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AN APPROXIMATION PROCESS OF KANTOROVICH TYPE

OCTAVIAN AGRATINI

Faculty of Mathematics and Computer Science, Babeş-Bolyai University
Kogălniceanu 1, 3400 Cluj-Napoca, Romania
`agratini@math.ubbcluj.ro`

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Abstract. The paper is devoted to the study of an approximation process on an unbounded interval representing an integral form in Kantorovich sense of K. Balazs operators. We establish the degree of approximation in some function spaces pointing out the relationship between the local smoothness of functions and the local approximation. By using the modulus of variation, the approximation property in discontinuity points is also examined.

Mathematical Subject Classification: 41A36

Keywords: Kantorovich-type operator, Bohman-Korovkin theorem, modulus of smoothness of first order, modulus of variation, local $Lip\alpha$ ($0 < \alpha \leq 1$)

1. Introduction

In [1] Katalin Balázs introduced and considered some approximation properties of the discrete linear operators defined by

$$(R_n f)(x) = \frac{1}{(1 + a_n x)^n} \sum_{k=0}^n \binom{n}{k} (a_n x)^k f\left(\frac{k}{b_n}\right), \quad x \geq 0, \quad n \in \mathbb{N}, \quad (1.1)$$

where a_n and b_n are suitably chosen positive numbers, independent of x , and the function f belongs to $[0, \infty)$. In the particular case

$$a_n = n^{\beta-1}, \quad b_n = n^\beta, \quad n \in \mathbb{N}, \quad 0 < \beta < 1, \quad (1.2)$$

the above operators will be denoted by $R_n^{[\beta]}$.

In [1], for the operators defined by (1.1) a Voronovskaja type formula was given under the hypothesis that $a_n = b_n/n \rightarrow 0$ and $\sqrt{n}/b_n \rightarrow 0$ ($n \rightarrow \infty$). In the same paper, for $R_n^{[2/3]}$ operators convergence theorems and the convergence of their derivatives to the derivative of the function were proved.

In [2], for $R_n^{[\beta]}$ operators, $0 < \beta \leq 2/3$, weighted estimates were given and certain questions of the uniform convergence of $R_n f$ on $(0, \infty)$ were also developed.

In his extensive paper [3], V. Totik settled the saturation properties of $R_n^{[\beta]} f$, $0 < \beta < 1$, and proved a general convergence theorem for R_n -like rational functions.

In 1989, for $R_n^{[\beta]}$, $0 < \beta < 1$, Biancamaria Della Vecchia [4] obtained some preservation properties and weak asymptotic relations generalizing the result of Totik. Regarding the same operators, the most recent paper is due to Ulrich Abel and B. Della Vecchia [5] who obtained the complete asymptotic expansion for $R_n^{[\beta]}f$ as n tends to infinity.

The aim of this paper is to investigate a generalization of R_n operators in Kantorovich sense. In the next section we present the integral extension and in the third part some approximation properties are revealed, establishing the rate of pointwise convergence for various classes of functions such as continuous functions, local *Lip α* ($0 < \alpha \leq 1$) functions and locally bounded functions having the one-sided limits at a given $x > 0$. In our investigation we use the modulus of smoothness of first order and the Chanturiya modulus of variation.

2. An integral extension

Throughout the paper e_n represents the monomial $e_n(x) = x^n$, where n belongs to $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Let us denote by $M_{loc}([0, \infty))$ the class of all functions measurable on $[0, \infty)$ and bounded on every compact subinterval of $[0, \infty)$.

Concerning the operators R_n , $n \in \mathbb{N}$, defined by (1) we consider that $b_n = na_n > 0$ holds for every $n \in \mathbb{N}$, a requirement already formulated in some previous papers. Now we modify these operators into integral form operators by replacing $f(k/b_n)$ with an integral mean of $f(x)$ over a small interval named $I_{n,k} = \left[\frac{k}{na_n}, \frac{k+1}{na_n}\right]$, as follows

$$(K_n f)(x) = na_n \sum_{k=0}^n r_{n,k}(x) \int_{I_{n,k}} f(t) dt, \quad x \geq 0, \quad n \in \mathbb{N}, \quad (2.1)$$

where $f \in M_{loc}([0, \infty))$ and

$$r_{n,k}(x) = \binom{n}{k} (a_n x)^k (1 + a_n x)^{-n}, \quad 0 \leq k \leq n. \quad (2.2)$$

We can see that K_n is a linear positive operator and it is stable in the Fejér-sense, i.e. $\inf_{t \in I_n} f(t) \leq (K_n f)(x) \leq \sup_{t \in I_n} f(t)$, $x \geq 0$, where $I_n = \left[0, \frac{n+1}{na_n}\right]$. Since

$\sum_{k=0}^n r_{n,k}(x) = 1$, we obtain

$$(K_n e_0)(x) = 1, \quad x \geq 0, \quad n \in \mathbb{N}, \quad (2.3)$$

in other words, K_n reproduces the constants.

Remarks. (i) Like any linear operators of discrete type, R_n , $n \in \mathbb{N}$, are not suitable for approximation to discontinuous functions of a general type because they

use information about f which comes from some certain points in the domain of f . On the other hand, the operators K_n , $n \in \mathbb{N}$, are constructed based on integral means of f over small intervals, so we may obtain better results in approximating the discontinuous functions.

(ii) We can denote more explicitly $K_n f$ by $K_n(a_n, \mathbf{r}_n, f; \cdot)$ where $\mathbf{r}_n = (r_{n,k})_{0 \leq k \leq n}$. We consider the function $\sigma : [0, 1) \rightarrow [0, \infty)$, $\sigma(t) = t/(1-t)$ and $\tilde{\mathbf{r}}_n = (\tilde{r}_{n,k})_{0 \leq k \leq n}$ where $\tilde{r}_{n,k}(x) := (r_{n,k} \circ \sigma)(x)$, $0 \leq x < 1$, and $\tilde{r}_{n,k}(1) := \delta_{n,k}$ (Kronecker symbol). If we choose $a_n = 1$, then by using (2.1) we obtain

$$K_{n+1}(1, \tilde{\mathbf{r}}_n, f; x) = (n+1) \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) dt,$$

which represents the n -th Kantorovich operator, a well-known approximation process in spaces of integrable functions. For more details we refer e.g. to the monograph [6; 5.3.7].

(iii) We can write the operator K_n as a singular integral of the type

$$(K_n f)(x) = \int_0^\infty k_n(x, t) f(t) dt, \quad x \geq 0,$$

where k_n is a non-negative kernel defined as follows

$$k_n(x, t) = na_n r_{n,k}(x) \text{ for } x \geq 0, k/(na_n) < t \leq (k+1)/(na_n) \text{ and}$$

$$k_n(x, t) = 0 \text{ for } x \geq 0, t \in \{0\} \cup \left(\frac{n+1}{na_n}, \infty\right).$$

Using (2.3) it is obvious that our kernel satisfies $\int_0^\infty k_n(x, t) dt = 1$.

Example. We choose $a_n = n^{\beta-1}$, $n \in \mathbb{N}$, where $0 < \beta < 1$ is fixed. The operator defined by (2.1) becomes the integral analogue (named $K_n^{[\beta]}$) of $R_n^{[\beta]}$ defined by (1.1) and (1.2). More precisely, we have

$$(K_n^{[\beta]} f)(x) = \frac{n^\beta}{(1 + n^{\beta-1}x)^n} \sum_{k=0}^n \binom{n}{k} (n^{\beta-1}x)^k \int_{k/n^\beta}^{(k+1)/n^\beta} f(t) dt, \quad x \geq 0. \quad (2.4)$$

3. Some approximation properties of the operators K_n

Setting $T_i(y) = \sum_{k=0}^n k^i \binom{n}{k} y^k$, $i \in \mathbb{N}_0$, the following elementary identities hold true

$$T_1(y) = ny(1+y)^{n-1} \quad \text{and} \quad T_2(y) = n(n-1)y^2(1+y)^{n-2} + T_1(y), \quad n \geq 1. \quad (3.1)$$

Taking into account (2.1) and (3.1), we have

$$(K_n e_1)(x) = \frac{1}{2na_n} \left(\frac{2T_1(a_n x)}{(1 + a_n x)^n} + 1 \right) = \frac{x}{1 + a_n x} + \frac{1}{2na_n},$$

and

$$\begin{aligned} (K_n e_2)(x) &= \frac{1}{3n^2 a_n^2} \left(\frac{3T_2(a_n x) + 3T_1(a_n x)}{(1 + a_n x)^n} + 1 \right) = \\ &= \frac{n-1}{n} \frac{x^2}{(1 + a_n x)^2} + \frac{2x}{na_n(1 + a_n x)} + \frac{1}{3n^2 a_n^2}, \end{aligned}$$

respectively.

Further, we consider $\mathcal{M}_{n,s}(x) := (K_n(e_1 - x e_0)^s)(x)$

$_{n,s}(x) := (K_n(e_1 - x e_0)^s)(x)$, $s \in \mathbb{N}$, representing the central moment of order s for K_n operator. Taking advantage of the identities regarding $K_n e_j$, $j \in \{0, 1, 2\}$, we obtain the following result.

Lemma 1. *If the operators K_n , $n \in \mathbb{N}$, are defined by (2.1), then we get*

$$\begin{aligned} i) \quad _{n,1}(x) &= -\frac{a_n x}{1 + a_n x} + \frac{1}{2na_n}, \\ ii) \quad _{n,2}(x) &= \frac{na_n^2 x^2 - 1}{n(a_n x + 1)^2} x^2 + \frac{1 - a_n x}{na_n(1 + a_n x)} x + \frac{1}{3n^2 a_n^2}. \end{aligned}$$

By using these relations we can state

Lemma 2. *Let K_n , $n \in \mathbb{N}$, be defined by (2.1) such that $a_n \rightarrow 0$, $na_n \rightarrow \infty$ as $n \rightarrow \infty$. Then we have*

$$\lim_{n \rightarrow \infty} _{n,j}(x) = 0, \quad \text{for every } x \geq 0 \quad \text{and} \quad j \in \{1, 2\}.$$

Theorem 1. *Let K_n , $n \in \mathbb{N}$, be defined by (2.1). If $a_n \rightarrow 0$, $na_n \rightarrow \infty$ as $n \rightarrow \infty$, then for every $f \in C([0, \infty))$ one has*

$$\lim_{n \rightarrow \infty} K_n f = f \quad \text{uniformly on any compact } K \subset [0, \infty).$$

Proof. Our assertion results directly from the well-known theorem of Bohman-Korovkin, relation (2.3) and Lemma 2 as well. $\square$

We are going to study the degree of approximation in terms of ω_1 , the modulus of smoothness of first order, defined as

$$\omega_1(f; s) = \sup\{|f(x) - f(y)| : x \geq 0, y \geq 0, |x - y| \leq s\}.$$

Theorem 2. *If the operators K_n , $n \in \mathbb{N}$, are defined by (2.1), then for each $x \geq 0$ the following inequality*

$$|(K_n f)(x) - f(x)| \leq 2\omega_1(f; \delta_{n,x})$$

holds, where

$$\delta_{n,x} = \left(\max \left\{ a_n x, \frac{1}{n} \right\} x^2 + \frac{x}{na_n} + \frac{1}{3n^2 a_n^2} \right)^{1/2}$$

and $f \in M_{loc}([0, \infty))$.

Proof. Since (2.3) holds and K_n is a positive operator, we can write

$$|(K_n f)(x) - f(x)| \leq a n_n \sum_{k=0}^n r_{n,k}(x) \int_{k/(na_n)}^{(k+1)/(na_n)} |f(t) - f(x)| dt. \quad (3.2)$$

On the other hand, the definition of the modulus of smoothness $\omega_1(f; \cdot)$ implies

$$|f(t) - f(x)| \leq \omega_1(f; |t - x|) \leq (1 + \delta^{-2}(t - x)^2) \omega_1(f; \delta), \quad \delta > 0. \quad (3.3)$$

We have also used both the fact that $\omega_1(f; \cdot)$ is an increasing function and the property $\omega_1(f; \lambda \delta) \leq (1 + \lambda^2) \omega_1(f; \delta)$ for every $\lambda > 0$ and $\delta > 0$.

Substituting (3.3) in (3.2) and taking into account (2.3) it results

$$|(K_n f)(x) - f(x)| \leq (1 + \delta^{-2}_{n,2}(x)) \omega_1(f; \delta), \quad \delta > 0. \quad (3.4)$$

At this step we consider $\delta = \frac{1}{n,2}(x)$. If we use Lemma 1 (ii) and the following relations

$$\left| \frac{a_n x - 1}{a_n x + 1} \right| \leq 1, \quad -\frac{1}{n} \leq \frac{n a_n^2 x^2 - 1}{n(a_n x + 1)^2} < a_n x, \quad n \in \mathbb{N}, \quad x \geq 0, \quad (3.5)$$

then we deduce that $_{n,2}(x) \leq \max \left\{ a_n x, \frac{1}{n} \right\} x^2 + \frac{x}{n a_n} + \frac{1}{3 n^2 a_n^2} := \delta_{n,x}^2$. The desired result follows. $\square$

Corollary. If the operators $K_n^{[\beta]}$, $n \in \mathbb{N}$, $0 < \beta < 1$, are defined by (6), then for each $x \geq 0$ and $\lambda > 0$ we have

$$|(K_n^{[\beta]} f)(x) - f(x)| \leq (1 + n^{-\lambda}) \omega_1(f; \theta_{n,x}),$$

where $\theta_{n,x} = (\max\{n^{\lambda+\beta-1}x, n^{\lambda-1}\}x^2 + n^{\lambda-\beta}x + n^{\lambda-2\beta})^{1/2}$ and $f \in M_{loc}([0, \infty))$.

Proof. We apply Theorem 2 for the special case $a_n = n^{\beta-1}$. Also, in (3.4) we choose $\delta = n^{\lambda/2} \frac{1}{n,2}(x)$ and further $\theta_{n,x} = n^{\lambda} \delta_{n,x}^2$. The proof is complete. $\square$

Remark. If the real number λ satisfies $0 < \lambda < \min\{1 - \beta, \beta\}$, then $\theta_{n,x} \rightarrow 0$ as $n \rightarrow \infty$.

Theorem 3. Let K_n be given by (2.1), $0 < \alpha \leq 1$ and E be any subset of $[0, \infty)$. If f is local Lip α on E , i.e.,

$$|f(x) - f(y)| \leq M_f |x - y|^\alpha, \quad (\forall) (x, y) \in [0, \infty) \times E, \quad (3.6)$$

then we have

$$|(K_n f)(x) - f(x)| \leq M_f \left(\varepsilon_n(x) + \frac{1}{(n a_n)^\alpha} + 2 d^\alpha(x, E) \right), \quad (3.7)$$

where M_f is a constant depending only on α and f ,

$$\varepsilon_n(x) = \left(\max \left\{ a_n x, \frac{1}{n} \right\} x^2 \right)^{\alpha/2} + \left(\frac{x}{na_n} \right)^{\alpha/2}$$

and $d(x, E)$ is the distance between x and E defined as $d(x, E) = \inf\{|x - y| : y \in E\}$.

Proof. It is clear that (3.6) holds for any $x \geq 0$ and $y \in \overline{E}$, the closure in $\mathbb{R}$ of the set E . Let $(x, x_0) \in [0, \infty) \times \overline{E}$ be such that $|x - x_0| = d(x, E)$. At this point we can write $|f - f(x)| \leq |f - f(x_0)| + |f(x_0) - f(x)|$ and applying the linear and positive operator K_n we have

$$\begin{aligned} |(K_n f)(x) - f(x)| &\leq K_n(|f - f(x_0)|, x) + |f(x) - f(x_0)| \leq \\ &\leq K_n(M_f |e_1 - x_0 e_0|^\alpha, x) + M_f |x - x_0|^\alpha \leq \\ &\leq M_f (K_n(|e_1 - x_0 e_0|^\alpha, x) + 2|x - x_0|^\alpha) \leq \\ &\leq M_f (\frac{\alpha}{n, 2}(x) + 2|x - x_0|^\alpha). \end{aligned} \quad (3.8)$$

We used the classical inequalities $|t - x_0|^\alpha \leq (|t - x| + |x - x_0|)^\alpha \leq |t - x|^\alpha + |x - x_0|^\alpha$, and $K_n(h^\alpha, x) \leq (K_n(h^2, x))^{\alpha/2}$, $0 < \alpha \leq 1$. The latter is a direct result of Hölder's inequality and relation (2.3).

On the other hand, Lemma 1 and relation (3.5) lead us to the following inequality

$$\frac{\alpha}{n, 2}(x) \leq \left(\max \left\{ a_n x, \frac{1}{n} \right\} \right)^{\alpha/2} x^\alpha + \left(\frac{x}{na_n} \right)^{\alpha/2} + \left(\frac{1}{na_n} \right)^\alpha = \varepsilon_n(x) + \frac{1}{(na_n)^\alpha}.$$

Returning to (3.8), the proof is complete. $\square$

In particular, for $E = [0, \infty)$ in (3.7) the term $d(x, E)$ vanishes.

In the final part of this section we present a general quantitative estimate for the rate of pointwise convergence of $(K_n f)(x)$, $x > 0$, for the functions $f \in M_{loc}([0, \infty))$ with the property that the limits $f(x+)$, $f(x-)$ exist. At first we recall that for a given positive integer k , the modulus of variation $v_k(h; Y)$ of a bounded function $h \in \mathbb{R}^{[0, \infty)}$ on the interval Y is defined as the upper bound of the set of all numbers $\sum_{j=1}^k |h(b_j) - h(a_j)|$ over all systems of k non-overlapping intervals (a_j, b_j) contained in Y , $j = \overline{1, k}$. If $k = 0$, we take $v_0(h; Y) = 0$. We mention that some basic properties of this modulus can be found e.g. in [7].

We consider that the following requirements

$$(H_1) \quad \lim_{n \rightarrow \infty} a_n = 0, \quad \lim_{n \rightarrow \infty} na_n = \infty,$$

(H_2) $x > 0$ such that the limits $f(x \pm)$ exist, are fulfilled.

Let A be an arbitrary positive number for which $x \leq A$. Our hypotheses imply:

$$(\exists) n_0, (\forall) n \geq n_0, \quad A < \frac{1}{\sqrt{a_n}}, \quad \frac{1}{3n^2 a_n^2} \leq \frac{1}{na_n} \quad (3.9)$$

and consequently $d_n(x) \leq (x^2 + x + 1)d_n$ where

$$d_n = \max \left\{ \frac{1}{n}, A\sqrt{a_n}, \frac{1}{na_n} \right\}. \quad (3.10)$$

We also consider the mappings g_x, sgn_x defined as follows

$$g_x(t) = \begin{cases} f(t) - f(x-), & t \in [0, x), \\ 0, & t = x, \\ f(t) - f(x+), & t > x, \end{cases} \quad \text{and} \quad \text{sgn}_x(t) = \begin{cases} -1, & t \in [0, x), \\ 0, & t = x, \\ 1, & t > x. \end{cases}$$

The estimate of $|(K_n f)(x) - f(x)|$ will be expressed in the terms of the modulus of variation of the function g_x .

Taking into account the above information and applying to our operators a result due to Grazyna Aniol, see [8; *Theorem 2*] we can state

Theorem 4. *Let $K_n, n \in \mathbb{N}$, be defined by (2.1) such that the hypotheses $(H_1), (H_2)$ are fulfilled and the function f satisfies the growth condition $|f| \leq \psi$ with a positive continuous function ψ . We also suppose that for all $n \geq n_0$ and $t \geq 0$, $(K_n \psi^2)(t) \leq \varphi(t)$ holds, where n_0 is defined by (3.9). For every $n \geq n_0$ such that $d_n \leq 1/2$, we have*

$$\begin{aligned} \left| (K_n f)(x) - \frac{f(x+) + f(x-)}{2} \right| &\leq 2P(x, A) \left\{ \sum_{i=1}^{\mu_n-1} \frac{1}{i^3} v_i(g_x; J_x(iAd_n)) + \right. \\ &\left. + \frac{1}{\mu_n^2} v_{\mu_n}(g_x; J_x(A)) \right\} + Q(x, A)d_n + \frac{1}{2}|f(x+) - f(x-)| |(K_n \text{sgn}_x)(x)|, \end{aligned} \quad (3.11)$$

where d_n is defined by (3.10), $\mu_n := [1/d_n]$, $J_x(h) := [x-h, x+h] \cap [0, \infty)$ for $h > 0$,

$$P(x, A) := 1 + 8(x^2 + x + 1)/A^2$$

and

$$Q(x, A) := A^{-1}((x^2 + x + 1)\varphi(x))^{1/2} + \frac{1}{2}A^{-2}(x^2 + x + 1)\psi(x).$$

Remarks. (i) If f is continuous in x , then the last term in (3.11) vanishes.

(ii) If we consider the operators $K_n^{[\beta]}$, $n \in \mathbb{N}$, defined by (2.4), then we can take $d_n = \max\{1, A\}n^{-\lambda}$ where $\lambda := \beta$ if $0 < \beta < 1/3$ and $\lambda := (1 - \beta)/2$ if $1/3 \leq \beta < 1$.

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ON THE BLOCK IMPLICIT LU ALGORITHM FOR LINEAR SYSTEMS OF EQUATIONS

ELENA BODON
University of Bergamo
Piazza Rosate 2, I-24129 Bergamo, Italy
E-mail:emilio@unibg.it

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Abstract. We theoretically describe different implementations of the block version of the implicit LU algorithm in the ABS class. We also consider the special block ABS algorithm proposed in [3], showing that it corresponds to a version of the block implicit LU algorithm, with blocksize equal to two, when the coefficient matrix is diagonally dominant. Numerical experiments show that the different block implementations have similar accuracy.

Mathematical Subject Classification: 65F05

Keywords: linear equations, ABS methods, implicit LU algorithm

1. Introduction

Define the linear system of equations

$$Ax = b, \quad (1.1)$$

where $A = (a_1, \dots, a_m)^T : a_i, x \in R^n, : b \in R^m, : m \leq n$.

The unscaled version of the ABS algorithm [2] for solving system (1.1) is given by the following procedure.

Algorithm (a)

Let $x_1 \in R^n$ be arbitrary, $H_1 \in R^{n \times n}$ arbitrary nonsingular.

(•) For $i = 1$ to m :

Compute the i -th component of the residual vector in x_i

$$\tau_i = a_i^T x_i - b^T e_i,$$

where e_i is the unit vector in R^n , and compute the vector

$$s_i = H_i a_i.$$

If $s_i = 0$ and $\tau_i = 0$, set $x_{i+1} = x_i, : H_{i+1} = H_i$ and go to (•).

If $s_i = 0$ and $\tau_i \neq 0$ stop, the system is incompatible.

Compute the search vector

$$p_i = H_i^T z_i, \quad (1.2)$$

where $z_i \in R^n$ is arbitrary save for the condition $a_i^T H_i^T z_i \neq 0$.

Update the approximation to the solution

$$x_{i+1} = x_i - \alpha_i p_i, \quad (1.3)$$

where the stepsize α_i is given by

$$\begin{aligned} \alpha_i &= \tau_i / \delta_i \\ \delta_i &= a_i^T p_i. \end{aligned}$$

If $i = m$ stop, x_{m+1} is a solution, otherwise update matrix H_i by

$$H_{i+1} = H_i - H_i a_i w_i^T H_i, \quad (1.4)$$

where $w_i \in R^n$ is arbitrary save for the condition $w_i^T H_i a_i = 1$.

We list some properties of algorithm (a) which will be used later (see [2] for the proofs and other properties):

- (P1) $H_i a_j = 0, j = 1, \dots, i-1$ (the vectors a_j span the null space of H_i).
- (P2) From (P1) we have: a_j is linearly independent of $a_1, \dots, a_{j-1}$ if $H_i a_j$ is linearly independent of $H_i a_1, \dots, H_i a_{j-1}, i < j$.
- (P3) Define $A_i = (a_1, \dots, a_i), P_i = (p_1, \dots, p_i), Q_i = (H_1^T w_1, \dots, H_i^T w_i), S_i = (s_1, \dots, s_i)$; then the following relations hold: $A_i^T P_i = L_i, A_i^T Q_i = L_i, A_i^T H_1^T = L_i S_i^T$, where L_i, L_i' are lower triangular matrices.
- (P4) The arbitrary parameter H_1 is redundant, since algorithm (a) can be defined in terms of $\tilde{H}_1 = I, \tilde{z}_i = H_1^T z_i, \tilde{w}_i = H_1^T w_i$, where I is the unit matrix in $R^{n \times n}$.
- (P5) The arbitrary parameters z_i, w_i , are redundant, since the sequences (1.2) and (1.3) can be defined in terms of only one parameter z_i or w_i .
- (P6) The matrix H_{i+1} (1.4) can be updated in the form $H_{i+1} = H_1 - S_i Q_i^T$. From (P3) it holds that $H_{i+1} = H_1 - H_1 A_i L_i'^{-T} Q_i^T$, and from (P4) and (P5) it holds that $H_{i+1} = I - A_i L_i'^{-T} P_i^T$.
- (P7) The solution of the system $A_i^T X = I_i$, where $X \in R^{n \times i}, I_i$ is the unit matrix in $R^{i \times i}$, given by algorithm (a) can be also written in the form $X = P_i L_i'^{-1}$.
- (P8) The representation of all solutions of the first i equations has the form

$$x' = x_{i+1} + H_{i+1}^T t \quad (1.5)$$

with $t \in R^n$ arbitrary.

2. The block ABS algorithm

The block ABS algorithm, due to Abaffy and Galántai [1] for the scaled ABS class, see also [2], and further developed in several papers by Galántai, is a block form of the ABS algorithm. The block unscaled version of algorithm (a) is given by the following procedure.

Assume for simplicity that A is full rank. Let $m_1, m_2, \dots, m_j$ be positive integers such that $m_1 + m_2 + \dots + m_j = m$. Set $A_k = (a_{m_1+m_2+\dots+m_{k-1}+1}, \dots, a_{m_1+m_2+\dots+m_k})$, $b_k = (b^T e_{m_1+m_2+\dots+m_{k-1}+1}, \dots, b^T e_{m_1+m_2+\dots+m_k})$, A_k^T is the k -th block of m_k rows of A , b_k is the k -th block of m_k components of b .

Let $x_1 \in R^n$ be arbitrary, and $H_1 \in R^{n \times n}$ arbitrary nonsingular.

For $k = 1$ to j :

Compute the matrix $P_k \in R^{n \times m_k}$ by

$$P_k = H_k^T Z_k ,$$

where $Z_k \in R^{n \times m_k}$ is arbitrary full rank save for the condition $A_k^T H_k^T Z_k$ nonsingular.

Update the approximation to the solution by

$$x_{k+1} = x_k - P_k d_k , \quad (2.1)$$

where d_k is the unique solution of the nonsingular system

$$A_k^T P_k d_k = r_k , \quad (2.2)$$

and $r_k = A_k^T x_k - b_k$ is the k -th block of m_k components of the residual in x_k .

If $k = j$ stop, x_{j+1} solves system (1.1), otherwise update matrix H_k by

$$H_{k+1} = H_k - H_k A_k W_k^T H_k ,$$

where $W_k \in R^{n \times m_k}$ is arbitrary full rank save for the condition

$$W_k^T H_k A_k = I_{m_k} . \quad (2.3)$$

The properties of block algorithm, similar to properties of algorithm (a), are given in [2].

If A is rank deficient, from property (P1), the linear dependent rows of A can be deleted from system (1.1) when solving system (2.3).

The choice $Z_k = W_k$ verifies the nonsingularity condition for the matrix $A_k^T H_k^T Z_k$ and implies $d_k = r_k$ in (2.2).

If $m_k = 1, : k = 1, \dots, j$ ($j = m$), we obtain algorithm (a), while if $m = n$ and $m_k = m_1 = m$ ($j = 1$) and if $H_1 = I$, we obtain the explicit computation of A^{-1} in (2.3) and if $Z_1 = W_1$, the solution (2.1) is computed by $x_2 = x_1 - A^{-1} r_1$.

3. Alternative formulations of the block ABS algorithm

We reformulate the block ABS algorithm for a general matrix A in the case $Z_k = W_k$, $k = 1, \dots, j$.

Algorithm (b1)

Give $x_1, H_1, m_1, m_2, \dots, m_j$ as above.

(•) For $k = 1$ to j :

Compute

$$S_k = H_k A_k, \quad (3.1)$$

$$r_k = A_k^T x_k - b_k. \quad (3.2)$$

If $S_k = 0$ and $r_k = 0$, set $x_{k+1} = x_k$, $H_{k+1} = H_k$ and go to (•).

If $S_k = 0$ and $r_k \neq 0$, stop, system (1.1) is incompatible, or if a column s_i of S_k and the corresponding element τ_i of r_k are such that $s_i = 0$, $\tau_i \neq 0$, stop, system (1.1) is incompatible.

