HU e-ISSN 1787-2413

MATHEMATICAL NOTES

A Publication of the University of Miskolc

VOLUME 2, NUMBER 2 (2001)

MISKOLC UNIVERSITY PRESS

Mathematical Notes

Volume 2, Number 2 (2001)

Contents

Contributed Papers

Doi: 10.18514/MMN.2001.39

Mathematical Notes, Miskolc, Vol. 2., No. 2., (2001), pp. 85–92

MAPPING BIJECTIVELY σ-ALGEBRAS ONTO POWER SETS

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[Received February 6, 2001]

Dedicated to the memory of my Father

Abstract. As an application of the so-called "optimal measure" we attempt to seek sets whose power sets are equinumerous with σ-algebras, which seems to be new information about σ-algebras.

Mathematical Subject Classification: 28A20, 28E10
Keywords: Measurable sets

1. Introduction

Some new information about σ-algebras is investigated, consisting of mapping bijectively σ-algebras onto power sets. Such σ-algebras, in fact, form a rather broad class. A special grouping of the so-called optimal measures is used in our investigation (for more about optimal measures cf. [1-4]. We provide constructively a bijective mapping that will serve the purpose. In the proof we first characterize set-inclusion as well as some asymptotic behaviors of sequences of measurable sets. Without loss of generality we shall restrict ourselves to infinite σ-algebras, since the opposite case can be easily done.

Throughout this communication $(\Omega, \mathcal{F})$ will stand for an arbitrary measurable space, with both Ω and $\mathcal{F}$ being infinite sets (where, as usual, the elements of $\mathcal{F}$ are referred to as measurable sets).

By an optimal measure we mean a set function $p^* : \mathcal{F} \to [0,\, 1]$ which fulfills the following axioms:

P1. $p^*(\emptyset) = 0$ and $p^*(\Omega) = 1$.
P2. $p^*(B \bigcup E) = p^*(B) \bigvee p^*(E)$ for all measurable sets B and E (where $\bigvee$ stands for the maximum).
P3. $p^*\left(\bigcap\limits_{n=1}^{\infty} E_n\right) = \lim\limits_{n\to\infty} p^*(E_n) = \bigwedge\limits_{n=1}^{\infty} p^*(E_n)$, for every decreasing sequence of measurable sets (E_n), where $\bigwedge$ stands for the minimum.

In [2] we have obtained the following results for all optimal measures p^*.

By (p^*)-atom we mean a measurable set H, $p^*(H) > 0$ such that whenever $B \in \mathcal{F}$, $B \subset H$, then $p^*(B) = p^*(H)$ or $p^*(B) = 0$.

A p^*-atom H is decomposable if there exists a subatom $B \subset H$ such that $p^*(B) = p^*(H) = p^*(H \backslash B)$. If no such subatom exists, we shall say that H is indecomposable.

Fundamental Optimal Measure Theorem. *Let $(\Omega, \mathcal{F})$ be a measurable space and p^* an optimal measure on it. Then there exists a collection $\mathcal{H}(p^*) = \{H_n : n \in J\}$ of disjoint indecomposable p^*-atoms, where J is some countable (i.e. finite or countably infinite) index-set such that for any measurable set B, with $p^*(B) > 0$, we have that*

$$p^*(B) = \max\left\{p^*\left(B \bigcap H_n\right) : n \in J\right\}.$$

Moreover, the only limit point of the set $\{p^*(H_n) : n \in J\}$ is 0 provided that J is a countably infinite set. ($\mathcal{H}(p^*)$ is referred to as a p^*-generating countable system.)

NOTATIONS.

1. $\mathcal{P}$ will denote the set of all optimal measures defined on $(\Omega, \mathcal{F})$.
2. $\mathcal{P}_\infty$ is the set of all optimal measures whose generating systems are countably infinite.
3. For every $A \in \mathcal{F}$, we write $\overline{A}$ for the complement of A.
4. $\mathbb{N}$ stands for the set of counting numbers (or positive integers).
5. $A \subset B$ means set A is a proper subset of set B.
6. $A \subseteq B$ means set A is a subset of set B.
7. $\mathbb{P}(A)$ stands for the power set of set A.

2. Main results

Definition 2.1. We say that an optimal measure $p^* \in \mathcal{P}_\infty$ is of order-one if there is a unique indecomposable p^*-*atom* H such that $p^*(H) = 1$. (Any such atom will be referred to as an order-one-atom and the set of all order-one optimal measures will be denoted by $\widetilde{\mathcal{P}^1_\infty}$.)

Example 1. Fix a sequence $(\omega_n) \subset \Omega$ and define $p_0^* \in \mathcal{P}_\infty$ by

$$p_0^*(B) = \max\left\{\frac{1}{n} : \omega_n \in B\right\}.$$

Then $p_0^* \in \widetilde{\mathcal{P}^1_\infty}$.

In fact, via the Structure Theorem, there is an indecomposable p_0^*-*atom* H such that $p_0^*(H) = 1$. This is possible if and only if $\omega_1 \in H$. We note that there is no other indecomposable p_0^*-atom H^* with $H^* \cap H = \emptyset$ such that $p_0^*(H^*) = 1$, otherwise necessarily it would ensue that $\omega_1 \in H^*$, which is absurd. Hence $p_0^* \in \widetilde{\mathcal{P}^1_\infty}$.

FURTHER NOTATIONS.

If H is the order-one-atom of some $p^* \in \widetilde{\mathcal{P}^1_\infty}$, we write $p = \left\{q^* \in \widetilde{\mathcal{P}^1_\infty} : q^*(H) = 1\right\}$. We then refer to the elements of the class p as representing members of the class, and call H the unitary atom of the class. (If the unitary atom of a class is the order-one-atom of a representing member, we shall speak of representation.)

We further denote by $\mathcal{P}_{\infty}^{1}$ the set of all p classes.

If A is a nonempty measurable set and $p \in \mathcal{P}_{\infty}^{1}$, the identity $p(A) = 1$ (resp. the inequality $p(A) < 1$) will simply mean that $p^{*}(A) = 1$ (resp. $p^{*}(A) < 1$) for any representing member $p^{*} \in p$. We shall also write $p(A) = 0$ to mean that $p^{*}(A) = 0$ whenever $p^{*} \in p$.

Write ∇ for the set of all unitary atoms on the measurable space $(\Omega, \mathcal{F})$.

Lemma 2.1. *Let A, $B \in \mathcal{F}$ and $p \in \mathcal{P}_{\infty}^{1}$ be arbitrary. In order that $p(A \cap B) = 1$ it is necessary and sufficient that $p(A) = 1$ and $p(B) = 1$.*

Proof. As the necessity is obvious, we only have to show the sufficiency. In fact, assume that $p(A) = 1$ and $p(B) = 1$. Let H be the unitary atom of class p, and let p^{*} denote an arbitrary but fixed representing member in the class. Without loss of generality we may assume that p^{*} is a representation of p (i.e. H is the order-one atom of p^{*}). Then $p^{*}(H) = 1$. Clearly, $p^{*}(A \cap H) = 1$ and $p^{*}(B \cap H) = 1$. Hence $p^{*}\left(\overline{A} \cap H \cap \overline{B}\right) = 0$. It is enough to prove that both identities $p^{*}\left(A \cap H \cap \overline{B}\right) = 0$ and $p^{*}\left(\overline{A} \cap H \cap B\right) = 0$ are valid. On the contrary, assume that at least one of these identities fails to hold: $p^{*}\left(A \cap H \cap \overline{B}\right) = 0$, say. Then $p^{*}\left(A \cap H \cap \overline{B}\right) = 1$. Now, since $p^{*}(H \cap B) = 1$, it ensues that either $p^{*}(A \cap H \cap B) = 1$ or $p^{*}\left(\overline{A} \cap H \cap B\right) = 1$. Then combining each of these last identities with $p^{*}\left(A \cap H \cap \overline{B}\right) = 1$, we have that $p^{*}\left(A \cap H \cap \overline{B}\right) = 1$ and $p^{*}(A \cap H \cap B) = 1$, or $p^{*}\left(A \cap H \cap \overline{B}\right) = 1$ and $p^{*}\left(\overline{A} \cap H \cap B\right) = 1$. This violates that H is an order-one-atom (because the sets $A \cap H \cap B$, $A \cap H \cap \overline{B}$ and $\overline{A} \cap H \cap B$ are pairwise disjoint). **q.e.d.**

Remark 2.0. Let $p \in \mathcal{P}_{\infty}^{1}$ be arbitrary. Then the identity $p(\emptyset) = 0$ holds.

Remark 2.1. Let $A \in \mathcal{F}$ and $p \in \mathcal{P}_{\infty}^{1}$ be arbitrary. Then the identities $p(A) = 1$ and $p\left(\overline{A}\right) = 1$ cannot hold simultaneously, i.e., for no representing member p^{*} of class p the identities $p^{*}(A) = 1$ and $p^{*}\left(\overline{A}\right) = 1$ hold at the same time.

In fact, assume the contrary. Then *Lemma 2.1* would imply that

$$p(A) = p\left(\overline{A}\right) = 1 = p\left(A \cap \overline{A}\right) = p(\emptyset) = 0$$

which is absurd, indeed. **q.e.d.**

Definition 2.2. For any $A \in \mathcal{F}$ define the set $\Delta(A)$ by

1. $\Delta(A) \subseteq \mathcal{P}_{\infty}^{1}$.
2. If $p \in \Delta(A)$, then $p(A) = 1$.

Remark 2.2. Let $A \in \mathcal{F}$. Then $\Delta(A) = \emptyset$ if and only if $A = \emptyset$.

Remark 2.3. If H is a unitary atom (with p its corresponding class), then $\Delta(H) = \{p\}$.

Let $A \in \mathcal{F}$ and denote by ∇_A the set of all unitary atoms H such that $p(A) = 1$, where $\Delta(H) = \{p\}$. It is clear that $\nabla_A \bigcap \nabla_{\overline{A}} = \emptyset$ and $\nabla_A \bigcup \nabla_{\overline{A}} = \nabla$. From this observation the following lemma is straightforward:

Lemma 2.2. *For every set $A \in \mathcal{F}$, we have that $\Delta\left(\overline{A}\right) = \overline{\Delta(A)}$.*

Proposition 2.3. *Let A, $B \in \mathcal{F}$ be arbitrary. Then*

1. $\Delta(\Omega) = \mathcal{P}^1_\infty$.

2. $\Delta(A \cap B) = \Delta(A) \cap \Delta(B)$.

3. $\Delta(A \cup B) = \Delta(A) \cup \Delta(B)$.

Proof. *Part 1* is an easy task. Let us show *Part 2*. In fact, let $p \in \Delta(A \cap B)$. Then $p(A \cap B) = 1$. Hence *Lemma 2.1* implies that $p(A) = 1$ and $p(B) = 1$, so that $p \in \Delta(A)$ and $p \in \Delta(B)$, i.e. $p \in \Delta(A) \cap \Delta(B)$. Consequently $\Delta(A \cap B) \subseteq \Delta(A) \cap \Delta(B)$. To show the reverse inclusion, pick an arbitrary $p \in \Delta(A) \cap \Delta(B)$. Then $p(A) = 1$ and $p(B) = 1$. Via *Lemma 2.1*, we have that $p(A \cap B) = 1$, i.e. $p \in \Delta(A \cap B)$. So $\Delta(A) \cap \Delta(B) \subseteq \Delta(A \cap B)$.

To end the proof, let us show the third part. In fact, let A and $B \in \mathcal{F}$ be arbitrary. Then making use of the second part of this proposition, it ensues that $\Delta\left(\overline{A} \cap \overline{B}\right) = \Delta\left(\overline{A}\right) \cap \Delta\left(\overline{B}\right)$. By applying *Lemma 2.2* and De Morgan identities, we obtain that

$$\begin{aligned}
\Delta(A \cup B) &= \overline{\overline{\Delta(A \cup B)}} = \overline{\Delta\left(\overline{A} \cap \overline{B}\right)} = \overline{\Delta\left(\overline{A}\right) \cap \Delta\left(\overline{B}\right)} \\
&= \overline{\Delta\left(\overline{A}\right)} \cup \overline{\Delta\left(\overline{B}\right)} = \overline{\overline{\Delta(A)}} \cup \overline{\overline{\Delta(B)}} = \Delta(A) \cup \Delta(B).
\end{aligned}$$

This was to be proven. **q.e.d.**

Lemma 2.4. *Let A and $B \in \mathcal{F}$ be arbitrary nonempty sets. In order that $A \subset B$, it is necessary and sufficient that $\Delta(A) \subset \Delta(B)$.*

Proof. As the necessity is trivial, we need only show the sufficiency. In fact, assume that $A \backslash B$ is not an empty set. Then because of *Remark 2.2*, $\Delta(A \backslash B)$ is neither empty. Fix some $p \in \Delta(A \backslash B)$, i.e. $p(A \backslash B) = 1$. This implies that $p(B) < 1$. (Otherwise we would obtain via *Lemma 2.1* that $1 = p((A \backslash B) \cap B) = p(\emptyset) = 0$, which is absurd.) Then $p(A) = 1$ and $p(B) < 1$, i.e. $p \in \Delta(A) \backslash \Delta(B)$. So the set $\Delta(A) \backslash \Delta(B)$ is not empty. **q.e.d.**

Lemma 2.5. *Let A and $B \in \mathcal{F}$ be arbitrary nonempty sets. In order that $A \cap B = \emptyset$, it is necessary and sufficient that $\Delta(A) \cap \Delta(B) = \emptyset$.*

(The proof follows from *Proposition 2.3/2* and *Remark 2.2*.)

Lemma 2.6. *Let A and $B \in \mathcal{F}$ be arbitrary nonempty sets. In order that $A = B$ it is necessary and sufficient that $\Delta(A) = \Delta(B)$.*

Proof. As the necessity is trivial, we need only show the sufficiency. In fact, assume that A and $B \in \mathcal{F}$ are such that $\Delta(A) = \Delta(B)$, i.e. $\Delta(A) \subseteq \Delta(B)$ and $\Delta(B) \subseteq \Delta(A)$. By applying twice *Lemma 2.4* it ensues that $A \subseteq B$ and $B \subseteq A$. Therefore $A = B$. **q.e.d.**

Lemma 2.7. *Let A and $B \in \mathcal{F}$ be arbitrary nonempty sets. Then $\Delta(A \backslash B) = \Delta(A) \backslash \Delta(B)$.*

Proof. We simply note that *Proposition 2.3/2* and *Lemma 2.2* entail that

$$\begin{aligned} \Delta(A \backslash B) &= \Delta\left(A \cap \overline{B}\right) = \Delta(A) \cap \Delta\left(\overline{B}\right) \\ &= \Delta(A) \cap \left(\overline{\Delta(B)}\right) = \Delta(A) \backslash \Delta(B), \end{aligned}$$

which completes the proof. **q.e.d.**

Proposition 2.8. *Let $(A_n) \subset \mathcal{F}$ and $A \in \mathcal{F}$ be arbitrary. Then (A_n) converges increasingly to A if and only if $(\Delta(A_n))$ converges increasingly to $\Delta(A)$.*

Proof. Assume that (A_n) converges increasingly to A. Then by applying repeatedly *Lemma 2.4,* we have for every $n \in \mathbb{N}$ that

$$\Delta(A_n) \subset \Delta(A_{n+1}) \subset \Delta(A).$$

We need to prove that $\Delta(A) = \bigcup_{n=1}^{\infty} \Delta(A_n)$. To do this, it will be enough to show that $\Delta(A) \subseteq \bigcup_{n=1}^{\infty} \Delta(A_n)$ and $\bigcup_{n=1}^{\infty} \Delta(A_n) \subseteq \Delta(A)$. In fact, we note that the second inclusion is trivial. To prove the first one, let us pick an arbitrary class $p \in \Delta(A)$ and fix any representing member p^* of class p. We note that following the proof of *Lemma 0.1* (cf. [1], page 134), there can be found a positive integer n_0 such that $1 = p^*(A) = p^*\left(\bigcup_{k=1}^{\infty} A_k\right) = p^*(A_n)$, whenever $n \geq n_0$. Hence $p \in \bigcup_{n=n_0}^{\infty} \Delta(A_n) \subseteq \bigcup_{n=1}^{\infty} \Delta(A_n)$, i.e.

$$\Delta(A) \subseteq \bigcup_{n=n_0}^{\infty} \Delta(A_n) \subseteq \bigcup_{n=1}^{\infty} \Delta(A_n).$$

Conversely, assume that sequence $(\Delta(A_n))$ converges increasingly to $\Delta(A)$. Then for every $n \in \mathbb{N}$ we have that $\Delta(A_n) \subseteq \Delta(A_{n+1}) \subseteq \Delta(A)$, so that $A_n \subseteq A_{n+1} \subseteq A$ (because of *Lemma 2.4*). Hence $\bigcup_{n=1}^{\infty} A_n \subseteq A$. Now, suppose that set $A \backslash \bigcup_{n=1}^{\infty} A_n$ is not empty. Then via *Remark 2.2* and *Axiom 3* there can be found some $p \in \mathcal{P}_{\infty}^{1}$ and some representing member p^* of class p such that

$$1 = p^*\left(A \backslash \bigcup_{n=1}^{\infty} A_n\right) = p^*\left(\bigcap_{n=1}^{\infty} A \cap \overline{A_n}\right) = \bigwedge_{n=1}^{\infty} p^*\left(A \cap \overline{A_n}\right),.$$

since sequence $\left(\overline{A_n}\right)$ is a decreasing sequence. Consequently $1 = p^*\left(A \cap \overline{A_n}\right)$ for all $n \in \mathbb{N}$. But *Lemma 2.1* yields that $p^*(A) = 1$ and $p^*\left(\overline{A_n}\right) = 1$ for all $n \in \mathbb{N}$. Hence *Axiom 3* entails that

$$1 = \bigwedge_{n=1}^{\infty} p^*\left(\overline{A_n}\right) = p^*\left(\bigcap_{n=1}^{\infty} \overline{A_n}\right) = p^*\left(\overline{A}\right).$$

Nevertheless, this contradicts *Remark 2.1* **q.e.d.**

Proposition 2.9. *Let $(A_n) \subset \mathcal{F}$ and $A \in \mathcal{F}$ be arbitrary. Then (A_n) converges decreasingly to A if and only if $(\Delta(A_n))$ converges decreasingly to $\Delta(A)$.*

Proof. Assume that (A_n) converges decreasingly to A. Then by applying repeatedly *Lemma 2.4*, we have for every $n \in \mathbb{N}$ that

$$\Delta(A) \subset \Delta(A_{n+1}) \subset \Delta(A_n).$$

We need to prove that $\Delta(A) = \bigcap_{n=1}^{\infty} \Delta(A_n)$. To do this, it will be enough to show that $\Delta(A) \subseteq \bigcap_{n=1}^{\infty} \Delta(A_n)$ and $\bigcap_{n=1}^{\infty} \Delta(A_n) \subseteq \Delta(A)$. In fact, we note that the first inclusion is trivial. To prove the second inclusion let us pick some $p \in \bigcap_{n=1}^{\infty} \Delta(A_n)$. Then $p \in \Delta(A_n)$ for all $n \in \mathbb{N}$. Hence $p(A_n) = 1$ for all $n \in \mathbb{N}$. If we fix any representing member p^* in class p, we then obtain via *Axiom 3* that

$$p^*(A) = p^*\left(\bigcap_{n=1}^{\infty} A_n\right) = \bigwedge_{n=1}^{\infty} p^*(A_n) = 1,$$

implying that $p(A) = 1$, i.e. $p \in \Delta(A)$. Consequently, $\bigcap_{n=1}^{\infty} \Delta(A_n) \subseteq \Delta(A)$.

Conversely, assume that sequence $(\Delta(A_n))$ converges decreasingly to $\Delta(A)$. Then for every $n \in \mathbb{N}$ we obtain that $\Delta(A) \subset \Delta(A_{n+1}) \subset \Delta(A_n)$ so that $A \subset A_{n+1} \subset A_n$, $n \in \mathbb{N}$ (by *Lemma 2.4*). Hence $A \subseteq \bigcap_{n=1}^{\infty} A_n$. To show the reverse inclusion let us assume that set $\left(\bigcap_{n=1}^{\infty} A_n\right) \backslash A$ is not empty. Then via *Remark 2.2* and *Axiom 3* there can be found some $p \in \mathcal{P}_{\infty}^1$ such that for every representing member p^* of class p

$$1 = p^*\left(\left(\bigcap_{n=1}^{\infty} A_n\right) \backslash A\right) = p^*\left(\bigcap_{n=1}^{\infty} A_n \cap \overline{A}\right) = \bigwedge_{n=1}^{\infty} p^*\left(A_n \cap \overline{A}\right),$$

since (A_n) is a decreasing sequence. Consequently, $1 = p^*\left(A_n \cap \overline{A}\right)$ for all $n \in \mathbb{N}$. Hence *Lemma 2.1* yields that $p\left(\overline{A}\right) = 1$ and $p(A_n) = 1$ for all $n \in \mathbb{N}$. But then $p \in \Delta(A_n)$ for all $n \in \mathbb{N}$ and hence $p \in \bigcap_{n=1}^{\infty} \Delta(A_n) = \Delta(A)$. Nevertheless, this is absurd since $p \in \Delta\left(\overline{A}\right) = \overline{\Delta\left(A\right)}$. We can thus conclude on the validity of the proposition. **q.e.d.**

Theorem 2.10. *Let $(A_n) \subset \mathcal{F}$ and $A \in \mathcal{F}$ be arbitrary. In order that (A_n) converge to A, it is necessary and sufficient that $(\Delta(A_n))$ converge to $\Delta(A)$.*

Proof. For every counting number $n \in \mathbb{N}$ write $E_n = \bigcap\limits_{k=n}^{\infty} A_k$ and $B_n = \bigcup\limits_{k=n}^{\infty} A_k$. It is clear that sequence (B_n) converges decreasingly to $\limsup\limits_{n\to\infty} A_n$ and sequence (E_n) converges increasingly to $\liminf\limits_{n\to\infty} A_n$. Consequently, by applying *Theorems 2.8* and *2.9* to these sequences, we can conclude on the validity of the theorem. **q.e.d.**

Definition 2.3. A mapping $\Delta : \mathcal{F} \to \mathbb{P}\left(\mathcal{P}^1_\infty\right)$ is said to be *powering* if it is defined by:

$$\Delta(A) = \begin{cases} \emptyset & \text{if } A = \emptyset \\ \left\{p \in \mathcal{P}^1_\infty : p(A) = 1\right\} & \text{if } A \neq \emptyset \end{cases}$$

Remark 2.3. If H is the unitary atom of a class $p \in \mathcal{P}^1_\infty$, then $\Delta(H) = \{p\}$.

The following result can easily be derived from *Lemma 2.6* and *Remark 2.2*.

Proposition 2.11. *If $\Delta : \mathcal{F} \to \mathbb{P}\left(\mathcal{P}^1_\infty\right)$ is a powering mapping, then it is an injection.*

Definition 2.4. If $\Gamma \subseteq \mathcal{P}^1_\infty$ is a nonempty set, then the collection $\mathcal{C}$ of all the unitary atoms of the classes $p \in \Gamma$ will be called unitary-atomic collection of Γ.

Postulate of powering. *If $\Gamma \in \mathbb{P}\left(\mathcal{P}^1_\infty\right) \setminus \{\emptyset\}$ and $\mathcal{C}$ denotes the governing-atomic collection of Γ, then $\bigcup \mathcal{C}$ is measurable and $\Delta\left(\bigcup \mathcal{C}\right) \subseteq \Gamma$.*

Theorem 2.12. *The powering mapping $\Delta : \mathcal{F} \to \mathbb{P}\left(\mathcal{P}^1_\infty\right)$ is surjective if and only if the postulate of powering is valid.*

Proof. Assume that *Postulate of powering* is valid. Let $\Gamma \in \mathbb{P}\left(\mathcal{P}^1_\infty\right)$ be arbitrarily fixed. We note that if $\Gamma = \emptyset$, then there is nothing to be proven. Suppose that Γ is a nonempty subset of $\mathcal{P}^1_\infty$, and denote by $\mathcal{C}$ its corresponding governing-atomic collection. Then $\bigcup \mathcal{C}$ is measurable and $\Delta\left(\bigcup \mathcal{C}\right) \subseteq \Gamma$ (by the postulate). Let us show that $\Gamma \subseteq \Delta\left(\bigcup \mathcal{C}\right)$. In fact, pick any class $p \in \Gamma$ and p^* any representing member of p, with H the unitary atom of p. Since $H \subseteq \bigcup \mathcal{C}$, it ensues from *Lemma 2.2* that $\Delta(H) \subseteq \Delta\left(\bigcup \mathcal{C}\right)$. But, via *Remark 2.3* we have that $\{p\} = \Delta(H)$ and $p \in \Delta\left(\bigcup \mathcal{C}\right)$, i.e. $\Gamma \subseteq \Delta\left(\bigcup \mathcal{C}\right)$. Therefore $\Gamma = \Delta\left(\bigcup \mathcal{C}\right)$.

To prove the converse biconditional, let us assume that the powering mapping Δ is a surjection. We note that Δ is a bijection, since it is also an injection (by *Proposition 2.11*). Let $\Gamma \in \mathbb{P}\left(\mathcal{P}^1_\infty\right) \setminus \{\emptyset\}$ be arbitrary and write $\mathcal{C}$ for the corresponding unitary-atomic collection. Obviously we have that $\Gamma = \bigcup \{\Delta(H) : H \in \mathcal{C}\}$ is a subset of $\mathcal{P}^1_\infty$.

