HU e-ISSN 1787-2413

MATHEMATICAL NOTES

A Publication of the University of Miskolc

VOLUME 3, NUMBER 2 (2002)

MISKOLC UNIVERSITY PRESS

Mathematical Notes

Volume 3, Number 2 (2002)

Contents

Contributed Papers

Doi: 10.18514/MMN.2002.53

Mathematical Notes, Miskolc, Vol. 3., No. 2., (2002), pp. 83–89

SOME RESULTS ON ONE-SIDED GENERALIZED LIE IDEALS WITH DERIVATION

Neşet Aydin
Mersin University, Faculty of Arts and Science, Department of Mathematics
Mersin, Turkey
neseta@mersin.edu.tr

Kâzim Kaya
18 Mart University, Faculty of Arts and Science, Department of Mathematics
Çanakkale, Turkey
kkaya@comu.edu.tr

Öznur Gölbaşi
Cumhuriyet University, Faculty of Arts and Science, Department of Mathematics
Sivas, Turkey
ogolbasi@cumhuriyet.edu.tr

[Received: June 6, 2001]

Abstract. Let R be a prime ring with a characteristic not equal to two, σ, τ be automorphisms of R, and d be a nonzero derivation of R commuting with σ and τ. It is proved that for any (σ,τ)-left Lie ideal U of R: (1) if $d(U) \subseteq Z$, then $\sigma(u)+\tau(u) \in Z$, for all $u \in U$, (2) if $d^2(U)=0$, then $\sigma(u)+\tau(u) \in Z$, for all $u \in U$, (3) if *char* $R \neq 2,3$, $d(U) \subseteq U$ and $d^2(U) \subseteq Z$, then $\sigma(u)+\tau(u) \in Z$, for all $u \in U$.

Mathematical Subject Classification: 16N60, 16W25, 16A72, 16U80
Keywords: prime ring, Lie ideal, generalized Lie ideal, derivation

1. **Introduction**

Let R be a ring and σ, τ be two mappings from R into itself. We write $[x, y]$, $[x, y]_{\sigma,\tau}$ for $xy - yx$ and $x\sigma(y) - \tau(y)x$, respectively, and make extensive use of basic commutator identities: $(xy, z) = x[y, z] + (x, z)y = x(y, z) - [x, z]y$, $[xy, z]_{\sigma,\tau} = x[y, z]_{\sigma,\tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma,\tau}y$.

An additive mapping $D : R \to R$ is called a *derivation* if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. A derivation D is *inner* if there exists an $a \in R$ such that $D(x) = [a, x]$ holds for all $x \in R$..

For subsets $A, B \subset R$, let $[A, B]$ ($[A, B]_{\sigma,\tau}$) be the additive subgroup generated by all $[a, b]$ ($[a, b]_{\sigma,\tau}$) for all $a \in A$ and $b \in B$. We recall that in a *Lie ideal*, L is an additive subgroup of R such that $[R, L] \subset L$. We first introduce the generalized Lie ideal in [6] as follows. Let U be an additive subgroup of R, $\sigma, \tau : R \to R$ two mappings. Then (i) U is a $(\sigma,\tau)-$right Lie ideal of R if $[U, R]_{\sigma,\tau} \subset U$. (ii) U is a $(\sigma,\tau)-$left Lie ideal of R if $[R, U]_{\sigma,\tau} \subset U$. (iii) U is both a $(\sigma,\tau)-$right Lie ideal and $(\sigma,\tau)-$left Lie ideal of R then U is a $(\sigma,\tau)-$Lie ideal of R. Every Lie ideal of R is a $(1,1)-$left Lie ideal of R, where $1 : R \to R$ is the identity map. As an example, let I be the set of integers,

$$R = \left\{ \begin{pmatrix} x & y \\ z & t \end{pmatrix} \mid x, y, z, t \in I \right\},$$
$$U = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \mid x, y \in I \right\} \subset R,$$

and $\sigma, \tau : R \to R$ the mappings defined by $\tau(x) = axa$, $\sigma(x) = bxb^{-1}$, where $a = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \in R$. Then U is a $(\sigma,\tau)-$left Lie ideal but not a Lie ideal of R. Some algebraic properties of $(\sigma,\tau)-$Lie ideals are considered in [2], [3] and [6], where further references can be found.

Let R be a prime ring with a characteristic not equal to two, $d : R \to R$ a nonzero derivation of R and U a Lie ideal of R. In [5] Bergen at all state that if $d^2(U) = 0$, then $U \subset Z$. Lee and Lee extended this result that if $d^2(U) \subset Z$, then $U \subset Z$ in [4]. Let d be a nonzero derivation such that $\sigma d = d\sigma, \tau d = d\tau$ and U a $(\sigma,\tau)-$Lie ideal of R. Aydın and Soytürk [3] proved that if $d^2(U) = 0$, then $U \subset Z$. In the present paper, we generalize this result on $(\sigma,\tau)-$left Lie ideal of R. Furthermore, we shall extend this theorem by proving that $d^2(U) \subset Z$ then $\sigma(u) + \tau(u) \in Z$, for all $u \in U$ in the case of a characteristic not equal to two and three.

Throughout, R will represent a prime ring with a characteristic not equal to 2 with automorphisms σ, τ and non-zero derivation d such that $\sigma d = d\sigma, \tau d = d\tau$ and Z the center of R, U a $(\sigma,\tau)-$left Lie ideal of R. Further, we often use the relations:

$$[xy, z]_{\sigma,\tau} = x[y, z]_{\sigma,\tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma,\tau}y.$$

2. **Results**

Lemma 1. *Let U a $(\sigma,\tau)-$left Lie ideal of R. $d^2(U) = 0$ and $d(U) \subset Z$ then $\sigma(u) + \tau(u) \in Z$, for all $u \in U$.*

Proof. If $U \subset Z$, then the proof is obvious. So, we assume that $U \not\subset Z$. For any $u \in U$ and $x \in R$, $\tau(u)[x, u]_{\sigma,\tau} = [\tau(u)x, u]_{\sigma,\tau} + [\tau(u), \tau(u)]x \in U$. By hypothesis, $0 = d^2(\tau(u)[x, u]_{\sigma,\tau}) = d(d(\tau(u))[x, u]_{\sigma,\tau} + \tau(u)d([x, u]_{\sigma,\tau})) = 2d(\tau(u))d([x, u]_{\sigma,\tau})$. Since $char R \neq 2$, we obtain $d(\tau(u))d([x, u]_{\sigma,\tau}) = 0$, for all $x \in R, u \in U$. Because of $d(U) \subset Z$ we have,

$$d(u) = 0 \qquad or \qquad d([x, u]_{\sigma,\tau}) = 0 \qquad \forall x \in R, u \in U. \tag{2.1}$$

Assume $d(u) \neq 0$. Then $d([x, u]_{\sigma,\tau}) = 0$, for all $x \in R$. Writing $x\sigma(u)$ by x in this equation, $0 = d([x\sigma(u), u]_{\sigma,\tau}) = d([x, u]_{\sigma,\tau}\sigma(u)) = d([x, u]_{\sigma,\tau})\sigma(u) + [x, u]_{\sigma,\tau}d(\sigma(u))$

we obtain

$$[x,u]_{\sigma,\tau}d(\sigma(u))=0 \qquad \forall x\in R. \tag{2.2}$$

Substituting $xy, y\in R$ for x in (2.2), we have $0=[xy,u]_{\sigma,\tau}d(\sigma(u))=x[y,u]_{\sigma,\tau}d(\sigma(u))+[x,\tau(u)]yd(\sigma(u))$ and so,

$$[R,\tau(u)]Rd(\sigma(u))=0.$$

By primeness of R, we obtain $u\in Z$. Thus, if we return to (2.1), then we get

$$d(u)=0 \qquad \textit{or} \qquad u\in Z.$$

Now, let us define the subsets $L=\{u\in U \mid u\in Z\}$ and $K=\{u\in U \mid d(u)=0\}$. Clearly, each L and K is an additive subgroup of U. Moreover, U is the set-theoretic union of L and K. But a group cannot be the set-theoretic union of two proper subgroups, hence $L=U$ or $K=U$. In the former case, $U\subset Z$, which is a contradiction. Therefore, it must be $d(U)=0$ and so,

$$0=d([x,u]_{\sigma,\tau})=[d(x),u]_{\sigma,\tau} \qquad \text{for all } x\in R,\ u\in U.$$

By [7, Lemma 1], we obtain $\sigma(u)+\tau(u)\in Z$, for all $u\in U$. Hence the proof is complete. □

Theorem 1. *Let U a $(\sigma,\tau)-$left Lie ideal of R. If $d(U)\subset Z$ then $\sigma(u)+\tau(u)\in Z$, for all $u\in U$.*

Proof. Assume that $U\not\subset Z$. For any $x,y\in R$ and $u,v\in U$, by hypothesis, $d([d(v)x,u]_{\sigma,\tau})$ $=d(d(v)[x,u]_{\sigma,\tau}+[d(v),\tau(u)]x)=d(d(v)[x,u]_{\sigma,\tau})\in Z$ and so,

$$d^2(v)[x,u]_{\sigma,\tau}+d(v)d([x,u]_{\sigma,\tau})\in Z$$

Since Z is a subring of R and $d(U)\subset Z$, we have

$$d^2(v)[x,u]_{\sigma,\tau}\in Z \qquad \forall x\in R, u,v\in U. \tag{2.3}$$

Replacing x by $x\sigma(u), u\in U$ in (2.3) and applying the above argument, we obtain

$$d^2(v)[x,u]_{\sigma,\tau}\sigma(u)\in Z \qquad \forall x\in R, u,v\in U.$$

Since $d^2(v)[x,u]_{\sigma,\tau}\in Z$ and R is prime ring, we get

$$d^2(v)[x,u]_{\sigma,\tau}=0 \qquad \textit{or} \qquad u\in Z.$$

If $d^2(v)[x,u]_{\sigma,\tau}=0$ for all $x\in R$. In this equation by taking $xy, y\in R$ for x and using this equation, we have $0=d^2(v)[xy,u]_{\sigma,\tau}=d^2(v)[x,u]_{\sigma,\tau}y+d^2(v)x[y,\sigma(u)]=d^2(v)x[y,\sigma(u)]$. By the primeness of R, it implies that $d^2(U)=0$ or $U\subset Z$. In the former case, we get $\sigma(u)+\tau(u)\in Z$, for all $u\in U$ by Lemma 1. Thus, we conclude that $\sigma(u)+\tau(u)\in Z$, for all $u\in U$. □

Now, suppose that U is a $(\sigma,\tau)-$left Lie ideal of R. Since for all $u,v\in U$ and $x\in R$,

$$\begin{aligned}[x,d(u)+v]_{\sigma,\tau}&=[x,d(u)]_{\sigma,\tau}+[x,v]_{\sigma,\tau}\\&=[x,d(u)]_{\sigma,\tau}+[d(x),u]_{\sigma,\tau}-[d(x),u]_{\sigma,\tau}+[x,v]_{\sigma,\tau}\\&=d([x,u]_{\sigma,\tau})-[d(x),u]_{\sigma,\tau}+[x,v]_{\sigma,\tau}\in d(U)+U.\end{aligned}$$

We conclude that $d(U)+U$ is a $(\sigma,\tau)-$left Lie ideal of R. Furthermore, if $d^2(U)=0$ then $d(d(U)+U)\subset d(U)\subset d(U)+U$ and $d^2(d(U)+U)=0$. Therefore without losing generality, we may assume that if U is a $(\sigma,\tau)-$left Lie ideal of such that $d^2(U)=0$, then $d(U)\subset U$.

Lemma 2. *Let U a $(\sigma,\tau)-$left Lie ideal of R. $d^2(U)=0$ and a be an element of R. If $ad([R,U]_{\sigma,\tau})=0$, then $a=0$ or $\sigma(u)+\tau(u)\in Z$, for all $u\in U$.*

Proof. For $x[\sigma(u),\sigma(u)]+[x,u]_{\sigma,\tau}\sigma(u)=[x\sigma(u),u]_{\sigma,\tau}\in[R,U]_{\sigma,\tau}$ by hypothesis $0=ad([x,u]_{\sigma,\tau}\sigma(u))=ad([x,u]_{\sigma,\tau})\sigma(u)+a[x,u]_{\sigma,\tau}d(\sigma(u))$ and so

$$a[x,u]_{\sigma,\tau}d(\sigma(u))=0,\forall x\in R,u\in U. \tag{2.4}$$

Since $d^2(U)=0$, from the above remark we may assume $d(U)\subset U$. So, replacing $u+d(v),v\in U$ by u in (2.4)

$$0=a[x,u+d(v)]_{\sigma,\tau}d(\sigma(u+d(v)).$$

Expanding the last equation and using $d^2(U)=0,\sigma d=d\sigma$ and (2.4), we get $a[x,d(v)]_{\sigma,\tau}d(\sigma(u))=0$, for all $u,v\in U,x\in R$. That is,

$$\sigma^{-1}(a[x,d(v)]_{\sigma,\tau})d(U)=0\,.$$

By [1, Theorem 2] we have $\sigma(u)+\tau(u)\in Z$, for all $u\in U$ or $a[x,d(v)]_{\sigma,\tau}=0$. Replacing $xy,y\in R$ in the last equation, we obtain $ax[y,\sigma(d(v)]=0$. Since R is a prime ring, we conclude $a=0$ or $d(U)\subset Z$. It gives $\sigma(u)+\tau(u)\in Z$, for all $u\in U$ from Theorem 1. This completes the proof. □

Theorem 2. *Let U a $(\sigma,\tau)-$left Lie ideal of R. If $d^2(U)=0$ then $\sigma(u)+\tau(u)\in Z$, for all $u\in U$.*

Proof. Assume that $U\nsubseteq Z$. There exists a $u_0\in U$ such that

$$\sigma(u_0)+\tau(u_0)\notin Z. \tag{2.5}$$

For $[x,u]_{\sigma,\tau}\sigma(u)\in U$,

$$\begin{aligned}0&=d^2([x,u]_{\sigma,\tau}\sigma(u))\\&=d^2([x,u]_{\sigma,\tau})\sigma(u)+2d([x,u]_{\sigma,\tau})d(\sigma(u))+[x,u]_{\sigma,\tau}d^2(\sigma(u)).\end{aligned}$$

In view of the hypothesis and $charR\neq 2$, we have

$$d([x,u]_{\sigma,\tau})d(\sigma(u))=0,\forall x\in R,u\in U. \tag{2.6}$$

Similarly for $\tau(u)[x,u]_{\sigma,\tau}\in U$, we get

$$d(\tau(u))d([x,u]_{\sigma,\tau})=0,\forall x\in R,u\in U. \tag{2.7}$$

By hypothesis $0=d^2([u,v]_{\sigma,\tau})=[d^2(u),v]_{\sigma,\tau}+2[d(u),d(v)]_{\sigma,\tau}+[u,d^2(v)]_{\sigma,\tau}$. Using $d^2(U)=0$ and $charR\neq 2$, we obtain

$$[d(u),d(v)]_{\sigma,\tau}=0,\forall u,v\in U.$$

That is

$$d(u)\sigma(d(v))=\tau(d(v))d(u),\forall u,v\in U. \tag{2.8}$$

Now, let us linearize (2.7) on $u = u + v$ and use (2.8), then we have

$$d(\tau(u))d([x, v]_{\sigma,\tau} + d(\tau(v))d([x, u]_{\sigma,\tau}) = 0, \forall x \in R, u, v \in U. \tag{2.9}$$

Multiply on the right by $d(\sigma(u))$ and use (2.8), (2.6), we obtain

$$(d(\tau(u)))^2 d([x, v]_{\sigma,\tau}) = 0, \forall x \in R, u, v \in U.$$

The last equation reduces to $(d(\tau(U)))^2 d([R, U]_{\sigma,\tau}) = 0$. By Lemma 2 and (2.5), we get $(d(U))^2 = 0$. Otherwise, writing $d(v)$ for v in (2.9) and using $d\tau = \tau d$, we see that

$$d(U)\tau^{-1}([d(x), d(v)]_{\sigma,\tau}) = 0, \forall x \in R, v \in U.$$

This means from [1, Theorem 2] $\sigma(u)+\tau(u) \in Z$, for all $u \in U$ or $[d(x), d(v)]_{\sigma,\tau} = 0$. By our assumption, we get $[d(x), d(v)]_{\sigma,\tau} = 0$, for all $x \in R, v \in U$. If we write $xd(u), u \in U$ for x in the last equation, we have $0 = [d(xd(u), d(v)]_{\sigma,\tau} = [d(x)d(u), d(v)]_{\sigma,\tau} = [d(x), \tau(d(v)]d(u)$ and so,

$$[d(R), \tau(d(U))]d(U) = 0.$$

From the above argument, we have $d(U) \subset Z$ by [1, Theorem 2]. That is $\sigma(u)+\tau(u) \in Z$, for all $u \in U$ from Theorem 1. $\square$

Theorem 3. *Let U a $(\sigma, \tau)-$left Lie ideal of R and $charR \neq 2, 3$. If $d(U) \subset U$ and $d^2(U) \subset Z$, then $\sigma(u) + \tau(u) \in Z$, for all $u \in U$.*

Proof. If $U \subset Z$, then the proof of the theorem is obvious. So, we assume that $U \not\subseteq Z$. That is,

$$\sigma(u_0) + \tau(u_0) \notin Z, \; \exists u_0 \in U. \tag{2.10}$$

Suppose that $d(Z) = 0$. Thus, we have

$$d^3(U) = d(d^2(U)) \subset d(Z) = 0.$$

Now, for $\tau(u)[x, u]_{\sigma,\tau} \in U$, where $x \in R$ and $u \in U$,

$$\begin{aligned} 0 &= d^3(\tau(u)[x, u]_{\sigma,\tau}) \\ &= 3(d^2(\tau(u))d([x, u]_{\sigma,\tau}) + d(\tau(u))d^2([x, u]_{\sigma,\tau}). \end{aligned}$$

Since $charR \neq 3$, we get

$$d^2(\tau(u))d([x, u]_{\sigma,\tau}) + d(\tau(u))d^2([x, u]_{\sigma,\tau}) = 0.$$

Taking $d(u)$ by u and using $\tau d = d\tau, d^3(U) = 0$, we obtain

$$d^2(\tau(u))d^2([x, d(u)]_{\sigma,\tau}) = 0.$$

Since $d^2(U) \subset Z$, the last equation gives us

$$d^2(u) = 0 \qquad or \qquad d^2([x, d(u)]_{\sigma,\tau}) = 0.$$

Let us define $K = \{u \in U \mid d^2(u) = 0\}$ and $L = \{u \in U \mid d^2([x, d(u)]_{\sigma,\tau}) = 0, \forall x \in R\}$. Clearly, both K and L are additive subgroups of U. Moreover, U is the set-theoretic union of K and L. But a group cannot be the set-theoretic union of two proper subgroups, hence $K = U$ or $L = U$. If $K = U$ then $\sigma(u) + \tau(u) \in Z$, for all $u \in U$ by Theorem 2 and it contradicts (2.10). So, we get $L = U$. That is,

$$d^2([x, d(u)]_{\sigma,\tau}) = 0, \forall x \in R, u \in U. \tag{2.11}$$

In this equation replace x by $\tau(d(u))x, u \in U, x \in R$, then we get

$$\begin{aligned} 0 &= d^2(\tau(d(u))[x,d(u)]_{\sigma,\tau}) \\ &= \tau(d^3(u))[x,d(u)]_{\sigma,\tau} + 2\tau(d^2(u))d([x,d(u)]_{\sigma,\tau}) + \tau(d(u))d^2([x,d(u)]_{\sigma,\tau}). \end{aligned}$$

Using (2.11) and $d^3(U)=0$, $char R \neq 2$, we obtain $\tau(d^2(u))d([x,d(u)]_{\sigma,\tau})=0$. Since $d^2(U) \subset Z$, we have

$$d^2(u) = 0 \qquad or \qquad d([x,d(u)]_{\sigma,\tau}) = 0.$$

Let $K = \{u \in U \mid d^2(u) = 0\}$ and $L = \{u \in U \mid d([x,d(u)]_{\sigma,\tau}) = 0, \forall x \in R\}$. Each of K and L is an additive subgroup of U such that $U = K \cup L$. The above trick gives us $U = K$ or $U = L$. In the former case, $d^2(U) = 0$, which forces $\sigma(u)+\tau(u) \in Z$, for all $u \in U$ by Theorem 2, which is a contradiction. Thus $U = L$ and hence $d([x,d(u)]_{\sigma,\tau}) = 0$ for all $u \in U$. Replacing $\tau(d(u))x, u \in U, x \in R$ by x we have $\tau(d^2(u))[x,d(u)]_{\sigma,\tau} = 0$. Since $d^2(U) \subset Z$, we obtain

$$d^2(u) = 0 \qquad or \qquad [x,d(u)]_{\sigma,\tau} = 0 \text{ for all } x \in R. \tag{2.12}$$

Again applying the above trick, we obtain $[x,d(u)]_{\sigma,\tau} = 0$. Taking $xy, y \in R$ in place of x and using (2.12), we have

$$0 = [xy,d(u)]_{\sigma,\tau} = x[y,d(u)]_{\sigma,\tau} + [x,\sigma(d(u))]y = [x,\sigma(d(u))]y\,.$$

Since R is a prime ring, we obtain $d(U) \subset Z$. By Theorem 1, it gives $\sigma(u)+\tau(u) \in Z$, for all $u \in U$, which is a contradiction. Thus, in the case of $d(Z) = 0$ the proof is completed.

Now, we would like to settle the problem when $d(Z)$ is different from zero. There is a non-zero $d(\alpha) \in d(Z)$ such that $\alpha \in Z$. In view of the hypothesis for $[\alpha x, u]_{\sigma,\tau} = \alpha[x,u]_{\sigma,\tau} \in U$,

$$d^2(\alpha[x,u]_{\sigma,\tau}) = d^2(\alpha)[x,u]_{\sigma,\tau} + 2d(\alpha)d([x,u]_{\sigma,\tau}) + \alpha d^2([x,u]_{\sigma,\tau}) \in Z\,.$$

Since $d^2(U) \subset Z$, the third term is in the center of R. So, we get

$$d^2(\alpha)[x,u]_{\sigma,\tau} + 2d(\alpha)d([x,u]_{\sigma,\tau}) \in Z, \forall x \in R, u \in U. \tag{2.13}$$

Replace x by $x\alpha$ in (2.13) to get

$$(d^2(\alpha)[x,u]_{\sigma,\tau} + 2d(\alpha)d([x,u]_{\sigma,\tau}))\alpha + 2d(\alpha)[x,u]_{\sigma,\tau}d(\alpha) \in Z.$$

However, in view of (2.13) and $\alpha \in Z$, this equation reduces to $2d(\alpha)[x,u]_{\sigma,\tau}d(\alpha) \in Z$. Since R is a prime ring, $char R \neq 2$ and $0 \neq d(\alpha) \in Z$, we have $[x,u]_{\sigma,\tau} \in Z$ for all $x \in R, u \in U$. By [8, Lemma 1], we obtain $\sigma(u)+\tau(u) \in Z$, for all $u \in U$. This completes the proof. □

REFERENCES

[1] Aydin, N.: *Notes on generalized Lie ideals*, Analele Universitatii din Timisoara Seria, Matematica-Informatica-Vol., **XXVI**(2), (1999), 7-13.

[2] Aydin, N. and Kandamar, H.: $(\sigma,\tau)-$*Lie ideals in prime rings*, Doğa Tr. J. of Math., **18**(2), (1994), 143-148.

[3] Aydin, N. and Soytürk, M.: $(\sigma,\tau)-$*Lie ideals in prime rings with derivations*, Doğa Tr. J. of Math. **19**, (1993), 239-244.

[4] Lee, P. H. and Lee, T. K.: *Lie ideals of prime rings with derivations*, Bull. Inst. Math. Acad. Sinica., **11**, (1983), 75-80.

[5] Bergen, J., Herstein, I. N. and Kerr, J. W.: *Lie ideals and derivation of prime rings*, J. of Algebra, **71**, (1981), 259-267.

[6] Kaya, K.: $(\sigma,\tau)-$*Lie ideals in prime rings*, An. Univ. Timisoara Stiinte Math., **30**(2-3), 251-255.

[7] Kaya, K., Gölbaşi, Ö. and Aydin, N.: *Some results for generalized Lie ideals in prime rings with derivation II.*, Applied Mathematics E-Notes, **1** (2001), 24-30.

[8] Soytürk, M.: $(\sigma,\tau)-$*Lie ideals in prime rings with derivations*, Doğa Tr. J. of Math., **18**, (1994), 280-283.

Doi: 10.18514/MMN.2002.55

Mathematical Notes, Miskolc, Vol. 3., No. 2., (2002), pp. 91–99

MULTI-POINT BOUNDARY VALUE PROBLEMS FOR LOWER SEMICONTINUOUS DIFFERENTIAL INCLUSIONS

Mouffak Benchohra
Laboratoire de Mathématiques, Université de Sidi Bel Abbès
BP 89, 22000 Sidi Bel Abbès, Algérie
benchohra@yahoo.com

Sotiris K. Ntouyas
Department of Mathematics, University of Ioannina
451 10 Ioannina, Greece
sntouyas@cc.uoi.gr

[Received: March 3, 2002]

Abstract. In this paper, Schaefer's fixed point theorem combined with a selection theorem due to Bressan and Colombo is used to investigate the existence of solutions to multi-point boundary value problems for second order differential inclusions with lower semicontinuous and nonconvex valued right-hand side.

Mathematical Subject Classification: 34A60, 34B10, 34B15
Keywords: multi-point boundary value problems, measurable selection, existence, fixed point, decomposable valued map.

1. **Introduction**

In this paper we shall prove existence results for multi-point boundary value problems (BVP for short). In Section 3 we study the three-point boundary value problem of the second order differential inclusion

$$y'' \in F(t, y), \quad t \in J = [0, 1] \tag{1.1}$$

$$y(0) = 0, \ y(\eta) = y(1), \tag{1.2}$$

where $F : J \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is a given multivalued map, $\eta \in (0, 1)$ and $\mathcal{P}(\mathbb{R})$ is the family of all subsets in $\mathbb{R}$.

Section 4 is devoted to studying the following four-point BVP

$$y'' \in F(t, y), \quad t \in J = [0, 1] \tag{1.3}$$

$$y(0) = y'(\eta), \ y(1) = y(\tau), \tag{1.4}$$

where F, η are as in the problem (1.1)-(1.2) and $\tau \in (0, 1)$.

The boundary value problems (1.1)-(1.2) and (1.3)-(1.4) in the case when the multivalued F has convex values were studied very recently by Benchohra and Ntouyas in [1], [3] by using a fixed point theorem for condensing multivalued maps due to Martelli [18]. The above problems in the case when the multivalued F has nonconvex values were studied by the same authors in [2], with the aid of a fixed point theorem for contraction multivalued maps due to Covitz and Nadler [7]. Here, by using Schaefer's fixed point theorem, combined with a selection theorem of Bressan and Colombo for lower semicontinuous and nonconvex valued multivalued operators with decomposable values, existence results are proposed for the problems (1.1)-(1.2) and (1.3)-(1.4). Possible generalizations are indicated in Section 4. For recent results on multi-point BVP for ordinary differential equations we refer to [11]-[14].

2. **Preliminaries**

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

$L^1(J,\mathbb{R})$ denotes the Banach space of measurable functions $y : J \longrightarrow E$ which are Lebesgue integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|\, dt \quad \text{for all} \quad y \in L^1(J,\mathbb{R}).$$

$AC^i([0,T],\mathbb{R})$ is the space of i-times differentiable functions $y : [0,T] \to \mathbb{R}$, whose i^{th} derivative, $y^{(i)}$, is absolutely continuous.

Let A be a subset of $J \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{I} \times D$ where $\mathcal{I}$ is Lebesgue measurable in J and D is Borel measurable in $\mathbb{R}$. A subset B of $L^1(J,\mathbb{R})$ is decomposable if, for all $u, v \in B$ and $\mathcal{I} \subset J$ measurable, the function $u\chi_{\mathcal{I}} + v\chi_{J-\mathcal{I}} \in B$, where χ denotes the characteristic function.

Let E be a separable Banach space, X a nonempty closed subset of E and $G : X \to \mathcal{P}(E)$ a multivalued operator with nonempty closed values. G is lower semicontinuous (l.s.c.) if the set $\{x \in X : G(x) \cap C \neq \emptyset\}$ is open for any open set C in E. G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

Definition 1. *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1(J,\mathbb{R}))$ be a multivalued operator. We say N has the property (BC) if*

1) *N is lower semi-continuous (l.s.c.);*
2) *N has nonempty closed and decomposable values.*

Let $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\mathcal{F} : C(J,\mathbb{R}) \to \mathcal{P}(L^1(J,\mathbb{R}))$$

by letting

$$\mathcal{F}(y) = \{w \in L^1(J, \mathbb{R}) : w(t) \in F(t, y(t)) \text{ for a.e. } t \in J\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated with F. We say F is the lower semi-continuous type (l.s.c. type) if its associated Niemytzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Next we state a selection theorem due to Bressan and Colombo.

Lemma 1. [6] *Let Y be separable metric space and let $N : Y \to \mathcal{P}(L^1(J, \mathbb{R}))$ be a multivalued operator which has the property (BC). Then N has a continuous selection, i.e. there exists a continuous function (single-valued) $g : Y \to L^1(J, \mathbb{R})$ such that $g(y) \in N(y)$ for every $y \in Y$.*

For more details on multivalued maps and the proof of known results cited in this section we refer to the books by Deimling [8], Gorniewicz [10], Hu and Papageorgiou [15] and Tolstonogov [20].

3. **Main Results**

By the help of Schaefer's theorem, combined with the selection theorem of Bressan and Colombo for lower semicontinuous maps with decomposable values, we shall present first an existence result for the problem (1.1)-(1.2). Before this, let us introduce the following hypotheses which are assumed hereafter:

(H1) $F : [0, 1] \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is a nonempty compact valued multivalued map such that:

a) $(t, y) \mapsto F(t, y)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;

b) $y \mapsto F(t, y)$ is lower semi-continuous for a.e. $t \in [0, 1]$;

(H2) F is integrably bounded, that is, there exists a function $H \in L^1(J, \mathbb{R})$ such that

$$\|F(t, y)\| := \sup\{\|v\| : v \in F(t, y)\} \leq H(t) \quad \text{for almost all } \ t \in J \ \text{ and all } \ y \in \mathbb{R}.$$

In the proof of our following theorem we will need for the auxiliary result:

Lemma 2. [9]. *Let $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty, compact values. Assume $(H1)$ and $(H2)$ hold. Then F is of l.s.c. type.*

Definition 2. *A function $y \in AC^1(J, \mathbb{R})$ is called a solution to the BVP (1.1)-(1.2) if y satisfies the differential inclusion (1.1) a.e. on J and the condition (1.2).*

Theorem 1. *Suppose that hypotheses (H1), (H2) hold. Then the boundary value problem (1.1)–(1.2) has at least one solution.*

Proof. (H1) and (H2) imply by Lemma 2 that F is of the lower semi-continuous type. Then from Lemma 1 there exists a continuous function $g : C(J, \mathbb{R}) \to L^1(J, \mathbb{R})$ such that $g(y) \in \mathcal{F}(y)$ for all $y \in C(J, \mathbb{R})$.

We consider the problem

$$y''(t) = g(y)(t), \quad \text{a.e. } \ t \in J = [0, 1], \tag{3.1}$$

$$y(0) = 0, \; y(\eta) = y(1). \tag{3.2}$$

Remark 1. *If $y \in C(J, \mathbb{R})$ is a solution to the problem (3.1)–(3.2), then y is a solution to the problem (1.1)-(1.2).*

Transform problem (3.1)-(3.2) into a fixed point problem. Consider the operator, $N : C(J, \mathbb{R}) \longrightarrow C(J, \mathbb{R})$ defined by:

$$\begin{aligned} N(y)(t) &= \int_0^t (t-s)g(y)(s)ds + \frac{t}{1-\eta}\int_0^\eta (\eta - s)g(y)(s)ds \\ &\quad - \frac{t}{1-\eta}\int_0^1 (1-s)g(y)(s)ds. \end{aligned}$$

We shall show that N is a compact operator.

Step 1: *N is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J, \mathbb{R})$. Then

$$\begin{aligned} |N(y_n)(t) - N(y)(t)| &= \int_0^t (t-s)|g(y_n)(s) - g(y)(s)|ds \\ &\quad + \frac{t}{1-\eta}\int_0^\eta (\eta - s)|g(y_n)(s) - g(y)(s)|ds \\ &\quad - \frac{t}{1-\eta}\int_0^1 (1-s)|g(y_n)(s) - g(y)(s)|ds. \end{aligned}$$

Since g is continuous, then

$$\|N(y_n) - N(y)\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

Step 2: *N is bounded on bounded sets of $C(J, \mathbb{R})$.*

Indeed, it is enough to show that there exists a positive constant c such that for each $h \in N(y), y \in B_r = \{y \in C(J, \mathbb{R}) : \|y\|_\infty \leq r\}$ one has $\|h\|_\infty \leq c$.

