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SOME GENERALIZED BIVARIATE BERNSTEIN OPERATORS

Dan Barbosu
Department of Mathematics and Computer Sciences, North University of Baia Mare
Baia Mare Victoriei 76, Romania
dbarbosu@univer.ubm.ro

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Abstract. A bivariate operator with knots which form a geometric progression is constructed and some approximation properties of this operator are established.

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1. Preliminaries

It is well known that the classical Bernstein operator associates the polynomial

$$B_n(f;x) = \sum_{r=0}^{n} \binom{n}{r} x^r (1-x)^{n-r} f\left(\frac{r}{n}\right) \tag{1.1}$$

to any function $f \in C[0,1]$.

In (1.1), the approximated function f is evaluated at equally spaced intervals. Further were given extensions of this operator to the case of multivariate functions by D. D. Stancu [10,11, 12], E. Dobrescu and I. Matei [6], C. Badea, I. Badea and H. H. Gonska [1], and D. Barbosu [2, 3, 4]. In all these extensions, the approximated function $f : [0,1]^n \to \mathbf{R}$ is evaluated at equally spaced intervals.

G. M. Phillips [7, 8, 9] constructed a generalization of the classical operator of Bernstein where the approximated function f is evaluated at intervals which form a geometric progression. We start by recalling some results due to G. M. Phillips containing q-integers.

Let $q > 0$ be any positive real number. For any non-negative integer i the so called "q-integer" was introduced, denoted by [i] and defined by

$$[i] = \begin{cases} \dfrac{1-q^i}{1-q}, & q \neq 1 \\ i, & q = 1. \end{cases} \tag{1.2}$$

A "q-factorial" was defined in the following obvious way:

$$[i]! = \begin{cases} [i][i-1]...[1], & i = 1, 2, ... \\ 1, & i = 0 \end{cases} \tag{1.3}$$

and a "q-binomial coefficient" was defined as:

$$\begin{bmatrix} k \\ r \end{bmatrix} = \frac{[k]!}{[r]![k-r]!}. \tag{1.4}$$

Note that the q-binomial coefficients satisfy – see [7] – the recurrence relations

$$\begin{bmatrix} k+1 \\ r \end{bmatrix} = q^{k-r+1} \begin{bmatrix} k \\ r-1 \end{bmatrix} + \begin{bmatrix} k \\ r \end{bmatrix} \tag{1.5}$$

and

$$\begin{bmatrix} k+1 \\ r \end{bmatrix} = \begin{bmatrix} k \\ r-1 \end{bmatrix} + q^{r} \begin{bmatrix} k \\ r \end{bmatrix}. \tag{1.6}$$

Next, a sequence of positive linear operators $B_n : C[0,1] \to C[0,1]$ was constructed, which associates for any positive integer n to any $f \in C[0,1]$ the polynomial

$$B_n(f;x) = \sum_{r=0}^{n} f_r \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1-q^s x). \tag{1.9}$$

In (1.9) an empty product denotes 1 and $f_r = f\left(\frac{[r]}{[n]}\right)$. Clearly, if $q = 1$ the operator (1.9) is reduced to the classical Bernstein operator.

It was proved ([7], [8], [9]) that the properties of the generalized Bernstein operator are similar to the properties of the classical Bernstein operator (1.1).

The aim of the present paper is to extend the operator (1.9) to the case of bivariate functions.

2. Main results

Let $I^2 = [0,1] \times [0,1]$ be the unit square and $\mathbf{R}^{I^2} = \{f \mid f : I^2 \to \mathbf{R}\}$ denote the space of real bivariate functions defined on this square.

For any function $f \in \mathbf{R}^{I^2}$ and any positive real numbers $q_1, q_2 > 0$ one denotes by

$$B^x_{n_1}(f;x,y) = \sum_{r_1=0}^{n_1} f_{r_1} \begin{bmatrix} n_1 \\ r_1 \end{bmatrix} x^{r_1} \prod_{s_1=0}^{n_1-r_1-1} (1-q_1^{s_1} x) \tag{2.1}$$

and

$$B^y_{n_2}(f;x,y) = \sum_{r_2=0}^{n_2} f_{r_2} \begin{bmatrix} n_2 \\ r_2 \end{bmatrix} y^{r_2} \prod_{s_2=0}^{n_2-r_2-1} (1-q_2^{s_2} y) \tag{2.2}$$

respectively, the parametric extensions of the operator (1.9). Note that in (2.1) and (2.2) an empty product denotes 1 and $f_{r_1} = f\left(\frac{[r_1]}{[n_1]}, y\right)$, $\quad f_{r_2} = f\left(x, \frac{[r_2]}{[n_2]}\right)$. Clearly, for $q_1 = 1$ and $q_2 = 1$ the parametric extensions (2.1) and (2.2) are reduced to the parametric extensions of the classical Bernstein operator.

From the definitions (2.1) and (2.2), it follows

Lemma 2.1. The parametric extensions (2.1) and (2.2) of the operator (1.9) are linear positive operators on $C(I^2)$.

Lemma 2.2. Let $f \in C(I^2)$. Then, for any $y \in [0,1]$ the operator (2.1) has the following interpolation properties

(i) $B^x_{n_1}(f;0,y) = f(0,y)$

(ii) $B^x_{n_2}(f;1,y) = f(1,y)$.

Proof.

(i) By the definition (2.1), we obtain:

$$B^x_{n_1}(f;x,y) = \sum_{r_1=0}^{n_1} f_{r_1} \begin{bmatrix} n_1 \\ r_1 \end{bmatrix} x^{r_1} \prod_{r_1=0}^{n_1-r_1-1} (1-q_1^{s_1}x) =$$

$$= f_0 \begin{bmatrix} n \\ 0 \end{bmatrix} x^0 \prod_{s_1=0}^{n_1-1} (1-q_1^{s_1}x) + f_1 \begin{bmatrix} n_1 \\ 1 \end{bmatrix} x \prod_{s=0}^{n_1-2} (1-q_1^{s_1}x) + ... + \tag{2.3}$$

$$+... + f_n \begin{bmatrix} n \\ n \end{bmatrix} x^n.$$

Here $f_i = f\left(\left[\frac{i}{n}\right] y\right)$, $i = \overline{0,n}$. Substituting $x = 0$, we get that

$$B^x_{n_1}(f;0,y) = f_0 \begin{bmatrix} n \\ 0 \end{bmatrix} = f\left(\frac{[0]}{[n]}, y\right) = f(0,y),$$

for any $y \in [0,1]$.

(ii) Substituting $x = 1$ in (2.3), one obtains $B^x_{n_1}(f;1,y) = f(1,y)$, for any $y \in [0,1]$.

In a similar way, we obtain

Lemma 2.3. For any $x \in [0,1]$ and any $f \in C(I^2)$ the operator (2.2) has the following interpolation properties

(i) $B^y_{n2}(f;x,0) = f(x,0)$

(ii) $B^y_{n_2}(f;x,1) = f(x,1)$.

Lemma 2.4. The operators $B^x_{n_1}, B^y_{n_2}$ commute on $C(I^2)$. Their product is the linear positive operator $B_{n_1,n_2} : C(I^2) \to C(I^2)$, which associates to any function $f \in C(I^2)$ the approximant

$$\begin{aligned} B_{n_1,n_2}(f;x,y) = \sum_{r_1=0}^{n_1} \sum_{r_2=0}^{n_2} f_{r_1,r_2} \begin{bmatrix} n_1 \\ r_1 \end{bmatrix} \begin{bmatrix} n_2 \\ r_2 \end{bmatrix} x^{r_1} y^{r_2} \\ \prod_{s_1=0}^{n_1-r_1-1} (1-q_1^{s_1}x) \prod_{s_2=0}^{n_2-r_2-1} (1-q_2^{s_2}y). \end{aligned} \tag{2.1}$$

Proof. By direct computation one obtains

$$B_{n_1}^x B_{n_2}^y(f;x,y) = B_{n_1}^x(B_{n_2}^y(f;x,y)) =$$

$$= B_{n_1}^x\left(\sum_{r_2=0}^{n_2} f_{r_2}\begin{bmatrix} n_2 \\ r_2\end{bmatrix} y^{r_2}\prod_{s_1=0}^{n_2-r_2-1}(1-q_2^{s_2}y)\right) =$$

$$= \sum_{r_2=0}^{n_2}\begin{bmatrix} n_2 \\ r_2\end{bmatrix} y^{r_2}\prod_{s_2=0}^{n_2-r_2-1}(1-q_2^{s_2}y)B_{n_1}^x(f_{r_2}) = \sum_{r_2=0}^{n_2}\begin{bmatrix} n_2 \\ r_2\end{bmatrix} y^{r_2}\prod_{s_2=0}^{n_2-r_2-1}(1-q_2^{s_2}y)$$

$$\left(\sum_{r_1=0}^{n_1}\begin{bmatrix} n_1 \\ r_1\end{bmatrix} x^{r_1} f_{r_1,r_2}\prod_{s_1=0}^{n_1-r_1-1}(1-q_1^{s_1}x)\right) =$$

$$= \sum_{r_1=0}^{n_1}\sum_{r_2=0}^{n_2}\begin{bmatrix} n_1 \\ r_1\end{bmatrix}\begin{bmatrix} n_2 \\ r_2\end{bmatrix} x^{r_1}y^{r_2}f_{r_1,r_2}\prod_{s_1=0}^{n_1-r_1-1}(1-q_1^{s_1}x)\prod_{s_2=0}^{n_2-r_2-1}(1-q_2^{s_2}y).$$

In a similar way, it follows

$$\begin{aligned} B_{n_2}^y B_{n_1}^x(f;x,y) \quad &= \quad \sum_{r_1=0}^{n_1}\sum_{r_2=0}^{n_2}\begin{bmatrix} n_1 \\ r_1\end{bmatrix}\begin{bmatrix} n_2 \\ r_2\end{bmatrix} x^{r_1}y^{r_2}f_{r_1,r_2}\times \\ &\quad \prod_{s_1=0}^{n_1-r_1-1}(1-q_1^{s_1}x)\prod_{s_2=0}^{n_2-r_2-1}(1-q_2^{s_2}y)\ . \end{aligned}$$

We obtain (2.4). The positivity and linearity of B_{n_1,n_2} follow from relation (2.4).

Lemma 2.5. The generalized bivariate Bernstein operator (2.4) interpolates the function f in the four corners of the unit square, i.e.,

$$\begin{cases} B_{n_1,n_2}(f;0,0) = f(0,0); & B_{n_1,n_2}(f;0,1) = f(0,1); \\ B_{n_1,n_2}(f;1,0) = f(1,0); & B_{n_1,n_2}(f;1,1) = f(1,1). \end{cases}$$

Proof. We apply lemma 2.4.

Lemma 2.6. Let $e_{ij} : I^2 \to I^2, e_{ij}(x,y) = x^i y^j \quad (0 \le i+j \le 2, i,j-\text{integers})$ be the test functions. Then, the following equalities

(i) $B_{n_1,n_2}(e_{00;}x,y) = e_{00}(x,y);$

(ii) $B_{n_1,n_2}(e_{10;}x,y) = e_{10}(x,y);$

(iii) $B_{n_1,n_2}(e_{01;}x,y) = e_{01}(x,y);$

(iv) $B_{n_1,n_2}(e_{11;}x,y) = e_{11}(x,y);$

(v) $B_{n_1,n_2}(e_{20;}x,y) = e_{20}(x,y) + \frac{x(1-x)}{[n_1]};$

(vi) $B_{n_1,n_2}(e_{02;}x,y) = e_{02}(x,y) + \frac{y(1-y)}{[n_2]},$

hold for any $(x,y)\in I^2$.

Proof. We show only the proof of (vi). By definition (2.4), one has

$$B_{n_1,n_2}(e_{02};x,y)=$$
$$=\sum_{r_1=0}^{n_1}\sum_{r_2=0}^{n_2}\left(\frac{[r_2]}{[n_2]}\right)^2\begin{bmatrix} n_1\\ r_1\end{bmatrix}\begin{bmatrix} n_2\\ r_2\end{bmatrix}\prod_{s_1=0}^{n_1-r_1-1}(1-q_1^{s_1}x)\prod_{s_2=0}^{n_2-r_2-1}(1-q_2^{s_2}x)=$$
$$=\left\{\sum_{r_1=0}^{n_1}\begin{bmatrix} n_1\\ r_1\end{bmatrix}x^{r_1}\prod_{s_1=0}^{n_1-r_1-1}(1-q_1^{s_1}x)\right\}\left\{\sum_{r_2=0}^{n_2}\left(\frac{[r_2]}{[n_2]}\right)^2 y^{r_2}\prod_{s_2=0}^{n_2-r_2-1}(1-q_2^{s_2}y)\right\}=$$
$$=B_{n_1}(e_{0;}x)\ B_{n_2}(e_{2;}y),$$

where B_{n_1}, B_{n_2} are the generalized Bernstein operators introduced by G. M. Phillips – see [7] – and $e_i: I\to I, e;(x)=x^i(i=0,1,2)$ are the one variate test functions. It is well known [7] that these operators satisfy the following equalities

$$B_{n_1}(e_{0;}x)=e_0(x), B_{n_2}(e_{2;}y)=e_2(y)+\frac{y(1-y)}{[n_2]}$$

for all $x\in I$ and $y\in I$. By using these two equalities, we get (vi).

Theorem 2.1. Let be $q_1=q_1(n_1), q_2=q_2(n_2)$, and let $q_1(n_1)\to 1, q_2(n_2)\to 1$ from below as $n_1\to\infty, n_2\to\infty$. Then, for any $f\in C(I^2)$ the sequence of bivariate generalized Bernstein polynomials defined at (2.4) converges uniformly to $f(x,y)$ on I^2.

Proof. By using relation (1.2) and the hypothesis that $q_1=q_1(n_1)\to 1$ as $n_1\to\infty$, one obtains that $[n_1]\to\infty$ as $n_1\to\infty$.

In a similar way, $[n_2]\to\infty$ as $n_2\to\infty$.

Next, using lemma 2.6, it follows that $B_{n_1,n_2}(e_{ij;}x,y)\Rightarrow e_{ij}$, uniformly on I^2 as $n_1,n_2\to\infty$.

Applying now the well-known test theorem of Korovkin for bivariate functions, one obtains that $B_{n_1,n_2}(f;x,y)$ converges to $f(x,y)$, uniformly on I^2, and the proof ends.

Theorem 2.2. For any $f: I^2\to\mathbf{R}$, bounded on I^2, the inequality

$$\left\|f-B_{n_1,n_2}(f)\right\|_\infty\le\frac{9}{4}\omega\left(\frac{1}{\sqrt{[n_1]}},\frac{1}{\sqrt{[n_2]}}\right)\tag{2.5}$$

holds.

In (2.5), $\|\cdot\|_\infty$ denotes the uniform norm and ω denotes the first order modulus of smoothness.

Proof. Let $f: I^2\to\mathbf{R}$ be a bivariate bounded function and denoted and let the set of all bounded bivariate functions defined on $\mathbf{I}^2$ be denoted by $\mathbf{R}^{I^2}$ Then, the first order modulus of smoothness is the function $\omega : \mathbf{R}_+^2 \to [0,1]$, defined for any $f \in \mathbf{R}^{I^2}$ and any $(\delta_1, \delta_2) \in \mathbf{R}_+^2$ by

$$\omega(\delta_1,\delta_2) = \sup\left\{|f(x,y) - f(x',y')|\,(x,y) \in I^2, (x',y') \in I^2, |x-x'| < \delta_1, |y-y'| < \delta_2\right\} \tag{2.6}$$

It is well known that ω, defined at (2.6), is a monotone increasing function with respect to the natural order of $\mathbf{R}^2$, i.e.

$$(\delta_1,\delta_2) \in \mathbf{R}_+^2, (\delta_1',\delta_2') \in \mathbf{R}_+^2, \delta_1 < \delta_1', \delta_2 < \delta_2' \Rightarrow \omega(\delta_1,\delta_2) \leq \omega(\delta_1',\delta_2'). \tag{2.7}$$

After this introduction, let us to begin the proof. Using lemma (2.6) (the equality (i)), we can write

$$\begin{aligned}
&\left|B_{n_1,n_2}(f;x,y) - f(x,y)\right| = \\
&= \sum_{r_1=0}^{n_1}\sum_{r_2=0}^{n_2} \begin{bmatrix} n_1 \\ r_1 \end{bmatrix} \begin{bmatrix} n_2 \\ r_2 \end{bmatrix} x^{r_1} y^{r_2} \times \\
&\prod_{s_1=0}^{n_1-r_1-1}(1-q_1^{s_1}x)\prod_{s_2=0}^{n_2-r_2-1}(1-q_2^{s_2}y)\left(f\left(\frac{[r_1]}{[n_1]},\frac{[r_2]}{[n_2]}\right) - f(x,y)\right) \leq \\
&\leq \sum_{r_1=0}^{n_1}\sum_{r_2=0}^{n_2} \begin{bmatrix} n_1 \\ r_1 \end{bmatrix} \begin{bmatrix} n_2 \\ r_2 \end{bmatrix} x^{r_1} y^{r_2} \times \\
&\prod_{s_1=0}^{n_1-r_1-1}(1-q_1^{s_1}x)\prod_{s_2=0}^{n_2-r_2-1}(1-q_2^{s_2}y)\left|f\left(\frac{[r_1]}{[n_1]},\frac{[r_2]}{[n_2]}\right) - f(x,y)\right|.
\end{aligned}$$

Taking into account the monotonicity of ω, one obtains

$$\begin{aligned}
&\left|f\frac{[r_1]}{[n_1]},\frac{[r_2]}{[n_2]} - f(x,y)\right| \leq \omega\left(\left|\frac{[r_1]}{[n_1]} - x\right|, \left|\frac{[r_2]}{[n_2]} - y\right|\right) \leq \\
&\leq \left(\left|\frac{[r_1]}{[n_1]} - x\right|\sqrt{[n_1]} + 1\right)\left(\left|\frac{[r_2]}{[n_2]} - y\right|\sqrt{[n_2]} + 1\right)\omega\left(\frac{1}{\sqrt{[n_1]}},\frac{1}{\sqrt{[n_2}}\right)
\end{aligned} \tag{2.8}$$

From (2.7), (2.8) follows the inequality

$$\left|B_{n_1,n_2}(f;x,y) - f(x,y)\right| \leq K \tag{2.9}$$

where

$$\begin{aligned}
&K = \omega\left(\frac{1}{\sqrt{[n_1]}},\frac{1}{\sqrt{[n_2}}\right) \times \\
&\left(\sqrt{[n_1]}\sum_{r_1=0}^{n_1}\left|\frac{[r_1]}{[n_1]} - 1\right| \begin{bmatrix} n_1 \\ r_1 \end{bmatrix} x^{r_1} \prod_{s_1=0}^{n_1-r_1-1}(1-q_1^{s_1}x) + 1\right) \times \\
&\left(\sqrt{[n_2]}\sum_{r_2=0}^{n_2}\left|\frac{[r_2]}{[n_2]} - 1\right| \begin{bmatrix} n_2 \\ r_2 \end{bmatrix} y^{r_2} \prod_{s_2=0}^{n_2-r_2-1}(1-q_2^{s_2}y) + 1\right).
\end{aligned}$$

Next, applying the Schwarz inequality, one obtains

$$\left(\sum_{r_1=0}^{n_1}\left(\frac{[r_1]}{[n_1]}-x\right)\left[\begin{array}{c} n_1 \\ r_1 \end{array}\right] x^{r_1}\prod_{s_1=0}^{n_1-r_1-1}(1-q_1^{s_1}x)\right)^2 =$$

$$= \sum_{r_1=0}^{n_1}\left|\frac{[r_1]}{[n_1]}-x\right|\left(\left[\begin{array}{c} n_1 \\ r_1 \end{array}\right] x^{r_1}\prod_{s_1=0}^{n_1-r_1-1}(1-q_1^{s_1}x)\right)^{\frac{1}{2}} \times$$

$$\left(\left(\left[\begin{array}{c} n_1 \\ r_1 \end{array}\right] x^{r_1}\prod_{s_1=0}^{n_1-r_1-1}(1-q_1^{s_1}x)\right)^{\frac{1}{2}}\right)^2 \leq$$

$$\leq \left(\sum_{r_1=0}^{n_1}\left(\frac{[r_1]}{[n_1]}-x\right)^2\left[\begin{array}{c} n_1 \\ r_1 \end{array}\right] x^{r_1}\prod_{s_1=0}^{n_1-r_1-1}(1-q_1^{s_1}x)\right) \times$$

$$\left(\sum_{r_1=0}^{n_1}\left[\begin{array}{c} n_1 \\ r_1 \end{array}\right] x^{r_1}\prod_{s_1=0}^{n_1-r_1-1}(1-q_1^{s_1}x)\right) =$$

$$= \left\{B_{n_1}(e_2;x) - 2x\cdot B_{n_1}(e_1;x) + B_{n_1}(e_0;x)\right\} B_{n_1}(e_0;x) =$$

$$= x^2 + \frac{x(1-x)}{[n_1]} - 2x^2 + x^2 = \frac{x(1-x)}{[n_1]} \leq \frac{1}{4[n_1]}\ . \tag{2.10}$$

In a similar way we obtain

$$\left(\sum_{r_2=0}^{n_2}\left(\frac{[r_2]}{[n_2]}-y\right)\left[\begin{array}{c} n_2 \\ r_2 \end{array}\right] y^{r_2}\prod_{s_2=0}^{n_2-r_2-1}(1-q_2^{s_2}y)\right)^2 \leq \frac{1}{4[n_2]}\ . \tag{2.11}$$

Combining now the inequalities (2.9), (2.10) and (2.11), one obtains

$$\left|B_{n_1,n_2}(f;x,y) - f(x,y)\right| \leq \omega\left(\frac{1}{\sqrt{[n_1]}}, \frac{1}{\sqrt{[n_2]}}\right) \times$$

$$\left(\sqrt{[n_1]}\cdot\frac{1}{2\sqrt{[n_1]}}+1\right)\left(\sqrt{[n_2]}\cdot\frac{1}{2\sqrt{[n_2]}}+1\right) = \frac{9}{4}\omega\left(\frac{1}{\sqrt{[n_1]}}, \frac{1}{\sqrt{[n_2]}}\right) \tag{2.12}$$

From (2.12), computing the supremum follows (2.5) and the proof ends.

Remarks 2.1.

(i) Clearly, if $q_1 = q_2 = 1$, then operator (2.4) is reduced to the classical Bernstein bivariate operator, considered first by D. D. Stancu [10].

In this case, the results contained in theorems 2.1 and 2.2 are reduced to the well-known results due to D. D. Stancu [10, 11].

(ii) If $q_1 = 1, q_2 \neq 1$ or $q_1 \neq 1, q_2 = 1$, other interesting operators of Bernstein type can be obtained. The approximation properties of these operators are similar to the properties of the classical bivariate operator of Bernstein.

(iii) All the results contained in the present paper can be extended to the case of n-variate functions.

Finally , we mention that a first form of this paper was presented at " microCAD 99"(in Miskolc, Hungary, February 1999), and we express our gratitude to all our colleagues from Miskolc, especially to Professor Aurél Galántai.

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RANK REDUCTION AND CONJUGATION

Aurél Galántai
Institute of Mathematics, University of Miskolc
3515 Miskolc – Egyetemváros, Hungary
matgal@gold.uni-miskolc.hu

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Abstract. We investigate Egerváry's rank reduction method and related conjugation algorithms. We give an exact characterization of the full rank factorization produced by the rank reduction algorithm and exploit this result concerning matrix decompositions and conjugation procedures.

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Keywords: Rank reduction, conjugation, matrix factorizations

1. Introduction

In the paper we investigate the rank reduction procedure of Egerváry [8], [10], [11], [24] and conjugation algorithms which are related to the following class of conjugate direction methods [25] for solving linear systems of the form

$$Ax = b \quad \left(A \in R^{m\times m},\ b \in R^m,\ \det(A) \neq 0\right). \tag{1.1}$$

ALGORITHM 1

Let $y_1 \in R^m$ be arbitrary.

for $k = 1 : m$
  $r_k = Ay_k - b$
  $\alpha_k = v_k^T r_k / (v_k^T A p_k)$
  $y_{k+1} = y_k - \alpha_k p_k$
end

Let $P = [p_1, p_2, \ldots, p_m] \in R^{m\times m}$ and $V = [v_1, v_2, \ldots, v_m] \in R^{m\times m}$ be nonsingular. The pair (P, V) is said to be A-conjugate if the matrix $L = V^T AP$ is lower triangular. The pair (P, V) is said to be A-biconjugate if $V^T AP = D$ is diagonal. Algorithm 1 terminates in m steps if $Ay_{m+1} = b$. Combining the results of Stewart [25] and Broyden [4] we can establish the following

Theorem 1 *Algorithm 1 terminates in m steps for any starting point y_1, if and only if the pair (P, V) is A-conjugate.*

A particular conjugate direction method is defined by the A-conjugate pair (P, V). In the paper we investigate conjugation procedures which are related to the rank reduction algorithm of Egerváry. Thus we deal with the ABS conjugation procedure of Abaffy, Broyden and Spedicato [1],[3], the conjugation procedure of Stewart [25], the two-sided Gram-Schmidt procedure of Hegedûs [16], [17], [18], Parlett [21], Chu, Funderlic and Golub [6] (for other A-conjugation procedures, see Voyevodin [26]).

It should be mentioned that the rank reduction algorithm was rediscovered concerning the ABS methods which were developed using quasi-Newton techniques [3].

We define A-conjugation for rectangular matrices as well. Let $A \in R^{m\times n}$, $V \in R^{m\times r}$ and $P \in R^{n\times r}$. The pair (P, V) is said to be A-conjugate if $L = V^T AP$ is nonsingular lower triangular. The pair (P, V) is A-biconjugate if $D = V^T AP$ is nonsingular diagonal.

