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A HIGHER-ORDER NONLOCAL THREE-POINT BOUNDARY VALUE PROBLEM FOR SEQUENTIAL FRACTIONAL DIFFERENTIAL EQUATIONS

BASHIR AHMAD AND SOTIRIS K. NTOUYAS

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Abstract. This paper investigates the existence of solutions for a sequential fractional differential equation of arbitrary order $\xi+1$, $\xi\in(n-1,n]$, $n\geq 2$ with three-point nonlocal boundary conditions. We make use of standard tools of fixed point theory to establish our existence results. An illustrative example and some special cases are also discussed.

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Keywords: differential equation, fractional order, three-point boundary conditions, existence, fixed point

1. Introduction

Initial and boundary value problems of fractional order have extensively been studied by several researchers in recent years. A variety of results ranging from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions have appeared in the literature. Fractional differential equations appear naturally in a number of fields such as physics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, etc. An excellent account in the study of fractional differential equations can be found in [15, 22–24]. For more details and examples, see [1–4, 6, 8–10, 12, 14, 17, 21, 25–27, 30] and references therein.

The concept of sequential fractional derivative is given, for example, on page 209 of the monograph [20]. There is a close connection between the sequential fractional derivatives and the non sequential Riemann-Liouville derivatives [28, 29]. For some recent work on sequential fractional differential equations, we refer the reader to the papers [7, 11, 16]. In [5], the authors studied a sequential fractional differential equation of order $\alpha+1$, $\alpha\in(1,2]$ with three-point boundary conditions.

In this paper, motivated by [5], we extend the study of sequential fractional differential equations to an arbitrary order. Precisely, for $\xi\in(n-1,n]$, $n\geq 2$, we consider

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a fractional boundary value problem of arbitrary order $\xi+1$ given by

$$ {}^{c}D^{\xi}(D+\lambda)x(t) = f(t,x(t)),\ \ 0<t<1, \tag{1.1}$$

$$\begin{aligned} &x(0)=0,\ x'(0)=0,\ x''(0)=0,\ \ldots,\ x^{(n-1)}(0)=0,\\ &x(1)=\alpha x(\sigma),\ \ 0<\sigma<1, \end{aligned} \tag{1.2}$$

where ${}^{c}D$ is the Caputo fractional derivative, D is the ordinary derivative, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$, λ is a positive real number and α is a real constant.

In Section 2, we present a basic result that lays the foundation for the sequel. The main results, based on Banach's contraction mapping principle, Krasnoselskii's fixed point theorem and nonlinear alternative of Leray-Schauder type, are presented in Section 3.

2. Basic result

Let us recall some basic definitions of fractional calculus [15,22].

Definition 1. For $(n-1)-$times absolutely continuous function $g:[0,\infty)\to\mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$ {}^{c}D^{q}g(t) = \frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1}g^{(n)}(s)ds,\ \ n-1<q<n, n=[q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2. The Riemann-Liouville fractional integral of order q is defined as

$$ I^{q}g(t) = \frac{1}{\Gamma(q)}\int_0^t \frac{g(s)}{(t-s)^{1-q}}ds,\ \ q>0,$$

provided the integral exists.

Definition 3. Sequential fractional derivative for a sufficiently smooth function $g(t)$ due to Miller-Ross [20] is defined as

$$ D^{\delta}g(t) = D^{\delta_1}D^{\delta_2}\ldots D^{\delta_k}g(t), \tag{2.1}$$

where $\delta=(\delta_1,\ldots,\delta_k)$ is a multi-index.

In general, the operator D^{δ} in (2.1) can either be Riemann-Liouville or Caputo or any other kind of integro-differential operator. For instance,

$$ {}^{c}D^{q}g(t) = D^{-(n-q)}\Big(\frac{d}{dt}\Big)^{n}g(t),\ n-1<q<n,$$

where $D^{-(n-q)}$ is the fractional integral operator of order $n-q$. Here we emphasize that $D^{-p}f(t)=I^{p}f(t),\ p=n-q$; for more details, see page 87 [22].

Before presenting an auxiliary lemma, we recall the following result [15].

Lemma 1. *For $q > 0$, the general solution of the fractional differential equation ${}^cD^q x(t) = 0$, $n-1 < q < n$, is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

where $c_0, c_1, \ldots, c_{n-1}$ are arbitrary real constants.

For the forthcoming analysis, we define

$$P(t) = P_o(t) = \frac{t^{n-1}}{\lambda} - \frac{(n-1)t^{n-2}}{\lambda^2} + \frac{(n-1)(n-2)t^{n-3}}{\lambda^3} - \ldots - \frac{(n-1)!t}{\lambda^{n-1}} + \frac{(n-1)!}{\lambda^n}(1-e^{-\lambda t}),\ n \text{ is odd}, \tag{2.2}$$

$$P(t) = P_e(t) = \frac{t^{n-1}}{\lambda} - \frac{(n-1)t^{n-2}}{\lambda^2} + \frac{(n-1)(n-2)t^{n-3}}{\lambda^3} - \ldots + \frac{(n-1)!t}{\lambda^{n-1}} - \frac{(n-1)!}{\lambda^n}(1-e^{-\lambda t}),\ n \text{ is even}. \tag{2.3}$$

Furthermore, we assume the non-resonance condition, that is, for $P = P_o$ and $P = P_e$, we choose α such that

$$P(1) - \alpha P(\sigma) \neq 0, \text{ for } 0 < \sigma < 1. \tag{2.4}$$

Lemma 2. *Assume that the non-resonance condition (2.4) holds. Then, for $v \in C[0,1]$, the unique solution of the equation*

$${}^cD^{\xi}(D+\lambda)x(t) = v(t),\ 0 < t < 1,\ n-1 < \xi \leq n,\ n \geq 2, \tag{2.5}$$

subject to the boundary conditions (1.2) is given by

$$\begin{aligned} x(t) = {} & \int_0^t e^{-\lambda(t-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)du \right) ds \\ & + \frac{P(t)}{P(1)-\alpha P(\sigma)} \left[\alpha \int_0^{\sigma} e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)du \right) ds \right. \\ & \left. - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)du \right) ds \right], \end{aligned} \tag{2.6}$$

where $P(t) = P_o(t)$ and $P(t) = P_e(t)$ are given by (2.2) and (2.3) respectively.

Proof. In view of the well known property: ${}^cD^{\xi} I^{\xi} x(t) = x(t)$ [16,19] and Lemma 1, observe that any solution of the equation

$$(D+\lambda)x(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1} + I^{\xi} v(t) \tag{2.7}$$

will satisfy (2.5), where $c_0, c_1, \ldots, c_{n-1}$ are arbitrary unknown constants. Rewriting (2.7), we have

$$D(e^{\lambda t} x(t)) = [c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1} + I^{\xi} v(t)] e^{\lambda t}.$$

Integrating from 0 to t and using $x(0)=0,\ x'(0)=0,\ x''(0)=0,\ \ldots,\ x^{(n-1)}(0)=0$, we get $c_0=0,\ c_1=0,\ c_2=0,\ c_{n-2}=0$ and

$$x(t)=c_{n-1}P_o(t)+\int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)}v(u)du\right)ds,\ n \text{ is odd}, \tag{2.8}$$

and

$$x(t)=\hat{c}_{n-1}P_e(t)+\int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)}v(u)du\right)ds,\ n \text{ is even}, \tag{2.9}$$

where $P_o(t)$ and $P_e(t)$ are given by (2.2) and (2.3) respectively. Using the condition $x(1)=\alpha x(\sigma)$ in (2.8), we find that

$$c_{n-1}=\frac{1}{P_o(1)-\alpha P_o(\sigma)}\left[\alpha\int_0^\sigma e^{-\lambda(\sigma-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)}v(u)du\right)ds - \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)}v(u)du\right)ds\right],$$

where P_o is given by (2.2). Similarly, using the condition $x(1)=\alpha x(\sigma)$ in (2.9), we get

$$\hat{c}_{n-1}=\frac{1}{P_e(1)-\alpha P_e(\sigma)}\left[\alpha\int_0^\sigma e^{-\lambda(\sigma-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)}v(u)du\right)ds - \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)}v(u)du\right)ds\right],$$

where P_e is given by (2.3). Substituting the values of $c_{n-1},\ \hat{c}_{n-1}$ respectively in (2.8) and (2.9), we obtain (2.6). This completes the proof. □

3. Existence of solutions

Let $\mathcal{C}=C([0,1],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,1]\to\mathbb{R}$ endowed with the sup norm defined by $\|x\|=\sup\{|x(t)|,t\in[0,1]\}<\infty$.

For the sake of convenience, we set

$$P_1=\max_{t\in[0,1]}\left|\frac{P(t)}{P(1)-\alpha P(\sigma)}\right|,\quad \rho=\left|\frac{(1+P_1)(1-e^{-\lambda})+P_1\alpha\sigma^{\xi}}{\lambda\Gamma(\xi+1)}\right|, \tag{3.1}$$

where $P(t)$ is given by (2.2), (2.3).

In view of Lemma 2, we transform problem (1.1) – (1.2) as

$$x=F(x), \tag{3.2}$$

where $F : \mathcal{C} \to \mathcal{C}$ is defined by

$$\begin{aligned}(Fx)(t) = &\int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} f(u,x(u))du\right)ds \\ &+ \frac{P(t)}{P(1)-\alpha P(\sigma)}\left[\alpha\int_0^\sigma e^{-\lambda(\sigma-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} f(u,x(u))du\right)ds\right. \\ &\left.- \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} f(u,x(u))du\right)ds\right].\end{aligned}$$

Observe that problem (1.1) – (1.2) has solutions if the operator equation (3.2) has fixed points.

Theorem 1. *Let $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ be a jointly continuous function satisfying the condition*

$$|f(t,x) - f(t,y)| \le L|x-y|,\ \forall t \in [0,1],\ x,y \in \mathbb{R},$$

where L is the Lipschitz constant. Further, the non-resonance condition (2.4) holds. Then the boundary value problem (1.1) – (1.2) has a unique solution if $\rho < 1/L$, where ρ is given by (3.1).

Proof. As a first step, we show that the operator F given by (3.2) maps $\mathcal{C}$ into itself. For that, we set $\sup_{t\in[0,1]}|f(t,0)| = M < \infty$. Then, for $x \in \mathcal{C}$, we have

$$\begin{aligned}&\|(Fx)\| \\ &= \sup_{t\in[0,1]}\left|\int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} f(u,x(u))du\right)ds\right. \\ &+ \frac{P(t)}{P(1)-\alpha P(\sigma)}\left[\alpha\int_0^\sigma e^{-\lambda(\sigma-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} f(u,x(u))du\right)ds\right. \\ &\left.\left.+ \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} f(u,x(u))du\right)ds\right]\right| \\ &\le \sup_{t\in[0,1]}\left(\int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)}(|f(u,x(u)) - f(u,0)|\right.\right. \\ &\left.\left.+ |f(u,0)|)du\right)ds\right) \\ &+ \sup_{t\in[0,1]}\left|\frac{P(t)}{P(1)-\alpha P(\sigma)}\right|\left[\alpha\int_0^\sigma e^{-\lambda(\sigma-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)}(|f(u,x(u))\right.\right.\end{aligned}$$

$$
\begin{aligned}
&- f(u,0)| + |f(u,0)|)du\Bigg)ds \\
&+ \int_0^1 e^{-\lambda(1-s)}\Bigg(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)}(|f(u,x(u)) - f(u,0)| \\
&+ |f(u,0)|)du\Bigg)ds\Bigg] \\
&\leq \sup_{t\in[0,1]}\left(\int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)}(L|x(u)| + |f(u,0)|)du\right)ds\right) \\
&+ \sup_{t\in[0,1]}\left|\frac{P(t)}{P(1)-\alpha P(\sigma)}\right|\Bigg[\alpha\int_0^\sigma e^{-\lambda(\sigma-s)}\Bigg(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)}(L|x(u)| \\
&+ |f(u,0)|)du\Bigg)ds \\
&+ \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)}(L|x(u)| + |f(u,0)|)du\right)ds\Bigg] \\
&\leq (L\|x\| + M)\Bigg[\sup_{t\in[0,1]}\left(\int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)}du\right)ds\right) \\
&+ \sup_{t\in[0,1]}\left|\frac{P(t)}{P(1)-\alpha P(\sigma)}\right|\Bigg\{\alpha\int_0^\sigma e^{-\lambda(\sigma-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)}du\right)ds \\
&+ \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)}du\right)ds\Bigg\}\Bigg] \\
&\leq (L\|x\| + M)\left(\frac{(1+P_1)(1-e^{-\lambda}) + P_1\alpha\sigma^\xi}{\lambda\Gamma(\xi+1)}\right) \\
&= (L\|x\| + M)\rho < \infty.
\end{aligned}
$$

This shows that F maps $\mathcal{C}$ into itself. Now, for $x, y \in \mathcal{C}$ and for each $t \in [0,1]$, we obtain

$$
\begin{aligned}
\|(Fx) - (Fy)\| &= \sup_{t\in[0,1]} |(Fx)(t) - (Fy)(t)| \\
&\leq \sup_{t\in[0,1]}\Bigg[\int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)}|f(u,x(u)) - f(u,y(u))|du\right)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\frac{P(t)}{P(1)-\alpha P(\sigma)}\right|\left\{\alpha\int_0^{\sigma}e^{-\lambda(\sigma-s)}\left(\int_0^{s}\frac{(s-u)^{\xi-1}}{\Gamma(\xi)}|f(u,x(u))\right.\right.\\
&\left.-f(u,y(u))|du\right)ds\\
&\left.+\int_0^{1}e^{-\lambda(1-s)}\left(\int_0^{s}\frac{(s-u)^{\xi-1}}{\Gamma(\xi)}|f(u,x(u))-f(u,y(u))|du\right)ds\right\}\Bigg]\\
&\le L\|x-y\|\Bigg[\sup_{t\in[0,1]}\left(\int_0^{t}e^{-\lambda(t-s)}\left(\int_0^{s}\frac{(s-u)^{\xi-1}}{\Gamma(\xi)}du\right)ds\right)\\
&+\sup_{t\in[0,1]}\left|\frac{P(t)}{P(1)-\alpha P(\sigma)}\right|\left\{\alpha\int_0^{\sigma}e^{-\lambda(\sigma-s)}\left(\int_0^{s}\frac{(s-u)^{\xi-1}}{\Gamma(\xi)}du\right)ds\right.\\
&\left.+\int_0^{1}e^{-\lambda(1-s)}\left(\int_0^{s}\frac{(s-u)^{\xi-1}}{\Gamma(\xi)}du\right)ds\right\}\Bigg]\\
&\le L\left|\frac{(1+P_1)(1-e^{-\lambda})+P_1\alpha\sigma^{\xi}}{\lambda\Gamma(\xi+1)}\right|\|x-y\|\\
&=\rho L\|x-y\|,
\end{aligned}
$$

where ρ is given by (3.1). As $\rho<1/L$, therefore, F is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle. This completes the proof. □

Now, we state a known result due to Krasnoselskii [18] which is needed to prove the existence of at least one solution of (1.1) – (1.2).

Theorem 2. *Let M be a closed, convex, bounded and nonempty subset of a Banach space X. Let $\mathcal{G}_1,\mathcal{G}_2$ be the operators such that: (i) $\mathcal{G}_1x+\mathcal{G}_2y\in M$ whenever $x,y\in M$; (ii) $\mathcal{G}_1$ is compact and continuous; (iii) $\mathcal{G}_2$ is a contraction mapping. Then there exists $z\in M$ such that $z=\mathcal{G}_1z+\mathcal{G}_2z$.*

Theorem 3. *Assume that $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a jointly continuous function and the following assumptions hold:*

(H₁) $|f(t,x)-f(t,y)|\le L|x-y|,\ \forall t\in[0,1],\ x,y\in\mathbb{R}$;

(H₂) $|f(t,x)|\le\mu(t),\ \ \forall(t,x)\in[0,1]\times\mathbb{R}$ *with* $\mu\in C([0,1],\mathbb{R})$.

Further, the non-resonance condition (2.4) holds. Then the boundary value problem (1.1)-(1.2) has at least one solution on $[0,1]$ *if*

$$\left|\frac{P_1(1-e^{-\lambda}+\alpha\sigma^{\xi})}{\lambda\Gamma(\xi+1)}\right|<1. \tag{3.3}$$

Proof. Letting $\sup_{t\in[0,1]}|\mu(t)| = \|\mu\|$, we fix

$$r \geq \left|\frac{(1+P_1)(1-e^{-\lambda})+P_1\alpha\sigma^{\xi}}{\lambda\Gamma(\xi+1)}\right|\|\mu\|, \tag{3.4}$$

and consider $\mathcal{B}_r = \{x \in \mathcal{C} : \|x\| \leq r\}$. Define the operators F_1 and F_2 on $\mathcal{B}_r$ as

$$(F_1x)(t) = \int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} f(u,x(u))du\right)ds,$$

$$(F_2x)(t) = \frac{P(t)}{P(1)-\alpha P(\sigma)}\left[\alpha\int_0^{\sigma} e^{-\lambda(\sigma-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} f(u,x(u))du\right)ds - \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} f(u,x(u))du\right)ds\right].$$

For $x, y \in \mathcal{B}_r$, it follows from (3.4) that

$$\|F_1x + F_2y\| \leq \left|\frac{(1+P_1)(1-e^{-\lambda})+P_1\alpha\sigma^{\xi}}{\lambda\Gamma(\xi+1)}\right|\|\mu\| \leq r.$$

Thus, $F_1x + F_2y \in \mathcal{B}_r$. In view of the condition (3.3), it can easily be shown that F_2 is a contraction mapping. The continuity of f implies that the operator F_1 is continuous. Also, F_1 is uniformly bounded on $\mathcal{B}_r$ as

$$\|F_1x\| \leq \frac{|1-e^{-\lambda}|\|\mu\|}{\lambda\Gamma(\xi+1)}.$$

Now we prove the compactness of the operator F_1. Setting $\Omega = [0,1]\times\mathcal{B}_r$, we define $\sup_{(t,x)\in\Omega}|f(t,x)| = M_r$, and consequently we have

$$\begin{aligned}
&\|(F_1x)(t_1)-(F_1x)(t_2)\| \\
&= \left\| \int_0^{t_1} e^{-\lambda(t_1-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} f(u,x(u))du\right)ds \right. \\
&\left. - \int_0^{t_2} e^{-\lambda(t_2-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} f(u,x(u))du\right)ds \right\| \\
&\leq \frac{M_r}{\lambda\Gamma(\xi+1)}\left(|t_1^{\xi}-t_2^{\xi}| + |t_1^{\xi}e^{-\lambda t_1} - t_2^{\xi}e^{-\lambda t_2}|\right),
\end{aligned}$$

which is independent of x and tends to zero as $t_2 \to t_1$. Thus, F_1 is relatively compact on $\mathcal{B}_r$. Hence, by the Arzelá-Ascoli Theorem, F_1 is compact on $\mathcal{B}_r$. Thus all the assumptions of Theorem 2 are satisfied and the conclusion of Theorem 2 implies

that the boundary value problem (1.1) – (1.2) has at least one solution on $[0,1]$. This completes the proof. □

In the next theorem we prove the existence of a solution for the boundary value problem (1.1) – (1.2) via Leray-Schauder nonlinear alternative.

Lemma 3 (Nonlinear alternative for single valued maps [13])**.** *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*
(ii) *there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

Theorem 4. *Suppose that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a jointly continuous function and the non-resonance condition (2.4) is satisfied. Further, it is assumed that the following conditions hold:*

($\mathbf{H_3}$) *There exist a function $\phi \in L^1([0,1], \mathbb{R}^+)$, and a nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t,x)| \le \phi(t)\psi(\|x\|)$, for all $(t,x) \in [0,1] \times \mathbb{R}$.*

($\mathbf{H_4}$) *There exists a constant $M > 0$ such that*

$$\frac{\|x\|}{\frac{\psi(\|x\|)}{\Gamma(\xi)}\left\{(1+P_1)\int_0^1 e^{-\lambda(1-s)}\phi(s)ds + \alpha P_1 \int_0^\sigma e^{-\lambda(\sigma-s)}\phi(s)ds\right\}} > 1.$$

Then the boundary value problem (1.1)-(1.2) has at least one solution on $[0,1]$.

Proof. Consider the operator $F : \mathcal{C} \to \mathcal{C}$ where

$$\begin{aligned}(Fx)(t) = {} & \int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} f(u,x(u))du\right)ds \\ & + \frac{P(t)}{P(1)-\alpha P(\sigma)}\left[\alpha\int_0^\sigma e^{-\lambda(\sigma-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} f(u,x(u))du\right)ds\right. \\ & \left. - \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} f(u,x(u))du\right)ds\right].\end{aligned}$$

We show that F maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$. For a positive number r, let $\mathcal{B}_r = \{x \in C([0,1],\mathbb{R}) : \|x\| \le r\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then

$$\begin{aligned}&|(Fx)(t)| \\ &\le \left|\int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} f(u,x(u))du\right)ds\right.\end{aligned}$$

$$
\begin{aligned}
&+\frac{P(t)}{P(1)-\alpha P(\sigma)}\left[\alpha\int_0^\sigma e^{-\lambda(\sigma-s)}\left(\int_0^s\frac{(s-u)^{\xi-1}}{\Gamma(\xi)}f(u,x(u))du\right)ds\right.\\
&\left.+\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s\frac{(s-u)^{\xi-1}}{\Gamma(\xi)}f(u,x(u))du\right)ds\right]\Bigg|\\
&\le\int_0^t e^{-\lambda(t-s)}\left(\int_0^s\frac{(s-u)^{\xi-1}}{\Gamma(\xi)}\phi(s)\psi(\|x\|)du\right)ds\\
&+\Big|\frac{P(t)}{P(1)-\alpha P(\sigma)}\Big|\left[\alpha\int_0^\sigma e^{-\lambda(\sigma-s)}\left(\int_0^s\frac{(s-u)^{\xi-1}}{\Gamma(\xi)}\phi(s)\psi(\|x\|)du\right)ds\right.\\
&\left.+\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s\frac{(s-u)^{\xi-1}}{\Gamma(\xi)}\phi(s)\psi(\|x\|)du\right)ds\right]\\
&\le\frac{\psi(\|x\|)}{\Gamma(\xi)}\left\{\int_0^1 e^{-\lambda(1-s)}\phi(s)ds+\alpha P_1\int_0^\sigma e^{-\lambda(\sigma-s)}\phi(s)ds\right.\\
&\left.+P_1\int_0^1 e^{-\lambda(1-s)}\phi(s)ds\right\}\\
&=\frac{\psi(\|x\|)}{\Gamma(\xi)}\left\{(1+P_1)\int_0^1 e^{-\lambda(1-s)}\phi(s)ds+\alpha P_1\int_0^\sigma e^{-\lambda(\sigma-s)}\phi(s)ds\right\}.
\end{aligned}
$$

Consequently,

$$
\|Fx\|\le\frac{\psi(r)}{\Gamma(\xi)}\left\{(1+P_1)\int_0^1 e^{-\lambda(1-s)}p(s)ds+\alpha P_1\int_0^\sigma e^{-\lambda(\sigma-s)}p(s)ds\right\}.
$$

Next we show that F maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$. Let $t_1,t_2\in[0,1]$ with $t_1<t_2$ and $x\in\mathcal{B}_r$, where $\mathcal{B}_r$ is a bounded set of $C([0,1],\mathbb{R})$. Then we obtain

$$
\begin{aligned}
&|(Fx)(t_2)-(Fx)(t_1)|\\
&=\left|\int_0^{t_1}\left(e^{-\lambda(t_2-s)}-e^{-\lambda(t_1-s)}\right)\left(\int_0^s\frac{(s-u)^{\xi-1}}{\Gamma(\xi)}f(u,x(u))du\right)ds\right.\\
&+\int_{t_1}^{t_2}e^{-\lambda(t_2-s)}\left(\int_0^s\frac{(s-u)^{\xi-1}}{\Gamma(\xi)}f(u,x(u))du\right)ds\\
&+\frac{(P(t_2)-P(t_1))}{P(1)-\alpha P(\sigma)}\left[\alpha\int_0^\sigma e^{-\lambda(\sigma-s)}\left(\int_0^s\frac{(s-u)^{\xi-1}}{\Gamma(\xi)}f(u,x(u))du\right)ds\right.
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} f(u,x(u))du \right) ds \Bigg] \Bigg| \\
&\leq \left| \int_0^{t_1} \left(e^{-\lambda(t_2-s)} - e^{-\lambda(t_1-s)} \right) \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} \psi(r)\phi(u)du \right) ds \right| \\
&+ \left| \int_{t_1}^{t_2} e^{-\lambda(t_2-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} \psi(r)\phi(u)du \right) ds \right| \\
&+ \left| \frac{(P(t_2)-P(t_1))}{P(1)-\alpha P(\sigma)} \left[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} \psi(r)\phi(u)du \right) ds \right. \right. \\
&\left. \left. - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} \psi(r)\phi(u)du \right) ds \right] \right|
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in \mathcal{B}_r$ as $t_2 - t_1 \to 0$. As F satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $F : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative (Lemma 3) once we have proved the boundendness of the set of all solutions to equations $x = \lambda Fx$ for $\lambda \in [0,1]$.

Let x be a solution. Then, for $t \in [0,1]$, and using the computations in proving that F is bounded, we have

$$
\begin{aligned}
|x(t)| &= |\lambda(Fx)(t)| \\
&\leq \frac{\psi(\|x\|)}{\Gamma(\xi)} \left\{ (1+P_1) \int_0^1 e^{-\lambda(1-s)} p(s)ds + \alpha P_1 \int_0^\sigma e^{-\lambda(\sigma-s)} p(s)ds \right\}.
\end{aligned}
$$

Consequently, we have

$$
\frac{\|x\|}{\dfrac{\psi(\|x\|)}{\Gamma(\xi)} \left\{ (1+P_1) \displaystyle\int_0^1 e^{-\lambda(1-s)} p(s)ds + \alpha P_1 \int_0^\sigma e^{-\lambda(\sigma-s)} p(s)ds \right\}} \leq 1.
$$

In view of $(\mathbf{H_4})$, there exists M such that $\|x\| \neq M$. Let us set

$$
U = \{x \in C([0,1],\mathbb{R}) : \|x\| < M\}.
$$

Note that the operator $F : \overline{U} \to C([0,1],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda F(x)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3), we deduce that F has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1)-(1.2). This completes the proof. □

Example 1. Consider the problem

$$\begin{cases} {}^{c}D^{7/2}(D+2)x(t) = \dfrac{L}{2}\Big(t^2+\cos t+1+x(t)+\tan^{-1}x(t)\Big),\ 0\le t\le 1,\\ x(0)=0,\ x'(0)=0,\ x''(0)=0,\ x'''(0)=0, x(1)=x(1/2). \end{cases} \tag{3.5}$$

Here, $\xi = 7/2,\ n=4,\ f(t,x(t)) = \dfrac{L}{2}\Big(t^2+\cos t+1+x(t)+\tan^{-1}x(t)\Big),\ \lambda = 2,\ \alpha=1,\ \sigma = 1/2$. Clearly

$$|f(t,x)-f(t,y)| \le L|x-y+\tan^{-1}x-\tan^{-1}y|/2 \le L|x-y|,$$

$P(t) = P_e(t) = t^3/2 - 3t(t-1)/4 - 3(1-e^{-2t})/8,\quad P_1 \approx 6.214821,\ \rho = [(1+P_1)(1-e^{-2})+2^{-7/2}P_1]/2\Gamma(9/2) \approx 0.2917760234.$

For $L < 1/\rho \approx 3.42728641$, it follows by Theorem 1 that problem (3.5) has a unique solution.

4. Conclusions

In this paper, we have obtained some existence results for a nonlocal three-point boundary value problem of sequential fractional differential equations of arbitrary order $\xi+1,\ \xi\in(n-1,n],\ n\ge 2$. Our results are new and generalize some earlier results. For instance, by taking $n=2$, Theorems 3.1 and 3.3 correspond to Theorems 4.1 and 4.3 of [5]. Letting $\xi = n$, we get the new existence results for the following nonlinear problem of ordinary differential equations:

$$D^n(D+\lambda)x(t) = f(t,x(t)),\ \ 0<t<1,$$
$$x(0)=0,\ x'(0)=0,\ x''(0)=0,\ \ldots,\ x^{(n-1)}(0)=0,$$
$$x(1)=\alpha x(\sigma),\ \ 0<\sigma<1.$$

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Authors' addresses

Bashir Ahmad

Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

E-mail address: bashirahmad_qau@yahoo.com

Sotiris K. Ntouyas

Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece

E-mail address: sntouyas@uoi.gr

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FIXED POINT THEOREMS FOR SOME GENERALIZED NONEXPANSIVE MAPPINGS IN BANACH SPACES

A. AMINI-HARANDI

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Abstract. In this paper, we first introduce the class of generalized nonexpansive mappings in Banach spaces. This class contains both the classes of nonexpansive and α-nonexpansive mappings. In addition, we obtain some fixed point and coincidence point theorems for generalized nonexpansive mappings in uniformly convex Banach spaces. Our results extend some well-known results in literature.

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1. INTRODUCTION AND PRELIMINARIES.

Recently, Aoyama and Kohsaka [2] introduced the class of α-nonexpansive mappings in Banach spaces and obtained a fixed point theorem for α-nonexpansive mappings in uniformly convex Banach spaces. The class of α-nonexpansive mappings contains the class of nonexpansive mappings and is related to the classes of firmly nonexpansive mappings and λ-hybrid mappings in Banach spaces, for more information on firmly nonexpansive mappings and λ-hybrid mappings see [3], [4], [5], [8], [1] and references therein.

In this paper, we introduce the class of generalized nonexpansive mappings in Banach spaces. This class contains the class of α-nonexpansive mappings. In addition, we obtain some fixed point and coincidence point theorems for generalized nonexpansive mappings in uniformly convex and p-uniformly convex Banach spaces. Our fixed point theorems generalize some of the results obtained in [2].

In the rest of this section, we recall some definitions and facts which will be used in the next section.

Throughout this paper, every Banach space is real. Let E be a Banach space and let C be a nonempty subset of E. We denote the fixed point set of T by $F(T)$. For a Banach space E, the norm of E is denoted by $\|.\|$. Strong convergence of a sequence

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$\{x_n\}$ in E to $x \in E$ is denoted by $x_n \to x$. For a Banach space E, we denote the unit sphere and the closed unit ball centered at the origin of E by S_E and B_E , respectively. We also denote the closed ball with radius $r > 0$ centered at the origin of E by rB_E. Let E be a Banach space with dimension $E \geq 2$. The modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) = \inf\{1 - \|\frac{x+y}{2}\| \ : \ \|x\| = 1, \|y\| = 1, \ \|x-y\| \geq \epsilon\}.$$

A Banach space E is said to be uniformly convex if for each $\epsilon \in (0,2]$, there exists $\delta > 0$ such that $\|\frac{(x+y)}{2}\| \leq 1-\delta$ whenever $x, y \in S_E$ and $\|x-y\| \geq \epsilon$. In other words, E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for each $\epsilon \in (0,2]$. Every uniformly convex Banach space is reflexive [9].
A Banach space E is called *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in (0,2]$. Notice that there is no p-uniformly convex Banach space for $p > 2$; see, for example [10].
In the sequel we will need the following lemmas.

Lemma 1 ([12]). *The Banach space E is uniformly convex if and only if $\|.\|^2$ is uniformly convex on bounded convex sets, i.e., for each $r > 0$ and $\epsilon \in (0, 2r]$, there exists $\delta > 0$ such that*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\delta,$$

for all $t \in (0,1)$ and for all $x, y \in rB_E$ with $\|x-y\| \geq \epsilon$.

Lemma 2 ([11]). *Let $1 < p \leq 2$ be a given real number. Let E be a p-uniformly convex Banach space. Then, there exists a constant $d > 0$ such that*

$$\|tx + (1-t)y\|^p \leq t\|x\|^p + (1-t)\|y\|^p - (t^p(1-t) + t(1-t)^p)d\|x-y\|^p,$$

for all $t \in (0,1)$ and for all $x, y \in E$.

A function g of a nonempty subset C of a Banach space E into $\mathbb{R}$ is said to be coercive if $g(z_n) \to \infty$ whenever $\{z_n\}$ is a sequence in C such that $\|z_n\| \to \infty$. Let l^∞ denotes the Banach space of bounded real sequences with the supremum norm. It is known that there exists a bounded linear functional μ on l^∞ such that the following three conditions hold:

(1) If $\{t_n\} \in l^\infty$ and $t_n \geq 0$ for every $n \in \mathbb{N}$, then $\mu(\{t_n\}) \geq 0$;
(2) If $t_n = 1$ for every $n \in \mathbb{N}$, then $\mu(\{t_n\}) = 1$;
(3) $\mu(\{t_{n+1}\}) = \mu(\{t_n\})$ for all $\{t_n\} \in l^\infty$.

Such a functional μ is called a Banach limit and the value of μ at $\{t_n\} \in l^\infty$ is denoted by $\mu_n t_n$ [9]. Let μ be a Banach limit and let $\{t_n\} \in l^\infty$ be such that $\lim_{n\to\infty} t_n = t$, then the Banach limit of $\{t_n\}$ is also t. It is known that the reflexivity of the Banach space E implies the following.

Lemma 3 ([9]). *Let E be a reflexive Banach space, let C be a nonempty, closed, and convex subset of E, and let $g : C \to \mathbb{R}$ be a convex, continuous, and coercive function. Then there exists $u \in C$ such that $g(u) = \inf g(C)$.*

Definition 1 ([2]). Let E be a Banach space, let C be a nonempty subset of E, and let α be a real number such that $0 \le \alpha < 1$. A mapping $T : C \to E$ is said to be α-nonexpansive if

$$\|Tx - Ty\|^2 \le \alpha\|Tx - y\|^2 + \alpha\|Ty - x\|^2 + (1-2\alpha)\|x - y\|^2$$

for all $x, y \in C$.

The following is the main result of Aoyama and Kohsaka [2].

Theorem 1. *Let E be a uniformly convex Banach space, let C be a nonempty, closed and convex subset of E, and let $T : C \to C$ be an α-nonexpansive mapping for some real number α such that $\alpha < 1$. Then $F(T)$ is nonempty if and only if there exists $x \in C$ such that $\{T^n x\}$ is bounded.*

2. Fixed Point Theory

We first give the definition of generalized nonexpansive mappings.

Definition 2. Let E be a Banach space, and let C be a nonempty subset of E. Let $p > 1$, $\alpha_1 \ge 0, \ldots, \alpha_m \ge 0$ with $\Sigma_{i=1}^m \alpha_i = 1$, and let $a, b, c, d \in \mathbb{R}$ with $b < \alpha_1$ for $m = 1$ and $b \le \alpha_1$ for $m > 1$, $a + c > 0$ and $a + b + c \le 1$. A mapping $T : C \to C$ is said to be generalized nonexpansive if

$$\Sigma_{i=1}^m \alpha_i \|T^i x - T^i y\|^p \le a\|x - y\|^p + b\|Ty - x\|^p + c\|y - Tx\|^p + d\|x - Tx\|^p$$

for all $x, y \in C$.

In the following, we give an example of a generalized nonexpansive mapping which is not an α-nonexpansive mapping.

Example 1. Let $E = \mathbb{R}$, $C = [\sqrt{2}, \sqrt{3}]$ and let Q denotes the set of rational numbers. Let $T : C \to C$ be defined as

$$Tx = \begin{cases} \sqrt{2}, & x \in Q \\ \sqrt{3}. & x \notin Q \end{cases}$$

Then $T^2 x = \sqrt{3}$ for each $x \in C$ and so

$$|T^2 x - T^2 y|^2 = 0 \le |x - y|^2, \text{ for each } x, y \in [\sqrt{2}, \sqrt{3}].$$

Thus T is a generalized nonexpansive map. Now, we show that T is not α-nonexpansive. On the contrary, assume that there exists $0 \le \alpha < 1$ such that

$$|Tx - Ty|^2 \le \alpha|Tx - y|^2 + \alpha|Ty - x|^2 + (1-2\alpha)|x - y|^2, \text{ for each } x, y \in [\sqrt{2}, \sqrt{3}].$$

Let $x \in Q$ and $y \notin Q$ with $\sqrt{2} < x < y < \sqrt{3}$. Then from the above we would have

$$(\sqrt{3}-\sqrt{2})^2 \leq \alpha|\sqrt{2}-y|^2+\alpha|\sqrt{3}-x|^2+(1-2\alpha)|x-y|^2$$
$$< \alpha(\sqrt{3}-\sqrt{2})^2+\alpha(\sqrt{3}-\sqrt{2})^2+(1-2\alpha)(\sqrt{3}-\sqrt{2})^2 = (\sqrt{3}-\sqrt{2})^2,$$

a contradiction.

Now, we are ready to state our first main result.

Theorem 2. *Let E be a Banach space, let C be a nonempty, closed, and convex subset of E, and let $T : C \to C$ be a generalized nonexpansive mapping. Assume that E is uniformly convex if $p = 2$ and assume that E is p-uniformly convex for $1 < p < 2$. Then $\bigcup_{i=1}^{m} F(T^i) \neq \varnothing$ if there exists $x_0 \in C$ such that $\{T^n x_0\}$ is bounded and either $d = 0$ or $\lim_{n\to\infty} \|T^n x_0 - T^{n+1}x_0\| = 0$. Moreover, if $\bigcup_{i=1}^{m} F(T^i) \neq \varnothing$ then there exists $x_0 \in C$ such that $\{T^n x_0\}$ is bounded.*

Proof. Notice first that if $x_0 \in \bigcup_{i=1}^{m} F(T^i)$ then there exists $1 \leq j \leq m$ such that $T^j x_0 = x_0$ and so $\{T^n x_0 : n \in \mathbb{N}\} = \{Tx_0, ..., T^j x_0\}$. Thus the sequence $\{T^n x_0\}$ is bounded. Now assume that there exists $x_0 \in C$ such that $\{T^n x_0\}$ is bounded. Let μ be a Banach limit and let $y \in C$ be given. For each bounded sequence $\{t_n\} \in l^\infty$ the value of μ at $\{t_n\} \in l^\infty$ is denoted by $\mu_n t_n$. Since T is generalized nonexpansive, we have

$$\Sigma_{i=1}^{m}\alpha_i\|T^{n+i}x_0 - T^i y\|^p$$
$$\leq a\|T^n x_0 - y\|^p + b\|Ty - T^n x_0\|^p$$
$$+ c\|y - T^{n+1}x_0\|^p + d\|T^n x_0 - T^{n+1}x_0\|^p,$$

for all $n \in \mathbb{N}$, where $p > 1$, $\alpha_i \geq 0$, $\Sigma_{i=1}^{m}\alpha_i = 1$, $b < \alpha_1$, $a + c > 0$ and $a + b + c \leq 1$. Since μ is a Banach limit, we have

$$\Sigma_{i=1}^{m}\alpha_i\mu_n\|T^{n+i}x_0 - T^i y\|^p$$
$$\leq a\mu_n\|T^n x_0 - y\|^p + b\mu_n\|Ty - T^n x_0\|^p$$
$$+ c\mu_n\|y - T^{n+1}x_0\|^p + d\mu_n\|T^n x_0 - T^{n+1}x_0\|^p.$$

Thus by our assumptions

$$(\frac{\alpha_1 - b}{a+c})\mu_n\|T^n x_0 - Ty\|^p + \Sigma_{i=2}^{m}\frac{\alpha_i}{a+c}\mu_n\|T^n x_0 - T^i y\|^p \leq \mu_n\|T^n x_0 - y\|^p \tag{2.1}$$

Let $g : C \to \mathbb{R}$ be a function defined by $g(y) = \mu_n\|T^n x_0 - y\|^p$ for all $y \in C$. Now we assert that g is a convex, continuous, and coercive function. The convexity of g follows immediately from Lemmas 1 and 2. We show that g is continuous. Let $\{y_m\}$ be a sequence in C such that $y_m \to y$. Then by the mean value theorem, we have

$$|\|T^n x_0 - y_m\|^p - \|T^n x_0 - y\|^p| = |\|T^n x_0 - y_m\| - \|T^n x_0 - y\||pc_{m,n}^{p-1}|,$$

for all $m,n \in \mathbb{N}$, where

$$\min\{\|T^n x_0 - y_m\|, \|T^n x_0 - y\|\} \le c_{m,n} \le \max\{\|T^n x_0 - y_m\|, \|T^n x_0 - y\|\}.$$

Hence

$$\begin{aligned}
&|\|T^n x_0 - y_m\|^p - \|T^n x_0 - y\|^p| \\
&\quad \le |\|T^n x_0 - y_m\| - \|T^n x_0 - y\||p(\|T^n x_0 - y_m\| + \|T^n x_0 - y\|)^{p-1} \\
&\quad \le \|y_m - y\| \sup\{p(\|T^n x_0 - y_m\| + \|T^n x_0 - y\|)^{p-1} : m,n \in \mathbb{N}\},
\end{aligned}$$

for all $m,n \in \mathbb{N}$. This shows that the function $h : C \to l^\infty$ defined by

$$h(z) = \{\|T^n x_0 - z\|^p\}_n, \quad z \in C$$

is continuous. Thus $g = \mu \circ h$ is also continuous. We next show that g is coercive. If $\{z_m\}$ is a sequence in C such that $\|z_m\| \to \infty$, then we have

$$\|T^n x_0 - z_m\|^p \ge (|\|z_m\| - \|T^n x_0\||)^p$$

and hence $g(z_m) \to \infty$.
It follows from Lemma 3 that there exists $u \in C$ such that $g(u) = \inf g(C)$. Now, we prove that such a point u is unique. Suppose that there exist $u_1, u_2 \in C$ such that $u_1 \ne u_2$ and $g(u_1) = g(u_2) = \inf g(C)$. If $p = 2$ then from Lemma 1 for $\epsilon = \| u_1 - u_2 \| > 0$, we have $\delta > 0$ such that

$$\|\frac{1}{2}(T^n x_0 - u_1) + \frac{1}{2}(T^n x_0 - u_2)\|^2 \le \frac{1}{2}\|T^n x_0 - u_1\|^2 + \frac{1}{2}\|T^n x_0 - u_2)\|^2 - \delta,$$

for all $n \in \mathbb{N}$. If $2 \ne p > 1$ then from Lemma 2, we get

$$\begin{aligned}
&\|\frac{1}{2}(T^n x_0 - u_1) + \frac{1}{2}(T^n x_0 - u_2)\|^p \\
&\qquad \le \frac{1}{2}\|T^n x_0 - u_1\|^p + \frac{1}{2}\|T^n x_0 - u_2)\|^p - (\frac{1}{2})^p d\,\|u_1 - u_2\|,
\end{aligned}$$

for all $n \in \mathbb{N}$. The above inequalities imply that $g(\frac{u_1+u_2}{2}) < \inf g(C)$. On the other hand, since $\frac{u_1+u_2}{2} \in C$, we have $\inf g(C) \le g(\frac{u_1+u_2}{2})$, a contradiction. Hence there exists a unique $u \in C$ such that $g(u) = \inf g(C)$. Now we show that there exists $j \in \{1,2,...,m\}$ such that $g(T^j u) \le g(u)$. On the contrary, assume that $g(u) < g(T^i u)$, for each $1 \le i \le m$. Since by our assumptions $\frac{\alpha_1 - b}{a+c} + \Sigma_{i=2}^m \frac{\alpha_i}{a+c} \ge 1$ then, we get

$$g(u) < \frac{\alpha_1 - b}{a+c} g(Tu) + \Sigma_{i=2}^m \frac{\alpha_i}{a+c} g(T^i)$$

which contradicts (2.1). Hence there exists $j \in \{1,2,...,m\}$ such that $g(T^j u) \le g(u)$. By the assumption on T, we also know that $T^j u \in C$, and so $T^j u = u$ for some $j \in \{1,2,...,m\}$. □

Theorem 2 immediately implies the following corollary.

Corollary 1. *Let E be a uniformly convex Banach space, and let C be a nonempty, closed, and convex subset of E. Let $T : C \to C$ be a mapping satisfying*

$$\|Tx - Ty\|^2 \leq a\|x - y\|^2 + b\|Tx - y\|^2 + c\|x - Ty\|^2 + d\|x - Tx\|^2,$$

for all $x, y \in C$, where $b < 1$, $a + c > 0$ and $a + b + c \leq 1$. Then $F(T)$ is nonempty if there exists $x_0 \in C$ such that $\{T^n x_0\}$ is bounded and either $d = 0$ or $\lim_{n\to\infty} \|T^n x_0 - T^{n+1} x_0\| = 0$.

The following corollary is a new coincident point result.

Corollary 2. *Let E be a uniformly convex Banach space, and let C be a nonempty, closed, bounded and convex subset of E. Let $T : C \to C$ and $S : C \to C$ be mappings such that $T(C) \subseteq S(C)$ and $S(C)$ is convex and closed. Assume that T and S satisfying*

$$\|Tx - Ty\|^2 \leq (1 - 2\alpha)\|Sx - Sy\|^2 + \alpha\|Tx - Sy\|^2 + \alpha\|Sx - Ty\|^2,$$

for all $x, y \in C$, where $0 \leq \alpha < 1$. Then T and S have a coincidence point, that is, there exists $u \in C$ such that $Tu = Su$.

Proof. We use the technique in [6]. There exists $D \subseteq C$ such that $S(D) = S(C)$ and $S : D \to C$ is one-to-one. Now, define a map $R : S(D) \to S(D)$ by $R(Sx) = Tx$. Since S is one-to-one on D and $T(C) \subseteq S(C)$, R is well-defined. Note that

$$\begin{aligned}\|R(Sx) - R(Sy)\|^2 &= \|Tx - Ty\|^2 \\ &\leq (1 - 2\alpha)\|Sx - Sy\|^2 + \alpha\|Tx - Sy\|^2 + \alpha\|Sx - Ty\|^2 \\ &= (1 - 2\alpha)\|Sx - Sy\|^2 + \alpha\|R(Sx) - Sy\|^2 + \alpha\|Sx - R(Sy)\|^2\end{aligned}$$

for all $Sx, Sy \in S(D)$. Since $S(D) = S(C)$ is convex, closed and bounded, by using Corollary 1, R has a fixed point in $S(C)$, that is, there exists $u \in C$ such that $R(Su) = Su$, and so $Tu = Su$. □

Corollary 3. *Let $1 < p < 2$, E be a p-uniformly convex Banach space, and let C be a nonempty, closed, and convex subset of E. Let $T : C \to C$ be a mapping satisfying*

$$\|Tx - Ty\|^p \leq (1 - 2\alpha)\|x - y\|^p + \alpha\|Tx - y\|^p + \alpha\|x - Ty\|^p,$$

for all $x, y \in C$, where $0 \leq \alpha < 1$. Then $F(T)$ is nonempty if and only if there exists $x_0 \in C$ such that $\{T^n x_0\}$ is bounded.

Corollary 4. *Let $1 < p < 2$, E be a p-uniformly convex Banach space, and let C be a nonempty, closed, and convex subset of E. Let $T : C \to C$ be a mapping satisfying*

$$\|Tx - Ty\|^p \leq (1 - 2\alpha)\|x - y\|^p + \alpha\|Tx - y\|^p + \alpha\|x - Ty\|^p,$$

for all $x, y \in C$, where $0 \leq \alpha < 1$. Then $F(T)$ is nonempty if and only if there exists $x_0 \in C$ such that $\{T^n x_0\}$ is bounded.

By the same technique as in the proof of Corollary 2, we can deduce the following coincidence point result from Corollary 4. For some previous studies of coincident point theory, see [7].

Corollary 5. *Let $1 < p < 2$, E be a p-uniformly convex Banach space, and let C be a nonempty, closed, bounded and convex subset of E. Let $T : C \to C$ and $S : C \to C$ be mappings such that $T(C) \subseteq S(C)$ and $S(C)$ is convex and closed. Assume that T and S satisfying*

$$\|Tx - Ty\|^p \leq (1-2\alpha)\|Sx - Sy\|^p + \alpha\|Tx - Sy\|^p + \alpha\|Sx - Ty\|^p,$$

for all $x, y \in C$, where $0 \leq \alpha < 1$. Then T and S have a coincidence point, that is, there exists $u \in C$ such that $Tu = Su$.

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Author's address

A. Amini-Harandi

Department of Mathematic, University of Isfahan, Isfahan, Iran

Current address: School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Iran

E-mail address: aminih_a@yahoo.com

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MULTIPLICATIVE ISOMORPHISMS AT INVERTIBLE MATRICES

A. ARMANDNEJAD AND M. JAMSHIDI

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Abstract. Let $\mathbf{M}_n$ be the algebra of all $n \times n$ matrices with entries in the field of real numbers. For a matrix $Z \in \mathbf{M}_n$, it is said that a linear map $T : \mathbf{M}_n \to \mathbf{M}_n$ is multiplicative at Z if $T(AB) = T(A)T(B)$ whenever $AB = Z$. In this paper we investigate some properties of multiplicative mapping at invertible matrices and also we characterize all multiplicative isomorphisms at invertible matrices. Then we give an example to show that multiplicativity at I doesn't imply multiplicativity on $\mathbf{M}_n$, where $I \in \mathbf{M}_n$ is the identity matrix.

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1. INTRODUCTION

Concerning linear maps on matrices, much work has been done on the problem described generally as follows. Let W be a space of matrices, and $Q(A)$ be a quantity or property of matrices $A \in W$. Then determine the structure of linear maps $T : W \to W$ which preserve Q [i.e. $Q(T(A)) = Q(A)$, for all $A \in W$]. Some aspects of this problem were surveyed in [4]. Moreover, when $R(A, B)$ is a relation between matrices $A, B \in W$, it is interesting to determine all linear maps $T : W \to W$ which preserve R [i.e. $R(Q(A), Q(B))$ holds whenever $R(A, B)$]. For example, several authors have discussed preserving linear maps of majorization on some spaces of matrices (see [1], [2] and references therein). A linear map $T : \mathbf{M}_n \to \mathbf{M}_n$ is called multiplicative on $\mathbf{M}_n$ if $T(AB) = T(A)T(B)$, for all $A, B \in \mathbf{M}_n$. A well known fact about a multiplicative linear map on $\mathbf{M}_n$ is the following theorem which has a short and interesting proof in [6].

Theorem 1. *Let $T : \mathbf{M}_n \to \mathbf{M}_n$ be a multiplicative bijective linear map. Then there exists an invertible matrix $S \in \mathbf{M}_n$ such that $T(A) = SAS^{-1}$, for every $A \in \mathbf{M}_n$.*

Proof. (From [6], Theorem 1.1) Choose and fix a pair of nonzero vectors $u, y \in \mathbb{R}^n$. Since T is injective, we can find $z \in \mathbb{R}^n$ such that $T(uy^t)z \neq 0$. Define $S : \mathbb{R}^n \to \mathbb{R}^n$ by $Sx = T(xy^t)z$, $x \in \mathbb{R}^n$. The linearity of S follows from the linearity of T.

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Moreover, S is nonzero since $Su \neq 0$. For arbitrary $A \in \mathbf{M}_n$ and $x \in \mathbb{R}^n$ we have

$$SAx = T((Ax)y^t)z = T(A(xy^t))z = T(A)T(xy^t)z = T(A)Sx,$$

and consequently,

$$SA = T(A)S.$$

Let w be any vector in $\mathbb{R}^n$. Since $Su \neq 0$ and because T is surjective we can find $B \in \mathbf{M}_n$ such that $T(B)Su = w = SBu$. Thus, S is surjective, and therefore invertible. It follows that $T(A) = SAS^{-1}$, $A \in \mathbf{M}_n$, as desired.

□

The concept of a linear multiplicative mapping at unity on $B(H)$ (where $B(H)$ is the algebra of all bounded linear operators on a Hilbert space H) was introduced in [3]. Also the authors characterized all linear multiplicative maps at unity which are weak-continuous and surjective on $B(H)$, where H is an infinite dimensional Hilbert space. In fact, they showed that, such maps are automorphisms on $B(H)$. In [5] the authors proved the following theorem which we use it to prove the main results of this paper.

Theorem 2. *Let $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ be a linear map. Then, T preserves the set of all invertible matrices, if and only if there exist invertible matrices A and B such that,*
(i) $T(X) = AXB, \ \forall \ X \in \boldsymbol{M}_n$, or
(ii) $T(X) = AX^tB, \forall \ X \in \boldsymbol{M}_n$.

2. Multiplicative isomorphism at invertible matrices

In this section we state some preliminaries about idempotent matrices to prove the main result of this paper. We recall that a matrix $P \in \mathbf{M}_n$ is idempotent if $P^2 = P$. The following proposition shows that the set of idempotent matrices generates $\mathbf{M}_n$. The notation $E_{i,j}$ is used for the matrix which has 1 as (i, j) entry and 0 elsewhere. It is easy to see that for every i and j $(1 \leq i \neq j \leq n)$ $E_{i,i}^2 = E_{i,i}$ and $E_{i,j}^2 = 0$.

Proposition 1. *There exists a basis for $\boldsymbol{M}_n$ which is contained in the set of idempotent matrices.*

Proof. Assume $\{E_{i,j} : 1 \leq i, j \leq n\}$ is the standard basis of $\mathbf{M}_n$. Put $P_i = E_{i,i}$ and $P_{i,j} = E_{i,i} + E_{i,j}$ for every i and j $(1 \leq j \neq i \leq n)$, then

$$P_{i,j}^2 = (E_{i,i} + E_{i,j})^2 = E_{i,i}^2 + E_{i,i}E_{i,j} + E_{i,j}E_{i,i} + E_{i,j}^2 = E_{i,i} + E_{i,j} = P_{i,j}.$$

Therefore P_i and $P_{i,j}$ are idempotent matrices. Put

$$\mathcal{B} = \{P_i \ : 1 \leq i \leq n\} \cup \{P_{i,j} : 1 \leq j \neq i \leq n\}.$$

It is easy to see that $\mathcal{B}$ is a basis for $\mathbf{M}_n$. □

Definition 1. Let $Z \in \mathbf{M}_n$ be given. A linear map $T : \mathbf{M}_n \to \mathbf{M}_n$ is called multiplicative at Z if $T(AB) = T(A)T(B)$ whenever $AB = Z$.

Now, we provide some lemmas to obtain the main result.

Lemma 1. *Let $T : \boldsymbol{M_n} \to \boldsymbol{M_n}$ be a linear map and let $Z \in \boldsymbol{M_n}$ be an invertible matrix. If T is multiplicative at Z, then for every $A \in \boldsymbol{M_n}$, the following assertions are true:*

(i) $T(Z)T(A) = T(ZA)T(I)$,
(ii) $T(A)T(Z) = T(I)T(AZ)$,
(iii) $T(Z)T(Z^{-1}AZ) = T(AZ)T(I)$,
(iv) $T(Z^2)T(Z^{-2}A) = T(AZ)T(Z^{-1})$.

Proof. (i) For every idempotent P,

$$(Z + ZP)(I - \frac{1}{2}P) = Z = (Z - \frac{1}{2}ZP)(I + P).$$

Since T is multiplicative at Z,

$$T(Z + ZP)T(I - \frac{1}{2}P) = T(Z - \frac{1}{2}ZP)T(I + P) \Rightarrow \tag{2.1}$$

$$[T(Z) + T(ZP)][T(I) - T(\frac{1}{2}P)] = [T(Z) - T(\frac{1}{2}ZP)][T(I) + T(P)] \Rightarrow$$

$$T(Z)T(P) = T(ZP)T(I). \tag{2.2}$$

Therefore by Proposition 1,

$$T(Z)T(A) = T(ZA)T(I), \quad \forall A \in \mathbf{M}_n.$$

(ii) For every idempotent P,

$$(I + P)(Z - \frac{1}{2}ZP) = Z = (I - \frac{1}{2}P)(Z + PZ).$$

An argument similar to that above implies $T(A)T(Z) = T(I)T(AZ)$.

(iii) In part (i), put $Z^{-1}AZ$ instead of A.

(iv) For every idempotent P,

$$(Z^2 + Z^2P)(Z^{-1} - \frac{1}{2}PZ^{-1}) = Z = (Z^2 - \frac{1}{2}Z^2P)(Z^{-1} + PZ^{-1}).$$

An argument similar to that above implies $T(Z^2)T(AZ^{-1}) = T(Z^2A)T(Z^{-1})$. Putting $Z^{-2}AZ$ instead of A, then $T(Z^2)T(Z^{-2}A) = T(AZ)T(Z^{-1})$. □

Lemma 2. *Let $T : \boldsymbol{M_n} \to \boldsymbol{M_n}$ be a bijective linear map that is multiplicative at an invertible matrix Z, then $T(I) = I$.*

Proof. Since T is multiplicative at Z,

$$T(Z)T(I) = T(Z) = T(I)T(Z). \qquad (v)$$

For every $A \in \mathbf{M}_n$,

$$\begin{aligned} T(A)T(Z)T(Z^{-1}) &= T(I)T(AZ)T(Z^{-1}) && \text{(by } (ii)\text{, Lemma 1)} \\ &= T(I)T(Z^2)T(Z^{-2}A) && \text{(by } (iv)\text{, Lemma 1)} \\ &= T(Z)T(Z)T(Z^{-2}A) && \text{(by } (ii)\text{, Lemma 1)} \\ &= T(Z)T(Z^{-1}A)T(I). && \text{(by } (i)\text{, Lemma 1)} \end{aligned}$$

And consequently,

$$\begin{aligned} T(I)T(AZ) &= T(A)T(Z) && \text{(by } (ii)\text{, Lemma 1)} \\ &= T(A)T(Z)T(I)^2 && \text{(by } (v)\text{)} \\ &= T(A)T(Z)T(Z^{-1})T(Z) && \text{(by } (ii)\text{, Lemma 1)} \\ &= T(Z)T(Z^{-1}A)T(I)T(Z) && \text{(by the above relation)} \\ &= T(Z)T(Z^{-1}A)T(Z) && \text{(by } (v)\text{)} \\ &= T(Z)T(I)T(Z^{-1}AZ) && \text{(by } (ii)\text{, Lemma 1)} \\ &= T(Z)T(Z^{-1}AZ) && \text{(by } (v)\text{)} \\ &= T(AZ)T(I) && \text{(by } (iii)\text{, Lemma 1)} \end{aligned}$$

Now, since T is surjective and $A \in \mathbf{M}_n$ is arbitrary, for every $B \in \mathbf{M}_n$, we have $BT(I) = T(I)B$. It implies that $T(I) = \lambda I$, for some nonzero $\lambda \in \mathbb{R}$. By (ii) of Lemma 1, $T(Z^{-1})T(Z) = T(I)^2 = \lambda^2 I$ and hence $T(Z)$ is invertible. Now using (v) implies that $T(I) = I$. □

The following theorems characterize the multiplicative isomorphism at invertible matrices.

Theorem 3. *Let $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ be a bijective linear map and let Z be an invertible non scalar matrix. Then T is multiplicative at Z if and only if T is an automorphism i.e. there exists an invertible matrix $B \in \boldsymbol{M}_n$ such that $T(X) = BXB^{-1}$ for all $X \in \boldsymbol{M}_n$.*

Proof. As the sufficiency of the condition is easy to see, only we prove the necessity of the condition. By the proof of Lemma 1, $T(Z)$ is invertible. For every invertible matrix $A \in \mathbf{M}_n$, $T(Z) = T(ZA^{-1}A) = T(ZA^{-1})T(A)$ and hence $T(A)$ is invertible. Therefore T preserves the set of invertible matrices and by Lemma 2, $T(I) = I$. Consequently, by Theorem 2, there exists an invertible matrix $B \in \mathbf{M}_n$ such that one of the following holds:

$$(i)\ \ T(X) = BXB^{-1},\ \ \forall\ \ X \in \mathbf{M}_n,\ \text{ or }\ (ii)\ \ T(X) = BX^tB^{-1},\ \ \forall\ X \in \mathbf{M}_n.$$

If $n = 1$ the statements (i) and (ii) are the same. So we may assume that $n \geq 2$ and hence there exists an invertible matrix $A \in \mathbf{M}_n$ such that $AZ \neq ZA$. If the statement

$(i\,i)$ holds, then

$$\begin{aligned} T(Z) = T(ZA^{-1}A) &= T(ZA^{-1})T(A) \Rightarrow \\ BZ^tB^{-1} &= B(ZA^{-1})^tB^{-1}BA^tB^{-1} \Rightarrow \\ AZ &= ZA. \end{aligned}$$

Which is a contradiction, so $T(X) = BXB^{-1},\ X \in \mathbf{M}_n$, as desired. □

Corollary 1. *Let $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ be a bijective linear map and let Z be an invertible non scalar matrix. Then T is multiplicative at Z if and only if T is multiplicative on $\boldsymbol{M}_n$.*

Theorem 4. *Let $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ be a bijective linear map and let Z be an invertible scalar matrix. Then T is multiplicative at Z if and only if T is an automorphism or anti automorphism i.e. there exists an invertible matrix $B \in \boldsymbol{M}_n$ such that $T(X) = BXB^{-1}$ for all $X \in \boldsymbol{M}_n$ or $T(X) = BX^tB^{-1}$ for all $X \in \boldsymbol{M}_n$.*

Proof. With an argument similar to the proof of Theorem 3, if T is multiplicative at an invertible matrix Z, then T is an automorphism or anti automorphism. It is easy to show that every anti automorphism is multiplicative at an invertible scalar matrix Z. □

The following example shows that multiplicativity at I doesn't imply multiplicativity on $\mathbf{M}_n$.

Example 1. Let $T : \mathbf{M}_2 \to \mathbf{M}_2$ be the linear map defined as follows :

$$T(X) = PX^tP^{-1},\ \forall X \in \mathbf{M}_2,\ where\quad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then T is multiplicative at I by Theorem 4. Consider the matrices:

$$X = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix} and\ \ Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that $T(XY) \neq T(X)T(Y)$. So T is not multiplicative on $\mathbf{M}_2$.

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Authors' addresses

A. Armandnejad
Vali-e-Asr University of Rafsanjan, Department of Mathematics, P. O. Box 7713936417, Rafsanjan, Iran
E-mail address: armandnejad@vru.ac.ir

M. Jamshidi
Graduate University of Advanced Technology, P. O. Box 7631133131, Kerman, Iran
E-mail address: m.jamshidi@kgut.ac.ir

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COMPLETENESS OF ONE TWO-INTERVAL BOUNDARY VALUE PROBLEM WITH TRANSMISSION CONDITIONS

K. AYDEMIR AND O. SH. MUKHTAROV

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Abstract. This paper presents a new approach to the two-interval Sturm-Liouville eigenfunction expansions, based essentially on the method of integral equations. We consider the Sturm-Liouville problem together with two supplementary transmission conditions at one interior point. Further we develop Green's function method for spectral analysis of the considered problem in modified Hilbert space.

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1. INTRODUCTION

The development of classical, rather than the operatoric, Sturm-Liouville theory in the years after 1950 can be found in various sources; in particular in the texts of Atkinson [2], Coddington and Levinson [5], Levitan and Sargsjan [9]. The operator theoretic development is given in the texts by Naimark [12] and Akhiezer and Glazman [1]. The subject of eigenfunction expansions is as old as operator theory. The Sturm-Liouville problem is also important because the solutions to a homogeneous boundary value problem (BVP) with homogeneous boundary conditions(BCs) produce a set of orthogonal functions. Such functions can be used to represent functions in Fourier series expansions. The completeness of classical systems of eigenfunction expansions was originally related to mechanical problems and boundary value problems for differential operators. Later the study of eigenfunctions expansions has gained an independent and abstract status. The expansion has an integral operator form whose kernel is a spectral function, the representation of which is the Green function of the operator. For the method of treating such problems see [8]. The method of Sturm expansions is widely used in calculations of the spectroscopic characteristics of atoms and molecules [13, 14]. The advantages of the method are both the possibility of adequate choice of the nature of the spectrum of an unperturbed problem in the calculation of quantum mechanical systems using the perturbation

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theory and the possibility of providing best convergence of corresponding expansions in basis functions. In this study we shall investigate one two-interval boundary value problem ,namely, the Sturm-Liouville equation

$$-p(x)y''(x)+q(x)y(x)=\lambda y(x) \tag{1.1}$$

to hold in disjoint intervals $(-\pi,0)$ and $(0,\pi)$ under boundary conditions

$$\cos\alpha y(-\pi)+\sin\alpha y'(-\pi)=0 \tag{1.2}$$

$$\cos\beta y(\pi)+\sin\beta y'(\pi)=0. \tag{1.3}$$

where singularity of the solution $y=y(x,\lambda)$ prescribed by transmission conditions

$$\beta_{11}^{-}y'(0-)+\beta_{10}^{-}y(0-)+\beta_{11}^{+}y'(0+)+\beta_{10}^{+}y(0+)=0 \tag{1.4}$$

$$\beta_{21}^{-}y'(0-)+\beta_{20}^{-}y(0-)+\beta_{21}^{+}y'(0+)+\beta_{20}^{+}y(0+)=0 \tag{1.5}$$

where $p(x)=p_1>0$ for $x\in[-\pi,0)$, $p(x)=p_2>0$ for $x\in(0,\pi]$, the potential $q(x)$ is given real-valued function which continuous in each of the intervals $[-\pi,0)$ and $(0,\pi]$, and has a finite limits $q(\mp 0)$, λ is a complex eigenparameter, $\beta_{ij}^{\pm}$, $(i=1,2$ and $j=0,1)$ are real numbers. In this study we introduce a new approach to the two-interval Sturm-Liouville eigenfunction expansions, based essentially on the method of integral equations. Note that in physics many problems arise in the form of boundary value problems involving second order ordinary differential equations. This derivation based on that in [7], however, is more thorough than that in most elementary physics texts; while most parameters such as density and other thermal properties are treated as constant in such treatments, the following allows fundamental properties of the bar to vary as a function of the bar's length, which will lead to a Sturm-Liouville problem of a more general nature. Detailed studies on spectral problems for ordinary differential operators depending on the parameter and/or with transmission conditions can be found in various publications, see e.g. [3,4,10,11,15], where further references and links to applications can be found.

2. Construction of the resolvent by means of Green's function

Let $u=\phi_1(x,\lambda)$ be solution of the equation (1.1) on left interval $[-\pi,0)$ satisfying the conditions

$$u(-\pi)=\sin\alpha,\ \ u'(-\pi)=-\cos\alpha$$

and $u=\phi_2(x,\lambda)$ be solution of same equation on right interval $(0,\pi]$ satisfying the conditions

$$u(0+,\lambda)=\frac{1}{\Delta_{12}}(\Delta_{23}\phi_1(0,\lambda)+\Delta_{24}\frac{\partial\phi_1(0,\lambda)}{\partial x})$$

$$u'(0+,\lambda)=\frac{-1}{\Delta_{12}}(\Delta_{13}\phi_1(0,\lambda)+\Delta_{14}\frac{\partial\phi_1(0,\lambda)}{\partial x})$$

Similarly we define by $u = \chi_2(x,\lambda)$ the solution of the equation (1.1) on the right interval $(0,\pi]$ by initial conditions

$$u(\pi) = -\sin\beta, \quad u'(\pi) = \cos\beta$$

and by $u = \chi_1(x,\lambda)$ the solution of the equation (1.1) on the left interval $[-\pi,0)$ by initial conditions

$$u(0-,\lambda) = \frac{-1}{\Delta_{34}}(\Delta_{14}\chi_2(0,\lambda) + \Delta_{24}\frac{\partial\chi_2(0,\lambda)}{\partial x})$$

$$u'(0-,\lambda) = \frac{1}{\Delta_{34}}(\Delta_{13}\chi_2(0,\lambda) + \Delta_{23}\frac{\partial\chi_2(0,\lambda)}{\partial x})$$

respectively, where Δ_{ij} $(1 \le i < j \le 4)$ denotes the determinant of the i-th and j-th columns of the matrix

$$T = \begin{bmatrix} \beta_{10}^{+} & \beta_{11}^{+} & \beta_{10}^{-} & \beta_{11}^{-} \\ \beta_{20}^{+} & \beta_{21}^{+} & \beta_{20}^{-} & \beta_{21}^{-} \end{bmatrix}.$$

By applying the method of [10] we can show that each of these solutions are entire functions of complex parameter λ. It is clear that each of the Wronskians

$$\omega_i(\lambda) = W(\phi_i(x,\lambda), \chi_i(x,\lambda)), \;\; i = 1,2$$

are independent of variable x. By using the definitions of solutions $\phi_i(x,\lambda)$ and $\chi_i(x,\lambda)$ we have

$$\begin{aligned} w_2(\lambda) &= \phi_2(0+,\lambda)\frac{\partial\chi_2(0+,\lambda)}{\partial x} - \frac{\partial\phi_2(0+,\lambda)}{\partial x}\chi_2(0+,\lambda) \\ &= \frac{\Delta_{34}}{\Delta_{12}}(\phi_1(0-,\lambda)\frac{\partial\chi_1(0-,\lambda)}{\partial x} - \frac{\partial\phi_1(0-,\lambda)}{\partial x}\chi_1(0-,\lambda)) \\ &= \frac{\Delta_{34}}{\Delta_{12}}w_1(\lambda). \end{aligned}$$

Introduce to the consideration the Green's function (see, [3]) $G(x,\xi,\lambda)$ of the problem (1.1) – (1.5) as

$$G(x,\xi;\lambda)=\begin{cases}\frac{\phi_1(x,\lambda)\chi_1(\xi,\lambda)}{\Delta_{34}p_1\omega_1(\lambda)}, & if\ x\in[-\pi,0),\ \xi\in[-\pi,x)\\ \frac{\chi_1(x,\lambda)\phi_1(\xi,\lambda)}{\Delta_{34}p_1\omega_1(\lambda)}, & if\ x\in[-\pi,0),\ \xi\in[x,0)\\ \frac{\chi_1(x,\lambda)\phi_2(\xi,\lambda)}{\Delta_{34}p_1\omega_1(\lambda)}, & if\ x\in[-\pi,0),\ \xi\in(0,\pi]\\ \frac{\phi_2(x,\lambda)\chi_2(\xi,\lambda)}{\Delta_{12}p_2\omega_2(\lambda)}, & if\ \ x\in(0,\pi],\ \xi\in(0,x]\\ \frac{\chi_2(x,\lambda)\phi_2(\xi,\lambda)}{\Delta_{12}p_2\omega_2(\lambda)}, & if\ \ x\in(0,\pi],\ \xi\in[x,\pi]\\ \frac{\phi_2(x,\lambda)\chi_1(\xi,\lambda)}{\Delta_{12}p_2\omega_2(\lambda)}, & if\ x\in(0,\pi],\ \xi\in[-\pi,0)\end{cases}\tag{2.1}$$

Let $f(x)$ be any function continuous in $[-\pi,0)$ and $(0,\pi]$, which has finite limits $f(\mp 0)$. Show that the function

$$u(x,\lambda)=\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}G(x,\xi;\lambda)f(\xi)d\xi+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}G(x,\xi;\lambda)f(\xi)d\xi\tag{2.2}$$

satisfies the nonhomogeneous equation

$$-p(x)u''+\{q(x)-\lambda\}u=f(x),\ x\in[-\pi,0)\cup(0,\pi]\tag{2.3}$$

and boundary-transmission conditions(BTCs) (1.2)-(1.5). Indeed, putting (2.1) in (2.2) and then differentiating twice we have

$$u''(x,\lambda)=\begin{cases}\frac{\chi_1''(x,\lambda)}{p_1\omega_1(\lambda)}\int_{-\pi}^{x}\phi_1(\xi,\lambda)f(\xi)d\xi+\frac{\phi_1''(x,\lambda)}{p_1\omega_1(\lambda)}\int_{x}^{0}\chi_1(\xi,\lambda)f(\xi)d\xi\\ +\frac{\phi_1''(x,\lambda)}{p_2\omega_2(\lambda)}\int_{0}^{\pi}\chi_2(\xi,\lambda)f(\xi)d\xi-\frac{f(x)p_1}{p_1\omega_1(\lambda)}W(\phi_1(x,\lambda),\chi_1(x,\lambda))\\ for\ x\in[-\pi,0)\\ \frac{\chi_2''(x,\lambda)}{p_1\omega_1(\lambda)}\int_{-\pi}^{0}\phi_1(\xi,\lambda)f(\xi)d\xi+\frac{\chi_2''(x,\lambda)}{\omega_2(\lambda)}\int_{0}^{x}\phi_2(\xi,\lambda)f(\xi)d\xi\\ +\frac{\phi_2''(x,\lambda)}{p_2\omega_2(\lambda)}\int_{x}^{\pi}\chi_2(\xi,\lambda)f(\xi)d\xi-\frac{f(x)p_2}{p_2\omega_2(\lambda)}W(\phi_2(x,\lambda),\chi_2(x,\lambda))\\ for\ x\in(0,\pi]\end{cases}$$

$$=(q(x)-\lambda)\begin{cases}\frac{\chi_1(x,\lambda)}{p_1\omega_1(\lambda)}\int_{-\pi}^{x}\phi_1(\xi,\lambda)f(\xi)d\xi+\frac{\phi_1(x,\lambda)}{p_1\omega_1(\lambda)}\int_{x}^{0}\chi_1(\xi,\lambda)f(\xi)d\xi\\ +\frac{\phi_1(x,\lambda)}{p_2\omega_2(\lambda)}\int_{0}^{\pi}\chi_2(\xi,\lambda)f(\xi)d\xi-f(x)\qquad for\ x\in[-\pi,0)\\ \frac{\chi_2(x,\lambda)}{p_1\omega_1(\lambda)}\int_{-\pi}^{0}\phi_1(\xi,\lambda)f(\xi)d\xi+\frac{\phi_2(x,\lambda)}{p_2\omega_2(\lambda)}\int_{x}^{\pi}\chi_2(\xi,\lambda)f(\xi)d\xi\\ +\frac{\chi_2(x,\lambda)}{p_2\omega_2(\lambda)}\int_{0}^{x}\phi_2(\xi,\lambda)f(\xi)d\xi-f(x)\qquad for\ x\in(0,\pi]\end{cases}$$

$$= (q(x)-\lambda)u(x,\lambda)-f(x)$$

i.e. the function $u(x,\lambda)$ satisfies the nonhomogeneous equation (2.3). The function $u(x,\lambda)$ also satisfies the transmission conditions (1.4)-(1.5) and both boundary conditions (1.2)-(1.3). Therefore $u(x,\lambda)$ forms an resolvent of the problem (1.1)-(1.5).

3. Expansion results for Green's function

Through in below we assume that the homogeneous equation

$$p(x)u''-q(x)u=0$$

under the same BTCs (1.2)-(1.5) has only the trivial solution $u=0$. This amounts to the assumption that $\lambda=0$ is not an eigenvalue of the considered BVTP (1.1)-(1.5). There is no less of generality in this assumption, since otherwise we may consider a new equation

$$-p(x)u''+\widetilde{q}(x)u=\lambda u$$

for $\widetilde{q}(x)=q(x)-\widetilde{\lambda}$ under the same BTCs (1.2)-(1.5). Obviously this problem has the same eigenfunctions as for the considered BVTP (1.1)-(1.5), all eigenvalues are shifted through $\widetilde{\lambda}$ to the right and therefore, we can chose $\widetilde{\lambda}$ such that $\lambda=0$ is not eigenvalue of the new problem. Now defining $G_0(x,\xi)=G(x,\xi;0)$ we see that the function

$$u_0(x)=\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}G_0(x,\xi)f(\xi)d\xi+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}G_0(x,\xi)f(\xi)d\xi$$

solves the nonhomogeneous equation $p(x)u''-q(x)u=f(x)$ and satisfies all boundary-transmission conditions (1.2)-(1.5). Rewritting (2.3) in the form

$$-p(x)u''-q(x)u=\lambda u-f(x)$$

We see that this equation under the same BTC's (1.2)-(1.5) reduces to the integral equation

$$u(x)+\lambda\{\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}G_0(x,\xi)u(\xi)d\xi+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}G_0(x,\xi)u(\xi)d\xi\}$$
$$=\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}G_0(x,\xi)f(\xi)d\xi+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}G_0(x,\xi)f(\xi)d\xi$$

Consequently the considered BVTP (1.1)-(1.5) can be written in equivalent integral equation form as

$$u(x)=-\lambda\{\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}G_0(x,\xi)u(\xi)d\xi+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}G_0(x,\xi)u(\xi)d\xi\} \tag{3.1}$$

By applying the same method as in [10] we can prove that the BVTP (1.1)-(1.5) has precisely denumerable many eigenvalues $\lambda_0,\lambda_1,\lambda_2,...$, with following asymptotic

behaviour as $n \to \infty$.
(i) If $\sin\beta \neq 0$ and $\sin\alpha \neq 0$, then

$$s_n = (\frac{n-1}{2}) + O\left(\frac{1}{n}\right)$$

(ii) If $\sin\beta \neq 0$ and $\sin\alpha = 0$, then

$$s_n = \frac{n}{2} + O\left(\frac{1}{n}\right),$$

(iii) If $\sin\beta = 0$ and $\sin\alpha \neq 0$, then

$$s_n = \frac{n}{2} + O\left(\frac{1}{n}\right),$$

(iv) If $\sin\beta = 0$ and $\sin\alpha = 0$, then

$$s_n = \frac{n}{2} + O\left(\frac{1}{n}\right),$$

as $n \to \infty$ where $\lambda_n = s_n^2$. It is evident that the functions $\phi_n(x) = \phi(x,\lambda_n)$ are eigenfunctions corresponding to the eigenvalues λ_n. Let

$$\varphi_n(x) = (\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}\phi_n^2(\xi)d\xi + \frac{\Delta_{12}}{p_2}\int_{0}^{\pi}\phi_n^2(x)dx)^{-\frac{1}{2}}\Phi_n(x)$$

It is obviously seen that the sequence $(\varphi_n(x))$ forms an orthonormal set of eigenfunctions in the sense of

$$\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}\varphi_n(x)\varphi_m(x)dx + \frac{\Delta_{12}}{p_2}\int_{0}^{\pi}\varphi_n(x)\varphi_m(x)dx = \delta_{nm}, \tag{3.2}$$

where δ_{nm} is the Kronecker delta.

Theorem 1. *The Green's function $G_0(x,\xi)$ can be expanded into an eigenfunction series*

$$G_0(x,\xi) = -\sum_{n=0}^{\infty}\lambda_n^{-1}\varphi_n(x)\varphi_n(\xi) \tag{3.3}$$

which converges absolutely and uniformly on $([-\pi,0)\cup(0,\pi])^2$.

Proof. By using the definition of eigenfunctions $\varphi_n(x)$ and asymptotic behaviour of eigenvalues it is not difficult to show that the series in (3.3) converges absolutely and uniformly and therefore represents a continuous function there. To prove the equality (3.3), suppose, it possible that, the function

$$\widetilde{G_0}(x,\xi) = G_0(x,\xi) + \sum_{n=0}^{\infty}\lambda_n^{-1}\varphi_n(x)\varphi_n(\xi)$$

is not identically zero. Taking in view that the Kernel $\widetilde{G_0}(x,\xi)$ is symmetric slightly modifying the method of proving of familiar theorem in the theory of integral equations (see, for example, [6]) which assert that any symmetric kernel which is not identically zero has at least one eigenfunction, we can prove that there is a real number $\mu_0 \neq 0$ and real valued function $\psi_0 \neq 0$ such that

$$\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}\widetilde{G}_0(x,\xi)\psi_0(\xi)d\xi+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}\widetilde{G}_0(x,\xi)\psi_0(\xi)d\xi=\mu_0\psi_0(x) \tag{3.4}$$

Multiplying by $\varphi_n(x)$ and make the necessary calculations we have

$$\begin{aligned}
&\mu_0\Big(\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}\psi_0(x)\varphi_n(x)dx+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}\psi_0(x)\varphi_n(x)dx\Big)\\
&=\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}\psi_0(\xi)\Big(\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}\widetilde{G_0}(x,\xi)\varphi_n(x)dx+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}\widetilde{G_0}(x,\xi)\varphi_n(x)dx\Big)d\xi\\
&\quad+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}\psi_0(\xi)\Big(\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}\widetilde{G_0}(x,\xi)\varphi_n(x)dx\\
&\quad+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}\widetilde{G_0}(x,\xi)\varphi_n(x)dx\Big)d\xi
\end{aligned} \tag{3.5}$$

Recalling that the set of eigenfunctions $(\varphi_n(x))$ is orthonormal in the sense of (3.2) it is easy to see that

$$\begin{aligned}
&\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}\widetilde{G_0}(x,\xi)\varphi_n(x)dx+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}\widetilde{G_0}(x,\xi)\varphi_n(x)dx\\
&=\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}G(x,\xi)\varphi_n(x)dx+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}G(x,\xi)\varphi_n(x)dx+\frac{\varphi_n(\xi)}{\lambda_n}
\end{aligned} \tag{3.6}$$

Substituting (3.6) in (3.5) and taking in view the fact that the eigenfunction $\varphi_n(x)$ satisfy the integral equation (3.1) for $\lambda=\lambda_n$ gives

$$\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}\psi_0(x)\varphi_n(x)dx+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}\psi_0(x)\varphi_n(x)dx=0 \tag{3.7}$$

for all $n=0,1,2...$, i.e. the function $\psi_0(x)$ is orthogonal to all eigenfunctions. On the other hand, from (3.4) and (3.7) it follows that

$$\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}G_0(x,\xi)\psi_0(\xi)d\xi+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}G_0(x,\xi)\psi_0(\xi)d\xi=\mu_0\psi_0(x),$$

so $\psi_0(x)$ is also an eigenfunction of BVTP (1.1)-(1.5) corresponding to eigenvalue $-\frac{1}{\mu_0}$. Since it is orthogonal to all eigenfunctions, it is orthogonal to itself, i.e.

$$\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}\psi_0^2(s)ds+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}\psi_0^2(s)ds=0$$

and hence $\psi_0(s)$ is identically zero. We get a contradiction, which complete the proof. □

4. Completeness of the eigenfunctions

To prove the completeness (φ_n) in square integrable function space at first we shall prove the next theorem.

Theorem 2. *Let $f(x)$ be any function on $[-\pi,0)\cup(0,\pi]$ satisfying the following conditions*
i) *f' and f'' are exist and continuous in both interval $[-\pi,0)$ and $(0,\pi]$*
ii) *There exist a finite one-hand side limits $f(\pm 0), f'(\pm 0)$ and $f''(\pm 0)$*
iii) *f satisfies the BTC's (1.2)-(1.5).*
Then $f(x)$ can be expanded into an absolutely and uniformly convergent series of eigenfunctions (φ_n), namely

$$f(x)=\sum_{n=0}^{\infty}\{\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}f(\xi)\varphi_n(\xi)d\xi+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}f(\xi)\varphi_n(\xi)d\xi\}\varphi_n(x)$$

Proof. Let f be satisfied all conditions of Theorem. Then by virtue of (3.1) the equality

$$\begin{aligned}f(x)=&\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}G_0(x,\xi)(f''(\xi)-q(\xi)f(\xi))d\xi\\&+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}G_0(x,\xi)(f''(\xi)-q(\xi)f(\xi))d\xi\end{aligned}$$

is hold. Substituting (3.3) in the right hand of this equality and integrating by parts twice yields the needed equality

$$f(x)=\sum_{n=0}^{\infty}\{\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}f(\xi)\varphi_n(\xi)d\xi+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}f(\xi)\varphi_n(\xi)dx\}\varphi_n(x)$$

The proof is complete. □

Corollary 1. *Let f be as in previous Theorem. Then the modified parseval equality*

$$\begin{aligned}\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}f^2(x)dx+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}f^2(x)dx=&\sum_{n=0}^{\infty}(\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}f(\xi)\varphi_n(\xi)d\xi\\&+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}f(\xi)\varphi_n(\xi)dx\}\varphi_n(\xi))^2\end{aligned}$$

is hold. This is also called the completeness relation.

We already know that the resolvent $u(x,\lambda)$ which determined by (2.2) for all λ are not eigenvalues, satisfies all conditions of the Theorem. By using this fact, denoting $\ell u(x) = p(x)u''(x) - q(x)u(x)$ and integrating by parts twice we have

$$\begin{aligned} c_n(\ell u(.,\lambda)) &= \frac{\Delta_{34}}{p_1}\int_{-\pi}^{0} \ell u(\xi,\lambda)\varphi_n(\xi)d\xi + \frac{\Delta_{12}}{p_2}\int_{0}^{\pi} \ell u(\xi,\lambda)\varphi_n(\xi)d\xi \\ &= \frac{\Delta_{34}}{p_1}\int_{-\pi}^{0} u(\xi,\lambda)\ell\varphi_n(\xi)d\xi + \frac{\Delta_{12}}{p_2}\int_{0}^{\pi} u(\xi,\lambda)\ell\varphi_n(\xi)d\xi \\ &= -\lambda_n c_n(u(.,\lambda)) \end{aligned}$$

where the Fourier coefficients c_n are defined as

$$\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0} f(\xi)\varphi_n(\xi)d\xi + \frac{\Delta_{12}}{p_2}\int_{0}^{\pi} f(\xi)\varphi_n(\xi)d\xi.$$

Since the resolvent $u(x,\lambda)$ satisfies the equation $\ell u(x,\lambda) + \lambda u(x,\lambda) = f(x)$ we have

$$\begin{aligned} c_n(f) &= c_n(\ell u(.,\lambda) + \lambda u(.,\lambda)) = c_n(\ell u(.,\lambda)) + \lambda c_n(u(.,\lambda)) \\ &= -\lambda_n c_n(u(.,\lambda)) + \lambda c_n(u(.,\lambda)) = (\lambda - \lambda_n)c_n(u(.,\lambda)). \end{aligned}$$

Thus if we know the expansion of given function $f(x)$ into an eigenfunctions, then the solution of homogeneous equation (2.3) satisfying BTC's (1.2)-(1.4) can be obtain by formula

$$u(x,\lambda) = \sum_{n=0}^{\infty} \frac{1}{\lambda - \lambda_n}\left(\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0} f(\xi)\varphi_n(\xi)d\xi + \frac{\Delta_{12}}{p_2}\int_{0}^{\pi} f(\xi)\varphi_n(\xi)d\xi)\varphi_n(x)\right)$$

for all λ are not eigenvalues.

5. Expansion of mean-square integrable functions into a series of eigenfunction

The expansion theorem will now be extended to the square integrable functions.

Theorem 3. *Let $f(x)$ be any square integrable function on $[-\pi,0)$ and $(0,\pi]$. Then f can be expanded into Fourier series of eigenfunctions in the sense of mean-square convergeness, namely the formula*

$$\begin{aligned} &\lim_{n\to\infty}\{\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}(f(x) - \sum_{k=0}^{n} c_k(f)\varphi_k(x))^2 dx \\ &+ \frac{\Delta_{12}}{p_2}\int_{0}^{\pi}(f(x) - \sum_{k=0}^{n} c_k(f)\varphi_k(x))^2 dx\} = 0 \end{aligned} \tag{5.1}$$

is hold.

Proof. Given any $\epsilon > 0$, there exist an infinitely differentiable function $g(x)$ which vanish in the neighborhoods $x = -\pi$, $x = 0$ and $x = \pi$ such that

$$\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}(f(x)-g(x))^2dx+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}(f(x)-g(x))^2dx<\epsilon \tag{5.2}$$

Denote, for shorting,

$$F_n(x)=\sum_{k=0}^{n}c_k(f)\varphi_k(x) \text{ and } G_n(x)=\sum_{k=0}^{n}c_k(g)\varphi_k(x)$$

By virtue of Theorem 2 there exist an integer N_1, depending on ϵ, such that

$$\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}(g(x)-G_n(x))^2dx+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}(g(x)-G_n(x))^2dx<\epsilon \tag{5.3}$$

for all $n \geq N_1$. By the well known Bessel inequality it can be shown easily that

$$\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}(G_n(x)-F_n(x))^2dx+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}(G_n(x)-F_n(x))^2dx<\epsilon \tag{5.4}$$

for all $\varepsilon > 0$. Finally, writing $f(x) - F_n(x)$ in the form $f(x) - F_n(x) = (f(x) - g(x)) + (g(x) - G_n(x)) + (G_n(x) - F_n(x))$ and using well-known Minkowski inequality from (5.2), (5.3) and (5.4) we can derive that

$$\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}(f(x)-F_n(x))^2dx+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}(f(x)-F_n(x))^2dx<3\epsilon$$

for $n \geq N_1$ which proving the formula (5.1). □

Corollary 2. *If f is as in previous theorem(i.e. $f \in L_2[-\pi,0) \oplus L_2(0,\pi]$) then the modified Parseval equality*

$$\frac{\Delta_{34}}{p_1}\int_{-\pi}^{0}f^2(x)dx+\frac{\Delta_{12}}{p_2}\int_{0}^{\pi}f^2(x)dx=\sum_{n=0}^{\infty}c_n^2(f).$$

is hold.

This is also called the completeness relation.

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Authors' addresses

K. Aydemir
Gaziosmanpaşa University, Department of Mathematics, Faculty of Science, 60250 Tokat, Turkey
E-mail address: kadriye.aydemir@gop.edu.tr

O. Sh. Mukhtarov
Institute of Mathematics and Mechanics, Azerbaijan National Academy of Sciences, Baku, Azerbaijan
Current address: Gaziosmanpaşa University, Department of Mathematics, Faculty of Science, 60250 Tokat, Turkey
E-mail address: omukhtarov@yahoo.com

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THE SECOND EDGE-WIENER INDEX OF SOME COMPOSITE GRAPHS

M. AZARI AND A. IRANMANESH

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Abstract. In this paper we study the behavior of the second edge-Wiener index under the join and corona product of graphs. Results are applied for some classes of graphs such as suspensions, bottlenecks, and thorny graphs.

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1. INTRODUCTION

In theoretical chemistry, the physico-chemical properties of chemical compounds are often modelled by means of molecular–graph–based structure–descriptors, which are also referred to as topological indices [14].

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The vertex version of the Wiener index is the first reported distance-based topological index which was introduced in 1947 by Wiener [16, 17], who used it for modeling the shape of organic molecules and for calculating several of their physico-chemical properties. The Wiener index $W(G)$ of G is defined as:

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v),$$

where $d_G(u,v)$ is the shortest path distance between vertices u and v in G. Details on the Wiener index can be found in [7, 8, 12, 14, 15].

Edge versions of the Wiener index based on the distance between all pairs of edges in a simple connected graph G were introduced in 2009 [9]. Two possible distances between the edges $g = uv$ and $f = zt$ of a graph G can be considered. Each of them gives rise to a corresponding edge-Wiener index. The first distance is the one based on the distance between the corresponding vertices in the line graph of G and obviously, its related edge-Wiener index is equal to the ordinary Wiener index of the line graph of G. The second distance $d_{e|G}(g,f)$ between the edges $g = uv$ and

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$f = zt$ of the graph G is defined as [9]:

$$d_{e|G}(g,f) = \begin{cases} 0 & g = f \\ \max\{d_G(u,z), d_G(u,t), d_G(v,z), d_G(v,t)\} & g \neq f \end{cases}$$

Related to this distance, the second edge-Wiener index $W_e(G)$ of G is defined as [9]:

$$W_e(G) = \sum_{\{g,f\}\subseteq E(G)} d_{e|G}(g,f).$$

We refer the reader to [6, 10] for more information on the edge-Wiener indices.

The first edge-Wiener index of some composite graphs was computed before [1, 2, 4]. In this paper, we are interested in the type of relationship that exists between the second edge-Wiener index of the join and corona product of graphs and their components. Results are applied for several classes of graphs by specializing components in these product graphs. So throughout the paper, by the edge-Wiener index of a graph G, we mean the second version, $W_e(G)$ and by distance between two edges g and f in G, we mean the second distance $d_{e|G}(g,f)$.

2. Preliminaries

Let $N_G(u)$ denote the neighborhood of a vertex u in G, i.e., the set of all vertices of G adjacent with u. The degree of u in G is the cardinality of $N_G(u)$ and is denoted by $deg_G(u)$. Let $\Delta(G)$ and $\Theta(G)$ denote the number of triangles in G and the number of subgraphs of G isomorphic to the 4-vertex complete graph K_4, respectively. It is easy to see that

$$\Delta(G) = \frac{1}{3} \sum_{uv\in E(G)} |N_G(u) \cap N_G(v)|,$$

$$\Theta(G) = \frac{1}{12} \sum_{uv\in E(G)} \sum_{z\in N_G(u)\cap N_G(v)} |N_G(u) \cap N_G(v) \cap N_G(z)|.$$

Corresponding to each triangle in G, there are 3 pairs of adjacent edges which are at distance 1 in G. So the number of such pairs of edges in G is equal to $3\Delta(G)$. Also, corresponding to each subgraph of G isomorphic to K_4, there are 3 pairs of nonadjacent edges which are at distance 1 in G. So the number of such pairs of edges in G is equal to $3\Theta(G)$. Hence, the total number of pairs of edges which are at distance 1 in G is equal to $3\,(\Delta(G) + \Theta(G))$.

Let x be a vertex of G and $g = uv$ be an edge of G. We define the distance $D_G(x,g)$ between the vertex x and the edge g of the graph G as [5]:

$$D_G(x,g) = \max\{d_G(x,u), d_G(x,v)\}.$$

The vertex-edge Wiener index $W_{ve}(G)$ of G is defined as the sum of such distances over all vertices $x \in V(G)$ and edges $g \in E(G)$ [5]:

$$W_{ve}(G) = \sum_{x \in V(G)} \sum_{g \in E(G)} D(x, g|G).$$

This index was studied in more details in [1, 3] under the name $Max(G)$. We refer the reader to these references for more information on the vertex-edge Wiener indices and for explicit formulas for $W_{ve}(G)$ of several classes of graphs.

3. Main results

Throughout this section, let G_1 and G_2 be two simple connected graphs and n_i and e_i denote the numbers of vertices and edges of G_i, respectively, where $i \in \{1,2\}$. Our aim is to compute the edge-Wiener index of join and corona product of G_1 and G_2.

3.1. *Join*

The join $G_1 + G_2$ of graphs G_1 and G_2 is defined as the graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set

$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup S,$$

where $S = \{u_1u_2 | u_1 \in V(G_1), u_2 \in V(G_2)\}$. All distinct vertices of $G_1 + G_2$ are either at distance 1 or 2. The vertices at distance 2 are precisely those of G_1 that are not adjacent in G_1, and those of G_2 that are not adjacent in G_2. So all distinct edges of $G_1 + G_2$ are either at distance 1 or 2. The join of two graphs is also known as their **sum**. Its definition can be extended inductively to more than two graphs in a straightforward manner. It is a commutative operation and hence both its components will appear symmetrically in any formula including distance-based invariants.

Theorem 1. *Let G_1 and G_2 be two simple connected graphs. Then*

$$W_e(G_1 + G_2) = 2\binom{n_1n_2}{2} + 2\binom{e_1}{2} + 2\binom{e_2}{2} + e_1n_2(2n_1 - 3) + n_1e_2(2n_2 - 3) - e_1e_2 - 3(n_2 + 1)\Delta(G_1) - 3(n_1 + 1)\Delta(G_2) - 3(\Theta(G_1) + \Theta(G_2)).$$

Proof. Let Q be the set of all pairs of edges of $G_1 + G_2$. We partition Q into six disjoint sets as follows:
$Q_1 = \{\{g, f\} | g, f \in E(G_1)\}$;
$Q_2 = \{\{g, f\} | g, f \in E(G_2)\}$;
$Q_3 = \{\{g, f\} | g \in E(G_1), f \in E(G_2)\}$;
$Q_4 = \{\{g, f\} | g \in E(G_1), f \in S\}$;
$Q_5 = \{\{g, f\} | g \in E(G_2), f \in S\}$;
$Q_6 = \{\{g, f\} | g, f \in S\}$.
The edge-Wiener index of $G_1 + G_2$ is obtained by summing the contributions of all

pairs of edges over those six sets. We proceed to evaluate their contributions in order of increasing complexity.

The case of Q_3 is the simplest. Let $\{g, f\} \in Q_3$, where $g = u_1v_1 \in E(G_1)$ and $f = u_2v_2 \in E(G_2)$. Then

$$\begin{aligned} d_{e|G_1+G_2}(g, f) =& \max\{d_{G_1+G_2}(u_1, u_2), d_{G_1+G_2}(u_1, v_2), d_{G_1+G_2}(v_1, u_2), \\ & d_{G_1+G_2}(v_1, v_2)\} \\ =& \max\{1, 1, 1, 1\} = 1. \end{aligned}$$

There are e_1e_2 such pairs of edges in Q_3 and each of them contributes 1 to the edge-Wiener index. Hence, the total contribution of pairs from Q_3 is equal to e_1e_2.

The set Q_6 contains pairs of edges from S. Let $\{g, f\} \in Q_6$ and $g = u_1u_2$, $f = v_1v_2$, where $u_1, v_1 \in V(G_1)$, $u_2, v_2 \in V(G_2)$. Then

$$\begin{aligned} d_{e|G_1+G_2}(g, f) =& \max\{d_{G_1+G_2}(u_1, v_1), d_{G_1+G_2}(u_1, v_2), d_{G_1+G_2}(u_2, v_1), \\ & d_{G_1+G_2}(u_2, v_2)\} \\ =& \max\{d_{G_1+G_2}(u_1, v_1), 1, 1, d_{G_1+G_2}(u_2, v_2)\}. \end{aligned}$$

It is easy to see that, if $u_1 = v_1, u_2v_2 \in E(G_2)$ or $u_2 = v_2, u_1v_1 \in E(G_1)$ or $u_1v_1 \in E(G_1), u_2v_2 \in E(G_2)$ then $d_{e|G_1+G_2}(g, f) = 1$, otherwise $d_{e|G_1+G_2}(g, f) = 2$. The total number of pairs of edges in Q_6 is equal to $\binom{n_1n_2}{2}$. Among them there are $n_1e_2 + n_2e_1 + 2e_1e_2$ pairs that contribute 1 to the edge-Wiener index, and all other pairs contribute 2. Hence the total contribution of pairs from Q_6 is equal to

$$2\binom{n_1n_2}{2} - n_1e_2 - n_2e_1 - 2e_1e_2.$$

Now, we compute the contribution of pairs from Q_4. Let $\{g, f\} \in Q_4$ and $g = u_1v_1 \in E(G_1)$ and $f = z_1u_2 \in S$, where $u_1, v_1, z_1 \in V(G_1)$, $u_2 \in V(G_2)$. Then

$$\begin{aligned} d_{e|G_1+G_2}(g, f) =& \max\{d_{G_1+G_2}(u_1, z_1), d_{G_1+G_2}(u_1, u_2), d_{G_1+G_2}(v_1, z_1), \\ & d_{G_1+G_2}(v_1, u_2)\} \\ =& \max\{d_{G_1+G_2}(u_1, z_1), 1, d_{G_1+G_2}(v_1, z_1), 1\}. \end{aligned}$$

It is easy to see that, if $z_1 = u_1$ or $z_1 = v_1$ or $u_1z_1, v_1z_1 \in E(G_1)$ then $d_{e|G_1+G_2}(g, f) = 1$, otherwise $d_{e|G_1+G_2}(g, f) = 2$.
The total number of pairs from Q_4 is equal to $e_1n_1n_2$. Among them there are $2e_1n_2 + 3n_2\Delta(G_1)$ pairs that contribute 1 to the edge-Wiener index, and all other pairs contribute 2. Hence the total contribution of pairs from Q_4 is equal to

$$2e_1n_1n_2 - 2e_1n_2 - 3n_2\Delta(G_1).$$

By symmetry, the total contribution of pairs from Q_5 is equal to

$$2n_1n_2e_2 - 2n_1e_2 - 3n_1\Delta(G_2).$$

It remains to compute the contributions of Q_1 and Q_2. Let $\{g, f\} \in Q_1$, where $g = u_1v_1$, $f = z_1t_1$. Then

$$d_{e|G_1+G_2}(g,f) = \max\{d_{G_1+G_2}(u_1,z_1), d_{G_1+G_2}(u_1,t_1), d_{G_1+G_2}(v_1,z_1), d_{G_1+G_2}(v_1,t_1)\}.$$

By definition of $G_1 + G_2$, the distances $d_{G_1+G_2}(u_1,z_1)$, $d_{G_1+G_2}(u_1,t_1)$, $d_{G_1+G_2}(v_1,z_1)$ and $d_{G_1+G_2}(v_1,t_1)$ are equal to 0, 1 or 2, so $d_{e|G_1+G_2}(g,f)$ is either equal to 1 or 2. The total number of pairs in Q_1 is equal to $\binom{e_1}{2}$. As mentioned before, $3(\Delta(G_1) + \Theta(G_1))$ pairs contribute 1 to the edge-Wiener index, and all other pairs contribute 2. Hence the total contribution of pairs from Q_1 is equal to

$$2\binom{e_1}{2} - 3(\Delta(G_1) + \Theta(G_1)).$$

Again, the total contribution of Q_2 is obtained by the symmetry as

$$2\binom{e_2}{2} - 3(\Delta(G_2) + \Theta(G_2)).$$

Now, the formula of the Theorem follows by adding the contributions of $Q_1, \ldots, Q_6$ and simplifying the resulting expression. □

As expected, G_1 and G_2 appear symmetrically in the above formula. It is interesting to note that the formula does not depend on the connectivity of G_1 and G_2. That allows us to compute the edge-Wiener index of joins of graphs that are not themselves connected.

3.2. *Corona product*

The corona product $G_1 \circ G_2$ of graphs G_1 and G_2 is obtained by taking one copy of G_1 and n_1 copies of G_2, and joining all vertices of the i-th copy of G_2 to the i-th vertex of G_1 for $i = 1, 2, \cdots, n_1$. Unlike join, corona is a non-commutative operation, and its component graphs appear in markedly asymmetric roles. More formally, we denote the copy of G_2 related to the vertex $x \in V(G_1)$ by $G_{2,x}$ and the edge set of $G_{2,x}$ by $S_{2,x}$. By definition of $G_1 \circ G_2$, the distance between two distinct vertices $u, v \in V(G_1 \circ G_2)$ is given by:

$$d_{G_1\circ G_2}(u,v) = \begin{cases} d_{G_1}(u,v) & u,v \in V(G_1) \\ d_{G_1}(u,x)+1 & u \in V(G_1), v \in V(G_{2,x}) \\ 1 & uv \in S_{2,x} \\ 2 & u,v \in V(G_{2,x}), uv \notin S_{2,x} \\ d_{G_1}(x,y)+2 & u \in V(G_{2,x}), v \in V(G_{2,y}), x \neq y \end{cases}$$

Theorem 2. *Let G_1 and G_2 be two simple connected graphs. Then*

$$W_e(G_1 \circ G_2) = W_e(G_1) + (n_2+e_2)^2 W(G_1) + (n_2+e_2) W_{ve}(G_1) - 6n_1\Delta(G_2)$$

$$-3n_1\Theta(G_2)+2\binom{n_1n_2}{2}+2\binom{n_1e_2}{2}+n_1e_1(n_2+e_2)$$
$$+n_1e_2(2n_1n_2-3).$$

Proof. It is obvious that the graph $G_1\circ G_2$ has $e_1+n_1e_2+n_1n_2$ edges. We partition the edge set of $G_1\circ G_2$ into three sets. The first one is the edge set of G_1, $S_1=E(G_1)$, the second one contains all edges in all copies of G_2, $S_2=\bigcup_{x\in V(G_1)}S_{2,x}$, and the third one contains all edges with one end in G_1 and the other end in some copy of G_2, $S_3=\bigcup_{x\in V(G_1)}S_{3,x}$, where $S_{3,x}=\{e|e=ux,u\in V(G_{2,x})\}$.
Now we start to compute the distances between the edges of these three sets. We consider the following six cases:
Case 1. $\{g,f\}\subseteq S_1$.
It is obvious that $d_{e|G_1\circ G_2}(g,f)=d_{e|G_1}(g,f)$, so

$$W_1=\sum_{\{g,f\}\subseteq S_1}d_{e|G_1\circ G_2}(g,f)=W_e(G_1).$$

Case 2. $\{g,f\}\subseteq S_2$, $g\in S_{2,x}$ and $f\in S_{2,y}$.
First, consider the case $x=y$ and let $g=u_{2,x}v_{2,x}$, $f=z_{2,x}t_{2,x}\in S_{2,x}$. Then

$$\begin{aligned}d_{e|G_1\circ G_2}(g,f)=&\max\{d_{G_1\circ G_2}(u_{2,x},z_{2,x}),d_{G_1\circ G_2}(u_{2,x},t_{2,x}),d_{G_1\circ G_2}(v_{2,x},z_{2,x}),\\&d_{G_1\circ G_2}(v_{2,x},t_{2,x})\}.\end{aligned}$$

Clearly, the above four distances are equal to 0, 1 or 2. So $d_{e|G_1\circ G_2}(g,f)=1$ or 2. In this case, the vertex x and its related copy, $G_{2,x}$, form a copy of K_1+G_2. So by the same reasoning as in the proof of Theorem 1, we obtain:

$$\sum_{\{g,f\}\subseteq S_{2,x}}d_{e|G_1\circ G_2}(g,f)=2\binom{e_2}{2}-3\left(\Delta(G_2)+\Theta(G_2)\right).$$

Now let $x\neq y$ and $g=u_{2,x}v_{2,x}$ and $f=u_{2,y}v_{2,y}$. Then

$$\begin{aligned}d_{e|G_1\circ G_2}(g,f)=&\max\{d_{G_1\circ G_2}(u_{2,x},u_{2,y}),d_{G_1\circ G_2}(u_{2,x},v_{2,y}),d_{G_1\circ G_2}(v_{2,x},u_{2,y}),\\&d_{G_1\circ G_2}(v_{2,x},v_{2,y})\}\\=&\max\{d_{G_1}(x,y)+2,d_{G_1}(x,y)+2,d_{G_1}(x,y)+2,d_{G_1}(x,y)+2\}\\=&d_{G_1}(x,y)+2.\end{aligned}$$

Now,

$$\begin{aligned}W_2=&\sum_{\{g,f\}\subseteq S_2}d_{e|G_1\circ G_2}(g,f)\\=&\sum_{x\in V(G_1)}\sum_{\{g,f\}\subseteq S_{2,x}}d_{e|G_1\circ G_2}(g,f)+\sum_{\{x,y\}\subseteq V(G_1)}\sum_{g\in S_{2,x}}\sum_{f\in S_{2,y}}d_{e|G_1\circ G_2}(g,f)\end{aligned}$$

$$= \sum_{x\in V(G_1)} \left(2\binom{e_2}{2} - 3\left(\Delta(G_2) + \Theta(G_2)\right)\right) + e_2^2 \sum_{\{x,y\}\subseteq V(G_1)} \left(d_{G_1}(x,y)+2\right)$$
$$= n_1\left(2\binom{e_2}{2} - 3\left(\Delta(G_2) + 3\Theta(G_2)\right)\right) + e_2^2\left(W(G_1) + 2\binom{n_1}{2}\right).$$

Case 3. $\{g,f\} \subseteq S_3$, $g \in S_{3,x}$ and $f \in S_{3,y}$.
If $x = y$ and $g = u_{2,x}x$, $f = v_{2,x}x$ where $u_{2,x}, v_{2,x} \in V(G_{2,x})$, then

$$\begin{aligned} d_{e|G_1\circ G_2}(g,f) =& \max\{d_{G_1\circ G_2}(u_{2,x},v_{2,x}), d_{G_1\circ G_2}(u_{2,x},x), d_{G_1\circ G_2}(x,v_{2,x}),\\ & d_{G_1\circ G_2}(x,x)\}\\ =& \max\{d_{G_1\circ G_2}(u_{2,x},v_{2,x}), 1, 1, 0\}. \end{aligned}$$

If $v_{2,x}$ is adjacent to $u_{2,x}$ in $G_{2,x}$, then $d_{G_1\circ G_2}(u_{2,x},v_{2,x}) = 1$, so $d_{e|G_1\circ G_2}(g,f) = 1$, otherwise $d(u_{2,x},v_{2,x}) = 2$, so $d_{e|G_1\circ G_2}(g,f) = 2$. Hence for each edge $g = u_{2,x}x \in S_{3,x}$, $deg_{G_{2,x}}(u_{2,x})$ edges of $S_{3,x}$ are at distance 1 from g and all other edges are at distance 2. So

$$\begin{aligned} &\sum_{\{g,f\}\subseteq S_{3,x}} d_{e|G_1\circ G_2}(g,f)\\ &= \frac{1}{2}\sum_{u_{2,x}\in V(G_{2,x})} \left(deg_{G_{2,x}}(u_{2,x}) + 2\left(n_2 - 1 - deg_{G_{2,x}}(u_{2,x})\right)\right)\\ &= \frac{1}{2}\left(2e_2 + 2n_2(n_2-1) - 4e_2\right) = n_2(n_2-1) - e_2. \end{aligned}$$

If $x \neq y$ and $g = u_{2,x}x \in S_{3,x}$, $f = u_{2,y}y \in S_{3,y}$, then

$$\begin{aligned} d_{e|G_1\circ G_2}(g,f) =& \max\{d_{G_1\circ G_2}(u_{2,x},u_{2,y}), d_{G_1\circ G_2}(u_{2,x},y), d_{G_1\circ G_2}(x,u_{2,y}),\\ & d_{G_1\circ G_2}(x,y)\}\\ =& \max\{d_{G_1}(x,y)+2, d_{G_1}(x,y)+1, d_{G_1}(x,y)+1, d_{G_1}(x,y)\}\\ =& d_{G_1}(x,y)+2. \end{aligned}$$

Now,

$$\begin{aligned} W_3 &= \sum_{\{g,f\}\subseteq S_3} d_{e|G_1\circ G_2}(g,f)\\ &= \sum_{x\in V(G_1)} \sum_{\{g,f\}\subseteq S_{3,x}} d_{e|G_1\circ G_2}(g,f) + \sum_{\{x,y\}\subseteq V(G_1)} \sum_{g\in S_{3,x}} \sum_{f\in S_{3,y}} d_{e|G_1\circ G_2}(g,f)\\ &= \sum_{x\in V(G_1)} (n_2(n_2-1) - e_2) + \sum_{\{x,y\}\subseteq V(G_1)} \sum_{g\in S_{3,x}} \sum_{f\in S_{3,y}} (d_{G_1}(x,y)+2)\\ &= n_1\left(n_2(n_2-1) - e_2\right) + n_2^2 \sum_{\{x,y\}\subseteq V(G_1)} (d_{G_1}(x,y)+2) \end{aligned}$$

$$= n_1\left(n_2(n_2-1)-e_2\right)+n_2^2W(G_1)+2n_2^2\binom{n_1}{2}$$

$$= 2\binom{n_1n_2}{2}-n_1e_2+n_2^2W(G_1).$$

Case 4. $g \in S_1, f \in S_2$.
Let $g = u_1v_1 \in S_1$, $f = u_{2,x}v_{2,x} \in S_{2,x}$, for some $x \in V(G_1)$. Then

$$\begin{aligned} d_{e|G_1\circ G_2}(g,f) =& \max\{d_{G_1\circ G_2}(u_1,u_{2,x}), d_{G_1\circ G_2}(u_1,v_{2,x}), d_{G_1\circ G_2}(v_1,u_{2,x}),\\ & d_{G_1\circ G_2}(v_1,v_{2,x})\}\\ =& \max\{d_{G_1}(u_1,x)+1, d_{G_1}(u_1,x)+1, d_{G_1}(v_1,x)+1,\\ & d_{G_1}(v_1,x)+1\}\\ =& \max\{d_{G_1}(u_1,x), d_{G_1}(v_1,x)\}+1\\ =& D_{G_1}(x,g)+1. \end{aligned}$$

Now,

$$\begin{aligned} W_4 &= \sum_{x\in V(G_1)}\sum_{f\in S_{2,x}}\sum_{g\in S_1} d_{e|G_1\circ G_2}(g,f) = \sum_{x\in V(G_1)}\sum_{f\in S_{2,x}}\sum_{g\in S_1}\left(D_{G_1}(x,g)+1\right)\\ &= e_2W_{ve}(G_1)+n_1e_1e_2 = e_2\left(W_{ve}(G_1)+n_1e_1\right). \end{aligned}$$

Case 5. $g \in S_1, f \in S_3$
Let $g = u_1v_1 \in S_1$, $f = u_{2,x}x \in S_{3,x}$, where $u_1, v_1, x \in V(G_1)$, $u_{2,x} \in V(G_{2,x})$. Then

$$\begin{aligned} d_{e|G_1\circ G_2}(g,f) =& \max\{d_{G_1\circ G_2}(u_1,u_{2,x}), d_{G_1\circ G_2}(u_1,x), d_{G_1\circ G_2}(v_1,u_{2,x}),\\ & d_{G_1\circ G_2}(v_1,x)\}\\ =& \max\{d_{G_1}(u_1,x)+1, d_{G_1}(u_1,x), d_{G_1}(v_1,x)+1, d_{G_1}(v_1,x)\}\\ =& \max\{d_{G_1}(u_1,x), d_{G_1}(v_1,x)\}+1\\ =& D_{G_1}(x,g)+1. \end{aligned}$$

Now,

$$\begin{aligned} W_5 &= \sum_{x\in V(G_1)}\sum_{f\in S_{3,x}}\sum_{g\in S_1} d_{e|G_1\circ G_2}(g,f) = \sum_{x\in V(G_1)}\sum_{f\in S_{3,x}}\sum_{g\in S_1}\left(D_{G_1}(x,g)+1\right)\\ &= n_2W_{ve}(G_1)+n_1e_1n_2 = n_2\left(W_{ve}(G_1)+n_1e_1\right). \end{aligned}$$

Case 6. $g \in S_2, f \in S_3$
Let $g = u_{2,x}v_{2,x} \in S_{2,x}$, $f = z_{2,x}x \in S_{3,x}$, where $x \in V(G_1)$ and $u_{2,x}, v_{2,x}, z_{2,x} \in V(G_{2,x})$. Then

$$\begin{aligned} d_{e|G_1\circ G_2}(g,f) =& \max\{d_{G_1\circ G_2}(u_{2,x},z_{2,x}), d_{G_1\circ G_2}(u_{2,x},x), d_{G_1\circ G_2}(v_{2,x},z_{2,x}),\\ & d_{G_1\circ G_2}(v_{2,x},x)\} \end{aligned}$$

$$=\max\{d_{G_1\circ G_2}(u_{2,x},z_{2,x}),1,d_{G_1\circ G_2}(v_{2,x},z_{2,x}),1\}.$$

By definition of $G_1\circ G_2$, the distances $d_{G_1\circ G_2}(u_{2,x},z_{2,x})$ and $d_{G_1\circ G_2}(v_{2,x},z_{2,x})$ are equal to 0, 1 or 2. It is easy to see that, if $z_{2,x}=u_{2,x}$ or $z_{2,x}=v_{2,x}$ or $u_{2,x}z_{2,x},v_{2,x}z_{2,x}\in S_{2,x}$, then $d_{e|G_1\circ G_2}(g,f)=1$, otherwise $d_{e|G_1\circ G_2}(g,f)=2$. So, for each edge $g=u_{2,x}v_{2,x}\in S_{2,x}$, the edges $u_{2,x}x,v_{2,x}x,z_{2,x}x\in S_{3,x}$, where $z_{2,x}\in N_{G_{2,x}}(u_{2,x})\cap N_{G_{2,x}}(v_{2,x})$ are at distance 1 from g and all other edges are at distance 2. So

$$\begin{aligned}
&\sum_{g\in S_{2,x}}\sum_{f\in S_{3,x}} d_{e|G_1\circ G_2}(g,f)\\
&=\sum_{u_{2,x}v_{2,x}\in S_{2,x}}\Big(2+|N_{G_{2,x}}(u_{2,x})\cap N_{G_{2,x}}(v_{2,x})|\\
&\qquad\qquad +2\big(n_2-2-|N_{G_{2,x}}(u_{2,x})\cap N_{G_{2,x}}(v_{2,x})|\big)\Big)\\
&=\sum_{u_{2,x}v_{2,x}\in S_{2,x}}\big(2n_2-2-|N_{G_{2,x}}(u_{2,x})\cap N_{G_{2,x}}(v_{2,x})|\big)=2e_2(n_2-1)-3\Delta(G_2).
\end{aligned}$$

If $g=u_{2,x}v_{2,x}\in S_{2,x}$, $f=u_{2,y}y\in S_{3,y}$, where $x,y\in V(G_1)$ and $x\neq y$, then

$$\begin{aligned}
d_{e|G_1\circ G_2}(g,f)=&\max\{d_{G_1\circ G_2}(u_{2,x},u_{2,y}),d_{G_1\circ G_2}(u_{2,x},y),d_{G_1\circ G_2}(v_{2,x},u_{2,y}),\\
&\quad d_{G_1\circ G_2}(v_{2,x},y)\}\\
=&\max\{d_{G_1}(x,y)+2,d_{G_1}(x,y)+1,d_{G_1}(x,y)+2,d_{G_1}(x,y)+1\}\\
=&d_{G_1}(x,y)+2.
\end{aligned}$$

Now,

$$\begin{aligned}
W_6=&\sum_{x,y\in V(G_1)}\sum_{g\in S_{2,x}}\sum_{f\in S_{3,y}} d_{e|G_1\circ G_2}(g,f)\\
=&\sum_{x\in V(G_1)}\sum_{g\in S_{2,x}}\sum_{f\in S_{3,x}} d_{e|G_1\circ G_2}(g,f)\\
&+\sum_{x\in V(G_1)}\sum_{y\in V(G_1)\setminus\{x\}}\sum_{g\in S_{2,x}}\sum_{f\in S_{3,y}} d_{e|G_1\circ G_2}(g,f)\\
=&\sum_{x\in V(G_1)}(2e_2(n_2-1)-3\Delta(G_2))\\
&+\sum_{x\in V(G_1)}\sum_{y\in V(G_1)\setminus\{x\}}\sum_{g\in S_{2,x}}\sum_{f\in S_{3,y}}(d_{G_1}(x,y)+2)\\
=&\,n_1(2e_2(n_2-1)-3\Delta(G_2))+n_2e_2\sum_{x\in V(G_1)}\sum_{y\in V(G_1)\setminus\{x\}}(d_{G_1}(x,y)+2)\\
=&\,n_1(2e_2(n_2-1)-3\Delta(G_2))+2n_2e_2(W(G_1)+n_1(n_1-1)).
\end{aligned}$$

Now the formula for the edge-Wiener index of $G_1 \circ G_2$ follows by adding all six contributions and simplifying the resulting expression. □

Again, it is interesting to note that the formula of Theorem 2 does not include any invariants of G_2 that depend on its connectivity. It is, hence, possible to apply Theorem 2 to corona products $G_1 \circ G_2$ with disconnected G_2.

4. Examples and corollaries

Now, we can obtain explicit formulas for the edge-Wiener indices of some classes of graphs by specializing components in joins and coronas. We start by computing the edge Wiener-index of a suspension of a graph G. For a given graph G, we call the graph $K_1 + G$ the **suspension** of G, where K_1 denotes the single vertex graph.

Corollary 1. *Let G be a simple graph of order n and size e. Then*

$$W_e(K_1 + G) = W_e(K_1 \circ G) = 2\binom{n+e}{2} - 3e - 6\Delta(G) - 3\Theta(G).$$

Now the formulas for the **wheel graph** $W_n = K_1 + C_n$ and for the **fan graph** $K_1 + P_n$ follow at once. Both graphs allow alternative representations as $K_1 \circ C_n$ and $K_1 \circ P_n$, respectively.

Corollary 2.

$$W_e(K_1 + C_n) = \begin{cases} 15 & n = 3 \\ 4n^2 - 5n & n \geq 4 \end{cases};$$

$$W_e(K_1 + P_n) = 4n^2 - 9n + 5, \quad n \geq 2.$$

Our next example is the **windmill graph**. The windmill graph $D_n^{(m)}$ is the graph obtained by taking m copies of the complete graph K_n with a vertex in common. The case $n = 3$ therefore corresponds to the **Dutch windmill graph**. One can easily see that the windmill graph is the suspension of m copies of K_{n-1}. Also, it is easy to see that, $\Delta(K_n) = \binom{n}{3}$ and $\Theta(K_n) = \binom{n}{4}$, for $n \geq 1$. So using Corollary 1 , we can get the formula for the edge-Wiener index of the windmill graph $D_n^{(m)}$.

Corollary 3. *For $n \geq 2$ and $m \geq 1$,*

$$W_e(D_n^{(m)}) = \frac{1}{4} m \binom{n}{2} \left(4m \binom{n}{2} - n^2 + n - 2 \right).$$

In particular, the edge-Wiener index of the Dutch windmill graph $D_3^{(m)}$ is given by:

Corollary 4. *For $m \geq 1$,*

$$W_e(D_3^{(m)}) = 3m(3m - 2).$$

Now, we turn our attention toward coronas. Coronas sometimes appear in chemical literature as plerographs of the usual hydrogen-suppressed molecular graphs known as kenographs; see [13] for definitions and more information. The **t-thorny graph** of a given graph G is obtained as $G \circ \overline{K}_t$, where $\overline{K}_t$ denotes the empty graph on t vertices [11]. For the t-thorny graph of a graph G, we obtain the following formula.

Corollary 5. *Let G be a simple connected graph of order n and size e. Then*

$$W_e(G \circ \overline{K}_t) = W_e(G) + t^2 W(G) + t W_{ve}(G) + nt(nt + e - 1).$$

Now, we present two formulas for the edge-Wiener indices of the **t-thorny cycle** $C_n \circ \overline{K}_t$ and the **t-thorny path** $P_n \circ \overline{K}_t$. We use known results for the edge-Wiener indices of paths and cycles [9] and our results on the vertex-edge Wiener indices of these graphs obtained in [1].

Corollary 6.

$$W_e(C_n \circ \overline{K}_t) = \begin{cases} \frac{n(t+1)}{8}[n^2(t+1) + 4n(2t+1) - (t+13)] & n \ is \ odd \\ \frac{n(t+1)}{8}[n^2(t+1) + 4n(2t+1) - 8] & n \ is \ even \end{cases};$$

$$W_e(P_n \circ \overline{K}_t) = \binom{n+1}{3}(t+1)^2 + (nt + n - 1)(nt - 1).$$

Interesting classes of graphs can also be obtained by specializing the first component in the corona product. For example, for a graph G, the graph $K_2 \circ G$ is called its **bottleneck graph**. For the bottleneck graph of a graph G, we obtain the following formula.

Corollary 7. *Let G be a simple graph of order n and size e. Then*

$$W_e(K_2 \circ G) = 5(n + e)^2 + 2(n - 2e) - 12\Delta(G) - 6\Theta(G).$$

Using Corollary 7, the formulas for the **bottleneck graph of a cycle** $K_2 \circ C_n$ and the **bottleneck graph of a path** $K_2 \circ P_n$ are easily obtained.

Corollary 8.

$$W_e(K_2 \circ C_n) = \begin{cases} 162 & n = 3 \\ 20n^2 - 2n & n \geq 4 \end{cases};$$

$$W_e(K_2 \circ P_n) = 20n^2 - 22n + 9.$$

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Authors' addresses

M. Azari
Department of Mathematics, Kazerun Branch, Islamic Azad University, P. O. Box: 73135-168, Kazerun, Iran
E-mail address: azari@kau.ac.ir

A. Iranmanesh
Department of pure Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, P. O. Box: 14115-137, Tehran, Iran
E-mail address: iranmanesh@modares.ac.ir

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GALOIS GROUP FOR SOME CLASS FIELD

ABDELMALEK AZIZI AND MOHAMMED TALBI

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Abstract. For some imaginary quartic cyclic fields $\mathbb{K}$, we will study the capitulation problem of the 2-class ideals of $\mathbb{K}$ and we will determine the structure of the Galois group of the second Hilbert 2-class field of $\mathbb{K}$ over $\mathbb{K}$.

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1. INTRODUCTION

Let $\mathbb{F}/\mathbb{L}$ a finite extension of number fields and $O_{\mathbb{L}}$ (resp. $O_{\mathbb{F}}$) the ring of integers of $\mathbb{L}$ (resp. of $\mathbb{F}$). We say, since Hilbert, that an ideal of $O_{\mathbb{L}}$ capitulates in $\mathbb{F}$, if it becomes principal by extension of scalars to $O_{\mathbb{F}}$, and, of course, when an ideal $\mathfrak{a}$ of $O_{\mathbb{L}}$ capitulates in $\mathbb{F}$, then the class $[\mathfrak{a}]$ of $\mathfrak{a}$ capitulates in $\mathbb{F}$ (ie. $[\mathfrak{a}O_{\mathbb{F}}]$ is trivial). Therefore, the study of the capitulation problem is precisely to describe the group of all classes of ideals of $\mathbb{L}$ which capitulate in $\mathbb{F}$, where $\mathbb{F}$ is an unramified abelian extension of $\mathbb{L}$.

Proposition 1 ([8]). *Let $\mathbb{G}$ a 2-group of finite order 2^m and $\mathbb{G}'$ its derived subgroup. Then $\mathbb{G}/\mathbb{G}'$ is of type $(2,2)$ if and only if $\mathbb{G}$ is isomorphic to one of 2-groups:*

$$\begin{aligned}
Q_m &= \langle \sigma, \tau \rangle \quad where \quad \sigma^{2^{m-2}} = \tau^2 = a, a^2 = 1, \tau^{-1}\sigma\tau = \sigma^{-1};\\
D_m &= \langle \sigma, \tau \rangle \quad where \quad \sigma^{2^{m-1}} = \tau^2 = 1, \tau^{-1}\sigma\tau = \sigma^{-1};\\
S_m &= \langle \sigma, \tau \rangle \quad where \quad \sigma^{2^{m-1}} = \tau^2 = 1, \tau^{-1}\sigma\tau = \sigma^{2^{m-2}-1};\\
(2,2) &= \langle \sigma, \tau \rangle \quad where \quad \sigma^2 = \tau^2 = 1, \tau^{-1}\sigma\tau = \sigma.
\end{aligned}$$

Where Q_m is the quaternionic group, D_m the dihedral group, S_m the semi-dihedral group of order 2^m and (2.2) is an abelian group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Suppose that $\mathbb{G}$ is a 2-group of finite order 2^m such that $\mathbb{G}/\mathbb{G}'$ is of type $(2,2)$, then $\mathbb{G}$ is isomorphic to Q_m, D_m, S_m or $(2,2)$ defined in the Proposition 1. Let $\{\sigma, \tau\}$ generates $\mathbb{G}$ such that the relationships cited in the Proposition 1 are verified, by a simple calculation we can see that the derived subgroup $\mathbb{G}' = [\mathbb{G}, \mathbb{G}] = \langle \sigma^2 \rangle$

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and $\mathbb{G}$ has three subgroups of index 2: $H_1 = \langle\sigma\rangle$, $H_2 = \langle\sigma^2, \tau\rangle$ and $H_3 = \langle\sigma^2, \sigma\tau\rangle$. Moreover if $\mathbb{G}$ is of type $(2,2)$, then the subgroups H_i are cyclic of order 2, if $\mathbb{G} \simeq Q_3$, then the subgroups H_i are cyclic of order 4 and if $\mathbb{G}$ is isomorphic to Q_m ($m > 3$), D_m or S_m, then H_1 is cyclic and H_2/H_2' and H_3/H_3' are of type $(2,2)$ where H_2' (resp. H_3') is the derived subgroup of H_2 (resp. H_3).

Throughout the remainder of this section we denote by $\mathbb{M}$ a number field, $\mathbb{M}_2^{(1)}$ the first 2-Hilbert class field of $\mathbb{M}$, $\mathbb{M}_2^{(2)}$ the second 2-Hilbert class field of $\mathbb{M}$, $C_{\mathbb{M}}$ the class group of $\mathbb{M}$, $C_{2,\mathbb{M}}$ the 2-part of $C_{\mathbb{M}}$, $\mathbb{G}$ the Galois group of $\mathbb{M}_2^{(2)}/\mathbb{M}$ and $\mathbb{G}'$ its derived sub-group. Then $\mathbb{G}' \simeq Gal(\mathbb{M}_2^{(2)}/\mathbb{M}_2^{(1)})$ and $\mathbb{G}/\mathbb{G}' \simeq Gal(\mathbb{M}_2^{(1)}/\mathbb{M})$, and we know from the class field theory that $Gal(\mathbb{M}_2^{(1)}/\mathbb{M}) \simeq C_{2,\mathbb{M}}$, so $\mathbb{G}/\mathbb{G}' \simeq C_{2,\mathbb{M}}$.

Definition 1. Let $\mathbb{F}$ be a cyclic unramified extension of $\mathbb{M}$ and j is the mapping of $C_{\mathbb{M}}$ into $C_{\mathbb{F}}$ that maps to the class of an ideal $\mathfrak{a}$ of $\mathbb{M}$ the class of the ideal generated by $\mathfrak{a}$ in $\mathbb{F}$. Then the extension $\mathbb{F}/\mathbb{M}$ is called:

- of type (A) if and only if $\#\ker j \cap N_{\mathbb{F}/\mathbb{M}}(C_{\mathbb{F}}) > 1$;
- of type (B) if and only if $\#\ker j \cap N_{\mathbb{F}/\mathbb{M}}(C_{\mathbb{F}}) = 1$.

Now suppose that $C_{2,\mathbb{M}} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then $\mathbb{G}/\mathbb{G}' \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and as $\mathbb{G}$ is a 2-finite group, then $\mathbb{G}'$ is cyclic. Thus the Hilbert 2-class field towers of $\mathbb{M}$ ends in $\mathbb{M}_2^{(2)}$.
In addition, we know that if $\mathbb{G}$ has order 2^m and $\mathbb{G}/\mathbb{G}' \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then $\mathbb{G}$ is isomorphic to Q_m, D_m, S_m or to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In all these cases, we have $\mathbb{G}' = \langle\sigma^2\rangle$ and the three subgroups of index 2 in $\mathbb{G}$ are: $H_1 = \langle\sigma\rangle$, $H_2 = \langle\sigma^2, \tau\rangle$ and $H_3 = \langle\sigma^2, \sigma\tau\rangle$, and if $\mathbb{G}' \neq 1$, then $\mathbb{M}_2^{(1)} \neq \mathbb{M}_2^{(2)}$ and $\langle\sigma^4\rangle$ is the only subgroup of $\mathbb{G}'$ of index 2.
Let $\mathbb{L}$ the subfield of $\mathbb{M}_2^{(2)}$ left fixed by $\langle\sigma^4\rangle$, $\mathbb{F}_i$ ($i = 1,2,3$) the subfield of $\mathbb{M}_2^{(2)}$ left fixed by H_i and j_i the mapping j defined for $\mathbb{F} = \mathbb{F}_i$.

Theorem 1 ([8]). *Assume that $\mathbb{G}/\mathbb{G}' \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, so we have*

(1) *If $\mathbb{M}_2^{(1)} = \mathbb{M}_2^{(2)}$, then the fields $\mathbb{F}_i$ are of type (A), $\#\ker j_i = 4$ for $i = 1,2,3$ and $\mathbb{G} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$;*
(2) *If $Gal(\mathbb{L}/\mathbb{M}) \simeq Q_3$, then the fields $\mathbb{F}_i$ are of type (A), $\#\ker j_i = 2$ for $i = 1,2,3$ and $\mathbb{G} \simeq Q_3$;*
(3) *If $Gal(\mathbb{L}/\mathbb{M}) \simeq D_3$, then the fields $\mathbb{F}_2$ and $\mathbb{F}_3$ are of type (B) and $\#\ker j_2 = \#\ker j_3 = 2$. Moreover, if $\mathbb{F}_1$ is of type (B) then $\#\ker j_1 = 2$ and $\mathbb{G} \simeq S_m$. If $\mathbb{F}_1$ is of type (A) and $\#\ker j_1 = 2$, then $\mathbb{G} \simeq Q_m$. Finally if $\mathbb{F}_1$ is of type (A) and $\#\ker j_1 = 4$, then $\mathbb{G} \simeq D_m$.*

Theorem 2 ([6]). *Let $\mathbb{L}/\mathbb{M}$ be an unramified cyclic extension of prime degree, then the number of classes that capitulate in $\mathbb{L}/\mathbb{M}$ is*

$$[\mathbb{L} : \mathbb{M}][E_{\mathbb{M}} : N_{\mathbb{L}/\mathbb{M}}(E_{\mathbb{L}})],$$

where $E_{\mathbb{M}}$ (resp. $E_{\mathbb{L}}$) is the group of units of $\mathbb{M}$ (resp. of $\mathbb{L}$).

For more details about the capitulation problem see [8, 11, 12].

2. Units of some number fields

Proposition 2 ([1]). *Let $\mathbb{K}_0$ a number field, abelian real and β an algebraic integer in $\mathbb{K}_0$, totally positive, without square factors. Assume that $\mathbb{F} = \mathbb{K}_0(\sqrt{-\beta})$ is a quadratic extension of $\mathbb{K}_0$, abelian over $\mathbb{Q}$ and $i = \sqrt{-1} \notin \mathbb{F}$. Let $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r\}$ be a fundamental system of units of $\mathbb{K}_0$. We choose, without limiting the generality, units ε_j positive. Then we have:*

1. *If there is a unit of $\mathbb{K}_0$ such that $\varepsilon = \varepsilon_1^{j_1}\varepsilon_2^{j_2}\ldots\varepsilon_{r-1}^{j_{r-1}}\varepsilon_r$ (close to a permutation), where the $j_k \in \{0, 1\}$, such that $\beta\varepsilon$ is a square in $\mathbb{K}_0$, then $\{\varepsilon_1, \ldots, \varepsilon_{r-1}, \sqrt{-\varepsilon}\}$ is a fundamental system of units of $\mathbb{F}$;*
2. *Otherwise $\{\varepsilon_1, \ldots, \varepsilon_r\}$ is a fundamental system of units of $\mathbb{F}$.*

Corollary 1 ([2]). *Let $\mathbb{L} = \mathbb{Q}(\sqrt{-n\varepsilon\sqrt{d}})$ a cyclic extension of degree 4 over $\mathbb{Q}$, where ε is the fundamental unit of $\mathbb{Q}(\sqrt{d})$ with d a positive integer squarefree and n a positive integer, then $\{\varepsilon\}$ is a fundamental system of units of $\mathbb{L}$.*

Theorem 3 ([2]). *Let $\mathbb{K}_0 = \mathbb{Q}(\sqrt{p}, \sqrt{p'})$ where p and p' are two different primes such that $p \equiv p' \equiv 1 \bmod 4$, ε_1 (resp. ε_2, ε_3) the fundamental unit of $\mathbb{Q}(\sqrt{p})$ (resp. $\mathbb{Q}(\sqrt{p'})$, $\mathbb{Q}(\sqrt{pp'})$) and $\mathbb{F} = \mathbb{K}_0(\sqrt{-n\varepsilon_1\sqrt{p}})$ where n is a positive integer squarefree. Then we have:*

1. *If ε_3 has norm 1, then $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_3}\}$ is a fundamental system of units of $\mathbb{K}_0$ and of $\mathbb{F}$;*
2. *Otherwise, $\{\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}, \varepsilon_2, \varepsilon_3\}$ is a fundamental system of units of $\mathbb{K}_0$ and of $\mathbb{F}$.*

Now, using the results of M. N. Gras [4], we'll define a fundamental system of units of real cyclic extension of degree 4 over $\mathbb{Q}$. Let $\mathbb{L} = \mathrm{k}(\sqrt{\varepsilon_0\sqrt{l}})$ where l is a prime number congruent to 1 modulo 8 and ε_0 is the fundamental unit of $\mathrm{k} = \mathbb{Q}(\sqrt{l})$, then $\mathbb{L}/\mathbb{Q}$ is a real cyclic extension of degree 4 with Galois group $H = \langle\gamma\rangle$ and quadratic subfield k. Since $\mathbb{L}$ has conductor $F_{\mathbb{L}} = l$, we have $\mathbb{L} \subset \mathbb{Q}^{(l)}$ and there exists a character χ' of $Gal(\mathbb{Q}^{(l)}/\mathbb{Q}) \simeq (\mathbb{Z}/l\mathbb{Z})^*$ such that $\ker\chi' = Gal(\mathbb{Q}^{(l)}/\mathbb{L})$.
Let $\chi = \chi' + \chi'^{-1}$, then χ is a rational character of $\mathbb{Q}^{(l)}$ and $\mathbb{L}$ is fixed by the common kernel of χ' and χ'^{-1}. Let $E_{\mathbb{L}}$ be the group of units of $\mathbb{L}$, E_χ the group of χ-relative units of $\mathbb{L}$, $|E_{\mathbb{L}}|$ (resp. $|E_\chi|$) the group of absolute values of $E_{\mathbb{L}}$ (resp.

E_χ), $|E^{\mathbb{L}}| = |E_{\mathbb{L}}| \oplus |E_\chi|$, $Q = [|E_{\mathbb{L}}| : |E^{\mathbb{L}}|]$ and ε_χ a generator of E_χ. Then we have the following result:

Theorem 4 ([4]). *Let* $\mathbb{L} = \mathrm{k}(\sqrt{\varepsilon_0\sqrt{l}})$ *where* l *is a prime number congruent to* 1 *modulo* 8 *and* ε_0 *the fundamental unit of* $\mathrm{k} = \mathbb{Q}(\sqrt{l})$*, then:*

(1) $Q = 2$*;*
(2) *There exists* ξ *in* $E_{\mathbb{L}}$ *such that* $\xi^2 = \pm\varepsilon_0\varepsilon_\chi^{1-\sigma}$ *and* $\{\xi, \xi^\sigma, \xi^{\sigma^2}\}$ *is a fundamental system of units of* $\mathbb{L}$.

Remark 1. Since $\xi^2 = \pm\varepsilon_0\varepsilon_\chi^{1-\sigma}$, then:

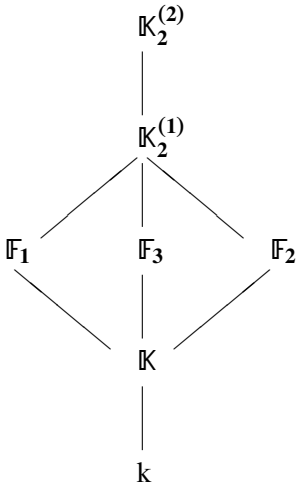
(1) $\xi^{1+\sigma} = \pm\varepsilon_\chi$;
(2) $\xi^{1+\sigma^2} = \pm\varepsilon_0$;
(3) $\xi^{1+\sigma+\sigma^2+\sigma^3} = \varepsilon_\chi^{1+\sigma^2} = \varepsilon_0^{1+\sigma}$;
(4) $N_{\mathrm{k}/\mathbb{Q}}(\varepsilon_0) = N_{\mathbb{L}/\mathrm{k}}(\varepsilon_\chi) = N_{\mathbb{L}/\mathbb{Q}}(\xi) = -1$.

Lemma 1. *With the same notation of Theorem 4,* $\{\xi, \xi^\sigma, \xi^{\sigma^2}\}$ *is a fundamental system of units of* $\mathbb{F} = \mathbb{L}(\sqrt{-n})$ *where* n *is an integer different from* 1*, relatively prime to* l *and square-free.*

Proof. By Proposition 2, to show that $\{\xi, \xi^\sigma, \xi^{\sigma^2}\}$ is a fundamental system of units of $\mathbb{F}$ it suffices to show that $n\mu$ is not a square in $\mathbb{L}$, for $\mu = {\xi'_1}^{j_1}{\xi'_2}^{j_2}\xi'_3$ where $\{\xi'_1, \xi'_2, \xi'_3\} = \{\xi, \xi^\sigma, \xi^{\sigma^2}\}$ and $j_1, j_2 \in \{0, 1\}$.
Indeed, if $\mu = \xi$, then if $n\xi = x^2$ in $\mathbb{L}$, so by calculating the norm in $\mathbb{L}/\mathrm{k}$, we find that $\xi^{1+\sigma^2} = \pm\varepsilon_0$ is a square in k, which is impossible.
If $\mu = \xi^\sigma$, then if $n\xi^\sigma = \pm n\frac{\varepsilon_\chi}{\xi} = x^2$ in $\mathbb{L}$, so by using the norm in $\mathbb{L}/\mathrm{k}$, we find that $\pm\frac{\varepsilon_0^{1+\sigma}}{\varepsilon_0} = \pm\varepsilon_0^{-1}$ is a square in k, which is absurd.
If $\mu = \xi^{\sigma^2}$, then if $n\xi^{\sigma^2} = \pm n\frac{\varepsilon_0}{\xi} = x^2$ in $\mathbb{L}$, so by calculating the norm in $\mathbb{L}/\mathrm{k}$, we find that $\pm\frac{\varepsilon_0^2}{\varepsilon_0} = \pm\varepsilon_0$ is a square in k, which is not the case.
If $\mu = \xi^{1+\sigma} = \pm\varepsilon_\chi$, then if $\pm n\varepsilon_\chi = x^2$ in $\mathbb{L}$, so by calculating the norm in $\mathbb{L}/\mathrm{k}$, we find that $\varepsilon_\chi^{1+\sigma^2} = \varepsilon_0^{1+\sigma} = -1$ is a square in k, which is absurd.
If $\mu = \xi^{1+\sigma^2} = \pm\varepsilon_0$, then $\pm n\varepsilon_0$ can not be a square in $\mathbb{L}$.
If $\mu = \xi^{\sigma+\sigma^2}$, then if $n\xi^{\sigma+\sigma^2} = x^2$ in $\mathbb{L}$, so by calculating the norm in $\mathbb{L}/\mathrm{k}$, we find that $\xi^{1+\sigma+\sigma^2+\sigma^3} = -1$ is a square in k, which is not the case.
If $\mu = \xi^{1+\sigma+\sigma^2}$, then if $n\xi^{1+\sigma+\sigma^2} = x^2$ in $\mathbb{L}$, so by calculating the norm in $\mathbb{L}/\mathrm{k}$, we find that $\xi^{1+\sigma+\sigma^2+\sigma^3}\xi^{1+\sigma^2} = \pm\varepsilon_0$ is a square in k, which is impossible, which completes the proof of the lemma. □

3. CAPITULATION OF THE 2-IDEAL CLASS OF $\mathbb{K}$ AND STRUCTURE OF $\mathcal{G}_2$

In the following, let $\mathbb{K} = \mathrm{k}(\sqrt{-pq\varepsilon_0\sqrt{l}})$ where ε_0 is the fundamental unit of $\mathrm{k} = \mathbb{Q}(\sqrt{l})$ with l is a prime number congruent to 1 modulo 8, p and q two prime numbers such that $p \equiv -q \equiv 1 \bmod 4$ and $(\frac{p}{l}) = (\frac{q}{l}) = -1$, $\mathbb{K}_2^{(1)}$ the Hilbert 2-class field of $\mathbb{K}$, $\mathbb{K}_2^{(2)}$ the Hilbert 2-class field of $\mathbb{K}_2^{(1)}$, $\mathcal{G}_2$ the Galois group of $\mathbb{K}_2^{(2)}/\mathbb{K}$ and for an ideal $\mathfrak{a}$ of $\mathbb{K}$, we note $[\mathfrak{a}]$ the class of $\mathfrak{a}$. Then, by [3], $C_{2,\mathbb{K}}$, the 2-class group of $\mathbb{K}$, is of type $(2,2)$. Using the Results of [7] for the calculation of the genus fields of an extension of degree 2^s on $\mathbb{Q}$, we find that $\mathbb{K}_2^{(1)} = \mathbb{K}^{(*)} = \mathbb{K}(\sqrt{p}, \sqrt{-q})$, where $\mathbb{K}^{(*)}$ is the genus field of $\mathbb{K}$, whose quadratic subfields over $\mathbb{K}$ are $\mathbb{F}_1 = \mathbb{K}(\sqrt{p})$, $\mathbb{F}_2 = \mathbb{K}(\sqrt{-q})$ and $\mathbb{F}_3 = \mathbb{K}(\sqrt{-pq})$.
We study the capitulation problem of the 2-ideal classes of $\mathbb{K}$ in different sub-quadratic extension $\mathbb{F}_i/\mathbb{K}$ of $\mathbb{K}_2^{(1)}/\mathbb{K}$, and hence we determine the structure of $\mathcal{G}_2$.



Proposition 3. *Let $\mathbb{K} = \mathrm{k}(\sqrt{-pq\varepsilon_0\sqrt{l}})$, $\mathcal{P}$ the prime ideal of $\mathbb{K}$ above p and $\mathcal{Q}$ that above q. Then the classes $[\mathcal{P}]$, $[\mathcal{Q}]$ and $[\mathcal{P}\mathcal{Q}]$ are of order 2 in $\mathbb{K}$, $C_{2,\mathbb{K}}$ is generated by the classes $[\mathcal{P}]$ and $[\mathcal{Q}]$. Also $\mathcal{P}$ capitulates in $\mathbb{F}_1$, $\mathcal{Q}$ capitulates in $\mathbb{F}_2$ and $\mathcal{P}\mathcal{Q}$ capitulates in $\mathbb{F}_3$.*

Proof. The class $[\mathcal{P}]$ has order 2, indeed, since p is inert in $\mathrm{k}/\mathbb{Q}$ and p ramifies in $\mathbb{K}/\mathbb{Q}$, then $\mathcal{P}^2 = (p)$. Assume that $\mathcal{P} = (\alpha)$ for some α in $\mathbb{K}$, which is equivalent to $(\alpha^2) = (p)$ in $\mathbb{K}$. So there is therefore a unit ε of K such that $p\varepsilon = \alpha^2$, but there exist a and b in k such that $\alpha = a + b\sqrt{-pq\varepsilon_0\sqrt{l}}$, thus $p\varepsilon = a^2 - pq\varepsilon_0\sqrt{l}b^2 + 2ab\sqrt{-pq\varepsilon_0\sqrt{l}}$ and as $\{\varepsilon_0\}$ is a fundamental system of units of $\mathbb{K}$ (Corollary 1) and

$i=\sqrt{-1}\notin\mathbb{K}$, then $p\varepsilon\in \mathrm{k}$, therefore a or $b=0$. If $b=0$, then $p\varepsilon=a^2$, thus, if ε has norm 1 (the norm in $\mathrm{k}/\mathbb{Q}$), p will be norm in $\mathrm{k}/\mathbb{Q}$ which is not the case because $(\frac{p}{l})=-1$, if ε has norm -1 we find that -1 is a square in $\mathbb{Q}$ which is impossible. Similarly, if $a=0$ we find that $\pm l$ is a square in $\mathbb{Q}$, thus $[\mathcal{P}]$ has order 2 and similarly one shows that $[\mathcal{Q}]$ and $[\mathcal{P}Q]$ orders are 2, therefore $C_{2,\mathbb{K}}$ is generated by $[\mathcal{P}]$ and $[\mathcal{Q}]$. To show that $\mathcal{P}$ capitulates in $\mathbb{F}_1$, it suffices to see that $\sqrt{p}\in\mathbb{F}_1$ and $(\sqrt{p}^2)=(p)$ in $\mathbb{F}_1$, so $\mathcal{P}$ capitulated in $\mathbb{F}_1$ and even $\mathcal{Q}$ capitulated in $\mathbb{F}_2$ and $\mathcal{P}\mathcal{Q}$ capitulated in $\mathbb{F}_3$. □

Proposition 4 ([5]). *Let $\mathbb{M}$ a number field which contains the m-th roots of unity, $\mathbb{L}$ a finite extension of $\mathbb{M}$, $\alpha\in\mathbb{M}^*$ and $\beta\in\mathbb{L}^*$. We denote by P a prime ideal of $\mathbb{M}$, and $\mathcal{P}$ a prime ideal of $\mathbb{L}$ above P. Then*

$$\prod_{\mathcal{P}}\left(\frac{\beta,\alpha}{\mathcal{P}}\right)_m=\left(\frac{N_{\mathbb{L}/\mathbb{M}}(\beta),\alpha}{P}\right)_m,$$

where the product is taken for all prime ideals of $\mathbb{L}$ that are above P.

Theorem 5. *Let $\mathbb{K}=\mathrm{k}(\sqrt{-pq\varepsilon_0\sqrt{l}})$ where ε_0 is the fundamental unit of $\mathrm{k}=\mathbb{Q}(\sqrt{l})$ with l is a prime number congruent to 1 modulo 8, p and q be two prime numbers such that $p\equiv -q\equiv 1 \bmod 4$ and $(\frac{p}{l})=(\frac{q}{l})=-1$, $\mathbb{F}_1=\mathbb{K}(\sqrt{p})$, $\mathbb{F}_2=\mathbb{K}(\sqrt{-q})$ and $\mathbb{F}_3=\mathbb{K}(\sqrt{-pq})$. Then in each extension $\mathbb{F}_i$, $i\in\{1,2,3\}$, there exist exactly two classes of $C_{2,\mathbb{K}}$ which capitulate and the group $\mathcal{G}_2$ is quaternionic of order 2^m with $m>3$.*

Proof. Let ε_2 (resp. ε_3) the fundamental unit of $\mathbb{Q}(\sqrt{p})$ (resp. $\mathbb{Q}(\sqrt{lp})$), $\mathcal{P}$ the prime ideal of $\mathbb{K}$ above p, $\mathcal{Q}$ that above q, then, by Theorem 3, $\{\sqrt{\varepsilon_0\varepsilon_2\varepsilon_3},\varepsilon_2,\varepsilon_3\}$ is a fundamental system of units of $\mathbb{F}_1$, since $N_{\mathbb{F}_1/\mathbb{K}}(\sqrt{\varepsilon_0\varepsilon_2\varepsilon_3})=\pm\varepsilon_0$ and $N_{\mathbb{F}_1/\mathbb{K}}(\varepsilon_2)=N_{\mathbb{F}_1/\mathbb{K}}(\varepsilon_3)=-1$, then $N_{\mathbb{F}_1/\mathbb{K}}(E_{\mathbb{F}_1})=E_{\mathbb{K}}$, therefore, by Theorem 2, we found that only two classes of $C_{2,\mathbb{K}}$ capitulate in $\mathbb{F}_1$, namely $[\mathcal{P}]$ and its square. Also the extension $\mathbb{F}_1/\mathbb{K}$ is of type (B), indeed, let $\mathbb{K}'=\mathrm{k}(\sqrt{-q\varepsilon_0\sqrt{l}})$, then we have $\mathbb{K}\mathbb{K}'=\mathbb{F}_1$, since $N_{\mathbb{K}/\mathrm{k}}(\mathcal{P})=p$ and p is unramified in $\mathbb{K}'/\mathrm{k}$, then to show that $\mathcal{P}$ is inert in $\mathbb{F}_1/\mathbb{K}$, it suffices to show that p is inert in $\mathbb{K}'/\mathrm{k}$ (translation theorem) and for this we compute the norm residue symbol $\left(\frac{p,-q\varepsilon_0\sqrt{l}}{p}\right)$. Since $p\in\mathbb{Q}$ is inert in $\mathrm{k}/\mathbb{Q}$ and $-q\varepsilon_0\sqrt{l}\in\mathrm{k}$, therefore using the Proposition 4, we find

$$\left(\frac{p,-q\varepsilon_0\sqrt{l}}{p}\right)=\left(\frac{p,N_{\mathrm{k}/\mathbb{Q}}(-q\varepsilon_0\sqrt{l})}{p}\right)=\left(\frac{p,lq^2}{p}\right)=(\frac{l}{p})=-1,$$

so p is inert in $\mathbb{K}'/\mathrm{k}$, which gives that $\mathcal{P}$ is inert in $\mathbb{F}_1/\mathbb{K}$, then $\mathbb{F}_1/\mathbb{K}$ is of type (B). Similarly, we show that extension $\mathbb{F}_2/\mathbb{K}$ is of type (B), using Theorem 1, only two classes of $C_{2,\mathbb{K}}$ capitulate in $\mathbb{F}_2$, namely $[\mathcal{Q}]$ and its square.

Using the notation of Theorem 4, we have that $\{\xi,\xi^{\sigma},\xi^{\sigma^2}\}$ is a fundamental system

of units of $\mathbb{F}_3$, since $N_{\mathbb{F}_3/\mathbb{K}}(\xi) = N_{\mathbb{F}_3/\mathbb{K}}(\xi^{\sigma^2}) = \pm\varepsilon_0$ and $N_{\mathbb{F}_3/\mathbb{K}}(\xi^{\sigma}) = \pm\varepsilon_0^{-1}$, then we have $N_{\mathbb{F}_3/\mathbb{K}}(E_{\mathbb{F}_3}) = E_{\mathbb{K}}$, using Theorem 2, we found that only two classes of $C_{2,\mathbb{K}}$ capitulate in $\mathbb{F}_3$, namely $[\mathcal{P}Q]$ and its square. Moreover, the extension $\mathbb{F}_3/\mathbb{K}$ is of type (A), indeed, let $\mathbb{L} = \mathrm{k}(\sqrt{\varepsilon_0\sqrt{l}})$, then we have $\mathbb{K}\mathbb{L} = \mathbb{F}_3$, $N_{\mathbb{K}/\mathrm{k}}(\mathcal{P}) = p$ and p is unramified in $\mathbb{L}/\mathrm{k}$, so to show that $\mathcal{P}$ is inert in $\mathbb{F}_3/\mathbb{K}$, it suffices to show that p is inert in $\mathbb{L}/\mathrm{k}$, for this, we compute the norm residue symbol $\left(\frac{p,\varepsilon_0\sqrt{l}}{p}\right)$. We have $p \in \mathbb{Q}$ is inert in $\mathrm{k}/\mathbb{Q}$ and $\varepsilon_0\sqrt{l} \in \mathrm{k}$, so by Proposition 4, we have $\left(\frac{p,\varepsilon_0\sqrt{l}}{p}\right) = \left(\frac{p,N_{\mathrm{k}/\mathbb{Q}}(\varepsilon_0\sqrt{l})}{p}\right) = \left(\frac{p,l}{p}\right) = (\frac{l}{p}) = -1$, therefore p is inert in $\mathbb{L}/\mathrm{k}$, what gives that $\mathcal{P}$ is inert in $\mathbb{F}_3/\mathbb{K}$, and similarly one shows that $\mathcal{Q}$ remains inert in $\mathbb{F}_3/\mathbb{K}$ and like $\mathcal{P}\mathcal{Q}$ capitulates in $\mathbb{F}_3/\mathbb{K}$, then by applying the Artin reciprocity law, we find that $\mathbb{F}_3/\mathbb{K}$ is of type (A), therefore, using Theorem 1, the group $\mathbb{G}_2$ is isomorphic to Q_m with $m > 3$. □

Example 1. Let $\mathbb{K} = \mathbb{Q}(\sqrt{-55(4+\sqrt{17})\sqrt{17}})$, $\mathbb{F}_1 = \mathbb{K}(\sqrt{5})$, $\mathbb{F}_2 = \mathbb{K}(\sqrt{-11})$ and $\mathbb{F}_3 = \mathbb{K}(\sqrt{-55})$. By Theorem 5, in each extension $\mathbb{F}_i$ there are exactly two classes of $C_{2,\mathbb{K}}$ which capitulate and $\mathbb{G}_2 \simeq Q_m$ with $m > 3$.

Corollary 2. *With the same notation of Theorem 5, we have* $\#\mathbb{G}_2 = 4h_2(\mathbb{K}_0)$ *where* $\mathbb{K}_0 = \mathbb{Q}(\sqrt{l}, \sqrt{-pq})$ *and* $h_2(\mathbb{K}_0)$ *its 2-class number.*

Proof. We have $\mathbb{K}_2^{(1)}/\mathbb{F}_3$ is an unramified extension and the extension $\mathbb{F}_3/\mathbb{K}$ is of type (A), then, according to [8], $C_{2,\mathbb{F}_3}$ is cyclic, thus $\mathbb{F}_3$ and $\mathbb{K}_2^{(1)}$ has the same Hilbert 2-class field which is $\mathbb{K}_2^{(2)}$, therefore $\#\mathbb{G}_2 = 2h_2(\mathbb{F}_3)$. Moreover $\mathbb{F}_3/\mathrm{k}$ is a biquadratic normal extension with Galois group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and quadratic subfields $\mathbb{K}$, $\mathbb{K}_0$ and $\mathbb{L} = \mathrm{k}(\sqrt{\varepsilon_0\sqrt{l}})$, so, by using [9], page 247, we find that

$$h_2(\mathbb{F}_3) = \frac{1}{2}q(\mathbb{F}_3/\mathrm{k})h_2(\mathbb{K})h_2(\mathbb{K}_0)h_2(\mathbb{L}),$$

where $q(\mathbb{F}_3/\mathrm{k})$ is the unit index of $\mathbb{F}_3/\mathrm{k}$ and $h_2(\mathbb{F})$ is the 2-class number of a number field $\mathbb{F}$. We have $h_2(\mathbb{K}) = 4$, according to [13] we have $h_2(\mathbb{L}) = 1$, by Corollary 1 we have $\{\varepsilon_0\}$ is a fundamental system of units of $\mathbb{K}$, using the Proposition 2 we show that $\{\varepsilon_0\}$ is a fundamental system of units of $\mathbb{K}_0$, according to Lemma 1, we have $\{\xi, \xi^{\sigma}, \xi^{\sigma^2}\}$ is a fundamental system of units of $\mathbb{L}$ and $\mathbb{F}_3$, which gives that $q(\mathbb{F}_3/\mathrm{k}) = 1$, thus $h_2(\mathbb{F}_3) = 2h_2(\mathbb{K}_0)$, therefore $\#\mathbb{G}_2 = 4h_2(\mathbb{K}_0)$. □

Corollary 3. *Let* $\mathbb{K} = \mathrm{k}(\sqrt{-pq\varepsilon_0\sqrt{l}})$ *where* ε_0 *is the fundamental unit of* $\mathrm{k} = \mathbb{Q}(\sqrt{l})$ *with* l *a prime number congruent to* 1 *modulo* 8, p *and* q *be two primes such that* $p \equiv -q \equiv 1 \bmod 4$ *and* $(\frac{p}{l}) = (\frac{q}{l}) = (\frac{p}{q}) = -1$, *then* $\mathbb{G}_2 \simeq Q_4$.

Proof. According to the Theorem 5, $\mathbb{G}_2$ is quaternionic of order 2^m with $m > 3$. Let $\mathbb{K}_0 = \mathbb{Q}(\sqrt{l}, \sqrt{-pq})$, since $(\frac{p}{l}) = (\frac{q}{l}) = (\frac{p}{q}) = -1$, then, according to [10], we have $h_2(\mathbb{K}_0) = 4$, so, using Corollary 2, we find that $\mathbb{G}_2 \simeq Q_4$. □

Example 2. Let $\mathbb{K} = \mathbb{Q}(\sqrt{-15(4+\sqrt{17})\sqrt{17}})$, $\mathbb{F}_1 = \mathbb{K}(\sqrt{5})$, $\mathbb{F}_2 = \mathbb{K}(\sqrt{-3})$ and $\mathbb{F}_3 = \mathbb{K}(\sqrt{-15})$. According to the Theorem 5, in each extension $\mathbb{F}_i$ there exist exactly two classes of $C_{2,\mathbb{K}}$ which capitulate and the group $\mathbb{G}_2$ is quaternionic of order 2^m with $m > 3$. Furthermore, since $(\frac{5}{3}) = -1$, then, according to the Corollary 3, $\mathbb{G}_2 \simeq Q_4$.

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Authors' addresses

Abdelmalek Azizi

Mohamed First University, Department of Mathematics and Computer Sciences, Faculty of Sciences, Oujda, Morocco

E-mail address: abdelmalekazizi@yahoo.fr

Mohammed Talbi
Regional Center of Education and Training, Oujda, Morocco
E-mail address: talbimm@yahoo.fr

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A NEW RESULT ON THE QUASI POWER INCREASING SEQUENCES

HÜSEYIN BOR, DANSHENG YU, AND PING ZHOU

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Abstract. In this paper, we prove a general theorem dealing with generalized absolute convolution Cesàro mean summability factors under weaker conditions by using a general class of increasing sequences instead of an almost increasing sequence. Some new results have also been obtained.

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Keywords: convolution Cesàro mean, power increasing sequences, summability factors, Hölder inequality, Minkowski inequality, infinite series

1. INTRODUCTION

A positive sequence (b_n) is said to be an almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants M and N such that $Mc_n \leq b_n \leq Nc_n$(see [1]). A positive sequence $X = (X_n)$ is said to be a quasi-σ-power increasing sequence if there exists a constant $K = K(\sigma, X) \geq 1$ such that $Kn^{\sigma}X_n \geq m^{\sigma}X_m$ for all $n \geq m \geq 1$ and $0 < \sigma < 1$. Every almost increasing sequence is quasi-σ-power increasing sequence for any nonnegative σ, but the converse is not true for $\sigma > 0$ (see[9]). Let $\sum a_n$ be a given infinite series. We denote by $t_n^{\alpha*\beta}$ the nth convolution Cesàro mean of order $(\alpha * \beta)$, with $\alpha + \beta > -1$, of the sequence (na_n), that is (see [5])

$$t_n^{\alpha*\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \tag{1.1}$$

where

$$A_n^{\alpha} = \frac{(\alpha+1)(\alpha+2)....(\alpha+n)}{n!} = O(n^{\alpha}), \quad A_{-n}^{\alpha} = 0 \quad for \quad n > 0. \tag{1.2}$$

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Let $(\theta_n^{\alpha*\beta})$ be a sequence defined by

$$\theta_n^{\alpha*\beta} = \begin{cases} \left|t_n^{\alpha*\beta}\right|, & \alpha = 1, \beta > -1 \\ \max_{1\le v\le n}\left|t_v^{\alpha*\beta}\right|, & 0 < \alpha < 1, \beta > -1 \end{cases} \tag{1.3}$$

The series $\sum a_n$ is said to be summable $\mid C, \alpha * \beta; \delta \mid_k$, $k \ge 1$ and $\delta \ge 0$, if (see [4])

$$\sum_{n=1}^{\infty} n^{\delta k-1} \mid t_n^{\alpha*\beta} \mid^k < \infty. \tag{1.4}$$

If we take $\delta = 0$, then $\mid C, \alpha * \beta; \delta \mid_k$ summability reduces to $\mid C, \alpha * \beta \mid_k$ summability (see [6]). Also, if we take $\beta = 0$ and $\delta = 0$, then $\mid C, \alpha * \beta; \delta \mid_k$ summability reduces to $\mid C, \alpha \mid_k$ summability (see [7]). If we set $\beta = 0$, then we get $\mid C, \alpha; \delta \mid_k$ summability (see [8]).

2. The known result

Theorem 1 ([4]). *Let (X_n) be an almost increasing sequence and let there be sequences (η_n) and (λ_n) such that*

$$\mid \Delta\lambda_n \mid \le \eta_n, \tag{2.1}$$

$$\eta_n \to 0 \quad as \quad n \to \infty, \tag{2.2}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\eta_n \mid X_n < \infty, \tag{2.3}$$

$$\mid \lambda_n \mid X_n = O(1) \quad as \quad n \to \infty. \tag{2.4}$$

If the condition

$$\sum_{n=1}^{m} n^{\delta k} \frac{(\theta_n^{\alpha*\beta})^k}{n} = O(X_m) \quad as \quad m \to \infty \tag{2.5}$$

satisfies, then the series $\sum a_n\lambda_n$ is summable $\mid C, \alpha * \beta; \delta \mid_k$, $0 < \alpha \le 1$, $\beta > -1$, $k \ge 1$, $\delta \ge 0$ and $(\alpha + \beta - \delta) > 0$.

It should be noted that if we take $\beta = 0$, then we get the result of Bor (see [2]).

3. The main result

The aim of this paper is to generalize Theorem 1 under weaker conditions to the $\mid C, \alpha * \beta; \delta \mid_k$ summability by using a quasi-σ-power increasing sequence, which is a wider class of sequences, instead of an almost increasing sequence.
We shall prove the following main theorem.

Theorem 2. *Let (X_n) be a quasi-σ-power increasing sequence for some σ $(0 < \sigma < 1)$ and let there be sequences (η_n) and (λ_n) such that conditions (2.1)-(2.4) of Theorem A are satisfied. If the condition*

$$\sum_{n=1}^{m} n^{\delta k} \frac{(\theta_n^{\alpha*\beta})^k}{n\ X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty \tag{3.1}$$

is satisfies, then the series $\sum a_n\lambda_n$ is summable $\mid C,\alpha*\beta;\delta \mid_k$ for $0 < \alpha \leq 1$, $\delta \geq 0$, $\beta > -1$, $k \geq 1$ and $(\alpha+\beta-\delta-1) > 0$.