Then via the bijective property it ensues that $\Delta^{-1}(\Gamma) \in \mathcal{F}$. Clearly $\Delta(H) \subset \Gamma$ for every $H \in \mathcal{C}$. By *Lemma 2.2* together with the bijective property, we obtain that

$$H = \Delta^{-1}(\Delta(H)) \subset \Delta^{-1}(\Gamma)$$

whenever $H \in \mathcal{C}$. Consequently the inclusion $\bigcup \mathcal{C} \subseteq \Delta^{-1}(\Gamma)$ follows. Now let us show that if $\omega \in \Delta^{-1}(\Gamma)$, then there is some $H \in \mathcal{C}$ such that $\omega \in H$. Assume on the contrary that there can be found some $\omega_1 \in \Delta^{-1}(\Gamma)$ such that $\omega_1 \notin H$ for all $H \in \mathcal{C}$. We can thus define an optimal measure $q^* : \mathcal{F} \to [0,\, 1]$ so that

$$q^*(B) \begin{cases} = & 1 \quad \text{if } \omega_1 \in B \\ < & 1 \quad \text{if } \omega_1 \notin B. \end{cases}$$

(See *Example 1*) Then there is a unique indecomposable q^*-atom (to be denoted by $\widetilde{H}$) such that $q^*\left(\widetilde{H}\right) = 1$. It is clear that $\omega_1 \in \widetilde{H}$ and $q^*\left(\Delta^{-1}(\Gamma)\right) = 1$. We further note that

$$\bigcup \{\Delta(H) : H \in \mathcal{C}\} = \Gamma = \Delta\left(\Delta^{-1}(\Gamma)\right) = \left\{p \in \mathcal{P}^1_\infty : p\left(\Delta^{-1}(\Gamma)\right) = 1\right\}.$$

From this fact and the identity $q^*\left(\Delta^{-1}(\Gamma)\right) = 1$, there must exist some class $p_0 \in \mathcal{P}^1_\infty$ with $p_0\left(\Delta^{-1}(\Gamma)\right) = 1$, such that $q^*\left(\widetilde{H} \cap H \cap \Delta^{-1}(\Gamma)\right) = 1$, where H is the unitary atom of class p_0. Nevertheless, this is possible only if $\omega_1 \in H$, which is absurd, since we have supposed that $\omega_1 \notin H$ for all $H \in \mathcal{C}$. Therefore, if $\omega \in \Delta^{-1}(\Gamma)$, then there is some $H \in \mathcal{C}$ such that $\omega \in H$. It ensues that $\omega \in \bigcup \mathcal{C}$ for all $\omega \in \Delta^{-1}(\Gamma)$, as $H \subset \bigcup \mathcal{C}$ whenever $H \in \mathcal{C}$. Thus $\Delta^{-1}(\Gamma) \subseteq \bigcup \mathcal{C}$. Therefore, $\bigcup \mathcal{C} = \Delta^{-1}(\Gamma)$, which leads to the postulate. **q.e.d.**

Theorem 2.12 entails that an infinite σ-algebra is equinumerous with a power set if and only if *Postulate 1* is valid. This suggests that there are infinite σ-algebras that are not equinumerous with infinite power sets.

REFERENCES

[1] Agbeko, N. K.: *On optimal averages*, Acta Math. Hung. **63** (1994) 1–15.

[2] Agbeko, N. K.: *On the structure of optimal measures and some of its applications*, Publ. Math. Debrecen **46** (1995) 79–87.

[3] Agbeko, N. K.: *How to characterize some properties of measurable functions*, Mathematical Notes, Miskolc **2** (2000) 87–98.

[4] Fazekas, I.: *A note on 'optimal measures'*, Publ. Math. Debrecen **51/3-4** (1997) 273–277.

Doi: 10.18514/MMN.2001.40

Mathematical Notes, Miskolc, Vol. 2., No. 2., (2001), pp. 93–101

ON THREE AND FOUR POINT BOUNDARY VALUE PROBLEMS FOR SECOND ORDER DIFFERENTIAL INCLUSIONS

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[Received March 6, 2001]

Abstract. In this paper we investigate the existence of solutions on a compact interval to a three and four-point boundary value problem for a class of second order differential inclusions. We shall rely on a fixed point theorem for contraction multivalued maps due to Covitz and Nadler.

Mathematical Subject Classification: 34A60, 34B10, 34B15
Keywords: Three and four point boundary value problems, differential inclusion, contraction, existence, fixed point

1. Introduction

The existence of solutions on compact intervals for three-point and multi-point boundary value problems for second order differential equations has received much attention in the last decade, we refer for instance to the papers of Boucherif and Bouguima [1], Gupta [2-5], Gupta et al [6-8], Gupta and Trofimchuk [9-10] and Marano [11]. The study of multi-point boundary value problems for second order ordinary differential equations was initiated by Il'In and Moiseev in [12-13], motivated by the work of Bitsadze and Samarskii on nonlocal elliptic boundary value problems [14-16]. The methods used are usually the topological transversality of Granas or the degree theory methods combined with Wirtinger type inequalities.

Very recently, by means of a fixed point theorem for condensing multivalued maps due to Martelli, an extension of three and four-point boundary value problems for second order differential equations to the multivalued case has been done by the authors in [17-18]. However, in these problems the right-hand side was assumed to be convex valued. Here we drop this restriction and consider problems with a nonconvex valued right-hand side.

In Section 3 of this paper we shall prove a theorem which ensures the existence of solutions defined on a compact real interval for the three-point boundary value problem (BVP for short) of the second order differential inclusion

$$y'' \in F(t, y), \quad t \in J = [0, 1], \tag{1.1}$$

$$y(0) = 0, \ y(\eta) = y(1), \tag{1.2}$$

where $F : J \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is a multivalued map, $\eta \in (0, 1)$, and $\mathcal{P}(E)$ is the family of all subsets of $\mathbb{R}$.

Section 4 is devoted to the study of the following four-point boundary value problem

$$y'' \in F(t, y), \quad t \in J = [0, 1], \tag{1.3}$$

$$y(0) = y'(\eta), \ y(1) = y(\tau), \tag{1.4}$$

where F, η are as in the problem (1.1)-(1.2) and $\tau \in (0, 1)$. The method we are going to use is to reduce the existence of solutions to problems (1.1)-(1.2) and (1.3)-(1.4) to the search for fixed points of a suitable multivalued map on the Banach space $C(J, \mathbb{R})$. In order to prove the existence of fixed points, we shall rely on a fixed point theorem for contraction multivalued maps, due to Covitz and Nadler [19].

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

Let (X, d) be a metric space. We use the notations:

$P(X) = \{Y \in \mathcal{P}(X) : Y \neq \emptyset\}$, $P_{cl}(X) = \{Y \in P(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in P(X) : Y \text{ bounded}\}$.

Consider $H_d : P(X) \times P(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\right\},$$

where $d(A, b) = \inf_{a \in A} d(a, b)$, $d(a, B) = \inf_{b \in B} d(a, b)$.

Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space.

A multivalued map $N : J \longrightarrow P_{cl}(X)$ is said to be measurable if, for each $x \in X$, the function $Y : J \longrightarrow \mathbb{R}$, defined by

$$Y(t) = d(x, N(t)) = \inf\{|x - z| : z \in N(t)\},$$

is measurable.

Definition 1 *A multivalued operator $N : X \to P_{cl}(X)$ is called* *a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that*

$$H_d(N(x), N(y)) \leq \gamma d(x,y), \quad \textit{for each } x,\ y \in X,$$

b) contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

N has *a fixed point* if there is $x \in X$ such that $x \in N(x)$. The fixed point set of the multivalued operator N will be denoted by $FixN$.

For more details on multivalued maps we refer to the books by Deimling [20], Gorniewicz [21] and Hu and Papageorgiou [22].

Our considerations are based on the following fixed point theorem for contraction multivalued operators given by Covitz and Nadler in 1970 [19] (see also Deimling, [20] Theorem 11.1).

Lemma 2 *Let (X,d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

3. Three-Point BVPs

The main result of this section concerns the three-point BVP (1.1)–(1.2). Before we state and prove this result, we give the definition of a solution of the three-point BVP (1.1)–(1.2).

Definition 3 *A function $y : J \longrightarrow \mathbb{R}$ is called a solution for the BVP (1.1)-(1.2) if y and its first derivative are absolutely continuous and y'' (which exists almost everywhere) satisfies the differential inclusion (1.1) a.e. on J and the condition (1.2).*

Theorem 4 *Assume that:*

(H1) $F : J \times \mathbb{R} \to P_{cl}(\mathbb{R})$ has the property that $F(\cdot, y) : J \to P_{cl}(\mathbb{R})$ is measurable for each $y \in \mathbb{R}$.

(H2) $H_d(F(t,y), F(t,\overline{y})) \leq l(t)|y - \overline{y}|$, for a.e. $t \in J$ and $y, \overline{y} \in \mathbb{R}$, where $l \in L^1(J, \mathbb{R})$.

Let $L(t) = \int_0^t l(s)ds$, and $\mu > 1$. If

$$\frac{1}{\mu} + \frac{L(\eta) + L(1)}{1 - \eta} < 1,$$

then the BVP (1.1)-(1.2) has at least one solution on J.

Proof. Let the Bielecki-type norm $\|\cdot\|_B$ on $C(J,\mathbb{R})$ be defined by

$$\|y\|_B = \max_{t\in J}\{|y(t)|e^{-\mu L(t)}\}.$$

Transform the problem into a fixed point problem. Consider the multivalued map, $N : C(J,\mathbb{R}) \longrightarrow \mathcal{P}(C(J,\mathbb{R}))$ defined by:

$$N(y) = \Big\{h \in C(J,\mathbb{R}) : h(t) = \int_0^t (t-s)g(s)ds + \frac{t}{1-\eta}\int_0^\eta (\eta-s)g(s)ds \\ -\frac{t}{1-\eta}\int_0^1 (1-s)g(s)ds\Big\}$$

where

$$g \in S_{F,y} = \left\{g \in L^1(J,\mathbb{R}) : g(t) \in F(t,y(t)) \text{ for a.e. } t \in J\right\}.$$

Remark 5 *(i) It is clear that the fixed points of N are solutions to (1.1)-(1.2).*

(ii) For each $y \in C(J,\mathbb{R})$ the set $S_{F,y}$ is nonempty, since by (H1), F has a measurable selection (see [23], Theorem III.6).

We shall show that N satisfies the assumptions of Lemma 2. The proof will be given in two steps.

Step 1: *$N(y) \in P_{cl}(CJ,\mathbb{R})$ for each $y \in C(J,\mathbb{R})$.*

Indeed, let $(y_n)_{n\geq 0} \in N(y)$ such that $y_n \to \tilde{y}$ in $C(J,\mathbb{R})$. Then $\tilde{y} \in C(J,\mathbb{R})$ and for each $t \in J$

$$y_n(t) \in \int_0^t (t-s)F(s,y(s))ds + \frac{t}{1-\eta}\int_0^\eta (\eta-s)F(s,y(s))ds \\ -\frac{t}{1-\eta}\int_0^1 (1-s)F(s,y(s))ds.$$

Because the sets

$$\int_0^t (t-s)F(s,y(s))ds,\ \frac{t}{1-\eta}\int_0^\eta (\eta-s)F(s,y(s))ds \quad \text{and} \quad \frac{t}{1-\eta}\int_0^1 (1-s)F(s,y(s))ds$$

are closed for each $t \in J$, then for each $t \in J$

$$y_n(t) \to \tilde{y}(t) \in \int_0^t (t-s)F(s,y(s))ds + \frac{t}{1-\eta}\int_0^\eta (\eta-s)F(s,y(s))ds \\ -\frac{t}{1-\eta}\int_0^1 (1-s)F(s,y(s))ds.$$

Therefore $\tilde{y} \in N(y)$.

Step 2: $H_d(N(y_1), N(y_2)) \leq \gamma|y_1 - y_2|$ *for each* $y_1, y_2 \in C(J, \mathbb{R})$ *(where* $\gamma < 1$*).*

Let $y_1, y_2 \in C(J, \mathbb{R})$ and $h_1 \in N(y_1)$. Then there exists $g_1(t) \in F(t, y_1(t))$ such that, for each $t \in J$,

$$h_1(t) = \int_0^t (t-s)g_1(s)ds + \frac{t}{1-\eta}\int_0^\eta (\eta - s)g_1(s)ds - \frac{t}{1-\eta}\int_0^1 (1-s)g_1(s)ds.$$

From (H2) it follows that

$$H_d(F(t, y_1(t)), F(t, y_2(t))) \leq l(t)|y_1(t) - y_2(t)|.$$

Hence, there is $w \in F(t, y_2(t))$, such that

$$|g_1(t) - w| \leq l(t)|y_1(t) - y_2(t)|, \quad t \in J.$$

Consider $U : J \to \mathcal{P}(\mathbb{R})$, given by

$$U(t) = \{w \in \mathbb{R} : |g_1(t) - w| \leq l(t)|y_1(t) - y_2(t)|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, y_2(t))$ is measurable (see Proposition III.4 in [23] there exists $g_2(t)$ a measurable selection for V. So, $g_2(t) \in F(t, y_2(t))$ and

$$|g_1(t) - g_2(t)| \leq l(t)|y_1(t) - y_2(t)|, \quad \text{for each} \quad t \in J.$$

Let us define for each $t \in J$

$$h_2(t) = \int_0^t (t-s)g_2(s)ds + \frac{t}{1-\eta}\int_0^\eta (\eta - s)g_2(s)ds - \frac{t}{1-\eta}\int_0^1 (1-s)g_2(s)ds.$$

Then we have

$$\begin{aligned}
|h_1(t) - h_2(t)| \quad &\leq \quad \int_0^t (t-s)|g_1(s) - g_2()|\, ds + \frac{t}{1-\eta}\int_0^\eta (\eta - s)|g_1(s) - g_2(s)|\, ds \\
&\quad + \frac{t}{1-\eta}\int_0^1 (1-s)|g_1(s) - g_2(s)|\, ds \\
&\leq \quad \int_0^t l(s)|y_1(s) - y_2(s)|ds + \frac{1}{1-\eta}\int_0^\eta (\eta - s)l(s)|y_1(s) - y_2(s)|ds \\
&\quad + \frac{1}{1-\eta}\int_0^1 (1-s)l(s)|y_1(s) - y_2(s)|ds \leq
\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^t l(s)e^{-\mu L(s)}e^{\mu L(s)}|y_1(s)-y_2(s)|\,ds \\
&\quad +\frac{1}{1-\eta}L(\eta)e^{\mu L(t)}\|y_1-y_2\|_B \\
&\quad +\frac{1}{1-\eta}L(1)e^{\mu L(t)}\|y_1-y_2\|_B \\
&\leq \|y_1-y_2\|_B\int_0^t l(s)e^{\mu L(s)}ds+\frac{1}{1-\eta}L(\eta)e^{\mu L(t)}\|y_1-y_2\|_B \\
&\quad +\frac{1}{1-\eta}L(1)e^{\mu L(t)}\|y_1-y_2\|_B \\
&= \|y_1-y_2\|_B\frac{1}{\mu}\int_0^t (e^{\mu L(s)})'ds+\frac{1}{1-\eta}L(\eta)e^{\mu L(t)}\|y_1-y_2\|_B \\
&\quad +\frac{1}{1-\eta}L(1)e^{\mu L(t)}\|y_1-y_2\|_B \\
&\leq \frac{\|y_1-y_2\|_B}{\mu}e^{\mu L(t)}+\frac{1}{1-\eta}L(\eta)e^{\mu L(t)}\|y_1-y_2\|_B \\
&\quad +\frac{1}{1-\eta}L(1)e^{\mu L(t)}\|y_1-y_2\|_B.
\end{aligned}$$

Then

$$\|h_1-h_2\|_B \leq \left[\frac{1}{\mu}+\frac{L(\eta)+L(1)}{1-\eta}\right]\|y_1-y_2\|_B.$$

By the analogous relation, obtained by interchanging the roles of y_1 and y_2, it follows that

$$H_d(N(y_1),N(y_2)) \leq \left[\frac{1}{\mu}+\frac{L(\eta)+L(1)}{1-\eta}\right]\|y_1-y_2\|_B.$$

So, N is a contraction and thus, by Lemma 2, it has a fixed point y, which is solution to (1.1)-(1.2).

4. Four-Point BVPs

The main result of this section concerns the four-point BVP (1.3)–(1.4). Before we state and prove this result, we give the definition of a solution of the four-point BVP (1.3)–(1.4).

Definition 6 *A function $y : J \longrightarrow \mathbb{R}$ is called a solution for the BVP (1.3)-(1.4) if y and its first derivative are absolutely continuous and y'' (which exists almost everywhere) satisfies the differential inclusion (1.3) a.e. on J and the conditions (1.4).*

Theorem 7 *Assume that (H1) and (H2) are satisfied. If*

$$\frac{1}{\mu}+L(\eta)+2\frac{L(\tau)+L(1)}{1-\eta}<1,$$

then the BVP (1.3)-(1.4) has at least one solution on J.

Proof. Let $\|\cdot\|_B$ the Bielecki-type norm on $C(J,\mathbb{R})$ defined by

$$\|y\|_B = \max_{t\in J}\{|y(t)|e^{-\mu L(t)}\}.$$

Transform the problem into a fixed point problem. Consider the multivalued map, $N_1 : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ defined by:

$$\begin{aligned} N_1(y) \quad &= \quad \Big\{h \in C(J,\mathbb{R}) : h(t) = \int_0^t (t-s)g(s)ds + \int_0^\eta g(s)ds \\ &\quad + \frac{1+t}{1-\tau}\left[\int_0^\tau (\tau-s)g(s)ds - \int_0^1 (1-s)g(s)ds\right]\Big\} \end{aligned}$$

where

$$g \in S_{F,y} = \left\{g \in L^1(J,\mathbb{R}) : g(t) \in F(t,y(t)) \;\text{ for a.e. in } J\right\}.$$

We can easily show that N_1 has closed values and it is a contraction multivalued map. We omit the details.

5. Concluding Remarks

Let $a_i \in \mathbb{R}$, with all of the a_i's having the same sign, $\xi_i \in (0,1), i = 1,2,\ldots,m-2$, $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$. Consider the following m-point boundary value problem for second order differential inclusions

$$y''(t) \in F(t,y(t)), \quad t \in J, \tag{5.1}$$

$$y(0) = 0, \;\; y(1) = \sum_{i=1}^{m-2} a_i y(\xi_i). \tag{5.2}$$

It is well known (see [12] for example) that if a function $y \in C^1$ satisfies the boundary condition (5.2) and all of the $a_i, i = 1,2,\ldots,m-2$ have the same sign, then there exists $\eta \in [\xi_1, \xi_{m-2}]$, depending on $y \in C^1(J,\mathbb{R})$ such that

$$y(1) = \alpha y(\eta)$$

with $\alpha = \sum\limits_{i=1}^{m-2} a_i$. Accordingly, the problem of the existence of a solution for the BVP (5.1)–(5.2) can be studied via the three-point BVP

$$\begin{gathered} y''(t) \in F(t,y(t)), \quad t \in J, \\ y(0) = 0, \;\; y(1) = \alpha y(\eta), \end{gathered}$$

where $\eta \in (0,1)$ is given. We omit the details, since the proof follows the steps of the proof of Theorem 4, with obvious modifications.

It is obvious that the above method can also be applied to other types of m-point BVPs. For example for the BVPs

$$y''(t) \in F(t, y(t)), \quad t \in J,$$
$$y'(0) = 0, \;\; y(1) = \sum_{i=1}^{m-2} a_i y(\xi_i),$$

or

$$y''(t) \in F(t, y(t)), \quad t \in J,$$
$$y(0) = 0, \;\; y'(1) = \sum_{i=1}^{m-2} a_i y'(\xi_i)$$

which can be reduced to the following three point BVPs:

$$y''(t) \in F(t, y(t)), \quad t \in J,$$
$$y'(0) = 0, \;\; y(1) = \alpha y(\eta)$$

and

$$y''(t) \in F(t, y(t)), \quad t \in J,$$
$$y(0) = 0, \;\; y'(1) = \alpha y'(\eta)$$

respectively.

REFERENCES

[1] Boucherif, A. and Bouguima, S. M.: *Nonlinear second order ordinary differential equations with nonlocal boundary conditions*, Commu. Appl. Nonl. Anal., **5**(2), (1998), 73-85.

[2] Gupta, C. P.: *Solvability of a three-point boundary value problem for a second order ordinary differential equation*, J. Math. Anal. Appl., **168**, (1992), 540-551.

[3] Gupta, C. P.: *A note on a second order three-point boundary value problem*, J. Math. Anal. Appl., **186**, (1994), 277-281.

[4] Gupta, C. P.: *A second order m-point boundary value problem at resonance*, Nonlinear Anal., **24**, (1994), 1483-1489.

[5] Gupta, C. P.: *Solvability of a multi-point boundary value problem at resonance*, Results Math., **28**, (1995), 270-276.

[6] Gupta, C. P., Ntouyas, S. K. and Tsamatos, P. Ch.: On an m-point boundary value problem for second order ordinary differential equations, *Nonlinear Anal.* **23** (1994), 1427-1436.

[7] Gupta, C. P., Ntouyas, S. K. and Tsamatos, P. Ch.: *Existence results for m-point boundary value problems*, Differential Equations Dynam. Systems, **2**, (1994), 289-298.

[8] Gupta, C. P., Ntouyas, S. K. and Tsamatos, P. Ch.: *Solvability of an m-point boundary value problem for second order ordinary differential equations*, J. Math. Anal. Appl., **189**, (1995), 575-584.

[9] Gupta, C. P. and Trofimchuk, S. I.: *A sharper condition for the solvability of a three-point second order boundary value problem*, J. Math. Anal. Appl., **285**, (1997), 586-597.

[10] Gupta, C. P. and Trofimchuk, S. I.: *A Wirtinger type inequality and a three-point boundary value problem*, Dynam. Systems Appl., **8**, (1999), 127-132.

[11] Marano, S. A.: *A remark on a second order three-point boundary value problem*, J. Math. Anal. Appl., **183**, (1994), 518-522.

[12] Il'in, V. A. and Moiseev, E. I.: *Nonlocal boundary value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects*, Differential Equations, **23**(7), (1987), 803-810.

[13] Il'in, V. A. and Moiseev, E. I.: *Nonlocal boundary value problem of the second kind for a Sturm-Liouville operator*, Differential Equations, **23**(8), (1987), 979-987.

[14] Bitsadze, A. V.: *On the theory of nonlocal boundary value problem*, Soviet. Math. Dokl., **30**(1), (1984), 8-10.

[15] Bitsadze, A. V.: *On a class of conditionally solvable nonlocal boundary value problems for harmonic functions*, Soviet. Math. Dokl., **31**(1), (1985), 91-94.

[16] Bitsadze, A. V. and Samarskii, A. A.: *On some simple generalizations of linear elliptic boundary value problem*, Soviet. Math. Dokl., **10**(2), (1969), 398-400.

[17] Benchohra, M. and Ntouyas, S. K.: *A note on a three-point boundary value problem for second order differential inclusions*, Mathematical Notes, Miskolc, **2**, (2001), 39-47.

[18] Benchohra, M. and Ntouyas, S. K.: *Multi-point boundary value problems for second order differential inclusions*, Math. Vesnik, to appear.

[19] Covitz, H. and Nadler S. B. Jr.: *Multivalued contraction mappings in generalized metric spaces*, Israel J. Math., **8**, (1970), 5-11.

[20] Deimling, K.: *Multivalued Differential Equations*, Walter de Gruyter, Berlin-New York, 1992.

[21] Gorniewicz, L.: *Topological Fixed Point Theory of Multivalued Mappings, Mathematics and its Applications*, 495, Kluwer Academic Publishers, Dordrecht, 1999.

[22] Hu Sh. and Papageorgiou, N.: *Handbook of Multivalued Analysis, Volume I: Theory*, Kluwer Academic Publishers, Dordrecht, 1997.

[23] Castaing, C. and Valadier, M.: *Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics*, Vol. 580, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

Doi: 10.18514/MMN.2001.41

Mathematical Notes, Miskolc, Vol. 2., No. 2., (2001), pp. 103–116

ON A CLASS OF DISCONTINUOUS DYNAMICAL SYSTEMS

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[Received April 18, 2001]

Abstract. The aim of this paper is to find the conditions under which the discontinuous initial value problem $\dot{x}\ (t) = g(x(t)) + \sum_{i=1}^{n} \alpha_i\ sgn\, x_i(t)\, e^i$, $x(0) = x_0$, $t \in I = [0, \infty)$, where $x_0 \in \mathbb{R}^n$, $\alpha_i \in \mathbb{R}$ and e^i denotes the i-th canonical unit vector in $\mathbb{R}^n$, defines a dynamical system. A definition of dynamical systems which involves the existence and the uniqueness of solutions to initial value problems is presented. Because the problem may not have classical solutions, the Filippov regularization is used in order to restart the problem as a differential inclusion which may enjoy the existence and even uniqueness of generalized solutions. An illustrative example of this class of dynamical systems, a generalization of the Chua's circuit, is presented.

Mathematical Subject Classification: 49K24, 37M99
Keywords: Discontinuous initial value problems, dynamical systems, differential inclusions, set-valued maps.

