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ABOUT BOUNDED SOLUTIONS OF LINEAR STOCHASTIC ITO SYSTEMS

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Abstract. We prove that a sufficient condition for stochastic Ito systems to be exponentially dichotomous on the semiaxis is that the nonhomogeneous system has mean square bounded solutions.

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1. An introductory remark

The present paper investigates what conditions should be fulfilled for an Ito system to be exponentially dichotomous on the semiaxis. After same preparation we shall prove a theorem which is the main result of the paper.

2. Preparations and the proof of our theorem

Let us consider a system of linear stochastic Ito differential equations,

$$\mathrm{d}x = A(t)x\mathrm{d}t + \sum_{i=1}^{m} B_i(t)x\mathrm{d}W_i(t), \tag{2.1}$$

where $t \geq 0,\ x \in \mathbf{R}^n,\ A(t), B_i(t)$ are matrices determinate, continuous, and bounded on the positive semiaxis, $W_i(t), i = \overline{1,\ldots m}$, are jointly independent scalar Winner processes defined on a probability space (Ω, F, P). As implied by [1, p. 230], for $x_0 \in \mathbf{R}^n$ system (2.1) has a unique strong solution of the Cauchy problem, $x(t, x_0),\ x(0, x_0) = x_0$, defined for $t \geq 0$ and such that the second moment of the solution is finite for $t \geq 0$.

Definition 1. *System (2.1) is called exponentially dichotomous in mean square on the semiaxis $t \geq 0$, if the space $\mathbf{R}^n$ can be represented as a direct sum of two subspaces R^-, R^+ such that an arbitrary solution $x(t, x_0)$ of system (2.1), where $x_0 \in R^-$,* *satisfies the inequality*

$$\mathrm{M}\,|x(t,x_0)|^2 \le K\exp\{-\gamma(t-\tau)\}\mathrm{M}\,|x(\tau,x_0)|^2, \tag{2.2}$$

for $t \ge \tau \ge 0$, and an arbitrary solution $x(t,x_0)$ of system (2.1), where $x_0 \in R^+$, satisfies the inequality

$$\mathrm{M}\,|x(t,x_0)|^2 \ge K_1\exp\{\gamma_1(t-\tau)\}\mathrm{M}\,|x(\tau,x_0)|^2 \tag{2.3}$$

for $t \ge \tau \ge 0$ with an arbitrary $\tau > 0$. Here $K, K1, \gamma,\ \gamma_1$ are positive constants independent of τ and x_0.

An example of such a system is exponentially stable in a mean square system (2.1) (in this case $R^+ = \{0\}$, and $R^- = \mathbf{R}^n$).

As opposed to the ordinary differential equations, where conditions for dichotomy are well known [2, p. 230], [3], the problems in stochastic systems remain open. The author knows only the results of [1, p. 296], where the conditions for exponential dichotomy in mean square are obtained for a system of type (2.1) and for stochastic systems with delay in case where the matrices $A(t),\ B(t)$ are constant or periodic. But the results of [1, p. 296] are obtained with the help of a system of ordinary differential equations written for second moments of solutions of system (2.1) or a system of matrix equations for the correlation matrix of the solutions. It leads to an analysis of systems of a dimension much greater than the dimension of the initial system, and moreover, this system may not be exponentially dichotomous, although the initial system is.

That is why there is an interest in studying dichotomy conditions for system (2.1), when the matrices $A(t)$ and $B(t)$ are not obligatorily constant or periodic, and the conditions can be obtained in terms of the initial system, without an analysis of an auxiliary system for other moments.

From the works cited it follows that for ordinary differential equations, the question of exponential dichotomy on the semiaxis is equivalent to the question of existence of solutions, bounded on the semiaxis, of a nonhomogeneous system. This work is devoted to a study of this problem for system (2.1). Another point of view on studying dichotomy with the use of quadratic forms is published in another paper.

In the sequel, we shall assume that there is only one scalar Winner process $W(t)$ in system (2.1), and system (2.1) is of the form

$$\mathrm{d}x = \ A(t)x\,\mathrm{d}t + \ B(t)x\,\mathrm{d}W(t). \tag{2.4}$$

Let us consider a system of linear nonhomogeneous equation,

$$\mathrm{d}x = [\ A(t)x + \alpha(t)]\,\mathrm{d}t + \ B(t)x\,\mathrm{d}W(t), \tag{2.5}$$

where $\alpha(t)$ is n-dimensional and measurable, and for each $t \ge 0,\ F_t$ is a measurable stochastic process. Here, F_t is a flow of σ-algebras involved in the definition of the solution of the initial systems. We will assume that $\sup_{t\ge0}\mathrm{M}\,|\alpha(t)|^2 < \infty$. With the norm $||\alpha||_2 = (\sup_{t\ge0}\mathrm{M}\,|\alpha(t)|^2)^{1/2}$, the set of stochastic processes becomes a Banach space. Denote it by B.

Theorem 1. *Let system (2.5) with an arbitrary stochastic process $\alpha(t) \in B$ have a positive solution $x(t, x_0)$, $x_0 \in \mathbf{R}^n$ bounded in mean square Then system (2.4) is exponentially dichotomic in mean square on the positive semiaxis.*

Proof. Let $G_1 \subset R^n$ be the set of initial values of solutions of system (2.4), which are bounded in mean square on the semiaxis. It follows from linearity of system (2.4) that G_1 is a subspace of $\mathbf{R}^n$. Let us show that it plays the role of R^- in the definition of the exponential dichotomy. Let us prove the lemma. □

Lemma 1. *Suppose that the conditions of Theorem 1 hold. Then, for each stochastic process $\alpha(t) \in B$, there exists a unique solution $x(t, x_0)$ of system (5) bounded in mean square such that $x(0, x_0) \in G_1^{\perp} = G_2$. ($G_1^{\perp}$ is the orthogonal complement.) This solution satisfies the estimate*

$$||x||_2 \le K||\alpha||_2, \tag{2.6}$$

where K is some positive constant, independent of $\alpha(t)$.

Proof. Let $\alpha(t)$ satisfy the condition of the Theorem. Then for this $\alpha(t)$, it follows from the conditions of the lemma that there is a solution $x(t, x_0)$ of system (2.5) bounded in mean square.

Let P_1, P_2 be a pair of complement projectors on G_1, G_2. Let $x_1(t)$ be a solution of equation (2.4), corresponding to equation (2.5), with the initial condition $x_1(0) = P_1 x_0$. From the definition of the subspace G_1, it follows that such a solution is bounded in mean square on the semiaxis $t \ge 0$. It is obvious that $x_2 = x(t, x_0) - x_1(t)$ is a solution of system (2.3). It is easy to see that it is bounded on the positive semiaxis. We have $x_2(0) = x_0 - P_1 x_0 = P - 2x_0 \in G_2$. So its initial condition belongs to G_2. The unicity of the solution follows from the fact that the difference of two such solutions is a solution of a homogeneous equation, bounded in mean square and starts in G_2, which is possible only for the zero solution.

Let us prove inequality (2.6). Let us consider the space B_1 of all solutions of the stochastic equation

$$x(t) = x(0) + \int_0^t (A(s)x(s) + \alpha(s))\mathrm{d}s + \int_0^t (B(s)x(s))\mathrm{d}W(s) \tag{2.7}$$

with the condition that $x(0) \in G_2, \alpha(t) \in B$ bounded in the norm $|| \, ||_2$.

This equation defines a one-to-one linear operator $F : B_1 \to B$ which $\forall x \in B_1$ defines $\alpha \in B$ such that $x(t)$ is a solution of equation (2.5) bounded in mean square. Indeed, if $x(t) \in B_1$, then it follows from the definition of this space that there exists $\alpha(t) \in B$ such that $x(t)$ is a solution of equation (2.7) with this $\alpha(t)$. Let there exist another $\alpha_1(t) \in B$ such that $x(t)$ is a solution of the equation

$$x(t) = x(0) + \int_0^t (A(s)x(s) + \alpha_1(s))\mathrm{d}s + \int_0^t (B(s)x(s))\mathrm{d}W(s). \tag{2.8}$$

Subtracting (2.7) from (2.8) we get

$$\int_0^t (\alpha(s) - \alpha_1(s))\,\mathrm{d}s = 0. \tag{2.9}$$

Then $\alpha(t) = \alpha_1(t)$, for $t \geq 0$, with probability 1, from which it follows that $\alpha(t)$ and $\alpha_1(t)$ are equal as elements of the space B. It was shown above that for an arbitrary $\alpha(t) \in B$ there exists only one solution, $x(t)$, of equation (2.7) such that $x(0) \in G_2,\ x(t) \in B_1$. The linearity of the operator F is obvious.

Let us introduce the norm

$$|||x||| = ||x|| + ||Fx||_2. \tag{2.10}$$

This at once implies continuity of the operator F. Let us prove completeness of the space B_1. Let $\{x_n(t)\}$ be a fundamental sequence. There is fundamentality in B, as follows from (10). Hence, there exists a limit $x(t) \in B$. So, $\forall t \geq 0,\ M\,|x_n(t) - x(t)|^2 \to 0,\ n \to \infty$. And thus $|x_n(0) - x(0)| \to 0, n \to \infty$. Since $x_n(0) \in G_2$ and G_2 is a subspace of in $\mathbf{R}^n$, $x(0) \in G_2$.

It follows from the inequality $||F(x_n - x_m)||_2 \leq ||F||\,|||x_n - x_m|||$ that the sequence $Fx_n = \alpha_n$ is fundamental in B, and so there is a limit $\alpha(t)$ such that $\sup_{t\geq 0} M\,|\alpha_n(t) - \alpha(t)|^2 \to 0$, as $n \to \infty$, and $\alpha(t) \in B$.

Let us show that $x(t)$ satisfies the equation

$$x(t) = x(0) + \int_0^t (A(s)x(s) + \ \alpha(s))\mathrm{d}s + \int_0^t (B(s)x(s))\mathrm{d}W(s). \tag{2.11}$$

Since $A(t)$ and $B(t)$ are continuous and bounded, $x(t) \in B$, we have that $x(t)$ is F_t-measurable and both integrals in (2.11) exist. Let us estimate, for each $t \geq 0$, the expression

$$\begin{aligned}
&M\left|x(t) - x(0) - \int_0^t (A(s)x(s) + \ \alpha(s))\mathrm{d}s - \int_0^t B(s)x(s)\mathrm{d}W(s)\right|^2 \\
&\quad \leq M\left(|x(t) - x_n(t)| + \left|x_n(t) - x(0) - \int_0^t (A(s)x(s) + \alpha(s))\,\mathrm{d}s\right.\right. \\
&\quad \left.\left. - \int_0^t (B(s)x(s))\mathrm{d}W(s)\right|\right)^2 \leq 2\left[M\,|x(t) - x_n(t)|^2 + M\left|x_n(t) - x(0)\right.\right. \\
&\quad \left.\left. - \int_0^t (A(s)x(s) + \alpha(s))\mathrm{d}s - \int_0^t (B(s)x(s))\mathrm{d}W(s)\right|^2\right].
\end{aligned} \tag{2.12}$$

The first summand in the last expression tends to zero when $n \to \infty$. Let us estimate the second summand. Since $x_n(t)$, for each n, belongs to B_1, it satisfies the equation

$$x_n(t) = x_n(0) + \int_0^t (A(s)x_n(s) + \alpha_n(s))\mathrm{d}s + \int_0^t B(s)x_n(s)\mathrm{d}W(s). \tag{2.13}$$

Let us substitute (2.13) into (2.12). We get that the second summand in (2.10) does not exceed the following expression

$$3\Bigg[M\,|x_n(0) - x(0)|^2 + M\left(\int_0^t (||A(s)||\,|x_n(s) - x(s)| + |\alpha_n(s) - \alpha(s)|)\mathrm{d}s\right)^2$$

$$+ M\left|\int_0^t B(s)(x_n(s) - x(s))\mathrm{d}W(s)\right|^2\Bigg]$$

$$\leq 3\Bigg[M\,|x_n(t) - x(0)|^2 + 2t\int_0^t ||A(s)||^2 M\,|x_n(s) - x(s)|^2\mathrm{d}s$$

$$+ 2t\int_0^t M\,|\alpha_n - \alpha(s)|^2\mathrm{d}s + \int_0^t ||B(s)||^2 M\,|x_n(s) - x(s)|^2\mathrm{d}s\Bigg].$$