Remark 1. It should also be noted that condition (3.1) is the same as condition (2.5) when k=1. When $k > 1$, condition (3.1) is weaker than condition (2.5) but the converse is not true. As in [10] we can show that if (2.5) is satisfied, then we get that

$$\sum_{n=1}^{m} n^{\delta k} \frac{(\theta_n^{\alpha*\beta})^k}{n\ X_n^{k-1}} = O(\frac{1}{X_1^{k-1}}) \sum_{n=1}^{m} n^{\delta k} \frac{(\theta_n^{\alpha*\beta})^k}{n} = O(X_m).$$

Also if (3.1) is satisfied, then for $k > 1$ we obtain that

$$\sum_{n=1}^{m} n^{\delta k} \frac{(\theta_n^{\alpha*\beta})^k}{n} = \sum_{n=1}^{m} \frac{(\theta_n^{\alpha*\beta})^k}{n\ X_n^{k-1}} X_n^{k-1} = O(X_m^{k-1}) \sum_{n=1}^{m} n^{\delta k} \frac{(\theta_n^{\alpha*\beta})^k}{n\ X_n^{k-1}}$$
$$= O(X_m^k) \neq O(X_m).$$

We need the following lemmas for the proof of our theorem.

Lemma 1 ([3]). *If $0 < \alpha \leq 1$, $\beta > -1$ and $1 \leq v \leq n$, then*

$$\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} a_p \mid \leq \max_{1\leq m\leq v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\beta} a_p \mid . \tag{3.2}$$

Lemma 2 ([9])**.** *Under the conditions on (X_n), (η_n) and (λ_n) as expressed in the statement of the theorem, we have the following ;*

$$nX_n\eta_n = O(1), \tag{3.3}$$

$$\sum_{n=1}^{\infty} \eta_n X_n < \infty. \tag{3.4}$$

4. Proof of the theorem

Let $(T_n^{\alpha*\beta})$ be the nth $(C,\alpha*\beta)$ mean of the sequence $(na_n\lambda_n)$. Then, by (1.1), we have

$$T_n^{\alpha*\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \lambda_v.$$

Applying Abel's transformation first and then using Lemma 1, we have that

$$T_n^{\alpha*\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} pa_p + \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} va_v,$$

thus,

$$\begin{aligned} | T_n^{\alpha*\beta} | &\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} | \Delta\lambda_v | | \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} pa_p | + \frac{| \lambda_n |}{A_n^{\alpha+\beta}} | \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} va_v | \\ &\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{(\alpha+\beta)} \theta_v^{\alpha*\beta} | \Delta\lambda_v | + | \lambda_n | \theta_n^{\alpha*\beta} \\ &= T_{n,1}^{\alpha*\beta} + T_{n,2}^{\alpha*\beta}. \end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{\delta k-1} | T_{n,r}^{\alpha*\beta} |^k < \infty, \quad for \quad r = 1,2.$$

Whenever $k > 1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$\begin{aligned} &\sum_{n=2}^{m+1} n^{\delta k-1} | T_{n,1}^{\alpha*\beta} |^k \\ &\leq \sum_{n=2}^{m+1} n^{\delta k-1} (A_n^{\alpha+\beta})^{-k} \{ \sum_{v=1}^{n-1} (A_v^{\alpha+\beta})^k (\theta_v^{\alpha*\beta})^k | \Delta\lambda_v |^k \} \times \{ \sum_{v=1}^{n-1} 1 \}^{k-1} \\ &= O(1) \sum_{n=2}^{m+1} n^{\delta k-2+k-(\alpha+\beta)k} \{ \sum_{v=1}^{n-1} v^{(\alpha+\beta)k} (\theta_v^{\alpha*\beta})^k \eta_v^k \} \\ &= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} (\theta_v^{\alpha*\beta})^k \eta_v^k \sum_{n=v+1}^{m+1} \frac{1}{n^{2+(\alpha+\beta-\delta-1)k}} \\ &= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} (\theta_v^{\alpha*\beta})^k \eta_v^k \int_v^{\infty} \frac{dx}{x^{2+(\alpha+\beta-\delta-1)k}} \\ &= O(1) \sum_{v=1}^{m} (\theta_v^{\alpha*\beta})^k \eta_v \eta_v^{k-1} v^{\delta k+k-1} \\ &= O(1) \sum_{v=1}^{m} (\theta_v^{\alpha*\beta})^k \eta_v \left(\frac{1}{vX_v} \right)^{k-1} v^{\delta k+k-1} \end{aligned}$$

$$= O(1)\sum_{v=1}^{m-1}\Delta(v\eta_v)\sum_{r=1}^{v} r^{\delta k}\frac{(\theta_r^{\alpha*\beta})^k}{rX_r^{k-1}} + O(1)m\eta_m\sum_{v=1}^{m} v^{\delta k}\frac{(\theta_v^{\alpha*\beta})^k}{vX_v^{k-1}}$$

$$= O(1)\sum_{v=1}^{m-1} \mid \Delta(v\eta_v) \mid X_v + O(1)m\eta_m X_m$$

$$= O(1)\sum_{v=1}^{m-1} v \mid \Delta\eta_v \mid X_v + O(1)\sum_{v=1}^{m-1}\eta_v X_v + O(1)m\eta_m X_m$$

$$= O(1) \quad as \quad m \to \infty,$$

by virtue of the hypotheses of the theorem and Lemma 2. Finally, we have that

$$\sum_{n=1}^{m} n^{\delta k-1} \mid T_{n,2}^{\alpha*\beta} \mid^k = \sum_{n=1}^{m} \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid n^{\delta k}\frac{(\theta_n^{\alpha*\beta})^k}{n}$$

$$= O(1)\sum_{n=1}^{m-1}\Delta \mid \lambda_n \mid \sum_{v=1}^{n} v^{\delta k}\frac{(\theta_v^{\alpha*\beta})^k}{vX_v^{k-1}}$$

$$+ O(1) \mid \lambda_m \mid \sum_{n=1}^{m} n^{\delta k}\frac{(\theta_n^{\alpha*\beta})^k}{nX_n^{k-1}}$$

$$= O(1)\sum_{n=1}^{m-1}\eta_n X_n + O(1) \mid \lambda_m \mid X_m = O(1) \quad as \quad m \to \infty,$$

by virtue of the hypotheses of the theorem and Lemma 2. This completes the proof of the theorem.

Remark 2. If we take (X_n) as an almost increasing sequence, $\beta = 0$ and $\delta = 0$, then we obtain a theorem dealing with the $\mid C,\alpha \mid_k$ summability factors. Also, if we take $\delta = 0$, then we get a new result concerning the $\mid C,\alpha*\beta \mid_k$ summability factors of infinite series.

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Authors' addresses

Hüseyin Bor
P. O. Box 121, 06502 Bahçelievler, Ankara Turkey
E-mail address: hbor33@gmail.com

Dansheng Yu
Department of Mathematics, Hangzhou Normal University, Hangzhou, Zhejiang 310036, China
E-mail address: dsyu_math@163.com

Ping Zhou
Department of Mathematics, Statistics and Computer Science, St. Francis Xavier University, Antigonish, Nova Scotia, B2G 2W5, Canada
E-mail address: pzhou@stfx.ca

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TWO DIVISORS OF $(n^2+1)/2$ SUMMING UP TO $\delta n+\varepsilon$, FOR δ AND ε EVEN

SANDA BUJAČIĆ

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Abstract. In this paper we are dealing with the problem of the existence of two divisors of $(n^2+1)/2$ whose sum is equal to $\delta n+\varepsilon$, in the case when δ and ε are even, or more precisely in the case in which $\delta \equiv \varepsilon+2 \equiv 0$ or $2 \pmod 4$. We will completely solve the cases $\delta = 2, \delta = 4$ and $\varepsilon = 0$.

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1. INTRODUCTION

In [1], Ayad and Luca have proved that there does not exist an odd integer $n>1$ and two positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1+d_2=n+1$. In [2], Dujella and Luca have dealt with a more general issue, where $n+1$ was replaced with an arbitrary linear polynomial $\delta n+\varepsilon$, where $\delta>0$ and ε are given integers. The reason that d_1 and d_2 are congruent to 1 modulo 4 comes from the fact that $(n^2+1)/2$ is odd and is a sum of two coprime squares $((n+1)/2)^2+((n-1)/2)^2$. Such numbers have the property that all their prime factors are congruent to 1 modulo 4. Since $d_1+d_2=\delta n+\varepsilon$, then there are two cases: it is either $\delta \equiv \varepsilon \equiv 1 \pmod 2$, or $\delta \equiv \varepsilon+2 \equiv 0$ or $2 \pmod 4$. In [2] authors have focused on the first case.

In this paper, we deal with the second case, the case where $\delta \equiv \varepsilon+2 \equiv 0$ or $2 \pmod 4$. We completely solve cases when $\delta = 2, \delta = 4$ and $\varepsilon = 0$. We prove that there exist infinitely many positive odd integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1+d_2=2n+\varepsilon$ for $\varepsilon \equiv 0 \pmod 4$ and we prove an analoguos result for $\varepsilon \equiv 2 \pmod 4$ and divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1+d_2=4n+\varepsilon$. In case when $\delta \geq 6$ is a positive integer of the form $\delta = 4k+2$, $k \in \mathbb{N}$ we prove that there does not exist an odd integer n such that there exists a pair of divisors d_1, d_2 of $\frac{n^2+1}{2}$ with the property $d_1+d_2=\delta n$. We also prove that there exist infinitely many odd integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1+d_2=2n$.

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2. The case $\delta = 2$

Theorem 1. *If $\varepsilon \equiv 0 \pmod 4$, then there exist infinitely many positive odd integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n + \varepsilon$.*

Proof. Let $\varepsilon \equiv 0 \pmod 4$. We want to find a positive odd integer n and positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n + \varepsilon$. Let $g = \gcd(d_1, d_2)$. We can write $d_1 = gd_1', d_2 = gd_2'$. Since $gd_1'd_2' = \operatorname{lcm}(d_1, d_2)$ divides $\frac{n^2+1}{2}$, we conclude that there exists a positive integer d such that

$$d_1 d_2 = \frac{g(n^2+1)}{2d}.$$

From the identity

$$(d_2 - d_1)^2 = (d_1 + d_2)^2 - 4d_1 d_2,$$

we can easily obtain

$$(d_2 - d_1)^2 = (2n+\varepsilon)^2 - 4\frac{g(n^2+1)}{2d},$$

$$(d_2 - d_1)^2 = 4n^2 + 4\varepsilon n + \varepsilon^2 - 2\frac{g(n^2+1)}{d},$$

$$d(d_2 - d_1)^2 = 4n^2 d + 4d\varepsilon n + \varepsilon^2 d - 2n^2 g - 2g,$$

$$d(d_2 - d_1)^2 = (4d - 2g)n^2 + 4d\varepsilon n + \varepsilon^2 d - 2g,$$

$$d(4d-2g)(d_2-d_1)^2 \tag{2.1}$$
$$= (4d-2g)^2 n^2 + 4(4d-2g)d\varepsilon n + 4d^2\varepsilon^2 - 8dg - 2\varepsilon^2 dg + 4g^2.$$

For $X = (4d-2g)n + 2d\varepsilon, Y = d_2 - d_1$, the equation (2.1) becomes

$$X^2 - d(4d-2g)Y^2 = 8dg + 2\varepsilon^2 dg - 4g^2.$$

For $g = 1$ the previous equation becomes

$$X^2 - 2d(2d-1)Y^2 = 8d + 2\varepsilon^2 d - 4,$$

$$X^2 - 2d(2d-1)Y^2 = 2d(4+\varepsilon^2) - 4. \tag{2.2}$$

The equation (2.2) is a Pellian equation. The right-hand side of (2.2) is nonzero.

Our goal is to make the right-hand side of (2.2) a perfect square. That condition can be satisfied by taking $d = \frac{1}{8}\varepsilon^2 - \frac{1}{2}\varepsilon + 1$. With this choice of d, we get

$$2d(4+\varepsilon^2) - 4 = 2\left(\frac{1}{8}\varepsilon^2 - \frac{1}{2}\varepsilon + 1\right)(4+\varepsilon^2) - 4 = \left(\frac{1}{2}(\varepsilon^2 - 2\varepsilon + 4)\right)^2.$$

Pellian equation (2.2) becomes

$$X^2-2d(2d-1)Y^2=\left(\frac{1}{2}(\varepsilon^2-2\varepsilon+4)\right)^2. \tag{2.3}$$

Now, like in [2], we are trying to solve (2.3). We let

$$X=\frac{1}{2}(\varepsilon^2-2\varepsilon+4)U,\ \ Y=\frac{1}{2}(\varepsilon^2-2\varepsilon+4)V.$$

The equation (2.3) becomes

$$U^2-2d(2d-1)V^2=1. \tag{2.4}$$

Equation (2.4) is a Pell equation which has infinitely many positive integer solutions (U,V), and consequently, there exist infinitely many positive integer solutions (X,Y) of (2.3). The least positive integer solution of (2.4) can be found using the continued fraction expansion of number $\sqrt{2d(2d-1)}$.

We can easily get $\sqrt{2d(2d-1)}=[2d-1;\overline{2,4d-2}]$. All positive solutions of (2.4) are given by (U_m,V_m) for some $m\geq 0$. The first few solutions are
$(U_0,V_0)=(1,0)$,
$(U_1,V_1)=(4d-1,2)$,
$(U_2,V_2)=(32d^2-16d+1,16d-4)$,
$(U_3,V_3)=(256d^3-192d^2+36d-1,128d^2-64d+6),\ldots.$

Generally, solutions of (2.4) are generated by recursive expressions

$$U_0=1,\ \ U_1=4d-1,\ \ U_{m+2}=2(4d-1)U_{m+1}-U_m,$$
$$V_0=0,\ \ V_1=2,\ \ V_{m+2}=2(4d-1)V_{m+1}-V_m,\ \ m\in\mathbb{N}_0. \tag{2.5}$$

By induction on m, one gets that $U_m\equiv 1 \pmod{(4d-2)}, m\geq 0$. Indeed, $U_0=1\equiv 1 \pmod{(4d-2)}$, $U_1=4d-1\equiv 1 \pmod{(4d-2)}$. Assume that $U_m\equiv U_{m-1}\equiv 1 \pmod{(4d-2)}$. For U_{m+1} we get

$$U_{m+1}=2(4d-1)U_m-U_{m-1}\equiv 2-1\equiv 1 \pmod{(4d-2)}.$$

Now, it remains to compute the corresponding values of n which arise from $X=(4d-2)n+2d\varepsilon$ and $X=\frac{1}{2}(\varepsilon^2-2\varepsilon+4)U$. We obtain

$$n=\frac{\frac{1}{2}(\varepsilon^2-2\varepsilon+4)U-2d\varepsilon}{4d-2}.$$

We want the above number n to be a positive integer.

From $d=\frac{1}{8}\varepsilon^2-\frac{1}{2}\varepsilon+1$, it follows $4d-2=\frac{1}{2}\varepsilon^2-2\varepsilon+2$. Note that ε is even. So, congruences

$$\frac{1}{2}(\varepsilon^2-2\varepsilon+4)U-2d\varepsilon\equiv 4d+\varepsilon-2-2d\varepsilon\equiv-(2d-1)\varepsilon\equiv 0 \pmod{(4d-2)},$$

show us that all numbers n generated in the specified way are integers.

The first few values of number n, which we get from U_1, U_2, U_3, are

$$\begin{cases} n = \frac{1}{2}(\varepsilon^2 - 3\varepsilon + 6), \\ d_1 = 1, \\ d_2 = \varepsilon^2 - 2\varepsilon + 5. \end{cases}$$

$$\begin{cases} n = \frac{1}{2}(\varepsilon^4 - 6\varepsilon^3 + 20\varepsilon^2 - 33\varepsilon + 34), \\ d_1 = \varepsilon^2 - 2\varepsilon + 5, \\ d_2 = \varepsilon^4 - 6\varepsilon^3 + 19\varepsilon^2 - 30\varepsilon + 29. \end{cases}$$

$$\begin{cases} n = \frac{1}{2}(\varepsilon^6 - 10\varepsilon^5 + 50\varepsilon^4 - 148\varepsilon^3 + 281\varepsilon^2 - 323\varepsilon + 198), \\ d_1 = \varepsilon^4 - 6\varepsilon^3 + 19\varepsilon^2 - 30\varepsilon + 29, \\ d_2 = \varepsilon^6 - 10\varepsilon^5 + 49\varepsilon^4 - 142\varepsilon^3 + 262\varepsilon^2 - 292\varepsilon + 169. \end{cases}$$

□

3. THE CASE $\delta = 4$

Theorem 2. *If $\varepsilon \equiv 2 \pmod 4$, then there exist infinitely many positive odd integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 4n + \varepsilon$.*

Proof. Proof of this theorem will be slightly different from the proof of Theorem 1. Instead of assuming that $\varepsilon \equiv 2 \pmod 4$, we will distiguish two cases: in one case we will be dealing with $\varepsilon \equiv 6 \pmod 8$ and we will apply strategies from [2] and in the other case we will be dealing with $\varepsilon \equiv 2 \pmod 8$ and we will use different methods in obtaining results.

We start with the case when $\varepsilon \equiv 6 \pmod 8$. We want to find odd positive integers n and positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 4n + \varepsilon$.

Let $g = \gcd(d_1, d_2)$, $d_1 = gd_1'$, $d_2 = gd_2'$ and d is a positive integer which satisfies the equation

$$d_1 d_2 = \frac{g(n^2+1)}{2d}.$$

From the identity

$$(d_2 - d_1)^2 = (d_1 + d_2)^2 - 4d_1 d_2,$$

we obtain

$$(d_2 - d_1)^2 = (4n + \varepsilon)^2 - 4\frac{g(n^2+1)}{2d},$$

$$d(d_2-d_1)^2=(16d-2g)n^2+8d\varepsilon n+\varepsilon^2 d-2g,$$

$$d(16d-2g)(d_2-d_1)^2 \tag{3.1}$$
$$=(16d-2g)^2n^2+8(16d-2g)d\varepsilon n+16d^2\varepsilon^2-32dg-2\varepsilon^2 dg+4g^2.$$

Let $X=(16d-2g)n+4d\varepsilon$, $Y=d_2-d_1$. Equation (3.1) becomes

$$X^2-2d(8d-g)Y^2=32dg+2\varepsilon^2 dg-4g^2. \tag{3.2}$$

For $g=1$ the previous expression becomes

$$X^2-2d(8d-1)Y^2=2d(16+\varepsilon^2)-4. \tag{3.3}$$

It is obvious that (3.3) is a Pellian equation. The right-hand side of (3.3) is nonzero.

Our goal is to make the right-hand side of (3.3) a perfect square. That condition can be satisfied by taking $d=\frac{1}{32}\varepsilon^2-\frac{1}{8}\varepsilon+\frac{5}{8}$. With this choice for d, we get

$$2d(16+\varepsilon^2)-4=2\left(\frac{1}{32}\varepsilon^2-\frac{1}{8}\varepsilon+\frac{5}{8}\right)(16+\varepsilon^2)-4=\left(\frac{1}{4}(\varepsilon^2-2\varepsilon+16)\right)^2.$$

So, Pellian equation (3.3) becomes

$$X^2-2d(8d-1)Y^2=\left(\frac{1}{4}(\varepsilon^2-2\varepsilon+16)\right)^2. \tag{3.4}$$

Let

$$X=\frac{1}{4}(\varepsilon^2-2\varepsilon+16)W,\quad Y=\frac{1}{4}(\varepsilon^2-2\varepsilon+16)Z.$$

The equation (3.4) becomes

$$W^2-2d(8d-1)Z^2=1. \tag{3.5}$$

The equation (3.5) is a Pell equation which has infinitely many positive integer solutions (W,Z), and consequently, there exist infinitely many positive integer solutions (X,Y) of (3.4). The least positive integer solution of (3.5) can be found using the continued fraction expansion of number $\sqrt{2d(8d-1)}$.

We can easily get

$$\sqrt{2d(8d-1)}=[4d-1;\overline{1,2,1,8d-2}].$$

All positive solutions of (3.5) are given by (W_m,Z_m) for some $m\geq 0$. The first few solutions are
$(W_0,Z_0)=(1,0)$,
$(W_1,Z_1)=(16d-1,4)$,
$(W_2,Z_2)=(512d^2-64d+1,128d-8),\ldots$.

Generally, solutions of (3.5) are generated by recursive expressions

$$W_0=1,\quad W_1=16d-1,\quad W_{m+2}=2(16d-1)W_{m+1}-W_m,$$
$$Z_0=0,\quad Z_1=4,\quad Z_{m+2}=2(16d-1)Z_{m+1}-Z_m,\quad m\in\mathbb{N}_0.$$

By induction on m, one gets that $W_m \equiv 1 \pmod{(16d-2)}, m \geq 0$. Indeed, $W_0 = 1 \equiv 1 \pmod{(16d-2)}$, $W_1 = 16d-1 \equiv 1 \pmod{(16d-2)}$. Assume that $W_m \equiv W_{m-1} \equiv 1 \pmod{(16d-2)}$. For W_{m+1} we get

$$W_{m+1} = 2(16d-1)W_m - W_{m-1} \equiv 2-1 \equiv 1 \pmod{(16d-2)}.$$

Now, it remains to compute the corresponding values of n which arise from $X = (16d-2)n + 4d\varepsilon$ and $X = \frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16)W$. We obtain

$$n = \frac{\frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16)W - 4d\varepsilon}{16d-2}.$$

We want to prove that number n is a positive integer.

From $d = \frac{1}{32}\varepsilon^2 - \frac{1}{8}\varepsilon + \frac{5}{8}$,it follows $8d-1 = \frac{1}{4}\varepsilon^2 - \varepsilon + 4$. Number $\frac{\varepsilon}{2}$ is an odd integer. Thus, the congruences

$$\frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16)W - 4d\varepsilon \equiv 8d - 1 + \frac{\varepsilon}{2} - 4d\varepsilon \equiv (8d-1)(1-\frac{\varepsilon}{2}) \equiv 0 \pmod{(16d-2)}$$

show us that all numbers n generated in the specified way are integers.

The first few values of number n, which we get from W_1, W_2, W_3, are

$$\begin{cases} n = \frac{1}{4}(\varepsilon^2 - 3\varepsilon + 18), \\ d_1 = 1 \\ d_2 = \varepsilon^2 - 2\varepsilon + 17. \end{cases}$$

$$\begin{cases} n = \frac{1}{4}(\varepsilon^4 - 6\varepsilon^3 + 44\varepsilon^2 - 105\varepsilon + 322), \\ d_1 = \varepsilon^2 - 2\varepsilon + 17, \\ d_2 = \varepsilon^4 - 6\varepsilon^3 + 43\varepsilon^2 - 102\varepsilon + 305. \end{cases}$$

$$\begin{cases} n = \frac{1}{4}(\varepsilon^6 - 10\varepsilon^5 + 86\varepsilon^4 - 388\varepsilon^3 + 1529\varepsilon^2 - 3155\varepsilon + 5778), \\ d_1 = \varepsilon^4 - 6\varepsilon^3 + 43\varepsilon^2 - 102\varepsilon + 305, \\ d_2 = \varepsilon^6 - 10\varepsilon^5 + 85\varepsilon^4 - 382\varepsilon^3 + 1486\varepsilon^2 - 3052\varepsilon + 5473. \end{cases}$$

Now, we deal with the case when $\varepsilon \equiv 2 \pmod 8$. Let $\varepsilon = 8k+2$, $k \in \mathbb{N}_0$. For $g = \frac{1}{4}\varepsilon^2 + 4$ and $g = d_1$, the equation (3.2) becomes

$$X^2 - 2d(8d-g)Y^2 = \frac{2d-1}{4}\varepsilon^4 + 8\varepsilon^2(2d-1) + 64(2d-1).$$

The right-hand side of the equation will be a perfect square if $2d-1$ is a perfect square. Motivated by the experimental data, we take

$$d=\frac{1}{512}\varepsilon^4-\frac{1}{64}\varepsilon^3+\frac{7}{64}\varepsilon^2-\frac{5}{16}\varepsilon+\frac{41}{32}.$$

We get

$$2d-1=16k^4+8k^2+1=(4k^2+1)^2.$$

So, the equation (3.2) becomes

$$X^2-2d(8d-g)Y^2=\left(\frac{1}{32}(\varepsilon^2+16)(\varepsilon^2-4\varepsilon+20)\right)^2. \tag{3.6}$$

We consider the corresponding Pell equation

$$U^2-2d(8d-g)V^2=1. \tag{3.7}$$

Let (U_0,V_0) be the least positive integer solution of (3.7). That equation has infinitely many solutions. From (3.7) we get that

$$U^2\equiv 1 \pmod{(16d-2g)}.$$

We deal with the case where $g=d_1=\frac{1}{4}\varepsilon^2+4$ and from the experimental data we can set

$$d_2=d_1^2-16kd_1,\ \ k\in\mathbb{N}_0.$$

For $Y=d_2-d_1$ we get

$$Y=\left(\frac{1}{4}\varepsilon^2+4\right)^2-(2\varepsilon-3)\left(\frac{1}{4}\varepsilon^2+4\right)=\frac{\varepsilon^4}{16}-\frac{\varepsilon^3}{2}+\frac{11\varepsilon^2}{4}-8\varepsilon+28.$$

From (3.6), we obtain:

$$X=\frac{(\varepsilon^2+16)(\varepsilon^6-16\varepsilon^5+140\varepsilon^4-768\varepsilon^3+3120\varepsilon^2-8704\varepsilon+14400)}{2048}.$$

We claim that X satisfies the congruence

$$X\equiv 4d\varepsilon \pmod{(16d-2g)}. \tag{3.8}$$

Indeed,

$$16d-2g=\frac{\varepsilon^4}{32}-\frac{\varepsilon^3}{4}+\frac{5\varepsilon^2}{4}-5\varepsilon+\frac{25}{2},$$

$$X-4d\varepsilon=\left(\frac{\varepsilon^4}{32}-\frac{\varepsilon^3}{4}+\frac{5\varepsilon^2}{4}-5\varepsilon+\frac{25}{2}\right)\left(\frac{\varepsilon^4}{64}-\frac{\varepsilon^3}{8}+\frac{13\varepsilon^2}{16}-\frac{9\varepsilon}{4}+9\right).$$

From $n=\frac{X-4d\varepsilon}{16d-2g}$, we get

$$n=\frac{\varepsilon^4}{64}-\frac{\varepsilon^3}{8}+\frac{13\varepsilon^2}{16}-\frac{9\varepsilon}{4}+9=64k^4+28k^2+7,$$

and we see that n is an odd integer. Thus, if we define

$$(X_0, Y_0) = \left(\frac{(\varepsilon^2+16)(\varepsilon^6-16\varepsilon^5+140\varepsilon^4-768\varepsilon^3+3120\varepsilon^2-8704\varepsilon+14400)}{2048}, \right.$$

$$\left. \frac{1}{16}(\varepsilon^2+16)(\varepsilon^2-8\varepsilon+28) \right),$$

we see that (X_0, Y_0) is a solution of (3.6) which satisfies the congruence (3.8). We have proved that for every $\varepsilon \equiv 2 \pmod 8$ there exists at least one odd integer n which satisfies the conditions of Theorem 2. Our goal is to prove that there exist infinitely many such integers n that satisfy the properties of Theorem 2.

If (X_0, Y_0) is a solution of (3.6), solutions of (3.6) are also

$$(X_i, Y_i) = \left(X_0 + \sqrt{2d(8d-g)}Y_0\right)\left(U_0 + \sqrt{2d(8d-g)}V_0\right)^{2i}, \quad i = 0, 1, 2, \ldots \tag{3.9}$$

From the equation (3.9), we get

$$X_i \equiv U_0^{2i} X_0 \equiv X_0 \equiv 4d\varepsilon \pmod{(16d-2g)}.$$

So, there are infinitely many solutions (X_i, Y_i) of (3.6) that satisfy the congruence (3.8). Therefore, by

$$n = \frac{X_i - 4d\varepsilon}{16d-2g},$$

we get infinitely many integers n with the required properties. It is easy to see that number n defined in this way is odd. Indeed, we have $16d-2g \equiv 2 \pmod 4$, $X_0 \equiv 2 \pmod 4$, and since (3.7) implies that U_0 is odd and V_0 is even, we get from (3.8) that

$$X_i - 4d\varepsilon \equiv X_i \equiv U_0^{2i} X_0 \equiv X_0 \equiv 2 \pmod 4,$$

so n is odd. □

4. The case $\varepsilon = 0$

Proposition 1. *There exist infinitely many positive odd integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n$. These solutions satisfy* $\gcd(d_1, d_2) = 1$ *and* $d_1 d_2 = \frac{n^2+1}{2}$.

Proof. We want to find a positive odd integer n and positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n$. Let $g = \gcd(d_1, d_2)$. Then $g|(2n)$ and $g|(n^2+1)$ which implies that $g|((2n)^2+4)$ so we can conclude that $g|4$. Because g is the greatest common divisor of d_1, d_2 and d_1, d_2 are odd numbers, we can also conclude that g is an odd number. So, $g = 1$. Like we did in the proofs of the previous

theorems, we define a positive integer d which satisfies the equation $d_1d_2 = \frac{n^2+1}{2d}$. From the identity

$$(d_2-d_1)^2 = (d_1+d_2)^2 - 4d_1d_2,$$

we can easily obtain

$$(d_2-d_1)^2 = (2n)^2 - 2\frac{(n^2+1)}{d},$$
$$d(d_2-d_1)^2 = 4n^2d - 2n^2 - 2.$$

Let $d_2-d_1 = 2y$, so we get

$$(4d-2)n^2 - 4dy^2 = 2,$$
$$(2d-1)n^2 - 2dy^2 = 1. \tag{4.1}$$

We will use the next lemma, which is Criterion 1 from [3] to check if there exists a solution for (4.1).

Lemma 1. *Let $a>1$, b be positive integers such that $gcd(a,b)=1$ and $D=ab$ is not a perfect square. Moreover, let (u_0,v_0) denote the least positive integer solution of the Pell equation*

$$u^2 - Dv^2 = 1.$$

Then equation $ax^2-by^2=1$ has a solution in positive integers x,y if and only if

$$2a|(u_0+1) \text{ and } 2b|(u_0-1).$$

□

We want to solve the Pell equation

$$U^2 - 2d(2d-1)V^2 = 1, \tag{4.2}$$

where $n=U$, $y=V$. The continued fraction expansion of the number $\sqrt{2d(2d-1)}$ is already known from Theorem 1 where we have obtained

$$\sqrt{2d(2d-1)} = [2d-1;\overline{2,4d-2}].$$

The least positive integer solution of the Pell equation (4.2) is $(4d-1,2)$. In our case, we want to find solutions of (4.1), so we apply Lemma 1 which gives us conditions that have to be fulfilled. It has to be that

$$2(2d-1)|4d \text{ and } 4d|(4d-2),$$

which is not true for $d\in\mathbb{N}$. So, for Pellian equation (4.1) there are no integer solutions (n,y) when $a=2d-1>1$. Finally, we have to check the remaining case for $a=1$, which is the case that is not included in Lemma 1.

If $a=2d-1=1$, then $d=1$. From (4.1) and $d=1$, we get the Pell equation

$$n^2 - 2y^2 = 1, \tag{4.3}$$

which has infinitely many solutions $n = U_m,\ \ y = V_m,\ \ m \in \mathbb{N}_0$ where

$$U_0 = 1,\ \ U_1 = 3,\ \ U_{m+2} = 6U_{m+1} - U_m,$$
$$V_0 = 0,\ \ V_1 = 2,\ \ V_{m+2} = 6V_{m+1} - V_m,\ \ m \in \mathbb{N}_0.$$

The first few values (U_i, V_i) are

$$(U_0, V_0) = (1,0),\ (U_1, V_1) = (3,2),\ (U_2, V_2) = (17,12),\ (U_3, V_3) = (99,70), \ldots.$$

From those solutions we can easily generate (n, d_1, d_2)

$$(n, d_1, d_2) = (3,1,5),\ \ (17,5,29),\ \ (99,29,169), \ldots.$$

We have proved that there exist infinitely many odd positive integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n$. We have also proved that $g = 1$ and $d = 1$, so we conclude that numbers d_1 and d_2 are coprime and that $d_1 d_2 = \frac{n^2+1}{2}$. □

Theorem 3. *Let $\delta \geq 6$ be a positive integer such that $\delta = 4k + 2, k \in \mathbb{N}$. Then there does not exist a positive odd integer n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = \delta n$.*

Proof. Suppose on the contrary that this is not so and let the number δ be the smallest positive integer $\delta = 4k + 2,\ \ k \in \mathbb{N}$ for which there exists an odd integer n and a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = \delta n$. Let $g = \gcd(d_1, d_2) > 1$. Since $d_1 = g d_1',\ \ d_2 = g d_2'$, it follows that $g|(n^2 + 1)$ and $g|(\delta n)$ and we conclude that $g|((\delta n)^2 + \delta^2)$, which implies that $g|\delta^2$. This means that g and δ have a common prime factor p. Let $d_1 = p d_1'', d_2 = p d_2'', \delta = p\delta''$. Then, we have $p d_1'' + p d_2'' = p\delta'' n$, so we can conclude $d_1'' + d_2'' = \delta'' n$ where d_1'', d_2'' are divisors of $\frac{n^2+1}{2}$. It is clear that $\delta'' < \delta$ and if it also satisfies $\delta'' \neq 2$, the existence of the number δ'' contradicts the minimality of δ. So, if $\delta'' \neq 2$, then we must have $g = 1$.

If $\delta'' = 2$, it follows from Proposition 1 that $\gcd(d_1'', d_2'') = 1$ and $d_1'' d_2'' = \frac{n^2+1}{2}$. But, $\gcd(d_1, d_2) = p d_1'' d_2''$ should be a divisor of $\frac{n^2+1}{2}$ which is not possible because $p > 1$. So, in this case we also conclude that $g = 1$.

From the identity

$$(d_2 - d_1)^2 = (d_1 + d_2)^2 - 4 d_1 d_2,$$

and using $g = 1$, we obtain

$$(d_2 - d_1)^2 = (\delta n)^2 - 2\frac{(n^2+1)}{d},$$
$$d(d_2 - d_1)^2 = \delta^2 n^2 d - 2n^2 - 2,$$
$$d(d_2 - d_1)^2 = (d\delta^2 - 2)n^2 - 2.$$

In the equation

$$(\delta^2 d-2)n^2-d(d_2-d_1)^2=2,$$

we set $(d_2-d_1)=2y$ (number d_2-d_1 is an even number because d_1,d_2 are odd integers), and we get

$$(\delta^2 d-2)n^2-4dy^2=2.$$

If we divide both sides of the above equation by 2, then it becomes

$$(2d(2k+1)^2-1)n^2-2dy^2=1.$$

Now, if we define $\delta'=\frac{\delta}{2}=2k+1$, we get

$$(2\delta'^2 d-1)n^2-2dy^2=1. \tag{4.4}$$

We will prove by applying Lemma 1 that the above Pell equation (4.4) has no solutions.

To be able to apply Lemma 1, we have to deal with an equation of the form

$$x^2-Dy^2=1.$$

We have $a=2d\delta'^2-1$, $a>1$ (because $\delta'\geq 3$) and $D=ab=2d(2\delta'^2 d-1)$ is not a perfect square because $2d(2\delta'^2 d-1)\equiv 2 \pmod 4$. We need to find the least positive integer solution of the equation

$$u^2-2d(2\delta'^2 d-1)v^2=1. \tag{4.5}$$

For that purpose we find the continued fraction expansion of the number

$$\sqrt{2d(2\delta'^2 d-1)},\ \ \delta'\geq 3.$$

We know that

$$\sqrt{2d(2\delta'^2 d-1)}=[a_0;\overline{a_1,a_2,\ldots,a_{l-1},2a_0}],$$

where we calculate numbers a_i recursively

$$a_i=\left\lfloor\frac{s_i+a_0}{t_i}\right\rfloor,\ \ s_{i+1}=a_i t_i-s_i,\ \ t_{i+1}=\frac{d-s_{i+1}^2}{t_i}.$$

In our case, we obtain

$$a_0=\lfloor\sqrt{2d(2\delta'^2 d-1)}\rfloor=2d\delta'-1,\ \ s_0=0,\ \ t_0=1;$$
$$s_1=2d\delta'-1,\ \ t_1=4d\delta'-2d-1,\ \ a_1=1;$$
$$s_2=2d\delta'-2d,\ \ t_2=2d,\ \ a_2=2\delta'-2;$$
$$s_3=2d\delta'-2d,\ \ t_3=4d\delta'-2d-1,\ \ a_3=1;$$
$$s_4=2d\delta'-1,\ \ t_4=1,\ \ a_4=2(2d\delta'-1)=2a_0.$$

We get

$$\sqrt{2d(2\delta'^2 d-1)}=[2d\delta'-1;\overline{1,2\delta'-2,1,2(2d\delta'-1)}].$$

Now, we can find the least positive integer solution of the equation (4.5). Because the length of the period of the expansion is $l = 4$, the least positive integer solution of (4.5) is (p_3, q_3), where numbers $p_i, q_i,\ i = 0,1,2,3$ are calculated recursively

$$p_0 = a_0,\ \ p_1 = a_0a_1 + 1,\ \ p_k = a_k p_{k-1} + p_{k-2},$$
$$q_0 = 1,\ \ q_1 = a_1,\ \ q_k = a_k q_{k-1} + q_{k-2},\ \ k = 2,3.$$

We obtain

$$(p_0, q_0) = (2d\delta' - 1, 1),\ \ (p_1, q_1) = (2d\delta', 1),\ \ (p_2, q_2) = (4\delta'^2 d - 2d\delta' - 1, 2\delta' - 1),$$
$$(p_3, q_3) = (4\delta'^2 d - 1, 2\delta').$$

So, the least positive integer solution is $(p_3, q_3) = (u_0, v_0) = (4\delta'^2 d - 1, 2\delta')$ and we apply Lemma 1.

In our case we have $a = 2\delta'^2 d - 1,\ \ b = 2d$. From Lemma 1 we get

$$(4\delta'^2 d - 2) | 4\delta'^2 d,\ \ 4d | (4\delta'^2 d - 2).$$

We can easily see that $4d | (4\delta'^2 d - 2)$ if and only if $4d | 2$ which is not possible because $d \in \mathbb{N}$. So, the equation (4.4) has no solutions. We have proved that there does not exist a positive odd integer n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = \delta n$. □

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Author's address

Sanda Bujačić
University of Rijeka, Department of Mathematics, Radmile Matejčić 2, 51 000 Rijeka, Croatia
E-mail address: sbujacic@math.uniri.hr

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APPROXIMATION BY A GENERALIZED SZÁSZ TYPE OPERATOR FOR FUNCTIONS OF TWO VARIABLES

NURSEL ÇETİN, SEVILAY KIRCI SERENBAY, AND ÇIĞDEM ATAKUT

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Abstract. In the present paper, we define a new Szász-Mirakjan type operator in exponential weighted spaces for functions of two variables having exponential growth at infinity using a method given by Jakimovski-Leviatan. This operator is a generalization of two variables of an operator defined by A. Ciupa [1]. In this study, we investigate approximation properties and also estimate the rate of convergence for this new operator.

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1. INTRODUCTION

For a real function of real variable $f : [0,\infty) \to \mathbb{R}$, the Szász-Mirakjan operators are defined in [2] as

$$S_n(f;x) = e^{-nx} \sum_{j=0}^{\infty} \frac{(nx)^j}{j!} f(\frac{j}{n}) \ , \ x \in [0,\infty),$$

where the convergence of $S_n(f;x)$ to $f(x)$ under the exponential growth condition on f that is $|f(x)| \leq Ce^{Bx}$, for all $x \in [0,\infty)$, with $C, B > 0$ was proved. Then, various modifications and further properties of the Szász-Mirakjan operators have been studied intensively by many authors (e.g. [1, 3–9]).

In [4], A. Jakimovski and D. Leviatan investigated approximation properties of a generalization of the Szász-Mirakjan operators which are stated as follows:

Let $g(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in the disk $|z| < R, R > 1$ and suppose $g(1) \neq 0$. Define the Appell polynomials $p_k(x) = p_k(x,g)$ $(k \geq 0)$ by

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k.$$

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For each function f defined in $[0,\infty)$, they considered the operators L_n defined by

$$L_n(f;x)=\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty}p_k(nx)\,f\left(\frac{k}{n}\right),n>0$$

and also the authors obtained several approximation properties of these operators. A. Ciupa [1] introduced a Szász-Mirakjan type operator that is a generalization of the operator defined by M. Lesniewicz and L. Rempulska [5] using the method given by Jakimovski-Leviatan. A. Ciupa studied the properties of approximation for functions of one variable in the space of continuous functions having an exponential growth at infinity.

In this paper, inspired by [1], for each function f defined in $[0,\infty)\times[0,\infty)$, we define the operators $L_{n,m}$ by

$$L_{n,m}(f;x,y)=\frac{e^{-nx}e^{-my}}{(g(1))^2}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}p_k(nx)\,p_j(my)\,f\left(\frac{k}{n},\frac{j}{m}\right)$$

where

$$g(u_1)e^{u_1x}g(u_2)e^{u_2y}=\sum_{k=0}^{\infty}p_k(x)\,u_1^k\sum_{j=0}^{\infty}p_j(y)\,u_2^j.$$

Now, we consider the function $g(x)=\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}=\sinh x$ where $\sinh x$ is the hyperbolic function of x and let p_k be the polynomials generated by relation

$$\sinh u_1\sinh(u_1x)\sinh u_2\sinh(u_2y)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}p_{2k}(x)\,p_{2j}(y)\,u_1^{2k}u_2^{2j}.$$

Using the following equalities

$$\sinh u_1\sinh(u_1x)=\frac{1}{2}\sum_{k=0}^{\infty}\frac{(1+x)^{2k}-(1-x)^{2k}}{(2k)!}u_1^{2k}$$

$$\sinh u_2\sinh(u_2y)=\frac{1}{2}\sum_{j=0}^{\infty}\frac{(1+y)^{2j}-(1-y)^{2j}}{(2j)!}u_2^{2j},$$

we have

$$p_{2k}(x)=\frac{(1+x)^{2k}-(1-x)^{2k}}{2(2k)!}\quad,\quad p_{2j}(y)=\frac{(1+y)^{2j}-(1-y)^{2j}}{2(2j)!}.$$

Let $C\left(R_1^2\right)$ be the set of all real-valued continuous functions of two variables on $R_1^2:=\{(x,y):x\geq1,y\geq1\}$.

For $p,q>0$ and $(x,y)\in R_1^2$, we define

$$w_{p,q}(x,y)=w_p(x)\,w_q(y)=e^{-px}e^{-qy}$$

$$C_{p,q}=\left\{f\in C\left(R_1^2\right):w_{p,q}f \text{ is uniformly continuous and bounded on } R_1^2\right\}$$

$$\|f\|_{p,q}=\sup_{(x,y)\in R_1^2} w_p(x)\,w_q(y)\,|f(x,y)|$$

and also for $h,k\geq 0$, $\delta\geq 0$, $f\in C_{p,q}$, the first order modulus of continuity given by

$$\omega\left(f,C_{p,q};\delta\right)=\sup_{0\leq h,k\leq\delta}\left\|\Delta_{h,k}f\right\|_{p,q}$$

where

$$\Delta_{h,k}f(x,y)=f(x+h,y+k)-f(x,y).$$

In this study, in the space $C_{p,q}$, $p,q>0$, we introduce the following positive linear operators

$$P_{n,m}(f;x,y)=\frac{1}{(\sinh 1)^2\sinh(nx)\sinh(my)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}p_{2k}(nx)\,p_{2j}(my)\,f\left(\frac{2k}{n},\frac{2j}{m}\right) \tag{1}$$

$n,m\in\mathbb{N}$, $(x,y)\in R_1^2$ and investigate the theorems on convergence of $P_{n,m}(f;x,y)$ operators to functions of two variables. We also estimate the rate of convergence for this new operator by using the modulus of continuity.

2. Auxiliary Results

In this section, we will give some useful results in order to study the convergence of the sequence $\left(P_{n,m}f\right)$ to the function $f\in C_{p,q}$.

Lemma 1. *If $(x,y)\in R_1^2$ and $n,m\in N$, we have*

$$P_{n,m}\left(e_{0,0};x,y\right)=1$$

$$P_{n,m}\left(e_{1,0};x,y\right)=\frac{1}{n}\coth 1+x\coth(nx)$$

$$P_{n,m}\left(e_{0,1};x,y\right)=\frac{1}{m}\coth 1+y\coth(my)$$

$$P_{n,m}\left(e_{1,0}^2+e_{0,1}^2;x,y\right)=\left(x^2+y^2\right)+\left(\frac{1}{n^2}+\frac{1}{m^2}\right)(1+\coth 1)$$
$$+(1+2\coth 1)\left(\frac{x}{n}\coth(nx)+\frac{y}{m}\coth(my)\right)$$

where $e_{i,j}(t_1,t_2)=t_1^i t_2^j$; $i,j\in\{0,1\}$ and $\coth u$ is the hyperbolic function of u.

Lemma 2. *If* $(x,y) \in R_1^2$, $p,q>0$ *and* $n,m \in N$, *then we have*

$$P_{n,m}\left(e^{pt_1};x,y\right) = \frac{1}{\sinh 1 \sinh(nx)} \sinh\left(e^{p/n}\right) \sinh\left(nxe^{p/n}\right)$$

$$P_{n,m}\left(e^{qt_2};x,y\right) = \frac{1}{\sinh 1 \sinh(my)} \sinh\left(e^{q/m}\right) \sinh\left(mye^{q/m}\right)$$

$$P_{n,m}\left(t_1 e^{pt_1};x,y\right) = \frac{e^{p/n}}{n} \frac{1}{\sinh 1 \sinh(nx)} \left\{\cosh\left(e^{p/n}\right) \sinh\left(nxe^{p/n}\right) + nx \sinh\left(e^{p/n}\right) \cosh\left(nxe^{p/n}\right)\right\}$$

$$P_{n,m}\left(t_2 e^{qt_2};x,y\right) = \frac{e^{q/m}}{m} \frac{1}{\sinh 1 \sinh(my)} \left\{\cosh\left(e^{q/m}\right) \sinh\left(mye^{q/m}\right) + my \sinh\left(e^{q/m}\right) \cosh\left(mye^{q/m}\right)\right\}$$

$$P_{n,m}\left(t_1^2 e^{pt_1};x,y\right) = \frac{1}{\sinh 1 \sinh(nx)} \left\{ \frac{e^{2p/n}}{n^2} \sinh\left(e^{p/n}\right) \sinh\left(nxe^{p/n}\right) + \frac{2x}{n} e^{2p/n} \cosh\left(e^{p/n}\right) \cosh\left(nxe^{p/n}\right) + x^2 e^{2p/n} \sinh\left(e^{p/n}\right) \sinh\left(nxe^{p/n}\right) + \frac{1}{n^2} e^{p/n} \cosh\left(e^{p/n}\right) \sinh(nxe^{p/n}) + \frac{x}{n} e^{p/n} \sinh\left(e^{p/n}\right) \cosh\left(nxe^{p/n}\right)\right\}$$

$$P_{n,m}\left(t_2^2 e^{qt_2};x,y\right) = \frac{1}{\sinh 1 \sinh(my)} \left\{ \frac{e^{2q/m}}{m^2} \sinh\left(e^{q/m}\right) \sinh\left(mye^{q/m}\right) + \frac{2y}{m} e^{2q/m} \cosh\left(e^{q/m}\right) \cosh\left(mye^{q/m}\right) + y^2 e^{2q/m} \sinh\left(e^{q/m}\right) \sinh\left(mye^{q/m}\right) + \frac{1}{m^2} e^{q/m} \cosh\left(e^{q/m}\right) \sinh(mye^{q/m}) + \frac{y}{m} e^{q/m} \sinh\left(e^{q/m}\right) \cosh\left(mye^{q/m}\right)\right\}.$$

Lemma 3. *For all* $(x,y) \in R_1^2$ *and* $n,m \in N$, *we have*

$$\begin{aligned} &P_{n,m}\left((t_1-x)^2 e^{pt_1};x,y\right) \\ &= \frac{1}{\sinh 1 \sinh nx} \left\{ x^2 \sinh\left(e^{p/n}\right) \sinh\left(nxe^{p/n}\right) \left[e^{p/n}-1\right]^2 \right. \\ &\quad + \sinh\left(nxe^{p/n}\right) \left[\frac{e^{2p/n}}{n^2} \sinh\left(e^{p/n}\right) + \frac{e^{p/n}}{n^2} \cosh\left(e^{p/n}\right) \right. \\ &\quad \left. \left. - \frac{2x}{n} e^{p/n} \cosh\left(e^{p/n}\right) \right] + \cosh\left(nxe^{p/n}\right) \left[\frac{2x}{n} e^{2p/n} \cosh\left(e^{p/n}\right) \right. \right. \end{aligned}$$

$$+\frac{x}{n}e^{p/n}\sinh\left(e^{p/n}\right)\Big]-2x^2e^{p/n}\sinh\left(e^{p/n}\right)e^{-nxe^{p/n}}\Big\}$$

$$P_{n,m}\left((t_2-y)^2e^{qt_2};x,y\right)$$
$$=\frac{1}{\sinh 1\sinh my}\left\{y^2\sinh\left(e^{q/m}\right)\sinh\left(mye^{q/m}\right)\left[e^{q/m}-1\right]^2\right.$$
$$+\sinh\left(mye^{q/m}\right)\left[\frac{e^{2q/m}}{m^2}\sinh\left(e^{q/m}\right)+\frac{e^{q/m}}{m^2}\cosh\left(e^{q/m}\right)\right.$$
$$-\frac{2y}{m}e^{q/m}\cosh\left(e^{q/m}\right)\Big]+\cosh\left(mye^{q/m}\right)\left[\frac{2y}{m}e^{2q/m}\cosh\left(e^{q/m}\right)\right.$$
$$\left.+\frac{y}{m}e^{q/m}\sinh\left(e^{q/m}\right)\Big]-2y^2e^{q/m}\sinh\left(e^{q/m}\right)e^{-mye^{q/m}}\right\}.$$

Lemma 4. *For all* $(x,y)\in R_1^2$ *and* $n,m\in N$, *we have*

$$P_{n,m}\left((t_1-x)^2;x,y\right)\le\frac{3(x+1)}{n}$$
$$P_{n,m}\left((t_2-y)^2;x,y\right)\le\frac{3(y+1)}{m}.$$

Proof. By Lemma 1, we get

$$P_{n,m}\left((t_1-x)^2;x,y\right)=(\coth(nx)-1)\,2x\left(\frac{1}{n}\coth 1-x\right)+\frac{x}{n}\coth nx$$
$$+\frac{1}{n^2}(1+\coth 1).$$

Thus for $(x,y)\in R_1^2$, we can write

$$P_{n,m}\left((t_1-x)^2;x,y\right)\le\frac{x-1}{n}+\frac{2x}{n}+\frac{3}{n^2}\le\frac{3(x+1)}{n}.$$

Similarly, we can easily obtain

$$P_{n,m}\left((t_2-y)^2;x,y\right)\le\frac{3(y+1)}{m}.$$

□

Lemma 5. *Let* $p,q>0$, $r>p$, $s>q$ *and let* $n_0=n_0(p,r)$, $m_0=m_0(q,s)$ *be fixed natural numbers such that* $n_0>p/(\ln r-\ln p)$ *and* $m_0>q/(\ln s-\ln q)$. *Then there exist positive constants* $C_{p,r}$ *and* $C_{q,s}$ *depending only on* p,r *and* q,s *such that*

$$w_r(x)\,P_{n,m}\left((t_1-x)^2e^{pt_1};x,y\right)\le C_{p,r}\frac{\sinh\left(e^{p/n}\right)}{\sinh 1}\frac{x+2}{n}$$

$$w_s(y)\, P_{n,m}\left((t_2-y)^2 e^{qt_2};x,y\right) \le C_{q,s}\frac{\sinh\left(e^{q/m}\right)}{\sinh 1}\frac{y+2}{m}$$

for all $(x,y)\in R_1^2$ *and* $n\ge n_0, m\ge m_0$.

Proof. Firstly, for $m,n\in\mathbb{N}$, we consider the sequence of real numbers (p_n) and (q_m),

$$p_n = n\left(e^{p/n}+1\right) \tag{2.1}$$

$$q_m = m\left(e^{q/m}+1\right) \tag{2.2}$$

which are decreasing and $\lim_{n\to\infty} p_n = p$, $\lim_{m\to\infty} q_m = q$. Thus

$$p < p_n < pe^{p/n} \le pe^{p} \tag{2.3}$$

$$q < q_m < qe^{q/m} \le qe^{q} \tag{2.4}$$

Since $n_0 > p/(\ln r - \ln p)$, we have $e^{p/n_0} < e^{\ln(r/p)} = r/p$ and $r > pe^{p/n_0} > p_{n_0} > p_n$ for $n\ge n_0$. Also, because $m_0 > q/(\ln s - \ln q)$, we get $e^{q/m_0} < e^{\ln(s/q)} = s/q$ and $s > qe^{q/m_0} > q_{m_0} > q_m$ for $m\ge m_0$.

Applying 2.1, we obtain

$$\begin{aligned}\sinh\left(nxe^{p/n}\right)(\sinh nx)^{-1} &\le 2e^{p_n x}\\ \cosh\left(nxe^{p/n}\right)(\sinh nx)^{-1} &\le e^{p_n x}\\ x^2(\sinh nx)^{-1} &\le \frac{x}{n}.\end{aligned} \tag{2.5}$$

Also using 2.2, we get

$$\begin{aligned}\sinh\left(mye^{q/m}\right)(\sinh my)^{-1} &\le 2e^{q_m y}\\ \cosh\left(mye^{q/m}\right)(\sinh my)^{-1} &\le e^{q_m y}\\ y^2(\sinh my)^{-1} &\le \frac{y}{m}.\end{aligned} \tag{2.6}$$

By writing the last inequalities in Lemma 2, we get respectively

$$P_{n,m}\left(e^{pt_1};x,y\right) \le \frac{\sinh\left(e^{p/n}\right)}{\sinh 1}2e^{p_n x}$$

$$P_{n,m}\left(e^{qt_2};x,y\right) \le \frac{\sinh\left(e^{q/m}\right)}{\sinh 1}2e^{q_m y}$$

and

$$\left\| P_{n,m}\left(e^{pt_1};x,y\right)\right\|_r \leq \sup \frac{\sinh\left(e^{p/n}\right)}{\sinh 1} 2e^{(p_n-r)x}$$

$$\left\| P_{n,m}\left(e^{qt_2};x,y\right)\right\|_s \leq \sup \frac{\sinh\left(e^{q/m}\right)}{\sinh 1} 2e^{(q_m-s)y}.$$

Taking into account Lemma 3 and 2.5, we obtain

$$\begin{aligned}
&P_{n,m}\left((t_1-x)^2 e^{pt_1};x,y\right)\\
&\leq \frac{1}{\sinh 1}\left\{2x^2 e^{p_n x}\frac{p_n^2}{n^2}\sinh\left(e^{p/n}\right)+\frac{2x}{n}\sinh\left(e^{p/n}\right)e^{\frac{p}{n}-nxe^{p/n}}\right.\\
&\quad+\frac{2}{n^2}e^{p_n x}e^{2p/n}\sinh\left(e^{p/n}\right)+\frac{2}{n^2}e^{p_n x}e^{p/n}\cosh\left(e^{p/n}\right)\\
&\quad-\frac{4x}{n}e^{p_n x}e^{p/n}\cosh\left(e^{p/n}\right)+\frac{2x}{n}e^{p_n x}e^{2p/n}\cosh\left(e^{p/n}\right)\\
&\quad\left.+\frac{x}{n}e^{p_n x}e^{p/n}\sinh\left(e^{p/n}\right)\right\}\\
&\leq \frac{\sinh\left(e^{p/n}\right)}{\sinh 1}\left\{2x^2\frac{p_n^2}{n^2}e^{p_n x}+\frac{2x}{n}e^{p/n}+\frac{2}{n^2}e^{p_n x}e^{2p/n}\right.\\
&\quad+\frac{2}{n^2}e^{p_n x}e^{p/n}\coth\left(e^{p/n}\right)+\frac{2x}{n}e^{p_n x}e^{2p/n}\coth\left(e^{p/n}\right)\\
&\quad\left.+\frac{x}{n}e^{p_n x}e^{p/n}\right\}
\end{aligned}$$

Since $\coth\left(e^{p/n}\right)\leq\coth 1<2$ and $e^{p/n}<e^p$, we can write

$$\begin{aligned}
P_{n,m}\left((t_1-x)^2 e^{pt_1};x,y\right)\leq\frac{\sinh\left(e^{p/n}\right)}{\sinh 1}&\left\{\frac{2x^2}{n^2}p_n^2 e^{p_n x}+\frac{2x}{n}e^p+\frac{2}{n^2}e^{p_n x}e^{2p}\right.\\
&\left.+\frac{4}{n^2}e^{p_n x}e^p+\frac{4x}{n}e^{p_n x}e^{2p}+\frac{x}{n}e^{p_n x}e^p\right\}.
\end{aligned}$$

Say $w_r(x)=e^{-rx}$. Thus, we get

$$\begin{aligned}
w_r(x)\,P_{n,m}\left((t_1-x)^2 e^{pt_1};x,y\right)\leq\frac{\sinh\left(e^{p/n}\right)}{\sinh 1}&\left\{\frac{2x}{n}e^{(p_n-r)x}\left(\frac{x}{n}p_n^2+\frac{e^p}{2}+2e^{2p}\right)\right.\\
&\left.+\frac{2x}{n}e^{p-rx}+\frac{2}{n^2}e^{(p_n-r)x}\left(e^{2p}+2e^p\right)\right\}.
\end{aligned}$$

Now, by using 2.1 and inequalities

$$\frac{x}{n}p_n^2 < \frac{x}{n}p^2e^{2p/n} < xp^2e^{2p},$$

it follows

$$\begin{aligned}
&w_r(x)\,P_{n,m}\left((t_1-x)^2e^{pt_1};x,y\right)\\
&\leq \frac{\sinh\left(e^{p/n}\right)}{\sinh 1}\left\{\frac{2x}{n}e^{(p_n-r)x}\left(xp^2e^{2p}+\frac{e^p}{2}+2e^{2p}\right)\right.\\
&\left.+\frac{2x}{n}e^p+\frac{2}{n^2}e^{(p_n-r)x}\left(e^{2p}+2e^p\right)\right\}.
\end{aligned}$$

Also, we have $r-p_n \geq r-p_{n_0} > 0$ and $xe^{-(r-p_n)x} \leq xe^{-(r-p_{n_0})x} \leq 1/(r-p_{n_0})$ for $n \geq n_0$. Applying $e^{(p_n-r)x} < 1$, we obtain

$$\begin{aligned}
&w_r(x)\,P_{n,m}\left((t_1-x)^2e^{pt_1};x,y\right)\\
&\leq \frac{\sinh\left(e^{p/n}\right)}{\sinh 1}\left\{\frac{2}{n}\frac{1}{r-p_{n_0}}\left(xp^2e^{2p}+2e^{2p}+\frac{e^p}{2}\right)\right.\\
&\left.+\frac{2x}{n}e^p+\frac{2}{n^2}\left(e^{2p}+2e^p\right)\right\}\\
&\leq \frac{\sinh\left(e^{p/n}\right)}{\sinh 1}\left\{\frac{2x}{n}\frac{e^{2p}}{r-p_{n_0}}p^2+\frac{2}{n}\frac{1}{r-p_{n_0}}\left(2e^{2p}+\frac{e^p}{2}\right)\right.\\
&\left.+\frac{2x}{n}e^p+\frac{2}{n^2}\left(e^{2p}+2e^p\right)\right\}\\
&\leq C_{p,r}\frac{\sinh\left(e^{p/n}\right)}{\sinh 1}\frac{x+2}{n}.
\end{aligned}$$

Similarly as above, applying 2.6 to Lemma 3, by simple calculations we easily obtain the required inequality

$$w_s(y)\,P_{n,m}\left((t_2-y)^2e^{qt_2};x,y\right) \leq C_{q,s}\frac{\sinh\left(e^{q/m}\right)}{\sinh 1}\frac{y+2}{m}.$$

□

Lemma 6. *If $p,q>0$, $r>p$, $s>q$ and $n_0=n_0(p,r)$, $m_0=m_0(q,s)$ be fixed natural numbers such that $n_0 > p/(\ln r-\ln p)$, $m_0 > q/(\ln s-\ln q)$ and if $f \in C_{p,q}$, then we have*

$$\|P_{n,m}(f;x,y)\|_{r,s} \leq 2\|f\|_{p,q}\frac{\sinh(e^p)\sinh(e^q)}{(\sinh 1)^2}.$$

Proof. By (1), we can write

$$e^{-rx}e^{-sy}\left|P_{n,m}(f;x,y)\right|$$
$$=\frac{e^{-rx}e^{-sy}}{(\sinh 1)^2\sinh(nx)\sinh(my)}\left|\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}p_{2k}(nx)\,p_{2j}(my)\,e^{-\frac{2kp}{n}}e^{-\frac{2jq}{m}}\right.$$
$$\left.\cdot f\left(\frac{2k}{n},\frac{2j}{m}\right)e^{\frac{2kp}{n}}e^{\frac{2jq}{m}}\right|.$$

Since $\|f\|_{p,q}=\sup_{(x,y)\in R_1^2}e^{-px}e^{-qy}\left|f(x,y)\right|$, it follows that

$$e^{-rx}e^{-sy}\left|P_{n,m}(f;x,y)\right|$$
$$\leq\frac{e^{-rx}e^{-sy}}{(\sinh 1)^2\sinh(nx)\sinh(my)}\|f\|_{p,q}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}p_{2k}(nx)\,p_{2j}(my)\,e^{\frac{2kp}{n}}e^{\frac{2jq}{m}}$$
$$=e^{-rx}e^{-sy}\|f\|_{p,q}\,P_{n,m}\left(e^{pt_1};x,y\right)P_{n,m}\left(e^{qt_2};x,y\right)$$
$$=e^{-rx}e^{-sy}\|f\|_{p,q}\frac{\sinh\left(e^{p/n}\right)\sinh\left(nxe^{p/n}\right)}{\sinh 1\sinh(nx)}\frac{\sinh\left(e^{q/m}\right)\sinh\left(mye^{q/m}\right)}{\sinh 1\sinh(my)}.$$

Using the notations in Lemma 5, the inequalities (2.5) and (2.6), we obtain

$$e^{-rx}e^{-sy}\left|P_{n,m}(f;x,y)\right|\leq\|f\|_{p,q}\,e^{-rx}e^{-sy}\frac{\sinh\left(e^{p/n}\right)}{\sinh 1}2e^{p_nx}\frac{\sinh\left(e^{q/m}\right)}{\sinh 1}2e^{q_my}$$
$$=4\|f\|_{p,q}\,e^{-x(r-p_n)}e^{-y(s-q_m)}\frac{\sinh\left(e^{p/n}\right)\sinh\left(e^{q/m}\right)}{(\sinh 1)^2}$$
$$\leq 4\|f\|_{p,q}\frac{\sinh(e^p)\sinh(e^q)}{(\sinh 1)^2}.$$

□

3. Approximation by $\mathbf{P}_{n,m}$ Operators

In this section, we give theorems on the degree of approximation of functions of two variables by these operators.

Theorem 1. *Let $p,q>0$, $r>p$, $s>q$ and $n_0=n_0(p,r)$, $m_0=m_0(q,s)$ be fixed natural numbers such that $n_0>p/(\ln r-\ln p)$ and $m_0>q/(\ln s-\ln q)$. If $f\in C^1_{p,q}$, where $C^1_{p,q}=\{f\in C_{p,q}:f_x,f_y\in C_{p,q}\}$, then there exists a positive constant $M_{p,q,r,s}$ depending only on p,q,r,s such that*

$$w_{r,s}(x,y)\left|P_{n,m}(f;x,y)-f(x,y)\right|$$
$$\leq M_{p,q,r,s}\left\{\left\|\frac{\partial f}{\partial x}\right\|_{p,q}\sqrt{\frac{x+2}{n}}+\left\|\frac{\partial f}{\partial y}\right\|_{p,q}\sqrt{\frac{y+2}{m}}\right\}$$

Proof. Let (x,y) be a fixed point in R_1^2. For $f\in C_{p,q}^1$ and $(t_1,t_2)\in R_1^2$ we have

$$f(t_1,t_2)-f(x,y)=\int_x^{t_1}\frac{\partial f}{\partial u}(u,t_2)\,du+\int_y^{t_2}\frac{\partial f}{\partial v}(x,v)\,dv.$$

Using $P_{n,m}(1;x,y)=1$, it results that

$$P_{n,m}(f(t_1,t_2);x,y)-f(x,y)$$
$$=P_{n,m}\left(\int_x^{t_1}\frac{\partial f}{\partial u}(u,t_2)\,du;x,y\right)+P_{n,m}\left(\int_y^{t_2}\frac{\partial f}{\partial v}(x,v)\,dv;x,y\right).$$

For $r>p, s>q$ and $m,n\in\mathbb{N}$, we have

$$w_{r,s}(x,y)\left|P_{n,m}(f(t_1,t_2);x,y)-f(x,y)\right|$$
$$\leq w_{r,s}(x,y)\,P_{n,m}\left(\left|\int_x^{t_1}\frac{\partial f}{\partial u}(u,t_2)\,du\right|;x,y\right)$$
$$+w_{r,s}(x,y)\,P_{n,m}\left(\left|\int_y^{t_2}\frac{\partial f}{\partial v}(x,v)\,dv\right|;x,y\right).$$

By using the following inequalities

$$\left|\int_x^{t_1}\frac{\partial f}{\partial u}(u,t_2)\,du\right|\leq\left\|\frac{\partial f}{\partial x}\right\|_{p,q}\left|\int_x^{t_1}\frac{1}{w_{p,q}(u,t_2)}du\right|$$
$$\leq\left\|\frac{\partial f}{\partial x}\right\|_{p,q}\frac{1}{w_q(t_2)}\left(\frac{1}{w_p(t_1)}+\frac{1}{w_p(x)}\right)|t_1-x|,$$

$$\left|\int_y^{t_2}\frac{\partial f}{\partial v}(x,v)\,dv\right|\leq\left\|\frac{\partial f}{\partial y}\right\|_{p,q}\left|\int_y^{t_2}\frac{1}{w_{p,q}(x,v)}dv\right|$$
$$\leq\left\|\frac{\partial f}{\partial y}\right\|_{p,q}\frac{1}{w_p(x)}\left(\frac{1}{w_q(t_2)}+\frac{1}{w_q(y)}\right)|t_2-y|$$

and Hölder inequality, we can write

$$
\begin{aligned}
& w_{r,s}(x,y)\, P_{n,m}\left(\left|\int_x^{t_1} \frac{\partial f}{\partial u}(u,t_2)\, du\right|; x, y\right) \\
& \leq \left\|\frac{\partial f}{\partial x}\right\|_{p,q} w_{r,s}(x,y)\, P_{n,m}\left(\frac{1}{w_q(t_2)}; x, y\right)\left\{P_{n,m}\left(\frac{|t_1-x|}{w_p(t_1)}; x, y\right)\right. \\
& \left. + \frac{1}{w_p(x)} P_{n,m}(|t_1-x|; x, y)\right\} \\
& \leq \left\|\frac{\partial f}{\partial x}\right\|_{p,q} \left\{\left[w_r(x) P_{n,m}\left((t_1-x)^2 e^{pt_1}; x, y\right)\right]^{1/2}\left[w_r(x) P_{n,m}\left(e^{pt_1}; x, y\right)\right]^{1/2}\right. \\
& \quad \left. + e^{(p-r)x}\left[P_{n,m}\left((t_1-x)^2; x, y\right)\right]^{1/2}\right\} w_s(y) P_{n,m}\left(e^{qt_2}; x, y\right),
\end{aligned}
$$

$$
\begin{aligned}
& w_{r,s}(x,y)\, P_{n,m}\left(\left|\int_y^{t_2} \frac{\partial f}{\partial v}(x,v)\, dv\right|; x, y\right) \\
& \leq \left\|\frac{\partial f}{\partial y}\right\|_{p,q} \frac{w_r(x)}{w_p(x)} P_{n,m}(1; x, y)\, w_s(y)\left\{P_{n,m}\left(\frac{|t_2-y|}{w_q(t_2)}; x, y\right)\right. \\
& \left. + \frac{1}{w_q(y)} P_{n,m}(|t_2-y|; x, y)\right\} \\
& \leq \left\|\frac{\partial f}{\partial y}\right\|_{p,q} \left\{\left[w_s(y)\, P_{n,m}\left((t_2-y)^2 e^{qt_2}; x, y\right)\right]^{1/2}\right. \\
& \left. \cdot\left[w_s(y)\, P_{n,m}\left(e^{qt_2}; x, y\right)\right]^{1/2} + \left[P_{n,m}\left((t_2-y)^2; x, y\right)\right]^{1/2}\right\}.
\end{aligned}
$$

Applying the inequalities 2.5 and 2.6 to Lemma 2, we obtain

$$
\begin{aligned}
w_r(x) P_{n,m}\left(e^{pt_1}; x, y\right) &\leq e^{-rx} \frac{\sinh\left(e^{p/n}\right)}{\sinh 1} 2 e^{p_n x} \\
&= 2\frac{\sinh\left(e^{p/n}\right)}{\sinh 1} e^{-x(r-p_n)} \\
&\leq 2\frac{\sinh\left(e^{p/n}\right)}{\sinh 1}
\end{aligned}
$$

and

$$
w_s(y) P_{n,m}\left(e^{qt_2}; x, y\right) \leq e^{-sy} \frac{\sinh\left(e^{q/m}\right)}{\sinh 1} 2 e^{q_m y}
$$

$$= 2\frac{\sinh\left(e^{q/m}\right)}{\sinh 1}e^{-y(s-q_m)}$$
$$\leq 2\frac{\sinh\left(e^{q/m}\right)}{\sinh 1}.$$

By these inequalities, Lemma 5 and Lemma 4, we get to

$$w_{r,s}(x,y)\,P_{n,m}\left(\left|\int_x^{t_1}\frac{\partial f}{\partial u}(u,t_2)\,du\right|;x,y\right)$$
$$\leq 2\left\|\frac{\partial f}{\partial x}\right\|_{p,q}\frac{\sinh\left(e^{p/n}\right)\sinh\left(e^{q/m}\right)}{(\sinh 1)^2}\sqrt{2C_{p,r}\frac{x+2}{n}}$$
$$+2\left\|\frac{\partial f}{\partial x}\right\|_{p,q}\frac{\sinh\left(e^{q/m}\right)}{\sinh 1}e^{-x(r-p)}\sqrt{\frac{3(x+1)}{n}}$$
$$\leq \left\|\frac{\partial f}{\partial x}\right\|_{p,q}M_{p,r}\sqrt{\frac{x+2}{n}},$$

and

$$w_{r,s}(x,y)\,P_{n,m}\left(\left|\int_y^{t_2}\frac{\partial f}{\partial v}(x,v)\,dv\right|;x,y\right)$$
$$\leq \left\|\frac{\partial f}{\partial y}\right\|_{p,q}\frac{\sinh\left(e^{q/m}\right)}{\sinh 1}\sqrt{2C_{q,s}\frac{y+2}{m}}+\left\|\frac{\partial f}{\partial y}\right\|_{p,q}\sqrt{\frac{3(y+1)}{m}}$$
$$\leq \left\|\frac{\partial f}{\partial y}\right\|_{p,q}M_{q,s}\sqrt{\frac{y+2}{m}}$$

for all $m \geq m_0$ and $n \geq n_0$. This proves the theorem. □

Theorem 2. *Suppose that $f \in C_{p,q}$ and p,r,q,s,n_0,m_0 satisfy the conditions of Theorem 1. Then there exists positive constant $M^* = M_{p,q,r,s}$ depending only on p,q,r,s such that*

$$w_{r,s}(x,y)\,|P_{n,m}(f;x,y)-f(x,y)| \leq M^*\omega\left(f,C_{p,q};\left(\frac{x+2}{n}\right)^{1/2},\left(\frac{y+2}{m}\right)^{1/2}\right)$$

for all $(x,y) \in R_1^2$ and $m \geq m_0$, $n \geq n_0$.

Proof. Similarly as in [5], we consider the Steklov means for $f \in C_{p,q}$

$$f_{h,\delta}(x,y) = \frac{1}{h\delta}\int_0^h\int_0^\delta f(x+u,y+v)\,du\,dv \quad , \quad h,\delta>0,(x,y)\in R_1^2.$$

We have

$$f_{h,\delta}(x,y) - f(x,y) = \frac{1}{h\delta}\int_0^h\int_0^\delta \Delta_{u,v} f(x,y)\,du\,dv,$$

$$\frac{\partial f_{h,\delta}}{\partial x}(x,y) = \frac{1}{h\delta}\int_0^\delta (f(x+h,y+v)-f(x,y+v))\,dv,$$

$$\frac{\partial f_{h,\delta}}{\partial y}(x,y) = \frac{1}{h\delta}\int_0^h (f(x+u,y+\delta)-f(x+u,y))\,du,$$

which implies $f_{h,\delta}\in C_{p,q}^1$ $(h,\delta>0)$ and

$$\|f_{h,\delta}-f\|_{p,q} \le w\left(f,C_{p,q};h,\delta\right),$$

$$\left\|\frac{\partial f_{h,\delta}}{\partial x}\right\|_{p,q} \le \sup_{(x,y)\in R_1^2} w_{p,q}(x,y)\frac{1}{h\delta}\int_0^\delta \left(\left|\Delta_{h,v} f(x,y)\right| + \left|\Delta_{0,v} f(x,y)\right|\right)dv$$

$$\le \frac{2}{h} w\left(f,C_{p,q};h,\delta\right)$$

and

$$\left\|\frac{\partial f_{h,\delta}}{\partial y}\right\|_{p,q} \le \frac{2}{\delta} w\left(f,C_{p,q};h,\delta\right)$$

for $h,\delta>0$.

For every fixed $(x,y)\in R_1^2$, $r>p$, $s>q$ and $n,m\in\mathbb{N}$, $h,\delta>0$ we have

$$\begin{aligned} &w_{r,s}(x,y)\left|P_{n,m}(f;x,y)-f(x,y)\right| \\ &\le w_{r,s}(x,y)\left\{\left|P_{n,m}\left(f-f_{h,\delta};x,y\right)\right|\right. \\ &\quad \left. + \left|P_{n,m}\left(f_{h,\delta};x,y\right)-f_{h,\delta}(x,y)\right| + \left|f_{h,\delta}(x,y)-f(x,y)\right|\right\}. \end{aligned}$$

By Lemma 5, one obtains

$$w_{r,s}(x,y)\left|P_{n,m}\left(f-f_{h,\delta};x,y\right)\right| \le 4w\left(f,C_{p,q};h,\delta\right)$$

for all $m\ge m_0$ and $n\ge n_0$. Since Theorem 1, we can write

$$w_{r,s}(x,y)\left|P_{n,m}\left(f_{h,\delta};x,y\right)-f_{h,\delta}(x,y)\right|$$

$$\leq M_{p,q,r,s} w\left(f, C_{p,q}; h, \delta\right)\left\{\frac{1}{h}\sqrt{\frac{x+2}{n}}+\frac{1}{\delta}\sqrt{\frac{y+2}{m}}\right\}$$

for $m \geq m_0$, $n \geq n_0$. Therefore

$$\begin{aligned}
&w_{r,s}(x,y)\left|P_{n,m}(f;x,y)-f(x,y)\right| \\
&\leq 4w\left(f, C_{p,q}; h, \delta\right)+M_{p,q,r,s} w\left(f, C_{p,q}; h, \delta\right)\left\{\frac{1}{h}\sqrt{\frac{x+2}{n}}+\frac{1}{\delta}\sqrt{\frac{y+2}{m}}\right\} \\
&\quad + w_{r,s}(x,y)\left|f_{h,\delta}(x,y)-f(x,y)\right| \\
&\leq w\left(f, C_{p,q}; h, \delta\right)\left(5+M_{p,q,r,s}\left\{\frac{1}{h}\sqrt{\frac{x+2}{n}}+\frac{1}{\delta}\sqrt{\frac{y+2}{m}}\right\}\right)
\end{aligned}$$

for all $h, \delta > 0$ and $m \geq m_0$, $n \geq n_0$. Setting $h = \sqrt{\frac{x+2}{n}}$, $\delta = \sqrt{\frac{y+2}{m}}$ we obtain the desired result. □

Corollary 1. *If $f \in C_{p,q}$, then for all $(x,y) \in R_1^2$*

$$\lim_{m,n\to\infty} P_{n,m}(f;x,y) = f(x,y).$$

Also, the convergence is uniform on every rectangle $1 \leq x \leq a$, $1 \leq y \leq b$.

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Authors' addresses

Nursel Çetin
Ankara University, Faculty of Science, Department of Mathematics, Tandogan 06100 Ankara, Turkey
E-mail address: ncetin@ankara.edu.tr

Sevilay Kirci Serenbay
Başkent University, Department of Mathematics Education, 06530 Ankara, Turkey
E-mail address: kirci@baskent.edu.tr

Çiğdem Atakut
Ankara University, Faculty of Science, Department of Mathematics, Tandogan 06100 Ankara, Turkey
E-mail address: atakut@science.ankara.edu.tr

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VARIETY OF ORTHOMODULAR POSETS

IVAN CHAJDA AND MIROSLAV KOLAŘÍK

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Abstract. Orthomodular posets play an important role in the so-called logical structure of a physical system as formerly pointed out by numerous authors. In particular, they play an essential role in the logic of quantum mechanics. To avoid usual problems with partial algebras, we define the so-called orthomodular directoid as an everywhere defined algebra and we show that every orthomodular poset can be converted into an orthomodular directoid and vice versa. Since orthomodular directoids are defined equationally, they form a variety having nice congruence properties.

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1. INTRODUCTION

By a **logical structure of a physical system** (see [1, 2, 14] or [13]) is meant a couple $(L;F)$, where L is a nonvoid set and F is a set of functions from L into the interval $[0,1]$ of real numbers satisfying the following axioms:

(I) If $p,q \in L$ and $f(p) = f(q)$ for every $f \in F$ then $p = q$.
(II) There exists an element $u \in L$ such that $f(u) = 1$ for each $f \in F$.
(III) For each $p \in L$, there exists an element $p' \in L$ such that $f(p) + f(p') = 1$ for every $f \in F$.

Let $\leq$ be the relation defined on L by

$$p \leq q \quad \text{if and only if} \quad f(p) \leq f(q) \text{ for every } f \in F.$$

Then $\leq$ is a partial order on L with the least element u' and the greatest element u. We say that $p,q \in L$ are **orthogonal** if $p \leq q'$ (which is equivalent to $q \leq p'$, see [1] for details).

We add one more axiom:

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(IV) For every orthogonal elements $p,q \in L$ there exists supremum $s = \sup(p,q)$ and $f(s) = f(p) + f(q)$ for each $f \in F$.

It is well-known that the system $(L;\leq,',0,1)$ is an orthomodular poset, the so-called **associated poset with the logical structure** $(L;F)$, see e.g. [1]. Hence, orthomodular posets serve as an axiomatic description of physical systems, see e.g. [5, 7]. If $\sup(p,q)$ exists for each couple p,q of elements of L, then $(L;\leq,',0,1)$ becomes an orthomodular lattice. Hence, the theory of orthomodular posets includes the theory of orthomodular lattices and, simultaneously, serves as an axiomatization of the logic of physical systems. In particular, it axiomatizes the logic of quantum mechanics, see [2, 7, 10, 14] and [13].

Due to the above mentioned properties, orthomodular posets were and are studied by numerous authors for several decades see e.g. [5, 7, 8, 12, 13]. However, up to now, orthomodular posets were treated as partial algebras where the binary operation of supremum is ensured only for orthogonal or comparable elements. In this paper, we try another approach, namely to introduce a certain everywhere defined algebra which can be assigned to every orthomodular poset in the way that the underlying poset coincides with the original one but its axioms can be expressed as identities. Hence, the class of these so-called orthomodular directoids forms a variety of algebras having nice algebraic properties. Moreover, every orthomodular poset can be recovered by means of this assigned algebra despite the fact that the assignment need not be done in a unique way.

2. Orthomodular directoids

Recall by [9] (see also [4]) that a groupoid $(A;+)$ is called a **commutative directoid** if it satisfies the following axioms:

$$x + x = x$$
$$x + y = y + x$$
$$x + ((x + y) + z) = (x + y) + z.$$

In what follows, we enrich the commutative directoid by a unary operation (orthocomplementation) and by two constants to get an algebra for our study. Since we need to ask for two more properties connected with orthomodular posets (namely the orthomodular law and the existence of suprema for orthogonal elements), we add two more axioms which caused that some other axioms for orthomodular directoids can follow from the remaining ones. Hence, we can define:

Definition 1. By an **orthomodular directoid** is called an algebra $\mathcal{D} = (D;+,',0,1)$ of type $(2,1,0,0)$ satisfying the following axioms:

(D1) $x + y = y + x$
(D2) $x + ((x + y) + z) = (x + y) + z$
(D3) $x + 0 = x$
(D4) $x + x' = 1$

(D5) $(((x+z)+(y+z)')'+(y+z)')+z'=z'$
(D6) $x+(x+(x+y)')'=x+y$.

Theorem 1. *The axioms (D1)–(D6) are independent.*

Proof. (a) Define $0'=1$, $a'=1$, $1'=0$, $0+a=1$, $a+a=a$, $1+a=1$ and $x+0=x$, $y+1=1$ for each $x,y\in\{0,a,1\}$. Then $(\{0,a,1\};+,',0,1)$ satisfies (D2)–(D6), but not (D1) because $a+0=a\neq 1=0+a$.

(b) Define $'$ and $+$ by the following tables

x	0	a	b	c	d	e	f	g	h	i	j	1
x'	1	b	a	d	c	f	e	h	g	j	i	0

$+$	0	a	b	c	d	e	f	g	h	i	j	1
0	0	a	b	c	d	e	f	g	h	i	j	1
a	a	a	1	g	d	1	1	g	d	1	1	1
b	b	1	b	b	1	1	1	1	b	1	1	1
c	c	g	b	c	1	1	1	g	b	1	1	1
d	d	d	1	1	d	1	1	1	d	1	1	1
e	e	1	1	1	1	e	1	i	h	i	h	1
f	f	1	1	1	1	1	f	f	1	1	f	1
g	g	g	1	g	1	i	f	g	1	i	f	1
h	h	d	b	b	d	h	1	1	h	1	h	1
i	i	1	1	1	1	i	1	i	1	i	1	1
j	j	1	1	1	1	h	f	f	h	1	j	1
1	1	1	1	1	1	1	1	1	1	1	1	1

Then $(\{0,a,b,c,d,e,f,g,h,i,j,1\};+,',0,1)$ satisfies (D1), (D3)–(D6), but not (D2) because

$$a+((a+c)+e)=a+i=1\neq i=(a+c)+e.$$

(c) Define $0'=1$, $1'=0$ and the constant operation $x+y=1$ for every $x,y\in\{0,1\}$. Then $(\{0,1\};+,',0,1)$ satisfies (D1), (D2) and (D4)–(D6), but not (D3) because $0+0=1\neq 0$.

(d) Suppose now that $+$ is commutative and $x+0=x=x+x$ for each $x\in\{0,a,b,1\}$. Further let $a+b=a+1=b+1=a$, $0'=a$, $a'=0$, $b'=1$ and $1'=b$. Then $(\{0,a,b,1\};+,',0,1)$ satisfies (D1)–(D3), (D5) and (D6), but not (D4) because $a+a'=a\neq 1$.

(e) Define $+$ and $'$ by the following tables

$+$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	1	c	d	1
b	b	1	b	c	d	1
c	c	c	c	c	1	1
d	d	d	d	1	d	1
1	1	1	1	1	1	1

x	0	a	b	c	d	1
x'	1	b	a	d	c	0

Then $(\{0,a,b,c,d,1\};+,',0,1)$ satisfies (D1)–(D4) and (D6), but not (D5) because

$$(((1+a)+(c+a)')'+(c+a)')+a'$$
$$=((1+c')'+c')+a'=(0+d)+b=d\neq b=a'.$$

(f) Finally, define $0'=1'=1$ and $0+0=0$, $0+1=1+0=1+1=1$. Then $(\{0,1\};+,',0,1)$ satisfies (D1)–(D5), but not (D6) because $0+(0+(0+0)')'=0+1=1\neq 0=0+0$. □

We can derive several more useful identities satisfied by orthomodular directoids.

Lemma 1. *Every orthomodular directoid satisfies the following:*

(a) $x''=x$
(b) $x+1=1$
(c) $x+x=x$
(d) $0'=1$ *and* $1'=0$
(e) $(x'+y)'+x=x$.

Proof. (a) Putting $x=0$ and $y=x$ in (D6) we obtain $0+(0+(0+x)')'=0+x$. Applying (D1) and (D3) four times we get $x''=x$.
(b) Put $y=x'$ and $z=0$ in (D2). We get $x+((x+x')+0)=(x+x')+0$ whence $x+1=1$ using (D3) and (D4).
(c) By (D2) where $y=z=0$ we have $x+((x+0)+0)=(x+0)+0$. Applying (D3) four times we conclude $x+x=x$.
(d) Putting $x=0$ in (D4) we obtain $1=0+0'=0'$ by (D1) and (D3). Hence, using (a), $0=1'$.
(e) Substitute y instead of x and x' instead of z in (D5). Using (a) we get

$$(((y+x')+(y+x')')'+(y+x')')+x=x$$

whence $(1'+(y+x')')+x=x$ by (D4). Applying (d), (D1) and (D3) we conclude $(x'+y)'+x=x$. □

Lemma 2. *Let $\mathcal{D}=(D;+,',0,1)$ be an orthomodular directoid. Define a binary relation $\leq$ on D as follows*

$$x\leq y \quad \textit{if and only if} \quad x+y=y. \tag{$*$}$$

Then $\leq$ is a partial order on D such that:

(a) $0 \le x \le 1$ *for each* $x \in D$
(b) $x \le x + y$, $y \le x + y$
(c) $x \le y$ *implies* $y' \le x'$
(d) *if* $x + y = 0$ *then* $x = y = 0$
(e) *if* $x + (x + y)' = 1$ *then* $y \le x$.

Proof. By (c) of Lemma 1 we see that $\le$ is reflexive. Assume $x \le y$ and $y \le x$. Then $x + y = y$ and $y + x = x$. Due to (D1) we infer $x = y$, i.e. $\le$ is antisymmetrical. If $x \le y$ and $y \le z$ then $x + y = y$, $y + z = z$ and, applying (D2), we have

$$x + z = x + (y + z) = x + ((x + y) + z) = (x + y) + z = y + z = z$$

showing $x \le z$, transitivity of $\le$. Hence, $\le$ is a partial order on D. By (D3) and (D1), $0 \le x$ and, by (b) of Lemma 1, $x \le 1$ for each $x \in D$.

If we put $z = 0$ in (D2), we obtain $x + (x + y) = x + y$ whence $x \le x + y$. Analogously $y \le y + x = x + y$.

Further, assume $x \le y$, i.e. $x + y = y$. Then, by (e) and (a) of Lemma 1, we infer $(x + y)' + x' = x'$ and hence $y' + x' = x'$, proving $y' \le x'$.

The assertion (d) follows from (b).

Finally, assume $x + (x + y)' = 1$. By (D6) we infer $x + y = x + (x + (x + y)')' = x + 1' = x + 0 = x$ thus $y \le x$. □

The partial order defined by $(*)$ will be referred to as the **induced order** of $\mathcal{D} = (D; +, ', 0, 1)$.

Now, we recall the concept of orthomodular poset (from [1]).

Definition 2. By an **orthomodular poset** is meant a structure $\mathcal{P} = (P; \le, ', 0, 1)$, where $\le$ is a partial order on P, $0 \le x \le 1$ for each $x \in P$, $x'' = x$, x' is a complement of x and $x \le y$ implies $y' \le x'$, and satisfying the following two conditions:

(i) if $x \le y'$ then the set $\{x, y\}$ has the supremum $x \vee y$ in $(P; \le)$
(ii) if $x \le y$ then $x \vee (x \vee y')' = y$.

Remark 1. (a) Since $x \le x$ for each $x \in P$, $x \vee x'$ exists and $x \vee x' = 1$.

(b) Since $x \le y$ implies $y' \le x'$, the existence of $x \vee y$ yields the existence of $x' \wedge y' = (x \vee y)'$, the infimum of x', y', by De Morgan laws. In particular, $x \vee x' = 1$ and $x'' = x$, $1' = 0$ get immediately $x' \wedge x = 0$ and hence x' is a complement of x.

(c) If $x \le y$ then, by (i), $x \vee y'$ exists. Since $x \le x \vee y'$, also $x \vee (x \vee y')'$ exists thus (ii) is correctly defined. By using De Morgan laws, (ii) can be read as follows:

$$x \le y \quad \Rightarrow \quad x \vee (x' \wedge y) = y \qquad \text{(OML)}$$

which is the **orthomodular law**. Hence, if $x \vee y$ exists for each $x, y \in P$ then $\mathcal{P} = (P; \le, ', 0, 1)$ is an **orthomodular lattice** (see [1, 11]).

By (i), if x, y are orthogonal then $x \vee y$ exists. Of course, $x \vee y$ exists also for comparable elements since $x \le y$ gets $x \vee y = y$.

If $\mathcal{P} = (P; \leq, ', 0, 1)$ is an orthomodular lattice then the orthomodular law (OML) can be expressed in the form of identity as follows:

$$x \vee (x' \wedge (x \vee y)) = x \vee y.$$

(d) If $x \leq a$ and $y \leq a'$ for some $a \in P$ then $x \vee y$ exists. Namely, $y \leq a'$ yields $a \leq y'$ thus $x \leq a \leq y'$ implies that x, y are orthogonal and, by (i) of Definition 2, $x \vee y$ exists in $(P; \leq)$.

Example 1. See [1]. Let M be a finite set with an even number of elements. Let P be the set of all subsets of M which have even number of elements ordered by inclusion and let $A' = M \setminus A$, the set-theoretical complementation. Then $\mathcal{P} = (P; \subseteq, ', \varnothing, M)$ is an orthomodular poset. If $|M| \geq 6$ then $\mathcal{P}$ is not a lattice.

Now, we are going to show that every orthomodular directoid is an orthomodular poset. For this, let us firstly prove the following lemma.

Lemma 3. *Let $\mathcal{D} = (D; +, ', 0, 1)$ be an orthomodular directoid and $\leq$ its induced order.*

$$\text{If } x \leq y' \text{ then } x + y = x \vee y. \tag{2.1}$$

Proof. In fact, we are going to prove that (2.1) is equivalent to the axiom (D5). At first we show that (2.1) is equivalent with the following condition:

$$(x \leq y' \text{ and } x, y \leq z) \text{ implies } x + y \leq z. \tag{2.2}$$

(2.1) $\Rightarrow$ (2.2): Assume $x \leq y'$ and let $x, y \leq z$. By (b) of Lemma 2 we have $x, y \leq x + y$ thus (2.1) yields (2.2) immediately.
(2.2) $\Rightarrow$ (2.1): Let $x \leq y'$ and $x, y \leq z$. By (2.2), $x + y$ is the least common upper bound of x, y thus $x + y = x \vee y$.

And now, the axiom (D5) is clearly equivalent to the condition

$$((x + z) + (y + z)')' + (y + z)' \leq z'. \tag{2.3}$$

It remains to show that (2.3) is equivalent to (2.2).
(2.2) $\Rightarrow$ (2.3): Assume that (2.2) holds and put $a' = (x + z) + (y + z)'$, $b' = y + z$. Then for $a = ((x + z) + (y + z)')'$ and $b = (y + z)'$ we have $(x + z) + (y + z)' \geq (y + z)'$ thus $a = ((x + z) + (y + z)')' \leq y + z = b'$. Further, $y + z \geq z$ thus $b = (y + z)' \leq z'$ and $(x + z) + (y + z)' \geq x + z \geq z$, whence $a = ((x + z) + (y + z)')' \leq z'$. Applying (2.2) for a, b and z' we conclude $((x + z) + (y + z)')' + (y + z)' \leq z'$ which is (2.3).
(2.3) $\Rightarrow$ (2.2): Assume that (2.3) holds and let $x \leq y'$, $x, y \leq z$. Then $y \leq x'$, i.e. $x' + y = x'$ and $z' \leq x', y'$, i.e. $x' + z' = x'$, $y' + z' = y'$. By (2.3) we compute

$$x + y = x'' + y = (x' + y)' + y = ((x' + z') + (y' + z')')' + (y' + z')' \leq z$$

proving (2.2). □

3. A REPRESENTATION OF ORTHOMODULAR POSETS

Now, we are ready to prove our first main theorem.

Theorem 2. *Let $\mathcal{D} = (D; +, ', 0, 1)$ be an orthomodular directoid and $\leq$ be its induced order. Then $\mathcal{P}(D) = (D; \leq, ', 0, 1)$ is an orthomodular poset where for orthogonal elements $x, y \in D$ we have*

$$x + y = x \vee y.$$

Proof. By Lemma 2, $(D; \leq, 0, 1)$ is a bounded poset where the mapping $x \mapsto x'$ is an antitone involution.

Assume x, y are orthogonal, i.e. $x \leq y'$. By Lemma 3 we have $x + y = x \vee y$.

Since $x \leq x$, the elements x, x' are orthogonal thus, by (D4), $x \vee x' = x + x' = 1$, i.e. x' is a complement of x.

It remains to prove the orthomodular law. Assume $x \leq y$. Then the elements x, y' are orthogonal and hence $x + y' = x \vee y'$ and, similarly, $x + (x \vee y')' = x \vee (x \vee y')'$. Because $x + y = y$, using (D6) we obtain

$$x \vee (x \vee y')' = x + (x + y')' = x + (x + (x + y)')' = x + y = y$$

which is (ii) of Definition 2. □

Moreover, we are able to prove the converse.

Theorem 3. *Let $\mathcal{P} = (P; \leq, ', 0, 1)$ be an orthomodular poset. Define a binary operation $+$ on P as follows:*

- *$x + y = x \vee y$ if $x \vee y$ exists*
- *$x + y = y + x$ is an arbitrary element of $U(x, y) = \{z \in P; x, y \leq z\}$ otherwise.*

Then $\mathcal{D}(P) = (P; +, ', 0, 1)$ is an orthomodular directoid.

Proof. By definition, $+$ satisfies (D1). Since $x + y \in U(x, y)$ in each case, it is elementary to check (D2). Since $0 \leq x$, we have $x = x \vee 0 = x + 0$ proving (D3). Since $x \vee x' = 1$, we obtain (D4). Since $x \leq x + y$, we obtain (D6) by using of the orthomodular law. It remains to prove (D5).

Put $a = ((x + z) + (y + z)')'$, $b = (y + z)'$. Then $a = ((x + z) + (y + z)')' \leq y + z = b'$, i.e. $a + b = a \vee b$. However, $(x + z) + (y + z)' \geq x + z \geq z$ thus $a \leq z'$. Evidently, $b = (y + z)' \leq z'$ and hence $a + b = a \vee b \leq z'$. Thus $(a + b) + z' = z'$ which is (D5). □

By Theorem 3, to every orthomodular poset $\mathcal{P} = (P; \leq, ', 0, 1)$ can be assigned an everywhere defined algebra which is an orthomodular directoid $\mathcal{D}(P) = (P; +, ', 0, 1)$. By Theorem 2, to the orthomodular directoid $\mathcal{D}(P)$ can be assigned an orthomodular poset $\mathcal{P}(\mathcal{D}(P))$. Since the underlying posets $(P; \leq)$ coincide in all $\mathcal{P}$, $\mathcal{D}(P)$ and $\mathcal{P}(\mathcal{D}(P))$ and the complementation is also the same, we conclude that $\mathcal{P} = \mathcal{P}(\mathcal{D}(P))$. Hence, although the directoid $\mathcal{D}(P)$ need not be assigned in a

unique way, it bears all the information on $\mathcal{P}$ because $\mathcal{P} = \mathcal{P}(\mathcal{D}(P))$ for every such a directoid.

On the contrary, if $\mathcal{D} = (D; +, ', 0, 1)$ is an orthomodular directoid, $\mathcal{P}(D)$ the assigned orthomodular poset and $\mathcal{D}(\mathcal{P}(D))$ the assigned orthomodular directoid then $\mathcal{D}$ and $\mathcal{D}(\mathcal{P}(D))$ need not be even isomorphic because the operation $+$ in $\mathcal{D}(\mathcal{P}(D))$ can be choosen differently than that in $\mathcal{D}$.

Theorem 4. *Let $\mathcal{D} = (D; +, ', 0, 1)$ be an orthomodular directoid, $\leq$ its induced order and $a \in D$. Then $([a, 1]; +, ^a, a, 1)$ for $x^a = x' + a$ is an orthomodular directoid.*

Proof. Assume $a \in D$ and $x \in [a, 1]$. Then $a \leq x$ and hence $a \vee x'$ exists and $x^a = x' + a = x' \vee a$. Since $x' \vee a \geq a$, also $a \vee (a \vee x')'$ exists and $x^{aa} = (x' \vee a)' \vee a = a$ by the orthomodular law. Moreover, $x \leq y$ for $x, y \in [a, 1]$ yields $y^a = y' \vee a \leq x' \vee a = x^a$ and $x^a = x' \vee a \geq a$, i.e. $x \mapsto x^a$ is an antitone involution on the interval $[a, 1]$. Further, $x^a = x' \vee a \geq x'$ thus $(x^a)' \vee x'$ exists and, by De Morgan laws, $(x^a)' \vee x' = (x^a \wedge x)' = ((x' \vee a) \wedge x)' = a'$ by the orthomodular law. Therefore $x^a \wedge x = a$ and $x^a \vee x = x^a \vee x^{aa} = (x \wedge x^a)^a = a^a = 1$, thus x^a is an orthocomplement of x in the interval $[a, 1]$. It is plain to check the orthomodular law in the poset $([a, 1]; \leq, ^a, a, 1)$ and hence the induced algebra $([a, 1]; +, ^a, a, 1)$ is an orthomodular directoid. □

4. The variety of orthomodular directoids

By Theorems 2 and 3, orthomodular posets can be represented by everywhere defined algebras, i.e. by orthomodular directoids. However, by Definition 1, these directoids are determined by the identities (D1)–(D6) and hence the class $\mathcal{K}$ of orthomodular directoids forms a variety of algebras. In what follows, we present several important properties of the variety $\mathcal{K}$.

Recall that an algebra $\mathcal{A}$ is **congruence distributive** if its congruence lattice $\mathrm{Con}\mathcal{A}$ is distributive. A variety $\mathcal{V}$ is congruence distributive if each $\mathcal{A} \in \mathcal{V}$ has this property. By a majority term is meant a ternary term $m(x, y, z)$ such that

$$m(x, x, y) = m(x, y, x) = m(y, x, x) = x.$$

It follows directly by the Jónsson characterization that a variety $\mathcal{V}$ having a majority term is congruence distributive, see e.g. [3].

Theorem 5. *The variety $\mathcal{K}$ of orthomodular directoids is congruence distributive.*

Proof. Consider the ternary term

$$m(x, y, z) = ((x \sqcap y) + (y \sqcap z)) + (x \sqcap z),$$

where $a \sqcap b = (a' + b')'$. It is immediately clear that $a \sqcap a = a$, $a \sqcap b = b \sqcap a$, $a \sqcap b \leq a$ and $a \sqcap b \leq b$, thus $(a \sqcap b) + a = a$ and $(a \sqcap b) + b = b$. Hence

$$m(x, x, y) = ((x \sqcap x) + (x \sqcap y)) + (x \sqcap y) = (x + (x \sqcap y)) + (x \sqcap y)$$

$$
\begin{aligned}
&= x + (x \sqcap y) = x \\
m(x,y,x) &= ((x \sqcap y) + (y \sqcap x)) + (x \sqcap x) = (x \sqcap y) + x = x \\
m(x,y,y) &= ((x \sqcap y) + (y \sqcap y)) + (x \sqcap y) = ((x \sqcap y) + y) + (x \sqcap y) \\
&= y + (x \sqcap y) = y
\end{aligned}
$$

proving that m is a majority term of $\mathcal{K}$. □

Let us recall (see e.g. [3]) that an algebra $\mathcal{A}$ is **congruence regular** if every congruence on $\mathcal{A}$ is determined by every its class, i.e. if for any $\Theta, \Phi \in \mathrm{Con}\mathcal{A}$ and each $a \in A$, if $[a]_\Theta = [a]_\Phi$ then $\Theta = \Phi$. A variety $\mathcal{V}$ is **congruence regular** if each $\mathcal{A} \in \mathcal{V}$ has this property. The following result was proved by B. Csákány, see [6].

Proposition 1. *A variety $\mathcal{V}$ is congruence regular if and only if there exists $n \geq 1$ and ternary terms $t_1, \ldots, t_n$ such that*

$$(t_1(x,y,z) = z \text{ and } \ldots \text{ and } t_n(x,y,z) = z) \text{ if and only if } x = y.$$

Using this, we prove the following result.

Theorem 6. *The variety of orthomodular directoids is congruence regular.*

Proof. At first, consider the term

$$x \triangle y = (x + (x + y)')' + (y + (x + y)')'.$$

If $x = y$ then clearly $(x + (x + y)')' = (x + x')' = 1' = 0$ and hence $x \triangle x = 0$. Conversely, assume $x \triangle y = 0$. By (d) of Lemma 2, $(x + (x + y)')' = 0$ and $(y + (x + y)')' = 0$, i.e. $x + (x + y)' = 1$ and $y + (x + y)' = 1$. By (e) of Lemma 2, $y \leq x$ and $x \leq y$ giving $x = y$.

Now, take $n = 2$ and consider the terms $t_1(x,y,z) = (x \triangle y) + z$, $t_2(x,y,z) = ((x \triangle y) + z')'$. If $x = y$ then

$$t_1(x,x,z) = (x \triangle x) + z = 0 + z = z$$

and

$$t_2(x,x,z) = ((x \triangle x) + z')' = (0 + z')' = z'' = z.$$

Conversely, assume $t_1(x,y,z) = t_2(x,y,z) = z$. Then $(x \triangle y) + z = z$, i.e. $x \triangle y \leq z$ and $((x \triangle y) + z')' = z$, that is, $(x \triangle y) + z' = z'$. Thus we get $x \triangle y \leq z'$, whence $z \leq (x \triangle y)'$. Together we have $x \triangle y \leq z \leq (x \triangle y)'$ which is possible if and only if $x \triangle y = 0$. As shown above, it gets $x = y$. Hence t_1, t_2 are the terms of the Proposition proving congruence regularity. □

Recall that a variety $\mathcal{V}$ is **permutable** if $\Theta \circ \Phi = \Phi \circ \Theta$ for every $\mathcal{A} \in \mathcal{V}$ and each $\Theta, \Phi \in \mathrm{Con}\mathcal{A}$. As proved by A.I. Mal'cev (see e.g. [3]), a variety $\mathcal{V}$ is permutable if and only if there exists a ternary term $p(x,y,z)$ such that $p(x,x,z) = z$ and $p(x,z,z) = x$.

Theorem 7. *The variety of orthomodular directoids is permutable.*

Proof. In Theorem 4, the involution x^a for $x \in [a,1]$ has been introduced. Since $y \leq x+y$, we have $x+y \in [y,1]$ and hence $(x+y)^y = (x+y)' + y$ is defined. As shown by Theorem 4, we have $(x+y)^{yy} = x+y$ and $(x+y)^y \geq y$ whence $(x+y)^y + y = (x+y)^y$. Define

$$p(x,y,z) = ((z+y)^y + x)^x \sqcap ((x+y)^y + z)^z,$$

where again $a \sqcap b = (a' + b')'$. Then

$$p(x,x,z) = ((z+x)^x + x)^x \sqcap ((x+x)^x + z)^z$$

$$= (z+x)^{xx} \sqcap (1+z)^z = (z+x) \sqcap z = z$$

and, analogously,

$$p(x,z,z) = ((z+z)^z + x)^x \sqcap ((x+z)^z + z)^z = x \sqcap (x+z) = x.$$

Thus p is a Mal'cev term, and hence the variety of orthomodular directoids is permutable. □

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Authors' addresses

Ivan Chajda
Palacký University Olomouc, Faculty of Science, Department of Algebra and Geometry, 17. listopadu 12, 77146 Olomouc, Czech Republic
E-mail address: ivan.chajda@upol.cz

Miroslav Kolařík
Palacký University Olomouc, Faculty of Science, Department of Computer Science, 17. listopadu 12, 77146 Olomouc, Czech Republic
E-mail address: miroslav.kolarik@upol.cz

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PRACTICAL STABILITY OF IMPULSIVE FUNCTIONAL DIFFERENTIAL SYSTEMS VIA ANALYSIS TECHNIQUES

SHUN-XUAN CHEN AND ZHI-QIANG ZHU

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Abstract. In this paper we consider the practical stability for a class of functional differential system with impulses. By making use of the analysis techniques, specially, the Bernoulli inequality, we obtain some criteria to guarantee the practical stability of our system, including the finite-time stability and infinite-time stability.

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1. Introduction and preliminaries

Since the system of functional differential equations with impulses can better describe the erupted process such as population, economics, control models and so on, there have been many research activities concerning the qualitative theory for the equations of this type, see, for example, the recent literature [2,3,5–7] and the others cited therein. This paper is concerned with the practical stability of the following system

$$\begin{cases} x'(t) = A(t)x(t) + f(t, x(t), x_t),\ t \geq 0, t \neq t_k, \\ x(t_k) = I_k x(t_k^-),\ k = 1, 2, 3, \ldots, \end{cases} \tag{1.1}$$

where $t_1 < t_2 < t_3 < \ldots$ and $t_k \to \infty$ as $k \to \infty$, x_t stands for the delay functions $x_t : [-\tau, 0] \to \mathbb{R}^n$ for a given positive constant $\tau > 0$ and any fixed t, and

$$x(t_k) = x(t_k^+) = \lim_{h \to 0^+} x(t_k + h),\ \ x(t_k^-) = \lim_{h \to 0^-} x(t_k + h).$$

Further, $I_k \in \mathbb{R}^{n \times n}$ is invertible for each k, $A \in C(\mathbb{R}^{n \times n}, \mathbb{R}^n)$, and $f : C([0, \infty) \times \mathbb{R}^n \times \mathcal{D}, \mathbb{R}^n)$, here $\mathcal{D}$ denotes the set of functions $\phi : [-\tau, 0] \to \mathbb{R}^n$ with the properties that $\phi(t)$ is continuous everywhere except for a countable number of points $\widetilde{t}$ at which $\phi(\widetilde{t} + 0)$ and $\phi(\widetilde{t} - 0)$ exist and $\phi(\widetilde{t} + 0) = \phi(\widetilde{t})$.

Note that the practical stability is different from the classical Lyapunov stability, see [1] for details. Roughly speaking, the practical stability means that the solutions

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of our system does not exceed some bounds on an advanced interval. Our motivation in this paper stems from the work by Stamova [6], who imposed the direct Lyapunov method to study the practical stability of (1.1) with special impulsive effects. We remark that, in general, it is difficult to find Lyapunov functions. In this paper we intend to avoid the trouble. For this purpose we require some hypotheses and notations as follows:

(H1) there exist two continuous functions $b_i : [0,\infty) \to (0,\infty)$ for $i = 1,2$, and a constant $\lambda > 0$ such that

$$|f(t,x,\phi)| \le b_1(t)|x|^\lambda + b_2(t)||\phi||^\lambda \ \text{ for all } \ (t,x,\phi) \in [0,\infty)\times\mathbb{R}^n\times\mathcal{D},$$

where $|\cdot|$ represents the norm of $\mathbb{R}^n$ and $||\cdot||$ the norm of $\mathcal{D}$ defined by $||\phi|| = \sup_{-\tau\le t\le 0}|\phi(t)|$;

(H2)

$$\int_0^\infty ||A(s)||\mathrm{d}s < \infty \ \text{ and } \ \int_0^\infty (b_1(s)+b_2(s))\mathrm{d}s < \infty,$$

where $||A(s)||$ is the norm of $A(s)$ induced by $|\cdot|$;

(H3) the functions $E(t)$ and $p(t)$ are defined, respectively, by

$$E(t) := \prod_{k:t_k\in[0,t]} I_k \ \text{ and } \ p(t) := \max\{||A(t)||,\ b_1(t)+b_2(t)\},$$

where $E(t)$ reduces an identity matrix when $t < t_1$;

(H4) there exists a constant $M_\infty > 0$ such that

$$M_\infty = \sup_{0\le t<\infty}\{||E(t)||, ||E^{-1}(t)||\},$$

where $E^{-1}(t)$ denotes the invertible of $E(t)$.

Let $\zeta > 0$ and $\phi \in \mathcal{D}$. By a solution $x(t) := x(t,\phi)$ of (1.1) defined on $[-\tau,\zeta)$ we mean that $x(t)$ is right continuous with $x(t) = \phi(t)$ on $[-\tau,0]$ and $x(t_k) = I_k x(t_k^-)$ for each possible k, and that $x(t)$ satisfies

$$x'(t) = A(t)x(t) + f(t,x(t),x_t),\ t \ne t_k \ \text{ and a.e. } \ t \ge 0. \tag{1.2}$$

Next we consider the relation of solutions between (1.1) and the following initial problem

$$\begin{cases} y'(t) = E^{-1}(t)[A(t)E(t)y(t) + f(t,E(t)y(t),(Ey)_t)], t \ge 0,\ t \ne t_k, \\ y(t) = \phi(t),\ t \in [-\tau,0], \end{cases} \tag{1.3}$$

where $(Ey)_t \in \mathcal{D}$ defined by $(Ey)_t(\theta) = E(t+\theta)y(t+\theta)$ for all $\theta \in [-\tau,0]$.

By a solution $y(t)$ of (1.3) we mean that $y(t)$ is continuous at points t_k, coincides with $\phi(t)$ on $[-\tau,0]$ and satisfies

$$y'(t) = E^{-1}(t)[A(t)E(t)y(t) + f(t,E(t)y(t),(Ey)_t)],\ t \ne t_k \ \text{ and a.e. } \ t \ge 0. \tag{1.4}$$

Let $x(t) := x(t,\phi)$ be a solution of (1.1) and $y(t) = E^{-1}(t)x(t)$. Then, a straightforward verification shows that $y(t)$ is continuous at points t_k for each possible k,

and that $y(t)$ is a solution of (1.3). To the contrary, for a solution $y(t)$ of (1.3) we can verify that $x(t) = E(t)y(y)$ meets (1.2) and the conditions $x(t_k) = I_k x(t_k^-)$. Hence we get a preliminary result as follows.

Lemma 1. *The solution $x(t) := x(t,\phi)$ of (1.1) implies that $y(t) = E^{-1}(t)x(t)$ is a solution of (1.3). Conversely, the solution $y(t)$ of (1.3) implies that $x(t) = E(t)y(t)$ is a solution of (1.1) satisfying the condition $x(t) = \phi(t)$ for all $t \in [-\tau, 0]$.*

With the preliminaries in hand we can now give the precise definition of practical stability. The system(1.1) is said to be practically stable with respect to (α, β, T) if for given $\beta > 0$ and $T > 0$, there exists a positive number $\alpha = \alpha(\beta, T) \leq \beta$ such that $||\phi|| \leq \alpha$ implies the solution $x(t) := x(t,\phi)$ of (1.1) fulfils $|x(t)| \leq \beta$ for all $t \in [0, T]$. Furthermore, if there exists a positive number $\alpha = \alpha(\beta) \leq \beta$ such that $||\phi|| \leq \alpha$ implies $|x(t)| \leq \beta$ for all $t \in [0, \infty)$, then the system (1.1) is said to be practically stable with respect to (α, β).

2. Main Results

We are now in a position to establish the practical stability criteria for (1.1). Referring to Hale's monograph[4, Chapter 2, Theorem 2.1], we note that the solution $x(t) := x(t,\phi)$ of (1.1) exists locally for each $\phi \in \mathcal{D}$. Further, the following holds.

Lemma 2. *Under Assumptions (H1)–(H4) and $\phi \in \mathcal{D}$, the solution $x(t) := x(t,\phi)$ of (1.1) exists on $[-\tau, \infty)$.*

Proof. Suppose that the solution $x(t) := x(t,\phi)$ of (1.1) exists on $[-\tau, \zeta)$, here $0 < \zeta \leq \infty$. Then, Lemma 1 implies that $y(t) = E^{-1}(t)x(t)$ is a solution of (1.3). We assert $x(t)$ is bounded on $[-\tau, \zeta)$. Indeed, from (1.4) we have

$$y(t) = \phi(0) + \int_0^t E^{-1}(s)[A(s)E(s)y(s) + f(s, E(s)y(s), (Ey)_s)]\mathrm{d}s,\ t \geq 0 \quad (2.1)$$

and hence, with the help of $x(t) = E(t)y(t)$ and Assumptions (H1) and (H4), it follows that

$$|x(t)| \quad (2.2)$$
$$\leq M_\infty||\phi|| + M_\infty^2 \int_0^t (||A(s)|| \cdot |x(s)| + b_1(s)|x(s)|^\lambda + b_2(s)||x_s||^\lambda)\mathrm{d}s,\ t \in [0, \zeta),$$

which infers that

$$||x_t|| \leq L + M_\infty^2 \int_0^t (||A(s)|| \cdot |x(s)| + b_1(s)|x(s)|^\lambda + b_2(s)||x_s||^\lambda)\mathrm{d}s,\ t \in [0, \zeta), \quad (2.3)$$

where $L := M_\infty||\phi||$. Let $u(t)$, $v(t)$ and $w(t)$ be defined, respectively, by

$$u(t) := t^\lambda,\ v(t) := L + M_\infty^2 \int_0^t (||A(s)|| \cdot |x(s)| + b_1(s)|x(s)|^\lambda + b_2(s)||x_s||^\lambda)\mathrm{d}s$$

and

$$w(t) := \int_L^t \frac{\mathrm{d}s}{u(s)},\ t \geq L.$$

Without loss of generality we can set $L \geq 1$. Then, from (2.2) and (2.3) we have

$$\frac{|x(t)|}{u(v(t))} \leq 1,\ \frac{u(|x(t)|)}{u(v(t))} \leq 1,\ \frac{u(||x_t||)}{u(v(t))} \leq 1,\ t \in [0,\zeta)$$

and hence, it follows that

$$\frac{||A(t)|| \cdot |x(s)| + b_1(t)u(|x(t)|) + b_2(t)u(||x_t||)}{u(|v(t)|)} \leq ||A(t)|| + b_1(t) + b_2(t),$$

which, with the aid of $w(t)$, produces

$$\frac{\mathrm{d}}{\mathrm{d}t} w(v(t)) \leq M_\infty^2(||A(t)|| + b_1(t) + b_2(t)). \tag{2.4}$$

Now integrating (2.4) from 0 to $t \in [0,\zeta)$ we obtain

$$w(v(t)) \leq M_\infty^2 \int_0^t (||A(s)|| + b_1(s) + b_2(s))\mathrm{d}s$$

and this, with the invertibility and monotonicity of w, results in

$$v(t) \leq w^{-1}\left(M_\infty^2 \int_0^t (||A(s)|| + b_1(s) + b_2(s))\mathrm{d}s\right),\ t \in [0,\zeta). \tag{2.5}$$

Invoking Assumption (H2), together with (2.2) and (2.5), it is clear that the solution $x(t) := x(t,\phi)$ of (1.1) is bounded on $[-\tau,\zeta)$.

Now for the solution $x(t) := x(t,\phi)$ of (1.1) we can set

$$|A(t)x(t)| \leq M_a \ \text{ and } \ |f(t,x(t),x_t)| \leq M_\phi \ \text{ for all } \ t \in [-\tau,\zeta).$$

We assert that the solution $x(t) := x(t,\phi)$ exists on $[-\tau,t_1]$. Otherwise, $\zeta \leq t_1$ and, from (2.1) we have

$$|x(s_1) - x(s_2)| \leq M_a|s_1 - s_2| + M_\phi|s_1 - s_2|,\ s_1,s_2 \in [0,\zeta),$$

which implies that $\lim_{t\to\zeta-} x(t)$ exists as a finite number. Thus $\varphi := x_\zeta \in \mathcal{D}$. Consequently, we can find a solution $z(t) := z(t;\zeta,\varphi)$ of (1.1) with $z_\zeta \equiv \varphi$, that is, the solution $x(t) := x(t,\phi)$ of (1.1) can be extended to a much larger interval than $[-\tau,\zeta)$, which leads to a contradiction.

Similarly, we can show that the solution $x(t) := x(t,\phi)$ of (1.1) exists on $[-\tau,t_k]$ for $k > 1$. Since $t_k \to \infty$ as $k \to \infty$, the desired result holds and the proof is complete. □

Theorem 1. *Suppose that Assumptions (H1) and (H3) are satisfied. Suppose further that $\beta > 0$, $T > 0$ and $M_T = \sup_{t\in[0,T]}\{||E(t)||, ||E^{-1}(t)||\}$. Then system (1.1) is practically stable with respect to (α,β,T) if one of the following conditions holds:*

(i) $\lambda = 1$ *and*

$$\alpha = \frac{\beta}{M_T e^{2M_T^2 \int_0^T p(s)\mathrm{d}s}}; \tag{2.6}$$

(ii) $\lambda > 1$ *and*

$$\alpha = \frac{1}{M_T \left(\left((\frac{1}{\beta})^{\lambda-1} + 1 \right) e^{M_T^2(\lambda-1)\int_0^T p(s)\mathrm{d}s} - 1 \right)^{\frac{1}{\lambda-1}}}; \tag{2.7}$$

(iii) $0 < \lambda < 1$ *and*

$$\alpha = \frac{1}{M_T} \left(\left(\beta^{1-\lambda} + 1 \right) e^{M_T^2(\lambda-1)\int_0^T p(s)\mathrm{d}s} - 1 \right)^{\frac{1}{1-\lambda}} \tag{2.8}$$

with

$$\beta > \left(e^{M_T^2(1-\lambda)\int_0^T p(s)\mathrm{d}s} - 1 \right)^{\frac{1}{1-\lambda}}.$$

Proof. Let $J := [0, T]$. Note that from (2.1) it follows that

$$|x(t)| \le M_T||\phi|| + M_T^2 \int_0^t (||A(s)|| \cdot |x(s)| + b_1(s)|x(s)|^\lambda + b_2(s)||x_s||^\lambda)\mathrm{d}s, \ t \in J, \tag{2.9}$$

which induces

$$||x_t|| \le M_T||\phi|| + M_T^2 \int_0^t (||A(s)|| \cdot |x(s)| + b_1(s)|x(s)|^\lambda + b_2(s)||x_s||^\lambda)\mathrm{d}s, \ t \in J. \tag{2.10}$$

Now if we set

$$R(t) = M_T||\phi|| + M_T^2 \int_0^t (||A(s)|| \cdot |x(s)| + b_1(s)|x(s)|^\lambda + b_2(s)||x_s||^\lambda)\mathrm{d}s, \ \ t \in J,$$

then $R(0) = M_T||\phi||$ and, together with (2.9) and (2.10),

$$R'(t) \le M_T^2 p(t) R(t) + M_T^2 p(t) R(t)^\lambda, \ t \in J, \tag{2.11}$$

where we have invoked the Assumption (H3). Next we proceed in steps.
(i) Case $\lambda = 1$. From (2.11) it follows that

$$R(t) \le R(0) e^{2M_T^2 \int_0^t p(s)\mathrm{d}s}, \ t \in J.$$

Since $|x(t)| \le R(t)$, to fulfill $|x(t)| \le \beta$ on J we turn to

$$R(0) e^{2M_T^2 \int_0^T p(s)\mathrm{d}s} \le \beta,$$

which implies that

$$R(0) = M_T||\phi|| \leq \frac{\beta}{e^{2M_T^2\int_0^T p(s)\mathrm{d}s}}.$$

That is, if we choose α as in (2.6), then system (1.1) is practically stable with respect to (α, β, T).

(ii) Case $\lambda > 1$. In this case we multiply (2.11) by $(1-\lambda)R(t)^{-\lambda}e^{M_T^2(\lambda-1)\int_0^t p(s)\mathrm{d}s}$ and obtain

$$\left((R(t)^{1-\lambda}+1)e^{M_T^2(\lambda-1)\int_0^t p(s)\mathrm{d}s}\right)' \geq 0,\ t \in J,$$

which means that

$$R(t)^{1-\lambda}+1 \geq (R(0)^{1-\lambda}+1)e^{M_T^2(1-\lambda)\int_0^t p(s)\mathrm{d}s},\ t \in J$$

and hence, when

$$R(0) < \frac{1}{\left(e^{M_T^2(\lambda-1)\int_0^T p(s)\mathrm{d}s}-1\right)^{\frac{1}{\lambda-1}}}, \tag{2.12}$$

it follows that

$$R(t) \leq \frac{1}{\left((R(0)^{1-\lambda}+1)e^{M_T^2(1-\lambda)\int_0^t p(s)\mathrm{d}s}-1\right)^{\frac{1}{\lambda-1}}},\ t \in J.$$

Now, to meet $|x(t)| \leq \beta$ on J we consider

$$\frac{1}{\left((R(0)^{1-\lambda}+1)e^{M_T(1-\lambda)\int_0^T p(s)\mathrm{d}s}-1\right)^{\frac{1}{\lambda-1}}} \leq \beta,$$

which deduces

$$R(0) \leq \frac{1}{\left(\left((\frac{1}{\beta})^{\lambda-1}+1\right)e^{M_T^2(\lambda-1)\int_0^T p(s)\mathrm{d}s}-1\right)^{\frac{1}{\lambda-1}}}. \tag{2.13}$$

Thus, from (2.12) and (2.13) we learn that when $\lambda > 1$ and α as in (2.7), system (1.1) is practically stable with respect to (α, β, T).

(iii) Case $0 < \lambda < 1$. Similarly, multiplying (2.11) by

$$(1-\lambda)R(t)^{-\lambda}e^{M_T^2(\lambda-1)\int_0^t p(s)\mathrm{d}s}$$

we have

$$\left((R(t)^{1-\lambda}+1)e^{M_T^2(\lambda-1)\int_0^t p(s)\mathrm{d}s}\right)' \leq 0,\ t \in J,$$

which means that

$$R(t) \leq \left((R(0)^{1-\lambda}+1)e^{M_T^2(1-\lambda)\int_0^t p(s)\mathrm{d}s}-1\right)^{\frac{1}{1-\lambda}},\ t \in J.$$

Solving

$$\left((R(0)^{1-\lambda}+1)e^{M_T^2(1-\lambda)\int_0^T p(s)\mathrm{d}s}-1\right)^{\frac{1}{1-\lambda}} \le \beta,$$

we obtain

$$R(0) \le \left(\left(\beta^{1-\lambda}+1\right)e^{M_T^2(\lambda-1)\int_0^T p(s)\mathrm{d}s}-1\right)^{\frac{1}{1-\lambda}},$$

where we require

$$\beta > \left(e^{M_T^2(1-\lambda)\int_0^T p(s)\mathrm{d}s}-1\right)^{\frac{1}{1-\lambda}}.$$

Hence, if we let α be defined as in (2.8), then system (1.1) can realize the practical stability with respect to (α,β,T).

In summary, the proof is complete. □

Note that by Assumptions (H2) and (H4) it holds that $\int_0^\infty p(s)\mathrm{d}s < \infty$. In addition, the Lemma 2 implies that for each $\phi \in \mathcal{D}$, the solution $x(t,\phi)$ of (1.1) exists on $[-\tau,\infty)$. Hence, it is reasonable to consider the practical stability on the interval $[-\tau,\infty)$. Indeed, The proof of Theorem 1 is valid for $T=\infty$. So the following results are clear and we give it to end our discussions.

Theorem 2. *Suppose that Assumptions (H1)–(H4) are satisfied and $\beta > 0$. Then system (1.1) is practically stable with respect to (α,β) if one of the following conditions holds:*

(i) $\lambda = 1$ and

$$\alpha = \frac{\beta}{M_\infty e^{2M_\infty^2\int_0^\infty p(s)\mathrm{d}s}};$$

(ii) $\lambda > 1$ and

$$\alpha = \frac{1}{M_\infty\left(\left((\frac{1}{\beta})^{\lambda-1}+1\right)e^{M_\infty^2(\lambda-1)\int_0^\infty p(s)\mathrm{d}s}-1\right)^{\frac{1}{\lambda-1}}};$$

(iii) $0 < \lambda < 1$ and

$$\alpha = \frac{1}{M_\infty}\left(\left(\beta^{1-\lambda}+1\right)e^{M_\infty^2(\lambda-1)\int_0^\infty p(s)\mathrm{d}s}-1\right)^{\frac{1}{1-\lambda}}$$

with

$$\beta > \left(e^{M_\infty^2(1-\lambda)\int_0^\infty p(s)\mathrm{d}s}-1\right)^{\frac{1}{1-\lambda}}.$$

Next we conclude this paper with an example.

Example 1. Let $\lambda > 0$ be constant, $x = (x_1, x_2)^T \in \mathbb{R}^2$ with the norm $|x| = |x_1| + |x_2|$, and $\phi \in \mathcal{D}$ with $\phi : [-1, 0] \to \mathbb{R}^2$ and $\phi = (\phi_1, \phi_2)^T$. Suppose in (1.1) that

$$A(t) = \begin{pmatrix} te^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix}, f(t, x, \phi) = \frac{1}{1+t^2} \begin{pmatrix} \sin^\lambda(x_1 + x_2) \\ \sin^\lambda(\phi_1(-1) + \phi_2(-1/2)) \end{pmatrix}$$

as well as

$$I_k = \begin{pmatrix} e^{\frac{(-1)^k}{k}} & 0 \\ 0 & e^{-\frac{(-1)^k}{k}} \end{pmatrix}, \ k = 1, 2, 3, \ldots.$$

Then

$$|f(t, x, \phi)| \le \frac{1}{1+t^2}(|x|^\lambda + ||\phi||^\lambda), \ \text{ for all } (t, x, \phi) \in [0, \infty) \times \mathbb{R}^2 \times \mathcal{D}$$

and hence, the Assumption (H1) is met. In this case the functions b_i and $p(t)$ in Assumptions (H1) and (H3) can be taken as

$$b_i(t) = \frac{1}{1+t^2}, \ p(t) = \frac{2}{1+t^2} \ \text{ for } \ i = 1, 2 \ \text{ and } \ t \ge 0.$$

On the other hand, we have

$$E(t) = \begin{pmatrix} c(t) & 0 \\ 0 & d(t) \end{pmatrix},$$

where

$$c(t) = e^{\sum_{k=1}^{[t]} \frac{(-1)^k}{k}}, \ d(t) = e^{-\sum_{k=1}^{[t]} \frac{(-1)^k}{k}}, \ t \ge 0$$

and $[t]$ indicates the integral part of t. Thus, a simple verification shows that

$$\frac{1}{3} \le c(t) \le 1 \ \text{ and } \ \frac{3}{2} \le d(t) \le 3 \ \text{ for } \ t \ge 0.$$

In other words, we can take $M_\infty = 3$ for the Assumption (H4). Note that $\int_0^\infty p(s)\mathrm{d}s = \pi$, for $\beta > 0$ we choose

$$\alpha = \frac{\beta}{3e^{18\pi}} \ \text{ when } \ \lambda = 1, \ \text{ or}$$

$$\alpha = \frac{1}{3\left(\left((\frac{1}{\beta})^{\lambda-1} + 1\right)e^{9\pi(\lambda-1)} - 1\right)^{\frac{1}{\lambda-1}}} \ \text{ when } \ \lambda > 1,$$

or, when $\lambda \in (0, 1)$,

$$\alpha = \frac{1}{3}\left(\left(\beta^{1-\lambda} + 1\right)e^{9\pi(\lambda-1)} - 1\right)^{\frac{1}{1-\lambda}} \ \text{ with } \ \beta > \left(e^{9\pi(1-\lambda)} - 1\right)^{\frac{1}{1-\lambda}}.$$

Then, by Theorem 2 we see that under our considerations, system (1.1) is practically stable with respect to (α, β).

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Authors' addresses

Shun-Xuan Chen

Guangdong Polytechnic Normal University, College of Computer Science, 510665 Guangzhou, P. R. China

E-mail address: chenshunxuan@126.com

Zhi-Qiang Zhu

Guangdong Polytechnic Normal University, Department of Computer Science, 510665 Guangzhou, P. R. China

E-mail address: z3825@163.com (corresponding author: Zhi-Qiang Zhu)

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MODULES OVER GROUP RINGS OF LOCALLY SOLUBLE GROUPS WITH A CERTAIN CONDITION OF MINIMALITY

O. YU. DASHKOVA

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Abstract. Let A be an $\mathbf{R}G$-module, where $\mathbf{R}$ is an associative ring, $A/C_A(G)$ is an infinite $\mathbf{R}$-module, $C_G(A) = 1$, G is a locally soluble group. Let $L_{nf}(G)$ be the system of all subgroups $H \leq G$ such that quotient modules $A/C_A(H)$ are infinite $\mathbf{R}$-modules. The author studies an $\mathbf{R}G$-module A such that $L_{nf}(G)$ satisfies the minimal condition as an ordered set. It is proved that a locally soluble group G with these conditions is soluble. The structure of G is described.

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1. INTRODUCTION

Let A be a vector space over a field F. The subgroups of the group $GL(F, A)$ of all automorphisms of A are called linear groups. If A has a finite dimension over F then $GL(F, A)$ can be considered as the group of non-singular $(n \times n)$-matrices, where $n = dim_F A$. Finite dimensional linear groups have played an important role in various fields of mathematics, physics and natural sciences, and have been studied many times. When A is infinite dimensional over F, the situation is totally different. Infinite dimensional linear groups have been investigated little. The study of this class of groups requires additional restrictions. In [5] it was introduced the definition of the central dimension of an infinite dimensional linear group. Let H be a subgroup of $GL(F, A)$. H acts on the quotient space $A/C_A(H)$ in a natural way. The authors define $centdim_F H$ to be $dim_F(A/C_A(H))$. The subgroup H is said to have a finite central dimension if $centdim_F H$ is finite and H has infinite central dimension otherwise. Let $G \leq GL(F, A)$. In [5] it was considered the system $L_{id}(G)$ of all subgroups of G of infinite central dimension. In order to investigate infinite dimensional linear groups that are close to finite dimensional, it is natural to consider the case where the system $L_{id}(G)$ is "very small". The authors have studied locally soluble infinite dimensional linear groups such that $L_{id}(G)$ satisfies the minimal condition as an ordered set [5].

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If $G \leq GL(F, A)$ then A can be considered as an FG-module. The natural generalization of this case is the consideration of an $\mathbf{R}$G-module A, where $\mathbf{R}$ is a ring whose structure is near to a field. At this point the generalization of the notion of the central dimension of a subgroup of a linear group is the notion of the cocentralizer of a subgroup. This notion was introduced in [8]. Let A be an $\mathbf{R}G$-module, $\mathbf{R}$ be an associative ring, G be a group. If $H \leq G$ then the quotient module $A/C_A(H)$ considered as an $\mathbf{R}$-module is called the cocentralizer of H in the module A.

Modules over group rings of finite groups have been considered by many authors. Recently this class of modules was investigated in [6]. Study of modules over group rings of infinite groups requires some additional restrictions as in the case of infinite dimensional linear groups. In [2] it was investigated an $\mathbf{R}G$-module A such that $\mathbf{R}$ is a dedekind domain and the cocentralizer of G in the module A is not an artinian $\mathbf{R}$-module. It was considered the system $L_{nad}(G)$ of all subgroups of G such that their cocentralizers in the module A are not artinian $\mathbf{R}$-modules which is ordered by the usual inclusion. It is investigated an $\mathbf{R}G$-module A such that the system $L_{nad}(G)$ satisfies the minimal condition as an ordered set, G is a locally soluble group, $C_G(A) = 1$. The analogous problem for the ring of integers $\mathbf{R}$ was investigated in [3].

In [1] we have studied an $\mathbf{R}G$-module A such that $\mathbf{R}$ is the ring of integers, the cocentralizer of G in the module A is not a noetherian $\mathbf{R}$-module and $C_G(A) = 1$. Let $L_{nnd}(G)$ be the system of all subgroups of G such that their cocentralizers in the module A are not noetherian $\mathbf{R}$-modules. It was investigated an $\mathbf{R}G$-module A such that $L_{nnd}(G)$ satisfies the minimal condition as an ordered set and G is locally soluble.

In [4] we have considered the similar problem where $\mathbf{R}$ is the ring of integers and the noetherian condition is replaced by the minimax condition.

In this paper we investigate $\mathbf{R}G$-module, where $\mathbf{R}$ is an associative ring, $A/C_A(G)$ is an infinite $\mathbf{R}$-module, $C_G(A) = 1$, G is a locally soluble group. Let $L_{nf}(G)$ be the system of all subgroups $H \leq G$ such that $A/C_A(H)$ are infinite $\mathbf{R}$-modules. We study an $\mathbf{R}G$-module A such that $L_{nf}(G)$ satisfies the minimal condition as an ordered set. It is proved that a locally soluble group G with these conditions is soluble and the structure of G is described.

The main results of this paper are Theorems 2 and 3.

2. Preliminary results

We reduce some elementary facts about $\mathbf{R}G$-modules.

Later on it is considered an $\mathbf{R}G$-module A such that $C_G(A) = 1$.

Let A be an $\mathbf{R}G$-module where G is a group, $\mathbf{R}$ is an associative ring. Recall that if $K \leq H \leq G$ and the cocentralizer of H in the module A is a finite $\mathbf{R}$-module then the cocentralizer of K in the module A is a finite $\mathbf{R}$-module also. If U, V are

subgroups of G such that their cocentralizers in the module A are finite $\mathbf{R}$-modules, then $A/(C_A(U) \cap C_A(V))$ is a finite $\mathbf{R}$-module also.

Suppose that a group G satisfies the condition $min-nf$. If $H_1 > H_2 > H_3 > \cdots$ is an infinite strictly descending chain of subgroups of G then there is the natural number n such that the cocentralizer of H_n in the module A is a finite $\mathbf{R}$-module. Moreover, if N is a normal subgroup of G and the cocentralizer of N in the module A is an infinite $\mathbf{R}$-module then G/N satisfies the minimal condition on subgroups.

Lemma 1. *Let A be an $\mathbf{R}G$-module, G be a group, $\mathbf{R}$ be an associative ring. Suppose that G satisfies the condition $min-nf$, X, H are subgroups of G and Λ is an index set such that*

(1) *$X = Dr_{\lambda\in\Lambda} X_\lambda$, where $1 \neq X_\lambda$ is an H-invariant subgroup of X, for each $\lambda \in \Lambda$.*

(2) *$H \cap X \leq Dr_{\lambda\in\Gamma} X_\lambda$ for some subset Γ of Λ.*

If the set $\Omega = \Lambda \setminus \Gamma$ is infinite, then the cocentralizer of H in the module A is a finite $\mathbf{R}$-module.

Proof. Suppose that the set Ω is infinite and let $\Omega_1 \supset \Omega_2 \supset \cdots$ be a strictly descending chain of subsets of the set Ω. Since $H \cap Dr_{\lambda\in\Omega} X_\lambda = 1$, the chain of subgroups $\langle H, X_\lambda | \lambda \in \Omega_1\rangle > \langle H, X_\lambda | \lambda \in \Omega_2\rangle > \cdots$ is strictly descending. It follows that for some natural number d the cocentralizer of the subgroup $\langle H, X_\lambda | \lambda \in \Omega_d\rangle$ in the module A is a finite $\mathbf{R}$-module. Therefore the cocentralizer of H in the module A is a finite $\mathbf{R}$-module also. □

Lemma 2. *Let A be an $\mathbf{R}G$-module, G be a group, $\mathbf{R}$ be an associative ring, G satisfy the condition $min-nf$, H, K be subgroups of G such that K is a normal subgroup of H. Suppose that there exists an index set Λ and subgroups H_λ of G such that $K \leq H_\lambda$ for all $\lambda \in \Lambda$, $H/K = Dr_{\lambda\in\Lambda} H_\lambda/K$, and the set Λ is infinite. Then the cocentralizer of H in the module A is a finite $\mathbf{R}$-module.*

Proof. Let Γ and Ω are infinite disjoint subsets of the set Λ such that $\Lambda = \Gamma \cup \Omega$. Let $U/K = Dr_{\lambda\in\Gamma} H_\lambda/K$, let $V/K = Dr_{\lambda\in\Omega} H_\lambda/K$, and let $\Gamma_1 \supset \Gamma_2 \supset \cdots$ be a strictly descending chain of subsets of the set Γ. Then we construct an infinite strictly descending chain of subgroups

$$\langle V, H_\lambda | \lambda \in \Gamma_1\rangle > \langle V, H_\lambda | \lambda \in \Gamma_2\rangle > \cdots .$$

It follows from the condition $min-nf$ that the cocentralizer of V in the module A is a finite $\mathbf{R}$-module. Likewise, we obtain that the cocentralizer of U in the module A is a finite $\mathbf{R}$-module. Since $H = UV$, it follows that the cocentralizer of H in the module A is a finite $\mathbf{R}$-module also. □

Lemma 3. *Let A be an $\mathbf{R}G$-module, G be a group, $\mathbf{R}$ be an associative ring, G satisfy the condition $min-nf$. If an element $g \in G$ has infinite order then the cocentralizer of $\langle g\rangle$ in the module A is a finite $\mathbf{R}$-module.*

Proof. Let p, q are distinct primes greater than 3 and let $u = g^p$, $v = g^q$. Then there is an infinite descending chain of subgroups $\langle u\rangle > \langle u^2\rangle > \langle u^4\rangle > \cdots$. It follows from the condition $min-nf$ that there exists the natural number k such that the cocentralizer of the subgroup $\langle u^{2^k}\rangle$ in the module A is a finite $\mathbf{R}$-module. Similarly, there exists a natural number l such that the cocentralizer of the subgroup $\langle v^{3^l}\rangle$ in the module A is a finite $\mathbf{R}$-module. Therefore the cocentralizer of the subgroup $\langle g\rangle = \langle u^{2^k}\rangle\langle v^{3^l}\rangle$ in the module A is a finite $\mathbf{R}$-module.

□

The following result gives an important information about the derived quotient group under the condition $min-nf$.

Lemma 4. *Let A be an $\mathbf{R}G$-module, G be a group, $\mathbf{R}$ be an associative ring. Suppose that the cocentralizer of G in the module A is an infinite $\mathbf{R}$-module, and G satisfies the condition $min-nf$. Then the quotient group G/G' is Chernikov.*

Proof. Suppose that the quotient group G/G' is not Chernikov group. Let $\mathfrak{S}$ be the family of all subgroups $H \leq G$ such that the quotient group H/H' is not Chernikov and the cocentralizer of H in the module A is an infinite $\mathbf{R}$-module. Since $G \in \mathfrak{S}$ then $\mathfrak{S} \neq \varnothing$. Since the set $\mathfrak{S}$ satisfies the minimal condition, then it has a minimal element. Let D be this minimal element. If U, V are proper subgroups of the group D such that $D = UV$ and $U \cap V = D'$, then at least one of these subgroups, U say, such that its cocentralizer in the module A is an infinite $\mathbf{R}$-module. The choice of D implies that U/U' is Chernikov. It follows with regard to the isomorphism $U/D' \simeq (U/U')/(D'/U')$ that U/D' is also Chernikov. Since the cocentralizer of U in the module A is an infinite $\mathbf{R}$-module it follows that the abelian quotient group D/U is also Chernikov. Hence the quotient group D/D' is Chernikov. Contrary to the choice of D. Therefore D/D' is indecomposable. Hence D/D' is isomorphic to a subgroup of quasi-cyclic group C_{q^∞}, for some prime q. Contradiction.

□

Let A be an $\mathbf{R}G$-module, G be a group, $\mathbf{R}$ be an associative ring. Let $FFD(G)$ be the set of all elements $x \in G$ such that the cocentralizer of $\langle x\rangle$ in the module A is a finite $\mathbf{R}$-module. Since $C_A(x^g) = C_A(x)g$ for all $x, g \in G$, it follows that $FFD(G)$ is a normal subgroup of G.

Lemma 5. *Let A be an $\mathbf{R}G$-module, G be a group, $\mathbf{R}$ be an associative ring. Suppose that the cocentralizer of G in the module A is an infinite $\mathbf{R}$-module, and G satisfies the condition $min-nf$. Then G is either periodic or $G = FFD(G)$.*

Proof. We suppose to the contrary that G is neither periodic nor $G \neq FFD(G)$. Let $\mathfrak{S}$ be the family of all subgroups $H \leq G$ such that H is not periodic and $H \neq FFD(H)$. $\mathfrak{S}$ is non-empty. If $H \neq FFD(H)$ then there is an element $h \in H$ such that the quotient module $A/C_A(h)$ is an infinite $\mathbf{R}$-module. Hence $\mathfrak{S} \subseteq L_{nf}(G)$, and

therefore $\mathfrak{S}$ satisfies the minimal condition. Let D be the minimal element of $\mathfrak{S}$, let $L = FFD(D)$. Note that $L \neq 1$, since D is not a periodic group. If $L \leq S \leq D$ and $S \neq D$, then $S = FFD(S)$ so $S \leq L$. Hence D/L has order q for some prime q. Let $x \in D \setminus L$. If an element a has infinite order, then the choice of D implies that $\langle x, a\rangle = D$. It follows that L is finitely generated and since $L = FFD(L)$, the quotient module $A/C_A(L)$ is a finite $\mathbf{R}$-module. Since the subgroup L is normal in D, then $C = C_A(L)$ is an $\mathbf{R}D$-submodule of A. It follows that A has the finite series of $\mathbf{R}D$-submodules

$$\langle 0\rangle \leq C \leq A,$$

such that A/C is a finite $\mathbf{R}$-module. Since A/C is a finite $\mathbf{R}$-module then $D/C_D(A/C)$ is finite. As $C = C_A(L)$ then $L \leq C_D(C)$. It follows that $D/C_D(C)$ is finite too.

Let $W = C_D(C) \cap C_D(A/C)$. By Remak theorem

$$D/W \leq D/C_D(C) \times D/C_D(A/C).$$

It follows that the quotient group D/W is finite. W acts trivially on each factor of the series $\langle 0\rangle \leq C \leq A$. Therefore W is abelian.

Let U be a normal subgroup of finite index of D. The subgroup U is not periodic and so $\langle U, x\rangle$ is neither periodic nor $\langle U, x\rangle \neq FFD(\langle U, x\rangle)$. The choice of D implies that $D = \langle U, x\rangle$ and hence the quotient group D/U is abelian. If E is the finite residual of D, it follows that the quotient group D/E is abelian. Since $E \leq W$ then D/W is also abelian. It follows that $D/(W \cap L)$ is abelian. Since $W \cap L \leq W$, then the subgroup $W \cap L$ is abelian, and so D is a finitely generated metabelian subgroup. By theorem of P.Hall (Theorem 9.51 [9]) D is residually finite. As above, D is therefore abelian. Since $D = U\langle x\rangle$ for every subgroup U of finite index, it follows that the group D is infinite cyclic. By Lemma 3 $D = FFD(D)$. We have the contradiction with the choice of D.

□

3. Locally soluble groups with the condition $min-nf$

Lemma 6. *Let A be an $\mathbf{R}G$-module, G be a periodic locally soluble group, $\mathbf{R}$ be an associative ring. Suppose that the cocentralizer of G in the module A is an infinite $\mathbf{R}$-module and G satisfies the condition $min-nf$. Then G either satisfies the minimal condition on subgroups or $G = FFD(G)$.*

Proof. We suppose to the contrary that G is neither satisfies the minimal condition on subgroups nor $G \neq FFD(G)$. Let $\mathfrak{S}$ be the family of all subgroups $H \leq G$ such that H does not satisfy the minimal condition on subgroups and $H \neq FFD(H)$. Then $\mathfrak{S} \neq \varnothing$. If $H \neq FFD(H)$ then the cocentralizer of H in the module A is an infinite $\mathbf{R}$-module and hence $\mathfrak{S} \subseteq L_{nf}(G)$. Therefore $\mathfrak{S}$ satisfies the minimal condition. Let D be the minimal element and let $L = FFD(D)$. There exists an infinite strictly descending chain of subgroups of D:

$$H_1 > H_2 > H_3 > \ \cdots \ .$$

Since D satisfies the condition $min-nf$ then there exists the natural number k such that the cocentralizer of H_k in the module A is a finite $\mathbf{R}$-module. Therefore $H_k \leq L$, and hence L does not satisfy the minimal condition. If $x \in D \setminus L$ then it follows from the choice of the subgroup D that $\langle x, L\rangle = D$. Hence the quotient group D/L has the order q for prime q. If it is necessary we replace x by the suitable power and obtain that x has the order q^r for some natural number r. Since the group D is not Chernikov then by D.I.Zaicev's theorem [10], D contains $\langle x\rangle$-invariant abelian subgroup $B = Dr_{n\in\mathbf{N}}\langle b_n\rangle$ and we may assume that the elements b_n have prime orders for all $n \in \mathbf{N}$. Let $1 \neq c_1 \in B$ and $C_1 = \langle c_1\rangle^{\langle x\rangle}$. Then C_1 is finite and there is the subgroup E_1 such that $B = C_1 \times E_1$. Let $U_1 = core_{\langle x\rangle} E_1$. Therefore U_1 has finite index in B. If $1 \neq c_2 \in U_1$ and $C_2 = \langle c_2\rangle^{\langle x\rangle}$ then C_2 is a finite $\langle x\rangle$-invariant subgroup and $\langle C_1, C_2\rangle = C_1 \times C_2$. Continuing in this manner, we can construct a family of subgroups $\{C_n | n \in \mathbf{N}\}$ for which $\langle C_n | n \in \mathbf{N}\rangle = Dr_{n\in\mathbf{N}} C_n$. By Lemma 1 $x \in L$. Contradiction.

□

From Lemmas 5 and 6 it follows the theorem.

Theorem 1. *Let A be an $\mathbf{R}G$-module, G be a locally soluble group, $\mathbf{R}$ be an associative ring. Suppose that the cocentralizer of G in the module A is an infinite $\mathbf{R}$-module, and G satisfies the condition $min-nf$. Then G either satisfies the minimal condition on subgroups or $G = FFD(G)$.*

Lemma 7. *Let A be an $\mathbf{R}G$-module, G be a locally soluble group. Suppose that the cocentralizer of G in the module A is a finite $\mathbf{R}$-module. Then G is almost abelian.*

Proof. Let $C = C_A(G)$. Then A has the series of $\mathbf{R}G$-submodules $\langle 0\rangle \leq C \leq A$, where A/C is a finite $\mathbf{R}$-module. Since $G \leq C_G(C)$ then $G/C_G(C)$ is trivial. As A/C is a finite $\mathbf{R}$-module then $G/C_G(A/C)$ is finite.

Let $H = C_G(C) \cap C_G(A/C)$. Each element of H acts trivially on every factor of the series $\langle 0\rangle \leq C \leq A/C$. By Kaluzhnin Theorem (p. 144 [7]) H is abelian. By Remak's Theorem

$$G/H \leq G/C_G(C) \times G/C_G(A/C).$$

It follows that G/H is finite. Then G is an almost abelian group.

□

Lemma 8. *Let A be an $\mathbf{R}G$-module, G be a locally soluble group, $\mathbf{R}$ be an associative ring, and if the cocentralizer of G in the module A is an infinite $\mathbf{R}$-module then G satisfies the condition $min-nf$. Then either G is soluble or G has an ascending series of normal subgroups $1 = W_0 \leq W_1 \leq \cdots \leq W_n \leq \ \cdots \ \leq W_\omega = \cup_{n\in\mathbf{N}} W_n \leq G$,*

such that the cocentralizer of each subgroup W_n in the module A is a finite **R***-module, the factors W_{n+1}/W_n are abelian for $n = 1, 2, \cdots$, and G/W_ω is a Chernikov group.*

Proof. If the quotient module $A/C_A(G)$ is a finite **R**-module then G is soluble by Lemma 7. Therefore it seemed reasonable to study locally soluble groups G such that $A/C_A(G)$ is an infinite **R**-module.

Later we consider the case when the cocentralizer of G in the module A is an infinite **R**-module. At first we prove that G is hyperabelian. To accomplish this we show that every non-trivial image of G contains a non-trivial normal abelian subgroup.

Let H be a proper normal subgroup of G. Suppose that the cocentralizer of H in the module A is an infinite **R**-module. Then G/H satisfies the minimal condition on subgroups. Therefore G/H is Chernikov group, and contains a non-trivial normal abelian subgroup. Now we suppose that the the cocentralizer of H in the module A is a finite **R**-module. Let $\mathfrak{L} = \{M_\sigma/H | \sigma \in \Sigma\}$ be the family of all non-trivial normal subgroups of the quotient group G/H. At first we consider the case when for all $\sigma \in \Sigma$ the cocentralizer of M_σ in the module A is an infinite **R**-module. We shall prove that the quotient group G/H satisfies the minimal condition on normal subgroups. Let $\{M_\delta/H\}$ be a non-empty subset of $\mathfrak{L}$. The cocentralizer of a subgroup M_δ in the module A is an infinite **R**-module for all δ. By the condition $min-nf$ the set $\{M_\delta\}$ has the minimal element M. Then M/H is the minimal element of subset $\{M_\delta/H\}$. Therefore G/H satisfies the minimal condition on normal subgroups. It follows that the quotient group G/H is hyperabelian and contains a non-trivial normal abelian subgroup. In the case when for some $\gamma \in \Sigma$ the cocentralizer of M_γ in the module A is a finite **R**-module, the subgroup M_γ is soluble. Then M_γ/H is a non-trivial normal soluble subgroup of G/H. Therefore the quotient group G/H contains a non-trivial normal abelian subgroup and so G is hyperabelian.

Let $1 = H_0 \leq H_1 \leq \cdots \leq H_\alpha \leq \cdots \leq G$ be a normal ascending series with abelian factors and let α be the least ordinal such that the cocentralizer of H_α in the module A is an infinite **R**-module. Then, as above, the subgroup H_β is soluble for all $\beta < \alpha$. Moreover, the quotient group G/H_α satisfies the minimal condition on subgroups, and so is a soluble Chernikov group.

At first we suppose that α is not a limit ordinal. Then the subgroup H_α is soluble and it follows that G is soluble also. Now we consider the case when α is a limit ordinal, and G is not soluble. For all natural numbers k there exists an ordinal β_k such that $\beta_k < \alpha$, H_{β_k} has derived length at least k. Moreover, we may assume that $\beta_i < \beta_{i+1}$ for all natural numbers i. Let $T_i = H_{\beta_i}$ for all natural numbers i. It follows that G has an ascending series of normal soluble subgroups $1 = T_0 \leq T_1 \leq \cdots \leq \cdots$. Then the subgroup $T_\omega = \cup_{n\in\mathbf{N}} T_n$ is not soluble and so $T_\omega = H_\alpha$. A series $1 = W_0 \leq W_1 \leq \cdots \leq W_n \leq \cdots \leq W_\omega = \cup_{n\in\mathbf{N}} W_n \leq G$ with the propeties referred in the theorem can be obtained from the series $1 = T_0 \leq T_1 \leq \cdots \leq T_\omega \leq G$.

□

Lemma 9. *Let A be an $\mathbf{R}G$-module, G be a group, $\mathbf{R}$ be an associative ring. Suppose that the cocentralizer of G in the module A is an infinite $\mathbf{R}$-module, G satisfies the condition $min-nf$ and $G = FFD(G)$. Then the quotient group $G/G^{\mathfrak{F}}$ is finite.*

Proof. We suppose for a contradiction that the quotient group $G/G^{\mathfrak{F}}$ is infinite. Then G has an infinite strictly descending series of normal subgroups $G > N_1 > N_2 > \cdots$, such that the quotient groups G/N_i are finite for each i. Therefore there exists k for which the quotient group G/N_k is finite and the cocentralizer of N_k in the module A is a finite $\mathbf{R}$-module. Since $G = FFD(G)$, there is the subgroup H such that its cocentralizer in the module A is a finite $\mathbf{R}$-module and $G = HN_k$. Hence the cocentralizer of G in the module A is a finite $\mathbf{R}$-module. Contradiction.

□

Lemma 10. *Let A be an $\mathbf{R}G$-module, G be a locally soluble group, $\mathbf{R}$ be an associative ring. Suppose that the cocentralizer of G in the module A is an infinite $\mathbf{R}$-module and G satisfies the condition $min-nf$. If G has an ascending series of normal subgroups $1 = W_0 \leq W_1 \leq \cdots \leq W_n \leq \cdots \leq \cup_{n\geq 1} W_n = G$, in which the cocentralizer of each subgroup W_n in the module A is a finite $\mathbf{R}$-module, and each factor W_{n+1}/W_n is abelian, then G is soluble.*

Proof. Since the quotient module $A/C_A(W_k)$ is a finite $\mathbf{R}$-module for each $k \in \mathbb{N}$ then there is the series of $\mathbf{R}G$-submodules $A = A_0 \geq A_1 \geq A_2 \geq \cdots \geq A_k \geq \cdots \geq A_\omega = C_A(G)$, such that $A_k = C_A(W_k)$ and each factor A_k/A_{k+1} is a finite $\mathbf{R}G$-module. Let $H = \cap_{j\geq 0} C_G(A_j/A_{j+1})$. Then $G/C_G(A_j/A_{j+1})$ is finite for each $j \in \mathbb{N}$. Since G/H embeds in the Cartesian product of the quotient groups $G/C_G(A_j/A_{j+1})$, it follows that G/H is residually finite. Moreover, G is a union of subgroups such that their cocentralizers in the module A are finite $\mathbf{R}$-modules. Hence $G = FFD(G)$. By Lemma 9 the quotient group G/H is finite.

Since $G = FFD(G)$ then the cocentralizer of H in the module A is an infinite $\mathbf{R}$-module. We shall prove that H is soluble. Let $L_j = C_H(A/A_j)$, $j = 1, 2, \cdots$. Let $H \neq L_j$ for some j. The quotient group H/L_j is finite for each $j = 1, 2, \cdots$. We suppose that there is the number j such that the cocentralizer of L_j in the module A is a finite $\mathbf{R}$-module. Let j be the least number with this property. It follows that the cocentralizer of L_{j-1} in the module A is an infinite $\mathbf{R}$-module. On the other hand since the quotient group L_{j-1}/L_j is finite and $G = FFD(G)$, then the cocentralizer of L_{j-1} in the module A is a finite $\mathbf{R}$-module. We have contradiction. Therefore the cocentralizer of each subgroup L_j in the module A is an infinite $\mathbf{R}$-module. Since H satisfies the condition $min-nf$ then there exists the number m such that $L_j = L_m$ for all $j \geq m$. From this fact and from the choice of subgroup L_j it follows that the subgroup L_m is soluble. Since the quotient group H/L_m is finite then H is also soluble. Then G is soluble.

□

From the obtained results it follows Theorem 2.

Theorem 2. *Let A be an $\mathbf{R}G$-module, G be a locally soluble group, $\mathbf{R}$ be an associative ring. Suppose that if the cocentralizer of G in the module A is an infinite $\mathbf{R}$-module, G satisfies the condition $min-nf$. Then G is soluble.*

Theorem 3. *Let A be an $\mathbf{R}G$-module, G be a locally soluble group, $\mathbf{R}$ be an associative ring. Suppose that the cocentralizer of G in the module A is an infinite $\mathbf{R}$-module and G satisfies the condition $min-nf$. Then G has the normal abelian subgroup H such that G/H is Chernikov.*

Proof. It should be noted that by Theorem 2 the group G is soluble. To accomplish this proof we consider the case when G is not Chernikov.

Let $G = D_0 \geq D_1 \geq D_2 \geq \cdots \geq D_n = 1$ be the derived series of G. There exists the number m such that the cocentralizer of D_m in the module A is an infinite $\mathbf{R}$-module but the cocentralizer of D_{m+1} in the module A is a finite $\mathbf{R}$-module. By Lemma 4 the quotient groups D_i/D_{i+1}, $i = 0, 1, ..., m$, are Chernikov. Let $U = D_{m+1}$. Then the quotient group G/U is Chernikov. Let $C = C_A(U)$. C is an $\mathbf{R}G$-submodule of A. Therefore there exists the series of -submodules

$$\langle 0 \rangle \leq C \leq A,$$

such that A/C is a finite $\mathbf{R}$-module. Then $G/C_G(A/C)$ is finite.

Let $H = C_G(C) \cap C_G(A/C)$. The subgroup H acts trivially on each factor of the series $\langle 0 \rangle \leq C \leq A$. Therefore H is abelian. Since the quotient group G/U is Chernikov and $U \leq C_G(C)$ then the quotient group $G/C_G(C)$ is also Chernikov. By Remak theorem $G/H \leq G/C_G(C) \times G/C_G(A/C)$. It follows that G/H is Chernikov. Therefore G contains the normal abelian subgroup H such that G/H is Chernikov. □

In the paper the author have used the methods of the proofs of [5].

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Author's address

O. Yu. Dashkova
The Branch of Moscow state university in Sevastopol, Department of Mathematics, 7 Sevastopol heroes street, 99001 Sevastopol, Russia
E-mail address: odashkova@yandex.ru

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NONLINEAR BERNSTEIN-TYPE OPERATORS PROVIDING A BETTER ERROR ESTIMATION

OKTAY DUMAN

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Abstract. In this paper, when approximating a continuos non-negative function on the unit interval, we present an alternative way to the classical Bernstein polynomials. Our new operators become nonlinear, however, for some classes of functions, they provide better error estimations than the Bernstein polynomials. Furthermore, we obtain a simultaneous approximation result for these operators.

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1. INTRODUCTION

Polynomials appear in a wide variety of areas of mathematics and science. We often need polynomials rather than more complicated functions as they are simply defined, can be calculated on computer systems and represent a tremendous variety of functions. However, sometimes it may be needed to modify polynomials in order to get a more efficient approximation. This paper concerns with the modification of the classical sequence of Bernstein polynomials (see [1]) which is one of the most widely used tools in approximation theory and numerical analysis. We show that our modified operators become nonlinear with respect to a given function, however, for some classes of functions, they provide better error estimations than the Bernstein polynomials. More precisely, we introduce the following operators:

$$L_{n,\alpha}(f;x) := \left\{ \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f^{\alpha}\left(\frac{k}{n}\right) \right\}^{1/\alpha}, \tag{1.1}$$

where f is a non-negative function on $[0,1]$, $x \in [0,1]$, $n \in \mathbb{N}$ and $\alpha > 0$. Observe that

$$L_{n,\alpha}(f;x) = \left(B_n\left(f^{\alpha};x\right)\right)^{1/\alpha},$$

where B_n denotes the classical Bernstein polynomials. We refer [2] for detailed investigations on the Bernstein polynomials. Observe that $L_{n,1}(f;x) = B_n(f,x)$. From (1.1), we see that operators $L_{n,\alpha}$ are positive but nonlinear, and also preserve

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the positive constant functions. If $e_\alpha(x) = x^\alpha$ $(\alpha > 0)$, then $L_{n,\frac{1}{\alpha}}$ preserves e_α, i.e., we get

$$L_{n,\frac{1}{\alpha}}(e_\alpha;x) = e_\alpha(x) = x^\alpha.$$

One can also check that

$$L_{n,\alpha}(f;0) = f(0) \text{ and } L_{n,\alpha}(f;1) = f(1) \text{ for any } \alpha > 0.$$

If $0 \le k \le f \le K$ on $[0,1]$ for some constants k and K, then it is clear that

$$k \le L_{n,\alpha}(f) \le K \text{ on } [0,1].$$

Using Hölder's inequality, we also get

$$L_{n,\frac{1}{\alpha}}(f;x) \le B_n(f,x) \le L_{n,\alpha}(f;x) \text{ for any } \alpha \ge 1. \tag{1.2}$$

For a fixed $\alpha > 0$, if f^α is monotonically increasing or decreasing on $[0,1]$, then so are all $L_{n,\alpha}(f)$.

2. Approximation properties of nonlinear Bernstein-type operators

Now, by $C([0,1],[0,+\infty))$ we denote the class of all non-negative and continuous functions on $[0,1]$.

Theorem 1. *For any $f \in C([0,1],[0,+\infty))$ and $\alpha > 0$, we have $L_{n,\alpha}(f) \rightrightarrows f$ on $[0,1]$, where the symbol $\rightrightarrows$ denotes the uniform convergence.*

Proof. Let $f \in C([0,1],[0,+\infty))$ and $\alpha > 0$. Assuming $0 \le f \le K$ on the interval $[0,1]$ define the function $g : [0,K^\alpha] \to [0,K]$ by $g(y) = y^{1/\alpha}$. Then, by the uniform continuity of g on $[0,K^\alpha]$ and the uniform convergence $B_n(f^\alpha)$ to f^α on $[0,1]$, we observe that $g \circ B_n(f^\alpha)$ is uniformly convergent to $g \circ f^\alpha$ on the interval $[0,1]$. This means that $L_{n,\alpha}(f) \rightrightarrows f$ on $[0,1]$. □

In Figure 1, it is indicated an approximation to the function

$$f(x) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{5\pi x}{2}\right)$$

by means of the operators $L_{n,3}(f;x)$ for $n = 15, 30, 50, 100$, respectively.

On the other hand, it follows from the general Leibniz rule

$$(u.v)^{(k)} = \sum_{i=0}^{k} \binom{k}{i} u^{(k-i)} v^{(i)}$$

that

$$(u_1 u_2 ... u_m)^{(k)} = \sum_{i_1=0}^{k} \sum_{i_2=0}^{i_1} ... \sum_{i_{m-1}=0}^{i_{m-2}} \binom{k}{i_1}\binom{i_1}{i_2}...\binom{i_{m-2}}{i_{m-1}} \times u_1^{(k-i_1)} u_2^{(i_1-i_2)} ... u_{m-1}^{(i_{m-2}-i_{m-1})} u_m^{(i_{m-1})},$$

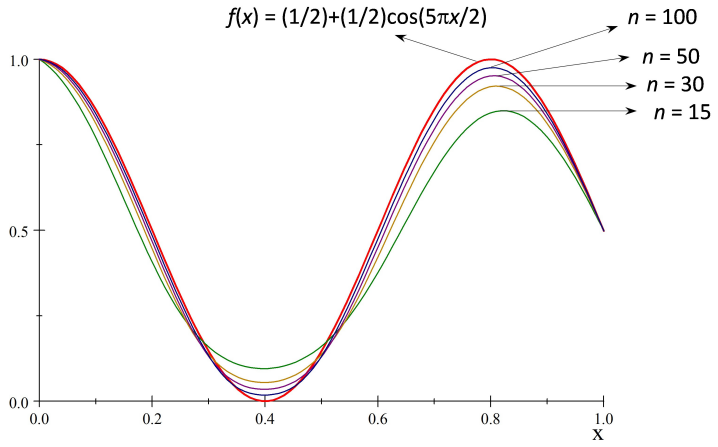

FIGURE 1. Approximation to $f(x) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{5\pi x}{2}\right)$ by $L_{n,3}(f;x)$ for $n = 15, 30, 50, 100$, respectively.

where, as usual, $f^{(k)} = \dfrac{d^k f}{dx^k}$. Now using this, we obtain the following simultaneous approximation result for $L_{n,\alpha}(f)$.