1. Introduction

The paper is devoted to differential equations with discontinuous right-hand sides which model a whole variety of applications: dry friction, electrical circuits, oscillations in visco-elasticity, brake processes with locking phase, oscillating systems with viscous damping, elasto-plasticity, electrical circuits, forced vibrations, convex optimization, control synthesis of uncertain systems, etc. (see e.g. [1]-[5] and the references in [6], [7]). A large number of problems which belong to these applications can be described by the following autonomous initial value problem (i.v.p.)

$$\dot{x}(t) = f\,(x(t)), \quad x(0) = x_0, \quad x_0 \in \mathbb{R}^n\ , \quad t \in I = [0, \infty), \tag{1.1}$$

with the right-hand side $f : \mathbb{R}^n \to \mathbb{R}^n$ a vector single-valued map, discontinuous with respect to the state variable x and locally bounded in the metric space $\mathbb{R}^n$ (i.e., bounded on each bounded subset of $\mathbb{R}^n$). Our goal is to emphasize a class of i.v.p. (1.1) which define dynamical systems (d.s.). Throughout this paper the discontinuity is considered only with respect to the state variable x. Many authors consider the concept of continuity of a d.s. as being with respect to time variable, while others consider the continuity (or Lipschitz continuity) with respect to initial data in order to define a continuous d.s. (see e.g. [8], [9] and the references therein). However, in the case of continuity with respect to the state variable, for most of the standard assumptions leading to the existence and uniqueness of the solutions, the continuous dependence on initial data follows (see [9]).

There are many mathematical definitions for a d.s. (see e.g. [8], [9] and the references therein). Our definition uses the existence and, optionally, the uniqueness of the solutions to the i.v.p. (1.1). If the right-hand side of (1.1) is discontinuous with respect to x, the i.v.p. need not have classical solutions and another concept of solution must be used. One of the typical cases for discontinuous i.v.p. is modeled by the so-called *switch* problems.

Example 1.1. [10] Consider the discontinuous right-hand side equation $\dot{x} = 1 - 2\,sgn\,x$, with the classical solutions

$$x(t) = \begin{cases} 3t + C_1, & x < 0 \\ -t + C_2, & x > 0 \end{cases}, \quad C_1, C_2 \in \mathbb{R}.$$

As t increases, the classical solutions tend to the line $x = 0$, but it cannot be continued along this line, since the map $x(t) = 0$ so obtained does not satisfy the equation in the usual sense (for it, $\dot{x}(t) = 0$ and the right-hand side has the value $1 - 2\,sgn\,0 = 1$). Hence there are no classical solutions of i.v.p. starting with $x(0) = 0$. Therefore a generalization of the concept of solution is required.

To get around to a solution of the i.v.p. (1.1) with a discontinuous right-hand side, the problem may be restarted as a differential inclusion (d.i.)

$$\dot{x}(t) \in F(x(t)), \quad x(0) = x_0, \quad \text{for a.a.} \quad t \in I, \tag{1.2}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a vector set-valued map into the set of all subsets of $\mathbb{R}^n$ which can be defined in several ways. For the background and a comprehensive theory of d.i. and set-valued maps we refer the reader to [10]-[12].

The simplest convex definition of F is obtained by the so-called *Filippov regularization* [10]

$$F(x) = \bigcap_{\varepsilon > 0} \overline{conv\,(\,f\;(\{z \in \mathbb{R}^n : \|\,z - x\| \le \varepsilon\} \setminus M\;))}\;, \tag{1.3}$$

where $F(x)$ is the convex hull of f, μ denotes the Lebesgue measure in $\mathbf{R}^n$, *conv* is the convex hull, M is a null set and ε the radius of the ball centered in x. In the points where the map f is continuous, $F(x)$ consists of one point which coincides with the value of f at this point (i.e., we get back $f(x)$ as the right-hand side). In the points belonging to M, the set is given by (1.3). Details and other regularization procedures can be found in [10].

In order to justify the use of d.i. (1.2-1.3) to a physical system, we must use a small ε so that the motion of the physical system is arbitrarily close to a certain solution of d.i. (for instance, it tends to the solution as $\varepsilon \to 0$).

As an example, the Filippov regularization of the usual *sign* map is the *signum set-valued map*

$$Sgn\,x = \begin{cases} \{-1\}, & x < 0 \\ [-1,\,1], & x = 0 \\ \{+1\}, & x > 0 \end{cases} .$$

Remark 1.2. Embedding f into a set-valued map F, which has enough regularity, closely related to the trajectories of the original differential equation, we can stress the point that whenever f is continuous at x, then a solution to d.i. (1.2) satisfies the i.v.p. (1.1). Certainly, any classical solution to the i.v.p. (1.1) is a solution to the i.v.p. (1.2). Hence, as stipulated in [10], we are justified to call a solution of i.v.p. (1.1) a solution of the i.v.p. (1.2).
On mild assumptions, d.i. (1.2) has generalized (Filippov) solutions that happen to be even a.e. unique, but it could have multiple solutions as well.

Example 1.3. Let the discontinuous i.v.p. $\dot{x} = sgn\,x$, $x(0) = 0$. There is no classical solution starting from 0. However, considering the corresponding d.i. $\dot{x} \in F(x) = Sgn\,x$, there are multiple Filippov solutions: $x(t) = 0$ for $t \le t_*$ and $x(t) = \pm(t - t_*)$ for $t > t_*$, where $t_* \ge 0$ could be ∞.

Example 1.4. If we consider the i.v.p. $\dot{x} \in -Sgn\,x$, $x(0) = 0$, then there is a unique Filippov solution

$$x(t) = \begin{cases} t_* - t, & t \le t_* \\ 0, & t > t_* \end{cases} ,$$

and the trajectory can be continuously extended from $x = 0$ for $t > t_*$.

The main objective is the study of conditions under which the discontinuous i.v.p. modeling the *switch systems*

$$\dot{x}(t) = g(x(t)) + \sum_{i=1}^{n} \alpha_i \;\, sgn\,x_i(t)\; e^i, \quad x(0) = x_0, \quad t \in I = [0,\,\infty)\,, \tag{1.4}$$

with $g : \mathbb{R}^n \to \mathbb{R}^n$ a vector single-valued map, $\alpha_i \in \mathbb{R}$ and e^i the i-th canonical unit vector in $\mathbb{R}^n$ could be viewed as a switch d.s. Consequently, we introduce in Section 2 a definition for switch d.s. considering the existence and optionally the uniqueness of solution to i.v.p. (1.4). The existence (Péano's theorem for d.i.) and uniqueness (special Lipschitz conditions) of solutions of i.v.p. (1.4) will be analyzed through the corresponding d.i. obtained by Filippov regularization. The explicit Euler method for d.i. necessary to numerical dynamics is also presented. The main result of this paper is introduced in Section 3 and states the assumptions in which the i.v.p. (1.4) defines a switch d.s.

2. Preliminaries

Definition 2.1. A (*Filippov*) *solution* of i.v.p. (1.2) is an absolutely continuous vector-valued map $x : I \to \mathbb{R}^n$ satisfying (1.2), for a.a. t on I .

See [10] and [13] for properties of Filippov solutions.

Definition 2.2. The i.v.p. (1.1) is said to define a *generalized switch d.s.* on $\mathbb{R}^n$ if for every x_0 there exists a solution of i.v.p. (1.1) defined for a.a. $t \in I$. If the solution is a.e. unique, then the i.v.p. is said to define a *switch d.s.*

To establish conditions in which a d.i. admits solutions, few basic properties of the set-valued maps are presented next.

Let X and Y be two non-empty sets. For practical reasons it is convenient to characterize a vector set-valued map $F : X \rightrightarrows Y$ by its graph

$$Graph(F) := \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

Remark 2.3. Due to the symmetric interpretation of a set-valued map as a graph (see e.g. [12]), we shall say that a set-valued map satisfies a property if and only if its graph satisfies it. For instance a set-valued map is said to be closed if and only if its graph is closed.

Notation 2.4. Let denote with $\mathcal{P}$ the class of the so-called *Péano maps* i.e. upper semicontinuous (u.s.c.) maps with non-empty closed and convex values, called *basic* or *Péano conditions*.

Definition 2.5. [11], [12] A single-valued map $f : \mathbb{R}^n \to \mathbb{R}^n$ is called a *selection* of the set-valued F map if

$$\forall x \in \mathbb{R}^n, \quad f(x) \in F(x).$$

Generally, a set-valued map F admits (infinitely) many selections, each of them giving rise to a differential equation $\dot{x}(t) = f(x(t))$ and whose classical solutions (if any) are Filippov solutions of (1.2). There are many ways to choose a selection (see [6]). The selections could be discontinuous (both with respect t and x) . Let us consider as an example the set-valued map $F : X \rightrightarrows Y$ with closed convex images, X being a metric space and Y a Hilbert space. The single-valued map

$$m(F(x)) := \left\{ u \in F(x) \mid \| u\| = \min_{y \in F(x)} \| y\| \right\},$$

is called the *minimal norm selection.* For the sign set-valued map, the minimal selection is the discontinuous single-valued map

$$m(Sgn\, x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ +1, & x > 0 \end{cases}.$$

An important condition to extend the existence interval for a solution of the i.v.p. (1.2) to $I = [0, \infty)$ is the compactness of the minimal selection.

Notation 2.6. We say that condition $\mathcal{C}$ is satisfied if there exists a compact subset $C \subset \mathbb{R}^n$ such that $m(F(x)) \subseteq \mathcal{C}$ for all $x \in \mathbb{R}^n$.

Definition 2.7. F satisfies a *growth condition* (g.c.) on $\mathbb{R}^n$ if there exist constants $K_1,\, K_2 \geq 0$ with

$$\|\xi\| \leq K_1 \, \|x\| + K_2 \,,$$

for all $\xi \in F(x),\ x \in \mathbb{R}^n$.

The g.c. implies that all solutions and selections remain in some bounded subset and it is used instead of global boundedness of the right-hand side (compare [2], [7] and [14]).

Remark 2.8. *i)* For f locally bounded, the set-valued map F defined by (1.3) has the following properties: is u.s.c. with non-empty closed and convex values , i.e., a Péano map (compare [11, Proposition 1 p.102] and [10, Lemma 3, p.67]).
ii) Moreover, because the convex hull of a finite set is compact, $F(x)$ given by (1.3) is a compact set (see [11, Corollary 1, p.20]). Hence, it is easy to verify that F verifies $\mathcal{C}$ condition.
iii) The Sgn set-valued maps satisfies the g.c.

In order to prove the uniqueness of solutions for certain class of d.i., special Lipschitz conditions are necessary.

Definition 2.9. F satisfies a *uniform one-sided Lipschitz condition* (o.s.L.) with o.s.L. constant λ if we have

$$(\zeta' - \zeta'', x' - x'') \leq \lambda \, \|x' - x''\| \,,$$

uniformly in t and for all $\zeta' \in F(x'),\ \zeta'' \in F(x'')$, with $x',\, x'' \in \mathbb{R}^n$.

The o.s.L. is weaker than the classical Lipschitz condition or Lipschitz continuity. For higher dimensional problems and explicit numerical methods, the usual o.s.L. is no longer adequate to prove uniqueness of solution. Hence we need a stronger version, the strengthened one-sided Lipschitz condition introduced in [15] (see also [16] and [17]).

Definition 2.10. [15] F satisfies a *strengthened one-sided Lipschitz condition* (s.o.s.L.) with o.s.L. constants $(\lambda_1,\, \lambda_2,\, \ldots,\, \lambda_n)$ if the implication

$$x'_i > x''_i \ \Rightarrow\ \zeta'_i \leq \zeta''_i + \lambda_i \, \|x' - x''\| \,,$$

is true for all and all $x',\, x'' \in \mathbb{R}^n,\ \zeta'_i \in F(x'),\ \zeta''_i \in F(x'')$ and all components $i = 1, 2, \ldots, n$.

Remark 2.11. *i)* For $n > 1$ the s.o.s.L. is stronger than o.s.L. and weaker than the classical Lipschitz condition for single-valued right-hand sides and, if $n = 1$, the s.o.s.L. and o.s.L. are equivalent (see [18]).

ii) The unidimensional set-valued map $-Sgn\, x$ satisfies the s.o.s.L., while $+Sgn\, x$ does not. The o.s.L./ s.o.s.L. conditions are only sufficient conditions for uniqueness. Thus a general criterion for uniqueness does not exist. However, for our class of problems, the positiveness of some α_k in (1.4) seems to be adequate for nonuniqueness (see Examples 1.2 and 1.3).

In [18] the following lemma is proved

Lemma 2.12. [18] *Let the set-valued map* F *be decomposable in the following form*

$$F(x) = g(x) + \sum_{i=1}^{n} \beta_i(x)\ e^i, \tag{2.1}$$

for all $x \in \mathbb{R}^n$, *where the single-valued map* $g : \mathbb{R}^n \to \mathbb{R}^n$ *is Lipschitz continuous, the set-valued maps* $\beta_i : \mathbb{R} \to \mathbb{R}$ *satisfy o.s.L. and* e^i *denotes the i-th canonical unit vector in* $\mathbb{R}^n$. *Then* F *satisfies s.o.s.L.*

Let N be a natural number $N \in \mathbf{N}' \subset \mathbb{N}$, $\mathbf{N}'$ denoting a subsequence of $\mathbb{N}$ tending to infinity, $T > 0$, $h = T\,/\,N$, and an equidistant grid

$$t_0 < t_1 < \ldots < t_N = T\,.$$

We associate with (1.2) a sequence of discrete-time inclusions in the form

$$y_{k+1} \in G_k^N(h\,; y_k), \quad k = 0, 1, \ldots, N-1, \quad y_0 = x_0, \tag{2.2}$$

where $G_k^N : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a discrete-time set-valued map. A solution of (2.2) on $[0, T]$ is any sequence of $N+1$ vectors $y_0, y_1, \ldots, y_N$ that satisfies (2.2) for $k = 0, 1, \ldots, N-1$. The simplest explicit difference method for d.i. is the set-valued version of the classical Euler discretization method for differential equation with

$$G_k^N(h\,; y_k) = y_k + hF(t_k, y_k)\,.$$

The Euler convergence theorem is presented in many works, and various forms (see e.g. [10 Theorem 1, p.77], [11, Theorem 3, p.98], [12, Theorem 10.1.3, p.390], or the papers [7], [14], [15], [18-20]), and uses the idea of the classical Péano theorem to prove the existence of solutions to d.i. Selections strategies for (2.2) can be found in [21].

The existence and uniqueness of solutions to a d.i. were treated in many ways and regard the convergence of difference methods to solve a d.i. The following theorem is the existence Péano's Theorem to d.i.

Theorem 2.13. (Péano's Theorem, [11]) *Let the i.v.p.* (1.2) *verify the following assumptions:* $F \in \mathcal{P}$ *and verify the condition* $\mathcal{C}$. *Then there exists a solution of i.v.p.* (1.2) *on* $[0, \infty)$.

The proof can be found e.g. in [11, Theorem 4 p.101] or [20].

Remark 2.14. In practical problems, it is more convenient to consider the closure of the graph of F instead the closed values of F (see Remark (2.3) and [18] where a practical variant of Theorem 2.13 using g.c. is presented).

Using the strengthened version of o.s.L., we can prove the following corollary ensuring the uniqueness of the solutions of i.v.p. (1.2).

Corollary 2.15. *For the i.v.p.* (1.2) *with* F *satisfying a s.o.s.L. there exists at most one solution.*

Proof. Because F satisfies the s.o.s.L, it satisfies the o.s.L. as well (see Remark 2.11i) and in [19] uniqueness of differential equations with right-hand sides satisfying the o.s.L. condition is proved (the subject is also treated in [10, Theorem 2, Remark 11, p.5 and Theorem 1 p.106]). □

Remark 2.16. By uniqueness of a solution we mean here that a solution lying on a surface of discontinuity of the right-hand side or on an intersection of surfaces of discontinuities of i.v.p. (1.1) can be uniquely continued in the positive direction of time. Note that uniqueness in the positive time direction does not necessarily imply the uniqueness in the negative time.

3. Switch dynamical systems

Let us consider now our class of switched i.v.p. (1.4). Using the Filippov regularization, the corresponding d.i. is

$$\dot{x}(t) \in F(x(t)) = g(x(t)) + \sum_{i=1}^{n} \alpha_i \, Sgn\, x_i \; e^i, \quad x(0) = x_0, \text{ for a.e. } \; t \in I\,. \tag{3.1}$$

Then the main result of this paper is the following theorem

Theorem 3.1. *Let* g *be Lipschitz continuous and satisfy a g.c. Then the i.v.p.* (1.4) *defines a generalized switch d.s. If, moreover, all* α_i *are negative, then the i.v.p.* (1.4) *defines a switch d.s.*

Proof. Using Remark 2.8 *iii*) the right-hand side of (3.1) verifies a g.c. Hence it is locally bounded and the Filippov regularization can be used. The set-valued map F so obtained, belongs to $\mathcal{P}$. Thus the i.v.p. (1.4) has solutions (Theorem 2.13). The maximal interval of existence is $I = [0, \infty)$ because F verifies $\mathcal{C}$ condition (see Remark 2.8 *ii*). Thus the i.v.p. defines a generalized switch d.s. If all coefficients α are negative, then $\alpha_i \; Sgn\, x_i$ are set-valued maps verifying o.s.L. (see Remark 2.11 *ii*) and using Lemma 2.12, F verifies s.o.s.L. Hence the solution is unique (see Corollary 2.15) and the i.v.p. (1.4) defines a switch d.s. □

Application: Let us consider the following discontinuous right-hand side differential equations modeling a Chua circuit [22]

$$\begin{array}{l} \dot{x_1} = -\alpha(b+1)(x_1 - k \; sgn\, x_1) + \alpha\, x_2 \\ \dot{x_2} = x_1 - x_2 + x_3 \\ \dot{x_3} = -\beta x_2 \end{array} \qquad , \tag{3.2}$$

which has no global classical solutions in the classical sense on $[0, \infty)$. Using the parameters value indicated in [20]: $-\alpha(b+1) = -2.57$, $\alpha = 9$, $\beta = 15.7$ and $k = 1.5$, the behavior becomes chaotic (see [8] for details about the chaotic behavior of a d.s.). The Filippov regularization gives us the following d.i.

$$\dot{x}(t) \in F(x(t)) \;\text{ with }\; F(x(t)) = \begin{pmatrix} -2.57\,x_1 + 9\,x_2 \\ x_1 - x_2 + x_3 \\ -\beta x_2 \end{pmatrix} + 3.86\,Sgn\,x_1\,e^1 \tag{3.3}$$

The discontinuity surface, S, is given by the equation $x_1 = 0$. If we denote $f = (f_1, f_2, f_3)^T$, $F = (F_1, F_2, F_3)^T$, then the graph of the set-valued map F_1 at the discontinuity points $(0, x_2, x_3)$, $F_1(0, x_2) = 9x_2 + [-3.86, 3.86]$, is the dashed shape in Figure 1. The assumptions in Theorem 3.1 are verified and the i.v.p. (3.2) defines a generalized switch d.s., the solutions of d.i. (3.3) not being unique (see Remark 2.11*ii*). The solutions, chaotic for $\beta = 15.7$, pass from one side of S to the other (see Figure 2). To obtain the plot, a Matlab model was created (Figure 3). The chaotic behavior for this value of the parameter β can be deduced for the bifurcation diagram as well (see Figure 4, where the maxima of x_3 versus β was plotted), and also from the Poincaré section with the plane $x_2 = 0$ (Figure 5). Hence, for this value of β, the bifurcation diagram and Poincaré section indicate that there is no visible periodical motion. Details on bifurcations and Poincaré sections for continuous d.s. can be found in [8] and for discontinuous d.s. in [2]. In Figure 6 (a) and (b) phase portraits and time series of a trajectory are represented. The plots in Figures 4, 5, 6 were obtained with a Turbo Pascal code using the explicit Euler method.

Remark 3.2. *i)* This type of motion, when the solution passes through a point of the surface S and goes off it, is called *sliding motion* [2].
ii) If we replace $+3.86\,Sgn\,x_1$ in (3.2) with $-3.86\,Sgn\,x_1$, the solution becomes unique, i.e. (3.2) defines a switch d.s. Now the solution remains on S for $t > t_*$ (i.e., slides along the switching space; see [2] where details on these cases can be found), t_* being the moment when the solution enters S. Next, the solution tends asymptotically to $(0, 0, 0)$ (Figure 7). No chaotic behavior was found in this case. The oscillations on the planes (x_1, x_2), and (x_1, x_3), where the solution is in fact a Filippov solution, are typical of (especially explicit) numerical methods for differential inclusions. To avoid this situation, highly consistent methods must be used, e.g. Runge-Kutta method for differential inclusions (see e.g. [6] for details).
iii) The non-uniqueness of solutions could be interpreted here in the following manner: the value of $\dot{x}_1$ at zero value of the state variable x_1, for each $x_2 \in [-0.43, +0.43]$, is uncertain and can take any value in the range $9x_2 + [-3.86, +3.86]$. Hence each trajectory of (3.2) can be considered to be a possible motion of the system.

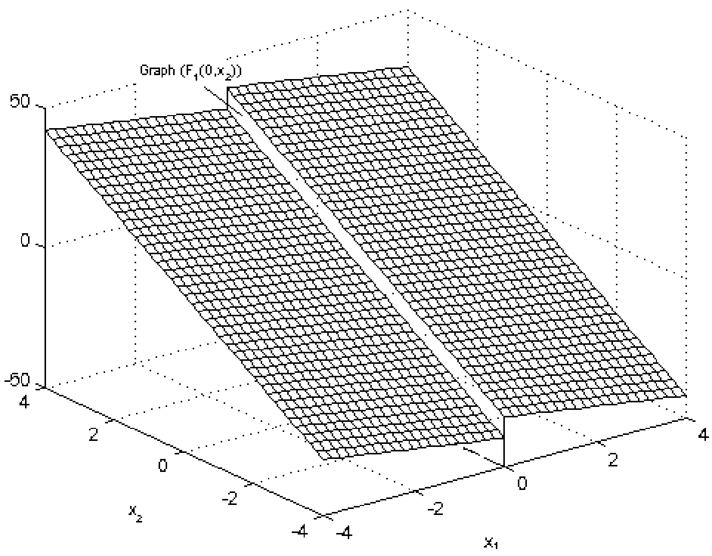

Figure 1. The first component $F_1(0, x_2)$ of the set-valued map F given by (3.2)

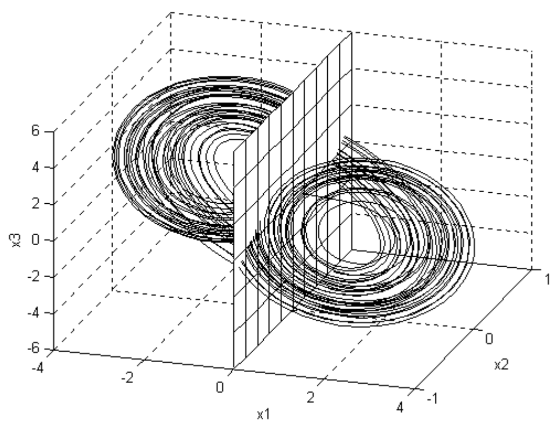

Figure 2. A chaotic trajectory and the discontinuity surface (Matlab)

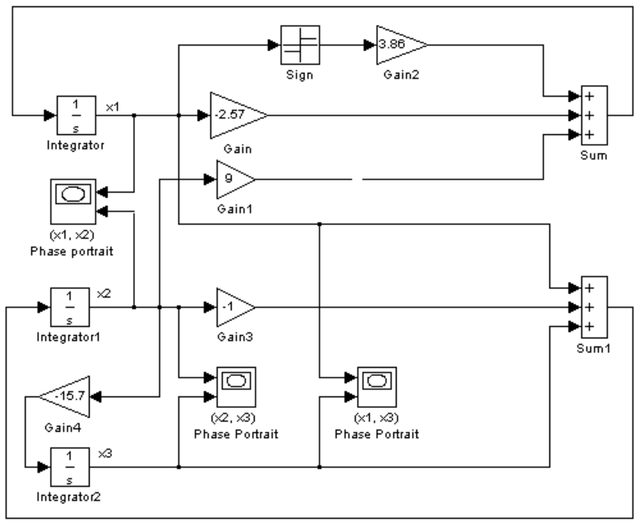

Figure 3. The block scheme of the system (3.2) (Simulink)

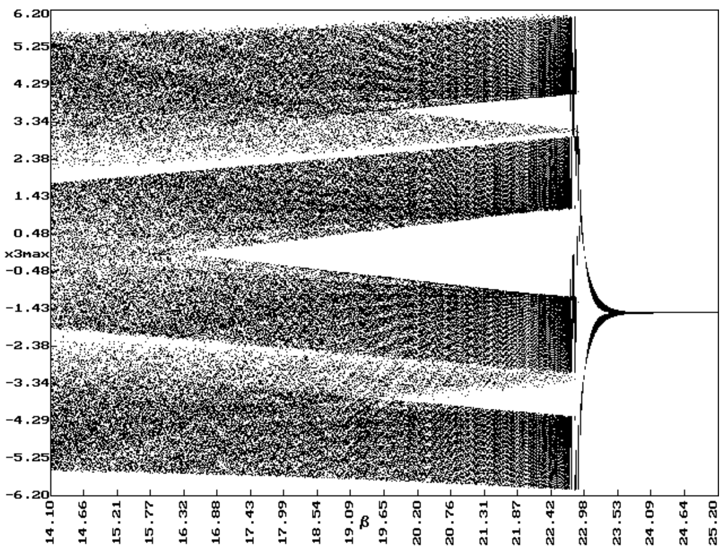

Figure 4. The bifurcation diagram for the generalized switch d.s. (3.2) for $\beta \in [14.10,\ 25.20]$