If $S_k \neq 0$, apply algorithm (a) for finding the search vectors of the matrix $S_k^T \in R^{m_k \times n}$ deleting the linearly dependent rows if they exist (from property (P2) the corresponding rows in A are linearly dependent from the previous rows). Set $S_{\hat{k}} \in R^{n \times l_k}$ the matrix formed by the linearly independent columns of S_k , $l_k \leq m_k$. Determine the solution of the system

$$S_{\hat{k}}^T W_k = I_{l_k} \quad (3.3)$$

by algorithm (a), where $W_k \in R^{n \times l_k}$.

Compute the matrix $P_k \in R^{n \times l_k}$ by

$$P_k = H_k^T W_k. \quad (3.4)$$

Update the approximation to the solution by

$$x_{k+1} = x_k - P_k r_{\hat{k}}, \quad (3.5)$$

where $r_{\hat{k}}$ consists of the l_k components of r_k corresponding to the l_k rows of $S_{\hat{k}}^T$.

If $l_k < m_k$, compute $\tau_i = a_i^T x_{k+1} - b^T e_i$ for the $m_k - l_k$ indices i corresponding to the linearly dependent rows of S_k^T . If $\tau_i \neq 0$ for at least one index i , stop, the system is incompatible.

If $k = j$, stop, x_{j+1} is a solution of system (1.1), otherwise update matrix H_k by

$$H_{k+1} = H_k - S_{\hat{k}} P_k^T. \quad (3.6)$$

If $m_k = 2$, $k = 1, \dots, j$, $j = m/2$, we obtain the following algorithm, also proposed by a different derivation in [3]. Suppose m even, if m is odd, the last step for $k = m/2 + 1$ corresponds to the last step of algorithm (a). Set $A_k = (a_{k1}, a_{k2})$, $b_k = (b_{k1}, b_{k2})$.

Algorithm (b2)

Let x_1, H_1 as above.

(•) For $k = 1$ to $m/2$:

Compute the vectors $S_k = (s_{k1}, s_{k2})$

$$\begin{aligned} s_{k1} &= H_k a_{k1} , \\ s_{k2} &= H_k a_{k2} . \end{aligned} \quad (3.7)$$

Compute the residuals $r_k = (\tau_{k1}, \tau_{k2})$

$$\begin{aligned} \tau_{k1} &= a_{k1}^T x_k - b_{k1} , \\ \tau_{k2} &= a_{k2}^T x_k - b_{k2} . \end{aligned} \quad (3.8)$$

If $s_{k1} = 0$, $s_{k2} = 0$, $\tau_{k1} = 0$, $\tau_{k2} = 0$, set $x_{k+1} = x_k$, $H_{k+1} = H_k$ and go to (•).

If $s_{k1} = 0$ and $\tau_{k1} \neq 0$, or $s_{k2} = 0$ and $\tau_{k2} \neq 0$, stop, the system is incompatible.

If $s_{k1} \neq 0$ and $s_{k2} = 0$, $\tau_{k2} = 0$, set $k_i = k_1$ and go to (••).

If $s_{k2} \neq 0$ and $s_{k1} = 0$, $\tau_{k1} = 0$, set $k_i = k_2$ and go to (••).

If $s_{k1} \neq 0$ and $s_{k2} \neq 0$, verify if s_{k1}, s_{k2} are linearly independent (for example applying algorithm (a) to the matrix $(s_{k1}, s_{k2})^T$).

If s_{k1}, s_{k2} are linearly independent, choose $W_k = (w_{k1}, w_{k2})$, $W_k \in R^{n \times 2}$ satisfying system $S_k^T W_k = I_2$

$$\begin{aligned} s_{k1}^T w_{k1} &= 1, & s_{k1}^T w_{k2} &= 0 , \\ s_{k2}^T w_{k1} &= 0, & s_{k2}^T w_{k2} &= 1 . \end{aligned} \quad (3.9)$$

Update the approximation to the solution by

$$x_{k+1} = x_k - H_k^T (\tau_{k1} w_{k1} + \tau_{k2} w_{k2}) . \quad (3.10)$$

If $k = m/2$, stop, $x_{\frac{m}{2}+1}$ is a solution of system (1.1), otherwise update matrix H_k by

$$H_{k+1} = H_k - s_{k1} w_{k1}^T H_k - s_{k2} w_{k2}^T H_k \quad (3.11)$$

and go to (•).

(*) If s_{k2} is linearly dependent from s_{k1} , set $k_i = k_2$ (row k_2 in A is linearly dependent from the previous rows).

(••) Choose $w_{k_i} \in R^n$ such that

$$s_{k_i}^T w_{k_i} = 1 .$$

Update the approximation to the solution by

$$x_{k+1} = x_k - \tau_{k_i} H_k^T w_{k_i} .$$

If it is case (*), compute $\tau_{k_2} = a_{k_2}^T x_{k+1} - b_{k_2}$ and if $\tau_{k_2} \neq 0$, stop, the system is incompatible. If $k = m/2$, stop, $x_{\frac{m}{2}+1}$ is a solution of system (1.1), otherwise update matrix H_k by

$$H_{k+1} = H_k - s_{k_i} w_{k_i}^T H_k$$

and go to (•).

We consider two equivalent forms of the block ABS algorithm (b1), other particular cases of block forms can be found in [2].

The first versions of the algorithm (b1) constructs at every step k , the same approximate solution x_k and the same matrices S_k, H_k avoiding the explicit computation of W_k . Rewrite (3.5) as $x_{k+1} = x_k - H_k^T W_k r_k$. Then by multiplying (3.3) by r_k we have $S_k^T W_k r_k = r_k$. Set $t_k = W_k r_k$, then instead of (3.3) we can solve $S_k^T t_k = r_k$ and compute $x_{k+1} = x_k - H_k^T t_k$. Rewrite (3.6) as $H_{k+1} = (I - S_k W_k^T) H_k$. Denote by $\tilde{H}_k$ the matrix obtained after m_k iterations of algorithm (a) applied to system $S_k^T t_k = r_k$. From (3.3) it holds that $\text{Null}(I - S_k W_k^T) = \text{Null}(\tilde{H}_k)$, hence from properties (P4),(P5),(P6),(P7) we can write $\tilde{H}_k$ as $\tilde{H}_k = I - S_k W_k^T$. Then the update of H_k in (3.6) is given by $H_{k+1} = \tilde{H}_k H_k$. The first alternative formulation of algorithm (b1) is given by the following procedure.

Algorithm (c)

Let $x_1, H_1, m_1, m_2, \dots, m_j$ as above.

(•) For $k = 1$ to j :

Compute

$$S_k = H_k A_k , \tag{3.12}$$

$$r_k = A_k^T x_k - b_k . \tag{3.13}$$

If $S_k = 0$ and $r_k = 0$ set $x_{k+1} = x_k, H_{k+1} = H_k$ and go to (•).

If $S_k = 0$ and $r_k \neq 0$, stop, the system is incompatible.

If $S_k \neq 0$, solve by algorithm (a) the system

$$S_k^T t_k = r_k . \tag{3.14}$$

Update the approximation to the solution by

$$x_{k+1} = x_k - H_k^T t_k . \tag{3.15}$$

If S_k is not full rank and at a step i system (3.14) is incompatible, then the corresponding equation i in system (1.1) is incompatible from the previous equations.

If $k = j$, stop, x_{j+1} is a solution of system (1.1), otherwise update matrix H_k by

$$H_{k+1} = \tilde{H}_k H_k, \quad (3.16)$$

where $\tilde{H}_k$ is the matrix (1.4) obtained after m_k steps of algorithm (a) applied to system (3.14).

The second version of algorithm (b1) is a modification of algorithm (c). It avoids the computation of the update H_{k+1} (a matrix-matrix product) but performs transformations on the matrix A .

Define $\bar{S}_k = H_k \bar{A}_k$ where $\bar{A}_k$ is the transpose of the matrix formed by the last $m_{k+1} + \dots + m_j$ rows of A (i.e. the last $j - k$ blocks of A), $\bar{r}_k = \bar{A}_k^T x_k - \bar{b}_k$ where $\bar{b}_k$ is the vector formed by the last $m_{k+1} + \dots + m_j$ components of b , and observe that $S_k = H_k A_k = \tilde{H}_{k-1} H_{k-1} A_k = \tilde{H}_{k-1} H_{k-1} (\bar{A}_{k-1})_{m_k} = \tilde{H}_{k-1} (\bar{S}_{k-1})_{m_k}$, $r_k = A_k^T x_k - b_k = A_k^T x_{k-1} - b_k - A_k^T H_{k-1}^T t_{k-1} = (\bar{A}_{k-1}^T)_{m_k} x_{k-1} - (\bar{b}_{k-1})_{m_k} - (\bar{A}_{k-1}^T)_{m_k} H_{k-1}^T t_{k-1} = (\bar{r}_{k-1})_{m_k} - (\bar{S}_{k-1}^T)_{m_k} t_{k-1}$, where $(\bar{A}_{k-1})_{m_k}$ is the submatrix formed by the first m_k columns of $\bar{A}_{k-1}$, $(\bar{S}_{k-1})_{m_k}$ is the submatrix formed by the first m_k columns of $\bar{S}_{k-1}$ and $(\bar{r}_{k-1})_{m_k}$ is the vector formed by the first m_k components of $\bar{r}_{k-1}$. Then algorithm (c) becomes

Algorithm (d)

Let $x_1, H_1, m_1, m_2, \dots, m_j$ as above.

Set $\bar{S}_1 = H_1 A$, $\bar{r}_1 = Ax_1 - b$

(•) For $k = 1$ to j :

Set $\bar{S}_k$ and $\bar{r}_k$ partitioned as follows:

$\bar{S}_k = (S_k, \bar{S}_k)$, S_k with dimension $(n \times m_k)$, $\bar{S}_k$ with dimension $(n \times (m_{k+1} + \dots + m_j))$

$\bar{r}_k = (r_k, \bar{r}_k)$, r_k with dimension m_k , $\bar{r}_k$ with dimension $m_{k+1} + \dots + m_j$

If $S_k = 0$ and $r_k = 0$, set $\bar{S}_k = \bar{S}_k$, $\bar{r}_k = \bar{r}_k$, $t_k = 0$, $\tilde{H}_k = I$ and go to (•).

If $S_k = 0$ and $r_k \neq 0$, stop, the system is incompatible.

If $S_k \neq 0$, solve by algorithm (a) the system

$$S_k^T t_k = r_k. \quad (3.17)$$

If S_k is not full rank and at a step i system (3.17) is incompatible, then the corresponding equation i in system (1.1) is incompatible from the previous equations.

If $k < j$, compute

$$\bar{S}_{k+1} = \tilde{H}_k \bar{S}_k, \quad (3.18)$$

$$\bar{r}_{k+1} = \bar{r}_k - \bar{S}_k^T t_k, \quad (3.19)$$

where $\tilde{H}_k$ is the matrix (1.4) obtained after m_k steps of algorithm (a) applied to system (3.17), and go to (•).

Estimate the solution by the following process:

Set $\bar{t}_j = t_j$, for $k = 1$ to $j - 1$ compute

$$\bar{t}_{j-k} = t_{j-k} + \tilde{H}_{j-k}^T \bar{t}_{j-k+1} , \quad (3.20)$$

$$x_{j+1} = x_1 - H_1^T \bar{t}_1 .$$

Algorithms (c) and (d) can be also derived from property (P8) if we consider at step k a solution of the k -th block and all the possible solutions given by formula (1.5), at step $k + 1$ the solutions of the $(k + 1)$ -th block satisfying relation (1.5).

The equivalent forms of algorithms (a), (b1), (b2), (c), (d) can differ in storage, overhead, computational timings, according to the type of ABS method, the type of representation of matrix H_i , and the type of implementation (sequential or parallel).

The block form of algorithm (b1) was tested in [4] on three algorithms of the ABS class: the implicit LU algorithm, the modified Huang algorithm, the implicit QR algorithm. The block form of algorithm (d) was tested in [5] on the implicit LU algorithm and on the modified Huang algorithm (using several representations of the matrix H_i) on parallel processor. In this paper we consider only the implicit LU method and a modification in [3] used in algorithm (b2).

4. The implicit LU algorithm and some of its block forms

The implicit LU algorithm corresponds to the parameter choices

$$H_1 = I, \quad z_i = e_i, \quad w_i = e_i / e_i^T H_i a_i .$$

It is well defined if A has nonsingular principal submatrices, otherwise column pivoting has to be performed. Matrix H_{i+1} has the structure

$$H_{i+1} = \begin{bmatrix} 0_{i \times i} & 0_{i \times (n-i)} \\ M_i & I_{n-i} \end{bmatrix} . \quad (4.1)$$

The first i rows are zero, the submatrix given by the last $n - i$ rows and columns is the identity matrix. The search vector p_{i+1} coincides with the $(i + 1)$ -th row of H_{i+1} , hence the matrix P_i in (P3) is unit upper triangular. The vector s_{i+1} has the first i components equal to zero. If $x_1 = 0$, only the first i components of x_{i+1} can be nonzero. Several tests on the implicit LU method can be found in [6].

The block generalization of the implicit LU method is obtained solving system (3.3), (3.9), (3.14), (3.17), of algorithms (b1), (b2), (c), (d), respectively, by the implicit LU method with zero starting point. Matrix H_{k+1} in (3.6), (3.11), (3.16) is equal to (1.4) after $m_1 + m_2 + \dots + m_k$ steps. Matrix W_k can have nonzeros only in m_k rows, which form the inverse of the $m_k \times m_k$ nonsingular submatrix of S_k^T

whose columns are chosen by the implicit LU method. Matrix P_k in (3.4), (3.10) has $n - (m_{k+1} + \dots + m_j)$ rows equal to zero and so x_{k+1} in (3.5), (3.10), (3.15). Matrix S_k in (3.1), (3.7), (3.12), (3.17) has $m_1 + \dots + m_{k-1}$ rows equal to zero and the partial solution t_k in (3.14), (3.17) can have only m_k components that are nonzero.

The methods tested in our experiments are listed below. For the block algorithms we have used a size n_b equal for all the blocks. The starting points are set to zero.

- Implicit LU - the method given above, where, for stability reasons, at every step i we take column j such that $\delta_j = \max(\delta_k = |e_k^T H_i a_i| : k = 1, \dots, n)$. From the structure of H_{i+1} (4.1), at every step i , the $n - i$ elements of a_i , not used to update the estimate solution (1.3), are overwritten by the elements of the last column of M_i , hence the storage required is the storage of A . Then the search vectors p_i (column i of H_i^T) are memorized in A and system (1.1) can be solved with different right hand-sides using the vectors $p_i, i = 1, \dots, n$ previously computed. The method requires $nm^2 - 2m^3/3$ multiplications for $m \leq n$ and $n^3/3$ for $m = n$.
- Block implicit LU (b) - the method given by algorithm (b1), where system (3.3) is solved by the implicit LU described above. For this method the $n_b \times n$ block A_k^T is overwritten by the $kn_b \times n_b$ nonzero submatrix of P_k and by the $n_b \times n - kn_b$ submatrix of S_k^T used to update H_k (3.6) and at step $k + i$ by a nonzero part of H_{k+i+1} , while for solving (3.3) we use a workspace of dimension $n_b \times n - (k - 1)n_b$. Thus the storage required is the storage of A plus $n_b \times n$. To solve (3.3) $(n - (k - 1)n_b)n_b^2 - 2n_b^3/3$ multiplications are needed for computing the search vectors and $2n_b^3/3$ for W_k . $(k - 1)n_b^3$ multiplications are needed for the product $H_k^T W_k$ in computing P_k (3.4), $(n - kn_b)kn_b^2$ for the product $S_k P_k^T$ in updating H_{k+1} (3.6), $(k - 1)n_b^2$ for r_k (3.2), kn_b^2 for x_{k+1} (3.5) and $(n - (k - 1)n_b)(k - 1)n_b^2$ for S_k (3.1). Hence the total cost for $m = n$ and n multiple of n_b is $n^3/3 + n^2n_b - nn_b^2/3 + O(n^2)$. The number of operations is an increasing function in n_b , minimum is $n^3/3$ for $n_b = 1$ (the algorithm corresponds to implicit LU) and maximum is n^3 for $n_b = n$ (the algorithm performs one step computing the inverse of A in (3.3)).
- Block implicit LU (c) - the method given by algorithm (c), where system (3.14) is solved by the implicit LU described above. For this method the $n_b \times n$ block A_k^T is overwritten by the $n_b \times n - (k - 1)n_b$ nonzero submatrix of S_k^T , then S_k is overwritten by the nonzero submatrix of $\tilde{H}_k$ when solving system (3.14) and then $\tilde{H}_k$ is overwritten with a nonzero part of H_{k+2} at next step $k + 1$ and by a nonzero part of H_{k+i+1} at step $k + i$. Hence the memory cost is the storage of A . The number of multiplications required is $(n - (k - 1)n_b)n_b^2 - 2n_b^3/3$ for solving system (3.14), $(k - 1)n_b^2$ for x_{k+1} in (3.15), $(k - 1)n_b^2$ for r_k in (3.13), $(n - (k - 1)n_b)(k - 1)n_b^2$ for computing S_k (3.12), $(k - 1)n_b^2(n - kn_b)$ for the product in H_{k+1} in (3.16). The total amount for $m = n$ and n multiple of n_b is $n^3/3 + O(n^2)$. Also in the case of blocks $m_1, m_2, \dots, m_j$ with different size the total amount is $n^3/3 + O(n^2)$.
- Block implicit LU (d) - the method given by algorithm (d), where system (3.17)

is solved by the implicit LU described above. For this method the $(n - kn_b) \times (n - (k - 1)n_b)$ nonzero submatrix of $\bar{S}_k$ is overwritten by the $(n - kn_b) \times (n - kn_b)$ nonzero submatrix of $\bar{S}_{k+1}$ and the $n_b \times (n - (k - 1)n_b)$ nonzero submatrix of S_k^T is overwritten by the $n_b \times (n - kn_b)$ nonzero submatrix of $\tilde{H}_k^T$ when solving system (3.17). Hence the memory is the storage of A . The number of multiplications required is $(n - (k - 1)n_b)n_b^2 - 2n_b^3/3$ for solving system (3.17), $(n - kn_b)n_b$ for computing r_{k+1} (3.19), $n_b(n - kn_b)^2$ for the product in $\bar{S}_{k+1}$ (3.18) and $n^2/2$ for the process in estimating the solution x_{k+1} (3.20). The total amount for $m = n$ and n multiple of n_b is $n^3/3 + O(n^2)$ also for blocks $m_1, m_2, \dots, m_j$ with different size.

Notice that the two versions of block implicit LU (c) and (d) have the same storage and overhead as implicit LU has.

5. The algorithm of Adib et al.

Here we call Algorithm [3] the method described by Adib et al. in [3]. We show that this method is algorithm (b2), where system (3.9) is solved applying twice the implicit LU.

Using implicit LU in (3.9) the search vectors p_{k_1}, p_{k_2} of the matrix $(s_{k_1}, s_{k_2})^T$ are constructed choosing two columns j_1, j_2 such that

$$\begin{aligned} \delta_{j_1} &= \max(\delta_i = |s_{k_1(i)}| \quad i = 1, \dots, n), \\ \delta_{j_2} &= \max(\delta_i = |s_{k_2(i)} - (s_{k_2(j_1)}s_{k_1(i)})/\delta_{j_1}| \quad i = 1, \dots, n), \end{aligned} \quad (5.1)$$

hence

$$\begin{aligned} p_{k_1(j_1)} &= 1, & p_{k_1(i)} &= 0, & i &= 1, \dots, n \quad i \neq j_1 \\ p_{k_2(j_1)} &= -s_{k_1(j_2)}/\delta_{j_1}, & p_{k_2(j_2)} &= 1, & p_{k_2(i)} &= 0 \quad i = 1, \dots, n \quad i \neq j_1, j_2. \end{aligned}$$

The solutions w_{k_1}, w_{k_2} can be nonzeros only in the components j_1, j_2

$$\begin{aligned} w_{k_1(j_1)} &= 1/\delta_{j_1} + (s_{k_2(j_1)}/(\delta_{j_1}\delta_{j_2}))(s_{k_1(j_2)}/\delta_{j_1}), \\ w_{k_1(j_2)} &= -s_{k_2(j_1)}/(\delta_{j_1}\delta_{j_2}), \quad w_{k_1(i)} = 0, \quad i = 1, \dots, n \quad i \neq j_1, j_2 \end{aligned} \quad (5.2)$$

$$w_{k_2(j_1)} = -s_{k_1(j_2)}/(\delta_{j_1}\delta_{j_2}), \quad w_{k_2(j_2)} = 1/\delta_{j_2}, \quad w_{k_2(i)} = 0, \quad i = 1, \dots, n \quad i \neq j_1, j_2 \quad (5.3)$$

and from property (4.1) of the update, rows j_1, j_2 in H_{k+1} (3.11) become zero.

In the method proposed in [3], the parameters w_{k_1}, w_{k_2} correspond to the two solutions obtained solving by implicit LU the two following systems

$$\begin{cases} s_{k_2}^T w_{k_1} = 0 \\ s_{k_1}^T w_{k_1} = 1 \end{cases}, \quad (5.4)$$

$$\begin{cases} s_{k_1}^T w_{k_2} = 0 \\ s_{k_2}^T w_{k_2} = 1 \end{cases}. \quad (5.5)$$

The solution w_{k_2} of the second system (5.5) is equal to the solution (5.3) since the sequence of the rows $s_{k_1}^T, s_{k_2}^T$ is the same as in (3.9) and so the choice of the pivot indices j_1, j_2 is (5.1). The solution w_{k_1} of the first system (5.4) is not (5.2) since in (5.4) the first considered row is $s_{k_2}^T$, then the columns taken for the pivot can be different from j_1, j_2 . Set

$$\delta_{j_3} = \max(\delta_i = |s_{k_2(i)}| \quad i = 1, \dots, n),$$

$$\delta_{j_4} = \max(\delta_i = |s_{k_1(i)} - (s_{k_1(j_3)} s_{k_2(i)}) / \delta_{j_3}| \quad i = 1, \dots, n).$$

The search vectors of (5.4) are

$$p_{k_1(j_3)} = 1, \quad p_{k_1(i)} = 0, \quad i = 1, \dots, n \quad i \neq j_3$$

$$p_{k_2(j_3)} = -s_{k_2(j_4)} / \delta_{j_3}, \quad p_{k_2(j_4)} = 1, \quad p_{k_2(i)} = 0, \quad i = 1, \dots, n \quad i \neq j_3, j_4$$

and the solutions w_{k_1} can be nonzeros only in the components j_3, j_4

$$w_{k_1(j_3)} = -s_{k_2(j_4)} / (\delta_{j_3} \delta_{j_4}), \quad w_{k_1(j_4)} = 1 / \delta_{j_4}, \quad w_{k_1(i)} = 0, \quad i = 1, \dots, n \quad i \neq j_3, j_4. \quad (5.6)$$

Since $H_1 = I$, rows j_1, j_2, j_3, j_4 in (3.11) can be different from the unit vector, and the process leads to a dense matrix H_k . If j_1, j_2 coincide with j_3, j_4 , solution (5.6) is equal to (5.2), the step corresponds to a step of the block implicit LU algorithm and rows j_1, j_2 of H_{k+1} become zero. The indices j_1, j_2 can be equal to j_3, j_4 if matrix $(s_{k_1}, s_{k_2})^T$ has two dominant elements in two different columns, thus if A is diagonally dominant, algorithm [3] becomes the block implicit LU (b) algorithm with blocksize $n_b = 2$.

The special structure of H_k depends on the pivot indices j_1, j_2, j_3, j_4 . A zero row appears if (s_{k_1}, s_{k_2}) has rank 1 (iteration k is an implicit LU iteration). Also zero rows appear at a step k when the number of a set of rows taken from the first $2k$ is equal to the number of different indices of columns selected as pivot in those rows (in this case all rows with indices in the set become zero in H_{k+1}).

Algorithm [3] can be generalized for blocks greater than 2, but it becomes very expensive.

The code written for numerical experiments uses an $n \times n$ array to memorize H_k since generally it is not known how sparse the matrix can be. The code performs at every iteration a search to find the set of pivot indices, used at previous iterations, which can give zero rows in H_k . If this set exists, these rows are ignored in the multiplications and so the number of operations is reduced. Then the amount of operations may vary according to the values of the elements of A , the minimum being $n^3/3$ when at every step the indices j_1, j_2 coincide with j_3, j_4 , maximum is $3n^3/2$ when at every step the indices j_1, j_2 are different from j_3, j_4 and there are no zero rows in H_k .

The following table reports storage requirement and number of multiplications for the above algorithms (only higher order terms are considered).

	storage	overhead
implicit LU	n^2	$n^3/3$
algorithm [3]	$2n^2$	$\min n^3/3, \max 3n^3/2$
block implicit LU (b) $n_b = 2$	n^2	$n^3/3$
block implicit LU (c) $n_b = 2$	n^2	$n^3/3$
block implicit LU (d) $n_b = 2$	n^2	$n^3/3$
block implicit LU (b)	$n^2 + n_b n$	$n^3/3 + n^2 n_b - n n_b^2/3$
block implicit LU (c)	n^2	$n^3/3$
block implicit LU (d)	n^2	$n^3/3$

6. Numerical testing and results

Codes for testing the above algorithms were written in Fortran 77, run in single precision and tested on square matrices with dimension $n = 100, n = 200$, using the same size n_b for all blocks, with values $n_b = 2, 5, 10, 20, 50$. The following families of problems were considered.

(M1) Matrices with random integer elements, solution vector x^* with random integer elements and right hand-side b computed exactly by $b = Ax^*$.

(M2) Ill conditioned matrices with random integer elements having two or more rows and/or columns equal save for an element a_{ij} set equal to $a_{ij} + 2^{-k}$, k integer, solution vector x^* with random integer elements and the component i (or j) equal to 2^k so that the right hand-side $b = Ax^*$ has integer elements.

(M3) Ill conditioned matrices formed by a product of a random matrix of family (M1) and an ill conditioned and small dimensioned matrix with integer elements taken from [7] (matrices of Wilson, Rutishauser, Morris, Pascal, Tanabe etc.). The solution vector x^* has random integer elements and b is computed by $b = Ax^*$.

(M4) A set of matrices listed in [3] and [8] having not all integer entries but exactly represented in the machine, the solution vector x^* having all components equal to 1.

The performance of the algorithms was measured by the relative error in the solution and in the residual with Euclidean norm defined respectively by $\|x - x^*\|/\|x^*\|$, $\|Ax - b\|/\|b\|$. Some of the results are reported in Tables 1, 2, 3, 4. Tables 5, 6 give a comparison of the algorithms on 200 problems. The elements u_{ij}/v_{ij} have the following meaning: u_{ij} indicates how many times the algorithm in row i has a smaller relative error than the algorithm in column j , v_{ij} how many times the relative error is equal (only the first two digits are considered). We find that all the algorithms have similar performance on most problems. The block implicit LU (b) seems to have the worst performance for great values of blocksize n_b , probably due to the computation of the inverse of a not small dimensioned matrix in (3.3). The original nonblock version gives generally the best performance.