Theorem 2. *If $\alpha = 1/m$, $m \in \mathbb{N}$, then, we get $\left\{L_{n,\frac{1}{m}}(f)\right\}^{(k)} \rightrightarrows f^{(k)}$ on $[0,1]$ for every $f \in C^k([0,1],[0,+\infty))$ with $k \leq nm$.*

Proof. First observe that $L_{n,\frac{1}{m}}(f;x)$ is a polynomial with degree $\leq nm$. Then, for each $k \leq nm$, we may write that

$$\begin{aligned}\left\{L_{n,\frac{1}{m}}(f)\right\}^{(k)} &= \left\{\left(B_n\left(f^{1/m}\right)\right)^m\right\}^{(k)} \\ &= \sum_{i_1=0}^{k}\sum_{i_2=0}^{i_1}\cdots\sum_{i_{m-1}=0}^{i_{m-2}}\binom{k}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{m-2}}{i_{m-1}} \\ &\quad\times\left\{B_n\left(f^{1/m}\right)\right\}^{(k-i_1)}\left\{B_n\left(f^{1/m}\right)\right\}^{(i_1-i_2)} \\ &\quad\times\ldots\left\{B_n\left(f^{1/m}\right)\right\}^{(i_{m-2}-i_{m-1})}\left\{B_n\left(f^{1/m}\right)\right\}^{(i_{m-1})}.\end{aligned}$$

Since $\{B_n(f)\}^{(k)} \rightrightarrows f^{(k)}$ on $[0,1]$, we see that

$$\left\{L_{n,\frac{1}{m}}(f)\right\}^{(k)} \rightrightarrows \sum_{i_1=0}^{k}\sum_{i_2=0}^{i_1}\cdots\sum_{i_{m-1}=0}^{i_{m-2}}\binom{k}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{m-2}}{i_{m-1}}$$

$$\times \left\{\left(f^{1/m}\right)\right\}^{(k-i_1)} \left\{\left(f^{1/m}\right)\right\}^{(i_1-i_2)}$$
$$\times \ldots \left\{\left(f^{1/m}\right)\right\}^{(i_{m-2}-i_{m-1})} \left\{\left(f^{1/m}\right)\right\}^{(i_{m-1})}$$
$$= \left(\underbrace{f^{1/m} f^{1/m} \ldots f^{1/m} f^{1/m}}_{m \text{ times}}\right)^{(k)}$$
$$= f^{(k)},$$

which completes the proof. □

Here, we give the following conjecture:

Conjecture 1. *For any $\alpha > 0$ and $f \in C^k([0,1],[0,+\infty))$, $\{L_{n,\alpha}(f)\}^{(k)} \rightrightarrows f^{(k)}$ on $[0,1]$.*

We also get the following approximation results.

Theorem 3. *If $\alpha = 1/m$, $m \in \mathbb{N}$, then, we get, for every $n \in \mathbb{N}$ and $f \in C([0,1],[0,+\infty))$, that*

$$\left\| L_{n,\frac{1}{m}}(f) - f \right\| \leq \frac{5}{4} m K^{\frac{m-1}{m}} \omega\left(f^{1/m}, \frac{1}{\sqrt{n}}\right),$$

where $K := \|f\|$ and ω denotes the classical modulus of continuity of f.

Proof. Using the identity

$$u^m - v^m = (u-v)(u^{m-1} + u^{m-2}v + \ldots + uv^{m-2} + v^{m-1}), \tag{2.1}$$

we see that

$$\left| L_{n,\frac{1}{m}}(f;x) - f(x) \right| = \left| \left(B_n(f^{1/m};x)\right)^m - \left(f^{1/m}(x)\right)^m \right|$$
$$\leq \left| B_n(f^{1/m};x) - f^{1/m}(x) \right|$$
$$\times \sum_{i=1}^{m} \left(B_n(f^{1/m};x)\right)^{m-i} \left(f^{1/m}(x)\right)^{i-1}.$$

Since $0 \leq f(x) \leq K$ for every $x \in [0,1]$, we obtain that

$$\left| L_{n,\frac{1}{m}}(f;x) - f(x) \right| \leq m K^{\frac{m-1}{m}} \left| B_n(f^{1/m};x) - f^{1/m}(x) \right|.$$

Now taking supremum over $x \in [0,1]$ and also using the approximation order of the classical Bernstein polynomials for the function $f^{1/m}$, the proof is completed. □

Theorem 4. *If $\alpha = m$, $m \in \mathbb{N}$, then, we get, for every $n \in \mathbb{N}$ and $f \in C([0,1],[0,+\infty))$ for which $\|f\| \geq k > 0$, that*

$$\|L_{n,m}(f) - f\| \leq \frac{5}{4mk^{\frac{m-1}{m}}}\omega\left(f^m, \frac{1}{\sqrt{n}}\right).$$

Proof. In (2.1) if we replace u and v with $u^{1/m}$ and $v^{1/m}$, respectively, then we have

$$u^{1/m} - v^{1/m} = \frac{u - v}{u^{\frac{m-1}{m}} + u^{\frac{m-2}{m}}v^{\frac{1}{m}} + \ldots + u^{\frac{1}{m}}v^{\frac{m-2}{m}} + v^{\frac{m-1}{m}}}.$$

Using the last equality, we see that

$$\begin{aligned}
|L_{n,m}(f;x) - f(x)| &= \left|\left(B_n(f^m;x)\right)^{1/m} - \left(f^m(x)\right)^{1/m}\right| \\
&= \frac{|B_n(f^m;x) - f^m(x)|}{\sum_{i=1}^{m}\left(B_n(f^m;x)\right)^{\frac{m-i}{m}}\left(f^m(x)\right)^{\frac{i-1}{m}}} \\
&\leq \frac{1}{mk^{\frac{m-1}{m}}}\,|B_n(f^m;x) - f^m(x)| \\
&\leq \frac{5}{4mk^{\frac{m-1}{m}}}\omega\left(f^m, \frac{1}{\sqrt{n}}\right),
\end{aligned}$$

whence the result. □

Thus, the following open problem arises.

Open Problem. As in Theorems 3 and 4, can we get an approximation order for any $\alpha > 0$?

The next result is useful when approximating a function f for which $f^{1/\alpha}$ $(\alpha \geq 1)$ is convex.

Theorem 5. *If $\alpha \geq 1$ and $f \in C([0,1],[0,+\infty))$ such that $f^{1/\alpha}$ is convex on $[0,1]$, then the error estimate $\left|L_{n,\frac{1}{\alpha}}(f;x) - f(x)\right|$ is better than $|B_n(f;x) - f(x)|$.*

Proof. It is well-known that if $f^{1/\alpha}$ is convex, then $B_n(f^{1/\alpha})$ is also convex, and $B_n(f^{1/\alpha};x) \geq f^{1/\alpha}(x)$ for every $x \in [0,1]$ (see, for instance, [3]). Thus, we get $L_{n,\frac{1}{\alpha}}(f;x) \geq f(x)$. Furthermore, it follows from (1.2) that

$$f(x) \leq L_{n,\frac{1}{\alpha}}(f;x) \leq B_n(f;x) \text{ for any } \alpha \geq 1,$$

whence the result. □

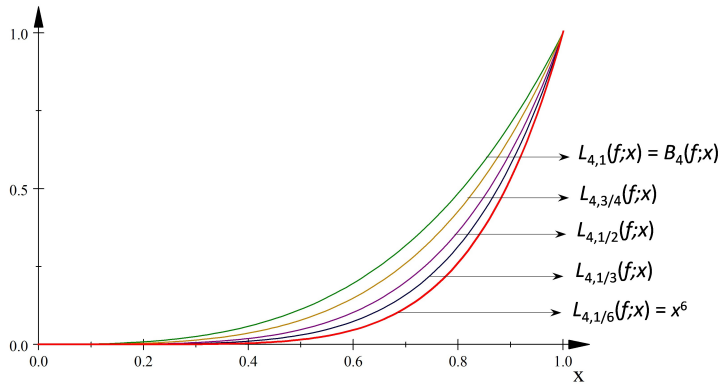

FIGURE 2. Approximation to $f(x) = x^6$ by $L_{4,\frac{1}{\alpha}}(f;x)$ for $\alpha = 1, \frac{4}{3}, 2, 3, 6$, respectively.

In Figure 2, taking $n = 4$ and $\alpha = 1, \dfrac{4}{3}, 2, 3, 6$, respectively, we graph $L_{4,\frac{1}{\alpha}}(f)$ for the function $f(x) = x^6$. Then, we observe that

$$\begin{aligned} f(x) = x^6 = L_{4,\frac{1}{6}}(f;x) &< L_{4,\frac{1}{3}}(f;x) \\ &< L_{4,\frac{1}{2}}(f;x) < L_{4,\frac{3}{4}}(f;x) \\ &< L_{4,1}(f;x) = B_4(f;x). \end{aligned}$$

Thus, in this example, for fixed x, n, we see that $L_{n,\alpha}(f;x)$ is getting close to $f(x) = x^6$ as α goes to $\dfrac{1}{6}$. Notice that $f^{1/\alpha}$ is convex for $0 < \alpha \leq 6$ and concave for $\alpha > 6$.

Similar result also holds for the approximation to concave functions.

Theorem 6. *If $\alpha \geq 1$ and $f \in C([0,1],[0,+\infty))$ such that f^{α} is concave on $[0,1]$, then the error estimate $|L_{n,\alpha}(f;x) - f(x)|$ is better than $|B_n(f;x) - f(x)|$.*

We get from Theorem 6 and Figure 3 that, for $n = 7$,

$$\begin{aligned} f(x) = \sin^{1/4}(\pi x) &> L_{7,4}(f;x) \\ &> L_{7,3}(f;x) > L_{7,2}(f;x) \\ &> L_{7,1}(f;x) = B_7(f;x). \end{aligned}$$

After Theorems 5 and 6, the following problem arises:

Open Problem. Is there any other class of functions satisfying a better error estimation as in Theorems 5 and 6 ?

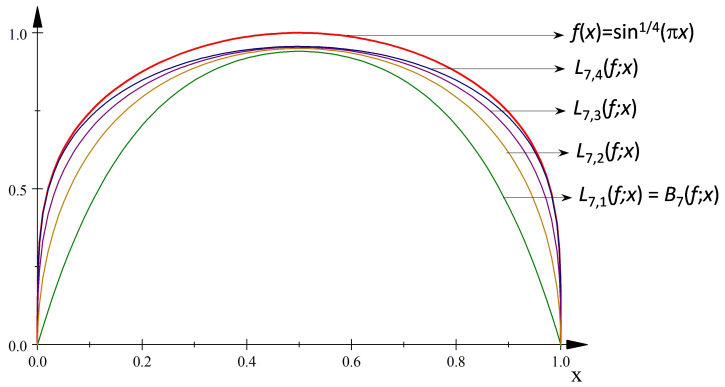

FIGURE 3. Approximation to $f(x) = \sin^{1/4}(\pi x)$ by $L_{7,\alpha}(f;x)$ for $\alpha = 1,2,3,4$, respectively.

3. EXTENSION TO THE MULTIVARIATE CASE

Our idea can be applied to the multivariate Bernstein polynomials in the following way. We first consider the standard unit simplex in $\mathbb{R}^m$ $(m \in \mathbb{N})$:

$$S_m = \{(x_1,...,x_m) : 0 \le x_i \le 1 \ (i = 1,...,m) \text{ and } x_1 + \cdots + x_m \le 1\}.$$

Now, instead of Cartesian coordinates, we denote barycentric coordinates by a boldface symbol:

$$\mathbf{x} = (x_0, x_1, ..., x_m) \text{ with } x_0 := 1 - x_1 - x_2 - \cdots - x_m.$$

Then, using the multi-index notations

$$\mathbf{x}^{\mathbf{k}} = x_0^{k_0} x_1^{k_1} \cdots x_m^{k_m},$$
$$|\mathbf{k}| = k_0 + k_1 + \cdots + k_m,$$
$$\binom{n}{\mathbf{k}} = \frac{n!}{k_0! k_1! \cdots k_m!},$$

for $\mathbf{x} \in \mathbb{R}^{m+1}$ and $\mathbf{k} = (k_0, k_1, ..., k_m) \in \mathbb{N}_0^{m+1}$, we define the following (nonlinear) multivariate operators:

$$L_{n,\alpha}(f;\mathbf{x}) = \left\{ \sum_{|\mathbf{k}|=n} \binom{n}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} f^{\alpha}\left(\frac{\mathbf{k}}{n}\right) \right\}^{1/\alpha}, \tag{3.1}$$

where $f : S_m \to [0, +\infty)$ is a function, $n \in \mathbb{N}$ and $\alpha > 0$.

In this case, as in Theorem 1, one can get the next result, immediately.

Theorem 7. *For any* $f \in C(S_m,[0,+\infty))$ *and* $\alpha > 0$, *we have* $L_{n,\alpha}(f;\mathbf{x}) \rightrightarrows f(\mathbf{x})$ *on* S_m.

The bivariate version of (3.1) with Cartesian coordinates may be written as follows:

$$L_{n,\alpha}(f;x,y) = \left\{ \sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{n!x^j y^k (1-x-y)^{n-j-k}}{j!k!(n-j-k)!} f^{\alpha}\left(\frac{j}{n},\frac{k}{n}\right) \right\}^{1/\alpha}$$

where $(x,y) \in S_2$, $n \in \mathbb{N}$, $\alpha > 0$ and $f \in C(S_2,[0,+\infty))$.

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Author's address

Oktay Duman
TOBB Economics and Technology University, Department of Mathematics, Söğütözü TR-06530, Ankara, Turkey
E-mail address: oduman@etu.edu.tr; okitayduman@gmail.com

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REMARKS ON QUASI-METRIC SPACES

N. V. DUNG

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Abstract. In this paper, we prove some properties of quasi-metric spaces and state some fixed point theorems in this setting. As applications, we show that most of recent results on G-metric spaces in [3, 10] may be also implied from certain fixed point theorems on metric spaces and quasi-metric spaces.

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1. Introduction and preliminaries

In recent time, many generalized metric spaces were introduced and the fixed point theory in these spaces was investigated. In [13], Mustafa and Sims introduced the concept of a G-metric space as a generalized metric space. After that, many fixed point theorems on G-metric spaces were stated, see [1,4,5,9,12,14,15] and references therein. But in [8], Jleli and Samet showed that most of the obtained fixed point theorems on G-metric spaces may be deduced immediately from fixed point theorems on metric spaces or quasi-metric spaces. The similar results can be found in [2, 17].

Very recently, Karapinar and Agarwal modified some existing results to suggest new fixed point theorems that fit with the nature of a G-metric space in [10]. Also, they asserted that for their results the techniques used in [8] and [17] are inapplicable. After that, this idea was continuously developed in [3, 7].

In this paper, we prove some properties of quasi-metric spaces and state some fixed point theorems in this setting. As applications, we show that most of recent results on G-metric spaces in [3, 10] may be also implied from certain fixed point theorems on metric spaces and quasi-metric spaces.

First, we recall notions and results which will be useful in what follows.

Definition 1 ([13], Definition 3)**.** Let X be a nonempty set and $G : X \times X \times X \longrightarrow [0, \infty)$ be a function such that, for all $x, y, z \in X$,

(1) $G(x, y, z) = 0$ if $x = y = z$.
(2) $0 < G(x, x, y)$ if $x \neq y \in X$.
(3) $G(x, x, y) \leq G(x, y, z)$ if $y \neq z$.

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(4) $G(x,y,z) = G(x,z,y) = G(y,x,z) = G(y,z,x) = G(z,x,y) = G(z,y,x)$.
(5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$.

Then G is called a *G-metric* on X and the pair (X,G) is called a *G-metric space*.

Definition 2 ([13], Definition 4)**.** The G-metric space (X,G) is called *symmetric* if $G(x,y,y) = G(x,x,y)$ for all $x,y \in X$.

Definition 3 ([13])**.** Let (X,G) be a G-metric space and $\{x_n\}$ be a sequence in X.

(1) For each $x_0 \in X$ and $r > 0$, the set
$$B_G(x_0,r) = \{x \in X : G(x_0,x,x) < r\}$$
is called a *G-ball with center* x_0 *and radius* r.
(2) The family of all G-balls forms a base of a topology $\tau(G)$ on X, and $\tau(G)$ is called the *G-metric topology*.
(3) $\{x_n\}$ is called *convergent* to x in X if $\lim_{n\to\infty} x_n = x$ in the G-metric topology $\tau(G)$.
(4) $\{x_n\}$ is called *Cauchy* in X if $\lim_{n,m,l\to\infty} G(x_n,x_m,x_l) = 0$.
(5) (X,G) is called a *complete G-metric* space if every Cauchy sequence is convergent.

Lemma 1 ([13], Proposition 6)**.** *Let (X,G) be a G-metric space. Then the following statements are equivalent.*

(1) x_n *is convergent to* x *in* X.
(2) $\lim_{n\to\infty} G(x_n,x_n,x) = 0$.
(3) $\lim_{n\to\infty} G(x_n,x,x) = 0$.
(4) $\lim_{n,m\to\infty} G(x_n,x_m,x) = 0$.

Lemma 2 ([13], Proposition 9)**.** *Let (X,G) be a G-metric space. Then the following statements are equivalent.*

(1) $\{x_n\}$ *is a Cauchy sequence.*
(2) $\lim_{n,m\to\infty} G(x_n,x_m,x_m) = 0$.

Definition 4 ([8], Definition 2.1)**.** Let X be a nonempty set and $d : X \times X \longrightarrow [0,+\infty)$ be a function such that, for all $x,y,z \in X$,

(1) $d(x,y) = 0$ if and only if $x = y$.
(2) $d(x,y) \leq d(x,z) + d(z,y)$.

Then d is called a *quasi-metric* and the pair (X,d) is called a *quasi-metric space*.

Definition 5 ([8]). Let (X,d) be a quasi-metric space and $\{x_n\}$ be a sequence in X.

(1) $\{x_n\}$ is called *convergent* to $x \in X$, written $\lim_{n\to\infty} x_n = x$, if
$$\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x,x_n) = 0.$$
(2) $\{x_n\}$ is called *left-Cauchy* if for each $\varepsilon > 0$ there exists $n(\varepsilon)$ such that $d(x_n,x_m) < \varepsilon$ for all $n \geq m > n(\varepsilon)$.
(3) $\{x_n\}$ is called *right-Cauchy* if for each $\varepsilon > 0$ there exists $n(\varepsilon)$ such that $d(x_n,x_m) < \varepsilon$ for all $m \geq n > n(\varepsilon)$.
(4) $\{x_n\}$ is called *Cauchy* if for each $\varepsilon > 0$ there exists $n(\varepsilon)$ such that $d(x_n,x_m) < \varepsilon$ for all $m,n > n(\varepsilon)$, that is, $\lim_{n,m\to\infty} d(x_n,x_m) = 0$.
(5) (X,d) is called *complete* if each Cauchy sequence in (X,d) is convergent.

Remark 1 ([8]). (1) Every metric is a quasi-metric.
(2) In a quasi-metric space, a sequence $\{x_n\}$ is Cauchy if and only if it is left-Cauchy and right-Cauchy.

The following examples show that the inversion of Remark 1.(1) does not hold.

Example 1. Let $X = \mathbb{R}$ and d be defined by
$$d(x,y) = \begin{cases} x-y & \text{if } x \geq y \\ 1 & \text{if } x < y. \end{cases}$$
Then d is a quasi-metric on X but d is not a metric on X.

Proof. It is clear that $d : X \times X \longrightarrow [0,+\infty)$ and $d(x,y) = 0$ if and only if $x = y$. For all $x,y,z \in X$, we consider two following cases.

Case 1. $x \geq y$. We have $d(x,y) = x-y$.
If $z < y$, then $d(x,z) = x-z$ and $d(z,y) = 1$.
If $y \leq z < x$, then $d(x,z) = x-z$ and $d(z,y) = z-y$.
If $x \leq z$, then $d(x,z) = 1$ and $d(z,y) = z-y$.
So we have $d(x,y) \leq d(x,z) + d(z,y)$.

Case 2. $x < y$. We have $d(x,y) = 1$.
If $z < x$, then $d(x,z) = x-z$ and $d(z,y) = 1$.
If $x \leq z < y$, then $d(x,z) = 1$ and $d(z,y) = 1$.
If $y \leq z$, then $d(x,z) = 1$ and $d(z,y) = z-y$.
So we have $d(x,y) \leq d(x,z) + d(z,y)$.

By the above, d is a quasi-metric on $\mathbb{R}$. Since $d(0,2) = 1 \neq d(2,0) = 2$, d is not a metric on $\mathbb{R}$. □

Example 2. Let $X = X_1 \cup X_2$, $X_1 \cap X_2 \neq \varnothing$ and d be defined by
$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 2 & \text{if } x \in X_1, y \in X_2 \\ 1 & \text{otherwise.} \end{cases}$$

Then d is a quasi-metric on X but d is not a metric on X.

For more quasi-metrics which are not metrics, see [16, Example 1.4].

In [8], Jleli and Samet showed that most of the obtained fixed point theorems on G-metric spaces may be deduced immediately from fixed point theorems on metric spaces or quasi-metric spaces. The main results in [8] are as follows.

Theorem 1 ([8], Theorem 2.2). *Let (X,G) be a G-metric space and $d_G : X \times X \longrightarrow [0,+\infty)$ be defined by $d_G(x,y) = G(x,y,y)$ for all $x,y \in X$. Then we have*

(1) *(X,d_G) is a quasi-metric space.*
(2) *A sequence $\{x_n\}$ is convergent to x in (X,G) if and only if $\{x_n\}$ is convergent to x in (X,d_G).*
(3) *A sequence $\{x_n\}$ is Cauchy in (X,G) if and only if $\{x_n\}$ is Cauchy in (X,d_G).*
(4) *The G-metric space (X,G) is complete if and only if the quasi-metric space (X,d_G) is complete.*

Theorem 2 ([8], Theorem 2.3). *Let (X,G) be a G-metric space and $\delta_G : X \times X \longrightarrow [0,+\infty)$ be defined by $\delta_G(x,y) = \max\{G(x,y,y), G(y,x,x)\}$ for all $x,y \in X$. Then we have*

(1) *(X,δ_G) is a metric space.*
(2) *A sequence $\{x_n\}$ is convergent to x in (X,G) if and only if $\{x_n\}$ is convergent to x in (X,δ_G).*
(3) *A sequence $\{x_n\}$ is Cauchy in (X,G) if and only if $\{x_n\}$ is Cauchy in (X,δ_G).*
(4) *The G-metric space (X,G) is complete if and only if the metric space (X,δ_G) is complete.*

Theorem 3 ([8], Theorem 3.2). *Let (X,d) be a complete quasi-metric space and $T : X \longrightarrow X$ be a map such that*

$$d(Tx,Ty) \leq d(x,y) - \varphi\big(d(x,y)\big) \tag{1.1}$$

for all $x,y \in X$, where $\varphi : [0,+\infty) \longrightarrow [0,+\infty)$ is continuous with $\varphi^{-1}\big(\{0\}\big) = \{0\}$. Then T has a unique fixed point.

Recently, in [16], Rajić proved the following result which is a generalization of Theorem 3.

Theorem 4 ([16], Theorem 2.2). *Let (X,d) be a complete quasi-metric space and $f,g : X \longrightarrow X$ be two maps such that*

$$\psi\big(d(fx,fy)\big) \leq \psi\big(d(gx,gy)\big) - \phi\big(d(gx,gy)\big) \tag{1.2}$$

for all $x,y \in X$, where $\psi : [0,+\infty) \longrightarrow [0,+\infty)$ is continuous, non-decreasing, $\psi^{-1}(0) = \{0\}$, $\phi : [0,+\infty) \longrightarrow [0,+\infty)$ is continuous and $\phi^{-1}(0) = \{0\}$. If the range of g contains the range of f and $f(X)$ or $g(X)$ is a closed subset of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly

compatible, that is, $fgx = gfx$ provided that $fx = gx$, then f and g have a unique common fixed point.

The main results of the paper are presented in Section 2 and Section 3. In Section 2, we prove some properties of the quasi-metric space and its modification. Then, by similar arguments as in metric spaces, we prove some analogues of fixed point theorems in quasi-metric spaces. In Section 3, we show that most of recent fixed point theorems on G-metric spaces in [3, 10] may be implied from certain fixed point theorems proved in Section 2.

2. Remarks on quasi-metric spaces

Note that every quasi-metric space (X,d) is a topological space with the topology induced by its convergence. Then $X \times X$ is a topological space with the product topology. The following result shows that the product space $X \times X$ is also a quasi-metric space.

Proposition 1. *Let (X,d_X) and (Y,d_Y) be two quasi-metric spaces. Then we have*

(1) *$d(x,y) = d_X(x_1,y_1) + d_Y(x_2,y_2)$ for all $x = (x_1,x_2), y = (y_1,y_2) \in X \times Y$ is a quasi-metric on $X \times Y$.*

(2) *$\lim_{n\to\infty}(x_n,y_n) = (x,y)$ in $(X \times Y,d)$ if and only if $\lim_{n\to\infty} x_n = x$ in (X,d_X) and $\lim_{n\to\infty} y_n = y$ in (Y,d_Y). In particular, the product topology on $X \times Y$ coincides the topology induced by d.*

(3) *$\{(x_n,y_n)\}$ is a Cauchy sequence in $(X \times Y,d)$ if and only if $\{x_n\}$ is a Cauchy sequence in (X,d_X) and $\{y_n\}$ is a Cauchy sequence in (Y,d_Y).*

(4) *$(X \times Y,d)$ is complete if and only if (X,d_X) and (Y,d_Y) are complete.*

Proof. (1). For all $x = (x_1,x_2), y = (y_1,y_2), z = (z_1,z_2) \in X \times Y$, we have $d(x,y) = 0$ if and only if $d_X(x_1,y_1) + d_Y(x_2,y_2) = 0$, that is, $d_X(x_1,y_1) = d_Y(x_2,y_2) = 0$. It is equivalent to $x_1 = y_1$ and $x_2 = y_2$, that is, $x = y$.

We also have

$$\begin{aligned} d(x,z) &= d_X(x_1,z_1) + d_Y(x_2,z_2) \\ &\le d_X(x_1,y_1) + d_X(y_1,z_1) + d_Y(x_2,y_2) + d_Y(y_2,z_2) \\ &= d_X(x_1,y_1) + d_Y(x_2,y_2) + d_X(y_1,z_1) + d_Y(y_2,z_2) \\ &= d(x,y) + d(y,z). \end{aligned}$$

By the above, d is a quasi-metric on $X \times Y$.

(2). $\lim_{n\to\infty}(x_n,y_n) = (x,y)$ in $(X \times Y,d)$ if and only if

$$\lim_{n\to\infty} d\big((x_n,y_n),(x,y)\big) = \lim_{n\to\infty}\big[d_X(x_n,x) + d_Y(y_n,y)\big] = 0$$

and

$$\lim_{n\to\infty} d\big((x,y),(x_n,y_n)\big) = \lim_{n\to\infty}\big[d_X(x,x_n) + d_Y(y,y_n)\big] = 0.$$

It is equivalent to

$$\lim_{n\to\infty} d_X(x_n,x) = \lim_{n\to\infty} d_Y(y_n,y) = \lim_{n\to\infty} d_X(x,x_n) = \lim_{n\to\infty} d_Y(y,y_n) = 0.$$

That is, $\lim_{n\to\infty} x_n = x$ in (X,d_X) and $\lim_{n\to\infty} y_n = y$ in (Y,d_Y).

(3). $\{(x_n,y_n)\}$ is a Cauchy sequence in $(X\times Y,d)$ if and only if

$$\lim_{n,m\to\infty} d\big((x_n,y_n),(x_m,y_m)\big) = \lim_{n,m\to\infty} \big[d_X(x_n,x_m)+d_Y(y_n,y_m)\big] = 0.$$

It is equivalent to

$$\lim_{n,m\to\infty} d_X(x_n,x_m) = \lim_{n,m\to\infty} d_Y(y_n,y_m) = 0.$$

That is, $\{x_n\}$ is a Cauchy sequence in (X,d_X) and $\{y_n\}$ is a Cauchy sequence in (Y,d_Y).

(4). It is a direct consequence of (2) and (3). □

In the proof of [8, Theorem 3.2], Jleli and Samet used the sequential continuity of a quasi-metric d without proving. From Proposition 1, we see that the sequential continuity and the continuity of d are equivalent and they are guaranteed by the following proposition.

Proposition 2. *Let (X,d) be a quasi-metric space. Then d is a continuous function.*

Proof. Suppose that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ in (X,d). We have

$$d(x_n,y_n) \le d(x_n,x)+d(x,y)+d(y,y_n).$$

It implies that

$$d(x_n,y_n)-d(x,y) \le d(x_n,x)+d(y,y_n). \tag{2.1}$$

Also, we have

$$d(x,y) \le d(x,x_n)+d(x_n,y_n)+d(y_n,y).$$

It implies that

$$d(x,y)-d(x_n,y_n) \le d(x,x_n)+d(y_n,y). \tag{2.2}$$

From (2.1) and (2.2), we have

$$0 \le |d(x,y)-d(x_n,y_n)| \le \max\{d(x_n,x)+d(y,y_n), d(x,x_n)+d(y_n,y)\}. \tag{2.3}$$

Taking the limit as $n\to\infty$ in (2.3), we obtain $\lim_{n\to\infty} |d(x,y)-d(x_n,y_n)| = 0$. That is, $\lim_{n\to\infty} d(x_n,y_n) = d(x,y)$. This proves that d is a continuous function. □

The following proposition proves that the topology of each quasi-metric space is metrizable. Then all topological properties of metric spaces hold on quasi-metric spaces.

Proposition 3. *Let (X,d) be a quasi-metric space and*

$$\delta_d(x,y) = \max\{d(x,y), d(y,x)\}$$

for all $x, y \in X$. Then we have

(1) *(X,δ_d) is a metric space.*
(2) *A sequence $\{x_n\}$ is convergent to x in (X,d) if and only if $\{x_n\}$ is convergent to x in (X,δ_d).*
(3) *A sequence $\{x_n\}$ is Cauchy in (X,d) if and only if $\{x_n\}$ is Cauchy in (X,δ_d).*
(4) *The quasi-metric space (X,d) is complete if and only if the metric space (X,δ_d) is complete.*

Proof. (1). See [8], page 3.
(2). We have $\lim_{n\to\infty} x_n = x$ in (X,d) if and only if

$$\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x,x_n) = 0.$$

It is equivalent to

$$\lim_{n\to\infty} \delta_d(x_n,x) = \lim_{n\to\infty} \max\{d(x_n,x), d(x,x_n)\} = 0.$$

That is, $\lim_{n\to\infty} x_n = x$ in (X,δ_d).
(3). A sequence $\{x_n\}$ is Cauchy in (X,d) if and only if

$$\lim_{n,m\to\infty} d(x_n,x_m) = \lim_{n,m\to\infty} d(x_m,x_n) = 0.$$

It is equivalent to

$$\lim_{n,m\to\infty} \delta_d(x_n,x_m) = \lim_{n,m\to\infty} \max\{d(x_n,x_m), d(x_m,x_n)\} = 0.$$

That is, $\{x_n\}$ is Cauchy in (X,δ_d).
(4). It is a direct consequence of (2) and (3). □

By modifying the notion of T-orbital completeness in [6], we introduce the notion of weak T-orbital completeness as follows.

Definition 6. Let (X,d) be a quasi-metric space and $T : X \longrightarrow X$ be a map. Then X is called *weak T-orbitally complete* if $\{T^n x\}$ is convergent in X provided that it is a Cauchy sequence in X.

Note that every T-orbitally complete quasi-metric space is a weak T-orbitally complete quasi-metric space for all maps $T : X \longrightarrow X$. The following example shows that the inversion does not hold, even when (X,d) is a metric space.

Example 3. Let $X = \{1,3,\ldots,2n+1,\ldots\} \cup \{\frac{1}{2}, \frac{1}{4}\ldots, \frac{1}{2n},\ldots\}$ with the usual metric and

$$T\frac{1}{2n} = 2n+1, T(2n-1) = \frac{1}{2n}$$

for all $n \in \mathbb{N}$. Since $\{T^n x\}$ is not Cauchy for all $x \in X$, (X,d) is weak T-orbitally complete. For $x = 1$, we have

$$\{T^n 1 : n \in \mathbb{N}\} = \Big\{1, \frac{1}{2}, 3, \frac{1}{4}, \ldots, 2n+1, \frac{1}{2n}, \ldots\Big\}.$$

Since $\Big\{\frac{1}{2n}\Big\}$ is a Cauchy sequence in (X,d) which is not convergent, (X,d) is not T-orbitally complete.

Following the proof of [11, Theorem 3.1], we get the following fixed point theorem on quasi-metric spaces.

Theorem 5. *Let (X,d) be a quasi-metric space and $T : X \longrightarrow X$ be a map such that*

(1) *X is weak T-orbitally complete.*
(2) *There exists $q \in [0,1)$ such that for all $x, y \in X$,*

$$\begin{aligned} d(Tx,Ty) \leq q \max \big\{ & d(x,y), d(y,x), d(x,Tx), d(Tx,x), d(y,Ty), \\ & d(x,Ty), d(y,Tx), d(Tx,y), d(T^2x,x), d(x,T^2x), d(T^2x,Tx), \\ & d(Tx,T^2x), d(T^2x,y), d(y,T^2x), d(T^2x,Ty)\big\}. \end{aligned} \tag{2.4}$$

Then we have

(1) *T has a unique fixed point x^* in X.*
(2) *$\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$.*
(3) *$\max\big\{d(T^n x, x^*), d(x^*, T^n x)\big\} \leq \dfrac{q^n}{1-q} \max\big\{d(x,Tx), d(Tx,x)\big\}$ for all $x \in X$ and $n \in \mathbb{N}$.*

Proof. (1). For each $x \in X$ and $1 \leq i \leq n-1$, $1 \leq j \leq n$, we have

$$\begin{aligned} & d(T^i x, T^j x) = d(TT^{i-1}x, TT^{j-1}x) \\ & \leq q \max \big\{ d(T^{i-1}x, T^{j-1}x), d(T^{j-1}x, T^{i-1}x), d(T^{i-1}x, TT^{i-1}x), \\ & \quad d(TT^{i-1}x, T^{i-1}x), d(T^{j-1}x, TT^{j-1}x), d(T^{i-1}x, TT^{j-1}x), d(T^{j-1}x, TT^{i-1}x), \\ & \quad d(TT^{i-1}x, T^{j-1}x), d(T^2T^{i-1}x, T^{i-1}x), d(T^{i-1}x, T^2T^{i-1}x), \\ & \quad d(T^2T^{i-1}x, TT^{i-1}x), d(TT^{i-1}x, T^2T^{i-1}x), d(T^2T^{i-1}x, T^{j-1}x), \\ & \quad d(T^{j-1}x, T^2T^{i-1}x), d(T^2T^{i-1}x, TT^{j-1}x)\big\} \\ & = q \max \big\{ d(T^{i-1}x, T^{j-1}x), d(T^{j-1}x, T^{i-1}x), d(T^{i-1}x, T^i x), d(T^i x, T^{i-1}x), \\ & \quad d(T^{j-1}x, T^j x), d(T^{i-1}x, T^j x), d(T^{j-1}x, T^i x), d(T^i x, T^{j-1}x), \\ & \quad d(T^{i+1}x, T^{i-1}x), d(T^{i-1}x, T^{i+1}x), d(T^{i+1}x, T^i x), d(T^i x, T^{i+1}x), \\ & \quad d(T^{i+1}x, T^{j-1}x), d(T^{j-1}x, T^{i+1}x), d(T^{i+1}x, T^j x)\big\} \end{aligned} \tag{2.5}$$

$\leq q\delta\big[O_T(x,n)\big]$

where $\delta\big[O_T(x,n)\big] = \max\big\{d(T^i x, T^j x) : 0 \leq i \leq n-1, 0 \leq j \leq n\big\}$.

From (2.5), since $0 \leq q < 1$, there exists $k_n(x) \leq n$ such that

$$d(x, T^{k_n(x)}x) = \delta\big[O_T(x,n)\big] \tag{2.6}$$

or there exists $k_n(x) \leq n-1$ such that

$$d(T^{k_n(x)}x, x) = \delta\big[O_T(x,n)\big]. \tag{2.7}$$

If (2.6) holds, we have

$$\begin{aligned} d(x, T^{k_n(x)}x) &\leq d(x, Tx) + d(Tx, T^{k_n(x)}x) \\ &\leq d(x, Tx) + q\delta\big[O_T(x,n)\big] \\ &= d(x, Tx) + qd(x, T^{k_n(x)}x). \end{aligned}$$

It implies that

$$\delta\big[O_T(x,n)\big] = d(x, T^{k_n(x)}x) \leq \frac{1}{1-q}d(x, Tx). \tag{2.8}$$

If (2.7) holds, we have

$$\begin{aligned} d(T^{k_n(x)}x, x) &\leq d(T^{k_n(x)}x, Tx) + d(Tx, x) \\ &\leq q\delta\big[O_T(x,n)\big] + d(Tx, x) \\ &= qd(T^{k_n(x)}x, x) + d(Tx, x). \end{aligned}$$

It implies that

$$\delta\big[O_T(x,n)\big] = d(T^{k_n(x)}x, x) \leq \frac{1}{1-q}d(Tx, x). \tag{2.9}$$

For all $n < m$, it follows from (2.4) and (2.8), (2.9) that

$$\begin{aligned} d(T^n x, T^m x) &= d(TT^{n-1}x, T^{m-n+1}T^{n-1}x) \\ &\leq q\delta\big[O_T(T^{n-1}x, m-n+1)\big] \\ &= qd(T^{n-1}x, T^{k_{m-n+1}(T^{n-1}x)}T^{n-1}x) \\ &= qd(TT^{n-2}x, T^{k_{m-n+1}(T^{n-1}x)+1}T^{n-2}x) \\ &\leq q^2\delta\big[O_T(T^{n-2}x, k_{m-n+1}(T^{n-1}x)+1)\big] \\ &\leq q^2\delta\big[O_T(T^{n-2}x, m-n+2)\big] \\ &\leq \cdots \\ &\leq q^n\delta\big[O_T(x,m)\big] \\ &\leq \frac{q^n}{1-q}\max\{d(x,Tx), d(Tx,x)\}. \end{aligned} \tag{2.10}$$

Since $\lim_{n\to\infty} q^n = 0$, by taking the limit as $n, m \to \infty$ in (2.10), we have

$$\lim_{n,m\to\infty} d(T^n x, T^m x) = 0. \tag{2.11}$$

This proves that $\{T^n x\}$ is a Cauchy sequence in X. Since X is weak T-orbitally complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(T^n x, x^*) = \lim_{n\to\infty} d(x^*, T^n x) = 0. \tag{2.12}$$

Therefore, by using (2.4) again, we have

$$\begin{aligned}
&d(x^*, Tx^*) \le d(x^*, T^{n+1}x) + d(T^{n+1}x, Tx^*) \qquad (2.13)\\
&= d(x^*, T^{n+1}x) + d(TT^n x, Tx^*)\\
&\le d(x^*, T^{n+1}x) + q\max\{d(T^n x, x^*), d(x^*, T^n x), d(T^n x, TT^n x),\\
&\quad d(TT^n x, T^n x), d(x^*, Tx^*), d(T^n x, Tx^*), d(x^*, TT^n x), d(TT^n x, x^*),\\
&\quad d(T^2T^n x, T^n x), d(T^n x, T^2T^n x), d(T^2T^n x, TT^n x), d(TT^n x, T^2T^n x),\\
&\quad d(T^2T^n x, x^*), d(x^*, T^2T^n x), d(T^2T^n x, Tx^*)\}\\
&= d(x^*, T^{n+1}x) + q\max\{d(T^n x, x^*), d(x^*, T^n x), d(T^n x, T^{n+1}x),\\
&\quad d(T^{n+1}x, T^n x), d(x^*, Tx^*), d(T^n x, Tx^*), d(x^*, T^{n+1}x), d(T^{n+1}x, x^*),\\
&\quad d(T^{n+2}x, T^n x), d(T^n x, T^{n+2}x), d(T^{n+2}x, T^{n+1}x), d(T^{n+1}x, T^{n+2}x),\\
&\quad d(T^{n+2}x, x^*), d(x^*, T^{n+2}x), d(T^{n+2}x, Tx^*)\}.
\end{aligned}$$

Taking the limit as $n \to \infty$ in (2.13), and using (2.11), (2.12) and Proposition 2, we get $d(x^*, Tx^*) \le q d(x^*, Tx^*)$. Since $q \in [0,1)$, $d(x^*, Tx^*) = 0$, that is, $x^* = Tx^*$. Then T has a fixed point.

Now, we prove the uniqueness of the fixed point of T. Let x^*, y^* be two fixed points of T. From (2.4), we have

$$\begin{aligned}
&d(x^*, y^*) = d(Tx^*, Ty^*)\\
&\le q\max\{d(x^*, y^*), d(y^*, x^*), d(x^*, Tx^*), d(Tx^*, x^*), d(y^*, Ty^*), d(x^*, Ty^*),\\
&\quad d(y^*, Tx^*), d(y^*, Tx^*), d(T^2x^*, x^*), d(x^*, T^2x^*), d(T^2x^*, Tx^*),\\
&\quad d(Tx^*, T^2x^*), d(T^2x^*, y^*), d(y^*, T^2x^*), d(T^2x^*, Ty^*)\}\\
&= q\max\{d(x^*, y^*), d(y^*, x^*)\}.
\end{aligned}$$

Since $q \in [0,1)$, we get

$$d(x^*, y^*) \le q d(y^*, x^*). \tag{2.14}$$

Again, from (2.4), we also have

$$\begin{aligned}
&d(y^*, x^*) = d(Ty^*, Tx^*)\\
&\le q\max\{d(y^*, x^*), d(x^*, y^*), d(y^*, Ty^*), d(Ty^*, y^*), d(x^*, Tx^*), d(y^*, Tx^*),
\end{aligned}$$

$$d(x^*,Ty^*),d(x^*,Ty^*),d(T^2y^*,y^*),d(y^*,T^2y^*),d(T^2y^*,Ty^*),$$
$$d(Ty^*,T^2y^*),d(T^2y^*,x^*),d(x^*,T^2y^*),d(T^2y^*,Tx^*)\}$$
$$= q\max\{d(y^*,x^*),d(x^*,y^*)\}.$$

Since $q \in [0,1)$, we get

$$d(y^*,x^*) \le qd(x^*,y^*). \tag{2.15}$$

From (2.14) and (2.15), since $q \in [0,1)$, we obtain $d(x^*,y^*) = 0$. That is, $x^* = y^*$. Then the fixed point of T is unique.

(2). It is proved by (2.12).

(3). Taking the limit as $m \to \infty$ in (2.10) and using Proposition 2, we get

$$d(T^nx,x^*) \le \frac{q^n}{1-q}\max\{d(x,Tx),d(Tx,x)\}.$$

Similarly, we have $d(x^*,T^nx) \le \frac{q^n}{1-q}\max\{d(x,Tx),d(Tx,x)\}$. Therefore,

$$\max\{d(T^nx,x^*),d(x^*,T^nx)\} \le \frac{q^n}{1-q}\max\{d(x,Tx),d(Tx,x)\}. \qquad \square$$

If d in Theorem 5 is a metric, then we have the following result. Note that Corollary 1 is a generalization of the following well-known result of Ćirić in [6]. This generalization is proper by [11, Example 3.6].

Corollary 1 ([11], Theorem 3.1)**.** *Let (X,d) be a metric space and $T : X \longrightarrow X$ be a map satisfying the following*

(1) *X is weak T-orbitally complete.*

(2) *There exists $q \in [0,1)$ such that for all $x,y \in X$,*

$$d(Tx,Ty) \le q\max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty), \tag{2.16}$$
$$d(y,Tx),d(T^2x,x),d(T^2x,Tx),d(T^2x,y),d(T^2x,Ty)\}.$$

Then we have

(1) *T has a unique fixed point x^* in X.*

(2) *$\lim_{n\to\infty} T^nx = x^*$ for all $x \in X$.*

(3) *$d(T^nx,x^*) \le \frac{q^n}{1-q}d(x,Tx)$ for all $x \in X$.*

Now, we modify the notion of a quasi-metric space in Definition 4 as follows.

Definition 7. Let X be a nonempty set and $d : X \times X \longrightarrow [0,+\infty)$ be a function such that, for all $x,y,z \in X$,

(1) $d(x,y) = 0$ if and only if $x = y$.

(2) $d(x,y) \le d(x,z) + d(y,z)$ if $z \ne x$.

Then d is also called a *quasi-metric* and the pair (X,d) is also called a *quasi-metric space*.

Note that if $z = x$ in Definition 7.(2), then $d(x,y) = d(y,x)$ for all $x,y \in X$. In this case, d deduces a metric on X.

The following proposition gives a way to set examples of quasi-metrics in the sense of Definition 7.

Proposition 4. *Let (X,G) be a G-metric space and $T : X \longrightarrow X$ be a map. For all $x,y \in X$, put*

$$d_{T,G}(x,y) = \begin{cases} 0 & \text{if } x = y \\ G(x,Tx,y) & \text{if } x \neq y. \end{cases}$$

If T has no any fixed point, then $d_{T,G}$ is a quasi-metric in the sense of Definition 7 on X.

Proof. For all $x,y,z \in X$ with $z \neq x$, note that $y \neq Ty$ for all $y \in X$, we have $G(y,z,z) \leq G(z,y,Ty)$ for all $y,z \in X$. Then

$$\begin{aligned} d_{T,G}(x,y) &\leq G(x,Tx,y) \\ &= G(y,x,Tx) \\ &\leq G(y,z,z) + G(x,Tx,z) \\ &\leq G(z,y,Ty) + G(x,Tx,z) \\ &= d_{T,G}(x,z) + d_{T,G}(y,z). \end{aligned}$$

This proves that $d_{T,G}$ is a quasi-metric in the sense of Definition 7 on X. □

Example 4. Let $X = \{1,2,3\}$ and

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 2 & \text{if } (x,y) = (1,2) \\ 1 & \text{otherwise.} \end{cases}$$

Then d is a quasi-metric in the sense of Definition 7 on X. For $(2,1),(1,2),(1,1) \in X \times X$, we have

$$\begin{aligned} d(2,1) + d(1,2) &= 3 \\ d(2,1) + d(1,1) &= 1 \\ d(1,1) + d(2,1) &= 1. \end{aligned}$$

This proves that $d(2,1) + d(1,2) > d(2,1) + d(1,1) + d(1,1) + d(2,1)$. So Proposition 1.(1) does not hold for quasi-metrics in the sense of Definition 7.

Proposition 5. *Let (X,d) be a quasi-metric space in the sense of Definition 7. For each $y \in X$, if $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} d(y,x_n) = d(y,x)$.*

Proof. **Case 1.** $y = x$. Then we have

$$d(y,x) = d(x,x) = 0 = \lim_{n\to\infty} d(x,x_n) = \lim_{n\to\infty} d(y,x_n).$$

Case 2. $y \neq x$. If $x_n = x$ for infinitely many n, then

$$\lim_{m\to\infty} d(y,x_n) = d(y,x).$$

So, we may assume that $x_n \neq x$ for n large enough. Also, $y \neq x_n$ for n large enough. Then we have, for all $n \in \mathbb{N}$,

$$d(y,x) \leq d(y,x_n) + d(x,x_n) \leq d(y,x) + d(x_n,x) + d(x,x_n). \tag{2.17}$$

Taking the limit as $n \to \infty$ in (2.17), we get $\lim_{n\to\infty} d(y,x_n) = d(y,x)$. □

With some minor changes in the proof of Theorem 5, we have the following result. Note that these changes mainly relate to Definition 7.(2).

Proposition 6. *Let (X,d) be a quasi-metric space in the sense of Definition 7 and $T : X \longrightarrow X$ be a map satisfying the following*

(1) *X is weak T-orbitally complete.*
(2) *There exists $q \in [0,1)$ such that for all $x, y \in X$,*

$$\begin{aligned} &d(Tx,Ty) \\ &\leq q \max\big\{d(x,y), d(y,x), d(Tx,x), d(y,Ty), d(Ty,y), d(x,Ty), d(Ty,x), \\ &\quad d(Tx,y), d(T^2y,y), d(y,T^2y), d(T^2y,Ty), d(Ty,T^2y), d(T^2y,x), \\ &\quad d(x,T^2y), d(Tx,T^2y)\big\}. \end{aligned} \tag{2.18}$$

Then we have

(1) *T has a unique fixed point x^* in X.*
(2) *$\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$.*
(3) *$\max\big\{d(T^n x, x^*), d(x^*, T^n x)\big\} \leq \dfrac{q^n}{1-q} \max\big\{d(x,Tx), d(Tx,x)\big\}$ for all $x \in X$ and $n \in \mathbb{N}$.*

Proof. As in the proof of Theorem 5.(1), there exists $k_n(x) \leq n$ such that

$$d(x, T^{k_n(x)}x) = \delta\big[O_T(x,n)\big] \tag{2.19}$$

or there exists $k_n(x) \leq n-1$ such that

$$d(T^{k_n(x)}x, x) = \delta\big[O_T(x,n)\big]. \tag{2.20}$$

If $Tx = x$, then T has a fixed point. So we may assume that $Tx \neq x$. If (2.19) holds, we have

$$\begin{aligned} d(x, T^{k_n(x)}x) &\leq d(x,Tx) + d(T^{k_n(x)}x, Tx) \\ &\leq d(x,Tx) + q\delta\big[O_T(x,n)\big] \end{aligned}$$

$$= d(x,Tx) + qd(x,T^{k_n(x)}x).$$

It implies that

$$\delta\big[O_T(x,n)\big] = d(x,T^{k_n(x)}x) \le \frac{1}{1-q}d(x,Tx). \tag{2.21}$$

If (2.20) holds and $T^{k_n(x)}x = Tx$, we have

$$\delta\big[O_T(x,n)\big] = d(T^{k_n(x)}x,x) = d(Tx,x) \le \frac{1}{1-q}d(Tx,x). \tag{2.22}$$

So, we may assume that $T^{k_n(x)}x \ne Tx$. Then

$$\begin{aligned} d(T^{k_n(x)}x,x) &\le d(T^{k_n(x)}x,Tx) + d(x,Tx) \\ &\le q\delta\big[O_T(x,n)\big] + d(x,Tx) \\ &= qd(T^{k_n(x)}x,x) + d(x,Tx). \end{aligned}$$

It implies that

$$\delta\big[O_T(x,n)\big] = d(T^{k_n(x)}x,x) \le \frac{1}{1-q}d(x,Tx). \tag{2.23}$$

As in the proof of Theorem 5.(1), we also have

$$\lim_{n,m\to\infty} d(T^n x, T^m x) = 0 \tag{2.24}$$

and there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(T^n x, x^*) = \lim_{n\to\infty} d(x^*, T^n x) = 0. \tag{2.25}$$

If $T^{n+1}x = Tx^*$ for infinitely many n, then $\lim_{n\to\infty} T^{n+1}x = Tx^* = x^*$. Then x^* is a fixed point of T. So, we may assume that $T^{n+1}x \ne x^*$ for n large enough. Therefore, by using (2.18) again, we have

$$\begin{aligned} &d(Tx^*,x^*) \le d(Tx^*,T^{n+1}x) + d(x^*,T^{n+1}x) \qquad (2.26)\\ &= d(x^*,T^{n+1}x) + d(Tx^*,TT^n x) \\ &\le d(x^*,T^{n+1}x) + q\max\big\{d(T^n x,x^*), d(x^*,T^n x), d(T^n x,TT^n x), \\ &\quad d(TT^n x,T^n x), d(Tx^*,x^*), d(Tx^*,T^n x), d(x^*,TT^n x), d(TT^n x,x^*), \\ &\quad d(T^2T^n x,T^n x), d(T^n x,T^2T^n x), d(T^2T^n x,TT^n x), d(TT^n x,T^2T^n x), \\ &\quad d(T^2T^n x,x^*), d(x^*,T^2T^n x), d(Tx^*,T^2T^n x)\big\} \\ &= d(x^*,T^{n+1}x) + q\max\big\{d(T^n x,x^*), d(x^*,T^n x), d(T^n x,T^{n+1}x), \\ &\quad d(T^{n+1}x,T^n x), d(Tx^*,x^*), d(Tx^*,T^n x), d(x^*,T^{n+1}x), d(T^{n+1}x,x^*), \\ &\quad d(T^{n+2}x,T^n x), d(T^n x,T^{n+2}x), d(T^{n+2}x,T^{n+1}x), d(T^{n+1}x,T^{n+2}x), \end{aligned}$$

$$d(T^{n+2}x, x^*), d(x^*, T^{n+2}x), d(Tx^*, T^{n+2}x)\}.$$

Taking the limit as $n \to \infty$ in (2.26), and using (2.24), (2.25) and Proposition 5, we get $d(Tx^*, x^*) \leq qd(Tx^*, x^*)$. Since $q \in [0,1)$, $d(Tx^*, x^*) = 0$, that is, $Tx^* = x^*$. Then T has a fixed point.

The remaining is similar as the proof of Theorem 5. □

With some minor changes in the proof of Theorem 3, we get the following result. Also, these changes mainly relate to Definition 7.(2).

Proposition 7. *Let (X,d) be a quasi-metric space in the sense of Definition 7 and $T : X \longrightarrow X$ be a map such that (X,d) is weak T-orbitally complete and*

$$d(Tx, Ty) \leq d(x,y) - \varphi\big(d(x,y)\big) \tag{2.27}$$

for all $x, y \in X$, where $\varphi : [0,+\infty) \longrightarrow [0,+\infty)$ is continuous with $\varphi^{-1}\big(\{0\}\big) = \{0\}$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ and define the sequence $x_{n+1} = Tx_n$ for all $n \geq 0$. From (2.27), we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq d(x_{n-1}, x_n) - \varphi\big(d(x_{n-1}, x_n)\big) \tag{2.28}$$

for all $n \geq 1$. This proves that $\{d(x_n, x_{n+1})\}$ is a non-increasing sequence of positive numbers. Then there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. Taking the limit as $n \to \infty$ in (2.28), we get $\varphi(r) = 0$, that is, $r = 0$. Then

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.29}$$

Using the same technique, we also have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{2.30}$$

Now, we will prove that $\{x_n\}$ is a Cauchy sequence in the quasi-metric space (X,d), that is, $\{x_n\}$ is left-Cauchy and right-Cauchy. By (2.29) and (2.30), since the sequences $\{d(x_{n+1}, x_n)\}$ and $\{d(x_n, x_{n+1})\}$ are non-increasing, we have $\{x_n\}$ is a Cauchy sequence if there exists n such that $d(x_{n+1}, x_n) = 0$ or $d(x_n, x_{n+1}) = 0$. Then, we may assume that, for all $n \in \mathbb{N}$,

$$d(x_{n+1}, x_n) \neq 0 \text{ and } d(x_n, x_{n+1}) \neq 0. \tag{2.31}$$

Now, suppose to the contrary that $\{x_n\}$ is not a left-Cauchy sequence. Then there exists $\varepsilon > 0$ such that for each $k \in \mathbb{N}$, there exist $n > m \geq k$ satisfying $d(x_n, x_m) \geq \varepsilon$. Put

$$n(1) = \min\big\{n : n > 1 \text{ and there exists } m \text{ with } 1 \leq m < n, d(x_n, x_m) \geq \varepsilon\big\}$$

$$m(1) = \max\big\{m : 1 \leq m < n(1) \text{ with } d(x_{n(1)}, x_m) \geq \varepsilon\big\}$$

$$n(2) = \min\big\{n : n > n(1), \text{ there exists } m \text{ with } n(1) \leq m < n, d(x_n, x_m) \geq \varepsilon\big\}$$

$$m(2) = \max\big\{m : n(1) \leq m < n(2) \text{ with } d(x_{n(2)}, x_m) \geq \varepsilon\big\}.$$

Note that $n(1) < n(2)$, $m(1) < m(2)$ and

$$d(x_{n(1)-1}, x_{m(1)}) < \varepsilon, d(x_{n(2)-1}, x_{m(2)}) < \varepsilon.$$

Continuing this process, we can find two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that, for all $k \in \mathbb{N}$, we have $n(k) > m(k) > k$ and

$$d(x_{n(k)}, x_{m(k)}) \geq \varepsilon, d(x_{n(k)-1}, x_{m(k)}) < \varepsilon. \tag{2.32}$$

Now, by (2.31) and (2.32), we have

$$\begin{aligned} \varepsilon &\leq d(x_{n(k)}, x_{m(k)}) \\ &\leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{m(k)}, x_{n(k)-1}) \\ &\leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{m(k)}, x_{m(k)-1}) + d(x_{n(k)-1}, x_{m(k)-1}) \\ &\leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{m(k)}, x_{m(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) \\ &\quad + d(x_{m(k)-1}, x_{m(k)}) \\ &< d(x_{n(k)}, x_{n(k)-1}) + d(x_{m(k)}, x_{m(k)-1}) + \varepsilon + d(x_{m(k)-1}, x_{m(k)}). \end{aligned} \tag{2.33}$$

Taking the limit as $k \to \infty$ in (2.33) and using (2.29), (2.30), we get

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{2.34}$$

Also, by (2.31), we have

$$\begin{aligned} d(x_{n(k)-1}, x_{m(k)-1}) &\leq d(x_{n(k)-1}, x_{n(k)}) + d(x_{m(k)-1}, x_{n(k)}) \\ &\leq d(x_{n(k)-1}, x_{n(k)}) + d(x_{m(k)-1}, x_{m(k)}) + d(x_{n(k)}, x_{m(k)}) \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} d(x_{n(k)}, x_{m(k)}) &\leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{m(k)}, x_{n(k)-1}) \\ &\leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{m(k)}, x_{m(k)-1}) + d(x_{n(k)-1}, x_{m(k)-1}). \end{aligned} \tag{2.36}$$

Taking the limit as $k \to \infty$ in (2.35) and (2.36) and using (2.29), (2.30), (2.34), we get

$$\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon. \tag{2.37}$$

Now, from (2.27), for all $k \in \mathbb{N}$, we have

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)-1}, x_{m(k)-1}) - \varphi\big(d(x_{n(k)-1}, x_{m(k)-1})\big). \tag{2.38}$$

Taking the limit as $k \to \infty$ in (2.38) and using (2.34), (2.37), we obtain $\varepsilon \leq \varepsilon - \varphi(\varepsilon)$. It implies that $\varepsilon = 0$. It is a contradiction. Then $\{x_n\}$ is a left-Cauchy sequence. Similarly, we can show that $\{x_n\}$ is a right-Cauchy sequence. Then $\{x_n\}$ is Cauchy. Since (X, d) is weak T-orbitally complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = \lim_{n\to\infty} d(x^*, x_n) = 0. \tag{2.39}$$

From (2.27), for all $n \in \mathbb{N}$, we have

$$d(Tx^*, x_{n+1}) = d(Tx^*, Tx_n) \leq d(x^*, x_n) - \varphi\big(d(x^*, x_n)\big) \tag{2.40}$$

Taking the limit as $n \to \infty$ in (2.40) and using (2.39), Proposition 5, we get $d(Tx^*, x^*) = 0$. It implies that $x^* = Tx^*$, that is, x^* is a fixed point of T.

The uniqueness of the fixed point is easy to see. □

Similar as the proof of [16, Theorem 2.2] and the proof of Proposition 7, we get the following result.

Proposition 8. *Let (X,d) be a weak T-orbitally complete quasi-metric space in the sense of Definition 7 and let $T : X \longrightarrow X$ be a map such that*

$$\psi\big(d(Tx,Ty)\big) \leq \psi\big(d(x,y)\big) - \varphi\big(d(x,y)\big) \tag{2.41}$$

where $\psi, \varphi : [0,+\infty) \longrightarrow [0,+\infty)$, ψ is continuous and non-decreasing, φ is lower semi-continuous, and $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

3. Applications to recent fixed point results in G-metric spaces

In this section, we show that most of recent results on G-metric spaces in [3, 10] may be also implied from certain fixed point theorems in metric spaces and quasi-metric spaces mentioned in Section 2. Notice that the authors of [10] forgot the assumption of completeness in [10, Theorems 3.1 & 3.2].

Corollary 2 ([10], Theorem 3.1)**.** *Let (X,G) be a complete G-metric space and $T : X \longrightarrow X$ be a map such that*

$$G(Tx,Ty,Tz) \leq kM(x,y,z) \tag{3.1}$$

for all $x,y,z \in X$, where $k \in \big[0,\frac{1}{2}\big)$ and

$$M(x,y,z) = \max\left\{\begin{array}{l} G(x,Tx,y), G(y,T^2x,Ty), G(Tx,T^2x,Ty), \\ G(y,Tx,Ty), G(x,Tx,z), G(z,T^2x,Tz), \\ G(Tx,T^2x,Tz), G(z,Tx,Ty), G(x,y,z), \\ G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz), \\ G(z,Tx,Tx), G(x,Ty,Ty), G(y,Tz,Tz). \end{array}\right\}$$

Then T has a unique fixed point.

Proof. Let d_G be the quasi-metric in Theorem 1. By choosing $z = y$ and using the axioms (G4) and (G5) in Definition 1, we have

$$M(x,y,y) = \max\left\{\begin{array}{l} G(x,Tx,y), G(y,T^2x,Ty), G(Tx,T^2x,Ty), \\ G(y,Tx,Ty), G(x,Tx,y), G(y,T^2x,Ty), \\ G(Tx,T^2x,Ty), G(y,Tx,Ty), G(x,y,y), \\ G(x,Tx,Tx), G(y,Ty,Ty), G(y,Ty,Ty), \\ G(y,Tx,Tx), G(x,Ty,Ty), G(y,Ty,Ty) \end{array}\right\}$$

$$= \max \left\{ \begin{array}{l} G(x,Tx,y), G(y,Ty,T^2x), G(Tx,Ty,T^2x), \\ G(y,Ty,Tx), G(x,Tx,y), G(y,Ty,T^2x), \\ G(T^2x,Tx,Ty), G(y,Ty,Tx), G(x,y,y), \\ G(x,Tx,Tx), G(y,Ty,Ty), G(y,Ty,Ty), \\ G(y,Tx,Tx), G(x,Ty,Ty), G(y,Ty,Ty) \end{array} \right\}$$

$$\leq \max \left\{ \begin{array}{l} G(x,Tx,Tx) + G(Tx,Tx,y), G(y,Ty,Ty) + G(Ty,Ty,T^2x), \\ G(Tx,Ty,Ty) + G(Ty,Ty,T^2x), \\ G(y,Ty,Ty) + G(Ty,Ty,Tx), G(x,Tx,Tx) + G(Tx,Tx,y), \\ G(y,Ty,Ty) + G(Ty,Ty,T^2x), \\ G(T^2x,Tx,Tx) + G(Tx,Tx,Ty), \\ G(y,Ty,Ty) + G(Ty,Ty,Tx), \\ G(x,y,y), G(x,Tx,Tx), G(y,Ty,Ty), G(y,Ty,Ty), \\ G(y,Tx,Tx), G(x,Ty,Ty), G(y,Ty,Ty) \end{array} \right\}$$

$$= \max \left\{ \begin{array}{l} d_G(x,Tx) + d_G(y,Tx), d_G(y,Ty) + d_G(T^2x,Ty), \\ d_G(Tx,Ty) + d_G(T^2x,Ty), d_G(y,Ty) + d_G(Tx,Ty), \\ d_G(x,Tx) + d_G(y,Tx), d_G(y,Ty) + d_G(T^2x,Ty), \\ d_G(T^2x,Tx) + d_G(Ty,Tx), d_G(y,Ty) + d_G(Tx,Ty), \\ d_G(x,y), d_G(x,Tx), d_G(y,Ty), d_G(y,Ty), d_G(y,Tx), \\ d_G(x,Ty), d_G(y,Ty) \end{array} \right\}$$

$$\leq 2\max\{d_G(x,Tx), d_G(y,Tx), d_G(y,Ty), d_G(T^2x,Ty), d_G(Tx,Ty), d_G(x,y), d_G(x,Ty)\}.$$

Then (3.1) becomes

$$d_G(Tx,Ty) \leq 2k \max\{d_G(x,Tx), d_G(y,Tx), d_G(y,Ty), d_G(T^2x,Ty), d_G(Tx,Ty), d_G(x,y), d_G(x,Ty)\}.$$

Since $0 \leq 2k < 1$, we have

$$d_G(Tx,Ty) \leq 2k \max\{d_G(x,y), d_G(x,Tx), d_G(y,Ty), d_G(x,Ty), d_G(y,Tx), d_G(T^2x,Ty)\}.$$

By Theorem 5, we see that T has a unique fixed point. □

Remark 2. The authors of [10] claimed that the proof of [10, Theorem 3.2] is the mimic of [10, Theorem 3.1]. But, by redoing the proof of [10, Theorem 3.1], we see that the equality (23) in the proof of [10, Theorem 3.1] becomes

$$G(x^*,Tx^*,Tx^*) \leq kG(x^*,Tx^*,Tx^*) \text{ or } G(x^*,Tx^*,Tx^*) \leq kG(x^*,x^*,Tx^*)$$

and the equality (25) in the proof of [10, Theorem 3.1] becomes

$$G(t^*,t^*,x^*) \leq kG(t^*,t^*,x^*) \text{ or } G(t^*,t^*,x^*) \leq kG(t^*,x^*,x^*).$$

In general, the second inequalities do not hold if $k \in [0,1)$.

Corollary 3 ([10], Theorem 3.3)**.** *Let (X,G) be a complete G-metric space and $T : X \longrightarrow X$ be a map such that*

$$\psi\big(G(Tx,T^2x,Ty)\big) \leq G(x,Tx,y) - \varphi\big(G(x,Tx,y)\big) \tag{3.2}$$

for all $x,y \in X$, where $\varphi : [0,+\infty) \longrightarrow [0,+\infty)$ is continuous with $\varphi^{-1}(\{0\}) = 0$. Then T has a unique fixed point.

Proof. It is easy to see that T has at most one fixed point. Suppose to the contrary that T has no any fixed point. Let $d_{T,G}$ be defined as in Proposition 4. Then, $d_{T,G}$ is a quasi-metric in the sense of Definition 7 on X. We prove that $(X,d_{T,G})$ is a weak T-orbitally complete quasi-metric space. Let $\{x_n\}$ be a Cauchy sequence in $(X,d_{T,G})$ where $x_0 \in X$ and $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. We have

$$\lim_{n,m\to\infty} d_{T,G}(x_n,x_m) = 0.$$

We may assume that $x_n \neq x_m$ for all $n \neq m \in \mathbb{N}$. Then

$$\begin{aligned} 0 &\leq \lim_{n,m\to\infty} G(x_n,x_m,x_m) \\ &\leq \lim_{n,m\to\infty} G(x_n,x_{n+1},x_m) \\ &= \lim_{n,m\to\infty} G(x_n,Tx_n,x_m) \\ &= \lim_{n,m\to\infty} d_{T,G}(x_n,x_m) \\ &= 0. \end{aligned}$$

It implies that $\lim_{n,m\to\infty} G(x_n,x_m,x_m) = 0$. By Lemma 2, $\{x_n\}$ is a Cauchy sequence in (X,G). Since (X,G) is complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$ in (X,G). Since $x_n \neq x_m$ for all $n \neq m \in \mathbb{N}$, we may assume that $x_n \neq x^*$ for all $n \in \mathbb{N}$. Therefore,

$$\lim_{n\to\infty} d_{T,G}(x_n,x^*) = \lim_{n\to\infty} G(x_n,Tx_n,x^*) = \lim_{n\to\infty} G(x_n,x_{n+1},x^*) = 0. \tag{3.3}$$

We also have

$$\begin{aligned} d_{T,G}(x^*,x_n) &= G(x^*,Tx^*,x_n) \qquad (3.4)\\ &\leq G(x^*,x_{n+1},x_{n+1}) + G(x_{n+1},Tx^*,x_n) \\ &= G(x^*,x_{n+1},x_{n+1}) + G(x_n,x_{n+1},Tx^*) \\ &= G(x^*,x_{n+1},x_{n+1}) + G(Tx_{n-1},T^2x_{n-1},Tx^*) \\ &\leq G(x^*,x_{n+1},x_{n+1}) + G(x_{n-1},Tx_{n-1},x^*) - \varphi\big(G(x_{n-1},Tx_{n-1},x^*)\big) \\ &\leq G(x^*,x_{n+1},x_{n+1}) + G(x_{n-1},x_n,x^*) - \varphi\big(G(x_{n-1},x_n,x^*)\big). \end{aligned}$$

Taking the limit as $n \to \infty$ in (3.4) and using Lemma 1, we obtain

$$\lim_{n\to\infty} d_{T,G}(x^*,x_n) = 0. \tag{3.5}$$

From (3.3) and (3.5), we get $\lim_{n\to\infty} x_n = x^*$ in $(X, d_{T,G})$. Then $(X, d_{T,G})$ is weak T-orbitally complete. Note that (3.2) becomes

$$\psi\big(d_{T,G}(Tx,Ty)\big) \le d_{T,G}(x,y) - \varphi\big(d_{T,G}(x,y)\big).$$

Therefore, by using Proposition 7, we conclude that T has a fixed point. It is a contradiction.

By the above, T has a unique fixed point. □

Corollary 4 ([3], Theorem 2.3)**.** *Let (X,G) be a complete G-metric space and $T : X \longrightarrow X$ be a map such that*

$$\psi\big(G(Tx,T^2x,Ty)\big) \le \psi\big(G(x,Tx,y)\big) - \varphi\big(G(x,Tx,y)\big) \tag{3.6}$$

for all $x, y \in X$, where $\psi : [0,+\infty) \longrightarrow [0,+\infty)$ is non-decreasing and continuous, $\varphi : [0,+\infty) \longrightarrow [0,+\infty)$ is lower semi-continuous and $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Proof. It is easy to see that T has at most one fixed point. Suppose to the contrary that T has no any fixed point. Using $d_{T,G}$ as in the proof of Corollary 3, then $d_{T,G}$ is a quasi-metric in the sense of Definition 7 on X. We prove that $(X, d_{T,G})$ is a weak T-orbitally complete quasi-metric space. Let $\{x_n\}$ be a Cauchy sequence in $(X, d_{T,G})$ where $x_0 \in X$ and $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. We have $\lim_{n,m\to\infty} d_{T,G}(x_n, x_m) = 0$. We may assume that $x_n \neq x_m$ for all $n \neq m \in \mathbb{N}$. Then

$$\begin{aligned} 0 &\le \lim_{n,m\to\infty} G(x_n, x_m, x_m) \\ &\le \lim_{n,m\to\infty} G(x_n, x_{n+1}, x_m) \\ &= \lim_{n,m\to\infty} G(x_n, Tx_n, x_m) \\ &= \lim_{n,m\to\infty} d_{T,G}(x_n, x_m) \\ &= 0. \end{aligned}$$

It implies that $\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0$. By Lemma 2, $\{x_n\}$ is a Cauchy sequence in (X,G). Since (X,G) is complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$ in (X,G). Since $x_n \neq x_m$ for all $n \neq m \in \mathbb{N}$, we may assume that $x_n \neq x^*$ for all $n \in \mathbb{N}$. Therefore,

$$\lim_{n\to\infty} d_{T,G}(x_n, x^*) = \lim_{n\to\infty} G(x_n, Tx_n, x^*) = \lim_{n\to\infty} G(x_n, x_{n+1}, x^*) = 0. \tag{3.7}$$

We also have

$$\begin{aligned} &d_{T,G}(x^*, x_n) = G(x^*, Tx^*, x_n) \qquad (3.8)\\ &\le G(x^*, x_{n+1}, x_{n+1}) + G(x_{n+1}, Tx^*, x_n) \\ &= G(x^*, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, Tx^*) \end{aligned}$$

$$
\begin{aligned}
&= G(x^*, x_{n+1}, x_{n+1}) + G(Tx_{n-1}, T^2x_{n-1}, Tx^*)\\
&\leq G(x^*, x_{n+1}, x_{n+1}) + \psi\big(G(x_{n-1}, Tx_{n-1}, x^*)\big) - \varphi\big(G(x_{n-1}, Tx_{n-1}, x^*)\big)\\
&\leq G(x^*, x_{n+1}, x_{n+1}) + \psi\big(G(x_{n-1}, x_n, x^*)\big) - \varphi\big(G(x_{n-1}, x_n, x^*)\big).
\end{aligned}
$$

Taking the limit as $n \to \infty$ in (3.8) and using Lemma 1, we get

$$\lim_{n\to\infty} d_{T,G}(x^*, x_n) = 0. \tag{3.9}$$

From (3.7) and (3.9), we get $\lim_{n\to\infty} x_n = x^*$ in $(X, d_{T,G})$. Then $(X, d_{T,G})$ is weak T-orbitally complete. Note that (3.6) becomes

$$\psi\big(d_{T,G}(Tx, Ty)\big) \leq \psi\big(d_{T,G}(x, y)\big) - \varphi\big(d_{T,G}(x, y)\big).$$

Therefore, by using Proposition 8, we conclude that T has a fixed point. It is a contradiction.

By the above, T has a unique fixed point. □

Remark 3. By using $d_{T,G}$ as in the proof of Corollary 3, we see that the inequality (30) in [3, Theorem 3.1] becomes

$$d_{T,G}(Tx, Ty) \geq \alpha d_{T,G}(x, y).$$

By similar arguments, we get analogues of the results in [18] for expansive maps on quasi-metric spaces and then we get [3, Theorem 3.1]. Also, similar arguments to the above may be possible for results in [7]. Note that for a complete G-metric space (X, G) with $|X| \geq 2$ and $T : X \longrightarrow X$ being the identify map, all assumptions of [3, Theorem 3.2] hold but T has more than one fixed point. This shows that the uniqueness of fixed points in [3, Theorem 3.2] is a gap.

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Author's address

N. V. Dung
Dong Thap University, Faculty of Mathematics and Information Technology Teacher Education, 783 Pham Huu Lau St., Cao Lanh City, Dong Thap Province, Viet Nam
E-mail address: nvdung@dthu.edu.vn

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SOME ITERATED CONVERGENCE AND FIXED POINT THEOREMS IN REAL LINEAR n-NORMED SPACES

HEMEN DUTTA

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Abstract. In this paper, we prove some iterated convergence theorems to fixed points of self mappings satisfying Z-type conditions and Z-operators by considering the base space as a real linear $n(>1)$-normed space. We also establish some common fixed point theorems for quadruple of self mappings satisfying certain weak conditions and Φ- contraction. The paper also demonstrates a way to construct convergence and fixed point theorems in real linear n-normed spaces as base space.

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1. INTRODUCTION AND PRELIMINARIES

In mathematics, a fixed point (also known as an invariant point) of a function is a point that is mapped to itself by the function. In many fields, equilibria or stability are fundamental concepts that can be described in terms of fixed points. Many applications of fixed point theorems can be found both on the theoretical side and on the applied side.

In the last three decades many papers have been published on the iterative approximation of fixed points for certain classes of operators, by using Picard, Krasnoselskij, Mann and Ishikawa iteration methods. They do basically differ due to their speed of convergence depending on the position of parameters involved.

The concept of 2-normed spaces was initially developed by Gähler [11] in the mid of 1960's, while that of n-normed spaces can be found in Misiak [19]. Since then, many others have studied this concept and obtained various results; see for instance Gunawan and Mashadi [12], Dutta [8, 9], and Dutta, Reddy and Cheng [10] etc.

Let $n \in N$ and X be a real vector space of dimension d, where $n \leq d$. A real-valued function $||.,. \ . \ ,.||$on X^n satisfying the following four conditions:

(N1) $||x_1, x_2, \ldots, x_n|| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,

(N2) $||x_1, x_2, \ldots, x_n||$is invariant under permutation,

(N3) $||\alpha x_1, x_2, \ldots, x_n|| = |\alpha|||x_1, x_2, \ldots, x_n||$, for any $\alpha \in R$,

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(N4) $||x+x', x_2, \ldots, x_n|| \leq ||x, x_2, \ldots, x_n|| + ||x', x_2, \ldots, x_n||$ is called an *n-norm* on X, and the pair $(X, ||.,. \ldots,.||)$ is called an n-normed space.

A trivial example of an n-normed space is $X=R^n$ equipped with the following Euclidean n-norm:

$$||x_1,x_2, \ldots ,x_n||_E = \text{abs}\begin{pmatrix} x_{11}\cdots x_{1n} \\ \vdots \ddots \vdots \\ x_{n1}\cdots x_{nn} \end{pmatrix}, \text{ where } x_i=(x_{i1},\ldots,x_{in}) \in R^n \text{ for each } i=1,2,\ldots,n.$$

If $(X,||.,. \ldots,.||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \ldots, a_n\}$ be a linearly independent set in X. Then the following function $||,.,. \ldots,.||_\infty$ on X^{n-1} defined by

$||x_1,x_2, \ldots ,x_{n-1}||_\infty = \max \{||x_1,x_2, \ldots ,x_{n-1},a_i||: i = 1, 2, \ldots,n\}$ defines an $(n\text{-}1)$ norm on X with respect to $\{a_1, a_2, \ldots, a_n\}$.

The standard n- norm on X, a real inner product space of dimension $d = n$ is as follows:

$$||x_1,\ldots,x_n||_S = \begin{vmatrix} \langle x_1,x_1\rangle \cdots \langle x_1,x_n\rangle \\ \vdots \ddots \vdots \\ \langle x_n,x_1\rangle \cdots \langle x_n,x_n\rangle \end{vmatrix}^{\frac{1}{2}}, \text{ where } \langle .,. \rangle \text{ denotes the inner product on } X. \text{ If } X=R^n, \text{ then this } n\text{-norm is exactly the same as the Euclidean } n\text{-norm } ||x_1,x_2, \ldots ,x_n||_E \text{ mentioned earlier. For } n = 1, \text{ this } n\text{-norm is the usual norm } ||x_1|| = \langle x_1,x_1\rangle^{\frac{1}{2}}.$$

A sequence (x_k) in an n-normed space $(X, ||., \ldots,.||)$ is said to *converge* to some $L \in X$ in the n-norm if

$\lim_{k\to\infty} \|x_k - L, u_2, \ldots\ldots, u_n\| = 0$, for every $u_2, \ldots, u_n \in X$.

A sequence (x_k) in an n-normed space $(X, ||., \ldots,.||)$ is said to be *Cauchy* with respect to the n-norm if

$\lim_{k,l\to\infty} \|x_k - x_l, u_2, \ldots\ldots, u_n\| = 0$, for every $u_2, \ldots, u_n \in X$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

If a term in the definition of n-norm represents the change of shape, and the n-norm stands for the associated area or center of gravity of the term, may be we can think of some applications of the notion of n-norm, and then the generalized convergence make sense. This is considered as the main issue in the use of n-normed structures here. Keeping this in mind, we consider real linear n-normed space as the base space.

The following iterative scheme was introduced by Mann [18] in 1953.

Let K be a non empty, closed, convex subset of a normed linear space E and $T : K \to K$ be a self map. For $x_0 \in K$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, n = 0, 1, 2, \ldots, \tag{1.1}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in $[0,1]$, is called the Mann iteration process. Many converging theorems and approximation results have been proved using the Mann iteration process. A two-step iteration process was introduced by Ishikawa [13] in 1974 as defined below:

For $x_0 \in K$, the sequence $\{x_n\}_{n=0}^{\infty}$ is given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, y_n = (1-\beta_n)x_n + \beta_n T x_n, n = 0,1,2,..., \quad (1.2)$$

where K be a non empty, closed, convex subset of a normed linear space E and $T : K \to K$ be a self map and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0,1]$. This iteration process is called as Ishikawa iteration process. When $\beta_n = 0$, this iteration process reduces to Mann iteration scheme given by (1.1).

We procure the following definitions in a metric space (X,d):

A mapping $T : X \to X$ is called an $a-$contraction if there exists $a \in [0,1)$ such that

(z_1) $d(Tx,Ty) \leq ad(x,y)$, for all $x,y \in X$.

The map T is called a Kannan mapping [16] if there exists $b \in [0,\frac{1}{2})$ such that

(z_2) $d(Tx,Ty) \leq b[d(x,Tx)+d(y,Ty)]$, for all $x,y \in X$.

A similar definition is due to Chatterjea [7] if there exists $c \in [0,\frac{1}{2})$ such that (z_3) $d(Tx,Ty) \leq c[d(x,Ty)+d(y,Tx)]$, for all $x,y \in X$.

The conditions (z_1), (z_2) and (z_3) are independent contractive conditions [24].

Combining these three definitions, Zamfirescu [27] obtained the following important fixed point theorem in 1972.

Theorem 1. *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping for which there exist real numbers a,b and c with $a \in [0,1)$, $b \in [0,\frac{1}{2})$and $c \in [0,\frac{1}{2})$ such that for all $x,y \in X$ at least one of the following conditions holds:*

$$(z_1) : d(Tx,Ty) \leq ad(x,y)$$
$$(z_2) : d(Tx,Ty) \leq b[d(x,Tx)+d(y,Ty)]$$
$$(z_3) : d(Tx,Ty) \leq c[d(x,Ty)+d(y,Tx)] \quad (1.3)$$

Then T has a unique fixed point x^* and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, n \in N$$

converges to x^* for any arbitrary $x_0 \in X$.

An operator T which satisfies at least one of the contractive conditions (z_1), (z_2) and (z_3) is called a Zamfirescu operator (Z-operator).

In 2004, Berinde [2] proved the strong convergence of Ishikawa iterative process given by (1.2) to approximate fixed points of Zamfirescu operators in an arbitrary Banach space E. In proving the theorem, he made use of the conditions,

$$\|Tx-Ty\| \leq \delta \|x,y\| + 2\delta \|x-Tx\| \quad (1.4)$$

which holds for any $x,y \in E$, where $0 \leq \delta < 1$.

In this paper, we define the following condition in order to establish our results of this paper in real linear n-normed spaces:

Let K be a non empty, closed, convex subset of a real linear n- normed space E and $T : K \to K$ be a self map. There exists a constant $L \geq 0$ such that for all $x, y, z_2, ..., z_n \in K$, we have

$$\|Tx - Ty, z_2, ..., z_n\|$$
$$\leq e^{L\|x-Tx,z_2,...,z_n\|} (2\delta \|x - Tx, z_2, ..., z_n\| + \delta \|x - y, z_2, ..., z_n\|) \tag{1.5}$$

where $0 \leq \delta < 1$ and e^x denotes the exponential function of $x \in K$. This condition will be recalled in the paper as generalized Z-type condition.

The above condition is similar to a condition introduced in [5] by generalizing (1.4) in normed linear spaces.

If $L = 0$, in the above condition, we obtain

$$\|Tx - Ty, z_2, ..., z_n\| \leq \delta \|x - y, z_2, ..., z_n\| + 2\delta \|x - Tx, z_2, ..., z_n\|$$

which is an n-norm extension of the Zamfirescu condition used by Berinde [3], where

$$\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}, 0 \leq \delta < 1,$$

where constants a, b and c are as defined in Theorem 1.

In a similar fashion, we can extend the Theorem 1 to real liner n-normed spaces by replacing the base space (metric spaces). For some relevant results in normed linear spaces, we refer to [5, 23], [26], etc. We extend some such results in the next section of this paper.

We shall use the following lemma in proving some results of this paper.

Lemma 1 (Berinde [4]). *Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$ be sequences of non-negative numbers satisfying $a_{n+1} \leq (1 - w_n)a_n + b_n + c_n$, for all $n \geq 0$, where $\{w_n\}_{n=0}^{\infty} \subset [0, 1]$.*
If $\sum_{n=0}^{\infty} w_n = \infty$, $b_n = O(w_n)$ and $\sum_{n=0}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n = 0$.

2. Convergence theorems

In this section, we shall prove some convergence theorems to fixed points. Throughout this section, K is taken as a non empty, closed, convex subset of an n-normed linear space E.

Theorem 2. *Let $T : K \to K$ and $S : K \to K$ be two self mappings with $F_T \bigcap F_S \neq \phi$ where F_T and F_S are the sets of fixed points of T and S respectively satisfying generalized Z - type conditions given by (1.5). For arbitrary $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by generalized Mann type iteration scheme (introduced by Owojori and Imoru [20])*

$$x_{n+1} = a_n x_n + b_n T x_n + c_n S x_n, n = 0, 1, 2, ...,$$

where $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *are real sequences in* $[0,1]$ *with* $a_n+b_n+c_n=1$ *and* $b_n+c_n=\alpha_n$ *and* $\sum_{n=0}^{\infty}\alpha_n=\infty$. *Then* $\{x_n\}_{n=0}^{\infty}$ *converges to a common fixed point of* T *and* S.

Proof. Since $F_T\bigcap F_S\neq\phi$, let us take $x^*\in K$ be a common fixed point of T and S. Since T and S satisfy generalized Z-type condition given by (1.5), we have that for all $x,y,z_2,...,z_n\in K$, the following inequalities hold

$$\begin{aligned}&\|Tx-Ty,z_2,...,z_n\|\\&\quad\le e^{L\|x-Tx,z_2,...,z_n\|}\left(2\delta\|x-Tx,z_2,...,z_n\|+\delta\|x-y,z_2,...,z_n\|\right)\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}&\|Sx-Sy,z_2,...,z_n\|\\&\quad\le e^{L\|x-Tx,z_2,...,z_n\|}\left(2\delta\|x-Sx,z_2,...,z_n\|+\delta\|x-y,z_2,...,z_n\|\right)\end{aligned}\tag{2.2}$$

where $L\ge 0$, $0\le\delta<1$ and $\delta=\max\left\{a,\frac{b}{1-b},\frac{c}{1-c}\right\}$.

Now let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by generalized Mann type iteration scheme and $x_0\in K$ be arbitrary. Then for all $z_2,...,z_n\in K$

$$\begin{aligned}\left\|x_{n+1}-x^*,z_2,...,z_n\right\|&=\left\|a_nx_n+b_nTx_n+c_nSx_n-x^*,z_2,...,z_n\right\|\\&=\left\|(1-\alpha_n)x_n+b_nTx_n+c_nSx_n-(a_n+b_n+c_n)x^*,z_2,...,z_n\right\|\\&=\left\|(1-\alpha_n)x_n+b_nTx_n+c_nSx_n-(1-\alpha_n)x^*-b_nx^*-c_nx^*,z_2,...,z_n\right\|\\&=\left\|(1-\alpha_n)(x_n-x^*)+b_n(Tx_n-x^*)+c_n(Sx_n-x^*),z_2,...,z_n\right\|\\&\le(1-\alpha_n)\left\|x_n-x^*,z_2,...,z_n\right\|+b_n\left\|Tx_n-x^*,z_2,...,z_n\right\|\\&\quad+c_n\left\|Sx_n-x^*,z_2,...,z_n\right\|\end{aligned}\tag{2.3}$$

Taking $x=x^*$ and $y=x_n$ in (2.1), we get

$$\begin{aligned}&\left\|Tx^*-Tx_n,z_2,...,z_n\right\|\\&\le e^{L\|x^*-Tx^*,z_2,...,z_n\|}\left(2\delta\left\|x^*-Tx^*,z_2,...,z_n\right\|+\delta\left\|x^*-x_n,z_2,...,z_n\right\|\right)\\&=e^{L\|x^*-x^*,z_2,...,z_n\|}\left(2\delta\left\|x^*-x^*,z_2,...,z_n\right\|+\delta\left\|x^*-x_n,z_2,...,z_n\right\|\right)\\&=e^{L(0)}\left(2\delta(0)+\delta\left\|x^*-x_n,z_2,...,z_n\right\|\right),\end{aligned}$$

This implies that

$$\left\|Tx_n-x^*,z_2,...,z_n\right\|\le\delta\left\|x_n-x^*,z_2,...,z_n\right\|\tag{2.4}$$

Similarly by taking $x=x^*$ and $y=x_n$ in (2.2), we obtain

$$\left\|Sx_n-x^*,z_2,...,z_n\right\|\le\delta\left\|x_n-x^*,z_2,...,z_n\right\|\tag{2.5}$$

Now using (2.4) and (2.5) in (2.3), we have

$$\begin{aligned}&\left\|x_{n+1}-x^*,z_2,...,z_n\right\|\le(1-\alpha_n)\left\|x_n-x^*,z_2,...,z_n\right\|\\&+b_n\delta\left\|x_n-x^*,z_2,...,z_n\right\|+c_n\delta\left\|cx_n-x^*,z_2,...,z_n\right\|\\&=(1-\alpha_n+b_n\delta+c_n\delta)\left\|x_n-x^*,z_2,...,z_n\right\|\end{aligned}$$

$$= (1-\alpha_n+\alpha_n\delta)\left\|x_n-x^*,z_2,...,z_n\right\|$$

Thus we have,

$$\left\|x_{n+1}-x^*,z_2,...,z_n\right\| \leq [1-\alpha_n(1-\delta)]\left\|x_n-x^*,z_2,...,z_n\right\|, n=0,1,2,...$$

Since $\{\alpha_n\}$ and δ satisfy the conditions of Lemma 1, we have our expected condition $\lim_{n\to\infty}\|x_{n+1}-x^*,z_2,...,z_n\|=0$, for every $z_2,...,z_n\in K$.

Thus $\{x_n\}_{n=0}^{\infty}$ converges to x^* which is the common fixed point of T and S. □

Corollary 1. *Let $T:K\to K$ and $S:K\to K$ be two Z-operators with $F_T\bigcap F_S\neq\phi$ where F_T and F_S are the sets of fixed points of T and S respectively. For any $x_0\in K$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence as defined in the statement of Theorem 2. Then $\{x_n\}_{n=0}^{\infty}$ converges to a common fixed point of T and S.*

Theorem 3. *Let $T:K\to K$ be a self maps satisfying generalized Z-type condition given by (1.5) with $F_T\neq\phi$, where F_T is the set of fixed points of T. For any $x_0\in K$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by the following two step iteration scheme (introduced by Thianwan [25])*

$$x_{n+1}=(1-\alpha_n)y_n+\alpha_nTy_n,$$
$$y_n=(1-\beta_n)x_n+\beta_nTx_n, n=0,1,2,...,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0,1]$ satisfying $\sum_{n=0}^{\infty}\alpha_n=\infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges to the unique fixed point of T.

Proof. The assumption $F_T\neq\phi$ guarantees that T has a fixed point in K, say x^*. Since T satisfies generalized Z-type condition given by (1.5), we have the inequality (2.1) holds for all $x,y,z_2,...,z_n\in K$. Now let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated as given in the statement and $x_0\in K$ be arbitrary. Then for all $z_2,...,z_n\in K$,

$$\left\|x_{n+1}-x^*,z_2,...,z_n\right\| = \left\|(1-\alpha_n)y_n+\alpha_nTy_n-x^*,z_2,...,z_n\right\|$$
$$= \left\|(1-\alpha_n)(y_n-x^*)+\alpha_n(Ty_n-x^*),z_2,...,z_n\right\|$$
$$\leq (1-\alpha_n)\left\|y_n-x^*,z_2,...,z_n\right\|+\alpha_n\left\|Ty_n-x^*,z_2,...,z_n\right\|$$
$$= (1-\alpha_n)\left\|y_n-x^*,z_2,...,z_n\right\|+\alpha_n\left\|Tx^*-Ty_n,z_2,...,z_n\right\|$$

Considering (2.1) by taking $x=x^*$ and $y=x_n$, the above inequality becomes

$$\left\|x_{n+1}-x^*,z_2,...,z_n\right\| \leq (1-\alpha_n)\left\|y_n-x^*,z_2,...,z_n\right\|$$
$$+\alpha_n\left[e^{L\|x^*-Tx^*,z_2,...,z_n\|}\left(2\delta\left\|x^*-Tx^*,z_2,...,z_n\right\|+\delta\left\|x^*-y_n,z_2,...,z_n\right\|\right)\right]$$
$$= (1-\alpha_n)\left\|y_n-x^*,z_2,...,z_n\right\|$$
$$+\alpha_n\left[e^{L\|x^*-x^*,z_2,...,z_n\|}\left(2\delta\left\|x^*-x^*,z_2,...,z_n\right\|+\delta\left\|x^*-y_n,z_2,...,z_n\right\|\right)\right]$$
$$= (1-\alpha_n)\left\|y_n-x^*,z_2,...,z_n\right\|+\alpha_n\left[e^{L(0)}\left(2\delta(0)+\delta\left\|x^*-y_n,z_2,...,z_n\right\|\right)\right]$$
$$= (1-\alpha_n)\left\|y_n-x^*,z\right\|+\alpha_n\delta\left\|y_n-x^*\right\|,$$

So,

$$\|x_{n+1}-x^*,z_2,...,z_n\| \leq (1-\alpha_n+\alpha_n\delta)\|y_n-x^*,z_2,...,z_n\| \tag{2.6}$$

We have

$$\begin{aligned}\|y_n-x^*,z_2,...,z_n\| &= \|(1-\beta_n)x_n+\beta_nTx_n-x^*,z_2,...,z_n\| \\ &= \|(1-\beta_n)(x_n-x^*)+\beta_n(Tx_n-x^*),z_2,...,z_n\| \\ &\leq (1-\beta_n)\|x_n-x^*,z_2,...,z_n\|+\beta_n\|Tx_n-x^*,z_2,...,z_n\| \end{aligned}\tag{2.7}$$

Using (2.7) and then (2.4) in (2.6), we get

$$\begin{aligned}&\|x_{n+1}-x^*,z_2,...,z_n\| \leq (1-\alpha_n+\alpha_n\delta)\|y_n-x^*,z_2,...,z_n\| \\ &\leq (1-\alpha_n+\alpha_n\delta)\left[(1-\beta_n)\|x_n-x^*,z_2,...,z_n\|+\beta_n\|Tx_n-x^*,z_2,...,z_n\|\right] \\ &\leq (1-\alpha_n+\alpha_n\delta)\left[(1-\beta_n)\|x_n-x^*,z_2,...,z_n\|+\beta_n\delta\|x_n-x^*,z_2,...,z_n\|\right] \\ &= (1-\alpha_n+\alpha_n\delta)(1-\beta_n+\beta_n\delta)\|x_n-x^*,z\|.\end{aligned}$$

Since,

$$\begin{aligned}(1-\alpha_n+\alpha_n\delta)(1-\beta_n+\beta_n\delta) &= 1-\alpha_n(1-\delta)-\beta_n(1-\delta)[1-\alpha_n(1-\delta)] \\ &\leq 1-\alpha_n(1-\delta),\end{aligned}$$

we have the inequality

$$\|x_{n+1}-x^*,z_2,...,z_n\| \leq [1-\alpha_n(1-\delta)]\|x_n-x^*,z_2,...,z_n\|, n=0,1,2,...$$

Since $0\leq\delta<1$, $\alpha_n\in[0,1]$ and $\sum_{n=0}^{\infty}\alpha_n=\infty$, setting $a_n=\|x_n-x^*,z_2,...,z_n\|$ (for arbitrarily fixed $z_2,...,z_n\in K$), $w_n=\alpha_n(1-\delta)$ and applying Lemma 1, we have $\lim_{n\to\infty}\|x_{n+1}-x^*,z_2,...,z_n\|=0$, for every $z_2,...,z_n\in K$.

Thus it follows that $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point of T.

To show the uniqueness of fixed point, let us assume that x_1^* and x_2^* be two distinct fixed points of T. Applying generalized Z-type condition given by (1.5) and using the fact that $0\leq\delta<1$, we obtain for every $z_2,...,z_n\in K$:

$$\begin{aligned}&\|x_1^*-x_2^*,z_2,...,z_n\| = \|Tx_1^*-Tx_2^*,z_2,...,z_n\| \\ &\leq e^{L\|x_1^*-Tx_1^*,z_2,...,z_n\|}\left(2\delta\|x_1^*-Tx_1^*,z_2,...,z_n\|+\delta\|x_1^*-x_2^*,z_2,...,z_n\|\right) \\ &= e^{L\|x_1^*-x_1^*,z_2,...,z_n\|}\left(2\delta\|x_1^*-x_1^*,z_2,...,z_n\|+\delta\|x_1^*-x_2^*,z_2,...,z_n\|\right) \\ &= e^{L(0)}\left(2\delta(0)+\delta\|x_1^*-x_2^*,z_2,...,z_n\|\right)=\delta\|x_1^*-x_2^*,z_2,...,z_n\| \\ &< \|x_1^*-x_2^*,z_2,...,z_n\|,\end{aligned}$$

a contradiction.

Therefore $x_1^*=x_2^*$. Thus $\{x_n\}_{n=0}^{\infty}$ converges to the unique fixed point of T. □

Corollary 2. *Let E be an arbitrary n-Banach space and $T : K \to K$ a Z-operator. For any $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be defined by two-step iteration process as in the statement of Theorem 3 with $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0,1]$ satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges to the unique fixed point of T.*

Theorem 4. *Let $T : K \to K$ and $S : K \to K$ be two self mappings of K satisfying generalized Z-type condition given by (1.5) with $F_T \bigcap F_S \neq \phi$ where F_T and F_S are the sets of fixed points of T and S respectively. For any $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by following two-step iteration scheme (introduced by Raphel and Pulickakunnel [23])*

$$x_{n+1} = (1-\alpha_n)y_n + \alpha_n S y_n,$$
$$y_n = (1-\beta_n)x_n + \beta_n T x_n, n = 0,1,2,...,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0,1]$ satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges to the common fixed point of T and S.

Proof. Since S satisfies generalized Z-type condition given by (1.5), we get that the inequality (2.2) holds for all $x, y, z_2, ..., z_n \in K$. As $F_T \bigcap F_S \neq \phi$, let x^* be the common fixed point of S and T in K. Now for every $z_2, ..., z_n \in K$,

$$\begin{aligned}
&\|x_{n+1} - x^*, z_2, ..., z_n\| = \|(1-\alpha_n)y_n + \alpha_n S y_n - x^*, z_2, ..., z_n\| \\
&= \|(1-\alpha_n)(y_n - x^*) + \alpha_n(Sy_n - x^*), z_2, ..., z_n\| \\
&\leq (1-\alpha_n)\|y_n - x^*, z_2, ..., z_n\| + \alpha_n\|Sy_n - x^*, z_2, ..., z_n\| \\
&= (1-\alpha_n)\|y_n - x^*, z_2, ..., z_n\| + \alpha_n\|Sx^* - Sy_n, z_2, ..., z_n\|.
\end{aligned}$$

Taking $x = x^*$ and $y = y_n$ in (2.2), we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*, z_2, ..., z_n\| \leq (1-\alpha_n)\|y_n - x^*, z_2, ..., z_n\| \\
&\quad + \alpha_n\left[e^{L\|x^* - Sx^*, z_2, ..., z_n\|}\left(2\delta\|x^* - Sx^*, z_2, ..., z_n\| + \delta\|x^* - y_n, z_2, ..., z_n\|\right)\right] \\
&= (1-\alpha_n)\|y_n - x^*, z_2, ..., z_n\| \\
&\quad + \alpha_n\left[e^{L\|x^* - x^*, z_2, ..., z_n\|}\left(2\delta\|x^* - x^*, z_2, ..., z_n\| + \delta\|x^* - y_n, z_2, ..., z_n\|\right)\right] \\
&= (1-\alpha_n)\|y_n - x^*, z_2, ..., z_n\| + \alpha_n\left[e^{L(0)}\left(2\delta(0) + \delta\|x^* - y_n, z_2, ..., z_n\|\right)\right] \\
&= (1-\alpha_n)\|y_n - x^*, z_2, ..., z_n\| + \alpha_n\delta\|y_n - x^*, z_2, ..., z_n\|.
\end{aligned}$$

Thus,

$$\|x_{n+1} - x^*, z_2, ..., z_n\| \leq (1 - \alpha_n + \alpha_n\delta)\|y_n - x^*, z_2, ..., z_n\|.$$

Now, applying iteration process as given in the statement and proceeding similarly as in the proof of Theorem 3, we obtain

$\lim_{n\to\infty}\|x_{n+1} - x^*, z_2, ..., z_n\| = 0$, for every $z_2, ..., z_n \in K$.

This completes the proof. □

Corollary 3. *Let $T : K \to K$ and $S : K \to K$ be two Z-operators with $F_T \bigcap F_S \neq \phi$ where F_T and F_S are the sets of fixed points of T and S respectively. For any $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence as defined in the statement of Theorem 4 with $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0, 1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges to a common fixed point of T and S.*

3. Fixed point theorems

In main aim of this section is to prove some common fixed point theorems for a quadruple of weak compatible self mappings on a real linear n-normed space satisfying a common $(E.A)$ property and a generalized Φ- contraction.

Definition 1. Two self mappings A and S of an n-normed space $(X, ||.,\ldots,||)$ are called compatible if $\lim_{m\to\infty} \|ASx_m, -SAx_m, z_2, ..., z_n\| = 0$ (for every z_2 ,...,$z_n \in X$), whenever $\{x_m\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_m = \lim_{n\to\infty} Sx_m = t$, for some t in X.

In 1986, the notion of compatible mappings which generalized commuting mappings, was introduced by Jungck [14]. This has proven useful for generalization of results in metric fixed point theory for single-valued as well as multi-valued mappings. Further in 1998, the more general class of mappings called weakly compatible mappings was introduced by Jungck and Rhoades [15]. Recall that self mappings S and T of a metric space (X, d) are called weakly compatible if $Sx = Tx$ for some $x \in X$ implies that $STx = TSx$. i.e., weakly compatible pair commute at coincidence points. For some other results on fixed point theory, one may refer to [21, 22], etc.

Aamri and Moutawakil [1] introduced the following notion of *the property* $(E.A)$ for a pair of self maps in metric spaces. We define this notion for real linear n-normed spaces as follows:

Definition 2. Two self mappings S and T on a real linear n-normed space $(X, ||.,\ldots,||)$ are said to satisfy the property $(E.A)$, if there exists a sequence $\{x_m\}$ in X such that for every z_2 ,...,$z_n \in X$ $\lim_{m\to\infty} \|Tx_m, z_2, ..., z_n\| = \lim_{m\to\infty} \|Sx_m, z_2, ..., z_n\|$ = t, for some $t \in X$.

Definition 3. Let $A, B, S, T : (X, \|.,...,.\|) \to (X, \|.,...,.\|)$. The pairs (A, S) and (B, T) are said to satisfy a common property $(E.A)$, if there exist two sequences $\{x_m\}$ and $\{y_m\}$ such that for every z_2 ,...,$z_n \in X$

$$\lim_{m\to\infty} \|Ax_m, z_2, ..., z_n\| = \lim_{m\to\infty} \|Sx_m, z_2, ..., z_n\| = \lim_{m\to\infty} \|By_m, z_2, ..., z_n\|$$

$$= \lim_{m\to\infty} \|Ty_m, z_2, ..., z_n\| = t \in X.$$

This definition reduces to the previous one, if we take, $B = A$ and $S = T$. Further this definition is given by Liu, Wu and Liu [17] for metric spaces.

We denote by Φ the collection of all functions $\varphi : [0,\infty) \to [0,\infty)$ which are upper semi-continuous from the right, non-decreasing and satisfy $\lim_{s\to t+} \sup \varphi(s) < t$, $\varphi(t) < t$ for all $t > 0$.

Let A, B, S and T be self-mappings on a real linear n-normed space $(X, ||.,\ldots,.||)$ such that for every z_2 ,...,$z_n \in X$

$$\begin{aligned}
&[\|Ax-By,z_2,...,z_n\|^p + a\,\|Sx-Ty,z_2,...,z_n\|^p]\,\|Ax-By,z_2,...,z_n\|^p \\
&\quad \le a\max\{\|Ax-Sx,z_2,...,z_n\|^p\,\|By-Ty,z_2,...,z_n\|^p, \\
&\qquad \|Ax-Ty,z_2,...,z_n\|^q\,\|By-Sx,z_2,...,z_n\|^{q'}\} \\
&\qquad + \max\{\varphi_1(\|Sx-Ty,z_2,...,z_n\|^{2p}), \\
&\qquad \varphi_2(\|Ax-Sx,z_2,...,z_n\|^r\,\|By-Ty,z_2,...,z_n\|^{r'}), \\
&\qquad \varphi_3(\|Ax-Ty,z_2,...,z_n\|^s\,\|By-Sx,z_2,...,z_n\|^{s'}), \\
&\qquad \varphi_4(\frac{1}{2}[\|Ax-Ty,z_2,...,z_n\|^l\,\|Ax-Sx,z_2,...,z_n\|^{l'} \\
&\qquad + \|By-Sx,z_2,...,z_n\|^l\,\|By-Ty,z_2,...,z_n\|^{l'}])\},
\end{aligned} \tag{3.1}$$

for all $x, y, z_2, ..., z_n \in X$, $\varphi_i \in \Phi (i = 1,2,3,4)$, $a, p, q, q', r, r', s, s', l, l' \ge 0$ and $2p = q + q' = r + r' = s + s' = l + l'$. The condition (3.1) is commonly called a generalized Φ- contraction.

Now we first prove a common fixed point theorem for a quadruple of weak compatible self mappings satisfying a common $(E.A)$ property and a generalized Φ-contraction and then present some other results as corollaries. Finally, we present an extension of this common fixed point theorem.

Theorem 5. *Let A, B, S and T be self mappings on a real linear n-normed space $(X, \|.,...,.\|)$ satisfying (3.1). If the pairs (A,S) and (B,T) satisfy a common $(E.A)$ property, weakly compatible and that $T(X)$ and $S(X)$ are closed subsets of X, then A, B, S and T have a unique common fixed point in X.*

Proof. Since (A,S) and (B,T) satisfy a common property $(E.A)$, we can find two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

for some $z \in X$.

Let us assume that $S(X)$ and $T(X)$ are closed subspaces of X. Then, $z = Su = Tv$ for some $u, v \in X$. Then using (3.1) by considering x in place of x_n and v in place of y, we have for all z_2,...,$z_n \in X$

$$\begin{aligned}
&[\|Ax_n-Bv,z_2,...,z_n\|^p + a\,\|Sx_n-Tv,z_2,...,z_n\|^p]\,\|Ax_n-Bv,z_2,...,z_n\|^p \\
&\quad \le a\max\{\|Ax_n-Sx_n,z_2,...,z_n\|^p\,\|Bv-Tv,z_2,...,z_n\|^p,
\end{aligned}$$

$$\|Ax_n - Tv, z_2, ..., z_n\|^q \|Bv - Sx_n, z_2, ..., z_n\|^{q'}\}$$
$$+ \max\{\varphi_1(\|Sx_n - Tv, z_2, ..., z_n\|^{2p}),$$
$$\varphi_2(\|Ax_n - Sx_n, z_2, ..., z_n\|^r \|Bv - Tv, z_2, ..., z_n\|^{r'}),$$
$$\varphi_3(\|Ax_n - Tv, z_2, ..., z_n\|^s \|Bv - Sx_n, z_2, ..., z_n\|^{s'}),$$
$$\varphi_4(\frac{1}{2}[\|Ax_n - Tv, z_2, ..., z_n\|^l \|Ax_n - Sx_n, z_2, ..., z_n\|^{l'}$$
$$+ \|Bv - Sx_n, z_2, ..., z_n\|^l \|Bv - Tv, z_2, ..., z_n\|^{l'}])\}.$$

Taking limit as $n \to \infty$, and $Tv = z$, we obtain

$$[\|z - Bv, z_2, ..., z_n\|^p + a\|z - z, z_2, ..., z_n\|^p]\|z - Bv, z_2, ..., z_n\|^p$$
$$\le a \max\{\|z - z, z_2, ..., z_n\|^p \|Bv - z, z_2, ..., z_n\|^p,$$
$$\|z - z, z_2, ..., z_n\|^q \|Bv - z, z_2, ..., z_n\|^{q'}\} + \max\{\varphi_1(\|z - z, z_2, ..., z_n\|^{2p}),$$
$$\varphi_2(\|z - z, z_2, ..., z_n\|^r \|Bv - z, z_2, ..., z_n\|^{r'}),$$
$$\varphi_3(\|z - z, z_2, ..., z_n\|^s \|Bv - z, z_2, ..., z_n\|^{s'}),$$
$$\varphi_4(\frac{1}{2}[\|z - z, z_2, ..., z_n\|^l \|z - z, z_2, ..., z_n\|^{l'}$$
$$+ \|Bv - z, z_2, ..., z_n\|^l \|Bv - z, z_2, ..., z_n\|^{l'}])\}$$

or

$$\|z - Bv, z_2, ..., z_n\|^{2p}$$
$$\le \max\{\varphi_1(0), \varphi_2(0), \varphi_3(0), \varphi_4(\frac{1}{2}\|Bv - z, z_2, ..., z_n\|^{l+l'})\},$$

or

$$\|z - Bv, z_2, ..., z_n\|^{2p}$$
$$\le \max\{\varphi_1(\|z - Bv, z_2, ..., z_n\|^{2p}), \varphi_2(\|z - Bv, z_2, ..., z_n\|^{r+r'}),$$
$$\varphi_3(\|z - Bv, z_2, ..., z_n\|^{s+s'}), \varphi_4(\frac{1}{2}\|Bv - z, z_2, ..., z_n\|^{l+l'})\}.$$

If $\varphi_i \in \Phi$ where $i \in I$ (some index set), then there exists a $\varphi \in \Phi$ such that $\max\{\varphi_i, i \in I\} \le \varphi(t)$ for all $t > 0$ (Chang [6]). This along with the above argument and the property $\varphi(t) < t$ for all $t > 0$; we have $\|z - Bv, z_2, ..., z_n\|^{2p} \le \varphi(\|z - Bv, z_2, ..., z_n\|^{2p}) < \|z - Bv, z_2, ..., z_n\|^{2p}$, a contradiction. This implies that $z = Bv$. Therefore $Tv = z = Bv$. Hence it follows by the weak compatibility of the pair (B, T) that $BTv = TBv$, that is $Bz = Tz$.

Now, we shall show that z is a common fixed point of B and T. For this, considering x_n in place of x and z in place of y, (3.1) becomes

$$[\|Ax_n - Bz, z_2, ..., z_n\|^p + a\|Sx_n - Tz, z_2, ..., z_n\|^p]\|Ax_n - Bz, z_2, ..., z_n\|^p$$
$$\le a \max\{\|Ax_n - Sx_n, z_2, ..., z_n\|^p \|Bz - Tz, z_2, ..., z_n\|^p,$$
$$\|Ax_n - Tz, z_2, ..., z_n\|^q \|Bz - Sx_n, z_2, ..., z_n\|^{q'}\}$$

$$+\max\{\varphi_1(\|Sx_n - Tz, z_2, ..., z_n\|^{2p}),$$
$$\varphi_2(\|Ax_n - Sx_n, z_2, ..., z_n\|^r \|Bz - Tz, z_2, ..., z_n\|^{r'}),$$
$$\varphi_3(\|Ax_n - Tz, z_2, ..., z_n\|^s \|Bz - Sx_n, z_2, ..., z_n\|^{s'}),$$
$$\varphi_4(\frac{1}{2}[\|Ax_n - Tz, z_2, ..., z_n\|^l \|Ax_n - Sx_n, z_2, ..., z_n\|^{l'}$$
$$+\|Bz - Sx_n, z_2, ..., z_n\|^l \|Bz - Tz, z_2, ..., z_n\|^{l'}])\}$$

Letting $n \to \infty$, and using the fact that $\lim_{n\to\infty} Ax_n = z = \lim_{n\to\infty} Sx_n$ and $Bz = Tz$, we get

$$[\|z - Bz, z_2, ..., z_n\|^p + a\|z - Tz, z_2, ..., z_n\|^p]\|z - Bz, z_2, ..., z_n\|^p$$
$$\le a\max\{\|z - z, z_2, ..., z_n\|^p \|Bz - Tz, z_2, ..., z_n\|^p,$$
$$\|z - Tz, z_2, ..., z_n\|^q \|Bz - z, z_2, ..., z_n\|^{q'}\} + \max\{\varphi_1(\|z - Tz, z_2, ..., z_n\|^{2p}),$$
$$\varphi_2(\|z - z, z_2, ..., z_n\|^r \|Bz - Tz, z_2, ..., z_n\|^{r'}),$$
$$\varphi_3(\|z - Tz, z_2, ..., z_n\|^s \|Bz - z, z_2, ..., z_n\|^{s'}),$$
$$\varphi_4(\frac{1}{2}[\|z - Tz, z_2, ..., z_n\|^l \|z - z, z_2, ..., z_n\|^{l'}$$
$$+\|Bz - z, z_2, ..., z_n\|^l \|Bz - Tz, z_2, ..., z_n\|^{l'}])\},$$

or $\|z - Bz, z_2, ..., z_n\|^{2p} + a\|z - Bz, z_2, ..., z_n\|^{2p}$
$$\le a\|Bz - z, z_2, ..., z_n\|^{q+q'} + \max\{\varphi_1(\|z - Bz, z_2, ..., z_n\|^{2p}), \varphi_2(0),$$
$$\varphi_3(\|z - Bz, z_2, ..., z_n\|^{s+s'}), \varphi_4(0)\},$$

or $(1 + a)\|z - Bz, z_2, ..., z_n\|^{2p} \le a\|Bz - z, z_2, ..., z_n\|^{q+q'}$
$$+\max\{\varphi_1(\|z - Bz, z_2, ..., z_n\|^{2p}),$$
$$\varphi_2(0), \varphi_3(\|z - Bz, z_2, ..., z_n\|^{s+s'}), \varphi_4(0)\},$$

or $\|z - Bz, z_2, ..., z_n\|^{2p} \le \frac{a}{1+a}\|Bz - z, z_2, ..., z_n\|^{q+q'}$
$$+\frac{1}{1+a}\max\{\varphi_1(\|z - Bz, z_2, ..., z_n\|^{2p}),$$
$$\varphi_2(0), \varphi_3(\|z - Bz, z_2, ..., z_n\|^{s+s'}), \varphi_4(0)\}$$

Since $2p = q + q'$, using the same argument as applied earlier, we have $\|z - Bz, z_2, ..., z_n\|^{2p} < \|z - Bz, z_2, ..., z_n\|^{2p}$ (for every $z_2, ..., z_n \in X$), a contradiction. So, $z = Bz = Tz$. Thus z is a common fixed point of B and T.

Similarly, we can prove that z is a common fixed point of A and S. Thus z is the common fixed point of A, B, S and T. The uniqueness of z as a common fixed point of A, B, S and T can easily be verified. □

Now we deduce some other relevant results as corollaries.

Corollary 4. *Let A, B and S be self-mappings of X such that (A,S) and (B,S) satisfy a common $(E.A)$ property and*

$$\begin{aligned}\|Ax-By,z_2,...,z_n\|^{2p} \le a\max\{&\|Ax-Sx,z_2,...,z_n\|^p \|By-Sy,z_2,...,z_n\|^p,\\ &\|Ax-Sy,z_2,...,z_n\|^q \|By-Sx,z_2,...,z_n\|^{q'}\}+\\ &\max\{\varphi_2(\|Ax-Sx,z_2,...,z_n\|^r \|By-Sy,z_2,...,z_n\|^{r'}),\\ &\varphi_3(\|Ax-Sy,z_2,...,z_n\|^s \|By-Sx,z_2,...,z_n\|^{s'}),\\ &\varphi_4(\frac{1}{2}[\|Ax-Sy,z_2,...,z_n\|^l \|Ax-Sx,z_2,...,z_n\|^{l'}\\ &+\|By-Sx,z_2,...,z_n\|^l \|By-Sy,z_2,...,z_n\|^{l'}])\},\end{aligned} \tag{3.2}$$

for all $x,y,z_2,...,z_n \in X$, $\varphi_i \in \Phi$ $(i=2,3,4)$, $a,p,q,q',r,r',s,s',l,l' \ge 0$ and $2p = q+q' = r+r' = s+s' = l+l'$. If the pairs (A,S) and (B,S) are weakly compatible and that $S(X)$ is closed, then A, B and S have a unique common fixed point in X.

Proof. This is an immediate consequence of Theorem 5 with $S=T$. □

Corollary 5. *Let A, B, S and T be self mappings of a real linear n-normed space $(X,\|.,...,.\|)$. If the pairs (A,S) and (B,T) satisfy a common $(E.A)$ property and*

$$\begin{aligned}\|Ax-By,z_2,...,z_n\|^{2p} \le h\max\{&\|Sx-Ty,z_2,...,z_n\|^{2p},\\ &\|Ax-Sx,z_2,...,z_n\|^r \|By-Ty,z_2,...,z_n\|^{r'},\\ &\|Ax-Ty,z_2,...,z_n\|^s \|By-Sx,z_2,...,z_n\|^{s'},\\ &\frac{1}{2}[\|Ax-Ty,z_2,...,z_n\|^l \|Ax-Sx,z_2,...,z_n\|^{l'}\\ &+\|By-Sx,z_2,...,z_n\|^l \|By-Ty,z_2,...,z_n\|^{l'}]\},\end{aligned} \tag{3.3}$$

for all $x,y,z_2,...,z_n \in X$, $p,q,q',r,r',s,s',l,l' \ge 0$ and $2p = q+q' = r+r' = s+s' = l+l'$. If the pairs (A,S) and (B,T) are weakly compatible and that $T(X)$ and $S(X)$ are closed, then A, B, S and T have a unique common fixed point in X.

Proof. If we take $a=0$ and $\varphi_i(t) = ht$ $(i=1,2,3,4)$, where $0<h<1$ in Theorem 5, we have the desired result. □

Corollary 6. *Let A and B be self mappings of a complete real linear n-normed space $(X,\|.,...,.\|)$ satisfying the following condition:*

$$\begin{aligned}[\|Ax-By,z_2,...,z_n\|^p + a\|x-y,z_2,...,z_n\|^p]\|Ax-By,z_2,...,z_n\|^p\\ \le a\max\{\|Ax-x,z_2,...,z_n\|^p \|By-y,z_2,...,z_n\|^p,\\ \|Ax-y,z_2,...,z_n\|^q \|By-x,z_2,...,z_n\|^{q'}\}+\\ \max\{\varphi_1(\|x-y,z_2,...,z_n\|^{2p}),\varphi_2(\|Ax-x,z_2,...,z_n\|^r \|By-y,z_2,...,z_n\|^{r'}),\\ \varphi_3(\|Ax-y,z_2,...,z_n\|^s \|By-x,z_2,...,z_n\|^{s'}),\end{aligned}$$

$$\varphi_4(\frac{1}{2}[\|Ax-y,z_2,...,z_n\|^l \|Ax-x,z_2,...,z_n\|^{l'}$$
$$+\|By-x,z_2,...,z_n\|^l \|By-y,z_2,...,z_n\|^{l'}])\},$$

for all $x,y,z_2,...,z_n \in X$, $\varphi_i \in \Phi$ $(i=1,2,3,4)$, $a,p,q,q',r,r',s,s',l,l' \geq 0$ *and* $2p=q+q'=r+r'=s+s'=l+l'$, *then* A *and* B *have a unique common fixed point in* X.

Proof. The proof is immediate, if we take $S=T=I_X$ (the identity mapping on X) in Theorem 5. □

Finally, we give the following result as an extension of Theorem 5.

Theorem 6. *Let* S, T *and* A_n $(n \in N)$ *be self mappings of a real linear n-normed space* $(X,\|.,...,.\|)$. *Suppose further that the pairs* (A_{2n-1},S) *and* (A_{2n},T) *are weakly compatible for any* $n \in N$ *and satisfying a common* $(E.A)$ *property. If* $S(X)$ *and* $T(X)$ *are closed and that for any* $i \in N$, *the following condition is satisfied for all* $x,y,z_2,...,z_n \in X$

$$[\|A_ix-A_{i+1}y,z_2,...,z_n\|^p + a\|Sx-Ty,z_2,...,z_n\|^p]\|A_ix-A_{i+1}y,z_2,...,z_n\|^p$$
$$\leq a\max\{\|A_ix-Sx,z_2,...,z_n\|^p \|A_{i+1}y-Ty,z_2,...,z_n\|^p$$
$$\|A_ix-Ty,z_2,...,z_n\|^q \|A_{i+1}y-Sx,z_2,...,z_n\|^{q'}\}$$
$$+\max\{\varphi_1(\|Sx-Ty,z_2,...,z_n\|^{2p}),$$
$$\varphi_2(\|A_ix-Sx,z_2,...,z_n\|^r \|A_{i+1}y-Ty,z_2,...,z_n\|^{r'}),$$
$$\varphi_3(\|A_ix-Ty,z_2,...,z_n\|^s \|A_{i+1}y-Sx,z_2,...,z_n\|^{s'}),$$
$$\varphi_4(\frac{1}{2}[\|A_ix-Ty,z_2,...,z_n\|^l \|A_ix-Sx,z_2,...,z_n\|^{l'}$$
$$+\|A_{i+1}y-Sx,z_2,...,z_n\|^l \|A_{i+1}y-Ty,z_2,...,z_n\|^{l'}])\},$$

where $\varphi_i \in \Phi$ $(i=1,2,3,4)$, $a,p,q,q',r,r',s,s',l,l' \geq 0$ *and* $2p=q+q'=r+r'=s+s'=l+l'$, *then* S, T *and* A_n $(n \in N)$ *have a common fixed point in* X.

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Author's address

Hemen Dutta
Gauhati University, Mathematics Department, Guwahati, 781014 Assam, India
E-mail address: hemen_dutta08@rediffmail.com

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NONLOCAL AND INTEGRAL CONDITIONS PROBLEMS FOR A MULTI-TERM FRACTIONAL-ORDER DIFFERENTIAL EQUATION

A. M. A. EL-SAYED AND E. O. BIN-TAHER

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Abstract. In this paper we study the existence of solution for a multi-term arbitrary (fractional) order differential equation with some nonlocal and integral conditions.

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1. INTRODUCTION

Problems with non-local conditions have been extensively studied by several authors in the last two decades. The reader is referred to [1,2,5] and references therein. In this work we study the existence of at least one absolutely continuous solution $x \in AC[0,1]$ for the nonlocal problem of the arbitrary (fractional) order differential equation

$$x'(t) = f(t, D^{\alpha_1}x(t), D^{\alpha_2}x(t), \cdots, D^{\alpha_n}x(t)), \alpha_i \in (0, 1), \text{ a.e. } t \in (0, 1) \quad (1.1)$$

with the nonlocal condition

$$\sum_{k=1}^{m} a_k\, x(\tau_k) = \beta \sum_{j=1}^{p} b_j\, x(\eta_j) \quad (1.2)$$

where $a_k, b_j > 0$, $\tau_k \in (a, c)$, $\eta_j \in (d, b)$, $0 < a < c \leq d < b < 1$, $\sum_{k=1}^{m} a_k \neq \beta \sum_{j=1}^{p} b_j$ and β is parameter.
As an application, we deduce the existence of solution for the nonlocal problem of the differential equation (1.1) with the nonlocal condition

$$\sum_{k=1}^{m} a_k x(\tau_k) = 0, \quad \tau_k \in (a,c) \subset (0,1), \quad (1.3)$$

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and the nonlocal integral conditions

$$\int_a^c x(s)\,ds = \beta \int_d^b x(s)\,ds,\ 0 < a < c \le d < b < 1, \tag{1.4}$$

and

$$\int_a^c x(s)\,ds = 0,\ (a,c) \subset (0,1). \tag{1.5}$$

are also considered.

2. PRELIMINARIES

Let $L^1(I)$ denotes the class of Lebesgue integrable functions on the interval $I = [0,1]$,where $0 \le a < b < \infty$ and let $\Gamma(.)$ denotes the gamma function. Recall that the operator T is compact if it is continuous and maps bounded sets into relatively compact ones. The set of all compact operators from the subspace $U \subset X$ into the Banach space X is denoted by $C(U, X)$.

Definition 1. The fractional-order integral of the function $f \in L_1[a,b]$ of order $\beta > 0$ is defined by (see [7])

$$I_a^\beta f(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\,ds.$$

Definition 2. The Riemann-Liouville fractional-order derivative of $f(t)$ of order $\alpha \in (0,1)$ is defined as (see [6] and [7])

$$D_a^\alpha f(t) = \frac{d}{dt} \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s)\,ds.$$

The following theorems will be needed.

Theorem 1 (Schauder fixed point theorem [3])**.** *Let E be a Banach space and Q be a convex subset of E, and $T : Q \longrightarrow Q$ is compact, continuous map, Then T has at least one fixed point in Q.*

Theorem 2 (Kolmogorov compactness criterion [4])**.** *Let $\Omega \subseteq L^p(0,1)$, $1 \le p < \infty$. If*

(i) *Ω is bounded in $L^p(0,1)$, and*
(ii) *$u_h \to u$ as $h \to 0$ uniformly with respect to $u \in \Omega$, then Ω is relatively compact in $L^p(0,1)$, where*

$$u_h(t) = \frac{1}{h} \int_t^{t+h} u(s)\,ds.$$

3. Main results

Consider firstly the fractional-order functional integral equation

$$y(t) = f(t,\ I^{1-\alpha_1} y(t), \cdots,\ I^{1-\alpha_n} y(t)), \tag{3.1}$$

Definition 3. The function y is called a solution of the fractional-order functional integral equation (3.1), if $y \in L_1[0,1]$ and satisfies (3.1).

Consider the following assumption:

(i) $f : [0,1] \times R_n \to R$ be a function with the following properties:
  (a) for each $t \in [0,1]$, $f(t,.)$ is continuous,
  (b) for each $x \in R_n$, $f(.,x)$ is measurable,

(ii) there exist an integral function a, $a \in L_1[0,1]$ and constants $q_i > 0$, $i = 1,\ 2$, such that

$$|f(t,x)| \ \leq\ a(t) \ +\ \sum_{i=1}^{n} q_i\ |x_i|,\ \text{ for each }\ t \in [0,1],\ \ x \in R_n,$$

Theorem 3. *Let the assumptions (i) and (ii) be satisfied.*

$$\textit{If} \quad \sum_{i=1}^{n} \frac{q_i}{\Gamma(2-\alpha_i)} \ <\ 1, \tag{3.2}$$

then the fractional-order functional integral equation (3.1) has at least one solution $y \in L_1[0,1]$, *where*

$$r \leq \frac{\|a\|}{1 - \sum_{i=1}^{n} \frac{q_i}{\Gamma(2-\alpha_i)}}$$

Proof. Define the operator T associated with equation (3.1) by

$$Ty(t) = f(t,\ I^{1-\alpha_1} y(t), \cdots,\ I^{1-\alpha_n} y(t))$$

Let $B_r = \{y \in L_1(I) : \|y\| < r,\ r > 0\}$ and let y be an arbitrary element in B_r. Then from the assumptions (i) and (ii), we obtain

$$\begin{aligned}
||Ty||_{L_1} &= \int_0^1 |Ty(t)|\ dt \\
&\leq \int_0^1 |f(t,\ I^{1-\alpha_1} y(t), \cdots,\ I^{1-\alpha_n} y(t))| dt \\
&\leq \int_0^1 |a(t)| dt \ +\ \sum_{i=1}^{n} q_i \int_0^1 \int_0^t \frac{(t-s)^{-\alpha_i}}{\Gamma(1-\alpha_i)} |y(s)|\ ds\ dt \\
&\leq \|a\| \ +\ \sum_{i=1}^{n} q_i \int_0^1 \int_s^1 \frac{(t-s)^{-\alpha_i}}{\Gamma(1-\alpha_i)} dt |y(s)|\ ds
\end{aligned}$$

$$\leq \|a\| + \sum_{i=1}^{n} q_i \int_0^1 -\frac{(t-s)^{1-\alpha_i}}{(1-\alpha_i)\Gamma(1-\alpha_i)}|_s^1 |y(s)|\, ds$$

$$\leq \|a\| + \sum_{i=1}^{n} q_i \int_0^1 \frac{1}{\Gamma(2-\alpha_i)}|y(s)|\, ds$$

$$\leq \|a\| + \sum_{i=1}^{n} \frac{q_i}{\Gamma(2-\alpha_i)}\|y\|_{L_1} \leq r,$$

which implies that the operator T maps B_r into itself.
Assumption (i) implies that T is continuous.
Now, we will show that T is compact, applying Theorem 1. So, let Ω be a bounded subset of B_r. Then $T(\Omega)$ is bounded in $L_1[0,1]$, i.e. condition (i) of Theorem 2 is satisfied. It remains to show that $(Ty)_h \to Ty$ in $L_1[0,1]$ as $h \to 0$, uniformly with respect to $Ty \in T\ \Omega$. Now

$$||(Ty)_h - Ty|| = \int_0^1 |(Ty)_h(t) - (Ty)(t)|\, dt$$

$$= \int_0^1 |\frac{1}{h}\int_t^{t+h} (Ty)(s)\, ds - (Ty)(t)|\, dt$$

$$\leq \int_0^1 \left(\frac{1}{h}\int_t^{t+h} |(Ty)(s) - (Ty)(t)|\, ds\right) dt$$

$$\leq \int_0^1 -\frac{1}{h}\int_t^{t+h} |f(s, I^{1-\alpha_1}y(t), ..., I^{1-\alpha_n}y(t))$$
$$- f(t, I^{1-\alpha_1}y(t), ..., I^{1-\alpha_n}y(t))|\, ds\, dt.$$

Now, $y \in \Omega$ implies (by assumption (ii)) that $f \in L_1(0,1)$, then

$$\frac{1}{h} - \int_t^{t+h} -|f(s, I^{1-\alpha_1}y(t), \cdots, I^{1-\alpha_n}y(t)) -$$
$$- f(t, I^{1-\alpha_1}y(t), \cdots, I^{1-\alpha_n}y(t))|ds \to 0$$

Therefore, by Theorem 2, we have that $T(\Omega)$ is relatively compact, that is, T is a compact operator, then the operator T has a fixed point B_r, which proves the existence of a positive solution $y \in (0,\ 1)$ of equation (3.1). □

Theorem 4. *Let the assumptions of Theorem 3 be satisfied. Then the nonlocal problem (1.1)- (1.2) has at least one solution* $x \in AC[0,1]$.

Proof. Consider the nonlocal fractional differential equation

$$x'(t) = f(t, D^{\alpha_1}x(t), D^{\alpha_2}x(t), \cdots, D^{\alpha_n}x(t)), \alpha_i \in (0, 1), \text{ a.e. } t \in (0, 1),$$

$$\sum_{k=1}^{m} a_k\, x(\tau_k) = \beta \sum_{j=1}^{p} b_j\, x(\eta_j), \qquad a_k, b_j > 0,\ \tau_k \in (0, c),\ \eta_j \in (d, 1),\ c \le d.$$

Let $y(t) = x'(t)$, then

$$x(t) = x(0) + Iy(t) \tag{3.3}$$

and y is the solution of the fractional-order integral equation (3.1).
Let $t = \tau_k$ in equation (3.3), we get

$$x(\tau_k) = \int_0^{\tau_k} y(s)\, ds + x(0)$$

$$\sum_{k=1}^{m} a_k x(\tau_k) = \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\, ds + x(0) \sum_{k=1}^{m} a_k$$

And let $t = \eta_j$ in equation (3.3), we get

$$x(\eta_j) = \int_0^{\eta_j} y(s)\, ds + x(0)$$

$$\sum_{j=1}^{p} b_j x(\eta_j) = \sum_{j=1}^{p} bj \int_0^{\eta_j} y(s)\, ds + x(0) \sum_{j=1}^{p} b_j$$

From equation (1.2), we get

$$\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\, ds + x(0) \sum_{k=1}^{m} a_k = \beta \sum_{j=1}^{p} b_j \int_0^{\eta_j} y(s)\, ds + x(0)\beta \sum_{j=1}^{p} b_j$$

Then we get

$$x(0) = A \left(\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\, ds - \beta \sum_{j=1}^{p} b_j \int_0^{\eta_j} y(s)\, ds \right),$$

where $A = (\beta \sum_{j=1}^{p} b_j - \sum_{k=1}^{m} a_k)^{-1}$
and

$$x(t) = A \left(\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\, ds - \beta \sum_{j=1}^{p} b_j \int_0^{\eta_j} y(s)\, ds \right) + \int_0^t y(s)\, ds \tag{3.4}$$

which, by Theorem 3, has at least one solution $x \in AC(0,1)$.
Now, from equation (3.4), we have

$$x(0) = \lim_{t \to 0^+} x(t) = A \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\, ds - A\beta \sum_{j=1}^{p} b_j \int_0^{\eta_j} y(s)\, ds.$$

Also

$$x(1) = \lim_{t \to 1^-} x(t) = A \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\, ds - A\beta \sum_{j=1}^{p} b_j \int_0^{\eta_j} y(s)\, ds + \int_0^1 y(s)\, ds$$

from which we deduce that equation (3.4) has at least one solution $x \in AC[0,1]$.
To complete the proof

$$\frac{dx}{dt} = y(t),$$

$$D^{\alpha_i} x(t) = I^{1-\alpha_i} \frac{d}{dt} x(t) = I^{1-\alpha_i} y(t)$$

where

$$x'(t) = f(t, x(t), D^{\alpha_1} x(t), D^{\alpha_2} x(t), \cdots, D^{\alpha_n} x(t)).$$

□

Now letting $\beta = 0$ in (1.2), we can easily prove the following theorem.

Theorem 5. *Let the assumptions (i) - (ii) be satisfied. Then the nonlocal problem*

$$x'(t) = f(t, D^{\alpha_1} x(t), D^{\alpha_2} x(t), \cdots, D^{\alpha_n} x(t)), \alpha_i \in (0, 1), \text{ a.e. } t \in (0, 1),$$

$$\sum_{k=1}^{m} a_k x(\tau_k) = 0, \quad \tau_k \in (a,c) \subset (0,1).$$

has at least one solution $x \in AC[0,1]$ *represented by*

$$x(t) = A \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\, ds - \int_0^t y(s)\, ds, \text{ where } A = (\sum_{k=1}^{m} a_k)^{-1}.$$

4. Nonlocal integral condition

Let $x \in AC[0,1]$ be the solution of the nonlocal problem (1.1-1.2).
Let $a_k = \tau_k - \tau_{k-1}$, $t_k \in (\tau_{k-1}, \tau_k)$, $a = \tau_0 < \tau_1 < \tau_2, \ldots < \tau_m = c$ and $b_j = \eta_j - \eta_{j-1}$, $t_j \in (\eta_{j-1}, \eta_j)$, $d = \eta_0 < \eta_1 < \eta_2, \ldots < \eta_p = b$ then the nonlocal condition (1.2) will be

$$\sum_{k=1}^{m} (\tau_k - \tau_{k-1})\, x(t_k) = \beta \sum_{j=1}^{p} (\eta_j - \eta_{j-1})\, x(t_j).$$

From the continuity of the solution x of the nonlocal problem (1.1-1.2) we can obtain

$$\lim_{m\to\infty}\sum_{k=1}^{m}(\tau_k-\tau_{k-1})\,x(t_k)\;=\;\beta\lim_{p\to\infty}\sum_{j=1}^{p}(\eta_j-\eta_{j-1})\,x(t_j).$$

and the nonlocal condition (1.2) is transformed to the integral one

$$\int_a^c x(s)\,ds\;=\;\beta\int_d^b x(s)\,ds\,. \tag{4.1}$$

Also from the continuity of the function $Iy(t)$, where y is the solution of the functional integral equation (3.1), we deduce that the solution (3.4) will be

$$x(t)=(\beta\,(b-d)-(c-a))^{-1}\left(\int_a^c\int_0^s y(\theta)d\theta\,ds-\beta\int_d^b\int_0^s y(\theta)\,d\theta\,ds\right)$$
$$+\int_0^t y(s)ds.$$

Now, we have the following theorem.

Theorem 6. *Let the assumptions of Theorem 4 be satisfied. Then there exist at least one solution $x\in AC[0,1]$ of the nonlocal problem with integral condition,*

$$x'(t)\;=\;f(t,\,D^{\alpha_1}x(t),\,D^{\alpha_2}x(t),\,\cdots,\,D^{\alpha_n}x(t))\,,\alpha_i\in(0,\,1),\;a.e.\;t\in(0,\,1),$$

$$\int_a^c x(s)\,ds\;=\;\beta\int_d^b y(s)\,ds\,,\;0\le a<c\le d<b\le 1,\;\;\beta\,(b-d)\neq(c-a).$$

Letting $\beta\;=\;0$ in (4.1), we can easily prove the following corollary.

Corollary 1. *Let the assumptions (i) - (ii) be satisfied. Then the nonlocal problem*

$$x'(t)\;=\;f(t,\,D^{\alpha_1}x(t),\,D^{\alpha_2}x(t),\,\cdots,\,D^{\alpha_n}x(t))\,,\alpha_i\in(0,\,1),\;a.e.\;t\in(0,\,1),$$

$$\int_a^c x(s)\,ds\;=\;0,\quad(a,c)\subset(0,1),$$

has at least one solution $x\in AC[0,1]$ represented by

$$x(t)\;=\;\int_0^t y(s)\,ds-(c-a)^{-1}\int_a^c\int_0^s y(\theta)\,d\theta\,ds.$$

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Authors' addresses

A. M. A. El-Sayed
Alexandria University, Faculty of Science, Alexandria, Egypt
E-mail address: amasayed@hotmail.com

E. O. Bin-Taher
Hadhramout Univeristy of Science and Technology, Faculty of Science, Hadhramout, Yemen
E-mail address: ebtsamsam@yahoo.com

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N-STABILITY OF THE θ-METHOD FOR REACTION-DIFFUSION PROBLEMS

I. FEKETE AND I. FARAGÓ

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Abstract. In this paper we investigate the N-stability notion in an abstract Banach space setting. The main result is that thanks to this notion we have an alternative opportunity for verifying the stability of the numerical solution for periodic initial-value reaction-diffusion problems.

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Keywords: numerical method, nonlinear problems, stability

1. INTRODUCTION

Many phenomena in nature such as physical, biological or chemical processes can be described by mathematical models. In chemistry one of the most investigated problems is the reaction-diffusion problem. Reaction-diffusion is a process in which two or more chemicals diffuse over a surface and react with one another to produce stable patterns. However (in general), the solution of such models cannot be given in a closed form, therefore we construct numerical models in order to approximate the exact solution. Generally, consistency in itself is not enough for convergence. To guarantee this property we introduce the notion of stability.

The stability property is verified for periodic initial-value reaction-diffusion problems in case of globally Lispchitz continuous forcing function f in several works, e.g. in Ascher [1] and Thomas [8]. Regarding the stability proof, these books use the fact that we know the eigenvalues of the standard matrix replacement of the second derivative operator with periodic boundary conditions. Further techniques are introduced e.g., discrete time Fourier transform [8].

In the recent years we dealt with the investigation of the numerical solution of nonlinear operator equations in an abstract (Banach space) setting. This work has been summarized in [3, 4]. In this paper our primary aim is to describe the discretizated version of the above mentioned problem and verify the N-stability in an abstract setting. Thanks to this method we can offer a different approach for verifying the stability of nonlinear equations.

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This paper is organized as follows. In Section 2 we give a short mathematical formulation of the general description of mathematical and numerical models. Moreover, we define the basic numerical notions (convergence, consistency). In Section 3 we introduce the notion of N-stability based on the work of Sans-Serna ([6,7]). In Section 3.1 we describe the connection with the notion of linear stability. In Section 3.2 we verify that the θ-method is N-stable both with and without the forcing term. In Section 4 we make some remarks and we draw conclusions.

2. Mathematical background

We consider the problem

$$F(u) = 0\,, \tag{2.1}$$

where $F : \mathcal{D} \to \mathcal{Y}$ is a (nonlinear) operator, $\mathcal{D} \subset \mathcal{X}$, $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces. In the theory of numerical analysis it is usually *assumed* that there exists a unique solution, which will be denoted by $\bar{u}$.

Definition 1. Problem (2.1) can be identified as a triplet $\mathscr{P} = (\mathcal{X}, \mathcal{Y}, F)$. We will refer to it as *problem* $\mathscr{P}$.

Definition 2. The sequence $\mathscr{N} = (\mathcal{X}_n, \mathcal{Y}_n, F_n)_{n\in\mathbb{N}}$ is called numerical method if it generates a sequence of problems

$$F_n(u_n) = 0\,, \quad n = 1,2,\dots\,, \tag{2.2}$$

where

- $\mathcal{X}_n, \mathcal{Y}_n$ are normed spaces,
- $\mathcal{D}_n \subset \mathcal{X}_n$ and $F_n : \mathcal{D}_n \to \mathcal{Y}_n$.

If there exists a unique solution of (2.2), it will be denoted by $\bar{u}_n$.

Remark 1. In the sequel we always assume the existence of $\bar{u}$ and $\bar{u}_n$.

Definition 3. We say that the sequence $\mathscr{D} = (\varphi_n, \psi_n, \Phi_n)_{n\in\mathbb{N}}$ is a *discretization* of problem $\mathscr{P}$ by numerical method $\mathscr{N}$, if

- the φ_n-s (respectively ψ_n-s) are restriction operators from $\mathcal{X}$ into $\mathcal{X}_n$ (respectively from $\mathcal{Y}$ into $\mathcal{Y}_n$),
- $\Phi_n : \{F : \mathcal{D} \to \mathcal{Y} \,|\, \mathcal{D} \subset \mathcal{X}\} \to \{F_n : \mathcal{D}_n \to \mathcal{Y}_n \,|\, \mathcal{D}_n \subset \mathcal{X}_n\}$.

In the sense of Definitions 1, 2 and 3 we can illustrate the general scheme, shown in Figure 1.
In our paper we always make the following assumption for the general discretization $\mathscr{D}$.

Assumption 1.

(a) The discretization $\mathscr{D}$ possesses the property $\psi_n(0) = 0$.
(b) The discretization $\mathscr{D}$ generates the numerical method $\mathscr{N}$ with the property $\dim \mathcal{X}_n = \dim \mathcal{Y}_n < \infty$.

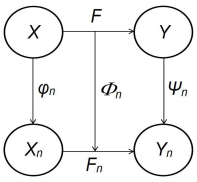

FIGURE 1. The general scheme of numerical methods.

(c) F_n is continuous on some ball $B_R(\varphi_n(\bar{u}))$.

Remark 2. Obviously, when ψ_n are linear operators, then Assumption 1 (a) is automatically satisfied.

Definition 4. The element $e_n = \varphi_n(\bar{u}) - \bar{u}_n \in \mathcal{X}_n$ is called global discretization error.

Definition 5. The discretization $\mathscr{D}$ applied to the problem $\mathscr{P}$ is called *convergent* if

$$\lim \|e_n\|_{\mathcal{X}_n} = 0$$

holds.

Definition 6. The element $l_n(v) = F_n(\varphi_n(v)) - \psi_n(F(v)) \in \mathcal{Y}_n$ is called local discretization error at the element $v \in \mathcal{D}$. The local discretization error on the solution, i.e., $l_n(\bar{u}) = F_n(\varphi_n(\bar{u})) - \psi_n(F(\bar{u})) = F_n(\varphi_n(\bar{u}))$ is called local discretization error.

Definition 7. The discretization $\mathscr{D}$ applied to problem $\mathscr{P}$ is called consistent on the element $v \in D$ if

- $\varphi_n(v) \in \mathcal{D}_n$ holds from some index,
- for these elements the relation

$$\lim \|l_n(v)\|_{\mathcal{Y}_n} = 0$$

holds.

3. N-STABILITY AND ITS ADVANTAGES

In numerical analysis one of the most important task is to guarantee the convergence of the sequence of the numerical solutions. Generally, consistency in itself is not enough for convergence. To guarantee this property we introduce the notion of stability.

According to Definition 4, the convergence yields that the global discretization error e_n tends to zero. Having consistency, we have information about the local discretization error, only. Intuitively, this means the following requirement. When $l_n(\bar{u}) = F_n(\varphi_n(\bar{u})) - F(\bar{u}_n)$ is small, then $e_n = \varphi(\bar{u}) - \bar{u}_n$ should be small, too.

Because $\bar{u}$ is unknown, therefore in first approach we require this property for any pairs in $\mathcal{D}_n$, i.e., for all $z_n,\ w_n \in \mathcal{D}_n$ we require the inequality

$$\|z_n - w_n\|_{\mathcal{X}_n} \leq S\,\|F_n(z_n) - F_n(w_n)\|_{\mathcal{Y}_n}\,, \tag{3.1}$$

where the stability constant S is indepedent of the mesh-size parameter.
This idea leads to make the first attempt to define the nonlinear stability notion.

Definition 8. The discretization $\mathscr{D}$ is called *N-stable* on the problem $\mathscr{P}$ if there exists a positive stability constant S, such that for each $z_n, w_n \in \mathcal{D}_n$, the estimation (3.1) holds.

Furthermore we will refer to this notion as the naive stability (N-stability).

Theorem 1. *We assume that*

- *there exists the solution of the problem* (2.1)-(2.2)*,*
- *the discretization $\mathscr{D}$ is consistent and N-stable at $\bar{u}$ with constant S on problem $\mathscr{P}$.*

Then the discretization $\mathscr{D}$ is convergent on problem $\mathscr{P}$.

Proof. Due to the N-stability, we have the relation

$$\|e_n\|_{\mathcal{X}_n} = \|\varphi_n(\bar{u}) - \bar{u}_n\|_{\mathcal{X}_n} \leq S\,\|F_n(\varphi_n(\bar{u})) - \underbrace{F_n(\bar{u}_n)}_{=0}\|_{\mathcal{Y}_n}$$

which leads to the estimation

$$\|e_n\|_{\mathcal{X}_n} \leq S\,\|F_n(\varphi_n(\bar{u}))\|_{\mathcal{Y}_n} = S\,\|l_n(\bar{u})\|_{\mathcal{Y}_n}$$

and hence, for consistent methods this implies the convergence. □

Remark 3. Formally, this statement can be written as the "basic theory of numerical analysis":

$$\text{Consistency} + \text{Stability} \Rightarrow \text{Convergence}$$

3.1. *Linear stability*

For studying the nonlinear stability, it is expedient to present the linear case. In this case the consistency and the stability together ensure the convergence. This result is well known as the Lax (or sometimes Lax-Richtmyer-Kantorovich [5]) theorem. From the formulation of the main theorem it turns out that these two, directly checkable conditions (i.e., the consistency and stability) serve together the convergence. Let

$$L_n(u_n) = 0\,, \quad n = 1, 2, \ldots\,, \tag{3.2}$$

where L_n is a linear oparator and $L_n : \mathcal{D}_n \to \mathcal{Y}_n$.

Definition 9. The discretization $\mathscr{D}$ is called *stable* on the problem $\mathscr{P}$ if there exists positive stability constant S, such that for each $v_n \in \mathcal{D}_n$

$$\|v_n\|_{\mathcal{X}_n} \leq S \, \|L_n(v_n)\|_{\mathcal{Y}_n} \tag{3.3}$$

holds.

The bound (3.3) implies three basic properties:

i, For any problems (3.2), the relation (3.3) shows that $L_n(v_n) = 0$ implies that $v_n = 0$, i.e., L_n is injective and hence L_n^{-1} exists on the entire space $\mathcal{Y}_n$ by Assumption 1 (b). Therefore, the stability bound implies the existence and uniqueness of solutions of (3.2).

ii, Due to the previous property i, using the estimate (3.3) we have

$$\left\| L_n^{-1} s_n \right\|_{\mathcal{X}_n} \leq S \, \|s_n\|_{\mathcal{Y}_n}$$

for arbitrary $s_n \in \mathcal{Y}_n$. Therefore the uniform norm estimation

$$\left\| L_n^{-1} \right\|_{Lin(\mathcal{Y}_n, \mathcal{X}_n)} \leq S$$

holds.

iii, In view of (3.3), we obtain the "basic theory of numerical analysis". In fact, due to the linearity of L_n, by the choice $e_n = \varphi_n(\bar{u}) - \bar{u}_n$, we have

$$\|\varphi_n(\bar{u}) - \bar{u}_n\|_{\mathcal{X}_n} \leq S \, \|L_n(\varphi_n(\bar{u})) - L_n(\bar{u}_n)\|_{\mathcal{Y}_n}$$

which leads to the estimation

$$\|e_n\|_{\mathcal{X}_n} = \|\varphi_n(\bar{u}) - \bar{u}_n\|_{\mathcal{X}_n} \leq S \, \|l_n(\bar{u})\|_{\mathcal{Y}_n}$$

and for consistent methods, this implies the convergence.

Remark 4.

(a) Definition 9 is special case of Definition 8 with the choice $F_n(u_n) = L_n(u_n)$,

(b) The reverse of property ii, is true. Namely, if the inverse of the linear operators L_n exist and they are uniformly bounded, then they are N-stable, too.

(c) In a straightforward way the above mentioned three basic properties can be generalized for the nonhomogeneous linear equation $F_n(u_n) \equiv L_n(u_n) - b_n = 0$.

3.2. *N-stability of the reaction-diffusion problem*

We showed in the previous section that N-stability is equivalent to the regular stability notion in the linear case. In addition, this notion has an advantageous property for nonlinear equations as well. Namely, it is useful from the application point of view. In the rest of the paper our aim is to verify the N-stability of the reaction-diffusion problem under certain conditions using the θ-method.

3.2.1. *Reaction-diffusion problem without forcing term*

Consider the following periodic initial-value reaction-diffusion problem:

$$\partial_t u(t,x) = \partial_{xx} u(t,x), \quad x \in \mathbb{R},\ t \in [0,T], \tag{3.4}$$

$$u(t,x) = u(t,x+1), \quad x \in \mathbb{R},\ t \in [0,T], \tag{3.5}$$

$$u(0,x) = u^0(x), \quad x \in \mathbb{R}, \tag{3.6}$$

where $T \in \mathbb{R}^+$. The conditions (3.5)-(3.6) are periodic boundary conditions and initial-value conditions, where u^0 is a given one-periodic function. It is easy to see that the continuous problem (3.4)-(3.6) can be rewritten in the form of (2.1). As we have mentioned, we assume the existence of unique, sufficiently smooth solution of the problem (3.4)-(3.6).

Remark 5. Since the solution is periodic, it is sufficient to determine the solution in one period, only.

To create the discretization $\mathscr{D}$ on the above mentioned problem we define both the spatial and time grids, as follows. The spatial grid points are

$$\{x_j = jh, \text{ where } j = 1,\dots,n,\ h = 1/n \text{ and } n \in \mathbb{N},\ n \geq 2\},$$

and the time levels are

$$\{t_k = k\delta, \text{ where } k = 0,\dots,K \text{ and } \delta = T/K\}.$$

Applying the θ-method to this reaction-diffusion problem, for $\theta \in [0,1]$, $j = 1,\dots,n$, and $k = 0,\dots,K-1$, we gain

$$\frac{u_j^{k+1} - u_j^k}{\delta} - (1-\theta)\frac{u_{j-1}^k - 2u_j^k + u_{j+1}^k}{h^2} - \theta\frac{u_{j-1}^{k+1} - 2u_j^{k+1} + u_{j+1}^{k+1}}{h^2} = 0, \tag{3.7}$$

where using the periodic boundary conditions it is obvious that $u_0^k = u_n^k$, $u_1^k = u_{n+1}^k$, $u_0^{k+1} = u_n^{k+1}$ and $u_1^{k+1} = u_{n+1}^{k+1}$. The initial-value condition can be written as

$$u_j^0 - u^0(x_j) = 0, \quad j = 1,\dots,n. \tag{3.8}$$

In the next step is that we rewrite (3.7)-(3.8) in the form of (2.2). To this aim we define the vector space of the grid functions $\mathcal{K}_n$, defined on the grid points $x_j : 1 \leq j \leq n$. If we consider u_j^k for the time level t_k for each k, then the denoted vector is $\mathbf{u}^k \in \mathcal{K}_n$. In Definition 3 we define φ_n, ψ_n as the grid restriction operators. Hence, (3.7)-(3.8) can be written as

$$\frac{\mathbf{u}^{k+1} - \mathbf{u}^k}{\delta} - (1-\theta)D_p^2\mathbf{u}^k - \theta D_p^2\mathbf{u}^{k+1} = 0, \quad k = 0,\dots,K-1, \tag{3.9}$$

$$\mathbf{u}^0 - \varphi_n(u^0) = 0, \tag{3.10}$$

where $\mathbf{u}^0 = (u^0(x_1), \dots, u^0(x_n)) \in \mathcal{K}_n$ and D_p^2 denotes the standard discretization matrix with periodic boundary conditions, i.e.,

$$D_p^2 = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}.$$

We choose in Definition 2: $\mathcal{X}_n = \mathcal{Y}_n = \underbrace{\mathcal{K}_n \times \ldots \times \mathcal{K}_n}_{K+1}$, hence $\mathbf{v}_n := (\mathbf{v}^0, \dots, \mathbf{v}^K) \in \mathcal{X}_n$. We introduce the following norms:

- in $\mathcal{K}_n$: $\left\|\mathbf{v}^k\right\|_{\mathcal{K}_n} = \max\limits_{1\le j\le n} |v^k(x_j)| = \left\|\mathbf{v}^k\right\|_\infty$,
- in $\mathcal{X}_n$: $\|\mathbf{v}_n\|_{\mathcal{X}_n} = \max\limits_{0\le k\le K} \left\|\mathbf{v}^k\right\|_{\mathcal{K}_n}$,
- in $\mathcal{Y}_n$: $\|\mathbf{v}_n\|_{\mathcal{Y}_n} = \left\|\mathbf{v}^0\right\|_{\mathcal{K}_n} + \sum\limits_{j=1}^{K} \delta \left\|\mathbf{v}^j\right\|_{\mathcal{K}_n}$.

Let $\mathbf{v}_n \in \mathcal{X}_n$ any element and we denote by $\eta_n = (\eta^0, \dots, \eta^K) \in \mathcal{Y}_n$ its image. Then the mapping $F_n : \mathcal{X}_n \to \mathcal{Y}_n$ can be written as $F_n(\mathbf{v}_n) = \eta_n$. Particularly, for our discretization (3.9)-(3.10) it yields the following:

$$\frac{\mathbf{v}^{k+1} - \mathbf{v}^k}{\delta} - (1-\theta) D_p^2 \mathbf{v}^k - \theta D_p^2 \mathbf{v}^{k+1} = \eta^{k+1},\ k = 0, \dots, K-1,$$

$$\mathbf{v}^0 = \eta^0.$$

The investigated method can be rewritten in the form

$$\mathbf{v}^{k+1} = Q_1^{-1} Q_2 \mathbf{v}^k + \delta Q_1^{-1} \eta^{k+1}, \tag{3.11}$$

where $Q_1 = I - \theta\delta D_p^2$ and $Q_2 = I + (1-\theta)\delta D_p^2$ are the subtransition matrices (which depend on h and δ). Introducing the notation $r = \delta/h^2$ we can write Q_1 as

$$Q_1 = \begin{pmatrix} 1+2r\theta & -r\theta & 0 & \cdots & 0 & 0 & -r\theta \\ -r\theta & 1+2r\theta & -r\theta & 0 & \cdots & 0 & 0 \\ 0 & -r\theta & 1+2r\theta & -r\theta & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -r\theta & 1+2r\theta & -r\theta & 0 \\ 0 & \cdots & \cdots & 0 & -r\theta & 1+2r\theta & -r\theta \\ -r\theta & 0 & 0 & \cdots & 0 & -r\theta & 1+2r\theta \end{pmatrix}.$$

Definition 10. An $A \in \mathbb{R}^{n\times n}$ matrix is an M-matrix, if

- $a_{ij} \le 0$ for all $i \ne j$,
- there exists a vector $g \in \mathbb{R}^n$, $g > 0$ such that $Ag > 0$.

Theorem 2. *Let $A \in \mathbb{R}^{n\times n}$ be an M-matrix. Then*

$$\left\|A^{-1}\right\|_\infty \le \frac{\|g\|_\infty}{\min\limits_{1\le i\le n}(Ag)_i}. \tag{3.12}$$

Proof. It can be found in [2]. □

Since $r > 0$ and $\theta \in [0,1]$, thus Q_1 is strictly diagonally dominant and $(Q_1)_{ij} \le 0$ for all $i \ne j$. Hence, Q_1 is an M-matrix with $g = (1,\dots,1)^T$. Due to (3.12), we have the estimate

$$\left\|Q_1^{-1}\right\|_\infty \le \frac{\|g\|_\infty}{\min\limits_{1\le i\le n}(Q_1^{-1}g)_i} = \frac{1}{1+2r\delta-r\delta-r\delta} = 1. \tag{3.13}$$

The matrix Q_2 can be written in the form

$$Q_2 = \begin{pmatrix} a & b & 0 & \cdots & 0 & 0 & b \\ b & a & b & 0 & \cdots & 0 & 0 \\ 0 & b & a & b & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b & a & b & 0 \\ 0 & \cdots & \cdots & 0 & b & a & b \\ b & 0 & 0 & \cdots & 0 & b & a \end{pmatrix},$$

where $a = 1-2r(1-\theta)$ and $b = r(1-\theta)$. It is easy to see that under the assumption $r \le \frac{1}{2(1-\theta)}$ we have

$$\|Q_2\|_\infty = 1. \tag{3.14}$$

Introducing the notation $Q = Q_1^{-1}Q_2$, the iteration (3.11) can be written as

$$\mathbf{v}^{k+1} = Q\mathbf{v}^k + \delta Q_1^{-1}\eta^{k+1}.$$

Applying the above recursion and putting $\mathbf{v}^0 = \eta^0$ for any $k = 0,1,\dots,K$, we get

$$\mathbf{v}^k = Q^k\mathbf{v}^0 + \sum_{j=1}^{k}\delta Q^{j-1}Q_1^{-1}\eta^{k+1-j} = Q^k\eta^0 + \delta Q^{k-1}Q_1^{-1}\eta^1 + \ldots + \delta Q_1^{-1}\eta^k.$$

Hence, according to the introduced norms, we obtain the estimation

$$\|\mathbf{v}_n\|_{\mathcal{X}_n} \le \max_{0\le k\le K}\left\{\max_{1\le j\le k}\left\{||Q^k||_\infty, ||Q^{j-1}Q_1^{-1}||_\infty\right\}\right\}\|\eta_n\|_{\mathcal{Y}_n}. \tag{3.15}$$

Using the relations (3.13) and (3.14), we get that

$$||Q||_\infty = ||Q_1^{-1}Q_2||_\infty \le ||Q_1^{-1}||_\infty||Q_2||_\infty \le 1,$$

thus $||Q^k||_\infty \le ||Q||_\infty^k \le 1$ and similarly $||Q^{j-1}Q_1^{-1}||_\infty \le ||Q||_\infty^{j-1}||Q_1^{-1}||_\infty \le 1$. Hence, obviously we have

$$\max_{0\le k\le K}\left\{\max_{1\le j\le k}\left\{||Q^k||_\infty, ||Q^{j-1}Q_1^{-1}||_\infty\right\}\right\} = 1.$$

Since $F_n(\mathbf{v}_n) = \eta_n$, we can rewrite (3.15) as

$$\|\mathbf{v}_n\|_{\mathcal{X}_n} \le \|\eta_n\|_{\mathcal{Y}_n} = \|F_n(\mathbf{v}_n)\|_{\mathcal{Y}_n}. \tag{3.16}$$

For any elements $\mathbf{z}_n, \mathbf{w}_n \in \mathcal{X}_n$ we denote by ϱ_n and ξ_n their image, i.e., $F_n(\mathbf{z}_n) = \varrho_n$ and $F_n(\mathbf{w}_n) = \xi_n$. This results in the relations

$$\frac{\mathbf{z}^{k+1}-\mathbf{z}^k}{\delta} - (1-\theta)D_p^2\mathbf{z}^k - \theta D_p^2\mathbf{z}^{k+1} = \varrho^{k+1},\ k = 0,\dots,K-1, \tag{3.17}$$

$$\mathbf{z}^0 - \varphi_n(u^0) = \varrho^0,$$

$$\frac{\mathbf{w}^{k+1}-\mathbf{w}^k}{\delta} - (1-\theta)D_p^2\mathbf{w}^k - \theta D_p^2\mathbf{w}^{k+1} = \xi^{k+1},\ k = 0,\dots,K-1, \tag{3.18}$$

$$\mathbf{w}^0 - \varphi_n(u^0) = \xi^0.$$

Substracting (3.18) from (3.17), we gain

$$\mathbf{z}^{k+1} - \mathbf{w}^{k+1} = Q(\mathbf{z}^k - \mathbf{w}^k) + \delta Q_1^{-1}(\varrho^{k+1} - \xi^{k+1}),\quad k = 0,\dots,K-1.$$

Using (3.16) by the notation $\mathbf{v}_n = \mathbf{z}_n - \mathbf{w}_n$, we gain

$$\|\mathbf{z}_n - \mathbf{w}_n\|_{\mathcal{X}_n} \le \|\varrho_n - \xi_n\|_{\mathcal{Y}_n} = \|F_n(\mathbf{z}_n) - F_n(\mathbf{w}_n)\|_{\mathcal{Y}_n}.$$

It is easy to see that the above estimation is in the form of (3.1) with $S = 1$. Hence, under the condition $r \le \frac{1}{2(1-\theta)}$ the reaction-diffusion problem is N-stable. Hence, the following statement is true.

Theorem 3. *Under the condition $r \le \frac{1}{2(1-\theta)}$ the θ-method is convergent for the periodic initial-value reaction-diffusion problem* (3.4)-(3.6).

Proof. The θ-method is consistent and N-stable for this problem. Hence, due to Theorem 1, it is convergent, too. □

3.2.2. *Reaction-diffusion problem with forcing term*

Further we consider the following periodic initial-value reaction-diffusion problem:

$$\partial_t u(t,x) = \partial_{xx}u(t,x) + f(u),\quad x \in \mathbb{R},\ t \in [0,T], \tag{3.19}$$

$$u(t,x) = u(t,x+1),\quad x \in \mathbb{R},\ t \in [0,T], \tag{3.20}$$

$$u(0,x) = u^0(x),\quad x \in \mathbb{R}, \tag{3.21}$$

where $T \in \mathbb{R}^+$. In equation (3.19) we assume that $f : \mathbb{R} \to \mathbb{R}$ is a given globally Lipschitz continuous function. The conditions (3.20)-(3.21) are periodic boundary conditions and initial-value conditions, where u^0 is a given one-periodic function. It

is easy to see that the continuous problem (3.19)-(3.21) can be rewritten in the form of (2.1). As we have mentioned, we assume the existence of unique, sufficiently smooth solution of the problem (3.19)-(3.21).

Remark 6. Since the solution is periodic, it is sufficient to determine the solution in one period, only.

Let us take the formerly introduced spatial and time grids and normes. We apply the IMEX-method based on the previous train of thought. For any elements $\mathbf{z}_n, \mathbf{w}_n \in \mathcal{X}_n$ we denote by ϱ_n and ξ_n their image, i.e., $F_n(\mathbf{z}_n) = \varrho_n$ and $F_n(\mathbf{w}_n) = \xi_n$. Then we consider the following two problems:

$$\frac{\mathbf{z}^{k+1} - \mathbf{z}^k}{\delta} - (1-\theta) D_p^2 \mathbf{z}^k - \theta D_p^2 \mathbf{z}^{k+1} - \mathbf{f}(\mathbf{z}^k) = \varrho^{k+1}, k = 0, \ldots, K-1, \quad (3.22)$$

$$\mathbf{z}^0 - \varphi_n(u^0) = \varrho^0,$$

$$\frac{\mathbf{w}^{k+1} - \mathbf{w}^k}{\delta} - (1-\theta) D_p^2 \mathbf{w}^k - \theta D_p^2 \mathbf{w}^{k+1} - \mathbf{f}(\mathbf{w}^k) = \xi^{k+1}, k = 0, \ldots, K-1, \quad (3.23)$$

$$\mathbf{w}^0 - \varphi_n(u^0) = \xi^0,$$

where $\mathbf{f}$ denotes the grid function defined on the grid points x_j, i.e., $[\mathbf{f}(\mathbf{z}^k)]_j = \varphi_n(f(x_j))$ for all $j = 1, \ldots, n$. Substracting (3.23) from (3.22), for $k = 0, \ldots, K-1$, we get

$$\mathbf{z}^{k+1} - \mathbf{w}^{k+1} = Q_1^{-1} Q_2 (\mathbf{z}^k - \mathbf{w}^k) + \delta Q_1^{-1} (\mathbf{f}(\mathbf{z}^k) - \mathbf{f}(\mathbf{w}^k)) + \delta Q_1^{-1} (\varrho^{k+1} - \xi^{k+1}), \quad (3.24)$$

where Q_1 and Q_2 are the earlier defined subtransition matrices. Since f is a given globally Lipschitz continuous function, this implies

$$\|\mathbf{f}(\mathbf{z}^k) - \mathbf{f}(\mathbf{w}^k)\|_{\mathcal{K}_n} \leq L \|\mathbf{z}^k - \mathbf{w}^k\|_{\mathcal{K}_n}. \quad (3.25)$$

Using the Lipschitz property (3.25) and applying the results (3.13) and (3.14) respectively, we get $\left\| Q_1^{-1} Q_2 \right\|_\infty \leq 1$. Then recursion (3.24) shows that for $k = 0, \ldots, K$ we have the estimate

$$\left\| \mathbf{z}^k - \mathbf{w}^k \right\|_{\mathcal{K}_n} \leq (1 + \delta L) \left\| \mathbf{z}^{k-1} - \mathbf{w}^{k-1} \right\|_{\mathcal{K}_n} + \delta \left\| \varrho^k - \xi^k \right\|_{\mathcal{K}_n}. \quad (3.26)$$

Applying recursion (3.26) and $\mathbf{z}^0 - \mathbf{w}^0 = \varrho^0 - \xi^0$, we get for $k = 0, \ldots, K$

$$\left\| \mathbf{z}^k - \mathbf{w}^k \right\|_{\mathcal{K}_n} \leq (1 + \delta L)^k \|\varrho_n - \xi_n\|_{\mathcal{Y}_n}.$$

Using that $\delta K = T$, we get the estimation

$$\|\mathbf{z}_n - \mathbf{w}_n\|_{\mathcal{X}_n} \leq (1 + \delta L)^K \|\varrho_n - \xi_n\|_{\mathcal{Y}_n} \leq e^{LT} \|F_n(\mathbf{z}_n) - F_n(\mathbf{w}_n)\|_{\mathcal{Y}_n}.$$

It is easy to see that the above estimation is in the form of (3.1) with $S = e^{LT}$. Hence, under the assumptions made, the reaction-diffusion problem is N-stable.

