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THE TOTAL IRREGULARITY OF GRAPHS UNDER GRAPH OPERATIONS

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Abstract. The *total irregularity* of a graph G is defined as $\operatorname{irr}_t(G) = \frac{1}{2}\sum_{u,v \in V(G)} |d_G(u) - d_G(v)|$, where $d_G(u)$ denotes the degree of a vertex $u \in V(G)$. In this paper we give (sharp) upper bounds on the total irregularity of graphs under several graph operations including join, lexicographic product, Cartesian product, strong product, direct product, corona product, disjunction and symmetric difference.

2010 *Mathematics Subject Classification:* 05C76; 05C07

Keywords: irregularity and total irregularity of graphs, graph operations

1. INTRODUCTION

Let G be a simple undirected graph with $|V(G)| = n$ vertices and $|E(G)| = m$ edges. The *degree* of a vertex v in G is the number of edges incident with v and it is denoted by $d_G(v)$. A graph G is *regular* if all its vertices have the same degree, otherwise it is *irregular*. However, in many applications and problems it is of great importance to know how irregular a given graph is. Several graph topological indices have been proposed for that purpose. Among the most investigated ones are: the irregularity of a graph introduced by Albertson [5], the variance of vertex degrees [8], and Collatz-Sinogowitz index [12].

The *imbalance* of an edge $e = uv \in E$, defined as $\operatorname{imb}(e) = |d_G(u) - d_G(v)|$, appeares implicitly in the context of Ramsey problems with repeat degrees [6], and later in the work of Chen, Erdős, Rousseau, and Schlep [11], where 2-colorings of edges of a complete graph were considered. In [5], Albertson defined the *irregularity* of G as

$$\operatorname{irr}(G) = \sum_{e \in E(G)} \operatorname{imb}(e). \tag{1.1}$$

It is shown in [5] that for a graph G, $\operatorname{irr}(G) < 4n^3/27$ and that this bound can be approached arbitrary closely. Albertson also presented upper bounds on irregularity for bipartite graphs, triangle-free graphs and a sharp upper bound for trees. Some claims about bipartite graphs given in [5] have been formally proved in [19]. Related to Albertson's result is the work of Hansen and Mélot [18], who characterized the graphs with n vertices and m edges with maximal irregularity. The irregularity measure irr also is related to the *first Zagreb index* $M_1(G)$ and the *second Zagreb index* $M_2(G)$, one of the oldest and most investigated topological graph indices, defined as follows:

$$M_1(G) = \sum_{v \in V(G)} d_G^2(v) \quad \text{and} \quad M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v).$$

Alternatively the first Zagreb index can be expressed as

$$M_1(G) = \sum_{uv \in E(G)} \left[d_G(u) + d_G(v) \right]. \tag{1.2}$$

Fath-Tabar [15] established new bounds on the first and the second Zagreb indices that depend on the irregularity of graphs as defined in (1.1). In line with the standard terminology of chemical graph theory, and the obvious connection with the first and the second Zagreb indices, Fath-Tabar named the sum in (1.1) the *third Zagreb index* and denoted it by $M_3(G)$. The graphs with maximal irregularity with 6, 7 and 8 vertices are depicted in Figure 1.

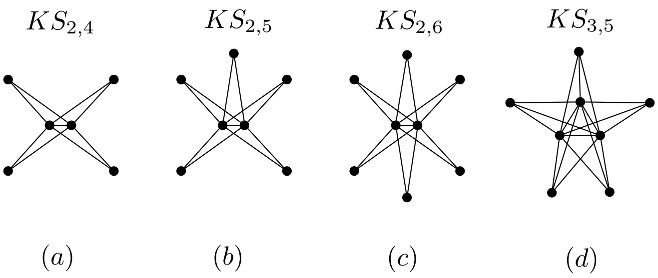

FIGURE 1. (a) The graph with 6 vertices with maximal irr. (b)The graph with 7 vertices with maximal irr. (c) and (d) Graphs with 8 vertices with maximal irr.

Two other most frequently used graph topological indices, that measure how irregular a graph is, are the *variance of degrees* and the *Collatz-Sinogowitz index* [12]. Let G be a graph with n vertices and m edges, and λ_1 be the index or largest eigenvalue of the adjacency matrix $A = (a_{ij})$ (with $a_{ij} = 1$ if vertices i and j are joined by an edge and 0 otherwise). Let n_i denotes the number of vertices of degree i for $i = 1, 2, \ldots, n-1$. Then, the variance of degrees and the Collatz-Sinogowitz index are respectively defined as

$$\mathrm{Var(G)} = \frac{1}{n} \sum_{i=1}^{n-1} n_i \left(i - \frac{2m}{n} \right)^2 \qquad \text{and} \qquad \mathrm{CS(G)} = \lambda_1 - \frac{2m}{n}. \tag{1.3}$$

Results of comparing irr, CS and Var are presented in [8, 13, 16].

There have been other attempts to determine how irregular graph is [2–4, 7, 9, 10, 20], but heretofore this has not been captured by a single parameter as it was done by the irregularity measure by Albertson.

The graph operation, especially graph products, plays significant role not only in pure and applied mathematics, but also in computer science. For example, the Cartesian product provide an important model for linking computers. In order to synchronize the work of the whole system it is necessary to search for Hamiltonian paths and cycles in the network. Thus, some results on Hamiltonian paths and cycles in Cartesian product of graphs can be applied in computer network design [23]. Many of the problems can be easily handled if the related graphs are regular or close to regular.

Recently in [1] a new measure of irregularity of a graph, so-called the *total irregularity*, that depends also on one single parameter (the pairwise difference of vertex degrees) was introduced. It was defined as

$$\operatorname{irr}_t(G) = \frac{1}{2} \sum_{u,v \in V(G)} |d_G(u) - d_G(v)|. \tag{1.4}$$

In the next theorem the upper bounds on the total irregularity of a graph are presented. Graphs with maximal total irregularity are depicted in Figure 2.

Theorem 1 ([1])**.** *For a simple undirected graph G with n vertices, it holds that*

$$\operatorname{irr}_t(G) \leq \begin{cases} \frac{1}{12}(2n^3 - 3n^2 - 2n) & n \text{ even}, \\ \\ \frac{1}{12}(2n^3 - 3n^2 - 2n + 3) & n \text{ odd}. \end{cases}$$

Moreover, the bounds are sharp.

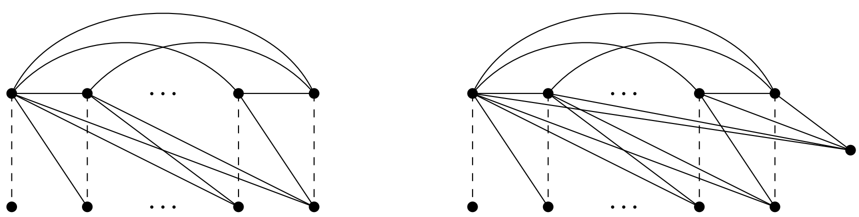

FIGURE 2. Graphs with maximal total irregularity for even and odd n, respectively (dashed edges are optional).

The motivation to introduce the total irregularity of a graph, as modification of the irregularity of graph, is twofold. First, in contrast to $\operatorname{irr}(G)$, $\operatorname{irr}_t(G)$ can be computed directly from the sequence of the vertex degrees (degree sequence) of G. Second, the most irregular graphs with respect to irr are graphs that have only two degrees (see Figure 1 for an illustration). On the contrary the most irregular graphs with respect to

irr_t, as it is shown in [1], are graphs with maximal number of different vertex degrees (graphs with all doted (optional) edges in Figure 2), which is much closer to what one can expect from "very" irregular graphs. Very recently, irregularity measures irr and irr_t were compared in [14], where it was shown that for a connected graph G with n vertices $\mathrm{irr}_t(G) \leq n^2\mathrm{irr}(G)/4$. Moreover, if G is a tree, then it was shown that $\mathrm{irr}_t(G) \leq (n-2)\mathrm{irr}(G)$.

The aim of this paper is to investigate the total irregularity of graphs under several graph operations including join, Cartesian product, direct product, strong product, lexicographic product, corona product, disjunction and symmetric difference. Detailed exposition on some graph operations one can find in [17].

2. RESULTS

We start with simple observations about the complement and the disjoint union.

The *complement* of a simple graph G with n vertices, denoted by $\overline{G}$, is a simple graph with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv \,|\, u,v \in V(G) \text{ and } uv \notin E(G)\}$. Thus, $uv \in E(G) \Longleftrightarrow uv \notin E(G)$. Obviously, $E(G) \cup E(\overline{G}) = E(K_n)$, and for a vertex u, we have $d_{\overline{G}}(u) = n-1-d_G(u)$. From $|d_{\overline{G}}(u)-d_{\overline{G}}(v)| = |n-1-d_G(u)-(n-1-d_G(v))| = |d_G(u)-d_G(v)|$ it follows that $\mathrm{irr}_t(\overline{G}) = \mathrm{irr}_t(G)$.

For two graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and disjoint edge sets $E(G_1)$ and $E(G_2)$ the *disjoint union* of G_1 and G_2 is the graph $G = G_1 \cup G_2$ with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$. Obviously, $\mathrm{irr}_t(G \cup H) \geq \mathrm{irr}_t(G) + \mathrm{irr}_t(H)$.

Next we present sharp upper bounds for join, lexicographic product, Cartesian product, strong product, direct product, corona product and upper bounds for disjunction and symmetric difference.

2.1. *Join*

The *join* $G+H$ of simple undirected graphs G and H is the graph with the vertex set $V(G+H) = V(G) \cup V(H)$ and the edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G),\ v \in V(H)\}$.

Theorem 2. *Let G and H be simple undirected graphs with $|V(G)| = n_1$ and $|V(H)| = n_2$ such that $n_1 \geq n_2$. Then,*

$$\mathrm{irr}_t(G+H) \leq \mathrm{irr}_t(G) + \mathrm{irr}_t(H) + n_2\,(n_1-1)\,(n_1-2).$$

Moreover, the bound is best possible.

Proof. The total irregularity of $G+H$ is

$$\mathrm{irr}_t(G+H) = \frac{1}{2} \sum_{u,v \in V(G+H)} |d_{G+H}(u) - d_{G+H}(v)|$$

$$
\begin{aligned}
&= \frac{1}{2} \sum_{u,v \in V(G)} |d_{G+H}(u) - d_{G+H}(v)| \\
&\quad + \frac{1}{2} \sum_{u,v \in V(H)} |d_{G+H}(u) - d_{G+H}(v)| \\
&\quad + \sum_{u \in V(G)} \sum_{v \in V(H)} |d_{G+H}(u) - d_{G+H}(v)| . \qquad (2.1)
\end{aligned}
$$

By definition, $|V(G+H)| = |V(G)| + |V(H)| = n_1 + n_2$. For vertices $u \in V(G)$ and $v \in V(H)$, it holds that $d_{G+H}(u) = d_G(u) + n_2$ and $d_{G+H}(v) = d_H(v) + n_1$. Thus, further we have

$$
\begin{aligned}
\operatorname{irr}_t(G+H) &= \frac{1}{2} \sum_{u,v \in V(G)} |d_G(u) - d_G(v)| + \frac{1}{2} \sum_{u,v \in V(H)} |d_H(u) - d_H(v)| \\
&\quad + \sum_{u \in V(G)} \sum_{v \in V(H)} |(d_G(u) + n_2) - (d_H(v) + n_1)| \qquad (2.2) \\
&= \operatorname{irr}_t(G) + \operatorname{irr}_t(H) + \sum_{u \in V(G)} \sum_{v \in V(H)} |n_1 - n_2 + d_H(v) - d_G(u)| .
\end{aligned}
$$

Under the constrains $n_1 \geq n_2$, $d_G(u) \leq n_1 - 1$, and $d_H(v) \leq n_2 - 1$, the double sum $\sum_{u \in V(G)} \sum_{v \in V(H)} |n_1 - n_2 + d_H(v) - d_G(u)|$ is maximal when H is a graph with maximal sum of vertex degrees, i.e., H is the complete graph K_{n_2}, and G is a graph with minimal sum of vertex degrees, i.e., G is a tree on n_1 vertices T_{n_1}. Thus,

$$
\begin{aligned}
&\sum_{u \in V(G)} \sum_{v \in V(H)} |n_1 - n_2 + d_H(v) - d_G(u)| \\
&\leq \sum_{u \in V(T_{n_1})} \sum_{v \in V(K_{n_2})} \left| n_1 - n_2 + d_{K_{n_2}}(v) - d_{T_{n_1}}(u) \right| \\
&= \sum_{u \in V(T_{n_1})} \sum_{v \in V(K_{n_2})} \left| n_1 - 1 - d_{T_{n_1}}(u) \right| \\
&= n_2 \sum_{u \in V(T_{n_1})} \Big(n_1 - 1 - d_{T_{n_1}}(u) \Big) \\
&= n_2 n_1 (n_1 - 1) - 2 n_2 (n_1 - 1) \\
&= n_2 (n_1 - 1)(n_1 - 2), \qquad (2.3)
\end{aligned}
$$

and

$$
\operatorname{irr}_t(G+H) \leq \operatorname{irr}_t(G) + \operatorname{irr}_t(H) + n_2(n_1 - 1)(n_1 - 2). \qquad (2.4)
$$

When $n_1 \leq 2$, $\mathrm{irr}_t(G) = \mathrm{irr}_t(H) = \mathrm{irr}_t(G+H) = 0$, and the claim of the theorem is fulfilled. From the derivation, it follows that (2.4) is equality when H is compete graph on n_2 vertices and G is any tree on n_1 vertices . $\square$

Example. Let denote by H_i a graph with $|V(H_i)| = i$ isolated vertices (vertices with degree zero). Then, the *bipartite graph* $K_{i,j}$ is a join of H_i and H_j. Analogously, the *complete* k*-partite graph* $G = K_{n_1,\cdots,n_k}$ is join of $H_{n_1},\ldots,H_{n_k}$. Straightforward calculation shows that $\mathrm{irr}_t(K_{n_i,n_j}) = n_i n_j \left|n_j - n_i\right|$, For the total irregularity of $K_{n_1,\cdots,n_k}$ we have

$$\begin{aligned}
\mathrm{irr}_t(K_{n_1,\cdots,n_k}) &= \frac{1}{2}\sum_{u,v\in V(K_{n_1,\cdots,n_k})} |d_G(u) - d_G(v)| \\
&= \sum_{i=1}^{k-1}\sum_{j=i+1}^{k}\left(\frac{1}{2}\sum_{u,v\in V(K_{n_i,n_j})} |d_G(u) - d_G(v)|\right) \\
&= \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} n_i n_j \left|n_j - n_i\right| = \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} \mathrm{irr}_t(K_{n_i,n_j}).
\end{aligned}$$

2.2. *Lexicographic product*

The *lexicographic product* $G \circ H$ (also known as the *graph composition*) of simple undirected graphs G and H is the graph with the vertex set $V(G \circ H) = V(G) \times V(H)$ and the edge set $E(G \circ H) = \{(u_i, v_k)(u_j, v_l) : [u_i u_j \in E(G)] \vee [(v_k v_l \in E(H)) \wedge (u_i = u_j)]\}$.

Theorem 3. *Let G and H be simple undirected graphs with $|V(G)| = n_1$, $|V(H)| = n_2$ then,*

$$\mathrm{irr}_t(G \circ H) \leq n_2^3\,\mathrm{irr}_t(G) + n_1^2\,\mathrm{irr}_t(H).$$

Moreover, this bound is sharp for infinitely many graphs.

Proof. By the definition of $G \circ H$, it follows that $|V(G \circ H)| = n_1 n_2$ and $d_{G\circ H}(u_i, v_j) = n_2 d_G(u_i) + d_H(v_j)$ for all $1 \leq i \leq n_1, 1 \leq j \leq n_2$. Applying those relations, we obtain

$$\begin{aligned}
\mathrm{irr}_t(G \circ H) &= \frac{1}{2}\sum_{\substack{(u_i,v_k)\in V(G\circ H)\\ (u_j,v_l)\in V(G\circ H)}} \left|d_{G\circ H}(u_i, v_k) - d_{G\circ H}(u_j, v_l)\right| \\
&= \frac{1}{2}\sum_{\substack{u_i,u_j\in V(G)\\ v_k,v_l\in V(H)}} \left|n_2 d_G(u_i) - n_2 d_G(u_j) + d_H(v_k) - d_H(v_l)\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2} \sum_{\substack{u_i,u_j \in V(G)\\ v_k,v_l \in V(H)}} \left(n_2 \left|d_G(u_i) - d_G(u_j)\right| + |d_H(v_k) - d_H(v_l)|\right) \\
&= \frac{1}{2} n_2^3 \sum_{u_i,u_j \in V(G)} \left|d_G(u_i) - d_G(u_j)\right| \\
&+ \frac{1}{2} n_1^2 \sum_{v_k,v_l \in V(H)} |d_H(v_k) - d_H(v_l)| \\
&= n_2^3 \operatorname{irr}_t(G) + n_1^2 \operatorname{irr}_t(H). \qquad (2.5)
\end{aligned}
$$

To prove that the presented bound is best possible, consider the lexicographic product $P_l \circ C_k$, $l \geq 1, k \geq 3$ (an illustration is given in Figure 3(b)). Straightforward calculations give that $\operatorname{irr}_t(P_l) = 2(l-2)$, $\operatorname{irr}_t(C_k) = 0$. The graph $P_l \circ C_k$ is comprised of $2k$ vertices of degree $k+2$, and $k(l-2)$ vertices of degree $2k+2$. Hence, $\operatorname{irr}_t(P_l \circ C_k) = 2k^3(l-2)$. On the other hand, the bound obtain here is $\operatorname{irr}_t(P_l \circ C_k) \leq k^3 \operatorname{irr}_t(P_l) + l^2 \operatorname{irr}_t(C_k) = 2k^3(l-2)$. □

2.3. *Cartesian product*

The *Cartesian product* $G \square H$ of two simple undirected graphs G and H is the graph with the vertex set $V(G \square H) = V(G) \times V(H)$ and the edge set $E(G \square H) = \{(u_i,v_k)(u_j,v_l) : [(u_iu_j \in E(G)) \wedge (v_k = v_l)] \vee [(v_kv_l \in E(H)) \wedge (u_i = u_j)]\}$. From the definition of the Cartesian product, it follows that $|V(G \square H)| = n_1 n_2$ and $d_{G \square H}(u_i,v_j) = d_G(u_i) + d_H(v_j)$. Since the derivation of the upper bound on $G \square H$ is similar to the case of a graph lexicographic product, we omit the proof and just state the result in Theorem 4. The best possible bound is obtained for $P_l \square C_k$, $l \geq 1, k \geq 3$, illustrated in Figure 3(c). The graph $P_l \square C_k$ is comprised of $2k$ vertices of degree 3, and $k(l-2)$ vertices of degree 4. Thus, $\operatorname{irr}_t(P_l \square C_k) = 2k^2(l-2)$. The bound obtain here is $\operatorname{irr}_t(P_l \square C_k) \leq k^2 \operatorname{irr}_t(P_l) + l^2 \operatorname{irr}_t(C_k) = 2k^2(l-2)$.

Theorem 4. *Let G and H be simple undirected graphs with $|V(G)| = n_1$, $|V(H)| = n_2$ then*

$$\operatorname{irr}_t(G \square H) \leq n_2^2 \operatorname{irr}_t(G) + n_1^2 \operatorname{irr}_t(H).$$

Moreover, this bound is sharp for infinitely many graphs.

2.4. *Strong product*

The *strong product* $G \boxtimes H$ of two simple undirected graphs G and H is the graph with the vertex set $V(G \boxtimes H) = V(G) \times V(H)$ and the edge set $E(G \boxtimes H) = \{(u_i,v_k)(u_j,v_l) : [(u_iu_j \in E(G)) \wedge (v_k = v_l)] \vee [(v_kv_l \in E(H)) \wedge (u_i = u_j)] \vee [(u_iu_j \in E(G)) \wedge (v_kv_l \in E(H))]\}$.

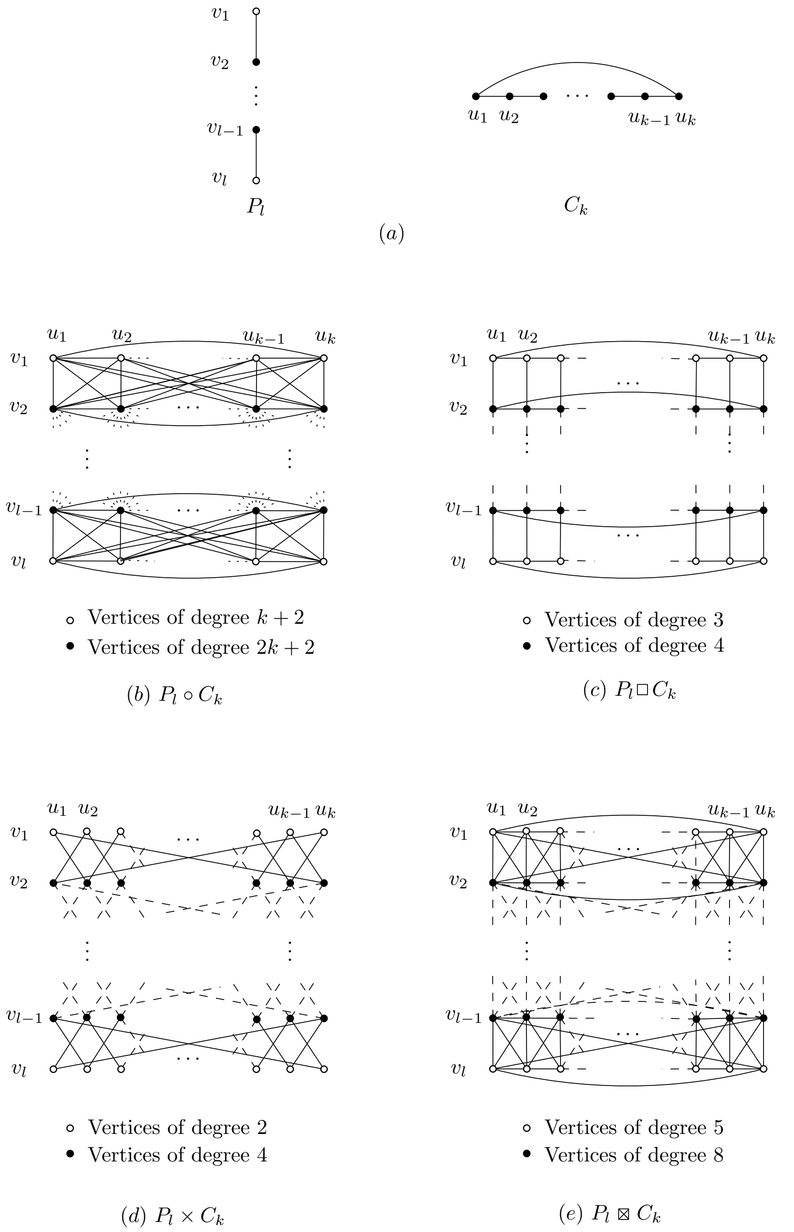

FIGURE 3. (a) Path graph on l vertices P_l, and cycle graph on k vertices C_k, (b) lexicographic product graph $P_l \circ C_k$, (c) Cartesian product graph $P_l \square C_k$, (d) direct product graph $P_l \times C_k$ and, (e) strong product graph $P_l \boxtimes C_k$.

Theorem 5. *Let G and H be simple undirected graphs with $|V(G)| = n_1$, $|E(G)| = m_1$, $|V(H)| = n_2$ and $|E(H)| = m_2$. Then,*

$$\mathrm{irr}_t(G \boxtimes H) \leq n_2(n_2 + 2m_2)\,\mathrm{irr}_t(G) + n_1(n_1 + 2m_1)\,\mathrm{irr}_t(H).$$

Moreover, this bound is best possible.

Proof. From the definiton of the strong product, it follows $|V(G \boxtimes H)| = n_1 n_2$, $|E(G \boxtimes H)| = m_1 n_2 + m_2 n_1 + 2m_1 m_2$, and $d_{G\boxtimes H}(u_i, v_k) = d_G(u_i) + d_H(v_k) + d_G(u_i)d_H(v_k)$. The total irregularity of $G \boxtimes H$ is

$$\begin{aligned}\mathrm{irr}_t(G \boxtimes H) &= \frac{1}{2} \sum_{(u_i,v_k),(u_j,v_l)\in V(G\boxtimes H)} \left| d_{G\boxtimes H}(u_i, v_k) - d_{G\boxtimes H}(u_j, v_l) \right| \\ &= \frac{1}{2} \sum_{u_i,u_j\in V(G),v_k,v_l\in V(H)} \left| d_{G\boxtimes H}(u_i, v_k) - d_{G\boxtimes H}(u_j, v_l) \right|. \qquad (2.6)\end{aligned}$$

Applying simple algebraic transformation and the triangle inequality, we obtain

$$\begin{aligned}\left| d_{G\boxtimes H}(u_i, v_k) - d_{G\boxtimes H}(u_j, v_l) \right| &= \big| (d_G(u_i) - d_G(u_j)) + (d_H(v_k) - d_H(v_l)) \\ &\quad + (d_G(u_i)d_H(v_k) - d_G(u_j)d_H(v_l)) \big| \\ &\leq \left| d_G(u_i) - d_G(u_j) \right| + |d_H(v_k) - d_H(v_l)| \\ &\quad + \frac{1}{2}(d_G(u_i) + d_G(u_j))\,|d_H(v_k) - d_H(v_l)| \\ &\quad + \frac{1}{2}(d_H(v_k) + d_H(v_l))\,\left| d_G(u_i) - d_G(u_j) \right|. \qquad (2.7)\end{aligned}$$

From (2.6) and (2.7), we obtain

$$\begin{aligned}\mathrm{irr}_t(G \boxtimes H) &\leq \frac{1}{2} \sum_{\substack{u_i,u_j\in V(G),\\ v_k,v_l\in V(H)}} \left[\left| d_G(u_i) - d_G(u_j) \right| + |d_H(v_l) - d_H(v_k)| \right] \\ &\quad + \frac{1}{4} \sum_{\substack{u_i,u_j\in V(G),\\ v_k,v_l\in V(H)}} (d_G(u_i) + d_G(u_j))\,|d_H(v_k) - d_H(v_l)| \\ &\quad + \frac{1}{4} \sum_{\substack{u_i,u_j\in V(G),\\ v_k,v_l\in V(H)}} (d_H(v_k) + d_H(v_l))\,\left| d_G(u_i) - d_G(u_j) \right| \\ &= n_2(n_2 + 2m_2)\,\mathrm{irr}_t(G) + n_1(n_1 + 2m_1)\,\mathrm{irr}_t(H). \qquad (2.8)\end{aligned}$$

To prove that the presented bound is best possible, consider the strong product $P_l \otimes C_k$, $l \geq 1, k \geq 3$, illustrated in Figure 3(e). We have, $\mathrm{irr}_t(P_l) = 2(l-2)$, $\mathrm{irr}_t(C_k) = 0$. The graph $P_l \boxtimes C_k$ is comprised of $2k$ vertices of degree 5, and $k(l-2)$ vertices of degree 8. Hence, $\operatorname{irr}_t(P_l \boxtimes C_k) = 6k^2(l-2)$. On the other hand, the bound obtain here, is $\operatorname{irr}_t(P_l \boxtimes C_k) \leq l(l+2(l-1))\operatorname{irr}_t(C_k) + k(k+2k)\operatorname{irr}_t(P_l) = 6k^2(l-2)$. □

2.5. *Direct product*

The *direct product* $G \times H$ (also know as the *tensor product*, the *Kronecker product* [22], *categorical product* [21] and *conjunctive product*) of simple undirected graphs G and H is the graph with the vertex set $V(G\times H) = V(G)\times V(H)$, and the edge set $E(G\times H) = \{(u_i,v_k)(u_j,v_l) : (u_i,u_j) \in E(G) \wedge (v_k,v_l) \in E(H)\}$. From the definition of the direct product, it follows $|V(G\times H)| = n_1n_2$, $|E(G\times H)| = 2m_1m_2$, and $d_{G\times H}(u_i,v_k) = d_G(u_i)d_H(v_k)$. The proof for the upper bound on $G\times H$ is similar as that of the strong product $G \boxtimes H$. Therefore, we show only that the bound in Theorem 6 is best possible, and omit the rest of the proof.

Theorem 6. *Let G and H be simple undirected graphs with $|V(G)| = n_1$, $|E(G)| = m_1$, $|V(H)| = n_2$ and $|E(H)| = m_2$. Then,*

$$\operatorname{irr}_t(G\times H) \leq 2n_2m_2\operatorname{irr}_t(G) + 2n_1m_1\operatorname{irr}_t(H).$$

Moreover, this bound is best possible.

To prove that the presented bound is best possible, we consider the direct product $P_l \times C_k$, $l \geq 1, k \geq 3$ (an illustration is given in Figure 3(d). Straightforward calculations give that $\operatorname{irr}_t(P_l) = 2(l-2)$, $\operatorname{irr}_t(C_k) = 0$. The graph $P_l \times C_k$ is comprised of $2k$ vertices of degree 2, and $k(l-2)$ vertices of degree 4. Thus, $\operatorname{irr}_t(P_l \times C_k) = 4k^2(l-2)$. On the other hand, the bound obtain by Proposition 6 is $\operatorname{irr}_t(P_l\times C_k) \leq 2n_2m_2\operatorname{irr}_t(G) + 2n_1m_1\operatorname{irr}_t(H) = 4k^2(l-2)$.

2.6. *Corona product*

The *corona* product $G \odot H$ of simple undirected graphs G and H with $|V(G)| = n_1$ and $|V(H)| = n_2$, is defined as the graph who is obtained by taking the disjoint union of G and n_1 copies of H and for each i, $1 \leq i \leq n_1$, inserting edges between the ith vertex of G and each vertex of the ith copy of H. Thus, the corona graph $G\odot H$ is the graph with the vertex set $V(G\odot H) = V(G) \cup_{i=1,\dots,n_1} V(H_i)$ and the edge set $E(G\odot H) = E(G)\cup_{i=1,\dots,n_1} E(H_i) \cup \{u_iv_j : u_i \in V(G), v_j \in V(H_i)\}$, where H_i is the ith copy of the graph H.

Theorem 7. *Let G and H be simple undirected graphs with $|V(G)| = n_1$ and $|V(H)| = n_2$. Then,*

$$\operatorname{irr}_t(G\odot H) \leq \operatorname{irr}_t(G) + n_1^2\operatorname{irr}_t(H) + n_1^2\left(n_2^2 + n_1n_2 - 4n_2 + 2\right).$$

Moreover, the bound is best possible.

Proof. The total irregularity of $G \odot H$ is

$$
\begin{aligned}
\operatorname{irr}_t(G \odot H) &= \frac{1}{2} \sum_{u,v \in V(G \odot H)} |d_{G\odot H}(u) - d_{G\odot H}(v)| \\
&= \frac{1}{2} \sum_{x,y \in V(G)} |d_{G\odot H}(x) - d_{G\odot H}(y)| \\
&+ \sum_{i=1}^{n_1} \left(\frac{1}{2} \sum_{z,t \in V(H_i)} |d_{G\odot H}(z) - d_{G\odot H}(t)| \right) \\
&+ \sum_{i=1}^{n_1-1} \sum_{j=i+1}^{n_1} \sum_{z \in V(H_i), t \in V(H_j)} |d_{G\odot H}(z) - d_{G\odot H}(t)| \\
&+ \sum_{i=1}^{n_1} \sum_{u \in V(G), v \in V(H)} |d_{G\odot H}(u) - d_{G\odot H}(v)|. \qquad (2.9)
\end{aligned}
$$

By the definition of $G \odot H$, $|V(G \odot H)| = |V(G)| + n_1 |V(H)| = n_1 + n_1 n_2$. For a vetrex $u \in V(G)$, it holds that $d_{G\odot H}(u) = d_G(u) + n_2$ and for a vertex $v \in V(H_i)$, $1 \le i \le n_2$, we have $d_{G\odot H}(v) = d_H(v) + 1$. Thus,

$$
\begin{aligned}
\operatorname{irr}_t(G \odot H) = \frac{1}{2} \sum_{x,y \in V(G)} |d_G(x) - d_G(y)| &+ \frac{1}{2} n_1 \sum_{z,t \in V(H)} |d_H(z) - d_H(t)| \\
&+ \sum_{i=1}^{n_1-1} \sum_{j=i+1}^{n_1} \sum_{z \in V(H_i), t \in V(H_j)} |d_H(z) + 1 - d_H(t) - 1| \\
&+ \sum_{i=1}^{n_1} \sum_{u \in V(G), v \in V(H)} |d_G(u) - d_H(v) + n_2 - 1| \\
&= \operatorname{irr}_t(G) + n_1 \operatorname{irr}_t(H) + n_1(n_1 - 1)\operatorname{irr}_t(H) \\
&+ \sum_{i=1}^{n_1} \sum_{u \in V(G), v \in V(H)} |d_G(u) - d_H(v) + n_2 - 1|. \qquad (2.10)
\end{aligned}
$$

Since $n_1 \ge n_2$, the sum $\sum\limits_{u \in V(G), v \in V(H)} |d_G(u) - d_H(v) + n_2 - 1|$ is maximal when $\sum_{u \in V(G)} d_G(u)$ is maximal, i.e., G is the complete graph K_{n_1}, and $\sum_{v \in V(H)} d_H(v)$ is minimal, i.e., H is a tree on n_2 vertices T_{n_2}. Thus,

$$
\sum_{u \in V(G), v \in V(H)} |d_G(u) - d_H(v) + n_2 - 1|
$$

$$
\begin{aligned}
&\leq \sum_{u\in V(K_{n_1})}\sum_{v\in V(T_{n_2})}\Big|d_{K_{n_1}}(u)-d_{T_{n_2}}(v)+n_2-1\Big|\\
&=\sum_{u\in V(K_{n_1})}\sum_{v\in V(T_{n_2})}\Big|n_1-1-d_{T_{n_2}}(v)+n_2-1\Big|\\
&=n_1\sum_{v\in V(T_{n_2})}\Big(n_1+n_2-2-d_{T_{n_2}}(v)\Big)\\
&=n_1n_2(n_1+n_2-2)-2n_1(n_2-1)\\
&=n_1(n_2^2+n_1n_2-4n_2+2). \qquad (2.11)
\end{aligned}
$$

Substituting (2.11) into (2.10), we obtain

$$\operatorname{irr}_t(G\odot H)\leq \operatorname{irr}_t(G)+n_1^2\operatorname{irr}_t(H)+n_1^2\left(n_2^2+n_1n_2-4n_2+2\right). \qquad (2.12)$$

From the derivation of the bound (2.12), it follows that the sharp bound is obtained when G is compete graph on n_1 vertices and H is any tree on n_2 vertices. □

2.7. *Disjunction*

The *disjunction* graph $G\vee H$ of simple undirected graphs G and H with $|V(G)|=n_1$ and $|V(H)|=n_2$ is the graph with the vertex set $V(G\vee H)=V(G)\times V(H)$ and the edge set $E(G\vee H)=\big\{(u_i,v_k)(u_j,v_l):\ u_iu_j\in E(G)\ \vee v_kv_l\in E(H)\big\}$. It holds that $|V(G\vee H)|=n_1n_2$, and $d_{G\vee H}(u_i,v_k)=n_2d_G(u_i)+n_1d_H(v_k)-d_G(u_i)d_H(v_k)$ for all i, k where, $1\leq i\leq n_1$, $1\leq k\leq n_2$.

Theorem 8. *Let G and H be simple undirected graphs with $|V(G)|=n_1$, $|E(G)|=m_1$, $|V(H)|=n_2$ and $|E(H)|=m_2$. Then,*

$$\operatorname{irr}_t(G\vee H)\leq n_2\,(n_2^2+2m_2)\operatorname{irr}_t(G)+n_1\,(n_1^2+2m_1)\operatorname{irr}_t(H).$$

Proof. The total irregularity of $G\vee H$ is

$$
\begin{aligned}
\operatorname{irr}_t(G\vee H)=&\ \frac{1}{2}\sum_{(u_i,v_k),(u_j,v_l)\in V(G\vee H)}\big|d_{G\vee H}(u_i,v_k)-d_{G\vee H}(u_j,v_l)\big|\\
=&\ \frac{1}{2}\sum_{u_i,u_j\in V(G),v_k,v_l\in V(H)}\big|d_{G\vee H}(u_i,v_k)-d_{G\vee H}(u_j,v_l)\big|. \qquad (2.13)
\end{aligned}
$$

Since $d_{G\vee H}(u_i,v_k)=n_2d_G(u_i)+n_1d_H(v_k)-d_G(u_i)d_H(v_k)$ for all i, k where, $1\leq i\leq n_1$, $1\leq k\leq n_2$. We obtain

$$
\begin{aligned}
\big|d_{G\vee H}(u_i,v_k)-d_{G\vee H}(u_j,v_l)\big|=&\ |n_2d_G(u_i)+n_1d_H(v_k)-d_G(u_i)d_H(v_k)\\
&-(n_2d_G(u_j)+n_1d_H(v_l)-d_G(u_j)d_H(v_l))\big| \qquad (2.14)
\end{aligned}
$$

Further, by simple algebraic manipulation and by the triangle inequality, we have

$$\begin{aligned}\left|d_{G\vee H}(u_i,v_k)-d_{G\vee H}(u_j,v_l)\right| \le\, & n_2\left|d_G(u_i)-d_G(u_j)\right|+n_1\left|d_H(v_k)-d_H(v_l)\right| \\ &+\frac{1}{2}(d_G(u_i)+d_G(u_j))\left|d_H(v_k)-d_H(v_l)\right| \\ &+\frac{1}{2}(d_H(v_k)+d_H(v_l))\left|d_G(u_i)-d_G(u_j)\right|.\end{aligned} \tag{2.15}$$

From (2.13) and (2.15), we obtain

$$\begin{aligned}\mathrm{irr}_t(G\vee H) \le\, & \frac{1}{2}\sum_{\substack{u_i,u_j\in V(G),\\ v_k,v_l\in V(H)}}\left[n_2\left|d_G(u_i)-d_G(u_j)\right|+n_1\left|d_H(v_l)-d_H(v_k)\right|\right] \\ &+\frac{1}{4}\sum_{\substack{u_i,u_j\in V(G),\\ v_k,v_l\in V(H)}}(d_G(u_i)+d_G(u_j))\left|d_H(v_k)-d_H(v_l)\right| \\ &+\frac{1}{4}\sum_{\substack{u_i,u_j\in V(G),\\ v_k,v_l\in V(H)}}(d_H(v_k)+d_H(v_l))\left|d_G(u_i)-d_G(u_j)\right|.\end{aligned} \tag{2.16}$$

The first sum in (2.16) is equal to $n_2^3\,\mathrm{irr}_t(G)+n_1^3\,\mathrm{irr}_t(H)$, the second to $2n_1m_1\,\mathrm{irr}_t(H)$, and the third to $2n_2m_2\,\mathrm{irr}_t(G)$. Hence,

$$\begin{aligned}\mathrm{irr}_t(G\vee H) \le\; & n_2^3\,\mathrm{irr}_t(G)+n_1^3\,\mathrm{irr}_t(H)+2n_1m_1\,\mathrm{irr}_t(H)+2n_2m_2\,\mathrm{irr}_t(G) \\ =\; & n_2\,(n_2^2+2m_2)\,\mathrm{irr}_t(G)+n_1\,(n_1^2+2m_1)\,\mathrm{irr}_t(H).\end{aligned}$$

□

2.8. *Symmetric difference*

The *symmetric difference* $G\oplus H$ of simple undirected graphs G and H with $|V(G)|=n_1$ and $|V(H)|=n_2$ is the graph with the vertex set $V(G\oplus H)=V(G)\times V(H)$ and the edge set

$$E(G\oplus H)=\left\{(u_i,v_k)(u_j,v_l):\text{ either } u_iu_j\in E(G)\text{ or } v_kv_l\in E(H)\right\}.$$

It holds that $|V(G\oplus H)|=n_1n_2$, and $d_{(G\oplus H)}(u_i,v_j)=n_2d_G(u_i)+n_1d_H(v_j)-2d_G(u_i)d_H(v_j)$ for all $1\le i\le n_1, 1\le j\le n_2$.

Much as in the previous case, we present only the bound on the total irregularity of symmetric difference of two graphs.

Theorem 9. *Let G and H be simple undirected graphs with $|V(G)|=n_1$, $|E(G)|=m_1$, $|V(H)|=n_2$ and $|E(H)|=m_2$. Then,*

$$\mathrm{irr}_t(G\oplus H)\le n_2\,(n_2^2+4m_2)\,\mathrm{irr}_t(G)+n_1\,(n_1^2+4m_1)\,\mathrm{irr}_t(H).$$

3. Conclusion

In this paper we consider the total irregularity of simple undirected graphs under several graph operations. We present sharp upper bounds for join, lexicographic product, Cartesian product, strong product, direct product and corona product. It is an open problem if the presented upper bounds on the total irregularity of disjunction and symmetric difference are the best possible.

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A NOTE ON *COMPLETE INTERSECTIONS GENERATED BY LINEAR FORMS

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Abstract. Let R be a commutative noetherian graded ring. In $R[Y_1,\dots,Y_m]$ we consider the linear forms $a_i = \sum_{j=1}^{m} a_{ji} Y_j$, $1 \le i \le n$, with a_{ji} homogeneous elements of R. We state a necessary and sufficient condition, in terms of *grades of the determinantal ideals of the matrix $A = (a_{ji})$, for the ideal $(a_1,\dots,a_n)$ to be a *complete intersection of *grade n in $R[Y_1,\dots,Y_m]$.

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INTRODUCTION

Let R be a commutative noetherian ring with unit. In [1] the author considered relation ideals of symmetric algebras of finitely generated R-modules and stated a condition to be a complete intersection. If R is a graded ring, similar properties can be established and studied. Definitions and results are inspired by the results known in the not graded case, however it is interesting to establish them in the graduate case. In fact, for the graded algebras good properties can be determined, as in the local case ([2–4]).

Let E be a finitely generated R-module and $Sym_R(E) = \bigoplus_{t=0}^{\infty} Sym_t(E)$ be its symmetric algebra. It is known that the symmetric algebra $Sym_R(E)$ has a presentation $R[Y_1,\dots,Y_m]/J$, where m is the number of the generators of E and J is the relation ideal generated by the linear forms $a_i = \sum_{j=1}^{m} a_{ji} Y_j$, $1 \le i \le n$ in the variables $Y_1,\dots,Y_m$. Theoretical properties of $Sym_R(E)$ (such as integrity, regularity, to be Cohen-Macaulay, Gorenstein) were studied by various authors. For a finitely generated module E this topic developed and culminated in the study of approximation complex of E. For best results the projective resolution of E can be examined, when it is finite. In particular, Avramov studied theoretic properties of $Sym_R(E)$ in terms of a condition on Fitting ideals of E, when the projective dimension of E is ≤ 1, and stated a condition for the presentation ideal of $Sym_R(E)$ to be a complete intersection.

The aim of this note is to investigate such properties in the graduate case.

The paper is organized as follows. In Section 1 some preliminary notions and properties on graded rings are introduced. Let R be a graded ring and I be an arbitrary ideal of R. I^* is the graded ideal generated by all homogeneous elements of I and it is the largest graded ideal contained in I. In particular, for each prime ideal $\wp \subset R$, the graded ideal $\wp^*$ is a prime ideal too. In [2] there are theorems that link theoretic properties of $\wp$ and $\wp^*$ by localizations of finitely generated graded R-modules, that is $dim E_{\wp} = dim E_{\wp^*} + 1$ and $depth E_{\wp} = depth E_{\wp^*} + 1$, for all not graded prime ideal $\wp$ of R. Now, we generalize the classic definitions for graded ideals. We define *grade of I the common length of the maximal homogeneous R-sequences in I^*. Then I is a *complete intersection if it can be generated by a number of homogeneous elements equal to its *grade. In Section 2 we consider homogeneous linear forms $a_i = \sum_{j=1}^{m} a_{ji} Y_j$, $1 \leq i \leq n$, of $R[Y_1, \dots, Y_m]$ with a_{ji} homogeneous elements of R. We state a necessary and sufficient condition, in terms of *grades of the determinantal ideals of the matrix $A = (a_{ji})$, such that $(a_1, \dots, a_n)$ is a *complete intersection of *grade n in $R[Y_1, \dots, Y_m]$.

1. Preliminary and notations

Let R be a commutative noetherian graded ring. In [2] there are some definitions related to the graded ideals of R and some properties on the dimension and the depth of some localizations of finitely generated graded R-modules.

Definition 1. Let $I \subset R$ be an ideal (not necessarily graded). I^* is the graded ideal of R generated by all the homogeneous elements of I (the homogeneous elements of I of degree j are $f \in I_j = I \cap R_j$).

The ideal I^* is the largest graded ideal contained in I. If I is homogeneous, then $I = I^*$.

Remark 1. Let $\wp \subset R$ be a prime ideal, then $\wp^*$ is a prime ideal too ([2], 1.5.6).

Example 1. Let $I = (X_1^2 - 2, X_1 X_2 - 2) \subset R = K[X_1, X_2]$. We have:

$$I_0 = I \cap R_0 = (0)$$

$$I_1 = I \cap R_1 = (0)$$

$$I_2 = I \cap R_2 = \langle X_1^2 - X_1 X_2 \rangle,$$

in fact $f = X_1^2 - 2 - (X_1 X_2 - 2) = X_1^2 - X_1 X_2 \in I_2$. In general, for $i > 2$:

$$I_i = I \cap R_i = \langle g(X_1^2 - X_1 X_2) \rangle,$$

with $\deg(g) = i - 2$. It follows that $I^* = (X_1^2 - X_1 X_2) \subset I$.

We recall the following well-known property ([2]).

Proposition 1. *Let R be a commutative noetherian graded domain and $I = (f) \subset R$ be a principal ideal. $I^* = (0) \iff f$ is not a homogeneous element.*

The previous proposition is not true if R is not a domain as the following example shows.

Example 2. Let $R =_8 [X]$ be the graded ring with standard graduation and I be the ideal of R generated by the not homogeneous element $f = X - 4$. In I there are homogeneous elements, hence $I^* \neq (0)$. In fact, let $a = 2X \in R_1$ be a homogeneous element of R, we have $af = 2X^2 \in I$ that is a homogeneous element of degree two. Hence $2X^2 \in I^*$.

Remark 2. Let R be a graded ring that is not a domain and $\wp = (f)$ be a prime ideal of R generated by a not homogeneous element. Since $\wp^*$ is a prime ideal, it follows that $\wp^* \neq (0)$ because (0) is not prime if R is not a domain. Hence in this case the condition $\Leftarrow$) of the Proposition 1 is not true.

In [2] the following definitions related to the graded ideals of R are given.

Definition 2. Let R be a graded ring. A graded ideal m of R is called *maximal if every graded ideal that properly contains m equals R.

Definition 3. A graded ring R is a *local ring if it has a unique *maximal ideal.

Example 3. Let R be a graded ring and $\wp$ be a prime ideal of R. Let S be the set of the homogeneous elements of R that don't lie in $\wp$. S is a multiplicatively closed set. Denote $R_{(\wp)}$ the homogeneous localization of R. $R_{(\wp)}$ is a graded ring and the graduation is defined setting $(R_{(\wp)})_i = \{\frac{x}{a} \in R_{(\wp)} | \deg(\frac{x}{a}) = i\}$, where $\deg(\frac{x}{a}) = \deg(x) - \deg(a)$, with x and a homogeneous elements. Then $R_{(\wp)}$ is a *local ring with *maximal ideal $\wp^* R_{(\wp)}$.

Now we introduce the following definitions.

Definition 4. Let R be a graded ring. A sequence $x_1, \ldots, x_n$ of homogeneous elements of R is called *regular sequence on R (or a homogeneous R-sequence) if the following conditions are satisfied:
1) $(x_1, \ldots, x_n)$ is a proper homogeneous ideal;
2) x_{i+1} is a regular homogeneous element in $R/(x_1, \ldots, x_i)$, for $i = 0, \ldots, n-1$.

Definition 5. Let R be a graded ring and E be a finitely generated graded R-module. A sequence $\underline{x} = \{x_1, \ldots, x_n\}$ of homogeneous elements of R is called E-*regular sequence (or a homogeneous E-sequence) if the following conditions are satisfied:
1) $\underline{x}E \neq E$;
2) x_i is a nonzero-divisor in $E/(x_1, \ldots, x_{i-1})E$, for $i = 1, \ldots, n$.

Definition 6. Let R be a graded ring, E be a finitely generated graded R-module and I be an ideal of R such that $IE \neq E$. The common length of the maximal homogeneous E-sequences in I^* is called the **grade* of I on E, denoted by $^*\mathrm{grade}(I, E)$.

A special situation occurs for *local rings:

Definition 7. Let (R,m) be a graded noetherian ring and E be a finitely generated graded R-module. The *grade of m on E is called the *depth of E, denoted by $^*\text{depth}(E)$.

Let I be an ideal of a graded ring R. It is customary to set: $^*\text{grade}(I) = {}^*\text{grade}(R/I) = {}^*\text{grade}(I,R)$. More precisely we give the following:

Definition 8. The *grade of I is the common length of the maximal homogeneous R-sequences in I^*, denoted $^*\text{grade}(I)$.

Definition 9. I is a *complete intersection if it can generated by a number of homogeneous elements equal to its *grade.

2. *COMPLETE INTERSECTIONS

Let R be a graded ring.

Definition 10. Let $A = (a_{ji})$ be a $m \times n$ matrix whose entries are homogeneous elements of R, for $i = 1,\dots,n$, $j = 1,\dots,m$. A is called a graded matrix on R.

Definition 11. Let $A = (a_{ji})$ be a graded $m \times n$ matrix on R for $i = 1,\dots,n$, $j = 1,\dots,m$. The ideal of R generated by the $t \times t$ minors of A, $1 \leq t \leq min(m,n)$ is called t-determinantal ideal, denoted $I_t(A)$.
By definition $I_0(A) = R$ and $I_t(A) = 0$ for $t > min(m,n)$.

Remark 3.
1) If the entries of the matrix A are homogenous elements of the same degree, $I_t(A)$ is a graded ideal.
2) If the elements of every column (row) are homogenous of the same degree, then the ideal $I_t(A)$ is graded.

Remark 4. If the matrix A is associated to a graded homomorphism φ of free graded modules, then $I_t(\varphi)$ denotes the ideal $I_t(A)$.

Theorem 1. *The homogeneous linear forms of* $R[Y_1,\dots,Y_m]$

$$a_i = \sum_{j=1}^{m} a_{ji} Y_j,\ 1 \leq i \leq n,$$

generate a *complete intersection of *grade n if and only if the determinantal ideals of $A = (a_{ji})$ satisfy the conditions

$$^*\text{grade}(I_k(A)) \geq n-k+1,\ 1 \leq k \leq n,$$

where A is a graded matrix whose entries are homogeneous elements of R of the same degree d_i for each column.

Proof. Set $S = R[Y_1, \dots, Y_m]$. Let $a_i = \sum_{j=1}^{m} a_{ji} Y_j \in S$.
Being $(a_1, \dots, a_n) \subset (Y_1, \dots, Y_m)$ and $grade(Y_1, \dots, Y_m) = m$, then we suppose $n \leq m$.
$\Rightarrow$) Let $(a_1, \dots, a_n)$ be a *complete intersection of *grade n in $R[Y_1, \dots, Y_m]$ with a_i homogeneous linear forms in the variables Y_j and bidegree $(g_i, 1)$. We consider the Koszul complex on $a_1, \dots, a_n$ that is a graded complex. By [1] (Proposition 1) the assert follows.
$\Leftarrow$) We use induction on n and we prove that $a_1, \dots, a_n$ generate a *complete intersection in S of *grade n.
$n = 1$: we prove that a_1 is a nonzero-divisor if and only if $^*\mathrm{grade}(a_{11}, a_{21}, \dots, a_{m1}) \geq 1$. If $^*\mathrm{grade}(a_{11}, a_{21}, \dots, a_{m1}) \geq 1$, then some homogenous element a_{j1} is nonzero-divisor. It follows that $a_{j1} Y_j$ is a nonzero-divisor. Choose an element $f \in S$ with $f \neq 0$, one has

$$fa_1 = f(a_{11}Y_1 + a_{21}Y_2 + \cdots + a_{m1}Y_m) \neq 0$$

because at least $fa_{j1}Y_j \neq 0$. Then a_1 is a nonzero-divisor. Conversely, suppose a_1 nonzero-divisor. For all nonzero element $f \in S$, one has $fa_1 \neq 0$. It follows

$$fa_1 = f(a_{11}Y_1 + a_{21}Y_2 + \cdots + a_{m1}Y_m) \neq 0$$

hence $fa_{j1}Y_j \neq 0$ for all f and for any $j = 1, \dots, m$. Being Y_j nonzero-divisor one has that each a_{j1} is nonzero-divisor and $^*\mathrm{grade}(a_{11}, a_{21}, \dots, a_{m1}) \geq 1$.
Suppose by induction that the assert is true for $n-1$ homogeneous elements. Let $\wp$ be a graded prime ideal of S associated to a maximal homogeneous S-regular sequence in $(a_1, \dots, a_n)$ and containing a_i's. Hence:

$$^*\mathrm{grade}(\wp) = {}^*\mathrm{grade}(\wp S_\wp) = {}^*\mathrm{grade}(a_1, \dots, a_n) S_\wp.$$

It is sufficient to prove that $^*\mathrm{grade}(\wp)$ can not be smaller than n. In fact from $^*\mathrm{grade}(\wp) \geq n$ it follows that $^*\mathrm{grade}(\wp) = n$. Assume that $^*\mathrm{grade}(\wp) < n$. By hypothesis for per $k = 1$ one has $^*\mathrm{grade}(I_1(A)) \geq n$, then some $a_{ji} = a$ is not in $\wp$. Consider the ring $R_{(a)}$, that is the homogeneous localization of R in the set of powers of a, with $a \notin \wp$. $R_{(a)}$ is a *local ring, whose graduation is given by $(R_{(a)})_i = \{\frac{x}{y} \in R_{(a)} | \deg(\frac{x}{y}) = i\}$, where $\deg(\frac{x}{y}) = \deg(x) - \deg(y)$ with x and y homogeneous elements of R (Example 3). Consider the graded ideal $J = (a_1, \dots, a_n)_{(a)}$ of $R_{(a)}$ and set a system of generators consisting of homogenous linear forms with coefficient in $R_{(a)}$. By elementary transformations over $R_{(a)}$ of the matrix A, which involve a linear change of the indeterminates $Y_1, \dots, Y_m$ to $T_1, \dots, T_m$, we can assume that J is generated by T_1 and $b_2, \dots, b_m$, with $b_i = \sum_{j=2}^{m} b_{ji} T_j \in R_{(a)}[T_2, \dots, T_m]$ and $b_{ji} \in R(a)$. We obtain a graded matrix $B = (b_{ji})$ whose entries are homogenous elements of the same degree in each column. Hence

$$I_k(A)_{(a)} = I_{k-1}(B),$$

with $I_k(A)_{(a)}$ graded ideal of $R_{(a)}$ obtained by a homogeneous localization of the graded ideal $I_k(A)$. Then:

$$^*\mathrm{grade}(I_{k-1}(B)) = {}^*\mathrm{grade}(I_k(A)_{(a)}) \geq {}^*\mathrm{grade}(I_k(A)) \geq n-k+1,$$

hence $^*\mathrm{grade}(I_{k-1}(B)) \geq (n-1)-(k-1)+1$.
By induction hypothesis one concludes that $b_2,\ldots,b_m$ generate a *complete intersection of *grade $n-1$ in $R(a)[T_2,\ldots,T_m]$. It is enough to prove that $T_1,b_2,\ldots,b_m$ generate a *complete intersection of grade n in $R(a)[T_2,\ldots,T_m]$. Being T_1 a nonzero-divisor in $R(a)[T_2,\ldots,T_m]$, it follows that $T_1,b_2,\ldots,b_m$ generate a *complete intersection. It follows $^*\mathrm{grade}(a_1,\ldots,a_n)_{(a)} = n$, hence

$$^*\mathrm{grade}(a_1,\ldots,a_n)S_\wp \geq n.$$

Being

$$^*\mathrm{grade}(\wp) = {}^*\mathrm{grade}(a_1,\ldots,a_n)S_\wp$$

one has $^*\mathrm{grade}(\wp) \geq n$, that is a contradiction. □

Now, let $R = K[X_1,\ldots,X_q]$ be the polynomial ring over a field K with standard graduation. Let $E = Rf_1 + \cdots + Rf_m$ be a finitely generated graded R-module, $d_j = \deg(f_j)$ for $j = 1,\ldots,m$, and $\mathrm{Sym_R}(E) = \bigoplus_{t\geq 0}\mathrm{Sym_t}(E)$ be its symmetric algebra.
If E has free graded presentation:

$$\bigoplus_{i=1}^{n} R(-c_i) \xrightarrow{\varphi} \bigoplus_{j=1}^{m} R(-d_j) \xrightarrow{\psi} E \to 0,$$

with $\varphi = (a_{ji})$ graded matrix on R, then the kernel J of the surjective homomorphism $Sym(\psi)$ induced by ψ

$$Sym(\psi) : R[Y_1,\ldots,Y_m] \to Sym_R(E) \to 0$$

is generated by linear forms in the variables Y_j

$$a_i = \sum_{j=1}^{m} a_{ji}Y_j,$$

for $i = 1,\ldots,n$ and such that $\sum_{j=1}^{m} a_{ji} f_j = 0$, $1 \leq i \leq n$.
Hence the symmetric algebra $Sym_R(E)$ has a presentation $R[Y_1,\ldots,Y_m]/J$. The ring $R[Y_1,\ldots,Y_m] = K[X_1,\ldots,X_q;Y_1,\ldots,Y_m]$ is bigraded by $\mathrm{bideg}(X_i) = (1,0)$ for $i = 1,\ldots,q$ and $\mathrm{bideg}(Y_j) = (0,d_j)$ for $j = 1,\ldots,m$.

Remark 5. If E is a free graded R-module of rank m, then its symmetric algebra is

$$\mathrm{Sym_R}(E) \cong R[Y_1,\ldots,Y_m],$$

with $E = \bigoplus_j R^m(-d_j)$.

By Theorem 1 it follows the following result for $Sym_R(E)$.

Corollary 1. *Let $E = Rf_1 + \cdots + Rf_m$ be a finitely generated graded module on $R = K[X_1, \ldots, X_q]$, with $d_j = \deg(f_j)$ for $j = 1, \ldots, m$. Let*

$$F_1 \xrightarrow{\varphi} F_0 \xrightarrow{\psi} E \to 0,$$

be a free graded presentation of E with F_0 and F_1 free graded modules of rank n and m, respectively, and $\varphi = (a_{ji})$ be a graded matrix whose columns have homogenous elements of R of the same degree. The kernel of the canonical epimorphism

$$Sym(\psi) : R[Y_1, \ldots, Y_m] \to Sym_R(E) \to 0$$

*is a *complete intersection of *grade n in $R[Y_1, \ldots, Y_m]$ if and only if*

$$^*\mathrm{grade}(I_k(\varphi)) \geq n - k + 1, \; 1 \leq k \leq n.$$

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CONVERGENCE THEOREMS FOR ADMISSIBLE PERTURBATIONS OF φ-PSEUDOCONTRACTIVE OPERATORS

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Abstract. We prove some convergence theorems for a Krasnoselskij type fixed point iterative method constructed as the admissible perturbation of a nonlinear φ-pseudocontractive operator defined on a convex and closed subset of a Hilbert space. These new results extend and unify several related results in the current literature established for contractions, strongly pseudocontractive operators and generalized pseudocontractions.

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1. Introduction and preliminaries

There exists a vast literature on the iterative approximation of fixed points of self and nonself single-valued and multi-valued mappings; see for example the recent monographs [3, 9] and references therein.

Let X be a space. Consider the generic fixed point problem

$$x = Tx, \tag{1.1}$$

where $T : K \to X$ is a given mapping and $K \subset X$.

A method that was intensively used to solve (1.1) is the well known *Picard iteration* or *sequence of successive approximations* $\{x_n\}_{n=0}^{\infty}$, defined by $x_0 \in K$ and

$$x_{n+1} = Tx_n, n \geq 0. \tag{1.2}$$

Picard iterative scheme (1.2) is known to converge to a solution of (1.1) under rather strong conditions on K, X and T, see the sample convergence theorems in [3, 9, 14].

In order to solve (1.1) under weaker assumptions and thus to broaden the scope of applications of fixed point theory, the researchers have introduced more reliable fixed point iterative methods; see [3] and [9].

We recall some of the most used fixed point iterative methods.

Let E be a real vector space and $T : E \to E$ a given operator. Let $x_0 \in E$ be arbitrary and $\{\alpha_n\} \subset [0, 1]$ a sequence of real numbers.

The sequence $\{x_n\}_{n=0}^{\infty} \subset E$ defined by $x_0 \in E$ and

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n\,, \quad n = 0,1,2,\dots \tag{1.3}$$

is called the *Mann iterative scheme*, in view of [13]. The sequence $\{x_n\}_{n=0}^{\infty} \subset E$ defined by

$$\begin{cases} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n\,, \quad n = 0,1,2,\dots \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n\,, \quad n = 0,1,2,\dots, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$, and $x_0 \in E$ is arbitrary, is called the *Ishikawa iterative scheme*, in view of [10].

Remark 1. For $\beta_n = 0$, (1.4) reduces to (1.3), while, for $\alpha_n = \lambda$ (constant), the Mann iteration scheme (1.3) reduces to the so called *Krasnoselskij iteration scheme*, [11].

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n\,, \quad n = 0,1,2,\dots \tag{1.5}$$

Picard iterative scheme (1.2) is obtained from it with $\lambda = 1$.

The above fixed point iterative methods have been extensively used to approximate fixed points of several classes of contractive type mappings; see for example the recent monographs [3, 9, 14] and references therein. One of the most important class of such mappings is that of *pseudocontractive* type mappings. There exist several concepts of pseudocontractive type mappings: pseudocontractive, strictly pseudocontractive, strongly pseudocontractive, generalized pseudocontractive mappings etc. We recall the definition of the most important ones for the purpose of the present paper.

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. An operator $T : K \subset H \to H$ is said to be:

- *pseudocontractive* if, for all $x, y \in K$,

$$\|Tx - Ty\|^2 \le \|x-y\|^2 + \|Tx - Ty - (x-y)\|^2\,. \tag{1.6}$$

  Condition (1.6) is equivalent to the following condition

$$\langle Tx - Ty, x - y\rangle \le \|x-y\|^2\,. \tag{1.7}$$

- *strongly pseudocontractive* if there exists a constant $k \in (0,1)$ such that, for all $x, y \in K$,

$$\langle Tx - Ty, x - y\rangle \le k\,\|x-y\|^2\,. \tag{1.8}$$

- *strictly pseudocontractive* if there exists a constant $k < 1$ such that, for all $x, y \in K$,

$$\|Tx - Ty\|^2 \le \|x-y\|^2 + k\,\|Tx - Ty - (x-y)\|^2\,. \tag{1.9}$$

- *generalized pseudocontractive* if there exists a constant $r > 0$ such that, for all $x, y \in K$,
$$\|Tx - Ty\|^2 \leq r^2 \|x - y\|^2 + \|Tx - Ty - r(x - y)\|^2. \tag{1.10}$$
Condition (1.10) is equivalent to
$$\langle Tx - Ty, x - y\rangle \leq r \|x - y\|^2. \tag{1.11}$$
- *of pseudocontractive type* if, for all $x, y \in K$,
$$\|Tx - Ty\|^2 \leq r^2 \|x - y\|^2 + \max\left\{\|Tx - x\|^2 + \|Ty - y\|^2, \|Tx - Ty - (x - y)\|^2\right\}, \tag{1.12}$$
- *generalized successively pseudocontractive* if there exists a constant $k > 0$ such that, for all $x, y \in K$ and for all $n \geq 1$,
$$\|T^n x - T^n y\|^2 \leq k^2 \|x - y\|^2 + \|T^n x - T^n y - k(x - y)\|^2. \tag{1.13}$$

The pseudocontractive type operators have been studied by many authors ([12, 17–21]) in view of their relationship with accretive operators: the operator $T := I - A$ is pseudocontractive if and only if A is accretive. In this context, Browder [8] has shown that the initial value evolution system
$$\frac{du}{dt} + Au = 0,\ u(0) = u_0,$$
is solvable if A is Lipschitzian and accretive.

In the iterative approximation of fixed point [9], in addition to finding zeros of accretive operators, we are also interesyed to solve the equivalent problem of computing fixed point of pseudocontractive type operators. In this context, some authors tried to unify various classes of pseudocontractive type operators.

The first author introduced in [2], the so called class of pseudo φ-contractions and established a general convergence theorem for the Krasnoselskij fixed point iterative method, while Kumar and Sharma [12] introduced and studied iterative methods for solving variational inequalities that involve Lipschitzian and generalized successively pseudocontractive operators.

On the other hand, Rus [15] considered a new approach to fixed point iterative methods, based on the concept of *admissible perturbation* of a self operator. Further contributions in this direction have been made in [4–6].

As the class of φ-pseudocontractions includes important classes of pseudocontractive type mappings, our aim in the present paper is to establish some convergence theorems for Krasnoselskij type fixed point iterations constructed as admissible perturbations of φ-pseudocontractions in Hilbert spaces, thus obtaining general results that unify and extend several results in literature (see [2, 3] and references therein).

The paper is organized as follows. In Section 2, we present concepts related to admissible mappings and admissible perturbations of a nonlinear operator [15]. Section 3 deals with main aspects and results related to φ-contractions. In Section 4, we prove our main results related to the approximation of fixed points of φ-pseudocontractive operators by means of general iterative methods defined as admissible perturbations of a nonlinear mapping.

2. Admissible perturbations of an operator

Definition 1 ([15]). Let X be a nonempty set. A mapping $G : X \times X \to X$ is called *admissible* if it satisfies the following two conditions:
(A_1) $G(x,x) = x$, for all $x \in X$;
(A_2) $G(x,y) = x$ implies $y = x$.

Definition 2 ([15]). Let X be a nonempty set. If $f : X \to X$ is a given mapping and $G : X \times X \to X$ is an admissible mapping, then the mapping $f_G : X \to X$, defined by

$$f_G(x) = G(x, f(x)), \forall x \in X, \tag{2.1}$$

is called the *admissible perturbation* of f.

Remark 2. The following property of admissible perturbations is fundamental in the iterative approximation of fixed points: if $f : X \to X$ is a given mapping and $f_G : X \to X$ denotes its admissible perturbation, then

$$Fix(f_G) = Fix(f) := \{x \in X | x = f(x)\}, \tag{2.2}$$

that is, the admissible perturbation f_G of f has the same set of fixed points as the operator f itself.

Note that, in general,

$$Fix(f_G^n) \neq Fix(f^n), n \geq 2. \tag{2.3}$$

Example 1 ([15]). Let $(V, +, \mathbb{R})$ be a real vector space, $X \subset V$ a convex subset, $\lambda \in (0,1)$, $f : X \to X$ and $G : X \times X \to X$ be defined by

$$G(x, f(x)) := (1-\lambda)x + \lambda f(x), \ x \in X.$$

Then f_G is an admissible perturbation of f. We shall denote f_G by f_λ and call it the Krasnoselskij perturbation of f.

Example 2 ([15]). Let $(V, +, \mathbb{R})$ be a real vector space, $X \subset V$ a convex subset, $\chi : X \times X \to (0,1)$, $f : X \to X$ and $G(x,y) := (1-\chi(x,y))x + \chi(x,y)y$.

Then f_G is an admissible perturbation of f which reduces to the Krasnoselskij perturbation in the case $\chi(x,y)$ is a constant function.

For other important examples of admissible mappings and admissible perturbations of nonlinear operators, see [15] (for the case of self mappings) and [7] (for the case of nonself mappings).

Definition 3 ([15]). Let $f : X \to X$ be a nonlinear mapping and $G : X \times X \to X$ an admissible mapping. Then the iterative scheme $\{x_n\}_{n=0}^{\infty}$ given by $x_0 \in X$ and

$$x_{n+1} = G(x_n, f(x_n)), n \geq 0, \tag{2.4}$$

is called the *Krasnoselskij iterative scheme* corresponding to G which we denote by *GK-iterative scheme for simplicity*.

Remark 3. In the particular case

$$G(x, y) := (1-\lambda)x + \lambda y, \, x, y \in X, \tag{2.5}$$

the GK-iterative scheme (2.4) reduces to the classical Krasnoselskij iterative scheme (1.5).

Definition 4. Let $G : E \times E \to E$ be an admissible operator on a normed space E. We say that G is *affine Lipschitzian* if there exists a constant $\mu \in [0, 1]$ such that

$$\|G(x_1, y_1) - G(x_2, y_2)\| \leq \|\mu(x_1 - x_2) + (1-\mu)(y_1 - y_2)\|, \tag{2.6}$$

for all $x_1, x_2, y_1, y_2 \in E$.

Note that the admissible operator $G : E \times E \to E$ given by (2.5) is affine Lipschitzian. Note also that in the very recent paper [4], Definition 3.7, we used a weaker concept of affine Lipschitzianity. It is easy to show that, if G is affine Lipschitzian in the sense of Definition 4 in the present paper, then it is also affine Lipschitzian in the sense of Definition 3.7 in [4], but the reverse implication is not true, in general.

3. Fixed point theorems for φ-contractions

In this section we present a minimal list of concepts and results on φ-contractions that will be needed in the sequel. They are mainly taken from the monograph [3].

Definition 5. Let (X, d) be a complete metric space. $T : X \to X$ is called a (*strict*) *Picard operator* if T has a unique fixed point, i.e, $Fix\, T = \{x^*\}$, and

$$T^n(x_0) \to x^* \quad \text{(uniformly) for all } x_0 \in X.$$

Example 3. *(Banach Contraction Principle)*
If (X, d) is a complete metric space, then any contraction $T : X \to X$, i.e., any mapping satisfying

$$d(Tx, Ty) \leq \alpha \cdot d(x, y), \quad \forall\, x, y \in X, \tag{3.1}$$

where $0 < \alpha < 1$, is a Picard operator.

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function. In connection with the function φ we consider the following properties:

(i_φ) φ is monotone increasing, i.e., $t_1 \leq t_2$ implies $\varphi(t_1) \leq \varphi(t_2)$;
(ii_φ) $\{\varphi^n(t)\}$ converges to 0 for all $t \geq 0$;
(iii_φ) $\sum_{n=0}^{\infty} \varphi^n(t)$ converges for all $t > 0$.

Definition 6. A function φ satisfying (i_φ) and (ii_φ) is said to be a *comparison function*;
2) A function φ satisfying (i_φ) and (iii_φ) is said to be a *(c)-comparison function*.

The following assertions are immediate consequences of the previous definition. For other properties see [3], Chapter 2.
1) Any (c)-comparison function is a comparison function ;
2) If φ is a (c)-comparison function, then the function

$$s : \mathbb{R}_+ \to \mathbb{R}_+, \; s(t) = \sum_{k=0}^{\infty} \varphi^k(t), \; t \in \mathbb{R}_+ \tag{3.2}$$

is continuous at 0 and satisfies (i_φ).