Each of the summands in the last expression tends to zero as $n \to \infty$. >From (2.12) it follows that $x(t)$ satisfies (2.11) with probability of 1 for each $t \geq 0$. So, the space B_1 is complete. That is why the linear continuous operator F defines a one-to-one mapping of the Banach space B_1 onto the Banach space B. By the Banach theorem, the inverse operator B^{-1} is also continuous. Then, for the solution of equation (2.3), we have the estimate

$$||x||_2 \leq |||x||| \leq ||F^{-1}||||\alpha||_2,$$

which is necessary estimate (2.4). The Lemma is proved. □

Let $x(t)$ be a nonzero solution of system (2.4), so that $x(0) \in G_1$. Let

$$y(t) = x(t)\int_0^t \frac{\beta(s)}{(\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}}}\,\mathrm{d}s, \tag{2.14}$$

where

$$\beta(t) = \begin{cases} 1, & 0 \leq t \leq t_0 + \tau, \\ 1 - (t - t_0 - \tau), & t_0 + \tau \leq t \leq t_0 + \tau + 1, \\ 0, & t \geq t_0 + \tau + 1. \end{cases}$$

It is obvious that $y(t)$ is F_t-dimensional and has a stochastic differential. Let us evaluate it,

$$\mathrm{d}y = \int_0^t \frac{\beta(s)}{(\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}}}\,\mathrm{d}s\mathrm{d}x + x(t)\frac{\beta(t)}{(\mathrm{M}\,|x(t)|^2)^{\frac{1}{2}}}\mathrm{d}t$$

$$= \int_0^t \frac{\beta(s)}{(\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}}}\,\mathrm{d}s(A(t)x\mathrm{d}t + B(t)x\mathrm{d}W(t)) + x(t)\frac{\beta(t)}{(\mathrm{M}\,|x(t)|^2)^{\frac{1}{2}}}\mathrm{d}t$$

$$= A(t)y\mathrm{d}t + x(t)\frac{\beta(t)}{(\mathrm{M}\,|x(t)|^2)^{\frac{1}{2}}}\mathrm{d}t + B(t)y\mathrm{d}W(t).$$

So $y(t)$ is a solution of equation (2.5) with $\alpha(t) = x(t)\dfrac{\beta(t)}{(\mathrm{M}\,|x(t)|^2)^{1/2}}$. Obviously, $||y||_2 < \infty$ and $\alpha(t) \in B$. And since $y(0) = 0 \in G_2$, the previous Lemma gives that

$$||y||_2 \leq K||\alpha||_2.$$

Whence,

$$(M|y(t)|^2)^{\frac{1}{2}} \leq K(\sup_{t\geq 0}\mathrm{M}\,|\alpha(t)|^2)^{\frac{1}{2}} \leq K$$

for $t \geq 0$. In particular, if $t = t_0 + \tau$, then

$$(\mathrm{M}\,|y(t)|^2)^{\frac{1}{2}} = (\mathrm{M}\,|x(t_0+\tau)|^2)^{\frac{1}{2}} \leq K. \tag{2.15}$$

Let us consider the function

$$\psi(t) = \int_{t_0}^t \frac{1}{(\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}}}\,\mathrm{d}s.$$

Since the second moments of system (2.4) satisfy a system of ordinary linear differential equations, see [1, p. 236], this function is continuously differentiable. Then (2.15) gives that

$$\frac{\psi'(t_0+\tau)}{\psi(t_0+\tau)} \geq \frac{1}{K}.$$

If we integrate the last inequality from 1 to τ , we get

$$\psi(t_0+\tau) \geq \psi(t_0+1)\exp\left\{\frac{\tau-1}{K}\right\} \tag{2.16}$$

for $\tau \geq 1$. Since $x(t)$ is a solution of system (2.4),

$$x(t) = x(t_0) + \int_{t_0}^t A(s)x(s)\,\mathrm{d}s + \int_{t_0}^t B(s)x(s)\,\mathrm{d}W(s). \tag{2.17}$$

And, hence, if $t \in [t_0\ t_0+1]$, we have

$$\mathrm{M}\,|x(t)|^2 \leq 3\left(\mathrm{M}\,|x(t_0)|^2 + \int\limits_{t_0}^{t_0+1} ||A(s)||^2 \mathrm{M}\,|x(s)|^2\,\mathrm{d}s + \int\limits_{t_0}^{t_0+1} ||B(s)||^2 \mathrm{M}\,|x(s)|^2\,\mathrm{d}s\right).$$

This and the Gronwall–Bellman inequality give

$$\mathrm{M}\,|x(t)|^2 \leq 3\mathrm{M}\,|x(t_0)|^2 \exp\{C\}, \tag{2.18}$$

where $C > 0$ is a constant independent of t_0. Therefore,

$$\psi(t_0+1) = \int\limits_{t_0}^{t_0+1} \frac{1}{(\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}}}\,\mathrm{d}s \geq \frac{1}{3^{\frac{1}{2}}}(\mathrm{M}\,|x(t_0)|^2)^{-\frac{1}{2}} \exp\left\{-\frac{C}{2}\right\}.$$

From this inequality using (2.15) and (2.16), we get for $\tau \geq 1$ that

$$(\mathrm{M}\,|x(t_0+\tau)|^2)^{\frac{1}{2}} \leq \frac{K}{\psi(t_0+\tau)} \leq N(\mathrm{M}\,|x(t_0)|^2)^{\frac{1}{2}} \exp\left\{-\frac{\tau}{K}\right\}, \tag{2.19}$$

where $N > 0$ is a constant independent of τ and t_0. If $\tau \leq 1$, it follows from inequality (2.18) that

$$\mathrm{M}\,|x(t_0+\tau)|^2 \leq 3\mathrm{M}\,|x(t_0)|^2 \exp\left\{\frac{2}{K} + C - \frac{2\tau}{K}\right\}. \tag{2.20}$$

Since $t_0 \geq 0$ is arbitrary, (2.19), (2.20) and the first inequality in Definition 1 hold with

$$\gamma = \frac{2}{K},\ K_1 = \max\left\{N^2; 3\exp\left\{\frac{2}{K} + C\right\}\right\}.$$

Let us prove the second inequality in Definition 1. Let $x(t)$ be a nonzero solution of equation (2.4) with $x(0) \in G_2$. It is easy to see that

$$y(t) = x(t)\int\limits_{t}^{\infty} \frac{\beta(s)}{(\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}}}\,\mathrm{d}s$$

is a solution of equation (2.5) with

$$\alpha(t) = -\frac{x(t)}{(\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}}}\beta(t)$$

and, since $t \geq t_0+\tau$, $\sup_{t\geq 0} \mathrm{M}\,|y(t)|^2 < \infty$. It is obvious that $y(0) \in G_2$. Therefore, because of the Lemma,

$$(\mathrm{M}\,|y(t)|^2)^{\frac{1}{2}} = (\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}} \int\limits_{t}^{\infty} \frac{\beta(s)}{\sqrt{\mathrm{M}\,|x(s)|^2}}\,\mathrm{d}s \leq K.$$

So $\forall \tau \geq 0$ and $\forall t \geq 0$,

$$\int\limits_{t}^{\infty} \frac{\beta(s)}{\sqrt{\mathrm{M}\,|x(s)|^2}}\mathrm{d}s \leq \frac{K}{\sqrt{\mathrm{M}\,|x(t)|^2}}. \tag{2.21}$$

The left-hand side of this inequality is monotone increasing for $\tau \geq 0$ and is bounded, so it has a limit for $\tau \to \infty$. But

$$\int_t^{\infty} \frac{\beta(s)}{\sqrt{(\mathrm{M}\,|x(s)|^2)}}\,\mathrm{d}s = \int_t^{t_0+\tau} \frac{1}{\sqrt{(\mathrm{M}\,|x(s)|^2)}}\,\mathrm{d}s + \int_{t_+\tau}^{t_0+\tau+1} \frac{\beta(s)}{\sqrt{(\mathrm{M}\,|x(s)|^2)}}\,\mathrm{d}s,$$

where the second summand tends to zero as $\tau \to \infty$ (since the integral is convergent), and so if $\tau \to \infty$, we get the inequality

$$\int_t^{\infty} \frac{1}{\sqrt{\mathrm{M}\,|x(s)|^2}}\,\mathrm{d}s \leq \frac{K}{\sqrt{\mathrm{M}\,|x(t)|^2}}. \tag{2.22}$$

Let

$$\psi(t) = \int_t^{\infty} \frac{1}{\sqrt{\mathrm{M}\,|x(s)|^2}}\,\mathrm{d}s.$$

Then it follows from (2.22) that

$$\psi'(t) \leq -\frac{1}{K}\psi(t).$$

Whence we get

$$\psi(t) \leq \psi(t_0)\exp\left\{-\frac{1}{K}(t-t_0)\right\}. \tag{2.23}$$

Since $x(t)$ is a solution of system (2.4), we have for $\tau \geq t$ that

$$\mathrm{M}\,|x(\tau)|^2 \leq C_1 \mathrm{M}|x(t)|^2 \exp\{L(\tau - t)\},$$

where L, C_1 are positive constants, independent of τ and t. From the last inequality, it follows that

$$\mathrm{M}\,|x(\tau)|^2 \leq 3\mathrm{M}\,|x(t)|^2 \exp\{3L(\tau - t + 1)(\tau - t)\}.$$

Therefore,

$$(\mathrm{M}\,|x(t)|^2)^{\frac{1}{2}}\psi(t) = (\mathrm{M}\,|x(t)|^2)^{\frac{1}{2}} \int_t^{\infty} \frac{1}{\sqrt{\mathrm{M}\,|x(s)|^2}}\,\mathrm{d}s$$

$$\geq \int_t^{\infty} \frac{1}{\sqrt{3}} \exp\left\{-\frac{3}{2}L(s-t+1)(s-t)\right\}\mathrm{d}s = l.$$

Here L is a positive constant. Then (2.22) and (2.23) give

$$(\mathrm{M}\,|x(t)|^2)^{\frac{1}{2}} \geq \frac{l}{\psi(t)} \geq \frac{l}{\psi(t_0)}\exp\left\{\frac{1}{K}(t-t_0)\right\} \geq \frac{l}{K}\exp\left\{\frac{1}{K}(t-t_0)\right\}(\mathrm{M}\,|x(t_0)|^2)^{\frac{1}{2}}.$$

This estimate is the second inequality, which figures in the definition of the exponential dichotomy. The theorem is proved.

In the theory of ordinary differential equation one proves a converse result that exponential dichotomy of a homogeneous system implies the existence of a bounded solution of the nonhomogeneous system and that

$$y(t) = \int\limits_0^\infty G(t,\tau)f(\tau)\,\mathrm{d}\tau, \tag{2.24}$$

where $G(t,\tau)$ is Green's function

$$G(t,\tau) = \begin{cases} \Phi(0,t)P_1(\Phi(0,\tau))^{-1}, & t \geq \tau, \\ -\Phi(0,t)P_2(\Phi(0,\tau))^{-1}, & t < \tau, \end{cases} \tag{2.25}$$

$\Phi(0,t)$ is the matriciant of the homogeneous system. For stochastic nonhomogeneous systems,

$$\mathrm{d}x = (A(t)x + \alpha(t))dt + \ (B(t)x + \beta(t))]\mathrm{d}W(t), \tag{2.26}$$

one can also write the representation

$$y(t) = \int\limits_0^\infty G(t,\tau)\alpha(\tau)\mathrm{d}\tau \ + \int\limits_0^\infty G(t,\tau)\beta(\tau)\mathrm{d}W(\tau), \tag{2.27}$$

however, in such a case, $y(t)$ will not be F_t-measurable any more. Thus, by using Green's function, one succeeds in getting a similar result only in the case where the homogeneous system is exponentially stable and the nonhomogeneous system has the form

$$\mathrm{d}x = [\ A(t)x + \alpha(t)]\mathrm{d}t + \ \beta(t)\mathrm{d}W(t). \tag{2.28}$$

Theorem 2. *Let the homogeneous system*

$$\mathrm{d}x = \ A(t)x\mathrm{d}t \tag{2.29}$$

be exponentially stable on the positive semiaxis. Then, for arbitrary $\alpha(t), \beta(t) \in B$, system (2.28) has a solution bounded in mean square on the positive semiaxis. In addition, all bounded solutions of system (2.28) are given by

$$x = \psi(t) + \int\limits_0^t \Phi(t,\tau)\alpha(\tau)\,\mathrm{d}\tau + \int\limits_0^t \Phi(t,\tau)\beta(\tau)\,\mathrm{d}W(\tau), \tag{2.30}$$

where $\psi(t)$ is an arbitrary solution of system (2.29), and $\Phi(t,\tau)$ is the matriciant of system (2.29), $\Phi(\tau,\tau) = E$.