Theorem 4. *Under the condition* $r \leq \frac{1}{2(1-\theta)}$*, for the Lipschitzian forcing term* f *the* θ*-method is convergent for the periodic initial-value reaction-diffusion problem* (3.19)-(3.21).

Proof. The θ-method is consistent and N-stable for this problem. Hence, due to Theorem 1, it is convergent, too. □

Remark 7. Due to these results we automatically get

(a) both the explicit finite difference method (for $\theta = 0$) under the condition $r \leq 0.5$ and the IMEX-method (for $\theta = 1$) without any condition are N-stable,
(b) if the given forcing term is in the form of $f(t,u)$ and it is a Lipschitz continuous function with respect to its second variable by the constant L, then we can similarly verify that the θ-method is convergent for the investigated problem.

4. Summary

In this paper we investigated the reaction-diffusion problem under certain conditions in an abstract setting. The base of our framework was the N-stablity notion. As we have shown in Section 3, it is a useful notion regarding both theoretical results and applications. We verified that the θ-method is N-stable for the reaction-diffusion problem both with and without the forcing term.

The main advantage of this notion is that we can offer another tool for verifying the stability property for nonlinear equations. In the near future it is worth dealing with the N-stability of different physical, biological or chemical processes.

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Authors' addresses

I. Fekete
Eötvös Loránd University, Department of Applied Analysis and Computational Mathematics, Pázmány P. s. 1/C, 1117 Budapest, Hungary
E-mail address: feipaat@cs.elte.hu

I. Faragó
MTA-ELTE Numerical Analysis and Large Networks Research Group, Pázmány P. s. 1/C, 1117 Budapest, Hungary
E-mail address: faragois@cs.elte.hu

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THE RIESZ-KOLMOGOROV THEOREM ON METRIC SPACES

PRZEMYSŁAW GÓRKA AND ANNA MACIOS

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Abstract. We study precompact sets in $L^p(X,\mu)$, where (X,μ) is a metric measure space. Using Vitali and maximal function theorems we establish a full characterization of such sets.

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1. INTRODUCTION

Compact sets are important object of analysis' research because of their important properties and huge applications. Therefore, question about full characterization of compact or precompact sets (sets such that their closure is compact) in different spaces is very important. Riesz-Kolmogorov theorem gives necessary and sufficient conditions for precompactness of subset of $L^p(\mathbb{R}^n)$ (see [6, 10]). This theorem plays a fundamental role in analysis. Especially, it is frequently used in the theory of function spaces, e.g. Sobolev and Besov spaces.

The goal of the paper is to give a characterization of precompact sets in $L^p(X,\mu)$, where (X,μ) is a metric space with *doubling measure*, i.e positive Borel measure satisfying condition

$$0 < \mu(B(x,2r)) \leq C\mu(B(x,r)) < \infty$$

for all $x \in X$, $r > 0$ and some constant $C > 0$ called *doubling constant*. We shall denote the average of locally integrable function f over the measurable set A in the following manner

$$(f)_A = \frac{1}{\mu(A)} \int_A f \, d\mu.$$

In this note we would like to present the following observation.

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Theorem 1. *Let μ be a doubling measure such that*

$$h(r) := \inf\{\mu(B(x,r)) : x \in X\} > 0 \quad \text{for each } r > 0$$

and assume that $1 < p < \infty$. Let $x_0 \in X$, then the subset $\mathcal{F}$ of $L^p(X,\mu)$ is relatively compact in $L^p(X,\mu)$ if and only if the following conditions are satisfied:

$$\mathcal{F} \quad \text{is bounded}, \tag{1.1}$$

$$\lim_{R\to\infty} \int_{X\setminus B(x_0,R)} |f(x)|^p d\mu(x) = 0, \quad \text{uniformly for } f \in \mathcal{F}, \tag{1.2}$$

$$\lim_{r\to 0} \int_X |f(x) - (f)_{B(x,r)}|^p d\mu(x) = 0 \quad \text{uniformly for } f \in \mathcal{F}. \tag{1.3}$$

We can find a bit similar theorem in [5], i.e. sufficient condition for precompactness of subset of $L^p(X,\mu)$ for $1 \le p < \infty$, where (X,μ) is a metric space with finite measure. The proof of our theorem is obtained by appropriate modifications in the proof of theorem mentioned above [5]. We apply Vitali convergence theorem. Moreover, Lebesgue differentiation theorem as well as Hardy-Littlewood maximal function theorem will be used.

Finally, let us mention about some generalizations of the Riesz-Kolmogorov theorem. For instance, Weil [11] showed the compactness theorem in $L^p(G)$, where G is a locally compact group. Pego [8] (see also [2]) formulated Kolmogorov theorem for $p = 2$ in terms of the Fourier transform. There also exists a characterization of relatively compact subsets of general Banach spaces [9].

2. Proof of Theorem

First of all we assume that conditions (1.1)-(1.3) hold. Let f_n be a sequence of elements from $\mathcal{F}$. We shall prove that f_n has a subsequence converging in $L^p(X,\mu)$. Since the set $\mathcal{F}$ is bounded and the space $L^p(X,\mu)$ is reflexive, we may assume that f_n converges weakly to some $f \in L^p(X,\mu)$. We can represent the space X in the following manner

$$X = \bigcup_{n=1}^{\infty} B(x_0,n).$$

Now, we shall show that for each n

$$f_k \chi_{B(x_0,n)} \underset{k\to\infty}{\longrightarrow} f\chi_{B(x_0,n)} \quad \text{in} \quad L^p(X,\mu).$$

We apply Vitali convergence theorem (see e.g. [7]). We first prove that the sequence $f_k\chi_{B(x_0,n)}$ is p-equi-integrable. For this purpose we fix $\epsilon > 0$. By (1.3), there exists r such that for each k

$$\int_X |f_k(x) - (f_k)_{B(x,r)}|^p d\mu(x) \le \frac{\varepsilon}{2}.$$

Moreover, we fix a measurable set A such that

$$\mu(A) \leq \left(\frac{\varepsilon}{2 \sup\limits_{k\in\mathbb{N}} ||f_k||_{L^p(X)}} \right)^p h(r).$$

Then, using Minkowski and Hölder inequality we get

$$\begin{aligned}
&\left(\int\limits_A |f_k \chi_{B(x_0,n)}(x)|^p d\mu(x) \right)^{\frac{1}{p}} \leq \left(\int\limits_A |f_k(x)|^p d\mu(x) \right)^{\frac{1}{p}} \\
&\leq \left(\int\limits_A |f_k(x) - (f_k)_{B(x,r)}|^p d\mu(x) \right)^{\frac{1}{p}} + \left(\int\limits_A |(f_k)_{B(x,r)}|^p d\mu(x) \right)^{\frac{1}{p}} \\
&\leq \frac{\varepsilon}{2} + \left(\int\limits_A \frac{1}{(\mu(B(x,r)))^p} \left| \int\limits_{B(x,r)} f_k(y) d\mu(y) \right|^p d\mu(x) \right)^{\frac{1}{p}} \\
&\leq \frac{\varepsilon}{2} + \left(\int\limits_A \frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} |f_k(y)|^p d\mu(y) d\mu(x) \right)^{\frac{1}{p}} \\
&\leq \frac{\varepsilon}{2} + \left(\frac{\mu(A)}{h(r)} \right)^{\frac{1}{p}} \sup_{k\in\mathbb{N}} ||f_k||_{L^p(X)} < \varepsilon.
\end{aligned}$$

This ends the proof of p-equi-integrability.

Our next claim is that $f_k \chi_{B(x_0,n)} \xrightarrow[k\to\infty]{\mu} f\chi_{B(x_0,n)}$ in measure. Let us take $\varepsilon > 0$. We note

$$\begin{aligned}
&\left\{x \in X : |f_k \chi_{B(x_0,n)}(x) - f\chi_{B(x_0,n)}(x)| > \varepsilon\right\} = \{x \in B(x_0,n) : |f_k(x) - f(x)| > \varepsilon\} \\
&\subset \left\{x \in B(x_0,n) : |f_k(x) - (f_k)_{B(x,r)}| > \frac{\varepsilon}{3}\right\} \\
&\quad \cup \left\{x \in B(x_0,n) : |(f_k)_{B(x,r)} - (f)_{B(x,r)}| > \frac{\varepsilon}{3}\right\} \\
&\quad \cup \left\{x \in B(x_0,n) : |f(x) - (f)_{B(x,r)}| > \frac{\varepsilon}{3}\right\} \\
&= A_1 \cup A_2 \cup A_3.
\end{aligned}$$

It suffices to prove that for each $\delta > 0$, there exists K such that for each $k \geq K$, the inequality

$$\mu(A_1) + \mu(A_2) + \mu(A_3) < \delta$$

holds. Using Markov's inequality we get

$$\mu(A_1) \leq \left(\frac{3}{\varepsilon}\right)^p \int\limits_{B(x_0,n)} |f_k(x) - (f_k)_{B(x,r)}|^p d\mu(x)$$
$$\leq \left(\frac{3}{\varepsilon}\right)^p \int\limits_{X} |f_k(x) - (f_k)_{B(x,r)}|^p d\mu(x).$$

Therefore, by assumption (1.3), there exists r_0 such that for each $r \leq r_0$ we get

$$\mu(A_1) < \frac{\delta}{3}.$$

Next, we consider the set A_2. We conclude from $f_k \xrightarrow[k\to\infty]{w} f$ that for each $x \in B(x_0,n)$ and $r > 0$

$$(f_k)_{B(x,r)} \to (f)_{B(x,r)}.$$

This gives $(f_k)_{B(x,r)} \xrightarrow{\mu} (f)_{B(x,r)}$ in measure on $B(x_0,n)$. By the definition, there exists $K(r)$ such that for each $k \geq K(r)$ the inequality $\mu(A_2) < \frac{\delta}{3}$ holds.

It remains to consider the set A_3. Using Markov's inequality we have

$$\mu(A_3) \leq D_r := \left(\frac{3}{\varepsilon}\right)^p \int\limits_{B(x_0,n)} |f(x) - (f)_{B(x,r)}|^p d\mu(x).$$

Thanks to Lebesgue differentiation theorem (see [4]) we get that $\lim\limits_{r\to 0} f_{B(x,r)} = f(x)$ a.e. on X. Direct calculations leads us to the following estimates

$$|f(x) - f_{B(x,r)}|^p \leq \left(|f(x)| + |(f)_{B(x,r)|}\right)^p \leq 2^{p-1}\left(|f(x)|^p + |(f)_{B(x,r)}|^p\right)$$
$$\leq 2^{p-1}\left(|f(x)|^p + |\mathcal{M}(f)(x)|^p\right),$$

where $\mathcal{M}(f)$ is a maximal function of f. By virtue of the Hardy-Littlewood maximal function theorem (see [4]) we have $\|\mathcal{M}(f)\|_{L^p(X,\mu)} \leq C(p)\|f\|_{L^p(X,\mu)}$, where $p > 1$. Therefore, since $|f|^p$ is integrable, the Lebesgue dominated convergence theorem gives

$$\lim_{r\to 0} D_r = \left(\frac{3}{\varepsilon}\right)^p \int\limits_{B(x_0,n)} \lim_{r\to 0} |f(x) - (f)_{B(x,r)}|^p d\mu(x) = 0$$

and hence there exists r_1, such that for $r \leq r_1$ we have

$$\left(\frac{3}{\varepsilon}\right)^p \int\limits_{B(x_0,n)} |f(x) - (f)_{B(x,r)}|^p d\mu(x) < \frac{\delta}{3}.$$

Finally, taking $\tilde{r} = \min\{r_0, r_1\}$, we obtain the inequality

$$\mu(\{x \in X : |f_k \chi_{B(x_0,n)}(x) - f\chi_{B(x_0,n)}(x)| > \varepsilon\}) < \delta$$

for each $k \geq K(\tilde{r})$. Now, we show that

$$f_k \underset{k\to\infty}{\longrightarrow} f \quad \text{in} \quad L^p(X,\mu).$$

Let us fix $\varepsilon > 0$. Since $f \in L^p(X,\mu)$ and by (1.2), there exists N such that

$$\int_{X\setminus B(x_0,N)} |f_k(x) - f(x)|^p d\mu(x) \leq \frac{\varepsilon}{2}.$$

Hence,

$$\begin{aligned}\|f_k - f\|^p_{L^p(X,\mu)} &= \int_X |f_k \chi_{B(x_0,N)}(x) - f\chi_{B(x_0,N)}(x)|^p d\mu(x) \\ &\quad + \int_{X\setminus B(x_0,N)} |f_k(x) - f(x)|^p d\mu(x) \leq \varepsilon.\end{aligned}$$

This ends the proof of relatively compactness of $\mathcal{F}$.

Now, we shall show the converse. We assume that the family $\mathcal{F}$ is relatively compact. Hence, the boundedness is straightforward. To establish condition (1.2), let us fix $\varepsilon > 0$ and let $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ be an ε-cover of $\mathcal{F}$. For each $k = 1, \ldots, n$ we can select $g_k \in U_k$ and $R_k > 0$ such that

$$\int_{X\setminus B(x_0,R_k)} |g_k(x)|^p d\mu(x) < \varepsilon.$$

Let us fix $f \in \mathcal{F}$. Since $\mathcal{U}$ is an ε-cover of $\mathcal{F}$, there exists $1 \leq k \leq n$ such that $f \in U_k$. Thus,

$$\int_X |f(x) - g_k(x)|^p d\mu(x) < \varepsilon$$

and for $R = \max\{R_i : 1 \leq i \leq n\}$ we get

$$\begin{aligned}\int_{X\setminus B(x_0,R)} |f(x)|^p d\mu(x) &\leq 2^{p-1} \int_{X\setminus B(x_0,R)} |f(x) - g_k(x)|^p d\mu(x) \\ &\quad + 2^{p-1} \int_{X\setminus B(x_0,R)} |g_k(x)|^p d\mu(x) < 2^p \varepsilon.\end{aligned}$$

So condition (1.2) holds.

Using g_k choosen as above, we obtain:

$$\begin{aligned}\int_X |f(x) - (f)_{B(x,r)}|^p d\mu(x) &\leq 2^{p-1} \int_X |f(x) - g_k(x)|^p d\mu(x) \\ &\quad + 2^{2(p-1)} \int_X |g_k(x) - (g_k)_{B(x,r)}|^p d\mu(x) \\ &\quad + 2^{2(p-1)} \int_X |(g_k)_{B(x,r)} - (f)_{B(x,r)}|^p d\mu(x) \\ &= 2^{p-1} I_1 + 2^{2(p-1)} I_2 + 2^{2(p-1)} I_3.\end{aligned}$$

We easily get $I_1 < \varepsilon$. Now, using the Hardy-Littlewood maximal function theorem we obtain

$$I_3 = \int_X \left| \frac{1}{\mu(B(x,r))} \int_{B(x,r)} (g_k(y) - f(y)) d\mu(y) \right|^p d\mu(x)$$
$$\leq \int_X |\mathcal{M}(g_k - f)(x)|^p d\mu(x) \leq C(p)^p \int_X |g_k(x) - f(x)|^p d\mu(x) \leq C(p)^p \varepsilon.$$

Now it remains to prove that I_2 is arbitrary small. By Lebesgue differentiation theorem we obtain that

$$\lim_{r \to 0} \left| g_k(x) - (g_k)_{B(x,r)} \right| = 0, \quad \text{a.e. on } X$$

Moreover,

$$|g_k(x) - (g_k)_{B(x,r)}|^p \leq 2^{p-1} (|g_k(x)|^p + |\mathcal{M}(g_k)(x)|^p).$$

Since $|g_k|^p$ and $|\mathcal{M}(g_k)|^p$ are integrable, we can apply Lebesgue dominated convergence theorem. Thus,

$$I_2 = \int_X |g_k(x) - (g_k)_{B(x,r)}|^p d\mu(x) < \varepsilon$$

and the proof is complete.

Let us remark that if μ is doubling and $\operatorname{diam} X < \infty$ or μ is continuous with respect to the metric ρ (see [1,3]) and X is compact, then for each $r > 0$, $h(r)$ is positive.

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Authors' addresses

Przemysław Górka
Warsaw University of Technology, Department of Mathematics and Information Sciences, Ul. Koszykowa 75, 00-662 Warsaw, Poland
E-mail address: P.Gorka@mini.pw.edu.pl

Anna Macios
Warsaw University of Technology, Department of Mathematics and Information Sciences, Ul. Koszykowa 75, 00-662 Warsaw, Poland

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ON MANNHEIM PARTNER CURVES IN THREE DIMENSIONAL LIE GROUPS

İSMAIL GÖK, O. ZEKI OKUYUCU, NEJAT EKMEKCI, AND YUSUF YAYLI

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Abstract. In this paper, we define Mannheim partner curves in a three dimensional Lie group G with a bi-invariant metric. The main result of the paper is given as (Theorem 4): A curve $\alpha : I \subset \mathbb{R} \to G$ with the Frenet apparatus $\{T, N, B, \kappa, \tau\}$ is a Mannheim partner curve if and only if

$$\lambda\kappa\left(1 + H^2\right) = 1$$

where λ, μ are constants and H is the harmonic curvature function of the curve α.

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1. INTRODUCTION

In the classical diferential geometry of curves, J. Bertrand studied curves in Euclidean 3-space whose principal normals are the principal normals of another curve. In [1] he showed that a necessary and sufficient condition for the existence of such a second curve is that a linear relationship with constant coefficients shall exist between the first and second curvatures of the given original curve. In other word, if we denote first and second curvatures of a given curve by k_1 and k_2 respectively, then for $\lambda, \mu \in \mathbb{R}$ we have $\lambda k_1 + \mu k_2 = 1$. Since the time of Bertrand's paper, pairs of curves of this kind have been called *Conjugate Bertrand Curves*, or more commonly *Bertrand Curves* (see [10]).

Another kind of associated curve whose principal normal vector field is the binormal vector field of another curve is called Mannheim curve. Mannheim partner curves was studied by Liu and Wang (see [12]) in Euclidean 3− space and in the Minkowski 3−space. After these papers lots of papers have been published about Mannheim curves in Euclidean 3−space, Minkowski 3−space, dual 3−space and Galilean spaces (see [7, 9, 15, 17, 18]). Matsuda and Yorozu [13] gave a definition of generalized Mannheim curve in Euclidean 4−space. They show some characterizations and examples of generalized Mannheim curves. Ersoy *et.al.* gave a definition of generalized Mannheim curve in Minkowski 4−space.

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The degenarete semi-Riemannian geometry of Lie group is studied by Çöken and Çiftçi [4]. Moreover, they obtanied a naturally reductive homogeneous semi-Riemannian space using the Lie group. Then Çiftçi [3] defined general helices in three dimensional Lie groups with a bi-invariant metric and obtained a generalization of Lancret's theorem and gave a relation between the geodesics of the so-called cylinders and general helices.

Recently, *Izumiya and Takeuchi* [8] have introduced the concept of slant helix in Euclidean 3-space. A slant helix in Euclidean space $\mathbb{E}^3$ was defined by the property that its principal normal vector field makes a constant angle with a fixed direction. Also, Izumiya and Takeuchi showed that α is a slant helix if and only if the geodesic curvature of spherical image of principal normal indicatrix (N) of a space curve α

$$\sigma_N(s) = \left(\frac{\kappa^2}{\left(\kappa^2+\tau^2\right)^{3/2}} \left(\frac{\tau}{\kappa}\right)' \right)(s)$$

is a constant function.

Harmonic curvature functions have been defined by Özdamar and Hacısalihoğlu [16]. Recently, many studies have been reported on generalized helices and slant helices using the harmonic curvatures in Euclidean spaces and Minkowski spaces [2, 6, 11]. Then, Okuyucu et al. [14] have defined slant helices in three dimensional Lie groups with a bi-invariant metric and obtained some characterizations using their harmonic curvature functions.

In this paper, first of all, we define Mannheim partner curves in a three dimensional Lie group G with a bi-invariant metric and we obtain the necessary and sufficient conditions for the Mannheim partner curves in a three dimensional Lie group G.

2. Preliminaries

Let G be a Lie group with a bi-invariant metric $\langle\,,\rangle$ and D be the Levi-Civita connection of Lie group G. If $\mathfrak{g}$ denotes the Lie algebra of G then we know that $\mathfrak{g}$ is issomorphic to T_eG where e is neutral element of G. If $\langle\,,\rangle$ is a bi-invariant metric on G then we have

$$\langle X,[Y,Z]\rangle = \langle [X,Y],Z\rangle \tag{2.1}$$

and

$$D_X Y = \frac{1}{2}[X,Y] \tag{2.2}$$

for all X, Y and $Z \in \mathfrak{g}$.

Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lenghted regular curve and $\{X_1, X_2, ..., X_n\}$ be an orthonormal basis of $\mathfrak{g}$. In this case, we write that any two vector fields W and Z along the curve α as $W = \sum_{i=1}^{n} w_i X_i$ and $Z = \sum_{i=1}^{n} z_i X_i$ where $w_i : I \to \mathbb{R}$ and $z_i : I \to \mathbb{R}$ are smooth functions. Also the Lie bracket of two vector fields W and Z

is given by

$$[W,Z]=\sum_{i=1}^{n} w_i z_i \left[X_i, X_j\right]$$

and the covariant derivative of W along the curve α with the notation $D_{\alpha'}W$ is given as follows

$$D_{\alpha'}W=\dot{W}+\frac{1}{2}[T,W] \tag{2.3}$$

where $T=\alpha'$ and $\dot{W}=\sum_{i=1}^{n}\dot{w}_i X_i$ or $\dot{W}=\sum_{i=1}^{n}\frac{dw}{dt}X_i$. Note that if W is the left-invariant vector field along to the curve α then $\dot{W}=0$ (see for details [5]).

Let G be a three dimensional Lie group and (T,N,B,κ,τ) denote the Frenet apparatus of the curve α. Then the Serret-Frenet formulas of the curve α satisfies:

$$D_T T=\kappa N,\quad D_T N=-\kappa T+\tau B,\quad D_T B=-\tau N$$

where D is Levi-Civita connection of Lie group G and $\kappa=\|\dot{T}\|$.

Definition 1. Let $\alpha: I\subset\mathbb{R}\to G$ be a parametrized curve. Then α is called a *general helix* if it makes a constant angle with a left-invariant vector field X. That is,

$$\langle T(s),X\rangle=\cos\theta \text{ for all } s\in I,$$

for the left-invariant vector field $X\in g$ is unit length and θ is a constant angle between X and T, which is the tangent vector field of the curve α (see [3]).

Definition 2. Let $\alpha: I\subset\mathbb{R}\to G$ be a parametrized curve with the Frenet apparatus (T,N,B,κ,τ) then

$$\tau_G=\frac{1}{2}\langle[T,N],B\rangle \tag{2.4}$$

or

$$\tau_G=\frac{1}{2\kappa^2\tau}\langle\ddot{T},[T,\dot{T}]\rangle+\frac{1}{4\kappa^2\tau}\|[T,\dot{T}]\|^2$$

(see [3]).

Proposition 1. *Let $\alpha: I\subset\mathbb{R}\to G$ be an arc length parametrized curve with the Frenet apparatus $\{T,N,B\}$. Then the following equalities*

$$[T,N]=\langle[T,N],B\rangle B=2\tau_G B$$
$$[T,B]=\langle[T,B],N\rangle N=-2\tau_G N$$

hold [14].

Definition 3. Let $\alpha: I\subset\mathbb{R}\to G$ be an arc length parametrized curve. Then α is called a *slant helix* if its principal normal vector field makes a constant angle with a left-invariant vector field X which is unit length. That is,

$$\langle N(s),X\rangle=\cos\theta \text{ for all } s\in I,$$

where $\theta \neq \frac{\pi}{2}$ is a constant angle between X and N which is the principal normal vector field of the curve α (see [14]).

Definition 4. Let $\alpha : I \subset \mathbb{R} \to G$ be an arc length parametrized curve with the Frenet apparatus $\{T, N, B, \kappa, \tau\}$. Then the *harmonic curvature function* of the curve α is defined by

$$H = \frac{\tau - \tau_G}{\kappa}$$

where $\tau_G = \frac{1}{2} \langle [T, N], B \rangle$ (see [14]).

Theorem 1. *Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized curve with the Frenet apparatus (T, N, B, κ, τ). The curve α is a general helix, if and only if,*

$$\tau = c\kappa + \tau_G$$

where c is a constant (see [3]) or using the definition of the harmonic curvature function of the curve α (see [14]) H is a constant function.

Theorem 2. *Let $\alpha : I \subset \mathbb{R} \to G$ be a unit speed curve with the Frenet apparatus (T, N, B, κ, τ). Then α is a slant helix if and only if*

$$\sigma_N = \frac{\kappa(1 + H^2)^{\frac{3}{2}}}{H'} = \tan\theta$$

is a constant where H is a harmonic curvature function of the curve α and $\theta \neq \frac{\pi}{2}$ is a constant (see [14]).

3. Mannheim partner curves in a three dimensional Lie groups

In this section, we define Mannheim partner curves and their characterizations are given in a three dimensional Lie group G with a bi-invariant metric $\langle , \rangle$. Also we give some characterizations of Mannheim partner curves using the special cases of G.

Definition 5. A curve α in 3-dimensional Lie group G is a *Mannheim curve* if there exists a special curve β in 3-dimensional Lie group G such that principal normal vector field of α is linearly dependent binormal vector field of β at corresponding point under ψ which is bijection from α to β. In this case β is called the *Mannheim partner curve* of α and (α, β) is called *Mannheim curve couple.*

The curve $\alpha : I \subset \mathbb{R} \to G$ in 3-dimensional Lie group G is parametrized by the arc-length parameter s and from the Definition 5 Mannheim mate curve of α is given $\beta : \overline{I} \subset \mathbb{R} \to G$ in 3-dimensional Lie group G with the help of Figure 1 such that

$$\beta(s) = \alpha(s) + \lambda(s) N(s), \; s \in I$$

where λ is a smooth function on I and N is the principal normal vector field of α.

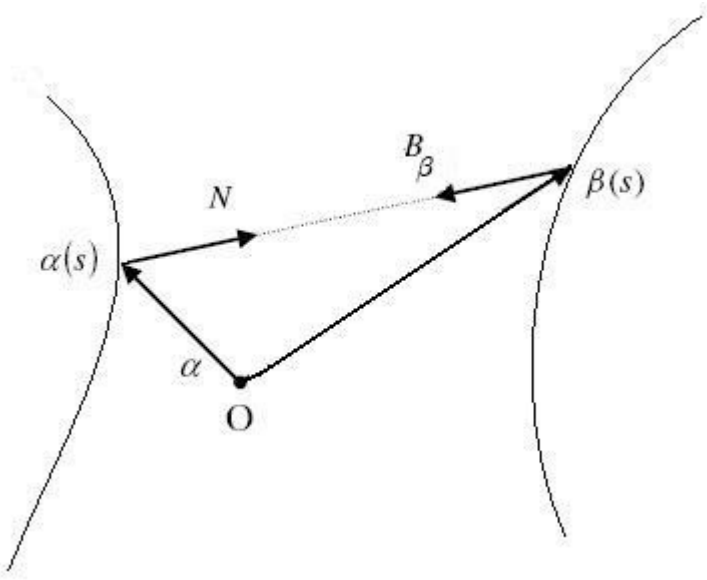

FIGURE 1. Mannheim Partner Curves

Theorem 3. *Let $\alpha : I \subset \mathbb{R} \to G$ and $\beta : \overline{I} \subset \mathbb{R} \to G$ be a Mannheim curve couple with arc-length parameter s and $\overline{s}$, respectively. Then corresponding points are a fixed distance apart for all $s \in I$, that is,*

$$d(\alpha(s), \beta(s)) = \text{constant}, \quad \text{for all } s \in I. \tag{3.1}$$

Proof. From Definition 5, we can simply write

$$\beta(s) = \alpha(s) + \lambda(s) N(s) \tag{3.2}$$

Differentiating Eq. (3.2) with respect to s and using Eq. (2.3), we get

$$\begin{aligned} \frac{d\beta(\overline{s})}{d\overline{s}} \frac{d\overline{s}}{ds} &= \frac{d\alpha(s)}{ds} + \lambda'(s) N(s) + \lambda(s) \dot{N}(s) \\ &= T(s) + \lambda'(s) N(s) + \lambda(s) \left[D_T N - \frac{1}{2} [T, N] \right] \end{aligned}$$

and with the help of Proposition 1 and Frenet equations, we obtain

$$\frac{d\beta(\overline{s})}{d\overline{s}} \frac{d\overline{s}}{ds} = (1 - \lambda(s)\kappa(s)) T(s) + \lambda'(s) N(s) + \lambda(s) ((\tau - \tau_G)(s)) B(s)$$

hence

$$T_\beta(\overline{s}) = \frac{ds}{d\overline{s}} \left[(1 - \lambda(s)\kappa(s)) T(s) + \lambda'(s) N(s) + \lambda(s) (\tau - \tau_G)(s) B(s) \right]. \tag{3.3}$$

And then, we know that $\{N(s), B_\beta(\overline{s})\}$ is a linearly dependent set, so we have

$$\langle T_\beta(\overline{s}), B_\beta(\overline{s})\rangle = \frac{ds}{d\overline{s}}\left[\begin{array}{c}(1-\lambda(s)\kappa(s))\langle T(s), B_\beta(\overline{s})\rangle + \lambda'(s)\langle N(s), B_\beta(\overline{s})\rangle \\ +\lambda(s)(\tau-\tau_G)(s)\langle B(s), B_\beta((\overline{s}))\rangle\end{array}\right]$$

hence

$$\lambda'(s) = 0$$

that is, $\lambda(s)$ is constant function on I. This completes the proof. □

Theorem 4. *Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized curve with arc length parameter s and the Frenet apparatus (T, N, B, κ, τ). Then, α is Mannheim curve if and only if*

$$\lambda\kappa\left(1+H^2\right) = 1, \text{ for all } s \in I \tag{3.4}$$

where λ is constant and H is the harmonic curvature function of the curve α.

Proof. Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized Mannheim curve with arc length parameter s then we can write

$$\beta(s) = \alpha(s) + \lambda N(s)$$

Differentiating the above equality with respect to s and by using the Frenet equations, we get

$$\frac{d\beta(s)}{ds} = (1-\lambda\kappa(s))T(s) + \lambda(\tau-\tau_G)(s)B(s).$$

On the other hand, we have

$$T_\beta(\overline{s}) = \frac{d\beta}{ds}\frac{ds}{d\overline{s}} = [(1-\lambda\kappa(s))T(s) + \lambda(\tau-\tau_G)(s)B(s)]\frac{ds}{d\overline{s}}.$$

By taking the derivative of this equation with respect to $\overline{s}$ and using the Frenet equations we obtain

$$\begin{aligned}\frac{dT_\beta}{d\overline{s}} &= \left[-\lambda\frac{d\kappa}{ds}T(s) + \left(\kappa-\lambda\kappa^2-\lambda(\tau-\tau_G)^2\right)N(s) + \lambda(\tau-\tau_G)'B(s)\right]\left(\frac{ds}{d\overline{s}}\right)^2 \\ &\quad + [(1-\lambda\kappa(s))T(s) + \lambda(\tau-\tau_G)(s)B(s)]\frac{d^2s}{d\overline{s}^2}\end{aligned}$$

From this equation we get

$$\left(\kappa-\lambda\kappa^2-\lambda(\tau-\tau_G)^2\right) = 0,$$

$$\lambda\kappa\left(1+H^2\right) = 1.$$

Conversely, if $\lambda\kappa\left(1+H^2\right) = 1$ then we can easily see that α is a Mannheim curve.

This completes the proof. □

Corollary 1. *If G is Abelian Lie group then $\tau_G = 0$. And so, if $\alpha : I \subset \mathbb{R} \to G$ is a parametrized Mannheim curve with arc length parameter s and the Frenet apparatus (T, N, B, κ, τ) in Abelian Lie group G. Then, α satisfy the following equality*

$$\lambda\left(\kappa^2 + \tau^2\right) = \kappa$$

Proof. If G is Abelian Lie group then using the $\tau_G = 0$ and Theorem 4 we have the result. □

So, the above Corollary shows that the study is a generalization of Mannheim curves defined by Liu and Wang [12] in Euclidean 3-space.

Theorem 5. *Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized curve with arc length parameter s. Then β is the Mannheim partner curve of α if and only if the curvature κ_β and the torsion τ_β of β satisfy the following equation*

$$\frac{d\kappa_\beta H_\beta}{d\bar{s}} = \frac{\kappa_\beta}{\mu}(1 + \mu^2 \kappa_\beta^2 H_\beta^2)$$

where μ is constant and H_β is the harmonic curvature function of the curve β.

Proof. Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized Mannheim curve with arc length parameter s then we can write

$$\alpha(\bar{s}) = \beta(\bar{s}) + \mu(\bar{s}) B_\beta(\bar{s})$$

for some function $\mu(\bar{s})$. By taking the derivative of this equation with respect to $\bar{s}$ and using the Frenet equations we obtain

$$T\frac{ds}{d\bar{s}} = T_\beta(\bar{s}) + \mu'(\bar{s}) B_\beta(\bar{s}) - \mu(\bar{s})\left(\tau_\beta - \tau_{G_\beta}\right)(\bar{s}) N_\beta(\bar{s})$$

hence

$$T\frac{ds}{d\bar{s}} = T_\beta(\bar{s}) + \frac{d\mu(\bar{s})}{d\bar{s}} B_\beta(\bar{s}) - \mu(\bar{s}) \kappa_\beta H_\beta N_\beta(\bar{s})$$

where H_β is the harmonic curvature function of the curve β. And then, we know that $\{N(s), B_\beta(\bar{s})\}$ is a linearly dependent set, so we have

$$\frac{d\mu(\bar{s})}{d\bar{s}} = 0.$$

This means that $\mu(\bar{s})$ is a constant function. Thus we have

$$T\frac{ds}{d\bar{s}} = T_\beta - \mu(\bar{s}) \kappa_\beta H_\beta N_\beta(\bar{s}). \tag{3.5}$$

On the other hand, we have

$$T = T_\beta \cos\theta + N_\beta \sin\theta \tag{3.6}$$

where θ is the angle between T and T_β at the corresponding points of the curves α and β. By taking the derivative of this equation with respect to $\bar{s}$ and using the Frenet equations we obtain

$$\kappa N\frac{ds}{d\bar{s}}=-\left(\kappa_\beta+\frac{d\theta}{d\bar{s}}\right)\sin\theta T_\beta+\left(\kappa_\beta+\frac{d\theta}{d\bar{s}}\right)\cos\theta N_\beta+\kappa_\beta H_\beta\sin\theta B_\beta.$$

From this equation and since $\{N(s), B_\beta(\bar{s})\}$ is a linearly dependent set, we get

$$\begin{cases}\left(\kappa_\beta+\frac{d\theta}{d\bar{s}}\right)\sin\theta=0\\ \left(\kappa_\beta+\frac{d\theta}{d\bar{s}}\right)\cos\theta=0.\end{cases}$$

For this reason we have

$$\frac{d\theta}{d\bar{s}}=-\kappa_\beta. \tag{3.7}$$

From Eq. (3.5), Eq. (3.6) and notice that T_β is orthogonal to B_β, we find that

$$\frac{ds}{d\bar{s}}=\frac{1}{\cos\theta}=-\frac{\mu\kappa_\beta H_\beta}{\sin\theta}.$$

Then we have

$$\mu\kappa_\beta H_\beta=-\tan\theta.$$

By taking the derivative of this equation and applying Eq. (3.7), we get

$$\mu\frac{d\kappa_\beta H_\beta}{d\bar{s}}=\kappa_\beta\left(1+\mu^2\kappa_\beta^2H_\beta^2\right)$$

that is

$$\frac{d\kappa_\beta H_\beta}{d\bar{s}}=\frac{\kappa_\beta}{\mu}\left(1+\mu^2\kappa_\beta^2H_\beta^2\right).$$

Conversely, if the curvature κ_β and torsion τ_β of the curve β in three dimensional Lie group G satisfy

$$\frac{d\kappa_\beta H_\beta}{d\bar{s}}=\frac{\kappa_\beta}{\mu}\left(1+\mu^2\kappa_\beta^2H_\beta^2\right)$$

for constant $\mu(\bar{s})$, then we define a curve by

$$\alpha(\bar{s})=\beta(\bar{s})+\mu B_\beta(\bar{s}) \tag{3.8}$$

and we will show that $\{\alpha,\beta\}$ is Mannheim curve couple in three dimensional Lie group G. By taking the derivative of Eq. (3.8) with respect to $\bar{s}$ twice, we get

$$T\frac{ds}{d\bar{s}}=T_\beta-\mu\kappa_\beta H_\beta N_\beta, \tag{3.9}$$

$$\kappa N\left(\frac{ds}{d\bar{s}}\right)^2+T\frac{d^2s}{d\bar{s}^2}=\mu\kappa_\beta^2H_\beta T_\beta+\left(\kappa_\beta-\mu\frac{d\kappa_\beta H_\beta}{d\bar{s}}\right)N_\beta-\mu\kappa_\beta^2H_\beta^2B_\beta, \tag{3.10}$$

respectively. Taking the cross product of Eq. (3.9) with Eq. (3.10) and noticing that

$$\kappa_\beta - \mu \frac{d\kappa_\beta H_\beta}{d\bar{s}} + \mu^2 \kappa_\beta^3 H_\beta^2$$

we have

$$\kappa B \left(\frac{ds}{d\bar{s}}\right)^3 = \mu^2 \kappa_\beta^3 H_\beta^3 T_\beta + \mu \kappa_\beta^2 H_\beta^2 N_\beta. \tag{3.11}$$

By taking the cross product of Eq. (3.9) with Eq. (3.11), we get

$$\kappa N \left(\frac{ds}{d\bar{s}}\right)^4 = -\mu \kappa_\beta^2 H_\beta^2 \left(1 + \mu^2 \kappa_\beta^2 H_\beta^2\right) B_\beta.$$

This means that the principal normal vector field of the curve α and binormal vector field of the curve β are linearly dependent set. And so α is a Mannheim curve and β is Mannheim partner curve of the curve α in three dimensional Lie group G. □

Proposition 2. *Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lenghted Mannheim curve with the Frenet vector fields $\{T, N, B\}$ and $\beta : \bar{I} \subset \mathbb{R} \to G$ be a Mannheim mate of α with the Frenet vector fields $\{T_\beta, N_\beta, B_\beta\}$. Then $\tau_G = \tau_{G_\beta}$ for the curves α and β where $\tau_G = \frac{1}{2}\langle [T, N], B\rangle$ and $\tau_{G_\beta} = \frac{1}{2}\langle [T_\beta, N_\beta], B_\beta\rangle$.*

Proof. Let (α, β) be a Mannheim curve couple. From Eq. (3.3) we have

$$T_\beta(\bar{s}) = [(1 - \lambda\kappa(s))\, T(s) + \lambda(\tau - \tau_G)(s) B(s)] \frac{ds}{d\bar{s}}.$$

We take the norm of this equation and by using Eq. (3.4), we obtain

$$\frac{d\bar{s}}{ds} = \lambda \kappa H \sqrt{1 + H^2}.$$

If we consider with together the last two equations, we get

$$T_\beta(\bar{s}) = \frac{H}{\sqrt{1+H^2}} T(s) + \frac{1}{\sqrt{1+H^2}} B(s). \tag{3.12}$$

Since (α, β) is a Mannheim curve couple we know $B_\beta(\bar{s}) = N(s)$. Then,

$$N_\beta(\bar{s}) = B_\beta(\bar{s}) \times T_\beta(\bar{s})$$

$$N_\beta(\bar{s}) = \frac{1}{\sqrt{1+H^2}} T(s) - \frac{H}{\sqrt{1+H^2}} B(s).$$

We know from Definition 2 $\langle [T_\beta, N_\beta], B_\beta\rangle = 2\tau_{G_\beta}$ for the curve β. Then with the help of above equations for $T_\beta(\bar{s})$, $N_\beta(\bar{s})$ and $B_\beta(\bar{s})$, we obtain

$$\left\langle \left[\frac{H}{\sqrt{1+H^2}} T + \frac{1}{\sqrt{1+H^2}} B, \frac{1}{\sqrt{1+H^2}} T(s) - \frac{H}{\sqrt{1+H^2}} B(s) \right], N \right\rangle = 2\tau_{G_\beta},$$

$$\frac{H^2}{1+H^2} \langle [T, N], B\rangle + \frac{1}{1+H^2} \langle [T, N], B\rangle = 2\tau_{G_\beta}.$$

Using the last equation of Proposition 1, we get

$$\tau_G = \tau_{G_\beta}.$$

This completes the proof. □

Theorem 6. *Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized Mannheim curve with curvature functions κ, τ and $\beta : \overline{I} \subset \mathbb{R} \to G$ be a Mannheim mate of α with curvatures functions κ_β, τ_β. Then the relations between these curvature functions are*

$$\kappa_\beta(\overline{s}) = \frac{H'(s)}{\lambda\kappa(s)H(s)\left(1+H^2(s)\right)^{3/2}},$$
$$\tau_\beta(\overline{s}) = \frac{1}{\lambda H(s)} + \tau_{G_\beta}.$$

Proof. If we differentiate Eq. (3.12) and using the Frenet formulas, we have

$$\kappa_\beta N_\beta \lambda\kappa H\sqrt{1+H^2} = \frac{H'}{\left(1+H^2\right)^{3/2}}(T - HB).$$

If we take the norm of the last equation, we get

$$\kappa_\beta = \frac{H'}{\lambda\kappa H\left(1+H^2\right)^{3/2}}.$$

Since $\{\alpha, \beta\}$ is a Mannheim curve couple, we know $B_\beta = N$. If we differentiate this equation and using the Frenet formulas, we have

$$-\left(\tau_\beta - \tau_{G_\beta}\right) N_\beta \lambda H\sqrt{1+H^2} = -T + HB.$$

If we take the norm of the last equation, we get

$$\tau_\beta = \frac{1}{\lambda H} + \tau_{G_\beta}.$$

This completes the proof. □

Theorem 7. *Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lenghted Mannheim curve and $\beta : \overline{I} \subset \mathbb{R} \to G$ be a Mannheim mate of α. The Mannheim curve α is a slant helix if and only if its Mannheim mate β is a general helix.*

Proof. If Mannheim curve α is a slant helix, then using Theorem 2 we have, σ_N is a constant function. From Theorem 6 for the curve β, we have

$$\frac{\tau_\beta - \tau_{G_\beta}}{\kappa_\beta} = \frac{\frac{1}{\lambda H}}{\frac{H'}{\lambda\kappa H\left(1+H^2\right)^{3/2}}}$$
$$\frac{\tau_\beta - \tau_{G_\beta}}{\kappa_\beta} = \frac{\kappa\left(1+H^2\right)^{3/2}}{H'}$$

$$\frac{\tau_\beta - \tau_{G_\beta}}{\kappa_\beta} = \sigma_N = \text{constant}.$$

So, the curve β which is Mannheim mate of α is a general helix.

Conversely, we assume that β, which is Mannheim mate of α, is a general helix. So we have

$$\frac{\tau_\beta - \tau_{G_\beta}}{\kappa_\beta} = \text{constant}.$$

From the last equation and Theorem 6 we can easily see that σ_N is a constant function. This completes the proof. □

Theorem 8. *Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lenghted Mannheim curve and $\beta : \overline{I} \subset \mathbb{R} \to G$ be a Mannheim mate of α. If α is a slant helix then the harmonic curvature function of the curve α: $H(s)$ is*

$$H(s) = \frac{1}{2}\left(ae^{bs} - \frac{1}{a}e^{-bs}\right) \tag{3.13}$$

for some nonzero constant a and b and s is the arc length parameter of the curve α. If we consider $a = b = 1$, we have the harmonic curvature funtion of the curve α is hyperbolic sine function in arc length s, that is, $H(s) = \sinh s$.

Proof. Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lenghted Mannheim curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ in three dimensional Lie group G. Since the curve α is a slant helix, we have

$$\langle N, X\rangle = \cos\theta,\ \theta \neq \frac{\pi}{2} \tag{3.14}$$

for left invariant vector field X. Differentiating Eq. (3.14) twice, we have

$$-\kappa\langle T, X\rangle + (\tau - \tau_G)\langle B, X\rangle = 0 \tag{3.15}$$

and

$$-\kappa'\langle T, X\rangle + (\tau - \tau_G)'\langle B, X\rangle = \left\{\kappa^2 + (\tau - \tau_G)^2\right\}\langle N, X\rangle.$$

Since α is a Mannheim curve using Theorem 4, we rewrite the last equation

$$-\kappa'\langle T, X\rangle + (\tau - \tau_G)'\langle B, X\rangle = \frac{\kappa}{\lambda}\cos\theta \tag{3.16}$$

where λ is a non-zero constant. By a direct calculation using Eq. (3.15) and Eq. (3.16), we obtain

$$\langle T, X\rangle = \frac{H}{\lambda H'}\cos\theta \tag{3.17}$$

and

$$\langle B, X\rangle = \frac{1}{\lambda H'}\cos\theta. \tag{3.18}$$

Differentiating Eq. (3.17) and Eq. (3.18), we have

$$\kappa = \frac{1}{\lambda}\left(1 - \frac{HH''}{(H')^2}\right),$$

$$\tau - \tau_G = \frac{H''}{\lambda\left(H'\right)^2},$$

respectively. These equations give that

$$H = \frac{\tau - \tau_G}{\kappa} = \frac{H''}{\left(H'\right)^2 - HH''}.$$

Then we have the following differential equation

$$\left(1 + H^2\right) H'' - \left(H'\right)^2 = 0.$$

Solving the last equation, we obtain Eq. (3.13). This completes the proof. □

Theorem 9. *Let (α, β) be a Mannheim curve couple in three dimensional Lie group G with bi-invariant metric. Then α is general helix if and only if β is a geodesic.*

Proof. If Mannheim curve α is a general helix, then its harmonic curvature H is constant function. And so from Theorem 6,

$$\kappa_\beta = 0.$$

So, the curve β is a geodesic.

Conversely we assume that β is a geodesic curve. From Theorem 6 we can easily see that

$$H'(s) = 0$$

and so

$$H(s) = \text{constant}.$$

This completes the proof. □

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Authors' addresses

İsmail Gök
Ankara University, Faculty of Sciences, Department of Mathematics, 06100 Ankara, Turkey
E-mail address: igok@science.ankara.edu.tr

O. Zeki Okuyucu
Bilecik Şeyh Edebali University, Faculty of Sciences and Arts, Department of Mathematics, 11210 Bilecik, Turkey
E-mail address: osman.okuyucu@bilecik.edu.tr

Nejat Ekmekci
Ankara University, Faculty of Sciences, Department of Mathematics, 06100 Ankara, Turkey
E-mail address: nekmekci@science.ankara.edu.tr

Yusuf Yaylı
Ankara University, Faculty of Sciences, Department of Mathematics, 06100 Ankara, Turkey
E-mail address: yayli@science.ankara.edu.tr

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NATURAL DENSITY OF RELATIVE COPRIME POLYNOMIALS IN $\mathbb{F}_q[x]$

XIANGQIAN GUO, FENGDAN HOU, AND XUEWEN LIU

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Abstract. Let $\mathbb{F}_q[x]$ be the polynomial ring over the finite field $\mathbb{F}_q$ containing q elements. We compute the probability that n polynomials in $\mathbb{F}_q[x]$ are k-wise relatively coprime, using the concept of natural density. As a special case, we get the probability that n polynomials in $\mathbb{F}_q[x]$ are pairwise coprime.

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1. Introduction and main results

Let $\mathbb{N}$ be the set of all positive integers. Dirichlet [2] first discovered an interesting result that relates the probability that two randomly chosen integers are relative prime to the Riemann's zeta function, and the probability turns out to be

$$\lim_{N\to\infty}\frac{|\{(m,n)\in\mathbb{N}^2\mid 1\le m,n\le N,\gcd(m,n)=1\}|}{N^2}=\zeta^{-1}(2)=\frac{6}{\pi^2},$$

where $\gcd(m,n)$ denotes the greatest common divisor of m and n, and $\zeta(s)$ is the Riemann's zeta function. This result was generalized to the case of several integers, that is, the probability of n randomly chosen integers to be coprime is given by

$$\lim_{N\to\infty}\frac{|\{(m_1,\dots,m_n)\in\mathbb{N}^n\mid 1\le m_1,\dots,m_n\le N,\gcd(m_1,\dots,m_n)=1\}|}{N^n}=\zeta^{-1}(n). \tag{1.1}$$

In [7], Kubota and Sugita gave a rigorous probabilistic interpretation to Dirichlet's theorem. Other probability problems over integers were also considered: L. Tóth [12] obtained that the probability of n positive integers to be pairwise coprime is $\prod_p(1-\frac{1}{p})^{n-1}(1+\frac{n-1}{p})$, where p is a prime number; Hu [4] showed that the probability of n positive integers to be k-wise relatively prime is $\prod_p(\sum_{m=0}^{k-1}\binom{n}{m}(\frac{1}{p})^m(1-\frac{1}{p})^{n-m})$. For deeper links between probability theory and number theory, please refer to Tenenbaum [11], Kubilius [6] and Kac [5].

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This notation of probability with respect to the uniform distribution over infinite sets $\mathbb{N}^n, n \in \mathbb{N}$, is also known as **natural density**, which can be defined for any subset A as

$$D(A) = \lim_{N\to\infty} \frac{|A \cap \{1,2,\cdots,N\}^n|}{N^n},$$

provided the limit exists, where $|\cdot|$ denotes the cardinality of the corresponding set. In [8], Maze, Rosenthal and Wagner computed the natural density of the set of $k \times n$ unimodular integer matrices for any positive integers $k \leq n$, where a $k \times n$ integer matrix is called unimodular if it can be extended to an invertible $n \times n$ matrix over the integers. Recently, Guo and Yang [3] generalized this result to the matrices of polynomials over finite fields.

Let $\mathbb{F}_q$ be the finite field consisting of q elements, and $\mathbb{F}_q[x]$ be the polynomial ring over $\mathbb{F}_q$, where q is a prime power. To define the concept of natural density for certain subsets, we need to enumerate polynomials in $\mathbb{F}_q[x]$. For convenience, denote the elements in $\mathbb{F}_q$ by $a_0 = 0, a_1, \cdots, a_{q-1}$. Let Σ be the set of all vectors $\alpha = (a_{m_0}, a_{m_1}, \cdots)$ with $m_i \in \{0,1,\cdots,q-1\}$ and $m_i = 0$ for sufficiently large i. Then there is a one-to-one map

$$\chi : \Sigma \to \mathbb{Z}_+ = \mathbb{N} \cup \{0\}, \quad \chi(a_{m_0}, a_{m_1}, \cdots) = \sum_{j=0}^{\infty} m_i q^i.$$

For all $j \in \mathbb{Z}_+$, we set

$$f_j(x) = \sum_{i=0}^{\infty} a_{m_i} x^i, \text{ with } \chi(a_{m_0}, a_{m_1}, \cdots) = j.$$

Then $\mathbb{F}_q[x] = \{f_j(x) \mid j \in \mathbb{Z}_+\}$.

From now on, we fix a prime power q and a positive integer $n \geq 2$. Denote $\mathcal{M} = (\mathbb{F}_q[x])^n$ for convenience and let $\mathcal{M}_N$ be the subset of $\mathcal{M}$ consisting of vectors with entries taken from $\{f_0, f_1, \cdots, f_N\}$. For any subset $S \subseteq \mathcal{M}$, we define the **natural density** of S in $\mathcal{M}$ as

$$D(S) = \lim_{N\to\infty} \frac{|S \cap \mathcal{M}_N|}{|\mathcal{M}_N|}.$$

Using a probabilistic method, Sugita and Takanobu [10] determined the probability of two polynomials over $\mathbb{F}_p$ to be coprime for a prime p. Recently, Morrison [9], Benjamin and Bennett [1] computed the probability that n polynomials over $\mathbb{F}_q$ are coprime, which is $1 - q^{1-n}$. They used natural density methods and Euclidean algorithm respectively. Then it is natural to consider the questions: what is the probability that n polynomials in $\mathbb{F}_q[x]$ are pairwise coprime? Generally, what is the probability that n polynomials in $\mathbb{F}_q[x]$ are k-wise relatively coprime?

Our main purpose in this paper is to compute the probabilities mentioned above. More precisely, we determined the natural density of the set of n-dimensional vectors

over $\mathbb{F}_q[x]$ whose entries are k-pairwise coprime, for any positive integer $k \leq n$. Our methods are conceptional and the main idea comes from [8] and [3].

Theorem 1. *Let k be a positive integer and $k \leq n$. Denote*

$$G = \{(g_1, \cdots, g_n) \in \mathcal{M} \mid \gcd(g_{i_1}, \cdots, g_{i_k}) = 1,\ \forall\ 1 \leq i_1 < \cdots < i_k \leq n\}.$$

Then the natural density of G is

$$\prod_{m=1}^{\infty} \Big(\sum_{i=0}^{k-1} \binom{n}{i} (\frac{1}{q^m})^i (1 - \frac{1}{q^m})^{n-i} \Big)^{\phi(m)}, \tag{1.2}$$

where $\phi(m)$ is the number of monic irreducible polynomials with degree m in $\mathbb{F}_q[x]$.

Remark 1. The result of Theorem 1 can be understood as follows: the probability that n polynomials in $\mathbb{F}_q[x]$ are k-wise relatively coprime is

$$\prod_{m=1}^{\infty} \Big(\sum_{i=0}^{k-1} \binom{n}{i} (\frac{1}{q^m})^i (1 - \frac{1}{q^m})^{n-i} \Big)^{\phi(m)}.$$

Take $k = n$, we get that the probability of n polynomials in $\mathbb{F}_q[x]$ being coprime is the $\prod_{m=1}^{\infty} (1 - \frac{1}{q^{mn}})^{\phi(m)}$. To see what this means, we introduce the following **q-zeta function**

$$\zeta_q(n) := \prod_f (1 - \frac{1}{q^{n \deg(f)}})^{-1} = \prod_{m=1}^{\infty} (1 - \frac{1}{q^{nm}})^{-\phi(m)}, \tag{1.3}$$

where f goes through all monic irreducible polynomials (not including the constant polynomials, as usual) in $\mathbb{F}_q[x]$. Recall the following interesting equation

$$\prod_f (1 - t^{\deg(f)})^{-1} = \sum_{l=0}^{\infty} q^l t^l = \frac{1}{1 - qt}. \tag{1.4}$$

For more details, see [9] and [3]. Putting $t = q^{-n}$ in (1.4), we get

$$\zeta_q^{-1}(n) = 1 - \frac{1}{q^{n-1}}. \tag{1.5}$$

Combining the equations (1.3) and (1.5), we get that the probability of n polynomials in $\mathbb{F}_q[x]$ being coprime is $\zeta_q^{-1}(n) = 1 - \frac{1}{q^{n-1}}$, which is just one of the main results of [9]. In particular, when $n = 2$, the probability that 2 polynomials in $\mathbb{F}_q[x]$ are coprime is $1 - \frac{1}{q}$, which is one of the main results of [1].

Taking $k = 2$ in Theorem 1, we have the following corollary.

Corollary 1. *Denote* $E = \{(g_1, \cdots, g_n) \in \mathcal{M} \mid \gcd(g_i, g_j) = 1,\ \forall\ 1 \le i < j \le n\}$. *Then*

$$D(E) = \prod_{m=1}^{\infty} \left((1 - \frac{1}{q^m})^{n-1}(1 + \frac{n-1}{q^m})\right)^{\phi(m)}. \tag{1.6}$$

Similarly, the value in (1.6) can be interpreted as the probability that n polynomials in $\mathbb{F}_q[x]$ are pairwise coprime.

2. Results

In this section, we will give the proof of Theorem 1. Before this, we need some preparations.

Fix a positive integer $k \le n$. Let T be a finite set of monic irreducible polynomials in $\mathbb{F}_q[x]$, denote

$$G_T = \{(g_1, \cdots, g_n) \in \mathcal{M} \mid f \nmid \gcd(g_{i_1}, \cdots, g_{i_k}), \\ \forall\ f \in T, 1 \le i_1 < \cdots < i_k \le n\}.$$

Clearly we have $G = \bigcap_T G_T$. Denote by $\langle f \rangle$ the ideal generated by $f \in \mathbb{F}_q[x]$.

Lemma 1. *Let* G_T *be defined as above, then we have*

$$D(G_T) = \prod_{f \in T} \sum_{i=0}^{k-1} \binom{n}{i} (\frac{1}{q^{\deg(f)}})^i (1 - \frac{1}{q^{\deg(f)}})^{n-i}.$$

Proof. Denote $f^{(T)} = \prod_{f \in T} f$ and $d_T = \deg(f^{(T)})$. Given $g \in \mathbb{F}_q[x]$ let $\bar{g}$ be its image in $\mathbb{F}_q[x]/\langle f^{(T)} \rangle$. Then for any positive integer N, we have the canonical maps

$$\pi :\ \mathcal{M}_N \to (\mathbb{F}_q[x]/\langle f^{(T)} \rangle)^n, \quad (g_1, \cdots, g_n) \mapsto (\bar{g}_1, \cdots, \bar{g}_n),$$

and

$$\varphi :\ \Big(\mathbb{F}_q[x]/\langle f^{(T)} \rangle\Big)^n \to \Big(\prod_{f \in T} \mathbb{F}_q[x]/\langle f \rangle\Big)^n \to \prod_{f \in T} \big(\mathbb{F}_q[x]/\langle f \rangle\big)^n,$$

where the first part of φ is induced from the isomorphism

$$\mathbb{F}_q[x]/\langle f^{(T)} \rangle \cong \prod_{f \in T} \mathbb{F}_q[x]/\langle f \rangle,$$

a consequence of the Chinese Remainder Theorem and the second part of φ is an obvious isomorphism of vector spaces.

First suppose that $N = mq^{d_T} - 1$ for some $m \in \mathbb{N}$. Then it is easy to see

$$\{f_l(x) \mid 0 \le l \le N\} = \{f_s(x)x^{d_T} + f_t(x) \mid 0 \le s \le m-1, 0 \le t \le q^{d_T} - 1\}.$$

For any fixed $0 \le s \le m-1$, the following projection is one-to-one:

$$\{f_s(x)x^{d_T} + f_t(x) \mid 0 \le t \le q^{d_T} - 1\} \longrightarrow \mathbb{F}_q[x]/\langle f^{(T)} \rangle,$$

and the canonical projection

$$\{f_l(x) \mid 0 \le l \le N\} \longrightarrow \mathbb{F}_q[x]/\langle f^{(T)} \rangle$$

is m-to-one. Thus the projection map π is m^n-to-one.

For any $f \in T$, let φ_f be the canonical projection from $(\mathbb{F}_q[x]/\langle f^{(T)} \rangle)^n$ to $(\mathbb{F}_q[x]/\langle f \rangle)^n$ via φ. Given any $A \in \mathcal{M}_N$, we see that $A \in G_T$ if and only if at most $k-1$ entries of $\varphi_f \circ \pi(A)$ is zero for all $f \in T$. Noticing that $|\mathbb{F}_q[x]/\langle f \rangle| = q^{\deg(f)}$, it is easy to deduce that

$$|\varphi \circ \pi(\mathcal{M}_N)| = \prod_{f \in T} \sum_{i=0}^{k-1} \binom{n}{i} (q^{\deg(f)} - 1)^{n-i}.$$

As a result we have

$$|G_T \bigcap \mathcal{M}_N| = m^n |\varphi \circ \pi(\mathcal{M}_N)|$$

$$= (mq^{d_T})^n \prod_{f \in T} \sum_{i=0}^{k-1} \binom{n}{i} (\frac{1}{q^{\deg(f)}})^i (1 - \frac{1}{q^{\deg(f)}})^{n-i}.$$

Now let N be any positive integer. There exist $m, r \in \mathbb{Z}_+$ such that $N+1 = mq^{d_T} + r$, where $0 \le r < q^{d_T}$ and m, r are not both 0. For convenience, set $\widetilde{N} = mq^{d_T} - 1$. Then by the definition of the natural density, we have

$$D(G_T) = \lim_{N \to \infty} \frac{|G_T \cap \mathcal{M}_N|}{|\mathcal{M}_N|}$$

$$= \lim_{N \to \infty} \frac{|G_T \bigcap \mathcal{M}_{\widetilde{N}}| + |G_T \bigcap (\mathcal{M}_N - \mathcal{M}_{\widetilde{N}})|}{|\mathcal{M}_N|}.$$

Note that $|\mathcal{M}_N - \mathcal{M}_{\widetilde{N}}| \le rn(N+1)^{n-1}$, that is

$$\lim_{N \to \infty} \frac{|G_T \bigcap (\mathcal{M}_N - \mathcal{M}_{\widetilde{N}})|}{|\mathcal{M}_N|} \le \lim_{N \to \infty} \frac{rn(N+1)^{n-1}}{(N+1)^n} = 0.$$

So, we obtain

$$D(G_T) = \lim_{N \to \infty} \frac{|G_T \bigcap \mathcal{M}_{\widetilde{N}}|}{(N+1)^n}$$

$$= \lim_{N \to \infty} \frac{(mq^{d_T})^n \prod_{f \in T} \sum_{i=0}^{k-1} \binom{n}{i} (\frac{1}{q^{\deg(f)}})^i (1 - \frac{1}{q^{\deg(f)}})^{n-i}}{(N+1)^n}$$

$$= \prod_{f \in T} \sum_{i=0}^{k-1} \binom{n}{i} (\frac{1}{q^{\deg(f)}})^i (1 - \frac{1}{q^{\deg(f)}})^{n-i}.$$

This completes the proof. □

Proof of Theorem 1.1. For any irreducible polynomial $f \in \mathbb{F}_q[x]$, denote

$$K_f = \{(g_1, \cdots, g_n) \mid f \,|\, \gcd(g_{i_1}, \cdots, g_{i_k}), 1 \le i_1 < \cdots < i_k \le n\}.$$

Let $q_f = q^{\deg(f)}$, then by Lemma 2.1 we have

$$\begin{aligned} D(K_f) &= 1 - D(G_{\{f\}}) \\ &= 1 - \sum_{i=0}^{k-1} \binom{n}{i} (\frac{1}{q_f})^i (1 - \frac{1}{q_f})^n \\ &\le 1 - (1 - \frac{n-1}{q_f})(1 + \frac{n-1}{q_f}) \\ &= (\frac{n-1}{q_f})^2. \end{aligned}$$

Let T_t be the set of all monic irreducible polynomials with degree no more than t, and denote $\widehat{T}$ the set of all monic irreducible polynomials in $\mathbb{F}_q[x]$. For convenience, we set $G_t = G_{T_t}$. Since

$$(G_t \setminus G) \subseteq \bigcup_{f \in \hat{T} \setminus T_t} K_f,$$

we have

$$\begin{aligned} \limsup_{N\to\infty} \frac{|(G_t \setminus G) \bigcap \mathcal{M}_N|}{|\mathcal{M}_N|} &\le \limsup_{N\to\infty} \frac{|(\bigcup_{f \in \hat{T} \setminus T_t} K_f) \bigcap \mathcal{M}_N|}{|\mathcal{M}_N|} \\ &\le \limsup_{N\to\infty} \frac{\sum_{f \in \hat{T} \setminus T_t} |K_f \bigcap \mathcal{M}_N|}{|\mathcal{M}_N|} \\ &\le \sum_{f \in \hat{T} \setminus T_t} \limsup_{N\to\infty} \frac{|K_f \bigcap \mathcal{M}_N|}{|\mathcal{M}_N|} \\ &= \sum_{f \in \hat{T} \setminus T_t} D(K_f) < \sum_{f \in \hat{T} \setminus T_t} (\frac{n-1}{q_f})^2 \\ &= \sum_{m=t+1}^{\infty} \frac{(n-1)^2}{q^{2m}} \phi(m), \end{aligned}$$

where $\phi(m)$ denotes the number of monic irreducible polynomials with degree m in $\mathbb{F}_q[x]$.

Since all irreducible polynomials with degree m can divide $x^{q^m}-x$, which has no multiple roots, thus $m\phi(m)\le q^m$ and

$$\limsup_{N\to\infty}\frac{|(G_t\setminus G)\bigcap\mathcal{M}_N|}{|\mathcal{M}_N|}\le\sum_{m=t+1}^{\infty}\frac{(n-1)^2}{mq^m}\le\frac{(n-1)^2}{q^t(q-1)}.$$

Note that $G\bigcap\mathcal{M}_N\subseteq G_t\bigcap\mathcal{M}_N$ and $G\bigcap\mathcal{M}_N=G_t\bigcap\mathcal{M}_N-(G_t\setminus G)\bigcap\mathcal{M}_N$, which imply that

$$\limsup_{N\to\infty}\frac{|G\bigcap\mathcal{M}_N|}{|\mathcal{M}_N|}\le\limsup_{N\to\infty}\frac{|G_t\bigcap\mathcal{M}_N|}{|\mathcal{M}_N|}\le D(G_t).$$

and

$$\begin{aligned}\liminf_{N\to\infty}\frac{|G\bigcap\mathcal{M}_N|}{|\mathcal{M}_N|}&\ge\liminf_{N\to\infty}\frac{|G_t\bigcap\mathcal{M}_N|}{|\mathcal{M}_N|}-\limsup_{N\to\infty}\frac{(G_t\setminus G)\bigcap\mathcal{M}_N}{|\mathcal{M}_N|}\\&\ge D(G_t)-\frac{(n-1)^2}{q^t(q-1)},\end{aligned}$$

for all $t\in\mathbb{N}$. Let t tend to ∞, from Lemma 2.1, we can conclude that

$$\begin{aligned}\lim_{N\to\infty}\frac{|G\bigcap\mathcal{M}_N|}{|\mathcal{M}_N|}&=\lim_{t\to\infty}D(G_t)\\&=\lim_{t\to\infty}\prod_{f\in T_t}\sum_{i=0}^{k-1}\binom{n}{i}(\frac{1}{q_f})^i(1-\frac{1}{q_f})^{n-i}\\&=\lim_{t\to\infty}\prod_{m=1}^{t}\sum_{i=0}^{k-1}\binom{n}{i}(\frac{1}{q^m})^i(1-\frac{1}{q^m})^{n-i}\\&=\prod_{m=1}^{\infty}\Big(\sum_{i=0}^{k-1}\binom{n}{i}(\frac{1}{q^m})^i(1-\frac{1}{q^m})^{n-i}\Big)^{\phi(m)}.\end{aligned}$$

This completes the proof. □

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Authors' addresses

Xiangqian Guo
Zhengzhou University, School of Mathematics and Statistics, 100 Science Road, 450001 Zhengzhou, P. R. China
E-mail address: guoxq@ZZu.edu.cn

Fengdan Hou
Zhengzhou University, School of Mathematics and Statistics, 100 Science Road, 450001 Zhengzhou, P. R. China
E-mail address: houfd@zzu.ps.edu.cn

Xuewen Liu
Zhengzhou University, School of Mathematics and Statistics, 100 Science Road, 450001 Zhengzhou, P. R. China
E-mail address: liuxw@zzu.edu.cn

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ON GREEN'S EQUIVALENCES IN Γ-GROUPOIDS

KOSTAQ HILA AND JANI DINE

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Abstract. This paper deals with Γ-groupoids which are generalizations of groupoids and Γ-semigroups. The main purpose of this paper is to extend Green's equivalences and Green's Lemma to suitably restricted Γ-groupoids. We study only Γ-groupoids satisfying some additional conditions and we show that these are sufficient for the statement of Green's equivalences in case of Γ-groupoids. Additional condition sufficient to prove Green's Lemma for Γ-groupoids is provided and some illustrative examples are presented.

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1. INTRODUCTION AND PRELIMINARIES

In 1964, N. Nobusawa [16] introduced the notion of a Γ-ring, more general than a ring. In 1966, W. E. Barnes [1] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa. Many fundamental results in ring theory have been extended to Γ-rings by different authors obtaining various generalization analogous to corresponding parts in ring theory. In 1981, M. K. Sen [19] and later in 1986, Sen and Saha [20] introduced the concept of the Γ-semigroup as a generalization of semigroup and ternary semigroup. Many classical notions and results of the theory of semigroups have been extended and generalized to Γ-semigroups by a lot of mathematicians. Green's relations for semigroups were introduced by J. A. Green in a paper of 1951 [7]. Green's relations for Γ-semigroups defined in [6, 18], play an important role in studying of the structure of Γ-semigroups as well as in case of the plain semigroups and become a familiar tool among Γ-semigroups. Several treatments and contributions concerning Green's relations for Γ-semigroups have been made by a lot of mathematicians, for instant [2–4, 6, 8–10, 12, 13, 15, 17, 18, 20]. Recently, in [11] we have introduced and studied the hyperversion of Green's relations in Γ-semihypergroups. The Green's equivalence relations, Green's Lemma and its corollaries are important tools in the theory of Γ-semigroups as well as in the case of the plain semigroups. The proof of those fundamental results depends on little more than the associativity of the Γ-operation defined in Γ-semigroups. However, when we remove this property, we find ourselves faced with the problem of obtaining

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similar results in Γ-groupoids. For this, we will study only Γ-groupoids satisfying some additional conditions and we show that these are sufficient for the statement of Green's equivalences in case of Γ-groupoids. Additional condition sufficient to prove Green's Lemma for Γ-groupoids is provided and some illustrative examples are presented. The main purpose of this paper is to extend Green's equivalences and Green's Lemma to suitably restricted Γ-groupoids and to obtain some results which are parallel to those obtained for groupoids and semigroups [5, 14].

We introduce below necessary notions and present a few auxiliary results that will be used throughout the paper.

Definition 1. Let M and Γ be two non-empty sets. Any map from $M \times \Gamma \times M \to M$ will be called a *Γ-multiplication* in M and denoted by $(\cdot)_\Gamma$. The result of this multiplication for $a, b \in M$ and $\alpha \in \Gamma$ is denoted by $a\alpha b$. A *Γ-groupoid* M is an ordered pair $(M, (\cdot)_\Gamma)$ where M and Γ are non-empty sets and $(\cdot)_\Gamma$ is a Γ-multiplication on M.

M is called a Γ-semigroup, if in addition, the following assertion is satisfied:

$$\forall (a, b, c, \alpha, \beta) \in M^3 \times \Gamma^2, (a\alpha b)\beta c = a\alpha(b\beta c).$$

Example 1. Let M be a semigroup and Γ be any nonempty set. If we define $a\gamma b = ab$ for all $a, b \in M$ and $\gamma \in \Gamma$. Then M is a Γ-semigroup.

Example 2. Let M be a set of all negative rational numbers. Obviously M is not a semigroup under usual product of rational numbers. Let $\Gamma = \{-\frac{1}{p} : p \text{ is prime}\}$. Let $a, b, c \in M$ and $\alpha \in \Gamma$. Now if $a\alpha b$ is equal to the usual product of rational numbers a, α, b, then $a\alpha b \in M$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$. Hence M is a Γ-semigroup.

Example 3. Let $M = \{-i, 0, i\}$ and $\Gamma = M$. Then M is a Γ-semigroup under the multiplication over complex numbers while M is not a semigroup under complex number multiplication.

Notice that every semigroup is a Γ-semigroup and Γ-semigroups are a generalization of semigroups. The same holds for Γ-groupoids.

A Γ-groupoid M is said to be *commutative* if for all $a, b \in M, \gamma \in \Gamma, a\gamma b = b\gamma a$.

2. On Green's relations in Γ-groupoids

Let M be a Γ-groupoid. If $\mathcal{E}$ is any binary relation on the set M and $a, b \in M$, then let $a\mathcal{E}b$ mean that a is $\mathcal{E}$-related to b and, whenever $\mathcal{E}$ is an equivalence relation on M, let the $\mathcal{E}$-equivalence class containing a be denoted by E_a, i.e., $E_a = \{x \in M | x\mathcal{E}a\}$. We define now two relations, the so-called Green's relations on a Γ-groupoid M.

Definition 2. Let M be a Γ-groupoid and $a, b \in M$. We define $a\mathcal{R}b$ if and only if either $a = b$ or there exist $x, y \in M$ and $\alpha, \beta \in \Gamma$, such that $a\alpha x = b$ and $b\beta y = a$. Dually, we define $c\mathcal{L}d$ if and only if either $c = d$ or there exist $u, v \in M$ and $\gamma, \delta \in \Gamma$ such that $u\gamma c = d$ and $v\delta d = c$.

When these are equivalence relations we will write R_a for the $\mathcal{R}$-class of a, and L_c for the $\mathcal{L}$-class of c.

When M is associative, that is, a Γ-semigroup, then it is known that the relations $\mathcal{R}$ and $\mathcal{L}$ are equivalence relations [6]. In this case we have $a\mathcal{R}b$ iff $a\Gamma M \cup \{a\} = b\Gamma M \cup \{b\}$.

For arbitrary Γ-groupoids, these two subsets of M need not have any particular relationship even though $a\mathcal{R}b$. For this, we give the following definition.

Definition 3. Let M be a Γ-groupoid. M is said to be *left consistent* if $H\gamma(x\alpha y) = (H\gamma x)\alpha y$ for any $x, y \in M$, $\gamma, \alpha \in \Gamma$ and any Γ-subgroupoid H of M. M is said to be *weakly left consistent* if the above holds just for $H = M$.

Definition 4. Let M be a Γ-groupoid. M is said to be *right consistent* if $(x\alpha y)\gamma H = x\alpha(y\gamma H)$ for any $x, y \in M$, $\alpha, \gamma \in \Gamma$ and any Γ-subgroupoid H of M. M is said to be *weakly right consistent* if the above holds just for $H = M$.

Definition 5. Let M be a Γ-groupoid. M is said to be [*weakly*]*consistent* if it is both [weakly] left and [weakly] right consistent.

Definition 6. Let M be a Γ-groupoid. M is said to be *intra-consistent* if $(x\gamma H)\alpha y = x\gamma(H\alpha y)$ for any $x, y \in M, \gamma, \alpha \in \Gamma$ and any Γ-subgroupoid H of M. M is said to be *weakly intra-consistent* if the above holds just for $H = M$.

Proposition 1. *Let M be a weakly right consistent or a weakly intra-consistent Γ-groupoid. Then $a\mathcal{R}b$ if and only if $a\Gamma M \cup \{a\} = b\Gamma M \cup \{b\}$ for $a, b \in M$.*

Proof. " $\Rightarrow$ ". Let M be weakly right consistent and assume $a\mathcal{R}b$. If $a = b$ the result is evident. Otherwise there exist $x, y \in M$ and $\alpha, \beta \in \Gamma$ such that $a\alpha x = b$ and $b\beta y = a$. Let $\gamma \in \Gamma$ and so $a\gamma M \subseteq a\Gamma M$. Then we have: $a\gamma M = (b\beta y)\gamma M = b\beta(y\gamma M) \subseteq b\beta M \subseteq b\Gamma M$, that is, $a\Gamma M \subseteq b\Gamma M$. On the other side, we have: $b\gamma M = (a\alpha x)\gamma M = a\alpha(x\gamma M) \subseteq a\alpha M \subseteq a\Gamma M$, that is $b\Gamma M \subseteq a\Gamma M$. Hence $a\Gamma M = b\Gamma M$. Since $a \in b\Gamma M$ and $b \in a\Gamma M$, the requested result follows. If M is weakly intra-consistent and $a\mathcal{R}b$ and $a \neq b$ we can show in a similar way that $a \in a\Gamma M = b\Gamma M$, and $b \in b\Gamma M$, and the result follows immediately.

" $\Leftarrow$ ". The converse is trivial. □

An immediate corollary of the above proposition is the following.

Corollary 1. *If M is either a weakly consistent or a weakly intra-consistent Γ-groupoids, $\mathcal{R}$ and $\mathcal{L}$ are equivalence relations. Indeed if M is weakly consistent, then $\mathcal{R}$ is a left congruence and $\mathcal{L}$ is a right congruence.*

Problem 1. *In general, $\mathcal{R}$ need not be a left congruence on a weakly intra-consistent Γ-groupoid. Can one finds an example of a non-trivial weakly intra-consistent Γ-groupoid in which $\mathcal{R}$ is not a left congruence?*

Let we consider now the case M is a commutative Γ-groupoid.

Proposition 2. *Let M be a commutative Γ-groupoid. Then M is [weakly] left consistent if and only if M is [weakly] right consistent and therefore [weakly] consistent.*

Proof. Let $a,b \in M, \alpha, \beta \in \Gamma$ and H be any Γ-subgroupoid of M. By the commutativity of M we have $(a\alpha b)\beta H = H\beta(b\alpha a)$ and $a\alpha(b\beta H) = (b\beta H)\alpha a = = (H\beta b)\alpha a$. The two equalities are linked if M is either [weakly] left or [weakly] right consistent and hence the conditions [with $H = M$] are equivalent. □

Remark 1. For the commutative groupoids, if they are [weakly] right consistent and therefore [weakly] consistent, in either case they are [weakly] intra-consistent. When we pass to Γ-groupoids, this property does not hold. In fact, if M is [weakly] right (or left) consistent Γ-groupoid, by the commutativity of M, we have: $(a\alpha H)\beta b = b\beta(a\alpha H) = (b\beta a)\alpha H = (a\beta b)\alpha H = a\beta(b\alpha H) = a\beta(H\alpha b)$, for any $a,b \in M, \alpha, \beta\Gamma$ and H is any Γ-subgroupoid of M [with $H = M$], which shows that M is not [weakly] intra-consistent in general.

Example 4. Let $M = \{x, y, z, t\}$ and $\Gamma = \{\alpha, \beta\}$ with the Γ-multiplication defined by

α	x	y	z	t
x	x	x	y	y
y	y	y	x	x
z	z	z	t	t
t	t	t	z	z

α	x	y	z	t
x	x	x	y	y
y	y	y	x	x
z	t	t	z	z
t	z	z	t	t

It can be easily verified that M is weakly left consistent and weakly intra-consistent but not weakly right consistent.

Problem 2. *Can one find an example of a non-trivial Γ-groupoid which is both weakly left and right consistent but not weakly intra-consistent or to prove that a weakly consistent Γ-groupoid is weakly intra-consistent?*

In order to prove Green's Lemma for Γ-groupoids we will need the following result, whose proof is straightforward.

Lemma 1. *Let M be a Γ-groupoid. If M is weakly right consistent, then for all $a \in M$ and $\rho \in \Gamma$, $a\rho M$ is a Γ-subgroupoid. Also, $a\Gamma M$ is a Γ-subgroupoid.*

Corollary 2. *Let M be a Γ-groupoid. If M is weakly right consistent, then for all $a \in M, \rho \in \Gamma$, $a\rho M \cup \{a\}$ is a Γ-subgroupoid. Also, $a\Gamma M \cup \{a\}$ is a Γ-subgroupoid.*

Let M be a Γ-groupoid. The mapping $\lambda_\alpha^s : M \to M$ defined by $\lambda_\alpha^s a = s\alpha a$ for all $s, a \in M$ and $\alpha \in \Gamma$ is called *left translation* of Γ-semigroup M. The mapping $\rho_{s'}^\beta : M \to M$ defined by $b\rho_{s'}^\beta = b\beta s'$ for all $s', b \in M$ and $\beta \in \Gamma$ is called *right translation* of Γ-semigroup M.

Theorem 1. *Let M be a consistent Γ-groupoid and suppose $c\mathcal{R}b$ for some $c \neq b$. Then there are $s, s' \in M, \alpha, \beta \in \Gamma$ such that $c\alpha s = b, b\beta s' = c$ and the right translations $\rho_s^\alpha, \rho_{s'}^\beta$, are, respectively, mappings from L_c into L_b and L_b into L_c, which are $\mathcal{R}$-class preserving, that is, for $x \in L_c, x\mathcal{R}x\rho_s^\alpha$ and for $y \in L_b, y\mathcal{R}y\rho_{s'}^\beta$.*

Proof. Let $c\mathcal{R}b$ for some $c \neq b$. Since $c \neq b$, the existence of s, s' follows from the Definition 2. Now let $a\mathcal{L}c$ and $d\mathcal{L}b$. By Corollary 1, $\mathcal{L}$ is a right congruence, thus we have $a\alpha s\mathcal{L}c\alpha s = b$ and $d\beta s'\mathcal{L}b\beta s' = c$. Thus we have $L_c\rho_s^\alpha \subseteq L_b$ and $L_b\rho_{s'}^\beta \subseteq L_c$.

Now, if $a \neq c$, then for any $\rho \in \Gamma$, $a\Gamma M \supseteq a\alpha(s\rho M) = (t\gamma c)\alpha(s\rho M)$ where $t\gamma c = a$ for some $t \in M, \gamma \in \Gamma$ by the Definition 2. Since $s\rho M$ is a Γ-subgroupoid by Lemma 1, we have $(t\gamma c)\alpha(s\rho M) = t\gamma[c\alpha(s\rho M)] = t\gamma[(c\alpha s)\rho M] = t\gamma(b\rho M)$. Thus $a = t\gamma c = t\gamma(b\beta s') \in t\gamma(b\Gamma M) \subseteq a\Gamma M$. Continuing we have $a\Gamma M \supseteq (a\alpha s)\rho M = t\gamma(b\rho M) \supseteq t\gamma[b\beta(s'\rho M)] = t\gamma[(b\beta s')\rho M] = t\gamma(c\rho M) = (t\gamma c)\rho M = a\rho M$, whence $a\Gamma M = (a\alpha s)\Gamma M$. But $a \in a\Gamma M$ and $a\alpha s \in a\Gamma M = (a\alpha s)\Gamma M$ so that we can conclude by Proposition 1 that $a\mathcal{R}a\alpha s$. If $a = c$, the preceding argument can be simplified to show that $c\mathcal{R}c\alpha s$. In a similar way, it can be shown $d\mathcal{R}d\beta s'$. □

Corollary 3. *If M is a consistent Γ-groupoid, then $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$ on M.*

Proof. If $a\mathcal{L}c\mathcal{R}b$, then the above Theorem yields an $s \in M$ such that $a\mathcal{R}a\gamma s\mathcal{L}b$ for some $\gamma \in \Gamma$, and so $\mathcal{L} \circ \mathcal{R} \subseteq \mathcal{R} \circ \mathcal{L}$. The reverse inclusion is proven dually. □

Now we give the following definition.

Definition 7. A Γ-groupoid M is said to be $\mathcal{D}$-Γ-groupoid if $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ on M.

In such cases we define $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$ and $\mathcal{D}$ is then clearly an equivalence relation.

Note that the consistent Γ-groupoids are $\mathcal{D} - \Gamma$-groupoid, while the converse is not necessarily true. The following example shows this.

Example 5. Let $M = \{x, y\}$ and $\Gamma = \{\alpha, \beta\}$ with the Γ-multiplication defined by

α	x	y
x	y	x
y	y	x

β	x	y
x	x	x
y	x	x

Here we have that $\mathcal{L} = \mathcal{R} = \omega$, the universal relation on M, so that M is certainly $\mathcal{D} - \Gamma$-groupoid, while $M\alpha(x\beta x) = y \neq x = (M\alpha y)\beta y$, that is, M is not a consistent Γ-groupoid.

3. Green's Lemma for Γ-groupoids

Theorem 1 shows us that for a consistent Γ-groupoid M certain right translations define maps between two $\mathcal{L}$-classes which are $\mathcal{R}$-class preserving. In general, it is not known if ρ_s^α and $\rho_{s'}^\beta$ are mutually inverse maps between L_c and L_b, as is the case of Γ-semigroups [6].

In the following results are provided additional conditions which suffice to guarantee this result.

Definition 8. Let M be a $\mathcal{D}$-Γ-groupoid. A $\mathcal{D}$-class, D of M is said to be *regular* if there is an γ-idempotent element ($x\gamma x = x$) in each $\mathcal{L}$ and $\mathcal{R}$-class of D.

Lemma 2. *Let M be a consistent Γ-groupoid. If $e\gamma e = e$ for some $\gamma \in \Gamma$, then $x\gamma e = x$ for all $x \in \mathcal{L}$-class, L_e, and $e\gamma y = y$ for all $y \in \mathcal{R}$-class R_e.*

Proof. Let $x \in L_e$. Then $x = t\alpha e$ for some $t \in M$ and $\alpha \in \Gamma$. Now $x\gamma e = (t\alpha e)\gamma e = t\alpha(e\gamma e) = t\alpha e = x$ since $\{e\}$ is a Γ-subgroupoid of M. The other result is dual. □

Proposition 3. *Let D be a regular $\mathcal{D}$-class of a consistent Γ-groupoid M. Then for any $a \in D$, there exist $t,t' \in M$ and $\gamma,\gamma',\alpha,\beta \in \Gamma$ such that $a = a\gamma(t\alpha a) = (a\beta t')\gamma' a$.*

Proof. Let $a \in D$. Since D is regular, there is an γ-idempotent $e \in L_a = L_e$. By Lemma 2, $a\gamma e = a$. Since $e\mathcal{L}a$, there is a $t \in M, \alpha \in \Gamma$ such that $t\alpha a = e$. Then $a = a\gamma e = a\gamma(t\alpha a)$. Dually one obtains $a = (a\beta t')\gamma' a$. □

Remark 2. A converse of Proposition 3 is false: a Γ-groupoid M may be consistent, and for every $a \in M$ may have $t,t' \in M$ and $\gamma,\gamma',\alpha,\beta \in \Gamma$ such that $a = a\gamma(t\alpha a) = (a\beta t')\gamma' a$, and yet M may have no idempotents. We have the following example.

Example 6. Let $M = \{x, y\}$ and $\Gamma = \{\alpha, \beta\}$ with the Γ-multiplication defined by

α	x	y
x	y	x
y	y	x

β	x	y
x	y	y
y	x	x

In the Γ-groupoid M there no idempotents and further, for example, $x = x\alpha(x\beta x) = (x\alpha x)\beta x$. Moreover, here we have $\mathcal{L} = \mathcal{R} = \omega$.

Definition 9. Let M be a Γ-groupoid. M is said to be *almost associative* if whenever H is a Γ-subgroupoid of M and $a,b,c \in M, \alpha,\beta,\gamma \in \Gamma$, we have $H\gamma[(a\alpha b)\beta c] = H\gamma[a\alpha(b\beta c)]$ and $[a\alpha(b\beta c)]\gamma H = [(a\alpha b)\beta c]\gamma H$.

Theorem 2. *A regular, consistent, almost associative Γ-groupoid is associative.*

Proof. Let $a,b,c \in M$. Then $(a\alpha b)\beta c = (a\alpha b)\beta(c\gamma e)$ for some $e\gamma e = e \in L_c, \gamma \in \Gamma$. But $(a\alpha b)\beta(c\gamma e) = [(a\alpha b)\beta c]\gamma e = [a\alpha(b\beta c)]\gamma e = a\alpha[(b\beta c)\gamma e] = a\alpha[b\beta(c\gamma e)] = a\alpha(b\beta c)$ since M is consistent and almost associative and $\{e\}$ is a Γ-subgroupoid. We thus have that $(a\alpha b)\beta c = a\alpha(b\beta c)$ for any $a,b,c \in M$ and $\alpha,\beta \in \Gamma$, i.e., M is associative. □

Now, based on the above results, we are ready to state and to prove the so-called Green's Lemma for Γ-groupoids.

Corollary 4. *(Green's Lemma). Let M be a consistent almost associative Γ-groupoid. If D is a regular $\mathcal{D}$-class of M and $c\mathcal{R}b$ for $c,b \in D$, then there exist $s,s' \in M, \alpha,\beta \in \Gamma$ such that $c\alpha s = b, b\beta s' = c$ and the right translations $\rho_s^\alpha, \rho_{s'}^\beta$, are mutually inverse bijections between L_c and L_b and are $\mathcal{R}$-class preserving.*

Proof. By Theorem 1 we need only to show that ρ_s^α and $\rho_{s'}^\beta$ are mutually inverse bijections. Let $f\gamma f = f \in R_c$ for some $\gamma \in \Gamma$, and $f\gamma u = c$ for some $u \in M$. Then $c = b\beta s' = (c\alpha s)\beta s' = ((f\gamma u)\alpha s)\beta s' = (f\gamma(u\alpha s))\beta s' = f\gamma((u\alpha s)\beta s') = f\gamma(u\alpha(s\beta s')) = (f\gamma u)\alpha(s\beta s') = c\alpha(s\beta s')$.

Now let $a \in L_c$ and $e\delta e = e \in L_c$ for some $\delta \in \Gamma$. There exist $t \in M$ and $\mu \in \Gamma$ such that $t\mu c = a$. Then $(a)\rho_s^\alpha \rho_{s'}^\beta = (a\alpha s)\beta s'$. Since $(a\alpha s)\beta s' \in L_a \cap R_a$ by Theorem 1, $(a\alpha s)\beta s' = [(a\alpha s)\beta s']\delta e = [a\alpha(s\beta s')]\delta e = [(t\mu c)\alpha(s\beta s')]\delta e = [t\mu(c\alpha(s\beta s'))]\delta e = [t\mu c]\delta e = a\delta e = a$ since $c = c\alpha(s\beta s')$ from above. Thus $\rho_s^\alpha \rho_{s'}^\beta$ is the identity map on L_c. Similarly $\rho_{s'}^\beta \rho_s^\alpha$ is the identity map on L_b. The result now follows. □

It is clear that, as it is shown in this paper, a consistent, almost associative Γ-groupoid with every $(\mathcal{R}\circ\mathcal{L})$-equivalence class regular is necessarily a Γ-semigroup and so the Green's Lemma takes its familiar form in Γ-semigroups [3,6].

From all the above, the following problems arise:

Problem 3. *What we can say about the validation of Green's Lemma for infinite Γ-groupoids?*

Problem 4. *How necessary is the regularity in the Corollary 4?*

Problem 5. *Is there any natural condition for Γ-groupoids weaker than that of almost associativity which can replace it leaving Corollary 4 true but Theorem 2 false?*

Problem 6. *Is there a regular consistent Γ-groupoid in which the right translations of Theorem 1 are not mutually inverse?*

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Authors' addresses

Kostaq Hila

Department of Mathematics & Computer Science, Faculty of Natural Sciences, University of Gjirokastra, Albania

E-mail address: kostaq_hila@yahoo.com

Jani Dine

Department of Mathematics & Computer Science, Faculty of Natural Sciences, University of Gjirokastra, Albania

E-mail address: jani_dine@yahoo.com

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SOME GENERAL BASKAKOV TYPE OPERATORS

ADRIAN D. INDREA AND OVIDIU T. POP

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Abstract. A general class of linear positive operators which generalizes Baskakov's operator is consrtucted. The operators of this type which preserve exactly two test functions from the set $\{e_0, e_1, e_2\}$ are determined in each case, and for the operators obtained, we give their approximation theorem, convergence theorem and Voronovskaja-type theorem.

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1. INTRODUCTION

Let $\mathbb{N}$ be a set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

In [6], J. P. King constructed and studied general operators which generalizes the classical Berstein operators. Some King-type operators were studied in [3–6], [8, 9].

In 1957, V. A. Baskakov [2], for $m \in \mathbb{N}$ has introduced the linear positive operator

$$(V_m f)(x) = (1+x)^{-m} \sum_{k=0}^{\infty} \binom{m+k-1}{k} \left(\frac{x}{1+x}\right)^k f\left(\frac{k}{m}\right) \tag{1.1}$$

defined for any $f \in C_2([0,+\infty)) = \left\{ f \in C([0,+\infty)) | \lim_{x \longrightarrow \infty} \frac{f(x)}{1+x^2} < +\infty \right\}$ and $x \in [0,+\infty)$. He proved that if $f \in C_2([0,+\infty))$ then $V_m f \longrightarrow f$ uniform on any compact $[a,b] \subset [0,+\infty)$. Note that the operators (1.1) preserve the test functions e_0 and e_1. Generalizations of the operators (1.1) were introduced by M.A.Özarslan, G.Duman and N.I.Mahmudov in [10] by the form

$$(T_m f)(x) = \sum_{k=0}^{\infty} \binom{m+k-1}{k} (u_m(x))^k (1+u_m(x))^{-m-k} f\left(\frac{k}{m}\right) \tag{1.2}$$

for $m \in \mathbb{N}, x \in [0,+\infty)$, and they show that if $u_m(x) \longrightarrow x$ on a compact $[a,b] \subset [0,+\infty)$, then $T_m f \longrightarrow f$ uniform on $[a,b]$ for all $f \in C_2([0,+\infty))$.

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A similar result was obtained in [9] by L. Rempulska and K. Tomczak for the case in which the modified operators of Baskakov type preserve the test functions e_0 and e_2.

In this paper, we introduce a general class of linear positive operators. We determine the operators of the general class which preserve only two test functions e_0 and e_1 or e_0 and e_2 or e_1 and e_2.

In all these cases we give approximation properties, convergence theorems and Voronovskaja-type theorems.

The paper is organized as follows. In Section 2 we recall some results obtained by O.T.Pop in [7] which are essentially used for obtaining the main results of the paper. Section 3 is devoted to the construction of the general class of linear and positive operators defined by infinite sum, which we announced in the start. For the constructed class we establish a convergence theorem and Voronovskaja type theorem. In Section 4 we prove that in the general class constructed in Section 3 exists a unique operator which preserve the test functions e_0 and e_1, the classical Baskakov operator. In Section 5 we obtain a King type operator, which is an operator that preserves the test functions e_0 and e_2 defined on semiaxis $[0,+\infty)$. We find here a result due the L. Rempulska and K. Tomczak [9].

Finally, in Section 6, we determine the operators from the general class which preserve the test function e_1 and e_2.

2. Preliminaries

In this section we recall some results from [7], which we shall use in the present paper. Let I, J be real intervals with the property $I \cap J$ is a nonempty interval. For any $m,k \in \mathbb{N}_0, m \neq 0$, we consider the functions $\varphi_{m,k} : J \longrightarrow \mathbb{R}$, with the property that $\varphi_{m,k}(x) \geq 0$, for any $x \in J$ and the linear positive functionals $A_{m,k} : E(I) \longrightarrow \mathbb{R}$.

For any $m \in \mathbb{N}$ we define the operator $L_m : E(I) \longrightarrow F(J)$, by

$$(L_m f)(x) = \sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}(f), \tag{2.1}$$

where $E(I)$ is a linear space of real valued functions defined on I, for which the operators (2.1) are convergent and $F(J)$ is a subset of real valued functions defined on J.

Remark 1. The operators $(L_m)_{m\in\mathbb{N}}$ are linear and positive on $E(I \cap J)$.

For $m \in \mathbb{N}$ and $i \in \mathbb{N}_0$, we define $T_{m,i}$ by

$$(T_{m,i} L_m)(x) = m^i (L_m \psi_x^i)(x) = m^i \sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}(\psi_x^i) \tag{2.2}$$

for any $x \in I \cap J$, where $\psi_x : I \longrightarrow \mathbb{R}, \psi_x(t) = t - x$.

In what follows $s \in \mathbb{N}_0$ is even and we assume that the following condition: there exist the smallest $\alpha_s, \alpha_{s+2} \in [0, +\infty)$, so that

$$\lim_{m \longrightarrow \infty} \frac{(T_{m,j} L_m)(x)}{m^{\alpha_j}} = B_j(x) \in \mathbb{R} \tag{2.3}$$

for any $x \in I \cap J$ and $j \in \{s, s+2\}$,

$$\alpha_{s+2} < \alpha_s + 2 \tag{2.4}$$

hold.

Theorem 1 ([7]). *Let $f \in E(I)$ be a function. If $x \in I \cap J$ and f is s times differentiable in a neighborhood of x, $f^{(s)}$ is continuous on x, then*

$$\lim_{m \longrightarrow \infty} m^{s-\alpha_s} \left((L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!} (T_{m,i} L_m)(x) \right) = 0. \tag{2.5}$$

Assume that f is s times differentiable on I. Let $K \subset I \cap J$ be a compact interval. For there one we assume that exist $m(s) \in \mathbb{N}$ and constant $k_j \in \mathbb{R}$ depending on K, such that for $m \geq m(s)$ and $x \in K$ the following relation

$$\frac{(T_{m,j} L_m)(x)}{m^{\alpha_j}} \leq k_j, \ j \in \{s, s+2\} \tag{2.6}$$

holds.

Following [7], the convergence expressed by (2.5) is uniform on K and

$$m^{s-\alpha_s} \left| (L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!} (T_{m,i} L_m)(x) \right| \leq \tag{2.7}$$

$$\leq \frac{1}{s!} (k_s + k_{s+2}) \omega \left(f^{(s)}; \frac{1}{\sqrt{m^{2+\alpha_s - \alpha_{s+2}}}} \right),$$

for any $x \in K, m \geq m(s)$, where $\omega(f; \delta)$ denotes the modulus of continuity of the function f.

In the following, we use the identity

$$(1+x)^{-m} = \sum_{k=0}^{\infty} (-1)^k \binom{m+k-1}{k} x^k \tag{2.8}$$

where $x \geq 0$ and $m \in \mathbb{N}$.

By differentiating the relation (2.8) and multiplying with $\frac{x}{m}$, we obtain

$$-x(1+x)^{-m-1} = \sum_{k=0}^{\infty} (-1)^k \binom{m+k-1}{k} x^k \frac{k}{m}. \tag{2.9}$$

Similarly, differentiating the relation (2.9) and multiplying with $\frac{x}{m}$ we get

$$\frac{x}{m}(mx-1)(1+x)^{-m-2}=\sum_{k=0}^{\infty}(-1)^k\binom{m+k-1}{k}x^k\left(\frac{k}{m}\right)^2, \tag{2.10}$$

where $x\geq 0$ and $m\in\mathbb{N}$.

3. The construction of a general linear and positive operators defined by infinite sum

Let $m_0\in\mathbb{N}$ be given, $\mathbb{N}_1=\{m\in\mathbb{N}|m\geq m_0\}$, the functions $\alpha_m: J\longrightarrow\mathbb{R}$ and $\beta_m: J\longrightarrow\mathbb{R}$ such that $\alpha_m(x)>0, \beta_m(x)>0, \beta_m(x)-\alpha_m(x)>0$ for any $x\in J$ and any $m\in\mathbb{N}_1$.

We define the operators of the following form

$$(P_m f)(x)=\sum_{k=0}^{\infty}\binom{m+k-1}{k}\alpha_m^k(x)\beta_m^{-m-k}(x)\,f\left(\frac{k}{m}\right), \tag{3.1}$$

for any $m\in\mathbb{N}_1, x\in J$ and $f\in E([0,+\infty))$, where $E([0,\infty))$ is a linear space of real valued functions defined on $[0,\infty)$, for which the operators defined by (3.1) are convergent.

If in (2.8)-(2.10), we substitute x by $-\frac{\alpha_m(x)}{\beta_m(x)}$, we obtain

$$(\beta_m(x)-\alpha_m(x))^{-m}=\sum_{k=0}^{\infty}\binom{m+k-1}{k}(\alpha_m(x))^k(\beta_m(x))^{-m-k} \tag{3.2}$$

$$\alpha_m(x)(\beta_m(x)-\alpha_m(x))^{-m-1}=\sum_{k=0}^{\infty}\binom{m+k-1}{k}(\alpha_m(x))^k(\beta_m(x))^{-m-k}\frac{k}{m} \tag{3.3}$$

$$\frac{1}{m}\alpha_m(x)(m\alpha_m(x)+\beta_m(x))(\beta_m(x)-\alpha_m(x))^{-m-2}= \tag{3.4}$$

$$=\sum_{k=0}^{\infty}\binom{m+k-1}{k}(\alpha_m(x))^k(\beta_m(x))^{-m-k}\left(\frac{k}{m}\right)^2, x\in J, m\in\mathbb{N}.$$

We impose the condition

$$(P_m e_0)(x)=1+u_m(x), \tag{3.5}$$

for any $m\in\mathbb{N}_1$ and any $x\in J$, where $u_m: J\longrightarrow\mathbb{R}, u_m(x)>-1$.

From (3.1), (3.2) and (3.5) follows the equality

$$\beta_m(x)-\alpha_m(x)=(1+u_m(x))^{-\frac{1}{m}} \tag{3.6}$$

for any $m\in\mathbb{N}_1$ and any $x\in J$.

Let us to impose the condition

$$(P_m e_1)(x) = x + v_m(x), \tag{3.7}$$

for any $m \in \mathbb{N}_1$ and any $x \in J$, where $v_m : J \longrightarrow \mathbb{R}, v_m(x) > -x$.

Taking (3.1), (3.3) and (3.7) into account, we get

$$\alpha_m(x)(\beta_m(x) - \alpha_m(x))^{-m-1} = x + v_m(x), m \in \mathbb{N}_1, x \in J. \tag{3.8}$$

From (3.6) and (3.8) it follows

$$\alpha_m(x) = \frac{x + v_m(x)}{1 + u_m(x)} (1 + u_m(x))^{-\frac{1}{m}} \tag{3.9}$$

and

$$\beta_m(x) = \left(1 + \frac{x + v_m(x)}{1 + u_m(x)}\right) (1 + u_m(x))^{-\frac{1}{m}}, \tag{3.10}$$

$m \in \mathbb{N}_1$, $x \in J$.

Taking (3.9) and (3.10) into account, the operator (3.1) becomes

$$(P_m f)(x) = (1 + u_m(x)) \sum_{k=0}^{\infty} \binom{m+k-1}{k} \left(\frac{x + v_m(x)}{1 + u_m(x)}\right)^k \cdot \tag{3.11}$$

$$\cdot \left(1 + \frac{x + v_m(x)}{1 + u_m(x)}\right)^{-m-k} f\left(\frac{k}{m}\right),$$

$m \in \mathbb{N}_1$, $x \in J$, $f \in E([0, +\infty))$.

From (3.1) and (3.4), we have

$$(P_m e_2)(x) = \frac{x + v_m(x)}{m} \left((m+1)\frac{x + v_m(x)}{1 + u_m(x)} + 1\right), \tag{3.12}$$

for any $m \in \mathbb{N}_1$ and any $x \in J$.

Next $(P_m \psi_x^2)(x) = (P_m e_2)(x) - 2x(P_m e_1)(x) + x^2(P_m e_0)(x)$ and taking (3.5), (3.7) and (3.12) into account we get

$$(P_m \psi_x^2)(x) = \frac{m(v_m(x) - x u_m(x))^2 + (x + v_m(x))^2 + (1 + u_m(x))(x + v_m(x))}{m(1 + u_m(x))} \tag{3.13}$$

for any $m \in \mathbb{N}_1$ and any $x \in J$.

Coming back to Theorem 1, for the operators (3.1), we have $I = [0, +\infty)$, $E(I) = C_2([0, +\infty))$

$$\varphi_{m,k}(x) = (1 + u_m(x)) \binom{m+k-1}{k} \left(\frac{x + v_m(x)}{1 + u_m(x)}\right)^k \left(1 + \frac{x + v_m(x)}{1 + u_m(x)}\right)^{-m-k} \tag{3.14}$$

and

$$A_{m,k}(f) = f\left(\frac{k}{m}\right), \tag{3.15}$$

for any $m \in \mathbb{N}_1, x \in J$ and $f \in C_2([0,+\infty))$.

In the following, let $K \subset I \cap J$ be a compact interval.

We suppose that there exists the sequences $\left(a_m(K)\right)_{m\in\mathbb{N}_1}, \left(b_m(K)\right)_{m\in\mathbb{N}_1}$, so that

$$\lim_{m\longrightarrow\infty} a_m(K) = \lim_{m\longrightarrow\infty} b_m(K) = 0, \tag{3.16}$$

$$|u_m(x)| \leq a_m(K), \tag{3.17}$$

$$|v_m(x)| \leq b_m(K), \tag{3.18}$$

for any $m \in \mathbb{N}_1$ and any $x \in K$.

In what follows, let us suppose that the following equality

$$\lim_{m\longrightarrow\infty} m(v_m(x) - xu_m(x)) = l(x) \tag{3.19}$$

holds for any $x \in J$, where $l : J \longrightarrow \mathbb{R}$ is a bounded function on K.

Remark 2. From (3.16) – (3.18) it results that if

$$\lim_{m\longrightarrow\infty} u_m(x) = \lim_{m\longrightarrow\infty} v_m(x) = 0, x \in K,$$

then

$$\lim_{m\longrightarrow\infty} m(v_m(x) - xu_m(x))^2 = \lim_{m\longrightarrow\infty} m(v_m(x) - xu_m(x)) \cdot$$
$$\cdot \lim_{m\longrightarrow\infty} (v_m(x) - xu_m(x)) = 0.$$

This Remark 2 implies that there exist $m_1 \in \mathbb{N}$ such that

$$(m(v_m(x) - xu_m(x))^2 \leq 1, m \in \mathbb{N}_1, m \geq m_1, x \in K. \tag{3.20}$$

Let us denote

$$M_1(K) = sup\{a_m(K) | m \in \mathbb{N}_1\},$$
$$M_2(K) = sup\{b_m(K) | m \in \mathbb{N}_1\}.$$

Now, let $\mathbb{N}_2 = \{m \in \mathbb{N} | m \geq max(m_0, m_1)\}$.

According to Theorem 1 one obtains $\alpha_0 = 0, \alpha_2 = 1$, $(T_{m,0}P_m)(x) = (P_m e_0)(x)$, for any $m \in \mathbb{N}_1$ and any $x \in K$.

From (3.16) one arrives at

$$\lim_{m\longrightarrow\infty} (T_{m,0}P_m)(x) = 1 = B_0(x), x \in K. \tag{3.21}$$

Consequently we get that exists $m(0) \in \mathbb{N}$ such that

$$(T_{m,0}P_m)(x) = 1 + u_m(x) \leq 1 + M_1(K) = k_0(K) \tag{3.22}$$

holds for any $m \geq max(m_0, m(0))$ and $x \in K$.

We have $(T_{m,2}P_m)(x) = m^2(P_m\psi_x^2)(x)$, $m \in \mathbb{N}_1$, $x \in J$. Taking (3.13), (3.19) and (3.20) into account, we get

$$\lim_{m\longrightarrow\infty} \frac{(T_{m,2}P_m)(x)}{m} = x(1+x) + l(x) = B_2(x), x \in K. \tag{3.23}$$

Also there exists $m(2) \in \mathbb{N}$ such that

$$\frac{(T_{m,2}P_m)(x)}{m} \leq b(1+b)+2 = k_2(K) \tag{3.24}$$

for any $m \geq max(m_0, m(2), m_1)$ and $x \in K$, where $maxK = b$.

Theorem 2. *Let $f \in C_2([0,+\infty))$. Then*

$$\lim_{m\longrightarrow\infty} P_m f = f \tag{3.25}$$

uniformly on K. There exists $m(0) \in \mathbb{N}, m(0)$ depending on K, so that the following inequalities

$$|(P_m f)(x) - (1+u_m(x))f(x)| \leq (k_0(K)+k_2(K))\omega\left(f; \frac{1}{\sqrt{m}}\right), \tag{3.26}$$

$$|(P_m f)(x) - f(x)| \leq |u_m(x)| \cdot |f(x)| + (k_0(K)+k_2(K))\omega\left(f; \frac{1}{\sqrt{m}}\right) \tag{3.27}$$

and

$$|(P_m f)(x) - f(x)| \leq a_m(K)M(K) + (k_0(K)+k_2(K))\omega\left(f; \frac{1}{\sqrt{m}}\right) \tag{3.28}$$

hold for any $m \in \mathbb{N}_2, m \geq m(0)$ and $x \in K$, where

$$M(K) = sup\{|f(x)| \mid x \in K\}.$$

Proof of Theorem 2. Applying the Theorem 1 for $\alpha = 0$ yields (3.25) and (3.26). Next, using the inequality $|a-c|-|b-c| \leq |a-b|$, (3.27) follows, and consequently (3.28) holds. □

Remark 3. The equations (3.26)-(3.28) are asymptotic formula for a class of approximation processes of King's type (see [1]).

Theorem 3. *Let $f \in C_2([0,+\infty))$. If $x \in K$, f is two times differentiable in x and $f^{(2)}$ is continuous in x, the following relations*

$$\lim_{m\longrightarrow\infty} m\left((P_m f)(x) - (1+u_m(x))f(x)\right) = l(x)f^{(1)}(x) + \frac{x(1+x)}{2}f^{(2)}(x) \tag{3.29}$$

holds.

Proof of Theorem 3. If $m \in \mathbb{N}_1, x \in K$, according Theorem 1 yields

$$(T_{m,1}P_m)(x) = m(P_m\psi_x)(x) = m((P_m e_1)(x) - x(P_m e_0)(x)).$$

Applying (3.1) and (3.5) it follows

$$(T_{m,1}P_m)(x) = m(v_m(x) - xu_m(x)). \tag{3.30}$$

Using Theorem 1 for $s = 2$, (3.22), (3.23) and (3.30) one arrives at (3.29). □

Remark 4. The relation (3.29) is a Voronovskaja-type theorem (see [11]).

4. $(P_m)_{m\geq m_0}$ OPERATORS PRESERVING TEST FUNCTIONS e_0 AND e_1

In the following, we consider $K=[a,b]$, where $b>0$. In this case $J=[0,+\infty)$ and $m_0=1$, then $\mathbb{N}_1=\mathbb{N}$. If the operators, $(P_m)_{m\in\mathbb{N}}$ preserve e_0 and e_1, we have $P_m e_0=e_0$ and $P_m e_1=e_1$, for any $m\in\mathbb{N}$. Taking (3.5) and (3.7) into account, it results that $u_m(x)=v_m(x)=0$ and $l(x)=0$ for any $m\in\mathbb{N}$ and any $x\in[0,+\infty)$.

In this case, we get again the classical Baskakov operators. One has $a_m([a,b])=b_m([a,b])=0$, for any $m\in\mathbb{N}$, $k_0([a,b])=1$ and $k_2([a,b])=b(1+b)+2$. Our statements turn into well known results.

Theorem 4 ([2]). *Let $f\in C_2([0,+\infty))$ one has*

$$\lim_{m\longrightarrow\infty} P_m f=f \tag{4.1}$$

uniformly on any compact interval $[a,b]\subset\mathbb{R}_+$ and then exists $m(0)\in\mathbb{N}$, $m(0)$ depending on b so that

$$|(P_m f)(x)-f(x)|\leq(3+b+b^2)\omega\left(f;\frac{1}{\sqrt{m}}\right),m\in\mathbb{N}_2,m\geq m(0),x\in[a,b]. \tag{4.2}$$

Theorem 5 ([2]). *Let $f\in C_2([0,+\infty))$. If $x\in[a,b]$, f is two times differentiable in x and $f^{(2)}$ is continuous in x, then*

$$\lim_{m\longrightarrow\infty} m((P_m f)(x)-f(x))=\frac{x(1+x)}{2}f^{(2)}(x). \tag{4.3}$$

5. $(P_m)_{m\geq m_0}$ OPERATORS PRESERVING THE TEST FUNCTIONS e_0 AND e_2

In this case $J=[0,+\infty)$ and $m_0=1$, then $\mathbb{N}_1=\mathbb{N}$. Because $P_m e_0=e_0$ and $P_m e_2=e_2$ for any $m\in\mathbb{N}$, taking (3.5) into account, it follows $u_m(x)=0$, for any $m\in\mathbb{N}$ and any $x\in[0,+\infty)$.

By using (3.12) yields

$$(m+1)(x+v_m(x))^2+(x+v_m(x))-mx^2=0 \tag{5.1}$$

for any $m\in\mathbb{N}$ and any $x\in[0,+\infty)$.

From (5.1) we get $v_m(x)=\frac{\sqrt{4m(m+1)x^2+1}-1}{2(m+1)}-x$, for any $m\in\mathbb{N}$ and any $x\in[0,+\infty)$, and then the operators from (3.8) become

$$(P_m f)(x)=\sum_{k=0}^{\infty}\binom{m+k-1}{k}\left(\frac{\sqrt{4m(m+1)x^2+1}-1}{2(m+1)}\right)^k\cdot \tag{5.2}$$

$$\cdot\left(1+\frac{\sqrt{4m(m+1)x^2+1}-1}{2(m+1)}\right)^{-m-k}f\left(\frac{k}{m}\right),$$

$m\in\mathbb{N},x\in[0,+\infty)$, $f\in C_2([0,+\infty))$.

So we came across the results obtained by L. Rempulska and K. Tomczak in [9].

Lemma 1. *We have that*

$$v_m(x) \leq \frac{\sqrt{4m(m+1)a^2+1}-1}{2(m+1)} - a, m \in \mathbb{N}, x \in K = [a,b] \tag{5.3}$$

and

$$\frac{\sqrt{4m(m+1)a^2+1}-1}{2(m+1)} - a \leq \sqrt{\frac{1}{2}a^2+\frac{1}{16}} - a, m \in \mathbb{N}. \tag{5.4}$$

Proof of Lemma 1. Since the function v_m is decreasing on $[a,b]$, it gets the maximum value in a and (5.3) follows. By direct computation, (5.4) is obtained. □

Lemma 2. *The following relation*

$$\lim_{m\longrightarrow\infty} m v_m(x) = -\frac{1+x}{2} \tag{5.5}$$

holds, where $x \in K$.

Proof of Lemma 2. We have

$$\lim_{m\longrightarrow\infty} m v_m(x) = \lim_{m\longrightarrow\infty} \frac{m}{2(m+1)}\left(-1+\sqrt{4m(m+1)x^2+1}-2(m+1)x\right) =$$

$$= \frac{1}{2}\left(-1+\lim_{m\longrightarrow\infty}\frac{-4mx^2-4x^2+1}{\sqrt{4m(m+1)x^2+1}+2(m+1)x}\right)$$

and (5.5) follows. □

According to the notations from Section 3, taking Lemma 1 and Lemma 2 into account we have $a_m([a,b]) = 0$, for any $m \in \mathbb{N}$, $b_m([a,b])$ $= \frac{\sqrt{4m(m+1)a^2+1}-1}{2(m+1)} - a$, $l(x) = -\frac{1+x}{2}$, for any $m \in \mathbb{N}$, any $x \in [a,b]$, $b_m([a,b]) \leq \sqrt{\frac{1}{2}a^2+\frac{1}{16}} - a = M_2([a,b])$, for any $m \in \mathbb{N}$ and then $M_1([a,b]) = 0$, $k_0([a,b]) = 1$, $k_2([a,b]) = b(1+b)+2$.

As consequences of Theorem 2 we get

Theorem 6. *For any $f \in C_2([0,+\infty))$ it follows*

$$\lim_{m\longrightarrow\infty} P_m f = f \tag{5.6}$$

uniformly on compact $[a,b]$ and there exists $m(0) \in \mathbb{N}$, $m(0)$ depending on b, so that

$$|(P_m f)(x) - f(x)| \leq (3+b(1+b))\,\omega\left(f;\frac{1}{\sqrt{m}}\right), m \in \mathbb{N}_2, m \geq m(0), x \in [a,b]. \tag{5.7}$$

Theorem 7. *Let $f \in C_2([0,+\infty))$. If $x \in [a,b]$, f is two times differentiable in x and $f^{(2)}$ is continuous in x, then*

$$\lim_{m\longrightarrow\infty} m((P_m f)(x) - f(x)) = -\frac{1+x}{2} f^{(1)}(x) + \frac{x(1+x)}{2} f^{(2)}(x). \tag{5.8}$$

Proof of Theorem 7. Taking Lemma 2 into account and applying (3.29), (5.8) is obtained. □

6. $(P_m)_{m\geq m_0}$ OPERATORS PRESERVING THE TEST FUNCTIONS e_1 AND e_2

In this case $m_0 \in \mathbb{N}$, $m_0 \geq 2$ is a fixed number and $J = \left[\frac{1}{m_0-1}, +\infty\right)$. If $P_m e_1 = e_1$, for any $m \in \mathbb{N}_1$, yields $v_m(x) = 0$, for any $m \in \mathbb{N}_1$ and any $x \in \left[\frac{1}{m_0-1}, +\infty\right)$. For $x \geq \frac{1}{m_0-1}$, we have $\frac{mx-1}{x+1} \geq \frac{m-m_0+1}{m_0}$ because the function $\frac{x+1}{mx-1}$ is decreasing on $\left[\frac{1}{m_0-1}, +\infty\right)$, from where $\frac{mx-1}{x+1} > 0$ for any $m \in \mathbb{N}_1$ and any $x \in \left[\frac{1}{m_0-1}, +\infty\right)$. Taking (3.12) into account, from $P_m e_1 = e_1$ and $P_m e_2 = e_2$ for any $m \in \mathbb{N}_1$, we have $\frac{m+1}{m}\frac{x^2}{1+u_m(x)} + \frac{x}{m} = x^2$, for any $x \in \left[\frac{1}{m_0-1}, +\infty\right)$, from where

$$u_m(x) = \frac{x+1}{mx-1}, m \in \mathbb{N}_1, x \in \left[\frac{1}{m_0-1}, +\infty\right). \tag{6.1}$$

Then the operators from (3.11) become

$$(P_m f)(x) \tag{6.2}$$
$$= \frac{(m+1)x}{mx-1}\sum_{k=0}^{\infty}\binom{m+k-1}{k}\left(\frac{mx-1}{m+1}\right)^k\left(1+\frac{x-1}{m+1}\right)^{-m-k} f\left(\frac{k}{m}\right)$$

for $m \in \mathbb{N}_1$, $x \in \left[\frac{1}{m_0-1}, +\infty\right)$ and $f \in C_2([0,+\infty))$.

According to the notations from Section 3, we have $b_m\left(\left[\frac{1}{m_0-1}, b\right]\right) = 0, l(x) = -1-x$, for any $m \in \mathbb{N}_1$, and because the function $u_m(x) = \frac{x+1}{mx-1}$ is decreasing on $\left[\frac{1}{m_0-1}, +\infty\right)$, we get that

$$u_m(x) \leq \frac{m_0}{m-m_0+1} = a_m\left(\left[\frac{1}{m_0-1}, b\right]\right)$$

for any $x \in \left[\frac{1}{m_0-1}, b\right)$ and $M_2\left(\left[\frac{1}{m_0-1}, b\right]\right) = 0$. Then $k_0 = 1+m_0$, $k_2 = b(1+b)+2$ and $M_1\left(\left[\frac{1}{m_0-1}, b\right]\right) = m_0$.

Theorem 8. *For any $f \in C_2([0,+\infty))$ it follows*

$$\lim_{m\longrightarrow\infty} P_m f = f \tag{6.3}$$

uniformly on the compact $\left[\frac{1}{m_0-1}, b\right]$ and there exists $m(0) \in \mathbb{N}$ depending on b, such that

$$|(P_m f)(x) - f(x)| \leq \frac{m_0}{m-m_0+1} M\left(\left[\frac{1}{m_0-1}, b\right]\right) + \tag{6.4}$$

$$+(3+m_0+b(1+b))\omega\left(f;\frac{1}{\sqrt{m}}\right)$$

for any $m\in\mathbb{N}_2$, $m\geq m(0)$ *and* $x\in\left[\frac{1}{m_0-1},b\right]$, *where*

$$M\left(\left[\frac{1}{m_0-1},b\right]\right)=sup\left\{|f(x)|\mid x\in\left[\frac{1}{m_0-1},b\right]\right\}.$$

Proof of Theorem 8. It results immediately from Theorem 2. □

Theorem 9. *Let* $f\in C_2([0,+\infty)$. *If* $x\in\left[\frac{1}{m_0-1},b\right]$, f *is two times differentiable in* x *and* $f^{(2)}$ *is continuous in* x, *then*

$$\lim_{m\longrightarrow\infty}m((P_mf)(x)-f(x))=\frac{1+x}{x}f(x)-(1+x)f^{(1)}(x)+\frac{x(1+x)}{2}f^{(2)}(x). \tag{6.5}$$

Proof of Theorem 9. We have $\lim_{m\longrightarrow\infty}mu_m(x)=\frac{1+x}{x},l(x)=-1-x$, for any $x\in\left[\frac{1}{m_0-1},b\right]$ and taking (3.29) into account, follows (6.5). □

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Authors' addresses

Adrian D. Indrea
Tehnological High School of Tarna Mare 22 tăşnadului Street 440114 Satu Mare, Romania
E-mail address: adrian.indrea@yahoo.com

Ovidiu T. Pop
National College "Mihai Eminescu" 5 Mihai Eminescu Street 440014 Satu Mare, Romania
E-mail address: ovidiutiberiu@yahoo.com

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ON GENERALIZED HUMBERT MATRIX POLYNOMIALS

LEVENT KARGIN AND VELI KURT

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Abstract. Aktaş et. al. in [3] introduced the generalized Humbert matrix polynomials (G-HMP) $P_n^A(m,x,y,c)$. In this paper we focus on some properties of these matrix polynomials such as matrix recurrence relations, matrix differential equation and an integral representation. We introduce generalized forms of operational rules associated with operators corresponding to a G-HMP expansions. Moreover, we obtain a series transformation formula involving Gegenbauer matrix polynomials. Then we provide a number of applications.

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1. INTRODUCTION

Classical Humbert polynomials, which are generalized forms of Gegenbauer, Legendre, second kind Chebyshev, Pincherle polynomials, are defined in [16] and studied by many mathematicians. Through the last two decades, matrix polynomials have comprised an emerging field of study with important results in both theory and applications continuing to appear in the literature. Extension of the matrix framework of the classical families of polynomials are introduced [12,13,18,20,23,29] and several properties of the these matrix polynomials are established [4–6,10,11,25,31,37]. Generalized Humbert matrix polynomials are introduced in [3]. Furthermore various families of bilinear and bilateral generating matrix functions and some relations of these matrix polynomials are presented [1,2]. By means of this study, choosing appropriate parameter we get both the known and the unknown properties of second kind Chebyshev, Legendre, Pincherle, Gegenbauer matrix polynomials which are introduced to date. The organization of this paper is as follows. In Section 2, starting from the generating matrix function, some matrix recurrence relations are given. Also, these polynomials appear as finite series solutions of $m\ th$-order matrix differential equation are demonstrated. Section 3 deals with an integral representation and operational identities for G-HMP. Finally in Section 4, a series transformation formula for Gegenbauer matrix polynomials is established and some applications are given.

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Throughout this paper, the zero matrix and identity matrix will be denoted by 0 and I, respectively. If A is a matrix in $\mathbb{C}^{r\times r}$, its spectrum $\sigma(A)$ denotes the set of all eigenvalues of A. Its 2-norm is denoted by $\|A\|$ and defined by

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2},$$

where for a vector y in $\mathbb{C}^r$, $\|y\|_2 = \left(y^T, y\right)^{\frac{1}{2}}$ is the Euclidean norm of y. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane and A is a matrix in $\mathbb{C}^{r\times r}$ such that $\sigma(A)\subset\Omega$, then from the properties of matrix functional calculus in [15, p. 558], it follows that $f(A)g(A) = g(A)f(A)$. For a matrix A in $\mathbb{C}^{r\times r}$ the authors give the following relation due to [21]

$$g(A) = (1-y)^{-A} = \sum_{n=0}^{\infty} \frac{(A)_n}{n!} y^n, \quad |y|<1, \tag{1.1}$$

where $(A)_n$ is the Pochhammer symbol or shifted factorial which is defined by

$$(A)_n = A(A+I)(A+2I)\ldots(A+(n-1)I), \quad n\geq 1, \tag{1.2}$$

with $(A)_0 = I$ (see [21]). Moreover, from [22] for a matrix A in $\mathbb{C}^{r\times r}$ such that $A+nI$ is invertible for every integer $n\geq 0$ we have

$$(A)_n = \Gamma(A+nI)\Gamma^{-1}(A), \tag{1.3}$$

where $\Gamma(A)$ is the gamma matrix function which is defined by [22]

$$\Gamma(A) = \int_0^{\infty} e^{-t} t^{A-I}\, dt. \tag{1.4}$$

Let A be an arbitrary matrix in $\mathbb{C}^{r\times r}$. Then the Humbert matrix polynomials $P_n^A(m,x,y,c)$ are defined by generating matrix function

$$\sum_{n=0}^{\infty} P_n^A(m,x,y,c)\, t^n = \left(c-mxt+yt^m\right)^{-A} = F(x,y,t,c,A), \tag{1.5}$$

where $|mxt-yt^m|<|c|$, m is a positive integer and the other parameters are unrestricted in general (see [3]). Also these matrix polynomials satisfy the following explicit expression

$$P_n^A(m,x,y,c) = \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-1)^k c^{-(A+(n-mk+k)I)} (A)_{n-mk+k}}{k!(n-mk)!} (mx)^{n-mk} y^k. \tag{1.6}$$

It is clear from (1.6) that

$$P_0^A(m,x,y,c) = c^{-A}, \quad P_1^A(m,x,y,c) = \begin{cases} c^{-(A+I)}Amx & ;m>1 \\ c^{-(A+I)}A(x-y) & ;m=1 \end{cases},$$

$$P_n^A(m,x,y,c) = c^{-A} P_n^A\left(m, \frac{x}{c}, \frac{y}{c}\right),$$

and

$$P_{nm}^A(m,0,y,c) = \frac{(-1)^n c^{-(A+nI)} (A)_n y^n}{n!},$$

$$P_n^A(m,x,0,c) = \frac{c^{-(A+nI)} (A)_n (mx)^n}{n!}.$$

For the special case of (1.6), we consider $P_n^A(m,x,y,c)$ as follows

$$P_n^A(2,x,y,1) = C_n^A(x,y),$$

$$P_n^{[1]_{1\times 1}}\left(2, x\sqrt{\frac{A}{2}}, y, 1\right) = U_n(x,y,A),$$

$$P_n^{[\frac{1}{2}]_{1\times 1}}\left(2, x\sqrt{\frac{A}{2}}, 1, 1\right) = P_n(x,A),$$

$$P_n^A(3,x,1,1) = h_n^A(x),$$

where $C_n^A(x,y)$: Gegenbauer matrix polynomials with two variables [19], $U_n(x,y,A)$: second kind Chebyshev matrix polynomials with two variables [7], $P_n(x,A)$: Legendre matrix polynomials [36], $h_n^A(x)$: Pincherle matrix polynomials [27].

2. Recurrence Relations

In this section we obtain matrix recurrence relations and $m\,th$-order matrix differential equation for G-HMP.

Let us get some matrix recurrence relations for G-HMP. Differentiating (1.5) with respect to c,x,y,t we have

$$\frac{\partial}{\partial c} F(x,y,t,c,A) = \frac{-A}{c-mxt+yt^m} F(x,y,t,c,A), \tag{2.1}$$

$$\frac{\partial}{\partial x} F(x,y,t,c,A) = \frac{t}{c-mxt+yt^m} mAF(x,y,t,c,A), \tag{2.2}$$

$$\frac{\partial}{\partial y} F(x,y,t,c,A) = \frac{-t^m}{c-mxt+yt^m} AF(x,y,t,c,A), \tag{2.3}$$

$$\frac{\partial}{\partial t} F(x,y,t,c,A) = \frac{\left(x-yt^{m-1}\right)}{c-mxt+yt^m} mAF(x,y,t,c,A). \tag{2.4}$$

We can rewrite (2.2) and (2.3) in the from

$$mA\left(c-mxt+yt^m\right)^{-(A+I)} = \sum_{n=0}^{\infty} \frac{\partial}{\partial x} P_{n+1}^A(m,x,y,c)\, t^n, \tag{2.5}$$

and

$$-A\left(c-mxt+yt^{m}\right)^{-(A+I)}=\sum_{n=0}^{\infty}\frac{\partial}{\partial y}P_{n+m}^{A}\left(m,x,y,c\right)t^{n}, \tag{2.6}$$

respectively. Applying (1.5) to the left-hand sides of equations (2.5) and (2.6) we have

$$\sum_{n=0}^{\infty}mAP_{n}^{A+I}\left(m,x,y,c\right)t^{n}=\sum_{n=0}^{\infty}\frac{\partial}{\partial x}P_{n+1}^{A}\left(m,x,y,c\right)t^{n},$$
$$\sum_{n=0}^{\infty}-AP_{n}^{A+I}\left(m,x,y,c\right)t^{n}=\sum_{n=0}^{\infty}\frac{\partial}{\partial y}P_{n+m}^{A}\left(m,x,y,c\right)t^{n}.$$

Comparing the coefficients of t^{n} in the above equations we arrive at

$$\frac{\partial}{\partial x}P_{n}^{A}\left(m,x,y,c\right)=mAP_{n-1}^{A+I}\left(m,x,y,c\right),\quad n\geq 1, \tag{2.7}$$
$$\frac{\partial}{\partial y}P_{n}^{A}\left(m,x,y,c\right)=-AP_{n-m}^{A+I}\left(m,x,y,c\right),\quad n\geq m$$

By iteration we obtain two differential relations for G-HMP

$$\frac{\partial^{r}}{\partial x^{r}}P_{n}^{A}\left(m,x,y,c\right)=m^{r}\left(A\right)_{r}P_{n-r}^{A+rI}\left(m,x,y,c\right), \tag{2.8}$$
$$\frac{\partial^{r}}{\partial y^{r}}P_{n}^{A}\left(m,x,y,c\right)=(-1)^{r}\left(A\right)_{r}P_{n-mr}^{A+rI}\left(m,x,y,c\right). \tag{2.9}$$

Therefore, we have

$$\frac{\partial^{r}}{\partial x^{r}}P_{n}^{A}\left(m,x,y,c\right)+(-1)^{r-1}m^{r}\frac{\partial^{r}}{\partial y^{r}}P_{n+(m-1)r}^{A}\left(m,x,y,c\right)=\mathbf{0}. \tag{2.10}$$

Considering equations (2.1) and (2.2), and applying the same method we get

$$\frac{\partial^{r}}{\partial x^{r}}P_{n}^{A}\left(m,x,y,c\right)=(-1)^{r}m^{r}\frac{\partial^{r}}{\partial c^{r}}P_{n-r}^{A}\left(m,x,y,c\right),\quad n\geq r. \tag{2.11}$$

Also, from equations (2.1) and (2.3) we have

$$\frac{\partial^{r}}{\partial y^{r}}P_{n}^{A}\left(m,x,y,c\right)=\frac{\partial^{r}}{\partial c^{r}}P_{n-mr}^{A}\left(m,x,y,c\right),\quad n\geq mr. \tag{2.12}$$

Let us consider the equations (2.2) and (2.4). Then we have

$$\left(x-yt^{m-1}\right)\frac{\partial}{\partial x}F\left(x,y,t,c,A\right)-t\frac{\partial}{\partial t}F\left(x,y,t,c,A\right)=\mathbf{0}.$$

Using (1.5) we get

$$\sum_{n=0}^{\infty}x\frac{\partial}{\partial x}P_{n}^{A}\left(m,x,y,c\right)t^{n}-\sum_{n=0}^{\infty}y\frac{\partial}{\partial x}P_{n}^{A}\left(m,x,y,c\right)t^{n+m-1}$$

$$= \sum_{n=1}^{\infty} n P_n^A(m,x,y,c)\, t^n.$$

Since $\frac{\partial}{\partial x} P_0^A(m,x,y,c) = 0$, comparing the coefficients of t^n yields

$$x \frac{\partial}{\partial x} P_n^A(m,x,y,c) - n P_n^A(m,x,y,c) = y \frac{\partial}{\partial x} P_{n+1-m}^A(m,x,y,c). \tag{2.13}$$

Furthermore, from equations (2.2) and (2.4), we get

$$\frac{mA(c-mxt+yt^m)^{-A}}{c-mxt+yt^m} = \sum_{n=0}^{\infty} \frac{\partial}{\partial x} P_n^A(m,x,y,c)\, t^{n-1} \tag{2.14}$$

and

$$\frac{\left(x-yt^{m-1}\right) mA(c-mxt+yt^m)^{-A}}{(c-mxt+yt^m)} = \sum_{n=1}^{\infty} n P_n^A(m,x,y,c)\, t^{n-1}, \tag{2.15}$$

respectively. Thus, multiplying (2.14) by $1-y(m-1)t^m$ and (2.15) by mt and substracting them we get

$$y(m-1) \frac{\partial}{\partial x} P_{n+1-m}^A(m,x,y,c) = \frac{\partial}{\partial x} P_{n+1}^A(m,x,y,c) - m(A+nI) P_n^A(m,x,y,c). \tag{2.16}$$

Using (2.13) in the above equation we have

$$\frac{\partial}{\partial x} P_{n+1}^A(m,x,y,c) = (m-1)x \frac{\partial}{\partial x} P_n^A(m,x,y,c) + (mA+nI) P_n^A(m,x,y,c). \tag{2.17}$$

Finally, applying (2.7) in equations (2.16) and (2.17) respectively we obtain two more matrix recurrence relations

$$m(A+nI) P_n^A(m,x,y,c) = mAP_n^{A+I}(m,x,y,c) - m(m-1)AyP_{n-m}^{A+I}(m,x,y,c), \quad n \geq m, \tag{2.18}$$

$$(mA+nI) P_n^A(m,x,y,c) = mAP_n^{A+I}(m,x,y,c) - m(m-1)AxP_{n-1}^{A+I}(m,x,y,c), \quad n \geq 1. \tag{2.19}$$

From (2.4) we get

$$\left(c-mxt+yt^m\right) \frac{\partial}{\partial t} F(x,y,t,c,A) = \left(x-yt^{m-1}\right) mAF(x,y,t,c,A).$$

Using (1.5) in the above equation we obtain

$$\sum_{n=1}^{\infty} \left(c-mxt+yt^m\right) nP_n^A(m,x,y,c)\, t^{n-1}$$

$$= \sum_{n=0}^{\infty} \left(x - yt^{m-1}\right) mAP_n^A(m,x,y,c)\, t^n.$$

By making appropriate changes of indices and comparing the coefficients of t^n, we obtain a three-term matrix recurrence relation for G-HMP

$$c(n+1)\, P_{n+1}^A(m,x,y,c) - mx(A+nI)\, P_n^A(m,x,y,c) \tag{2.20}$$
$$+ y[mA + (n-m+1)I]\, P_{n+1-m}^A(m,x,y,c) = \mathbf{0}, \quad n \geq m-1.$$

Summary of these results is given in the following theorem.

Theorem 1. *Let A be an arbitrary matrix in $\mathbb{C}^{r\times r}$ and m be a positive integer. Then G-HMP satisfy the differential relation (2.10) and recurrence relations (2.18), (2.19) and (2.20).*

Before concluding this section we discuss the matrix differential equation of G-HMP. Let the sequence $(f_r)_{r=0}^n$ be given by $f_r = f(r)$, where

$$f(t) = (n-t)\left(\frac{(n-t)I + m(A+tI)}{m}\right)_{m-1}.$$

Also we introduce two standard difference operators: the forward difference operator Δ and the displacement (or shift) operator E are given as

$$\Delta f_r = f_{r+1} - f_r \quad \text{and} \quad Ef_r = f_{r+1}$$

and their powers are given by

$$\Delta^0 f_r = f_r, \quad \Delta^k f_r = \Delta\left(\Delta^{k-1} f_r\right), \quad E^k f_r = f_{r+k}.$$

Theorem 2. *Let A be an arbitrary matrix in $\mathbb{C}^{r\times r}$ and m be a positive integer. Then G-HMP are a particular solution of the m th-order matrix differential equation in the form:*

$$c^{m-1} y \frac{\partial^m}{\partial x^m} P_n^A(m,x,y,c) + \sum_{s=0}^{m} a_s x^s \frac{\partial^s}{\partial x^s} P_n^A(m,x,y,c) = \mathbf{0}. \tag{2.21}$$

where the coefficients a_s are given by

$$a_s = \frac{m^{m-1}}{s!} \Delta^s f_0. \tag{2.22}$$

Proof. Let $n = mp + q$, where $p = \left[\frac{n}{m}\right]$ and $0 \leq q \leq m-1$. Differentiating (1.6) with respect to x, we get

$$x^s \frac{\partial^s}{\partial x^s} P_n^A(m,x,y,c) = \sum_{k=0}^{\left[\frac{n-s}{m}\right]} \frac{(-1)^k c^{-(A+(n-mk+k)I)} (A)_{n-(m-1)k}}{k!(n-mk-s)!} (mx)^{n-mk} y^k,$$

and

$$\frac{\partial^m}{\partial x^m} P_n^A(m,x,y,c) = \sum_{k=0}^{p-1} \frac{(-1)^k c^{-(A+(n-mk+k)I)} (A)_{n-(m-1)k} m^m}{k!(n-m(k+1))!} (mx)^{n-m(k+1)} y^k,$$

where $\left[\frac{n-s}{m}\right] = \begin{cases} p & ;s \le q \\ p-1 & ;s > q \end{cases}$. Substituting these expressions in the differential equation (2.21) and comparing the corresponding coefficients, we obtain

$$\sum_{s=0}^{m} \binom{n-mk}{s} s! a_s = m^m k (A + (n-(m-1)k) I)_{m-1} \tag{2.23}$$

where $k = 0, 1, 2, \cdots p-1$, and

$$\sum_{s=0}^{q} \binom{n-mp}{s} s! a_s = m^m p (A + (n-(m-1)k) I)_{m-1}. \tag{2.24}$$

(2.24) can be written as

$$\sum_{s=0}^{q} \binom{q}{s} m^{m-1} \Delta^s f_0 = m^{m-1} (n-q) \left(A + \left(q + \frac{n-q}{m}\right) I\right)_{m-1}.$$

This equality holds because it is equivalent to

$$(1+\Delta)^q f_0 = E^q f_0 = f_q = f(q).$$

For $k = 0, 1, 2, \cdots p-1$,(2.23) can be written in the form

$$\sum_{s=0}^{m} \binom{n-mk}{s} \Delta^s f_0 = f_{n-mk}. \tag{2.25}$$

Since $f(t)$ is a polynomial of degree m, the last equality holds and (2.25) is a forward difference formula for f at the point $t = n-mk$. Thus, the proof is completed. □

Corollary 1. *Pincherle matrix polynomials are a solution of the third-order matrix differential equation in the form:*

$$\left(1 - \frac{32}{27}x^3\right) \frac{d^3}{dx^3} h_n^A(x) - \frac{16}{9}(2A+3I) x^2 \frac{d^2}{dx^2} h_n^A(x) - \frac{8}{27}(3n(2A+(n+1)I) - (3A+2I)(3A+5I)) x \frac{d}{dx} h_n^A(x) + \frac{8}{27} n (3A+nI)(3A+(n+1)I) h_n^A(x) = \mathbf{0}.$$

3. An Integral Representation and Its Applications

Firstly, let us discuss the extension of classical Lahiri generalized Hermite polynomials (G-LHP) which are specified by the series definition

$$H_{n,m}(x,y) = n! \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-1)^k y^k (mx)^{n-mk}}{k!(n-mk)!} \tag{3.1}$$

and by the generating function

$$\sum_{n=0}^{\infty} H_{n,m}(x,y) \frac{t^n}{n!} = \exp\left(mxt - yt^m\right). \tag{3.2}$$

It is easy to show that

$$H_{n,m}(x,y) = y^{\frac{n}{m}} H_n\left(\frac{x}{\sqrt[m]{y}}\right) \text{ and } H_{n,m}(x,1) = H_{n,m}(x),$$

where $H_{n,m}(x)$ is the G-LHP given in [30]. In particular, for $m=2$, $H_{n,2}(x,1) = H_n(x)$, where $H_n(x)$ denotes the ordinary Hermite polynomials [35]. Moreover, $H_{n,m}(x,y) = g_n^m(mx,-y)$, where $g_n^m(x,y)$ is the Gould–Hopper generalized Hermite polynomials (G-HGHP) [17].

Using the summation formula for G-HGHP in [28, Theorem 2.1 p. 1538] G-LHP satisfy

$$H_{n+s,m}(w,y) = \sum_{k,r=0}^{n,s} \binom{n}{k}\binom{s}{r} m^{k+r} (w-x)^{k+r} H_{n+s-k-r,m}(x,y). \tag{3.3}$$

In the following theorem we give an integral representation for G-HMP including G-LHP.

Theorem 3. *Let A be a matrix in $\mathbb{C}^{r\times r}$ such that $\operatorname{Re}(z) > 0$ for all $z \in \sigma(A)$, m be a positive integer and c be a complex number such that $\operatorname{Re}(c) > 0$. Then G-HMP have the integral representation*

$$P_n^A(m,x,y,c) = \frac{\Gamma^{-1}(A)}{n!} \int_0^{\infty} e^{-ct} t^{A+(n-1)I} H_{n,m}\left(x, \frac{y}{t^{m-1}}\right) dt. \tag{3.4}$$

Proof. Using (3.1) in the right-hand side of (3.4) we have

$$\begin{aligned}
&\frac{\Gamma^{-1}(A)}{n!} \int_0^{\infty} e^{-ct} t^{A+(n-1)I} H_{n,m}\left(x, \frac{y}{t^{m-1}}\right) dt \\
&= \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-1)^k c^{-(A+(n-mk+k)I)} (mx)^{n-mk} y^k}{k!(n-mk)!} \Gamma^{-1}(A)\, \Gamma(A+(n-mk+k)).
\end{aligned}$$

From (1.3) we get the proof. □

As an application of this theorem we give the following proposition for G-HMP.

Proposition 1. *G-HMP satisfy the summation formula*

$$P_{n+s}^{A}(m,w,y,c)=\sum_{k,r=0}^{n,s}\frac{\binom{n}{k}\binom{s}{r}(A)_{k+r}\,m^{k+r}(w-x)^{k+r}}{\binom{n+s}{k+r}(k+r)!}P_{n+s-k-r}^{A+(k+r)I}(m,x,y,c). \tag{3.5}$$

Proof. Replacing n with $n+s$ in (3.4) and using the summation formula (3.3) we deduce

$$P_{n+s}^{A}(m,w,y,c)=\sum_{k,r=0}^{n,s}\frac{(-1)^{k+r}\binom{n}{k}\binom{s}{r}(n+s-k-r)!m^{k+r}(w-x)^{k+r}}{(n+s)!}\frac{\partial^{k+r}}{\partial c^{k+r}}P_{n+s-k-r}^{A}(m,x,y,c).$$

From (2.8) we obtain the result. □

Taking $k=0$ in assertion (3.5) of Proposition 1, we deduce the following consequence of Proposition 1.

Corollary 2. *The following summation formula for the G-HMP holds*

$$P_{n}^{A}(m,w,y,c)=\sum_{k=0}^{n}\frac{(A)_k\,m^k(w-x)^k}{k!}P_{n-k}^{A+kI}(m,x,y,c). \tag{3.6}$$

Remark 1. Replacing w by $w+x$ in (3.6), we obtain

$$P_{n}^{A}(m,x+w,y,c)=\sum_{k=0}^{n}\frac{(A)_k\,(mw)^k}{k!}P_{n-k}^{A+kI}(m,x,y,c).$$

Now, we try to understand more deeply the role played by the integral transform connecting G-LHP and G-HMP. It is obvious that both $H_{n,m}(x,y)$ and $P_n^A(m,x,y,c)$ reduce to ordinary form for $y=1$.

It is easy to find that

$$P_{n}^{A}(m,x,c)=\frac{\Gamma^{-1}(A)}{n!}\int_0^\infty e^{-ct}t^{A+(\frac{n}{m}-1)I}H_{n,m}\left(\frac{xt}{\sqrt[m]{t}}\right)dt. \tag{3.7}$$

Using the fact that

$$\exp\left(\lambda x\frac{d}{dx}\right)f(x)=f(x\exp\lambda),$$

where $f(x)$ is an appropriate function, we have

$$P_{n}^{A}(m,x,c)=\frac{\Gamma^{-1}(A)}{n!}\int_0^\infty e^{-ct}t^{A+(\frac{n}{m}+(1-\frac{1}{m})x\frac{d}{dx}-1)I}dtH_{n,m}(x).$$

From (1.4) we acquire the following theorem.

Theorem 4. *Let A be a matrix in $\mathbb{C}^{r\times r}$ such that $\operatorname{Re}(z)>0$ for all $z\in\sigma(A)$, m be a positive integer and c be a complex number such that $\operatorname{Re}(c)>0$. Then G-HMP satisfy*

$$\Gamma(A)\,n!P_n^A(m,x,c)=\exp\left(-\widehat{Q}\ln c\right)\Gamma\left(\widehat{Q}\right)H_{n,m}(x), \tag{3.8}$$

where $\widehat{Q}=\left[A+\left(\frac{n}{m}+\left(1-\frac{1}{m}\right)x\frac{d}{dx}\right)I\right]$.

We conclude this section giving another representation for the G-HMP.

Theorem 5. *Let A be an arbitrary matrix in $\mathbb{C}^{r\times r}$ and m be a positive integer. Then G-HMP satisfy the identity*

$$P_n^A(m,x,y,c)=\frac{(A)_n}{n!}\exp\left[y(-m)^{-m}\widehat{D}_c^{-(m-1)}\frac{\partial^m}{\partial x^m}\right]c^{-(A+nI)}(mx)^n, \tag{3.9}$$

where $\widehat{D}_c^{-1}$ denotes the inverse derivative operator (see [14] for details).

Proof. Identities (2.11) and (2.12) allow us to conclude that

$$(-m)^m\frac{\partial^m}{\partial c^{m-1}\partial y}P_n^A(m,x,y,c)=\frac{\partial^m}{\partial x^m}P_n^A(m,x,y,c).$$

The last identity and the fact that

$$P_n^A(m,x,0,c)=\frac{(A)_n}{n!}c^{-(A+nI)}(mx)^n,$$

give the proof. □

We note that taking $A=[1]_{1\times1}$, $m=2$ and replacing xI by $x\sqrt{\frac{A}{2}}$, the expressions (3.8) and (3.9) coincide with the formulas which was given in [24] for the second kind Chebyshev matrix polynomials.

4. Some Results for Gegenbauer Matrix Polynomials

In this section we deal with the integral representation for Gegenbauer matrix polynomials including classical Hermite polynomials. Using this representation we give a generating function of Gegenbauer matrix polynomials with two-variables. As a main theorem of this section we obtain a series transformation formula involving Gegenbauer matrix polynomials. We then provide a number of applications using appropriate binomial transformations.

Taking $c=1$, $m=2$ and replacing n with $n+p$ in (3.4) we deduce

$$C_{n+p}^A(x,y)=\frac{\Gamma^{-1}(A)}{(n+p)!}\int_0^\infty e^{-t}t^{A+(n+p-1)I}H_{n+p}\left(x,\frac{y}{t}\right)dt,$$

Multiplying both hand sides of above equation by $\frac{(n+p)!}{n!p!}z^n$ and summing over n from zero to infinity, we get

$$\sum_{n=0}^{\infty}\frac{(n+p)!}{n!p!}C_{n+p}^{A}(x,y)\,z^{n}$$
$$=\frac{\Gamma^{-1}(A)}{p!}\int_{0}^{\infty}e^{-t}t^{A+(p-1)I}\sum_{n=0}^{\infty}H_{n+p}\left(x,\frac{y}{t}\right)\frac{(zt)^{n}}{n!}dt.$$

Using identity [32]

$$\sum_{n=0}^{\infty}H_{n+p}(x,y)\frac{z^{n}}{n!}=\exp\left(2xz-yz^{2}\right)H_{p}(x-yz,y),\tag{4.1}$$

we have

$$\sum_{n=0}^{\infty}\frac{(n+p)!}{n!p!}C_{n+p}^{A}(x,y)\,z^{n}$$
$$=\frac{\Gamma^{-1}(A)}{p!}\int_{0}^{\infty}e^{-t\left(1-2xz+yz^{2}\right)}t^{A+(p-1)I}H_{p}\left(x-yz,\frac{y}{t}\right)dt.$$

This gives the following theorem for Gegenbauer matrix polynomials with two-variables.

Theorem 6. *Let A be a matrix in $\mathbb{C}^{r\times r}$ such that $\mathrm{Re}(z)>0$ for all $z\in\sigma(A)$, p be a positive integer. Then Gegenbauer matrix polynomials with two-variables have the generating function*

$$\sum_{n=0}^{\infty}\binom{n+p}{p}C_{n+p}^{A}(x,y)\,z^{n}\tag{4.2}$$
$$=\left(1-2xz+yz^{2}\right)^{-(A+pI)}C_{p}^{A}\left(x-yz,\left(1-2xz+yz^{2}\right)y\right)$$

where $\left|2xz-yz^{2}\right|<1$.

We note that for the case $y=1$ the expression (4.2) coincides with the formula which was proved by Khammash ([26]) for the Gegenbauer matrix polynomials.

Before giving the main theorem of this section we need the following lemma.

Lemma 1 ([8, p. 386, Theorem 1.1])**.** *The following series transformation formula holds*

$$\sum_{n=0}^{\infty}a_{n}H_{n}(x)\frac{t^{n}}{n!}=e^{2xt-t^{2}}\sum_{n=0}^{\infty}(-1)^{n}H_{n}(x-t)\frac{t^{n}}{n!}\left\{\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}a_{k}\right\},$$

where

$$f(t)=\sum_{k=0}^{\infty}a_{k}t^{k},$$

is an arbitrary function, analytical in a neighborhood of zero.

Theorem 7. *Let A be a matrix in $\mathbb{C}^{r\times r}$ such that $\operatorname{Re}(z)>0$ for all $z\in\sigma(A)$. Then Gegenbauer matrix polynomials have the series transformation formula*

$$\sum_{n=0}^{\infty} a_n C_n^A(x) z^n = \rho^{-A} \sum_{n=0}^{\infty} (-1)^n C_n^A\left(\frac{x-z}{\rho}, \frac{1}{\rho}\right) z^n \left\{\sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k\right\}, \tag{4.3}$$

where a_k are the Maclauren coefficients of an arbitrary function $f(t)$, $\rho = 1-2xz+yz^2$ and $|1-\rho|<1$.

Proof. For $c=1$ and $m=2$,(3.7) reduce to

$$C_n^A(x) = \frac{\Gamma^{-1}(A)}{n!} \int_0^{\infty} e^{-t} t^{A+(\frac{n}{2}-1)I} H_n\left(x\sqrt{t}\right) dt.$$

Multiplying both hand sides of the above equation by $a_n z^n$ and summing over n from zero to infinity, we get

$$\sum_{n=0}^{\infty} a_n C_n^A(x) z^n = \sum_{n=0}^{\infty} (-1)^n C_n^A(x-z,\rho) z^n \left\{\sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k\right\},$$

Using this equation and

$$C_n^A(x,c) = c^{-A} C_n^A\left(\frac{x}{c}, \frac{1}{c}\right),$$

for $c = 1-2xz+yz^2$, we get the proof. □

Note that $C_n^A\left(\frac{x-z}{\rho}, \frac{1}{\rho}\right)$ are Gegenbauer matrix polynomials with two-variables. Our first application is the following.

Corollary 3. *Let $B\in\mathbb{C}^{r\times r}$ and C be matrix in $\mathbb{C}^{r\times r}$ such that $C+kI$ is invertible for all integer $k\geq 0$. Then we obtain bilateral series with hypergeometric matrix functions and Gegenbauer matrix polynomials*

$$\sum_{n=0}^{\infty} C_n^A(x) (B)_n [(C)_n]^{-1} (zy)^n = \rho^{-A} \sum_{n=0}^{\infty} (-1)^n C_n^A\left(\frac{x-z}{\rho}, \frac{1}{\rho}\right) F(-nI, B; C; y) z^n. \tag{4.4}$$

Proof. Hypergeometric matrix functions have the representation [21]

$$F(A,B;C;y) = \sum_{k\geq 0} \frac{(A)_k (B)_k [(C)_k]^{-1}}{k!} y^k \quad |y|<1,$$

where A, B, C are matrices in $\mathbb{C}^{r\times r}$ such that $C+kI$ is invertible for all integer $k \geq 0$. Setting $A = -nI$, where n is a natural number, we get

$$F(-nI, B; C; y) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (B)_k [(C)_k]^{-1} y^k$$

Applying (4.3) with $a_k = (B)_k [(C)_k]^{-1} y^k$ we obtain (4.4). □

Now, we can obtain bilateral generating function with hypergeometric matrix functions and Gegenbauer matrix polynomials with two-variables. Taking $C = 2A$ in (4.4) and using the generating function [5]

$$\sum_{n=0}^{\infty} (B)_n [(2A)_n]^{-1} C_n^A(x) r^n = (1-xr)^{-B} F(\frac{B}{2}, \frac{B}{2}+\frac{I}{2}; A+\frac{I}{2}; \frac{(x^2-1)r^2}{(1-xr)^2}),$$

we have

$$\sum_{n=0}^{\infty} (-1)^n F(-nI, B; 2A; y) C_n^A\left(\frac{x-z}{\rho}, \frac{1}{\rho}\right) z^n$$

$$= (1-xyz)^{-B} \exp(A\ln\rho) F(\frac{B}{2}, \frac{B}{2}+\frac{I}{2}; A+\frac{I}{2}; \frac{(x^2-1)(yz)^2}{(1-xyz)^2}),$$

where $AB = BA$.

For the next corollary we use the binomial transform

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{p+k}{k} = (-1)^n \binom{p}{n}$$

which is a version of the Vandermonde convolution formula (see [33]). Here, p is any complex number. The generating function for $a_k = \binom{p+k}{k}$ is

$$(1-t)^{-p-1} = \sum_{k=0}^{\infty} \binom{p+k}{k} t^k.$$

According to (4.3) we obtain the following corollary.

Corollary 4. *For any complex number p*

$$\sum_{n=0}^{\infty} \binom{p+n}{n} C_n^A(x) z^n = \rho^{-A} \sum_{n=0}^{\infty} \binom{p}{n} C_n^A\left(\frac{x-z}{\rho}, \frac{1}{\rho}\right) z^n.$$

It is interesting that when p is a positive integer, the right-hand side is finite. So, we have the closed form evaluation

$$\sum_{n=0}^{\infty} \binom{p+n}{n} C_n^A(x) z^n = \rho^{-A} \sum_{n=0}^{p} \binom{p}{n} C_n^A\left(\frac{x-z}{\rho}, \frac{1}{\rho}\right) z^n. \tag{4.5}$$

We have the equation for Stirling numbers of the second kind extended for complex argument [9]

$$(-1)^n n! \left\{ \begin{matrix} \alpha \\ n \end{matrix} \right\} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k k^{\alpha},$$

where $0 \neq \alpha \in \mathbb{C}$. Using this equation with $a_k = k^{\alpha}$ we have the following corollary.

Corollary 5. *For any complex number* $\alpha \neq 0$

$$\sum_{n=0}^{\infty} n^{\alpha} C_n^A(x) z^n = \rho^{-A} \sum_{n=0}^{\infty} \left\{ \begin{matrix} \alpha \\ n \end{matrix} \right\} n! C_n^A\left(\frac{x-z}{\rho}, \frac{1}{\rho}\right) z^n.$$

Setting $\alpha = m \in \mathbb{Z} \cup \{0\}$ we have $\left\{ \begin{matrix} m \\ n \end{matrix} \right\} = 0$ for $m < n$. Therefore, we obtain the closed form evaluation

$$\sum_{n=0}^{\infty} n^{\alpha} C_n^A(x) z^n = \rho^{-A} \sum_{n=0}^{m} \left\{ \begin{matrix} m \\ n \end{matrix} \right\} n! C_n^A\left(\frac{x-z}{\rho}, \frac{1}{\rho}\right) z^n. \tag{4.6}$$

We note that taking $A = [\gamma]_{1\times 1}$ the expression (4.6) coincides with the formula which was given by Srivastava ([34]) for the classical Gegenbauer polynomials by different means.

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Authors' addresses

Levent Kargin
Akdeniz University, Akseki Vocational School, TR-07630 Antalya, Turkey
E-mail address: lkargin@akdeniz.edu.tr

Veli Kurt
Akdeniz University, Faculty of Science Department of Mathematics, TR-07058 Antalya, Turkey
E-mail address: vkurt@akdeniz.edu.tr

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PRINCIPAL FUNCTIONS OF MATRIX STURM–LIOUVILLE OPERATORS WITH BOUNDARY CONDITIONS DEPENDENT ON THE SPECTRAL PARAMETER

DENIZ KATAR, MURAT OLGUN, AND CAFER COSKUN

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Abstract. Let L denote operator generated in $L_2(\mathbb{R}_+, E)$ by the differential expression

$$l(y) = -y'' + Q(x)y,\ x \in \mathbb{R}_+ := [0, \infty),$$

and the boundary condition $(A_0 + A_1\lambda)Y'(0,\lambda) - (B_0 + B_1\lambda)Y(0,\lambda) = 0$, where Q is a matrix-valued function and A_0, A_1, B_0, B_1 are non-singular matrices, with $A_0B_1 - A_1B_0 \neq 0$. In this paper, we investigate the principal functions corresponding to the eigenvalues and the spectral singularities of L.

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Keywords: eigenvalues, spectral singularities, spectral analysis, Sturm–Liouville operator, non-selfadjoint matrix operator, principal functions

1. Introduction

Let us consider the boundary value problem (BVP)

$$-u'' + q(x)u = \lambda^2 u\ , \quad x \in \mathbb{R}_+\ , \tag{1.1}$$

$$u(0) = 0\ , \tag{1.2}$$

in $L^2(\mathbb{R}_+)$, where q is a complex-valued function. The spectral theory of the BVP (1.1)–(1.2) with continuous and point spectrum was investigated by Naimark [20]. He showed the existence of the spectral singularities in the continuous spectrum of the BVP (1.1)–(1.2). Note that the eigenfunctions and the associated functions (principal functions) corresponding to the spectral singularities are not the elements of $L^2(\mathbb{R}_+)$. Also, the spectral singularities belong to the continuous spectrum and are the poles of the resolvent's kernel, but are not the eigenvalues of the BVP (1.1)–(1.2). The spectral singularities in the spectral expansion of the BVP (1.1)–(1.2) in terms of the principal functions have been investigated in [19]. The spectral analysis of the quadratic pencil of Schrödinger, Dirac and Klein-Gordon operators with spectral singularities were studied in [2–18]. The spectral analysis of the non-selfadjoint

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operator, generated in $L^2(\mathbb{R}_+)$ by (1.1) and the boundary condition

$$\frac{y'(0)}{y(0)} = \frac{\beta_1\lambda+\beta_0}{\alpha_1\lambda+\alpha_0},$$

where $\alpha_i,\ \beta_i \in \mathbb{C},\ i=0,\ 1$ with $\alpha_0\beta_1-\alpha_1\beta_0 \neq 0$ were investigated in detail by Bairamov et al. [9]. The all above mentioned papers related with the differential and difference equations are scalar coefficients. Spectral analysis of the self-adjoint differential and difference equations with matrix coefficients are studied in [3,10–13].

Let E be an n-dimensional ($n<\infty$) Euclidian space with the norm $\|.\|$ and let us introduce the Hilbert space $L^2(\mathbb{R}_+,E)$ consisting of vector-valued functions with the values in E. We will consider the BVP

$$-y''+Q(x)y=\lambda^2 y\ ,\quad x\in\mathbb{R}_+\ , \tag{1.3}$$

$$y(0)=0, \tag{1.4}$$

in $L^2(\mathbb{R}_+,E)$ where Q is a non-selfadjoint matrix-valued function (i. e., $Q\neq Q^*$). It is clear that, the BVP (1.3)–(1.4) is nonselfadjoint. In [14, 21] discrete spectrum of the non-selfadjoint matrix Sturm–Liouville operator and properties of the principal functions corresponding to the eigenvalues and the spectral singularities were investigated.

Let us consider the BVP in $L_2(\mathbb{R}_+,E)$

$$-y''+Q(x)y=\lambda^2 y,\quad x\in\mathbb{R}_+, \tag{1.5}$$

$$(A_0+A_1\lambda)y'(0,\lambda)-(B_0+B_1\lambda)y(0,\lambda)=0, \tag{1.6}$$

where Q is a non-singular matrix-valued function and $A_0,\ A_1,\ B_0,\ B_1$ are non-selfadjoint matrices such $A_0B_1-A_1B_0\neq 0$. In this paper, we aim to investigate the properties of the principal functions corresponding to the eigenvalues and the spectral singularities of the BVP (1.5)–(1.6).

2. Jost Solution of (1.5)

We will denote the solution of (1.5) satisfying the condition

$$\lim_{x\to\infty} y(x,\lambda)e^{-i\lambda x}=I,\qquad \lambda\in\overline{\mathbb{C}}_+:=\{\lambda:\lambda\in\mathbb{C},\operatorname{Im}\lambda\geq 0\} \tag{2.1}$$

by $E(x,\lambda)$. The solution $E(x,\lambda)$ is called the Jost solution of (1.5).

Under the condition

$$\int_0^\infty x\,\|Q(x)\|\,dx<\infty \tag{2.2}$$

the Jost solution has a representation

$$E(x,\lambda)=e^{i\lambda x}I+\int_x^\infty K(x,t)e^{i\lambda t}\,dt, \tag{2.3}$$

for $\lambda \in \bar{\mathbb{C}}_+$, where the kernel matrix function $K(x,t)$ satisfies

$$K(x,t) = \frac{1}{2}\int_{\frac{x+t}{2}}^{\infty} Q(s)ds + \frac{1}{2}\int_{x}^{\frac{x+t}{2}}\int_{t+x-s}^{t+s-x} Q(s)K(s,v)dvds$$
$$+\frac{1}{2}\int_{\frac{x+t}{2}}^{\infty}\int_{s}^{t+s-x} Q(s)K(s,v)dvds. \tag{2.4}$$

Moreover, $K(x,t)$ is continuously differentiable with respect to its arguments and

$$\|K(x,t)\| \le c\sigma(\frac{x+t}{2}), \tag{2.5}$$
$$\|K_x(x,t)\| \le \frac{1}{4}\left\|Q(\frac{x+t}{2})\right\| + c\sigma(\frac{x+t}{2}), \tag{2.6}$$
$$\|K_t(x,t)\| \le \frac{1}{4}\left\|Q(\frac{x+t}{2})\right\| + c\sigma(\frac{x+t}{2}), \tag{2.7}$$

where $\sigma(x) = \int_x^\infty \|Q(s)\|\,ds$ and $c > 0$ is a constant. Therefore, $E(x,\lambda)$ is analytic with respect to λ in $\mathbb{C}_+ := \{\lambda : \lambda \in \mathbb{C}, \operatorname{Im}\lambda > 0\}$ and continuous on the real axis [1].

Let $\hat{E}^{\pm}(x,\lambda)$ denote the solutions of (1.5) subject to the conditions

$$\lim_{x\to\infty} \hat{E}^{\pm}(x,\lambda)e^{\pm i\lambda x} = I, \quad \lim_{x\to\infty} \hat{E}^{\pm}_x(x,\lambda)e^{\pm i\lambda x} = \pm i\lambda I, \quad \lambda \in \bar{\mathbb{C}}_{\pm}. \tag{2.8}$$

Then

$$W\left[E(x,\lambda), \hat{E}^{\pm}(x,\lambda)\right] = \mp 2i\lambda I, \quad \lambda \in \mathbb{C}_{\pm}, \tag{2.9}$$
$$W\left[E(x,\lambda), E(x,-\lambda)\right] = -2i\lambda I, \quad \lambda \in \mathbb{R}, \tag{2.10}$$

where $W[f_1, f_2]$ is the Wronskian of f_1 and f_2.

Let $\varphi(x,\lambda)$ denote the solution of (1.5) subject to the initial conditions $\varphi(0,\lambda) = A_0 + A_1\lambda$, $\varphi'(0,\lambda) = B_0 + B_1\lambda$. Therefore $\varphi(x,\lambda)$ is an entire function of λ.