Given any matrix A, the matrices $A^{\overline{k}}$, $A^{|k}$, $A^{\underline{k}}$ and $A^{k|}$ will denote the submatrices consisting of the first k rows, the first k columns, the last k rows and the last k columns, respectively. Thus $A^{|\overline{k}}$ is the leading principal submatrix of order k. A nonsingular matrix $A \in R^{m\times m}$ is said to be strongly nonsingular, if A has an LU decomposition. It is well known that a nonsingular matrix A has an LU decomposition, if and only if each $A^{|\overline{k}}$ is nonsingular for $k = 1, \ldots, m-1$.

In Section 2 we introduce the rank reduction procedure. In Sections 3 and 4 we investigate the necessary and sufficient conditions for performing the rank reduction procedure without breakdown. In Section 5 we give an exact characterization for the components of the full rank factorization $H_1 = Q_r D_r^{-1} P_r^T \in R^{m\times n}$ produced by the rank reduction algorithm and derive various factorization results. It is also shown that the components of the full rank factorization are biconjugate. More precisely the pair (Q_r, P_r) is H_1^--biconjugate, where H_1^- denotes any g-inverse of H_1. The ABS conjugation procedure, which produces I-conjugate pairs (X, P), is investigated in Section 6. Here we show that ABS conjugation is essentially a full rank factorization by Egerváry's rank reduction algorithm. The equivalence of Stewart's conjugation algorithm and the ABS conjugation is shown in Section 7. In Section 8 we investigate the TSGS biconjugation process of Hegedûs [16], [17], [18], Parlett [21], Chu, Funderlic and Golub [6]. It is shown to be a special case of Egerváry's rank reduction algorithm. Section 9 reviews some of the TSGS related results of Chu, Funderlic and Golub [6] giving new proofs as well.

2. Egerváry's rank reduction procedure

The rank reduction procedure is based on the following theorem of Egerváry [10] and Wedderburn [28].

Theorem 2 *(Egerváry [10]) Let $A \in R^{m\times n}$ be arbitrary with rank $(A) \geq 1$. Then*

$$rank\left(A - bc^T\right) = rank\,(A) - 1 \qquad (b \in R^m,\ c \in R^n) \tag{2.1}$$

holds if and only if $bc^T = Auv^T A / v^T Au$, where $u \in R^m$, $v \in R^n$ are arbitrary vectors subject only to the restriction $v^T Au \neq 0$.

It was Wedderburn who proved the if part of the theorem first [28]. Therefore we call this result the Egerváry-Wedderburn theorem. For other proofs and related matters we refer to Elsner-Rózsa [12] and Ouellette [20]. It is also noted that some authors mistakenly attribute Theorem 2 only to Wedderburn ([19]), to Householder ([20]) or to Wedderburn and Householder ([7]).

The rank reduction procedure of Egerváry [10], [11] is the following.

Let $H_1 \in R^{m\times n}$ be an arbitrary matrix of rank r and let

$$H_{k+1} = H_k - H_k x_k y_k^T H_k / y_k^T H_k x_k, \quad k = 1, \dots, r \tag{2.2}$$

for any vectors $x_k \in R^n$ and $y_k \in R^m$ for which $y_k^T H_k x_k \neq 0$.

As $rank\,(H_{k+1}) = rank\,(H_k) - 1$ the procedure terminates in r steps, that is $H_{r+1} = 0$. Using the notation $D_k = diag\left(y_1^T H_1 x_1, \dots, y_k^T H_k x_k\right)$ we can write H_k and H_1 in outer-product form

$$H_{k+1} = H_1 - [H_1 x_1, \dots, H_k x_k]\, D_k^{-1} \begin{bmatrix} y_1^T H_1 \\ \vdots \\ y_k^T H_k \end{bmatrix} \tag{2.3}$$

and

$$H_1 = [H_1 x_1, \dots, H_r x_r]\, D_r^{-1} \begin{bmatrix} y_1^T H_1 \\ \vdots \\ y_r^T H_r \end{bmatrix}, \tag{2.4}$$

respectively. It follows from recursion (2.2) that $Range\,(H_{k+1}) \subset Range\,(H_k)$ and $Range\left(H_{k+1}^T\right) \subset Range\left(H_k^T\right)$. If $X = [x_1, \dots, x_r]$ and $Y = [y_1, \dots, y_r]$, then for $m = n = r$

$$Null\,(H_k) = Span\left(X^{|k-1}\right), \quad Range\,(H_k) = Span^{\perp}\left(Y^{|k-1}\right). \tag{2.5}$$

3. The canonical form of Egerváry

We seek for conditions under which k steps of successive simple rank reduction operations (2.2) can be performed. We need the following result of Guttman [15] (see also Ouellette [20]).

Lemma 3 *(Guttman [15]) Let $A \in R^{m\times n}$ be partitioned in the form*

$$A = \begin{bmatrix} E & F \\ G & H \end{bmatrix}, \tag{3.1}$$

where $E \in R^{k\times k}$ is nonsingular. Then

$$rank\,(A) = rank\,(E) + rank\left(H - GE^{-1}F\right). \tag{3.2}$$

If both A and E are nonsingular, then the Schur complement $H - GE^{-1}F$ is also nonsingular. The if part of the following basic theorem was first proved by Egerváry [10], [11], while the only if part was first proved by Abaffy, Broyden and Spedicato [1], [3].

Theorem 4 *The first k steps of rank reduction algorithm (2.2) can be carried out if and only if $Y^{|kT} H_1 X^{|k}$ is strongly nonsingular. If $Y^{|kT} H_1 X^{|k}$ is strongly nonsingular, then*

$$H_{k+1} = H_1 - H_1 X^{|k} \left(Y^{|kT} H_1 X^{|k} \right)^{-1} Y^{|kT} H_1. \tag{3.3}$$

Proof. We first assume that k successive steps were performed, which means that $y_i^T H_i x_i \neq 0$ for $i = 1, \dots, k$. Let $Q_k = [H_1 x_1, \dots, H_k x_k]$ and $P_k = \left[H_1^T y_1, \dots H_k^T y_k \right]$. Then

$$H_{k+1} = H_1 - Q_k D_k^{-1} P_k^T.$$

Let us observe that $Y^{|kT} Q_k = \left[y_i^T H_j x_j \right]_{i,j=1}^k = L_k = \tilde{L}_k D_k$ is nonsingular lower triangular and $P_k^T X^{|k} = \left[y_i^T H_i x_j \right]_{i,j=1}^k = U_k = D_k \tilde{U}_k$ is nonsingular upper triangular. Here $\tilde{L}_k$ and $\tilde{U}_k$ are unit lower and upper triangular, respectively. We can also observe that

$$H_1 X^{|k} = \left(H_1 - H_{k+1} \right) X^{|k} = Q_k D_k^{-1} P_k^T X^{|k} = Q_k D_k^{-1} U_k = Q_k \tilde{U}_k$$

from which $Q_k = H_1 X^{|k} U_k^{-1} D_k = H_1 X^{|k} \tilde{U}_k^{-1}$ follows. Similarly we obtain that

$$Y^{|kT} H_1 = Y^{|kT} \left(H_1 - H_{k+1} \right) = Y^{|kT} Q_k D_k^{-1} P_k^T = L_k D_k^{-1} P_k^T = \tilde{L}_k P_k^T$$

and $P_k^T = D_k L_k^{-1} Y^{|kT} H_1 = \tilde{L}_k^{-1} Y^{|kT} H_1$. Hence

$$H_{k+1} = H_1 - H_1 X^{|k} \tilde{U}_k^{-1} D_k^{-1} \tilde{L}_k^{-1} Y^{|kT} H_1 = H_1 - H_1 X^{|k} \left(\tilde{L}_k D_k \tilde{U}_k \right)^{-1} Y^{|kT} H_1.$$

As

$$Y^{|kT} H_1 X^{|k} = Y^{|kT} Q_k D_k^{-1} P_k^T X^{|k} = \tilde{L}_k D_k \tilde{U}_k \tag{3.4}$$

we proved that $Y^{|kT} H_1 X^{|k}$ is strongly nonsingular. By substitution we obtain formula (3.3). Let us suppose that $Y^{|kT} H_1 X^{|k}$ is strongly nonsingular. We must prove that $y_i^T H_i x_i \neq 0$ for $i = 1, \dots, k$. As by assumption $y_1^T H_1 x_1 \neq 0$ the matrix $H_2 = H_1 - H_1 x_1 y_1^T H_1 / y_1^T H_1 x_1$ is defined. Let us assume that we have defined H_i for $i \leq k$. Then

$$H_i = H_1 - H_1 \left(X^{|i-1} \right)^T \left(\left(Y^{|i-1} \right)^T H_1 X^{|i-1} \right)^{-1} \left(Y^{|i-1} \right)^T H_1$$

and

$$y_i^T H_i x_i = y_i^T H_1 x_i - y_i^T H_1 \left(X^{|i-1}\right)^T \left(\left(Y^{|i-1}\right)^T H_1 X^{|i-1}\right)^{-1} \left(Y^{|i-1}\right)^T H_1 x_i.$$

Let us observe that $y_i^T H_i x_i$ is the Schur complement of the bordered matrix

$$Y^{|iT} H_1 X^{|i} = \begin{bmatrix} \left(Y^{|i-1}\right)^T H_1 X^{|i-1} & \left(Y^{|i-1}\right)^T H_1 x_i \\ y_i^T H_1 X^{|i-1} & y_i^T H_1 x_i \end{bmatrix}.$$

As $rank\left(Y^{|iT} H_1 X^{|i}\right) = i$ and $rank\left(\left(Y^{|i-1}\right)^T H_1 X^{|i-1}\right) = i-1$, the Guttman lemma implies that $rank\left(y_i^T H_i x_i\right) = 1$ and $y_i^T H_i x_i \neq 0$. Hence H_{i+1} is defined and we completed the proof of the theorem. ■

The first part of the proof is a slight modification of Theorems 4.2 and 4.3 in [3]. The second part of the proof is new.

We recall that for general bordered matrices of the form

$$B' = \begin{bmatrix} B & z \\ w^T & \lambda \end{bmatrix} \quad \left(B \in R^{k\times k},\ z, w \in R^k\right)$$

the identity $\det(B') = \det(B)\,(\lambda - w^T B^{-1} z)$ holds, if B is nonsingular. For $i > 1$ we can write

$$y_i^T H_i x_i = \det\left(Y^{|iT} H_1 X^{|i}\right) / \det\left(\left(Y^{|i-1}\right)^T H_1 X^{|i-1}\right). \tag{3.5}$$

For $i > 1$ it easily follows that

$$\prod_{j=1}^{i} y_j^T H_j x_j = \det\left(Y^{|iT} H_1 X^{|i}\right). \tag{3.6}$$

This formula is well-known concerning the bordering inversion method, where $1/(\lambda - w^T B^{-1} z)$ appears in place of λ in the inverse of B' (see, e.g. [23], [6]).

Using the canonical form (3.3) we can easily prove

Proposition 5 *Let H_1^- be any g-inverse of H_1. Then*

$$H_k H_1^- H_j = H_j H_1^- H_k = H_k \quad (j \leq k). \tag{3.7}$$

This result was first observed by Abaffy, Broyden and Spedicato for nonsingular H_1 [1], [3]. Let us notice that $H_1^- H_k$'s are projectors. Particularly, for $H_1 = I \in R^{m\times m}$ all H_k's are projectors.

For $k = r$ the canonical form (3.3) leads to the decomposition

$$H_1 = H_1 X \left(Y^T H_1 X\right)^{-1} Y^T H_1, \tag{3.8}$$

where $Z = X\left(Y^T H_1 X\right)^{-1} Y^T$ satisfies $H_1 Z H_1 = H_1$ and $Z H_1 Z = Z$. Hence Z is a reflexive g-inverse of H_1. If $H_1 = FG$ is a full rank factorization, $X = G^T M$ and $Y = FN$ ($M, N \in R^{r\times r}$ any nonsingular matrices), then $Z = X\left(Y^T H_1 X\right)^{-1} Y^T$ is the Moore-Penrose inverse of H_1. For other properties of the reflexive g-inverse Z we refer to Egerváry [9].

Proposition 6 *If $H_1 = FG$ is a full rank factorization, $X = G^T\left(GG^T\right)^{-1}$ and $Y = F\left(F^T F\right)^{-1}$, then H_{k+1} can be factorized in the form*

$$H_{k+1} = F\left(I - I^{|k} I^{\overline{k}}\right) G \quad \left(I \in R^{r\times r}\right). \tag{3.9}$$

4. Block rank reduction

Egerváry considered formula (3.3) as one block rank reduction [10], [11]. Block rank reduction was also investigated by Cline and Funderlic [7], and Ouellette [20] (see also Rao [22]). Here we recall the result of Cline and Funderlic ([7], Corollary 3.1) and give a new proof which exploits a technique of Ouellette ([20], Thms. 2.6a and 2.6b).

Theorem 7 *(Cline-Funderlic [7]) Let $H, S \in R^{m\times n}$, $rank\,(H) \geq rank\,(S) = k$ and let $UR^{-1}V^T$ ($U \in R^{m\times k}$, $R \in R^{k\times k}$, $V \in R^{n\times k}$) be a full rank factorization of S. Then*

$$rank\,(H - S) = rank\,(H) - rank\,(S) \tag{4.1}$$

if and only if

$$U = HX, \quad V^T = Y^T H, \quad Y^T H X = R \tag{4.2}$$

for some matrices $X \in R^{n\times k}$ and $Y \in R^{m\times k}$.

Proof. We first prove the if part. Let us consider the matrix

$$B = \begin{bmatrix} Y^T H X & Y^T H \\ HX & H \end{bmatrix} = \begin{bmatrix} Y^T \\ I_m \end{bmatrix} H\,[X, I_n]\,.$$

From the Sylvester law of nullity it follows that $rank\,(H) = rank\,(B)$. Applying the Guttman lemma we obtain

$$rank\left(H - HX\left(Y^T H X\right)^{-1} Y^T H\right) = rank\,(H) - rank\,(S)\,.$$

The only if part is demonstrated in the following way. Let us assume that $rank\,(H - S) < rank\,(H) - k + 1$ and let

$$B = \begin{bmatrix} R & V^T \\ U & H \end{bmatrix}.$$

As by the Guttman lemma

$$rank\,(H) \le rank\,(B) = rank\,(R) + rank\left(H - UR^{-1}V^T\right) < rank\,(H) + 1,$$

we obtain that $rank\,(H) = rank\,(B)$. Hence there exist matrices X and Y such that

$$\left[R, V^T\right] = Y^T\left[U, H\right], \quad \left[\begin{array}{c} R \\ U \end{array}\right] = \left[\begin{array}{c} V^T \\ H \end{array}\right] X.$$

From here we obtain that $V^T = Y^T H$, $U = HX$ and $R = V^T X = Y^T U = Y^T HX$. Thus we proved the theorem. ■

Remark 8 *The case $k = 1$ gives the Wedderburn-Egerváry theorem. The if part of Theorem 7 was first proved by Egerváry [10], [11] and also appears in Rao [22]. Ouellette investigated the block rank reduction*

$$H - HX\left(Y^T HX\right)^{-} Y^T H$$

with rectangular $Y^T HX$ and any choice of generalized inverse [20].

The block rank reduction and the successive simple rank reductions are not equivalent. Egerváry proved that for strongly nonsingular $Y^{|kT} H_1 X^{|k}$ the block reduction (3.3) is obtainable by k successive simple rank operations [10], [11]. This is not the case if $Y^{|kT} H_1 X^{|k}$ is not strongly nonsingular as shown by the following example

$$H_1 = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right], \quad X = Y = I, \quad k = 2.$$

So the conditions of block rank reduction are less restrictive. We can easily extend the canonical formula of Egerváry to the block rank reduction algorithm

$$H_{k+1} = H_k - H_k X_k \left(Y_k^T H_k X_k\right)^{-1} Y_k^T H_k, \quad k = 1, \ldots, t, \tag{4.3}$$

where $X_k \in R^{n \times l_k}$, $Y_k \in R^{m \times l_k}$, $l_k \ge 1$ and $rank\,(H_1) \ge \sum_{k=1}^{t} l_k$.

Let $m = \sum_{i=1}^{k} l_i$. A matrix $A \in R^{m \times m}$ is said to be partitioned with respect to the partition $\{l_1, \ldots, l_k\}$, if $A = [A_{ij}]_{i,j=1}^{k}$ and $A_{ij} \in R^{l_i \times l_j}$. A matrix A is said to be block strongly nonsingular with respect to the partition $\{l_1, \ldots, l_k\}$ if there exist nonsingular block lower triangular L and nonsingular block upper triangular U such that $A = LU$, and both matrices are partitioned with respect to $\{l_1, \ldots, l_k\}$. Let $\widetilde{X}^{|k} = [X_1, \ldots, X_k]$, $\widetilde{Y}^{|k} = [Y_1, \ldots, Y_k]$. Then

$$Range\,(H_k) = Span^{\perp}\left(\widetilde{Y}^{|k-1}\right), \quad Null\,(H_k) = Span\left(\widetilde{X}^{|k-1}\right). \tag{4.4}$$

The following result can be proved in the same way as Theorem 4.

Theorem 9 *The first k steps of block rank reduction algorithm (4.3) can be carried out if and only if $\widetilde{Y}^{|kT} H_1 \widetilde{X}^{|k}$ is block strongly nonsingular. If $\widetilde{Y}^{|kT} H_1 \widetilde{X}^{|k}$ is block strongly nonsingular, then*

$$H_{k+1} = H_1 - H_1 \widetilde{X}^{|k} \left(\widetilde{Y}^{|kT} H_1 \widetilde{X}^{|k} \right)^{-1} \widetilde{Y}^{|kT} H_1. \tag{4.5}$$

The only if part of the theorem first appeared in [2] with a slightly different proof (see, also [3]). If $\widetilde{Y}^{|kT} H_1 \widetilde{X}^{|k}$ is strongly nonsingular in the usual sense, then it is also block strongly nonsingular for any partition. In this case the if part of the theorem follows from Theorem 4, which is a special case of Theorem 9. We also note that all the results of subsequent sections which follow from Theorem 4 can also be established for the block case in a straightforward way.

5. Matrix factorizations and rank reduction

We determine the components of the full rank factorization (2.4) in terms of the parameters H_1, X and Y and derive important consequences such as matrix decompositions, orthogonalization and similarity transformations to Hessenberg and tridiagonal forms. It is also shown that the pair (Q_r, P_r) is H_1^--biconjugate, where H_1^- is any g-inverse.

Let $B = L_B D_B U_B$ be the unique LDU-decomposition of matrix $B \in R^{m \times m}$ with unit lower triangular L_B, diagonal D_B and unit upper triangular U_B. We later use the following simple observations.

If Y is nonsingular lower triangular, then $L_{YA} = Y L_A D_Y^{-1}$, $D_{YA} = D_Y D_A$ and $U_{YA} = U_A$. If X is nonsingular upper triangular, then $L_{AX} = L_A$, $D_{AX} = D_A D_X$ and $U_{AX} = D_X^{-1} U_A X$. Let Y be a nonsingular lower triangular matrix. A is strongly nonsingular, if and only if YA is also strongly nonsingular. Similarly, let X be a nonsingular upper triangular matrix. Then A is strongly nonsingular if and only if AX is also strongly nonsingular.

If two nonsingular matrices A and B have the LU-decompositions $A = L_1 U$ and $B = L_2 U$ with the same upper triangular U, then there is a unique lower triangular matrix L such that $A = LB$. Similarly, if $A = LU_1$ and $B = LU_2$ hold with the same lower triangular L, then there is a unique upper triangular matrix U such that $A = BU$.

From the proof of Theorem 4 (formula (3.4)) we can easily establish that

$$\tilde{L}_k = L_{Y^{|kT} H_1 X^{|k}}, \quad D_k = D_{Y^{|kT} H_1 X^{|k}}, \quad \widetilde{U}_k = U_{Y^{|kT} H_1 X^{|k}}, \tag{5.1}$$

$$P_k^T = L^{-1}_{Y^{|kT} H_1 X^{|k}} Y^{|kT} H_1, \quad Q_k = H_1 X^{|k} U^{-1}_{Y^{|kT} H_1 X^{|k}}. \tag{5.2}$$

For $k = r$ we obtain the components of Egerváry's full rank factorization in terms of the parameters X, Y and H_1.

Theorem 10 *Let $H_1 \in R^{m\times n}$ of rank r and let $Y^T H_1 X$ be strongly nonsingular. The components of the full rank factorization $H_1 = Q_r D_r^{-1} P_r^T$ are*

$$P_r = H_1^T Y L_{Y^T H_1 X}^{-T}, \quad Q_r = H_1 X U_{Y^T H_1 X}^{-1}, \quad D_r = D_{Y^T H_1 X}, \tag{5.3}$$

and H_1 can be expressed in the form

$$H_1 = \left(H_1 X U_{Y^T H_1 X}^{-1}\right) D_{Y^T H_1 X}^{-1} \left(L_{Y^T H_1 X}^{-1} Y^T H_1\right). \tag{5.4}$$

Expression (5.3) for P_r first appeared in [13], [14] with a different proof. It was not realized until writing this paper that the proof of Theorem 4 can be modified so that expressions (5.3)-(5.4) could be derived easily.

The matrices P_r and Q_r can be written in the form $P_r = H_1^T Y U_{X^T H_1^T Y}^{-1}$ and $Q_r = H_1 X L_{X^T H_1^T Y}^{-T}$, respectively. The role of P_r can be changed with Q_r by transposing H_1. As

$$H_1^T = \left(H_1^T Y U_{X^T H_1^T Y}^{-1}\right) D_{X^T H_1 Y}^{-1} \left(L_{X^T H_1^T Y}^{-1} X^T H_1^T\right) \tag{5.5}$$

several results on P_r can be formulated for Q_r if we replace H_1 by H_1^T, X and Y by each other.

We also note that for nonsingular upper triangular U_1 and U_2 the transformations $Y \to YU_1$ and $X \to XU_2$ change the factorization (5.4) to

$$H_1 = \left(H_1 X U_{Y^T H_1 X}^{-1} D_{U_2}\right) \left(D_{U_2}^{-1} D_{Y^T H_1 X}^{-1} D_{U_1}^{-1}\right) \left(D_{U_1} L_{Y^T H_1 X}^{-1} Y^T H_1\right). \tag{5.6}$$

Hence the effect of these transformations is only a diagonal scaling.

The following observation enlightens a basic property of the full rank factorization (5.4).

Proposition 11 *Let H_1^- denote any g-inverse of H_1. Then the pair (Q_r, P_r) is H_1^--biconjugate.*

Proof. By definition

$$P_r^T H_1^- Q_r = L_{Y^T H_1 X}^{-1} Y^T \left(H_1 H_1^- H_1\right) X U_{Y^T H_1 X}^{-1} = D_{Y^T H_1 X}.$$

The rank reduction procedure defines a full rank factorization (5.4) of matrix H_1. The next observation, which follows from Theorem 10, shows that all full rank factorizations can be generated by Egerváry's rank reduction procedure.

Proposition 12 *Let $H_1 \in R^{m\times n}$ be of rank r and let $H_1 = FG$ be any full rank factorization. Let $X = G^T \left(GG^T\right)^{-1}$ and $Y = F\left(F^T F\right)^{-1}$. Then $Q_r = F$, $D_r = I$ and $P_r^T = G$.*

We mention here two simple examples.

Example 13 *Let the singular value decomposition of $H_1 \in R^{m\times n}$ be given in the form $H_1 = U\left(\Sigma V^T\right)$, where $U \in R^{m\times r}$, $V \in R^{m\times r}$, $U^TU = I$, $V^TV = I$ and $\Sigma \in R^{r\times r}$ is nonsingular. The choice $F = U$, $G = \Sigma V^T$ yields $X = V\Sigma^{-1}$ and $Y = U$. Thus $Q_r = U$ and $P_r = \Sigma V^T$.*

The polar decomposition theorem says that every nonsingular matrix $A \in R^{m\times m}$ can be written in the form $A = QP$, where Q is an orthogonal and P is a symmetric positive definite matrix.

Example 14 *Let $H_1 \in R^{m\times m}$ be nonsingular and let $H_1 = QP$ be its polar decomposition. The choice $F = P$, $G = Q$ yields $X = P^{-1}$ and $Y = Q$. Thus $Q_m = Q$ and $P_m^T = P$.*

We consider now the matrix factorizations we can obtain from Theorem 10. For the rest of this section we assume that all matrices $H_1, X, Y \in R^{m\times m}$ are nonsingular so that X^TH_1Y are strongly nonsingular. Then Q_m and P_m are also nonsingular, $Q_m = H_1P_m^{-T}D_m$ and formula (5.4) can be written in the form

$$H_1 = \left(Y^{-T}L_{Y^TH_1X}D_{Y^TH_1X}\right)D_{Y^TH_1X}^{-1}\left(D_{Y^TH_1X}U_{Y^TH_1X}X^{-1}\right). \tag{5.7}$$

Proposition 15 *P_m^T is upper triangular if and only if X is upper triangular. P_m^T is lower triangular if and only if Y^TH_1 is lower triangular. Q_m is upper triangular if and only if H_1X is upper triangular. Q_m is lower triangular if and only if Y^T is lower triangular.*

Proof. From the relation $P_m^T = \left(D_{Y^TH_1X}U_{Y^TH_1X}\right)X^{-1} = \widetilde{U}$, where $\widetilde{U}$ is upper triangular, it follows that $X^{-1} = U_{Y^TH_1X}^{-1}D_{Y^TH_1X}^{-1}\widetilde{U}$ is also upper triangular. Hence X is upper triangular. If $P_m^T = L_{Y^TH_1X}^{-1}Y^TH_1 = \widetilde{L}$, where $\widetilde{L}$ is lower triangular, then $Y^TH_1 = L_{Y^TH_1X}\widetilde{L}$ is also lower triangular. Similarly, $Q_m = H_1XU_{Y^TH_1X}^{-1} = \widetilde{U}$, if and only if H_1X is upper triangular. Finally, $Q_m = Y^{-T}L_{Y^TH_1X}D_{Y^TH_1X} = \widetilde{L}$, if and only if Y^T is lower triangular. ■

Corollary 16 *Q_m is lower triangular and P_m^T is upper triangular, if and only if X and Y are upper triangular.*

In this case factorization (5.4) simplifies to the LU factorization

$$H_1 = \left(L_{H_1}D_{H_1}D_X\right)\left(D_X^{-1}D_{H_1}^{-1}D_{Y^T}^{-1}\right)\left(D_{Y^T}D_{H_1}U_{H_1}\right)$$

in agreement with (5.6). For $X = Y = I$ this result was first observed by Egerváry [8], [10] (for ABS conjugation, see [3]).