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Table 1 - Test matrices (M1)

Method	dimension n=100		Dimension n=200	
	Solution error	Residual error	Solution error	Residual error
impl.LU	.24E-05	.44E-06	.39E-05	.85E-06
algorithm[3]	.37E-05	.71E-06	.73E-05	.12E-05
block impl.LU (b) $n_b = 2$	.59E-05	.61E-06	.14E-04	.12E-05
block impl.LU (c) $n_b = 2$	.30E-05	.49E-06	.12E-04	.11E-05
block impl.LU (d) $n_b = 2$	.34E-05	.10E-05	.82E-05	.25E-05
block impl.LU (b) $n_b = 5$	.33E-05	.54E-06	.14E-04	.11E-05
block impl.LU (c) $n_b = 5$	.54E-05	.57E-06	.13E-04	.13E-05
block impl.LU (d) $n_b = 5$	.80E-05	.86E-06	.14E-04	.22E-05
block impl.LU (b) $n_b = 10$	.39E-05	.71E-06	.10E-04	.14E-05
block impl.LU (c) $n_b = 10$	.55E-05	.66E-06	.25E-04	.12E-05
block impl.LU (d) $n_b = 10$	.33E-05	.11E-05	.69E-05	.26E-05
block impl.LU (b) $n_b = 20$	.60E-05	.13E-05	.21E-04	.29E-05
block impl.LU (c) $n_b = 20$	.48E-05	.78E-06	.10E-04	.14E-05
block impl.LU (d) $n_b = 20$	.85E-05	.74E-06	.93E-05	.19E-05
block impl.LU (b) $n_b = 50$	.16E-04	.39E-05	.16E-04	.42E-05
block impl.LU (c) $n_b = 50$	.96E-05	.82E-06	.20E-04	.14E-05
block impl.LU (d) $n_b = 50$	.96E-05	.82E-06	.28E-04	.15E-05
impl.LU	.26E-05	.60E-06	.62E-05	.94E-06
algorithm[3]	.35E-05	.76E-06	.84E-05	.13E-05
block impl.LU (b) $n_b = 2$	.40E-05	.63E-06	.15E-04	.12E-05
block impl.LU (c) $n_b = 2$	.47E-05	.57E-06	.98E-05	.12E-05
block impl.LU (d) $n_b = 2$	.55E-05	.92E-06	.85E-05	.21E-05
block impl.LU (b) $n_b = 5$	.13E-04	.71E-06	.81E-05	.12E-05
block impl.LU (c) $n_b = 5$	.11E-04	.59E-06	.68E-05	.11E-05
block impl.LU (d) $n_b = 5$	.95E-05	.10E-05	.14E-04	.30E-05
block impl.LU (b) $n_b = 10$	.24E-05	.76E-06	.73E-05	.14E-05
block impl.LU (c) $n_b = 10$	.53E-05	.62E-06	.99E-05	.11E-05
block impl.LU (d) $n_b = 10$	.78E-05	.74E-06	.84E-05	.19E-05
block impl.LU (b) $n_b = 20$	.98E-05	.15E-05	.85E-05	.18E-05
block impl.LU (c) $n_b = 20$	.25E-05	.59E-06	.99E-05	.13E-05
block impl.LU (d) $n_b = 20$	.60E-05	.88E-06	.42E-05	.16E-05
block impl.LU (b) $n_b = 50$	.11E-04	.75E-05	.22E-04	.58E-05
block impl.LU (c) $n_b = 50$	.19E-04	.86E-06	.46E-05	.15E-05
block impl.LU (d) $n_b = 50$	.19E-04	.86E-06	.10E-04	.16E-05
impl.LU	.50E-05	.56E-06	.46E-05	.96E-06
algorithm[3]	.46E-05	.71E-06	.13E-03	.11E-05
block impl.LU (b) $n_b = 2$	.18E-04	.56E-06	.60E-04	.11E-05
block impl.LU (c) $n_b = 2$	.12E-04	.66E-06	.12E-03	.12E-05
block impl.LU (d) $n_b = 2$	.84E-05	.95E-06	.86E-04	.20E-05
block impl.LU (b) $n_b = 5$	.45E-05	.64E-06	.13E-03	.11E-05
block impl.LU (c) $n_b = 5$	.17E-04	.63E-06	.11E-03	.12E-05
block impl.LU (d) $n_b = 5$	.52E-05	.93E-06	.50E-04	.26E-05
block impl.LU (b) $n_b = 10$	.54E-05	.80E-06	.15E-03	.16E-05
block impl.LU (c) $n_b = 10$	.45E-05	.55E-06	.10E-03	.12E-05
block impl.LU (d) $n_b = 10$	.55E-05	.67E-06	.13E-03	.17E-05
block impl.LU (b) $n_b = 20$	.55E-05	.21E-05	.23E-04	.28E-05
block impl.LU (c) $n_b = 20$	.15E-04	.62E-06	.10E-03	.13E-05
block impl.LU (d) $n_b = 20$	.61E-05	.75E-06	.61E-04	.15E-05
block impl.LU (b) $n_b = 50$	.19E-04	.52E-05	.19E-03	.37E-04
block impl.LU (c) $n_b = 50$	.19E-04	.64E-06	.77E-04	.12E-05
block impl.LU (d) $n_b = 50$	.19E-04	.64E-06	.33E-04	.14E-05
impl.LU	.50E-05	.41E-06	.21E-04	.84E-06
algorithm[3]	.13E-04	.69E-06	.92E-05	.94E-06
block impl.LU (b) $n_b = 2$	.13E-04	.57E-06	.16E-04	.96E-06
block impl.LU (c) $n_b = 2$	.18E-04	.57E-06	.18E-04	.95E-06
block impl.LU (d) $n_b = 2$	.51E-05	.82E-06	.10E-04	.17E-05
block impl.LU (b) $n_b = 5$	.71E-05	.60E-06	.11E-04	.10E-05
block impl.LU (c) $n_b = 5$	.31E-05	.49E-06	.14E-04	.96E-06
block impl.LU (d) $n_b = 5$	.63E-05	.82E-06	.26E-04	.16E-05
block impl.LU (b) $n_b = 10$	.97E-05	.79E-06	.98E-05	.11E-05
block impl.LU (c) $n_b = 10$	.53E-05	.50E-06	.71E-05	.99E-06
block impl.LU (d) $n_b = 10$	.98E-05	.79E-06	.13E-04	.19E-05
block impl.LU (b) $n_b = 20$	.46E-05	.13E-05	.87E-05	.24E-05
block impl.LU (c) $n_b = 20$	.67E-05	.61E-06	.70E-05	.12E-05
block impl.LU (d) $n_b = 20$	.17E-04	.50E-06	.21E-04	.17E-05
block impl.LU (b) $n_b = 50$	.20E-04	.12E-04	.53E-04	.97E-05
block impl.LU (c) $n_b = 50$	.15E-04	.62E-06	.25E-04	.13E-05
block impl.LU (d) $n_b = 50$	.15E-04	.62E-06	.31E-04	.13E-05

Table 2 - Test matrices (M2)

Method	dimension n=100		Dimension n=200	
	Solution error	Residual error	Solution error	Residual error
impl.LU	.17E-01	.46E-06	.22E-04	.74E-06
algorithm[3]	.43E-01	.72E-06	.69E-04	.10E-05
block impl.LU (b) $n_b = 2$	.18E-02	.61E-06	.24E-03	.11E-05
block impl.LU (c) $n_b = 2$	.61E-02	.53E-06	.10E-03	.11E-05
block impl.LU (d) $n_b = 2$	.32E-02	.72E-06	.16E-03	.21E-05
block impl.LU (b) $n_b = 5$	.21E-01	.66E-06	.94E-04	.12E-05
block impl.LU (c) $n_b = 5$	.81E-02	.55E-06	.15E-03	.10E-05
block impl.LU (d) $n_b = 5$	.27E-04	.71E-06	.12E-03	.17E-05
block impl.LU (b) $n_b = 10$	.84E-02	.14E-05	.14E-03	.12E-05
block impl.LU (c) $n_b = 10$	.42E-01	.55E-06	.16E-03	.12E-05
block impl.LU (d) $n_b = 10$	.19E-01	.12E-05	.16E-03	.14E-05
block impl.LU (b) $n_b = 20$	.87E-02	.41E-05	.20E-03	.12E-04
block impl.LU (c) $n_b = 20$	.25E-01	.71E-06	.51E-04	.12E-05
block impl.LU (d) $n_b = 20$	.31E-02	.65E-06	.52E-04	.14E-05
block impl.LU (b) $n_b = 50$	.11E-01	.27E-04	.21E-03	.24E-04
block impl.LU (c) $n_b = 50$	.28E-01	.74E-06	.47E-04	.13E-05
block impl.LU (d) $n_b = 50$	.28E-01	.74E-06	.55E-04	.15E-05
impl.LU	.32E-03	.52E-06	.79E-03	.86E-06
algorithm[3]	.17E-04	.61E-06	.60E-03	.13E-05
block impl.LU (b) $n_b = 2$	.21E-04	.63E-06	.10E-02	.10E-05
block impl.LU (c) $n_b = 2$	.25E-03	.63E-06	.24E-03	.99E-06
block impl.LU (d) $n_b = 2$	.81E-03	.79E-06	.84E-04	.19E-05
block impl.LU (b) $n_b = 5$	.23E-03	.84E-06	.52E-03	.11E-05
block impl.LU (c) $n_b = 5$	.74E-03	.61E-06	.65E-03	.11E-05
block impl.LU (d) $n_b = 5$	.88E-05	.65E-06	.22E-03	.19E-05
block impl.LU (b) $n_b = 10$	.48E-03	.68E-05	.61E-03	.11E-05
block impl.LU (c) $n_b = 10$	.43E-03	.56E-06	.31E-03	.11E-05
block impl.LU (d) $n_b = 10$	.80E-03	.81E-06	.50E-03	.19E-05
block impl.LU (b) $n_b = 20$	.31E-03	.60E-04	.22E-03	.15E-05
block impl.LU (c) $n_b = 20$	.99E-04	.64E-06	.61E-03	.11E-05
block impl.LU (d) $n_b = 20$	.48E-03	.72E-06	.40E-03	.17E-05
block impl.LU (b) $n_b = 50$	.30E-03	.16E-03	.22E-03	.49E-05
block impl.LU (c) $n_b = 50$	.86E-04	.54E-06	.44E-04	.14E-05
block impl.LU (d) $n_b = 50$	.86E-04	.54E-06	.43E-04	.17E-05
impl.LU	.18E-03	.60E-06	.43E-04	.10E-05
algorithm[3]	.12E-02	.94E-06	.46E-03	.91E-06
block impl.LU (b) $n_b = 2$	.41E-03	.80E-06	.87E-04	.11E-05
block impl.LU (c) $n_b = 2$	.56E-03	.73E-06	.56E-03	.11E-05
block impl.LU (d) $n_b = 2$	.32E-04	.14E-05	.13E-04	.28E-05
block impl.LU (b) $n_b = 5$	.12E-02	.71E-05	.29E-03	.98E-06
block impl.LU (c) $n_b = 5$	.13E-02	.68E-06	.14E-03	.11E-05
block impl.LU (d) $n_b = 5$	.58E-03	.10E-05	.12E-03	.22E-05
block impl.LU (b) $n_b = 10$	.36E-03	.69E-05	.25E-03	.12E-05
block impl.LU (c) $n_b = 10$	.20E-02	.87E-06	.17E-03	.11E-05
block impl.LU (d) $n_b = 10$	.94E-03	.78E-06	.42E-03	.19E-05
block impl.LU (b) $n_b = 20$	.14E-02	.16E-04	.28E-03	.16E-05
block impl.LU (c) $n_b = 20$	.76E-05	.67E-06	.48E-04	.12E-05
block impl.LU (d) $n_b = 20$	.89E-03	.81E-06	.22E-03	.17E-05
block impl.LU (b) $n_b = 50$	.98E-03	.39E-04	.18E-03	.76E-05
block impl.LU (c) $n_b = 50$	.16E-02	.75E-06	.14E-03	.14E-05
block impl.LU (d) $n_b = 50$	.16E-02	.75E-06	.13E-03	.15E-05
impl.LU	.94E-03	.48E-06	.16E-04	.87E-06
algorithm[3]	.11E-02	.54E-06	.61E-03	.90E-06
block impl.LU (b) $n_b = 2$	.51E-03	.53E-06	.25E-03	.10E-05
block impl.LU (c) $n_b = 2$	.40E-03	.53E-06	.18E-02	.11E-05
block impl.LU (d) $n_b = 2$	.29E-03	.89E-06	.24E-03	.21E-05
block impl.LU (b) $n_b = 5$	.10E-02	.56E-06	.31E-03	.11E-04
block impl.LU (c) $n_b = 5$	.25E-02	.56E-06	.21E-03	.11E-05
block impl.LU (d) $n_b = 5$	.49E-03	.71E-06	.78E-03	.18E-05
block impl.LU (b) $n_b = 10$	.31E-03	.63E-06	.42E-03	.16E-04
block impl.LU (c) $n_b = 10$	.18E-02	.58E-06	.37E-04	.11E-05
block impl.LU (d) $n_b = 10$	.24E-03	.55E-06	.12E-03	.18E-05
block impl.LU (b) $n_b = 20$	.13E-02	.93E-06	.18E-03	.14E-04
block impl.LU (c) $n_b = 20$	.74E-03	.51E-06	.24E-03	.12E-05
block impl.LU (d) $n_b = 20$	.59E-03	.71E-06	.41E-03	.19E-05
block impl.LU (b) $n_b = 50$	.14E-02	.89E-04	.13E-02	.67E-04
block impl.LU (c) $n_b = 50$	.14E-03	.68E-06	.39E-03	.14E-05
block impl.LU (d) $n_b = 50$	.14E-03	.68E-06	.40E-03	.16E-05

Table 3 - Test matrices (M3)

Method	dimension n=100		Dimension n=200	
	Solution error	Residual error	Solution error	Residual error
impl.LU	.34E-02	.68E-06	.13E-02	.51E-06
algorithm[3]	.76E-02	.17E-05	.75E-03	.66E-06
block impl.LU (b) $n_b = 2$	.22E-01	.37E-06	.10E-02	.93E-06
block impl.LU (c) $n_b = 2$	.18E-01	.41E-06	.38E-02	.64E-06
block impl.LU (d) $n_b = 2$	.44E-03	.25E-06	.10E-02	.61E-06
block impl.LU (b) $n_b = 5$	.22E-01	.28E-05	.96E-03	.97E-06
block impl.LU (c) $n_b = 5$	.85E-02	.51E-06	.45E-02	.45E-06
block impl.LU (d) $n_b = 5$	.23E-02	.90E-06	.13E-02	.42E-06
block impl.LU (b) $n_b = 10$	.89E-02	.43E-05	.63E-03	.45E-06
block impl.LU (c) $n_b = 10$	.44E-02	.85E-06	.78E-03	.36E-06
block impl.LU (d) $n_b = 10$	.29E-02	.48E-06	.15E-02	.48E-06
block impl.LU (b) $n_b = 20$	.10E+00	.32E-04	.41E-02	.30E-05
block impl.LU (c) $n_b = 20$	.73E-02	.85E-06	.66E-03	.46E-06
block impl.LU (d) $n_b = 20$	.53E-02	.32E-06	.21E-02	.77E-06
block impl.LU (b) $n_b = 50$	.22E+00	.52E-03	.48E-02	.14E-04
block impl.LU (c) $n_b = 50$	.54E-03	.79E-06	.18E-02	.63E-06
block impl.LU (d) $n_b = 50$	.54E-03	.79E-06	.18E-02	.64E-06
impl.LU	.36E-03	.39E-06	.51E-03	.20E-06
algorithm[3]	.33E-03	.20E-06	.91E-03	.59E-06
block impl.LU (b) $n_b = 2$	.72E-03	.30E-06	.38E-03	.27E-06
block impl.LU (c) $n_b = 2$	.20E-02	.54E-06	.34E-03	.31E-06
block impl.LU (d) $n_b = 2$	.53E-04	.14E-06	.29E-03	.32E-06
block impl.LU (b) $n_b = 5$	.95E-03	.37E-06	.37E-03	.39E-06
block impl.LU (c) $n_b = 5$	.12E-02	.38E-06	.38E-03	.26E-06
block impl.LU (d) $n_b = 5$	.41E-04	.11E-06	.24E-03	.22E-06
block impl.LU (b) $n_b = 10$	.24E-02	.49E-06	.96E-03	.14E-05
block impl.LU (c) $n_b = 10$	.14E-02	.51E-06	.30E-03	.41E-06
block impl.LU (d) $n_b = 10$	.15E-03	.21E-06	.35E-03	.18E-06
block impl.LU (b) $n_b = 20$	.66E-02	.19E-05	.39E-03	.79E-06
block impl.LU (c) $n_b = 20$	.54E-03	.34E-06	.48E-03	.24E-06
block impl.LU (d) $n_b = 20$	.20E-03	.48E-06	.24E-03	.17E-06
block impl.LU (b) $n_b = 50$	.21E-01	.33E-05	.23E-02	.42E-05
block impl.LU (c) $n_b = 50$	.11E-02	.30E-06	.78E-03	.74E-06
block impl.LU (d) $n_b = 50$	.11E-02	.30E-06	.78E-03	.74E-06
impl.LU	.10E-02	.25E-06	.53E-02	.39E-06
algorithm[3]	.92E-03	.17E-06	.28E-02	.30E-06
block impl.LU (b) $n_b = 2$	.46E-02	.24E-06	.22E-02	.80E-06
block impl.LU (c) $n_b = 2$	.53E-03	.38E-06	.86E-02	.83E-06
block impl.LU (d) $n_b = 2$	.46E-04	.19E-06	.15E-02	.18E-06
block impl.LU (b) $n_b = 5$	.47E-02	.49E-06	.12E-01	.18E-05
block impl.LU (c) $n_b = 5$	.31E-02	.30E-06	.47E-02	.54E-06
block impl.LU (d) $n_b = 5$	.26E-03	.15E-06	.14E-02	.47E-06
block impl.LU (b) $n_b = 10$	.37E-02	.69E-06	.98E-02	.25E-05
block impl.LU (c) $n_b = 10$	.24E-02	.29E-06	.26E-02	.32E-06
block impl.LU (d) $n_b = 10$	.12E-03	.16E-06	.20E-02	.32E-06
block impl.LU (b) $n_b = 20$	.18E-02	.27E-05	.43E-01	.37E-05
block impl.LU (c) $n_b = 20$	.39E-02	.23E-06	.19E-02	.36E-06
block impl.LU (d) $n_b = 20$	.44E-03	.13E-06	.21E-02	.69E-06
block impl.LU (b) $n_b = 50$	.47E+00	.29E-04	.22E+00	.29E-04
block impl.LU (c) $n_b = 50$	.95E-02	.32E-06	.23E-02	.33E-06
block impl.LU (d) $n_b = 50$	.95E-02	.32E-06	.23E-02	.33E-06
impl.LU	.21E-01	.23E-06	.82E-01	.37E-06
algorithm[3]	.13E-01	.27E-06	.56E-01	.15E-06
block impl.LU (b) $n_b = 2$	.37E-02	.24E-06	.30E-01	.31E-06
block impl.LU (c) $n_b = 2$	.72E-02	.25E-06	.25E-02	.25E-06
block impl.LU (d) $n_b = 2$	.42E-03	.21E-06	.72E-01	.11E-06
block impl.LU (b) $n_b = 5$	.16E+00	.15E-05	.72E-01	.28E-06
block impl.LU (c) $n_b = 5$	.98E-02	.19E-06	.52E-01	.16E-06
block impl.LU (d) $n_b = 5$	.11E-02	.12E-06	.43E-01	.16E-06
block impl.LU (b) $n_b = 10$	.58E-01	.11E-05	.47E+00	.20E-05
block impl.LU (c) $n_b = 10$	.55E-02	.17E-06	.94E-01	.20E-06
block impl.LU (d) $n_b = 10$	.32E-02	.13E-06	.73E-01	.29E-06
block impl.LU (b) $n_b = 20$	.42E+00	.64E-05	.26E+00	.22E-05
block impl.LU (c) $n_b = 20$	.71E-02	.26E-06	.18E-01	.26E-06
block impl.LU (d) $n_b = 20$	.79E-02	.26E-06	.32E-01	.17E-06
block impl.LU (b) $n_b = 50$	.21E+01	.32E-04	.25E+00	.12E-04
block impl.LU (c) $n_b = 50$	.11E-01	.27E-06	.98E-01	.33E-06
block impl.LU (d) $n_b = 50$	.11E-01	.27E-06	.98E-01	.33E-06

Table 4 - Test matrices (M4)

Method	Dimension n=100		Dimension n=200	
	Solution error	Residual error	Solution error	Residual error
impl.LU	.69E-05	.11E-06	.19E-04	.12E-06
algorithm[3]	.64E-05	.97E-07	.26E-04	.19E-06
block impl.LU (b) $n_b = 2$	.13E-04	.24E-06	.55E-04	.47E-06
block impl.LU (c) $n_b = 2$	.13E-04	.16E-06	.52E-04	.27E-06
block impl.LU (d) $n_b = 2$	.35E-05	.44E-06	.46E-05	.35E-06
block impl.LU (b) $n_b = 5$	.12E-04	.10E-06	.44E-04	.33E-06
block impl.LU (c) $n_b = 5$	.82E-05	.16E-06	.67E-04	.46E-06
block impl.LU (d) $n_b = 5$	.29E-05	.35E-06	.50E-05	.87E-06
block impl.LU (b) $n_b = 10$	.20E-04	.25E-06	.61E-04	.65E-06
block impl.LU (c) $n_b = 10$	.18E-04	.65E-06	.92E-04	.12E-05
block impl.LU (d) $n_b = 10$	.33E-05	.57E-07	.36E-05	.23E-06
block impl.LU (b) $n_b = 20$	.14E-04	.26E-06	.12E-03	.11E-05
block impl.LU (c) $n_b = 20$	.15E-04	.14E-06	.10E-03	.57E-06
block impl.LU (d) $n_b = 20$	.40E-05	.49E-07	.78E-05	.13E-06
block impl.LU (b) $n_b = 50$	.23E-04	.43E-06	.16E-03	.95E-06
block impl.LU (c) $n_b = 50$	.13E-04	.21E-06	.46E-04	.85E-06
block impl.LU (d) $n_b = 50$	.13E-04	.21E-06	.18E-04	.11E-06
impl.LU	.28E-04	.87E-07	.21E-04	.28E-06
algorithm[3]	.28E-04	.25E-06	.24E-04	.90E-07
block impl.LU (b) $n_b = 2$	.38E-04	.36E-06	.62E-04	.31E-06
block impl.LU (c) $n_b = 2$	.17E-04	.29E-06	.57E-04	.25E-06
block impl.LU (d) $n_b = 2$	.42E-05	.49E-06	.65E-05	.42E-07
block impl.LU (b) $n_b = 5$	.13E-04	.11E-06	.67E-04	.26E-06
block impl.LU (c) $n_b = 5$	.22E-04	.28E-06	.12E-03	.73E-06
block impl.LU (d) $n_b = 5$	.42E-05	.52E-06	.59E-05	.27E-06
block impl.LU (b) $n_b = 10$	.32E-04	.12E-06	.98E-04	.40E-06
block impl.LU (c) $n_b = 10$	.28E-04	.27E-06	.92E-04	.44E-06
block impl.LU (d) $n_b = 10$	.50E-05	.35E-06	.88E-05	.12E-05
block impl.LU (b) $n_b = 20$	.58E-04	.84E-06	.11E-03	.15E-05
block impl.LU (c) $n_b = 20$	.16E-04	.49E-06	.73E-04	.72E-06
block impl.LU (d) $n_b = 20$	.50E-05	.29E-06	.10E-04	.61E-06
block impl.LU (b) $n_b = 50$	.23E-03	.28E-05	.97E-04	.41E-06
block impl.LU (c) $n_b = 50$	.32E-04	.30E-06	.12E-03	.15E-05
block impl.LU (d) $n_b = 50$	.32E-04	.30E-06	.18E-04	.17E-06
impl.LU	.68E-03	.29E-06	.37E-02	.35E-06
algorithm[3]	.37E-03	.13E-06	.14E-02	.11E-06
block impl.LU (b) $n_b = 2$	.10E-02	.17E-06	.78E-02	.26E-06
block impl.LU (c) $n_b = 2$	.83E-03	.21E-06	.72E-02	.23E-06
block impl.LU (d) $n_b = 2$	.28E-03	.13E-06	.16E-02	.20E-06
block impl.LU (b) $n_b = 5$	.59E-03	.15E-06	.58E-02	.22E-06
block impl.LU (c) $n_b = 5$	.10E-02	.15E-06	.63E-02	.20E-06
block impl.LU (d) $n_b = 5$	.33E-03	.19E-06	.22E-02	.26E-06
block impl.LU (b) $n_b = 10$	.84E-03	.19E-06	.68E-02	.24E-06
block impl.LU (c) $n_b = 10$	.87E-03	.20E-06	.54E-02	.23E-06
block impl.LU (d) $n_b = 10$	.32E-03	.22E-06	.19E-02	.39E-06
block impl.LU (b) $n_b = 20$	.13E-02	.50E-06	.64E-02	.23E-06
block impl.LU (c) $n_b = 20$	.54E-03	.23E-06	.42E-02	.28E-06
block impl.LU (d) $n_b = 20$	.32E-03	.22E-06	.15E-02	.22E-06
block impl.LU (b) $n_b = 50$	.12E-01	.25E-05	.12E-01	.11E-05
block impl.LU (c) $n_b = 50$	.65E-03	.29E-06	.41E-02	.48E-06
block impl.LU (d) $n_b = 50$	.65E-03	.29E-06	.14E-02	.53E-06
impl.LU	.49E-03	.15E-06	.20E-02	.22E-06
algorithm[3]	.50E-03	.14E-06	.22E-02	.20E-06
block impl.LU (b) $n_b = 2$	.10E-02	.18E-06	.37E-02	.26E-06
block impl.LU (c) $n_b = 2$	.88E-03	.16E-06	.42E-02	.24E-06
block impl.LU (d) $n_b = 2$	.15E-03	.19E-06	.48E-03	.22E-06
block impl.LU (b) $n_b = 5$	.12E-02	.17E-06	.37E-02	.25E-06
block impl.LU (c) $n_b = 5$	.13E-02	.18E-06	.34E-02	.25E-06
block impl.LU (d) $n_b = 5$	.20E-03	.15E-06	.88E-03	.23E-06
block impl.LU (b) $n_b = 10$	.14E-02	.19E-06	.57E-02	.28E-06
block impl.LU (c) $n_b = 10$	.11E-02	.17E-06	.34E-02	.26E-06
block impl.LU (d) $n_b = 10$	.49E-03	.15E-06	.11E-02	.23E-06
block impl.LU (b) $n_b = 20$	.28E-01	.40E-05	.91E-01	.37E-05
block impl.LU (c) $n_b = 20$	.87E-03	.21E-06	.35E-02	.26E-06
block impl.LU (d) $n_b = 20$	.73E-03	.18E-06	.15E-02	.21E-06
block impl.LU (b) $n_b = 50$	.32E+00	.45E-04	.55E+00	.29E-04
block impl.LU (c) $n_b = 50$	.82E-03	.15E-06	.36E-02	.24E-06
block impl.LU (d) $n_b = 50$	.82E-03	.15E-06	.25E-02	.23E-06

Table 5 - Comparison of solution error

	n_b	imp. LU	algo- rithm [3]	Block implicit					
				LU(b)	LU(c)	LU(d)	LU(b)	LU(c)	LU(d)
				2	2	2	5	5	5
imp.LU			106/9	107/9	113/10	72/10	122/9	115/11	78/10
algorithm [3]		85/9		98/8	108/8	66/6	102/11	106/6	72/7
imp.LU(b)	2	84/9	94/8		92/14	54/12	99/12	92/14	62/10
imp.LU(c)	2	77/10	84/8	94/14		48/14	95/13	95/11	59/7
imp.LU(d)	2	118/10	128/6	134/12	138/14		135/12	127/9	99/10
imp.LU(b)	5	69/9	87/11	89/12	92/13	53/12		95/12	55/10
imp.LU(c)	5	74/11	88/6	94/14	94/11	64/9	93/12		61/11
imp.LU(d)	5	112/10	121/7	128/10	134/7	91/10	135/10	128/11	
imp.LU(b)	10	62/8	70/8	77/10	88/9	58/6	79/10	83/9	58/10
imp.LU(c)	10	73/11	89/9	99/7	105/9	66/9	97/10	90/11	68/10
imp.LU(d)	10	107/11	119/9	126/10	121/8	67/11	123/13	119/11	73/11
imp.LU(b)	20	50/7	58/5	57/8	63/10	43/9	59/8	61/6	42/8
imp.LU(c)	20	70/11	96/7	101/8	93/9	52/10	100/12	102/13	54/11
imp.LU(d)	20	96/12	107/7	105/8	118/7	64/7	119/10	110/9	67/12
imp.LU(b)	50	29/7	42/5	30/11	35/9	20/8	36/10	39/11	19/7
imp.LU(c)	50	79/11	94/8	88/10	96/10	61/7	98/12	89/14	62/10
imp.LU(d)	50	81/11	95/9	91/10	96/10	62/7	99/12	92/12	63/10

		Block implicit								
		LU(b)	LU(c)	LU(d)	LU(b)	LU(c)	LU(d)	LU(b)	LU(c)	LU(d)
		10	10	10	20	20	20	50	50	50
alg.[3]	n_b	130/8	116/11	82/11	143/7	119/11	92/12	164/7	110/11	108/11
		122/8	102/9	72/9	137/5	97/7	86/7	153/5	98/8	96/9
imp.LU(b)	2	113/10	94/7	64/10	135/8	91/8	87/8	159/11	102/10	99/10
imp.LU(c)	2	103/9	86/9	71/8	127/10	98/9	75/7	156/9	94/10	94/10
imp.LU(d)	2	136/6	125/9	122/11	148/9	138/10	129/7	172/8	132/7	131/7
imp.LU(b)	5	111/10	93/10	64/13	133/8	88/12	71/10	154/10	90/12	89/12
imp.LU(c)	5	108/9	99/11	70/11	133/6	85/13	81/9	150/11	97/14	96/12
imp.LU(d)	5	132/10	122/10	116/11	150/8	135/11	121/12	174/7	128/10	127/10
imp.LU(b)	10		79/14	65/8	126/9	80/11	72/9	151/8	82/10	81/10
imp.LU(c)	10	107/14		69/16	127/8	88/14	80/11	149/10	79/13	79/13
imp.LU(d)	10	127/8	115/16		140/8	115/15	112/15	166/6	117/15	117/15
imp.LU(b)	20	65/9	65/8	52/8		62/7	50/9	138/12	56/8	57/8
imp.LU(c)	20	109/11	98/14	70/15	131/7		65/16	161/6	88/13	87/13
imp.LU(d)	20	119/9	109/11	73/15	141/9	119/16		161/6	115/14	113/14
imp.LU(b)	50	41/8	41/10	28/6	50/12	33/6	33/6		32/10	31/10
imp.LU(c)	50	108/10	108/13	68/15	136/8	99/13	71/14	158/10		5/187
imp.LU(d)	50	109/10	108/13	68/15	135/8	100/13	73/14	159/10	8/187	