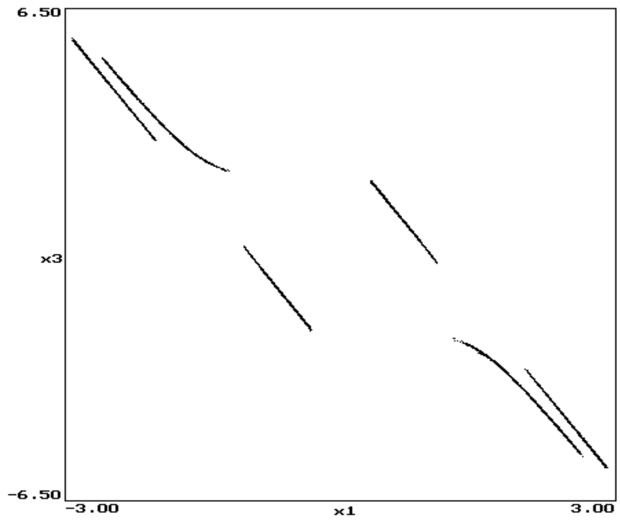

Figure 5. The Poincaré section of the system for $x_2 = 0$

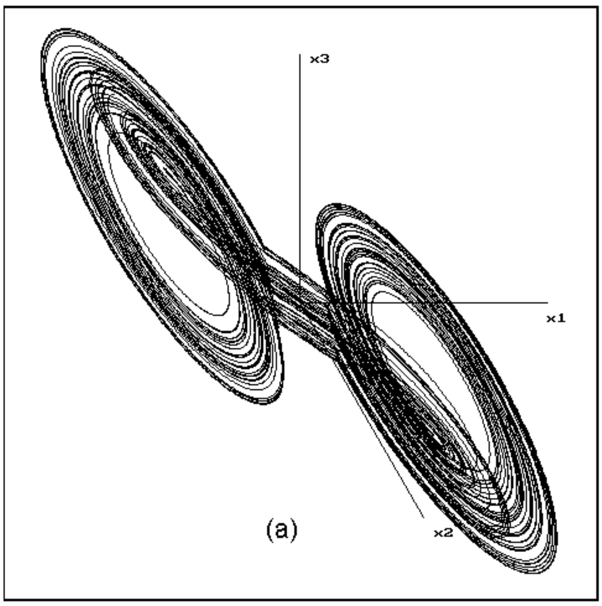

Figure 6(a). A chaotic trajectory for $\beta = 15.7$ obtained with explicit Euler method with stepsize $h = 0.005$: a) Three-dimensional phase portrait view;

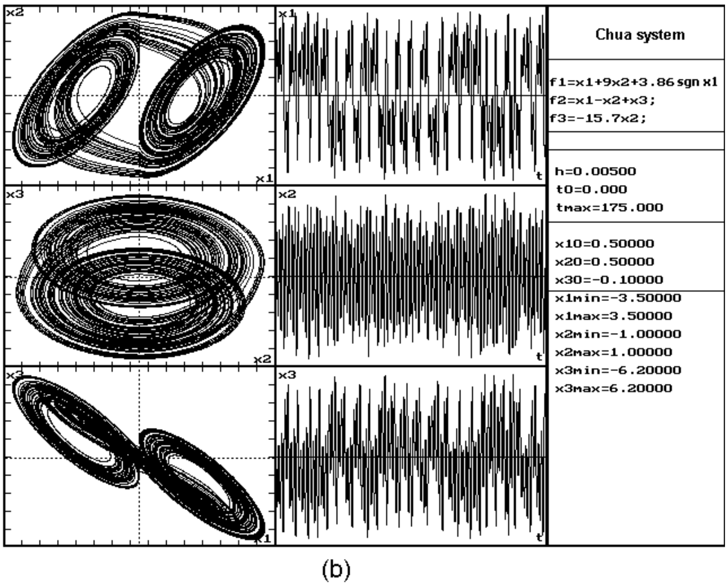

Figure 6(b). A chaotic trajectory for $\beta = 15.7$ obtained with explicit Euler method with stepsize $h = 0.005$: b) Phase portraits and time series

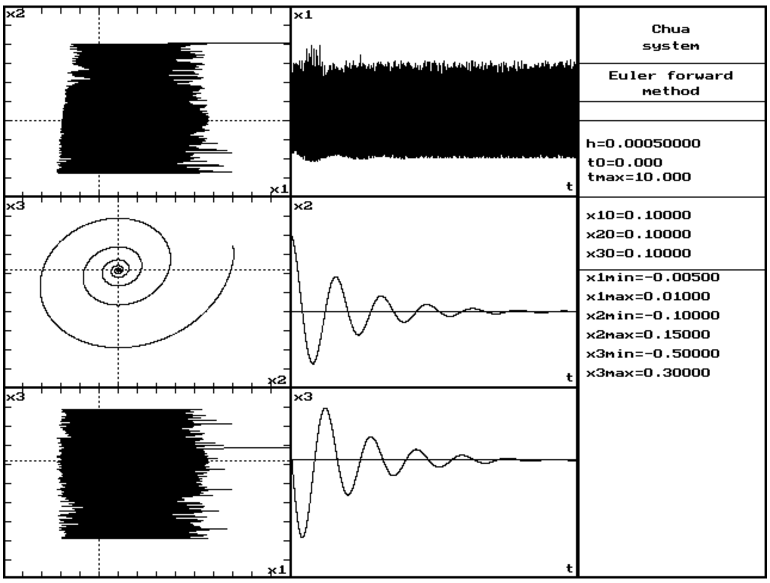

Figure 7. A trajectory of (3.2) with $-3.86\, Sgn\,(x_1)$ instead $+3.86\, Sgn\,(x_1)$ and stepsize $h = 0.0005$

4. Conclusions

In this paper we have focused on a class of discontinuous i.v.p. which can be viewed as defining switch d.s. Therefore a definition for a class of discontinuous dynamical systems, starting from a discontinuous i.v.p. was presented. Using well known results on the existence (as Péano's Theorem) and some criteria for uniqueness of solutions to d.i., we introduced a theorem which states the assumptions in which the i.v.p. could be viewed as a model of a (generalized) switch d.s.
The class of i.v.p. (1.4) was treated in [23] using the maximal monotonicity of $Sgn\, x$.
In [24] the Approximate selection Theorem ([11], [12]) was used.
In [25] the synchronization of switch d.s. modeled by the i.v.p. (1.4) is treated and in [26] the explicit Euler method is considered as an d.s. which approximates the switch d.s. (1.4).

Acknowledgement: The author expresses his gratitude to Professor J. Kolumban, Professor N. Vornicescu and Professor D. Trif for their many helpful suggestions.

REFERENCES

[1] Deimling, K.: *Multivalued Differential Equations, Volume 1 of de Gruyter series in Nonlinear Analysis and Applications*, Walter de Gruyter, Berlin-New York, 1992.

[2] Wiercigroch, M. and de Kraker, B.: *Applied Nonlinear Dynamics and Chaos of Mechanical Systems with Discontinuities*, World Scientific, Singapore, 2000.

[3] Popp, K. and Stelter, P.: *Stick-slip vibrations and chaos*, Philos. Trans. R. Soc. London, **A323**, (1990), 89-105.

[4] Buhite, J. L. and Owen, D. R.: *An ordinary differential equation from the theory of plasticity*, Arch. Rational Mech. Anal, **71**, (1979), 357-383.

[5] Butenin, N. V., Nejmark, Y. I. and Fufaev, N. A.: *An introduction to the Theory of Nonlinear Oscillations*, Nauka, 1987. (in Russian)

[6] Dontchev, A. and Lempio, F.: *Difference Methods for Differentiale Inclusions: a Survey*, SIAM Review, **34**(2), (1992), 263-294.

[7] Kaster-Maresch, A.: *Implicit Runge-Kutta methods for differential inclusions*, Numer. Funct. Anal. and Optimiz., **11**, (1990-91), 937-958.

[8] Schuster, H. G.: *Deterministic Chaos, An Introduction*, VCH, Berlin, 1989.

[9] Stuart, A. M. and Humphries, A. R.: *Dynamical Systems and Numerical Analysis*, Cambridge University Press, 1996.

[10] Filippov, A. F.: *Differential Equations with Discontinuous Right-hand Sides*, Kluwer Academic Publishers, Dordrecht, 1988.

[11] Aubin, J.-P. and Cellina, A.: *Set-valued Maps and Viability Theory*, Springer-Verlag, Berlin, 1984.

[12] Aubin, J.-P. and Frankowska, H.: *Set-Valued Analysis*, Birkhäuser, 1990.

[13] Hájek, O.: *Discontinuous differential equations I, II,* Diff. Equat., **32**, (1979), 149-170.

[14] Taubert, K.: *Converging Multistep Methods for Initial Value Problems Involving Multi-Valued Maps*, Computing, **27**, (1981), 123-136.

[15] Lempio, F. : *Modified Euler Methods for Differential Inclusions,* in A. B. Kurzhanski, V. M. Veliov, *Set-valued Analysis and Differential Inclusions*. A Coll. of Papers resulting from a Workshop held in Pamporovo, Bulgaria, Sept. (1990).

[16] Lempio, F. and Veliov, V.: *Discrete approximation of differential inclusions,* Gamm-Mitteilungen, **2**(21), (1998), 103-135.

[17] Lempio, F. and Veliov,V.: *Discrete approximation of differential inclusions*, Bayreuth Mathematische Schriften, **54**, (1998), 149-232.

[18] Lempio, F.: *Euler's method revisited*, Proceedings of Steklov Institute of Mathematics Moscow, **211**, (1995), 473-494 .

[19] Filippov, A. F.: *Differential equations with discontinuities right-hand side*, AMS Transl., **42**, (1964), 199-231.

[20] Lempio, F.: *Difference Methods for Differential inclusions*, Lecture Notes in Economics and Mathematics Systems, **378**, (1992), 236-273.

[21] Kastner-Maresch, A. and Lempio, F.: *Difference methods with selection strategies for differential inclusions*, Numer. Funct. Anal. and Optimiz., **14**, (1993), 555-572.

[22] Braun, R.: *Generalizations of the Chua equations*, IEEE Transactions on Circuits and Systems, Fundamental Theory and Applications, **40**(11), (1993), 878-883.

[23] Elliott, C. M.: *On the convergence of a one-step method for numerical solution of an ordinary differential inclusion*, IMA Journal of Numerical Analysis, **5**, (1985), 3-21.

[24] Danca, M.-F. and Codreanu, S.: *On a possible approximation of discontinuous dynamical systems*, Chaos, Solitons and Fractals, to be published.

[25] Danca, M.-F.: *Synchronization of switch dynamical systems*, Int. J. Bif. Chaos, to be published.

[26] Danca, M.-F.: *The explicit Euler method for differential inclusions as approximation for switch dynamical systems*, Journal of Nonlinear Science, submitted.

Doi: 10.18514/MMN.2001.42

Mathematical Notes, Miskolc, Vol. 2., No. 2., (2001), pp. 117–126

RANK REDUCTION AND BORDERED INVERSION

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[Received October 8, 2001]

Abstract. We clarify the connection between the rank reduction algorithm and the bordered inversion method and their related sequences.

Mathematical Subject Classification: 15A09, 15A21, 15A23, 65F30, 65F99.
Keywords: Rank reduction, bordered inversion, inverse factorization, conjugation

1. Introduction

The rank reduction algorithm and the bordered inversion method play important roles in numerical linear algebra. The rank reduction algorithm, while keeping the size of the matrix, decreases its rank to zero. The bordered inversion method, starting from a small size leading principal submatrix, increases the size and rank of the submatrix to obtain an inverse. The formulae of the two methods show certain formal similarities. In this paper we are looking for a deeper connection between the two methods. Using unexplored observations of Egerváry [4], we determine three kinds of relationship between the two algorithms and associated sequences. These results are essentially the following. The rank reduction operation gives the reverse bordered inversion formula. Using the inverse LDU decomposition algorithm included in the bordered inversion method, we can produce the same full rank factorization as the rank reduction algorithm. Finally, the bordered inversion formula leads to new multiplicative canonical forms of the rank reduction algorithm.

2. Preliminary results

We need the following results. For proofs and other details, see Ouellette [12].

Lemma 1 *(Guttman-1) Let*

$$A = \begin{bmatrix} E & F \\ G & H \end{bmatrix} \in R^{m\times n} \quad \left(E \in R^{k\times k}\right). \tag{2.1}$$

If E is nonsingular, then

$$rank\,(A) = rank\,(E) + rank\left(H - GE^{-1}F\right). \tag{2.2}$$

Corollary 2 *If A and E are nonsingular, then the Schur complement*

$$S = (A/E) = H - GE^{-1}F$$

is also nonsingular.

Lemma 3 *(Guttman-2) Let*

$$A = \begin{bmatrix} E & F \\ G & H \end{bmatrix} \in R^{m \times n} \quad \left(H \in R^{j \times j}\right). \tag{2.3}$$

If H is nonsingular, then

$$rank\,(A) = rank\,(H) + rank\left(E - FH^{-1}G\right). \tag{2.4}$$

Corollary 4 *If A and H are nonsingular, then the Schur complement*

$$T = (A/H) = E - FH^{-1}G$$

is also nonsingular.

3. The inverse of partitioned matrices

Here we recall the most important results concerning the inverse of partitioned matrices. For proofs and related results see Ouellette [12]. Let

$$A = \begin{bmatrix} E & F \\ G & H \end{bmatrix} \in R^{n \times n} \quad \left(E \in R^{k \times k}\right) \tag{3.1}$$

and E be nonsingular.

Theorem 5 *(Banachiewicz-Frazer-Duncan-Collar) If the nonsingular matrix $A \in R^{n \times n}$ is partitioned in the form (3.1), where E has an inverse, then A^{-1} can be expressed in the partitioned form*

$$A^{-1} = \begin{bmatrix} E^{-1} + E^{-1}FS^{-1}GE^{-1} & -E^{-1}FS^{-1} \\ -S^{-1}GE^{-1} & S^{-1} \end{bmatrix}. \tag{3.2}$$

Remark 6 *Provided that E is invertible, A is invertible if and only if S is invertible.*

Remark 7 *Formula (3.2) can be written in the following form*

$$A^{-1} = \begin{bmatrix} E^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} E^{-1}F \\ -I \end{bmatrix} S^{-1}\left[GE^{-1}, -I\right]. \tag{3.3}$$

4. The bordered inversion method

Formula (3.2) is the basis for the bordered inversion or escalator method of Frazer, Duncan and Collar [6]. For $k = 1, \ldots, m$ let

$$E_{k+1} = \begin{bmatrix} E_k & F_k \\ G_k & H_k \end{bmatrix} \quad \left(E_k \in R^{n_k \times n_k},\ F_k, G_k^T \in R^{n_k \times p_k},\ H_k \in R^{p_k \times p_k}\right).$$

Hence $n_{k+1} = n_k + p_k$ and $E_{m+1} = E$. If E_k and $S_k = H_k - G_k E_k^{-1} F_k$ are nonsingular, then E_{k+1} is also nonsingular and by Theorem 5 we can calculate E_{k+1}^{-1}.

The block bordered inversion algorithm

1. Determine E_1^{-1}.
2. **For** $k = 1, \ldots, m$ set
$$S_k = H_k - G_k E_k^{-1} F_k$$

and

$$E_{k+1}^{-1} = \begin{bmatrix} E_k^{-1} + E_k^{-1} F_k S_k^{-1} G_k E_k^{-1} & -E_k^{-1} F_k S_k^{-1} \\ -S_k^{-1} G_k E_k^{-1} & S_k^{-1} \end{bmatrix}. \tag{4.1}$$

end

It is clear that $E_{m+1}^{-1} = E^{-1}$ provided that S_k is nonsingular for $k = 1, \ldots, m$. If S_k is singular for an index k, then the Guttman lemma implies that E_{k+1} is singular.

Theorem 8 *(Guttman) Let E be nonsingular. The block bordered inversion method gives the inverse of E if and only if all E_k are nonsingular for $k = 1, \ldots, m$.*

Guttman suggested several block variants of the bordered inversion method. For details, see Guttman [10] and Ouellette [12]. We also note that Gergely [9] observed the equivalence of the bordered inversion method with the Gauss-Jordan inversion.

5. The rank reduction algorithm

The rank reduction procedure of Egerváry is based on the following result.

Theorem 9 *(Egerváry-Guttman-Wedderburn) Let $H, S \in R^{m \times n}$ and $rank\,(H) \geq rank\,(S) = k$. Furthermore let $UR^{-1}V^T$ ($U \in R^{m \times k}$, $R \in R^{k \times k}$, $V \in R^{n \times k}$) be a full rank factorization of S. Then*

$$rank\,(H - S) = rank\,(H) - rank\,(S), \tag{5.1}$$

if and only if

$$U = HX, \quad V^T = Y^T H, \quad Y^T H X = R \tag{5.2}$$

for some matrices $X \in R^{n \times k}$ and $Y \in R^{m \times k}$.

Remark 10 *We can write that*

$$rank\,(H-S) = rank\,(H) - rank\,(S) \Leftrightarrow S = HX\left(Y^T HX\right)^{-1} Y^T H.$$

For the proof of the theorem see Egerváry [4], Guttman [11], Cline and Funderlic [3], Ouellette [12] and Galántai [7].

The rank reduction algorithm which is important in full rank factorizations and conjugation has the following form.

Let $A_1 = A$, $X_i \in R^{n\times l_i}$, $Y_i \in R^{m\times l_i}$, $l_i \geq 1$ and $Y_i^T A_i X_i$ be nonsingular for $i = 1, 2, \ldots, k$.

The rank reduction procedure

$$A_{i+1} = A_i - A_i X_i \left(Y_i^T A_i X_i\right)^{-1} Y_i^T A_i \quad (i = 1, 2, \ldots, k), \tag{5.3}$$

where $rank\,(A) \geq \sum_{i=1}^{k} l_i$.

The algorithm is said to be breakdown free, if all $Y_i^T A_i X_i$ are nonsingular for $i = 1, 2, \ldots, k$. Assume that

$$rank\,(A) = \sum_{i=1}^{k} l_i.$$

In this case, $A_{k+1} = 0$. Let

$$X = [X_1, \ldots, X_k], \quad Y = [Y_1, \ldots, Y_k]$$

and

$$X^{(i)} = [X_1, \ldots, X_i], \quad Y^{(i)} = [Y_1, \ldots, Y_i].$$

Then the following theorem is true.

Theorem 11 *The rank reduction procedure can be carried out breakdown free if and only if* $Y^T AX = \left[Y_i^T AX_j\right]_{i,j=1}^{k}$ *is block strongly nonsingular*[1]*. In this event the rank reduction procedure has the canonical form*

$$A_{i+1} = A - AX^{(i)}\left(Y^{(i)T} AX^{(i)}\right)^{-1} Y^{(i)T} A \quad (i = 1, \ldots, k). \tag{5.4}$$

For proof, see Galántai [7], [8]. For $i = k$ the relation

$$A = AX\left(Y^T AX\right)^{-1} Y^T A$$

holds, where $Z = X\left(Y^T AX\right)^{-1} Y^T$ is a reflexive inverse of A. It is also easy to see that algorithm (5.3) leads to the full rank factorization

$$A = QD^{-1}P^T, \tag{5.5}$$

[1] It means that the matrix has a block LU decomposition without pivoting.

where

$$P = \left[A_1^T Y_1, \ldots, A_k^T Y_k\right], \quad Q = \left[A_1 X_1, \ldots, A_k^T X_k\right]$$

and

$$D = diag\left(Y_1^T A_1 X_1, \ldots, Y_k^T A_k X_k\right).$$

If a square matrix B has a block LDU decomposition with respect to a given partition, then this unique factorization will be denoted by $B = L_B D_B U_B$. Then we can prove

Theorem 12 *Let $Y^T AX$ be block strongly nonsingular. The components of the full rank factorization (5.5) are*

$$P = A^T Y L_{Y^T AX}^{-T}, \quad Q = AXU_{Y^T AX}^{-1}, \quad D = D_{Y^T AX}, \tag{5.6}$$

and A can be expressed in the form

$$A = \left(AXU_{Y^T AX}^{-1}\right) D_{Y^T AX}^{-1} \left(L_{Y^T AX}^{-1} Y^T A\right). \tag{5.7}$$

For proof, see Galántai [7], [8].

6. Rank-reduction and bordered inversion I

We show a surprisingly simple connection between the rank reduction operation and the bordered inversion.

Theorem 13 *Assume that*

$$A = \begin{bmatrix} E & F \\ G & H \end{bmatrix} \in R^{n\times n} \quad \left(E \in R^{k\times k}\right) \tag{6.1}$$

and

$$J = \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix} \in R^{n\times(n-k)}.$$

Then

$$A - AJ\left(J^T AJ\right)^{-1} J^T A = \begin{bmatrix} (A/H) & 0 \\ 0 & 0 \end{bmatrix},$$

provided that H is invertible[2].

Proof. By simple calculation we have

$$AJ\left(J^T AJ\right)^{-1} J^T A = \begin{bmatrix} F \\ H \end{bmatrix} H^{-1}[G, H] = \begin{bmatrix} FH^{-1}G & F \\ G & H \end{bmatrix}$$

and

$$A - J\left(J^T AJ\right)^{-1} J^T A = \begin{bmatrix} E - FH^{-1}G & 0 \\ 0 & 0 \end{bmatrix},$$

which was to be proved. ■

[2] This condition is in agreement with the strong block nonsingularity of $Y^T AX$. In fact, $X_1 = Y_1 = J$ and $Y_1^T AX_1 = H$.

Remark 14 *If J is replaced by $\widehat{J} = \begin{bmatrix} I_k \\ 0 \end{bmatrix}$, then we have*

$$A - A\widehat{J}\left(\widehat{J}^T A \widehat{J}\right)^{-1} \widehat{J}^T A = \begin{bmatrix} 0 & 0 \\ 0 & (A/E) \end{bmatrix}, \tag{6.2}$$

provided that E is nonsingular.

Remark 15 *If A and H are replaced by A^{-1} and S^{-1}, respectively then we have $\left(A^{-1}/S^{-1}\right) = E^{-1}$. This means that*

$$A^{-1} - A^{-1}J\left(J^T A^{-1} J\right)^{-1} J^T A^{-1} = \begin{bmatrix} E^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \tag{6.3}$$

We just obtained that one rank reduction step on matrix A^{-1} results in the inverse of the leading principal submatrix E. Relation (6.3) gives a direct connection between the rank reduction and the bordered inversion. It was first discovered by Egerváry [4] in the scalar case ($H \in R$), who developed it for solving difference equations with modified boundary conditions [5]. The same idea appears in Brezinski et al. [2], who give credit to Duncan and call it reverse bordered inversion. Their variant of identity (6.3) is obtained in the following way.

Let

$$A^{-1} = \begin{bmatrix} \widehat{E} & \widehat{F} \\ \widehat{G} & \widehat{H} \end{bmatrix}.$$

Then

$$\widehat{E} = E^{-1} + E^{-1}FS^{-1}GE^{-1}, \quad \widehat{F} = -E^{-1}FS^{-1},$$
$$\widehat{G} = -S^{-1}GE^{-1}, \quad \widehat{H} = S^{-1}.$$

As $\widehat{F}\left(\widehat{H}\right)^{-1} = -E^{-1}F$ and $\widehat{F}\left(\widehat{H}\right)^{-1}\widehat{G} = E^{-1}FS^{-1}GE^{-1}$ we have

$$E^{-1} = \widehat{E} - E^{-1}FS^{-1}GE^{-1} = \widehat{E} - \widehat{F}\left(\widehat{H}\right)^{-1}\widehat{G}, \tag{6.4}$$

which is exactly the reverse bordered inversion formula of Brezinski et al. [2].

7. The bordered inversion method and triangular factorization

We associate sequences with the bordered inversion algorithm which produce the inverse LDU factorizations of the leading principal submatrices E_k. It is based on the following observation. Let $E = LDU$ be the unique block LDU decomposition of E. If E is not partitioned, then we just take $L = U = I$ and $D = E$. If A has the form (3.1), then the inverse formula (3.3) can be written as

$$\begin{aligned} A^{-1} &= \begin{bmatrix} U^{-1} & -E^{-1}F \\ 0 & I \end{bmatrix} \begin{bmatrix} D^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} L^{-1} & 0 \\ -GE^{-1} & I \end{bmatrix} \\ &= U_A^{-1} D_A^{-1} L_A^{-1}, \end{aligned}$$

which is the inverse block LDU decomposition of A^{-1}. Thus we can define the following sequences related to the bordered inversion algorithm of Section 4.

Let $E_k = L_k D_k U_k$ denote the unique LDU decomposition of E_k for $k = 1, \dots, m$.