Example 4 (Example 2.8 in [3]). 1) $\varphi(t) = at$, $t \in \mathbb{R}_+$, $a \in [0,1)$ satisfies all conditions (i_φ)-(iii_φ);
2) $\varphi(t) = \dfrac{t}{1+t}$, $t \in \mathbb{R}_+$ is a comparison function but not a (c)-comparison function;
3) $\varphi(t) = \dfrac{1}{2}t$, if $0 \le t \le 1$ and $\varphi(t) = t - \dfrac{1}{3}$, if $t > 1$ is a discontinuous (c)-comparison function.

Definition 7. Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be a **'**-*contraction* if there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$d(Tx, Ty) \le \varphi(d(x,y)), \; \text{ for all } x, y \in X. \tag{3.3}$$

A general fixed point principle for φ-contractions is given by the well-known Matkowski-Rus theorem (see [16], page 31), established for comparison functions (part (i) and (ii)) and strict comparison functions (part (iii)), respectively. Here we consider a particular version of it which additionally provides an posteriori error estimate for Picard iteration.

Theorem 1 (Theorem 2.8 in [3])**.** *Let (X,d) be a complete metric space and $T : X \to X$ a $\varphi-$contraction with φ a (c)-comparison function. Then*
(i) $Fix(T) = \{x^*\}$;
(ii) The Picard iterative scheme $\{x_n\} = \{T^n x_0\}_{n \in \mathbb{N}}$ converges to x^* (as $n \to \infty$), for any $x_0 \in X$;
(iii) The following estimate holds

$$d(x_n, x^*) \le s(d(x_n, x_{n+1})), \; n = 0, 1, 2, \ldots, \tag{3.4}$$

where $s(t) = \sum\limits_{k=0}^{\infty} \varphi^k(t)$ is the sum of the comparison series.

Note that if φ is the comparison function given in Example 4, then by Theorem 1, we obtain the well known Banach Contraction Mapping Principle.

It is possible to extend Theorem 1, the so called generalized φ-contractions by replacing (3.3) with the following (see [3], Chapter 2):
there exists a 5-dimensional comparison function $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq \varphi(d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)).$$

4. Approximating fixed points of φ-pseudocontractions

Definition 8 ([2])**.** Let H be a real Hilbert. An operator $T : K \subset H \to H$ is said to be (strictly) φ-pseudocontractive if, for given $a, b, c \in \mathbb{R}_+$ with $a + b + c = 1$, there exists a (comparison) function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$a \cdot \|x - y\|^2 + b\langle Tx - Ty, x - y\rangle + c\|Tx - Ty\|^2 \leq \varphi^2\big(\|x - y\|\big), \tag{4.1}$$

holds, for all $x, y \in K$.

The class of φ-pseudocontractions includes various other important classes of contractive and pseudocontractive type operators, as partially illustrated by the following examples.

Example 5. Any pseudocontractive operator T is also $\varphi-$pseudocontractive with $a = 0, b = 1, c = 0$ and $\varphi(t) = t$, by virtue of the equivalent condition (1.7).

Example 6. Any generalized pseudocontractive operator T is a φ-pseudocontractive operator, with $a = 0, b = 1, c = 0$ and $\varphi(t) = r \cdot t,\ r > 0$.

Example 7. Any strongly pseudocontractive operator T is a strictly φ-pseudocontractive operator, with $a = 0, b = 1, c = 0$ and $\varphi(t) = k \cdot t,\ k \in (0,1)$.

Example 8. Any strictly pseudocontractive operator T is a φ-pseudocontractive operator, with $a = \dfrac{k-1}{2k},\ b = 1,\ c = \dfrac{1-k}{2k}$ and $\varphi(t) = t$.

Example 9. Any Lipschitzian operator T (and hence any contraction and any nonexpansive operator) is φ-pseudocontractive with $a = 0, b = 0, c = 1$ and $\varphi(t) = L \cdot t, L > 0$.

The main result of this paper is the following convergence theorem, which extends Theorem 1 from the case of Krasnoselskij iterative scheme (1.5) to the case of more general GK-iterative scheme (2.4).

Theorem 2. *Let K be a nonempty closed and convex subset of a real Hilbert space H and $T : K \to K$ a strictly φ-pseudocontractive. Then*
(i) T has a unique fixed point p in K;
(ii) If $G : K \times K \to K$ is an affine Lipschitzian admissible operator with constant $\lambda \in (0,1)$, then the GK-iterative scheme $\{x_n\}_{n=0}^{\infty}$ given by x_0 in K and

$$x_{n+1} = G(x_n, f(x_n)), n \geq 0, \tag{4.2}$$

converges (strongly) to p, for any $x_0 \in K$;

(iii) If, additionally φ is a (c)-comparison function, then the following estimate

$$\|x_n - p\| \leq s\big(\|x_n - x_{n-1}\|\big), \quad n = 1, 2, \ldots$$

holds, with $s(t) = \sum_{k=0}^{\infty} \varphi^k(t)$.

Proof. We consider the admissible perturbation operator $F : K \to K$, associated with T and defined by

$$Fx = G(x, Tx), \, x \in K. \tag{4.3}$$

By the fundamental property of an admissible mapping in (2.2), we have

$$Fix(F) = Fix(T). \tag{4.4}$$

Moreover, as the admissible mapping G is affine Lipschitzian, so there exists a constant $\lambda \in [0, 1]$ such that

$$\|G(x, Tx) - G(y, Ty)\| \leq \|(1-\lambda)(x - y) + \lambda(Tx - Ty)\|,$$

for all $x, y \in K$. Then

$$\begin{aligned} \|Fx - Fy\|^2 &= \|G(x, Tx) - G(y, Ty)\|^2 \\ &\leq \|(1-\lambda)(x - y) + \lambda(Tx - Ty)\|^2 = (1-\lambda)^2 \cdot \|x - y\|^2 + 2\lambda(1-\lambda) \cdot \\ &\qquad \cdot \langle Tx - Ty, x - y\rangle + \lambda^2 \cdot \|Tx - Ty\|^2 . \end{aligned}$$

Now put $a = (1-\lambda)^2 \geq 0$, $b = 2\lambda(1-\lambda) \geq 0$ and $c = \lambda^2 \geq 0$. We have $a + b + c = (1-\lambda)^2 + 2\lambda(1-\lambda) + \lambda^2 = 1$ and since T is a strictly φ-pseudocontractive, it results that there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$\|Fx - Fy\|^2 \leq \varphi^2\big(\|x - y\|\big), \quad \forall\, x, y \in K,$$

which yields

$$\|Fx - Fy\| \leq \varphi\big(\|x - y\|\big), \quad \forall\, x, y \in K.$$

This means that F is a φ-contraction, and hence by Theorem 1, the conclusions of our theorem follow. □

Our result unifies and generalizes many convergence theorems in the existing literature.

Remark 4. 1) If T is strongly pseudocontractive, that is, T satisfies (1.8) with $k \in (0, 1)$, then T is a strictly φ-pseudocontraction and hence, by Theorem 2, T has a unique fixed point p in K and the GK-iterative scheme converges strongly to p.

2) If T is a generalized pseudocontractive mapping with constant $r < 1$ and Lipschitzian with constant $s \geq 1$, then by Theorem 2, we obtain a generalization of the main result in [6].

3) In case $G(x, y)$ is the admissible mapping given by (2.5), then by Theorem 2, we obtain the main results in [2] (see also Theorem 3.8 in [3]).

Corollary 1. *Let K be a nonempty closed and convex subset of a Hilbert space H and let $T : K \to K$ be a generalized pseudocontractive and Lipschitzian operator with the corresponding constants $r < 1$ and $s \geq 1$, respectively. Then*

(i) T has an unique fixed point p in K;

(ii) If $G : K \times K \to K$ is an affine Lipschitzian admissible operator with constant $\lambda \in (0,1)$, then the GK-iterative scheme $\{x_n\}_{n=0}^{\infty}$ given by x_0 in K and

$$x_{n+1} = G(x_n, f(x_n)), n \geq 0,$$

converges (strongly) to p for all $\lambda \in (0,1)$ satisfying

$$0 < \lambda < 2(1-r)/(1-2r+s^2).$$

(iii) The priori

$$\| x_n - p\| \leq \frac{\theta^n}{1-\theta} \cdot \| x_1 - x_0\| \ , \ n = 1,2,\ldots$$

and the posteriori

$$\| x_n - p\| \leq \frac{\theta}{1-\theta} \cdot \| x_n - x_{n-1}\| \ , \ \ n = 1,2,\ldots$$

estimates hold, with

$$\theta = \left((1-\lambda)^2 + 2\lambda(1-\lambda)r + \lambda^2 s^2\right)^{1/2}.$$

In case $G(x,y)$ is the admissible mapping given by (2.5), then by Corollary 1, we obtain the main results in [1] and [20] (see also [3], Theorem 3.6).

A more general result than the one given in Theorem 2, can be similarly proven, if we consider generalized φ-pseudocontractive operators instead of φ-pseudocontractions, as shown by the next theorem.

Theorem 3. *Let K be a nonempty closed and convex subset of a real Hilbert space H and $T : K \to K$ a strictly generalized φ-pseudocontraction with $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ satisfying:*

(i) *$\psi(t) = \varphi(t,t,t,t,t)$ is a continuous comparison function;*

(ii) *$h(t) = t - \psi(t)$ is increasing and bijective.*

Then

(i) *T has a unique fixed point p in K;*

(ii) *If $G : K \times K \to K$ is an affine Lipschitzian admissible operator with constant $\lambda \in (0,1)$, then the GK-iterative scheme $\{x_n\}_{n=0}^{\infty}$ given by*

$$x_{n+1} = G(x_n, Tx_n)\,, \quad n = 0,1,2,\ldots$$

converges strongly to p, for all $\lambda \in (0,1)$.

(iii) $\qquad \|x_n - p\| \leq \psi^n\left(h^{-1}(\|x_0 - x_1\|)\right), \quad n \geq 1.$

Proof. The proof is similar to that of Theorem 2 with the only difference that, we apply the generalized φ-contraction principle ([3],Theorem 2.10) instead of Theorem 2. $\square$

Remark 5. If T is not a strictly φ-pseudocontraction, then the conclusion of Theorem 2 (and 1) is no longer valid. If for example, $\varphi(t) = t$, then the admissible perturbation of T, $Fx = G(x, Tx)$, is nonexpansive but in general it is not a contraction (or a φ-contraction). This problem will be considered in a forthcoming paper (cf. [4]).

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NONTRANSITIVE DICE SETS REALIZING THE PALEY TOURNAMENTS FOR SOLVING SCHÜTTE'S TOURNAMENT PROBLEM

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Abstract. The problem of a multiple player dice tournament is discussed and solved in the paper. A die has a finite number of faces with real numbers written on each. Finite dice sets are proposed which have the following property, defined by Schütte for tournaments: for an arbitrary subset of k dice there is at least one die that beats each of the k with a probability greater than $1/2$. It is shown that the proposed dice set realizes the Paley tournament, that is known to have the Schütte property (for a given k) if the number of vertices is large enough. The proof is based on Dirichlet's theorem, stating that the sum of quadratic nonresidues is strictly larger than the sum of quadratic residues.

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1. INTRODUCTION

The game of rock, paper, scissors is one of the simplest nontransitive tournaments. The relation between any two elements from the set {rock, paper, scissors} is well defined. However, the directed graph of three vertices —representing the three objects, with edges leading from rock to scissors (since rock beats scissors), from scissors to paper, and from paper to rock —is a directed three-cycle. In other words, there is no optimal choice among the three alternatives, since any of them is beaten by another one. This game is directly defined to be a paradox.

Similar types of nontransitivity can be observed with certain dice sets. Let us have a finite set of dice, each die having a finite number of faces (not necessarily with the same probabilities) and each face having a real number written upon it. Die i beats die j if, when rolled together repeatedly, the number of cases when the rolled face of die i is greater than the rolled face of die j is, in the long run, greater than the half of the number of total cases. In terms of probability theory, dice are considered as independent discrete random variables, and die X beats die Y if $P(X > Y) > 1/2$.

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Definition 1. (see, e.g., [28, Def. 5.3]) Let G be a tournament on n vertices. Dice set $\{X_1, X_2, \ldots, X_n\}$ *realizes* G if $P(X_i > X_j) > 1/2$ if and only if $i \to j$ is an edge of G.

Section 2 introduces quadratic residues, the *Paley tournament* and the *Schütte property*. The Paley tournament [21], denoted by P_p hereafter, is a complete directed graph of p vertices without loops, where $p = 4m + 3 \geq 7$ is a prime number. If $j - i$ is a quadratic residue modulo p, then a directed edge is drawn from vertex i to vertex j, otherwise it is drawn in the opposite direction. Note that the Paley tournament can be defined not only for prime numbers, and not only for prime numbers in the form of $p = 4m + 3$, however, those variants are not used in the paper.

For a given positive integer k, the Schütte property, denoted by S_k, is defined for directed graphs, and, as a special case, it is applicable to tournaments: for every k vertices there exists at least one vertex, from which edges go to each of the k vertices [10]. It follows from the definition that a tournament with property S_k fulfills property $S_{k'}$ as well, when $k > k'$.

Several lower and upper bounds are known for the minimal number n such that there exists a tournament on n vertices that fulfills property S_k. One of them is directly applied in the paper, provided by Graham and Spencer [14]: if $p > k^2 2^{2k-2}$, then P_p fulfills property S_k.

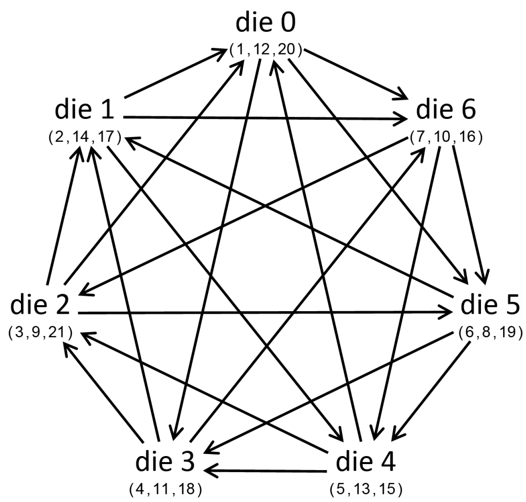

FIGURE 1. van Deventer's dice set realizes P_7.

Section 3 summarizes a number of nontransitive dice sets, starting with the one by Steinhaus and Trybula [30]. The multiple player dice tournament problem requires property S_k: for every k dice there exists at least one die in the set that beats each of the k. A nontransitive dice set for three players (having property S_2) is proposed by van Deventer [5]. Each die in Figure 1 has three faces with equal probabilities, and the numbers written on them are given in parentheses. A directed edge is going from die i to die j if and only if die i wins over die j with probability greater than 1/2, the tournament resulted in is P_7, that is, P_7 is realized by the dice set in Figure 1.

Departing from van Deventer's dice set, the results of the paper are presented in Section 4. Dice set D_p with p dice is proposed for all $p = 4m + 3$ ($m \geq 1$) prime numbers and it shown that D_p realizes P_p.

In terms of probability theory again, for any k, a constructive definition of the set of independent discrete random variables $X_1, X_2, \ldots, X_p$ is given such that each variable of an arbitrarily chosen subset $X_{i_1}, X_{i_2}, \ldots, X_{i_k}$ can be simultaneously beaten by at least one of the other variables. That is, there exists a random variable $X_{i_{k+1}}$ in the set such that $P(X_{i_{k+1}} > X_{i_1}) > 1/2,\ P(X_{i_{k+1}} > X_{i_2}) > 1/2,\ \ldots,\ P(X_{i_{k+1}} > X_{i_k}) > 1/2$. Moreover, as an unintended secondary result, all probabilities above are equal.

Section 5 concludes and discusses further research of realizations having fewer faces and possible relations to voting theory.

2. Quadratic residues, Schütte's tournament problem and the Paley tournaments

Let $p = 4m+3$ ($m \geq 1$) be a prime number. q is called a quadratic residue modulo p, if q is congruent to a perfect square modulo p —otherwise it is called a quadratic nonresidue. Three properties of quadratic residues are applied in the paper. The first two properties refer to the sets of quadratic residues and of quadratic nonresidues (see, e.g., Theorem 3.1. in the book of Niven and Zuckerman [20]). If q is a quadratic residue modulo p, then $p - q$ is a quadratic nonresidue, and vice versa. The set of nonzero quadratic residues is closed for multiplication (modulo p) by any nonzero quadratic residue (and the set of quadratic nonresidues is closed for multiplication by any nonzero quadratic residue). The third property is related to the distribution of quadratic residues proven by Dirichlet [6–8], starting from Jacobi's observation [17]:

Theorem 1. *For $p \geq 7$, the sum of the residues minus the sum of the nonresidues is a negative number, and it is an odd multiple of p.*

Probably the shortest proof of Theorem 1 is due to Moser [19]. For the illustrative example $p = 7$, the sum of quadratic residues is $(0+)1 + 2 + 4 = 7$, while the sum of quadratic nonresidues is $3 + 5 + 6 = 14$, and their difference is $7 - 14 = -1 \cdot 7$.

Erdős writes in one of his more than twenty papers from 1963 [10, p. 221]:

> "*The problem was recently put to me by Professor Schütte in its graph-theoretic form: If $\mathcal{G}^{(n)}$ is a complete directed graph, with n vertices, which has the property that for every k vertices of $\mathcal{G}^{(n)}$ there is at least one vertex from which edges go out to each of the k, we shall say that $\mathcal{G}^{(n)}$ has the property S_k. Schütte's problem is to show that for every k there is a $\mathcal{G}^{(n)}$ with the property S_k and to find the least possible n for a given k.*"

Let us keep the notations above and let $f(k)$ denote the minimal number of n such that there exists a tournament on n vertices having property S_k.

That $f(k) < \infty$ (for all $k = 1, 2, \ldots$) is proved first by Erdős [10], using the probabilistic method, while a geometric proof is given by Alon, Brightwell, Kierstead, Kostochka and Winkler [1]. It can be noted that Schütte's problem has several variants, e.g., a bipartite version discussed by Borowiecki, Grytczuk, Haluszczak and Tuza [4].

Lower and upper bounds for $f(k)$ are also known:

$$f(k) \geq 2^{k+1} - 1 \quad \text{(Erdős [10, p. 221])}$$

$$f(k) \geq 2^{k-1}(k+2) - 1 \quad \text{(Szekeres and Szekeres [32, p. 292])}$$

$$\limsup_k \frac{f(k)}{2^k k^2} \geq \log 2 \quad \text{(Erdős [10, p. 221])}$$

However, exact values of $f(k)$ are known only for small k. Szekeres and Szekeres [32, p. 293] showed that $f(1) = 3$, $f(2) = 7$, $f(3) = 19$.

Definition 2. Let $p = 4m + 3 \geq 7$ be a prime number. The *Paley tournament* on p vertices, denoted by P_p, is defined by its incidence matrix $\mathbf{M}$ of size $p \times p$ as follows:

$$\mathbf{M}_{i,j} := \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{if } i - j \text{ is a quadratic residue modulo } p, \\ -1 & \text{if } j - i \text{ is a quadratic residue modulo } p. \end{cases}$$

Following from the properties of quadratic residues, P_p is well defined. It is a simple complete directed graph of p vertices (without loops), in which there is a directed edge from vertex i to vertex j if and only if $j - i$ is a quadratic residue modulo p. Otherwise (that is, if $i - j$ is a quadratic residue) the directed edge goes from vertex j to vertex i. A rotational symmetry of P_p can be also observed: the $(i+1)$-th row is a shifted copy of the i-th row, and the last element is relocated to the first position.

The construction is originally given in a linear algebraic way by Paley [21], and, according to Goethals and Seidel [13, p. 1002], it is represented as graphs by Sachs [26] and Erdős and Rényi [11]. Graham and Spencer [14] draw P_p as directed graph and give a sufficient condition for fulfilling S_k:

Theorem 2. *If $p > k^2 2^{2k-2}$, then P_p has property S_k.*

Graham and Spencer [14, p. 47] also propose that P_{67} has property S_4 and so, consequently, $f(4) \le 67$. Their conjecture is that $f(4) = 67$. According to Reid, McRae, Hedetniemi and Hedetniemi [24] it was computationally verified by Fisher in 1996 that

Proposition 1. *Among the Paley tournaments:*

(1) *P_{67} is the smallest one having property S_4;*
(2) *P_{331} is the smallest one having property S_5;*
(3) *P_{1163} is the smallest one having property S_6.*

Fisher's results are published in [24] as private communication. We tried to confirm Fisher's results and succeeded for the first two regarding S_4 and S_5. However, the third result was only confirmed in part: it was shown that P_p does not have property S_6 if $4m+3 = p < 1163$ is a prime number.

Tyszkiewicz [36] proposed tournaments fulfilling property S_k and they are different from the Paley tournaments.

3. Nontransitive dice sets

Steinhaus and Trybula [30, p. 69] constructed three independent random variables X,Y,Z as follows: let $c > 4$ be an arbitrary constant and $P(X = c-5/3) = 3/8$, $P(X = c+1) = 5/8$, $P(Y = c+5/3) = 3/8$, $P(Y = c-1) = 5/8$, $P(Z = c-2/3\sqrt{30}) = 1/16$, $P(Z = c+2/3\sqrt{30}) = 1/16$, $P(Z = c) = 7/8$. Then $P(X > Z) = P(Z > Y) = P(Y > X) = 39/64 > 1/2$. As far as we know, it is the first nontransitive dice set, even if the word 'dice' is not mentioned in the paper of Steinhaus and Trybula.

Another construction by Trybula [34, p. 321] is interpreted here as a nontransitive dice set, shown in Table 1. Let $\varphi = \frac{\sqrt{5}+1}{2}$ denote the golden ratio, then its reciprocal can be written as $1/\varphi = \frac{\sqrt{5}-1}{2} \approx 0.618$.

	die 1	die 2	die 3
face 1 with probability $1/\varphi$	3	2	1
face 2 with probability $1-1/\varphi$	0	2	4

TABLE 1. Nontransitive dice set by Trybula.

Die 1 beats die 2, die 2 beats die 3 and die 3 beats die 1, each with probability $1/\varphi > 1/2$. It is also stated in [30] and shown in [34] that if the random variables X,Y,Z are independent and $P(X > Y) = P(Y > Z) = P(Z > X)$, then $1/\varphi$ is a sharp upper bound for this probability. If the three probabilities are not necessary equal, then upper bounds can be given for the sum and product of them. Results are extended to n variables in [35].

In the rest of the paper, all faces have the same probabilities and, therefore, probabilities are not denoted explicitly in the tables.

Moon and Moser [18, p. 531] constructed an example for three teams, each including three players of different skill levels, such that any team is beaten by another team. A comparison between two teams is calculated from all the individual matches played by pairs of players. Their construction can also be interpreted as a nontransitive dice set, where a die is associated with a team, and the numbers written on the faces represent the skill levels of players on that team. One can verify from Table 2 that die 1 beats die 2, die 2 beats die 3 and die 3 beats die 1, with probability $5/9$.

	die 1	die 2	die 3
face 1	2	1	3
face 2	6	5	4
face 3	7	9	8

TABLE 2. Nontransitive dice set by Moon and Moser.

Efron's dice set was published by Gardner [12] and, together with dice sets by Schwenk [29] and Rowett [25], is well explained by Peterson[23] and Pegg [22].

Tenney and Foster [33], Székely [31], Batakci and Singer [2] and Epstein [9, pp. 195–199] analyzed a number of dice sets and winning probabilities. Grime [15] constructed and discussed several nontransitive dice sets.

As we saw, the role of the golden ratio had already been realized at the birth of nontransitive dice sets. The reader will probably not be surprised by the fact that Fibonacci numbers are also involved in some of the constructions for nontransitive dice sets. See Savage's paper [27] and recent results by Schaefer and Schweig [28] for more details.

As we have seen in Section 1, van Deventer's dice set realizes P_7 having property S_2. In other words, it is a *three-player* nontransitive dice set: after each of the two opponents pick a die of their choice, the third player will find a die among the remaining dice which beats both opponents' dice. The dice set is originally presented as dice cubes of 6 faces, but they are in fact doubles of 3 faces as in Figure 1. The numbers written on the faces can be divided into three groups 1-7, 8-14 and 15-21, and they are rewritten with additive terms in Table 3. We keep the way of additive writing in the rest of the paper.

	die 0	die 1	die 2	die 3	die 4	die 5	die 6	add
face 1	0	1	2	3	4	5	6	+1+0·7
face 2	4	6	1	3	5	0	2	+1+1·7
face 3	5	2	6	3	0	4	1	+1+2·7

TABLE 3. van Deventer's dice set in additive form.

Note that the dice set presented in [1, p. 377], is equivalent to van Deventer's dice set. Jackson [16] proposed a dice set of five dice for the three-player game.

4. CONSTRUCTION OF DICE SET $\mathbf{D}_p$ THAT REALIZES P_p

Hereafter, a construction of dice set $\mathbf{D}_p$ of p dice is sketched, then it is shown that it realizes P_p. Remember that $p = 4m+3 \geq 7$ is a prime number. Each die has $p(p-1)/2$ faces having equal probabilities. The dice set is written as an array of $p(p-1)/2$ rows and p columns. Each column is associated with a die and faces are organized in rows. Quadratic residues modulo p are denoted by $q_0 = 0, q_1 = 1, q_2, q_3, \ldots, q_{(p-1)/2}$ as before. Since $q_1 = 1$ for any p, the first $p \times p$ array can be written as in Table 4.

	die 0	die 1	die 2	…	die $p-1$	add
face 1	**0**	**1**	**2**	…	$\boldsymbol{p-1}$	$+0 \cdot p$
face 2	1	2	3	…	0	$+1 \cdot p$
face 3	2	3	4	…	1	$+2 \cdot p$
⋮	⋮	⋮	⋮	⋱	⋮	⋮
face p	$p-1$	0	1	…	$p-2$	$+(p-1) \cdot p$

TABLE 4. The first $p \times p$ array of dice set $\mathbf{D}_p$ generated by $q_1 = 1$.

Table 4 is continued by $(p-3)/2$ arrays, each of which are of size $p \times p$. For a quadratic residue q_m $(1 \leq m \leq (p-1)/2)$, the construction of the key row is as follows. 0 is written on face $(m-1)p+1$ of die d_0 such that $0 \leq d_0 \leq p-1$ and $d_0 \equiv 0q_m \pmod{p}$, that is $d_0 = 0$. Then, for $i = 1, 2, \ldots, p-1$, i is written on face $(m-1)p+1$ of die d_i such that $0 \leq d_i \leq p-1$ and $d_i \equiv iq_m \pmod{p}$;

Each key face generates $p-1$ additional faces by its shifted copies. The corresponding $p \times p$ array is as in Table 5:

face	die 0	…	die q_m	…	die $2q_m$	…	die $p-1$	add
$\boldsymbol{(m-1)p+1}$	$\boldsymbol{d_0{=}0}$	…	**1**	…	**2**	…	…	$+(m-1)p^2$
$(m-1)p+2$	1	…	2	…	3	…	…	$+(m-1)p^2+p$
$(m-1)p+3$	2	…	3	…	4	…	…	$+(m-1)p^2+2p$
⋮	⋮	⋮	⋮	⋮	⋮	⋱	⋮	⋮
$mp-1$	$p-2$	…	$p-1$	…	0	…	…	$+mp^2-2p$
mp	$p-1$	…	0	…	1	…	…	$+mp^2-p$

TABLE 5. The m-th $p \times p$ array of dice set $\mathbf{D}_p$ generated by q_m.

The illustrative example $\mathbf{D}_7$ is given in the Appendix.

Proposition 2. *Dice set $\mathbf{D}_p$ constructed in Tables 4-5 realizes P_p.*

Proof. Note that P_p is isomorphic to its reversed version. Consequently, if a dice set realizes the Paley tournament, then it also realizes the reversed Paley tournament after an appropriate reindexing of the dice. An alternative way of reversing a tournament realized by a set of dice, presumably containing only nonnegative values on their faces, is to multiply every value by -1, then all edges of the tournament reverse. On the other hand, it is sufficient to calculate the odds of die 0 vs. all other dice, due to the rotational symmetry of P_p. The rest of the proof is built up from four steps:

Lemma 1. *The probability that die* 0 *beats die* i *is smaller than, equal to, larger than* $1/2$*, if and only if the probability that die* 0 *beats die* i*, restricted to the cases where their same-indexed faces compete to each other, is smaller than, equal to, larger than* $1/2$*, respectively.*

Proof of Lemma 1. Due to the additive terms in the construction $s > t$ implies that face s beats face t for an arbitrary pair of dice. By symmetry, the number of cases when different faces of die 0 and die i compete, is equal to the number of cases when different faces of die i and die 0 compete. The remaining cases are exactly the ones when the same-indexed faces compete. □

Lemma 2. *The odds of die* 0 *vs. die* i*, restricted to the cases when their same-indexed faces compete against each other, are equal to* $\left[\sum_{\ell=1}^{(p-1)/2} v_{\ell,i}\right] : : \left[\sum_{\ell=1}^{(p-1)/2} (p - v_{\ell,i})\right]$*, where* $v_{\ell,i}$ *denotes the value written on* ℓ*-th key face of die* i*, and the sums are taken over the* $(p-1)/2$ *key faces.*

Proof of Lemma 2. First consider only one square array, corresponding to a fixed key face. Let us restrict the dice to this array only. Then the value written on the key face of die i shows the number of cases out of p when die i beats die 0, because the values follow each other in the same order, and differ only in the rotation. Count the number of cases when die 0 beats die i for every square array corresponding to a key face and take their sum. □

Lemma 3. *Nonzero quadratic residues are written on key faces of die* i*, where* i *is a nonzero quadratic residue. Equivalently, quadratic nonresidues are written on key faces of dice* j*, where* j *is from the set of quadratic nonresidues.*

Proof of Lemma 3. Nonzero quadratic residues are written on key faces of dice indexed by nonzero quadratic residues. By analogy, quadratic nonresidues are written on key faces of dice indexed by quadratic nonresidues. □

Lemma 4. *The probability that die* 0 *beats die* i *is greater than* $1/2$ *if and only if* i *is a quadratic nonresidue. The probability that die* 0 *beats die* i *is smaller than* $1/2$ *if and only if* i *is a nonzero quadratic residue.*

Proof of Lemma 4. Let i be a quadratic nonresidue. It follows from Lemmas 2 and 3 that the odds of die 0 vs. die i, restricted to the cases when their same-indexed faces compete to each other, are greater than 1. The number of cases when different faces die 0 vs. die i compete, is equal to the number of cases when different faces of die i vs. die 0 compete. Taking the sum, we get that the odds of die 0 vs. die i are greater than 1, consequently, the probability that die 0 beats die i is greater than $1/2$. The case of nonzero quadratic residue is analogous, the probability that die 0 beats die i is smaller than $1/2$. As we see, the crux is in the relation between the sum of quadratic residues and the sum of quadratic nonresidues. It should be also noted that the odds of these dice are never equal. Thus, Lemma 1 could be restated without the case of equality. □

Lemma 4 can be specified for any fixed value of p, but an explicit formula for the odds is unknown. In case of a minimal tournament P_p fulfilling property S_k given in Proposition 1, the differences of the sum of quadratic residues and the sum of quadratic nonresidues are $-1\cdot 7, -1\cdot 19, -1\cdot 67, -3\cdot 331, -7\cdot 1163$.

Lemmas 1, 2, 3 and 4 complete the proof of Proposition 2. □

Remark 1. It follows from the construction that the sum of faces is the same, $(p^5-2p^4+p^3-2p^2+2p)/8$ for all dice, i.e., each die has the same expected value $\frac{p^4-2p^3+p^2-2p+2}{4(p-1)}$.

Proposition 3. *If $p > k^2 2^{2k-2}$, then $\mathbf{D}_p$ is a nontransitive dice set for $k+1$ players.*

Proof. The claim follows from Theorem 2 and Proposition 2. □

5. Conclusions

Research can be continued in several directions, one of them has already been started: finding dice sets that realize P_p and have essentially fewer than $p(p-1)/2$ faces, preliminary results are summarized in [3]. The assumption of the equal probability of the faces can be relaxed, which is supposed to decrease the number of faces. One would expect that replacing discrete random variables with continuous ones requires a different methodology. The historical overview of nontransitive dice refer to the common roots with voting situations, therefore, both areas would benefit from further comparative studies.

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APPENDIX: DICE SET $\mathbf{D}_7$

The construction provided in Section 4 is illustrated for $p = 7$.

	die 0	die 1	die 2	die 3	die 4	die 5	die 6	add
face 1	**0**	**1**	**2**	**3**	**4**	**5**	**6**	+0·7
face 2	1	2	3	4	5	6	0	+1·7
face 3	2	3	4	5	6	0	1	+2·7
face 4	3	4	5	6	0	1	2	+3·7
face 5	4	5	6	0	1	2	3	+4·7
face 6	5	6	0	1	2	3	4	+5·7
face 7	6	0	1	2	3	4	5	+6·7
face 8	**0**	**4**	**1**	**5**	**2**	**6**	**3**	+7·7
face 9	1	5	2	6	3	0	4	+8·7
face 10	2	6	3	0	4	1	5	+9·7
face 11	3	0	4	1	5	2	6	+10·7
face 12	4	1	5	2	6	3	0	+11·7
face 13	5	2	6	3	0	4	1	+12·7
face 14	6	3	0	4	1	5	2	+13·7
face 15	**0**	**2**	**4**	**6**	**1**	**3**	**5**	+14·7
face 16	1	3	5	0	2	4	6	+15·7
face 17	2	4	6	1	3	5	0	+16·7
face 18	3	5	0	2	4	6	1	+17·7
face 19	4	6	1	3	5	0	2	+18·7
face 20	5	0	2	4	6	1	3	+19·7
face 21	6	1	3	5	0	2	4	+20·7

TABLE 6. Dice set $\mathbf{D}_7$.

Remark 2. Dice set $\mathbf{D}_7$ is obviously much less efficient in minimizing the number of faces than van Deventer's dice set, since his dice have 3 faces only (faces 1, 19, 13 in Table 6 are essentially the same as faces 1, 2, 3 in Table 3 —they differ only in the additive term).

Remark 3. Table 6 is built up from three 7×7 arrays, each of which are latin squares if the additional $k \times 7$ is not considered.

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EXISTENCE RESULTS FOR FIRST ORDER BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH FOUR-POINT INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this paper, we study the existence of solutions for boundary value problems of fractional differential equations of order $q \in (0,1]$ with four-point integral boundary conditions. Existence and uniqueness results are obtained by using well known fixed point theorems. Some illustrative examples are also discussed.

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1. INTRODUCTION

In recent years, boundary value problems for nonlinear fractional differential equations have been addressed by several researchers. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes, see [15]. These characteristics of the fractional derivatives make the fractional-order models more realistic and practical than the classical integer-order models. As a matter of fact, fractional differential equations arise in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. [13, 15–17]. For some recent development on the topic, see [1–11] and the references therein.

In this paper, we discuss the existence and uniqueness of solutions for a boundary value problem of nonlinear fractional differential equations of order $q \in (0,1]$ with four-point integral boundary conditions given by

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \ \ 0 < t < 1, \ \ 0 < q \le 1, \\ x(0) + \alpha \displaystyle\int_\mu^\nu x(s)ds = x(1), \ \ 0 < \mu < \nu < 1 \ (\mu \ne \nu), \end{cases} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous and $\alpha\in\mathbb{R}\setminus\{0\}$. We denote by $\mathcal{C}=C([0,1],\mathbb{R})$ the Banach space of all continuous functions from $[0,1]\to\mathbb{R}$ endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|$.

The boundary condition in the problem (1.1) can be regarded as four-point nonlocal boundary condition, which reduces to the typical integral boundary condition in the limit $\mu\to 0,\ \nu\to 1$.

We prove some new existence and uniqueness results by using a variety of fixed point theorems. In Theorem 1 we prove an existence and uniqueness result by using Banach's contraction principle, in Theorem 2 we prove the existence of a solution by using Krasnoselskii's fixed point theorem, while in Theorem 3 we prove the existence of a solution via Leray-Schauder nonlinear alternative.

It is worth mention that the methods used in this paper are standard, however their exposition in the framework of problem (1.1) is new.

2. Preliminaries on Fractional Calculus

Let us recall some basic definitions of fractional calculus [13, 17].

Definition 1. For a continuous function $g:[0,\infty)\to\mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$ {}^cD^q g(t)=\frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1}g^{(n)}(s)ds,\ \ n-1<q<n, n=[q]+1, $$

where $[q]$ denotes the integer part of the real number q.

Definition 2. The Riemann-Liouville fractional integral of order q is defined as

$$ I^q g(t)=\frac{1}{\Gamma(q)}\int_0^t \frac{g(s)}{(t-s)^{1-q}}ds,\ \ q>0, $$

provided the integral exists.

Definition 3. The Riemann-Liouville fractional derivative of order q for a continuous function $g(t)$ is defined by

$$ D^q g(t)=\frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n\int_0^t \frac{g(s)}{(t-s)^{q-n+1}}ds,\ \ n=[q]+1, $$

provided the right hand side is pointwise defined on $(0,\infty)$.

Lemma 1 ([13])**.** *For $q>0$, the general solution of the fractional differential equation ${}^cD^q x(t)=0$ is given by*

$$ x(t)=c_0+c_1t+c_2t^2+\ldots+c_{n-1}t^{n-1}, $$

where $c_i\in\mathbb{R}$, $i=0,1,2,\ldots,n-1$ ($n=[q]+1$).

In view of Lemma 1, it follows that

$$I^{q\ c}D^{q}x(t)=x(t)+c_0+c_1t+c_2t^2+\ldots+c_{n-1}t^{n-1}, \tag{2.1}$$

for some $c_i\in\mathbb{R},\ i=0,1,2,\ldots,n-1$ $(n=[q]+1)$.

Lemma 2. *For a given $g\in C([0,1],\mathbb{R})$ the unique solution of the boundary value problem*

$$\begin{cases} {}^{c}D^{q}x(t)=g(t),\ \ 0<t<1,\ \ 0<q\le 1,\\ x(0)+\alpha\displaystyle\int_{\mu}^{\nu}x(s)ds=x(1),\ \ 0<\eta<1, \end{cases}$$

is given by

$$\begin{aligned} x(t)= &\ \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}g(s)ds+\frac{1}{\alpha(\nu-\mu)\Gamma(q)}\int_0^1(1-s)^{q-1}g(s)ds\\ &-\frac{1}{(\nu-\mu)\Gamma(q)}\int_{\mu}^{\nu}\left(\int_0^s(s-m)^{q-1}g(m)dm\right)ds,\quad t\in[0,1]. \end{aligned} \tag{2.2}$$

Proof. For some constant $c_0\in\mathbb{R}$, we have

$$x(t)=I^{q}g(t)-c_0=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}g(s)ds-c_0. \tag{2.3}$$

We have $x(0)=-c_0$,

$$\begin{aligned} \alpha\int_{\mu}^{\nu}x(s)ds&=\alpha\int_{\mu}^{\nu}\left(\int_0^s\frac{(s-m)^{q-1}}{\Gamma(q)}g(m)dm-c_0\right)ds\\ &=\alpha\int_{\mu}^{\nu}\left(\int_0^s\frac{(s-m)^{q-1}}{\Gamma(q)}g(m)dm\right)ds-\alpha c_0(\nu-\mu), \end{aligned}$$

and

$$x(1)=\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}g(s)ds-c_0,$$

which imply that

$$c_0=\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\left(\int_0^s\frac{(s-m)^{q-1}}{\Gamma(q)}g(m)dm\right)ds-\frac{1}{\alpha(\nu-\mu)}\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}g(s)ds.$$

Substituting the value of c_0 in (2.3) we obtain the solution (2.2). $\square$

3. Existence Results

In view of Lemma 2, we define an operator $\mathbf{F} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned}(\mathbf{F}x)(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,x(s))ds \\ &+\frac{1}{\alpha(\nu-\mu)\Gamma(q)}\int_0^1 (1-s)^{q-1} f(s,x(s))ds \\ &-\frac{1}{(\nu-\mu)\Gamma(q)}\int_\mu^\nu \left(\int_0^s (s-m)^{q-1} f(m,x(m))dm\right) ds, \quad t\in[0,1].\end{aligned} \tag{3.1}$$

In the following we use the norm $\|x\| = \sup_{t\in[0,1]} |x(t)|$ and for convenience, we set

$$\Lambda = \frac{1}{\Gamma(q+1)}\left(1+\frac{\delta_1[q+1+|\alpha|(\nu^{q+1}-\mu^{q+1})]}{|\alpha|(q+1)}\right), \quad \delta_1 = \frac{1}{|\nu-\mu|}. \tag{3.2}$$

Our first result is based on Banach's contraction principle.

Theorem 1. *Assume that $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a continuous function and satisfies the assumption*

(A_1) $|f(t,x)-f(t,y)| \le L|x-y|, \forall t \in [0,1],\ L>0,\ x,y\in\mathbb{R},$

with $L < 1/\Lambda$, where Λ is given by (3.2). Then the boundary value problem (1.1) has a unique solution.

Proof. Setting $\sup_{t\in[0,1]}|f(t,0)| = M$ and choosing $r \ge \dfrac{\Lambda M}{1-L\Lambda}$, we show that $\mathbf{F}B_r \subset B_r$, where $B_r = \{x \in \mathcal{C} : \|x\| \le r\}$. For $x \in B_r$, we have

$$\begin{aligned}&\|(\mathbf{F}x)(t)\| \\ &\le \sup_{t\in[0,1]}\left\{\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,x(s))|ds\right. \\ &+\frac{1}{|\alpha(\nu-\mu)|\Gamma(q)}\int_0^1 (1-s)^{q-1}|f(s,x(s))|ds \\ &\left.+\frac{1}{|\nu-\mu|\Gamma(q)}\int_\mu^\nu\left(\int_0^s (s-m)^{q-1}|f(m,x(m))|dm\right)ds\right\} \\ &\le \sup_{t\in[0,1]}\left\{\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}(|f(s,x(s))-f(s,0)|+|f(s,0)|)\|ds\right. \\ &+\frac{1}{|\alpha(\nu-\mu)|\Gamma(q)}\int_0^1 (1-s)^{q-1}(|f(s,x(s))-f(s,0)|+|f(s,0)|)ds \\ &+\frac{1}{|\nu-\mu|\Gamma(q)}\int_\mu^\nu\Big(\int_0^s (s-m)^{q-1}(|f(m,x(m))-f(m,0)|\end{aligned}$$

$$
\begin{aligned}
&+|f(m,0)|)dm\Big)ds\Big\} \\
&\leq (Lr+M)\sup_{t\in[0,1]}\Big\{\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds + \frac{1}{|\alpha(\nu-\mu)|\Gamma(q)}\int_0^1 (1-s)^{q-1}ds \\
&+\frac{1}{|\nu-\mu|\Gamma(q)}\int_\mu^\nu \Big(\int_0^s (s-m)^{q-1}dm\Big)ds\Big\} \\
&\leq \frac{(Lr+M)}{\Gamma(q+1)}\left(1+\frac{\delta_1[q+1+|\alpha|(\nu^{q+1}-\mu^{q+1})]}{|\alpha|(q+1)}\right) \\
&= (Lr+M)\Lambda \leq r.
\end{aligned}
$$

Now, for $x, y \in \mathcal{C}$ and for each $t \in [0,1]$, we obtain

$$
\begin{aligned}
&\|(\mathbf{F}x)(t)-(\mathbf{F}y)(t)\| \\
&\leq \sup_{t\in[0,1]}\Big\{\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,x(s))-f(s,y(s))|ds \\
&+\frac{1}{|\alpha(\nu-\mu)|\Gamma(q)}\int_0^1 (1-s)^{q-1}|f(s,x(s))-f(s,y(s))|ds \\
&+\frac{1}{|\nu-\mu|\Gamma(q)}\int_\mu^\nu\left(\int_0^s (s-m)^{q-1}|f(m,x(m))-f(m,y(m))|dm\right)ds\Big\} \\
&\leq L\|x-y\|\sup_{t\in[0,1]}\Big\{\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds + \frac{1}{|\alpha(\nu-\mu)|\Gamma(q)}\int_0^1 (1-s)^{q-1}ds \\
&+\frac{1}{|\nu-\mu|\Gamma(q)}\int_\mu^\nu\left(\int_0^s (s-m)^{q-1}dm\right)ds\Big\} \\
&\leq \frac{L}{\Gamma(q+1)}\left(1+\frac{\delta_1[q+1+|\alpha|(\nu^{q+1}-\mu^{q+1})]}{|\alpha|(q+1)}\right)\|x-y\| = L\Lambda\|x-y\|,
\end{aligned}
$$

where Λ is given by (3.2). Observe that Λ depends only on the parameters involved in the problem. As $L < 1/\Lambda$, therefore $\mathbf{F}$ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Now, we prove the existence of solution of (1.1) by applying Krasnoselskii's fixed point theorem.

Lemma 3 ([14], Krasnoselskii's fixed point theorem)**.** *Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A, B be the operators such that:*

(i) *$Ax + By \in M$ whenever $x, y \in M$;*
(ii) *A is compact and continuous;*

(iii) *B is a contraction mapping.*
Then there exists $z \in M$ such that $z = Az + Bz$.

Theorem 2. *Let $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ be a continuous function satisfying assumption (A_1). Moreover we assume that*

(A_2) $|f(t,x)| \le \mu(t)$, $\forall (t,x) \in [0,1]\times\mathbb{R}$, *and* $\mu \in C([0,1],\mathbb{R}^+)$.

If

$$\frac{L}{\Gamma(q+1)}\left(\frac{\delta_1[q+1+|\alpha|(\nu^{q+1}-\mu^{q+1})]}{|\alpha|(q+1)}\right) < 1, \tag{3.3}$$

then the boundary value problem (1.1) has at least one solution on $[0,1]$.

Proof. Letting $\sup_{t\in[0,1]}|\mu(t)| = \|\mu\|$, we fix

$$\bar{r} \ge \frac{\|\mu\|}{\Gamma(q+1)}\left(1+\frac{\delta_1[q+1+|\alpha|(\nu^{q+1}-\mu^{q+1})]}{|\alpha|(q+1)}\right),$$

and consider $B_{\bar{r}} = \{x \in \mathcal{C} : \|x\| \le \bar{r}\}$. We define the operators $\mathcal{P}$ and $\mathcal{Q}$ on $B_{\bar{r}}$ as

$$(\mathcal{P}x)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,u(s))ds, \;\; t \in [0,1]$$

$$\begin{aligned}(\mathcal{Q}x)(t) &= \frac{1}{\alpha(\nu-\mu)\Gamma(q)}\int_0^1 (1-s)^{q-1} f(s,x(s)ds \\ &\quad - \frac{1}{(\nu-\mu)\Gamma(q)}\int_\mu^\nu \left(\int_0^s (s-m)^{q-1} f(m,x(m))dm\right) ds, \; t \in [0,1].\end{aligned}$$

For $x, y \in B_{\bar{r}}$, we find that

$$\|\mathcal{P}x+\mathcal{Q}y\| \le \frac{\|\mu\|}{\Gamma(q+1)}\left(1+\frac{\delta_1[q+1+|\alpha|(\nu^{q+1}-\mu^{q+1})]}{|\alpha|(q+1)}\right) \le \bar{r}.$$

Thus, $\mathcal{P}x+\mathcal{Q}y \in B_{\bar{r}}$. It follows from the assumption (A_1) together with (3.3) that $\mathcal{Q}$ is a contraction mapping. Continuity of f implies that the operator $\mathcal{P}$ is continuous. Also, $\mathcal{P}$ is uniformly bounded on $B_{\bar{r}}$ as

$$\|\mathcal{P}x\| \le \frac{\|\mu\|}{\Gamma(q+1)}.$$

Now we prove the compactness of the operator $\mathcal{P}$.

In view of (A_1), we define $\sup_{(t,x)\in[0,1]\times B_{\bar{r}}}|f(t,x)| = \overline{f}$, and consequently we have

$$\begin{aligned}&\|(\mathcal{P}x)(t_1)-(\mathcal{P}x)(t_2)\| \\ &= \sup_{(t,x)\in[0,1]\times B_{\bar{r}}}\left|\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]f(s,x(s))ds\right.\end{aligned}$$

$$+\int_{t_1}^{t_2}(t_2-s)^{q-1}f(s,x(s))ds\Big|$$
$$\leq \frac{\overline{f}}{\Gamma(q+1)}|2(t_2-t_1)^q+t_1^q-t_2^q|,$$

which is independent of x and tends to zero as $t_2-t_1\to 0$. Thus, $\mathcal{P}$ is equicontinuous. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{P}$ is compact on $B_{\overline{r}}$. Thus all the assumptions of Lemma 3 are satisfied. So the conclusion of Lemma 3 implies that the boundary value problem (1.1) has at least one solution on $[0,1]$. □

Our next result is based on Leray-Schauder Nonlinear Alternative.

Lemma 4 ([12], Nonlinear alternative for single valued maps)**.** . *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0\in U$. Suppose that $F:\overline{U}\to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*
(ii) *there is a $u\in\partial U$ (the boundary of U in C) and $\lambda\in(0,1)$ with $u=\lambda F(u)$.*

Theorem 3. *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Assume that:*

(A_3) *There exist a function $p\in L^1([0,1],\mathbb{R}^+)$, and $\psi:\mathbb{R}^+\to\mathbb{R}^+$ nondecreasing such that $|f(t,x)|\leq p(t)\psi(\|x\|)$, $\forall(t,x)\in[0,1]\times\mathbb{R}$.*

(A_4) *There exists a constant $M>0$ such that*

$$\frac{M}{\frac{\psi(M)}{\Gamma(q)}\left\{\left(1+\frac{\delta_1}{|\alpha|}\right)\int_0^1(1-s)^{q-1}p(s)\,ds+\delta_1\int_\mu^\nu\left(\int_0^s(s-m)^{q-1}p(s)\,ds\right)ds\right\}}>1,$$

where δ_1 is given by (3.2).

Then the boundary value problem (1.1) has at least one solution on $[0,1]$.

Proof. Consider the operator $F:\mathcal{C}\to\mathcal{C}$ given by (3.1).

We prove that the operator F is completely continuous. First we prove that Fx *maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$.* Indeed, it is enough to show that there exists a positive constant ℓ such that, for each $x\in B_r=\{x\in C([0,1],\mathbb{R}):\|x\|\leq r\}$, we have $\|Fx\|\leq\ell$. From (A_3) we have

$$|(Fx)(t)|$$
$$=\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}|f(s,x(s))|ds$$
$$+\frac{1}{|\alpha(\nu-\mu)|\Gamma(q)}\int_0^1(1-s)^{q-1}|f(s,x(s))|ds$$
$$+\frac{1}{|\nu-\mu|\Gamma(q)}\int_\mu^\nu\left(\int_0^s(s-m)^{q-1}|f(m,x(m))|dm\right)ds$$

$$\begin{aligned}
&\le \frac{\psi(\|x\|)}{\Gamma(q)}\int_0^t (t-s)^{q-1}p(s)\,ds + \frac{\psi(\|x\|)}{|\alpha(\nu-\mu)|\Gamma(q)}\int_0^1 (1-s)^{q-1}p(s)\,ds \\
&+ \frac{\psi(\|x\|)}{|\nu-\mu|\Gamma(q)}\int_\mu^\nu \left(\int_0^s (s-m)^{q-1}p(s)\,ds\right)ds \\
&\le \frac{\psi(\|x\|)}{\Gamma(q)}\left\{\left(1+\frac{\delta_1}{|\alpha|}\right)\int_0^1 (1-s)^{q-1}p(s)\,ds \right.\\
&\qquad \left. +\delta_1\int_\mu^\nu \left(\int_0^s (s-m)^{q-1}p(s)\,ds\right)ds\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|(Fx)\| \le \frac{\psi(r)}{\Gamma(q)}&\left\{\left(1+\frac{\delta_1}{|\alpha|}\right)\int_0^1 (1-s)^{q-1}p(s)\,ds \right.\\
&\left. +\delta_1\int_\mu^\nu \left(\int_0^s (s-m)^{q-1}p(s)\,ds\right)ds\right\} := \ell.
\end{aligned}$$

Next we show that Fx *maps bounded sets into equicontinuous sets of* $C([0,1],\mathbb{R})$. Let $t',t'' \in [0,1]$ with $t' < t''$ and $x \in B_r$, where B_r is a bounded set of $C([0,1],\mathbb{R})$. Then

$$\begin{aligned}
|(Fx)(t'')-(Fx)(t')| &= \left|\frac{1}{\Gamma(q)}\int_0^{t''}(t''-s)^{q-1}f(s,x(s))ds \right.\\
&\qquad \left. -\frac{1}{\Gamma(q)}\int_0^{t'}(t'-s)^{q-1}f(s,x(s))ds\right| \\
&\le \left|\frac{\psi(r)}{\Gamma(q)}\int_0^{t'}[(t''-s)^{q-1}-(t'-s)^{q-1}]p(s)ds\right| \\
&\quad + \left|\frac{\psi(r)}{\Gamma(q)}\int_{t'}^{t''}(t''-s)^{q-1}p(s)ds\right|.
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_r$ as $t''-t' \to 0$. Therefore it follows by the Arzelá-Ascoli theorem that $F : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous.

Now let $\lambda \in (0,1)$ and let $x = \lambda Fx$. Then for $x \in [0,1]$, using the previous computations in proving that Fx is bounded, we have

$$\begin{aligned}
|x(t)| &= |\lambda(Fx)(t)| \\
&\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,x(s))|ds \\
&+ \frac{1}{|\alpha(\nu-\mu)|\Gamma(q)}\int_0^1 (1-s)^{q-1}|f(s,x(s))|ds
\end{aligned}$$

$$+\frac{1}{|\nu-\mu|\Gamma(q)}\int_{\mu}^{\nu}\left(\int_0^s (s-m)^{q-1}|f(m,x(m))|\,dm\right)ds$$
$$\frac{\psi(\|x\|)}{\Gamma(q)}\left\{\left(1+\frac{\delta_1}{|\alpha|}\right)\int_0^1 (1-s)^{q-1}p(s)\,ds\right.$$
$$\left.+\delta_1\int_{\mu}^{\nu}\left(\int_0^s (s-m)^{q-1}p(s)\,ds\right)ds\right\}.$$

and consequently

$$\frac{\|x\|}{\frac{\psi(\|x\|)}{\Gamma(q)}\left\{\left(1+\frac{\delta_1}{|\alpha|}\right)\int_0^1 (1-s)^{q-1}p(s)\,ds+\delta_1\int_{\mu}^{\nu}\left(\int_0^s (s-m)^{q-1}p(s)\,ds\right)ds\right\}}\leq 1.$$

In view of (A_4), there exists M such that $\|x\|\neq M$. Let us set

$$U=\{x\in C([0,1],X):\|x\|<M\}.$$

Note that the operator $F:\overline{U}\to C([0,1],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x\in\partial U$ such that $x=\lambda F(x)$ for some $\lambda\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 4), we deduce that F has a fixed point $x\in\overline{U}$ which is a solution of the problem (1.1). This completes the proof. □

In the special case when $p(t)=1$ and $\psi(|x|)=k|x|+N$ we have the following corollary.

Corollary 1. *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$. Assume that:*

(A_5) *there exist constants $0\leq\kappa<\frac{1}{\Lambda}$, where Λ is given by (3.2) and $M>0$ such that*

$$|f(t,x)|\leq\kappa|x|+M,\quad \text{for all}\quad t\in[0,1],x\in C[0,1].$$

Then the boundary value problem (1.1) has at least one solution.

4. Examples

Example 1. Consider the following four-point integral fractional boundary value problem

$$\begin{cases} {}^cD^{1/2}x(t)=\dfrac{1}{(t+9)^2}\dfrac{|x|}{1+|x|}, & t\in[0,1],\\ x(0)+\displaystyle\int_{1/4}^{3/4}x(s)ds=x(1). \end{cases}\tag{4.1}$$

Here, $q = 1/2$, $\alpha = 1$, $\mu = 1/4$, $\nu = 3/4$, and $f(t,x) = \dfrac{1}{(t+9)^2}\dfrac{|x|}{1+|x|}$. As $|f(t,x) - f(t,y)| \leq \dfrac{1}{81}|x-y|$, therefore, (A_1) is satisfied with $L = \dfrac{1}{81}$. Further,

$$L\Lambda = \frac{L}{\Gamma(q+1)}\left(1 + \frac{\delta_1[q+1+|\alpha|(\nu^{q+1}-\mu^{q+1})]}{|\alpha|(q+1)}\right) = \frac{17+3\sqrt{3}}{729\sqrt{\pi}} < 1.$$

Thus, by the conclusion of Theorem 1, the boundary value problem (4.1) has a unique solution on $[0,1]$.

Example 2. We consider the following boundary value problem

$$\begin{cases} {}^cD^{1/2}x(t) = \dfrac{1}{(16\pi)}\sin(2\pi x) + \dfrac{|x|}{1+|x|}, \quad t \in [0,1], \\ x(0) + \displaystyle\int_{1/4}^{3/4} x(s)ds = x(1). \end{cases} \tag{4.2}$$

Here, $q = 1/2$, $\alpha = 1$, $\mu = 1/4$, $\nu = 3/4$, and

$$\Big|f(t,x)\Big| = \left|\frac{1}{(16\pi)}\sin(2\pi x) + \frac{|x|}{1+|x|}\right| \leq \frac{1}{8}|x| + 1.$$

Clearly $M = 1$ and

$$\kappa = \frac{1}{8} < \frac{1}{\Lambda} = \frac{9\sqrt{\pi}}{17+3\sqrt{3}}.$$

Thus, all the conditions of Corollary 1 are satisfied and consequently the problem (4.2) has at least one solution.

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SHARP BOUNDS FOR THE JENSEN DIVERGENCE WITH APPLICATIONS

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Abstract. In this paper we provide sharp bounds for the Jensen divergence generated by different classes of functions including functions of bounded variation, absolutely continuous, Lipschitz continuous, convex functions and differentiable functions whose derivatives enjoy various properties as mentioned above. The bounds are expressed in terms of known and simpler divergence measures that are of importance in various applications such as the analysis of diversity as between and within populations and to cluster analysis.

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1. INTRODUCTION

For a function Φ defined on an interval I of the real line $\mathbb{R}$, following Burbea and Rao [2], we consider the *Jensen divergence* (or simply $\mathcal{J}$*-divergence*) between x and y from I^n (where $n \geq 1$) defined by the quantity

$$\mathcal{J}_{n,\Phi}(x,y) := \sum_{i=1}^{n} \left\{ \frac{1}{2} [\Phi(x_i) + \Phi(y_i)] - \Phi\left(\frac{x_i + y_i}{2}\right) \right\}, \quad (x,y) \in I^n \times I^n.$$

For a *probability distribution* $p_1, \dots, p_k \geq 0$ $(k \geq 2)$ with $\sum_{j=1}^{k} p_j = 1$ the authors of [2] also considered the *generalized mutual information measure* defined by

$$\mathcal{J}_{n,\Phi}^{p}\left(y^1, \dots, y^k\right) := \sum_{i=1}^{n} \left\{ \sum_{j=1}^{k} p_j \Phi\left(y_i^j\right) - \Phi\left(\sum_{j=1}^{k} p_j y_i^j\right) \right\}$$

where $\left(y^1, \dots, y^k\right) \in I^n \times \dots \times I^n$.

For the convex function $\Phi : [0,\infty) \to \mathbb{R}$, $\Phi(t) := t \log t$ with the usual convention that $0 \log 0 = 0$, $\mathcal{J}_{n,\Phi}^{p}$ which is nonnegative, is the same as the *mutual information* (*trans-information*) $\mathcal{J}_n^p$ defined in information theory as a measure of information on a k-input channel for input distribution $p = (p_1, \dots, p_k)$.

For a discussion on the properties of $\mathcal{J}_n^p$ see Gallager [3, p. 16] and Aczél and Daróczy [1, 196-199].

In biological work, $\mathcal{J}_n^p$ is defined to be the information radius on the probability distributions associated with $y^1,...,y^k$ (see [12]). Applications of this concept to cluster analysis are discussed in [12] and [5]. For applications of $\mathcal{J}_n^p$ in the analysis of diversity as between and within populations, see [6] and [10].

Define

$$S_n := \left\{ (x_1,...,x_n) \in I_0^n, \sum_{i=1}^n x_i = 1 \right\}, I_0 := (0,1).$$

We denote by

$$\overline{S}_n := \left\{ (x_1,...,x_n) \in \overline{I}_0^n, \sum_{i=1}^n x_i = 1 \right\}, \overline{I}_0 := [0,1]$$

the closure of S_n.

Utilizing the family of functions

$$\Phi_\alpha(t) := \begin{cases} (\alpha-1)^{-1}(t^\alpha - t), & \alpha \neq 1, \\ t\log t & \alpha = 1, \end{cases}$$

by Havrda and Charvát in [4] to introduce their *entropies of degree* α we mention, as examples, the following family of Jensen divergences considered in [2]

$$\mathcal{J}_{n,\alpha}(x,y) := \begin{cases} (\alpha-1)^{-1}\sum_{i=1}^n \left[\frac{1}{2}\left(x_i^\alpha + y_i^\alpha\right) - \left(\frac{x_i+y_i}{2}\right)^\alpha\right], & \alpha \neq 1 \\ \log\left[\frac{\prod_{i=1}^n \left(x_i^{x_i} y_i^{y_i}\right)^{1/2}}{\prod_{i=1}^n \left(\frac{x_i+y_i}{2}\right)^{\frac{x_i+y_i}{2}}}\right], & \alpha = 1 \end{cases} \tag{1.1}$$

that can be extended on $\overline{S}_n \times \overline{S}_n$ with the usual convention that $0\log 0 = 0$.

The divergence $\mathcal{J}_{n,1}$, written in its equivalent form as

$$\mathcal{J}_{n,1}(x,y) := \frac{1}{2}\sum_{i=1}^n \left\{ x_i \log x_i + y_i \log y_i - (x_i+y_i)\log\left(\frac{x_i+y_i}{2}\right)\right\},$$

is also known in the literature as the *Jensen-Shannon divergence*. Its important applications in various fields of Mathematics and Statistics can be found, for instance in [8, 9, 11, 13] and the references therein.

The convexity of divergence measures is an attractive feature. The following result concerning this property holds:

Theorem 1 (Burbea-Rao, 1982, [2])**.** *Let Φ be a C^2 function on the interval of real numbers I. Then $\mathcal{J}_{n,\Phi}$ is convex (concave) on $I^n \times I^n$ if and only if Φ is convex (concave) and $\frac{1}{\Phi''}$ is concave (convex) on I. Further, in this case $\mathcal{J}_{n,\Phi}^p$ is also convex (concave) on I^{nk} for any given probability distribution p.*

In this paper we endeavour to provide sharp upper and lower bounds for the Jensen divergence for various classes of functions Φ, including functions of bounded variation, absolutely continuous functions, Lipschitzian continuous functions, convex functions and differentiable functions whose derivative belong to the above mentioned classes. The bounds will be expressed in terms of known and simpler divergence measures that have been employed in various applications as mentioned above.

2. Some General Results

The following result may be stated

Proposition 1. *If $\Phi : [a,b] \to \mathbb{R}$ is a bounded function with $-\infty < m \le \Phi(t) \le M < \infty$ for any $t \in [a,b]$, then*

$$\left|\mathcal{J}_{n,\Phi}(x,y)\right| \le n(M-m). \tag{2.1}$$

For a fixed n, the multiplicative constant 1 *in front of $M-m$ cannot be replaced by a smaller quantity.*

Proof. For the sake of completeness, we present a short proof.

Since Φ is bounded, we have $m \le \Phi(x) \le M$, $m \le \Phi(y) \le M$ and $-M \le -\Phi\left(\frac{x+y}{2}\right) \le -m$, which gives, by addition that

$$-(M-m) \le \frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right) \le M-m,$$

for each $x, y \in [a,b]$, i.e.,

$$\left|\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right| \le M-m \tag{2.2}$$

which implies the desired inequality (2.1).

Let us prove the sharpness of the constant for the case $n=1$.

If $\Phi : [a,b] \to \mathbb{R}$, $\Phi(t) = \left|t - \frac{a+b}{2}\right|$, then $\Phi(a) = \Phi(b) = \frac{b-a}{2}$, $\Phi\left(\frac{a+b}{2}\right) = 0$, $M = \frac{b-a}{2}$ and $m = 0$ and the inequality (2.2) becomes an equality with both terms equal to $\frac{b-a}{2}$. □

Proposition 2. *Let $\Phi : [a,b] \to \mathbb{R}$ be a function of bounded variation on the compact interval $[a,b]$ of real numbers $\mathbb{R}$. Then for any $x = (x_1, ..., x_n)$, $y =$*

$(y_1,...,y_n) \in [a,b]^n$ *we have the inequality*

$$\left|\mathcal{J}_{n,\Phi}(x,y)\right| \le \frac{1}{2}\sum_{i=1}^{n}\left|\bigvee_{x_i}^{y_i}(\Phi)\right| \le \frac{1}{2}n\bigvee_{a}^{b}(\Phi) \tag{2.3}$$

where $\bigvee_a^b(\Phi)$ *denotes the total variation of* Φ *on* $[a,b]$.

The constant $\frac{1}{2}$ *is best possible in both inequalities from (2.3).*

Proof. It suffices to prove the inequality for $n = 1$.

Assume that $x < y$. Then

$$\begin{aligned}&\left|\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right|\\&\le\frac{1}{2}\left[\left|\Phi\left(\frac{x+y}{2}\right)-\Phi(x)\right|+\left|\Phi(y)-\Phi\left(\frac{x+y}{2}\right)\right|\right]\le\frac{1}{2}\bigvee_{x}^{y}(\Phi).\end{aligned}$$

By symmetry reasons, we deduce a similar inequality when $y < x$. Therefore, for any $x, y \in I$ we have

$$\left|\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right| \le \frac{1}{2}\left|\bigvee_{x}^{y}(\Phi)\right| \tag{2.4}$$

which implies the desired inequality (2.3).

The last part of (2.3) is obvious since $\left|\bigvee_{x_i}^{y_i}(\Phi)\right| \le \bigvee_a^b(\Phi)$ for any selection of vectors $x = (x_1,...,x_n)$, $y = (y_1,...,y_n) \in [a,b]^n$.

Now, if we take the function $\Phi : [0,1] \to \mathbb{R}$, $\Phi(x) = sgn\left(x-\frac{1}{2}\right)$, then this function is of bounded variation on $[0,1]$ and if we take $x = 0$ and $x = 1$ then we have $\bigvee_0^1(\Phi) = 2$ and we get in both sides of (2.4) the same quantity 1, which proves the sharpness of the constant $\frac{1}{2}$ in both inequalities (2.3). □

Corollary 1. *Let* $\Phi : [a,b] \to \mathbb{R}$ *be a L-Lipschitzian function on* $[a,b]$, *i.e. we recall that* Φ *satisfies the condition*

$$|\Phi(t)-\Phi(s)| \le L|t-s| \text{ for any } t,s \in [a,b]$$

where $L > 0$ *is given.*

Then for any $x = (x_1,...,x_n)$, $y = (y_1,...,y_n) \in [a,b]^n$ *we have the inequality*

$$\left|\mathcal{J}_{n,\Phi}(x,y)\right| \le \frac{1}{2}L\sum_{i=1}^{n}|x_i-y_i| = L\delta(x,y), \tag{2.5}$$

where $\delta(x,y) := \frac{1}{2}\sum_{i=1}^{n}|x_i-y_i|$ *is known in the literature as the statistical distance between* x *and* y.

The constant $\frac{1}{2}$ *is best possible in (2.5).*

Proof. It is well known that, any L-Lipschitzian function on $[a,b]$ is of bounded variation on $[a,b]$ and $\bigvee_a^b(\Phi) \leq L(b-a)$. Applying this property to the inequality (2.3), we deduce the desired result (2.5).

We prove the sharpness esof the constant $\frac{1}{2}$ for the case $n=1$, i.e.,

$$\left|\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right| \leq \frac{1}{2}L|x-y| \tag{2.6}$$

is a sharp inequality provided $\Phi:[a,b]\to\mathbb{R}$ is a L-Lipschitzian function on $[a,b]$.

If we consider the function $\Phi:[a,b]\to[0,\infty)$ defined by $\Phi(t)=\left|t-\frac{a+b}{2}\right|$, then Φ is Lipschitzian with the constant $L=1$ and if we use this function and $x=a$, $y=b$ in (2.6) we obtain the same quantity $\frac{1}{2}(b-a)$ in both sides.

This proves the desired result. □

The following lemma may be stated.

Lemma 1. *Let $u:[a,b]\to\mathbb{R}$ and $\gamma,\Gamma\in\mathbb{R}$ with $\Gamma>\gamma$. The following statements are equivalent:*

(i) *The function $u-\frac{\gamma+\Gamma}{2}\cdot e$, where $e(t)=t$, $t\in[a,b]$, is $\frac{1}{2}(\Gamma-\gamma)$-Lipschitzian;*

(ii) *We have the inequality:*

$$\gamma \leq \frac{u(t)-u(s)}{t-s} \leq \Gamma \quad \textit{for each} \quad t,s\in[a,b] \quad \textit{with } t\neq s;$$

(iii) *We have the inequality:*

$$\gamma(t-s) \leq u(t)-u(s) \leq \Gamma(t-s) \quad \textit{for each} \quad t,s\in[a,b] \quad \textit{with } t>s.$$

Following [7], we can introduce the concept:

Definition 1. A function $u:[a,b]\to\mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) is said to be (γ,Γ)-Lipschitzian on $[a,b]$.

Notice that in [6], the definition was introduced on utilizing the statement (iii) and only the equivalence (i) $\Leftrightarrow$ (iii) was considered.

Utilizing *Lagrange's mean value theorem*, we can state the following result that provides practical examples of (γ,Γ)-Lipschitzian functions.

Proposition 3. *Let $u:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If*

$$-\infty<\gamma:=\inf_{t\in(a,b)}u'(t), \qquad \sup_{t\in(a,b)}u'(t)=:\Gamma<\infty$$

then u is (γ,Γ)-Lipschitzian on $[a,b]$.

We can improve the inequality (2.6) as follows:

Corollary 2. *Let $\Phi : [a,b] \to \mathbb{R}$ be (γ,Γ)-Lipschitzian on $[a,b]$ with $\gamma < \Gamma$. Then for any $x = (x_1,...,x_n)$, $y = (y_1,...,y_n) \in [a,b]^n$ we have the inequality*

$$\left|\mathcal{J}_{n,\Phi}(x,y)\right| \le \frac{1}{4}(\Gamma-\gamma)\sum_{i=1}^{n}|x_i-y_i| = \frac{1}{2}(\Gamma-\gamma)\delta(x,y). \tag{2.7}$$

The constant $\frac{1}{4}$ is best possible in (2.7).

In particular, if $\Phi : [a,b] \to \mathbb{R}$ is differentiable on (a,b) and the derivative Φ' satisfies the inequality $-\infty < \gamma \le \Phi'(t) \le \Gamma < \infty$ for each $t \in (a,b)$, then (2.7) is valid as well.

Proof. Follows by Corollary 1 on taking into account that

$$\mathcal{J}_{n,\Phi-\frac{\gamma+\Gamma}{2}\cdot e}(x,y) = \mathcal{J}_{n,\Phi}(x,y)$$

and the details are omitted. □

We recall that a function $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ if and only if it is differentiable almost everywhere in $[a,b]$. The derivative f' is Lebesgue integrable on this interval and $f(y) - f(x) = \int_x^y f'(t)\,dt$ for any $x,y \in [a,b]$.