Let us define the following functions:

$$D_{\pm}(\lambda) = \varphi(0,\lambda)E_x(0,\pm\lambda) - \varphi'(0,\lambda)E(0,\pm\lambda) \quad \lambda \in \bar{\mathbb{C}}_{\pm}, \tag{2.11}$$

where $\bar{\mathbb{C}}_{\pm} = \{\lambda : \lambda \in \mathbb{C}, \ \pm\operatorname{Im}\lambda \ge 0\}$. It is obvious that the functions $D_+(\lambda)$ and $D_-(\lambda)$ are analytic in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively, and continuous on the real axis. The functions D_+ and D_- are called Jost functions of L.

3. Eigenvalues and Spectral Singularities of L

The resolvent of L defined by

$$R_\lambda(L)f = \int_0^\infty G(x,t;\lambda)g(t)dt, \qquad g \in L_2(\mathbb{R}_+, E), \tag{3.1}$$

where

$$G(x,t;\lambda) = \begin{cases} G_+(x,t;\lambda), & \lambda \in \mathbb{C}_+ \\ G_-(x,t;\lambda), & \lambda \in \mathbb{C}_-, \end{cases} \tag{3.2}$$

and

$$G_\pm(x,t;\lambda) = \begin{cases} -E(x,\pm\lambda)D_\pm^{-1}(\lambda)\varphi^T(t,\lambda), & 0 \le t \le x \\ -\varphi(x,\lambda)\left[D_+^T(\pm\lambda)\right]^{-1}E^T(t,\pm\lambda), & x \le t < \infty. \end{cases} \tag{3.3}$$

We will show the set of eigenvalues and the set of spectral singularities of the operator L by σ_d and σ_{ss}, respectively.

Let us suppose that

$$H_\pm(\lambda) = \det D_\pm(\lambda). \tag{3.4}$$

From (2.3) and (3.1)–(3.4), we get

$$\begin{aligned} \sigma_d &= \{\lambda : \lambda \in \mathbb{C}_+,\ H_+(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathbb{C}_-,\ H_-(\lambda) = 0\} \\ \sigma_{ss} &= \left\{\lambda : \lambda \in \mathbb{R}^*,\ H_+(\lambda) = 0\right\} \cup \left\{\lambda : \lambda \in \mathbb{R}^*,\ H_-(\lambda) = 0\right\}, \end{aligned} \tag{3.5}$$

where $\mathbb{R}^* = \mathbb{R}\setminus\{0\}$.

We see from that, the functions

$$K^+(\lambda) = \frac{\hat{D}_+(\lambda)}{2i\lambda}E(x,\lambda) - \frac{D_+(\lambda)}{2i\lambda}\hat{E}^+(x,\lambda), \quad \lambda \in \mathbb{C}_+, \tag{3.6}$$

$$K^-(\lambda) = \frac{\hat{D}_-(\lambda)}{2i\lambda}E(x,-\lambda) - \frac{D_-(\lambda)}{2i\lambda}\hat{E}^-(x,\lambda), \quad \lambda \in \mathbb{C}_-, \tag{3.7}$$

$$K(\lambda) = \frac{D_+(\lambda)}{2i\lambda}E(x,-\lambda) - \frac{D_-(\lambda)}{2i\lambda}E(x,\lambda), \quad \lambda \in \mathbb{R}^*, \tag{3.8}$$

are the solutions of the boundary problem (1.5)–(1.6), where

$$\hat{D}_\pm(\lambda) = (A_0 + A_1\lambda)\hat{E}_x^\pm(0,\lambda) - (B_0 + B_1\lambda)\hat{E}^\pm(0,\lambda). \tag{3.9}$$

Now let us assume that

$$Q \in AC(\mathbb{R}_+)\,,\ \lim_{x\to\infty} Q(x) = 0\,,\ \sup_{x\in[0,\infty)}\left[e^{\varepsilon\sqrt{x}}\left\|Q'(x)\right\|\right] < \infty,\ \ \varepsilon > 0. \tag{3.10}$$

Theorem 1. *Under the condition (3.10), the operator L has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

4. Principal Functions of L

Under the condition (3.10), let $\lambda_1,...,\lambda_j$ and $\lambda_{j+1},...,\lambda_k$ denote the zeros H^+ in $\mathbb{C}_+$ and H^- in $\mathbb{C}_-$ (which are the eigenvalues of L) with multiplicities $m_1,...,m_j$ and $m_{j+1},...,m_k$, respectively. It is obvious that from the definition of the Wronskian

$$\left\{\frac{d^n}{d\lambda^n}W\left[K^+(x,\lambda),E(x,\lambda)\right]\right\}_{\lambda=\lambda_p}=\left\{\frac{d^n}{d\lambda^n}D_+(\lambda)\right\}_{\lambda=\lambda_p}=0 \tag{4.1}$$

for $n=0,1,...,m_p-1$, $p=1,2,...,j$, and

$$\left\{\frac{d^n}{d\lambda^n}W\left[K^-(x,\lambda),E(x,-\lambda)\right]\right\}_{\lambda=\lambda_p}=\left\{\frac{d^n}{d\lambda^n}D_-(\lambda)\right\}_{\lambda=\lambda_p}=0 \tag{4.2}$$

for $n=0,1,...,m_p-1$, $p=j+1,...,k$.

Theorem 2. *The following formulae:*

$$\left\{\frac{\partial^n}{\partial\lambda^n}K^+(x,\lambda)\right\}_{\lambda=\lambda_p}=\sum_{m=0}^{n}F_m(\lambda_p)\left\{\frac{\partial^m}{\partial\lambda^m}E(x,\lambda)\right\}_{\lambda=\lambda_p}, \tag{4.3}$$

$n=0,1,...,m_p-1$, $p=1,2,...,j$, *where*

$$F_m(\lambda_p)=\binom{n}{m}\left\{\frac{\partial^{n-m}}{\partial\lambda^{n-m}}\hat{D}_+(\lambda)\right\}_{\lambda=\lambda_p}, \tag{4.4}$$

$$\left\{\frac{\partial^n}{\partial\lambda^n}K^-(x,\lambda)\right\}_{\lambda=\lambda_p}=\sum_{m=0}^{n}N_m(\lambda_p)\left\{\frac{\partial^m}{\partial\lambda^m}E(x,-\lambda)\right\}_{\lambda=\lambda_p}, \tag{4.5}$$

$n=0,1,...,m_p-1$, $p=j+1,...,k$, *where*

$$N_m(\lambda_p)=\binom{n}{m}\left\{\frac{\partial^{n-m}}{\partial\lambda^{n-m}}\hat{D}_-(\lambda)\right\}_{\lambda=\lambda_p} \tag{4.6}$$

hold.

Proof. We will prove only (4.3) using the method of induction, because the case of (4.5) is similar. Let be $n=0$. Since $K^+(x,\lambda)$ and $E(x,\lambda)$ are linearly dependent from (4.1), we get

$$K^+(x,\lambda_p)=f_0(\lambda_p)E(x,\lambda_p), \tag{4.7}$$

where $f_0(\lambda_p)\neq 0$. Let us assume that $1\le n_0\le m_p-2$, (4.7) holds; that is,

$$\left\{\frac{\partial^{n_0}}{\partial\lambda^{n_0}}K^+(x,\lambda)\right\}_{\lambda=\lambda_p}=\sum_{m=0}^{n_0}F_m(\lambda_p)\left\{\frac{\partial^m}{\partial\lambda^m}E(x,\lambda)\right\}_{\lambda=\lambda_p}. \tag{4.8}$$

We will prove that (4.3) holds for n_0+1. If $Y(x,\lambda)$ is a solution of (1.5), then $\frac{\partial^n}{\partial\lambda^n}Y(x,\lambda)$ satisfies

$$\left[-\frac{d^2}{dx^2}+Q(x)-\lambda^2\right]\frac{\partial^n}{\partial\lambda^n}Y(x,\lambda)=2\lambda n\frac{\partial^{n-1}}{\partial\lambda^{n-1}}Y(x,\lambda)+n(n-1)\frac{\partial^{n-2}}{\partial\lambda^{n-2}}Y(x,\lambda). \tag{4.9}$$

Writing (4.9) for $K^+(x,\lambda)$ and $E(x,\lambda)$, and using (4.8), we find

$$\left[-\frac{d^2}{dx^2}+Q(x)-\lambda^2\right]g_{n_0+1}(x,\lambda_p)=0, \tag{4.10}$$

where

$$g_{n_0+1}(x,\lambda_p)=\left\{\frac{\partial^{n_0+1}}{\partial\lambda^{n_0+1}}K^+(x,\lambda)\right\}_{\lambda=\lambda_p}-\sum_{m=0}^{n_0+1}F_m(\lambda_p)\left\{\frac{\partial^m}{\partial\lambda^m}E(x,\lambda)\right\}_{\lambda=\lambda_p}. \tag{4.11}$$

From (4.1), we have

$$W\left[g_{n_0+1}(x,\lambda_p),E(x,\lambda_p)\right]=\left\{\frac{d^{n_0+1}}{d\lambda^{n_0+1}}W\left[K^+(x,\lambda),E(x,\lambda)\right]\right\}_{\lambda=\lambda_p}=0. \tag{4.12}$$

Hence there exists a constant $f_{n_0+1}(\lambda_p)$ such that

$$g_{n_0+1}(x,\lambda_p)=f_{n_0+1}(\lambda_p)E(x,\lambda_p). \tag{4.13}$$

This shows that (4.3) holds for $n=n_0+1$. □

Using (4.3) and (4.5), define the functions

$$U_{n,p}(x)=\left\{\frac{\partial^n}{\partial\lambda^n}K^+(x,\lambda)\right\}_{\lambda=\lambda_p}=\sum_{m=0}^{n}F_m(\lambda_p)\left\{\frac{\partial^m}{\partial\lambda^m}E(x,\lambda)\right\}_{\lambda=\lambda_p}, \tag{4.14}$$

$n=0,1,...,m_p-1$, $p=1,2,...,j$ and

$$U_{n,p}(x)=\left\{\frac{\partial^n}{\partial\lambda^n}K^-(x,\lambda)\right\}_{\lambda=\lambda_p}=\sum_{m=0}^{n}N_m(\lambda_p)\left\{\frac{\partial^m}{\partial\lambda^m}E(x,-\lambda)\right\}_{\lambda=\lambda_p}, \tag{4.15}$$

$n=0,1,...,m_p-1$, $p=j+1,...,k$.

Then for $\lambda=\lambda_p$, $p=1,2,...,j,j+1,...,k$,

$$\begin{aligned} l(U_{0,p})&=0,\\ l(U_{1,p})+\frac{1}{1!}\frac{\partial}{\partial\lambda}l(U_{0,p})&=0,\\ l(U_{n,p})+\frac{1}{1!}\frac{\partial}{\partial\lambda}l(U_{n-1,p})+\frac{1}{2!}\frac{\partial^2}{\partial\lambda^2}l(U_{n-2,p})&=0, \end{aligned} \tag{4.16}$$

$n = 2,3,...,m_p - 1$,
hold, where $l(u) = -u'' + Q(x)u - \lambda^2 u$ and $\frac{\partial^m}{\partial\lambda^m} l(u)$ denote the differential expressions whose coefficients are the m-th derivatives with respect to λ of the corresponding coefficients of the differential expression $l(u)$. (4.16) shows that $U_{0,p}$ is the eigenfunction corresponding to the eigenvalue $\lambda = \lambda_p$; $U_{1,p}, U_{2,p}, ...U_{m_p-1,p}$ are the associated functions of $U_{0,p}$ [16, 17].

$U_{0,p}, U_{1,p}, ...U_{m_p-1,p}$, $p = 1,2,...,j,j+1,...,k$ are called the principal functions corresponding to the eigenvalue $\lambda = \lambda_p$, $p = 1,2,...,j,j+1,...,k$ of L.

Theorem 3.

$$U_{n,p} \in L_2(\mathbb{R}_+, E), \quad n = 0,1,...m_p - 1, \ p = 1,2,...,j,j+1,...,k. \tag{4.17}$$

Proof. Let be $0 \le n \le m_p - 1$ and $1 \le p \le j$. Using (2.5) and (3.10), we obtain that

$$\|K(x,t)\| \le ce^{-\sqrt{\frac{x+t}{2}}}. \tag{4.18}$$

From (2.3), we get

$$\left\| \left\{ \frac{\partial^n}{\partial\lambda^n} E(x,\lambda) \right\}_{\lambda=\lambda_p} \right\| \le x^n e^{-x \operatorname{Im}\lambda_p} + c \int_x^\infty t^n e^{-\sqrt{\frac{x+t}{2}}} e^{-t \operatorname{Im}\lambda_p} dt, \tag{4.19}$$

where $c > 0$ is a constant. Since $\operatorname{Im}\lambda_p > 0$ for the eigenvalues λ_p, $p = 1,2,...,j$, of L, implies that

$$\left\{ \frac{\partial^n}{\partial\lambda^n} E(x,\lambda) \right\}_{\lambda=\lambda_p} \in L_2(\mathbb{R}_+, E), \quad n = 0,1,...m_p - 1, \ p = 1,2,...,j. \tag{4.20}$$

So we get $U_{n,p} \in L_2(\mathbb{R}_+, E)$ from (4.14) and (4.20) Similarly we prove the results for $0 \le n \le m_p - 1$, $j+1 \le p \le k$. This completes the proof. □

Let $\mu_1, ..., \mu_v$ and $\mu_{v+1}, ..., \mu_l$ be the zeros of D_+ and D_- in $\mathbb{R}^*$ with multiplicities $n_1, ..., n_v$ and $n_{v+1}, ..., n_l$, respectively. We can show

$$\left\{ \frac{\partial^n}{\partial\lambda^n} K(x,\lambda) \right\}_{\lambda=\mu_p} = \sum_{m=0}^{n} C_m(\lambda_p) \left\{ \frac{\partial^m}{\partial\lambda^m} E(x,\lambda) \right\}_{\lambda=\mu_p}, \tag{4.21}$$

$n = 0,1,...,n_p - 1$, $p = 1,2,...,v$, where

$$C_m(\mu_p) = -\binom{n}{m} \left\{ \frac{\partial^{n-m}}{\partial\lambda^{n-m}} D_-(\lambda) \right\}_{\lambda=\mu_p}, \tag{4.22}$$

$$\left\{ \frac{\partial^n}{\partial\lambda^n} K(x,\lambda) \right\}_{\lambda=\mu_p} = \sum_{m=0}^{n} R_m(\mu_p) \left\{ \frac{\partial^m}{\partial\lambda^m} E(x,-\lambda) \right\}_{\lambda=\mu_p},$$

$n=0,1,...,n_p-1,\ p=v+1,...,l,$
where

$$R_m(\mu_p)=\binom{n}{m}\left\{\frac{\partial^{n-m}}{\partial\lambda^{n-m}}D_+(\lambda)\right\}_{\lambda=\mu_p}. \tag{4.23}$$

Now define the generalized eigenfunctions and generalized associated functions corresponding to the spectral singularities of L by the following :

$$V_{n,p}(x)=\left\{\frac{\partial^n}{\partial\lambda^n}K(x,\lambda)\right\}_{\lambda=\mu_p}=\sum_{m=0}^{n}C_m(\mu_p)\left\{\frac{\partial^m}{\partial\lambda^m}E(x,\lambda)\right\}_{\lambda=\mu_p}, \tag{4.24}$$

$n=0,1,...,n_p-1,\ p=1,2,...,v,$

$$V_{n,p}(x)=\left\{\frac{\partial^n}{\partial\lambda^n}K(x,\lambda)\right\}_{\lambda=\mu_p}=\sum_{m=0}^{n}R_m(\mu_p)\left\{\frac{\partial^m}{\partial\lambda^m}E(x,-\lambda)\right\}_{\lambda=\mu_p}, \tag{4.25}$$

$n=0,1,...,n_p-1,\ p=v+1,...,l.$

Then $V_{n,p}, n=0,1,...,n_p-1,\ p=1,2,...,v,v+1,...,l,$ also satisfy the equations analogous to (4.16).

$V_{0,p},V_{1,p},...,V_{n_p-1,p},\ p=1,2,...,v,v+1,...,l$ are called the principal functions corresponding to the spectral singularities $\lambda=\mu_p, p=1,2,...,v,v+1,...,l$ of L.

Theorem 4.

$$V_{n,p}\notin L_2(\mathbb{R}_+,E),\quad n=0,1,...n_p-1,\ p=1,2,...,v,v+1,...,l.$$

Proof. For $0\le n\le n_p-1$ and $1\le p\le v$ using (2.3), we obtain

$$\left\|\left\{\frac{\partial^n}{\partial\lambda^n}E(x,\lambda)\right\}_{\lambda=\mu_p}\right\|\le\left\|(ix)^n e^{i\mu_p x}I+\int_x^\infty(it)^n K(x,t)e^{i\mu_p t}dt\right\|$$

since $\operatorname{Im}\mu_p=0,\ p=1,2,...,v,$ we find that

$$\int_0^\infty\left\|(ix)^n e^{i\mu_p x}I\right\|^2dx=\int_0^\infty x^{2n}=\infty.$$

So we obtain $V_{n,p}\notin L_2(\mathbb{R}_+,E), n=0,1,...n_p-1,\ p=1,2,...,v.$ Using the similar way, we may also prove the results for $0\le n\le n_p-1,\ v+1\le p\le l.$ □

Now define the Hilbert spaces of vector-valued functions with values in E by

$$H_n:=\left\{f:\int_0^\infty(1+|x|)^{2n}\|f(x)\|^2dx<\infty\right\},\ n=1,2,..., \tag{4.26}$$

$$H_{-n} := \left\{ g : \int_0^\infty (1+|x|)^{-2n} \|g(x)\|^2 \, dx < \infty \right\}, \quad n = 1,2,..., \tag{4.27}$$

with the norms

$$\|f\|_n^2 := \int_0^\infty (1+|x|)^{2n} \|f(x)\|^2 \, dx,$$

and

$$\|g\|_{-n}^2 := \int_0^\infty (1+|x|)^{-2n} \|g(x)\|^2 \, dx,$$

respectively. Then

$$H_{n+1} \subsetneqq H_n \subsetneqq L_2(\mathbb{R}_+, E) \subsetneqq H_{-n} \subsetneqq H_{-(n+1)}, \quad n = 1,2,..., \tag{4.28}$$

and H_{-n} is isomorphic to the dual of H_n.

Theorem 5.

$$V_{n,p} \in H_{-(n+1)}, \quad n = 0,1,...n_p - 1, \ p = 1,2,...,v,v+1,...,l.$$

Proof. For $0 \le n \le n_p - 1$ and $1 \le p \le v$ using (2.3) and (4.24), we get

$$\int_0^\infty (1+|x|)^{-2(n+1)} \left\| V_{n,p} \right\|^2 dx$$

$$\le M \int_0^\infty (1+|x|)^{-2(n+1)} \left\{ \left\| \{E(x,\lambda)\}_{\lambda=\mu_p} \right\|^2 + ... + \left\| \left\{ \frac{\partial^n}{\partial \lambda^n} E(x,\lambda) \right\}_{\lambda=\mu_p} \right\|^2 \right\}$$

where $M > 0$ is a constant. Using (2.3), we have

$$\int_0^\infty (1+|x|)^{-2(n+1)} \left\| (ix)^n e^{i\mu_p x} I \right\|^2 dx < \infty,$$

and

$$\int_0^\infty (1+|x|)^{-2(n+1)} \left\| \int_x^\infty (it)^n K(x,t) e^{i\mu_p t} dt \right\|^2 dx < \infty.$$

Consequently $V_{n,p} \in H_{-(n+1)}$ for $0 \le n \le n_p - 1$ and $1 \le p \le v$. Similarly, we obtain $V_{n,p} \in H_{-(n+1)}$ for $0 \le n \le n_p - 1$ and $v+1 \le p \le l$. □

Let us choose

$$n_0 = \max\{n_1, n_2, ..., n_v, n_{v+1}, ..., n_l\}.$$

By (4.28), we get the following

Theorem 6.

$$V_{n,p} \in H_{-n_0}, \quad n = 0, 1, ...n_p - 1, p = 1, 2, ..., v, v+1, ..., l.$$

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Authors' addresses

Deniz Katar
Ankara University, Faculty of Sciences, Department of Mathematics, Ankara, Turkey
E-mail address: deniz.ktr@hotmail.com

Murat Olgun
Ankara University, Faculty of Sciences, Department of Mathematics, Ankara, Turkey
E-mail address: olgun@ankara.edu.tr

Cafer Coskun
Ankara University, Faculty of Sciences, Department of Mathematics, Ankara, Turkey
E-mail address: ccoskun@ankara.edu.tr

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RECOGNITION BY CHARACTER DEGREE GRAPH AND ORDER OF SIMPLE GROUPS OF ORDER LESS THAN 6000

BEHROOZ KHOSRAVI, BEHNAM KHOSRAVI, BAHMAN KHOSRAVI,
AND ZAHRA MOMEN

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Abstract. Let G be a finite group. The character degree graph of G, which is denoted by $\Gamma(G)$, is the graph whose vertices are the prime divisors of the character degrees of the group G and two vertices p_1 and p_2 are joined by an edge if $p_1 p_2$ divides some character degree of G. In this paper we prove that if G is a simple group of order less that 6000, then G is uniquely determined by its character degree graph and its order. Also by an example we show that this result is not true for all simple groups.

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1. Introduction and Preliminary Results

Let G be a finite group, $\mathrm{Irr}(G)$ be the set of irreducible characters of G, and denote by $\mathrm{cd}(G)$, the set of irreducible character degrees of G.

There are several graphs attached to the set of character degrees of a finite group that have been studied (see [3, 5]). Results on these graphs are often useful to prove results that provide structural information of the group from some property of the set of character degrees. The graph that has been most commonly studied is the graph $\Gamma(G)$ whose vertices are the prime divisors of the character degrees of the group G and two vertices are joined by an edge if the product of the primes divides some character degree of G. This graph is called *character degree graph* and was introduced in [5]. Later this graph has been widely studied (see [4, 6]).

A finite group G is called a K_3-group if $|G|$ has exactly three distinct prime divisors. Recently Chen et. al. in [7] proved that these groups are uniquely determined by their orders and one or both of their largest and second largest irreducible character degrees. It is proved that A_5, $L_2(7)$, $L_2(17)$, $L_3(3)$ and $U_4(2)$ can be uniquely determined by their orders and the largest degrees of their irreducible characters. Also it is proved that A_6 is characterized by its order and the second largest degree of

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its irreducible characters. Finally it is proved that $L_2(8)$ and $U_3(3)$ are characterizable by their orders and the largest and the second largest degrees of their irreducible characters.

The goal of this paper is to introduce a new characterization for the simple groups S such that $|S| < 6000$. In fact we prove the following result:

Let S be a simple group such that $|S| < 6000$. If G is a finite group such that $|G| = |S|$ and $\Gamma(G) = \Gamma(S)$, then $G \cong S$.

We note that this result is not true for all simple groups. In fact we give a solvable group of order 6048 such that its order and its character degree graph are the same as $U_3(3)$ and so the simple group of order 6048 is not characterized by its degree graph and its order.

If $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$, then the inertia group of θ in G is $I_G(\theta) = \{g \in G \mid \theta^g = \theta\}$. If the character $\chi = \sum_{i=1}^{k} e_i \chi_i$, where for each $1 \leq i \leq k$, $\chi_i \in \mathrm{Irr}(G)$ and e_i is a natural number, then each χ_i is called an irreducible constituent of χ.

Lemma 1 ((Ito's Theorem) [2, Theorem 6.15])**.** *Let $A \trianglelefteq G$ be abelian. Then $\chi(1)$ divides $|G : A|$, for all $\chi \in Irr(G)$.*

Lemma 2 ([2, Theorems 6.2, 6.8, 11.29])**.** *Let $N \trianglelefteq G$ and let $\chi \in$ Irr(G). Let θ be an irreducible constituent of χ_N and suppose $\theta_1 = \theta, \ldots, \theta_t$ are the distinct conjugates of θ in G. Then $\chi_N = e \sum_{i=1}^{t} \theta_i$, where $e = [\chi_N, \theta]$ and $t = |G : I_G(\theta)|$. Also $\theta(1) \mid \chi(1)$ and $\chi(1)/\theta(1) \mid |G : N|$.*

Lemma 3 ([7, Lemma])**.** *Let G be a nonsolvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$.*

Lemma 4 ((Palfy's Theorem) [4, Theorem 4.1])**.** *Let $\rho(G)$ be the set of all prime divisors of the character degrees of G. Let G be a solvable group and $\pi \subseteq \rho(G)$. If $|\pi| \geq 3$, then there exist primes $p, q \in \pi$ and a degree $a \in \mathrm{cd}(G)$ so that pq divides a. In other words, any three primes in $\rho(G)$ must have an edge in $\Gamma(G)$ that is incident to two of those primes.*

If n is an integer and r is a prime number, then we write $r^\alpha \| n$, when $r^\alpha \mid n$ but $r^{\alpha+1} \nmid n$. Also if r is a prime number we denote by $\mathrm{Syl}_r(G)$, the set of Sylow r-subgroups of G and we denote by $n_r(G)$, the number of elements of $\mathrm{Syl}_r(G)$. If H is a characteristic subgroup of G, we write H ch G. All other notations are standard and we refer to [1].

2. Main Results

Theorem 1. *If G is a finite group of order 60 such that $\Gamma(G) = \Gamma(A_5)$, then $G \cong A_5$.*

Proof. We know that $\mathrm{cd}(A_5) = \{1, 3, 4, 5\}$. Therefore $\Gamma(G)$ is a graph with vertex set $\{2, 3, 5\}$ and there is no edge in $\Gamma(G)$. So there exists $\chi \in \mathrm{Irr}(G)$, such that

$5 \mid \chi(1)$. Also $\chi(1)^2 < |G| = 60$, implies that $\chi(1) = 5$. On the other hand, the order and the character degree graph of G show that 5 is the largest character degree of G. Therefore $G \cong A_5$, by [7]. □

Theorem 2. *Let G be a finite group of order* 168 *and $\Gamma(G) = \Gamma(L_2(7))$. Then $G \cong L_2(7)$.*

Proof. By [1], We know that $\mathrm{cd}(L_2(7)) = \{1, 3, 6, 7, 8\}$. Therefore $\Gamma(G)$ is a graph with vertex set $\{2, 3, 7\}$ and there exists an edge between 2 and 3. Similarly to the proof of Theorem 1 there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = 7$. Also there exists $\beta \in \mathrm{Irr}(G)$ such that $6 \mid \beta(1)$. Therefore $\beta(1) = 2^\alpha 3$, for some $\alpha > 0$. Then $|G| = 168$ shows that $\beta(1) = 6$ or $\beta(1) = 12$. If $\beta(1) = 12$, then $1 + \beta(1)^2 + \chi(1)^2 > |G|$, which is a contradiction.

If $O_7(G) \neq 1$, then $O_7(G)$ is an abelian normal subgroup of order 7 and we get a contradiction by Ito's theorem, since $\chi(1) = 7$. Similarly it follows that $O_3(G) = 1$.

Now we prove that G is a nonsolvable group. On the contrary let G be a solvable group. Let M be a normal minimal subgroup of G. By the above discussion M is a 2-elementary abelian subgroup of G. Also $|M| \neq 8$, by Ito's theorem, since $\beta(1) = 6$. Hence $|M| = 2^k$, where $1 \leq k \leq 2$. Hence if $Q/M \in \mathrm{Syl}_7(G/M)$, then $Q/M \lhd G/M$. Also if $P \in \mathrm{Syl}_7(Q)$, then P ch Q, which implies that $P \lhd G$, and so $O_7(G) \neq 1$, which is a contradiction.

Therefore G is nonsolvable and so G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. By the classification of finite simple groups and [1], it follows that K/H is isomorphic to $L_2(7)$ and so $H = 1$ and $G = K$. Therefore $G \cong L_2(7)$. □

Theorem 3. *Let G be a finite group such that $|G| = |A_6| = 2^3 3^2 5$ and $\Gamma(G) = \Gamma(A_6)$. Then $G \cong A_6$.*

Proof. By [1], We know that $\mathrm{cd}(A_6) = \{1, 5, 8, 9, 10\}$. Therefore $\Gamma(G)$ is a graph with vertex set $\{2, 3, 5\}$ and there is an edge between 2 and 5. Similarly to the proof of Theorem 2 we conclude that G has an irreducible character χ such that $\chi(1) = 10$. Also by Ito's theorem it follows that $O_5(G) = 1$.

Let G be a solvable group and M be a normal minimal subgroup of G. Then M is a p-elementary abelian subgroup, where $p = 2$ or $p = 3$. Now we consider two cases:

Case 1. Let M be a 3-group. Then $|M| = 3$ or $|M| = 9$. If $|M| = 9$, then since 3 is a vertex of $\Gamma(G)$, we get a contradiction by Ito's theorem. Therefore $|M| = 3$. Let H/M be a Hall subgroup of G/M of order $2^3 5$. Then $|G/M : H/M| = 3$ and so $(G/M)/(L/M) \hookrightarrow S_3$, where $L/M = \mathrm{Core}_{G/M}(H/M)$. Therefore $5 \mid |L/M|$ and $|L/M| \mid 40$, which implies that $Q/M \lhd L/M$, where $Q/M \in \mathrm{Syl}_5(L/M)$. Hence $Q \lhd G$ and $|Q| = 15$. Therefore $O_5(G) \neq 1$, which is a contradiction.

Case 2. Let M be a 2-group. Since $\chi(1) = 10$, we get that $|M| = 2$ or $|M| = 4$.

(2.1) Let $|M| = 4$. Then $|G/M| = 2 \times 3^2 \times 5$.

Let H/M be a Hall subgroup of G/M of order $3^2 5$, which is a normal subgroup of G/M. Also $Q/M \lhd H/M$, where $Q/M \in \mathrm{Syl}_5(H/M)$, and so $Q/M \lhd G/M$. Therefore $Q \lhd G$ and $|Q| = 20$. If $P \in \mathrm{Syl}_5(Q)$, then $P \lhd G$, which implies that $O_5(G) \neq 1$ and this is a contradiction.

(2.2) Therefore $|M| = 2$ and $|G/M| = 2^2 3^2 5$. Let H/M be a Hall subgroup of G/M of order $3^2 5$. Then $|G/M : H/M| = 4$, and so if $L/M = \mathrm{Core}_{G/M}(H/M)$, then $(G/M)/(L/M) \hookrightarrow S_4$, which implies that $5 \mid |L/M|$. Also $|L/M| \mid |H/M| = 3^2 5$. Hence if $Q/M \in \mathrm{Syl}_5(L/M)$, then similarly to the above, $Q/M \lhd G/M$ and so $Q \lhd G$. Then $|Q| = 10$ and if $P \in \mathrm{Syl}_5(Q)$, then $P \lhd G$, which implies that $O_5(G) \neq 1$, and this is a contradiction.

Therefore G is nonsolvable and so G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. By the classification of finite simple groups and [1], it follows that K/H is isomorphic to A_5 or A_6.

(i) Let $K/H \cong A_5$. Then $|H| \cdot |G/K| = 6$ and $|G/K| \mid 2$.

(a) Let $|G/K| = 1$. Then $|H| = 6$. Now we consider two cases:

(a.1) Let $H \cong S_3$.

Then $H \lhd G$ and so $C_G(H) \lhd N_G(H) = G$. Also $H \cap C_G(H) = 1$. Therefore $G \cong H \times C_G(H)$. Also $C_G(H) \cong G/H \cong A_5$ and hence $G \cong S_3 \times A_5$. Now G has an irreducible character of degree 6, which is a contradiction, since 2 and 3 are nonadjacent in $\Gamma(G)$.

(a.2) Let $H \cong \mathbb{Z}_6$.

Then there exists $\theta \in \mathrm{Irr}(H)$ such that $e = [\chi_H, \theta] \neq 0$, where $\chi(1) = 10$. Hence $10 = et$, where $t = |G : I_G(\theta)|$. We know that $t \mid |\mathrm{Aut}(H)| = 2$. If $t = 1$, then $e = 10$, and so $[\chi_H, \chi_H] = e^2 t = 100 < |G : H| = 60$, which is a contradiction. Therefore $t = 2$ and $e = 5$. Since $H \subseteq I_G(\theta)$, it follows that $G/H \cong A_5$ has a subgroup $I_G(\theta)/H$ such that $|G/H : I_G(\theta)/H| = 2$, and this is a contradiction.

(b) Let $|G/K| = 2$. Then $G/H \cong S_5$ and we get a contradiction, since S_5 has an irreducible character of degree 6.

(ii) Therefore $K/H \cong A_6$ and so $H = 1$ and $G \cong A_6$. □

Theorem 4. *Let G be a finite group such that $|G| = |L_3(3)| = 2^4 3^3 13$ and $\Gamma(G) = \Gamma(L_3(3))$. Then $G \cong L_3(3)$.*

Proof. By [1], We know that $\Gamma(G)$ is a complete graph on the vertex set $\{2, 3, 13\}$. Then there exists $\chi \in \mathrm{Irr}(G)$, such that $\chi(1) = 39$. So by Ito's theorem we get that $O_{13}(G) = 1$.

Let G be a solvable group. Let M be a normal minimal subgroup of G. Then M is a p-elementary abelian subgroup where $p = 2$ or $p = 3$. Since there exists $\beta \in \mathrm{Irr}(G)$ such that $6 \mid \beta(1)$, it follows that $|M| \neq 3^3$ and $|M| \neq 2^4$.

(i) Let M be a 3-group. Then $|M| = 3^k$, where $1 \leq k \leq 2$. Then $|G/M| = 2^4 3^{3-k} 13$. Let H/M be a Hall subgroup of G/M of order $2^4 13$. Then $|G/M : H/M| = 3^{3-k}$ and so $G/H_G \hookrightarrow S_9$ which implies that $13 \mid |H_G|$. Let $Q/M \in$

$\mathrm{Syl}_{13}(H_G/M)$. Then $Q/M \lhd G/M$ and so $Q \lhd G$. If $P \in \mathrm{Syl}_{13}(Q)$, then P ch $Q \lhd G$, which implies that $O_{13}(G) \neq 1$, a contradiction.

(ii) Let M be a 2-group and $|M| = 2^k$, where $1 \leq k \leq 3$. Let H/M be a Hall subgroup of G/M of order $3^3 13$. Then $|G/M : H/M| = |G : H| = 2^r \leq 8$, and so $G/H_G \hookrightarrow S_8$. Therefore $13 \mid |H_G|$. Let $Q/M \in \mathrm{Syl}_{13}(H_G/M)$. Also $|H_G/M| \mid |H/M| = 3^3 13$. If $|H_G/M| \neq 3^3 13$, then Q/M ch $H_G/M \lhd G/M$ and so $Q \lhd G$. Let $P \in \mathrm{Syl}_{13}(Q)$, where $|Q| = 2^k 13$, $1 \leq k \leq 3$. Then $P \lhd G$, which implies that $O_{13}(G) \neq 1$ and this is a contradiction. Therefore $H_G/M = H/M$, i.e. $H/M \lhd G/M$ and $|H| = 2^k 3^3 13$. Let $\theta \in \mathrm{Irr}(H)$ such that $e = [\chi_H, \theta] \neq 0$. then $39 = et\theta(1)$, where $t = |G : I_G(\theta)|$. We know that e and t are divisors of $|G : H| = 2^{4-k}$. Therefore $e = t = 1$ and so $\theta(1) = 39$ and $\chi_H = \theta$. Since $\theta(1)^2 = 1521 < |H|$, it follows that $k = 3$ and $|M| = 8$, $|H| = 2^3 3^3 13$. Hence $M \lhd G$ is a 2-elementary abelian subgroup of G of order 8 and $M \subseteq H$. Let $\eta \in \mathrm{Irr}(M)$ such that $e' = [\theta_M, \eta] \neq 0$. Therefore $39 = e't'$, where $t' = |H : I_H(\eta)|$. Also M has 8 linear characters and so $t' \leq 8$. Therefore one of the following occurs: $(e', t') = (13, 3)$ or $(39, 1)$. But in each case we have

$$507 = 13^2 3 \leq [\theta_M, \theta_M] = e'^2 t' \leq |H : M| = 3^3 13,$$

which is a contradiction. Therefore G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. By the classification of finite simple groups and [1], it follows that K/H is isomorphic to $L_3(3)$. Hence $G \cong L_3(3)$. □

Theorem 5. *Let $q = 8$ or 16. Let G be a finite group such that $|G| = |L_2(q)|$ and $\Gamma(G) = \Gamma(L_2(q))$. Then $G \cong L_2(q)$.*

Proof. **Case 1.** Let $q = 8$.
We know that $\mathrm{cd}(L_2(8)) = \{1, 7, 8, 9\}$. Therefore $\Gamma(G)$ is a graph with vertex set $\{2, 3, 7\}$ and there is no edge in $\Gamma(G)$. So by Lemma 4 we get that G is a nonsolvable group. Therefore G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. By the classification of finite simple groups and [1], it follows that K/H is isomorphic to $L_2(7)$ or $L_2(8)$. Since $6 \in \mathrm{cd}(L_2(7))$, we get that $K/H \cong L_2(8)$ and so $G \cong L_2(8)$.

Case 2. Let $q = 16$.
Similarly to Case 1, by Lemma 4 we get that G is nonsolvable and we get that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong A_5$ or $L_2(16)$. If $K/H \cong A_5$, then $|H| \cdot |G/K| = 2^2 17$ and $|G/K| \mid 2$. Let $P \in \mathrm{Syl}_{17}(H)$. Then P ch $H \lhd G$, which implies that $O_{17}(G) \neq 1$. Now we get a contradiction by Ito's theorem since 17 is a vertex of $\Gamma(G)$. Therefore $K/H \cong L_2(16)$ and so $G \cong L_2(16)$. □

Theorem 6. *Let G be a finite group such that $|G| = |A_7| = 2^3 3^2 5 \times 7$ and $\Gamma(G) = \Gamma(A_7)$. Then $G \cong A_7$.*

Proof. The charcter degree graph of A_7 is a complete graph on the vertex set $\{2,3,5,7\}$. Also there exists $\chi \in \mathrm{Irr}(G)$, such that $21 \mid \chi(1)$.

Let G be a solvable group. Then let H be a Hall subgroup of G of order $2^3 3^2 7$. Then $|G:H| = 5$ and so $G/H_G \hookrightarrow S_5$. The order of a solvable subgroup of S_5 which is divisible by 5 is a divisor of 20. Let $\theta \in \mathrm{Irr}(H_G)$ such that $[\chi_{H_G}, \theta] \neq 0$. Then $\chi(1) = et\theta(1)$ and e and t are divisors of $|G:H_G|$, which implies that $e \mid 20$ and $t \mid 20$. Therefore $21 \mid \theta(1)$ and since $\theta(1)^2 < |H_G|$, it follows that $|H_G| = 504$ and so $H = H_G \lhd G$. On the other hand, there exists $\eta \in \mathrm{Irr}(G)$ such that $\eta(1) = 35$. Let $\varphi \in \mathrm{Irr}(H)$ such that $[\eta_H, \varphi] \neq 0$. Then $35 = \chi(1) = et\varphi(1)$, where $\varphi(1) \mid |H|$ and $et \mid |G:H| = 5$. Therefore $\theta(1) = 7$ and $et = 5$. Since $[\eta_H, \eta_H] = e^2 t \leq |G:H|$, it follows that $e = 1$ and $t = 5$. Hence H has 5 irreducible characters of degree 7. Therefore $1 + 21^2 + 5 \times 7^2 \leq |H| = 504$, which is a contradiction. So G is a nonsolvable group and by Lemma 3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. By the classification of finite simple groups and [1], it follows that K/H is isomorphic to A_5, A_6, A_7, $L_2(7)$ or $L_2(8)$.

If $K/H \cong A_5$ or A_6, then $7 \mid |H|$ and since $|H| \mid 42$, we get that $O_7(G) \neq 1$. Now by Ito's theorem we get a contradiction, since 7 is a vertex of $\Gamma(G)$.

If $K/H \cong L_2(7)$ or $L_2(8)$, then $5 \mid |H|$ and $|H| \mid 15$, which implies that $O_5(G) \neq 1$ and we get a contradiction by Ito's theorem.

Finally $K/H \cong A_7$ and so $G \cong A_7$. □

Theorem 7. *Let $p \in \{11, 13, 17, 19\}$. If G is a finite group such that $|G| = |L_2(p)|$ and $\Gamma(G) = \Gamma(L_2(p))$. Then $G \cong L_2(p)$.*

Proof. Since $p^2 \nmid |G|$, and p is a vertex in the character degree graph of G, by Ito's theorem it follows that $O_p(G) = 1$.

Case 1. Let $p = 11$. Then $|G| = 2^2 \times 3 \times 5 \times 11$.

If G is solvable, let H be a Hall subgroup of G of order $2^2 \times 5 \times 11$. Then $G/H_G \hookrightarrow S_3$. Hence $11 \mid |H_G|$. Now if $P \in \mathrm{Syl}_{11}(G)$, then $P \lhd G$, which implies that $O_{11}(G) \neq 1$ and this is a contradiction. Therefore G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is isomorphic to A_5 or $L_2(11)$. If $K/H \cong A_5$, then $|H||G/K| = 11$ and $|G/K| \mid 2$. Then $|H| = 11$ and so $O_{11}(G) \neq 1$, which is a contradiction. Therefore $K/H \cong L_2(11)$ and so $G \cong L_2(11)$.

Case 2. Let $p = 13$. Then if G is a solvable group, let H be a Hall subgroup of G of index 7. Then $13 \mid |H_G|$ and similarly to above $O_{13}(G) \neq 1$, which is a contradiction. Now similarly to above we get that $G \cong L_2(13)$.

Case 3. Let $p = 17$. Then if G is a solvable group, let H be a Hall subgroup of G of index 9. Then $17 \mid |H_G|$ and similarly to the above $O_{17}(G) \neq 1$, which is a contradiction. Now similarly to the above we get that $G \cong L_2(17)$.

Case 4. If $p = 19$, then similarly we get the result. □

Remark 1. The first simple group of order greater than 6000 is $U_3(3)$ of order 6048. In the sequel we show that this group is not characterizable by its character degree graph and its order. We know that the character degrees of $U_3(3)$ are $\{1,6,7,14,21,27,28,32\}$. So $\Gamma(U_3(3))$ is a complete graph on the three vertices 2, 3 and 7. Now we construct a solvable group G with $|G| = 2^5 3^3 \times 7$ and $\Gamma(G) = \Gamma(U_3(3))$. Let H be the Frobenius group of order $56 = 8 \times 7 = 2^3 \times 7$, and observe that $\mathrm{cd}(H) = \{1,7\}$. Let P be an extraspecial group of order 3^3 and exponent 3. It is not difficult to see that P has an automorphism σ of order 2 that does not centralize the center of P. Let K be the semi-direct product of $\langle\sigma\rangle$ acting on P. It is not difficult to see that $|K| = 2 \times 3^3$ and $\mathrm{cd}(K) = \{1,2,6\}$. Let Z be a cyclic group of order 2, and take $G = H \times K \times Z$. Then $|G| = 2^5 3^3 \times 7$ and $\mathrm{cd}(G) = \{1,2,6,7,14,42\}$. Thus, $\Gamma(G)$ is a complete graph on three vertices.

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Authors' addresses

Behrooz Khosravi

School of Mathematics, Institute for Research in Fundamental sciences (IPM), P.O.Box: 19395–5746, Tehran, Iran

Current address: Dept. of Pure Math., Faculty of Math. and Computer Sci., Amirkabir University of Technology (Tehran Polytechnic), 424, Hafez Ave., Tehran 15914, Iran

E-mail address: khosravibbb@yahoo.com

Behnam Khosravi
Department of Mathematics, Institute for Advanced Studies in Basic Sciences, Zanjan 45137-66731, Iran
E-mail address: behnam_kho@yahoo.com

Bahman khosravi
Department of Mathematics, Faculty of Science, Qom University of Technology, Qom, Iran
E-mail address: bahman_kho@yahoo.com

Zahra Momen
Dept. of Pure Math., Faculty of Math. and Computer Sci., Amirkabir University of Technology (Tehran Polytechnic), 424, Hafez Ave., Tehran 15914, Iran
E-mail address: zahramomen@yahoo.com

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A PROOF OF A CONJECTURE OF MARQUES AND TROJOVSKY

EMRAH KILIÇ

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Abstract. In this paper, we consider Marques and Trojovsky's conjecture involving Fibonomial coefficients, Fibonacci and Lucas numbers. After rewriting it in terms of Gaussian q-binomial coefficients and q-Pochhammer symbols, we give a solution of the conjecture. The approach is to use q-analysis to prove the claimed formulas.

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1. Introduction

The *Fibonomial coefficient* is, for $n \geq m \geq 1$, defined by

$$\begin{Bmatrix} n \\ m \end{Bmatrix}_F := \frac{F_1 F_2 \dots F_n}{(F_1 F_2 \dots F_{n-m})(F_1 F_2 \dots F_m)}$$

with $\begin{Bmatrix} n \\ n \end{Bmatrix} = \begin{Bmatrix} n \\ 0 \end{Bmatrix} = 1$ where F_n is the nth Fibonacci number.

Recently as an interesting generalization of the binomial coefficients, the Fibonomial coefficients have taken the interest of several authors. For their properties, we refer to [2, 3, 5].

The Gaussian q–binomial coefficient $\begin{bmatrix} n \\ m \end{bmatrix}_q$ is defined, for all real n and integers m with $m \geq 0$, by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}}$$

and as zero otherwise, where the q-Pochhammer symbol is defined as

$$(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1}).$$

Thus, $\begin{bmatrix} n \\ m \end{bmatrix}_q$ is a rational function of the parameter q (for more details, see [1, 10.2.2(c)]).

Let

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

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Then the well known Binet forms give

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad L_n = \alpha^n + \beta^n,$$

and thus

$$F_n = \alpha^{n-1}\frac{1-q^n}{1-q}, \qquad L_n = \alpha^n(1+q^n),$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\mathbf{i} = \alpha\sqrt{q}$, where F_n is the nth Fibonacci number and L_n is the nth Lucas number.

All the identities we are going to prove hold for general q, and results about Fibonacci and Lucas numbers come out as corollaries for the special choice of q.

The link between the Fibonomial and Gaussian q–binomial coefficients is

$$\begin{Bmatrix} n \\ m \end{Bmatrix}_F = \alpha^{m(n-m)} \begin{bmatrix} n \\ m \end{bmatrix}_q \quad \text{with } q = -\alpha^{-2}.$$

For the reader's convenience and later use, we recall Rothe's formula [1, 10.2.2(c)]:

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} x^k = (x;q)_n .$$

In a recent paper, Marques and Trojovsky [7] provided various sums including Fibonomial coefficients, Fibonacci and Lucas numbers (for various sums properties of the coefficients, we refer to [4, 6]). For example, for positive integers m and n, the authors [7] showed that

$$\sum_{j=0}^{4m+2} (-1)^{\frac{j(j+1)}{2}} \begin{Bmatrix} 4m+2 \\ j \end{Bmatrix}_F L_{2m+1-j} = -\begin{Bmatrix} 4m+2 \\ 4n+3 \end{Bmatrix}_F \frac{F_{4n+3}}{F_{2m+1}},$$

$$\sum_{j=0}^{4m+2} (-1)^{\frac{j(j-1)}{2}} \begin{Bmatrix} 4m \\ j \end{Bmatrix}_F L_{2m-j} = -\begin{Bmatrix} 4m \\ 4n+1 \end{Bmatrix}_F \frac{F_{4n+1}}{F_{2m}},$$

$$\sum_{j=0}^{4m+2} (-1)^{\frac{j(j-1)}{2}} \begin{Bmatrix} 4m \\ j \end{Bmatrix}_F F_{n+4m-j} = \frac{1}{2} F_{2m+n} \sum_{j=0}^{4m} (-1)^{\frac{j(j-1)}{2}} \begin{Bmatrix} 4m \\ j \end{Bmatrix}_F L_{2m-j},$$

$$\sum_{j=0}^{4m+2} (-1)^{\frac{j(j+1)}{2}} \begin{Bmatrix} 4m+2 \\ j \end{Bmatrix}_F F_{n+4m+2-j} =$$

$$= \frac{1}{2} F_{2m+n+1} \times \sum_{j=0}^{4m+2} (-1)^{\frac{j(j+1)}{2}} \begin{Bmatrix} 4m+2 \\ j \end{Bmatrix}_F L_{2m+1-j}.$$

They also proved that

$$\sigma_1(n) + \sigma_2(n) - \sigma_3(n) - \sigma_4(n) = 0, \tag{1.1}$$

where

$$\sigma_1(n)=\sum_{j=0}^{m}\left\{\begin{matrix}4m\\4j\end{matrix}\right\}_F F_{n+4m-4j},\ \sigma_2(n)=\sum_{j=0}^{m-1}\left\{\begin{matrix}4m\\4j+1\end{matrix}\right\}_F F_{n+4m-4j-1},$$

$$\sigma_3(n)=\sum_{j=0}^{m-1}\left\{\begin{matrix}4m\\4j+2\end{matrix}\right\}_F F_{n+4m-4j-2},\ \sigma_4(n)=\sum_{j=0}^{m-1}\left\{\begin{matrix}4m\\4j+3\end{matrix}\right\}_F F_{n+4m-4j-3}.$$

Also they conjectured three complex sum-product formulas involving the Fibonomial coefficients, Fibonacci and Lucas numbers. We recall this conjecture:

Conjecture 1. *Let m,n be any positive integers. Then*

$$i)\ \sigma_1(n)+\sigma_2(n)=F_{4m+n}\prod_{k=1}^{2m-1}L_{2k},$$

$$ii)\ \sigma_1(n)-\sigma_3(n)=(-1)^m F_{2m+n}L_{2m}\prod_{k=1}^{2m-1}L_k^2,$$

$$iii)\ \sigma_1(n)-\sigma_4(n)=F_n\prod_{k=1}^{2m-1}L_{2k}.$$

Finally by solving the system of linear equations in (1.1) and the Conjecture, they state their last result as

Corollary 1. *Let m,n be any positive integers. Then*

$$i)\ \sigma_1(n)=\frac{1}{2}F_{2m+n}L_{2m}\left((-1)^m\prod_{k=1}^{2m-1}L_k^2+\prod_{k=1}^{2m-1}L_{2k}\right),$$

$$ii)\ \sigma_2(n)=\frac{1}{2}\left((-1)^{m+1}F_{2m+n}L_{2m}\prod_{k=1}^{2m-1}L_k^2+L_{2m+n}F_{2m}\prod_{k=1}^{2m-1}L_{2k}\right),$$

$$iii)\ \sigma_3(n)=\frac{1}{2}F_{2m+n}L_{2m}\left((-1)^{m+1}\prod_{k=1}^{2m-1}L_k^2+\prod_{k=1}^{2m-1}L_{2k}\right),$$

$$iv)\ \sigma_4(n)=\frac{1}{2}\left((-1)^m F_{2m+n}L_{2m}\prod_{k=1}^{2m-1}L_k^2+L_{2m+n}F_{2m}\prod_{k=1}^{2m-1}L_{2k}\right).$$

In this paper, we shall prove the results of Corollary 1 rather than the Conjecture because if one can prove the results of Corollary 1, then the results of Conjecture could be easily proved by some simple algebra.

The present paper is organized as follows: (i) Before the proof of Corollary, we will give some auxiliary Lemmas. (ii) We rewrite it via the Gaussian q-binomial

coefficients and the q-Pochhammer symbol. (iii) Finally we give a solution for the formulas mentioned in (ii). The approach is to use q-analysis to prove the claimed formulas.

2. The corollary in terms of Gaussian q-binomial Coefficients

In this section, before the Corollary 1, we give some auxiliary Lemmas :

Lemma 1. *For nonnegative integer n and arbitrary integer c,*

$$\prod_{k=0}^{n}\left(1+q^{k+c+1}\right)=\frac{1+q^{n+c+1}}{1+q^{c}}\prod_{k=0}^{n}\left(1+q^{k+c}\right).$$

Proof. (Induction on n) If $n=0$, then the claim is easily seen follows from

$$1+q^{k+c+1}=\frac{1+q^{n+c+1}}{1+q^{c}}\left(1+q^{c}\right).$$

Suppose that the claim is true for $n>0$. Then we show that the claim is true for $n+1$. Consider

$$\prod_{k=0}^{n+1}\left(1+q^{k+c+1}\right)=\left(1+q^{n+c+2}\right)\prod_{k=0}^{n}\left(1+q^{k+c+1}\right),$$

which, by the induction hypothesis, equals

$$\begin{aligned}\prod_{k=0}^{n+1}\left(1+q^{k+c+1}\right)&=\left(1+q^{n+c+2}\right)\frac{\left(1+q^{n+c+1}\right)}{\left(1+q^{c}\right)}\prod_{k=0}^{n}\left(1+q^{k+c}\right)\\&=\frac{\left(1+q^{n+c+2}\right)}{\left(1+q^{c}\right)}\prod_{k=0}^{n+1}\left(1+q^{k+c}\right),\end{aligned}$$

as claimed. □

By using the Lemma 1 and the definition of q-Pochhammer symbol, we have the following lemmas:

Lemma 2. *For any positive integer m and arbitrary integer n,*
i)

$$\begin{aligned}&q^{-2m-\frac{n-1}{2}}\left(-q^{-2m+1};q\right)_{4m}-q^{2m+\frac{n+1}{2}}\left(-q^{-2m};q\right)_{4m}\\&\qquad=2q^{-\frac{1}{2}(n+4m^{2}+2m-1)}\left(-q;q\right)_{2m}^{2}\frac{\left(1-q^{n+2m}\right)}{\left(1+q^{2m}\right)},\end{aligned}$$

ii)

$$q^{-2m-\frac{n-1}{2}}\left(-\mathbf{i}q^{-2m+1};q\right)_{4m}+q^{-2m-\frac{n-1}{2}}\left(\mathbf{i}q^{-2m+1};q\right)_{4m}$$

$$-q^{2m+\frac{n+1}{2}}\left(-\mathbf{i}q^{-2m};q\right)_{4m}-q^{2m+\frac{n+1}{2}}\left(\mathbf{i}q^{-2m};q\right)_{4m}$$
$$=2(-1)^m q^{-\frac{1}{2}(n+4m^2+2m-1)}\left(1+q^{2m}\right)\left(1-q^{n+2m}\right)\left(-q^2;q^2\right)_{2m-1}.$$

Proof. To prove (i), we rewrite it as in the clear form

$$q^{-2m-\frac{n-1}{2}}\prod_{k=0}^{4m-1}\left(1+q^{k-2m+1}\right)-q^{2m+\frac{n+1}{2}}\prod_{k=0}^{4m-1}\left(1+q^{k-2m}\right)$$
$$=2q^{-\frac{n+4m^2+2m-1}{2}}\frac{\left(1-q^{n+2m}\right)}{\left(1+q^{2m}\right)}\sum_{k=1}^{2m}\left(1+q^k\right)^2$$

or

$$q^{m(2m-1)}\prod_{k=0}^{4m-1}\left(1+q^{k-2m+1}\right)-q^{2m^2+3m+n}\prod_{k=0}^{4m-1}\left(1+q^{k-2m}\right)$$
$$=2\frac{\left(1-q^{n+2m}\right)}{\left(1+q^{2m}\right)}\sum_{k=1}^{2m}\left(1+q^k\right)^2.$$

Now we shall consider the left hand side of the above equation:

$$q^{m(2m-1)}\prod_{k=0}^{4m-1}\left(1+q^{k-2m+1}\right)-q^{2m^2+3m+n}\prod_{k=0}^{4m-1}\left(1+q^{k-2m}\right).$$

By Lemma 1, we rewrite it as

$$q^{m(2m-1)}\left(\frac{1+q^{4m-1-2m+1}}{1+q^{-2m}}\right)\prod_{k=0}^{4m-1}\left(1+q^{k-2m}\right)$$
$$-q^{2m^2+3m+n}\prod_{k=0}^{4m-1}\left(1+q^{k-2m}\right)$$
$$=\left(q^{m(2m-1)}\left(\frac{1+q^{2m}}{1+q^{-2m}}\right)-q^{2m^2+3m+n}\right)\prod_{k=0}^{4m-1}\left(1+q^{k-2m}\right)$$
$$=q^{m(2m+1)}\left(1-q^{2m+n}\right)\prod_{k=0}^{4m-1}\left(1+q^{k-2m}\right).$$

Since

$$\prod_{k=0}^{4m-1}\left(1+q^{k-2m}\right)=\frac{2}{q^{m(2m+1)}\left(1+q^{2m}\right)}\sum_{k=1}^{2m}\left(1+q^k\right)^2,$$

we get

$$\begin{aligned}
&q^{m(2m-1)}\prod_{k=0}^{4m-1}\left(1+q^{k-2m+1}\right)-q^{2m^2+3m+n}\prod_{k=0}^{4m-1}\left(1+q^{k-2m}\right)\\
&=\frac{2q^{m(2m+1)}\left(1-q^{2m+n}\right)}{q^{m(2m+1)}\left(1+q^{2m}\right)}\sum_{k=1}^{2m}\left(1+q^{k}\right)^{2}\\
&=2\frac{\left(1-q^{2m+n}\right)}{\left(1+q^{2m}\right)}\left(-q;q\right)_{2m}^{2},
\end{aligned}$$

as claimed.

The second claim is similarly obtained. □

The Lemma 2 will be required to prove the first claim of Corollary 1. We give the following Lemma to prove the second claim of Corollary 1 without proof:

Lemma 3. *For any positive integer m and arbitrary integer n,*
i)

$$\begin{aligned}
&q^{2m-1}\left(-q^{-2m+1};q\right)_{4m}-q^{n+6m-1}\left(-q^{-2m};q\right)_{4m}\\
&=\frac{2\left(1-q^{n+2m}\right)\left(1+q^{2m}\right)}{q^{(m-1)(2m-1)}}\left(-q;q\right)_{2m-1}^{2}
\end{aligned}$$

and
ii)

$$\begin{aligned}
&\mathbf{i}q^{2m-1}\left(\left(\mathbf{i}q^{-2m+1};q\right)_{4m}-\left(-\mathbf{i}q^{-2m+1};q\right)_{4m}\right)\\
&-\mathbf{i}q^{n+6m-1}\left(\left(\mathbf{i}q^{-2m};q\right)_{4m}-\left(-\mathbf{i}q^{-2m};q\right)_{4m}\right)\\
&=2\left(-1\right)^{m+1}q^{(1-m)(2m-1)}\left(1+q^{n+2m}\right)\left(1-q^{2m}\right)\left(-q^{2};q^{2}\right)_{2m-1}.
\end{aligned}$$

3. Main Results

We will need to rewrite the Corollary 1 in terms of the Gaussian q-binomial coefficients and the q-Pochhammer symbol. Firstly we prove the first claim of the Corollary. It is stated as

$$\sum_{k=0}^{m}\left\{\begin{matrix}4m\\4k\end{matrix}\right\}_{F}F_{n+4m-4k}=\frac{1}{2}F_{n+2m}L_{2m}\left((-1)^{m}\sum_{k=1}^{2m-1}L_{k}^{2}+\sum_{k=1}^{2m-1}L_{2k}\right). \quad (3.1)$$

Before proving it, we convert the formula (3.1) into q-notation. The left hand side of it is:

$$\sum_{k=0}^{m}\left\{\begin{matrix}4m\\4k\end{matrix}\right\}_{F}F_{n+4m-4k}$$

$$= \frac{\alpha^{4m+n-1}}{(1-q)} \sum_{k=0}^{m} \begin{bmatrix} 4m \\ 4k \end{bmatrix}_q \alpha^{-4k+16km-16k^2} \left(1-q^{n+4m-4k}\right)$$

$$= \frac{\mathbf{i}^{n-1} q^{-\frac{n-1}{2}-2m}}{(1-q)} \sum_{k=0}^{m} \begin{bmatrix} 4m \\ 4k \end{bmatrix}_q q^{8k^2+2k-8km}$$

$$- \frac{\mathbf{i}^{n-1} q^{\frac{n+1}{2}+2m}}{(1-q)} \sum_{k=0}^{m} \begin{bmatrix} 4m \\ 4k \end{bmatrix}_q q^{8k^2-2k-8km}. \tag{3.2}$$

The right hand side of (3.1) is

$$\frac{1}{2} F_{n+2m} L_{2m} \left((-1)^m \sum_{k=1}^{2m-1} L_k^2 + \sum_{k=1}^{2m-1} L_{2k} \right) = \frac{1}{2} \mathbf{i}^{n-1} q^{-\frac{1}{2}(n+4m^2+2m-1)}$$

$$\times \left(1+q^{2m}\right) \frac{\left(1-q^{n+2m}\right)}{(1-q)} \left(\sum_{k=1}^{2m-1} \left(1+q^k\right)^2 + (-1)^m \sum_{k=1}^{2m-1} \left(1+q^{2k}\right) \right).$$

Thus we must show that

$$\left(q^{-\frac{n-1}{2}-2m} \sum_{k=0}^{m} \begin{bmatrix} 4m \\ 4k \end{bmatrix}_q q^{8k^2+2k-8km} - q^{\frac{n+1}{2}+2m} \sum_{k=0}^{m} \begin{bmatrix} 4m \\ 4k \end{bmatrix}_q q^{8k^2-2k-8km} \right)$$

$$\times \frac{\mathbf{i}^{n-1}}{1-q} = \frac{1}{2} \mathbf{i}^{n-1} q^{-\frac{1}{2}(n+4m^2+2m-1)} \frac{\left(1+q^{2m}\right)\left(1-q^{n+2m}\right)}{(1-q)}$$

$$\times \left((-q;q)_{2m-1}^2 + (-1)^m \left(-q^2;q^2\right)_{2m-1} \right).$$

Now we consider the first sums on the left hand side of the above equation and then compute it:

$$\sum_{k=0}^{m} \begin{bmatrix} 4m \\ 4k \end{bmatrix}_q q^{8k^2+2k-8km} = \frac{1}{4} \sum_{k=0}^{4m} \begin{bmatrix} 4m \\ k \end{bmatrix}_q q^{\binom{k+1}{2}-2mk} \left(1+(-1)^k+(-\mathbf{i})^k+\mathbf{i}^k\right),$$

which, by Rothe's formula, equals

$$\frac{1}{4} \frac{\mathbf{i}^{n-1} q^{-\frac{1}{2}(4m+n-1)}}{(1-q)} \left(\left(-q^{-2m+1};q\right)_{4m} + \left(q^{-2m+1};q\right)_{4m} \right.$$

$$\left. + \left(-\mathbf{i}q^{-2m+1};q\right)_{4m} + \left(\mathbf{i}q^{-2m+1};q\right)_{4m} \right),$$

which, since $\left(q^{-2m+1};q\right)_{4m} = 0$, gives us

$$\sum_{k=0}^{m} \begin{bmatrix} 4m \\ 4k \end{bmatrix}_q q^{8k^2+2k-8km} = \frac{1}{4} \frac{\mathbf{i}^{n-1} q^{-\frac{1}{2}(4m+n-1)}}{(1-q)}$$

$$\times\left(\left(-q^{-2m+1};q\right)_{4m}+\left(-\mathbf{i}q^{-2m+1};q\right)_{4m}+\left(\mathbf{i}q^{-2m+1};q\right)_{4m}\right).$$

Now we consider the second sums on the left hand side of (3.2):

$$\sum_{k=0}^{m}\begin{bmatrix}4m\\4k\end{bmatrix}_q q^{8k^2-2k-8km}=\frac{1}{4}\sum_{k=0}^{4m}\begin{bmatrix}4m\\4k\end{bmatrix}_q q^{\binom{k}{2}-2mk}\left(1+(-1)^k+(-\mathbf{i})^k+\mathbf{i}^k\right)$$

$$=\frac{\mathbf{i}^{n-1}q^{2m+\frac{n+1}{2}}}{4(1-q)}\left(\left(-q^{-2m};q\right)_{4m}+\left(q^{-2m};q\right)_{4m}+\left(-\mathbf{i}q^{-2m};q\right)_{4m}+\left(\mathbf{i}q^{-2m};q\right)_{4m}\right),$$

which, since $\left(q^{-2m};q\right)_{4m}=0$, equals

$$\frac{\mathbf{i}^{n-1}q^{2m+\frac{n+1}{2}}}{4(1-q)}\left(\left(-q^{-2m};q\right)_{4m}+\left(-iq^{-2m};q\right)_{4m}+\left(iq^{-2m};q\right)_{4m}\right).$$

Thus our original result takes the form:

$$\frac{\mathbf{i}^{n-1}}{(1-q)}\left(q^{-\frac{n-1}{2}-2m}\sum_{k=0}^{m}\begin{bmatrix}4m\\4k\end{bmatrix}_q q^{8k^2+2k-8km}-q^{\frac{n+1}{2}+2m}\sum_{k=0}^{m}\begin{bmatrix}4m\\4k\end{bmatrix}_q q^{8k^2-2k-8km}\right)$$

$$=\frac{\mathbf{i}^{n-1}q^{-2m-\frac{n-1}{2}}}{4(1-q)}\left(\left(-q^{-2m+1};q\right)_{4m}+\left(-\mathbf{i}q^{-2m+1};q\right)_{4m}+\left(\mathbf{i}q^{-2m+1};q\right)_{4m}\right)$$

$$-\frac{1}{4}\frac{\mathbf{i}^{n-1}q^{2m+\frac{n+1}{2}}}{(1-q)}\left(\left(-q^{-2m};q\right)_{4m}+\left(-\mathbf{i}q^{-2m};q\right)_{4m}+\left(\mathbf{i}q^{-2m};q\right)_{4m}\right)$$

$$=\frac{\mathbf{i}^{n-1}}{4(1-q)}\left(q^{-2m-\frac{n-1}{2}}\left(-q^{-2m+1};q\right)_{4m}+q^{-2m-\frac{n-1}{2}}\left(-\mathbf{i}q^{-2m+1};q\right)_{4m}\right.$$

$$+q^{-2m-\frac{n-1}{2}}\left(\mathbf{i}q^{-2m+1};q\right)_{4m}-q^{2m+\frac{n+1}{2}}\left(-q^{-2m};q\right)_{4m}$$

$$\left.-q^{2m+\frac{n+1}{2}}\left(-\mathbf{i}q^{-2m};q\right)_{4m}-q^{2m+\frac{n+1}{2}}\left(\mathbf{i}q^{-2m};q\right)_{4m}\right),$$

which, by Lemma 2, equals

$$\frac{1}{2}\mathbf{i}^{n-1}q^{-\frac{1}{2}\left(n+4m^2+2m-1\right)}\frac{\left(1-q^{n+2m}\right)}{(1-q)}$$

$$\times\left(\frac{1}{\left(1+q^{2m}\right)}(-q;q)_{2m}^2+(-1)^m\left(1+q^{2m}\right)\left(-q^2;q^2\right)_{2m-1}\right)$$

$$=\frac{1}{2}\mathbf{i}^{n-1}q^{-\frac{1}{2}\left(n+4m^2+2m-1\right)}\left(1+q^{2m}\right)\frac{\left(1-q^{n+2m}\right)}{(1-q)}$$

$$\times\left((-q;q)_{2m-1}^2+(-1)^m\left(-q^2;q^2\right)_{2m-1}\right),$$

as claimed.

The other claims in the Corollary 1 takes the following forms via the Gaussian q-binomial coefficients and the q-Pochhammer symbol. The second claim

$$\sum_{j=0}^{m-1} \left\{ \begin{matrix} 4m \\ 4j+1 \end{matrix} \right\}_F F_{n+4m-4j-1} = \frac{1}{2}\left((-1)^{m+1} F_{2m+n} L_{2m} \prod_{k=1}^{2m-1} L_k^2 + L_{2m+n} F_{2m} \prod_{k=1}^{2m-1} L_{2k} \right)$$

is converted into the q-form as

$$\begin{aligned} &\sum_{k=0}^{m-1} \begin{bmatrix} 4m \\ 4k+1 \end{bmatrix}_q q^{8k^2+6k-8km} \left(1-q^{n+4m-4k-1}\right) \\ &= \frac{1}{2}(-1)^{m+1} q^{-2m^2+3m-1} \Big((-1)^{m+1} \left(1-q^{2m+n}\right)\left(1+q^{2m}\right)(-q;q)_{2m-1}^2 \\ &\quad + \left(1+q^{2m+n}\right)\left(1-q^{2m}\right)\left(-q^2;q^2\right)_{2m-1} \Big). \end{aligned}$$

To prove the above claim, consider the left hand side of it:

$$\begin{aligned} &\sum_{k=0}^{m-1} \begin{bmatrix} 4m \\ 4k+1 \end{bmatrix}_q q^{8k^2+6k-8km} \left(1-q^{n+4m-4k-1}\right) \\ &= \sum_{k=0}^{m-1} \begin{bmatrix} 4m \\ 4k+1 \end{bmatrix}_q q^{8k^2+6k-8km} - q^{n+4m-1} \sum_{k=0}^{m-1} \begin{bmatrix} 4m \\ 4k+1 \end{bmatrix}_q q^{8k^2+2k-8km} \\ &= \frac{1}{4} \sum_{k=0}^{4m-4} \begin{bmatrix} 4m \\ k+1 \end{bmatrix}_q q^{\frac{1}{2}k(k-4m+3)} \left(1+(-1)^k+\mathbf{i}^k+(-\mathbf{i})^k\right) \\ &- \frac{1}{4} q^{n+4m-1} \sum_{k=0}^{4m-4} \begin{bmatrix} 4m \\ k+1 \end{bmatrix}_q q^{\frac{1}{2}k(k-4m+1)} \left(1+(-1)^k+\mathbf{i}^k+(-\mathbf{i})^k\right). \end{aligned}$$

Let $f(k) = q^{\frac{1}{2}k(k-4m+3)} \left(1+(-1)^k+\mathbf{i}^k+(-\mathbf{i})^k\right)$. Since $f(-1) = 0$, for the first sum in the above equation, we can write

$$\begin{aligned} &\sum_{k=0}^{4m-4} \begin{bmatrix} 4m \\ 4k+1 \end{bmatrix}_q q^{\frac{1}{2}k(k-4m+3)} \left(1+(-1)^k+\mathbf{i}^k+(-\mathbf{i})^k\right) \\ &= \sum_{k=0}^{4m-4} \begin{bmatrix} 4m \\ k+1 \end{bmatrix}_q f(k) = \sum_{k=1}^{4m-3} \begin{bmatrix} 4m \\ k \end{bmatrix}_q f(k-1) \end{aligned}$$

$$= \sum_{k=0}^{4m-3} \begin{bmatrix} 4m \\ k \end{bmatrix}_q f(k-1) - f(-1) = \sum_{k=0}^{4m-3} \begin{bmatrix} 4m \\ k \end{bmatrix}_q f(k-1)$$
$$= \sum_{k=0}^{4m-3} \begin{bmatrix} 4m \\ k \end{bmatrix}_q q^{\frac{1}{2}(k-1)(k-4m+2)} \left(1 + (-1)^{k-1} + \mathbf{i}^{k-1} + (-\mathbf{i})^{k-1}\right).$$

Similarly for the second sum in the same equation, we write

$$\sum_{k=0}^{4m-4} \begin{bmatrix} 4m \\ k+1 \end{bmatrix}_q q^{\frac{1}{2}k(k-4m+1)} \left(1 + (-1)^{k} + \mathbf{i}^{k} + (-\mathbf{i})^{k}\right)$$
$$= \sum_{k=0}^{4m-3} \begin{bmatrix} 4m \\ k \end{bmatrix}_q q^{\frac{1}{2}(k-1)(k-4m)} \left(1 + (-1)^{k-1} + \mathbf{i}^{k-1} + (-\mathbf{i})^{k-1}\right).$$

Consequently we get

$$\sum_{k=0}^{m-1} \begin{bmatrix} 4m \\ 4k+1 \end{bmatrix}_q q^{8k^2+6k-8km} \left(1 - q^{n+4m-4k-1}\right)$$
$$= \frac{1}{4} q^{2m-1} \sum_{k=0}^{4m-3} \begin{bmatrix} 4m \\ k \end{bmatrix}_q q^{\binom{k}{2}+(1-2m)k} \left(1 + (-1)^{k-1} + \mathbf{i}^{k-1} + (-\mathbf{i})^{k-1}\right)$$
$$- \frac{1}{4} q^{n+6m-1} \sum_{k=0}^{4m-3} \begin{bmatrix} 4m \\ k \end{bmatrix}_q q^{\binom{k}{2}-2mk} \left(1 + (-1)^{k-1} + \mathbf{i}^{k-1} + (-\mathbf{i})^{k-1}\right).$$

Let

$$A = \frac{q^{1-2m}\left(1-q^{4m}\right)\left(1-q^{4m-1}\right)}{\left(q^2-1\right)(q-1)} - \frac{\left(1-q^{4m}\right)}{q-1} + q^{2m},$$
$$B = \frac{q^{1-2m}\left(1-q^{4m}\right)\left(1-q^{4m-1}\right)}{\left(1-q^2\right)(1-q)} - \frac{\left(1-q^{4m}\right)}{(1-q)} + q^{2m},$$
$$C = -\frac{q^{1-2m}\left(1-q^{4m}\right)\left(1-q^{4m-1}\right)}{\left(1-q^2\right)(1-q)} - \mathbf{i}\frac{\left(1-q^{4m}\right)}{(1-q)} + q^{2m},$$
$$D = -\frac{q^{1-2m}\left(1-q^{4m}\right)\left(1-q^{4m-1}\right)}{\left(1-q^2\right)(1-q)} + \mathbf{i}\frac{\left(1-q^{4m}\right)}{(1-q)} + q^{2m},$$

and

$$A_1 = \frac{q^{3-6m}\left(1-q^{4m}\right)\left(1-q^{4m-1}\right)}{\left(1-q^2\right)(1-q)} + \frac{q^{1-4m}\left(1-q^{4m}\right)}{1-q} + q^{-2m}$$
$$B_1 = \frac{q^{3-6m}\left(1-q^{4m}\right)\left(1-q^{4m-1}\right)}{\left(1-q^2\right)(1-q)} - \frac{q^{1-4m}\left(1-q^{4m}\right)}{1-q} + q^{-2m}$$

$$C_1 = -\frac{q^{3-6m}\left(1-q^{4m}\right)\left(1-q^{4m-1}\right)}{\left(1-q^2\right)(1-q)} - \mathbf{i}\frac{q^{1-4m}\left(1-q^{4m}\right)}{(1-q)} + q^{-2m}$$

$$D_1 = -\frac{q^{3-6m}\left(1-q^{4m}\right)\left(1-q^{4m-1}\right)}{\left(1-q^2\right)(1-q)} + \mathbf{i}\frac{q^{1-4m}\left(1-q^{4m}\right)}{(1-q)} + q^{-2m}.$$

Then clearly $-A+B+\mathbf{i}C-\mathbf{i}D=0$ and $-A_1+B_1+\mathbf{i}C_1-\mathbf{i}D_1=0$. Thus

$$\begin{aligned}
&\frac{1}{4}q^{2m-1}\sum_{k=0}^{4m-3}\begin{bmatrix}4m\\k\end{bmatrix}_q q^{\binom{k}{2}+(1-2m)k}\left(1+(-1)^{k-1}+\mathbf{i}^{k-1}+(-\mathbf{i})^{k-1}\right)\\
&\quad-\frac{1}{4}q^{n+6m-1}\sum_{k=0}^{4m-3}\begin{bmatrix}4m\\k\end{bmatrix}_q q^{\binom{k}{2}-2mk}\left(1+(-1)^{k-1}+\mathbf{i}^{k-1}+(-\mathbf{i})^{k-1}\right)\\
&=\frac{1}{4}q^{2m-1}(\left(\left(-q^{1-2m};q\right)_{4m}-A\right)-\left(\left(q^{1-2m};q\right)_{4m}-B\right)\\
&\quad-\mathbf{i}\left(\left(-\mathbf{i}q^{1-2m};q\right)_{4m}-C\right)+\mathbf{i}\left(\left(\mathbf{i}q^{1-2m};q\right)_{4m}-D\right))\\
&\quad-\frac{1}{4}q^{n+6m-1}(\left(\left(-q^{-2m};q\right)_{4m}-A_1\right)-\left(\left(q^{-2m};q\right)_{4m}-B_1\right)\\
&\quad-\mathbf{i}\left(\left(-\mathbf{i}q^{-2m};q\right)_{4m}-C_1\right)+\mathbf{i}\left(\left(\mathbf{i}q^{-2m};q\right)_{4m}-D_1\right)),
\end{aligned}$$

which, since $\left(q^{1-2m};q\right)_{4m}=0$ and $\left(q^{-2m};q\right)_{4m}=0$, gives us

$$\begin{aligned}
&=\frac{1}{4}q^{2m-1}\left(\left(-q^{1-2m};q\right)_{4m}-\mathbf{i}\left(-\mathbf{i}q^{1-2m};q\right)_{4m}+\mathbf{i}\left(\mathbf{i}q^{1-2m};q\right)_{4m}\right)\\
&\quad-\frac{1}{4}q^{n+6m-1}\left(\left(-q^{-2m};q\right)_{4m}-\mathbf{i}\left(-\mathbf{i}q^{-2m};q\right)_{4m}+\mathbf{i}\left(\mathbf{i}q^{-2m};q\right)_{4m}\right).
\end{aligned}$$

By Lemma 3, we write the last equation as

$$\begin{aligned}
&=\frac{\left(1-q^{n+2m}\right)\left(1+q^{2m}\right)}{2q^{(m-1)(2m-1)}}\prod_{k=1}^{2m-1}\left(1+q^k\right)^2+\\
&\qquad(-1)^{m+1}\frac{\left(1+q^{n+2m}\right)\left(1-q^{2m}\right)}{2q^{(m-1)(2m-1)}}\prod_{k=1}^{2m-1}\left(1+q^{2k}\right),
\end{aligned}$$

as claimed.

As we state as before, if one have the Corollary 1, then the claims of Conjecture could be easily derived. As an example, since we have first two claims of Corollary 1, we can obtain the first claim of Conjecture :

$$\sigma_1(n)+\sigma_2(n)=\frac{1}{2}F_{2m+n}L_{2m}\left((-1)^m\prod_{k=1}^{2m-1}L_k^2+\prod_{k=1}^{2m-1}L_{2k}\right)$$

$$+\frac{1}{2}\left((-1)^{m+1}F_{2m+n}L_{2m}\prod_{k=1}^{2m-1}L_k^2+L_{2m+n}F_{2m}\prod_{k=1}^{2m-1}L_{2k}\right)$$
$$=\frac{1}{2}\left(F_{2m+n}L_{2m}+L_{2m+n}F_{2m}\right)\prod_{k=1}^{2m-1}L_{2k}$$
$$=F_{4m+n}\prod_{k=1}^{2m-1}L_{2k},$$

as claimed in the Conjecture.

After that we only state the other claims of Corollary 1 in q-notation and leave the proofs of them to the readers. They are similarly proved as we did previously.

The claims (iii) and (iv) in Corollary 1 take the following forms in q-notation:

$$iii)\qquad \sum_{k=0}^{m-1}\begin{bmatrix}4m\\4k+2\end{bmatrix}_q q^{2k(4k-4m+5)}\left(1-q^{n+4m-4k-2}\right)$$
$$=\frac{1}{2}(-1)^{m+1}q^{-(m-1)(2m-3)}\left(1-q^{2m+n}\right)\left(1+q^{2m}\right)$$
$$\times\left((-1)^{m+1}(-q;q)_{2m-1}^2+\left(-q^2;q^2\right)_{2m-1}\right)$$

and

$$\sum_{k=0}^{m-1}\begin{bmatrix}4m\\4k+3\end{bmatrix}_q q^{2k(4k-4m+7)}\left(1-q^{n+4m-4k-3}\right)$$
$$=\frac{1}{2}(-1)^m q^{-2m^2+7m-6}\left((-1)^m\left(1-q^{2m+n}\right)\left(1+q^{2m}\right)(-q;q)_{2m-1}^2\right.$$
$$\left.+\left(1+q^{2m+n}\right)\left(1-q^{2m}\right)\left(-q^2;q^2\right)_{2m-1}\right).$$

By following the proof way of the claims (i) and (ii), the above claims could be proved similarly.

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Author's address

Emrah Kılıç
TOBB Economics and Technology University, Mathematics Department, 06560 Ankara, Turkey
E-mail address: ekilic@etu.edu.tr

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A NOTE ON (σ,τ)-DERIVATIONS OF RINGS WITH INVOLUTION

EMINE KOÇ AND ÖZNUR GÖLBAŞI

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Abstract. Let R be a 2−torsion free simple *−ring and $D : R \to R$ be an additive mapping satisfiying $D(xx^*) = D(x)\sigma(x^*) + \tau(x)D(x^*)$, for all $x \in R$. Then D is a (σ,τ)−derivation of R or R is S_4 ring. Also, if R is a 2−torsion free semiprime ring and $G : R \to R$ is an additive mapping related with some (σ,τ)−derivation D of R such that $G(xx^*) = G(x)\sigma(x^*) + \tau(x)D(x^*)$, for all $x \in R$, then G is generalized (σ,τ)−derivation of R.

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1. INTRODUCTION

Let R be an associative ring with center Z. Recall that a ring R is prime if $xRy = \{0\}$ implies $x = 0$ or $y = 0$. A ring R is semiprime if $xRx = \{0\}$ implies $x = 0$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. A left (right) centralizer of R is an additive mapping $T : R \to R$ which satisfies $T(xy) = T(x)y$ $(T(xy) = xT(y))$ for all $x, y \in R$. If $a \in R$, then $L_a(x) = ax$ is a left centralizer and $R_a(x) = xa$ is a right centralizer. Inspired by the definition derivation and left (right) centralizer, the notion of (σ,τ)−derivation and σ−centralizer were extended as follow:

Let σ and τ be any two functions of R. An additive mapping $d : R \to R$ is called a (σ,τ)−derivation if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ holds for all $x, y \in R$. A left (right) σ−centralizer of R is an additive mapping $T : R \to R$ which satisfies $T(xy) = T(x)\sigma(y)$ $(T(xy) = \sigma(x)T(y))$ for all $x, y \in R$. Of course a $(1,1)$−derivation where 1 is the identity map on R is a derivation and a left (right) 1−centralizer is a left (right) centralizer. An additive mapping $x \to x^*$ on a ring R is called an involution if $(x^*)^* = x$ and $(xy)^* = y^*x^*$ holds for all $x, y \in R$. A ring equipped with an involution is called a ring with involution, or a *−ring. Let $S = \{x \in R \mid x^* = x\}$ be the set of symmetric elements of R and $K = \{x \in R \mid x^* = -x\}$ the set of skew elements of R. If A and B are nonempty subsets of R, then AB and $[A, B]$ will be additive subgroups of R generated respectively by ab and $[a,b] = ab - ba$ for all $a \in A, b \in B$.

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It is well known that a prime ring R satisfies the standard identity

$$S_{2n}(x_1, x_2, ..., x_{2n}) = \sum_{\sigma \in S_{2n}} (-1)^{\sigma} x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(2n)}$$

if and only if R is an order in a simple algebra of dimension at most n^2 over its center. Since such a condition appears from time to time in what follows, we shall say that "R satisfies S_{2n}" for simplicity. Otherwise, as in Lanski [13], R will be called $S_{2n}-$free.

Recently, in [4], Bresar introduced the following definition: An additive mapping $f : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that

$$f(xy) = f(x)y + xd(y), \text{ for all } x, y \in R.$$

One may observe that the concept of generalized derivation includes the concept of derivations and the left centralizers when $d = 0$. The main examples are the derivations and generalized inner derivations a functions $f_{a,b} : R \to R$, the type $f_{a,b}(x) = ax + xb$ for some fixed $a, b \in R$. Given an arbitrary mapping $f : R \to R$ and additive mapping $d : R \to R$ of a semiprime (or prime) ring R such that $f(xy) = f(x)y + xd(y)$, for all $x, y \in R$, we note that f is uniquely defined by d, which should be a derivation by [4, Remark 2]. The notion of generalized derivation was extended as follows: Let σ, τ two functions of R. An additive mapping $f : R \to R$ is called a generalized $(\sigma, \tau)-$derivation on R if there exists a $(\sigma, \tau)-$derivation $d : R \to R$ such that

$$f(xy) = f(x)\sigma(y) + \tau(x)d(y), \text{ for all } x, y \in R.$$

On the other hand, an additive mapping $d : R \to R$ is called a Jordan derivation if $d(x^2) = d(x)x + xd(x)$ holds for all $x \in R$. Every derivation is a Jordan derivation. The converse is false in general. Herstein's result [8] states that each Jordan derivation of a prime 2-torsion free ring is a derivation. M. Bresar extended this result to the case of Jordan derivations of a semiprime 2-torsion free rings in [5]. In [6], under same conditions it was shown that each of Jordan $(\sigma, \tau)-$derivation of a prime 2-torsion free ring is a $(\sigma, \tau)-$derivation. C. Lanski showed the same theorem for semiprime rings in [14]. Following [2], M. Ashraf and N. Rehman proved it for a generalized derivation of a prime 2-torsion free ring.

I. N. Herstein proved that if R is a simple $^*-$ring with characteristic different from two, $dim_Z R > 4$ and an additive mapping $D : R \to R$ such that $D(xx^*) = D(x)x^* + xD(x^*)$, for all $x \in R$, then D must be a derivation in [10, Theorem 4.1.3]. M. N. Daif and M. S. Tammam El-Sayiad extended this result for additive mapping $G : R \to R$ related with some derivation D of R such that $G(xx^*) = G(x)x^* + xD(x^*)$, for all x in a $2-$torsion free semiprime $^*-$ring R in [7]. They showed that G is a Jordan derivation of R. Also, in [15], Vukman and Kosi-Ulbl proved that R is a $2-$torsion free semiprime $^*-$ring and an additive mapping $T : R \to R$ is an additive mapping such that $T(xx^*) = T(x)x^*$ $(T(x^*x) = x^*T(x))$ is fulfilled for all

$x \in R$, then T is a left (right) centralizer. This result was extended for a left (right) $\sigma-$centralizer of R in [1].

The first purpose of this paper is to prove the theorem in [10, Theorem 4.1.3] for $(\sigma,\tau)-$derivation of R. The second aim is to show the theorem in [7, Theorem 2.1] for generalized $(\sigma,\tau)-$derivation of R.

2. Results

Throughout the present paper, σ and τ are automorphisms of R. In order to prove the theorems, we shall require the following lemmas.

Lemma 1 ([15], Lemma 1)**.** *Let R be a semiprime $^*-$ring. Suppose there exists an element $a \in R$ such that $ax^* = ax$ for all $x \in R$. In this case $a \in Z$.*

Lemma 2 ([9], Theorem 1.3)**.** *Let R be a simple ring of characteristic differrent from two and U be a Lie ideal of R. Then either $U \subset Z$ or $[R,R] \subset U$.*

Lemma 3 ([9], Corollary, p.6)**.** *If R is a noncommutative simple ring of characteristic differrent from two. Then the subring generated by $[R,R]$ in R.*

Lemma 4 ([9], Lemma 2.1)**.** *Let R be any ring with involution $R = S + K$, then K^2 is a Lie ideal of R.*

Lemma 5 ([10], Theorem 2.1.2)**.** *Let R be a $2-$torsion free semiprime ring and suppose that A is both a subring of R and a Lie ideal of R. Then $A \subset Z$ or A contains a nonzero ideal of R.*

Lemma 6 ([12], Lemma 2)**.** *Let R be any semiprime ring with involution. If $[K^2,K^2] = (0)$, then R satisfies S_4.*

Lemma 7 ([14], Theorem 2)**.** *Let R be a $2-$torsion free semiprime ring and d a Jordan $(\phi,\theta)-$derivation with ϕ or θ an automorphism of R. Then d is a $(\phi,\theta)-$derivation of R.*

Lemma 8 ([1], Theorem 2.2)**.** *Let R be a $2-$torsion free semiprime ring and α be an automorphism of R. If $T : R \to R$ is an additive mapping such that $T(x^2) = T(x)\sigma(x)$ for all $x \in R$, then T is a left $\alpha-$centralizer.*

Lemma 9 ([3], Lemma 4)**.** *Let R be a $2-$torsion free prime ring, U is a Lie ideal of R and $a,b \in R$. If $aUb = (0)$, then $a = 0$ or $b = 0$ or $U \subset Z$.*

Lemma 10 ([11], 1.1 Lemma)**.** *Let R be a prime ring with characteristic not two and U a nonzero Lie ideal of R. If d is a nonzero $(\sigma,\tau)-$derivation of R such that $d(U) = 0$, then $U \subseteq Z$.*

The following theorem gives a generalization of [10, Theorem 4.1.3] for $(\sigma,\tau)-$derivation of R.

Theorem 1. *Let R be a $2-$torsion free simple $*-$ring. Suppose there exists an additive mapping $D : R \to R$ such that*

$$D(xx^*) = D(x)\sigma(x^*) + \tau(x)D(x^*), \text{ for all } x \in R. \tag{2.1}$$

Then D is $(\sigma,\tau)-$derivation of R or R is S_4 ring.

Proof. Assume that R is S_4 free. A linearization of (2.1) yields that

$$\begin{aligned} D(xx^* + xy^* + yx^* + yy^*) = {} & D(x)\sigma(x^*) + D(x)\sigma(y^*) + D(y)\sigma(x^*) \\ & + D(y)\sigma(y^*) + \tau(x)D(x^*) + \tau(y)D(x^*) \\ & + \tau(x)D(y^*) + \tau(y)D(y^*), \text{ for all } x, y \in R. \end{aligned}$$

Using D is an additive mapping and (2.1), we arrive at

$$D(xy^* + yx^*) = D(x)\sigma(y^*) + D(y)\sigma(x^*) + \tau(y)D(x^*) + \tau(x)D(y^*). \tag{2.2}$$

Taking x^* instead of y in (2.2), we get

$$D(x^2 + (x^*)^2) = D(x)\sigma(x) + D(x^*)\sigma(x^*) + \tau(x^*)D(x^*) + \tau(x)D(x),$$

and so

$$D(x^2) - D(x)\sigma(x) - \tau(x)D(x) + D((x^*)^2) - D(x^*)\sigma(x^*) - \tau(x^*)D(x^*) = 0.$$

This relation reduces to

$$A(x) + A(x^*) = 0, \text{ for all } x \in R$$

where $A(x)$ stands for $A(x) = D(x^2) - D(x)\sigma(x) - \tau(x)D(x)$. Replacing y by $xy^* + yx^*$ in (2.2), we obtain that

$$\begin{aligned} D(x(xy^* + yx^*)^* + (xy^* + yx^*)x^*) = {} & D(x)\sigma((xy^* + yx^*)^*) \\ & + \tau(x)D((xy^* + yx^*)^*) + D(xy^* + yx^*)\sigma(x^*) + \tau(xy^* + yx^*)D(x^*), \end{aligned}$$

and so

$$\begin{aligned} D(xyx^* + x^2y^* + xy^*x^* + y(x^*)^2) = {} & D(x)\sigma(yx^*) + D(x)\sigma(xy^*) \\ & + \tau(x)D(yx^* + xy^*) + D(xy^* + yx^*)\sigma(x^*) + \tau(xy^*)D(x^*) + \tau(yx^*)D(x^*). \end{aligned}$$

Using (2.2), we have

$$\begin{aligned} D(x(y + y^*)x^*) + D(x^2y^* + y(x^*)^2) = {} & D(y)\sigma(x^*)\sigma(x^*) + D(x)\sigma(y^*)\sigma(x^*) \\ & + \tau(y)D(x^*)\sigma(x^*) + \tau(x)D(y^*)\sigma(x^*) \\ & + D(x)\sigma(yx^*) + D(x)\sigma(xy^*) \\ & + \tau(xy^*)D(x^*) + \tau(yx^*)D(x^*) \\ & + \tau(x)D(y)\sigma(x^*) + \tau(x)D(x)\sigma(y^*) \\ & + \tau(xy)D(x) + \tau(x^2)D(y^*). \end{aligned}$$

Now the relation (2.2) reduces to $D(x^2y^*+y(x^2)^*)=D(y)\sigma(x^2)^*+D(x^2)\sigma(y^*)$ $+\tau(y)D((x^*)^2)+\tau(x^2)D(y^*)$. Using this in the last equation, we arrive at

$$\begin{aligned}D(x(y+y^*)x^*) &= -A(x)\sigma(y^*)-\tau(y)A(x^*)+D(x)\sigma(y)\sigma(x^*)\\ &\quad +D(x)\sigma(y^*)\sigma(x^*)+\tau(x)D(y^*)\sigma(x^*)+\tau(x)\tau(y^*)D(x^*)\\ &\quad +\tau(x)D(y)\sigma(x^*)+\tau(x)\tau(y)D(x^*).\end{aligned}$$

We can write the last equation such as

$$\begin{aligned}D(x(y+y^*)x^*) &= -A(x)\sigma(y^*)-\tau(y)A(x^*)+D(x)\sigma(y+y^*)\sigma(x^*)\\ &\quad +\tau(x)\tau(y+y^*)D(x^*)+\tau(x)D(y+y^*)\sigma(x^*). \qquad (2.3)\end{aligned}$$

Replacing $y-y^*$ by y in (2.3), we get

$$-A(x)\sigma(y^*-y)-\tau(y-y^*)A(x^*)=0,$$

and so

$$A(x)\sigma(y^*)-A(x)\sigma(y)+\tau(y)A(x^*)-\tau(y^*)A(x^*)=0.$$

Using $A(x)+A(x^*)=0$, for all $x\in R$ in the last equation, we arrive at

$$A(x)\sigma(y^*)+\tau(y^*)A(x)=A(x)\sigma(y)+\tau(y)A(x), \quad \text{for all } x,y\in R. \qquad (2.4)$$

Now, writting $k\in K$ by y in (2.4), we have

$$-A(x)\sigma(k)-\tau(k)A(x)=A(x)\sigma(k)+\tau(k)A(x),$$

and so

$$2(A(x)\sigma(k)+\tau(k)A(x))=0.$$

Since R is 2−torsion free ring, we get

$$A(x)\sigma(k)+\tau(k)A(x)=0, \quad \text{for all } k\in K, x\in R. \qquad (2.5)$$

Multipliying (2.5) from the right by $\sigma(t), t\in K$ and using (2.5), we obtain that

$$[A(x),kt]_{\sigma,\tau}=0,$$

and so

$$[A(x),K^2]_{\sigma,\tau}=0. \qquad (2.6)$$

We know that K^2 is a Lie ideal of R by Lemma 4. So, in view of Lemma 2, we have either $K^2\subset Z$ or $[R,R]\subset K^2$. If $K^2\subset Z$, then R is S_4 ring by Lemma 6. Since R is S_4 free, we have $[R,R]\subset K^2$. Also, by Lemma 3 and (2.6), we conclude that

$$[A(x),R]_{\sigma,\tau}=(0).$$

Hence we obtain that $A(x)\in C_{\sigma,\tau}$, for all $x\in R$. Returning (2.5) and using $A(x)\in C_{\sigma,\tau}$, we arrive at

$$2A(x)\sigma(k)=0, \quad \text{for all } k\in K, x\in R,$$

and so

$$\sigma^{-1}(A(x))K^2=(0), \quad \text{for all } x\in R.$$

Using $[R,R] \subset K^2$ and Lemma 3, we have $\sigma^{-1}(A(x))R = (0)$, and so $A(x) = (0)$, for all $x \in R$ by the semiprimeness of R.

$$D(x^2) = D(x)\sigma(x) + \tau(x)D(x), \text{ for all } x \in R.$$

Thus we obtain that D is a Jordan $(\sigma,\tau)-$derivation of R, and so D is $(\sigma,\tau)-$derivation of R by Lemma 7. □

Theorem 2. *Let R be a $2-$torsion free prime $^*-$ring. Suppose there exists an additive mapping $D : R \to R$ such that*

$$D(xx^*) = D(x)\sigma(x^*) + \tau(x)D(x^*), \text{ for all } x \in R.$$

Then D is a $(\sigma,\tau)-$derivation of R or R is S_4 ring.

Proof. Using the same methods in the proof of Theorem 1, we have

$$A(x)\sigma(k) + \tau(k)A(x) = 0, \text{ for all } k \in K, x \in R. \tag{2.7}$$

Multipliying (2.7) from the right by $\sigma(t), t \in K$ and using (2.5), we obtain that

$$[A(x), kt]_{\sigma,\tau} = 0,$$

and so

$$[A(x), K^2]_{\sigma,\tau} = (0).$$

This yields that $d_{A(x)}\left(K^2\right) = (0)$, where $d_{A(x)} : R \to R$, $d_{A(x)}(y) = [A(x), y]_{\sigma,\tau}$ is an inner $(\sigma,\tau)-$derivation of R. Since K^2 is a Lie ideal of R, we have $K^2 \subset Z$ or $d_{A(x)} = 0$, for all $x \in R$ by Lemma 10. In the first case, R is S_4 ring by Lemma 6. So, we get $d_{A(x)} = 0$, for all $x \in R$. Hence we obtain that $A(x) \in C_{\sigma,\tau}$, for all $x \in R$. Returning (2.7) and using $A(x) \in C_{\sigma,\tau}$, we get $2A(x)\sigma(k) = 0$, and so $A(x)\sigma(ktw) = 0$, for all $k,t,w \in K, x \in R$. Hence $\sigma^{-1}(A(x))K^2\sigma^{-1}(w) = 0$, for all $w \in K, x \in R$. By Lemma 9, we have $A(x) = 0$, for all $x \in R$ or $K = (0)$. If $K = (0)$, then $K^2 \subset Z$, and so R is S_4 ring by Lemma 6. So, we get $A(x) = 0$, for all $x \in R$. That is

$$D(x^2) = D(x)\sigma(x) + \tau(x)D(x), \text{ for all } x \in R.$$

Thus we obtain that D is a Jordan $(\sigma,\tau)-$derivation of R, and so D is $(\sigma,\tau)-$derivation of R by Lemma 7. □

The following theorem extends [7, Theorem 2.1] to generalized $(\sigma,\tau)-$derivations of R.

Theorem 3. *Let R be a semiprime $2-$torsion free $*-$ring. Suppose there exists an additive mapping $G : R \to R$ related with some $(\sigma,\tau)-$derivation D such that*

$$G(xx^*) = G(x)\sigma(x^*) + \tau(x)D(x^*), \text{ for all } x \in R. \tag{2.8}$$

Then G is generalized $(\sigma,\tau)-$derivation of R.

Proof. A linearization of (2.8) yields that

$$G(xx^*+xy^*+yx^*+yy^*)=G(x+y)\sigma(x^*+y^*)+\tau(x+y)D(x^*+y^*),$$

for all $x,y\in R$. Using G is an additive mapping and (2.8), we arrive at

$$G(xy^*+yx^*)=G(y)\sigma(x^*)+G(x)\sigma(y^*)+\tau(y)D(x^*)+\tau(x)D(y^*). \tag{2.9}$$

Substituting x^* for y in (2.9), we get

$$G(x^2+(x^*)^2)=G(x^*)\sigma(x^*)+G(x)\sigma(x)+\tau(x^*)D(x^*)+\tau(x)D(x),$$

and so

$$G(x^2)-G(x)\sigma(x)-\tau(x)D(x)+G((x^*)^2)-G(x^*)\sigma(x^*)-\tau(x^*)D(x^*)=0.$$

This relation reduces to

$$A(x)+A(x^*)=0,\ \text{for all } x\in R$$

where $A(x)$ stands for $A(x)=G(x^2)-G(x)\sigma(x)-\tau(x)D(x)$. Replacing y by xy^*+yx^* in (2.9), we obtain that

$$\begin{aligned}G(x(xy^*+yx^*)^*+(xy^*+yx^*)x^*)&=G(x)\sigma((xy^*+yx^*)^*)\\&\quad+G(xy^*+yx^*)\sigma(x^*)+\tau(xy^*+yx^*)D(x^*)+\tau(x)D((xy^*+yx^*)^*),\end{aligned}$$

and so

$$\begin{aligned}G(x(yx^*+xy^*)+(xy^*+yx^*)x^*)&=G(xy^*+yx^*)\sigma(x^*)+G(x)\sigma(yx^*+xy^*)\\&\quad+\tau(xy^*+yx^*)D(x^*)+\tau(x)D(yx^*+xy^*).\end{aligned}$$

Using (2.9) and D is $(\sigma,\tau)-$derivation of R, we have

$$\begin{aligned}G(xyx^*+x^2y^*+xy^*x^*+y(x^*)^2)&=(G(y)\sigma(x^*)+G(x)\sigma(y^*)+\tau(y)D(x^*)\\&\quad+\tau(x)D(y^*))\sigma(x^*)+G(x)\sigma(yx^*+xy^*)\\&\quad+\tau(xy^*+yx^*)D(x^*)+\tau(x)(D(y)\sigma(x^*)\\&\quad+\tau(y)D(x^*)+D(x)\sigma(y^*)+\tau(x)D(y^*).\end{aligned}$$

Again using

$$G(x^2y^*+y(x^2)^*)=G(y)\sigma(x^2)^*+G(x^2)\sigma(y^*)+\tau(y)D((x^*)^2)+\tau(x^2)D(y^*)$$

in the last equation, we arrive at

$$\begin{aligned}G(x(y+y^*)x^*)&=-G(x^2)\sigma(y^*)+G(x)\sigma(y^*)\sigma(x^*)+\tau(x)D(y^*)\sigma(x^*)\\&\quad+G(x)\sigma(y)\sigma(x^*)+G(x)\sigma(x)\sigma(y^*)+\tau(x)\tau(y^*)D(x^*)\\&\quad+\tau(x)D(y)\sigma(x^*)+\tau(x)\tau(y)D(x^*)+\tau(x)D(x)\sigma(y^*),\end{aligned}$$

and so

$$G(x(y+y^*)x^*)=-A(x)\sigma(y^*)+G(x)\sigma(y+y^*)\sigma(x^*)+\tau(x)D((y+y^*)x^*). \tag{2.10}$$

Replacing $y-y^*$ by y in (2.10), we get

$$A(x)\sigma(y)=A(x)\sigma(y^*),$$

and so

$$\sigma^{-1}(A(x))y=\sigma^{-1}(A(x))y^*, \text{ for all } x,y\in R.$$

By Lemma 1 and σ is an automorphism of R, we have $A(x)\in Z$, for all $x\in R$.

Now, writing y^* by y in (2.9), we have

$$G(xy+y^*x^*)=G(y^*)\sigma(x^*)+G(x)\sigma(y)+\tau(y^*)D(x^*)+\tau(x)D(y). \quad (2.11)$$

Taking xy instead of y in (2.11), we conclude that

$$\begin{aligned}G(x^2y+y^*(x^*)^2)&=G(y^*x^*)\sigma(x^*)+G(x)\sigma(x)\sigma(y)+\tau(y^*)\tau(x^*)D(x^*)\\&\quad+\tau(x)D(x)\sigma(y)+\tau(x)\tau(x)D(y). \quad (2.12)\end{aligned}$$

Replacing x^2 by x in (2.11), we get

$$\begin{aligned}G(x^2y+y^*(x^*)^2)&=G(y^*)\sigma((x^*)^2)+G(x^2)\sigma(y)+\tau(y^*)D(x^*)\sigma(x^*)\\&\quad+\tau(y^*)\tau(x^*)D(x^*)+\tau(x)\tau(x)D(y). \quad (2.13)\end{aligned}$$

Comparing (2.12) and (2.13), we obtain that

$$A(x)\sigma(y)+(G(y^*)\sigma(x^*)+\tau(y^*)D(x^*)-G(y^*x^*)\sigma(x^*))=0.$$

Replacing y by x in this equation, we get

$$A(x)\sigma(x)+(G(x^*)\sigma(x^*)+\tau(x^*)D(x^*)-G(x^*x^*)\sigma(x^*))=0,$$

and so

$$A(x)\sigma(x)-A(x^*)\sigma(x^*)=0.$$

Using $A(x)+A(x^*)=0$, we arrive at

$$A(x)\sigma(x+x^*)=0 \quad (2.14)$$

Returning $A(x)\sigma(y)=A(x)\sigma(y^*)$ and writing x by y in this equation, we have

$$A(x)\sigma(x-x^*)=0. \quad (2.15)$$

Combining (2.14) and (2.15), we arrive at $A(x)\sigma(x)=0$, and so $\sigma(x)A(x)=0$. A linearization of this equation yields that

$$A(x+y)\sigma(x+y)=0,$$

and so

$$\begin{aligned}&(G(x^2)-G(x)\sigma(x)-\tau(x)D(x)+G(xy+yx)-G(x)\sigma(y)-G(y)\sigma(x)\\&+G(y^2)-G(y)\sigma(y)-\tau(y)D(y)-\tau(x)D(y)-\tau(y)D(x))\sigma(x+y)=0.\end{aligned}$$

Defining $B(x,y)=G(xy+yx)-G(x)\sigma(y)-G(y)\sigma(x)-\tau(x)D(y)-\tau(y)D(x)$, we arrive at

$$(A(x)+B(x,y)+A(y))\sigma(x+y)=0.$$

Expanding this equation and using $A(x)\sigma(x)=0$, we have

$$B(x,y)\sigma(x)+A(y)\sigma(x)+A(x)\sigma(y)+B(x,y)\sigma(y)=0. \tag{2.16}$$

Taking $-x$ instead of x in (2.16) and using $A(-x)=A(x)$, $B(-x,y)=-B(x,y)$, we conclude that

$$B(x,y)\sigma(x)-A(y)\sigma(x)+A(x)\sigma(y)-B(x,y)\sigma(y)=0. \tag{2.17}$$

Adding (2.16) and (2.17), we obtain that

$$2B(x,y)\sigma(x)+2A(x)\sigma(y)=0.$$

Since R is 2−torsion free ring, we get $B(x,y)\sigma(x)+A(x)\sigma(y)=0$, for all $x,y\in R$. Multipliying this equation with $A(x)$ from the right and using $\sigma(x)A(x)=0$, we find that

$$A(x)\sigma(y)A(x)=0, \text{ for all } x,y\in R.$$

Since R is semiprime ring, we have $A(x)=0$, and so

$$G(x^2)=G(x)\sigma(x)+\tau(x)D(x), \text{ for all } x\in R.$$

Hence G is a generalized (σ,τ)−Jordan derivation of R.

Now, let assume $T=G-D$. We get

$$\begin{aligned} T(x^2)=(G-D)(x^2) &= G(x)\sigma(x)+\tau(x)D(x)-D(x)\sigma(x)-\tau(x)D(x)\\ &= G(x)\sigma(x)-D(x)\sigma(x)\\ &= (G-D)(x)\sigma(x)=T(x)\sigma(x). \end{aligned}$$

Hence we find that T is a Jordan σ−centralizer. In view of Lemma 8, T is left σ−centralizer. On the other hand, since $G=D+T$, we have

$$\begin{aligned} G(xy)=(D+T)(xy) &= D(x)\sigma(y)+\tau(x)D(y)+T(x)\sigma(y)\\ &= (D+T)(x)\sigma(y)+\tau(x)D(y), \end{aligned}$$

and so

$$G(xy)=G(x)\sigma(y)+\tau(x)D(y), \text{ for all } x,y\in R.$$

Hence we obtain that G is generalized (σ,τ)−derivation of R. □

Corollary 1. *Let R be a 2−torsion free simple *−ring. Suppose there exists an additive mapping $G:R\to R$ related with some additive mapping D such that*

$$D(xx^*)=D(x)\sigma(x^*)+\tau(x)D(x^*), \textit{ for all } x\in R$$

and

$$G(xx^*)=G(x)\sigma(x^*)+\tau(x)D(x^*), \textit{ for all } x\in R.$$

Then G is generalized (σ,τ)−derivation of R or R is S_4 ring.

Proof. By Theorem 1, we find that D is a (σ,τ)−derivation of R or R is S_4 ring. Hence G is generalized (σ,τ)−derivation of R by Theorem 3. □

Corollary 2. *Let R be a $2-$torsion free prime $*-$ring. Suppose there exists an additive mapping $G : R \to R$ related with some additive mapping D such that*

$$D(xx^*) = D(x)\sigma(x^*) + \tau(x)D(x^*), \text{ for all } x \in R$$

and

$$G(xx^*) = G(x)\sigma(x^*) + \tau(x)D(x^*), \text{ for all } x \in R.$$

Then G is generalized $(\sigma,\tau)-$derivation of R or R is S_4 ring.

In particular, if we take $D = 0$ in Theorem 3, we have the following corollary which is in [15].

Corollary 3. *Let R be a $2-$torsion free semiprime $*-$ring. Suppose there exists an additive mapping $T : R \to R$ an additive mapping such that*

$$T(xx^*) = T(x)\sigma(x^*), \text{ for all } x \in R.$$

Then T is left $\sigma-$centralizer.

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Authors' addresses

Emine Koç
Cumhuriyet University, Department of Mathematics, Sivas, TURKEY
E-mail address: eminekoc@cumhuriyet.edu.tr

Öznur Gölbaşı
Cumhuriyet University, Department of Mathematics, Sivas, TURKEY
E-mail address: ogolbasi@cumhuriyet.edu.trt

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CHARACTERIZATION OF $V(\mathbb{Z}\mathbf{C}_n^+)$ OF RANK ≤ 4

BÜLENT KÖKLÜCE AND NESIBE TÜFEKÇI

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Abstract. There are only a few cases for cyclic groups $\mathbf{C}_n$, where the unit group of their integral group rings are determined. In this article, we have completed the characterization of torsion-free part $V(\mathbb{Z}\mathbf{C}_n^+)$ of the integral group ring of cyclic group $\mathbf{C}_n$, for $n = 11, 15, 16, 20, 24$ and 30 where $\rho \leq 4$. We explicitly find all the generators.

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1. INTRODUCTION

Let G be a finite group and $\mathcal{U}(\mathbb{Z}G)$ be the unit group of integral group ring $\mathbb{Z}G$. We denote the group of normalized units of $\mathbb{Z}G$ by $\mathcal{U} = V(\mathbb{Z}G)$ and the subgroup of the central units of $\mathcal{U}$ by $\mathcal{Z}(\mathcal{U})$. By $\mathcal{N}_{\mathcal{U}}(G)$ we mean the normalizer of G in $\mathcal{U}$. The structure of the unit group $\mathcal{U}(\mathbb{Z}G)$ has been of a fundamental interest after the G. Higman's thesis written in 1939. Problem 43 in [17] asks whether the normalizer property; $\mathcal{N}_{\mathcal{U}}(G) = G\mathcal{Z}(\mathcal{U})$, holds for any finite group G.

In [5] it is shown by Coleman in particular that, this property holds for any finite nilpotent group. Jackowski and Marciniak [8] extended this to finite groups of odd order and the groups having normal Sylow 2-subgroup. Later in [14] Li et al. have shown that if the intersection of non-normal subgroups of G is non-trivial then, G satisfies the normalizer property. Meanwhile, the normalizer property is verified by Li [13] for some metabelian groups and by Marciniak and Roggenkamp [15] for finite metabelian groups with an abelian Sylow 2-subgroup. We refer the reader to [2] for a general information about the theory of group rings and to [6,7,9,11,16] for a survey on results on the unit group of integral group rings.

An explicit characterization of the normalizer of $D_n = \langle a, b : a^n = b^2 = 1, b^{-1}ab = a^{-1}\rangle$, dihedral group of order $2n$, in the normalized units of $\mathbb{Z}D_n$ has recently been given by Bilgin [4]. In the present paper, we give the characterization of torsion-free part of the integral group ring of some cyclic groups. Before we give the main theorems we obtained, we introduce some notations and give some basic facts.

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Let $\mathbf{C}_n =< a : a^n = 1 >$ be a cyclic group of order n, $\mathbb{Z}\mathbf{C}_n$ be its integral group ring and $V(\mathbb{Z}\mathbf{C}_n)$ be its normalized units. Now, let us consider the following subring of symmetric elements of $\mathbb{Z}\mathbf{C}_n$

$$\mathbb{Z}\mathbf{C}_n^+ = \left\{ \gamma = \sum_{i=0}^{n-1} \gamma_i a^i : \gamma_i = \gamma_{n-i} \right\}.$$

If we denote the unit group of $\mathbb{Z}\mathbf{C}_n^+$ by $\mathcal{U}(\mathbb{Z}\mathbf{C}_n^+)$ and its normalized units by $V(\mathbb{Z}\mathbf{C}_n^+)$ then we can state Higman's following result :

Theorem 1. $\mathcal{N}_{\mathcal{U}}(D_n) = D_n \times V(\mathbb{Z}\mathbf{C}_n^+)$.

When we modify Higman's result [17] for a cyclic group of order n instead of a finite abelian group, we obtain the following corollary:

Corollary 1. $V(\mathbb{Z}\mathbf{C}_n^+) = \mathcal{Z}(D_n) \times F$, *where* F *is a free abelian group of rank*

$$\rho = \frac{1}{2}\varphi(n) - 1.$$

Corollary 1 reduces the characterization of $V(\mathbb{Z}\mathbf{C}_n)$ to the construction of generator(s) of F. So, $V(\mathbb{Z}\mathbf{C}_n)$ is trivial when $n = 1, 2, 3, 4$ or 6. Torsion free part of $V(\mathbb{Z}\mathbf{C}_n)$ has a unique generator if $n = 5, 8, 10$ or 12, and it has been characterized for $n = 5, 8$ and 12 in [3, 10] as follows

$$V(\mathbb{Z}\mathbf{C}_5) = \mathbf{C}_5 \times \left\langle -1 + a + a^4 \right\rangle, \tag{1.1}$$

$$V(\mathbb{Z}\mathbf{C}_8) = \mathbf{C}_8 \times \left\langle -1 - (a + a^{-1}) + (a^3 + a^{-3}) + 2a^4 \right\rangle, \tag{1.2}$$

$$\begin{aligned} V(\mathbb{Z}\mathbf{C}_{12}) &= \mathbf{C}_{12} \\ &\times \left\langle 3 + 2(a + a^{-1}) + (a^2 + a^{-2}) - (a^4 + a^{-4}) - 2(a^5 + a^{-5}) - 2a^6 \right\rangle. \end{aligned} \tag{1.3}$$

$\rho = 2$ when $n = 7, 9, 14$ or 18. For $n = 7$, F was characterized by Karpilovsky [10] and for the cases $n = 7$ and 9, it was characterized by Aleev [1]. A different characterization for $n = 7$ and 9 was also given by Kokluce and Kelebek [12] as

$$V(\mathbb{Z}\mathbf{C}_7) = \mathbf{C}_7 \times \left\langle -1 + (a + a^{-1}) \right\rangle \times \left\langle -1 + 2(a + a^{-1}) - (a^2 + a^{-2}) \right\rangle, \tag{1.4}$$

$$\begin{aligned} V(\mathbb{Z}\mathbf{C}_9) = \mathbf{C}_9 &\times \left\langle -1 + (a + a^{-1}) - (a^4 + a^{-4}) \right\rangle \\ &\times \left\langle -1 + (a + a^{-1}) + (a^2 + a^{-2}) + (a^3 + a^{-3}) - 2(a^4 + a^{-4}) \right\rangle. \end{aligned} \tag{1.5}$$

There is a strong relationship between F and $V(\mathbb{Z}\mathbf{C}_n^+)$. It is clear that $V(\mathbb{Z}\mathbf{C}_n^+)$ and F have the same rank. In some cases they are the same. Thus the characterization of $V(\mathbb{Z}\mathbf{C}_n^+)$ encourages one to make the characterization of $V(\mathbb{Z}\mathbf{C}_n)$.

In this study, we have completed the characterization of normalizers of dihedral groups where the rank $\rho \leq 4$. The following table lists the orders of dihedral groups corresponding to rank $\rho \leq 4$.

TABLE 1.

rank(ρ)	order(n)
0	1,2,3,4,6
1	5,8,10,12
2	7,9,14,18
3	15,16,20,24,30
4	11,22

Let $\omega = e^{\frac{2\pi i}{n}}$ and $\alpha = \omega + \omega^{-1}$. The minimal polynomial $\min_Q(\alpha, x)$ of α over Q can be obtained by a simple calculation. The image of $\mathbb{Z}\mathbf{C}_n^+$ under the ring homomorphism

$$\begin{array}{rccc} \psi: & \mathbb{Z}\mathbf{C}_n & \to & \mathbb{Z}[] \\ & \sum \gamma_i a^i & \mapsto & \sum \gamma_i \omega^i \end{array}$$

gives the ring of integers $\mathbb{Z}[]$. In determination of the normalized units $V(\mathbb{Z}\mathbf{C}_n^+)$ of $\mathbb{Z}\mathbf{C}_n^+$ we firstly need to find the fundamental units of $\mathbb{Z}[]$ whose minimal polynomial is $\min_Q(\alpha, x)$. The following Table 2 list the all fundamental units of $\mathbb{Z}[]$ for necessary cases.

TABLE 2.

ρ	n	$\min_Q(\alpha, x)$	Fundamental Units
3	15	$x^4 - x^3 - 4x^2 + 4x + 1$	$\varepsilon_1 = \alpha - 1$ $\varepsilon_2 = \alpha^2 - 3$ $\varepsilon_3 = \alpha^3 - 3\alpha$
3	16	$x^4 - 4x^2 + 2$	$\varepsilon_1 = \alpha - 1$ $\varepsilon_2 = \alpha^2 - 1$ $\varepsilon_3 = \alpha^2 + \alpha - 1$
3	20	$x^4 - 5x^2 + 5$	$\varepsilon_1 = \alpha^2 - 2$ $\varepsilon_2 = \alpha^2 + \alpha - 2$ $\varepsilon_3 = \alpha^3 - \alpha^2 - 3\alpha + 3$
4	11	$x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$	$\varepsilon_1 = \alpha$ $\varepsilon_2 = \alpha + 1$ $\varepsilon_3 = \alpha^2 - 2$ $\varepsilon_4 = \alpha^4 + \alpha^3 - 3\alpha^2 - 3\alpha$

2. Motivation for Contruction of $V(\mathbb{Z}\mathbf{C}_n^+)$.

Lemma 1. *Any $\gamma \in V(\mathbb{Z}\mathbf{C}_n^+)$ can be written as*

$$\gamma = \gamma_0 + \sum_{i=1}^{k} \gamma_i C_i.$$

where $C_i = a^i + a^{-i}$.

Proof. Let us denote $C_i = a^i + a^{-i}$ and $\gamma \in V(\mathbb{Z}\mathbf{C}_n^+)$. If $n = 2k+1$ then $\gamma = \gamma_0 + \sum_{i=1}^{k} \gamma_i C_i$. If $n = 2k$ then $\gamma = \gamma_0 + \gamma_k' a^k + \sum_{i=1}^{k-1} \gamma_i C_i$. By considering their augmentations, we have

$$\varepsilon(\gamma) = \begin{cases} \gamma_0 + 2\sum_{i=1}^{k} \gamma_i & , \ n = 2k+1 \\ \gamma_0 + \gamma_k' + 2\sum_{i=1}^{k-1} \gamma_i, & n = 2k. \end{cases}$$

by modulo 2 ,we obtain

$$\varepsilon(\gamma) \equiv 1(mod\,2) \Rightarrow \begin{cases} \gamma_0 \equiv 1(mod\,2), & n = 2k+1 \\ \gamma_0 + \gamma_k' \equiv 1(mod\,2), & n = 2k. \end{cases}$$

By choosing γ_0 as an odd integer in both cases we obtain $\gamma_k' = 2\gamma_k$, for some $\gamma_k \in \mathbb{Z}$ we have

$$\gamma_k' a^k = 2\gamma_k a^k = \gamma_k(a^k + a^{-k}) = \gamma_k C_k.$$

So, $\gamma \in V(\mathbb{Z}\mathbf{C}_n^+)$ can be written as

$$\gamma = \gamma_0 + \sum_{i=1}^{k} \gamma_i C_i$$

in both cases. □

Proposition 1. *Let H be a subgroup of a finite abelian group G. We can define a group epimorphism:*

$$\begin{aligned} \varphi : \quad & G \to G/H \\ & g \mapsto gH. \end{aligned}$$

If we extend φ linearly over $\mathbb{Z}$, then we can get the natural ring homomorphism

$$\begin{aligned} \overline{\varphi} : \quad & \mathbb{Z}G \to \mathbb{Z}(G/H) \\ & \sum \gamma_g g \mapsto \sum \gamma_g (gH). \end{aligned}$$

If $G/H \cong \mathbf{C}_2, \mathbf{C}_3, \mathbf{C}_4$ or $\mathbf{C}_6$, then for any torsion-free unit $\gamma \in V(\mathbb{Z}G)$, $\overline{\varphi}(\gamma) = H$.

Remark 1. If $\gamma \in V(\mathbb{Z}\mathbf{C}_n^+)$ and $n = 2k$ then $a^k\gamma \in V(\mathbb{Z}\mathbf{C}_n^+)$. The coefficient of identity of either γ or $a^k\gamma$ is odd. $\gamma \in V(\mathbb{Z}\mathbf{C}_n^+)$ can be chosen as a generator if the coefficient of its identity is odd.

Proposition 2. *Let n be an odd integer then $\mathbf{C}_{2n} =< a : a^{2n} = 1 >$ and $H =< a^2 >$. Then $V(\mathbb{Z}\mathbf{C}_n^+)$ and $V(\mathbb{Z}H^+)$ have the same rank.*

Proof. If we extend group epimorphism linearly over $\mathbb{Z}$

$$\begin{array}{rl} f: & \mathbf{C}_{2n} \to H \\ & a^i \mapsto a^{2i}. \end{array}$$

then, we obtain the following ring epimorphism:

$$\begin{array}{rl} \overline{f}: & \mathbb{Z}\mathbf{C}_{2n} \to \mathbb{Z}H \\ & \sum_{i=0}^{2n-1} \gamma_i a^i \mapsto \sum_{i=0}^{2n-1} \gamma_i a^{2i}. \end{array} \tag{2.1}$$

If we restrict $\overline{f}$ to multiplicative torsion-free group we have

$$\widetilde{f}: \quad V(\mathbb{Z}\mathbf{C}_n^+) \to V(\mathbb{Z}H^+).$$

Since

$$\begin{aligned} \rho_{2n} &= \frac{1}{2}\varphi(2n) - 1 \\ &= \frac{1}{2}\varphi(2)\varphi(n) - 1 \\ &= \frac{1}{2}\varphi(n) - 1 \\ &= \rho_n, \end{aligned}$$

$V(\mathbb{Z}\mathbf{C}_n^+)$ and $V(\mathbb{Z}H^+)$ have the same rank. □

Remark 2. Let be n be an odd integer. If $\gamma = \gamma_0 + \sum_{i=0}^{k} \gamma_i C_i$ is a generator of $V(\mathbb{Z}H^+) \subset \mathbb{Z}\mathbf{C}_{2n}$, then by Proposition 2 $\gamma = \gamma_0 + \sum_{i=0}^{k} \gamma_i a^{2i}$ is a generator of $V(\mathbb{Z}\mathbf{C}_n^+)$.

By using Remark 2 and considering (1.1), (1.4) and (1.5), the unit groups $V(\mathbb{Z}\mathbf{C}_{10})$, $V(\mathbb{Z}\mathbf{C}_{14})$ and $V(\mathbb{Z}\mathbf{C}_{18})$ can as follows be given as follows:

$$V(\mathbb{Z}\mathbf{C}_{10}) = \mathbf{C}_{10} \times \left\langle -1 + a^2 + a^8 \right\rangle,$$

$$V(\mathbb{Z}\mathbf{C}_{14}) = \mathbf{C}_{14} \times \left\langle -1 + (a^2 + a^{-2}) \right\rangle \times \left\langle -1 + 2(a^2 + a^{-2}) - (a^4 + a^{-4}) \right\rangle,$$

$$\begin{aligned} V(\mathbb{Z}\mathbf{C}_{18}) = \mathbf{C}_{18} &\times \left\langle -1 + (a^2 + a^{-2}) - (a^8 + a^{-8}) \right\rangle \\ &\times \left\langle -1 + (a^2 + a^{-2}) + (a^4 + a^{-4}) + (a^6 + a^{-6}) - 2(a^8 + a^{-8}) \right\rangle. \end{aligned}$$

3. Characterization of $V(\mathbb{Z}\mathbf{C}_n^+)$.

Characterization of $V(\mathbb{Z}\mathbf{C}_{15}^+)$ and $V(\mathbb{Z}\mathbf{C}_{30}^+)$

Theorem 2.

$$V(\mathbb{Z}\mathbf{C}_{15}^+) =< -1+C_1-C_2+C_3-C_4+C_5, -1+C_2-C_4+C_5+C_6-C_7, -1+C_3 > .$$

Proof. Let $\gamma \in V(\mathbb{Z}\mathbf{C}_{15}^+)$ be a generator of torsion-free unit. Then, by Lemma 1 we have

$$\gamma = \gamma_0 + \sum_{i=1}^{7} \gamma_i C_i .$$

Let us consider the subgroups $H_1 = \langle a^3 \rangle$ and $H_2 = \langle a^5 \rangle$ of prime order. Since $\mathbf{C}_{15}/H_1 \cong \mathbf{C}_3$ and $\mathbf{C}_{15}/H_2 \cong \mathbf{C}_5$ by Propositon 1, we have $\overline{\varphi}_j(\gamma) = H_j (j = 1,2)$. For $\overline{\varphi}_1(\gamma) = H_1$, we obtain

$$\begin{aligned} \gamma_0 + 2\gamma_3 + 2\gamma_6 &= 1, \\ \gamma_1 + \gamma_2 + \gamma_4 + \gamma_5 + \gamma_7 &= 0. \end{aligned} \tag{3.1}$$

and for $\overline{\varphi}_2(\gamma) = H_2$, we obtain

$$\begin{aligned} \gamma_0 + 2\gamma_5 &= 1, \\ \gamma_1 + \gamma_4 + \gamma_6 &= 0, \\ \gamma_2 + \gamma_3 + \gamma_7 &= 0. \end{aligned} \tag{3.2}$$

Substituting $\gamma_1 = p, \gamma_2 = q, \gamma_4 = r, \gamma_7 = s$ in (3.1) and (3.2), we obtain

$$\begin{aligned} \gamma_0 &= 1 + 2(p+q+r+s), \\ \gamma_3 &= -q-s, \\ \gamma_5 &= -(p+q+r+s), \\ \gamma_6 &= -p-r. \end{aligned} \tag{3.3}$$

Denoting $\omega = e^{\frac{2\pi i}{15}}$ and $\alpha = \omega + \omega^{-1}$ we get the minimal polynomial of α over $\mathbb{Q}$ as follows:

$$\min_{\mathbb{Q}}(\alpha, x) = x^4 - x^3 - 4x^2 + 4x + 1. \tag{3.4}$$

Now, let us consider the following ring homomorphism:

$$\begin{aligned} \psi : \quad \mathbb{Z}\mathbf{C}_{15} &\to \mathbb{Z}[] \\ \textstyle\sum \gamma_i a^i &\mapsto \textstyle\sum \gamma_i \omega^i . \end{aligned}$$

By (3.3) and (3.4), the image of the unit is

$$\begin{aligned} \psi(\gamma) &= \psi(\gamma_0 + \sum_{i=1}^{7} \gamma_i C_i) \\ &= \gamma_0 + \gamma_1(\alpha) + \gamma_2(\alpha^2 - 2) + \gamma_3(\alpha^3 - 3\alpha) + \gamma_4(\alpha^3 - 4\alpha + 1) \end{aligned}$$

$$+\gamma_5(-1)+\gamma_6(-\alpha^3+3\alpha-1)+\gamma_7(-\alpha^3-\alpha^2+3\alpha+2)$$
$$=(1+4p+q+5r+5s)+(-2p+3q-7r+6s)\alpha+(q-s)\alpha^2$$
$$+(p-q+2r-2s)\alpha^3.$$

Since $\psi(\gamma)\in\mathcal{U}(\mathbb{Z}[\alpha])=\{\sum_{i=0}^{3}a_i\alpha^i:\alpha^4-\alpha^3-4\alpha^2+4\alpha+1=0\}$ by Table 2,

$$\mathcal{U}(\mathbb{Z}[\alpha])=\langle\alpha-1,\alpha^2-3,\alpha^3-3\alpha\rangle.$$

As a consequence of the calculations we see that $\gamma\in U(\mathbb{Z}[a+a^{-1}])$ can be obtained in the following single product of the fundamental units or their inverses;

$$\varepsilon_1^2,\varepsilon_2^2,\varepsilon_1^{-2},\varepsilon_2^{-2}.$$

One can easily see that

$$\psi(\gamma)=\ \varepsilon_1^2\Rightarrow\gamma=-1+C_1-C_2+C_3-C_4+C_5,$$
$$\psi(\gamma)=\varepsilon_2^2\ \Rightarrow\gamma=-1+C_2-C_4+C_5+C_6-C_7.$$

Since $H_1=\langle a^3\rangle$ is a cyclic group of order 5, by (1.1), its unit group of integral group ring $V(\mathbb{Z}H_1)=H_1\times\langle-1+(a^3+a^{-3})\rangle$, so the third unit is $-1+C_3$. □

Corollary 2.

$$V(\mathbb{Z}\mathbf{C}_{30}^+)$$
$$=<-1+C_2-C_4+C_5+C_6-C_7,-1-C_1+C_3+C_4+C_5-C_7,-1+C_6>.$$

Proof. By considering Theorem 2 with Remark 2 we get the required result. □

Characterization of $V(\mathbb{Z}\mathbf{C}_{16}^+)$

Theorem 3.

$$V(\mathbb{Z}\mathbf{C}_{16}^+)=<1-C_2-C_3+C_5+C_6,1-C_1+C_2-C_6+C_7,-1-C_2+C_6+C_8>$$

Proof. Let $\gamma\in V(\mathbb{Z}\mathbf{C}_{16}^+)$ be a generator of torsion-free unit. Then, by Lemma 1 we have

$$\gamma=\gamma_0+\sum_{i=1}^{8}\gamma_iC_i.$$

Let us consider the prime subgroup $H=\langle a^8\rangle$. Since $\mathbf{C}_{16}/H\cong\mathbf{C}_8$ by Propositon 1 $\mathcal{U}(\mathbb{Z}(\mathbf{C}_{16}/H))$ is trivial. For any $\gamma\in V(\mathbb{Z}\mathbf{C}_{16}^+)$ we get

$$\overline{\varphi}(\gamma)=H\Rightarrow\gamma_0+\gamma_8=1, \tag{3.5}$$
$$\gamma_i+\gamma_{8-i}=0\text{ for }i=1,2,3,4.$$

Substituting $\gamma_0=p,\gamma_1=q,\gamma_2=r,\gamma$ in (3.5), we have

$$\gamma_4=0,\gamma_5=-s,\gamma_6=-r,\gamma_7=-q,\gamma_8=1-p.$$

Denote $\omega = e^{\frac{2\pi i}{16}}$ and $\alpha = \omega + \omega^{-1}$ then the minimal polynomial of α over $\mathbb{Q}$ can be obtained as follows

$$\min_{\mathbb{Q}}(\alpha, x) = x^4 - 4x^2 + 2. \tag{3.6}$$

By considering the following ring homomorphism

$$\begin{aligned} \psi : \quad \mathbb{Z}\mathbf{C}_{16} &\to \mathbb{Z}[] \\ \textstyle\sum \gamma_i a^i &\mapsto \textstyle\sum \gamma_i \omega^i, \end{aligned}$$

and (3.6), the image of the unit can be given as

$$\begin{aligned} \psi(\gamma) &= \psi(\gamma_0 + \sum_{i=1}^{8} \gamma_i C_i) \\ &= \gamma_0 + \gamma_1(\alpha) + \gamma_2(\alpha^2 - 2) + \gamma_3(\alpha^3 - 3\alpha) + \gamma_4(0) \\ &+ \gamma_5(-\alpha^3 + 3\alpha) + \gamma_6(-\alpha^2 + 2) + \gamma_7(-\alpha) - \gamma_8 \\ &= (-1 + 2p - 4r) + (2q - 6s)\alpha + (2r)\alpha^2 + (2s)\alpha^3. \end{aligned}$$

Since $\psi(\gamma) \in \mathcal{U}(\mathbb{Z}[\alpha]) = \{\sum_{i=0}^{4} a_i \alpha^i : \alpha^4 - 4\alpha^2 + 2 = 0\}$, by Table 2 we have

$$\mathcal{U}(\mathbb{Z}[\alpha]) = \left\langle \alpha - 1, \alpha^2 - 1, \alpha^2 + \alpha - 1 \right\rangle.$$

The calculations shows that $\gamma \in U(\mathbb{Z}[a + a^{-1}])$ can be obtained in the following double products of the fundamental units or their inverses;

$$\varepsilon_1^2\varepsilon_2^{-1}, \varepsilon_1^2\varepsilon_2, \varepsilon_1^{-2}\varepsilon_2, \varepsilon_1^{-2}\varepsilon_2^{-1}, \varepsilon_3^2\varepsilon_2, \varepsilon_3^{-2}\varepsilon_2, \varepsilon_3^{-2}\varepsilon_2^{-1}, \varepsilon_3^2\varepsilon_2^{-1}.$$

One can easily see that these units can be generated three units. The first one can be chosen as

$$\psi(\gamma) = \varepsilon_2\varepsilon_1^{-2} \Rightarrow \gamma = C_2 - C_3 + C_5 - C_6 + a^8.$$

By considering Remark 1 we may write the first unit as follows

$$a^8\gamma = 1 - C_2 - C_3 + C_5 + C_6$$

The second generator is

$$\psi(\gamma) = \varepsilon_2\varepsilon_3^{-2} \Rightarrow \gamma = 1 - C_1 + C_2 - C_6 + C_7.$$

Since $K = \langle a^2 \rangle$ is a cyclic group of order 8, by (1.2), its unit group of integral group ring $V(\mathbb{Z}K) = K \times \langle -1 - (a^2 + a^{-2}) + (a^6 + a^{-6}) + 2a^8 \rangle$, so the third generator is $-1 - C_2 + C_6 + 2a^8$. By Lemma 1, the third generator can be written as $-1 - C_2 + C_6 + C_8$ which finishes the proof. □

Characterization of $V(\mathbb{Z}\mathbf{C}_{20}^+)$

Theorem 4.

$$V(\mathbb{Z}\mathbf{C}_{20}^+) =< 1+C_1+C_2+C_3-C_7-C_8-C_9-C_{10},$$
$$1-C_1+C_3-C_4+C_6-C_7+C_9-C_{10}, -1+C_4 >$$

Proof. Let $\gamma \in V(\mathbb{Z}\mathbf{C}_{20}^+)$ be a generator of torsion-free unit. By Lemma 1 we have

$$\gamma = \gamma_0 + \sum_{i=1}^{10} \gamma_i C_i .$$

Let us consider subgroups $H_1 = \langle a^{10} \rangle$ and $H_2 = \langle a^4 \rangle$ of prime indices. Since $\mathbf{C}_{20}/H_1 \cong \mathbf{C}_{10}$ and $\mathbf{C}_{20}/H_2 \cong \mathbf{C}_4$ by Propositon 1 we have $\overline{\varphi}_j(\gamma) = H_j$ $(j = 1,2)$.For $\overline{\varphi}_1(\gamma) = H_1$, we obtain

$$\begin{aligned} \gamma_0 + \gamma_{10} &= 1, \\ \gamma_i + \gamma_{10-i} &= 0, \qquad (i = 1,2,3,4,5). \end{aligned} \tag{3.7}$$

and for $\overline{\varphi}_2(\gamma) = H_2$, we get

$$\begin{aligned} \gamma_0 + 2\gamma_4 + 2\gamma_8 &= 1, \\ \gamma_1 + \gamma_3 + \gamma_5 + \gamma_7 + \gamma_9 &= 0, \\ 2\gamma_2 + 2\gamma_6 + \gamma_{10} &= 0. \end{aligned} \tag{3.8}$$

If we substitude $\gamma_1 = p, \gamma_2 = q, \gamma_3 = r, \gamma_4 = s$ in (3.7) and (3.8), we can write

$$\begin{aligned} \gamma_0 &= 1 + 2q - 2s, \gamma_5 = 0, \\ \gamma_6 &= -s, \gamma_7 = -r, \gamma_8 = -q, \\ \gamma_9 &= -p, \gamma_{10} = -2q + 2s. \end{aligned} \tag{3.9}$$

By taking $\omega = e^{\frac{2\pi i}{20}}$ and $\alpha = \omega + \omega^{-1}$ we can write the minimal polynomial of α over $\mathbb{Q}$ as follows;

$$\min_{\mathbb{Q}}(\alpha, x) = x^4 - 5x^2 + 5. \tag{3.10}$$

Now consider the following ring homomorphism,

$$\begin{aligned} \psi : \quad \mathbb{Z}\mathbf{C}_{20} &\to \mathbb{Z}[] \\ \textstyle\sum \gamma_i a^i &\mapsto \textstyle\sum \gamma_i \omega^i \end{aligned}$$

with equations (3.9) and (3.10). This gives the image of the unit as

$$\begin{aligned} \psi(\gamma) &= \psi(\gamma_0 + \sum_{i=1}^{10} \gamma_i C_i), (C_{10} = 2a^3) \\ &= \gamma_0 + \gamma_1(\alpha) + \gamma_2(\alpha^2 - 2) + \gamma_3(\alpha^3 - 3\alpha) + \gamma_4(\alpha^4 - 4\alpha^2 + 2) + \gamma_5(0), \\ &+ \gamma_6(-\alpha^4 + 4\alpha^2 - 2) + \gamma_7(-\alpha^3 + 3\alpha) + \gamma_8(-\alpha^2 + 2) + \gamma_9(-\alpha) - \gamma_{10} \end{aligned}$$

$$= (1-10s) + (2p-6r)\alpha + (2q+2s)\alpha^2 + (2r)\alpha^3.$$

Since $\psi(\gamma) \in U(\mathbb{Z}[\alpha]) = \{\sum_{i=0}^{4} a_i\alpha^i : \alpha^4 - 5\alpha^2 + 5 = 0\}$. By using Table 2

$$U(\mathbb{Z}[\alpha]) = \left\langle \alpha^2 - 2, \alpha^2 + \alpha - 2, \alpha^3 - \alpha^2 - 3\alpha + 3 \right\rangle.$$

$\gamma \in U(\mathbb{Z}[a+a^{-1}])$ can be obtained in the following single product of the fundamental units or their inverses;

$$\varepsilon_2^2, \varepsilon_3^2, \varepsilon_2^{-2}, \varepsilon_3^{-2}.$$

The two generators are obtained as follows:

$$\psi(\gamma) = \varepsilon_2^2 \Rightarrow \gamma = -1 - C_1 - C_2 - C_3 + C_7 + C_8 + C_9 + 2C_{10}$$
$$\psi(\gamma) = \varepsilon_3^2 \Rightarrow \gamma = -1 + C_1 - C_3 + C_4 - C_6 + C_7 - C_9 + 2C_{10}.$$

Their augmentations are negative, to make normalized units we must multiply by -1 and also by regarding Lemma 1 we get

$$\psi(\gamma) = \varepsilon_2^2 \Rightarrow \gamma = 1 + C_1 + C_2 + C_3 - C_7 - C_8 - C_9 - C_{10}$$
$$\psi(\gamma) = \varepsilon_3^2 \Rightarrow \gamma = 1 - C_1 + C_3 - C_4 + C_6 - C_7 + C_9 - C_{10}.$$

Since $H_2 = \langle a^4 \rangle$ is a cyclic group of order 5, by (1.1), its unit group of integral group ring $V(\mathbb{Z}H_2) = H_2 \times \left\langle -1 + (a^4 + a^{-4}) \right\rangle$, so the third generator is clearly $-1 + C_4$. □

Characterization of $V(\mathbb{Z}\mathbf{C}_{24}^{+})$

Theorem 5.

$$V(\mathbb{Z}\mathbf{C}_{24}^{+}) = < -5 - 2C_1 - 4C_3 - 3C_4 + 2C_5 - 2C_7 + 3C_8 + 4C_9 + 2C_{10} + 3C_{12},$$
$$3 + 2C_2 + C_4 - C_8 - 2C_{10} - C_{12}, -1 + C_1 - C_4 + C_5 - C_7 + C_8 - C_{11} + C_{12} >$$

Proof. Let $\gamma \in V(\mathbb{Z}\mathbf{C}_{24}^{+})$ be a generator of torsion-free unit. Then, by Lemma 1 we have

$$\gamma = \gamma_0 + \sum_{i=1}^{12} \gamma_i C_i$$

Consider the subgroups $H_1 = \langle a^{12} \rangle$ and $H_2 = \langle a^8 \rangle$ of prime orders. Since $\mathbf{C}_{24}/H_1 \cong \mathbf{C}_{12}$ and $\mathbf{C}_{24}/H_2 \cong \mathbf{C}_8$, by Propositon 1 we have, $\overline{\varphi}_j(\gamma) = H_j$ $(j = 1, 2)$.

For $\overline{\varphi}_1(\gamma) = H_1$, we obtain

$$\gamma_0 + \gamma_{12} = 1, \tag{3.11}$$
$$\gamma_i + \gamma_{12-i} = 0, \qquad (i = 1, 2, 3, 4, 5, 6).$$

and for $\overline{\varphi}_2(\gamma) = H_2$, we get

$$\begin{aligned} \gamma_0 + 2\gamma_8 &= 1, \\ \gamma_1 + \gamma_7 + \gamma_9 &= 0, \\ \gamma_2 + \gamma_6 + \gamma_{10} &= 0, \\ \gamma_3 + \gamma_5 + \gamma_{11} &= 0, \\ 2\gamma_4 + \gamma_{12} &= 0. \end{aligned} \tag{3.12}$$

Substituting $\gamma_2 = p, \gamma_3 = q, \gamma_4 = r, \gamma_5 = s$ in (3.11) and (3.12), we have

$$\begin{aligned} &\gamma_0 = 1 + 2r, \quad \gamma_1 = q + s, \quad \gamma_6 = 0 \\ &\gamma_7 = -s, \quad \gamma_8 = -r, \quad \gamma_9 = -q, \\ &\gamma_{10} = -p, \quad \gamma_{11} = -q - s, \quad \gamma_{12} = -2r \end{aligned} \tag{3.13}$$

On the other hand, if we denote $\omega = e^{\frac{2\pi i}{24}}$ and $\alpha = \omega + \omega^{-1} = 2\cos(\frac{\pi}{12})$ then we get the minimal polynomial of α over $\mathbb{Q}$ as

$$\min_{\mathbb{Q}}(\alpha, x) = x^4 - 4x^2 + 1. \tag{3.14}$$

Considering the ring homomorphism

$$\begin{aligned} \psi : \quad \mathbb{Z}\mathbf{C}_{24} &\to \mathbb{Z}[] \\ \textstyle\sum \gamma_i a^i &\mapsto \textstyle\sum \gamma_i \omega^i \end{aligned}$$

with (3.13) and (3.14), the image of the unit can be obtained as

$$\begin{aligned} \psi(\gamma) &= \psi(\gamma_0 + \sum_{i=1}^{11} \gamma_i C_i) \\ &= (\gamma_0 - \gamma_{12}) + (\gamma_1 - \gamma_{11})(\frac{\sqrt{6}+\sqrt{2}}{2}) + (\gamma_2 - \gamma_{10})(\sqrt{3}) \\ &+ (\gamma_3 - \gamma_9)(\sqrt{2}) + (\gamma_4 - \gamma_8)(1) + (\gamma_5 - \gamma_7)(\frac{\sqrt{6}-\sqrt{2}}{2}) \\ &= (1 + 6r) + 3q\sqrt{2} + 2p\sqrt{3} + (q + 2s)\sqrt{6}. \end{aligned}$$

Since $\psi(\gamma) \in \mathcal{U}(\mathbb{Z}[\sqrt{2}]) \times \mathcal{U}(\mathbb{Z}[\sqrt{3}]) \times \mathcal{U}(\mathbb{Z}[\sqrt{6}])$,

$$\begin{aligned} i)\psi(\gamma) \in \mathcal{U}(\mathbb{Z}[\sqrt{2}]) \quad &\Rightarrow p = 0, q = -2s \\ &\Rightarrow \psi(\gamma) = (1 + 6r) - 6s\sqrt{2} = \pm(1 \pm \sqrt{2})^k, (k \in \mathbb{Z}) \\ &\Rightarrow (1 + 6r) - 6s\sqrt{2} = \pm(17 \pm 12\sqrt{2}), (k = 4) \\ &\Rightarrow \gamma = -5 - 2C_1 - 4C_3 - 3C_4 + 2C_5 - 2C_7 + 3C_8 + 4C_9 \\ &\qquad +2C_{11} + 6a^{12}, \end{aligned}$$

$$\begin{aligned} ii)\psi(\gamma) \in \mathcal{U}(\mathbb{Z}[\sqrt{3}]) &\Rightarrow q = s = 0 \\ &\Rightarrow \psi(\gamma) = (1+6r) + 2p\sqrt{3} = \pm(2 \pm \sqrt{3})^k, (k \in \mathbb{Z}) \\ &\Rightarrow (1+6r) + 2p\sqrt{3} = \pm(7 \pm 4\sqrt{3}), (k = 2) \\ &\Rightarrow \gamma = 3 + 2C_2 + C_4 - C_8 - 2C_{10} - 2a^{12}, \end{aligned}$$

$$\begin{aligned} iii)\psi(\gamma) \in U(\mathbb{Z}[\sqrt{6}]) &\Rightarrow p = q = 0 \\ &\Rightarrow \psi(\gamma) = (1+6r) + 2s\sqrt{6} = \pm(5 \pm 2\sqrt{6})^k, (k \in \mathbb{Z}) \\ &\Rightarrow \psi(\gamma) = (1+6r) + 2s\sqrt{6} = \pm 5 \pm 2\sqrt{6}, (k = 1) \\ &\Rightarrow \gamma = -1 + C_1 - C_4 + C_5 - C_7 + C_8 - C_{11} + 2a^{12}. \end{aligned} ,$$

By considering Lemma 1 the generators can be written respectively as follows

$$\begin{aligned} &-5 - 2C_1 - 4C_3 - 3C_4 + 2C_5 - 2C_7 + 3C_8 + 4C_9 + 2C_{11} + 3C_{12}, \\ &3 + 2C_2 + C_4 - C_8 - 2C_{10} - C_{12}, \\ &-1 + C_1 - C_4 + C_5 - C_7 + C_8 - C_{11} + C_{12}. \end{aligned}$$

□

Characterization of $V(\mathbb{Z}\mathbf{C}_{11}^{+})$ and $V(\mathbb{Z}\mathbf{C}_{22}^{+})$

Theorem 6. *The normalized units of* $V(\mathbb{Z}\mathbf{C}_{11}^{+}) \subset \mathbb{Z}\mathbf{C}_{11}$ *are generated by the set*

$$V(\mathbb{Z}\mathbf{C}_{11}^{+}) = < -1 + C_1, -1 + C_2, -1 - C_3, -1 + C_4 > .$$

Proof. Let $\gamma \in V(\mathbb{Z}\mathbf{C}_{11}^{+})$ be a generator of torsion-free unit. Then, we have

$$\gamma = \gamma = \gamma_0 + \sum_{i=1}^{5} \gamma_i C_i$$

and

$$\gamma_0 + 2\gamma_1 + 2\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 = 1. \tag{3.15}$$

If we substitute $\gamma_1 = p, \gamma_2 = q, \gamma_3 = r, \gamma_4 = s, , \gamma_5 = t$ in (3.15), we can write

$$\gamma_0 = 1 - 2(p + q + r + s + t). \tag{3.16}$$

On the other hand, if we denote $\omega = e^{\frac{2\pi i}{11}}$ and $\alpha = \omega + \omega^{-1}$ then we can get the minimal polynomial of α over $\mathbb{Q}$ as follows;

$$\min_{\mathbb{Q}}(\alpha, x) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1. \tag{3.17}$$

By regarding the following ring homomorphism:

$$\begin{aligned} \psi : \quad \mathbb{Z}C_{11} &\to \mathbb{Z}[] \\ \textstyle\sum \gamma_i a^i &\mapsto \textstyle\sum \gamma_i \omega^i \end{aligned}$$

with equations (3.16) and (3.17), the image of the unit is

$$\begin{aligned}\psi(\gamma) &= \psi\Big(\gamma_0 + \sum_{i=1}^{5}\gamma_i C_i\Big)\\ &= \gamma_0 + \gamma_1\alpha + \gamma_2(\alpha^2-2) + \gamma_3(\alpha^3-3\alpha) + \gamma_4(\alpha^4-4\alpha^2+2)\\ &+ \gamma_5(\alpha^5-5\alpha^3+5\alpha)\\ &= (1-2p-4q-2r-3t) + (p-3r+2t)\alpha + (q-4s+3t)\alpha^2\\ &+ (r-t)\alpha^3 + (s-t)\alpha^4.\end{aligned}$$

Since $\psi(\gamma) \in U(\mathbb{Z}[\alpha]) = \{\sum_{i=0}^{4} a_i\alpha^i : \alpha^5+\alpha^4-4\alpha^3-3\alpha^2+3\alpha+1=0\}$. By Table 2

$$U(\mathbb{Z}[\alpha]) = \left\langle \alpha, \alpha+1, \alpha^2-2, \alpha^4+\alpha^3-3\alpha^2-3\alpha \right\rangle.$$

$\gamma \in U(\mathbb{Z}[a+a^{-1}])$ can be obtained in the following single product of the fundamental units or their inverses;

$$-\varepsilon_1\varepsilon_2^{-1}\varepsilon_3\varepsilon_4^{-1}, \varepsilon_1^{-1}\varepsilon_2^{-1}\varepsilon_4, \varepsilon_1\varepsilon_4, -\varepsilon_1^{-1}\varepsilon_2\varepsilon_3^{-2}.$$

Here

$$\begin{aligned}\psi(\gamma) &= -\varepsilon_1\varepsilon_2^{-1}\varepsilon_3\varepsilon_4^{-1} \Rightarrow \gamma = -1 + C_1\\ \psi(\gamma) &= \varepsilon_1^{-1}\varepsilon_2^{-1}\varepsilon_4 \quad \Rightarrow \gamma = -1 + C_2\\ \psi(\gamma) &= \varepsilon_1\varepsilon_4 \quad \Rightarrow \gamma = -1 + C_3\\ \psi(\gamma) &= -\varepsilon_1^{-1}\varepsilon_2\varepsilon_3^{-2} \quad \Rightarrow \gamma = -1 + C_4\end{aligned}$$

□

Corollary 3.

$$V(\mathbb{Z}\mathbf{C}_{22}^+) = < -1+C_2, -1+C_4, -1+C_6, -1+C_8 > .$$

Proof. We obtain desired result by considering Theorem 6 with Remark 2. □

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Authors' addresses

Bülent Köklüce
Mathematics Education, Faculty of Education, Fatih University, Istanbul, Turkey
E-mail address: bkokluce@fatih.edu.tr

Nesibe Tüfekçi
Department of Mathematics, Faculty of Arts and Sciences, Fatih University, Istanbul, Turkey
E-mail address: nesibe.tufekci@fatih.edu.tr

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NEW INTEGRAL INEQUALITIES INVOLVING BETA FUNCTION VIA P-CONVEXITY

WENJUN LIU

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Abstract. In this note we establish some estimates, involving the Euler Beta function, of the integral $\int_a^b (x-a)^p (b-x)^q f(x)dx$ for functions when a power of the absolute value is $P-$convex. An extension to functions of several variables is also obtained.

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1. INTRODUCTION

Let I be an interval in $\mathbb{R}$. Then $f : I \to \mathbb{R}$ is said to be convex if

$$f\left(tx+(1-t)\,y\right) \leq tf\left(x\right)+(1-t)\,f\left(y\right)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

The notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f : [a,b] \to \mathbb{R}$ is said to be quasi-convex on $[a,b]$ if

$$f(tx+(1-t)y) \leq \max\{f(x), f(y)\}$$

holds for any $x, y \in [a,b]$ and $t \in [0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [11]).

The generalized quadrature formula of Gauss-Jacobi type has the form

$$\int_a^b (x-a)^p (b-x)^q f(x)dx = \sum_{k=0}^{m} B_{m,k} f(\gamma_k) + \mathcal{R}_m[f] \tag{1.1}$$

for certain $B_{m,k}, \gamma_k$ and rest term $\mathcal{R}_m[f]$ (see [22]).

In [17], Özdemir et al. established several integral inequalities concerning the left-hand side of (1.1) via some kinds of convexity. Especially, they discussed the following result connecting with quasi-convex function:

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Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ such that $f\in L([a,b])$, $0\le a<b<\infty$. If f is quasi-convex on $[a,b]$, then for some fixed $p,q>0$, we have*

$$\int_a^b (x-a)^p(b-x)^q f(x)dx \le (b-a)^{p+q+1}\beta(p+1,q+1)\max\{f(a),f(b)\},$$

where $\beta(x,y)$ is the Euler Beta function.

Recently, Liu [12] established some new integral inequalities for quasi-convex functions as follows:

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ such that $f\in L([a,b])$, $0\le a<b<\infty$ and let $k>1$. If $|f|^{\frac{k}{k-1}}$ is quasi-convex on $[a,b]$, for some fixed $p,q>0$, then*

$$\begin{aligned}&\int_a^b (x-a)^p(b-x)^q f(x)dx\\ &\le (b-a)^{p+q+1}[\beta(kp+1,kq+1)]^{\frac{1}{k}}\left(\max\left\{|f(a)|^{\frac{k}{k-1}},|f(b)|^{\frac{k}{k-1}}\right\}\right)^{\frac{k-1}{k}}.\end{aligned}$$

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ such that $f\in L([a,b])$, $0\le a<b<\infty$ and let $l\ge 1$. If $|f|^l$ is quasi-convex on $[a,b]$, for some fixed $p,q>0$, then*

$$\begin{aligned}&\int_a^b (x-a)^p(b-x)^q f(x)dx\\ &\le (b-a)^{p+q+1}\beta(p+1,q+1)\left(\max\left\{|f(a)|^l,|f(b)|^l\right\}\right)^{\frac{1}{l}}.\end{aligned}$$

On the other hand, Dragomir et al. in [6] defined the following class of functions of P-convex.

Definition 1. Let $I\subseteq\mathbb{R}$ be an interval. The function $f:I\to\mathbb{R}$ is said to belong to the class $P(I)$ (or to be P-convex) if it is nonnegative and, for all $x,y\in I$ and $t\in[0,1]$, satisfies the inequality

$$f(tx+(1-t)y)\le f(x)+f(y).$$

Note that $P(I)$ contain all nonnegative convex and quasiconvex functions. Since then numerous articles have appeared in the literature reflecting further applications in this category; see [1,2,4,5,7–10,13–16,18–21,23–26] and references therein.

The main purpose of this note is to establish some new estimates of the integral $\int_a^b (x-a)^p(b-x)^q f(x)dx$ for functions when a power of the absolute value is $P-$convex. An extension to functions of several variables is also obtained. That is, this study is a continuation and further generalization of [12,17] via P-convexity.

2. New integral inequalities via P-convexity

In this section we generalize Theorems 1-3 with a P-convex function setting. For this purpose, we need the following lemma (see [17, Lemma 2.1]):

Lemma 1. *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be continuous on $[a,b]$ such that $f\in L([a,b])$, $a<b$. Then the equality*

$$\int_a^b (x-a)^p(b-x)^q f(x)dx=(b-a)^{p+q+1}\int_0^1 (1-t)^p t^q f(ta+(1-t)b)dt \tag{2.1}$$

holds for some fixed $p,q>0$.

The next theorem gives a new result for P-convex functions.

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ such that $f\in L([a,b])$, $0\le a<b<\infty$. If $|f|$ is P-convex on $[a,b]$, for some fixed $p,q>0$, then*

$$\begin{aligned}&\int_a^b (x-a)^p(b-x)^q f(x)dx\\ &\le(b-a)^{p+q+1}\beta(p+1,q+1)(|f(a)|+|f(b)|),\end{aligned} \tag{2.2}$$

where $\beta(x,y)$ is the Euler Beta function.

Proof. By Lemma 1, the Beta function which is defined for $x,y>0$ as

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt$$

and the fact that f is P-convex on $[a,b]$, we have

$$\begin{aligned}\int_a^b (x-a)^p(b-x)^q f(x)dx&\le(b-a)^{p+q+1}\int_0^1 (1-t)^p t^q|f(ta+(1-t)b)|dt\\ &\le(b-a)^{p+q+1}\int_0^1 (1-t)^p t^q(|f(a)|+|f(b)|)dt\\ &=(b-a)^{p+q+1}\beta(q+1,p+1)(|f(a)|+|f(b)|),\end{aligned}$$

which completes the proof. □

The corresponding version for powers of the absolute value is incorporated in the following result.

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ such that $f\in L([a,b])$, $0\le a<b<\infty$ and let $k>1$. If $|f|^{\frac{k}{k-1}}$ is P-convex on $[a,b]$, for some fixed $p,q>0$, then*

$$\int_a^b (x-a)^p(b-x)^q f(x)dx$$

$$\leq (b-a)^{p+q+1}\left[\beta(kp+1,kq+1)\right]^{\frac{1}{k}}\left(|f(a)|^{\frac{k}{k-1}}+|f(b)|^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}. \tag{2.3}$$

Proof. By Lemma 1, Hölder's inequality, the definition of Beta function and the fact that $|f|^{\frac{k}{k-1}}$ is P-convex on $[a,b]$, we have

$$\begin{aligned}
&\int_a^b (x-a)^p(b-x)^q f(x)dx\\
\leq&(b-a)^{p+q+1}\left[\int_0^1 (1-t)^{kp}t^{kq}dt\right]^{\frac{1}{k}}\left[\int_0^1 |f(ta+(1-t)b)|^{\frac{k}{k-1}}dt\right]^{\frac{k-1}{k}}\\
\leq&(b-a)^{p+q+1}\left[\beta(kq+1,kp+1)\right]^{\frac{1}{k}}\left[\int_0^1 \left(|f(a)|^{\frac{k}{k-1}}+|f(b)|^{\frac{k}{k-1}}\right)dt\right]^{\frac{k-1}{k}}\\
=&(b-a)^{p+q+1}\left[\beta(kq+1,kp+1)\right]^{\frac{1}{k}}\left(|f(a)|^{\frac{k}{k-1}}+|f(b)|^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}},
\end{aligned}$$

which completes the proof. □

A more general inequality using Lemma 1 is as follows:

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ such that $f\in L([a,b])$, $0\leq a<b<\infty$ and let $l>1$. If $|f|^l$ is P-convex on $[a,b]$, for some fixed $p,q>0$, then*

$$\begin{aligned}
&\int_a^b (x-a)^p(b-x)^q f(x)dx\\
\leq&(b-a)^{p+q+1}\beta(p+1,q+1)\left(|f(a)|^l+|f(b)|^l\right)^{\frac{1}{l}}.
\end{aligned}\tag{2.4}$$

Proof. By Lemma 1, Hölder's inequality, the definition of Beta function and the fact that $|f|^l$ is P-convex on $[a,b]$, we have

$$\begin{aligned}
&\int_a^b (x-a)^p(b-x)^q f(x)dx\\
=&(b-a)^{p+q+1}\int_0^1 \left[(1-t)^p t^q\right]^{\frac{l-1}{l}}\left[(1-t)^p t^q\right]^{\frac{1}{l}} f(ta+(1-t)b)dt\\
\leq&(b-a)^{p+q+1}\left[\int_0^1 (1-t)^p t^q dt\right]^{\frac{l-1}{l}}\left[\int_0^1 (1-t)^p t^q |f(ta+(1-t)b)|^l dt\right]^{\frac{1}{l}}\\
\leq&(b-a)^{p+q+1}\left[\beta(q+1,p+1)\right]^{\frac{l-1}{l}}\left[\left(|f(a)|^l+|f(b)|^l\right)\beta(q+1,p+1)\right]^{\frac{1}{l}}\\
=&(b-a)^{p+q+1}\beta(p+1,q+1)\left(|f(a)|^l+|f(b)|^l\right)^{\frac{1}{l}},
\end{aligned}$$

which completes the proof. □

3. AN EXTENSION TO FUNCTIONS OF SEVERAL VARIABLES

In this section some new integral inequalities for functions of several variables on convex subsets of $\mathbb{R}^n$ will be given. First we recall the notion of P-convexity for functions on a convex subset U of $\mathbb{R}^n$.

Definition 2 ([3, Definition 3.1])**.** The function $f : U \to \mathbb{R}$ is said to be P-convex on U if it is nonnegative and, for all $x, y \in U$ and $\lambda \in [0,1]$, satisfies the inequality

$$f(\lambda x + (1-\lambda)y) \leq f(x) + f(y).$$

The following proposition will be used throughout this section.

Proposition 1 ([3, Proposition 3.2])**.** *Let $U \subseteq \mathbb{R}$ be a convex subset of $\mathbb{R}$ and $f : U \to \mathbb{R}$ be a function. Then f is P-convex on U if and only if, for every $x, y \in U$, the function $\varphi : [0,1] \to \mathbb{R}$, defined by*

$$\varphi(t) := f((1-t)x + ty),$$

is P-convex on I with $I = [0,1]$.

We have the following inequalities for functions of several variables on convex subsets of $\mathbb{R}^n$.

Theorem 7. *Let $U \subseteq \mathbb{R}$ be a convex subset of $\mathbb{R}$. Assume that $f : U \to \mathbb{R}^+$ is a P-convex function on U. Then, for every $x, y \in U$ and every $[a,b] \in [0,1]$ with $a < b$, the following inequality holds:*

$$\begin{aligned}&\int_a^b (t-a)^p (b-t)^q f((1-t)x + ty)dt \\ &\leq (b-a)^{p+q+1}\beta(p+1, q+1)[f((1-a)x + ay) + f((1-b)x + by)]. \end{aligned} \tag{3.1}$$

Proof. Let $x, y \in U$ and every $[a,b] \in [0,1]$ with $a < b$. Since $f : U \to \mathbb{R}^+$ is a P-convex function, by Proposition 1 the function $\varphi : [0,1] \to \mathbb{R}^+$ defined by

$$\varphi(t) := f((1-t)x + ty),$$

is P-convex on I with $I = [0,1]$. Applying Theorem 4 to the function φ implies that

$$\begin{aligned}&\int_a^b (t-a)^p (b-t)^q \varphi(t)dt \\ &\leq (b-a)^{p+q+1}\beta(p+1, q+1)(|\varphi(a)| + |\varphi(b)|),\end{aligned}$$

and we deduce that (3.1) holds. □

Similarly, we have

Theorem 8. *Let $U \subseteq \mathbb{R}$ be a convex subset of $\mathbb{R}$ and let $k > 1$. Assume that $f^{\frac{k}{k-1}} : U \to \mathbb{R}^+$ is a P-convex function on U. Then, for every $x, y \in U$ and every $[a,b] \in [0,1]$ with $a < b$, the following inequality holds:*

$$\begin{aligned}&\int_a^b (t-a)^p (b-t)^q f((1-t)x+ty)dt\\ &\leq (b-a)^{p+q+1} [\beta(kp+1, kq+1)]^{\frac{1}{k}}\\ &\left[f^{\frac{k}{k-1}}((1-a)x+ay) + f^{\frac{k}{k-1}}((1-b)x+by)\right]^{\frac{k-1}{k}}.\end{aligned}$$

Theorem 9. *Let $U \subseteq \mathbb{R}$ be a convex subset of $\mathbb{R}$ and let $l > 1$. Assume that $f^l : U \to \mathbb{R}^+$ is a P-convex function on U. Then, for every $x, y \in U$ and every $[a,b] \in [0,1]$ with $a < b$, the following inequality holds:*

$$\begin{aligned}&\int_a^b (t-a)^p (b-t)^q f((1-t)x+ty)dt\\ &\leq (b-a)^{p+q+1} \beta(p+1, q+1)[f^l((1-a)x+ay) + f^l((1-b)x+by)]^{\frac{1}{l}}.\end{aligned}$$

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Author's address

Wenjun Liu
College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China
E-mail address: wjliu@nuist.edu.cn

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DIAGONAL COMMON QUADRATIC LYAPUNOV FUNCTIONS FOR SETS OF POSITIVE LTI SYSTEMS

XIU LIU, LAN SHU, AND XIUYONG DING

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Abstract. This paper focuses on the problems of a diagonal common quadratic Lyapunov function (DCQLF) existence for sets of stable positive linear time-invariant (LTI) systems. We derive the equivalent algebraic conditions to verify the existence of a DCQLF, namely that the finite number Hurwitz Mezler matrices at least have a common diagonal Stein solution. Finally some reduced cases are considered.