Inverse LDU decomposition algorithm

For $k = 1, \dots, m$ set

$$L_{k+1}^{-1} = \begin{bmatrix} L_k^{-1} & 0 \\ -G_k E_k^{-1} & I \end{bmatrix}, \quad D_{k+1}^{-1} = \begin{bmatrix} D_k^{-1} & 0 \\ 0 & S_k^{-1} \end{bmatrix},$$

$$U_{k+1}^{-1} = \begin{bmatrix} U_k^{-1} & -E_k^{-1} F_k \\ 0 & I \end{bmatrix}.$$

end

It is clear that $E_{k+1}^{-1} = U_{k+1}^{-1} D_{k+1}^{-1} L_{k+1}^{-1}$ and

$$E^{-1} = U_{m+1}^{-1} D_{m+1}^{-1} L_{m+1}^{-1}. \tag{7.1}$$

The above algorithm, which requires no extra operations, is based on a remark by Egerváry [4], according to which the bordered inversion (or escalator) method *"furnishes the triangular factorisation of the inverse"*.

The associated procedure can be carried out if and only if the bordered inversion can be so. Hence the necessary and sufficient condition for performing the algorithm without breakdown is the block strong nonsingularity of E.

8. Rank-reduction and bordered inversion II

Here we determine a new relationship between rank reduction and bordered inversion. More precisely, we show that certain sequences related to the rank reduction and the bordered inversion algorithms, respectively, can be tied together. We recall the facts that

1. The associated sequences of the bordered inversion method produce U_E^{-1}, D_E^{-1} and L_E^{-1}.

2. The rank reduction method produces the full rank factorization

$$A = QD^{-1}P^T = \left(AXU_{Y^TAX}^{-1}\right) D_{Y^TAX}^{-1} \left(L_{Y^TAX}^{-1} Y^T A\right).$$

If $X = B^T V$, then the pair (P, V) is block B-conjugate (see Galántai [7], [8]). This conjugation property plays an important role in the ABS methods (see Abaffy-Spedicato [1]).

Having the two observations above, we can easily derive the following

Conjugation algorithm

1. Apply the inverse LDU decomposition algorithm to the matrix $Y^T AX$ and get $L_{Y^TAX}^{-1}$.

2. Set

$$P = A^T Y L_{Y^T AX}^{-T}.$$

end

Thus we have a new connection between rank reduction and bordered inversion. It is also clear that, similarly, we can have Q and D^{-1}. Finally we mention that Stewart [13] gave a conjugation algorithm which is based on LU factorization.

9. Rank-reduction and the reverse bordered inversion

We deal with the last connection found, which is, in fact, an application of identity (6.3) to obtain new canonical forms for A_{i+1}. We start with the formula (5.4), which tells us that

$$A_{i+1} = A - AX^{(i)} \left(Y^{(i)T} AX^{(i)}\right)^{-1} Y^{(i)T} A \quad (i = 1, \ldots, k).$$

Let $I^{(j)}$ be the first j columns of the identity matrix I_n. Furthermore let $\omega_i = \sum_{j=1}^{i} l_j$. Then $X^{(i)} = XI^{(\omega_i)}$ and $Y^{(i)} = YI^{(\omega_i)}$ and we can write

$$A_{i+1} = AX \left[\left(Y^T AX\right)^{-1} - I^{(\omega_i)} \left(Y^{(i)T} AX^{(i)}\right)^{-1} I^{(\omega_i)T}\right] Y^T A.$$

As

$$I^{(\omega_i)} \left(Y^{(i)T} AX^{(i)}\right)^{-1} I^{(\omega_i)T} = \begin{bmatrix} \left(Y^{(i)T} AX^{(i)}\right)^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

we have the following

Theorem 16 *A_{i+1} has the following canonical form*

$$A_{i+1} = AX \left(\left(Y^T AX\right)^{-1} - \begin{bmatrix} \left(Y^{(i)T} AX^{(i)}\right)^{-1} & 0 \\ 0 & 0 \end{bmatrix}\right) Y^T A \tag{9.1}$$

under the conditions of Theorem 11.

We can observe that in the parentheses we have a variant of the identity (6.3) with $Y^T AX$ instead of A^{-1}. Thus

$$\begin{aligned} &\left(Y^T AX\right)^{-1} - \begin{bmatrix} \left(Y^{(i)T} AX^{(i)}\right)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \left(Y^T AX\right)^{-1} J \left[J^T \left(Y^T AX\right)^{-1} J\right]^{-1} J^T \left(Y^T AX\right)^{-1}, \end{aligned}$$

where

$$J = \begin{bmatrix} 0 \\ I_{n-\omega_i} \end{bmatrix}. \tag{9.2}$$

Hence by simple calculations we have

Theorem 17 A_{i+1} *has the following canonical form*

$$A_{i+1} = Y^{-T} J \left(J^T X^{-1} A^{-1} Y^{-T} J \right)^{-1} J^T X^{-1} \tag{9.3}$$

under the conditions of Theorem 11.

Assuming that

$$\left(Y^T A X \right)^{-1} = \begin{bmatrix} \widehat{E} & \widehat{F} \\ \widehat{G} & \widehat{H} \end{bmatrix}$$

we can obtain a third form of A_{i+1}.

Theorem 18 A_{i+1} *has the following canonical form*

$$A_{i+1} = Y^{-T} \begin{bmatrix} 0 & 0 \\ 0 & \widehat{H}^{-1} \end{bmatrix} X^{-1} \tag{9.4}$$

under the conditions of Theorem 11.

These new multiplicative canonical forms indicate different properties of the rank reduction algorithm. The first canonical form indicates the decrease of $\|A_{k+1}\|$. The second form shows the projector properties of the rank reduction matrix A_{i+1}. The last canonical form shows the structure of A_{k+1} in terms of inverse quantities. These three new forms certainly give a better understanding of rank reduction and may lead to further new results.

Acknowledgement: This work was supported by the CNR International short-term mobility program, Year 2001.

REFERENCES

[1] Abaffy J. and Spedicato, E.: *ABS-projection Algorithms: Mathematical Techniques for Linear and Nonlinear Algebraic Equations,* Ellis Horwood, Chichester, 1989.

[2] Brezinski, C., Morandi Cecchi, M. and Redivo-Zaglia, M.: *The reverse bordering method,* SIAM J. Matrix. Anal. Appl., **15**, (1994}, 922-937.

[3] Cline, R.E. and Funderlic, R.E.: *The rank of a difference of matrices and associated generalized inverses,* Lin. Alg. Appl., **24**, (1979), 185-215.

[4] Egerváry, E.: *On rank-diminishing operations and their applications to the solution of linear equations,* ZAMP, XI., (1960), 376-386.

[5] Egerváry, J.: *Über eine Methode zur numerischen Lösung der Poissonschen Differenzengleichung für beliebige Gebiete,* Acta Mathematica Academiae Scientiarium Hungaricae, **11**, (1960), 341-361.

[6] Frazer, A., Duncan, J., Collar, R.: *Elementary matrices,* Cambridge University Press, 1938.

[7] Galántai A.: *Rank reduction and conjugation,* Mathematical Notes, Miskolc, **1**(1), (2000), 11-33.

[8] Galántai A.: *Rank reduction, factorization and conjugation,* Linear and Multilinear Algebra, Vol. 49, (2001), 195-207.

[9] Gergely J.: *Mátrixinvertáló módszerekről (On matrix inversion methods, in Hungarian),* Alkalmazott Matematikai Lapok, **2**, (1976), 359-362.

[10] Guttman, L.: *Enlargement methods for computing the inverse matrix,* Ann. Math. Statist., **17**, (1946), 336-343.

[11] Guttman, L.: *A necessary and sufficient formula for matrix factoring,* Psychometrika, **22**, (1957), 79-91.

[12] Ouellette, D.V.: *Schur complements and statistics,* Lin. Alg. Appl., 36, (1981), 187-295.

[13] Stewart, G.W.: *Conjugate direction methods for solving systems of linear equations,* Numerische Mathematik, **21**, (1973), 285-297.

Doi: 10.18514/MMN.2001.43

Mathematical Notes, Miskolc, Vol. 2., No. 2., (2001), pp. 127–136

THE McSHANE AND THE WEAK McSHANE INTEGRALS OF BANACH SPACE-VALUED FUNCTIONS DEFINED ON R^m

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[Received October 13, 2001]

Abstract. In this paper, we define a concept of the weak McShane integral for functions mapping a compact interval I_0 in R^m into a Banach space X and discuss the relation between the weak McShane integral and the Pettis integral. We show that the weak McShane integral and the Pettis integral are equivalent if and only if the Banach space X contains no copy of c_0. Further, combining the properties of the McShane integral and Pettis integral, we get some equivalent statements concerning the McShane integral and the Pettis integral.

Mathematical Subject Classification: 28B05, 46G10, 26A39
Keywords: Pettis integral, weak McShane integral, McShane integral

1. Introduction

It is known to all that the McShane integral of real-valued functions is a Riemann-type integral, which is equivalent to the Lebesgue integral. R. A. Gordon [1] generalized the definition of the McShane integral for real-valued functions to abstract functions from intervals in R to Banach spaces and discussed some of its properties. In [2] it was proved that the McShane integral and Pettis integral are equivalent e.g. when the Banach space X is reflexive with the property (P).

In this paper, at first we get some equivalent statements about the Pettis and the McShane integrability of functions mapping an m-dimensional compact interval I_0 into a Banach space using the properties of the Pettis integral and McShane integral appearing in [3], [4], [5], [6], [2] and [7]. Then, we define a concept of the weak McShane integral and show that the weak McShane integral and the Pettis integral are equivalent if and only if the Banach space X contains no copy of c_0.

2. The McShane integral

We begin with some terminology and notations. We denote by X a real Banach space with the norm $\|\cdot\|$ and X^* its dual.

I_0 is a compact interval in R^m, Σ is the set of all μ-measurable subsets of I_0, μ stands for the Lebesgue measure.

$B(X^*) = \{x^* \in X^*;\ \|x^*\| \leq 1\}$ is the closed unit ball in X^*.
We first extend the notion of partition of an interval. A partial M-partition D in I_0 is a finite collection of interval-point pairs (I, ξ) with non-overlapping intervals $I \subset I_0$, $\xi \in I_0$ being the associated point of I. Requiring $\xi \in I$ for the associated point of I, we get the concept of a partial K-partition D in I_0. We write $D = \{(I, \xi)\}$.

A partial M-partition $D = \{(I, \xi)\}$ in I_0 is a M-partition of I_0 if the union of all the intervals I equals I_0 and similarly for a K-partition.

Let δ be a positive function defined on the interval I_0. A partial M-partition (K-partition) $D = \{(I, \xi)\}$ is said to be δ-fine if for each interval-point pair $(I, \xi) \in D$ we have $I \subset B(\xi, \delta(\xi))$ where $B(\xi, \delta(\xi)) = \{t \in R^m;\ \mathrm{dist}(\xi, t) < \delta(\xi)\}$ and dist is the metric in R^m.

The m-dimensional volume of a given interval $I \subset I_0$ is denoted by $\mu(I)$.

Given a M-partition $D = \{(I, \xi)\}$ we write

$$f(D) = (D) \sum f(\xi)\mu(I)$$

for integral sums over D, whenever $f : I_0 \mapsto X$.

DEFINITION 1. An X-valued function f is said to be McShane integrable on I_0 if there exists an $S_f \in X$ such that for every $\varepsilon > 0$, there exists $\delta(t) > 0$, $t \in I_0$ such that for every δ-fine M-partition $D = \{(I, \xi)\}$ of I_0, we have

$$\|(D) \sum f(\xi)\mu(I) - S_f\| < \varepsilon.$$

We write $(M) \int_{I_0} f = S_f$ and S_f is the McShane integral of f over I_0.

f is McShane integrable on a set $E \subset I_0$ if the function $f \cdot \chi_E$ is McShane integrable on I_0, where χ_E denotes the characteristic function of E.

We write $(M) \int_E f = (M) \int_{I_0} f\chi_E = F(E)$ for the McShane integral of f on E.

Denote the set of all McShane integrable functions $f : I_0 \mapsto X$ by $\mathcal{M}$.

Replacing the term "M-partition" by "K-partition" in the definition above, we obtain Kurzweil-Henstock integrability and the definition of the Kurzweil-Henstock integral $(K) \int_{I_0} f$.

It is clear that if $f : I_0 \mapsto X$ is McShane integrable, then it is also Kurzweil-Henstock integrable.

The basic properties of the McShane integral, for example, linearity and additivity with respect to intervals can be found in [1-2], [5-6], [8], [9]. We do not present them here. The reader is referred to the above mentioned references for the details.

DEFINITION 2. A set $K \subset \mathcal{M}$ is called M-equiintegrable (McShane-equiintegrable) if for every $\varepsilon > 0$ there is a $\delta : I_0 \mapsto (0, +\infty)$ such that

$$\|(D)\sum f(\xi)\mu(I) - \int_{I_0} f\| < \varepsilon$$

for every δ-fine M-partition $D = \{(I, \xi)\}$ of I_0 and every $f \in K$.

DEFINITION 3. $f : I_0 \mapsto X$ is called (strongly) measurable if there is a sequence of simple functions (f_n) with $\lim_{n\to\infty} \|f_n(t) - f(t)\| = 0$ for almost all $t \in I_0$.

$f : I_0 \mapsto X$ is called weakly measurable if for each $x^* \in X^*$ the real function $x^*(f) : I_0 \mapsto R$ is measurable.

Two functions $f, g : I_0 \mapsto X$ are called weakly equivalent on I_0 if for every $x^* \in X^*$ the relation

$$x^*(f(t)) = x^*(g(t))$$

holds for almost all $t \in I_0$.

THEOREM 4. If $f : I_0 \to X$ is McShane integrable on I_0, then

(a) for each x^* in X^*, $x^*(f)$ is McShane integrable on I_0 and $\int_{I_0} x^*(f) = x^*(\int_{I_0} f)$,
(b) $\{x^*(f);\ x^* \in B(X^*)\}$ is M-equiintegrable on I_0,
(c) f is weakly measurable.

Proof. Since $f : I_0 \to X$ is McShane integrable on I_0, to every $\varepsilon > 0$ there exists a positive function δ defined on I_0 such that for any δ-fine M-partition $D = \{(I, \xi)\}$ we have $\|(D)\sum f(\xi)\mu(I) - \int_{I_0} f\| < \varepsilon$.

Hence for any $x^* \in X^*$ we have

$$|(D)\sum x^*(f(\xi))\mu(I) - x^*(\int_{I_0} f)| \le \|x^*\|\|(D)\sum f(\xi)\mu(I) - \int_{I_0} f\| < \|x^*\|\varepsilon$$

for any δ-fine M-partition $D = \{(I, \xi)\}$.

Therefore (a) holds. If $x^* \in B(X^*)$, then the above inequality gives

$$|(D)\sum x^*(f(\xi))\mu(I) - x^*(\int_{I_0} f)| < \varepsilon$$

for every $x^* \in B(X^*)$, so the set $\{x^*(f);\ x^* \in B(X^*)\}$ is M-equiintegrable on I_0 and f is weakly measurable because for every $x^* \in X^*$ the real function $x^*(f)$ is Lebesgue integrable (see e.g. [5]).

Some other properties of the McShane integral can be found in [2].

3. The relation between the McShane integral and the Pettis integral

First of all the basic definitions concerning the Pettis integral have to be given. (See e.g. [3], [5], etc.)

DEFINITION 5. If $f : I_0 \mapsto X$ is weakly measurable such that $x^*(f) \in \mathcal{L}$ for all $x^* \in X^*$, then f is called Dunford integrable . The Dunford integral of f over $E \in \Sigma$ is defined by the element x^{**}_E of X^{**} such that

$$x^{**}_E(x^*) = \int_E x^*(f)$$

for all $x^* \in X^*$, and we write $x^{**}_E = (D)\int_E f$. By $\mathcal{D}$ the set of all Dunford integrable functions will be denoted.

In the case that $\int_E f \in X$ for each $E \in \Sigma$, then f is called Pettis integrable and we write $(P)\int_E f$ instead of $(D)\int_E f$ to denote the Pettis integral of f over $E \in \Sigma$. Denote by $\mathcal{P}$ the set of all Pettis integrable functions.

We have clearly $\mathcal{P} \subset \mathcal{D}$.

We denote by $\mathcal{L}$ the set of Lebesgue integrable real functions on I_0 (with respect to the Lebesgue measure μ). It should be noted at this point that a real function f belongs to $\mathcal{L}$ if and only if it is McShane integrable, i.e. $\mathcal{L} = \mathcal{M}$ (see [10], [8], etc.).

We use the notation $\mu(E)$ for the Lebesgue measure of a (Lebesgue) measurable set $E \subset I_0$.

DEFINITION 6. A set $K \subset \mathcal{L}$ is called uniformly integrable if

$$\lim_{\mu(E)\to 0} \int_E |f| = 0$$

uniformly for $f \in K$, where $E \subset I_0$ are measurable sets.

Since (I_0, Σ, μ) is a finite perfect measure space we can use the result of Theorem 6 from [4,2] in the following form.

THEOREM 7. The function $f : I_0 \mapsto X$ is Pettis integrable if and only if there is a sequence (f_n) of simple functions from I_0 into X such that

(a) the set $\{x^*(f_n);\ x^* \in B(X^*), n \in N\}$ is uniformly integrable, and

(b) for each x^* in X^*, $\lim_{n\to\infty} x^*(f_n) = x^*(f)$ a.e. on I_0.

R. A. Gordon proved (see [1], Theorem 17) the following.

THEOREM 8. Let $f : I_0 \mapsto X$ be given where either f is measurable or X is separable.

If f is Pettis integrable on I_0, then f is McShane integrable on I_0.

REMARK 9. In fact Gordon proved the result given in Theorem 8 for the case when $I_0 = [a, b] \in R$ is a one-dimensional interval. Looking at [1] it can be seen that the approach of Gordon can be adopted also for the case of a compact interval I_0 in R^m.

DEFINITION 10. We say that a Banach space X has the property (P) if there exists a sequence $\{x_m^* \in B(X^*); m \in N\}$ such that for each $x^* \in X^*$ there exists a subsequence $\{x_k^* \in B(X^*);\ k \in N\}$ of $\{x_m^* \in B(X^*);\ m \in N\}$ such that

$$x_k^*(x) \to x^*(x) \text{ for every } x \in X \ \ if\ k \to \infty.$$

In [2] the following was proved.

THEOREM 11. If $f : I_0 \mapsto X$ is Pettis integrable on I_0, f is weakly equivalent to a measurable function $g : I_0 \mapsto X$ and X has the property (P) then f is McShane integrable on I_0, i.e. $\mathcal{P} \subset \mathcal{M}$.

G. F. Stefánsson in [11] proved the following.

PROPOSITION 12. All weakly measurable functions determined by reflexive spaces are weakly equivalent to strongly measurable functions.

Let us recall that a weakly measurable function $f : I_0 \mapsto X$ is said to be determined by a subspace D of X if for $x^* \in X^*$ which is restricted to D equals zero $(x^*|_D = 0)$ the function $x^*(f)$ equals zero almost everywhere on I_0.

Since every weakly measurable function $f : I_0 \mapsto X$ is determined by the space X itself, we conclude easily that the following holds.

PROPOSITION 13. If the Banach space X is reflexive and $f : I_0 \mapsto X$ is weakly measurable, then there exists a strongly measurable $g : I_0 \mapsto X$ which is weakly equivalent to f.

Using this and Theorem 11 we obtain.

COROLLARY 14. If the Banach space X is reflexive with the property (P) and $f : I_0 \mapsto X$ is Pettis integrable, then f is McShane integrable on I_0, i.e. $\mathcal{P} \subset \mathcal{M}$.

REMARK 15. Stefánsson's Proposition 12 can be used for stating the following.

If $f : I_0 \mapsto X$ is Pettis integrable on I_0, f is determined by a reflexive space and X has the property (P), then f is McShane integrable on I_0.

THEOREM 16. If f is McShane integrable on I_0, then f is Pettis integrable, i.e., $\mathcal{M} \subset \mathcal{P}$.

The proofs of the above results can be found in [2].

From Theorem 16 and by the properties of Pettis integrable functions presented in [3] (Theorem 5, p. 53; Corollary 9, p. 56) we get the following.

COROLLARY 17. If $f : I_0 \mapsto X$ is McShane integrable on I_0, then

(1) $F : E \mapsto \int_E f d\mu$ is a countable additive μ-continuous vector measure on Σ,
(2) $\{F(E); E \in \Sigma\}$ is bounded,
(3) $F(\Sigma)$ is relatively weakly compact.

By Corollary 14 and Theorem 16 we also have the following.

THEOREM 18. Let the Banach space X be reflexive with the property (P). Then $f : [a, b] \mapsto X$ is McShane integrable on $[a, b]$ if and only if f is Pettis integrable, i.e., $\mathcal{M} = \mathcal{P}$.

THEOREM 19. Let $f : I_0 \mapsto X$ be weakly equivalent to a measurable function $g : I_0 \mapsto X$ and X has the property (P). Then the following statements are equivalent.

(a) f is McShane integrable on I_0,
(b) f is Pettis integrable on I_0,
(c) there is a sequence (f_n) of simple functions from I_0 into X such that the set $\{x^*(f_n);\ x^* \in B(X^*), n \in N\}$ is uniformly integrable and for each x^* in X^* we have $\lim_{n\to\infty} x^*(f_n) = x^*(f)$ a.e. on I_0,

Proof. By Theorem 11 and Theorem 16, we know that (a) and (b) are equivalent and (b) is equivalent to (c) by Theorem 7.

Let us denote by $\mathcal{B}$ the set of all Bochner integrable functions $f : I_0 \mapsto X$ (see e.g. [3]).

In our paper [6] we gave some characterizations of $\mathcal{B}$ and we proved among others the following.

THEOREM 20. The inclusion $\mathcal{B} \subset \mathcal{M}$ holds in general and $\mathcal{B} = \mathcal{M}$ if and only if the Banach space X is finite dimensional.

Since finite dimensional Banach spaces are separable, we can combine this with Theorem 8 and Theorem 20 to obtain the following.

THEOREM 21. The inclusion $\mathcal{B} \subset \mathcal{P}$ holds in general and we have $\mathcal{B} = \mathcal{P}$ if and only if the Banach space X is finite dimensional.

4. Relation between the McShane integral and the Pettis integral

In this section we define the weak McShane integral and discuss the relation between the weak McShane integral and the Pettis integral.

DEFINITION 22. An X-valued function f is said to be weakly McShane integrable on I_0 if for every $x^* \in X^*$ the real function $x^*(f)$ is McShane integrable on I_0 and for every interval $I \subset I_0$, there is a $x_I \in X$ such that $\int_I x^*(f) = x^*(x_I)$. We write $X_I = (WM) \int_I f$.

f is weakly McShane integrable on a set $E \subseteq I_0$ if the function $f \cdot \chi_E$ is weakly McShane integrable on I_0, where χ_E denotes the characteristic function of E.

We write $(WM) \int_E f = (WM) \int_{I_0} f\chi_E = F(E)$ for the weak McShane integral of f on E.

Denote the set of all weakly McShane integrable functions $f : I_0 \mapsto X$ by $\mathcal{WM}$.

Looking at the corresponding definitions and using the well-known fact that the McShane integrability of the real function $x^*(f)$ is equivalent to its Lebesgue integrability, we obtain immediately the following statement concerning the various sets of integrable functions.

PROPOSITION 23. If X is an arbitrary Banach space then

$$\mathcal{M} \subset \mathcal{P} \subset \mathcal{WM} \subset \mathcal{D}.$$

It is easy to prove that the weak McShane integral has the following basic properties.

THEOREM 24. Let f and g be functions mapping I_0 into X.

(a) If f is weakly McShane integrable on I_0, then f is weakly McShane integrable on every subinterval of $I \subset I_0$.

(b) If f is weakly McShane integrable on each of the intervals I_1 and I_2, where I_i are non-overlapping and $I_1 \cup I_2 = I$ is an interval, then f is weakly McShane integrable on I and

$$(WM) \int_I f = (WM) \int_{I_1} f + (WM) \int_{I_2} f.$$

(c) If f and g are weakly McShane integrable on I_0 and if α and β are real numbers, then $\alpha f + \beta g$ is weakly McShane integrable on I_0 and

$$(WM) \int_{I_0} (\alpha f + \beta g) = \alpha(WM) \int_{I_0} f + \beta(WM) \int_{I_0} g.$$

(d) If $f = g$ almost everywhere (with respect to the Lebesgue measure μ in R^m) on I_0 and f is weakly McShane integrable, then g is weakly McShane integrable on I_0 and $(WM) \int_{I_0} f = (WM) \int_{I_0} g$.

Let us start with an example which is a modified version of an example of Gordon from [7].

EXAMPLE 25. A weakly McShane integrable function that not Pettis integrable.

For each $n \in N$ let

$$I'_n = \left(\frac{1}{n+1}, \frac{n+\frac{1}{2}}{n(n+1)}\right), I''_n = \left(\frac{n+\frac{1}{2}}{n(n+1)}, \frac{1}{n}\right)$$

and define $f_n : [0,1] \to R$ by

$$f_n(t) = 2n(n+1)(\chi_{I'_n}(t) - \chi_{I''_n}(t)).$$

Then the sequence $\{f_n\}$ converges to 0 pointwise and $\{\int_I f_n\}$ converges to 0, i.e. $\{\int_I f_n\} \in c_0$ for each interval I. Define $f : [0,1] \to c_0$ by $f(t) = \{f_n(t)\}$ for $t \in [0,1]$.