By (H2) we have for each $t \in J$ that

$$|h(t)| \leq \int_0^t H(s)ds + \frac{1}{1-\eta}\int_0^\eta (\eta - s)H(s)ds + \frac{1}{1-\eta}\int_0^1 (1-s)H(s)ds.$$

Then

$$\|h\|_\infty \leq \int_0^1 H(s)ds + \frac{1}{1-\eta}\int_0^\eta (\eta - s)H(s)ds + \frac{1}{1-\eta}\int_0^1 (1-s)H(s)ds := c.$$

Step 3: *N sends bounded sets of $C(J, \mathbb{R})$ into equicontinuous sets.*

Let $t_1, t_2 \in J,\ t_1 < t_2$ and B_r be a bounded set of $C(J,\mathbb{R})$. Then we obtain

$$\begin{aligned}|h(t_2)-h(t_1)| &\leq \int_0^{t_2}(t_2-s)|g(y)(s)|ds + \int_{t_1}^{t_2}(t_1-s)|g(y)(s)|ds\\ &+\frac{t_2-t_1}{1-\eta}\int_0^{\eta}(\eta-s)|g(y)(s)|ds + \frac{t_2-t_1}{1-\eta}\int_0^1(1-s)|g(y)(s)|ds\\ &\leq \int_0^{t_2}(t_2-s)H(s)ds + \int_{t_1}^{t_2}(t_1-s)H(s)ds\\ &+\frac{t_2-t_1}{1-\eta}\int_0^{\eta}(\eta-s)H(s)ds + \frac{t_2-t_1}{1-\eta}\int_0^1(1-s)H(s)ds.\end{aligned}$$

As $t_2 \longrightarrow t_1$ the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem we can conclude that N is completely continuous.

In order to apply Schaefer's theorem, it remains to show that

Step 4: *The set*

$$\Omega := \{y \in C(J,\mathbb{R}) : \lambda y = N(y) \ \ \textit{for some} \ \ \lambda > 1\}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y = N(y)$ for some $\lambda > 1$ and

$$\begin{aligned}y(t) &= \lambda^{-1}\int_0^t (t-s)g(y)(s)ds + \lambda^{-1}\frac{t}{1-\eta}\int_0^{\eta}(\eta-s)g(y)(s)ds\\ &-\lambda^{-1}\frac{t}{1-\eta}\int_0^1(1-s)g(y)(s)ds,\ \ t \in J.\end{aligned}$$

This implies by (H2) that for each $t \in J$ we have

$$|y(t)| \leq \int_0^t (t-s)H(s)ds + \frac{1}{1-\eta}\int_0^{\eta}(\eta-s)H(s)ds + \frac{1}{1-\eta}\int_0^1(1-s)H(s)ds.$$

Thus

$$\|y\|_\infty \leq \int_0^1 (1-s)H(s)ds + \frac{1}{1-\eta}\int_0^{\eta}(\eta-s)H(s)ds + \frac{1}{1-\eta}\int_0^1(1-s)H(s)ds := K.$$

This shows that Ω is bounded.

As a consequence of Schaefer's theorem (see [19] p. 29) we deduce that N has a fixed point which is a solution to (3.1)-(3.2) and hence from Remark 1 a solution to the problem (1.1)-(1.2).

The next result concerns the four-point BVP (1.3)–(1.4). Before stating and proving this result, we give the definition of a solution of the four-point BVP (1.3)–(1.4).

Definition 3. *A function $y \in AC^1(J,\mathbb{R})$ is called a solution to the BVP (1.3)-(1.4) if y satisfies the differential inclusion (1.3) a.e. on J and the conditions (1.4).*

Theorem 2. *Assume that (H1), (H2) are satisfied. Then the BVP (1.3)-(1.4) has at least one solution on J.*

Proof. (H1) and (H2) imply by Lemma 2 that F is of the lower semi-continuous type. Then from Lemma 1 there exists a continuous function $g : C(J,\mathbb{R}) \to L^1(J,\mathbb{R})$ such that $g(y) \in \mathcal{F}(y)$ for all $y \in C(J,\mathbb{R})$.

We consider the problem

$$y''(t) = g(y)(t), \quad \text{a.e.} \quad t \in J = [0,1], \tag{3.3}$$

$$y(0) = y'(\eta), \ \ y(1) = y(\tau). \tag{3.4}$$

Transform problem (3.3) - (3.4) into a fixed point problem. Consider the multivalued map, $N_1 : C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$ defined by:

$$\begin{aligned} N_1(y)(t) \quad &= \int_0^t (t-s)g(y)(s)ds + \int_0^\eta g(y)(s)ds \\ &+\frac{1+t}{1-\tau}\left[\int_0^\tau (\tau-s)g(y)(s)ds - \int_0^1 (1-s)g(y)(s)ds\right]. \end{aligned}$$

We can easily show that N_1 has a fixed point, following the steps of Theorem 1. We omit the details.

4. **Concluding Remarks**

Let $a_i \in \mathbb{R}$, with all of the $a_i^{\prime}s$ having the same sign, $\xi_i \in (0,1), i = 1,2,\ldots,m-2$, $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$. Consider the following m-point boundary value problem for second order differential inclusions

$$y''(t) \in F(t,y(t)), \quad t \in J, \tag{4.1}$$

$$y(0) = 0, \ \ y(1) = \sum_{i=1}^{m-2} a_i y(\xi_i). \tag{4.2}$$

It is well known (see [16] for example), that if a function $y \in C^1(J,\mathbb{R})$ satisfies the boundary condition (4.2) and all of the $a_i, i = 1,2,\ldots,m-2$ have the same sign, then there exists $\eta \in [\xi_1, \xi_{m-2}]$, depending on $y \in C^1(J,\mathbb{R})$ such that

$$y(1) = \alpha y(\eta),$$

with $\alpha = \sum\limits_{i=1}^{m-2} a_i$. Accordingly, the problem of the existence of a solution to the BVP (4.1)–(4.2) can be studied via the three-point BVP

$$y''(t) \in F(t,y(t)), \quad t \in J,$$

$$y(0) = 0, \ \ y(1) = \alpha y(\eta),$$

where $\eta \in (0,1)$ is given. We omit the details, since the proof follows the steps of the proof of Theorem 1, with obvious modifications.

It is obvious that the above method can also be applied to other types of m-point BVPs. For example for the BVPs

$$\begin{cases} y''(t) \in F(t, y(t)), \quad t \in J, \\ y'(0) = 0, \quad y(1) = \sum\limits_{i=1}^{m-2} a_i y(\xi_i), \end{cases}$$

or

$$\begin{cases} y''(t) \in F(t, y(t)), \quad t \in J, \\ y(0) = 0, \quad y'(1) = \sum\limits_{i=1}^{m-2} a_i y'(\xi_i), \end{cases}$$

which can be reduced to the following three point BVPs:

$$\begin{cases} y''(t) \in F(t, y(t)), \quad t \in J, \\ y'(0) = 0, \ \ y(1) = \alpha y(\eta), \end{cases}$$

and

$$\begin{cases} y''(t) \in F(t, y(t)), \quad t \in J, \\ y(0) = 0, \ \ y'(1) = \alpha y'(\eta), \end{cases}$$

respectively.

Let $b_j \in \mathbb{R}$ having the same sign, $\zeta_i \in (0, 1)$, $i = 1, 2, \ldots, n-2$, $0 < \zeta_1 < \zeta_2 < \ldots < \zeta_{n-2} < 1$. Then the following general m-point BVP

$$\begin{cases} y''(t) \in F(t, y(t)), \quad t \in J, \\ y(0) = \sum\limits_{i=1}^{m-2} a_i y'(\xi_i), \quad y(1) = \sum\limits_{j=1}^{n-2} b_j y(\zeta_j), \end{cases}$$

$$\begin{cases} y''(t) \in F(t, y(t)), \quad t \in J, \\ y(0) = \sum\limits_{i=1}^{m-2} a_i y'(\xi_i), \quad y'(1) = \sum\limits_{j=1}^{n-2} b_j y'(\zeta_j), \end{cases}$$

$$\begin{cases} y''(t) \in F(t, y(t)), \quad t \in J, \\ y(0) = \sum\limits_{i=1}^{m-2} a_i y(\xi_i), \quad y(1) = \sum\limits_{j=1}^{n-2} b_j y(\zeta_j) \end{cases}$$

and

$$\begin{cases} y''(t) \in F(t, y(t)), \quad t \in J, \\ y(0) = \sum\limits_{i=1}^{m-2} a_i y(\xi_i), \quad y'(1) = \sum\limits_{j=1}^{n-2} b_i y'(\zeta_j) \end{cases}$$

which can be reduced to the following four point BVP

$$\begin{cases} y''(t) \in F(t, y(t)), \quad t \in J, \\ y(0) = \alpha y'(\eta), \ \ y(1) = \beta y(\tau), \end{cases}$$

$$\begin{cases} y''(t) \in F(t, y(t)), \quad t \in J, \\ y(0) = \alpha y'(\eta), \ \ y'(1) = \beta y'(\tau), \end{cases}$$

$$\begin{cases} y''(t) \in F(t, y(t)), \quad t \in J, \\ y(0) = \alpha y(\eta), \ \ y(1) = \beta y(\tau), \end{cases}$$

and

$$\begin{cases} y''(t) \in F(t, y(t)), \quad t \in J, \\ y(0) = \alpha y(\eta), \quad y'(1) = \beta y'(\tau), \end{cases}$$

respectively with $\alpha = \sum_{i=1}^{m-2} a_i$, $\beta = \sum_{j=1}^{n-2} b_j$.

REFERENCES

[1] Benchohra, M. and Ntouyas, S. K.: *A note on a three point boundary value problem for second order differential inclusions*, Mathematical Notes, Miskolc, **2**(1), (2001), 39-47.

[2] Benchohra, M. and Ntouyas, S. K.: *On a three and four point boundary value problem for second order differential inclusions*, Mathematical Notes, Miskolc, **2**(2), (2001), 93-101.

[3] Benchohra, M. and Ntouyas, S. K.: *Multi point boundary value problem for second order differential inclusions*, Mathematicki Vesnik, **53**, (2001), 51-58.

[4] Bitsadze, A. V.: *On a class of conditionally solvable nonlocal boundary value problems for harmonic functions*, Soviet. Math. Dokl., **31**(1), (1985), 91-94.

[5] Bitsadze, A. V. and Samarskii, A. A.: *On some simple generalizations of linear elliptic boundary value problem*, Soviet. Math. Dokl., **10**(2), (1969), 398-400.

[6] Bressan, A. and Colombo, G.: *Extensions and selections of maps with decomposable values*, Studia Math., **90**, (1988), 69-86.

[7] Covitz, H. and Nadler S. B. Jr.: *Multivalued contraction mappings in generalized metric spaces*, Israel J. Math., **8**, (1970), 5-11.

[8] Deimling, K.: *Multivalued Differential Equations*, Walter de Gruyter, Berlin-New York, 1992.

[9] Frigon, M. and Granas, A.: *Théorèmes d'existence pour des inclusions différentielles sans convexité*, C. R. Acad. Sci. Paris, Ser. I., **310**, (1990), 819-822.

[10] Gorniewicz, L.: *Topological Fixed Point Theory of Multivalued Mappings, Mathematics and its Applications*, 495, Kluwer Academic Publishers, Dordrecht, 1999.

[11] Gupta, C. P., Ntouyas, S. K. and Tsamatos P. Ch.: *On an m-point boundary value problem for second order ordinary differential equations*, Nonlinear Anal., **23**, (1994), 1427-1436.

[12] Gupta, C. P., Ntouyas, S. K. and Tsamatos P. Ch.: *Existence results for m-point boundary value problems*, Differential Equations Dynam. Systems, **2**, (1994), 289-298.

[13] Gupta, C. P., Ntouyas, S. K. and Tsamatos P. Ch.: *On the solvability of some m-point boundary value problems*, Appl. Math., **41**, (1996), 1-17.

[14] Gupta, C. P., Ntouyas, S. K. and Tsamatos P. Ch.: *Existence results for multi-point boundary value problem for second order ordinary differential equations*, Bull. Greek Math. Soc., **43**, (2000), 105-123.

[15] Hu, Sh. and Papageorgiou N.: *Handbook of Multivalued Analysis, Volume I: Theory*, Kluwer Academic Publishers, Dordrecht–Boston–London, 1997.

[16] Il'in, V.A. and Moiseev E.I.: *Nonlocal boundary value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects*, Differential Equations **23**(7), (1987), 803-810.

[17] Il'in, V.A. and Moiseev E.I.: *Nonlocal boundary value problem of the second kind for a Sturm-Liouville operator*, Differential Equations, **23**(8), (1987), 979-987.

[18] Martelli, M.: *A Rothe's type theorem for non-compact acyclic-valued map*, Boll. Un. Mat. Ital., **11**(3), (1975), 70-76.

[19] Smart D. R.: *Fixed Point Theorems*, Cambridge Univ. Press, Cambridge, 1974.

[20] Tolstonogov, A. A.: *Differential Inclusions in a Banach Space*, Kluwer Academic Publishers, Dordrecht, 2000.

Doi: 10.18514/MMN.2002.56

Mathematical Notes, Miskolc, Vol. 3., No. 2., (2002), pp. 101–112

UNIFORM APPROXIMATION BY MEANS OF SOME PIECEWISE LINEAR FUNCTIONS

Zoltán Finta
Babeş - Bolyai University, Department of Mathematics
1, M. Kogălniceanu St., 3400 Cluj-Napoca, Romania
fzoltan@math.ubbcluj.ro

[Received: September 21, 2001]

Abstract. In the present paper we construct new piecewise linear functions using a special partition of the interval $[-1, 1]$. This construction leads to the definition of some new linear operators and we shall obtain global estimates for the remainder in approximating continuous functions by these operators using the second order modulus of smoothness of Ditzian - Totik.

Mathematical Subject Classification: 41A10, 41A35
Keywords: piecewise linear function, Ditzian - Totik modulus of smoothness

1. **Introduction**

Let $(\delta_k)_{k=-n}^{n}$ be a sequence such that $\delta_k = \delta_{-k}$ $(k = 1, 2, \ldots, n)$ and $1/n^2 = \delta_n < \delta_{n-1} < \ldots < \delta_1 < \delta_0 < c_0/n$. Furthermore, let $(x_k)_{k=-n}^{n} : -1 = x_{-n} < x_{-n+1} < \ldots < x_{n-1} < x_n = 1$ be a partition of the interval $[-1, 1]$ with the properties:

: (i) $c_1\ \delta_k \ \le\ x_{k+1} - x_k \ \le\ c_2\ \delta_k$ $(k = -n, \ldots, n-1)$
: (ii) for any $u \in [x_k, x_{k+1}]$ $(k = -n+1, \ldots, n-2)$ we have $x_{k+1} - x_k \le c \cdot \frac{\sqrt{1-u^2}}{n}$.

Here and throughout c_0, c_1, c_2 and c denote absolute constants and the value of c may vary with each occurrence, even on the same line.

The existence of $(\delta_k)_{k=-n}^{n}$ and $(x_k)_{k=-n}^{n}$ is guaranteed by the constructive proof of DeVore and Yu given in [1, p. 326 and p. 329]. This proof establishes a pointwise estimate of the Timan - Teljakovski type for monotone polynomial approximation [1, p. 324, Theorem 1]. The idea of DeVore and Yu was successfully applied by Leviatan [3, p. 3, Theorem 1'] to give a global estimate on monotone approximation. Both proofs are based on a two - stage approximation. At first the function $f \in C[-1, 1]$ is approximated by a piecewise linear function $S_n f$ which interpolates f at the points x_k $(k = -n, -n+1, \ldots, n)$. By Newton's formula we have

$$(S_n f)(x) - f(x) \ = \ (x - x_k)\ (x_{k+1} - x)\ [\ x_k,\ x,\ x_{k+1}\ ;\ f\ ],$$

$x_k \leq x \leq x_{k+1}$ $(k = -n, \dots, n-1)$, where the square brackes denote the divided difference of f at x_k, x, x_{k+1}. Using the proof of [3, p. 7, Theorem 7] we can deduce the following result:

$$\|f - S_n f\| \ \leq \ c\ \omega_\varphi^2(f, n^{-1}), \tag{1.1}$$

where $\|\cdot\|$ is the sup - norm on $[-1,1]$, $\varphi(x) = \sqrt{1-x^2}$, $x \in [-1,1]$ and $\omega_\varphi^2(f,t) = \sup_{0<h\leq t} \ \|\Delta^2_{h\varphi(x)} f(x)\|$ is the Ditzian - Totik modulus of smoothness [2], where

$$\Delta^2_{h\varphi(x)} f(x) = f(x - h\varphi(x)) - 2f(x) + f(x + h\varphi(x)),$$

if $x \pm h\varphi(x) \in [-1,1]$ and $\Delta^2_{h\varphi(x)} f(x) = 0$, otherwise.

In this note we are interested in uniform approximation by piecewise linear functions different from $S_n f$.

2. Approximation by piecewise linear functions

Using a representation theorem given by Popoviciu for the operator S_n [5], we consider the following piecewise linear function:

$$\begin{aligned}(U_n f)(x) =& \\ =& \ \frac{x_{-n+1} - x + |x_{-n+1} - x|}{2(x_{-n+1} - x_{-n})} \cdot f(x_{-n}) \ + \\ +& \sum_{k=-n+1}^{n-1} \frac{x_{k+1} - x_{k-1}}{2} \cdot [\ x_{k-1},\ x_k,\ x_{k+1}\ ;\ |t-x|\ ]_t \ \cdot \\ & \qquad \cdot \left\{ f(x_k + n^{-\gamma}\varphi(x_k)) - f(x_k) + f(x_k - n^{-\gamma}\varphi(x_k)) \right\} + \\ +& \ \frac{x - x_{n-1} + |x - x_{n-1}|}{2(x_n - x_{n-1})} \cdot f(x_n),\end{aligned}$$

where $f \in C[-1,1]$ and $\gamma \geq 1$ such that $c_1\delta_{-n} = c_1 n^{-2} \geq n^{-\gamma}$. Here we denote by $[\ x_{k-1},\ x_k,\ x_{k+1}\ ;\ |t-x|]_t$ the fact that the divided difference is applied to the variable t.

The operator $U_n : C[-1,1] \to C[-1,1]$ is linear which preserves the linear functions. The function $U_n f$ interpolates f at the endpoints of the interval $[-1,1]$. Moreover, if $f \in C[-1,1]$ is a positive, convex function, then $U_n f$ is also positive. Indeed, for $k = -n+1, \dots, n-1$ we have $f(x_k + n^{-\gamma}\varphi(x_k)) - f(x_k) + f(x_k - n^{-\gamma}\varphi(x_k)) \geq f(x_k) \geq 0$. Hence, by definition of $U_n f$ we obtain $(U_n f)(x) \geq (S_n f)(x) \geq 0$, $x \in [-1,1]$ because $S_n : C[-1,1] \to C[-1,1]$ is a positive linear operator.

Our first result is the following:

Theorem 1. *If $f \in C[-1,1]$, then $\|f - U_n f\| \ \leq \ c\ \omega_\varphi^2(f, n^{-1})$.*

Proof. By (i) we get $n^{-\gamma}\varphi(x_{-n+1}) \leq n^{-\gamma} \leq c_1\delta_{-n} \leq x_{-n+1} - x_{-n}$. Hence $x_{-n} \leq x_{-n+1} - n^{-\gamma}\varphi(x_{-n+1}) \leq x_{-n+1}$. Again, by (i) and by the properties of $(\delta_k)_{k=-n}^n$ we have $n^{-\gamma}\varphi(x_k) \leq n^{-\gamma} \leq c_1\delta_{-n} \leq c_1\delta_k \leq x_{k+1} - x_k$ $(k = -n+1, \dots, n-1)$. Then

$x_k \le x_k + n^{-\gamma}\varphi(x_k) \le x_{k+1}$ $(k = -n+1, \ldots, n-1)$. With the same hypotheses we obtain $n^{-\gamma}\varphi(x_{k-1}) \le n^{-\gamma} \le c_1\delta_{-n} \le c_1\delta_{k-1} \le x_k - x_{k-1}$ $(k = -n+2, \ldots, n)$. Hence $x_{k-1} \le x_k - n^{-\gamma}\varphi(x_{k-1}) \le x_k$ $(k = -n+2, \ldots, n)$. This means that $U_n f$ is well - defined.

Now, in view of Popoviciu's representation theorem [5] we have

$$(S_n f)(x) = \frac{x_{-n+1} - x + |x_{-n+1} - x|}{2(x_{-n+1} - x_{-n})} \cdot f(x_{-n}) \; +$$

$$+ \quad \sum_{k=-n+1}^{n-1} \frac{x_{k+1} - x_{k-1}}{2} \cdot [\; x_{k-1},\; x_k,\; x_{k+1} \;;\; |t - x| \;]_t \cdot f(x_k) \; +$$

$$+ \quad \frac{x - x_{n-1} + |x - x_{n-1}|}{2(x_n - x_{n-1})} \cdot f(x_n).$$

Hence

$$\begin{aligned}
(U_n f)(x) - (S_n f)(x) = & \\
= \quad & \frac{x - x_{-n}}{x_{-n+1} - x_{-n}} \cdot \big\{ f(x_{-n+1} + n^{-\gamma}\varphi(x_{-n+1})) - \\
- \quad & 2f(x_{-n+1}) + f(x_{-n+1} - n^{-\gamma}\varphi(x_{-n+1})) \big\}
\end{aligned} \tag{2.1}$$

for $x_{-n} \le x \le x_{-n+1}$;

$$\begin{aligned}
(U_n f)(x) - (S_n f)(x) = & \\
= \quad & \frac{x_{k+1} - x}{x_{k+1} - x_k} \cdot \big\{ f(x_k + n^{-\gamma}\varphi(x_k)) - \\
- \quad & 2\; f(x_k) + f(x_k - n^{-\gamma}\varphi(x_k)) \big\} \; + \\
+ \quad & \frac{x - x_k}{x_{k+1} - x_k} \cdot \big\{ f(x_{k+1} + n^{-\gamma}\varphi(x_{k+1})) - \\
- \quad & 2\; f(x_{k+1}) + f(x_{k+1} - n^{-\gamma}\varphi(x_{k+1})) \big\}
\end{aligned} \tag{2.2}$$

for $x_k \le x \le x_{k+1}$ $(k = -n+1, \ldots, n-1)$;

$$\begin{aligned}
(U_n f)(x) - (S_n f)(x) = & \\
= \quad & \frac{x_n - x}{x_n - x_{n-1}} \cdot \big\{ f(x_{n-1} + n^{-\gamma}\varphi(x_{n-1})) - \\
- \quad & 2\; f(x_{n-1}) + f(x_{n-1} - n^{-\gamma}\varphi(x_{n-1})) \big\}
\end{aligned} \tag{2.3}$$

for $x_{n-1} \le x \le x_n$.

On the other hand, by definition of the Ditzian-Totik modulus of smoothness we obtain from (2), (3), (4) and $\gamma \ge 1$ the estimates

$$\begin{aligned}
|(U_n f)(x) - (S_n f)(x)| = & \\
= \quad & \frac{x - x_{-n}}{x_{-n+1} - x_{-n}} \cdot |\Delta^2_{n^{-\gamma}\varphi(x_{-n+1})} f(x_{-n+1})| \\
\le \quad & \|\Delta^2_{n^{-\gamma}\varphi(x)} f(x)\| \le \omega^2_{\varphi}(f, n^{-1}),
\end{aligned}$$

where $x_{-n} \le x \le x_{-n+1}$;

$$
\begin{aligned}
&|(U_nf)(x) - (S_nf)(x)| \le \\
&\qquad \le \frac{x_{k+1}-x}{x_{k+1}-x_k} \cdot |\Delta^2_{n^{-\gamma}\varphi(x_k)} f(x_k)| + \\
&\qquad + \frac{x-x_k}{x_{k+1}-x_k} \cdot |\Delta^2_{n^{-\gamma}\varphi(x_{k+1})} f(x_{k+1})| \\
&\qquad \le \|\Delta^2_{n^{-\gamma}\varphi(x)} f(x)\| \le \omega^2_\varphi(f, n^{-1}),
\end{aligned}
$$

where $x_k \le x \le x_{k+1}$;

$$
\begin{aligned}
|(U_nf)(x) - (S_nf)(x)| &= \frac{x_n - x}{x_n - x_{n-1}} \cdot |\Delta^2_{n^{-\gamma}\varphi(x_{n-1})} f(x_{n-1})| \\
&\le \|\Delta^2_{n^{-\gamma}\varphi(x)} f(x)\| \;\le\; \omega^2_\varphi(f, n^{-1}),
\end{aligned}
$$

where $x_{n-1} \le x \le x_n$. Hence $\|U_nf - S_nf\| \le \omega^2_\varphi(f, n^{-1})$. Using (1) we obtain the assertion of the theorem. □

Our next piecewise linear function is the following:

$$
\begin{aligned}
(V_nf)(x) &= \frac{x_{k+1}-x}{x_{k+1}-x_k} \cdot \frac{1}{c_1\delta_k} \int_{x_k}^{x_k+c_1\delta_k} f(u)\,du \;+ \\
&+ \frac{x-x_k}{x_{k+1}-x_k} \cdot \frac{1}{c_1\delta_k} \int_{x_{k+1}-c_1\delta_k}^{x_{k+1}} f(u)\,du,
\end{aligned}
$$

if $x_k < x < x_{k+1}$ $(k = -n, \ldots, n-1)$ and $(V_nf)(x) = f(x_k)$, if $x = x_k$ $(k = -n, \ldots, n)$.

Then $V_n : C[-1,1] \to L_\infty[-1,1]$ is a linear, positive operator such that V_nf interpolates the function f at the points x_k $(k = -n, \ldots, n)$. The main difference between V_n and S_n is that V_nf is not necessarily continuous at x_k $(k = -n+1, \ldots, n-1)$. Moreover, $V_ne_0 = e_0$ and $V_ne_1 = e_1 + O(n^{-1})$ (here $e_0(x) = 1$ and $e_1(x) = x$ for $x \in [-1,1]$). Indeed, the first statement is obvious and for the second one we have for $x_k < x < x_{k+1}$ $(k = -n, \ldots, n-1)$:

$$
\begin{aligned}
(V_ne_1)(x) &= \frac{x_{k+1}-x}{x_{k+1}-x_k} \cdot \frac{1}{c_1\delta_k} \cdot \frac{1}{2}\left(2x_kc_1\delta_k + c_1^2\delta_k^2\right) \;+ \\
&\quad + \frac{x-x_k}{x_{k+1}-x_k} \cdot \frac{1}{c_1\delta_k} \cdot \frac{1}{2}\left(2x_{k+1}c_1\delta_k - c_1^2\delta_k^2\right) \\
&= x + \frac{1}{2}\,c_1\delta_k \left(\frac{x_{k+1}-x}{x_{k+1}-x_k} - \frac{x-x_k}{x_{k+1}-x_k}\right).
\end{aligned}
$$

Hence, by properties of $(\delta_k)_{k=-n}^n$ we obtain

$$
|(V_ne_1)(x) - e_1(x)| \;\le\; \frac{1}{2}\,c_1\delta_k \;\le\; \frac{c_0c_1}{2n}, \qquad n \ge 1.
$$

Furthermore, we denote the set of all algebraic polynomials of degree at most n by Π_n and the best uniform approximation on $[-1,1]$ by $E_n(f) = \inf\{\|f-p\| \ : \ p \in \Pi_n\}$. Our next results are the following:

Theorem 2. *Let $f \in C[-1,1]$ and V_n $(n \in N, \ n \geq 2)$ be defined as above. Then*
$\|f - V_n f\| \ \leq$

$$\leq \ c \ \left\{ \omega_\varphi^2(f, n^{-1}) \ + \ n^{-1} \int_{\frac{1}{n}}^{\frac{1}{2}} \frac{\omega_\varphi^2(f,t)}{t^3} \, dt \ + \ n^{-1} \ E_0(f) \right\}.$$

Corollary 1. *Let $f \in C[-1,1]$, $n \in N$, $n \geq 2$. Then*

$$\|f - V_n f\| \ \leq \ c \ n^{-1} \ \left\{ \int_{\frac{1}{n}}^{\frac{1}{2}} \frac{\omega_\varphi^2(f,t)}{t^3} \, dt \ + \ \|f\| \right\}.$$

To prove our statements we need a lemma:

Lemma 1. *For $g \in C^2[-1,1]$, $n \geq 2$ we have*

$$\|g - V_n g\| \ \leq \ c \ \left\{ n^{-1} \ \|g'\| \ + \ n^{-2} \ \|\varphi^2 g''\| \right\}.$$

Proof. If $x_k < x < x_{k+1}$ $(k = -n+1, \ldots, n-2)$, then
$(V_n g)(x) - g(x) =$

$$\begin{aligned}
&= \frac{x_{k+1} - x}{x_{k+1} - x_k} \cdot \\
&\cdot \left\{ \frac{1}{c_1 \delta_k} \int_{x_k}^{x_k + c_1 \delta_k} [g(u) - g(x_k)] \ du + g(x_k) - g(x) \right\} \\
&+ \frac{x - x_k}{x_{k+1} - x_k} \cdot \\
&\cdot \left\{ \frac{1}{c_1 \delta_k} \int_{x_{k+1} - c_1 \delta_k}^{x_{k+1}} [g(u) - g(x_{k+1})] \ du + g(x_{k+1}) - g(x) \right\}.
\end{aligned}$$

Simple computations show that

$$\int_\alpha^\beta [g(u) - g(\alpha)] \ du \ = \ \int_\alpha^\beta (\beta - u) \ g'(u) \ du$$

and

$$\int_\alpha^\beta [g(\beta) - g(u)] \ du \ = \ \int_\alpha^\beta (u - \alpha) \ g'(u) \ du.$$

Hence

$$\begin{aligned}
(V_n g)(x) - g(x) &= \\
&= \frac{x_{k+1}-x}{x_{k+1}-x_k} \cdot \\
&\cdot \left\{ \frac{1}{c_1\delta_k} \int_{x_k}^{x_k+c_1\delta_k} (x_k + c_1\delta_k - u)\ g'(u)\ du - \int_{x_k}^{x} g'(u)\ du \right\} \\
&+ \frac{x-x_k}{x_{k+1}-x_k} \cdot \left\{ \frac{1}{c_1\delta_k} \cdot \right. \\
&\cdot \left. \int_{x_{k+1}-c_1\delta_k}^{x_{k+1}} (x_{k+1} - c_1\delta_k - u)\ g'(u)\ du + \int_{x}^{x_{k+1}} g'(u)\ du \right\} \\
&= \frac{x_{k+1}-x}{x_{k+1}-x_k} \cdot \frac{1}{c_1\delta_k} \cdot \\
&\cdot \left\{ \int_{x_k}^{x_k+c_1\delta_k} (x_k + c_1\delta_k - u)\ g'(u)\ du - \int_{x_k}^{x} c_1\delta_k g'(u)\ du \right\} \\
&+ \frac{x-x_k}{x_{k+1}-x_k} \cdot \frac{1}{c_1\delta_k} \cdot \left\{ \int_{x_{k+1}-c_1\delta_k}^{x_{k+1}} (x_{k+1} - c_1\delta_k - u)\ g'(u)\ du + \right. \\
&+ \left. \int_{x}^{x_{k+1}} c_1\delta_k g'(u)\ du \right\} \\
&= \frac{x_{k+1}-x}{x_{k+1}-x_k} \cdot \frac{1}{c_1\delta_k} \\
&\cdot \left\{ \int_{x_k}^{x} (x_k - u) g'(u)\ du + \int_{x}^{x_k+c_1\delta_k} (x_k + c_1\delta_k - u) g'(u)\ du \right\} \\
&+ \frac{x-x_k}{x_{k+1}-x_k} \cdot \frac{1}{c_1\delta_k} \cdot \left\{ \int_{x_{k+1}-c_1\delta_k}^{x} (x_{k+1} - c_1\delta_k - u) g'(u)\ du + \right. \\
&+ \left. \int_{x}^{x_{k+1}} (x_{k+1} - u) g'(u)\ du \right\}.
\end{aligned}$$

By partial integration we obtain

$$\begin{aligned}
(V_n g)(x) - g(x) &= \\
&= \frac{x_{k+1}-x}{x_{k+1}-x_k} \cdot \frac{1}{c_1\delta_k} \left\{ -\frac{1}{2}\ (x_k - x)^2\ g'(x) + \right. \\
&+ \frac{1}{2} \int_{x_k}^{x} (x_k - u)^2 g''(u)\ du + \frac{1}{2}\ (x_k + c_1\delta_k - x)^2\ g'(x)\ + \\
&+ \left. \frac{1}{2} \int_{x}^{x_k+c_1\delta_k} (x_k + c_1\delta_k - u)^2 g''(u)\ du \right\} +
\end{aligned}$$

$$+ \quad \frac{1}{2} \int_x^{x_k+c_1\delta_k} (x_k + c_1\delta_k - u)^2 g''(u)\, du \Bigg\} \; +$$

$$+ \quad \frac{x - x_k}{x_{k+1} - x_k} \cdot \frac{1}{c_1\delta_k} \; \Bigg\{ -\frac{1}{2} \, (x_{k+1} - c_1\delta_k - x)^2 g'(x) \; +$$

$$+ \quad \frac{1}{2} \int_{x_{k+1}-c_1\delta_k}^{x} (x_{k+1} - c_1\delta_k - u)^2 g''(u)\, du \; +$$

$$+ \quad \frac{1}{2} \, (x_{k+1} - x)^2 g'(x) + \frac{1}{2} \int_x^{x_{k+1}} (x_{k+1} - u)^2 g''(u)\, du \Bigg\}. \tag{2.4}$$