Proposition 17 *P_m is upper Hessenberg if and only if $H_1^T Y$ is upper Hessenberg. Q_m is upper Hessenberg if and only if $H_1 X$ is upper Hessenberg.*

Proof. We exploit the fact that the upper Hessenberg form is invariant under multiplication by upper triangular matrix from both sides. By definition $P_m = H_1^T Y L_{Y^T H_1 X}^{-T} = F$, where F is upper Hessenberg if and only if $H_1^T Y = F L_{Y^T H_1 X}^T$ is upper Hessenberg. Similarly, $Q_m = H_1 X U_{Y^T H_1 X}^{-1} = F$ is upper Hessenberg if and only if $H_1 X = F U_{Y^T H_1 X}$ is upper Hessenberg. ■

Proposition 18 *Let $B \in R^{m\times m}$ be a symmetric and positive definite matrix. P_m is B-orthogonal if and only if $X = BH_1^T YU$ holds with any nonsingular Y and a nonsingular upper triangular U.*

Proof. P_m is B-orthogonal, if and only if

$$P_m^T B P_m = L_{Y^T H_1 X}^{-1} Y^T H_1 B H_1^T Y L_{Y^T H_1 X}^{-T} = D,$$

where D is diagonal. This holds exactly if $Y^T H_1 B H_1^T Y = L_{Y^T H_1 X} D L_{Y^T H_1 X}^T$. This implies $L_{Y^T H_1 B H_1^T Y} = L_{Y^T H_1 X}$. Hence a nonsingular upper triangular U exists such that $Y^T H_1 B H_1^T Y U = Y^T H_1 X$, from which $X = BH_1^T YU$ follows. As B is symmetric and positive definite, the nonsingularity of Y and U is the condition for the strong nonsingularity of $Y^T H_1 X$. ■

We can drop the positive definiteness condition of B if we assume that $Y^T H_1 X = Y^T H_1 B H_1^T Y$ is strongly nonsingular.

Corollary 19 *P_m is orthogonal up to a diagonal scaling and Q_m is lower triangular if and only if $X = H_1^T Y U$ holds with any nonsingular upper triangular U and Y. In such a case, factorization (5.4) is the* LQ*-factorization of H_1.*

Proposition 20 *Let $B \in R^{m\times m}$ be a symmetric and positive definite matrix. Q_m is B-orthogonal, if and only if $Y = BH_1 X L^T$ holds with any nonsingular X and a nonsingular lower triangular L.*

Proof. Q_m is B-orthogonal if and only if

$$Q_m^T B Q_m = U_{Y^T H_1 X}^{-T} X^T H_1^T B H_1 X U_{Y^T H_1 X}^{-1} = D,$$

where D is diagonal. This holds exactly if $X^T H_1^T B H_1 X = U_{Y^T H_1 X}^T D U_{Y^T H_1 X}$ from which $U_{X^T H_1^T B H_1 X} = U_{Y^T H_1 X}$ follows. Hence a nonsingular lower triangular matrix L exists such that $L X^T H_1^T B H_1 X = Y^T H_1 X$. This implies $Y = BH_1 X L^T$. As B is symmetric and positive definite, the nonsingularity of X and L is the condition for the strong nonsingularity of $Y^T H_1 X$. ■

Corollary 21 *Q_m is orthogonal up to a diagonal scaling and P_m^T is upper triangular if and only if $Y = H_1 X L^T$ holds with any nonsingular lower triangular L and upper triangular X. In such a case, factorization (5.4) is the* QR*-factorization of H_1.*

We note that special cases of Corollaries 19 and 21 appear in the ABS conjugation [3].

Given a matrix $H_1 \in R^{m\times m}$, a GR decomposition of H_1 is a factorization of H_1 into a product $B = GR$, where G is nonsingular and R is upper triangular [27]. By Proposition 12 any GR decomposition can be obtained by the rank reduction algorithm of Egerváry. For special cases such as the LU and QR factorizations we refer to Corollaries 16 and 21.

Finally we investigate the special case $H_1 = I$. Then

$$I = \left(XU_{Y^TX}^{-1}\right) D_{Y^TX}^{-1} \left(L_{Y^TX}^{-1} Y^T\right) \tag{5.8}$$

and the factors $Q_m = XU_{Y^TX}^{-1} = Y^{-T}L_{Y^TX}D_{Y^TX}$, $D_m = D_{Y^TX}$ and $P_m = YL_{Y^TX}^{-T} = X^{-T}U_{Y^TX}^{T}D_{Y^TX}$ are such that $Q_m^T = \left(P_m D_m^{-1}\right)^{-1}$.

The following table gives some LU factorization related results for $H_1 = I$ and the parameters given therein.

X	Y	Q_m	D_m^{-1}	P_m^T
A	I	$L_A D_A$	D_A^{-1}	L_A^{-1}
A^T	I	$U_A^T D_A$	D_A^{-1}	U_A^{-T}
I	A	L_A^{-T}	D_A^{-1}	$D_A L_A^T$
I	A^T	U_A^{-1}	D_A^{-1}	$D_A U_A$

It is noted that the parameters $X = A^T$ and $Y = I$ define the implicit LU ABS method (see [3]).

The following observations are related to the Lánczos reductions (see, e.g. [19]). Let $y_1 \in R^m$ and $y_i = A^{i-1}y_1$ $(i = 2,\dots,m)$. Let $y_1 \in R^m$ be such that the Krylov matrix $Y = \left[y_1, Ay_1, \dots, A^{m-1}y_1\right]$ is nonsingular. Then $A^m y = \sum_{i=1}^m \gamma_i A^{i-1}y_1$ and $AY = YF$, where the companion matrix F is defined by

$$F = \begin{bmatrix} 0 & 0 & \dots & 0 & \gamma_1 \\ 1 & 0 & & \vdots & \vdots \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \gamma_{m-1} \\ 0 & \dots & 0 & 1 & \gamma_m \end{bmatrix}.$$

We exploit that $Q_m^T = \left(P_m D_m^{-1}\right)^{-1}$ for $H_1 = I$. Let us consider the similarity transformation

$$\begin{aligned} Q_m^T A\left(P_m D_m^{-1}\right) &= D_{Y^TX}L_{Y^TX}^T\left(Y^{-1}AY\right)L_{Y^TX}^{-T}D_{Y^TX}^{-1} \\ &= D_{Y^TX}L_{Y^TX}^T F L_{Y^TX}^{-T}D_{Y^TX}^{-1}, \end{aligned} \tag{5.9}$$

where the upper Hessenberg matrix F is multiplied by two upper triangular matrices. This multiplication keeps the upper Hessenberg form. Hence Egerváry's rank reduction algorithm defines a similarity transformation of matrix A to upper Hessenberg form and gives the transformation matrices Q_m^T and Q_m^{-T} simultaneously. Let us assume again that $x_1 \in R^m$ is such that the Krylov matrix $X = \left[x_1, A^T x_1, \dots, \left(A^T\right)^{m-1} x_1\right]$ is nonsingular. Then $A^T X = X F_1$ holds with a suitable companion matrix F_1. With this choice of X, relation (5.9) becomes

$$Q_m^T A\left(P_m D_m^{-1}\right) = U_{Y^T X}^{-T}\left(X^T A X^{-T}\right) U_{Y^T X}^T = U_{Y^T X}^{-T} F_1^T U_{Y^T X}^T,$$

where the lower Hessenberg matrix F_1^T is multiplied by two lower triangular matrices. This multiplication keeps the lower Hessenberg form. As $Q_m^T A\left(P_m D_m^{-1}\right)$ is of both upper and lower Hessenberg form, the matrix must be tridiagonal. Hence the rank reduction algorithm also defines a similarity transformation of matrix A to tridiagonal form giving the transformation matrices Q_m^T and Q_m^{-T} simultaneously.

Theorem 22 *Let $y_1 \in R^m$ be such that $Y = \left[y_1, Ay_1, \dots, A^{m-1}y_1\right]$ is nonsingular. If $H_1 = I$ and X is such that $Y^T X$ is strongly nonsingular, then $Q_m^T A\left(P_m D_m^{-1}\right)$ defines an upper Hessenberg matrix which is similar to A. If, in addition $X = \left[x_1, A^T x_1, \dots, \left(A^T\right)^{m-1} x_1\right]$, then $Q_m^T A\left(P_m D_m^{-1}\right)$ is a tridiagonal matrix, which is similar to A.*

Similar results appeared in Stewart [25], and Abaffy and Spedicato [3] in association with the A-conjugation algorithms. What is really new here is the simultaneous computation of the transformation matrix and its inverse.

6. The *ABS* conjugation procedure

In this section we investigate the *ABS* conjugation procedure of Abaffy, Broyden and Spedicato [3]. This algorithm, which is based on the rank reduction procedure, produces A-conjugate pairs (X, P). We assume that $H_1, A \in R^{m\times m}$ are nonsingular.

ALGORITHM 2

$H_1 \in R^{m\times m}$ is arbitrary nonsingular

for $k = 1 : m$
  $p_k = H_k^T z_k \quad (z_k \in R^m,\ x_k^T p_k \neq 0)$
  $H_{k+1} = H_k - H_k x_k y_k^T H_k / y_k^T H_k x_k \quad \left(y_k \in R^m,\ y_k^T H_k x_k \neq 0\right)$
end

As for $i < j$ the relation $x_i \in Null\left(H_j\right)$ implies $x_i^T I p_j = x_i^T H_j^T p_j = 0$, the matrix $L = X^T P$ is nonsingular lower triangular. Hence Algorithm 2 produces the I-conjugate pair (P, X). The substitution $x_i = A^T v_i$ $(i = 1, \dots, m)$ or $X = A^T V$ yields $L = X^T P = V^T A P$. So the resulting pair (P, V) is A-conjugate. Therefore it is enough to formulate and prove statements only for I-conjugate pairs.

We can also define the conjugate pair (Q, Y) by setting $q_k = H_k w_k$ $(k = 1, \dots, m)$ and $Q = [q_1, \dots, q_m]$. For $i < j$, $y_i^T I q_j = y_i^T H_j w_j = 0$. Hence the pair (Q, Y) is indeed I-conjugate.

For further use let $Z = [z_1, \dots, z_m]$. We seek for conditions under which Algorithm 2 produces nonsingular I-conjugate pairs (P, X). This happens if and only if

$$x_k^T p_k \neq 0, \quad y_k^T H_k x_k \neq 0 \quad (i = 1, \dots, m)\,. \tag{6.1}$$

The second condition holds if and only if $X^T H_1 Y$ is strongly nonsingular. By definition $x_1^T p_1 = x_1^T H_1^T z_1$ and

$$x_{k+1}^T p_{k+1} = x_{k+1}^T H_1^T z_{k+1} - x_{k+1}^T H_1^T Y^{|k} \left(X^{|kT} H_1^T Y^{|k} \right)^{-1} X^{|kT} H_1^T z_{k+1},$$

which is the Schur complement of the bordered matrix

$$\left(X^{|k+1} \right)^T H_1^T \left[Y^{|k}, z_{k+1} \right] = \begin{bmatrix} X^{|kT} H_1^T Y^{|k} & X^{|kT} H_1^T z_{k+1} \\ x_{k+1}^T H_1^T Y^{|k} & x_{k+1}^T H_1^T z_{k+1} \end{bmatrix}.$$

By the Guttman lemma $x_{k+1}^T p_{k+1} \neq 0$ holds if and only if

$$rank \left(x_{k+1}^T p_{k+1} \right) = rank \left(\left(X^{|k+1} \right)^T H_1^T \left[Y^{|k}, z_{k+1} \right] \right) - rank \left(X^{|kT} H_1^T Y^{|k} \right) = 1.$$

This happens exactly if $rank \left(\left(X^{|k+1} \right)^T H_1^T \left[Y^{|k}, z_{k+1} \right] \right) = k + 1$, that is if

$$\det \left(\left(X^{|k+1} \right)^T H_1^T \left[Y^{|k}, z_{k+1} \right] \right) \neq 0.$$

Thus we have

Theorem 23 *Algorithm 2 produces a nonsingular I-conjugate pair (P, X) if and only if the matrix $Y^T H_1 X$ is strongly nonsingular, $x_1^T H_1^T z_1 \neq 0$ and*

$$\det \left(\left(X^{|k+1} \right)^T H_1^T \left[Y^{|k}, z_{k+1} \right] \right) \neq 0 \quad (k = 1, \dots, m-1)\,. \tag{6.2}$$

If we choose $z_i = y_i$ $(i = 1, \dots, m)$, then condition (6.1) changes to

$$x_k^T p_k = y_k^T H_k x_k \neq 0 \quad (i = 1, \dots, m). \tag{6.3}$$

Hence, the strong nonsingularity of $Y^T H_1 X$ is the necessary and sufficient condition for producing nonsingular I-conjugate pairs (X, P). The choice $Z = Y$ defines the generalized implicit LU or $GILUABS$ subclass of the ABS methods [13], [14]. Let us notice that $p_i = H_i^T y_i$ is the ith component of P_m, the transpose of which we investigated in Section 5. As $P = P_m$ we can conclude that $GILUABS$ conjugation is producing the pair (X, P_m), which is I-conjugate. So the $GILUABS$ conjugation is nothing else but a full rank factorization by Egerváry's rank reduction procedure. Similarly, we have that pair (Q_m, Y) is also I-conjugate. Theorems 4 and 10 imply

Theorem 24 *Let $Z = Y$. Algorithm 2 produces a nonsingular I-conjugate pair (P, X) if and only if the matrix $Y^T H_1 X$ is strongly nonsingular. If $Y^T H_1 X$ is strongly nonsingular, then $P = H_1^T Y L_{Y^T H_1 X}^{-T} = H_1^T Y U_{X^T H_1^T Y}^{-1}$.*

We prove the following characterizations of Algorithm 2.

Theorem 25 *For any I-conjugate pair (P, X) there exist an orthogonal matrix Q and a nonsingular diagonal matrix D such that Algorithm 2 with $H_1 = I$, $Y = Z = QD$ generates the pair (P, X).*

Proof. Let $P = QDR_1$ be a QR-factorization of P with diagonal D and unit upper triangular R_1. As $X^T P = X^T QDR_1 = L$ implies $X^T QD = LR_1^{-1}$, the matrix $X^T QD$ is strongly nonsingular. Hence by Theorem 24 Algorithm 2 produces exactly the pair (P, X) for the parameters $H_1 = I$, $Z = Y = QD$. ■

For a given X and orthogonal Q there does not necessarily exist an I-conjugate pair (P, X) produced by a *GILUABS* conjugation. For example, let $H_1 = I$,

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{bmatrix} \quad \left(\alpha^2 + \beta^2 = 1\right).$$

Then

$$Y^T H_1 X = Q^T X = \begin{bmatrix} 0 & 1 & 0 \\ \alpha & \alpha & -\beta \\ \beta & \beta & \alpha \end{bmatrix}$$

is not strongly nonsingular.

Proposition 26 *Let $Z = Y$. Algorithm 2 produces a nonsingular I-biconjugate pair (P, X), if and only if the matrix $Y^T H_1 X$ is lower triangular.*

Proof. The pair (P, X) is I-biconjugate, if $X^T P = D$ is diagonal. As $X^T P = X^T H_1^T Y U_{X^T H_1^T Y}^{-1} = L_{X^T H_1^T Y} D_{X^T H_1^T Y}$ this may happen exactly if $L_{X^T H_1^T Y}$ is also diagonal. ■

Theorem 27 *Let $B \in R^{m \times m}$ be a symmetric positive definite matrix and $Z = Y$. Algorithm 2 produces a nonsingular I-conjugate pair (P, X) such that P is B-orthogonal, if and only if $X = BH_1^T Y L^T$, where Y is any nonsingular and L is any nonsingular lower triangular matrix. In such a case $P = H_1^T Y U_{Y^T H_1 B H_1^T Y}^{-1}$.*

The same P can be obtained, if we select $X = BP$ in Algorithm 2. Then $X^T P = P^T BP = L$ implies that $L = D$ is diagonal,

$$U_{X^T H_1^T Y}^{-T} Y^T H_1 B H_1^T Y U_{X^T H_1^T Y}^{-1} = D,$$

$Y^TH_1BH_1^TY = U^T_{X^TH_1^TY}DU_{X^TH_1^TY}$ and $X = BH_1^TYL^T$ with a nonsingular lower triangular L. In fact, the choice $X = BP = BH_1^TYU^{-1}_{X^TH_1^TY}$ corresponds to $L^T = U^{-1}_{X^TH_1^TY}$.

The above result appeared first in Stewart [25] for the special case $X = A^TV$, $V = AP$, $H_1 = Z = Y = I$ and $L^T = I$, when $P = YU^{-1}_{Y^TA^TAY}$ (see also [3]).

7. Stewart 's conjugation procedure

Stewart's A-conjugation process is based on the LU factorization $V^TAQ = LS$, where $Q = [q_1, \dots, q_m]$ is the parameter [25]. Clearly, $\widetilde{P} = QS^{-1}$ defines the A-conjugate pair $\left(\widetilde{P}, V\right)$. We prove that Stewart's A-conjugation procedure is equivalent with the $GILUABS$ conjugation procedure. Hence it is also related to the rank reduction algorithm.

STEWART'S CONJUGATION ALGORITHM

$\tilde{p}_1 = q_1$
for $k = 1 : m$
  if $k > 1$
    $s_k = \left(L^{|\overline{k-1}}\right)^{-1}\left(V^{|k-1}\right)^T Aq_k$
    $\tilde{p}_k = \sigma^{-1}_{kk}\left(q_k - \tilde{P}^{|k-1}s_k\right), \quad \sigma_{kk} \neq 0$
  end
  $L^{|\overline{k}} = \left(V^{|k}\right)^T A\tilde{P}^{|k}$
end

It is easy to see that

$$\tilde{p}_{k+1} = \sigma^{-1}_{k+1,k+1}\left(I - \tilde{P}^{|k}\left(V^{|kT}A\tilde{P}^{|k}\right)^{-1}V^{|kT}A\right)q_{k+1}.$$

As $\tilde{P} = QS^{-1}$ we can write $\tilde{P}^{|k} = Q^{|k}\left(S^{-1}\right)^{|\overline{k}}$ and

$$\tilde{P}^{|k}\left(V^{|kT}A\tilde{P}^{|k}\right)^{-1}V^{|kT}A = Q^{|k}\left\{V^{|kT}AQ^{|k}\right\}^{-1}V^{|kT}A.$$

By the simple substitution $y_k = \sigma^{-1}_{kk}q_k$ we obtain that $\tilde{p}_k = H_k^Ty_k$ for $k = 1, \dots, m$. Thus we proved that Stewart's conjugation procedure is algebraically equivalent with Algorithm 2 in the special case $Z = Y$, $H_1 = I$ and $X = A^TV$. In fact, the Stewart algorithm implicitly uses the canonical form (3.3).

8. The two-sided Gram-Schmidt method

In this section we deal with the two-sided Gram-Schmidt (TSGS) process, which produces an H_1-biconjugate pair (U, V). Let $H_1 \in R^{m\times n}$ be of rank r, $X = [x_1, \dots, x_r]$

and $Y = [y_1, \dots, y_r]$. The two-sided Gram-Schmidt algorithm is defined by

$$u_1 = x_1, \; v_1 = y_1 \tag{8.1}$$

$$u_k = x_k - \sum_{i=1}^{k-1} \frac{v_i^T H_1 x_k}{v_i^T H_1 u_i} u_i \quad (k = 2, \dots, r) \tag{8.2}$$

$$v_k = y_k - \sum_{i=1}^{k-1} \frac{y_k^T H_1 u_i}{v_i^T H_1 u_i} v_i \quad (k = 2, \dots, r) \tag{8.3}$$

where u_k and v_k are the projections of x_k and y_k, respectively.

This process was developed and rediscovered by several authors, the first of which is C. Hegedûs [16], [17],[18]. Hegedûs proved that the pair (U, V) is H_1-biconjugate, $H_1 = H_1 U \left(V^T H_1 U\right)^{-1} V^T H_1$, and $U \left(V^T H_1 U\right)^{-1} V^T$ is a reflexive g-inverse of H_1 if the process is carried out. Parlett [21] defined algorithm (8.1)-(8.3) for $H_1 = I \in R^{m \times m}$. Chu, Funderlic and Golub [6] defined and investigated the TSGS biconjugation process in association with the Egerváry rank reduction algorithm. Finally we mention Broyden [5], who gave a block generalization of the Gram-Schmidt algorithm, which contains the TSGS biconjugation process of Hegedûs as a special case.

Using Theorem 10 we give a unique characterization of U and V in terms of the parameters H_1, X and Y. We show that TSGS biconjugation is a special case of Egerváry's rank reduction algorithm.

Chu, Funderlic and Golub [6] proved the following result in association with the rank reduction algorithm (2.2).

Theorem 28 *(Chu-Funderlic-Golub [6]) Provided that process (8.1)-(8.3) is carried out,* $H_1 u_k = H_k x_k$, $v_k^T H_1 = y_k^T H_k$, $y_k^T H_k x_k = v_k^T H_1 u_k$ *for all* k *and* $v_i^T H_1 u_j = 0$ *for all* $i \neq j$.

Let $U = [u_1, \dots, u_r]$, $V = [v_1, \dots, v_r]$ and $\Omega = diag\,(\omega_i)$. Then $V^T H_1 U = \Omega$ is nonsingular diagonal by Theorem 28. Hence the pair (U, V) is H_1-biconjugate. We can observe that U and V also satisfy the relations $Q_r = H_1 U$, $\Omega = D_r$ and $P_r = H_1^T V$. Thus from the representation (5.3)-(5.4) we can assume that

$$U = X U_{Y^T H_1 X}^{-1}, \quad V = Y L_{Y^T H_1 X}^{-T} = Y U_{X^T H_1^T Y}^{-1}. \tag{8.4}$$

Using the proof of Theorem 4 we can prove this and more.

Theorem 29 *Let* $H_1 \in R^{m \times n}$ *be of rank* r, $X \in R^{n \times r}$ *and* $Y \in R^{m \times r}$ *The TSGS biconjugation algorithm can be carried out if and only if Egerváry's rank reduction algorithm can be carried out. In such a case, when* $Y^T H_1 X$ *is strongly nonsingular, relation (8.4) holds.*

Proof. Let us assume first that $Y^T H_1 X$ is strongly nonsingular. For $k = 1$, $u_1 = X U_{Y^T H_1 X} e_1 = X e_1 = x_1$ and $v_1 = Y L_{Y^T H_1 X}^{-T} e_1 = Y e_1 = y_1$. Assuming the statement for $1 \leq i \leq k-1$ we can write

$$v_i^T H_1 u_i = e_i^T V^T H_1 U e_i = e_i^T L_{Y^T H_1 X}^{-1} Y^T H_1 X U_{Y^T H_1 X}^{-1} e_i = e_i^T D_{Y^T H_1 X} e_i = \omega_i.$$

By definition we have

$$\begin{aligned}
u_k &= x_k - \sum_{i=1}^{k-1} \frac{v_i^T H_1 x_k}{v_i^T H_1 u_i} u_i \\
&= X e_k - \sum_{i=1}^{k-1} \frac{e_i^T L_{Y^T H_1 X}^{-1} Y^T H_1 X e_k}{\omega_i} X U_{Y^T H_1 X}^{-1} e_i \\
&= X U_{Y^T H_1 X}^{-1} \left(I - \sum_{i=1}^{k-1} \frac{e_i e_i^T D_{Y^T H_1 X}}{\omega_i} \right) U_{Y^T H_1 X} e_k \\
&= X U_{Y^T H_1 X}^{-1} e_k.
\end{aligned}$$

Similarly we have

$$v_k = y_k - \sum_{i=1}^{k-1} \frac{y_k^T H_1 u_i}{v_i^T H_1 u_i} v_i = Y L_{Y^T H_1 X}^{-T} e_k.$$

It is also clear that $v_i^T H_1 u_j = e_i^T L_{Y^T H_1 X}^{-1} Y^T H_1 X U_{Y^T H_1 X}^{-1} e_j = e_i^T D_{Y^T H_1 X} e_j$ is zero, if $i \neq j$ and ω_i for $i = j$ $(1 \leq i, j \leq k)$. Let us assume now that $Y^T H_1 X$ is not strongly nonsingular. Then there is an index t such that $y_t^T H_t x_t = 0$ and $y_i^T H_i x_i \neq 0$ for $i < t$ (whenever $t > 1$). Hence we can calculate $\{u_i, v_i\}_{i=1}^t$ with the *TSGS* algorithm. We cannot define, however, u_{t+1} and v_{t+1} because $y_t^T H_t x_t = v_t^T H_1 u_t = 0$. Thus we proved the theorem. ■

Let us notice that we also proved Theorem 28.

The following part of the above theorem was formulated and proved in a different way by Chu, Funderlic and Golub [6].

Theorem 30 *([6], Thm. 2.2) Let $H_1 \in R^{m \times n}$ be of rank r, $X \in R^{n \times r}$ and $Y \in R^{m \times r}$. The TSGS biconjugation algorithm can be carried out, if and only if $Y^T H_1 X$ is strongly nonsingular. In such a case, relation (8.4) holds.*

This Theorem altogether with Theorem 4 also yields Theorem 29. The close relationship between the rank reduction and TSGS biconjugation is not at all surprising. Next we show that TSGS is a special case of Egerváry's rank reduction algorithm. We have two possibilities for doing this.