We use the following notations for Lebesgue integrable functions:

$$\|g\|_{[x,y],p} := \left|\int_x^y |g(s)|^p\,ds\right|^{1/p} \text{ if } 1 \le p < \infty,\ x,y \in [a,b] \text{ and } g \in L_p[a,b];$$

and for $g \in L_\infty[a,b]$ we denote

$$\|g\|_{[x,y],\infty} := \begin{cases} \operatorname{ess\,sup}_{s\in[x,y]}|g(s)| & \text{if } x < y \\ \\ \operatorname{ess\,sup}_{s\in[y,x]}|g(s)| & \text{if } y < x. \end{cases}$$

Theorem 2. *Assume that $\Phi : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$. Then we have the bounds*

$$\left|\mathcal{J}_{n,\Phi}(x,y)\right| \tag{2.8}$$
$$\le \frac{1}{2}\times\begin{cases} \sum_{i=1}^{n}|y_i-x_i|\,\|\Phi'\|_{[x_i,y_i],\infty} & \text{if } \Phi' \in L_\infty[a,b] \\ \\ \sum_{i=1}^{n}|y_i-x_i|^{\frac{p-1}{p}}\,\|\Phi'\|_{[x_i,y_i],p} & \text{if } \Phi' \in L_p[a,b],\ p>1 \\ \\ \sum_{i=1}^{n}\|\Phi'\|_{[x_i,y_i],1} \end{cases}$$
$$\le \frac{1}{2}\times\begin{cases} \|\Phi'\|_{[a,b],\infty}\sum_{i=1}^{n}|y_i-x_i| & \text{if } \Phi' \in L_\infty[a,b] \\ \\ \|\Phi'\|_{[a,b],p}\sum_{i=1}^{n}|y_i-x_i|^{\frac{p-1}{p}} & \text{if } \Phi' \in L_p[a,b],\ p>1 \\ \\ n\,\|\Phi'\|_{[a,b],1} \end{cases}$$

for any $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in [a,b]^n$.

Moreover, if the modulus of the derivative is convex, then we have the inequality

$$\begin{aligned} \left| \mathcal{J}_{n,\Phi}(x,y) \right| &\leq \frac{1}{4} \sum_{i=1}^{n} |y_i - x_i| \left[\left| \Phi'\left(\frac{x_i + y_i}{2} \right) \right| + \frac{|\Phi'(x_i)| + |\Phi'(y_i)|}{2} \right] \\ &\leq \frac{1}{4} \sum_{i=1}^{n} |y_i - x_i| \left[\left| \Phi'(x_i) \right| + \left| \Phi'(y_i) \right| \right] \\ &\left(\leq \left\| \Phi' \right\|_{[a,b],\infty} \delta(x,y) \right), \end{aligned} \tag{2.9}$$

for any $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in [a,b]^n$.

The constant $\frac{1}{4}$ *is best possible in both inequalities.*

Proof. Assume that $x < y$. Then we have

$$\begin{aligned} \left| \frac{\Phi(x) + \Phi(y)}{2} - \Phi\left(\frac{x+y}{2} \right) \right| &= \left| \frac{1}{2} \int_x^{\frac{x+y}{2}} \Phi'(s)\, ds + \frac{1}{2} \int_{\frac{x+y}{2}}^{y} \Phi'(s)\, ds \right| \\ &\leq \frac{1}{2} \left| \int_x^{\frac{x+y}{2}} \Phi'(s)\, ds \right| + \frac{1}{2} \left| \int_{\frac{x+y}{2}}^{y} \Phi'(s)\, ds \right| \\ &\leq \frac{1}{2} \int_x^{\frac{x+y}{2}} \left| \Phi'(s) \right| ds + \frac{1}{2} \int_{\frac{x+y}{2}}^{y} \left| \Phi'(s) \right| ds \\ &= \frac{1}{2} \int_x^{y} \left| \Phi'(s) \right| ds. \end{aligned}$$

If $y < x$, then we also get

$$\left| \frac{\Phi(x) + \Phi(y)}{2} - \Phi\left(\frac{x+y}{2} \right) \right| \leq \frac{1}{2} \int_x^{y} \left| \Phi'(s) \right| ds.$$

Therefore, for any $x, y \in [a,b]$ we have the inequality

$$\left| \frac{\Phi(x) + \Phi(y)}{2} - \Phi\left(\frac{x+y}{2} \right) \right| \leq \frac{1}{2} \left| \int_x^{y} \left| \Phi'(s) \right| ds \right|. \tag{2.10}$$

Further, on making use of Hölder's integral inequality, we also have that

$$\left| \int_x^{y} \left| \Phi'(s) \right| ds \right| \leq \begin{cases} |y - x| \, \|\Phi'\|_{[x,y],\infty} & \text{if } \Phi' \in L_\infty[a,b] \\ \\ |y - x|^{\frac{p-1}{p}} \|\Phi'\|_{[x,y],p} & \text{if } \Phi' \in L_p[a,b],\, p > 1. \end{cases} \tag{2.11}$$

The above two inequalities (2.10) and (2.11) provide the result

$$\left| \frac{\Phi(x) + \Phi(y)}{2} - \Phi\left(\frac{x+y}{2} \right) \right| \tag{2.12}$$

$$\leq \frac{1}{2}\times\begin{cases} |y-x|\,\|\Phi'\|_{[x,y],\infty} & \text{if } \Phi'\in L_{\infty}[a,b] \\ |y-x|^{\frac{p-1}{p}}\,\|\Phi'\|_{[x,y],p} & \text{if } \Phi'\in L_p[a,b],\, p>1 \\ \|\Phi'\|_{[x,y],p} \end{cases} \tag{2.13}$$

for any $x,y\in[a,b]$, which imply the first inequality in (2.8). The second inequality is obvious.

In order to prove the inequality (2.9) we use the following refinement of the celebrated Hermite-Hadamard inequality

$$\begin{aligned}\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} f(t)\,dt &\leq \frac{1}{2}\left[f\left(\frac{\alpha+\beta}{2}\right)+\frac{f(\alpha)+f(\beta)}{2}\right]\\ &\leq \frac{f(\alpha)+f(\beta)}{2}\end{aligned}$$

that works for a convex function $f:[a,b]\to\mathbb{R}$ and any $\alpha,\beta\in[a,b]$ with $\alpha\neq\beta$.

Applying this inequality for the convex function $|\Phi'|$ we have

$$\begin{aligned}\left|\int_x^y \left|\Phi'(s)\right| ds\right| &\leq \frac{1}{2}\left[\left|\Phi'\left(\frac{x+y}{2}\right)\right|+\frac{|\Phi'(x)|+|\Phi'(y)|}{2}\right]|y-x|\\ &\leq \frac{1}{2}\left[\left|\Phi'(x)\right|+\left|\Phi'(y)\right|\right]|y-x|\end{aligned}$$

which together with (2.10) produces the required result (2.9).

Let us prove the sharpness of the constant $\frac{1}{4}$ for $n=1$, meaning that we need to prove that the inequalities

$$\begin{aligned}&\left|\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right| \\ &\leq \frac{1}{4}\left[\left|\Phi'\left(\frac{x+y}{2}\right)\right|+\frac{|\Phi'(x)|+|\Phi'(y)|}{2}\right]|y-x| \\ &\leq \frac{1}{4}\left[\left|\Phi'(x)\right|+\left|\Phi'(y)\right|\right]|y-x|\end{aligned} \tag{2.14}$$

reduce to equality for some function f and some numbers $x,y\in[a,b]$.

Consider $\Phi:[a,b]\to\mathbb{R}$, $\Phi(t)=\frac{1}{2}\left(t-\frac{a+b}{2}\right)^2$. Then $\Phi'(t)=t-\frac{a+b}{2}$ and, obviously, $|\Phi'|$ is a convex function.

If we replace this function in (2.14) and choose $x=a$ and $y=b$, then we get in all sides the same quantity $\frac{1}{4}(b-a)^2$. □

3. Bounds for Convex Functions

The case of convex functions is as follows:

Theorem 3. *Let $\Phi : I \to \mathbb{R}$ be a convex function on the interval I of real numbers $\mathbb{R}$. Then for any $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in \mathring{I}^n$, where $\mathring{I}$ denotes the interior of I, we have the inequalities*

$$\begin{aligned} &\frac{1}{4}\sum_{i=1}^{n}(x_i - y_i)\left(\Phi'_{+(-)}(x_i) - \Phi'_{+(-)}(y_i)\right) \\ &\geq \mathcal{J}_{n,\Phi}(x, y) \\ &\geq \frac{1}{4}\sum_{i=1}^{n}|x_i - y_i|\left(\Phi'_{+}\left(\frac{x_i + y_i}{2}\right) - \Phi'_{-}\left(\frac{x_i + y_i}{2}\right)\right) \geq 0 \end{aligned} \tag{3.1}$$

where $\Phi'_{+(-)}(z)$ denote the right (left) derivative of Φ in z.

The constant $\frac{1}{4}$ is best possible in both inequalities.

Proof. It suffices to prove the inequality for $n = 1$.

It is well know that, if $\Phi : I \to \mathbb{R}$ is a convex function on the interval I, then for any $x, y \in \mathring{I}$ we have the *gradient inequality*

$$\mu(y)(y - x) \geq \Phi(y) - \Phi(x) \geq \lambda(x)(y - x) \tag{3.2}$$

where $\mu(y) \in \left[\Phi'_{-}(y), \Phi'_{+}(y)\right]$ and $\lambda(x) \in \left[\Phi'_{-}(x), \Phi'_{+}(x)\right]$.

Now, making use of the second inequality in (3.2) and assuming that $x > y$ we can write that

$$\Phi(x) - \Phi\left(\frac{x + y}{2}\right) \geq \frac{1}{2}\Phi'_{+}\left(\frac{x + y}{2}\right)(x - y)$$

and

$$\Phi(y) - \Phi\left(\frac{x + y}{2}\right) \geq -\frac{1}{2}\Phi'_{-}\left(\frac{x + y}{2}\right)(x - y).$$

If we multiply both inequalities with $\frac{1}{2}$ and add, we obtain

$$\frac{1}{2}[\Phi(x) + \Phi(y)] - \Phi\left(\frac{x + y}{2}\right) \geq \frac{1}{4}(x - y)\left[\Phi'_{+}\left(\frac{x + y}{2}\right) - \Phi'_{-}\left(\frac{x + y}{2}\right)\right].$$

By symmetry reasons, if $y > x$ we also have

$$\frac{1}{2}[\Phi(x) + \Phi(y)] - \Phi\left(\frac{x + y}{2}\right) \geq \frac{1}{4}(y - x)\left[\Phi'_{+}\left(\frac{x + y}{2}\right) - \Phi'_{-}\left(\frac{x + y}{2}\right)\right].$$

Therefore, for any $x, y \in \mathring{I}$ we have

$$\begin{aligned} \frac{1}{2}[\Phi(x) + \Phi(y)] - \Phi\left(\frac{x + y}{2}\right) &\geq \frac{1}{4}|y - x|\left[\Phi'_{+}\left(\frac{x + y}{2}\right) - \Phi'_{-}\left(\frac{x + y}{2}\right)\right] \\ &\geq 0, \end{aligned} \tag{3.3}$$

and the second inequality in (3.1) is proven.

In order to prove the sharpness of the constant $\frac{1}{4}$ in (3.3), we consider the function $\Phi : [a,b] \to \mathbb{R}$, $\Phi(x) = \left| x - \frac{a+b}{2} \right|$. This function is convex and

$$\Phi'_+\left(\frac{a+b}{2}\right) = 1 \text{ while } \Phi'_-\left(\frac{a+b}{2}\right) = -1.$$

If we write the inequality (3.3) for this convex function and $x = a, y = b$ we obtain in both sides of this inequality the same quantity $\frac{1}{2}(b-a)$ which proves the desired result.

Now, making use of the first inequality in (3.2) and assuming that $x, y \in \mathring{I}$ we can write that

$$\frac{1}{2}\Phi'_{+(-)}(x)(x-y) \geq \Phi(x) - \Phi\left(\frac{x+y}{2}\right)$$

and

$$-\frac{1}{2}\Phi'_{+(-)}(y)(x-y) \geq \Phi(y) - \Phi\left(\frac{x+y}{2}\right).$$

If we multiply both inequalities with $\frac{1}{2}$ and add, we obtain

$$\frac{1}{4}\left[\Phi'_{+(-)}(x) - \Phi'_{+(-)}(y)\right](x-y) \geq \frac{1}{2}\left[\Phi(x) + \Phi(y)\right] - \Phi\left(\frac{x+y}{2}\right) \tag{3.4}$$

for any $x, y \in \mathring{I}$ and the first inequality in (3.1) is also proven.

If we consider the same function $\Phi : [a,b] \to \mathbb{R}$, $\Phi(x) = \left| x - \frac{a+b}{2} \right|$, then we observe that $\Phi'_-(b) = 1, \Phi'_+(a) = -1$ and if we write the inequality (3.4) for this function and for $x = b, y = a$ we obtain in both sides of this inequality the same quantity $\frac{1}{2}(b-a)$. □

Corollary 3. *Let $\Phi : I \to \mathbb{R}$ be a differentiable convex function on the interval I of real numbers $\mathbb{R}$. Then for any $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in \mathring{I}^n$, we have the inequalities*

$$\frac{1}{4}\sum_{i=1}^{n}(x_i - y_i)\left(\Phi'(x_i) - \Phi'(y_i)\right) \geq \mathcal{J}_{n,\Phi}(x,y) \geq 0 \tag{3.5}$$

Remark 1. We observe that if Φ' is $r - H$-Hölder continuous on I, i.e., there exist the constants $r \in (0,1]$ and $H > 0$ such that $|\Phi'(s) - \Phi'(t)| \leq H|s-t|^r$ for any $s, t \in I$, then by (3.5) we have the upper bound

$$\frac{1}{4}H\sum_{i=1}^{n}|x_i - y_i|^{r+1} \geq \mathcal{J}_{n,\Phi}(x,y) \geq 0$$

and, in particular, for $r = 1$, we have the bound

$$H\mathcal{J}_{n,2}(x,y) \geq \mathcal{J}_{n,\Phi}(x,y) \geq 0$$

where, by (1.1),

$$\mathcal{J}_{n,2}(x,y) = \sum_{i=1}^{n}\left[\frac{1}{2}\left(x_i^2+y_i^2\right)-\left(\frac{x_i+y_i}{2}\right)^2\right] = \frac{1}{4}\sum_{i=1}^{n}(x_i-y_i)^2 .$$

For two vectors $x = (x_1,...,x_n)$, $y = (y_1,...,y_n) \in I^n$ we say that $x \le y$ if for all $i \in \{1,...,n\}$ we have that $x_i \le y_i$. For $x \le y$, we call the set $[x,y] := \{g = (g_1,...,g_n) \mid \text{with } x_i \le g_i \le y_i \text{ for all } i \in \{1,...,n\}\}$ the generalized interval generated by x and y.

Theorem 4. *Let $\Phi : I \to \mathbb{R}$ be a convex function on the interval I of real numbers $\mathbb{R}$.*

(i) *If $x, y, z \in I^n$ are so that $x \le y \le z$, then*

$$0 \le \mathcal{J}_{n,\Phi}(x,y) + \mathcal{J}_{n,\Phi}(y,z) \le \mathcal{J}_{n,\Phi}(x,z), \tag{3.6}$$

i.e., $\mathcal{J}_{n,\Phi}$ is superadditive as a functional of the generalized interval;

(ii) *If $x, y, z, u \in I^n$ are so that $x \le y \le z \le u$, then*

$$0 \le \mathcal{J}_{n,\Phi}(y,z) \le \mathcal{J}_{n,\Phi}(x,u), \tag{3.7}$$

i.e., $\mathcal{J}_{n,\Phi}$ is monotonic nondecreasing as a functional of the generalized interval.

Proof. (i) It suffices to prove it for $n = 1$.

So, assume that $x, y, z \in I$ with $x \le y \le z$. We claim that, if $\Phi : I \to \mathbb{R}$ is a convex function on the interval I, then

$$\frac{1}{2}[\Phi(x)+\Phi(y)] - \Phi\left(\frac{x+y}{2}\right) + \frac{1}{2}[\Phi(y)+\Phi(z)] - \Phi\left(\frac{y+z}{2}\right)$$
$$\le \frac{1}{2}[\Phi(x)+\Phi(z)] - \Phi\left(\frac{x+z}{2}\right).$$

Observe that, this inequality actually reduces to

$$\Phi(y) - \Phi\left(\frac{x+y}{2}\right) \le \Phi\left(\frac{y+z}{2}\right) - \Phi\left(\frac{x+z}{2}\right). \tag{3.8}$$

If either $x = y$ or $y = z$, then (3.8) reduces to an equality, so we can suppose that $x < y < z$.

Now, for a convex function $\varphi : I \subset \mathbb{R} \to \mathbb{R}$, where I is an interval, and any real numbers t_1, t_2, s_1 and s_2 from I and with the properties that $t_1 \le s_1$ and $t_2 \le s_2$ we have that

$$\frac{\varphi(t_1)-\varphi(t_2)}{t_1-t_2} \le \frac{\varphi(s_1)-\varphi(s_2)}{s_1-s_2}. \tag{3.9}$$

Indeed, since φ is convex on I then for any $a \in I$ the function $\psi : I \setminus \{a\} \to \mathbb{R}$

$$\psi(t) := \frac{\varphi(t)-\varphi(a)}{t-a}$$

is monotonic nondecreasing where it is defined. Utilizing this property repeatedly we have

$$\begin{aligned}\frac{\varphi(t_1)-\varphi(t_2)}{t_1-t_2} &\leq \frac{\varphi(s_1)-\varphi(t_2)}{s_1-t_2} = \frac{\varphi(t_2)-\varphi(s_1)}{t_2-s_1} \\ &\leq \frac{\varphi(s_2)-\varphi(s_1)}{s_2-s_1} = \frac{\varphi(s_1)-\varphi(s_2)}{s_1-s_2}\end{aligned}$$

which proves the inequality (3.9).

Now, if we choose

$$t_1 = y, t_2 = \frac{x+y}{2}, s_1 = \frac{y+z}{2} \text{ and } s_2 = \frac{x+z}{2}$$

then we have $t_1 < s_1$ and $t_2 < s_2$ and by (3.9) we can write that

$$\frac{\Phi(y)-\Phi\left(\frac{x+y}{2}\right)}{y-\frac{x+y}{2}} \leq \frac{\Phi\left(\frac{y+z}{2}\right)-\Phi\left(\frac{x+z}{2}\right)}{\frac{y+z}{2}-\frac{x+z}{2}}$$

which is equivalent to the desired result (3.8).

(ii). We have from (i) that

$$\mathcal{J}_{n,\Phi}(x,y)+\mathcal{J}_{n,\Phi}(y,z)+\mathcal{J}_{n,\Phi}(z,u) \leq \mathcal{J}_{n,\Phi}(x,u)$$

and since $\mathcal{J}_{n,\Phi}(x,y), \mathcal{J}_{n,\Phi}(y,z) \geq 0$ we deduce the desired result (3.7). □

Remark 2. With the assumptions of Theorem 4, we have the bound

$$\sup_{x\leq y\leq z\leq u} \mathcal{J}_{n,\Phi}(y,z) = \mathcal{J}_{n,\Phi}(x,u).$$

For a constant $c \in \mathbb{R}$ we denote the vector having all components equal to this constant by $\overline{c}$, i.e., $\overline{c} = (c,...,c) \in \mathbb{R}^n$. With this notation we have:

Corollary 4. *Let $\Phi : I \to \mathbb{R}$ be a convex function on the interval I of real numbers $\mathbb{R}$.*

(i) *If $m, M \in I$ and $x \in I^n$ are such that $m \leq x_i \leq M$ for all $i \in \{1,...,n\}$, then*

$$0 \leq \mathcal{J}_{n,\Phi}(\overline{m},x)+\mathcal{J}_{n,\Phi}\left(x,\overline{M}\right) \leq n\left[\frac{1}{2}[\Phi(m)+\Phi(M)]-\Phi\left(\frac{m+M}{2}\right)\right];$$

(ii) *If $m, M \in I$ and $x, y \in I^n$ are such that $m \leq x_i \leq y_i \leq M$ for all $i \in \{1,...,n\}$, then*

$$0 \leq \mathcal{J}_{n,\Phi}(x,y) \leq n\left[\frac{1}{2}[\Phi(m)+\Phi(M)]-\Phi\left(\frac{m+M}{2}\right)\right].$$

4. Inequalities for Differentiable Functions

The following result holds:

Theorem 5. *Let $\Phi : [a,b] \to \mathbb{R}$ be a differentiable function on the interval $[a,b]$ of real numbers $\mathbb{R}$.*

(i) *If the derivative Φ' is of bounded variation on $[a,b]$, then*

$$\left|\mathcal{J}_{n,\Phi}(x,y)\right| \le \frac{1}{4}\sum_{i=1}^{n}|y_i - x_i|\left|\bigvee_{x_i}^{y_i}(\Phi')\right| \le \frac{1}{4}\bigvee_{a}^{b}(\Phi')\sum_{i=1}^{n}|y_i - x_i| \tag{4.1}$$
$$= \frac{1}{2}\bigvee_{a}^{b}(\Phi')\,\delta(x,y)$$

for any $x = (x_1,...,x_n)$, $y = (y_1,...,y_n) \in [a,b]^n$.

The constant $\frac{1}{4}$ is best possible in both inequalities (4.1).

(ii) *If the derivative Φ' is K-Lipschitzian on $[a,b]$ with the constant $K > 0$, then*

$$\left|\mathcal{J}_{n,\Phi}(x,y)\right| \le \frac{1}{8}K\sum_{i=1}^{n}(y_i - x_i)^2 = \frac{1}{2}K\mathcal{J}_{n,2}(x,y) \tag{4.2}$$

for any $x = (x_1,...,x_n)$, $y = (y_1,...,y_n) \in [a,b]^n$.

The constant $\frac{1}{8}$ is best possible in (4.2).

Proof. For $x, y \in [a,b]$ with $x < y$ we consider the kernel $K : [x,y] \to \mathbb{R}$ defined by

$$K_{x,y}(s) := \begin{cases} s - x & \text{if } x \le s \le \frac{x+y}{2} \\ \\ y - s & \text{if } \frac{x+y}{2} < s \le y. \end{cases}$$

Integrating by parts in the Riemann-Stieltjes integral we have that

$$\begin{aligned}\int_x^y K_{x,y}(s)\,d\left[\Phi'(s)\right] &= \int_x^{\frac{x+y}{2}}(s-x)\,d\left[\Phi'(s)\right] + \int_{\frac{x+y}{2}}^{y}(y-s)\,d\left[\Phi'(s)\right] \\ &= (s-x)\Phi'(s)\Big|_x^{\frac{x+y}{2}} - \int_x^{\frac{x+y}{2}}\Phi'(s)\,dt \\ &+ (y-s)\Phi'(s)\Big|_{\frac{x+y}{2}}^{y} + \int_{\frac{x+y}{2}}^{y}\Phi'(s)\,dt \\ &= \frac{y-x}{2}\Phi'\left(\frac{x+y}{2}\right) - \left[\Phi\left(\frac{x+y}{2}\right) - \Phi(x)\right] \\ &- \frac{y-x}{2}\Phi'\left(\frac{x+y}{2}\right) + \left[\Phi(y) - \Phi\left(\frac{x+y}{2}\right)\right]\end{aligned}$$

$$= 2\left[\frac{\Phi(x) + \Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right]$$

i.e., we have the following representation of interest

$$\frac{\Phi(x) + \Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right) = \frac{1}{2}\int_x^y K_{x,y}(s)\, d\left[\Phi'(s)\right].$$

If $y < x$, then we also get

$$\frac{\Phi(x) + \Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right) = \frac{1}{2}\int_y^x K_{y,x}(s)\, d\left[\Phi'(s)\right].$$

(i) It is well known that, if $p : [\alpha, \beta] \to \mathbb{R}$ is continuous and $v : [\alpha, \beta] \to \mathbb{R}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_\alpha^\beta p(s)\, dv(s)$ exists and

$$\left|\int_\alpha^\beta p(s)\, dv(s)\right| \le \sup_{s\in[\alpha,\beta]} |p(s)| \bigvee_\alpha^\beta (v).$$

If $x < y$, then on utilizing this property of the Riemann-Stieltjes integral we have successively

$$\begin{aligned}
&\left|\frac{\Phi(x) + \Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right| \\
&= \frac{1}{2}\left|\int_x^y K_{x,y}(s)\, d\left[\Phi'(s)\right]\right| \\
&= \frac{1}{2}\left|\int_x^{\frac{x+y}{2}} (s-x)\, d\left[\Phi'(s)\right] + \int_{\frac{x+y}{2}}^y (y-s)\, d\left[\Phi'(s)\right]\right| \\
&\le \frac{1}{2}\left|\int_x^{\frac{x+y}{2}} (s-x)\, d\left[\Phi'(s)\right]\right| + \frac{1}{2}\left|\int_{\frac{x+y}{2}}^y (y-s)\, d\left[\Phi'(s)\right]\right| \\
&\le \frac{y-x}{4}\bigvee_x^{\frac{x+y}{2}}(\Phi') + \frac{y-x}{4}\bigvee_{\frac{x+y}{2}}^y(\Phi') = \frac{1}{4}(y-x)\bigvee_x^y(\Phi')
\end{aligned}$$

and, similarly, if $y < x$, then

$$\left|\frac{\Phi(x) + \Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right| \le \frac{1}{4}(x-y)\bigvee_y^x(\Phi').$$

Therefore, for any $x, y \in [a, b]$ we have the following inequality

$$\left|\frac{\Phi(x) + \Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right| \le \frac{1}{4}|y-x|\left|\bigvee_x^y(\Phi')\right|,$$

which written for $x_i, y_i \in [a,b]$, $i \in \{1,...,n\}$ produces by summation the desired result (4.1).

In order to prove the sharpness of the constant $\frac{1}{4}$ in both inequalities from (4.1), assume that there exists a constant $G > 0$ such that

$$\left|\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right| \le G(b-a)\bigvee_a^b(\Phi'). \tag{4.3}$$

Consider the function $\Phi : [a,b] \to \mathbb{R}$, $\Phi(t) = \left|t-\frac{a+b}{2}\right|$. This function is absolutely continuous, $\Phi'(t) = \operatorname{sgn}\left(t-\frac{a+b}{2}\right)$, $t \in [a,b] \setminus \left\{\frac{a+b}{2}\right\}$ and $\bigvee_a^b(\Phi') = 2$. Thus, (4.3) becomes

$$\frac{b-a}{2} \le 2G(b-a),$$

which implies that $G \ge \frac{1}{4}$.

(ii) We utilize the fact that for an $L-$Lipschitzian function, $p : [\alpha,\beta] \to \mathbb{R}$ and a Riemann integrable function $v : [\alpha,\beta] \to \mathbb{R}$, the Riemann-Stieltjes integral $\int_\alpha^\beta p(s)\,dv(s)$ exists and

$$\left|\int_\alpha^\beta p(s)\,dv(s)\right| \le L\int_\alpha^\beta |p(s)|\,ds.$$

If $x < y$, then on utilizing this property of the Riemann-Stieltjes integral we have successively

$$\begin{aligned}
&\left|\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right| \\
&= \frac{1}{2}\left|\int_x^y K_{x,y}(s)\,d\left[\Phi'(s)\right]\right| \\
&= \frac{1}{2}\left|\int_x^{\frac{x+y}{2}} (s-x)\,d\left[\Phi'(s)\right] + \int_{\frac{x+y}{2}}^y (y-s)\,d\left[\Phi'(s)\right]\right| \\
&\le \frac{1}{2}\left|\int_x^{\frac{x+y}{2}} (s-x)\,d\left[\Phi'(s)\right]\right| + \frac{1}{2}\left|\int_{\frac{x+y}{2}}^y (y-s)\,d\left[\Phi'(s)\right]\right| \\
&\le \frac{1}{2}K\left[\int_x^{\frac{x+y}{2}} (s-x)\,ds + \int_{\frac{x+y}{2}}^y (y-s)\,ds\right] = \frac{1}{8}K(y-x)^2.
\end{aligned}$$

The same inequality holds if $y < x$, and the desired inequality (4.2) is thus obtained.

To prove the sharpness of the constant $\frac{1}{8}$ in (4.2), let us assume that there exist $U > 0$ so that

$$\left|\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right| \le UK(b-a)^2. \tag{4.4}$$

Consider $\Phi : [a,b] \to \mathbb{R}$, $\Phi(t) = \frac{1}{2}\left(t - \frac{a+b}{2}\right)^2$. Then $\Phi'(t) = t - \frac{a+b}{2}$ is Lipschitzian with the constant $K = 1$ and (4.4) becomes

$$\frac{1}{8}(b-a)^2 \le U(b-a)^2,$$

which implies that $U \ge \frac{1}{8}$. □

Corollary 5. *Let $\Phi : [a,b] \to \mathbb{R}$ be a differentiable function on the interval $[a,b]$ of real numbers $\mathbb{R}$. If the derivative Φ' is (Δ,δ)-Lipschitzian on $[a,b]$ with the constant $\Delta > \delta$, then*

$$\left|\mathcal{J}_{n,\Phi}(x,y) - \frac{\Delta+\delta}{4}\mathcal{J}_{n,2}(x,y)\right| \le \frac{1}{16}(\Delta-\delta)\sum_{i=1}^{n}(y_i - x_i)^2 = \frac{1}{4}(\Delta-\delta)\mathcal{J}_{n,2}(x,y), \tag{4.5}$$

for any $x = (x_1,...,x_n)$, $y = (y_1,...,y_n) \in [a,b]^n$. The constant $\frac{1}{16}$ is best possible in (4.5).

In particular, if Φ is twice differentiable and the second derivative Φ'' satisfies the inequality $-\infty < \delta \le \Phi''(t) \le \Delta < \infty$ for each $t \in (a,b)$, then (4.5) is valid as well.

Proof. Follows by the statement (ii) from Theorem 5 on noticing that the function $\Phi - \frac{\Delta+\delta}{4}e^2$ has the property that $\Phi' - \frac{\Delta+\delta}{2}e$ is $\frac{\Delta-\delta}{2}$-Lipschitzian on $[a,b]$. The details are omitted. □

Theorem 6. *Let $\Phi : [a,b] \to \mathbb{R}$ be a differentiable function on the interval $[a,b]$ of real numbers $\mathbb{R}$.*

(i) *If the derivative Φ' is absolutely continuous on $[a,b]$, then*

$$\left|\mathcal{J}_{n,\Phi}(x,y)\right| \tag{4.6}$$

$$\le \begin{cases} \frac{1}{16}\sum_{i=1}^{n}(y_i - x_i)^2 \\ \times\left[\|\Phi''\|_{\left[x_i,\frac{x_i+y_i}{2}\right],\infty} + \|\Phi''\|_{\left[\frac{x_i+y_i}{2},y_i\right],\infty}\right] & \text{if } \Phi'' \in L_\infty[a,b] \\ \\ \frac{1}{(q+1)^{1/q}2^{2+\frac{1}{q}}}\sum_{i=1}^{n}|y_i - x_i|^{1+\frac{1}{q}} & \text{if } \Phi'' \in L_p[a,b], \\ \times\left[\|\Phi''\|_{\left[x_i,\frac{x_i+y_i}{2}\right],p} + \|\Phi''\|_{\left[\frac{x_i+y_i}{2},y_i\right],p}\right] & p > 1, \frac{1}{p}+\frac{1}{q} = 1, \end{cases}$$

$$\leq \begin{cases} \frac{1}{8}\left\|\Phi''\right\|_{[a,b],\infty}\sum_{i=1}^{n}(y_i-x_i)^2 & \text{if } \Phi''\in L_\infty[a,b] \\ \frac{1}{(q+1)^{1/q}2^{1+\frac{1}{q}}}\left\|\Phi''\right\|_{[a,b],p}\sum_{i=1}^{n}\left|y_i-x_i\right|^{1+\frac{1}{q}} & \text{if } \Phi''\in L_p[a,b], \\ & p>1, \frac{1}{p}+\frac{1}{q}=1, \end{cases}$$

for any $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)\in[a,b]^n$.

(ii) *If the absolute value of the second derivative* $|\Phi''|$ *is convex on* $[a,b]$, *then*

$$\begin{aligned} &\left|\mathcal{J}_{n,\Phi}(x,y)\right| \qquad (4.7)\\ &\leq \frac{1}{12}\sum_{i=1}^{n}(y_i-x_i)^2\left[\frac{|\Phi''(y_i)|+|\Phi''(x_i)|}{4}+\left|\Phi''\left(\frac{x_i+y_i}{2}\right)\right|\right]\\ &\leq \frac{1}{16}\sum_{i=1}^{n}(y_i-x_i)^2\left[\left|\Phi''(y_i)\right|+\left|\Phi''(x_i)\right|\right]\\ &\leq \frac{1}{2}\left\|\Phi''\right\|_{[a,b],\infty}\mathcal{J}_{n,2}(x,y), \end{aligned}$$

for any $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)\in[a,b]^n$.

Proof. (i) Since Φ' is absolutely continuous on $[a,b]$, then the Riemann-Stieltjes integral in the proof of Theorem 5 can be replaced with the Lebesgue integral and therefore we have

$$\begin{aligned} &\left|\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right| \qquad (4.8)\\ &=\frac{1}{2}\left|\int_x^y K_{x,y}(s)\Phi''(s)\,ds\right|\\ &=\frac{1}{2}\left|\int_x^{\frac{x+y}{2}}(s-x)\Phi''(s)\,ds+\int_{\frac{x+y}{2}}^{y}(y-s)\Phi''(s)\,ds\right|\\ &\leq\frac{1}{2}\left|\int_x^{\frac{x+y}{2}}(s-x)\Phi''(s)\,ds\right|+\frac{1}{2}\left|\int_{\frac{x+y}{2}}^{y}(y-s)\Phi''(s)\,ds\right|\\ &\leq\frac{1}{2}\left[\int_x^{\frac{x+y}{2}}(s-x)\left|\Phi''(s)\right|ds+\int_{\frac{x+y}{2}}^{y}(y-s)\left|\Phi''(s)\right|ds\right] \end{aligned}$$

for any $x,y\in[a,b]$ with $x<y$.

Utilizing Hölder's integral inequality, we can state that

$$\int_x^{\frac{x+y}{2}}(s-x)\left|\Phi''(s)\right|ds \qquad (4.9)$$

$$\leq \begin{cases} \frac{(y-x)^2}{8} \|\Phi''\|_{\left[x,\frac{x+y}{2}\right],\infty} & \text{if } \Phi'' \in L_\infty[a,b] \\ \\ I(q)\|\Phi''\|_{\left[x,\frac{x+y}{2}\right],p} & \text{if } \Phi'' \in L_p[a,b],\ p>1 \end{cases}$$

where

$$I(q) = \left[\int_x^{\frac{x+y}{2}} (s-x)^q\, ds\right]^{1/q}$$

and $\frac{1}{q}+\frac{1}{p}=1, p>1$.

Observe that

$$I(q) = \left[\left.\frac{(s-x)^{q+1}}{q+1}\right|_x^{\frac{x+y}{2}}\right]^{1/q} = \frac{1}{(q+1)^{1/q}}\left(\frac{y-x}{2}\right)^{1+\frac{1}{q}} = \frac{1}{(q+1)^{1/q}\,2^{1+\frac{1}{q}}}(y-x)^{1+\frac{1}{q}}. \tag{4.10}$$

In a similar manner we also have the inequality

$$\int_{\frac{x+y}{2}}^{y} (y-s)\left|\Phi''(s)\right| ds \tag{4.11}$$
$$\leq \begin{cases} \frac{(y-x)^2}{8} \|\Phi''\|_{\left[\frac{x+y}{2},y\right],\infty} & \text{if } \Phi'' \in L_\infty[a,b] \\ \\ I(q)\|\Phi''\|_{\left[\frac{x+y}{2},y\right],p} & \text{if } \Phi'' \in L_p[a,b],\ p>1 \end{cases}$$

where $I(q)$ is given in (4.10).

Utilizing (4.8), (4.9) and (4.11) we can state that

$$\left|\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right| \tag{4.12}$$
$$\leq \begin{cases} \frac{(y-x)^2}{16}\left[\|\Phi''\|_{\left[x,\frac{x+y}{2}\right],\infty} + \|\Phi''\|_{\left[\frac{x+y}{2},y\right],\infty}\right] & \text{if } \Phi'' \in L_\infty[a,b] \\ \\ \frac{1}{(q+1)^{1/q}2^{2+\frac{1}{q}}}|y-x|^{1+\frac{1}{q}} & \text{if } \Phi'' \in L_p[a,b], \\ \times\left[\|\Phi''\|_{\left[x,\frac{x+y}{2}\right],p} + \|\Phi''\|_{\left[\frac{x+y}{2},y\right],p}\right] & p>1, \frac{1}{p}+\frac{1}{q}=1, \end{cases}$$

for any $x, y \in [a,b]$.

(ii). It is well know that, if $\varphi : [\alpha, \beta] \to \mathbb{R}$ is a convex function, then for each $s \in [\alpha, \beta]$ we have the inequality

$$\varphi(s) \leq \frac{(s-\alpha)\varphi(\beta) + (\beta - s)\varphi(\alpha)}{\beta - \alpha}.$$

Utilizing this property and the fact that $|\Phi''|$ is convex, we have for $x < y$ that

$$\left|\Phi''(s)\right| \leq \frac{(s-x)\left|\Phi''\left(\frac{x+y}{2}\right)\right| + \left(\frac{x+y}{2} - s\right)\left|\Phi''(x)\right|}{\frac{y-x}{2}}$$

for any $s \in \left[x, \frac{x+y}{2}\right]$ and

$$\left|\Phi''(s)\right| \leq \frac{\left(s - \frac{x+y}{2}\right)\left|\Phi''(y)\right| + (y-s)\left|\Phi''\left(\frac{x+y}{2}\right)\right|}{\frac{y-x}{2}}$$

for any $s \in \left[\frac{x+y}{2}, y\right]$.

These imply the inequalities

$$(s-x)\left|\Phi''(s)\right| \leq \frac{(s-x)^2\left|\Phi''\left(\frac{x+y}{2}\right)\right| + \left(\frac{x+y}{2} - s\right)(s-x)\left|\Phi''(x)\right|}{\frac{y-x}{2}} \tag{4.13}$$

for any $s \in \left[x, \frac{x+y}{2}\right]$ and

$$(y-s)\left|\Phi''(s)\right| \leq \frac{\left(s - \frac{x+y}{2}\right)(y-s)\left|\Phi''(y)\right| + (y-s)^2\left|\Phi''\left(\frac{x+y}{2}\right)\right|}{\frac{y-x}{2}} \tag{4.14}$$

for any $s \in \left[\frac{x+y}{2}, y\right]$.

Integrating (4.13) on $\left[x, \frac{x+y}{2}\right]$ and (4.14) on $\left[\frac{x+y}{2}, y\right]$ we get

$$\begin{aligned}\int_x^{\frac{x+y}{2}} (s-x)\left|\Phi''(s)\right| ds \leq{}& \frac{2\left|\Phi''\left(\frac{x+y}{2}\right)\right|}{y-x}\int_x^{\frac{x+y}{2}} (s-x)^2\, ds \\ &+ \frac{2\left|\Phi''(x)\right|}{y-x}\int_x^{\frac{x+y}{2}} \left(\frac{x+y}{2} - s\right)(s-x)\, ds\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}\int_{\frac{x+y}{2}}^{y} (y-s)\left|\Phi''(s)\right| ds \leq{}& \frac{2\left|\Phi''(y)\right|}{y-x}\int_{\frac{x+y}{2}}^{y} \left(s - \frac{x+y}{2}\right)(y-s)\, ds \\ &+ \frac{2\left|\Phi''\left(\frac{x+y}{2}\right)\right|}{y-x}\int_{\frac{x+y}{2}}^{y} (y-s)^2\, ds.\end{aligned} \tag{4.16}$$

Since simple calculations reveal that:

$$\int_x^{\frac{x+y}{2}} (s-x)^2\,ds = \frac{(y-x)^3}{24},$$

$$\int_x^{\frac{x+y}{2}} \left(\frac{x+y}{2}-s\right)(s-x)\,ds = \frac{(y-x)^3}{48}$$

$$\int_{\frac{x+y}{2}}^{y} \left(s-\frac{x+y}{2}\right)(y-s)\,ds = \frac{(y-x)^3}{48},$$

$$\text{and } \int_{\frac{x+y}{2}}^{y} (y-s)^2\,ds = \frac{(y-x)^3}{24},$$

then (4.15) and (4.16) become

$$\int_x^{\frac{x+y}{2}} (s-x)\left|\Phi''(s)\right| ds \le \frac{(y-x)^2}{12}\left|\Phi''\left(\frac{x+y}{2}\right)\right| + \frac{(y-x)^2}{24}\left|\Phi''(x)\right|$$

and

$$\int_{\frac{x+y}{2}}^{y} (y-s)\left|\Phi''(s)\right| ds \le \frac{(y-x)^2}{24}\left|\Phi''(y)\right| + \frac{(y-x)^2}{12}\left|\Phi''\left(\frac{x+y}{2}\right)\right|.$$

Further on, by making use of the inequality (4.8) we deduce

$$\begin{aligned} &\left|\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right| \qquad (4.17)\\ &\le \frac{1}{2}\left[\int_x^{\frac{x+y}{2}} (s-x)\left|\Phi''(s)\right| ds + \int_{\frac{x+y}{2}}^{y} (y-s)\left|\Phi''(s)\right| ds\right]\\ &\le \frac{1}{12}(y-x)^2\left[\frac{|\Phi''(y)|+|\Phi''(x)|}{4} + \left|\Phi''\left(\frac{x+y}{2}\right)\right|\right] \end{aligned}$$

for any $x, y \in [a,b]$.

The first inequality in (4.7) follows then easily from (4.17). The other two inequalities are obvious. □

5. Applications

For the convex function $\Phi : I \subset (0,\infty) \to \mathbb{R}$ with $\Phi(t) = -\ln t$ we define the *Jensen divergence*

$$\mathcal{J}_{n,0}(x,y) := \sum_{i=1}^{n}\left\{\ln\left(\frac{x_i+y_i}{2}\right) - \frac{1}{2}\left[\ln(x_i)+\ln(y_i)\right]\right\}, (x,y) \in I^n \times I^n$$

which can be written as

$$\mathcal{J}_{n,0}(x,y) = \ln\left[\prod_{i=1}^{n}\left(\frac{\frac{x_i+y_i}{2}}{\sqrt{x_i y_i}}\right)\right], (x,y) \in I^n \times I^n.$$

If one wants to compare the *Jensen divergence* for a convex function $\Phi : [m,M] \to \mathbb{R}$ with $\delta(x,y) := \frac{1}{2}\sum_{i=1}^{n}|x_i - y_i|$, the statistical distance between x and y, then one has from (2.7) the following inequality

$$0 \le \mathcal{J}_{n,\Phi}(x,y) \le \frac{1}{2}\left(\Phi'_{-}(M) - \Phi'_{+}(m)\right)\delta(x,y) \tag{5.1}$$

for any $x,y \in [m,M]^n$, provided the lateral derivatives $\Phi'_{-}(M)$ and $\Phi'_{+}(m)$ are finite.

For instance, if we apply the inequality (5.1) to the function $\Phi(t) = -\ln t$ defined on the interval $[m,M] \subset (0,\infty)$, then we get

$$0 \le \mathcal{J}_{n,0}(x,y) \le \frac{M-m}{2mM}\cdot\delta(x,y) \tag{5.2}$$

for any $x,y \in [m,M]^n$.

The same inequality (5.1) applied for the convex function $\Phi(t) = t\ln t$ defined on the interval $[m,M] \subset (0,\infty)$ produces the result

$$0 \le \mathcal{J}_{n,1}(x,y) \le \ln\sqrt{\frac{M}{m}}\cdot\delta(x,y) \tag{5.3}$$

for any $x,y \in [m,M]^n$.

As another example, we can consider the convex function $\Phi(t) = \exp t, t \in \mathbb{R}$. If we apply the inequality (5.1) to this function, we get

$$0 \le \mathcal{J}_{n,\exp}(x,y) \le \frac{1}{2}(\exp(M) - \exp(m))\,\delta(x,y) \tag{5.4}$$

for any $x,y \in [m,M]^n$.

Now, if one wants to compare the *Jensen divergence* for a twice differentiable convex function $\Phi : [m,M] \to \mathbb{R}$ satisfying the condition $0 \le \delta \le \Phi''(t) \le \Delta < \infty$ for any $t \in (m,M)$, with

$$\mathcal{J}_{n,2}(x,y) = \sum_{i=1}^{n}\left[\frac{1}{2}\left(x_i^2 + y_i^2\right) - \left(\frac{x_i+y_i}{2}\right)^2\right] = \frac{1}{4}\sum_{i=1}^{n}(x_i - y_i)^2,$$

that one has from (4.5) the following double inequality

$$\frac{1}{2}\delta\mathcal{J}_{n,2}(x,y) \le \mathcal{J}_{n,\Phi}(x,y) \le \frac{1}{2}\Delta\mathcal{J}_{n,2}(x,y), \tag{5.5}$$

for any $x,y \in [m,M]^n$.

If we apply the inequality (5.5) to the function $\Phi(t) = -\ln t$ defined on the interval $[m, M] \subset (0, \infty)$, then we get

$$\frac{1}{2M^2}\mathcal{J}_{n,2}(x, y) \le \mathcal{J}_{n,0}(x, y) \le \frac{1}{2m^2}\mathcal{J}_{n,2}(x, y) \tag{5.6}$$

for any $x, y \in [m, M]^n$.

The same inequality (5.5) applied to the convex function $\Phi(t) = t \ln t$ defined on the interval $[m, M] \subset (0, \infty)$ produces the result

$$\frac{1}{2M}\mathcal{J}_{n,2}(x, y) \le \mathcal{J}_{n,1}(x, y) \le \frac{1}{2m}\mathcal{J}_{n,2}(x, y), \tag{5.7}$$

for any $x, y \in [m, M]^n$.

Finally, if we apply the inequality (5.5) to the convex function $\Phi(t) = \exp t$ on the interval $[m, M] \subset \mathbb{R}$, then we get the bounds

$$\frac{1}{2}\mathcal{J}_{n,2}(x, y)\exp m \le \mathcal{J}_{n,\exp}(x, y) \le \frac{1}{2}\mathcal{J}_{n,2}(x, y)\exp M, \tag{5.8}$$

for any $x, y \in [m, M]^n$.

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FIXED SETS AND ENDPOINTS OF SET-VALUED ASYMPTOTIC CONTRACTIONS IN METRIC SPACES

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Abstract. By introducing a new concept called "set-valued asymptotic contraction of final type" in metric spaces, the existence and uniqueness of compact fixed sets for such mappings have been obtained. Furthermore, by adding additional conditions, we prove the existence and uniqueness of endpoints for these maps.

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Keywords: fixed set, endpoint, asymptotic contraction of the final type, approximate endpoint property, topological contraction

1. INTRODUCTION AND PRELIMINARIES

In 2003, Kirk [8] introduced the notion of asymptotic contraction on metric spaces and proved a fixed point theorem for such contractions. Later on, Suzuki [11] introduced the notion of asymptotic contraction of Meir-Keeler and extended Kirk's fixed point theorem. Recently, Suzuki [12] introduced the concept of asymptotic contraction of final type for single valued mappings and proved the following result.

Theorem 1 (Suzuki [12]). *Let (X,d) be a complete metric space and let T be a continuous mapping on X. Assume that T is asymptotic contraction of final type (ACF, for short), i.e., the following contractions hold:*

(C1) $\lim_{\delta\to+0}\sup\{\limsup_{n\to\infty} d(T^n(x),T^n(y)) : d(x,y)<\delta\}=0.$

(C2) *For each $\varepsilon>0$, there exists $\delta>0$ such that for $x,y\in X$ with $\varepsilon<d(x,y)<\varepsilon+\delta$, there exists $\nu\in\mathbb{N}$ such that $d(T^\nu(x),T^\nu(y))\le\varepsilon$.*

(C3) *For $x,y\in X$ with $x\neq y$, there exists $\nu\in\mathbb{N}$ such that*
$$d(T^\nu(x),T^\nu(y))<d(x,y).$$

(C4) *For $x\in X$ and $\varepsilon>0$, there exist $\delta>0$ and $\nu\in\mathbb{N}$ such that*
$$\epsilon<d(T^i(x),T^j(x))<\epsilon+\delta \text{ implies } d(T^\nu\circ T^i(x),T^\nu\circ T^j(x))\le\epsilon$$

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for all $i, j \in \mathbb{N}$.

Then T *has a unique fixed point* $z \in X$. *Moreover,* $\lim_n T^n(x) = z$ *holds for every* $x \in X$.

Suzuki also mentioned that this result is the final generalization in some senses.

In recent years, many authors have studied the existence and uniqueness of fixed sets and endpoints for set-valued maps in metric spaces and topological spaces; see [1–7, 9, 10, 13–18] and references therein. Recently Fakhar [6] extended the Kirk's asymptotic contraction to set-valued maps and proved the existence and uniqueness of endpoints for those contractions which have the approximate endpoint property. Here, we introduce the notion of set-valued asymptotic contraction of final type in metric spaces and the existence and uniqueness of compact fixed set of these contractions are proved. Then, by using this result, we prove the existence and uniqueness of endpoint for such mappings which are topological contraction or they have the approximate endpoint property.

Let us introduce some definitions and facts which will be used in the sequel. Let X be a nonempty set and $T : X \to 2^X$ be a set-valued map with nonempty values. A subset A of X is said to be a fixed set of T, if $T(A) = A$. We denote the set of all fixed set of T with $Fix(T)$. An element $x \in X$ is said to be an endpoint (or stationary point or strict fixed point) of T, if $T(x) = \{x\}$. The set of all endpoints of T is denoted by $End(T)$. Suppose that X is a topological space, a set-valued map $T : X \to 2^X$ is said to be a topological contraction if for every nonempty compact subset A of X with $T(A) = A$, A is singleton, i.e., A is an endpoint of T. Let (X, d) be a metric space and $CB(X)$ and $K(X)$ denote the family of all nonempty closed and bounded subset of X and the family of all nonempty compact subset of X respectively. Then the Hausdorff metric on $CB(X)$ is given by

$$H(A, B) = \max\{e(A, B), e(B, A)\}$$

where

$$e(A, B) = \sup_{a \in A} d(a, B), e(B, A) = \sup_{b \in B} d(b, A) \quad \text{and} \qquad d(a, B) = \inf_{b \in B} d(a, b).$$

It is well known that if (X, d) is a complete metric space, then the pair $(K(X), H)$ is a complete generalized metric space. We say that a set-valued map T on a metric space X has the approximate endpoint property if

$$\inf_{x \in X} \sup_{y \in Tx} d(x, y) = 0.$$

2. Main Results

In this section we assume that (X, d) is a metric space. In the first step we introduce the set-valued asymptotic contraction of the final type in metric spaces as follows.

Definition 1. A set-valued map $T : X \to CB(X)$ is said to be set-valued asymptotic contraction of final type, if the following conditions hold:

(D1) $\lim_{\delta\to 0^+} \sup\{\limsup_{n\to\infty} H(T^n(x), T^n(y)) : d(x, y) < \delta\} = 0$.

(D2) For each $\varepsilon > 0$, there exists $\delta > 0$ such that for $x, y \in X$ with $d(x, y) < \varepsilon + \delta$, there exists $\nu \in \mathbb{N}$ such that $H(T^\nu(x), T^\nu(y)) < \varepsilon$.

(D3) For $x \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $\nu \in \mathbb{N}$ such that
$$H(T^i(x), T^j(x)) < \varepsilon + \delta \quad \text{implies} \quad H(T^\nu \circ T^i(x), T^\nu \circ T^j(x)) < \varepsilon$$
for all $i, j \in \mathbb{N}$.

It's easy to see that, if $lim_{n\to\infty} H(T^n(x), T^n(y)) = 0$ for every $x, y \in X$ and $\{T^n(x)\}$ is a Cauchy sequence respect to Hausdorff metric, then T is set-valued asymptotic contraction of final type. According to this fact we give the following example of a set-valued asymptotic contraction of final type as follows.

Example 1. Let $X = \mathbb{R}$ and $T : X \to 2^X$ be defined as follows:
$$T(x) = \begin{cases} \{0\} & x < 0; \\ [1, 2 - \frac{x}{2}] & 0 \le x < 1; \\ \{3\} & 1 \le x \le 2; \\ [3, 4] & x > 2. \end{cases}$$
Since $T^n(x) = [3, 4]$ for $x \in X$ and $n \ge 4$. Therefore,
$$lim_{n\to\infty} H(T^n(x), T^n(y)) = 0$$
for every $x, y \in X$ and $\{T^n(x)\}$ is a Cauchy sequence respect to Hausdorff metric, then T is asymptotic contraction of final type.

Definition 2 ([6])**.** A set-valued map $T : X \to CB(X)$ is said to be a set-valued Kirk's asymptotic contraction, if there exists a continuous function φ from $[0, \infty)$ into itself and a sequence φ_n of functions from $[0, \infty)$ into itself satisfying the following:

(A1) $\varphi(0) = 0$.

(A2) $\varphi(r) < r$ for $r \in (0, \infty)$

(A3) $\{\varphi_n\}$ is uniformly convergent to φ

(A4) $H(T^n(x), T^n(y)) < \varphi_n(d(x, y))$ for all $n \in \mathbb{N}$ and $x, y \in X$.

In the following result we show that any set-valued Kirk's asymptotic contraction is a set-valued asymptotic contraction of final type.

Proposition 1. *Suppose that T is a set-valued Kirk's asymptotic contraction on X. Then T is a set-valued asymptotic contraction of final type.*

Proof. Similar to Proposition 3 in [11], we put $E = \{d(x, y) : x, y \in X\}$ and define a sequence (ψ_n) of function from $[0, \infty)$ into itself by
$$\psi_n(t) = \begin{cases} \varphi_n(t) + \frac{t}{n} & \text{if } t \in E, \\ 0 & \text{if } t \notin E, \end{cases}$$

when the sequence (φ_n) is defined in Definition 2 corresponding to T.
Now, we prove the following conditions :

- (B1) $\limsup_n \psi_n(\varepsilon) \leq \varepsilon$ for all $\varepsilon \geq 0$,
- (B2) For each $\varepsilon > 0$, there exist $\delta > 0$ and $\nu \in \mathbb{N}$ such that $\psi_\nu(t) \leq \varepsilon$ for all $t \in [0, \varepsilon + \delta]$,
- (B3) $H(T^n(x), T^n(y)) < \psi_n(d(x,y))$ for all $n \in \mathbb{N}$ and $x, y \in X$ with $x \neq y$.

It is obvious that (ψ_n) satisfies (B1) and (B3). Let us prove (B2). Fix $\varepsilon > 0$. Since $\varphi(\varepsilon) < \varepsilon$ and φ is continuous, we can choose $\delta > 0$ such that $\delta < \frac{(\varepsilon - \varphi(\varepsilon))}{2}$ and $\varphi(t) - \varphi(\varepsilon) < \frac{(\varepsilon - \varphi(\varepsilon))}{2}$ for $t \in [\varepsilon - \delta, \varepsilon + \delta]$. Also for $t \in (0, \varepsilon - \delta]$, we have $\varphi(t) < t < \varepsilon - \delta$. For such δ, we also choose $\nu \in \mathbb{N}$ such that

$$\frac{(\varepsilon + \delta)}{\nu} < \frac{\delta}{2} \quad \text{and} \quad \sup\{\,|\varphi_\nu(t) - \varphi(t)| : t \in E\} < \frac{\delta}{2}.$$

We fix $t \in (0, \varepsilon + \delta]$. In the case of $t \notin E$, we have $\psi_\nu(t) = 0 < \varepsilon$. For the case $t \in E$ and $t \in (0, \varepsilon - \delta]$, we have

$$\psi_\nu(t) = \varphi_\nu(t) + \frac{t}{\nu} \leq \varphi(t) + \frac{\delta}{2} + \frac{\delta}{2} \leq \varepsilon - \delta + \frac{2\delta}{2} = \varepsilon.$$

Also, in the case $t \in E$ and $t \in [\varepsilon - \delta, \varepsilon + \delta]$, we have

$$\begin{aligned}
\psi_\nu(t) &= \varphi_\nu(t) + \frac{t}{\nu} \\
&\leq \varphi(t) + \frac{\delta}{2} + \frac{\delta}{2} \\
&\leq \varphi(\varepsilon) + \frac{(\varepsilon - \varphi(\varepsilon))}{2} + \frac{2\delta}{2} \\
&\leq \varphi(\varepsilon) + \frac{2(\varepsilon - \varphi(\varepsilon))}{2} = \varepsilon,
\end{aligned}$$

which completes our assertion. Now, we prove T fulfills the conditions (D1), (D2) and (D3). Since

$$\begin{aligned}
&\lim_{\delta \to 0^+} \sup\{\limsup_{n \to \infty} H(T^n(x), T^n(y)) : d(x,y) < \delta\} \\
&\qquad \leq \lim_{\delta \to 0^+} \sup\{\limsup_{n \to \infty} \psi_n(t) : t < \delta\} \\
&\qquad \leq \lim_{\delta \to 0^+} = 0,
\end{aligned}$$

hence we obtain (D1). Let $\varepsilon > 0$, from (B2), there exist $\delta > 0$ and $\nu \in \mathbb{N}$ such that $\psi_\nu(t) \leq \varepsilon$ for all $t \in [0, \varepsilon + \delta]$. Then for each $x, y \in X$ with $d(x,y) < \varepsilon + \delta$, from (B3), we have

$$H(T^\nu(x), T^\nu(y)) < \psi_\nu(d(x,y)) \leq \varepsilon,$$

which implies (D2).
For (D3), let $x \in X$ such that $H(T^i(x), T^j(x)) < \varepsilon + \delta$ for all $i, j \in \mathbb{N}$. So

$$e(T^i(x), T^j(x)) < \varepsilon + \delta$$

and $e(T^j(x), T^i(x)) < \varepsilon + \delta$. By the first inequality, we have

$$\forall z \in T^i(x),\ \exists w \in T^j(x) \quad \text{such that} \quad d(z, w) < \delta + \varepsilon.$$

Therefore, by (D2), for all $z \in T^i(x)$, there exists $\nu \in \mathbb{N}$ such that

$$H(T^\nu(z), T^\nu(w)) < \varepsilon,$$

that deduces $e(T^\nu(z), T^\nu(w)) < \varepsilon$, so for all $z \in T^i(x)$, $e(T^\nu(z), T^\nu(T^j(x))) < \varepsilon$. Then, $e(T^\nu(T^i(x)), T^\nu(T^j(x))) < \varepsilon$. By the same argument, we have

$$e(T^\nu(T^j(x)), T^\nu(T^i(x))) < \varepsilon.$$

Therefore,

$$H(T^\nu(T^i(x)), T^\nu(T^j(x))) < \varepsilon,$$

which implies (D3). □

Example 1 and Theorem 9 of [12] and Proposition 3 of [11], show that set-valued asymptotic contraction of the final type is strictly weaker than set-valued Kirk's asymptotic contraction.

Now, we give the main result as follows.

Theorem 2. *Let (X, d) be a complete metric space. Suppose that T is a set-valued asymptotic contraction of final type. Assume that T is continuous and compact valued. Then T has a unique fixed set $A \in K(X)$. Moreover,* $\lim_n H(T^n(B), A) = 0$ *holds for every $B \in K(X)$.*

Proof. We consider function $F : K(X) \to K(X)$ defined by $F(A) = T(A) = \cup_{x \in A} T(x)$ for all $A \in K(X)$. Since T is continuous and compact valued, then F is well defined. Also, the continuity of T implies the continuity of F. We will claim that F also fulfills condition (D1), (D2) and (D3). Fix $\varepsilon > 0$. From (D1) for T, there exists $\delta > 0$ such that

$$\sup\{\limsup_{n\to\infty} H(T^n(x), T^n(y)) : d(x, y) < \delta\} < \varepsilon.$$

So for each $x, y \in X$ such that $d(x, y) < \delta$, there exists $m_0 > 1$ such that for $n > m_0$, $H(T^n(x), T^n(y)) < \varepsilon$. Let $A, B \in K(X)$ such that $H(A, B) < \delta$, we show that $H(F^n(A), F^n(B)) < \varepsilon$. If $H(A, B) < \delta$, then $e(A, B) < \delta$ and $e(B, A) < \delta$. So by the first inequality,

$$\forall z \in A,\ \exists w \in B \quad \text{such that} \ d(z, w) < \delta,$$

It follows that for $n > m_0$, $H(T^n(z), T^n(w)) < \varepsilon$. Thus, $e(T^n(z), T^n(w)) < \varepsilon$. Therefore, $e(T^n(z), T^n(B)) < \varepsilon$ for all $z \in A$. We deduce that

$$e(T^n(A), T^n(B)) < \varepsilon.$$

From $e(B,A) < \delta$, by the same argument, we obtain $e(T^n(B),T^n(A)) < \varepsilon$. Hence, $H(T^n(A),T^n(B)) < \varepsilon$ and so,

$$\sup\{\limsup_{n\to\infty} H(T^n(A),T^n(B)) : H(A,B) < \delta\} < \varepsilon.$$

Let $\varepsilon > 0$, from condition (D2), there exists $\delta > 0$ such that for $x,y \in X$ with $d(x,y) < \varepsilon+\delta$, there exists $\nu \in \mathbb{N}$ such that $H(T^\nu(x),T^\nu(y)) \leq \varepsilon$. Let $A,B \in K(X)$ such that $H(A,B) < \delta+\varepsilon$, we show that $H(F^\nu(A),F^\nu(B)) < \varepsilon$. If $H(A,B) < \delta+\varepsilon$, then $e(A,B) < \delta+\varepsilon$ and $e(B,A) < \delta+\varepsilon$. So by the first inequality,

$$\forall z \in A,\ \exists w \in B \quad \text{such that} \quad d(z,w) < \delta+\varepsilon.$$

Hence, there exists $\nu \in \mathbb{N}$ such that

$$H(T^\nu(z),T^\nu(w)) < \varepsilon.$$

Thus, $e(T^\nu(z),T^\nu(w)) < \varepsilon$, it follows that, for all $z \in A$, $e(T^\nu(z),T^\nu(B)) < \varepsilon$, we deduce that $e(T^\nu(A),T^\nu(B)) < \varepsilon$. From $e(B,A) < \delta+\varepsilon$, by the same argument, we obtain $e(T^\nu(B),T^\nu(A)) < \varepsilon$. Hence, $H(T^\nu(A),T^\nu(B)) < \varepsilon$ and so

$$H(F^\nu(A),F^\nu(B)) < \varepsilon,$$

which shows that F satisfies condition (D2).
Now, from condition (D3), there exists $\delta > 0$ and $\nu \in \mathbb{N}$ such that for $x \in X$ with

$$H(T^i(x),T^j(x)) < \varepsilon+\delta \ \text{ implies } \ H(T^\nu \circ T^i(x),T^\nu \circ T^j(x)) \leq \varepsilon \ \text{ for all } \ i,j \in \mathbb{N}.$$

Let $A \in K(X)$ such that $H(F^i(A),F^j(A)) < \varepsilon+\delta$, we will show that

$$H(F^\nu \circ (A),F^\nu \circ (A)) < \varepsilon.$$

Since $H(T^i(A),T^j(A)) < \varepsilon+\delta$, then

$$e(T^i(A),T^j(A)) < \varepsilon+\delta$$

and

$$e(T^i(A),T^j(A)) < \varepsilon+\delta.$$

So by the first inequality,

$$\forall z \in T^i(A),\ \exists w \in T^j(A) \quad \text{such that} \quad d(z,w) < \delta+\varepsilon.$$

Hence, from condition (D2), there exists $\nu \in \mathbb{N}$ such that

$$H(T^\nu(z),T^\nu(w)) < \varepsilon.$$

Thus, $e(T^\nu(z),T^\nu(w)) < \varepsilon$. It follows that $e(T^\nu(z),T^\nu(T^j(A))) < \varepsilon$ for all $z \in T^i(A)$, which deduces that $e(T^\nu(T^i(A)),T^\nu(T^j(A))) < \varepsilon$. From $e(T^i(A),T^j(A)) < \varepsilon+\delta$, by the same argument, we obtain

$$e(T^\nu(T^j(A)),T^\nu(T^i(A))) < \varepsilon.$$

Hence, $H(T^\nu(T^i(A)),T^\nu(T^j(A))) < \varepsilon$ and so

$$H(F^\nu \circ F^i(A)),F^\nu \circ F^j(A))) < \varepsilon,$$

which completes our claim. Therefore, F is a single-valued asymptotic contraction of final type. Hence, by Theorem 1, F has a unique fixed point $A \in K(X)$, i.e., $T(A) = F(A) = A$. Moreover, $\lim_n H(F^n(B), A) = 0$ holds for every $B \in K(X)$. □

As a consequence of Proposition 1 and Theorem 2 we obtain the following result.

Corollary 1. *Let (X,d) be a complete metric space and T be a set-valued Kirk's asymptotic contraction on X. If T is continuous and compact valued, then T has a compact fixed set.*

Theorem 3. *Let (X,d) be a complete metric space and T satisfying in the conditions of the above theorem. Suppose that T satisfies one of the following conditions:*

(i) *T has the approximate endpoint property.*
(ii) *T is a topological contraction.*

Then T has the unique endpoint x_0.

Proof. By Theorem 2, there is a compact fixed set $A \in K(X)$ such that $T(A) = A$. Let T have the approximate endpoint property. Then

$$C_n = \{x \in A : \ H(x, T(x)) = \sup_{y \in Tx} d(x, y)\} \neq \varnothing,$$

for each $n \in \mathbb{N}$. It is clear that for each $n \in \mathbb{N}$, $C_n \supseteq C_{n+1}$. Since the map $x \to H(x, T(x))$ is continuous, we have that C_n is closed, so C_n is compact. Therefore, $\cap_{n\in\mathbb{N}} C_n = C \neq \varnothing$. Thus, for $x \in C$, $H(x, T(x)) = \sup_{y \in Tx} d(x, y) = 0$, and so $T(x) = \{x\}$. If T is a topological contraction then by definition, A is a singleton that is there exists $x_0 \in X$ such that $T(x_0) = \{x_0\} = A$. Since T has a unique fixed set A in $K(X)$, then T has a unique endpoint. □

By the above theorem and Proposition 1, we have the following result.

Corollary 2. *Let (X,d) be a complete metric space and T be a set-valued Kirk's asymptotic contraction. Assume that T is continuous and compact valued. Suppose that T satisfies one of the following conditions:*

(i) *T has the approximate endpoint property.*
(ii) *T is a topological contraction.*

Then T has the unique endpoint x_0.

Remark 1. Since continuous set-valued map with compact valued and compact domain is uniformly continuous, the above corollary extends Theorem 2.3 in [6].

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FURTHER RESULTS REGARDING THE DEGREE KIRCHHOFF INDEX OF GRAPHS

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Abstract. Let G be a connected graph with vertex set $V(G)$. The degree Kirchhoff index of G is defined as $S'(G) = \sum_{\{u,v\}\subseteq V(G)} d(u)d(v)R(u,v)$, where $d(u)$ is the degree of vertex u, and $R(u,v)$ denotes the resistance distance between vertices u and v. In this paper we obtain some upper and lower bounds for the degree Kirchhoff index of graphs. We also obtain some bounds for the Nordhaus-Gaddum-type result for the degree Kirchhoff index.

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1. INTRODUCTION

In this paper all graphs considered are assumed to be connected. Let $G = (V(G), E(G))$ be a simple undirected graph with $n = |V(G)|$ vertices and $m = |E(G)|$ edges. For a vertex $u \in V(G)$, $d(u)$ denote the degree of u. $\Delta = \max\{d(u)|u \in V(G)\}$, $\delta = \min\{d(u)|u \in V(G)\}$. The distance $d(v,u) = d(u,v|G)$ between the vertices v and u of the graph G is defined as the length of a shortest path between v and u. The diameter of G, denoted by D, is the maximum distance between all pairs of vertices.

The *Wiener index* is defined as the sum of distances between all unordered pairs of vertices

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u,v).$$

This molecular structure descriptor is one of the most used topological indices, well correlated with many physical and chemical properties of a variety of classes of chemical compounds. For details, see the survey paper [9].

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In [16], a weighted version of the Wiener index is established, called the Gutman index [22, 23], which is defined as

$$S(G) = \sum_{\{u,v\}\subseteq V(G)} d(u)\, d(v)\, d(u,v)\ .$$

For tree [16] it holds that $S(G) = 4W(G) - (2n-1)(n-1)$.

The Gutman index of graphs attracted attention only quite recently. Several bounds on the Gutman index are established in [1]. In [6] an asymptotic upper bound for $S(G)$ was reported. In [6] relations between the edge–Wiener index and Gutman index were established. The maximal and minimal Gutman indices of unicyclic and bicyclic graphs are determined in [4, 10] and [3, 12], respectively.

In 1993 Klein and Randić [19] introduced a new distance function named resistance distance, based on the theory of electrical networks. They viewed G as an electrical network N by replacing each edge of G with a unit resistor. The resistance distance between the vertices u and v of the graph G, denoted by $R(u,v) = R(u,v|G)$, is then defined to be the effective resistance between the nodes u and v in N. Similar to the long recognized shortest–path distance, the resistance distance is also intrinsic to the graph, not only with some nice purely mathematical properties, but also with a substantial potential for chemical applications [18, 19, 25, 26].

The *Kirchhoff index* (or resistance index) is defined in analogy to the Wiener index as:

$$Kf(G) = \sum_{\{u,v\}\subseteq V(G)} R(u,v).$$

The Kirchhoff index is also much studied in the literature. Extremal unicyclic graphs with respect to the Kirchhoff index were determined in [15, 27, 28]. Deng also studied the Kirchhoff index of fully loaded unicyclic graphs [15] and graphs with many cut edges [8]. Zhou [30] characterized the extremal graphs with given matching number, connectivity, and minimal Kirchhoff index. Wang et al. [24] determined the first three minimal Kirchhoff indices among cacti.

Recently, a new index named *degree Kirchhoff index* was put forward in [2]. It is defined as

$$S'(G) = \sum_{\{u,v\}\subseteq V(G)} d(u)\, d(v)\, R(u,v).$$

Apparently, we can see that the degree Kirchhoff index may be viewed as the resistance–distance analogue of the Gutman index.

Let $L = (L_{uv})_{n\times n}$ be the Laplacian matrix of the (connected) graph G. Its entry is

$$L_{uv} = \begin{cases} d(u), & \text{if } u = v; \\ -1, & \text{if } u \sim v; \\ 0, & \text{otherwise.} \end{cases}$$

Suppose the Laplacian eigenvalues of $L(G)$ are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$. For details on Laplacian eigenvalues see [13, 14, 21]. Then a long time known result for the Kirchhoff index is [17]:

$$Kf(G) = n \sum_{i=1}^{n-1} \frac{1}{\mu_i} . \tag{1.1}$$

In [5] Fan Chung defined the *normalized Laplacian* matrix $\mathscr{L}(G) = (\mathscr{L}_{uv})_{n\times n}$ of a graph G as follows:

$$\mathscr{L}_{uv} = \begin{cases} 1, & \text{if } u = v; \\ -\frac{1}{\sqrt{d(u)d(v)}}, & \text{if } u \sim v; \\ 0, & \text{otherwise,} \end{cases}$$

We call eigenvalues of $\mathscr{L}(G)$ the *normalized Laplacian eiganvalues* of G, and the eigenvalues of $\mathscr{L}(G)$ are ordered by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0$. One can find that $\mathscr{L} = D^{-1/2}LD^{-1/2}$.