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1. INTRODUCTION

In dynamical systems, an additional frequent and inherent constraint is the nonnegativity of the states. Many physical systems in the real world involve variables that have nonnegative sign: population levels, absolute temperature, concentration of substances, and so on. Such systems are referred to as positive systems which means that any trajectory of the system starting at an initial state in the positive orthant remains forever (see [4, 9]). This feature makes analysis and synthesis of positive systems a challenging and interesting job (see, for example, [3, 5, 6, 12] and references therein).

In this paper we are interested in the general stability properties of switched positive systems. Our objective in this paper is on the diagonal quadratic Lyapunov function existence for such systems with stable subsystems. Diagonal quadratic Lyapunov functions play a central role in the study of positive linear time-invariant(LTI) systems. For general LTI systems $\Sigma_A : \dot{x}(t) = Ax(t)$ with $A \in \mathbb{R}^{n\times n}$ and the corresponding Stein matrix inequalities

$$A^T P + PA < 0,$$

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where $P \in \mathbb{R}^{n\times n}$ is positive definite symmetric matrix, we can find a P satisfying the Stein matrix inequalities if and only if A is stable, i.e., each eigenvalue of A has negative real part. We call such a positive definite matrix Stein solution for A. If we can find a Stein solution $D = P$ in the set of diagonal matrices, we say A is diagonally stable, and $V(x) = x^T Dx$ is called a diagonal quadratic Lyapunov function for system Σ_A. We also say a set of matrices is simultaneously diagonal stable if there exists a diagonal matrix that is a Stein solution D for all matrices belonging such a set. Here $V(x) = x^T Dx$ is called a DCQLF for the LTI systems with these matrices. In practice, the diagonal Lyapunov functions are required to assure the stability of quantized systems if a finite precision arithmetic is used to calculate the states of systems[14]. The basic results about the diagonal Lyapunov functions were presented by [15]. This question has attracted a great deal of attention in the past [1,2,10,11,13,16–20]. In general, the solution approaches on diagonal stability problem follows three distinct lines of enquiry. One line can be concluded from the work of [11,20] and others. By checking the negative diagonal entry existence of the matrix AX for all positive semi-definite X, the authors attempted to derive algebraic conditions to verify the existence of a DCQLF. The second line is mainly based on the structure of several convex cones that arise in the study of DCQLF[13]. The authors considered a pair of positive LTI systems and demonstrated how this structure can be used to derive conditions for the existence of such functions. The authors of [17–19] provided the third approach. Through making use of both A and A^{-1}, the author derived the diagonal stability conditions. In particular, conditions for diagonal stability are given in his paper in terms of a common quadratic Lyapunov function existence problem for a pair of lower dimensional systems constructed from A and A^{-1}.

The purpose of this paper is to present some results about the simultaneous diagonal stability of Hurwitz Metzler matrices. By showing a common diagonal Stein solution existence for a finite number Hurwitz Metzler matrices, we object to determine some tractable conditions for the existence of a DCQLF for sets of positive LTI systems. The organization of this paper is as follows. The preparations will be made in Section 2. Section 3 is dedicated to derive some checkable necessary and sufficient conditions on the problem of DCQLF existence for a finite number of positive LTI systems. Some concluding remarks are presented in Section 4.

2. PRELIMINARIES

Throughout, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}^n$ stands for the n-dimensional real vector space and $\mathbb{R}^{n\times n}$ is the space of $n\times n$ matrices. For A in $\mathbb{R}^{n\times n}$, a_{kl} denotes the element in the (k,l) position of A. $A \succeq 0 (\preceq 0)$ means that all elements of matrix A are nonnegative (nonpositive) and $A \succ 0 (\prec 0)$ means that all elements of matrix A are positive (negative). $A \succeq B (A \succ B)$ implies that all elements of matrix $A - B$ are nonnegative (positive). The notion $A > 0 (< 0)$ means that A is a symmetric positive

(negative) definite matrix and $A \succeq 0 (\preceq 0)$ means that matrix A is symmetrical positive (negative) semi-definite. $A = 0$ implies that all elements of A are zero and $A \neq 0$ stands for non-zero matrix. A^T (A^{-1}) represents the transpose (inverse) of matrix A. Let $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$. $\lambda(A)$ represents the eigenvalue of A and $\mu(A)$ denotes the real part of the eigenvalue $\lambda(A)$ of A. Also, when referring to the linear switched systems in this paper, stability shall be used to denote uniform asymptotic stability under arbitrary switching signals. Before proceeding, we recall some facts which are relevant for the work of this paper.

Firstly, Perron-Frobenius Theorem pointed in the following guarantees that the eigen-space of irreducible matrix[7] A corresponding to $\lambda(A)$ is one dimensional.

Lemma 1. *Given an irreducible matrix $A \succeq 0$ in $\mathbb{R}^{n\times n}$, then*

(1) *$\lambda(A)$ is an eigenvalue of A with algebraic multiplicity one.*
(2) *There exists a vector $x \succ 0$ such that $Ax = \lambda(A)x$.*

In [6], Farina and Rinaldi described the basic theory and several applications of positive linear systems. Such systems relate to Metzler and Hurwitz matrices. A matrix A is called a Metzler matrix if and only if $a_{kl} \geq 0$ for $k \neq l$, and Hurwitz matrix if and only if the real part $\mu(A)$ of the eigenvalue $\lambda(A)$ satisfies $\mu(A) < 0$. Now we collect some facts to introduce positive systems.

Theorem 1. *For $A \in \mathbb{R}^{n\times n}$, consider the associated LTI system $\Sigma_A : x(t) = Ax(t)$ with the initial condition $x_0 = x(0)$, then the following statements are equivalent.*

(1) *The LTI system Σ_A is positive.*
(2) *For all $t \geq 0$, the condition $x_0 \succeq 0$ implies that $x(t) \succeq 0$.*
(3) *A is a Metzler matrix.*

Theorem 1 actually gives the definition of positive systems. Furthermore, a classic result guarantees the stability of positive LTI systems.

Theorem 2. *The positive LTI system Σ_A is stable if and only if $A \in \mathbb{R}^{n\times n}$ is a Hurwitz Metzler matrix.*

Therefore, from Theorem 1 and 2, when referring to Hurwitz Metzler matrices in the sequel, this is equivalent to the stable positive LTI systems. Applying the properties of Hurwitz Metzler matrix, some results follows straightforward.

Lemma 2 ([13])**.** *Assume $A \in \mathbb{R}^{n\times n}$ is a Metzler matrix such that $a_{kl} \neq 0$ for $1 \leq k, l \leq n$, and a diagonal matrix $D \in \mathbb{R}^{n\times n}$ satisfies $D \geq 0$ and $D \neq 0$. Then*

(1) *The matrix $A^T D + DA$ is Metzler.*
(2) *The matrix $A^T D + DA$ is irreducible.*

Here if replacing $a_{kl} \neq 0$ with irreducible and Hurwitz properties of A, we can also derive that $A^T D + DA$ are Metzler and irreducible.

Lemma 3. *Let matrices A, B in $\mathbb{R}^{n\times n}$ be Hurwitz Metzler, for any diagonal matrix $D \in \mathbb{R}^{n\times n}$, then $A^T D + DA = 0$ if and only if $B^T D + DB = 0$.*

Lemma 4 ([8])**.** *Assume that A, B in $\mathbb{R}^{n\times n}$ are Metzler matrices with $a_{kl}, b_{kl} \neq 0$ for $1 \leq k, l \leq n$, respectively. If $A \succeq B$ and A is Hurwitz, then B is also Hurwitz.*

Paper [8] pointed that, for two Hurwitz Metzler matrices A, B and any positive number δ, the Hurwitz of matrix pencil $A + \delta B$ is equivalent to its non-singular property. This fact can be easy to extend the finite numbers case as follows.

Lemma 5. *Suppose that $A_1, \cdots, A_m$ are Hurwitz Metzler in $\mathbb{R}^{n\times n}$, for all positive real numbers $\delta_1, \cdots, \delta_m$, then the following statements are equivalent.*

(1) $\sum_{j=1}^{m} \delta_j A_j$ *is a Hurwitz matrix.*
(2) *The determination* $\det \sum_{j=1}^{m} \delta_j A_j \neq 0$.

As stable LTI systems have diagonal Lyapunov functions, it is natural to ask under what conditions families of such systems will possess a DCQLF. The next section shall focus on this problem.

3. Diagonal stability

In this section, a necessary and sufficient condition for the existence of a DCQLF will be derived for sets of stable positive LTI systems. To be more precise, assume that $A_1, \cdots, A_m$ in $\mathbb{R}^{n\times n}$ are Hurwitz Metzler matrices with no zero entries, Theorem 3 provides a equivalent condition for the existence of a positive definite diagonal matrix D in $\mathbb{R}^{n\times n}$ such that $A_j^T D + DA_j < 0$ for all $j = 1, \cdots, m$.

However, before stating that theorem, we need the following technical result.

Lemma 6. *Assume that $A_1, \cdots, A_m \in \mathbb{R}^{n\times n}$ are Hurwitz Metzler matrices satisfying $a_{jkl} \neq 0$ for $1 \leq k, l \leq n$ and $1 \leq j \leq m$, then there exist some diagonal matrices $D_j > 0$ in $\mathbb{R}^{n\times n}$ such that*

$$\sum_{j=1}^{m} \det D_j A_j D_j = 0 \tag{3.1}$$

holds if the following conditions are satisfied:

(1) *The positive LTI systems $\Sigma_{A_1}, \cdots, \Sigma_{A_m}$ have no DCQLF.*
(2) *There is a diagonal matrix $\tilde{D} \in \mathbb{R}^{n\times n}$ satisfying $\tilde{D} \geq 0$ and $\tilde{D} \neq 0$ such that*

$$A_j^T \tilde{D} + \tilde{D} A_j \leq 0, \ \forall j = 1, \cdots, m. \tag{3.2}$$

Proof. The proof of this lemma is broken into three steps.

Step 1: We claim that there exist m vectors $y_j \succ 0$ in $\mathbb{R}^n$ such that

$$(A_j^T \tilde{D} + \tilde{D} A_j) y_j = 0, \ \forall j = 1, \cdots, m. \tag{3.3}$$

For simplicity, write $\tilde{R}_j = A_j^T \tilde{D} + \tilde{D} A_j$. In fact, for each j, since $A_j \in \mathbb{R}^{n\times n}$ with $a_{jkl} \neq 0$ $(1 \leq k,l \leq n)$ is Metzler matrices, and diagonal matrix $\tilde{D} \geq 0, \tilde{D} \neq 0$, it is easy to check from Lemma 2 that $\tilde{R}_j$ is also irreducible Metzler matrices. Then in this case, there is $\alpha > 0$ large enough such that $\tilde{R}_j + \alpha I \succeq 0$. Obviously, $\tilde{R}_j + \alpha I$ is also irreducible, together (3.2) and Lemma 1 yields that $\lambda(\tilde{R}_j + \alpha I) = \alpha$ is an eigenvalue of algebraic multiplicity one for $\tilde{R}_j + \alpha I$. This implies that $\lambda(\tilde{R}_j) = 0$ is an eigenvalue of algebraic multiplicity one for $\tilde{R}_j$, Therefore, the rank of $\tilde{R}_j$ is $n-1$. Finally, by Lemma 1, we can conclude that there exist m vectors $y_j \succ 0$ such that (3.3) holds for all j as the claim.

Step 2: We wish to show that there is no diagonal matrix $\hat{D} \in \mathbb{R}^{n\times n}$ satisfying

$$y_j^T (A_j^T \hat{D} + \hat{D} A_j) y_j < 0, \ \forall j = 1, \cdots, m, \tag{3.4}$$

where y_j defined by (3.3).

To this end, conversely, for any vectors $x_j \succ 0$, suppose that there are some diagonal $\hat{D}$ satisfying (3.4) for all j. We shall find a diagonal $D > 0$ in $\mathbb{R}^{n\times n}$ such that

$$A_j^T D + D A_j < 0 \tag{3.5}$$

simultaneously hold for all j, which would contradict with statement (1).

With this in mind, for all j, consider the sets

$$\Omega_j = \{y \in \mathbb{R}^n : y^T y = 1 \ \text{ and } \ y^T (A_j^T \hat{D} + \hat{D} A_j) y \geq 0\}. \tag{3.6}$$

Now distinguish two cases.

Case 1: $\Omega_j = \varnothing$ for all j.

Under this case, $y^T (A_j^T \hat{D} + \hat{D} A_j) y < 0$ with $y^T y = 1$, together with (3.2) yields that for any positive constant $\epsilon_j > 0$,

$$y^T \left(A_j^T (\tilde{D} + \epsilon_j \hat{D}) + (\tilde{D} + \epsilon_j \hat{D}) A_j \right) y < 0. \tag{3.7}$$

Since the inequality (3.7) is unchange if we scale y by any non-zero real number. That is, each $\tilde{D} + \epsilon_j \hat{D} > 0$ results from the Hurwitz of A_j. (3.5) thus holds by setting $D = \tilde{D} + \epsilon \hat{D}$ with $\epsilon = \min_{j=1}^m \{\epsilon_j\}$. A contradiction occurs.

Case 2: $\Omega_j \neq \varnothing$ for all j.

For vector $y \in \mathbb{R}^n$ with $y^T y = 1$, note that (3.4) is true as assumption, then $y_j \in \mathbb{R}^n / \Omega_j$, it follows from (3.3) that for any $y \in \Omega_j$

$$y^T (A_j^T \tilde{D} + \tilde{D} A_j) y < 0. \tag{3.8}$$

Now consider two subcases: $y \in \Omega_j$ and $y \in \mathbb{R}^n / \Omega_j$.

On the one hand, if $y \in \Omega_j$, then

$$y^T (A_j^T \hat{D} + \hat{D} A_j) y \geq 0. \tag{3.9}$$

From (3.8) and (3.9), one can guarantee the following inequality

$$y^T\left(A_j^T(\tilde{D}+\epsilon_j^{'}\hat{D})+(\tilde{D}+\epsilon_j^{'}\hat{D})A_j\right)y<0 \tag{3.10}$$

holds, where $\epsilon_j^{'}<|\tilde{M}_j|/\hat{M}_j$ with

$$\hat{M}_j=\max\{y^T(A_j^T\hat{D}+\hat{D}A_j)y|y\in\Omega_j\},$$

$$\tilde{M}_j=\max\{y^T(A_j^T\tilde{D}+\tilde{D}A_j)y|y\in\Omega_j\}.$$

Then it is easy to see that (3.10) holds for any $y\in\mathbb{R}^n$. This means that $\tilde{D}+\epsilon_j^{'}\hat{D}>0$. Therefore, (3.5) holds by setting $D=\tilde{D}+\epsilon^{'}\hat{D}$ with $\epsilon^{'}=\min\{\epsilon_j^{'},j=1,\cdots,m\}$, which contradict with statement (1).

On the other hand, if $y\in\mathbb{R}^n/\Omega_j$, then $y^T(A_j^T\hat{D}+\hat{D}A_j)y<0$, taking (3.2) into account yields that for $\epsilon_j^{'}$

$$y^T\left(A_j^T(\tilde{D}+\epsilon_j^{'}\hat{D})+(\tilde{D}+\epsilon_j^{'}\hat{D})A_j\right)y<0. \tag{3.11}$$

With the same discussion as case $y\in\Omega_j$, a contradiction shall occur.

Now based on the argument above, we thus know that (3.4) is not true.

Step 3: Now it remains to show that there exist m diagonal matrices $D_j>0$ ($j=1,\ldots,m$) such that (3.1) holds.

In Step 2, we have shown that there is no diagonal solution to (3.4) for all j. Now for diagonal $\hat{D}$, Hurwitz Metzler matrices A_j and vector $y_j\succ 0$, it follows that

$$\begin{aligned}&y_{j_k}^T(A_{j_k}^T\hat{D}+\hat{D}A_{j_k})y_{j_k}<0, 1\le k\le l-1\\ \Leftrightarrow&y_{j_k}^T(A_{j_k}^T\hat{D}+\hat{D}A_{j_k})y_{j_k}>0, l\le k\le m\end{aligned} \tag{3.12}$$

with $j_k\in\{1,\cdots,m\}$ and $1\le l<m$. On the other hand, from Lemma 3 we can know

$$y_1^T(A_1^T\hat{D}+\hat{D}A_1)y_1=0\Leftrightarrow\cdots\Leftrightarrow y_m^T(A_m^T\hat{D}+\hat{D}A_m)y_m=0. \tag{3.13}$$

Furthermore, taking (3.13) and (3.12) into account, we can obtain that for a constant $c>0$

$$\sum_{k=1}^{l-1}y_{j_k}^T(A_{j_k}^T\hat{D}+\hat{D}A_{j_k})y_{j_k}=-c\sum_{k=l}^{m}y_{j_k}^T(A_{j_k}^T\hat{D}+\hat{D}A_{j_k})y_{j_k}. \tag{3.14}$$

From (3.14) we may replace y_{j_k} with $y_{j_k}/\sqrt{c}$, in this case, due to $y_j\succ 0$, now fix a vector y_{j_u} ($1\le u\le l-1$), then for any vector y_j ($j\neq j_u$), we can find a associated diagonal matrix $D_j>0$ for all j such that

$$D_{j_u}y_{j_u}=D_jy_j. \tag{3.15}$$

Moreover, substituting (3.15) into (3.14) yields that

$$y_{j_u}^T(A_{j_u}^T\hat{D}+\hat{D}A_{j_u})y_{j_u}+\sum_{k\neq u,k=1}^{l-1}y_{j_u}^T D_{j_u}D_{j_k}^{-1}(A_{j_k}^T\hat{D}+\hat{D}A_{j_k})D_{j_k}^{-1}D_{j_u}y_{j_u}$$
$$=-\sum_{k=l}^{m}y_{j_u}^T D_{j_u}D_{j_k}^{-1}(A_{j_k}^T\hat{D}+\hat{D}A_{j_k})D_{j_k}^{-1}D_{j_u}y_{j_u}. \quad (3.16)$$

By induction, (3.16) implies

$$\det\left[(A_{j_u}^T\hat{D}+\hat{D}A_{j_u})+\sum_{k\neq u,k=1}^{m}D_{j_u}D_{j_k}^{-1}(A_{j_k}^T\hat{D}+\hat{D}A_{j_k})D_{j_k}^{-1}D_{j_u}\right]=0. \quad (3.17)$$

Now pre- and post-multiplying (3.17) by $D_{j_u}^{-1}$, respectively, then (3.17) is equivalent to

$$\det\left[\sum_{j=1}^{m}D_j^{-1}(A_j^T\hat{D}+\hat{D}A_j)D_j^{-1}\right]=2\det\sum_{j=1}^{m}D_j^{-1}A_j^T\hat{D}D_j^{-1}=0. \quad (3.18)$$

As diagonal matrices are commutative, for all j, replacing D_j^{-1} in (3.18) by D_j, (3.1) follows immediately. This completes the proof. □

Theorem 3. *Given m Hurwitz Metzler matrices $A_1,\cdots,A_m\in\mathbb{R}^{n\times n}$ with $a_{jkl}\neq 0$ for $1\leq k,l\leq n$ and $j=1,\cdots,m$, then the following statements are equivalent:*

(1) *The positive LTI systems $\Sigma_{A_1},\cdots,\Sigma_{A_m}$ share a DCQLF.*
(2) *For any m diagonal matrices $D_j>0$ in $\mathbb{R}^{n\times n}$,*

$$\det\sum_{j=1}^{m}D_jA_jD_j\neq 0. \quad (3.19)$$

Proof. (1)⇒ (2) According to definition of DCQLFs, the statement implies that for all j and diagonal matrix $D>0$

$$A_j^TD+DA_j<0 \quad (3.20)$$

simultaneous hold. By (3.20), for any diagonal $D_j>0$, we have

$$(D_jA_jD_j)^TD+D(D_jA_jD_j)=D_j(A_j^TD+DA_j)D_j<0. \quad (3.21)$$

One can further get from (3.21) that

$$(\sum_{j=1}^{m}D_jA_jD_j)^TD+D(\sum_{j=1}^{m}D_jA_jD_j)=\sum_{j=1}^{m}D_j(A_j^TD+DA_j)D_j<0. \quad (3.22)$$

That is, $\sum_{j=1}^{m}D_jA_jD_j$ is a Hurwitz matrix and thus non-singular.

(2)⇒ (1). We shall prove this by contradiction. First of all, suppose that the

statement (1) is not true, i.e. $\Sigma_{A_1},\cdots,\Sigma_{A_m}$ have no DCQLF. In this case, for all j, if choose sufficiently large $\xi_j > 0$, then there is always a DCQLF for $\Sigma_{A_1-\xi_1 I},\cdots,\Sigma_{A_m-\xi_m I}$. Now, for any $j = 1,\cdots,m$, define

$$\tilde{\xi}_j = \inf\left\{\xi_j > 0 | \Sigma_{A_1-\xi_1 I},\cdots,\Sigma_{A_m-\xi_m I} \text{ have a DCQLF}\right\},$$

and further let $\tilde{\xi} = \min\{\tilde{\xi}_j, j = 1,\cdots,m\}$. Then according to Lemma 3, one can check that the matrices $A_1-\tilde{\xi}I,\cdots,A_m-\tilde{\xi}I$ satisfy the conditions of Lemma 6. Then there exist m diagonal matrices $D_j > 0$ such that

$$\det\sum_{j=1}^{m} D_j(A_j-\tilde{\xi}I)D_j = 0. \tag{3.23}$$

Moreover, from (3.23), matrix $\sum_{j=1}^{m} D_j A_j D_j$ has a positive real eigenvalue and thus is not Hurwitz. Note that all $D_j A_j D_j$ are Hurwitz Metzler matrices, it follows from Lemma 5 that there exist positive real numbers $\delta_1,\cdots,\delta_m$ such that

$$\det\sum_{j=1}^{m} \delta_j D_j A_j D_j = 0. \tag{3.24}$$

Finally, for all j, replacing $\sqrt{\delta_j}D_j$ in (3.20) by D_j, it turn out that the statement (2) is not true in this case. This completes the proof. □

The above result follows from the problem of determining whether a DCQLF exists for sets of positive LTI systems. We have presented a equivalent condition for such problem through algebraic argument. Obviously, if sets of positive LTI systems share a DCQLF, each individual LTI system must have a DCQLF.

Note that the comments below Lemma 2, in Theorem 3, if we replace the condition of $a_{jkl} \neq 0$ with the irreducibility of $A_1,\cdots,A_m$, then the necessary and sufficient condition between statements (1) and (2) also holds. Next we shall consider some reduced cases. These results are straightforward from Theorem 3.

Corollary 1. *If $A_1,\cdots,A_m \in \mathbb{R}^{n\times n}$ are Hurwitz diagonal matrices, then LTI systems $\Sigma_{A_1},\cdots,\Sigma_{A_m}$ are positive and thus share a DCQLF.*

Under diagonal matrices, Corollary 1 just restricts the Hurwitz properties of system matrices. In fact, as Hurwitz diagonal, the positivity and irreducibility are naturally satisfied. Next we consider upper and lower triangular matrices.

Corollary 2. *Suppose that $A_1,\cdots,A_m \in \mathbb{R}^{n\times n}$ with $a_{jkl} \neq 0$ for $1 \leq k,l \leq n$ are Hurwitz Metzler matrix. If $A_1,\cdots,A_m \in \mathbb{R}^{n\times n}$ are upper (lower) triangular, then positive LTI systems $\Sigma_{A_1},\cdots,\Sigma_{A_m}$ have a DCQLF.*

Applying Lemma 4, Corollary 3 and 4 follow immediately.

Corollary 3. *For $i, j = 1, \cdots, m$, given m Metzler matrices $A_1, \cdots, A_m \in \mathbb{R}^{n\times n}$ with $a_{jkl} \neq 0$ for $1 \leq k, l \leq n$. If there is a matrix A_j such that A_j is Hurwitz and satisfies $A_j \succeq A_i$ for $i \neq j$, then positive LTI systems $\Sigma_{A_1}, \cdots, \Sigma_{A_m}$ have a DCQLF.*

Corollary 4. *For $i, j = 1, \cdots, m$, given m Metzler matrices $A_1, \cdots, A_m \in \mathbb{R}^{n\times n}$ with $a_{jkl} \neq 0$ for $1 \leq k, l \leq n$, and some diagonal $D_j > 0$. If there is a matrix A_j such that A_j is Hurwitz and satisfies $D_j A_j D_j \succeq A_i$ for $i \neq j$, then positive LTI systems $\Sigma_{A_1}, \cdots, \Sigma_{A_m}$ have a DCQLF.*

For 2×2 Hurwitz matrices with negative diagonal entries, this is equivalent to the determinants being positive. We thus obtain the following result.

Corollary 5. *If Hurwitz Metzler matrices $A_1, \cdots, A_m$ with $a_{jkl} \neq 0$ $(1 \leq k, l \leq n)$ are in $\mathbb{R}^{2\times 2}$, then positive LTI systems $\Sigma_{A_1}, \cdots, \Sigma_{A_m}$ share a DCQLF.*

4. Conclusions

In this paper, a necessary and sufficient condition for the existence of a DCQLF has been derived for families of positive LTI systems. By algebraic argument, we answered that under what conditions a finite number Hurwitz Metzler matrices have a common diagonal Stein solution. In other words, the simultaneous diagonal stability can be achieved for system matrices of positive LTI systems. Furthermore some results for reduced cases followed from such a result.

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Authors' addresses

Xiu Liu
Southwest Jiaotong University, School of Mathematics, 611756, Chengdu, PR China
E-mail address: xiuliu@outlook.com

Lan Shu
University of Electronic Science and Technology of China, School of Mathematical Sciences, 611731, Chengdu, PR China
E-mail address: shul@uestc.edu.cn

Xiuyong Ding
Southwest Jiaotong University, School of Mathematics, 611756, Chengdu, PR China
E-mail address: dingxingzhi@hotmail.com

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CONTROLLABILITY OF NONAUTONOMOUS BOUNDARY INTEGRODIFFERENTIAL CAUCHY PROBLEMS

M. MOUSSI

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Abstract. In this paper, we show the existence of mild solutions to nonautonomous boundary integrodifferential Cauchy problems. Moreover, we establish a controllability result. An example illustrates the obtained results.

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1. INTRODUCTION

We consider in the present paper the following nonlinear boundary integrodifferential Cauchy problem:

$$\begin{cases} \frac{d}{dt}x(t) = A_{\max}(t)x(t), \quad 0 \le \tau \le t \le T, \\ L(t)x(t) = f\left(t, x(t), \int_\tau^t g(t,\sigma,x(\sigma))\,d\sigma\right), \quad 0 \le \tau \le t \le T, \\ x(\tau) = x_0, \end{cases} \tag{1.1}$$

where $A_{\max}(t)$ are closed operators on a Banach space X endowed with a maximal domain $D(A_{\max}(t))$, $L(t) : D(A_{\max}(t)) \to \partial X$ with a 'boundary Banach space' ∂X, a function $f : \mathbb{R}_+ \times X \times X \to \partial X$ and a function $g : \mathbb{R}_+ \times \mathbb{R}_+ \times X \to X$.

In recent years, such abstract integrodifferential equations have attracted the interest of many authors. Questions as existence of solutions, perturbations, controllability and the asymptotic behavior are the subject of many works. We cite among others [1, 3, 4, 6–8, 10–12, 14–17, 20–23] and the references therein.

Our aim is to study the wellposedness of (1.1) and to present a controllability result.

In [2] we have studied the boundary Cauchy problem in the case that the second equation in (1.1) is replaced by an equation $L(t)x(t) = f(t)$. For this type of equation we established mild solutions given by a variation of constants formula. In this

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work, we firstly extend this to a variation of constants formula for problem (1.1) using the contraction fixed point theorem, see Section 2.

In Section 3 we present sufficient conditions to obtain exact controllability of the following boundary integrodifferential control system:

$$\begin{cases} \frac{d}{dt}x(t) = A_{\max}(t)x(t), \quad t \in [0,T], \\ L(t)x(t) = f\left(t, x(t), \int_\tau^t g(t,\sigma,x(\sigma))\,d\sigma\right) + B(t)u(t) \quad t \in [0,T], \\ x(0) = x_0. \end{cases} \tag{1.2}$$

Here the input function $u(\cdot)$ takes values in a Banach space U (the control space) and the nonautonomous control operator $B(t), t \in [0,T]$, are bounded from U into ∂X.

To illustrate these assumptions and our results, we present in Section 4 a population equation as an example.

2. Mild solution of nonlinear boundary integrodifferential equation

Consider the following nonlinear integrodifferential boundary Cauchy problem

$$\begin{cases} \frac{d}{dt}x(t) = A_{\max}(t)x(t), \quad 0 \le \tau \le t \le T, \\ L(t)x(t) = f\left(t, x(t), \int_\tau^t g(t,\sigma,x(\sigma))\,d\sigma\right), \quad \tau \le t \le T, \\ x(\tau) = x_0 \in X, \end{cases} \tag{2.1}$$

where X is a Banach space, the operators $A_{\max}(t) \in \mathcal{L}(D,X)$ and $L(t) \in \mathcal{L}(D,\partial X)$ for $t \ge 0$, with D and ∂X Banach spaces such that D is dense and continuously embedded in X.

The operators $A_{\max}(t)$ and $L(t)$ are supposed to satisfy the following hypotheses:

(H1) There are positive constants C_1, C_2 such that

$$C_1\|x\|_D \le \|x\| + \|A_{\max}(t)x\| \le C_2\|x\|_D$$

for all $x \in D$ and $t \ge 0$;

(H2) for each $x \in D$, the mapping $\mathbb{R}_+ \ni t \mapsto A_{\max}(t)x \in X$ is continuously differentiable;

(H3) the operators $L(t) : D \to \partial X, t \ge 0$, are surjective;

(H4) for each $x \in D$, the mapping $\mathbb{R}_+ \ni t \mapsto L(t)x \in \partial X$ is continuously differentiable;

(H5) there exist constants $\gamma > 0$ and $\omega \in \mathbb{R}$ such that

$$||L(t)x||_{\partial X} \ge \gamma^{-1}(\lambda - \omega)||x||_X,$$

for $x \in \ker(\lambda - A_{\max}(t)), \lambda > \omega$ and $t \ge 0$;

(H6) the family of operators $(A(t))_{0\le t\le T}$, with $A(t) := A_{\max}(t)|_{\ker L(t)}$, is stable, that is, there are constants $M \ge 1$ and $\omega \in \mathbb{R}$ such that $(\omega,\infty) \subset \rho(A(t))$ (the *resolvent set* of $A(t)$) $\forall\ 0 \le t \le T$ and

$$\left\| \prod_{i=1}^{k} R(\lambda, A(t_i)) \right\| \le M(\lambda-\omega)^{-k}$$

for $\lambda > \omega$ and any finite sequence $0 \le t_1 \le \cdots \le t_k \le T$, and $R(\lambda, A(t)) := (\lambda - A(t))^{-1}$.

In the following lemma we cite consequences of the above assumptions from [9, Lemma 1.2] which will be needed below.

Lemma 1. *The restriction* $L(t)|_{\ker(\lambda - A_{\max}(t))}$ *is an isomorphism from* $\ker(\lambda - A_{\max}(t))$ *into* ∂X *and its inverse* $L_{\lambda,t} := [L(t)|_{\ker(\lambda - A_{\max}(t))}]^{-1} : \partial X \to \ker(\lambda - A_{\max}(t))$ *satisfies*

$$\|L_{\lambda,t}\| \le \gamma(\lambda-\omega)^{-1} \quad \text{for } \lambda > \omega.$$

Recall that in case $f \equiv 0$ the problem (2.1) is reduced to the linear boundary Cauchy problem

$$\begin{cases} \frac{d}{dt}x(t) = A_{\max}(t)x(t), \quad \tau \le t \le T, \\ L(t)x(t) = 0, \quad \tau \le t \le T, \\ x(\tau) = x_0 \end{cases} \tag{2.2}$$

which was studied by Kellermann [13] and Nguyen Lan [18]. In particular, the authors proved that under assumptions (H1)–(H6) the problem (2.2) has a unique solution given by an evolution family $(U(t,\tau))_{\tau\le t\le T}$ satisfying

$$\|U(t,\tau)\| \le Me^{\omega(t-\tau)}, \quad \forall\ \tau \le t \le T, \tag{2.3}$$

where M and ω are the stability constants of $A(t)$.

In this section we study the existence of mild solutions of the nonlinear problem (2.1) and assume for the nonlinear part:

(H7) The nonlinear function $f : [0,T] \times X \times X \to \partial X$ is continuous and there exists a positive constant ℓ_f such that one has the following global Lipschitz estimate

$$\|f(t,x,y) - f(t,\bar{x},\bar{y})\| \le \ell_f(\|x-\bar{x}\| + \|y-\bar{y}\|)$$

for all $x, y, \bar{x}, \bar{y} \in X$ and $t \in [0,T]$;

(H8) the nonlinear function $g : \Delta_T \times X \to X$ is continuous and there exists a positive constant ℓ_g such that one has the global Lipschitz estimate

$$\|g(t,s,x) - g(t,s,\bar{x})\| \le \ell_g\|x-\bar{x}\|$$

for all $x, \bar{x} \in X$ and $(t,s) \in \Delta_T$. Here Δ_T is defined by:

$$\Delta_T := \{(t,s) : 0 \le s \le t \le T\}.$$

Under the assumptions (H1)–(H8) and using the usual contraction argument we shall show the existence of a unique mild solution given by the following definition.

Definition 1. A continuous function $x(\cdot):[\tau,T]\to X$ is called *mild solution of* the problem (2.1) if it satisfies the variation of constants formula

$$x(t)=U(t,\tau)x_0+\lim_{\lambda\to\infty}\int_\tau^t U(t,\sigma)\lambda L_\lambda f\left(\sigma,x(\sigma),\int_\tau^\sigma g(\sigma,\alpha,x(\alpha))d\alpha\right)d\sigma \tag{2.4}$$

for all $\tau\le t\le T$.

Remark 1. The limit in equation (2.4) is well-defined, for details we refer the reader to [2].

In the following theorem we prove the existence of a mild solution.

Theorem 1. *Let the assumptions (H1)-(H8) be satisfied. Then for every $x_0\in X$ and $\tau\ge 0$ the boundary integrodifferential equation (2.1) has a unique mild solution on $[\tau,T]$.*

Proof. Let $x_0\in X$ be fixed. Let $Y:=C([0,T],X)$ be the Banach space of all continuous functions from $[0,T]$ into X. Define an operator

$$(\Phi v)(t):=U(t,\tau)x_0+\lim_{\lambda\to\infty}\int_\tau^t U(t,\sigma)\lambda L_{\lambda,\sigma} f\left(\sigma,v(\sigma),\int_\tau^\sigma g(\sigma,\alpha,v(\alpha))\,d\alpha\right)d\sigma$$

for all $\tau\le t\le T$ and $v\in Y$. It is clear that Φ maps Y into itself.

Now for v_1,v_2 in Y we have

$$\begin{aligned}\|(\Phi v_1)(t)-(\Phi v_2)(t)\|&\le\Bigg\|\lim_{\lambda\to\infty}\int_\tau^t U(t,\sigma)\lambda L_{\lambda,\sigma} f\left(\sigma,v_1(\sigma),\int_\tau^\sigma g(\sigma,\alpha,v_1(\alpha))\,d\alpha\right)\\&\quad-f\left(\sigma,v_2(\sigma),\int_\tau^\sigma g(\sigma,\alpha,v_2(\alpha))\,d\alpha\right)d\sigma\Bigg\|.\end{aligned}$$

Using (2.3), assumption (H7) and assumption (H8) we obtain

$$\begin{aligned}\|(\Phi v_1)(t)-(\Phi v_2)(t)\|&\le\int_\tau^t\gamma Me^{\omega(t-\sigma)}\ell_f\Bigg(\|v_1(\sigma)-v_2(\sigma)\|\\&\quad+\left\|\int_\tau^\sigma g(\sigma,\alpha,v_1(\alpha))-g(\sigma,\alpha,v_2(\alpha))\,d\alpha\right\|\Bigg)d\sigma\\&\le\int_\tau^t\gamma Me^{\omega(t-\sigma)}\ell_f\left(\|v_1-v_2\|+\ell_g(\sigma-\tau)\|v_1-v_2\|\right)d\sigma\\&\le e^{\omega(t-\tau)}\gamma M(t-\tau)\ell_f\left(\|v_1-v_2\|+\ell_g(t-\tau)\|v_1-v_2\|\right)\\&\le e^{\omega(t-\tau)}\gamma M\ell_f(t-\tau)(1+\ell_g T)\|v_1-v_2\|.\end{aligned}$$

$$\leq e^{\omega T}\gamma M\ell_f T(1+\ell_g T)\|v_1-v_2\|.$$

By induction, one obtains

$$\|\Phi^n v_1-\Phi^n v_2\|\leq e^{\omega T}\frac{(\gamma M\ell_f T(1+\ell_g T))^n}{n!}\|v_1-v_2\|$$

which implies that, for n sufficiently large, Φ^n is a contraction map on the Banach space Y. Hence by the Banach fixed point theorem there exists a unique function $v\in Y$ satisfying $\Phi v=v$ which is the mild solution of the problem (2.1). □

3. Boundary intergrodifferential controlled system

In this section we consider the following boundary control system

$$\begin{cases}\frac{d}{dt}x(t)=A_{\max}(t)x(t), & t\in[0,T],\\ L(t)x(t)=f\left(t,x(t),\int_0^t g(t,\sigma,x(\sigma))\,d\sigma\right)+B(t)u(t) & t\in[0,T],\\ x(0)=x_0.\end{cases}\tag{3.1}$$

Here $A_{\max}(t),L(t),f(t,\cdot,\cdot),g(t,\cdot,\cdot)$ satisfy the hypotheses (H1)-H(8), the control function $u(\cdot)$ takes values in a Banach space U and $B(t)$ are bounded operators from U into ∂X. As in the previous section, one can show that the control system (3.1) admits a unique mild solution $x(\cdot)$ given by

$$\begin{aligned}x(t)=U(t,0)x_0+\lim_{\lambda\to\infty}\int_0^t U(t,\sigma)\lambda L_\lambda\\ \times\left[f\left(\sigma,x(\sigma),\int_0^\sigma g(\sigma,\alpha,x(\alpha))d\alpha\right)+B(\sigma)u(\sigma)\right]d\sigma\end{aligned}\tag{3.2}$$

for $t\geq 0$ and $x_0\in X$.

Our aim in this section is to present sufficient conditions to obtain the exact controllability of the controlled system (3.1). We first recall the following definition.

Definition 2. The control system (3.1) is said to be exactly controllable on the interval $[0,T]$ for some $T>0$, if for all x_0, $y\in X$, there exists a control $u\in L^2([0,T],U)$ such that the mild solution $x(t)$ of (3.1) corresponding to u satisfies $x(T)=y$.

In order to reach the goal of this section, we further assume the following additional hypothesis.

(H9) The operator $W:L^2([0,T],U)\longrightarrow X$ defined by

$$Wu:=\lim_{\lambda\to\infty}\int_0^T U(t,\sigma)\lambda L_\lambda B(\sigma)u(\sigma)\,d\sigma\tag{3.3}$$

has a pseudo inverse operator W^{-1} which takes values in $L^2([0,T],U)/ker(W)$ and there exist positive constants k_1, k_2 such that

$$\|B(\cdot)\| \leq k_1 \text{ and } \|W^{-1}\| \leq k_2. \tag{3.4}$$

Here, $ker(W)$ denotes the kernel of the operator W.

Remark 2. For the construction of W and W^{-1} we refer to the paper [19].

Our overall approach is the usual fixed point method. Therefore, define the operator $F : C([0,T],X) \longrightarrow C([0,T],X)$ by

$$\begin{aligned}(Fx)(t) := U(t,0)x_0 + \lim_{\lambda\to\infty}\int_0^t U(t,\sigma)\lambda L_\lambda \\ \times\left[f\left(\sigma, x(\sigma), \int_0^\sigma g(\sigma,\alpha,x(\alpha))d\alpha\right) + B(\sigma)u(\sigma)\right]d\sigma.\end{aligned} \tag{3.5}$$

Here the control $u(\cdot)$ is chosen for $x(\cdot) \in C([0,T],X)$ and $y \in X$ as

$$\begin{aligned}u(t) := W^{-1}\left[y - U(T,0)x_0 - \lim_{\lambda\to\infty}\int_0^T U(t,\sigma)\lambda L_\lambda \right. \\ \left. \times f\left(\sigma, x(\sigma), \int_0^\sigma g(\sigma,\alpha,x(\alpha))d\alpha\right)d\sigma\right](t).\end{aligned} \tag{3.6}$$

We have the following proposition.

Proposition 1. *The mapping F is a contraction from the Banach space $C([0,T],X)$ into itself provided that $(1+T\ell_g)(\gamma MT\ell_f + \gamma^2 M^2 k_1 k_2 \ell_f T) < 1$.*

Proof. Let $x_1, x_2 \in C([0,T],X)$ and choose control functions u_1, u_2 respectively. We have

$$\begin{aligned}(Fx_1)(t) - (Fx_2)(t) = \lim_{\lambda\to\infty}\int_0^t U(t,\sigma)\lambda L_\lambda\left[f\left(\sigma, x_1(\sigma), \int_0^\sigma g(\sigma,\alpha,x_1(\alpha))\,d\alpha\right)\right. \\ \left. - f\left(\sigma, x_2(\sigma), \int_0^\sigma g(\sigma,\alpha,x_2(\alpha))d\alpha\right) + B(\sigma)(u_1(\sigma) - u_2(\sigma))\right]d\sigma.\end{aligned}$$

For simplicity, we put $M := \sup_{(t,s)\in\Delta_T} U(t,s)$. Then using assumptions (H7), (H8) and (H9) we obtain

$$\begin{aligned}\|(Fx_1)(t) - (Fx_2)(t)\| \leq \gamma M\int_0^t \left\| f\left(\sigma, x_1(\sigma), \int_0^\sigma g(\sigma,\alpha,x_1(\alpha))\,d\alpha\right)\right. \\ \left. - f\left(\sigma, x_2(\sigma), \int_0^\sigma g(\sigma,\alpha,x_2(\alpha))\,d\alpha\right)\right\| d\sigma \\ + \gamma M\|B(\cdot)\|\|W^{-1}\|\gamma M\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^T \left\| f\left(\alpha, x_1(\alpha), \int_0^\alpha g(\alpha,\tau,x_1(\tau))\,d\tau\right)\right.\\
&\left. - f\left(\alpha, x_2(\alpha), \int_0^\alpha g(\alpha,\tau,x_2(\tau))d\tau\right) d\alpha \right\|\\
&\le \gamma M \int_0^t \ell_f \left(\|x_1(\sigma)-x_2(\sigma)\| \right.\\
&\left. + \left\| \int_0^\sigma g(\sigma,\tau,x_1(\tau)) - g(\sigma,\tau,x_2(\alpha))\,d\tau \right) \right\| d\sigma\\
&+ \gamma M \|B(\cdot)\| \|W^{-1}\| \gamma M \int_0^T \ell_f \left(\|x_1(\alpha)-x_2(\alpha)\| \right.\\
&\left. + \left\| \int_0^\alpha g(\alpha,\tau,x_1(\tau)) - g(\alpha,\tau,x_2(\alpha))\,d\tau \right) \right\| d\alpha\\
&\le \gamma M T \ell_f \left(\|x_1(\cdot)-x_2(\cdot)\| + T\ell_g \|x_1(\cdot)-x_2(\cdot)\| \right)\\
&+ \gamma M \|B(\cdot)\| \|W^{-1}\| \gamma M T \ell_f \left(\|x_1(\cdot)-x_2(\cdot)\| \right)\\
&+ \gamma M \|B(\cdot)\| \|W^{-1}\| \gamma M T \ell_f \left(T\ell_g \|x_1(\cdot)-x_2(\cdot)\| \right)\\
&\le \gamma M T \ell_f (1+T\ell_g) \|x_1(\cdot)-x_2(\cdot)\|\\
&+ \gamma^2 M^2 k_1 k_2 \ell_f T (1+T\ell_g) \|x_1(\cdot)-x_2(\cdot)\|\\
&= (1+T\ell_g)(\gamma M T \ell_f + \gamma^2 M^2 k_1 k_2 \ell_f T) \|x_1(\cdot)-x_2(\cdot)\|.
\end{aligned}
$$

Since, by hypotheses, $(1+T\ell_g)(\gamma M T \ell_f + \gamma^2 M^2 k_1 k_2 \ell_f T) < 1$, then F is a contraction mapping. □

We are now ready to state the main result of this section.

Theorem 2. *Assume that the hypotheses (H1)-(H9) are satisfied and suppose that* $(1+T\ell_g)(\gamma M T \ell_f + \gamma^2 M^2 k_1 k_2 \ell_f T) < 1$ *for some* $T > 0$. *Then the boundary integrodifferential control system (3.1) is exactly controllable on the interval* $[0,T]$.

Proof. Let $x_0, y \in X$. From the above proposition we obtain that the operator F defined in (3.5) has a fixed point $x(\cdot)$ which is a mild solution of the system (3.1). Furthermore, for $x(\cdot)$ choose a control $u(\cdot)$ as in (3.6). Then one can see that $x(\cdot)$ satisfies $x(T) = y$, giving the exact controllability. □

4. Application

To illustrate the previous general assumptions and abstract results we consider in this section the following controlled population equation.

$$\begin{cases} \frac{\partial}{\partial t}v(t,a) = -\dfrac{\partial}{\partial a}v(t,a) - \mu(t,a)v(t,a), \quad t \in [0,T], a \geq 0, \\ v(t,0) = \displaystyle\int_0^t \int_0^\infty K(t,\tau,v(t,a))\, da\, d\tau + \int_0^\infty \beta(t,a,v(t,a))\, da + b(t)u(t), \\ u(0,a) = \varphi(a), \quad a \geq 0. \end{cases} \tag{4.1}$$

Here the function $v(t,a)$ represents the density of individuals of the population of age a at time t. The functions μ and β correspond to the aging and the birth rates, respectively. We note that this equation is a special case of the general population equation investigated in [5].

We impose the following conditions:

(i) $\mu \in C^1([0,T], L^\infty(\mathbb{R}_+))$ and there exists a constant $\eta > 0$ such that $\mu(t,a) > \eta$ for all $t \geq 0$ and $a.e.a \in \mathbb{R}_+$.

(ii) β is a positive and continuous function defined on $[0,T] \times \mathbb{R}_+ \times \mathbb{R}$. Moreover, there exists a constant $\alpha > 0$ such that

$$|\beta(t,a,x) - \beta(t,a,\bar{x})| \leq \alpha |x - \bar{x}| \tag{4.2}$$

for all $t \in [0,T]$; $x, \bar{x} \in \mathbb{R}$ and a.e. $a \in \mathbb{R}_+$.

(iii) $b(\cdot) \in L^2(0,T)$ and $\varphi \in L^1(\mathbb{R}_+)$.

(iv) K is a continuous function from $\Delta_T \times \mathbb{R}$ into $\mathbb{R}$ and there exists a constant $k > 0$ such that

$$|K(t,\tau,x) - K(t,\tau,\bar{x})| \leq k|x - \bar{x}| \tag{4.3}$$

for all $(t,\tau) \in \Delta_T$ and $x, \bar{x} \in \mathbb{R}$.

We first write the system (4.1) as a boundary integrodifferential controlled system of the form (3.1) satisfying the hypotheses (H1)-(H6). For this purpose, we define the Banach spaces $X := L^1(\mathbb{R}^+)$, $U = \partial X := \mathbb{R}$ and $D := W^{1,1}(\mathbb{R}^+)$.

For each $t \geq 0$ we define the operator $A_{\max}(t) : X \to X$ by

$$(A_{\max}(t)\varphi)(a) = -\frac{\partial}{\partial a}\varphi(a) - \mu(t,a)\varphi(a) \tag{4.4}$$

with domain $D(A_{\max}(t)) = D$ equipped with the norm

$$\|\varphi\|_D := \|\varphi\|_X + \|\varphi'\|_X$$

and the operator $L(t) : D \longrightarrow \partial X$ by

$$L(t)\varphi = \varphi(0) \tag{4.5}$$

for all $\varphi \in D$.

The functions g and f are given by

$$\begin{aligned} g &: \Delta_T \times X \longrightarrow X, \\ g(t,\tau,\varphi)(a) &= K(t,\tau,\varphi(a)) \end{aligned} \tag{4.6}$$

for all $(t,\tau)\in\Delta_T$ and $\varphi\in X$;

$$f:[0,T]\times X\times X\longrightarrow \partial X$$

$$f(t,\varphi_1,\varphi_2)=\int_0^\infty \varphi_2(a)\,da+\int_0^\infty \beta(t,a,\varphi_1(a))\,da \tag{4.7}$$

for all $t\in[0,T]$ and $\varphi_1,\varphi_2\in X$.

Remark 3. Using the Fubini's theorem, one can see that

$$f\left(t,\varphi,\int_0^t g(t,\tau,\varphi)\,d\tau\right)=\int_0^t\int_0^\infty K(t,\tau,\varphi(a)\,da\,d\tau+\int_0^\infty \beta(t,a,\varphi(a))da$$

for all $\varphi\in X$.

The system (4.1) is then a concrete case of the abstract boundary integrodifferential system (3.1).

Let us now check the assumptions (H1)–(H6).

Verification of (H1): Set $\mu_\infty:=\sup_{t\in[0,T]}\|\mu(t,\cdot)\|_\infty$ and let $\varphi\in D$ be arbitrary. From the definition of $\|\cdot\|_D$, we have

$$\begin{aligned}\|\varphi\|_D&=\int_0^\infty|\varphi(a)|\,da+\int_0^\infty|\varphi'(a)|\,da\\&\le\int_0^\infty|\varphi(a)|\,da+\int_0^\infty|\varphi'(a)-\mu(t,a)\varphi(a)|\,da+\int_0^\infty|\mu(t,a)\varphi(a)|\,da\\&\le\|\varphi\|_X+\|A_{\max}(t)\varphi\|_X+\int_0^{+\infty}|\mu(t,a)\varphi(a)|\,da\\&\le(1+\mu_\infty)(\|\varphi\|_X+\|A_{\max}(t)\varphi\|_X).\end{aligned}$$

On the other hand,

$$\begin{aligned}\|\varphi\|_X+\|A_{\max}(t)\varphi\|_X&=\|\varphi\|_X+\|\varphi'-\mu(t,\cdot)\varphi\|_X\\&\le(1+\mu_\infty)\|\varphi\|_D.\end{aligned}$$

This shows the assumption (H1) with $C_1=(1+\mu_\infty)^{-1}$ and $C_2=(1+\mu_\infty)$.

Verification of (H2): From (4.4) and the assumptions on μ, we derive that the map $t\mapsto A_{\max}(t)\varphi$ is continuously differentiable for each fixed $\varphi\in D$.

Verification of (H3): Since $\varphi(0)=\int_0^{+\infty}\frac{\partial}{\partial a}\varphi(a)\,da$ for all $\varphi\in D$, it follows that

$$\begin{aligned}|L(t)\varphi|&=|\varphi(0)|\\&\le\int_0^{+\infty}\left|\frac{\partial}{\partial a}\varphi(a)\right|da\\&\le\|\varphi\|_D.\end{aligned}$$

This shows the boundedness of $L(t)$. To prove the surjectivity of $L(t)$, let $x \in \partial X$ be arbitrary. Define

$$\varphi(a) := e^{-a}x \qquad \text{for all } a \in \mathbb{R}_+.$$

We have $\varphi \in D$. Furthermore, one can easily see that $L(t)\varphi = x$ and therefore $L(t)$ is surjective.

Verification of (H4): From (4.5), we see that $L(t)$ is independent of t. Hence the mapping from $[0,T] \to \partial X, t \mapsto L(t)\varphi$ is continuously differentiable for each fixed φ in D.

Verification of (H5): Let $\lambda > -\eta$, and $\varphi \in ker(\lambda - A_{max}(t))$, i.e,

$$\lambda\varphi(\cdot) + \frac{\partial}{\partial a}\varphi(\cdot) + \mu(t,\cdot)\varphi(\cdot) = 0.$$

We have

$$\begin{aligned} |L(t)\varphi| &= |\varphi(0)| \\ &= \int_0^{+\infty} \left| \frac{\partial}{\partial a}\varphi(a) \right| da \\ &= \int_0^{+\infty} (\lambda + \mu(t,a))|\varphi(a)|\, da. \\ &\geq (\lambda + \eta)|\varphi|_X. \end{aligned}$$

Then (H5) is satisfied with $\omega = -\eta$ and $\gamma = 1$.

Verification of (H6): Let $\lambda > 0$. One can show that the resolvent operator of $A(t) := A_{max}(t)|_{kerL(t)}$ is given by

$$R(\lambda, A(t))\varphi = \int_0^{\cdot} e^{-\int_\tau^{\cdot} \lambda + \mu(t,\sigma)\, d\sigma} \varphi(\tau)\, d\tau \quad \forall \varphi \in X.$$

Then for $\lambda > -\eta$ we have

$$\begin{aligned} \|R(\lambda, A(t))\varphi\| &= \int_0^{+\infty} |(R(\lambda, A(t))\varphi)(a)|\, da \\ &= \int_0^{+\infty} \left| \int_0^a e^{-\int_\tau^a \lambda + \mu(t,\sigma)\, d\sigma} \varphi(\tau) d\tau \right| da \\ &\leq \int_0^{\infty} \int_0^a e^{-\int_\tau^a \lambda + \mu(t,\sigma)\, d\sigma} |\varphi(\tau)|\, d\tau\, da \\ &= \int_0^{+\infty} |\varphi(\tau)| \int_\tau^{+\infty} e^{-\int_\tau^a \lambda + \mu(t,\sigma)\, d\sigma}\, da\, d\tau \\ &\leq \int_0^{+\infty} |\varphi(\tau)| \int_\tau^{+\infty} e^{-\int_\tau^a (\lambda + \eta)\, d\sigma}\, da\, d\tau \\ &\leq \frac{1}{\lambda + \eta} \int_0^{+\infty} |\varphi(\tau)|\, d\tau \end{aligned}$$

$$= \frac{1}{\lambda+\eta}\|\varphi\|.$$

Hence, we obtain that $(A(t))_{t\in[0,T]}$ is stable with stability constants $M = 1$ and $\omega = -\eta$. □

We conclude this section by the following controllability result of the controlled population equation (4.1).

Proposition 2. *Under the above assumptions and if we assume that*

$$(1+Tk)(T(1+\alpha)+\|b(\cdot)\|\|W^{-1}\|(1+\alpha)T) < 1,$$

then the controlled population problem (4.1) is exactly controllable.

Proof. From (4.7) and (4.2) one can verify that the function f satisfies (H7) with constant $(1+\alpha)$. Using (4.6) and (4.3), one can also verify that the function g satisfies (H8) with constant k. The result is then a direct application of Theorem 2. □

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Author's address

M. Moussi
University Mohamed I, Faculty of Sciences, Department of Mathematics and Informatics, BV Mohamed VI, 60000 Oujda, Morocco
E-mail address: mohmoussi@hotmail.com

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BEST PROXIMITY POINT THEOREMS FOR F-CONTRACTIVE NON-SELF MAPPINGS

M. OMIDVARI, S. M. VAEZPOUR, AND R. SAADATI

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Abstract. In this article, we prove the existence of a best proximity point for $F-$contractive non-self mappings and state some results in the complete metric spaces. Also we define two kinds of $F-$proximal contraction and extend some best proximity theorems and improve the recent results.

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1. INTRODUCTION

Let us assume that A, B be two nonempty subsets of a metric space (X, d) and $T : A \longrightarrow B$. Clearly $T(A) \cap A \neq \varnothing$ is a necessary condition for the existence of a fixed point of T. Now if $T(A) \cap A = \varnothing$ then to find an element $x \in A$ such that $d(x, Tx) = d(A, B)$ which called best proximity point is the idea of best proximity point theorems. In the other words, we determine an approximate solution x such that the error of equation $d(x, Tx) = 0$ is minimum. Several authors such as Prolla [2], Reich [4], Sehgal and Singh [6, 7], Vertivel, Veermani and Bhattacharyya [8] and others[1] generalized and extended the best proximity point theorems in many directions. In 2011, Sadiq Basha [5] stated the best proximity points theorems for proximal contractions. On the other hand, Wardowski [9] introduced a new type of contraction which called $F-$contraction and proved a fixed point result in complete metric spaces. In this paper, by using Wardowski's contraction, we prove the existence of a best proximity point. Moreover we define $F-$proximal contractions of the first and second kind and establish the best proximity point theorems in spire of Wardowski's contraction.

2. PRELIMINARY

Let A, B be two nonempty subsets of a metric space X. The following notations will be used throughout this paper:

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$d(y, A) := \inf\{d(x, y) : x \in A\}$,

$d(A, B) := \inf\{d(x, y) : x \in A \text{ and } y \in B\}$,

$A_0 := \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\}$,

$B_0 := \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}$.

We recall that $x \in A$ is a best proximity point of the mapping $T : A \longrightarrow B$ if $d(x, Tx) = d(A, B)$. It can be observed that a best proximity reduces to a fixed point if the underlying mapping is a self-mapping.

Definition 1 ([3])**.** Let (A, B) be a pair of nonempty subsets of a metric space X with $A \neq \varnothing$. Then the pair (A, B) is said to have the P-property if and only if

$$\left.\begin{array}{l} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{array}\right\} \Longrightarrow d(x_1, x_2) = d(y_1, y_2)$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

It is clear that, for any nonempty subset A of X, the pair (A, A) has the P-property.

Definition 2 ([5])**.** A is said to be approximatively compact with respect to B if every sequence $\{x_n\}$ of A satisfying the condition that $d(y, x_n) \longrightarrow d(y, A)$ for some y in B has a convergent subsequence.

It is easy to see that every set is approximatively compact with respect to itself.

Definition 3 ([5])**.** Given $T : A \longrightarrow B$ and an isometry $g : A \longrightarrow A$, the mapping T is said to preserve isometric distance with respect to g if

$$d(Tgx_1, Tgx_2) = d(Tx_1, Tx_2)$$

for all x_1 and x_2 in A.

Recently Wardowski [9] defined the following contraction which was called $F-$contraction.

Definition 4. Let $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a mapping satisfying:

(F1) F is strictly increasing, i.e. for all $a, b \in \mathbb{R}_+$ such that $\alpha < \beta \Longrightarrow F(\alpha) < F(\beta)$;

(F2) For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

A mapping $T : X \longrightarrow X$ is said to be an $F-$contraction if there exists $\tau > 0$ such that

$$For\ all\ \ x, y \in X, (d(Tx, Ty)) > 0 \Longrightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)). \quad (2.1)$$

Now, we consider the mapping F in the different types, and we obtain the variety of contractions.

Example 1 ([9]). Let $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be given by the formula $F(\alpha) = \ln\alpha$. It is clear that F satisfies the (F1)-(F3) ((F3) for any $k \in (0,1)$) conditions of Definition 4. Each mapping $T : X \longrightarrow X$ satisfying (2.1) is an $F-$contraction such that

$$d(Tx,Ty) \leq e^{-\tau}d(x,y), \quad for\ all\ \ x,y \in X,\ \ Tx \neq Ty.$$

It is clear that for $x, y \in X$ such that $Tx = Ty$ the inequality $d(Tx,Ty) \leq e^{-\tau}d(x,y)$ also holds, i.e. T is a Banach contraction.

3. Main Results

Now, let us state our main result.

Theorem 1. *Let A and B be non-empty, closed subsets of a complete metric space X such that A_0 is nonempty. Let $T : A \longrightarrow B$ be an $F-$contraction non-self mapping such that $T(A_0) \subseteq B_0$. Assume that the pair (A, B) has the P-property. Then there exists a unique x^* in A such that $d(x^*,Tx^*) = d(A,B)$.*

Proof. Choose $x_0 \in A_0$. Since $Tx_0 \in T(A_0) \subseteq B_0$, there exists $x_1 \in A_0$ such that $d(x_1,Tx_0) = d(A,B)$. Again, since $Tx_1 \in T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that $d(x_2,Tx_1) = d(A,B)$. Continuing this process, we can find a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1},Tx_n) = d(A,B), \quad for\ all\ n \in \mathbb{N}. \tag{3.1}$$

(A,B) satisfies the P-property, therefore from (3.1) we obtain

$$d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n), \quad for\ all\ n \in \mathbb{N}. \tag{3.2}$$

We will prove that the sequence $\{x_n\}$ is convergent in A_0. If there exists $n_0 \in \mathbb{N}$ such that $d(Tx_{n_0-1},Tx_{n_0}) = 0$, then by (3.2) we have $d(x_{n_0},x_{n_0+1}) = 0$ that implies $x_{n_0} = x_{n_0+1}$. Therefore

$$Tx_{n_0} = Tx_{n_0+1} \Longrightarrow d(Tx_{n_0},Tx_{n_0+1}) = 0 \tag{3.3}$$

From (3.2) and (3.3) we receive that

$$d(x_{n_0+2},x_{n_0+1}) = d(Tx_{n_0+1},Tx_{n_0}) = 0 \Longrightarrow x_{n_0+2} = x_{n_0+1}$$

Therefore $x_n = x_{n_0},\ for\ all\ n \geq n_0$ and $\{x_n\}$ is convergent in A_0.
Now let $d(Tx_{n-1},Tx_n) \neq 0,\ for\ all\ n \in \mathbb{N}$. T is a $F-$contraction and (3.2) holds, hence for any positive integer n we have

$$\tau + F(d(Tx_n,Tx_{n-1})) \leq F(d(x_n,x_{n-1}))$$

$$\Longrightarrow F(d(x_{n+1},x_n)) \leq F(d(x_n,x_{n-1})) - \tau$$
$$\vdots$$
$$\leq F(d(x_1,x_0)) - n\tau. \tag{3.4}$$

Put $\beta_n := d(x_{n+1},x_n)$.
From (3.4), we obtain $\lim_{n\to\infty} F(\beta_n) = -\infty$ that together with $(F2)$ gives

$$\lim_{n\to\infty} \beta_n = 0 \tag{3.5}$$

Also from $(F3)$ we have

$$\exists k \in (0,1) \ \ such\ that \ \ \beta_n^k F(\beta_n) = 0 \tag{3.6}$$

By (3.3), the following holds for all $n \in \mathbb{N}$:

$$F(\beta_n) - F(\beta_0) \leq -n\tau.$$

Therefore

$$\beta_n^k F(\beta_n) - \beta_n^k F(\beta_0) \leq -n\beta_n^k \tau \leq 0.$$

Letting $k \to \infty$ in the above inequality and using (3.5),(3.6), we obtain

$$\lim_{n\to\infty} n\beta_n^k = 0.$$

Hence there exists $n_1 \in \mathbb{N}$ such that $n\beta_n^k \leq 1$ for all $n \geq n_1$. Therefore for any $n \geq n_1$,

$$\beta_n \leq \frac{1}{n^{\frac{1}{k}}}. \tag{3.7}$$

This means that series $\sum_{i=1}^{\infty} \beta_i$ is convergent.
Now let $m \geq n \geq n_1$. By the triangular inequality and (3.7), we have

$$d(x_m,x_n) \leq \beta_{m-1} + \beta_{m-2} + \ldots + \beta_n \leq \sum_{i=n}^{\infty} \beta_i.$$

Therefore $\{x_n\}$ is a Cauchy sequence in A. Since (X,d) is complete and A is a closed subset of X, there exist $x^* \in A$ such that

$$\lim_{n\to\infty} x_n = x^*$$

Since T is continuous, we have $Tx_n \longrightarrow Tx^*$. Hence continuity of the metric function d which implies that $d(x_{n+1},Tx_n) \longrightarrow d(x^*,Tx^*)$. From (3.1), $d(x^*,Tx^*) = d(A,B)$. So we show that x^* is a best proximity of T.
The uniqueness of the best proximity point follows from the condition that T is $F-$ contraction. That is, suppose that

$$x_1,x_2 \in A \ \ such\ that \ \ x_1 \neq x_2 \ \ and \ \ d(x_1,Tx_1) = d(x_2,Tx_2) = d(A,B).$$

Then by the P-property of (A,B), we have $d(x_1,x_2) = d(Tx_1,Tx_2)$. Also

$$x_1 \neq x_2 \Longrightarrow d(x_1, x_2) \neq 0.$$

Therefore

$$F(d(x_1, x_2)) = F(d(Tx_1, Tx_2)) \leq F(d(x_1, x_2)) - \tau < F(d(x_1, x_2)),$$

which is a contraction. Hence the best proximity point is unique. □

The following result is a special case of Theorem 1, obtained by setting $A = B$.

Corollary 1. *Let (X, d) be a complete metric space and A be a nonempty closed subset of X. Let $T : A \longrightarrow A$ be a $F-$contractive self-map. Then T has a unique fixed point x^* in A.*

The next result is an immediate consequence of Theorem 1 by taking $F(\alpha) = \ln\alpha$.

Corollary 2 (Banach Contraction Principle)**.** *Let (X, d) be a complete metric space and A be a nonempty closed subset of X. Let $T : A \longrightarrow A$ be a contractive self-map. Then T has a unique fixed point x^* in A.*

Let F be the function as in Definition 4, we define the proximal contractions.

Definition 5. A mapping $T : A \longrightarrow B$ is said to be a $F-$proximal contraction of the first kind if there exists a $\tau > 0$ such that

$$\left.\begin{array}{c} d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \\ d(u_1, u_2), d(x_1, x_2) > 0 \end{array}\right\} \Longrightarrow \tau + F(d(u_1, u_2)) \leq F(d(x_1, x_2))$$

where $u_1, u_2, x_1, x_2 \in A$.

Remark 1. If $T : A \longrightarrow B$ is a $F-$proximal contraction of the first kind and (A, B) has the P-property then T is a $F-$contractive non-self mapping.

Definition 6. A mapping $T : A \longrightarrow B$ is said to be a $F-$proximal contraction of the second kind if there exists a $\tau > 0$ such that

$$\left.\begin{array}{c} d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \\ d(Tu_1, Tu_2), d(Tx_1, Tx_2) > 0 \end{array}\right\} \Longrightarrow \tau + F(d(Tu_1, Tu_2)) \leq F(d(Tx_1, Tx_2))$$

where $u_1, u_2, x_1, x_2 \in A$.

The following theorem is a best proximity point theorem for non-self $F-$proximal contraction of the first kind.

Theorem 2. *Let A and B be non-empty, closed subsets of a complete metric space X such that A_0 is non-empty. Let $T : A \longrightarrow B$ and $g : A \longrightarrow A$ satisfy the following conditions:*

(a) *T is a continuous $F-$proximal contraction of the first kind.*
(b) *g is an isometry.*
(c) *$T(A_0) \subseteq B_0$.*

(d) $A_0 \subseteq g(A_0)$.

Then, there exists a unique element $x \in A$ such that

$$d(gx, Tx) = d(A, B).$$

Proof. Choose $x_0 \in A_0$. Since $Tx_0 \in T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists $x_1 \in A_0$ such that $d(gx_1, Tx_0) = d(A, B)$. If $x_0 = x_1$ then put $x_n := x_1$ for all $n \geq 2$. Also, since $Tx_1 \in T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists $x_2 \in A_0$ such that $d(gx_2, Tx_1) = d(A, B)$. If $x_1 = x_2$ then put $x_n := x_2$ for all $n \geq 3$. Continuing this process, we can find a sequence $\{x_n\}$ in A_0 such that

$$d(gx_{n+1}, Tx_n) = d(A, B), \quad for\ all\ n \in \mathbb{N}. \tag{3.8}$$

We will prove the convergence of the sequence $\{x_n\}$ in A. If there exists $n_0 \in \mathbb{N}$ such that $d(gx_{n_0}, gx_{n_0+1}) = 0$ then it is clear that the sequence $\{x_n\}$ is convergent. Hence let $d(gx_n, gx_{n+1}) \neq 0$, $for\ all\ n \in \mathbb{N}$. T is a $F-$proximal contraction of the first kind and (3.8) holds, hence for any positive integer n we have

$$\tau + F(d(gx_n, gx_{n+1})) \leq F(d(x_{n-1}, x_n))$$

$$\Longrightarrow F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)) - \tau$$

$$\vdots$$

$$\leq F(d(x_0, x_1)) - n\tau. \tag{3.9}$$

Similarly as the process in the proof of Theorem 1, $\{x_n\}$ is a Cauchy sequence in A. Since X is complete metric space and A is closed subset of X, there exists $x \in A$ such that $\lim_{n\to\infty} x_n = x$.

Since, T, g and d are continuous, therefore with letting $n \longrightarrow \infty$ in (3.8), we obtain

$$d(gx, Tx) = d(A, B).$$

Now, x^* be in A such that

$$d(gx^*, Tx^*) = d(A, B).$$

We show that $x = x^*$. Suppose to the contrary, that $x \neq x^*$. Hence $d(x, x^*) \neq 0$. Since T is a $F-$proximal contraction of the first kind and g is an isometry,

$$F(d(x, x^*)) = F(d(gx, gx^*)) \leq F(d(x, x^*)) - \tau < F(d(x, x^*)),$$

which is a contraction. Therefore $x = x^*$ and this completes the proof of theorem. □

The following result is a special case of Theorem 2, if g is the identity mapping.

Corollary 3. *Let A and B be non-empty, closed subsets of a complete metric space X such that A is approximatively compact with respect to B. Further, suppose that A_0 is non-empty. Let $T : A \longrightarrow B$ satisfies the following conditions:*

(a) *T is a continuous $F-$proximal contraction of the first kind.*
(b) $T(A_0) \subseteq B_0$.
Then T has a unique best proximity point in A.

The following theorem is a best proximity point theorem for non-self $F-$proximal contraction of the second kind.

Theorem 3. *Let A and B be non-empty, closed subsets of a complete metric space X such that A is approximatively compact with respect to B. Further, suppose that A_0 is non-empty. Let $T : A \longrightarrow B$ and $g : A \longrightarrow A$ satisfy the following conditions:*
(a) *T is a continuous $F-$proximal contraction of the second kind.*
(b) *g is an isometry.*
(c) $T(A_0) \subseteq B_0$.
(d) $A_0 \subseteq g(A_0)$.
(e) *T preserves isometric distance with respect to g.*
Then, there exists an element $x \in A$ such that

$$d(gx, Tx) = d(A, B).$$

Moreover, if x^* is another element of A such that $d(gx^*, Tx^*) = d(A, B)$ then $Tx = Tx^*$.

Proof. Choose $x_0 \in A_0$. Since $Tx_0 \in T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists $x_1 \in A_0$ such that $d(gx_1, Tx_0) = d(A, B)$. If $Tx_0 = Tx_1$ then put $x_n := x_1$ for all $n \geq 2$. Otherwise again since $Tx_1 \in T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists $x_2 \in A_0$ such that $d(gx_2, Tx_1) = d(A, B)$. If $Tx_1 = Tx_2$) then put $x_n := x_2$ for all $n \geq 3$. Continuing this process, we can find a sequence $\{x_n\}$ in A_0 such that

$$d(gx_{n+1}, Tx_n) = d(A, B), \quad for\ all\ n \in \mathbb{N}. \tag{3.10}$$

We will prove the convergence of the sequence $\{Tx_n\}$ in B. If there exists $n_0 \in \mathbb{N}$ such that $d(Tgx_{n_0}, Tgx_{n_0+1}) = 0$ then it is clear that the sequence $\{Tx_n\}$ is convergent. Hence let $d(Tgx_n, Tgx_{n+1}) \neq 0$, $for\ all\ n \in \mathbb{N}$. T is a $F-$proximal contraction of the second kind, T preserves isometric distance with respect to g and (3.10) holds, hence for any positive integer n we have

$$\tau + F(d(Tgx_n, Tgx_{n+1})) \leq F(d(Tx_{n-1}, Tx_n))$$

$$\Longrightarrow F(d(Tx_n, Tx_{n+1})) \leq F(d(Tx_{n-1}, Tx_n)) - \tau$$

$$\vdots$$

$$\leq F(d(Tx_0, Tx_1)) - n\tau. \tag{3.11}$$

Similarly as the process in the proof of Theorem 1, we receive that $\{Tx_n\}$ is a Cauchy sequence in B. Since X is complete metric space and B is closed subset of X, there exists $y \in B$ such that $\lim_{n\to\infty} Tx_n = y$.
By the triangular inequality, we have,

$$\begin{aligned} d(y,A) \leq d(y,gx_n) &\leq d(y,Tx_{n-1}) + d(Tx_{n-1},gx_n) \\ &= d(y,Tx_{n-1}) + d(A,B) \\ &\leq (y,Tx_{n-1}) + d(y,A) \end{aligned}$$

Letting $k \longrightarrow \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} d(y,gx_n) = d(y,A).$$

Since A is approximatively compact with respect to B, there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that converging to some $z \in A$. Therefore

$$d(z,y) = \lim_{k\to\infty} d(gx_{n_k}, Tx_{n_k-1}) = d(A,B).$$

This implies that $z \in A_0$. Since $A_0 \subseteq g(A_0)$, there exists $x \in A_0$ such that $z = gx$. As $\lim_{n\to\infty} g(x_{n_k}) = g(x)$ and g is an isometry, we have

$$\lim_{n\to\infty} x_{n_k} = x.$$

T is continuous and $\{Tx_n\}$ is convergent to y, Therefore

$$\lim_{n\to\infty} Tx_{n_k} = Tx = y.$$

Thus, it follows that

$$d(gx,Tx) = \lim_{n\to\infty} d(gx_{n_k}, Tx_{n_k}) = d(A,B).$$

Now let x^* be another element in A such that

$$d(gx^*, Tx^*) = d(A,B).$$

We will show that $Tx = Tx^*$. Suppose to the contrary, that $Tx \neq Tx^*$. Hence $d(Tx,Tx^*) \neq 0$. Since T preserves isometric distance with respect to g and T is a $F-$proximal contraction of the second kind,

$$F(d(Tx,Tx^*)) = F(d(Tgx,Tgx^*)) \leq F(d(Tx,Tx^*)) - \tau < F(d(Tx,Tx^*)),$$

which is a contraction. Therefore $Tx = Tx^*$. □

The next result is an immediate consequence of the Theorem 3, if g is the identity mapping.

Corollary 4. *Let A and B be non-empty, closed subsets of a complete metric space X such that A is approximatively compact with respect to B. Further, suppose that A_0 is non-empty. Let $T : A \longrightarrow B$ satisfies the following conditions:*

(a) *T is a continuous $F-$proximal contraction of the second kind.*

(b) *$T(A_0) \subseteq B_0$.*

Then, T has a best proximity point in A. Moreover, if x^* is another best proximity point of T then $Tx = Tx^*$.

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Authors' addresses

M. Omidvari
Department of Mathematics, Science and Research Branch, Islamic Azad University(IAU), Tehran, Iran
E-mail address: mehdi.omidvari@gmail.com

S. M. Vaezpour
Department of Mathematics and Computer Science, Amirkabir University of Technology, Hafez Ave., P. O. Box 15914, Tehran, Iran
E-mail address: vaez@aut.ac.ir

R. Saadati
Department of Mathematics, Iran University of Science and Technology, Tehran, Iran
E-mail address: rsaadati@iust.ac.ir

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A KOROVKIN-TYPE APPROXIMATION THEOREM FOR DOUBLE SEQUENCES OF POSITIVE LINEAR OPERATORS OF TWO VARIABLES IN STATISTICAL A-SUMMABILITY SENSE

SEVDA ORHAN, FADIME DIRIK, AND KAMIL DEMIRCI

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Abstract. In this paper, using the concept of statistical A-summability which is stronger than classical convergence and A-statistical convergence, we obtain a Korovkin-type approximation theorem for double sequences of positive linear operators of two variables from $H_w(K)$ to $C_B(K)$. Also, we give an example such that our new approximation result works but its classical and A-statistical cases do not work.

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1. INTRODUCTION

The study of the Korovkin-type approximation theory is an area of active research, which deals with the problem of approximating a function by means of a sequence of positive linear operators. Later many researchers investigated approach conditions for various operators defined on different spaces. Using the concept of statistical convergence in approximation theory provides us with many advantages. In particular, the matrix summability methods of Cesáro type are strong enough to correct the lack of convergence of various sequences of linear operators such as the interpolation operator of Hermite-Fejér [15], because these types of operators do not converge at points of simple discontinuity. Furthermore, in recent years, with the help of the concept of uniform statistical convergence, which is a regular (non-matrix) summability transformation, various statistical approximation results have been proved [1–3, 5, 6, 9, 14]. Then, it was demonstrated that those results are more powerful than the classical Korovkin theorem. Also, Erkuş and Duman [8] have studied a Korovkin type approximation theorem via $A-$statistical convergence in the space $H_\omega(K)$ (was given in [10]) where $K = [0,\infty)\times[0,\infty)$. Later, using the concept of the equi-statistical convergence which is more powerful than classical convergence and statistical convergence, Karakuş and Demirci have obtained the Korovkin-type approximation theorem for positive linear operators defined on $H_\omega(K)$ in [13].

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Our primary interest in the present paper is to obtain a general Korovkin type approximation theorem for double sequences of positive linear operators of two variables from $H_\omega(K)$ to $C_B(K)$ via statistical A-summability.

Let us first remind of the concept of A-statistical convergence for double sequences.

A double sequence $x=(x_{m,n})$ is said to be convergent in Pringsheim's sense if, for every $\varepsilon>0$, there exists $N=N(\varepsilon)\in\mathbb{N}$, the set of all natural numbers, such that $|x_{m,n}-L|<\varepsilon$ whenever $m,n>N$, where L is called the Pringsheim limit of x and denoted by $P-\lim_{m,n} x_{m,n}=L$ (see [20]). We shall call such an x, briefly, "P-convergent". A double sequence is called bounded if there exists a positive number M such that $|x_{m,n}|\leq M$ for all $(m,n)\in\mathbb{N}^2=\mathbb{N}\times\mathbb{N}$. Note that in contrast to the case for single sequences, a convergent double sequence need not to be bounded. Let $A=(a_{j,k,m,n})$ be a four-dimensional summability matrix. For a given double sequence $x=(x_{m,n})$, the $A-$transform of x, denoted by $Ax:=((Ax)_{j,k})$, is given by

$$(Ax)_{j,k}=\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}x_{m,n} \tag{1.1}$$

provided the double series converges in Pringsheim's sense for every $(j,k)\in\mathbb{N}^2$.

A two dimensional matrix transformation is said to be regular if it maps every convergent sequence in to a convergent sequence with the same limit. The well-known characterization for two dimensional matrix transformations which are regular is known as Silverman-Toeplitz conditions (see, for instance, [12]). In 1926, Robison [21] presented a four dimensional analog of the regularity by considering an additional assumption of boundedness. This assumption was made because a double P-convergent sequence is not necessarily bounded. The definition and the characterization of regularity for four dimensional matrices is known as Robison-Hamilton conditions, or briefly, RH-regularity (see [11, 21]).

Recall that a four dimensional matrix $A=(a_{j,k,m,n})$ is said to be $RH-$regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. The Robinson-Hamilton conditions state that a four dimensional matrix $A=(a_{j,k,m,n})$ is RH-regular if and only if

$$(i)\ \ P-\lim_{j,k} a_{j,k,m,n}=0 \text{ for each } (m,n)\in\mathbb{N}^2,$$

$$(ii)\ \ P-\lim_{j,k}\sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n}=1,$$

$$(iii)\ \ P-\lim_{j,k}\sum_{m\in\mathbb{N}} \left|a_{j,k,m,n}\right|=0 \text{ for each } n\in\mathbb{N},$$

$$(iv)\ \ P-\lim_{j,k}\sum_{n\in\mathbb{N}} \left|a_{j,k,m,n}\right|=0 \text{ for each } m\in\mathbb{N},$$

(v) $\sum_{(m,n)\in\mathbb{N}^2} \left|a_{j,k,m,n}\right|$ is $P-$convergent for every $(j,k)\in\mathbb{N}^2$,

(vi) there exist finite positive integers A and B such that $\sum_{m,n>B} \left|a_{j,k,m,n}\right| < A$ holds for every $(j,k)\in\mathbb{N}^2$.

Now let $A=(a_{j,k,m,n})$ be a non-negative $RH-$regular summability matrix, and let $K\subset\mathbb{N}^2$. Then the $A-$density of K is given by

$$\delta_2^A\{K\} := P-\lim_{j,k}\sum_{(m,n)\in K(\varepsilon)} a_{j,k,m,n}$$

where

$$K(\varepsilon) := \left\{(m,n)\in\mathbb{N}^2 : |x_{m,n}-L|\geq\varepsilon\right\}$$

provided that the limit on the right-hand side exists in Pringsheim's sense. A real double sequence $x=(x_{m,n})$ is said to be $A-$statistically convergent to a number L if, for every $\varepsilon>0$,

$$\delta_2^A\{(m,n)\in\mathbb{N}^2 : |x_{m,n}-L|\geq\varepsilon\}=0.$$

In this case, we write $st_2^A-\lim_{m,n} x=L$. Clearly, a P-convergent double sequence is A-statistically convergent to the same value but its converse is not always true. Also, note that an A-statistically convergent double sequence need not to be bounded. For example, consider the double sequence $x=(x_{m,n})$ given by

$$x_{m,n}=\begin{cases}1, & \text{if } m \text{ and } n \text{ are squares,}\\ 0, & \text{otherwise.}\end{cases}$$

We should note that if we take $A=C(1,1)$, which is the double Cesáro matrix, then $C(1,1)$-statistical convergence coincides with the notion of statistical convergence for double sequence, which was introduced in [17, 19]:

If $E\subset\mathbb{N}^2$ is a two-dimensional subset of positive integers, then $E_{j,k}$ denotes the set $(m,n)\in E : m\leq j, n\leq k$ and $\left|E_{j,k}\right|$ denotes the cardinality of $E_{j,k}$. The double natural density of E [17, 19] is given by

$$\delta_2(E) := P-\lim_{j,k}\frac{1}{jk}\left|E_{j,k}\right|,$$

if it exists. The number sequence $x=(x_{m,n})$ is statistically convergent to L provided that for every $\varepsilon>0$, the set $E:=E(\varepsilon):=\{m\leq j, n\leq k : \ |x_{m,n}-L|\geq\varepsilon\}$ has natural density zero; i.e.,

$$P-\lim_{j,k}\frac{1}{jk}\left|\{m\leq j, n\leq k : \ |x_{m,n}-L|\geq\varepsilon\}\right|=0,$$

in that case we write $st_2-\lim_{m,n} x_{m,n}=L$.

Finally, if we replace the matrix A by the identity matrix for four dimensional matrices, then A-statistical convergence reduces to the Pringsheim convergence.

Recently, the idea of statistical $(C,1)$-summability was introduced in [16] and of statistical $(H,1)$-summability in [18] by Moricz. Then these statistical summability methods were generalized by defining the statistical A-summability in [7]. The concept of statistical A-summability for double sequences has defined by Karakuş and Demirci [4].

Hence, we know that if a double sequence is bounded and A-statistical convergent to L, then it is A-summable to L, and hence statistically A-summable to L but not conversely [4].

Now we recall the following definition for a double sequence.

Definition 1 ([4])**.** Let $A=(a_{j,k,m,n})$ be a non-negative RH-regular summability matrix and $x=(x_{m,n})$ be a double sequence. We say that x is statistically $A-$summable to L if for every $\varepsilon>0$,

$$\delta_2\{(j,k)\in\mathbb{N}^2 : \left|(Ax)_{j,k}-L\right|\geq\varepsilon\}=0,$$

i.e.,

$$P-\lim_{m,n}\frac{1}{mn}\left|\{j\leq m,k\leq n : \left|(Ax)_{j,k}-L\right|\geq\varepsilon\}\right|=0,$$

where $(Ax)_{j,k}$ is as in (1.1).

Thus, the double sequence x is statistically $A-$summable to L if and only if $Ax:=\left((Ax)_{j,k}\right)$ is statistically convergent to L.

2. A Korovkin-type approximation theorem

We denote by $C_B(K)$ the space of all continuous and bounded real valued functions on $K=[0,\infty)\times[0,\infty)$. This space is equipped with the supremum norm

$$\|f\|=\sup_{(x,y)\in K}|f(x,y)|,\ (f\in C_B(K)).$$

Consider the space $H_\omega(K)$ of all real valued functions f on K and satisfying

$$|f(u,v)-f(x,y)|\leq\omega\left(\left|\frac{u}{1+u}-\frac{x}{1+x}\right|,\left|\frac{v}{1+v}-\frac{y}{1+y}\right|\right)$$

where ω be a function of the type of the modulus of continuity given by, for $\delta,\delta_1,\delta_2>0$,

(1) ω is non-negative increasing function on K with respect to δ_1,δ_2,
(2) $\omega(\delta,\delta_1+\delta_2)\leq\omega(\delta,\delta_1)+\omega(\delta,\delta_2)$,
(3) $\omega(\delta_1+\delta_2,\delta)\leq\omega(\delta_1,\delta)+\omega(\delta_2,\delta)$,
(4) $\lim_{\delta_1,\delta_2\to0}\omega(\delta_1,\delta_2)=0$.

Then it is obvious, by (4) that any function in $H_\omega(K)$ is continuous on K. Also, any function $f \in H_\omega(K)$ satisfies the inequality

$$|f(x,y)| \leq |f(0,0)| + \omega(1,1), \quad x,y \geq 0$$

and therefore is bounded on K. So, $H_\omega(K) \subset C_B(K)$.

We also use the following test functions

$$f_0(u,v) = 1,\ f_1(u,v) = \frac{u}{1+u},\ f_2(u,v) = \frac{v}{1+v}$$

$$\text{and } f_3(u,v) = \left(\frac{u}{1+u}\right)^2 + \left(\frac{v}{1+v}\right)^2.$$

Now we have the following result.

Theorem 1. *Let $\{L_{m,n}\}$ be a sequence of positive linear operators from $H_\omega(K)$ into $C_B(K)$ and let $A = (a_{j,k,m,n})$ be a nonnegative $RH-$regular summability matrix. Assume that the following conditions hold:*

$$st_2 - \lim \left\| \sum_{(m,n)} a_{j,k,m,n} L_{m,n}(f_i) - f_i \right\| = 0,\ i = 0,1,2,3. \tag{2.1}$$

Then, for any $f \in H_\omega(K)$,

$$st_2 - \lim \left\| \sum_{(m,n)} a_{j,k,m,n} L_{m,n}(f) - f \right\| = 0. \tag{2.2}$$

Proof. Assume that (2.1) holds. Let $f \in H_\omega(K)$ and $(x,y) \in K$ be fixed. Since $f \in H_\omega(K)$, for all $(u,v) \in K$, we write

$$|f(u,v) - f(x,y)| \leq \varepsilon + \frac{2N}{\delta^2}\left\{\left(\frac{u}{1+u} - \frac{x}{1+x}\right)^2 + \left(\frac{v}{1+v} - \frac{y}{1+y}\right)^2\right\}$$

where $N := \|f\|$. After some simple calculations, using the linearity and positivity of the operators $L_{m,n}$, we obtain

$$\begin{aligned}
&\left| \sum_{(m,n)} a_{j,k,m,n} L_{m,n}(f;x,y) - f(x,y) \right| \\
&\leq \varepsilon + C\left\{ \left| \sum_{(m,n)} a_{j,k,m,n} L_{m,n}(f_0;x,y) - f_0(x,y) \right| \right. \\
&\quad \left. + \left| \sum_{(m,n)} a_{j,k,m,n} L_{m,n}(f_1;x,y) - f_1(x,y) \right| \right.
\end{aligned}$$

$$+\left|\sum_{(m,n)} a_{j,k,m,n} L_{m,n}(f_2;x,y) - f_2(x,y)\right|$$
$$+\left|\sum_{(m,n)} a_{j,k,m,n} L_{m,n}(f_3;x,y) - f_3(x,y)\right|\Bigg\}$$

where $C := \max\left\{\varepsilon + N + \frac{2N}{\delta^2}\left(\left(\frac{A}{1+A}\right)^2 + \left(\frac{B}{1+B}\right)^2\right), \frac{4N}{\delta^2}\frac{A}{1+A}, \frac{4N}{\delta^2}\frac{B}{1+B}, \frac{2N}{\delta^2}\right\}$.
Then, taking supremum over $(x,y) \in K$ we get

$$\left\| \sum_{(m,n)} a_{j,k,m,n} L_{m,n}(f) - f \right\| \leq \varepsilon + C\sum_{i=0}^{3} \left\| \sum_{(m,n)} a_{j,k,m,n} L_{m,n}(f_i) - f_i \right\|. \quad (2.3)$$

For a given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon < r$. Then, for each $i = 0,1,2,3$, setting

$$U := \left\{(j,k) : \left\| \sum_{(m,n)} a_{j,k,m,n} L_{m,n}(f) - f \right\| \geq r\right\},$$
$$U_i := \left\{(j,k) : \left\| \sum_{(m,n)} a_{j,k,m,n} L_{m,n}(f_i) - f_i \right\| \geq \frac{r-\varepsilon}{4C}\right\}, \ (i = 0,1,2,3),$$

it follows from (2.3) that

$$U \subset \bigcup_{i=0}^{3} U_i,$$

which gives, for all $(j,k) \in \mathbb{N}^2$,

$$\delta_2(U) \leq \sum_{i=0}^{3} \delta_2(U_i).$$

From (2.1), we obtain (2.2). The proof is complete. □

If we take $A = I$, which is the identity matrix for four dimensional matrices, in Theorem 1, we get the following statistical version of Theorem 1.

Corollary 1. *Let $\{L_{m,n}\}$ be a sequence of positive linear operators from $H_\omega(K)$ into $C_B(K)$. Assume that the following conditions hold:*

$$st_2 - \lim \| L_{m,n}(f_i) - f_i \| = 0,\ i = 0,1,2,3.$$

Then, for any $f \in H_\omega(K)$,

$$st_2 - \lim \| L_{m,n}(f) - f \| = 0.$$

In Corollary 1, if the statistical convergence ($C(1,1)-$statistical convergence) replace with Pringsheim convergence, we obtain the following classical version of Theorem 1.

Corollary 2. *Let $\{L_{m,n}\}$ be a sequence of positive linear operators from $H_\omega(K)$ into $C_B(K)$. Assume that the following conditions hold:*

$$P-\lim \| L_{m,n}(f_i) - f_i \| = 0,\ i = 0,1,2,3.$$

Then, for any $f \in H_\omega(K)$,

$$P-\lim \| L_{m,n}(f) - f \| = 0.$$

Remark 1. We now show that our result Theorem 1 is stronger than its classical version Corollary 2 and statistical version Corollary 1. To see this first consider the following Bleimann, Butzer and Hahn operators of two variables:

$$T_{m,n}(f;x,y) = \frac{1}{(1+x)^m (1+y)^n} \sum_{k=0}^{m} \sum_{l=0}^{n} f\left(\frac{k}{m-k+1}, \frac{l}{n-l+1}\right) \binom{m}{k} \binom{n}{l} x^k y^l, \tag{2.4}$$

where $f \in H_\omega(K)$, and $K = [0,\infty) \times [0,\infty)$. It is known [8] that

$$\begin{aligned}
T_{m,n}(f_0;x,y) &= 1,\\
T_{m,n}(f_1;x,y) &= \frac{m}{m+1}\frac{x}{1+x},\\
T_{m,n}(f_2;x,y) &= \frac{n}{n+1}\frac{y}{1+y},\\
T_{m,n}(f_3;x,y) &= \frac{m(m-1)}{(m+1)^2}\left(\frac{x}{1+x}\right)^2 + \frac{m}{(m+1)^2}\frac{x}{1+x}\\
&\quad + \frac{n(n-1)}{(n+1)^2}\left(\frac{y}{1+y}\right)^2 + \frac{n}{(n+1)^2}\frac{y}{1+y}.
\end{aligned} \tag{2.5}$$

Now take $A = C(1,1)$ and define a double sequence $u := \{u_{m,n}\}$ by

$$u_{m,n} = (-1)^{m+n} \tag{2.6}$$

In this case, observe that

$$st_2 - \lim C(1,1)(u) = 0. \tag{2.7}$$

However, the double sequence u is not P-convergent and statistical convergent. Now using (2.4) and (2.6), we define the following double positive linear operators on $H_\omega(K)$ as follows:

$$L_{m,n}(f;x,y) = \left(1+u_{m,n}\right) T_{m,n}(f;x,y). \tag{2.8}$$

Then, observe that the double sequence of positive linear operators $\{L_{m,n}\}$ defined by (2.8) satisfy all hypotheses of Theorem 1. Hence, by (2.5) and (2.7), we have, for all $f \in H_\omega(K)$,

$$st_2 - \lim \left\| \sum_{(m,n)} a_{j,k,m,n} L_{m,n}(f) - f \right\| = 0.$$

Since u is not P-convergent and statistical convergent, the sequence $\{T_{m,n}(f)\}$ can not uniformly convergence to f on K or statistical sense.

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Authors' addresses

Sevda Orhan
Sinop University, Faculty of Sciences and Arts, Department of Mathematics, 57000 Sinop, Turkey
E-mail address: orhansevda@gmail.com

Fadime Dirik
Sinop University, Faculty of Sciences and Arts, Department of Mathematics, 57000 Sinop, Turkey
E-mail address: dirikfadime@gmail.com

Kamil Demirci
Sinop University, Faculty of Sciences and Arts, Department of Mathematics, 57000 Sinop, Turkey
E-mail address: kamild@sinop.edu.tr

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INEQUALITIES FOR MAPPINGS WHOSE SECOND DERIVATIVES ARE QUASI-CONVEX OR h-CONVEX FUNCTIONS

M. EMİN ÖZDEMİR, MUSTAFA GÜRBÜZ, AND ÇETİN YILDIZ

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Abstract. In this paper, we establish some new integral inequalities for quasi-convex and $h-$convex functions by using a kernel and fairly elementary analysis.

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1. INTRODUCTION

One of the most famous inequality for convex functions is so called Hermite-Hadamard's inequality as follows: Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$

be a convex function defined on the interval I of real numbers and $a,b \in I$, with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

For recent results, refinements, counterparts, generalizations and new *Hermite-Hadamard* type inequalities see [9], [2],[4] and [11].

Definition 1 (See [7])**.** A function $f : [a,b] \to \mathbb{R}$ is said quasi-convex on $[a,b]$ if

$$f(\lambda x + (1-\lambda)y) \leq \sup\{f(x), f(y)\}, \qquad (QC)$$

holds for all $x, y \in [a,b]$ and $\lambda \in [0,1]$.

Definition 2 (see [10])**.** Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a non-negative function. We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is an $h-$convex function or that f belongs to the class $SX(h,I)$, if f is nonnegative and for all $x, y \in I$ and $\alpha \in [0,1]$ we have

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha) f(x) + h(1-\alpha) f(y).$$

In recent years, several authors have been studied on integral inequalities. One of the well known of these inequalities is Simpson's inequality as following:

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Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_\infty=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then, the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{1}{2880}\left\|f^{(4)}\right\|_\infty(b-a)^4.$$

For recent refinements, counterparts, generalizations and new Simpson's type inequalities, see the papers [8], [1], [3], [5].

In [6], Özdemir *et al.* proved the following lemma:

Lemma 1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I° such that $f''\in L_1[a,b]$, where $a,b\in I$ with $a<b$ and $r\in\mathbb{R}^+$ then the following equality holds:*

$$\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx \tag{1.1}$$

$$=(b-a)^2\int_0^1 k(t)\,f''(tb+(1-t)a)\,dt$$

where

$$k(t)=\begin{cases}\frac{t}{r}\left(\frac{1}{r+1}-t\right), & t\in\left[0,\frac{1}{2}\right)\\ (1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right), & t\in\left[\frac{1}{2},1\right]\end{cases}.$$

In this article, by using functions whose second derivatives' absolute values are quasi-convex or $h-$convex, we obtained new inequalities related to the left hand side of Simpson inequality. Also new inequalities are proved.

2. Results for Quasi-Convex Functions

Theorem 2. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I° such that $f''\in L_1[a,b]$ and $|f''|^q$ is quasi-convex, where $a,b\in I$ with $a<b$, $r\geq1$ and $q\geq1$, then one has the following inequality;*

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right|$$

$$\leq\frac{r^3-3r+6}{12r(r+1)^3}(b-a)^2\left(\sup\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}}. \tag{2.1}$$

Proof. Suppose that $q = 1$, from Lemma 1 and by using quasi-convexity of $|f''|$, we obtain

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r}\int_0^1 f\left(tb+(1-t)a\right)dt\right|$$

$$\leq (b-a)^2\left\{\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left|f''(tb+(1-t)a)\right|dt\right.$$

$$\left.+\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left|f''(tb+(1-t)a)\right|dt\right\}$$

$$\leq (b-a)^2\left\{\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\sup\left\{\left|f''(a)\right|,\left|f''(b)\right|\right\}dt\right.$$

$$\left.+\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\sup\left\{\left|f''(a)\right|,\left|f''(b)\right|\right\}dt\right\}$$

$$=\frac{r^3-3r+6}{12r\,(r+1)^3}(b-a)^2\sup\left\{\left|f''(a)\right|,\left|f''(b)\right|\right\}.$$

Now suppose that $q > 1$, from Lemma 1 we have

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r}\int_0^1 f\left(tb+(1-t)a\right)dt\right|$$

$$\leq (b-a)^2\int_0^1|k(t)|\left|f''(tb+(1-t)a)\right|dt$$

By using Power mean inequality we get

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r}\int_0^1 f\left(tb+(1-t)a\right)dt\right|$$

$$\leq (b-a)^2\left(\int_0^1|k(t)|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1|k(t)|\left|f''(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f''|^q$ is quasi-convex, we have

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r}\int_0^1 f\left(tb+(1-t)a\right)dt\right|$$
$$\leq \frac{r^3-3r+6}{12r\,(r+1)^3}(b-a)^2\left(\sup\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}}.$$

This completes the proof. □

Corollary 1. *Under the assumptions of Theorem 2, if we choose $r=1$ in (2.1), we have the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)^2}{24}\left(\sup\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}}.$$

Corollary 2. *If we take $q=1$ and $r=1$ in (2.1), we obtain an inequality which includes the left-hand side of the Corollary 3 in [6]:*

$$\left|\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)^2}{24}\sup\left\{|f''(a)|,|f''(b)|\right\}$$
$$\leq \frac{(b-a)^2}{24}\left(|f''(a)|+|f''(b)|\right)$$

Corollary 3. *If we choose $r=2$ in (2.1), we obtain*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{81}\left(\sup\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}}$$
$$\leq \frac{(b-a)^2}{81}\left(|f''(a)|^q+|f''(b)|^q\right)^{\frac{1}{q}}$$

Let $a_1=|f''(a)|^q$, $b_1=|f''(b)|^q$. Here $0<\frac{1}{q}<1$ for $q>1$, using the fact that $\sum_{k=1}^{n}(a_k+b_k)^s\leq\sum_{k=1}^{n}(a_k)^s+\sum_{k=1}^{n}(b_k)^s$ for $0\leq s\leq 1$, $a_1,a_2,...,a_n\geq 0$,

$b_1, b_2, ..., b_n \geq 0$, *we get*

$$\left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx \right|$$
$$\leq \frac{(b-a)^2}{81}\left(\left|f''(a)\right| + \left|f''(b)\right|\right).$$

Theorem 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I° such that $f'' \in L_1[a,b]$. If $|f''|^p$ is quasi-convex on $[a,b]$,where $a, b \in I$ with $a < b$, $r \geq 1$ and $p > 1$, then the following inequality hold*

$$\left| \frac{1}{r(r+1)}[f(a)+f(b)] + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{2}{r(b-a)}\int_a^b f(x)\,dx \right| \tag{2.2}$$
$$\leq \frac{(b-a)^2}{4}\left(\sup\left\{\left|f''(a)\right|^p, \left|f''(b)\right|^p\right\}\right)^{\frac{1}{p}}$$
$$\times \left(\frac{1}{2}\right)^{\frac{1}{p}} \left(\frac{1}{r(r+1)^{\frac{q+2}{q}}}\right)\left\{\frac{(r-1)^{q+1}\left[2(q+2)-(r-1)(q+1)+2^{2+q}\right]}{(q+1)(q+2)}\right\}^{\frac{1}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using Lemma 1 and triangle inequality, we have

$$\left| \frac{1}{r(r+1)}[f(a)+f(b)] + \frac{2}{r+1} f(raca+b2) - \frac{2}{r(b-a)}\int_a^b f(x)\,dx \right|$$
$$\leq (b-a)^2\left\{\left|\int_0^{\frac{1}{2}} \frac{t}{r}\left(\frac{1}{r+1}-t\right) f''(tb+(1-t)a)\,dt\right|\right.$$
$$\left. + \left|\int_{\frac{1}{2}}^1 (1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right) f''(tb+(1-t)a)\,dt\right|\right\}$$
$$= (b-a)^2\left\{\left|\int_0^{\frac{1}{2}} f''(tb+(1-t)a)\left(\frac{1}{r+1}-t\right)\frac{t}{r}dt\right|\right.$$

$$+\left|\int_{\frac{1}{2}}^{1} f''(tb+(1-t)a)\left(\frac{t}{r}-\frac{1}{r+1}\right)(1-t)dt\right|\Bigg\}.$$

By using weighted version of Hölder's inequality which is described as

$$\left|\int_I f(s)g(s)h(s)ds\right| \leq \left(\int_I |f(s)|^p h(s)ds\right)^{\frac{1}{p}} \left(\int_I |g(s)|^q h(s)ds\right)^{\frac{1}{q}}$$

for $p>1$, $p^{-1}+q^{-1}=1$, h is non-negative on I and provided all the other integrals exist and are finite; we have

$$\begin{aligned}
&\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right| \\
&\leq (b-a)^2 \\
&\times\Bigg\{\left(\int_0^{\frac{1}{2}}\left|f''(tb+(1-t)a)\right|^p\frac{t}{r}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\frac{1}{r+1}-t\right|^q\frac{t}{r}dt\right)^{\frac{1}{q}} \\
&+\left(\int_{\frac{1}{2}}^1\left|f''(tb+(1-t)a)\right|^p(1-t)dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|\frac{t}{r}-\frac{1}{r+1}\right|^q(1-t)dt\right)^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

By using the quasi-convexity of $|f''|^p$, we get

$$\begin{aligned}
&\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right| \\
&\leq (b-a)^2\left(\sup\left\{\left|f''(a)\right|^p,\left|f''(b)\right|^p\right\}\right)^{\frac{1}{p}} \\
&\times\Bigg\{\left(\int_0^{\frac{1}{2}}\frac{t}{r}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\frac{1}{r+1}-t\right|^q\frac{t}{r}dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\int_{\frac{1}{2}}^{1}(1-t)dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left|\frac{t}{r}-\frac{1}{r+1}\right|^{q}(1-t)dt\right)^{\frac{1}{q}}\Bigg\}$$

$$=\frac{(b-a)^2}{4}\left(\sup\left\{\left|f''(a)\right|^p,\left|f''(b)\right|^p\right\}\right)^{\frac{1}{p}}$$

$$\times\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\frac{1}{r\,(r+1)^{\frac{q+2}{q}}}\right)\left\{\frac{(r-1)^{q+1}\left[2(q+2)-(r-1)(q+1)+2^{2+q}\right]}{(q+1)\,(q+2)}\right\}^{\frac{1}{q}}.$$

So the proof is complete. □

Corollary 4. *Under the assumptions of Theorem 3, if we choose $r=1$ in the inequality (2.2), we have the following inequalities:*

$$\left|\frac{f(a)+f\,(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b f\,(x)\,dx\right|$$

$$\leq\frac{(b-a)^2}{16}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\sup\left\{\left|f''(a)\right|^p,\left|f''(b)\right|^p\right\}\right)^{\frac{1}{p}}$$

$$\leq\frac{(b-a)^2}{16}\left(\sup\left\{\left|f''(a)\right|^p,\left|f''(b)\right|^p\right\}\right)^{\frac{1}{p}}$$

where $\left(\frac{1}{2}\right)^{\frac{1}{q}}\leq 1$, $q\in(1,\infty)$.

Corollary 5. *If we choose $r=2$ in (2.2), we obtain*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq\frac{(b-a)^2}{24}\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\frac{1}{3}\right)^{\frac{2}{q}}\left(\frac{(q+3)+2^{q+2}}{(q+1)(q+2)}\right)\left(\sup\left\{\left|f''(a)\right|^p,\left|f''(b)\right|^p\right\}\right)^{\frac{1}{p}}.$$

3. Results for h-Convex Functions

Theorem 4. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I° such that $f''\in L_1[a,b]$ and $|f''|$ is $h-$convex, where $a,b\in I$ with $a<b$ and $r\geq 1$, then one has the following inequality;*

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r\,(b-a)}\int_a^b f\,(x)\,dx\right| \tag{3.1}$$

$$\leq (b-a)^2 K(h,r)\left(\left|f''(a)\right|+\left|f''(b)\right|\right)$$

where

$$K(h,r)=\frac{3r^2-9r+8}{480r^2(r+1)^2}+\int_0^1 h^2(t)dt.$$

Proof. From Lemma 1 and properties of absolute value, we have

$$\begin{aligned}
&\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right| \\
&\leq (b-a)^2\int_0^1 |k(t)|\left|f''(tb+(1-t)a)\right|dt \\
&=(b-a)^2\left\{\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left|f''(tb+(1-t)a)\right|dt\right. \\
&\qquad\left.+\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left|f''(tb+(1-t)a)\right|dt\right\}.
\end{aligned}$$

Since $|f''|$ is $h-$convex, we get

$$\begin{aligned}
&\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right| \\
&\leq (b-a)^2\left[\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left|h(t)f''(b)+h(1-t)f''(a)\right|dt\right. \\
&\qquad\left.+\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left|h(t)f''(b)+h(1-t)f''(a)\right|dt\right] \\
&\leq (b-a)^2\left[\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left\{h(t)\left|f''(b)\right|+h(1-t)\left|f''(a)\right|\right\}dt\right.
\end{aligned}$$

$$+\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left\{h(t)\left|f''(b)\right|+h(1-t)\left|f''(a)\right|\right\}dt\Bigg]$$

$$=(b-a)^2\left[J_1+J_2\right]. \tag{3.2}$$

So we can write

$$J_1=\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left\{h(t)\left|f''(b)\right|+h(1-t)\left|f''(a)\right|\right\}dt$$

$$=\left|f''(a)\right|\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|h(1-t)\,dt+\left|f''(b)\right|\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|h(t)\,dt.$$

Since $cd\leq\frac{1}{2}\left(c^2+d^2\right)$ for $c,d\in\mathbb{R}$ we have

$$J_1\leq\left|f''(a)\right|\left\{\frac{1}{2}\left[\int_0^{\frac{1}{2}}\left(\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right)^2dt+\int_0^{\frac{1}{2}}h^2(1-t)dt\right]\right\}$$

$$+\left|f''(b)\right|\left\{\frac{1}{2}\left[\int_0^{\frac{1}{2}}\left(\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right)^2dt+\int_0^{\frac{1}{2}}h^2(t)dt\right]\right\}$$

$$=\left|f''(a)\right|\left\{\frac{1}{2}\left[\frac{3r^2-9r+8}{480r^2(r+1)^2}+\int_{\frac{1}{2}}^{1}h^2(t)dt\right]\right\} \tag{3.3}$$

$$+\left|f''(b)\right|\left\{\frac{1}{2}\left[\frac{3r^2-9r+8}{480r^2(r+1)^2}+\int_0^{\frac{1}{2}}h^2(t)dt\right]\right\}.$$

By similar calculation we get

$$J_2\leq\left|f''(a)\right|\left\{\frac{1}{2}\left[\frac{3r^2-9r+8}{480r^2(r+1)^2}+\int_0^{\frac{1}{2}}h^2(t)dt\right]\right\} \tag{3.4}$$

$$+\left|f''(b)\right|\left\{\frac{1}{2}\left[\frac{3r^2-9r+8}{480r^2(r+1)^2}+\int_{\frac{1}{2}}^{1}h^2(t)dt\right]\right\}.$$

If we use the inequalities (3.3) and (3.4) in (3.2) we get the desired result. □

Corollary 6. *In the inequality (3.1), if we choose $h(t)=1$ and $r=1$, we get*

$$\left|\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\frac{37.13}{2^5.3.5}(b-a)^2\left(\frac{|f''(a)|+|f''(b)|}{2}\right).$$

Corollary 7. *In Theorem 6, if we choose $h(t)=t$ and $r=2$ we have*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\frac{11.131}{2^5.3^3.5}(b-a)^2\left(\frac{|f''(a)|+|f''(b)|}{2}\right).$$

Theorem 5. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I° such that $f''\in L_1[a,b]$. If $|f''|^p$ is $h-$convex on $[a,b]$,where $a,b\in I$ with $a<b$, $r\geq 1$ and $p>1$, then the following inequality hold*

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right| \tag{3.5}$$
$$\leq\frac{(b-a)^2}{4r}\left\{\frac{2^{q+2}+(r-1)^{q+1}(qr+q+r+3)}{2(r+1)^{q+2}(q+1)(q+2)}\right\}^{\frac{1}{q}}$$
$$\times\left\{\left[\left|f''(b)\right|^p A(h,t)+\left|f''(a)\right|^p B(h,t)\right]^{\frac{1}{p}}\right.$$
$$\left.+\left[\left|f''(b)\right|^p B(h,t)+\left|f''(a)\right|^p A(h,t)\right]^{\frac{1}{p}}\right\}$$

where $A(h,t)=\frac{1}{24}+\int_0^{\frac{1}{2}}h^2(t)\,dt$ and $B(h,t)=\frac{1}{24}+\int_{\frac{1}{2}}^{1}h^2(t)\,dt$.

Proof. By using Lemma 1, triangle inequality and weighted version of Hölder's inequality we have

$$\begin{aligned}
&\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right| \\
&\leq (b-a)^2 \\
&\times\left\{\left(\int_0^{\frac{1}{2}}\left|f''(tb+(1-t)a)\right|^p\frac{t}{r}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\frac{1}{r+1}-t\right|^q\frac{t}{r}dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1\left|f''(tb+(1-t)a)\right|^p(1-t)dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|\frac{t}{r}-\frac{1}{r+1}\right|^q(1-t)dt\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

By using $h-$convexity of $|f''|^p$ we have

$$\begin{aligned}
&\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right| \\
&\leq (b-a)^2 \\
&\times\left\{\left(\int_0^{\frac{1}{2}}\left[h(t)\left|f''(b)\right|^p+h(1-t)\left|f''(a)\right|^p\right]\frac{t}{r}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\frac{1}{r+1}-t\right|^q\frac{t}{r}dt\right)^{\frac{1}{q}}\right. \\
&+\left(\int_{\frac{1}{2}}^1\left[h(t)\left|f''(b)\right|^p+h(1-t)\left|f''(a)\right|^p\right](1-t)dt\right)^{\frac{1}{p}} \\
&\left.\left(\int_{\frac{1}{2}}^1\left|\frac{t}{r}-\frac{1}{r+1}\right|^q(1-t)dt\right)^{\frac{1}{q}}\right\} \\
&=(b-a)^2
\end{aligned}$$

$$\times\left\{\left(\frac{|f''(b)|^p}{r}\int_0^{\frac{1}{2}} th(t)\,dt+\frac{|f''(a)|^p}{r}\int_0^{\frac{1}{2}} th(1-t)\,dt\right)^{\frac{1}{p}}\right.$$

$$\left(\int_0^{\frac{1}{2}}\left|\frac{1}{r+1}-t\right|^q t\,dt\right)^{\frac{1}{q}}$$

$$+\left(\left|f''(b)\right|^p\int_{\frac{1}{2}}^1 (1-t)h(t)\,dt+\left|f''(a)\right|^p\int_{\frac{1}{2}}^1 (1-t)h(1-t)\,dt\right)^{\frac{1}{p}}$$

$$\left.\left(\int_{\frac{1}{2}}^1\left|\frac{t}{r}-\frac{1}{r+1}\right|^q (1-t)dt\right)^{\frac{1}{q}}\right\}.$$

Since $cd\le\frac{1}{2}\left(c^2+d^2\right)$ for $c,d\in\mathbb{R}$ we have

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right|$$

$$\le(b-a)^2\left\{\left(\int_0^{\frac{1}{2}}\left|\frac{1}{r+1}-t\right|^q t\,dt\right)^{\frac{1}{q}}\right.$$

$$\times\left[\frac{|f''(b)|^p}{2r}\left(\int_0^{\frac{1}{2}} t^2dt+\int_0^{\frac{1}{2}} h^2(t)\,dt\right)\right.$$

$$\left.\left.+\frac{|f''(a)|^p}{2r}\left(\int_0^{\frac{1}{2}} t^2dt+\int_0^{\frac{1}{2}} h^2(1-t)\,dt\right)\right]^{\frac{1}{p}}\right.$$

$$+\left(\int_{\frac{1}{2}}^{1}\left|\frac{t}{r}-\frac{1}{r+1}\right|^{q}(1-t)dt\right)^{\frac{1}{q}}\times\left[\frac{|f''(b)|^{p}}{2}\left(\int_{\frac{1}{2}}^{1}(1-t)^{2}dt+\int_{\frac{1}{2}}^{1}h^{2}(t)\,dt\right)\right.$$
$$\left.+\frac{|f''(a)|^{p}}{2}\left(\int_{\frac{1}{2}}^{1}(1-t)^{2}dt+\int_{\frac{1}{2}}^{1}h^{2}(1-t)\,dt\right)\right]^{\frac{1}{p}}\Bigg\}$$

If we simplify the expression, we have

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_{a}^{b}f(x)\,dx\right|$$
$$\leq\frac{(b-a)^{2}}{4r}\left\{\frac{2^{q+2}+(r-1)^{q+1}(qr+q+r+3)}{2(r+1)^{q+2}(q+1)(q+2)}\right\}^{\frac{1}{q}}$$
$$\times\left\{\left[|f''(b)|^{p}\left(\frac{1}{24}+\int_{0}^{\frac{1}{2}}h^{2}(t)\,dt\right)+|f''(a)|^{p}\left(\frac{1}{24}+\int_{\frac{1}{2}}^{1}h^{2}(t)\,dt\right)\right]^{\frac{1}{p}}\right.$$
$$\left.+\left[|f''(b)|^{p}\left(\frac{1}{24}+\int_{\frac{1}{2}}^{1}h^{2}(t)\,dt\right)+|f''(a)|^{p}\left(\frac{1}{24}+\int_{0}^{\frac{1}{2}}h^{2}(t)\,dt\right)\right]^{\frac{1}{p}}\right\}$$

then the proof is complete. □

Corollary 8. *If we choose $h(t)=1$, $r=1$ in the inequality (3.5) we get*

$$\left|\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_{a}^{b}f(x)\,dx\right|$$
$$\leq\frac{(b-a)^{2}}{4}\left\{\frac{1}{(q+1)(q+2)}\right\}^{\frac{1}{q}}\left\{\frac{13}{12}\left(|f''(a)|^{p}+|f''(b)|^{p}\right)\right\}^{\frac{1}{p}}.$$

Corollary 9. *If we choose $h(t)=1$, $r=2$ in (3.5), we get*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|$$

$$\leq \frac{(b-a)^2}{4}\left\{\frac{2^{q+2}+(3q+5)}{2.3^{q+2}(q+1)(q+2)}\right\}^{\frac{1}{q}}\left\{\frac{13}{24}\left(\left|f''(a)\right|^p+\left|f''(b)\right|^p\right)\right\}^{\frac{1}{p}}.$$

Corollary 10. *If we choose $h(t)=t$, $r=2$ in the inequality (3.5), we have*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{8}\left\{\frac{2^{q+2}+(3q+5)}{2.3^{q+2}(q+1)(q+2)}\right\}^{\frac{1}{q}}$$
$$\times\left\{\left(\frac{|f''(a)|^p}{12}+\frac{|f''(b)|^p}{3}\right)^{\frac{1}{p}}+\left(\frac{|f''(a)|^p}{3}+\frac{|f''(b)|^p}{12}\right)^{\frac{1}{p}}\right\}.$$

Corollary 11. *In addition to the conditions given above, if we choose $p=q=2$, we have*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{48\sqrt{2}}\left\{\left(\frac{|f''(a)|^2}{12}+\frac{|f''(b)|^2}{3}\right)^{\frac{1}{2}}+\left(\frac{|f''(a)|^2}{3}+\frac{|f''(b)|^2}{12}\right)^{\frac{1}{2}}\right\}.$$

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Authors' addresses

M. Emin Özdemir
Atatürk University, K. K. Education Faculty, Department of Mathematics, 25240, Campus, Erzurum, Turkey
E-mail address: emos@atauni.edu.tr

Mustafa Gürbüz
Graduate School of Natural and Applied Sciences, Ağri İbrahim Çeçen University, Ağri, Turkey
E-mail address: mgurbuz@agri.edu.tr

Çetin Yildiz
Atatürk University, K. K. Education Faculty, Department of Mathematics, 25240, Campus, Erzurum, Turkey
E-mail address: cetin@atauni.edu.tr

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RANK FUNCTION EQUATIONS AND THEIR SOLUTION SETS

PIOTR POKORA

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Abstract. We examine so-called rank function equations and their solutions consisting of non-nilpotent matrices. Secondly, we present some geometrical properties of the set of solutions to certain rank function equations in the nilpotent case. The main results are Theorem 3, Theorem 4 and Theorem 6.

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1. Introduction to the subject

Conjugacy classes of matrices were studied by many authors in various settings, in particular in the context of algebraic geometry. There are many deep geometrical results describing some properties of conjugacy classes. We are interested here in some analogous of theorems due to Gerstenhaber – see for example [3]. These theorems can be formulated using the notion of rank functions of matrices $r_A(m) := \mathrm{rk}(A^m)$ as introduced by Eisenbud and Saltman [1]. Gerstenhaber investigated closures of conjugacy classes. For nilpotent matrices he obtained the following well-known theorem [3].

Theorem. *Let A, B be $n \times n$ nilpotent matrices with entries in an arbitrary algebraically closed field of characteristic zero and let $\mathcal{O}(B)$ denote the conjugacy class of B under the standard action of the general linear group by conjugation, see* (•). *Then*

$$A \in \overline{\mathcal{O}(B)} \text{ if and only if } r_A(m) \leq r_B(m) \text{ for all } m \geq 0.$$

In this note we extend this result to the Cartesian product set up, i.e. we obtain the following result, see also Theorem 6.

Theorem. *Let $A_1, ..., A_k, B_1, ..., B_k$ be $n \times n$ nilpotent matrices matrices with entries in $\mathbb{C}$. Then $(A_1, ..., A_k) \in \overline{\mathcal{O}(B_1) \times ... \times \mathcal{O}(B_k)}$ if and only if $r_{A_i}(m) \leq r_{B_i}(m)$ for all $m \geq 0$ and $i \in \{1, ..., k\}$.*

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Keeping the notion from the above theorem and taking $B = B_1 \oplus ... \oplus B_k$ one can be interested in solutions to the following inequality

$$(\bullet\bullet) \quad r_{A_1}(m) + ... + r_{A_k}(m) \leq r_B(m).$$

Finding all solutions to $(\bullet\bullet)$ without additional assumptions on the matrices $A_1, ..., A_k, B$ seems to be a quite complicated task. However if we replace the inequality by the equality in $(\bullet\bullet)$, then more can be said. In fact we can generalize the equation

$$r_{A_1}(m) + ... + r_{A_k}(m) = r_B(m)$$

to the following problem

$$f(r_{A_1}(m)) + ... + f(r_{A_k}(m)) = g(r_B(m))$$

with arbitrary functions $f, g : \mathbb{N} \to \mathbb{N}$. This is called the *rank function equation*, see Definition 4. A somewhat technical statement concerning solutions of certain rank function equations is formulated in Theorem 3 and in Theorem 4, which are the other main results of this note.

2. Preliminaries

Throughout this paper we assume that $\mathbb{F}$ is an arbitrary field of characteristic zero. We denote by $\mathbb{N}_0$ the set of all positive integers and by $\mathbb{N}$ the set of all non-negative integers. For $n \in \mathbb{N}_0$ we define $M_{n\times n}(\mathbb{F})$ to be the ring of all $n \times n$ matrices whose entries are elements of the field $\mathbb{F}$. This ring has a natural structure of a n^2-dimensional $\mathbb{F}$-vector space. We denote the zero matrix by O_n. The set of all nonsingular $n \times n$ matrices over $\mathbb{F}$ will be denoted by $GL(n, \mathbb{F})$. Finally, let $\mathfrak{N}_n$ be the set of all nilpotent $n \times n$ matrices over $\mathbb{F}$. The group $GL(n, \mathbb{F})$ acts on $M_{n\times n}(\mathbb{F})$ and $\mathfrak{N}_n$ by conjugation. The conjugacy class $\mathcal{O}(A)$ of a matrix $A \in M_{n\times n}(\mathbb{F})$ is defined by

$$(\bullet) \quad \mathcal{O}(A) = \{U^{-1}AU : U \in GL(n, \mathbb{F})\}.$$

By $\overline{\mathcal{O}(A)}$ we denote the Zariski closure of the conjugacy class of a matrix A in $M_{n\times n}(\mathbb{F})$.

We refer to [2] for matrix theory and to [6] for algebraic geometry.

Definition 1. The matrix

$$N_k = \begin{pmatrix} 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 0 & 1 \\ 0 & 0 & 0 & ... & 0 & 0 \end{pmatrix} \in M_{k\times k}(\mathbb{F})$$

is called the *Jordan nilpotent block* of size k.

Such matrices are building blocks of all nilpotent matrices as the following classical results shows.

Proposition 1. *Let $A \in M_{n\times n}(\mathbb{F})$ be a nilpotent matrix. Then there exist $U \in GL(n,\mathbb{F})$, $\ell \in \mathbb{N}_0$, and a weakly decreasing sequence $(k_1,...,k_\ell)$ of positive integers such that $U^{-1}AU = N_{k_1} \oplus ... \oplus N_{k_\ell}$. Moreover, ℓ and $(k_1,...,k_\ell)$ are uniquely determined by the matrix A.*

Definition 2. The matrix $N_{k_1} \oplus ... \oplus N_{k_\ell}$ is referred to as the *Jordan canonical form* of A and the related sequence $\mathrm{jp}(A) = (k_1,...,k_\ell)$ is called the *Jordan partition*.

For a nilpotent matrix A with Jordan canonical form $N_{k_1} \oplus ... \oplus N_{k_\ell}$, we denote by $\tilde{A}$ the direct sum of all non-trivial nilpotent blocks, i.e. those with $k_j \geq 2$.

Now we recall some facts related to rank functions and rank function equations. For more details we refer to [5] and [7].

Definition 3. The function $r_A : \mathbb{N} \to \mathbb{N}$ defined by

$$r_A(m) = \mathrm{rk}(A^m)$$

is called the *rank function* of a matrix $A \in M_{n\times n}(\mathbb{F})$.

Proposition 2. *For a matrix $A \in M_{n\times n}(\mathbb{F})$ its rank function satisfies the following conditions:*

(1) $r_A(0) = n$,
(2) *the function r_A is weakly decreasing,*
(3) *A is nilpotent if and only if $r_A(n) = 0$,*
(4) *if $r_A(m_0) = r_A(m_0+1)$ for some integer $m_0 \in \mathbb{N}$, then $r_A(m_0) = r_A(m_0+i)$ for every $i \in \mathbb{N}$,*
(5) *$r_{U^{-1}AU}(m) = r_A(m)$ for every $m \in \mathbb{N}$ and every $U \in GL(n,\mathbb{F})$,*
(6) *if $A = A_1 \oplus A_2$, where $A_i \in M_{n_i\times n_i}(\mathbb{F})$, $i = 1,2$, and $\oplus$ is the standard direct sum of matrices, then $r_A(m) = r_{A_1}(m) + r_{A_2}(m)$ for all $m \in \mathbb{N}$.*

Rank functions are characterized in the class of all non-negative integer-valued sequences by the following result, see Theorem 2 in [7].

Theorem 1. *A function $r : \mathbb{N} \to \mathbb{N}$ with $r(0) = n$ is the rank function of a matrix $A \in M_{n\times n}(\mathbb{F})$ if and only if it is weakly decreasing and satisfies the following convexity condition*

$$\forall m \in \mathbb{N} : r(m) + r(m+2) \geq 2r(m+1).$$

Now, we define the main object of our interest.

Definition 4. Let $k, n \in \mathbb{N}\setminus\{0,1\}$. For fixed functions $f, g : \mathbb{N} \to \mathbb{N}$ and a nonempty set $S \subseteq \mathbb{N}_0$, a *rank function equation* is the equation

$$f(r_{A_1}(m)) + ... + f(r_{A_k}(m)) = g(r_B(m)) \tag{2.1}$$

for all $m \in S$. The indeterminates are matrices $A_1,...,A_k, B \in M_{n\times n}(\mathbb{F})$.

For $f = g = \mathrm{id}_{\mathbb{N}}$, equation (2.1) reduces to

$$r_{A_1}(m) + ... + r_{A_k}(m) = r_B(m). \tag{2.2}$$

In this note we will consider only *non-trivial solutions*, which means sequences $(A_1, ..., A_k, B)$ consisting of nonzero matrices. In [5] we proved the following theorem.

Theorem 2. *Consider a strictly increasing convex function $f : [0, +\infty) \longrightarrow \mathbb{R}$. Assume that $f(\mathbb{N}) \subseteq \mathbb{N}$ and $f(0) = 0$. For nilpotent matrices $A_1, ..., A_k \in \mathfrak{N}_n$ define $r(m) := f(r_{A_1}(m)) + ... + f(r_{A_k}(m))$. Then the following conditions are equivalent:*

(1) *there exists a matrix $B \in M_{n\times n}(\mathbb{F})$ such that $(A_1, ..., A_k, B)$ is a solution to equation*

$$r(m) = r_B(m) \tag{2.3}$$

for all $m \in S = \{1, ..., n\}$ (hence for all $m \in \mathbb{N}_0$)

(2) $2r(1) - r(2) \leq n$.

Moreover, if the above conditions are satisfied, then the matrix B is nilpotent and unique up to conjugation.

3. Rank Function Equations for non-nilpotent matrices

Recall that for a matrix $A \in M_{n\times n}(\mathbb{F})$ the number $r_A(n) \in \mathbb{N}$ is called the *stable rank*. We will need the following fact, Proposition 4 in [7].

Proposition 3. *For nilpotent matrices $A, B \in \mathfrak{N}_n(\mathbb{F})$ the following conditions are equivalent:*

- *$B = U^{-1}AU$ for a certain $U \in GL(n, \mathbb{F})$,*
- *$r_A(m) = r_B(m)$, $m \in \mathbb{N}$.*

The following result characterizes explicitely non-trivial solutions of equation (2.2).

Theorem 3. *Let us consider equation (2.2) with $S = \{1, ..., n\}$. Assume that $(A_1, ..., A_k, B)$ is a solution to (2.2), where $A_j, B \in M_{n\times n}(\mathbb{F})$, $r_{A_j}(n) = q_j \in \mathbb{N}_0$ for $j = 1, ..., k$ and $n \geq q := q_1 + ... + q_k$. Then B is similar to $C \oplus D$, where $D \in GL(q, \mathbb{F})$ and $C \in \mathfrak{N}_{n-q}(\mathbb{F})$ is a nilpotent matrix such that its nonzero nilpotent blocks in the Jordan canonical form are conjugate to the direct sum of all nonzero nilpotent blocks contained in the Jordan canonical forms of $A_1, ..., A_k$.*

Proof. By [7, Theorem 1] without loss of generality we may assume that $A_j = B_j \oplus S_j$, where $B_j \in \mathfrak{N}_{n-q_j}(\mathbb{F})$ are nilpotent matrices and $S_j \in GL(q_j, \mathbb{F})$ for all $j \in \{1, ..., k\}$. Let $\tilde{B_j}$ be the direct sum of all nonzero nilpotent blocks that appear in the Jordan canonical form of the matrix B_j. Then there exist $U_j \in GL(n, \mathbb{F})$ such that $U_j^{-1} A_j U_j = \tilde{B_j} \oplus S_j \oplus O_{n-q_j-d_j}$ with $d_j \in \mathbb{N}$, which depends on the matrices B_j. Since $(A_1, ..., A_k, B)$ is a solution to the equation (2.2), then for all $m \in \{1, ..., n\}$ we have

$$r_B(m) = \sum_{j=1}^{k} r_{A_j}(m) = \sum_{j=1}^{k} r_{U_j^{-1}A_jU_j}(m) = \sum_{j=1}^{k} r_{\tilde{B}_j \oplus S_j \oplus O_{n-q_j-d_j}}(m) = \\ = \sum_{j=1}^{k} r_{\tilde{B}_j}(m) + q.$$

Obviously $r_B(n) = q$, thus B is similar to $C \oplus D$ with $C \in \mathfrak{N}_{n-q}(\mathbb{F})$ and $D \in GL(q,\mathbb{F})$. Then there exists a matrix $V \in GL(n,\mathbb{F})$ such that $V^{-1}BV = \tilde{C} \oplus D \oplus O_{n-q-d_C}$, where d_C depends on the matrix C. Since for $D \in GL(q,\mathbb{F})$ we have $r_D(m) = q$ for all $m \in \mathbb{N}$, then

$$\sum_{j=1}^{k} r_{\tilde{B}_j}(m) + q = r_{\tilde{B}_1 \oplus \ldots \oplus \tilde{B}_k}(m) + q = r_B(m) = r_{V^{-1}BV}(m) = \\ = r_{\tilde{C} \oplus D \oplus O_{n-q-d_C}}(m) = r_{\tilde{C}}(m) + q.$$

Now we can focus on the conditions:

$$\begin{cases} \forall\, m \in \mathbb{N}_0 : r_{\tilde{B}_1 \oplus \ldots \oplus \tilde{B}_k}(m) = r_{\tilde{C}}(m), \\ r_{\tilde{B}_1 \oplus \ldots \oplus \tilde{B}_k}(n) = r_{\tilde{C}}(n) = 0. \end{cases}$$

Thus $r_{\tilde{B}_1 \oplus \ldots \oplus \tilde{B}_k}(0) = r_{\tilde{C}}(0) = d_C$ and by Proposition 3 we obtain $\tilde{B}_1 \oplus \ldots \oplus \tilde{B}_k = W^{-1}\tilde{C}W$ for a certain $W \in GL(d_C,\mathbb{F})$, what ends the proof. □

Remark 1. Notice that solutions in the above case are not unique (the invertible matrix D can be chosen arbitrarily).

The next result generalizes Theorem 2.8.

Theorem 4. *Consider the rank function equation (2.3) with $S = \{1,\ldots,n\}$. Let $f : [0,\infty) \longrightarrow \mathbb{R}$ be a strictly increasing convex function such that $f(\mathbb{N}) \subseteq \mathbb{N}$ and $f(0) = 0$. Let $A_1,\ldots,A_k \in M_{n\times n}(\mathbb{F})$ with $r_{A_j}(n) = q_j \in \mathbb{N}_0$ for $j \in \{1,\ldots,k\}$. Then the following conditions are equivalent:*

(1) *there exists a matrix $B \in M_{n\times n}$ such that $(A_1,\ldots,A_k,B)$ is a solution to (2.3),*
(2) *$2r(1) - r(2) \leq n$, where $r(m) := f(r_{A_1}(m)) + \ldots + f(r_{A_k}(m))$ for $m \in \mathbb{N}$.*

Proof. By [7, Theorem 1] A_j's are similar to $\bar{A}_j \oplus D_j$, where $D_j \in GL(q_j,\mathbb{F})$ and $\bar{A}_j \in \mathfrak{N}_{n-q_j}(\mathbb{F})$ are nilpotent matrices for all $j \in \{1,\ldots,k\}$. Obviously

$$r(n) = f(r_{\bar{A}_1 \oplus D_1}(n)) + \ldots + f(r_{\bar{A}_k \oplus D_k}(n)) = f(r_{\bar{A}_1}(n) + r_{D_1}(n)) + \ldots + \\ f(r_{\bar{A}_k}(n) + r_{D_k}(n)) = f(r_{D_1}(n)) + \ldots + f(r_{D_k}(n)) = f(q_1) + \ldots + f(g_k).$$

In the virtue of Theorem 1, condition (i) holds true iff the function $r_B : \mathbb{N} \longrightarrow \mathbb{N}$ defined by

$$r_B(m) = \begin{cases} n & \text{for } m = 0, \\ r(m) & \text{for } m \in \{1, ..., n\}, \\ \sum_{j=1}^{k} f(q_j) & \text{for } m > n \end{cases}$$

is weakly decreasing and such that $r_B(m) + r_B(m+2) \geq 2r_B(m+1)$ for all $m \in \mathbb{N}$. By [5, Lemma 3.2] we see that r_B is a rank function and thus (i) is satisfied iff $n \geq r_B(1)$ and $n + r_B(2) \geq 2r_B(1)$. By the monotonicity of r_B, the last two inequalities hold iff $n - r_B(1) \geq r_B(1) - r_B(2)$, and this is the condition (ii). □

4. Some consequences of Gerstenhaber theorem

For nilpotent matrices $A, B \in \mathfrak{N}_n(\mathbb{F})$ the dominance $\prec$ is defined in the following way

$$A \prec B \iff \mathrm{rk}(A^m) \leq \mathrm{rk}(B^m) \text{ for all } m \in \mathbb{N}.$$

It can be shown that $\prec$ is a partial order.

From now on we fix a function $f : [0, \infty) \longrightarrow \mathbb{R}$, which is convex, strictly increasing, $f(0) = 0$ and maps all non-negative integers to non-negative integers. Let us define the following set

$$\mathrm{Sol} = \{ (A_1, ..., A_k, B) \in \underbrace{\mathfrak{N}_n \times ... \times \mathfrak{N}_n}_{k+1} : (A_1, ..., A_k, B) \text{ form a solution to the equation (2.3) with fixed } f \text{ and } S = \{1, ..., n\} \}.$$

For $(A_1, ..., A_k) \in \mathcal{A}$, where $\mathcal{A} \subseteq \underbrace{\mathfrak{N}_n \times ... \times \mathfrak{N}_n}_{k}$ is an arbitrary subset, we define the rank matrix

$$\mathrm{Rk}(A_1, ..., A_k) = \begin{pmatrix} r_{A_1}(0) & r_{A_1}(1) & .. & r_{A_1}(n-1) & r_{A_1}(n) \\ r_{A_2}(0) & r_{A_2}(1) & .. & r_{A_2}(n-1) & r_{A_2}(n) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{A_k}(0) & r_{A_k}(1) & .. & r_{A_k}(n-1) & r_{A_k}(n) \end{pmatrix}.$$

Let us denote by $\mathrm{Rank}(\mathcal{A}) \subseteq M_{k \times (n+1)}(\mathbb{F})$ the set, which consists of all matrices of the above form. Note that the set $\mathrm{Rank}(\mathcal{A})$ is always finite.

For matrices $\mathrm{Rk}(A_1, ..., A_k) = [a_{ij}]$ and $\mathrm{Rk}(A'_1, ..., A'_k) = [a'_{ij}]$, which belong to the set $\mathrm{Rank}(\mathcal{A})$ we define the relation:

$$\mathrm{Rk}(A_1, ..., A_k) \preceq \mathrm{Rk}(A'_1, ..., A'_k) \iff \text{for all } i \in \{1, ..., k\} \text{ we have } a_{ij} \leq a'_{ij} \text{ for any } j \in \{1, ..., n+1\}.$$

It is easy to see that the relation $\preceq$ is a partial order, which is compatible with the dominance in the sense that for a fixed $i \in \{1, ..., k\}$ we have

$$a_{ij} \leq a'_{ij} \text{ for all } j \in \{1, ..., n+1\} \text{ iff } A_i \prec A'_i.$$

Example 1. Let us consider the rank function equation (2.2) with $k = 2$. Let $A_1, A_1', A_2, A_2', B, B' \in \mathfrak{N}_{10}(\mathbb{F})$ be nilpotent matrices, such that

$$\begin{aligned} \mathrm{jp}(A_1) &= (2,2,1,1,1,1,1,1), \\ \mathrm{jp}(A_2) &= (3,2,1,1,1,1,1), \\ \mathrm{jp}(A_1') &= (4,1,1,1,1,1,1), \\ \mathrm{jp}(A_2') &= (4,2,1,1,1,1), \\ \mathrm{jp}(B) &= (3,2,2,2,1), \\ \mathrm{jp}(B') &= (4,4,2). \end{aligned}$$

Then (A_1, A_2, B) and (A_1', A_2', B') form solutions to the equation (2.2) with $S = \{1, \ldots, 10\}$. The rank matrices are the following:

$$\mathrm{Rk}(A_1, A_2, B) = \begin{pmatrix} 10 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{Rk}(A_1', A_2', B') = \begin{pmatrix} 10 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 4 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 7 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Of course, $\mathrm{Rk}(A_1, A_2, B) \preceq \mathrm{Rk}(A_1', A_2', B')$.

It is quite easy to see that the set Rank(Sol) may not be totally ordered. Indeed,

Example 2. Consider the rank function equation (2.2) with $k = 2$. Let $A_1, A_1', A_2, A_2', B, B' \in \mathfrak{N}_8(\mathbb{F})$ be nilpotent matrices, with

$$\begin{aligned} \mathrm{jp}(A_1) &= (2,1,1,1,1,1,1), \\ \mathrm{jp}(A_2) &= (3,1,1,1,1,1), \\ \mathrm{jp}(A_1') &= (2,2,1,1,1,1), \\ \mathrm{jp}(A_2') &= (2,2,1,1,1,1), \\ \mathrm{jp}(B) &= (3,2,1,1,1), \\ \mathrm{jp}(B') &= (2,2,2,2). \end{aligned}$$

The triplets (A_1, A_2, B) and (A_1', A_2', B') form solutions to equation (2.2) with $S = \{1, \ldots, 8\}$, and the rank matrices are the following:

$$\mathrm{Rk}(A_1, A_2, B) = \begin{pmatrix} 8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{Rk}(A_1', A_2', B') = \begin{pmatrix} 8 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We see that $\mathrm{Rk}(A_1, A_2, B)$ and $\mathrm{Rk}(A_1', A_2', B')$ are not comparable.

We denote by Rk_i the i-th row of a matrix $\mathrm{Rk} \in \mathrm{Rank(Sol)}$. To a fixed matrix $\mathrm{Rk} \in \mathrm{Rank(Sol)}$ we associate a sequence of matrices $(M(\mathrm{Rk}_1), ..., M(\mathrm{Rk}_{k+1}))$, such that $M(\mathrm{Rk}_i) \in \mathfrak{N}_n$ is in the Jordan canonical form, which is defined by the vector Rk_i in the obvious manner.

From now on we will work over the field of complex numbers $\mathbb{C}$.

We recall a well-known theorem due to Gerstenhaber [3].

Theorem 5 (Gerstenhaber)**.** *Let $A, B \in \mathfrak{N}_n$. Then $A \in \overline{\mathcal{O}(B)}$ if and only if $A \prec B$.*

Lemma 1. *Let $n \in \mathbb{N}_0$ and let $A, B \subseteq \mathbb{C}^n$ be constructible sets. The following equality holds*

$$\overline{A} \times \overline{B} = \overline{A \times B}.$$

Proof. This is an immediate consequence of the well-known fact that the Zariski and the Euclidean closure of a constructible set coincide. □

The following result generalizes Gerstenhaber's theorem.

Theorem 6. *Let $A_1, ..., A_k, B_1, ..., B_k \in \mathfrak{N}_n$. Then*

(1) $(A_1, ..., A_k) \in \overline{\mathcal{O}(B_1) \times ... \times \mathcal{O}(B_k)}$ *iff* $\mathrm{Rk}(A_1, ..., A_k) \preceq \mathrm{Rk}(B_1, ..., B_k)$
(2) $\mathcal{O}(A_1) \times ... \times \mathcal{O}(A_k) = \mathcal{O}(B_1) \times ... \times \mathcal{O}(B_k)$ *iff* $\mathrm{Rk}(A_1, ..., A_k) = \mathrm{Rk}(B_1, ..., B_k)$.

Proof. Ad(1). Using Theorem 5 we have $\mathrm{Rk}(A_1, ..., A_k) \preceq \mathrm{Rk}(B_1, ..., B_k)$ if and only if $(A_1, ..., A_k) \in \overline{\mathcal{O}(B_1)} \times ... \times \overline{\mathcal{O}(B_k)}$. Since $\mathrm{Rk}(A_1, ..., A_k) \preceq \mathrm{Rk}(B_1, ..., B_k)$, thus by Lemma 1

$$(A_1, ..., A_k) \in \overline{\mathcal{O}(B_1) \times ... \times \mathcal{O}(B_k)},$$

and the proof of the implication "$\Leftarrow$" in (i) is completed.

Suppose that $(A_1, ..., A_k) \in \overline{\mathcal{O}(B_1) \times ... \times \mathcal{O}(B_k)}$. By Lemma 1 we have the equality

$$\overline{\mathcal{O}(B_1)} \times ... \times \overline{\mathcal{O}(B_k)} = \overline{\mathcal{O}(B_1) \times ... \times \mathcal{O}(B_k)},$$

which implies that $A_j \in \overline{\mathcal{O}(B_j)}$ for all $j \in \{1, ..., k\}$, and thus $r_{A_j}(m) \leq r_{B_j}(m)$ for all $m \in \mathbb{N}$.

Ad(2). Implication "$\Rightarrow$" is obvious, the other implication is a simple consequence of Proposition 3. □

Since for constructible sets $A, B \subseteq \mathbb{F}^n$, where $\mathbb{F}$ is an arbitrary field, the equality $\overline{A \times B} = \overline{A} \times \overline{B}$ does not hold in general, the proof of Theorem 6 breaks. Nevertheless it would be interesting to know if the result of Theorem 6 remains true.

5. GEOMETRY OF THE SET OF SOLUTIONS

Recall that a set $\mathcal{U}$ is $GL(n,\mathbb{C})$-invariant if $\mathcal{U} \supseteq \bigcup_{A\in\mathcal{U}} \mathcal{O}(A)$. Thus we see (compare [5]) that the set Sol is $GL(n,\mathbb{C})$-invariant in the following sense:

$$\mathrm{Sol} = \bigcup_{(A_1,\ldots,A_k,B)\in \mathrm{Sol}} \mathcal{O}(A_1)\times\ldots\times\mathcal{O}(A_k)\times\mathcal{O}(B).$$

Next, wee see that the set Sol is a cone, i.e. $\mathrm{Sol} \neq \varnothing$ and

$$\mathrm{Sol} \supseteq \mathbb{C}\mathrm{Sol} := \{\lambda(A_1,\ldots,A_k,B) : \lambda \in \mathbb{C}, (A_1,\ldots,A_k,B) \in \mathrm{Sol}\}.$$

Recall also (see for example [9]) that for a matrix $A \in \mathfrak{N}_n$ the dimension of $\overline{\mathcal{O}(A)}$ can be computed by the following formula

$$\dim(\overline{\mathcal{O}(A)}) = n^2 - \sum_{j=0}^{\infty}(r_A(j) - r_A(j+1))^2.$$

We denote by $\mathrm{Sol}_{\mathrm{id}}$ the set of all nilpotent solutions to the rank function equation (2.2) with $S = \{1,\ldots,n\}$.

Example 3. It is easy to see that the set $\overline{\mathrm{Sol}}_{\mathrm{id}}$ with $n = 2k$ is irreducible and has the form

$$\overline{\mathrm{Sol}}_{\mathrm{id}} = \overline{\underbrace{\mathcal{O}(N_2\oplus O_{2k-2})\times\ldots\times\mathcal{O}(N_2\oplus O_{2k-2})}_{k}\times\mathcal{O}(\underbrace{N_2\oplus\ldots\oplus N_2}_{k})}.$$

The dimension of this set is equal $\dim\overline{\mathrm{Sol}}_{\mathrm{id}} = 6k^2 - 2k$ (we omit some dull computations).

Example 4. Let us consider the set $\overline{\mathrm{Sol}}_{\mathrm{id}}$ with $n = 2k+1$. It is quite easy to see that this set is reducible and has the form

$$\overline{\mathrm{Sol}}_{\mathrm{id}} = \bigcup_{i=1}^{k} \overline{\mathcal{O}(N_2\oplus O_{2k-1})\times\ldots\times\mathcal{O}(N_3\oplus O_{2k-2})^{[i]}\times\ldots} \overline{\ldots\times\mathcal{O}(N_2\oplus O_{2k-1})\times\mathcal{O}(N_3\oplus\underbrace{N_2\oplus\ldots\oplus N_2}_{k-1})},$$

where $[i]$ denotes the i-th position, counted from the left-hand side, on which the conjugacy class $\mathcal{O}(N_3\oplus O_{2k-2})$ appears. We see that all irreducible components have the same dimension, and $\dim\overline{\mathrm{Sol}}_{\mathrm{id}} = 6k^2 + 8k - 2$.

One of the most important consequences of Theorem 5 is that if $\mathcal{U} \subseteq M_{n\times n}(\mathbb{F})$ is a $GL(n,\mathbb{F})$-invariant set of nilpotent matrices over an algebraic closed field of characteristic zero, then there is a bijective correspondence between irreducible components of $\mathcal{U}$ and maximal elements of the set $R(\mathcal{U}) := \{r_A : A \in \mathcal{U}\}$ in the sense of order $\prec$. The same can be shown in our case.

We will follow [8]. We denote by $\mathcal{Z}(\mathrm{Sol})$ the set of all maximal elements of the set $\mathrm{Rank}(\mathrm{Sol})$ in the sense of order $\preceq$ and by $\mathcal{I}(\mathrm{Sol})$ the familly of all irreducible components of $\overline{\mathrm{Sol}}$.

It is well known that conjugacy classes of matrices are irreducible constructible sets, and thus the cartesian products of them are irreducible. By Theorem 6 and the fact that for each matrix $\mathrm{Rk} \in \mathrm{Rank}(\mathrm{Sol})$ there exists an $\mathrm{Rk}_0 \in \mathcal{Z}(\mathrm{Sol})$ such that $\mathrm{Rk} \preceq \mathrm{Rk}_0$ we obtain the following result.

Theorem 7. *The maps*

$$\mathcal{Z}(\mathrm{Sol}) \ni \mathrm{Rk} \mapsto \overline{\mathcal{O}(M(\mathrm{Rk}_1)) \times \ldots \times \mathcal{O}(M(\mathrm{Rk}_{k+1}))} \in \mathcal{I}(\mathrm{Sol}),$$

$$\mathcal{I}(\mathrm{Sol}) \ni W \mapsto \max \mathrm{Rank}(W) \in \mathcal{Z}(\mathrm{Sol})$$

are well defined mutually inverse bijections.

Corollary 1. *For the set* $\mathrm{Sol} \subset \underbrace{\mathfrak{N}_n \times \ldots \times \mathfrak{N}_n}_{k+1}$ *the following conditions are equivalent:*

- $\overline{\mathrm{Sol}}$ *is an irreducible set,*
- *there is a greatest element in* $\mathrm{Rank}(\mathrm{Sol})$ *with respect to* $\preceq$.

6. The linear capacity

At the end of the note we would like to formulate some remarks about the linear capacity of solution sets, which will be denoted by $\Lambda(\overline{\mathrm{Sol}})$.

Definition 5. Let $\mathcal{E} \subset M_{n\times n}(\mathbb{F})$, where $\mathbb{F}$ is an algebraically closed field of characteristic zero. The linear capacity $\Lambda(\mathcal{E}) \in \mathbb{N} \cup \{-\infty\}$ is defined by the formula

$$\Lambda(\mathcal{E}) = \max\{\dim \mathcal{L} : \mathcal{L} \text{ is a linear subspace of } M_{n\times n}(\mathbb{F}),\ \mathcal{L} \subseteq \overline{\mathcal{E}}\}.$$

Proposition 4. *Let* $\mathcal{E}_1, \mathcal{E}_2 \subseteq M_{n\times n}(\mathbb{F})$*, with* $\mathbb{F}$ *as above, be such that* $\overline{\mathcal{E}_1}, \overline{\mathcal{E}_2}$ *are irreducible algebraic cones. Then*

(1) $\Lambda(\overline{\mathcal{E}_1}) = \Lambda(\mathcal{E}_1)$.
(2) $\Lambda(\mathcal{E}_1 \times \mathcal{E}_2) = \Lambda(\mathcal{E}_1) + \Lambda(\mathcal{E}_2)$

Theorem 8 (Gerstenhaber, Chavey - Brualdi)**.** *Let* $\mathbb{F}$ *be an algebraically closed field of characteristic zero. If* $B \in \mathfrak{N}_n$ *is a nilpotent matrix, then*

$$\Lambda(\mathcal{O}(B)) = \frac{1}{2}\dim(\overline{\mathcal{O}(B)}).$$

Example 5. We compute the linear capacity of

$$\overline{\mathrm{Sol}}_{\mathrm{id}} \subseteq \underbrace{M_{n\times n}(\mathbb{C}) \times \ldots \times M_{n\times n}(\mathbb{C})}_{\mathrm{k+1}}$$

with $n = 2k$, namely

$$\Lambda(\overline{\mathrm{Sol}}_{\mathrm{id}}) = \Lambda(\mathrm{Sol}_{\mathrm{id}}) = \Lambda(\mathcal{O}(A_1)\times\ldots\times\mathcal{O}(A_k)\times\mathcal{O}(B)) = \Lambda(\mathcal{O}(A_1))+\ldots+\Lambda(\mathcal{O}(A_k))+\Lambda(\mathcal{O}(B)) = (\dim\overline{\mathcal{O}(A_1)}+\ldots+\dim\overline{\mathcal{O}(A_k)}+\dim\overline{\mathcal{O}(B)})/2 = 3k^2-k.$$

All these facts and reducibility of the set $\overline{\mathrm{Sol}}$ lead to the following modification of the above definition of the linear capacity

$$\Lambda(\overline{\mathrm{Sol}}) = \max\{\dim\mathcal{L} : \mathcal{L} \text{ is a linear subspace } \underbrace{M_{n\times n}(\mathbb{C})\times\ldots\times M_{n\times n}(\mathbb{C})}_{\text{k+1}},$$
$$\mathcal{L}\subseteq\overline{\mathcal{O}(A_1^i)\times\ldots\times\mathcal{O}(A_k^i)\times\mathcal{O}(B^i)}, i\in I\},$$

where $I\subset\mathbb{N}_0$ is such that $\#I$ is equal to the number of irreducible components of $\overline{\mathrm{Sol}}$. Of course, if $\overline{\mathrm{Sol}}$ is irreducible, then by Corollary 1 the above maximum of dimension can be attained. Moreover, it is quite easy to find the following useful inclusion.

Proposition 5. *For the set* $\overline{\mathrm{Sol}}$ *there exist matrices* $C_1,\ldots,C_k,D\in\mathfrak{N}_n$ *such that*

$$\overline{\mathrm{Sol}} = \bigcup_{(A_1,\ldots,A_k,B)\in\mathrm{Sol}}\overline{\mathcal{O}(A_1)\times\ldots\times\mathcal{O}(A_k)\times\mathcal{O}(B)}$$
$$\subseteq\overline{\mathcal{O}(C_1)\times\ldots\times\mathcal{O}(C_k)\times\mathcal{O}(D)}\subsetneq\mathfrak{N}_n^{k+1}.$$

Proof. If the set $\overline{\mathrm{Sol}}$ is irreducible, there is nothing to prove. Suppose that the set $\overline{\mathrm{Sol}}$ is reducible. It is enough to prove that for a set Rank(Sol) we can find an element $\mathrm{Rk}_0\in\mathrm{Rank}(\mathfrak{N}_n)$ such that $\mathrm{Rk}\preceq\mathrm{Rk}_0$ for every $\mathrm{Rk}\in\mathrm{Rank}(\mathrm{Sol})$. We construct the matrix $\mathrm{Rk}_0\in\mathrm{Rank}(\mathfrak{N}_n)$ using the below procedure:

(1) For a fixed $n\in\mathbb{N}_0$ - this number depends on the set $\overline{\mathrm{Sol}}$ - by Theorem 5 we can construct the diagram of all posible Jordan partitions ordered by the domination $\prec$ (the precise construction can be found in [4, Example 2.12].
(2) We consider the set of all nilpotent matrices from the first coordinate of the set $\overline{\mathrm{Sol}}$ and we denote it by Cr_1. For all matrices in Cr_1 we find their Jordan partitions. Notice that the set of all such Jordan partitions is finite.
(3) Using the diagram from step 1 we find a matrix $C_1\in\mathfrak{N}_n$, which dominates all matrices from the set Cr_1 - of course such a matrix always exists. Moreover, the matrix C_1 can be chosen in such a way that its Jordan partition is different from (n) (see [5, Proposition 4.2]).
(4) We continue this procedure for another coordinates. In the consequence we find matrices $C_1,\ldots,C_k,D\in\mathfrak{N}_n$, such that

$$\mathrm{Rk}\preceq\mathrm{Rk}(C_1,\ldots,C_k,D) := \mathrm{Rk}_0,\ \forall\,\mathrm{Rk}\in\mathrm{Rank}(\mathrm{Sol}).$$

Thus Theorem 6 gives us the desired inclusion.

The strict containment $\overline{\mathcal{O}(C_1)\times\ldots\times\mathcal{O}(C_k)\times\mathcal{O}(D)}\subsetneq\mathfrak{N}_n^{k+1}$ is the consequence of step (iii) in the above procedure. □

Using the above proposition we find a quite obvious relation

$$\Lambda(\overline{\mathrm{Sol}}) \leq \frac{1}{2}\left(\dim(\overline{\mathcal{O}(C_1)}) + \ldots + \dim(\overline{\mathcal{O}(C_k)}) + \dim(\overline{\mathcal{O}(D)}) \right).$$

7. Further Problems

In two consecutive notes we examined some rank function equations in the case of singular matrices. However our all attention was focused on the case with f satisfying some natural properties and g as the identity function and thus we are curious what can happen if $g(m) \neq m$. These lead us to the following problem.

Problem 1. *Describe all possible nilpotent solutions (i.e. a solution consisting of nilpotent matrices) to the equation (2.1) with a fixed function f satisfying the same properties as in the hypothesis of Theorem 2, and a fixed function $g(m) \neq m$ and $S = \{1, \ldots, n\}$.*

The easiest example of such an equation is

$$[r_A(m)]^2 + [r_B(m)]^2 = [r_C(m)]^2 \tag{7.1}$$

with $A, B, C \in \mathfrak{N}_n$. This equation is somehow connected with the famous Pythagorean equation and, what is not so surprising, we can construct some nilpotent solutions to (7.1) using Pythagorean triples – see Example 2.7 & 2.8 in [5]. It is natural to formulate the following question.

Problem 2. *Are there other solutions to the equation (7.1) than mentioned above?*

Of course, the same questions can be formulated for singular non-nilpotent matrices.

In Section 4, we investigated a certain generalization of Theorem 5 in the special case $\mathbb{F} = \mathbb{C}$. Since the method of proving this theorem breaks in the case of other fields, it is really interesting whether this theorem is still valid for an arbitrarily chosen algebraically closed field $\mathbb{F}$.

If $g(m) \neq m$, then some of methods presented in these notes break and thus we need new tools and ideas, even for the equation (7.1), because there is no natural rank function, which allows to mimic our argumentations.

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Author's address

Piotr Pokora
Pedagogical University of Cracow, Institute of Mathematics, Podchorążych 2, 30-084 Kraków, Poland
E-mail address: piotrpkr@gmail.com

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A UNIFIED PROOF OF SEVERAL INEQUALITIES AND SOME NEW INEQUALITIES INVOLVING NEUMAN-SÁNDOR MEAN

FENG QI AND WEN-HUI LI

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Abstract. In the paper, by finding linear relations of differences between some means, the authors supply a unified proof of several double inequalities for bounding Neuman-Sándor means in terms of the arithmetic, harmonic, and contra-harmonic means and discover some new sharp inequalities involving Neuman-Sándor, contra-harmonic, root-square, and other means of two positive real numbers.

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1. INTRODUCTION

It is well known that the quantities

$$A(a,b)=\frac{a+b}{2}, \qquad G(a,b)=\sqrt{ab},$$
$$H(a,b)=\frac{2ab}{a+b}, \qquad \overline{C}(a,b)=\frac{2(a^2+ab+b^2)}{3(a+b)},$$
$$C(a,b)=\frac{a^2+b^2}{a+b}, \qquad P(a,b)=\frac{a-b}{4\arctan\sqrt{a/b}-\pi},$$
$$Q(a,b)=\sqrt{\frac{a^2+b^2}{2}}, \qquad T(a,b)=\frac{a-b}{2\arctan\frac{a-b}{a+b}}$$

are respectively called in the literature the arithmetic, geometric, harmonic, centroidal, contra-harmonic, first Seiffert, root-square, and second Seiffert means of two positive real numbers a and b with $a\neq b$.

For $a,b>0$ with $a\neq b$, Neuman-Sándor mean $M(a,b)$ is defined in [11] by

$$M(a,b)=\frac{a-b}{2\operatorname{arcsinh}\frac{a-b}{a+b}},$$

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where $\operatorname{arcsinh} x = \ln(x+\sqrt{x^2+1})$ is the inverse hyperbolic sine function. At the same time, a chain of inequalities

$$G(a,b) < L_{-1}(a,b) < P(a,b) < A(a,b) < M(a,b) < T(a,b) < Q(a,b)$$

were given in [11], where

$$L_p(a,b) = \begin{cases} \left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & p \neq -1,0 \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & p = 0 \\ \dfrac{b-a}{\ln b - \ln a}, & p = -1 \end{cases}$$

is the p-th generalized logarithmic mean of a and b with $a \neq b$.

In [11, 12], it was established that

$$A(a,b) < M(a,b) < T(a,b), \quad P(a,b) < M(a,b) < T^2(a,b),$$
$$A(a,b)T(a,b) < M^2(a,b) < \frac{A^2(a,b)+T^2(a,b)}{2}$$

for $a,b>0$ with $a \neq b$.

For $0 < a,b < \frac{1}{2}$ with $a \neq b$, Ky Fan type inequalities

$$\frac{G(a,b)}{G(1-a,1-b)} < \frac{L_{-1}(a,b)}{L_{-1}(1-a,1-b)} < \frac{P(a,b)}{P(1-a,1-b)} < \frac{A(a,b)}{A(1-a,1-b)} < \frac{M(a,b)}{M(1-a,1-b)} < \frac{T(a,b)}{T(1-a,1-b)}$$

were presented in [11, Proposition 2.2].

In [8], it was showed that the double inequality

$$L_{p_0}(a,b) < M(a,b) < L_2(a,b)$$

holds for all $a,b>0$ with $a \neq b$ and for $p_0 = 1.843\ldots$, where p_0 is the unique solution of the equation $(p+1)^{1/p} = 2\ln(1+\sqrt{2})$.

In [10], Neuman proved that the double inequalities

$$\alpha Q(a,b) + (1-\alpha)A(a,b) < M(a,b) < \beta Q(a,b) + (1-\beta)A(a,b)$$

and

$$\lambda C(a,b) + (1-\lambda)A(a,b) < M(a,b) < \mu C(a,b) + (1-\mu)A(a,b) \tag{1.1}$$

hold for all $a,b>0$ with $a \neq b$ if and only if

$$\alpha \leq \frac{1-\ln(1+\sqrt{2})}{(\sqrt{2}-1)\ln(1+\sqrt{2})} = 0.3249\ldots, \quad \beta \geq \frac{1}{3}$$

and

$$\lambda \le \frac{1-\ln(1+\sqrt{2})}{\ln(1+\sqrt{2})} = 0.1345\ldots, \quad \mu \ge \frac{1}{6}.$$

In [20, Theorems 1.1 to 1.3], it was found that the double inequalities

$$\alpha_1 H(a,b)+(1-\alpha_1)Q(a,b) < M(a,b) < \beta_1 H(a,b)+(1-\beta_1)Q(a,b),$$
$$\alpha_2 G(a,b)+(1-\alpha_2)Q(a,b) < M(a,b) < \beta_2 G(a,b)+(1-\beta_2)Q(a,b),$$

and

$$\alpha_3 H(a,b)+(1-\alpha_3)C(a,b) < M(a,b) < \beta_3 H(a,b)+(1-\beta_3)C(a,b) \tag{1.2}$$

hold for all $a,b>0$ with $a\ne b$ if and only if

$$\alpha_1 \ge \frac{2}{9} = 0.2222\ldots, \quad \beta_1 \le 1-\frac{1}{\sqrt{2}\ln(1+\sqrt{2})} = 0.1977\ldots,$$
$$\alpha_2 \ge \frac{1}{3} = 0.3333\ldots, \quad \beta_2 \le 1-\frac{1}{\sqrt{2}\ln(1+\sqrt{2})} = 0.1977\ldots,$$
$$\alpha_3 \ge 1-\frac{1}{2\ln(1+\sqrt{2})} = 0.4327\ldots, \quad \beta_3 \le \frac{5}{12} = 0.4166\ldots$$

In [19, Theorem 3.1], it was established that the double inequality

$$\alpha I(a,b)+(1-\alpha)Q(a,b) < M(a,b) < \beta I(a,b)+(1-\beta)Q(a,b)$$

holds for all $a,b>0$ with $a\ne b$ if and only if

$$\alpha \ge \frac{1}{2} \quad \text{and} \quad \beta \le \frac{e\left[\sqrt{2}\ln(1+\sqrt{2})-1\right]}{(\sqrt{2}e-2)\ln(1+\sqrt{2})} = 0.4121\ldots$$

For more information on this topic, please refer to [1–3, 5, 7–10, 12–14, 16–18, 20] and plenty of references cited therein.

The first goal of this paper is, by finding linear relations of differences between some means, to supply a unified proof of inequalities (1.1) and (1.2).

The second purpose of this paper is to establish some new sharp inequalities involving Neuman-Sándor, centroidal, contra-harmonic, and root-square means of two positive real numbers a and b with $a\ne b$.

2. Lemmas

In order to attain our aims, the following lemmas are needed.

Lemma 1 ([15, Lemma 1.1]). *Suppose that the power series* $f(x)=\sum_{n=0}^{\infty} a_n x^n$ *and* $g(x)=\sum_{n=0}^{\infty} b_n x^n$ *have the radius of convergence* $r>0$ *and* $b_n>0$ *for all* $n\in \mathbb{N}=\{0,1,2,\ldots\}$. *Let* $h(x)=\frac{f(x)}{g(x)}$. *Then the following statements are true.*

(1) *If the sequence* $\{\frac{a_n}{b_n}\}_{n=0}^{\infty}$ *is (strictly) increasing (decreasing), then* $h(x)$ *is also (strictly) increasing (decreasing) on* $(0,r)$.
(2) *If the sequence* $\{\frac{a_n}{b_n}\}$ *is (strictly) increasing (decreasing) for* $0<n\le n_0$ *and (strictly) decreasing (increasing) for* $n>n_0$, *then there exists* $x_0\in(0,r)$ *such that* $h(x)$ *is (strictly) increasing (decreasing) on* $(0,x_0)$ *and (strictly) decreasing (increasing) on* (x_0,r).

Lemma 2. *Let*

$$h_1(x)=\frac{\sinh x-x}{2x\sinh^2 x}. \tag{2.1}$$

Then $h_1(x)$ *is strictly decreasing on* $(0,\infty)$ *and has the limit* $\lim_{x\to0^+}h_1(x)=\frac{1}{12}$.

Proof. Let $f_1(x)=\sinh x-x$ and $f_2(x)=2x\sinh^2 x=x\cosh 2x-x$. Using the power series

$$\sinh x=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}\quad\text{and}\quad\cosh x=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}, \tag{2.2}$$

we can express the functions $f_1(x)$ and $f_2(x)$ as

$$f_1(x)=\sum_{n=0}^{\infty}\frac{x^{2n+3}}{(2n+3)!}\quad\text{and}\quad f_2(x)=\sum_{n=0}^{\infty}\frac{2^{2n+2}x^{2n+3}}{(2n+2)!}. \tag{2.3}$$

Hence, we have

$$h_1(x)=\frac{\sum_{n=0}^{\infty}a_nx^{2n}}{\sum_{n=0}^{\infty}b_nx^{2n}}, \tag{2.4}$$

where $a_n=\frac{1}{(2n+3)!}$ and $b_n=\frac{2^{2n+2}}{(2n+2)!}$. Let $c_n=\frac{a_n}{b_n}$. Then $c_n=\frac{1}{(2n+3)2^{2n+2}}$ and

$$c_{n+1}-c_n=\frac{-(6n+17)}{(2n+3)(2n+5)2^{2n+4}}<0.$$

As a result, by Lemma 1, it follows that the function $h_1(x)$ is strictly decreasing on $(0,\infty)$.