Let $H_1 = A \in R^{m \times n}$ be of rank r, $X \in R^{n \times r}$ and $Y \in R^{m \times r}$ be such that $Y^T A X$ is strongly nonsingular. Let $A^- = X \left(Y^T A X\right)^{-1} Y^T$ be the reflexive g-inverse of A

produced implicitly by the rank reduction procedure (2.2). Let us define the rank reduction process

$$\begin{aligned} \widehat{H}_1 &= A^- && (8.5)\\ \widehat{H}_{k+1} &= \widehat{H}_k - \widehat{H}_k A x_k y_k^T A \widehat{H}_k / y_k^T A \widehat{H}_k A x_k \quad (k=1,\ldots,r) && (8.6) \end{aligned}$$

and the related quantities

$$\widehat{p}_k = \widehat{H}_k^T A^T y_k, \quad \widehat{q}_k = \widehat{H}_k A x_k \quad (k=1,\ldots,r). \tag{8.7}$$

As $\left(A^T Y\right)^T \widehat{H}_1 (AX) = Y^T A A^- A X = Y^T A X$, the rank reduction process can be carried out, and by the special choice of A^- we have $\widehat{Q}_r = A^- A X U_{Y^T AX}^{-1} = X U_{Y^T AX}^{-1}$ and $\widehat{P}_r = \left(A^-\right)^T A^T Y L_{Y^T AX}^{-T} = Y L_{Y^T AX}^{-T}$. These are exactly U and V defined in (8.4). We have the following recursions

$$\widehat{p}_{k+1} = \widehat{H}_1^T A^T y_{k+1} - \sum_{i=1}^{k} \frac{\widehat{q}_i^T A^T y_{k+1}}{y_i^T A \widehat{H}_i A x_i} \widehat{p}_i$$

and

$$\widehat{q}_{k+1} = \widehat{H}_1 A x_{k+1} - \sum_{i=1}^{k} \frac{\widehat{p}_i^T A x_{k+1}}{y_i^T A \widehat{H}_i A x_i} \widehat{q}_i.$$

As for reflexive g-inverses $\left(A^-\right)^- = A$, we can write

$$y_i^T A \widehat{H}_i A x_i = y_i^T A \widehat{H}_i \widehat{H}_1^- \widehat{H}_i A x_i = \widehat{p}_i^T \widehat{H}_1^- \widehat{q}_i = \widehat{p}_i^T A \widehat{q}_i.$$

Also, by the special choice of A^-, we have $\widehat{H}_1^T A^T y_{k+1} = y_{k+1}$ and $\widehat{H}_1 A x_{k+1} = x_{k+1}$. Thus we have the following recursion

$$\widehat{p}_{k+1} = y_{k+1} - \sum_{i=1}^{k} \frac{y_{k+1}^T A \widehat{q}_i}{\widehat{p}_i^T A \widehat{q}_i} \widehat{p}_i, \quad \widehat{q}_{k+1} = x_{k+1} - \sum_{i=1}^{k} \frac{\widehat{p}_i^T A x_{k+1}}{\widehat{p}_i^T A \widehat{q}_i} \widehat{q}_i. \tag{8.8}$$

Letting $u_k = \widehat{q}_k$ and $v_k = \widehat{p}_k$ we obtain the TSGS recursion (8.1)-(8.3).

Theorem 31 *The rank reduction process (8.5)-(8.7) can be written as TSGS biconjugation algorithm.*

We can obtain the TSGS biconjugation process without using a special reflexive g-inverse, if we use two rank reduction processes as follows. Let I_m denote the $m \times m$ unit matrix. Let

$$\begin{aligned} \widehat{H}_1 &= I_n && (8.9)\\ \widehat{H}_{k+1} &= \widehat{H}_k - \widehat{H}_k A^T y_k x_k^T \widehat{H}_k / x_k^T \widehat{H}_k A^T y_k \quad (k=1,\ldots,r) && (8.10)\\ \widehat{p}_k &= \widehat{H}_k^T x_k && (8.11) \end{aligned}$$

and

$$\begin{aligned} H_1 &= I_m && (8.12)\\ \widetilde{H}_{k+1} &= \widetilde{H}_k - \widetilde{H}_k A x_k y_k^T \widetilde{H}_k / y_k^T \widetilde{H}_k A x_k \quad (k = 1, \ldots, r) && (8.13)\\ \widetilde{p}_k &= \widetilde{H}_k^T y_k. && (8.14) \end{aligned}$$

As $Y^T AX$ is strongly nonsingular, both recursions can be carried out. By Theorem 10 $\widehat{P}_r = XL^{-T}_{X^T A^T Y} = XU^{-1}_{Y^T AX}$ and $\widetilde{P}_r = YL^{-T}_{Y^T AX}$. These are exactly U and V defined by (8.4). Let us notice that the related quantities $\widehat{Q}_r$ and $\widetilde{Q}_r$ are not equal to V or U. For $\widehat{P}_r$ and $\widetilde{P}_r$ we have the following recursions

$$\widehat{p}_{k+1} = x_{k+1} - \sum_{i=1}^{k} \frac{y_i^T A \widehat{H}_i^T x_{k+1}}{x_i^T \widehat{H}_i A^T y_i} \widehat{p}_i$$

and

$$\widetilde{p}_{k+1} = y_{k+1} - \sum_{i=1}^{k} \frac{x_i^T A^T \widetilde{H}_i^T y_{k+1}}{y_i^T \widetilde{H}_i A x_i} \widetilde{p}_i.$$

Noticing that $A\widehat{H}_i^T = \widetilde{H}_i A$ we can write

$$y_i^T A \widehat{H}_i^T x_{k+1} = \widetilde{p}_i^T A x_{k+1}, \quad x_i^T A^T \widetilde{H}_i^T y_{k+1} = y_{k+1}^T A \widehat{p}_i$$
$$x_i^T \widehat{H}_i A^T y_i = y_i^T \widetilde{H}_i A x_i = y_i^T \widetilde{H}_i \widetilde{H}_i A x_i = y_i^T \widetilde{H}_i A \widehat{H}_i^T x_i = \widetilde{p}_i^T A \widehat{p}_i.$$

By substitution we obtain the following recursion

$$\widehat{p}_{k+1} = x_{k+1} - \sum_{i=1}^{k} \frac{\widetilde{p}_i^T A x_{k+1}}{\widetilde{p}_i^T A \widehat{p}_i} \widehat{p}_i, \quad \widetilde{p}_{k+1} = y_{k+1} - \sum_{i=1}^{k} \frac{y_{k+1}^T A \widehat{p}_i}{\widetilde{p}_i^T A \widehat{p}_i} \widetilde{p}_i.$$

Letting $u_k = \widehat{p}_k$ and $v_k = \widetilde{p}_k$ we obtain the TSGS biconjugation process.

Theorem 32 *The two rank reduction procedures (8.9)-(8.11) and (8.12)-(8.14) can be written as TSGS algorithm.*

Finally, we note that Abaffy and Spedicato proved that the TSGS algorithm of Hegedűs (and also of Bodócs) is a special case of the ABS conjugation algorithm ([3], Thm. 8.30).

9. Biconjugation and factorizations

Although the TSGS biconjugation algorithm is a special case of the rank reduction algorithm, it is useful to review the following main results of Chu-Funderlic and Golub [6]. It is noted that we formulate and prove these results differently.

Proposition 33 *([6], Thm. 3.2) Let $H_1 \in R^{n\times n}$. For $X = Y = I$, then the TSGS algorithm can be carried out if and only if H_1 is strongly nonsingular. In this case the main diagonals of AH_1, $V^T H_1$, and Ω are identical and $H_1 = V^{-T}\Omega U^{-1}$ is the unique* LDU *factorization of H_1.*

Proof. The first part of the theorem follows from Theorem 29. If $X = I$ and $Y = I$, then $U = U_{H_1}^{-1}$, $V = L_{H_1}^{-T}$ and $\Omega = D_{H_1}$. Thus $H_1 U = H_1 U_{H_1}^{-1} = L_{H_1} D_{H_1}$, $V^T H_1 = L_{H_1}^{-1} H_1 = D_{H_1} U_{H_1}$ showing that all three matrices have the same diagonal. The product $V^{-T}\Omega U^{-1} = L_{H_1} D_{H_1} U_{H_1}$ is indeed an LDU factorization of H_1. ■

Proposition 34 *([6], Thm. 3.3) If $H_1 \in R^{n\times n}$ is symmetric and $X \in R^{n\times n}$ is such that $X^T H_1 X$ is strongly nonsingular, then the resulting H_1-biconjugate pair (U, V) has $U = V$. In this case, $V^T H_1 U = \Omega$ is the canonical form of H_1 with respect to congruence and the columns of U are A-orthogonal.*

Proof. By Theorem 29 $U = XU_{X^T H_1 X}^{-1}$ and $V = XL_{X^T H_1 X}^{-T} = XU_{X^T H_1 X}^{-1}$. Thus $U = V$ and

$$V^T H_1 U = V^T H_1 V = L_{X^T H_1 X}^{-1} X^T H_1 X U_{X^T H_1 X} = D_{X^T H_1 X}.$$

Proposition 35 *([6], Thm. 3.6) Let $H_1 \in R^{m\times n}$ have full column rank, $X = I$ and $Y = H_1$. Then the TSGS biconjugation can be carried out and the resulting H_1-biconjugate pair (U, V) is such that $Q = VD_{H_1^T H_1}^{-1/2}$ and $R = D_{H_1^T H_1}^{1/2} U$ defines a* QR-*factorization of H_1.*

Proof. The $Y^T H_1 X = H_1^T H_1$ is symmetric and positive definite. Hence $U = U_{H_1^T H_1}^{-1}$, $V = H_1 U_{H_1^T H_1}^{-1}$. It is clear that $Q = H_1 U_{H_1^T H_1}^{-1} D_{H_1^T H_1}^{-1/2}$ is orthogonal, and Q and $R = D_{H_1^T H_1}^{1/2} U_{H_1^T H_1}$ define the QR-factorization of H_1. ■

For quadratic matrices this result appears in several forms (see, e.g. Stewart [25], Abaffy-Spedicato [3] and Section 5 of this paper).

Proposition 36 *([6], Thm. 3.7) Let $H_1 \in R^{m\times n}$ have the singular value decomposition $H_1 = Y\Sigma X^T$, where $Y^T Y = I = X^T X$ and $\Sigma \in R^{r\times r}$ is nonsingular. For these X and Y the TSGS biconjugation algorithm produces the H_1-biconjugate pair (X, Y).*

Proof. By definition $Y^T H_1 X = \Sigma$ is strongly nonsingular. So the biconjugation can be carried out. Also by definition $U = XU_{Y^T H_1 X}^{-1} = X$ and $V = YL_{Y^T H_1 X}^{-T} = Y$. ■

This result is a special case of Proposition 12.

Proposition 37 *([6], Thm. 3.8) Let the nonzero singular values of H_1 be ordered such that $\omega_1 \geq \omega_2 \geq \ldots \geq \omega_r > 0$. If in the rank reducing process $\omega_{k+1} = \|H_{k+1}\|_2$,* *then the corresponding* x_{k+1} *and* y_{k+1} *vectors may be chosen to be the corresponding singular vectors of* H_1*, and the largest singular value of* H_{k+1} *is the* $(k+1)$*th largest singular value of* H_1.

The result is a special case of Proposition 6 with the choice $F = X$ and $G = \Sigma X^T$.

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NUMERICAL EVALUATION OF A GENERALIZED CAUCHY PRINCIPAL VALUE

András Nyíri and László Baranyi
Department of Fluid and Heat Engineering, University of Miskolc
3515 Miskolc – Egyetemváros, Hungary
arambl@gold.uni-miskolc.hu

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Abstract. A method is presented for the evaluation of a generalized Cauchy principal value of an improper integral. The integral is transformed into a Riemann integral which is evaluated using Romberg integration. This method is very convenient for computational purposes. The classical Cauchy principal value can be considered as a special case of the present one.

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Keywords: Cauchy principal value, improper integral, Romberg integration

1. Introduction

Various methods have been developed for the evaluation of principal value integrals [1-5]. Some of these are based on the Gaussian quadrature [2-4]. Methods of this type usually yield high accuracy for a given number of function evaluations. It may be difficult, however, to choose the weighing functions and the abscissas where the function is to be evaluated. Also if one desires to improve the accuracy by going to a higher order method, the calculation must be done from scratch. Kutt [5] presented a method for evaluation principal values by using finite-part integration. The authors [6] calculated a certain type of improper integrals occurring in fluid flow computations. That method has been extended and is applicable to a more general type Cauchy principal value integrals. The p.v. integral is evaluated here by the computation of a Riemann integral by using Romberg integration [7]. Although the number of function evaluations in this case is higher than that of the Gaussian quadrature, a simple and reliable method is presented by which high accuracy can be achieved within a fraction of a second CPU time even on a Personal Computer.

2. Evaluation of generalized Cauchy principal value

Let us consider the following improper integral

$$I = p.v. \int_{x_0-a}^{x_0+a} \frac{f(x)}{h(x)-h(x_0)} \mathrm{d}x \tag{2.1}$$

where
- $h(x) \neq h(x_0)$ for $x \neq x_o$ while $x \in [x_0 - a, x_0 + a]$,
- a is a positive constant,
- $f(x)$ is continuous at $x = x_0$ and $f(x_0) \neq 0$,
- $f(x)$ is Riemann integrable over the interval $[x_0 - a, x_0 + a]$,
- $h(x) \in C^1$ in the vicinity of x_0,
- there is an interval around x_0 where the functions $f'(x)$ and $h''(x)$ exist, therefore the following left and right side limits can be written

$$\begin{array}{ll} \lim\limits_{\epsilon \to 0} f'(x_0 - \epsilon) = f'^{-}(x_0) & \lim\limits_{\epsilon \to 0} f'(x_0 + \epsilon) = f'^{+}(x_0) \\ \lim\limits_{\epsilon \to 0} h''(x_0 - \epsilon) = h''^{-}(x_0) & \lim\limits_{\epsilon \to 0} h''(x_0 + \epsilon) = h''^{+}(x_0). \end{array} \tag{2.2}$$

In this expression the single and double primes denote the first and second derivatives, respectively. In equation (2.1) p.v. stands for principal value.

Theorem 2.1 If the conditions above hold true, the improper integral (2.1) is equivalent to the following Riemann integral

$$I = \int_0^a g(x_0, u) du \tag{2.3}$$

where

$$g(x_0, u) = \frac{f(x_0 - u)}{h(x_0 - u) - h(x_0)} + \frac{f(x_0 + u)}{h(x_0 + u) - h(x_0)}; \quad \text{if } u \neq 0 \tag{2.4}$$

$$g(x_0, u) = \frac{f'^{-}(x_0) + f'^{+}(x_0)}{h'(x_0)} - \frac{f(x_0)}{2[h'(x_0)]^2}[h''^{-}(x_0) + h''^{+}(x_0)]; \quad \text{if } u = 0. \tag{2.5}$$

Proof Let us resolve equation (2.1) into two parts

$$I = p.v. \int\limits_{x_0 - a}^{x_0 + a} \frac{f(x)}{h(x) - h(x_0)} dx = \lim_{\epsilon \to 0} \left[\int\limits_{x_0 - a}^{x_0 - \epsilon} \frac{f(x)}{h(x) - h(x_0)} dx + \int\limits_{x_0 + \epsilon}^{x_0 + a} \frac{f(x)}{h(x) - h(x_0)} dx \right]. \tag{2.6}$$

By introducing the transformation

$$u = |x - x_0|,$$

equation (2.6) can be reshaped to yield

$$I = \int\limits_0^a \left[\frac{f(x_0 - u)}{h(x_0 - u) - h(x_0)} + \frac{f(x_0 + u)}{h(x_0 + u) - h(x_0)} \right] du. \tag{2.7}$$

It can be seen that we arrived at equations (2.3) and (2.4) for the case when $u \neq 0$. The integrand $g(x_0, u)$ occurring in equation (2.7) is Riemann integrable and bounded in the interval $(0, a]$. When u tends to zero, integrand $g(x_0, u)$ becomes an indeterminate of the form $0/0$. Now let us examine the limiting case

$$\lim_{u \to 0} g(x_0, u).$$

By using Taylor's theorem [8], functions occurring in the integrand of equation (2.7) can be written as follows

$$\begin{aligned}
f(x_0 + u) &= f(x_0) + f'(b)u, \text{ where } b \in (x_0, x_0 + u);\\
f(x_0 - u) &= f(x_0) - f'(c)u, \text{ where } c \in (x_0 - u, x_0);\\
h(x_0 + u) - h(x_0) &= h'(x_0)u + h''(d)\frac{u^2}{2}, \text{ where } d \in (x_0, x_0 + u);\\
h(x_0 - u) - h(x_0) &= -h'(x_0)u + h''(e)\frac{u^2}{2}, \text{ where } e \in (x_0 - u, x_0).
\end{aligned}$$

The substitution of these expressions into equation (2.4) yields

$$g(x_0, u) = \frac{f(x_0) - f'(c)u}{-h'(x_0)u + h''(e)\frac{u^2}{2}} + \frac{f(x_0) + f'(b)u}{h'(x_0)u + h''(d)\frac{u^2}{2}}; \; u \neq 0. \tag{2.8}$$

By introducing a common denominator for the fractions in equation (2.8) and applying some simple algebra, the equation can be reshaped

$$g(x_0, u) = \frac{[f'(b) + f'(c)]h'(x_0) - \frac{f(x_0)}{2}[h''(e) + h''(d)] - [f'(b)h''(e) - f'(c)h''(d)]\frac{u}{2}}{[h'(x_0)]^2 + h'(x_0)[h''(d) - h''(e)]\frac{u}{2} - h''(e)h''(d)\frac{u^2}{2}} \quad \text{if } u \neq 0\,. \tag{2.9}$$

Bearing in mind equation (2.2), expressions in equation (2.9) in the limit of u tending to zero will have the following forms

$$\begin{aligned}
\lim_{u \to 0} f'(b) &= f'^{+}(x_0); \quad & \lim_{u \to 0} f'(c) &= f'^{-}(x_0);\\
\lim_{u \to 0} h''(d) &= h''^{+}(x_0); \quad & \lim_{u \to 0} h''(e) &= h''^{-}(x_0).
\end{aligned}$$

Carrying out now the limit of equation (2.9) when u tends to zero gives equation (2.5) to be proved

$$\lim_{u \to 0} g(x_0, u) = \frac{f'^{-}(x_0) + f'^{+}(x_0)}{h'(x_0)} - \frac{f(x_0)}{2[h'(x_0)]^2}[h''^{-}(x_0) + h''^{+}(x_0)].$$

With this theorem **2.1** is proved.

Remark 2.1 By applying the theorem proved above the evaluation of the generalized Cauchy principal value can be reduced to that of a Riemann integral by using various types of numerical quadrature. The advantage of the method lies in its simplicity. Computational experience gained shows that there is no need to evaluate the integrand at the close proximity of the pole. In this way the error due to the summing up of two large numbers of different signs can be avoided.

Remark 2.2 In a case when the derivatives $f'(x)$ and $h''(x)$ are continuous at $x = x_0$, equation (2.5) will have the following form

$$g(x_0, u) = 2\frac{f'(x_0)}{h'(x_0)} - \frac{f(x_0)}{[h'(x_0)]^2}h''(x_0); \quad \text{if } u = 0.$$

Remark 2.3 The classical Cauchy principal value is obtained by substituting

$$h(x) \equiv x$$

into equation (2.1). In this case equations (2.4) and (2.5) are simplified to the following equations

$$g(x_0, u) = \frac{f(x_0 + u) - f(x_0 - u)}{u}, \quad \text{if } u \neq 0;$$
$$g(x_0, u) = f'^{-}(x_0) + f'^{+}(x_0), \quad \text{if } u = 0.$$

Remark 2.4 The method shown in [6] worked out for the computation of incompressible inviscid flow around airfoils can also be considered as a special case of the present method. It can easily be seen that the evaluation of the integral in [6]

$$I = p.v. \int_{x_0 - a}^{x_0 + a} \frac{f(x)}{\cos x - \cos x_0}\,\mathrm{d}x \tag{2.10}$$

can be carried out by the method shown here. The substitution of

$$h(x) \equiv \cos x$$

into equation (2.1) leads to equation (2.10). Now it is straightforward that the integrands occurring in the two cases coincide

$$\begin{aligned} g(x_0, u) &= \frac{f(x_0 - u)}{\cos(x_0 - u) - \cos x_0} + \frac{f(x_0 + u)}{\cos(x_0 + u) - \cos x_0}, \;\; u \neq 0; \\ g(x_0, u) &= -\frac{f'^{-}(x_0) + f'^{+}(x_0)}{\sin x_0} + f(x_0)\frac{\cos x_0}{\sin^2 x_0}, \;\; u = 0, \; \sin x_0 \neq 0. \end{aligned}$$

3. Examples

The computations of the following three examples were performed in double precision on an IBM PC. The evaluation of the integral (2.3) was carried out by dividing the domain of integration into n equidistant part-intervals and applying the Romberg integration. Calculations were repeated for several different values of n; hence, its effect on the convergence of the result can also be studied (see Tables 3.1-3.3).

(a)

$$I = p.v. \int_{0.5}^{1.5} \frac{\mathrm{d}x}{x^3 - 1}\ . \tag{3.1}$$

Comparing equations (2.1) and (3.1) it can easily be seen that in this case

$$f(x) \equiv 1;\ h(x) = x^3;\ a = 0.5.$$

The primitive function $F(x)$ of the integrand in (3.1) is also available in analytic form [9]

$$F(x) = \frac{1}{6} \log \left[\frac{(x-1)^2}{|x^2+x+1|} \right] + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{-2x-1}{\sqrt{3}} \right).$$

Hence integral (3.1) can be evaluated as

$$I = p.v. \int_{0.5}^{1.5} \frac{dx}{x^3-1} = \lim_{\epsilon \to 0} \left\{ [F(x)]_{0.5}^{1-\epsilon} + [F(x)]_{1+\epsilon}^{1.5} \right\} = -0.342563258354480.$$

This accurate result can be compared to that of the one obtained from the present numerical approximation. The computational results for different values of the part-intervals n are shown in Table 3.1

Table 3.1

n	I
4	−0.342563143279927
8	−0.342563258270464
16	−0.342562258354677
32	−0.342563258354481
64	−0.342563258354480

(b)

$$I = p.v. \int_{-1}^{1} \frac{e^x}{x} dx\ .$$

Although as far as our knowledge goes, the primitive function does not exist in this case, Table 3.2 displays a rapid convergence.

Table 3.2

n	I
4	2.11450191410513
8	2.11450175080131
16	2.11450175075146
32	2.11450175075146
64	2.11450175075146

(c)

$$I = p.v. \int_{0.5}^{1.5} \frac{x^2}{x^4-1} \mathrm{d}x\ . \tag{3.2}$$

The primitive function $F(x)$ of the integrand in the equation (3.2) is also available in analytical form [9]

$$F(x) = \frac{1}{4}\left[\log\left|\frac{x-1}{x+1}\right| + 2\tan^{-1} x\right].$$

Hence integral (3.2) can be evaluated as

$$I = p.v.\int_{0.5}^{1.5} \frac{x^2}{x^4-1}\mathrm{d}x = \lim_{\epsilon\to 0}\left\{[F(x)]_{0.5}^{1-\epsilon} + [F(x)]_{1+\epsilon}^{1.5}\right\} = 0.131866651181764.$$

This accurate result shows good agreement with the results of our computations belonging to different values of n, contained in Table 3.3

Table 3.3

n	I
4	0.131866662407660
8	0.131866651984160
16	0.131866651180677
32	0.131866651181764
64	0.131866651181764

The fast convergence of the results can be observed in Tables 3.1-3.3. The advantages of the method are its simplicity, high accuracy and reliability. Although there may be faster methods, the actual CPU times for the evaluation of the sample cases above were small fractions of a second even on a PC.

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ON THE INVESTIGATION OF SOME BOUNDARY VALUE PROBLEMS WITH NON-LINEAR CONDITIONS

Andrei Ronto
Institute of Mathematics, Ukrainian Academy of Sciences
3 Tereshchenkovskaya Str., 01601 Kiev
ar@imath.kiev.ua

Miklós Rontó
Institute of Mathematics, University of Miskolc
3515 Miskolc – Egyetemváros, Hungary
matronto@gold.uni-miskolc.hu

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Abstract. A scheme of the numerical-analytic method based upon successive approximations is given for the investigation of solutions of nonlinear differential equations with nonlinear two-point boundary conditions.

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1. Introduction

The so-called numerical-analytic method of successive approximations [1,2,3,4] is widely used for studying the solvability of non-linear boundary value problems and constructing approximate solutions. For a survey of applications and development of the method to problems of higher order ordinary differential and impulsive differential systems with affine (periodic, two- and multi-point) conditions, one can refer to the appropriate parts of the series of survey papers [5, 6]

In [7,3], the methodology of the numerical-analytic method is extended in order to make it possible to study the non-linear two-point boundary problem of the form

$$y'(t) = f(t, y(t)), \qquad t \in [0, T], \tag{1.1}$$

$$g(y(0), y(T)) = 0, \tag{1.2}$$

for which purpose a general non-linear change of variable was introduced in Eq. (1.1). Here, we use a simpler substitution, which, as is shown, essentially facilitates the subsequent application of the numerical-analytic method. In particular, all the assumptions are formulated in terms of the original problem, and not the transformed one.

2. A reduction to the problem with linear conditions

Consider the boundary value problem (1.1), (1.2), where the functions $f : [0, T] \times G \to \mathbb{R}^n$ and $g : G \times G \to \mathbb{R}^n$ are continuous, $G \subset \mathbb{R}^n$ being the closure of a bounded domain. Assume that, for $t \in [0, T]$ fixed, f satisfies the Lipschitz condition

$$|f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2|, \qquad \{y_1, y_2\} \subset G, \tag{2.1}$$

where the square matrix K is supposed to have non-negative components. In (2.1), the absolute value sign and inequality are understood component-wise.