A remarkable analogy between the Kirchhoff and degree Kirchhoff indices is the formula [2]:

$$S'(G) = 2m \sum_{i=1}^{n-1} \frac{1}{\lambda_i}. \tag{1.2}$$

For the following consideration it is important to recall that $R(u,v) = R(v,u)$, $R(u,u) = 0$ and that [19] $d(u,v) \geq R(u,v)$ with equality if and only if there is a unique path linking the vertices u and v. Therefore, for trees, $Kf(G) = W(G)$ and $S'(G) = S(G)$.

In [31] the authors obtained the following result.

Theorem 1. *Let G be a connected (molecular) bipartite graph with $n \geq 2$ vertices and m edges. Then*

$$S'(G) \geq (2n-3)m$$

with equality if and only if $G \cong K_{r,n-r}$ for $1 \leq r \leq \lfloor \frac{n}{2} \rfloor$.

In [11] the upper and lower bounds the degree Kirchhoff index of unicyclic graphs is established. In this paper, we further study the degree Kirchhoff index of graphs and obtain several upper and lower bounds on the degree Kirchhoff index. We also obtain some bounds for the Nordhaus-Gaddum-type result for the degree Kirchhoff index.

2. Lemmas and results

Lemma 1 ([5])**.** *Let G be a graph on n vertices. Then for $n \geq 2$,*

$$\lambda_1 \geq \frac{n}{n-1},$$

the equality holds if and only if $G \cong K_n$.

Lemma 2 ([29]). *Let G be a graph with the largest degree Δ. Then*

$$\lambda_1(G) \geq 1 + \frac{1}{\Delta},$$

with equality if and only if G is a complete graph.

Recall that the *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors such that G can be colored in a way such that no two adjacent vertices have the same color.

Lemma 3 ([5]). *Let G be a graph on n vertices with chromatic number χ. Then*

$$\lambda_1(G) \geq \frac{\chi}{\chi - 1}.$$

Lemma 4 ([7]). *Let G be a connected graph of order $n > 2$. Then $\lambda_2 = \cdots = \lambda_{n-1}$ if and only if $G \cong K_n$ or $G \cong K_{p,q}$.*

Theorem 2. *Let G be a connected graph on $n > 2$ vertices and m edges. Then*

$$S'(G) \geq 2m\left(n - 2 + \frac{1}{n}\right),$$

$$S'(G) \geq 2m\left(\frac{\Delta}{\Delta+1} + \frac{(n-2)^2}{n-1-\frac{1}{\Delta}}\right),$$

$$S'(G) \geq 2m\left(\frac{\chi}{\chi+1} + \frac{(n-2)^2}{n-1-\frac{1}{\chi}}\right).$$

Each of the above equalities holds if and only if $G \cong K_n$.

Proof. From the definition, one has

$$\begin{aligned}
\frac{1}{2m}S' &= \sum_{i=1}^{n-1} \frac{1}{\lambda_i} \\
&= \frac{1}{\lambda_1} + \sum_{i=2}^{n-1} \frac{1}{\lambda_i} \\
&\geq \frac{1}{\lambda_1} + \frac{(n-2)^2}{\sum_{i=2}^{n-1} \lambda_i} \\
&= \frac{1}{\lambda_1} + \frac{(n-2)^2}{n-\lambda_1}.
\end{aligned}$$

Now we consider the function

$$f(x) = \frac{1}{x} + \frac{(n-2)^2}{n-x}.$$

It is easy to check that $\frac{df(x)}{dx} \geq 0$ for $x \geq \frac{n}{n-1}$, with equality if and only if $x = \frac{n}{n-1}$. Therefore it follows that $f(x)$ is not decreasing with respect to x. Since $x \geq \frac{n}{n-1}$ from Lemma 1, we have

$$f(x) \geq f\left(\frac{n}{n-1}\right) = n-2+\frac{1}{n}.$$

This implies the result.

If the equality holds, then all the inequalities in the proof must be equalities. It follows that $\lambda_2 = \cdots = \lambda_{n-1}$, and from Lemmas 4 and 1 we get the result. The reverse is easy to check.

Note that from the above proof and Lemmas 2, 3 we can get

$$f(x) \geq f\left(\frac{\Delta+1}{\Delta}\right) = \frac{\Delta}{\Delta+1} + \frac{(n-2)^2}{n-1-\frac{1}{\Delta}},$$
$$f(x) \geq f\left(\frac{\chi}{\chi-1}\right) = \frac{\chi}{\chi+1} + \frac{(n-2)^2}{n-1-\frac{1}{\chi}}.$$

Therefore we obtain the desired two bounds involving the maximum degree or the chromatic number, which is larger than $f\left(\frac{n}{n-1}\right)$. □

Fan Chung [5] obtained the following formula for the number of spanning trees of a graph based on the eigenvalues of the normalized Laplacian:

$$t(G) = \frac{\prod_v d(v)}{\sum_v d(v)} \prod_{i=1}^{n-1} \lambda_i.$$

Theorem 3. *Let G be a connected graph with $n \geq 3$ vertices, t spanning trees and maximum vertex degree Δ. Then*

$$S'(G) \geq 2m\left(\frac{\Delta}{\Delta+1} + (n-2)\left(\frac{(\Delta+1)\prod_v d(v)}{2mt\Delta}\right)^{\frac{1}{n-2}}\right),$$

with equality if and only if $G \cong K_n$.

Proof. From the definition, one has

$$\begin{aligned}
\frac{1}{2m}S' &= \sum_{i=1}^{n-1} \frac{1}{\lambda_i} \\
&= \frac{1}{\lambda_1} + \sum_{i=2}^{n-1} \frac{1}{\lambda_i} \\
&\geq \frac{1}{\lambda_1} + \frac{n-2}{(\prod_{i=2}^{n-1} \lambda_i)^{\frac{1}{n-2}}}
\end{aligned}$$

$$= \frac{1}{\lambda_1} + (n-2)\left(\frac{\lambda_1}{\prod_{i=1}^{n-1}\lambda_i}\right)^{\frac{1}{n-2}}$$
$$= \frac{1}{\lambda_1} + (n-2)\left(\frac{\lambda_1\prod_v d(v)}{2mt}\right)^{\frac{1}{n-2}}.$$

Let
$$g(x) = \frac{1}{x} + (n-2)\left(\frac{x\prod_v d(v)}{2mt}\right)^{\frac{1}{n-2}}.$$

Then we have
$$\frac{dg(x)}{dx} = -\frac{1}{x^2} + \left(\frac{x\prod_v d(v)}{2mt}\right)^{\frac{1}{n-2}-1}\left(\frac{\prod_v d(v)}{2mt}\right) = -\frac{1}{x^2} + \frac{1}{x}\left(\frac{x\prod_v d(v)}{2mt}\right)^{\frac{1}{n-2}}.$$

Therefore $g(x)$ is non-decreasing for
$$x \geq \left(\frac{2mt}{\prod_v d(v)}\right)^{\frac{1}{n-1}}.$$

Note that
$$\left(\frac{2mt}{\prod_v deg(v)}\right)^{\frac{1}{n-1}} = \left(\prod_{i=1}^{n-1}\lambda_i\right)^{\frac{1}{n-1}} \leq \frac{\sum_{i=1}^{n-1}\lambda_i}{n-1} = \frac{n}{n-1} \leq \frac{\Delta+1}{\Delta} \leq \lambda_1,$$

it follows that
$$g(x) \geq g\left(\frac{\Delta+1}{\Delta}\right) = \frac{\Delta}{\Delta+1} + (n-2)\left(\frac{(\Delta+1)\prod_v d(v)}{2mt\Delta}\right)^{\frac{1}{n-2}}.$$

This implies the result. The equality case is easy to see from Lemma 4. □

One may note from the above proof that
$$\frac{1}{2m}S' = \sum_{i=1}^{n-1}\frac{1}{\lambda_i} \geq \frac{n-1}{(\prod_{i=1}^{n-1}\lambda_i)^{\frac{1}{n-1}}} = (n-1)\left(\frac{\prod_v d(v)}{2mt}\right)^{\frac{1}{n-1}}, \tag{2.1}$$

with equality if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1}$, that is $G \cong K_n$.

The above expression implies that $S'(G)$ is strictly decreasing with respect to $t(G)$.

Before listing an upper bound for $t(G)$, we first recall one definition. For $k \geq 1$, a graph G is k-connected if either G is a complete graph K_{k+1}, or G has at least $k+2$ vertices and contains no $(k-1)$-vertex cut. The maximal value of k for which a connected graph G is k-connected is the *connectivity* of G, denoted by $\kappa(G)$. If G is disconnected, we define $\kappa(G) = 0$.

If G is a graph of order n, then $\kappa(G) \leq n-1$, with equality holding if and only if $G \cong K_n$.

Recently, the following result for the spanning trees of graphs was obtained in [20].

Let G be a connected graph of order n with connectivity $\kappa(G) = k$. Then

$$t(G) \le k n^{k-1}(n-1)^{n-k-2}.$$

The equality holds if and only if $G \cong (K_1 \cup K_{n-k-1}) \vee K_k$.

By using this bound and expression (2.1), we can get the following bound involving connectivity.

Theorem 4. *Let G be a connected graph with $n \ge 3$ vertices. If G has connectivity $\kappa(G) = k$, then*

$$S' \ge 2m(n-1)\left(\frac{\prod_v d(v)}{2mkn^{k-1}(n-1)^{n-k-2}}\right)^{\frac{1}{n-1}},$$

with equality if and only if $G \cong K_n$.

At last, we consider the Nordhaus-Gaddum-type result for the degree Kirchhoff index.

Theorem 5. *Let G be a connected (molecular) graph on $n \ge 5$ vertices with a connected complement $\overline{G}$. Then*

$$S'(G) + S'(\overline{G}) \ge \frac{(n-1)^3}{2}.$$

Proof. From the result in Theorem 2, we have

$$\begin{aligned} S'(G) + S'(\overline{G}) &\ge 2m\left(n-2+\frac{1}{n}\right) + \left(\binom{n}{2} - 2m\right)\left(n-2+\frac{1}{n}\right) \\ &= \binom{n}{2}\left(n-2+\frac{1}{n}\right) = \frac{(n-1)^3}{2}. \end{aligned}$$

This implies the result. □

Lemma 5 ([2]). *Let G be a graph of order n, with m edges and diameter D. For all $v_i, v_j \in V(G)(i \ne j)$, then*

$$R(v_i, v_j) \le 2mD\left(\frac{1}{d(v_i)} + \frac{1}{d(v_j)}\right).$$

Theorem 6. *Let G be a connected (molecular) graph on $n \ge 5$ vertices with diameter D. Then*

$$S'(G) \le 4D(n-1)m^2. \tag{2.2}$$

Proof. From the definition, we have

$$\begin{aligned}
S'(G) &= \sum_{\{u,v\}\subseteq V(G)} d(u)\,d(v)\,R(u,v) \\
&\le \sum_{\{u,v\}\subseteq V(G)} d(u)\,d(v)\,2mD\left(\frac{1}{d(u)}+\frac{1}{d(v)}\right) \\
&= 2mD \sum_{\{u,v\}\subseteq V(G)} (d(u)+d(v)) \\
&= 2mD(n-1)\sum_{u\in V(G)} d(u) \\
&= 4D(n-1)m^2.
\end{aligned}$$

This implies the result. □

Theorem 7. *Let G be a graph of order n with maximum degree Δ, and maximal Gutman index $S(G)$. Then the following holds:*

$$S'(G) \le \binom{n+1}{3}\Delta^3. \tag{2.3}$$

Proof. Since $d(u,v) \ge R(u,v)$ with equality if and only if there is a unique path linking the vertices u and v [19], we immediately get

$$S'(G) = \sum_{\{u,v\}\subseteq V(G)} d(u)\,d(v)\,R(u,v) \le \sum_{\{u,v\}\subseteq V(G)} d(u)\,d(v)\,d(u,v) = S(G).$$

In [1] it is obtained that under the hypothesis of the theorem,

$$S(G) \le \binom{n+1}{3}\Delta^3.$$

The desired result is obtained. □

We now present one example to illustrate that the above two bounds are incomparable. For the star graph $H = K_{1,n-1}$, bound (2.2) reads $8(n-1)^3$, while bound (4) is $(n-1)^3\binom{n+1}{3}$, showing that (2.2) is better than (2.3). For the path P_n, bound (3) reads $4(n-1)^4$, while bound (2.3) is $8\binom{n+1}{3}$, showing that (2.3) is better than (2.2). Therefore the above two bounds are incomparable.

In light of Theorems 6, 7, we can establish the following two upper bounds for the Nordhaus-Gaddum-type results.

Theorem 8. *Let G be a connected (molecular) graph on $n \geq 5$ vertices with a connected complement $\overline{G}$. Then*

$$S'(G) + S'(\overline{G}) \leq 4(n-1)^2 \left(m^2 + \left(\binom{n}{2} - m \right)^2 \right).$$

Proof. Note that the diameter of any graph is at most $n-1$. From the result in Theorem 6, we have

$$S'(G) + S'(\overline{G}) \leq 4(n-1)^2 \left(m^2 + \left(\binom{n}{2} - m \right)^2 \right).$$

This implies the result. □

Theorem 9. *Let G be a connected (molecular) graph on $n \geq 5$ vertices with a connected complement $\overline{G}$. Then*

$$S'(G) + S'(\overline{G}) \leq \binom{n+1}{3} \left(\Delta^3 + (n-1-\delta)^3 \right).$$

Proof. From the result in Theorem 7, we have the result. □

3. Further discussion

As stated in Section 1, formulae (1.1) and (1.2) reveal that those two resistance distance based molecular descriptors have a close connection with the eigenvalues of some matrices associated with graphs. For a edge weighted connected graph G with weight c_{uv} on the edge uv, and $c(u) = \sum_{uw \in E(G)} c_{uw}$, we define $\overline{S} = \sum_{\{u,v\} \subseteq V(G)} c(u)c(v)R(u,v)$. Then we may naturally ask that whether there exists a real symmetric and positive semidefinite matrix P with eigenvalues

$$\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_{n-1} > \gamma_n = 0$$

such that $\overline{S} = \sum_{i=1}^{n-1} \frac{1}{\gamma_i}$?

Conversely, in general, let M be a real symmetric and positive semidefinite matrix. Suppose its eigenvalues are

$$\rho_1 \geq \rho_2 \geq \ldots \geq \rho_{n-1} > \rho_n = 0,$$

with their corresponding mutually orthogonal unit eigenvector

$$\xi_1, \xi_2, \ldots \xi_{n-1}, \xi_n.$$

Now, we construct the matrix

$$B = M + \frac{1}{a} \xi_n (\xi_n)^t,$$

where $a = (\xi_n)^t \xi_n$. We claim that the eigenvalues of B are

$$\rho_1, \rho_2, \ldots, \rho_{n-1}, 1.$$

In fact, we can check that, for $1 \le i \le n-1$,

$$\begin{aligned} B\xi_i &= (M + \frac{1}{a}\xi_n(\xi_n)^t)\xi_i \\ &= M\xi_i + \frac{1}{a}\xi_n(\xi_n)^t\xi_i \\ &= M\xi_i \\ &= \rho_i\xi_i. \end{aligned}$$

Similarly,

$$B\xi_n = M\xi_n + \frac{1}{a}\xi_n(\xi_n)^t\xi_n = \frac{1}{a}\xi_n\big[(\xi_n)^t\xi_n\big] = \xi_n.$$

Since all the eigenvalues of B are not zero, B is non-singular. Therefore, its inverse B^{-1} exists and its eigenvalues are $\frac{1}{\rho_1}, \frac{1}{\rho_2}, \ldots, \frac{1}{\rho_{n-1}}, 1$.

If we take $M = L = D - A$, and $\xi_n = \mathbf{1}$, the all one vector, then we get Theorem 1 and Theorem 2 in [26]. And the formulae (1) is established.

If we take $M = \mathscr{L} = D^{-1/2}LD^{-1/2}$, and $\xi_n = D^{\frac{1}{2}}\mathbf{1}$, then we get some part of Theorem 3.1 in [2]. And the formulae (2) is established.

Another natural question is: for general M described above, is there a connection between $\sum_{i=1}^{n-1}\frac{1}{\rho_i}$ and some (resistance) distance based molecular descriptor of graphs?

We leave it for further research.

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A NOTE ON REFERENCE SOLUTION BASED HP-ADAPTIVE PDE SOLVERS

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Abstract. In this note we suggest an improvement for a family of hp-adaptive finite element methods. We point out the necessity of the new procedure by constructing model problems for which certain standard hp-adaptive algorithms fail to work properly. It is verified also in the corresponding simulations that the algorithm can terminate even though the numerical solution still contains a sizable computational error.

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Keywords: reference solution, counterexample, hp-adaptivity

1. INTRODUCTION

There are several hp-adaptive finite element algorithms in the literature for the numerical solution of PDE's (see the collection [8] and the references therein). These are based on the following scheme

> **Initialize**: solve the initial problem with small polynomial degree p on a coarse grid,
> **Repeat**:
> S1 estimate the error,
> S2 if the error is small then stop,
> S3 else determine on which elements in the grid and how to refine/derefine,
> S4 compute the new solution and go to S1.

The main differences between the different methods are in the error estimation and refinement procedures. Here we will focus on the method introduced by Demkowicz et al. [2] and also used (with a small modification) by Šolín et al. [11]. In step S1 this method uses the so called "reference solution" to compute the error. This reference solution is calculated using a uniform refinement in space and by increasing the degree of polynomials on *every* element in order to calculate a finer solution that will be considered as the reference solution. The error is defined as the difference between the reference solution and original solution.

Despite the fact that this method can be used for many different problems, such as time dependent problems [4, 13], Maxwell's equations [12, 14] or coupled highly nonlinear problems [5], the efficiency of the error estimation has not been justified rigorously. We illustrate this by the so-called antenna example [3, Section 15.3.] where the convergence of the method seems to be broken because the estimated error does not decrease during refinements. However, after further iteration steps it decreases again until the estimated error becomes small enough. In that case the error estimator works properly because it forces the refinements until the estimated error is small enough.

The aim of this paper is to give a counterexample where the reference solution is the same as the original one, hence the computed error will be zero, whereas the error between the exact solution and the approximate solution is large. Furthermore we suggest a modification of the method that can be used to avoid these kinds of difficulties.

2. NOTATIONS

We investigate the elliptic boundary-value problem

$$-\operatorname{div}(K\nabla u) + Cu = f \quad \text{in } \Omega, \tag{2.1}$$

$$u = g \quad \text{on } \Gamma, \tag{2.2}$$

and its weak form. Find $u \in H^1(\Omega)$ such that $u = u_D + \phi$ where $u_D = g$ on $\partial\Omega$, $\phi \in H_0^1(\Omega)$

$$a(\phi, v) = \int_\Omega f v - a(u_D, v) \quad \forall v \in H_0^1(\Omega), \tag{2.3}$$

$$a(\phi, v) := \int_\Omega K\nabla\phi \cdot \nabla v + \int_\Omega C\phi v,$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 1$, $\Gamma = \partial\Omega$, $f \in L_2(\Omega)$, $C \in L_\infty(\Omega)$, $K = \{k_{i,j}\}_{i,j=1}^d$ is a continuous, matrix valued, symmetric function on $\overline{\Omega}$ and $v \in H_0^1(\Omega)$ is a test function.

Let us denote by τ_h a tessallation of Ω into elements. The discretization of (2.3) takes the following form. Let us denote by $V_{h,p} \subset H^1(\Omega)$ a finite dimensional subspace. Find $u_{h,p} \in V_{h,p}$ such that $u_{h,p} = u_{D_{h,p}} + \phi_{h,p}$ where $u_{D_{h,p}} \in V_{h,p}$ approximates g on $\partial\Omega$, $\phi \in V_{h,p}$

$$a(\phi_{h,p}, v_{h,p}) = \int_\Omega f v_{h,p} - a(u_{D_{h,p}}, v_{h,p}) \quad \forall v \in V_{h,p}. \tag{2.4}$$

$V_{h,p}$ contains piecewise polynomials whose degrees can vary from element to element.

$V_{h/2,p+1}$ denotes the approximation space where the mesh is refined and the degree of the local polynomials is increased.

The reference solution based methods described in [2] and [11] use the following basic idea:

S1 compute $u_{h,p} \in V_{h,p}$ by solving (2.4),
S2 compute the reference solution $u_{h/2,p+1} \in V_{h/2,p+1}$ by solving (2.4) in the enriched space $V_{h/2,p+1}$,
S3 use $u_{h/2,p+1}$ as a more accurate solution and define $u_{h,p} - u_{h/2,p+1}$ as an error indicator,
S4 compute one of the following quantities on all $T \in \tau_h$

- $\eta_T = \dfrac{|u_{h,p} - u_{h/2,p+1}|_{H^1(T)}}{|u_{h/2,p+1}|_{H^1(T)}}$
- $\eta_T = \dfrac{\|u_{h,p} - u_{h/2,p+1}\|_{H^1(T)}}{\|u_{h/2,p+1}\|_{H^1(T)}}$
- $\eta_T = |u_{h,p} - u_{h/2,p+1}|_{H^1(T)}$
- $\eta_T = \|u_{h,p} - u_{h/2,p+1}\|_{H^1(T)}$

S5 if $\sqrt{\sum_T \eta_T^2} < \mathtt{TOL}$ stop, else refine where it is needed, and go to S1.

The difference between [2] and [11] lies in the choice of η_T and [2] does a refinement all over the edges not only all over the elements.

3. Counterexample

The construction of the counterexample is based on the following simple idea. Let us suppose that we can find a function $f \neq 0$ such that $\int_\Omega f v_{h,p} = 0$, $\forall v_{h,p} \in V_{h,p}$. In this case (2.4) simplifies to $a(\phi_{h,p}, v_{h,p}) = a(u_{D_{h,p}}, v_{h,p})$. Suppose that it has an exact solution $u_0 \in V_{h,p}$ so $u_{h,p} = u_0$ (i.e. if $g = 0$ then $u_D = 0$ yielding $u_0 = 0$).

If $\int_\Omega f v_{h/2,p+1} = 0$, $\forall v_{h/2,p+1} \in V_{h/2,p+1}$ then using the fact $V_{h,p} \subset V_{h/2,p+1}$ we have that $\int_\Omega f v_{h,p} = 0$ also holds. In this case $u_{h,p} = u_{h/2,p+1} = u_0$, and therefore the computed error is zero.

Now we show how to create a function f satisfying the above assumptions. For any $T \in \tau_h$ we define $u_c : \Omega \to \mathbb{R}$ such that $\operatorname{supp}(u_c) = T$, $u_c(x,y) = \chi_T(x,y) b_T^2(x,y) p(x,y)$, where χ_T is the characteristic function of T, $b_T : \Omega \to \mathbb{R}$ is a bubble function on T, so $b_T = 0$ on ∂T and

$$p(x,y) = \sum_{k=0}^{m-1} c_k x^{a_k} y^{b_k}. \tag{3.1}$$

Here $a_k, b_k \in \mathbb{N} \cup \{0\}$ and $c_k \in \mathbb{R}$, m is a fixed integer (that will be determined later). The polynomial $p(x,y)$ is chosen so that

$$\int_\Omega (-\operatorname{div}(K \nabla u_c) + C u_c) v = 0 \qquad \forall v : v \in V_{h/2,p+1}, \operatorname{supp}(v) = T, \tag{3.2}$$

or, since $\operatorname{supp}(u_c) = T$

$$\int_T (-\operatorname{div}(K\nabla u_c) + C u_c)v = 0 \qquad \forall v : v \in V_{h/2,p+1}. \tag{3.3}$$

In order to compute the coefficient vector $c = (c_0, \dots, c_{m-1})$ we have to solve the linear system $Ac = 0$ where the entries of $A \in \mathbb{R}^{n\times m}$ are given by

$$A_{i,j} = \int_T (-\operatorname{div}(K\nabla b_T^2 c_j x^{a_j} y^{b_j}) + C b_T^2 c_j x^{a_j} y^{b_j}) v_i \, dT \qquad \forall i \in \{1,2,\dots,n\}.$$

Here we have used the notation $\dim V_{h/2,p+1}|_T = n$. To find a nonzero solution c, A must have more columns than rows, so $m := n+1$. It is sufficient to find a submatrix $\overline{A} \in \mathbb{R}^{r(A)\times r(A)+1}$ where $r(A)$ is the rank of A, and solve the reduced system $\overline{A}\overline{c} = 0$.

If we have such a solution then we have at least one free parameter to define c. We set this parameter to an arbitrary number, i.e. 1. This can be used as coefficients c in the definition of u_c.

For any $C_0 \in \mathbb{R}$ the test function $u_0 + C_0 u_c$ will give $u_{h,p} = u_{h/2,p+1} = u_0$ and the algorithm will terminate, even though the error $C_0 u_c$ can be arbitrarily large.

Remark 1. If we use implicit a posteriori error estimation first we solve (2.4) which gives $u_{h,p} = u_0$. Then a local Neumann problem is solved on every element $\widetilde{T}$

$$-\operatorname{div}(K\nabla e) + Ce = f + \operatorname{div}(K\nabla u_{h,p}) - C u_{h,p} \quad \text{in } \widetilde{T}, \tag{3.4}$$

$$\frac{\partial e}{\partial \eta} = -\frac{1}{2}\left[\frac{\partial u_{hp}}{\partial \eta}\right] \quad \text{on } \partial\widetilde{T} \setminus \partial\Omega, \tag{3.5}$$

$$e = 0 \quad \text{on } \partial\widetilde{T} \cap \partial\Omega, \tag{3.6}$$

where e is an estimator of $u - u_{h,p}$, $\left[\frac{\partial u_{hp}}{\partial \eta}\right]$ is the jump of the outward normal derivative of the numerical solution on an interior edge (see Chapter 3 in [1] for details). If $u_0 = 0$ on T then the r.h.s. of (3.4) will be f. The estimated boundary condition will also be zero. The solution of (3.4)-(3.6) depends on the local finite element space $W_{\widetilde{T}}$. If $W_{\widetilde{T}} \subseteq V_{h/2,p+1}$, then we will again have $e = 0$ on $\widetilde{T}$.

4. Numerical results

By courtesy of William F. Mitchell the procedure described above was tested numerically using his PHAML code [7]. The code was supplied with the following initial mesh:

The problem was a simple Poisson equation, $K \equiv 1$, $C \equiv 0$, with Dirichlet boundary condition in (2.1)-(2.2)

$$-\triangle u(x,y) = f(x,y) \quad \text{in } \Omega,$$
$$u(x,y) = u_D(x,y) \quad \text{on } \Gamma,$$

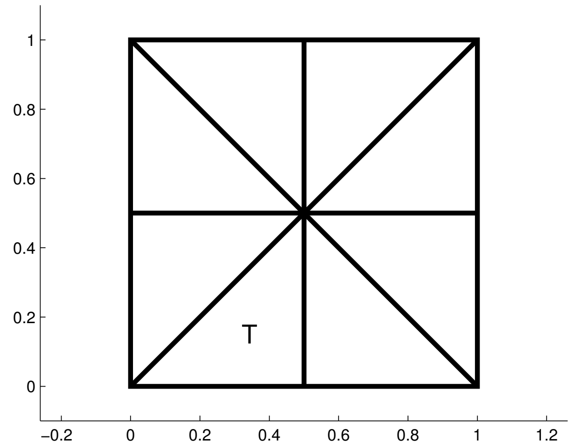

FIGURE 1. The initial mesh and triangle T.

with $\Omega = (0,1)^2$, and $b_k = 0$ (see (3.1)). The polynomial degree p was 1 at the initial step. For computing the reference solution PHAML used bisected triangles.

We used as our counterexample

$$\begin{aligned} u_c(x,y) = x + C_0 \chi_T(x,y)(&-3200(x-y)^2 y^2 (2x-1)^2(-1+4x)^2 \cdot \\ &(1810432x^7 - 4313088x^6 + 4323072x^5 - 2356224x^4 \\ &+ 751088x^3 - 139176x^2 + 13747x - 549)), \end{aligned} \tag{4.1}$$

and $f_c(x,y) := -\triangle u_c(x,y)$, $u_D(x,y) = x|_\Gamma$, $C_0 = 10^5$.

The second term of u_c was calculated by the method described above. Theoretically we should have $u_{h,p} = u_{h/2,p+1} = x$ according to the previous section, yielding that the real error $\|u - u_{h,p}\|_{H^1}$ can be arbitrary and we can control it by our choice of C_0.

When we implemented this model problem in PHAML we encountered problems with numerical integration. One can verify that $\int_{T_j} f_c(x,y) \cdot x^k y^l = 0$ if $0 \le k, l \le 2$, $k + l \le 2$ for all T_j that are a subtriangle of T and for any C_0. However, even when the highest available order quadrature was used it was different from zero and the r.h.s. of (2.3) became nonzero.

We obtained $\|u_{h/2,p+1} - u_{h,p}\|_{H^1(\Omega)} \approx 10^{-9}$, which was our main aim. The addition of x was necessary. Without it $\|u_{h/2,p+1}\|_{H^1(\Omega)} \approx 10^{-9}$ and the relative error was $O(1)$. The addition of 1 would not make any difference if we used a seminorm instead of a norm.

For all possible stopping criteria mentioned in Section 2 we could achieve $\sqrt{\sum_T \eta_T^2} \approx 10^{-9}$. This means that the algorithm terminated at the initial step whenever $\mathtt{TOL} > 10^{-8}$, even though the computational error can be almost arbitrary, depending only on C_0.

5. Conclusion

We can easily fix this problem by building a back-up estimator into the code. For example, we can use the residual-based error estimator

$$|||u - u_{h,p}|||^2 \leq \eta_{\mathrm{res}}^2 = C_{\mathrm{res}} \left(\sum_{T \in \tau_h} h_T^2 \|r\|_{L_2(T)}^2 + \sum_{\gamma \in \partial T} h_T \|R\|_{L_2(\gamma)}^2 \right), \tag{5.1}$$

where r is the interior residual $r = f + \operatorname{div}(K \nabla u_{h,p}) - C u_{h,p}$, $R = \left[\frac{\partial u_{hp}}{\partial \eta} \right]$ is the jump of the derivative of the numerical solution on the interior edges, $|||u|||^2 = a(u,u)$ is the energy norm (see [1] for details), and C_{res} is a constant which does not depend on h.

It is well known that reference solution based methods are a very effective class of adaptive techniques. The inequality (5.1) supplies a guaranteed upper bound; on the other hand, its use for the purpose of hp-adaptivity is a little bit complicated. We should modify our algorithm as follows:

Initialize: solve the initial problem with small polynomial degree p on a coarse grid

Repeat:

- S1 compute the error using one of the quantities from Step 4 of the algorithm described at the end of Section 2.
- S2 if the error is small then use (5.1)
  - S2a if $\eta_{\mathrm{res}} < \mathtt{TOL}$ terminate
  - S2b else do a brute-force adaptive step (both h and p) and go to step S4. (See Remark 3.)
- S3 else determine on which elements in the grid and how to refine/derefine
- S4 compute the new solution and go to S1

Remark 2. By adopting this strategy we can avoid the need for building an effective hp-adaptive technique that impinges on a residual-based estimator. We propose to use a brute-force adaptive step in order to try to avoid a major modification of the original algorithm. It is possible to use the hp-adaptive technique from [6] that provides lower degrees of freedom, however, the method would become more complicated.

Remark 3. There exists another possible correction that was published in [9, 10]. Those papers introduce an adaptive method that is proved to be convergent. The key idea is the measurement of the oscillation of f. Elements should be checked twice:

first according to the error estimator and after that according to the oscillation of the right hand side. If the refinement involves all problematic elements the method will converge. In our case the only question would be how to decide which refining option should be used over the elements where there is considerable oscillation: they could be refined both in h and p.

The reference solutions are characterized by one fundamental issue: they assume a local behavior of the error. This means that in most cases the error can be located easily if it is caused by a rough mesh or low polynomial degree at the place where it is detected. Otherwise, when the error is spread around the reference solution methods fail. However, above we have seen that it can also fail to converge to the exact solution even when the error is caused by low polynomial degree on an element.

The idea of creating the counterexample, can be used for many applications. Although, we should note that the way of the correction is a more difficult question. The residual-based error estimator is hard to carry out for every type of PDEs, especially with the proper constant. On the other hand, the reference solution based algorithm is easy-to-use for wide range of problems.

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NOTES ON GENERALIZED DERIVATIONS OF $*$-PRIME RINGS

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Abstract. Let R be a $*$-prime ring with characteristic different from two and $U \neq 0$ be a square closed $*$-Lie ideal of R. An additive mapping $F : R \to R$ is called an generalized derivation if there exits a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$. In the present paper, it is shown that $U \subseteq Z$ if R is a $*$-prime ring which admits a generalized derivation satisfying several conditions that are associated with a derivation commuting with $*$.

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Keywords: prime rings, derivations, generalized derivations, Lie ideals

1. INTRODUCTION, ETC

Let R will be an associative ring with center Z. For any $x, y \in R$, denote the commutator $xy - yx$ by $[x, y]$ and the anti-commutator $xy + yx$ by $x \circ y$. Recall that a ring R is prime if $xRy = 0$ implies $x = 0$ or $y = 0$. An additive mapping $* : R \to R$ is called an involution if $(xy)^* = y^* x^*$ and $(x^*)^* = x$ for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or $*-$ring. A ring with an involution is said to $*-$prime if $xRy = xRy^* = 0$ or $xRy = x^*Ry = 0$ implies that $x = 0$ or $y = 0$. Every prime ring with an involution is $*-$prime but the converse need not hold general. An example due to Oukhtite [6] justifies the above statement. Suppose that R is a prime ring, $S = R \times R^o$ where R^o is the opposite ring of R. Define involution $*$ on S as $*(x, y) = (y, x)$. S is $*-$prime, but not prime. This example shows that every prime ring can be injected in a $*-$prime ring and from this point of view $*-$prime rings constitute a more general class of prime rings. In all that follows the symbol $S_{a_*}(R)$, first introduced by Oukhtite, will denote the set of symmetric and skew symmetric elements of R, i.e. $S_{a_*}(R) = \{x \in R \mid x^* = \pm x\}$.

An additive subgroup U of R is said to be a Lie ideal of R if $[U, R] \subseteq U$. A Lie ideal is said to be a $*$-Lie ideal if $U^* = U$. If U is a Lie (resp. $*$-Lie) ideal of R, then U is called a square closed Lie (resp. $*$-Lie) ideal of R if $x^2 \in U$ for all $x \in U$.

An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. In particular, for fixed $a \in R$, the mapping $I_a : R \to R$ given by $I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation. An additive function $F : R \to R$ is called a generalized inner derivation if $F(x) = ax + xb$ for fixed $a, b \in R$. For such mapping F, it is easy to see that

$$F(xy) = F(x)y + x[y,b] = F(x)y + xI_b(y), \text{ for all } x, y \in R.$$

This observation leads to the following definition, an additive mapping $F : R \to R$ is called a generalized derivation associated with a derivation $d : R \to R$ if $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.

Familiar examples of generalized derivations are derivations and generalized inner derivations, and the latter includes left multipliers. Since the sum of two generalized derivation is a generalized derivation, every map of the form $F(x) = cx + d(x)$, where c is a fixed element of R and d a derivation of R is a generalized derivation, and if R has multiplicative identity, then all generalized derivations have this form.

Over the past thirty years, there has been an ongoing interest concerning the relationship between the commutativity of a prime ring R and the behavior of a special mapping on that ring. Recently, some well-known results concerning prime rings have been proved for $*-$prime rings by Oukhtite et al. (see [4–8], where further references can be found). In the year 2005, Ashraf et al. [1] proved some commutativity theorems for prime rings. Recently the first author with Al-Omary [3] obtained the commutativity of $*$-prime ring R admitting generalized derivations satisfying several conditions. Motivated by the above results, in this paper we shall discuss the situation when a $*$-prime ring R which admits a generalized derivation F associated with a derivation d satisfying any one of the following properties: (i) $d(u)oF(v) = [u,v]$, (ii) $[d(u), F(v)] = (u \circ v)$ (iii) $d(u)F(v) = [u,v]$ (iv) $d(u)F(v) = (u \circ v)$ (v) $[d(u), F(v)] = uv$ and (vi) $(d(u) \circ F(v)) = uv$.

2. Results

2.1. *Preliminary considerations*

The followings are some useful identities which hold for every $x, y, z \in R$. We will use them in the proof of our theorems.

- $[x, yz] = y[x,z] + [x, y]z$
- $[xy, z] = [x, z]y + x[y, z]$
- $xo(yz) = (xoy)z - y[x,z] = y(xoz) + [x, y]z$
- $(xy) \circ z = x(y \circ z) - [x,z]y = (x \circ z)y + x[y,z]$.

We begin our discussion with the following results.

Lemma 1 ([7, Lemma 4])**.** *Let R be a $*-$prime ring with characteristic not two, U be a nonzero $*-$Lie ideal of R and $a, b \in R$. If $aUb = aUb^* = 0$, then $a = 0$ or $b = 0$ or $U \subseteq Z$.*

Lemma 2 ([2, Lemma 2.7])**.** *Let R be a $*-$prime ring with characteristic not two, U be a nonzero $*-$Lie ideal of R. If $a \in R$ such that $[a, U] \subseteq Z$, then either $a \in Z$ or $U \subseteq Z$.*

The following Lemma is immediate consequences of Lemma 2.

Lemma 3. *Let R be a $*-$prime ring with characteristic not two and U be a nonzero $*-$Lie ideal of R. Suppose that $[U, U] \subseteq Z$, then $U \subseteq Z$.*

Lemma 4 ([5, Lemma 2.4])**.** *Let R be a $*-$prime ring with characteristic not two, d be a nonzero derivation of R which commutes with $*$ and U be a nonzero $*-$Lie ideal of R. If $d(U) \subseteq Z$, then $U \subseteq Z$.*

Lemma 5 ([5, Lemma 2.5])**.** *Let R be a $*-$prime ring with characteristic not two, d be a nonzero derivation of R which commutes with $*$ and U be a nonzero $*-$Lie ideal of R. If $a \in R$ and $ad(U) = 0$ $(d(U)a = 0)$, then $a = 0$ or $U \subseteq Z$.*

Lemma 6 ([5, Theorem 1.1])**.** *Let R be a $*-$prime ring with characteristic not two, d be a nonzero derivation of R which commutes with $*$ and U be a nonzero $*-$Lie ideal of R. If $d^2(U) = 0$, then $U \subseteq Z$.*

2.2. *Something else*

Theorem 1. *Let R be a $*-$prime ring with characteristic not two and U be a nonzero square closed $*-$Lie ideal of R. Suppose that R admits a generalized derivation F associated with nonzero derivation d which commutes with $*$ such that*

(i) $(d(u) \circ F(v)) = [u, v]$ for all $u, v \in U$, or
(ii) $[d(u), F(v)] = (u \circ v)$ for all $u, v \in U$, or
(iii) $[d(u), F(v)] = uv$ for all $u, v \in U$, or
(iv) $d(u) \circ F(v) = uv$ for all $u, v \in U$, or

If $F = 0$ or $d \neq 0$, then $U \subseteq Z$.

Proof. Suppose on the contrary that $U \nsubseteq Z$. Write $L = [U, U]$, then it is easy to show that L is a Lie ideal and $d(L) \subseteq U$. Moreover, since $U \nsubseteq Z$, then $L \nsubseteq Z$ by Lemma 3.

(i) If $F = 0$, then by Lemma 3, we get $U \subseteq Z$, a contradiction.
Henceforth, we shall assume that $d \neq 0$. We have

$$(d(u) \circ F(v)) = [u, v] \text{ for all } u, v \in U. \tag{2.1}$$

Replacing v by $2vw$ in (2.1) and using (2.1), and the fact that $charR \neq 2$, we conclude that

$$-F(v)[d(u), w] + v(d(u) \circ d(w)) + [d(u), v]d(w) = v[u, w], \tag{2.2}$$

for all $u,v,w \in U$. Now, choose $w = d(u)$ for all $u \in L$ in the above expression we find that

$$v\left(d\left(u\right) \circ d^2\left(u\right)\right) + \left[d\left(u\right), v\right] d^2\left(u\right) = v\left[u, d\left(u\right)\right]. \tag{2.3}$$

Again, replace v by $2vw$ in (2.3) and use (2.3), to get

$$\left[d\left(u\right), v\right] U d^2\left(u\right) = 0 \text{ for all } v \in U, u \in L. \tag{2.4}$$

Since U is a nonzero square closed $*-$Lie ideal of R, L is a nonzero square closed $*-$Lie ideal of R, too. Hence using $*d = d*$, we get

$$\left[d\left(u\right), v\right] U (d^2\left(u\right))^* = 0 \text{ for all } v \in U, u \in L \cap S_{a_*}(R).$$

Thus, by Lemma 1, we get either $[d(u), v] = 0$, for all $v \in U$ or $d^2(u) = 0$ for each $u \in L \cap S_{a_*}(R)$. Let $u \in L$, as $u + u^*, u - u^* \in L \cap S_{a_*}(R)$ and $[d(u \mp u^*), v] = 0$, for all $v \in U$ or $d^2(u \mp u^*) = 0$. Hence, we have $\left[d\left(u\right), v\right] = 0$ or $d^2\left(u\right) = 0$, for all $v \in U, u \in L$. We obtain that L is the set theoretic union of two its proper subgroups viz.

$$A = \{u \in L \mid [d(u), v] = 0\}$$

and

$$B = \{u \in L \mid d^2(u) = 0\}.$$

But a group cannot be the set-theoretic union of two proper subgroups, hence $A = L$ or $B = L$. If $A = L$, then $\left[d\left(u\right), U\right] = 0$ for all $u \in L$, and hence by Lemma 2 we find that $d(u) \in Z$ for all $u \in L$. Thus, by Lemma 4 and Lemma 3, we have $U \subseteq Z$, a contradiction. On the other hand, if $B = L$, then $d^2(L) = 0$ that is, $L \subseteq Z$ by Lemma 6, and so again using Lemma 3, we get the required result.

(ii) If $F = 0$, then $u \circ v = 0$, for all $u,v \in U$. Replacing v by $2vw$, $w \in U$ in the last equation, we get $U\left[u, w\right] = 0$, for all $u,w \in U$. Hence we arrive at

$$vU[u,w] = 0 \text{ for all } u,v,w \in U. \tag{2.5}$$

Since U is a nonzero $*-$Lie ideal of R yields that $v^*U[u,w] = 0$ for all $u,v,w \in U$. Hence, we have $vU[u,w] = v^*U[u,w] = 0$ for all $u,w,v \in U$. By Lemma 1, we get either $[u,w] = 0$, for all $u,w \in U$ or $v = 0$ for each $v \in U$. And so, $[u,w] = 0$, for all $u,w \in U$. By Lemma 3, we obtain $U \subseteq Z$.

Therefore, we shall assume that $d \neq 0$. We have

$$[d(u), F(v)] = u \circ v \text{ for all } u,v \in U. \tag{2.6}$$

Replacing v by $2vw$ in (2.6 and using (2.6), we conclude that

$$F\left(v\right)\left[d\left(u\right), w\right] + v[d\left(u\right), d\left(w\right)] + \left[d\left(u\right), v\right] d\left(w\right) = -v\left[u, w\right], \tag{2.7}$$

for all $u,v,w \in U$. For any $u \in L$, replace w by $d(u)$ in (2.7), to get

$$v[d\left(u\right), d^2\left(u\right)] + \left[d\left(u\right), v\right] d^2\left(u\right) = -v\left[u, d\left(u\right)\right], \tag{2.8}$$

for all $v \in U, u \in L$. Now, replacing v by $2vw$ in (2.8) and using (2.8), we see that $[d(u), v]Ud^2(u) = 0$ for all $v \in U, u \in L$. Notice that the arguments given in the proof of (i) after equation (2.4) are still valid in the present situation and hence repeating the same process, we the required result.

(iii)If $F = 0$, then $uv = 0$, for all $u, v \in U$ and hence $u[v, w] = 0$ for all $u, v, w \in U$. Now using the similar arguments as used in the proof of (ii) that follows equation (2.5), we get the required result.

Therefore, we shall assume that $d \neq 0$. For any $u, v \in U$, we have

$$[d(u), F(v)] = uv. \tag{2.9}$$

Replacing v by $2vw$, in (2.9) and using (2.9), we arrive at

$$F(v)[d(u), w] + [d(u), v]d(w) + v[d(u), d(w)] = 0 \text{ for all } u, v, w \in U.$$

For any $u \in L$, replace w by $d(u)$, we obtain that

$$[d(u), v]d^2(u) + v\left[d(u), d^2(u)\right] = 0 \text{ for all } u \in L, v \in U. \tag{2.10}$$

Now, replace v by $2vw$, in (2.10), to get

$$2\left([d(u), v]wd^2(u) + v[d(u), w]d^2(u) + vw\left[d(u), d^2(u)\right]\right) = 0,$$

for all $u \in L, v, w \in U$. In the view of (2.10) the above expression yields that $2[d(u), v]wd^2(u) = 0$, for all $u \in L, v, w \in U$. Since $charR \neq 2$, we find that $[d(u), v]wd^2(u) = 0$ for all $u \in L, v, w \in U$. that is, $[d(u), v]Ud^2(u) = 0$ for all $v \in U, u \in L$, and hence using the similar arguments as used in the proof of (i) that follows equation (2.4), we find the required result.

(iv) If $F = 0$, then $uv = 0$, for all $u, v \in U$. Using the same arguments as we used in the proof of (iii), we get the required result.

Therefore, we shall assume that $d \neq 0$. For any $u, v \in U$, we have

$$(d(u) \circ F(v)) = uv. \tag{2.11}$$

Replacing v by $2vw$, in (2.11) and using (2.11), we obtain that

$$-F(v)[d(u), w] + [d(u), v]d(w) + v(d(u)od(w)) = 0 \text{ for all } u, v, w \in U.$$

Now, replace w by $d(u)$ for all $u \in L$, to get

$$[d(u), v]d^2(u) + v\left(d(u)od^2(u)\right) = 0 \text{ for all } u \in L, v \in U.$$

Now, applying similar technique as the one used after equation (2.10) in the proof of (iii), we get the required result. $\square$

Theorem 2. *Let R be a $*-$prime ring with characteristic not two and U be a nonzero square closed $*-$Lie ideal of R. Suppose that R admits a generalized derivation F associated with nonzero derivation d which commutes with $*$ such that*

(i) $d(u)F(v) = [u,v]$ for all $u,v \in U$, or

(ii) $d(u)F(v) = u \circ v$ for all $u,v \in U$.

If $F = 0$ or $d \neq 0$, then $U \subseteq Z$.

Proof. (i)If $F = 0$, then $[u,v] = 0$ for all $u,v \in U$. Thus, by Lemma 3, we obtain that $U \subseteq Z$.
Suppose on contrary that $U \nsubseteq Z$. Therefore, we shall assume that $d \neq 0$. We have

$$d(u)F(v) - [u,v] = 0 \text{ for all } u,v \in U. \tag{2.12}$$

Replacing u by $2uw, w \in U$ in (2.12), we get

$$2(d(u)wF(v) + ud(w)F(v) - u[w,v] - [u,v]w) = 0.$$

Using (2.12) and $charR \neq 2$, we see that

$$d(u)wF(v) - [u,v]w = 0 \text{ for all } u,v,w \in U. \tag{2.13}$$

Substituting v for u in (2.13), we have

$$d(u)wF(u) = 0 \text{ for all } u,w \in U.$$

That is,

$$d(u)UF(u) = 0 \text{ for all } u \in U.$$

Using $*d = d*$, we get

$$(d(u))^* UF(u) = 0, \text{ for all } u \in U \cap S_{a_*}(R).$$

By Lemma 1, we get either $d(u) = 0$ or $F(u) = 0$, for each $u \in U \cap S_{a_*}(R)$. Let $u \in U$, as $u+u^*, u-u^* \in U \cap S_{a_*}(R)$ and $d(u \mp u^*) = 0$ or $F(u \mp u^*) = 0$. Hence, we obtain that $d(u) = 0$ or $F(u) = 0$, for all $u \in U$. Let

$$K = \{u \in U \mid d(u) = 0\}$$

and

$$L = \{u \in U \mid F(u) = 0\}$$

of additive subgroups of U. Now using the same argument, we get $U = K$ or $U = L$. If $U = K$, then $U \subseteq Z$ by Lemma 4, a contradiction. If $U = L$, then $0 = F(uv) = F(u)v + ud(v) = ud(v)$, and so $Ud(U) = (0)$. Hence $U \subseteq Z$ by Lemma 5, a contradiction. This completes the proof.

(ii) If $F = 0$, then $uov = 0$, for all $u,v \in U$, and hence using the same arguments as used in the proof of Theorem 1(ii), we get $U \subseteq Z$.

Suppose on the contrary that $U \nsubseteq Z$. Henceforth, we shall assume that $d \neq 0$. We have

$$d(u)F(v) - u \circ v = 0 \text{ for all } u, v \in U. \tag{2.14}$$

Writing $2uw, w \in U$ by u in (2.14) and using $charR \neq 2$, we find that

$$d(u)wF(v) + ud(w)F(v) - u(wov) + [u,v]w = 0.$$

Applying (2.14), we obtain

$$d(u)wF(v) + [u,v]w = 0 \text{ for all } u, v, w \in U.$$

Using the same arguments as used in the proof of (i) after equation (2.13), we get the required result. □

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THE INTEGRAL REPRESENTATION OF THE SOLUTION OF SOME MIXED PROBLEM FOR EVOLUTIONARY EQUATIONS

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Abstract. By means of the method of hybrid integral transform of Euler-Fourier-Bessel type with spectral parameter the integral representation of exact analytical solution of mixed problem for the system of equations of parabolic type in the three-part segment $[0, R]$ is obtained under the assumption that the limits are soft. Modeling of evolutionary processes is performed by the method of hybrid differential Euler-Fourier-Bessel operators.

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1. INTRODUCTION

The theory of boundary value problems for partial differential equations is an important section of the modern theory of differential equations which is intensively developing in the present time. The popularity of the problem is the consequence of the significance of its results in the development of many mathematical problems, as well as of its numerous applications in mathematical modeling of different processes and phenomenon of physics, chemistry, mechanics, biology, medicine, economics, engineering.

It is well known that the complexity of a boundary-value problem significantly depends on the coefficients of equations (different types of degeneracy and features) and the geometry of domain (smoothness of the boundary, the presence of corner points, etc.) in which the problem is considered. The dependence of the properties of solutions of boundary value problems for linear, quasi-linear, and certain classes of nonlinear equations in homogeneous domains on the above-mentioned properties of the coefficients of equations and geometry of domain are studied in detail, and functional spaces of correctness of problems for some domains are constructed [2, 13].

However, many important applied problems of thermophysics, thermodynamics, theory of elasticity, theory of electrical circuits, theory of vibrations lead to boundary value problems for partial differential equations not only in homogeneous domains (when the coefficients of the equations are continuous), but also in inhomogeneous and piecewise homogeneous domains if the coefficients of the equations are piecewise continuous or piecewise constant [1, 14].

Besides the method of separation of variables [17], which is one of the important and effective methods of studying linear boundary value problems for partial differential equations, there is the method of integral transforms, which makes it possible to obtain analytical solutions of some boundary-value problems via their integral images. It is also worth mentioning that for a rather wide class of problems (in piecewise homogeneous domains) there exists the effective method of hybrid integral transforms. They are generated by differential operators, when on each of the components of the piecewise homogeneous domain one considers different differential operators or differential operators of the same form, but with different sets of coefficients [3–5, 7].

In theoretical investigations and in applied problems the most frequently used differential operators are of second order, in particular the Fourier differential operator [16]

$$F = \frac{d^2}{dr^2}, \tag{1.1}$$

the Euler differential operator [9]

$$B_\alpha^* = r^2\frac{d^2}{dr^2} + (2\alpha+1)r\frac{d}{dr} + \alpha^2, \tag{1.2}$$

the Bessel differential operator [11]

$$B_{\nu,\alpha} = \frac{d^2}{dr^2} + \frac{2\alpha+1}{r}\frac{d}{dr} - \frac{\nu^2-\alpha^2}{r^2}, \tag{1.3}$$

the generalized Legendre differential operator [6]

$$\Lambda_{(\mu)} = \frac{d^2}{dr^2} + cthr\frac{d}{dr} + \frac{1}{4} + \frac{1}{2}\left(\frac{\mu_1^2}{1-chr} + \frac{\mu_2^2}{1+chr}\right),\ \mu = (\mu_1, \mu_2) \tag{1.4}$$

and Kontorovich-Lebedev differential operator [10]

$$B_\alpha = r^2\frac{d^2}{dr^2} + (2\alpha+1)r\frac{d}{dr} + \alpha^2 - \lambda^2 r^2. \tag{1.5}$$

If $\theta(x)$ is the Heaviside step function [15] and L_j is one of listed differential operators, then we can always create hybrid differential operator that corresponds to the geometric structure of piecewise homogeneous domain.

For example, for the piecewise homogeneous interval $(R_0, R_1)\cup(R_1, R_2)\cup(R_2, R)$ it is possible to create hybrid differential operator (HDO)

$$\begin{aligned} M = {} & \theta(r-R_0)\theta(R_1-r)a_1^2L_1 + \theta(r-R_1)\theta(R_2-r)a_2^2L_2 + \\ & +\theta(r-R_2)\theta(R-r)a_3^2L_3;\ a_j^2 = const. \end{aligned} \tag{1.6}$$

It is obvious that operator L_1 is defined in the interval (R_0, R_1), operator L_2 is defined in the interval (R_1, R_2), and operator L_3 is defined in the interval (R_2, R). It is clear, that if we change the order of operators L_1, L_2, L_3 we get other hybrid differential operator.

2. Formulation of the problem

Here we consider the problem of the structure of the solution which is bounded in the set $D_2 = \{(t,r) : t \in (0;+\infty);\, r \in I_2 = (0,R_1)\cup(R_1,R_2)\cup(R_2,R);\, R < +\infty\}$ for the system of evolutionary partial differential equations of parabolic type [17]

$$\begin{aligned} &\frac{\partial u_1}{\partial t} + \gamma_1^2 u_1(t,r) - a_1^2 B^*_{\alpha_1}[u_1(t,r)] = f_1(t,r),\, r \in (0,R_1),\\ &\frac{\partial u_2}{\partial t} + \gamma_2^2 u_2(t,r) - a_2^2 F[u_2(t,r)] = f_2(t,r),\, r \in (R_1,R_2),\\ &\frac{\partial u_3}{\partial t} + \gamma_3^2 u_3(t,r) - a_3^2 B_{\nu,\alpha_2}[u_3(t,r)] = f_3(t,r),\, r \in (R_2,R) \end{aligned} \tag{2.1}$$

with zero initial conditions, boundary conditions

$$\lim_{r\to 0} \frac{\partial^k u_1(t,r)}{\partial r^k} = 0, \quad L^3_{22}[u_3(t,r)]\Big|_{r=R} = g_R(t) \tag{2.2}$$

and conjugate conditions

$$\Big(L^k_{j1}[u_k(t,r)] - L^k_{j2}[u_{k+1}(t,r)]\Big)\Big|_{r=R_k} = \omega_{jk}(t);\; j,k = 1,2. \tag{2.3}$$

Here $B^*_{\alpha_1}$ is Euler differential operator (1.2) with parameter $\alpha = \alpha_1$, F is Fourier differential operator (1.1), B_{ν,α_2} is Bessel differential operator (1.3) with parameter $\alpha = \alpha_2$; $2\alpha_j + 1 > 0$, $\nu \geq \alpha_2$.

In the boundary conditions (2.2) at the point $r = R$ and in the conjugate conditions (2.3) the differential operators

$$L^m_{jk} = \Big(\alpha^m_{jk} + \delta^m_{jk}\frac{\partial}{\partial t}\Big)\frac{\partial}{\partial r} + \beta^m_{jk} + \gamma^m_{jk}\frac{\partial}{\partial t};\; j,k = 1,2;\, m = 1,2,3,$$

take part.

We assume that the following conditions on the coefficients are true: $\alpha^m_{jk} \geq 0$, $\beta^m_{jk} \geq 0$, $\delta^m_{jk} \geq 0$, $\gamma^m_{jk} \geq 0$, $a_j > 0$, $c_{j1,k} = \alpha^k_{2j}\beta^k_{1j} - \alpha^k_{1j}\beta^k_{2j}$, $c_{11,k}\cdot c_{21,k} > 0$; $\alpha^3_{22} + \beta^3_{22} \neq 0$, $c_{j2,k} = \delta^k_{2j}\gamma^k_{1j} - \delta^k_{1j}\gamma^k_{2j} = 0$, $\alpha^k_{11}\gamma^k_{21} - \alpha^k_{21}\gamma^k_{11} = \beta^k_{11}\delta^k_{21} - \beta^k_{21}\delta^k_{11}$, $\alpha^k_{12}\gamma^k_{22} - \alpha^k_{22}\gamma^k_{12} = \beta^k_{12}\delta^k_{22} - \beta^k_{22}\delta^k_{12}$, $j,k = 1,2$.

Remark 1. The presence of the operator $\dfrac{\partial}{\partial t}$ in the boundary conditions at the point $r = R$ and in the conjugate conditions can be interpreted as the softness of the boundary of the domain when it reflects heat waves.

Remark 2. In the case of hard boundary of the domain for the reflection of waves ($\delta_{jk}^m = 0$, $\gamma_{jk}^m = 0$), we have mixed a problem with classical boundary conditions and classical conjugate conditions. Its solution can be derived from the solution of the problem (2.1)-(2.3) as a particular case.

3. HYBRID EULER-FOURIER-BESSEL INTEGRAL TRANSFORM WITH SPECTRAL PARAMETER

Let's consider in the set I_2 the hybrid differential operator

$$M_{\nu,(\alpha)} = \theta(r)\theta(R_1 - r)B_{\alpha_1}^* + \theta(r - R_1)\theta(R_2 - r)F + \theta(r - R_2)\theta(R - r)B_{\nu,\alpha_2}. \tag{3.1}$$

Here $\theta(x)$ is the Heaviside step function, $(\alpha) = (\alpha_1, \alpha_2)$.

Definition 1. The domain of definition of the HDO $M_{\nu,(\alpha)}$ is the set G of functions $g(r) = \{g_1(r); g_2(r); g_3(r)\}$ with the following properties:
1) the function $f(r) = \{B_{\alpha_1}^*[g_1(r)]; g_2''(r); B_{\nu,\alpha_2}[g_3(r)]\}$ is continuous in the set I_2;
2) functions $g_j(r)$ satisfy the boundary conditions

$$\lim_{r\to 0} \frac{d^k u_1(r)}{dr^k} = 0;\ k = 0,1;\quad \left(\tilde{\alpha}_{22}^3 \frac{d}{dr} + \tilde{\beta}_{22}^3\right) g_3(r)\Big|_{r=R} = 0 \tag{3.2}$$

and conjugate conditions

$$\left[\left(\tilde{\alpha}_{j1}^k \frac{d}{dr} + \tilde{\beta}_{j1}^k\right) g_k(r) - \left(\tilde{\alpha}_{j2}^k \frac{d}{dr} + \tilde{\beta}_{j2}^k\right) g_{k+1}(r)\right]\Big|_{r=R_k} = 0;\ j,k = 1,2. \tag{3.3}$$

Here $\tilde{\alpha}_{jm}^k = \alpha_{jm}^k - (\beta^2 + \gamma^2)\delta_{jm}^k$, $\tilde{\beta}_{jm}^k = \beta_{jm}^k - (\beta^2 + \gamma^2)\gamma_{jm}^k$, $\beta \in (0, +\infty)$ is the spectral parameter, $\gamma^2 = \max\{\gamma_1^2; \gamma_2^2; \gamma_3^2\}$.

From the conjugate conditions (3.3) by direct calculations we obtain the basic identity for the two functions $u(r) = \{u_1(r); u_2(r); u_3(r)\} \in G$ and $v(r) = \{v_1(r); v_2(r); v_3(r)\} \in G$:

$$\begin{aligned} &\left[u_k'(r)v_k(r) - u_k(r)v_k'(r)\right]\Big|_{r=R_k} = \\ &= \frac{c_{21,k}}{c_{11,k}}\left[u_{k+1}'(r)v_{k+1}(r) - u_{k+1}(r)v_{k+1}'(r)\right]\Big|_{r=R_k};\ k = 1,2. \end{aligned} \tag{3.4}$$

Let's define the values

$$a_1^2\sigma_1 = 1,\quad a_2^2\sigma_2 = \frac{c_{21,1}}{c_{11,1}} R_1^{2\alpha_1+1},\quad a_3^2\sigma_3 = \frac{c_{21,1}c_{21,2}}{c_{11,1}c_{11,2}} \frac{R_1^{2\alpha_1+1}}{R_2^{2\alpha_2+1}},$$

the weight function

$$\begin{aligned} \sigma(r) = {} & \theta(r)\theta(R_1 - r)\sigma_1 r^{2\alpha_1 - 1} + \theta(r - R_1)\theta(R_2 - r)\sigma_2 \\ & + \theta(r - R_2)\theta(R - r)\sigma_3 r^{2\alpha_2 + 1} \end{aligned} \tag{3.5}$$

and the scalar product

$$\Big(u(r),v(r)\Big)=\int\limits_0^R u(r)v(r)\sigma(r)dr\equiv\int\limits_0^{R_1}u_1(r)v_1(r)\sigma_1 r^{2\alpha_1-1}dr+$$
$$+\int\limits_{R_1}^{R_2}u_2(r)v_2(r)\sigma_2dr+\int\limits_{R_2}^{R}u_3(r)v_3(r)\sigma_3 r^{2\alpha_2+1}dr. \tag{3.6}$$

We now check, that HDO $M_{\nu,(\alpha)}$ is a self-conjugate operator, namely

$$\Big(M_{\nu,(\alpha)}[u],v\Big)=\Big(u,M_{\nu,(\alpha)}[v]\Big). \tag{3.7}$$

According to the equality (3.6) we have, that

$$\Big(M_{\nu,(\alpha)}[u],v(r)\Big)=\int\limits_0^{R_1}\Big(a_1^2B^*_{\alpha_1}[u_1]\Big)v_1\sigma_1r^{2\alpha_1-1}dr+$$
$$+\int\limits_{R_1}^{R_2}\Big(a_2^2\frac{d^2u_2}{dr^2}\Big)v_2(r)\sigma_2dr+\int\limits_{R_2}^{R}\Big(a_3^2B_{\nu,\alpha_2}[u_3(r)]v_3(r)\sigma_3r^{2\alpha_2+1}dr. \tag{3.8}$$

Let's integrate in (3.8) by parts twice. We get:

$$\Big(M_{\nu,(\alpha)}[u],v\Big)=a_1^2\sigma_1\Big[r^{2\alpha_1+1}\Big(\frac{du_1}{dr}v_1-u_1\frac{dv_1}{dr}\Big)\Big]\Big|_0^{R_1}+\int\limits_0^{R_1}u_1(r)\Big(a_1^2B^*_{\alpha_1}[v_1]\Big)\times$$
$$\times\sigma_1r^{2\alpha_1-1}dr+a_2^2\sigma_2\Big(\frac{du_2}{dr}v_2-u_2\frac{dv_2}{dr}\Big)\Big|_{R_1}^{R_2}+\int\limits_{R_1}^{R_2}u_2(r)\Big(a_2^2\frac{d^2v_2}{dr^2}\Big)\sigma_2dr+ \tag{3.9}$$
$$+a_3^2\sigma_3\Big[r^{2\alpha_2+1}\Big(\frac{du_3}{dr}v_3-u_3\frac{dv_3}{dr}\Big)\Big]\Big|_{R_2}^{R}+\int\limits_{R_2}^{R}u_3(r)\Big(a_3^2B_{\nu,\alpha_2}[v_3]\Big)\sigma_3r^{2\alpha_2+1}dr.$$

Due to the first condition of (3.2), the term outside the integral at the point $r=0$ is equal to zero.

At the point $r=R_1$, because of the basic identity (3.4) when $k=1$ we have

$$a_1^2\sigma_1R_1^{2\alpha_1+1}\Big(u_1'v_1-u_1v_1'\Big)\Big|_{r=R_1}-a_2^2\sigma_2\Big(u_2'v_2-u_2v_2'\Big)\Big|_{r=R_1}=$$
$$=\Big(a_1^2\sigma_1R_1^{2\alpha_1+1}\frac{c_{21,1}}{c_{11,1}}-a_2^2\sigma_2\Big)\Big(u_2'v_2-u_2v_2'\Big)\Big|_{r=R_1}=$$

$$= \Big(\frac{c_{21,1}}{c_{11,1}} R_1^{2\alpha_1+1} - \frac{c_{21,1}}{c_{11,1}} R_1^{2\alpha_1+1}\Big)\Big(u_2' v_2 - u_2 v_2'\Big)\Big|_{r=R_1} = 0.$$

At the point $r = R_2$, because of the basic identity (3.4) when $k = 2$ we have

$$a_2^2\sigma_2\Big(u_2'v_2 - u_2v_2'\Big)\Big|_{r=R_2} - a_3^2\sigma_3 R_2^{2\alpha_2+1}\Big(u_3'v_3 - u_3v_3'\Big)\Big|_{r=R_2} =$$

$$= \Big(a_2^2\sigma_2\frac{c_{21,2}}{c_{11,2}} - a_3^2\sigma_3 R_2^{2\alpha_2+1}\Big)\Big(u_3'v_3 - u_3v_3'\Big)\Big|_{r=R_2} =$$

$$= \Big(\frac{c_{21,1}}{c_{11,1}}\frac{c_{21,2}}{c_{11,2}} R_1^{2\alpha_1+1} - \frac{c_{21,1}c_{21,2}}{c_{11,1}c_{11,2}}\frac{R_1^{2\alpha_1+1}}{R_2^{2\alpha_2+1}} R_2^{2\alpha_2+1}\Big(u_3'v_3 - u_3v_3'\Big)\Big|_{r=R_2} =$$

$$= \frac{c_{21,1}c_{21,2}}{c_{11,1}c_{11,2}} R_1^{2\alpha_1+1}(1-1)\Big(u_3'v_3 - u_3v_3'\Big)\Big|_{r=R_2} \equiv 0.$$

If $\tilde{\alpha}_{22}^3 \neq 0$, then expression

$$\Big(\frac{du_3}{dr}v_3 - u_3\frac{dv_3}{dr}\Big)\Big|_{r=R} = (\tilde{\alpha}_{22}^3)^{-1}\Big[\Big(\tilde{\alpha}_{22}^3\frac{du_3}{dr} + \tilde{\beta}_{22}^3 u_3\Big)\Big|_{r=R}\cdot v_3(R) - \Big(\tilde{\alpha}_{22}^3\frac{dv_3}{dr} +$$

$$+ \tilde{\beta}_{22}^3 v_3\Big)\Big|_{r=R}\cdot u_3(R)\Big] = (\tilde{\alpha}_{22}^3)^{-1}\Big(0\cdot v_3(R) - 0\cdot u_3(R)\Big) = 0.$$

Therefore, in the equation (3.9) the terms outside the integral are equal to zero and we have that

$$\Big(M_{\nu,(\alpha)}[u], v\Big) = \Big(u, M_{\nu,(\alpha)}[v]\Big).$$

So HDO $M_{\nu,(\alpha)}$ is self-conjugate. Hence its eigenvalues form a real spectrum. Since the HDO $M_{\nu,(\alpha)}$ has one singular point $r = 0$, then it's spectrum is continuous [9]. We can assume, that spectral parameter $\beta \in (0, +\infty)$. Real spectral function

$$V_{\nu,(\alpha)}(r,\beta) = \sum_{k=1}^{3}\theta(r - R_{k-1})\theta(R_k - r)V_{\nu,(\alpha);k}(r,\beta), \quad R_0 = 0,\ R_3 = R, \tag{3.10}$$

corresponds to it.

Herewith functions $V_{\nu,(\alpha);k}(r,\beta)$ must satisfy respectively the differential equations

$$\begin{aligned}\Big(B_{\alpha_1}^* + b_1^2\Big)V_{\nu,(\alpha);1}(r,\beta) = 0,\ r \in (0, R_1),\\ \Big(F + b_2^2\Big)V_{\nu,(\alpha);2}(r,\beta) = 0,\ r \in (R_1, R_2),\\ \Big(B_{\nu,\alpha_2} + b_3^2\Big)V_{\nu,(\alpha);3}(r,\beta) = 0,\ r \in (R_2, R),\end{aligned} \tag{3.11}$$

boundary conditions (3.2) and conjugate conditions (3.3); $b_j^2 = a_j^{-2}(\beta^2 + k_j^2)$, $k_j^2 \geq 0$, $j = \overline{1,3}$.

The fundamental system of solutions for Euler differential equation $(B_{\alpha_1}^* + b_1^2)v = 0$ is formed by functions $r^{-\alpha_1}\cos(b_1 \ln r)$ and $r^{-\alpha_1}\sin(b_1 \ln r)$ [16]; the fundamental system of solutions for Fourier differential equation $(F + b_2^2)v = 0$ is formed by functions $\cos b_2 r$ and $\sin b_2 r$ [16]; the fundamental system of solutions for Bessel differential equation $(B_{\nu,\alpha_2}+b_3^2)v=0$ is formed by functions $J_{\nu,\alpha_2}(b_3 r)$ and $N_{\nu,\alpha_2}(b_3 r)$ [11].

The presence of the fundamental system of solutions allows us to put

$$\begin{gathered} V_{\nu,(\alpha);1}(r,\beta)=A_1 r^{-\alpha_1}\cos(b_1\ln r)+B_1 r^{-\alpha_1}\sin(b_1\ln r), \\ V_{\nu,(\alpha);2}(r,\beta)=A_2\cos(b_2 r)+B_2\sin(b_2 r), \\ V_{\nu,(\alpha);3}(r,\beta)=A_3 J_{\nu,\alpha_2}(b_3 r)+B_3 N_{\nu,\alpha_2}(b_3 r). \end{gathered} \tag{3.12}$$

Conjugate conditions (3.3) and boundary condition at the point $r=R$ for the determination of the six variables $A_j,\ B_j\ (j=\overline{1,3})$ give homogeneous algebraic system of five equations:

$$\begin{gathered} Y^{11}_{\alpha_1;j1}(b_1,R_1)A_1+Y^{12}_{\alpha_1;j1}(b_1,R_1)B_1-v^{11}_{j2}(b_2R_1)A_2-v^{12}_{j2}(b_2R_1)B_2=0,\ j=1,2; \\ v^{21}_{j1}(b_2R_2)A_2+v^{22}_{j1}(b_2R_2)B_2-u^{21}_{\nu,\alpha_2;j2}(b_3R_2)A_3-u^{22}_{\nu,\alpha_2;j2}(b_3R_2)B_3=0; \\ u^{31}_{\nu,\alpha_2;22}(b_3R)A_3+u^{32}_{\nu,\alpha_2;22}(b_3R)B_3=0. \end{gathered} \tag{3.13}$$

Algebraic system (3.13)) is compatible. It's solution is constructed in standard way.