On the other hand, we get for $x_k \le u \le x_{k+1}$ by (i) and (ii)

$$\left| -\tfrac{1}{2} \, (x_k - x)^2 + \tfrac{1}{2} \, (x_k + c_1\delta_k - x)^2 \right| =$$

$$\begin{aligned}
&= \quad |c_1\delta_k(x_k - x) + \frac{1}{2} \, (c_1\delta_k)^2| = c_1\delta_k |(x_k - x) + \frac{1}{2} \, c_1\delta_k| \\
&\le \quad c_1\delta_k(x - x_k + \frac{1}{2} \, c_1\delta_k) \;\le\; \frac{3}{2} \, c_1\delta_k \; (x_{k+1} - x_k) \\
&\le \quad \frac{3}{2} \, c_1\delta_k \cdot c \, \frac{\sqrt{1-u^2}}{n} \;\le\; c \;\cdot \frac{c_1\delta_k}{n};
\end{aligned} \tag{2.5}$$

we have for $x_k \le u \le x < x_{k+1}$, by (ii)

$$(x_k - u)^2 \le (x_{k+1} - x_k)^2 \le c \, \frac{1-u^2}{n^2} \tag{2.6}$$

and for $x_k < x \le u \le x_k + c_1\delta_k \le x_{k+1}$ or $x_k \le x_k + c_1\delta_k \le u \le x < x_{k+1}$ we have in view of (i) and (ii)

$$(x_k + c_1\delta_k - u)^2 \le 2 \, (x_k - u)^2 + 2 \, (c_1\delta_k)^2 \le 4 \, (x_{k+1} - x_k)^2 \le c \frac{1-u^2}{n^2} \, . \tag{2.7}$$

Using the same arguments we obtain

$$\left| -\frac{1}{2} \, (x_{k+1} - c_1\delta_k - x)^2 + \frac{1}{2} \, (x_{k+1} - x)^2 \right| \;\le\; c \frac{c_1\delta_k}{n} \tag{2.8}$$

for $x_k \le u \le x_{k+1}$;

$$(x_{k+1} - c_1\delta_k - u)^2 \le c \, \frac{1-u^2}{n^2} \tag{2.9}$$

for $x_k \le x_{k+1} - c_1\delta_k \le u \le x < x_{k+1}$ or $x_k < x \le u \le x_{k+1} - c_1\delta_k \le x_{k+1}$ and

$$(x_{k+1} - u)^2 \le c \frac{1-u^2}{n^2} \tag{2.10}$$

for $x_k < x \le u \le x_{k+1}$. Then (5) – (11) and (ii) imply

$$
\begin{aligned}
&|(V_n g)(x) - g(x)| \le \\
&\le \quad \frac{x_{k+1}-x}{x_{k+1}-x_k}\cdot\frac{1}{c_1\delta_k}\ \Big\{|-\frac{1}{2}\ (x_k-x)^2+\frac{1}{2}\ (x_k+c_1\delta_k-x)^2|\cdot \\
&\cdot \quad |g'(x)|+\frac{1}{2}\ \int_{x_k}^{x}(u-x_k)^2|g''(u)|\ du\ + \\
&+ \quad \frac{1}{2}\ \left|\int_x^{x_k+c_1\delta_k}(x_k+c_1\delta_k-u)^2|g''(u)|\ du\right|\Big\}\ + \\
&+ \quad \frac{x-x_k}{x_{k+1}-x_k}\cdot\frac{1}{c_1\delta_k}\ \Big\{|-\frac{1}{2}\ (x_{k+1}-c_1\delta_k-x)^2+\frac{1}{2}\ (x_{k+1}-x)^2|\cdot \\
&\cdot \quad |g'(x)|+\frac{1}{2}\ \left|\int_{x_{k+1}-c_1\delta_k}^{x}(x_{k+1}-c_1\delta_k-u)^2|g''(u)|\ du\right|\ + \\
&+ \quad \frac{1}{2}\ \int_x^{x_{k+1}}(x_{k+1}-u)^2|g''(u)|\ du\Big\} \\
&\le \quad \frac{x_{k+1}-x}{x_{k+1}-x_k}\cdot\frac{1}{c_1\delta_k}\ \Big\{c\cdot\frac{c_1\delta_k}{n}\cdot\|g'\|\ + \\
&+ \quad \frac{1}{2}\ \int_{x_k}^{x}c\cdot\frac{1-u^2}{n^2}\cdot|g''(u)|\ du\ + \\
&+ \quad \frac{1}{2}\ \left|\int_x^{x_k+c_1\delta_k}c\cdot\frac{1-u^2}{n^2}|g''(u)|\ du\right|\Big\}\ + \\
&+ \quad \frac{x-x_k}{x_{k+1}-x_k}\cdot\frac{1}{c_1\delta_k}\ \Big\{c\cdot\frac{c_1\delta_k}{n}\|g'\|\ + \\
&+ \quad \frac{1}{2}\ \left|\int_{x_{k+1}-c_1\delta_k}^{x}c\cdot\frac{1-u^2}{n^2}|g''(u)|\ du\right|\ + \\
&+ \quad \frac{1}{2}\ \int_x^{x_{k+1}}c\cdot\frac{1-u^2}{n^2}|g''(u)|\ du\Big\}.
\end{aligned}
\tag{2.11}
$$

But

$$
\int_{x_k}^{x}\ +\ \left|\int_x^{x_k+c_1\delta_k}\right| = \int_{x_k}^{x}\ +\ \int_x^{x_k+c_1\delta_k}\ = \int_{x_k}^{x_k+c_1\delta_k}
$$

if $x_k < x \le x_k + c_1\delta_k$ or

$$
\int_{x_k}^{x}\ +\ \left|\int_x^{x_k+c_1\delta_k}\right| = \int_{x_k}^{x}\ +\ \int_{x_k+c_1\delta_k}^{x}\ \le\ \int_{x_k}^{x}+\int_{x_k}^{x}\ =\ 2\ \int_{x_k}^{x}
$$

if $x_k + c_1\delta_k \le x < x_{k+1}$ and

$$
\left|\int_{x_{k+1}-c_1\delta_k}^{x}\right| + \int_x^{x_{k+1}}\ =\ \int_{x_{k+1}-c_1\delta_k}^{x}\ +\ \int_x^{x_{k+1}}\ =\ \int_{x_{k+1}-c_1\delta_k}^{x_{k+1}}
$$

if $x_{k+1} - c_1\delta_k \le x < x_{k+1}$ or

$$\left|\int_{x_{k+1}-c_1\delta_k}^{x}\right| + \int_{x}^{x_{k+1}} = \int_{x}^{x_{k+1}-c_1\delta_k} + \int_{x}^{x_{k+1}} \le$$
$$\le \int_{x}^{x_{k+1}} + \int_{x}^{x_{k+1}} = 2\int_{x}^{x_{k+1}}$$

if $x_k < x \le x_{k+1} - c_1\delta_k$. So we obtain in view of (12) that

$|(V_n g)(x) - g(x)| \le$

$$\begin{aligned}
&\le \frac{x_{k+1}-x}{x_{k+1}-x_k}\cdot\frac{1}{c_1\delta_k}\left\{c\cdot\frac{c_1\delta_k}{n}\,\|g'\| + \right.\\
&\qquad\qquad \left. +c\cdot\frac{\|\varphi^2 g''\|}{n^2}\cdot\max\left(c_1\delta_k; x-x_k\right)\right\}\\
&+ \frac{x-x_k}{x_{k+1}-x_k}\cdot\frac{1}{c_1\delta_k}\left\{c\cdot\frac{c_1\delta_k}{n}\|g'\| + \right.\\
&\qquad\qquad \left. +c\cdot\frac{\|\varphi^2 g''\|}{n^2}\cdot\max(c_1\delta_k; x_{k+1}-x)\right\}\\
&\le c\left\{n^{-1}\,\|g'\| + n^{-2}\,\|\varphi^2 g''\|\cdot\max\left(1;\frac{x_{k+1}-x_k}{c_1\delta_k}\right)\right\}.
\end{aligned}$$

By (i) we have $x_{k+1} - x_k \le c_2\delta_k$. Therefore

$$|(V_n g)(x) - g(x)| \le c\left\{n^{-1}\|g'\| + n^{-2}\|\varphi^2 g''\|\right\}. \tag{2.12}$$

If $x_{-n} < x < x_{-n+1}$ then

$$\begin{aligned}
(V_n g)(x) - g(x) &= \\
&= \frac{x_{-n+1}-x}{x_{-n+1}-x_{-n}}\cdot\left\{\frac{1}{c_1\delta_{-n}}\cdot\right.\\
&\cdot \left.\int_{x_{-n}}^{x_{-n}+c_1\delta_{-n}}[g(u)-g(x_{-n})]\,du + g(x_{-n}) - g(x)\right\}\\
&+ \frac{x-x_{-n}}{x_{-n+1}-x_{-n}}\cdot\left\{\frac{1}{c_1\delta_{-n}}\cdot\right.\\
&\quad \left.\int_{x_{-n+1}-c_1\delta_{-n}}^{x_{-n+1}}[g(u)-g(x_{-n+1})]\,du + g(x_{-n+1}) - g(x)\right\}
\end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{x_{-n+1}-x}{x_{-n+1}-x_{-n}} \cdot \left\{ \frac{1}{c_1\delta_{-n}} \cdot \right. \\
\cdot \quad & \left. \int_{x_{-n}}^{x_{-n}+c_1\delta_{-n}} [g(u)-g(x_{-n})] \ du - \int_{x_{-n}}^{x} g'(u) \ du \right\} \\
+ \quad & \frac{x-x_{-n}}{x_{-n+1}-x_{-n}} \cdot \left\{ \frac{1}{c_1\delta_{-n}} \cdot \right. \\
\cdot \quad & \left. \int_{x_{-n+1}-c_1\delta_{-n}}^{x_{-n+1}} [g(u)-g(x_{-n+1})] \ du + \int_{x}^{x_{-n+1}} g'(u) \ du \right\} \\
= \quad & \frac{x_{-n+1}-x}{x_{-n+1}-x_{-n}} \cdot \left\{ \frac{1}{c_1\delta_{-n}} \cdot \right. \\
\cdot \quad & \left. \int_{x_{-n}}^{x_{-n}+c_1\delta_{-n}} \left[\int_{x_{-n}}^{u} g'(v) \ dv \right] \ du - \int_{x_{-n}}^{x} g'(u) \ du \right\} \\
+ \quad & \frac{x-x_{-n}}{x_{-n+1}-x_{-n}} \cdot \left\{ -\frac{1}{c_1\delta_{-n}} \cdot \right. \\
\cdot \quad & \left. \int_{x_{-n+1}-c_1\delta_{-n}}^{x_{-n+1}} \left[\int_{u}^{x_{-n+1}} g'(v) \ dv \right] \ du + \int_{x}^{x_{-n+1}} g'(u) \ du \right\}.
\end{aligned}
$$

Hence

$|(V_n g)(x) - g(x)| \leq$

$$
\begin{aligned}
\leq \quad & \frac{x_{-n+1}-x}{x_{-n+1}-x_{-n}} \cdot \left\{ \frac{1}{c_1\delta_{-n}} \cdot \right. \\
\cdot \quad & \left. \int_{x_{-n}}^{x_{-n}+c_1\delta_{-n}} \left[\int_{x_{-n}}^{u} |g'(v)| \ dv \right] \ du + \int_{x_{-n}}^{x} |g'(u)| \ du \right\} \\
+ \quad & \frac{x-x_{-n}}{x_{-n+1}-x_{-n}} \cdot \left\{ \frac{1}{c_1\delta_{-n}} \cdot \right. \\
\cdot \quad & \left. \int_{x_{-n+1}-c_1\delta_{-n}}^{x_{-n+1}} \left[\int_{u}^{x_{-n+1}} |g'(v)| \ dv \right] \ du + \int_{x}^{x_{-n+1}} |g'(u)| \ du \right\} \\
\leq \quad & \frac{x_{-n+1}-x}{x_{-n+1}-x_{-n}} \cdot \left\{ \frac{1}{c_1\delta_{-n}} \cdot \right. \\
\cdot \quad & \left. \int_{x_{-n}}^{x_{-n}+c_1\delta_{-n}} (u-x_{-n}) \ du + (x-x_{-n}) \right\} \ \|g'\| \\
+ \quad & \frac{x-x_{-n}}{x_{-n+1}-x_{-n}} \cdot \left\{ \frac{1}{c_1\delta_{-n}} \cdot \right. \\
\cdot \quad & \left. \int_{x_{-n+1}-c_1\delta_{-n}}^{x_{-n+1}} (x_{-n+1}-u) \ du + (x_{-n+1}-x) \right\} \ \|g'\|.
\end{aligned}
$$

In view of (i) we have $x_{-n} \le u \le x_{-n} + c_1\delta_{-n} \le x_{-n+1}$, $x_{-n} \le x_{-n+1} - c_1\delta_{-n} \le u \le x_{-n+1}$ and $x_{-n} \le x \le x_{-n+1}$, respectively. So

$$
\begin{aligned}
|(V_n g)(x) - g(x)| \le & \\
\le \quad & \frac{x_{-n+1} - x}{x_{-n+1} - x_{-n}} \left\{ \frac{1}{c_1\delta_{-n}} \cdot \right. \\
\cdot \quad & \left. \int_{x_{-n}}^{x_{-n}+c_1\delta_{-n}} (x_{-n+1} - x_{-n})\, du + (x_{-n+1} - x_{-n}) \right\} \|g'\| \\
+ \quad & \frac{x - x_{-n}}{x_{-n+1} - x_{-n}} \left\{ \frac{1}{c_1\delta_{-n}} \cdot \right. \\
\cdot \quad & \left. \int_{x_{-n+1}-c_1\delta_{-n}}^{x_{-n+1}} (x_{-n+1} - x_{-n})\, du + (x_{-n+1} - x_{-n}) \right\} \|g'\| \\
\le \quad & 2\, (x_{-n+1} - x_{-n})\, \|g'\|.
\end{aligned}
$$

Again, (i) implies $x_{-n+1} - x_{-n} \le c_2\delta_{-n}$. This means that

$$
|(V_n g)(x) - g(x)| \;\le\; 2\, c_2\, \delta_{-n}\, \|g'\| \;\le\; \frac{c}{n^2}\, \|g'\|. \tag{2.13}
$$

Analogously

$$
|(V_n g)(x) - g(x)| \;\le\; \frac{c}{n^2}\, \|g'\| \tag{2.14}
$$

for $x_{n-1} < x < x_n$. In conclusion (13), (14) and (15) imply

$$
\|g - V_n g\| \;\le\; c\, \left\{ n^{-1}\, \|g'\| \;+\; n^{-2}\, \|\varphi^2 g''\| \right\},
$$

which completes the proof. $\square$

Proof of Theorem 2. We have

$$
\begin{aligned}
|(V_n f)(x)| \quad \le \quad & \frac{x_{k+1} - x}{x_{k+1} - x_k} \cdot \frac{1}{c_1\delta_k} \int_{x_k}^{x_k + c_1\delta_k} |f(u)|\, du \\
+ \quad & \frac{x - x_k}{x_{k+1} - x_k} \cdot \frac{1}{c_1\delta_k} \int_{x_{k+1} - c_1\delta_k}^{x_{k+1}} |f(u)|\, du \;\le\; \|f\|
\end{aligned}
$$

for $x_k < x < x_{k+1}$ $(k = -n, \ldots, n-1)$. Thus

$$
\|V_n f\| \;\le\; \|f\|. \tag{2.15}
$$

On the other hand, let us denote by $p_n \in \Pi_n$ the best nth degree polynomial approximation to f. Then we know for $f \in C[-1,1]$ (see [2, p. 79, Theorem 7.2.1]) that

$$
E_n(f) \;=\; \|f - p_n\| \;\le\; c\, \omega_\varphi^2(f, n^{-1}). \tag{2.16}
$$

Moreover, we have the following Bernstein type inequality [2, p. 84, Theorem 7.3.1] for the best approximation polynomial

$$
\|\varphi^2 p_n''\| \;\le\; n^2\, \omega_\varphi^2(f, n^{-1}). \tag{2.17}
$$

Then, using (16), Lemma 1, (17), (18) and the proof of [4, pp. 83 - 84, Theorem 3.2] we obtain the conclusion of our theorem. □

Proof of Corollary 1. In view of [4, p. 86, Remark 3.4], the proof is a direct consequence of the fact that we can drop the first term on the right hand side of the estimate given in Theorem 2 because of

$$\int_{\frac{1}{n}}^{\frac{1}{2}} \frac{\omega_\varphi^2(f,t)}{t^3}\,dt \;\geq\; \omega_\varphi^2(f,n^{-1}) \cdot \int_{\frac{1}{n}}^{\frac{1}{2}} \frac{dt}{t^3} \;\geq\; c\,n^2\,\omega_\varphi^2(f,n^{-1}), \quad n > 2.$$

□

Acknowledgement. This work was done with the financial support of *Domus Hungarica Scientiarum et Artium.*

REFERENCES

[1] DeVore, R. A. and Yu, X. M.: *Pointwise estimates for monotone polynomial approximation*, Constr. Approx., **1**, (1985), 323-331.

[2] Ditzian, Z. and Totik, V.: *Moduli of Smoothness*, Springer - Verlag, Berlin Heidelberg New York London, 1987.

[3] Leviatan, D.: *Monotone and Comonotone Polynomial Approximation, Revisited*, J. Approx. Theory, **53**, (1988), 1-16.

[4] Mache, D. H. and Zhou, D. X.: *Best direct and converse results for Lagrange - type operators*, Approx. Theory Appl., **11**(2), (1995), 76-93.

[5] Popoviciu, T.: *Course of Mathematical Analysis, Part III,* Babeş-Bolyai University, Cluj, 1974.

Doi: 10.18514/MMN.2002.57

Mathematical Notes, Miskolc, Vol. 3., No. 2., (2002), pp. 113–122

ON POLYNOMIAL APPROXIMATIONS TO SOLUTIONS OF IMPLICIT DIFFERENTIAL EQUATIONS

Ihor Korol
Mathematical Faculty, Uzhgorod National University
Pidhirna 49, 88000 Uzhgorod, Ukraine
math1@univ.uzhgorod.ua

[Received: April 4, 2002]

Abstract. In this paper the possibility to present by a polynomial an independent variable for the approximate solutions of the systems of implicit ordinary differential equations under multi-point boundary conditions is substantiated.

Mathematical Subject Classification: 34A09, 34B10
Keywords: multi-point boundary value problems, implicit differential equations, numerical-analytic method, determining equation.

1. Introduction

There is a large number of methods which mathematicians elaborated for studying boundary value problems (BVPs). In the papers [1], [2] the numerical-analytic method based upon successive approximations was introduced. The polynomial version of this method in which the successive approximations are polynomials was proposed in [1] and then developed in [3], [4] for three- and multi-point boundary conditions. In this paper the issue of existence and approximate construction of the solutions of multi-point boundary conditions for the systems of implicit ordinary differential equations of the first order are studied by using polynomial approximations.

2. Construction of successive polynomial approximations

Let us consider a system of implicit equations

$$\frac{dx}{dt} = f(t, x, \frac{dx}{dt}), \tag{2.1}$$

with a multi-point boundary conditions

$$A_0 x(0) + \sum_{k=1}^{q} A_k x(t_k) + A_{q+1} x(T) = d, \tag{2.2}$$

where $x, d \in R^n$, $\quad f : [0, T] \times D_1 \times D_2 \to R^n$, D_1, D_2 are closed bounded domains in R^n, $0 = t_0 < t_1 < \ldots < t_q < t_{q+1} = T$, $\; A_k \quad (k = 0, 1, \ldots, q + 1)$ - are $n \times n$ matrices so that $\det \left[\sum_{k=1}^{q=1} A_k t_k\right] \neq 0$.

First of all we will introduce some notations [1].

It is known that for $f(t) \in C\,[0,T]$ there is a unique polynomial $P_m^0(t)$ among all the polynomials $P_m(t)$ with no more than m degree which is the best approximation for $f(t)$:

$$E_m(f) \equiv \left\| f(t) - P_m^0(t) \right\| = \inf_{P_m(t)} \left\| f(t) - P_m(t) \right\|.$$

Let us set in the interval $[0,T]$ the nodes

$$\tau_i = \frac{T}{2}\left(\cos\frac{(2i-1)\,\pi}{2(p+1)} + 1\right), \qquad i = 1,2,\ldots,p+1, \tag{2.3}$$

which are obtained by the substitution $\tau = \frac{T}{2}\left(\tau' + 1\right)$ from the corresponding zeroes $\tau_i' \in [-1,1]$ of the Chebyshev polynomials

$$T_{p+1}(t) = \cos\left((p+1)\arccos t\right).$$

For arbitrary continuous function $x_r(t)$ by $f^p\left(t, x_r(t), y_r(t)\right)$ we denote the Lagrange interpolation polynomial with p degree and with respect to the points (2.3):

$$f^p\left(t, x_r(t,x_0), y_r(t,x_0)\right) = \left(f_1^p\left(t, x_r(t,x_0), y_r(t,x_0)\right), \ldots, f_n^p\left(t, x_r(t,x_0), y_r(t,x_0)\right)\right),$$

where $y_r(t) := \frac{dx_r(t)}{dt}$, $f_j^p\left(t, x_r(t,x_0), y_r(t,x_0)\right) = a_{0j}^r + a_{1j}^r t + \ldots + a_{pj}^r$, $j = 1,2,\ldots,n$, $f_j^p\left(\tau_i, x_r(\tau_i), y_r(\tau_i)\right) = f_j(\tau_i, x_r(\tau_i), y_r(\tau_i)$, $i = 1,2,\ldots,p+1$.

Let us denote by

$$\overline{\mathcal{L}}\left(f,x,y,t,x_0\right) = f\left(t, x(t,x_0), y(t,x_0)\right) - \frac{1}{T}\int_0^T f\left(s, x(s,x_0), y(s,x_0)\right) ds,$$

$$\mathcal{L}\left(f,x,y,t,x_0\right) = \int_0^t \left(f\left(\tau, x(\tau,x_0), y(\tau,x_0)\right) - \frac{1}{T}\int_0^T f\left(s, x(s,x_0), y(s,x_0)\right) ds \right) d\tau.$$

We assume that the following conditions hold for the BVP (2.1), (2.2):

a) the vector-function $f(t,x,y)$ is continuous in $\Omega = [0,T] \times D_1 \times D_2$ (and therefore it is bounded by some vector M) and Lipschitzian in x and y, i.e.,

$$|f(t,x,y)| \leq M, \ \ |f(t,x,y) - f(t,\overline{x},\overline{y})| \leq K_1\,|x-\overline{x})| + K_2\,|y-\overline{y})|\,, \tag{2.4}$$

where M and $n \times n$ matrices K_1, K_2 have non-negative components. The absolute value sign and the inequalities we understand component-wise;

b) domains D_1 and D_2 satisfy the conditions

$$D_{\beta_1} := \{x \in R^n \mid B\left(x, \beta_1(x)\right) \subset D_1\} \neq \varnothing, \quad B\left(0, \beta_2(x)\right) \subset D_2,$$

where $B(x,\rho(x))$ is the ball of radius $\rho(x)$ with center x and

$$\beta_1(x) = \left(\frac{T}{2}E+G\right)\cdot(M'+L_p)+T\,|d(x)|\,,\quad G=T\cdot\sum_{k=1}^{q}|HA_k|\cdot\alpha_1(t_k)\,,$$

$$\beta_2(x) = 2\,(M+L_p)+\frac{1}{T}G\,(M'+L_p)+|d(x)|\,,\quad H=\left[\sum_{k=1}^{q+1}A_kt_k\right]^{-1},$$

$$d(x)=H\cdot\left(d-\sum_{k=0}^{q+1}A_kx\right),\quad \alpha_1(t)=2t\left(1-\frac{t}{T}\right),$$

$$M'=\frac{1}{2}\left[\max_{(t,x,y)\in\Omega}f(t,x,y)-\min_{(t,x,y)\in\Omega}f(t,x,y)\right],$$

$$L_p=(5+\lg p)\,\max_r E_p\left(f\left(t,x_r^{p+1}(t,x_0)\,,y_r^p(t,x_0)\right)\right)=$$

$$=(5+\lg p)\cdot\left(\max_r E_p\left(f_1\left(t,x_r^{p+1}(t,x_0)\,,y_r^p(t,x_0)\right)\right),\ldots\right.$$

$$\left.\ldots,\max_r E_p\left(f_n\left(t,x_r^{p+1}(t,x_0)\,,y_r^p(t,x_0)\right)\right)\right);$$

c) the eigenvalues $\lambda_j(Q)$ of the matrix $Q=K_1\left(\frac{T}{2}E+G\right)+K_2\left(2E+\frac{1}{T}G\right)$ satisfy the inequalities

$$|\lambda_j(q)|<1,\quad j=1,\ldots,n. \tag{2.5}$$

Let us introduce the sequence of polynomials with $p+1$ degree

$$\begin{aligned} x_m^{p+1}(t,x_0)&=x_0+\mathcal{L}\left(f^p,x_{m-1}^{p+1},y_{m-1}^p,t,x_0\right)+tHd(x_0)-\\ -tH\sum_{k=1}^{q}A_k\mathcal{L}&\left(f^p,x_{m-1}^{p+1},y_{m-1}^p,t_k,x_0\right),\ x_0^{p+1}(t,x_0)=x_0,\ m=1,2,\ldots \end{aligned} \tag{2.6}$$

Their derivatives look as follows:

$$\begin{aligned} y_m^p(t,x_0)&=\overline{\mathcal{L}}\left(f^p,x_{m-1}^{p+1},y_{m-1}^p,t,x_0\right)+Hd(x_0)-\\ -H\sum_{k=1}^{q}A_k\mathcal{L}&\left(f^p,x_{m-1}^{p+1},y_{m-1}^p,t_k,x_0\right),\ y_0^p(t,x_0)=0,\ m=1,2,\ldots \end{aligned} \tag{2.7}$$

Here the above index means that this expression is a polynomials of a correspondent degree. It is easy to see that all the members of the sequence (2.6) satisfy the boundary condition (2.2) for arbitrary $x_0\in D_{\beta_1}$.
The next theorem establishes the convergence of the sequence (2.6) and the properties of the limit functions.

Theorem 1. *Let BVP (2.1), (2.2) satisfy the conditions a)-c). Then:*

(1) *the sequences (2.6) and (2.7) converge to the functions $x^*(t,x_0)$ and $y^*(t,x_0)$, respectively, as $m\to\infty$, uniformly in $(t,x_0)\in[0,T]\times D_{\beta_1}$:*

$$x^*(t,x_0)=\lim_{m\to\infty}x_m^{p+1}(t,x_0),\quad y^*(t,x_0)=\lim_{m\to\infty}y_m^p(t,x_0),$$

where $y^*(t,x_0)=\frac{dx^*(t,x_0)}{dt}$;

(2) *the limit function* $x^*(t,x_0)$ *satisfies the "perturbed" BVP*

$$\begin{cases} \frac{dx}{dt}=f\left(t,x,\frac{dx}{dt}\right)+\Delta(x_0), \\ A_0x(0)+\sum\limits_{k=1}^{q}A_kx(t_k)+A_{q+1}x(T)=d, \end{cases} \tag{2.8}$$

where

$$\Delta(x_0)=-\frac{1}{T}\int\limits_0^T f^p\left(s,x^*\left(s,x_0\right),y^*\left(s,x_0\right)\right)ds+Hd(x_0)- \\ -H\sum_{k=1}^{q}A_k\mathcal{L}\left(f^p,x^*,y^*,t_k,x_0\right), \tag{2.9}$$

with the initial value $x^*\left(0,x_0\right)=x_0$;

(3) *the following error estimations hold:*

$$\left|x^*\left(t,x_0\right)-x_m^{p+1}\left(t,x_0\right)\right|\le\left(\alpha_1(t)E+G\right)\cdot W_{m-1}^p, \tag{2.10}$$

$$\left|y^*\left(t,x_0\right)-y_m^p\left(t,x_0\right)\right|\le\left(2E+\frac{1}{T}G\right)\cdot W_{m-1}^p, \tag{2.11}$$

where

$$W_{m-1}^p=\left[\sum_{i=0}^{m-1}Q^i\right]\cdot L_p+Q^{m-1}(E-Q)^{-1}\cdot \\ \cdot\left[K_1\left\{\left(\frac{T}{2}E+G\right)M'+T\left|d(x_0)\right|\right\}+K_2\left\{2M+\frac{1}{T}GM'+\left|d(x_0)\right|\right\}\right].$$

Proof. In addition to (2.6), (2.7) let us introduce the sequence of functions.

$$x_m(t,x_0)=x_0+\mathcal{L}\left(f,x_{m-1},y_{m-1},t,x_0\right)+tHd(x_0)- \\ -tH\sum_{k=1}^{q}A_k\mathcal{L}\left(f,x_{m-1},y_{m-1},t_k,x_0\right),\quad x_0\left(t,x_0\right)=x_0,\quad m=1,2,\ldots, \tag{2.12}$$

$$y_m(t,x_0):=\frac{dx_m(t,x_0)}{dt}=\overline{\mathcal{L}}\left(f,x_{m-1},y_{m-1},t,x_0\right)+Hd(x_0)- \\ -H\sum_{k=1}^{q}A_k\mathcal{L}\left(f,x_{m-1},y_{m-1},t_k,x_0\right),\quad y_0\left(t,x_0\right)=0,\quad m=1,2,\ldots \tag{2.13}$$

Also we introduce some notations:

$$x_m:=x_m(t,x_0),\ x_m^{p+1}:=x_m^{p+1}(t,x_0),\ r_{m+1}(t,x_0):=\left|x_{m+1}(t,x_0)-x_m(t,x_0)\right|,$$

$$y_m:=y_m(t,x_0),\ y_m^p:=y_m^p(t,x_0),\ \widehat{r}_{m+1}(t,x_0):=\left|y_{m+1}(t,x_0)-y_m(t,x_0)\right|.$$

We note [1] that

$$\left|f^p(t,x_m^{p+1},y_m^p)-f(t,x_m^{p+1},y_m^p)\right|\le L_p, \tag{2.14}$$

and making use of (2.4) we get

$$\left|f^p(t,x_m^{p+1},y_m^p)-f(t,x_m,y_m)\right|\le\left|f^p(t,x_m^{p+1},y_m^p)-f(t,x_m^{p+1},y_m^p)\right|+ \\ +\left|f(t,x_m^{p+1},y_m^p)-f(t,x_m,y_m)\right|\le L_p+K_1\left|x_m^{p+1}-x_m\right|+K_2\left|y_m^p-y_m\right|. \tag{2.15}$$

Using Lemma 3 of [5] we have that

$$|\mathcal{L}(f,x,y,t,x_0)| \leq \alpha_1(t)M' \leq \frac{T}{2}M', \tag{2.16}$$

$$\left|TH\sum_{k=1}^{q} A_k\mathcal{L}(f,x,y,t_k,x_0)\right| \leq GM', \tag{2.17}$$

$$\left|\mathcal{L}\left(f^p, x_m^{p+1}, y_m^p, t, x_0\right) - \mathcal{L}\left(f, x_m^{p+1}, y_m^p, t, x_0\right)\right| \leq \alpha_1(t)L_p, \tag{2.18}$$

$$\left|TH\sum_{k=1}^{q} A_k\left[\mathcal{L}\left(f^p, x_m^{p+1}, y_m^p, t_k, x_0\right) - \mathcal{L}\left(f, x_m^{p+1}, y_m^p, t_k, x_0\right)\right]\right| \leq GL_p. \tag{2.19}$$