Let us introduce the traditional (see, e.g.,[8]) substitution

$$y(t) = x(t) + w, \tag{2.2}$$

where $w \in \Omega \subset \mathbb{R}^n$ is an unknown parameter. The domain, Ω, of w is chosen so that $D + \Omega \subset G$, whereas the new variable, x, is supposed to have range in D, the closure of a bounded subdomain of G.

Substitution (2.2) allows one to rewrite (1.1), (1.2) as

$$x'(t) = f(t, x(t) + w), \qquad t \in [0, T], \tag{2.3}$$

$$Ax(0) + Bx(T) = \phi(x(0) + w, x(T) + w) - [A + B]w, \tag{2.4}$$

where $\phi(u, v) := Au + Bv + g(u, v)$.

It is natural to determine the parameter w from the additional equation

$$\phi(x(0) + w, x(T) + w) - [A + B]w = 0$$

or, equivalently,

$$Ax(0) + Bx(T) + g(x(0) + w, x(T) + w) = 0. \tag{2.5}$$

Thus, the essentially non-linear problem (1.1), (1.2) turns out to be equivalent to problem (2.3), (2.4), and (2.5).

On the other hand, system (2.3), (2.4), (2.5) can be regarded as a collection of problems

$$x'(t) = f(t, x(t) + w), \tag{2.6}$$

$$Ax(0) + Bx(T) = 0 \tag{2.7}$$

parametrised by w belonging to a certain (unknown) set. The essential advantage obtained thereby is that the boundary condition (2.7) is linear. Therefore, problems of the family (2.6), (2.7) can be studied by using the numerical-analytic method developed in [2,3].

Assume that

$$D_\beta := \{x \in \mathbb{R}^n \mid B(x, \beta(x)) \subset D\} \neq \emptyset, \tag{2.8}$$

where

$$\beta(x) := \frac{T}{4}\delta_G(f) + |(B^{-1}A + I_n)x|. \tag{2.9}$$

Here,

$$\delta_G(f) := \max_{(t,y)\in[0,T]\times G} f(t,y) - \min_{(t,y)\in[0,T]\times G} f(t,y). \tag{2.10}$$

Moreover, we suppose that

$$r(K) < \frac{10}{3T}\,. \tag{2.11}$$

Introduce the sequence of functions

$$\begin{aligned} x_{m+1}(t,w,z) := z + \int_0^t f(s, x_m(s,w,z)+w)ds \\ - \frac{t}{T}\int_0^T f(s, x_m(s,w,z)+w)ds - \frac{t}{T}[B^{-1}A + I_n]z, \end{aligned} \tag{2.12}$$

where $m = 0, 1, \ldots$ and $x_0(t,w,z) \equiv z$. It is easily seen that all the members of sequence (2.12) satisfy condition (2.7) for every $z \in D_\beta$ and $w \in \Omega$. Furthermore, $x_m(0,w,z) = z$ for all $m \in \mathbb{N}$.

By virtue of (2.7), a solution x of (2.6), (2.7) satisfies

$$x(T) = -B^{-1}Ax(0)$$

and, therefore, Eq. (2.5) can be rewritten as

$$g(x(0)+w, -B^{-1}Ax(0)+w) = 0. \tag{2.13}$$

Summarising the above, we easily conclude that problem (2.6), (2.7), (2.5) is equivalent to system (2.6), (2.7), (2.13). We suggest solving it sequentially: first solve (2.6), (2.7), and then try to clarify whether (2.13) can be fulfilled.

Theorem 1 *Assume conditions* (2.1), (2.8), *and* (2.11). *Then*

1. *Sequence* (2.12) *has a limit $x^*(\cdot,w,z)$ uniform in $(t,w,z) \in [0,T]\times\Omega\times D_\beta$;*
2. *The limit function $x^*(\cdot,w,z)$ satisfies the 'perturbed' boundary value problem*

$$\begin{gathered} x'(t) = f(t, x(t)+w) + \Delta(w,z), \\ Ax(0) + Bx(T) = 0, \end{gathered}$$

where

$$\Delta(w,z) := -\frac{1}{T}[B^{-1}A + I_n]z - \frac{1}{T}\int_0^T f(s, x^*(s,w,z)+w)ds.$$

Furthermore, $x^*(0,w,z) = z$.

3. The error estimate

$$|x^*(t,w,z) - x_m(t,w,z)| \le \frac{20}{9} t \left(1 - \frac{t}{T}\right) Q^{m-1}(I_n - Q)^{-1}\left[\frac{1}{2}Q\delta_G(f) + K|(B^{-1}A + I_n)z|\right] \quad (2.14)$$

holds, where $Q := \frac{3T}{10}K$ *and* $\delta_G(f)$ *is as in* (2.9).

Proof. It is carried out similarly to the proof of Theorem 1 from [9] and Theorem 2.1 from [3, p. 32]. ■

The following statement shows the relation of the function $x^*(\cdot, w, z)$ to problem (2.6), (2.7).

Theorem 2 *Under assumptions* (2.1), (2.8), *and* (2.11), *the function* $x^*(\cdot, w, z^*)$ *is a solution of the boundary value problem* (2.6), (2.7) *if, and only if* $z = z^*$ *satisfies the 'determining equation',*

$$[B^{-1}A + I_n]z + \int_0^T f(s, x^*(s,w,z) + w)ds = 0,$$

where w *is considered as a parameter.*

Proof. Analogous to that of Theorem 2.4 from [3, p. 36]. ■

Theorem 3 *Assume conditions* (2.1), (2.8), *and* (2.11). *Then, for the function* $x^*(\cdot, w, z^*) + w^*$ *to be a solution of the boundary value problem* (1.1), (1.2), *it is sufficient that the parameters* $z = z^*$, $w = w^*$ *satisfy the system of determining equations*

$$[B^{-1}A + I_n]z + \int_0^T f(s, x^*(s,w,z) + w)ds = 0, \quad (2.15)$$

$$g(z + w, -B^{-1}Az + w) = 0. \quad (2.16)$$

In this case,

$$y^*(t) = x^*(t, w^*, z^*) + w^* \quad (2.17)$$

is a solution of the boundary value problem (1.1), (1.2).

Proof. It is obvious from the form of substitution (2.2) that equations (2.15) and (2.16) hold whenever the transformed boundary value problem (2.6), (2.7), (2.13) coincides with (1.1), (1.2). ■

Remark 4 *In [7,3], a change of variable more general than* (2.2) *was used,*

$$y(t) = x(t) + h(t, w). \tag{2.18}$$

However, when $\partial h(t,w)/\partial t \not\equiv 0$*, substitution of* (2.18) *leads to a more complicated transformed boundary value problem, as well as more complicated recursion sequence and determining equations. Furthermore, the basic assumptions should then be made on the transformed differential equation.*

Remark 5 *Considering* (2.17)*, one can treat the function*

$$y_m(t) = x_m(t, w_m, z_m) + w_m \tag{2.19}$$

as the m*th approximation to the exact solution,* $y^*(\cdot, w^*, z^*)$*, of the boundary value problem* (1.1), (1.2)*. In* (2.19)*,* w_m *and* z_m *are solutions of the 'approximate determining equations',*

$$[B^{-1}A + I_n]z + \int_0^T f(s, x_m(s, w, z) + w)ds = 0, \tag{2.20}$$

$$g(z + w, -B^{-1}Az + w) = 0, \tag{2.21}$$

and $x_m(\cdot, z, w)$ *is given by* (2.12)*. We do not consider the strict substantiation, referring the reader to [3] where similar techniques are described.*

Remark 6 *It follows from the consideration in [6] that the convergence of the recursion sequence* (2.12) *can be proved under the condition* $r(K) < 3.4161\ldots \cdot T^{-1}$*, which is weaker than* (2.11)*. However, estimate* (2.14) *does not hold in this case.*

Remark 7 *By using an idea from [4], one can obtain similar statements based on replacing* (2.12) *by the sequence*

$$\begin{aligned} x_{m+1}(t, w, z) := z - (A + B)z + \int_0^t f(s, x_m(s, w, z) + w)ds \\ - \Phi(t) \int_0^T f(s, x_m(s, w, z) + w)ds, \end{aligned}$$

where Φ *is an arbitrary continuous matrix-valued function such that*

$$A\Phi(0) + B\Phi(T) = I_n.$$

In this case, the restrictive conditions of type (2.11) *can be avoided. However, an analogue of condition* (2.8) *is more difficult toverify unless* f *in Eq.* (1.1) *is globally Lipschitzian.*

Example 8 *Consider the differential equation*

$$y''(t) + \frac{t}{16}y(t) + \frac{t}{4}(y'(t))^2 = \frac{t^2}{128}, \qquad t \in [0, 1] \tag{2.22}$$

under the non-linear boundary conditions

$$\begin{aligned} y'(1)y'(0) + \frac{1}{4}y(0) &= 0, \\ y(1)y'(0) - \frac{1}{2}y'(1) &= -\frac{7}{128}. \end{aligned} \tag{2.23}$$

We reduce Eq. (2.22) *to the first order system*

$$\begin{aligned} y_1'(t) &= y_2(t), \\ y_2'(t) &= \frac{t^2}{128} - \frac{t}{16}y_1(t) - \frac{t}{4}y_2^2(t); \end{aligned} \tag{2.24}$$

conditions (2.23) *then rewrite as*

$$\begin{aligned} y_2(1)y_2(0) + \frac{1}{4}y_1(0) &= 0, \\ y_1(1)y_2(0) - \frac{1}{2}y_2(1) &= -\frac{7}{128}. \end{aligned}$$

Put $A = B = I_2$ *and* $T = 1$. *It is not difficult to verify that the assumptions of the theorems above hold in the domain* $G := \{(y_1, y_2) \mid |y_1| \le 1,\ |y_2| \le \frac{1}{2}\}$.

Namely, (2.8) *holds when* β *from* (2.9) *satisfies the component-wise inequality* $\beta(x_1, x_2) \le \begin{pmatrix} \frac{1}{4} \\ \frac{17}{256} \end{pmatrix} + 2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. *The vector function in the right-hand side of* (2.24) *satisfies* (2.1) *in* G *with* $K = \begin{pmatrix} 0 & 1 \\ \frac{1}{16} & \frac{1}{4} \end{pmatrix}$. *Since* $r(K) = \frac{1+\sqrt{2}}{4}$, *it is obvious that* (2.11) *holds.*

Substitution (2.2) *in* (2.24) *leads us to the system*

$$\begin{aligned} x_1'(t) &= x_2(t) + w_2, \\ x_2'(t) &= \frac{t^2}{128} - \frac{t}{16}[x_1(t) + w_1] - \frac{t}{4}[x_2(t) + w_2]^2. \end{aligned}$$

In this case, the determining equation (2.21) *has the form*

$$\begin{aligned} w_2^2 - z_2^2 + \frac{1}{4}(z_1 + w_1) &= 0, \\ (w_1 - z_1)(w_2 + z_2) - \frac{1}{2}(w_2 - z_2) &= -\frac{7}{128}. \end{aligned} \tag{2.25}$$

Let us find the first approximation, $x_1 = \begin{pmatrix} x_{1,1} \\ x_{1,2} \end{pmatrix}$, *in the sense of Remark 5. According to* (2.12), *we have*

$$\begin{aligned} x_{1,1}(t, w_1, w_2, z_1, z_2) &= z_1 - 2tz_1, \\ x_{1,2}(t, w_1, w_2, z_1, z_2) &= z_2 + \frac{t^3}{384} - \frac{t^2}{32}(z_1 + w_1) - \frac{t^2}{8}(z_1 + w_2)^2 - \frac{t}{384} \\ &\quad + \frac{t}{32}(z_1 + w_1) + \frac{t}{8}(z_2 + w_2)^2 - 2tz_2. \end{aligned}$$

The first approximate determining equation (2.20) *(i.e., that corresponding to* $m = 1$*), is equivalent to* $\Delta_1(w, z) = 0$*, where* $\Delta_1 = \begin{pmatrix} \Delta_{1,1} \\ \Delta_{1,2} \end{pmatrix}$ *is given by*

$$\Delta_{1,1}(w_1, w_2, z_1, z_2) = \frac{385}{192} z_1 - \frac{1}{1536} + \frac{1}{192} w_1 + \frac{1}{48}(z_2 + w_2)^2 + w_2 \qquad (2.26)$$

and

$$\begin{aligned}\Delta_{1,2}(w_1, w_2, z_1, z_2) = &- \frac{1}{15360} z_1 z_2 \, w_2 + \frac{36863}{14155776} + \frac{107531}{10321920} z_1 \\ &+ \frac{46079}{23040} z_2 - \frac{322549}{10321920} w_1 + \frac{1}{5760} w_2 - \frac{1}{15360} w_1 z_2 w_2 \\ &- \frac{1}{15360} {z_2}^4 - \frac{1}{15360} {w_2}^4 - \frac{107509}{2580480} {z_2}^2 + \frac{107531}{1290240} z_2 w_2 \\ &- \frac{322549}{2580480} {w_2}^2 - \frac{1}{245760} {z_1}^2 - \frac{1}{122880} z_1 w_1 - \frac{1}{30720} z_1 {z_2}^2 \\ &- \frac{1}{30720} z_1 {w_2}^2 - \frac{1}{245760} {w_1}^2 - \frac{1}{30720} w_1 {z_2}^2 - \frac{1}{30720} w_1 {w_2}^2 \\ &- \frac{1}{3840} {z_2}^3 w_2 - \frac{1}{2560} {z_2}^2 {w_2}^2 - \frac{1}{3840} z_2 {w_2}^3 + \frac{1}{3840} z_2 z_1 \\ &+ \frac{1}{3840} z_2 w_1 - \frac{1}{320} {z_2}^2 w_2 - \frac{3}{320} z_2 {w_2}^2 - \frac{1}{768} w_2 z_1 \\ &- \frac{1}{768} w_2 w_1 + \frac{1}{960} {z_2}^3 - \frac{1}{192} {w_2}^3. \end{aligned} \qquad (2.27)$$

Solving system (2.25) *together with the approximate determining equations*

$$\Delta_{1,1}(w_1, w_2, z_1, z_2) = 0, \qquad \Delta_{1,2}(w_1, w_2, z_1, z_2) = 0,$$

where $\Delta_{1,1}$ *and* $\Delta_{1,2}$ *are given by* (2.26) *and* (2.27)*, we obtain the following values for* $z_1 = \begin{pmatrix} z_{1,1} \\ z_{1,2} \end{pmatrix}$ *and* $w_1 = \begin{pmatrix} w_{1,1} \\ w_{1,2} \end{pmatrix}$*:*

$$z_{1,1} \approx -0.06206890391, \qquad z_{1,2} \approx -0.00001967788609,$$
$$w_{1,1} \approx -0.0002202244703, \qquad w_{1,2} \approx 0.1247889518.$$

With these values of parameters, the first approximation, x_1*, has the form*

$$x_{1,1}(t) \approx -0.06228912838 + 0.1241378078\, t,$$
$$x_{1,2}(t) \approx -0.196778861 \cdot 10^{-4} + \frac{1}{384} t^3 + 0.613798 \cdot 10^{-6} t^2 - 0.2565424693 \cdot 10^{-2} t.$$

Consequently, according to substitution (2.2)*, the solution,* y*, of problem* (2.22)*,* (2.23)*, up to the first approximation* $y_1(t) = x_1(t) + w_1$*, has the form*

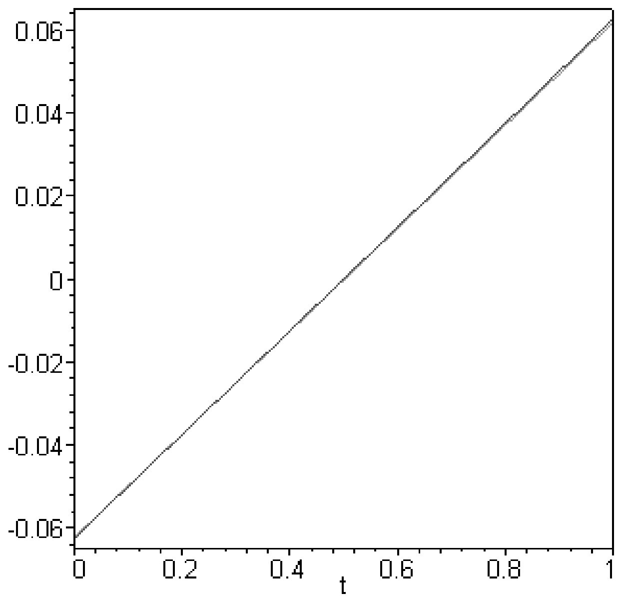

Figure 1.

$$
\begin{aligned}
y_{1,1}(t) &\approx -0.06250935285 + 0.1241378078\,t, \\
y_{1,2}(t) &\approx 0.1247692739 + \frac{1}{384}t^3 + 0.613798 \cdot 10^{-6} t^2 \\
&\quad - 0.2565424693 \cdot 10^{-2} t.
\end{aligned}
$$

Note that the function $y^*(t) = \frac{t}{8} - \frac{1}{16}$ *is an exact solution of problem* (2.22), (2.23). *As is seen from Figures 1 and 2, the graphs of the exact solution and the first approximation almost coincide, whereas the deviation of their derivatives does not exceed* 0.001. *Thus, even the first iteration of the method gives a satisfactory approximation for the solution of the problem considered.*

3. Separated non-linear boundary conditions

It turns out that, in the case of *separated*, in a sense, boundary conditions, the numerical-analytic method can be applied without any change of variable.

Consider the two-point boundary value problem

$$x'(t) = f(t, x(t)), \qquad t \in [0, T], \tag{3.1}$$

$$x(T) = a(x(0)), \tag{3.2}$$

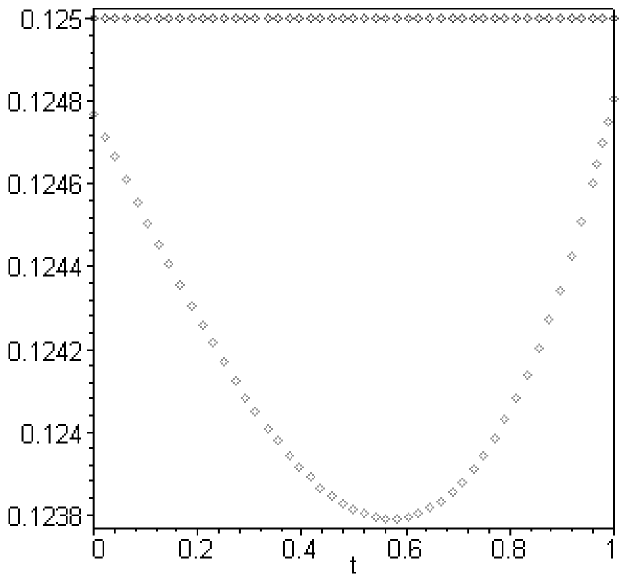

Figure 2.

where $a : D \to \mathbb{R}^n$ is continuous and f satisfies conditions (2.1) and (2.11) in the closure, D, of a bounded domain from $\mathbb{R}^n$.

Put $\gamma(x) := \frac{T}{4}\delta_D(f) + |a(x) - x|$ and assume that

$$D_\gamma \neq \emptyset, \tag{3.3}$$

where the set D_γ is defined similarly to (2.8), whereas the number $\delta_D(f)$ is given by (2.10).

Introduce the sequence of functions

$$\begin{aligned} x_{m+1}(t,z) := z + \int_0^t f(s, x_m(s,z))ds \\ - \frac{t}{T}\int_0^T f(s, x_m(s,z))ds + \frac{t}{T}[a(z) - z], \end{aligned} \tag{3.4}$$

where $m = 0, 1, \ldots$ and $x_0(t,z) \equiv z \in D_\gamma$. Obviously, all the members of sequence (3.4) satisfy condition (3.1) for every $z \in D_\gamma$.

Theorem 9 *Assume conditions* (2.1), (2.11), *and* (3.3). *Then:*

1. *Sequence* (3.4) *converges uniformly in* $[0, T] \times D_\gamma$,

$$\lim_{n\to+\infty} \sup_{(t,z)\in[0,T]\times D_\gamma} \|x_n(t, z) - x^*(t, z)\| = 0.$$

2. *The function* $x^*(\cdot, z)$ *is a solution of the 'modified' problem,*

$$\begin{aligned} &x'(t) = f(t, x(t)) + \Delta(z), \qquad t \in [0, T],\\ &x(T) = a(x(0)), \end{aligned}$$

where

$$T\Delta(z) := a(z) - z - \int_0^T f(s, x^*(s, z))ds.$$

Moreover, $x^*(\cdot, z)$ *satisfies* $x^*(0, z) = x_m(0, z) = z$ $(\forall m \in \mathbb{N})$.

3. *The error estimate*

$$|x^*(t, w, z) - x_m(t, w, z)| \le \frac{20}{9} t \left(1 - \frac{t}{T}\right) Q^{m-1}(I_n - Q)^{-1} \left[\frac{1}{2} Q\delta_D(f) + K|a(z) - z|\right] \tag{3.5}$$

holds, where $Q := \frac{3T}{10} K$ *and* $\delta_D(f)$ *is defined by* (2.10).

Proof. Analogous to that of Theorem 2.1 from [3, p. 32]. ■

Theorem 10 *Under conditions of Theorem 9, the limit function of sequence* (3.4) *is a solution of problem (3.1), (3.2) if, and only if the parameter* $z = z^*$ *(which stands for the initial value of the solution at* $t = 0$*) satisfies the 'determining equation,'*

$$a(z) - z = \int_0^T f(s, x^*(s, z))ds.$$

Proof. Similar to that of Theorem 2.3 from [3, p. 36]. ■

Remark 11 *When, in* (3.1), $a(z) \equiv z$, *then (3.1), (3.2) is nothing but the* T*-periodic boundary value problem. In this case, sequence* (3.4) *is reduced to the well-known [1]* T*-periodic successive approximations,'*

$$x_{m+1}(t, z) := z + \int_0^t f(s, x_m(s, z))ds - \frac{t}{T}\int_0^T f(s, x_m(s, z))ds.$$

Remark 12 *Estimate* (3.5) *implies that the function* $t \mapsto x_m(t, z_m)$ *is natural to be taken as the* m*th approximation of the exact solution of problem (3.1), (3.2), when* $z = z_m$ *is a root of the '*m*th approximate determining equation,'*

$$a(z) - z = \int_0^T f(s, x_m(s, z))ds. \tag{3.6}$$

Example 13 *Let us apply the technique described above to the problem*

$$
\begin{aligned}
&x''(t) - \frac{t}{8}x'(t) + \frac{1}{2}(x'(t))^2 = \frac{1}{4}, \qquad t \in [0,1],\\
&x(1) = \frac{1}{2}(x'(0))^2 + \frac{1}{16},\\
&x'(1) = -x(0) + \frac{3}{16}.
\end{aligned} \tag{3.7}
$$

As usual, we rewrite the second order problem (3.7) *in the form*

$$
\begin{aligned}
&x_1'(t) = x_2,\\
&x_2'(t) = \frac{1}{4} - \frac{t}{8}x_2(t) - \frac{1}{2}x_2^2(t), \qquad t \in [0,1],\\
&x_1(1) = \frac{1}{2}x_2^2(0) + \frac{1}{16},\\
&x_2(1) = -x_1(0) + \frac{3}{16}.
\end{aligned} \tag{3.8}
$$

Consider (3.8) *in the domain*

$$
(t, x_1, x_2) \in [0,1] \times D, \quad D = \left\{(x_1, x_2) \mid |x_1| \le 1,\ |x_2| \le \tfrac{3}{4}\right\}. \tag{3.9}
$$

It is not difficult to verify that the conditions of Theorem 9 are fulfilled with $K = \left(\begin{smallmatrix} 0 & 1 \\ 0 & \frac{7}{8} \end{smallmatrix}\right)$, $\gamma(x) \le \left(\begin{smallmatrix} \frac{7}{16}+\frac{1}{2}x_2^2-x_1 \\ \frac{1}{2}-x_1-x_2 \end{smallmatrix}.\right)$ *The corresponding quantity* (2.10) *is estimated as* $\delta_D(f) \le \left(\begin{smallmatrix} \frac{3}{4} \\ \frac{5}{8} \end{smallmatrix}\right)$, *and* K *satisfies* (2.11) *because* $r(K) = \frac{7}{8}$.

According to (3.4), *we have the following formulæ for* $x_1 = \left(\begin{smallmatrix} x_{1,1} \\ x_{1,2} \end{smallmatrix}\right)$*:*

$$
x_{1,1}(t, \zeta_1, \zeta_2) = \zeta_1 + t\left(\frac{1}{2}\zeta_2^2 - \zeta_1 + \frac{1}{16}\right), \tag{3.10}
$$

$$
x_{1,2}(t, \zeta_1, \zeta_2) = \zeta_2 + \frac{t^2}{16}\zeta_2 - \frac{1}{16}t\zeta_2 - t\left(\zeta_1 + \zeta_2 - \frac{3}{16}\right). \tag{3.11}
$$

The first approximate determining equations (3.6) has the form

$$
\Delta_{1,1}(\zeta_1, \zeta_2) = 0, \Delta_{1,2}(\zeta_1, \zeta_2) = 0, \tag{3.12}
$$

where

$$
\begin{aligned}
\Delta_{1,1}(\zeta_1, \zeta_2) &= \frac{1}{2}\zeta_2^2 - \frac{47}{96}\zeta_2 - \frac{1}{2}\zeta_1 - \frac{1}{32},\\
\Delta_{1,2}(\zeta_1, \zeta_2) &= -\frac{49}{48}\zeta_1 - \frac{33}{512} - \frac{88165}{89088}\zeta_2 + \frac{827}{5120}{\zeta_2}^2 - \frac{31}{192}\zeta_2\zeta_1 + \frac{1}{6}{\zeta_1}^2.
\end{aligned}
$$

The solution, $\zeta = \left(\begin{smallmatrix} \zeta_1 \\ \zeta_2 \end{smallmatrix}\right)$, of system (3.12) has the form

$$
\zeta_1 = -\frac{1}{16}, \qquad \zeta_2 = 0. \tag{3.13}
$$

Inserting (3.13) into (3.10) and (3.11), we obtain the first approximation

$$x_{1,1}(t) = \frac{t}{8} - \frac{1}{16}, \qquad x_{1,2}(t) = \frac{t}{4}.$$

Note that the vector function with components

$$x_1^*(t) = \frac{t^2}{8} - \frac{1}{16}, \qquad x_2^*(t) = \frac{t}{4} \tag{3.14}$$

is an exact solution of problem (3.8).