Table 6 - Comparison of residual error

		imp. LU	algo- rithm [3]	Block implicit					
				LU(b)	LU(c)	LU(d)	LU(b)	LU(c)	LU(d)
	n_b			2	2	2	5	5	5
imp.LU			136/10	149/14	138/16	146/8	162/9	133/10	144/11
algorithm [3]		54/10		99/12	83/13	131/10	125/8	84/16	127/8
imp.LU(b)	2	37/14	89/12		70/21	128/9	130/14	65/18	127/10
imp.LU(c)	2	46/16	104/13	109/21		139/9	137/13	85/18	138/12
imp.LU(d)	2	46/8	59/10	63/9	52/9		88/8	47/9	81/18
imp.LU(b)	5	29/9	67/8	56/14	50/13	104/8		44/15	98/11
imp.LU(c)	5	57/10	100/16	117/18	97/18	144/9	141/15		140/12
imp.LU(d)	5	45/11	65/8	63/10	50/12	101/18	91/11	48/12	
imp.LU(b)	10	12/6	30/9	21/7	15/8	65/11	31/12	13/10	57/11
imp.LU(c)	10	43/13	98/11	98/17	71/23	140/9	120/17	75/22	136/11
imp.LU(d)	10	47/10	70/8	68/10	46/9	110/18	96/9	48/9	97/20
imp.LU(b)	20	7/6	8/7	5/6	4/7	15/8	9/6	4/6	12/8
imp.LU(c)	20	38/12	83/9	78/15	66/13	139/11	121/12	57/15	135/12
imp.LU(d)	20	49/9	74/11	75/9	54/10	127/15	102/10	55/11	121/13
imp.LU(b)	50	4/6	6/6	1/7	2/6	1/7	2/8	4/6	0/6
imp.LU(c)	50	40/13	79/12	85/12	61/14	134/10	111/14	54/15	135/10
imp.LU(d)	50	42/13	80/12	87/12	63/13	135/10	113/14	56/14	136/11

	n_b	Block implicit								
		LU(b)	LU(c)	LU(d)	LU(b)	LU(c)	LU(d)	LU(b)	LU(c)	LU(d)
		10	10	10	20	20	20	50	50	50
imp.LU		182/6	144/13	143/10	187/6	150/12	142/9	190/6	147/13	145/13
alg. [3]		161/9	91/11	122/8	185/7	108/9	115/11	188/6	109/12	108/12
imp.LU(b)	2	172/7	85/17	122/10	189/6	107/15	116/9	192/7	103/12	101/12
imp.LU(c)	2	177/8	106/23	145/9	189/7	121/13	136/10	192/6	125/14	124/13
imp.LU(d)	2	124/11	51/9	72/18	177/8	50/11	58/15	192/7	56/10	55/10
imp.LU(b)	5	157/12	63/17	95/9	185/6	67/12	88/10	190/8	75/14	73/14
imp.LU(c)	5	177/10	103/22	143/9	190/6	128/15	134/11	190/6	131/15	130/14
imp.LU(d)	5	132/11	53/11	83/20	180/8	53/12	66/13	194/6	55/10	53/11
imp.LU(b)	10		19/14	56/7	175/7	26/12	48/7	190/7	31/10	30/9
imp.LU(c)	10	167/14		140/15	187/8	104/18	130/11	190/6	116/16	116/15
imp.LU(d)	10	137/7	45/15		182/8	61/10	71/14	192/6	61/12	59/13
imp.LU(b)	20	18/7	5/8	10/8		4/9	7/8	180/7	5/7	6/6
imp.LU(c)	20	162/12	78/18	129/10	187/9		115/13	192/6	101/13	101/12
imp.LU(d)	20	145/7	59/11	115/14	185/8	72/13		191/6	67/14	66/16
imp.LU(b)	50	3/7	4/6	2/6	13/7	2/6	3/6		1/7	0/7
imp.LU(c)	50	159/10	68/16	127/12	188/7	86/13	119/14	192/7		10/187
imp.LU(d)	50	161/9	69/15	128/13	188/6	87/12	118/16	193/7	3/187	

MONOTONE ITERATIONS FOR DIFFERENTIAL PROBLEMS

TADEUSZ JANKOWSKI

Department of Differential Equations, Technical University of Gdansk
11/12 Narutowicz Str., 80–952 Gdansk, Poland
tjank@mif.pg.gda.pl

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Abstract. The purpose of this survey paper is to formulate existence results for parameterized non-linear boundary value problems employing the method of upper and lower solutions and the method of quasilinearization.

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1. Introduction

Let us consider the differential equation of first order with a parameter of the form

$$\begin{cases} x'(t) = f(t, x(t), \lambda), & t \in J = [0, b] \\ x(0) = k_0 \in R & G(x(b), \lambda) = 0 \end{cases} \quad (1.1)$$

where $f \in C(J \times R \times R, R)$, and $G \in C(R \times R, R)$. By a solution of (1.1) we mean a pair $(x, \lambda) \in C^1(J, R) \times R$ for which problem (1.1) is satisfied. Problems with a parameter have been considered for many years. Some of them appeared as mathematical models of physical systems (see, for example [16]).

An important area of research in the qualitative theory of differential equations is the study of existence of solutions. Existence/uniqueness theorems for problem (1.1) can be formulated under the assumption that f and G satisfy the Lipschitz condition with respect to the last two variables with suitable Lipschitz constants or Lipschitz functions (see [1, 2, 8, 15]). In the above mentioned papers the method of successive approximations and comparison technique are used to obtain sufficient conditions on existence/uniqueness solutions for problems with a parameter. An interesting technique for proving existence results is the method based on upper and lower solutions (for details, see [9]). The purpose of this paper is to formulate existence results for problem (1.1) employing the method of upper and lower solutions. As we see one-sided Lipschitz conditions are imposed on f and G (Theorems 1 and 2). If we apply the method of quasilinearization (for details, see [10]), then we can construct monotone sequences which converge quadratically to the unique solution of problem (1.1). We show that if we replace f and G by the sum of two corresponding functions,

then monotone iterations converge quadratically to the unique solution of problem (3.1) (see Theorem 3). Theorem 4 generalizes this result. Theorem 5 concerns a more general case when some Lipschitz functions are added to the right-hand sides of problem (3.1). In this case the convergence is only semi-quadratic. This is a survey paper connected with the above technique for problems of type (1.1).

2. Extremal solutions of problem (1.1)

A pair $(v, \alpha) \in C^1(J, R) \times R$ is said to be a lower solution of (1.1) if

$$\begin{cases} v'(t) \leq f(t, v(t), \alpha), & t \in J \\ v(0) = k_0 & 0 \leq G(v(b), \alpha), \end{cases}$$

and an upper solution of (1.1) if the above inequalities are reversed.

The next two theorems give constructive sufficient conditions when problem (1.1) has minimal and maximal solutions. Note that one-sided Lipschitz conditions are imposed on f and G .

Theorem 1 (see [4]). Assume that $f \in C(J \times R \times R, R)$, $G \in C(R \times R, R)$ and

1° $(y_0, \lambda_0), (z_0, \gamma_0) \in C^1(J, R) \times R$ are lower and upper solutions of problem (1.1) such that $y_0(t) \leq z_0(t)$, $t \in J$ and $\lambda_0 \leq \gamma_0$,

2° f is nondecreasing with respect to the last variable,

2° $f(t, \bar{u}, \lambda) - f(t, u, \lambda) \geq -M(\bar{u} - u) =$ for $y_0 \leq u \leq \bar{u} \leq z_0$ with $M \geq 0$,

4° $G(\bar{u}, \lambda) - G(u, \lambda) \geq Q(\bar{u} - u)$ for $y_0 \leq u \leq \bar{u} \leq z_0$, $\lambda_0 \leq \lambda \leq \gamma_0$ with $Q \geq 0$,

5° $G(u, \bar{\lambda}) - G(u, \lambda) \geq -N(\bar{\lambda} - \lambda)$ for $\lambda_0 \leq \lambda \leq \bar{\lambda} \leq \gamma_0$, $y_0 \leq u \leq z_0$ with $N > 0$.

Then there exist monotone sequences $\{y_n, \lambda_n\}$, $\{z_n, \gamma_n\}$ such that $y_n(t) \rightarrow y(t)$, $z_n(t) \rightarrow z(t)$, $t \in J$ and $\lambda_n \rightarrow \lambda$, $\gamma_n \rightarrow \gamma$ as $n \rightarrow \infty$ and this convergence is uniform and monotone on J . Moreover, (y, λ) and (z, γ) are minimal and maximal solutions of problem (1.1), respectively.

Remark 1. We observe that the special cases when f is monotone nondecreasing with respect to the second variable and G is monotone nondecreasing with respect to the first and second variables are covered by Theorem 1. To see this, it is enough to put $M = 0$ in condition 3°, and $Q = 0$ in condition 4°. If G is monotone nondecreasing with respect to the second variable, then there exists $N > 0$ such that for $\bar{\lambda} \geq \lambda$ we have

$$G(u, \bar{\lambda}) - G(u, \lambda) \geq 0 \geq -N(\bar{\lambda} - \lambda),$$

which proves that condition 5° holds also.

Example (see [4]). Let $J = [0, 1]$,

$$\begin{cases} x'(t) = x^2(t) + \frac{1}{3}[\sin^2 x(t) + t^2 + g(\lambda)] \equiv f(t, x(t), \lambda), & t \in J, x(0) = 0 \\ 0 = x(1.1) + \frac{1}{10} \sin[x(1.1) + \lambda] - \lambda \equiv G(x(1.1), \lambda) \end{cases}$$

where

$$g(\lambda) = \begin{cases} 0 & \text{if } \lambda \leq 0, \\ \lambda^2 & \text{if } 0 < \lambda < 1, \\ 1 & \text{if } 1 \leq \lambda. \end{cases}$$

Since

$$\frac{1}{3}t^2 \leq f(t, x, \lambda) \leq 1 + x^2,$$

then

$$y_0(t) = \frac{1}{9}t^3, \quad z_0(t) = \tan t, \quad t \in J.$$

Similarly,

$$\frac{1}{9} - \frac{1}{10} - \lambda \leq x(1.1) - \frac{1}{10} - \lambda \leq G(x(1.1), \lambda) \leq x(1.1) + \frac{1}{10} - \lambda \leq \tan 1 + \frac{1}{10} - \lambda,$$

then

$$\lambda_0 = \frac{1}{90} \approx 0.01, \quad \gamma_0 = \tan 1 + \frac{1}{10} \approx 1.66.$$

Note that (y_0, λ_0) , (z_0, γ_0) are lower and upper solutions of our problem, respectively, and $y_0(t) \leq z_0(t)$, $t \in J$, $\lambda_0 < \gamma_0$. Condition 1° of Theorem 1 holds since g is nondecreasing in λ . Let $\bar{u} \geq u$ and $\bar{\lambda} \geq \lambda$. Using the mean value theorem we see that

$$f(t, \bar{u}, \lambda) - f(t, u, \lambda) = (\bar{u})^2 - u^2 + \frac{1}{3}[\sin^2 \bar{u} - \sin^2 u] \geq \frac{1}{3}[\sin^2 \bar{u} - \sin^2 u] \geq -\frac{1}{3}(\bar{u} - u),$$

$$G(\bar{u}, \lambda) - G(u, \lambda) = \bar{u} - u + \frac{1}{10}[\sin(\bar{u} + \lambda) - \sin(u + \lambda)] \geq \frac{9}{10}(\bar{u} - u),$$

$$G(u, \bar{\lambda}) - G(u, \lambda) = \frac{1}{10}[\sin(u + \bar{\lambda}) - \sin(u + \lambda)] - (\bar{\lambda} - \lambda) \geq -\frac{11}{9}(\bar{\lambda} - \lambda).$$

All assumptions of Theorem 1 hold, its assertion is satisfied for this problem.

Theorem 2 (see [4]). Let $f \in C(J \times R \times R, R)$, $G \in C(R \times R, R)$ and }

1° $(y_0, \lambda_0), (z_0, \gamma_0) \in C^1(J, R) \times R$ are lower and upper solutions of problem (1.1) such that $y_0(t) \leq z_0(t)$, $t \in J$ and $\lambda_0 \leq \gamma_0$,

2° $f(t, u, \bar{\lambda}) - f(t, u, \lambda) \geq Q(\bar{\lambda} - \lambda)$ for $\lambda_0 \leq \lambda \leq \bar{\lambda} \leq \gamma_0$, $y_0 \leq u \leq z_0$ with $Q \geq 0$,

3° $f(t, \bar{u}, \lambda) - f(t, u, \lambda) \geq -M(\bar{u} - u)$ for $y_0 \leq u \leq \bar{u} \leq z_0$ with $M \geq 0$,

4° G is nondecreasing with respect to the first variable,

5° $G(u, \bar{\lambda}) - G(u, \lambda) \geq -N(\bar{\lambda} - \lambda)$ for $\lambda_0 \leq \lambda \leq \bar{\lambda} \leq \gamma_0$, $y_0 \leq u \leq z_0$, with $N > 0$.

Then there exist monotone sequences $\{y_n, \lambda_n\}$, $\{z_n, \gamma_n\}$ such that $y_n(t) \rightarrow y(t)$, $z_n(t) \rightarrow z(t)$, $t \in J$ and $\lambda_n \rightarrow \lambda$, $\gamma_n \rightarrow \gamma$ as $n \rightarrow \infty$ and this convergence is uniform and monotone on J . Moreover, (y, λ) and (z, γ) are minimal and maximal solutions of problem (1.1), respectively.

Remark 2. Our results may be extended to a finite system of differential equations of type (1.1). Although this extension follows similar ideas, such a case requires special

care since we need to split variables as it is evident in the technique of monotone iterations.

3. Quasilinearization method

Let $y_0, z_0 \in C^1(J, R)$ and $\lambda_0, \gamma_0 \in R$ such that $y_0(t) \leq z_0(t)$ on J and $\lambda_0 \leq \gamma_0$. Define the closed sets:

$$\begin{aligned}\Omega &= \{(t, y, \lambda) : t \in J, y_0(t) \leq y \leq z_0(t), \lambda_0 \leq \lambda \leq \gamma_0\}, \\ \bar{\Omega} &= \{(y, \lambda) : y_0(b) \leq y \leq z_0(b), \lambda_0 \leq \lambda \leq \gamma_0\}.\end{aligned}$$

Now, instead of (1.1), we shall consider the following problem

$$\begin{cases} x'(t) = f(t, x(t), \lambda) + g(t, x(t), \lambda), & t \in J, \quad x(0) = k_0 \in R, \\ G(x(b), \lambda) + H(x(b), \lambda) = 0, \end{cases} \quad (3.1)$$

where $f, g \in C(J \times R \times R, R)$, $G, H \in C(R \times R, R)$.

The method of quasilinearization offers monotone sequences of approximate solutions that converge quadratically to the unique solution. This problem is considered in the next theorems using less restrictive assumptions.

Theorem 3 (see [5]). Let $f, g, f_y, g_y, f_\lambda, g_\lambda \in C(\Omega, R)$, $G, H, G_y, H_y, G_\lambda, H_\lambda \in C(\bar{\Omega}, R)$. Assume that:

1° $(y_0, \lambda_0), (z_0, \gamma_0) \in C^1(J, R) \times R$ are lower and upper solutions of problem (3.1), respectively, and such that $y_0(t) \leq z_0(t)$, $t \in J$ and $\lambda_0 \leq \gamma_0$,

2° $-G_\lambda(u, v) \geq K_1$, $-H_\lambda(u, v) \geq K_2$, $K = K_1 + K_2 > 0$, $\bar{L}_1 \leq G_y(u, v) \leq L_1$, $\bar{L}_2 \leq H_y(u, v) \leq L_2$, $\bar{L} = \bar{L}_1 + \bar{L}_2 \geq 0$, $\bar{M}_1 \leq f_\lambda(t, \bar{u}, \bar{v}) \leq M_1$, $\bar{M}_2 \leq g_\lambda(t, \bar{u}, \bar{v}) \leq M_2$, $\bar{M} = \bar{M}_1 + \bar{M}_2 \geq 0$

for $(u, v) \in \bar{\Omega}$, $(t, \bar{u}, \bar{v}) \in \Omega$, and put $L = L_1 + L_2$, $M = M_1 + M_2$,

3° $|f_y(t, u, v)| \leq N_1$, $|g_y(t, u, v)| \leq N_2$ for $(t, u, v) \in \Omega$, and put $N = N_1 + N_2$,

4° $S(b) < 1$, where

$$S(t) = \begin{cases} \frac{ML}{K}t & \text{if } N = 0, \\ \frac{ML}{KN}[\exp(Nt) - 1] & \text{if } N > 0, \end{cases}$$

5° $f_{yy}, f_{y\lambda}, f_{\lambda y}, f_{\lambda\lambda}, g_{yy}, g_{y\lambda}, g_{\lambda y}, g_{\lambda\lambda}$ exist, are continuous and satisfy the relations:

$$f_{yy}(t, u, v) \geq 0, \quad f_{y\lambda}(t, u, v) \geq 0, \quad f_{\lambda\lambda}(t, u, v) \geq 0 \quad \text{for } (t, u, v) \in \Omega,$$

$$g_{yy}(t, u, v) \leq 0, \quad g_{y\lambda}(t, u, v) \leq 0, \quad g_{\lambda\lambda}(t, u, v) \leq 0 \quad \text{for } (t, u, v) \in \Omega,$$

6° $G_{yy}, G_{y\lambda}, G_{\lambda y}, G_{\lambda\lambda}, H_{yy}, H_{y\lambda}, H_{\lambda y}, H_{\lambda\lambda}$ exist, are continuous and satisfy the relations:

$$G_{yy}(u, v) \geq 0, \quad G_{y\lambda}(u, v) \geq 0, \quad G_{\lambda\lambda}(u, v) \geq 0 \quad \text{for } (u, v) \in \bar{\Omega},$$

$$H_{yy}(u, v) \leq 0, \quad H_{y\lambda}(u, v) \leq 0, \quad H_{\lambda\lambda}(u, v) \leq 0 \quad \text{for } (u, v) \in \bar{\Omega}.$$

Then there exist monotone sequences $\{y_n\}, \{z_n\}, \{\lambda_n\}, \{\gamma_n\}$ which converge uniformly to the unique solution (x, λ) of problem (3.1) and the convergence is quadratic, i.e.,

$$\begin{aligned} \max_{t \in J} |x(t) - y_{n+1}(t)| &\leq a_1 \max_{t \in J} |x(t) - y_n(t)|^2 + a_2 \max_{t \in J} |x(t) - z_n(t)|^2 \\ &\quad + a_3 |\lambda - \lambda_n|^2 + a_4 |\lambda - \gamma_n|^2, \\ |\lambda - \lambda_{n+1}| &\leq b_1 \max_{t \in J} |x(t) - y_n(t)|^2 + b_2 \max_{t \in J} |x(t) - z_n(t)|^2 \\ &\quad + b_3 |\lambda - \lambda_n|^2 + b_4 |\lambda - \gamma_n|^2, \\ \max_{t \in J} |x(t) - z_{n+1}(t)| &\leq \bar{a}_1 \max_{t \in J} |x(t) - y_n(t)|^2 + \bar{a}_2 \max_{t \in J} |x(t) - z_n(t)|^2 \\ &\quad + \bar{a}_3 |\lambda - \lambda_n|^2 + \bar{a}_4 |\lambda - \gamma_n|^2, \\ |\lambda - \gamma_{n+1}| &\leq \bar{b}_1 \max_{t \in J} |x(t) - y_n(t)|^2 + \bar{b}_2 \max_{t \in J} |x(t) - z_n(t)|^2 \\ &\quad + \bar{b}_3 |\lambda - \lambda_n|^2 + \bar{b}_4 |\lambda - \gamma_n|^2 \end{aligned}$$

for some nonnegative constants $a_i, b_i, \bar{a}_i, \bar{b}_i$, $i = 1, 2, 3, 4$.

4. Generalization of the quasilinearization method

We will use the following notation $f \in C^{0,2,2}(\Omega, R)$ which means that $f, f_y, f_\lambda, f_{yy}, f_{y\lambda}, f_{\lambda y}, f_{\lambda\lambda} \in C(\Omega, R)$. The next theorem generalizes the result of Theorem 3.

Theorem 4 (see [6]). Let $f, g \in C^{0,2,2}(\Omega, R)$, $G, H \in C^2(\bar{\Omega}, R)$. Assume that:

1° $(y_0, \lambda_0), (z_0, \gamma_0) \in C^1(J, R) \times R$ are lower and upper solutions of problem (3.1), respectively, and such that $y_0(t) \leq z_0(t)$, $t \in J$ and $\lambda_0 \leq \gamma_0$,

2° $-G_\lambda(u, v) \geq K_1$, $-H_\lambda(u, v) \geq K_2$ and $K = K_1 + K_2 > 0$,

$$\bar{L}_1 \leq G_y(u, v) \leq L_1, \quad \bar{L}_2 \leq H_y(u, v) \leq L_2, \quad \bar{L} = \bar{L}_1 + \bar{L}_2,$$

$\bar{M}_1 \leq f_\lambda(t, \bar{u}, \bar{v}) \leq M_1$, $\bar{M}_2 \leq g_\lambda(t, \bar{u}, \bar{v}) \leq M_2$, $\bar{M} = \bar{M}_1 + \bar{M}_2$ for $(u, v) \in \bar{\Omega}$, $(t, \bar{u}, \bar{v}) \in \Omega$, and put $L = L_1 + L_2$, $M = M_1 + M_2$,

3° $|f_y(t, u, v)| \leq N_1$, $|g_y(t, u, v)| \leq N_2$ for $(t, u, v) \in \Omega$, and put $N = N_1 + N_2$,

4° $S(b) < 1$, where S is defined as in condition 4° of Theorem 3,

5° $\Psi, \Phi \in C^{0,2,2}(\Omega, R)$ and satisfy the relations for $(t, u, v) \in \Omega$ with $F = f + \Phi$, $T = g + \Psi$

$$\Phi_{yy}(t, u, v) \geq 0, \quad \Phi_{y\lambda}(t, u, v) \geq 0, \quad \Phi_{\lambda\lambda}(t, u, v) \geq 0,$$

$$F_{yy}(t, u, v) \geq 0, \quad F_{y\lambda}(t, u, v) \geq 0, \quad F_{\lambda\lambda}(t, u, v) \geq 0,$$

$$\Psi_{yy}(t, u, v) \leq 0, \quad \Psi_{y\lambda}(t, u, v) \leq 0, \quad \Psi_{\lambda\lambda}(t, u, v) \leq 0,$$

$$T_{yy}(t, u, v) \leq 0, \quad T_{y\lambda}(t, u, v) \leq 0, \quad T_{\lambda\lambda}(t, u, v) \leq 0$$

6° $\Delta, \Gamma \in C^2(\bar{\Omega}, R)$ and satisfy the relations for $(u, v) \in \bar{\Omega}$ with $P = G + \Delta, \quad Q = H + \Gamma$

$$\begin{aligned} \Delta_{yy}(u, v) &\geq 0, \quad \Delta_{y\lambda}(u, v) \geq 0, \quad \Delta_{\lambda\lambda}(u, v) \geq 0, \\ P_{yy}(u, v) &\geq 0, \quad P_{y\lambda}(u, v) \geq 0, \quad P_{\lambda\lambda}(u, v) \geq 0, \\ \Gamma_{yy}(u, v) &\leq 0, \quad \Gamma_{y\lambda}(u, v) \leq 0, \quad \Gamma_{\lambda\lambda}(u, v) \leq 0, \\ Q_{yy}(u, v) &\leq 0, \quad Q_{y\lambda}(u, v) \leq 0, \quad Q_{\lambda\lambda}(u, v) \leq 0, \end{aligned}$$

7° $\bar{\eta} \leq \Gamma_y(u, v) \leq \eta, \quad \bar{\zeta} \leq \Delta_y(u, v) \leq \zeta, \quad \bar{L} + \bar{\zeta} - \zeta + \bar{\eta} - \eta \geq 0, \quad \bar{\vartheta} \leq \Phi_\lambda(t, \bar{u}, \bar{v}) \leq \vartheta, \quad \bar{\mu} \leq \Psi_\lambda(t, \bar{u}, \bar{v}) \leq \mu, \quad \bar{M} + \bar{\vartheta} - \vartheta + \bar{\mu} - \mu \geq 0$ for $(u, v) \in \bar{\Omega}, \quad (t, \bar{u}, \bar{v}) \in \Omega$.

Then there exist monotone sequences $\{y_n\}, \{z_n\}, \{\lambda_n\}, \{\gamma_n\}$ which converge uniformly to the unique solution of problem (3.1) and the convergence is quadratic.

Remark 3. Note that Theorem 4 contains, as a special case, the result of Theorem 3 if we put $\Phi(t, u, v) = \Psi(t, u, v) = 0$ for $(t, u, v) \in \Omega$, and $\Delta(\bar{u}, \bar{v}) = \Gamma(\bar{u}, \bar{v}) = 0$ for $(\bar{u}, \bar{v}) \in \bar{\Omega}$. Finally, we study the situation when problem (3.1) is replaced by the following one

$$\begin{cases} x'(t) = \bar{f}(t, x(t), \lambda), & t \in J, \quad x(0) = k_0, \\ 0 = \bar{G}(x(b), \lambda) \end{cases} \quad (4.1)$$

with $\bar{f} = f + \tilde{f} + g + \tilde{g}, \quad \bar{G} = G + \tilde{G} + H + \tilde{H}$. We get weak quadratic convergence only if we assume that $f, g \in C^{0,2,2}(\Omega, R), \quad G, H \in C^2(\bar{\Omega}, R), \quad \tilde{f}, \tilde{g} \in C^{0,1,1}(\Omega, R), \quad \tilde{G}, \tilde{H} \in C^1(\bar{\Omega}, R)$. This general case is considered in the next theorem.

Theorem 5 (see [7]). Let $f, g \in C^{0,2,2}(\Omega, R), \quad G, H \in C^2(\bar{\Omega}, R), \quad \tilde{f}, \tilde{g} \in C(\Omega, R), \quad \tilde{G}, \tilde{H} \in C(\bar{\Omega}, R)$. Assume that:

1° $(y_0, \lambda_0), (z_0, \gamma_0) \in C^1(J, R) \times R$ are lower and upper solutions of problem (4.1), respectively, and such that $y_0(t) \leq z_0(t), \quad t \in J$ and $\lambda_0 \leq \gamma_0$,

2° conditions 5° and 6° of Theorem 4 hold,

3° $-G_\lambda(u, v) \geq K_1, \quad -H_\lambda(u, v) \geq K_2, \quad \bar{L}_1 \leq G_y(u, v) \leq L_1, \quad \bar{L}_2 \leq H_y(u, v) \leq L_2, \quad \bar{M}_1 \leq f_\lambda(t, \bar{u}, \bar{v}) \leq M_1, \quad \bar{M}_2 \leq g_\lambda(t, \bar{u}, \bar{v}) \leq M_2$ for $(u, v) \in \bar{\Omega}, \quad (t, \bar{u}, \bar{v}) \in \Omega; \quad |f_y(t, u, v)| \leq N_1, \quad |g_y(t, u, v)| \leq N_2$ for $(t, u, v) \in \Omega$,

5° $\tilde{F}_y, \tilde{F}_\lambda, \tilde{\Phi}_y, \tilde{\Phi}_\lambda, \tilde{P}_y, \tilde{P}_\lambda, \tilde{\Delta}_y, \tilde{\Delta}_\lambda$ exist, are nondecreasing in the last two variables, while $\tilde{T}_y, \tilde{T}_\lambda, \tilde{\Psi}_y, \tilde{\Psi}_\lambda, \tilde{Q}_y, \tilde{Q}_\lambda, \tilde{\Gamma}_y, \tilde{\Gamma}_\lambda$ exist and are nonincreasing in the last two variables with $\tilde{F} = \tilde{f} + \tilde{\Phi}, \quad \tilde{T} = \tilde{g} + \tilde{\Psi}, \quad \tilde{P} = \tilde{G} + \tilde{\Delta}, \quad \tilde{Q} = \tilde{H} + \tilde{\Gamma}$,

6° $\bar{M}_3 \leq \tilde{f}_\lambda(t, \bar{u}, \bar{v}) \leq M_3, \quad \bar{M}_4 \leq \tilde{g}_\lambda(t, \bar{u}, \bar{v}) \leq M_4$,

$K_3 \leq -\tilde{G}_\lambda(u, v) \leq \bar{K}_3, \quad K_4 \leq -\tilde{H}_\lambda(u, v) \leq \bar{K}_4, \quad \bar{L}_3 \leq \tilde{G}_y(u, v) \leq L_3, \quad \bar{L}_4 \leq \tilde{H}_y(u, v) \leq L_4, \quad |\tilde{f}_y(t, \bar{u}, \bar{v})| \leq N_3, \quad |\tilde{g}_y(t, \bar{u}, \bar{v})| \leq N_4, \quad \bar{\eta}_1 \leq \tilde{\Gamma}_y(u, v) \leq \eta_1, \quad \bar{\zeta}_1 \leq \tilde{\Delta}_y(u, v) \leq \zeta_1, \quad \bar{\vartheta}_1 \leq \tilde{\Phi}_\lambda(t, \bar{u}, \bar{v}) \leq \vartheta_1, \quad \bar{\mu}_1 \leq \tilde{\Psi}_\lambda(t, \bar{u}, \bar{v}) \leq \mu_1$ for $(u, v) \in \Omega, \quad (t, \bar{u}, \bar{v}) \in \bar{\Omega}$

7° put $L = L_1 + L_2 + L_3 + L_4$, $K = K_1 + K_2 + K_3 + K_4 > 0$, $M = M_1 + M_2 + M_3 + M_4$, $N = N_1 + N_2 + N_3 + N_4$, $\bar{L} = \bar{L}_1 + \bar{L}_2 + \bar{L}_3 + \bar{L}_4 + \bar{\zeta}_1 - \zeta_1 + \bar{\eta}_1 - \eta_1$, $\bar{M} = \bar{M}_1 + \bar{M}_2 + \bar{M}_3 + \bar{M}_4 + \bar{\vartheta}_1 - \vartheta_1 + \bar{\mu}_1 - \mu_1$,

8° condition 7° of Theorem 4 holds,

9° $S(b) < 1$, where S is defined as in condition 4° of Theorem 3.