We have to show that (2.6) is a Cauchy sequence in the space of continuous vector functions. To begin with, we establish for arbitrary $(t,x_0) \in [0,T] \times D_{\beta_1}$, and $m = 0,1,2,\dots$ that $x_m^{p+1}(t,x_0) \in D_1$ and $y_m^p(t,x_0) \in D_2$ by using (2.16)-(2.19):

$$\left|x_1^{p+1} - x_0\right| \leq \left|\mathcal{L}\left(f^p, x_0^{p+1}, y_0^p, t, x_0\right)\right| + \left|TH\sum_{k=1}^{q} A_k\mathcal{L}\left(f^p, x_0^{p+1}, y_0^p, t_k, x_0\right)\right| +$$

$$+T\,|d(x_0)| \leq \left|\mathcal{L}\left(f^p, x_0^{p+1}, y_0^p, t, x_0\right) - \mathcal{L}(f, x_0, y_0, t, x_0)\right| + |\mathcal{L}(f, x_0, y_0, t, x_0)| +$$

$$+T\,|d(x_0)| + \left|TH\sum_{k=1}^{q} A_k\left[\mathcal{L}\left(f^p, x_0^{p+1}, y_0^p, t_k, x_0\right) - \mathcal{L}(f, x_0, y_0, t_k, x_0)\right]\right| +$$

$$+\left|TH\sum_{k=1}^{q} A_k\mathcal{L}(f, x_0, y_0, t_k, x_0)\right| \leq (\alpha_1(t)E + G)(L_p + M') + T\,|d(x_0)| \leq \beta_1(x_0),$$

$$+T\,|d(x_0)| \leq \left|\mathcal{L}\left(f^p, x_0^{p+1}, y_0^p, t, x_0\right) - \mathcal{L}(f, x_0, y_0, t, x_0)\right| + |\mathcal{L}(f, x_0, y_0, t, x_0)| +$$

$$+T\,|d(x_0)| + \left|TH\sum_{k=1}^{q} A_k\left[\mathcal{L}\left(f^p, x_0^{p+1}, y_0^p, t_k, x_0\right) - \mathcal{L}(f, x_0, y_0, t_k, x_0)\right]\right| +$$

$$+\left|TH\sum_{k=1}^{q} A_k\mathcal{L}(f, x_0, y_0, t_k, x_0)\right| \leq (\alpha_1(t)E + G)(L_p + M') + T\,|d(x_0)| \leq \beta_1(x_0),$$

$$|y_1^p| \leq \left|\overline{\mathcal{L}}\left(f^p, x_0^{p+1}, y_0^p, t, x_0\right)\right| + |d(x_0)| + \left|H\sum_{k=1}^{q} A_k\mathcal{L}\left(f^p, x_0^{p+1}, y_0^p, t_k, x_0\right)\right| \leq$$

$$\leq \left|\overline{\mathcal{L}}\left(f^p, x_0^{p+1}, y_0^p, t, x_0\right) - \overline{\mathcal{L}}(f, x_0, y_0, t, x_0)\right| + \left|\overline{\mathcal{L}}(f, x_0, y_0, t, x_0)\right| + |d(x_0)| +$$

$$+\left|H\sum_{k=1}^{q} A_k\left[\mathcal{L}\left(f^p, x_0^{p+1}, y_0^p, t_k, x_0\right) - \mathcal{L}(f, x_0, y_0, t_k, x_0)\right]\right| +$$

$$+\left|H\sum_{k=1}^{q} A_k\mathcal{L}(f, x_0, y_0, t_k, x_0)\right| \leq 2(M + L_p) + G(M' + L_p) + |d(x_0)| \leq \beta_2(x_0).$$

It follows that $x_1^{p+1}(t,x_0) \in D_1$, $y_1^p(t,x_0) \in D_2$. By induction in a similar way we can establish that

$$\left|x_m^{p+1} - x_0\right| \leq \beta_1(x_0), \quad |y_m^p| \leq \beta_2(x_0).$$

Now we consider the differences $x_m - x_m^{p+1}$ and $y_m - y_m^p$. For $m=1$ we have

$$\begin{gathered}\left|x_1 - x_1^{p+1}\right| \le \left|\mathcal{L}\left(f, x_0, y_0, t, x_0\right) - \mathcal{L}\left(f^p, x_0^{p+1}, y_0^p, t, x_0\right)\right| + \\ + \left|TH\sum_{k=1}^{q} A_k\left[\mathcal{L}\left(f, x_0, y_0, t_k, x_0\right) - \mathcal{L}\left(f^p, x_0^{p+1}, y_0^p, t_k, x_0\right)\right]\right| \le \\ \le \left(\alpha_1(t)E + G\right)L_p,\end{gathered} \tag{2.20}$$

$$\begin{gathered}\left|y_1 - y_1^p\right| \le \left|\overline{\mathcal{L}}\left(f, x_0, y_0, t, x_0\right) - \overline{\mathcal{L}}\left(f^p, x_0^{p+1}, y_0^p, t, x_0\right)\right| + \\ + \left|H\sum_{k=1}^{q} A_k\left[\mathcal{L}\left(f, x_0, y_0, t_k, x_0\right) - \mathcal{L}\left(f^p, x_0^{p+1}, y_0^p, t_k, x_0\right)\right]\right| \le \\ \le \left(2E + \tfrac{1}{T}G\right)L_p.\end{gathered} \tag{2.21}$$

Using (2.14)-(2.21) and Lemma 4 of [5] we get

$$\begin{gathered}\left|x_2 - x_2^{p+1}\right| \le \left|\mathcal{L}\left(f, x_1, y_1, t, x_0\right) - \mathcal{L}\left(f^p, x_1^{p+1}, y_1^p, t, x_0\right)\right| + \\ + \left|TH\sum_{k=1}^{q} A_k\left[\mathcal{L}\left(f, x_1, y_1, t_k, x_0\right) - \mathcal{L}\left(f^p, x_1^{p+1}, y_1^p, t_k, x_0\right)\right]\right| \le \\ \le \left[\alpha_1(t)E + K_1\left(\alpha_2(t)E + \alpha_1(t)G\right) + \alpha_1(t)K_2\left(2E + \tfrac{1}{T}G\right)\right]L_p + \\ + \left|TH\sum_{k=1}^{q} A_k\left[\alpha_1(t_k)E + K_1\left(\alpha_2(t_k)E + \alpha_1(t_k)G\right) + \alpha_1(t_k)K_2\left(2E + \tfrac{1}{T}G\right)\right]L_p\right| \le \\ \le \left(\alpha_1(t)E + G\right)\left[E + K_1\left(\tfrac{T}{3}E + G\right) + K_2\left(2E + \tfrac{1}{T}G\right)\right]L_p \le \\ \le \left(\alpha_1(t)E + G\right)\left[E + Q\right]L_p,\\ \left|y_2 - y_2^p\right| \le \left|\overline{\mathcal{L}}\left(f, x_1, y_1, t, x_0\right) - \overline{\mathcal{L}}\left(f^p, x_1^{p+1}, y_1^p, t, x_0\right)\right| + \\ + \left|H\sum_{k=1}^{q} A_k\left[\mathcal{L}\left(f, x_1, y_1, t_k, x_0\right) - \mathcal{L}\left(f^p, x_1^{p+1}, y_1^p, t_k, x_0\right)\right]\right| \le \\ \le 2\max_{t\in[0,T]}\left|f(t, x_1, y_1) - f^p(t, x_1^{p+1}, y_1^p)\right| + \\ + \left|H\sum_{k=1}^{q} A_k\left|\mathcal{L}_1\left(f, x_1, y_1, t_k, x_0\right) - \mathcal{L}_1\left(f^p, x_1^{p+1}, y_1^p, t_k, x_0\right)\right|\right| \le \\ \le \left(2E + \tfrac{1}{T}G\right)\left[E + Q\right]L_p.\end{gathered}$$

We can obtain by induction that

$$\left|x_m(t, x_0) - x_m^{p+1}(t, x_0)\right| \le \left(\alpha_1(t)E + G\right)\left[\sum_{i=1}^{m-1} Q^i\right]L_p, \tag{2.22}$$

$$\left|y_m(t, x_0) - y_m^p(t, x_0)\right| \le \left(2E + \frac{1}{T}G\right)\left[\sum_{i=1}^{m-1} Q^i\right]L_p. \tag{2.23}$$

Now we have to estimate $r_{m+1}(t,x_0)$ and $\widehat{r}_{m+1}(t,x_0)$ for every $m = 0,1,2,\ldots$ by using Lemmas 3 and 4 of [5]:

$$r_1(t,x_0) \le |\mathcal{L}(f,x_0,y_0,t,x_0)| + T\,|d(x_0)| +$$

$$+\left|TH\sum_{k=1}^{q} A_k\mathcal{L}(f,x_0,y_0,t_k,x_0)\right| \le \left(\tfrac{T}{2}E+G\right)M' + T\,|d(x_0)| \equiv \gamma_1(x_0),$$

$$\widehat{r}_1(t,x_0) \le \left|\overline{\mathcal{L}}(f,x_0,y_0,t,x_0)\right| + |d(x_0)| + \left|H\sum_{k=1}^{q} A_k\mathcal{L}(f,x_0,y_0,t_k,x_0)\right| \le$$

$$\le 2M + |d(x_0)| + \tfrac{1}{T}GM' \equiv \gamma_2(x_0),$$

$$r_2(t,x_0) \le |\mathcal{L}(f,x_1,y_1,t,x_0) - \mathcal{L}(f,x_0,y_0,t,x_0)| +$$

$$+\left|TH\sum_{k=1}^{q} A_k\left[\mathcal{L}(f,x_1,y_1,t_k,x_0) - \mathcal{L}(f,x_0,y_0,t_k,x_0)\right]\right| \le$$

$$\le \left(1-\tfrac{t}{T}\right)\int_0^t \left[K_1 r_1(\tau,x_0) + K_2\widehat{r}_1(\tau,x_0)\right]d\tau + \tfrac{t}{T}\int_t^T \left[K_1 r_1(\tau,x_0) + K_2\widehat{r}_1(\tau,x_0)\right]d\tau +$$

$$+\left|TH\sum_{k=1}^{q} A_k\left[\left(1-\tfrac{t_k}{T}\right)\int_0^{t_k}\left[K_1 r_1(\tau,x_0) + K_2\widehat{r}_1(\tau,x_0)\right]d\tau +\right.\right.$$

$$\left.\left.+\tfrac{t_k}{T}\int_{t_k}^T \left[K_1 r_1(\tau,x_0) + K_2\widehat{r}_1(\tau,x_0)\right]d\tau\right]\right| \le (\alpha_1(t)E+G)\cdot\left[K_1\gamma_1(x_0) + K_2\gamma_2(x_0)\right],$$

$$\widehat{r}_2(t,x_0) \le \left|\overline{\mathcal{L}}(f,x_1,y_1,t,x_0) - \overline{\mathcal{L}}(f,x_0,y_0,t,x_0)\right| +$$

$$+\left|TH\sum_{k=1}^{q} A_k\left[\mathcal{L}(f,x_1,y_1,t_k,x_0) - \mathcal{L}(f,x_0,y_0,t_k,x_0)\right]\right| \le$$

$$\le 2\max_{t\in[0,T]}|K_1 r_1(\tau,x_0) + K_2\widehat{r}_1(\tau,x_0)| +$$

$$+\left|H\sum_{k=1}^{q} A_k\left[\left(1-\tfrac{t_k}{T}\right)\int_0^{t_k}\left[K_1 r_1(\tau,x_0) + K_2\widehat{r}_1(\tau,x_0)\right]d\tau +\right.\right.$$

$$\left.\left.+\tfrac{t_k}{T}\int_{t_k}^T \left[K_1 r_1(\tau,x_0) + K_2\widehat{r}_1(\tau,x_0)\right]d\tau\right]\right| \le \left(2E+\tfrac{1}{T}G\right)\cdot\left[K_1\gamma_1(x_0) + K_2\gamma_2(x_0)\right].$$

Similarly,

$$r_3(t,x_0) \le \Big\{ \left(1-\tfrac{t}{T}\right) \int\limits_0^t \left[K_1\left(\alpha_1(\tau)E+G\right)+K_2\left(2E+\tfrac{1}{T}G\right)\right] d\tau +$$

$$+\tfrac{t}{T}\int\limits_t^T \left[K_1\left(\alpha_1(\tau)E+G\right)+K_2\left(2E+\tfrac{1}{T}G\right)\right] d\tau +$$

$$+\Big| TH\sum_{k=1}^{q} A_k \Big[\left(1-\tfrac{t_k}{T}\right)\int\limits_0^{t_k} \left[K_1\left(\alpha_1(\tau)E+G\right)+K_2\left(2E+\tfrac{1}{T}G\right)\right] d\tau +$$

$$+\tfrac{t_k}{T}\int\limits_{t_k}^T \left[K_1\left(\alpha_1(\tau)E+G\right)+K_2\left(2E+\tfrac{1}{T}G\right)\right] d\tau \Big]\Big|\Big\} \cdot \left[K_1\gamma_1(x_0)+K_2\gamma_2(x_0)\right] \le$$

$$\le \left(\alpha_1(t)E+G\right)\cdot Q\cdot \left[K_1\gamma_1(x_0)+K_2\gamma_2(x_0)\right],$$

$$\widehat{r}_3(t,x_0) \le \Big\{ 2\max_{t\in[0,T]} \left| K_1\left(\alpha_1(\tau)E+G\right)+K_2\left(2E+\tfrac{1}{T}G\right)\right| +$$

$$+\Big| H\sum_{k=1}^{q} A_k \Big[\left(1-\tfrac{t_k}{T}\right)\int\limits_0^{t_k} \left[K_1\left(\alpha_1(\tau)E+G\right)+K_2\left(2E+\tfrac{1}{T}G\right)\right] d\tau +$$

$$+\tfrac{t_k}{T}\int\limits_{t_k}^T \left[K_1\left(\alpha_1(\tau)E+G\right)+K_2\left(2E+\tfrac{1}{T}G\right)\right] d\tau \Big]\Big|\Big\} \cdot \left[K_1\gamma_1(x_0)+K_2\gamma_2(x_0)\right] \le$$

$$\le \left(2E+\tfrac{1}{T}G\right)\cdot Q\cdot \left[K_1\gamma_1(x_0)+K_2\gamma_2(x_0)\right].$$

We can show by induction that for arbitrary $m = 0, 1, 2, \ldots$

$$r_{m+1}(t,x_0) \le \left(\alpha_1(t)E+G\right)\cdot Q^{m-1}\cdot \left[K_1\gamma_1(x_0)+K_2\gamma_2(x_0)\right], \tag{2.24}$$

$$\widehat{r}_{m+1}(t,x_0) \le \left(2E+\frac{1}{T}G\right)\cdot Q^{m-1}\cdot \left[K_1\gamma_1(x_0)+K_2\gamma_2(x_0)\right]. \tag{2.25}$$

From (2.24) and assumption c) we obtain the inequality

$$\begin{gathered} \left|x_{m+j}\left(t,x_0\right)-x_m\left(t,x_0\right)\right| \le \sum_{i=0}^{j-1}\left|x_{m+i+1}\left(t,x_0\right)-x_{m+i}\left(t,x_0\right)\right| \le \\ \le \sum_{i=0}^{j-1} r_{m+i+1}\left(t,x_0\right) \le \sum_{i=0}^{j-1}\left(\alpha_1(t)E+G\right)Q^{m+i-1}\left[K_1\gamma_1(x_0)+K_2\gamma_2(x_0)\right] \le \\ \le \left(\alpha_1(t)E+G\right)\cdot Q^{m-1}\left(E-Q\right)^{-1}\cdot\left[K_1\gamma_1(x_0)+K_2\gamma_2(x_0)\right]. \end{gathered} \tag{2.26}$$

For the derivatives $y_m\left(t,x_0\right)$ from (2.25) in a similar way we have:

$$\begin{gathered} \left|y_{m+j}\left(t,x_0\right)-y_m\left(t,x_0\right)\right| \le \\ \le \left(2E+\tfrac{1}{T}G\right)Q^{m-1}\left(E-Q\right)^{-1}\left[K_1\gamma_1(x_0)+K_2\gamma_2(x_0)\right]. \end{gathered} \tag{2.27}$$

It follows that (2.12) and (2.13) are uniformly convergent sequences:

$$\lim_{m\to\infty} x_m\left(t,x_0\right) = x^*\left(t,x_0\right), \ \lim_{m\to\infty} y_m\left(t,x_0\right) = y^*\left(t,x_0\right).$$

Taking the limit as $j\to\infty$ in (2.26) and (2.27) we get the error estimates

$$|x^*(t,x_0) - x_m\left(t,x_0\right)| \le (\alpha_1(t)E + G)\cdot Q^{m-1}\left(E-Q\right)^{-1}\cdot\left[K_1\gamma_1(x_0) + K_2\gamma_2(x_0)\right],$$

$$|y^*(t,x_0) - y_m\left(t,x_0\right)| \le \left(2E + \tfrac{1}{T}G\right)\cdot Q^{m-1}\left(E-Q\right)^{-1}\cdot\left[K_1\gamma_1(x_0) + K_2\gamma_2(x_0)\right].$$

Combining the last two inequalities with (2.22) and (2.23), we get the error estimates (2.10) and (2.11). Passing to the limit as $m\to\infty$ in (2.6) we obtain that $x^*(t,x_0)$ satisfies the integral equation

$$x(t) = x_0 + \mathcal{L}\left(f,x,y,t,x_0\right) + tHd(x_0) - tH\sum_{k=1}^{q} A_k\mathcal{L}\left(f,x,y,t_k,x_0\right).$$

While differentiating it, we get that $x^*(t,x_0)$ is a solution of the perturbed BVP (2.8)-(2.9). □

The following statement gives necessary and sufficient conditions for the existence of a solution of the BVP (2.1)-(2.2).

Theorem 2. *Under the conditions of Theorem 1, the limit function $x^*(t,x_0^*)$ is a solution of the BVP (2.1)-(2.2) if and only if x_0^* verifies the determining equation*

$$\begin{aligned}\Delta(x_0) = -\tfrac{1}{T}\int_0^T f\left(s,x^*\left(s,x_0\right),y^*\left(s,x_0\right)\right)ds + Hd(x_0) + \\ +H\sum_{k=1}^{q} A_k\mathcal{L}\left(f,x^*,y^*,t_k,x_0\right) = 0.\end{aligned} \tag{2.28}$$

Proof. The proof can be carried out in the same way as for the corresponding statements from [2] (Theorem 2.3). □

3. **Sufficient existence conditions**

Consider the m-th approximation to the determining equation (2.28)

$$\begin{aligned}\Delta_m^p(x_0) = -\tfrac{1}{T}\int_0^T f^p\left(s,x_m^{p+1}\left(s,x_0\right),y_m^p\left(s,x_0\right)\right)ds + Hd(x_0) + \\ +H\sum_{k=1}^{q} A_k\mathcal{L}\left(f^p,x_m^{p+1},y_m^p,t_k,x_0\right) = 0.\end{aligned} \tag{3.1}$$

Theorem 3. *Suppose that the conditions of Theorem 1 hold. Furthermore, assume that*

d) *there exists a closed, convex subset $D' = D_1' \times D_2' \subset D_1 \times D_2$ so that for arbitrary m and fixed p the approximate determining equation (3.1) has only one solution $x_0 = x_{0m}^p$ with non-zero topological index;*

e) *on the boundary ∂D of the subset D the inequality*

$$\inf_{x_0\in\partial D}\left|\Delta_m^p\left(x_0\right)\right| > \left(E+\frac{1}{T}G\right)W_m^p$$

holds.

Then there exists a solution $x = x^*(t)$ to the BVP (2.1)-2.2) with the initial value $x^*(0) = x_0^*$, where $x_0^* \in D_1'$.

Proof. Similarly to (2.15) and making use of (2.10) and (2.11), we get

$$\left|f(t,x^*,y^*)-f^p(t,x_m^{p+1},y_m^p)\right| \le \left[K_1\left(\alpha_1\left(t\right)E+G\right)+K_2\left(2E+\frac{1}{T}G\right)\right]W_{m-1}^p+L_p.$$

For the deviation of the exact and approximate determining functions we have that

$$\left|\Delta(x_0)-\Delta_m^p(x_0)\right| \le \tfrac{1}{T}\int_0^T\left|f^p\left(s,x^*\left(s,x_0\right),y^*\left(s,x_0\right)\right)-\right.$$

$$\left.-f^p\left(s,x_m^{p+1}\left(s,x_0\right),y_m^p\left(s,x_0\right)\right)\right|+H\sum_{k=1}^{q}A_k\left|\mathcal{L}\left(f^p,x^*,y^*,t_k,x_0\right)-\right.$$

$$\left.-\mathcal{L}\left(f^p,x_m^{p+1},y_m^p,t_k,x_0\right)\right| \le \left(E+\tfrac{1}{T}G\right)\left(QW_{m-1}^p+L_p\right) \le \left(E+\tfrac{1}{T}G\right)W_m^p.$$

Similarly to Theorem 3.1 of [2], one can prove that the vector fields $\Delta(x_0)$ and $\Delta_m^p(x_0)$ are homotopic, which completes the proof of Theorem 3. □

REFERENCES

[1] SAMOILENKO, A. M. and RONTÓ, N. I.: *Numerical-Analytic Methods of Investigating Solutions of Boundary Value Problems* , Naukova Dumka, Kiev, 1985 (in Russian).

[2] SAMOILENKO, A. M. and RONTÓ, N. I.: *Numerical-Analytic Methods in the Theory of Boundary Value Problems*, Naukova Dumka, Kiev, 1992 (in Russian).

[3] RONTÓ, M. and SAMOILENKO, A. M.: *Numerical-Analytic Methods in the Theory of Boundary Value Problems, World Scientific*, Singapore, 2000.

[4] KOROL, I. I. and KOROL, I. YU.: *Using of polynomial approximation method for solving of multi-point BVPs*, Naukovij Visnik Uzhgorods'koho Universitetu, Matematika, **4**, (1999), 71-78.

[5] RONTÓ, M. and MÉSZÁROS, J.: *Some remarks on the convergence analysis of the numerical-analytic method based upon successive approximations*, Ukrainskij Matematicheskij Zhurnal, **48**(1), (1996), 90-95.

Doi: 10.18514/MMN.2002.58

Mathematical Notes, Miskolc, Vol. 3., No. 2., (2002), pp. 123–132

WENDROFF TYPE INTEGRO–SUM INEQUALITIES AND APPLICATIONS

Anatolij Samoilenko
Institute of Mathematics, Ukrainian National Academy of Sciences
3 Tereshenkovskaya Str., 01601 KIEV, Ukraine
sam@imath.kiev.ua

Sergiy Borysenko
Department of Mathematics, National Technical University of Ukraine
37 Peremohy Str., 03056 KIEV, Ukraine
borys@mbox.com.ua

Ettore Laserra
Department of Mathematics and Informatics, University of Salerno, Italy
S. Allende Str., 84081 BARONISSI, Italy
laserra@unisa.it

Giovanni Matarazzo
Department of Engineering and Applied Mathematics, University of Salerno, Italy
S.Allende Str., 84081 BARONISSI, Italy
matarazzo@bridge.diima.unisa.it

[Received: June 20, 2002]

Abstract. We present some new nonlinear Wendroff type integral inequalities with respect to two independent variables for discontinuous functions. The results obtained are applied to investigate properties of solutions of a certain class of hyperbolic equations with impulse influence on certain surfaces.

Mathematical Subject Classification: 34B15
Keywords: differential equations with impulse influence, integro-sum inequalities

1. **Introduction**

Differential models of various real processes are described by systems of ordinary differential equations with impulse influence [1-3]. Among the methods of investigating the properties of these systems (such as existence, uniqueness, boundedness, and stability of solutions), the method of integro-sum inequalities [4-15] plays an important role. Note that applications and generalizations of this method to estimate solutions of systems of partial differential equations with impulse perturbations were not considered.

The study of this kind of problems is motivated by models of real processes in the dynamics of hydromechanical systems which are described by certain classes of hyperbolic partial differential equations with impulse perturbations, where the impulse effect is concentrated on surfaces transversal to characteristics.

Integro-sum representations of the solutions of such systems contain Lebesgue–Stiltjes measure concentrated on the curves of jumps of the solutions.

2. **Main results**

Let us assume that:

(a) D^* is an open set in R^2 $(D^* \subset R^2)$;
(b) $D^* = D \setminus \Gamma$, where $D = \bigcup_j D_j$, $j = 1, 2, \dots$;
(c) $\Gamma = \bigcup_j \Gamma_j$, $\Gamma_j = \{(x,y) : \varphi_j(x,y) = 0, j = 1, 2, \dots\}$;
(d) $\varphi_j(x,y)$ are real-valued continuously differentiable functions such that $\operatorname{grad} \varphi_j(x,y) > 0$ for all $j = 1, 2, \dots$;
(e) $D_1 \stackrel{\text{def}}{=} \{(x,y) : x \geq 0,\ y \geq 0,\ \varphi_1(x,y) < 0\}$;
$D_k \stackrel{\text{def}}{=} \{(x,y) : x \geq 0,\ y \geq 0,\ \varphi_{k-1}(x,y) \geq 0, \varphi_k(x,y) < 0,\ \ \forall k > 2,\ k \in N\}$;
(f) $G_j = \{(u,v) : (x,y) \in D_j,\ 0 \leq u \leq x,\ 0 \leq v \leq y,\ j \in N\}$;
(g) μ_{φ_k} is the Lebesgue-Stiltjes measure concentrated on the curves Γ_k.

Let us consider a real-valued nonnegative continuous function $u(x,y)$ on D^*, which has finite jumps on the curves $\{\Gamma_j\}$.

Denote by $\Phi(x,y,u)$ a nonnegative, continuous, and nondecreasing function of u, with $x, y \in D^*$ fixed. We shall consider the functions Φ of one of the following types $\Phi = \Phi_j$ $(j = 1, 4)$:

(i) $\Phi_1(x,y,u) = f_1(x,y)u(x,y)$,
(ii) $\Phi_2(x,y,u) = f_2(x,y)u(x,y) + f_3(x,y)$,
(iii) $\Phi_3(x,y,u) = f_4(x,y)u^\alpha(x,y)$, $\alpha = \text{const} > 0, \alpha \neq 1$,
(iv) $\Phi_4(x,y,u) = f_5(x,y)u(x,y) + f_6(x,y)u^\alpha(x,y)$, where $f_j(x,y) (j = \overline{1,6})$ are continuous nonnegative functions in $R^2_+ = \{(x,y) : x \geq 0,\ y \geq 0\}$.

Let $W(x,y,u(x,y))$ denote a function of one of the following two types W_1 and W_2:

$$W_1 = \beta_j u(x,y), \quad j \in N$$

where $\beta_j \geq 0$ are constants, and

$$W_2 = \beta_j(x,y)u(x,y), \quad j \in N,$$

where $\beta_j(x,y)$ are continuous functions nonnegative for all $(x,y) \in R^2_+$.

Let $g(x,y)$ be a positive, nondecreasing continuous function in R^2_+. Assume that $u(x,y)$ satisfies the following integro-sum inequality in D^*:

$$u(x,y) \leq g(x,y) + \iint_{G_n} \Phi(\tau, s, u(\tau,s)) d\tau ds + \sum_{j=1}^{n-1} \int_{\Gamma_j \cap G_n} W(x,y,u(x,y)) d\mu_{\varphi_j}, \quad (2.1)$$

which is satisfied for all $(x, y) \in D^*$. Then the following results are true.

Proposition 1. *The estimate*

$$u(x,y) \le g(x,y)\exp[F_1(x,y)]\prod(\beta_j(x,y)),\ \ \textit{if}\ \Phi=\Phi_1,\ W=W_2; \tag{2.2}$$

$$u(x,y) \le g(x,y)\exp[F_1(x,y)]\prod(\beta_j),\ \ \textit{if}\ \Phi=\Phi_1,\ W=W_1$$

holds for all $(x, y) \in D^*$. *Here,*

$$\int_0^x\int_0^y f_i(\tau,s)d\tau ds \overset{\text{def}}{=} F_i(x,y),\ \ i=1,3,5,$$

$$\prod_{j=1}^{n-1}\int_{\Gamma_j\cap G_n}(1+\beta_j(x,y))d\mu_{\varphi_j} \overset{\text{def}}{=} \prod(\beta_j(x,y)),\qquad \prod_{j=1}^{n-1}(1+\beta_j)\bigvee_{\Gamma_j\cap G_n}(\mu_{\varphi_j}) \overset{\text{def}}{=} \prod(\beta_j),$$

where $\bigvee_{\Gamma_j\cap G_n}(\mu_{\varphi_j})$ *is the complete variation of* μ_{φ_j}.

Proposition 2. *The estimate*

$$u(x,y) \le g(x,y)\exp[F_2(x,y)]\prod(\beta_j(x,y))\times$$
$$\times\left(1+\int_0^x\int_0^y \frac{f_3(\tau,s)}{g(\tau,s)}\exp[-F_2(\tau,s)]d\tau ds\right)$$

holds if $\Phi = \Phi_2$, $W = W_2$, *and*

$$u(x,y) \le g(x,y)\exp[F_2(x,y)]\prod(\beta_j)\left(1+\int_0^x\int_0^y \frac{f_3(\tau,s)}{g(\tau,s)}\exp[-F_2(\tau,s)]d\tau ds\right),$$

if $\Phi = \Phi_2$, $W = W_1$.

Proposition 3. *The following assertions hold.*

(A) *The estimate*

$$u(x,y) \le g(x,y)\prod(\beta_j(x,y)\left[1+(1-\alpha)\int_0^x\int_0^y f_4(\tau,s)g^{\alpha-1}(\tau,s)d\tau ds\right]^{\frac{1}{1-\alpha}}$$

is true if $0 < \alpha < 1$, $\Phi = \Phi_3$, *and* $W = W_2$. *When* $\beta_j(x, y) = \beta_j = \text{const} > 0$, *the expression* $\prod(\beta_j(x, y))$ *in estimate (A) is replaced by* $\prod(\beta_j)$.

(B) *The estimate*

$$u(x,y) \le g(x,y)\prod(\beta_j(x,y))\Big[1-(\alpha-1)\times$$
$$\times\prod^{\alpha-1}\beta_j(x,y)\int_0^x\int_0^y f_y(\tau,s)g^{\alpha-1}(\tau,s)d\tau, ds\Big]^{-\frac{1}{\alpha-1}}$$

holds for $\alpha > 1$ *and for arbitrary* $(x, y) \in D^*$ *such that*

$$\int_0^x\int_0^y f_4(\tau,s)g^{\alpha-1}(\tau,s)d\tau ds < [(\alpha-1)\prod^{\alpha-1}(\beta_j(x,y))]^{-1}.$$

If $\beta_j(x, y) = \beta_j = \text{const} > 0$, *then* $\prod(\beta_j(x, y))$ *in estimate (B) is replaced by* $\prod(\beta_j)$.

Proposition 4. *The following assertions hold.*

(C) *The estimate*

$$u(x,y) \leq g(x,y)\prod(\beta_j(x,y))\exp[F_5(x,y)] \times \left[1+(1-\alpha)\int_0^x\int_0^y f_6(\tau,s)g^{\alpha-1}(\tau,s)\times \right.$$

$$\left. \times \exp[(\alpha-1)F_5(\tau,s)]d\tau ds\right]^{\frac{1}{1-\alpha}}$$

holds for $0<\alpha<1$, $\Phi=\Phi_4$, $W=W_2$. *If* $\beta_j(x,y)=\beta_j$, *the expression* $\prod(\beta_j(x,y))$ *in (C) changes to* $\prod(\beta_j)$.

(D) *The estimate*

$$u(x,y) \leq g(x,y)\prod(\beta_j(x,y))\exp[F_5(x,y)]\left[1-(\alpha-1)\prod^{\alpha-1}(\beta_j(x,y))\times \right.$$

$$\left. \times\int_0^x\int_0^y f_6(\tau,s)\times g^{\alpha-1}(\tau,s)\exp[(\alpha-1)F_5(\tau,s)]d\tau ds\right]^{-\frac{1}{\alpha-1}}$$

is true for $\alpha>1$ *and arbitrary* $(x,y)\in D^*$ *such that*

$$\int_0^x\int_0^y f_6 g^{\alpha-1}\exp[(\alpha-1)F_5]d\tau ds < \left[(\alpha-1)\prod^{\alpha-1}(\beta_j(x,y))\right]^{-1}.$$

If $\beta_j(x,y)=\beta_j$, *the expression* $\prod(\beta_j(x,y))$ *in (D) changes to* $\prod(\beta_j)$.

The proofs of Propositions 1–4 are obtained by using the induction method.

Proof. Let us establish Proposition 1 (Propositions 2–4 are proved analogously).

Let $(x,y)\in D_1$. Inequality (2.1) reduces to the form

$$u(x,y)\leq g(x,y)+\iint_{G_1}\Phi_1(\tau,s,u(\tau,s))d\tau ds.$$

We can suppose that $g(x,y)=C>0$, a constant function, because otherwise, if $g(x,y)\neq C$ then, by using the fact that g is positive and nondecreasing, it is possible to obtain the comparison inequality

$$\mu(x,y)\leq 1+\iint_{G_1}\Phi_1(\tau,s,\mu(\tau,s))d\tau ds,$$

where $\mu=\frac{u}{g}$.

Obviously, the following estimate holds in D_1:

$$u(x,y)\leq C\exp[F_1(x,y)].$$

Consider the domain D_2. Let $G_2=G_2^1\cup G_2^2$, where

$$G_2^2=\{(x,y):(x,y)\in D_2\cap G_2\}.$$

Then, for all $(x,y)\in D_2$, the inequality

$$u(x,y) \le C + \iint_{G_2^1} \Phi_1(\tau,s,u(\tau,s))d\tau ds + \int_{\Gamma_1\cap G_2} \beta_1(x,y)u(x,y)d\mu_{\varphi_1} + \iint_{G_2^2} \Phi_1(\tau,s,u(\tau,s))d\tau ds$$

holds.

