. *The transversality condition*

$$\left[\frac{d(\operatorname{Re}\lambda(\tau))}{d\tau} \right] \Big|_{\tau=\tau_j} > 0.$$

Proof. Substituting $\lambda(\tau)$ into Eq.(2.3) and taking the derivative with respect to τ , we have

$$\left(\frac{d\lambda(\tau)}{d\tau} \right)^{-1} = \frac{[3\lambda^2 + 2(\alpha + \beta)\lambda + \alpha\beta]e^{\lambda\tau}}{2\alpha\lambda} - \frac{\tau}{\lambda},$$

by (2.7), we get

$$\begin{aligned} \left[\frac{d(\operatorname{Re}\lambda(\tau))}{d\tau} \right] \Big|_{\tau=\tau_j} &= \operatorname{Re} \left[\frac{[3\lambda^2 + 2(\alpha + \beta)\lambda + \alpha\beta]e^{\lambda\tau}}{2\alpha\lambda} \right] \Big|_{\tau=\tau_j} \\ &= \operatorname{Re} \left[\frac{[-3\omega^2 + 2i(\alpha + \beta)\omega + \alpha\beta](\cos\omega_k\tau_k^j + i\sin\omega_k\tau_k^j)}{2i\alpha\omega} \right] \Big|_{\tau=\tau_j} \\ &= \frac{\alpha^2\beta^2 + 3\omega^4 + 2(\alpha^2 + \beta^2)\omega^2}{4\alpha^2} > 0. \end{aligned} \quad (2.13)$$

□

By above Lemmas and the conditions for Hopf bifurcation theorem [6], we have the following result:

Theorem 1. *Assume that (H_1) holds.*

$$\tau_k^j = \frac{1}{\omega_k} \left\{ \arccos \left(\frac{(\alpha + \beta)\omega_k^2}{2\alpha} \right) + 2j\pi \right\} \quad (2.14)$$

are Hopf bifurcation values at E_+ , where $i\omega_k$ ($k = 1, 2$) are the roots of (2.3). And E_+ is locally asymptotically stable for $\tau \in [0, \tau_1^0)$ and unstable where $\tau > \tau_1^0$.

3. THE PROPERTY OF LOCAL HOPF BIFURCATION

In the previous part, it is shown that system (2.1) undergoes Hopf bifurcation under certain conditions. In the following part, we will derive explicit formulae determining the direction of Hopf bifurcation and the stability of the periodic solutions bifurcating from E_+ at τ^j ($j = 0, 1, 2, \dots$), by employing center manifold theory and normal form method. For convenience, denote τ^j by $\tilde{\tau}$ and $\tau = \tilde{\tau} + \mu$, $\mu \in R$, then $\mu = 0$ is the Hopf bifurcation value for system (2.1).

The discussion will be divided into five steps.

Step 1. Transform system (2.1) into the abstract ODE.

The system (2.1) can locally be represented as the following DDE in $C = C([-1, 0], \mathbb{R}^3)$

$$\dot{u}(t) = L_\mu(u_t) + F(\mu, u_t), \quad (3.1)$$

where $u(t) = (u_1(t), u_2(t), u_3(t))^T$, $u_t(\theta) = u(t + \theta)$, $L_\mu : C \rightarrow \mathbb{R}$ is a bounded linear operator and $F : \mathbb{R} \times C \rightarrow \mathbb{R}$ is continuous and differentiable with

$$L_\mu \phi = (\tilde{\tau} + \mu) \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\beta & -\sqrt{\alpha} \\ 0 & 0 & -\alpha \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2\sqrt{\alpha} & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \end{pmatrix},$$

and

$$F(\mu, \phi) = (\tilde{\tau} + \mu) \begin{pmatrix} 0 \\ -\phi_1(-1)\phi_3(0) \\ \phi_1^2(-1) \end{pmatrix},$$

where $\phi = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in C$.

By the Riese representation theorem, there exists a 3×3 matrix whose elements are bounded variation function $\eta(\theta, \mu)$ in $\theta \in [-1, 0]$ such that

$$L_\mu \phi = \int_{-1}^0 d\eta(\theta, \mu) \phi(\theta), \quad \phi \in C,$$

where $\eta(\theta, \mu)$ can be chosen as

$$\eta(\theta, \mu) = (\tilde{\tau} + \mu) \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\beta & -\sqrt{\alpha} \\ 0 & 0 & -\alpha \end{pmatrix} \delta(\theta) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2\sqrt{\alpha} & 0 & 0 \end{pmatrix} \delta(\theta + 1),$$

where $\delta(\theta)$ is a Dirac delta function [12] and $\theta \in [-1, 0]$.

For $\phi \in C$, let

$$A(\mu)\phi(\theta) = \begin{cases} d\phi(\theta)/d\theta, & \theta \in [-1, 0), \\ \int_{-1}^0 d\eta(\mu, \theta)\phi(\theta), & \theta = 0, \end{cases}$$

$$R(\mu)\phi(\theta) = \begin{cases} 0, & \theta \in [-1, 0), \\ F(\mu, \phi), & \theta = 0, \end{cases}$$

then system (3.1) is equivalent to the following abstract operator equation

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t. \quad (3.2)$$

Step 2. Calculate the eigenfunctions of $A = A(0)$ and the adjoint operator A^* corresponding to $i\omega_0\tilde{\tau}$ and $-i\omega_0\tilde{\tau}$.

For $\psi \in C([0, 1], (C^3)^*)$, where $(C^3)^*$ is the 3-dimensional complex space of row vectors, we define the adjoint operator A^* of A

$$A^*\psi(s) = \begin{cases} -d\psi(s)/ds, & s \in (0, 1], \\ \int_{-1}^0 d\eta^T(\mu, t)\psi(-t), & s = 0, \end{cases}$$

and the bilinear form is given by

$$\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0)\phi(0) - \int_{-\tau}^0 \int_{\xi=0}^{\theta} \bar{\psi}^T(\xi - \theta) d\eta(\theta) \phi(\xi) d\xi,$$

where $\eta(\theta) = \eta(\theta, 0)$. Then $A = A(0)$ and $A^*(0)$ are adjoint operators.

By [8], $\pm i\omega_0\tau$ are eigenvalues of $A(0)$, so they are also eigenvalues of $A^*(0)$. Suppose that $q(\theta) = (1, \tilde{\alpha}, \tilde{\beta})^T e^{i\omega_0\theta}$ is the eigenfunction of $A(0)$ corresponding to the eigenvalue $i\omega_0\tau$ and $q^*(s) = G(1, \tilde{\alpha}^*, \tilde{\beta}^*) e^{i\omega_0 s}$ is the eigenfunction of A^* corresponding to the eigenvalue $-i\omega_0\tau$, where

$$\tilde{\alpha} = i\omega_k, \quad \tilde{\beta} = \frac{2\sqrt{\alpha}e^{i\omega\tau}}{i\omega_k + \alpha}, \quad \tilde{\alpha}^* = \frac{1}{\beta - i\omega_k}, \quad \tilde{\beta}^* = \frac{\sqrt{\alpha}}{(\omega_k^2 - \alpha\beta) + i\omega_k(\alpha + \beta)},$$

$$G = (1 + \tilde{\alpha}\tilde{\alpha}^* + \tilde{\beta}\tilde{\beta}^* + \tau_k\tilde{\alpha}\tilde{\alpha}^* e^{-i\omega\tau})^{-1},$$

which assures that $\langle q^*(s), q(\theta) \rangle = 1$, $\langle q^*(s), \bar{q}(\theta) \rangle = 0$.

Step 3. Obtain the reduced system on the center manifold.

In this part, we will use the same notations as in [6] and compute the coordinates to describe the center manifold C_0 at $\mu = 0$ (A local center manifold is in general not unique). Let $u_t \in C$ be the solution of system (3.2) when $\mu = 0$, and define

$$z(t) = \langle q^*, u_t \rangle, \quad W(t, \theta) = u_t(\theta) - z(t)q(\theta) - \bar{z}(t)\bar{q}(\theta), \quad (3.3)$$

where z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. On the center manifold C_0 , we have $W(t, \theta) = W(z(t), \bar{z}(t), \theta)$, where

$$W(z, \bar{z}, \theta) = W_{20}(\theta)z^2/2 + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\bar{z}^2/2 + \dots \quad (3.4)$$

The existence of center manifold enables us to reduce (3.2) to an ODE on C_0 . Note that W is real if u_t is real, we consider only real solutions. For solution $u_t \in C_0$ of system (3.2) at $\mu = 0$,

$$\begin{aligned} \dot{z}(t) &= \langle q^*, \dot{u}_t \rangle = \langle q^*, A(u_t) + R(u_t) \rangle = \langle A^*(q^*), u_t \rangle + \langle q^*, R(u_t) \rangle \\ &= i\omega_0\tau z(t) + \bar{q}^*(0) \cdot f(0, W(z, \bar{z}, \theta) + 2\text{Re}\{z(t)q(\theta)\}), \\ &= i\omega_0\tau z(t) + \bar{q}^*(0) \cdot f(0, u_t), \end{aligned} \quad (3.5)$$

with

$$\bar{q}^*(0) \cdot f(0, u_t) \triangleq g(z, \bar{z}).$$

Rewrite (3.5), we obtain the reduced system on C_0 is described by

$$\dot{z}(t) = i\omega_0\tau z(t) + g(z, \bar{z}), \quad (3.6)$$

where

$$g(z, \bar{z}) = g_{20}(\theta)z^2/2 + g_{11}(\theta)z\bar{z} + g_{02}(\theta)\bar{z}^2/2 + g_{21}(\theta)z^2\bar{z}/2 + \dots \quad (3.7)$$

We will mainly discuss Eq. (3.6) in the following part.

Step 4. Get the value of $g_{20}, g_{11}, g_{02}, g_{21}$ in (3.7).

In this part, we will calculate the coefficient $W_{20}(\theta), W_{11}(\theta), W_{02}(\theta), \dots$, then substitute them in (3.5), and get the reduced system (3.6) on C_0 .

It follows from (3.3) that

$$\begin{aligned} u_t(\theta) &= u(t + \theta) = W(t, \theta) + 2\operatorname{Re}\{z(t), q(\theta)\} \\ &= W_{20}(\theta)z^2/2 + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\bar{z}^2/2 + (1, \tilde{\alpha})^T e^{i\omega_0\tilde{\tau}\theta} z \\ &\quad + (1, \tilde{\alpha})^T e^{-i\omega_0\tilde{\tau}\theta} \bar{z} + \dots. \end{aligned}$$

We obtain

$$\begin{aligned} \phi_3(0) &= z\tilde{\beta} + \bar{z}\tilde{\beta} + W^{(3)}(0) + \dots, \\ \phi_1(-1) &= ze^{-i\omega_0\theta} + \bar{z}e^{i\omega_0\theta} + W^{(1)}(-1) + \dots, \end{aligned} \quad (3.8)$$

It follows together with $F(\mu, \phi)$, we get

$$f(0, u_t) = \tilde{\tau} \begin{pmatrix} 0 \\ -\phi_1(-1)\phi_3(0) \\ \phi_1^2(-1) \end{pmatrix}. \quad (3.9)$$

Substituting (3.8) into (3.9) (3.5) and comparing the coefficients with (3.7), we obtain

$$\begin{aligned} g_{20} &= 2\bar{D}\tau_k(-\tilde{\alpha}^*\tilde{\beta}e^{-i\omega_0\theta} + \tilde{\beta}^*e^{-2i\omega_0\theta}), \\ g_{11} &= \bar{D}\tau_k[-\tilde{\alpha}^*(\tilde{\beta}e^{i\omega_0\theta} + \tilde{\beta}e^{-i\omega_0\theta}) + 2\tilde{\beta}^*], \\ g_{02} &= 2\bar{D}\tau_k(-\tilde{\alpha}^*\tilde{\beta}e^{i\omega_0\theta} + \tilde{\beta}^*e^{2i\omega_0\theta}), \\ g_{21} &= 2\bar{D}\tau_k\{-\tilde{\alpha}^*e^{-i\omega_0\theta}W_{11}^{(3)}(0) + \tilde{\beta}^*[2e^{-i\omega_0\theta}W_{20}^{(1)}(-1)]\}. \end{aligned} \quad (3.10)$$

Since there are $W_{20}(\theta)$ and $W_{11}(\theta)$ in g_{21} , we still need to compute them.