Then there exist monotone sequences $\{y_n\}, \{z_n\}, \{\lambda_n\}, \{\gamma_n\}$ which converge uniformly and monotonically to the unique solution of problem (4.1) and the convergence is semi-quadratic, i.e.,

$$\begin{aligned} \max_{t \in J} |x(t) - y_{n+1}(t)| &\leq a_1 \max_{t \in J} |x(t) - y_n(t)|^2 + a_2 \max_{t \in J} |x(t) - z_n(t)|^2 + a_3 |\lambda - \lambda_n|^2 \\ &\quad + a_4 |\lambda - \gamma_n|^2 + a_5 \max_{t \in J} |x(t) - y_n(t)| + a_6 |\lambda - \lambda_n|, \\ |\lambda - \lambda_{n+1}| &\leq b_1 \max_{t \in J} |x(t) - y_n(t)|^2 + b_2 \max_{t \in J} |x(t) - z_n(t)|^2 + b_3 |\lambda - \lambda_n|^2 \\ &\quad + b_4 |\lambda - \gamma_n|^2 + b_5 \max_{t \in J} |x(t) - y_n(t)| + b_6 |\lambda - \lambda_n|, \\ \max_{t \in J} |x(t) - z_{n+1}(t)| &\leq \bar{a}_1 \max_{t \in J} |x(t) - y_n(t)|^2 + \bar{a}_2 \max_{t \in J} |x(t) - z_n(t)|^2 + \bar{a}_3 |\lambda - \lambda_n|^2 \\ &\quad + \bar{a}_4 |\lambda - \gamma_n|^2 + \bar{a}_5 \max_{t \in J} |x(t) - z_n(t)| + \bar{a}_6 |\lambda - \gamma_n|, \\ |\lambda - \gamma_{n+1}| &\leq \bar{b}_1 \max_{t \in J} |x(t) - y_n(t)|^2 + \bar{b}_2 \max_{t \in J} |x(t) - z_n(t)|^2 + \bar{b}_3 |\lambda - \lambda_n|^2 \\ &\quad + \bar{b}_4 |\lambda - \gamma_n|^2 + \bar{b}_5 \max_{t \in J} |x(t) - z_n(t)| + \bar{b}_6 |\lambda - \gamma_n|, \end{aligned}$$

for some nonnegative constants $a_i, b_i, \bar{a}_i, \bar{b}_i$, $i = 1, 2, 3, 4, 5, 6$.

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A NOTE ON A THREE POINT BOUNDARY VALUE PROBLEM FOR SECOND ORDER DIFFERENTIAL INCLUSIONS

MOUFFAK BENCHOHRA

Département de Mathématiques, Université de Sidi Bel Abbès
BP 89, 22000 Sidi Bel Abbès, Algérie
benchohra@yahoo.com

SOTIRIS K. NTOUYAS

Department of Mathematics, University of Ioannina
451 10 Ioannina, Greece
sntouyas@cc.uoi.gr

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Abstract. In this paper we investigate the existence of solutions on a compact interval to a three-point boundary value problem for a class of second order differential inclusions. We shall rely on a fixed point theorem for condensing maps due to Martelli.

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Keywords: Three point boundary value problems, differential inclusion, existence, fixed point

1. Introduction

The existence of solutions on compact intervals for three-point boundary value problems for second order differential equations has received much attention in the last decade. We refer for instance to the papers of Boucherif and Bouguima [1], Gupta [2], [3], [4], [5], Gupta et al [6], [7], [8], Gupta and Trofimchuk [7], [8] and Marano [9]. However, to our knowledge, this type of problem has not been studied for the multivalued case.

The study of multi-point boundary value problems for second order ordinary differential equations was initiated by Il'in and Moiseev in [10], [11] motivated by the work of Bitsadze and Samarskii on nonlocal elliptic boundary value problems, [12], [13], [14].

The methods used are usually the topological transversality of Granas or the degree theory methods combined with Wirtinger type inequalities.

In this paper we shall prove a theorem which assures the existence of solutions defined on a compact real interval for the three-point boundary value problem (BVP

for short) of the second order differential inclusion

$$y'' \in F(t, y), \quad t \in J = [0, 1] \quad (1.1)$$

$$y(0) = 0, y(\eta) = y(1), \quad (1.2)$$

where $F : J \times \mathbf{R} \longrightarrow 2^{\mathbf{R}}$ is a multivalued map with compact convex values and $\eta \in (0, 1)$.

The method we are going to use is to reduce the existence of solutions to problem (1.1)-(1.2) to the search for fixed points of a suitable multivalued map on the Banach space $C(J, \mathbf{R})$. In order to prove the existence of fixed points, we shall rely on a fixed point theorem for condensing maps due to Martelli [15].

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

Let $(X, \|\cdot\|)$ be a Banach space. A multivalued map $G : X \longrightarrow 2^X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x \in B} G(x)$ is bounded in X for any bounded set B of X (i.e. $\sup_{x \in B} \{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

G is called upper semicontinuous (u.s.c.) on X if for each $x_* \in X$ the set $G(x_*)$ is a nonempty, closed subset of X , and if for each open set B of X containing $G(x_*)$, there exists an open neighbourhood V of x_* such that $G(V) \subseteq B$.

G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$, $y_n \in Gx_n$ imply $y_* \in Gx_*$).

G has a fixed point if there is $x \in X$ such that $x \in Gx$.

In the following $CC(X)$ denotes the set of all nonempty compact and convex subsets of X .

A multivalued map $G : J \longrightarrow CC(E)$ is said to be measurable if for each $x \in X$ the function $Y : J \longrightarrow \mathbf{R}$ defined by

$$Y(t) = d(x, G(t)) = \inf\{|x - z| : z \in G(t)\}$$

is measurable.

Definition 1 A multivalued map $F : J \times \mathbf{R} \longrightarrow 2^{\mathbf{R}}$ is said to be an L^1 -Carathéodory if

- (i) $t \longmapsto F(t, y)$ is measurable for each $y \in \mathbf{R}$;
- (ii) $y \longmapsto F(t, y)$ is upper semicontinuous for almost all $t \in J$;

(iii) For each $k > 0$, there exists $h_k \in L^1(J, \mathbf{R}_+)$ such that

$$\|F(t, y)\| = \sup\{\|v\| : v \in F(t, y)\} \leq h_k(t) \text{ for all } |y| \leq k$$

and for almost all $t \in J$.

An upper semi-continuous map $G : X \longrightarrow 2^X$ is said to be condensing if for any subset $B \subseteq X$ with $\alpha(B) \neq 0$, we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompactness. For properties of the Kuratowski measure, we refer to Banas and Goebel [16].

We remark that a completely continuous multivalued map is the easiest example of a condensing map. For more details on multivalued maps see the books by Deimling [17] and Hu and Papageorgiou [18].

We will need the following hypotheses:

(H1) $F : J \times \mathbf{R} \longrightarrow CC(\mathbf{R})$ is an L^1 -Carathéodory multivalued map.

(H2) There exists a function $H \in L^1(J, \mathbf{R}_+)$ such that

$$\|F(t, y)\| := \sup\{\|v\| : v \in F(t, y)\} \leq H(t) \text{ for almost all } t \in J \text{ and all } y \in \mathbf{R}.$$

Definition 2 A function $y : J \longrightarrow \mathbf{R}$ is called a solution for the BVP (1.1)-(1.2) if y and its first derivative are absolutely continuous and y'' (which exists almost everywhere) satisfies the differential inclusion (1.1) a.e. on J and the condition (1.2).

For the multivalued map F and for each $y \in C(J, \mathbf{R})$ we define $S_{F,y}^1$ by

$$S_{F,y}^1 = \{v \in L^1(J, \mathbf{R}) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in J\}.$$

Our considerations are based on the following lemmas.

[19]. Let I be a compact real interval and X be a Banach space. Let F be a multivalued map satisfying (H1) and $S_{F,y}^1 \neq \emptyset$ for any $y \in C(I, X)$. Let Γ be a linear continuous mapping from $L^1(I, X)$ to $C(I, X)$, then the operator

$$\Gamma \circ S_F : C(I, X) \longrightarrow CC(C(I, X)), \quad y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(I, X) \times C(I, X)$.

Lemma 3 [15]. Let X be a Banach space and $N : X \longrightarrow CC(X)$ be a condensing map. If the set

$$\Omega := \{y \in X : \lambda y \in N(y) \text{ for some } \lambda > 1\}$$

is bounded, then N has a fixed point.

3. Main Result

Now, we are able to state and prove our main theorem.

Theorem 4 *Assume that hypotheses (H1)-(H2) hold. Then the BVP (1.1)-(1.2) has at least one solution on J .*

Proof. Let $C(J, \mathbf{R})$ be the Banach space provided with the norm

$$\|y\|_\infty := \sup\{\|y(t)\| : t \in J\}, \text{ for } y \in C(J, \mathbf{R}).$$

Transform the problem into a fixed point problem. Consider the multivalued map, $N : C(J, \mathbf{R}) \longrightarrow 2^{C(J, \mathbf{R})}$ defined by

$$N(y) = \left\{ h \in C(J, \mathbf{R}) : h(t) = \int_0^t (t-s)g(s)ds + \frac{t}{1-\eta} \int_0^\eta (\eta-s)g(s)ds - \frac{t}{1-\eta} \int_0^1 (1-s)g(s)ds \right\},$$

where

$$g \in S_{F,y} = \{g \in L^1(J, \mathbf{R}) : g(t) \in F(t, y(t)) \text{ for a.e. } t \in J\}.$$

Remark 5 (i) *It is clear that the fixed points of N are solutions to (1.1)-(1.2).*

(ii) *For each $y \in C(J, \mathbf{R})$ the set $S_{F,y}$ is nonempty (see Lasota and Opial [19]).*

We shall show that N satisfies the assumptions of Lemma 3. The proof will be given in several steps.

Step 1: $N(y)$ is convex for each $y \in C(J, \mathbf{R})$.

Indeed, if h_1, h_2 belong to $N(y)$, then there exist $g_1, g_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$h_i(t) = \int_0^t (t-s)g_i(s)ds + \frac{t}{1-\eta} \int_0^\eta (\eta-s)g_i(s)ds - \frac{t}{1-\eta} \int_0^1 (1-s)g_i(s)ds, \quad i = 1, 2.$$

Let $0 \leq \alpha \leq 1$. Then for each $t \in J$ we have

$$\begin{aligned} (\alpha h_1 + (1-\alpha)h_2)(t) &= \int_0^t (t-s)[\alpha g_1(s) + (1-\alpha)g_2(s)]ds \\ &+ \frac{t}{1-\eta} \int_0^\eta (\eta-s)[\alpha g_1(s) + (1-\alpha)g_2(s)]ds - \frac{t}{1-\eta} \int_0^1 (1-s)[\alpha g_1(s) + (1-\alpha)g_2(s)]ds. \end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), then

$$\alpha h_1 + (1-\alpha)h_2 \in N(y).$$

Step 2: N is bounded on bounded sets of $C(J, \mathbf{R})$.

Indeed, it is enough to show that there exists a positive constant c such that for each $h \in N(y)$, $y \in B_r = \{y \in C(J, \mathbf{R}) : \|y\|_\infty \leq r\}$ one has $\|h\|_\infty \leq c$.

If $h \in N(y)$, then there exists $g \in S_{F,y}$ such that for each $t \in J$ we have

$$h(t) = \int_0^t (t-s)g(s)ds + \frac{t}{1-\eta} \int_0^\eta (\eta-s)g(s)ds - \frac{t}{1-\eta} \int_0^1 (1-s)g(s)ds.$$

By (H1) we have for each $t \in J$ that

$$\|h(t)\| \leq \int_0^t h_r(s)ds + \frac{1}{1-\eta} \int_0^\eta (\eta-s)h_r(s)ds + \frac{1}{1-\eta} \int_0^1 (1-s)h_r(s)ds$$

Then

$$\|h\|_\infty \leq \int_0^1 h_r(s)ds + \frac{1}{1-\eta} \int_0^\eta (\eta-s)h_r(s)ds + \frac{1}{1-\eta} \int_0^1 (1-s)h_r(s)ds = c.$$

Step 3: N sends bounded sets of $C(J, \mathbf{R})$ into equicontinuous sets.

Let $t_1, t_2 \in J$, $t_1 < t_2$ and B_r be a bounded set of $C(J, \mathbf{R})$. For each $y \in B_r$ and $h \in N(y)$, there exists $g \in S_{F,y}$ such that

$$h(t) = \int_0^t (t-s)g(s)ds + \frac{t}{1-\eta} \int_0^\eta (\eta-s)g(s)ds - \frac{t}{1-\eta} \int_0^1 (1-s)g(s)ds, \quad t \in J.$$

Thus we obtain

$$\begin{aligned} \|h(t_2) - h(t_1)\| &\leq \int_0^{t_2} (t_2-s)|g(s)|ds + \int_{t_1}^{t_2} (t_1-s)\|g(s)\|ds + \\ &\quad + \frac{t_2-t_1}{1-\eta} \int_0^\eta (\eta-s)\|g(s)\|ds + \frac{t_2-t_1}{1-\eta} \int_0^1 (1-s)\|g(s)\|ds \\ &\leq \int_0^{t_2} (t_2-s)h_r(s)ds + \int_{t_1}^{t_2} (t_1-s)h_r(s)ds \\ &\quad + \frac{t_2-t_1}{1-\eta} \int_0^\eta (\eta-s)h_r(s)ds + \frac{t_2-t_1}{1-\eta} \int_0^1 (1-s)h_r(s)ds. \end{aligned}$$

As $t_2 \rightarrow t_1$ the right-hand side of the above inequality tends to zero.

As a consequence of Step 2, Step 3 together with the Ascoli-Arzelà theorem we can conclude that N is completely continuous.

Step 4: N has a closed graph.

Let $y_n \rightarrow y_*$, $h_n \in N(y_n)$, and $h_n \rightarrow h_*$. We shall prove that $h_* \in N(y_*)$.

$h_n \in N(y_n)$ means that there exists $g_n \in S_{F,y_n}$ such that

$$h_n(t) = \int_0^t (t-s)g_n(s)ds + \frac{t}{1-\eta} \int_0^\eta (\eta-s)g_n(s)ds - \frac{t}{1-\eta} \int_0^1 (1-s)g_n(s)ds, \quad t \in J.$$

We must prove that there exists $g_* \in S_{F,y_*}$ such that

$$h_*(t) = \int_0^t (t-s)g_*(s)ds + \frac{t}{1-\eta} \int_0^\eta (\eta-s)g_*(s)ds - \frac{t}{1-\eta} \int_0^1 (1-s)g_*(s)ds, \quad t \in J.$$

Now, we consider the linear continuous operator

$$\Gamma : L^1(J, \mathbf{R}) \longrightarrow C(J, \mathbf{R})$$

$$g \longmapsto \Gamma(g)(t) = \int_0^t (t-s)g(s)ds + \frac{t}{1-\eta} \int_0^\eta (\eta-s)g(s)ds - \frac{t}{1-\eta} \int_0^1 (1-s)g(s)ds, \quad t \in J.$$

From Lemma , it follows that $\Gamma \circ S_F$ is a closed graph operator.

Moreover, from the definition of Γ we have

$$h_n(t) \in \Gamma(S_{F,y_n}).$$

Since $y_n \longrightarrow y_*$, it follows from Lemma that

$$h_*(t) = \int_0^t (t-s)g_*(s)ds + \frac{t}{1-\eta} \int_0^\eta (\eta-s)g_*(s)ds - \frac{t}{1-\eta} \int_0^1 (1-s)g_*(s)ds, \quad t \in J$$

for some $g_* \in S_{F,y_*}$.

Step 5: *The set*

$$\Omega := \{y \in C(J, \mathbf{R}) : \lambda y \in N(y) \text{ for some } \lambda > 1\}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that

$$\begin{aligned} y(t) &= \lambda^{-1} \int_0^t (t-s)g(s)ds + \lambda^{-1} \frac{t}{1-\eta} \int_0^\eta (\eta-s)g(s)ds - \\ &\quad - \lambda^{-1} \frac{t}{1-\eta} \int_0^1 (1-s)g(s)ds, \quad t \in J. \end{aligned}$$

This implies by (H2) that for each $t \in J$ we have

$$\|y(t)\| \leq \int_0^t (t-s)H(s)ds + \frac{1}{1-\eta} \int_0^\eta (\eta-s)H(s)ds + \frac{1}{1-\eta} \int_0^1 (1-s)H(s)ds.$$

Thus

$$\|y\|_\infty \leq \int_0^1 (1-s)H(s)ds + \frac{1}{1-\eta} \int_0^\eta (\eta-s)H(s)ds + \frac{1}{1-\eta} \int_0^1 (1-s)H(s)ds := K.$$

This shows that Ω is bounded. Set $X := C(J, \mathbf{R})$. As a consequence of Lemma 3 we deduce that N has a fixed point which is a solution of (1.1)-(1.2) on J . ■

4. Concluding Remarks

Let $a_i \in \mathbf{R}$, with all of the a_i 's having the same sign, $\xi_i \in (0, 1)$, $i = 1, 2, \dots, m-2$, $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$. Consider the following m-point boundary value problem for second order differential inclusions

$$y''(t) \in F(t, y(t)), \quad t \in [0, 1] \quad (4.1)$$

$$y(0) = 0, \quad y(1) = \sum_{i=1}^{m-2} a_i y(\xi_i). \quad (4.2)$$

It is well known (see [10] for example) that if a function $y \in C^1$ satisfies the boundary condition (4.2) and all of the a_i , $i = 1, 2, \dots, m-2$ have the same sign, then there exists $\eta \in [\xi_1, \xi_{m-2}]$, depending on $y \in C^1[0, 1]$ such that

$$y(1) = \alpha y(\eta)$$

with $\alpha = \sum_{i=1}^{m-2} a_i$. Accordingly, the problem of the existence of a solution for the BVP (4.1)- (4.2) can be studied via the use of the a priori estimates for the three point BVP

$$\begin{aligned} y''(t) &\in F(t, y(t)), \quad t \in [0, 1] \\ y(0) &= 0, \quad y(1) = \alpha y(\eta), \end{aligned}$$

where $\eta \in (0, 1)$ is given. We omit the details, since the proof follows the steps of the proof of Theorem 4, with obvious modifications.

It is obvious that the above method can also be applied to other types of m-point BVPs. For example for the BVPs

$$\begin{aligned} y''(t) &\in F(t, y(t)), \quad t \in [0, 1] \\ y'(0) &= 0, \quad y(1) = \sum_{i=1}^{m-2} a_i y(\xi_i) \end{aligned}$$

or

$$\begin{aligned} y''(t) &\in F(t, y(t)), \quad t \in [0, 1] \\ y(0) &= 0, \quad y'(1) = \sum_{i=1}^{m-2} a_i y'(\xi_i) \end{aligned}$$

which can be reduced to the following three point BVPs:

$$\begin{aligned} y''(t) &\in F(t, y(t)), \quad t \in [0, 1] \\ y'(0) &= 0, \quad y(1) = \alpha y(\eta) \end{aligned}$$

and

$$\begin{aligned} y''(t) &\in F(t, y(t)), \quad t \in [0, 1] \\ y(0) &= 0, \quad y'(1) = \alpha y(\eta), \end{aligned}$$

respectively.

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ON THE STABILITY OF INTEGRAL SETS OF IMPULSIVE DIFFERENTIAL SYSTEMS

NIKOLAI A. PERESTYUK, OLGA S. CHERNIKOVA

Department of Mechanics and Mathematics, Kiev National Taras Shevchenko University
Volodimirska 64, 01033 Kiev, Ukraine
pmo@mechmat.univ.kiev.ua

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Abstract. The concept of stability of integral sets is introduced for impulsive differential systems of a general type (with nonfixed times of impulse effect). Sufficient conditions for the stability of integral sets are established.

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1. Preliminaries

Consider the impulsive differential system

$$\begin{aligned} \frac{dx}{dt} &= f(t, x), \quad t \neq t_i(x), \\ \Delta x|_{t=t_i(x)} &= I_i(x), \end{aligned} \quad (1.1)$$

where $x \in R^n$, $t \in R_+$, $f(t, x)$, $I_i(x)$, $i = 1, 2, \dots$ are continuous vector-functions, functions $t_i(x)$ are continuous and such that $0 < t_1(x) < t_2(x) < \dots$, $x \in R^n$, $t_i(x) \rightarrow \infty$ as $i \rightarrow \infty$ uniformly on $x \in R^n$. We assume that for any point $(t_0, x_0) \in R_+ \times R^n$ there exists unique solution $x(t) = x(t, t_0, x_0)$ of (1.1) on $[t_0, \infty)$ satisfying $x(t_0) = x_0$, and that the integral curve of each solution meets each of the hypersurfaces $t = t_i(x)$ only once, i.e., the phenomenon of “beating” is not observed. Sufficient conditions for the absence of the phenomenon of “beating” are pointed out in [1,2].

We shall recall [1,2] that any solution $x(t, t_0, x_0)$ of (1.1) is a piecewise continuous function with points of discontinuity

$$\begin{aligned} \theta_i^x, \theta_i^x &= t_i(x(\theta_i^x)) \quad i = 1, 2, \dots; \\ x(\theta_i^x, t_0, x_0) &= x(\theta_i^x - 0, t_0, x_0), \\ x(\theta_i^x + 0, t_0, x_0) &= x(\theta_i^x, t_0, x_0) + I_i(x(\theta_i^x, t_0, x_0)). \end{aligned}$$

We shall say that set $M \subset R_+ \times R^n$ is an integral set of the system (1.1), if for any point $(t_0, x_0) \in M$ it follows that $(t, x(t, t_0, x_0)) \in M$ for $t \geq t_0$. Note that together with point $(t_0, x_0) \in M$ belonging to the surface of discontinuity ($t_0 = t_i(x_0)$) the set M contains also point $(t_0, x_0 + I_i(x_0))$.

For each $i = 1, 2, \dots$ denote by Λ_i the set of time moments when the integral curves belonging to M meet hypersurface $t = t_i(x)$, i.e., the set of solutions of equation $\tau = t_i(x(\tau))$, $(\tau, x(\tau)) \in M$. If, for instance, the integral set is the integral curve $\{(t, x) : t \in R_+, x = x(t, 0, x_0)\}$, then Λ_i consists of one point: $\Lambda_i = \{\theta_i^x\}$. In the case when $t_i(x) = \tau_i$, $i = 1, 2, \dots$, each set Λ_i also consists of one element: $\Lambda_i = \{\tau_i\}$.

Let $M(t) = \{x \in R^n : (t, x) \in M\}$. If $t = \theta_i \in \Lambda_i$, $(\theta_i, x) \in M$ and $\theta_i = t_i(x)$, then, as it was noted above, the set M contains both points: (θ_i, x) and $(\theta_i, x + I_i(x))$, therefore, the section $M(\theta_i)$ contains both points x and $x + I_i(x)$.

For $\theta_i \in \Lambda_i$ we make a difference between the two subsets of the section: $M_i(\theta_i)$ and $M_{i+1}(\theta_i)$. The one belonging to $M_i(\theta_i)$ or $M_{i+1}(\theta_i)$ for point $x \in R^n$ will be defined the following way. Let $\Omega_i = \{(t, x) \in R_+ \times R^n : t_{i-1}(x) < t \leq t_i(x)\}$, $i = 1, 2, \dots$, $t_0(x) \equiv 0$. If $(\theta_i, x) \in M \cap \Omega_i$, then $x \in M_i(\theta_i)$; if $(\theta_i, x) \in M \cap \Omega_{i+1}$ or (θ_i, x) is a boundary point for $M \cap \Omega_{i+1}$, then $x \in M_{i+1}(\theta_i)$; $M(\theta_i) = M_i(\theta_i) \cup M_{i+1}(\theta_i)$ ($\theta_i \in \Lambda_i$). It is possible that $M_i(\theta_i) \cap M_{i+1}(\theta_i) \neq \emptyset$.

Assume that $M(t) \neq \emptyset$, $M(t) \subseteq Q \subset R^n$, $t \in R_+$, where Q is a compact set. Furthermore we shall suppose that $Q_{\varepsilon_0} \subset B_h$, where $Q_{\varepsilon_0} = \{x \in R^n : \rho(x, Q) < \varepsilon_0\}$, $B_h = \{x \in R^n : \|x\| \leq h\}$, for some $\varepsilon_0 > 0$, $h > 0$.

In the case when an impulsive system (1.1) is a system with impulses at fixed times, i.e., $t_i(x) = \tau_i$, $i = 1, 2, \dots$, we make a difference between the two subsets of the section of integral set M only for points τ_i : if $(\tau_i, x) \in M \cap \Omega_i$, then $x \in M_i(\tau_i)$; if $(\tau_i, x) \in M$ and (τ_i, x) is a boundary point of $\Omega_{i+1} \cap M$, then $x \in M_{i+1}(\tau_i)$.

We shall say that the solution $y(t)$ of (1.1) belongs to ε -neighbourhood of the integral set M for $t \geq t_0$ if there exists a sequence $\{\theta_i\}$, $i = 1, 2, \dots$, $\theta_i \in \Lambda_i$, such that $|\theta_i^y - \theta_i| < \varepsilon$ and $\rho(y(t, t_0, y_0), M(t)) < \varepsilon$ for $t \geq t_0$, $t \notin \bigcup_{i=1}^{\infty} (\theta_i - \varepsilon, \theta_i + \varepsilon)$ (here θ_i^y , $i = 1, 2, \dots$, are points of discontinuity of the solution $y(t)$ and $\rho(y, M(t))$ is a distance between y and the set $M(t)$).

With every point $(t_0, y_0) \in R_+ \times R^n$ we connect a set $M_{y_0}(t_0)$ as follows:

$$M_{y_0}(t_0) = \begin{cases} M(t_0), & \text{if } t_0 \notin \bigcup_{i=1}^{\infty} \Lambda_i, \\ M_i(t_0), & \text{if } t_0 \in \Lambda_i, (t_0, y_0) \in \Omega_i, \\ M_{i+1}(t_0), & \text{if } t_0 \in \Lambda_i, (t_0, y_0) \in \Omega_{i+1}. \end{cases}$$

We shall give the following definitions of stability and asymptotic stability of an integral set.

The integral set M of system (1.1) is stable if for any $\varepsilon > 0$ and $t_0 \in R_+$ there exists such a number $\delta = \delta(\varepsilon, t_0) > 0$ that for any solution $y(t) = y(t, t_0, y_0)$ of system (1.1), $\rho(y_0, M_{y_0}(t_0)) < \delta$ implies that $y(t, t_0, y_0)$ belongs to ε -neighbourhood of the integral set M for $t \geq t_0$ (it is assumed that $t_0 \neq \theta_i^y$).

The stable integral set M of system (1.1) is asymptotically stable if there exists such a number $\delta_0 > 0$ that for any solution $y(t) = y(t, t_0, y_0)$ of (1.1) that satisfies the inequality $\rho(y_0, M_{y_0}(t_0)) < \delta_0$, the following holds: $\lim_{t \rightarrow \infty} \rho(y(t), M(t)) = 0$.