Let's suppose, that $A_3=-A_0u^{32}_{\nu,\alpha_2;22}(b_3,R)$, $B_3=A_0u^{31}_{\nu,\alpha_2;22}(b_3,R)$, here A_0 is subjected to be defined. At such choice of A_3, B_3 the last equation of the system becomes the identity. For determination of the values A_2, B_2 we obtain the algebraic system:

$$\begin{gathered} v^{21}_{j1}(b_2R_2)A_2+v^{22}_{j1}(b_2R_2)B_2=-A_0\Big[u^{21}_{\nu,\alpha_2;j2}(b_3R_2)u^{32}_{\nu,\alpha_2;22}(b_3R)- \\ -u^{22}_{\nu,\alpha_2;j2}(b_3R_2)u^{31}_{\nu,\alpha_2;22}(b_3R)\Big]\equiv -A_0\delta_{\nu,\alpha_2;j2}(b_3R_2,b_3R),\ j=1,2. \end{gathered} \tag{3.14}$$

Functions

$$\begin{gathered} A_2=\frac{A_0}{c_{11,2}b_2}\Big[\delta_{\nu,\alpha_2;22}(b_3R_2,b_3R)v^{22}_{11}(b_2R_2)-\delta_{\nu,\alpha_2;12}(b_3R_2,b_3R)v^{22}_{21}(b_2R_2)\Big], \\ B_2=\frac{A_0}{c_{11,2}b_2}\Big[\delta_{\nu,\alpha_2;12}(b_3R_2,b_3R)v^{21}_{21}(b_2R_2)-\delta_{\nu,\alpha_2;22}(b_3R_2,b_3R)v^{21}_{11}(b_2R_2)\Big] \end{gathered} \tag{3.15}$$

are the solution of the algebraic system (3.14).

Considering A_2, B_2 to be known let's consider the algebraic system on values A_1, B_1:

$$Y^{11}_{\alpha_1;j1}(b_1,R_1)A_1+Y^{12}_{\alpha_1;j1}(b_1,R_1)B_1=A_0c_{11,2}b_2a_{\nu,\alpha_2;j}(\beta),\ j=1,2. \tag{3.16}$$

In the system (3.16) we accept the denotation:

$$\begin{gathered} \delta_{jk}(b_2R_1,b_2R_2)=v^{11}_{j2}(b_2R_1)v^{22}_{k1}(b_2R_2)-v^{12}_{j2}(b_2R_1)v^{21}_{k1}(b_2R_2),\ j,k=1,2; \\ a_{\nu,\alpha_2;j}(\beta)=\delta_{\nu,\alpha_2;22}(b_3R_2,b_3R)\delta_{j1}(b_2R_1,b_2R_2)- \end{gathered}$$

$$-\delta_{\nu,\alpha_2;12}(b_3R_2,b_3R)\delta_{j2}(b_2R_1,b_2R_2).$$

Algebraic system (3.16) has the unique solution [8]:

$$A_0 = c_{11,1}b_1R_1^{-(2\alpha_1+1)}c_{11,2}b_2;\quad A_1 = \omega_{\nu,(\alpha);2}(\beta),\quad B_1 = -\omega_{\nu,(\alpha);1}(\beta);\quad (3.17)$$

$$\omega_{\nu,(\alpha);j}(\beta) = a_{\nu,\alpha_2;1}(\beta)Y^{1j}_{\alpha_1;21}(b_1,R_1) - a_{\nu,\alpha_2;2}(\beta)Y^{1j}_{\alpha_1;11}(b_1,R_1);\ j=1,2.$$

Substituting values A_j, B_j to equality (3.12) according to the formulas (3.15) and (3.17), we obtain the functions

$$V_{\nu,(\alpha);1}(r,\beta) = \omega_{\nu,(\alpha);2}(\beta)r^{-\alpha_1}\cos(b_1\ln r) - \omega_{\nu,(\alpha);1}(\beta)r^{-\alpha_1}\sin(b_1\ln r),$$

$$V_{\nu,(\alpha);2}(r,\beta) = \frac{c_{11,1}b_1}{R_1^{2\alpha_1+1}}\Big[\delta_{\nu,\alpha_2;22}(b_3R_2,b_3R)\varphi^2_{11}(b_2R_2,b_2r) -$$

$$-\delta_{\nu,\alpha_2;12}(b_3R_2,b_3R)\varphi^2_{21}(b_2R_2,b_2r)\Big], \qquad (3.18)$$

$$\varphi^2_{j1}(b_2R_2,b_2r) = v^{22}_{j1}(b_2R_2)\cos b_2r - v^{21}_{j1}(b_2R_2)\sin b_2r;$$

$$V_{\nu,(\alpha);3}(r,\beta_n) = \frac{c_{11,1}b_1}{R_1^{2\alpha_1+1}}c_{11,2}b_2\Big[u^{31}_{\nu,\alpha_2;22}(b_3R)N_{\nu,\alpha_2}(b_3r) -$$

$$-u^{32}_{\nu,\alpha_2;22}(b_3R)J_{\nu,\alpha_2}(b_3r)\Big].$$

According to the formula (3.10) spectral function $V_{\nu,(\alpha)}(r,\beta)$ becomes known (defined).

Let's introduce a spectral density into consideration

$$\Omega_{\nu,(\alpha)}(\beta) = \beta[b_1(\beta)]^{-1}\Big(\Big[\omega_{\nu,(\alpha);1}(\beta)\Big]^2 + \Big[\omega_{\nu,(\alpha);2}(\beta)\Big]^2\Big)^{-1}. \qquad (3.19)$$

The presence of spectral function $V_{\nu,(\alpha)}(r,\beta)$, weight function $\sigma(r)$ and spectral density $\Omega_{\nu,(\alpha)}(\beta)$ makes it possible to determine the direct $H_{\nu,(\alpha)}$ and inverse $H^{-1}_{\nu,(\alpha)}$ hybrid integral transform (HIT) of Euler - Fourier - Bessel type with the spectral parameter, generated in the set I_2 by HDO $M_{\nu,(\alpha)}$ [12]:

$$H_{\nu,(\alpha)}[g(r)] = \int_0^R g(r)V_{\nu,(\alpha)}(r,\beta)\sigma(r)dr \equiv \tilde{g}(\beta), \qquad (3.20)$$

$$H^{-1}_{\nu,(\alpha)}[\tilde{g}(\beta)] = \frac{2}{\pi}\int_0^\infty \tilde{g}(\beta)V_{\nu,(\alpha)}(r,\beta)\Omega_{\nu,(\alpha)}(\beta)d\beta \equiv g(r). \qquad (3.21)$$

Next statement is a mathematical justification of rules (3.20), (3.21).

Theorem 1. *If the function*

$$f(r)=\Big[\theta(r)\theta(R_1-r)r^{\alpha_1-1/2}+\theta(r-R_1)\theta(R_2-r)\cdot 1+$$
$$+\theta(r-R_2)\theta(R-r)r^{\alpha_2+1/2}\Big]g(r)$$

is continuous, absolutely summable and has bounded variation in the set $(0,R)$*, then for any* $r\in I_2$ *integral representation is true*

$$g(r)=\frac{2}{\pi}\int\limits_0^\infty V_{\nu,(\alpha)}(r,\beta)\int\limits_0^R g(\rho)V_{\nu,(\alpha)}(\rho,\beta)\sigma(\rho)d\rho\Omega_{\nu,(\alpha)}(\beta)d\beta. \tag{3.22}$$

The proof of theorem 1 is performed by the method of delta-shaped sequence (Cauchy kernel or Dirihle kernel) [12].

Let's define the numbers and functions:

$$d_1=a_1^2\sigma_1R_1^{2\alpha_1+1}\cdot c_{11,1}^{-1},\ d_2=a_2^2\sigma_2\cdot c_{11,2}^{-1},\ \tilde g_1(\beta)=\int\limits_0^{R_1}g_1(r)V_{\nu,(\alpha);1}(r,\beta)\sigma_1r^{2\alpha_1-1}dr,$$

$$\tilde g_2(\beta)=\int\limits_{R_1}^{R_2}g_2(r)V_{\nu,(\alpha);2}(r,\beta)\sigma_2dr,\ \tilde g_3(\beta)=\int\limits_{R_2}^{R}g_3(r)V_{\nu,(\alpha);3}(r,\beta)\sigma_3r^{2\alpha_2+1}dr,$$

$$Z^k_{\nu,(\alpha);i2}(\beta)=\Big(\tilde\alpha^k_{i2}\frac{d}{dr}+\tilde\beta^k_{i2}\Big)V_{\nu,(\alpha);k+1}(r,\beta)\Big|_{r=R_k},\quad i,k=1,2.$$

Theorem 2. *If the function* $\{B^*_{\alpha_1}[g_1(r)];g_2''(r);B_{\nu,\alpha_2}[g_3(r)]\}$ *is continuous in the set* I_2*, and functions* $g_j(r)$ *satisfy the boundary conditions*

$$\lim_{r\to0}\Big[r^{2\alpha_1+1}\Big(g_1'V_{\nu,(\alpha);1}-g_1V'_{\nu,(\alpha);1}\Big)\Big]=0,\ \Big(\tilde\alpha^3_{22}\frac{d}{dr}+\tilde\beta^3_{22}\Big)V_{\nu,(\alpha);3}(r,\beta)\Big|_{r=R}=g_R \tag{3.23}$$

and the conjugate conditions

$$\Big[\Big(\tilde\alpha^k_{j1}\frac{d}{dr}+\tilde\beta^k_{j1}\Big)g_k(r)-\Big(\tilde\alpha^k_{j2}\frac{d}{dr}+\tilde\beta^k_{j2}\Big)g_{k+1}(r)\Big]\Big|_{r=R_k}=\omega_{jk};\ j,k=1,2, \tag{3.24}$$

then the basic identity for the integral transform of the HDO $M_{\nu,(\alpha)}$ *defined by equality* (3.1) *is true:*

$$H_{\nu,(\alpha)}\Big[M_{\nu,(\alpha)}[g(r)]\Big]=-\beta^2\tilde g(\beta)-\sum_{i=1}^3k_i^2\tilde g_i(\beta)+(\tilde\alpha^3_{22})^{-1}V_{\nu,(\alpha);3}(R,\beta)\times$$
$$\times a_3^2\sigma_3R^{2\alpha_2+1}g_R+\sum_{k=1}^2d_k\Big[Z^k_{\nu,(\alpha);12}(\beta)\omega_{2k}-Z^k_{\nu,(\alpha);22}(\beta)\omega_{1k}\Big]. \tag{3.25}$$

Proof of Theorem 2. According the rule (3.20) we have that

$$H_{\nu,(\alpha)}\Big[M_{\nu,(\alpha)}[g(r)]\Big]=\int\limits_0^R M_{\nu,(\alpha)}[g(r)]V_{\nu,(\alpha)}(r,\beta)\sigma(r)dr\equiv$$

$$\equiv\int\limits_0^{R_1}\Big(a_1^2B^*_{\alpha_1}[g_1(r)]\Big)V_{\nu,(\alpha);1}(r,\beta)\sigma_1 r^{2\alpha_1-1}dr+\int\limits_{R_1}^{R_2}\Big(a_2^2\frac{d^2g_2}{dr^2}\Big)V_{\nu,(\alpha);2}(r,\beta)\sigma_2 dr+$$
$$+\int\limits_{R_2}^{R}\Big(a_3^2B_{\nu,\alpha_2}[g_3(r)]\Big)V_{\nu,(\alpha);3}(r,\beta)\sigma_3 r^{2\alpha_2+1}dr. \tag{3.26}$$

Let's integrate by parts twice in (3.26):

$$H_{\nu,(\alpha)}\Big[M_{\nu,(\alpha)}[g(r)]\Big]=a_1^2\sigma_1\Big[r^{2\alpha_1+1}\Big(\frac{dg_1}{dr}V_{\nu,(\alpha);1}(r,\beta)-g_1(r)\frac{dV_{\nu,(\alpha);1}}{dr}\Big)\Big]\Big|_{R_0}^{R_1}+$$
$$+\int\limits_0^{R_1}g_1(r)\Big(a_1^2B^*_{\alpha_1}[V_{\nu,(\alpha);1}(r,\beta)]\Big)\sigma_1 r^{2\alpha_1-1}dr+a_2^2\sigma_2\Big(\frac{dg_2}{dr}V_{\nu,(\alpha);2}(r,\beta)-$$
$$-g_2(r)\frac{dV_{\nu,(\alpha);2}}{dr}\Big)\Big|_{R_1}^{R_2}+\int\limits_{R_1}^{R_2}g_2(r)\Big(a_2^2\frac{d^2V_{\nu,(\alpha);2}}{dr^2}\Big)\sigma_2 dr+$$
$$+a_3^2\sigma_3\Big[r^{2\alpha_2+1}\Big(\frac{dg_3}{dr}V_{\nu,(\alpha);3}(r,\beta)-g_3(r)\frac{dV_{\nu,(\alpha);3}}{dr}\Big)\Big]\Big|_{R_2}^{R}+$$
$$+\int\limits_{R_2}^{R}g_3(r)\Big(a_3^2B_{\nu,\alpha_2}[V_{\nu,(\alpha);3}(r,\beta)]\Big)\sigma_3 r^{2\alpha_2+1}dr. \tag{3.27}$$

Due to the boundary condition in the point $r=0$ we obtain that

$$\lim_{r\to 0}\Big[r^{2\alpha_1+1}\Big(\frac{dg_1}{dr}V_{\nu,(\alpha);1}(r,\beta)-g_1(r)\frac{dV_{\nu,(\alpha);1}}{dr}\Big)\Big]=0.$$

Let's use the basic identity (3.4) for the case if the conjugate conditions are not homogeneous

$$\Big[g_k'V_{\nu,(\alpha);k}(r,\beta)-g_kV'_{\nu,(\alpha);k}(r,\beta)\Big]\Big|_{r=R_k}=\frac{c_{21,k}}{c_{11,k}}\Big[g'_{k+1}(r)V_{\nu,(\alpha);k+1}-g_{k+1}\times$$
$$\times V'_{\nu,(\alpha);k+1}\Big]\Big|_{r=R_k}+c_{11,k}^{-1}\Big[Z^k_{\nu,(\alpha);12}(\beta)\omega_{2k}-Z^k_{\nu,(\alpha);22}(\beta)\omega_{1k}\Big]. \tag{3.28}$$

Due to the basic identity (3.28) in the point $r = R_1$ for k=1 we have that

$$a_1^2\sigma_1 R_1^{2\alpha_1+1}\left(g_1' V_{\nu,(\alpha);1} - g_1 V_{\nu,(\alpha);1}'\right)\Big|_{R_1} - a_2^2\sigma_2\left(g_2' V_{\nu,(\alpha);2} - g_2 V_{\nu,(\alpha);2}'\right)\Big|_{R_1} =$$
$$= \left(a_1^2\sigma_1 R_1^{2\alpha_1+1}\frac{c_{21,1}}{c_{11,1}} - a_2^2\sigma_2\right)\left(g_2' V_{\nu,(\alpha);2} - g_2 V_{\nu,(\alpha);2}'\right)\Big|_{r=R_1} + a_1^2\sigma_1 R_1^{2\alpha_1+1}\cdot c_{11,1}^{-1}\times \tag{3.29}$$
$$\times\left(Z^1_{\nu,(\alpha);12}(\beta)\omega_{21} - Z^1_{\nu,(\alpha);22}(\beta)\omega_{11}\right) = d_1\left(Z^1_{\nu,(\alpha);12}(\beta)\omega_{21} - Z^1_{\nu,(\alpha);22}(\beta)\omega_{11}\right),$$

because due to the choice of numbers σ_1, σ_2 expression

$$\left(a_1^2\sigma_1 R_1^{2\alpha_1+1}\frac{c_{21,1}}{c_{11,1}} - a_2^2\sigma_2\right) = \frac{c_{21,1}}{c_{11,1}} R_1^{2\alpha_1+1} - \frac{c_{21,1}}{c_{11,1}} R_1^{2\alpha_1+1} =$$
$$= \frac{c_{21,1}}{c_{11,1}} R_1^{2\alpha_1+1}(1-1) \equiv 0.$$

Due to the basic identity (3.28) in the point $r = R_2$ for k=2 we have that

$$a_2^2\sigma_2\left(g_2' V_{\nu,(\alpha);2} - g_2 V_{\nu,(\alpha);2}'\right)\Big|_{r=R_2} - a_3^2\sigma_3 R_2^{2\alpha_2+1}\left(g_3' V_{\nu,(\alpha);3} - g_3 V_{\nu,(\alpha);3}'\right)\Big|_{r=R_2} =$$
$$= \left(a_2^2\sigma_2\frac{c_{21,2}}{c_{11,2}} - a_3^2\sigma_3 R_2^{2\alpha_2+1}\right)\left(g_3' V_{\nu,(\alpha);3} - g_3 V_{\nu,(\alpha);3}'\right)\Big|_{r=R_2} + a_2^2\sigma_2 c_{11,2}^{-1}\times \tag{3.30}$$
$$\times\left(Z^2_{\nu,(\alpha);12}(\beta)\omega_{22} - Z^2_{\nu,(\alpha);22}(\beta)\omega_{12}\right) = d_2\left(Z^2_{\nu,(\alpha);12}(\beta)\omega_{22} - Z^2_{\nu,(\alpha);22}(\beta)\omega_{12}\right),$$

because due to the choice of numbers σ_2 and σ_3 expression

$$a_2^2\sigma_2\frac{c_{21,2}}{c_{11,2}} - a_3^2\sigma_3 R_2^{2\alpha_2+1} = \frac{c_{21,1}}{c_{11,1}}\frac{c_{21,2}}{c_{11,2}} R_1^{2\alpha_1+1} - \frac{c_{21,1}c_{21,2}}{c_{11,1}} c_{11,2}\frac{R_1^{2\alpha_1+1}}{R_2^{2\alpha_2+1}} R_2^{2\alpha_2+1} =$$
$$= \frac{c_{21,1}c_{21,2}}{c_{11,1}} c_{11,2} R_1^{2\alpha_1+1}(1-1) \equiv 0.$$

If $\tilde{\alpha}^3_{22} \neq 0$, then

$$a_3^2\sigma_3 R^{2\alpha_2+1}\left(\frac{dg_3}{dr} V_{\nu,(\alpha);3} - g_3\frac{dV_{\nu,(\alpha);3}}{dr}\right)\Big|_{r=R} = a_3^2\sigma_3(\tilde{\alpha}^3_{22})^{-1} R^{2\alpha_2+1}\times$$
$$\times\left[\left(\tilde{\alpha}^3_{22}\frac{d}{dr} + \tilde{\beta}^3_{22}\right) g_3\Big|_{r=R}\cdot V_{\nu,(\alpha);3}(R,\beta) - g_3(R)\left(\tilde{\alpha}^3_{22}\frac{d}{dr} + \tilde{\beta}^3_{22}\right)\times\right.$$
$$\left.\times V_{\nu,(\alpha);3}(R,\beta)\Big|_{r=R}\right] = (\tilde{\alpha}^3_{22})^{-1} V_{\nu,(\alpha);3}(R,\beta) a_3^2\sigma_3 R^{2\alpha_2+1} g_R. \tag{3.31}$$

From the differential identities

$$\left[a_1^2 B^*_{\alpha_1} + (\beta^2 + k_1^2)\right] V_{\nu,(\alpha);1}(r,\beta) = 0,\ \left[a_2^2 F + (\beta^2 + k_2^2)\right] V_{\nu,(\alpha);2}(r,\beta) = 0,$$
$$\left[a_3^2 B_{\nu,\alpha_2} + (\beta^2 + k_3^2)\right] V_{\nu,(\alpha);3}(r,\beta) = 0$$

we find the functional dependence:

$$a_1^2 B_{\alpha_1}^*\Big[V_{\nu,(\alpha);1}(r,\beta)\Big] = -(\beta^2+k_1^2)V_{\nu,(\alpha);1}(r,\beta),\ a_2^2\frac{d^2}{dr^2}\Big[V_{\nu,(\alpha);2}(r,\beta)\Big] =$$

$$= -(\beta^2+k_2^2)V_{\nu,(\alpha);2}(r,\beta),\ a_3^2 B_{\nu,\alpha_2}\Big[V_{\nu,(\alpha);3}(r,\beta)\Big] = -(\beta^2+k_3^2)V_{\nu,(\alpha);3}(r,\beta). \tag{3.32}$$

Let's substitute the obtained dependence (3.29)-(3.32) to the (3.27). We get:

$$H_{\nu,(\alpha)}\Big[M_{\nu,(\alpha)}[g(r)]\Big] = -\sum_{i=1}^{3}(\beta^2+k_i^2)\tilde{g}_i(\beta) + \sum_{k=1}^{2} d_k\Big[Z^k_{\nu,(\alpha);12}(\beta)\omega_{2k}-$$

$$-Z^k_{\nu,(\alpha);22}(\beta)\omega_{1k}\Big] + (\tilde{\alpha}^3_{22})^{-1}V_{\nu,(\alpha);3}(R,\beta)a_3^2\sigma_3 R^{2\alpha_2+1}g_R. \tag{3.33}$$

Since the

$$\sum_{i=1}^{3}(\beta^2+k_i^2)\tilde{g}_i(\beta) = \beta^2\sum_{i=1}^{3}\tilde{g}_i(\beta) + \sum_{i=1}^{3}k_i^2\tilde{g}_i(\beta) = \beta^2\tilde{g}(\beta) + \sum_{i=1}^{3}k_i^2\tilde{g}_i(\beta),$$

then equality (3.33) coincides with equality (3.25). The theorem is proved. □

4. Solution of the problem

Let's construct the integral representation of the exact analytical solution of parabolic boundary value problem of conjugation (2.1) - (2.3) by the method of Euler-Fourier-Bessel hybrid integral transform with spectral parameter.

Let's write the system (2.1) and the zero initial conditions in matrix form:

$$\begin{bmatrix} \left(\frac{\partial}{\partial t}+\gamma_1^2-a_1^2B^*_{\alpha_1}\right)u_1(t,r) \\ \left(\frac{\partial}{\partial t}+\gamma_2^2-a_2^2F\right)u_2(t,r) \\ \left(\frac{\partial}{\partial t}+\gamma_3^2-a_3^2B_{\nu,\alpha_2}\right)u_3(t,r) \end{bmatrix} = \begin{bmatrix} f_1(t,r) \\ f_2(t,r) \\ f_3(t,r) \end{bmatrix},$$

$$\begin{bmatrix} u_1(t,r) \\ u_2(t,r) \\ u_3(t,r) \end{bmatrix}\Bigg|_{t=0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4.1}$$

The integral operator $H_{\nu,(\alpha)}$ is represented as an operator matrix row due to the rule (3.20):

$$H_{\nu,(\alpha)}[\cdots] = \Big[\int_0^{R_1}\cdots V_{\nu,(\alpha);1}(r,\beta)\sigma_1 r^{2\alpha_1-1}dr \quad \int_{R_1}^{R_2}\cdots V_{\nu,(\alpha);2}(r,\beta)\sigma_2 dr$$

$$\int\limits_{R_2}^{R} \cdots V_{\nu,(\alpha);3}(r,\beta)\sigma_3 r^{2\alpha_2+1}dr\Big]. \tag{4.2}$$

Let's apply the operator matrix-row (4.2) to the problem (4.1) according to the matrix multiplication rule. As a result of the main identity (3.25), we get a Cauchy problem [16]:

$$\Big(\frac{d}{dt}+\beta^2+\gamma^2\Big)\tilde{u}(t,\beta) \tag{4.3}$$

$$= \tilde{f}(t,\beta)+\sum_{k=1}^{2} d_k\Big[Z^k_{\nu,(\alpha);12}(\beta)\omega_{2k}(t)-Z^k_{\nu,(\alpha);22}(\beta)\omega_{1k}(t)\Big]+ \tag{4.4}$$

$$+(\tilde{\alpha}^3_{22})^{-1}V_{\nu,(\alpha);3}(R,\beta)a_3^2\sigma_3 R^{2\alpha_2+1}g_R(t);\ \gamma^2=\max\{\gamma_1^2;\gamma_2^2;\gamma_3^2\};\ \tilde{u}\Big|_{t=0}=0. \tag{4.5}$$

Function

$$\tilde{u}(t,\beta)=\int\limits_0^t e^{-(\beta^2+\gamma^2)(t-\tau)}\Big[\tilde{f}(\tau,\beta)+\sum_{k=1}^{2}\Big(Z^k_{\nu,(\alpha);12}(\beta)\omega_{2k}(\tau)-Z^k_{\nu,(\alpha);22}(\beta)\times$$

$$\times\omega_{1k}(\tau)\Big)+(\tilde{\alpha}^3_{22})^{-1}V_{\nu,(\alpha);3}(R,\beta)a_3^2\sigma_3R^{2\alpha_2+1}g_R(\tau)\Big]d\tau. \tag{4.6}$$

is the solution of Cauchy problem (4.5).

Integral operator $H^{-1}_{\nu,(\alpha)}$ due to the rule (3.21), as inverse to (4.2), we represent as the operator matrix-column:

$$H^{-1}_{\nu,(\alpha)}[\cdots]=\begin{bmatrix} \frac{2}{\pi}\int\limits_0^\infty \cdots V_{\nu,(\alpha);1}(r,\beta)\Omega_{\nu,(\alpha)}(\beta)d\beta \\ \frac{2}{\pi}\int\limits_0^\infty \cdots V_{\nu,(\alpha);2}(r,\beta)\Omega_{\nu,(\alpha)}(\beta)d\beta \\ \frac{2}{\pi}\int\limits_0^\infty \cdots V_{\nu,(\alpha);3}(r,\beta)\Omega_{\nu,(\alpha)}(\beta)d\beta \end{bmatrix}. \tag{4.7}$$

Let's apply operator matrix column (4.7) to matrix element due to matrices multiplication rule, to the matrix-element $[\tilde{u}(t,\beta)]$, where the function $\tilde{u}(t,\beta)$ is defined by formula (4.6). As a result of elementary transformations, we get the integral representation of the only analytical solution of parabolic problem (2.1)-(2.3):

$$u_j(t,r)=\int\limits_0^t W_{\nu,(\alpha);3j}(t-\tau,r)g_R(\tau)d\tau$$

$$+\sum_{k=1}^{2} d_k \int_0^t \Big[R^{k,j}_{\nu,(\alpha);12}(t-\tau,r)\omega_{2k}(\tau) - R^{k,j}_{\nu,(\alpha);22}(t-\tau,r)\omega_{1k}(\tau)\Big]d\tau$$

$$+\sum_{k=1}^{3} \int_0^t \int_{R_{k-1}}^{R_k} H_{\nu,(\alpha);jk}(t-\tau,r,\rho) f(\tau,\rho)\varphi_k(\rho)\sigma_k\, d\rho\, d\tau. \tag{4.8}$$

Here $R_0=0$, $R_3=R$, $\varphi_1(\rho)=\rho^{2\alpha_1-1}$, $\varphi_2(\rho)=1$, $\varphi_3(\rho)=\rho^{2\alpha_2+1}$, $j=\overline{1,3}$.

In equalities (4) there are principal solutions of parabolic problem (2.1)-(2.3):

1) Green's functions generated by boundary condition at point $r=R$

$$W_{\nu,(\alpha);3j}(t,r) = \frac{2}{\pi}\int_0^\infty e^{-(\beta^2+\gamma^2)t}(\tilde{\alpha}^3_{22})^{-1} V_{\nu,(\alpha);3}(R,\beta) V_{\nu,(\alpha);j}(r,\beta)\times$$
$$\times \Omega_{\nu,(\alpha)}(\beta)\, d\beta\, a_3^2\sigma_3 R^{2\alpha_1+1}, \tag{4.9}$$

2) Green's functions generated by inhomogeneity of the conjugate conditions

$$R^{k,j}_{\nu,(\alpha);i2}(t,r) = \frac{2}{\pi}\int_0^\infty e^{-(\beta^2+\gamma^2)t} Z^k_{\nu,(\alpha);i2}(\beta) V_{\nu,(\alpha);j}(r,\beta)\Omega_{\nu,(\alpha)}(\beta)\, d\beta; \tag{4.10}$$

$i,k=1,2$; $j=\overline{1,3}$,

3) the influence functions generated by the inhomogeneity of system (2.1)

$$H_{\nu,(\alpha);jk}(t,r,\rho) = \frac{2}{\pi}\int_0^\infty e^{-(\beta^2+\gamma^2)t} V_{\nu,(\alpha);j}(r,\beta) V_{\nu,(\alpha);k}(\rho,\beta)\Omega_{\nu,(\alpha)}(\beta)\, d\beta; \tag{4.11}$$

$j,k=\overline{1,3}$.

We get the following theorem as the summary of the above results.

Theorem 3. *Let us suppose that the next conditions are true:*
1) functions $f_j(t,r)$, $g_R(t)$ and $\omega_{jk}(t)$ are originals by Laplace;
2) functions $f_j(t,r)$ and $g_j(r)$ satisfy the conjugate conditions;
3) functions $f(t,r)=\{f_1(t,r), f_2(t,r), f_3(t,r)\}$ and $g(r)=\{g_1(r),g_2(r),g_3(r)\}$ are bounded, continuous, absolutely summable with the weight function $\sigma(r)$ and have the bounded variation in the set I_2;
4) function $F(t,r)=\left\{\frac{\partial}{\partial r}B^*_{\alpha_1}[f_1(t,r)], \frac{\partial}{\partial r}F[f_2(t,r)], \frac{\partial}{\partial r}B_{\nu,\alpha_2}[f_3(t,r)]\right\}$ is continuously differentiable by t and continuous by r in the set D_2.

Then in the class of functions $u(t,r)=\{u_1(t,r),u_2(t,r),u_3(t,r)\}$, which are continuously differentiable by variable t and continuously differentiable by variable r twice in the set D_2 and satisfy conditions 1), 3), parabolic mixed boundary-value *problem* (2.1)-(2.3) *has unique bounded solution, which is determined by the formula* (4).

Remark 3. If $\gamma^2 = \gamma_1^2 > 0$, then $k_1^2 = 0$, $k_2^2 = \gamma_1^2 - \gamma_2^2 \geq 0$, $k_3^2 = \gamma_1^2 - \gamma_3^2 \geq 0$; if $\gamma^2 = \gamma_2^2 > 0$, then $k_1^2 = \gamma_2^2 - \gamma_1^2 \geq 0$, $k_2^2 = 0$, $k_3^2 = \gamma_2^2 - \gamma_3^2 \geq 0$; if $\gamma^2 = \gamma_3^2 > 0$, then $k_1^2 = \gamma_3^2 - \gamma_1^2 \geq 0$, $k_2^2 = \gamma_3^2 - \gamma_2^2 \geq 0$, $k_3^2 = 0$.

Remark 4. If initial conditions are not zero, namely

$$u_j(t,r)\Big|_{t=0} = g_j(r),\, r \in (R_{j-1}, R_j),\, j = \overline{1,3},\, R_0 = 0,\, R_3 = R,$$

then we go to new function $v(t,r) = \{v_1(t,r); v_2(t,r); v_3(t,r)\}$ by formulas

$$u_j(t,r) = v_j(t,r) + g_j(r), \quad j = \overline{1,3}.$$

Then it is clear that $v_j(t,r)\Big|_{t=0} = 0$, $j = \overline{1,3}$.

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QUASILINEARIZATION FOR FRACTIONAL DIFFERENTIAL EQUATIONS OF RIEMANN-LIOUVILLE TYPE

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Abstract. In this paper, we deal with the quasilinearization for Riemann-Liouville fractional differential equations with two point boundary condition. By establishing a new comparison principle we get a monotone sequence which converges quadratically to the unique solution of the fractional differential equations.

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1. INTRODUCTION

Recently, the theory of fractional differential equations become a hot topic in many fields. It is known that many physical system can be represented more accurately through fractional derivative formulation. There are many fields of applications where we can use the fractional calculus as the mathematical model of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, viscoelasticity, heat conduction, electricity mechanics, control theory, and so forth. For more details on the topics one can see for instance, [7–9, 11–13, 15, 16, 20] and the reference therein..

Many authors have studied fractional differential equations from two aspects, one is the theoretical aspects of the existence and uniqueness of solutions [1, 5, 6, 22, 23], the other is the development of analytic and numerical methods for finding solutions. The numerical-analytical technique based on successive approximations leads us the approximate solutions to differential equations. Now, it has been studied widely in the theory of integer differential equations, see for instance [17–19].

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It is well known that the method of quasilinearization [10] offers an approach for obtaining approximate solutions to differential equations. This method not only applies to integer differential equations, but also applies to the fractional differential equations, which combining with the lower and upper solution method enables us to get a monotone sequence which converge quadratically to the solution of the differential equations(cf. [2–4, 14, 21]).

In [2, 4], the authors studied the following Caputo fractional differential equations by using the quasilinearization method:

$$\begin{cases} {}^cD^q_{0+}x(t) = f(t, x(t)), \ t \in J = [t_0, T], \\ x(t_0) = x_0, \end{cases}$$

where $0 < q \leq 1, \ f \in C(J \times R, \ R)$.

Later on, in [3], Vasundhara Devi and Radhika developed the quasilinearization method for the Caputo fractional impulsive differential systems as:

$$\begin{cases} {}^cD^q_{0+}x(t) = f(t, x(t))), \ t \neq t_k, \\ x(t_k^+) = I_k(x(t_k)), \ k = 1, 2, \cdots n-1, \\ x(t_0) = x_0, \end{cases}$$

where $0 < q \leq 1, \ f \in PC([t_0, T] \times R, \ R), \ I_k : \ R \to R$. The comparison theorem they used is developed from differential equation.

Motivated by the works mentioned above, we investigate the following nonlinear two point boundary value problem for Riemann-Liouville fractional differential equations:

$$\begin{cases} D^q_{0+}x(t) = f(t, x(t)), \ t \in (0, T], \\ \tilde{x}(0) = g(\tilde{x}(T)), \end{cases} \tag{1.1}$$

where $J \in [0, T], \ \frac{1}{2} < q \leq 1, \ f \in C(J \times R, \ R), \ g \in C(R, R)$ and $\tilde{x}(0) = t^{1-q}x(t)\mid_{t=0}, \ \tilde{x}(T) = t^{1-q}x(t)\mid_{t=T}$. We develop a new comparison principle and the method of quasilinearization for Riemann-Liouville fractional differential equations.

Significant progresses have been made to the quasilinearization of the Caputo fractional differential equations (cf.[2–4]). However, to our best knowledge, the quasilinearization for Riemann-Liouville fractional differential equations is still an untreated topics in the literature and this fact is the motivation of the present work. Our aim in this paper is to provide some suitable sufficient conditions for the existence and uniqueness of solutions and approximate results for Riemann-Liouville fractional differential equations.

This paper is organized as follows. In Section 2, we provide some definitions, lemmas and establish a new comparison principle by integral equations. In Section 3, The lower and upper solutions method and quasilinearization method are used to construct a monotone sequence which converge uniformly and quadratically to the unique solution of (1.1).

2. Preliminaries

In this section, we introduce some notations, definitions and preliminary facts which are used throughout this paper.

Let $C_{1-\alpha}(J,R)=\{x\in C(0,T]:t^{1-\alpha}x(t)\in C[0,T]\}$ with the norm $\| x \|_{C_{1-\alpha}}= max_{t\in J}|t^{1-\alpha}x(t)|$. Obviously, the space $C_{1-\alpha}(J,R)$ is a Banach space. The following definitions can be found from [8, 16]:

Definition 1. The integral

$$I_{0+}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(s)}{(t-s)^{1-\alpha}}dt, \qquad \alpha>0,$$

is called Riemann-Liouville fractional integral of order α, where Γ is the gamma function.

Definition 2. For a function $f(t)$, the Riemann-Liouville derivative of order α can be written as

$$D_{0+}^{\alpha}f(t)=(\frac{d}{dt})^n(I_{0+}^{n-\alpha}f(t))=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t(t-s)^{n-\alpha-1}f(s)ds,$$

where $n-1<\alpha\le n$.

Lemma 1 ([8]). *Let $n-1<\alpha\le n$ and let $f_{n-\alpha}(t)=I_{0+}^{n-\alpha}f(t)$ be the fractional integral of order $n-\alpha$. If $f(t)\in L(0,T)$ and $f_{n-\alpha}(t)\in AC^n[0,T]$, then we have the following equality*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}f(t)=f(t)-\sum_{i=1}^{n}\frac{f_{n-\alpha}^{(n-i)}(0)}{\Gamma(\alpha-i+1)}t^{\alpha-i}.$$

Lemma 2. *Let $\sigma\in C_{1-q}(J,R)$. $x\in C_{1-q}(J,R)$ is a solution of the following linear initial value problem:*

$$\begin{cases} D_{0+}^{q}x(t)=Mx(t)+\sigma(t), & t\in(0,T], 0<q\le 1,\\ \tilde{x}(0)=N\tilde{x}(T)+r, \quad r\in R, \end{cases} \tag{2.1}$$

if and only if $x(t)$ is a solution of the following integral equation:

$$x(t)=(N\tilde{x}(T)+r)t^{q-1}+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}(\sigma(s)+Mx(s))ds, \tag{2.2}$$

where M, N are constants.

Proof. Assume $x(t)$ satisfies (2.1). From the first equation of (2.1) and Lemma 1, we have

$$\begin{aligned} x(t) &= \frac{I_{0+}^{1-q}x(t)|_{t=0}t^{q-1}}{\Gamma(q)} + I_{0+}^{q}(\sigma(t)+Mx(t)) \\ &= (N\tilde{x}(T)+r)t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}(\sigma(s)+Mx(s))ds. \end{aligned}$$

Conversely, assume that $x(t)$ satisfies (2.2). It is easy to check that $x(t) \in C_{1-q}(J,R)$. Applying the operator D_{0+}^{q} to both sides of (2.2), we have

$$D_{0+}^{q}x(t) = Mx(t) + \sigma(t).$$

In addition, by simple calculation, we conclude $\tilde{x}(0) = t^{1-q}x(t)|_{t=0} = N\tilde{x}(T) + r$. □

Lemma 3. *Assume that M, $N \geq 0$ are constants and the following inequality holds*

$$N + \frac{MT^q\Gamma(q)}{\Gamma(2q)} < 1, \tag{2.3}$$

then (1.1) has a unique solution.

Proof. We firstly define an operator $F : C_{1-q}(J,R) \to C_{1-q}(J,R)$ by

$$(Fx)(t) = (N\tilde{x}(T)+r)t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}(\sigma(s)+Mx(s))ds.$$

It is easy to check that $t^{1-q}(Fx)(t) \in C(J,R)$. Hence the operator F is well defined on $C_{1-q}(J,R)$.
For any $x, y \in C_{1-q}(J,R)$, we have

$$\begin{aligned} &\| Fx - Fy \|_{C_{1-q}} \\ &= \max_{t\in J} |t^{1-q}[(Fx)(t) - (Fy)(t)]| \\ &\leq \max_{t\in J} |N(\tilde{x}(T) - \tilde{y}(T))| + \max_{t\in J} \frac{t^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}Ms^{q-1}s^{1-q}|x(s)-y(s)|ds \\ &\leq N \| x-y \|_{C_{1-q}} + \max_{t\in J}\frac{Mt^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^{q-1}ds \| x-y \|_{C_{1-q}} \\ &\leq \left(N + \frac{MT^q\Gamma(q)}{\Gamma(2q)}\right) \| x-y \|_{C_{1-q}}. \end{aligned}$$

According to (2.3) and the Banach fixed point theorem, (1.1) has a unique solution. □

Definition 3. A function $\alpha_0 \in C_{1-q}(J,R)$ is called a lower

$$\begin{cases} D^q_{0+}\alpha_0 \le \tilde{\alpha_0}(0)t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,\alpha_0(s))ds, \\ \tilde{\alpha_0}(0) \le g(\tilde{\alpha_0}(T)). \end{cases}$$

Analogously, $\beta_0 \in C_{1-q}(J,R)$ is called an upper solution of (1.1) if

$$\begin{cases} D^q_{0+}\beta_0 \ge \tilde{\beta_0}(0)t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,\beta_0(s))ds, \\ \tilde{\beta_0}(0) \ge g(\tilde{\beta_0}(T)). \end{cases}$$

In what follows, we assume that

$$\alpha_0(t) \le \beta_0(t),\ t \in (0,T],\ \tilde{\alpha_0}(0) \le \tilde{\beta_0}(0).$$

Let $[\alpha_0,\beta_0] = \{x \in C_{1-q}(J,R) : \alpha_0(t) \le x(t) \le \beta_0(t), \forall t \in (0,T],\ \tilde{\alpha_0}(0) \le \tilde{x_0}(0) \le \tilde{\beta_0}(0)\}$.

Lemma 4. *Suppose that there are two constants $M,N \ge 0$ such that*

$$0 \le f_x(t,\eta(t)) \le M,\ \ 0 \le g'(\tilde{\eta}(T)) \le N, \forall \eta \in [\alpha_0,\beta_0].$$

If (2.3) holds and $p(t) \in C_{1-q}(J,R)$ satisfies

$$\begin{cases} p(t) \le \tilde{p}(0)t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f_x(s,\eta(s))p(s)ds, \\ \tilde{p}(0) \le g'(\tilde{\eta}(T))\tilde{P}(T), \end{cases}$$

then $p(t) \le 0$ for all $t \in (0,T]$ and $\tilde{p}(0) \le 0$.

Proof. Suppose that the inequality $p(t) \le 0, \forall t \in (0,T]$ is not true. So there exists at least a $t_* \in (0,T]$ such that $t_*^{1-q}p(t_*) > 0$. Without loss of generality, we assume that $t_*^{1-q}p(t_*) = \max\{t^{1-q}p(t) : t \in (0,T]\} = \rho > 0$.
Then we have that

$$\begin{aligned} t^{1-q}p(t) &\le \tilde{p}(0) + \frac{t^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1} f_x(s,\eta(s))p(s)ds \\ &\le g'(\tilde{\eta}(T))\tilde{P}(T) + \frac{t^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1} f_x(s,\eta(s))p(s)ds \\ &\le N\rho + \frac{t^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^{q-1} f_x(s,\eta(s))s^{1-q}p(s)ds. \end{aligned}$$

Let $t = t_*$, we have

$$\rho \le \left(N + \frac{MT^q\Gamma(q)}{\Gamma(2q)}\right)\rho.$$

So

$$N + \frac{MT^q\Gamma(q)}{\Gamma(2q)} \ge 1.$$

Which contradicts (2.3). Hence $p(t) \leq 0$ for all $t \in (0,T]$.
For $t = 0$, $\tilde{p}(T) \leq 0$, we have that

$$\tilde{p}(0) \leq g'(\tilde{\eta}(T))\tilde{P}(T) \leq 0.$$

□

3. QUASILINEARIZATION

In this section, we use the method of quasilinearization to construct a monotone sequence which converge quadratically to the solution of fractional differential equations with two point boundary condition.

Theorem 1. *Let α_0, β_0 be a lower and upper solutions of (1.1),respectively. Assume that (2.3) holds and that*

(1) $0 \leq g'(\tilde{\eta}(T)) \leq N$, $0 \leq g'(x) - g'(y) \leq L_1(x-y)$,
(2) $0 \leq f_x(t,\eta(t)) \leq M$, $0 \leq f_x(t,x) - f_x(t,y) \leq L_2(x-y)$, *where* L_1, $L_2 \geq 0$, $\eta \in [\alpha_0, \beta_0]$ *and* $\alpha_0 \leq y \leq x \leq \beta_0$,
(3) $1 - \Gamma(q)E_{q,q}(MT^q)N > 0$.

Then there exist two monotone sequences $\{\alpha_n\}$,$\{\beta_n\} \subset [\alpha_0, \beta_0]$ both of which converge uniformly to the unique solution of (1.1) and the convergence is quadratic.

Proof. For any $\eta \in [\alpha_0, \beta_0]$, consider the following linear fractional differential equation:

$$\begin{cases} D^q_{0+}x(t) = f(t,\eta(t)) + f_x(t,\eta(t))(x(t) - \eta(t)), \\ \tilde{x}(0) = g(\tilde{\eta}(T)) + g'(\tilde{\eta}(T))(\tilde{x}(T) - \tilde{\eta}(T)). \end{cases}$$

Obviously, by Lemma 3, the problem above has a unique solution which satisfies

$$\begin{cases} x(t) = \tilde{x}(0)t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[f(s,\eta(s)) + f_x(s,\eta(s))(x(s) - \eta(s))]ds, \\ \tilde{x}(0) = g(\tilde{\eta}(T)) + g'(\tilde{\eta}(T))(\tilde{x}(T) - \tilde{\eta}(T)). \end{cases}$$

Replacing η, x by α_0, α_1,respectively, we obtain

$$\begin{cases} \alpha_1(t) & = \tilde{\alpha_1}(0)t^{q-1} \\ & \quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[f(s,\alpha_0(s)) + f_x(s,\alpha_0(s))(\alpha_1(s) - \alpha_0(s))]ds, \\ \tilde{\alpha_1}(0) & = g(\tilde{\alpha_0}(T)) + g'(\tilde{\alpha_0}(T))(\tilde{\alpha_1}(T) - \tilde{\alpha_0}(T)). \end{cases}$$

Setting $p(t) = \alpha_0(t) - \alpha_1(t)$, we obtain that

$$\begin{aligned} p(t) &= \alpha_0(t) - \alpha_1(t) \\ &\leq \tilde{\alpha_0}(0)t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,\alpha_0(s))ds - \tilde{\alpha_1}(0)t^{q-1} \\ &- \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[f(s,\alpha_0(s)) + f_x(s,\alpha_0(s))(\alpha_1(s) - \alpha_0(s))]ds \end{aligned}$$

$$\leq \tilde{p}(0)t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f_x(s,\alpha_0(s))p(s)ds.$$

And we have that

$$\begin{aligned}\tilde{p}(0) &= \tilde{\alpha_0}(0) - \tilde{\alpha_1}(0)\\ &\leq g(\tilde{\alpha_0}(T)) - g(\tilde{\alpha_0}(T)) - g'(\tilde{\alpha_0}(T))(\tilde{\alpha_1}(T) - \tilde{\alpha_0}(T))\\ &= g'(\tilde{\alpha_0}(T))\tilde{p}(T).\end{aligned}$$

By Lemma 4, we know $p(t) \leq 0$ for all $t \in (0,T]$ and $\tilde{p}(0) \leq 0$. So $t^{1-q}\alpha_0(t) \leq t^{1-q}\alpha_1(t)$ for all $t \in J$.
Now replacing $\eta,\ x$ by $\beta_0,\ \beta_1$, we have that

$$\begin{cases}\beta_1(t) = & \tilde{\beta_1}(0)t^{q-1}\\ & +\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[f(s,\beta_0(s)) + f_x(s,\alpha_0(s))(\beta_1(s) - \beta_0(s))]ds,\\ \tilde{\beta_1}(0) = & g(\tilde{\beta_0}(T)) + g'(\tilde{\alpha_0}(T))(\tilde{\beta_1}(T) - \tilde{\beta_0}(T)).\end{cases}$$

Similarly, we can get $t^{1-q}\beta_1(t) \leq t^{1-q}\beta_0(t)$ for all $t \in J$.
Next we set $p(t) = \alpha_1(t) - \beta_1(t)$, we can obtain

$$\begin{aligned} p(t) &= \alpha_1(t) - \beta_1(t)\\ &= \tilde{\alpha_1}(0)t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[f(s,\alpha_0(s)) + f_x(s,\alpha_0(s))(\alpha_1(s) - \alpha_0(s))ds]\\ &\quad -\tilde{\beta_1}(0)t^{q-1} - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[f(s,\beta_0(s)) + f_x(s,\alpha_0(s))(\beta_1(s) - \beta_0(s))]ds\\ &\leq \tilde{p}(0)t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f_x(s,\alpha_0(s))p(s)ds.\end{aligned}$$

And we have that

$$\begin{aligned}\tilde{p}(0) &= \tilde{\alpha_1}(0) - \tilde{\beta_1}(0)\\ &= g(\tilde{\alpha_0}(T)) - g(\tilde{\beta_0}(T)) + g'(\tilde{\alpha_0}(T))[\tilde{\alpha_1}(T) - \tilde{\alpha_0}(T) - \tilde{\beta_1}(T) + \tilde{\beta_0}(T)]\\ &\leq g'(\tilde{\alpha_0}(T))\tilde{p}(T).\end{aligned}$$

By Lemma 4, we know $p(t) \leq 0$ for all $t \in (0,T]$ and $\tilde{p}(0) \leq 0$. So $t^{1-q}\alpha_1(t) \leq t^{1-q}\beta_1(t)$ for all $t \in J$.
Hence, we have $t^{1-q}\alpha_0(t) \leq t^{1-q}\alpha_1(t) \leq t^{1-q}\beta_1(t) \leq t^{1-q}\beta_0(t)$ for all $t \in J$.

Now suppose that

$$t^{1-q}\alpha_0(t) \leq t^{1-q}\alpha_{k-1}(t) \leq t^{1-q}\alpha_k(t) \leq t^{1-q}\beta_k(t) \leq t^{1-q}\beta_{k-1}(t) \leq t^{1-q}\beta_0(t).$$

To show $t^{1-q}\alpha_k(t) \le t^{1-q}\alpha_{k+1}(t) \le t^{1-q}\beta_{k+1}(t) \le t^{1-q}\beta_k(t)$.
It's easy to get that $\alpha_k(t)$ is the lower solution of (1.1) and $\alpha_{k+1}(t)$ satisfies

$$\begin{cases} \alpha_{k+1}(t) & = \tilde{\alpha_{k+1}}(0)t^{1-q} \\ & \quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[f(s,\alpha_k(s)) + f_x(s,\alpha_k(s))(\alpha_{k+1}(s)-\alpha_k(s))]ds, \\ \tilde{\alpha_{k+1}}(0) & = g(\tilde{\alpha_k}(T)) + g'(\tilde{\alpha_k}(T))(\tilde{\alpha_{k+1}}(T)-\tilde{\alpha_k}(T)). \end{cases}$$

So using the above method we can obtain $t^{1-q}\alpha_k(t) \le t^{1-q}\alpha_{k+1}(t)$.
Similarly, $\beta_k(t)$ is the upper solution of (1.1) and $\beta_{k+1}(t)$ satisfies

$$\begin{cases} \beta_{k+1}(t) & = \tilde{\beta_{k+1}}(0)t^{q-1} \\ & \quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[f(s,\beta_k(s)) + f_x(s,\alpha_k(s))(\beta_{k+1}(s)-\beta_k(s))]ds, \\ \tilde{\beta_{k+1}}(0) & = g(\tilde{\beta_k}(T)) + g'(\tilde{\alpha_k}(T))(\tilde{\beta_{k+1}}(T)-\tilde{\beta_k}(T)). \end{cases}$$

Hence $t^{1-q}\beta_{k+1}(t) \le t^{1-q}\beta_k(t)$.
To show $t^{1-q}\alpha_{k+1}(t) \le t^{1-q}\beta_{k+1}(t)$, we set $p(t) = \alpha_{k+1}(t) - \beta_{k+1}(t)$.

$$\begin{aligned} p(t) &= \alpha_{k+1}(t) - \beta_{k+1}(t) \\ &= \tilde{\alpha_{k+1}}(0)t^{q-1} \\ &\quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[f(s,\alpha_k(s)) + f_x(s,\alpha_k(s))(\alpha_{k+1}(s)-\alpha_k(s))ds] \\ &\quad - \tilde{\beta_{k+1}}(0)t^{q-1} \\ &\quad - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[f(s,\beta_k(s)) + f_x(s,\alpha_k(s))(\beta_{k+1}(s)-\beta_k(s))]ds \\ &\le \tilde{p}(0)t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f_x(s,\alpha_k(s))p(s)ds. \end{aligned}$$

Then we have that

$$\begin{aligned} \tilde{p}(0) &= \tilde{\alpha_{k+1}}(0) - \tilde{\beta_{k+1}}(0) \\ &= g(\tilde{\alpha_k}(T)) - g(\tilde{\beta_k}(T)) + g'(\tilde{\alpha_k}(T))[\tilde{\alpha_{k+1}}(T) - \tilde{\alpha_k}(T) - \tilde{\beta_{k+1}}(T) + \tilde{\beta_k}(T)] \\ &\le g'(\tilde{\alpha_k}(T))\tilde{p}(T). \end{aligned}$$

By Lemma 4, we know $t^{1-q}\alpha_{k+1}(t) \le t^{1-q}\beta_{k+1}(t)$.
By induction, we easily get $\{t^{1-q}\alpha_n\}$, $\{t^{1-q}\beta_n\}$ which satisfy the relation

$$t^{1-q}\alpha_0 \le t^{1-q}\alpha_1 \le \cdots \le t^{1-q}\alpha_n \le \cdots \le t^{1-q}\beta_n \le \cdots \le t^{1-q}\beta_1 \le t^{1-q}\beta_0.$$

Obviously, the sequences $\{t^{1-q}\alpha_n\}$, $\{t^{1-q}\beta_n\}$ are uniformly bounded and equicontinuous. Hence by the Ascoli-Arzela Theorem that the sequences $\{t^{1-q}\alpha_n\}$, $\{t^{1-q}\beta_n\}$ converge uniformly on J with

$$\lim_{n\to\infty} t^{1-q}\alpha_n = t^{1-q}\alpha, \quad \lim_{n\to\infty} t^{1-q}\beta_n = t^{1-q}\beta.$$

Then we can easily show that α and β are the solutions of (1.1) in $[\alpha_0, \beta_0]$.

To prove quadratic convergence of $\{t^{1-q}\alpha_n\}$, $\{t^{1-q}\beta_n\}$ to the solution, we consider

$$p_{k+1} = x - \alpha_{k+1}, \;\; r_{k+1} = \beta_{k+1} - x.$$

We can obtain

$$\begin{aligned}
\tilde{p_{k+1}}(0) &= \tilde{x}(0) - \tilde{\alpha_{k+1}}(0) \\
&= g(\tilde{x}(T)) - g(\tilde{\alpha_k}(T)) - g'(\tilde{\alpha_k}(T))(\tilde{\alpha_{k+1}}(T) - \tilde{\alpha_k}(T)) \\
&= g'(\tilde{\theta}(T))\tilde{p_k}(T) - g'(\tilde{\alpha_k}(T))(\tilde{p_k}(T) - \tilde{p_{k+1}}(T)) \\
&\le L_1|\tilde{\theta}(T) - \tilde{\alpha_k}(T)|\tilde{p_k}(T) + g'(\tilde{\alpha_k}(T))\tilde{p_{k+1}}(T) \\
&\le L_1 \parallel p_k(t) \parallel^2_{C_{1-q}} + N \parallel p_{k+1}(t) \parallel_{C_{1-q}}.
\end{aligned}$$

Then we have that

$$\begin{gathered}
p_{k+1}(t) = x(t) - \alpha_{k+1}(t) \\
= \tilde{p_{k+1}}(0)t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[f(s,x(s)) - f(s,\alpha_k(s)) \\
- f_x(s,\alpha_k(s))(\alpha_{k+1}(s) - \alpha_k(s))]ds \\
= \tilde{p_{k+1}}(0)t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[f_x(s,\rho(s))p_k(s) - f_x(s,\alpha_k(s))p_k(s) \\
+ f_x(s,\alpha_k(s))p_{k+1}(s))]ds \\
\le \tilde{p_{k+1}}(0)t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[L_2(\rho(s) - \alpha_k(s))p_k(s) + Mp_{k+1}(s)]ds \\
\le \tilde{p_{k+1}}(0)t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[L_2(x(s) - \alpha_k(s))p_k(s) + Mp_{k+1}(s)]ds \\
\le \tilde{p_{k+1}}(0)t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}(L_2 s^{2q-2} \parallel p_k(t) \parallel^2_{C_{1-q}} + Mp_{k+1}(s))ds.
\end{gathered}$$

Then using the method of successive approximations we get that

$$p_{k+1}(t) \tag{3.1}$$

$$\le \Gamma(q)\tilde{p_{k+1}}(0)t^{q-1}E_{q,q}(Mt^q) \tag{3.2}$$

$$+L_2 \parallel p_k(t) \parallel^2_{C_{1-q}} \int_0^t (t-s)^{q-1}s^{2q-2}E_{q,q}(M(t-s)^q)ds$$

$$\le \Gamma(q)E_{q,q}(MT^q)\tilde{p_{k+1}}(0)t^{q-1} \tag{3.3}$$

$$+L_2 \parallel p_k(t) \parallel^2_{C_{1-q}} E_{q,q}(M(T)^q)\frac{\Gamma(q)\Gamma(2q-1)}{\Gamma(3q-1)}t^{3q-2}.$$

$$t^{1-q}p_{k+1}(t) \le \Gamma(q)\tilde{p_{k+1}}(0)E_{q,q}(MT^q) \tag{3.4}$$

$$
\begin{aligned}
&+L_2 \parallel p_k(t) \parallel_{C_{1-q}}^2 E_{q,q}(M(T)^q)\frac{\Gamma(q)\Gamma(2q-1)}{\Gamma(3q-1)}T^{2q-1} \\
&\leq (L_1 \parallel p_k(t) \parallel_{C_{1-q}}^2 + N \parallel p_{k+1}(t) \parallel_{C_{1-q}})\Gamma(q)E_{q,q}(MT^q) \\
&+L_2 \parallel p_k(t) \parallel_{C_{1-q}}^2 E_{q,q}(M(T)^q)\frac{\Gamma(q)\Gamma(2q-1)}{\Gamma(3q-1)}T^{2q-1} \\
&= \left(L_1\Gamma(q) + L_2\frac{\Gamma(q)\Gamma(2q-1)}{\Gamma(3q-1)}T^{2q-1}\right)E_{q,q}(MT^q) \parallel p_k(t) \parallel_{C_{1-q}}^2 \\
&+\Gamma(q)E_{q,q}(MT^q)N \parallel p_{k+1}(t) \parallel_{C_{1-q}},
\end{aligned}
$$

where $E_q(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(qk+1)}$, $E_{q,q}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(qk+q)}$.
Thus we have the estimate

$$
\parallel p_{k+1}(t) \parallel_{C_{1-q}} \leq \Omega \parallel p_k(t) \parallel_{C_{1-q}}^2
$$

where

$$
\Omega = \frac{\left(L_1\Gamma(q) + L_2\frac{\Gamma(q)\Gamma(2q-1)}{\Gamma(3q-1)}T^{2q-1}\right)E_{q,q}(MT^q)}{1-\Gamma(q)E_{q,q}(MT^q)N}.
$$

Similarly

$$
\parallel r_{k+1}(t) \parallel_{C_{1-q}} \leq \Omega \parallel r_k(t) \parallel_{C_{1-q}}^2 .
$$

□

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THE CAUCHY-SCHWARZ INEQUALITY IN CAYLEY GRAPH AND TOURNAMENT STRUCTURES ON FINITE FIELDS

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Abstract. The Cayley graph construction provides a natural grid structure on a finite vector space over a field of prime or prime square cardinality, where the characteristic is congruent to 3 modulo 4, in addition to the quadratic residue tournament structure on the prime subfield. Distance from the null vector in the grid graph defines a Manhattan norm. The Hermitian inner product on these spaces over finite fields behaves in some respects similarly to the real and complex case. An analogue of the Cauchy-Schwarz inequality is valid with respect to the Manhattan norm. With respect to the non-transitive order provided by the quadratic residue tournament, an analogue of the Cauchy-Schwarz inequality holds in arbitrarily large neighborhoods of the null vector, when the characteristic is an appropriate large prime.

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1. MANHATTAN NORMS AND GRID GRAPHS

We consider the finite fields $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$ of prime and prime square cardinality, where $p \equiv 3 \mod 4$. The field $\mathbb{F}_{p^2}$ has a natural graph structure with the field elements as vertices, two distinct vertices u, z being adjacent if $(z-u)^4 = 1$. The subfield $\mathbb{F}_p$ of $\mathbb{F}_{p^2}$ then induces a subgraph in which x and y are adjacent if and only if $(y-x)^2 = 1$. The graph $\mathbb{F}_{p^2}$ is isomorphic to the Cartesian square $C_p^2 = C_p \square C_p$, where C_p is a p-cycle and within $\mathbb{F}_{p^2}$ the induced subgraph $\mathbb{F}_p$ is itself a p-cycle. Clearly the graph $\mathbb{F}_{p^2}$ is not planar, but can be drawn as a grid on the torus.

For any connected graph whose vertex set is a group, the distance of any vertex z from the identity element of the group is called the *norm* of z, denoted $N(z)$. In general, distances and norms measured in connected subgraphs induced by subgroups can be larger than distances and norms measured with reference to the whole graph. However, with respect to the distance-preserving subgraph induced by $\mathbb{F}_p$ in $\mathbb{F}_{p^2}$, the norm of any $z \in \mathbb{F}_p$ is the same as its norm with respect to the whole graph $\mathbb{F}_{p^2}$: this is simply the length of the shortest path from 0 to z in the cycle induced by $\mathbb{F}_p$.

For $q = p$ or $q = p^2$, the n-dimensional vector space $\mathbb{F}_q^n$ is also endowed with the Cartesian product graph structure $\mathbb{F}_q \square \cdots \square \mathbb{F}_q$ isomorphic to C_p^n or C_p^{2n}. The norm of a vector $\mathbf{v} = (v_1, \ldots, v_n)$ in $\mathbb{F}_q^n$ is then equal to the sum $N(v_1) + \cdots + N(v_n)$ and we also write $N(\mathbf{v})$ for this vector norm.

The Gaussian integers $\mathbb{Z}[i]$ also constitute a graph in which u and z are adjacent if and only if $(z-u)^4 = 1$.

It is easy to see that the norm in this *infinite Manhattan grid* satisfies the triangle and submultiplicative inequalities

$$N(u+z) \leq N(u) + N(z)$$
$$N(uz) \leq N(u)N(z)$$

To emphasize that the norms on $\mathbb{F}_{p^2}$, $\mathbb{F}_{p^2}^n$ and $\mathbb{Z}[i]$ are understood with reference to the specific grid graphs defined above, we call these norms *Manhattan norms*. Throughout this paper we think of $\mathbb{F}_{p^2}$ as the ring quotient $\mathbb{Z}[i]/(p)$.

2. Graph quotients and Cayley graphs

Given a graph G (undirected, with possible loops) on vertex set V and an equivalence relation $\equiv$ on V, the *quotient graph* $G/\equiv$ is defined as follows: the vertices of $G/\equiv$ are the equivalence classes of $\equiv$, and classes A, B are adjacent if for some $a \in A$, $b \in B$, the elements a, b are adjacent in G. Note that the distance of A to B in the quotient graph is at most equal to, but possibly less than the minimum of the distances a to b for all $a \in A$, $b \in B$. Note also that $G/\equiv$ can have loops even if G has not.

Given a group G with identity element e and a set Γ of group elements that generates G, the *(left) Cayley graph* $\mathcal{C}(G, \Gamma)$ of G with respect to Γ has vertex set G, elements $a, b \in G$ being considered adjacent if ab^{-1} or ba^{-1} belongs to Γ. For each congruence $\equiv$ of the group G, corresponding to some normal subgroup H, Γ yields a generating set $\Gamma_{\equiv}$ of $G/\equiv$ consisting with those classes of $\equiv$ that intersect Γ. The graph quotient of $\mathcal{C}(G, \Gamma)$ by the equivalence $\equiv$ coincides with the Cayley graph of the quotient graph $G/\equiv$ with respect to $\Gamma_{\equiv}$. For $R \subseteq G$ inducing a connected subgraph $[R]$ in $\mathcal{C}(G, \Gamma)$, denote by $d_R(x, y)$ the distance function of the subgraph $[R]$. Denoting by xH the H-coset of any $x \in G$, this relates to norms in $\mathcal{C}(G, \Gamma)$ and $\mathcal{C}(G, \Gamma)/\equiv$ as follows: for all $x \in R$,

$$d_R(x, e) \geq N(x) \geq N(xH)$$

Both inequalities can be strict. However, we have:

Cayley Graph Quotient Lemma. *Let a group G with identity e be generated by $\Gamma \subseteq G$, and consider any normal subgroup H with corresponding congruence $\equiv$. There is a set $R \subseteq G$ having exactly one element in common with each congruence* *class modulo H, and such that for every $x \in R$*

$$d_R(x,e) = N(x) = N(xH)$$

Proof. We can define the unique (representative) element $r(A) \in R \cap A$ for each coset A by induction on the distance $d(H,A)$ of A from H in $\mathcal{C}(G,\Gamma)/\equiv$. Let $r(H) = e$. Assuming $r(A)$ defined for all A with $d(H,A) \leq m$, let a coset B have distance $m+1$ from H. Choose any coset A adjacent to B with $d(H,A) = m$ and elements $a \in A$, $b \in B$ that are adjacent in $\mathcal{C}(G,\Gamma)$. Let $r(B) = ba^{-1}r(A)$. □

We can apply the above lemma in the case where $G = \mathbb{Z}[i]$, $\Gamma = \{1,i\}$ and $H = p\mathbb{Z}[i] = \{pa + pbi : a,b \in \mathbb{Z}\}$ for a prime integer $p \equiv 3 \mod 4$. Now $\mathcal{C}(G,\Gamma)$ and $\mathcal{C}(G,\Gamma)/\equiv$ are the Manhattan grid graphs on $\mathbb{Z}[i]$ and $\mathbb{Z}[i]/H = \mathbb{F}_{p^2}$, respectively. Referring to the set R of representatives in the lemma, for any H-cosets X,Y let x,y be the unique elements in $X \cap R$, $Y \cap R$. As $xy \in XY$, we have $N(XY) \leq N(xy)$. By the submultiplicative inequality in $\mathbb{Z}[i]$ we have $N(xy) \leq N(x)N(y)$. Using the lemma we have $N(x)N(y) = N(X)N(Y)$. This yields a submultiplicative inequality in $\mathbb{F}_{p^2}$ and a similar reasoning on the coset $X+Y$ yields a triangle inequality:

Triangle and Submultiplicative Inequalities in $\mathbb{F}_{p^2}$. *For all u,z in $\mathbb{F}_{p^2}$*

$$N(u+z) \leq N(u) + N(z)$$
$$N(uz) \leq N(u)N(z)$$

This indicates that Manhattan distance provides a well-behaved notion of neighborhood of 0 in the finite fields $\mathbb{F}_{p^2}$.

3. Squares in $\mathbb{F}_p$ and non-transitive order

For each prime $p \equiv 3 \mod 4$ the *quadratic residue tournament* on $\mathbb{F}_p$ is the directed graph with vertex set $\mathbb{F}_p$ in which there is an arrow from vertex x to vertex y if $y - x$ is a non-zero square in $\mathbb{F}_p$, in which case we write $x <_p y$. We write $x \leq_p y$ if $x <_p y$ or $x = y$. The relation $\leq_p$ is reflexive, anti-symmetric but not transitive, and for every $x \neq y$ exactly one of $x \leq_p y$ or $y \leq_p x$ holds. Using Dirichlet's theorem on primes in arithmetic progressions, Kustaanheimo showed [4] that for every positive integer k, there is a prime $p \equiv 3 \mod 4$, such that $\leq_p$ is a transitive (and linear) order relation on $\{0,1,\ldots,k\} \subseteq \mathbb{F}_p$, that is, all positive integers up to k are quadratic residues $\mod p$. Obviously k cannot exceed $(p-1)/2$. Implications of [4] and related questions were investigated by Järnefelt, Kustaanheimo, Quist [3,5], in particular with a view to discrete models in physics, also in subsequent application-oriented work between the 1950's (Coish [1]) and the 1980's (Nambu [6]). For further references see [2]. In particular [4] implies that for every positive integer k, there is a prime $p \equiv 3 \mod 4$, such that all $z \in \mathbb{F}_{p^2}$ with $N(z) \leq k$ are squares in $\mathbb{F}_{p^2}$. (Note that all elements of the prime subfield $\mathbb{F}_p$ are squares in $\mathbb{F}_{p^2}$.) To emphasise the analogy of the relation $\leq_p$ with the ordinary inequality relation $\leq$ among numbers, we say that a non-zero $z \in \mathbb{F}_{p^2}$ is *positive* if $z \in \mathbb{F}_p$ and $0 \leq_p z$.

4. Inner products compared in non-transitive order

The only non-trivial automorphism of the field $\mathbb{F}_{p^2}$ associates to each $z \in \mathbb{F}_{p^2}$ its *conjugate* $\overline{z}$. The *inner product* $\mathbf{v}\cdot\mathbf{w}$ of vectors $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ in $\mathbb{F}_{p^2}^n$ is defined as the scalar $v_1\overline{w_1} + \cdots + v_n\overline{w_n} \in \mathbb{F}_{p^2}$. This inner product is left and right distributive over vector addition, satisfies $\mathbf{v}\cdot\mathbf{w} = \overline{\mathbf{w}\cdot\mathbf{v}}$, $c(\mathbf{v}\cdot\mathbf{w}) = (c\mathbf{v})\cdot\mathbf{w} = \mathbf{v}\cdot(\overline{c}\mathbf{w})$ for all $c \in \mathbb{F}_{p^2}$. However, while $\mathbf{v}\cdot\mathbf{v}$ belongs to the prime subfield $\mathbb{F}_p$, $\mathbf{v}\cdot\mathbf{v}$ is not necessarily positive, and can be 0 even if $\mathbf{v} \neq \mathbf{0}$. Still, a conditional version of positive definiteness holds locally:

Theorem 1. *For every $k \geq 1$ there is a prime $p \equiv 3 \mod 4$, such that for all $n \geq 1$ and for all vectors $\boldsymbol{v} \in \mathbb{F}_{p^2}^n$ of Manhattan norm $N(\boldsymbol{v}) \leq k$, we have $0 \leq_p \boldsymbol{v}\cdot\boldsymbol{v}$ with equality if and only if $\boldsymbol{v} = \boldsymbol{0}$.*

Proof. By Kustaanheimo's result in [4] there is a prime integer $p \equiv 3 \mod 4$ such that $0, 1, \ldots, 2k^3$ are all quadratic residues mod p. For $\mathbf{v} = (v_1, \ldots, v_n)$ in $\mathbb{F}_{p^2}^n$, let $v_j = a_j + b_j i$, where $i^2 = -1$. If $N(\mathbf{v}) \leq k$ then for all j, $N(a_j) \leq k$ and $N(b_j) \leq k$, $v_j\overline{v_j} = a_j^2 + b_j^2$ belongs to the set of squares $\{0, \ldots, 2k^2\}$. Since v_j can be non-zero for at most k indices $1 \leq j \leq n$ only, the sum of the corresponding terms $a_j^2 + b_j^2$ belongs to the set of squares $\{0, 1, \ldots, 2k^3\}$. □

Note that for all vectors $\mathbf{v}, \mathbf{w} \in \mathbb{F}_{p^2}^n$

$$(\mathbf{v}\cdot\mathbf{w})(\mathbf{w}\cdot\mathbf{v}) = (\mathbf{v}\cdot\mathbf{w})(\overline{\mathbf{v}\cdot\mathbf{w}}) \in \mathbb{F}_p \quad \text{and}$$
$$(\mathbf{v}\cdot\mathbf{v})(\mathbf{w}\cdot\mathbf{w}) \in \mathbb{F}_p.$$

If $\mathbf{v}$ and $\mathbf{w}$ are *proportional*, i.e. if there exists a scalar c in $\mathbb{F}_{p^2}$ such that $\mathbf{v} = c\mathbf{w}$ or $\mathbf{w} = c\mathbf{v}$, then the above two products are equal. Generally, they are related in the quadratic residue tournament of $\mathbb{F}_p$ as follows.