Let us construct the second approximation. By (3.4), we have

$$\begin{aligned} x_{2,1}(t,\zeta_1,\zeta_2) &= \zeta_1 + \frac{1}{48}\zeta_2 t^3 - \frac{1}{2}t^2\zeta_1 + \frac{3}{32}t^2 - \frac{17}{32}t^2\zeta_2 \\ &\quad + \frac{49}{96}\zeta_2 t - \frac{1}{2}t\zeta_1 - \frac{1}{32}t + \frac{1}{2}t\zeta_2^2, \end{aligned} \tag{3.15}$$

$$\begin{aligned} x_{2,2}(t,\zeta_1,\zeta_2) &= \frac{95}{512}t - \frac{31}{192}t\zeta_2\zeta_1 + z_2 + \frac{17}{768}\zeta_2 t^3 - \frac{1}{32}t^2\zeta_2 - \frac{49}{48}t\zeta_1 \\ &\quad - \frac{1733}{5120}t\zeta_2^2 + \frac{1}{512}t^3 - \frac{3041}{3072}\zeta_2 t - \frac{1}{2560}\zeta_2^2 t^5 + \frac{1}{64}t^4\zeta_2\zeta_1 \\ &\quad - \frac{17}{48}t^3\zeta_2\zeta_1 + \frac{1}{2}t^2\zeta_2\zeta_1 - \frac{1}{1024}t^4\zeta_2 + \frac{17}{1024}t^4\zeta_2^2 + \frac{1}{48}t^3\zeta_1 \\ &\quad - \frac{107}{512}t^3\zeta_2^2 - \frac{1}{6}t^3\zeta_1^2 + \frac{17}{32}t^2\zeta_2^2 + \frac{1}{6}t\zeta_1^2. \end{aligned} \tag{3.16}$$

It is not difficult to verify that inserting into (3.15) and (3.16) even the roots (3.13) of the *first* determining system (3.12), we arrive at the exact solution (3.14) of problem (3.8).

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BUFFERNESS PHENOMENON IN DISTRIBUTED OSCILLATORY SYSTEMS

Nikolai ROZOV
Moscow State University, Mechanical and Mathematical Department
119899 MOSCOW, Russia
rozov@rozov.mccme.ru

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Abstract. A theoretical approach to the investigation of the bufferness phenomenon in systems of differential equations is presented.

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Keywords: Partial differential equations, singular perturbation, periodic solution, bufferness, oscillation

1. Introduction

It is widely known that the stable limit cycles of the ordinary differential equations play an exclusive role in the analysis of mathematical models of various *oscillatory systems with lumped parameters* (in physics, biology, economics, engineering etc.). However, it is useful to remember that the concept of a limit cycle was introduced by the great scientist, A. Poincaré, purely theoretically, in the early 80s of the 19th century. And only in the late 1920s did the outstanding Soviet physicist, A. A. Andronov, discovere that a stable limit cycle gives an adequate mathematical description of a steady periodic regime.

There are models where the matter of principle is the existence of unique stable limit cycle. In other models, however, it is interesting to discover the multiplicity of such cycles. Everything depends on the properties of concrete applied problem under consideration. Moreover, it is clear that, generally speaking, there exist various number of stable limit cycles, depending on the values of parameters in the system of differential equations. (These parameters describe actual properties of the object under investigation.)

It is possible to give examples of systems of ordinary differential equations, where, with appropriate choice of parameter values, one can guarantee the existence of an arbitrary, beforehand prescribed finite number of stable limit cycles. For example, let us consider the following second order system (in polar coordinates)

$$\dot{r} = r\sin(1/\alpha r), \dot{\varphi} = 1,$$

where $\alpha, 0 < \alpha < 1/\pi$, is a parameter. For any natural number N it is easy to indicate the value of parameter α, positive and close to zero, providing exactly N stable limit cycles, which are located in the area $r > 1$ of phase plane.

As it was mentioned above, the mathematical models of oscillatory systems with lumped parameters are usually described by ordinary differential equations. However, nowadays in physics, chemistry, biology, new engineering and modern technologies the so-called *oscillatory systems with distributed parameters* or, more shortly, *the distributed oscillatory systems* are widely spread. These are objects, the state of which depends on both time and space variables; and the state of each point of space changes periodically with time. Thus, periodic auto-oscillations are generated, or *auto-wave processes*, following the term introduced by R. V. Khokhlov.

The dynamics of these objects is simulated, as a rule, by the partial differential equations (with boundary and initial conditions). Each actual auto-oscillatory regime corresponds to a *stable cycle* (it is a solution of the appropriate equation, periodic on time and satisfying the boundary conditions). The partial differential equation, being an adequate mathematical model of an oscillatory system with distributed parameters, can have one or several stable cycles. It is natural that the number of such cycles, generally speaking, may be different depending on the values of parameters in the equation.

It is said that *the bufferness phenomenon* takes place in the distributed oscillatory system if the system possesses the following property. The partial differential equation, which is the mathematical model of the system, possesses, under the appropriate values of parameters, an arbitrary, beforehand prescribed, finite number of different stable cycles. Theoretically speaking, for any natural number N one can choose physical characteristics of the system so that it will have N auto-oscillatory regimes.

The typical example of a distributed oscillatory system is the classical autogenerator containing a long line. The problem of investigation of its periodic (with time) regimes was initially formulated by A. A. Witt [1, 2]. It is appropriate here to say more about this scientist, because his name is hardly known to mathematicians. Alexander A. Witt was one of the bright scientists representing the remarkable Soviet school of the oscillation theory (L. I. Mandelstam, N. D. Papalexi, A. A. Andronov, V. V. Migulin etc.). He carried out a number of interesting and perspective theoretical research. In particular, he was one of the co-authors of a classical book "The Oscillation Theory" [3]. However, as A. A. Witt was innocently convicted (and was killed in 1937), his name did not appear in the book [3] and was rehabilitated only after many years had passed [4, 5].

Let's point out that A. A. Witt proposed, in his work [2], the following hypothesis. There is a principal possibility that the oscillator with long line may possess several stable cycles simultaneously. (However, the bufferness phenomenon was not formulated yet). The fact that the number of auto-oscillating regimes may increase under the variation of the parameter values of the object was first observed during the physical experiment carried out by the research group of V. V. Migulin (see [6]). As for the mathematical investigation of the bufferness phenomenon, it began on Yu. S. Kolesov's initiative. He studied this phenomenon in parabolic reaction-diffusion systems [7] using numerical methods; later he completed the theoretical research for hyperbolic equations [8] (see also monograph [9]).

2. Bufferness phenomenon for periodic solutions

Let's consider one mathematical result which theoretically proves the existence of bufferness phenomenon in so-called $LCRG-$autogenerator composed on a long line with a tunnel diode. Omitting the description of physical details and the deduction of the necessary equations (it is standard enough), we consider the boundary value problem (see [10]) to be its mathematical model (under a number of conventional assumptions — in particular, under cubic approximation of the diode characteristics):

$$\begin{aligned} &u_x = -Ri - Li_t, \quad i_x = -Gu - Cu_t, \\ &i|_{x=1} = 0, \quad u|_{x=0} + k_0\, u|_{x=1} + k_2\, u^2\big|_{x=1} + k_3\, u^3\big|_{x=1} = 0, \end{aligned} \tag{2.1}$$

where t is time, x is the coordinate along a line, $u(t,x)$ and $i(t,x)$ are the voltage and the current in a line; all the autogenerator parameters (such as the distributed resistance R, inductance L, capacity C and conductivity G) are constant. It is well known that this problem can be reduced to a boundary value problem for the so-called *telegraph equation.*

It is shown in [11] that the problem of searching for auto-oscillations in this model can be reformulated (after technical transforms and changes of variables) as a question of existence and stability of periodic solutions of a nonlinear differential-difference equation

$$\begin{aligned} &z'(t) - (1-\alpha\epsilon)z'(t-h) + (1-\epsilon)z(t) + (1-\alpha\epsilon)(1+\epsilon)z(t-h) = \\ &\qquad = a[z(t) - (1-\alpha\epsilon)z(t-h)]^2 - [z(t) - (1-\alpha\epsilon)z(t-h)]^3; \end{aligned} \tag{2.2}$$

here $\epsilon > 0$ is a small parameter, the parameter $\alpha > 0$ has the order of unit, the parameters $h = const > 0$, $a = const$.

Let's denote by $\omega_1 < \omega_2 < \dots$ all the positive roots of the equation

$$\omega \tan \frac{\omega h}{2} = 1.$$

Lemma. *All the roots of a quasi-polynomial*

$$P(\lambda) = \lambda[1 - (1-\alpha\epsilon)e^{-\lambda h}] + (1-\epsilon) + (1-\alpha\epsilon)(1+\epsilon)e^{-\lambda h},$$

corresponding to zero equilibrium state of equation (2), *for* $0 < \epsilon \ll 1$ *and* $\alpha > 2(1+\omega_1^2)^{-1}$ *lie in a half-plane* $Re\lambda < 0$. *If the parameter* $\alpha > 0$ *is decreasing, and when it becomes equal to the values* $2(1+\omega_n^2)^{-1}, n \geq 1$, *then the roots of a quasi-polynomial* $P(\lambda)$ *transfer sequentially, one after another, to the half-plane* $Re\lambda > 0$.

Theorem. *Let for some natural* $n \geq 1$ *the conditions*

$$0 < \alpha < \frac{2}{1 + 2\omega_n^2 - \omega_1^2}, \qquad 1 + 10\omega_n^2 - 15\omega_n^4 \neq 0$$

be fulfilled. Then there exists $\epsilon_n > 0$, *such that for any* $\epsilon \in [0, \epsilon_n]$ *the equation* (2) *has an exponentially orbital-stable* (*in metrics of the phase space* $W_2^1(-h, 0)$) *periodic solution* $z_n(\tau, \epsilon)$:

$$z_n(\tau + \frac{2\pi}{\omega_n}, \epsilon) = z_n(\tau, \epsilon), \quad \tau = (1 + \delta_n(\epsilon))t.$$

For this solution the following asymptotic (*when* $\epsilon \to 0$) *representation is valid:*

$$z_n(\tau, \epsilon) = \sqrt{\epsilon}\frac{1 + \omega_n^2}{\sqrt{6}}\sqrt{\frac{2}{1 + \omega_n^2} - \alpha}\cos\omega_n\tau + \mathcal{O}(\epsilon), \quad \delta_n(\epsilon) = \mathcal{O}(\epsilon).$$

It follows from the above Theorem (the proof of which is rather nontrivial), that the following dynamics of equation (2) takes place under the appropriate decrease of the parameters ϵ and α.

Corollary. *Zero equilibrium state of equation* (2) *is exponentially stable for all* $\alpha > 2(1 + \omega_1^2)^{-1}$. *When the parameter* α *is equal to* $2(1 + \omega_1^2)^{-1}$, *then a stable cycle branches from this equilibrium state* (*the Andronov — Hopf bifurcation*). *For the further decrease of* α, *when it becomes equal to each value* $2(1 + \omega_n^2)^{-1}, n \geq 2$, *an unstable cycle appears* (*the secondary Andronov — Hopf bifurcation*), *which becomes stable for* $\alpha > 2(1 + 2\omega_n^2 - \omega_1^2)^{-1}$.

Thus, under appropriate decrease of parameters ϵ and α, one can guarantee the existence of any beforehand prescribed number of stable cycles for equation (2). It means that the bufferness phenomenon takes place for the system (1). That is, the mathematical model of the $LCRG-$autogenerator with a long line, under suitable choice of the values of its parameters, may possess an arbitrary number of stable periodic regimes.

Let's point out that the last conclusion can be illustrated by the real experiment (see [12]).

To summarize all the above information, let us formulate the specific features of the bufferness phenomenon from the physical point of view. A distributed oscillatory system possessing the bufferness phenomenon has a set of stable periodic regimes. Note that under the appropriate choice of the system parameters, this set may contain an arbitrary large number of such regimes. When the values of parameters are fixed, then the concrete regime, one of the potentially possible auto-oscillatory regimes, comes into realization depending on the initial conditions or external factors. The spontaneous transition of the system to some other periodic regime is impossible.

The further research concerning the bufferness phenomenon and associated problems is discussed in the monograph [13].

As a conclusion we will consider one more rather simple concrete example of a system with a bufferness property — an example from mechanics.

Let's consider an auto-oscillatory system consisting of the homogeneous string of the length l with fixed ends, to the middle of which a generator of mechanical oscillations is connected. Let's assume that the oscillating string is characterized by density ρ, tension T and condensation of friction forces h. As for the generator, it is represented by a resonator consisting of the mass M, a spring with the elasticity k and nonlinear element of friction $h_*(v) = \lambda(v^2 - 1), \lambda > 0$.

Omitting the rather simple deduction of the equations for the string and generator, we shall consider only the final mathematical model (after necessary normalization of the variables). It is the following boundary value problem in the segment $0 \le x \le 1$:

$$u_{tt} + \varepsilon u_t = u_{xx},$$
$$u|_{x=0} = 0, \quad [u_{tt} + \varepsilon\alpha(u^2 - 1)u_t + \beta u]|_{x=1} = -\gamma u_x|_{x=1}.$$

Here the required function $u(t, x)$ characterizes the shift of an element of the string, $0 < \varepsilon \ll 1$, and parameters $\alpha, \beta, \gamma > 0$ have the order of unit.

It appears (under certain assumptions, which are not formulated here for brevty's sake), that, with the appropriate increase of parameter α and proper decrease of parameter ε, one can guarantee that this boundary value problem possesses an arbitrary beforehand prescribed finite number of stable solutions, periodic with t; i.e. the bufferness phenomenon is observed.

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PERIODIC BOUNDARY VALUE PROBLEM FOR SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

Svatoslav Staněk
Department of Mathematical Analysis, Faculty of Science, Palacký University
Tomkova 40, 779 00 Olomouc, Czech Republic
stanek@risc.upol.cz

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Abstract. The periodic boundary value problem for a functional differential equation is considered. The existence of a solution and upper and lower solutions is proved under assumptions that there exist lower and upper functions and the right side of the functional differential equation satisfies one-sided growth conditions. Results are proved by the Borsuk antipodal theorem and the Leray-Schauder degree.

Mathematical Subject Classification: 34K10, 34B15
Keywords: Periodic boundary value problem, functional differential equation, upper and lower functions, existence, upper and lower solutions, Borsuk antipodal theorem, Leray-Schauder degree

1. Introduction

Let $J = [0, T]$ be a compact interval and, for any positive constant K, $\mathcal{L}^K(J) = \{x : x \in L_1(J), |x(t)| \le K \text{ for each } t \in J\}$. We write $x_n \to x$ as $n \to \infty$ in $\mathcal{L}^K(J)$ if $\lim_{n\to\infty} x_n(t) = x(t)$ for a.e. $t \in J$. We denote by $\|x\| = \max\{|x(t)| : t \in J\}$ and $\|x\|_{L_1} = \int_a^b |x(t)|\,dt$ the norms in the Banach spaces $C^0(J)$ and $L_1(J)$, respectively. Set $\mathcal{L}(J) = \bigcup_{K>0} \mathcal{L}^K(J)$.

Consider the periodic boundary value problem (PBVP for short)

$$(g(x'(t)))' = F(x, x(t), x'(t))(t), \tag{1.1}$$

$$x(0) = x(T), \quad x'(0) = x'(T). \tag{1.2}$$

Here $g : \mathbb{R} \to \mathbb{R}$ is an increasing homomorphism with inverse $g^{-1} : \mathbb{R} \to \mathbb{R}$, $g(0) = 0$ and $F : C^0(J) \times \mathbb{R}^2 \to L_1(J)$, $(x, a, b) \longmapsto F(x, a, b)(t)$ is the operator having the following properties:

(i) $F(x, y(t), z(t))(t) \in L_1(J)$ for $x, y \in C^0(J)$ and $z \in \mathcal{L}(J)$;

(ii) for any $K > 0$, $\lim_{n\to\infty}(x_n, y_n, z_n) = (x, y, z)$ in $C^0(J) \times C^0(J) \times \mathcal{L}^K(J) \Rightarrow$

$$\lim_{n\to\infty} F(x_n, y_n(t), z_n(t))(t) = F(x, y(t), z(t))(t) \text{ in } L_1(J);$$

(iii) $a, b \in \mathbb{R}$, $x_1, x_2 \in C^0(J)$, $x_1(t) \le x_2(t)$ for $t \in J \Rightarrow$
$F(x_1, a, b)(t) \ge F(x_2, a, b)(t)$ for a.e. $t \in J$;

(iv) for any $r > 0$ there exists $h_r \in L_1(J)$ such that $(x, a, b) \in C^0(J) \times \mathbb{R}^2$, $\|x\| + |a| + |b| \le r \Rightarrow$
$|F(x, a, b)(t)| \le h_r(t)$ for a.e. $t \in J$.

The special cases of the operator F having the properties (i)-(iv) are, for example, the operators

$$F_1(x, a, b)(t) = \int_{\varphi_1(t)}^{\varphi_2(t)} f_1(s, \xi(t)x(s), a, b)\, ds + f_2(t, a, b)$$

and

$$F_2(x, a, b)(t) = f_1(t, (Sx)(t), a, b),$$

where f_1 (resp. f_2) satisfies the local Carathéodory conditions on $J \times \mathbb{R}^3$ (resp. $J \times \mathbb{R}^2$), $f_1(t, \cdot, a, b)$ is nonincreasing on $\mathbb{R}$ for a.e. $t \in J$ and each $(a, b) \in \mathbb{R}^2$, $\varphi_1, \varphi_2, \xi \in C^0(J)$, $0 \le \varphi_1(t) \le \varphi_2(t) \le T$, $\xi(t) \ge 0$ for $t \in J$ and the operator $S : C^0(J) \to C^0(J)$ is bounded continuous and $x, y \in C^0(J)$, $x(t) \le y(t)$ for $t \in J$ $\Rightarrow (Sx)(t) \le (Sy)(t)$ for $t \in J$. For the operators F_1 and F_2, equation (1) has the form

$$(g(x'(t)))' = \int_{\varphi_1(t)}^{\varphi_2(t)} f_1(s, \xi(t)x(s), x(t), x'(t))\, ds + f_2(t, x(t), x'(t))$$

and

$$(g(x'(t)))' = f_1(t, (Sx)(t), x(t), x'(t)).$$

Under *a solution of PBVP* (1.1), (1.2) we mean a function $x \in C^1(J)$ such that $g(x'(t))$ is absolutely continuous on J ($AC(J)$ for short), equality (1.1) is satisfied a.e. on J and x satisfies the periodic boundary conditions (1.2).

The purpose of this paper is to give sufficient conditions formulated by upper and lower functions of PBVP (1.1), (1.2) and one-sided growth restrictions on the operator F ensuring the existence of a solution and upper and lower solutions for PBVP (1.1), (1.2) in a subset of $C^1(J)$.

Let α, β be lower and upper functions of PBVP (1.1), (1.2) (see Section 2), $\alpha(t) \le \beta(t)$ for $t \in J$, and set $\mathcal{M}_{\alpha\beta} = \{x : x \in C^1(J),\ \alpha(t) \le x(t) \le \beta(t) \text{ for } t \in J\}$. The existence results for PBVP (1.1), (1.2) in the set $\mathcal{M}_{\alpha\beta}$ are proved by the Borsuk antipodal theorem and the Leray-Schauder degree (see e.g. [2], [9]). Denote by $\mathcal{D}_{\alpha\beta}$ the set of all solutions of PBVP (1.1), (1.2) in the set $\mathcal{M}_{\alpha\beta}$. The proof of the existence of upper and lower solutions for PBVP (1.1), (1.2) in the set $\mathcal{D}_{\alpha\beta}$ is based on the following two facts: a) for any $u, v \in \mathcal{D}_{\alpha\beta}$ there exist $x, y \in \mathcal{D}_{\alpha\beta}$

such that $x(t) \le \min\{u(t), v(t)\}$, $\max\{u(t), v(t)\} \le y(t)$ for $t \in J$, b) the functional $\varphi : \mathcal{D}_{\alpha\beta} \to \mathbb{R}$, $\varphi(x) = \int_0^T x(t)\,dt$ achieves its extremal values at upper and lower solutions of PBVP (1.1), (1.2).

We observe that there are many papers devoted to PBVP

$$x''(t) = f(t, x(t), x'(t)) \tag{1.3}$$

with the scalar function f where the existence results have been proved by the method of upper and lower functions combined with a priori estimates for solutions (see, e.g., [1], [3-8], [10-15] and references cited therein). To obtain a priori estimates for solutions of the above PBVP authors usually assumed that the function f satisfies either the two-sided Bernstein-Nagumo growth condition ([1], [3, 4], [8], [10], [11], [13-15]) or f satisfies some sign conditions ([12]) or f satisfies one-sided growth conditions ([5-7]). Our results generalized those of [7].

2. Notation, lemmas

Let $G, Q \in C^0([0,\infty))$ be defined on $[0,\infty)$ by the formulas

$$G(u) = \max\{-g(-u), g(u)\}, \quad Q(u) = \max\{-g^{-1}(-u), g^{-1}(u)\}. \tag{2.1}$$

Then

$$|g(u)| \le G(|u|), \quad |g^{-1}(u)| \le Q(|u|), \qquad u \in \mathbb{R}. \tag{2.2}$$

Clearly, $G(u) = g(u)$, $Q(u) = g^{-1}(u)$ on $[0,\infty)$ provided g is an odd function (for example $g(u) = |u|^{p-2}u$ with $p > 1$).

Lemma 1 *Let $x \in C^1(J)$, $g(x') \in AC(J)$, x satisfy the periodic boundary conditions (1.2) and set*

$$\mathcal{C}_1 = \{t : t \in J, x'(t) > 0\}, \quad \mathcal{C}_2 = \{t : t \in J, x'(t) < 0\}.$$

Assume that there exist $\sigma_k \in \{-1, 1\}$ $(k = 1, 2)$ and positive-valued functions $h \in L_1(J)$ and $w \in C^0(\mathbb{R})$ such that

$$\int_{-\infty}^0 \frac{dt}{w(g^{-1}(t))} = \int_0^\infty \frac{dt}{w(g^{-1}(t))} = \infty \tag{2.3}$$

and

$$\sigma_k(g(x'(t)))' \le (h(t) + |x'(t)|)w(x'(t)) \quad \text{for a.e. } t \in \mathcal{C}_k \ (k = 1, 2). \tag{2.4}$$

Let P be a positive constant satisfying the inequality

$$\min\Big\{\int_{g(-P)}^0 \frac{dt}{w(g^{-1}(t))}, \int_0^{g(P)} \frac{dt}{w(g^{-1}(t))}\Big\} > 2(\|h\|_{L_1} + 2\|x\|). \tag{2.5}$$

Then

$$\|x'\| < P. \tag{2.6}$$

Proof. Let

$$\mathcal{B} = \{t : t \in J, x'(t) = 0\}$$

and $\|x'\| = |x'(\xi)|$, $\xi \in J$. By (2), $\mathcal{B} \neq \emptyset$ and, without loss of generality, we can assume that $\xi \in (0,T]$. For $\|x'\| = 0$ the inequality (2.6) is satisfied. Let $\|x'\| > 0$. We first assume that $x'(\xi) > 0$. The next part of the proof is divided into two cases according to whether $\sigma_1 = 1$ or $\sigma_1 = -1$.

a) Let $\sigma_1 = 1$. If $\mathcal{B} \cap [0,\xi) \neq \emptyset$ then there exists $t_1 \in \mathcal{B} \cap [0,\xi)$ such that $x'(t) > 0$ for $t \in (t_1, \xi]$ and (cf. (2.4))

$$(g(x'(t)))' \le (h(t) + x'(t))w(x'(t)) \tag{2.7}$$

for a.e. $t \in [t_1, \xi]$, and consequently

$$\begin{aligned} \int_0^{g(\|x'\|)} \frac{dt}{w(g^{-1}(t))} &= \int_{g(x'(t_1))}^{g(x'(\xi))} \frac{dt}{w(g^{-1}(t))} = \int_{t_1}^{\xi} \frac{(g(x'(t)))'}{w(x'(t))}\,dt \\ &\le \int_{t_1}^{\xi} (h(t) + x'(t))\,dt \le \|h\|_{L_1} + 2\|x\|. \end{aligned} \tag{2.8}$$

Let $\mathcal{B} \cap [0,\xi) = \emptyset$. Then $x'(t) > 0$ on $[0,\xi]$, $x'(T) > 0$ (by (1.2)) and there exists $t_2 \in \mathcal{B}$ such that $x'(t)) > 0$ for $t \in (t_2, T]$. Then (2.7) is satisfied for a.e. $t \in [0,\xi] \cup [t_2, T]$, and so

$$\begin{aligned} \int_0^{g(x'(0))} \frac{dt}{w(g^{-1}(t))} &= \int_{g(x'(t_2))}^{g(x'(T))} \frac{dt}{w(g^{-1}(t))} = \int_{t_2}^{T} \frac{(g(x'(t)))'}{w(x'(t))}\,dt \\ &\le \int_{t_2}^{T} (h(t) + x'(t))\,dt \le \|h\|_{L_1} + 2\|x\|, \\ \int_{g(x'(0))}^{g(x'(\xi))} \frac{dt}{w(g^{-1}(t))} &= \int_0^{\xi} \frac{(g(x'(t)))'}{w(x'(t))}\,dt \le \int_0^{\xi} (h(t) + x'(t))\,dt \le \|h\|_{L_1} + 2\|x\|. \end{aligned}$$