From (2.4), it is easy to see that $\lim_{x\to0^+}h_1(x)=\frac{a_0}{b_0}=\frac{1}{12}$. The proof of Lemma 2 is complete. □

Lemma 3. *Let*

$$h_2(x)=\frac{1-\frac{\sinh x}{x}+\frac{\sinh^2 x}{3}}{\cosh x-\frac{\sinh x}{x}}. \tag{2.5}$$

Then $h_2(x)$ *is strictly increasing on* $(0,\infty)$ *and has the limit* $\lim_{x\to0^+}h_2(x)=\frac{1}{2}$.

Proof. Let

$$f_3(x) = 1 - \frac{\sinh x}{x} + \frac{\sinh^2 x}{3} = 1 - \frac{\sinh x}{x} + \frac{\cosh 2x - 1}{6}$$

and

$$f_4(x) = \cosh x - \frac{\sinh x}{x}.$$

Making use of the power series in (2.2) shows that

$$f_3(x) = \sum_{n=0}^{\infty} \frac{(2n+3)2^{2n+2} - 6}{6(2n+3)!} x^{2n+2} \quad \text{and} \quad f_4(x) = \sum_{n=0}^{\infty} \frac{2n+2}{(2n+3)!} x^{2n+2}.$$

Therefore, we have

$$h_2(x) = \frac{\sum_{n=0}^{\infty} a_n x^{2n+2}}{\sum_{n=0}^{\infty} b_n x^{2n+2}}, \tag{2.6}$$

where $a_n = \frac{(2n+3)2^{2n+2}-6}{6(2n+3)!}$ and $b_n = \frac{2n+2}{(2n+3)!}$. Let $c_n = \frac{a_n}{b_n}$. Then

$$c_n = \frac{(2n+3)2^{2n+1} - 3}{6(n+1)}$$

and

$$c_{n+1} - c_n = \frac{3 + 7 \cdot 2^{2n+2} + 21n \cdot 2^{2n+1} + 3n^2 \cdot 2^{2n+2}}{6(n+1)(n+2)} > 0.$$

Accordingly, by Lemma 1, it follows that the function $h_2(x)$ is strictly increasing on $(0, \infty)$.

It is clear that $\lim_{x \to 0^+} h_2(x) = \frac{a_0}{b_0} = \frac{1}{2}$. The proof of Lemma 3 is complete. □

Lemma 4. *Let*

$$h_3(x) = \frac{\cosh x - \frac{\sinh x}{x}}{1 + \sinh^2 x - \frac{\sinh x}{x}}. \tag{2.7}$$

Then $h_3(x)$ is strictly decreasing on $(0, \infty)$ and has the limit $\lim_{x \to 0^+} h_3(x) = \frac{2}{5}$.

Proof. Let

$$f_5(x) = \cosh x - \frac{\sinh x}{x}$$

and

$$f_6(x) = 1 + \sinh^2 x - \frac{\sinh x}{x} = 1 - \frac{\sinh x}{x} + \frac{\cosh 2x - 1}{2}.$$

Utilizing the power series in (2.2) gives

$$f_5(x) = \sum_{n=0}^{\infty} \frac{2n+2}{(2n+3)!} x^{2n+2} \quad \text{and} \quad f_6(x) = \sum_{n=0}^{\infty} \frac{(2n+3)2^{2n+1} - 1}{(2n+3)!} x^{2n+2}.$$

This implies that

$$h_3(x)=\frac{\sum_{n=0}^{\infty}a_nx^{2n+2}}{\sum_{n=0}^{\infty}b_nx^{2n+2}}, \tag{2.8}$$

where $a_n=\frac{2n+2}{(2n+3)!}$ and $b_n=\frac{(2n+3)2^{2n+1}-1}{(2n+3)!}$. Let $c_n=\frac{a_n}{b_n}$. Then

$$c_n=\frac{2n+2}{(2n+3)2^{2n+1}-1}$$

and

$$c_{n+1}-c_n=-\frac{2(1+7\cdot2^{2n+2}+21n\cdot2^{2n+1}+3n^2\cdot2^{2n+2})}{(3\cdot2^{2n+1}+n\cdot2^{2n+2}-1)(5\cdot2^{2n+3}+n\cdot2^{2n+4}-1)}<0.$$

In light of Lemma 1, we obtain that the function $h_3(x)$ is strictly decreasing on $(0,\infty)$.

It is obvious that $\lim_{x\to0^+}h_3(x)=\frac{a_0}{b_0}=\frac{2}{5}$. The proof of Lemma 4 is complete. □

3. A UNIFIED PROOF OF INEQUALITIES (1.1) AND (1.2)

Now we are in a position to supply a unified proof of inequalities (1.1) and (1.2) and, as corollaries, to establish some new inequalities involving Neuman-Sándor, contra-harmonic, centroidal, and root-square means of two positive real numbers a and b with $a\neq b$.

It is not difficult to see that the inequalities (1.1) and (1.2) can be rearranged respectively as

$$\lambda-1<\frac{M(a,b)-C(a,b)}{C(a,b)-A(a,b)}<\mu-1 \tag{3.1}$$

and

$$-\alpha_3<\frac{M(a,b)-C(a,b)}{C(a,b)-H(a,b)}<-\beta_3. \tag{3.2}$$

The denominators in (3.1) and (3.2) meet

$$2[C(a,b)-A(a,b)]=C(a,b)-H(a,b)=\frac{(a-b)^2}{a+b} \tag{3.3}$$

which were presented in [4, Eq. (4.4)]. This implies that the inequalities (1.1) and (1.2) are identical up to a scalar. Therefore, it is sufficient to prove one of the two inequalities (1.1) and (1.2).

By a direct calculation, we also find

$$\begin{aligned}6[\overline{C}(a,b)-A(a,b)]&=3[C(a,b)-\overline{C}(a,b)]=2[A(a,b)-H(a,b)]\\&=\frac{3}{2}[\overline{C}(a,b)-H(a,b)]=\frac{(a-b)^2}{a+b}\triangleq CH(a,b).\end{aligned} \tag{3.4}$$

So, it is natural to raise a problem: what are the best constants α and β such that the double inequality

$$\alpha < \frac{M(a,b)-C(a,b)}{CH(a,b)} < \beta \tag{3.5}$$

holds for all $a,b>0$ with $a \neq b$? The following theorem gives a solution to this problem.

Theorem 1. *The double inequality (3.5) holds for all $a,b>0$ with $a \neq b$ if and only if*

$$\alpha \leq \frac{1}{2\ln(1+\sqrt{2})}-1 = -0.4327\ldots \quad \text{and} \quad \beta \geq -\frac{5}{12} = -0.4166\ldots$$

Proof. Without loss of generality, we assume that $a>b>0$. Let $x=\frac{a}{b}$. Then $x>1$ and

$$\frac{M(a,b)-C(a,b)}{CH(a,b)} = \frac{\frac{x-1}{2\operatorname{arcsinh}\frac{x-1}{x+1}}-\frac{x^2+1}{x+1}}{\frac{(x-1)^2}{x+1}}.$$

Let $t=\frac{x-1}{x+1}$. Then $t\in(0,1)$ and

$$\frac{M(a,b)-C(a,b)}{CH(a,b)} = \frac{\frac{t}{\operatorname{arcsinh} t}-t^2-1}{2t^2}.$$

Let $t=\sinh\theta$ for $\theta\in\left(0,\ln\left(1+\sqrt{2}\right)\right)$. Then

$$\frac{M(a,b)-C(a,b)}{CH(a,b)} = \frac{\frac{\sinh\theta}{\theta}-\sinh^2\theta-1}{2\sinh^2\theta} = \frac{\sinh\theta-\theta}{2\theta\sinh^2\theta}-\frac{1}{2}.$$

In virtue of Lemma 2, Theorem 1 is thus proved. □

Corollary 1. *The double inequality*

$$\alpha CH(a,b)+M(a,b) < C(a,b) < \beta CH(a,b)+M(a,b) \tag{3.6}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha \leq \frac{5}{12} = 0.4166\ldots$ and

$$\beta \geq 1-\frac{1}{2\ln(1+\sqrt{2})} = 0.4327\ldots$$

Corollary 2. *The double inequality*

$$\alpha CH(a,b)+M(a,b) < \overline{C}(a,b) < \beta CH(a,b)+M(a,b) \tag{3.7}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha \leq \frac{1}{12} = 0.0833\ldots$ and

$$\beta \geq \frac{2}{3}-\frac{1}{2\ln(1+\sqrt{2})} = 0.0993\ldots$$

4. Some new inequalities involving Neuman-Sándor mean

Finally we further establish some new inequalities involving Neuman-Sándor, centroidal, root-square, and other means.

Theorem 2. *The inequality*

$$M(a,b) > \lambda CH(a,b) \tag{4.1}$$

holds for all $a,b>0$ with $a \neq b$ if and only if $\lambda \le \frac{1}{2\ln(1+\sqrt{2})} = 0.5672\ldots$.

Proof. It is clear that

$$\frac{M(a,b)}{CH(a,b)} = \frac{(a-b)(a+b)}{(a-b)^2 2\operatorname{arcsinh}\frac{a-b}{a+b}} = \frac{a+b}{a-b}\frac{1}{2\operatorname{arcsinh}\frac{a-b}{a+b}}.$$

Without loss of generality, we assume that $a>b>0$. Let $x = \frac{a-b}{a+b}$. Then $x \in (0,1)$ and

$$\frac{M(a,b)}{CH(a,b)} = \frac{1}{2x\operatorname{arcsinh} x} \triangleq f(x).$$

Differentiating $f(x)$ yields

$$f'(x) = -\frac{\frac{x}{\sqrt{1+x^2}} + \operatorname{arcsinh} x}{2x^2 \operatorname{arcsinh}^2 x} \le 0$$

which means that function $f(x)$ is decreasing for $x \in (0,1)$.

It is apparent that

$$\lim_{x\to 1^-} f(x) = \frac{1}{2\ln(1+\sqrt{2})}.$$

The proof of Theorem 2 is thus complete. □

Theorem 3. *The double inequality*

$$\alpha Q(a,b) + (1-\alpha)M(a,b) < \overline{C}(a,b) < \beta Q(a,b) + (1-\beta)M(a,b) \tag{4.2}$$

holds for all $a,b>0$ with $a \neq b$ if and only if $\alpha \le \frac{1}{2}$ and

$$\beta \ge \frac{3-4\ln(1+\sqrt{2})}{3[1-\sqrt{2}\ln(1+\sqrt{2})]} = 0.7107\ldots$$

Proof. It is sufficient to show

$$\alpha < \frac{\overline{C}(a,b)-M(a,b)}{Q(a,b)-M(a,b)} < \beta.$$

Without loss of generality, we assume that $a>b>0$. Let $x = \frac{a}{b}$. Then $x>1$ and

$$\frac{\overline{C}(a,b)-M(a,b)}{Q(a,b)-M(a,b)} = \frac{\frac{2(x^2+x+1)}{3(x+1)} - \frac{x-1}{2\operatorname{arcsinh}\frac{x-1}{x+1}}}{\sqrt{\frac{x^2+1}{2}} - \frac{x-1}{2\operatorname{arcsinh}\frac{x-1}{x+1}}}.$$

Let $t = \frac{x-1}{x+1}$. Then $t \in (0,1)$ and

$$\frac{\overline{C}(a,b)-M(a,b)}{Q(a,b)-M(a,b)} = \frac{\frac{t^2}{3}+1-\frac{t}{\operatorname{arcsinh} t}}{\sqrt{1+t^2}-\frac{t}{\operatorname{arcsinh} t}}.$$

Let $t = \sinh\theta$ for $\theta \in \left(0, \ln\left(1+\sqrt{2}\right)\right)$. Then

$$\frac{\overline{C}(a,b)-M(a,b)}{Q(a,b)-M(a,b)} = \frac{\frac{\sinh^2\theta}{3}+1-\frac{\sinh\theta}{\theta}}{\cosh\theta-\frac{\sinh\theta}{\theta}}.$$

By Lemma 3, we obtain Theorem 3. □

Theorem 4. *The double inequality*

$$\alpha C(a,b)+(1-\alpha)M(a,b) < Q(a,b) < \beta C(a,b)+(1-\beta)M(a,b) \tag{4.3}$$

holds for all $a,b>0$ with $a \neq b$ if and only if

$$\alpha \leq \frac{\sqrt{2}\ln(1+\sqrt{2})-1}{2\ln(1+\sqrt{2})-1} = 0.3231\ldots \quad \textit{and} \quad \beta \geq \frac{2}{5}.$$

Proof. The double inequalities (4.3) is the same as

$$\alpha < \frac{Q(a,b)-M(a,b)}{C(a,b)-M(a,b)} < \beta.$$

Without loss of generality, we assume that $a > b > 0$. Let $x = \frac{a}{b}$. Then $x > 1$ and

$$\frac{Q(a,b)-M(a,b)}{C(a,b)-M(a,b)} = \frac{\sqrt{\frac{x^2+1}{2}}-\frac{x-1}{2\operatorname{arcsinh}\frac{x-1}{x+1}}}{\frac{x^2+1}{x+1}-\frac{x-1}{2\operatorname{arcsinh}\frac{x-1}{x+1}}}.$$

Let $t = \frac{x-1}{x+1}$. Then $t \in (0,1)$ and

$$\frac{Q(a,b)-M(a,b)}{C(a,b)-M(a,b)} = \frac{\sqrt{1+t^2}-\frac{t}{\operatorname{arcsinh} t}}{1+t^2-\frac{t}{\operatorname{arcsinh} t}}.$$

Let $t = \sinh\theta$ for $\theta \in \left(0, \ln\left(1+\sqrt{2}\right)\right)$. Then

$$\frac{Q(a,b)-M(a,b)}{C(a,b)-M(a,b)} = \frac{\cosh\theta-\frac{\sinh\theta}{\theta}}{1+\sinh^2\theta-\frac{\sinh\theta}{\theta}}.$$

According to Lemma 4, the proof of Theorem 3 is complete. □

Remark 1. This paper is a slightly revised version of the preprint [6].

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Authors' addresses

Feng Qi

Department of Mathematics, School of Science, Tianjin Polytechnic University, Tianjin City, 300387, China, College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China, Institute of Mathematics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

E-mail address: qifeng618@gmail.com, qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@hotmail.com, qifeng618@qq.com, qifeng618@qq.com

URL: qifeng618.wordpress.com

Wen-Hui Li

Department of Mathematics, School of Science, Tianjin Polytechnic University, Tianjin City, 300387, China

E-mail address: wen.hui.li102@gmail.com, wen.hui.li102@gmail.com, wen.hui.li@foxmail.com, wen.hui.li@foxmail.com

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GENERALIZED STIRLING TRANSFORM

MOURAD RAHMANI

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Abstract. In this paper algorithms are developed for computing the Stirling transform and the inverse Stirling transform; specifically, we investigate a class of sequences satisfying a two-term recurrence. We derive a general identity which generalizes the usual Stirling transform and investigate the corresponding generating functions also. In addition, some interesting consequences of these results related to classical sequences like Fibonacci, Bernoulli and the numbers of derangements have been derived.

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1. INTRODUCTION

The Stirling numbers arise frequently in mathematics, especially in enumerative problems. This is the reason of their important role in combinatorial analysis, number theory, probability, graph theory, calculus of finite differences and interpolation. The notations for these numbers have never been standardized, this paper follows the notation of Riordan for the signed Stirling numbers of the first kind $s(n,k)$ and Knuth's notation for the Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$.

The Stirling transform of a sequence (a_n) is the the sequence (b_n) given by

$$b_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} a_k, \tag{1.1}$$

and the inverse transform is

$$a_n = \sum_{k=0}^{n} s(n,k)\, b_k. \tag{1.2}$$

The identity (1.1) has a combinatorial interpretation given in [2]. If a_n is the number of objects in some class with points labeled $1,2,\ldots,n$ (with all labels distinct) then b_n is the number of objects with points labeled $1,2,\ldots,n$ (with repetitions allowed).

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In this paper algorithms are developed for computing the Stirling transform and the inverse Stirling transform; specifically, we investigate a class of sequences satisfying a two-term recurrence. We derive a general identity which generalizes the usual Stirling transform and investigate the corresponding generating functions also.

Given a sequence $a_m := a_{0,m}$ $(m \geq 0)$. We construct an infinite matrix $\mathcal{S} := \left(a_{n,m}\right)$ as follows:

The first row $a_{0,m}$ of the matrix is the initial sequence; the first column $b_n := a_{n,0}$ $(n \geq 0)$ is called the final sequence and, each entry $a_{n,m}$ is given recursively by

$$a_{n+1,m} = a_{n,m+1} + m a_{n,m}. \tag{1.3}$$

Conversely, if we start with the final sequence, the matrix $\mathcal{S}$ can be recovered by the recursive relations

$$a_{n,m+1} = a_{n+1,m} - m a_{n,m}. \tag{1.4}$$

2. Definitions and notation

In this section, we introduce some definitions and notations which are useful in the rest of the paper. $\mathbb{N}$ being the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The falling and rising factorials are defined, respectively by

$$(x)_n = x(x-1)\cdots(x-n+1), (x)_0 = 1$$

and

$$\langle x \rangle_n = x(x+1)\cdots(x+n-1), \langle x \rangle_0 = 1.$$

The (signed) Stirling numbers $s(n,k)$ of the first kind, are usually defined by

$$(x)_n = \sum_{k=0}^{n} s(n,k) x^k, \tag{2.1}$$

or by the following generating function

$$\frac{1}{k!}(\ln(1+x))^k = \sum_{n \geq k} s(n,k) \frac{x^n}{n!}. \tag{2.2}$$

It follows from (2.1) or (2.2) that

$$s(n+1,k) = s(n,k-1) - n\, s(n,k) \tag{2.3}$$

and that

$$s(n,0) = \delta_{n,0} \quad (n \in \mathbb{N}),\ s(n,k) = 0 \quad (k > n \text{ or } k < 0),$$

where $\delta_{n,m}$ denotes the Kronecker symbol.

The Stirling numbers $\left\{ {n \atop k} \right\}$ of the second kind count the number of possible partitions of a set of n objects into k disjoint blocks. These numbers can be defined explicitly by

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_k \,.$$

For any positive $r \in \mathbb{N}$ the quantity $\left\{ {n \atop k} \right\}_r$ denotes the number of partitions of a set of n objects into exactly k nonempty, disjoint subsets, such that the first r elements are in distinct subsets. These numbers obey the recurrence relation

$$\begin{aligned} \left\{ {n \atop k} \right\}_r &= 0, && n < r, \\ \left\{ {n \atop k} \right\}_r &= \delta_{k,r}, && n = r, \\ \left\{ {n \atop k} \right\}_r &= k \left\{ {n-1 \atop k} \right\}_r + \left\{ {n-1 \atop k-1} \right\}_r, && n > r, \end{aligned} \tag{2.4}$$

The exponential generating function is given by

$$\sum_{n \geq k} \left\{ {n+r \atop k+r} \right\}_r \frac{x^n}{n!} = \frac{1}{k!} e^{rx} \left(e^x - 1 \right)^k . \tag{2.5}$$

The properties

$$\left\{ {n \atop r} \right\}_r = r^{n-r}$$

and

$$\left\{ {n+r \atop k+r} \right\}_r = \left\{ {n+r \atop k+r} \right\}_{r-1} - (r-1) \left\{ {n+r-1 \atop k+r} \right\}_{r-1} \tag{2.6}$$

are given in [3], which one can consult for more details on r-Stirling numbers.

3. Combinatorial identities

Theorem 1. *Given an initial sequence $\left(a_{0,m}\right)_{m \geq 0}$, define the matrix $\mathcal{S}$ by (1.3). Then, the entries of the infinite matrix $\mathcal{S}$ are given by*

$$a_{n,m} = \sum_{k=0}^{n} \left\{ {n+m \atop k+m} \right\}_m a_{0,m+k}. \tag{3.1}$$

Proof. We prove by induction on n, the result clearly holds for $n=0$. By induction hypothesis

$$\begin{aligned}
a_{n,m+1}+ma_{n,m} &= \sum_{k=0}^{n}\left\{\begin{matrix} n+m+1 \\ k+m+1 \end{matrix}\right\}_{m+1} a_{0,m+k+1}+m\left\{\begin{matrix} n+m \\ m \end{matrix}\right\}_{m} a_{0,m} \\
&\quad +m\sum_{k=1}^{n-1}\left\{\begin{matrix} n+m \\ k+m \end{matrix}\right\}_{m} a_{0,m+k} \\
&= \sum_{k=0}^{n}\left\{\begin{matrix} n+m+1 \\ k+m+1 \end{matrix}\right\}_{m+1} a_{0,m+k+1}+m\left\{\begin{matrix} n+m \\ m \end{matrix}\right\}_{m} a_{0,m} \\
&\quad +m\sum_{k=0}^{n-2}\left\{\begin{matrix} n+m \\ k+m+1 \end{matrix}\right\}_{m} a_{0,m+k+1} \\
&= \left\{\begin{matrix} n+m+1 \\ n+m+1 \end{matrix}\right\}_{m+1} a_{0,m+n+1}+\left\{\begin{matrix} n+m+1 \\ n+m \end{matrix}\right\}_{m+1} a_{0,m+n} \\
&\quad +\sum_{k=0}^{n-2}\left\{\begin{matrix} n+m+1 \\ k+m+1 \end{matrix}\right\}_{m+1} a_{0,m+k+1} \\
&\quad +m\left\{\begin{matrix} n+m \\ m \end{matrix}\right\}_{m} a_{0,m}+m\sum_{k=0}^{n-2}\left\{\begin{matrix} n+m \\ k+m+1 \end{matrix}\right\}_{m} a_{0,m+k+1}.
\end{aligned}$$

From (2.6) and after some rearrangements, we get

$$\begin{aligned}
a_{n,m+1}+ma_{n,m} &= \sum_{k=0}^{n+1}\left\{\begin{matrix} n+m+1 \\ k+m \end{matrix}\right\}_{m} a_{0,m+k}. \\
&= a_{n+1,m}.
\end{aligned}$$

□

Theorem 2. *Given a final sequence* $\left(a_{n,0}\right)_{n\geq 0}$, *define the matrix* $\mathcal{S}$ *by (1.4). Then, the entries of the infinite matrix* $\mathcal{S}$ *are given by*

$$a_{n,m}=\sum_{k=0}^{m} s\left(m,k\right)a_{n+k,0}. \tag{3.2}$$

Proof. We prove by induction on m, the result clearly holds for $n=0$. By induction hypothesis and (2.3), we have

$$a_{n+1,m}-ma_{n,m}=\sum_{k=1}^{m+1} s\left(m,k-1\right)a_{n+k,0}-m\sum_{k=0}^{m} s\left(m,k\right)a_{n+k,0}$$

$$
\begin{aligned}
&= s(m,m)\,a_{n+m+1,0} + \sum_{k=1}^{m} s(m,k-1)\,a_{n+k,0} \\
&- m s(m,0)\,a_{n,0} - m \sum_{k=1}^{m} s(m,k)\,a_{n+k,0} \\
&= s(m,m)\,a_{n+m+1,0} + \sum_{k=1}^{m} \left(s(m,k-1) - m s(m,k)\right) a_{n+k,0} \\
&- m s(m,0)\,a_{n,0} \\
&= a_{n,m+1}.
\end{aligned}
$$

□

Corollary 1. *For $n,m \in \mathbb{N}_0$, we have*

$$\sum_{k=0}^{m} s(m,k)\,b_{n+k} = \sum_{k=0}^{n} \left\{ {n+m \atop k+m} \right\}_{m} a_{m+k}. \tag{3.3}$$

The last identity can be viewed as the generalized Stirling transform which reduced, for $m = 0$, to the Stirling transform (1.1) of the sequence a_n, and for $n = 0$ reduces to the inverse Stirling transform (1.2) of the sequence b_m. We may now formulate the following algorithms

Algorithm 1. Stirling transform
Input: a_n
Output: b_n
Set $X_m = a_{n-m,m}$
for $n = 0,1,\ldots$ **do**
  $X_n := a_n$
  for $m = n, n-1, \ldots, 0$ **do**
  $X_{m-1} := (m-1)\,X_{m-1} + X_m$
  end do
  $b_n := X_0$
end do

Algorithm 2. inverse Stirling transform
Input: b_m
Output: a_m
Set $Y_n = b_{n,m-n}$
for $m = 0,1,\ldots$ **do**
  $Y_m := b_m$
  for $n = m, m-1, \ldots, 0$ **do**

$Y_{n-1} := Y_n - (m-n)\, Y_{n-1}$
end do
$a_m := Y_0$
end do

Example 1. Setting $a_{0,m} = 1$ in (3.3), we get the well known identity [7]

$$\sum_{k=0}^{n} \left\{ \begin{matrix} n+m \\ k+m \end{matrix} \right\}_m = \sum_{k=0}^{m} s\,(m,k)\, B_{n+k},$$

where B_n is the nth Bell number.

Example 2. Let $(F_n)_{n\in\mathbb{N}_0}$ be the Fibonacci sequence given by Binet's formula

$$F_n = \frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right),$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. If the initial sequence

$$a_{0,m} = \frac{(-1)^m}{\sqrt{5}}\left(\langle -\alpha\rangle_m - \langle -\beta\rangle_m\right),$$

then we get the following matrix

$$\mathcal{S} = \begin{pmatrix} 0 & 1 & 0 & 1 & -4 & 19 & -108 & \cdots \\ 1 & 1 & 1 & -1 & 3 & -13 & 71 & \\ 1 & 2 & 1 & 0 & -1 & 6 & -37 & \\ 2 & 3 & 2 & -1 & 2 & -7 & 34 & \\ 3 & 5 & 3 & -1 & 1 & -1 & -3 & \\ 5 & 8 & 5 & -2 & 3 & -8 & 31 & \\ 8 & 13 & 8 & -3 & 4 & -9 & 28 & \\ 13 & 21 & 13 & -5 & 7 & -17 & 59 & \\ \vdots & & & & & & & \end{pmatrix}.$$

From this matrix we observe that $a_{n,0} = a_{n,2} = -a_{n+2,3} = F_n$, and $a_{n+3,4} = L_n$, where $(L_n)_{n\in\mathbb{N}_0}$ the Lucas sequence given by Binet's formula

$$L_n = \alpha^n + \beta^n.$$

It is well known that the F_n and L_n are connected by the formula

$$L_n = F_{n-1} + F_{n+1},\ (n \in \mathbb{N}).$$

By (3.3), one can deduce that

$$\sum_{k=0}^{m} s\,(m,k)\, F_{n+k} = \frac{1}{\sqrt{5}} \sum_{k=0}^{n} (-1)^{m+k} \left\{ \begin{matrix} n+m \\ k+m \end{matrix} \right\}_m \left(\langle -\alpha\rangle_{m+k} - \langle -\beta\rangle_{m+k}\right),$$

and by Theorem 2, we get

$$F_n = \sum_{k=0}^{2} s(2,k)\, F_{n+k} = -F_{n+1} + F_{n+2}$$
$$= -\sum_{k=0}^{3} s(3,k)\, F_{n+2+k} = -2F_{n+3} + 3F_{n+4} - F_{n+5},$$

and for $n \in \mathbb{N}_0$, we have

Corollary 2.

$$L_n = \sum_{k=0}^{4} s(4,k)\, F_{n+3+k} = -6F_{n+4} + 11F_{n+5} - 6F_{n+6} + F_{n+7}.$$

By Theorem 1, one easily gets the following explicit formulas.

Corollary 3. *For $n \in \mathbb{N}_0$, we have*

$$F_n = \frac{1}{\sqrt{5}} \sum_{k=0}^{n} (-1)^k \left\{ {n \atop k} \right\} \left(\langle -\alpha \rangle_k - \langle -\beta \rangle_k \right)$$
$$= \frac{1}{\sqrt{5}} \sum_{k=0}^{n} (-1)^k \left\{ {n+2 \atop k+2} \right\}_2 \left(\langle -\alpha \rangle_{k+2} - \langle -\beta \rangle_{k+2} \right)$$
$$= \frac{1}{\sqrt{5}} \sum_{k=0}^{n+2} (-1)^k \left\{ {n+5 \atop k+3} \right\}_3 \left(\langle -\alpha \rangle_{k+3} - \langle -\beta \rangle_{k+3} \right)$$

and

$$L_n = \frac{1}{\sqrt{5}} \sum_{k=0}^{n+3} (-1)^k \left\{ {n+7 \atop k+4} \right\}_4 \left(\langle -\alpha \rangle_{k+4} - \langle -\beta \rangle_{k+4} \right).$$

4. Generating function

Theorem 3. *Suppose that the initial sequence $a_{0,m+r}$ has the following exponential generating function $A_r(z) = \sum_{k\geq 0} a_{0,k+r} \frac{z^k}{k!}$. Then the sequence $\{a_{n,r}\}_n$ of the rth columns of the matrix $\mathcal{S}$ has an exponential generating function $\mathcal{B}_r(z) = \sum_{n\geq 0} a_{n,r} \frac{z^n}{n!}$ given by*

$$B_r(z) = e^{rz} A_r\left(e^z - 1\right) \tag{4.1}$$

Proof. We have

$$\begin{aligned} B_r(z) &= \sum_{k\geq 0} a_{0,r+k} \sum_{n\geq 0} \left\{ {n+r \atop k+r} \right\}_r \frac{z^n}{n!} \\ &= \sum_{k\geq 0} a_{0,r+k} \frac{1}{k!} e^{rz} \left(e^z-1\right)^k \\ &= e^{rz} \sum_{k\geq 0} a_{0,r+k} \frac{(e^z-1)^k}{k!} \\ &= e^{rz} A_r\left(e^z-1\right). \end{aligned}$$

□

Theorem 4. *Suppose that the final sequence $a_{n+r,0}$ has the following exponential generating function $\mathcal{B}_r(z) = \sum_{k\geq 0} a_{k+r,0} \frac{z^k}{k!}$. Then the sequence $\{a_{r,m}\}_m$ of the rth rows of the matrix $\mathcal{S}$ has an exponential generating function $\mathcal{A}_r(z) = \sum_{m\geq 0} a_{r,m} \frac{z^m}{m!}$ given by*

$$\mathcal{A}_r(z) = \mathcal{B}_r(\ln(1+z)). \tag{4.2}$$

Proof. We have

$$\begin{aligned} \mathcal{A}_r(z) &= \sum_{k\geq 0} a_{r+k,0} \sum_{m\geq 0} s(m,k) \frac{z^m}{m!} \\ &= \sum_{k\geq 0} a_{r+k,0} \frac{(\ln(1+z))^k}{k!} \\ &= \mathcal{B}_r(\ln(1+z)). \end{aligned}$$

□

Example 3. A derangement on a set $\{1,2,\ldots,m\}$ is a permutation $\pi = i_1 i_2 \cdots i_m$ such that $i_k \neq k$ for $k = 1,2,\ldots m$. The number of derangements on $\{1,2,\ldots,m\}$ is denoted by D_m and given by $D_m = \left[\frac{m!}{e}\right]$, where $[x]$ the nearest integer function.

Now, if the initial sequence $a_{0,m} = (-1)^m D_m$, then we get the following matrix

$$\mathcal{S} = \begin{pmatrix} 1 & 0 & 1 & -2 & 9 & -44 & \cdots \\ 0 & 1 & 0 & 3 & -8 & 45 & \\ 1 & 1 & 3 & 1 & 13 & -39 & \\ 1 & 4 & 7 & 16 & 13 & 76 & \\ 4 & 11 & 30 & 61 & 128 & 159 & \\ 11 & 41 & 121 & 311 & 671 & 1381 & \\ \vdots & & & & & & \end{pmatrix}$$

The generating function of the sequence $a_{0,m}$ is $A_0(z) = \frac{e^z}{1+z}$. It follows from (4.1) that $B_0(z) = exp(e^z - z - 1)$, and we notice that $a_{n,0}$ is the number v_n of partitions of $\{1,2,\ldots,n\}$ without singletons (see for instance [8]). By (3.3), one can then deduce that

Corollary 4. *For $n,m \in \mathbb{N}_0$, we have*

$$\sum_{k=0}^{m} s(m,k)\, v_{n+k} = \sum_{k=0}^{n} (-1)^{m+k} \left\{ {n+m \atop k+m} \right\}_m D_{m+k}.$$

If $m = 0$, we get the well-known identity (cf. [9], p.1569)

$$v_n = \sum_{k=0}^{n} (-1)^k \left\{ {n \atop k} \right\} D_k.$$

Example 4. The exponential generating function of the Bernoulli polynomials $B_n(x)$ is

$$\mathcal{B}_0(z) := \frac{ze^{xz}}{e^z - 1} = \sum_{n\geq 0} B_n(x) \frac{z^n}{n!}.$$

By Theorem 4, we have

$$\mathcal{A}_0(z) = \frac{(1+z)^x \ln(1+z)}{z}.$$

It is not difficult to show that

$$[z^m]\, \mathcal{A}_0(z) = \sum_{i=0}^{m} (-1)^{m-i} \frac{(x)_i}{m-i+1},$$

where $[z^n]\, f(z)$ denote the operation of extracting the coefficient of z^n in the formal power series $f(z) = \sum f_n z^n$. Now, let us consider $\mathcal{S}$ defined by (1.4) with the final sequence $a_{n,0} = B_n(x)$, by (3.3), we obtain the following corollary

Corollary 5. *For $n,m\in\mathbb{N}_0$, we have*

$$\sum_{k=0}^{m}s\left(m,k\right)B_{n+k}\left(x\right)=\sum_{k=0}^{n}\left\{ n+m\atop k+m\right\}\sum_{i=0}^{m+k}\left(-1\right)^{m+k-i}\frac{\left(x\right)_{i}}{m+k-i+1}. \tag{4.3}$$

Notice that the Todorov theorem [10, Eq. 49] is obtained by setting $n=0$ in (4.3).

Example 5. Catalan and Motzkin numbers naturally appear in a large number of combinatorial objects. It is well known that the Catalan number $C_n=\frac{1}{n+1}\binom{2n}{n}$ and Motzkin number $M_n=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{1}{k+1}\binom{n}{2k}\binom{2k}{k}$ are connected by [1]

$$C_{n+1}=\sum_{k=0}^{n}\binom{n}{k}M_k\Longleftrightarrow M_n=\sum_{k=0}^{n}\left(-1\right)^{n-k}\binom{n}{k}C_{k+1}.$$

Using the generalized Stirling transform, we can show that the Catalan numbers are related to the Motzkin numbers in terms of Stirling numbers by

$$\sum_{k=0}^{n}s\left(n,k\right)M_k=\sum_{k=0}^{n+1}s\left(n+1,k\right)C_k, \tag{4.4}$$

and

$$C_n=\delta_{n,0}+\sum_{k=1}^{n}\sum_{i=0}^{k-1}\left\{ n\atop k\right\}s\left(k-1,i\right)M_i\Longleftrightarrow M_n=\sum_{k=0}^{n}\sum_{i=0}^{k+1}\left\{ n\atop k\right\}s\left(k+1,i\right)C_i. \tag{4.5}$$

Setting the final sequence $a_{n,0}=C_n$, we get the following matrix

$$\mathcal{S}=\begin{pmatrix}1 & 1 & 1 & 1 & 0 & 1 & -5 & 29 & \cdots\\ 1 & 2 & 3 & 3 & 1 & 0 & -1 & 7 & \\ 2 & 5 & 9 & 10 & 4 & -1 & 1 & -1 & \\ 5 & 14 & 28 & 34 & 15 & -4 & 5 & -11 & \\ 14 & 42 & 90 & 117 & 56 & -15 & 19 & -42 & \\ 42 & 132 & 297 & 407 & 209 & -56 & 72 & -160 & \\ 132 & 429 & 1001 & 1430 & 780 & -208 & 272 & -614 & \\ \vdots & & & & & & & & \end{pmatrix}.$$

Since

$$\mathcal{B}_0\left(z\right)=\sum_{n\geq0}C_n\frac{z^n}{n!}=\ {}_1F_1\left(\begin{matrix}1/2\\2\end{matrix};4z\right),$$

where ${}_1F_1\left(\begin{matrix} p \\ q \end{matrix};z\right)$ denotes the Kummer confluent hypergeometric function defined by

$$\sum_{n\geq 0}\frac{\langle p\rangle_n}{\langle q\rangle_n}\frac{z^n}{n!}.$$

It follows from (4.2) that

$$\mathcal{A}_0(z)=\sum_{n\geq 0}R_n\frac{z^n}{n!}={}_1F_1\left(\begin{matrix} 1/2 \\ 2 \end{matrix};4\ln(1+z)\right)$$

and

$$\sum_{k=0}^{m}s(m,k)\,C_{n+k}=\sum_{k=0}^{n}\left\{\begin{matrix} n+m \\ k+m \end{matrix}\right\}_m R_{m+k}. \tag{4.6}$$

Now, if the initial sequence $a_{0,m}=R_{m+1}$, we get the following matrix

$$\mathcal{T}=\begin{pmatrix} 1 & 1 & 1 & 0 & 1 & -5 & 29 & -196 & \cdots \\ 1 & 2 & 2 & 1 & -1 & 4 & -22 & 146 & \\ 2 & 4 & 5 & 2 & 0 & -2 & 14 & -100 & \\ 4 & 9 & 12 & 6 & -2 & 4 & -16 & 93 & \\ 9 & 21 & 30 & 16 & -4 & 4 & -3 & -26 & \\ 21 & 51 & 76 & 44 & -12 & 17 & -44 & 172 & \\ 51 & 127 & 196 & 120 & -31 & 41 & -92 & 282 & \\ \vdots & & & & & & & & \end{pmatrix}.$$

From this matrix we observe that $a_{n,0}=M_n$. We prove this observation using generating functions. We have

$$\begin{aligned} A_0(z)&=\sum_{n\geq 0}R_{n+1}\frac{z^n}{n!} \\ &=\frac{d}{dz}\left(\sum_{n\geq 0}R_n\frac{z^n}{n!}\right) \\ &=\frac{1}{1+z}\,{}_1F_1\left(\begin{matrix} 3/2 \\ 3 \end{matrix};4\ln(1+z)\right). \end{aligned}$$

From (4.1), we get

$$\begin{aligned} B_0(z)&={}_1F_1\left(\begin{matrix} 3/2 \\ 3 \end{matrix};4z\right)e^{-z} \\ &=\frac{d}{dz}\left(\sum_{n\geq 0}C_n\frac{z^n}{n!}\right)\sum_{n\geq 0}(-1)^n\frac{z^n}{n!} \end{aligned}$$

$$= \sum_{n\geq 0}\left(\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}C_{k+1}\right)\frac{z^n}{n!}$$
$$= \sum_{n\geq 0} M_n \frac{z^n}{n!}.$$

It follows

$$\sum_{k=0}^{m} s(m,k)\, M_{n+k} = \sum_{k=0}^{n}\left\{\begin{matrix} n+m \\ k+m \end{matrix}\right\}_m R_{m+k+1}. \tag{4.7}$$

Combining results (4.6) and (4.7) gives (4.4) and (4.5).

5. Hankel transform

The Hankel transform of a sequence α_n is the sequence of Hankel determinants $\det\left(\alpha_{i+j}\right)_{0\leq i,j\leq n}$. A number of methods for computing the Hankel determinants have been widely investigated [4, 5]. It is well known that the Hankel transform of sequences α_n and β_n are equal under the binomial transform [6]

$$\beta_n = \sum_{k=0}^{n}\binom{n}{k}\alpha_k.$$

A natural question arises: "What about the Hankel transform of the sequences a_n and b_n under the Stirling transform?" In this section we show that there is a connection between the generalized Stirling transform and the Hankel determinants.

Theorem 5. *For $n \in \mathbb{N}_0$, we consider a matrix $(a_{i,j})_{0\leq i,j\leq n}$ of order n arising from (1.3) with initial sequence $a_{0,j}$ and the final sequence $b_i := a_{i,0}$, then*

$$\det\left(a_{i,j}\right)_{0\leq i,j\leq n} = \det\left(b_{i+j}\right)_{0\leq i,j\leq n},$$

where $\det\left(b_{i+j}\right)_{0\leq i,j\leq n}$ is the Hankel transform of the sequence (b_i).

Proof. We can write

$$\det\left(b_{i+j}\right)_{0\leq i,j\leq n} = \begin{vmatrix} a_{0,0} & a_{1,0} & a_{2,0} & a_{3,0} & \cdots & a_{n,0} \\ a_{1,0} & a_{2,0} & a_{3,0} & a_{4,0} & \cdots & a_{n+1,0} \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_{n,0} & a_{n+1,0} & a_{n+2,0} & a_{n+3,0} & \cdots & a_{2n,0} \end{vmatrix},$$

after applying (1.4), the determinant is unchanged

$$\det\left(b_{i+j}\right)_{0\leq i,j\leq n} =$$

$$\begin{vmatrix} a_{0,0} & a_{1,0} & a_{2,0}-a_{1,0} & a_{3,0}-a_{2,0}-2\left(a_{2,0}-a_{1,0}\right) & \cdots \\ a_{1,0} & a_{2,0} & a_{3,0}-a_{2,0} & a_{4,0}-a_{3,0}-2\left(a_{3,0}-a_{2,0}\right) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ a_{n,0} & a_{n+1,0} & a_{n+2,0}-a_{n+1,0} & a_{n+3,0}-a_{n+1,0}-2\left(a_{n+2,0}-a_{n+1,0}\right) & \cdots \end{vmatrix}.$$

Using (3.2), we get

$$\det\left(b_{i+j}\right)_{0\le i,j\le n} = \begin{vmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} & \cdots & a_{0,n} \\ a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots & & \ddots & \\ a_{n,0} & a_{n,1} & a_{n,2} & a_{n,3} & & a_{n,n} \end{vmatrix},$$

from which the relation follows. □

The answer to the previous question is given in the following

Corollary 6. *For $n \in \mathbb{N}_0$, we have*

$$\det\left(b_{i+j}\right)_{0\le i,j\le n} = \det\left(\sum_{k=0}^{i} \left\{ {i+j \atop k+j} \right\} a_{k+j}\right)_{0\le i,j\le n}.$$

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Author's address

Mourad Rahmani
USTHB, Faculty of Mathematics, P.O. Box 32, El Alia,16111, Algiers, Algeria
E-mail address: mrahmani@usthb.dz

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EXISTENCE SOLUTION OF NEUTRAL FRACTIONAL DIFFERENTIAL INCLUSIONS WITH FRACTIONAL OPERATOR

M. H. M. RASHID AND A. AL-OMARI

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Abstract. This paper is mainly concerned with existence of mild solutions for a first order impulsive neutral fractional integrodifferential inclusions with nonlocal initial conditions in α-norm. We assume that the linear part generates an analytic semigroup and transform it into an integral inclusion. By using a fixed point theorem for multi-valued maps due to Dhage, a main existence theorem is established. Finally, we present an example to illustrate this main theorem.

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1. INTRODUCTION

In recent years, impulsive differential and partial differential equations have become more important in some mathematical models of real phenomena, especially in control, biological, and medical domains. In these models, the investigated simulating processes and phenomena usually have short-time perturbations during their evolution. The perturbations are performed discretely and their duration is negligible in comparison with the total duration of the processes. That is why the perturbations are considered to take place instantaneously in the form of impulses. The theory of impulsive differential equations has seen considerable development; see the monographs of Bainov and Simeonov [1], Lakshmikantham et al. [12], and Samoilenko and Perestyuk [17], where numerous properties of their solutions are studied and detailed bibliographies are given.

The starting point of this paper is the work in paper [9], where the authors have investigated the existence of solutions for the neutral partial differential equations in α-norm

$$\frac{d}{dt}[u(t)-F(t,u(h_1(t)))] = -A[u(t)-F(t,u(h_1(t)))]+G(t,u(h_2(t))), \quad t \in J$$
$$u(0)+g(u)=u_0.$$

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Furthermore, in [5], authors studied the following impulsive neutral partial differential equations in α-norm

$$\frac{d}{dt}[u(t)-F(t,u(h_1(t)))] \in Au(t)+\int_0^t K(t,s)G(s,u(h_2(s)))\,ds,$$
$$t \in J \setminus \{t_1,t_2,\cdots,t_m\}, \quad \text{where } J=[0,a],$$
$$\Delta u|_{t_k} = I_k(u(t_k^-)), \quad k=1,2,\cdots,m.$$
$$u(0)=g(u)\in X.$$

Motivated by the above-mentioned works, in this paper we study the existence of mild solutions for the following system in a general Banach space X with α-norm:

$${}^cD^q_{a+}[u(t)-F(t,u(h_1(t)))] \in -Au(t)+\int_0^t K(t,s)G(s,u(h_2(s)))\,ds,$$
$$t \in J \setminus \{t_1,t_2,\cdots,t_m\}, \quad \text{where } J=[0,a], 0<q\le 1, \tag{1.1}$$
$$\Delta u|_{t_k} = I_k(u(t_k^-)), \quad k=1,2,\cdots,m.$$
$$u(0)+g(u)=u_0\in X.$$

Here A is the infinitesimal generator of a compact, analytic semigroup $\{T(t),t>0\}$, G is a multi-valued map and $\Delta u|_{t=t_k}=u(t_k^+)-u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$represent the right and left limits of $u(t)$ at $t=t_k$. Let $K:D\to\mathbb{R}$, $D=\{(t,s)\in J\times J : 0\le s\le t\}$ and $F,G,g,I_k(k=1,2,\cdots,m)$ and h_1,h_2 are given functions to be specified later.

The nonlocal Cauchy problem was considered by Byszewski [2] and the importance of nonlocal conditions in different fields has been discussed in [2,7] and the references therein. For example, in [7] the author described the diffusion phenomenon of a small amount of gas in a transparent tube by using the formula

$$g(u)=\sum_{j=0}^{p} c_j u(t_j),$$

where $c_j, j=0,1,\cdots,p$ are given constants and $0<t_0<t_1<\cdots<t_p<a$. In this case the above equation allows the additional measurement at $t_j, j=0,1,\cdots,p$. In the past several years theorems about existence, uniqueness and stability of differential and functional differential abstract evolution Cauchy problem with nonlocal conditions have been studied by Byszewski and Lakshmikantham [4], by Byszewski [2,3], by Fu [10], by Fu and Ezzinbi [11] and recently, Cheng et al. [5] studied the existence results for impulsive neutral integrodifferential equations with nonlocal conditions.

2. PRELIMINARIES

In this section, we introduce some definitions, notations and preliminary facts from multi-valued analysis which are used throughout this paper.

Let $(X, \|.\|)$ be a Banach space. $\mathcal{C}(J, X)$ is the Banach space of continuous functions from J to X with the norm $\|u\|_J = \sup\{u(t) : t \in J\}$. $B(X)$ denotes the Banach space of bounded linear operators from X to X, with norm $\|N\|_{B(X)} = \sup\{\|N(x)\| : \|x\| = 1\}$. A measurable function $v : J \to X$ is Bochner integrable if and only if $\|v\|$ is Lebesgue integrable (for properties of the Bochner integral see Yosida [18]). $L^1(J, X)$ denotes the Banach space of Bochner integrable functions $v : J \to X$ with norm $\|v\|_{L^1} = \int_0^a \|v(t)\|\, dt$ for all $x \in L^1(J, X)$.

We use the notations $\mathcal{P}(X)$ for the family of all subsets of X. Let $P_{b,cl}(X)$ and $P_{cp,c}(X)$ denote respectively the classes of all bounded-closed and compact-convex subsets of X. Similarly, $P_{bcc}(X)$ denotes the classes of all bounded, closed and convex subsets of X. A multi-valued map $G : X \to P(X)$ is convex (closed) for all $x \in X$. G is said to be bounded on bounded sets if $G(B) = \bigcup_{x \in B} G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e.,

$\sup_{x \in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$). G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open subset $\mathcal{U}$ of X containing $G(x_0)$, there exists an open neighborhood Γ of x_0 such that $G(\Gamma) \subset \mathcal{U}$.

G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*, y_n \to y_*, y_n \in G(x_n)$ imply $y_* \in G(x_*)$. We say that G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

A multi-valued map $G : J \to P_{cl}(X)$ is said to be measurable if for each $x \in X$ the function $Y : J \to \mathbb{R}$ defined by $Y(t) = d(x, G(t)) = \inf\{\|x - z\| : z \in G(t)\}$ is measurable. For more details on multi-valued maps we refer to the book by Deimling [6].

Let $-A$ be the infinitesimal generator of an analytic semigroup $\{T(t), t \geq 0\}$ in X. We note that if $-A$ is the infinitesimal generator of an analytic semigroup then $-(A + \alpha I)$ is invertible and generates a bounded analytic semigroup for $\alpha > 0$ large enough. This allows us to reduce the general case in which $-A$ is the infinitesimal generator of an analytic semigroup to the case in which the semigroup is bounded and the generator is invertible. Hence for convenience, we suppose that $\|T(t)\| \leq M$ for $t \geq 0$.

We assume without loss of generality that $0 \in \rho(-A)$ where $\rho(-A)$ is the resolvent set of $-A$. This allows us to define the fractional power A^α for $0 < \alpha < 1$, as a closed linear operator on its domain $D(A^\alpha)$ with inverse $A^{-\alpha}$. We have the following basic properties A^α.

Theorem 1 ([16])**.**

(1) $X_\alpha = D(A^\alpha)$ *is a Banach space with norm* $\|x\|_\alpha = \|A^\alpha x\|$ *for* $x \in D(A^\alpha)$.
(2) $T(t) : X \to X_\alpha$ *for each* $x > 0$.
(3) $A^\alpha T(t)x = T(t)A^\alpha X$ *for each* $x \in D(A^\alpha)$ *and* $t > 0$.
(4) *For every* $t > 0$, $A^\alpha T(t)$ *is bounded on* X *and there exist* $C_\alpha > 0$ *and* $\delta > 0$ *such that*
$$\|A^\alpha T(t)\| \le C_\alpha t^{-\alpha} e^{-\delta t} \le C_\alpha t^{-\alpha}.$$
(5) $A^{-\alpha}$ *is a bounded linear operator in* X *with* $D(A^\alpha) = im(A^{-\alpha})$.
(6) *If* $0 < \alpha \le \beta$, *then* $D(A^\beta) \hookrightarrow D(A^\alpha)$.

Remark 1. Observe as in [14] that by Theorem 1 (2) and (3), the restriction $T_\alpha(t)$ of $T(t)$ to X_α is exactly the part of $T(t)$ in X_α. Let $x \in X_\alpha$. Since
$$\|T(t)x\|_\alpha = \|A^\alpha T(t)x\| = \|T(t)A^\alpha x\| \le \|T(t)\|\,\|A^\alpha x\| = \|T(t)\|\,\|x\|_\alpha$$
and as t decreases to 0
$$\|T(t)x - x\|_\alpha = \|A^\alpha T(t)x - A^\alpha x\| = \|T(t)A^\alpha x - A^\alpha x\| \to 0$$
for all $x \in X_\alpha$ it follows that $(T(t)_{t\ge 0})$ is a family of strongly continuous semigroup on X_α and $\|T_\alpha(t)\| \le \|T(t)\|$ for all $t \ge 0$.

To begin with the analysis we need some basic definitions and properties from the fractional calculus theory.

Definition 1. A real function $f(x), x \ge 0$ is said to be space C_μ, $\mu \in \mathbb{R}$ if there exist a real number $p(> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1 \in C[0,\infty)$ and it is said to be in the space C_μ^m if and only if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

Definition 2. The Riemann-Liouville fractional integral operator of order $0 \le \alpha$, of a function $f \in C_\mu$, $\mu \ge -1$ is defined as
$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0, x > 0,$$
$$I^0 f(x) = f(x).$$

While modeling some real world phenomena with fractional differential equation the Riemann-Liouville derivative has certain disadvantage. Therefore, we shell introduce a modified fractional differential operator D_*^α proposed by M. Caputo.

Definition 3. The fractional derivative of $f(x)$ in the Caputo sense is defined as
$$DI_x^\alpha f(x) = I^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} D^m f(t)dt,$$
$$\text{for} \quad m-1 \le \alpha < m, m \in \mathbb{N}, x > 0, f \in C_{-1}^m.$$

In order to define the solution of system (1.1) we shall consider the space

$$\Omega = \{u : [0,a] \to X_\alpha : u \in \mathcal{C}((t_{k-1}, t_k], X), k = 1,2,\cdots,m \quad \text{and there exist}$$
$$u(t_k^-) \quad \text{and} \quad u(t_k^+), k = 1,2,\cdots,m$$
$$\text{with} \quad u(t_k^-) = u(t_k), u(0) + g(u) = u_0\}.$$

which is a Banach space with the norm

$$\|u\|_\Omega = \sup_{t\in J} \|u(t)\|_\alpha .$$

Now we define the mild solution for the system (1.1).

Definition 4. A continuous function u, such that

(i) $u(0) + g(u) = u_0$ for each $0 \le t \le a$;
(ii) $\Delta u|_{t=t_k} = I_k(u(t_k^-)), k = 1,2,\cdots,m$, the restriction of $u(.)$ to the interval $[0,a) \setminus \{t_1, t_2, \cdots, t_k\}$ is continuous.

satisfying the integro-differential equation

$$\begin{aligned} u(t) = {} & T(t)\,[u_0 - g(u) - F(0, u(h_1(0)))] + F(t, u(h_1(t))) \\ & + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} T(t-s) F(s, u(h_1(s)))\, ds \\ & + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} T(t-s) \int_0^s K(s,w) v(w)\, dw ds \\ & + \sum_{k=1}^{m} T(t-t_k) I_k(u(t_k^-)), \qquad t \in J,\ v(t) \in G(t, u(h_2(t))) \end{aligned}$$

is called a mild solution of problem (1.1).

Our main results are based on the following lemmas.

Lemma 1 ([13]). *Let X be a Banach space. Let $G : J \times X \to P_{b,cl,c}(X)$ satisfied that*

(i) *For each $u \in X$, $(t,u) \to G(t,u)$ is measurable with respect to t;*
(ii) *For each $t \in J$, $(t,u) \to G(t,u)$ is u.s.c. with respect to u;*
(iii) *For each fixed $u \in \mathcal{C}(J,X)$, the set*

$$S_{G,u} = \left\{v \in L^1(J,X) : v(t) \in G(t, u(h_2(t))) \quad \text{for a.e } t \in J\right\}$$

is nonempty. Let Γ be a linear continuous mapping from $L^1(J,X)$ to $\mathcal{C}(J,X)$. Then the operator

$$\Gamma \circ S_G : \mathcal{C}(J,X) \to P_{cp,c}(\mathcal{C}(J,X)), \qquad u \to (\Gamma \circ S_G)(u) := \Gamma(S_{G,u})$$

is a closed graph operator in $\mathcal{C}(J,X) \times \mathcal{C}(J,X)$.

Lemma 2 ([8]). *Let $B(0,r)$ and $B[0,r]$ denote respectively the open and closed balls in a Banach space X centered at the origin and of radius r and let $\mathcal{A} : X \to P_{bcc}(X)$ and $\mathcal{B} : B[0,r] \to P_{cp,c}(X)$ be two multi-valued operators satisfying:*

(a) *$\mathcal{A}$ is a multi-valued contraction, and*

(b) *$\mathcal{B}$ is upper semicontinuous and completely continuous. Then either*

(i) *the operator inclusion $u \in \mathcal{A}u + \mathcal{B}u$ has a solution in $B[0,r]$, or*

(ii) *there exists a $w \in X$ with $\|w\| = r$ such that $\delta w \in \mathcal{A}w + \mathcal{B}w$ for some $\delta > 1$.*

We have the following result from [14].

Lemma 3. *$T(t)_{t\geq 0}$ is an immediately compact semigroup in X_α, and hence it is immediately norm-continuous.*

3. Main result

In this section, we state and prove the existence theorem for the system (1.1). Let us list the following hypothesis: for some $\alpha \in (0,1)$,

(H1) There exists a constant $\beta \in (0,1)$ with $\alpha \leq \beta \leq 1$ such that $F : [0,a] \times X_\alpha \to X_\beta$ is a continuous function, and $A^\beta F : [0,a] \times X_\alpha \to X_\alpha$ satisfies the Lipschitz condition, that is, there exists a constant $L > 0$ such that

$$\left\| A^\beta F(t_1,u_1) - A^\beta F(t_2,u_2) \right\| \leq L\left(|t_1 - t_2| + \|u_1 - u_2\|_\alpha\right),$$

for any $(t_1,u_1),(t_2,u_2) \in [0,a] \times X_\alpha$. Moreover, there exists a constant $L_1 > 0$ such that the inequality

$$\left\| A^\beta F(t,u) \right\|_\alpha \leq L_1(\|u\|_\Omega + 1)$$

holds for any $u \in X_\alpha$ with

$$L_0 = \left[(M+1)L + \frac{LM_{\alpha-\beta}a^{q+\alpha-\beta}}{\Gamma(q)(q+\alpha-\beta)} \right] < 1. \tag{3.1}$$

(H2) The multi-valued map $G : J \times X_\alpha \to P_{c,cp}(X)$ satisfies the following conditions:

(i) For each $t \in J$, the function $G(t,.) : X_\alpha \to P_{c,cp}(X)$ is u.s.c.; and for each $x \in X_\alpha$, the function $G(.,u)$ is measurable. And for each fixed $u \in \Omega$ the set

$$S_{G,u} = \left\{ v \in L^1(J,X) : v(t) \in G(t,u) \quad \text{for a.e } t \in J \right\}$$

is nonempty.

(ii) For each positive number $l > 0$, there exists a positive function $w(l)$ dependent on l such that

$$\sup_{\|u\|_\alpha \leq l} \|G(t,u)\| \leq w(l),$$

where

$$\|G(t,u)\| = \sup\{\|v\| : v \in G(t,u)\}, \quad \|u\|_\alpha = \sup_{0\le s\le a} \|u(s)\|_\alpha .$$

(H3) $h_i \in \mathcal{C}(J,J), i = 1,2$. $g : \Omega \to X_\alpha$ satisfies that

(i) There exist positive constants L_2 and L_3 such that

$$\|g(u)\|_\alpha < L_2 \|u\|_\Omega + L_3 \qquad \text{for all } u \in \Omega.$$

(ii) g is a completely continuous map.

(H4) For each $t \in J$, $K(t,s)$ is measurable on J and

$$K(t) = ess\sup\{|K(t,s)|, 0 \le s \le t\}$$

is bounded on J. The map $t \to K_t$ is continuous from J to $L^\infty(J,\mathbb{R})$, here $K_t(s) = K(t,s)$ and $K = \sup_{t\in J} K(t)$.

(H5) $I_k \in \mathcal{C}(X_\alpha, X_\alpha), k = 1,2,\cdots,m$, and there exist nondecreasing functions $\psi_k \in \mathcal{C}(J,\mathbb{R}_+), k = 1,2,\cdots,m$ such that

$$\|I_k(u)\|_\alpha \le \psi_k(\|u\|_\alpha) \le N \|u\|_\Omega, \quad \text{for } u \in X_\alpha, N \in \mathbb{R}_+$$

(H6) There exists a real number $r > 0$ such that

$$r \le \frac{2 + M(\|u_0\|_\alpha + L_3) + \dfrac{M_\alpha a^{q-\alpha}(1 + aw(r)K)}{(q-\alpha)\Gamma(q)}}{1 - \left[M(L_2 + mN) + 2L_1 + \dfrac{M_\alpha L_1 a^{q-\alpha}}{(q-\alpha)\Gamma(q)}\right]}, \tag{3.2}$$

where

$$\left[M(L_2 + mN) + 2L_1 + \frac{M_\alpha L_1 a^{q-\alpha}}{(q-\alpha)\Gamma(q)}\right] < 1. \tag{3.3}$$

Theorem 2. *Let $u_0 \in X_\alpha$. If the hypotheses $(H1)-(H6)$ are satisfied, then the system (1.1) admits at least one mild solution on J.*

Proof. Consider the operator $N : \Omega \to \mathcal{P}(\Omega)$ defined by

$$\begin{aligned} N(u) = \{w \in \Omega : w(t) &= T(t)[u_0 - g(u) - F(0,u(h_1(0)))] + F(t,u(h_1(t))) \\ &+ \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} T(t-s) F(s,u(h_1(s)))\,ds \\ &+ \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} T(t-s) \int_0^s K(s,w(s)v(w)\,dw\,ds \\ &+ \sum_{k=1}^m T(t-t_k) I_k(u(t_k^-)), \quad t \in J, v(t) \in G(t,u(h_2(t)))\}. \end{aligned}$$

Clearly the fixed point of N are solutions of the system (1.1). Let

$$\mathcal{A}(u) = F(t,u(h_1(t))) - T(t)F(0,u(h_1(0)))$$

$$+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}T(t-s)F(s,u(h_1(s)))\,ds$$

and

$$\begin{aligned}\mathcal{B}(u)=\{w\in\Omega : w(t)&=T(t)[u_0-g(u)]\\&+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}T(t-s)\int_0^s K(s,w(s)v(w)\,dw\,ds\\&+\sum_{k=1}^{m}T(t-t_k)I_k(u(t_k^-)),\ v\in S_{G,u}\}.\end{aligned}$$

In order to apply Lemma 2, we give the proof in several steps.

Step 1. $\mathcal{A}$ is a contraction.

Take $x, y \in B_l$. Then for each $t \in J$ and by condition $(H1)$, we have that

$$\begin{aligned}&\|\mathcal{A}(x)(t)-\mathcal{A}(y)(t)\|_\alpha\\&\le \|F(t,x(h_1(t)))-F(t,y(h_1(t)))\|_\alpha+\|T(t)[F(0,x(h_1(0)))-F(0,y(h_1(0)))]\|_\alpha\\&+\left\|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}T(t-s)F(t,x(h_1(t)))-F(t,y(h_1(t)))\right\|_\alpha\\&=\|A^\alpha[F(t,x(h_1(t)))-F(t,y(h_1(t)))]\|\\&+\|A^\alpha T(t)[F(0,x(h_1(0)))-F(0,y(h_1(0)))]\|\\&+\left\|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}A^{\alpha-\beta}T(t-s)A^\beta[F(t,x(h_1(t)))-F(t,y(h_1(t)))]\,ds\right\|\\&\le\left[(M+1)L+\frac{LM_{\alpha-\beta}a^{q+\alpha-\beta}}{\Gamma(q)(q+\alpha-\beta)}\right]\sup_{s\in J}\|x(s)-y(s)\|_\alpha\\&=L_0\sup_{s\in J}\|x(s)-y(s)\|_\alpha.\end{aligned}$$

Thus,

$$\|\mathcal{A}(x)-\mathcal{A}(y)\|\le L_0\|x-y\|_\alpha.$$

Therefore, by assumption $0<L_0<1$, we can see that $\mathcal{A}$ is a contraction.

Step 2. $\mathcal{B}(u)$ is convex for each $u\in\Omega$.

Indeed, if $w_1, w_2 \in \mathcal{B}(x)$, then there exist $v_1, v_2 \in S_{G,x}$, such that for each $t \in J$ we have

$$\begin{aligned}w_i&=T(t)[u_0-g(u)]+\sum_{k=1}^{m}T(t-t_k)I_k(u(t_k^-))\\&+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}T(t-s)\int_0^s K(s,w)v_i(w)\,dw\,ds,\qquad i=1,2.\end{aligned}$$

Let $0 \leq \lambda \leq 1$. Then for each $t \in J$ we have

$$\lambda w_1 + (1-\lambda)w_2 = T(t)[u_0 - g(u)] + \sum_{k=1}^{m} T(t-t_k) I_k(u(t_k^-))$$
$$+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} T(t-s) \int_0^s K(s,w)[\lambda v_1(w) + (1-\lambda)v_2(w)]\, dw ds.$$

Since $S_{G,u}$ is convex (because G has convex values), then $\lambda w_1 + (1-\lambda)w_2 \in N(u)$.

Step 3. $\mathcal{B}$ is bounded on bounded sets of Ω.

Let $B_l = \{u \in \Omega : \|u\|_\alpha \leq l\}$ be a bounded set in Ω. Now for each $u \in B_l, x \in \mathcal{B}(u)$, there exists a function $v \in S_{G,u}$ such that for each $t \in J$. We have

$$\begin{aligned}
\|x(t)\|_\alpha &\leq \|T(t)(x_0 - g(x))\|_\alpha \\
&+ \left\| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} T(t-s) \int_0^s K(s,x)v(x)\, dw ds \right\|_\alpha \\
&+ \sum_{k=1}^{m} \left\| T(t-t_k) I_k(x(t_k^-)) \right\|_\alpha \\
&\leq \|T(t)\| \, \|(x_0 - g(x))\|_\alpha + \\
&\left\| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} A^\alpha T(t-s) \int_0^s K(s,x)v(x)\, dw ds \right\| \\
&+ \sum_{k=1}^{m} \left\| A^\alpha T(t-t_k) I_k(x(t_k^-)) \right\| \\
&\leq M(L_2 \|x\|_\Omega + L_3 + \|x_0\|_\alpha) \\
&+ \frac{a w(l) \sup_{t\in J} K(t)}{\Gamma(q)} \int_0^t \frac{M_\alpha (t-s)^{q-1}}{(t-s)^\alpha}\, ds + M \sum_{k=1}^{m} \psi_k(\|x\|_\alpha) \\
&\leq M(l L_2 + L_3 + \|x_0\|_\alpha) + \frac{w(l) \sup_{t\in J} K(t) M_\alpha}{(q-\alpha)\Gamma(q)} a^{q-\alpha+1} + M \sum_{k=1}^{m} \psi_k(l).
\end{aligned}$$

Hence $\mathcal{B}$ is bounded on bounded sets of Ω for each $x \in \mathcal{B}(B(l))$.

Step 4. $\mathcal{B}$ sends bounded sets into equicontinuous sets on Ω.

Let $\tau_1, \tau_2 \in J, \tau_1 < \tau_2$. Let $u \in B_l$ and $x \in \mathcal{B}(u)$, then there exists $v \in S_{G,u}$ such that, for each $t \in J$, we have

$$x(t) = T(t)(x_0 - g(x)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} T(t-s) \int_0^s K(s,x)v(x)\, dx ds$$

$$+\sum_{k=1}^{m} T(t-t_k)I_k(x(t_k^-)).$$

Then

$$\begin{aligned}
&\|x(\tau_2)-x(\tau_1)\|_\alpha \\
&\le \|[T(\tau_2)-T(\tau_1)](x_0-g(x))\|_\alpha \\
&+\left\| \frac{1}{\Gamma(q)}\int_0^{\tau_1} [(\tau_2-s)^{q-1}T(\tau_2-s)-(\tau_1-s)^{q-1}T(\tau_1-s)] \right. \\
&\qquad \left. \int_0^s K(s,x)v(x)\,dx\,ds \right\|_\alpha \\
&+\left\| \frac{1}{\Gamma(q)}\int_{\tau_1}^{\tau_2} (\tau_2-s)^{q-1}T(\tau_2-s)\int_0^s K(s,x)v(x)\,dx\,ds \right\|_\alpha \\
&+\left\| \sum_{0<t_k<\tau_1} [T(\tau_2-t_k)-T(\tau_1-t_k)]I_k(x(t_k^-)) \right\|_\alpha \\
&+\left\| \sum_{\tau_1<t<\tau_2} T(\tau_2-t_k)I_k(x(t_k^-)) \right\|_\alpha \\
&\le A_1+A_2+A_3+A_4+A_5+A_6,
\end{aligned}$$

Actually, A_1, A_2, A_3, A_4, A_5 and A_6 tend to 0 independently of B_l when $\tau_2-\tau_1 \to 0$. Indeed, let $x \in B_l$, we have

$$\begin{aligned}
A_1 &= \|[T(\tau_2)-T(\tau_1)](x_0-g(x))\|_\alpha \\
&\le \|T(\tau_2)-T(\tau_1)\|_\alpha \,\|(x_0-g(x))\|_\alpha \\
&\le \|T(\tau_2)-T(\tau_1)\|_\alpha \,(\|x_0\|_\alpha + L_2\|x\|_\Omega + L_3)
\end{aligned}$$

from which we deduce that $\lim_{\tau_2\to\tau_1} A_1 = 0$ since by Lemma 3 the function $t \to \|T(t)\|_\alpha$ for $t \in (0,a]$.

$$\begin{aligned}
A_2 &= \left\| \frac{1}{\Gamma(q)}\int_0^{\tau_1} [(\tau_2-s)^{q-1}[T(\tau_2-s)-T(\tau_1-s)]\int_0^s K(s,x)v(x)\,dx\,ds \right\|_\alpha \\
&\le \frac{a^{q-1}}{\Gamma(q)}\int_0^{\tau_1} \left\| [T(\tau_2-s)-T(\tau_1-s)]\int_0^s K(s,x)v(x) \right\|_\alpha dx\,ds \\
&\le \frac{a^{q-1}}{\Gamma(q)}\int_0^{\tau_1} \left\| [T(\frac{\tau_2-\tau_1}{2}+\frac{\tau_2-s}{2})-T(\frac{\tau_1-s}{2})]A^\alpha T(\frac{\tau_1-s}{2}) \right. \\
&\qquad \left. \int_0^s K(s,x)v(x) \right\| dx\,ds
\end{aligned}$$

$$\leq \frac{a^{q-\alpha-1} M_\alpha \sup_{t\in J} K(t) w(l)}{\Gamma(q)} \int_0^{\tau_1} [T(\frac{\tau_2-\tau_1}{2}+\frac{\tau_2-s}{2}) - T(\frac{\tau_1-s}{2})] ds.$$

Therefore, the continuity of the function $t \to \|T(t)\|_\alpha$ for $t \in (0,a]$ allows us to conclude that $\lim_{\tau\to\tau_1} A_2 = 0$.

$$\begin{aligned}
A_3 &= \left\| \frac{1}{\Gamma(q)} \int_0^{\tau_1} [(\tau_2-s)^{q-1} - (\tau_1-s)^{q-1}] T(\tau_1-s) \int_0^s K(s,x) v(x)\, dx ds \right\|_\alpha \\
&\leq \frac{1}{\Gamma(q)} \int_0^{\tau_1} [(\tau_2-s)^{q-1} - (\tau_1-s)^{q-1}] \left\| A^\alpha T(\tau_1-s) \int_0^s K(s,x) v(x)\, dx \right\| ds \\
&\leq \frac{M_\alpha a^{-\alpha}}{\Gamma(q)} \int_0^{\tau_1} [(\tau_2-s)^{q-1} - (\tau_1-s)^{q-1}] \left\| \int_0^s K(s,x) v(x)\, dx \right\| ds \\
&\leq \frac{M_\alpha a^{-\alpha} \sup_{t\in J} w(l)}{\Gamma(q)} \int_0^{\tau_1} [(\tau_2-s)^{q-1} - (\tau_1-s)^{q-1}] ds \\
&\leq \frac{M_\alpha a^{-\alpha} \sup_{t\in J} w(l)}{q\Gamma(q)} |\tau_2-\tau_1|^q.
\end{aligned}$$

Hence $\lim_{\tau_2\to\tau_1} A_3 = 0$.

$$\begin{aligned}
A_4 &= \left\| \frac{1}{\Gamma(q)} \int_{\tau_1}^{\tau_2} (\tau_2-s)^{q-1} T(\tau_2-s) \int_0^s K(s,x) v(x)\, dx ds \right\|_\alpha \\
&\leq \frac{a^{q-1} \sup_{t\in J} K(t) w(l)}{\Gamma(q)} \int_{\tau_1}^{\tau_2} \|A^\alpha T(\tau_2-s)\|\, ds \\
&\leq \frac{M_\alpha a^{q-1} \sup_{t\in J} K(t) w(l)}{\Gamma(q)} \int_{\tau_1}^{\tau_2} (\tau_2-s)^{-\alpha} ds \\
&\leq \frac{M_\alpha a^{q-1} \sup_{t\in J} K(t) w(l)}{(1-\alpha)\Gamma(q)} |\tau_2-\tau_1|^{1-\alpha}.
\end{aligned}$$

Since $1-\alpha > 0$, we deduce that $\lim_{\tau_2\to\tau_1} A_4 = 0$.

$$\begin{aligned}
A_5 &= \left\| \sum_{0<t_k<\tau_1} [T(\tau_2-t_k) - T(\tau_1-t_k)] I_k(x(t_k^-)) \right\|_\alpha \\
&\leq \sum_{0<t_k<\tau_1} \|[T(\tau_2-t_k) - T(\tau_1-t_k)]\|_\alpha \left\| I_k(x(t_k^-)) \right\|_\alpha \\
&\leq \sum_{0<t_k<\tau_1} \psi_k(\|x\|_\alpha) \|[T(\tau_2-t_k) - T(\tau_1-t_k)]\|_\alpha .
\end{aligned}$$

Hence, we deduce $\lim_{\tau_2\to\tau_1} A_5 = 0$.

$$A_6 = \left\| \sum_{\tau_1<t<\tau_2} T(\tau_2-t_k) I_k(x(t_k^-)) \right\|_\alpha$$

$$\leq \sum_{\tau_1<t<\tau_2} \psi_k(\|x\|_\alpha \|T(\tau_2-t_k)\|_\alpha .$$

Therefore, $\lim_{\tau_2\to\tau_1} A_6 = 0$.

Step 5. $(\mathcal{B}B_l)(t)$ is relatively compact for each $t \in J$, where $(\mathcal{B}B_l)(t) = \{x(t); x \in \mathcal{B}B_l, t \in J\}$. Obviously, by condition $(H3)(ii)$, $(\mathcal{B}B_l)(t)$ is relatively compact in Ω for $t = 0$. Let $0 < t \leq a$ be fixed and $0 < \epsilon < t$. For $u \in B_l$ and $x \in \mathcal{B}(u)$, there exists a function $v \in S_{G,u}$ such that

$$x(t) = T(t)(x_0 - g(x)) + \frac{1}{\Gamma(q)} \int_0^{t-\epsilon} (t-s)^{q-1} T(t-s) \int_0^s K(s,x)v(x)\,dx\,ds$$
$$+ \frac{1}{\Gamma(q)} \int_{t-\epsilon}^{t} (t-s)^{q-1} T(t-s) \int_0^s K(s,x)v(x)\,dx\,ds + \sum_{0<t_k<t} T(t-t_k) I_k(x(t_k^-)).$$

Define

$$x_\epsilon(t) = T(t)(x_0 - g(x)) + \frac{1}{\Gamma(q)} \int_0^{t-\epsilon} (t-s)^{q-1} T(t-s) \int_0^s K(s,x)v(x)\,dx\,ds$$
$$+ \sum_{0<t_k<t-\epsilon} T(t-t_k) I_k(x(t_k^-))$$
$$= T(t)(x_0 - g(x)) + \frac{T(\epsilon)}{\Gamma(q)} \int_0^{t-\epsilon} (t-s)^{q-1} T(t-\epsilon-s) \int_0^s K(s,x)v(x)\,dx\,ds$$
$$+ \sum_{0<t_k<t-\epsilon} T(t-t_k) I_k(x(t_k^-)).$$

Since $\{T(t), t > 0\}$ is compact, the set $U_\epsilon = \{x(t) : x \in \mathcal{B}(B_l)\}$ is relatively compact in Ω for every $\epsilon, 0 < \epsilon < t$. Moreover, for every $x \in \mathcal{B}(B_l)$,

$$\|x(t) - x_\epsilon(t)\|_\alpha = \frac{1}{\Gamma(q)} \left\| \int_{t-\epsilon}^{t} (t-s)^{q-1} T(t-s) \int_0^s K(s,x)v(x)\,dx\,ds \right\|_\alpha$$
$$+ M \sum_{t-\epsilon<t_k<t} \psi(l).$$

Therefore, letting $\epsilon \to 0$, we see that, there are relatively compact sets arbitrarily close to the set $\{x(t) : x \in \mathcal{B}(B_l)\}$. Hence the set $x(t); x \in \mathcal{B}(B_l)$ is relatively compact in Ω. As a consequence of steps $(3-5)$ together with the Arzela-Ascoli theorem, we can conclude that $\mathcal{B}$ is a compact multivalued map.

Step 6. $\mathcal{B}$ has a closed graph.

Let $u_n \to u_*, u_n \in B_l$, $\{x_n \in \mathcal{B}(u_n)\}$, and $x_n \to x_*$. We shall prove that $x_* \in \mathcal{B}(u_*)$. $x_n \in \mathcal{B}(u_n)$ means that there exists $v_n \in S_{G,u_n}$ such that, for

each $t \in J$,

$$x_n(t) = T(t)(u_0 - g(u_n)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} T(t-s) \int_0^s K(s,x) v_n(x)\, dx ds + \sum_{0<t_k<t} T(t-t_k) I_k(u_n(t_k^-)).$$

We must prove that there exists $v_* \in S_{G,u_*}$ such that, for each $t \in J$,

$$x_*(t) = T(t)(u_0 - g(u_*)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} T(t-s) \int_0^s K(s,x) v_*(x)\, dx ds + \sum_{0<t_k<t} T(t-t_k) I_k(u_*(t_k^-)).$$

Clearly since $I_k, k = 1,2,\cdots,m$ and g are continuous we have that

$$\| \left(x_n - T(t)(u_0 - g(u_n)) - \sum_{0<t_k<t} T(t-t_k) I_k(u_n(t_k^-)) \right) - \left(x_* - T(t)(u_0 - g(u_*)) - \sum_{0<t_k<t} T(t-t_k) I_k(u_*(t_k^-)) \right) \|_{\Omega} \quad \to \quad 0$$

as $n \to \infty$. Consider the linear continuous operator

$$\Phi : L^1(J,X) \to \mathcal{C}(J,X) \quad v \to \Phi(v)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} T(t-s) \int_0^s K(s,x) v_n(x)\, dx ds$$

From Lemma 1, it follows that $\Phi \circ S_{G,u}$ is a closed graph operator. Moreover we have that

$$x_n(t) - T(t)(u_0 - g(u_n)) - \sum_{0<t_k<t} T(t-t_k) I_k(u_n(t_k^-)) \in \Phi(S_{G,u_n}).$$

Since $u_n \to u_*$ it follows from Lemma 1 that

$$x_*(t) - T(t)(u_0 - g(u_*)) - \sum_{0<t_k<t} T(t-t_k) I_k(u_*(t_k^-)) \in \Phi(S_{G,u_*}),$$

that is, there must exist a $v_* \in S_{G,u_*}$ such that

$$x_*(t) - T(t)(u_0 - g(u_*)) - \sum_{0<t_k<t} T(t-t_k) I_k(u_*(t_k^-)) = \Phi((v_*)(t)) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} T(t-s) \int_0^s K(s,x) v_*(x)\, dx ds.$$

Therefore, $\mathcal{B}$ has a closed graph and so $\mathcal{B}$ is u.s.c.

Step 7. The operator inclusion $u \in \mathcal{A}(u) + \mathcal{B}(u)$ has a solution in $B[0,r]$.
Define an open ball $B(0,r)$ in Ω, where the real number r satisfies the inequality given in condition (H6). As a consequence of steps $(1-6)$, we can see that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all conditions of Lemma 2. Now we show that the second assertion of Lemma 2 is not true. Let $u \in \Omega$ be a possible solution for $\lambda x \in \mathcal{A}x + \mathcal{B}x$ for some $\lambda > 1$ with $\|x\|_\alpha = r$. Then we have,

$$\begin{aligned} x(t) &= \lambda^{-1} T(t)[u_0 - g(u) - F(0, u(h_1(0)))] + \lambda^{-1} F(t, u(h_1(t))) \\ &\quad + \frac{\lambda^{-1}}{\Gamma(q)} \int_0^t (t-s)^{q-1} T(t-s) F(s, x(h_1(s)))\, ds \\ &\quad + \frac{\lambda^{-1}}{\Gamma(q)} \int_0^t (t-s)^{q-1} T(t-s) \int_0^s K(s,x) v(x)\, dw ds \\ &\quad + \lambda^{-1} \sum_{k=1}^{m} T(t-t_k) I_k(u(t_k^-)). \end{aligned}$$

Thus by $(H1)-(H5)$

$$\begin{aligned} \|x(t)\|_\alpha &\le \lambda^{-1} \|T(t)[u_0 - g(u) - F(0, u(h_1(0)))]\|_\alpha + \lambda^{-1} \|F(t, u(h_1(t)))\|_\alpha \\ &\quad + \frac{\lambda^{-1}}{\Gamma(q)} \int_0^t \left\| (t-s)^{q-1} T(t-s) F(s, x(h_1(s)))\, ds \right\|_\alpha \\ &\quad + \frac{\lambda^{-1}}{\Gamma(q)} \int_0^t \left\| (t-s)^{q-1} T(t-s) \int_0^s K(s,x) v(x)\, dw ds \right\|_\alpha \\ &\quad + \lambda^{-1} \sum_{k=1}^{m} \left\| T(t-t_k) I_k(u(t_k^-)) \right\|_\alpha \\ &\le I_1 + I_2 + I_3 + I_4 + I_5 \end{aligned}$$

where,

$$\begin{aligned} I_1 &= \lambda^{-1} \|T(t)[u_0 - g(u) - F(0, u(h_1(0)))]\|_\alpha \\ &\le M \|u_0 - g(u)\|_\alpha + \|A^\alpha F(0, u(h_1(0)))\| \\ &\le M(\|u_0\|_\alpha + \|g(u)\|_\alpha) + (L_1 \|u\|_\Omega + 1) \\ &\le M(\|u_0\|_\alpha + L_2 \|u\|_\Omega + L_3) + (L_1 \|u\|_\Omega + 1). \end{aligned}$$

$$\begin{aligned} I_2 &= \lambda^{-1} \|F(t, u(h_1(t)))\|_\alpha \\ &\le \|A^\alpha F(t, u(h_1(t)))\| \\ &\le (L_1 \|u\|_\Omega + 1). \end{aligned}$$

$$
\begin{aligned}
I_3 &= \frac{\lambda^{-1}}{\Gamma(q)} \int_0^t \left\| (t-s)^{q-1} T(t-s) F(s, x(h_1(s)))\, ds \right\|_\alpha \\
&\leq \frac{1}{\Gamma(q)} \int_0^t \left\| (t-s)^{q-1} A^\alpha T(t-s) F(s, x(h_1(s)))\, ds \right\| \\
&\leq \frac{M_\alpha (L_1 \|u\|_\Omega + 1)}{\Gamma(q)} \int_0^t (t-s)^{q-1-\alpha}\, ds \\
&\leq \frac{M_\alpha (L_1 \|u\|_\Omega + 1) a^{q-\alpha}}{\Gamma(q)(q-\alpha)}
\end{aligned}
$$

$$
\begin{aligned}
I_4 &= \frac{\lambda^{-1}}{\Gamma(q)} \int_0^t \left\| (t-s)^{q-1} T(t-s) \int_0^s K(s,x) v(x)\, dw\, ds \right\|_\alpha \\
&\leq \frac{a w(\|u\|_\alpha) \sup_{t\in J} K(t)}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|A^\alpha T(t-s)\|\, ds \\
&\leq \frac{w(\|u\|_\alpha) M_\alpha a^{q-\alpha+1} \sup_{t\in J} K(t)}{(q-\alpha)\Gamma(q)}
\end{aligned}
$$

$$
\begin{aligned}
I_5 &= \lambda^{-1} \sum_{k=1}^m \left\| T(t-t_k) I_k(u(t_k^-)) \right\|_\alpha \\
&\leq \sum_{k=1}^m \left\| T(t-t_k) I_k(u(t_k^-)) \right\|_\alpha \\
&\leq M \sum_{k=1}^m \psi_k(\|u\|_\alpha).
\end{aligned}
$$

Taking the supremum over t we obtain,

$$
\begin{aligned}
\|x(t)\|_\Omega \leq{}& M(\|u_0\|_\alpha + L_2 \|u\|_\Omega + L_3) + (L_1 \|u\|_\Omega + 1) + (L_1 \|u\|_\Omega + 1) \\
&+ \frac{M_\alpha (L_1 \|u\|_\Omega + 1) a^{q-\alpha}}{\Gamma(q)(q-\alpha)} + \frac{w(\|u\|_\Omega) M_\alpha a^{q-\alpha+1} \sup_{t\in J} K(t)}{(q-\alpha)\Gamma(q)} \\
&+ M \sum_{k=1}^m \psi_k(\|u\|_\Omega).
\end{aligned}
$$

Substituting $\|u\|_\Omega = r$ in the above inequality and noting that (3.2) holds, we have

$$
\begin{aligned}
r \leq{}& M(\|u_0\|_\alpha + L_2 r + L_3) + (L_1 r + 1) + (L_1 r + 1) + \frac{M_\alpha (L_1 r + 1) a^{q-\alpha}}{\Gamma(q)(q-\alpha)} \\
&+ \frac{w(r) M_\alpha a^{q-\alpha+1} \sup_{t\in J} K(t)}{(q-\alpha)\Gamma(q)} + M \sum_{k=1}^m \psi_k(r),
\end{aligned}
$$

and hence

$$r \leq \frac{2 + M(\|u_0\|_\alpha + L_3) + \dfrac{M_\alpha a^{q-\alpha}(1 + aw(r)K)}{(q-\alpha)\Gamma(q)}}{1 - \left[M(L_2 + mN) + 2L_1 + \dfrac{M_\alpha L_1 a^{q-\alpha}}{(q-\alpha)\Gamma(q)}\right]}$$

which is contradiction to (3.2). Hence the operator inclusions $u \in \mathcal{A}(u) + \mathcal{B}(u)$ has a solution in $B[0,r]$. It further implies that the system (1.1) has at least one mild solution x in Ω. The proof is complete.

□

4. An example

As an application of previous sections, we study the fractional neutral differential equation of order q, where $0 < q \leq 1$.

$$\begin{aligned}
&\frac{\partial^q}{\partial t^q}\left[u(t,x) - \int_0^\pi \Theta(t,y,x)[u(\sin t, y) + \frac{\partial}{\partial y}u(\sin t, y)]\,dy\right] \\
&= \frac{\partial^2}{\partial x^2}u(t,x) + \int_0^s K(t,s)g\left(t, u(\sin t, x), \frac{\partial}{\partial x}u(\sin t, x)\right)ds, \\
&0 \leq t \leq 1, 0 \leq x \leq \pi 0 \leq s \leq t, t \neq t_k, \quad k = 1, \cdots, m \\
&\qquad u(t,0) = u(t,\pi) = 0. \qquad (4.1) \\
&u(t_k^+) - u(t_k^-) = I_k(u(t_k^-)) \quad k = 1, \cdots, m \\
&u(0,x) + \sum_{i=0}^{p}\int_0^\pi k_i(y,x)u(s_i,y)\,dy = u_0(x), \quad 0 \leq x \leq \pi,
\end{aligned}$$

where $0 < q \leq 1$, $0 < t_1 < \cdots, t_p, u_0 \in X = L^2([0,1],\mathbb{R})$ equipped with the L^2-norm $\|.\|_2$ and $k_i(.,.) \in L^2([0,1]\times[0,1],\mathbb{R})$ for $i = 1,2$.

Consider the operator $A : D(A) \subset X \to X$ defined by

$$\begin{cases} D(A) = H_0^2([0,\pi])) = \{f \in X : f, f'' \in X, f(0) = f(\pi) = 0\}, \\ Af = -f''. \end{cases}$$

Then $-A$ generates a compact, analytic semigroup $T(.)$ of uniformly bounded linear operators. It is well known that $0 \in \rho(A)$ and so the fractional powers of A are well defined. Moreover, the eigenvalues of A are n^2 and the corresponding normalized eigenvectors are $e_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx), n = 1,2,\cdots$. The following results are well known:

(A1) If $z \in D(A)$, then

$$Az = \sum_{n=1}^{\infty} n^2\langle z, e_n\rangle e_n.$$

(A2) For each $z \in X$,

$$A^{-\frac{1}{2}} z = \sum_{n=1}^{\infty} \frac{1}{n} \langle z, e_n \rangle e_n.$$

(A3) The operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}} z = \sum_{n=1}^{\infty} n \langle z, e_n \rangle e_n$$

for each $z \in D(A^{\frac{1}{2}}) := \left\{ f \in X : \sum_{n=1}^{\infty} n \langle z, e_n \rangle e_n \in X \right\}$.

We assume that the following conditions hold:

(i) The function Θ is measurable and

$$\sup_{0 \le t \le 1} \int_0^{\pi} \int_0^{\pi} \Theta^2(t, y, x)\, dy dx < \infty.$$

(ii) For each $t \in J$, $K(t,s)$ is measurable on J and
$K(t) = ess \sup \{|K(t,s)|, 0 \le s \le t\}$, is bounded on J.

(iii) The function $\dfrac{\partial^2}{\partial x^2}$ is measurable, $\Theta(t, y, 0) = \Theta(t, y, \pi) = 0$, and

$$\sup_{0 \le t \le 1} \left[\int_0^{\pi} \int_0^{\pi} \left(\frac{\partial^2}{\partial x^2} \Theta^2(t, y, x) \right)^2 dy dx \right]^{\frac{1}{2}} < \infty.$$

(iv) For the function $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ the following three conditions are satisfied:

(a) For each $t \in J, g(t, ., .)$ is continuous.

(b) For each $u \in X_{\frac{1}{2}}$, $g(., u, u')$ is measurable.

(c) There is a positive number ν such that

$$\left\| g(t, u, u') \right\| \le \nu \left\| u \right\|, \qquad \text{for all } (t, u) \in J \times X_{\frac{1}{2}}.$$

(v) $I_k \in \mathcal{C}(X_\alpha, X_\alpha)$, $k = 1, 2, \cdots, m$ and there exist nondecreasing function $\psi_k \in \mathcal{C}(J, \mathbb{R}_+)$ such that for each $u \in X_\alpha$

$$\| I_k(u) \| \le \psi_k(\| u \|_\alpha).$$

Here we choose $\alpha = \beta = \frac{1}{2}$. It follows from [15] that if $u \in X_{\frac{1}{2}}$, then u is absolutely continuous, $u' \in X$, and $u(0) = u(\pi) = 0$. In view of this result, for $(t, u) \in J \times X_{\frac{1}{2}}$ $w \in \Omega$, we can define respectively that

$$F(t,u)(x) = \int_0^{\pi} \Theta(t, y, x)[u(y) + u'(y)]\, dy',$$

$$G(t,u)(x) = g(t, u(x), u'(x)) \qquad \text{and}$$

$$g(w(t)) = \sum_{i=0}^{p} K_i(u)(x)w(t_i) = \sum_{i=0}^{p} \int_0^{\pi} k_i(y,x)u(y)\,dy, \qquad w \in \Omega.$$

So $G : J \times X_{\frac{1}{2}} \to X$. And it is easy to see that $F : J \times X_{\frac{1}{2}} \to X_{\frac{1}{2}}$, $A^{\frac{1}{2}}F : J \times X_{\frac{1}{2}} \to X_{\frac{1}{2}}$. In fact, for each $t \in J$, we have

$$\langle F(t,u), u_n \rangle = \frac{1}{n}\sqrt{\frac{2}{\pi}} \left(\int_0^{\pi} \frac{\partial}{\partial x} \Theta(t,y,x)[u(y) + u'(y)]\,dy, \cos(n\pi) \right)$$

and also

$$\langle A^{\frac{1}{2}}F(t,u), u_n \rangle = \frac{1}{n^2}\sqrt{\frac{2}{\pi}} \left(\int_0^{\pi} \frac{\partial^2}{\partial x^2} \Theta(t,y,x)[u(y) + u'(y)]\,dy, \sin(n\pi) \right).$$

It shows that F and $A^{\frac{1}{2}}F$ both take values in $X_{\frac{1}{2}}$ in terms of properties $A1$ and $A3$, and so does the function g. Since, for any $x_1, x_2 \in X_{\frac{1}{2}}$,

$$\begin{aligned} \|x_2 - x_1\|^2 &= \sum_{n=1}^{\infty} \langle x_2 - x_1, e_n \rangle^2 \\ &\le \sum_{n=1}^{\infty} n^2 \langle x_2 - x_1, e_n \rangle^2 \\ &\le \|x_2 - x_1\|_{\frac{1}{2}}. \end{aligned}$$

from which and conditions (iii) we see that $(H1)$ and $(H2)$ are satisfied. In addition, G satisfies condition $(H4)$ while g verifies $(H5)$. From (i) it is clear that $F(t,u)$ is bounded linear operator on X. Let $h_1(t) = h_2(t) = \sin t$. If conditions (3.1)-(3.3) are satisfied, then from Theorem 2, system (4.1) admits a mild solution on $[0,1]$ under the above assumptions.

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Authors' addresses

M. H. M. Rashid
Department of Mathematics, Statistics, Faculty of Science, Mutah University, P.O. Box(7) Alkarak-Jordan
E-mail address: malik_okasha@yahoo.com

A. Al-Omari
Department of Mathematics, Faculty of Sciences, Al al-Bayt University, P.O. Box 130095, Mafraq 25113, Jordan
E-mail address: omarimutah1@yahoo.com

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AN EXISTENCE THEOREM FOR AN ORDINARY DIFFERENTIAL EQUATION IN MENGER PROBABILISTIC METRIC SPACE

ABDOLRAHMAN RAZANI

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Abstract. In this article an existence theorem for an ordinary differential equation in probabilistic metric space is proved.

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Keywords: contractive map, fixed point, Menger probabilistic metric space, ordinary differential equations

1. Introduction

It is well known that the theory of probabilistic metric space is a new frontier branch between probability theory and functional analysis and has an important background, which contains the common metric space as a special case. Chang et al. [2] generalized contraction mapping principle in probabilistic metric space and as an application of this generalization, an existence theorem of solution of some kind of ordinary differential equations in probabilistic metric space is proved. Here, we consider an ordinary differential equations in the Menger probabilistic metric space, then we prove the existence of solution for this problem. Due to do this and for the sake of convenience, some definitions and notations are recalled from [1, 3–9].

Definition 1. A mapping $F : R \to R^+$ is called a *distribution function* if it is nondecreasing and left-continuous and it has the following properties:

(1) $\inf_{t\in R} F(t) = 0$,
(2) $\sup_{t\in R} F(t) = 1$.

Let D^+ be the set of all distribution functions F such that $F(0) = 0$ (F is a nondecreasing, left-continuous mapping from R into $[0, 1]$ such that $\sup_{t\in R} F(t) = 1$). Also denote by H the distribution function

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

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Definition 2. A *probabilistic metric space* (briefly, *PM-space*) is an ordered pair (S,F) where S is a nonempty set and $F : S \times S \to D^+\Big(F(p,q)$ is denoted by $F_{p,q}$ for every $(p,q) \in S \times S\Big)$ satisfies the following conditions:

(1) $F_{p,q}(t) = 1$ for all $t > 0$ if and only if $p = q$ $(p,q \in S)$.
(2) $F_{p,q}(t) = F_{q,p}(t)$ for all $p,q \in S$ and $t \in R$.
(3) If $F_{p,q}(t_1) = 1$ and $F_{q,r}(t_2) = 1$ then $F_{p,r}(t_1 + t_2) = 1$ for $p,q,r \in S$ and $t_1, t_2 \in R^+$.

If only (2) and (3) hold, the ordered pair (S,F) is known as a *probabilistic semimetric space*.

Definition 3. A mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a *triangular norm* (abbreviated, *t-norm*) if the following conditions are satisfied:

(i) $\Delta(a,1) = a$ for every $a \in [0,1]$,
(ii) $\Delta(a,b) = \Delta(b,a)$ for every $a,b \in [0,1]$,
(iii) $a \geq b,\ c \geq d \to \Delta(a,c) \geq \Delta(b,d)$ $(a,b,c,d \in [0,1])$,
(iv) $\Delta(a,\Delta(b,c)) = \Delta(\Delta(a,b),c)$ $(a,b,c \in [0,1])$

Definition 4. A *Menger probabilistic metric space* (briefly, *Menger PM-space*) (see [6]) is a triple (S,F,Δ), where (S,F) is a probabilistic metric space, Δ is a t-norm and the following inequality holds:

$$F_{p,q}(t_1 + t_2) \geq \Delta(F_{p,r}(t_1), F_{r,q}(t_2)), \tag{1.1}$$

for all $p,q,r \in S$ and every $t_1 > 0,\ t_2 > 0$.

Definition 5. A triple (S,F,Δ) is called a *Menger probabilistic normed space* (briefly, *Menger PN-space*) (see [2]) if S is a real vector space, F is a mapping from S into D (for $x \in S$, the distribution function $F(x)$ is denoted by F_x and $F_x(t)$ is the value of F_x and $t \in R$) and Δ is a t-norm satisfying the following conditions:

(i) $E_x(0) = 0$,
(ii) $F_x(t) = H(t)$ for all $t > 0$ if and only if $x = 0$,
(iii) $F_{\alpha x}(t) = F_x(\frac{t}{|\alpha|})$ for all $\alpha \in R, \alpha \neq 0$,
(iv) $F_{x+y}(t_1 + t_2) \geq \Delta(F_x(t_1), F_y(t_2))$ for all $x,y \in E$ and $t_1, t_2 \in R^+$.

Schweizer, Sklar and Thorp [7] proved that if (S,F,Δ) is a Menger PM-space with $\sup_{0<t<1} \Delta(t,t) = 1$, then (S,F,Δ) is a Hausdorff topological space in the topology τ induced by the family of $(\epsilon,\lambda)-$neighborhoods

$$\{U_p(\epsilon,\lambda) : p \in S, \epsilon > 0, \lambda > 0\},$$

where

$$U_p(\epsilon,\lambda) = \{u \in S : F_{u,p}(\epsilon) > 1 - \lambda\}.$$

Definition 6. Let (S, F, Δ) be a Menger PM-space with $\sup_{0<t<1} \Delta(t,t) = 1$.

(1) A sequence $\{u_n\}$ in S is said to be $\tau-$convergent to $u \in S$ (we write $u_n \to u$) if for any given $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N = N(\epsilon, \lambda)$ such that $F_{u_n,u}(\epsilon) > 1-\lambda$ whenever $n \geq N$.
(2) A sequence $\{u_n\}$ in S is called a $\tau-$Cauchy sequence if for any $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N = N(\epsilon, \lambda)$ such that $F_{u_n,u_m}(\epsilon) > 1-\lambda$, whenever $n, m \geq N$.
(3) A Menger PM-space (S, F, Δ) is said to be τ-complete if each τ-Cauchy sequence in S is τ-convergent to some point in S.

Finally, we state a fixed point theorem form [2]. This theorem prove the existence of a fixed point in a Menger PM-space.

Theorem 1. *Let (S, F, Δ) be a complete Menger PM-space with a t-norm Δ satisfying $\Delta(t,t) \geq t$ for all $t \in [0,1]$. Let $T : S \to S$ be a mapping satisfying the following conditions:*

$$F_{Tx,Ty}(t) \geq F_{x,y}(\frac{t}{k(\alpha,\beta)}),$$

for all $x, y \in S, t \geq 0$ and $\alpha, \beta \in (0, +\infty)$ with $F_{x,y}(\alpha) > 0$ and $F_{x,y}(\beta) < 1$, where $k(\alpha,\beta) : (0,+\infty)^2 \to (0,1)$ is a function. Then T has exactly one fixed point in S.

In the next section, the existence of a solution for an ordinary differential equation in the Menger PN-space is proved.

2. An existence theorem

In this section we prove some lemmas to obtain the existence of solution for an ordinary differential equation in the Menger PN-space.

Lemma 1. *If (R, F, Δ) is a Menger PN-space, then*

$$|x| < |y| \Longrightarrow F_x(t) > F_y(t),$$

for all $x, y \in R$ and $t \geq 0$.

Proof. Note that

$$F_x(t) = F_{\frac{x}{y}y}(t) = F_y(\frac{t}{|\frac{x}{y}|}) \geq F_y(t).$$

The last inequality holds, because $F_x(.)$ is a nondecreasing function. □

Definition 7. Let (R, F, Δ) be a Menger PN-space and $(E, \|.\|_E)$ be a normed real vector space, we define a mapping $\tilde{F} : E \to D^+$ by

$$\tilde{F}_x(t) = F_{\|x\|_E}(t)$$

and denote the distribution function by $\tilde{F}_x$ and the value of $\tilde{F}_x$ at $t \in R$ by $\tilde{F}_x(t)$.

Proposition 1. *Let (R,F,Δ) be a Menger PN-space, then $(E,\tilde{F},\Delta)$ is also a Menger PN-space.*

Proof. Note that

$$\tilde{F}_x(t) = F_{\|x\|_E}(t) \in D^+.$$

Secondly, $\tilde{F}_x(t)$ satisfies the next four conditions of the Definition 5.

(i) $\tilde{F}_x(0) = F_{\|x\|_E}(0) = 0$,
(ii) when $t>0$, $\tilde{F}_x(t) = 1 \Longleftrightarrow F_{\|x\|_E}(t) = 1 \Longleftrightarrow \|x\|_E = 0 \Longleftrightarrow x = 0$,
(iii) $\tilde{F}_{\alpha x}(t) = F_{\|\alpha x\|_E} = F_{|\alpha|\|x\|_E}(t) = F_{\|x\|_E}(\frac{t}{|\alpha|}) = \tilde{F}_x(\frac{t}{|\alpha|})$,

and finally, by Lemma 1

$$\begin{aligned}(iv) \qquad \tilde{F}_{x+y}(t_1+t_2) &= F_{\|x+y\|_E}(t_1+t_2) \geq F_{\|x\|_E+\|y\|_E}(t_1+t_2)\\ &\geq \Delta(F_{\|x\|_E}(t_1), F_{\|y\|_E}(t_2))\\ &= \Delta(\tilde{F}_x(t_1), \tilde{F}_y(t_2)).\end{aligned}$$

This ends the proof. □

Proposition 2. *Let (R,F,Δ) be a Menger PN-space, then (R,G,Δ) is a PM-space, where*

$$G_{x,y}(t) = F_{x-y}(t)$$

Proposition 3. *Let (R,F,Δ) be a complete Menger PN-space with a continuous t-norm Δ. Then $(C([0,T]),R),\tilde{F},\Delta)$ is also a complete Menger PN-space.*

Now, we can state the existence theorem for the solution of an ordinary differential equation in the probabilistic metric space as follow:

Theorem 2. *Let (R,F,Δ) be a complete Menger PN-space with a t-norm Δ satisfying $\Delta(t,t) \geq t$ for all $t \in [0,1]$. Let f be Lipschitz mapping from $R \times \overline{U_{x_0}(\epsilon,\lambda)}$ into R. Then the following differential equation*

$$\begin{cases} \dot{x}(t) = f(t,x(t)),\\ x(0) = x_0. \end{cases}$$

has a solution in $C([0,\delta_0],R)$, where δ_0 is small enough.

Proof. Note that

$$\begin{cases} \dot{x}(t) = f(t,x(t)),\\ x(0) = x_0, \end{cases}$$

is equivalent to the following integral equation

$$x(t) = x_0 + \int_0^t f(s,x(s))ds.$$

Let $C([0,\delta_0],R)$ be the same as above, and

$$Tx(t) = x_0 + \int_0^t f(s,x(s))ds,$$

for all $x(.) \in C([0,\delta_0], \overline{U_{x_0}(\epsilon,\lambda)})$. Note that T is continuous and $T : C([0,\delta_0], \overline{U_{x_0}(\epsilon,\lambda)}) \to C([0,\delta_0], \overline{U_{x_0}(\epsilon,\lambda)})$ is contraction. In order to do this, we have

$$\begin{aligned}\tilde{F}_{Tx-Ty}(t) &= \tilde{F}_{\int_0^t (f(s,x(s))-f(s,y(s)))ds}(t)\\ &= F_{\|\int_0^t f(s,x(s))-f(s,y(s))ds\|_\infty}(t)\\ &\geq F_{\delta_0 k\|x-y\|_\infty}(t) = F_{\|x-y\|_\infty}(\frac{t}{\delta_0 K}) = \tilde{F}_{x(t)-y(t)}(\frac{t}{\delta_0 K})\end{aligned}$$

It means that

$$G_{Tx,Ty}(t) \geq G_{x,y}(\frac{t}{\delta_0 K}).$$

In addition, Proposition 3 shows $(C([0,\delta_0],R), G, \Delta)$ is a complete Menger PM-space and T is contraction, so by Theorem 1, T has exactly one fixed point in $C([0,\delta_0],R)$, i.e. there exists a unique $x(.) \in C([0,\delta_0],R)$ such that

$$Tx(t) = x(t).$$

It means that

$$x(t) = x_0 + \int_0^t f(s,x(s))ds.$$

Thus $x(t)$ is a solution for

$$\begin{cases} \dot{x}(t) = f(t,x(t)),\\ x(0) = x_0, \end{cases}$$

Moreover, this solution is unique. □

Example 1. Consider

$$\begin{cases} \dot{x}(t) = t + x(t),\\ x(0) = x_0, \end{cases} \tag{2.1}$$

By Theorem 2 the ordinary differential equation (2.1) for any $\delta_0 < 1$ has a unique solution.

Example 2. Consider

$$\begin{cases} \dot{x}(t) = t + sinx(t),\\ x(0) = x_0, \end{cases} \tag{2.2}$$

By Theorem 2 the ordinary differential equation (2.2) for any $\delta_0 < 1$ has a unique solution.

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Author's address

Abdolrahman Razani
Imam Khomeini International University, Department of Mathematics, P.O.Box 34149-16818, Qazvin, Iran
E-mail address: razani@sci.ikiu.ac.ir

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ON IDEALS WITH SKEW DERIVATIONS OF PRIME RINGS

NADEEM UR REHMAN AND MOHD ARIF RAZA

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Abstract. Let R be a prime ring and set $[x,y]_1 = [x,y] = xy - yx$ for all $x, y \in R$ and inductively $[x,y]_k = [[x,y]_{k-1}, y]$ for $k > 1$. We apply the theory of generalized polynomial identities with automorphism and skew derivations to obtain the following result: Let R be a prime ring and I a nonzero ideal of R. Suppose that (δ, φ) is a skew derivation of R such that $\delta([x,y]) = [x,y]_n$ for all $x, y \in I$, then R is commutative.

2010 *Mathematics Subject Classification:* 16N20; 16W25; 16N55; 16N60

Keywords: skew derivation, automorphism, generalized polynomial identity (GPI), prime ring, ideal

1. Introduction, Notation and Statements of the Results

Throughout this paper, unless specifically stated, R is always an associative prime ring with center $Z(R)$, Q its Martindale quotient ring. Note that Q is also prime and the center C of Q, which is called the extended centroid of R, is field (we refer the reader to [1] for the definitions and related properties of these objects). For any $x, y \in R$, the symbol $[x,y]$ stands for the commutator $xy - yx$. Recall that a ring R is called prime if for any $x, y \in R$, $xRy = \{0\}$ implies that either $x = 0$ or $y = 0$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $F : R \longrightarrow R$ is called a generalized derivation if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, denoted by (F, d). Hence, the concept of generalized derivations covers both the concepts of a derivation and of a left multiplier.

Given any automorphism φ of R, an additive mapping $\delta : R \to R$ satisfying $\delta(xy) = \delta(x)y + \varphi(x)\delta(y)$ for all $x, y \in R$ is called a φ-derivation of R, or a skew derivation of R with respect to φ, denoted by (δ, φ). It is easy to see if $\varphi = 1_R$, the identity map of R, then a φ-derivation is merely an ordinary derivation. And if $\varphi \neq 1_R$, then $\varphi - 1_R$ is a skew derivation. Thus the concept of skew derivations can be regard as a generalization of both derivations and automorphism. When $\delta(x) = \varphi(x)b - bx$ for some $b \in Q$, then (δ, φ) is called an inner skew derivation, and otherwise it is outer. Any skew derivation (δ, φ) extends uniquely to a skew derivation of Q [12] via extensions of each map to Q. Thus we may assume that any skew derivation of

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R is the restriction of a skew derivation of Q. Recall that φ is called an inner automorphism if when acting on Q, $\varphi(q) = uqu^{-1}$ for some invertible $u \in Q$. When φ is not inner, then it is called an outer automorphism. The skew derivations have been extensively studied by many researchers from various views (see for instance [5] and [12], where further references can be found).

Let $Q *_C C\{X\}$ be the free product of Q and the free algebra $C\{X\}$ over C on an infinite set X, of indeterminate. Elements of $Q *_C C\{X\}$ are called generalized polynomials and a typical element in $Q *_C C\{X\}$ is a finite sum of monomials of the form $\alpha a_{i_0} x_{j_1} a_{i_1} x_{j_2} \cdots x_{j_n} a_{i_n}$ where $\alpha \in C$, $a_{ik} \in Q$ and $x_{jk} \in X$. We say that R satisfies a nontrivial generalized polynomial identity (abbreviated as GPI) if there exists a nonzero polynomial $\phi(x_i) \in Q *_C C\{X\}$ such that $\phi(r_i) = 0$ for all $r_i \in R$. By a generalized polynomial identity with automorphisms and skew derivations, we mean an identity of R expressed as the form $\phi(\varphi_j(x_i), \delta_k(x_i))$, where each φ_j is an automorphism, each δ_k is a skew derivation of R and $\phi(y_{ij}, z_{ik})$ is a generalized polynomial in distinct indeterminates y_{ij}, z_{ik}.

We need some well-known facts which will be used in the sequel.

Fact 1 ([5, Theorem 1]). Let R be a prime ring with an automorphism φ. Suppose that (δ, φ) is a Q-outer derivation of R. Then any generalized polynomial identity of R in the form $\phi(x_i, \delta(x_i)) = 0$ yields the generalized polynomial identity $\phi(x_i, y_i) = 0$ of R, where x_i, y_i are distinct indeterminates.

Fact 2 ([5, Theorem 1]). Let R be a prime ring with an automorphism φ. Suppose that (δ, φ) is a Q-outer derivation of R. Then any generalized polynomial identity of R in the form $\phi(x_i, \varphi(x_i), \delta(x_i)) = 0$ yields the generalized polynomial identity $\phi(x_i, y_i, z_i) = 0$ of R, where x_i, y_i, z_i are distinct indeterminates.

Fact 3 ([14, Proposition]). Let R be a prime algebra over an infinite field k and let K be a field extension over k. Then R and $R \otimes_k K$ satisfy the same generalized polynomial identities with coefficients in R.

The next result is a slight generalization of [13, Lemma 2] and can be obtained directly by the proof of [13, Lemma 2] and Fact 3.

Fact 4. Let R be a non-commutative simple algebra, finite dimensional over its center Z. Then $R \subseteq M_n(F)$ with $n > 1$ for some field F and R and $M_n(F)$ satisfy the same generalized polynomial identities with coefficients in R.

In 1992, Daif and Bell [6, Theorem 3], showed that if in a semiprime ring R there exists a nonzero ideal I of R and a derivation d such that $d([x, y]) = [x, y]$ for all $x, y \in I$, then $I \subseteq Z(R)$. If R is a prime ring, this implies that R is commutative. Later in 2011, Huang [8, Theorem 2.1], prove that if R is a prime ring, I a nonzero ideal of R and d a derivation of R such that $d([x, y])^m = [x, y]_n$ for all $x, y \in I$, then R is commutative. At this point the natural question is what happens in case the derivation is replaced by a generalized derivation. In [16], Quadri et. al., generalize

Daif and Bell result for generalized derivation, they showed that if R is a prime ring, I a nonzero ideal of R and (F,d) a generalized derivation with $d \neq 0$ such that $F([x,y]) = [x,y]$ for all $x,y \in I$, then R is commutative. In 2013, Huang and Davvaz [9], generalized Quadri et. al., results, more precisely they proved that if R be a prime ring, m,n are fixed positive integers, and (F,d) a generalized derivation with $d \neq 0$ such that $(F([x,y]))^m = [x,y]^n$ for all $x,y \in R$, then R is commutative.

Here we will continue the study of analogue problems on ideals of a prime ring involving skew derivations. The goal of this paper is to extend Daif and Bell theorem [6], and Huang theorem [8], in a systematic way by using the theory of generalized polynomial identities with automorphisms and skew derivations as developed by Kharchenko [11], Chuang [3,4] and recently by Chuang and Lee [5].

Explicitly we shall prove the following theorem.

Theorem 1. *Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer. Suppose that (δ,φ) is a skew derivation of R such that $\delta([x,y]) = [x,y]_n$ for all $x,y \in I$, then R is commutative.*

When $\delta = \varphi - 1_R$, we obtain the following

Corollary 1. *Let R be a prime ring, I a nonzero ideal of R, and n a fixed positive integer. If φ is a non-identity automorphism of R such that $\varphi([x,y]) = [x,y]_n$ for all $x,y \in I$, then R is commutative.*

Let R be a unital ring. For a unit $u \in R$, the map $\varphi_u : x \to uxu^{-1}$ defines an automorphism of R. If d is a derivation of R, then it is easy to see that the map $ud : x \to ud(x)$ defines a φ_u-derivation of R. So we have

Corollary 2. *Let R be a prime unital ring, u a unit in R, I a nonzero ideal of R, and n a fixed positive integer. Suppose that φ_u is a derivation of R such that $\varphi_u([x,y]) = [x,y]_n$ for all $x,y \in I$, then R is commutative.*

2. Main Result

Now, we are in a position to prove the main result:

Theorem 2. *Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer. Suppose that (δ,φ) is a skew derivation of R such that $\delta([x,y]) = [x,y]_n$ for all $x,y \in I$, then R is commutative.*

Proof. If $\delta = 0$, then $[x,y]_n = 0$ for all $x,y \in I$, which can be rewritten as

$$[x,y]_n = 0 = [I_x(y),y]_{n-1} \text{ for all } x,y \in I.$$

By Lanski [13, Theorem 1], either R is commutative or $I_x = 0$, i.e., $I \subseteq Z(R)$ in which case R is also commutative by Mayne [15, Lemma 3].

Now we assume that $\delta \neq 0$ and $\delta([x,y]) = [x,y]_n$ for all $x,y \in I$, which can be rewritten as

$$(\delta(x)y + \varphi(x)\delta(y)) - (\delta(y)x + \varphi(y)\delta(x)) = [x,y]_n. \tag{2.1}$$

In the light of Kharchenko's theory [11], we split the proof into two cases:

Case 1. Let δ is Q-outer, then I satisfies the polynomial identities

$$(sy+\varphi(x)t)-(tx+\varphi(y)s)=[x,y]_n, \text{ for all } x,y,s,t\in I. \tag{2.2}$$

Firstly, we assume that φ is not Q-inner, then for all $x,y,s,t,u,v\in I$, we have

$$(sy+ut)-(tx+vs)=[x,y]_n, \text{ for all } x,y,s,t,u,v\in I.$$

In particular $s=t=0$, then I satisfied the polynomial identity $[x,y]_n=0$, for all $x,y\in I$, so by Lanski [13, Theorem 1], R is commutative.

Secondly, if φ is Q-inner, then there exist an invertible element $T\in Q$, $\varphi(x)=TxT^{-1}$ for all $x\in R$. Thus from (2.2), we have

$$(sy+TxT^{-1}t)-(tx+TyT^{-1}s)=[x,y]_n \text{ for all } x,y,s,t\in I.$$

In particular $s=t=0$, and using the same argument presented as above, R is commutative.

Case 2. Let δ is Q-inner, then $\delta(x)=\varphi(x)q-qx$ for all $x\in R$, $q\in Q$. From (2.1), we have

$$\begin{aligned}(\varphi(x)q-qx)y+\varphi(x)(\varphi(y)q-qy)-(\varphi(y)q-qy)x-\varphi(y)(\varphi(x)q-qx)\\ =[x,y]_n \quad \text{for all } x,y\in I.\end{aligned} \tag{2.3}$$

If φ is not Q-inner, then I satisfies the polynomial identity

$$\begin{aligned}(uq-qx)y+u(vq-qy)-(vq-qy)x-v(uq-qx)\\ =[x,y]_n \quad \text{for all } x,y,u,v\in I.\end{aligned}$$

In particular $u=v=0$, then I satisfied the following polynomial identity

$$(-qxy+qyx)=[x,y]_n, \text{ for all } x,y\in I.$$

By Chuang [5, Theorem 1 and Theorem 2], shows that Q satisfies this polynomial identity and hence R as well. Note that this is a polynomial identity and hence there exist a field $\mathbb{F}$ such that $R\subseteq M_k(\mathbb{F})$, the ring of $k\times k$ matrices over a field $\mathbb{F}$, where $k\geq 1$. Moreover, R and $M_k(\mathbb{F})$ satisfy the same polynomial identity[2], that is $M_k(\mathbb{F})$ satisfy

$$(qyx-qxy)=[x,y]_n.$$

Denote e_{ij} the usual matrix unit with 1 in (i,j)-entry and zero elsewhere. By choosing $x=e_{12}$, $y=e_{22}$, $q=e_{12}$, we see that

$$\begin{aligned}0=(q[y,x])-[x,y]_n&=(e_{12}[e_{22},e_{12}])-[e_{12},e_{22}]_n\\ &=-e_{12}\neq 0, \text{ a contradiction.}\end{aligned}$$

Now consider, if φ is Q-inner, then there exist an invertible element $T\in Q$, $\varphi(x)=TxT^{-1}$ for all $x\in R$. From (2.3) we can write,

$$\begin{aligned}(TxT^{-1}q-qx)y+TxT^{-1}(TyT^{-1}q-qy)-(TyT^{-1}q-qy)x\\ -TyT^{-1}(TxT^{-1}q-qx)=[x,y]_n \text{ for all } x,y\in I.\end{aligned}$$

We can see easily that if $T^{-1}q \in C$, then

$$\delta(x) = TxT^{-1}q - qx = T(xT^{-1}q - T^{-1}qx) = T[x, T^{-1}q] = 0, \text{ a contradiction.}$$

Thus $T^{-1}q \notin C$. with this,

$$\begin{aligned}\phi(x,y) = {} & (TxT^{-1}q - qx)y + TxT^{-1}(TyT^{-1}q - qy) \\ & - (TyT^{-1}q - qy)x - TyT^{-1}(TxT^{-1}q - qx) - [x,y]_n. \end{aligned} \tag{2.4}$$

Since by [2] or [1, Theorem 6.4.4], I and Q satisfy the same generalized polynomial identities, with this we can see easily that $\phi(x,y) = 0$ is a nontrivial generalized polynomial identity of Q. Let $\mathcal{F}$ be the algebraic closure of C if C is infinite, otherwise let $\mathcal{F}$ be C. By Fact 3, $\phi(x,y)$ is also a generalized polynomial identity of $Q \otimes_C \mathcal{F}$. Moreover, in view of [7, Theorem 3.5], $Q \otimes_C \mathcal{F}$ is a prime ring with $\mathcal{F}$ as its extended centroid. Thus $Q \otimes_C \mathcal{F}$ is a prime ring satisfies a nontrivial generalized polynomial identity and its extended centroid $\mathcal{F}$ is either an algebraically closed field or a finite field. Since both Q and $Q \otimes_C \mathcal{F}$ are prime and centrally closed [7, Theorem 3.5], we may replace R by Q or $Q \otimes_C \mathcal{F}$. Thus we may assume that R is centrally closed and the field $\mathcal{F}$ which is either algebraically closed or finite and R satisfies generalized polynomial identity (2.4). By Martindale's theorem [1, Corollary 6.1.7], R is a primitive ring having nonzero socle with the field $\mathcal{D}$ as its associated division ring. By Jacobson theorem [10, p.75], R is isomorphic to a dense subring of the ring of linear transformations on a vector space V over $\mathcal{D}$(or $End(V_{\mathcal{D}})$ in brief), containing nonzero linear transformations of finite rank.

We assume that $dim(V_{\mathcal{D}}) \geq 2$, otherwise we are done.

Step 1. We want to show that w and $T^{-1}qw$ are linearly $\mathcal{D}$-dependent for all $w \in \mathcal{V}$. If $T^{-1}qw = 0$ then $\{w, T^{-1}qw\}$ is linearly $\mathcal{D}$-dependent. Suppose on contrary that w_0 and $T^{-1}qw_0$ are linearly $\mathcal{D}$-independent for some $w_0 \in \mathcal{D}$.

If $T^{-1}w_0 \notin Span_{\mathcal{D}}\{w_0, T^{-1}qw_0\}$ then $\{w_0, T^{-1}qw_0, T^{-1}w_0\}$ are linearly $\mathcal{D}$-independent. By the density of R there exist $x, y \in R$ such that

$$\begin{array}{lll} xw_0 = 0, & xT^{-1}qw_0 = T^{-1}w_0, & xT^{-1}w_0 = 0 \\ yw_0 = w_0, & yT^{-1}qw_0 = 0, & yT^{-1}w_0 = T^{-1}w_0. \end{array}$$

With all these, we obtained from (2.4),

$$\begin{aligned} -w_0 = \big(& (TxT^{-1}q - qx)y + TxT^{-1}(TyT^{-1}q - qy) - (TyT^{-1}q - qy)x \\ & - TyT^{-1}(TxT^{-1}q - qx) - [x,y]_n \big) w_0, \text{ a contradiction.} \end{aligned}$$

If $T^{-1}w_0 \in Span_{\mathcal{D}}\{w_0, T^{-1}qw_0\}$ then $T^{-1}w_0 = w_0\beta + T^{-1}qw_0\gamma$ for some $\beta, \gamma \in \mathcal{D}$ and $\beta \neq 0$. Since w_0 and $T^{-1}qw_0$ are linearly $\mathcal{D}$-independent, by the density of R there exist $x, y \in R$ such that

$$\begin{array}{ll} xw_0 = 0, & xT^{-1}qw_0 = w_0\beta + T^{-1}qw_0\gamma \\ yw_0 = w_0, & yT^{-1}qw_0 = 0. \end{array}$$

The application of (2.4) implies that

$$0 = \big((TxT^{-1}q - qx)y + TxT^{-1}(TyT^{-1}q - qy) - (TyT^{-1}q - qy)x - TyT^{-1}(TxT^{-1}q - qx) - [x,y]_n\big)w_0 = -Tw_0\beta = -w_0\beta \neq 0,$$

and we arrive at a contradiction. So we conclude that $\{w_0, T^{-1}w_0\}$ are linearly $\mathcal{D}$-dependent, for all $w_0 \in \mathcal{V}$ as claimed.

Step 2. By using the arguments presented above, we prove that $T^{-1}qw_0 = w_0\mu(w)$, for all $w \in \mathcal{V}$, where $\mu(w) \in \mathcal{D}$ depends on $w \in \mathcal{V}$. In fact, it is easy to check that $\mu(w)$ is independent of choice $w \in \mathcal{V}$. Indeed, for any $w, z \in \mathcal{V}$, in view of above situation, there exist $\mu(w), \mu(z), \mu(w+z) \in \mathcal{D}$ such that

$$T^{-1}qw = w\mu(w),\ T^{-1}qz = z\mu(z),\ T^{-1}q(w+z) = (w+z)\mu(w+z)$$

and therefore,

$$w\mu(w) + z\mu(z) = T^{-1}q(w+z) = (w+z)\mu(w+z).$$

Hence,

$$w(\mu(w) - \mu(w+z)) + z(\mu(z) - \mu(w+z)) = 0.$$

Since w and z are $\mathcal{D}$-independent, then $\mu(w) = \mu(z) = \mu(w+z)$. Otherwise, w and z are $\mathcal{D}$-dependent, say $w = \lambda z$ for some $\lambda \in \mathcal{D}$. Thus,

$$w\mu(w) = T^{-1}qw = T^{-1}q\lambda z = \lambda T^{-1}qz = \lambda z\mu(z) = w\mu(z)$$

i.e., $\mathcal{V}(\mu(w) - \mu(z)) = 0$. Since $\mathcal{V}$ is faithful, we get $\mu(w) = \mu(z)$. Hence, we conclude that there exists $\chi \in \mathcal{D}$ such that $T^{-1}qw = w\chi$ for all $w \in \mathcal{V}$.

At last, we want to show that $\chi \in Z(\mathcal{D})$ (the center of $\mathcal{D}$). Indeed, for any $\eta \in \mathcal{D}$, we have

$$T^{-1}q(w\eta) = (w\eta)\chi = w(\eta\chi),$$

and on the other hand,

$$T^{-1}q(w\eta) = (T^{-1}qw)\eta = (w\chi)\eta = w(\chi\eta).$$

Therefore, $\mathcal{V}(\eta\chi - \chi\eta) = 0$ and thus, $\eta\chi = \chi\eta$, which implies that $\chi \in Z(\mathcal{D})$. Hence, $T^{-1}q \in C$, a contradiction. With this completes the proof of the theorem. □

The following example demonstrates that the hypothesis of primeness of R is essential in Theorem 1.

Example 1. Let S be the set of all integers. Consider

$R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in S \right\}$ and $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in S \right\}$. Define maps $\varphi : R \to R$ by $\varphi \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & 0 \end{pmatrix}$ and $\delta : R \to R$ by $\delta \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & -2b \\ 0 & 0 \end{pmatrix}$. The fact that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$ implies that R is not

prime. It is easy to check that I is a nonzero ideal of R and (δ, φ) is a skew derivation of R such that $\delta([x, y]) = [x, y]_n$ for all $x, y \in I$. However, R is not commutative.

Remark 1. In view of the above result, it is an obvious question, what about the commutativity of R, if $\delta([x, y])^m = [x, y]_n$ for all $x, y \in I$ (or a Lie ideal L). Unfortunately, we are unable to solve it and leave as an open question whether or not this result can be prove.

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Authors' addresses

Nadeem ur Rehman
Department of Mathematics, Aligarh Muslim University, 202002, Aligarh, India
E-mail address: nu.rehman.mm@amu.ac.in

Mohd Arif Raza
Department of Mathematics, Aligarh Muslim University, 202002, Aligarh, India
E-mail address: arifraza03@gmail.com

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CONSTRUCTIVE EXISTENCE ANALYSIS OF SOLUTIONS OF NON-LINEAR INTEGRAL BOUNDARY VALUE PROBLEMS

MIKLÓS RONTÓ AND YANA VARHA

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Abstract. Sufficient conditions guaranteeing the solvability of non-linear integral boundary value problems for a system of non-linear ordinary differential equations are obtained using a special successive approximation technique. The efficiency of the suggested approach is shown on an example of a non-linear integral boundary value problem possessing at least two solutions.

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1. INTRODUCTION

In [9], a new approach has been suggested for the investigation of existence and approximate construction of solutions of non-local boundary value problems for ordinary differential equations. The purpose of the present paper is to apply this technique to obtain a scheme for the constructive solvability analysis of integral boundary value problems in the case where the non-local boundary conditions depend involve both the space variable and its derivative. Note that our approach is easier to apply compared with those used earlier [3, 4, 6, 12, 14, 15] for more special cases. At first, the given problem is reduced to a certain "model-type" one with a very simple two-point separated linear boundary condition depending on parameters. The transformed problem is then replaced by the Cauchy problem for suitably perturbed system containing some artificially introduced vector parameters the numerical values of which are to be determined later. The functional perturbation term, together with the given integral boundary conditions, generates a system of finite-dimensional system of algebraic or transcendental "determining" equations from which the numerical values of introduced parameters should be found. The solvability of the determining system, in turn, may be checked by studying some approximations constructed explicitly. We use for that purpose topological degree techniques similarly to [6, 7, 15].

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2. Notation and symbols

We will use mainly the notation from [9]. For any vector $x = \mathrm{col}(x_1, \dots, x_n) \in \mathbb{R}^n$, the obvious notation $|x| = \mathrm{col}(|x_1|, \dots, |x_n|)$ is used and the inequalities between vectors are understood componentwise. The same convention is adopted implicitly for operations "max", "min", etc. applied to vector-valued functions. The symbol $\mathbb{1}_n$ stands for the unit matrix of dimension n and $r(K)$ denotes the maximal, in modulus, eigenvalue of a square matrix K.

Definition 1. For any non-negative vector $\rho \in \mathbb{R}^n$ under the componentwise ρ-neighbourhood of a point $z \in \mathbb{R}^n$ we understand the set

$$B(z, \rho) := \{\xi \in \mathbb{R}^n : |\xi - z| \leq \rho\}.$$

Similarly, for the given bounded connected set $\Omega \subset \mathbb{R}^n$, we define its componentwise ρ-neighbourhood by putting

$$B(\Omega, \rho) := \bigcup_{\xi \in \Omega} B(\xi, \rho).$$

Definition 2. For given two bounded connected sets $D_a \subset \mathbb{R}^n$ and $D_b \subset \mathbb{R}^n$, introduce the set

$$D_{a,b} := \{(1-\theta)z + \theta\eta : z \in D_a,\ \eta \in D_b,\ \theta \in [0,1]\} \tag{2.1}$$

and its componentwise ρ-neighbourhood

$$D := B(D_{a,b}, \rho). \tag{2.2}$$

For a set $D \subset \mathbb{R}^n$, closed interval $[a,b] \subset \mathbb{R}$, Carathéodory function $f : [a,b] \times D \to \mathbb{R}^n$, $n \times n$ matrix K with non-negative entries, we write

$$f \in \mathrm{Lip}_K(D) \tag{2.3}$$

if the inequality

$$|f(t,u) - f(t,v)| \leq K\,|u - v|$$

holds for all $\{u, v\} \subset D$ and a. e. $t \in [a,b]$.

Finally, with a function $f : [a,b] \times D \to \mathbb{R}^n$, we associate the vector

$$\delta_{[a,b],D}(f) := \frac{1}{2}\left(\mathrm{ess\,sup}_{(t,x)\in[a,b]\times D} f(t,x) - \mathrm{ess\,inf}_{(t,x)\in[a,b]\times D} f(t,x)\right). \tag{2.4}$$

3. Problem setting and reduction to a model boundary condition

We consider the following non-linear integral boundary value problem which was studied in [13]

$$\frac{dx}{dt} = f(t,x), \quad t \in [a,b], \tag{3.1}$$

$$\int_a^b [g(s,x(s)) + h(s, f(s,x(s)))]\,ds = d. \tag{3.2}$$

Let D_a and D_b be a convex subsets of $\mathbb{R}^n$ where one looks for respectively the initial value $x(a)$ and the value $x(b)$ of the solution of the boundary value problem (3.1), (3.2).

Based on the sets D_a and D_b according to (2.1) we introduce the convex set $D_{a,b}$ and its componentwise ρ-neighbourhood D as in (2.2). Thus, the domain of the space variables in the given problem (3.1), (3.2) is the set D defined according to (2.2).

Here we suppose that the functions $f : [a,b] \times D \to \mathbb{R}^n$, $g : [a,b] \times D \to \mathbb{R}^n$, and $h : [a,b] \times D \to \mathbb{R}^n$ satisfy the Carathéodory and the Lipschitz condition in the domain D and d is a given vector.

Finally we suppose that the vector ρ in (2.2) is chosen so that

$$\rho \geq \frac{b-a}{2}\delta_{[a,b],D}(f), \tag{3.3}$$

where $\delta_{[a,b],D}(f)$ is given in (2.4). We also assume that the maximal in modulus eigenvalue of the matrix

$$Q := \frac{3(b-a)}{10}K \tag{3.4}$$

satisfies the inequality

$$r(Q) < 1. \tag{3.5}$$

It is important to emphasize that $D \subset \mathbb{R}^n$ is supposed to be bounded and, thus, the Lipschitz condition is not assumed globally.

The problem is to find and establish the existence of an absolutely continuous solution $x : [a,b] \to D$ of the problem (3.1), (3.2) with initial value $x(a) \in D_a$.

At first we simplify the boundary condition (3.2) and reduce it to a suitable two-point separated linear one. To do so, similarly to [5, 10, 11, 14] we apply an appropriate "freezing" technique. Namely, we introduce the vectors of parameters

$$z = \mathrm{col}(z_1, z_2, \ldots, z_n), \qquad \eta = \mathrm{col}(\eta_1, \eta_2, \ldots, \eta_n) \tag{3.6}$$

by formally putting

$$z = x(a), \qquad \eta = x(b). \tag{3.7}$$

Now, instead of integral problem (3.1), (3.2) we will consider the following "model-type" two-point boundary value problem with separated parameterized conditions:

$$\frac{dx}{dt} = f(t,x), \quad t \in [a,b], \tag{3.8}$$

$$x(a) = z, \qquad x(b) = \eta. \tag{3.9}$$

The parameterization technique that we are going to use suggest that, instead of the original boundary value problem with nonlinear integral boundary conditions (3.2), we study the family of parametrized boundary value problems (3.8), (3.9), where the boundary restrictions are linear and separated. Later, we return to the original problem by choosing the values of the introduced parameters appropriately.

Remark 1. The set of solutions of the non-linear integral boundary value problem (3.1), (3.2) coincides with the set of the solutions of the parametrized problem (3.8), (3.9) with separated restrictions, satisfying additional conditions (3.9).

4. Some results from [13]

Similarly to [9,13] let us associate with the two-point parametrized boundary value problem (3.8), (3.9) with separated boundary conditions the sequence of functions

$$x_{m+1}(t,z,\eta) = z + \int_a^t f(s,x_m(s,z,\eta))ds - \frac{t-a}{b-a}\int_a^b f(\tau,x_m(\tau,z,\eta))d\tau + \frac{t-a}{b-a}[\eta - z], \quad t \in [a,b], \quad m = 1,2,\ldots, \tag{4.1}$$

satisfying (3.9) for arbitrary $z,\eta \in \mathbb{R}^n$, where

$$x_0(t,z,\eta) = z + \frac{t-a}{b-a}[\eta - z] = \left(1 - \frac{t-a}{b-a}\right)z + \frac{t-a}{b-a}\eta, \quad t \in [a,b], \tag{4.2}$$

and $z \in D_a$, $\eta \in D_b$ are considered as parameters.

It is obvious from (4.2) that $x_0(t,z,\eta)$ is a convex combination of vectors z and η for any $t \in [a,b]$.

The following Theorems 1, 2, and 3 were proved in [13].

Theorem 1 ([13, Theorem 1])**.** *Let there exists a non-negative vector ρ such that inequality (3.3) holds and $f \in \mathrm{Lip}_K(D)$ for D given by (2.2) with the given value of ρ. Let, moreover, the matrix K be such that (3.5) holds with Q given by (3.4). Then, for all fixed $(z,\eta) \in D_a \times D_b$:*

1. *The functions of sequence (4.1) are absolutely continuous on $[a,b]$, satisfy the two-point separated boundary conditions (3.9), and have values in D.*
2. *The sequence of functions (4.1) converges as $m \to \infty$ to the limit function*

$$x_\infty(t,z,\eta) := \lim_{m\to\infty} x_m(t,z,\eta) \tag{4.3}$$

   uniformly in $t \in [a,b]$.
3. *The limit function satisfies the initial condition*

$$x_\infty(a,z,\eta) = z$$

   and the two-point separated boundary conditions (3.9).
4. *The function $x_\infty(\cdot,z,\eta)$ is a unique absolutely continuous solution of the integral equation*

$$x(t) = z + \int_a^t f(s,x(s))ds - \frac{t-a}{b-a}\int_a^b f(s,x(s))ds + \frac{t-a}{b-a}[\eta - z].$$

In other words, $x_\infty(\cdot,z,\eta)$ satisfies the Cauchy problem for the modified system of integro-differential equations:

$$\frac{dx}{dt} = f(t,x) + \frac{1}{b-a}\Delta(z,\eta), \quad t \in [a,b],$$
$$x(a) = z,$$

where $\Delta : D_a \times D_b \to \mathbb{R}^n$ is the mapping given by formula

$$\Delta(z,\eta) := \eta - z - \int_a^b f(s, x_\infty(s,z,\eta))ds. \tag{4.4}$$

5. *The following error estimate holds:*

$$|x_\infty(t,z,\eta) - x_m(t,z,\eta)| \le \frac{10}{9}\alpha_1(t,a,b)\,Q^m\,(\mathbb{1}_n - Q)^{-1}\,\delta_{[a,b],D}(f), \tag{4.5}$$

for any $t \in [a,b]$ and $m \ge 0$, where $\delta_{[a,b],D}(f)$ is given in (2.4) and

$$\alpha_1(t,a,b) := 2(t-a)\left(1 - \frac{t-a}{b-a}\right). \tag{4.6}$$

Remark 2. It is easy to verify that $\max_{t\in[a,b]}\alpha_1(t,a,b) = (b-a)/2$.

Theorem 2 ([13, Theorem 3]). *Let $(z,\eta) \in D_a \times D_b$. Under the assumptions of Theorem 1, the limit function $x_\infty(\cdot,z,\eta)$ of sequence (4.1) is an absolutely continuous solution of the integral boundary value problem (3.1), (3.2) if and only if the pair of parameters (z,η) satisfies the system of $2n$ algebraic determining equations*

$$\Delta(z,\eta) = 0, \tag{4.7}$$
$$\Lambda(z,\eta) = 0, \tag{4.8}$$

with $\Delta : D_a \times D_b \to \mathbb{R}^n$ given by formula (4.4) and

$$\Lambda(z,\eta) := \int_a^b [g(s, x_\infty(s,z,\eta)) + h(s, f(s, x_\infty(s,z,\eta)))]\,ds - d. \tag{4.9}$$

The next statement proves that the system of determining equations (4.7), (4.8) defines all possible solutions of the original non-linear integral boundary value problem (3.1), (3.2).

Theorem 3 ([13, Theorem 4]). *Assume that conditions of Theorem 1 is satisfied. If there exists a pair of vectors $(z^0,\eta^0) \in D_a \times D_b$ satisfying the system of determining equations (4.7), (4.8), then the integral boundary value problem (3.1), (3.2) has a solution $x^0(\cdot)$ such that*

$$x^0(a) = z^0, \qquad x^0(b) = \eta^0$$

and, moreover, $x^0(\cdot) = x_\infty(\cdot,z^0,\eta^0)$.

Conversely, if the integral boundary value problem (3.1), (3.2) has a solution $x^0(\cdot)$ with $(x^0(a),x^0(b)) \in D_a \times D_b$ and $\{x^0(t) : t \in [a,b]\} \subset D$, then $x^0(\cdot)$ necessarily

has the form $x^0(\cdot) = x_\infty(\cdot, x^0(a), x^0(b))$, *and the system of determining equations* (4.7), (4.8) *is satisfied with* $z = x^0(a)$, $\eta = x^0(b)$.

5. Solvability analysis based on the approximate determining system

The solvability of the determining system (4.7), (4.8), in turn, may be checked by using the so-called *approximate determining equations*

$$\Delta_m(z,\eta) = 0, \tag{5.1}$$

$$\Lambda_m(z,\eta) = 0, \tag{5.2}$$

where m is fixed and

$$\Delta_m(z,\eta) := \eta - z - \int_a^b f(s, x_m(s,z,\eta))ds, \tag{5.3}$$

$$\Lambda_m(z,\eta) := \int_a^b \left[g(s, x_m(s,z,\eta)) + h(s, f(s, x_m(s,z,\eta)))\right] ds - d. \tag{5.4}$$

Note that, unlike (4.7) and (4.8), equations (5.1) and (5.2) can be constructed explicitly.

In view of Theorem 1, it is natural to expect that, under suitable conditions, systems (4.7), (4.8) and (5.1), (5.2) are close enough to one another for m sufficiently large.

Lemma 1. *Assume that the conditions of Theorem 1 are satisfied and, moreover,* $g \in \mathrm{Lip}_{K_g}(D)$ *and* $h \in \mathrm{Lip}_{K_h}(\tilde{D})$, *where*

$$\tilde{D} := \{f(t,y) : t \in [a,b],\ y \in D\}, \tag{5.5}$$

with some non-negative square matrices K_g *and* K_h *of dimension* n. *Then the exact and approximatathe determining functions defined by* (4.4), (4.9) *and* (5.3), (5.2) *satisfy the following estimates for any* $(z,\eta) \in D_a \times D_b$ *and* $m \geq 1$*:*

$$|\Delta(z,\eta) - \Delta_m(z,\eta)| \leq \frac{10(b-a)^2}{27} K Q^m (\mathbb{1}_n - Q)^{-1} \delta_{[a,b],D}(f), \tag{5.6}$$

$$|\Lambda(z,\eta) - \Lambda_m(z,\eta)| \leq \frac{10(b-a)^2}{27} (K_g + K_h K) Q^m (\mathbb{1}_n - Q)^{-1} \delta_{[a,b],D}(f), \tag{5.7}$$

where the matrix Q *and the vector* $\delta_{[a,b],D}(f)$ *are given in* (3.4) *and* (2.4) *respectively.*

Proof. Let us fix an arbitrary $(z,\eta) \in D_a \times D_b$. Using the Lipschitz condition (2.3), estimate (4.5) and the equality

$$\int_a^b \alpha_1(t,a,b)dt = \frac{(b-a)^2}{3}, \tag{5.8}$$

where $\alpha_1(\cdot,a,b)$ is given in (4.6), we have

$$\begin{aligned}|\Delta(z,\eta)-\Delta_m(z,\eta)| &= \left|\int_a^b f(s,x_m(s,z,\eta))ds-\int_a^b f(s,x_\infty(s,z,\eta))ds\right| \\ &\le K\int_a^b \frac{10}{9}\alpha_1(s,a,b)Q^m(\mathbb{1}_n-Q)^{-1}\delta_{[a,b],D}(f)ds \\ &= \frac{10(b-a)^2}{27}KQ^m(\mathbb{1}_n-Q)^{-1}\delta_{[a,b],D}(f),\end{aligned}$$

which proves the inequality in (5.6).

From (4.8) and (5.2) using the Lipschitz condition for g and the relations (4.5), (5.8) we obtain

$$\begin{aligned}|\Lambda(z,\eta)-\Lambda_m(z,\eta)| &= \Big|\int_a^b [g(s,x_\infty(s,z,\eta))+h(s,f(s,x_\infty(s,z,\eta))) \\ &\quad -g(s,x_m(s,z,\eta))-h(s,f(s,x_m(s,z,\eta)))]ds\Big| \\ &\le K_g\int_a^b |x_\infty(s,z,\eta)-x_m(s,z,\eta)|\,ds \\ &\quad +K_hK\int_a^b |x_\infty(s,z,\eta)-x_m(s,z,\eta)|\,ds \\ &\le (K_g+K_hK)\int_a^b \frac{10}{9}\alpha_1(s,a,b)Q^m(\mathbb{1}_n-Q)^{-1}\delta_{[a,b],D}(f)ds \\ &\le \frac{10(b-a)^2}{27}(K_g+K_hK)Q^m(\mathbb{1}_n-Q)^{-1}\delta_{[a,b],D}(f),\end{aligned}$$

that is, estimate (5.7) also holds. □

Based on the exact an approximate determining systems (4.7), , (4.8) and (5.1), (5.2), let us fix an m and introduce the mappings $H: D_a\times D_b\to\mathbb{R}^{2n}$ and $H_m: D_a\times D_b\to\mathbb{R}^{2n}$ by setting

$$H(z,\eta):=\begin{pmatrix}\eta-z-\int_a^b f(s,x_\infty(s,z,\eta))ds \\ \int_a^b [g(s,x_\infty(s,z,\eta))+h(s,f(s,x_\infty(s,z,\eta)))]\,ds-d\end{pmatrix} \tag{5.9}$$

and

$$H_m(z,\eta):=\begin{pmatrix}\eta-z-\int_a^b f(s,x_m(s,z,\eta))ds \\ \int_a^b [g(s,x_m(s,z,\eta))+h(s,f(s,x_m(s,z,\eta)))]\,ds-d\end{pmatrix} \tag{5.10}$$

for any $(z,\eta) \in D_a \times D_b$. Recall that $x_\infty(\cdot, z^*, \eta^*)$ is the limit function of the recurrence sequence (4.1).

We see from Theorem 2 that the critical points of the vector field H determine solutions of the integral boundary value problem (3.1), (3.2). The next statement establishes a similar result based upon properties of the vector field H_m explicity known from (5.10).

Theorem 4. *Assume that the conditions of Lemma 1 hold. Let, moreover, one can specify an $m \geq 1$ and a set $\Omega \subset \mathbb{R}^{2n}$ of form*

$$\Omega := D_1 \times D_2 \tag{5.11}$$

where $D_1 \subset D_a$, $D_2 \subset D_b$ are certain bounded open sets such that the mapping H_m satisfies the relation

$$|H_m| \rhd_{\partial\Omega} \frac{10\,(b-a)^2}{27} \begin{pmatrix} KQ^m\,(\mathbb{1}_n - Q)^{-1}\,\delta_{[a,b],D}(f) \\ (K_g + K_h K)\,Q^m\,(\mathbb{1}_n - Q)^{-1}\,\delta_{[a,b],D}(f) \end{pmatrix} \tag{5.12}$$

on $\partial\Omega$. If, in addition,

$$\deg(H_m, \Omega, 0) \neq 0, \tag{5.13}$$

then there exists a pair (z^*, η^*) from $D_1 \times D_2$ for which the function

$$x^*(\cdot) := x_\infty(\cdot, z^*, \eta^*) \tag{5.14}$$

is a solution of the integral boundary value problem (3.1), (3.2).

In (5.12), $\partial\Omega$ is the boundary of Ω and the binary relation $\rhd_{\partial\Omega}$ is defined [2] as follows: functions $u = (u_i)_{i=1}^{2n} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and $v = (v_i)_{i=1}^{2n} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ are said to satisfy the relation $u \rhd_{\partial\Omega} v$ if and only if there exists a function $k : \partial\Omega \to \{1, 2, \ldots, 2n\}$ such that $u_{k(z)}(z) > v_{k(z)}(z)$ at every point $z \in \partial\Omega$.

Remark 3. The degree in (5.13) is the Brouwer degree because all the vectors fields are finite-dimensional. Likewise, all the terms in the right-hand side of (5.12) are computed explicitly (e. g., by using computer algebra systems).

Proof of Theorem 4. We shall use Lemma 1 stated above. By analogy to [2, 4, 8], we shall prove that the vector fields H and H_m, given by (5.9) and (5.10) are homotopic. For this purpose, we consider the linear deformation determined by the family of mappings

$$\Psi_\vartheta(z,\eta) = H_m(z,\eta) + \vartheta\,[H(z,\eta) - H_m(z,\eta)], \qquad (z,\eta) \in \partial\Omega, \tag{5.15}$$

for $\vartheta \in [0,1]$. Obviously, Ψ_ϑ is a continuous mapping on $\partial\Omega$ for every $\vartheta \in [0,1]$ and, moreover,

$$\Psi_0 = H_m, \qquad \Psi_1 = H$$

on $\partial\Omega$.

For arbitrary $\vartheta \in [0,1]$ and $(z,\eta) \in \partial\Omega$, we have

$$\begin{aligned} |\Psi_\vartheta(z,\eta)| &= |H_m(z,\eta) + \vartheta\,[H(z,\eta) - H_m(z,\eta)]| \\ &\geq |H_m(z,\eta| - |H(z,\eta) - H_m(z,\eta)|\,. \end{aligned} \tag{5.16}$$

On the other hand, by estimates (5.6) and (5.7) of Lemma 1, we have the componentwise inequalities

$$|H(z,\eta) - H_m(z,\eta))| \leq \frac{10\,(b-a)^2}{27} \begin{pmatrix} KQ^m\,(\mathbb{1}_n - Q)^{-1}\,\delta_{[a,b],D}(f), \\ (K_g + K_h K)\,Q^m\,(\mathbb{1}_n - Q)^{-1}\,\delta_{[a,b],D}(f) \end{pmatrix}.$$

Therefore, it follows from (5.12) and (5.16) that

$$|\Psi_\vartheta| \rhd_{\partial\Omega} 0 \tag{5.17}$$

for any $\vartheta \in [0,1]$. Here, 0 stands for the zero column vector of dimension $2n$. Relation (5.17) implies that Ψ_ϑ dos not vanish on $\partial\Omega$ for any θ, i. e., deformation (5.15) is non-degenerate. Thus, H is homotopic to H_m and, using assumption (5.13) and the property of invariance of the Brouwer degree under homotopy, we conclude that

$$\deg(H,\Omega,0) = \deg(H_m,\Omega,0) \neq 0.$$

The classical topological result (see, e. g., [1, Theorem A2.4]) then guarantees the existence of a pair $(z^*,\eta^*) \in \Omega$ satisfying the equation $H(z^*,\eta^*) = 0$. Therefore, the pair (z^*,η^*) satisfies the system of determining equations (4.7), (4.8).

Applying now Theorem 2, we find that the function (5.14) is an absolutely continuous solution of integral boundary value problem (3.1), (3.2). □

Remark 4. Theorem 1 does not guarantee the convergence of sequence (4.1) without assumption (3.5). In the case where condition (3.5) for the matrix Q given in (3.4) does not hold, the limitation can be overcome by using a suitable parametrization and applying the interval halving technique introduced in [5,8] for periodic boundary value problems.

Using this approach, the smallness condition (3.5) can be weakened to the inequality

$$r\left(\frac{3\,(b-a)}{20}K\right) < 1.$$

6. Scheme of analysis of the problem

on Theorem 4 consists in carrying out the following:

1. Choose the vector ρ and compute the vector $\delta_{[a,b],D}(f)$ according to (2.4).
2. Check the fulfilment of inequality (3.3), construct the function $x_m(\cdot,z,\eta)$ analytically (e. g., by using computer algebra systems) for a certain fixed value $m = m_0$, keeping z and η as parameters.

3. Select a suitable set Ω and verify conditions (5.12), (5.13) for $m = m_0$.

   To verify condition (5.12) of Theorem 4 one has to use the reccurence formula (4.1) to compute the functions $x_m(\cdot, z, \eta)$ analytically, depending on the parameters z and η, and, at every point (z, η) of $\partial\Omega$, verify whether at least one component of the $2n$-dimensional vector $|H_m(z, \eta)|$ is strictly greater than the corresponding component of the vector at the right-hand side of (5.12).

4. Verify the validity of (5.13).

   This is rather difficult problem in general. However, there are sufficient conditions applicable in a number of important cases. In the smooth case, it follows directly from the definition of the topological degree [1, Definition A2.1] that if the Jacobian matrix of the function H_m in (5.10) is non-singular at its isolated zero $(\bar{z}_m, \bar{\eta}_m)$, i. e.,

$$\det \frac{\partial}{\partial(z, \eta)} H_m(\bar{z}_m, \bar{\eta}_m) \neq 0, \tag{6.1}$$

   then inequality (5.13) holds. In (6.1), the symbol $\partial/\partial(z, \eta)$ means the derivative with respect to the vector of variables $(z_1, \ldots, z_n, \eta_1, \ldots, \eta_n)$.

   In particular, when H_m is odd mapping, i.e.

$$H_m(-z, -\eta) = -H_m(z, \eta)$$

   for all $(z, \eta) \in \Omega$, then, according to Borsuk theorem (see, e. g., [1, Theorem A2.12]), its Brouwer degree is an odd number and, therefore, is different from zero.

Theorem 1 can be complemented by the following natural observation. Let $(\hat{z}, \hat{\eta}) \in D_a \times D_b$ be a root of the approximate determining system (5.1), (5.2) for a certain m. The observation in Section 5 suggests to consider $\hat{z}$ and $\hat{\eta}$ as approximations of the values of a solution of problem (3.1), (3.2), respectively, at a and b. Furthermore, the function

$$\tilde{x}(t) := x_m(t, \hat{z}, \hat{\eta}), \quad t \in [a, b], \tag{6.2}$$

defined according to (4.1) can be regarded as the mth approximation to a solution of the integral boundary value problem (3.1), (3.2). The latter is justified by the estimate

$$|x_\infty(t, \hat{z}, \hat{\eta}) - x_m(t, \hat{z}, \hat{\eta}))| \leq \frac{10}{9} \alpha_1(t, a, b) Q^m (\mathbb{1}_n - Q)^{-1} \delta_{[a,b],D}(f) \tag{6.3}$$

implied directly by inequality (4.5) of Theorem 1 for any $t \in [a, b]$ and $m \geq 0$. Recall that, in (6.3), Q and $\delta_{[a,b],D}(f)$ are given by (3.4) and (2.4) respectively.

7. An example with two-solutions

Let us apply the numerical-analytic approach desribed above to the system of differential equations

$$\begin{cases} x_1'(t) = x_2^2(t) - \dfrac{t}{5}x_1(t) + \dfrac{t^3}{100} - \dfrac{t^2}{25}, \\ x_2'(t) = \dfrac{t^2}{10}x_2(t) + \dfrac{t}{8}x_1(t) - \dfrac{21}{800}t^3 + \dfrac{1}{16}t + \dfrac{1}{5}, \quad t \in \left[0, \dfrac{1}{2}\right], \end{cases} \tag{7.1}$$

considered under the integral boundary conditions

$$\begin{cases} \displaystyle\int_0^{\frac{1}{2}} s x_1(s) x_2(s) ds = -\frac{197}{48000}, \\ \displaystyle\int_0^{\frac{1}{2}} s^2 x_2^2(s) ds = \frac{1}{4000}. \end{cases} \tag{7.2}$$

This boundary value problem was considered in [9]. Clearly, (7.1) is a particular case of (3.1), (3.2) with $a := 0$, $b := 1/2$,

$$f(t, x_1, x_2) := \begin{pmatrix} x_2^2 - \dfrac{t}{5}x_1 + \dfrac{t^3}{100} - \dfrac{t^2}{25} \\ \dfrac{t^2}{10}x_2 + \dfrac{t}{8}\left(x_1 + \dfrac{1}{2}\right) - \dfrac{21}{800}t^3 + \dfrac{1}{5} \end{pmatrix}, \tag{7.3}$$

$g(t, x_1, x_2) := \begin{pmatrix} t x_1 x_2 \\ t^2 x_2^2 \end{pmatrix}$, $h := 0$, and $d := \begin{pmatrix} -197/48000 \\ 1/4000 \end{pmatrix}$.

Following to (3.6), (3.7), introduce the parameters $z = \operatorname{col}(z_1, z_2)$ and $\eta = \operatorname{col}(\eta_1, \eta_2)$. Let us consider the following choice of the convex subsets D_a and D_b, where one looks the values $x(a)$ and $x(b)$:

$$D_a = D_b = \{(x_1, x_2) : -0.55 \le x_1 \le 0.45,\ -0.2 \le x_2 \le 0.15\}. \tag{7.4}$$

In this case, according to (2.1), we have

$$D_{a,b} = D_a = D_b. \tag{7.5}$$

For ρ involved in (2.2) and (3.3), we choose the value

$$\rho := \operatorname{col}(0.2, 0.2). \tag{7.6}$$

Then, in view of (7.4)–(7.6), the set (2.2) takes the form

$$D = \{(x_1, x_2) : -0.75 \le x_1 \le 0.65,\ -0.4 \le x_2 \le 0.35\} \tag{7.7}$$

A direct computation shows that the Lipschitz condition (2.3) for f given by (7.3) on D of form (7.7) holds with

$$K = \begin{pmatrix} 1/10 & 9/10 \\ 1/16 & 1/40 \end{pmatrix}$$

and, therefore, by (3.4), $Q = \frac{3}{20}\begin{pmatrix} 1/10 & 9/10 \\ 1/16 & 1/40 \end{pmatrix}$ and $r(Q) = 0.045 < 1$. Furthermore, in view of (7.7) and (2.4),

$$\delta_{[a,b],D}(f) = \frac{1}{2}\left(\max_{(t,x)\in[0,\frac{1}{2}]\times D} f(t,x) - \min_{(t,x)\in[0,\frac{1}{2}]\times D} f(t,x) \right) = \begin{pmatrix} 0.15 \\ 0.053125 \end{pmatrix}$$

and, by (7.6),

$$\frac{b-a}{2}\delta_{[a,b],D}(f) = \begin{pmatrix} 0.0375 \\ 0.01328125 \end{pmatrix} \leq \rho.$$

Note that f is continuous.

We thus see that all the conditions of Theorem 1 are fulfilled, and the sequence of functions (4.1) for this example is convergent.

Using (4.1) and applying *Maple 13* at the first iteration ($m = 1$) we get [9]

$$\begin{aligned} x_{11}(t,z,\eta) = z_1 + \frac{t^4}{400} + \frac{1}{3}\left((-2z_2+2\eta_2)^2 + \frac{2}{5}(z_1-\eta_1) - \frac{1}{25} \right) t^3 \\ + \frac{1}{2}\left(2z_2(-2z_2+2\eta_2) - \frac{1}{5}z_1 \right) t^2 + z_2^2 t - 2t\left(-\frac{29}{19200} \right. \\ + \frac{1}{24}(-2z_2+2\eta_2)^2 - \frac{1}{120}z_1 - \frac{1}{60}\eta_1 \\ \left. + \frac{1}{4}z_2(-2z_2+2\eta_2) + \frac{1}{2}z_2^2 \right) + 2t(\eta_1 - z_1) \end{aligned} \tag{7.8}$$

and

$$\begin{aligned} x_{12}(t,z,\eta) = z_2 + \frac{1}{5}t + \frac{1}{4}\left(\frac{1}{5}(\eta_2 - z_2) - \frac{21}{800} \right) t^4 \\ + \frac{1}{3}\left(\frac{1}{4}(\eta_1 - z_1) + \frac{1}{10}z_2 \right) t^3 + \frac{1}{16}\left(z_1 + \frac{1}{2} \right) t^2 \\ - 2t\left(\frac{5499}{51200} + \frac{1}{960}z_2 + \frac{1}{320}\eta_2 \right. \\ \left. + \frac{1}{192}z_1 + \frac{1}{96}\eta_1 \right) + 2t(\eta_2 - z_2), \end{aligned} \tag{7.9}$$

where we use the notation $x_{mi} = \mathrm{col}(x_{m,1}, x_{m,2})$ for any m. Using (7.8) and (7.9) to form the approximate determining system (5.1), (5.2) for $m = 1$ and solving the latter numerically, we find its roots $z_1 = \mathrm{col}(z_{11}, z_{12})$ and $\eta_1 = \mathrm{col}(\eta_{11}, \eta_{12})$:

$$\begin{aligned} z_{11} &\approx -0.5000145056, \quad z_{12} \approx 5.750026703 \cdot 10^{-7}, \\ \eta_{11} &\approx -0.4875143149, \quad \eta_{12} \approx 0.1000004007. \end{aligned} \tag{7.10}$$

Substituting (7.10) into (7.8), we obtain the first and second components of the first approximation $\tilde{x}_1 = \operatorname{col}(\tilde{x}_{11}, \tilde{x}_{12})$ to a solution of the given integral boundary value problem (7.1), (7.2):

$$\begin{aligned}\tilde{x}_{11}(t) &= -0.5000145056 + (1/400)t^4 - 0.001666738533t^3 \\ &\quad + 0.05000156555t^2 + 0.00010378326t, \\ \tilde{x}_{12}(t) &= 5.750026703\cdot 10^{-7} + 0.1999349926t - 0.001562508715t^4 \\ &\quad + 0.0010417017t^3 - 9.066\cdot 10^{-7}t^2.\end{aligned} \tag{7.11}$$

Let us now show that problem (7.1), (7.2) indeed has a solution in a neighbourhood of $\tilde{x}_1$. For this purpose, we analyse the corresponding properties of first approximation given in (7.8) and use Theorem 4.

We choose the domain $\Omega := D_1 \times D_2$ in (5.11) as a direct product of rectangles

$$D_1 := \{(z_1, z_2) : -0.55 \le z_1 \le 0.1;\ -0.1 \le z_2 \le 0.05\} \subset D_a, \tag{7.12}$$

$$D_2 := \{(\eta_1, \eta_2) : -0.55 \le \eta_1 \le 0;\ 0 \le \eta_2 \le 0.12\} \subset D_b. \tag{7.13}$$

The the boundaries ∂D_1 and ∂D_2 thus have, respectively, the equations

$$\begin{aligned}&\{z_1 = -0.55, \quad -0.1 \le z_2 \le 0.05\}, \\ &\{z_2 = 0.05, \quad -0.55 \le z_1 \le 0.1\} \\ &\{z_1 = 0.1, \quad -0.1 \le z_2 \le 0.05\} \\ &\{z_2 = -0.1, \quad -0.55 \le z_1 \le 0.1\}\end{aligned} \tag{7.14}$$

and

$$\begin{aligned}&\{\eta_1 = -0.55, \quad 0 \le \eta_2 \le 0.12\}, \\ &\{\eta_2 = 0.12, \quad -0.55 \le \eta_1 \le 0\}, \\ &\{\eta_1 = 0, \quad 0 \le \eta_2 \le 0.12\}, \\ &\{\eta_2 = 0, \quad -0.55 \le \eta_1 \le 0\}.\end{aligned} \tag{7.15}$$

Consequently, the fulfilment of the relation (5.12) for the first iteration ($m = 1$) should be checked for all values (7.14) and (7.15).

Direct computations for this example give that $g \in \operatorname{Lip}_{K_g}(D)$ with D given by (7.7) and

$$K_g = \begin{pmatrix} 0.2 & 0.375 \\ 0 & 0.2 \end{pmatrix}.$$

Furthermore,

$$\frac{10(b-a)^2}{27}\begin{pmatrix} KQ\,(\mathbb{1}_n - Q)^{-1}\,\delta_{[a,b],D}(f) \\ K_g\,Q\,(\mathbb{1}_n - Q)^{-1}\,\delta_{[a,b],D}(f) \end{pmatrix} = \begin{pmatrix} 2.32707024\cdot 10^{-4} \\ 6.065016891\cdot 10^{-4} \\ 2.406173922\cdot 10^{-4} \\ 3.155040783\cdot 10^{-5} \end{pmatrix}. \tag{7.16}$$

Comparing the terms involved in (5.12) at the points determined by equations (7.14) and (7.15) and using (7.16), we find that relation (5.12) is satisfied in this case.

In order to verify condition (5.13), we use the differentiability of f. We form the Jacobi matrix

$$\frac{\partial H_1}{\partial(z,\eta)} = \begin{pmatrix} \frac{\partial H_{11}}{\partial z_1} & \frac{\partial H_{11}}{\partial z_2} & \frac{\partial H_{11}}{\partial \eta_1} & \frac{\partial H_{11}}{\partial \eta_2} \\ \frac{\partial H_{12}}{\partial z_1} & \frac{\partial H_{12}}{\partial z_2} & \frac{\partial H_{12}}{\partial \eta_1} & \frac{\partial H_{12}}{\partial \eta_2} \\ \frac{\partial H_{13}}{\partial z_1} & \frac{\partial H_{13}}{\partial z_2} & \frac{\partial H_{13}}{\partial \eta_1} & \frac{\partial H_{13}}{\partial \eta_2} \\ \frac{\partial H_{14}}{\partial z_1} & \frac{\partial H_{14}}{\partial z_2} & \frac{\partial H_{14}}{\partial \eta_1} & \frac{\partial H_{14}}{\partial \eta_2} \end{pmatrix}, \tag{7.17}$$

where

$$\begin{aligned} H_{11}(z,\eta) &= \eta_1 - z_1 - \int_a^b f_1(t, x_{11}(t,z,\eta), x_{12}(t,z,\eta))dt, \\ H_{12}(z,\eta) &= \eta_2 - z_2 - \int_a^b f_2(t, x_{11}(t,z,\eta), x_{12}(t,z,\eta))dt, \\ H_{13}(z,\eta) &= \int_a^b g_1(t, x_{11}(t,z,\eta), x_{12}(t,z,\eta))dt + \frac{197}{48000}, \\ H_{14}(z,\eta) &= \int_a^b g_2(t, x_{11}(t,z,\eta), x_{12}(t,z,\eta))dt - \frac{1}{4000} \end{aligned}$$

with f and g as above. At point (7.10), matrix (7.17) has the value

$$\begin{pmatrix} -0.991664946 & 0.0165680208 & 1.016639475 & 0.0334006242 \\ -0.005205078 & -1.001032832 & -0.0104752604 & 0.9967402959 \\ -0.00208135681 & -0.02066615277 & -0.00609620114 & -0.041047519 \\ -0.00000185526 & -0.00124495269 & 0.00001482908 & 0.0049824817 \end{pmatrix}.$$

The determinant of the above matrix is equal to $5.0813401 \cdot 10^{-5}$ and, in particular, is different from 0. Recalling Section 6, we conclude that (6.1) holds with $\bar{z}_1 = \mathrm{col}(z_{11}, z_{12})$ and $\bar{\eta}_1 = \mathrm{col}(\eta_{11}, \eta_{12})$ given by (7.10), whence (5.13) follows. Thus, by virtue of Theorem 4, there exist a pair $(z^*, \eta^*) \in D_1 \times D_2$ such that function (5.14) is a solution of the integral boundary value problem (7.1), (7.2).