Theorem 2 (Cauchy-Schwarz Inequality). *For every $k \geq 1$ there is a prime $p \equiv 3 \mod 4$, such that for all $n \geq 1$ and for all vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{F}_{p^2}^n$ of Manhattan norm at most k,*

$$(\boldsymbol{v}\cdot\boldsymbol{w})(\boldsymbol{w}\cdot\boldsymbol{v}) \leq_p (\boldsymbol{v}\cdot\boldsymbol{v})(\boldsymbol{w}\cdot\boldsymbol{w}).$$

Proof. For $n = 1$ the inequality holds trivially as the two sides are equal. Assume $n \geq 2$, $\mathbf{v} = (v_1, \ldots, v_n)$, $\mathbf{w} = (w_1, \ldots, w_n)$. For all $1 \leq i \leq n$, $N(v_i) \leq k$, $N(w_i) \leq k$. By Kustaanheimo's result [4] there is a prime $p \equiv 3 \mod 4$ such that all positive integers up to $4k^6$ are quadratic residues modulo p. For each of the $\binom{n}{2}$ pairs $\{i, j\} \subseteq \{1, \ldots, n\}$, $i \neq j$, by the triangle and submultiplicative inequalities in $\mathbb{F}_{p^2}$

$$N[(v_i w_j - v_j w_i)(\overline{v}_i\overline{w}_j - \overline{v}_j\overline{w}_j)] \leq (k^2 + k^2)^2 = 4k^4$$

Thus the element

$$(v_i w_j \overline{v}_i \overline{w}_j + v_j w_i \overline{v}_j \overline{w}_i) - (v_i w_j \overline{v}_j \overline{w}_i + v_j w_i \overline{v}_i \overline{w}_j) = (v_i w_j - v_j w_i)(\overline{v}_i\overline{w}_j - \overline{v}_j\overline{w}_j)$$

is a square of Manhattan norm at most $4k^4$ in $\mathbb{F}_p$, and it is non-zero for at most $\binom{k}{2} \leq k^2$ pairs $\{i,j\}$. Summing over all pairs $\{i,j\}$, all but at most $\binom{k}{2} \leq k^2$ terms vanish in the sum

$$\sum [(v_i w_j \overline{v}_i \overline{w}_j + v_j w_i \overline{v}_j \overline{w}_i) - (v_i w_j \overline{v}_j \overline{w}_i + v_j w_i \overline{v}_i \overline{w}_j)]$$

which therefore has Manhattan norm at most $4k^6$ and it must also be a square in $\mathbb{F}_p$. But this sum is equal to the difference of products

$$\sum_{i=1}^{n} v_i \overline{v}_i \sum_{j=1}^{n} w_j \overline{w}_j - \sum_{i=1}^{n} v_i \overline{w}_i \sum_{j=1}^{n} \overline{v}_j w_j = (\mathbf{v}\cdot\mathbf{v})(\mathbf{w}\cdot\mathbf{w}) - (\mathbf{v}\cdot\mathbf{w})(\mathbf{w}\cdot\mathbf{v})$$

which is consequently a square in $\mathbb{F}_p$. □

Remark. From the proof it is clear that, in analogy with the classical Cauchy-Schwarz inequality, for vectors $\mathbf{v},\mathbf{w}$ of norm not exceeding k in $\mathbb{F}_{p^2}^n$, where p is related to k as stipulated above, the Cauchy-Schwarz inequality with respect to $\leq_p$ holds with equality if and only if $v_i w_j - v_j w_i = 0$ for all i,j, i.e. if and only if $\mathbf{v},\mathbf{w}$ are proportional.

We note that the inequality established above is conditional, it holds only in a specified Manhattan neighborhood of the null vector. Every non-zero element of $\mathbb{F}_p$ can be written as a sum of two squares, in particular there are $a,b \in \mathbb{F}_p$, such that $a^2+b^2=-1$. For $z=a+bi$ we have $z\overline{z}=-1$. As soon as $n \geq 2$, in $\mathbb{F}_{p^2}^n$ let

$$\mathbf{v} = (a,b,0,\dots,0) \quad \text{and} \quad \mathbf{w} = (bz,-az,0,\dots,0)$$

The inequality $(\mathbf{v}\cdot\mathbf{w})(\mathbf{w}\cdot\mathbf{v}) \leq_p (\mathbf{v}\cdot\mathbf{v})(\mathbf{w}\cdot\mathbf{w})$ fails because the left-hand side is 0 and the right-hand side is -1. In fact if $n \geq 3$, the inequality can be invalidated with vectors $\mathbf{v},\mathbf{w}$ in $\mathbb{F}_p^n$ as follows. Taking again $a,b \in \mathbb{F}_p$ with $a^2+b^2=-1$, let

$$\mathbf{v} = (1,a,b,0,\dots,0) \quad \text{and} \quad \mathbf{w} = (1,0,0,0,\dots,0)$$

However, the Cauchy-Schwarz inequality holds unconditionally in the 2-dimensional case for vectors with components in $\mathbb{F}_p$:

Special case of $\mathbb{F}_p^2$. *Let p be a prime congruent 3 modulo 4. For all vectors $\boldsymbol{v},\boldsymbol{w}$ in $\mathbb{F}_p^2$*

$$(\boldsymbol{v}\cdot\boldsymbol{w})(\boldsymbol{w}\cdot\boldsymbol{v}) \leq_p (\boldsymbol{v}\cdot\boldsymbol{v})(\boldsymbol{w}\cdot\boldsymbol{w}).$$

Proof. Now the conjugation appearing in the inner products is the identity. Written in components,

$$(\mathbf{v}\cdot\mathbf{v})(\mathbf{w}\cdot\mathbf{w}) - (\mathbf{v}\cdot\mathbf{w})(\mathbf{w}\cdot\mathbf{v}) = (v_1^2+v_2^2)(w_1^2+w_2^2) - (v_1w_1+v_2w_2)^2 =$$
$$= v_1^2w_2^2 + v_2^2w_1^2 - 2v_1w_1v_2w_2 = (v_1w_2 - v_2w_1)^2$$

□

5. MANHATTAN NORM OF INNER PRODUCT

The Manhattan norm can be seen to be submultiplicative not only on the ring $\mathbb{Z}[i]$ and its quotient field $\mathbb{F}_{p^2}$, but on all vector spaces $\mathbb{F}_{p^2}^n$, with respect to the inner product:

Cauchy-Schwarz Inequality for Manhattan Norm on $\mathbb{F}_{p^2}^n$. *Consider any prime* $p \equiv 3 \mod 4$ *and let* $n \geq 1$. *For all* $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{F}_{p^2}^n$

$$N(\boldsymbol{v} \cdot \boldsymbol{w}) \leq N(\boldsymbol{v})N(\boldsymbol{w}).$$

Proof. Let $\mathbf{v} = (v_1, \ldots, v_n), \mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{F}_{p^2}^n$. Then $\mathbf{v} \cdot \mathbf{w} = \sum v_j \overline{w_j}$. Clearly $N(z) = N(\overline{z})$ for any $z \in \mathbb{F}_{p^2}$. By the triangle and submultiplicative inequalities in $\mathbb{F}_{p^2}$ we have

$$\begin{aligned} N(\mathbf{v} \cdot \mathbf{w}) = N\left(\sum v_j \overline{w_j}\right) &\leq \sum N\left(v_j \overline{w_j}\right) \leq \sum N(v_j) N(w_j) \leq \\ &\leq \sum N(v_j) \sum N(w_j) = N(\mathbf{v}) N(\mathbf{w}) \end{aligned}$$

□

Remark. The inequality $N(\mathbf{v} \cdot \mathbf{w}) \leq N(\mathbf{v})N(\mathbf{w})$ is easily interpreted and continues to hold for $\mathbf{v}, \mathbf{w}$ in the module $(\mathbb{Z}[i]/m\mathbb{Z}[i])^n$ for any positive integer m. As soon as m is composite, or a prime not congruent to 3 modulo 4, the ring $\mathbb{Z}[i]/m\mathbb{Z}[i]$ fails to be an integral domain.

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NECESSARY AND SUFFICIENT CONDITION FOR THE BOUNDEDNESS OF THE GERBER-SHIU FUNCTION IN DEPENDENT SPARRE ANDERSEN MODEL

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Abstract. In this paper we investigate the boundedness of the Gerber-Shiu expected discounted penalty function applied in insurance mathematics. We give a necessary and sufficient condition for its boundedness. This condition is essentially the boundedness of the conditional expectation given the ruin occurs at the first claim. It assures that the integral equation satisfied by the Gerber-Shiu function has a unique bounded solution and it is exactly the Gerber-Shiu function. By the help of the proven necessary and sufficient condition we can also simplify the theorem concerning the necessary and sufficient condition for the limit property. We do not assume independence between the inter-claim time and the claim size but the results can be applied to the usually investigated independent case as well. We present examples to show the applicability of the condition.

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1. INTRODUCTION

In insurance mathematics the following stochastic process is frequently investigated ([1],[5],[7],[8])

$$U(t) = x + ct - \sum_{k=1}^{N(t)} Y_k.$$

$U(t)$ stands for the surplus of the insurance company as a function of time t, x is the initial surplus of the insurance company, c is the insurer's rate of the premium income per unit time, $N(t)$ is the number of claims up to time t and Y_k is the random size of the kth claim. The time periods between the consequtive claims are called inter-claim times. As an explanation, ct stands for the income of the company to t, while $\sum_{k=1}^{N(t)} Y_k$ is its expenses to t paid for the insurers. The process $U(t)$ is briefly called as Sparre Andersen risk process.

In risk theory the main topics of interest are the probability of ruin, the moments of the time of ruin, the distribution of the surplus before ruin and the deficit at ruin.

Ruin means that the surplus falls below zero, time of ruin is the time point when it first happens. To answer these questions, the Gerber-Shiu expected discounted penalty function was introduced by Gerber and Shiu in [6],[7] and was investigated in lots of papers (for example [1],[5],[8],[10]).

In the papers dealing with determination of the Gerber-Shiu function, usually an integral equation is set up for it and solved with a boundary condition (e.g.[1], [8]). Researchers use the boundary condition to select the Gerber-Shiu function from the solutions of the integral equation. In [1],[8], the authors use the boundary condition that the expected discounted penalty function tends to zero if its argument tends to infinity. This implies the boundedness of the function investigated, as well. However, the mentioned limit property does not always hold, moreover, the Gerber-Shiu function may be unbounded as well. It strictly depends on the penalty funtion. Examples for these cases are presented in [9]. The identification of the Gerber-Shiu function in the set of solutions is rather connected to its boundedness than to its limit. In this paper we present such a condition (condition A in Section 2) which assures the boundedness of the Gerber-Shiu function. This condition for the boundedness of the Gerber-Shiu function coincides with the condition for uniqueness of the bounded solution of the integral equation. It concerns the penalty function and the joint density function of inter-claim times and claim sizes but it is not a condition on the solution itself.

Providing a necessary and sufficient condition for the limit property in [9], boundedness of the Gerber-Shiu function was assumed. In applications it was checked in each case by different methods based on estimations. Now condition A makes easier to check the boundedness.

Researchers usually assume independence between the inter-claim times and claim size but recently some results have been published for models that allow for dependence between claim size and claim-time ([2],[3],[4],[5],[11]). In this paper we examine the generalized case.

Let us consider the risk model with the following notations: $T_0 = 0$, T_k, $k = 1,2,3,..$ are the inter-arrival times, Y_k is the kth claim size, (T_k, Y_k) has joint distribution function $H(t, y)$ and density function $h(t, y)$ for any value of k. We suppose that (T_k, Y_k) are independent for $k = 1,2,3,...$ but the random variables T_k and Y_k may be dependent for fixed values of $k = 1,2,3....$ The marginal distribution function of T_k is denoted by $F(t)$, its density function by $f(t)$, where T_k is nonnegative with finite expectation μ_f. The marginal distribution function of Y_k is denoted by $G(y)$, its density function by $g(y)$, where Y_k is nonnegative with finite expectation μ_g. The number of claims up to time t is given by

$$N(t) = \begin{cases} 0, & if\ t \leq T_1 \\ n, & if\ \sum_{k=1}^{n} T_k < t\ and\ t \leq \sum_{k=1}^{n+1} T_k. \end{cases}$$

The ultimate probability of ruin is

$$\Psi(x) = P\left(x + ct - \sum_{k=1}^{N(t)} Y_k < 0 \ \ for\ some\ 0 \le t\right) .$$

The time of ruin is

$$T = \begin{cases} \inf\{t : U(t) < 0\}, & if\ there\ exists\ 0 \le t, such\ that\ U(t) < 0, \\ \infty, & if\ \ 0 \le U(t)\ \ for\ all\ 0 \le t. \end{cases}$$

Let $w : \mathbb{R}_0^+ \mathrm{x} \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be the penalty function.

The Gerber-Shiu expected discounted penalty function is defined as

$$m_\delta(x) = \mathbb{E}\left(e^{-\delta T} \cdot w\left(U(T^-), |U(T)|\right) \cdot 1_{T<\infty} | U(0) = x\right),$$

if the expectation is finite. Here $0 \le x,\ 0 \le \delta$ and $U(T^-)$ is the surplus immediately before ruin, $|U(T)|$ is the deficit at ruin. Note that $0 \le m_\delta(x)$.

Applying appropriate penalty functions, the Gerber-Shiu function is suitable for investigating some important characteristics of the process. For example if we use penalty function $w(x, y) \equiv 1$ and substitute $\delta = 0$ we get

$$m_0(x) = \mathbb{E}\left(1_{T<\infty} | U(0) = x\right) = \Psi(x),$$

furthermore $\frac{\partial m_\delta(x)}{\partial \delta}|_{\delta=0} = -\mathbb{E}\left(1_{T<\infty}\right)$. If we apply penalty function $w(x, y) = x^n$ and substitute $\delta = 0$, $m_0(x) = \mathbb{E}\left(U(T^-)^n \cdot 1_{T<\infty} | U(0) = x\right)$, which is the nth moments of the surplus before ruin. If $w(x, y) = y$ then

$$m_0(x) = \mathbb{E}\left(|U(T)| \cdot 1_{T<\infty} | U(0) = x\right)$$

is the expectation of deficit at ruin.

For the sake of simplicity in this paper we suppose that h and w are continuous functions on $[0, \infty)\mathrm{x}[0, \infty)$ and so is $m_\delta(x)$. We note that $U(0) = x$ follows from the definition of $U(t)$, hence in the following we do not indicate it.

In the next section of the paper we prove a necessary and sufficient condition for the boundedness of $m_\delta(x)$ and we present some examples to show the applicability of the theorem. Then we conclude the consequences of the theorem concerning the integral equation satisfied by the Gerber-Shiu function and we present a necessary and sufficient condition for the limit property $\lim\limits_{x\to\infty} m_\delta(x) = 0$.

2. A NECESSARY AND SUFFICIENT CONDITION FOR THE BOUNDEDNESS

Theorem 1. *Suppose*

$$S(x) = \int_0^\infty \int_{x+ct}^\infty e^{-\delta t} w(x + ct, y - ct - x) h(t, y)\, dy\, dt < \infty$$

for all $0 \le x$ *and let* $0 < \delta$ *be fixed.*

Then, the Gerber-Shiu function $m_\delta(x)$ *is bounded if and only if*

$$\sup_{0 \le x} S(x) = \sup_{0 \le x} \int_0^\infty \int_{x+ct}^\infty e^{-\delta t} w(x + ct, y - ct - x) h(t, y)\, dy\, dt < \infty. \qquad \text{(A)}$$

Proof. First we suppose that $m_\delta(x)$ is a bounded function of x, and we prove that

$$\kappa = \sup_{0\le x} S(x) < \infty.$$

The non-negativity of the functions h and w implies the non-negativity of S as well. Let A_i be the event that the ruin occurs at the ith claim, $i = 1,2,3,....$ Notice that

$$1_{T<\infty} = \sum_{i=1}^{\infty} 1_{A_i}.$$

Now

$$m_\delta(x) = \mathbb{E}\left(e^{-\delta T}\cdot w\left(U(T^-),|U(T)|\right)\cdot 1_{T<\infty}\right) \ge \mathbb{E}\left(e^{-\delta T}\cdot w(U(T^-),|U(T)|)\cdot 1_{A_1}\right) = \\ = \int_0^\infty \int_{x+ct}^\infty e^{-\delta t} w(x+ct, y-ct-x)h(t,y)dydt = S(x).$$

Therefore, if $m_\delta(x) \le K$, then $S(x) \le K$, consequently $\kappa = \sup_{0\le x} S(x) \le K < \infty$. Conversely, we suppose that $\kappa = \sup_{0\le x} S(x) < \infty$. We prove that in this case $m_\delta(x)$ is bounded.

Now

$$m_\delta(x) = \mathbb{E}\left(e^{-\delta T}\cdot w\left(U(T^-),|U(T)|\right)\cdot\left(\sum_{i=1}^{\infty} 1_{A_i}\right)\right) = \\ = \sum_{i=1}^{\infty} \mathbb{E}\left(e^{-\delta T}\cdot w\left(U(T^-),|U(T)|\right)\cdot 1_{A_i}\right).$$

Let

$$m_\delta(x;A_i) = \mathbb{E}\left(e^{-\delta T}\cdot w\left(U(T^-),|U(T)|\right)\cdot 1_{A_i}\right),\ \ i = 1,2,3,...$$

Then

$$m_\delta(x;A_1) = \int_0^\infty \int_{x+ct}^\infty e^{-\delta t} w(x+ct, y-ct-x)h(t,y)dydt = S(x) < \infty$$

according to condition (A).

If A_2 holds, then

$$T = T_1 + T_2,\ \ U(T^-) = x + cT_1 - Y_1 + cT_2,$$

$$U(T) = x + cT_1 - Y_1 + cT_2 - Y_2$$

and

$$m_\delta(x;A_2) = \int_0^\infty \int_0^{x+ct_1} e^{-\delta t_1} \cdot \int_0^\infty \int_{x+ct_1-y_1+ct_2}^\infty e^{-\delta t_2} \cdot w(x+ct_1-y_1+ct_2, y_2-(x+ct_1-y_1+ct_2)) \cdot \cdot h(t_2,y_2)dy_2dt_2 \cdot h(t_1,y_1)dy_1dt_1$$

If we use the notation $z = x+ct_1-y_1$, then $0 \le z$, as A_1 does not hold. The inner integral of the previous formula has the form

$$\int_0^\infty \int_{x+ct_1-y_1+ct_2}^\infty e^{-\delta t_2} w(x+ct_1-y_1+ct_2, y_2-(x+ct_1-y_1+ct_2))h(t_2,y_2)dy_2dt_2$$
$$= \int_0^\infty \int_{z+ct_2}^\infty e^{-\delta t_2} w(z+ct_2, y_2-(z+ct_2))h(t_2,y_2)dy_2dt_2 = m_\delta(x+ct_1-y_1;A_1).$$

Now we can use the following estimate:

$$\begin{aligned} m_\delta(x;A_2) &= \int_0^\infty \int_0^{x+ct_1} e^{-\delta t_1} m_\delta(x+ct_1-y_1;A_1)h(t_1,y_1)dy_1dt_1 \\ &\le \int_0^\infty \int_0^\infty e^{-\delta t_1} \cdot \kappa \cdot h(t_1,y_1)dy_1dt_1 = \kappa \int_0^\infty e^{-\delta t_1} f(t_1)dt_1 = \kappa \hat{f}(\delta), \end{aligned}$$

where $\hat{f}(\delta)$ is the Laplace transform of the density function f. Note that if $0<\delta$, then $\hat{f}(\delta) < 1$.
In general, for $2 \le i$, the following recursion holds:

$$m_\delta(x;A_i) = \int_0^\infty \int_0^{x+ct_1} e^{-\delta t_1} m_\delta(x+ct_1-y_1;A_{i-1})h(t_1,y_1)dy_1dt_1.$$

This formula can be proved with the following argument: take the conditional expectation on the first claim time and size. If A_i holds for some $2 \le i$, then the ruin does not occur at the first claim, consequently $Y_1 \le x+cT_1$. After the first claim, the "initial" surplus is $x+cT_1-Y_1$, the number of claims to ruin is decreased. If $T_1 = t_1$ and $Y_1 = y_1$, then the expectation of the function $e^{-\delta T} \cdot w(U(T^-), |U(T)|)$ from this time point t_1 and with initial surplus $x+ct_1-y_1$ is $m_\delta(x+ct_1-y_1;A_{i-1})$. The time to ruin from zero is t_1 more than the time to ruin from the time T_1, which is the reason for multiplication by $e^{-\delta t_1}$. Consequently,

$$\begin{aligned} m_\delta(x;A_i) &= \mathbb{E}\left(e^{-\delta T} w\left(U(T^-),|U(T)|\right)\cdot 1_{A_i}\right) = \\ &= \mathbb{E}\left(\mathbb{E}\left(e^{-\delta T} w\left(U(T^-),|U(T)|\right)\cdot 1_{A_i}|T_1,Y_1\right)\right) = \\ &= \int_0^\infty \int_0^{x+ct_1} e^{-\delta t_1} m_\delta(x+ct_1-y_1;A_{i-1})h(t_1,y_1)dy_1dt_1 \end{aligned}$$

Now if

$$m_\delta(x+ct_1-y_1;A_{i-1}) \leq K_{i-1},$$

then

$$m_\delta(x;A_i) \leq \int_0^\infty \int_0^\infty e^{-\delta t_1} m_\delta(x+ct_1-y_1;A_{i-1})h(t_1,y_1)dy_1dt_1 \leq K_{i-1}\hat{f}(\delta)$$

We have seen that

$$m_\delta(x;A_2) \leq \kappa \hat{f}(\delta),$$

hence

$$m_\delta(x;A_i) \leq \kappa\left(\hat{f}(\delta)\right)^{i-1},\ i\geq 3$$

also holds, consequently the inequality

$$m_\delta(x) \leq \sum_{i=1}^{\infty} \kappa\left(\hat{f}(\delta)\right)^{i-1} = \frac{\kappa}{1-\hat{f}(\delta)}$$

is satisfied, which implies the boundedness of $m_\delta(x)$. □

Remark 1. Notice that the necessary and sufficient condition for the boundedness of is $m_\delta(x)$ essentially the boundedness of the conditional expectation given the ruin occurs at the first claim.

Remark 2. In insurance mathematics the net profit condition $(0 < c\mu_f - \mu_g)$ is usually assumed. Net profit condition expresses that the income per unit time is more than the average expenses per unit time. Condition A can be satisfied even in the case when the usual net profit condition does not hold. Consequently, the net profit condition is not necessary for the boundedness of the Gerber-Shiu function if δ>0.

Let us investigate some examples when the boundedness of $m_\delta(x)$ holds and when it does not hold. In Examples 1,2,3 we suppose that

$$h(t,y) = f(t)g(y),$$

that is we assume the independence between the inter-claim times and claim sizes. In Example 4 we deal with a dependent case.

Example 1. Let

$$w(x,y) = x^n,\ 1 \le n.$$

If

$$\lim_{y\to\infty} y^n(1-G(y))$$

is finite, then $m_\delta(x)$ is bounded.
Let us investigate $S(x)$ and prove its boundedness.

$$\begin{aligned}
S(x) &= \int_0^\infty \int_{x+ct}^\infty e^{-\delta t} w(x+ct, y-ct-x)h(t,y)dydt = \\
&= \int_0^\infty \int_{x+ct}^\infty e^{-\delta t}(x+ct)^n f(t)g(y)dydt = \\
&= \int_0^\infty e^{-\delta t}(x+ct)^n f(t)\left(\int_{x+ct}^\infty g(y)dy\right)dt = \\
&= \int_0^\infty e^{-\delta t}(x+ct)^n f(t)\,(1-G(x+ct))\,dt \le \\
&\le K\int_0^\infty e^{-\delta t} f(t)dt
\end{aligned}$$

as

$$\lim_{y\to\infty} y^n(1-G(y)) \le K_1$$

implies

$$(x+ct)^n\,(1-G(x+ct)) \le K$$

for appropriate value of K.
Consequently $\sup\limits_{0\le x} S(x)$ is finite, hence applying Theorem 1 we can conclude that $m_\delta(x)$ is bounded.
We note that if the nth moment of Y_k is finite, then $\lim\limits_{y\to\infty} y^n(1-G(y)) = 0$. Hence in that case the Gerber-Shiu function is bounded.

Example 2. Let

$$w(x,y) = y.$$

If

$$\lim_{y\to\infty} y(1-G(y))$$

is finite, then $m_\delta(x)$ is bounded.
Let us investigate $S(x)$ again and prove its boundedness.

$$S(x) = \int_0^\infty \int_{x+ct}^\infty e^{-\delta t} w(x+ct, y-ct-x)h(t,y)dydt =$$

$$
\begin{aligned}
&= \int_0^\infty \int_{x+ct}^\infty e^{-\delta t}(y-ct-x)f(t)g(y)dydt = \\
&= \int_0^\infty e^{-\delta t} f(t) \left(\int_{x+ct}^\infty (y-ct-x)g(y)dy \right) dt \le \\
&\le \int_0^\infty e^{-\delta t} f(t) \left(\mu_g - (x+ct)(1-G(x+ct)) \right) dt \le \\
&\le K \int_0^\infty e^{-\delta t} f(t) dt = K \cdot \hat{f}(\delta)
\end{aligned}
$$

as $\lim\limits_{y\to\infty} y(1-G(y)) \le K_1$ implies the inequality

$$0 \le \mu_g - (x+ct)(1-G(x+ct)) \le K$$

for appropriate value of K.

We point out that in both cases the tails of the distribution function of the claim sizes determine the boundedness of the Gerber-Shiu function. So does in our next example as well.

Example 3. Let

$$w(x,y) = x^3$$

and

$$h(t,y) = f(t) \cdot \frac{2}{(1+y)^3}, \quad y \ge 0,$$

and let $f(t)$ be an arbitrary continuous density function. In this case $m_\delta(x)$ is not bounded.

Let us investigate $S(x)$ again and prove that condition A does not hold.

$$
\begin{aligned}
S(x) &= \int_0^\infty \int_{ct+x}^\infty e^{-\delta t} (x+ct)^3 f(t)g(y)dydt = \\
&= \int_0^\infty e^{-\delta t} (x+ct)^3 f(t) \frac{1}{(1+ct+x)^2} dt \ge \\
&\ge \int_0^\infty e^{-\delta t} f(t)(x+ct-2)dt \ge \\
&\ge (x-2) \int_0^\infty e^{-\delta t} f(t)dt = (x-2)\, \hat{f}(\delta),
\end{aligned}
$$

consequently

$$\sup_{0\le x} S(x) = \infty$$

Applying Theorem 1 we can see that $m_\delta(x)$ is not bounded.

Similar arguments can be given for other heavy tailed distributions of claim sizes with penalty function

$$w(x,y) = x^n$$

with appropriate large exponent n.

Finally, we present an example which shows the applicability of Theorem 1 in the case of dependence between inter-claim times and claim sizes.

Example 4. Let

$$w(x,y) = x^n$$

and

$$H(t,y) = F_1(t)\left(e^{-\beta t}G_1(y) + \left(1 - e^{-\beta t}\right)G_2(y)\right)$$

(mentioned in [5]), where $F_1(t), G_1(y), G_2(y)$ are distribution functions with density functions $f_1(t)$, $g_1(y)$, $g_2(y)$ respectively. If $G_1(y)$ and $G_2(y)$ have finite nth moments, then the function $m_\delta(x)$ is bounded. This can be seen as follows: first we note that finite nth moments imply

$$\lim_{y\to\infty} y^n(1 - G_1(y)) = 0$$

and

$$\lim_{y\to\infty} y^n(1 - G_2(y)) = 0,$$

hence

$$0 \le y^n(1 - G_1(y)) \le K_1$$

and

$$0 \le y^n(1 - G_2(y)) \le K_2$$

hold with appropriate constant values. Moreover,

$$\begin{aligned} h(t,y) = \frac{\partial^2 H(t,y)}{\partial t \partial y} = f_1(t)\left(e^{-\beta t}g_1(y) + \left(1 - e^{-\beta t}\right)g_2(y)\right) + \\ F_1(t)\left(-\beta e^{-\beta t}g_1(y) + \beta e^{-\beta t}g_2(y)\right), \end{aligned}$$

hence

$$h(t,y) \le \left(f_1(t) + \beta e^{-\beta t}\right)(g_1(y) + g_2(y)).$$

We can see that

$$\sup_{0\le x} S(x)$$

is finite, as

$$\begin{aligned} S(x) &= \int_0^\infty \int_{ct+x}^\infty e^{-\delta t}(x+ct)^n h(t,y)dydt \\ &\le \int_0^\infty e^{-\delta t}(x+ct)^n \left(f_1(t) + \beta e^{-\beta t}\right)\left(\int_{ct+x}^\infty g_1(y) + g_2(y)dy\right)dt \end{aligned}$$

$$= \int_0^\infty e^{-\delta t}(x+ct)^n \left(f_1(t)+\beta e^{-\beta t}\right)(1-G_1(x+ct)+1-G_2(x+ct))\,dt$$
$$\le \int_0^\infty e^{-\delta t}\left(f_1(t)+\beta e^{-\beta t}\right)(K_1+K_2)\,dt$$
$$\le (K_1+K_2)\left(\hat{f}_1(\delta)+\frac{\beta}{\beta+\delta}\right).$$

Applying Theorem 1 we can conclude that $m_\delta(x)$ is bounded.

We note that examples similar to Example 2 and Example 3 can be constructed applying appropriate dependent joint density function in the case of penalty functions $w(x,y)=y$ and $w(x,y)=x^3$.

Remark 3. In [9], in Theorem 2 the authors prove that if condition (A) holds, then the integral equation satisfied by $m_\delta(x)$ (Eq. (3) in that paper, which is a generalization of the Eq.(1) in [1]) ,

$$\Phi_\delta(x) = \int_0^\infty \int_0^{x+ct} e^{-\delta t_1}\Phi_\delta(x+ct-y)\cdot h(t,y)dydt+ \\ + \int_0^\infty \int_{x+ct}^\infty e^{-\delta t}\cdot w(x+ct,y-(x+ct))dydt \tag{2.1}$$

has a unique solution in the set of bounded functions. Now Theorem 1 states that if condition (A) holds, then $m_\delta(x)$ is bounded. Consequently if condition (A) holds, then the unique bounded solution of the above integral equation equals $m_\delta(x)$. If condition (A) does not hold, the bounded solution of the equation (if it exists) is not the Gerber-Shiu function. Consequently, condition (A) is suitable for selecting the function $m_\delta(x)$ from the bounded and unbounded solutions of the integral equation satisfied by the Gerber-Shiu function.

Finally, we use Theorem 1 to simplify the theorem giving a necessary and sufficient condition for the limit property $\lim_{x\to\infty} m_\delta(x)=0$. In this version we do not have any assumption concerning boundedness of $m_\delta(x)$.

First we recall Theorem 3 from [9].

Proposition 1. *Let $0<\delta$ be fixed and suppose that $m_\delta(x)$ is bounded. Then $\lim_{x\to\infty} m_\delta(x)=0$ if and only if*

$$\limsup_{0\le x} S(x) = 0. \tag{B}$$

Theorem 2. *Let $S(x)$ be finite for all $0 \leq x$ and let $0<\delta$ be fixed. Then*

$$\lim_{x\to\infty} m_\delta(x) = 0$$

if and only if

$$\limsup_{0\leq x} S(x) = 0.$$

Proof. It is obvious that condition (B) implies condition (A). Consequently, by Theorem 1, we can conclude that $m_\delta(x)$ is bounded if condition B holds. Taking the previous Proposition into account, we can state that $\lim_{x\to\infty} m_\delta(x) = 0$ holds as well.

Conversely, $\lim_{x\to\infty} m_\delta(x) = 0$ implies boundedness of the function $m_\delta(x)$, as $m_\delta(x)$ is continuous. Recalling the previous Proposition again, we can conclude that $\limsup_{0\leq x} S(x) = 0$. □

3. Conclusion

In this paper a necessary and sufficient condition was proved for the boundedness of the frequently used Gerber-Shiu function. This condition is essentially the boundedness of the conditional expectation given the ruin occurs at the first claim. It assures that the integral equation satisfied by the Gerber-Shiu function (Eq.2.1) has a unique bounded solution and it is exactly the Gerber-Shiu function. The condition implies that the bounded property of the Gerber-Shiu function is the consequence of the choice of the penalty function and the density function, moreover it can be checked in the knowledge of the penalty function and the joint density function.

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ON IYENGAR-TYPE INEQUALITIES VIA QUASI-CONVEXITY AND QUASI-CONCAVITY

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Abstract. In this paper, we obtain some new estimations of Iyengar-type inequality in which quasi-convex(quasi-concave) functions are involved. These estimations are improvements of some recently obtained estimations. Some error estimations for the trapezoidal formula are given. Applications for special means are also provided.

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1. Introduction

If it is necessary to bound one quantity by another, the classical inequalities are very useful for this purpose. This first book called "Inequalities" written by Hardy, Littlewood and Polya at Cambridge University Press in 1934 represents the first effort to systemize a rapidly expanding domain. In this sense, the second important book "Classical and New Inequalities in Analysis" is written by D.S. Mitrinović, J.E.Pećarič and A.M. Fink. The third book "Analytic Inequalities" is written by D.S. Mitrinović, and the other book "Means and Their Inequalities" is written by Bullen, D.S. Mitrinović, D.S. Vasic, P.M.

Today inequalities play a significant role in the development in all fields of Mathematics. They have applications in a variety of applied Mathematics. For example, convex functions are tractable in optimization because local optimality guarantees global optimality. In recent years a number of authors have discovered new integral inequalities for convex, $s-$convex, logarithmic convex, $h-$convex, $quasi$-convex, $m-$convex, $(\alpha,m)-$convex, co-ordinated convex, Godunova-Levin and $P-$functions.

On November 22, 1881, Hermite (1822-1901) sent a letter to the Journal Mathesis. This letter was published in Mathesis 3 (1883,p.82) and in this letter an inequality presented which is well-known in the literature as Hermite-Hadamard integral inequality :

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is a convex function on the interval I of a real numbers and $a,b\in I$ with $a<b$. If the function f is concave, the inequality in (1.1) is reversed. That is

$$f\left(\frac{a+b}{2}\right)\geq\frac{1}{b-a}\int_a^b f(x)\,dx\geq\frac{f(a)+f(b)}{2}.$$

For recent results, generalizations and new inequalities related to the inequality (1.1) see ([1, 3–5, 7, 8, 10–15, 18–23]).

Then left hand side of Hermite-Hadamard inequality (LHH) can also be estimated by the inequality of Iyengar.

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{M(b-a)}{4}-\frac{[f(b)-f(a)]^2}{4M(b-a)} \tag{1.2}$$

where

$$M=\sup\left\{\left|\frac{f(x)-f(y)}{x-y}\right|;x\neq y\right\}$$

In [9], Daniel Alexandru Ion proved the following inequalities of Iyengar type for differentiable $quasi-$convex functions:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{(b-a)}{4}\left(\sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\right) \tag{1.3}$$

where $f:[a,b]\to\mathbb{R}$ is differentiable function on (a,b), and $|f'|$ is $quasi-$convex on $[a,b]$ with $a<b$.

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{1.4}$$

$$\leq\frac{(b-a)}{2(p+1)^{\frac{1}{p}}}\left(\sup\left\{\left|f'(a)\right|^{\frac{p}{p-1}},\left|f'(b)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}$$

where $f:[a,b]\to\mathbb{R}$ is differentiable function on (a,b), and $|f'|^{\frac{p}{p-1}}$ is $quasi-$convex on $[a,b]$ with $a<b$.

We give some necessary definitions and mathematical preliminaries for $quasi-$convex functions which are used throughout this paper.

Definition 1 (see [16]). A function $f:[a,b]\to\mathbb{R}$ is said to be $quasi-$convex on $[a,b]$ if

$$f(\lambda x+(1-\lambda)y)\leq\max\{f(x),f(y)\}, \tag{1.5}$$

holds for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. For additional results on $quasi-$convexity, see [17]. Clearly, any convex function is $quasi-$convex function. Furthermore, there exists $quasi-$convex functions which are not convex. See [9] :

$$g(t)=\begin{cases}1, & t\in[-2,-1]\\ t^2, & t\in(-1,2]\end{cases}$$

is not a convex function on $[-2,2]$,but it is a $quasi-$convex function on $[-2,2]$.If we choose $g:[-2,2]\to\mathbb{R}$, $g(-2)=1$, $g(2)=4$ and for $\alpha=\frac{1}{2}$, $a=-2$, $b=0$, we get $g(\alpha a+(1-\alpha)b)=g(-1)=1$ and $\alpha g(a)+(1-\alpha)g(b)=\frac{1}{2}g(-2)+\frac{1}{2}g(0)=\frac{1}{2}$. Thus it is not convex but it is $quasi-$convex function for all $\alpha\in[0,1]$,

$$g(-\alpha 2+(1-\alpha)2)\le\max\{g(-2),g(2)\}=\max\{1,4\}=4.$$

The main purpose of this paper is to point out new estimations of the inequality in (1.2), but now for the class of $quasi-$convex functions.

In order to prove our main results we need the following lemma (see [2]).

Lemma 1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I°, $a,b\in I$ with $a<b$ and f'' be integrable on $[a,b]$. Then the following equality holds:*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{(b-a)^2}{2}\int_0^1 t(1-t)f''(ta+(1-t)b)\,dt.$$

The main results of this paper are given by the following theorems.

2. Results

Theorem 1. *Let $f:I^{\circ}\subset[0,\infty)\to\mathbb{R}$, be a twice differentiable mapping on I°, such that $f''\in L[a,b]$, $a,b\in I$ with $a<b$. If $|f''|^q$ is $quasi-$convex on $[a,b]$ for $q>1$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (2.1)\\ &\le\frac{(b-a)^2}{2}\left(\frac{q-1}{2q-p-1}\right)^{\frac{q-1}{q}}(\beta(p+1,q+1))^{\frac{1}{q}}\\ &\times\left(\max\left\{\left|f''(a)\right|^q,\left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $\beta(\,,)$ is Euler Beta Function:

$$\beta(x\,,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt,\quad x,y>0.$$

Theorem 2. *Let $f : I^\circ \subset [0,\infty) \to \mathbb{R}$, be a twice differentiable mapping on I°, such that $f'' \in L[a,b]$, $a,b \in I$ with $a < b$. If $|f''|^q$ is quasi$-$convex on $[a,b]$ for $q \geq 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \tag{2.2}$$
$$\leq \frac{(b-a)^2}{4}\left(\frac{2}{(q+1)(q+2)}\right)^{\frac{q-1}{q}}\left(\max\left\{\left|f''(a)\right|^q,\left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}.$$

Theorem 3. *With the assumptions of Theorem 1, we obtain another*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{2^{1+\frac{1}{q}}}\left(\beta(2,p+1)\right)^{\frac{1}{p}}\left(\max\left\{\left|f''(a)\right|^q,\left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}.$$

Proof of Theorem 1. Using Lemma 1 and the well known Hölder's inequality for $q > 1$,

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{2}\left(\int_0^1 t^{\frac{q-p}{q-1}}dt\right)^{\frac{q-1}{q}}\left[\int_0^1 t^p(1-t)^q\left|f''(ta+(1-t)b)\right|^q dt\right]^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

On the other hand, since $|f''|^q$ is quasi$-$convex on $[a,b]$, we know that for any $t \in [0,1]$

$$\left|f''(ta+(1-t)b)\right|^q \leq \max\left\{\left|f''(a)\right|^q,\left|f''(b)\right|^q\right\}.$$

Therefore, we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{2}\left(\int_0^1 t^{\frac{q-p}{q-1}}dt\right)^{\frac{q-1}{q}}\left[\int_0^1 t^p(1-t)^q\left|f''(ta+(1-t)b)\right|^q dt\right]^{\frac{1}{q}}$$
$$\leq \frac{(b-a)^2}{2}\left(\int_0^1 t^{\frac{q-p}{q-1}}dt\right)^{\frac{q-1}{q}}\left[\int_0^1 t^p(1-t)^q\left(\max\left\{\left|f''(a)\right|^q,\left|f''(b)\right|^q\right\}\right)dt\right]^{\frac{1}{q}}$$
$$= \frac{(b-a)^2}{2}\left(\frac{q-1}{2q-p-1}\right)^{\frac{q-1}{q}}\left(\beta(p+1,q+1)\right)^{\frac{1}{q}}\left(\max\left\{\left|f''(a)\right|^q,\left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}},$$

which completes the proof. □

Corollary 1. *In Theorem 1, if we choose* $M = Sup_{x\in(a,b)}|f''(x)| < \infty$*, we get*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)^2}{2} M \left(\frac{q-1}{2q-p-1}\right)^{\frac{q-1}{q}} \left(\beta(p+1,q+1)\right)^{\frac{1}{q}}.$$

Proof of Theorem 2. From Lemma 1 and the well known power-mean inequality we obtain

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)^2}{2}\int_0^1 t(1-t)\left|f''(ta+(1-t)b)\right|dt$$
$$\leq \frac{(b-a)^2}{2}\left(\int_0^1 t\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)^q\left|f''(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{(b-a)^2}{2}\left(\int_0^1 t\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)^q\left(\max\left\{\left|f''(a)\right|^q,\left|f''(b)\right|^q\right\}\right)dt\right)^{\frac{1}{q}}$$
$$= \frac{(b-a)^2}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{(q+1)(q+2)}\right)^{\frac{1}{q}}\left(\max\left\{\left|f''(a)\right|^q,\left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$= \frac{(b-a)^2}{4}\left(\frac{2}{(q+1)(q+2)}\right)^{\frac{1}{q}}\left(\max\left\{\left|f''(a)\right|^q,\left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}.$$

The proof of Theorem 2 is completed. □

Corollary 2. *Under the assumptions of Theorem 2,*

Case i: *Since* $\lim_{q\to\infty}\left(\frac{2}{(q+1)(q+2)}\right)^{\frac{1}{q}} = 1$ *and* $\lim_{q\to 1^+}\left(\frac{2}{(q+1)(q+2)}\right)^{\frac{1}{q}} = \frac{1}{3}$*, we have*

$$\frac{1}{3} < \left(\frac{2}{(q+1)(q+2)}\right)^{\frac{1}{q}} < 1, \quad q \in [1,\infty).$$

Therefore,

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{4}\left(\max\left\{\left|f''(a)\right|^q,\left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}. \tag{2.3}$$

In (2.3),

- *if $|f''|^q$ is decreasing, we get*
$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\frac{(b-a)^2}{4}\left|f''(a)\right|,$$
- *if $|f''|^q$ is increasing , we get*
$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\frac{(b-a)^2}{4}\left|f''(b)\right|.$$

Case ii: *If we choose $M=Sup_{x\in(a,b)}|f''(x)|<\infty$ in (2.2), then the inequality in (2.1) is better than the inequality in (2.2).*

Proof of Theorem 3. From Lemma 1 with properties of modulus we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.4}$$
$$\le\frac{(b-a)^2}{2}\int_0^1 t(1-t)\left|f''(ta+(1-t)b)\right|dt.$$

Now, if we use the following weighted version of Hölder's inequality [6, p. 117]:

$$\left|\int_I f(s)g(s)h(s)ds\right|\le\left(\int_I|f(s)|^p h(s)ds\right)^{\frac{1}{p}}\left(\int_I|g(s)|^q h(s)ds\right)^{\frac{1}{q}} \tag{2.5}$$

for $p>1,p^{-1}+q^{-1}=1$, h is nonnegative on I and provided all the other integrals exist and are finite.

If we rewrite the inequality (2.4) with respect to (2.5) with $|f''|^q$ is $quasi-$convex on $[a,b]$ for all $t\in[0,1]$, we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\
\le&\frac{(b-a)^2}{2}\int_0^1 t(1-t)\left|f''(ta+(1-t)b)\right|dt\\
=&\frac{(b-a)^2}{2}\int_0^1 (1-t)\left|f''(ta+(1-t)b)\right|t\,dt\\
\le&\frac{(b-a)^2}{2}\left(\int_0^1(1-t)^p\,t\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''(ta+(1-t)b)\right|^q t\,dt\right)^{\frac{1}{q}}\\
=&\frac{(b-a)^2}{2}\left(\beta(2,p+1)\right)^{\frac{1}{p}}\left(\frac{\max\left\{|f''(a)|^q,|f''(b)|^q\right\}}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

The proof of Theorem 3 is completed. □

Corollary 3. *In Theorem 3, if we choose* $M = Sup_{x\in(a,b)}\left|f''(x)\right| < \infty$, *we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^2}{2^{1+\frac{1}{q}}}M\left(\beta\left(2,p+1\right)\right)^{\frac{1}{p}}.$$

Remark 1. From Theorems 1-3, we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \min\{v_1,v_2,v_3\}$$

where

$$v_1 =$$

$$\frac{(b-a)^2}{2}\left(\frac{q-1}{2q-p-1}\right)^{\frac{q-1}{q}}\left(\beta\left(p+1,q+1\right)\right)^{\frac{1}{q}}\times\left(\max\left\{\left|f''(a)\right|^q,\left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}},$$

$$v_2 = \frac{(b-a)^2}{4}\left(\frac{2}{(q+1)(q+2)}\right)^{\frac{q-1}{q}}\left(\max\left\{\left|f''(a)\right|^q,\left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}$$

and

$$v_3 = \frac{(b-a)^2}{2^{1+\frac{1}{q}}}\left(\beta\left(2,p+1\right)\right)^{\frac{1}{p}}\left(\max\left\{\left|f''(a)\right|^q,\left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}.$$

3. Error Estimates for the Trapezoidal Rule

Let d be a partition $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ of the interval $[a,b]$ and consider the quadrature formula

$$\int_a^b f(x)dx = T_i(f,d)+E_i(f,d), \quad i = 1,2,\ldots,n-1 \tag{3.1}$$

where

$$T_1(f,d) = \sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i)$$

for the Trapezoidal version and

$$T_2(f,d) = \sum_{i=0}^{n-1} f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)$$

for the Midpoint formula and $E_i(f,d)$ denotes the associated approximation errors.

Proposition 1. *Suppose that all the assumptions of Theorem 1 are satisfied for every division d of $[a,b]$, we have*

$$|E(f,d)| \le \frac{1}{2}\left(\frac{q-1}{2q-p-1}\right)^{\frac{q-1}{q}}\left(\beta\left(p+1,q+1\right)\right)^{\frac{1}{q}}$$

$$\times \sum_{i=0}^{n-1} (x_{i+1}-x_i)^3 \left(\max\left\{\left|f''(x_i)\right|^q, \left|f''(x_{i+1})\right|^q\right\}\right)^{\frac{1}{q}}.$$

Proof. Applying Theorem 1 on the subinterval (x_{i+1}, x_i), $i = 1,2,...,n-1$ of the partition and by using the $quasi-$convexity of $|f''|^q$, we obtain

$$\left|\frac{f(x_i)+f(x_{i+1})}{2} - \frac{1}{(x_{i+1}-x_i)}\int_{x_i}^{x_{i+1}} f(x)dx\right|$$
$$\leq \frac{(x_{i+1}-x_i)^2}{2}\left(\frac{q-1}{2q-p-1}\right)^{\frac{q-1}{q}} (\beta(p+1,q+1))^{\frac{1}{q}}$$
$$\times \left(\max\left\{\left|f''(x_i)\right|^q, \left|f''(x_{i+1})\right|^q\right\}\right)^{\frac{1}{q}}.$$

Hence in (3.1), we have

$$\left|\int_a^b f(x)dx - T(f,d)\right| = \left|\sum_{i=0}^{n-1}\left\{\int_{x_i}^{x_{i+1}} f(x)dx - \frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i)\right\}\right|$$
$$\leq \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(x)dx - \frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i)\right|$$
$$\leq \frac{1}{2}\left(\frac{q-1}{2q-p-1}\right)^{\frac{q-1}{q}} (\beta(p+1,q+1))^{\frac{1}{q}}$$
$$\times \sum_{i=0}^{n-1} (x_{i+1}-x_i)^3 \left(\max\left\{\left|f''(x_i)\right|^q, \left|f''(x_{i+1})\right|^q\right\}\right)^{\frac{1}{q}}.$$

□

Proposition 2. *Suppose that all the assumptions of Theorem 2 are satisfied for every division d of* $[a,b]$, *we have*

$$|E(f,d)| \leq \frac{1}{4}\left(\frac{2}{(q+1)(q+2)}\right)^{\frac{1}{q}}$$
$$\times \sum_{i=0}^{n-1} (x_{i+1}-x_i)^3 \left(\max\left\{\left|f''(x_i)\right|^q, \left|f''(x_{i+1})\right|^q\right\}\right)^{\frac{1}{q}}.$$

Proof. The proof immediately follows from Theorem 2 and by applying a similar argument to Proposition 1. □

Proposition 3. *Suppose that all the assumptions of Theorem 3 are satisfied for every division d of* $[a,b]$, *we have*

$$|E(f,d)| \leq \frac{1}{2}(\beta(1,p+1))^{\frac{1}{p}}$$

$$\times \sum_{i=0}^{n-1} (x_{i+1}-x_i)^2 \left(\frac{\max\left\{ |f''(a)|^q, |f''(b)|^q \right\}}{2} \right)^{\frac{1}{q}}.$$

Proof. The proof immediately follows from Theorem 3 and by applying a similar argument to Proposition 1. □

4. Applications to special means

Let us consider the special means for real numbers a,b $(a \neq b)$. We take

1. Arithmetic mean:

$$A(a,b) = \frac{a+b}{2}, \quad a,b \in \mathbb{R}.$$

2. Logarithmic mean:

$$L(a,b) = \frac{a-b}{\ln|a| - \ln|b|}, \quad |a| \neq |b|, a,b \neq 0, a,b \in \mathbb{R}.$$

3. Generalized log-mean:

$$L_n(a,b) = \left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)} \right]^{\frac{1}{n}}, \quad n \in \mathbb{Z}\backslash\{-1,0\}, a,b \in \mathbb{R}, a \neq b.$$

Proposition 4. *Let $a,b \in \mathbb{R}$, $a < b$ and $n \in \mathbb{N}$, $n \geq 2$. Then we have*

$$|A(a^n,b^n) - L_n^n(a,b)| \leq \frac{n(n-1)(b-a)^2}{2}$$
$$\times \left(\frac{q-1}{2q-p-1} \right)^{\frac{q-1}{q}} (\beta(p+1,q+1))^{\frac{1}{q}} \left(\max\left\{ |a|^{(n-2)q}, |b|^{(n-2)q} \right\} \right)^{\frac{1}{q}}.$$

Proof. The assertion follows from Theorem 1 applied to the *quasi*−convex mapping $f(x) = x^n$, $x \in \mathbb{R}$. □

Proposition 5. *Let $a,b \in \mathbb{R}$, $a < b$ and $n \in \mathbb{N}$, $n \geq 2$. Then we have*

$$|A(a^n,b^n) - L_n^n(a,b)| \leq \frac{n(n-1)(b-a)^2}{4}$$
$$\times \left(\frac{2}{(q+1)(q+2)} \right)^{\frac{1}{q}} \left(\max\left\{ |a|^{(n-2)q}, |b|^{(n-2)q} \right\} \right)^{\frac{1}{q}}.$$

Proof. The assertion follows from Theorem 2 applied to the *quasi*−convex mapping $f(x) = x^n$, $x \in \mathbb{R}$. □

Proposition 6. *Let $a,b \in \mathbb{R}$, $a < b$ and $n \in \mathbb{N}$, $n \geq 2$. Then we have*

$$|A(a^n,b^n) - L_n^n(a,b)| \leq \frac{n(n-1)(b-a)^2}{2^{1+\frac{1}{q}}}$$

$$\times\left(\beta\left(2,q+1\right)\right)^{\frac{1}{p}}\left(\max\left\{|a|^{(n-2)q},|b|^{(n-2)q}\right\}\right)^{\frac{1}{q}}.$$

Proof. The assertion follows from Theorem 3 applied to the $quasi-$convex mapping $f(x)=x^n$, $x\in\mathbb{R}$. □

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COMMON FIXED POINT THEOREMS FOR STRICT OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS IN COMPACT METRIC SPACES

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Abstract. We prove a common fixed point theorem for four two pairs of hybrid mappings in compact metric space satisfying an implicit relations using the concept of strict occasionally weak compatibility which generalize theorems of [1, 4, 7, 28]. As an application we obtain a general fixed point theorem for hybrid pairs satisfying a contractive condition of integral type, which is a new result in compact metric spaces.

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1. INTRODUCTION

Let (X,d) be a metric space. Denote by $B(X)$ the set of all nonempty sets of X. As in [10, 11] we define the functions $D(A,B)$ and $\delta(A,B)$ by:

$$D(A,B) = \inf\{d(a,b) : a \in A, b \in B\},$$
$$\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\}.$$

for $A, B \in B(X)$.

If A consists of a single point "a" we write $\delta(A,B) = \delta(a,B)$.

If B consists also of a single point "b" we write $\delta(A,B) = d(a,b)$.

It follows immediately from the definition of δ that

$$\delta(A,B) = \delta(B,A) \geq 0, \forall A, B \in B(X),$$

$$\delta(A,B) = 0 \text{ implies } A = B = \{a\}.$$

Definition 1 ([10, 11])**.** A sequence $\{A_n\}$ of nonempty sets of (X,d) is said to be convergent to a set A of X if

(i) each point $a \in A$ is the limit of a convergent sequence $\{a_n\}$, where $a_n \in A_n$ for all $n \in \mathbb{N}$,

(ii) for any arbitrary $\varepsilon > 0$, there exists an integer $m > 0$ such that $A_n \subset A_\varepsilon$ for $n > m$, where A_ε denote the set of all points $x \in X$ for which there exists a point $a \in X$, depending on x, such that $d(x,a) < \varepsilon$.

A is said to be the limit of the sequence $\{A_n\}$.

Lemma 1 ([10])**.** *If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ convergent to A and B, respectively, then $\delta(A_n, B_n) \to \delta(A, B)$.*

Lemma 2 ([10])**.** *Let $\{A_n\}$ be a sequence in $B(X)$ and $y \in X$ such that $\delta(A_n, y) \to 0$. Then the sequence $\{A_n\}$ converges to the set $\{y\}$ in $B(X)$.*

Definition 2. A set valued mapping $F : X \to B(X)$ is said to be continuous at $x \in X$ if the sequence $\{Fx_n\} \in B(X)$ converges to $\{Fx\}$, whenever $\{x_n\}$ is a sequence in X converging to x in X.

F is said to be continuous at X if it is continuous at every point in X.

Let A and S be self mappings of a metric space (X,d). Jungck [12] defined A and S to be compatible if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

A point $x \in X$ is a coincidence point of A and S if $Ax = Sx$. We denote by $C(A,S)$ the set of all coincidence points of A and S.

In [23], Pant defined A and S to be pointwise R - weakly commuting if for all $x \in X$, there exists $R > 0$ such that $d(SAx, ASx) \leq Rd(Ax, Sx)$. It has been proved in [24] that pointwise R - weakly commuting is equivalent to commutativity at coincidence points.

Definition 3 ([17])**.** A and S is said to be weakly compatible if $SAu = ASu$ for $u \in C(A,S)$.

Definition 4 ([2])**.** A and S is said to be occasionally weakly compatible mappings (briefly owc) if $ASu = SAu$ for some $u \in C(A,S)$.

Remark 1. If A and S are weakly compatible and $C(A,S) \neq 0$ then A and S are owc, but the converse is not true (Example, [2]).

Some fixed point theorems for occasionally weakly compatible mappings are proved in [2, 6–8, 16, 22, 30–32] and in other papers.

Definition 5. Let $f : (X,d) \to (X,d)$ and $F : (X,d) \to B(X)$ be. Then:

1) a point $x \in X$ is said to be a coincidence point of f and F if $fx \in Fx$. We denote by $C(f,F)$ the set of all coincidence points of f and F.
2) a point $x \in X$ is said to be a strict coincidence point of f and F if $\{fx\} = Fx$. We denote by $SC(f,F)$ the set of all strict coincidence points of f and F.
3) a point $x \in X$ is said to be a fixed point of F if $x \in Fx$.
4) a point $x \in X$ is said to be a strict fixed point of F if $\{x\} = Fx$.

Definition 6 ([14])**.** The mappings $f : X \to X$ and $F : X \to B(X)$ is said to be δ - compatible if $\lim_{n\to\infty} \delta(Ffx_n, fFx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $Fx_n \in B(X)$, $fx_n \to t$, $Fx_n \to \{t\}$ for some $t \in X$.

Definition 7 ([15])**.** The hybrid pair $f : X \to X$ and $F : X \to B(X)$ is weakly compatible if for each $x \in SC(f,F)$, $Ffx = fFx$.

Remark 2. If the pair (f,F) is δ - compatible, then it is weakly compatible but the converse is not true [15].

Definition 8. The hybrid pair $f : X \to X$ and $F : X \to B(X)$ is strict occasionally weakly compatible (briefly sowc) if there exists $x \in SC(f,F)$ such that $Ffx = fFx$.

Remark 3. If $C(f,F) \neq \phi$ and the pair (f,F) is weakly compatible then the pair (f,F) is owc.

There exists sowc pairs which are not weakly compatible.

Example 1 ([6]). Let $X = [0,2]$ with usual metric. Define $f : X \to X$ and $F : X \to B(X)$ by

$$fx = \begin{cases} x, x = 0 \\ 2-x, x \neq 0 \end{cases}$$

and

$$Fx = \begin{cases} [0,x], x \leq 1 \\ [0,2x], x > 1 \end{cases}$$

Clearly, $C(f,F) = \{0,1\}$, $SC(f,F) = \{0\}$, $Ff0 = fF0 = \{0\}$ and $Ffx \neq fFx$ for all $x \in (0,2]$. Hence, the pair (f,F) is sowc, but it is not weakly compatible.

Remark 4. It is obviously $\{f0\} = F0 = \{0\}$ and $F1 = [0,1]$. Therefore 0 and 1 are fixed points for f and F and only 0 is a strict point of f and F.

2. Preliminaries

In [9], Branciari established the following result

Theorem 1. *Let (X,d) be a complete metric space, $c \in (0,1)$ and $f : X \to X$ be a mapping such that for all $x,y \in X$*

$$\int_0^{d(fx,fy)} h(t)dt \leq c \int_0^{d(x,y)} h(t)dt,$$

where $h : [0,\infty) \to [0,\infty)$ is a Lebesgue measurable mapping which is summable (i.e. with a finite integral) on each compact subset of $[0,\infty)$ such that for $\varepsilon > 0$, $\int_0^{\varepsilon} h(t)dt > 0$. Then f has a unique fixed point $z \in X$ such that for each $x \in X$, $\lim_{n\to\infty} f^n x = z$.

Recently, Kumar et al. [20] extended Theorem 1 for two compatible mappings.

Theorem 2. *Let $f, g : (X, d) \to (X, d)$ compatible mappings satisfying the following conditions:*

1) *g is continuous,*
2) *$f(X) \subset g(X)$ and*

$$\int_0^{d(fx,fy)} h(t)dt \leq c \int_0^{d(gx,gy)} h(t)dt,$$

for all $x, y \in X$, $c \in (0,1)$, where h is as in Theorem 1.
Then f and g have a unique common fixed point.

Definition 9. Let X be a nonempty set. A symmetric on X is a nonnegative real valued function D on $X \times X$ such that

(i) $D(x, y) = 0$ if and only if $x = y$,
(ii) $D(x, y) = D(y, x)$ for any $x, y \in X$.

Some fixed point theorems in metric and symmetric spaces for compatible, weakly compatible and occasionally weakly compatible mappings satisfying a contractive condition of integral type have been established in [3, 12, 19, 21, 29, 35] and in other papers.

Let (X, d) be a metric space and $D(x, y) = \int_0^{d(x,y)} h(t)dt$, where $h(t)$ is as in Theorem 1. It is proved in [21] and [29] that $D(x, y)$ is a symmetric on X. It has also been proved in [21] and [29] that the study of fixed points for mappings satisfying a contractive condition of integral type is reduced to the study of fixed points in symmetric spaces.

The method is not applicable for hybrid pairs.

Definition 10. An altering distance is a mapping $\psi : [0, \infty) \to [0, \infty)$ which satisfies:

$(\psi_1) : \psi$ is increasing and continuous,
$(\psi_2) : \psi(t) = 0$ if and only if $t = 0$.

In [18] a fixed point result involving altering distances have been obtained. Fixed point problem involving altering distances have been studied in [28, 37, 38] and in other papers.

Definition 11. A weakly altering distance is a mapping $\psi : [0, \infty) \to [0, \infty)$ which satisfies:

$(\psi_1) : \psi$ is increasing,
$(\psi_2) : \psi(t) = 0$ if and only if $t = 0$.

Lemma 3. *The function $\psi(t) = \int_0^t h(x)dx$, where $h(x)$ is as in Theorem 1 is a weakly altering distance.*

Proof. The proof follows from Lemma 2.5 [30]. □

Several classical fixed point theorems and common fixed point theorems have recently unified by considering a general condition expressed by an implicit relation [25, 26] and other papers.

Actually, the method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, convex metric spaces, reflexive metric spaces, compact metric spaces, paracompact metric spaces, in two and three metric spaces, for single valued functions, hybrid pairs of functions, set - valued functions.

Quite recently, the method is used in the study of fixed points for mappings satisfying a contractive condition of integral type, in fuzzy metric spaces and intuitionistic metric spaces.

In [30] a general fixed point theorem for compatible mappings satisfying an implicit relation has been proved.

In [13] the results from [30] have been improved relaxing compatibility to weak compatibility.

In [27] a general fixed point theorem for weakly compatible mappings in compact metric spaces satisfying an implicit relation is proved.

In [28] a common fixed point theorem for four weakly compatible mappings in compact metric spaces involving an altering distance was proved, which extends the main results of [4] and [37].

Theorem 3 ([28])**.** *Let f, g, S and T be self mappings of a compact metric space (X, d) such that*

a) $f(X) \subset T(X)$ *and* $g(X) \subset S(X)$,
b) *the pairs* (f, T) *is compatible and the pair* (g, S) *is weakly compatible,*
c) f *and* S *are continuous,*
d) $\psi(d(fx, gy)) \leq a\psi(d(Sx, Ty)) + b[\psi(d(fx, Sx) + \psi(d(gy, Ty)] + +c[\psi(d(Sx, gy) \cdot \psi(d(fx, Ty)]^{1/2}$ *for all* $x, y \in X$, $a, b, c \geq 0$, $a + 2b < 1$, $a + c < 1$, *and* ψ *is an altering distance.*

Then f, g, S and T have a unique common fixed point in X.

Recently, in [5] the authors have proved a new fixed point theorem for mappings satisfying a new type of implicit relation.

The results from [5] are extended in [30] for owc mappings involving altering distances.

In [1] the following theorem is proved.

Theorem 4. *Let I, J be two single valued functions from a compact metric space (X, d) into itself and $F, G : X \to B(X)$ two set-valued functions with $\cup G(X) \subset I(X)$ and $\cup F(X) \subset J(X)$ such that*

$$\begin{aligned}\psi(\delta(Fx, Gy)) < \max\{&\psi(d(Ix, Jy)), \psi(\delta(Ix, Fx)), \psi(\delta(Jy, Gy)\},\\ &\min\{\psi(D(Ix, Gy)), \psi(D(Jy, Fx))\}\\ &-\omega(\max\{\psi(d(Ix, Jy)), \psi(\delta(Ix, Fx)), \psi(\delta(Jy, Gy)\},\end{aligned}$$

$$\min\{\psi(D(Ix,Gy)),\psi(D(Jy,Fx))\})$$

for all $x,y\in X$*, where the right hand side of inequality is positive,* ψ *is an altering distance and* $\omega:[0,\infty)\to[0,\infty)$ *is a continuous function satisfying* $0<\omega(r)<r$ *for* $r>0$.

If the pairs (I,F) *and* (J,G) *are weakly compatible and the functions* F,I *are continuous, then there exists a unique point* $p\in X$ *such that* $\{p\}=\{Ip\}=Fp=\{Jp\}=Gp$.

Remark 5. In the proof of this theorem is used the fact that the function $r-\omega(r)$ is a non-decreasing function.

Some fixed point theorems for hybrid pair in compact metric spaces are proved in [33, 34, 36] and in other papers.

The purpose of this paper is to extend Theorem 3, Theorem 4 and Theorem 2 [4] for strictly owc mappings satisfying implicit relations and to transfer the study of fixed points for hybrid pairs of mappings satisfying a contractive condition of integral type in compact metric spaces to the study of fixed points in compact metric spaces by altering distances.

3. Implicit relations

Let $\mathcal{F}_c$ be the family of all real functions $F:\mathbb{R}_+^6\to\mathbb{R}$ satisfying the following conditions:

(ϕ_1) F is increasing in variable t_1 and nonincreasing in variables t_2 and t_4,
(ϕ_2) If $u\geq 0$, $v>0$, $w\geq 0$ such that
(ϕ_{2a}) $F(u,v,v,u,w,0)\leq 0$ or
(ϕ_{2b}) $F(u,v,u,v,0,w)\leq 0$,
then $u<v$ and $u=0$ if $v=0$.
(ϕ_3) $F(t,t,0,0,t,t)>0$, $\forall t>0$.

Example 2. $F(t_1,...,t_6)=t_1-at_2-b(t_3+t_4)-c(t_5t_6)^{1/2}$, where $a>0$, $b,c\geq 0$, $a+2b<1$ and $a+c<1$.

(ϕ_1) : Obviously.
(ϕ_2) : Let $u,v>0$, $w\geq 0$ and $F(u,v,v,u,w,0)=u-av-b(u+v)\leq 0$. Then $u\leq\frac{a+b}{1-b}v<v$. Similarly, $F(u,v,u,v,0,w)\leq 0$ implies $u<v$. If $u=0$, $v>0$, $w>0$, then $u<v$.
(ϕ_3) : $F(t,t,0,0,t,t)=t(1-(a+c))>0$, $\forall t>0$.

Example 3. $F(t_1,...,t_6)=t_1^2-at_2^2-b\frac{t_3^2+t_4^2}{1+\min\{t_5,t_6\}}$, where $a>0$ and $a+2b<1$.

(ϕ_1) : Obviously.
(ϕ_2) : Let $u>0,v>0$, $w>0$ and $F(u,v,v,u,w,0)=u^2-av^2-b(u^2+v^2)\leq 0$ which implies $u^2\leq\frac{a+b}{1-b}v^2$, hence $u<v$. Similarly, $F(u,v,u,v,0,w)\leq 0$ implies $u<v$. If $u=0$, $v>0$ then $u<v$. If $v=0$ then $u=0$.

(ϕ_3) : $F(t,t,0,0,t,t) = t^2(1-a) > 0$, $\forall t > 0$.

Example 4. $F(t_1,...,t_6) = t_1^2 - at_2^2 - b\frac{t_3t_4}{1+t_5t_6}$, where $a > 0$ and $a+b < 1$.

(ϕ_1) : Obviously.

(ϕ_2) : Let $u \geq 0, v > 0$, $w \geq 0$ and $F(u,v,v,u,w,0) = u^2 - av^2 - buv \leq 0$. Then $f(t) = t^2 - bt - a \leq 0 \leq$ where $t = \frac{u}{v}$. Since $f(0) = -a < 0$ and $f(1) = 1-(a+b) > 0$, there exists $h \in (0,1)$ such that $f(t) < 0$ for $t < h$. Hence $u < v$. Similarly, $F(u,v,u,v,0,w) \leq 0$ implies $u < v$. If $v = 0$ then $u = 0$.

(ϕ_3) : $F(t,t,0,0,t,t) = t^2(1-a) > 0$, $\forall t > 0$.

Let $\omega : [0,\infty) \to [0,\infty)$ with $0 < \omega(r) < r$ for $r > 0$, $\omega(0) = 0$ and $r - \omega(r)$ is non decreasing.

Example 5.

$$F(t_1,...,t_6) = t_1 - \max\{t_2,t_3,t_4,\min\{t_5,t_6\}\} + \omega(\max\{t_2,t_3,t_4,\min\{t_5,t_6\}\}).$$

(ϕ_1) : It follows from the fact that $t - \omega(t) > 0$ is a non decreasing function.

(ϕ_2) : Let $u \geq 0, v > 0, w \geq 0$ and $F(u,v,v,u,w,0) = u - \max\{u,v\} + \omega(\max\{u,v\}) \leq 0$ which implies $u - \max\{u,v\} < 0$. If $v = 0$ then $u - [u - \omega(u)] \leq 0$ which implies $\omega(u) \leq 0$, a contradiction if $u > 0$, hence $u = 0$. Similarly, $F(u,v,u,v,0,w) \leq 0$ implies $u < v$ if $v > 0$. If $v = 0$ then $u = 0$.

(ϕ_3) : $F(t,t,0,0,t,t) = t - (t - \omega(t)) = \omega(t) > 0$, $\forall t > 0$.

4. Fixed points for sowc mappings in compact metric spaces

Theorem 5. *Let $I : (X,d) \to (X,d)$ and $F : (X,d) \to B(X)$ be sowc mappings. If I and F have a unique point of strict coincidence $\{z\} = \{Ix\} = Fx$, then z is the unique common fixed point of I and F which is a strict fixed point for F.*

Proof. Since I and F are sowc, there exists a point $x \in X$ such that $\{z\} = \{Ix\} = Fx$ implies $IFx = FIx$. Then, $\{Iz\} = \{IIx\} \in IFx = FIx$. Then $u = Ix$ is a point of strict coincidence of I and F. By hypothesis $u = z$ and $\{z\} = \{Iz\} = Fz$. Hence z is a common fixed point for I and F. Suppose that $v \neq z$ is another common fixed point of I and F, which is a strict fixed point for F. Then $\{v\} = \{Iv\} = Fv$. Hence v is a point of strict coincidence of I and F, by hypothesis $v = z$. □

Theorem 6. *Let (X,d) be a metric space and let $I,J : X \to X$ and $F,G : X \to B(X)$ such that*

$$\begin{aligned}&\phi(\psi(\delta(Fx,Gy)),\psi(d(Ix,Jy)),\psi(\delta(Ix,Fx)),\\&\psi(\delta(Jy,Gy)),\psi(D(Ix,Gy)),\psi(D(Jy,Fx))) \leq 0\end{aligned} \tag{4.1}$$

holds for all $x,y \in X$, where ϕ satisfies condition (ϕ_3) and ψ is weakly altering distance. Suppose that there exists $x,y \in X$ such that $\{u\} = \{Ix\} = Fx$ and $\{v\} = \{Jy\} = Gy$. Then u is the unique point of strict coincidence of I and F and v is the unique point of strict coincidence of J and G.