Proof. Since system (2.29) is exponentially stable, its matriciant satisfies the estimate

$$||\Phi(t,\tau)|| \leq K\exp\{-\gamma(t-\tau)\}, \tag{2.31}$$

for $t \geq \tau \geq 0$, with some positive K and γ. Let us show that $x(t)$, defined by (2.30), is bounded in mean square for $t \geq 0$. To do that, it is sufficient to prove the boundedness of each of its summands. Indeed, $\psi(t)$ is a function bounded on the semiaxis. Let us estimate the second summand. From the Cauchy–Bunyakovskii inequality, we have

$$M \left| \int_0^t \Phi(t,\tau)\alpha(\tau)\,\mathrm{d}\tau \right|^2 \le M \left(\int_0^t ||\Phi(t,\tau)|||\alpha(\tau)|\,\mathrm{d}\tau| \right)^2$$

$$\le K^2 M \left(\int_0^t \exp\left\{ \frac{-\gamma(t-\tau)}{2} \right\} \exp\left\{ \frac{-\gamma(t-\tau)}{2} \right\} |\alpha(\tau)\,\mathrm{d}\tau \right)^2$$

$$\le K^2 \int_0^t \exp\{-\gamma(t-\tau)\}\,\mathrm{d}\tau \int_0^t \exp\{-\gamma(t-\tau)\} M\,|\alpha(\tau)|^2\,\mathrm{d}\tau) < C,$$

where $C > 0$ is a constant, since $\alpha(t) \in B$. Let us use the properties of the stochastic integral

$$M \left| \int_0^t \Phi(t,\tau)\beta(\tau)\,\mathrm{d}W(\tau) \right|^2 \le \int_0^t ||\Phi(t,\tau)||^2 M\,|\beta(\tau)|^2\,\mathrm{d}\tau$$

$$\le K^2 \int_0^t \exp\{-2\gamma(t-\tau)\}\,\mathrm{d}\tau \sup_{t\ge 0} M\,|\beta(t)|^2 < C_1, \qquad C_1 > 0.$$

So the function $x(t)$ in (2.30) is bounded in mean square. It is obvious that it is F_t-dimensional, which follows from [1, p. 234], and gives a solution of system (2.28). The theorem is proved. □

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HOPF BIFURCATION IN MODELS FOR MICRO – AND MACROPARASITICAL DISEASES

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Abstract. The purpose of this article is to establish, via Hopf bifurcation, the occurrence of attracting periodical orbits, in two models for microparasitical and macroparasitical diseases, due to Diekmann and Krezschmar, and described by systems of two ordinary differential equations. This proof is achieved by making use, after a suitable change of coordinates, of a result due to Marsden and McCracken.

Mathematical Subject Classification: 92B05
Keywords: Biomathematics, infectious diseases, host-parasite, S-I system, periodic orbit, Hopf bifurcation.

1. Introduction

In this work we will establish the occurrence of attracting periodical orbits, via Hopf bifurcation, in two epidemiological models introduced in [1]; one of them for microparasitical diseases of susceptible-infected S-I and the other for macroparasitical diseases of the host-parasite N-P type.

The first model is described by the system

$$\begin{cases} \dfrac{dS}{dt} = \beta \dfrac{S^2 + 2\xi IS + \xi^2 IS}{I+S} - \mu S - \dfrac{KI}{c+S+I} S \\ \dfrac{dI}{dt} = \dfrac{KI}{c+S+I} S - \mu I - \alpha I. \end{cases} \tag{1.1}$$

and the macroparasitical model is represented by

$$\begin{cases} \dfrac{dN}{dt} = -\mu N - \alpha P + \beta N \Big(\dfrac{kN}{kN + P(1-\xi)} \Big)^k \\ \dfrac{dP}{dt} = -(\mu+\sigma)P + \dfrac{KPN}{c+N} - \alpha N \Big(\dfrac{P}{N} + \Big(\dfrac{P}{N}\Big)^2 \Big(\dfrac{k+1}{k}\Big)\Big). \end{cases} \tag{1.2}$$

where in the model (1.1) (model (1.2)), the parameters $\alpha, \beta, \xi, \mu, \sigma, c, k, K$ stand for :
β: per capita natural birth rate (of hosts)
μ: per capita natural death rate (of hosts), $\mu < \beta$
α: additional mortality rate due to disease (by one parasite)
ξ: parameter describing the reduction of fertility of an infected individual (host) due to the disease (to one parasite), $0 \leq \xi \leq 1$
K: contact rate between infectives and susceptibles (between hosts and infective stages of the parasites, such as eggs, cysts, spores, chrysalis, parasites)
σ: death rate of parasites
k: "clumping" parameter (a small k indicates high clumping, i.e., few hosts carry a large part of the parasites , while a large part of the hosts have very few parasites; for $k \to \infty$, the parasites are randomly distributed over the host population).

The change of variables $x = \dfrac{1}{I+S}$ and $y = \dfrac{I}{I+S}$, transforms (1.1) into

$$\begin{cases} \dfrac{dx}{dt} = x\Big(\mu + \alpha y - \beta(1-(1-\xi)y)^2\Big) \\ \dfrac{dy}{dt} = y\Big(\Big(\dfrac{K}{cx+1} - \alpha\Big)(1-y) - \beta(1-(1-\xi)y)^2 \Big) \end{cases} \tag{1.3}$$

and $x = \dfrac{1}{N}$, $y = \dfrac{P}{N}$, transforms (1.2) into

$$\begin{cases} \dfrac{dx}{dt} = x\Big(\mu + \alpha y - \beta\Big(\dfrac{k}{(1-\xi)y+k}\Big)^k\Big) \\ \dfrac{dy}{dt} = y\left(\dfrac{K}{cx+1} - (\sigma+\alpha) - \dfrac{\alpha}{k}y - \beta\dfrac{k}{(1-\xi)y+k}^k\right) \end{cases} \tag{1.4}$$

Each one of these models will be treated with unified notation:

$$\begin{cases} \dot{x} = xF(\xi, y) \\ \dot{y} = yG(K, \xi, x, y) \end{cases}$$

in the corresponding region $M = \{(x,y) \in \mathbb{R}^2 : x \geq 0,\ 0 \leq y < b\}$, where b = 1 or ∞ according to the model being (1.3) or (1.4).

The following proposition of [1] synthesizes common properties of both systems and plays an important role in the determination and classification of critical points in both cases. It also ensures the existence of at most one critical point $(\overline{x}, \overline{y})$ in the interior of M.

Proposition 1: *Consider each one of systems (1.3) and (1.4) in its respective domain M with $0 \leq \xi \leq 1$ and $0 < \mu < \beta$. Suppose further that for system (1.4), $\beta < \alpha + \mu$. Then,*

a) $\dfrac{\partial F}{\partial y} > 0$ *and there is a unique* $\overline{y} = \overline{y}(\xi)$ *in* $[0, b[$ *rendering* $F(\xi, \overline{y}(\xi)) = 0$*;*

b) $\dfrac{\partial G}{\partial x} < 0$ *and there is a unique* $x = g(K, \xi, y)$ *rendering* $G(K,\xi,g(K,\xi,y),y)=0$*;*

c) $\dfrac{\partial G}{\partial K} > 0$ *and* $G(0, \xi, x, y) < 0$ *for all* $(x, y) \in M$.

In Section 2 we will give the preliminary results needed for the study of each one of the systems (1.3) and (1.4) and also introduce the fundamental parameters $\xi = \xi_p$ and $K = K_t(\xi_p)$.

Further, we present some phase portrait simulations , obtained by means of the software MAPLE V, whereby the occurrence of Hopf bifurcation becomes apparent.

In Section 3, guided by a result of [4], we prove the theorem below, which is also stated in [1].

Theorem: *Consider the parameter restrictions* $0 < \mu < \beta, 0 < \xi < 1$ *on systems (1.3), (1.4) and the additional restriction on (1.4):* $\beta < \alpha + \mu$, $0 < \alpha < k\beta$. *Under the assumption that a value* K, $K > K_t(\xi_p)$, *has been fixed it follows that system (1.3) admits a supercritical Hopf bifurcation at the critical point inside* M, *with respect to the parameter* ξ *at the value* ξ_p. *Otherwise, there exists a value* μ_0, $0 < \mu_0 < \beta$, *such that occurs the same for system (1.4) if* $\mu > \mu_0$.

As a consequence we have a Hopf bifurcation for each one of the systems (1.1) and (1.2). Thus, for each value of the parameter ξ in a certain region , there is one periodic (hyperbolic) attracting orbit and, by results of Kooij and Zegeling (see [2], [3]), there are no other periodic orbits.

2. The parameters ξ_p and $K_t(\xi_p)$

Now we consider models (1.3) and (1.4) in their corresponding domains M, with the restrictions on the parameters as in the assumptions of the foregoing theorem.

In this article our attention will be focused solely upon critical points, which are solutions of the system:

$$\begin{cases} F(\xi, y) = 0\,, \\ G(K, \xi, x, y) = 0\,. \end{cases}$$

Given ξ, $0 < \xi < 1$, consider

$$\overline{y}(\xi) = \frac{2(1-\xi)\beta + \alpha - \sqrt{(2(1-\xi)\beta + \alpha)^2 + 4(1-\xi)^2\beta(\mu - \beta)}}{2(1-\xi)^2\beta}$$

the unique value for y in $]0, 1[$ as to render $F(\xi, \overline{y}(\xi)) = 0$.

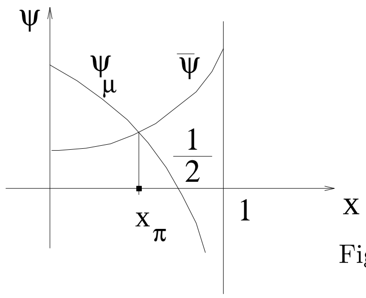
The points $(x, y) \in M$ such that $G(K, \xi, x, y) = 0$ are given by

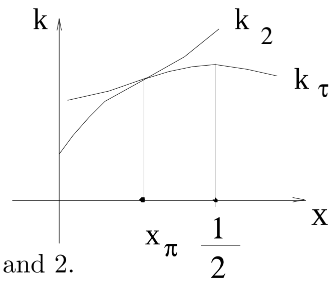
$$x = g(K, \xi, y) = -\frac{1}{c}\left(1 - \frac{K(1-y)}{\beta(1-(1-\xi)y)^2 + \alpha(1-y)}\right).$$

The critical points are obtained by intercepting the straight line $y = \overline{y}(\xi)$ and the graph of $x = g(K, \xi, y)$.

Let ξ_p denote the (unique) value of the parameter ξ for which $\overline{y}(\xi) = y_m(\xi)$, where $y_m(\xi) = 2 - \dfrac{1}{1-\xi}$ is the point of maximum of $x = g(K, \xi, y)$ (see Figure 1): For each fixed ξ, the following values of the parameter K are relevant: $K_t(\xi) = \alpha + 4\beta\xi(1-\xi)$

for which the graph of $x = g(K, \xi, y)$ is tangent to the y-axis and $K_2(\xi)$ which renders





Figures 1 and 2.

$g(K, \xi, \overline{y}(\xi)) = 0$. Since $K_2(\xi)$ is determined by

$$\begin{cases} \beta(1-(1-\xi)\overline{y}(\xi))^2 + (\alpha - K)(1-\overline{y}(\xi)) = 0 \\ \beta(1-(1-\xi)\overline{y}(\xi))^2 - \alpha\overline{y}(\xi) - \mu = 0 \end{cases}$$

and $\overline{y}'(\xi) > 0$, it follows that $K_2(\xi)$ is derivable and increasing. Further, $K_2(\xi) > K_t(\xi)$ for $\xi \neq \xi_p$, provided that the function $x = g(K, \xi, y)$ is increasing with respect to K, as long as $y < 1$ (see Figure 2).

Thus we may state that for any $K, K > K_t(\xi_p) = K_2(\xi_p)$, there is a neighborhood of ξ_p where $K > K_2(\xi) \geq K_t(\xi)$; therefore, for such values of ξ there is one and only one critical point $(\overline{x}, \overline{y}) = (\overline{x}(\xi), \overline{y}(\xi))$ in the interior of M. Here we denote $\overline{x}(\xi) = g(K, \xi, \overline{y}(\xi))$ (see Figure 6, where, for fixed ξ, near ξ_p the graphs of $x = g(K, \xi, y)$ and $y = \overline{y}(\xi)$ are displayed).

Next we present some simulations of system (1.3), generated by the software MAPLE V. This suggests the existence of Hopf bifurcations.

Let us now regard system (4)

Given ξ between 0 and 1, let $\overline{y}(\xi)$ be the unique value of y in $]0, \infty[$ satisfying $F(\xi, \overline{y}(\xi)) = 0$, that is, $\alpha\overline{y} + \mu = \beta\Big(\dfrac{k}{(1-\xi)\overline{y} + k}\Big)^k$ (see Figure 7). Deriving this equation implicitly, we obtain $\overline{y}'(\xi) > 0$.

The points $(x, y) \in M$ satisfying $G(K, \xi, x, y) = 0$ are:

$$x = g(K, \xi, y) = -\frac{1}{c}\Big(1 - \frac{K}{\phi(\xi, y)}\Big)$$

where $\phi(\xi, y) = \sigma + \alpha + \dfrac{\alpha y}{k} + \beta\left(\dfrac{k}{(1-\xi)y + k}\right)^k$.