From (3.2) and (3.3), we have

$$\dot{W} = \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q} = \begin{cases} AW - 2\operatorname{Re}\{gq(\theta)\}, & \theta \in [-1, 0), \\ AW - 2\operatorname{Re}\{gq(0)\} + f_0, & \theta = 0, \end{cases} \quad (3.11)$$

where

$$f_0 = f_{z^2}z^2/2 + f_{z\bar{z}}z\bar{z} + f_{\bar{z}^2}\bar{z}^2/2 + f_{z^2\bar{z}}z^2\bar{z}/2 + \dots.$$

On the other hand, near the origin, on the center manifold C_0 , according to (3.4) we obtain

$$\begin{aligned} \dot{W} &= W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}} = [W_{20}(\theta)z + W_{11}(\theta)\bar{z}]\dot{z} + [W_{11}(\theta)z + W_{02}(\theta)\bar{z}]\dot{\bar{z}} \\ &= [W_{20}(\theta)z + W_{11}(\theta)\bar{z}](i\omega_0z + g(z, \bar{z})) \\ &\quad + [W_{11}(\theta)z + W_{02}(\theta)\bar{z}](\bar{g}(z, \bar{z}) - i\omega_0\bar{z}) + \dots. \end{aligned} \quad (3.12)$$

Substituting (3.4) into right side of (3.11), equating terms of $\frac{z^2}{2}$ and $z\bar{z}$ of (3.11) with (3.12), we obtain

$$(2i\omega_0 I - A)W_{20}(\theta) = \begin{cases} -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), & \theta \in [-\tilde{\tau}, 0), \\ -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + f_{z^2}, & \theta = 0, \end{cases} \quad (3.13)$$

$$-AW_{11}(\theta) = \begin{cases} -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta), & \theta \in [-\tilde{\tau}, 0), \\ -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + f_{z\bar{z}}, & \theta = 0. \end{cases} \quad (3.14)$$

According to the definition of A and from (3.13)(3.14) for $\theta \in [-\tilde{\tau}, 0)$, we get

$$\begin{aligned} \dot{W}_{20}(\theta) &= 2i\omega_0 W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta), \\ \dot{W}_{11}(\theta) &= g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta), \end{aligned}$$

solving for $W_{20}(\theta)$ and $W_{11}(\theta)$, we obtain

$$W_{20}(\theta) = i g_{20}/\omega_0 \cdot q(0)e^{i\omega_0\theta} + i \bar{g}_{02}/3\omega_0 \cdot \bar{q}(0)e^{-i\omega_0\theta} + E_1 e^{2i\omega_0\theta}, \quad (3.15)$$

$$W_{20}(\theta) = -i g_{11}/\omega_0 \cdot q(0)e^{i\omega_0\theta} + i \bar{g}_{11}/\omega_0 \cdot \bar{q}(0)e^{-i\omega_0\theta} + E_2, \quad (3.16)$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)})^T \in \mathbb{R}^3$ and $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)})^T \in \mathbb{R}^3$ are constant vectors.

In what follows we shall seek appropriate E_1 and E_2 in (3.15) and (3.16), respectively. According to the definition of A and (3.13)(3.14) for $\theta = 0$, we have

$$\int_{-\tilde{\tau}}^0 d\eta(\theta) W_{20}(\theta) = 2i\omega_0 W_{20}(0) + g_{20}q(0) + \bar{g}_{02}\bar{q}(0) - f_{z^2}, \quad (3.17)$$

$$\int_{-\tilde{\tau}}^0 d\eta(\theta) W_{11}(\theta) = g_{11}q(0) + \bar{g}_{11}\bar{q}(0) - f_{z\bar{z}}, \quad (3.18)$$

where $\eta(\theta) = \eta(0, \theta)$ and

$$f_{z^2} = \begin{pmatrix} 0 \\ \tilde{\beta}e^{-i\omega_0\theta} \\ e^{-2i\omega_0\theta} \end{pmatrix}, \quad f_{z\bar{z}} = \begin{pmatrix} 0 \\ \tilde{\beta}e^{i\omega_0\theta} + \bar{\tilde{\beta}}e^{-i\omega_0\theta} \\ 2 \end{pmatrix}.$$

Substituting (3.15) into (3.17), we obtain

$$\left(2i\omega_0 I - \int_{-\tilde{\tau}}^0 e^{2i\omega_0\theta} d\eta(\theta) \right) E_1 = f_{z^2},$$

that is

$$\begin{pmatrix} 2i\omega_0 & 1 & 0 \\ 0 & 2i\omega_0 & \sqrt{\alpha} \\ -2\sqrt{\alpha}e^{-i\omega_0\tau} & 0 & 2i\omega_0 + \alpha \end{pmatrix} E_1 = f_{z^2}. \quad (3.19)$$

Similarly, substituting (3.16) into (3.18), we get

$$\int_{-\tilde{\tau}}^0 d\eta(\theta) E_2 = f_z \bar{z},$$

that is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \sqrt{\alpha} \\ 2\sqrt{\alpha}e^{-i\omega_0\tau} & 0 & \alpha \end{pmatrix} E_2 = f_z \bar{z}. \quad (3.20)$$

We have got the value of E_1 and E_2 as (3.19) and (3.20) shown and the reduced system (3.6) finally.

Step 5. Obtain the key values μ_2, β_2, T_2 to judge the property of the Hopf bifurcation.

Similar to calculate the Hopf bifurcation parameter of the ODE and as in [6], according to the analysis above and the expressions of g_{20}, g_{11}, g_{02} and g_{21} , we can compute the following values:

$$\begin{aligned} c_1(0) &= i/2\omega_0\tilde{\tau}(g_{11}g_{20} - 2|g_{11}|^2 - |g_{02}|^2/3) + g_{21}/2, \\ \mu_2 &= -\operatorname{Re}\{c_1(0)\}/\alpha'(\tilde{\tau}), \\ \beta_2 &= 2\operatorname{Re}\{c_1(0)\}, \\ T_2 &= -[\operatorname{Im}\{c_1(0)\} + \mu_2\omega'(\tilde{\tau})]/\omega_0, \end{aligned} \quad (3.21)$$

where $\lambda(\tau) = \alpha(\tau) \pm i\omega(\tau)$ is the characteristic root of (2.3), which is a continuous differentiable family. $\alpha'(\tilde{\tau})$ and $\omega'(\tilde{\tau})$ can be obtained by taking the derivation of two sides of (2.3) and taking values at $\tilde{\tau}$.

These formulae give a description of the Hopf bifurcation periodic solution of system (1.1) at $\tau = \tau^j$ ($j = 0, 1, 2, \dots$) on the center manifold. Thus, we can obtain the following results according to the discussion about properties of Hopf bifurcating periodic solutions of dynamical system in [6]:

Theorem 2. For system (1.3), we assume that the conditions (H_1) holds for the parameters.

- (i) If $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical);
- (ii) If $\beta_2 < 0$ ($\beta_2 > 0$), then bifurcating periodic solution is stable (unstable);
- (iii) If $T_2 > 0$ ($T_2 < 0$), then periods of periodic solutions increase (decrease).

4. NUMERICAL SIMULATIONS AND CONCLUSIONS

In this section, we shall give some numerical simulations to support the theoretical analysis discussed in previous section. In addition, we will give out our actual conclusions and the shortcomings.

Firstly, we study the following specific model

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = x(t-\tau) - y(t) - x(t-\tau)z(t), \\ \dot{z}(t) = -2z(t) + x^2(t-\tau), \end{cases} \quad (4.1)$$

with initial value is $(x(t), y(t), z(t)) = (0.1, 0.1, 0.1)$ which satisfies (H_1) . By computing, $E_+ = (1.414, 0, 1)$ and $h(z) = z^3 + 5z^2 + 4z - 16$ has only one positive root $z \approx 1.307$, get $\omega_1 \approx -1.143$, $\omega_2 \approx 1.143$, $\tau^0 \approx 0.174$. By Theorem 1, we get that E_+ is asymptotically stable when $0 \leq \tau < \tau^0$, as Figure 2 illustrated, and E_+ is unstable when $\tau > \tau^0$, as shown in Figure 3. Figure 2–3 are drawn with dde23 [14] which is a MatLab [10] package that integrates DDEs.

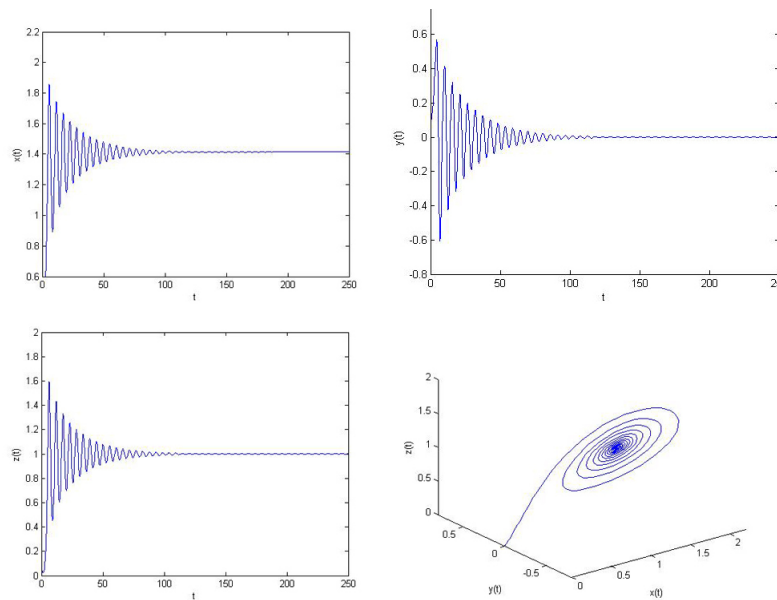


FIGURE 2. The equilibrium $E_+ = (1.414, 0, 1)$ is locally asymptotically for system (2.1) with $\tau = 0.1 < \tau_0$.

In addition, the bifurcation diagram is shown by Figure 4, it is drawn using DDEBifTool [4][13] which is also a MatLab software package. The command window shows the equilibrium is $E_0 = (1.4140, 0, 1.0000)$, Hopf Point 1 is at $\tau = 0.1744$ with $\omega = 1.1432$, Hopf Point 2 is at $\tau = 5.6705$ with $\omega = 1.1432$, Hopf Point 3 is at $\tau = 11.1667$ with $\omega = 1.1432$.

We take the $\tau - x$ space as example to explain the bifurcation meaning. At $\tau = \tau^0 = 0.1744$ (the position of (a) dashed vertical line), the stable equilibrium point loses stability and becomes unstable, Hopf bifurcation occurs and generated the

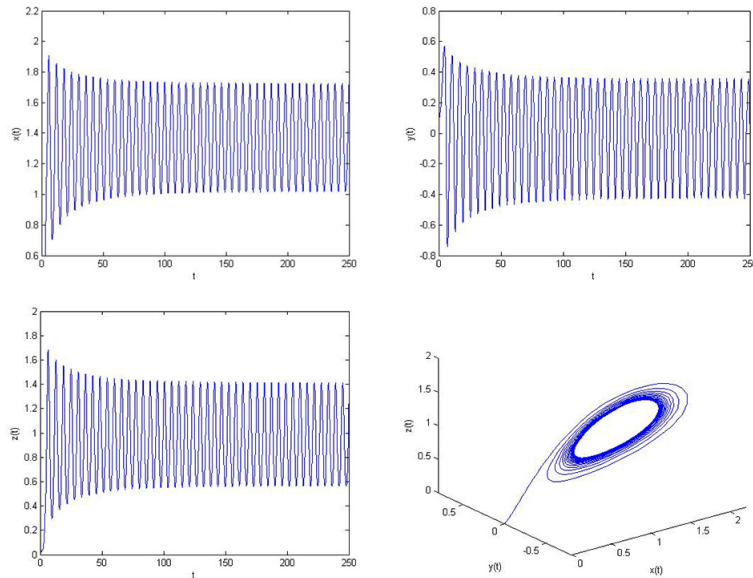


FIGURE 3. Bifurcating stable periodic solution for system (2.1) at $E_+ = (1.414, 0, 1)$ with $\tau_0 < \tau = 0.2$.

stable periodic orbit. After 2π , that is at $\tau = \tau^1 = 5.6705$ (the position of (b) dashed vertical line), a Hopf bifurcation occurs and generates the unstable periodic orbit. At $\tau = \tau^2 = 11.1667$ (the position of (c) dashed vertical line), the Hopf bifurcation occurs and generated the unstable periodic orbit again. The second graph of Figure 4 is chosen the value of delay till 0.6, which makes more clear to study the bifurcation diagram. In this paper, we study the Shimizu-Morioka system at Hopf bifurcation occurs and the stability of equilibrium. We also investigate the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions, by using the center manifold theory and normal form method. Our theoretical results and numerical simulations show that the chaos phenomena of system (1.1) can be controlled by delay. As the delay increases further, the numerical simulations show that the periodic solution disappears and the chaos attractor appears again. The obtained results can also be applied to the control and anticontrol of chaos phenomena of system (1.1). The other kinds of dynamical behaviors of system (4.1) out of the assumptions on the parameters have been disregarded. There are still abundant and complex dynamical behaviors and the topological structure of the new system should be completely and thoroughly investigated and exploited. We leave these as our future work.