Thus

$$\begin{aligned} \int_0^{g(\|x'\|)} \frac{dt}{w(g^{-1}(t))} &= \int_0^{g(x'(0))} \frac{dt}{w(g^{-1}(t))} + \int_{g(x'(0))}^{g(x'(\xi))} \frac{dt}{w(g^{-1}(t))} \\ &\le 2(\|h\|_{L_1} + 2\|x\|). \end{aligned} \tag{2.9}$$

b) Let $\sigma_1 = -1$. Assume $\xi < T$. If $\mathcal{B} \cap (\xi, T] \neq \emptyset$ then there exists $t_3 \in \mathcal{B} \cap (\xi, T]$ such that $x'(t) > 0$ on $[\xi, t_3)$ and (cf. (2.4))

$$(g(x'(t)))' \ge -(h(t) + x'(t))w(x'(t)) \tag{2.10}$$

for a.e. $t \in [\xi, t_3]$, which yields

$$\begin{aligned} \int_0^{g(\|x'\|)} \frac{dt}{w(g^{-1}(t))} &= -\int_{\xi}^{t_3} \frac{(g(x'(t)))'}{w(x'(t))}\,dt \le \int_{\xi}^{t_3} (h(t) + x'(t))\,dt \\ &\le \|h\|_{L_1} + 2\|x\|. \end{aligned} \tag{2.11}$$

Let $\mathcal{B} \cap (\xi, T] = \emptyset$. Then $x'(t) > 0$ on $[\xi, T]$, $x'(0) = x'(T) > 0$, and there exists $t_4 \in \mathcal{B}$ such that $x'(t) > 0$ on $[0, t_4)$. Therefore (2.10) is satisfied for a.e. $t \in [0, t_4] \cup [\xi, T]$, and so

$$\begin{aligned}
\int_0^{g(x'(0))} \frac{dt}{w(g^{-1}(t))} &= -\int_0^{t_4} \frac{(g(x'(t)))'}{w(x'(t))}\,dt \le \int_0^{t_4} (h(t) + x'(t))\,dt \\
&\le \|h\|_{L_1} + 2\|x\|, \\
\int_{g(x'(0))}^{g(x'(\xi))} \frac{dt}{w(g^{-1}(t))} &= \int_{g(x'(T))}^{g(x'(\xi))} \frac{dt}{w(g^{-1}(t))} = -\int_\xi^T \frac{(g(x'(t)))'}{w(x'(t))}\,dt \\
&\le \int_\xi^T (h(t) + x'(t))\,dt \le \|h\|_{L_1} + 2\|x\|;
\end{aligned}$$

hence

$$\begin{aligned}
\int_0^{g(\|x'\|)} \frac{dt}{w(g^{-1}(t))} &= \int_0^{g(x'(0))} \frac{dt}{w(g^{-1}(t))}\,dt + \int_{g(x'(0))}^{g(x'(\xi))} \frac{dt}{w(g^{-1}(t))} \\
&\le 2(\|h\|_{L_1} + 2\|x\|).
\end{aligned} \tag{2.12}$$

Assume $\xi = T$. Since $x'(0) = x'(T) > 0$, there exists $t_5 \in \mathcal{B}$ such that $x'(t) > 0$ on $[0, t_5)$ and (2.10) is satisfied for a.e. $t \in [0, t_5]$. Consequently

$$\begin{aligned}
\int_0^{g(\|x'\|)} \frac{dt}{w(g^{-1}(t))} &= -\int_0^{t_5} \frac{(g(x'(t)))'}{w(x'(t))}\,dt \le \int_0^{t_5} (h(t) + x'(t))\,dt \\
&\le \|h\|_{L_1} + 2\|x\|.
\end{aligned} \tag{2.13}$$

Summarizing, we have (cf. (2.8), (2.9) and (2.12)-(2.13))

$$\int_0^{g(\|x\|)} \frac{dt}{w(g^{-1}(t))} \le 2(\|h\|_{L_1} + 2\|x\|) \tag{2.14}$$

provided $x'(\xi) > 0$. Analogously (by (2.4) with $k = 2$) we can prove

$$\int_{g(-\|x\|)}^{0} \frac{dt}{w(g^{-1}(t))} \le 2(\|h\|_{L_1} + 2\|x\|) \tag{2.15}$$

provided $x'(\xi) < 0$. The inequality (2.6) follows immediately from (2.5), (2.14) and (2.15). $\square$

Remark 2 *Lemma 1 is a generation of Lemma 2.5 in [7] which is proved for $g(u) \equiv u$.*

For any number $V \ge 0$ and any function $z : J \to \mathbb{R}$, we define the functions $r_{\mathrm{V}} z$ and $z\Big|_{-V}^{V}$ by

$$(r_{\mathrm{V}} z)(t) = \begin{cases} z(t) - V & \text{if } z(t) > V \\ 0 & \text{if } |z(t)| \le V \\ z(t) + V & \text{if } z(t) < -V \end{cases} \tag{2.16}$$

and

$$z(t)\Big|_{-V}^{V} = \begin{cases} V & \text{if } z(t) > V \\ z(t) & \text{if } |z(t)| \le V \\ -V & \text{if } z(t) < -V. \end{cases} \tag{2.17}$$

Let k, l, V be positive constants, $\varrho_1 \in L_1(J)$ be a positive-valued function and set

$$\begin{aligned} \Omega^* = \Big\{(x,a,b) :\ & (x,a,b) \in C^1(J) \times \mathbb{R}^2,\ \|x\| < V+2, \\ & \|x'\| < k,\ |a| < V+2,\ |b| < l\Big\}. \end{aligned} \tag{2.18}$$

Define the operator

$$H : [0,1] \times \overline{\Omega}^* \to C^1(J) \times \mathbb{R}^2$$

by

$$\begin{aligned} H(\lambda, x, a, b) = \Bigg(a + \int_0^t \Big[(1-\lambda)\Big(b + \int_0^s \varrho_1(\nu)(r_{\mathrm{V}} x)(\nu)\, d\nu\Big) \\ + \lambda g^{-1}\Big(b + \int_0^s \varrho_1(\nu)(r_{\mathrm{V}} x)(\nu)\, d\nu\Big)\Big]\, ds, \\ a + x(0) - x(T),\ b + x'(0) - x'(T) \Bigg) \end{aligned} \tag{2.19}$$

Lemma 3 $\mathrm{D}(I - H(1,\cdot), \Omega^*, 0) \neq 0$, *where I is the identical operator on $C^1(J) \times \mathbb{R}^2$ and "D" stands for the Leray-Schauder degree.*

Proof. Since $r_{\mathrm{V}}(-x) = -r_{\mathrm{V}} x$ for $x \in C^1(J)$, we have

$$H(0, -x, -a, -b) = -H(0, x, a, b), \quad (x,a,b) \in \overline{\Omega}^*$$

and so $H(0,\cdot)$ is an odd operator.

Let $H(\lambda_0, x_0, a_0, b_0) = (x_0, a_0, b_0)$ for some $(\lambda_0, x_0, a_0, b_0) \in [0,1] \times \partial\Omega^*$. Then

$$\begin{aligned} x_0(t) = a_0 + \int_0^t \Big[(1-\lambda_0)\Big(b_0 + \int_0^s \varrho_1(\nu)(r_{\mathrm{V}} x_0)(\nu)\, d\nu\Big) \\ + \lambda_0 g^{-1}\Big(b_0 + \int_0^s \varrho_1(\nu)(r_{\mathrm{V}} x_0)(\nu)\, d\nu\Big)\Big]\, ds, \end{aligned} \tag{2.20}$$

$$x_0(0) = x_0(T), \quad x_0'(0) = x_0'(T). \tag{2.21}$$

Assume $\|x_0\| > V$. Then there exists $\xi \in [0,T)$ (see (2.21)) such that $|x_0(\xi)| = \|x_0\|$. Let $x_0(\xi) > V$ (the case $x_0(\xi) < -V$ can be treated analogously). Then $x_0(t) > V$ on

$[\xi, t_1]$ with a $t_1 \in (\xi, T]$. From the definition (2.16) of the function $r_{\mathrm{V}} u$, the equalities $x_0'(\xi) = 0$ and (cf. (2.20))

$$\begin{aligned} x_0'(t) &= (1-\lambda_0)\Big(b_0 + \int_0^t \varrho_1(s)(r_{\mathrm{V}} x_0)(s)\, ds\Big) \\ &+\lambda_0 g^{-1}\Big(b_0 + \int_0^t \varrho_1(s)(r_{\mathrm{V}} x_0)(s)\, ds\Big), \quad t \in J \end{aligned} \tag{2.22}$$

we see that

$$x_0'(t) = x_0'(t) - x_0'(\xi) = (1-\lambda_0)\int_\xi^t \varrho_1(s)(r_{\mathrm{V}} x_0)(s)\, ds$$
$$+\lambda_0\Big[g^{-1}\Big(b_0 + \int_0^t \varrho_1(s)(r_{\mathrm{V}} x_0)(s)\, ds\Big) - g^{-1}\Big(b_0 + \int_0^\xi \varrho_1(s)(r_{\mathrm{V}} x_0)(s)\, ds\Big)\Big] > 0$$

for $t \in (\xi, t_1]$, and consequently x_0 is increasing on $[\xi, t_1]$, a contradiction. Hence $\|x_0\| \le V$ and then (cf. (2.22))

$$x_0'(t) = (1-\lambda_0)b_0 + \lambda_0 g^{-1}(b_0).$$

Since $x_0(0) = x_0(T)$ implies $x_0'(\tau) = 0$ for a $\tau \in (0, T)$, we infer

$$(1-\lambda_0)b_0 + \lambda_0 g^{-1}(b_0) = 0,$$

which is satisfied if and only if $b_0 = 0$. So $x_0(t) = a_0$. We have proved: $\|x_0\| \le V$, $\|x_0'\| = 0$, $|a_0| \le V$ and $b_0 = 0$, a contradiction. Thus

$$H(\lambda, x, a, b) \ne (x, a, b) \quad \text{for } (\lambda, x, a, b) \in [0,1] \times \partial\Omega^*.$$

Let $(\lambda_n, x_n, a_n, b_n) \to (\lambda, x, a, b)$ as $n \to \infty$ in $[0,1] \times C^1(J) \times \mathbb{R}^2$. Then $\lim_{n\to\infty} r_{\mathrm{V}} x_n = r_{\mathrm{V}} x$ in $C^0(J)$ and we see that

$$\lim_{n\to\infty} H(\lambda_n, x_n, a_n, b_n) = H(\lambda, x, a, b).$$

Hence H is a continuous operator. Let $\{(\lambda_n, x_n, a_n, b_n)\} \subset [0,1] \times \overline{\Omega}^*$ and set $(u_n, A_n, B_n) = H(\lambda_n, x_n, a_n, b_n)$. Then

$$\begin{aligned} u_n(t) = a_n + \int_0^t \Big[(1-\lambda_n)\Big(b_n + \int_0^s \varrho_1(\nu)(r_{\mathrm{V}} x_n)(\nu)\, d\nu\Big) \\ +\lambda_n g^{-1}\Big(b_n + \int_0^s \varrho_1(\nu)(r_{\mathrm{V}} x_n)(\nu)\, d\nu\Big)\Big]\, ds, \end{aligned}$$

$$A_n = a_n + x_n(0) - x_n(T), \quad B_n = b_n + x_n'(0) - x_n'(T)$$

for $t \in J$ and $n \in \mathbb{N}$. It is easily seen that (cf. (2.1), (2.2) and (2.16))

$$\begin{gathered} |u_n(t)| \leq V + 2 + T(l + 2\|\varrho_1\|_{L_1}) + TQ(l + 2\|\varrho_1\|_{L_1}), \\ |u_n'(t)| \leq l + 2\|\varrho_1\|_{L_1} + Q(l + 2\|\varrho_1\|_{L_1}), \\ A_n \leq 3(V + 2), \quad |B_n| \leq 2k + l \end{gathered}$$

for $t \in J$ and $n \in \mathbb{N}$.

Fix $\varepsilon > 0$. Then there exists δ_1, $0 < \delta_1 < \varepsilon$, such that $|g^{-1}(v_1) - g^{-1}(v_2)| < \varepsilon$ for $v_1, v_2 \in [-l - 2\|\varrho\|_{L_1}, l + 2\|\varrho\|_{L_1}]$, $|v_1 - v_2| < \delta_1$. Let $\delta_2 > 0$ be a number such that $\Big|\int_{t_1}^{t_2} \varrho_1(s)\,ds\Big| < \dfrac{\delta_1}{2}$ for $t_1, t_2 \in J$, $|t_1 - t_2| < \delta_2$. Then (for $t_1, t_2 \in J$, $|t_1 - t_2| < \delta_2$)

$$\Big|\int_0^{t_1} \varrho_1(s)(r_{\mathrm{V}} x_n)(s)\,ds - \int_0^{t_2} \varrho_1(s)(r_{\mathrm{V}} x_n)(s)\,ds\Big| \leq 2\Big|\int_{t_1}^{t_2} \varrho_1(s)\,ds\Big| < \delta_1,$$

and so

$$\begin{gathered} |u_n'(t_1) - u_n'(t_2)| \leq \Big|\int_{t_1}^{t_2} \varrho_1(s)(r_{\mathrm{V}} x_n)(s)\,ds\Big| \\ +\Big|g^{-1}\Big(b_n + \int_0^{t_1} \varrho_1(s)(r_{\mathrm{V}} x_n)(s)\,ds\Big) - g^{-1}\Big(b_n + \int_0^{t_2} \varrho_1(s)(r_{\mathrm{V}} x_n)(s)\,ds\Big)\Big| \\ < \delta_1 + \varepsilon < 2\varepsilon, \end{gathered}$$

which implies that $\{u_n'(t)\}$ is equicontinuous on J. Now, by the Arzelà-Ascoli theorem and the Bolzano-Weierstrass theorem, there exists a subsequence of $\{(u_n, A_n, B_n)\}$ converging in $C^1(J) \times \mathbb{R}^2$. Hence H is a compact operator. By the Borsuk theorem and the homotopy, $\mathrm{D}(I - H(0,\cdot), \Omega^*, 0) \neq 0$ and $\mathrm{D}(I - H(1,\cdot), \Omega^*, 0) = \mathrm{D}(I - H(0,\cdot), \Omega^*, 0)$. The proof is complete. $\square$

Let $\gamma, \delta \in AC(J)$ and $\gamma(t) \leq \delta(t)$ for $t \in J$. For each $u \in AC(J)$, define the truncation function $\Delta_{\gamma\delta} u : J \to \mathbb{R}$, the penalty function $p_{\gamma\delta} u : J \to [-1, 1]$ and the sets $\mathcal{A}_1(\gamma, \delta)$, $\mathcal{A}_2(\gamma, \delta)$, $\mathcal{A}(\gamma, \delta)$ by

$$(\Delta_{\gamma\delta} u)(t) = \begin{cases} \delta(t) & \text{if } u(t) > \delta(t) \\ u(t) & \text{if } \gamma(t) \leq u(t) \leq \delta(t) \\ \gamma(t) & \text{if } u(t) < \gamma(t), \end{cases} \tag{2.23}$$

$$(p_{\gamma\delta} u)(t) = \begin{cases} 1 & \text{if } u(t) > \delta(t) + 1 \\ u(t) - \delta(t) & \text{if } \delta(t) < u(t) \leq \delta(t) + 1 \\ 0 & \text{if } \gamma(t) \leq u(t) \leq \delta(t) \\ u(t) - \gamma(t) & \text{if } \gamma(t) - 1 \leq u(t) < \gamma(t) \\ -1 & \text{if } u(t) < \gamma(t) - 1, \end{cases} \tag{2.24}$$

$$\begin{aligned}
\mathcal{A}_1(\gamma,\delta) &= \Big\{(x,y,b) : (x,y,b) \in C^0(J)\times C^0(J)\times\mathbb{R},\ \gamma(t)\le x(t)\le\delta(t),\\
&\qquad \gamma(t)\le y(t)\le\delta(t) \text{ for } t\in J,\ b>0\Big\},\\
\mathcal{A}_2(\gamma,\delta) &= \Big\{(x,y,b) : (x,y,b) \in C^0(J)\times C^0(J)\times\mathbb{R},\ \gamma(t)\le x(t)\le\delta(t),\\
&\qquad \gamma(t)\le y(t)\le\delta(t) \text{ for } t\in J,\ b<0\Big\},\\
\mathcal{A}(\gamma,\delta) &= \Big\{(x,y,b) : (x,y,b) \in C^0(J)\times C^0(J)\times\mathbb{R},\ \gamma(t)\le x(t)\le\delta(t),\\
&\qquad \gamma(t)\le y(t)\le\delta(t) \text{ for } t\in J,\ b\in\mathbb{R}\Big\}.
\end{aligned} \tag{2.25}$$

It is easy to check that $(\Delta_{\gamma\delta}u)(t) = \max\{\gamma(t), \min\{\delta(t), u(t)\}\}$ for $t \in J$. Since $\min\{a,b\} = \frac{1}{2}(a+b-|a-b|)$, $\max\{a,b\} = \frac{1}{2}(a+b+|a-b|)$ for $a,b\in\mathbb{R}$, we see that $\Delta_{\gamma\delta} : AC(J) \to AC(J)$ and

$$(\Delta_{\gamma\delta}u)'(t) = \begin{cases} \delta'(t) & \text{if } u(t)>\delta(t)\\ u'(t) & \text{if } \gamma(t)\le u(t)\le\delta(t)\\ \gamma'(t) & \text{if } u(t)<\gamma(t). \end{cases} \tag{2.26}$$

From $(\Delta_{\gamma\delta}u)' \in L_1(J)$ it may be concluded that $(\Delta_{\gamma\delta}u)'\Big|_{-V}^{V} \in L_1(J)$ for any $u\in AC(J)$ and $V\ge 0$.

A function $\alpha \in C^1(J)$ is called *a lower function of PBVP* (1.1), (1.2) if $g(\alpha')$ is absolutely continuous on J,

$$(g(\alpha'(t)))' \ge F(\alpha,\alpha(t),\alpha'(t))(t) \quad \text{for a.e. } t\in J$$

and

$$\alpha(0)=\alpha(T), \quad \alpha'(0)\ge\alpha'(T).$$

An upper function of PBVP (1.1), (1.2) is defined by reversing the above inequalities.

The following assumption will be needed throughout the paper.

(H_1) There exist lower and upper functions α and β of PBVP (1.1), (1.2), respectively, and

$$\alpha(t)\le\beta(t) \quad \text{for } t\in J;$$

(H_2) There exist $\sigma_k \in \{-1,1\}$ $(k=1,2)$ and positive-valued functions $h\in L_1(J)$, $w\in C^0(\mathbb{R})$ such that (2.3) is satisfied, $w(u)$ and $w(-u)$ are nondecreasing on $[0,\infty)$ and

$$\sigma_k F(x,y(t),b)(t) \le (h(t)+|b|)w(b) \tag{2.27}$$

for a.e. $t\in J$ and each $(x,y,b)\in\mathcal{A}_k(\alpha,\beta)$ $(k=1,2)$.

Set

$$V = \max\{\|\alpha\|, \|\beta\|\} \tag{2.28}$$

and let $P \geq V$ be a positive number such that

$$\min\Big\{\int_{g(-P)}^{0} \frac{dt}{w(g^{-1}(t))}, \int_{0}^{g(P)} \frac{dt}{w(g^{-1}(t))}\Big\} > 2(\|h\|_{L_1} + 2V). \tag{2.29}$$

By the property (iv) of F, there exists a positive-valued function $\varrho \in L_1(J)$ such that

$$\Big|F\Big(\Delta_{\alpha\beta}x, (\Delta_{\alpha\beta}x)(t), (\Delta_{\alpha\beta}x)'(t)\Big|_{-P}^{P}\Big)(t)\Big| \leq \varrho(t) \tag{2.30}$$

for a.e. $t \in J$ and each $x \in C^1(J)$.

Consider the one-parameter family of the functional differential equations

$$\begin{aligned}(g(x'(t)))' \quad &= \quad \lambda F\Big(\Delta_{\alpha\beta}x, (\Delta_{\alpha\beta}x)(t), (\Delta_{\alpha\beta}x)'(t)\Big|_{-P}^{P}\Big)(t) \\ &\quad\quad + \lambda(p_{\alpha\beta}x)(t) + \varrho(t)(r_{\mathrm{V}}x)(t)\end{aligned} \tag{2.31}$$

depending on the parameter $\lambda \in [0, 1]$.

Lemma 4 *Let assumptions* (H_1) *and* (H_2) *be satisfied. If* $u(t)$ *is a solution of PBVP (2.31)*$_\lambda$*, (1.2) for some* $\lambda \in [0, 1]$ *then*

$$\|u\| \leq V + 1, \quad \|u'\| \leq Q(2\|\varrho\|_{L_1} + T). \tag{2.32}$$

Proof. Assume $\|u\| = u(\xi) > V + 1$, $\xi \in J$. From (2) we conclude that $u'(\xi) = 0$ and we can assume $\xi \in [0, T)$. Then there is a $t_0 > \xi$ such that $u(t) > V + 1$ for $t \in [\xi, t_0]$, and so (cf. (2.16), (2.24), (2.28) and (2.30))

$$(g(u'(t)))' \geq \lambda(1 - \varrho(t)) + \varrho(t)(u(t) - V) > \lambda(1 - \varrho(t)) + \varrho(t) > 0$$

for a.e. $t \in [\xi, t_0]$. Hence $g(u'(t))$ is increasing on $[\xi, t_0]$ and since $g(u'(\xi)) = 0$ we have $u'(t) > 0$ for $[\xi, t_0]$, a contradiction.

Analogously for $\|u\| = -u(\varepsilon) > V + 1$, $\varepsilon \in J$. Thus $\|u\| \leq V + 1$. Then

$$|(g(u'(t)))'| \leq \lambda(1 + \varrho(t)) + \varrho(t) \leq 2\varrho(t) + 1$$

for a.e. $t \in J$. From the equality $u(0) = u(T)$ it follows that $u'(\nu) = 0$ for a $\nu \in (0, T)$, and consequently

$$|g(u'(t))| = |g(u'(t)) - g(u'(\nu))| \leq \Big|\int_{\nu}^{t} (2\varrho(s) + 1)\, ds\Big| \leq 2\|\varrho\|_{L_1} + T$$

for $t \in J$. Hence (cf. (2.2)) $\|u'\| \leq Q(2\|\varrho\|_{L_1} + T)$. $\square$

3. Existence results

Theorem 5 *Let assumptions* (H_1) *and* (H_2) *be satisfied. Then PBVP (1.1), (1.2) has a solution* $u(t)$ *satisfying the inequalities*

$$\alpha(t) \le u(t) \le \beta(t) \quad \text{for } t \in J. \tag{3.1}$$

Proof. Let V be given by (2.28) and let the constant $P \ge V$ satisfy inequality (2.29). Set

$$\Omega = \Big\{(x,a,b) :\ (x,a,b) \in C^1(J) \times \mathbb{R}^2,\ \|x\| < V+2, \\ \|x'\| < Q(2\|\varrho\|_{L_1} + T) + 1,\ |a| < V+2, \\ |b| < G[Q(2\|\varrho\|_{L_1} + T) + 1]\Big\}$$

and $K : [0,1] \times \overline{\Omega} \to C^1(J) \times \mathbb{R}^2$,

$$K(\lambda,x,a,b) = \Bigg(a + \int_0^t g^{-1}\Big(b + \int_0^s \Big[\lambda F\Big(\Delta_{\alpha\beta}x, (\Delta_{\alpha\beta}x)(\nu), (\Delta_{\alpha\beta}x)'(\nu)\Big|_{-P}^{P}\Big)(\nu) \\ + \lambda(p_{\alpha\beta}x)(\nu) + \varrho(\nu)(r_{\mathrm{V}}x)(\nu)\Big]\, d\nu\Big)\, ds,\ a + x(0) - x(T),\ b + x'(0) - x'(T)\Bigg)$$

where $\varrho \in L_1(J)$ is a positive-valued function satisfying (2.30) and the functions G, Q are defined by (2.1).

We now prove that

$$\mathrm{D}(I - K(1,\cdot), \Omega, 0) \ne 0. \tag{3.2}$$

Since (cf. (2.19) with $\varrho = \varrho_1$) $K(0,\cdot) = H(1,\cdot)$, Lemma 2 (with $k = Q(2\|\varrho\|_{L_1} + T) + 1$ and $l = G[Q(2\|\varrho\|_{L_1} + T) + 1]$ in Ω^* defined by (2.18)) implies

$$\mathrm{D}(I - K(0,\cdot), \Omega, 0) \ne 0. \tag{3.3}$$

Hence to prove (3.2) it is sufficient to verify that

(j) K is a compact operator, and

(jj) $K(\lambda,x,a,b) \ne (x,a,b)$ for $(\lambda,x,a,b) \in [0,1] \times \partial\Omega$.

Let $\lim_{n\to\infty}(\lambda_n, x_n, a_n, b_n) = (\lambda, x, a, b)$ in the Banach space $[0,1] \times C^1(J) \times \mathbb{R}^2$. Then $\lim_{n\to\infty} \Delta_{\alpha\beta}x_n = \Delta_{\alpha\beta}x$ in $C^0(J)$, $\{(\Delta_{\alpha\beta}x_n)'\Big|_{-P}^{P}\} \subset \mathcal{L}^P(J)$,

$$\lim_{n\to\infty} (\Delta_{\alpha\beta}x_n)'(t)\Big|_{-P}^{P} = (\Delta_{\alpha\beta}x)'(t)\Big|_{-P}^{P} \quad \text{a.e. on } J$$

$\Big(\Rightarrow (\Delta_{\alpha\beta}x_n)'\Big|_{-P}^{P} \to (\Delta_{\alpha\beta}x)'\Big|_{-P}^{P}$ in $\mathcal{L}^P(J)$ as $n \to \infty\Big)$, $\lim_{n\to\infty} p_{\alpha\beta}x_n = p_{\alpha\beta}x$ and $\lim_{n\to\infty} r_{\mathrm{V}}x_n = r_{\mathrm{V}}x$ in $C^0(J)$, and consequently (see property (ii) of F)

$$\lim_{n\to\infty} K(\lambda_n, x_n, a_n, b_n) = K(\lambda, x, a, b) in C^1(J) \times \mathbb{R}^2$$

which shows that K is a continuous operator.