Let $y_m(\xi) = \dfrac{k}{1-\xi}\left[\Big(\dfrac{k\beta(1-\xi)}{\alpha}\Big)^{\frac{1}{k+1}} - 1\right]$ be the point of minimum of $\phi(\xi, y)$, that is, the point of maximum of $g(K, \xi, y)$. We have $y_m(0) > 0$, since $\alpha < k\beta$, and $y_m(\xi)$ being independent on μ.

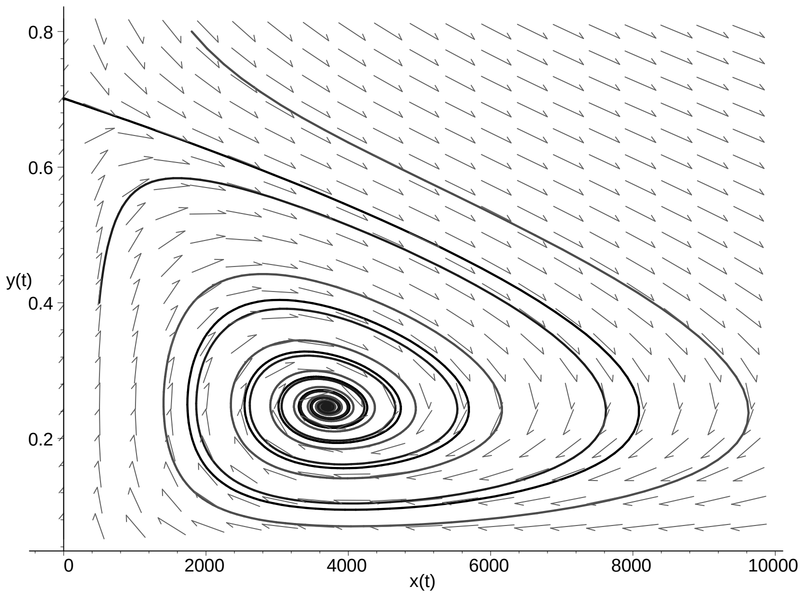


Figure 3. Phase portrait of system (3) with $\mu = 1.2$, $\alpha = 1.0$, $\beta = 1.8$, $\xi = 0.58$, $c = 0.0001$, $\kappa = 4.0$, $\kappa_t = 2.75$, $\kappa_1 = 2.8$, $\kappa_2 = 3.03$

There exists $\mu_0,\ 0 < \mu_0 < \beta$, with $\mu_0 < \mu < \beta \Rightarrow 0 < \overline{y}(0) < y_m(0)$. In fact, if $\alpha y_m(0) \geq \beta\Big(\dfrac{k}{y_m(0)+k}\Big)^k$, $\mu_0 = 0$; otherwise, $\mu_0 = \beta\Big(\dfrac{k}{y_m(0)+k}\Big)^k - \alpha y_m(0)$ (see Figure 8).

In this case there is a unique $\xi = \xi_p \in \left]0, 1 - \dfrac{\alpha}{k\beta}\right[$ with $\overline{y}(\xi_p) = y_m(\xi_p)$ (see Figure 9).

For each fixed ξ let $K_t(\xi)$ be the value of K for which the graph of $x = g(K, \xi, y)$ is tangent to the axis y and $K_2(\xi)$ the value giving $g(K, \xi, \overline{y}(\xi)) = 0$. We have, $K_t(\xi) = \phi(\xi, y_m(\xi)) \ \Rightarrow\ K_t'(\xi) = \phi_\xi + \phi_y y' = \phi_\xi = \beta y_m(\xi)\Big(\dfrac{k}{(1-\xi)y_m + k}\Big)^{k+1} > 0$.

Furthermore, $K_2(\xi)$ is derivable and increasing, since it is determined by the system

$$\begin{cases} FK - \sigma - \alpha - \dfrac{\alpha y}{k} - \beta\Big(\dfrac{k}{(1-\xi)y + k}\Big)^k = 0 \\ \alpha y + \mu - \beta\Big(\dfrac{k}{(1-\xi)y + k}\Big)^k = 0 \end{cases}$$

and $\overline{y}'(\xi) > 0$ holds.

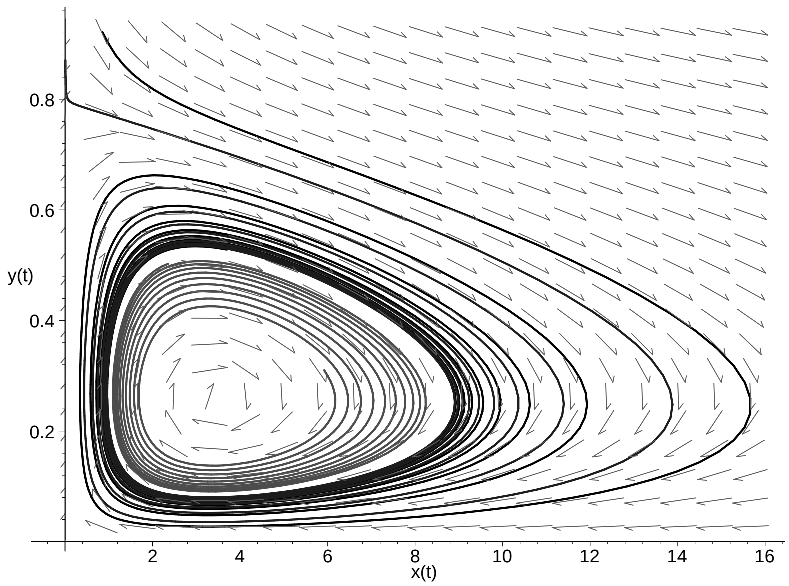


Figure 4. Phase portrait of system (3) with $\mu = 1.2$, $\alpha = 1.0$, $\beta = 1.8$, $\xi = 0.44$, $c = 0.1$, $\kappa = 4$, $\kappa_t = 2.77$, $\kappa_1 = 3.0$, $\kappa_2 = 2.9$

Proceeding in a similar way to what was done in the study of model (3), we get: $K_2(\xi) > K_t(\xi)$ for $\xi \neq \xi_p$ and for K being fixed, $K > K_t(\xi_p)$, we see that there is a neighbourhood of ξ_p where $K > K_2(\xi) > K_t(\xi)$ for $\xi \neq \xi_p$ (see Figure 10).

In Figure 11 we display the graph of $x = g(K, \xi, y)$ and $y = \overline{y}(\xi)$, according to the variation of K. Here the value of ξ is kept fixed and close to ξ_p.

3. **The Hopf Bifurcation. Proof of the Theorem**

The Jacobian matrix at the critical point under consideration has for both systems the expression

$$J(\overline{x}, \overline{y}) = \begin{bmatrix} 0 & \overline{x}F_y(\xi, \overline{y}) \\ \overline{y}G_x(K, \xi, \overline{x}, \overline{y}) & \overline{y}G_y(K, \xi, \overline{x}, \overline{y}) \end{bmatrix}, \text{ with } \det J(\overline{x}, \overline{y}) \neq 0.$$

We have also trace $J(\overline{x}, \overline{y}) = -\overline{y}G_x(K, \xi, \overline{x}, \overline{y})g_y(K, \xi, \overline{y})$ since $G(K, \xi, g(K, \xi, y), y), y) = 0 \Rightarrow\ G_x g_y + G_y = 0$.

Thus, for a fixed $K, K > K_t(\xi_p)$, and ξ sufficiently close to ξ_p:
$\xi < \xi_p \Rightarrow g(K, \xi, \overline{y}(\xi)) > 0 \Rightarrow trJ(\overline{x}, \overline{y}) > 0 \Rightarrow (\overline{x}, \overline{y})$ is an unstable spiral point,
$\xi > \xi_p \Rightarrow g(K, \xi, \overline{y}(\xi)) < 0 \Rightarrow trJ(\overline{x}, \overline{y}) < 0 \Rightarrow (\overline{x}, \overline{y})$ is a stable spiral point.

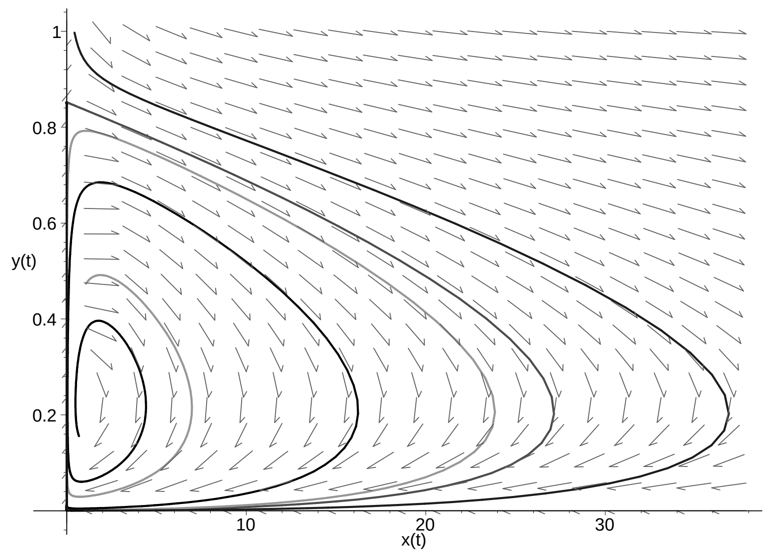


Figure 5. Phase portrait of system (3) with $\mu = 1.2$, $\alpha = 1.0$, $\beta = 1.8$, $\xi = 0.3$, $c = 0.1$, $\kappa = 3.2$, $\kappa_t = 2.51$, $\kappa_1 = 3.0$, $\kappa_2 = 2.83$.

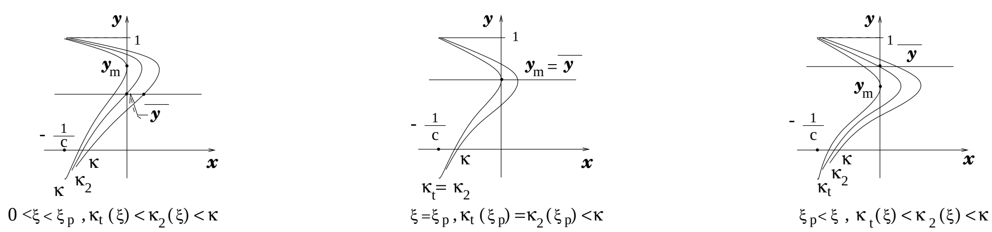
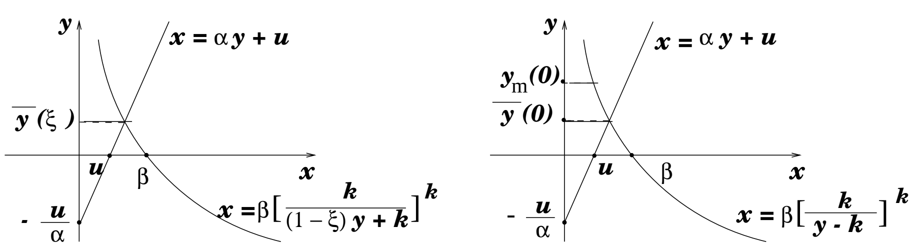
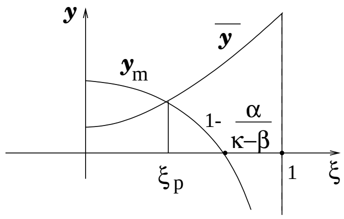
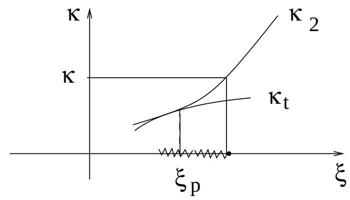


Figure 6.

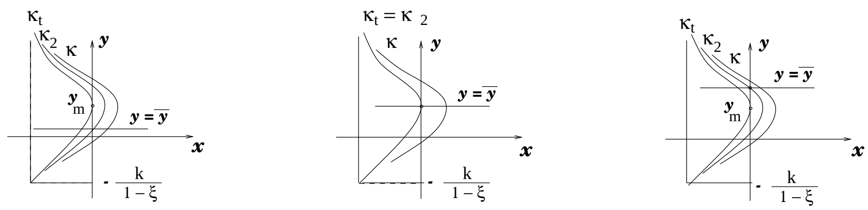


Figures 7 and 8.





Figures 9 and 10.



$0 < \xi < \xi_p,\ \kappa_t(\xi) < \kappa_2(\xi) < \kappa$ $\quad$ $\xi = \xi_p,\ \kappa_t(\xi_p) = \kappa_2(\xi_p) < \kappa$ $\quad$ $\xi_p = \xi,\ \kappa_t(\xi) = \kappa_2(\xi) < \kappa$

Figure 11.

The eigenvalues of $J(\overline{x},\overline{y})$ are $\lambda(\xi) = \dfrac{\overline{y}G_y(K,\xi,\overline{x},\overline{y}) \pm \sqrt{\overline{y}^2G_y^2 + 4\overline{x}\,\overline{y}F_yG_x}}{2}$, where $\lambda(\xi_p) = \pm i\sqrt{\overline{x}\,\overline{y}F_y(-G_x)}\,$.