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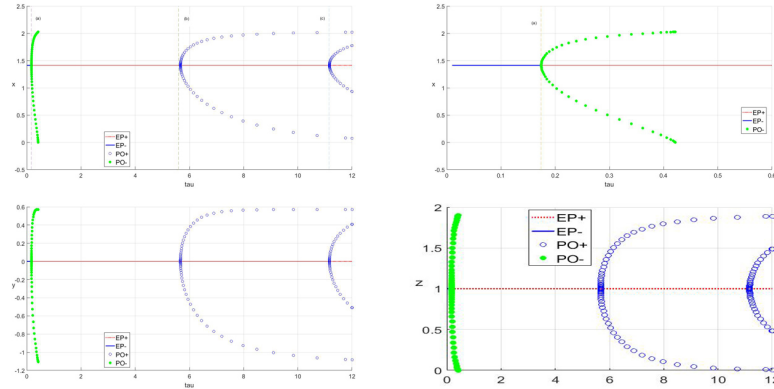


FIGURE 4. The bifurcation diagram for system (4.1). The blue solid line represents stable equilibrium point, and the red dotted line is unstable equilibrium point. Filled green circle represents stable periodic orbit, and open blue circles are unstable periodic orbits

REFERENCES

- [1] L. J. Alvarez-Vázquez, F. J. Fernández, and R. Muñoz Sola, “Analysis of a multistate control problem related to food technology,” *J. Differential Equations*, vol. 245, no. 1, pp. 130–153, 2008, doi: [10.1016/j.jde.2008.03.022](https://doi.org/10.1016/j.jde.2008.03.022). [Online]. Available: <https://doi.org/10.1016/j.jde.2008.03.022>
- [2] K. Cooke, P. van den Driessche, and X. Zou, “Interaction of maturation delay and nonlinear birth in population and epidemic models,” *J. Math. Biol.*, vol. 39, no. 4, pp. 332–352, 1999, doi: [10.1007/s002850050194](https://doi.org/10.1007/s002850050194). [Online]. Available: <https://doi.org/10.1007/s002850050194>
- [3] M. M. El-Dessoky, M. T. Yassen, and E. S. Aly, “Bifurcation analysis and chaos control in Shimizu-Morioka chaotic system with delayed feedback,” *Appl. Math. Comput.*, vol. 243, pp. 283–297, 2014, doi: [10.1016/j.amc.2014.05.072](https://doi.org/10.1016/j.amc.2014.05.072). [Online]. Available: <https://doi.org/10.1016/j.amc.2014.05.072>
- [4] K. Engelborghs, T. Luzyanina, and D. Roose, “Numerical bifurcation analysis of delay differential equations using DDE-BIFTOOL,” *ACM Trans. Math. Software*, vol. 28, no. 1, pp. 1–21, 2002, doi: [10.1145/513001.513002](https://doi.org/10.1145/513001.513002). [Online]. Available: <https://doi.org/10.1145/513001.513002>
- [5] J. Hale, *Theory of functional differential equations*, 2nd ed. Springer, 1977, applied Mathematical Sciences, Vol. 3.
- [6] B. D. Hassard, N. D. Kazarinoff, and Y. H. Wan, *Theory and applications of Hopf bifurcation*, ser. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge-New York, 1981, vol. 41.
- [7] D. C. Hill and D. S. Shafer, “Asymptotics and stability of the delayed Duffing equation,” *J. Differential Equations*, vol. 265, no. 1, pp. 33–68, 2018, doi: [10.1016/j.jde.2018.02.029](https://doi.org/10.1016/j.jde.2018.02.029). [Online]. Available: <https://doi.org/10.1016/j.jde.2018.02.029>
- [8] G.-P. Hu, W.-T. Li, and X.-P. Yan, “Hopf bifurcations in a predator-prey system with multiple delays,” *Chaos Solitons Fractals*, vol. 42, no. 2, pp. 1273–1285, 2009, doi: [10.1016/j.chaos.2009.03.075](https://doi.org/10.1016/j.chaos.2009.03.075). [Online]. Available: <http://dx.doi.org/10.1016/j.chaos.2009.03.075>
- [9] X. Liao, F. Xu, P. Wang, and P. Yu, “Chaos control and synchronization for a special generalized Lorenz canonical system—the SM system,” *Chaos Solitons Fractals*,

- vol. 39, no. 5, pp. 2491–2508, 2009, doi: [10.1016/j.chaos.2007.07.029](https://doi.org/10.1016/j.chaos.2007.07.029). [Online]. Available: <https://doi.org/10.1016/j.chaos.2007.07.029>
- [10] MATLAB, *version 9.1.0 (R2016b)*. Natick, Massachusetts: The MathWorks Inc.
- [11] A. V. Rezounenko and J. Wu, “A non-local PDE model for population dynamics with state-selective delay: local theory and global attractors,” *J. Comput. Appl. Math.*, vol. 190, no. 1-2, pp. 99–113, 2006, doi: [10.1016/j.cam.2005.01.047](https://doi.org/10.1016/j.cam.2005.01.047). [Online]. Available: <https://doi.org/10.1016/j.cam.2005.01.047>
- [12] J. I. Richards and H. K. Youn, *The theory of distributions: a nontechnical introduction*. CUP Archive, 1995.
- [13] L. F. Shampine and S. Thompson, “Solving DDEs in MATLAB,” *Appl. Numer. Math.*, vol. 37, no. 4, pp. 441–458, 2001, doi: [10.1016/S0168-9274\(00\)00055-6](https://doi.org/10.1016/S0168-9274(00)00055-6). [Online]. Available: [https://doi.org/10.1016/S0168-9274\(00\)00055-6](https://doi.org/10.1016/S0168-9274(00)00055-6)
- [14] L. F. Shampine, S. Thompson, and J. Kierzenka, “Solving delay differential equations with dde23,” URL <http://www.runet.edu/~thompson/webddes/tutorial.pdf>, 2000.
- [15] A. L. Shilprime nikov, “On bifurcations of the Lorenz attractor in the Shimizu-Morioka model,” *Phys. D*, vol. 62, no. 1-4, pp. 338–346, 1993, homoclinic chaos (Brussels, 1991), doi: [10.1016/0167-2789\(93\)90292-9](https://doi.org/10.1016/0167-2789(93)90292-9). [Online]. Available: [https://doi.org/10.1016/0167-2789\(93\)90292-9](https://doi.org/10.1016/0167-2789(93)90292-9)
- [16] T. Shimizu and N. Morioka, “On the bifurcation of a symmetric limit cycle to an asymmetric one in a simple model,” *Phys. Lett. A*, vol. 76, no. 3-4, pp. 201–204, 1980, doi: [10.1016/0375-9601\(80\)90466-1](https://doi.org/10.1016/0375-9601(80)90466-1). [Online]. Available: [https://doi.org/10.1016/0375-9601\(80\)90466-1](https://doi.org/10.1016/0375-9601(80)90466-1)
- [17] H. Smith, “An introduction to delay differential equations with applications to the life sciences,” vol. 57, pp. xii+172, 2011, doi: [10.1007/978-1-4419-7646-8](https://doi.org/10.1007/978-1-4419-7646-8). [Online]. Available: <https://doi.org/10.1007/978-1-4419-7646-8>
- [18] G. Tigan and D. Turaev, “Analytical search for homoclinic bifurcations in the Shimizu-Morioka model,” *Phys. D*, vol. 240, no. 12, pp. 985–989, 2011, doi: [10.1016/j.physd.2011.02.013](https://doi.org/10.1016/j.physd.2011.02.013). [Online]. Available: <https://doi.org/10.1016/j.physd.2011.02.013>
- [19] S. Wiggins, *Introduction to applied nonlinear dynamical systems and chaos*. Springer Science & Business Media, 2003, vol. 2.
- [20] W. Zhang, J. Li, and M. Chen, “Global exponential stability and existence of periodic solutions for delayed reaction-diffusion BAM neural networks with Dirichlet boundary conditions,” *Bound. Value Probl.*, pp. 2013:105, 23, 2013, doi: [10.1186/1687-2770-2013-105](https://doi.org/10.1186/1687-2770-2013-105). [Online]. Available: <https://doi.org/10.1186/1687-2770-2013-105>
- [21] X. Zhang, Z. Zhang, and M. J. Wade, “Dynamical analysis of a competition and cooperation system with multiple delays,” *Bound. Value Probl.*, pp. Paper No. 111, 13, 2018, doi: [10.1186/s13661-018-1032-9](https://doi.org/10.1186/s13661-018-1032-9). [Online]. Available: <https://doi.org/10.1186/s13661-018-1032-9>

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A NOTE ON BOUNDED SOLUTIONS OF AN ITERATIVE EQUATION

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Abstract. In this paper, we use Schauder and Banach fixed point theorems to study the existence, uniqueness and stability of bounded nonhomogeneous iterative functional differential equations of the form

$$x'(t) = \lambda_1 x(t) + \lambda_2 x^{[2]}(t) + \dots + \lambda_n x^{[n]}(t) + f(t).$$

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1. INTRODUCTION

Differential equations of the form

$$x'(t) = f(t, x(t), x(t - \tau_1(t)), \dots, x(t - \tau_k(t)))$$

were discussed in [1] and [5]. In particular, the delay functions $\tau_j(z)$, $j = 0, 1, \dots, k$ depend not only on unknown function, but also state, $\tau_j(z, x(z))$, $j = 0, 1, \dots, k$ have been studied in many literatures. In [2], Cooke points out that it is highly desirable to establish the existence and stability of periodic solutions for equations of the form

$$x'(t) + ax(t - h(t, x(t))) = F(t),$$

in which the lag $h(t, x(t))$ implicitly involves $x(t)$. Eder [3] considered the iterative functional differential equation

$$x'(t) = x^{[2]}(t)$$

and obtains that every solution either vanishes identically or is strictly monotonic. Fečkan [4] studied the equation

$$x'(t) = f(x^{[2]}(t))$$

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by obtaining an existence theorem for solutions satisfying $x(0) = 0$. Staněk [9] studies the global properties of solutions of functional differential equation $x'(t) = x(t) + x(x(t))$, and shows that every solution either vanishes identically or is strictly monotonic. Later, in [6], Si, Li and Cheng considered the equation

$$x'(t) = x^{[m]}(t)$$

and establish sufficient conditions for the existence of analytic solutions. Si and Wang [7] discuss the smooth solutions of equation

$$x'(t) = \lambda_1 x(t) + \lambda_2 x^{[2]}(t) + \dots + \lambda_n x^{[n]}(t) + f(t). \quad (1.1)$$

Particularly, in 2017, Zeng, Zhang, Lu and Zhang ([11]) considered a general n -dimensional mixed type differential equation with state dependence, give the existence of solutions without requiring the return condition, and apply the result to differential equations with iterates. For some various properties of solutions for several delay functional differential equations, we refer the interested reader to [8, 10].

In this paper, we consider the existence of bounded solutions of Eq. (1.1).

For $L > 0$, define the set of all real valued continuous functions from $\mathbb{R}$ into $\mathbb{R}$

$$\mathcal{B}(L) = \left\{ x \in C(\mathbb{R}, \mathbb{R}) : \|x\| \leq L, \forall t \in \mathbb{R} \right\}.$$

Then $\mathcal{B}(L)$ is a Banach space with the norm

$$\|x\| = \max_{t \in \mathbb{R}} |x(t)|.$$

For $P > 0, L \geq 0$, define the sets

$$\mathcal{B}(L, P) = \left\{ x \in \mathcal{P}_T : \|x\| \leq L, |x(t_2) - x(t_1)| \leq P|t_2 - t_1|, \forall t_1, t_2 \in \mathbb{R} \right\},$$

which are closed convex and bounded subset of $\mathcal{B}(L)$, and we wish to find bounded functions $x \in \mathcal{B}(L, P)$ satisfying (1.1).

2. EXISTENCE OF BOUNDED SOLUTIONS

In this section, the existence of bounded solutions of Eq. (1.1) will be proved. Let us state the Schauder fixed point theorem, which will be used to prove our main theorem.

Theorem 1 (Schauder). *Let Ω be a closed convex compact subset of a Banach space. Suppose that $A : \Omega \rightarrow \Omega$ is continuous. Then there exists $z \in \Omega$ with $z = Az$.*

Throughout this paper, we assume that all functions are continuous with respect to their arguments and the following condition holds.

We begin with the following lemma.

Lemma 1. For any $\varphi, \psi \in \mathcal{B}(L, P)$,

$$\|\varphi^{[n]} - \psi^{[n]}\| \leq \sum_{j=0}^{n-1} P^j \|\varphi - \psi\|, \quad n = 1, 2, \dots \quad (2.1)$$

Proof. The result follows from the definition of $\mathcal{B}(L, P)$. $\square$

Now we rewrite (1.1) as a fixed point equation.