On the curves $\Gamma_1 \cap G_2$, we select some points $A_i(x_i^1, y_i^1),\ i=\overline{0,\ n-1}$, and consider the inequality

$$u(x,y) \le \sum_{i=0}^{n-1}\Big(C + \int_0^{x_1^1}\int_0^{y_i^1} \Phi_1(\tau,s,u(\tau,s))d\tau ds + u(x_i^1,y_i^1)\beta_1(x_i^1,y_i^1)\Delta\mu_{\varphi_1^i} + \int_{x_i^1}^{x}\int_{y_i^1}^{y} \Phi_1(\tau,s,u(\tau,s))d\tau ds\Big),$$

where $\Delta\mu_{\varphi_1^i}$ is the variation of the measure function φ_1 on the segment A_iA_{i+1}. Using (2.2), we obtain

$$\begin{aligned} u(x,y) &\le \sum_{i=0}^{n-1}\{C\exp[F_1(x_i^1,y_i^1)] + C\beta_1(x_i^1,y_i^1)\Delta\mu^i_{\varphi_1}\exp[F_1(x_i^1,y_i^1)]+ \\ &+ \int_{x_i^1}^{x}\int_{y_i^1}^{y}\Phi_1 d\tau ds\} \le \sum_{i=0}^{n-1}\{C(1+\beta_1(x_i^1,y_i^1)\Delta\mu^i_{\varphi_1}\exp[F_1(x_i^1,y_i^1)]+ \\ &+ \int_{x_i^1}^{x}\int_{y_i^1}^{y}\Phi_1 d\tau ds)\} \le \sum_{i=0}^{n-1}[C(1+\beta_1(x_i^1,y_i^1))\Delta^i_{\varphi_1}\exp[F_1(x,y)]]. \end{aligned} \tag{2.3}$$

When $\max_{0\le i\le n-1}\Delta\mu^i_{\varphi_1} \to 0$, we obtain

$$u(x,y) \le C\exp[F_1(x,y)]\int_{\Gamma_1\cap G}(1+\beta_1(x,y))d\mu_{\varphi_1}.$$

Let (2.2) be satisfied for all $(x,y)\in D_k$. Let us consider $(x,y)\in D_{k+1}$ and put $G = G^1_{k+1}\cup G^2_{k+1}$,

$$G^1_{k+1} = \{(x,y) : (x,y)\in D_{k+1}\cap G\},\ G^2_{k+1} = G\setminus G^1_{k+1}.$$

Then

$$u(x,y) \le C + \iint_{G^1_{k+1}}\Phi_1 d\tau ds + \int_{\Gamma_k\cap G}\beta_k(x,y)u(x,y)d\mu_{\varphi_k} + \iint_{G^2_{k+1}}\Phi_1 d\tau ds.$$

On the curve $\Gamma_k \cap G$, we select some points $B_i(x_i^k, y_i^k), i=\overline{0,\ n-1}$. Denote by $\Delta\mu^i_{\varphi_k}$ the variation of the measure on the segment B_iB_{i+1}. Similarly to the consideration above, we obtain

$$u(x,y) \le \sum_{i=0}^{n-1}[C + \int_0^{x_i^k}\int_0^{y_i^k}\Phi_1 d\tau ds + u(x_i^k,y_i^k)\beta_k(x_i^k,y_i^k)\Delta^i_{\varphi_k}+$$

$$+u(x_i^k, y_i^k)\beta_k(x_i^k, y_i^k)\Delta^i_{\varphi_k} + \int_{x_i^k}^{x}\int_{y_i^k}^{y} \Phi_1 d\tau ds].$$

Then

$$u(x,y) \le \sum_{i=0}^{n-1}[C(1+\beta_k(x_i^k, y_i^k)) \times \Delta\mu^i_{\varphi_k} \exp[F_1(x,y)] \prod_{j=1}^{n-1} \int_{\Gamma_j \cap G} (1+\beta_j(x,y)) d\mu_{\varphi_j}].$$

When $\max_{0\le i \le n-1} \Delta\mu^i_{\varphi_K} \to 0$, we get

$$u(x,y) \le C\exp[F_1(x,y)] \prod_{j=1}^{k} \int_{\Gamma_j \cap G} (1+\beta_j(x,y)) d\mu_{\varphi_j}$$

for all $(x,y) \in D_{k+1}$. This completes the proof of Proposition 1. □

3. Applications

Consider the model of a hydraulic system consisting of a tank, a delivery pipeline, an auger centrifugal pump, and a pressure head pipeline giving a liquid in gasogenerator and the chamber output pressure. Such systems were considered in [16, 17, 20]. Note that there may be auto-oscillations stipulated by the feedback in a hydraulic part of the system due to the fact that the system is located on a stand.

Experiments showed that, in certain cases, there may be a significant increase of the amplitude of auto-oscillations of the stand. Thus, the oscillation frequencies of the liquid in the tank are characterized by the eigenfrequencies of the stand that coincide with the eigen-oscillations of the output pressure. The amplitude of the oscillations may have white-noise-like-behaviour.

The main problem in the study of such systems is to describe the deviation of the system state from the equilibrium.

3.1. Dynamics of fluid in the pipeline.

In the one-dimensional case, the dynamics of a compressed liquid in the homogeneous pipeline is usually described by a system consisting of the following equations:

A) Equation of motion of the liquid:

$$\frac{\partial V}{\partial t} + u\frac{\partial V}{\partial x} + \frac{1}{\gamma}\frac{\partial p}{\partial x} + \frac{\lambda}{2d}V|V| = 0.$$

Here, λ is the resistance coefficient depending on the Reynolds number, x represents the coordinate of an axis of the pipeline, γ is the density of the liquid, d is the diameter of the pipeline, p, V are the instantaneous pressure and speed of the liquid, respectively.

B) Equation of continuity:

$$\frac{\partial \gamma}{\partial t} + V\frac{\partial \gamma}{\partial x} + \gamma\frac{\partial V}{\partial x} = 0.$$

C) The equation of state for the liquid, whose role is played by Hook's law, is

$$\gamma = \gamma_0 \left(1 + \frac{p - p_0}{\gamma_0 c^2}\right) .$$

For the majority of hydraulic systems, the convection terms can be neglected. For the turbulence analysis, the nonlinear term admits linearization, and the equations with the distributed parameters reduce to the form

$$-\frac{\partial p}{\partial x} = \frac{\gamma}{F_\tau}\left(\frac{\partial q}{\partial t} + kq\right),$$

$$\frac{\partial p}{\partial t} = \frac{c^2\gamma}{F_\tau}\frac{\partial q}{\partial x}.$$

Here, q is the volumetric intensity of the liquid.

3.2. **The description of the centrifugal pump and pressure head sprocket.** Among the main characteristic features describing the mode of operation of the cavitational centrifugal pump is the dependence of the sizes of the cavitational cavities V_k on the input and output pressure $V_k = f(p, q)$. This dependence is related to the cavitation number $k(V_k, q)$ defined as the ratio of the pressure difference to the velocity head $p = p_n + k(V_k, q)\gamma\frac{\omega^2}{2}$, where p_n is the pressure of the saturated steam of the liquid, ω is the speed of the blade-to-blade flow in the auger channels, γ is the density of the liquid steam. When the liquid passes through the auger, the pressure varies according to the rule

$$p_n = (a_i n^2 + b_i n q^{(1)} + c_i (q^{(1)})^2) f_1(V),$$

where n is the number of revolutions of the shaft, a_i, b_i, c_i are the parameters describing the operation of the pump and the pressure head sprocket, $f_1(V)$ is an experimentally obtained function describing the influence of the size of the cavitational cavities on the operation of the auger of the pump and the pressure of the head sprocket.

Thus, a simplified model of the dynamics of the system is as follows:

tank → delivery pipeline → auger of the pump → pressure head pipeline → chamber of combustion positioned on the sliding stand.

This reduces to the following mathematical problems.

3.3. **Model without sharing cavitational cavities.** The corresponding system contains the following equations:

1) Equation of mechanical oscillations of the stand:

$$\ddot{z}_i + b_i(t)\dot{z}_i + \omega_i^2(t) z_i = \frac{R(t)}{m(t)\overline{p}_6} p_6, \quad i = 1, 2, \ldots, 6;$$

2) Equation of motion of the liquid in the pipeline:

$$-\frac{\partial p_1}{\partial x} = \frac{\gamma_1}{F_1}\frac{\partial q_1}{\partial t},$$

$$-\frac{\partial p_1}{\partial t} = \frac{c^2\gamma_1}{F_1}\frac{\partial q_1}{\partial x},$$

with the boundary conditions

$$p_1(0,t) + R_1 q_1(0,t) = \frac{\gamma_1 H(t)}{g}\sum_{i=1}^{6}\alpha_{il}\ddot{z}_i + \beta\gamma_1 F_1\sum_{i=1}^{6}\alpha_{i2}\dot{z}_i,$$

$$p_1(l_1,t) = B_1 V_k + B_2 q_1(l_1,t) + E\dot{V}_k + I_k\dot{q}(l_1,t).$$

Here, l_1 is the length of the pipeline and, in the quasistationary model, $E = I_k = 0$.

3) Equation of motion of the liquid in the pressure head pipeline:

$$-\frac{\partial p_2}{\partial x} = \frac{\gamma_2}{F_2}\frac{\partial q_2}{\partial t},$$

$$-\frac{\partial p_2}{\partial t} = \frac{c^2\gamma_2}{F_2}\frac{\partial q_2}{\partial x}$$

with the boundary conditions:

$$p_2(l_1+0,t) + R_2 q_2(l_1+0,t) = p_1(l_1-0,t) + s_1 q_1(l_1-0,t) + \varepsilon V_k, \tag{3.1}$$

$$\tau\dot{p}_2(l_1+l_2,t) + p_2(l_1+l_2,t) = A_2 q_2(l_1+l_2,t),$$

$$p_2(l_1+l_2,t) = p_2(t).$$

4) Equation of material balance describing summarized size of cavitational cavities in the pump:

$$\gamma_1\dot{V}_k = q_2(l_1+0,t) - q_1(l_1-0,t). \tag{3.2}$$

Relations (3.1), (3.2) can be considered to be the impulse effect concentrated on the curves transversal to the characteristics. Thus, we have to study solutions of the hyperbolic equations with impulse effect concentrated on surfaces transversal to characteristics [18, 19].

3.4. **Reduction to the integral equations and inequalities.** The reduced model is represented by an equation of hyperbolic type with, generally speaking, a nonlinear right-hand member and boudary conditions acting as the impulse effect.

Let us assume $s_i = c_i$ and $z_i = \frac{c_i F_i}{\gamma_i}$. The general solution of the partial differential equations considered has the form $p_i(x,t) = \nu(x - s_i t) + u(x + s_i t)$ and $q_i(x,t) = \frac{1}{z_i}(\nu(x - s_i t) - u(x + s_i t))$. In view of the boundary conditions at $t = 0$, by making the standard substitution $\xi = x - st$, and $\eta = x + st$, applying the d'Alembert formula in the domain $0 < \xi + \eta < 2l_1$, we obtain the following representation of the solutions:

$$u_1(\xi,\eta) = \frac{\varphi_1(\xi) + \varphi_1(\eta)}{2} + \frac{1}{2z_i}\int_{\xi}^{\eta}\psi_1(s)ds + \frac{z_i}{2}\iint_D f_1(s,t)u_1(s,t)|u_1(s,t)|dsdt.$$

Here, the functions φ, ψ are determined from the initial and boundary conditions. The integration is carried out in the domain

$$D_1 = \{s,t : s+t > 0, \quad s < \xi, \quad t < \eta, \ \xi + \eta < 2l_1\}.$$

Similarly, in the domain $2l_1 < \xi + \eta < 2l_2$, the formula of representation for solutions has the form

$$u_2(\xi,\eta) = \frac{\varphi_2(\xi)+\varphi_2(\eta)}{2} + \frac{1}{2z_i}\int_\xi^\eta \psi_2(s)ds + \frac{z_i}{2}\iint_{D_2} f_2(s,t)u_2(s,t)|u_2(s,t)|dsdt + \\ + \int_{D_2\cap\gamma} u(s,t)\varphi_2(s,t)d\mu.$$

Here,

$$D_2 = \{s,t : s+t > 0, \quad s < \xi, \quad t < \eta, \quad 2l_1 < \xi + \eta < 2l_2, \}$$

the curve $\gamma = \{\xi,\ \eta : \xi + \eta = 2l_1\}$ represents the curve of rupture of the stream stipulated by the presence of the pump. The functions are constant along the characteristics in the domain $D_2 \setminus D_1$; their change is determined by a Stieltjes integral on the curve γ and depends on the constant characteristics of the pump. We do not consider the process of reflection of a wave from the blades of the pump.

Thus, the problem considered can be studied by using the multi-dimensional integral inequalities considered in Section 2.

REFERENCES

[1] Samoilenko, A. M. and Perestyuk, N. A.: *Differential Equations with Impulse Effect*, Vyshcha Shkola, Kyiv, 1987. (in Russian)

[2] Samoilenko, A., Borysenko, S., Cattani C. and Yasinsky V.: *Differential Models: Construction, Representations & Applications*, Nauk. Dumka, Kyiv, 2001.

[3] Samoilenko A., Borysenko, S., Cattani, C., Matarazzo, G. and Yasinsky, V.: *Differential Models: Stability, Inequalities & Estimates*, Nauk. Dumka, Kyiv, 2001.

[4] Samoilenko, A. M. and Borysenko, S. D.: *Integro-sum inequalities and stability of processes with discrete disturbances*, Proc. of the Third International Conference "Differential Equations and Their Applications", Russia, (1987), 377–380.

[5] Samoilenko, A. M. and Borysenko, S. D.: *On functional inequalities of Bihari type for Discontinuous functions*, Uspekhi Mat. Nauk, **53**(4), (1998), 147–148.

[6] Borysenko, S. D., Kosolapov, V.I. and Obolensky A.Yu.: *Stability of Processes under Continuous and Discrete Perturbations*, Nauk. Dumka, Kyiv, 1988. (in Russian)

[7] Borysenko, S. D.: *Construction of Mathematical Models*, NTUU "KPI"(National Technical University "Kiev, Polytechnical Institute"); (1995). (in Russian)

[8] Borysenko, S. D.: *Integro-sum inequalities for functions of many independent variables*, Differents. Uravneniya, **23**(9), (1989), 1638–1641.

[9] Borysenko, S. D.: *On some integro-sum inequalities and their applications*, Vest. Kiev. Univ. Mat. i Mekh., **31**, (1989), 7–12.

[10] Borysenko, S. D.: *On integro-sum functional inequalities*, Differents. Uravneniya, **34**(6), (1998), p. 850.

[11] Borysenko, S. D.: *On some Bihari inequalities for discontinuous functions*, Nauk. Visti NTUU "KPI", **1** (1998), 147–151.

[12] Borysenko, S. D.: *Multidimensional integro-sum inequalities*, Ukr. Nat. Zh., **50**, No. 2 (1998), 172–177.

[13] Borysenko, S. D.: *Integro-sum inequalities and stability problems of systems with impulse excitation*, Author's Abstract of the DSc Dissertation (Phys. & Math.), Inst. of Math., NAS of Ukraine, Kyiv, 1998.

[14] Borysenko, S. D., Matarazzo, G. and Piltyay M. M.: *One-dimensional integro-sum inequalities. The Bellman–Bihari–Rakhmatullina method*, Nauk. Visti NTUU "KPI", **1** (2000), 111–115.

[15] Borysenko, S. D., Matarazzo, G. and Piltyay M. M.: *Problem of reduction of multidimensional functional integral inequalities to one-dimensional ones*, Neliniyni Kolyvannya, **2**, (2000), 18–24.

[16] Slyusarchuk, V. E.: *General theorems on existence solutions of differential equations with impulse influence*, Ukr. Mat. Zh., **52**(7), (2000), 954–964.

[17] Pilipenko, V. V., Zadontsev, V. A. and Natanzon M. S.: *Cavitational Auto-Oscillations and Dynamics of Hydrosystems*, Mashinostroenie, Moscow, 1977. (in Russian)

[18] Pilipenko, V. V., Zadontsev, V. A. and Chalyi N. P.: *Theoretical Stability Investigation of the Auger – Centrifugal Pump – Pipeline System for Systems on the Stand. Cavitational Auto-Oscillations and Dynamics of Hydrosystems*, Nauk. Dumka, Kiev, 1977. (in Russian)

[19] Hale, J.: *Theory of Functional-Differential Equations*, Mir, Moscow, 1984.

[20] Kolmanovsky, V. G. and Nosov, V. P.: *Stability and Periodic Solutions of Delay Systems*, Nauka, Moscow, 1981. (in Russian)

[21] Martinyuk, A. A., Obolensky, A. Yu. and Chernitskaya L. N.: *Mathematical models of hydromechanical systems*, in: *Stability of Hydromechanical Systems*, Inst. of Mechanics, Akad. Nauk Ukr. SSR, Dep. VINITI (1986), 5–36.

Doi: 10.18514/MMN.2002.54

Mathematical Notes, Miskolc, Vol. 3., No. 2., (2002), pp. 133–156

SIGN-CHANGING SOLUTIONS AND W-SOLUTIONS TO SINGULAR DIRICHLET BOUNDARY VALUE PROBLEMS

Svatoslav Staněk
Department of Mathematical Analysis, Faculty of Science
Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic
stanek@risc.upol.cz

[Received: May 9, 2002]

Abstract. The singular Dirichlet problem $(r(x)x')' = q(t)f(t,x)$, $x(0) = x(T) = 0$, $\lambda \max\{x(t) : 0 \le t \le T\} = -\min\{x(t) : 0 \le t \le T\}$ is considered. Here f is singular at the point $x = 0$ of the phase variable x and λ is a positive parameter. The notions of a solution and a w-solution of the above problem changing its sign exactly once on $(0,T)$ are introduced. Effective conditions for the existence and multiplicity results are presented. Next, the notion of an exceptional n-sign-changing w-solution of our problem with $\lambda = 1$ is given and for such solutions existence and multiplicity results are proved.

Mathematical Subject Classification: 34B15, 34B16
Keywords: singular Dirichlet problem, sign-changing solution, w-solution, exceptional solution, existence, multiplicity

1. Introduction

Consider the problem

$$(r(x(t))x'(t))' = q(t)f(t,x(t)), \tag{1.1}$$

$$x(0) = 0, \quad x(T) = 0, \tag{1.2}$$

$$\lambda \max\{x(t) : 0 \le t \le T\} = -\min\{x(t) : 0 \le t \le T\}, \tag{1.3}$$

where T is a positive number, λ is a positive parameter and f is singular at the point $x = 0$ of the phase variable x in the following sense

$$\lim_{x\to 0^-} f(t,x) = -\infty, \quad \lim_{x\to 0^+} f(t,x) = \infty \quad \text{for } t \in [0,T]. \tag{1.4}$$

Definition 1.1. We say that $x \in C^1([0,T])$ is a *solution of problem* (1.1) $-$ (1.3) if x has precisely one zero t_0 on $(0,T)$, $r(x)x' \in C^1((0,T) \setminus \{t_0\})$, (1.1) is satisfied for $t \in (0,T) \setminus \{t_0\}$, x fulfils (1.2) and there exists $\lambda_0 \in (0,\infty)$ such that (1.3) holds with $\lambda = \lambda_0$.

Besides a solution of problem (1.1)-(1.3), we introduce in accordance with [15] the notion of a w-solution of problem (1.1)-(1.3).

Definition 1.2. Let $\lambda \in (0,\infty)$. A function $x \in C^0([0,T])$ is called a *w-solution of problem* $(1.1)-(1.3)$ if x has precisely one zero $t_0 \in (0,T)$, $x \in C^1([0,T] \setminus \{t_0\})$, there exist finite $\lim_{t\to t_0^-} x'(t)$ and $\lim_{t\to t_0^+} x'(t)$, $r(x)x' \in C^1((0,T) \setminus \{t_0\})$, x fulfils (1.2), (1.3), and (1.1) holds on $(0,T) \setminus \{t_0\}$.

We note that in contrast to a solution x of problem (1.1)-(1.3) which belongs to the class $C^1([0,T])$ and satisfies (1.3) with a suitable value of λ, a w-solution x of problem (1.1)-(1.3) is continuous on $[0,T]$, has continuous derivative on $[0,t_0) \cup (t_0,T]$ where t_0 is the unique zero of x in $(0,T)$ and (1.3) holds with a given value of λ. Naturaly, any solution of problem (1.1)-(1.3) is also a w-solution of this problem.

In the paper we will use the following assumptions:

(H_1) $r \in C^0(\mathbb{R})$, $r(x) \ge r_0 > 0$ for $x \in \mathbb{R}$;

(H_2) $q \in C^0((0,T))$, $q(t) < 0$ for $t \in (0,T)$ and $Q = \sup\{|q(t)| : 0 \le t \le T\} < \infty$;

(H_3) $f \in C^0([0,T] \times D)$, where $D = (-\infty,0) \cup (0,\infty)$, $f(t,\cdot)$ is nonincreasing on D for $t \in [0,T]$ and

$$0 < f(t,x)\operatorname{sign} x \le g(x) \quad \text{for } (t,x) \in [0,T] \times D,$$

where $g \in C^0(D)$ and

$$\int^0 g(s)\,ds < \infty, \qquad \int_0 g(s)\,ds < \infty;$$

(H_4) for each $(t_0,x_0,x_1) \in (0,T) \times D \times \mathbb{R}$, there exists a unique solution x of (1.1) satisfying the initial conditions $x(t_0) = x_0$, $x'(t_0) = x_1$ defined in a neighbourhood of $t = t_0$.

Remark 1.3. If f satisfies (H_3) then for each $M > 0$ there exists a positive function $k_M \in C^0([0,T])$ such that

$$0 < k_M(t) \le f(t,x)\operatorname{sign} x \le g(x) \quad \text{on } (t,x) \in [0,T] \times ([-M,0) \cup (0,M]).$$

Next under the assumption that f is a locally Lipschitz function on $(0,T) \times D$, assumption (H_4) is satisfied.

In many papers (see, e.g., [1]–[13], [16]–[22] and references therein) only positive (negative) solutions on $(0,T)$ of the Dirichlet boundary value problems with the singularity at the point $x = 0$ of the phase variable x in nonlinearities of considered second-order differential equations have been studied. Solutions were considered either in the class $C^0([0,T]) \cap C^2((0,T))$ ([1]–[3], [7], [11], [12], [18], [19]) or $C^1([0,T]) \cap C^2((0,T))$ ([4]–[6], [12], [13], [16]–[19], [22]) or $C^0([0,T]) \cap AC^1_{\rm loc}((0,T))$ ([8]–[10], [20], [21]). Here $AC^1_{\rm loc}((0,T))$ denotes the set of functions having absolutely continuous first derivatives on any compact subintervals of $(0,T)$. The nonlinearities of equations are usually nonpositive ([1], [2], [6]–[8], [11], [12], [16]–[20], [22]), but in [3]–[5], [9], [10], [13] and [21] this assumption is overcome.

For the first time in [14] solutions of singular Dirichlet boundary value problems changing their signs exactly once on $(0,T)$ were considered. Here differential equations of the form

$$(r(x(t))x'(t))' = \mu q(t)f(t,x(t)) \tag{1.5}$$

together with the condition

$$\max\{x(t) : 0 \le t \le T\}\min\{x(t) : 0 \le t \le T\} < 0\,, \tag{1.6}$$

were studied where μ is a positive parameter and f is singular at the point $x = 0$ of the phase variable x. A function $x \in C^1([0,T])$ is called a solution of problem $(1.5), (1.2), (1.6)$ if x has precisely one zero t_0 on $(0,T)$, $r(x)x' \in C^1((0,T) \setminus \{t_0\})$, x fulfils (1.2) and (1.6) and there exists $\mu_0 > 0$ such that (1.5) with $\mu = \mu_0$ is satisfied for $t \in (0,T)\setminus\{t_0\}$. In [14] under assumptions $(H_1)-(H_3)$ it is proved among others that for each $A \in (0,\infty)$ there exists a solution x of problem $(1.5), (1.2), (1.6)$ such that $\max\{x(t) : 0 \le t \le T\} = A$. We see that any solution of problem $(1.5), (1.2), (1.6)$ depends on a value of the parameter μ in equation (1.5) unlike our definition of a solution of problem (1.1)-(1.3) depending on a value of the parameter λ appearing in condition (1.3).

A generalization of the notion of a solution of problem $(1.5), (1.2), (1.6)$ was given in [15]. Here $x \in C^0([0,T])$ is said to be a w-solution of problem (1.5), (1.2), (1.6) if x has precisely one zero t_0 in $(0,T)$, $x \in C^1([0,T]\setminus\{t_0\})$, there exist finite $\lim_{t\to t_{0-}} x'(t)$, $\lim_{t\to t_{0+}} x'(t)$, $r(x)x' \in C^1((0,T) \setminus \{t_0\})$, x fulfils (1.2) and (1.6), and finally there exists $\mu_0 > 0$ such that (1.5) with $\mu = \mu_0$ is satisfied for $t \in (0,T) \setminus \{t_0\}$. It is proved among others that under assumptions $(H_1) - (H_3)$ for $A > 0$ and $t_0 \in (0,T)$ problem $(1.5), (1.2), (1.6)$ has just two w-solutions vanishing at t_0 and having their maximum values on $[0,T]$ equal to A.

This paper is a continuation of [15] and in comparison with (1.5) our equation (1.1) does not depend on the parameter μ. By our definitions any solution as well as any w-solution x of problem (1.1)-(1.3) have precisely one zero in $(0,T)$ where they change their signs. Hence any solution and any w-solution of problem (1.1)-(1.3) 'pass through' the singularity of f at a point of the interval $(0,T)$.

The paper is organized as follows. In Section 2 we define functions Λ_+, Φ_+, Λ_- and Φ_- by (2.6)-(2.9) and present some of their important properties. By these functions we prove existence and uniqueness results for w-solutions of problem (1.1)-(1.3) in Section 3 (Theorems 3.1 and 3.2). Section 4 is devoted to the study of existence and multiplicity results for solutions of problem (1.1)-(1.3) (Theorem 4.5). In Section 5 we first give the notion of an exceptional n-sign-changing w-solution x of problem $(1.1), (1.2), (5.1)$ which changes its sign exactly n times on $[0,T]$ and on each maximum subinterval of $[0,T]$ where x keeping its sign the function $|x|$ has the same maximum value. Existence and multiplicity results for the n-sign-changing w-solution of problem $(1.1), (1.2), (5.1)$ are given in Theorem 5.7.

2. Lemmas, notation

Let $0 \le a < b \le T$. In our consideration we will work with the following auxiliary boundary conditions

$$x(a) = x(b) = 0, \quad x(t) > 0 \quad \text{for } t \in (a,b), \tag{2.1}$$

$$x(a) = x(b) = 0, \quad x(t) < 0 \quad \text{for } t \in (a,b) \tag{2.2}$$

and we will use the function $H : \mathbb{R} \to \mathbb{R}$ defined by

$$H(u) = \int_0^u r(s)\,ds \tag{2.3}$$

with r occurring in (1.1) and satisfying assumption (H_1). Clearly, $H \in C^1(\mathbb{R})$ is increasing on $\mathbb{R}$ and the inverse function to H denoted by H^{-1} is increasing on $\mathbb{R}$.

We say that x is *a solution of problem* (1.1), (j), j $\in \{2.1,\ 2.2\}$ if $x \in C^1([a,b])$, $r(x)x' \in C^1((a,b))$, x satisfies the boundary conditions (j) and (1.1) is fulfilled for $t \in (a,b)$.

Remark 2.1. Let the function $\tilde{q} : (0,T) \to (-\infty, 0)$ and $\tilde{f} : [0,T] \times D \to \mathbb{R}$ be defined by

$$\tilde{q}(t) = q(T-t), \quad \tilde{f}(t,x) = f(T-t,x).$$

Then

$$0 < \tilde{f}(t,x)\operatorname{sign} x < g(x), \quad (t,x) \in [0,T] \times D$$

and assumptions (H_2)-(H_4) are satisfied with $\tilde{q}$ and $\tilde{f}$ instead of q and f. If we consider the differential equation

$$(r(x(t))x'(t))' = \tilde{q}(t)\tilde{f}(t,x(t)), \tag{2.4}$$

we see that a function x is a solution of problem (1.1), (j) with $a = 0$, $b = c\,(< T)$ and j$\in \{2.1,\ 2.2\}$ if the function $\tilde{x}(t) = x(T-t)$, $t \in [T-c, T]$, is a solution of problem (2.4), (j) with $a = T-c$, $b = T$. Conversely, if $\tilde{x}$ is a solution of problem (2.4), (j) with $a = T - c\,(> 0)$, $b = T$ and j$\in \{2.1, 2.2\}$, then the function $x(t) = \tilde{x}(T-t)$, $t \in [0,c]$, is a solution of problem (1.1), (j) with $a = 0$, $b = c$.

Remark 2.2. Let $r^* : \mathbb{R} \to [r_0, \infty)$, $f_* : [0,T] \times D \to \mathbb{R}$ and $g^* : D \to \mathbb{R}$ be defined by the formulas (see [14])

$$r^*(x) = r(-x), \quad f^*(t,x) = -f(t,-x), \quad g^*(x) = g(-x).$$

Then

$$0 < f^*(t,x)\operatorname{sign} x \le g^*(x), \quad (t,x) \in [0,T] \times D$$

and assumptions $(H_1) - (H_4)$ are satisfied with r^*, f^* and g^* instead of r, f and g. It is easily seen that a function x is a solution of problem (1.1), (j), j$\in \{2.1,\ 2.2\}$, if and only if $x^* = -x$ on $[a,b]$ is a solution of problem (2.5), (j), where

$$(r^*(x(t))x'(t))' = q(t)f^*(t,x(t)). \tag{2.5}$$

Lemma 2.3. Let assumptions $(H_1)-(H_3)$ be satisfied. Then for each a, $b \in [0,T]$, $a < b$, there exists a unique solution of problem (1.1), (2.1).

Proof. The assertion of our lemma follows from Theorem 2.1 in [14] with $\mu = 1$. □

Corollary 2.4. Under assumptions of Lemma 2.3, for each a, $b \in [0,T]$, $a < b$, there exists a unique solution of problem (1.1), (2.2).

Proof. Fix $0 \le a < b \le T$. Since assumptions $(H_1)-(H_3)$ are satisfied with the functions r^*, f^* and g^* defined in Remark 2.2 instead of r, f and g, problem (2.5), (2.1) has a unique solution $\tilde{x}$ by Lemma 2.3. Now the function $x = -\tilde{x}$ on $[a,b]$ is the unique solution of problem (1.1), (2.2). □

For each $\alpha \in (0,T]$ and $\beta \in [0,T)$, we denote throughout this paper by u_α and v_β the unique solution of problem (1.1), (2.1) with $a = 0$, $b = \alpha$ and $a = \beta$, $b = T$, respectively. Next by $\overline{u}_\alpha$ and $\overline{v}_\beta$ we denote the unique solution of problem (1.1), (2.2) with $a = 0$, $b = \alpha$ and $a = \beta$, $b = T$, respectively. The existence and uniqueness of u_α, v_β and $\overline{u}_\alpha$, $\overline{v}_\beta$ follow from Lemma 2.3 and Corollary 2.4, respectively.

Lemma 2.5. (Lemma 2.7 in [14]. Let assumptions $(H_1)-(H_3)$ be satisfied and let $0 < \alpha_1 < \alpha_2 \le T$. Then

$$u_{\alpha_1}(t) \le u_{\alpha_2}(t) \quad for\ t \in [0,\alpha_1].$$

By the solutions u_α, v_β, $\overline{u}_\alpha$ and $\overline{v}_\beta$ define the functions $\Lambda_+ : (0,T] \to (0,\infty)$, $\Phi_+ : [0,T) \to (0,\infty)$, $\Lambda_- : (0,T] \to (-\infty,0)$, $\Phi_- : [0,T) \to (-\infty,0)$ by the formulas

$$\Lambda_+(\alpha) = \max\{u_\alpha(t) : 0 \le t \le \alpha\}, \tag{2.6}$$

$$\Phi_+(\beta) = \max\{v_\beta(t) : \beta \le t \le T\}, \tag{2.7}$$

$$\Lambda_-(\alpha) = \min\{\overline{u}_\alpha(t) : 0 \le t \le \alpha\} \tag{2.8}$$

and

$$\Phi_-(\beta) = \min\{\overline{v}_\beta(t) : \beta \le t \le T\}. \tag{2.9}$$

Properties of the functions Λ_+, Φ_+, Λ_- and Φ_- are presented in the following lemmas.