To assure that there is a Hopf bifurcation at ξ_p we will show that $\dfrac{d}{d\xi}$ Re $\lambda(\xi)|_{\xi_p} \neq 0$.

In fact,

$$2\frac{d}{d\xi}\mathrm{Re}\lambda(\xi)|_{\xi_p} = \frac{d}{d\xi}\overline{y}G_y(K,\xi,\overline{x},\overline{y})|_{\xi_p} = \overline{y}\{G_{\xi y} + G_{xy}g_\xi + G_{yy}\overline{y}'\}|_{\xi_p}$$

since

$$g_y(K,\xi_p,\overline{y}(\xi_p) = 0 \ \Rightarrow \left\{ \begin{array}{c} G_y(K,\xi_p,\overline{x},\overline{y}) = 0 \\ \overline{x}'(\xi_p) = g_\xi(K,\xi_p,\overline{y}(\xi_p)) \end{array} \right. .$$

For system (3):

$$G_x = -(1-y)\frac{Kc}{(cx+1)^2}, \qquad G_{xx} = \frac{2c^2K(1-y)}{(cx+1)^3}, \qquad G_{xxy} = -\frac{2c^2K}{(cx+1)^3},$$

$$G_y = -\Big(\frac{K}{cx+1} - \alpha\Big) + 2\beta(1-\xi)(1-(1-\xi)y), \qquad G_{\xi y} = 2\beta((1-\xi)y-1),$$

$$G_{yy} = -2\beta(1-\xi)^2, \qquad G_{xy} = \frac{Kc}{(cx+1)^2}, \qquad g_\xi = -\frac{2\beta K(1-y)(1-(1-\xi)y)}{[(\beta(1-(1-\xi)y)^2 + \alpha(1-\overline{y})]^2}$$

For system (4):

$$G_y = \frac{\alpha}{k} + \beta(1-\xi)\Big(\frac{k}{(1-\xi)y+k}\Big)^{k+1},$$

$$G_{\xi y} = \beta((1-\xi)y-1)\Big(\frac{k}{(1-\xi)y+k}\Big)^{k+2},\ G_{xy} = 0,$$

$$G_{yy} = -\beta(1-\xi)^2\frac{k+1}{k}\Big(\frac{k}{(1-\xi)y+k}\Big)^{k+2},$$

$$g_\xi = -\frac{K}{c}\Big(\frac{k}{(1-\xi)y+k}\Big)^{k+1}\frac{\beta y}{\Big(\sigma+\alpha+\frac{\alpha}{k}y+\beta\Big(\frac{k}{(1-\xi)y+k}\Big)^k\Big)^2}.$$

Therefore, in both cases:

$$2\frac{d}{d\xi}\mathrm{Re}\lambda(\xi)|_{\xi_p} = \overline{y}(G_{\xi y}+\overline{y}G_{xy}g_\xi + G_{yy}\overline{y}')|_{\xi_p} < 0.$$

Next we engage in showing, in this following closely [4], that the critical point is a vague attractor at $\xi = \xi_p$. Thus we will conclude the existence of attracting periodical orbits, for ξ sufficiently close to ξ_p , $\xi < \xi_p$ (see Figure 12).

Let us consider the system for a fixed $K > K_t(\xi_p)$, and $\xi = \xi_p$, and let us, for the sake of simplicity, omit the notation for those parameters.

Initially we apply a translation, carrying the critical point to the origin, thereby transforming any of the systems in

$$\begin{cases} \dot{\tilde{X}} = \tilde{X}F(\tilde{Y}+\overline{y}) + \overline{x}F(\tilde{Y}+\overline{y}) \\ \dot{\tilde{Y}} = \tilde{Y}G(\tilde{X}+\overline{x},\tilde{Y}+\overline{y}) + \overline{y}G(\tilde{X}+\overline{x},\tilde{Y}+\overline{y}) \end{cases} \tag{3.1}$$

whose Jacobian matrix at the origin is

$$J(0,0) = \begin{bmatrix} 0 & \overline{x}F'_y(\overline{y}) \\ \overline{y}G_x(\overline{x},\overline{y}) & 0 \end{bmatrix}.$$

Applying the coordinate change:

$$X = \tilde{X}, Y = \frac{\overline{x}F'(\overline{y})\tilde{Y}}{\sqrt{-\overline{x}\,\overline{y}F'(\overline{y})G_x(\overline{x},\overline{y})}}$$

system (3.1) becomes

$$\begin{cases} \dot{X} = Z_1(X,Y) = XF(aY+\overline{y}) + xF(aY+\overline{y}) \\ \dot{Y} = Z_2(X,Y) = YG(X+\overline{x},aY+\overline{y}) + \frac{\overline{y}}{a}G(X+\overline{x},aY+\overline{y}) \end{cases} \tag{3.2}$$

where

$$a = \frac{\sqrt{-\overline{xy}F'(\overline{y})G_x(\overline{x},\overline{y})}}{\overline{x}F'(\overline{y})}.$$

By showing that $V'''(0) < 0$ where $V(X) = P(X) - X, P(X)$ being the local Poincaré transformation at $(0,0)$ we may conclude that $(0,0)$ is a vague attractor for (6) whereas $(\overline{x},\overline{y})$ is a vague attractor for (3) or (4).

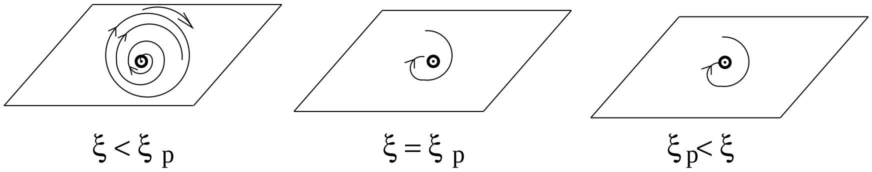


Figure 12.

To this end we use the following expression, taken from [4]:

$$V'''(0) = \frac{3\pi}{4|\lambda(\xi_p)|}\Big(\frac{\partial^3 Z_1}{\partial X^3} + \frac{\partial^3 Z_1}{\partial X \partial Y^2} + \frac{\partial^3 Z_2}{\partial X^2 \partial Y} + \frac{\partial^3 Z_2}{\partial Y^3}\Big) +$$
$$+ \frac{3\pi}{4|\lambda(\xi_p)|^2}\Bigg(-\frac{\partial^2 Z_1}{\partial X^2}\frac{\partial^2 Z_1}{\partial X \partial Y} + \frac{\partial^2 Z_2}{\partial Y^2}\frac{\partial^2 Z_2}{\partial X \partial Y} + \frac{\partial^2 Z_2}{\partial X^2}\frac{\partial^2 Z_2}{\partial X \partial Y} -$$
$$-\frac{\partial^2 Z_1}{\partial Y^2}\frac{\partial^2 Z_1}{\partial X \partial Y} + \frac{\partial^2 Z_1}{\partial X^2}\frac{\partial^2 Z_2}{\partial X^2} - \frac{\partial^2 Z_1}{\partial Y^2}\frac{\partial^2 Z_2}{\partial Y^2}\Bigg).$$

For both systems:

$$\frac{\partial Z_1}{\partial X}(0,0) = F(\overline{y}) = 0\ , \quad \frac{\partial^2 Z_1}{\partial X^2}(0,0) = \frac{\partial^3 Z_1}{\partial X^3}(0,0) = 0,$$
$$\frac{\partial Z_1}{\partial Y}(0,0) = a\overline{x}F(\overline{y})\ , \quad \frac{\partial^2 Z_1}{\partial Y^2}(0,0) = \overline{x}a^2F''(\overline{y}),$$
$$\frac{\partial^3 Z_1}{\partial X \partial Y^2}(0,0) = a^2F''(\overline{y}) \quad \frac{\partial^2 Z_1}{\partial X \partial Y}(0,0) = aF'(\overline{y}),$$
$$\frac{\partial Z_2}{\partial Y}(0,0) = 0 \quad \frac{\partial^2 Z_2}{\partial Y^2}(0,0) = a\overline{y}G_{yy}(\overline{x},\overline{y}),$$

$$\frac{\partial Z_2}{\partial Y}(0,0) = 0, \quad \frac{\partial^2 Z_2}{\partial Y^2}(0,0) = a\overline{y}G_{yy}(\overline{x},\overline{y}),$$
$$\frac{\partial^3 Z_2}{\partial Y^3}(0,0) = 3a^2G_{yy}(\overline{x},\overline{y})\ , \quad \frac{\partial^2 Z_2}{\partial X \partial Y}(0,0) = G_x(\overline{x},\overline{y}) + \overline{y}G_{xy}(\overline{x},\overline{y}),$$
$$\frac{\partial Z_2}{\partial X}(0,0) = \frac{\overline{y}G_x(\overline{x},\overline{y})}{a}\ , \quad \frac{\partial^2 Z_2}{\partial X^2}(0,0) = \frac{\overline{y}}{a}G_{xx}(\overline{x},\overline{y}),$$
$$\frac{\partial^3 Z_2}{\partial X^3}(0,0) = \frac{\overline{y}G_{xxx}(\overline{x},\overline{y})}{a}\ , \quad \frac{\partial^3 Z_2}{\partial X^2 \partial Y}(0,0) = G_{xx}(\overline{x},\overline{y}) + \overline{y}G_{yxx}(\overline{x},\overline{y}).$$

Thus, taking into account:

$$a = \frac{|\lambda(\xi_p)|}{\overline{x}F'(\overline{y})}\, , \frac{aF'(\overline{y}) + \overline{y}aG_{yy}(\overline{x},\overline{y})}{|\lambda(\xi_p)|} = \frac{1}{x} + \frac{\overline{y}G_{yy}(\overline{x},\overline{y})}{\overline{x}F'(\overline{y})},$$
$$G_{yxx}(\overline{x},\overline{y}) - \frac{G_{xx}(\overline{x},\overline{y})G_{xy}(\overline{x},\overline{y})}{G_x(\overline{x},\overline{y})} = 0$$

we obtain

$$V'''(0) = \frac{3\pi}{4|\lambda(\xi_p)|}\Big(-\frac{2\overline{y}G_x(\overline{x},\overline{y})G_{yy}(\overline{x},\overline{y})}{\overline{x}F'(\overline{y})} + \frac{\overline{y}^2G_{yy}(\overline{x},\overline{y})G_{xy}(\overline{x},\overline{y})}{\overline{x}F'(\overline{y})} - \frac{\overline{y}a^2F''(\overline{y})G_{yy}(\overline{x},\overline{y})}{F'(\overline{y})}\Big).$$

Finally, we get from this:

$$\left.\begin{array}{c} G_x(\overline{x},\overline{y}) < 0 \; G_{yy}(\overline{x},\overline{y}) < 0\,, \\ F'(\overline{y}) > 0\,, G_{xy}(\overline{x},\overline{y}) \geq 0\,, \\ F''(\overline{y}) < 0 \end{array}\right\} \Rightarrow V'''(0) < 0.$$

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PROBLEM OF CAUCHY FOR LINEAR SINGULARLY PERTURBED IMPULSIVE SYSTEMS

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Abstract. An initial value problem for singularly perturbed systems of ordinary differential equations is considered in a critical case. A unique asymptotic expansion of the solution is constructed by the method of boundary functions and generalized inverse matrices and projectors under some additional conditions.

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1. **Formulation of the problem**

A singularly perturbed system

$$\varepsilon\frac{dx}{dt} = Ax + \varepsilon A_1(t)x + \varphi(t),\ t \in [a,b],\ t \neq \tau_i,\ i = \overline{1,p},\ 0 < \varepsilon << 1, \tag{1.1}$$

$$a \equiv \tau_0 < \tau_1 < \cdots < \tau_p < \tau_{p+1} \equiv b$$

is considered. The coefficients of system (1.1) satisfy the following conditions

H1: A is $n \times n$ matrix with constant elements. It has an eigenvalue $\lambda = 0$, whose multiplicity is r and r linear independent eigenvectors correspond to this eigenvalue. The remaining $(n-r)$ eigenvalues have negative real parts, i.e.,

$$\lambda_j \in \sigma(A), \quad Re\lambda_j < 0, \qquad j = \overline{1, n-r},\ \ \lambda_j = 0, j = \overline{n-r+1, n}.$$

Condition H1 shows that system (1.1) is considered in a critical case [9].

H2: $A_1(t)$ is $n \times n$ matrix. Its elements are continuously differentiable functions of class $C^\infty[a,b]$.