Lemma 2. Suppose $\lambda_1 > 0$. Then $x(t)$ is a solution of (1.1) if and only if

$$x(t) = - \sum_{l=2}^n \lambda_l \int_t^{+\infty} x^{[l]}(s) e^{\lambda_1(t-s)} ds - \int_t^{+\infty} f(s) e^{\lambda_1(t-s)} ds. \quad (2.2)$$

Proof. The proof is well-known but we present it here for the reader convenience. It is easy to see that Eq. (1.1) can be written in the form of

$$x'(t) e^{-\lambda_1 t} - \lambda_1 x(t) e^{-\lambda_1 t} = \left(\sum_{l=2}^n \lambda_l x^{[l]}(t) + f(t) \right) e^{-\lambda_1 t}.$$

If $x(t)$ is a solution of (1.1), then integrating the above equality from t to $+\infty$, we obtain

$$x(t) = - \sum_{l=2}^n \lambda_l \int_t^{+\infty} x^{[l]}(s) e^{\lambda_1(t-s)} ds - \int_t^{+\infty} f(s) e^{\lambda_1(t-s)} ds.$$

This completes the proof. $\square$

Lemma 3. Suppose $\lambda_1 < 0$. Then $x(t)$ is a solution of (1.1) if and only if

$$x(t) = \sum_{l=2}^n \lambda_l \int_{-\infty}^t x^{[l]}(s) e^{\lambda_1(t-s)} ds + \int_{-\infty}^t f(s) e^{\lambda_1(t-s)} ds. \quad (2.3)$$

Proof. As in Lemma 2, Eq. (1.1) can be written in the form of

$$x'(t) e^{-\lambda_1 t} - \lambda_1 x(t) e^{-\lambda_1 t} = \left(\sum_{l=2}^n \lambda_l x^{[l]}(t) + f(t) \right) e^{-\lambda_1 t}.$$

If $x(t)$ is a solution of (1.1), then integrating the above equality from $-\infty$ to t , we obtain

$$x(t) = \sum_{l=2}^n \lambda_l \int_{-\infty}^t x^{[l]}(s) e^{\lambda_1(t-s)} ds + \int_{-\infty}^t f(s) e^{\lambda_1(t-s)} ds.$$

This completes the proof. $\square$

Now we will need to construct a mapping satisfying the hypotheses of Theorem 1. To this aim, we consider maps $A, A_1 : \mathcal{B}(L, P) \rightarrow \mathbb{R}$ defined as follows:

$$(Ax)(t) = - \sum_{l=2}^n \lambda_l \int_t^{+\infty} x^{[l]}(s) e^{\lambda_1(t-s)} ds - \int_t^{+\infty} f(s) e^{\lambda_1(t-s)} ds, \quad (2.4)$$

$$(A_1x)(t) = \sum_{l=2}^n \lambda_l \int_{-\infty}^t x^{[l]}(s) e^{\lambda_1(t-s)} ds + \int_{-\infty}^t f(s) e^{\lambda_1(t-s)} ds, \quad (2.5)$$

Lemma 4. *Operator A and A_1 are Lipschitz continuous.*

Proof. Take $x, y \in \mathcal{B}(L, P), t \in \mathbb{R}$, then by (2.1) and (2.4), we have

$$\begin{aligned} |(Ax)(t) - (Ay)(t)| & \leq \sum_{l=2}^n |\lambda_l| \left| \int_t^{+\infty} (x^{[l]}(s) - y^{[l]}(s)) e^{\lambda_1(t-s)} ds \right| \\ & \leq \frac{1}{\lambda_1} \sum_{l=2}^n \sum_{j=0}^{l-1} |\lambda_l| P^j \|x - y\|. \end{aligned}$$

This proves A is Lipschitz continuous.

Using the same method, it is easy to prove A_1 is Lipschitz continuous. This completes the proof. $\square$

It is easy to see by Arzela-Ascoli theorem that $\mathcal{B}(L, P)$ is compact. Now, we are ready to prove the following existence result.

Theorem 2. *Suppose $f \in \mathcal{B}(\widetilde{L}, \widetilde{P})$, $\lambda_1 > 0$, $\sum_{l=2}^n |\lambda_l| < \min\{\lambda_1, \frac{1}{2}\}$ and the following inequalities hold*

$$\widetilde{L} \leq (\lambda_1 - \sum_{l=2}^n |\lambda_l|) L, \quad (2.6)$$

$$\widetilde{P} \leq (\frac{1}{2} - \sum_{l=2}^n |\lambda_l|) P, \quad (2.7)$$

then Eq. (1.1) has a solution in $\mathcal{B}(L, P)$.

Proof. First, for any $x, y \in \mathcal{B}(L, P)$, by (2.6), we have

$$\begin{aligned} |(Ax)(t)| & \leq \sum_{l=2}^n |\lambda_l| \left| \int_t^{+\infty} x^{[l]}(s) e^{\lambda_1(t-s)} ds \right| + \left| \int_t^{+\infty} f(s) e^{\lambda_1(t-s)} ds \right| \\ & \leq \frac{L}{\lambda_1} \sum_{l=2}^n |\lambda_l| + \frac{\widetilde{L}}{\lambda_1} \end{aligned}$$

$$\leq L.$$

Next, assuming $t_1 \leq t_2$, by (2.7), we obtain

$$\begin{aligned} & \left| \int_{t_2}^{+\infty} x^{[l]}(s) e^{\lambda_1(t_2-s)} ds - \int_{t_1}^{+\infty} x^{[l]}(s) e^{\lambda_1(t_1-s)} ds \right| \\ & \leq \left| \int_{t_2}^{+\infty} x^{[l]}(s) (e^{\lambda_1 t_2} - e^{\lambda_1 t_1}) e^{-\lambda_1 s} ds \right| + \left| \int_{t_1}^{t_2} x^{[l]}(s) e^{\lambda_1(t_1-s)} ds \right| \\ & \leq \frac{P}{\lambda_1} (1 - e^{-\lambda_1(t_2-t_1)}) + \frac{P}{\lambda_1} (1 - e^{-\lambda_1(t_2-t_1)}) \\ & \leq \frac{2P}{\lambda_1} e^{\xi} (t_2 - t_1) \lambda_1 \\ & \leq 2P(t_2 - t_1), \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} & \left| \int_{t_2}^{+\infty} f(s) e^{\lambda_1(t_2-s)} ds - \int_{t_1}^{+\infty} f(s) e^{\lambda_1(t_1-s)} ds \right| \\ & \leq \left| \int_{t_2}^{+\infty} f(s) (e^{\lambda_1 t_2} - e^{\lambda_1 t_1}) e^{-\lambda_1 s} ds \right| + \left| \int_{t_1}^{t_2} f(s) e^{\lambda_1(t_1-s)} ds \right| \\ & \leq \frac{2\tilde{P}}{\lambda_1} e^{\xi} (t_2 - t_1) \lambda_1 \\ & \leq 2\tilde{P}(t_2 - t_1), \end{aligned} \tag{2.9}$$

where $-\lambda_1(t_2 - t_1) < \xi < 0$,

By (2.8), (2.9) and (2.7), we have

$$\begin{aligned} & \left| (Ax)(t_2) - (Ax)(t_1) \right| \\ & \leq \sum_{l=2}^n |\lambda_l| \left| \int_{t_2}^{+\infty} x^{[l]}(s) e^{\lambda_1(t_2-s)} ds - \int_{t_1}^{+\infty} x^{[l]}(s) e^{\lambda_1(t_1-s)} ds \right| \\ & \quad + \left| \int_{t_2}^{+\infty} f(s) e^{\lambda_1(t_2-s)} ds - \int_{t_1}^{+\infty} f(s) e^{\lambda_1(t_1-s)} ds \right| \\ & \leq 2 \left(\sum_{l=2}^n |\lambda_l| P + \tilde{P} \right) (t_2 - t_1) \\ & \leq P(t_2 - t_1). \end{aligned}$$

So by Lemma 4, we see that all conditions of Schauder's theorem are satisfied on $\mathcal{B}(L, P)$. Thus there exists a fixed point x in $\mathcal{B}(L, P)$ such that $x = Ax$, from Lemma 2, x is a bounded solution of Eq. (1.1). This completes the proof. $\square$

Theorem 3. Suppose $f \in \mathcal{B}(\tilde{L}, \tilde{P})$, $\lambda_1 < 0$, $\sum_{l=2}^n |\lambda_l| < \min\{-\lambda_1, \frac{1}{2}\}$,

$$\tilde{L} \leq (-\lambda_1 - \sum_{l=2}^n |\lambda_l|)L, \quad (2.10)$$

and (2.7) holds, then Eq. (1.1) has a solution in $\mathcal{B}(L, P)$.

Proof. First, for any $x, y \in \mathcal{B}(L, P)$, by (2.10), we have

$$\begin{aligned} |(A_1 x)(t)| &\leq \sum_{l=2}^n |\lambda_l| \left| \int_{-\infty}^t x^{[l]}(s) e^{\lambda_1(t-s)} ds \right| + \left| \int_{-\infty}^t f(s) e^{\lambda_1(t-s)} ds \right| \\ &\leq -\frac{L}{\lambda_1} \sum_{l=2}^n |\lambda_l| - \frac{\tilde{L}}{\lambda_1} \\ &\leq L. \end{aligned}$$

Next, assuming $t_1 \leq t_2$, we obtain

$$\begin{aligned} &\left| \int_{-\infty}^{t_2} x^{[l]}(s) e^{\lambda_1(t_2-s)} ds - \int_{-\infty}^{t_1} x^{[l]}(s) e^{\lambda_1(t_1-s)} ds \right| \\ &\leq \left| \int_{-\infty}^{t_1} x^{[l]}(s) (e^{\lambda_1 t_2} - e^{\lambda_1 t_1}) e^{-\lambda_1 s} ds \right| + \left| \int_{t_1}^{t_2} x^{[l]}(s) e^{\lambda_1(t_2-s)} ds \right| \\ &\leq -\frac{P}{\lambda_1} (1 - e^{\lambda_1(t_2-t_1)}) - \frac{P}{\lambda_1} (1 - e^{\lambda_1(t_2-t_1)}) \\ &\leq \frac{2P}{-\lambda_1} e^{\xi_1} (t_2 - t_1) (-\lambda_1) \\ &\leq 2P(t_2 - t_1), \end{aligned} \quad (2.11)$$

and

$$\begin{aligned} &\left| \int_{-\infty}^{t_2} f(s) e^{\lambda_1(t_2-s)} ds - \int_{-\infty}^{t_1} f(s) e^{\lambda_1(t_1-s)} ds \right| \\ &\leq \left| \int_{-\infty}^{t_1} f(s) (e^{\lambda_1 t_2} - e^{\lambda_1 t_1}) e^{-\lambda_1 s} ds \right| + \left| \int_{t_1}^{t_2} f(s) e^{\lambda_1(t_2-s)} ds \right| \\ &\leq \frac{2\tilde{P}}{-\lambda_1} e^{\xi_1} (t_2 - t_1) (-\lambda_1) \\ &\leq 2\tilde{P}(t_2 - t_1), \end{aligned} \quad (2.12)$$

where $\lambda_1(t_2 - t_1) < \xi_1 < 0$.

By (2.11), (2.12) and (2.10), we have

$$\begin{aligned} & \left| (Ax)(t_2) - (Ax)(t_1) \right| \\ & \leq \sum_{l=2}^n |\lambda_l| \left| \int_{-\infty}^{t_2} x^{[l]}(s) e^{\lambda_1(t_2-s)} ds - \int_{-\infty}^{t_1} x^{[l]}(s) e^{\lambda_1(t_1-s)} ds \right| \\ & \quad + \left| \int_{-\infty}^{t_2} f(s) e^{\lambda_1(t_2-s)} ds - \int_{-\infty}^{t_1} f(s) e^{\lambda_1(t_1-s)} ds \right| \\ & \leq 2 \left(\sum_{l=2}^n |\lambda_l| P + \widetilde{P} \right) (t_2 - t_1) \\ & \leq P(t_2 - t_1). \end{aligned}$$

So by Lemma 4, we see that all conditions of Schauder's theorem are satisfied on $\mathcal{B}(L, P)$. Thus there exists a fixed point x in $\mathcal{B}(L, P)$ such that $x = Ax$, from Lemma 3, x is a bounded solution of Eq. (1.1). This completes the proof. $\square$

From Theorems 2 and 3 we have the following theorem.

Theorem 4. Suppose $f \in \mathcal{B}(\widetilde{L}, \widetilde{P})$, $\lambda_1 \neq 0$, $\sum_{l=2}^n |\lambda_l| < \min\{|\lambda_1|, \frac{1}{2}\}$ and the following inequalities hold

$$\widetilde{L} \leq (|\lambda_1| - \sum_{l=2}^n |\lambda_l|) L, \quad (2.13)$$

$$\widetilde{P} \leq \left(\frac{1}{2} - \sum_{l=2}^n |\lambda_l| \right) P, \quad (2.14)$$

then Eq. (1.1) has a solution in $\mathcal{B}(L, P)$.

3. UNIQUENESS AND STABILITY

In this section, uniqueness and stability of (1.1) will be proved.

Theorem 5. In addition to the assumption of Theorem 4, suppose that

$$\Gamma = \frac{1}{|\lambda_1|} \sum_{l=2}^n \sum_{j=0}^{l-1} |\lambda_l| P^j < 1, \quad (3.1)$$

then Eq. (1.1) has a unique solution in $\mathcal{B}(L, P)$.