Lemma 2.6. Let assumptions $(H_1)-(H_3)$ be satisfied. Then Λ_+ is continuous nondecreasing on $(0,T]$ and

$$\lim_{\alpha \to 0^+} \Lambda_+(\alpha) = 0.$$

Proof. As a direct consequence of Lemma 2.5 we get that Λ_+ is nondecreasing on $(0,T]$. Suppose that Λ_+ is discontinuous on the right at a point $\alpha_0 \in (0,T)$, i.e. there is a decreasing sequence $\{\alpha_n\} \subset (\alpha_0, T)$ such that $\lim_{n\to\infty} \alpha_n = \alpha_0$ and

$$\lim_{n\to\infty} \Lambda_+(\alpha_n) > \Lambda_+(\alpha_0). \tag{2.10}$$

Consider the sequence $\{u_{\alpha_n}\}$. Since $(r(u_{\alpha_n}(t))u'_{\alpha_n}(t))' = q(t)f(t,u'_{\alpha_n}(t)) < 0$ for $t \in (0,\alpha_n)$, $r(u_{\alpha_n})u'_{\alpha_n}$ is decreasing on $[0,\alpha_n]$ and therefore there exists a (unique) $\xi_n \in (0,\alpha_n)$ such that $u'_{\alpha_n} > 0$ on $[0,\xi_n)$, $u'_{\alpha_n} < 0$ on $(\xi_n,\alpha_n]$ and $u'_{\alpha_n}(\xi_n) = 0$. Integrating the inequalities

$$\begin{aligned}(r(u_{\alpha_n}(t))u'_{\alpha_n}(t))' r(u_{\alpha_n}(t))u'_{\alpha_n}(t)\\ \geq -Qg(u_{\alpha_n}(t))r(u_{\alpha_n}(t))u'_{\alpha_n}(t), \quad t \in (0,\xi_n)\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}(r(u_{\alpha_n}(t))u'_{\alpha_n}(t))' r(u_{\alpha_n}(t))u'_{\alpha_n}(t)\\ \leq -Qg(u_{\alpha_n}(t))r(u_{\alpha_n}(t))u'_{\alpha_n}(t), \quad t \in (\xi_n,\alpha_n)\end{aligned} \tag{2.12}$$

over $[0,\xi_n]$ and $[\xi_n,\alpha_n]$, we obtain

$$(r(0)u'_{\alpha_n}(0))^2 \leq 2Q\int_0^{\Lambda_+(\alpha_n)} g(s)r(s)\,ds \leq 2Q\int_0^{\Lambda_+(\alpha_1)} g(s)r(s)\,ds$$

and

$$(r(0)u'_{\alpha_n}(\alpha_n))^2 \leq 2Q\int_0^{\Lambda_+(\alpha_n)} g(s)r(s)\,ds \leq 2Q\int_0^{\Lambda_+(\alpha_1)} g(s)r(s)\,ds,$$

respectively. Hence

$$\begin{aligned}|r(u_{\alpha_n}(t))u'_{\alpha_n}(t)| &\leq r(0)\max\{u'_{\alpha_n}(0), |u'_{\alpha_n}(\alpha_n)|\}\\ &\leq \sqrt{2Q\int_0^{\Lambda_+(\alpha_1)} g(s)r(s)\,ds}, \quad t \in [0,\alpha_n],\ n \in \mathbb{N}\end{aligned} \tag{2.13}$$

and

$$|u'_{\alpha_n}(t)| \leq \frac{1}{r_0}\sqrt{2Q\int_0^{\Lambda_+(\alpha_1)} g(s)r(s)\,ds} \quad \text{for } t \in [0,\alpha_n],\ n \in \mathbb{N}. \tag{2.14}$$

In addition, by Lemma 2.5,

$$u_{\alpha_0}(t) \leq u_{\alpha_n}(t) \leq u_{\alpha_{n+1}}(t) \quad \text{for } t \in [0,\alpha_0],\ n \in \mathbb{N}. \tag{2.15}$$

From (2.14) and (2.15) we deduce that $\{u_{\alpha_n}(t)\}$ is uniformly convergent on $[0,\alpha_0]$ and let $\lim_{n\to\infty} u_{\alpha_n}(t) = u(t)$, $t \in [0,\alpha_0]$. Then $u \in C^0([0,\alpha_0])$, $u(0) = 0$, $u(t) \geq u_{\alpha_0}(t) > 0$ for $t \in (0,\alpha_0)$. Moreover, $u(\alpha_0) = 0$, since in the case that $u(\alpha_0) > 0$ it may be concluded from

$$u(\alpha_0) \leq u_{\alpha_n}(\alpha_0) = u_{\alpha_n}(\alpha_0) - u_{\alpha_n}(\alpha_n) = u'_{\alpha_n}(\eta_n)(\alpha_0 - \alpha_n),$$

where $\eta_n \in (\alpha_0,\alpha_n)$ that

$$\lim_{n\to\infty} u'_{\alpha_n}(\eta_n) \leq \lim_{n\to\infty} \frac{u(\alpha_0)}{\alpha_0 - \alpha_n} = -\infty,$$

contrary to (2.14). As

$$0 < f(t, u_{\alpha_{n+1}}(t)) \le f(t, u_{\alpha_n}(t)), \quad \lim_{n\to\infty} f(t, u_{\alpha_n}(t) = f(t, u(t)), \quad t \in (0, \alpha_0)$$

and (see (2.13))

$$\begin{aligned} 0 \quad &> \quad \int_0^{\alpha_0} q(t)f(t, u_{\alpha_n}(t))\,dt = r(u_{\alpha_n}(\alpha_0))u'_{\alpha_n}(\alpha_0) - r(0)u'_{\alpha_n}(0) \\ &\ge \quad -2\sqrt{2Q\int_0^{\Lambda_+(\alpha_1)} g(s)r(s)\,ds} \end{aligned}$$

for $n \in \mathbb{N}$, Fatou's and Levi's theorems give $q(\cdot)f(\cdot, u(\cdot)) \in L_1([0, \alpha_0])$ and

$$\lim_{n\to\infty} \int_0^t q(s)f(s, u_{\alpha_n}(s)\,ds = \int_0^t q(s)f(s, u(s)\,ds, \quad t \in [0, \alpha_0].$$

By (2.14), $\{u'_{\alpha_n}(0)\}$ is bounded and we may assume that it is convergent. Let $\lim_{n\to\infty} u'_{\alpha_n}(0) = A$. Letting $n \to \infty$ in

$$H(u_{\alpha_n}(t)) = r(0)u'_{\alpha_n}(0)t + \int_0^t \int_0^s q(v)f(v, u_{\alpha_n}(v))\,dv\,ds \quad \text{for } t \in [0, \alpha_0], \qquad (2.16)$$

where H is given by (2.3), we get

$$H(u(t)) = r(0)At + \int_0^t \int_0^s q(v)f(v, u(v))\,dv\,ds, \quad t \in [0, \alpha_0].$$

Then

$$u(t) = H^{-1}\Big(r(0)At + \int_0^t \int_0^s q(v)f(v, u(v)\,dv\,ds\Big),$$

and so $u \in C^1([0, \alpha_0])$. Now from $r(u(t))u'(t) = r(0)A + \int_0^t q(s)f(s, u(s))\,ds$, $t \in [0, \alpha_0]$, and the above proved properties of u, we see that u is a solution of problem (1.1), (2.1) with $a = 0$ and $b = \alpha_0$, and consequently $u = u_{\alpha_0}$ by Lemma 2.3. We have proved that $\lim_{n\to\infty} u_{\alpha_n}(t) = u_{\alpha_0}(t)$ uniformly on $[0, \alpha_0]$, which implies $\lim_{n\to\infty} \Lambda_+(\alpha_n) = \Lambda_+(\alpha_0)$, contrary to (2.10). Hence Λ_+ is continuous on the right on $(0, T)$.

Assume now that Λ_+ is discontinuous on the left at a point $\alpha_0 \in (0, T]$, i.e., there is an increasing sequence $\{\alpha_n\} \subset (0, \alpha_0)$ such that $\lim_{n\to\infty} \alpha_n = \alpha_0$ and

$$\lim_{n\to\infty} \Lambda_+(\alpha_n) < \Lambda_+(\alpha_0). \qquad (2.17)$$

Then

$$u_{\alpha_n}(t) \le u_{\alpha_{n+1}}(t) \le u_{\alpha_0}(t) \quad \text{for } t \in [0, \alpha_n],\ n \in \mathbb{N} \qquad (2.18)$$

by Lemma 2.5 and as above it can be verified that

$$
\begin{aligned}
|r(u_{\alpha_n}(t))u'_{\alpha_n}(t)| &\le \sqrt{2Q\int_0^{\Lambda_+(\alpha_0)} g(s)r(s)\,ds}, \\
|u'_{\alpha_n}(t)| &\le \frac{1}{r_0}\sqrt{2Q\int_0^{\Lambda_+(\alpha_0)} g(s)r(s)\,ds}
\end{aligned}
\tag{2.19}
$$

and

$$
0 \ge \int_0^t q(s)f(s,u_{\alpha_n}(s))\,ds \ge -2\sqrt{2Q\int_0^{\Lambda_+(\alpha_0)} g(s)r(s)\,ds} \tag{2.20}
$$

for $t \in [0,\alpha_n]$ and $n \in \mathbb{N}$. By (2.18) and (2.19), $\{u_{\alpha_n}(t)\}$ is locally uniformly convergent on $[0,\alpha_0)$ and let $\lim_{n\to\infty} u_{\alpha_n}(t) = u(t)$, $t \in [0,\alpha_0)$. Then $u \in C^0([0,\alpha_0))$, $u(0)=0$, $0 < u(t) \le u_{\alpha_0}(t)$ for $t \in [0,\alpha_0)$ and $\lim_{t\to\alpha_0^-} u(t) = 0$. From the inequalities $f(t,u_{\alpha_n}(t)) \ge f(t,u_{\alpha_{n+1}}(t)) \ge f(t,u_{\alpha_0}(t)) \ge 0$, (2.20), $\lim_{n\to\infty} f(t,u_{\alpha_n}(t)) = f(t,u(t))$ for $t \in (0,\alpha_0)$ and Fatou's theorem we obtain $q(\cdot)f(\cdot,u(\cdot)) \in L_1([0,\alpha_0])$. Define $u^* \in C^0([0,\alpha_0])$ by

$$
u^*(t) = \begin{cases} u(t) & \text{for } t \in [0,\alpha_0) \\ 0 & \text{for } t = \alpha_0. \end{cases}
$$

Without violating generality, we can assume that $\{u'_{\alpha_n}(0)\}$ is convergent and let $\lim_{n\to\infty} u'_{\alpha_n}(0) = B$. Taking the limit as $n \to \infty$ in (2.16) which now holds on $[0,\alpha_n]$, we obtain

$$
H(u^*(t)) = r(0)Bt + \int_0^t \int_0^s q(v)f(v,u^*(v))\,dv\,ds \quad \text{for } t \in [0,\alpha_0].
$$

Then $u^* \in C^1([0,\alpha_0])$ and u^* is a solution of problem (1.1), (2.1) with $a=0$ and $b=\alpha_0$. Hence $u^* = u_{\alpha_0}$ and from $\lim_{n\to\infty} u_{\alpha_n}(t) = u_{\alpha_0}(t)$ locally uniformly on $[0,\alpha_0)$ we deduce that $\lim_{n\to\infty}\Lambda_+(\alpha_n) = \Lambda_+(\alpha_0)$, contrary to (2.17). It follows that Λ_+ is continuous on the left on $(0,T]$. Consequently, Λ_+ is continuous on $(0,T]$.

Finally, assume that $\lim_{\alpha\to 0^+}\Lambda_+(\alpha) = \mu > 0$. Let $\{\alpha_n\} \subset (0,T)$ be a decreasing sequence and $\lim_{n\to\infty}\alpha_n = 0$. Let $\Lambda_+(\alpha_n) = u_{\alpha_n}(\xi_n)$ with a $\xi_n \in (0,\alpha_n)$. Then $u_{\alpha_n}(\xi_n) \ge \mu$ and from $\mu \le u_{\alpha_n}(\xi_n) = u_{\alpha_n}(\xi_n) - u_{\alpha_n}(0) = u'_{\alpha_n}(\tau_n)\xi_n$, where $\tau_n \in (0,\xi_n)$, we have $u'_{\alpha_n}(\tau_n) \ge \mu/\xi_n$ for $n \in \mathbb{N}$. Therefore $\lim_{n\to\infty} u'_{\alpha_n}(\tau_n) \ge \lim_{n\to\infty}\mu/\xi_n = \infty$, contrary to (2.14). Hence $\lim_{\alpha\to 0^+}\Lambda_+(\alpha) = 0$. □

Lemma 2.7. Let assumptions $(H_1)-(H_3)$ be satisfied. Then Φ_+ is continuous nonincreasing on $[0,T)$ and

$$
\lim_{\beta\to T^-}\Phi_+(\beta) = 0.
$$

Proof. By Remark 2.1, for each $\gamma \in (0,T]$ the function $\tilde{u}_\gamma(t) = v_\gamma(T-t)$, $t \in [0,T-\gamma]$, is a (unique) solution of problem (2.4), (2.1) with $a=0$ and $b=T-\gamma$. Set $\tilde{\Lambda}_+(\gamma) = \max\{\tilde{u}_\gamma(t) : 0 \le t \le \gamma\}$ for $\gamma \in (0,T]$. Applying Lemma 2.6 to equation (2.4), we see that $\tilde{\Lambda}_+$ is continuous and nondecreasing on $(0,T]$ and $\lim_{\gamma\to 0^+}\tilde{\Lambda}_+(\gamma) =$

0. The assertion of our lemma now follows from the equality $\Phi_+(\beta) = \tilde{\Lambda}_+(T-\beta)$ for $\beta \in [0,T)$. □

Lemma 2.8. Let assumptions $(H_1)-(H_3)$ be satisfied. Then Λ_- is continuous nonincreasing on $(0,T]$, Φ_- is continuous nondecreasing on $[0,T)$ and

$$\lim_{\alpha\to 0^+} \Lambda_-(\alpha) = \lim_{\beta\to T^-} \Phi_-(\beta) = 0.$$

Proof. Let $\tilde{\Lambda}_+$ and $\tilde{\Phi}_+$ be associated to problem (2.5), (2.1) analogously as Λ_+ and Φ_+ are to problem (1.1), (2.1). Then $\tilde{\Lambda}_+$ is continuous and nondecreasing on $(0,T]$, $\tilde{\Phi}_+$ is continuous and nonincreasing on $[0,T)$ and $\lim_{\alpha\to 0^+}\tilde{\Lambda}_+(\alpha) = \lim_{\beta\to T^-}\tilde{\Phi}_+(\beta) = 0$ by Lemmas 2.6 and 2.7. The assertions of the lemma follow immediately from the equalities $\Lambda_- = -\tilde{\Lambda}_+$ on $(0,T]$ and $\Phi_- = -\tilde{\Phi}_+$ on $[0,T)$ which we get applying Remark 2.2. □

Lemma 2.9. Let assumptions $(H_1)-(H_4)$ be satisfied. Then for each $\alpha_1,\ \alpha_2 \in (0,T]$, $\alpha_1 < \alpha_2$, the inequality

$$u_{\alpha_1}(t) < u_{\alpha_2}(t) \quad \textit{for } t \in (0,\alpha_1] \tag{2.21}$$

holds.

Proof. Fix $0 < \alpha_1 < \alpha_2 \le T$. Then $0 = u_{\alpha_1}(\alpha_1) < u_{\alpha_2}(\alpha_1)$ and, by Lemma 2.5, $u_{\alpha_1}(t) \le u_{\alpha_2}(t)$ for $t \in (0,\alpha_1]$. If $u_{\alpha_1}(\xi) = u_{\alpha_2}(\xi)$ for some $\xi \in (0,\alpha_1)$, then $u'_{\alpha_1}(\xi) = u'_{\alpha_2}(\xi)$, and consequently $u_{\alpha_1} = u_{\alpha_2}$ in a neighbourhood of $t=\xi$ by assumption (H_4). Repeated application of this result enables us to prove that $u_{\alpha_1} = u_{\alpha_2}$ on $[0,\alpha_1)$, which is impossible. Hence (2.21) holds. □

Lemma 2.10. Under assumptions $(H_1)-(H_4)$, Λ_+ is increasing on $(0,T]$, Φ_+ is decreasing on $[0,T)$, Λ_- is decreasing on $(0,T]$ and Φ_- is increasing on $[0,T)$.

Proof. By Lemma 2.9, for each $0 < \alpha_1 < \alpha_2 \le T$, inequality (2.21) holds and from the definition of Λ_+ we have $\Lambda_+(\alpha_1) < \Lambda_+(\alpha_2)$. Hence Λ_+ is increasing on $(0,T]$. The other three assertions of the lemma can be verified from strict inequalities between solutions v_{α_1}, v_{α_2}; $\overline{u}_{\alpha_1}$, $\overline{u}_{\alpha_2}$ and $\overline{v}_{\alpha_1}$, $\overline{v}_{\alpha_2}$ with different α_1 and α_2. □

3. Existence results for w-solutions of problem (1.1)-(1.3)

Theorem 3.1. Let assumptions $(H_1)-(H_3)$ be satisfied. Then for each $\lambda \in (0,\infty)$ there exist at least two w-solutions of problem (1.1)-(1.3).

Proof. Fix $\lambda \in (0,\infty)$. By Lemmas 2.6 and 2.8, the function $\lambda\Lambda_+ + \Phi_-$ is continuous and nondecreasing on $(0,T)$ and $\lim_{\alpha\to 0^+}(\lambda\Lambda_+(\alpha) + \Phi_-(\alpha)) = \Phi_-(0) < 0$,

$\lim_{\alpha\to T^-}(\lambda\Lambda_+(\alpha)+\Phi_-(\alpha)) = \lambda\Lambda_+(T) > 0$. Hence the equation $\lambda\Lambda_+(\alpha)+\Phi_-(\alpha) = 0$ has at least one solution $\alpha_1 \in (0,T)$. Setting

$$x_1(t) = \begin{cases} u_{\alpha_1}(t) & \text{for } t \in [0,\alpha_1] \\ \overline{v}_{\alpha_1}(t) & \text{for } t \in (\alpha_1,T], \end{cases} \tag{3.1}$$

x_1 is a w-solution of problem (1.1)-(1.3). Analogously, the equation $\lambda\Phi_+(\alpha)+\Lambda_-(\alpha) = 0$ has at least one solution $\alpha_2 \in (0,T)$ since $\lambda\Phi_+ + \Lambda_-$ is continuous and nonincreasing on $(0,T)$ and $\lim_{\alpha\to 0^+}(\lambda\Phi_+(\alpha)+\Lambda_-(\alpha)) = \lambda\Phi_+(0) > 0$, $\lim_{\alpha\to T^-}(\lambda\Phi_+(\alpha)+\Lambda_-(\alpha)) = \Lambda_-(T) < 0$ by Lemmas 2.7 and 2.8. Then setting

$$x_2(t) = \begin{cases} \overline{u}_{\alpha_2}(t) & \text{for } t \in [0,\alpha_2] \\ v_{\alpha_2}(t) & \text{for } t \in (\alpha_2,T], \end{cases} \tag{3.2}$$

x_2 is the second w-solution of problem (1.1)-(1.3). From $x_1 > 0$ on $(0,\alpha_1)$ and $x_2 < 0$ on $(0,\alpha_2)$ we see that $x_1 \neq x_2$. □

Theorem 3.2. Let assumptions $(H_1) - (H_4)$ be satisfied. Then for each $\lambda \in (0,\infty)$ there exist precisely two w-solutions of problem (1.1)-(1.3).

Proof. Fix $\lambda \in (0,\infty)$. It follows from Lemma 2.10 and the properties of the functions Λ_+, Φ_+, Λ_- and Φ_- given in Lemmas 2.6–2.8 that the equations $\lambda\Lambda_+(\alpha) + \Phi_-(\alpha) = 0$ and $\lambda\Phi_+(\alpha) + \Lambda_-(\alpha) = 0$ have in $(0,T)$ the unique solutions α_1 and α_2, respectively. Now x_1 and x_2 defined by (3.1) and (3.2) are unique w-solutions of problem (1.1)-(1.3). □

4. Existence results for solutions of problem (1.1)–(1.3)

Let assumptions $(H_1) - (H_4)$ be satisfied. By Theorem 3.2, for each $\lambda \in (0,\infty)$ there exist precisely two w-solutions $x_1(t;\lambda)$ and $x_2(t;\lambda)$ of problem (1.1)-(1.3). If c_λ is the (unique) solution of the equation $\lambda\Lambda_+(c) + \Phi_-(c) = 0$ and α_λ is the (unique) solution of the equation $\lambda\Phi_+(\alpha) + \Lambda_-(\alpha) = 0$, then

$$x_1(t;\lambda) = \begin{cases} u_{c_\lambda}(t) & \text{for } t \in [0,c_\lambda] \\ \overline{v}_{c_\lambda}(t) & \text{for } t \in (c_\lambda,T] \end{cases}$$

and

$$x_2(t;\lambda) = \begin{cases} \overline{u}_{\alpha_\lambda}(t) & \text{for } t \in [0,\alpha_\lambda] \\ v_{\alpha_\lambda}(t) & \text{for } t \in (\alpha_\lambda,T]. \end{cases}$$

Here solutions u_α, v_β, $\overline{u}_\alpha$ and $\overline{v}_\beta$ were defined in Section 2. Of course,

$$\lambda \max\{u_{c_\lambda}(t) : 0 \le t \le c_\lambda\} = -\min\{\overline{v}_{c_\lambda} : c_\lambda \le t \le T\},$$
$$\lambda \max\{v_{\alpha_\lambda}(t) : \alpha_\lambda \le t \le T\} = -\min\{\overline{u}_{\alpha_\lambda} : 0 \le t \le \alpha_\lambda\}$$

and c_λ (resp. α_λ) is the (unique) zero of $x_1(t;\lambda)$ (resp. $x_2(t;\lambda)$) in $(0,T)$.

Lemma 4.1. Let assumptions $(H_1) - (H_4)$ be satisfied. Then *a)* c_λ is continuous and decreasing on $(0,\infty)$,

$$\lim_{\lambda\to\infty} c_\lambda = 0, \quad \lim_{\lambda\to 0^+} c_\lambda = T,$$

b) $u_{c_{\lambda_1}}(t) > u_{c_{\lambda_2}}(t)$ for $t \in (0, c_{\lambda_2}]$ and $0 < \lambda_1 < \lambda_2$.

Proof. We know (see the proof of Theorem 3.2) that c_λ is the (unique) solution of the equation $\lambda\Lambda_+(c)+\Phi_-(c) = 0$. Hence the equality $\lambda\Lambda_+(c_\lambda)+\Phi_-(c_\lambda) = 0$ holds for $\lambda \in (0,\infty)$. Let $0 < \lambda_1 < \lambda_2$. If $c_{\lambda_1} \le c_{\lambda_2}$, then from the properties of the functions Λ_+ and Φ_- given in Lemmas 2.6–2.10 it follows that $0 = \lambda_1\Lambda_+(c_{\lambda_1}) + \Phi_-(c_{\lambda_1}) < \lambda_2\Lambda_+(c_{\lambda_2})+\Phi_-(c_{\lambda_2})$, contrary to $\lambda_2\Lambda_+(c_{\lambda_2})+\Phi_-(c_{\lambda_2}) = 0$. Therefore c_λ is decreasing on $(0,\infty)$.

Assume that c_λ is discontinuous at a point $\lambda_0 \in (0,\infty)$. Then there is a sequence $\{\lambda_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \lambda_n = \lambda_0$ such that $\lim_{n\to\infty} c_{\lambda_n} = \mu_0 \ne c_{\lambda_0}$. Letting $n \to \infty$ in the equalities $\lambda_n\Lambda_+(c_{\lambda_n}) + \Phi_-(c_{\lambda_n}) = 0$, $n \in \mathbb{N}$, we get

$$\lambda_0\Lambda_+(\mu_0) + \Phi_-(\mu_0) = 0 \tag{4.1}$$

since Λ_+ and Φ_- are continuous. But the equation $\lambda_0\Lambda_+(c) + \Phi_-(c) = 0$ has the unique solution $c = c_{\lambda_0}$, contrary to (4.1). Hence c_λ is continuous on $(0,\infty)$.

Suppose $\lim_{\lambda\to\infty} c_\lambda = \mu > 0$. Then $\Lambda_+(c_\lambda) \ge \Lambda_+(\mu) > 0$ and $\Phi_-(c_\lambda) \le \Phi_-(\mu) < 0$ for $\lambda \in (0,\infty)$, and so $\lim_{\lambda\to\infty}(\lambda\Lambda_+(c_\lambda) + \Phi_-(c_\lambda)) = \infty$, contrary to

$$\lambda\Lambda_+(c_\lambda) + \Phi_-(c_\lambda)) = 0 \quad \text{for } \lambda \in (0,\infty). \tag{4.2}$$

Therefore $\lim_{\lambda\to\infty} c_\lambda = 0$. If $\lim_{\lambda\to 0^+} c_\lambda = \varrho < T$, then $\lim_{\lambda\to 0^+} \Lambda_+(c_\lambda) = \Lambda_+(\varrho) > 0$, $\lim_{\lambda\to 0^+} \Phi_-(c_\lambda) = \Phi_-(\varrho) < 0$, and so $\lim_{\lambda\to 0^+}(\lambda\Lambda_+(c_\lambda) + \Phi_-(c_\lambda)) = \Phi_-(\varrho) < 0$, contrary to (4.2). Hence $\lim_{\lambda\to 0^+} c_\lambda = T$.

Finally, if $0 < \lambda_1 < \lambda_2$, then $c_{\lambda_1} > c_{\lambda_2}$ and $u_{c_{\lambda_1}}(t) > u_{c_{\lambda_2}}(t)$ for $t \in (0, c_{\lambda_2}]$ by Lemma 2.9. □

Lemma 4.2. Let assumptions $(H_1) - (H_4)$ be satisfied and let $\{\lambda_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \lambda_n = \lambda_0 > 0$. Then

$$\lim_{n\to\infty} u_{c_{\lambda_n}}(t) = u_{c_{\lambda_0}}(t) \quad \textit{locally uniformly on } [0, c_{\lambda_0}).$$

Proof. First from (2.14) it follows that

$$|u'_{c_{\lambda_n}}(t)| \le \frac{1}{r_0}\sqrt{2Q\int_0^{\Lambda_+(T)} g(s)r(s)\,ds} \quad \text{for } t \in [0, c_{\lambda_n}],\ n \in \mathbb{N}. \tag{4.3}$$

Now from (4.3) and using the fact that for $\{\lambda_n\}$ decreasing, $\{c_{\lambda_n}\}$ is increasing and

$$u_{c_{\lambda_{n+1}}}(t) > u_{c_{\lambda_n}}(t), \quad t \in [0, c_{\lambda_n}],\ n \in \mathbb{N}$$

and for $\{\lambda_n\}$ increasing, $\{c_{\lambda_n}\}$ is decreasing and

$$u_{c_{\lambda_{n+1}}}(t) < u_{c_{\lambda_n}}(t), \quad t \in [0, c_{\lambda_{n+1}}],\ n \in \mathbb{N}$$

(see Lemma 4.1), we deduce the assertion of our lemma. □

Define the function $S_+ : (0,\infty) \to (-\infty, 0]$ by the formula

$$S_+(\lambda) = u'_{c_\lambda}(c_\lambda),$$

where $u'_{c_\lambda}(c_\lambda)$ denotes the derivative of $u_{c_\lambda}(t)$ on the left at the point $t = c_\lambda$.

Lemma 4.3. Let assumptions $(H_1) - (H_4)$ be satisfied. Then S_+ is continuous on $(0,\infty)$ and

$$\lim_{\lambda\to\infty} S_+(\lambda) = 0, \quad \limsup_{\lambda\to 0^+} S_+(\lambda) < 0.$$

Proof. Assume, on the contrary, that S_+ is discontinuous at a point $\lambda_0 \in (0,\infty)$. Then there exist $\varepsilon_0 > 0$ and a sequence $\{\lambda_n\} \subset (\lambda_0/2, 2\lambda_0)$, $\lim_{n\to\infty} \lambda_n = \lambda_0$ such that $|S_+(\lambda_n) - S_+(\lambda_0)| \geq \varepsilon_0$ for $n \in \mathbb{N}$, that is

$$|u'_{c_{\lambda_n}}(c_{\lambda_n}) - u'_{c_{\lambda_0}}(c_{\lambda_0})| \geq \varepsilon_0 \quad \text{for } n \in \mathbb{N}. \tag{4.4}$$

We claim that there exists $\nu > 0$ such that

$$|u'_{c_{\lambda_n}}(t) - u'_{c_{\lambda_n}}(c_{\lambda_n})| < \frac{\varepsilon_0}{2} \quad \text{for } t \in [c_{\lambda_n} - \nu, c_{\lambda_n}],\ n \in \mathbb{N}. \tag{4.5}$$

If not, without restriction of generality we can assume that there is a sequence $\{\tau_n\} \subset (0,T)$, $\tau_n < c_{\lambda_n}$, $\lim_{n\to\infty}(\tau_n - c_{\lambda_n}) = 0$ such that

$$|u'_{c_{\lambda_n}}(\tau_n) - u'_{c_{\lambda_n}}(c_{\lambda_n})| = \frac{\varepsilon_0}{2} \quad \text{for } n \in \mathbb{N}. \tag{4.6}$$

If $u'_{c_{\lambda_n}}(\tau_n) \leq 0$, then $r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t) \leq 0$ for $t \in [\tau_n, c_{\lambda_n}]$ and integrating the inequality

$$(r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t))' r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t) \leq -Qg(u_{c_{\lambda_n}}(t))r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t) \tag{4.7}$$

from τ_n to c_{λ_n} we get

$$\begin{aligned} &(0 \leq)\, (r(0)u'_{c_{\lambda_n}}(c_{\lambda_n}))^2 - (r(u_{c_{\lambda_n}}(\tau_n))u'_{c_{\lambda_n}}(\tau_n))^2 \\ &\leq -2Q \int_{u_{c_{\lambda_n}}(\tau_n)}^{0} g(s)r(s)\,ds = 2Q \int_0^{u_{c_{\lambda_n}}(\tau_n)} g(s)r(s)\,ds. \end{aligned} \tag{4.8}$$

If $u'_{c_{\lambda_n}}(\tau_n) > 0$, then there exists $\xi_n \in (\tau_n, c_{\lambda_n})$ such that $u'_{c_{\lambda_n}}(\xi_n) = 0$ and then integrating the inequality

$$(r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t))' r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t) \geq -Qg(u_{c_{\lambda_n}}(t))r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t)$$

from τ_n to ξ_n and inequality (4.7) from ξ_n to c_{λ_n}, we have

$$(r(u_{c_{\lambda_n}}(\tau_n))u'_{c_{\lambda_n}}(\tau_n))^2 \leq 2Q \int_{u_{c_{\lambda_n}}(\tau_n)}^{u_{c_{\lambda_n}}(\xi_n)} g(s)r(s)\,ds \tag{4.9}$$

and

$$(r(0)u'_{c_{\lambda_n}}(c_{\lambda_n}))^2 \leq 2Q \int_0^{u_{c_{\lambda_n}}(\xi_n)} g(s)r(s)\,ds. \tag{4.10}$$

Let $\mathbb{N}_+$ be the set of all $n \in \mathbb{N}$ such that $u'_{c_{\lambda_n}}(\tau_n) > 0$. Assume $\mathbb{N}_+$ is infinite. Using (4.3) and the equalities $\lim_{n\to\infty}(\xi_n - c_{\lambda_n}) = 0$, $\lim_{n\to\infty}(\xi_n - \tau_n) = 0$, we have

$$\lim_{n\in\mathbb{N}_+,\, n\to\infty} (u_{c_{\lambda_n}}(\xi_n) - u_{c_{\lambda_n}}(\tau_n)) = 0,$$

$$\lim_{n\in\mathbb{N}_+,\, n\to\infty} u_{c_{\lambda_n}}(\xi_n) = \lim_{n\in\mathbb{N}_+,\, n\to\infty} (u_{c_{\lambda_n}}(\xi_n) - u_{c_{\lambda_n}}(c_{\lambda_n})) = 0,$$

and so (4.9), (4.10) and $r(x) \ge r_0 > 0$ for $x \in \mathbb{R}$ yield

$$\lim_{n\in\mathbb{N}_+,\, n\to\infty} u'_{c_{\lambda_n}}(\tau_n) = \lim_{n\in\mathbb{N}_+,\, n\to\infty} u'_{c_{\lambda_n}}(c_{\lambda_n}) = 0,$$

contrary to (4.6). Hence $\mathbb{N}_+$ is finite and there is no loss of generality in assuming $u'_{c_{\lambda_n}}(\tau_n) \le 0$ for $n \in \mathbb{N}$ and then (see (4.8))

$$(0 \le)\, (r(0)u'_{c_{\lambda_n}}(c_{\lambda_n}))^2 - (r(u_{c_{\lambda_n}}(\tau_n))u'_{c_{\lambda_n}}(\tau_n))^2 \le 2Q \int_0^{u_{c_{\lambda_n}}(\tau_n)} g(s)r(s)\, ds \qquad (4.11)$$

for $n \in \mathbb{N}$. From Lemma 1.2 in [14] (with $\mu = 1$) it follows that

$$u'_{c_{\lambda_n}}(c_{\lambda_n}) \le -\frac{2K}{Vc_{\lambda_0/2}} \quad \text{for } n \in \mathbb{N}\,, \qquad (4.12)$$

where

$$V = \max\Big\{r(x) : 0 \le x \le \max\{u_T(t) : 0 \le t \le T\}\Big\}, \qquad (4.13)$$

$$K = \min\Big[\min\Big\{\int_0^{t/2} s|q(s)|k(s)\, ds, \int_{t/2}^{t} (t-s)|q(s)|k(s)\, ds\Big\} : \frac{\lambda_0}{2} \le t \le 2\lambda_0\Big]$$

and $k \in C^0([0,T])$ is a positive function such that

$$0 < k(t) \le f(t,x)\operatorname{sign} x \quad \text{for } (t,x) \in [0,T] \times ([-\|u_T\|, 0) \cup (0, \|u_T\|]) \qquad (4.14)$$

with $\|u_T\| = \max\{u_T(t) : 0 \le t \le T\}$ (for the function k see Remark 1.3). By (4.3), $\{u'_{c_{\lambda_n}}(c_{\lambda_n})\}$ and $\{u'_{c_{\lambda_n}}(\tau_n)\}$ are bounded, and so going if necessary to subsequences, we can assume that they are convergent, say

$$\lim_{n\to\infty} u'_{c_{\lambda_n}}(c_{\lambda_n}) = A, \quad \lim_{n\to\infty} u'_{c_{\lambda_n}}(\tau_n) = B.$$