H3: Vector-function $\varphi(t) : [a, b] \to \mathbf{R}^n$ is partially continuous with break points of the first kind $\tau_i,\ i = \overline{1, p}$, i.e.,

$$\varphi(t) = \varphi_i(t),\ t \in (\tau_{i-1}, \tau_i],\ i = \overline{1, p+1},\ \ \varphi(a) = \varphi_1(\tau_0),$$
$$\varphi(b) = \varphi_{p+1}(\tau_{p+1}),\ \ \varphi_{i+1}(\tau_i) = \lim_{t \to \tau_i + 0} \varphi(t),\ i = \overline{1, p}.$$

An n-dimensional vector-function $x(t, \varepsilon)$ is sought for such that $x(\cdot, \varepsilon)$ is continuously differentiable in every subinterval $[a, \tau_0],\ (\tau_{i-1}, \tau_i],\ i = \overline{2, p+1},\ x(t, \cdot) \in C(0, \varepsilon_0]$ and satisfying system (1.1), the generalized initial condition

$$Dx(a) = v \tag{1.2}$$

and the generalized impulse conditions in fixed moments of time

$$N_i x(\tau_i + 0) + M_i x(\tau_i - 0) = h_i,\ i = \overline{1, p}. \tag{1.3}$$

The matrix D is known $s \times n$ matrix with constant elements, v is given column vector from $\mathbf{R}^s$ and $M_i,\ N_i,\ i = \overline{1, p}$, satisfy the following condition

H4: $M_i,\ N_i,\ i = \overline{1, p}$ are known $k_i \times n$ matrices with constant elements, $h_i \in \mathbf{R}^{k_i}$ are given column vectors.

If $\varepsilon = 0$ in (1.1), then the degenerate system is obtained

$$Ax + \varphi(t) = 0, \tag{1.4}$$

which has a solution

$$x_0(t) = P_A^r \alpha_0(t) - A^+ \varphi(t),\ t \in (\tau_{i-1}, \tau_i],\ i = \overline{1, p+1}, \tag{1.5}$$

if and only if

H5: $P_{A^*}^r \varphi(t) = 0,\ t \in (\tau_{i-1}, \tau_i],\ i = \overline{1, p+1}$.

Here $\alpha_0(t)$ is a partially continuous arbitrary r-dimensional vector-function. A^+ denotes a unique Moore - Penrose inverse matrix of the matrix A. According to H1 rank$A = n - r$, then rank$P_A =$ rank$P_{A^*} = r$, where P_A and P_{A^*} are projectors

$$P_A : \mathbf{R}^n \to \ker A,\ \ P_{A^*} : \mathbf{R}^n \to \ker A^*,\ \ A^* = A^T.$$

In $n \times n$ matrix P_A there exist r linear independent columns and in $n \times n$ matrix P_{A^*} there exist r linear independent rows. P_A^r denotes $n \times r$ matrix consisting of arbitrary r linear independent columns of the matrix P_A and $P_{A^*}^r$ denotes $r \times n$ matrix consisting of arbitrary r linear independent rows of the matrix P_{A^*}.

The asymptotic expansion of the solution of the problem (1.1-1.3) is sought for so that under $\varepsilon \to 0$ it tends to solution (1.5) of the degenerate system (1.4) when $t \in (\tau_{i-1}, \tau_i]$. $i = \overline{1, p+1}$.

The essential methods for investigating linear and nonlinear impulsive systems are presented in [2], [7].

Initial and boundary-value problems for singularly perturbed systems of the kind

$$\frac{dx}{dt} = f(t, x, y),\ \ \varepsilon \frac{dy}{dt} = g(t, x, y) \tag{1.6}$$

are considered in monography [8]. [9] considers a critical case for systems of the form $\varepsilon\frac{dx}{dt} = A(t)x + \varepsilon f(t, x, \varepsilon)$, with initial condition of the form $x(a) = v$. If impulse conditions of the form

$$\Delta x|_{t=\tau_i} = S_i x + a_i, \quad i = \overline{1, p} \tag{1.7}$$

are added to (1.6), a system investigated in [1] is obtained. In the same monography initial problems for systems of the form

$$\varepsilon\frac{dx}{dt} = f(t, x, \varepsilon) \tag{1.8}$$

are considered in a critical case. In [7], [1] the fundamental matrix of solutions of the impulsive system

$$\frac{dx}{dt} = A(t)x, \;\; t \neq \tau_i, \;\; \Delta x|_(t = \tau_i) = S_i x, \;\; i = \overline{1, p}$$

is essential under condition $\det(S_i + E) \neq 0$. Making use of the fundamental matrix solutions of system (1.6), (1.7) and system (1.8), (1.7) are constructed in [1].

In the present work generalized impulse conditions are considered and additional requirements are not set for the matrices M_i, N_i. Therefore the fundamental matrix from [7], [1] can not be used in this case.

In this paper the method of boundary functions is used to construct an asymptotic expansion of the solution of the singular problem posed. The generalized inverse matrices and projectors are also used [4], [5], [6].

2. **Asymptotic expansion**

The asymptotic expansion of the solution of problem (1.1-1.3) is sought for in the form

$$x(t, \varepsilon) \equiv x^i(t, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k (x_k^i(t) + \Pi_k^i(\nu_i)), \;\; \nu_i = \frac{t - \tau_{i-1}}{\varepsilon}, \tag{2.1}$$

where $t \in [\tau_0, \tau_1]$ and $t \in (\tau_{i-1}, \tau_i]$, under $i = \overline{2, p+1}$. The elements $x_k^i(t)$ of the expansion generate regular series and $\Pi_k^i(\nu_i)$ are boundary functions in the right neighborhood of the points τ_{i-1}, $i = \overline{1, p+1}$ and generate singular series of the solution.

For the elements of the regular series the following systems are obtained

$$\begin{aligned} &Ax_0^i(t) = -\varphi_i(t), \\ &Ax_k^i(t) = \dot{x}_{k-1}^i(t) - A_1(t)x_{k-1}^i(t), \quad k = 1, 2, 3, \cdots, \end{aligned} \tag{2.2}$$

and for the elements of the singular series - the systems

$$\frac{d\Pi_k^i(\nu_i)}{d\nu_i} = A\Pi_k^i(\nu_i) + f_k^i(\nu_i), \;\; k = 0, 1, 2, \cdots, \;\; i = \overline{1, p+1}, \tag{2.3}$$

where the functions $f_k^i(\nu_i)$, $i = \overline{1, p+1}$ have the presentation

$$f_k^i(\nu_i) = \begin{cases} 0, & k = 0 \\ \sum_{j=0}^{k-1} \frac{A_1^{(k-j-1)}(\tau_{i-1})}{(k-j-1)!} \nu_i^{k-j-1} \Pi_j^i(\nu_i), & k = 1, 2, \cdots \end{cases}$$

Systems (2.2) have solutions of the form

$$\begin{aligned} x_0^i(t) &= P_A^r \alpha_0^i(t) - A^+ \varphi_i(t), \\ x_k^i(t) &= P_A^r \alpha_k^i(t) + A^+ (L x_{k-1}^i)(t), \quad k = 1, 2, 3, \cdots , \end{aligned} \tag{2.4}$$

if and only if H5 and

(I): $P_{A^*}^r (L x_{k-1}^i)(t) = 0, \ k = 1, 2, 3, \cdots , \ i = \overline{1, p+1}$
are fulfilled. The condition of solvability (I) will be used for the determination of r - dimensional vector - functions $\alpha_k^i(t)$, $k = 1, 2, 3, \cdots$, $i = \overline{1, p+1}$. $(Lx)(t) = \left(\frac{d}{dt}x - A_1(t)x\right)(t)$.

$X(t)$ denotes a normal fundamental matrix of solutions of the homogeneous system $\frac{dx}{dt} = Ax$. According to condition H1 matrix T exists such that

$$\det T \neq 0 \quad \text{and} \quad A = T \begin{pmatrix} \overline{A} & 0 \\ 0 & 0 \end{pmatrix} T^{-1},$$

where $\overline{A}$ is $(n-r) \times (n-r)$ matrix, whose eigenvalues have negative real parts. Then

$$\exp(At) = T \begin{pmatrix} \exp(\overline{A}t) & 0 \\ 0 & E \end{pmatrix} T^{-1} \quad or \quad \exp(At)T = T \begin{pmatrix} \exp(\overline{A}t) & 0 \\ 0 & E \end{pmatrix}.$$

Let the matrix T also be presented in a block form $T = [T_1 \ T_2]$, where T_1 is $n \times (n-r)$ matrix and T_2 is $n \times r$ matrix. Then $\exp(At)T_1 = T_1 \exp(\overline{A}t)$ and $\exp(At)T_2 = T_2$.

A solution of system (2.3) under $k = 0$ has the form

$$\Pi_0^i(\nu_i) = X(\nu_i)c_0^i, \ \ c_0^i \in \mathbf{R}^n.$$

In order to realize a condition $\Pi_0(\nu_i) \to 0$ under $\varepsilon \to 0$ the last r components of the vector c_0^i are taken to be equal to zero. The solution of (2.3) under $k = 0$ becomes the following

$$\Pi_0^i(\nu_i) = X_{n-r}(\nu_i)c_0^i, \ \ c_0^i \in \mathbf{R}^{n-r}, \tag{2.5}$$

where $X_{n-r}(\nu_i) = \exp(A\nu_i)T_1$ is $n \times (n-r)$ matrix. The solution of (2.3) under $k = 1, 2, \cdots$ is

$$\Pi_k^i(\nu_i) = X_{n-r}(\nu_i)c_k^i + \int_0^\infty K(\nu_i, s) f_k^i(s) ds,$$

where

$$K(\nu_i, s) = \begin{cases} X(\nu_i) P X^{-1}(s), & 0 \le s \le \nu_i < \infty, \\ -X(\nu_i)(I - P) X^{-1}(s), & 0 \le \nu_i \le s < \infty, \end{cases}$$

P is a spectral projector of the matrix A on the left semi-plane.

A choice of $K(\nu_i, s)$ guarantees that the partial solutions of (2.3) are bounded exponentially.

For definition of the vector-functions $\alpha_k^i(t)$ the form of $x_k^i(t)$ from (2.4) is substituted in the condition of solvability (I) and the following systems are obtained

$$\overline{D}\dot{\alpha}_k^i(t) = B(t)\alpha_k^i(t) + g_k^i(t) = 0, \ \ k = 0, 1, 2, \cdots , \ \ i = \overline{1, p}, \tag{2.6}$$

where

$$\overline{D} = P^r_{A^*} P^r_A, g^i_k(t) = \begin{cases} -P^r_{A^*}\left(LA^+\varphi_i\right)(t), & k=0 \\ -P^r_{A^*}\left(LA^+(Lx^i_{k-1})(t)\right)(t), & k=1,2,\cdots \end{cases},$$
$$B(t) = P^r_{A^*} A_1(t) P^r_A.$$

According to condition H1 rank$A = r$. It is easy to prove rank$\overline{D} = r$.

A general solution of (2.6) in every subinterval $]\tau_{i-1}, \tau_i]$, $i = \overline{1, p+1}$ is

$$\alpha^i_k(t) = \Phi(t)\Phi^{-1}(\tau_{i-1})\eta^i_k + \int_{\tau_{i-1}}^{t} \Phi(t)\Phi^{-1}(s)\overline{D}^{-1} g^i_k(s)ds, \tag{2.7}$$

where $\Phi(t)$ is $r \times r$ fundamental matrix of solutions of the homogeneous system $\dot{x} = \overline{D}^{-1}B(t)x$ and η^i_k is r-dimensional unknown constant vector.

The initial and impulse conditions will be used for definition of unknown constant vectors η^i_k and c^i_k. Because of the generalized character of the impulsive and initial conditions, the solution $x^i(\varepsilon, t)$ in every subinterval $(\tau_{i-1}, \tau_i]$, $i = \overline{1, p+1}$ depends on an arbitrary constant vector ξ_i under successive gradation from interval to interval. These constants take part in the elements of the singular series and also in the elements of the regular series. Therefore in the interval $(\tau_{i-1}, \tau_i]$ it is got accumulation of the constants $\xi_1,\ \xi_2, \cdots, \xi_i$ in the solution $x^i(\varepsilon, t)$. A dependence between constants ξ_i, $i = \overline{1, p+1}$ exists because of the recurrent relation between the elements $x^i_k(t)$ of the regular series and between the elements of the singular series. Difficulties are obvious if we work by successive gradation from interval to interval. For this reason a modification of the problem (1.1-1.3) similar to this one in [3] is needed.