Proof. We know from the proof of Theorem 2 that $A : \mathcal{B}(L, P) \rightarrow \mathcal{B}(L, P)$. Moreover, by Lemma 2, we get

$$\|A\varphi - A\psi\| \leq \Gamma \|\varphi - \psi\|, \quad \varphi, \psi \in \mathcal{B}(L, P)$$

(3.1) means $\Gamma < 1$, so the fixed point must be unique by the Banach fixed point theorem. $\square$

Theorem 6. *The unique solution obtained in Theorem 5 depends continuously on the given constants λ_l and function $f(t)$, for $l = 1, \dots, n$.*

Proof. First, assuming $\lambda_1, \mu_1 > 0$. Let constants $\lambda_l, \mu_l, l = 2, \dots, n$ and functions $f(t), \tilde{f}(t) \in \mathcal{B}(\tilde{L}, \tilde{P})$ be given. Then we consider the corresponding operators $A, \tilde{A}$, defined by (2.4), respectively. Assuming corresponding conditions (2.6), (2.7) and (3.1), there are two unique corresponding functions $x(t)$ and $\tilde{x}(t)$ in $\mathcal{B}(L, P)$ such that

$$x = Ax, \quad \tilde{x} = \tilde{A}\tilde{x}.$$

Then we have

$$\|x - \tilde{x}\| \leq \|Ax - A\tilde{x}\| + \|A\tilde{x} - \tilde{A}\tilde{x}\| \leq \Gamma \|x - \tilde{x}\| + \|A\tilde{x} - \tilde{A}\tilde{x}\|,$$

which implies

$$\|x - \tilde{x}\| \leq \frac{\|A\tilde{x} - \tilde{A}\tilde{x}\|}{1 - \Gamma}. \quad (3.2)$$

If $\lambda_1 \geq \mu_1 > 0$, then

$$\begin{aligned} & \int_t^{+\infty} |e^{\lambda_1(t-s)} - e^{\mu_1(t-s)}| ds \\ &= \int_t^{+\infty} (e^{\mu_1(t-s)} - e^{\lambda_1(t-s)}) ds = \frac{1}{\mu_1} - \frac{1}{\lambda_1} \end{aligned} \quad (3.3)$$

If $\mu_1 \geq \lambda_1 > 0$, then

$$\begin{aligned} & \int_t^{+\infty} |e^{\lambda_1(t-s)} - e^{\mu_1(t-s)}| ds \\ &= \int_t^{+\infty} (e^{\lambda_1(t-s)} - e^{\mu_1(t-s)}) ds = \frac{1}{\lambda_1} - \frac{1}{\mu_1} \end{aligned} \quad (3.4)$$

From (3.3) and (3.4), we have

$$\begin{aligned} & |(A\tilde{x})(t) - (\tilde{A}\tilde{x})(t)| \\ & \leq \sum_{l=2}^n |\lambda_l - \mu_l| \left| \int_t^{+\infty} \tilde{x}^{[l]}(s) e^{\lambda_1(t-s)} ds \right| \\ & \quad + \sum_{l=2}^n |\mu_l| \int_t^{+\infty} |\tilde{x}^{[l]}(s)| |e^{\lambda_1(t-s)} - e^{\mu_1(t-s)}| ds \\ & \quad + \int_t^{+\infty} |f(s) - \tilde{f}(s)| e^{\lambda_1(t-s)} ds + \int_t^{+\infty} |\tilde{f}(s)| |e^{\lambda_1(t-s)} - e^{\mu_1(t-s)}| ds \end{aligned}$$

$$\begin{aligned}
&\leq \frac{P}{\lambda_1} \sum_{l=2}^n |\lambda_l - \mu_l| + P \sum_{l=2}^n |\mu_l| \left| \frac{1}{\lambda_1} - \frac{1}{\mu_1} \right| + \frac{1}{\lambda_1} \|f - \tilde{f}\| + P \left| \frac{1}{\lambda_1} - \frac{1}{\mu_1} \right| \\
&= (P + P \sum_{l=2}^n |\mu_l|) \left| \frac{1}{\lambda_1} - \frac{1}{\mu_1} \right| + \frac{P}{\lambda_1} \sum_{l=2}^n |\lambda_l - \mu_l| + \frac{1}{\lambda_1} \|f - \tilde{f}\|,
\end{aligned}$$

by (3.2), $\|x - \tilde{x}\|$ tends to 0 when μ_l tends to λ_l and $\tilde{f}$ tends to f .

For the case $\lambda_1, \mu_1 < 0$, we have the similar proof. This completes the proof. $\square$

4. EXAMPLES

First, we show that the conditions in Theorem 2 do not self-contradict. Consider the following equation:

Example 1.

$$x'(t) = \frac{1}{3}x(t) + \frac{1}{4}x^{[2]}(t) + \frac{1}{12}\sin t, \quad (4.1)$$

where $\lambda_1 = \frac{1}{3}, \lambda_2 = \frac{1}{4}, f(t) = \frac{1}{12}\sin t$. Take $P = 1, L = 1, \tilde{P} = \frac{1}{12}, \tilde{L} = \frac{1}{12}$. A simple calculation yields

$$\begin{aligned}
\tilde{L} &= \frac{1}{12} \leq \frac{1}{12} = (\lambda_1 - \sum_{l=2}^n |\lambda_l|)L, \\
\tilde{P} &= \frac{1}{12} \leq \frac{1}{4} = (\frac{1}{2} - \sum_{l=2}^n |\lambda_l|)P.
\end{aligned}$$

By Theorem 4, Eq. (4.1) has a bounded solution x such that $|x(t)| \leq 1$, and $|x(t_2) - x(t_1)| \leq |t_2 - t_1|, \forall t_1, t_2 \in \mathbb{R}$.

Next, we apply Theorem 3.

Example 2. Consider equation:

$$x'(t) = \frac{1}{3}x(t) + \frac{1}{4}x^{[2]}(t) + \frac{1}{16}\sin t, \quad (4.2)$$

where $\lambda_1 = \frac{1}{3}, \lambda_2 = \frac{1}{4}, f(t) = \frac{1}{16}\sin t$. Take $P = \frac{7}{24}, L = 1, \tilde{P} = \frac{1}{16}, \tilde{L} = \frac{1}{16}$. A simple calculation yields

$$\begin{aligned}
\tilde{L} &= \frac{1}{16} \leq \frac{1}{12} = (-\lambda_1 - \sum_{l=2}^n |\lambda_l|)L, \\
\tilde{P} &= \frac{1}{16} \leq \frac{7}{96} = (\frac{1}{2} - \sum_{l=2}^n |\lambda_l|)P.
\end{aligned}$$

By Theorem 2, Eq. (4.1) has a bounded solution x such that $|x(t)| \leq 1$, and $|x(t_2) - x(t_1)| \leq |t_2 - t_1|$, $\forall t_1, t_2 \in \mathbb{R}$. Furthermore,

$$\Gamma = \frac{1}{|\lambda_1|} \sum_{l=2}^n \sum_{j=0}^{l-1} |\lambda_l| P^j = \frac{31}{32} < 1,$$

by Theorem 5, we know this bounded solution is unique.

Example 3.

$$x'(t) = \frac{1}{3}x(t) + \frac{1}{4}x^{[2]}(t) + \delta \sin t, \quad (4.3)$$

where $\delta > 0$ is a parameter. As Example 1 or 2, here $\lambda_1 = \frac{1}{3}, \lambda_2 = \frac{1}{4}, f(t) = \delta \sin t$ and $\tilde{P} = \delta, \tilde{L} = \delta$. Next, we consider $L(\delta)$ and $P(\delta)$ as variables to be defined by δ . Then (2.6) and (2.7) have the forms

$$\delta \leq \frac{1}{12}L(\delta), \quad (4.4)$$

$$\delta \leq \frac{1}{4}P(\delta), \quad (4.5)$$

$$\frac{3}{4}(1 + P(\delta)) < 1, \quad (4.6)$$

First, by (4.4) and (4.5) we have

$$L(\delta) \geq 12\delta \quad (4.7)$$

and

$$P(\delta) \geq 4\delta. \quad (4.8)$$

Next, (4.6) shows us

$$P(\delta) < \frac{1}{3}. \quad (4.9)$$

By (4.8) and (4.9), we have

$$4\delta \leq P(\delta) < \frac{1}{3}. \quad (4.10)$$

and

$$0 < \delta \leq \frac{1}{12} \doteq 0.083333. \quad (4.11)$$

Thus, if we taking $P(\delta)$ and $L(\delta)$ satisfy (4.7) and (4.8), Theorem 2 are satisfied and Eq. (4.3) has a 2π -periodic solution such that $|x(t)| \leq P(\delta)$ and $|x(t_2) - x(t_1)| \leq L(\delta)|t_2 - t_1|$, $\forall t_1, t_2 \in \mathbb{R}$. Furthermore, if we taking $L(\delta)$ and $P(\delta)$ satisfy (4.7)-(4.11), by Theorem 5, we know the 2π -periodic solution of (4.3) is a unique one.

Remark 1. Obviously, Example 1 satisfies conditions (4.7) and (4.8), Example 2 satisfies conditions (4.7)-(4.11).

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REFERENCES

- [1] R. Bellman and K.L. Cooke, *Differential-Difference Equations*. New York: Academic Press, 1963.
- [2] K. Cooke, *Functional differential systems: some models and perturbation problems*, in: *Proceedings of the International Symposium on Differential Equations and Dynamical Systems*. New York: Academic Press, 1967.
- [3] E. Eder, "The functional differential equation $x'(t) = x(x(t))$." *J. Differential Equations.*, vol. 54, pp. 390–400, 1984, doi: [10.1016/0022-0396\(84\)90150-5](https://doi.org/10.1016/0022-0396(84)90150-5).
- [4] M. Fečkan, "On a certain type of functional differential equations." *Math. Slovaca.*, vol. 43, pp. 39–43, 1993.
- [5] J. Hale, *Theory of Functional Differential Equations*. New York: Academic Press, 1977.
- [6] J.G. Si, W.R. Li, and S.S. Cheng, "Analytic solutions of an iterative functional differential equation." *Comput. Math. Appl.*, vol. 33, pp. 47–51, 1997, doi: [10.1016/S0898-1221\(97\)00030-8](https://doi.org/10.1016/S0898-1221(97)00030-8).
- [7] J.G. Si and X.P. Wang, "Smooth solutions of a nonhomogeneous iterative functional differential equation with variable coefficients." *J. Math. Anal. Appl.*, vol. 226, pp. 377–392, 1998, doi: [10.1006/jmaa.1998.6086](https://doi.org/10.1006/jmaa.1998.6086).
- [8] S.N. Chow, "Existence of periodic solutions of autonomous functional differential equations." *J. Differential Equations.*, vol. 15, pp. 350–378, 1974, doi: [10.1016/0022-0396\(74\)90084-9](https://doi.org/10.1016/0022-0396(74)90084-9).
- [9] S. Staněk, "On global properties of solutions of functional differential equation $x'(t) = x(t) + x(x(t))$." *Dynam. Systems Appl.*, vol. 4, pp. 263–278, 1995.
- [10] T.A. Burton, *Stability by Fixed Point Theory for Functional Differential Equations*. New York: Dover, 2006.
- [11] Y.Y. Zeng, P.P. Zhang, T.T. Lu, and W.N. Zhang, "Existence of solutions for a mixed type differential equation with state-dependence." *J. Math. Anal. Appl.*, vol. 453, pp. 629–644, 2017, doi: [10.1016/j.jmaa.2017.04.020](https://doi.org/10.1016/j.jmaa.2017.04.020).

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STABLE ANALYSIS FOR A DELAYED POPULATION MODEL WITH SURVIVAL RATE

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Abstract. In this paper, we consider a discrete population model and obtain a couple of criteria to guarantee our model being stable, including the global attractor, the extreme stability, and the periodicity.

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Keywords: delayed population model, invariant set, extreme stability, periodic solution, fixed point theorem

1. INTRODUCTION

This paper is devoted to consider the stable behavior for a discrete population model with survival rate given by

$$x(t+1) = \gamma(t)x(t) + x(\tau(t))e^{r(t)\left(1-\frac{x(\tau(t))}{k(t)}\right)}, \quad t \in \mathbb{Z}_+, \quad (1.1)$$

where $\mathbb{Z}_+$ is the set of nonnegative integers, $\gamma(t) \in (0, 1)$ is the survival rate for each $t \in \mathbb{Z}_+$, $\tau(t) \in \mathbb{Z}_+$ denotes the delay in t , and $r(t)$ and $k(t)$ stand respectively, for the intrinsic growth rate and the carrying capacity of the habitat.

The motivation of our considerations stems from the papers [2, 3, 5, 6, 8]. Concretely speaking, Diagana in [2] studied the periodic problem of population models of the form

$$x(t+1) = \gamma(t)x(t) + f(t, x(t)), \quad t \in \mathbb{Z},$$

where $\mathbb{Z}$ is the set of integers. Hamaya considered the almost periodic solutions in [3] for the discrete Ricker delay model

$$x(t+1) = x(t)e^{f(t, x(t-\tau(t)))}, \quad t \in \mathbb{Z}_+.$$

In [5, 8] Liz and Saker et al. discussed respectively, the qualitative behavior of periodic Ricker equation

$$x(t+1) = x(t)e^{r(t)-x(t)}$$

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and in [6], the authors investigated the extreme stability of the following discrete logistic equation

$$x(t+1) = x(t)e^{r(t)\left(1-\frac{x(t)}{k(t)}\right)}, \quad t \in \mathbb{Z}_+.$$

We observe that the stability is of vital importance when considering the periodicity of a model, see, for example, [3, 6, 9]. In this paper, we will concentrate our attention on the stability of (1.1), including the global attractor, the extremely stability and the periodicity.