It is easy to verify that the pair of functions

$$x_1^*(t) = \frac{t^2}{20} - \frac{1}{2}, \qquad x_2^*(t) = \frac{t}{5} \tag{7.18}$$

is a solution of the integral boundary value problem (7.1), (7.2) and its values at 0 and $1/2$,

$$\begin{aligned} x_1^*(0) &= -0.5, \quad x_2^*(0) = 0, \\ x_1^*\left(\frac{1}{2}\right) &= -\frac{39}{80}, \quad x_2^*\left(\frac{1}{2}\right) = \frac{1}{10}, \end{aligned}$$

belong to the respective sets D_1 and D_2 of form (7.12) and (7.13).

Comparing (7.11) with (7.18), we find that the error of the first approximation is estimated as

$$\max_{t\in[0,\frac{1}{2}]}\left|x_1^*(t)-x_{11}(t)\right|\le 2\cdot 10^{-5},\qquad \max_{t\in[0,\frac{1}{2}]}\left|x_2^*(t)-x_{12}(t)\right|=6\cdot 10^{-6}.$$

The graphs of the first approximation and the exact solution of the given boundary value problem are shown on Figure 1.

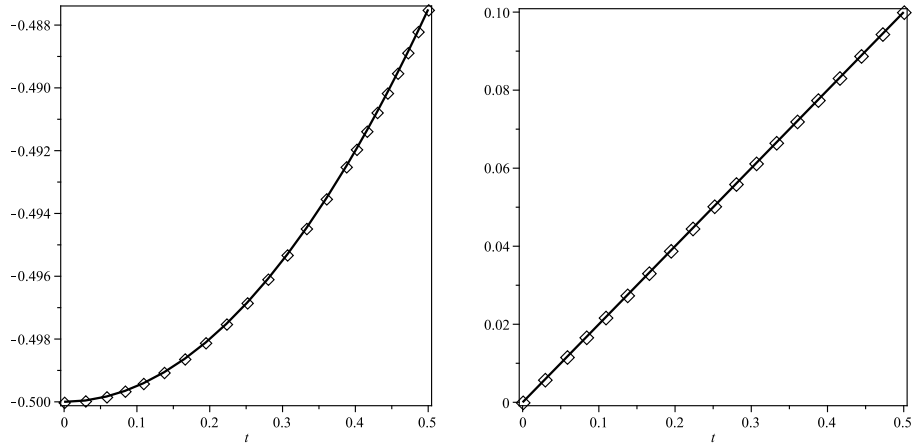

FIGURE 1. The components of the exact solution (solid line) and its first approximation (drawn with dots)

According to Theorems 2 and 3, the number of solutions of the algebraic detemining system (4.7), (4.8) coincides with the number of solutions of the given integral boundary value problem in the domain under consideration. Computations show that, along with solution (7.10), the approximate determining system of algebraic equations (5.1), (5.2) for $m=1$ has another solution

$$\begin{aligned}&\hat{z}_{11}\approx 0.3923536713,\quad \hat{z}_{12}\approx -0.1570525052,\\ &\hat{\eta}_{11}\approx 0.3868493960,\quad \hat{\eta}_{12}\approx -0.04383992217.\end{aligned}\tag{7.19}$$

Instead of $\Omega := D_1\times D_2$ defined by (7.12), (7.13), we now choose $\tilde{\Omega}=D_3\times D_4$ with

$$D_3=\{(z_1,z_2): 0\le z_1\le 0.42,\ -0.2\le z_2\le -0.12\},\tag{7.20}$$

$$D_4=\{(\eta_1,\eta_2): 0\le \eta_1\le 0.41,\ -0.1\le \eta_2\le 0.05\}.\tag{7.21}$$

The equations of ∂D_3 and ∂D_4 then have the form

$$\begin{aligned} &\{z_1 = 0, && -0.2 \le z_2 \le -0.12\},\\ &\{z_2 = -0.12, && 0 \le z_1 \le 0.42\},\\ &\{z_1 = 0.42, && -0.2 \le z_2 \le -0.12\},\\ &\{z_2 = -0.2, && 0 \le z_1 \le 0.42\} \end{aligned} \tag{7.22}$$

and

$$\begin{aligned} &\{\eta_1 = 0, && -0.1 \le \eta_2 \le 0.05\},\\ &\{\eta_2 = 0.05, && 0 \le \eta_1 \le 0.41\},\\ &\{\eta_1 = 0.41, && -0.1 \le \eta_2 \le 0.05\},\\ &\{\eta_2 = -0.1, && 0 \le \eta_1 \le 0.41\}. \end{aligned} \tag{7.23}$$

Checking (5.12) for the first iteration of form (7.8) at the boundary points (7.22) and (7.23), we find that (5.12) is satisfied. A direct computation shows that (5.13) also holds. Consequently, by Theorem 4, there exist a pair $(\hat{z}^*, \hat{\eta}^*) \in D_3 \times D_4$ such that the function

$$\hat{x}^*(t) = x_\infty(t, \hat{z}^*, \hat{\eta}^*), \qquad t \in [0, 1/2], \tag{7.24}$$

is a solution of the integral boundary value problem (7.1), (7.2) with $(\hat{x}_1^*(0), \hat{x}_2^*(0)) \in D_3$, $(\hat{x}_1^*(1/2), \hat{x}_2^*(1/2) \in D_4$. By substituting (7.19) into the first iteration (7.8), we obtain the following first approximation to solution (7.24) of (7.1), (7.2):

$$\begin{aligned} \hat{x}_{11}(t) &= 0.3923536713 + (1/400)t^4 + 0.004490021967t^3\\ &\quad - 0.07479600670t^2 + 0.02495444725t,\\ \hat{x}_{12}(t) &= -0.1570525052 + 0.2000752910t - 0.0009018708475t^4\\ &\quad - 0.005693773113t^3 + 0.05577210445t^2. \end{aligned} \tag{7.25}$$

We see that the values of (7.24) at 0 and $1/2$,

$$\begin{aligned} &\hat{x}_{11}(0) = 0.39235367135, && \hat{x}_{12}(0) = -0.1570525052,\\ &\hat{x}_{11}\Big(\frac{1}{2}\Big) = 0.3868493959, && \hat{x}_{12}\Big(\frac{1}{2}\Big) = -0.04383992216 \end{aligned}$$

belong to D_3 and D_4 of form (7.20) and (7.21) respectively. By analogy, we can obtain the second and third approximations ($m = 2, 3$) to solution (7.24).

The residual obtained as a result of substitution of the third approximation to solution (7.24) into the given differential system (7.1) is estimated as follows:

$$\max_{t\in[0,\frac{1}{2}]} \Big| x'_{31}(t) - x_{32}^2(t) + \frac{t}{5}x_{31}(t) - \frac{t^3}{100} + \frac{t^2}{25} \Big| = 3.230806 \cdot 10^{-8},$$

$$\max_{t\in[0,\frac{1}{2}]} \Big| x'_{32}(t) - \frac{t^2}{10}x_{32}(t) - \frac{t}{8}x_{31}(t) + \frac{21}{800}t^3 - \frac{1}{16}t - \frac{1}{5} \Big| = 5.55695 \cdot 10^{-9}.$$

The graphs of first and third approximations to the second solution of the given boundary value problem are shown on Figure 2.

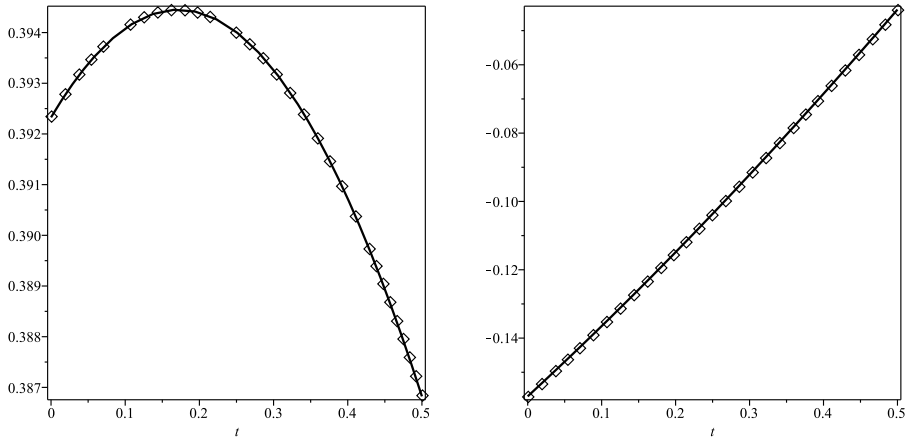

FIGURE 2. The components of the first (drawn with dots) and third (solid line) approximations to the second solution

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Authors' addresses

Miklós Rontó
Department of Analysis, University of Miskolc, 3515 Miskolc-Egyetemváros
E-mail address: matronto@uni-miskolc.hu

Yana Varha
Uzhgorod National University, 14 Universitetska St., 88000 Uzhgorod, Ukraine
E-mail address: jana.varha@mail.ru

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MONOTONE POSITIVE SOLUTION OF NONLINEAR THIRD-ORDER TWO-POINT BOUNDARY VALUE PROBLEM

YONGPING SUN, MIN ZHAO, AND SHUHONG LI

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Abstract. In this paper, we are concerned with the existence of monotone positive solution to the nonlinear third-order two-point boundary value problem

$$\begin{cases} x'''(t) + f(x(t)) = 0, \quad t \in (0,1), \\ x(0) = x''(0) = x'(1) = 0. \end{cases}$$

Under suitable assumptions on f and by using a fixed point theorem of cone expansion and compression of functional type due to Avery, Anderson and Krueger, we establish the existence of at least one positive solution of the boundary value problem.

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Keywords: monotone positive solution, nonlinear boundary value problem, fixed point theorem, existence

1. Introduction

In this paper, we are concerned with the existence of single monotone positive solution to nonlinear third-order two-point boundary value problem (BVP):

$$\begin{cases} x'''(t) + f(x(t)) = 0, \quad t \in (0,1), \\ x(0) = x''(0) = x'(1) = 0, \end{cases} \tag{1.1}$$

where $f : [0,\infty) \to [0,\infty)$ is continuous. Here, by a positive solution u^* of BVP (1.1), we mean a function u^* satisfies (1.1) and $u^*(t) > 0$, for $0 < t < 1$.

Third-order equations arise in a variety of different areas of applied mathematics and physics, such as the deflection of a curved beam having a constant or varying cross section, three layer beam, electromagnetic waves or gravity-driven flows and so on [17]. The existence of positive solutions of third-order equations with two-point or three-point boundary conditions have been studied by several authors using the fixed point index theorem, Leray-Schauder continuation theorem, nonlinear alternative of Leray, upper solution and lower solutions method, and comparison principles and so

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on. One may see [3,4,7–11,13–16,18,20–24,26] and the references therein for some related results. In a recent paper [25], by applying a monotone iterative method, Sun studied the boundary value problem

$$\begin{cases} x'''(t) + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ x(0) = x''(0) = x'(1) = 0, \end{cases}$$

where $f : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous, and $q : (0,1) \to [0,\infty)$ is a continuous function such that $0 < \int_0^1 (1-s)q(s)ds < \infty$. The iterative schemes for approximating the solutions are obtained.

One of the most frequently used tools for proving the existence of positive solutions to the boundary value problems is Krasnosel'skii's theorem on cone expansion and compression and its norm-type version due to Guo [19]. Many authors have used this approach or a variation to obtain positive solution of boundary value problem; see [1, 2]. Recently, Anderson and Avery in [5], Avery, Anderson and Krueger in [6] generalized the fixed point theorem of cone expansion and compression of norm type by replacing the norms with two functionals satisfying certain conditions to produce a fixed point theorem of cone expansion and compression of functional type, and they applied the fixed point to verify the existence of one positive solution to a discrete second-order conjugate boundary value problem and the right focal second order boundary value problem. Ehrke [12] applied this fixed point to verify the existence of one positive solution to the left focal second order boundary value problem. According to [5], the flexibility of using functionals instead of norms allows the theorem to be used in a wider variety of situations especially in applications to boundary value problems it allows for improved sufficiency conditions for the existence of a positive solution.

Motivated greatly by the above-mentioned works, in this paper we will try using this fixed point theorem to consider the existence of monotone positive solution to BVP (1.1). This paper is organized as follows. In Section 2, we present some definitions and background results on cones and completely continuous operators. We also state the fixed point theorem of cone expansion and compression of functional type. Some lemmas and the main result will be given in Section 3. In Section 4, an explicit example is given to illustrate our main result.

2. Preliminaries

In this section, we present some definitions and background results on cones and completely continuous operators. We also state the fixed point theorem of cone expansion and compression of functional type.

Definition 1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a *cone* of E if it satisfies the following two conditions:

(1) $x \in P$, $\lambda > 0$ implies $\lambda x \in P$;
(2) $x \in P$, $-x \in P$ implies $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Definition 2. A cone P of a real Banach space E is said to be normal if there exists $\delta > 0$ such that $\|x + y\| \geq \delta$ for all $x, y \in P$ with $\|x\| = \|y\| = 1$.

It is well known that for a cone P in a real Banach space E, P is normal if and only if the norm of the Banach space E is semimonotone. That is, there exists a constant $N > 0$ such that $0 \leq x \leq y$ implies that $\|x\| \leq N\|y\|$. A proof of this result can be found in [19].

Definition 3. Let E be a real Banach space. An operator $T : E \to E$ is said to be completely continuous if it is continuous and maps bounded sets into precompact sets. Additionally, the operator T is called increasing on a domain D, provided that $Tx_1 \leq Tx_2$ for all $x_1, x_2 \in D$ with $x_1 \leq x_2$.

Let α and γ are non-negative continuous functionals on P; then, for positive real numbers r and R, we define the following sets:

$$P(\alpha, r) = \{x \in P : \alpha(x) < r\},$$
$$P(\gamma, R) = \{x \in P : \gamma(x) < R\},$$
$$P(\gamma, \alpha, r, R) = \{x \in P : r < \alpha(x) \text{ and } \gamma(x) < R\}.$$

Now we state the fixed point theorem of cone expansion and compression of functional type due to Anderson, Avery, and Krueger in [6].

Theorem 1. *Let P be a normal cone in a real Banach space E, and let α and γ are nonnegative continuous functionals on P. Assume $P(\gamma, \alpha, r, R)$ as above is a nonempty bounded subset of P,*

$$T : \overline{P(\gamma, \alpha, r, R)} \to P$$

is a completely continuous operator and

$$\overline{P(\alpha, r)} \subseteq P(\gamma, R).$$

If one of the two conditions,

(H1) $\alpha(Tx) \leq r$ *for all* $x \in \partial P(\alpha, r)$, $\gamma(Tx) \geq R$ *for all* $x \in \partial P(\gamma, R)$,

$$\inf_{x \in \partial P(\gamma, R)} \|Tx\| > 0,$$

and for all $y \in \partial P(\alpha, r)$, $z \in \partial P(\gamma, R)$, $\lambda \geq 1$, *and* $\mu \in (0, 1]$, *the functionals satisfy the properties*

$$\alpha(\lambda y) \geq \lambda \alpha(y), \ \gamma(\mu z) \leq \mu \gamma(z), \text{ and } \alpha(0) = 0,$$

$(H2)$ $\alpha(Tx) \geq r$, *for all* $x \in \partial P(\alpha, r)$, $\gamma(Tx) \leq R$, *for all* $x \in \partial P(\gamma, R)$,

$$\inf_{x \in \partial P(\alpha,r)} \|Tx\| > 0,$$

and for all $y \in \partial P(\alpha, r)$, $z \in \partial P(\gamma, R)$, $\lambda \in (0,1]$, *and* $\mu \geq 1$, *the functionals satisfy the properties*

$$\alpha(\lambda y) \leq \lambda\alpha(y),\ \gamma(\mu z) \geq \mu\gamma(z),\ \text{and}\ \gamma(0) = 0,$$

is satisfied, then T *has at least one fixed point* x^* *such that*

$$r \leq \alpha(x^*)\ \text{and}\ \gamma(x^*) \leq R.$$

Moreover, suppose there exist $x_l,\ x_u \in P$ *such that* $\overline{P(\gamma, \alpha, r, R)} \subseteq [x_l, x_u]$.

$(E1)$ *If there exists an increasing, completely continuous operator* $U : [x_l, x_u] \to P$, *such that* $Tx \leq Ux$ *for all* $x \in [x_l, x_u]$ *and* $U^2 x_u \leq Ux_u$, *then*

$$x^* \leq x_u^* \leq U^n x_u,$$

where $n \in N$ *and* $x_u^* = \lim_{n\to\infty} U^n x_u$.

$(E2)$ *If there exists an increasing, completely continuous operator* $L : [x_l, x_u] \to P$, *such that* $Lx \leq Tx$ *for all* $x \in [x_l, x_u]$ *and* $Lx_l \leq L^2 x_l$, *then*

$$L^n x_l \leq x_l^* \leq x^*,$$

where $n \in N$ *and* $x_l^* = \lim_{n\to\infty} L^n x_l$.

3. Main result

In this section, we discuss the existence of positive solution of BVP (1.1). We shall consider the Banach space $E = C[0,1]$ equipped with norm $\|u\| = \max_{0\leq t\leq 1} |u(t)|$. Denote

$$C^+[0,1] = \{u \in C[0,1] : u(t) \geq 0,\ t \in [0,1]\}.$$

In arriving our result, we need the following three preliminary lemmas.

Lemma 1 ([23]). *Let* $x(t) \in C[0,1]$, *then the BVP:*

$$\begin{cases} u'''(t) + f(x(t)) = 0, \quad 0 \leq t \leq 1, \\ u(0) = u''(0) = u'(1) = 0, \end{cases} \tag{3.1}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s) f(x(s)) ds,$$

where

$$G(t,s) = \begin{cases} t(1-s), \quad 0 \leq t \leq s \leq 1, \\ t(1-s) - \dfrac{1}{2}(t-s)^2, \quad 0 \leq s \leq t \leq 1. \end{cases} \tag{3.2}$$

By simple computation we can get the following results.

Lemma 2. *For any $\eta \in (0,1)$, the Green's fuction $G(t,s)$ defined by* (3.2) *have the following properties:*

$$\int_0^1 G(t,s)ds = -\frac{1}{6}t^3 + \frac{1}{2}t, \qquad t \in [0,1];$$

$$\int_\eta^1 G(t,s)ds = \begin{cases} \dfrac{1}{2}(1-\eta)^2 t, & t \in [0,\eta], \\ -\dfrac{1}{6}(t-\eta)^3 + \dfrac{1}{2}(1-\eta)^2 t, & t \in [\eta,1]. \end{cases}$$

Particularly,

$$\int_0^1 G(1,s)ds = \frac{1}{3}, \quad \int_\eta^1 G(\eta,s)ds = \frac{1}{2}\eta(1-\eta)^2.$$

Define the cone P by

$$P = \{x \in C^+[0,1] : x(t) \text{ is concave and nondecreasing}\}.$$

Then P is a normal cone of E. Define the integral operator $T : P \to C^+[0,1]$ by

$$(Tx)(t) = \int_0^1 G(t,s) f(x(s))ds. \tag{3.3}$$

By Lemma 1, BVP (1.1) has a positive solution $x^* = x^*(t)$ if and only if x^* is a fixed point of T.

Lemma 3. *For any $x \in P$, $\eta \in (0,1)$, the following properties hold:*

(1) $x(t) \geq t\|x\|$, $t \in [0,1]$;
(2) $x(t) \geq \eta\|x\|$, $t \in [\eta,1]$;
(3) $\|x\| = x(1)$.

Lemma 4. *Suppose that $f \in C([0,\infty),[0,\infty))$, then $T : P \to P$ is completely continuous.*

Proof. For $x \in P$, let $w(t) = (Tx)(t)$. Then $w(0) = w''(0) = w'(1) = 0$ and $w'''(t) = -f(x(t)) \leq 0$. Hence $w''(t)$ is decreasing and since $w''(0) = 0$, we know $w''(t) \leq 0$ for all $t \in [0,1]$, thus $w(t)$ is concave and $w'(t)$ is decreasing. By the boundary conditions $w'(1) = 0$ we have $w'(t) \geq 0$ for all $t \in [0,1]$, thus $w(t)$ is increasing and $w(t) \geq 0$ by the boundary condition $w(0) = 0$. Hence $w(t) \in P$. So $T : P \to P$. The operator T is completely continuous by an application of Ascoli-Arzela Theorem (see [27]). □

Define continuous functionals $\alpha, \gamma : P \to [0,\infty)$ by

$$\alpha(x) := \min_{t\in[\eta,1]} x(t) = x(\eta), \text{ and } \gamma(x) := \max_{t\in[0,1]} x(t) = \|x\| = x(1).$$

It is clear that

$$\alpha(\lambda y) = \lambda\alpha(y), \ \gamma(\mu z) = \mu\gamma(z), \text{ and } \gamma(0) = 0,$$

for all $y \in \partial P(\alpha, r)$, $z \in \partial P(\gamma, R)$, $\lambda \in (0,1]$, and $\mu \geq 1$.

Now we discuss the existence of at least one monotone positive solution for the BVP (1.1). We obtain the following existence result.

Theorem 2. *Suppose that $f \in C([0,\infty),[0,\infty))$, $\eta \in (0,1)$ and there exist two positive real numbers r, R with $r < \eta R$ such that the following conditions are satisfied:*

$(A1)$ $f(w) \geq \dfrac{2r}{\eta(1-\eta)^2}$, *for* $w \in [r,R]$,

$(A2)$ $f(w) \leq 3R$, *for* $w \in [0,R]$.

Then BVP (1.1) *has at least one monotone positive solution $x^* \in P$ such that*

$$\frac{rt}{\eta} \leq x^*(t) \leq \frac{1}{2}(-t^3+3t)R, \quad t \in [0,\eta],$$

and

$$\left(-\frac{(t-\eta)^3}{3\eta(1-\eta)^2}+\frac{t}{\eta}\right)r \leq x^*(t) \leq \frac{1}{2}(-t^3+3t)R, \quad t \in [\eta,1].$$

Proof. From Lemma 4 we know $T : P \to P$ is completely continuous. Also, the cone P is normal since the norm of the Banach space E is semi-monotone. We verify a series of claims for the result.

Claim 1. $\overline{P(\alpha,r)} \subset P(\gamma,R)$.

Let $x \in \overline{P(\alpha,r)}$, from Lemma 3 we have

$$r \geq \alpha(x) = x(\eta) \geq \eta\|x\| = \eta\gamma(x).$$

Thus $R > \frac{1}{\eta}r \geq \gamma(x)$, i.e. $x \in P(\gamma,R)$, Therefore, $\overline{P(\alpha,r)} \subset P(\gamma,R)$.

Claim 2. For $x \in \partial P(\alpha,r)$ (i.e. $\alpha(x) = x(\eta) = r$) we have $\alpha(Tx) \geq r$.

Let $x \in \partial P(\alpha,r)$, then $x(s) \geq \alpha(x) = r$, $s \in [\eta,1]$. Thus it follows from $(A1)$ and Lemma 2,

$$\alpha(Tx) = \min_{t\in[\eta,1]}(Tx)(t) = (Tx)(\eta) = \int_0^1 G(\eta,s)f(x(s))ds \geq \int_\eta^1 G(\eta,s)f(x(s))ds$$
$$\geq \frac{2r}{\eta(1-\eta)^2}\int_\eta^1 G(\eta,s)ds = \frac{2r}{\eta(1-\eta)^2}\cdot\frac{\eta(1-\eta)^2}{2} = r.$$

The above arguments also yield

$$\inf_{x\in\,\partial P(\alpha,r)}\|Tx\| \geq r > 0.$$

Claim 3. For $x \in \partial P(\gamma, R)$ (i.e. $\gamma(x) = x(1) = R$) we have $\gamma(Tx) \leq R$.

Let $x \in \partial P(\gamma, R)$, then $x(s) \leq \gamma(x) = R,\ s \in [0,1]$. Thus from $(A2)$ and Lemma 2 we have

$$\begin{aligned}\gamma(Tx) &= \max_{t\in[0,1]}(Tx)(t) = (Tx)(1) = \int_0^1 G(1,s)f(x(s))ds \\ &\leq 3R\int_0^1 G(1,s)ds = 3R\cdot\frac{1}{3} = R.\end{aligned}$$

By Theorem 1 we know that T has a fixed point $x^* \in P$ such that

$$r \leq \alpha(x^*) \quad \text{and} \quad \gamma(x^*) \leq R.$$

Now define the increasing completely continuous operators $L : P \to P$ and $U : P \to P$ by

$$(Lx)(t) := \int_\eta^1 G(t,s)\frac{2r}{\eta(1-\eta)^2}ds$$

and

$$(Ux)(t) := \int_0^1 G(t,s)3Rds$$

respectively. Then for all $x \in P(\gamma, \alpha, r, R)$, we have

$$Lx \leq Tx \leq Ux.$$

Moreover, if we define $x_l,\ x_u \in P$ by

$$x_l(t) := \begin{cases} \dfrac{rt}{\eta}, & t \in [0,\eta], \\ r, & t \in [\eta, 1], \end{cases} \quad \text{and} \quad x_u(t) := R, \quad t \in [0,1],$$

then as a consequence of the concavity of P, we have $P(\gamma, \alpha, r, R) \subset [x_l, x_u]$. Clearly, by the the definition of U and L, for any $t \in [0,1]$, we have

$$(Ux_u)(t) = \int_0^1 G(t,s)3Rds \leq 3R\int_0^1 G(1,s)ds = R = x_u(t),$$

and

$$
\begin{aligned}
(Lx_l)(t) &= \int_\eta^1 G(t,s)\frac{2r}{\eta(1-\eta)^2}ds = \frac{2r}{\eta(1-\eta)^2}\int_\eta^1 G(t,s)ds \\
&= \frac{2r}{\eta(1-\eta)^2}\begin{cases} \dfrac{1}{2}(1-\eta)^2 t, & t\in[0,\eta], \\ -\dfrac{1}{6}(t-\eta)^3+\dfrac{1}{2}(1-\eta)^2 t, & t\in[\eta,1] \end{cases} \\
&\geq \frac{2r}{\eta(1-\eta)^2}\begin{cases} \dfrac{1}{2}(1-\eta)^2 t, & t\in[0,\eta], \\ \dfrac{1}{2}\eta(1-\eta)^2, & t\in[\eta,1] \end{cases} \\
&= \begin{cases} \dfrac{rt}{\eta}, & t\in[0,\eta], \\ r, & t\in[\eta,1] \end{cases} = x_l(t).
\end{aligned}
$$

This gives

$$
(Lx_l)(t) = \frac{2r}{\eta(1-\eta)^2}\int_\eta^1 G(t,s)ds \leq x^*(t) \leq 3R\int_0^1 G(t,s)ds = (Ux_u)(t).
$$

It follows from Lemma 2,

$$
\frac{rt}{\eta} \leq x^*(t) \leq \frac{1}{2}(-t^3+3t)R, \quad t\in[0,\eta],
$$

and

$$
\left(-\frac{(t-\eta)^3}{3\eta(1-\eta)^2}+\frac{t}{\eta}\right)r \leq x^*(t) \leq \frac{1}{2}(-t^3+3t)R, \quad t\in[\eta,1].
$$

This completes the proof. □

4. Example

In this section, in order to illustrate our result, we consider one simple example.

Example 1. Consider the BVP

$$
\begin{cases} x'''(t)+\dfrac{1}{64}x^2(t)+\dfrac{1}{8}x(t)+\dfrac{32\sqrt[3]{x(t)}}{1+\sqrt[3]{x(t)}}=0, & t\in(0,1), \\ x(0)=x''(0)=x'(1)=0. \end{cases} \tag{4.1}
$$

Set $f(x)=\frac{1}{64}x^2+\frac{1}{8}x+\frac{32\sqrt[3]{x}}{1+\sqrt[3]{x}}$. Then $f\in C([0,\infty),[0,\infty))$.

Let $\eta=\frac{1}{2}$, $r=1$, $R=8$. For $x\in[0,R]$, we have

$$
f(x)\leq f(R)=f(8)=\frac{70}{3}<24=3R,
$$

and for $x \in [r, R]$, we have

$$f(x) \geq f(r) = f(1) = \frac{1033}{64} > 16 = \frac{2r}{\eta(1-\eta)^2}.$$

Hence, by Theorem 2, BVP 4.1 has at least one monotone positive solution $x^*(t)$ such that

$$2t \leq x^*(t) \leq -4t^3 + 12t, \quad t \in [0, \frac{1}{2}],$$

and

$$-\frac{8}{3}(t-\frac{1}{2})^3 + 2t \leq x^*(t) \leq -4t^3 + 12t, \quad t \in [\frac{1}{2}, 1].$$

Remark 1. In this example, the function f satisfy condition $f(0) = 0$, thus the problem has a trivial solution.

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Authors' addresses

Yongping Sun
School of Electron and Information, Zhejiang University of Media and Communications, Hangzhou, Zhejiang 310018, China
E-mail address: sunyongping@126.com

Min Zhao
School of Electron and Information, Zhejiang University of Media and Communications, Hangzhou, Zhejiang 310018, China
E-mail address: zhaomin666@sina.com

Shuhong Li
Department of Mathematics, Lishui University, Lishui, Zhejiang 323000, China
E-mail address: mathlsh@126.com

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GENERAL DECAY OF SOLUTIONS FOR A NONLINEAR VISCOELASTIC WAVE EQUATION WITH NONLOCAL BOUNDARY DAMPING

FARAMARZ TAHAMTANI AND AMIR PEYRAVI

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Abstract. In this work we show that under weaker assumptions on the memory kernel g, exponential and polynomial decay rates of the solution energy in Li and Zhao [8] are only special cases. Our result improves earlier results in the literature.

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1. INTRODUCTION

In this article, we investigate the following initial value problem

$$\begin{cases} u_{tt} - k_0\Delta u + \displaystyle\int_0^t g(t-s)\mathrm{div}\big[a(x)\nabla u(s)\big]ds + b(x)h(u_t) = 0, \\ \qquad (x,t) \in \Omega \times (0,\infty), \\ -k_0\dfrac{\partial u}{\partial \nu} + \displaystyle\int_0^t g(t-s)\big(a(x)\nabla u(s)\big)\,.\,\nu ds = f(u), \quad (x,t) \in \Gamma_1 \times (0,\infty), \\ u(x,t) = 0, \quad (x,t) \in \Gamma_0 \times (0,\infty), \\ u(x,0) = u_0(x) \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{1.1}$$

where $k_0 > 0$, Ω is a bounded domain in $R^n (n \geq 1)$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1, \overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \varnothing$, Γ_0 and Γ_1 are closed with positive measures, ν is the unit outward normal to $\partial\Omega$, g denotes the memory kernel and a, b, h and f are real valued functions which satisfy appropriate conditions.

This problem arises in the study of motion of viscoelastic materials. We refer to [7,18] for mathematical analysis on the motions of materials with memory. The above problem with dirichlet boundary conditions has been considered by many authors. In this regard we recall the pioneer works by Cavalcanti et al. [1, 3], Santos [19] and Rivera et al. [15, 16].

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In [8], Li and Zhao studied the problem (1.1) and proved exponential and polynomial decay results under weaker assumptions on g which improved [17]. In fact, in [17], the authors studied problem (1.1) with nonlinear boundary damping when $f(u) = |u|^\gamma u$ and $b(x) = 1$ on Ω. Assuming that the kernel g in the memory term decays exponentially, they showed exponential energy decay by using the perturbed energy method provided that $\|g\|_{L^1[0,\infty)}$ is small enough. In [9] Li et al. considered a related problem with nonlinear boundary dissipation. Under suitable conditions on the initial data and relaxation function, they established existence and uniqueness of global solutions by means of Galerkin method and showed that the energy decays exponentially if the decay rate of the memory kernel is also exponentially. These results have been recently improved by Wu and Chen [21] where the authors considered a nonlinear wave equation with boundary dissipation in presence of a local damping term, $b(x)u_t$, in Ω. The authors used Lyapunov functions to establish general decay rate of solution energy which is not necessarily of exponential or polynomial type. However, in order to prove main results, they supposed that the function $a(x)$ satisfies

$$|\nabla a(x)|^2 \leq \alpha_1^2 |a(x)|. \tag{1.2}$$

For more related results about the boundary stabilization we refer to Cavalcanti et al.[2], Liu and Yu [10], Lu et al. [11], Messaoudi and Soufyane [13]. We can also recall some other pioneer papers in connecting with the viscoelasticity such as Sobrinho and Rivera [20], Fabrizio and Polidro [6], Rivera et al. [14] and Dafermos [4].

In this work, we study problem (1.1). We show that for a certain class of relaxation functions, the decay rate of the energy is similar to that of g. Therefore, our result improves earlier results in [8, 9] where only the exponential and polynomial decay rates are obtained. The main point of the contribution is based on an inequality given by Martinez [12]. In this way, we are allowed to weaken some technical assumptions for the kernel g or even for the function $a(x)$ (such as inequality (1.2) which has been considered in [21]).

2. Preliminaries

In this section we present some materials that will be needed throughout the paper. We begin by presenting the precise hypotheses on the problem (1.1).

(H_1) $a, b : \Omega \to R$ are positive functions so that $a, b \in L^\infty(\Omega)$ and

$$b(x) \geq b_0 > 0. \tag{2.1}$$

(H_2) $f : R \to R$ satisfies

$$f(s)s \geq 2F(s) \geq 0, \quad \forall s \in R, \tag{2.2}$$

where

$$F(z) = \int_0^z f(s)ds. \tag{2.3}$$

(H_3) $h : R \to R$ is a nondecreasing function, such that for some positive constants α and β, satisfies

$$h(s)s \geq \alpha|s|^2, \qquad |h(s)| \leq \beta|s|, \qquad \forall s \in R. \tag{2.4}$$

(H_4) $g : [0,\infty) \to [0,\infty)$ is a non-increasing C^1 function such that

$$g(0) > 0, \qquad k_0 - \|a\|_\infty \int_0^{+\infty} g(s)ds = l > 0, \tag{2.5}$$

and there exists a non-increasing positive differentiable function ξ such that

$$g'(t) \leq -\xi(t)g(t), \qquad \forall t \geq 0, \qquad \int_0^{+\infty} \xi(s)ds = \infty. \tag{2.6}$$

We will also consider the Hilbert space

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) \,:\, u = 0 \text{ on } \Gamma_0\}.$$

Lemma 1 (Poincaré inequality)**.** *There exists a positive constant B such that*

$$\|u\|_{L^2(\Omega)} \leq B\|\nabla u\|_{L^2(\Omega)},$$

for all $u \in H^1_{\Gamma_0}(\Omega)$.

Referring to [5, 17], we state the following existence and uniqueness theorem.

Theorem 1. *If $(u_0, u_1) \in \big(H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\big) \times H^1_{\Gamma_0}(\Omega)$, then the problem* (1.1) *has a unique solution satisfying*

$$u \in L^\infty_{loc}\big(0,\infty; H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega)\big), \quad u_t \in L^\infty_{loc}\big(0,\infty; H^1_{\Gamma_0}(\Omega)\big),$$

$$u_{tt} \in L^\infty_{loc}\big(0,\infty; L^2(\Omega)\big).$$

Moreover

$$u \in C\big([0,\infty); H^1_0(\Omega)\big), \qquad u_t \in C\big([0,\infty); L^2(\Omega)\big).$$

Finally, we present the following lemma by Martinez [12] which plays important role in our proof.

Lemma 2. *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonincreasing function and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a C^2 increasing function such that $\psi(0) = 0$ and $\lim_{t\to+\infty} \psi(t) = +\infty$. Assume that there exists $c > 0$ for which*

$$\int_t^{+\infty} \psi'(s)E(s)ds \leq cE(t), \qquad \forall t \geq 0, \tag{2.8}$$

then

$$E(t) \leq \lambda E(0)e^{-\omega\psi(t)}, \tag{2.9}$$

for some positive constants ω and λ.

3. Energy decay

In this section we state and prove our main result. First, we define the energy related to problem (1.1) as in [8]

$$E(t)=\frac{1}{2}\int_\Omega |u_t(t)|^2dx+\frac{1}{2}\int_\Omega \Big(k_0-a(x)\int_0^t g(s)ds\Big)|\nabla u(t)|^2dx + \frac{1}{2}(g\circ\nabla u)(t)+\int_{\Gamma_1}F(u)d\Gamma, \tag{3.1}$$

where

$$(g\circ\nabla u)(t)=\int_\Omega\int_0^t g(t-s)a(x)|\nabla u(t)-\nabla u(s)|^2dsdx.$$

We also observe that

$$\int_\Omega\Big(k_0-a(x)\int_0^t g(s)ds\Big)|\nabla u(t)|^2dx\geq \Big(k_0-\|a\|_\infty\int_0^t g(t)dt\Big)\|\nabla u(t)\|_2^2\geq l\|\nabla u(t)\|_2^2.$$

Lemma 3 (Lemma 2.3 of [8])**.** *The energy function $E(t)$ satisfies $E(t)\geq 0$ and*

$$E'(t)=\frac{1}{2}(g'\circ\nabla u)(t)-\frac{1}{2}g(t)\int_\Omega a(x)|\nabla u(t)|^2dx-\int_\Omega b(x)h(u_t)u_tdx\leq 0. \tag{3.2}$$

Theorem 2. *Assume that $(u_0,u_1)\in\big(H^2(\Omega)\cap H^1_{\Gamma_0}(\Omega)\big)\times H^1_{\Gamma_0}(\Omega)$, $(H_1)-(H_3)$ and (2.5) hold, and*

$$a(x)\geq a_0>0. \tag{3.3}$$

(i) If

$$g'(t)\leq -cg(t),\quad c>0, \tag{3.4}$$

then, the energy $E(t)$ of problem (1.1) satisfies the decay rate

$$E(t)\leq 4E(0)e^{-Ct},\qquad \forall t\geq T.$$

(ii) If

$$g'(t)\leq -cg^{1+\frac{1}{p}}(t),\quad p>2,\quad c>0, \tag{3.5}$$

then, the energy $E(t)$ of problem (1.1) satisfies the decay rate

$$E(t)\leq\frac{C}{(1+t)^p},\qquad \forall t\geq T,$$

for some $C,T>0$.

Proof. See [8], Theorems 3.1 and 3.2. □

In the next theorem, we state our main result. We extend the above rates of decay to a more general case which is similar to that of g. In fact, we use the assumption (2.6) which is weaker than (3.4) and (3.5) .

Theorem 3. *Assume that* $(H_1)-(H_4)$ *hold. If the initial data* $(u_0,u_1)\in\big(H^2(\Omega)\cap H^1_{\Gamma_0}(\Omega)\big)\times H^1_{\Gamma_0}(\Omega)$, *then the solution of problem* (1.1) *satisfies*

$$E(t)\le KE(0)e^{-\kappa\int_0^t\xi(s)ds},\tag{3.6}$$

for some $K,\kappa>0$.

Proof. Multiplying $(1.1)_1$ by $\xi(t)u$ and integrating over $\Omega\times[t_1,t_2]$ $(0\le t_1\le t_2)$, we get

$$\begin{aligned}&\int_{t_1}^{t_2}\xi(t)\int_\Omega uu_{tt}dxdt+k_0\int_{t_1}^{t_2}\xi(t)\int_\Omega|\nabla u(t)|^2dxdt\\&+\int_{t_1}^{t_2}\xi(t)\int_\Omega b(x)uh(u_t)dxdt+\int_{t_1}^{t_2}\xi(t)\int_{\Gamma_1}uf(u)d\Gamma dt\\&-\int_{t_1}^{t_2}\xi(t)\int_\Omega\nabla u(t)\,.\int_0^t g(t-s)\big(a(x)\nabla u(s)\big)dsdxdt=0.\end{aligned}\tag{3.7}$$

For the last term in the right hand side of (3.7) we have

$$\begin{aligned}&\int_\Omega\nabla u(t)\,.\int_0^t g(t-s)\big(a(x)\nabla u(s)\big)dsdx\\&=\int_\Omega\nabla u(t)\,.\int_0^t g(t-s)a(x)\big(\nabla u(s)-\nabla u(t)\big)dsdx\\&+\int_0^t g(s)ds\int_\Omega a(x)|\nabla u(t)|^2dx.\end{aligned}\tag{3.8}$$

Substituting (3.8) in (3.7) and using (2.2) , (2.3) and (3.1) we obtain

$$\begin{aligned}&2\int_{t_1}^{t_2}\xi(t)E(t)dt\\&\le-\int_{t_1}^{t_2}\xi(t)\int_\Omega uu_{tt}dxdt+\int_{t_1}^{t_2}\xi(t)\|u_t\|_2^2dt-\int_{t_1}^{t_2}\xi(t)\int_\Omega b(x)uh(u_t)dxdt\\&+\int_{t_1}^{t_2}\xi(t)\int_\Omega\nabla u(t)\,.\int_0^t g(t-s)a(x)\big(\nabla u(s)-\nabla u(t)\big)dsdxdt\\&+\int_{t_1}^{t_2}\xi(t)(g\circ\nabla u)(t)dt.\end{aligned}\tag{3.9}$$

For the first term in the right hand side of (3.9) we have

$$\begin{aligned}&-\int_{t_1}^{t_2}\xi(t)\int_\Omega uu_{tt}dxdt\\&=-\int_\Omega\xi(t)uu_t dx\Big|_{t_1}^{t_2}+\int_{t_1}^{t_2}\xi'(t)\int_\Omega uu_t dxdt+\int_{t_1}^{t_2}\xi(t)\|u_t\|_2^2dt.\end{aligned}\tag{3.10}$$

By Lemma 1 and (3.1) we have

$$\begin{aligned}\left|-\int_\Omega \xi(t)uu_t dx\Big|_{t_1}^{t_2}\right| &\le \sum_{i=1}^{2}\left|\xi(t)\int_\Omega uu_t dx\right|_{t=t_i}\\ &\le \sum_{i=1}^{2}\left[\xi(t)\left(\frac{B^2}{2}\|\nabla u\|_2^2+\frac{1}{2}\|u_t\|_2^2\right)\right]_{t=t_i}\\ &\le \sum_{i=1}^{2}\left[\left(\frac{B^2}{l}+1\right)\xi(t)E(t)\right]_{t=t_i}\le 2\left(\frac{B^2+l}{l}\right)\xi(t_1)E(t_1). \quad (3.11)\end{aligned}$$

Similarly,

$$\begin{aligned}\left|\int_{t_1}^{t_2}\xi'(t)\int_\Omega uu_t dxdt\right| &\le \int_{t_1}^{t_2}|\xi'(t)|\left(\frac{B^2}{2}\|\nabla u\|_2^2+\frac{1}{2}\|u_t\|_2^2\right)\\ &\le -\left(\frac{B^2+l}{l}\right)\int_{t_1}^{t_2}\xi'(t)E(t)dt\le\left(\frac{B^2+l}{l}\right)\xi(t_1)E(t_1). \quad (3.12)\end{aligned}$$

To estimate the last term in the right-hand side of (3.10), we use (3.2), (2.1) and (2.4) to obtain

$$E'(t)\le -\int_\Omega b(x)h(u_t)u_t dx\le -b_0\int_\Omega h(u_t)u_t dx\le -b_0\alpha\|u_t\|_2^2.$$

Therefore,

$$\int_{t_1}^{t_2}\xi(t)\|u_t\|_2^2dt\le -\frac{1}{b_0\alpha}\int_{t_1}^{t_2}\xi(t)E'(t)dt\le\frac{1}{b_0\alpha}\xi(t_1)E(t_1). \quad (3.13)$$

From (3.10)-(3.13) we get

$$\left|-\int_{t_1}^{t_2}\xi(t)\int_\Omega uu_{tt}dxdt\right|+\int_{t_1}^{t_2}\xi(t)\|u_t\|_2^2dt\le\left(3l^{-1}(B^2+l)+\frac{2}{b_0\alpha}\right)\xi(t_1)E(t_1). \quad (3.14)$$

By (2.4), Young's inequality, (3.1) and (3.2) we arrive at

$$\begin{aligned}\left|-\int_{t_1}^{t_2}\xi(t)\int_\Omega b(x)uh(u_t)dxdt\right| &\le \beta\int_{t_1}^{t_2}\xi(t)\int_\Omega b(x)\left(\frac{\delta B^2}{2}|\nabla u|^2+\frac{1}{2\delta}|u_t|^2\right)dxdt\\ &\le \frac{\delta}{2}\beta B^2\|b\|_\infty\int_{t_1}^{t_2}\xi(t)\|\nabla u\|_2^2dt-\frac{\beta}{2\delta\alpha}\int_{t_1}^{t_2}\xi(t)E'(t)dt\\ &\le \frac{\delta}{l}\beta B^2\|b\|_\infty\int_{t_1}^{t_2}\xi(t)E(t)dt+\frac{\beta}{2\delta\alpha}\xi(t_1)E(t_1). \quad (3.15)\end{aligned}$$

Also, we have

$$\int_\Omega a(x)\nabla u(t)\,.\int_0^t g(t-s)\big(\nabla u(s)-\nabla u(t)\big)dsdx$$

$$\leq \delta \int_\Omega a(x)|\nabla u(t)|^2 dx + \frac{1}{4\delta}\int_\Omega a(x)\left(\int_0^t g(t-s)\big(\nabla u(s)-\nabla u(t)\big)ds\right)^2 dx$$

$$\leq \delta\|a\|_\infty \|\nabla u\|_2^2 + \frac{1}{4\delta}\int_0^t g(s)ds \int_\Omega \int_0^t g(t-s)a(x)|\nabla u(s)-\nabla u(t)|^2 ds dx$$

$$\leq \frac{2\delta}{l}\|a\|_\infty E(t) + \frac{k_0-l}{4\delta\|a\|_\infty}(g\circ\nabla u)(t). \tag{3.16}$$

By (2.6) we have

$$\xi(t)(g\circ\nabla u)(t) \leq -(g'\circ\nabla u)(t) \leq -2E'(t). \tag{3.17}$$

Finally, using (3.14)-(3.17), the estimate (3.9) takes the form

$$\left[2-\frac{\delta}{l}\Big(\beta B^2\|b\|_\infty + 2\|a\|_\infty\Big)\right]\int_{t_1}^{t_2}\xi(t)E(t)dt$$

$$\leq \left(3l^{-1}(B^2+l)+\frac{2}{b_0\alpha}+\frac{\beta}{2\delta\alpha}\right)\xi(t_1)E(t_1) - \left(\frac{k_0-l}{2\delta\|a\|_\infty}+2\right)\int_{t_1}^{t_2}E'(t)dt$$

$$\leq \left[\left(3l^{-1}(B^2+l)+\frac{2}{b_0\alpha}+\frac{\beta}{2\delta\alpha}\right)\xi(0)+\frac{k_0-l}{2\delta\|a\|_\infty}+2\right]E(t_1) \tag{3.18}$$

Choosing δ small enough and letting t_2 go to infinity, we rewrite (3.18) as

$$\int_t^{+\infty}\xi(t)E(t)dt \leq \lambda E(t), \qquad \forall t\geq 0, \tag{3.19}$$

for some $\lambda > 0$. Then, the assumptions of Lemma 2 satisfied with $\psi(t) = \int_0^t \xi(s)ds$. Therefore (3.6) is established and the proof of Theorem 3 is now complete. □

Remark 1. We note that similar rates of decay were given in [21]. However, we did not use (3.3) which has been supposed in [8]-H_1 and [21]-A_3.

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Authors' addresses

Faramarz Tahamtani
Department of Mathematics, College of Sciences, Shiraz University, Shiraz, 71457-44776, Iran
E-mail address: tahamtani@shirazu.ac.ir

Amir Peyravi
Department of Mathematics, College of Sciences, Shiraz University, Shiraz, 71457-44776, Iran
E-mail address: peyravi@shirazu.ac.ir

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WEAKLY LASKERIAN MODULES AND WEAK COFINITENESS

BAHRAM VAKILI AND JAFAR AZAMI

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Abstract. Let R be a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R. It is shown that if $\mathfrak{a} = (x_1, \ldots, x_t)$, and M is an R-module, then $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is weakly Laskerian for all i iff $\operatorname{Tor}^R_i(R/\mathfrak{a}, M)$ is weakly Laskerian for all i iff the Koszul cohomology module $H^i(x_1, \ldots, x_t; M)$ is weakly Laskerian for all i. Furthermore, each of these coditions imply that $M/\mathfrak{a}^n M$ is weakly Laskerian for all $n \in \mathbb{N}$. In Section 3, we show that if M is an R-module with $\operatorname{Supp} M \subseteq V(\mathfrak{a})$, then M is $\mathfrak{a}$-weakly cofinite, in the following cases:
a) there exists $x \in \mathfrak{a}$ such that $0 :_M x$ and M/xM are both $\mathfrak{a}$-weakly cofinite.
b) there exists $x \in \sqrt{\mathfrak{a}}$ such that $0 :_M x$ and M/xM are both weakly Laskerian.

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1. Introduction

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity, and $\mathfrak{a}$ will be an ideal of R. Let M be an R-module. The $\mathfrak{a}$-*torsion submodule* of M is defined as $\Gamma_\mathfrak{a}(M) = \bigcup_{n\geq 1}(0 :_M \mathfrak{a}^n)$. The i^{th} *local cohomology functor* $\mathrm{H}^i_\mathfrak{a}(.)$ is defined as the i^{th} right derived functor $\Gamma_\mathfrak{a}(.)$. It is known that for each $i \geq 0$ there is a natural isomorphism of R-modules

$$\mathrm{H}^i_\mathfrak{a}(M) \cong \varinjlim_{n\geq 1} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M).$$

We refer the reader to [5] or [1] for the basic properties of local cohomology.

The notions of weakly Laskerian modules and $\mathfrak{a}$-weakly cofinite modules were introduced by Divaani-Aazar and Mafi in [3] and [4]. An R module M is said to be *weakly Laskerian* if the set of associated primes of any quotient module of M is finite. An R module M is said to be $\mathfrak{a}$-*weakly cofinite* if $\operatorname{Supp} M \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is weakly Laskerian for all $i \geq 0$.

Divaani-Aazar and Mafi in [4, Theorem 2.10] have shown using change of rings principle and spectral sequence that if M is an $\mathfrak{a}$-weakly cofinite R-module , then

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$M/\mathfrak{a}M$ is weakly Laskerian. In Section 2, without using change of rings principle and spectral sequence, we prove that if M is an R-module such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is a weakly Laskerian R-module for all $i \geq 0$, then $M/\mathfrak{a}^n M$ is weakly Laskerian for all $n \in \mathbb{N}$. One of the main results of this article is to prove that if $\mathfrak{a} = (x_1, \ldots, x_t)$, and M is an R-module, then the following statements are equivalent:
(ii) $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is a weakly Laskerian R-module for all i.
(ii) $\operatorname{Tor}^R_i(R/\mathfrak{a}, M)$ is a weakly Laskerian R-module for all i.
(iii) The Koszul cohomology module $H^i(x_1, \ldots, x_t; M)$ is weakly Laskerian R-module for all i.

In Section 3, we obtain a sufficient condition for $\mathfrak{a}$-weakly cofinite modules. In fact, we prove that if M is an R-module with $\operatorname{Supp} M \subseteq V(\mathfrak{a})$, then M is $\mathfrak{a}$-weakly cofinite, in the following cases:
a) there exists $x \in \mathfrak{a}$ such that $0 :_M x$ and M/xM are both $\mathfrak{a}$-weakly cofinite.
b) there exists $x \in \sqrt{\mathfrak{a}}$ such that $0 :_M x$ and M/xM are both weakly Laskerian.

In Section 4, we prove that if $\mathfrak{b}$ is a second ideal of R with $\mathfrak{b} \supseteq \mathfrak{a}$ and $\operatorname{cd}(\mathfrak{b}) = 1$ and M is a weakly Laskerian R-module, then for every finitely generated R-module L with $\operatorname{Supp} L \subseteq V(\mathfrak{b})$, the R-module $\operatorname{Ext}^{\mathrm{j}}_{\mathrm{R}}(\mathrm{L}, \mathrm{H}^{\mathrm{i}}_{\mathfrak{a}}(\mathrm{M}))$ is weakly Laskerian for all i and j. In particular, the R-module $H^i_{\mathfrak{a}}(M)/\mathfrak{b}^n H^i_{\mathfrak{a}}(M)$ is weakly Laskerian for all i and n.

2. Weakly Laskerian modules and $\mathfrak{a}$-weakly cofinite modules

To prove the main results of this paper, we need to the following two lemmas.

Lemma 1. *Let M be an R-module such that $0 :_M \mathfrak{a}$ is a weakly Laskerian R-module. Then $0 :_M \mathfrak{a}^n$ is weakly Laskerian for all $n \in \mathbb{N}$.*

Proof. Consider the exact sequence

$$0 \to 0 :_M \mathfrak{a} \to 0 :_M \mathfrak{a}^n \xrightarrow{f} a_1(0 :_M \mathfrak{a}^n) \oplus \cdots \oplus a_t(0 :_M \mathfrak{a}^n),$$

where $\mathfrak{a} = (a_1, \ldots, a_t)$ and f is defined by $f(x) = (a_1 x, \ldots, a_t x)$. The result is followed by induction on n and [3, Lemma 2.3 (i)]. Note that $a_i(0 :_M \mathfrak{a}^n)$ is a submodule of $0 :_M \mathfrak{a}^{n-1}$ for all $i = 1, 2, \ldots, t$. □

Lemma 2. *Let M be an R-module such that $M/\mathfrak{a}M$ is a weakly Laskerian R-module. Then $M/\mathfrak{a}^n M$ is weakly Laskerian for all $n \in \mathbb{N}$.*

Proof. Consider the exact sequence

$$(M/\mathfrak{a}^{n-1}M)^t \xrightarrow{f} M/\mathfrak{a}^n M \xrightarrow{g} M/\mathfrak{a}M \to 0,$$

where $\mathfrak{a} = (a_1, \ldots, a_t)$, g is the canonical map, and f is defined by

$$f(m_1 + \mathfrak{a}^{n-1}M, \cdots, m_t + \mathfrak{a}^{n-1}M) = a_1 m_1 + \cdots + a_t m_t + \mathfrak{a}^n M.$$

Now, the result is followed by induction on n and [3, Lemma 2.3 (i)]. □

Divaani-Aazar and Mafi in [4, Theorem 2.10] have shown using change of rings principle and spectral sequence that if M is an $\mathfrak{a}$-weakly cofinite R-module , then $M/\mathfrak{a}M$ is weakly Laskerian. We generalize this result and give a direct proof without using change of rings principle and spectral sequence.

Theorem 1. *Let M be an R-module such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is a weakly Laskerian R-module for all $i \geq 0$. Then $M/\mathfrak{a}^n M$ is weakly Laskerian for all $n \in \mathbb{N}$.*

Proof. By Lemma 2, it is enough to prove that $M/\mathfrak{a}M$ is weakly Laskerian. To do this, let $\mathfrak{a} = (x_1, \ldots, x_n)$. Then $M/\mathfrak{a}M \simeq H^n(x_1, \ldots, x_n; M)$, where $H^n(x_1, \ldots, x_n; M)$ denotes the n^{th} Koszul cohomology module. Consider the co-Koszul complex

$$K^\bullet(\mathbf{x}, M) : 0 \to \operatorname{Hom}(K_0(\mathbf{x}), M) \to \operatorname{Hom}(K_1(\mathbf{x}), M) \to \cdots \to \operatorname{Hom}(K_n(\mathbf{x}), M) \to 0.$$

Then $H^i(x_1, \cdots, x_n; M) = Z^i/B^i$, where B^i and Z^i are the modules of coboundaries and cocycles of the complex $K^\bullet(\mathbf{x}, M)$, respectively. Let $\mathcal{W}$ be the class of all R modules N such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, N)$ is weakly Laskerian for all $i \geq 0$. By induction we claim that $B^j \in \mathcal{W}$ for all j. We have $B^0 = 0 \in \mathcal{W}$. Now, let $B^t \in \mathcal{W}$. Put $C^i = \operatorname{Hom}(K_i(\mathbf{x}), M)/B^i$. Since $K_t(\mathbf{x})$ is a finitely generated free R-module, it follows that $\operatorname{Hom}(K_t(\mathbf{x}), M)$ is a direct sum of finitely many copies of M. Therefore, $\operatorname{Hom}(K_t(\mathbf{x}), M) \in \mathcal{W}$ by [3, Lemma 2.3 (i)]. Now, since $B^t \in \mathcal{W}$ and $\operatorname{Hom}(K_t(\mathbf{x}), M) \in \mathcal{W}$, we have $C^t \in \mathcal{W}$ by [3, Lemma 2.3 (i)]. Hence $0 :_{C^t} \mathfrak{a} \simeq \operatorname{Hom}_R(R/\mathfrak{a}, C^t)$ is weakly Laskerian. But since $\mathfrak{a}H^t(x_1, \ldots, x_n; M) = 0$, it follows that $H^t(x_1, \ldots, x_n; M) \subseteq 0 :_{C^t} \mathfrak{a}$, and so $H^t(x_1, \ldots, x_n; M)$ is weakly Laskerian. Next, from the short exact sequence

$$0 \to H^t(x_1, \ldots, x_n; M) \to C^t \to B^{t+1} \to 0$$

and [3, Lemma 2.3 (i)] we deduce that $B^{t+1} \in \mathcal{W}$. Hence, by induction we have proved that $B^j \in \mathcal{W}$ for all j. Now, since $B^n \in \mathcal{W}$ and $\operatorname{Hom}(K_n(\mathbf{x}), M) \in \mathcal{W}$, we obtain that $C^n \in \mathcal{W}$. Hence $0 :_{C^n} \mathfrak{a} \simeq \operatorname{Hom}_R(R/\mathfrak{a}, C^n)$ is weakly Laskerian. Thus $H^n(x_1, \ldots, x_n; M) \subseteq 0 :_{C^n} \mathfrak{a}$ implies that $H^n(x_1, \ldots, x_n; M)$ is weakly Laskerian. Since $M/\mathfrak{a}M = H^n(x_1, \ldots, x_n; M)$, it follows that $M/\mathfrak{a}M$ is weakly Laskerian. □

Corollary 1. *Let M be a $\mathfrak{a}$-weakly cofinite R-module. Then $M/\mathfrak{a}^n M$ is weakly Laskerian for all $n \in \mathbb{N}$.*

Proof. The assertion follows from the definition and Theorem 1. □

Corollary 2. *Let $\mathfrak{a}$ be an ideal of R, and let M be an R-module such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-weakly cofinite for all i. Then $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is weakly Laskerian for all i, and $M/\mathfrak{a}^n M$ is weakly Laskerian for all $n \in \mathbb{N}$.*

Proof. By Theorem 1, it is sufficient to prove that $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is weakly Laskerian for all i. The case $i = 0$ is clear, so let $i > 0$ and we do induction on i. We first reduce to the case $\Gamma_{\mathfrak{a}}(M) = 0$. To do this, let $\bar{M} = M/\Gamma_{\mathfrak{a}}(M)$, then we have the exact sequence

$$\ldots \to \mathrm{Ext}^i_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \to \mathrm{Ext}^i_R(R/\mathfrak{a}, M) \to \mathrm{Ext}^i_R(R/\mathfrak{a}, \bar{M}) \to \mathrm{Ext}^{i+1}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \to \ldots$$

and isomorphism $\mathrm{H}^i_{\mathfrak{a}}(M) \cong \mathrm{H}^i_{\mathfrak{a}}(\bar{M})$ for all $i > 0$. Since $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-weakly cofinite, so in view of [3, Lemma 2.3 (i)], we may assume that M is $\mathfrak{a}$-torsion free. Let E be the injective envelope of M and put $L = E/M$. Then $H^i_{\mathfrak{a}}(E) = 0$, and we therefore get the isomorphisms $\mathrm{H}^i_{\mathfrak{a}}(L) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(M)$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, L) \cong \mathrm{Ext}^{i+1}_R(R/\mathfrak{a}, M)$ for all $i \geq 0$. Now the assertion follows by induction. □

Theorem 2. *Let $\mathfrak{a} = (x_1, \ldots, x_t)$ be an ideal of R, and let M be an R-module. Then the following statements are equivalent:*

(i) $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ *is a weakly Laskerian R-module for all i.*

(ii) $\mathrm{Tor}^R_i(R/\mathfrak{a}, M)$ *is a weakly Laskerian R-module for all i.*

(iii) *The Koszul cohomology module $H^i(x_1, \ldots, x_t; M)$ is weakly Laskerian R-module for all i.*

Furthermore, each of these coditions imply that $M/\mathfrak{a}^n M$ is weakly Laskerian for all $n \in \mathbb{N}$.

Proof. (i) $\Rightarrow$ (ii) Let

$$\mathbb{F}_\bullet : \cdots \to F_2 \to F_1 \to F_0 \to 0$$

be a free resolution of finitely generated R-modules for $R/\mathfrak{a}$. We have $\mathrm{Tor}^R_i(R/\mathfrak{a}, M) = Z_i/B_i$, where B_i and Z_i are the modules of boundaries and cycles of the complex $\mathbb{F}_\bullet \otimes_R M$, respectively. Let $\mathcal{W}$ be the class of all R modules N such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, N)$ is weakly Laskerian for all $i \geq 0$. By induction we claim that $Z_j \in \mathcal{W}$ for all j. We have $Z_0 = F_0 \otimes_R M \in \mathcal{W}$. Now, let $Z_t \in \mathcal{W}$. Consider the exact sequence

$$(\dagger) \qquad 0 \to C_{i+1} \to Z_i \to \mathrm{Tor}^R_i(R/\mathfrak{a}, M) \to 0,$$

where $C_i = F_i \otimes_R M/Z_i$. Hence we obtain the exact sequence

$$Z_i/\mathfrak{a}Z_i \to \mathrm{Tor}^R_i(R/\mathfrak{a}, M) \to 0.$$

Therefore, $\mathrm{Tor}^R_t(R/\mathfrak{a}, M)$ is a homomorphic image of $Z_t/\mathfrak{a}Z_t$. Since $Z_t \in \mathcal{W}$, it follows from Theorem 1 that $Z_t/\mathfrak{a}Z_t$ is weakly Laskerian, and so $\mathrm{Tor}^R_t(R/\mathfrak{a}, M)$

is weakly Laskerian. Hence, we deduce by (†) that $C_{t+1} \in \mathcal{W}$, and so $Z_{t+1} \in \mathcal{W}$. Hence by induction we have proved that $Z_j \in \mathcal{W}$ for all j. It follows from Theorem 1 that $Z_i/\mathfrak{a}Z_i$ is weakly Laskerian for all i, and so $\operatorname{Tor}_i^R(R/\mathfrak{a}, M)$ is weakly Laskerian for all i.

To prove the implication (ii) $\Rightarrow$ (iii), since

$$H^i(x_1,\dots,x_t;M) \simeq H_{t-i}(x_1,\dots,x_t;M),$$

so it is sufficient to show that $H_i(x_1,\dots,x_t;M)$ is weakly Laskerian for all i. Let $\mathbf{x} = x_1,\dots,x_t$. Consider the Koszul complex

$$K_\bullet(\mathbf{x}) : 0 \to K_t(\mathbf{x}) \to K_{t-1}(\mathbf{x}) \to \cdots \to K_1(\mathbf{x}) \to K_0(\mathbf{x}) \to 0,$$

We have $H_i(x_1,\dots,x_t;M) = Z_i/B_i$, where B_i and Z_i are the modules of boundaries and cycles of the complex $K_\bullet(\mathbf{x}) \otimes_R M$, respectively. Let $\mathcal{W}$ be the class of all R modules N such that $\operatorname{Tor}_i^R(R/\mathfrak{a}, N)$ is weakly Laskerian for all $i \geq 0$. Consider the exact sequence

$$0 \to C_{i+1} \to Z_i \to H_i(x_1,\dots,x_t;M) \to 0,$$

where $C_i = K_i(\mathbf{x}) \otimes_R M/Z_i$. Hence we obtain the exact sequence

$$Z_i/\mathfrak{a}Z_i \to H_i(x_1,\dots,x_t;M) \to 0.$$

By using a similar proof as in the proof of the implication (i) $\Rightarrow$ (ii), $Z_i \in \mathcal{W}$ for all i. It follows that $Z_i/\mathfrak{a}Z_i = \operatorname{Tor}_0^R(R/\mathfrak{a}, Z_i)$ is weakly Laskerian for all i, and so $H_i(x_1,\dots,x_t;M)$ is weakly Laskerian for all i.

To prove the implication (iii) $\Rightarrow$ (i), let

$$\mathbb{F}_\bullet : \ \cdots \to F_2 \to F_1 \to F_0 \to 0$$

be a free resolution of finitely generated R-modules for $R/\mathfrak{a}$. We have $\operatorname{Ext}_R^i(R/\mathfrak{a}, M) = Z^i/B^i$, where B^i and Z^i are the modules of coboundaries and cocycles of the complex $\operatorname{Hom}_R(\mathbb{F}_\bullet, M)$, respectively. Let $\mathcal{W}$ be the class of all R modules N such that $H^i(x_1,\dots,x_t;N)$ is weakly Laskerian for all $i \geq 0$. Consider the short exact sequence

$$0 \to \operatorname{Ext}_R^i(R/\mathfrak{a}, M) \to C^i \to B^{i+1} \to 0,$$

where $C^i = \operatorname{Hom}_R(F_i, M)/B^i$. By using a similar proof as in the proof of the Theorem 1, $B^i \in \mathcal{W}$ for all i. Thus $C^i \in \mathcal{W}$ for all i. Now, since

$$\operatorname{Ext}_R^i(R/\mathfrak{a}, M) \subseteq 0 :_{C^i} \mathfrak{a} \simeq \operatorname{Hom}_R(R/\mathfrak{a}, C^i) \simeq H^0(x_1,\dots,x_t;C^i)$$

and $H^0(x_1,\dots,x_t;C^i)$ is weakly Laskerian, so $\operatorname{Ext}_R^i(R/\mathfrak{a}, M)$ is weakly Laskerian for all i.

Finally, the end part is followed by Theorem 1.

□

The first part of the next result has been proved using Gruson's Theorem by Divaani-Aazar and Mafi [4, Theorem 2.8] by using the same proof as that used in Delfino and Marley [2, Proposition 1]. We give a direct proof for this.

Theorem 3. *Let M be an R-module such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is a weakly Laskerian R-module for all $i \geq 0$. Then for any finitely generated R-module L with $\operatorname{Supp} L \subseteq V(\mathfrak{a})$, the R-modules $\operatorname{Ext}^i_R(L, M)$ and $\operatorname{Tor}^R_i(L, M)$ are weakly Laskerian for all $i \geq 0$.*

Proof. We have $V(\operatorname{Ann}_R L) = \operatorname{Supp} L \subseteq V(\mathfrak{a})$. Hence there exists $n \in \mathbb{N}$ such that $\mathfrak{a}^n L = 0$. It follows that $\mathfrak{a}^n \operatorname{Ext}^i_R(L, M) = 0$ and $\mathfrak{a}^n \operatorname{Tor}^R_i(L, M) = 0$ for all i. Let

$$\mathbb{F}_\bullet : \ \cdots \to F_2 \to F_1 \to F_0 \to 0$$

be a free resolution of finitely generated R-modules for L. Then $\operatorname{Ext}^i_R(L, M)$ $= Z^i/B^i$, where B^i and Z^i are the modules of coboundaries and cocycles of the complex $\operatorname{Hom}_R(\mathbb{F}_\bullet, M)$, respectively. Let $\mathcal{C}$ be the class of all R modules N such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, N)$ is weakly Laskerian for all $i \geq 0$. Consider the short exact sequence

$$0 \to \operatorname{Ext}^i_R(L, M) \to C^i \to B^{i+1} \to 0,$$

where $C^i = \operatorname{Hom}_R(F_i, M)/B^i$. By using a similar proof as in the proof of the Theorem 1 and using Lemma 1, we have that $B^i \in \mathcal{C}$ for all i. (Note that $\operatorname{Ext}^i_R(L, M) \subseteq 0 :_{C^i} \mathfrak{a}^n$.) Thus $C^i \in \mathcal{C}$ for all i. Hence $0 :_{C^i} \mathfrak{a}$ is weakly Laskerian for all i, and so it follows from Lemma 1 that $0 :_{C^i} \mathfrak{a}^n$ is weakly Laskerian for all i. Now, since $\operatorname{Ext}^i_R(L, M) \subseteq 0 :_{C^i} \mathfrak{a}^n$, $\operatorname{Ext}^i_R(L, M)$ is weakly Laskerian for all i.

Also, we have $\operatorname{Tor}^R_i(L, M) = Z_i/B_i$, where B_i and Z_i are the modules of boundaries and cycles of the complex $\mathbb{F}_\bullet \otimes_R M$, respectively. Let $\mathcal{C}'$ be the class of all R modules N such that $\operatorname{Tor}^R_i(R/\mathfrak{a}, N)$ is weakly Laskerian for all $i \geq 0$. In view of Theorem 2 and our assumption, $M \in C'$. Consider the exact sequence

$$0 \to C_{i+1} \to Z_i \to \operatorname{Tor}^R_i(L, M) \to 0,$$

where $C_i = F_i \otimes_R M/Z_i$. As $\mathfrak{a}^n \operatorname{Tor}^R_i(L, M) = 0$ for all i, we obtain the exact sequence

$$Z_i/\mathfrak{a}^n Z_i \to \operatorname{Tor}^R_i(L, M) \to 0.$$

Now, by using a similar proof as in the proof of the Theorem 2((i) $\Rightarrow$ (ii)) and using Lemma 2, we have $Z_i \in \mathcal{C}$ for all i. Therefore, it follows from Lemma 2 that $Z_i/\mathfrak{a}^n Z_i$ is weakly Laskerian for all i, and $\operatorname{Tor}^R_i(L, M)$ is weakly Laskerian for all i. □

The change of ring principle for weak cofiniteness has been proved by using a spectral sequence argument by Divaani-Aazar and Mafi [4, Theorem 2.9]. We give a direct proof for it.

Theorem 4. *Let the ring T be a homomorphic image of R, and let M be a T-module. Then M is an $\mathfrak{a}T$-weakly cofinite as a T-module if and only if M is an $\mathfrak{a}$-weakly cofinite as an R-module.*

Proof. Assume that $T = R/I$ for some ideal I of R and let N be a T-module. Then $\mathfrak{p} \in \operatorname{Ass}_R N$ if and only if $\mathfrak{p}/I \in \operatorname{Ass}_T N$, and so N is weakly Laskerian as a T-module if and only if N is weakly Laskerian as an R-module. Also, since $\mathfrak{p} \in \operatorname{Supp}_R N$ if and only if $\mathfrak{p}/I \in \operatorname{Supp}_T N$, it follows that $\operatorname{Supp}_T M \subseteq V(\mathfrak{a}T)$ if and only if $\operatorname{Supp}_R M \subseteq V(\mathfrak{a})$.

Now, let $\mathfrak{a} = (x_1, \ldots, x_t)$ and let $\varphi : R \to T$ be the natural epimorphism. As $\mathfrak{a}T = (\varphi(x_1), \ldots, \varphi(x_t))$, it follows from Theorem 2 that $\operatorname{Ext}^i_T(T/\mathfrak{a}T, M)$ is weakly Laskerian T-module for all i if and only if the Koszul cohomology modules $H^i(\varphi(x_1), \ldots, \varphi(x_t); M)$ are weakly Laskerian T-modules for all i. But, by above $H^i(\varphi(x_1), \ldots, \varphi(x_t); M)$ is a weakly Laskerian T-module if and only if $H^i(\varphi(x_1), \ldots, \varphi(x_t); M)$ is a weakly Laskerian R-module. On the other hand,

$$H^i(\varphi(x_1), \ldots, \varphi(x_t); M) \cong H^i(x_1, \ldots, x_t; M).$$

Now, the result follows from Theorem 2. □

3. A SUFFICIENT CONDITION FOR $\mathfrak{a}$-WEAKLY COFINITE MODULES

Theorem 5. *Let $f : M \to N$ be an R-homomorphism such that the modules $\operatorname{Ext}^i_R(R/\mathfrak{a}, \operatorname{Ker} f)$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, \operatorname{Coker} f)$ are both weakly Laskerian for all i. Then $\operatorname{Ker}\operatorname{Ext}^i_R(\mathrm{id}_{R/\mathfrak{a}}, f)$ and $\operatorname{Coker}\operatorname{Ext}^i_R(\mathrm{id}_{R/\mathfrak{a}}, f)$ are also weakly Laskerian for all i.*

Proof. Consider the exact sequences

$$0 \to \operatorname{Ker} f \to M \xrightarrow{g} \operatorname{Im} f \to 0 \quad \text{and} \quad 0 \to \operatorname{Im} f \xrightarrow{\iota} N \to \operatorname{Coker} f \to 0,$$

where $\iota \circ g = f$. Hence we obtain the following two exact sequences

$$\cdots \to \operatorname{Ext}^i_R(R/\mathfrak{a}, \operatorname{Ker} f) \to \operatorname{Ext}^i_R(R/\mathfrak{a}, M) \to \operatorname{Ext}^i_R(R/\mathfrak{a}, \operatorname{Im} f) \to \cdots$$

and

$$\cdots \to \operatorname{Ext}^i_R(R/\mathfrak{a}, \operatorname{Im} f) \to \operatorname{Ext}^i_R(R/\mathfrak{a}, N) \to \operatorname{Ext}^i_R(R/\mathfrak{a}, \operatorname{Coker} f) \to \cdots$$

Now, since $\operatorname{Ext}^{i+1}_R(R/\mathfrak{a}, \operatorname{Ker} f)$ is weakly Laskerian, it follows from the first exact sequence that $\operatorname{Coker}\operatorname{Ext}^i_R(id_{R/\mathfrak{a}}, g)$ and $\operatorname{Ker}\operatorname{Ext}^{i+1}_R(id_{R/\mathfrak{a}}, g)$ are both weakly Laskerian for all i. Also, as $\operatorname{Ext}^i_R(R/\mathfrak{a}, \operatorname{Coker} f)$ is weakly Laskerian, the second exact sequence implies that the R-modules $\operatorname{Coker}\operatorname{Ext}^i_R(id_{R/\mathfrak{a}}, \iota)$ and $\operatorname{Ker}\operatorname{Ext}^{i+1}_R(id_{R/\mathfrak{a}}, \iota)$ are weakly Laskerian for all i. Therefore, the assertion follows from the exact sequences

$$0 \to \operatorname{Ker}\operatorname{Ext}^i_R(id_{R/\mathfrak{a}}, g) \to \operatorname{Ker}\operatorname{Ext}^i_R(id_{R/\mathfrak{a}}, f) \to \operatorname{Ker}\operatorname{Ext}^i_R(id_{R/\mathfrak{a}}, \iota)$$

and

$$\operatorname{Coker}\operatorname{Ext}^i_R(id_{R/\mathfrak{a}}, g) \to \operatorname{Coker}\operatorname{Ext}^i_R(id_{R/\mathfrak{a}}, f) \to \operatorname{Coker}\operatorname{Ext}^i_R(id_{R/\mathfrak{a}}, \iota) \to 0. \quad \square$$

Corollary 3. *Let M be an R-module with $\operatorname{Supp} M \subseteq V(\mathfrak{a})$. Suppose that $x \in \mathfrak{a}$ is such that $0 :_M x$ and M/xM are both $\mathfrak{a}$-weakly cofinite. Then M is also $\mathfrak{a}$-weakly cofinite.*

Proof. Put $f = x1_M$. Then $\operatorname{Ker} f = 0 :_M x$ and $\operatorname{Coker} f = M/xM$. Hence in view of Theorem 5, the R-module $\operatorname{Ker}\operatorname{Ext}^i_R(1_{R/\mathfrak{a}}, f)$ is weakly Laskerian. But since $\operatorname{Ext}^i_R(1_{R/\mathfrak{a}}, f) = 0$, so $\operatorname{Ker}\operatorname{Ext}^i_R(1_{R/\mathfrak{a}}, f) = \operatorname{Ext}^i_R(R/\mathfrak{a}, M)$. This completes the proof. $\square$

Corollary 4. *Let M be an R-module. Suppose that $x \in \sqrt{\mathfrak{a}}$ is such that $0 :_M x$ and M/xM are both weakly Laskerian. Then $\operatorname{Ext}^i_R(R/\mathfrak{a}, \Gamma_x(M))$ is also weakly Laskerian for all i.*

Proof. We have $x^n \in \mathfrak{a}$ for some $n \in \mathbb{N}$. Put $f = x^n 1_{\Gamma_x(M)}$. Then, $\operatorname{Ker} f = 0 :_{\Gamma_x(M)} x^n = 0 :_M x^n$ and $\operatorname{Coker} f = \Gamma_x(M)/x^n\Gamma_x(M)$. Consider the exact sequence

$$0 \to \operatorname{Coker} f \to M/x^nM.$$

As M/xM is weakly Laskerian, it follows from Lemma 2 that M/x^nM is weakly Laskerian, and so $\operatorname{Coker} f$ is weakly Laskerian. Therefore, in view of [3, Lemma 2.3 (i)] and Theorem 5, $\operatorname{Ker}\operatorname{Ext}^i_R(1_{R/\mathfrak{a}}, f)$ is weakly Laskerian. But $x^n \in \mathfrak{a}$ implies that $\operatorname{Ext}^i_R(1_{R/\mathfrak{a}}, f) = 0$, and so $\operatorname{Ker}\operatorname{Ext}^i_R(1_{R/\mathfrak{a}}, f) = \operatorname{Ext}^i_R(R/\mathfrak{a}, \Gamma_x(M))$. This completes the proof. $\square$

Corollary 5. *Let M be an R-module with $\operatorname{Supp} M \subseteq V(\mathfrak{a})$. Suppose that $x \in \sqrt{\mathfrak{a}}$ is such that $0 :_M x$ and M/xM are both weakly Laskerian. Then M is $\mathfrak{a}$-weakly cofinite.*

Proof. The result follows from Corollary 4. $\square$

4. Cohomological dimension and weakly Laskerian modules

Before bringing the next result we recall that, for an R-module M, the *cohomological dimension of M with respect to an ideal* $\mathfrak{a}$ of R is defined as

$$\operatorname{cd}(\mathfrak{a}, M) = \sup\{i \in \mathbb{Z} \mid H^i_{\mathfrak{a}}(M) \neq 0\}.$$

Proposition 1. *Let* $\operatorname{cd}(\mathfrak{a}) = 1$, *and let M be a weakly Laskerian R-module. Then $\mathrm{H}^j_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-weakly cofinite for all i.*

Proof. Since $\mathrm{H}^0_{\mathfrak{a}}(M)$ is a submodule of M, it follows that $\mathrm{H}^0_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-weakly cofinite. Also, $\mathrm{cd}(\mathfrak{a}) = 1$ implies that $\mathrm{H}^i_{\mathfrak{a}}(M) = 0$ for all $i > 1$. Therefore, the result follows from [4, Theorem 3.1]. □

Proposition 2. *Let $\mathfrak{b} \supseteq \mathfrak{a}$ be two ideals of R with $\mathrm{cd}(\mathfrak{b}) = 1$, and let M be an R-module with $\Gamma_{\mathfrak{a}}(M) = 0$. Then*

$$\mathrm{H}^j_{\mathfrak{b}}(\mathrm{H}^i_{\mathfrak{a}}(M)) \cong \begin{cases} \mathrm{H}^1_{\mathfrak{b}}(M), & \text{if } \mathrm{j} = 0, \mathrm{i} = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. See the proof of [6, Proposition 3.15]. □

Corollary 6. *Let $\mathfrak{b} \supseteq \mathfrak{a}$ be two ideals of R with $\mathrm{cd}(\mathfrak{b}) = 1$, and let M be a weakly Laskerian R-module. Then $\mathrm{H}^j_{\mathfrak{b}}(H^i_{\mathfrak{a}}(M))$ is $\mathfrak{b}$-weakly cofinite for all i and j.*

Proof. Since $\mathrm{cd}(\mathfrak{b}) = 1$, it follows from Proposition 1 that $\mathrm{H}^j_{\mathfrak{b}}(\Gamma_{\mathfrak{a}}(M))$ is $\mathfrak{b}$-weakly cofinite for all j. Now, let $i > 0$. As $\mathrm{H}^i_{\mathfrak{a}}(M) \cong \mathrm{H}^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))$, we may therefore assume that $\Gamma_{\mathfrak{a}}(M) = 0$. Thus, the result follows from Propositions 1 and 2. □

Corollary 7. *Let $\mathfrak{b} \supseteq \mathfrak{a}$ be two ideals of R with $\mathrm{cd}(\mathfrak{b}) = 1$, and let M be a weakly Laskerian R-module. Then for every finitely generated R-module L with $\mathrm{Supp}\, L \subseteq V(\mathfrak{b})$, the R-modules $\mathrm{Ext}^{\mathrm{j}}_{\mathrm{R}}(\mathrm{L}, \mathrm{H}^{\mathrm{i}}_{\mathfrak{a}}(\mathrm{M}))$ and $\mathrm{Tor}^{\mathrm{R}}_{\mathrm{j}}(\mathrm{L}, \mathrm{H}^{\mathrm{i}}_{\mathfrak{a}}(\mathrm{M}))$ are weakly Laskerian for all i and j. In particular, the R-modules $H^i_{\mathfrak{a}}(M)/\mathfrak{b}^n H^i_{\mathfrak{a}}(M)$ are weakly Laskerian for all i and n.*

Proof. By Corollary 6, $\mathrm{H}^j_{\mathfrak{b}}(H^i_{\mathfrak{a}}(M))$ is $\mathfrak{b}$-weakly cofinite for all i and j. Therefore, it follows from Corollary 2 that the R-modules $\mathrm{Ext}^{\mathrm{j}}_{\mathrm{R}}(\mathrm{R}/\mathfrak{b}, \mathrm{H}^{\mathrm{i}}_{\mathfrak{a}}(\mathrm{M}))$ are weakly Laskerian for all i and j. Thus, the result follows from Theorem 3. □

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Authors' addresses

Bahram Vakili
Department of Mathematics, Islamic Azad University , Shabestar Branch, Shabestar, Iran
E-mail address: bvakil2004@yahoo.com and bvakil@iaushab.ac.ir

Jafar Azami
Department of Mathematics, University of Mohaghegh Ardabili, Ardabil, Iran
E-mail address: jafar.azami@gmail.com and azami@uma.ac.ir

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ON CONVERGENCE OF AN IMPLICIT ALGORITHM FOR MULTIVALUED MAPPINGS IN BANACH SPACES

ISA YILDIRIM

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Abstract. In this paper, we introduce an implicit algorithm for finding a common fixed point of two quasi-nonexpansive multivalued mappings in Banach spaces. We also prove some convergence theorems for the purposed algorithm under some control conditions.

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1. INTRODUCTION

Let X be a Banach space. A subset $E \subset X$ is called proximinal if for each $x \in X$, there exists an element $y \in E$ such that

$$d(x, y) = \inf\{\|x - z\| : z \in E\} = d(x, E).$$

It is known that every closed convex subset of a uniformly convex Banach space is proximinal.

We denote by $CB(E), K(E)$ and $P(E)$ the collection of all nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximal bounded subsets of E respectively. The Hausdorff metric H on $CB(X)$ is defined by

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}$$

for all $A, B \in CB(X)$.

Let $T : X \longrightarrow 2^X$ be a multivalued mapping. An element $x \in X$ is said to be a fixed point of T, if $x \in Tx$. Denote by $F(T)$ the set of fixed points of T and by $F := F(S) \cap F(T)$ the set of common fixed points of the mappings S and T.

Definition 1. A multivalued mapping $T : X \longrightarrow CB(X)$ is called
(i) contraction if there exists a constant $k \in [0, 1)$ such that for any $x, y \in X$,

$$H(Tx, Ty) \leq k \|x - y\|,$$

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(ii) nonexpansive if

$$H(Tx,Ty) \leq \|x-y\|$$

for all $x, y \in X$,

(iii) quasi-nonexpansive if $F(T) \neq \varnothing$ and

$$H(Tx,Tp) \leq \|x-p\|$$

for all $x \in X$ and all $p \in F(T)$.

It is clear that every nonexpansive multivalued mappings T with $F(T) \neq \varnothing$ is quasi-nonexpansive. But there exist quasi-nonexpansive mappings that are not nonexpansive, see [13]. It is known that if T is a quasi-nonexpansive multivalued mappings, then $F(T)$ is closed.

The fixed point theory of multivalued nonexpansive mappings is much more complicated and diffucult than the corresponding theory of single-valued nonexpansive mappings. However, some classical fixed point theorems for single-valued nonexpansive mappings have already been extended to multivalued mappings.

The study of fixed points for multivalued contractions and nonexpansive mappings using the Hausdorff metric was initiated by Markin [7] (see also [8]). Later, an interesting and rich fixed point theory for such maps was developed which has applications in control theory, convex optimization, differential inclusion and economics (see, [2] and references cited therein). Moreover, the existence of fixed points for multivalued nonexpansive mappings in uniformly convex Banach spaces was proved by Lim [6].

In 1999, Sahu [11] obtained the strong convergence theorems of the nonexpansive type and nonself multivalued mappings for the following algorithm:

$$x_n = t_n u + (1-t_n)\, y_n, \quad n \geq 0, \tag{1.1}$$

where $y_n \in Tx_n, u \in E, t_n \in (0,1)$ and $\lim_{n\to\infty} t_n = 0$. He proved that $\{x_n\}$ converges strongly to some fixed points of T. Xu [15] extended Theorem 1.3 to a multivalued nonexpansive nonself mapping and obtained the fixed theorem in 2001.

Recently, He et al. [3] obtained common fixed points of a nonexpansive multivalued mapping $T : E \longrightarrow CB(E)$ satisfying certain conditions. To achive this, they employed the following Mann type implicit algorithm:

$$\begin{cases} x_0 \in E, \\ x_n = \alpha_n x_{n-1} + (1-\alpha_n)\, y_n, \quad n \geq 1, \end{cases} \tag{1.2}$$

where $y_n \in Tx_n$ and $\alpha_n \in [0,1]$. They proved some strong convergence theorems of the sequence $\{x_n\}$ defined by (1.2) for nonexpansive multivalued mappings in Banach spaces.

In this paper, we introduce an iterative algorithm for common fixed points of two quasi-nonexpansive multivalued mappings. Let $S, T : E \longrightarrow CB(E)$ be two quasi-nonexpansive multivalued mappings with common fixed point p. Our algorithm is as

follows:

$$\begin{cases} x_0 \in E, \\ x_n = \alpha_n x_{n-1} + \beta_n y_n + \gamma_n z_n, \quad n \geq 1, \end{cases} \tag{1.3}$$

where $y_n \in Sx_n$, $z_n \in Tx_n$ and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequence of numbers in $[0,1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$.

Using the implicit algorithm (1.3), we prove some weak and strong convergence theorems for approximating common fixed points of two quasi-nonexpansive multivalued mappings in a uniformly convex Banach space. These results improve and extend the corresponding results of Khan et al. [5], Soltuz [14] and Xu and Ori [16] to the case of multivalued mappings. Our results also improve the corresponding results of He et al. [3] for two quasi-nonexpansive multivalued mappings.

We shall use the condition $Sp = Tp = \{p\}$ for any $p \in F := F(S) \cap F(T)$ in order to prove main results of this paper. Below is an example of two quasi-nonexpansive multivalued mappings satisfying this condition.

Example 1. Let $E = [0,1]$ be endowed with the Euclidean metric. Let $S, T : E \longrightarrow CB(E)$ be defined by $Sx = [0,x]$ and $Tx = \left[0, \frac{x}{2}\right]$. Thus

$$\begin{aligned} H(Sx, Sy) &= \max\{|x-y|, 0\} \\ &\leq |x-y|. \end{aligned}$$

In a similar way, we obtain that

$$\begin{aligned} H(Tx, Ty) &= \max\left\{\left|\frac{x}{2} - \frac{y}{2}\right|, 0\right\} \\ &\leq \left|\frac{x}{2} - \frac{y}{2}\right| \\ &\leq |x-y|. \end{aligned}$$

Since $F(S) = [0,1]$ and $F(T) = \{0\}$ are nonempty sets, S and T are quasi-nonexpansive multivalued mappings and $Sp = Tp = \{p\}$ for any $p \in F$.

2. Preliminaries

A Banach space X is called uniformly convex if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in X$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, $\|x + y\| \leq 2(1-\delta)$ holds. The modulus of convexity of X is defined by

$$\delta_X(\epsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : \|x\|, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\},$$

for all $\epsilon \in [0,2]$. X is said to be uniformly convex if $\delta_X(0) = 0$, and $\delta(\epsilon) > 0$ for all $0 < \epsilon \leq 2$.

A Banach space X is said to satisfy Opial's condition [9] if for any sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$.

Examples of Banach spaces satisfying this condition are Hilbert spaces and all l^p spaces $(1 < p < \infty)$. On the other hand, $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial's condition.

A multivalued mapping $T : E \to P(X)$ is called demiclosed at $y \in E$ if for any sequence $\{x_n\}$ in E which is weakly convergent to an element x and $y_n \in Tx_n$ with $\{y_n\}$ converges strongly to y, we have $y \in Tx$.

A multivalued mapping $T : E \to CB(E)$ is said to be satisfy Condition (A) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$d(x, Tx) \geq f(d(x, F(T))$$

for all $x \in E$. Khan and Fukhar-ud-din [4] introduced the so-called Condition (A') and gave a bit improved version in [1]. The following is the multivalued version of Condition (A').

Two multivalued nonexpansive mappings $S, T : E \to CB(E)$ where E a subset of X, are said to satisfy Condition (A') if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\text{either } d(x, Tx) \geq f(d(x, F(S) \cap F(T)) \text{ or } d(x, Sx) \geq f(d(x, F(S) \cap F(T))$$

for all $x \in E$. The Condition (A') reduces to the Condition (A) when $S = T$.

Next, we state the following useful lemma.

Lemma 1 ([12]). *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers n. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

3. Main Results

In order to prove some strong and weak convergence theorems, we need the following lemmas. By means of the iterative algorithm (1.3), we shall prove the following lemmas.

Lemma 2. *Let X be a normed space and E be a nonempty closed convex subset of X. Let $S, T : E \to CB(E)$ be two quasi-nonexpansive multivalued mappings and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be three real sequences in $[0, 1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1, 0 < a \leq \alpha_n$ where a is a constant. Let $\{x_n\}$ be the sequence as defined in (1.3). If $F \neq \varnothing$ and $Sp = Tp = \{p\}$ for any $p \in F$ then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$.*

Proof. Let $p \in F$. It follows from (1.3) that

$$\begin{aligned}\|x_n - p\| &\le \alpha_n \|x_{n-1} - p\| + \beta_n \|y_n - p\| + \gamma_n \|z_n - p\| \\ &\le \alpha_n \|x_{n-1} - p\| + \beta_n d(Sx_n, p) + \gamma_n d(Tx_n, p) \\ &\le \alpha_n \|x_{n-1} - p\| + \beta_n H(Sx_n, Sp) + \gamma_n H(Tx_n, Tp) \\ &\le \alpha_n \|x_{n-1} - p\| + \beta_n \|x_n - p\| + \gamma_n \|x_n - p\|\end{aligned}$$

and implies that $(1-\beta_n-\gamma_n)\|x_n - p\| \le \alpha_n \|x_{n-1} - p\|$ or $\alpha_n \|x_n - p\| \le \alpha_n \|x_{n-1} - p\|$. Since $\alpha_n \ge a > 0$, therefore

$$\|x_n - p\| \le \|x_{n-1} - p\|. \tag{3.1}$$

We get that $\{\|x_n - p\|\}$ is a decreasing sequence, so $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F$. □

Lemma 3. *Let X be a uniformly convex Banach space and E be a nonempty closed convex subset of X. Let $S, T : E \to CB(E)$ be two quasi-nonexpansive multivalued mappings and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be three real sequences in $[0,1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1, 0 < a \le \alpha_n, \beta_n, \gamma_n \le b < 1$ where a, b are some constants. Let $\{x_n\}$ be the sequence as defined in (1.3). If $F \ne \varnothing$ and $Sp = Tp = \{p\}$ for any $p \in F$ then $\lim_{n\to\infty} d(x_n, Sx_n) = 0 = \lim_{n\to\infty} d(x_n, Tx_n)$.*

Proof. From Lemma 2, $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F$. We suppose that $\lim_{n\to\infty} \|x_n - p\| = d$ for some $d \ge 0$. Then

$$\begin{aligned}\lim_{n\to\infty} \|x_n - p\| &= \lim_{n\to\infty} \|\alpha_n(x_{n-1} - p) + \beta_n (y_n - p) + \gamma_n (z_n - p)\| \\ &= \lim_{n\to\infty} \left\| (1-\gamma_n)\left[\frac{\alpha_n}{1-\gamma_n}(x_{n-1} - p) + \frac{\beta_n}{1-\gamma_n}(y_n - p)\right] + \gamma_n (z_n - p) \right\| \\ &= d.\end{aligned} \tag{3.2}$$

Since T is a quasi-nonexpansive mapping and $F \ne \varnothing$, we have

$$\|y_n - p\| = d(y_n, Sp) \le H(Sx_n, Sp) \le \|x_n - p\|$$

for each $p \in F$. Taking lim sup on both sides, we obtain

$$\limsup_{n\to\infty} \|y_n - p\| \le d. \tag{3.3}$$

Similarly,

$$\limsup_{n\to\infty} \|z_n - p\| \le d. \tag{3.4}$$

Now using (3.1), we have

$$\begin{aligned}&\limsup_{n\to\infty} \left\| \frac{\alpha_n}{1-\gamma_n}(x_{n-1} - p) + \frac{\beta_n}{1-\gamma_n}(y_n - p) \right\| \\ &\le \limsup_{n\to\infty} \left[\frac{\alpha_n}{1-\gamma_n} \|x_{n-1} - p\| + \frac{\beta_n}{1-\gamma_n} \|y_n - p\| \right]\end{aligned}$$

$$\begin{aligned}
&\leq \limsup_{n\to\infty}\left[\frac{\alpha_n}{1-\gamma_n}\|x_{n-1}-p\|+\frac{\beta_n}{1-\gamma_n}\|x_n-p\|\right]\\
&=\limsup_{n\to\infty}\|x_{n-1}-p\|\\
&=d. \qquad (3.5)
\end{aligned}$$

Using (3.2), (3.3), (3.5) and Lemma 1, we get

$$\lim_{n\to\infty}\left\|\frac{\alpha_n}{1-\gamma_n}(x_{n-1}-p)+\frac{\beta_n}{1-\gamma_n}(y_n-p)-(z_n-p)\right\|=0.$$

This means that

$$\begin{aligned}
&\lim_{n\to\infty}\left\|\frac{\alpha_n}{1-\gamma_n}x_n+\frac{\beta_n}{1-\gamma_n}y_n-z_n\right\|\\
&=\lim_{n\to\infty}\left(\frac{1}{1-\gamma_n}\right)\|\alpha_n x_n+\beta_n y_n-(1-\gamma_n)z_n\|\\
&=0.
\end{aligned}$$

Since $0<a\leq\gamma_n\leq b<1$, we have $\frac{1}{1-a}\leq\frac{1}{1-\gamma_n}\leq\frac{1}{1-b}$. Thus,

$$\lim_{n\to\infty}\|x_n-z_n\|=0.$$

In a similar way, we can show that

$$\lim_{n\to\infty}\|x_n-y_n\|=0.$$

Therefore,

$$d(x_n,Tx_n)\leq d(x_n,z_n)$$

and

$$d(x_n,Sx_n)\leq d(x_n,y_n)$$

gives $d(x_n,Tx_n)$, and $d(x_n,Sx_n)\to 0$ as $n\to\infty$. This completes the proof of the lemma. □

Now, we give some strong convergence theorems. Our first strong convergence theorem is in general real Banach spaces. We then apply this theorem to obtain a result in uniformly convex Banach spaces.

Theorem 1. *Let X be a real Banach space and E, $\{x_n\}$ S,T be as in Lemma 3. If $F\neq\varnothing$ and $Sp=Tp=\{p\}$ for any $p\in F$ then $\{x_n\}$ converges strongly to a common fixed point of S and T if and only if $\liminf_{n\to\infty}d(x_n,F)=0$.*

Proof. The necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty}d(x_n,F)=0$. From Lemma 2, we know that

$$\|x_n-p\|\leq\|x_{n-1}-p\|,$$

which gives

$$d(x_n, F) \leq d(x_{n-1}, F).$$

This implies that $\lim_{n\to\infty} d(x_n, F)$ exists and so by the hypothesis, $\liminf_{n\to\infty} d(x_n, F) = 0$. Therefore we must have $\lim_{n\to\infty} d(x_n, F) = 0$.

Now we show that $\{x_n\}$ is a Cauchy sequence in E. Let $\varepsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, F) = 0$, there exists a constant n_0 such that for all $n \geq n_0$, we have

$$d(x_n, F) < \frac{\varepsilon}{4}.$$

In particular, $\inf\{\|x_{n_0} - p\| : p \in F\} < \frac{\varepsilon}{4}$. There must exist a $p^* \in F$ such that

$$\|x_{n_0} - p^*\| < \frac{\varepsilon}{2}.$$

For $m, n \geq n_0$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \\ &\leq 2\|x_{n_0} - p^*\| \\ &< 2\left(\frac{\varepsilon}{2}\right) = \varepsilon. \end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence in a closed subset E of a Banach space E. So it must converge in E. We suppose that $\lim_{n\to\infty} x_n = q$. Now

$$\begin{aligned} d(q, Sq) &\leq d(q, x_n) + d(x_n, Sx_n) + H(Sx_n, Sq) \\ &\leq d(q, x_n) + d(x_n, y_n) + d(x_n, q) \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

gives that $d(q, Sq) = 0$ which implies that $q \in Sq$. Similarly,

$$\begin{aligned} d(q, Tq) &\leq d(q, x_n) + d(x_n, Tx_n) + H(Tx_n, Tq) \\ &\leq d(q, x_n) + d(x_n, z_n) + d(x_n, q) \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

implies that $q \in Tq$. Consequently, $q \in F$. □

In our next theorem, we assume that $S, T : E \to CB(E)$ satisfy condition (A'). In contrast to Theorem 3.8 [10], we do not impose the condition of proximinality on F. We now apply the above theorem to obtain the following.

Theorem 2. *Let X be a uniformly convex Banach space and $E, \{x_n\}$ be as in Lemma 2. Let $S, T : E \to CB(E)$ be two quasi-nonexpansive multivalued mappings satisfying Condition (A'). If $F \neq \varnothing$ and $Sp = Tp = \{p\}$ for any $p \in F$ then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof. From Lemma 3, $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$. We suppose that $\lim_{n\to\infty}\|x_n - p\| = d$ for some $d \geq 0$. If $d = 0$, there is nothing to prove. Suppose $d > 0$. Now $\|x_n - p\| \leq \|x_{n-1} - p\|$ gives that

$$\inf_{p\in F}\|x_n - p\| \leq \inf_{p\in F}\|x_{n-1} - p\|$$

which means that $d(x_n, F) \leq d(x_{n-1}, F)$ and so $\lim_{n\to\infty} d(x_n, F)$ exists. By using condition (A'), either

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0$$

or

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} d(x_n, Sx_n) = 0.$$

In both the cases, we have

$$\lim_{n\to\infty} f(d(x_n, F)) = 0.$$

Since f is a nondecreasing function and $f(0) = 0$, so it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. Now applying the above theorem, we obtain the result. □

Finally, here we will approximate common fixed points of the mappings S and T through the weak convergence of the sequence $\{x_n\}$ defined in (1.3).

Theorem 3. *Let X be a uniformly convex Banach space satisfying the Opial's condition and E, S, T and $\{x_n\}$ be as taken in Lemma 3. If $F \neq \varnothing$ and $Sp = Tp = \{p\}$ for any $p \in F$, $I - S$ and $I - T$ are demiclosed with respect to zero, then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Proof. Let $p \in F$. From the proof of Lemma 2, $\lim_{n\to\infty}\|x_n - p\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in F. To prove this, let z_1 and z_2 be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemma 3, there exists $y_n \in Sx_n$ such that $\lim_{n\to\infty}\|x_n - y_n\| = 0$ and $I - S$ is demiclosed with respect to zero, therefore we obtain $z_1 \in Sz_1$. Similarly, $z_1 \in Tz_1$. Again in the same way, we can prove that $z_2 \in F$.

Next, we prove uniqueness. For this, suppose that $z_1 \neq z_2$. Then by the Opial's condition, we have

$$\begin{aligned}
\lim_{n\to\infty}\|x_n - z_1\| &= \lim_{n_i\to\infty}\|x_{n_i} - z_1\| \\
&< \lim_{n_i\to\infty}\|x_{n_i} - z_2\| \\
&= \lim_{n\to\infty}\|x_n - z_2\| \\
&= \lim_{n_j\to\infty}\|x_{n_j} - z_2\| \\
&< \lim_{n_j\to\infty}\|x_{n_j} - z_1\|
\end{aligned}$$

$$= \lim_{n\to\infty} \|x_n - z_1\|,$$

which is a contradiction. Hence $\{x_n\}$ converges weakly to a point in F. □

The compactness assumption is quite strong, since it is easy to find a sequence in the domain which converges to a fixed point of the mapping. Therefore, we give the following result.

Corollary 1. *Let E be a nonempty compact convex subset of a uniformly convex Banach space X satisfying Opial's condition, $S,T : E \to K(E)$ be two quasi-nonexpansive multivalued mappings where $K(E)$ is the family of nonempty compact subsets of E and $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ be three real sequences in $[0,1]$ satisfying $\alpha_n+\beta_n+\gamma_n = 1, 0 < a \le \alpha_n,\beta_n,\gamma_n \le b < 1$ where a,b are some constants. Let $\{x_n\}$ be the sequence as defined in (1.3). If $F \neq \varnothing$ and $Sp = Tp = \{p\}$ for any $p \in F$ then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

The algorithm (1.3) reduces to the algorithm (1.2) when either $S = T$ or $\beta_n = 0$ or $\gamma_n = 0$. Therefore, we obtain the following results.

Corollary 2. *Let X be a uniformly convex Banach space and E be a nonempty closed convex subset of X. Let $T : E \to CB(E)$ be a quasi-nonexpansive multivalued mapping satisfying Condition (A) and $\{\alpha_n\}$ be a real sequence in $[0,1]$ satisfying $0 < a \le \alpha_n \le b < 1$ where a,b are some constants. Let $\{x_n\}$ be the sequence as defined in (1.2). If $F \neq \varnothing$ and $Tp = \{p\}$ for any $p \in F$ then $\{x_n\}$ converges strongly to a fixed point of T.*

Corollary 3. *Let X be a uniformly convex Banach space satisfying the Opial's condition and E be a nonempty closed convex subset of X. Let $T : E \to CB(E)$ be a quasi-nonexpansive multivalued mapping and $\{\alpha_n\}$ be a real sequence in $[0,1]$ satisfying $0 < a \le \alpha_n \le b < 1$ where a,b are some constants. Let $\{x_n\}$ be the sequence as defined in (1.2). If $F \neq \varnothing$ and $Tp = \{p\}$ for any $p \in F$, $I - T$ is demiclosed with respect to zero, then $\{x_n\}$ converges weakly to a fixed point of T.*

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Author's address

Isa Yildirim
Department of Mathematics, Faculty of Science, Ataturk University, 25240 Erzurum, Turkey
E-mail address: isayildirim@atauni.edu.tr

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THE FOURIER SPECTRAL APPROXIMATION FOR KOLMOGOROV-SPIEGEL-SIVASHINSKY EQUATION

XIAOPENG ZHAO, BO LIU, AND PENG ZHANG

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Abstract. In this paper, we consider the Fourier spectral approximation for numerically solving the Kolmogorov-Spiegel-Sivashinsky equation. The semi-discrete and fully discrete schemes are established. Moreover, the existence, uniqueness and the optimal error bound are also considered.

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1. INTRODUCTION

In this paper, we study the initial-boundary value problem for the Kolmogorov-Spiegel-Sivashinsky(KSS) equation

$$u_t + ku_{xxxx} + \beta u + \gamma u_x^2 + \alpha u_{xx} - \delta(u_x^3)_x = 0,\ (x,t) \in Q_T, \tag{1.1a}$$

$$u_x(0,t) = u_x(1,t) = u_{xxx}(0,t) = u_{xxx}(1,t) = 0, \tag{1.1b}$$

$$u(x,0) = u_0(x), \quad x \in (0,1). \tag{1.1c}$$

where $Q_T = (0,1)\times(0,T)$, k, α, β, γ and δ are positive parameters.

Equation (1a), which was derived by Sivashinsky [10], is a fourth-order nonlinear parabolic equation, which models the effective negative viscosity in a certain direction x of a large-scale flow. It is easy to check that if $\beta = 0$ and $\delta = 0$, the equation (1a) is the classical Kuramoto-Sivashinsky equation (see [1, 7, 9, 11, 15]).

In [8], Nicolaenko presents mathematical and computational investigations of the finite dimensional behavior of the solutions for the above equation, and points out the existence of the global attractor and inertial manifold for the equation.

In [13], by discarding the linear damping term, Ünal and Suhubi obtained the KSS model's periodic, quasi-periodic and solitary wave solutions analytically to a certain degree of approximation. Melnikov analysis had also been carried out to identify

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the homoclinic bifurcation. Transient spatiotemporal chaos had been observed. Ünal and Suhubi[12] also studied the group invariant solutions to the KSS equation. And a local analysis of the dynamical system obtained by the group theoretical means were performed by employing normal form analysis.

Recently, Guo and Wang[5] made a simple transform for the equation (1a). Differentiating the equation with respect to x and setting $u_x = v$, they obtained

$$v_t + k v_{xxxx} + \beta v + \gamma (v^2)_x + \alpha v_{xx} - \delta (v^3)_{xx} = f(x). \tag{1.2}$$

Therefore it is interesting to study the periodic BVP of (1.2) in multidimensional version. The authors first established the existence and uniqueness of the global solution, and then showed the existence of the global attractor, which has finite Hausdorff and fractal dimensions. Finally, they derived the Gevrey class regularity for the equation and constructed approximate inertial manifolds.

Fourier spectral approximations are essentially discretization methods for the approximate solution of partial differential equations. They have the natural advantage in keeping the physical properties of primitive problems. During the past years, many papers have already been published to study Fourier spectral method, for example [3, 6, 16, 17].

In this paper, we consider the Fourier spectral method for Kolmogorov-Spiegel-Sivashinsky equation (1a) with Neumann boundary condition (1b) and the initial condition (1c). Based on Sobolev's embedding theorem and some important inequalities, we obtain the error result $O((\Delta t)^2 + N^{-s})$ $(s = 2)$. Noticing that the existence of a solution locally in time is proved by the standard Picard iteration, global existence results are obtained by proving a priori estimates for the appropriate norms of $u(x,t)$. Adjusted to our needs, similar to the proof in [4, 18], the following results on global existence and uniqueness of solution to problem (1)-(3) are given in the following form:

Theorem 1. *Assume that* $u_0 \in H_E^2(0,1) = \{w; w \in H^2, w_x(0,t) = w_x(1,t) = 0\}$. *Then there exists a unique global solution* $u(x,t)$ *such that*

$$u(x,t) \in L^\infty(0,T; H_E^2(0,1)) \bigcap L^2(0,T; H^4(0,1)).$$

This paper is organized as follows. In the next section, we consider a semi-discrete Fourier spectral approximation, prove its existence and uniqueness of the numerical solution and derive the error bound. In Section 3, we consider the full-discrete approximation for problem (1). Furthermore, we prove convergence to the solution of the associated continuous problem. In Section 4, some numerical experiments which confirm our results are performed. In the last section, conclusions are given.

Throughout this paper, we denote the L^2, L^p, L^∞, H^k norms in $(0,1)$ simply by $\|\cdot\|$, $\|\cdot\|_p$, $\|\cdot\|_\infty$ and $\|\cdot\|_{H^k}$.

2. SEMI-DISCRETE APPROXIMATION

In this section, we consider the semi-discrete approximation for problem (1). First of all, we recall some basic results on the Fourier spectral method which will be used throughout this paper. For any integer $N > 0$, we introduce the finite dimensional subspace of $H_E^2(0,1)$:

$$S_N = \text{span}\{\cos k\pi x; k = 0, 1, \cdots, N\}.$$

Let $P_N : L^2(0,1) \to S_N$ be an orthogonal projecting operator which satisfies:

$$(u - P_N u, v) = 0, \quad \forall v \in S_N. \tag{2.1}$$

For the operator P_N, we have the following result (see [2, 16]):
(B1) P_N commutes with derivation on $H_E^2(0,1)$, i.e.,

$$P_N u_{xx} = (P_N u)_{xx}, \quad \forall u \in H_E^2(0,1).$$

Using the same method as previous papers [2, 14] , we can obtain the following result (B2) for problem (1):
(B2) For any real $0 \le \mu \le 2$, there is a constant c, such that

$$\|u - P_N u\|_\mu \le c N^{\mu-2} \|u_{xx}\|, \quad \forall u \in H_E^2(0,1).$$

We define the Fourier spectral approximation for problem (1): Find $u_N(t) = \sum_{j=1}^N a_j(t) \cos j\pi x \in S_N$ such that

$$\begin{aligned} &(\frac{\partial u_N}{\partial t}, v_N) + k(u_{Nxx}, v_{Nxx}) + \beta(u_N, v_N) + \gamma(u_{Nx}^2, v_N) - \alpha(u_{Nx}, v_{Nx}) \\ &- \delta((u_{Nx}^3)_x, v_N) = 0, \quad \forall v_N \in S_N, \end{aligned} \tag{2.2}$$

for all $T \ge 0$ with $u_N(0) = P_N u_0$.

Now, we are going to establish the existence and uniqueness of the Fourier spectral approximation solution $u_N(t)$ for all $T \ge 0$.

Lemma 1. *Let $u_0 \in H_E^2(0,1)$, then (2.2) has a unique solution $u_N(t)$ satisfying the following inequalities:*

$$\|u_N(t)\|_{H_E^2}^2 \le c_1 \|u_0\|_{H_E^2}^2, \quad \int_0^T \|u_N(\tau)\|_{H_E^4}^2 d\tau \le c_1' \|u_0\|_{H_E^2}^2, \tag{2.3}$$

where c_1 and c_1' are positive constants depend only on α, β, γ, δ, T and $\|u_0\|_{H_E^2}$, independents of N.

Proof. Set $v_N = \cos j\pi x$ in (2.2) for each j $(1 \le j \le N)$ to obtain

$$\frac{d}{dt} a_j(t) = f_j(a_1(t), a_2(t), \cdots, a_N(t)), \quad j = 1, 2, \cdots, N, \tag{2.4}$$

where all $f_j : \mathbb{R}^N \to \mathbb{R}$ $(1 \le j \le N)$ are smooth and locally Lipschitz continuous. Note that $u_N(0) = P_N u_0$. Therefore

$$a_j(0) = (u_0, \cos j\pi x), \quad j = 1, 2, \cdots, N. \tag{2.5}$$

Using the theory of initial-value problems of the ordinary differential equations, there is a time $T_N > 0$ such that the initial-value problem (2.4)-(2.5) has a unique smooth solution $(a_1(t), a_2(t), \cdots, a_N(t))$ for $t \in [0, T_N]$. Therefore, there are three steps for us to prove the lemma:

Step 1. Setting $v_N = u_N$ in (2.2), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_N\|^2 + k\|u_{Nxx}\|^2 + \beta\|u_N\|^2 + \delta\|u_{Nx}\|_4^4 = -\gamma(u_{Nx}^2, u_N) - \alpha(u_{Nxx}, u_N).$$

Noticing that

$$-\gamma(u_{Nx}^2, u_N) \le \delta\|u_{Nx}\|_4^4 + \frac{\gamma^2}{4\delta}\|u_N\|^2,$$

and

$$-\alpha(u_{Nxx}, u_N) \le \frac{k}{2}\|u_{Nxx}\|^2 + \frac{\alpha^2}{2k}\|u_N\|^2.$$

Hence, by a simple calculation, we get

$$\frac{d}{dt}\|u_N\|^2 + k\|u_{Nxx}\|^2 = (\frac{\gamma^2}{2\delta} + \frac{\alpha^2}{k} - 2\beta)\|u_N\|^2. \tag{2.6}$$

Using Gronwall's inequality, we deduce that

$$\|u_N\|^2 \le e^{(\frac{\gamma^2}{2\delta} + \frac{\alpha^2}{k} - 2\beta)t}\|u_N(0)\|^2 \le e^{(\frac{\gamma^2}{2\delta} + \frac{\alpha^2}{k} - 2\beta)T}\|u_0\|^2, \quad \forall t \in [0, T]. \tag{2.7}$$

Integrating (2.6) from 0 to T, we obtain

$$\begin{aligned}\int_0^T \|u_{Nxx}\|^2 dt \le& \frac{1}{k}[(\frac{\gamma^2}{2\delta} + \frac{\alpha^2}{k} - 2\beta)\int_0^T \|u_N\|^2 dt + \|u_N(0)\|] \\ \le& \frac{1}{k}[(\frac{\gamma^2}{2\delta} + \frac{\alpha^2}{k} - 2\beta)e^{(\frac{\gamma^2}{2\delta} + \frac{\alpha^2}{k} - 2\beta)T} + 1]\|u_0\|^2.\end{aligned} \tag{2.8}$$

Step 2. Setting $v_N = -u_{Nxx}$ in (2.2), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_{Nx}\|^2 + k\|u_{Nxxx}\|^2 + \beta\|u_{Nx}\|^2 - \gamma(u_{Nx}^2, u_{Nxx}) \\ &-\alpha\|u_{Nxx}\|^2 + 3\delta(u_{Nx}^2 u_{Nxx}, u_{Nxx}) = 0.\end{aligned} \tag{2.9}$$

Noticing that

$$\gamma(u_{Nx}^2, u_{Nxx}) = 0, \quad \delta(u_{Nx}^2 u_{Nxx}, u_{Nxx}) \ge 0, \quad \alpha\|u_{Nxx}\|^2 = -\alpha(u_{Nx}, u_{Nxxx}).$$

Then, we have

$$\frac{1}{2}\frac{d}{dt}\|u_{Nx}\|^2 + k\|u_{Nxxx}\|^2 + \beta\|u_{Nx}\|^2 \le \alpha\|u_{Nxx}\|^2 \le \frac{k}{2}\|u_{Nxxx}\|^2 + \frac{\alpha^2}{2k}\|u_{Nx}\|^2,$$

that is

$$\frac{d}{dt}\|u_{Nx}\|^2 + k\|u_{Nxxx}\|^2 \le (\frac{\alpha^2}{k} - 2\beta)\|u_{Nx}\|^2. \tag{2.10}$$

Using Gronwall's inequality, we get

$$\|u_{Nx}\|^2 \le e^{(\frac{\alpha^2}{k}-2\beta)t}\|u_{Nx}(0)\|^2 \le e^{(\frac{\alpha^2}{k}-2\beta)T}\|u_{x0}\|^2, \quad \forall t \in [0,T]. \tag{2.11}$$

Integrating (2.10) from 0 to T, we obtain

$$\begin{aligned}\int_0^T \|u_{Nxxx}\|^2 dt \le& \frac{1}{k}[(\frac{\alpha^2}{k} - 2\beta)e^{(\frac{\alpha^2}{k}-2\beta)T}\|u_{x0}\|^2 + \|u_{Nx}(0)\|^2] \\ \le& \frac{1}{k}[(\frac{\alpha^2}{k} - 2\beta)e^{(\frac{\alpha^2}{k}-2\beta)T} + 1]\|u_{x0}\|^2.\end{aligned} \tag{2.12}$$

Step 3. Setting $v_N = u_{Nxxxx}$ in (2.2), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_{Nxx}\|^2 + k\|u_{Nxxxx}\|^2 + \beta\|u_{Nxx}\|^2 \\ &+ \gamma((u_{Nx})^2, u_{Nxxxx}) + \alpha(u_{Nxx}, u_{Nxxxx}) - 3\delta((u_{Nx})^2 u_{Nxx}, u_{Nxxxx}) = 0.\end{aligned}$$

By Nirenberg's inequality, we deduce that

$$\|u_{Nx}\|_8^2 \le c'\|u_{Nxxxx}\|^{\frac{1}{4}}\|u_{Nx}\|^{\frac{7}{4}},$$

$$\|u_{Nx}\|_4^4 \le c'\|u_{Nxxxx}\|^{\frac{1}{3}}\|u_{Nx}\|^{\frac{11}{3}},$$

and

$$\|u_{Nxx}\|_4 \le c'\|u_{Nxxxx}\|^{\frac{5}{12}}\|u_{Nx}\|^{\frac{7}{12}},$$

where c' is a positive constant dependents only on the domain. Hence,

$$\begin{aligned}3\delta((u_{Nx})^2 u_{Nxx}, u_{Nxxxx}) \le& 3\delta\|u_{Nx}\|_8^2\|u_{Nxx}\|\|u_{Nxxxx}\| \\ \le& 3\delta(c')^2\|u_{Nxxxx}\|^{\frac{5}{3}}\|u_{Nx}\|^{\frac{7}{3}} \le \frac{k}{6}\|u_{Nxxxx}\|^2 + c_2,\end{aligned}$$

where c_2 is a positive constant depends only on $k, \alpha, \beta, \delta, T$ and $\|u_0\|_{H^1}$, independents of N. On the other hand, we have

$$-\gamma((u_{Nx})^2, u_{Nxxxx}) \le \frac{k}{12}\|u_{Nxxxx}\|^2 + \frac{3\gamma^2}{k}\|u_{Nx}\|_4^4 \le \frac{k}{6}\|u_{Nxxxx}\|^2 + c_3,$$

where c_3 is a positive constant depends only on k, α, β, T and $\|u_0\|_{H^1}$, independents of N. We also have

$$-\alpha(u_{Nxx}, u_{Nxxxx}) \le \frac{k}{6}\|u_{Nxxxx}\|^2 + \frac{3\alpha^2}{2k}\|u_{Nxx}\|^2.$$

Summing up, we get

$$\frac{d}{dt}\|u_{Nxx}\|^2 + k\|u_{Nxxxx}\|^2 \le (\frac{3\alpha^2}{k} - 2\beta)\|u_{Nxx}\|^2 + 2(c_2 + c_3). \tag{2.13}$$

Using Gronwall's inequality, we obtain

$$\begin{aligned}\|u_{Nxx}\|^2 \leq & e^{(\frac{3\alpha^2}{k}-2\beta)t}(\|u_{Nxx}(0)\|^2+2(c_2+c_3)t)\\ \leq & e^{(\frac{3\alpha^2}{k}-2\beta)T}(\|u_{xx0}\|^2+2(c_2+c_3)T)=c_4,\quad \forall t\in[0,T],\end{aligned} \tag{2.14}$$

where c_4 is a positive constant depends only on k, α, β, γ, δ, T and $\|u_0\|_{H^2}$, independents of N. Integrating (2.13) from 0 to T, we obtain

$$\begin{aligned}&\int_0^T \|u_{Nxxxx}\|^2 dt\\ \leq & \frac{1}{k}[(\frac{3\alpha^2}{k}-2\beta)\int_0^T \|u_{Nxx}\|^2 dt+2(c_2+c_3)T+\|u_{Nxx}(0)\|^2]\\ \leq & \frac{1}{k}[(\frac{3\alpha^2}{k}-2\beta)c_4^2T+2(c_2+c_3)T+\|u_{xx0}\|^2]=c_5.\end{aligned} \tag{2.15}$$

where c_5 is a positive constant depends only on k, α, β, γ, δ, T and $\|u_0\|_{H^2}$, independents of N.

Combining (2.7), (2.8), (2.11), (2.12), (2.14), (2.15) together, we get the result of Lemma 1.

□

Hence, we have the following theorem on the existence and uniqueness of global solution for problem (2.2).

Theorem 2. *Let $u_0 \in H_E^2(0,1)$, then for any $T>0$, problem (2.2) admits a unique global solution $u_N(x,t)$, such that*

$$u_N(x,t)\in L^\infty\left(0,T;H_E^2(0,1)\right)\bigcap L^2\left(0,T;H^4(0,1)\right).$$

Proof. We are going to apply the Leray-Schauder fixed point theorem to complete the proof. Define the linear space

$$\begin{aligned}X=\Big\{&u_N\in L^\infty\left(0,T;H_E^2(0,1)\right)\bigcap L^2\left(0,T;H^4(0,1)\right);\\ &u_{Nx}(0,t)=u_{Nx}(1,t)=0, u_N(x,0)=u_0(x)\Big\}.\end{aligned}$$

Clearly, X is a Banach space. Define the associated operator T,

$$T: X\to X,\quad u_N\mapsto w,$$

where w is determined by the following linear problem:

$$\begin{aligned}&\frac{\partial w}{\partial t}+kw_{xxxx}+\beta w+\alpha w_{xx}=-\gamma u_{Nx}^2+\delta(u_{Nx}^3)_x,\quad x\in(0,1),\\ &w_x(0,t)=w_x(1,t)=w_{xxx}(0,t)=w_{xxx}(1,t)=0,\quad t\in(0,T),\\ &w(x,0)=u_0(x).\end{aligned}$$

Form the discussions in Lemmas 1 and by the contraction mapping principle, T has a unique fixed point u_N, which is the desired solution of problem (1). Since the proof of the uniqueness of the solution is easy, we omit it here. Then, Theorem 2 is proved. □

Now, we give the following theorem.

Theorem 3. *Let $u_0 \in H_E^2(0,1)$, $u(x,t)$ is the solution of problem (1) and $u_N(x,t)$ is the solution of semi-discrete approximation (2.2). Then, there exists a constant c, depends on k, α, β, γ, δ, T and $\|u_0\|_{H_E^2}$, independent of N, such that*

$$\|u(x,t)-u_N(x,t)\| \le c(N^{-2}+\|u_0-u_N(0)\|).$$

Proof. Denote $\eta_N = u(t)-P_Nu(t)$ and $e_N = P_Nu(t)-u_N(t)$. It then follows from (1a) and (2.2) that

$$\begin{aligned}&(e_{Nt},v_N)+k(e_{Nxx},v_{Nxx})+\beta(e_N,v_N)+\gamma(u_x^2-u_{Nx}^2,v_N)\\&+\alpha(e_N,v_{Nxx})+\delta(u_x^3-u_{Nx}^3,v_{Nx})=0,\quad \forall v_N\in S_N.\end{aligned}\tag{2.16}$$

Setting $v_N = e_N$ in (2.16), we derive that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|e_N\|^2+k\|e_{Nxx}\|^2+\beta\|e_N\|^2\\&=-\gamma(u_x^2-u_{Nx}^2,e_N)-\alpha(e_N,e_{Nxx})-\delta(u_x^3-u_{Nx}^3,e_{Nx}).\end{aligned}\tag{2.17}$$

By Theorem 1, we have

$$\|u(x,t)\|_{H_E^2} \le c(k,\alpha,\beta,\gamma,\delta,\|u_0\|_{H_E^2}),$$

Using Sobolev's embedding theorem, we get

$$\|u(x,t)\|_{W^{1,\infty}} \le c(k,\alpha,\beta,\gamma,\delta,\|u_0\|_{H^2}).$$

We also have

$$\|e_N\|_\infty^2 \le c\|e_N\|_{H^1}^2 \le c(\|e_N\|^2+\|e_{Nx}\|^2) \le c'(\|e_N\|^2+\|e_{Nxx}\|^2),$$

and

$$\|e_{Nx}\|_\infty^2 \le c\|e_N\|_{H_E^2}^2 \le c(\|e_N\|^2+\|e_{Nx}\|^2+\|e_{Nxx}\|^2) \le c'(\|e_N\|^2+\|e_{Nxx}\|^2).$$

Then

$$\begin{aligned}&-\gamma(u_x^2-u_{Nx}^2,e_N)\\=&-\gamma((u_x+u_{Nx})(e_{Nx}+\eta_{Nx}),e_N)\\=&\gamma(e_N+\eta_N,(u_x+u_{Nx})e_{Nx}+(u_{xx}+u_{Nxx})e_N)\\\le&\gamma(\|e_N\|\|e_{Nx}\|\|u_x+u_{Nx}\|_\infty+\|\eta_N\|\|e_{Nx}\|\|u_x+u_{Nx}\|_\infty\\&+\|e_N\|\|u_{xx}+u_{Nxx}\|\|e_N\|_\infty+\|\eta_N\|\|u_{xx}+u_{Nxx}\|\|e_N\|_\infty)\\\le&\frac{k}{6}\|e_{Nxx}\|^2+c_6(\|e_N\|^2+\|\eta_N\|^2),\end{aligned}$$

and

$$\begin{aligned}
&-\delta(u_x^3-u_{Nx}^3,e_{Nx})\\
=&-\delta((e_{Nx}+\eta_{Nx})(u_x^2+u_xu_{Nx}+u_{Nx}^2),e_{Nx})\\
=&\delta(e_N+\eta_N,(u_x^2+u_xu_{Nx}+u_{Nx}^2)e_{Nxx})\\
&\delta(e_N+\eta_N,(2u_xu_{xx}+u_xu_{Nxx}+u_{xx}u_{Nx}+2u_{Nxx})e_{Nx})\\
\le&\delta(\|e_N\|+\|\eta_N\|)\|e_{Nxx}\|\|u_x^2+u_xu_{Nx}+u_{Nx}^2\|_\infty\\
&+\delta(\|e_N\|+\|\eta_N\|)\|2u_xu_{xx}+u_xu_{Nxx}+u_{xx}u_{Nx}+2u_{Nxx}\|\|e_{Nx}\|_\infty\\
\le&\frac{k}{6}\|e_{Nxx}\|^2+c_7(\|e_N\|^2+\|\eta_N\|^2),
\end{aligned}$$

where c_6 and c_7 are positive constants depends only on k, α, β, γ, δ, T and $\|u_0\|_{H^2}$, independents of N. We also have

$$\alpha(e_N,e_{Nxx})\le\frac{k}{6}\|e_{Nxx}\|^2+\frac{3\alpha^2}{2k}\|e_N\|^2.$$

Summing up, we deduce that

$$\frac{d}{dt}\|e_N\|^2+k\|e_{Nxx}\|^2\le(2c_6+2c_7+\frac{3\alpha^2}{k}-2\beta)\|e_N\|^2+(2c_6+2c_7)\|\eta_N\|^2.$$

Noticing that

$$\|\eta_N\|^2\le cN^{-4}\|u_{xx}\|^2\le c_8(k,\alpha,\beta,\gamma,\delta,T,\|u_0\|_{H^2})N^{-4}.$$

Therefore

$$\frac{d}{dt}\|e_N\|^2+k\|e_{Nxx}\|^2\le(2c_6+2c_7+\frac{3\alpha^2}{k}-2\beta)\|e_N\|^2+c_8(2c_6+2c_7)N^{-4}.$$

Using Gronwall's inequality, we complete the proof. □

3. FULLY DISCRETE SCHEME

Let Δt be the time-step, the full-discretization spectral method for problem (1) is read as: find $u_N^j\in S_N$ $(j=0,1,2,\cdots,N)$ such that

$$\begin{aligned}
&(\frac{u_N^{j+1}-u_N^j}{\Delta t},v_N)+k(\bar{u}_{Nxx}^{j+\frac12},v_{Nxx})+\beta(\bar{u}_N^{j+\frac12},v_N)+\gamma\left((\bar{u}_{Nx}^{j+\frac12})^2,v_N\right)\\
&+\alpha(\bar{u}_N^{j+\frac12},v_{Nxx})+\delta\left((\bar{u}_{Nx}^{j+\frac12})^3,v_{Nx}\right)=0,\quad\forall v_N\in S_N,
\end{aligned}\tag{3.1}$$

with $u_N(0)=P_Nu_0$, where $\bar{u}_N^{j+\frac12}=\frac12(u_N^j+u_N^{j+1})$.

The solution u_N^j has the following property:

Lemma 2. *Assume that $u_0 \in H_E^2(0,1)$. Suppose that u_N^j is a solution of problem (3.1), then there exists positive constants c_9, c_{10} depend only on k, α, β, γ, δ, T and $\|u_0\|_{H_E^2}$, independent of N, such that*

$$\|u_N^j\|_{H_E^2} \le c_9, \quad \|u_N^j\|_{W_E^{1,\infty}} \le c_{10}.$$

Proof. We can use the same method as Lemma 1 to prove this lemma. Since the proof is so easy, we omit it here. □

In the following, we analyze the error estimates between the numerical solution u_N^j and the exact solution $u(t_j)$. According to the properties of the projection operator P_N, we only need to analyze the error between $P_N u(t_j)$ and u_N^j. Denoted by $u^j = u(t_j)$, $e^j = P_N u^j - u_N^j$ and $\eta^j = u^j - P_N u^j$. Therefore

$$u^j - u_N^j = \eta^j + e^j.$$

If no confusion occurs, we denote the average of the two instant errors e^n and e^{n+1} by $\bar{e}^{n+\frac{1}{2}}$, where $\bar{e}^{n+\frac{1}{2}} = \frac{e^n + e^{n+1}}{2}$. On the other hand, we let $\bar{\eta}^{j+\frac{1}{2}} = \frac{\eta^j + \eta^{j+1}}{2}$.

Firstly, we give the following error estimates for the full discretization scheme.

Lemma 3 (see [3])**.** *For the instant errors e^{j+1} and e^j, we have*

$$\begin{aligned} \|e^{j+1}\|^2 \le & \|e^j\|^2 + 2\Delta t\Big(u_t(t_{j+\frac{1}{2}}) - \frac{u_N^{j+1} - u_N^j}{\Delta t}, \bar{e}^{j+\frac{1}{2}}\Big) \\ & + \frac{1}{320}(\Delta t)^4 \int_{t_j}^{t_{j+1}} \|u_{ttt}\|^2 dt + \Delta t \|\bar{e}^{j+\frac{1}{2}}\|^2. \end{aligned} \tag{3.2}$$

of (1a) with $\bar{e}^{j+\frac{1}{2}}$, letting $t = t_{j+\frac{1}{2}}$, we obtain

$$\begin{aligned} & (u_t^{j+\frac{1}{2}}, \bar{e}^{j+\frac{1}{2}}) + (k u_{xx}^{j+\frac{1}{2}}, \bar{e}_{xx}^{j+\frac{1}{2}}) + \beta(u^{j+\frac{1}{2}}, \bar{e}^{j+\frac{1}{2}}) \\ & + \gamma((\bar{u}_x^{j+\frac{1}{2}})^2, \bar{e}^{j+\frac{1}{2}}) + \alpha(\bar{u}^{j+\frac{1}{2}}, \bar{e}_{xx}^{j+\frac{1}{2}}) + \delta((u_x^{j+\frac{1}{2}})^3, \bar{e}_x^{j+\frac{1}{2}}) = 0 \end{aligned}$$

Taking $v_N = \bar{e}^{n+\frac{1}{2}}$ in (3.1), we obtain

$$\begin{aligned} & \Big(\frac{u_N^{j+1} - u_N^j}{\Delta t}, \bar{e}^{j+\frac{1}{2}}\Big) + k(\bar{u}_{Nxx}^{j+\frac{1}{2}}, \bar{e}_{xx}^{j+\frac{1}{2}}) + \beta(\bar{u}_N^{j+\frac{1}{2}}, \bar{e}^{j+\frac{1}{2}}) \\ & + \gamma((\bar{u}_{Nx}^{j+\frac{1}{2}})^2, \bar{e}^{j+\frac{1}{2}}) + \alpha(\bar{u}_N^{j+\frac{1}{2}}, \bar{e}_{xx}^{j+\frac{1}{2}}) + \delta((u_{Nx}^{j+\frac{1}{2}})^3, \bar{e}_x^{j+\frac{1}{2}}) = 0 \end{aligned}$$

Comparing the above two equations, we get

$$\begin{aligned}&(u_t^{j+\frac{1}{2}}-\frac{u_N^{j+1}-u_N^j}{\Delta t},\bar{e}^{j+\frac{1}{2}})\\ =&-k(u_{xx}^{j+\frac{1}{2}}-\bar{u}_{Nxx}^{j+\frac{1}{2}},\bar{e}_{xx}^{j+\frac{1}{2}})-\beta(u^{j+\frac{1}{2}}-\bar{u}_N^{j+\frac{1}{2}},\bar{e}^{j+\frac{1}{2}})-\gamma((u_x^{j+\frac{1}{2}})^2\\ &-(\bar{u}_{Nx}^{j+\frac{1}{2}})^2,\bar{e}^{j+\frac{1}{2}})-\alpha(u^{j+\frac{1}{2}}-\bar{u}_N^{j+\frac{1}{2}},\bar{e}_{xx}^{j+\frac{1}{2}})-\delta((u_x^{j+\frac{1}{2}})^3-(\bar{u}_{Nx}^{j+\frac{1}{2}})^3,\bar{e}_x^{j+\frac{1}{2}}).\end{aligned}$$

Now, we investigate the error estimates of the five items in the right-hand side of previous equation.

Lemma 4. *Assume that $u_0\in H_E^2(0,1)$, u is the solution for problem (1) and u_N^j is the solution for problem (3.1). Then*

$$-k(u_{xx}^{j+\frac{1}{2}}-\bar{u}_{Nxx}^{j+\frac{1}{2}},\bar{e}_{xx}^{j+\frac{1}{2}})\le-\frac{k}{2}\|\bar{e}_{xx}^{j+\frac{1}{2}}\|^2+\frac{k(\Delta t)^3}{192}\int_{t_j}^{t_{j+1}}\|u_{xxtt}\|^2dt.$$

Proof. Using Taylor's expansion, we obtain

$$\begin{aligned}u^j&=u^{j+\frac{1}{2}}-\frac{\Delta t}{2}u_t^{j+\frac{1}{2}}+\int_{t_j}^{t_{j+\frac{1}{2}}}(t-t_j)u_{tt}dt,\\ u^{j+1}&=u^{j+\frac{1}{2}}+\frac{\Delta t}{2}u_t^{j+\frac{1}{2}}+\int_{t_{j+\frac{1}{2}}}^{t_{j+1}}(t_j-t)u_{tt}dt.\end{aligned}$$

Hence

$$\frac{1}{2}(u^j+u^{j+1})-u^{j+\frac{1}{2}}=\frac{1}{2}\left(\int_{t_j}^{t_{j+\frac{1}{2}}}(t-t_j)u_{tt}dt+\int_{t_{j+\frac{1}{2}}}^{t_{j+1}}(t_j-t)u_{tt}dt\right).$$

By Hölder's inequality, we have

$$\begin{aligned}&\|u_{xx}^{j+\frac{1}{2}}-\frac{1}{2}(u_{xx}^j+u_{xx}^{j+1}))\|^2\\ =&\frac{1}{4}\left(\|\Big(\int_{t_j}^{t_{j+\frac{1}{2}}}(t-t_j)u_{tt}dt+\int_{t_{j+\frac{1}{2}}}^{t_{j+1}}(t_j-t)u_{tt}dt\Big)_{xx}\|^2\right)\\ \le&\frac{(\Delta t)^3}{96}\int_{t_j}^{t_{j+1}}\|u_{xxtt}\|^2dt.\end{aligned}$$

Noticing that $(\bar{\eta}_{xx}^{j+\frac{1}{2}}, \bar{e}_{xx}^{j+\frac{1}{2}}) = 0$. Therefore

$$\begin{aligned}
&-(u_{xx}^{j+\frac{1}{2}} - \bar{u}_{Nxx}^{j+\frac{1}{2}}, \bar{e}_{xx}^{j+\frac{1}{2}}) \\
=&-(u_{xx}^{j+\frac{1}{2}} - \frac{u_{xx}^{j} + u_{xx}^{j+1}}{2}, \bar{e}_{xx}^{j+\frac{1}{2}}) - (\frac{u_{xx}^{j+1} + u_{xx}^{j}}{2} - \frac{u_{N}^{j+1} + u_{N}^{j}}{2}, \bar{e}_{xx}^{j+\frac{1}{2}}) \\
\le&\|u_{xx}^{j+\frac{1}{2}} - \frac{u_{xx}^{j} + u_{xx}^{j+1}}{2}\|\|\bar{e}_{xx}^{j+\frac{1}{2}}\| - (\bar{\eta}_{xx}^{j+\frac{1}{2}} + \bar{e}_{xx}^{j+\frac{1}{2}}, \bar{e}_{xx}^{j+\frac{1}{2}}) \\
\le&\left(\frac{(\Delta t)^3}{96}\int_{t_j}^{t_{j+1}} \|u_{xxtt}\|^2 dt\right)^{\frac{1}{2}} \|\bar{e}_{xx}^{j+\frac{1}{2}}\| - (\bar{\eta}_{xx}^{j+\frac{1}{2}}, \bar{e}_{xx}^{j+\frac{1}{2}}) - \|\bar{e}_{xx}^{j+\frac{1}{2}}\|^2. \\
\le&\frac{(\Delta t)^3}{192}\int_{t_j}^{t_{j+1}} \|u_{xxtt}\|^2 dt - \frac{1}{2}\|\bar{e}_{xx}^{j+\frac{1}{2}}\|^2.
\end{aligned}$$

Then, Lemma 4 is proved. □

Lemma 5. *Assume that $u_0 \in H_E^2(0,1)$, u is the solution for problem (1) and u_N^j is the solution for problem (3.1). Then*

$$\begin{aligned}
&-\beta(u^{j+\frac{1}{2}} - \bar{u}_N^{j+\frac{1}{2}}, \bar{e}^{j+\frac{1}{2}}) - \alpha(u^{j+\frac{1}{2}} - \bar{u}_N^{j+\frac{1}{2}}, \bar{e}_{xx}^{j+\frac{1}{2}}) \\
\le&\frac{k}{6}\|\bar{e}_{xx}^{j+\frac{1}{2}}\|^2 + \frac{3\alpha^2}{k}\|\bar{e}^{j+\frac{1}{2}}\|^2 + c_{11}N^{-4},
\end{aligned}$$

where c_{11} is a positive constant depends only on k, α, β, γ, δ, T and $\|u_0\|_{H_E^2}$, independent of N.

Proof. Noticing that

$$\|\bar{\eta}^{j+\frac{1}{2}}\| \le c(k,\alpha,\beta,\gamma,\delta,T,\|u_0\|_{H_E^2})N^{-2}.$$

Hence

$$\begin{aligned}
&-\beta(u^{j+\frac{1}{2}} - \bar{u}_N^{j+\frac{1}{2}}, \bar{e}^{j+\frac{1}{2}}) - \alpha(u^{j+\frac{1}{2}} - \bar{u}_N^{j+\frac{1}{2}}, \bar{e}_{xx}^{j+\frac{1}{2}}) \\
=&-\beta(\bar{e}^{j+\frac{1}{2}} + \bar{\eta}^{j+\frac{1}{2}}, \bar{e}^{j+\frac{1}{2}}) - \alpha(\bar{e}^{j+\frac{1}{2}} + \bar{\eta}^{j+\frac{1}{2}}, \bar{e}_{xx}^{j+\frac{1}{2}}) \\
=&-\alpha\|\bar{e}^{j+\frac{1}{2}}\|\|\bar{e}_{xx}^{j+\frac{1}{2}}\| - \alpha\|\bar{\eta}^{j+\frac{1}{2}}\|\|\bar{e}_{xx}^{j+\frac{1}{2}}\| - \beta\|\bar{e}^{j+\frac{1}{2}}\|^2 - \beta\|\bar{\eta}^{j+\frac{1}{2}}\|\|\bar{e}^{j+\frac{1}{2}}\| \\
\le&\frac{k}{6}\|\bar{e}_{xx}^{j+\frac{1}{2}}\|^2 + \frac{3\alpha^2}{k}(\|\bar{e}^{j+\frac{1}{2}}\|^2 + \|\bar{\eta}^{j+\frac{1}{2}}\|^2) + \frac{\beta}{4}\|\bar{\eta}^{j+\frac{1}{2}}\|^2 \\
\le&\frac{k}{6}\|\bar{e}_{xx}^{j+\frac{1}{2}}\|^2 + \frac{3\alpha^2}{k}\|\bar{e}^{j+\frac{1}{2}}\|^2 + c_{11}N^{-4}.
\end{aligned}$$

Then, Lemma 5 is proved. □

Lemma 6. *Assume that $u_0 \in H_E^2(0,1)$, u is the solution for problem (1) and u_N^j is the solution for problem (3.1). Then*

$$\begin{aligned}
&-\gamma((u_x^{j+\frac{1}{2}})^2-(\bar{u}_{Nx}^{j+\frac{1}{2}})^2,\bar{e}^{j+\frac{1}{2}})\\
\le&\frac{k}{6}\|\bar{e}_{xx}^{j+\frac{1}{2}}\|^2+c_{12}\left(\frac{(\Delta t)^3}{96}\int_{t_j}^{t_{j+1}}\|u_{xtt}\|^2dt+\|\bar{e}^{j+\frac{1}{2}}\|^2+N^{-4}\right),
\end{aligned}$$

where c_{12} is a positive constant depends only on k, α, β, γ, δ, T and $\|u_0\|_{H_E^2}$, independent of N.

Proof. Noticing that

$$\|\bar{e}_{Nx}^{j+\frac{1}{2}}\|^2\le\varepsilon\|\bar{e}_{Nxx}^{j+\frac{1}{2}}\|^2+\frac{1}{4\varepsilon}\|\bar{e}^{j+\frac{1}{2}}\|^2,$$

$$\|\bar{\eta}^{j+\frac{1}{2}}\|\le c(k,\alpha,\beta,\gamma,\delta,T,\|u_0\|_{H_E^2})N^{-2}.$$

Hence, using the integration by parts, we have

$$\begin{aligned}
&-\gamma((u_x^{j+\frac{1}{2}})^2-(\bar{u}_{Nx}^{j+\frac{1}{2}})^2,\bar{e}^{j+\frac{1}{2}})\\
=&-\gamma((u_x^{j+\frac{1}{2}})^2-(\bar{u}_x^{j+\frac{1}{2}})^2,\bar{e}^{j+\frac{1}{2}})-\gamma((\bar{u}_x^{j+\frac{1}{2}})^2-(\bar{u}_{Nx}^{j+\frac{1}{2}})^2,\bar{e}^{j+\frac{1}{2}})\\
=&-\gamma((u_x^{j+\frac{1}{2}}+\bar{u}_x^{j+\frac{1}{2}})(u_x^{j+\frac{1}{2}}-\bar{u}_x^{j+\frac{1}{2}}),\bar{e}^{j+\frac{1}{2}})\\
&-\gamma((\bar{u}_x^{j+\frac{1}{2}}+\bar{u}_{Nx}^{j+\frac{1}{2}})(\bar{e}_x^{j+\frac{1}{2}}+\bar{\eta}_x^{j+\frac{1}{2}}),\bar{e}^{j+\frac{1}{2}})\\
=&-\gamma((u_x^{j+\frac{1}{2}}+\bar{u}_x^{j+\frac{1}{2}})(u_x^{j+\frac{1}{2}}-\bar{u}_x^{j+\frac{1}{2}}),\bar{e}^{j+\frac{1}{2}})\\
&+\gamma\left(\bar{e}^{j+\frac{1}{2}}+\bar{\eta}^{j+\frac{1}{2}},(\bar{u}_{xx}^{j+\frac{1}{2}}+\bar{u}_{Nxx}^{j+\frac{1}{2}})\bar{e}^{j+\frac{1}{2}}+(\bar{u}_x^{j+\frac{1}{2}}+\bar{u}_{Nx}^{j+\frac{1}{2}})\bar{e}_x^{j+\frac{1}{2}}\right).
\end{aligned}$$

By Hölder's inequality and Sobolev's embedding theorem, we get

$$\begin{aligned}
&\gamma((u_x^{j+\frac{1}{2}})^2-(\bar{u}_{Nx}^{j+\frac{1}{2}})^2,\bar{e}^{j+\frac{1}{2}})\\
\le&\gamma\|u_x^{j+\frac{1}{2}}+\bar{u}_x^{j+\frac{1}{2}}\|_\infty\|u_x^{j+\frac{1}{2}}-\bar{u}_x^{j+\frac{1}{2}}\|\|\bar{e}^{j+\frac{1}{2}}\|\\
&+\gamma(\|\bar{e}^{j+\frac{1}{2}}\|+\|\eta^{j+\frac{1}{2}}\|)\|\bar{u}_{xx}^{j+\frac{1}{2}}+\bar{u}_{Nxx}^{j+\frac{1}{2}}\|\|\bar{e}^{j+\frac{1}{2}}\|_\infty\\
&+\gamma(\|\bar{e}^{j+\frac{1}{2}}\|+\|\eta^{j+\frac{1}{2}}\|)\|\bar{u}_x^{j+\frac{1}{2}}+\bar{u}_{Nx}^{j+\frac{1}{2}}\|_\infty\|\bar{e}_x^{j+\frac{1}{2}}\|\\
\le&c\gamma\left(\frac{(\Delta t)^3}{96}\int_{t_j}^{t_{j+1}}\|u_{xtt}\|^2dt\right)^{\frac{1}{2}}\|\bar{e}^{j+\frac{1}{2}}\|+c\gamma(\|\bar{e}^{j+\frac{1}{2}}\|+\|\eta^{j+\frac{1}{2}}\|)\|\bar{e}_x^{j+\frac{1}{2}}\|\\
\le&\frac{k}{6}\|\bar{e}_{xx}^{j+\frac{1}{2}}\|^2+c_{12}\left(\frac{(\Delta t)^3}{96}\int_{t_j}^{t_{j+1}}\|u_{xtt}\|^2dt+\|\bar{e}^{j+\frac{1}{2}}\|^2+N^{-4}\right).
\end{aligned}$$

Hence, Lemma 6 is proved. □

Lemma 7. *Assume that $u_0 \in H_E^2(0,1)$, u is the solution for problem (1) and u_N^j is the solution for problem (3.1). Then*

$$\begin{aligned}&-\delta((u_x^{j+\frac{1}{2}})^3-(\bar{u}_{Nx}^{j+\frac{1}{2}})^3,\bar{e}_x^{j+\frac{1}{2}})\\ \le&\frac{k}{6}\|\bar{e}_{xx}^{j+\frac{1}{2}}\|^2+c_{13}\left(\frac{(\Delta t)^3}{96}\int_{t_j}^{t_{j+1}}\|u_{xtt}\|^2dt+\|\bar{e}^{j+\frac{1}{2}}\|^2+N^{-4}\right).\end{aligned}$$

where c_{13} is a positive constant depends only on k, α, β, γ, δ, T and $\|u_0\|_{H_E^2}$, independent of N.

Proof. Noticing that

$$\|\bar{\eta}^{j+\frac{1}{2}}\| \le c(k,\alpha,\beta,\gamma,\delta,T,\|u_0\|_{H_E^2})N^{-2}.$$

Hence, using the integration by parts, we have

$$\begin{aligned}&-\delta((u_x^{j+\frac{1}{2}})^3-(\bar{u}_{Nx}^{j+\frac{1}{2}})^3,\bar{e}_x^{j+\frac{1}{2}})\\ =&-\delta((u_x^{j+\frac{1}{2}})^3-(\bar{u}_x^{j+\frac{1}{2}})^3,\bar{e}_x^{j+\frac{1}{2}})+\delta((\bar{u}_x^{j+\frac{1}{2}})^3-(\bar{u}_{Nx}^{j+\frac{1}{2}})^3,\bar{e}_x^{j+\frac{1}{2}})\\ =&-\delta\left((u_x^{j+\frac{1}{2}}+\bar{u}_x^{j+\frac{1}{2}})((u_x^{j+\frac{1}{2}})^2+(\bar{u}_x^{j+\frac{1}{2}})^2+\bar{u}_x^{j+\frac{1}{2}}\bar{u}_{Nx}^{j+\frac{1}{2}}),\bar{e}_x^{j+\frac{1}{2}}\right)\\ &+\delta\Big(\bar{u}^{j+\frac{1}{2}}-\bar{u}_N^{j+\frac{1}{2}},\\ &\qquad(2\bar{u}_x^{j+\frac{1}{2}}\bar{u}_{xx}^{j+\frac{1}{2}}+\bar{u}_x^{j+\frac{1}{2}}\bar{u}_{Nxx}^{j+\frac{1}{2}}+\bar{u}_{xx}^{j+\frac{1}{2}}\bar{u}_{Nx}^{j+\frac{1}{2}}+2\bar{u}_{Nx}\bar{u}_{Nxx})\bar{e}_x^{j+\frac{1}{2}}\Big)\\ &+\delta\left(\bar{u}^{j+\frac{1}{2}}-\bar{u}_N^{j+\frac{1}{2}},[(\bar{u}_x^{j+\frac{1}{2}})^2+\bar{u}_x^{j+\frac{1}{2}}\bar{u}_{Nx}^{j+\frac{1}{2}}+(\bar{u}_{Nx}^{j+\frac{1}{2}})^2]\bar{e}_{xx}^{j+\frac{1}{2}}\right).\end{aligned}$$

By Hölder's inequality and Sobolev's embedding theorem, we immediately obtain

$$
\begin{aligned}
&-\delta((u_x^{j+\frac{1}{2}})^3-(\bar{u}_{Nx}^{j+\frac{1}{2}})^3,\bar{e}_x^{j+\frac{1}{2}})\\
\le&\delta\|(u_x^{j+\frac{1}{2}})^2+(\bar{u}_x^{j+\frac{1}{2}})^2+\bar{u}_x^{j+\frac{1}{2}}\bar{u}_{Nx}^{j+\frac{1}{2}}\|_\infty\|u_x^{j+\frac{1}{2}}-\bar{u}_x^{j+\frac{1}{2}}\|\|\bar{e}_x^{j+\frac{1}{2}}\|\\
&+\delta(\|\bar{e}^{j+\frac{1}{2}}\|+\|\bar{\eta}^{j+\frac{1}{2}}\|)\|2\bar{u}_x^{j+\frac{1}{2}}\bar{u}_{xx}^{j+\frac{1}{2}}+\bar{u}_x^{j+\frac{1}{2}}\bar{u}_{Nxx}^{j+\frac{1}{2}}+\bar{u}_{xx}^{j+\frac{1}{2}}\bar{u}_{Nx}^{j+\frac{1}{2}}\\
&+2\bar{u}_{Nx}\bar{u}_{Nxx}\|\|\bar{e}_x^{j+\frac{1}{2}}\|_\infty\\
&+\delta(\|\bar{e}^{j+\frac{1}{2}}\|+\|\bar{\eta}^{j+\frac{1}{2}}\|)\|(\bar{u}_x^{j+\frac{1}{2}})^2+\bar{u}_x^{j+\frac{1}{2}}\bar{u}_{Nx}^{j+\frac{1}{2}}+(\bar{u}_{Nx}^{j+\frac{1}{2}})\|_\infty\|\bar{e}_{xx}^{j+\frac{1}{2}}\|\\
\le&c\delta\left(\frac{(\Delta t)^3}{96}\int_{t_j}^{t_{j+1}}\|u_{xtt}\|^2dt\right)^{\frac{1}{2}}\|\bar{e}^{j+\frac{1}{2}}\|+c\delta(\|\bar{e}^{j+\frac{1}{2}}\|+\|\eta^{j+\frac{1}{2}}\|)\|\bar{e}_{xx}^{j+\frac{1}{2}}\|\\
\le&\frac{k}{6}\|\bar{e}_{xx}^{j+\frac{1}{2}}\|^2+c_{13}\left(\frac{(\Delta t)^3}{96}\int_{t_j}^{t_{j+1}}\|u_{xtt}\|^2dt+\|\bar{e}^{j+\frac{1}{2}}\|^2+N^{-4}\right).
\end{aligned}
$$

Then, Lemma 7 is proved. □

Thus, we obtain the following theorem.

Theorem 4. *Assume that $u_0\in H_E^2(0,1)$, $u(x,t)$ is the solution for problem (1) satisfying*

$$u\in L^\infty(0,T;H_E^2(0,1)),\ \ u_{tt}\in L^2(0,T;H_E^2(0,1)),\ \ u_{ttt}\in L^2(0,T;L^2(0,1)).$$

Assume further that u_N^j is the solution for problem (3.1). Then, if Δt is sufficiently small, there exists positive constants c_{14} depends on k, α, β, γ δ, T, $\|u_0\|_{H^2}$, independent of N, and c_{15} depends on k, α, β, γ δ, T, $\|u_0\|_{H^2}$, $\int_0^T\|u_{tt}\|_{H^2}^2dt$, $\int_0^T\|u_{ttt}\|^2dt$, independent of N, such that, for $j=0,1,2,\cdots,N$,

$$\|e^{j+1}\|\le c_{14}(N^{-2}+\|e^0\|)+c_{15}(\Delta t)^2.$$

Proof. By Lemmas 3-7, we obtain

$$
\begin{aligned}
\|e^{j+1}\|^2\le&\|e^j\|^2+\Delta t c_{16}(\|e^{j+1}\|^2+\|e^j\|^2+N^{-4})\\
&+(\Delta t)^4c_{17}\int_{t_j}^{t_{j+1}}(\|u_{tt}\|^2+\|u_{xtt}\|^2+\|u_{xxtt}\|^2+\|u_{ttt}\|^2)dt,
\end{aligned}
$$

where c_{16} and c_{17} are positive constants depend only on k,a,b,T and $\|u_0\|_{H^2}$. For Δt being sufficiently small, such that $c_{17}\Delta t\le\frac{1}{2}$, setting $c_{18}=2(c_{16}+c_{17})$, we get

$$\|e^{j+1}\|^2\le(1+c_{18}\Delta t)\|e^j\|^2+c_{18}(\Delta tN^{-4}+(\Delta t)^4B^j),$$

where

$$B^j=\int_{t_j}^{t_{j+1}}(\|u_{xtt}\|^2+\|u_{xxtt}\|^2+\|u_{ttt}\|^2)dt.$$

Using Gronwall's inequality for the discrete form, we have

$$\|e^{j+1}\|^2 \le e^{c_{18}(j+1)\Delta t}\left(\|e^0\|^2 + c_{18}\big(j\Delta t N^{-4} + (\Delta t)^4 \sum_{i=0}^{j} B^i\big)\right).$$

Directly computation shows that

$$\sum_{i=0}^{j} B^i \le \int_0^{t_{j+1}} (\|u_{tt}\|^2 + \|u_{xxtt}\|^2 + \|u_{ttt}\|^2)dt.$$

Thus, Theorem 4 is proved. □

Furthermore, we have the following theorem.

Theorem 5. *Assume that $u_0 \in H_E^2(0,1)$, $u(x,t)$ is the solution for problem (1) satisfying*

$$u \in L^\infty(0,T;H_E^2(0,1)),\ \ u_{tt} \in L^2(0,T;H_E^2(0,1)),\ \ u_{ttt} \in L^2(0,T;L^2(0,1)).$$

Assume further that $u_N^j \in S_N$ $(j = 0,1,2,\cdots)$ is the solution for problem (3.1) and the initial value u_N^0 satisfies

$$\|e^0\| = \|P_N u_0 - u_N^0\| \le cN^{-2}\|u_{xx}\|.$$

Then, there exists positive constants c' depends on k, α, β, γ δ, T, $\|u_0\|_{H^2}$, independent of N, and c'' depends on k, α, β, γ δ, T, $\|u_0\|_{H^2}$, $\int_0^T \|u_{tt}\|_{H^2}^2 dt$, $\int_0^T \|u_{ttt}\|^2 dt$, independent of N, such that

$$\|u(x,t_j) - u_N^j\| \le c'N^{-2} + c''(\Delta t)^2, \quad j = 0,1,2,\cdots,N.$$

4. Numerical result

In this section, using the Fourier spectral method described in (3.1), we carry out some numerical computations to illustrate out results in previous section. The full-discretization spectral method is read as: For $v_l = \sin l\pi x$, $l = 1,\cdots,N$, find

$$u_N^n = \sum_{i=0}^{N} \alpha_i^n \cos i\pi x.$$

such that (3.1) hold.

As an example, we choose $k = 1$, $\beta = 2$, $\gamma = 1$, $\alpha = 1$, $\delta = 1$, $u_0 = (1-x)^4x^4 + 0.001$, $\Delta t = 0.001/2, 0.001/4$, $N = 32$, and get the solution which evolves from $t = 0$ to $t = 0.1$ (cf. Figure 1) .

Now, we consider the variation of error. Since no exact solution to problem (1) is known for us, we make a comparison between the solution of (3.1) on a coarse mesh and on a fine mesh.

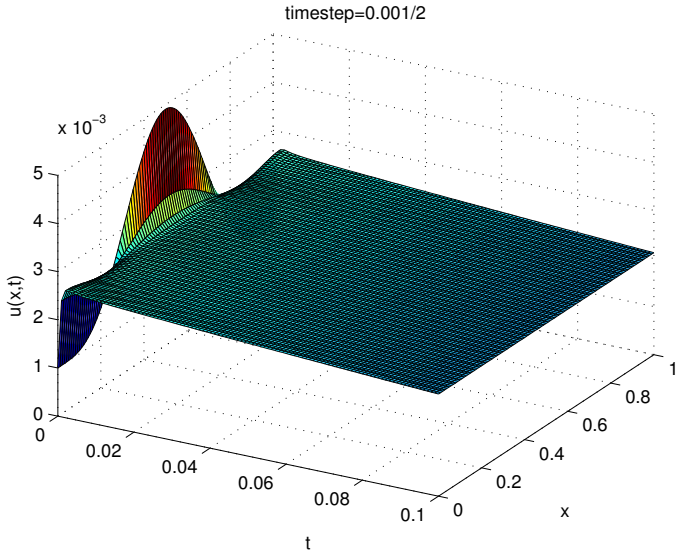

FIGURE 1. The expanded property solution for $N = 32$, $\Delta t = 0.0005$.

We choose $\Delta t = 0.001, 0.001 \times \frac{1}{2}, 0.001 \times \frac{1}{4}, 0.001 \times \frac{1}{8}, 0.001 \times \frac{1}{16}$, respectively to solve (3.1). Set $u_N^{\min}(x, 0.1)$ as the solution for $\Delta t_{\min} = 0.001 \times \frac{1}{32}$. Denote

$$err(0.1, \Delta t) = \left(\int_0^1 (u_N^k(x, 0.1) - u_N^{\min}(x, 0.1) \right)^2 dx)^{\frac{1}{2}}, k = 1, 2, \ldots, 6. \quad (4.1)$$

Then the error is showed in the Table 1 at $t = 0.1$.

TABLE 1. The error for difference time step at $t = 0.1$

Δt	$err(0.1, \Delta t)$	$\frac{err(0.1, \Delta t)}{(\Delta t)^2}$
0.001	2.3691×10^{-6}	2.3691
$0.001 \times \frac{1}{2}$	3.7397×10^{-7}	1.4959
$0.001 \times \frac{1}{4}$	5.6875×10^{-8}	0.9100
$0.001 \times \frac{1}{8}$	8.4589×10^{-9}	0.5413
$0.001 \times \frac{1}{16}$	1.0349×10^{-9}	0.1035

In Table 1, it is easy to see that the third column $\frac{err(0.1, \Delta t)}{(\Delta t)^2}$ is monotone decreasing along with the time step's waning. Hence, we can find a positive constant $C = 2.3691$, such that

$$\frac{err(0.1, \Delta t)}{(\Delta t)^2} \leq C, \quad k = 1, 2, \ldots, 5,$$

which means the order of error estimates is $O((\Delta t)^2)$ proved in Theorem 5.

On the other hand, Choose $N = 32, 40, 48, 56$, $\Delta t_0 = 0.001 \times \frac{1}{32}$, respectively to solve (3.1).

TABLE 2. The errors of different basic function numbers at t = 0.1

N	$err(0.1, \Delta t_0)$	$\frac{err(0.1,\Delta t_0)}{N^{-2}}$
32	1.8637×10^{-10}	1.908×10^{-7}
40	1.4862×10^{-10}	2.378×10^{-7}
48	8.6941×10^{-11}	2.003×10^{-7}
56	4.1655×10^{-11}	1.306×10^{-7}

Then the error is showed in Table 2 at $t = 0.1$. In Table 2, it is easy to see that the third column $\frac{err(0.1,\Delta t_0)}{(N)^{-2}}$ is almost monotone decreasing along with the time step's waning. Hence, we can find a positive constant $C = 2.378 \times 10^{-7}$, such that

$$\frac{err(0.1, \Delta t)}{(\Delta t)^2} \leq C, \quad N = 32, 40, 48, 56,$$

which means the order of error estimates is $O(N^{-2})$ proved in Theorem 5.

5. CONCLUSIONS

Since the tools we have used work for the periodic boundary values, this result is also valid for the 1D Kolmogorov-Spiegel-Sivashinsky equation with the periodic boundary conditions. That is, for any $u_0 \in H^2_{per}(0,1)$, choose the finite dimensional subspace of $H^2_{per}(0,1)$:

$$S_N = \text{span}\{e^{ikx}; -N/2 \leq k \leq N/2\},$$

the existence, uniqueness and optimal error bonds for semi-discrete and fully discrete schemes can also be proved under the periodic boundary conditions

$$\partial^j u(0,t) = \partial^j u(1,t), \quad t > 0, j = 0, 1, 2, 3.$$

Since the original Kolmogorov-Spiegel-Sivashinsky(KSS) equation which arises in physical systems such as the Kolmogorov flow (a turbulent system of small-scale eddies supported by external energy sources) and the large-scale structure of compressible non-Boussinesqian convection (large scale turbulent solar convection) is formulated in $\mathbb{R}^n$. Here, we only consider the 1D case of the equation. If we want to understand the properties of this model better, we should study the numerical solutions for the multi-dimensional KSS equation, which is our intention in the future.

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Authors' addresses

Xiaopeng Zhao
Jiangnan University, School of Science, 214122 Wuxi, China
E-mail address: zxp032@126.com

Bo Liu
Jilin University, College of Mathematics, 130012 Changchun, China

Peng Zhang
Chinese Academy of Sciences, Institute of Software, 100190 Beijing, China

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matfs@uni-miskolc.hu

Zoltán Füredi, A. Rényi Institute of Mathematics, Hungarian Academy of Sciences; H-1364, Budapest, Hungary, P. O .Box 127;
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gyori@almos.vein.hu

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hatvani@math.u-szeged.hu

Attila Házy, Institute of Mathematics, University of Miskolc; H-3515 Miskolc–Egyetemváros, Hungary;
matha@uni-miskolc.hu

Nino Partsvania, Andrea Razmadze Mathematical Institute; 2 University St., Tbilisi 0186 GEORGIA;
ninopa@rmi.ge

Nikolai Perestyuk, National Shevchenko University of Kiev, Faculty of Mechanics and Mathematics; 64 Vladimirskaya St., 01006 Kiev, Ukraine;
pmo@univ.kiev.ua

Adrian Petrusel, Department of Mathematics, Babes-Bolyai University of Cluj-Napoca; Mihail Kogalniceanu Street, No. 1, 400084, Cluj-Napoca, Romania;
petrusel@ubbcluj.ro

András Rontó, Institute of Mathematics, Academy of Sciences of Czech Republic; Žižkova 22, 616 62 Brno, Czech Republic;
ronto@math.cas.cz

Nikolai Rozov, Moscow State University, Faculty of Mechanics and Mathematics; The Main Building, Vorobiovy Gory, 119899 Moscow, Russia;
rozov@rozov.mccme.ru

Anatoly Samoilenko, Institute of Mathematics, National Academy of Sciences of Ukraine; 3 Tereshenkovskaya St., 01601 Kiev, Ukraine;
sam@imath.kiev.ua

Jenő Szigeti, Institute of Mathematics, University of Miskolc; H-3515 Miskolc–Egyetemváros, Hungary;
matszj@gold.uni-miskolc.hu

Jiří Šremr, Institute of Mathematics, Academy of Sciences of Czech Republic; Žižkova 22, 616 62 Brno, Czech Republic;
sremr@ipm.cz

Sergei Trofimchuk, Instituto de Matemática y Física, Universidad de Talca; 2 Norte 685, Talca, Chile;
trofimch@inst-mat.utalca.cl

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