Proof. First we prove that $Ix = Jy$. Suppose that $Ix \neq Jy$. Then by (4.1) we obtain

$$\phi(\psi(d(Ix,Jy)),\psi(d(Ix,Jy)),0,0,\psi(d(Ix,Jy)),\psi(d(Ix,Jx))) \leq 0,$$

a contradiction of (ϕ_3). Hence $d(Ix,Jy) = 0$ which implies $Ix = Jy$. Thus $\{u\} = \{Ix\} = Fx = Gy = \{Jy\}$. Suppose that $z \in X, z \neq x$ such that $\{w\} = \{Iz\} = Fz$. Then by (4.1) we obtain

$$\phi(\psi(d(Iz,Jy)),\psi(d(Iz,Jy)),0,0,\psi(d(Iz,Jy)),\psi(d(Iz,Jx))) \leq 0,$$

a contradiction of (ϕ_3). Hence $\{w\} = \{Iz\} = Fz = \{Jy\} = Gy = Fx = \{Ix\} = \{u\}$ and u is the unique point of strict coincidence of I and F. Similarly, v is the unique point of strict coincidence of J and G. □

Theorem 7. *Let (X,d) be a compact metric space, $I,J : X \to X$ and $F,G : X \to B(X)$ satisfying the inequality (4.1) for all $x,y \in X$, $\phi \in \mathcal{F}_c$ satisfies condition (ϕ_3) and ψ is weakly altering distance such that $Fx \subset J(X)$ and $Gx \subset I(X)$, $\forall x \in X$ and the functions I and F are continuous. Then*

3) F and I have a strict coincidence point,

4) G and J have a strict coincidence point.

Moreover, if the pairs (I,F) and (J,G) are strict owc, then I,J,F and G have an unique common fixed point which is a strict fixed point for F and G.

Proof. Let $m = \inf\{\delta(Ix,Fx) : x \in X\}$. Because (X,d) is compact and F and I are continuous as in [1, 33, 34] there exists $x_0 \in X$ such that $\delta(Ix_0,Fx_0) = m$. We prove that $m = 0$. Suppose that $m > 0$. Since $Fx \subset JX, \forall x \in X$, there exists $Jy_0 \in Fx_0$ and $d(Ix_0,Jy_0) \leq \delta(Ix_0,Fx_0) = m$.

By (4.1) we have

$$\phi(\psi(\delta(Fx_0,Gy_0)),\psi(d(Ix_0,Jy_0)),\psi(\delta(Ix_0,Fx_0)),$$
$$\psi(\delta(Jy_0,Gy_0)),\psi(D(Ix_0,Gy_0)),\psi(D(Jy_0,Fx_0))) \leq 0.$$

By (ϕ_1) we obtain

$$\phi(\psi(d(Jy_0,Gy_0)),\psi(m),\psi(m),$$
$$\psi(\delta(Jy_0,Gy_0)),\psi(D(Ix_0,Gy_0)),0) \leq 0. \tag{4.2}$$

Since $\psi(m) > 0$, by (ϕ_{2a}) we obtain

$$\psi(\delta(Jy_0,Gy_0)) < \psi(m).$$

Since $Gx \subset IX$, $\forall x \in X$, there exists a point $z_0 \in X$ such that $Iz_0 \in Gy_0$ and $d(Iz_0,Jy_0) \leq m$. We obtain $\psi(m) \leq \psi(\delta(Iz_0,Fz_0)) \leq \psi(\delta(Fz_0,Gy_0))$.

Then by (4.1) we have

$$\phi(\psi(\delta(Fz_0,Gy_0)),\psi(d(Iz_0,Jy_0)),\psi(\delta(Iz_0,Fx_0)),$$
$$\psi(\delta(Jy_0,Gy_0)),\psi(D(Iz_0,Gy_0)),\psi(D(Jy_0,Fz_0))) \leq 0.$$

By (ϕ_1) we obtain

$$\phi(\psi(\delta(Iz_0, Fz_0)), \psi(m), \psi(\delta(Iz_0, Fz_0)),$$
$$\psi(m), 0, \psi(D(Jy_0, Fz_0))) \leq 0.$$

By (ϕ_{2b}) we have

$$\psi(\delta(Iz_0, Fz_0)) < \psi(m).$$

Hence, $\psi(m) \leq \psi(\delta(Iz_0, Fz_0)) < \psi(m)$, a contradiction. Hence $m = 0$ and $\psi(m) = 0$. By (4.2) $\psi(\delta(Jy_0, Gy_0)) = 0$ which implies $\{Jy_0\} = Gy_0$. Therefore $\{Ix_0\} = Fx_0 = \{Jy_0\} = Gy_0 = \{p\}$. Hence, x_0 is a strict coincidence point of I and F and y_0 is a strict coincidence point of J and G.

By Theorem 6, p is the unique point of strict coincidence of I and F and also p is the unique point of strict coincidence of J and G.

If (I,F) and (J,G) are sowc, then by Theorem 5, p is the unique common fixed point of I, J, F and G, which is a strict fixed point for F and G. □

Remark 6.
(1) By Example 2 and Theorem 7 we obtain a generalization of Theorem 3.
(2) By Example 5 and Theorem 7 we obtain a generalization of Theorem 4.

If $\psi(t) = t$ by Theorem 7 we obtain

Theorem 8. *Let (X,d) be a compact metric space, $I, J : X \to X$ and $F, G : X \to B(X)$ satisfying the following conditions:*

a) *$Fx \subset J(X)$ and $Gx \subset I(X)$, $\forall x \in X$,*
b) *the functions I and F are continuous,*
c) *$\phi(\delta(Fx, Gy), d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy), D(Ix, Gy), D(Jy, Fx)) \leq 0$, for all $x, y \in X$ and $\phi \in \mathcal{F}_c$. Then:*
d) *F and I have a strict coincidence point,*
e) *G and J have a strict coincidence point.*

Moreover, if the pairs (I,F) and (J,G) are strict owc, then I, J, F and G have an unique common fixed point which is a strict fixed point for F and G.

Remark 7. If I, J, F and G are self mappings of (X,d) then by Theorem 7 we obtain a generalization of Theorem 4.1 [7].

Corollary 1. *Let (X,d) be a compact metric space, $I, J : X \to X$ and $F, G : X \to B(X)$ satisfying the following conditions:*

a) *$F(X) \subset J(X)$ and $G(X) \subset I(X)$, $\forall x \in X$,*
b) *the functions I and F are continuous,*
c) *$\phi(\delta(Fx, Gy)) \leq$
$ad(Ix, Jy) + b[\delta(Ix, Fx) + \delta(Jy, Gy)] + c[D(Ix, Gy) \cdot D(Jy, Fx)]^{1/2}$,
for all $x, y \in X$, where $a > 0$, $b, c \geq 0$, $a + 2b < 1$ and $a + c < 1$. Then*
d) *F and I have a strict coincidence point,*

e) *G and J have a strict coincidence point.*

Moreover, if the pairs (I,F) and (J,G) are strict owc, then I,J,F and G have an unique common fixed point which is a strict fixed point for F and G.

Proof. The proof follows by Theorem 8 and Example 2. $\square$

Example 6. Let $X=[0,1]$ endowed with the Euclidean metric d. We define

$$Fx=\left\{\frac{1}{2}\right\}, x\in[0,1] \qquad Gx=\begin{cases}\frac{1}{2}, x\in\left[0,\frac{1}{2}\right]\\ \left(\frac{1}{4},\frac{1}{2}\right], x\in\left(\frac{1}{2},1\right]\end{cases}$$

$$Ix=\begin{cases}\frac{2x+1}{4}, x\in\left[0,\frac{1}{2}\right]\\ \frac{1}{2}, x\in\left(\frac{1}{2},1\right]\end{cases} \qquad Jx=\begin{cases}1-x, x\in\left[0,\frac{1}{2}\right]\\ 0, x\in\left(\frac{1}{2},1\right]\end{cases}$$

Then we have

$$FX=\left\{\frac{1}{2}\right\}, \qquad GX=\left[\frac{1}{4},\frac{1}{2}\right], \qquad IX=\left[\frac{1}{4},\frac{1}{2}\right], \qquad JX=\{0\}\cup\left[\frac{1}{2},1\right].$$

Hence $F(X)\subset J(X)$, $G(X)\subset I(X)$.
I and F are continuous.

$$J\left(\frac{1}{2}\right)=G\left(\frac{1}{2}\right)=\left\{\frac{1}{2}\right\}, \qquad I\left(\frac{1}{2}\right)=F\left(\frac{1}{2}\right)=\left\{\frac{1}{2}\right\},$$

$$IF\left(\frac{1}{2}\right)=I\left(\frac{1}{2}\right)=\frac{1}{2}, \qquad FI\left(\frac{1}{2}\right)=F\left(\frac{1}{2}\right)=\left\{\frac{1}{2}\right\},$$

$$JG\left(\frac{1}{2}\right)=J\left(\frac{1}{2}\right)=\frac{1}{2}, \qquad GJ\left(\frac{1}{2}\right)=G\left(\frac{1}{2}\right)=\left\{\frac{1}{2}\right\}.$$

Hence, (I,F) and (J,G) are strict owc.
If $x\in[0,1]$ and $y\in\left[0,\frac{1}{2}\right]$ then $\delta(Fx,Gy)=0$.
If $y\in\left(\frac{1}{2},1\right]$, then $\delta(Fx,Gy)=\delta\left(\left\{\frac{1}{2}\right\},\left(\frac{1}{4},\frac{1}{2}\right]\right)=\frac{1}{4}$ and $d(Ix,Jy)=d\left(\left\{\frac{1}{2}\right\},0\right)=\frac{1}{2}$.

Hence the condition $c)$ of Corollary 1 is satisfied for $a>\frac{1}{2}$, $a+2b<1$, $a+c<1$.

Hence I,J,F and G have an unique common fixed point $x=\frac{1}{2}$, which is a strict fixed point for F and G.

5. Altering distance and fixed points for hybrid pairs satisfying a contractive condition of integral type

Theorem 9. *Let (X,d) be a compact metric space, $I,J:(X,d)\to(X,d)$ and $F,G:X\to B(X)$ satisfying the following conditions:*

1) *$Fx\subset J(X)$ and $Gx\subset I(X)$, $\forall x\in X$,*
2) *the functions I and F are continuous,*

3) $\phi\Big(\int_0^{\delta(Fx,Gy)} h(t)dt, \int_0^{d(Ix,Jy)} h(t)dt, \int_0^{\delta(Ix,Fx)} h(t)dt,$
$\int_0^{\delta(Jy,Gy)} h(t)dt, \int_0^{D(Ix,Gy)} h(t)dt, \int_0^{D(Jy,Fx)} h(t)dt\Big) \leq 0,$
for all $x, y \in X$, where $\phi \in \mathcal{F}_c$ and $h(t)$ is as in Theorem 1. Then:

4) *F and I have a strict coincidence point,*
5) *G and J have a strict coincidence point.*

Moreover, if the pairs (I, F) and (J, G) are strict owc, then I, J, F and G have an unique common fixed point which is a strict fixed point for F and G.

Proof. As in Lemma 3 we have

$$\int_0^{\delta(Fx,Gy)} h(t)dt = \psi(\delta(Fx,Gy)), \qquad \int_0^{d(Ix,Jy)} h(t)dt = \psi(d(Ix,Jy)),$$
$$\int_0^{\delta(Ix,Fx)} h(t)dt = \psi(\delta(Ix,Fx)), \qquad \int_0^{\delta(Jy,Gy)} h(t)dt = \psi(\delta(Jy,Gy)),$$
$$\int_0^{D(Ix,Gy)} h(t)dt = \psi(D(Ix,Gy)), \qquad \int_0^{D(Jy,Fx)} h(t)dt = \psi(D(Jy,Fx)).$$

Then by 3) we obtain

$$\phi(\psi(\delta(Fx,Gy)), \psi(d(Ix,Jy)), \psi(\delta(Ix,Fx)),$$
$$\psi(\delta(Jy,Gy)), \psi(D(Ix,Gy)), D(Jy,Fx)) \leq 0.$$

By Lemma 3 $\psi(t)$ is a weakly altering distance. Hence the conditions of Theorem 7 are satisfied and the conclusion of Theorem 9 follows from Theorem 7. □

Remark 8. If $h(t) = 1$, by Theorem 9 we obtain Theorem 8.

By Theorem 9 and Example 2 - 5 we obtain particular results for mappings satisfying implicit relations in compact metric space. For example, by Theorem 9 and Example 2 we obtain

Corollary 2. *Let (X, d) be a compact metric space, $I, J : (X, d) \to (X, d)$, $F, G : X \to B(X)$ satisfying conditions (1) and (2) of Theorem 9 and*

$$\int_0^{\delta(Fx,Gy)} h(t)dt \leq a\int_0^{d(Ix,Jy)} h(t)dt + b[\int_0^{\delta(Ix,Fx)} h(t)dt +$$
$$\int_0^{\delta(Jy,Gy)} h(t)dt] + c(\int_0^{D(Ix,Gy)} h(t)dt \int_0^{D(Jy,Fx)} h(t)dt)^{1/2} \leq 0,$$

for all $x, y \in X$, where $h(t)$ is as in Theorem 1. Then:

a) *F and I have a strict coincidence point,*
b) *G and J have a strict coincidence point.*

Moreover, if the pairs (I, F) and (J, G) are strict owc, then I, J, F and G have an unique common fixed point which is a strict fixed point for F and G.

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ASYMPTOTIC PROPERTIES OF EIGENVALUES AND EIGENFUNCTIONS OF A STURM-LIOUVILLE PROBLEM WITH DISCONTINUOUS WEIGHT FUNCTION

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Abstract. In this paper, we extend some spectral properties of regular Sturm-Liouville problems to those which consist of a Sturm-Liouville equation with discontinuous weight at two interior points together with spectral parameter-dependent boundary conditions. By modifying some techniques of [C. T. Fulton, Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions, Proc. Roy. Soc. Edinburgh Sect. A 77 (1977) 293-308; O. Sh. Mukhtarov and M. Kadakal, Some spectral properties of one Sturm-Liouville type problem with discontinuous weight, Siberian Mathematical Journal, 46 (2005) 681-694], we give an operator-theoretic formulation for the considered problem and obtain asymptotic formulas for the eigenvalues and eigenfunctions.

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1. Introduction

Sturmian theory is one of the most extensively developing fields in theoretical and applied mathematics The literature is voluminous and we refer to [1–25]. Particularly, there has been an increasing interest in the spectral analysis of boundary-value problems with eigenvalue-dependent boundary conditions [1–3, 5–10, 12–14, 16, 17, 19–22, 24, 25].

In this paper following [12] we consider the boundary value problem for the differential equation

$$\tau u := -u'' + q(x)u = \lambda\omega(x)u \tag{1.1}$$

for $x \in [-1,h_1) \cup (h_1,h_2) \cup (h_2,1]$ (i.e., x belongs to $[-1,1]$ but the two inner points $x = h_1$ and $x = h_2$), where $q(x)$ is a real valued function, continuous in $[-1,h_1)$, (h_1,h_2) and $(h_2,1]$ with the finite limits $q(\pm h_1) = \lim_{x\to\pm h_1}$, $q(\pm h_2) = \lim_{x\to\pm h_2}$; $\omega(x)$ is a discontinuous weight function such that $\omega(x) = \omega_1^2$ for $x \in [-1,h_1)$, $\omega(x) = \omega_2^2$ for $x \in (h_1,h_2)$ and $\omega(x) = \omega_3^2$ for $x \in (h_2,1]$, $\omega > 0$ together with the standart boundary condition at $x=-1$

$$L_1u := \cos\alpha u(-1)+\sin\alpha u'(-1)=0, \tag{1.2}$$

the spectral parameter dependent boundary condition at $x=1$

$$L_2u := \lambda\left(\beta_1' u(1)-\beta_2' u'(1)\right)+\left(\beta_1 u(1)-\beta_2 u'(1)\right)=0, \tag{1.3}$$

and the four transmission conditions at the points of discontinuity $x=h_1$ and $x=h_2$

$$L_3u := \gamma_1 u(h_1-0)-\delta_1 u(h_1+0)=0, \tag{1.4}$$

$$L_4u := \gamma_2 u'(h_1-0)-\delta_2 u'(h_1+0)=0, \tag{1.5}$$

$$L_5u := \gamma_3 u(h_2-0)-\delta_3 u(h_2+0)=0, \tag{1.6}$$

$$L_6u := \gamma_4 u'(h_2-0)-\delta_4 u'(h_2+0)=0, \tag{1.7}$$

in the Hilbert space $L_2(-1,h_1)\oplus L_2(h_1,h_2)\oplus L_2(h_2,1)$ where $\lambda\in\mathbb{C}$ is a complex spectral parameter; and all coefficients of the boundary and transmission conditions are real constants. We assume naturally that $|\alpha_1|+|\alpha_2|\neq 0$, $\left|\beta_1'\right|+\left|\beta_2'\right|\neq 0$ and $|\beta_1|+|\beta_2|\neq 0$. Moreover, we will assume that $\rho := \beta_1'\beta_2-\beta_1\beta_2'>0$. A Sturm-Liouville problem with eigenparameter contained in the boundary condition arise upon separation of variables in the one-dimensional wave and heat equations for a varied assortment of physical problems, e.g. in the diffusion of water vapour through a porous membrane and several electric circuit problems involving long cables. (for example, see [3, 13]), vibrating string problems when the string loaded additionally with point masses (for example, see [18]), and a thermal conduction problem for a thin laminated plate (for example, see [23]).

2. Operator-Theoretic Formulation of the Problem

In this section, we introduce a special inner product in the Hilbert space $(L_2(-1,h_1)\oplus L_2(h_1,h_2)\oplus L_2(h_2,1))\oplus\mathbb{C}$ and define a linear operator A in it so that the problem (1.1)-(1.7) can be interpreted as the eigenvalue problem for A. To this end, we define a new Hilbert space inner product on

$$H := (L_2(-1,h_1)\oplus L_2(h_1,h_2)\oplus L_2(h_2,1))\oplus\mathbb{C}$$

by

$$\begin{aligned}\langle F,G\rangle_H &= \omega_1^2\int_{-1}^{h_1} f(x)\overline{g(x)}dx+\omega_2^2\frac{\delta_1\delta_2}{\gamma_1\gamma_2}\int_{h_1}^{h_2} f(x)\overline{g(x)}dx\\ &+\omega_3^2\frac{\delta_1\delta_2\delta_3\delta_4}{\gamma_1\gamma_2\gamma_3\gamma_4}\int_{h_2}^{1} f(x)\overline{g(x)}dx+\frac{\delta_1\delta_2\delta_3\delta_4}{\rho\gamma_1\gamma_2\gamma_3\gamma_4}f_1\overline{g_1}\end{aligned}$$

for $F=\begin{pmatrix} f(x)\\ f_1\end{pmatrix}$ and $G=\begin{pmatrix} g(x)\\ g_1\end{pmatrix}\in H$. For convenience we will use the notations

$$R_1(u) := \beta_1 u(1)-\beta_2 u'(1),\quad R_1'(u) := \beta_1' u(1)-\beta_2' u'(1).$$

In this Hilbert space we construct the operator $A : H \to H$ with domain

$$D(A) = \Big\{ F = \begin{pmatrix} f(x) \\ f_1 \end{pmatrix} \mid f(x), f'(x) \text{ are absolutely continuous in } [1,h_1] \cup [h_1,h_2] \cup [h_2,1]; \text{ has finite limits } f(h_1 \pm 0),\ f(h_2 \pm 0), f'(h_1 \pm 0), f'(h_2 \pm 0); \tau f \in L_2(-1,h_1) \oplus L_2(h_1,h_2) \oplus L_2(h_2,1); L_1 f = L_3 f = L_4 f = L_5 f = L_6 f = 0, f_1 = R_1'(f) \Big\} \tag{2.1}$$

which acts by the rule

$$AF = \begin{pmatrix} \frac{1}{\omega(x)}[-f'' + q(x)f] \\ -R_1(f) \end{pmatrix} \quad \text{with} \quad F = \begin{pmatrix} f(x) \\ R_1'(f) \end{pmatrix} \in D(A). \tag{2.2}$$

Thus we can pose the boundary-value-transmission problem (1.1)-(1.7) in H as

$$AU = \lambda U, \quad U := \begin{pmatrix} u(x) \\ R_1'(u) \end{pmatrix} \in D(A). \tag{2.3}$$

It is readily verified that the eigenvalues of A coincide with those of the problem (1.1)-(1.7).

Theorem 1. *The operator A is symmetric.*

Proof. Let $F = \begin{pmatrix} f(x) \\ R_1'(f) \end{pmatrix}$ and $G = \begin{pmatrix} g(x) \\ R_1'(g) \end{pmatrix}$ be arbitrary elements of $D(A)$. Twice integrating by parts we find

$$\begin{aligned} \langle AF, G \rangle_H - \langle F, AG \rangle_H &= W(f,\overline{g};h_1-0) - W(f,\overline{g};-1) \\ &+ \frac{\delta_1\delta_2}{\gamma_1\gamma_2}\left(W(f,\overline{g};h_2-0) - W(f,\overline{g};h_1+0)\right) \\ &+ \frac{\delta_1\delta_2\delta_3\delta_4}{\gamma_1\gamma_2\gamma_3\gamma_4}\left(W(f,\overline{g};1) - W(f,\overline{g};h_2+0)\right) \\ &+ \frac{\delta_1\delta_2\delta_3\delta_4}{\rho\gamma_1\gamma_2\gamma_3\gamma_4}\left(R_1'(f)R_1(\overline{g}) - R_1(f)R_1'(\overline{g})\right) \end{aligned} \tag{2.4}$$

where, as usual, $W(f,g;x)$ denotes the Wronskian of f and g; i.e.,

$$W(f,g;x) := f(x)g'(x) - f'(x)g(x).$$

Since $F, G \in D(A)$, the first components of these elements, i.e. f and g satisfy the boundary condition (1.2). From this fact we easily see that

$$W(f,\overline{g};-1) = 0, \tag{2.5}$$

since $\cos\alpha$ and $\sin\alpha$ are real. Further, as f and g also satisfy both transmission conditions, we obtain

$$W(f,\overline{g};h_1-0)=\frac{\delta_1\delta_2}{\gamma_1\gamma_2}W(f,\overline{g};h_1+0) \tag{2.6}$$

$$W(f,\overline{g};h_2-0)=\frac{\delta_1\delta_2\delta_3\delta_4}{\gamma_1\gamma_2\gamma_3\gamma_4}W(f,\overline{g};h_2+0) \tag{2.7}$$

Moreover, the direct calculations give

$$R_1'(f)R_1(\overline{g})-R_1(f)R_1'(\overline{g})=-\rho W(f,\overline{g};1) \tag{2.8}$$

Now, inserting (2.5)-(2.8) in (2.4), we have

$$\langle AF,G\rangle_H=\langle F,AG\rangle_H \qquad (F,G\in D(A))$$

and so A is symmetric. □

Recalling that the eigenvalues of (1.1)-(1.7) coincide with the eigenvalues of A, we have the next corollary:

Corollary 1. *All eigenvalues of (1.1)-(1.7) are real.*

Since all eigenvalues are real it is enough to study only the real-valued eigenfunctions. Therefore we can now assume that all eigenfunctions of (1.1)-(1.7) are real-valued.

3. Asymptotic Formulas for Eigenvalues and Fundamental Solutions

Let us define fundamental solutions

$$\phi(x,\lambda)=\begin{cases}\phi_1(x,\lambda), & x\in[-1,h_1),\\ \phi_2(x,\lambda), & x\in(h_1,h_2),\\ \phi_3(x,\lambda), & x\in(h_2,1]\end{cases}$$

and

$$\chi(x,\lambda)=\begin{cases}\chi_1(x,\lambda), & x\in[-1,h_1),\\ \chi_2(x,\lambda), & x\in(h_1,h_2),\\ \chi_3(x,\lambda), & x\in(h_2,1]\end{cases}$$

of (1.1) by the following procedure. We first consider the next initial-value problem:

$$-u''+q(x)u=\lambda\omega_1^2u,\ x\in[-1,h_1] \tag{3.1}$$

$$u(-1)=\sin\alpha, \tag{3.2}$$

$$u'(-1)=-\cos\alpha \tag{3.3}$$

By virtue of ([19], Theorem 1.5) the problem (3.1)-(3.3) has a unique solution $u=\phi_1(x,\lambda)$ which is an entire function of $\lambda\in\mathbb{C}$ for each fixed $x\in[-1,h_1]$. Similarly,

$$-u'' + q(x)u = \lambda\omega_2^2 u, \quad x \in [h_1, h_2] \tag{3.4}$$

$$u(h_1) = \frac{\gamma_1}{\delta_1}\phi_1(h_1, \lambda), \tag{3.5}$$

$$u'(h_1) = \frac{\gamma_2}{\delta_2}\phi_1'(h_1, \lambda), \tag{3.6}$$

has a unique solution $u = \phi_2(x, \lambda)$ which is an entire function of $\lambda \in \mathbb{C}$ for each fixed $x \in [h_1, h_2]$. Continuing in this manner

$$-u'' + q(x)u = \lambda\omega_3^2 u, \quad x \in [h_2, 1] \tag{3.7}$$

$$u(h_2) = \frac{\gamma_3}{\delta_3}\phi_2(h_2, \lambda), \tag{3.8}$$

$$u'(h_2) = \frac{\gamma_4}{\delta_4}\phi_2'(h_2, \lambda), \tag{3.9}$$

has a unique solution $u = \phi_3(x, \lambda)$ which is an entire function of $\lambda \in \mathbb{C}$ for each fixed $x \in [h_2, 1]$. Slightly modifying the method of ([19], Theorem 1.5) we can prove that the initial-value problem

$$-u'' + q(x)u = \lambda\omega_3^2 u, \ x \in [h_2, 1] \tag{3.10}$$

$$u(1) = \beta_2'\lambda + \beta_2, \tag{3.11}$$

$$u'(1) = \beta_1'\lambda + \beta_1 \tag{3.12}$$

(3.10)-(3.13) has a unique solution $u = \chi_3(x, \lambda)$ which is an entire function of spectral parameter $\lambda \in \mathbb{C}$ for each fixed $x \in [h_2, 1]$. Similarly,

$$-u'' + q(x)u = \lambda\omega_2^2 u, \quad x \in [h_1, h_2] \tag{3.13}$$

$$u(h_2) = \frac{\delta_3}{\gamma_3}\chi_3(h_2, \lambda), \tag{3.14}$$

$$u'(h_2) = \frac{\delta_4}{\gamma_4}\chi_3'(h_2, \lambda), \tag{3.15}$$

has a unique solution $u = \chi_2(x, \lambda)$ which is an entire function of $\lambda \in \mathbb{C}$ for each fixed $x \in [h_1, h_2]$. Continuing in this manner

$$-u'' + q(x)u = \lambda\omega_3^2 u, \quad x \in [-1, h_1] \tag{3.16}$$

$$u(h_1) = \frac{\delta_1}{\gamma_1}\chi_2(h_1, \lambda), \tag{3.17}$$

$$u'(h_1) = \frac{\delta_2}{\gamma_2}\chi_2'(h_1, \lambda), \tag{3.18}$$

has a unique solution $u = \chi_1(x,\lambda)$ which is an entire function of $\lambda \in \mathbb{C}$ for each fixed $x \in [-1,h_1]$.

By virtue of (3.2) and (3.3) the solution $\phi(x,\lambda)$ satisfies the first boundary condition (1.2). Moreover, by (3.5), (3.6), (3.8) and (3.9), $\phi(x,\lambda)$ satisfies also transmission conditions (1.4)-(1.7). Similarly, by (3.11), (3.12), (3.14), (3.15), (3.17) and (3.18) the other solution $\chi(x,\lambda)$ satisfies the second boundary condition (1.3) and transmission conditions (1.4)-(1.7). It is well-known from the theory of ordinary differential equations that each of the Wronskians $\Delta_1(\lambda) = W(\phi_1(x,\lambda),\chi_1(x,\lambda))$, $\Delta_2(\lambda) = W(\phi_2(x,\lambda),\chi_2(x,\lambda))$ and $\Delta_3(\lambda) = W(\phi_3(x,\lambda),\chi_3(x,\lambda))$ are independent of x in $[-1,h_1]$, $[h_1,h_2]$ and $[h_2,1]$ respectively.

Lemma 1. *The equality* $\Delta_1(\lambda) = \frac{\delta_1\delta_2}{\gamma_1\gamma_2}\Delta_2(\lambda) = \frac{\delta_1\delta_2\delta_3\delta_4}{\gamma_1\gamma_2\gamma_3\gamma_4}\Delta_3(\lambda)$ *holds for each* $\lambda \in \mathbb{C}$.

Proof. Since the above Wronskians are independent of x, using (3.8), (3.9), (3.11), (3.12), (3.14), (3.15), (3.17) and (3.18) we find

$$\begin{aligned}
\Delta_1(\lambda) &= \phi_1(h_1,\lambda)\chi_1'(h_1,\lambda) - \phi_1'(h_1,\lambda)\chi_1(h_1,\lambda)\\
&= \left(\frac{\delta_1}{\gamma_1}\phi_2(h_1,\lambda)\right)\left(\frac{\delta_2}{\gamma_2}\chi_2'(h_1,\lambda)\right) - \left(\frac{\delta_2}{\gamma_2}\phi_2'(h_1,\lambda)\right)\left(\frac{\delta_1}{\gamma_1}\chi_2(h_1,\lambda)\right)\\
&= \frac{\delta_1\delta_2}{\gamma_1\gamma_2}\Delta_2(\lambda) = \left(\frac{\delta_1\delta_3}{\gamma_1\gamma_3}\phi_3(h_2,\lambda)\right)\left(\frac{\delta_2\delta_4}{\gamma_2\gamma_4}\chi_3'(h_2,\lambda)\right)\\
&\quad - \left(\frac{\delta_2\delta_4}{\gamma_2\gamma_4}\phi_3'(h_2,\lambda)\right)\left(\frac{\delta_1\delta_3}{\gamma_1\gamma_3}\chi_3(h_2,\lambda)\right) = \frac{\delta_1\delta_2\delta_3\delta_4}{\gamma_1\gamma_2\gamma_3\gamma_4}\Delta_3(\lambda).
\end{aligned}$$

□

Corollary 2. *The zeros of* $\Delta_1(\lambda)$, $\Delta_2(\lambda)$ *and* $\Delta_3(\lambda)$ *coincide.*

In view of Lemma 3.1 we denote $\Delta_1(\lambda)$, $\frac{\delta_1\delta_2}{\gamma_1\gamma_2}\Delta_2(\lambda)$ and $\frac{\delta_1\delta_2\delta_3\delta_4}{\gamma_1\gamma_2\gamma_3\gamma_4}\Delta_3(\lambda)$ by $\Delta(\lambda)$. Recalling the definitions of $\phi_i(x,\lambda)$ and $\chi_i(x,\lambda)$, we can state the next corollary.

Corollary 3. *The function* $\Delta(\lambda)$ *is an entire function.*

Theorem 2. *The eigenvalues of (1.1)-(1.7) are the roots of* $\Delta(\lambda) = 0$.

Proof. Let $\Delta(\lambda_0) = 0$. Then $W(\phi_1(x,\lambda_0),\chi_1(x,\lambda_0)) = 0$ for all $x \in [-1,h_1]$. Consequently, the functions $\phi_1(x,\lambda_0)$ and $\chi_1(x,\lambda_0)$ are linearly dependent, i.e. $\chi_1(x,\lambda_0) = k\phi_1(x,\lambda_0)$, $x \in [-1,h_1]$, for some $k \neq 0$. By (3.2) and (3.3), from this equality, we have

$$\cos\alpha\chi(-1,\lambda_0) + \sin\alpha\chi'(-1,\lambda_0) = \cos\alpha\chi_1(-1,\lambda_0) + \sin\alpha\chi_1'(-1,\lambda_0)$$

$$= k\left(\cos\alpha\phi_1(-1,\lambda_0) + \sin\alpha\phi_1'(-1,\lambda_0)\right) = k(\cos\alpha\sin\alpha + \sin\alpha(-\cos\alpha)) = 0,$$

and so $\chi(x,\lambda_0)$ satisfies the first boundary condition (1.2). Recalling that the solution $\chi(x,\lambda_0)$ also satisfies the other boundary condition (1.3) and transmission conditions (1.4)-(1.7). We conclude that $\chi(x,\lambda_0)$ is an eigenfunction of (1.1)-(1.7); i.e., λ_0 is an eigenvalue. Thus, each zero of $\Delta(\lambda)$ is an eigenvalue. Now let λ_0 be an eigenvalue and let $u_0(x)$ be an eigenfunction with this eigenvalue. Suppose that $\Delta(\lambda_0) \neq 0$. Whence $W(\phi_1(x,\lambda_0),\chi_1(x,\lambda_0)) \neq 0$, $W(\phi_2(x,\lambda_0),\chi_2(x,\lambda_0)) \neq 0$ and $W(\phi_3(x,\lambda_0),\chi_3(x,\lambda_0)) \neq 0$. From this, by virtue of the well-known properties of Wronskians, it follows that each of the pairs $\phi_1(x,\lambda_0)$, $\chi_1(x,\lambda_0)$; $\phi_2(x,\lambda_0)$, $\chi_2(x,\lambda_0)$ and $\phi_3(x,\lambda_0)$, $\chi_3(x,\lambda_0)$ is linearly independent. Therefore, the solution $u_0(x)$ of (1.1) may be represented as

$$u_0(x) = \begin{cases} c_1\phi_1(x,\lambda_0) + c_2\chi_1(x,\lambda_0), & x \in [-1,h_1), \\ c_3\phi_2(x,\lambda_0) + c_4\chi_2(x,\lambda_0), & x \in (h_1,h_2), \\ c_5\phi_3(x,\lambda_0) + c_6\chi_3(x,\lambda_0), & x \in (h_2,1], \end{cases}$$

where at least one of the coefficients c_i $\left(i = \overline{1,6}\right)$ is not zero. Considering the true equalities

$$L_\upsilon(u_0(x)) = 0, \quad \upsilon = \overline{1,6}, \tag{3.19}$$

as the homogenous system of linear equations in the variables c_i $\left(i = \overline{1,6}\right)$ and taking (3.5), (3.6), (3.8), (3.9), (3.14), (3.15), (3.17) and (3.18) into account, we see that the determinant of this system is equal to $-\frac{(\delta_1\delta_2\delta_3\delta_4)^2}{\gamma_1\gamma_2\gamma_3\gamma_4}\Delta^4(\lambda_0)$ and so it does not vanish by assumption. Consequently the system (3.19) has the only trivial solution $c_i = 0$ $\left(i = \overline{1,6}\right)$. This is a contradiction. And the proof is complete. □

Theorem 3. *Let* $\lambda = \mu^2$ *and* $\operatorname{Im}\mu = t$. *Then the following asymptotic equalities hold as* $|\lambda| \to \infty$:

(1) In case $\sin\alpha \neq 0$

$$\phi_1^{(k)}(x,\lambda) = \sin\alpha \frac{d^k}{dx^k}\cos[\mu\omega_1(x+1)] + O\left(\frac{1}{|\mu|^{1-k}}\exp(|t|\omega_1(x+1))\right), \tag{3.20}$$

$$\phi_2^{(k)}(x,\lambda) = \frac{\gamma_1}{\delta_1}\sin\alpha\frac{d^k}{dx^k}\cos[\mu(\omega_2 x + \omega_1 h_1 + \omega_1)] + O\left(\frac{1}{|\mu|^{1-k}}\exp(|t|(\omega_2 x + \omega_1 h_1 + \omega_1))\right), \tag{3.21}$$

$$\phi_3^{(k)}(x,\lambda) = \frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\frac{d^k}{dx^k}\cos[\mu(\omega_3 x + \omega_2 h_2 + \omega_1)] + O\left(\frac{1}{|\mu|^{1-k}}\exp(|t|(\omega_3 x + \omega_2 h_2 + \omega_1))\right). \tag{3.22}$$

(2) In case $\sin\alpha = 0$

$$\phi_1^{(k)}(x,\lambda) = \frac{-1}{\mu\omega_1}\cos\alpha\frac{d^k}{dx^k}\sin[\mu\omega_1(x+1)] + O\left(\frac{1}{|\mu|^{2-k}}\exp(|t|\omega_1(x+1))\right), \tag{3.23}$$

$$\begin{aligned}\phi_2^{(k)}(x,\lambda) = &-\frac{\gamma_1}{\mu\delta_1}\cos\alpha\frac{d^k}{dx^k}\sin[\mu(\omega_2 x+\omega_1 h_1+\omega_1)] \\ &+ O\left(\frac{1}{|\mu|^{2-k}}\exp(|t|(\omega_2 x+\omega_1 h_1+\omega_1))\right),\end{aligned} \tag{3.24}$$

$$\begin{aligned}\phi_3^{(k)}(x,\lambda) = &-\frac{\gamma_1\gamma_3}{\mu\delta_1\delta_3}\cos\alpha\frac{d^k}{dx^k}\sin[\mu(\omega_3 x+\omega_2 h_2+\omega_1)] \\ &+ O\left(\frac{1}{|\mu|^{2-k}}\exp(|t|(\omega_3 x+\omega_2 h_2+\omega_1))\right).\end{aligned} \tag{3.25}$$

for $k = 0$ *and* $k = 1$. *Moreover, each of these asymptotic equalities holds uniformly for* x.

Proof. Asymptotic formulas for $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$ are found in ([19], Lemma 1.7) and ([12], Theorem 3.2) respectively. But the formulas for $\phi_3(x,\lambda)$ need individual considerations, since this solution is defined by the initial condition with some special nonstandart form. The initial-value problem (3.7)-(3.9) can be transformed into the equivalent integral equation

$$\begin{aligned}u(x) = &\frac{\gamma_3}{\delta_3}\phi_2(h_2,\lambda)\cos\mu\omega_3 x + \frac{\gamma_4}{\mu\omega_3\delta_4}\phi_2'(h_2,\lambda)\sin\mu\omega_3 x \\ &+ \frac{\omega_3}{\mu}\int_{h_2}^{x}\sin[\mu\omega_3(x-y)]q(y)u(y)\,dy\end{aligned} \tag{3.26}$$

Let $\sin\alpha \neq 0$. Inserting (3.21) in (3.26) we have

$$\begin{aligned}\phi_3(x,\lambda) = &\frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\cos[\mu(\omega_3 x+\omega_2 h_2+\omega_1)] \\ &+ \frac{\omega_3}{\mu}\int_{h_2}^{x}\sin[\mu\omega_3(x-y)]q(y)\phi_3(y,\lambda)\,dy \\ &+ O\left(\frac{1}{|\mu|}\exp(|t|(\omega_3 x+\omega_2 h_2+\omega_1))\right).\end{aligned} \tag{3.27}$$

Multiplying this by $\exp(-|t|(\omega_3 x+\omega_2 h_2+\omega_1))$ and denoting

$$F(x,\lambda) = \exp(-|t|(\omega_3 x+\omega_2 h_2+\omega_1))\phi_3(x,\lambda),$$

we have the following integral equation

$$F(x,\lambda) = \frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\exp\left(-|t|\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\right)\cos\left[\mu\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\right]$$
$$+\frac{\omega_3}{\mu}\int_{h_2}^{x}\sin\left[\mu\omega_3(x-y)\right]\exp\left(-|t|\,\omega_3(x-y)\right)q(y)\,F(y,\lambda)dy+O\left(\frac{1}{\mu}\right).$$

Putting $M(\lambda) = \max_{x\in[h_2,1]}|F(x,\lambda)|$, from the last equation we derive that

$$M(\lambda) \le M_0\left(\left|\frac{\gamma_1\gamma_3}{\delta_1\delta_3}\right|+\frac{1}{\mu}\right)$$

for some $M_0 > 0$. Consequently, $M(\lambda) = O(1)$ as $|\lambda| \to \infty$, and so $\phi_3(x,\lambda) = O\left(\exp\left(|t|\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\right)\right)$ as $|\lambda| \to \infty$. Inserting the integral term of (3.27) yields (3.22) for $k = 0$. The case $k = 1$ of (3.22) follows at once on differentiating (3.21) and making the same procedure as in the case $k = 0$. The proof of (3.25) is similar to that of (3.22). □

Theorem 4. *Let $\lambda = \mu^2$, $\mu = \sigma + it$. Then the following asymptotic formulas hold for the eigenvalues of the boundary-value-transmission problem(1.1)-(1.7):*

Case 1: $\beta_2' \neq 0$, $\sin\alpha \neq 0$

$$\mu_n = \frac{\pi(n-1)}{\omega_3+\omega_2 h_2+\omega_1}+O\left(\frac{1}{n}\right), \tag{3.28}$$

Case 2: $\beta_2' \neq 0$, $\sin\alpha = 0$

$$\mu_n = \frac{\pi\left(n-\frac{1}{2}\right)}{\omega_3+\omega_2 h_2+\omega_1}+O\left(\frac{1}{n}\right), \tag{3.29}$$

Case 3: $\beta_2' = 0$, $\sin\alpha \neq 0$

$$\mu_n = \frac{\pi\left(n-\frac{1}{2}\right)}{\omega_3+\omega_2 h_2+\omega_1}+O\left(\frac{1}{n}\right), \tag{3.30}$$

Case 4: $\beta_2' = 0$, $\sin\alpha = 0$

$$\mu_n = \frac{\pi n}{\omega_3+\omega_2 h_2+\omega_1}+O\left(\frac{1}{n}\right), \tag{3.31}$$

Proof. Let us consider only the case 1. Putting $x = 1$ in

$$\Delta_3(\lambda) = \phi_3(x,\lambda)\,\chi_3'(x,\lambda)-\phi_3'(x,\lambda)\,\chi_3(x,\lambda)$$

and inserting $\chi_3(1,\lambda) = \beta_2'\lambda+\beta_2$, $\chi_3'(1,\lambda) = \beta_1'\lambda+\beta_1$ we have the following representation for $\Delta_3(\lambda)$:

$$\Delta_3(\lambda) = \left(\beta_1'\lambda+\beta_1\right)\phi_3(1,\lambda)-\left(\beta_2'\lambda+\beta_2\right)\phi_3'(1,\lambda). \tag{3.32}$$

Putting $x=1$ in (3.22) and inserting the result in (3.32), we derive now that

$$\begin{aligned}\Delta_3(\lambda) &= \frac{\delta_2\delta_4}{\gamma_2\gamma_4}\omega_3\beta_2'(\sin\alpha)\mu^3\sin[\mu(\omega_3+\omega_2h_2+\omega_1)] \\ &\quad + O\left(|\mu|^2\exp(2|t|(\omega+\omega_2h_2+\omega_1))\right). \qquad (3.33)\end{aligned}$$

By applying the Rouché Theorem, it follows that $\Delta_3(\lambda)$ has the same number of zeros inside the contour as the leading term in (3.33). Hence, if $\lambda_0<\lambda_1<\lambda_2...$ are the zeros of $\Delta_3(\lambda)$ and $\mu_n^2=\lambda_n$, we have

$$\frac{\pi(n-1)}{\omega_3+\omega_2h_2+\omega_1}+\delta_n \qquad (3.34)$$

for sufficiently large n, where $|\delta_n|<\frac{\pi}{4(\omega_3+\omega_2h_2+\omega_1)}$ for sufficiently large n. By putting in (3.33) we have $\delta_n=O\left(\frac{1}{n}\right)$, and the proof is completed in Case 1. The proofs for the other cases are similar. □

Theorem 5. *The following asymptotic formulas hold for the eigenfunctions*

$$\phi_{\lambda_n}(x)=\begin{cases}\phi_1(x,\lambda_n), & x\in[-1,h_1),\\ \phi_2(x,\lambda_n), & x\in(h_1,h_2),\\ \phi_3(x,\lambda_n), & x\in(h_2,1]\end{cases}$$

of (1.1)-(1.7):

Case 1: $\beta_2'\neq 0$, $\sin\alpha\neq 0$

$$\phi_{\lambda_n}(x)=\begin{cases}\sin\alpha\cos\left[\frac{\omega_1\pi(n-1)(x+1)}{\omega_2+\omega_1}\right]+O\left(\frac{1}{n}\right), & x\in[-1,h_1),\\ \frac{\gamma_1}{\delta_1}\sin\alpha\cos\left[\frac{(\omega_2x+\omega_1h_1+\omega_1)\pi(n-1)}{\omega_2+\omega_1h_1+\omega_1}\right]+O\left(\frac{1}{n}\right), & x\in(h_1,h_2),\\ \frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\cos\left[\frac{(\omega_3x+\omega_2h_2+\omega_1)\pi(n-1)}{\omega_3+\omega_2h_2+\omega_1}\right]+O\left(\frac{1}{n}\right), & x\in(h_2,1].\end{cases}$$

Case 2: $\beta_2'\neq 0$, $\sin\alpha=0$

$$\phi_{\lambda_n}(x)$$

$$=\begin{cases}-\frac{\omega_1+\omega_2}{\omega_1}\frac{\cos\alpha}{\pi\left(n-\frac{1}{2}\right)}\sin\left[\frac{\omega_1\pi\left(n-\frac{1}{2}\right)(x+1)}{\omega_2+\omega_1}\right]+O\left(\frac{1}{n^2}\right), & x\in[-1,h_1),\\ \frac{-\gamma_1}{\delta_1}\frac{\omega_1+\omega_2}{\omega_1}\frac{\cos\alpha}{\pi\left(n-\frac{1}{2}\right)}\sin\left[\frac{(\omega_2x+\omega_1h_1+\omega_1)\pi\left(n-\frac{1}{2}\right)}{\omega_2+\omega_1h_1+\omega_1}\right]+O\left(\frac{1}{n^2}\right), & x\in(h_1,h_2),\\ \frac{-\gamma_1\gamma_3}{\delta_1\delta_3}\frac{\omega_1+\omega_2}{\omega_1}\frac{\cos\alpha}{\pi\left(n-\frac{1}{2}\right)}\sin\left[\frac{(\omega_3x+\omega_2h_2+\omega_1)\pi\left(n-\frac{1}{2}\right)}{\omega_3+\omega_2h_2+\omega_1}\right]+O\left(\frac{1}{n^2}\right), & x\in(h_2,1].\end{cases}$$

Case 3: $\beta_2' = 0$, $\sin\alpha \neq 0$

$$\phi_{\lambda_n}(x) = \begin{cases} \sin\alpha\cos\left[\frac{\omega_1\pi(n-\frac{1}{2})(x+1)}{\omega_2+\omega_1}\right] + O\left(\frac{1}{n}\right), & x \in [-1,h_1), \\ \frac{\gamma_1}{\delta_1}\sin\alpha\cos\left[\frac{(\omega_2 x+\omega_1 h_1+\omega_1)\pi(n-\frac{1}{2})}{\omega_2+\omega_1 h_1+\omega_1}\right] + O\left(\frac{1}{n}\right), & x \in (h_1,h_2), \\ \frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\cos\left[\frac{(\omega_3 x+\omega_2 h_2+\omega_1)\pi(n-\frac{1}{2})}{\omega_3+\omega_2 h_2+\omega_1}\right] + O\left(\frac{1}{n}\right), & x \in (h_2,1]. \end{cases}$$

Case 4: $\beta_2' = 0$, $\sin\alpha = 0$

$$\phi_{\lambda_n}(x) = \begin{cases} -\frac{\omega_1+\omega_2}{\omega_1}\frac{\cos\alpha}{\pi n}\sin\left[\frac{\omega_1\pi n(x+1)}{\omega_2+\omega_1}\right] + O\left(\frac{1}{n^2}\right), & x \in [-1,h_1), \\ \frac{-\gamma_1}{\delta_1}\frac{\omega_1+\omega_2}{\omega_1}\frac{\cos\alpha}{\pi n}\sin\left[\frac{(\omega_2 x+\omega_1 h_1+\omega_1)\pi n}{\omega_2+\omega_1 h_1+\omega_1}\right] + O\left(\frac{1}{n^2}\right), & x \in (h_1,h_2), \\ \frac{-\gamma_1\gamma_3}{\delta_1\delta_3}\frac{\omega_1+\omega_2}{\omega_1}\frac{\cos\alpha}{\pi n}\sin\left[\frac{(\omega_3 x+\omega_2 h_2+\omega_1)\pi n}{\omega_3+\omega_2 h_2+\omega_1}\right] + O\left(\frac{1}{n^2}\right), & x \in (h_2,1]. \end{cases}$$

All these asymptotic formulas hold uniformly for x.

Proof. Let us consider only the Case 1. Inserting (3.22) in the integral term of (3.27), we easily see that

$$\int_{h_2}^{x} \sin[\mu\omega_3(x-y)]\, q(y)\,\phi_3(y,\lambda)\,dy = O\left(\exp\left(|t|\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\right)\right).$$

Inserting in (3.20) yields

$$\begin{aligned} \phi_3(x,\lambda) = {} & \frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\cos\left[\mu\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\right] \\ & + O\left(\frac{1}{|\mu|}\exp|t|\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\right). \end{aligned} \tag{3.35}$$

We already know that all eigenvalues are real. Furthermore, putting $\lambda = -H$, $H > 0$ in (3.33) we infer that $\omega(-H) \to \infty$ as $H \to +\infty$, and so $\omega(-H) \neq 0$ for sufficiently large $R > 0$. Consequently, the set of eigenvalues is bounded below. Letting $\sqrt{\lambda_n} = \mu_n$ in (3.35) we now obtain

$$\phi_3(x,\lambda_n) = \frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\cos\left[\mu_n\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\right] + O\left(\frac{1}{\mu_n}\right)$$

since $t_n = lm\mu_n$ for sufficiently large n. After some calculation, we easily see that

$$\cos\left[\mu_n\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\right] = \cos\left[\frac{\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\pi(n-1)}{\omega_3+\omega_2 h_2+\omega_1}\right] + O\left(\frac{1}{n}\right).$$

Consequently,

$$\phi_3(x,\lambda_n) = \frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\cos\left[\frac{\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\pi(n-1)}{\omega_3+\omega_2 h_2+\omega_1}\right] + O\left(\frac{1}{n}\right).$$

In a similar method, we can deduce that

$$\phi_2(x,\lambda_n) = \frac{\gamma_1}{\delta_1}\sin\alpha\cos\left[\frac{(\omega_2 x+\omega_1 h_1+\omega_1)\pi(n-1)}{\omega_2+\omega_1 h_1+\omega_1}\right] + O\left(\frac{1}{n}\right),$$

and

$$\phi_1(x,\lambda_n) = \sin\alpha\cos\left[\frac{\omega_1\pi(n-1)(x+1)}{\omega_2+\omega_1}\right] + O\left(\frac{1}{n}\right).$$

Thus the proof of the theorem completed in Case 1. The proofs for the other cases are similar. □

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ALMOST PERIODIC SOLUTIONS OF DIFFERENCE EQUATIONS WITH DISCRETE ARGUMENT ON METRIC SPACE

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Abstract. We obtain conditions for existence of almost periodic solutions of difference equations with discrete argument on metric space.

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1. INTRODUCTION

Let M be a metric space with metric ρ_M. Fix an arbitrary element $a \in M$. Denote by $\mathfrak{M}$ the metric space of sequences $\mathbf{x} = (\mathbf{x}(n))_{n\in\mathbb{Z}}$ for which

$$\sup_{n\in\mathbb{Z}} \rho_M(\mathbf{x}(n), a) < \infty$$

with metric

$$\rho_{\mathfrak{M}}(\mathbf{x}_1, \mathbf{x}_2) = \sup_{n\in\mathbb{Z}} \rho_M(\mathbf{x}_1(n), \mathbf{x}_2(n)).$$

Define the shift operator $S_m : \mathfrak{M} \to \mathfrak{M}$, $m \in \mathbb{Z}$, by the formulae

$$(S_m\mathbf{x})(n) = \mathbf{x}(n+m), \quad n \in \mathbb{Z}.$$

Definition 1. The sequence $\mathbf{x} \in \mathfrak{M}$ is called almost periodic if the set $\overline{\{S_m\mathbf{x} : m \in \mathbb{Z}\}}$ is compact in $\mathfrak{M}$.

Denote by $\mathfrak{B}$ the metric space of sequences $\mathbf{x} \in \mathfrak{M}$ which are almost periodic.

Let $\mathcal{K}$ be the set of compact sets K of the metric space M and let $R(\mathbf{x})$ be the set $\{\mathbf{x}(n) : n \in \mathbb{Z}\}$. Fix an arbitrary compact set $K \subset \mathcal{K}$. We denote by $\mathfrak{D}_K$ the set of all elements of $\mathbf{x} \in \mathfrak{M}$ for each of which $R(\mathbf{x}) \subset K$.

Definition 2. The operator $\mathbf{H} : \mathfrak{M} \to \mathfrak{M}$ is called almost periodic if for every set $K \in \mathcal{K}$ and a sequence $(m_k)_{k\geqslant 1}$ of whole numbers there exists a subsequence $(m_{k_l})_{l\geqslant 1}$, which

$$\lim_{l_1\to\infty,\ l_2\to\infty} \sup_{\mathbf{x}\in\mathfrak{D}_K} \rho_{\mathfrak{M}}\left(S_{m_{l_1}}\mathbf{H}S_{-m_{l_1}}\mathbf{x}, S_{m_{l_2}}\mathbf{H}S_{-m_{l_2}}\mathbf{x}\right) = 0.$$

Note that the almost periodic operator $\mathbf{H} : \mathfrak{M} \to \mathfrak{M}$ can not be almost periodic by Bochner [2, 3]. However, the almost periodic by Bochner operator $\mathbf{H} : \mathfrak{M} \to \mathfrak{M}$ be almost periodic by Definition 2.

Let Λ be a bounded subset of the space M. Define the diameter $\operatorname{diam}\Lambda$ of the set Λ by the equality

$$\operatorname{diam}\Lambda = \sup\{\rho_M(x, y) : x, y \in \Lambda\}.$$

Consider the almost periodic difference operator $\mathbf{F} : \mathfrak{M} \to \mathfrak{M}$ defined by the formulae

$$(\mathbf{F}\mathbf{x})(n) = F(n, \mathbf{x}(n), \mathbf{x}(n + m_1), \ldots, \mathbf{x}(n + m_k)), \ \ n \in \mathbb{Z},$$

where $\mathbf{x} \in \mathfrak{M}$, $k \in \mathbb{N}$, $m_1, \ldots, m_k \in \mathbb{Z}$ and $F : \mathbb{Z} \times M^{k+1} \to M$ is operator such that

$$\operatorname{diam} F(\mathbb{Z} \times M_1 \times \ldots \times M_{k+1}) < +\infty$$

for all bounded sets $M_1, \ldots, M_{k+1}$.

Consider the difference equation

$$\mathbf{F}\mathbf{x} = \mathbf{h}, \tag{1.1}$$

where $\mathbf{h} \in \mathfrak{B}$.

The aim of this work is to find conditions under which the bounded solutions of equation (1.1) are almost periodic.

In the study of equation (1.1) will use a functional defined on the set of solutions of an equation of the sets of values which are subsets of compact sets.

2. Functional δ

Fix an arbitrary set $K \in \mathcal{K}$. Let $N(\mathbf{F}, K)$ be the set of all solutions of equation (1.1), each of which $R(\mathbf{x}) \subset \mathcal{K}$ and $\overline{R(\mathbf{x})} \neq K$. Suppose that $N(\mathbf{F}, K) \neq \varnothing$.

Fix an arbitrary element $\mathbf{x}^* \in N(\mathbf{F}, K)$. Let

$$r(\mathbf{x}^*, K) = \sup\left\{\rho_M(x, y) : x \in \overline{R(\mathbf{x}^*)}, y \in K\right\}.$$

Due to the inequality $N(\mathbf{F}, K) \neq \varnothing$

$$r(\mathbf{x}^*, K) > 0.$$

Also fix the arbitrary number $\varepsilon \in [0, r(\mathbf{x}^*, K)]$. We denote by $\Omega(\mathbf{x}^*, K, \varepsilon)$ the set of all elements of $\mathbf{y} \in \mathfrak{M}$, each of which

$$R(\mathbf{y}) \subset K$$

and

$$\rho_{\mathfrak{M}}(\mathbf{y}, \mathbf{x}^*) \geq \varepsilon.$$

Consider the functional

$$\delta(\mathbf{x}^*, K, \varepsilon) = \inf_{\mathbf{y} \in \Omega(\mathbf{x}^*, K, \varepsilon)} \rho_{\mathfrak{M}}(\mathbf{F}\mathbf{y}, \mathbf{F}\mathbf{x}^*). \tag{2.1}$$

First analogous functionals have been proposed by the author in the papers [7,8,10] for the study of nonlinear almost periodic equations

$$x(t+1) = f(t, x(t)), \;\; t \in \mathbb{R},$$
$$\frac{dx(t)}{dt} = f(t, x(t)), \;\; t \in \mathbb{R},$$

and

$$f(t, x(t)) = 0, \;\; t \in \mathbb{R},$$

with continuous operator $f : \mathbb{R} \times E \to E$. Here E is a Banach space. Analogous functional for nonlinear difference equation

$$x(n+1) = g(n, x(n)), \;\; n \in \mathbb{Z},$$

used in [9].

3. Main Result

We give conditions for the existence of almost periodic solutions of equation (1.1), in contrast to the well-known theorem Amerio of almost periodic solutions of nonlinear differential equations (see [1, 4]) do not use the H-class of equation (1.1). In the case of linear differential equations using H-classes of these equations is essential [5, 6].

The main result of this paper reads as follows

Theorem 1. *Let us suppose that* $K \in \mathcal{K}$, $\mathbf{z} \in N(\mathbf{F}, K)$, $\operatorname{diam} R(\mathbf{z}) \neq 0$ *and*

$$\delta(\mathbf{z}, K, \varepsilon) > 0 \tag{3.1}$$

for each $\varepsilon \in (0, r(\mathbf{z}, K))$. *Then solution* $\mathbf{z}$ *of equation* (1.1) *is almost periodic.*

Proof. Assume that the solution $\mathbf{z} \in N(\mathbf{F}, K)$ of the equation (1.1) is not element of the space $\mathfrak{B}$. Then there exists a sequence $(S_{m_p}\mathbf{z})_{p\geq 1}$ such that each subsequence $(S_{k_p}\mathbf{z})_{p\geq 1}$ is divergent. Consequently, for some numbers $p_r \in \mathbb{N}$, $q_r \in \mathbb{N}$, $r \geq 1$, and $\gamma \in (0, \operatorname{diam} R(\mathbf{z}))$

$$\rho_{\mathfrak{M}}(S_{k_{p_r}}\mathbf{z}, S_{k_{q_r}}\mathbf{z}) \geq \gamma, \;\; r \geq 1.$$

Because

$$\rho_{\mathfrak{M}}(\mathbf{z}, S_{-k_{p_r}} S_{k_{q_r}}\mathbf{z}) \geq \gamma, \;\; r \geq 1,$$

and therefore

$$S_{-k_{p_r}} S_{k_{q_r}}\mathbf{z} \in \Omega(\mathbf{z}, K, \gamma), \;\; r \geq 1. \tag{3.2}$$

Based on the inclusion of $\mathbf{h} \in \mathfrak{B}$, without loss of generality of the proof, we can assume that

$$\lim_{r\to\infty} \rho_{\mathfrak{M}}\left(S_{-k_{p_r}}\mathbf{h}, S_{-k_{q_r}}\mathbf{h}\right) = 0. \tag{3.3}$$

Note that $\operatorname{diam} R(\mathbf{z})) \leq r(\mathbf{z}, K))$. Without loss of generality we can assume that the sequence $(S_{k_p}\mathbf{F}S_{-k_p}\mathbf{x})_{p\geq 1}$ converges uniformly on $\mathfrak{D}_K$. Then

$$\lim_{p,q\to\infty} \sup_{\mathbf{x}\in\mathfrak{D}_K} \rho_{\mathfrak{M}}(S_{k_p}\mathbf{F}S_{-k_p}\mathbf{x}, S_{k_q}\mathbf{F}S_{-k_q}\mathbf{x}) = 0. \tag{3.4}$$

We show that

$$\delta(\mathbf{z}, K, \gamma) = 0. \tag{3.5}$$

It is obvious that by (2.1) and (3.4)

$$\delta(\mathbf{z}, K, \gamma) = \inf_{\mathbf{y}\in\Omega(\mathbf{z},K,\gamma)} \rho_{\mathfrak{M}}(\mathbf{F}\mathbf{y}, \mathbf{F}\mathbf{z}) \leq \rho_{\mathfrak{M}}(\mathbf{F}S_{-k_{pr}}S_{k_{qr}}\mathbf{z}, \mathbf{F}\mathbf{z}), \quad r \geq 1. \tag{3.6}$$

Note that

$$\begin{aligned}
&\rho_{\mathfrak{M}}(\mathbf{F}S_{-k_{pr}}S_{k_{qr}}\mathbf{z}, \mathbf{F}\mathbf{z}) = \\
&= \rho_{\mathfrak{M}}\left(S_{-k_{pr}}\left(S_{k_{pr}}\mathbf{F}S_{-k_{pr}}\right)S_{k_{qr}}\mathbf{z}, S_{-k_{qr}}\left(S_{k_{qr}}\mathbf{F}S_{-k_{qr}}\right)S_{k_{qr}}\mathbf{z}\right) \\
&\leq \rho_{\mathfrak{M}}\left(S_{-k_{pr}}\left(S_{k_{pr}}\mathbf{F}S_{-k_{pr}}\right)S_{k_{qr}}\mathbf{z}, S_{-k_{pr}}\left(S_{k_{qr}}\mathbf{F}S_{-k_{qr}}\right)S_{k_{qr}}\mathbf{z}\right) \\
&+ \rho_{\mathfrak{M}}\left(S_{-k_{pr}}\left(S_{k_{qr}}\mathbf{F}S_{-k_{qr}}\right)S_{k_{qr}}\mathbf{z}, S_{-k_{qr}}\left(S_{k_{qr}}\mathbf{F}S_{-k_{qr}}\right)S_{k_{qr}}\mathbf{z}\right) \\
&= \rho_{\mathfrak{M}}\left(\left(S_{k_{pr}}\mathbf{F}S_{-k_{pr}}\right)S_{k_{qr}}\mathbf{z}, \left(S_{k_{qr}}\mathbf{F}S_{-k_{qr}}\right)S_{k_{qr}}\mathbf{z}\right) + \rho_{\mathfrak{M}}\left(S_{-k_{pr}}S_{k_{qr}}\mathbf{h}, S_{-k_{qr}}S_{k_{qr}}\mathbf{h}\right) \\
&\leq \sup_{\mathbf{x}\in\mathfrak{D}_K} \rho_{\mathfrak{M}}\left(S_{k_{pr}}\mathbf{F}S_{-k_{pr}}\mathbf{x}, S_{k_{qr}}\mathbf{F}S_{-k_{qr}}\mathbf{x}\right) + \rho_{\mathfrak{M}}\left(S_{-k_{pr}}\mathbf{h}, S_{-k_{qr}}\mathbf{h}\right), \quad r \geq 1.
\end{aligned}$$

Therefore, based on (3.3), (3.4) and (3.6) the equality (3.5) is true.

This relation contradicts (3.1).

Thus, the assumption that the solution $\mathbf{z} \in N(\mathbf{F}, K)$ of the equation (1.1) is not element of the space $\mathfrak{B}$, is false.

So, the proof is complete. □

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CORRIGENDUM TO "ON THE DIOPHANTINE EQUATION $X^2+7^\alpha\cdot 11^\beta=Y^N$" [MISKOLC MATH. NOTES, VOL. 13 (2012) NO. 2, PP. 515-527.]

GÖKHAN SOYDAN

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Abstract. This note presents some corrections to (Miskolc Math. Notes, Vol. 13 (2012) No. 2, pp. 515-527.)

2010 *Mathematics Subject Classification:* 11D41; 11D61

Keywords: exponential diophantine equations, primitive divisors of Lucas sequences

The author regrets some technical mistakes in the proof of Lemma 3 specifically: In page 524, between the lines 9 and 11 statement that "So $\pm 11^{\beta_1}\equiv 1 \pmod 8$, showing that the sign on the left hand side is positive and β_1 is odd, or the sign on the left hand side is negative and β_1 is even." must be deleted.

It should be written as "So $\pm 11^{\beta_1}\equiv 1 \pmod 8$, showing that the sign on the left hand side is positive and β_1 is even."

In page 524, between the lines 12 and 16 the statement that "Assume first that $\beta_1=2\beta_0+1$ be odd. We get

$$11V^2=5U^4-70U^2+49,$$

where $(U,V)=(u/v,11^{\beta_0}/v^2)$ is a $\{7\}$-integral point on the above elliptic curve. We get that the only such points on the above curve are $(U,V)=(\pm 7,\pm 28)$. This does not lead to solutions of our original equation." must be deleted.

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LIMIT CYCLES AND INVARIANT PARABOLAS FOR AN EXTENDED KUKLES SYSTEM

BÉLA SZABÓ AND IVÁN SZÁNTÓ

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Abstract. A class of polynomial systems of odd degree with limit cycles, invariant parabolas and invariant straight lines, is examined. The limit cycles can be obtain as a bifurcation of a non hyperbolic focus at the origin as Hopf bifurcations. We will also obtain the necessary and sufficient conditions for the critical point at the interior of bounded region to be a center.

2010 *Mathematics Subject Classification:* 92D25; 34C; 58F14; 58F21

Keywords: limit cycle, center, bifurcation

1. INTRODUCTION

Let us consider a real autonomous system of ordinary differential equations on the plane with polynomial nonlinearities.

$$\begin{cases} \dot{x} = P(x,y) = \sum_{i+j=0}^{n} a_{ij} x^i y^j \\ \dot{y} = Q(x,y) = \sum_{i+j=0}^{n} b_{ij} x^i y^j \end{cases}, \quad \text{with } a_{ij},\, b_{ij} \in \mathbb{R} \tag{1.1}$$

Suppose that the origin of (1.1) is a critical point of center-focus type. We are concerned with two closely related questions, both of which are significant elements in work on Hilbert's 16th problem. The first is the number of limit cycles (that is, isolated periodic solutions) which bifurcate from a critical point and the second is the derivation of necessary and sufficient conditions for a critical point to be a center (that is, all orbits in the neighborhood of the critical point are closed).

In order to describe Hilbert's 16th problem more precisely, let S_n be the collection of systems of form (1.1), with P and Q of degree at most n, and let $\pi(P,Q)$ be the number of limit cycles of (1.1). We let (P,Q) denote system (1.1). Define the so-called Hilbert numbers by

$$H_n = \mathrm{Sup}\{\pi(P,Q);\ (P,Q) \in S_n\}\,.$$

The problem consists of estimating H_n in terms of n and obtaining the possible relative configurations of limit cycles.
This is the second part of the 16th problem, which is contained in the famous list of

problems proposed by Hilbert at the International Congress of Mathematicians held in Paris in 1900.

Let us assume that the origin is a critical point of (1.1) and transform the system to canonical form

$$\dot{x} = \lambda x + y + p(x,y)\ , \qquad \dot{y} = -x + \lambda y + q(x,y)\ ,$$

where p, q are polynomials without linear terms. For the origin to be a center we must have $\lambda = 0$. If $\lambda = 0$ and the origin is not a center, it is said to be a *fine focus.*

The necessary conditions for a center are obtained by computing the *focal values.* These are polynomials in the coefficients arising in p and q and are defined as follows. There is a function V, analytic in a neighbourhood of the origin, such that the rate of change along orbits, $\dot{V}$, is of the form

$$\eta_2 r^2 + \eta_4 r^4 + \cdots\ , \quad \text{where } r^2 = x^2 + y^2\ .$$

The focal values are the η_{2k}, and the origin is a center if and only if they are all zero. However, since they are polynomials, the ideal they generate has a finite basis, so there is M such that $\eta_{2\ell} = 0$, for $\ell \leq M$, implies that $\eta_{2\ell} = 0$ for all ℓ. The value of M is not known *a priori*, so it is not clear in advance how many focal values should be calculated.

The software Mathematica [4] is used to calculate the first few focal values. These are then 'reduced' in the sense that each is computed modulo the ideal generated by the previous ones: that is, the relations $\eta_2 = \eta_4 = \cdots = \eta_{2l} = 0$ are used to eliminate some of the variables in η_{2l+2}. The reduced focal value η_{2l+2}, with strictly positive factors removed, is known as the *Liapunov quantity* L_l. Common factors of the reduced focal values are removed and the computation proceeds until it can be shown that the remaining expressions cannot be zero simultaneously. The circumstances under which the calculated focal values are zero yield the necessary center conditions. The origin is a fine focus of *order* l if $L_i = 0$ for $i = 0, 1, ..., l-1$ and $L_l \neq 0$. At most l limit cycles can bifurcate out of a fine focus of order l; these are called *small amplitude* limit cycles.
Various methods used to prove the sufficiency of the possible center conditions. Of particular interest to us in this paper is the symmetry.

The limit cycles problem and the center problem is concentrated on specific classes of systems, those may be systems of a given degree (quadratic systems or a cubic systems, for example), or they may be of a particular form; for instance, much has been written on Liénard systems, that is, systems of the form $\dot{x} = y - F(x)$, $\dot{y} = -g(x)$ and a Kukles systems - that is, a systems of the form

$$\dot{x} = -y\ , \qquad \dot{y} = x + \lambda y + g(x,y)\ .$$

Recall that systems of this type were studied, for the first time, by Isaak Solomonovich Kukles [2], who studied a linear center with cubic nonhomogeneous nonlinearities.
A closely related problem is the derivation of conditions under which the system is

integrable and other questions of interest relate to the existence of algebraic invariant curves.

Let $n = \max(\partial P, \partial Q)$, where the symbol ∂ denotes 'degree of'. A function C is said to be *invariant* with respect to (1.1) if there is a polynomial L called cofactor, with $\partial L < m$, such that $\dot{C} = CL$. Here $\dot{C} = C_x P + C_y Q$ is the rate of change of C along orbits. It is well known that the existence of invariant curves has significant repercussions on the possible phase-portraits of the system. For example, in the case of quadratic systems ($n = 2$) the existence of an invariant ellipse or hyperbola implies that there are no limit cycles other than, possibly, the ellipse itself.

Recently in [1], consider a Kukles extended system of degree three, namely, a system of the form

$$\begin{cases} \dot{x} = P(x,y) = \lambda x + y + kxy \\ \dot{y} = Q(x,y) = -x + \lambda y + \sum\limits_{i+j=2}^{3} a_{ij} x^i y^j \end{cases}, \quad \text{with } k,\ a_{ij} \in \mathbb{R}$$

and the authors obtained some conditions for the existence of centre and limit cycles.

In this paper we consider another class of extended Kukles system of degree $2n+5$, $n \geq 1$ with an invariant non–degenerate conic and two invariant straight line. For these kind of second-order differential systems, we show that for certain values of the parameters, the invariant conic and the invariant straight lines, can coexist with at least n small amplitude limit cycles which are constructed by Hopf bifurcation. We conclude with some numerical simulations of the results

2. Main results

Let us consider the extended Kukles of degree $2n+5$

$$X_\mu : \begin{cases} \dot{x} = & -yL1(x)L2(x) \\ \dot{y} = & \frac{1}{k^2}\big[2xL1^2(x)L2^2(x) \\ & +P1(x,y)P2(x,y)\left(k^2x - 2x + k^2Ly + \sum_{i=1}^{n} a_{2i+1} y^{2i+1}\right)\big] \end{cases} \tag{2.1}$$

where $\mu = (k, \lambda, a_3, ..., a_{2n+1}) \in \mathbb{R}^{n+2}$, $k \neq 0$ and

$$P1(x,y) = -1 + k^2x^2 - k^2y\,, \qquad L1(x) = 1 - kx\,,$$
$$P2(x,y) = -1 + k^2x^2 + k^2y\,, \qquad L2(x) = 1 + kx.$$

Lemma 1. *For all $n \geq 1$ and for all $\mu = (k, \lambda, a_3, ..., a_{2n+1}) \in \mathbb{R}^{n+2}, k \neq 0$ the straight lines $L1(x) = 1 - kx$, $L2(x) = 1 + kx$ and the parabolas $P1(x,y) = -1 + k^2x^2 - k^2y$, $P2(x,y) = -1 + k^2x^2 + k^2y$ are invariant algebraic curves of* (2.1).