2. Reduction to a system with impulses at fixed times

Suppose that in (1.1) functions $f(t, x)$ and $I_i(x)$ satisfy the conditions:

$$\|f(t, x) - f(t, y)\| + \|I_i(x) - I_i(y)\| \leq K\|x - y\| \quad (2.1)$$

uniformly with respect to $t \in R_+$, $i = 1, 2, \dots$ for all $x, y \in R^n$, and

$$\sup_{t \geq 0} \|f(t, 0)\| + \sup_{t \geq 1} \|I_i(0)\| = N < +\infty. \quad (2.2)$$

Suppose that for $x \in B_h$ the equations of surfaces of discontinuity of solutions can be written in the form:

$$t = t_i(x) = \tau_i + \tau_i(x), \quad i = 1, 2, \dots, \quad (2.3)$$

and the following conditions are satisfied

$$\begin{aligned} \tau_{i+1} - \tau_i &\geq d > 0, \quad 0 \leq \tau_i(x) \leq l < d, \quad i = 1, 2, \dots, \\ |\tau_i(x) - \tau_i(y)| &\leq l\|x - y\|, \end{aligned} \quad (2.4)$$

where l is a sufficiently small number.

Under such assumptions investigation of the stability of an integral set of impulsive differential system (1.1) can be reduced to an investigation of the stability of an integral set of some differential system with impulses at fixed times.

Let M be an integral set of the system (1.1). Consider set $\hat{M}$ which differs from set M in points (t, x) disposed between surfaces $t = \tau_i$ and $t = t_i(x)$, $i = 1, 2, \dots$, as follows. Each point $(t^*, x^*) \in M$, such that $\tau_i < t^* \leq t_i(x^*)$, is replaced by point $(t^*, \tilde{x}(t^*, \theta_i^x, x(\theta_i^x + 0)))$, where $\tilde{x}(t)$ is a solution of system $\frac{d\tilde{x}}{dt} = f(t, \tilde{x})$ with initial condition $\tilde{x}(\theta_i^x) = x(\theta_i^x + 0) = x(\theta_i^x + 0, t^*, x^*)$ (θ_i^x denotes the moment at which the integral curve $\{t, x(t, t^*, x^*)\}$, $t \geq t^*$, of (1.1) meets a surface $t = t_i(x)$).

Let $x(t)$ be a solution of (1.1) with initial condition $x(\tau_i) = x$. Consider function

$$G_i(x) = \int_{\theta_i^x}^{\tau_i} f(\tau, \tilde{x}(\tau)) d\tau + I_i(x(\theta_i^x)) + \int_{\tau_i}^{\theta_i^x} f(\tau, x(\tau)) d\tau.$$

Proceeding from the construction of set $\hat{M}$ one can easily make sure that the following statement is true.

Lemma 1. If M is an integral set of system (1.1), then $\hat{M}$ is an integral set of the differential system with impulses at fixed times

$$\begin{aligned} \frac{dx}{dt} &= f(t, x), \quad t \neq \tau_i, \\ \Delta x|_{t=\tau_i} &= G_i(x). \end{aligned} \quad (2.5)$$

Applying the Gronwall-Bellman inequality and taking into consideration conditions (2.1)-(2.4) we set the following statement .

Lemma 2. Let $x \in B_h$. Then the solution $x(t)$ satisfies the inequality

$$\|x(t)\| \leq H, t \in [\tau_i, \tau_i + l],$$

where $H = \{(1 + K)(h + Nl)e^{Kl} + N(1 + l)\}e^{Kl}$.

Lemma 3. [3]. Let θ_i^x and θ_i^y be the points of discontinuity of solutions $x(t)$ and $y(t)$ of (1.1) with initial conditions $x(\tau_i) = x$, $y(\tau_i) = y$ ($x, y \in B_h$). Then

$$|\theta_i^x - \theta_i^y| \leq \frac{le^{Kl}}{1 - l(N + KH)} \|x - y\|$$

($l < (N + KH)^{-1}$).

Theorem 1. The integral set M of the differential impulsive system (1.1) is stable if the integral set $\hat{M}$ of differential system (2.5) with impulses at fixed times is stable.

Proof. Let integral set $\hat{M}$ of system (2.5) be stable. Choose and fix $t_0 \in R^+$.

Consider at first a case when $t_0 \notin \bigcup_{i=1}^{\infty} [\tau_i, T_i]$, where $T_i = \sup \Lambda_i$, $i = 1, 2, \dots$

According to the definition of the stability of an integral set for a differential system with impulses at fixed times, for arbitrary number $\varepsilon > 0$ ($\varepsilon < \varepsilon_0$) there exists such a number $\delta > 0$ that for any solution $\hat{y}(t, t_0, y_0)$ of (2.5) inequality $\rho(y_0, \hat{M}(t_0)) < \delta$ implies

$$\rho(\hat{y}(t, t_0, y_0), \hat{M}(t)) < \varepsilon \quad (2.6)$$

for $t \geq t_0$ ($t \neq \tau_i$) and $\rho(\hat{y}(\tau_i, t_0, y_0), \hat{M}_i(\tau_i)) < \varepsilon$ ($\tau_i \geq t_0$). The last means that for each i such that $\tau_i \geq t_0$, there exists $x_i \in \hat{M}_i(\tau_i)$ which satisfies inequality $\|\hat{y}(\tau_i, t_0, y_0) - x_i\| < \varepsilon$. Because solution $\hat{y}(t) = \hat{y}(t, t_0, y_0)$ of system (2.5) coincides with solution $y(t) = y(t, t_0, y_0)$ of system (1.1) in all of points except points from

$\bigcup_{i: \tau_i \geq t_0} (\tau_i, \theta_i^y)$ and sets M and $\hat{M}$ differ from one another only for $t \in \bigcup_{i=1}^{\infty} (\tau_i, T_i)$, where $T_i = \sup \Lambda_i$, $i = 1, 2, \dots$, then $\rho(y(t), M(t)) < \varepsilon$ for $t \notin \bigcup_{i: \tau_i \geq t_0} (\tau_i, \gamma_i)$,

$\gamma_i = \max(\theta_i^y, T_i)$. Estimate distance $\rho(y(t), M(t))$ for $t \in (\tau_i, \gamma_i)$, supposing, for example, that $\gamma_i = T_i$. Let $x(t)$, $t \geq \tau_i$, be a solution of system (1.1) with initial condition $x(\tau_i) = x_i$. In obedience to lemma 3 for time moments θ_i^x , θ_i^y ($\theta_i^x \in \Lambda_i$) we have the following

$$|\theta_i^x - \theta_i^y| \leq \frac{le^{KL}}{1 - l(N + KH)} \varepsilon. \quad (2.7)$$

Without loss of generality, we may suppose that $\theta_i^y < \theta_i^x$. It is easy to verify that for $\tau_i < t \leq \theta_i^y$ the following inequality is fulfilled

$$\|y(t) - x(t)\| \leq \|y(\tau_i) - x(\tau_i)\| e^{K(t - \tau_i)} < \varepsilon e^{K(\theta_i^y - \tau_i)} \quad (2.8)$$

and also for $\theta_i^x < t \leq T_i$ - inequality

$$\|y(t) - x(t)\| \leq \left[(1 + K)\varepsilon e^{Kl} + (2 + K)(KH + N)(\theta_i^x - \theta_i^y) \right] e^{K(i - \theta_i^x)} \quad (2.9)$$

Let ε_1 be an arbitrary sufficiently small number ($\varepsilon_1, \varepsilon_0$) and $\varepsilon < \frac{\varepsilon_1}{A}$, where

$$A = \max \left\{ e^{2Kl} \left[1 + K + \frac{(2+K)(N+KH)l}{1-l(N+KH)} \right], \frac{le^{Kl}}{1-l(N+KH)} \right\}.$$

From the stability of integral set $\hat{M}$ for system (2.5) and inequalities (2.6) – (2.9) it follows that for any solution $y(t, t_0, y_0)$ of system (1.1), which satisfies inequality $\rho(y_0, M(t_0)) < \delta$, inequality $\rho(y(t, t_0, y_0), M(t)) < \varepsilon_1$ holds for

$$t \geq t_0, \quad t \notin \bigcup_{i=1}^{\infty} (\theta_i - \varepsilon_1, \theta_i + \varepsilon_1),$$

where θ_i is some point of set Λ_i , $i = 1, 2, \dots$, i.e., it follows that integral set M of system (1.1) is stable.

It is not difficult to show that if $t_0 \in \bigcup_{i=1}^{\infty} [\tau_i, T_i)$, then from $\rho(y_0, M_{y_0}(t_0)) < \delta_1$ where $\delta_1 < \frac{\delta}{A}$, follows the inequality $\rho(y(i, i_0, y_0), M(t)) < \delta$ on $[i_0, \tau_i + l]$, except, perhaps, of points of some interval $(\alpha_i^{y_0}, \beta_i^{y_0})$, the length of which does not exceed δ . Taking into account this fact and the stability of the set $\hat{M}$ for system (2.5), we conclude that the integral set M of system (1.1) is stable.

The next assertion is also true:

Theorem 2. The integral set M of differential impulsive system (1.1) is asymptotically stable if the integral set $\hat{M}$ of differential system (2.5) with impulses at fixed times is asymptotically stable.

3. Stability of the integral set for a system with impulses at fixed times

We shall give sufficient conditions for the stability (asymptotic stability) of an integral set of an impulsive differential system with fixed time moments sequence of impulse action

$$\begin{aligned} \frac{dx}{dt} &= F(t, x), \quad t \neq \tau_i, \\ \Delta x|_{t=\tau_i} &= J_i(x), \end{aligned} \quad (3.1)$$

where $x \in R^n$, $t \in R_+$, $i = 1, 2, \dots$, $0 = \tau_0 < \tau_1 < \tau_2 < \dots$, $\tau_i \rightarrow \infty$ as $i \rightarrow \infty$.

For the establishment of the stability conditions of integral sets of an impulsive differential system (3.1) we will employ a method of auxiliary functions (the second Lyapunov method), which, as it is known, is effectively applicable for the study of stability of sets in the theory of usual differential equations, in particular, the stability of invariant sets of dynamic systems [4-8], and also for research of various stability questions in the theory of impulse differential equations. We will apply means analogous to those which were applied in [1, 2, 9-15].

As above, assume that system (3.1) satisfies the conditions of existence and uniqueness of solutions, and that any solution of the system (3.1) is defined on $[t_0, +\infty)$.

Let M be an integral set of system (3.1). As above, suppose, that for each $t \in R_+$ section $M(t)$ is not empty and is contained in compact set $Q \subset R^n$; $Q_{\varepsilon_0} \subseteq B_h$.

Consider continuously differentiable function $V(t, x)$ defined in domain $Z = \{(t, x) : t \in R_+, x \in Q_{\varepsilon_0}\}$, which satisfies such properties:

$$V(t, x) = 0, (t, x) \in M; \quad V(t, x) > 0, (t, x) \notin M; \quad (3.2)$$

$$V(t, x) \geq a(\rho(x, M(t))), \quad (3.3)$$

where $a(s)$ ($s \geq 0$) is a continuous increasing function, $a(0) = 0$.

Assume that there exists such $\mu > 0$ ($0 < \mu < \varepsilon_0$), that for $x \in Q_\mu$ will be $x + J_i(x) \in Q_{\varepsilon_0}$, $i = 1, 2, \dots$.

Theorem 3. If for system (3.1) there exists a continuously differentiable function $V(t, x)$ satisfying in domain Z the conditions (3.2), (3.3) and conditions

$$\frac{\partial V(t, x)}{\partial t} + \langle \text{grad}_x V(t, x), F(t, x) \rangle \leq 0, \quad (3.4)$$

$$V(\tau_i, x + J_i(x)) \leq V(\tau_i, x), \quad i = 1, 2, \dots, \quad (3.5)$$

then the set M is stable with respect to system (3.1).

Proof. Let $t_0 \in R_+$ and $\varepsilon > 0$ be a sufficiently small number. Taking into consideration properties of the function $V(t, x)$ we can indicate such $\delta > 0$, that $V(t_0, x) < a(\varepsilon)$ if $\rho(x, M(t_0)) < \delta$. Let $\rho(x_0, M(t_0)) < \delta$ and $x(t) = x(t, t_0, x_0)$ be a solution of the system (3.1). Evidently, that function $v(t) = V(t, x(t))$ under conditions (3.4), (3.5) is unincreasing; $v(t) \leq v(t_0)$, $t \geq t_0$. The assumption that at some time moment $t^* > t_0$ we shall have $\rho(x(t^*), M(t^*)) \geq \varepsilon$, leads to the contradictory inequalities:

$$a(\varepsilon) \leq a(\rho(x(t^*), M(t^*))) \leq v(t^*) \leq v(t_0) < a(\varepsilon).$$

This contradiction proves Theorem 3.

Theorem 4. Assume that for system (3.1) there exists a continuously differentiable function $V(t, x)$ satisfying in domain Z the conditions (3.2), (3.3) and conditions

$$\frac{\partial V(t, x)}{\partial t} + \langle \text{grad}_x V(t, x), F(t, x) \rangle \leq -\alpha(t)\varphi(V(t, x)) \quad (3.6)$$

$$V(\tau_i, x + J_i(x)) \leq \psi_i(V(\tau_i, x)), \quad i = 1, 2, \dots, \quad (3.7)$$

where $\alpha(t)$, ($t \in R_+$) is continuous nonnegative function, $\varphi(s), \psi_i(s)$ are continuous functions, $\varphi(s) > 0$, $\psi_i(s) > 0$ for $s > 0$ and $\varphi(0) = \psi_i(0) = 0$.

If there exists a positive number a_0 such, that for all $a \in (0, a_0]$ the inequalities

$$\int_a^{\psi_i(a)} \frac{ds}{\varphi(s)} \leq \int_{\tau_{i-1}}^{\tau_i} \alpha(t) dt \quad i = 1, 2, \dots \quad (3.8)$$

hold, then the set M is stable with respect to system (3.1).

Proof. Let $\varepsilon > 0$ be an arbitrary number such that $\varepsilon < \mu$, $a(\varepsilon) < a_0$, and let $t_0 \in (\tau_j, \tau_{j+1}]$. For arbitrary $\delta > 0$, denote by $m_\delta = \sup_{\rho(x, M(t_0)) < \delta} V(t_0, x)$. Choose $\delta > 0$ so small that inequalities $m_\delta < a(\varepsilon)$, $\sup_{0 \leq s \leq m_\delta} \psi_{j+1}(s) < a(\varepsilon)$ hold.

Let $x(t) = x(t, t_0, x_0)$ be an arbitrary solution of system (3.1), such that $\rho(x_0, M(t_0)) < \delta$. Prove that $\rho(x(t), M(t)) < \varepsilon$ for $t \geq t_0$. Suppose the contrary: in some time moment $t^* > t_0$ will be $\rho(x(t^*), M(t^*)) \geq \varepsilon$. First such a time moment can be only a moment of impulse action. Assume that τ_k is such time moment. Evidently, $k \geq j+2$. Consider function $v(t) = V(t, x(t))$. In obedience to our supposition as for behaviour of solution $x(t)$ the next inequalities hold $v(t) < a(\varepsilon)$, $t_0 \leq t \leq \tau_k$, and $v(\tau_k + 0) \geq a(\varepsilon)$. In view of the conditions of theorem

$$\int_{v(\tau_i)}^{v(\tau_i+0)} \frac{ds}{\varphi(s)} \leq \int_{v(\tau_i)}^{\psi_i(v(\tau_i))} \frac{ds}{\varphi(s)} \leq \int_{\tau_{i-1}}^{\tau_i} \alpha(t) dt, \quad i = j+2, \dots, k$$

and

$$\int_{v(\tau_{i-1}+0)}^{v(\tau_i)} \frac{ds}{\varphi(s)} \leq - \int_{\tau_{i-1}}^{\tau_i} \alpha(t) dt \quad i = j+2, \dots, k.$$

So,

$$\int_{v(\tau_{j+1}+0)}^{v(\tau_k+0)} \frac{ds}{\varphi(s)} = \sum_{j+2}^k \int_{v(\tau_{i-1}+0)}^{v(\tau_i)} \frac{ds}{\varphi(s)} + \sum_{j+2}^k \int_{v(\tau_i)}^{v(\tau_i+0)} \frac{ds}{\varphi(s)} \leq \int_{\tau_{j+1}}^{\tau_k} \alpha(t) dt - \int_{\tau_{j+1}}^{\tau_k} \alpha(t) dt = 0$$

or $v(\tau_k + 0) \leq v(\tau_{j+1} + 0)$ and $v(\tau_k + 0) < a(\varepsilon)$. The contradiction obtained proves the theorem.

Consider a case when sequence $\{\tau_i\}$, $i = 1, 2, \dots$, satisfies a condition

$$\tau_{i+1} - \tau_i \geq \theta, \quad \theta > 0. \quad (3.9)$$

Theorem 5. Let for system (3.1) there exist a continuously differentiable function $V(t, x)$ satisfying in domain Z the conditions (3.2), (3.3) and condition

$$\frac{\partial V(t, x)}{\partial t} + \langle \text{grad}_x V(t, x), F(t, x) \rangle \leq -\varphi(V(t, x)), \quad (3.10)$$

$$V(\tau_i, x + J_i(x)) \leq \psi(V(\tau_i, x)), \quad i = 1, 2, \dots, \quad (3.11)$$

where $\varphi(s)$, $\psi(s)$ are continuous functions, $\varphi(s) > 0$, $\psi(s) > 0$ for $s > 0$ i $\varphi(0) = \psi(0) = 0$.

If sequence $\{\tau_i\}$, $i = 1, 2, \dots$, satisfies condition (3.9) and there exists a positive number a_0 such that for arbitrary $a \in (0, a_0]$

$$\int_a^{\psi(a)} \frac{ds}{\varphi(s)} \leq \theta,$$

then the set M is stable with respect to system (3.1).

Assertion of Theorem 5, which extends the results [10] about the stability of a trivial solution of an impulsive differential system to the case of stability of integral sets, immediately follows from Theorem 4.

Theorem 6. Let for system (3.1) there exist a continuously differentiable function $V(t, x)$ satisfying in domain Z the conditions (3.2), (3.3) and conditions

$$\frac{\partial V(t, x)}{\partial t} + \langle \text{grad}_x V(t, x), F(t, x) \rangle \leq -\alpha(t)\varphi(V(t, x)) \quad (3.12)$$

$$V(\tau_i, x + J_i(x)) \leq \psi_i(V(\tau_i, x)), \quad i = 1, 2, \dots, \quad (3.13)$$

where $\alpha(t)$ ($t \in R_+$) is a continuous nonnegative function, $\varphi(s)$, $\psi_i(s)$, $i = 1, 2, \dots$, are continuous functions, $\varphi(s) > 0$, $\psi_i(s) > 0$ for $s > 0$ and $\varphi(0) = \psi_i(0) = 0$.

If there exists a positive number a_0 such that for all $a \in (0, a_0]$ the following inequalities

$$\int_{\psi_i(a)}^a \frac{ds}{\varphi(s)} \geq \int_{\tau_i}^{\tau_{i+1}} \alpha(t) dt, \quad i = 1, 2, \dots, \quad (3.14)$$

hold, then the set M is stable with respect to system (3.1).

Proof. Let $\varepsilon > 0$ be an arbitrary sufficiently small number ($\varepsilon < \mu$, $a(\varepsilon) < a_0$), and let $t_0 \in (\tau_j, \tau_{j+1}]$, $j \geq 1$. For arbitrary $\delta > 0$ note $m_\delta = \sup_{\rho(x, M(t_0)) < \delta} V(t_0, x)$.

Choose $\delta > 0$ so small that inequality $m_\delta < \min\{a(\varepsilon), \psi_j(a(\varepsilon))\}$ holds. Let $x(t) = x(t, t_0, x_0)$ be an arbitrary solution of system (3.1) such that $\rho(x_0, M(t_0)) < \delta$. Prove that $\rho(x(t), M(t)) < \varepsilon$ for $t \geq t_0$. Suppose the contrary, i.e., that there exists $t^* > t_0$, $t^* \in (\tau_k, \tau_{k+1}]$, $k \geq j$, such that $\rho(x(t^*), M(t^*)) \geq \varepsilon$. Suppose at first that $k = j$ and $t^* \in (t_0, \tau_{j+1}]$. In view of (3.12) for function $v(t) = V(t, x(t))$ have:

$$\int_{v(t_0)}^{v(t^*)} \frac{ds}{\varphi(s)} \leq \int_{t_0}^{t^*} \alpha(s) ds \leq \int_{\tau_j}^{\tau_{j+1}} \alpha(s) ds.$$

On the other hand, $v(t^*) \geq a(\varepsilon)$, and

$$\int_{v(t_0)}^{v(t^*)} \frac{ds}{\varphi(s)} > \int_{\psi_j(a(\varepsilon))}^{a(\varepsilon)} \frac{ds}{\varphi(s)},$$

so,

$$\int_{\psi_j(a(\varepsilon))}^{a(\varepsilon)} \frac{ds}{\varphi(s)} < \int_{\tau_j}^{\tau_{j+1}} \alpha(s) ds,$$

that contradicts condition (3.14). Thus $v(t) < a(\varepsilon)$, $t \in [t_0, \tau_{j+1}]$ (in particular, $v(\tau_{j+1}) < a(\varepsilon)$).

Assume now that $t^* \in (\tau_k, \tau_{k+1}]$, $k \geq j+1$, $\rho(x(t), M(t)) < \varepsilon$ for $t_0 \geq t \geq \tau_k$, and that $v(\tau_k) < a(\varepsilon)$. Evidently, one can think that $\varepsilon \leq \rho(x(t^*), M(t^*)) < \varepsilon_0$; also $\rho(x(t), M(t)) < \varepsilon_0$ for $t \in [t_0, t^*]$.

In view of conditions of theorem the next correlations hold for $t \in (\tau_k, \tau_{k+1}]$:

$$\begin{aligned} \int_{v(\tau_k)}^{v(t^*)} \frac{ds}{\varphi(s)} &= \int_{v(\tau_k)}^{v(\tau_k+0)} \frac{ds}{\varphi(s)} + \int_{v(\tau_k+0)}^{v(t^*)} \frac{ds}{\varphi(s)} \leq \\ &\leq \int_{v(\tau_k)}^{\psi_k(v(\tau_k))} \frac{ds}{\varphi(s)} + \int_{v(\tau_k+0)}^{v(t^*)} \frac{ds}{\varphi(s)} \leq - \int_{\tau_k}^{\tau_{k+1}} \alpha(s) ds + \int_{\tau_k}^{\tau_{k+1}} \alpha(s) ds = 0, \end{aligned}$$

so $v(t^*) \leq v(\tau_k)$, $t \in (\tau_k, \tau_{k+1}]$. Supposing that $\rho(x(t^*), M(t^*)) \geq \varepsilon$ we obtain a contradiction: $a(\varepsilon) \leq v(t^*) \leq v(\tau_k) < a(\varepsilon)$. Thus $\rho(x(t), M(t)) < \varepsilon$ for $t \in (\tau_k, \tau_{k+1}]$. From the reasoning above we conclude that $v(t) \leq v(t_0)$ on $[t_0, \infty)$ and also $\rho(x(t), M(t)) < \varepsilon$, $t \geq t_0$. The theorem is proved.

Suppose now that the time moments of impulsive action satisfy condition

$$\tau_{i+1} - \tau_i \leq \theta_1, \quad \theta_1 > 0, \quad i = 1, 2, \dots \quad (3.15)$$

Theorem 7. Let for system (3.1) there exists a continuously differentiable function $V(t, x)$ satisfying in domain Z conditions (3.2), (3.3) and conditions

$$\frac{\partial V(t, x)}{\partial t} + \langle \text{grad}_x V(t, x), F(t, x) \rangle \leq \varphi(V(t, x)), \quad (3.16)$$

$$V(\tau_i, x + J_i(x)) \leq \psi(V(\tau_i, x)), \quad i = 1, 2, \dots, \quad (3.17)$$

where $\varphi(s)$, $\psi_i(s)$, $i = 1, 2, \dots$, are continuous functions, $\varphi(s) > 0$, $\psi_i(s) > 0$ for $s > 0$ and $\varphi(0) = \psi_i(0) = 0$.

If there exists a positive number a_0 such that for all $a \in (0, a_0]$ the following inequality

$$\int_{\psi(a)}^a \frac{ds}{\varphi(s)} \geq \theta_1$$

holds, then the set is M stable with respect to system (3.1).

Verification of the assertion of Theorem 7 evidently follows from Theorem 6.

The next assertion also follows from Theorems 4 and 6.

Theorem 8. Let for system (3.1) there exist a continuously differentiable function $V(t, x)$ satisfying in domain Z conditions (3.2), (3.3), (3.10), (3.11).

Assume that sequence $\{\tau_i\}$, $i = 1, 2, \dots$ is such that for sufficiently large t ($t > T_0 \geq t_0$)

$$\frac{i(t, t_0)}{t - t_0} \leq p, \quad p = \text{const},$$

where by $i(t, t_0)$, $t \geq t_0$, a number of impulses are denoted on $[t_0, t]$.

Then if there exists such $a_0 > 0$ that for all $a \in (0, a_0]$

$$\int_a^{\psi(a)} \frac{ds}{\varphi(s)} \leq \frac{1}{p},$$

then the set M is stable with respect to system (3.1).

Theorem 9. Let for system (3.1) there exist a continuously differentiable function $V(t, x)$ satisfying in domain Z conditions (3.2), (3.3), (3.16), (3.17).

Assume that sequence $\{\tau_i\}$, $i = 1, 2, \dots$ is such that for sufficiently large t ($t > T_0 \geq t_0$)

$$\frac{i(t, t_0)}{t - t_0} \geq p, \quad p > 0, \quad p = \text{const}.$$

Then if there exists such $a_0 > 0$ that for all $a \in (0, a_0]$

$$\int_{\psi(a)}^a \frac{ds}{\varphi(s)} \geq \frac{1}{p},$$

then the set M is stable with respect to system (3.1).

We shall present the sufficient conditions for the asymptotic stability of an integral set.

Theorem 10. Let for system (3.1) there exist a continuously differentiable function $V(t, x)$ satisfying in domain Z conditions (3.2), (3.3), (3.6), (3.7).

Then if there exists such $a_0 > 0$ that for all $a \in (0, a_0]$

$$\int_{(a)}^{\psi_i(a)} \frac{ds}{\varphi(s)} \leq \int_{\tau_{i-1}}^{\tau_i} \alpha(t) dt - \gamma_i \quad i = 1, 2, \dots$$

where $\gamma_i \geq 0$, $\sum_{i=1}^{\infty} \gamma_i = \infty$, then the set M is asymptotically stable with respect to system (3.1).

Proof. Since the conditions of Theorem 4 are fulfilled, then the integral set of system (3.1) is stable. So for given t_0 (let $t_0 \in (\tau_j, \tau_{j+1}]$) and arbitrary sufficiently small $\varepsilon > 0$) there exists such $\delta > 0$ that for arbitrary solution $x(t) = x(t, t_0, x_0)$ of

(3.1) which satisfies $\rho(x_0, M(t_0)) < \delta$ the inequality $\rho(x(t), M(t)) < \varepsilon$, $t \geq t_0$, holds. Reasoning as well as proof of Theorem 4, brings us to inequalities

$$\int_{\nu(\tau_i+0)}^{\nu(\tau_{i+1}+0)} \frac{ds}{\varphi(s)} \leq -\gamma_{i+1}, \quad i = j+1, j+2, \dots, \quad (3.18)$$

which implies that sequence $\{v(\tau_{j+k}+0)\}$, $k = 1, 2, \dots$, is nonincreasing. Let $\beta = \lim_{k \rightarrow \infty} v(\tau_{j+k}+0)$. Assume that $\beta > 0$. Under such assumption from (3.18) we obtain:

$$v(\tau_{j+n}+0) \leq v(\tau_{j+1}+0) - c \sum_{i=j+2}^n \gamma_i, \quad (3.19)$$

where $c = \min_{\beta \leq s \leq a(\epsilon)} \varphi(s)$. Since $v(\tau_{j+1}+0) - c \sum_{i=j+2}^n \gamma_i \rightarrow -\infty$ as $n \rightarrow \infty$, then inequality (3.19) is contradictory. The reason for the contradiction obtained is the assumption that $\beta > 0$. So, $\beta = 0$. In view of conditions of the theorem ($v(t)$, decreases on each interval of continuity $(\tau_i, \tau_{i+1}]$, $i = j+1, j+2, \dots$, so $v(t) \rightarrow 0$ as $t \rightarrow \infty$, and, simultaneously, $\lim_{t \rightarrow \infty} \rho(x(t), M(t)) = 0$. The theorem is proved.

Theorem 11. Let for system (3.1) there exist a continuously differentiable function $V(t, x)$ satisfying in domain Z conditions (3.2), (3.3), (3.10), (3.11).

If the sequence $\{\tau_i\}$, $i = 1, 2, \dots$ satisfies inequality (3.9) and if there exist such positive numbers a_0 and γ that for all $a \in (0, a_0]$

$$\int_a^{\psi(a)} \frac{ds}{\varphi(s)} \leq \theta - \gamma,$$

then the set M is asymptotically stable with respect to system (3.1). Theorem 11 is a corollary of Theorem 10.