In what follows, we always assume that

(H1) there exist γ_0 and γ_1 in $(0, 1)$, and an integer $\tau_0 > 0$ such that

$$\inf_{t \in \mathbb{Z}_+} \gamma(t) = \gamma_0, \quad \sup_{t \in \mathbb{Z}_+} \gamma(t) = \gamma_1$$

and

$$t - \tau_0 \leq \tau(t) \leq t \quad \text{for } t \in \mathbb{Z}_+;$$

(H2) there exist r_i and k_i in $(0, \infty)$ for $i = 0, 1$, such that

$$r_0 \leq r(t) \leq r_1 \quad \text{and} \quad k_0 \leq k(t) \leq k_1 \quad \text{for } t \in \mathbb{Z}_+,$$

where

$$\inf_{t \in \mathbb{Z}_+} r(t) = r_0, \quad \sup_{t \in \mathbb{Z}_+} r(t) = r_1, \quad \inf_{t \in \mathbb{Z}_+} k(t) = k_0, \quad \sup_{t \in \mathbb{Z}_+} k(t) = k_1;$$

(H3) the functions $L(x)$ and $U(x)$, and the constants M and m are defined, respectively, by

$$L(x) = xe^{r_0 - \frac{r_1 x}{k_0}} \quad \text{and} \quad U(x) = xe^{r_1 - \frac{r_0 x}{k_1}} \quad \text{for } x \geq 0$$

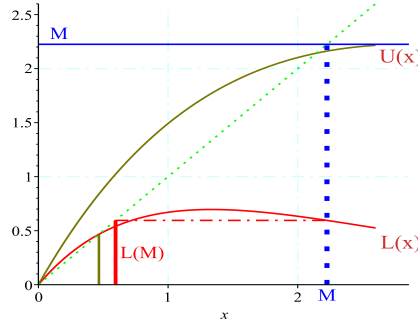
as well as

$$M = U\left(\frac{k_1}{r_0}\right) \quad \text{and} \quad m = L\left(\frac{M}{1 - \gamma_1}\right);$$

(H4) for $\delta \geq 1$, the constant Δ is defined by

$$\Delta = \begin{cases} L(\delta M(1 - \gamma_1)^{-1}) = \frac{\delta M}{1 - \gamma_1} e^{r_0 - \frac{\delta r_1 M}{k_0(1 - \gamma_1)}}, & L(\delta M(1 - \gamma_1)^{-1}) < \frac{r_0 k_0}{r_1}, \\ \frac{r_0 k_0}{r_1}, & \text{otherwise,} \end{cases}$$

where $\frac{r_0 k_0}{r_1}$ is the fixed point of $L(x)$. See the following diagram when allowing $\delta = 1$ and $(1 - \gamma_1)^{-1} = 1$:



Let $\mathcal{D}[-\tau_0, 0]$ be the set of all discrete functions $\varphi : \{-\tau_0, -\tau_0 + 1, \dots, 0\} \rightarrow \mathbb{R}$ with the norm $|\varphi| = \max_{-\tau_0 \leq t \leq 0} |\varphi(t)|$. For a given $\varphi \in \mathcal{D}[-\tau_0, 0]$, from (1.1) we can get a unique sequence $\{x(t)\}_{t \geq -\tau_0}$ satisfying $x(t) = \varphi(t)$ for $t = -\tau_0, -\tau_0 + 1, \dots, 0$. This sequence is called a solution of (1.1), and sometimes we denote this solution by $\{x(t; \varphi)\}_{t \geq -\tau_0}$.

To continue our discussions in the sequel, we require some notions as follows.

Definition 1 ([9]). (i) The set $S \subset \mathbb{R}$ is said to be invariant relative to (1.1) if for $\varphi \in \mathcal{D}[-\tau_0, 0]$ with $\varphi(t) \in S$ for $t \in [-\tau_0, 0]$, the solution $\{x(t; \varphi)\}_{t \geq -\tau_0}$ of (1.1) satisfies $x(t; \varphi) \in S$ for all $t \geq -\tau_0$.

(ii) The set $S \subset \mathbb{R}$ is said to be a global attractor of (1.1) if for any $\varepsilon > 0$ and positive function $\varphi \in \mathcal{D}[-\tau_0, 0]$, there exists an integer $N(\varepsilon, \varphi) > 0$ such that the solution $\{x(t; \varphi)\}_{t \geq -\tau_0}$ of (1.1) satisfies

$$\min_{s \in S} |x(t; \varphi) - s| < \varepsilon \quad \text{for all } t \geq N(\varepsilon, \varphi).$$

Definition 2 ([6]). We say (1.1) to be extremely stable provided any two positive solutions $\{x(t)\}_{t \geq -\tau_0}$ and $\{y(t)\}_{t \geq -\tau_0}$ of (1.1) satisfy

$$\lim_{t \rightarrow \infty} |x(t) - y(t)| = 0.$$

Definition 3 ([7]). A sequence $\{x(t)\}_{t \geq -\tau_0}$ is said to be asymptotically σ -periodic if there exist two sequences $\{p(t)\}_{t \geq -\tau_0}$ and $\{q(t)\}_{t \geq -\tau_0}$ such that

$$x(t) = p(t) + q(t),$$

where $\{p(t)\}_{t \geq -\tau_0}$ is a σ -periodic sequence, and $q(t) \rightarrow 0$ as $t \rightarrow \infty$.

2. STABILITY OF (1.1)

In this section we study the stability of (1.1). For this end, we first note that any solution $\{x(t; \varphi)\}_{t \geq -\tau_0}$ of (1.1) verifies that

$$x(t; \varphi) = \prod_{v=0}^{t-1} \gamma(v) \varphi(0) + \sum_{v=0}^{t-1} \left[\prod_{s=v}^{t-2} \gamma(s+1) \right] f(v, x(\tau(v))), \quad t \geq 1, \quad (2.1)$$

where

$$f(v, u) = ue^{r(v)\left(1-\frac{u}{k(v)}\right)}, \quad \prod_{v=0}^{t-1} \gamma(v) = \gamma(0)\gamma(1)\dots\gamma(t-1),$$

and $x(\tau(v))$ expresses the term $x(\tau(v); \varphi)$.

Since (1.1) denotes a realistic model of populations, we only consider the positive initial values $\varphi \in \mathcal{D}[-\tau_0, 0]$. Then, from (2.1) it follows that the solution $\{x(t; \varphi)\}_{t \geq -\tau_0}$ of (1.1) is positive.

In addition, in view of the assumption (H2), we have

$$ue^{r_0 - \frac{r_1 u}{k_0}} \leq f(v, u) \leq ue^{r_1 - \frac{r_0 u}{k_1}} \quad \text{for } v \in \mathbb{Z}_+ \text{ and } u \geq 0, \quad (2.2)$$

which results in

$$f(v, u) \leq M = \frac{k_1}{r_0} e^{r_1 - 1} \quad \text{for } v \in \mathbb{Z}_+ \text{ and } u \geq 0. \quad (2.3)$$

Now, for the positive initial value $\varphi \in \mathcal{D}[-\tau_0, 0]$, in combination with the assumption (H1), we have from (2.1) and (2.3) that

$$\begin{aligned} x(t; \varphi) &\leq \gamma_1^t \varphi(0) + M \sum_{v=0}^{t-1} \gamma_1^{t-v-1} \\ &= \gamma_1^t \varphi(0) + M(1 - \gamma_1)^{-1}(1 - \gamma_1^t), \quad t \in \mathbb{Z}_+. \end{aligned} \quad (2.4)$$

Hence, for $|\varphi| \in (0, \delta M(1 - \gamma_1)^{-1}]$ with $\delta \geq 1$, coupled with (2.4), it holds that

$$x(t; \varphi) \leq \delta M(1 - \gamma_1)^{-1}, \quad t \in \mathbb{Z}_+. \quad (2.5)$$

On the other hand, by the assumptions (H3)–(H4) and the left inequality in (2.2) we have

$$f(v, u) \geq \Delta \quad \text{for } v \in \mathbb{Z}_+ \text{ and } u \in [\Delta(1 - \gamma_0)^{-1}, \delta M(1 - \gamma_1)^{-1}], \quad (2.6)$$

where the constant Δ defined as in (H4). Consequently, similar to (2.4) it follows from (2.1) and (2.6) that when $|\varphi| \in [\Delta(1 - \gamma_0)^{-1}, \delta M(1 - \gamma_1)^{-1}]$,

$$\begin{aligned} x(t; \varphi) &\geq \gamma_0^t \varphi(0) + \Delta \sum_{v=0}^{t-1} \gamma_0^{t-v-1} \\ &= \gamma_0^t \varphi(0) + \Delta(1 - \gamma_0)^{-1}(1 - \gamma_0^t) \\ &\geq \Delta(1 - \gamma_0)^{-1}, \quad t \in \mathbb{Z}_+. \end{aligned} \quad (2.7)$$

Summarizing (2.5) and (2.7) we obtain the following result:

Theorem 1. *Under the assumptions (H1)–(H4), the set $[\Delta(1 - \gamma_0)^{-1}, \delta M(1 - \gamma_1)^{-1}]$ is invariant relative to (1.1), here $\delta \geq 1$, and Δ is defined as in (H4).*

Let $\tilde{m} = \min\{m, \frac{r_0 k_0}{r_1}\}$. Note that in Theorem 1, $\Delta = \tilde{m}$ as $\delta = 1$. Note further that, since γ_0 and $\gamma_1 \in (0, 1)$, for any given $\varepsilon > 0$ and positive function $\varphi \in \mathcal{D}[-\tau_0, 0]$, there exists an $N(\varepsilon, \varphi) > 0$ for which it follows that

$$\gamma_1^t |\varphi(0) - M(1 - \gamma_1)^{-1}| < \varepsilon, \quad \gamma_0^t |\varphi(0) - \tilde{m}(1 - \gamma_0)^{-1}| < \varepsilon \quad \text{for } t \geq N(\varepsilon, \varphi).$$

Consequently, from (2.4) and (2.7) we have

$$x(t; \varphi) < M(1 - \gamma_1)^{-1} + \varepsilon \quad \text{for } t \geq N(\varepsilon, \varphi)$$

and

$$x(t; \varphi) > \tilde{m}(1 - \gamma_0)^{-1} - \varepsilon \quad \text{for } t \geq N(\varepsilon, \varphi).$$

This induces our second result:

Theorem 2. *Under the assumptions (H1)–(H3), the set $[\tilde{m}(1 - \gamma_0)^{-1}, M(1 - \gamma_1)^{-1}]$ is a global attractor of (1.1), here $\tilde{m} = \min\{m, \frac{r_0 k_0}{r_1}\}$.*

Next we continue to employ $\tilde{m} = \min\{m, \frac{r_0 k_0}{r_1}\}$ and consider the extreme stability with respect to (1.1). To do this, we first show the following conclusion.

Lemma 1. *Suppose that $\{x(t; \varphi)\}_{t \geq -\tau_0}$ ($\{x(t)\}$ for short) is a solution of (1.1) and*

$$\frac{\tilde{m}}{k_1(1 - \gamma_0)} > 1. \quad (2.8)$$

Then, under the assumptions (H1)–(H3), it follows that

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \left| 1 - \frac{r(t)x(\tau(t))}{k(t)} \right| e^{r(t)\left(1 - \frac{x(\tau(t))}{k(t)}\right)} \\ & \leq \max \left\{ \left| 1 - \frac{r_0 \tilde{m}}{k_1(1 - \gamma_0)} \right|, \left| 1 - \frac{r_1 M}{k_0(1 - \gamma_1)} \right| \right\}. \end{aligned} \quad (2.9)$$

Proof. We proceed in steps.

Assertion 1:

$$\limsup_{t \rightarrow \infty} \left| 1 - \frac{r(t)x(\tau(t))}{k(t)} \right| \leq \max \left\{ \left| 1 - \frac{r_0 \tilde{m}}{k_1(1 - \gamma_0)} \right|, \left| 1 - \frac{r_1 M}{k_0(1 - \gamma_1)} \right| \right\}.$$

Indeed, for a given solution $\{x(t; \varphi)\}_{t \geq -\tau_0}$ of (1.1), Theorem 2 indicates that for any given $\varepsilon > 0$, there exists an $N = N(\varepsilon, \varphi)$ such that

$$\frac{\tilde{m}}{1 - \gamma_0} - \varepsilon < x(\tau(t)) < \frac{M}{1 - \gamma_1} + \varepsilon \quad \text{for } t \geq N. \quad (2.10)$$

Hence, from (2.10) we have for $t \geq N$,

$$1 - \frac{r(t)x(\tau(t))}{k(t)} < 1 - \frac{r_0}{k_1} \left(\frac{\tilde{m}}{1 - \gamma_0} - \varepsilon \right) \quad (2.11)$$

and

$$1 - \frac{r(t)x(\tau(t))}{k(t)} > 1 - \frac{r_1}{k_0} \left(\frac{M}{1 - \gamma_1} + \varepsilon \right). \quad (2.12)$$

Since ε is arbitrary, with the aid of (2.11)–(2.12), we see that our statement is true.