Proof. It is easy to verify that

$$\dot{L1} = L1_x \dot{x} + L1_y \dot{y} = kyL1L(x,y)\,,$$

where the cofactor is $L(x,y) = ky(1+kx)$,

$$\dot{L2} = L2_x \dot{x} + L2_y \dot{y} = -kyL2L(x,y) ,$$

where the cofactor is $L(x,y) = -ky(1-kx)$,

$$\dot{P1} = P1_x \dot{x} + P1_y \dot{y} = P1(x,y)L(x,y) ,$$

where the cofactor is $L(x,y) = -x + k^2x^3 - Ly - 2xy + k^2xy + k^2Lx^2y + k^2Ly^2 + \frac{1}{k^2}P1(x,y)\left(\sum_{i=1}^{n} a_{2i+1}y^{2i+1}\right)$ and

$$\dot{P2} = P2_x \dot{x} + P2_y \dot{y} = P2(x,y)L(x,y) ,$$

where the cofactor is $L(x,y) = -x + k^2x^3 - Ly + 2xy - k^2xy + k^2Lx^2y - k^2Ly^2 - \frac{1}{k^2}P2(x,y)\left(\sum_{i=1}^{n} a_{2i+1}y^{2i+1}\right)$.

Then straight lines $L1(x)$, $L2(x)$ and the parabolas $P1(x,y)$ and $P2(x,y)$ are invariant algebraic curves of (2.1). □

Theorem 1. *If $a_{2j+1} = 0$, $\forall j = 1,2,...,n-1$ and $a_{2n+1} \neq 0$, system (2.1) at the origin has a weak focus of order n.*

Proof. The linear part of (2.1) at the singularity $(0,0)$ is

$$DX_\mu(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & \lambda \end{pmatrix}.$$

If $\lambda = 0$, we have $\operatorname{div} X_\mu(0,0) = 0$ and $\det DX_\mu(0,0) = 1$, then the critical point $(0,0)$ is a weak focus.

As $\operatorname{div} X_\mu(0,0) = \lambda$, we have the Liapunov quantities $L_0 = \lambda$, and using Mathematica Software [4], we are able to compute the Liapunov quantities L_l, for $l \geq 1$.

If $\lambda = 0$, $L_0 = 0$ and $L_1 = \dfrac{3a_3}{8k^2}$, if $a_3 = 0$, $L_1 = 0$ and $L_2 = \dfrac{5a_5}{16k^2}$, if $a_5 = 0$, $L_2 = 0$ and $L_3 = \dfrac{35a_7}{128k^2}$ and so on. Then, if $\lambda = a_3 = a_5 = ... = a_{2n-1} = 0$, and $a_{2n+1} \neq 0$, we have $L_0 = L_1 = L_3 = ... = L_{n-1} = 0$, and for $n \geq 1$, $L_n = a_{2n+1}\dfrac{(2n+1)!!}{(2n+2)!!k^2}$. Finally, for all $l \geq n$, $L_l = a_{2l+1}R_l(k)$ where $R_l(k)$ is a polynomial on k, then system (2.1) at the origin has a weak focus of order n. □

Theorem 2. *System (2.1) has a local center at the origin if and only if* $\lambda = a_3 = a_5 = ... = a_{2n+1} = 0$.

Proof. From the linear part at the origin, it is clear that the condition $\lambda = 0$ is necessary for a center. By Theorem 1, the Liapunov quantities are given by $L_0 = \lambda$ and for $n \geq 1$, $L_n = a_{2n+1}\dfrac{(2n+1)!!}{(2n+2)!!k^2}$. As all calculated Liapunov quantities are

zero, this shows that the conditions are necessary.
If $a_{2j+1} = 0$, $j = 1,2,3,...,n$, the system (2.1) is of degree 5 and is given by

$$\begin{cases} \dot{x} = & P(x,y) = -y + k^2x^2y \\ \dot{y} = & Q(x,y) = x(1 - 2k^2x^2 + k^4x^4 + 2k^2y^2 - k^4y^2) \end{cases} \tag{2.2}$$

As the symmetries $P(-x,y) = P(x,y)$ and $Q(-x,y) = -Q(x,y)$ are satisfied, this proves the sufficiency that system (2.2) has a center at the origin. $\square$

In order to illustrate the result stated in Theorem 1, Figure 1 shows a numerical simulation of the system (2.1) of degree five with $k = 1$, $a_3 = 0$ and $\lambda = 0$, which corresponds to the case of center. The simulation was obtained using the Pplane7 Software with MATLAB [3]

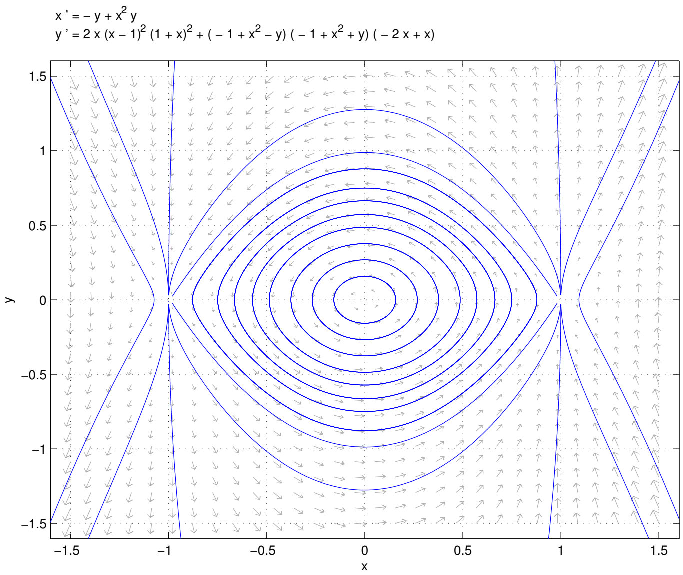

FIGURE 1. Local Center.

Theorem 3. *In the parameters space $\mathbb{R}^{n+2}$, there exists an open set $\mathcal{N} \neq \phi$, such that for all $\mu = (k,\lambda,a_3,...,a_{2n+1}) \in \mathcal{N}$, $k \neq 0$, system (2.1) has at least n small-amplitude limit cycle bounded by the invariant conic.*

Proof. If $\lambda = 0$ and $a_{2j-1} = 0$, $\forall j = 2,3,...,n$ and $a_{2n+1} \neq 0$, by Theorem 1, system (2.1) has at the singularity $(0,0)$ a repelling or attracting weak focus of order n.

Let us consider the case $a_{2n+1} > 0$ and n odd number, that is an attracting weak focus of order n. Taking $a_{2n-1} < 0$, so that $L_n > 0$ and $L_{n-1} < 0$, the origin of (2.1)

is reversed and a hyperbolic repelling small amplitude limit cycle is created (Hopf bifurcation), the limit cycle created persist under new small perturbation. Furthermore the singularity at the origin is an attracting weak focus of order $n-1$.
If we adjust the parameter a_{2n-3}, so that $a_{2n-3} > 0$ small enough and $L_n > 0$, $L_{n-1} < 0$ and $L_{n-2} > 0$, the stability at the origin is reversed and new hyperbolic attracting small amplitude limit cycle is created, that persists under new small perturbation. Furthermore the singularity at the origin is an repelling weak focus of order $n-2$. By following the same process until all Liapunov quantities are non-zero, such that in each step the stability of the singularity is inverted, n hyperbolic small amplitude limit cycles are created and we obtain $n+1$ inequalities which define an a open set $\mathcal{N}$ in the parameter space, given by $\mathcal{N} = \{(k,\lambda,a_3,...,a_{2n+1}) \in \mathbb{R}^{n+2} \mid k \neq 0, L_n > 0, L_{n-1} < 0, L_{n-2} > 0,..., L_2 < 0, L_1 > 0, L_0 < 0\}$. □

In order to illustrate the result stated in Theorem 3, Figure 2 shows a numerical simulation of the system (2.1) with $k = 1$, $a_3 = -1$ and $\lambda = 0.1$, which corresponds to the case of attracting limit cycle. The simulation was obtained using the Pplane7 Software with MATLAB [3].

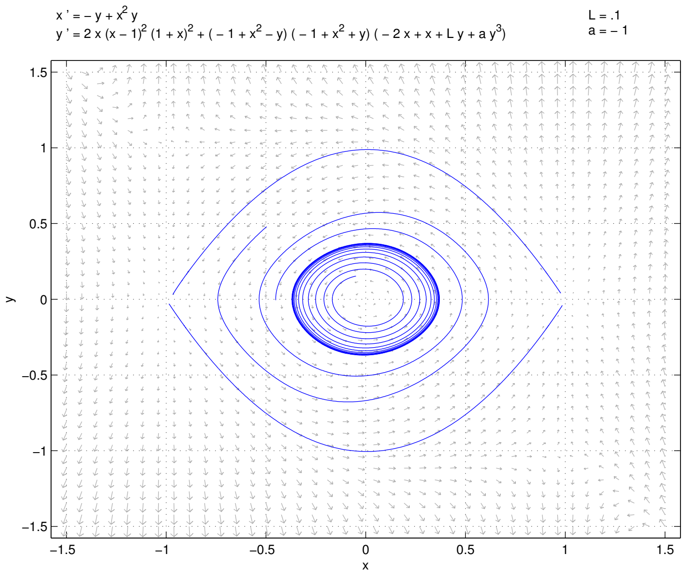

FIGURE 2. Attracting limit cycle ($n = 1$).

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EXACT SOLUTIONS FOR DIFFERENTIAL-ALGEBRAIC EQUATIONS

HECTOR VAZQUEZ-LEAL

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Abstract. This work presents the application of series method to find solutions of differential-algebraic equations systems (DAEs). We present two case studies to show that series method generates approximate solutions for DAEs. The type of tested equations are a linear and a non-linear DAEs of index-3. What is more, we present the post-processing of the series solutions with Laplace-Padé (LP) resummation method as a useful strategy to find exact solutions or to extend the domain of convergence of the power series solutions.

2010 *Mathematics Subject Classification:* 34L30

Keywords: series method, resummation methods, differential-algebraic equations

1. INTRODUCTION

Solving differential equations is an important issue in sciences because many physical phenomena are modelled using such equations. Modern methods like homotopy perturbation method (HPM) [7, 8, 14, 16, 22, 23, 35, 41–43], homotopy analysis method (HAM)[13, 15, 32], variational iteration method (VIM) [1, 5, 20], among others, are powerful tools to approximate nonlinear dynamic problems. Nevertheless, series method [9, 10, 17] is a well known classic procedure from literature that can be applied successfully to solve differential equations. This method establishes that the solution of a differential equation can be expressed as a power series of the independent variable. Therefore, in this work, we apply series method to solve two differential-algebraic equations. Additionally, we present the use of Laplace-Padé (LP) resummation method as an useful strategy to obtain exact solutions or approximations possessing a large domain of convergence.

This paper is organized as follows. In Section 2, we introduce the basic concept of the series method. The concept about Laplace-Padé resummation method is explained in Section 3. The solution of two differential algebraic equations is presented

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in Section 4. In Section 5, numerical simulations and a discussion about the results are provided. Finally, a brief conclusion is given in Section 6.

2. Basic concept of series method

It can be considered that a nonlinear differential equation can be expressed as

$$A(u) - f(t) = 0, \qquad t \in \Omega, \tag{2.1}$$

having as boundary condition

$$B(u, \frac{\partial u}{\partial \eta}) = 0, \qquad t \in \Gamma, \tag{2.2}$$

where A is a general operator, $f(t)$ is a known analytic function, B is a boundary operator, and Γ is the boundary of domain Ω.

The series method establishes that the solution of a differential equation can be written as

$$u = \sum_{i=0}^{\infty} u_i t^i, \tag{2.3}$$

where $u_0, u_1, \ldots$ are unknowns to be determined by series method.

The basic process of series method can be described as:

(1) Equation (2.3) is substituted into (2.1), then we regroup equation in terms of t-powers.
(2) We equate each coefficient of the resulting polynomial to zero.
(3) The boundary conditions of (2.1) are substituted into (2.3) to generate an equation for each boundary condition.
(4) Aforementioned steps generate a nonlinear algebraic equation system (NAEs) in terms of the unknowns of (2.3).
(5) Finally, we solve the NAEs to obtain $u_0, u_1, \ldots$, coefficients.

3. Laplace-Padé resummation method

Several approximate methods provide power series solutions (polynomial). Nevertheless, sometimes, this type of solutions lacks of large domains of convergence. Therefore, Laplace-Padé [3, 6, 11, 18, 21, 26–28, 31, 33] resummation method is used in literature to enlarge the domain of convergence of solutions or inclusive to find exact solutions.

The Laplace-Padé method can be explained as follows:

(1) First, Laplace transformation is applied to power series (2.3).
(2) Next, s is substituted by $1/t$ in the resulting equation.
(3) After that, we convert the transformed series into a meromorphic function by forming its Padé approximant of order $[N/M]$. N and M are arbitrarily chosen, but they should be of smaller value than the order of the power series.

In this step, the Padé approximant extends the domain of the truncated series solution to obtain better accuracy and convergence.

(4) Then t is substituted by $1/s$.

(5) Finally, by using the inverse Laplace s transformation, we obtain the exact or approximate solution.

This process is known as Laplace-Padé series method (LPSM).

4. Case Studies

In this section we will solve two DAE problems in order to depict the LPSM method.

4.1. *Hessenberg index-3 DAE*

Consider the following DAE [34]

$$\begin{aligned} x_1' + x_1 - tx_3 + x_4 &= 0,\\ x_2' - x_1 + x_2 - t^2x_3 + tx_4 &= 0,\\ x_3' - t^3x_1 + t^2x_2 - x_3 &= 0,\\ tx_1 - x_2 + tx_3 - x_4 &= 0,\\ x_1(0) = x_3(0) &= 1,\\ x_2(0) = x_4(0) &= 0, \end{aligned} \tag{4.1}$$

where prime denotes derivative with respect to t.

We suppose that solution for (4.1) has the following fourth order expression

$$\begin{aligned} x_1(t) &= \sum_{i=0}^{4} x_{1i}t^i, & x_2(t) &= \sum_{i=0}^{4} x_{2i}t^i,\\ x_3(t) &= \sum_{i=0}^{4} x_{3i}t^i, & x_4(t) &= \sum_{i=0}^{4} x_{4i}t^i, \end{aligned} \tag{4.2}$$

Substituting (4.2) into (4.1), rearranging and equating terms having the same t-powers, we obtain

$$x_{11} + x_{40} + x_{10} + \left(x_{41} + x_{11} + 2x_{12} - x_{30}\right)t + \cdots = 0.$$

$$x_{21} + x_{20} - x_{10} + \left(x_{40} + 2x_{22} + x_{21} - x_{11}\right)t + \cdots = 0.$$

$$x_{31} - x_{30} + \left(-x_{31} + 2x_{32}\right)t + \cdots = 0.$$

$$-x_{20} - x_{40} + \left(x_{10} - x_{41} - x_{21} + x_{30}\right)t + \cdots = 0. \tag{4.3}$$

Next, equating the coefficients of (4.3) to zero, we obtain the following system of algebraic equations

$$\begin{aligned} t^0 &: x_{11} + x_{40} + x_{10} = 0, \\ t^1 &: x_{41} + x_{11} + 2x_{12} - x_{30} = 0, \\ &\vdots \\ t^3 &: \cdots \end{aligned} \tag{4.4}$$

$$\begin{aligned} t^0 &: x_{21} + x_{20} - x_{10} = 0, \\ t^1 &: x_{40} + 2x_{22} + x_{21} - x_{11} = 0, \\ &\vdots \\ t^3 &: \cdots \end{aligned} \tag{4.5}$$

$$\begin{aligned} t^0 &: x_{31} - x_{30} = 0, \\ t^1 &: -x_{31} + 2x_{32} = 0, \\ &\vdots \\ t^3 &: \cdots \end{aligned} \tag{4.6}$$

$$\begin{aligned} t^0 &: -x_{20} - x_{40} = 0, \\ t^1 &: x_{10} - x_{41} - x_{21} + x_{30} = 0, \\ &\vdots \\ t^4 &: -x_{24} + x_{33} + x_{13} - x_{44} = 0 \end{aligned} \tag{4.7}$$

Now, in order to consider the initial condition of (4.1), we substitute them into (4.2) to obtain

$$\begin{aligned} x_{10} &= 1, & x_{20} &= 0, \\ x_{30} &= 1, & x_{40} &= 0, \end{aligned} \tag{4.8}$$

It is important to notice that, from (4.4)-(4.6), we use only the powers t^i $(i = 0, 1, 2, 3)$, because the rest of the information needed is taken from (4.8). From (4.1), we can observe that there is not an explicit equation for variable x_4. Therefore, from (4.7), we use coefficients of powers t^i $(i = 1,2,3,4)$ to collect enough information to compensate the presence of x_4 in those equations. Furthermore, order zero term of (4.7) possesses redundant information that can be ignored. Hence, for (4.7), we can use powers t^i $(i = 1,2,3,4)$. Then, solving the NAEs composed by (4.4), (4.5), (4.6), (4.7), and (4.8) results the following approximate solution

$$\begin{aligned}
x_1(t) &= 1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4, \\
x_2(t) &= t - t^2 + \frac{1}{2}t^3 - \frac{1}{6}t^4, \\
x_3(t) &= 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4, \\
x_4(t) &= t + t^2 + \frac{1}{2}t^3 + \frac{1}{6}t^4.
\end{aligned} \tag{4.9}$$

Then, Laplace transformation is applied to (4.9) and then $1/t$ is written in place of s. Afterwards, Padé approximant of order $[2/2]$ is applied and $1/s$ is written in place of t for each variable. Finally, by using the inverse Laplace s transformation, we obtain the exact solution for (4.1)

$$\begin{aligned}
x_1(t) &= \exp(-t), \\
x_2(t) &= t\exp(-t), \\
x_3(t) &= \exp(t), \\
x_4(t) &= t\exp(t).
\end{aligned} \tag{4.10}$$

4.2. *Index-three nonlinear differential-algebraic equation system*

Consider the following nonlinear DAE [2]

$$\begin{aligned}
y_1' &= 2y_1y_2z_1z_2, \\
y_2' &= -y_1y_2z_2^2, \\
z_1' &= (y_1y_2 + z_1z_2)u, \\
z_2' &= -y_1y_2^2z_2^2u, \\
y_1y_2^2 &= 1, \\
y_1(0) &= y_2(0) = 1, \\
z_1(0) &= z_2(0) = 1, \\
u(0) &= 1,
\end{aligned} \tag{4.11}$$

where prime denotes derivative with respect to t.

We suppose that solution for (4.11) has the following fourth order expression

$$
\begin{aligned}
y_1(t) &= \sum_{i=0}^{4} y_{1i} t^i, & y_2(t) &= \sum_{i=0}^{4} y_{2i} t^i,\\
z_1(t) &= \sum_{i=0}^{4} z_{1i} t^i, & z_2(t) &= \sum_{i=0}^{4} z_{2i} t^i,\\
u(t) &= \sum_{i=0}^{4} u_i t^i. & & \qquad (4.12)
\end{aligned}
$$

Substituting (4.12) into (4.11), rearranging and equating terms having the same t-powers, we obtain

$$
\begin{aligned}
&y_{11} - 2y_{10}y_{20}z_{10}z_{20} + \Big(-2y_{10}y_{20}z_{11}z_{20} + 2y_{12} - 2y_{10}y_{21}z_{10}z_{20}\\
&-2y_{11}y_{20}z_{10}z_{20} - 2y_{10}y_{20}z_{10}z_{21}\Big)t + \cdots = 0.\\
&y_{21} + y_{10}y_{20}z_{20}^2 + \Big(y_{10}y_{21}z_{20}^2 + 2y_22 + 2y_{10}y_{20}z_{20}z_{21} + y_{11}y_{20}z_{20}^2\Big)t + \cdots = 0.\\
&z_{11} - u_0y_{10}y_{20} - u_0z_{10}z_{20} + \Big(-u_0y_{10}y_{21} - u_1z_{10}z_{20}\\
&-u_0y_{11}y_{20} - u_0z_{10}z_{21} - u_0z_{11}z_{20} - u_1y_{10}y_{20} + 2z_{12}\Big)t + \cdots = 0.\\
&y_{10}y_{20}^2z_{20}^2u_0 + z_{21} + \Big(2z_{22} + 2y_{10}y_{20}^2z_{20}z_{21}u_0\\
&+ y_{11}y_{20}^2z_{20}^2u_0 + y_{10}y_{20}^2z_{20}^2u_1 + 2y_{10}y_{20}y_{21}z_{20}^2u_0\Big)t + \cdots = 0.\\
&-1 + y_{10}y_{20}^2 + \Big(y_{11}y_{20}^2 + 2y_{10}y_{20}y_{21}\Big)t + \cdots = 0. \qquad (4.13)
\end{aligned}
$$

Next, equating the coefficients of (4.13) to zero, we obtain the following system of nonlinear algebraic equations

$t^0 : y_{11} - 2y_{10}y_{20}z_{10}z_{20} = 0,$

$t^1 : -2y_{10}y_{20}z_{11}z_{20} + 2y_{12} - 2y_{10}y_{21}z_{10}z_{20} - 2y_{11}y_{20}z_{10}z_{20}$

$$
\begin{aligned}
&\quad -2y_{10}y_{20}z_{10}z_{21} = 0,\\
&\ \vdots\\
&t^3 : \cdots
\end{aligned}
\tag{4.14}
$$

$$
\begin{aligned}
&t^0 : y_{21} + y_{10}y_{20}z_{20}^2 = 0,\\
&t^1 : y_{10}y_{21}z_{20}^2 + 2y_22 + 2y_{10}y_{20}z_{20}z_{21} + y_{11}y_{20}z_{20}^2 = 0,\\
&\ \vdots\\
&t^3 : \cdots
\end{aligned}
\tag{4.15}
$$

$$
\begin{aligned}
&t^0 : z_{11} - u_0y_{10}y_{20} - u_0z_{10}z_{20} = 0,\\
&t^1 : -u_0y_{10}y_{21} - u_1z_{10}z_{20} - u_0y_{11}y_{20} - u_0z_{10}z_{21} - u_0z_{11}z_{20}\\
&\quad -u_1y_{10}y_{20} + 2z_{12} = 0,\\
&\ \vdots\\
&t^4 : \cdots
\end{aligned}
\tag{4.16}
$$

$$
\begin{aligned}
&t^0 : y_{10}y_{20}^2z_{20}^2u_0 + z_{21} = 0,\\
&t^1 : 2z_{22} + 2y_{10}y_{20}^2z_{20}z_{21}u_0 + y_{11}y_{20}^2z_{20}^2u_0 + y_{10}y_{20}^2z_{20}^2u_1\\
&\quad + 2y_{10}y_{20}y_{21}z_{20}^2u_0 = 0,\\
&\ \vdots\\
&t^4 : \cdots
\end{aligned}
\tag{4.17}
$$

$$
\begin{aligned}
&t^0 : -1 + y_{10}y_{20}^2 = 0,\\
&t^1 : y_{11}y_{20}^2 + 2y_{10}y_{20}y_{21} = 0,\\
&t^2 : 2y_{10}y_{20}y_{22} + y_{10}y_{21}^2 + 2y_{11}y_{20}y_{21} + y_{12}y_{20}^2 = 0,\\
&t^3 : y_{11}y_{21}^2 + 2y_{11}y_{20}y_{22} + 2y_{10}y_{20}y_{23} + 2y_{10}y_{21}y_{22} + 2y_{12}y_{20}y_{21}\\
&\quad + y_{13}y_{20}^2 = 0,\\
&t^4 : y_14y_{20}^2 + 2y_{11}y_{20}y_{23} + 2y_{13}y_{20}y_{21} + y_{12}y_{21}^2 + 2y_{12}y_{20}y_{22} + 2y_{10}y_{21}y_{23}\\
&\quad + y_{10}y_{22}^2 + 2y_{10}y_{20}y_{24} + 2y_{11}y_{21}y_{22} = 0.
\end{aligned}
\tag{4.18}
$$

Now, in order to consider the initial condition of (4.11), we substitute them into (4.12) to obtain

$$\begin{aligned} y_{10} &= 1, & y_{20} &= 1, \\ z_{10} &= 1, & z_{20} &= 1, \\ u_0 &= 1. \end{aligned} \tag{4.19}$$

It is important to notice that, from (4.14) and (4.15), we use only the powers t^i $(i = 0, 1, 2, 3)$, because the rest of the information needed is taken from (4.19). From (4.11), we can observe that there is not an explicit equation for variable u. Instead, u is implicitly included in equations for z_1' and z_2'. Therefore, from (4.16) and (4.17), we use coefficients of powers t^i $(i = 0, 1, 2, 3, 4)$ to collect enough information to compensate the presence of u in those equations. Furthermore, powers t^i $(i = 0, 1, 2)$ of (4.18) possesses redundant information that can be ignored. Hence, for (4.18), we can use only powers t^3 and t^4. Then, solving the NAEs composed by (4.14), (4.15), (4.16), (4.17), (4.18), and (4.19) results the following approximate solution

$$\begin{aligned} y_1(t) &= 1 + 2t + 2t^2 + \frac{4}{3}t^3 + \frac{2}{3}t^4, \\ y_2(t) &= 1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4, \\ z_1(t) &= 1 + 2t + 2t^2 + \frac{4}{3}t^3 + \frac{101}{174}t^4, \\ z_2(t) &= 1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{59}{696}t^4, \\ u(t) &= 1 + t + \frac{1}{2}t^2 - \frac{1}{174}t^3 - \frac{25}{58}t^4. \end{aligned} \tag{4.20}$$

Then, Laplace transformation is applied to (4.20) and then $1/t$ is written in place of s. Afterwards, Padé approximant of order $[2/2]$ is applied and $1/s$ is written in place of t for each variable. Finally, by using the inverse Laplace s transformation, we obtain the exact solution for (4.11)

$$\begin{aligned} y_1(t) &= \exp(2t), \\ y_2(t) &= \exp(-t), \\ z_1(t) &= \exp(2t), \\ z_2(t) &= \exp(-t), \\ u(t) &= \exp(t). \end{aligned} \tag{4.21}$$

5. Numerical Simulation and Discussion

Using LPSM method, we obtained the exact solution of two DAEs problems, one linear and other one nonlinear. It should be noticed that the high complexity of both problems was effectively handled by LPSM method due to the malleability of series method. Moreover, by inspection, redundant information in (4.7) and (4.18) was detected and neglected. Therefore, we obtained the series solutions (4.9) and (4.20). Next using the LP resummation method we obtained the exact solutions (4.10) and (4.21) for both case studies. The differential-algebraic problems are of relevance on several fields of sciences including microelectronics and chemistry. What is more, there is no any standard analytical method to solve this type of equations, converting the LPSM method into an attractive tool to solve DAEs problems. Finally, if the DAE problem do not possesses exact solution then the LP resummation method will provide a larger domain of convergence as reported in literature [3, 6, 11, 18, 21, 26–28, 31, 33].

On one side, semi-analytic methods like: homotopy perturbation method [7, 8, 14, 16, 22, 23, 35, 41–43], homotopy analysis method[13, 15, 32], variational iteration method [1, 5, 20], among others [19], need an initial approximation for the sought solutions and the calculus of one or several adjustment parameters. If initial approximation is properly chosen the results can be highly accurate, nonetheless, no general method is available to choose such initial approximation. This issue motivates the use of adjustment parameters obtained by minimizing the least-squares error respect to the numerical solution. On the other side, series method or LPSM method do not need any trial equation as requirement for the method. Besides, series method obtain its coefficients using a straight forward procedure. Furthermore, at least for lower orders of approximation, the solution can be easily obtained using "solve" or "fsolve" commands of MAPLE or equivalent routines in Mathematica or MATLAB softwares. If high order approximations are required we propose the use of homotopy continuation methods [4, 12, 24, 25, 29, 30, 36–40, 44–46] by its well know ability to solve highly nonlinear equations.

6. CONCLUSIONS

This work presented LPSM method as a combination of the classic series method and a resummation method based on the Laplace and Padé transforms. By solving two problems, we presented the LPSM as a useful tool with high potential to solve linear/nonlinear differential-algebraic equations. The proposed method possesses a straightforward procedure, suitable for engineers and physicists. Finally, further research should be performed to solve other kind of related problems like fractional/partial nonlinear differential-algebraic equations, among others.

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A NOTE ON RAZUMIKHIN THEOREMS IN UNIFORM ULTIMATE BOUNDEDNESS

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Abstract. Uniform boundedness and uniform ultimate boundedness of solutions of retarded functional differential equations are studied by Liapunov functions. The obtained result in this work improves Razumikhin theorems on uniform ultimate boundedness. Some examples are given to illustrate the advantage of the obtained result.

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1. INTRODUCTION

Retarded functional differential equations (RFDEs) are a general type of equations and they include ordinary differential equations and differential difference equations [4]. Significance of boundedness (BD) and ultimate boundedness (UB) of partial solutions of RFDEs is widely known. For some problems, one may be interested only in some variables, and it often is very difficult to solve once for all the problems of whole variables about some complex systems. On the other hand, the BD and the UB of whole variables can be obtained by the BD and the UB of partial variables [4, 6, 7].

In fact, the BD and the UB of partial solutions were studied for ordinary differential equations (ODEs) and difference equations. For some results in the area see, for example, [2, 5–7, 9–11, 13–19] and the references cited therein.

On the results of ODEs, in [5] the authors obtained two Lyapunov-like theorems for the BD and the UB of partial solutions of nonlinear dynamical system

$$\begin{aligned}\dot{x}_1(t) &= f_1(x_1(t),\, x_2(t)),\\ \dot{x}_2(t) &= f_2(x_1(t),\, x_2(t)),\end{aligned}$$

where $x_1 \in D$, $D \subseteq \mathbb{R}^{n_1}$ is an open set, $x_2 \in \mathbb{R}^{n_2}$, $f_1 : D \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$, $f_2 : D \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ (see Theorem 3.1 and Theorem 3.2). In [6] the author obtained four

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results (sufficient conditions) for the BD and the UB (dissipation) of partial solutions of the nonlinear autonomous system

$$\begin{aligned}\tfrac{dy}{dt} &= col\left(\sum_{j=1}^{n} f_{1\,j}(x_j),\cdots,\sum_{j=1}^{n} f_{m\,j}(x_j)\right),\\ \tfrac{dz}{dt} &= col\left(\sum_{j=1}^{n} f_{m+1\,j}(x_j),\cdots,\sum_{j=1}^{n} f_{n\,j}(x_j)\right)\end{aligned}$$

(see Theorem 1 – Theorem 4), where $y = col(x_1,\cdots,x_m)$, $z = col(x_{m+1},\cdots,x_n)$; $f_{i\,j}(x_j) \in C(-\infty,+\infty)$. In [7] the authors obtained four results (sufficient conditions) for the uniform ultimate boundedness (uniform dissipation) of partial solutions of the nonlinear nonautonomous system

$$\begin{aligned}\tfrac{dy}{dt} &= col\left(\sum_{j=1}^{n} f_{1\,j}(t,x_j),\cdots,\sum_{j=1}^{n} f_{m\,j}(t,x_j)\right),\\ \tfrac{dz}{dt} &= col\left(\sum_{j=1}^{n} f_{m+1\,j}(t,x_j),\cdots,\sum_{j=1}^{n} f_{n\,j}(t,x_j)\right)\end{aligned}$$

(see Theorem 1 – Theorem 4), where $f_{i\,j}(t,x_j) \in C([t_0,+\infty)\times\mathbb{R},\mathbb{R})$. In monograph [8] the author obtained some results (sufficient and necessary conditions, and sufficient conditions) for the BD of partial solutions of the nonlinear ordinary differential equation

$$\dot{x}(t) = f(t,x(t))$$

(see Theorem 8.38 and Theorem 8.39), where $f \in C(I\times\mathbb{R}^n,\mathbb{R}^n)$.

On the results of the BD and the UB of partial solutions of difference equations, in [18] the authors obtained a result (sufficient conditions) for the uniform ultimate boundedness (UUB) of partial solutions of the large-scale discrete system

$$x_i(\tau+1) = f_i[\tau, x_i(\tau)] + g_i[\tau, x_1(\tau),\cdots,x_m(\tau)] \ \ (i = 1,2,\cdots,m),$$

where $\tau \in I = \{\tau_0 + k : k = 0,1,2,\cdots\}, x_i \in \mathbb{R}^{n_i}$, $f_i : I\times\mathbb{R}^{n_i} \to \mathbb{R}^{n_i}, g_i : I\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_m} \to \mathbb{R}^{n_i}$, $\sum_{i=1}^{m} n_i = n$ (see Theorem 3). In [15] the authors gave some results (sufficient conditions) for the BD and the UB of partial solutions of discrete-time nonlinear dynamical system

$$\begin{aligned}x_1(k+1) &= f_1(x_1(k),\ x_2(k)),\\ x_2(k+1) &= f_2(x_1(k),\ x_2(k)),\end{aligned}$$

where $k \in \{0,1,2,\cdots\}, x_1 \in D,\ D \subseteq \mathbb{R}^{n_1}$ is an open set, $x_2 \in \mathbb{R}^{n_2}, f_1 : D\times\mathbb{R}^{n_2} \to \mathbb{R}^{n_1}, f_2 : D\times\mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ (see Theorem 3.2 and Theorem 3.3). In [9] the author obtained some results for the BD and the UB of partial solutions of the difference equation

$$z(\tau+1) = f(\tau, z(\tau))$$

(see Theorem 1-5), where $z \in \mathbb{R}^n$, $\tau \in I = \{0, 1, 2, \cdots\}$, $f : I \times \mathbb{R}^n \to \mathbb{R}^n$.

On the BD and the UB of partial solutions of ODEs and difference equations, one obtained quite a few results. Nonetheless, the results for uniform BD and UUB of partial solutions of RFDEs are still very few [3, 4]. In this paper, using Liapunov function (Liapunov functions are simpler than Liapunov functionals), a result of uniform BD and UUB of the partial coordinates or all the coordinates of the solutions of RFDEs is given. Our result improves the well-known Razumikhin theorems on UUB [4]. The result is easier to apply; its application area is widened. Moreover, some examples are given to illustrate the application and advantage of the obtained result at the end.

2. Preliminaries

Suppose $r \geq 0$ is a given real number, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R}^n$ is an n-dimensional linear vector space over the reals with norm $|\cdot|$ (in this paper, if $x \in \mathbb{R}^n$ is a column vector, then $|x|$ denotes the Euclidean length of x: $|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$), $C = C([-r, 0], \mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[-r, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. We designate the norm of an element $\phi = (\phi_1, \phi_2, \cdots, \phi_n)^T$ in C by $|\phi| = \sup_{-r\leq\theta\leq 0} |\phi(\theta)|$. If

$$\sigma \in \mathbb{R}, A \geq 0 \quad \text{and} \quad x \in C([\sigma - r, \sigma + A], \mathbb{R}^n),$$

then for any $t \in [\sigma, \sigma + A]$, we let $x_t \in C$ be defined by $x_t(\theta) = x(t+\theta)$, $\theta \in [-r, 0]$.

Suppose $f : \mathbb{R} \times C \to \mathbb{R}^n$ is continuous and consider the retarded functional differential equation [1,4]

$$\dot{x}(t) = f(t, x_t). \tag{2.1}$$

We will assume that there is a unique solution $x(t, t_0, \phi)$ $(x(t_0, \phi)(t))$ of the Equation (2.1) through $(t_0, \phi) \in \mathbb{R} \times C$. Let

$$\begin{aligned} &x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n, \phi = (\phi_1, \phi_2, \cdots, \phi_n)^T \in C, \\ &x_{i\sim j} = (x_i, x_{i+1}, \cdots, x_j)^T \in \mathbb{R}^{j+1-i} \quad (1 \leq i \leq j \leq n), \\ &x(t) = x(t, t_0, \phi), \ x_{i\sim j}(t) = x_{i\sim j}(t, t_0, \phi), \end{aligned}$$

$$\begin{aligned} &y(t) = x_{1\sim m}(t), \ z(t) = x_{m+1\sim n}(t); \\ &C_{i\sim j} = C([-r, 0], \mathbb{R}^{j+1-i}), \phi_{i\sim j} = (\phi_i, \phi_{i+1}, \cdots, \phi_j)^T \in C_{i\sim j}. \end{aligned}$$

If $V:\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^{n-m}\to\mathbb{R}$ is a continuous function, then $\dot V(t,\phi_{1\sim m}(0),\phi_{m+1\sim n}(0))$, the derivative of V along the solutions of Equation (2.1) is defined to be

$$\begin{aligned}&\dot V(t,\phi_{1\sim m}(0),\phi_{m+1\sim n}(0)) =\\ &\quad\overline{\lim_{h\to 0^+}}\frac{1}{h}[V(t+h,x_{1\sim m}(t,\phi)(t+h),x_{m+1\sim n}(t,\phi)(t+h))-\\ &\qquad\qquad V(t,\phi_{1\sim m}(0),\phi_{m+1\sim n}(0))].\end{aligned}$$

In this paper, suppose that if $(x_{1\sim m},x_{m+1\sim n})^T\neq 0$, then $V(t,x_{1\sim m},x_{m+1\sim n})>0$ for all $t\in\mathbb{R}$.

Definition 1. The partial solutions $x_{1\sim m}(t,t_0,\phi)$ $(1\le m\le n)$ of the Equation (2.1) are uniformly bounded if , for any $B_1>0$, there is a $B_2=B_2(B_1)>0$ such that, for all $t_0\in\mathbb{R}$, $\phi\in C$, and $|\phi|\le B_1$, we have

$$|x_{1\sim m}(t,t_0,\phi)|\le B_2(B_1)$$

for all $t\ge t_0$.

Definition 2. The partial solutions $x_{1\sim m}(t,t_0,\phi)$ of the Equation (2.1) are uniformly ultimately bounded if there is a $B>0$ such that, for any $B_3>0$, there is a constant $T(B_3)>0$ such that

$$|x_{1\sim m}(t,t_0,\phi)|\le B$$

for $t\ge t_0+T(B_3)$ for all $t_0\in\mathbb{R}$, $\phi\in C,|\phi|\le B_3$.

3. Main result

Without loss of generality, we suppose $r>0$. Suppose $f:\mathbb{R}\times C\to\mathbb{R}^n$ takes $\mathbb{R}\times$ (bounded sets of C) into bounded sets of $\mathbb{R}^n$. Suppose $u,v,w,q:\mathbb{R}^+\to\mathbb{R}^+$ are continuous, nondecreasing functions, $u(s),v(s),w(s)$ positive for $s>0$, $u(0)=v(0)=w(0)=q(0)=0$, $u(s)\to\infty$ as $s\to\infty$. Suppose there is a constant $c>0$ such that $\dot q(s)\ge c\ for\ s\ge 0$.

Theorem 1. *Let the following conditions hold.*

(1) *There are positive integers m and k with $1\le m\le k\le n$. There is a continuous function $V:\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^{n-m}\to\mathbb{R}$ such that*

$$u(|x_{1\sim m}|)\le V(t,x_{1\sim m},x_{m+1\sim n})\le v(|x_{1\sim k}|)\tag{3.1}$$

for $t\in\mathbb{R},x\in\mathbb{R}^n$.

(2) *There is a continuous, nondecreasing function $p:\mathbb{R}^+\to\mathbb{R}^+$ with*

$$p(s)>q(s)\ for\ s>0.\tag{3.2}$$

(3) *Let* $x_{m+1\sim n} \in \mathbb{R}^{n-m}$ *be a vector, and let*

$$x(t) = x_{1\sim n}(t) = (y^T(t), z^T(t))^T = (x^T_{1\sim m}(t), x^T_{m+1\sim n}(t))^T$$

be the solution of the Equation (2.1). *If there is a constant* $H > 0$ *and if* $|x_{1\sim m}(t)| \geq H$ *for all* $t \geq t_0$ *and*

$$q(V(s, y(s), x_{m+1\sim n})) < p(V(t, y(t), x_{m+1\sim n}))$$

for all $t \geq t_0,\ s \in [t-r, t]$, *then*

$$\dot{V}_{(2.1)}(t, y(t), z(t)) \leq -w(|y(t)|) \tag{3.3}$$

for all $t \geq t_0$.

Under these conditions, the coordinates $x_{1\sim m}(t, t_0, \phi)$ *of the solutions of the Equation* (2.1) *are uniformly bounded and uniformly ultimately bounded.*

Remark 1. The well-known Razumikhin theorems on UUB may be deduced from the above Theorem 1 (the function $q(s) \equiv s$, $m = k = n$).

4. Proof of Theorem 1

In order to prove Theorem 1 in this paper, we need three lemmas.

Although $|x_{1\sim n}(t)| \geq H$ is assumed in Lemma 2 and Lemma 3 below, this paper is interested only in the issues of the uniform boundedness and the uniform ultimate boundedness of the partial solutions $x_{1\sim m}(t)$ ($m \leq n$). Therefore, condition $|x_{1\sim m}(t)| \geq H$ is assumed in Theorem 1. In fact, if $|x_{1\sim m}(t)| \geq H$, then $|x_{1\sim n}(t)| \geq H$ ($|x_{1\sim n}(t)| \geq |x_{1\sim m}(t)| \geq H$). Consequently, the Theorem 1 in this paper can be proved by Lemma 3 below (see page 11).

Let $\mathbb{R}_0 = [t_0, \infty)$. If

$$\overline{V}(t, y_t, z_t) = \sup_{\theta \in [-r, 0]} V(t+\theta, y(t+\theta), z(t)) \tag{4.1}$$

for $(t, x(t+\theta)) \in \mathbb{R}_0 \times C$, $\theta \in [-r, 0]$ ($x(t+\theta) = (y_t^T, z_t^T)^T \in C$), then there is a $\theta_0 = \theta_0(t)$ in $[-r, 0]$ such that

$$\overline{V}(t, y_t, z_t) = V(t+\theta_0(t), y(t+\theta_0(t)), z(t)) \tag{4.2}$$

and either $\theta_0(t) = 0$ or $\theta_0(t) < 0$ and

$$V(t+\theta, y(t+\theta), z(t)) < V(t+\theta_0(t), y(t+\theta_0(t)), z(t)) \tag{4.3}$$

if $\theta_0(t) < \theta \leq 0$.

Lemma 1. *Let $(y^T(t), z^T(t))^T$ be the solution of Equation (2.1). If $\theta_0 = \theta_0(t) < 0$ for some $t \in \mathbb{R}_0$, then*

$$\dot{\overline{V}}(t, y_t, z_t) = 0. \tag{4.4}$$

Proof. If

$$V(t+h+\theta, y(t+h+\theta), z(t+h)) > V(t+\theta_0, y(t+\theta_0), z(t))$$

for $\theta \in (\theta_0, 0]$, and sufficiently small $h > 0$, then we would have

$$\lim_{h\to 0+} V(t+h+\theta, y(t+h+\theta), z(t+h)) \geq \lim_{h\to 0+} V(t+\theta_0, y(t+\theta_0), z(t)),$$

and using the facts that V, $y(t+h+\theta)$, and $z(t+h)$ are continuous, we get

$$V(t+\theta, y(t+\theta), z(t)) \geq V(t+\theta_0, y(t+\theta_0), z(t))$$

for $\theta \in (\theta_0, 0]$, which contradicts the (4.3). So

$$V(t+h+\theta, y(t+h+\theta), z(t+h)) \leq V(t+\theta_0, y(t+\theta_0), z(t)) \tag{4.5}$$

for $\theta \in (\theta_0, 0]$, and sufficiently small $h > 0$. From this, we have

$$\overline{V}(t+h, y_{t+h}, z_{t+h}) = \overline{V}(t, y_t, z_t) \tag{4.6}$$

for sufficiently small $h > 0$ [4]. Therefore, $\dot{\overline{V}}(t, y_t, z_t) = 0$. The proof of the Lemma 1 is therefore complete. □

Lemma 2. *Let the following conditions hold.*

(1) *There is a continuous function $V : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}$ such that*

$$u(|x_{1\sim m}|) \leq V(t, x_{1\sim m}, x_{m+1\sim n}) \leq v(|x_{1\sim k}|) \quad (1 \leq m \leq k \leq n) \tag{4.7}$$

for $t \in \mathbb{R}$, $x \in \mathbb{R}^n$.

(2) *Let $x(t) = x_{1\sim n}(t) = (y^T(t), z^T(t))^T = (x^T_{1\sim m}(t), x^T_{m+1\sim n}(t))^T$ be the solution of the Equation (2.1). If $|x_{1\sim n}(t)| \geq H$ for all $t \geq t_0$ and*

$$V(t+\theta, y(t+\theta), z(t)) \leq V(t, y(t), z(t))$$

for all $t \geq t_0$, $\theta \in [-r, 0]$, then

$$\dot{V}_{(2.1)}(t, y(t), z(t)) \leq 0 \tag{4.8}$$

for all $t \geq t_0$.

Under these conditions, the coordinates $x_{1\sim m}(t, t_0, \phi)$ of the solutions of the Equation (2.1) are uniformly bounded .

Proof. Let $B_1 \geq H$ be given, $t_0 \in \mathbb{R}$, $\phi \in C$, and $|\phi| \leq B_1$. Choose $B_2 > B_1$ such that $u(B_2) \geq v(B_1)$. Consider any solution $x(t)$ of (2.1). Using (4.2), (4.7) and our choice of $u(B_2)$, we get

$$\begin{aligned}&\overline{V}(t_0, y_{t_0}, z_{t_0}) = V(t_0+\theta_0, y(t_0+\theta_0), z(t_0)) \leq \\ &v(|(y^T(t_0+\theta_0), x^T_{m+1\sim k}(t_0))^T|) \leq v(|\phi|) \leq v(B_1) \leq u(B_2).\end{aligned} \tag{4.9}$$

We claim that

$$\overline{V}(t, y_t, z_t) \le u(B_2) \quad \text{for all } t \ge t_0. \tag{4.10}$$

If this were not so, then there would exist a t^*, $t^* > t_0$, such that

$$\overline{V}(t^*, y_{t^*}, z_{t^*}) > u(B_2). \tag{4.11}$$

Since (4.9), (4.11) and $\overline{V}(t, y_t, z_t)$ is continuous in t, there exists $t_1 \in [t_0, t^*]$ such that

$$\overline{V}(t_1, y_{t_1}, z_{t_1}) = u(B_2) < \overline{V}(t^*, y_{t^*}, z_{t^*}). \tag{4.12}$$

Therefore the mean value theorem yields there exists $\bar{t} \in [t_1,\ t^*]$ such that

$$\dot{\overline{V}}(\bar{t}, y_{\bar{t}}, z_{\bar{t}}) > 0. \tag{4.13}$$

If $\theta_0 < 0$, then using Lemma 1, we have $\dot{\overline{V}}(\bar{t}, y_{\bar{t}}, z_{\bar{t}}) = 0$, which contradicts the (4.13). If $\theta_0 = 0$, then using (4.2), (4.7), (4.12), and our choice of $u(B_2)$, we have

$$v(B_1) \le u(B_2) \le \overline{V}(\bar{t}, y_{\bar{t}}, z_{\bar{t}}) = V(\bar{t}, y(\bar{t}), z(\bar{t})) \le v(|x_{1\sim k}(\bar{t})|) \le v(|x(\bar{t})|).$$

Using the fact that v is nondecreasing, we get $B_1 \le |x(\bar{t})|$ $\ (\,|x(\bar{t})| \ge B_1 \ge H\,)$. If $\theta_0 = 0$, then using (4.1) and (4.2), we have

$$\begin{aligned} &V(\bar{t}+\theta, y(\bar{t}+\theta), z(\bar{t})) \le \sup_{\theta\in[-r,\,0]} V(\bar{t}+\theta, y(\bar{t}+\theta), z(\bar{t})) = \\ &\overline{V}(\bar{t}, y_{\bar{t}}, z_{\bar{t}}) = V(\bar{t}+\theta_0, y(\bar{t}+\theta_0), z(\bar{t})) = V(\bar{t}, y(\bar{t}), z(\bar{t})) \end{aligned} \tag{4.14}$$

for $\theta \in [-r, 0]$, $\bar{t} \ge t_0$. From (4.8), (4.14) and $|x(\bar{t})| \ge H$, we have $\dot{\overline{V}}(\bar{t}, y_{\bar{t}}, z_{\bar{t}}) = \dot{V}(\bar{t}, y_{\bar{t}}, z_{\bar{t}}) \le 0$, which contradicts the (4.13). So $\overline{V}(t, y_t, z_t) \le u(B_2)$ for all $t \ge t_0$.

From (4.1), (4.7) and (4.10), we obtain

$$\begin{aligned} &u(|y(t)|) \le V(t, y(t), z(t)) = \\ &V(t+0, y(t+0), z(t)) \le \sup_{\theta\in[-r,\,0]} V(t+\theta, y(t+\theta), z(t)) = \\ &\overline{V}(t, y_t, z_t) \le u(B_2) \end{aligned}$$

for all $t \ge t_0$. Using the fact that u is nondecreasing, we get $|y(t)| \le B_2$ for all $t \ge t_0$. The proof of the Lemma 2 is therefore complete. □

Lemma 3. *Let the following conditions hold.*

(1) *There are positive integers m and k with $1 \le m \le k \le n$. There is a continuous function $V : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}$ such that*

$$u(|x_{1\sim m}|) \le V(t, x_{1\sim m}, x_{m+1\sim n}) \le v(|x_{1\sim k}|) \tag{4.15}$$

for $t \in \mathbb{R}, x \in \mathbb{R}^n$.

(2) *There is a continuous, nondecreasing function $p : \mathbb{R}^+ \to \mathbb{R}^+$ with*

$$p(s) > q(s) \ for \ s > 0. \tag{4.16}$$

(3) *Let* $x(t) = x_{1\sim n}(t) = (y^T(t), z^T(t))^T = (x^T_{1\sim m}(t), x^T_{m+1\sim n}(t))^T$ *be the solution of the Equation* (2.1). *If* $|x_{1\sim n}(t)| \geq H$ *for all* $t \geq t_0$ *and*

$$q(V(t+\theta, y(t+\theta), z(t))) < p(V(t, y(t), z(t)))$$

for all $t \geq t_0$, $\theta \in [-r, 0]$, *then*

$$\dot{V}_{(2.1)}(t, y(t), z(t)) \leq -w(H) \tag{4.17}$$

for all $t \geq t_0$.

Under these conditions, the coordinates $x_{1\sim m}(t, t_0, \phi)$ *of the solutions of the Equation* (2.1) *are uniformly bounded and uniformly ultimately bounded.*

Proof. First, we show the uniform boundedness.

If $|x(t)| \geq H$ and $V(t+\theta, y(t+\theta), z(t)) \leq V(t, y(t), z(t))$ for $\theta \in [-r, 0]$, $t \geq t_0$, then using (4.16) and the fact that q is nondecreasing, we have

$$q(V(t+\theta, y(t+\theta), z(t))) \leq q(V(t, y(t), z(t))) < p(V(t, y(t), z(t))) \tag{4.18}$$

for $\theta \in [-r, 0]$, $t \geq t_0$; this follows since $V(t, y(t), z(t)) > 0$ $(|x(t)| \geq H > 0)$. Hypothesis (4.17) implies $\dot{V}(t, y(t), z(t)) \leq -w(H) \leq 0$. Lemma 2 implies the coordinates $x_{1\sim m}(t, t_0, \phi)$ of the solutions of the Equation (2.1) are uniformly bounded.

We now show the uniform ultimate boundedness. Choose $B > H$ such that $u(B) > v(H)$. Let $B_3 \geq H$ be given. The same argument as in the proof of Lemma 2 shows there is a $B_4 > B$ such that for any $t_0 \in \mathbb{R}, t \geq t_0$, and $\phi \in C$ with $|\phi| \leq B_3$, the B_4 and $u(B_4)$ satisfy

$$\begin{aligned} &u(B) < u(B_4),\ \ \overline{V}(t, y_t, z_t) \leq u(B_4)\ \text{ and} \\ &|y(t)| \leq B_4 \ \text{ for all }\ t \geq t_0\ \ (\text{see } (4.10)). \end{aligned} \tag{4.19}$$

If $\tilde{x} = (B_4, 0, \cdots, 0)^T \in \mathbb{R}^n$, then using (4.15) and (4.19), we have

$$\begin{aligned} &0 < u(B) < u(B_4) = u(|\tilde{x}_{1\sim m}|) \leq \\ &V(t, \tilde{x}_{1\sim m}, \tilde{x}_{m+1\sim n}) \leq v(|\tilde{x}_{1\sim k}|) = v(B_4). \end{aligned} \tag{4.20}$$

From the properties of the functions $p(s)$ and $q(s)$, there is a number $a > 0$ such that

$$p(s) - q(s) > a \ \text{ for }\ u(B) \leq s \leq v(B_4). \tag{4.21}$$

Let N be the first nonnegative integer such that

$$q(u(B)) + Na \geq q(v(B_4)), \tag{4.22}$$

and let

$$T_l = \frac{l\,q(v(B_4))}{c\,w(H)} \quad (l = 0, 1, 2, \cdots, N). \tag{4.23}$$

First, we show that

$$q(V(t, y(t), z(t))) \leq q(u(B)) + (N-l)a \tag{4.24}$$

for all $t \geq t_0 + lr + T_l$, $l = 0, 1, 2, \cdots, N$. From (4.19)and (4.20), we have

$$V(t, y(t), z(t)) \leq \overline{V}(t, y_t, z_t) \leq u(B_4) \leq v(B_4) \text{ for } t \geq t_0. \tag{4.25}$$

If $l = 0$, then using (4.22) and (4.25), we have

$$q(V(t, y(t), z(t))) \leq q(\overline{V}(t, y_t, z_t)) \leq q(v(B_4)) \leq$$
$$q(u(B)) + Na = q(u(B)) + (N-0)a$$

for all $t \geq t_0 = t_0 + 0 \cdot r + T_0$. That is, the inequality (4.24) is true for the $l = 0$.

We now show that inequality (4.24) is true for $l = 1$. If

$$q(u(B)) + (N-1)a < q(V(\tilde{t}_0, y(\tilde{t}_0), z(\tilde{t}_0))) \tag{4.26}$$

for $\tilde{t}_0 \geq t_0$, then, since $q(u(B)) \leq q(u(B)) + (N-1)a < q(V(\tilde{t}_0, y(\tilde{t}_0), z(\tilde{t}_0)))$, it follows that

$$u(B) < V(\tilde{t}_0, y(\tilde{t}_0), z(\tilde{t}_0)). \tag{4.27}$$

From (4.25) and (4.27), we obtain

$$u(B) < V(\tilde{t}_0, y(\tilde{t}_0), z(\tilde{t}_0)) \leq v(B_4). \tag{4.28}$$

Using (4.15), (4.27), and our choice of $v(H)$, we have

$$v(H) < u(B) < V(\tilde{t}_0, y(\tilde{t}_0), z(\tilde{t}_0)) \leq v(|x_{1\sim k}(\tilde{t}_0)|) \leq v(|x(\tilde{t}_0)|), \tag{4.29}$$

and using the fact that v is nondecreasing, we get $H \leq |x(\tilde{t}_0)|$. Using (4.21), (4.22), (4.25), (4.26) and (4.28), we get

$$p(V(\tilde{t}_0, y(\tilde{t}_0), z(\tilde{t}_0))) > q(V(\tilde{t}_0, y(\tilde{t}_0), z(\tilde{t}_0))) + a >$$
$$q(u(B)) + (N-1)a + a = q(u(B)) + Na \geq q(v(B_4)) \geq$$
$$q(\overline{V}(\tilde{t}_0, y_{\tilde{t}_0}, z_{\tilde{t}_0}) \geq q(V(\tilde{t}_0 + \theta, y(\tilde{t}_0 + \theta), z(\tilde{t}_0))) \tag{4.30}$$

for θ in $[-r, 0]$. Hypothesis (4.17) implies

$$\dot{V}(\tilde{t}_0, y(\tilde{t}_0), z(\tilde{t}_0)) \leq -w(H) < 0 \tag{4.31}$$

for $\tilde{t}_0 \geq t_0$. Therefore, if there is a $t_0^* \in [t_0, t_0 + T_1)$ such that

$$q(V(t_0^*, y(t_0^*), z(t_0^*))) \leq q(u(B)) + (N-1)a,$$

then using (4.26) and (4.31), we have

$$q(V(t, y(t), z(t))) \leq q(u(B)) + (N-1)a \tag{4.32}$$

for $t \geq t_0 + r + T_1$. If

$$q(u(B)) + (N-1)a < q(V(t, y(t), z(t)))$$

for all $t \in [t_0, t_0 + T_1)$, then using (4.23) and (4.31), we have

$$q(V(t_0 + T_1, y(t_0 + T_1), z(t_0 + T_1))) =$$
$$q(V(t_0, y(t_0), z(t_0))) + \dot{q}(V(\xi, y(\xi), z(\xi)))\, \dot{V}(\xi, y(\xi), z(\xi))\, T_1 =$$

$$
\begin{aligned}
&q(V(t_0,y(t_0),z(t_0)))+\dot{q}(V(\xi,y(\xi),z(\xi)))\,\dot{V}(\xi,y(\xi),z(\xi))\,\frac{q(v(B_4))}{c\,w(H)}\le\\
&q(V(t_0,y(t_0),z(t_0)))-\dot{q}(V(\xi,y(\xi),z(\xi)))\,\frac{q(v(B_4))\,w(H)}{c\,w(H)}=\\
&q(V(t_0,y(t_0),z(t_0)))-\dot{q}(V(\xi,y(\xi),z(\xi)))\,\frac{q(v(B_4))}{c}\le\\
&q(V(t_0,y(t_0),z(t_0)))-c\,\frac{q(v(B_4))}{c}=\\
&q(V(t_0,y(t_0),z(t_0)))-q(v(B_4))\le q(v(B_4))-q(v(B_4))=0\le\\
&q(u(B))+(N-1)a.
\end{aligned}
$$

Therefore, we have

$$
q(V(t,y(t),z(t)))\le q(u(B))+(N-1)a \quad \text{for all } \ t\ge t_0+r+T_1. \tag{4.33}
$$

Thus, the inequality (4.24) is true for the $l=1$.

The arguments are essentially the same as before, we are able to show that

$$
q(V(t,y(t),z(t)))\le q(u(B))+(N-K)a
$$

for all $t\ge t_0+Kr+T_K, K=2,3,\cdots,N$ [4, 12]. Therefore, we have

$$
q(V(t,y(t),z(t)))\le q(u(B))+(N-l)a
$$

for all $t\ge t_0+lr+T_l, l=0,1,2,\cdots,N$. For $l=N$, we have

$$
q(V(t,y(t),z(t)))\le q(u(B))+(N-N)a=q(u(B))
$$

for all $t\ge t_0+Nr+T_N$. Hypothesis (4.15) implies

$$
q(u(|y(t)|))\le q(V(t,y(t),z(t)))\le q(u(B))
$$

for all $t\ge t_0+Nr+T_N$ and hence

$$
|(x_1(t,t_0,\phi),\ x_2(t,t_0,\phi),\ \cdots,x_m(t,t_0,\phi))^T|\le B
$$

for all $t\ge t_0+Nr+T_N$. Let $T=Nr+T_N$. Thus, this completes the proof of uniform ultimate boundedness. The proof of the Lemma 3 is therefore complete. □

Now we are in the position to prove our Theorem 1.

Proof of Theorem 1. If

$$
|x_{1\sim m}(t)|\ge H \tag{4.34}
$$

for all $t\ge t_0$ and

$$
q(V(s,y(s),x_{m+1\sim n}))<p(V(t,y(t),x_{m+1\sim n})) \tag{4.35}
$$

for all $t\ge t_0, s\in[t-r,t],\ x_{m+1\sim n}\in\mathbb{R}^{n-m}$, then using (c), we have

$$
\dot{V}_{(2.1)}(t,y(t),z(t))\le -w(|y(t)|) \tag{4.36}
$$

for all $t \geq t_0$. From (4.34) and (4.35), we obtain

$$|x_{1\sim n}(t)| \geq |x_{1\sim m}(t)| \geq H \tag{4.37}$$

for all $t \geq t_0$ and

$$q(V(t+\theta, y(t+\theta), z(t))) < p(V(t, y(t), z(t))) \tag{4.38}$$

for all $t \geq t_0$, $\theta \in [-r, 0]$. Using (4.36), (4.37), and the fact that w is nondecreasing, we get

$$\dot{V}_{(2.1)}(t, y(t), z(t)) \leq -w(|y(t)|) = -w(|x_{1\sim m}(t)|) \leq -w(H) \tag{4.39}$$

for all $t \geq t_0$. Lemma 3 implies the coordinates $x_{1\sim m}(t, t_0, \phi)$ of the solutions of the Equation (2.1) are uniformly bounded and uniformly ultimately bounded (see the (4.37), the (4.38), and the (4.39)). The proof of Theorem 1 is therefore complete. $\square$

5. Examples

We give the following examples in order to illustrate the application and advantage of the Theorem 1 in this paper [4, 20, 21]. It is easy to see that the well-known Razumikhin theorems on uniform ultimate boundedness cannot apply to the following examples.

Example 1. Consider the scalar equation

$$\begin{aligned}\dot{x}(t) = &-\sum_{j=1}^{n} b_j(t)x^{2k-1}(t)\,[\alpha \arctan \beta x^{2m}(t-\tau_j(t)) + \gamma x^{2m}(t-\tau_j(t))] \\ &-a(t)x^{2k-1}(t)\,[\alpha \arctan \beta x^{2m}(t) + \gamma x^{2m}(t)] + \tilde{p}(t),\end{aligned} \tag{5.1}$$

where $\tilde{p}(t)$, $a(t)$, $b_j(t)$, and $\tau_j(t)$ are bounded continuous functions on $\mathbb{R}$, $\alpha, \beta, \gamma =$ const., $\alpha\beta > 0, \gamma > 0,\ k = 1, 2, \cdots, K,\ m = 1, 2, \cdots, M$.

We make the following assumptions on Equation (5.1): there is a $q_0 \in (0, 1)$ such that

$$a(t) \geq \delta > 0,\ \sum_{j=1}^{n} |b_j(t)| \leq \delta q_0,\ 0 \leq \tau_j(t) \leq r \quad (j = 1, 2, \cdots, n)$$

for all $t \in \mathbb{R}$, $q_0, \delta, r =$ const.

Under the above hypotheses, we will show that the solutions of the Equation (5.1) are uniformly bounded and uniformly ultimately bounded.

In fact, since $q_0 \in (0,1)$, there is a $q_1 > 1$ such that $q_0 q_1 < 1$. If

$$p(s) = q_1 q(s),\ q(s) = \alpha \arctan \beta s + \gamma s,\ V(x) = x^{2m},$$

$$|x(t)| \geq H = \text{ const. } > 0, \text{ and}$$

$$p(V(x(t))) > q(V(x(s))),\ s \in [t-r,\ t]\ (\Rightarrow q_1(\alpha \arctan \beta x^{2m}(t) +$$

$$\gamma x^{2m}(t)) > \alpha \arctan \beta x^{2m}(t-\tau_j(t)) + \gamma x^{2m}(t-\tau_j(t)),\ j = 1,2,\cdots,n),$$

then

$$\begin{aligned}
\dot{V}_{(5.1)}(x(t)) &= 2mx^{2(k+m-1)}(t)\{-a(t)[\alpha \arctan \beta x^{2m}(t) + \gamma x^{2m}(t)] \\
&\quad - \sum_{j=1}^{n} b_j(t)[\alpha \arctan \beta x^{2m}(t-\tau_j(t)) + \gamma x^{2m}(t-\tau_j(t))]\} \\
&\quad + 2m\tilde{p}(t)x^{2m-1}(t) \\
&\leq 2mx^{2(k+m-1)}(t)\{-a(t)[\alpha \arctan \beta x^{2m}(t) + \gamma x^{2m}(t)] \\
&\quad + \sum_{j=1}^{n} |b_j(t)| \cdot q_1[\alpha \arctan \beta x^{2m}(t) + \gamma x^{2m}(t)]\} + 2m\tilde{p}(t)x^{2m-1}(t) \\
&= 2mx^{2(k+m-1)}(t)[\alpha \arctan \beta x^{2m}(t) + \gamma x^{2m}(t)] \times \\
&\quad [-a(t) + q_1 \sum_{j=1}^{n} |b_j(t)|] + 2m\tilde{p}(t)x^{2m-1}(t) \\
&\leq 2mx^{2(k+m-1)}(t)[\alpha \arctan \beta x^{2m}(t) + \gamma x^{2m}(t)] \times \\
&\quad [-\delta + q_0 q_1 \delta] + 2m\tilde{p}(t)x^{2m-1}(t) \\
&\leq -2m\delta(1 - q_0 q_1)x^{2(k+m-1)}(t)[\alpha \arctan \beta x^{2m}(t) + \gamma x^{2m}(t)] \\
&\quad + 2m\tilde{p}(t)x^{2m-1}(t).
\end{aligned}$$

By choosing $H_1 \geq H = \text{const} > 0$ appropriately ($\tilde{p}(t)$ is bounded continuous function on $\mathbb{R}$, constant $\delta(1-q_0 q_1) > 0, \alpha\beta > 0, \gamma > 0$), we obtain a positive constant μ such that

$$2m \left\{ \delta(1 - q_0 q_1) - \frac{\tilde{p}(t)}{x^{2k-1}(t)\left[\alpha \arctan \beta x^{2m}(t) + \gamma x^{2m}(t)\right]} \right\} \geq \mu$$

for $|x(t)| \geq H_1$ and $p(V(x(t))) > q(V(\phi(\theta))),\ \theta \in [-r,\ 0]$. From this, we have

$$\begin{aligned}
\dot{V}_{(5.1)}(x(t)) &\leq -2m\delta(1 - q_0 q_1)x^{2(k+m-1)}(t)[\alpha \arctan \beta x^{2m}(t) + \gamma x^{2m}(t)] \\
&\quad + 2m\tilde{p}(t)x^{2m-1}(t)
\end{aligned}$$

$$= -2m\left\{\delta(1-q_0q_1) - \frac{\tilde{p}(t)}{x^{2k-1}(t)\left[\alpha\arctan\beta x^{2m}(t)+\gamma x^{2m}(t)\right]}\right\}$$
$$\times x^{2(k+m-1)}(t)[\alpha\arctan\beta x^{2m}(t)+\gamma x^{2m}(t)]$$
$$\leq -\mu x^{2(k+m-1)}(t)[\alpha\arctan\beta x^{2m}(t)+\gamma x^{2m}(t)]$$

for $|x(t)| \geq H_1$ and $p(V(x(t))) > q(V(\phi(\theta)))$, $\theta \in [-r, 0]$. Therefore, the Theorem 1 implies the solutions of the Equation (5.1) are uniformly bounded and uniformly ultimately bounded.

Example 2. Consider the second-order system

$$\dot{x}(t) = 2y(t),$$

$$\dot{y}(t) = -f(x(t)) - \Phi(t, y(t))a^{y^2(t)} \tag{5.2}$$

$$+\tilde{p}(t) + y(t)\int_{-r}^{0} g(x(t+\theta))a^{y^2(t+\theta)}d\theta,$$

where $a =$ constant > 1.

We make the following assumptions on System (5.2):

(a) $\Phi : \mathbb{R}^2 \to \mathbb{R}$ is continuous, Φ takes $\mathbb{R}\times$ (bounded sets of $\mathbb{R}$) into bounded sets and there are constants $a_0 > 0$, $H > 0$, such that

$$\frac{\Phi(t, y)}{y} > a_0 > 0 \text{ for } t \in \mathbb{R},\ |y| \geq H.$$

(b) $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x)sgnx \to \infty\ as\ |x| \to \infty$.

(c) $\tilde{p} : \mathbb{R} \to \mathbb{R}$ is bounded and continuous.

(d) $g : \mathbb{R} \to \mathbb{R}$ is continuous and there is a constant $L > 0$, such that

$$|g(x)| \leq L \text{ for all } x \in \mathbb{R}.$$

(e) $Lr < a_0$.

It is always assumed that a uniqueness result holds for the solutions of System (5.2). Under the above hypotheses, we will show that the second coordinate of the solutions of the System (5.2) is uniformly bounded and uniformly ultimately bounded.

In fact, since $Lr < a_0$, there is a $q_1 > 1$ such that $q_1 Lr < a_0$. If

$q(s) = a^s - 1,\ p(s) = q_1 q(s),\ V(x,y) = F(x) + y^2,\ F(x) = \int_0^x f(s)ds,$

$|y(t)| \geq H$, and

$q(V(x,y(s))) < p(V(x,y(t))),\ s \in [t-r,\ t]\ (q(V(x,y(t+\theta))) <$

$p(V(x,y(t))),\ \theta \in [-r,\ 0])$

$$\Big(\Rightarrow a^{F(x)+y^2(t+\theta)} - 1 < q_1(a^{F(x)+y^2(t)} - 1)$$

$$\Rightarrow a^{F(x)+y^2(t+\theta)} < a^{F(x)+y^2(t+\theta)} + (q_1 - 1) < q_1 a^{F(x)+y^2(t)}$$

$$\Rightarrow a^{y^2(t+\theta)} < q_1 a^{y^2(t)} \Big),$$

then

$$\begin{aligned}
\dot{V}_{(5.2)}(x(t),y(t)) &= 2y(t)[-\Phi(t,y(t))a^{y^2(t)} + \tilde{p}(t) + \\
&\quad y(t)\int_{-r}^{0} g(x(t+\theta))a^{y^2(t+\theta)}d\theta] \\
&\leq 2y^2(t)[-a_0 a^{y^2(t)} + \int_{-r}^{0} |g(x(t+\theta))|a^{y^2(t+\theta)}d\theta)] \\
&\quad +2|y(t)|\,|\tilde{p}(t)| \\
&\leq -2(a_0 - q_1 Lr)y^2(t)a^{y^2(t)} + 2|y(t)|\,|\tilde{p}(t)|,
\end{aligned}$$

By choosing $H_1 \geq H$ appropriately ($\tilde{p}(t)$ is bounded continuous function, constant $(a_0 - q_1 Lr) > 0$), we obtain a positive constant μ such that

$$2\left[(a_0 - q_1 Lr) - \frac{|\tilde{p}(t)|}{|y(t)|\, a^{y^2(t)}}\right] \geq \mu$$

for $|y(t)| \geq H_1$ and $q(V(x,y(s))) < p(V(x,y(t))),\ s \in [t-r,\ t]$. From this, we have

$$\begin{aligned}
\dot{V}_{(5.2)}(x(t),y(t)) &\leq -2(a_0 - q_1 Lr)y^2(t)a^{y^2(t)} + 2|y(t)|\,|\tilde{p}(t)| \\
&= -2\left[(a_0 - q_1 Lr) - \frac{|\tilde{p}(t)|}{|y(t)|\, a^{y^2(t)}}\right] y^2(t)a^{y^2(t)} \\
&\leq -\mu y^2(t)a^{y^2(t)}
\end{aligned}$$

for $|y(t)| \geq H_1$ and $q(V(x,y(s))) < p(V(x,y(t))),\ s \in [t-r,\ t]$. Therefore, the Theorem 1 implies the second coordinate of the solutions of the System (5.2) is uniformly bounded and uniformly ultimately bounded.

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