By virtue of (4.6), we have

$$|A - B| = \frac{\varepsilon_0}{2}. \qquad (4.15)$$

In addition, $\lim_{n\to\infty} u_{c_{\lambda_n}}(\tau_n) = \lim_{n\to\infty}(u_{c_{\lambda_n}}(\tau_n) - u_{c_{\lambda_n}}(c_{\lambda_n})) = 0$ since (4.3) holds and $\lim_{n\to\infty}(\tau_n - c_{\lambda_n}) = 0$. Letting $n \to \infty$ in (4.11), we get $0 \le (r(0))^2(A^2 - B^2) = 0$. Therefore $A^2 - B^2 = 0$ and since $A \le -2K/(Vc_{\lambda_0/2})$ by (4.12) and $B \le 0$, we have $A = B$, contrary to (4.15). We have proved that (4.5) holds. Let

$$|u'_{c_{\lambda_0}}(t) - u'_{c_{\lambda_0}}(c_{\lambda_0})| \le \frac{\varepsilon_0}{4} \quad \text{for } t \in [c_{\lambda_0} - \gamma, c_{\lambda_0}]\,, \qquad (4.16)$$

where γ is a positive constant. Set $\kappa = \min\{\nu, \gamma\}$. Using (4.4) and (4.5) we have

$$|u'_{c_{\lambda_n}}(t) - u'_{c_{\lambda_0}}(c_{\lambda_0})| \ge |u'_{c_{\lambda_n}}(c_{\lambda_n}) - u'_{c_{\lambda_0}}(c_{\lambda_0})| - |u'_{c_{\lambda_n}}(t) - u'_{c_{\lambda_n}}(c_{\lambda_n})| > \frac{\varepsilon_0}{2}$$

for $t \in [c_{\lambda_n} - \kappa, c_{\lambda_n}]$ and $n \in \mathbb{N}$. Assume that

$$\mathbb{N}^* = \Big\{ n : n \in \mathbb{N},\ u'_{c_{\lambda_n}}(t) - u'_{c_{\lambda_0}}(c_{\lambda_0}) > \frac{\varepsilon_0}{2} \ \text{for } t \in [c_{\lambda_n} - \kappa, c_{\lambda_n}] \Big\}$$

is an infinite set (analogously for $\mathbb{N} \setminus \mathbb{N}^*$ infinite). Then

$$u_{c_{\lambda_n}}(t) = \int_{c_{\lambda_n}}^t u'_{c_{\lambda_n}}(s)\,ds \le \Big(u'_{c_{\lambda_0}}(c_{\lambda_0}) + \frac{\varepsilon_0}{2} \Big)(t - c_{\lambda_n}) \tag{4.17}$$

for $t \in [c_{\lambda_n} - \kappa, c_{\lambda_n}]$ and $n \in \mathbb{N}^*$. On the other hand, (4.16) gives

$$\begin{aligned} &\Big(u'_{c_{\lambda_0}}(c_{\lambda_0}) + \frac{\varepsilon_0}{4} \Big)(t - c_{\lambda_0}) \le u_{c_{\lambda_0}}(t) \\ &= \int_{c_{\lambda_0}}^t u'_{c_{\lambda_0}}(s)\,ds \le \Big(u'_{c_{\lambda_0}}(c_{\lambda_0}) - \frac{\varepsilon_0}{4} \Big)(t - c_{\lambda_0}) \end{aligned} \tag{4.18}$$

for $t \in [c_{\lambda_0} - \kappa, c_{\lambda_0})$. Since $\lim_{n\to\infty}(c_{\lambda_n} - \kappa) = c_{\lambda_0} - \kappa$, there exists $n_0 \in \mathbb{N}$ such that for $n \in \mathbb{N}^*$, $n \ge n_0$, we have $c_{\lambda_n} - \kappa \le c_{\lambda_0} - \kappa/2$ and then letting $n \to \infty$ in (4.17) and using Lemmas 4.1 and 4.2,

$$u_{c_{\lambda_0}}(t) \le \Big(u'_{c_{\lambda_0}}(c_{\lambda_0}) + \frac{\varepsilon_0}{2} \Big)(t - c_{\lambda_0}), \quad t \in \Big[c_{\lambda_0} - \frac{\kappa}{2}, c_{\lambda_0} \Big],$$

contrary to (4.18). We have proved that S_+ is continuous on $(0, \infty)$.

Let $\{\lambda_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \lambda_n = \infty$. Then $\lim_{n\to\infty} c_{\lambda_n} = 0$ by Lemma 4.1, and there exists $\{\xi_n\}$, $0 < \xi_n < c_{\lambda_n}$, such that $u'_{c_{\lambda_n}}(\xi_n) = 0$ and $r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t) < 0$ on $(\xi_n, c_{\lambda_n}]$ and $n \in \mathbb{N}$. Integrating (4.7) from ξ_n to c_{λ_n} we get

$$(r(0)u'_{c_{\lambda_n}}(c_{\lambda_n}))^2 \le 2Q \int_0^{u_{c_{\lambda_n}}(\xi_n)} g(s)r(s)\,ds \quad \text{for } n \in \mathbb{N}. \tag{4.19}$$

By Lemma 1.2 in [14] (with $\mu = 1$), $u_{c_{\lambda_n}}(t) \le L_n$ for $t \in [0, c_{\lambda_n}]$ where $L_n > 0$ is an arbitrary constant satisfying the inequality

$$1 \le \frac{2\Big(\int_0^{L_n} r(s)\,ds \Big)^2}{(c_{\lambda_n})^2 Q \int_0^{L_n} g(s)r(s)\,ds}.$$

From the last inequality we see that L_n can be chosen such that $\lim_{n\to\infty} L_n = 0$ and then (4.19) yields $\lim_{n\to\infty} u'_{c_{\lambda_n}}(c_{\lambda_n}) = 0$. Hence $\lim_{\lambda\to\infty} S_+(\lambda) = 0$.

Let $\{\lambda_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \lambda_n = 0$. Then $\lim_{n\to\infty} c_{\lambda_n} = T$ by Lemma 4.1, and from Lemma 1.2 in [14] (with $\mu = 1$) we deduce that for each $n \in \mathbb{N}$ such that $c_{\lambda_n} \ge T/2$,

$$u'_{c_{\lambda_n}}(c_{\lambda_n}) \le -\frac{2K_1}{VT}, \quad n \in \mathbb{N},$$

where V is given by (4.13) and

$$K_1 = \min \Big[\min \Big\{ \int_0^{t/2} s|q(s)|k(s)\,ds,\ \int_{t/2}^t (t-s)|q(s)|k(s)\,ds \Big\} : \frac{T}{2} \le t \le T \Big]$$

with $k \in C^0([0,T])$ satisfying (4.14). Hence $\limsup_{\lambda\to 0^+} S_+(\lambda) \le -2K_1/(VT)$. □

Define the functions $S_- : (0,\infty) \to (0,\infty)$, $Z_+ : (0,\infty) \to (0,\infty)$ and $Z_- : (0,\infty) \to (-\infty,0)$ by the formulas

$$\begin{aligned} S_-(\lambda) &= \overline{u}'_{\alpha_\lambda}(\alpha_\lambda),\\ Z_+(\lambda) &= v'_{c_\lambda}(\alpha_\lambda),\\ Z_-(\lambda) &= \overline{v}'_{\alpha_\lambda}(c_\lambda). \end{aligned}$$

We observe that c_λ (resp. α_λ) is the (unique) solution of the equation $\lambda\Lambda_+(c) + \Phi_-(c) = 0$ (resp. $\lambda\Phi_+(\alpha) + \Lambda_-(\alpha) = 0$). From the properties of the functions Λ_+, Λ_-, Φ_+, Φ_-, using Remarks 2.1 and 2.2 and applying procedures analogical to those in the proofs of Lemmas 4.1–4.3, we can show properties of the functions S_-, Z_+ and Z_- which are given in the following lemma.

Lemma 4.4. Let assumptions $(H_1) - (H_4)$ be satisfied. Then the functions S_-, Z_+ and Z_- are continuous on $(0,\infty)$ and

$$\begin{aligned} &\liminf_{\lambda\to 0^+} S_-(\lambda) > 0, \quad \lim_{\lambda\to\infty} S_-(\lambda) = 0,\\ &\lim_{\lambda\to 0^+} Z_+(\lambda) = 0, \quad \liminf_{\lambda\to\infty} Z_+(\lambda) > 0,\\ &\lim_{\lambda\to 0^+} Z_-(\lambda) = 0, \quad \limsup_{\lambda\to\infty} Z_-(\lambda) < 0. \end{aligned}$$

Theorem 4.5. Let assumptions $(H_1) - (H_4)$ be satisfied. Then problem (1.1)-(1.3) has at least two solutions.

Proof. Define the function k, $p : (0,\infty) \to \mathbb{R}$ by

$$k(\lambda) = S_+(\lambda) - Z_-(\lambda), \quad p(\lambda) = S_-(\lambda) - Z_+(\lambda).$$

By Lemmas 4.3 and 4.4, the functions k and p are continuous on $(0,\infty)$ and

$$\begin{aligned} &\limsup_{\lambda\to 0^+} k(\lambda) < 0, \quad \liminf_{\lambda\to\infty} k(\lambda) > 0,\\ &\liminf_{\lambda\to 0^+} p(\lambda) > 0, \quad \limsup_{\lambda\to\infty} p(\lambda) < 0. \end{aligned}$$

Hence there exist λ_1, $\lambda_2 \in (0,\infty)$ such that $k(\lambda_1) = 0$ and $p(\lambda_2) = 0$, that is $S_+(\lambda_1) = Z_-(\lambda_1)$ and $S_-(\lambda_2) = Z_+(\lambda_2)$. Then the functions

$$x_1(t) = \begin{cases} u_{c_1}(t) & \text{for } t \in [0, c_1]\\ \overline{v}_{c_1}(t) & \text{for } t \in (c_1, T] \end{cases}$$

and

$$x_2(t) = \begin{cases} \overline{u}_{\alpha_1}(t) & \text{for } t \in [0, \alpha_1]\\ v_{\alpha_1}(t) & \text{for } t \in (\alpha_1, T] \end{cases}$$

are solutions of problem (1.1)-(1.3), where c_1 (resp. α_1) is the (unique) solution of the equation $\lambda_1\Lambda_+(c) + \Phi_-(c) = 0$ (resp. $\lambda_2\Phi_+(\alpha) + \Lambda(\alpha) = 0$). Clearly, $x_1 \neq x_2$. □

5. Exceptional n-sign-changing w-solutions of problem (1.1), (1.2), (5.1)

Let $c \in (0, T]$. In this Section we will use the following conditions

$$\max\{x(t) : 0 \le t \le T\} = -\min\{x(t) : 0 \le t \le T\}, \tag{5.1}$$

$$x(0) = 0, \quad x(c) = 0 \tag{5.2}$$

and

$$\max\{x(t) : 0 \le t \le c\} = -\min\{x(t) : 0 \le t \le c\}. \tag{5.3}$$

We note that (5.1) is (1.3) with $\lambda = 1$.

Definition 5.1. Let $n \in \mathbb{N}$, $n \ge 2$. We say that x is an *n-sign-changing w-solution of problem* (1.1), (5.2), (5.3) if x has precisely $n-1$ zeros $t_1 < t_2 < \cdots < t_{n-1}$ in $(0, c)$, $x \in C^0([0, c]) \cap C^1([0, c] \setminus \{t_1, t_2, \cdots, t_{n-1}\})$, there exist finite $\lim_{t \to t_i^-} x'(t)$, $\lim_{t \to t_i^+} x'(t)$ for $i = 1, 2, \cdots, n-1$, $r(x)x' \in C^1([0, c] \setminus \{t_1, t_2, \cdots, t_{n-1}\})$, x satisfies (5.2), equality (1.1) holds on $(0, c) \setminus \{t_1, t_2, \cdots, t_{n-1}\}$ and finally

$$\max\{x(t) : t_i \le t \le t_{i+2}\} \min\{x(t) : t_i \le t \le t_{i+2}\} < 0$$

for $i = 0, 1, \cdots, n-2$ with $t_0 = 0$ and $t_n = c$.

If, in addition,

$$\max\{|x(t)| : 0 \le t \le t_1\} = \max\{|x(t)| : t_j \le t \le t_{j+1}\}$$

for $j = 1, 2, \cdots, n-1$, we say that x is an *exceptional n-sign-changing w-solution of problem* (1.1), (5.2), (5.3). In case of $c = T$, x is called an *exceptional n-sign-changing w-solution of problem* (1.1), (1.2), (5.1).

Remark 5.2. We observe, that the notion of the w-solution of problem (1.1)-(1.3) with $\lambda = 1$ stated in Section 1 corresponds to the notion of exceptional 2-sign-changing w-solution of problem (1.1), (1.2), (5.1).

Before we give existence results for exceptional n-sign-changing w-solutions of problem (1.1), (1.2), (5.1), we will define a function Δ_+ whose properties are important in our next considerations.

Let assumptions $(H_1) - (H_4)$ be satisfied. Then, by Theorem 3.2 (with $\mu = 1$) and its proof, for each $c \in (0, T]$ there exists the unique w-solution of problem (1.1), (5.2), (5.3), which is positive in the right neighbourhood of $t = 0$. This solution we will denote throughout this Section by w_c. Using w_c we define the function $\Delta_+ : (0, T] \to (0, \infty)$ by

$$\Delta_+(c) = \max\{w_c(t) : 0 \le t \le c\}.$$

Lemma 5.3. Let assumptions $(H_1) - (H_4)$ be satisfied. Then Δ_+ is continuous and increasing on $(0, T]$ and

$$\lim_{c \to 0^+} \Delta_+(c) = 0. \tag{5.4}$$

Proof. To prove that Δ_+ is increasing on $(0, T]$ we assume, on the contrary, that $\Delta_+(a) \geq \Delta_+(b)$ for some $0 < a < b \leq T$. Let $w_a(t_a) = 0$ and $w_b(t_b) = 0$ with unique $t_a \in (0, a)$ and $t_b \in (0, b)$. From Lemma 2.9 and our assumption $\Delta_+(a) \geq \Delta_+(b)$ we deduce that $t_a \geq t_b$. We claim that

$$w_b(t) < w_a(t) \quad \text{for } t \in (t_a, a]. \tag{5.5}$$

If not, since $0 = w_a(t_a) \geq w_b(t_a)$ and $0 = w_a(a) > w_b(a)$ we have either $w_a(\xi) = w_b(\xi)$ for some $\xi \in (t_a, a)$ and $w_b(t) \leq w_a(t)$ for $t \in [t_a, a]$ or there exist $t_a \leq \nu < \tau < a$ such that $w_a(\nu) = w_b(\nu)$, $w_a(\tau) = w_b(\tau)$ and $w_b(t) > w_a(t)$ for $t \in (\nu, \tau)$. In the first case $w_a'(\xi) = w_b'(\xi)$ and $w_a = w_b$ in a neighbourhood of $t = \xi$ by (H_4), and then by repeated application of (H_4) we get $w_a = w_b$ on (t_a, a), which is impossible. In the second case, we have $r(w_a(\nu))w_a'(\nu) \leq r(w_b(\nu))w_b'(\nu)$ and $f(t, w_a(t)) \geq f(t, w_b(t))$ for $t \in (\nu, \tau]$. Hence

$$\Big(\int_{w_a(t)}^{w_b(t)} r(s)\, ds \Big)'' = q(t)(f(t, w_b(t)) - f(t, w_a(t))) \geq 0, \quad t \in (\nu, \tau],$$

and so $\Big(\int_{w_a(t)}^{w_b(t)} r(s)\, ds \Big)'$ is nondecreasing on $[\nu, \tau]$ and then the equalities $w_a(\nu) = w_b(\nu)$, $w_a(\tau) = w_b(\tau)$ imply $\int_{w_a(t)}^{w_b(t)} r(s)\, ds = 0$ for $t \in [\nu, \tau]$, contrary to $w_b > w_a$ on (ν, τ). Now (5.5) yields $\min\{w_b(t) : 0 \leq t \leq b\} < \min\{w_a(t) : 0 \leq t \leq a\}$, hence $\Delta_+(b) > \Delta_+(a)$, contrary to our assumption $\Delta_+(a) \geq \Delta_+(b)$. We have proved that Δ_+ is increasing on $(0, T]$.

Suppose that Δ_+ is discontinuous on the right at a point $c_0 \in (0, T)$, i.e., there is a decreasing sequence $\{c_n\} \subset (c_0, T)$ such that $\lim_{n\to\infty} c_n = c_0$ and

$$\lim_{n \to \infty} \Delta_+(c_n) = \mu > \Delta_+(c_0). \tag{5.6}$$

Let $w_{c_n}(t_n) = 0$ for the (unique) $t_n \in (0, c_n)$, $n \in \mathbb{N} \cup \{0\}$. Since $\mu < \Delta_+(c_{n+1}) < \Delta_+(c_n)$ for $n \in \mathbb{N}$, Lemma 2.9 shows that $t_0 < t_{n+1} < t_n$ for $n \in \mathbb{N}$. There is no loss of generality in assuming $t_1 < c_0$. Moreover, $\Delta_+(c_n) = \Lambda_+(t_n)$ for $n \in \mathbb{N} \cup \{0\}$ and from $\Delta_+(c_0) < \mu < \Lambda_+(t_n)$ and the continuity of Λ_+ by Lemma 2.6, we see that $\lim_{n\to\infty} t_n = t_* > t_0$. Applying the procedure as in the proof of Lemma 2.6 (now on $[t_n, c_n]$), we get

$$|r(w_{c_n}(t))w_{c_n}'(t)| \leq \sqrt{2Q \int_0^{\Lambda_+(t_n)} g(s)r(s)\, ds} \leq \sqrt{2Q \int_0^{\Lambda_+(t_1)} g(s)r(s)\, ds} \tag{5.7}$$

for $t \in [t_n, c_n]$, $n \in \mathbb{N}$, and then

$$0 \le \int_{t_n}^{t} q(s) f(s, w_{c_n}(s))\, ds = r(w_{c_n}(t)) w'_{c_n}(t) - r(0) w'_{c_n}(t_n) \le 2\sqrt{2Q \int_0^{\Lambda_+(t_1)} g(s) r(s)\, ds} \tag{5.8}$$

for $t \in [t_n, c_n]$ and $n \in \mathbb{N}$. By (5.7),

$$|w'_{c_n}(t)| \le S \quad \text{for } t \in [t_n, c_n],\ n \in \mathbb{N}, \tag{5.9}$$

where

$$S = \frac{1}{r_0} \sqrt{2Q \int_0^{\Lambda_+(t_1)} g(s) r(s)\, ds}. \tag{5.10}$$

From (5.9), $0 \ge w_{c_n}(t) \ge -\Lambda_+(t_1)$ for $t \in [t_n, c_n]$, $n \in \mathbb{N}$, and the Arzelà–Ascoli theorem, we deduce that there exists a subsequence of $\{w_{c_n}\}$, which we denote by $\{w_{c_n}\}$ again, such that $\lim_{n\to\infty} w_{c_n}(t) = w(t)$ locally uniformly on $(t_*, c_0]$. Then $w \in C^0((t_*, c_0])$, $w \le 0$ on $(t_*, c_0]$ and $w(c_0) = 0$ since in the case that $w(c_0) < 0$ from the relations

$$\frac{w(c_0)}{2} \ge w_{c_n}(c_0) = w_{c_n}(c_0) - w_{c_n}(c_n) = w'_{c_n}(\xi_n)(c_0 - c_n)\,,$$

which are satisfied for sufficiently large n and where $\xi_n \in (c_0, c_n)$, we obtain

$$\lim_{n\to\infty} w'_{c_n}(\xi_n) \ge \lim_{n\to\infty} \frac{w(c_0)}{2(c_0 - c_n)} = \infty,$$

contrary to (5.9). We are going to show that $w(t) < 0$ on (t_*, c_0) and

$$\lim_{t\to t_*^+} w(t) = 0. \tag{5.11}$$

By (H_3) (see Remark 1.3), there exists a positive function $k \in C^0([0,T])$ such that

$$k(t) \le f(t,x) \operatorname{sign} x \qquad \text{for } (t,x) \in [0,T] \times [-\Delta_+(c_1), 0) \cup (0, \Delta_+(c_1)].$$

Now using our Remark 2.2 and Lemma 1.2 in [14] with $\mu = 1$, we get (for $n \in \mathbb{N}$)

$$w_{c_n}(t) \le \begin{cases} H^{-1}\Big(-\dfrac{2K_n(t-t_n)}{c_n - t_n}\Big) & \text{for } t \in \Big[t_n, \dfrac{t_n + c_n}{2}\Big] \\ H^{-1}\Big(-\dfrac{2K_n(c_n - t)}{c_n - t_n}\Big) & \text{for } t \in \Big(\dfrac{t_n + c_n}{2}, c_n\Big], \end{cases} \tag{5.12}$$

where

$$K_n = \min\Big\{ \int_{t_n}^{(t_n+c_n)/2} (s - t_n)|q(s)| k(s)\, ds,\ \int_{(t_n+c_n)/2}^{c_n} (c_n - s)|q(s)| k(s)\, ds \Big\}$$

and H^{-1} is the inverse to H given by (2.3). Let $t_n \le (3t_* + c_0)/4$ and $t_n + c_n \le (t_* + 3c_0)/2$ for $n \ge n_1$ with an $n_1 \in \mathbb{N}$. Then (for $n \ge n_1$)

$$\int_{t_n}^{(t_n+c_n)/2} (s - t_n)|q(s)| k(s)\, ds \ge \int_{(3t_*+c_0)/4}^{(t_*+c_0)/2} \Big(s - \frac{3t_* + c_0}{4}\Big)|q(s)| k(s)\, ds,$$

$$\int_{(t_n+c_n)/2}^{c_n}(c_n-s)|q(s)|k(s)\,ds \ge \int_{(t_*+3c_0)/4}^{c_0}(c_0-s)|q(s)|k(s)\,ds$$

and from (5.12) it follows that

$$w_{c_n}(t) \le \begin{cases} H^{-1}\Big(-\dfrac{2K(t-t_n)}{c_n-t_n}\Big) & \text{for } t\in\Big[t_n,\dfrac{t_n+c_n}{2}\Big] \\ H^{-1}\Big(-\dfrac{2K(c_n-t)}{c_n-t_n}\Big) & \text{for } t\in\Big(\dfrac{t_n+c_n}{2},c_n\Big], \end{cases}$$

where

$$K=\min\Big\{\int_{(3t_*+c_0)/4}^{(t_*+c_0)/2}\Big(s-\frac{3t_*+c_0}{4}\Big)|q(s)|k(s)\,ds,\ \int_{(t_*+3c_0)/4}^{c_0}(c_0-s)|q(s)|k(s)\,ds\Big\}.$$

Consequently,

$$w(t)=\lim_{n\to\infty}w_{c_n}(t)\le\begin{cases} H^{-1}\Big(-\dfrac{2K(t-t_*)}{c_0-t_*}\Big)<0 & \text{for } t\in\Big(t_*,\dfrac{t_*+c_0}{2}\Big] \\ H^{-1}\Big(-\dfrac{2K(c_0-t)}{c_0-t_*}\Big)<0 & \text{for } t\in\Big(\dfrac{t_*+c_0}{2},c_0\Big), \end{cases}$$

and we see that $w<0$ on (t_*,c_0). If (5.11) is not true, then there exist $\delta<0$ and a decreasing sequence $\{\nu_n\}\subset(t_*,c_0)$ such that $\lim_{n\to\infty}\nu_n=t_*$ and $w(\nu_n)\le\delta$ for $n\in\mathbb{N}$. Now let

$$\nu_{n_*}<t_*-\frac{r_0\delta}{4\sqrt{2Q\int_0^{\Lambda_+(t_1)}g(s)r(s)\,ds}}$$

for some $n_*\in\mathbb{N}$. Then there exists $n_2\in\mathbb{N}$ such that $w_{c_n}(\nu_{n_*})<\delta/2$ and

$$\frac{\delta}{2}>w_{c_n}(\nu_{n_*})=w_{c_n}(\nu_{n_*})-w_{c_n}(t_n)=w'_{c_n}(\varphi_n)(\nu_{n_*}-t_n)$$

for $n\ge n_2$, where $\varphi_n\in(t_n,\nu_{n_*})$. Therefore

$$w'_{c_n}(\varphi_n)<\frac{\delta}{2(\nu_{n_*}-t_n)}<\frac{\delta}{2(\nu_{n_*}-t_*)}<-\frac{2}{r_0}\sqrt{2Q\int_0^{\Lambda_+(t_1)}g(s)r(s)\,ds}$$

for $n\ge n_2$, contrary to (5.9).

Define $w_*:[t_*,c_0]\to(-\infty,0]$ by

$$w_*(t)=\begin{cases} w(t) & \text{for } t\in(t_*,c_0] \\ 0 & \text{for } t=t_*. \end{cases}$$

Then $w_*\in C^0([t_*,c_0])$, $w_*(t_*)=w_*(c_0)=0$ and $w_*<0$ for $t\in(t_*,c_0)$. We are now in a position to show that w_* is a solution of problem (1.1), (2.2) with $a=t_*$ and $b=c_0$. For this, we define for each $n\in\mathbb{N}$ the function $p_n:[t_*,c_0]\to(-\infty,0]$ by

$$p_n(t)=\begin{cases} f(t,w_{c_n}(t)) & \text{for } t\in(t_n,c_0] \\ 0 & \text{for } t\in[t_*,t_n]. \end{cases}$$

Since

$$0 \le \int_{t_*}^{c_0} q(t)p_n(t)\,dt = \int_{t_n}^{c_0} q(t)f(t,w_{c_n}(t))\,dt \le 2\sqrt{2Q\int_0^{\Lambda_+(t_1)} g(s)r(s)\,ds}$$

by (5.8) and $\lim_{n\to\infty} p_n(t) = f(t,w_*(t))$ for $t \in (t_*,c_0)$, Fatou's theorem gives $q(\cdot)f(\cdot,w_*(\cdot)) \in L_1([t_*,c_0])$. Fix $\beta \in (t_1,c_0)$. Going if necessary to a subsequence, we can assume that $\{r(w_{c_n}(\beta))w'_{c_n}(\beta)\}$ is convergent, $\lim_{n\to\infty} r(w_{c_n}(\beta))w'_{c_n}(\beta) = A$. Letting $n \to \infty$ in

$$\int_{w_{c_n}(\beta)}^{w_{c_n}(t)} r(s)\,ds = r(w_{c_n}(\beta))w'_{c_n}(\beta)(t-\beta) + \int_\beta^t\int_\beta^s q(v)f(v,w_{c_n}(v))\,dv\,ds,\ t \in [t_n,c_n]$$

and using the Lebesgue dominated theorem, we get

$$\int_{w_*(\beta)}^{w_*(t)} r(s)\,ds = A(t-\beta) + \int_\beta^t\int_\beta^s q(v)f(v,w_*(v))\,dv\,ds, \quad t \in [t_*,c_0].$$

Whence

$$w_*(t) = H^{-1}\Big(H(w_*(\beta)) + A(t-\beta) + \int_\beta^t\int_\beta^s q(v)f(v,w_*(v))\,dv\,ds\Big), \quad t \in [t_*,c_0]$$

and we see that $w_* \in C^1([t_*,c_0])$ and $(r(w_*(t))w'_*(t))' = q(t)f(t,w_*(t))$ for $t \in (t_*,c_0)$. Therefore w_* is a solution of problem (1.1), (2.2) with $a = t_*$ and $b = c_0$. Finally, applying Remarks 2.1 and 2.2 to Lemma 2.9, we have $w_{c_0}(t) < w_*(t)$ for $t \in [t_*,c_0)$, and consequently

$$-\Delta_+(c_0) = \min\{w_{c_0}(t) : t_0 \le t \le c_0\} < \min\{w_*(t) : t_* \le t \le c_0\} = -\mu,$$

contrary to (5.6). Hence Δ_+ is continuous on the right on $(0,T)$.

The continuity of Δ_+ on the left on $(0,T]$ can be proved similarly. □

Now, define the 'dual' function $\Delta_- : (0,T] \to (0,\infty)$ to Δ_+ by the formula

$$\Delta_-(c) = \min\{\overline{w}_c(t) : 0 \le t \le c\},$$

where $\overline{w}_c$ is the unique solution of problem (1.1), (5.2), (5.3) such that $\overline{w}_c < 0$ in the right neighbourhood of $t = 0$.

Lemma 5.4. Let assumptions $(H_1) - (H_4)$ be satisfied. Then Δ_- is continuous decreasing on $(0,T]$ and

$$\lim_{c\to 0^+} \Delta_-(c) = 0.$$

Proof. Since the proof of the lemma is similar to that of Lemma 5.3, we will omit it. □

Lemma 5.5. Let assumptions $(H_1) - (H_4)$ be satisfied. Then there exist exactly two exceptional 3-sign-changing w-solutions of problem (1.1), (1.2), (5.1).

Proof. By Lemmas 2.7, 2.10 and 5.3, Φ_+ is continuous decreasing on $[0,T)$, Δ_+ is continuous increasing on $(0,T]$ and $\lim_{c\to T^-}\Phi_+(c)=\lim_{c\to 0^+}\Delta_+(c)=0$. Set $p_+(c)=\Delta_+(c)-\Phi_+(c)$ for $c\in(0,T)$. Then $p_+\in C^0((0,T))$, $\lim_{c\to 0^+}p_+(c)=-\Phi_+(0)<0$, $\lim_{c\to T^-}p_+(c)=\Delta_+(T)>0$ and since p_+ is increasing on $(0,T)$, there is the unique $c_+\in(0,T)$ such that $p_+(c_+)=0$. Hence the function

$$x_+(t)=\begin{cases} w_{c_+}(t) & \text{for } t\in[0,c_+]\\ v_{c_+}(t) & \text{for } t\in(c_+,T]\end{cases}$$

is an exceptional 3-sign-changing w-solution of problem (1.1), (1.2), (5.1), which is positive in the right neighbourhood of $t=0$. Assume that $\overline{x}_+$ is an additional exceptional 3-sign-changing w-solution of problem (1.1), (1.2), (5.1) having positive values in the right neighbourhood of $t=0$ and let $\overline{x}_+(t_j)=0$, $j=1,2$, with $0<t_1<t_2<T$. Since $\overline{x}_+\neq x_+$, it is necessary $t_2\neq c_+$, say $t_2>c_+$. Then $\Delta_+(t_2)-\Phi_+(t_2)>0$ and from this inequality we deduce that

$$\max\{\overline{x}_+(t):0\le t\le t_2\}=\Delta_+(t_2)>\Phi_+(t_2)=\max\{\overline{x}_+(t):t_2\le t\le T\},$$

contrary to the definition of an exceptional 3-sign-changing w-solution of problem (1.1), (1.2), (5.1).

By Lemmas 2.8, 2.10 and 5.4, Φ_- is continuous increasing on $[0,T)$, Δ_- is continuous decreasing on $(0,T]$ and $\lim_{c\to T^-}\Phi_-(c)=\lim_{c\to 0^+}\Delta_-(c)=0$. Set $p_-(c)=\Delta_-(c)-\Phi_-(c)$ for $c\in(0,T)$. Then $p_-\in C^0((0,T))$, $\lim_{c\to 0^+}p_-(c)=-\Phi_-(0)>0$, $\lim_{c\to T^-}p_-(c)=\Delta_-(T)<0$ and since p_- is decreasing on $(0,T)$, there is the unique $c_-\in(0,T)$ such that $p_-(c_-)=0$. The function

$$x_-(t)=\begin{cases} \overline{w}_{c_-}(t) & \text{for } t\in[0,c_-]\\ \overline{v}_{c_-}(t) & \text{for } t\in(c_-,T]\end{cases}$$

is an exceptional 3-sign-changing w-solution of problem in the right neighbourhood of $t=0$ and x_- is the unique exceptional 3-sign-changing w-solution of problem (1.1), (1.2), (5.1) having negative value in the right neighbourhood of $t=0$. □

Remark 5.6. Let assumptions $(H_1)-(H_4)$ be satisfied and let $c\in(0,T]$. Then $(H_1)-(H_4)$ are satisfied with c instead of T, and so there exist exactly two exceptional 3-sign-changing w-solutions of problem (1.1), (5.2), (5.3) by Lemma 5.5.