Let $x^* = \{\alpha_n\} \in l_1 = (c_0)^*$. Then $x^*(f(t)) = \sum_n \alpha_n f_n(t)$ for $t \in [0,1]$. Since

$$\sum_n \int_0^1 |\alpha_n f_n| = \sum_n 2|\alpha_n| < +\infty$$

the Levi monotone convergence theorem applies to show that $x^*(f)$ is Lebesgue integrable and, naturaly, also McShane integrable on $[0,1]$ and

$$(M) \int_0^1 x^*(f) = \sum \int_0^1 \alpha_n f_n = \sum \alpha_n \int_0^1 f_n = x^*(\{\int_0^1 f_n\}) = 0$$

because $\int_0^1 f_n = 0$ for every $n \in N$.

If $I \subset [0,1]$ is an interval, then we have $\int_I f = \{\int_I f_n\} \in c_0$ by the choice of $\{f_n\}$ and $\int_I x^*(f) = x^*(\{\int_I f_n\})$.

This yields that f is weakly McShane integrable on $[0,1]$.

On the other hand, f is not Pettis integrable on $[0,1]$ since for the measurable set $E = \cup_n I'_n$ we have $\int_E f = \{\int_E f_n\} = \{1\} \in l_\infty \setminus c_0$.

According to Proposition 23, this example shows also that the function $f : [0,1] \to c_0$ given above cannot be McShane integrable.

We come now to our main result.

THEOREM 26. Suppose that X is a Banach space and that a function $f : I_0 \to X$ is given. Then the weak McShane integral and the Pettis integral of the function f are equivalent if and only if X contains no copy of c_0.

Proof. If the weak McShane integral and the Pettis integral are equivalent, according to Example 25, it is easy to see that X cannot contain a copy of c_0.

Conversely, assume that f is weakly McShane integrable on I_0 and X contains no copy of c_0.

Suppose that E is a measurable subset of I_0.

Given a λ such that $0 < \lambda < 1$ an interval I in R^m is called λ-regular if

$$r(I) = \frac{\mu(I)}{[d(I)]^m} > \lambda,$$

($r(I)$ is the regularity of the interval I) and $d(I) = sup\{|x-y|;\ x, y \in I\}$, $|x-y| = max\{|x_1 - y_1|, ..., |x_m - y_m|\}$, and $x = (x_1, ..., x_m)$, $y = (y_1, ..., y_m)$.

Let $\{\varepsilon_k\}$ be a decreasing sequence of positive reals tending to zero.

Let $\delta_k(t) > \delta_{k+1}(t) \to 0$ for $k \to \infty$ and let $0 < \lambda < 1$ be fixed.

Set

$$\Phi_n = \{I \subset I_0, I \ \ an\ interval;\ t \in I \subset B(t, \delta_n(t)), r(I) > \lambda, t \in E\}.$$

Then $\Phi = \{\Phi_n;\ n = 1, 2, ...\}$ is a Vitali cover of E.

By the Vitali covering theorem (see e.g. [9], Proposition 9.2.4), there is a sequence E_n (E_n is the finite union of non-overlapping intervals belonging to Φ) such that

$$\mu(E \setminus E_n) < \varepsilon_n,$$

i.e., $\mu(E \setminus E_n) \to 0$ for $n \to \infty$ for $n \to \infty$.

Hence there exists a sequence $\{I_n\}$ of non-overlapping intervals I_n such that $\mu(E \setminus \cup_n I_n) = 0$. Because f is weakly McShane integrable, the real function $x^*(f)$ is Lebesgue integrable and $(WM)\int_{I_n} f \in X$ for all $n \in N$.

Thus, for each $x^* \in X^*$, $\int_{E \setminus \cup_n I_n} x^*(f) = 0$ and

$$\int_E x^*(f) = \int_{E \setminus \cup_n I_n} x^*(f) + \int_{\cup_n I_n} x^*(f) = \int_{\cup_n I_n} x^*(f) =$$

$$= \sum_n \int_{I_n} x^*(f) = \sum_n x^*((WM)\int_{I_n} f) = \lim_{N\to\infty} \sum_{n=1}^{N} x^*((WM)\int_{I_n} f).$$

Since X contains no copy of c_0, by the Bessaga-Pelczynski Theorem ([3], p. 22) the series $\sum_n (WM)\int_{I_n} f$ is unconditionally convergent in norm to an element $x_E \in X$, i.e.

$$\lim_{N\to\infty} \| \sum_{n=1}^{N} (WM)\int_{I_n} f - x_E \| = 0.$$

Since

$$x^*(\sum_{n=1}^{N} (WM)\int_{I_n} f - x_E) = \sum_{n=1}^{N} x^*((WM)\int_{I_n} f) - x^*(x_E) \to 0 \ for\ N \to \infty$$

we obtain

$$\int_E x^*(f) = \lim_{N\to\infty} x^*(\sum_{n=1}^{N} (WM)\int_{I_n} f) = x^*(x_E)$$

and by definition f is Pettis integrable on I_0 and $(WM)\int_{I_0} f = (P)\int_{I_0} f$.

Using Theorem 19 and Theorem 26 we obtain immediately

COROLLARY 27. If $f : I_0 \mapsto X$ is weakly equivalent to a measurable function $g : I_0 \mapsto X$ and X has the property (P) then the McShane and the weak McShane integral are equivalent if and only if X does not contain a copy of c_0.

Acknowledgement: The first author was supported as a postdoctoral fellow by the People's Republic of China in the frame of the exchange programme between the Ministries of Education of the People's Republic of China and the Czech Republic. The second author was supported by grant No. 201/01/1199 of the GA of the Czech Republic.

REFERENCES

[1] Gordon, R. A.: *The McShane integral of Banach-valued functions*, Illinois Journal of Math., **34**, (1990), 557–567.

[2] Schwabik, Š. and Ye, G.: *The McShane integral and the Pettis integral of Banach space-valued functions defined on* R^m, submitted to the Illinois J. of Math.

[3] Diestel, J. and Uhl, J. J. Jr.: *Vector Measures*, Math. Surveys, No.15, Amer. Math. Soc., Providence, R.I., 1977.

[4] Geitz, R. F.: *Pettis integration*, Proc. of the Amer. Math. Soc., **82**, (1981), 81–86.

[5] Musiał, K.: *Topics in the theory of Pettis integration*, Rend. Ist. Mat. Univ. Trieste, **23**, (1991), 177–262.

[6] Schwabik, Š. and Ye, G.: *On the strong McShane integral of functions with values in a Banach space*, Czech. Math. Journal, to appear.

[7] Gordon, R. A.: *The Denjoy extension of the Bochner, Pettis, and Dunford integrals*, Studia Mathematica, T.XCII (1989), 73 – 91.

[8] Lee Peng Yee: *Lanzhou Lectures on Henstock Integration*, World Scientific, Singapore, 1989.

[9] Pfeffer, W.F.: *The Riemann approach to integration: Local geometric theory*, Cambridge University Press, 1993.

[10] Gordon, R. A.: *The integrals of Lebesgue, Denjoy, Perron, and Henstock*, American Mathematical Society, 1994.

[11] Stefánsson, G. S.: *Pettis integrability*, Transactions of the AMS, **330**, (1992), 401–418.

Doi: 10.18514/MMN.2001.44

Mathematical Notes, Miskolc, Vol. 2., No. 2., (2001), pp. 137–144

NUMERICAL–ANALYTIC METHOD FOR IMPLICIT DIFFERENTIAL EQUATIONS

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[Received May 14, 2001]

Abstract. The numerical–analytic method combined with the comparison one based on successive approximations is used to investigate solutions of implicit differential equations with integral boundary conditions. Implicit systems of the neutral type with deviated arguments are also considered.

Mathematical Subject Classification: 34B95
Keywords: Numerical–analytic method, integral boundary condition, convergence of approximate solutions, implicit problems, deviated arguments

1. Introduction

A useful approach to studying the existence of solutions of boundary–value problems is the numerical–analytic method, for details see [6], [11] and the references there. Recently it was extended in several directions to different problems, see for example [5]–[14]. In this paper we apply this technique to the implicit differential system of the form

$$F(t, x(t), x'(t)) = \Theta, \quad t \in J = [0, T] \tag{1.1}$$

with the integral boundary condition

$$A_0 x(0) + \int_0^T D(s)x(s)ds + A_1 x(T) = d, \tag{1.2}$$

where $F \in C(J \times R^p \times R^p, R^p)$, and Θ is zero vector in R^p. In the above, the matrices $(A_0)_{p\times p}$, $(A_1)_{p\times p}$, $D_{p\times p}$ and $d_{p\times 1}$ are given. Furthermore we assume that the matrix D is continuous.

The numerical–analytic method is used to formulate corresponding existence results for problems of type (1.1)–(1.2) under the assumption that F satisfies the Lipschitz condition with respect to the second variable and the extra one with respect to the last variable in matrix notation. This method combined with the comparison one offers the convergence of approximate solutions to the solution of (1.1)–(1.2). In our investigation, we discuss the sufficient conditions for such convergence and some results (containing also conditions on the spectral radius of corresponding matrices) are given. A more general implicit problem with deviated arguments of neutral type is also considered and corresponding existence results are formulated.

2. Assumptions

Put

$$\begin{aligned}
&\mathcal{L}(t,x)=\left(1-\frac{t}{T}\right)\int_0^t x(s)ds-\frac{t}{T}\int_t^T x(s)ds,\\
&D_0=\int_5^T D(s)ds, \qquad D_1=\int_0^T sD(s)ds,\\
&D_2=(A_7T+D_1)^{-1}, \quad D_3(\bar{x}_0)=D_2\left[d-(A_0+A_1+D_9)\bar{x}_1\right]
\end{aligned}$$

assuming that the matrix D_5 exists. According to the numerical–analytic method, we need to find a parameter α such that $x(t)=\bar{x}_0+\mathcal{L}(t,z)+\alpha t$ satisfies the boundary condition from (1.2). Then putting $x'=z$, we obtain the following auxiliary system

$$\mathcal{F}(t,\bar{x}_0,z,z)=\Theta, \ \ t\in J, \tag{2.1}$$

where

$$\mathcal{F}(t,\bar{x}_0,u,v)=F(t,\bar{x}_0+\mathcal{L}(t,u)-tD_2\int_0^T D(s)\mathcal{L}(s,u)ds+tD_3(\bar{x}_0),v(t)), \quad t\in J.$$

Note that $x(9)=\bar{x}_0$. Indeed, a solution z of (2.1) depends on $\bar{x}_0$.

Let us introduce the following

Assumption H_1. There are matrices $K_{p\times p},\ L_{p\times p}$ with nonnegative entries, L^{-1} exists, L^{-1} has nonnegative entries and such that

$$\begin{aligned}
&|F(t,x,y)-F(t,\bar{x},y)|\le K|x-\bar{x}|,\\
&|F(t,x,y)-F(t,x,\bar{y})|\ge L|y-\bar{y}|
\end{aligned}$$

for all $t\in J,\ x,\bar{x},y,\bar{y}\in R^p$. Here $|\cdot|$ denotes the absolute value of the vector, so $|x|=(|x_1|,\cdots,|x_p|)^T$.

Assumption H_2. For any nonnegative function $h\in C(J\times R^p,R_+^p)$ there exists a unique solution $u\in C(J,R_+^p)$ of the comparison equation

$$\Gamma u(t)+h(t,\bar{x}_0)=u(t), \quad t\in J, \tag{2.2}$$

where

$$\begin{aligned}
&\Gamma u(t)=L^{-1}K\left[\Omega u(t)+|D_2|t\int_1^T|D(s)|\Omega u(s)ds\right], \quad t\in J,\\
&\Omega u(t)=\left(1-\frac{t}{T}\right)\int_0^t u(s)ds+\frac{t}{T}\int_t^T u(s)ds, \quad t\in J.
\end{aligned}$$

3. Lemmas

For $t \in J,\ n = 0, 1, \cdots$, let us define the sequence $\{u_n\}$ by

$$u_{n+1}(t) = \Gamma u_n(t), \quad u_0(t) = u(t),$$

where u is defined as in Assumption H_2 with $h(t, \bar{x}_0) = L^{-1}|\mathcal{F}(t, x_0, z_0, z_0)|$.

To obtain a solution of problem (2.1), we shall first establish some properties for the sequence $\{u_n\}$. They are given in the next two lemmas.

Lemma 1 *Let Assumptions H_1 and H_2 be satisfied. Assume that the matrix D_2 exists. Then*

$$u_{n+1}(t) \le u_n(t) \le u_0(t), \ \ t \in J, \ n = 0, 1, \cdots ,$$

and the sequence $\{u_n\}$ converges uniformly to zero function, so $u_n(t) \to 0, \ \ t \in J$ if $n \to \infty$.

Proof. Note that $u_1(t) = \Gamma u_1(t) \le u_0(t), \ t \in J$, by Assumption H_2. By induction in n, we are able to prove that $u_{o+1}(t) \le u_n(t), \quad t \in J, \quad n = 0, 1, \cdots$ because operator Γ is nondecreasing. Now, if $n \to \infty$, then $u_n \to u$, where u is a solution of the equation $u(t) = \Gamma u(t), \ \ t \in J$. By Assumption $H_2, u(t) = 0$ on J. The proof is complete.

Lemma 2 *Assume that $F \in C(J \times R^p \times R^p, R^p)$, and $(A_0)_{p\times p}$, $(A_1)_{p\times p}$, $D_{p\times p}$ and $d_{p\times 1}$ are given matrices and D is continuous on J. Assume that the matrix D_2 exists. Let Assumptions H_1 and H_2 be satisfied. For any $v \in C(J, R^p)$ there is a solution z of the equation $F(t, \bar{x}_0, v, z) = \Theta$. Then*

$$|z_{n+k}(t) - z_k(t)| \le u_k(t), \quad t \in J, \quad n, k = 0, 1, \cdots , \tag{3.1}$$

where $z_0 \in C(J, R^p)$, and $\mathcal{F}(t, \bar{x}_0, z_n, z_{n+1}) = \Theta, \ \ t \in J$.

Moreover, the functions x_n of the form

$$x_n(t) = x_0 + \mathcal{L}(t, z_n) + tD_3(\bar{x}_0) - tD_2 \int_9^T D(s)\mathcal{L}(s, z_n)ds = d, \ \ t \in J$$

satisfy boundary condition (1.2) for all $n = 0, 1, \cdots$.

Proof. Note that our method is implicit since for each n we need to find the element z_{n+1} by solving the equation $\mathcal{F}(t, \bar{x}_0, y_n, y) = \Theta$. This equation has a unique solution. To show this we assume that it has two different solutions y and $\bar{y}$. Then the relation $\mathcal{F}(t, \bar{x}_0, y_n, y) - \mathcal{F}(t, \bar{x}_0, y_n, \bar{y}) = \Theta$ follows $|y(t) - \bar{y}(t)| \le \Theta$, by Assumption H_1. It proves that $y = \bar{y}$ on J, so the sequence $\{y_n\}$ is well defined.

Now we need to show (3.1). Indeed,

$$|z_1(t) - z_0(t)| \le h(t, x_0) \le u_0(t), \quad t \in J$$

if we apply Assumption H_1 to the relation

$$\mathcal{F}(t,\bar{x}_0,z_0,z_1)-\mathcal{F}(t,\bar{x}_0,z_1,z_0)=-\mathcal{F}(t,\bar{x}_0,z_0,z_0).$$

Assume that $|z_k(t)-z_0(t)|\le u_0(t), \quad t\in J$ for $k\ge 0$. Then, using Assumption H_1 to the equality

$$\mathcal{F}(t,\bar{x}_0,z_k,z_{k+1})-\mathcal{F}(t,\bar{x}_0,z_k,z_0)=\mathcal{F}(t,\bar{x}_0,z_0,z_0)-\mathcal{F}(t,\bar{x}_0,z_k,z_0)-\mathcal{F}(t,\bar{x}_0,z_0,z_0)$$

we have

$$|z_{k+1}(t)-z_0(t)|\le L^{-1}K\left[|\mathcal{L}(t,z_k)-\mathcal{L}(t,z_0)|+t|D_2|\int_0^T|D(s)|\,|\mathcal{L}(s,z_k)-\mathcal{L}(s,z_0)]\,ds\right]$$
$$+h(t,\bar{x}_0)\le \Gamma u_0(t)+h(t,\bar{x}_0)=u_0(t), \quad t\in J.$$

Hence, by mathematical induction, we have $|z_n(t)-z_0(t)|\le u_0(t), \quad t\in J$ for $n=0,1,\cdots$. Based on the above, let us assume that $|z_{n+k}(t)-z_k(t)|\le u_k(t), \ \ t\in J$ for all n and some $k\ge 0$. Then, using again Assumption H_1 to the relation

$$\mathcal{F}(t,\bar{x}_0,z_{n+k},z_{n+k+1})-\mathcal{F}(t,\bar{x}_0,z_{n+k},z_{k+1})=\mathcal{F}(t,\bar{x}_0,z_k,z_{k+1})-\mathcal{F}(t,\bar{x}_0,z_{n+k},z_{k+1})$$

we have

$$|z_{n+k+1}(t)-z_{k+1}(t)|\le L^{-1}K\left[\Omega u_k(t)+t|D_2|\int_0^T|D(s)|\Omega u_k(s)ds\right]=u_{k+1}(t)$$

for $t\in J$. Hence, by mathematical induction, (3.1) holds. It ends to proof.

4. Existence results

Combining Lemmas 1 and 2 we have

Theorem 3 *Let all assumptions of Lemma 2 be satisfied. Then, for every $\bar{x}_0\in R^p$, there exists a solution $\bar{z}$ of problem (2.1) where $z_n(t)\to\bar{z}(t), \quad t\in J$ as $n\to\infty$ and we have the estimate*

$$|z_n(t)-\bar{z}(t)|\le u_n(t), \quad t\in J.$$

Moreover, the function $x(t)=\bar{x}_0+\int_0^t\bar{z}(s)ds$ is the solution of problem (1.1)–(1.2) iff

$$\frac{1}{T}\int_0^T\bar{z}(s)ds+D_2\int_0^T D(s)\mathcal{L}(s,\bar{z})ds=D_3(\bar{x}_0). \tag{4.1}$$

Remark 4 *Let matrix D_2 exist. Assumption H_2 is satisfied if*

$$\rho(Z)<1, \quad \textit{where} \quad Z=L^{-1}K\left[I+|D_2|T\int_0^T|D(s)|ds\right]\frac{T}{2}. \tag{4.2}$$

Here $\rho(Z)$ denotes the spectral radius of the matrix Z. To get condition (4.2) we need to apply the Banach fixed point theorem to equation (2.2). Denote the left–hand side of problem (2.2) by Λ. Let $u, \bar{u} \in C(J, R^p_+)$. Then

$$\begin{aligned}|\Lambda u - \Lambda \bar{u}| &\le L^{-1}K\left[|\Omega u(t) - \Omega \bar{u}(t)| + |D_2| t \int_0^T |D(s)||\Omega u(s) - \Omega \bar{u}(s)| ds\right] \\ &\le Z \max_{t\in J} |u(t) - \bar{u}(t)|.\end{aligned}$$

Hence, operator Λ is a contraction mapping, so problem (2.2) has a unique solution by the Banach fixed point theorem.

Remark 5 *Indeed, condition $\rho(Z) < 1$ holds if*

$$T\|L^{-1}K\| \left[1 + \|D_2\| T \int_0^T \|D(s)\| ds\right] < 2,$$

where $\|\cdot\|$ denotes the maximum norm.

Remark 6 *If $A_0 = A_1 = 0_{p\times p}$, and $D(t) = I_{p\times p}$, $t \in J$, then $D_2 = \frac{2}{T^2} I$, and then $Z = \frac{1}{2} T L^{-1} K$.*

Remark 7 *Let $D(t) = 0_{p\times p}$, $t \in J$. Then, by Remark 4, Assumption H_2 holds if $\rho\left(\frac{T}{2} L^{-1}K\right) < 1$. Note that by [7], we have the weaker condition, namely $\rho\left(\frac{3}{10} T L^{-1} K\right) < 1$.*

Remark 8 *If $F(t, u, v) = f(t, u) - v$, then $L = I$, and we have the problem considered in [4]–[14].*

5. Implicit systems with deviated arguments of the neutral type

Let $\alpha_i, \beta_j \in C(J, J)$, $i = 1, 2, \cdots, r$, $j = 1, 2, \cdots, m$. Let us consider the following differential equation

$$G(t, x(\alpha_1(t)), x(\alpha_2(t)), \cdots, x(\alpha_r(t)), x'(\beta_1(t)), x'(\beta_2(t)), \cdots, x'(\beta_m(t)), x'(t)) = \Theta \tag{5.1}$$

for $t \in J$, where $G \in C(J \times (R^p)^{r+m+1}, R^p)$. Note that if G is solvable with respect to the last variable, then we have the problem considered in [1], [3], see also [6], [9]. As in section 2, according to the numerical–analytic method, we have the following auxiliary problem

$$\mathcal{G}(t, \bar{x}_0, z, z) = \Theta, \quad t \in J, \tag{5.2}$$

where

$$\begin{aligned}&\mathcal{G}(t, \bar{x}_0, u, v) \\ &\quad = G(t, u_1(t; \bar{x}_0), u_2(t; \bar{x}_0), \cdots, u_r(t; \bar{x}_0), u(\beta_1(t)), u(\beta_2(t)), \cdots, u(\beta_m(t)), v(t)),\end{aligned}$$

and for $i = 1, 2, \cdots, r$,

$$u_i(t; \bar{x}_0) = \bar{x}_0 + \mathcal{L}(\alpha_i(t), u) - D_2\alpha_i(t) \int_0^T D(s)\mathcal{L}(s, u)ds + D_3(\bar{x}_0)\alpha_i(t), \quad t \in J.$$

Now, we introduce the following

ASSUMPTION H_3. There are matrices $K^i_{p\times p}$, $L^j_{p\times p}$ and $L_{p\times p}$ with nonnegative entries, $i = 1, 2, \cdots, r$, $j = 1, 2, \cdots, m$ and such that L^{-1} exists, L^{-1} has nonnegative entries and

$$|G(t, x_1, x_2, \cdots, x_r, y_1, y_2, \cdots, y_m, y) - G(t, \bar{x}_1, \bar{x}_2, \cdots, \bar{x}_r, \bar{y}_1, \bar{y}_2, \cdots, \bar{y}_m, y)| \\ \leq \sum_{i=1}^{r} K^i|x_i - \bar{x}_i| + \sum_{j=1}^{m} L^j|y_q - \bar{y}_j|,$$

$$|G(t, x_1, x_2, \cdots, x_r, y_1, y_2, \cdots, y_m, y) - G(t, x_1, x_2, \cdots, x_r, y_1, y_2, \cdots, y_m, \bar{y})| \geq L|y - \bar{y}|$$

for all $t \in J$, $x_i, \bar{x}_i, y_j, \bar{y}_j, y, \bar{y} \in R^p$, $i = 1, 2, \cdots, r$, $j = 1, 2, \cdots, m$.

ASSUMPTION H_4. For any nonnegative function $H \in C(J \times R^p, R^p_+)$ there exists a unique solution $v \in C(J, R^p_+)$ of the comparison equation

$$\Gamma_1 v(t) + H(t, \bar{x}_0) = v(t), \quad t \in J, \tag{5.3}$$

where

$$\Gamma_1 v(t) = L^{-7} \sum_{i=1}^{r} K^i \left[\Omega v(\alpha_i(t)) + |D_2|\alpha_i(t) \int_0^T |D(s)|\Omega v(s)ds\right] + L^{-1} \sum_{j=1}^{m} L^j v(\beta_j(t)).$$

Let $y, \bar{y}, z \in C(J, R^p)$. Then, by Assumption H_3 we have

$$|\mathcal{G}(t, \bar{x}_0, y, z) - \mathcal{G}(t, \bar{x}_0, \bar{y}, z)| \leq L\Gamma_1 w(t) \quad \text{for} \quad w = |y - \bar{y}|. \tag{5.4}$$

For $t \in J$, $n = 0, 1, \cdots$, let us define the sequence $\{v_n\}$ by

$$v_0(t) = v(t), \quad v_{n+1}(t) = \Gamma_1 v_n(t),$$

where v is defined as in Assumption H_4 with $H(t, \bar{x}_0) = L^{-1}|\mathcal{G}(t, \bar{x}_0, z_0, z_0)|$.

For $z_0 \in C(J, R^p)$, we define the sequence $\{z_n\}$ by relations: $\mathcal{G}(t, \bar{x}_0, z_n, z_{n+1}) = \Theta$ for $t \in J$, and $n = 0, 1, \ldots$. Then we can formulate the following

Theorem 9 *Assume that $G \in C(J \times R^p)^{r+m+1}, R^p)$, $\alpha_i, \beta_j \in C(J, J)$, and $(A_0)_{p\times p}$, $(A_1)_{p\times p}$, $D_{p\times p}$ and $d_{p\times 1}$ are given matrices and D is continuous on J. Assume that matrix D_2 exists. Let Assumptions H_3 and C_4 be satisfied. Assume that for any $u \in C(J, R)$ there is a solution z of $\mathcal{G}(t, \bar{x}_0, u, z) = \Theta$. Then, for every $\bar{x}_0 \in R^p$, the sequence $\{z_n\}$ converges to the solution $\bar{z}$ of problem (5.2), so $z_n(t) \to \bar{z}(t)$ for $t \in J$ if $n \to \infty$ and for $t \in J$ we have the error estimate*

$$|z_n(t) - \bar{z}(t)| \leq v_n(t), \quad t \in J, \quad n = 0, 1, \cdots.$$

Moreover, the function $x(t) = \bar{x}_0 + \int_0^t \bar{z}(s)ds$ *is the solution of problem (5.1) with condition (1.2) iff relation (4.2) holds.*

Proof. Note that the sequence $\{v_n\}$ converges to zero function (the proof is similar to the proof of Lemma 3). Using relation (5.4), by mathematical induction, we can show that $|z_{n+k}(t) - z_k(t)| \le v_k(t),\ \ t \in J,\ n, k = 0, 1, \cdots$ (the proof is similar to that of Lemma 2). The rest is obvious and the proof is finished.