Let $\{(\lambda_n, x_n, a_n, b_n)\} \subset [0,1] \times \overline{\Omega}$ and set

$$(u_n, A_n, B_n) = K(\lambda_n, x_n, a_n, b_n), \quad n \in \mathbb{N}.$$

Then

$$u_n(t) = a_n + \int_0^t g^{-1}\Big(b_n + \int_0^s \Big[\lambda_n F\Big(\Delta_{\alpha\beta}x_n, (\Delta_{\alpha\beta}x_n)(\nu), (\Delta_{\alpha\beta}x_n)'(\nu)\Big|_{-P}^{P}\Big)(\nu)$$
$$+\lambda_n(p_{\alpha\beta}x_n)(\nu) + \varrho(\nu)(r_{\mathrm{V}}x_n)(\nu)\Big]\, d\nu\Big)\, ds,$$

$$A_n = a_n + x_n(0) - x_n(T), \quad B_n = b_n + x_n'(0) - x_n'(T),$$

and consequently

$$|u_n(t)| \le V + 2 + TQ\Big(G[Q(2\|\varrho\|_{L_1} + T) + 1] + 3\|\varrho\|_{L_1} + 1\Big),$$
$$|u_n'(t)| \le Q\Big(G(Q(2\|\varrho\|_{L_1} + T) + 1) + 3\|\varrho\|_{L_1} + 1\Big),$$
$$|g(u_n'(t_1)) - g(u_n'(t_2))| \le 3\Big|\int_{t_1}^{t_2} \varrho(s)\, ds\Big| + |t_1 - t_2|$$

for $t, t_1, t_2 \in J$ and $n \in \mathbb{N}$. Then $\{u_n\}$ is bounded in $C^1(J)$, $\{u_n'(t)\}$ is equicontinuous on J since g is continuous and increasing on $\mathbb{R}$ and $\{A_n\}$, $\{B_n\}$ are bounded (in $\mathbb{R}$), and so there exists a subsequence $\{(u_{k_n}, A_{k_n}, B_{k_n})\}$ converging in $C^1(J) \times \mathbb{R}^2$. Thus K is a compact operator.

Assume $K(\lambda_0, x_0, a_0, b_0) = (x_0, a_0, b_0)$ for some $(\lambda_0, x_0, a_0, b_0) \in [0,1] \times \partial\Omega$. Then $x_0(t)$ is a solution of PBVP $(2.31)_{\lambda_0}, (1.2)$ and $a_0 = x_0(0)$, $b_0 = g(x_0'(0))$. By Lemma 3, $\|x_0\| \le V + 1$, $\|x_0'\| \le Q(2\|\varrho\|_{L_1} + T)$, and so

$$|a_0| \le V + 1, \quad |b_0| \le G[Q(2\|\varrho\|_{L_1} + T)]$$

which contradicts $(x_0, a_0, b_0) \in \partial\Omega$. We have proved that K satisfies (j) and (jj); hence (3.2) holds. Then there exists a solution $u(t)$ of PBVP $(2.31)_1, (1.2)$, that is

$$\begin{aligned}(g(u'(t)))' = F\Big(\Delta_{\alpha\beta}u, (\Delta_{\alpha\beta}u)(t), (\Delta_{\alpha\beta}u)'(t)\Big|_{-P}^{P}\Big)(t)\\ +(p_{\alpha\beta}u)(t) + \varrho(t)(r_{\mathrm{V}}u)(t)\end{aligned} \tag{3.4}$$

for a.e. $t \in J$ and

$$u(0) = u(T), \quad u'(0) = u'(T). \tag{3.5}$$

Set $w(t) = u(t) - \beta(t)$ for $t \in J$. Then

$$w(0) = w(T), \quad w'(0) \geq w'(T). \tag{3.6}$$

Let $\max\{w(t) : t \in J\} = w(\xi) > 0$ for some $\xi \in J$. Then (cf. (3.6)) $w'(\xi) = 0$ and without loss of generality we can assume $\xi \in [0, T)$. Since on any interval of the type $[\xi, t_1] \subset J$ where $w(t) > 0$ we have (see property (iii) of F)

$$\begin{aligned}(g(u'(t)))' - (g(\beta'(t)))' &\geq F\Big(\Delta_{\alpha\beta} u, (\Delta_{\alpha\beta} u)(t), (\Delta_{\alpha\beta} u)'(t)\Big|_{-P}^{P}\Big)(t) \\ &\quad + (p_{\alpha\beta} u)(t) + \varrho(t)(r_{\mathrm{V}} u)(t) - F(\beta, \beta(t), \beta'(t))(t) \\ &\geq F(\beta, \beta(t), \beta'(t))(t) + \min\{w(t), 1\} - F(\beta, \beta(t), \beta'(t))(t) > 0,\end{aligned}$$

and so from the equality $g(u'(t)) - g(\beta'(t)) = \displaystyle\int_{\xi}^{t} (g(u'(s)) - g(\beta'(s)))'\, ds$ we deduce $w'(t) > 0$ on a right neighbourhood of $t = \xi$, a contradiction. Hence $w(t) \leq 0$ on J and $u(t) \leq \beta(t)$ for $t \in J$.

Set $z(t) = u(t) - \alpha(t)$, $t \in J$. Then

$$z(0) = z(T), \quad z'(0) \leq z'(T). \tag{3.7}$$

Let $\min\{z(t) : t \in J\} = z(\nu) < 0$ with a $\nu \in J$. Then (cf. (3.7)) $z'(\nu) = 0$ and without loss of generality we can assume that $\nu \in [0, T)$. Hence there exists $t_1 \in (\nu, T]$ such that $z(t) < 0$ on $[\nu, t_1]$. Then

$$\begin{aligned}(g(u'(t)))' - (g(\alpha'(t)))' &\leq F(\alpha, \alpha(t), \alpha'(t))(t) + (p_{\alpha\beta} u)(t) + \varrho(t)(r_{\mathrm{V}} u)(t) \\ &\quad - F(\alpha, \alpha(t), \alpha'(t))(t) \leq (p_{\alpha\beta} u)(t) = \max\{z(t), -1\} < 0\end{aligned}$$

for $t \in [\nu, t_1]$, and consequently $g(u'(t)) - g(\alpha'(t)) < 0$ on $(\nu, t_1]$ since $g(u'(\nu)) - g(\alpha'(\nu)) = 0$. Then $z'(t) = u'(t) - \alpha'(t) < 0$ for $t \in (\nu, t_1]$, a contradiction. Hence $z(t) \geq 0$ on J.

From (3.1) now it follows

$$(g(u'(t)))' = F\Big(u, u(t), u'(t)\Big|_{-P}^{P}\Big)(t) \quad \text{for a.e. } t \in J.$$

Then (cf. (H_2))

$$\sigma_k (g(u'(t)))' \leq \Big(h(t) + |u'(t)|\Big|_{0}^{P}\Big) w\Big(u'(t)\Big|_{-P}^{P}\Big) \leq (h(t) + |u'(t)|) w(u'(t))$$

for a.e. $t \in \mathcal{C}_k$ ($k = 1, 2$), where $\mathcal{C}_k$ is defined in Lemma 1 (with $x = u$). Applying Lemma 1 and using (2.29) and $\|u\| \leq V$, we see that $\|u'\| \leq P$. Thus $(g(u'(t)))' = F(u, u(t), u'(t))(t)$ and u is a solution of PBVP (1.1), (1.2) satisfying the inequalities (3.1). $\square$

If assumptions (H_1) and (H_2) are satisfied then there exists a solution $u(t)$ of PBVP $(1.1),(1.2)$ in the set

$$\mathcal{D}_{\alpha\beta} = \{x : x \text{ is a solution of PBVP } (1),(2) \text{ and } \alpha(t) \le x(t) \le \beta(t) \text{ for } t \in J\},$$

by Theorem 1.

We say that solutions $u_*(t)$ and $u^*(t)$ are respectively *lower and upper solutions of PBVP* $(1.1),(1.2)$ *in the set* $\mathcal{D}_{\alpha\beta}$ if $u_*, u^* \in \mathcal{D}_{\alpha\beta}$ and

$$u_*(t) \le u(t) \le u^*(t), \quad t \in J$$

for any $u \in \mathcal{D}_{\alpha\beta}$.

Theorem 6 *Let assumptions* (H_1) *and* (H_2) *be satisfied. Then there exist lower and upper solutions of PBVP* (1), (2) *in the set* $\mathcal{D}_{\alpha\beta}$.

Proof. By Theorem 1, $\mathcal{D}_{\alpha\beta} \ne \emptyset$. Let V be given by (2.28) and $P \ge V$ satisfies (2.29). From the assumption (H_2) and applying Lemma 1 we conclude (see the proof of Theorem 1) that $\|u'\| \le P$ for every $u \in \mathcal{D}_{\alpha\beta}$. Let $u_1, u_2 \in \mathcal{D}_{\alpha\beta}$ and set

$$w_-(t) = \min\{u_1(t), u_2(t)\}, \quad w_+(t) = \max\{u_1(t), u_2(t\}, \quad t \in J.$$

Then $\alpha(t) \le w_-(t) \le w_+(t) \le \beta(t)$ on J, $w_-, w_+ \in AC(J)$ and $\|u_1'\| \le P$, $\|u_2'\| \le P$. We now show that there exist $u, v \in \mathcal{D}_{\alpha\beta}$ satisfying the inequalities

$$u(t) \le w_-(t), \quad w_+(t) \le v(t), \quad t \in J. \tag{3.8}$$

Consider the one-parameter family of the functional differential equations

$$(g(x'(t)))' = \lambda(S_+x)(t) + \lambda(p_{w_+\beta}x)(t) + 5\varrho(t)(r_V x)(t), \quad \lambda \in [0,1], \tag{3.9}$$

where

$$S_+ : C^1(J) \to L_1(J),$$

$$\begin{aligned}
(S_+x)(t) &= F\Big(\Delta_{w_+\beta}x, (\Delta_{w_+\beta}x)(t), (\Delta_{w_+\beta}x)'(t)\Big|_{-P}^{P}\Big)(t) \\
&- \Big|F\Big(\Delta_{w_+\beta}x, (\Delta_{w_+\beta}x)(t), (\Delta_{w_+\beta}x)'(t)\Big|_{-P}^{P}\Big)(t) \\
&\quad -F\Big(\Delta_{w_+\beta}x, (\Delta_{u_2\beta}x)(t), (\Delta_{u_2\beta}x)'(t)\Big|_{-P}^{P}\Big)(t)\Big| \\
&- \Big|F\Big(\Delta_{w_+\beta}x, (\Delta_{w_+\beta}x)(t), (\Delta_{w_+\beta}x)'(t)\Big|_{-P}^{P}\Big)(t) \\
&\quad -F\Big(\Delta_{w_+\beta}x, (\Delta_{u_1\beta}x)(t), (\Delta_{u_1\beta}x)'(t)\Big|_{-P}^{P}\Big)(t)\Big|
\end{aligned}$$

and $\varrho \in L_1(J)$ is a positive-valued function satisfying (2.30) for a.e. $t \in J$ and each $x \in C^1(J)$.

Let

$$\Omega = \Big\{(x, a, b) : \ (x, a, b) \in C^1(J) \times \mathbb{R}^2, \|x\| < V + 2, \\ \|x'\| < Q(10\|\varrho\|_{L_1} + T) + 1, \ |a| < V + 2, \\ |b| < G[Q(10\|\varrho\|_{L_1} + T) + 1]\Big\},$$

and $L_+ : [0,1] \times \overline{\Omega} \to C^1(J) \times \mathbb{R}^2$ be defined by the formula

$$L_+(\lambda, x, a, b) = \\ = \bigg(a + \int_0^t g^{-1}\Big(b + \int_0^s \Big[\lambda(S_+x)(\nu) + \lambda(p_{w_+\beta}x)(\nu) + 5\varrho(\nu)(r_{\mathrm{V}}x)(\nu)\Big]\, d\nu\Big)\, ds, \\ a + x(0) - x(T), \, b + x'(0) - x'(T)\bigg).$$

Here the function Q and G are defined by (2.1). We see that (cf. (2.19)) $L_+(0, \cdot) = H(1, \cdot)$ (with $\varrho_1 = 5\varrho$ in (2.19) and $k = Q(10\|\varrho\|_{L_1} + T) + 1$, $l = G[Q(10\|\varrho\|_{L_1} + T) + 1]$ in Ω^* given by (2.18)), and so

$$\mathrm{D}(I - L_+(0, \cdot), \Omega, 0) \neq 0$$

by Lemma 2. Let $L_+(\lambda_0, x_0, a_0, b_0) = (x_0, a_0, b_0)$ for some $(\lambda_0, x_0, a_0, b_0) \in [0,1] \times \partial\Omega$. Then $a_0 = x_0(0)$, $b_0 = g(x_0'(0))$ and $x_0(t)$ is a solution of PBVP $(3.9)_{\lambda_0}$, (1.2). Since (cf. (2.30)) $|(S_+x)(t)| \le 5\varrho(t)$ for a.e. $t \in J$ and each $x \in C^1(J)$, we can proceed analogously to the proof of Lemma 3 to verify that

$$\|x_0\| \le V + 1, \quad \|x_0'\| \le Q(10\|\varrho\|_{L_1} + T),$$

and so $|a_0| \le V + 1$, $|b_0| \le G(Q(10\|\varrho\|_{L_1} + T))$. Hence $(x_0, a_0, b_0) \notin \partial\Omega$, which is impossible.

Analysis similar to that of the proof of Theorem 1 shows that L_+ is a compact operator. Thus

$$\mathrm{D}(I - L_+(1, \cdot), \Omega, 0) = \mathrm{D}(I - L_+(0, \cdot), \Omega, 0) \neq 0$$

and then, of course, there exists a fixed point (v, a, b) of the operator $L_+(1, \cdot)$. Clearly, $v(t)$ is a solution of PBVP $(3.9)_1$, (1.2). Assume $\max\{v(t) - \beta(t) : t \in J\} = v(\xi) - \beta(\xi) > 0$ for some $\xi \in J$. Then $v'(\xi) = \beta'(\xi)$ and we can assume $\xi \in [0, T)$. Let

$t_1 \in (\xi, T]$ be a point such that $v(t) - \beta(t) > 0$ for $t \in [\xi, t_1]$. Then (for a.e. $t \in [\xi, t_1]$)

$$\begin{aligned}
(g(v'(t)))' - (g(\beta'(t)))' \geq \\
&\geq (S_+v)(t) + (p_{w_+\beta}v)(t) + 5\varrho(t)(r_{\mathrm{V}}v)(t) - F(\beta, \beta(t), \beta'(t))(t) \\
&= F\Big(\Delta_{w_+\beta}v, (\Delta_{w_+\beta}v)(t), (\Delta_{w_+\beta}v)'(t)\Big|_{-P}^{P}\Big)(t) + (p_{w_+\beta}v)(t) \\
&\quad +5\varrho(t)(r_{\mathrm{V}}v)(t) - F(\beta, \beta(t), \beta'(t))(t) \\
&> F(\Delta_{w_+\beta}v, \beta(t), \beta'(t))(t) - F(\beta, \beta(t), \beta'(t))(t) \geq 0
\end{aligned} \tag{3.10}$$

since $F(\Delta_{w_+\beta}v, \beta(t), \beta'(t))(t) \geq F(\beta, \beta(t), \beta'(t))(t)$ for a.e. $t \in J$ by property (iii) of F. From (3.10) and the equality $v'(\xi) = \beta'(\xi)$ we deduce $v'(t) > \beta'(t)$ for $t \in (\xi, t_1]$, a contradiction. Hence

$$v(t) \leq \beta(t), \quad t \in J.$$

Assume $\min\{v(t) - u_k(t) : t \in J\} < 0$ for some $k \in \{1, 2\}$. Then there exist $\tau_k \in [0, T)$ and $t_k \in (\tau_k, T]$ such that $v(\tau_k) - u_k(\tau_k) = \min\{v(t) - u_k(t) : t \in J\}$, $v'(\tau_k) = u_k'(\tau_k)$ and $v(t) < u_k(t)$ for $t \in (\tau_k, t_k]$. Then (for a.e. $t \in [\tau_k, t_k]$)

$$\begin{aligned}
(g(v'(t)))' - (g(u_k'(t)))' = \\
&= (S_+v)(t) + (p_{w_+\beta}v)(t) + 5\varrho(t)(r_{\mathrm{V}}v)(t) - F(u_k, u_k(t), u_k'(t))(t) \\
&< F(\Delta_{w_+\beta}v, w_+(t), w_+'(t))(t) - F(u_k, u_k(t), u_k'(t))(t) \\
&\quad -\Big|F(\Delta_{w_+\beta}v, w_+(t), w_+'(t))(t) - F(\Delta_{w_+\beta}v, u_k(t), u_k'(t))(t)\Big| \\
&\leq F(\Delta_{w_+\beta}v, w_+(t), w_+'(t))(t) - F(\Delta_{w_+\beta}v, u_k(t), u_k'(t))(t) \\
&\quad -\Big|F(\Delta_{w_+\beta}v, w_+(t), w_+'(t))(t) - F(\Delta_{w_+\beta}v, u_k(t), u_k'(t))(t)\Big| \leq 0,
\end{aligned}$$

and consequently $v'(t) < u_k'(t)$ on $(\tau_k, t_k]$, a contradiction. Hence

$$v(t) \geq u_k(t), \quad t \in J \ (k = 1, 2).$$

We have verified that $w_+(t) \leq v(t) \leq \beta(t)$ for $t \in J$. Then $\|v\| \leq V$ and

$$(g(v'(t)))' = (S_+v)(t) + (p_{w_+\beta}v)(t) + 5\varrho(t)(r_{\mathrm{V}}v)(t) = F\Big(v, v(t), v'(t)\Big|_{-P}^{P}\Big)(t)$$

for a.e. $t \in J$. As in the proof of Theorem 1 we can show that $\|v'\| \leq P$. Hence v is a solution of PBVP (1), (2).

To prove the existence of a solution $u(t)$ of PBVP (1), (2) satisfying the inequalities $\alpha(t) \leq u(t) \leq w_-(t)$, $t \in J$, we consider the operator $L_- : [0, 1] \times \overline{\Omega} \to C^1(J) \times \mathbb{R}^2$ given by the formula

$$\begin{aligned}
L_-(\lambda, x, a, b) = \\
&= \Big(a + \int_0^t g^{-1}(b + \int_0^s [\lambda(S_-x)(\nu) + \lambda(p_{\alpha w_-}x)(\nu) + 5\varrho(\nu)(r_{\mathrm{V}}x)(\nu)]\, d\nu)\, ds, \\
&\quad a + x(0) - x(T),\ b + x'(0) - x'(T)\Big).
\end{aligned}$$

where $S_- : C^1(J) \to L_1(J)$,

$$\begin{aligned}(S_-x)(t) &= F\Big(\Delta_{\alpha w_-}x, (\Delta_{\alpha w_-}x)(t), (\Delta_{\alpha w_-}x)'(t)\Big|_{-P}^{P}\Big)(t)\\ &+ \Big|F\Big(\Delta_{\alpha w_-}x, (\Delta_{\alpha w_-}x)(t), (\Delta_{\alpha w_-}x)'(t)\Big|_{-P}^{P}\Big)(t)\\ &\quad -F\Big(\Delta_{\alpha w_-}x, (\Delta_{\alpha u_2}x)(t), (\Delta_{\alpha u_2}x)'(t)\Big|_{-P}^{P}\Big)(t)\Big|\\ &+ \Big|F\Big(\Delta_{\alpha w_-}x, (\Delta_{\alpha w_-}x)(t), (\Delta_{\alpha w_-}x)'(t)\Big|_{-P}^{P}\Big)(t)\\ &\quad -F\Big(\Delta_{\alpha w_-}x, (\Delta_{\alpha u_1}x)(t), (\Delta_{\alpha u_1}x)'(t)\Big|_{-P}^{P}\Big)(t)\Big|.\end{aligned}$$

The proof is similar to that of the first part of this theorem and therefore it is omitted.

It is easy to check that $\mathcal{D}_{\alpha\beta}$ is a closed set in $C^1(J)$. Since for each $x \in \mathcal{D}_{\alpha\beta}$ and $t_1, t_2 \in J$ we have

$$\|x\| \le V, \quad \|x'\| \le P, \quad |g(x'(t_1)) - g(x'(t_2))| \le \Big|\int_{t_1}^{t_2} \varrho(s)\, ds\Big|,$$

$\mathcal{D}_{\alpha\beta}$ is a compact set in $C^1(J)$. We shall prove only the existence of the maximal solution u^* of PBVP (1.1), (1.2) in the set $\mathcal{D}_{\alpha\beta}$ since the case of the minimal solution u_* of PBVP (1.1), (1.2) in $\mathcal{D}_{\alpha\beta}$ is very similar. By the compactness of $\mathcal{D}_{\alpha\beta}$, there exists $u^* \in \mathcal{D}_{\alpha\beta}$ such that

$$\int_0^T u^*(t)\, dt \ge \int_0^T x(t)\, dt \quad \text{for } x \in \mathcal{D}_{\alpha\beta}. \tag{3.11}$$

Suppose that there exist $x_0 \in \mathcal{D}_{\alpha\beta}$ and $t_0 \in (0, T)$ such that $x_0(t_0) > u^*(t_0)$. By the first part of the proof of our theorem, there exists $x_1 \in \mathcal{D}_{\alpha\beta}$ such that

$$\max\{u^*(t), x_0(t)\} \le x_1(t) \le \beta(t), \quad t \in J.$$

Then

$$\int_0^T x_1(t)\, dt > \int_0^T u^*(t)\, dt,$$

contrary to (3.11). Hence u^* is the maximal solution of PBVP (1.1), (1.2) in the set $\mathcal{D}_{\alpha\beta}$. □

Corollary 7 *Let assumption (H_1) be satisfied and let there exist positive-valued functions $h \in L_1(J)$ and $w \in C^0(J)$ such that (5) is satisfied and $w(u)$, $w(-u)$ are nondecreasing on $[0, \infty)$. If at least one of the following inequalities*

$$\begin{aligned}F(x, y(t), b)(t) &\le (h(t) + |b|)w(b), &(3.12)\\ F(x, y(t), b)(t) &\ge -(h(t) + |b|)w(b), &(3.13)\\ F(x, y(t), b)(t)\,\mathrm{sign}\, b &\le (h(t) + |b|)w(b) &(3.14)\end{aligned}$$

and

$$F(x, y(t), b)(t) \operatorname{sign} b \geq -(h(t) + |b|)w(b) \tag{3.15}$$

is satisfied for a.e. $t \in J$ and each $(x, y, b) \in \mathcal{A}(\alpha, \beta)$, then PBVP (1.1), (1.2) is solvable and there exist lower and upper solutions in the set $\mathcal{D}_{\alpha\beta}$.

Proof. Corollary 1 follows at once from Theorems 1 and 2 assuming that (H_2) $\sigma_1 = \sigma_2 = 1$ for (3.12), $\sigma_1 = \sigma_2 = -1$ for (3.13), $\sigma_1 = -\sigma_2 = 1$ for (3.14) and $\sigma_1 = -\sigma_2 = -1$ for (3.15). $\square$

Example 8 *Consider the functional differential equation*

$$\begin{aligned}(|x'(t)|^{p-2}x'(t))' = \int_{\frac{t}{2}}^{t} f_1(s, x(T-s), x(t), x'(t))\, ds \\ +f_2(t, x(t), x'(t))(x(t))^{2m-1} + \sigma f_3(t, x(t), x'(t))(x'(t))^n\end{aligned} \tag{3.16}$$

where $p > 1$, f_1 (resp. f_2, f_3) satisfies the local Carathéodory conditions on $J \times \mathbb{R}^3$ (resp. $J \times \mathbb{R}^2$), $f_1(t, \cdot, a, b)$ is nonincreasing on $\mathbb{R}$ for a.e. $t \in J$ and each $a, b \in \mathbb{R}$, $m, n \in \mathbb{N}$ and $\sigma \in \{-1, 1\}$. In addition, there exist positive constants ε and l such that the inequalities

$$|f_1(t, u, x, y)| \leq \frac{2l}{T}(1 + |y|^p), \quad \varepsilon \leq f_2(t, x, y) \leq l(1 + |y|^p), \quad f_3(t, x, y) \geq 0$$

are satisfied for a.e. $t \in J$ and each $u, x \in \left[-\sqrt[2m-1]{\frac{l}{\varepsilon}}, \sqrt[2m-1]{\frac{l}{\varepsilon}}\right]$, $y \in \mathbb{R}$. Set

$$\alpha = -\sqrt[2m-1]{\frac{l}{\varepsilon}} \quad \text{and} \quad \beta = \sqrt[2m-1]{\frac{l}{\varepsilon}}.$$

Then $\alpha(t) \equiv \alpha$ and $\beta(t) \equiv \beta$ are lower and upper functions of PBVP (3.16), (1.2) and

$$\begin{aligned}F(x, y(t), z(t))(t) = \int_{\frac{t}{2}}^{t} f_1(s, x(T-s), y(t), z(t))\, ds \\ +f_2(t, y(t), z(t))(y(t))^{2m-1} + \sigma f_3(t, y(t), z(t))(z(t))^n\end{aligned}$$

satisfies assumption (H_2) with $h(t) = 1$, $\omega(y) = l(1 + \frac{l}{\varepsilon})(1 + |y|^{1-p})$, $g(u) = |u|^{p-2}u$ and

$$\sigma_k = \begin{cases} -\sigma & \text{if } n \text{ is even} \\ (-1)^k\sigma & \text{if } n \text{ is odd.} \end{cases}$$

Theorems 1 and 2 show that PBVP (3.16), (1.2) is solvable and has upper and lower solutions in the set $\mathcal{M}_{\alpha\beta} = \{x : x \in C^1(J), \alpha \leq x(t) \leq \beta \text{ for } t \in J\}$.

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