Assertion 2:

$$\limsup_{t \rightarrow \infty} e^{r(t)\left(1 - \frac{x(\tau(t))}{k(t)}\right)} \leq 1.$$

Indeed, by (2.8) and the left inequality in (2.10), we can derive

$$\frac{x(\tau(t))}{k(t)} > \frac{\tilde{m}}{k_1(1-\gamma_0)} - \frac{1}{k_1}\varepsilon > 1 - \frac{1}{k_1}\varepsilon \quad \text{for } t \geq N,$$

which lead to the result

$$\liminf_{t \rightarrow \infty} \frac{x(\tau(t))}{k(t)} \geq 1.$$

This implies the assertion 2 is valid.

In summary, the conclusion (2.9) is effective. The proof is complete. $\square$

We now consider the extreme stability of (1.1).

Theorem 3. Let $\tilde{m} = \min\{m, \frac{r_0 k_0}{r_1}\}$. Suppose that condition (2.8) in Lemma 1 is verified and

$$\max \left\{ \left| 1 - \frac{r_0 \tilde{m}}{k_1(1-\gamma_0)} \right|, \left| 1 - \frac{r_1 M}{k_0(1-\gamma_1)} \right| \right\} < 1 - \gamma_1. \quad (2.13)$$

Then (1.1) is extremely stable under the assumptions (H1)–(H3).

Proof. By the condition (2.13), we can choose a real number $\widetilde{M}$ so that

$$1 - \gamma_1 > \widetilde{M} > \max \left\{ \left| 1 - \frac{r_0 \tilde{m}}{k_1(1-\gamma_0)} \right|, \left| 1 - \frac{r_1 M}{k_0(1-\gamma_1)} \right| \right\}. \quad (2.14)$$

Let $\{x(t)\}$ and $\{y(t)\}$ be any two positive solutions of (1.1). Then, both $\{x(t)\}$ and $\{y(t)\}$ verify (1.1). Thereby, it follows that for any integer $N \geq 0$,

$$x(t) = \prod_{v=N}^{t-1} \gamma(v)x(N) + \sum_{v=N}^{t-1} \left[\prod_{s=v}^{t-2} \gamma(s+1) \right] x(\tau(v)) e^{r(v)\left(1 - \frac{x(\tau(v))}{k(v)}\right)}, \quad t \geq N$$

as well as

$$y(t) = \prod_{v=N}^{t-1} \gamma(v)y(N) + \sum_{v=N}^{t-1} \left[\prod_{s=v}^{t-2} \gamma(s+1) \right] y(\tau(v)) e^{r(v)\left(1 - \frac{y(\tau(v))}{k(v)}\right)}, \quad t \geq N.$$

Hence, it holds that

$$\begin{aligned} x(t) - y(t) &= \prod_{v=N}^{t-1} \gamma(v)(x(N) - y(N)) \\ &\quad + \sum_{v=N}^{t-1} \left[\prod_{s=v}^{t-2} \gamma(s+1) \right] \left\{ x(\tau(v)) e^{r(v)\left(1 - \frac{x(\tau(v))}{k(v)}\right)} \right. \\ &\quad \left. - y(\tau(v)) e^{r(v)\left(1 - \frac{y(\tau(v))}{k(v)}\right)} \right\} \end{aligned}$$

$$-y(\tau(v))e^{r(v)\left(1-\frac{y(\tau(v))}{k(v)}\right)}\}, t \geq N,$$

which derives

$$\begin{aligned} |x(t) - y(t)| &\leq \gamma_1^{N-t} |x(N) - y(N)| \\ &+ \sum_{v=N}^{t-1} \gamma_1^{t-v-1} \left| 1 - r(v) \frac{\xi(\tau(v))}{k(v)} \right| e^{r(v)\left(1-\frac{\xi(\tau(v))}{k(v)}\right)} |x(\tau(v)) - y(\tau(v))|, t \geq N, \end{aligned} \quad (2.15)$$

where $\xi(\tau(v))$ is between $x(\tau(v))$ and $y(\tau(v))$. Hence, Lemma 1 implies that there exists an N for which

$$\left| 1 - r(v) \frac{\xi(\tau(v))}{k(v)} \right| e^{r(v)\left(1-\frac{\xi(\tau(v))}{k(v)}\right)} \leq \widetilde{M} \text{ for } t \geq N. \quad (2.16)$$

Note that Theorem 1–2 imply that $\{x(t) - y(t)\}$ is bounded. Thus, we can set

$$\limsup_{t \rightarrow \infty} |x(t) - y(t)| = w.$$

Consequently, for any $\varepsilon > 0$, there exists an $N_1 = N_1(\varepsilon)$ large enough such that

$$|x(\tau(t)) - y(\tau(t))| < w + \varepsilon \text{ for } t \geq N_1. \quad (2.17)$$

For convenience, we deem that (2.16)–(2.17) hold whenever $t \geq N_1$. Then, by invoking (2.15)–(2.17) we have

$$\begin{aligned} |x(t) - y(t)| &\leq \gamma_1^{t-N_1} |x(N_1) - y(N_1)| \\ &+ \frac{1 - \gamma_1^{t-N_1}}{1 - \gamma_1} \widetilde{M}(w + \varepsilon), t \geq N_1, \end{aligned}$$

which results in

$$w \leq \frac{1}{1 - \gamma_1} \widetilde{M}(w + \varepsilon).$$

Now we impose (2.14) on the inequality above and then obtain $w = 0$. This shows that $\lim_{t \rightarrow \infty} |x(t) - y(t)| = 0$. The proof is complete. $\square$

Next we give an example to illustrate our results above.

Example 1. Let

$$\gamma(t) = 0.34 + 0.14 \sin t, r(t) = 0.51 + 0.01 \sin t, k(t) = 1.1 + 0.1 \sin t$$

and consider

$$x(t+1) = \gamma(t)x(t) + x(t-2)e^{r(t)\left(1-\frac{x(t-2)}{k(t)}\right)}, t \in \mathbb{Z}_+. \quad (2.18)$$

Then

$$\gamma_0 = 0.2, \gamma_1 = 0.48; r_0 = 0.5, r_1 = 0.52; k_0 = 1, k_1 = 1.2;$$

and, for $M = \frac{k_1}{r_0} e^{r_1-1}$ and $m = \frac{M}{1-\gamma_1} e^{r_0-r_1 \frac{M}{k_0(1-\gamma_1)}}$,

$$\tilde{m} = \min \left\{ m, \frac{r_0 k_0}{r_1} \right\} = \frac{r_0 k_0}{r_1} = \frac{50}{52}$$

as well as

$$1 - \frac{r_1}{k_0} \cdot \frac{M}{1-\gamma_1} = -0.485080140, \quad 1 - \frac{r_0}{k_1} \cdot \frac{\tilde{m}}{1-\gamma_0} = 0.4991987179.$$

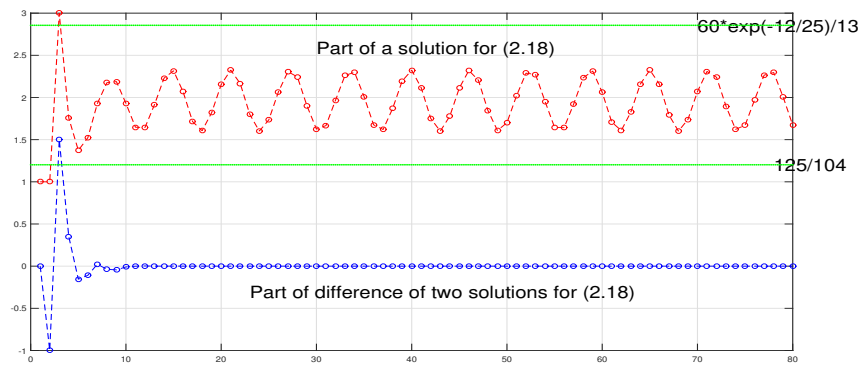
Therefore, we have

$$\frac{\tilde{m}}{k_1(1-\gamma_0)} = 1.001602564,$$

and

$$\max \left\{ \left| 1 - \frac{r_0 \tilde{m}}{k_1(1-\gamma_0)} \right|, \left| 1 - \frac{r_1 M}{k_0(1-\gamma_1)} \right| \right\} < 1 - \gamma_1.$$

So far, Theorem 2 implies that the interval $\left[\frac{125}{104}, \frac{60}{13} e^{-\frac{12}{25}} \right]$ is a global attractor of (2.18), and Theorem 3 shows that (2.18) is extremely stable. The following diagram supports partly our inferences.



3. PERIODICITY OF (1.1)

This section is devoted to consider the existence of periodic solutions of (1.1). For this purpose, we consider the Banach space

$$l^\infty := \{x = \{x(t)\}_{t \geq -\tau_0} : \sup_{t \geq -\tau_0} |x(t)| < \infty\}$$

equipped with the norm defined by $\|x\| = \sup_{t \geq -\tau_0} |x(t)|$. Let

$$\tilde{m} = \min \left\{ m, \frac{r_0 k_0}{r_1} \right\}$$

and the set $\mathcal{S} \subset l^\infty$ be defined by

$$\mathcal{S} := \{\{x(t)\}_{t \geq -\tau_0} \in l^\infty : \widetilde{m}(1 - \gamma_0)^{-1} \leq \|x\| \leq M(1 - \gamma_1)^{-1}\}.$$

The following result is similar to Lemma 1, and since the proof is analogous, we neglect it.

Lemma 2. Suppose that $\{x(t)\} \in \mathcal{S}$ and

$$\frac{\widetilde{m}}{k_1(1 - \gamma_0)} \geq 1.$$

Then, under the assumptions (H1)–(H2), it follows that

$$\left| 1 - \frac{r(t)x(\tau(t))}{k(t)} \right| e^{r(t)\left(1 - \frac{x(\tau(t))}{k(t)}\right)} \leq \max \left\{ \left| 1 - \frac{r_0 \widetilde{m}}{k_1(1 - \gamma_0)} \right|, \left| 1 - \frac{r_1 M}{k_0(1 - \gamma_1)} \right| \right\}, \quad t \geq 0.$$

We now establish a criterion to guarantee the existence of periodic solutions of (1.1). As we shall see below, our work is due to the celebrated Krasnoselskii fixed-point theorem:

Lemma 3 ([1, 4]). Let $\mathcal{S}$ be a closed, convex, and bounded subset of a Banach space $\mathcal{X}$ and $T, U : \mathcal{S} \rightarrow \mathcal{X}$ be two operators such that

- (i) T is a contraction;
- (ii) U is completely continuous, and
- (iii) $Tx + Uy \in \mathcal{S}$ for all $x, y \in \mathcal{S}$.

Then $T + U$ has a fixed point in $\mathcal{S}$.

Theorem 4. Suppose that the assumptions (H1)–(H3) hold and $\{\gamma(t)\}$, $\{r(t)\}$ and $\{k(t)\}$ are all periodic with the common period σ . Suppose further that

$$\tau(t + \sigma) = \tau(t) + \sigma \text{ for all integer } t \geq 0, \quad (3.1)$$

$$\frac{\widetilde{m}}{k_1(1 - \gamma_0)} \geq 1 \quad (3.2)$$

and

$$\max \left\{ \left| 1 - \frac{r_0 \widetilde{m}}{k_1(1 - \gamma_0)} \right|, \left| 1 - \frac{r_1 M}{k_0(1 - \gamma_1)} \right| \right\} < 1 - \gamma_1. \quad (3.3)$$

Then the following is true:

- (S1). equation (1.1) admits an unique σ -periodic solution;
- (S2). any solution of (1.1) is asymptotically σ -periodic.