The following denotations are introduced.

$$\begin{aligned} &Q_0 = [D\ \Theta_1 \cdots \Theta_p]^T,\ Q_1 = [\Theta_0\ N_1 \Theta_2 \cdots \Theta_p]^T, \ldots, Q_i = [\Theta_0 \cdots \Theta_{i-1}\ N_i\ \Theta_{i+1} \cdots \Theta_p]^T, \ldots \\ &Q_p = [\Theta_0 \cdots \Theta_{p-1}\ N_p]^T, \\ &R_1 = [\Theta_0\ M_1\ \Theta_2 \cdots \Theta_p]^T, \ldots, R_i = [\Theta_0 \cdots \Theta_{i-1}\ M_i\ \Theta_{i+1} \cdots \Theta_p]^T, \ldots \\ &R_p = [\Theta_0\ \cdots \Theta_{p-1}\ M_p]^T,\ h = [v\ h_1\ \cdots h_p]^T, \end{aligned}$$

where Q_i, $i = \overline{0, p}$, R_i, $i = \overline{1, p}$ are $\nu \times n$ matrices, $\nu = s + k_1 + k_2 + \cdots + k_p$, Θ_0 is $s \times n$ matrix with zero elements, $\Theta_i,\ i = 1, 2, \cdots p$, are $k_i \times n$ matrices with zero elements, h is ν-dimensional vector. Then the initial and impulse conditions are rewritten as follows

$$\sum_{i=0}^{p} Q_i x^{i+1}(\tau_i, \varepsilon) + \sum_{i=1}^{p} R_i x^i(\tau_i, \varepsilon) = h\,. \tag{2.8}$$

The coefficients before identical powers of ε are equalized in (2.8). Then

$$\begin{aligned} &\textstyle\sum_{i=0}^p Q_i \left(x^{i+1}_0(\tau_i) + \Pi^{i+1}_0(0)\right) + \sum_{i=1}^p R_i \left(x^i_0(\tau_i) + \Pi^i_0\left(\frac{\tau_i - \tau_{i-1}}{\varepsilon}\right)\right) = h, \\ &\textstyle\sum_{i=0}^p Q_i \left(x^{i+1}_k(\tau_i) + \Pi^{i+1}_k(0)\right) + \sum_{i=1}^p R_i \left(x^i_k(\tau_i) + \Pi^i_k\left(\frac{\tau_i - \tau_{i-1}}{\varepsilon}\right)\right) = 0, \\ &k = 1, 2, 3, \cdots. \end{aligned} \tag{2.9}$$

In system (2.9) under $k=0$ are substituted $x_0^i(t)$ and $\Pi_0^i(\nu_i)$, $i=\overline{1,p+1}$ from (2.4) and (2.5), respectively. Then system (2.9) under $k=0$ takes the form

$$\sum_{i=1}^{p+1}\left(l_i\alpha_0^i(\cdot)+D_i(\varepsilon)c_0^i\right)=h+\sum_{i=0}^{p}Q_iA^+\varphi_{i+1}(\tau_i)+\sum_{i=1}^{p}R_iA^+\varphi_i(\tau_i), \tag{2.10}$$

where

$$l_ix(\cdot)=Q_{i-1}P_A^rx(\tau_{i-1})+R_iP_A^rx(\tau_i), i=\overline{1,p}, l_{p+1}x(\cdot)=Q_pP_A^rx(\tau_{i-1})$$

are $\nu\times r$ vector functionals,

$$D_i(\varepsilon)=Q_{i-1}X_{n-r}(0)+R_iX_{n-r}\left(\frac{\tau_i-\tau_{i-1}}{\varepsilon}\right), i=\overline{1,p}, D_{p+1}(\varepsilon)=Q_pX_{n-r}(0)$$

are $(\nu\times(n-r))$ matrices.

Let $\overline{D}(\varepsilon)=[D_1(\varepsilon)\cdots D_{p+1}(\varepsilon)]$ be a $(\nu\times((p+1)(n-r)))$ matrix,
$l(\cdot)=[l_1(\cdot)\cdots l_{p+1}(\cdot)]$ is $(\nu\times((p+1)r))$ a vector functional,
$\overline{h}_0=h+\sum_{i=0}^{p}Q_iA^+\varphi_{i+1}(\tau_i)+\sum_{i=1}^{p}R_iA^+\varphi_i(\tau_i)$ is a ν-dimensional vector,
$\alpha_0(t)=\left[\alpha_0^1(t)\cdots\alpha_0^{p+1}(t)\right]^T$ is a $(p+1)r$-dimensional vector,
$c_0=\left[c_0^1\cdots c_0^{p+1}\right]^T$ is a $(p+1)(n-r)$-dimensional vector.

With these notations system (2.10) may be written as follows

$$l\alpha_0(\cdot)=\overline{h}-\overline{D}(\varepsilon)c_0. \tag{2.11}$$

In the last equality are substituted $\alpha_0^i(t)$, $i=\overline{1,p+1}$ from (2.7). Then we obtain

$$P\eta_0+\overline{D}(\varepsilon)c_0=\overline{\overline{h}}_0, \tag{2.12}$$

where $P=[P_1\cdots P_{p+1}]$ is $(\nu\times(n+1)r)$ matrix, $P_i=l_i\Phi(\cdot)\Phi^{-1}(\tau_{i-1})$, $i=\overline{1,p+1}$ are $(\nu\times r)$ matrices, $\overline{\overline{h}}_0=\overline{h}_0-\sum_{i=1}^{p+1}l_i\int_{\tau_{i-1}}^{(}\cdot)\Phi(\cdot)\Phi^{-1}(s)\overline{D}^{-1}g_0^i(s)ds$ is ν-dimensional vector and $\eta_0=\left[\eta_0^1\cdots\eta_0^{p+1}\right]^T$ is $(p+1)r$-dimensional vector.

The matrix $\overline{D}(\varepsilon)$ has a structure $\overline{D}(\varepsilon)=\overline{D}_0+O\left(\varepsilon^q\exp\left(-\frac{\alpha}{\varepsilon}\right)\right)$, $q\in N$, α is a positive constant, $\overline{D}_0$ is $(\nu\times(p+1)(n-r))$ matrix with constant elements. The exponentially small elements in the matrix $\overline{D}(\varepsilon)$ are rejected and system (2.12) takes the form

$$M\begin{bmatrix}\eta_0\\ c_0\end{bmatrix}=\overline{\overline{h}}_0, \tag{2.13}$$

where $M=\left[P\ \overline{D}_0\right]$ is $(\nu\times(p+1)n)$ matrix. Let the following condition be fulfilled.

H6: $\operatorname{rank}M=m_1\le\min(\nu,(p+1)n)$.

Then system (2.13) has a solution

$$\begin{bmatrix}\eta_0\\ c_0\end{bmatrix}=P_M^k\xi_0+M^+\overline{\overline{h}}_0, \tag{2.14}$$

if and only if

H7: $P^d_{M^*}\overline{\overline{h}}_0 = 0$.
P^k_M designates a matrix consisting of $k = (p+1) - m_1$ linear independent columns of the matrix projector P_M, $P_M : \mathbf{R}^{(p+1)n} \to \ker M$ and $P^d_{M^*}$ denotes a matrix consisting of $d = \nu - m_1$ linear independent rows of the matrix projector P_{M^*}, $P_{M^*} : \mathbf{R}^{\nu} \to \ker M^*$, $\xi_0 \in \mathbf{R}^k$ and M^+ is a unique Moore-Penrose inverse matrix of the matrix M.

Let $(p+1)r = \overline{r}$ and $(p+1)(n-r) = \overline{n}$. Then

$$\eta_0 = \left[P^k_M\right]_{\overline{r}}\xi_0 + \left[M^+\overline{\overline{h}}_0\right]_{\overline{r}}, \quad c_0 = \left[P^k_M\right]_{\overline{n}}\xi_0 + \left[M^+\overline{\overline{h}}_0\right]_{\overline{n}},$$

where index $\overline{r}$ means the first $\overline{r}$ rows of the matrix P^k_M and the vector $M^+\overline{\overline{h}}_0$ and index $\overline{n}$ means the last $\overline{n}$ rows of the matrix P^k_M and the vector $M^+\overline{\overline{h}}_0$.

According to denotations of η_0 and c_0 above, we obtain

$$\eta^i_0 = \left[P^k_M\right]^{r_i}_{\overline{r}}\xi_0 + \left[M^+\overline{\overline{h}}_0\right]^{r_i}_{\overline{r}}, \quad c_0 = \left[P^k_M\right]^{n_i}_{\overline{n}}\xi_0 + \left[M^+\overline{\overline{h}}_0\right]^{n_i}_{\overline{n}}, \quad i = \overline{1, p+1}, \tag{2.15}$$

where the index r_i means that under $i = 1$ we take the first r rows of the matrix $\left[P^k_M\right]_{\overline{r}}$ and the vector $\left[M^+\overline{\overline{h}}_0\right]_{\overline{r}}$, under $i = 2$ the second r rows of the same matrix and the same vector and etc. The index n_i means that under $i = 1$ we take the first $(n-r)$ rows of the matrix $\left[P^k_M\right]_{\overline{n}}$ and the vector $\left[M^+\overline{\overline{h}}_0\right]_{\overline{n}}$, under $i = 2$- the second $(n-r)$ rows and etc.

According to (2.15) the forms of $x^i_0(t)$ and $\Pi^i_0(\nu_i)$ become the following

$$\begin{gathered}
x^i_0(t) = \Phi^{r_i}_r(t)\xi_0 + \overline{x}^i_0(t), \ \Pi^i_0(\nu_i) = X^{n_i}_{n-r}(\nu_i)\xi_0 + \overline{\Pi}^i_0(\nu_i) \ \ i = \overline{1, p+1}, \\
\Phi^{r_i}_r(t) = P^r_A\Phi(t)\Phi^{-1}(\tau_{i-1})\left[P^k_M\right]^{r_i}_{\overline{r}}, X^{n_i}_{n-r}(\nu_i) = X_{n-r}(\nu_i)\left[P^k_M\right]^{r_i}_{\overline{r}}, \\
\overline{\Pi}^i_0(\nu_i) = X_{n-r}(\nu_i)\left[M^+\overline{\overline{h}}_0\right]^{n_i}_{\overline{n}}, \\
\overline{x}^i_0(t) = P^r_A\Phi(t)\Phi^{-1}(\tau_{i-1})\left[M^+\overline{\overline{h}}_0\right]^{r_i}_{\overline{r}} + P^r_A\int_{\tau_{i-1}}^{t}\Phi(t)\Phi^{-1}(s)\overline{D}^{-1}g^i_0(s) - A^+\varphi_i(t).
\end{gathered} \tag{2.16}$$

On analogy of system (2.11) the following system is obtained

$$l\alpha_1(\cdot) + \overline{D}(\varepsilon)c_1 = \overline{h}_1(\varepsilon), \tag{2.17}$$

where $\alpha_1(t) = \left[\alpha^1_1(t)\cdots\alpha^{p+1}_1(t)\right]^T$ is $(p+1)r$-dimensional vector, $c_1 = \left[c^1_1\cdots c^{p+1}_1\right]^T$ is $(p+1)(n-r)$-dimensional vector.

Keeping in mind the expressions from (2.16), the ν-dimensional vector $\overline{h}_1(\varepsilon)$ may be written as follows

$$\overline{h}_1(\varepsilon) = \left(A_{11} + A_{12}(\varepsilon)\right)\xi_0 + a_1(\varepsilon), \tag{2.18}$$

where

$$A_{11} = -\sum_{i=0}^{p+1} Q_iA^+\left(L\Phi^{r_{i+1}}_r\right)(\tau_i) - \sum_{i=1}^{p} R_iA^+\left(L\Phi^{r_i}_r\right)(\tau_i),$$

$$A_{12}(\varepsilon)=-\sum_{i=0}^{p+1}Q_i\int_0^\infty K(0,s)A_1(\tau_i)X_{n-r}^{n_{i+1}}(s)ds-$$
$$-\sum_{i=1}^{p}R_i\int_0^\infty K\left(\frac{\tau_i-\tau_{i-1}}{\varepsilon},s\right)A_1(\tau_i)X_{n-r}^{n_i}(s)ds$$

$$a_1(\varepsilon)=-\sum_{i=0}^{p+1}Q_iA^+\left(L\overline{x}_0^{i+1}\right)(\tau_i)-\sum_{i=1}^{p}R_iA^+\left(L\overline{x}_0^{i}\right)(\tau_i)-$$
$$-\sum_{i=0}^{p+1}Q_i\int_0^\infty K(0,s)A_1(\tau_i)\overline{\Pi}_0^{i+1}(s)ds-\sum_{i=1}^{p}R_i\int_0^\infty K\left(\frac{\tau_i-\tau_{i-1}}{\varepsilon},s\right)A_1(\tau_i)\overline{\Pi}_0^{i}(s)ds.$$

According to (2.6) the functions $\alpha_1^i(t)$, $i=\overline{1,p+1}$ are defined from the systems

$$\overline{D}\dot{\alpha}_1^i(t)=B(t)\alpha_1^i(t)+g_1^i(t),\ i=\overline{1,p+1}.$$

The functions $g_1^i(t),\ i=\overline{1,p+1}$ are presented in the form

$$g_1^i(t)=B_{11}^i(t)\xi_0+b_1^i(t), \tag{2.19}$$

where $B_{11}^i(t)=-P_{A^*}^r\left(LA^+\left(L\Phi_r^{r_i}\right)(t)\right)(t)$, $b_1^i(t)=-P_{A^*}^r\left(LA^+\left(L\overline{x}_0^i\right)(t)\right)(t)$.