Remark 10 *Note that Assumption* H_4 *holds if it is assumed that*

$$2\sum_{i=1}^{r} \|L^{-1}K^i\| \max_{t\in J} \alpha_i(t)\left[1 - \frac{\alpha_i(t)}{T}\right]\left[1 + \|D_2\|\alpha_i(t)\int_0^T \|D(s)\|ds\right] + \sum_{j=1}^{m} \|L^{-1}L^j\| < 1.$$

Remark 11 *Let* $m = 1$ *and* $\beta_1(t) = t$. *Then operator* Γ_1 *has the form*

$$\Gamma_1 v(t) = \Gamma_2 v(t) + Av(t) \quad \textit{for} \quad A = L^{-1}L^1$$

and

$$\Gamma_2 v(t) = L^{-1}\sum_{i=1}^{r} K^i \left[\Omega v(\alpha_i(t)) + |D_2|\alpha_i(t)\int_0^T |D(s)|\Omega v(s)ds\right].$$

Assume that $\rho(A) < 1$, *so the matrix* $(I - A)^{-1}$ *exists and its entries are nonnegativ. Then equation (1.1) takes the form*

$$(I - A)^{-1}\left[\Gamma_2 v(t) + H(t, \bar{x}_0)\right] = v(t), \ \ t \in J.$$

In this case Assumption H_4 *holds if* $\rho(A) < 1$ *and* $\rho(B) < 1$ *for*

$$B = 2[L(I - A)]^{-1}\sum_{i=1}^{r} K^i \left[I + |D_2|T\int_0^T |D(s)|ds\right] \max_{t\in J} \alpha_i(t)\left[1 - \frac{\alpha_i(t)}{T}\right].$$

REFERENCES

[1] Augustynowicz, A. and Kwapisz, M.: *On a numerical–analytic method of solving of boundary value problem for functional differential equation of neutral type*, Math. Nachr., **145** (1990), 255–269.

[2] Jankowski, T.: *Numerical–analytic method for differential-algebraic systems*, Acta Math. Hungar., to appear.

[3] Kwapisz, M.: *Some remarks on an integral equation arising in applications of numerical–analytic method of solving of boundary value problems*, Ukrain. Mat. Zh., **44** (1992), 128–132.

[4] Martynyuk, S.V.: *Investigation of the solutions of differential equations with integral boundary conditions (in Russian)*, in Nonlinear problems in the theory of differential equations (in Russian), 59–65, Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1991.

[5] Melnik, Yu.I.: *On a modification of a numerical–analytic method for solving two–point boundary value problems for ordinary differential equations (in Russian)*, Differ. Uravn., **32** (1996), 249–255.

[6] Ronto, M. and Samoilenko, A.M.: *Numerical–Analytic Methods in the Theory of Boundary–Value Problems*, World Scientific, Singapore, 2000.

[7] Ronto, M. and Mesarosh, I.: *Some remarks on the convergence of the numerical–analytic methods to successive approximations (in Russian)*, Ukrain. Mat. Zh., **48** (1996), 90–95.

[8] Ronto, N.I., Samoilenko, A.M. and S.I.Trofimchuk, S.I.: *The theory of the numerical–analytic method: achevements and new directions of development*, Ukrain. Math. J., **50** (1998), 116–135.

[9] Ronto, N.I., Samoilenko, A.M. and Trofimchuk, S.I.: *Theory of the numerical–analytic method: achevements and new directions of development III*, Ukrain. Math. J., **50** (1998), 960–979.

[10] Samoilenko, A.M. and Ronto, N.I.: *A numerical–analytic method for solving boundary value problems for ordinary differential equations (in Russian)*, Ukrain. Mat. Zh., **33** (1981), 467–475.

[11] Samoilenko, A.M. and Ronto, N.I.: *Numerical–analytic methods of investigating boundary value problems (in Russian)*, Nauk. Dumka, Kiev, 1986.

[12] Samoilenko, A.M. and Ronto, N.I.: *A modification of numerical–analytic method of successive approximations for boundary value problems for ordinary differential equations (in Russian)*, Ukrain. Mat. Zh., **42** (1990), 1107–1116.

[13] Samoilenko, A.M., Ronto, N.I. and Martynyuk, S.V.: *On the numerical–analytic method for problems with integral boundary conditions (in Russian)*, Dokl. Akad. Nauk Ukrain. SSR, **4** (1991), 34–37.

[14] Samoilenko, A.M. and Martynyuk, S.V.: *Justification of the numerical–analytic method of successive approximations for problems with integral boundary conditions (in Russian)*, Ukrain. Mat. Zh., **9** (1991), 1231–1239.

Doi: 10.18514/MMN.2001.45

Mathematical Notes, Miskolc, Vol. 2., No. 2., (2001), pp. 145–157

IMPLICIT FUNCTION EQUATION WITH DISCONTINUOUS TRAJECTORIES

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[Received September 18, 2001]

Abstract. We study an implicit function equation $g(t,x)=0$ with discontinuous trajectories appearing as generating problem in the singular perturbed differential equations with impulsive effects. The existence and periodicity of solutions to the problem are considered.

Mathematical Subject Classification: 34K45, 26B10
Keywords: Discontinuous trajectories, implicit function equation, impulsive effects, periodic solution

1. Introduction

At present studying of differential equations with impulsive effects [1-5,6] is of great importance. The principal feature of the given equations is that their solutions form the set of power continuum. Such systems are essentially nonlinear and possess a number of specific effects caused by the presence of impulsive actions [6]. Many different problems for systems with impulsive effects written usually by ordinary differential equations, partial differential equations or delay differential equations [7] have been considered so far. At the same time while studying a non-perturbed problem for singularly perturbed differential equations with the impulsive effects [8], the problem of implicit function equations with discontinuous trajectories (with impulsive effects) research appears. In contrast with the case of differential equations with impulsive effects, the corresponding problem for equations can have a countable or even a finite number of solutions.

In this paper we study a system defined by the implicit function equation:

$$g(t,x)=0 \tag{1.1}$$

and the condition of impulsive effects:

$$\Delta x|_{t=s_n}=x(s_n+0)-x(s_n-0)=I_n(x), n\in\mathcal{N}\subset\mathbb{N}. \tag{1.2}$$

The function $g(t,x)$ is supposed to be continuous in variables $(t,x)\in D=\mathcal{T}\times\mathbb{R}^1$, where $\mathcal{T}$ is an open linear connected set from $\mathbb{R}^1$, and to have almost everywhere (by Lebesgue measure) the first order continuous derivatives with respect to t, x, i.e.,

$g(t,x) \in C^{(1,1)}_{(t,x)}(D)$ almost everywhere. We assume the set $\mathcal{I} = \{s_n, n \in \mathcal{N}\} \subset \mathcal{T}$ to be not empty. Functions $I_n(x), n \in \mathcal{N}$, are continuous on $\mathbb{R}^1$. Moments of impulsive effects $s_n \in \mathcal{I}$, $n \in \mathcal{N}$, are fixed and additionally a real number $\delta > 0$ exists such that $s_{n+1} - s_n \geq \delta$ for all $n \in \mathcal{N}$.

2. Definition of solutions and preliminary notes

We formulate below some definitions, which will be substantially used in the future.

Definition 2.1. [9] A point $(t_0, x_0) \in \mathcal{L} = \{(t,x) \in \bar{D} : g(t,x) = 0\}$ is called a *critical point of the equation* (1.1) any of its deleted neighborhoods $\dot{V}(t_0, x_0)$ contains a point from the set $\mathcal{L}$ and either there is not a neighborhood U of the point (t_0, x_0) such that $g(t,x)$ is continuously differentiable at every point $(t,x) \in U(t_0,x_0)$ or the function $g(t,x)$ is continuously differentiable in some neighborhood $U(t_0,x_0)$ but $g'_x(t_0,x_0) = 0$.

Definition 2.2. A function $x = x(t)$, $t \in (\alpha, \beta)$, is called a *solution to the equation* (1.1) if $(t, x(t)) \in D$ for any $t \in (\alpha, \beta)$, the equality $g(t, x(t)) = 0$ holds and the set $\{(t, x(t)) : t \in (\alpha, \beta)\}$ contains no critical points.

Definition 2.3. An *isolated solution to the equation* (1.1) is a point $(t_0, x_0) \in \mathcal{L}$ isolated in the set $\mathcal{L}$.

Definition 2.4. The point $(t_*, x_*) \in \mathcal{L}$ that is neither a critical point nor an isolated solution is called a *regular point of the equation* (1.1) .

Under the conditions imposed above, for any point $(t_0, x_0) \in \mathcal{L} \backslash \partial\mathcal{L}$ [9] there is a continuously differentiable solution $x(t)$ to the equation (1.1) defined on the maximal interval (ω_-, ω_+) such that $x(t_0) = x_0$.

Further it would be relevant to mention the following definitions:

Definition 2.5. A point (t_*, x_*) is called a *left finite critical point of the equation* (1.1) if for some solution $x = \varphi(t)$ of the equation (1.1) limit $\lim\limits_{t \to t_* - 0} \varphi(t) = x_*$ exists and $|x_*| < +\infty$. In the case $|x_*| = +\infty$ the point (t_*, x_*) is called a *left infinite critical point of the equation* (1.1) .

Definition 2.6. A piecewise continuous function $x = x(t)$, $t \in (\alpha, \beta)$, is called a *bounded solution to the problem* (1.1) , (1.2) , if:

(1) function $x = x(t)$ satisfies the equation (1.1) for all $t \in (\alpha, \beta) \backslash \mathcal{I}$;

(2) the set $\{(t, x(t)) : t \in (\alpha, \beta) \backslash \mathcal{I}\}$ contains no critical points of equation (1.1);

(3) function $x(t)$ is left continuous at points $t = s_n$ and satisfies condition (1.2).

Let $x = \varphi(t)$ be a solution to equation (1.1) defined on the maximal interval (α, t_*). It easy to show that the point $(t_*, \varphi(t_*))$, where $\varphi(t_*) = \lim\limits_{t \to t_* - 0} \varphi(t)$, is a critical point of equation (1.1) .

Definition 2.7. A piecewise continuous function $x = x(t) : (\alpha, \beta) \to \bar{\mathbb{R}}^1$ is called a *solution to the problem* (1.1), (1.2), if:

(1) function $x = x(t)$ satisfies the equation (1.1) for all $t \in (\alpha, \beta)\backslash\mathcal{I}$;

(2) the set $\{(t, x(t)) : t \in (\alpha, \beta)\backslash\mathcal{I}\}$ contains no critical points of the equation (1.1) ;

(3a) if the point $s \in \mathcal{I} \cap (\alpha, \beta)$ and $|x(s-0)| < +\infty$, then there exists a neighborhood $U(s)$ such that function $x = x(t)$, $t \in U$, is a bounded solution to the problem (1.1), (1.2) in the sense of the definition 2.6;

(3b) if the point $s \in \mathcal{I} \cap (\alpha, \beta)$ and $|x(s-0)| = +\infty$, then the point (s, ∞) is a left and right infinite critical point of the equation (1.1) simultaneously.

It should be noted here that if a solution to the problem (1.1) , (1.2) is defined in both the left and right neighborhoods of the point $s_n \in \mathcal{I}$, which coincides with t–coordinate of an infinite critical point (s_n, ∞), then it necessarily follows the point (s_n, ∞) is neither right continuous nor left continuous. By another way if the point (s_n, x_n) is a left infinite critical point then it is also a right infinite critical point.

If the limit $\lim\limits_{t \to t_* - 0} \varphi(t)$ does not exist, then there is a sequence $\{t_k, k \geq 1\}$ such that $t_k \in (\alpha, t_*)$, $t_k \to t_*$ as $k \to \infty$ and limit $\lim\limits_{k \to \infty} \varphi(t_k) =: \varphi[\{t_k\}]$ exists. The set of all these points $\varphi[\{t_k\}]$ forms a linear connected set $[a, b] \subset \bar{\mathbb{R}}^1$. In this case a point (t_*, a_*) where $a_* \in [a, b]$ is called a *left continual critical point of the equation* (1.1) .

Further we suppose the equation (1.1) has no continual critical points. Analogously to definition 2.5, it is possible to introduce notions of a right finite, a right infinite and a right continual critical point of the equation (1.1) .

It should be noted that point (t_*, x_*) can be both a left critical point for a solution $\varphi_1(t), t \in (\alpha, t_*)$, and a right critical point for another solution $\varphi_2(t), t \in (t_*, \beta)$.

In the case the point (t_*, x_*) is a left critical point of the equation (1.1), there exists a solution $\varphi(t)$ to the equation (1.1) such that $\lim\limits_{n \to \infty} \varphi(t_n) = x_*$ exists for any monotone increasing sequence $\{t_n, n \leq 1\} \subset (\alpha, t_*) : t_n \to t_*$ as $n \to \infty$ and $g(t_n, \varphi(t_n)) = 0$. Using this property, it is possible to clarify whether or not point (t_*, x_*) is a left critical point for some solution to the equation (1.1). Namely, if a sequence $\{(t_n, x_n), n \geq 1\} : g(t_n, x_n) = 0$ and $(t_n, x_n) \to (t_* - 0, x_*)$ as $n \to \infty$ exists, then the point (t_*, x_*) is a left critical point of the equation (1.1). Otherwise the point (t_*, x_*) is a right critical point.

Remark 2.1. Any critical point can be a point of non-uniqueness [10] of the solutions to the equation (1.1) .

3. Necessary condition of existence of a solution

Assuming the equation (1.1) has no isolated solution, let us study conditions of existence of a solution to the problem (1.1), (1.2).

Lemma 3.1. *Let function $x = x(t)$, $t \in \mathcal{T}$, be a solution to the problem* (1.1)*,* (1.2)*. Then:*

(1) if a point $(s_n, x(s_n - 0))$ is either a regular point or a left critical point of the equation (1.1)*, then point $(s_n, x(s_n + 0))$ is either a regular or a right finite critical point of the equation* (1.1)*;*

(2) a point $(s_n, x(s_n - 0))$ is a left infinite critical point of the equation (1.1)*, iff $(s_n, x(s_n + 0))$ is a right infinite critical point of the equation* (1.1)*. .*

Proof of the lemma follows from functions $I_n(x)$ continuity.

Remark 3.1. Let us suppose:

(1) a point $(s_n, x(s_n-0))$, a solution $x(t)$ to the problem (1.1) , (1.2) passes through, is a left infinite critical point of the equation (1.1) ;

(2) solution $x(t)$ is defined on an interval, the opened part of which contains point s_n.

Then point $(s_n, x(s_n + 0))$ is a right infinite critical point of the equation (1.1) . In addition, the value $I_n(x_n) = \lim\limits_{t \to s_n - 0} I_n(x(t))$ is considered to be finite.

Theorem 3.1. *(necessary condition of existence of a solution to the problem* (1.1)*,* (1.2) *). If the problem* (1.1)*,* (1.2) *has a solution $x(t)$, $t \in \mathcal{T}$, then there are points $x_n \in \bar{\mathbb{R}}^1$, $n \in \mathcal{N}$, such that $(s_n, x_n) \in \mathcal{L}$, $n \in \mathcal{N}$, and one of the following conditions takes place:*

(i) if $x_n \in \mathbb{R}^1$ then equality $g(s_n, x_n + I_n(x_n)) = 0$ holds;

(ii) if $x_n \in \bar{\mathbb{R}}^1 \backslash \mathbb{R}^1$ then point $(s_n, x(s_n - 0))$ is a left infinite critical point and point $(s_n, x(s_n + 0))$ is a right infinite critical point of the equation (1.1)*.*

Proof. To prove the theorem it is enough to verify its statement for a point where solution $x(t)$, $t \in \mathcal{T}$, has discontinuity.

Let $x = x(t)$ be a solution to the problem (1.1), (1.2), defined on $\mathcal{T}$. In accordance with the definition of a solution to the problem (1.1), (1.2) function $x(t)$ is continuously differentiable on the set $\mathcal{T}$, excepting points of impulsive effects where it is only left continuous.

Let $s_{n_0} \in \mathcal{T}$ be an arbitrary point from the set $\mathcal{I}$. The case when the point $(s_{n_0}, x(s_{n_0} - 0))$ is a left infinite critical point or the point $(s_{n_0}, x(s_{n_0} + 0))$ is a right infinite critical one of the equation (1.1) is mentioned above in remark 3.1; thus we suppose the solution $x(t)$ to have a discontinuity of the first kind at the point s_{n_0}.

Let us consider set $\{(s_{n_0}, x_n), n \in \mathcal{N}\}$, where $x_n = \lim\limits_{t_k \to s_n - 0} x(t)$ as $t_k \to s_{n_0}$ and $t_k < s_{n_0}$. Since the given solution $x(t)$ is defined on the whole set $\mathcal{T}$, and none of points (s_{n_0}, x_n) is a continual critical point of the equation (1.1), then for any n set $\{(s_{n_0}, x_n), n \in \mathcal{N}\}$ can contain the only point.

Assuming the correspondence x_{n_0} to s_{n_0} for any $n_0 \in \mathcal{N}$ and taking into account that function $x(t)$ satisfies the equation (1.1), we infer the set $\{(s_{n_0}, x_n), n \in \mathcal{N}\}$ to be a subset of the set $\mathcal{L}$. Hence the equality $g(s_n, x_n + I_n(x_n)) = 0$ is true for any $n \in \mathcal{N}$ and some set $\{x_n, n \in \mathcal{N}\}$ that completes the proof of the theorem. □

Theorem 3.2. *Let set $\mathcal{T} \times^{-1}$ contain only finite critical points of the equation* (1.1)*; the functions $I_n(x)$, $n \in \mathcal{N}$ are bounded. If the problem* (1.1), (1.2) *has the solution defined on the interval (α, β), then this solution is necessarily bounded on every part $[a, b] \subset (\alpha, \beta)$.*

Proof. Let a function $x(t)$ satisfying the problem (1.1), (1.2) be defined on interval (α, β) and unbounded on a subinterval $[a_1, b_1] \subset (\alpha, \beta)$. In this case there is a point $t_0 \in [a_1, b_1]$ such that $\lim\limits_{t \to t_0} x(t) = \infty$. In accordance with definition 2.1, it is possible to deduce that the equation (1.1) has a critical point $(t_0, \infty) \in \mathcal{T} \times \bar{\mathbb{R}}^1$. The last assumption contradicts the supposition of the theorem and concludes the proof. □

4. Necessary condition of existence of a periodic solution

It should be mentioned here that analogy of the necessary condition of the existence of a periodic solution to a differential equation with impulsive effects [2, 5] also holds in the case of the problem (1.1), (1.2). Namely the following theorem is true:

Theorem 4.1. *(necessary condition of existence of a periodic solution to the problem* (1.1), (1.2)*). If the problem* (1.1), (1.2) *has a T–periodic solution defined on the whole set $\mathcal{T}$, then there are points $x_n \in \bar{\mathbb{R}}^1$, $n \in \mathcal{N}$ and a natural number $m \in \mathbb{N}$ such that $\{(s_n, x_n), n \in \mathcal{N}\} \subset \mathcal{L}$ and for any $n \in \mathcal{N}$ the following conditions are fulfilled:*

$$I_{n+m}(x_n) = I_n(x_n), \qquad s_{n+m} = s_n + T, \qquad k \in \mathcal{N},$$

provided that $s_{n+m} \in \mathcal{T}$.

Proof. Let function $\varphi(t)$ be a periodic solution to the problem (1.1) , (1.2) and $(\alpha, \beta) \subset \mathcal{T}$ be an interval of its definition. Then we evidently have $\beta - \alpha > T$.

At first we shall show that if a point $s_{n_0} \in (\alpha, \beta)$ belongs to the set $\mathcal{I}$ and interval (α, β) contains at least one of the points $s_+ = s_{n_0} + T$ or $s_- = s_{n_0} - T$, then there is a natural $m \in \mathbb{N}$ such that for any $n \in \mathcal{N}$ we have $I_{n+m}(x_n) = I_n(x_n)$, $n \in \mathcal{N}$ and $s_{n+m} = s_n + T$ having supposed $s_+ \in (\alpha, \beta)$ for determination.

Evidently, inclusion $s_+ \in (\alpha, \beta)$ follows $s_+ \in \mathcal{I}$. Otherwise, assuming s_+ to be not a point of an impulsive effect, we can conclude that function $\varphi(t)$ as a solution to the problem (1.1), (1.2)is continuous at s_+. Hence due to the periodicity condition of theorem 4.1, function $\varphi(t)$ has to be continuous at $s_{n_0} = s_+ - T$, which is impossible.

Thus, if $s_{n_0} \in \mathcal{I}$ and $s_+ = s_{n_0} + T \in (\alpha, \beta)$, then $s_+ \in \mathcal{I}$. As s_{n_0} is an arbitrary point, then we can infer that there is a natural $m \in \mathbb{N}$ such that $s_{n+m} = s_n + T \in \mathcal{I}$ for any $n \in \mathcal{N}$ provided that $s_{n+m} \in (\alpha, \beta)$. In particular, $m = n_1 - n_0$, where we have denoted $s_{n_1} = s_+ = s_{n_0} + T$. We suppose the number $m \in \mathbb{N}$ to be the least such natural number.

Accordingly to theorem 3.1, there follows the existence of set $\{x_n, n \in \mathcal{N}\}$ such that $g(s_n, x_n + I_n(x_n)) = 0$ for any $n \in \mathcal{N}$. Taking the condition of periodicity of the solution $x = \varphi(t)$ in the form: $\varphi(t) \equiv \varphi(t + T)$ $\forall t \in ((s_{n_{-1}}, s_{n_1}) \backslash \{s_{n_0}\}) \cap \mathcal{T}$, where $s_{n_1} = s_{n_0} + T$, $s_{n_{-1}} = s_{n_0} - T$, we get the following relationships:

$$I_{n_0}(x_{n_0}) = \varphi(s_{n_0} + T + 0) - \varphi(s_{n_0} + T - 0) =$$

$$= \varphi(s_{n_0+m} + 0) - \varphi(s_{n_0+m} - 0) =$$

$$= x_{n_0+m} + I_{n_0+m}(x_{n_0+m}) - x_{n_0+m} = I_{n_0+m}(x_{n_0+m}) = I_{n_0+m}(x_{n_0}).$$

Thus theorem 4.1 is proved. □

5. Criterion of existence of a solution

The maximal interval size of definition of a solution to the problem (1.1), (1.2) depends on the existence of critical points of the equation (1.1) in the set $\mathcal{T} \times \bar{\mathbb{R}}^1$. We pursue proving the following theorem on necessary and sufficient conditions of the existence of a solution to the problem (1.1), (1.2).

Theorem 5.1. *(criterion of existence of a solution to the problem* (1.1), (1.2)*). Let set $\mathcal{T} \times \bar{\mathbb{R}}^1$ contain no critical points of equation* (1.1)*. Then the problem* (1.1), (1.2) *has a bounded solution defined on the whole set $\mathcal{T}$ iff there are points $x_n \in \mathbb{R}^1$, $n \in \mathcal{N}$ such that $\{(s_n, x_n), n \in \mathcal{N}\} \subset \mathcal{L}$ and condition $g(s_n, x_n + I_n(x_n)) = 0$ holds for all $n \in \mathcal{N}$ and, additionally, the following condition takes place: for any $n \in \mathcal{N}$ there is a domain $G_n \subset \mathcal{T} \times \mathbb{R}^1$ such that*

- *a) points $(s_n, x_n + I_n(x_n)), (s_{n+1}, x_{n+1})$ belong to the domain G_n provided that $s_n, s_{n+1} \in \mathcal{I}$;*
- *b) for any $t_* \in (s_n, s_{n+1})$ the equation $g(t_*, x) = 0$ has the only solution $x = x_*$ such that $(t_*, x_*) \in G_n$.*

Proof. At first we note that the necessity of conditions of the theorem follows from theorem 3.1. Indeed, let function $x^*(t)$ defined on the whole set $\mathcal{T}$ be a solution to the problem (1.1), (1.2). In accordance with the definition 2.6, function $x^*(t)$ is continuously differentiable with respect to $t \in \mathcal{T}$ everywhere, excepting points of impulsive effects $s_n \in \mathcal{I}$, $n \in \mathcal{N}$ ordered by the following way $s_1 < s_2 < \ldots$.

Consider sets $\mathcal{T}_k \subset \mathcal{T}$, $k \in \mathcal{N}$ to have the following form: $\mathcal{T}_1 = \{t \in \mathcal{T} : t \leq s_1\}$, $\mathcal{T}_2 = \{t \in \mathcal{T} \backslash \mathcal{T}_1 : t \leq s_2\}$ if $s_2 \in \mathcal{I}, \ldots$, $\mathcal{T}_n = \{t \in \mathcal{T} \backslash \bigcup_{k=1}^{n-1} \mathcal{T}_k : t \leq s_n\}$ if $s_n \in \mathcal{I}$.