Proof. To prove the first statement (S1), we introduce a subset $l_{ap}^\infty \subset l^\infty$ defined by

$$l_{ap}^\infty := \{x \in l^\infty : x \text{ is asymptotically } \sigma\text{-periodic}\}.$$

Then l_{ap}^∞ becomes a Banach space according to the norm $\|\cdot\|$ of l^∞ . Let

$$\mathcal{S}_{ap} := \{\{x(t)\}_{t \geq -\tau_0} \in l_{ap}^\infty : \tilde{m}(1 - \gamma_0)^{-1} \leq \|x\| \leq M(1 - \gamma_1)^{-1}\}.$$

It is clear that $\mathcal{S}_{ap}$ is bounded, convex and closed. Now we define two operators $T, U : \mathcal{S}_{ap} \rightarrow l_{ap}^\infty$ as follows:

$$(Tx)(t) = \begin{cases} 0, & t = -\tau_0, -\tau_0 + 1, \dots, 0, \\ \gamma(t-1)x(t-1) + f(t-1, x(\tau(t-1))), & t \geq 1 \end{cases}$$

and

$$(Ux)(t) = \begin{cases} x(t), & t = -\tau_0, -\tau_0 + 1, \dots, 0, \\ 0, & t \geq 1, \end{cases}$$

where $f(t-1, u) = ue^{r(t-1)(1-\frac{u}{k(t-1)})}$. Then, for any $x, y \in \mathcal{S}_{ap}$, we have from (H1) and (H3) that

$$\begin{aligned} (Tx)(t) + (Uy)(t) &= \gamma(t-1)x(t-1) + f(t-1, x(\tau(t-1))) \\ &\leq \gamma_1 \frac{M}{1 - \gamma_1} + M \\ &= M(1 - \gamma_1)^{-1}, \quad t \geq 1 \end{aligned}$$

as well as

$$\begin{aligned} (Tx)(t) + (Uy)(t) &\geq \gamma_0 \frac{\tilde{m}}{1 - \gamma_0} + \tilde{m} \\ &= \tilde{m}(1 - \gamma_0)^{-1}, \quad t \geq 1, \end{aligned}$$

where the relation $\tilde{m}(1 - \gamma_0)^{-1} \leq x(t-1) \leq M(1 - \gamma_1)^{-1}$ have been employed for the first and second inequality. Hence, we have shown that

$$\tilde{m}(1 - \gamma_0)^{-1} \leq (Tx)(t) + (Uy)(t) \leq M(1 - \gamma_1)^{-1} \text{ for all } t \geq -\tau_0. \quad (3.4)$$

Next we proceed in several steps.

(i) Assertion 1: if $\{x(t)\}$ is asymptotically σ -periodic, then $\{f(t, x(\tau(t)))\}$ is also too.

Indeed, since $\{x(t)\}$ is asymptotically σ -periodic, we have the following decomposition

$$x(t) = p(t) + q(t), \quad t \geq -\tau_0, \quad (3.5)$$

where $\{p(t)\}$ is σ -periodic and $q(t) \rightarrow 0$ as $t \rightarrow \infty$. In addition,

$f(t, u) = ue^{r(t)(1-\frac{u}{k(t)})}$ implies that $\frac{\partial f(t, u)}{\partial u} = \left(1 - \frac{r(t)}{k(t)}u\right)e^{r(t)(1-\frac{u}{k(t)})}$. Then, by

the mean value theorem, it holds that

$$\begin{aligned} |f(t, x(t)) - f(t, p(t))| &= \left| x(t)e^{r(t)\left(1-\frac{x(t)}{k(t)}\right)} - p(t)e^{r(t)\left(1-\frac{p(t)}{k(t)}\right)} \right| \\ &= \left| 1 - \frac{r(t)\xi(t)}{k(t)} \right| e^{r(t)\left(1-\frac{\xi(t)}{k(t)}\right)} |x(t) - p(t)|, \end{aligned} \quad (3.6)$$

where, $\xi(t)$ is between $x(t)$ and $p(t)$.

Hence, in the light of (3.5)–(3.6) we see that $\{f(t, x(t))\}$ is asymptotically σ -periodic, which, together with (3.1), infers the assertion 1 is true.

(ii) Assertion 2: $Tx + Uy \in \mathcal{S}_{ap}$ for any $x, y \in \mathcal{S}_{ap}$.

It is clear that the assertion 1, together with (3.4), results in $Tx \in \mathcal{S}_{ap}$ for any $x \in \mathcal{S}_{ap}$. The belonging $Uy \in \mathcal{S}_{ap}$ is obvious. Hence, with the aid of (3.4), we have shown that the assertion 2 is valid.

(iii) Assertion 3: T is contractive.

Indeed, for any $x, y \in \mathcal{S}_{ap}$ we have

$$\begin{aligned} (Tx)(t) - (Ty)(t) &= \gamma(t-1)(x(t-1) - y(t-1)) + \\ &\quad (f(t-1, x(\tau(t-1))) - f(t-1, y(\tau(t-1)))), \quad t \geq 1, \end{aligned}$$

which amounts to

$$\begin{aligned} &|(Tx)(t) - (Ty)(t)| \\ &\leq \gamma_1 \|x - y\| + \\ &\quad \left| 1 - \frac{r(t-1)\xi(\tau(t-1))}{k(t-1)} \right| e^{r(t-1)\left(1-\frac{\xi(\tau(t-1))}{k(t-1)}\right)} \|x - y\|, \quad t \geq 1, \end{aligned} \quad (3.7)$$

where $\xi(\tau(t-1))$ is between $x(\tau(t-1))$ and $y(\tau(t-1))$. Now by (3.3) we can choose a $\varepsilon_0 > 0$ so that

$$\max \left\{ \left| 1 - \frac{r_0 \tilde{m}}{k_1(1-\gamma_0)} \right|, \left| 1 - \frac{r_1 M}{k_0(1-\gamma_1)} \right| \right\} \leq 1 - \gamma_1 - \varepsilon_0.$$

Then, in association with (3.3) and Lemma 2, from (3.7) it holds that

$$|(Tx)(t) - (Ty)(t)| \leq (1 - \varepsilon_0) \|x - y\|, \quad t \geq 1,$$

which implies that T is a contraction.

(iv) Assertion 4: U is completely continuous.

As a matter of fact, $U\mathcal{S}_{ap} \subset \mathbb{R}^{\tau_0}$ and the boundedness of $U\mathcal{S}_{ap}$ mean that it is relatively compact. Consequently, U is completely continuous.

In summary, the Krasnoselskii fixed-point theorem implies that, there exists $\tilde{x} \in \mathcal{S}_{ap}$ so that

$$\tilde{x}(t) = (T\tilde{x})(t) + (U\tilde{x})(t), \quad t \geq -\tau_0,$$

which indicates that

$$\tilde{x}(t+1) = \gamma(t)\tilde{x}(t) + f(t, \tilde{x}(\tau(t))), \quad t \geq 0. \quad (3.8)$$

In other words, $\tilde{x}$ is an asymptotically σ -periodic solution of (1.1).

(v) Assertion 5: Equation (1.1) has a unique σ -periodic solution.

Indeed, note that $\tilde{x} = \{\tilde{x}(t)\}$ can be decomposed by

$$\tilde{x}(t) = \tilde{p}(t) + \tilde{q}(t), \quad t \geq -\tau_0,$$

where $\{\tilde{p}(t)\}$ is σ -periodic, and $\tilde{q}(t) \rightarrow 0$ as $t \rightarrow \infty$. Then, from (3.8) we have

$$\tilde{p}(t+1) \rightarrow \gamma(t)\tilde{p}(t) + \tilde{p}(\tau(t))e^{r(t)\left(1-\frac{\tilde{p}(\tau(t))}{k(t)}\right)} \quad \text{as } t \rightarrow \infty. \quad (3.9)$$

Note that $\{\gamma(t)\tilde{p}(t) + \tilde{p}(\tau(t))e^{r(t)\left(1-\frac{\tilde{p}(\tau(t))}{k(t)}\right)}\}$ is σ -periodic. Hence, from (3.9) it follows that

$$\tilde{p}(t+1) = \gamma(t)\tilde{p}(t) + \tilde{p}(\tau(t))e^{r(t)\left(1-\frac{\tilde{p}(\tau(t))}{k(t)}\right)}, \quad t \geq 0.$$

That is, $\{\tilde{p}(t)\}$ is a σ -periodic solution of (1.1).

Let $\tilde{y}$ be another σ -periodic solution of (1.1). Then Theorem 3 shows that

$$\lim_{t \rightarrow \infty} |\tilde{p}(t) - \tilde{y}(t)| = 0,$$

which, induces

$$\tilde{y}(t) = \tilde{p}(t) \quad \text{for all } t \geq -\tau_0.$$

That is, the first statement (S1) is true.

To prove the second statement (S2), assume that $\{x(t)\}$ is any solution of (1.1). Then, invoking Theorem 3 again, we arrive at

$$\lim_{t \rightarrow \infty} |x(t) - \tilde{p}(t)| = 0,$$

which leads to

$$x(t) = \tilde{p}(t) + q(t), \quad t \geq -\tau_0,$$

where $q(t) \rightarrow 0$ as $t \rightarrow \infty$. That is, $\{x(t)\}$ is an asymptotically σ -periodic solution of (1.1). The proof is complete. $\square$

Example 2. Let

$$\gamma(t) = 0.34 + 0.14 \sin t\pi, \quad r(t) = 0.51 + 0.01 \sin \frac{t\pi}{2}, \quad k(t) = 1.1 + 0.1 \sin \frac{t\pi}{2}$$

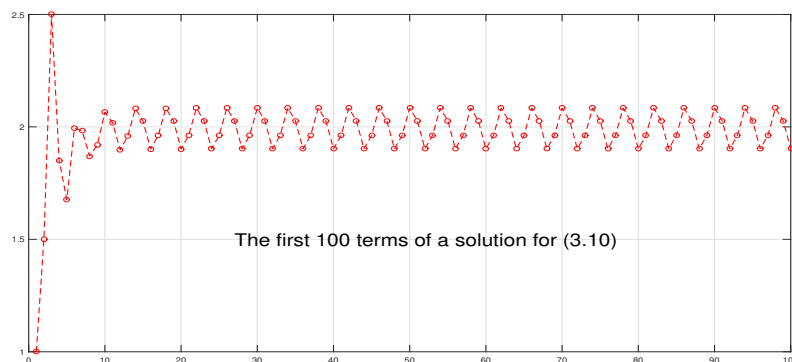
and consider

$$x(t+1) = \gamma(t)x(t) + x(\tau(t))e^{r(t)\left(1-\frac{x(\tau(t))}{k(t)}\right)}, \quad t \in \mathbb{Z}_+, \quad (3.10)$$

where

$$\tau(t) = \begin{cases} t-3, & t \text{ is even;} \\ t-2, & \text{otherwise.} \end{cases}$$

Then, referring to example 1 we see that all the conditions in Theorem 4 are verified. Hence, (3.10) has a unique 4-periodic solution, and others are asymptotically 4-periodic. The facticity can be confirmed partly by the following:



where we choose $x(-2) = 1$, $x(-1) = 1.5$ and $x(0) = 2.5$ as the initial values of (3.10).

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REFERENCES

- [1] B. C. Dhage, "Remarks on two fixed-point theorems involving the sum and the product of two operators." *Comput. Math. Appl.*, vol. 46, no. 12, pp. 1779–1785, 2003, doi: [10.1016/S0898-1221\(03\)00403-6](https://doi.org/10.1016/S0898-1221(03)00403-6).
- [2] T. Diagana, "Almost automorphic solutions to a Beverton–Holt dynamic equation with survival rate." *Appl. Math. Lett.*, vol. 36, no. 2, pp. 19–24, 2014, doi: [10.1016/j.aml.2014.04.011](https://doi.org/10.1016/j.aml.2014.04.011).
- [3] Y. Hamaya, "Existence of almost periodic solutions of discrete ricker delay models." *Int. J. Difference Equ.*, vol. 9, no. 2, pp. 187–205, 2014, doi: [10.1007/978-3-662-44140-4_9](https://doi.org/10.1007/978-3-662-44140-4_9).
- [4] M. A. Krasnoselskii, "Some problems of nonlinear analysis." *Amer. Math. Soc. Trans.*, vol. 10, no. 2, pp. 345–409, 1958, doi: [10.1090/trans2/010/13](https://doi.org/10.1090/trans2/010/13).
- [5] E. Liz, "On the global stability of periodic Ricker maps." *Electronic J. Qual. Theory of Diff. Equ.*, vol. 2016, no. 76, p. 8 PP., 2016, doi: [10.14232/ejqtde.2016.1.76](https://doi.org/10.14232/ejqtde.2016.1.76).
- [6] S. Mohamad and K. Gopalsamy, "Extreme stability and almost periodicity in a discrete logistic equation." *Tohoku Math. J.*, vol. 52, no. 1, pp. 107–125, 2000, doi: [10.2748/tmj/1178224661](https://doi.org/10.2748/tmj/1178224661).
- [7] J. Popena and E. Schmeidel, "On the asymptotically periodic solution of some linear difference equations." *Arch. Math. (Brno)*, vol. 35, no. 1, pp. 13–19, 1999, doi: <http://eudml.org/doc/248367>.
- [8] B. Ryals and R. J. Sacker, "Global stability in the 2D Ricker equation." *J. Difference Equations Appl.*, vol. 21, no. 11, pp. 1068–1081, 2015, doi: [10.1080/10236198.2015.1065825](https://doi.org/10.1080/10236198.2015.1065825).
- [9] Z.-Q. Zhu and S. S. Cheng, "Stability analysis for multistep computational schemes." *Comput. Math. Appl.*, vol. 55, no. 12, pp. 2753–2761, 2008, doi: [10.1016/j.camwa.2007.10.024](https://doi.org/10.1016/j.camwa.2007.10.024).

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