Theorem 12. Let for system (3.1) there exist a continuously differentiable function $V(t, x)$ satisfying in domain Z conditions (3.2), (3.3), (3.12), (3.13).

If there exists a positive number a_0 such that for all $a \in (0, a_0]$ for some $\gamma_i > 0$, $i = 1, 2, \dots$, inequalities

$$\int_{\psi_i(a)}^a \frac{ds}{\varphi(s)} \geq \int_{\tau_{i-1}}^{\tau_i} \alpha(t) dt + \gamma_i, \quad i = 1, 2, \dots,$$

hold, where $\sum_{i=1}^{\infty} \gamma_i = \infty$, then the set M is asymptotically stable with respect to system (3.1).

The proof of Theorem 12 resembles the proof of Theorem 10.

Theorem 13. Let for system (3.1) there exist a continuously differentiable function $V(t, x)$ satisfying in domain Z conditions (3.2), (3.3), (3.16), (3.17). If sequence $\{\tau_i\}$, $i = 1, 2, \dots$, satisfies (3.15) and there exist positive numbers a_0 and γ such that for arbitrary $a \in (0, a_0]$ inequality

$$\int_{\psi(a)}^a \frac{ds}{\varphi(s)} \geq \theta_1 + \gamma,$$

holds, then the set M is asymptotically stable with respect to system (3.1).

The assertion of Theorem 13 follows from Theorem 12.

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ALGEBRAIC MODELS FOR DISCRETE SYSTEMS' ANALYSIS

VOLODIMYR G. SKOBELEV

Institute for Applied Mathematics and Mechanics
of the National Academy of Sciences of Ukraine, Donetsk, Ukraine
`skbv@iamm.ac.donetsk.ua`

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Abstract. The present paper deals with algebraic models and methods sufficient to solve effectively problems of investigation of two basic classes of control systems, namely, finite automata and boolean functions. Suggested models for finite automata are based on finite groups and result in establishing basic algebraic characteristics, developing a general scheme for estimating exponential lower bounds and in the design of nonstationary secret locks of arbitrary high complexity. It is also shown that the problem of the identification of boolean vector-functions may be effectively solved via the methods of the Theory of Vector Spaces over $GF(2)$.

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1. Introduction

It is well known that solving of the majority of problems of investigation of finite automata and boolean functions is based on exhaustive searching. The main disadvantage of these methods is their high complexity. Besides, many important properties of the structure of investigated objects are not taken into account at all since the applied tools are very limited. On the other hand, many problems of analysis, description, classification, estimation and effective algorithm design may be solved by powerful methods of Modern Algebra. Thus it is natural to present finite automata and boolean functions via basic algebraic structures. This gives the possibility to explore effectively algebraic characteristics of these structures in solving the investigated problems. Three basic problems are solved in what follows.

The first one is the presentation of mappings performed by finite automata via finite groups. This results in an exhaustive investigation of algebraic characteristics and the extraction of a special subclass of permutation automata. This subclass is basic for the design of nonstationary secret locks of arbitrary high complexity.

The second one is the elaboration of a general scheme for establishing the exponential lower bounds. The suggested scheme is based on the selection of permutation with special characteristics. This directly results in establishing the lower bounds for Shannon's functions estimating the maximum of the minimal lengths of distin-

guishing and synchronizing sequences for weakly initialized automata as well as of the complexity of the secret locks designed.

The third one is the identification of boolean vector-functions via the methods of the Theory of Vector Spaces over $GF(2)$. The suggested scheme is based on presenting the graph of any boolean vector-function via the union of special subspaces of $GF^{m+n}(2)$. This results in the design of a characteristic function in the form of a set of special matrices over the field $GF(2)$.

All notions that are not determined are the same as in [1-7].

2. Automata and groups

An automaton is a quintuple $A = (Q, X, Y, \delta, \lambda)$ (Q is a set of states, X and Y are input and output alphabets, $\delta : Q \times X \rightarrow Q$ and $\lambda : Q \times X \rightarrow Y$ are transition and output mappings). It is suggested that Q , X and Y are finite sets. Usually an automaton is interpreted as a model of a discrete device or as an algorithm. It is also evident that an automaton is a *Heterogeneous Algebra*. Thus algebraic methods are directly applied in the Automata Theory. Indeed, the investigation of congruences of the free semigroup X^* leads to the *Krohn-Rhodes Decomposition Theory*. Similarly, the investigation under conditions that δ and λ are presented via linear transformations of vector spaces leads to a restricted but very important class of *Linear Sequential Machines*. It is natural to present δ and λ via operations in arbitrary finite groups since the last ones are one of the best examined algebraic systems. Basic properties of this presentation were investigated in [1,2].

Let $\mathcal{A}_{mnk}(m, n, k \in \mathbb{N}, k \geq 2)$ be the set of all automata $A = (Q_k, X_m, Y_n, \delta_A, \lambda_A)$, where $Q_k = \{q_1, \dots, q_k\}$, $X_m = \{x_1, \dots, x_m\}$, $Y_n = \{y_1, \dots, y_n\}$ and $\mathcal{G}_1 = (G_1, \circ)$ and $\mathcal{G}_2 = (G_2, \star)$ be finite groups. $\mathcal{F}_{mnk}(\mathcal{G}_1, \mathcal{G}_2)$ denotes the set of all 6-tuples $F = (f_1, f_2, f_3, f_4, f_5, f_6)$ of mappings, where $f_1 : Q_k \rightarrow G_1$, $f_2 : X_m \rightarrow G_1$, $f_3 : G_1 \rightarrow Q_k$, $f_4 : Q_k \rightarrow G_2$, $f_5 : X_m \rightarrow G_2$, $f_6 : G_2 \rightarrow Y_n$. An automaton $B_F = (Q_k, X_m, Y_n, \delta_F, \lambda_F)$ ($F \in \mathcal{F}_{mnk}(\mathcal{G}_1, \mathcal{G}_2)$) is determined by the identities

$$\delta_F(q, x) = f_3(f_1(q) \circ f_2(x)), \lambda_F(q, x) = f_6(f_4(q) \star f_5(x)) \quad (q \in Q_k, x \in X_m).$$

Let $\mathcal{A}_{mnk}(\mathcal{G}_1, \mathcal{G}_2) = \{A_F | F \in \mathcal{F}_{mnk}(\mathcal{G}_1, \mathcal{G}_2)\}$. It is evident that $\emptyset \neq \mathcal{A}_{mnk}(\mathcal{G}_1, \mathcal{G}_2) \subseteq \mathcal{A}_{mnk}$ for any finite groups $\mathcal{G}_1$ and $\mathcal{G}_2$. Moreover, the following characteristics are established.

Theorem 1 [1]. If sequences of finite groups $\{\mathcal{G}_1^{(j)} | j \in \mathbb{N}\}$ and $\{\mathcal{G}_2^{(j)} | j \in \mathbb{N}\}$ are under the conditions $\mathcal{G}_i^{(1)} \leq \mathcal{G}_i^{(2)} \leq \dots \mathcal{G}_i^{(j)} \leq \dots$ ($i = 1, 2$), then

$$\emptyset \neq \mathcal{A}_{mnk}(\mathcal{G}_1^{(1)}, \mathcal{G}_2^{(1)}) \subseteq \mathcal{A}_{mnk}(\mathcal{G}_1^{(2)}, \mathcal{G}_2^{(2)}) \subseteq \dots \subseteq \mathcal{A}_{mnk}(\mathcal{G}_1^{(j)}, \mathcal{G}_2^{(j)}) \subseteq \dots \subseteq \mathcal{A}_{mnk}.$$

Theorem 2 [1]. If there exist cyclic groups $\mathcal{Z}_{l_1} = (\mathbf{Z}_{l_1}, \oplus_{\text{mod } l_1})$ and $\mathcal{Z}_{l_2} = (\mathbf{Z}_{l_2}, \oplus_{\text{mod } l_2})$ such that $\mathcal{Z}_{l_i} \leq \mathcal{G}_i$ ($i = 1, 2$) and $\min\{l_1, l_2\} \geq mk$ then $\mathcal{A}_{mnk}(\mathcal{G}_1, \mathcal{G}_2) = \mathcal{A}_{mnk}$.

It is worth noting that Theorem 2 justifies ROM-realizations of finite automata and extracts a sufficiently wide class of these realizations in an explicit form.

If $A = B_F$ ($A \in \mathcal{A}_{mnk}, B_F \in \mathcal{A}_{mnk}(\mathcal{G}_1, \mathcal{G}_2)$) then B_F is $(\mathcal{G}_1, \mathcal{G}_2)$ -presentation of A . At every instant of time computations performed by an automaton B_F realize the scheme

$$\text{coding} \rightarrow \text{operations in groups} \rightarrow \text{decoding}.$$

This scheme is inconvenient if for an automaton $A = (Q_k, X_m, Y_n, \delta_A, \lambda_A)$ transition and output mappings' extensions on the set $Q_k \times X_m^*$ are determined by the identities

$$\begin{aligned} \delta_A(q, \Lambda) = q, \lambda_A(q, \Lambda) = \Lambda, \delta_A(q, px) = \delta_A(\delta_A(q, p), x), \lambda_A(q, px) = \\ = \lambda_A(q, p) \lambda_A(\delta_A(q, p), x), \end{aligned}$$

where $q \in Q_k, p \in X_m^*, x \in X_m$ and Λ is the empty string. It is more preferable to code a state in the initial instant of time and to decode an element of a group into a state in the final instant of time only. Thus it is natural to extend a mapping $f_2 : X_m \rightarrow G_1$ onto the set X_m^* and extract the following subset $\mathcal{F}_{mnk}^c(\mathcal{G}_1, \mathcal{G}_2)$ of the set $\mathcal{F}_{mnk}(\mathcal{G}_1, \mathcal{G}_2)$: $F = (f_1, \dots, f_6) \in \mathcal{F}_{mnk}^c(\mathcal{G}_1, \mathcal{G}_2)$ if $\delta_F(q, p) = f_3(f_1(q) \circ f_2(p))$ for all $q \in Q_k$ and $p \in X_m^*$. The subset $\mathcal{F}_{mnk}^c(\mathcal{G}_1, \mathcal{G}_2)$ is determined by the following criteria.

Theorem 3 [2]. $F = (f_1, \dots, f_6) \in \mathcal{F}_{mnk}^c(\mathcal{G}_1, \mathcal{G}_2)$ if and only if there exists an extension of the mapping f_2 onto the set X_m^* such that the identities

$$f_3(f_1(q) \circ f_2(\Lambda)) = q, \quad f_3(f_1(q) \circ f_2(px)) = f_3(f_1(f_3(f_1(q) \circ f_2(p))) \circ f_2(x))$$

hold for all $q \in Q_k, p \in X_m^*, x \in X_m$.

The following basic characteristics of the subset $\mathcal{F}_{mnk}^c(\mathcal{G}_1, \mathcal{G}_2)$ are established.

Theorem 4 [2]. $\mathcal{F}_{mnk}^c(\mathcal{G}_1, \mathcal{G}_2) = \emptyset$ for any group $\mathcal{G}_1 = (G_1, \circ)$ such that $|G_1| < k$.

Theorem 5 [2]. There exists a group $\mathcal{G}_1 = (G_1, \circ)$ such that $|G_1| = k$ and $\mathcal{F}_{mnk}^c(\mathcal{G}_1, \mathcal{G}_2) \neq \emptyset$.

Theorem 6 [2]. Let $\mathcal{G}_1 = (G_1, \circ)$ be any group such that $|G_1| = k$ and $\mathcal{F}_{mnk}^c(\mathcal{G}_1, \mathcal{G}_2) \neq \emptyset$. Let $F = (f_1, \dots, f_6) \in \mathcal{F}_{mnk}^c(\mathcal{G}_1, \mathcal{G}_2)$ and f_2 be extended onto the set X_m^* . Then B_F is a permutation automaton consisting of $k/|Val f_2|$ strongly connected components, each of $|Val f_2|$ states.

Theorem 7 [2]. Let $F = (f_1, \dots, f_6) \in \mathcal{F}_{mnk}^c(\mathcal{G}_1, \mathcal{G}_2)$ and f_2 be extended onto the set X_m^* . If $f_2(\Lambda)$ is the identity element of the group $\mathcal{G}_1$ and $Val f_1 \circ Val f_2 = Val f_1$, then f_2 is a homeomorphism of the free semigroup X_m^* into the group $\mathcal{G}_1$.

Theorem 6 extracts permutation automata with strongly connected components consisting of the same number of states. It is natural to present these automata via well examined groups. For autonomous automata this problem was investigated in [2,3].

Let $A_i = (Z_{k_i}, \{x\}, \delta_i)$ ($i = 1, \dots, n$) be a sequence of counters, i.e., $\delta_i(z, x) = z + 1 \pmod{k_i}$ for all $z \in Z_{k_i}$ and $\chi(A_1, \dots, A_n) = (A_1 \times \dots \times A_n, \{x\}, \delta)$, where

$$\delta(z_1, \dots, z_n) = (\delta_1(z_1, x), \dots, \delta_n(z_n, x))$$

for all $(z_1, \dots, z_n) \in A_1 \times \dots \times A_n$.

Theorem 8 [2,3]. The automaton $\chi(A_1, \dots, A_n)$ consists of $GCD(k_1, \dots, k_n)$ strongly connected components, each of $LCM(k_1, \dots, k_n)$ states.

Theorem 8 implies that an autonomous automaton B with k states consisting of u strongly connected components, each of v states, is isomorphic to an automaton $\chi(A_1, \dots, A_l)$ if and only if canonical forms $k = p_1^{\gamma_1} \dots p_r^{\gamma_r}$, $u = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, $v = p_1^{\beta_1} \dots p_r^{\beta_r}$, $k_i = p_1^{\mu_{i1}} \dots p_r^{\mu_{ir}}$ ($i = 1, \dots, l$) satisfy the conditions

$$\alpha_j + \beta_j = \gamma_j \quad (j = 1, \dots, r),$$

$$\mu_{1j} + \dots + \mu_{lj} = \gamma_j \quad (j = 1, \dots, r),$$

$$\min\{\mu_{1j}, \dots, \mu_{lj}\} = \alpha_j \quad (j = 1, \dots, r),$$

$$\max\{\mu_{1j}, \dots, \mu_{lj}\} = \beta_j \quad (j = 1, \dots, r).$$

3. Exponential lower bounds

The estimation of Shannon functions' lower bounds is one of the fundamental problems of Discrete Mathematics. Indeed, it is an integral part of algorithms' complexity analysis. The complexity of this problem is justified by the fact that the unique method for solving it is to design an object on which this estimation is accessible explicitly, as a rule. Moreover, the higher the lower bound is, the more difficult is to design the corresponding object. Thus a natural way to reduce the complexity of lower bounds' estimation is to elaborate abstract mathematical schemes with prescribed values of parameters and to give an interpretation in terms of specific discrete problems. In the case when lower bounds are exponents this problem was solved in [4].

Let a permutation $g \in S_n$ be presented as the product of cycles, i.e., $g = D_{r_1} \dots D_{r_l}$, where D_h ($h = r_1, \dots, r_l$) is h -cycle. Then $order(g) = LCM(r_1, \dots, r_l)$. Thus if $f = D_{r+1} \dots D_{2r} \in S_n$, then $order(f) = LCM(r+1, \dots, 2r) = LCM(1, \dots, 2r)$. Since [5] $e^{c_1 x} < LCM(1, \dots, x) < e^{c_2 x}$ ($x \geq 2$), where c_1, c_2 ($0 < c_1 < c_2$) are constants, then $order(f) = e^{O(r)}$ ($r \rightarrow \infty$).

Theorem 9. [4]. If $f = D_{r+1} \dots D_{2r} \in S_n$ ($r = \lfloor \frac{1}{6} \sqrt{(24n+1)-1} \rfloor$), then

$$order(f) = e^{O(\sqrt{n})} \quad (n \rightarrow \infty).$$

This scheme was applied in [6] for estimating the lengths of minimal distinguishing and synchronizing sequences for weakly initialized automata and in [3] for the design of recursive locks of arbitrary high complexity.

A *weakly initialized automaton* (via) is a pair (A, Q_{in}) , where $A \in \mathcal{A}_{mnk}$ and $Q_{in} \subseteq Q_k$ ($|Q_{in}| \geq 2$). For a via (A, Q_{in}) a sequence $p \in X_m^*$ is:

- 1) *distinguishing*, if $\delta_A(q, p) = \delta_A(q', p) \implies \lambda_A(q, p) = \lambda_A(q', p)$ for all $q, q' \in Q_{in}$;
- 2) *synchronizing*, if $|\delta_A(Q_{in}, p)| = 1$.

Let there be selected pair-wise disjoint subsets $W_1, \dots, W_{r-1}$ of the set $Q_k \setminus \{q_1\}$, where

$$W_i = \{q_{\alpha(i)+j} | j = 0, 1, \dots, l+i-1\} \quad (i = 1, \dots, r-1)$$

and $\alpha(i) = 2+l(i-1)+0.5i(i-1)$. These subsets exist for all $r \in \{2, \dots, \lfloor 0.5(\sqrt{8k-7}+1) \rfloor\}$ and $l \in \{0, 1, \dots, \lfloor \frac{1}{2(r-1)}(2(k-1) - r(r-1)) \rfloor\}$. Let (A, Q_{in}) be the via, where $A \in \mathcal{A}_{2nk}$, $Q_{in} = \{q_1, q_{\alpha(1)}, \dots, q_{\alpha(r-1)}\}$ and

$$\begin{aligned} \delta_A(q_u, x_v) &= \begin{cases} q_{u+1}, & \text{if } v = 1 \text{ and } q_u \in W_i \setminus \{q_{\alpha(i+1)-1}\} \quad (i = 1, \dots, r-1), \\ q_{\alpha(i)}, & \text{if } v = 1 \text{ and } u = \alpha(i+1) - 1 \quad (i = 1, \dots, r-1), \\ q_1, & \text{if } v = 2 \text{ and } q_u \in W_1 \cup \dots \cup W_{r-1}, \\ q_1, & \text{if } v = 1, 2 \text{ and } u = 1, \end{cases} \\ \lambda_A(q_u, x_v) &= \begin{cases} y_1, & \text{if } v = 1 \text{ and } q_u \in W_1 \cup \dots \cup W_{r-1}, \\ y_1, & \text{if } v = 2 \text{ and } q_u \in W_i \setminus \{q_{\alpha(i+1)-1}\} \quad (i = 1, \dots, r-1), \\ y_{i+1}, & \text{if } v = 2 \text{ and } u = \alpha(i+1) - 1 \quad (i = 1, \dots, r-1), \\ y_1, & \text{if } v = 1, 2 \text{ and } u = 1. \end{cases} \end{aligned}$$

Similarly, let there be selected pair-wise disjoint subsets $U_1, \dots, U_{r-1}$ of the set $Q_k \setminus \{q_1, q_2\}$, where $U_i = \{q_{\beta(i)+j} | j = 0, 1, \dots, l+i-1\}$ ($i = 1, \dots, r-1$) and $\beta(i) = \alpha(i) + 1$. These subsets exist for all $r \in \{2, \dots, \lfloor 0.5(\sqrt{8k-15}+1) \rfloor\}$. Let (B, Q_{in}) be the via, where $B \in \mathcal{A}_{2nk}$, $Q_{in} = \{q_1, q_{\beta(1)}, \dots, q_{\beta(r-1)}\}$ and

$$\begin{aligned} \delta_B(q_u, x_v) &= \begin{cases} q_{u+1}, & \text{if } v = 1 \text{ and } q_u \in U_i \setminus \{q_{\beta(i+1)-1}\} \quad (i = 1, \dots, r-1), \\ q_{\beta(i)}, & \text{if } v = 1 \text{ and } u = \beta(i+1) - 1 \quad (i = 1, \dots, r-1), \\ q_2, & \text{if } v = 2 \text{ and } q_u \in U_i \setminus \{q_{\beta(i+1)-1}\}, \\ q_1, & \text{if } v = 2 \text{ and } u = \beta(i+1) - 1 \quad (i = 1, \dots, r-1), \\ q_u, & \text{if } v = 1, 2 \text{ and } u = 1, 2. \end{cases} \end{aligned}$$

$\delta_A|_{W_1 \cup \dots \cup W_{r-1}}$ and $\delta_B|_{U_1 \cup \dots \cup U_{r-1}}$ are permutations of the sets $W_1 \cup \dots \cup W_{r-1}$ and $U_1 \cup \dots \cup U_{r-1}$, correspondingly, and consist of the cycles of the lengths $l+1, \dots, l+r-1$. Thus the sequence $x_1^\mu x_2$, where $\mu = LCM(l, l+1, \dots, l+r-2)$ is the single minimal distinguishing one for the via (A, Q_{in}) and the single minimal synchronizing one for the via (B, Q_{in}) . To establish that the lower bound for the maximal lengths of minimal distinguishing and synchronizing sequences for automata $A \in \mathcal{A}_{2nk}$ is $e^{O(\sqrt{k})}$ ($k \rightarrow \infty$), it is sufficient to set $l = r-2$ and $r = O(\sqrt{k})$ ($k \rightarrow \infty$).

A *recursive secret lock* [3] is designed in the following way. Let $A_i = (Z_{k_i}, \{x\}, \delta_i)$ ($i = 1, \dots, l$) be fixed sequence of counters, where $k_1 \dots k_l = k$. The key is an

automaton $\chi(A_1, \dots, A_l)$ which states are identified with the states of its isomorphic image $B \in \mathcal{A}_{1nk}$, such that

$$\delta_B(q_{r\nu+j}, x) = \begin{cases} q_{r\nu+j+1}, & \text{if } j = 1, \dots, \nu - 1 \\ q_{r\nu+1}, & \text{if } j = \nu \end{cases} \quad (r = 0, 1, \dots, \mu - 1)$$

where $\mu = GCD(k_1, \dots, k_l)$ and $\nu = LCM(k_1, \dots, k_l)$. It is suggested that the key may be installed instantly at any of the states $q_{r\nu+1}$ ($r = 0, 1, \dots, \mu - 1$). Let $\varphi : N \rightarrow \{\nu, 2\nu, \dots, \mu\nu\}$ be any recursive piecewise constant function with the length of any step (i.e. interval of permanent values) higher than ν . A *recursive secret lock* is a predicate $P_\varphi : \mathbf{N} \times Q_k \times \mathbf{N} \rightarrow \{0, 1\}$, where

$$P_\varphi(t, q, h) = 1 \iff (q = q_{r\nu+1}) \& (\delta_B(q_{r\nu+1}, x^h) = q_{\varphi(t+h)}).$$

Thus to open the lock it is necessary to install the key correctly and to apply the sequence x^h , where h is multiple of $\nu - 1$. Let $k_i = l + i$ ($i = 1, \dots, l$). The minimal admissible value of h is under the condition $h = e^{O(l)}$ ($l \rightarrow \infty$). This implies that the suggested method gives the possibility to design nonstationary secret locks of arbitrary high level of complexity.

4. Identification of boolean functions

On-line control of discrete devices is often reduced to real-time analysis of corresponding input-output pairs. If the analyzed device is a combinational circuit it realizes the prescribed boolean vector-function $f : \{0, 1\}^m \rightarrow \{0, 1\}^n$. Thus the Problem is formulated as follows: for the given boolean vector-function f and a set $\Omega \subset \{0, 1\}^{m+n}$ it is necessary to check if the inclusion $\Omega \subseteq \text{graph}(f)$ holds. This Problem may be easily reduced to the classic Problem of boolean functions' identification. Thus classic methods based on searching are directly applied. The main disadvantage of these methods is their high complexity. To reduce the complexity it is natural to replace searching by algebraic operations wherever it is possible. This problem was solved in [7]. The main idea of the suggested approach is to explore the fact that $\{0, 1\}^{m+n}$ is the vector space $GF^{m+n}(2)$.

Let $P_{m,n}$ be the set of all boolean vector-functions $f : \{0, 1\}^m \rightarrow \{0, 1\}^n$. Any function $f \in P_{m,n}$ may be presented in the form $f = (f_1, \dots, f_n)$, where $f_i \in P_{m,1}$ ($i = 1, \dots, n$). Thus $P_{m,n} = (P_{m,1})^n$. Let $T_0(m) = \{f \in P_{m,1} | f(0, \dots, 0) = 0\}$ and $L(m)$ be the set of all linear functions $f \in P_{m,1}$.

A set $\chi_f = \{M_i | i = 1, \dots, l\}$ of matrices over the field $GF(2)$ is called a *characteristic function* for the set $\text{graph}(f)$ ($f \in P_{m,n}$) if it satisfies the condition

$$(\forall \mathbf{a} \in \{0, 1\}^{m+n})(\exists M_i \in \chi_f)(\mathbf{a}M_i = \mathbf{0}) \iff \mathbf{a} \in \text{graph}(f).$$

Thus $\mathbf{a} \in \text{graph}(f) \iff \mathbf{0} \in \chi_f(\mathbf{a})$, where $\chi_f(\mathbf{a}) = \{\mathbf{a}M_i | i = 1, \dots, l\}$.

The basic characteristics of the structure of the set $\text{graph}(f)$ ($f \in P_{m,n}$) are the following.

Theorem 10 [7]. The set $\text{graph}(f)$ ($f \in P_{m,n}$) is a subspace of the space $GF^{m+n}(2)$ if and only if $f \in (T_0(m) \cap L(m))^n$.

Since $V = \{\mathbf{0}, \mathbf{a}\}$ is a subspace of $GF^{m+n}(2)$ for any $\{\mathbf{a}\} \in \{0, 1\}^{m+n} \setminus \{\mathbf{0}\}$, the notion of $\text{lin graph}(f)$ ($f \in P_{m,n}$) as the set of all maximal (relatively to the inclusion relation) subspaces of $GF^{m+n}(2)$ which are subsets of $\text{graph}(f)$ is justified.

Theorem 11 [7]. $\text{lin graph}(f) \neq \emptyset$ ($f \in P_{m,n}$) if and only if $f \in T_0^{(m)}(m)$.

The significance of Theorem 11 is justified by the fact that $g = f + f(0, \dots, 0) \in T_0^{(m)}(m)$ for any $f \in P_{m,n}$.

Theorem 12 [7]. For any $f \in T_0^{(m)}(m)$ there holds the identity

$$\text{graph}(f) = \bigcup_{V \in \text{lin graph}(f)} V.$$

Theorem 12 suggests an effective method for the design of a characteristic function $\chi_f \in T_0^{(m)}(m)$. Indeed, let V be a subspace of $GF^{m+n}(2)$ and $\{\mathbf{e}_1, \dots, \mathbf{e}_{m+n-\text{Dim}(V)}\}$ be any basis of its orthogonal complement $V^\perp$. M_V denotes the $(m+n) \times (m+n-\text{Dim}(V))$ -matrix whose columns are $\mathbf{e}_1, \dots, \mathbf{e}_{m+n-\text{Dim}(V)}$.

Theorem 13 [7]. For any $f \in T_0^{(m)}(m)$ the set $\chi_f = \{M_V | V \in \text{lin graph}(f)\}$ is a characteristic function for the set $\text{graph}(f)$.

Theorem 13 implies that the investigated problem may be solved via applying the following algorithm

Step 1. If $f \notin T_0^{(m)}(m)$, then $f := f + f(0, \dots, 0)$, $\Omega := \Omega + f(0, \dots, 0)$.

Step 2. Design the characteristic function $\chi_f = \{M_V | V \in \text{lin graph}(f)\}$.

Step 3. Compute $\chi_f(\mathbf{a})$ for all $\mathbf{a} \in \Omega$.

Step 4. If $\mathbf{0} \in \chi_f(\mathbf{a})$ for all $\mathbf{a} \in \Omega$, then $\Omega \subseteq \text{graph}(f)$, else $\Omega \not\subseteq \text{graph}(f)$.

The space and time complexities of the suggested algorithm are equal to

$$V = O((m+n)((m+n)|\text{lin graph}(f)| + |\Omega| - \sum_{V \in \text{lin graph}(f)} \text{Dim}(V)) \quad (m, n \rightarrow \infty),$$

$$T = O(|\Omega|(m+n)((m+n)|\text{lin graph}(f)| - \sum_{V \in \text{lin graph}(f)} \text{Dim}(V)) \quad (m, n \rightarrow \infty).$$

5. Conclusions

The presented results illustrate the significance of joining algebraic models and methods with searching for solutions of basic problems of the investigation of fundamental

classes of discrete control systems, namely, finite automata and boolean functions. Further investigation of the properties of the suggested presentation of finite automata via detalization of the structure of finite groups gives the possibility to establish subtle relations between The Automata Theory and The Theory of Groups, one of the best examined branch of Modern Algebra. The suggested approach for the identification of boolean vector-functions may be naturally generalized for the case of arbitrary discrete functions. The subtle difference is that Theorem 12 does not hold in the general case.