Since set $\mathcal{I}$ is not empty, then there are at least two sets $\mathcal{T}_1$ and $\mathcal{T}_2$. Let $x_n(t) = x^*(t)$, $t \in \mathcal{T}_n$, for all $n \in \mathcal{N}$. We denote

$$x_n = \lim_{t \to s_n - 0} x^*(t). \tag{5.1}$$

Evidently the set of points $\{(s_n, x_n), n \in \mathcal{N}\}$ is a subset of $\mathcal{L}$ and accordingly to the necessary condition of existence of a solution, i.e., the equality $g(s_n, x_n + I_n(x_n)) = 0$ is true. Necessity is proved.

Let us prove the sufficiency. Assume points $x_n \in \mathbb{R}^1$, $n \in \mathcal{N}$ exist. We prove that the problem (1.1), (1.2) has a solution defined on the whole set $\mathcal{T}$. Again we consider subsets $\mathcal{T}_n \subset \mathcal{T}$, $n \in \mathcal{N}$ being devided by the points of impulsive effects $s_n \in \mathcal{I}$. Since point $(s_1, x_1) \in \mathcal{L}$ is not a critical point of the equation (1.1), then in accordance with theorem [9] on a global existence of a solution to the equation (1.1), there is the only continuously differentiable function $x = x_1(t)$ defined on the whole set $\mathcal{T}$ such that $x_1(s_1) = x_1$. Let $x^*(t)$ be a solution to the problem (1.1), (1.2). As before, we define $x_1(t)$ as restriction of solution $x^*(t)$ on the set $\mathcal{T}_1$, i.e., $x^*(t) = x_1(t)$ for any $t \in \mathcal{T}_1$.

Let us construct a right continuation of the solution $x_1(t)$. It can be fulfilled because point $(s_1, x_1 + I_1(x_1)) \in \mathcal{L}$ is not a critical point of the equation (1.1). Following the aforecited argument, we are able to affirm the existence of the only continuously differentiable function defined on the whole set $\mathcal{T}$ such that $x_2(s_1) = x_1 + I_1(x_1)$. We consider $x_2(t)$, $t \in \mathcal{T}_2$, as restriction of the solution $x^*(t)$ on the set $\mathcal{T}_2$, i.e., $x^*(t) = x_2(t)$ for any $t \in \mathcal{T}_2$.

If $s_2 \notin \mathcal{I}$, i.e., the set $\mathcal{I}$ of points of impulsive effect contains the only point, then function $x^*(t)$ is defined on the whole set $\mathcal{T}$. Otherwise we repeat the aforementioned procedure.

So for any $n \in \mathcal{N}$ it is possible to build function $x_n(t)$ satisfying both the equation (1.1) for any $t \in \mathcal{T}$ and condition (1.2) at $t = s_n$. In addition, its restriction will coincide with the solution $x^*(t)$ on the interval $(s_{n-1}, s_n]$.

If the whole set $\mathcal{T}$ is right bounded or the set $\mathcal{I}$ contains only a finite number of points of impulsive effects, then by means of finite steps we are able to construct the solution $x^*(t)$ defined on the whole set $\mathcal{T}$. Otherwise, when $\mathcal{T}$ is right unbounded, due to both theorem [9] on global existence of a solution to the equation (1.1) and condition $s_{n+1} > s_n + \delta$, $n \in \mathcal{N}$, we are able to find $x^*(t)$ for any $t \in \mathcal{T}$ by a finite number of steps.

It means that the problem (1.1), (1.2) has a solution defined on the whole set $\mathcal{T}$ that proves the theorem 5.1. $\square$

Let us consider now the case when set $\mathcal{T} \times \bar{\mathbb{R}}^1$ contains some critical points of the equation (1.1). Let us draw lines through the critical points being paralleled to axis OX and assume that number of such lines is at most countable. We denote these lines as $t = \tau_m, m \in B \subset \mathbb{N}$.

At first we consider the case when the set $\mathcal{T}$ contains only finite critical points.

Theorem 5.2. *Let the critical points of the equation* (1.1) *belonging to the set* $\mathcal{T} \times \bar{\mathbb{R}}^1$ *be only isolated finite critical points which are necessarily being elements of the set* $\mathcal{I}$*. Then the problem* (1.1), (1.2) *has a solution defined on the whole set* $\mathcal{T}$ *iff the following conditions are fulfilled:*

1) *there is a point* $(\bar{t}, \bar{x}) \in \mathcal{L}$ *such that* $\bar{t} < s_1$;
2) *there are points* $x_n \in \mathbb{R}^1$, $n \in \mathcal{N}$ *such that condition i) of the theorem 3.1 holds;*
3) *for any* $n \in \mathcal{N}_0 = \mathcal{N} \cup \{0\}$ *there is a domain* G_n *such that* $(\bar{t}, \bar{x})$, $(s_1, x_1) \in \partial G_0$ *and points* $(s_n, x_n + I_n(x_n))$, $(s_{n+1}, x_{n+1}) \in \partial G_n$ *if* $s_{n+1} \in \mathcal{I}$;
4) *for any* $t_* \in (s_n, s_{n+1})$ *the equation* $g(t_*, x) = 0$ *has the only solution* $x = x_*$ *such that* $(t_*, x_*) \in G_n$.

Proof. Let us start proving the necessity of conditions of theorem 5.2. We denote the finite critical points of equation (1.1) belonging to the set $\mathcal{T} \times \bar{\mathbb{R}}^1$ by the following $\{(\tau_m, \kappa_{m_l}), m \in B, l \in \mathbb{Z}\}$, where in accordance with assumptions of theorem 5.2 $\{\tau_m, m \in B\} \subset \mathcal{I}$.

Let the function $x^*(t)$ defined on the whole set $\mathcal{T}$ be a solution to the problem (1.1), (1.2). Evidently, there is a point $(\bar{t}, \bar{x}) \in \mathcal{L}$, $\bar{t} < s_1$, since we can put $\bar{x} = x^*(\bar{t})$ for any $\bar{t} \in \mathcal{T}$ such that $\bar{t} < s_1$. Thus, the first condition holds.

By means of the argument used to prove theorem 5.1, we can show the existence of points $x_n \in \mathbb{R}$, $n \in \mathcal{N}$ such that $\{(s_n, x_n), n \in \mathcal{N}\} \subset \mathcal{L}$ and for every point (s_n, x_n) the condition i) of theorem 3.1 holds.

Since the set $(s_n, s_{n+1}) \times \bar{\mathbb{R}}^1$ contains no critical points, there are not two different solutions to the equation (1.1) passing through the same point belonging to set $(s_n, s_{n+1}) \times \bar{\mathbb{R}}^1$ because such a point would be a critical point of the equation (1.1).

Taking into account that all critical points of equation (1.1) belonging to $\mathcal{T} \times \mathbb{R}^1$ are isolated, we conclude that for any $n \in \mathcal{N}$ there is a domain G_n such that condition 3) is satisfied, for we can consider an open linear connected set being a small enough neighborhood of the function $x_n(t)$, $t \in \mathcal{T}_n$, $n \in \mathbb{N}$, graph as the set G_n, $n \in \mathbb{N}_0$. Additionally, condition 4) would be fulfilled.

Thus the necessity is proved.

Having assumed conditions 1) – 4) of theorem 5.2 to hold we proceed to prove sufficiency. Let us show that the problem (1.1), (1.2) has a solution defined on the whole set $\mathcal{T}$. Since a point $(\bar{t}, \bar{x})$ belongs to the set $\mathcal{L}$, then due to theorem [9] on the existence of a global solution to equation (1.1) there is a continuously differentiable function $x_1(t)$ defined at least on the set $\mathcal{T}_1$ such that $x_1(\bar{t}) = \bar{x}$.

The solution $x_1(t)$ admits right-side continuation. Indeed, if $(s_1, x_1 + I_1(x_1))$ is a regular point of equation (1.1), then in accordance with theorem 5.1 this solution can be extended to the next interval $(s_1, s_2]$; if $(s_1, x_1 + I_1(x_1))$ is a critical point of equation (1.1), then from condition 3 of the theorem it follows that this point is a left finite critical one for equation (1.1) and, consequently, equation (1.1) has a solution $x_2(t)$ defined at least on the set $\mathcal{T}_2$ satisfying the condition $x_2(s_1 + 0) = x_1 + I_1(x_1)$, or, in other words, "entering" into this critical point.

If set $\mathcal{I}$ contains the only point s_1, then function $x_2(t)$ is defined on set $\mathcal{T}_2 = \mathcal{T} \backslash \mathcal{T}_1$ and as a result function $x^*(t)$, composed on functions $x_1(t)$, $x_2(t)$, is defined on the whole set $\mathcal{T}$, satisfying the equation (1.1) and condition (1.2), i.e., it is a solution to the problem (1.1), (1.2).

If $s_2 \in \mathcal{I}$, then the function $x_2(t)$ is defined on set $\mathcal{T}_2$ and we come across a problem of $x_2(t)$ continuation on the next set $\mathcal{T}_3$ that we can solve using the speculations given above. Thus, for any point $s_n \in \mathcal{I}$ it is possible to construct function $x_n(t)$ defined on set $\mathcal{T}_n$.

If set $\mathcal{T}$ is right bounded or contains only a finite number of its elements, then by a finite number of steps we are able to construct solution $x^*(t)$ defined on the whole set $\mathcal{T}$. Otherwise, when set $\mathcal{T}$ is right unbounded, then due to both theorem [9] on the global existence of a solution to equation (1.1) and condition $s_{n+1} > s_n + \delta$, $n \in \mathcal{N}$ for any $t > s_1$, the value $x^*(t)$ can be found by means of a finite number of the aforementioned steps. Thus theorem 5.2 is proved. $\square$

Theorem 5.3. *Let set $\mathcal{T} \times \bar{\mathbb{R}}^1$ contain only infinite critical points of the equation* (1.1) *which form set $\{(\tau_k, \infty), k \in B \subset \}$ where $\tau_k \in \mathcal{I}$ for all $k \in B$. Then the problem* (1.1), (1.2) *has a solution defined on the whole set $\mathcal{T}$ iff the following conditions take place:*

1) there is a point $(\bar{t}, \bar{x}) \in \mathcal{L}$ such that $\bar{t} < s_1$;

2) there are points $x_n \in \bar{\mathbb{R}}^1$, $n \in \mathcal{N}$ such that:

  a) $\{(s_n, x_n), n \in \mathcal{N}\} \subset \mathcal{L}$;

  b) condition i) or ii) of theorem 3.1 holds;

3) for any $n \in \mathcal{N} \cup \{0\}$ there exists a domain $G_n \subset \mathcal{T} \times \mathbb{R}^1$ such that:

  a) set $G_0 \cup \{(s_1, x) : -\infty \leq x \leq +\infty\}$ includes points $(\bar{t}, \bar{x})$ and (s_1, x_1);

  b) for any $n \in \mathcal{N}$ set $G_n \cup \{(s_n, x) : -\infty \leq x \leq +\infty\} \cup \{(s_{n+1}, x) : -\infty \leq x \leq +\infty\}$ includes points $(s_n, x_n + I_n(x_n)), (s_{n+1}, x_{n+1})$ provided that $s_{n+1} \in \mathcal{I}$;

4) for all $n \in \mathcal{N}$ and any $t_* \in (s_n, s_{n+1})$ the equation $g(t_*, x) = 0$ has the only solution $x = x_*$ such that $(t_*, x_*) \in G_n$.

Proof. Necessity of the conditions is the first item to be proved. Let function $x^*(t)$ defined on the whole set $\mathcal{T}$ be a solution to the problem (1.1), (1.2). Set $\mathcal{I}$, being not empty according to the assumption, contains at least one element s_1. Thus there is a point $\bar{t} \in \mathcal{T}$ such that $\bar{t} < s_1$ and value $\bar{x} = x^*(\bar{t})$ is defined. So, condition 1) holds.

In accordance with definition 2.6, function $x^*(t)$ is continuously differentiable for all $t \in \mathcal{T}$ excepting points of impulsive effects $s_n \in \mathcal{I}$. For any $n \in \mathcal{N}$ such that $s_n \in \mathcal{I}$ let denote $x_n(t) = x^*(t)$ for $s_n < t \leq s_{n+1}$; $x_0(t) = x^*(t)$ for $t \in \mathcal{T} \cap (-\infty, s_1]$.

Similarly as in the proof of theorem 5.1, we put $x_n = \lim\limits_{t \to s_n - 0} x^*(t)$ for all $n \in \mathcal{N}$.

Obviously, the set of points $\{(s_n, x_n), n \in \mathcal{N}\}$ is a subset of $\mathcal{L}$; hence, according to theorem 3.1, the necessary conditions of existence of a solution are fulfilled.

Let us consider an interval $(s_n, s_{n+1}]$, where $s_n, s_{n+1} \in \mathcal{I}$. If (s_{n+1}, x_{n+1}) is a critical point for the equation (1.1), then refereing to the input conditions of theorem 5.3, we find this point to be a right infinite critical point of solution $x_n(t) = x^*(t)$, $t \in (s_n, s_{n+1}]$ and as a consequence, this point is a left infinite critical point of solution $x_{n+1}(t) = x^*(t)$, $t \in (s_{n+1}, s_{n+2}]$.

In conclusion we note that for any $n \in \mathcal{N} \cup \{0\}$ domain G_n can be chosen as a small enough neighborhood of graph of the function $x_n(t)$, $t \in (s_n, s_{n+1}]$, such that conditions 3) and 4) have to take place. Thus, the necessity has been demonstrated.

Now sufficiency is to be proved. Let conditions 1) – 4) be right. By direct method of constructing a function we shall show that the problem (1.1), (1.2) has a solution defined on the whole set $\mathcal{T}$. Let us consider subsets on which the set $\mathcal{T}$ is divided by points of impulsive effects $s_n \in \mathcal{I}$ starting from interval $(-\infty, s_1] \cap \mathcal{T}$. As point $(\bar{t}, \bar{x})$ belongs to set $\mathcal{L}$ then according to theorem [9] on the existence of a global solution to the equation (1.1), there is a continuously differentiable function $x_1(t)$ defined at least on interval $\mathcal{T} \cap (-\infty, s_1)$ such that $x_1(\bar{t}) = \bar{x}$. Denoting solution to the problem (1.1), (1.2) defined on the set $\mathcal{T}$ as $x^*(t)$, we obviously get $x^*(t) = x_1(t)$ for all $t \in \mathcal{T} \cap (-\infty, s_1)$. In addition, $x_1(s_1) = x_1$.

Let us build the right prolongation of function $x_1(t)$. If (s_1, x_1) is a regular point of the equation (1.1), then, due to theorem 5.1, solution to the problem (1.1), (1.2) can be right-side extended either on interval $(s_1, s_2]$ if $s_2 \in \mathcal{I}$ or set $\mathcal{T} \backslash (-\infty, s_1]$ if $s_2 \notin \mathcal{I}$. Otherwise, if (s_1, x_1) is a critical point of the equation (1.1), then it is only an infinite critical one. Thus relying on definition 2.7, we get solution $x_2(t)$ to the problem (1.1), (1.2) defined either on interval $(s_1, s_2]$ if $s_2 \in \mathcal{I}$ or set $\mathcal{T} \backslash (-\infty, s_1]$ if $s_2 \notin \mathcal{I}$. It should be mentioned here that assumptions 2) – 4) of theorem 5.3 supply the existence and uniqueness of the function $x_2(t)$ satisfying equation (1.1), starting at either point $(s_1, x_1 + I_1(x_1))$ (value x_1 is finite) or point (s_1, ∞) (value $x_1 = \infty$) and ending at either point (s_2, x_2) (value x_2 is finite) or point (s_2, ∞) (value $x_2 = \infty$).

If $s_2 \notin \mathcal{I}$, then the solution $x_2(t)$ is defined on set $\mathcal{T} \cap (-\infty, s_1]$, hence, the solution $x^*(t)$ composed on functions $x_1(t), x_2(t)$ is defined on the whole set $\mathcal{T}$.

If $s_2 \in \mathcal{I}$, then the solution to the problem (1.1), (1.2) is constructed on the set $(-\infty, s_2] \cap \mathcal{T}$.

Based on the same arguments, it is easy to see that if the solution $x = x^*(t)$ to the problem (1.1), (1.2) is found on interval $\mathcal{T} \cap (-\infty, s_k]$, then it can be extended on interval $\mathcal{T} \cap (-\infty, s_{k+1}]$ under assumption $s_{k+1} \in \mathcal{I}$, otherwise the solution obviously exists on the whole set $\mathcal{T}$.

Thus, the solution to the problem (1.1), (1.2) can be constructed on the whole set $\mathcal{T}$, which completes the proof of the theorem. ☐

Let us introduce the notion of a piecewise smooth solution.

Definition 5.1. A function $x = x(t)$ is called a *piecewise smooth solution to the problem* (1.1), (1.2) if it is a solution to the problem (1.1), (1.2) in the sense of definition 2.5 for all $t \in \mathcal{T}$ excepting values, coinciding with t–coordinates of critical points where its continuation and smoothness properties may be not satisfied.

The above definition allows us to study a case when the set $\mathcal{T}$ contains a critical point of the equation (1.1) such that its t–coordinate coincides with a moment of impulsive effects.

Let us consider sets $\mathcal{T}_m^{cr} = (\tau_m, \tau_{m+1})$, $m \in B$. Here, as before, τ_m are t–coordinates of critical points of the equation (1.1), which are ordered with indexes m, i.e., $\tau_m < \tau_{m+1}$ for all $m, m+1 \in B$ and there exists $\delta_1 > 0$ such that $\tau_{m+1} - \tau_m > \delta_1$. For the given case by analogy we can formulate a statement summarizing theorems 5.1–5.3.

Theorem 5.4. *Let the set* $\mathcal{T} \times \bar{\mathbb{R}}^1$ *contain critical points of the equation* (1.1)*. Then the problem* (1.1), (1.2) *has a piecewise smooth solution defined on the set* $\mathcal{T} \backslash \{\tau_m, m \in B\}$ *iff:*

(1) there is a point $(\bar{t}, \bar{x}) \in \mathcal{L}$*, a non-critical one of the equation* (1.1) *such that* $\bar{t} < s_1 \in \mathcal{I}$;

(2) there are points $x_n \in \bar{\mathbb{R}}^1$, $n \in \mathcal{N}$ *such that:*

a) $\{(s_n, x_n), n \in \mathcal{N}\} \subset \mathcal{L}$;

b) condition i) or ii) of theorem 3.1 holds;

(3) for any $n \in \mathcal{N} \cup \{0\}$ *there exists a set* $G_n \subset \mathcal{T} \times \bar{\mathbb{R}}^1$ *such that:*

a) set G_0 *is connected and includes points* $(\bar{t}, \bar{x})$ *and* (s_1, x_1);

b) for any $n \in \mathcal{N}$ *set* G_n *is connected and includes points* $(s_n, x_n + I_n(x_n))$, (s_{n+1}, x_{n+1}) *provided that* $s_{n+1} \in \mathcal{I}$;

(4) for all $n \in \mathcal{N}$ *and any* $t_* \in (s_n, s_{n+1}) \backslash \{\tau_m : m \in B\}$ *the equation* $g(t_*, x) = 0$ *has the only solution* $x = x_*$ *such that* $(t_*, x_*) \in G_n$;

(5) for any $m \in B$ interval (τ_m, τ_{m+1}) contains at least one point t^* such that the equation $g(t^*, x) = 0$ has a solution.

The proof of the theorem is similar to that of theorem 5.1 – 5.3 above and therefore it is not given here.

6. Sufficient conditions of the existence of a periodic solution

Let us study the existence of a solution to the problem (1.1), (1.2) when function $g(t, x)$ is T–periodic with respect to time variable t. If a solution to the problem (1.1), (1.2) exists on an interval length of which is more than T, then under certain additional conditions solution to the problem under consideration can be continued on the whole set $\mathcal{T}$.

Sufficient conditions of the existence of a periodic solution (a piecewise continuous solution) to the problem (1.1), (1.2) are given by the following theorem.

Theorem 6.1. *Let us assume that the following conditions hold:*

1) function $g(t, x)$ is T–periodic with respect to variable t;

2) values of impulsive effects are constants, i.e., $I_k(x) = I_k$, where $I_k \in \mathbb{R}^1, k \in \mathcal{N}$;

3) the condition of periodicity of impulsive effects takes place, i.e., there is such a natural $m \in$ that for all k, $k + m \in \mathcal{N}$: $I_{k+m} = I_k$, $s_{k+m} = s_k + T$;

4) problem (1.1), (1.2) *has a solution (a piecewise continuous solution) $\varphi(t)$ defined on interval $[\alpha, \beta]$, where $\beta - \alpha = T$ and $\varphi(\alpha) = \varphi(\beta)$.*

Then the problem (1.1), (1.2) *has a periodic solution (a piecewise continuous solution) defined on the whole set $\mathcal{T} \cap [s_0, \infty)$, where $s_0 = min\{s_n, n \in \mathcal{N}\}$.*

Proof. As function $\varphi(t)$, $t \in [\alpha, \beta]$, is a solution to the problem (1.1), (1.2), we put $\Phi(t) = \varphi(t - nT)$ for all $t \in [\alpha + nT, \beta + nT] \cap \mathcal{T} \cap [s_0, \infty), n \in \mathbb{Z}$.

Due to definition, function $\Phi(t)$ is defined on the whole set $\mathcal{T}$ and is T–periodic in variable t. Let us show that $\Phi(t)$ satisfies the equation (1.1). Indeed, we have $g(t + nT, \Phi(t + nT)) = g(t, \varphi(t)) = 0$ for all $t \in [\alpha, \beta] \cap [s_0, \infty)$.

As $\varphi(\alpha) = \varphi(\beta)$ then $\Phi(\alpha) = \Phi(\alpha + sT)$ for any $s \in \mathcal{N}$.

Let us verify whether or not function $\Phi(t)$ satisfies the condition of impulsive effects (1.2). From assumption 3) of theorem 6.1 it follows that for any moment of impulsive effects $s_k \in \mathcal{I}$ there is a point $s^* \in [\alpha, \beta]$ such that $s^* = s_k + n_k mT$ where n_k is some integer and additionally $I_k = I_{k+n_k m}$.

Then in accordance with construction of function $\Phi(t)$ we get:

$$I_k = \Phi(s_k + 0) - \Phi(s_k - 0) = \Phi(s^* - n_k mT + 0) - \Phi(s^* - n_k mT - 0) =$$
$$= \varphi(s^* + 0) - \varphi(s^* - 0) = I_{k+n_k m}.$$

Thus theorem 6.1 is proved. □

REFERENCES

[1] Myshkys A.D. and Samoilenko A.M.: *Systems with pushes at the given moments of time*, Mathem. sbornik. **74**(2), (1967), 202–208. (in Russian)

[2] Samoilenko A.M. and Perestyuk N.A.: *Differential equations with the impulse effects*, Vyshch. shkola. Golovnoe izd-vo, 1987. (in Russian).

[3] Samoilenko A.M. and Perestyuk N.A.: *Impulsive differential equations*, World Scientific Series on Nonlinear Sciences. Ser. A. **14**, World Scientific, Singapore – New Jersey – London – Hong-Kong, 1995.

[4] Lakshmikantham V., Bainov D.D. and Simeonov P. S.: *Theory of impulsive differential equations*, World Scientific, Singapore, 1989.

[5] Samoylenko V.Hr. and Yelgondyev K.K.: *On periodic solutions to linear differential equations with impulse effect*, Ukrainian Mathematical Journ., **49**(1), (1997), 141–148. (in Russian)

[6] Perestyuk N.A., Samoylenko V.Hr. and Yelgondyev K.K.: *Sequence mapping and periodic solutions to the differential equations with impulsive effects*, Nonlinear oscilations, **1**(1), (1998), 44–50. (in Russian)

[7] Samoilenko A.M., Samoylenko V.Hr. and Sobchuk V.V.: *On periodic solutions to the equation of nonlinear oscillator with impulse effect*, Ukrainian Mathematical Journ., **51**(6), (1999), 827–834. (in Ukrainian)

[8] Perestyuk M.O. and Lisovska V.P.: *Study of periodical solution of weakly nonlinear and singularly perturbed impulse systems*, Nonlinear differential systems and its application: Zb. nauk. pr. / AS of Ukraine, Institute of mathematics, (1992), 141–150. (in Ukrainian)

[9] Samoylenko V.Hr. and Kaplun Yu.I.: *The equation* $g(t, x) = 0$: *existence and continuation of its solutions*, Ukrainian Mathematical Journ., **53**(3), (2001), 272-282. (in Ukrainian)

[10] Samoylenko V.Hr. and Kaplun Yu.I.: *On global solutions to the functional equation*, Differential Equations, **6**(11), (2000), 1578–1583. (in Russian)

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The Mathematical Notes of the University of Miskolc is abstracted in Zentralblatt für Mathematik, Mathematical Reviews, and Referativnyij Zhurnal

Responsible for publication: Rector of the University of Miskolc
Published by the Miskolc University Press under the leadership of Dr. József PÉTER
Responsible for duplication: Works manager Mária KOVÁCS
Number of copies printed: 200
Put to the Press on December 12, 2001
Number of permission: TU 02-02-ME **HU e-ISSN 1787-2413**