Theorem 5.7. Let assumptions $(H_1)-(H_4)$ be satisfied. Then for each $n\in\mathbb{N}$, $n\ge 2$, there exist exactly two exceptional n-sign-changing w-solutions of problem $(1.1),(1.2),(5.1)$.

Proof. We proceed by induction. By Theorem 3.2 (with $\lambda=1$) and Lemma 5.5, there exist exactly two exceptional j-sign-changing w-solutions of problem (1.1), $(1.2),(5.1)$, $j=2,3$. In addition (see Remark 5.6), for each $c\in(0,T]$ there exists exactly two exceptional 3-sign-changing w-solutions of problem $(1.1),(5.2),(5.3)$. Assuming that this result holds for $n=k\ge 3$, we will prove it for $n=k+1$. Since the proof is similar to that of Lemma 5.5, we give only its main ideas.

First, we assume that for each $c \in (0,T]$ there exist exactly two exceptional k-sign-changing w-solutions w_{kc} and $\overline{w}_{kc}$ of problem (1.1), (5.2), (5.3) such that w_{kc} is positive and $\overline{w}_{kc}$ is negative in the right neighbourhood of $t=0$. Now define $\Delta_+^k : (0,T] \to (0,\infty)$ and $\Delta_-^k : (0,T] \to (0,\infty)$ by

$$\Delta_+^k(c) = \max\{w_{kc}(t) : 0 \le t \le c\}$$

and

$$\Delta_-^k(c) = \max\{\overline{w}_{kc}(t) : 0 \le t \le c\}.$$

Further, we can proceed as in the proof of Lemmas 5.3 and 5.4 to verify that Δ_+^k and Δ_-^k are continuous increasing on $(0,T]$, $\lim_{c\to 0^+} \Delta_+^k(c) = \lim_{c\to 0^+} \Delta_-^k(c) = 0$. Finally, if k is an even positive integer, set

$$p_k^+(c) = \Delta_+^k(c) - \Phi_+(c), \quad p_k^-(c) = \Delta_-^k(c) + \Phi_-(c) \quad \text{for } c \in (0,T).$$

Since p_k^+ and p_k^- are continuous increasing on $(0,T)$, $\lim_{c\to 0^+} p_k^+(c) = -\Phi_+(0) < 0$, $\lim_{c\to T^-} p_k^+(c) = \Delta_+^k(T) > 0$, $\lim_{c\to 0^+} p_k^-(c) = \Phi_-(0) < 0$ and $\lim_{c\to T^-} p_k^-(c) = \Delta_-^k(T) > 0$, there exists the unique solution c_+ (resp. c_-) of the equation $p_k^+(c)=0$ (resp. $p_k^-(c)=0$). Then

$$x_+(t) = \begin{cases} w_{kc_+}(t) & \text{for } t \in [0,c_+] \\ v_{c_+}(t) & \text{for } t \in (c_+,T], \end{cases}$$

$$x_-(t) = \begin{cases} \overline{w}_{kc_-}(t) & \text{for } t \in [0,c_-] \\ \overline{v}_{c_-}(t) & \text{for } t \in (c_-,T] \end{cases}$$

are the unique two exceptional $(k+1)$-sign-changing w-solutions of problem (1.1), (1.2), (5.1).

If k is an odd positive integer, we now set

$$p_k^+(c) = \Delta_+^k(c) + \Phi_-(c), \quad p_k^-(c) = \Delta_-^k(c) - \Phi_+(c) \quad \text{for } c \in (0,T).$$

Then p_k^+ and p_k^- are continuous increasing on $(0,T)$, the equations $p_k^+(c)=0$ and $p_k^-(c)=0$ have the unique solutions c_+ and c_-, respectively, and

$$x_+(t) = \begin{cases} w_{kc_+}(t) & \text{for } t \in [0,c_+] \\ \overline{v}_{c_+}(t) & \text{for } t \in (c_+,T], \end{cases}$$

and

$$x_-(t) = \begin{cases} \overline{w}_{kc_-}(t) & \text{for } t \in [0,c_-] \\ v_{c_-}(t) & \text{for } t \in (c_-,T] \end{cases}$$

are the unique two exceptional $(k+1)$-sign-changing w-solutions of problem (1.1), (1.2), (5.1). $\square$

Acknowledgement. Supported by grant No. 201/01/1451 of the Grant Agency of the Czech Republic and by the Council of the Czech Government J14/98:153100011

REFERENCES

[1] Agarwal, R. P. and O'Regan, D.: *Singular boundary value problems for superlinear second order ordinary and delay differential equations*, J. Differential Equations, **130**, (1996), 333-355.

[2] Agarwal, R. P. and O'Regan, D.: *Positive solutions to superlinear singular boundary value problems*, J. Comput. Appl. Math., **88**, (1998), 129–147.

[3] Agarwal, R. P. and O'Regan, D.: *Some new results for singular problems with sign changing nonlinearities*, J. Comput. Appl. Math., **113**, (2000), 1–15.

[4] Agarwal, R. P., O'Regan, D. and Lakshmikantham V.: *An upper and lower solution approach for nonlinear singular boundary value problems with y' dependence*, J. Inequal. & Appl., to appear

[5] Agarwal, R. P. and Staněk, S.: *Nonnegative solutions of singular boundary value problems with sign changing nonlinearities*, Math. Modelling and Computation (in press).

[6] Gatica, J. A., Oliker, V. and Waltman, P.: *Singular nonlinear boundary value problems for second order ordinary differential equations*, J. Differential Equations, **79**, (1989), 62–78.

[7] Kannan, R. and O'Regan, D.: *Singular and nonsingular boundary value problems with sign changing nonlinearities*, J. Inequal. & Appl., **5**, (2000), 621–637.

[8] Kiguradze, I. T. and Shekhter, B. L.: *Singular boundary value problems for second order ordinary differential equations*. In "Current Problems in Mathematics: Newest Results", **30**, 105–201, Moscow, 1987 (in Russian); English translation: J. Soviet Math. **43**, (1988), 2340–2417.

[9] Lomtatidze, A.: *Positive solutions of boundary value problems for second order differential equations with singular points*, Differentsial'nye Uravneniya, **23**, (1987), 1685–1692. (in Russian)

[10] Lomtatidze, A. and Torres, P.: *On a two-point boundary value problem for second order singular equations*, Czech. Math. J. (in press).

[11] O'Regan, D.: *Theory of Singular Boundary Value Problems*, World Scientific Publishing, Singapore, 1994.

[12] O'Regan, D.: *Existence principles and theory for singular Dirichlet boundary value problems*, Differential Eqns. Dyn. Systems, **3**, (1995), 289–304.

[13] O'Regan, D. and Agarwal, R. P.: *Singular problems: an upper and lower solution approach*, J. Math. Anal. Appl. **251**, (2000), 230–250.

[14] Rachunková, I. and Staněk, S.: *Sign-changing solutions of singular Dirichlet boundary value problems*, J. Inequal. & Appl. (in press)

[15] Rachunková, I. and Staněk, S.: *Connection between the type of singularity and smoothness of solutions for Dirichlet BVPs*, Dyn. Conti. Discrete Impulsive Syst. (in press)

[16] Staněk, S.: *Positive solutions of singular Dirichlet boundary value problems*, Math. Computer Modelling, **33**, (2001), 341–351.

[17] Staněk, S.: *Positive solutions of singular Dirichlet and periodic boundary value problems*, Computers Math. Appl., **43**, (2002), 681–692.

[18] TALIAFERRO, S. D.: *A nonlinear singular boundary value problem*. Nonlin. Anal., **3**, (1979), 897–904.

[19] TINEO, A.: *Existence theorems for a singular two-point Dirichlet problem*, Nonlin. Anal., **19**, (1992), 323–333.

[20] WANG, J.: *Solvability of singular nonlinear two-point boundary value problems*, Nonlin. Anal., **24**, (1995), 555-561.

[21] WANG, J. and GAO, W.: *A note on singular nonlinear two-point boundary-value problems*, Nonlin. Anal., **39**, (2000), 281–287.

[22] ZENGQUIN, Z.: *Uniqueness of positive solutions for singular nonlinear second-order boundary-value problems*, Nonlin. Anal., **23**, (1994), 755-765.

Doi: 10.18514/MMN.2002.59

Mathematical Notes, Miskolc, Vol. 3., No. 2., (2002), pp. 157–164

SHARP INEQUALITIES INVOLVING THE SYMMETRIC MEAN

Tiberiu Trif
Babeş - Bolyai University, Department of Mathematics
1, M. Kogălniceanu St., 3400 Cluj-Napoca, Romania
ttrif@math.ubbcluj.ro

[Received: September 9, 2001]

Abstract. In this paper we prove several sharp inequalities involving the symmetric mean of two arguments. They refine already known inequalities and relate to Hölder means as well as to some special Gini means.

Mathematical Subject Classification: 26D99
Keywords: means, symmetric mean, inequalities

1. **Introduction and notation**

Given the positive real numbers x and y, let $A(x,y) := \frac{x+y}{2}$, $G(x,y) := \sqrt{xy}$,

$$L(x,y) := \begin{cases} \frac{x-y}{\ln x - \ln y} & \text{if } x \neq y \\ x & \text{if } x = y, \end{cases}$$

and

$$H_p(x,y) := \begin{cases} \left(\frac{x^p+y^p}{2}\right)^{1/p} & \text{if } p \neq 0 \\ \sqrt{xy} & \text{if } p = 0 \end{cases}$$

denote their arithmetic, geometric, logarithmic, and Hölder means, respectively. It is well known that if $x \neq y$, then H_p is strictly increasing in p. The symmetric mean of x and y is defined by (see [4, pp. 44–45])

$$S_\alpha(x,y) := \frac{x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha}{2} \qquad \alpha \in \mathbf{R}.$$

Since $S_\alpha(x,y) = S_{1-\alpha}(x,y)$, it suffices to consider $\alpha \geq 1/2$. Following A. O. Pittenger [8], we write the symmetric mean into the form

$$S_\delta(x,y) := \frac{x^{\frac{1+\sqrt{\delta}}{2}} y^{\frac{1-\sqrt{\delta}}{2}} + x^{\frac{1-\sqrt{\delta}}{2}} y^{\frac{1+\sqrt{\delta}}{2}}}{2} \qquad \delta \geq 0.$$

It is easily seen that if $x \neq y$, then S_δ is strictly increasing in δ, so

$$G(x,y) = S_0(x,y) < S_\delta(x,y) < S_1(x,y) = A(x,y) \tag{1.1}$$

for all $\delta \in \,]0,1[$.

A. O. Pittenger [8, Theorem 1 and Theorem 2] proved that for all positive real numbers $x \neq y$, the symmetric mean satisfies also the following inequalities:

$$S_{1/3}(x,y) < L(x,y), \tag{1.2}$$
$$S_\delta(x,y) < H_\delta(x,y) \qquad \text{if } 0 < \delta < 1, \tag{1.3}$$
$$S_\delta(x,y) > H_\delta(x,y) \qquad \text{if } \delta > 1. \tag{1.4}$$

At this point we present a very short proof of the intriguing inequality (1.2), which is sharp in the sense that 1/3 cannot be replaced by any greater constant. In order to prove (1.2), let us apply the Gauss quadrature formula with two knots (see [1, pp. 343–344], [2, p. 36])

$$\int_0^1 f(t)dt = \frac{1}{2}f\left(\frac{1}{2}+\frac{1}{2\sqrt{3}}\right)+\frac{1}{2}f\left(\frac{1}{2}-\frac{1}{2\sqrt{3}}\right)+\frac{1}{4320}f^{(4)}(\xi) \qquad 0<\xi<1,$$

to the function $f(t) = x^t y^{1-t}$. Since $\int_0^1 f(t)dt = L(x,y)$ (this integral representation of $L(x,y)$ was pointed out for the first time by E. Neuman [6]), we get

$$L(x,y) = S_{1/3}(x,y) + \frac{1}{4320}x^\xi y^{1-\xi}(\ln x - \ln y)^4 \qquad 0<\xi<1.$$

Therefore

$$\frac{\min(x,y)}{4320}(\ln x - \ln y)^4 < L(x,y) - S_{1/3}(x,y) < \frac{\max(x,y)}{4320}(\ln x - \ln y)^4.$$

This inequality yields (1.2) and estimates the sharpness of (1.2).

It is the main purpose of the present paper to point out other sharp inequalities involving the symmetric mean. In Section 2 we are concerned with sharp "interpolating inequalities" related to (1.1) and (1.2). In Section 3 we establish two sharp inequalities involving the symmetric and special Gini means.

2. Sharp interpolating inequalities related to (1.1) and (1.2)

In the next theorem we prove a sharp interpolating inequality related to (1.1):

THEOREM 2.1. *If* $0 < \delta < 1$, *then for all positive real numbers* $x \neq y$ *it holds that*

$$S_\delta(x,y) < \delta A(x,y) + (1-\delta)G(x,y). \tag{2.1}$$

Moreover, (2.1) *is the best possible inequality of the type*

$$S_\delta(x,y) < \lambda A(x,y) + (1-\lambda)G(x,y) \qquad 0<\lambda<1, \tag{2.2}$$

in the sense that if $0 < \lambda < \delta$, *then* (2.2) *cannot be true for all positive real numbers* $x \neq y$.

Proof. Letting $x/y = e^{2t}$ ($t \in \mathbf{R}$) and multiplying both sides by e^{-t}, we see that (2.2) is equivalent to $f_\lambda(t) < 0$ for all $t \neq 0$, where $f_\lambda : \mathbf{R} \to \mathbf{R}$ is the function defined by $f_\lambda(t) := \cosh(\sqrt{\delta}t) - \lambda\cosh t - 1 + \lambda$. By using the power series expansion

$$\cosh z = 1 + \sum_{n=1}^{\infty}\frac{1}{(2n)!}z^{2n}, \tag{2.3}$$

we get

$$f_\lambda(t) = \sum_{n=1}^{\infty} \frac{a_n(\lambda)}{(2n)!} t^{2n}, \qquad \text{where} \qquad a_n(\lambda) := \delta^n - \lambda.$$

If $0 < \lambda < \delta$, then $a_1(\lambda) = \delta - \lambda > 0$, hence $f_\lambda(t) > 0$ for $|t|$ sufficiently small. Therefore, (2.2) cannot be true for all positive real numbers $x \neq y$.

On the other hand, $a_1(\delta) = 0$ and $a_n(\delta) < 0$ for all $n \geq 2$. Consequently, $f_\delta(t) < 0$ for all $t \neq 0$. This proves the validity of (2.1). □

Remark. Obviously, (2.1) is a refinement of the second inequality in (1.1). It would be also interesting to compare (2.1) with (1.3). Thus, for $0 < \delta < 1/2$ the inequality (2.1) is weaker than (1.3), for $\delta = 1/2$ it coincides with (1.3), while for $1/2 < \delta < 1$ it is stronger than (1.3). This assertion is proved by the following theorem.

Theorem 2.2. *For all positive real numbers $x \neq y$ it holds that*

$$\delta A(x,y) + (1-\delta)G(x,y) > H_\delta(x,y) \qquad \text{if} \quad 0 < \delta < 1/2 \tag{2.4}$$

and

$$\delta A(x,y) + (1-\delta)G(x,y) < H_\delta(x,y) \qquad \text{if} \quad 1/2 < \delta < 1. \tag{2.5}$$

Proof. Due to the symmetry, we may assume in both (2.4) and (2.5) that $x > y$. Letting $x/y = e^{2t}$ ($t > 0$) and multiplying both sides by e^{-t}, we see that (2.4) is equivalent to $f_\delta(t) > 1$ for all $t > 0$, where $f_\delta : \mathbf{R} \to \mathbf{R}$ is the function defined by

$$f_\delta(t) := \frac{\delta \cosh t + 1 - \delta}{(\cosh(\delta t))^{1/\delta}}.$$

We have

$$f'_\delta(t) = \frac{g(t)}{(\cosh(\delta t))^{1+1/\delta}}, \qquad \text{where} \qquad g(t) = \delta \sinh((1-\delta)t) - (1-\delta)\sinh(\delta t).$$

By using the power series expansion

$$\sinh z = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} \tag{2.6}$$

we get

$$g(t) = \sum_{n=1}^{\infty} \frac{\delta(1-\delta)\left[(1-\delta)^{2n} - \delta^{2n}\right]}{(2n+1)!} t^{2n+1}.$$

Since $0 < \delta < 1/2$, it follows that $g(t) > 0$ for all $t > 0$, hence f_δ is strictly increasing on $]0, \infty[$. Therefore $f_\delta(t) > 1$ for all $t > 0$, because $f_\delta(0) = 1$. This completes the proof of (2.4). The proof of (2.5) is analogous. □

From (1.1) and (1.2) it follows that $G(x,y) < S_\delta(x,y) < L(x,y)$ for all $0 < \delta \leq 1/3$. Related to this, the following sharp interpolating inequality holds.

Theorem 2.3. *If $0 < \delta \leq 1/3$, then for all positive real numbers $x \neq y$ it holds that*

$$S_\delta(x,y) < 3\delta L(x,y) + (1-3\delta)G(x,y). \tag{2.7}$$

Moreover, (2.7) *is the best possible inequality of the type*

$$S_\delta(x,y) < \lambda L(x,y) + (1-\lambda)G(x,y) \qquad 0 < \lambda \leq 1, \tag{2.8}$$

in the sense that if $0 < \lambda < 3\delta$, then (2.8) *cannot be true for all positive real numbers $x \neq y$.*

Proof. Due to the symmetry, we may assume in both (2.7) and (2.8) that $x > y$. Letting $x/y = e^{2t}$ $(t > 0)$ and multiplying both sides by te^{-t}, we see that (2.8) is equivalent to $f_\lambda(t) < 0$ for all $t > 0$, where $f_\lambda :]0,\infty[\to \mathbf{R}$ is the function defined by $f_\lambda(t) := t\cosh(\sqrt{\delta}t) - \lambda \sinh t - (1-\lambda)t$. From (2.3) and (2.6) we get

$$f_\lambda(t) = \sum_{n=1}^{\infty} \frac{a_n(\lambda)}{(2n+1)!} t^{2n+1}, \qquad \text{where} \qquad a_n(\lambda) := (2n+1)\delta^n - \lambda.$$

If $0 < \lambda < 3\delta$, then $a_1(\lambda) = 3\delta - \lambda > 0$, hence $f_\lambda(t) > 0$ for $t > 0$ sufficiently small. Therefore, (2.8) cannot be true for all positive real numbers $x \neq y$.

On the other hand, $a_1(3\delta) = 0$ and $a_n(3\delta) = \delta[(2n+1)\delta^{n-1} - 3] < 0$ for all $n \geq 2$, because

$$(2n+1)\delta^{n-1} - 3 \leq \frac{2n+1}{3^{n-1}} - 3 < 0 \qquad \text{for all } n \geq 2.$$

The inequality $3^n > 2n+1$ $(n \geq 2)$ can be easily proved by induction. Consequently, $f_{3\delta}(t) < 0$ for all $t > 0$. This proves the validity of (2.7). □

Remark. The inequality (2.7) is a refinement of (2.1), because

$$3\delta L(x,y) + (1-3\delta)G(x,y) < \delta A(x,y) + (1-\delta)G(x,y).$$

Indeed, this inequality is equivalent to

$$L(x,y) < \frac{1}{3}A(x,y) + \frac{2}{3}G(x,y),$$

a well known inequality due to E. B. Leach and M. C. Sholander [5].

From (1.1) and (1.3) it follows that $G(x,y) < S_\delta(x,y) < H_\delta(x,y)$ for all $0 < \delta < 1$. It is not difficult to see that we cannot have a sharp inequality of the type

$$S_\delta(x,y) < \lambda H_\delta(x,y) + (1-\lambda)G(x,y) \qquad 0 < \lambda < 1.$$

3. Sharp inequalities involving the symmetric and special Gini means

Given the positive real numbers x and y, the Lehmer mean $L_r(x,y)$ of x and y is defined for each $r \in \mathbf{R}$ by (see [9])

$$L_r(x,y) := \frac{x^{r+1} + y^{r+1}}{x^r + y^r}.$$

In this section we deal also with the mean

$$M(x,y) := \exp\left(\frac{x\ln x + y\ln y}{x+y}\right) = x^{x/(x+y)}y^{y/(x+y)}.$$

It should be noted that both the Lehmer means and the mean M are special cases of Gini means (see [3], [7]).

In the next theorem a sharp inequality involving the symmetric and Lehmer means is established.

THEOREM 3.1. *If $r > 0$, then for all positive real numbers $x \neq y$ it holds that*

$$S_{2r+1}(x,y) > L_r(x,y). \tag{3.1}$$

Moreover, (3.1) *is the best possible inequality of the type*

$$S_\delta(x,y) > L_r(x,y), \tag{3.2}$$

in the sense that if $0 < \delta < 2r+1$, then (3.2) *cannot be true for all positive real numbers $x \neq y$.*

Proof. Letting $x/y = e^{2t}$ $(t \in \mathbf{R})$, after a little bit of algebra we see that

$$\begin{aligned}
\frac{S_\delta(x,y)}{L_r(x,y)} - 1 &= \frac{\left[\left(\frac{x}{y}\right)^{\frac{1+\sqrt{\delta}}{2}} + \left(\frac{x}{y}\right)^{\frac{1-\sqrt{\delta}}{2}}\right]\left[\left(\frac{x}{y}\right)^{r}+1\right]}{2\left[\left(\frac{x}{y}\right)^{r+1}+1\right]} \\
&= \frac{\cosh(\sqrt{\delta}t)\cosh(rt)}{\cosh((r+1)t)} - 1 \\
&= \frac{f_\delta(t)}{2\cosh((r+1)t)},
\end{aligned}$$

where $f_\delta : \mathbf{R} \to \mathbf{R}$ is the function defined by

$$f_\delta(t) := \cosh((r+\sqrt{\delta})t) + \cosh((r-\sqrt{\delta})t) - 2\cosh((r+1)t).$$

By using the power series expansion (2.3) we get

$$f_\delta(t) = \sum_{n=1}^{\infty} \frac{a_n(\delta)}{(2n)!}t^{2n}, \qquad \text{where} \qquad a_n(\delta) := (r+\sqrt{\delta})^{2n} + (r-\sqrt{\delta})^{2n} - 2(r+1)^{2n}.$$

Suppose first that $0 < \delta < 2r+1$. Since $a_1(\delta) = 2(\delta - 2r - 1)$, it follows that $f_\delta(t) < 0$ for $|t|$ sufficiently small. Consequently, $S_\delta(x,y) < L_r(x,y)$ for $x \neq y$ sufficiently close, hence (3.2) cannot be true for all positive real numbers $x \neq y$.

Next we show that (3.1) holds true. Set, for brevity,

$$a_n = a_n(2r+1) = (r+\sqrt{2r+1})^{2n} + (r-\sqrt{2r+1})^{2n} - 2(r+1)^{2n}$$

for each positive integer n. Then $a_1 = 0$ and, in order to establish the validity of (3.1), it suffices to prove that $a_n > 0$ for every $n \geq 2$.

Set $x_n := (r+\sqrt{2r+1})^{2n} + (r-\sqrt{2r+1})^{2n}$ for each positive integer n. Then the sequence (x_n) satisfies

$$x_{n+1} = 2(r+1)^2 x_n - (r^2-2r-1)^2 x_{n-1} \qquad \text{for every } n \geq 2. \tag{3.3}$$

We claim that

$$(r+1)^2 x_n > (r^2-2r-1)^2 x_{n-1} \tag{3.4}$$

for all $n \geq 2$. Since $x_1 = 2(r+1)^2$ and $x_2 = (r+\sqrt{2r+1})^4 + (r-\sqrt{2r+1})^4$, for $n=2$ the inequality (3.4) is equivalent to

$$(r+\sqrt{2r+1})^4 + (r-\sqrt{2r+1})^4 > 2(r^2-2r-1)^2 = 2(r+\sqrt{2r+1})^2(r-\sqrt{2r+1})^2,$$

which is true because $u^2+v^2 > 2uv$ for $u \neq v$. Assuming that (3.4) holds for $n \geq 2$, we prove that it holds also for $n+1$. By virtue of (3.3) we have

$$\begin{aligned}
&(r+1)^2 x_{n+1} - (r^2-2r-1)^2 x_n \\
&\quad = \left[2(r+1)^4 - (r^2-2r-1)^2\right] x_n - (r+1)^2(r^2-2r-1)^2 x_{n-1} \\
&\quad = (r^4+12r^3+10r^2+4r+1)x_n - (r+1)^2(r^2-2r-1)^2 x_{n-1} \\
&\quad > (r+1)^4 x_n - (r+1)^2(r^2-2r-1)^2 x_{n-1} \\
&\quad > 0.
\end{aligned}$$

Consequently, (3.4) holds true as claimed.

Next, we prove by induction that

$$x_n > 2(r+1)^{2n} \tag{3.5}$$

for all $n \geq 2$. Indeed, $x_2 = 2(r+1)^4 + 16r^3 + 8r^2 > 2(r+1)^4$. Assuming that (3.5) holds for $n \geq 2$, we show that it holds also for $n+1$. By virtue of (3.3) and (3.4) we have

$$x_{n+1} > (r+1)^2 x_n > 2(r+1)^{2n+2}.$$

From (3.5) it follows that $a_n > 0$ for every $n \geq 2$, completing the proof of the theorem. □

REMARKS. 1. The inequality (3.1) is a refinement of $S_{2r+1}(x,y) > H_{2r+1}(x,y)$ (see (1.4)), by virtue of the inequality $L_r(x,y) > H_{2r+1}(x,y)$ (see Z. Páles [7, Theorem 3], K. B. Stolarski [9, Theorem 1]).

2. For $-1/2 < r < 0$ the means S_{2r+1} and L_r are not comparable. Indeed, with the notations in the proof of Theorem 3.1, we have $a_1 = 0$ and $a_2 = 8r^2(2r+1) < 0$, hence $f_{2r+1}(t) < 0$ for $|t|$ sufficiently small. Consequently, $S_{2r+1}(x,y) < L_r(x,y)$ for $x \neq y$ sufficiently close. On the other hand, since $\frac{1+\sqrt{2r+1}}{2} > r+1$, it follows that

$$\frac{S_{2r+1}(x,y)}{L_r(x,y)} = \frac{\left[\left(\frac{x}{y}\right)^{\frac{1+\sqrt{2r+1}}{2}} + \left(\frac{x}{y}\right)^{\frac{1-\sqrt{2r+1}}{2}}\right]\left[\left(\frac{x}{y}\right)^r + 1\right]}{2\left[\left(\frac{x}{y}\right)^{r+1} + 1\right]} > 1$$

for x/y sufficiently large.

Now we prove a sharp inequality involving the means S_δ and M.

THEOREM 3.2. *For all positive real numbers $x \neq y$ it holds that*

$$S_2(x,y) > M(x,y). \tag{3.6}$$

Moreover, (3.6) *is the best possible inequality of the type*

$$S_\delta(x,y) > M(x,y), \tag{3.7}$$

in the sense that if $0 \leq \delta < 2$, *then* (3.7) *cannot be true for all positive real numbers* $x \neq y$.

Proof. Letting $x/y = e^{2t}$ ($t \in \mathbf{R}$) and multiplying both sides by e^{-t}, we see that (3.7) is equivalent to $f_\delta(t) > 0$ for all $t \neq 0$, where $f_\delta : \mathbf{R} \to \mathbf{R}$ is the function defined by $f_\delta(t) := \ln(\cosh(\sqrt{\delta}t)) - t\tanh t$. We have

$$\tanh z = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_{2n} z^{2n-1} \qquad |z| < \frac{\pi}{4}$$

and

$$\ln(\cosh z) = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{2n(2n)!} B_{2n} z^{2n} \qquad |z| < \frac{\pi}{4},$$

where B_k is the kth Bernoulli number. Consequently, for $|t|$ sufficiently small, the inequality $f_\delta(t) > 0$ is equivalent to

$$\sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{2n(2n)!} B_{2n} \delta^n t^{2n} > \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_{2n} t^{2n}.$$

If $0 \leq \delta < 2$, then the last inequality cannot be true for all $t \neq 0$ with $|t|$ sufficiently small, because the coefficient of t^2 in the right side is greater than the coefficient of t^2 in the left side. Therefore, (3.7) cannot be true for all positive real numbers $x \neq y$.

Next we show that (3.6) holds true. We have just seen that (3.6) is equivalent to

$$f(t) := f_2(t) = \ln(\cosh(\sqrt{2}t)) - t\tanh t > 0 \qquad \text{for all } t \neq 0. \tag{3.8}$$

We have

$$f'(t) = \sqrt{2}\,\frac{\sinh(\sqrt{2}t)}{\cosh(\sqrt{2}t)} - \frac{\sinh t}{\cosh t} - \frac{t}{\cosh^2 t} = \frac{g(t)}{\cosh(\sqrt{2}t)\cosh^2 t},$$

where $g(t) = \sqrt{2}\sinh(\sqrt{2}t)\cosh^2 t - \sinh t \cosh t \cosh(\sqrt{2}t) - t\cosh(\sqrt{2}t)$. By means of certain usual hyperbolic identities we obtain

$$g(t) = \frac{\sinh(\sqrt{2}t)}{\sqrt{2}} - t\cosh(\sqrt{2}t) + \frac{2-\sqrt{2}}{4\sqrt{2}}\sinh((2+\sqrt{2})t) - \frac{2+\sqrt{2}}{4\sqrt{2}}\sinh((2-\sqrt{2})t).$$

Taking into account the power series expansions (2.3) and (2.6) we get

$$g(t) = \sum_{n=0}^{\infty} \frac{a_n}{(2n+1)!} t^{2n+1}, \qquad \text{where} \qquad a_n := \frac{(2+\sqrt{2})^{2n} - (2-\sqrt{2})^{2n}}{2\sqrt{2}} - n2^{n+1}.$$

Note that $a_1 = 0$ and that

$$\begin{aligned} a_n &> \frac{(2+\sqrt{2})^{2n}-1}{2\sqrt{2}} - n2^{n+1} > \frac{2^{2n}+2n\sqrt{2}2^{2n-1}-1}{2\sqrt{2}} - n2^{n+1} \\ &= \frac{2^{2n}-1+2n\sqrt{2}(2^{2n-1}-2^{n+1})}{2\sqrt{2}} > 0 \end{aligned}$$

for all $n \geq 2$. Therefore, $g(t) < 0$ for $t < 0$ and $g(t) > 0$ for $t > 0$, hence $f'(t) < 0$ for $t < 0$ and $f'(t) > 0$ for $t > 0$. Consequently, f is strictly decreasing on $]-\infty, 0[$ and strictly increasing on $]0, \infty[$. Since $f(0) = 0$, we can conclude that (3.8) holds true. $\square$

REMARKS. 1. The inequality (3.6) is a refinement of $S_2(x,y) > L_{1/2}(x,y)$ (see (3.1)), by virtue of the inequality $M(x,y) > L_{1/2}(x,y)$ (see Z. Páles [7, Theorem 3]).

2. From (3.8) we deduce that

$$\frac{x}{\sqrt{2}} \tanh \frac{x}{\sqrt{2}} \leq \ln(\cosh x) \qquad \text{for all } x \in \mathbf{R}.$$

This inequality is complementary to the inequality

$$\ln(\cosh x) \leq x \tanh \frac{x}{2} \qquad \text{for all } x \in \mathbf{R},$$

established by K. B. Stolarsky [9, Theorem 3].

REFERENCES

[1] DAVIS, P. J.: *Interpolation and Approximation*, Blaisdell, New York, 1963.

[2] DAVIS, P. J. and RABINOWITZ, P.: *Numerical Integration*, Blaisdell, Massachusetts-Toronto-London, 1967.

[3] GINI C.: *Di una formula comprensiva delle medie*, Metron, **13**, (1938), 3–22.

[4] G. H. HARDY, G. H., LITTLEWOOD, J. E. and PÓLYA, G.: *Inequalities*, Cambridge Univ. Press, Cambridge, 1934.

[5] LEACH, E. B. and SHOLANDER, M. C.: *Extended mean values II*, J. Math. Anal. Appl., **92**, (1983), 207–223.

[6] NEUMAN, E.: *The weighted logarithmic mean*, J. Math. Anal. Appl., **188**, (1994), 885–900.

[7] PÁLES, Z.: *Inequalities for sums of powers*, J. Math. Anal. Appl., **131**, (1988), 265–270.

[8] PITTENGER, A. O.: *The symmetric, logarithmic and power means*, Univ. Beograd. Publ. Elektrotehn. Fak., Ser. Mat. Fiz., **No. 681**, (1980), 19–23.

[9] STOLARSKY, K. B.: *Hölder means, Lehmer means, and $x^{-1}\log\cosh x$*, J. Math. Anal. Appl., **202**, (1996), 810–818.

The Mathematical Notes of the University of Miskolc is abstracted in Zentralblatt für Mathematik, Mathematical Reviews, and Referativnyij Zhurnal

Responsible for publication: Rector of the University of Miskolc
Published by the Miskolc University Press under the leadership of Dr. József PÉTER
Responsible for duplication: Works manager Mária KOVÁCS
Number of copies printed: 200
Put to the Press on December 9, 2002
Number of permission: TU 2002-1260-ME **HU e-ISSN 1787-2413**