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A NOTE ON PERIODIC SOLUTIONS IN NON-SYMMETRIC MAY-LEONARD SYSTEMS

ZSOLT SZENTKUTI

Budapest University of Technology and Economics, Institute of Mathematics
H-1521 Budapest, P.O.B.91., Hungary
szs@math.bme.hu

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Abstract. We shall give a proof in this extension of our previous paper for the conjecture about the occurrence of periodic orbits by means of Hopf bifurcation in non-symmetric May-Leonard systems.

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Keywords: Periodic orbits, Hopf bifurcation

1. Introduction

This paper is the continuation of our previous considerations in which we have studied the occurrence of periodic orbits by means of Hopf bifurcation in non-symmetric May-Leonard systems ([5], [8]). In the previous work we gave a conjecture through graphical analysis of the null-clines and we shall give a proof in the present work. In the meantime the valuable results of Chia-Wei Chi, Sze-Bi Hsu and Lih-Ing Wu ([7]) were published in which they observed the global behavior of a more general three-dimensional non-symmetric system than it was shown in our paper. Their results contain our former findings about a special case of the non-symmetric May-Leonard model.

We have analyzed in the previous paper when a Hopf bifurcation occurs in the following special case of the Lotka-Volterra model ([1]):

$$\left. \begin{aligned} \frac{dx_1}{dt} &= x_1 \cdot (1 - x_1 - \alpha \cdot x_2 - \gamma \cdot x_3) \\ \frac{dx_2}{dt} &= x_2 \cdot (1 - \beta \cdot x_1 - x_2 - \alpha \cdot x_3) \\ \frac{dx_3}{dt} &= x_3 \cdot (1 - \alpha \cdot x_1 - \beta \cdot x_2 - x_3) \end{aligned} \right\} \quad (1.1)$$

with the conditions:

$$0 < \alpha < 1, \quad \beta > 1, \quad \gamma > 1 \quad (1.2)$$

May and Leonard [2] first observed the symmetric case (it means that $\gamma = \alpha$) of the system (1.1) and they showed:

1. if $\alpha + \beta < 2$, then there is an internal fixed point P in the positive cone R_+^3 and P is stable,

2. if $\alpha + \beta \geq 2$ and also $\alpha > 1$ and $\beta > 1$, then the system converges to a fixed point, denoted by R_i ($i = 1, 2, 3$) lying on any of the coordinate axes. Fixed points of this type are called axial fixed points,
3. if $\alpha + \beta \geq 2$ and also $\alpha < 1$ and $\beta > 1$ (or $\alpha > 1$ and $\beta < 1$, which is a totally symmetric case), then no stable fixed point exists.

May, Leonard and Schuster, Sigmund and Wolf [2], [3] showed that in the last case the system exhibits a general class of solutions with non-periodic oscillations of bounded amplitude and increasing cycle time.

The third is the most interesting case for our recent considerations and we call the system May-Leonard system if the conditions in the third case hold. (We supposed the condition (1.2) because of this.) We will analyze the system and sketch the possible phase-portraits for this model in Section 2. We shall give a necessary and sufficient condition for the occurrence of Hopf bifurcation. We will use by this consideration the classification theory of M. L. Zeeman for three-dimensional Lotka-Volterra models [4], [6]. We will give a summary about the global behavior of the general case of non-symmetric May-Leonard systems in Section 3.

2. Occurrence of periodic orbits in the special case

We will study in this section the global behavior of system (1.1) with condition (1.2) on the carrying simplex. For the sake of completeness we will observe the case $0 < \gamma < 1$ at the end of the section. Let us first compute the equilibrium points of the model:

1. the origin: $O = \text{col}[0, 0, 0]$,
2. the axial fixed points: $R_1 : \text{col}[1, 0, 0]$, $R_2 : \text{col}[0, 1, 0]$ and $R_3 : \text{col}[0, 0, 1]$,
3. the planar fixed points: $Q_1 : \frac{1}{\alpha \cdot \beta - 1} \text{col}[0, \alpha - 1, \beta - 1]$, $Q_2 : \frac{1}{\alpha \cdot \gamma - 1} \text{col}[\gamma - 1, 0, \alpha - 1]$ and $Q_3 : \frac{1}{\alpha \cdot \beta - 1} \text{col}[\alpha - 1, \beta - 1, 0]$,
4. the interior fixed point:

$$P : \frac{1}{\det A} \text{col} [\alpha^2 - \alpha\beta - \alpha + 1 + \gamma\beta - \gamma, 1 - \alpha - \gamma\alpha - \beta + \alpha^2 + \gamma\beta, \\ 1 - \alpha\beta + \alpha^2 - \alpha - \beta + \beta^2] = \frac{1}{\det A} (a, b, c)$$

where $\det A = 1 - 2\alpha\beta - \gamma\alpha + \alpha^3 > 0$ and a, b, c are the coordinates of the interior fixed point. A later computation shows that $a > 0$, $b > 0$, $c > 0$, $\det A > 0$ and because of this P is lying in the positive cone.

Considering conditions (1.2) we can see that by these assumptions the planar fixed points do not lie in the positive cone. Computing the R_{ij} and Q_{ii} expressions ($i, j = 1, 2, 3$) for the classification of M. L. Zeeman ([4]) we get:

$$\begin{array}{lll} R_{12} = 1, & R_{21} = -1, & R_{31} = 1, \\ R_{13} = -1, & R_{23} = 1, & R_{32} = -1. \end{array}$$

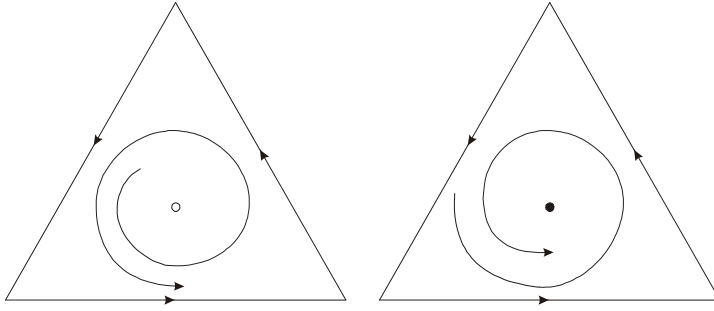


Figure 1. Possible phase

On the basis of the preceding considerations we can classify the system into a stable null-cline class and we can draw two possible phase portraits for the system on the carrying simplex (FIGURE 1) representing the global long-term behavior (it is represented for simplicity on the unit simplex which is homeomorphic with the real phase portrait).

If we want to say something about the occurrence of Hopf bifurcation, we have to compute the varying system. The varying system of system (1.1) is the following:

$$\dot{y} = \frac{1}{\det A} \begin{bmatrix} -a & -\alpha a & -\gamma a \\ -\beta b & -b & -\alpha b \\ -\alpha c & -\beta b & -c \end{bmatrix} \cdot y = \frac{1}{\det A} \begin{bmatrix} -a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -c \end{bmatrix} \cdot \begin{bmatrix} 1 & \alpha & \gamma \\ \beta & 1 & \alpha \\ \alpha & \beta & 1 \end{bmatrix} \cdot y, \quad (2.1)$$

where $\det A > 0$ and $a, b, c > 0$. Computing the eigenvalues of the varying system (2.1):

$$\begin{aligned} \lambda_1 &= -1 \\ \lambda_{2,3} &= \frac{1}{2} \left[(1 - a - b - c) \pm \sqrt{(a + b + c - 1)^2 - 4abc(\det A)} \right] \end{aligned} \quad (2.2)$$

By studying the (2.2) eigenvalues of the varying system we can find the following:

Theorem 1 *In system (1.1) there occurs a Hopf bifurcation if and only if the expression*

$$\gamma = 1 - \frac{(\alpha - 1)^3}{(\beta - 1)^2}$$

holds.

Proof. Let us consider how the conditions of the Andronov-Hopf theorem [1] hold.

1. Condition $\text{Re}\lambda_1 < 0$ is fulfilled because $\lambda_1 = -1$.
2. Secondly, we need to observe the two other eigenvalues. Let us use the following notations:

$$\Delta = \det A, \quad \Delta_1 = \det \begin{bmatrix} 1 & \alpha & \gamma \\ 1 & 1 & \alpha \\ 1 & \beta & 1 \end{bmatrix}, \quad \Delta_2 = \det \begin{bmatrix} 1 & 1 & \gamma \\ \beta & 1 & \alpha \\ \alpha & 1 & 1 \end{bmatrix},$$

$$\Delta_3 = \det \begin{bmatrix} 1 & \alpha & 1 \\ \beta & 1 & 1 \\ \alpha & \beta & 1 \end{bmatrix}$$

and $A = 1 - \alpha > 0$, $B = \beta - 1 > 0$, $C = \gamma - 1 > 0$. Using these notations a simple computation yields:

$$\begin{aligned} \Delta &= B^2C + 2BC + B^2 + 2AB + AC + A^2(3 - A) > 0, \\ \Delta_1 &= A^2 + AB + BC > 0, \\ \Delta_2 &= A^2 + AC + BC > 0, \\ \Delta_3 &= A^2 + AB + B^2 > 0. \end{aligned}$$

We can easily see with these results that the interior fixed point is lying in the positive cone:

$$P : (a, b, c) = \left(\frac{\Delta_1}{\Delta}, \frac{\Delta_2}{\Delta}, \frac{\Delta_3}{\Delta} \right) > 0$$

The real part of $\lambda_{2,3}$ is $\frac{1}{2}(1 - a - b - c)$ because

$$(a + b + c - 1)^2 - 4abc(\det A) < 0$$

Let us prove this inequality:

$$\begin{aligned} (a + b + c - 1)^2 - 4abc(\det A) &= \frac{1}{\Delta^2} \left[(\Delta_1 + \Delta_2 + \Delta_3 - \Delta)^2 - 4\Delta_1\Delta_2\Delta_3 \right] = \\ &= \frac{1}{\Delta^2} \left[(B^2C - A^3)^2 - 4(A^2 + AB + BC)(A^2 + AC + BC)(A^6 + AB + B^2) \right] = \\ &= \frac{1}{\Delta^2} [B^4C^0 + A^6 - 2A^2B^2C - 4(B^4C^2 + A^6 + G(A, B, C))] < 0 \end{aligned}$$

where G is a homogeneous polynomial of A, B, C and $G > 0$. Since the inequality holds, $\lambda_{2,3}$ has not only real parts. Let us observe the following bifurcation parameter:

$$\mu = 1 - (a + b + c)$$

The real part of $\lambda_{2,3}$ is with this parameter:

$$\text{Re}\lambda_{2,3} = \frac{\mu}{2} \quad \text{and} \quad \frac{d\text{Re}\lambda_{2,3}}{d\mu} = \frac{1}{6} > 0.$$

All of the conditions of the Andronov-Hopf theorem are fulfilled.

■

3. Summary

We could see previously a simple generalization of the symmetrical May-Leonard model, which neglects considering symmetry. The papers mentioned in the introduction give stability results also for models more general than system (1.1). In our previous and current paper we examined which conditions should hold for periodic orbits to occur in the model by means of Hopf bifurcation. We gave a necessary and sufficient condition for Hopf bifurcation to occur in the above non-symmetric system, and our results coincided with the more general case restricted to model (1.1) with (1.2). Based on this fact we can say that the stability loss of the equilibrium state of the system by the specified conditions is due to the presence of periodic orbits occurring by means of Hopf bifurcation.

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LEFT SUBHEREDITARY SUPERNILPOTENT RADICAL CLASSES

D. ISABEL C. MENDES
Universidade da Beira Interior
Covilhã, Portugal
dmendes@noe.ubi.pt

SODNOMKHORLOO TUMURBAT
University of Mongolia, P.O.Box 75
Ulaan Baatar 20, Mongolia
tumur@www.com

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Abstract. (Left) subhereditary radicals were introduced in [5]. Here, we continue to study such radicals and give some characterizations of supernilpotent subhereditary radicals.

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1. Preliminaries

In this note rings are associative, not necessarily with unit elements. As usual, $I \triangleleft A$ and $L \triangleleft_\ell A$ denote, respectively, that I is an ideal and L is a left ideal in the ring A .

Let us recall that a (Kurosh–Amitsur) *radical* γ is a class of rings, which is closed under homomorphisms, extensions (I and A/I in γ imply A in γ), and has the inductive property (if $I_1 \subseteq \cdots \subseteq I_\lambda \cdots$ is a chain of ideals of the ring $A = \cup I_\lambda$ and each I_λ is in γ , then A is in γ). Every ring A contains a unique largest γ -ideal (i.e. an ideal which is in γ), $\gamma(A)$, which is called the γ -radical of A .

A radical γ determines its semisimple class

$$S\gamma = \{A \mid \gamma(A) = 0\}$$

which is always a hereditary class ($I \triangleleft A \in S\gamma$ implies $I \in S\gamma$).

A class $\mathcal{M}$ of rings is said to be *regular*, if every nonzero ideal of a ring in $\mathcal{M}$ has a nonzero homomorphic image in $\mathcal{M}$. Starting from a regular (in particular, *hereditary*) class $\mathcal{M}$ of rings, the *upper radical operator* $\mathcal{U}$ yields a radical class:

$$\mathcal{U}\mathcal{M} = \{A \mid A \text{ has no nonzero homomorphic image in } \mathcal{M}\}.$$

Remember that the *Baer radical* β is the upper radical determined by all prime rings. A radical γ containing all nilpotent rings is called a *hypernilpotent radical*. A

radical γ is said to be *supernilpotent* if it is hereditary and hypernilpotent. A left ideal L of a ring A is said to be *essential* if $I \cap L \neq 0$ for every $0 \neq I \triangleleft A$ and is denoted by $L \triangleleft_\ell^\bullet A$. A radical γ is said to be *(left) subhereditary* if

$$0 \neq L \triangleleft_\ell A \in \gamma \text{ implies } \gamma(L) \neq 0,$$

or equivalently

$$0 \neq L \triangleleft_\ell^\bullet A \text{ and } \gamma(L) = 0 \text{ imply } \gamma(A) = 0$$

(see [5]).

A class $\mathcal{M}$ of prime (semi-prime) rings is *special* (*weakly special*) if it is hereditary and closed *under essential extensions* (if $I \in \mathcal{M}$ is an essential ideal of a ring A , then also $A \in \mathcal{M}$).

The upper radical $\mathcal{UM}$ of a special (weakly special) class $\mathcal{M}$ of rings is called a *special radical* (*weakly special radical*).

2. Subhereditary and hypernilpotent radicals

PROPOSITION 1. *If γ is a subhereditary radical and A is any semiprime ring, then every γ -semisimple left ideal L of A can be extended to a unique largest ideal I of A such that $\gamma(A) \cap I = 0$.*

Proof. First, we shall prove that $\gamma(A) \cap L = 0$. Suppose that this is not the case. Since γ is subhereditary and

$$\gamma(A) \cap L \triangleleft_\ell \gamma(A) \in \gamma,$$

it follows that $\gamma(\gamma(A) \cap L) \neq 0$. By $\gamma(A) \cap L \triangleleft L$, also

$$0 \neq \gamma(\gamma(A) \cap L) \subseteq \gamma(L),$$

contradicting the assumption on L . Thus $\gamma(A) \cap L = 0$ holds, implying $\gamma(A)L = 0$. Define

$$K = \sum (L \triangleleft_\ell A / 0 \neq L \in S\gamma) \triangleleft_\ell A.$$

We also have $\gamma(A)K = 0$. The ideal I of A generated by K is $I = K + KA$. Now $\gamma(A) \cap I \triangleleft A$ and

$$(\gamma(A) \cap I)^2 \subseteq \gamma(A)I = \gamma(A)(K + KA) = 0.$$

Since A is semiprime, we conclude that $\gamma(A) \cap I = 0$. □

PROPOSITION 2. *Let γ be a hypernilpotent radical. If in every semiprime ring A each γ -semisimple left ideal L can be extended to a unique largest ideal I of A such that $\gamma(A) \cap I = 0$, then γ is subhereditary.*

Proof. Let $0 \neq L \triangleleft_\ell^\bullet A$ and $\gamma(L) = 0$. We prove the validity of condition ($\diamond$). If the ring A is semiprime, then, by the assumption, there exists an ideal I of A such that

$L \subseteq I$ and $\gamma(A) \cap I = 0$. Hence also $\gamma(A) \cap L = 0$. Since L is an essential left ideal of A , it follows that $\gamma(A) = 0$.

If A is not semiprime, then $\beta(A) \neq 0$. Clearly $\beta(A) \cap L \triangleleft L$ and $\beta(A) \cap L \triangleleft_\ell \beta(A)$. Since $L \triangleleft_\ell^\bullet A$, we have by the left hereditariness of β that

$$0 \neq \beta(A) \cap L \subseteq \beta(L).$$

Since γ is hypernilpotent, also $\gamma(L) \neq 0$, contradicting $\gamma(L) = 0$. $\square$

COROLLARY 3. *Let γ be a subhereditary radical. If a semiprime ring A has a nonzero γ -semisimple left ideal L , then either $\gamma(A) = 0$ or A is not a subdirectly irreducible ring.* $\square$

THEOREM 4. *Let γ be a hypernilpotent radical. Then the following are equivalent:*

- i) γ is subhereditary.
- ii) If A is a semiprime ring, $L \triangleleft_\ell A$ and $\gamma(A)L \neq 0$, then $\gamma(L) \neq 0$.
- iii) In every semiprime ring A each γ -semisimple left ideal L can be extended to a unique largest ideal I of A such that $\gamma(A) \cap I = 0$.

Proof. The equivalence i) $\Leftrightarrow$ iii) follows from Propositions 1 and 2.

i) $\Rightarrow$ ii): Let γ be a subhereditary radical, A a semiprime ring and $L \triangleleft_\ell A$, such that $\gamma(A)L \neq 0$. Clearly $\gamma(A)L \triangleleft L$ and $\gamma(A)L \triangleleft_\ell \gamma(A)$. Since γ is subhereditary, $0 \neq \gamma(\gamma(A)L) \triangleleft L$ and so $\gamma(L) \neq 0$.

ii) $\Rightarrow$ i): Let $0 \neq L \triangleleft_\ell \gamma(A)$. Suppose that $\gamma(L) = 0$. Let $I \triangleleft \gamma(A)$ such that I is maximal with respect to $I \cap L = 0$. Then $\gamma(A)/I$ is a semiprime ring. Indeed, if not, then $\beta(\gamma(A)/I) \neq 0$ and so $\beta(\gamma(A)/I) \cap (L + I)/I \neq 0$. Clearly

$$L = L/L \cap I \cong (L + I)/I = \overline{L}.$$

Hence $0 \neq \beta(\overline{L})$ and so $\beta(L) \neq 0$.

Now we get $(\gamma(A)/I)\overline{L} \neq 0$ and so $\gamma(\overline{L}) \neq 0$. Since $\overline{L} \cong L$, we have $\gamma(L) \neq 0$, a contradiction. $\square$

For a subset S of a ring A , we denote the left (right) annihilator of S in A by $\ell(A, S)$, (and $r(A, S)$, respectively). Further, let $\mathcal{P}$ denote the class of all prime rings.

THEOREM 5. *For a special class $\mathcal{M}$, the following conditions are equivalent:*

- i) $\gamma = \mathcal{UM}$ is subhereditary.
- ii) If A is a semiprime ring with $\gamma(A) \neq 0$ and $L \triangleleft_\ell^\bullet A$, then $\gamma(L)L \neq 0$.
- iii) Let $B = \sum (B_\lambda \in \gamma \cap \mathcal{P} \mid \text{there exists } 0 \neq L_\lambda \triangleleft_\ell B_\lambda \text{ with } L_\lambda/r(L_\lambda, L_\lambda) \in \mathcal{M})$. If $L \triangleleft_\ell^\bullet B$ and $L \in \mathcal{S}\gamma$, then $B \in \mathcal{S}\gamma$.

Proof. i) $\Rightarrow$ ii). Suppose that γ is subhereditary, $L \triangleleft_\ell^\bullet A$ and $\gamma(A) \neq 0$. Since $L \triangleleft_\ell^\bullet A$, $\gamma(A) \cap L \neq 0$. Clearly $\gamma(A) \cap L \triangleleft L$ and $\gamma(A) \cap L \triangleleft_\ell \gamma(A)$. Since γ is subhereditary,

$$0 \neq \gamma(\gamma(A) \cap L) \subseteq \gamma(L),$$

it follows $\gamma(L) \neq 0$. If $\gamma(L)L = 0$, then $\ell(A, L) \neq 0$. Since $L \triangleleft_\ell^\bullet A$,

$$0 \neq \ell(A, L) \cap L \triangleleft_\ell A \text{ and so } (\ell(A, L) \cap L)^2 = 0.$$

Therefore $\beta(A) \neq 0$, contradicting A is semiprime. Thus $\gamma(L)L \neq 0$.

ii) $\Rightarrow$ iii) Suppose $L \triangleleft_\ell^\bullet B$ and $L \in S\gamma$. Since B_λ is prime for each index λ , B is semiprime. If $\gamma(B) \neq 0$, then $\gamma(L)L \neq 0$, and so $\gamma(L) \neq 0$, a contradiction.

iii) $\Rightarrow$ i). Let $0 \neq L \triangleleft_\ell A$, where $A \in \gamma$. Suppose that $L \in S\gamma$. Without loss of generality we can choose $L \triangleleft_\ell^\bullet A$. Since γ is a special radical, L is a subdirect sum of prime rings $0 \neq L/I_\lambda \in \mathcal{M}$, where $\cap I_\lambda = 0$. By Lemma 3 [1], there exist prime ideals A_λ of A , for each λ such that

$$I_\lambda^2 \subseteq L \cap A_\lambda \subseteq I_\lambda.$$

Put $B_\lambda = A/A_\lambda$ and $L_\lambda = (L + A_\lambda)/A_\lambda$. Therefore $\gamma(L_\lambda) = \gamma((L + A_\lambda)/A_\lambda) \cong \gamma(L/L \cap A_\lambda) = I_\lambda/L \cap A_\lambda \in \beta$, where β is Baer radical. It is easy to see $L_\lambda \gamma(L_\lambda) = 0$ and so $\gamma(L_\lambda) = r(L_\lambda, L_\lambda)$. We show that $\cap A_\lambda = 0$. If $0 \neq \cap A_\lambda$, then $0 \neq L \cap (\cap A_\lambda) = \cap (L \cap A_\lambda) \subseteq \cap I_\lambda = 0$, a contradiction. Hence $A = \sum_{\text{subdirect sum}} B_\lambda$, and clearly $B_\lambda \in \gamma$. Since $L \in S\gamma$, by assumption $A \in S\gamma \cap \gamma = 0$, contradicting $0 \neq L$. $\square$

DEFINITION 6. A class $\mathcal{M}$ of rings is called a *left special* (*left weakly special*) class of rings if the following conditions are satisfied:

- i) $\mathcal{M}$ consists of prime (semiprime) rings.
- ii) $\mathcal{M}$ is hereditary.
- iii) If A is a semiprime ring, $0 \neq L \triangleleft_\ell^\bullet A$ and $L \in \mathcal{M}$, then $A \in \mathcal{M}$.

Let P denote the class of all prime rings.

THEOREM 7. i) A hypernilpotent radical γ is subhereditary if and only if $S\gamma$ is a left weakly special class of rings.

ii) A hypernilpotent radical γ is subhereditary and special if and only if $S\gamma$ is left weakly special and $\gamma = \mathcal{U}(S\gamma \cap P)$.

Proof. i) Let γ be a subhereditary radical. Then by $(\diamond)$, if $L \in S\gamma$ and $L \triangleleft_\ell^\bullet A$, then $\gamma(A) = 0$, and so $A \in S\gamma$. Since γ is hypernilpotent, $S\gamma$ consists of semiprime rings, and $S\gamma$ is hereditary.

Suppose $S\gamma$ is a left weakly special class. If $L \in S\gamma$ and $L \triangleleft_\ell^\bullet A$ then $A \in S\gamma$. Again by $(\diamond)$, γ is subhereditary.

ii) $\Rightarrow$: Since γ is a special, clearly $\gamma = \mathcal{U}(S\gamma \cap P)$. By i) $S\gamma$ is a left weakly special class.

$\Leftarrow$: Again by i) γ is a subhereditary. Since $S\gamma$ is a left weakly special class, $S\gamma \cap P$ is a special class and so γ is special. $\square$

LEMMA 8. Let $\{H_i : i \in \Lambda\}$ be a family of subhereditary and weakly special radicals.

Then $\mathcal{L}(\cup H_i)$, the lower radical class determined by $\cup H_i$, is also a subhereditary and weakly special radical class.

Proof. We know that $\mathcal{L}(\cup H_i)$ is a weakly special radical. By [5, Proposition 2.2] $\mathcal{L}(\cup H_i)$ is subhereditary. $\square$

In a similar way to ([3], Theorem 4) we can use Lemma 8 to prove the following theorem:

THEOREM 9. *A radical class $\mathcal{H}$ contains the largest subhereditary special radical class if and only if it contains the class $\mathcal{Z}$ of all zero rings.*

The above theorem and ([2], Proposition 2) give rise to the following:

COROLLARY 10. *A radical class $\mathcal{H}$ contains the largest subhereditary radical class if and only if $\mathcal{H}$ contains the largest weakly special radical class.*

The following theorem can be proved in a similar way to ([3], Theorem 5).

THEOREM 11. *The following conditions are equivalent for a class $\mathcal{M}$ of rings:*

- (i) $\mathcal{UM}$ is a radical class containing a subhereditary special radical class.
- (ii) $\mathcal{UM}$ is a radical class containing a subhereditary weakly special radical class.
- (iii) (a) Every $0 \neq A \in \mathcal{M}$ has some non-zero image in SUM .
- (b) $SUM \subseteq \mathcal{E}$ where $\mathcal{E}$ denotes the class of all semi-prime rings.

COROLLARY 12. (see [4], Corollary 1) *The following conditions are equivalent for an arbitrary class $\mathcal{M}$ of rings:*

- (i) $\mathcal{UM}$ is a radical class containing a subhereditary special radical class.
- (ii) $\mathcal{UM}$ is a radical class containing a subhereditary weakly special radical class.
- (iii) $\mathcal{UM}$ is a radical class containing a special left strong radical class.
- (iv) $\mathcal{UM}$ is a radical class containing a weakly special left strong radical class.
- (v) $\mathcal{UM}$ is a radical class containing a hereditary left special radical class.
- (vi) $\mathcal{UM}$ is a radical class containing a hereditary left weakly special radical class.
- (vii) $\mathcal{UM}$ is a radical class containing a left special radical class.
- (viii) $\mathcal{UM}$ is a radical class containing a weakly special radical class.
- (ix) $\mathcal{UM}$ is a hypernilpotent radical class. $\square$

For a hereditary class $\mathcal{M}$ of semiprime rings, let us define

$$\mathcal{M}_k = \{A \mid A \text{ has an essential left ideal in } SUM\}.$$

We terminate this paper with the following theorem:

THEOREM 13. *Let $\mathcal{M}$ be a hereditary class of semiprime rings. Then the following statements are equivalent:*

- (i) $\mathcal{UM}$ is a left subhereditary radical class.
- (ii) $\mathcal{UM} = \mathcal{UM}_k$.
- (iii) $\mathcal{UM} \cap \mathcal{M}_k = 0$.
- (iv) $\mathcal{M}_k \subseteq \mathcal{SUM}$.
- (v) Every $0 \neq R \in \mathcal{M}_k$ has some non-zero image in $\mathcal{SUM}$.
- (vi) $\mathcal{UM}$ has the intersection property relative to $\mathcal{M}_k$; that is, for any ring R , the radical $\mathcal{UM}(R) = \cap \{I : R/I \in \mathcal{M}_k\}$.

Proof. First we show that (i) implies (ii). Since $\mathcal{M} \subseteq \mathcal{M}_k$, we have that $\mathcal{UM}_k \subseteq \mathcal{UM}$. Now if $R \in \mathcal{M}_k$, then R has an essential left ideal in $\mathcal{SUM}$ and, if $\mathcal{UM}$ is left subhereditary, it follows that $R \in \mathcal{SUM}$ and so $\mathcal{UM} \subseteq \mathcal{UM}_k$. To show that (ii) implies (i), suppose that $\mathcal{UM} = \mathcal{UM}_k$. Let $R \in \mathcal{UM}$ and $0 \neq L \triangleleft_\ell R$. If $L \in \mathcal{SUM}$ and $I \triangleleft R$ is maximal with respect to $I \cap L = 0$, then $L \cong (L + I)/I \triangleleft_\ell^* R/I$ and, obviously, $R/I \in \mathcal{M}_k$. But this is a contradiction with $R \in \mathcal{UM} = \mathcal{UM}_k$. It is easy to prove that (ii) implies (iii), (iii) implies (iv), (iv) implies (v) and (v) implies (ii). The proofs of (iv) implies (vi) and (vi) implies (v) are analogous to the proofs of (iv) implies (vi) and (vi) implies (v) of ([3], Theorem 6). $\square$

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