Analogously to system (2.13) the following system is obtained

$$M\begin{bmatrix}\eta_1\\ c_1\end{bmatrix}=\overline{\overline{h}}_1(\varepsilon), \tag{2.20}$$

where $\eta_1=\left(\eta_1^1\cdots\eta_1^{p+1}\right)^T$ is $(p+1)r$-dimensional vector and

$$\overline{\overline{h}}_1(\varepsilon)=\overline{A}_{11}(\varepsilon)\xi_0+\overline{a}_1(\varepsilon), \tag{2.21}$$

$\overline{A}_{11}(\varepsilon)=A_{11}+A_{12}(\varepsilon)-\sum_{i=1}^{p+1}l_i\int_{\tau_{i-1}}^{(\cdot)}\Phi(\cdot)\Phi^{-1}(s)B_{11}^i(s)ds,$
$\overline{a}_1(\varepsilon)=a_1(\varepsilon)-\sum_{i=1}^{p+1}l_i\int_{\tau_{i-1}}^{(\cdot)}\Phi(\cdot)\Phi^{-1}(s)b_1^i(s)ds.$

System (2.20) has solution

$$\eta_1^i=\left[P_M^k\right]_{\overline{r}}^{r_i}\xi_1+\left[M^+\overline{\overline{h}}_1\right]_{\overline{r}}^{r_i},\ c_1^i=\left[P_M^k\right]_{\overline{n}}^{n_i}\xi_1+\left[M^+\overline{\overline{h}}_1\right]_{\overline{n}}^{n_i},\ i=\overline{1,p+1}, \tag{2.22}$$

if and only if $P_{M^*}^d\overline{\overline{h}}_1=0$.

Keeping in mind the form of $\overline{\overline{h}}_1(\varepsilon)$ from (2.21), the last condition may be written as follows

$$R(\varepsilon)\xi_0=\overline{\overline{a}}_1(\varepsilon), \tag{2.23}$$

where $R(\varepsilon)=P_{M^*}^d\overline{A}_{11}(\varepsilon)$ is $d\times\nu$ matrix and $\overline{\overline{a}}_1(\varepsilon)=-P_{M^*}^d\overline{a}_1(\varepsilon)$ is d-dimensional vector.

The matrix $R(\varepsilon)$ has a structure $R(\varepsilon)=R_0+O\left(\varepsilon^s\exp\left(-\frac{\alpha}{\varepsilon}\right)\right)$ and the vector $\overline{\overline{a}}_1(\varepsilon)$ has a structure $\overline{\overline{a}}_1(\varepsilon)=\overline{\overline{a}}_{10}+O\left(\varepsilon^q\exp\left(-\frac{\alpha}{\varepsilon}\right)\right)$, $s,\ q\in N$, α-positive constant, R_0-$d\times\nu$ constant matrix, $\overline{\overline{a}}_{10}$- d-dimensional constant vector.

The exponentially small elements in $R(\varepsilon)$ and $\overline{\overline{a}}_1(\varepsilon)$ are rejected and system (2.23) takes the form

$$R_0\xi_0 = \overline{\overline{a}}_{10}. \tag{2.24}$$

Let the following condition be fulfilled

H8: $\mathrm{rank} R_0 = \nu < d$.
Then system (2.24) has a unique solution

$$\xi_0 = R_0^+\overline{\overline{a}}_{10}, \tag{2.25}$$

if and only if $P_{R_0^*}P_{M^*}^d\overline{\overline{a}}_{10} = 0$.

The last requirement is always fulfilled if the following condition is real

H9: $P_{R_0^*}P_{M^*}^d = 0$.

According to (2.25), the equalities (2.16) take the representation

$$x_0^i(t) = \Phi_r^{r_i}(t)R_0^+\overline{\overline{a}}_{10} + \overline{x}_0^i(t),\ \Pi_0^i(\nu_i) = X_{n-r}^{n_i}(\nu_i)R_0^+\overline{\overline{a}}_{10} + \overline{\Pi}_0^i(\nu_i),\ i = \overline{1, p+1}. \tag{2.26}$$

From (2.4), (2.5) and (2.22) the following is obtained

$$x_1^i(t) = \Phi_r^{r_i}(t)\xi_1 + \overline{x}_1^i(t),\ \Pi_1^i(\nu_i) = X_{n-r}^{n_i}(\nu_i)\xi_1 + \overline{\Pi}_1^i(\nu_i),\ i = \overline{1, p+1}, \tag{2.27}$$

where $\overline{x}_1^i(t) = P_A^r\Phi(t)\Phi^{-1}(\tau_{i-1})\left[M^+\overline{\overline{h}}_1\right]_{\overline{r}}^{r_i} + P_A^r\int_{\tau_{i-1}}^t \Phi(t)\Phi^{-1}(s)\overline{D}^{-1}g_1^i(s)ds + A^+\left(Lx_0^i\right)(t)$,
$\overline{\Pi}_1^i(\nu_i) = X_{n-r}(\nu_i)\left[M^+\overline{\overline{h}}_1\right]_{\overline{n}}^{n_i} + \int_0^\infty K(\tau,s)A_1(\tau_i)\Pi_0^i(s)ds$.

Analogously to system (2.11) the following system is obtained

$$l\alpha_2(\cdot) + \overline{D}(\varepsilon)c_2 = \overline{h}_2(\varepsilon), \tag{2.28}$$

where $\alpha_2(t) = \left(\alpha_2^1(t)\cdots\alpha_2^{p+1}(t)\right)^T$ is $(p+1)r$-dimensional vector, $c_2 = (c_2^1\cdots c_2^{p+1})^T$ is
$(p+1)(n-r)$- dimensional vector and $\overline{h}_2(\varepsilon) = \left(A_{11} + A_{12}(\varepsilon)\right)\xi_1 + a_2(\varepsilon)$ is ν-dimensional vector, $a_2(\varepsilon) = -\sum_{i=0}^p Q_iA^+\left(L\overline{x}_1^{i+1}\right)(\tau_i) - \sum_{i=1}^p R_iA^+\left(L\overline{x}_1^i\right)(\tau_i) -$
$-\sum_{i=0}^p Q_i\int_0^\infty K(0,s)\left(A_1(\tau_i)\overline{\Pi}_1^{i+1}(s) + A_1'(\tau_i)s\Pi_0^{i+1}(s)\right)ds -$
$-\sum_{i=1}^p R_i\int_0^\infty K\left(\frac{\tau_i-\tau_{i-1}}{\varepsilon}, s\right)\left(A_1(\tau_i)\overline{\Pi}_1^i(s) + A_1'(\tau_i)s\Pi_0^i(s)\right)ds.$

According to (2.7), the functions $\alpha_2^i(t)$, $i = \overline{1, p+1}$ have the form

$$\alpha_2^i(t) = \Phi(t)\Phi^{-1}(\tau_{i-1})\eta_2^i + \int_{\tau_{i-1}}^t \Phi(t)\Phi^{-1}(s)\overline{D}^{-1}g_2^i(s)ds,$$

where $g_2^i(t)$ has the form

$$g_2^i(t) = B_{11}^i(t)\xi_1 + b_2^i(t), \tag{2.29}$$
$$b_2^i(t) = -P_{A^*}^r\left(LA^+\left(L\overline{x}_1^i\right)(t)\right)(t).$$

A system

$$M\begin{bmatrix}\eta_2\\ c_2\end{bmatrix} = \overline{h}_2(\varepsilon,\xi_1), \tag{2.30}$$

is obtained analogously to system (2.11). From solvability condition $P^d_{M^*}\overline{h}_2(\varepsilon,\xi_1)=0$ of system (2.30) and by analogy to (2.23), the following system is obtained

$$R(\varepsilon)\xi_1=\overline{\overline{a}}_2(\varepsilon), \tag{2.31}$$

where $\overline{\overline{a}}_2(\varepsilon)=-P^d_{M^*}\overline{a}_2(\varepsilon)$, $\overline{a}_2(\varepsilon)=a_2(\varepsilon)-\sum_{i=1}^{p+1}l_i\int_{\tau_{i-1}}^{(\cdot)}\Phi(\cdot)\Phi^{-1}(s)\overline{D}^{-1}b_2^i(s)ds$, and $\overline{\overline{a}}_2(\varepsilon)=\overline{\overline{a}}_{20}+O\left(\varepsilon^s\exp\left(-\frac{\alpha}{\varepsilon}\right)\right)$ $s\in N$, α - positive constant, $\overline{\overline{a}}_{20}$ - d-dimensional vector.

The exponentially small elements in $R(\varepsilon)$ and $\overline{\overline{a}}_2(\varepsilon)$ are rejected and system (2.31) takes the form

$$R_0\xi_1=\overline{\overline{a}}_{20}.$$

The last system under condition H8 has a unique solution

$$\xi_1=R_0^+\overline{\overline{a}}_{20}.$$

if and only if condition H9 is fulfilled. The last equality is substituted in (2.27). Then the following expressions for $x_1^i(t)$ and $\Pi_1^i(\nu_i)$ are obtained

$$x_1^i(t)=\Phi_r^{r_i}(t)R_0^+\overline{\overline{a}}_{20}+\overline{x}_1^i(t),\ \Pi_1^i(\nu_i)=X_{n-r}^{n_i}R_0^+\overline{\overline{a}}_{20}+\overline{\Pi}_1^i(\nu_i),\ i=\overline{1,p+1}. \tag{2.32}$$

Analogously to the statement above for $k=2,3,\cdots$ the following is obtained

$$x_k^i(t)=\Phi_r^{r_i}(t)R_0^+\overline{\overline{a}}_{k+1,0}+\overline{x}_k^i(t),\ \Pi_k^i(\nu_i)=X_{n-r}^{n_i}R_0^+\overline{\overline{a}}_{k+1,0}+\overline{\Pi}_k^i(\nu_i),\ i=\overline{1,p+1}. \tag{2.33}$$

$$\overline{x}_k^i(t)=P_A^r\Phi(t)\Phi^{-1}(\tau_{i-1})\left[M^+\overline{\overline{h}}_k\right]_{\overline{r}}^{r_i}+P_A^r\int_{\tau_{i-1}}^t\Phi(t)\Phi^{-1}(s)\overline{D}^{-1}g_k^i(s)ds+A^+\left(Lx_{k-1}^i\right)(t),$$

$$\overline{\Pi}_k^i(\nu_i)=X_{n-r}(\nu_i)\left[M^+\overline{\overline{h}}_k\right]_{\overline{n}}^{n_i}+\int_0^\infty K(\tau,s)\sum_{j=0}^{k-1}\frac{A_1^{k-j-1}(\tau_i)}{(k-j-1)!}s^{k-j-1}\Pi_j^i(s)ds,$$

$$\overline{\overline{a}}_k(\varepsilon)=-P^d_{M^*}\overline{a}_k(\varepsilon),\overline{a}_k(\varepsilon)=a_k(\varepsilon)-\sum_{i=1}^{p+1}l_i\int_{\tau_{i-1}}^{(\cdot)}\Phi(\cdot)\Phi^{-1}(s)\overline{D}^{-1}b_k^i(s)ds,$$

$b_k^i(t)=-P^r_{A^*}\left(LA^+\left(L\overline{x}_{k-1}^i\right)(t)\right)(t)$, $a_k(\varepsilon)=-\sum_{i=0}^pQ_iA^+\left(L\overline{x}_{k-1}^{i+1}\right)(\tau_i)-$
$-\sum_{i=1}^pR_iA^+\left(L\overline{x}_{k-1}^i\right)(\tau_i)-\sum_{i=0}^pQ_i\int_0^\infty K(0,s)\left(\sum_{j=0}^{k-2}\frac{A_1^{(k-j-1)}(\tau_i)}{(k-j-1)!}s^{k-j-1}\Pi_j^{i+1}(s)+\right.$
$\left.+A_1(\tau_i)\overline{\Pi}_{k-1}^{i+1}(s)\right)ds-\sum_{i=1}^pR_i\int_0^\infty K\left(\frac{\tau_i-\tau_{i-1}}{\varepsilon},s\right)\left(\sum_{j=0}^{k-2}\frac{A_1^{(k-j-1)}(\tau_i)}{(k-j-1)!}s^{k-j-1}\Pi_j^{i+1}(s)+\right.$
$\left.+A_1(\tau_i)\overline{\Pi}_{k-1}^i(s)\right)ds$, $\overline{\overline{h}}_k(\varepsilon)=\overline{A}_{11}(\varepsilon)\xi_{k-1}+\overline{a}_k(\varepsilon)$, $\overline{h}_k(\varepsilon)=\left(A_{11}+A_{12}(\varepsilon)\right)\xi_{k-1}+a_k(\varepsilon)$,
$g_k^i(t)=B_{11}^i(t)R_0^+\overline{\overline{a}}_{k+1,0}+b_k^i(t)$.

Let

$$u(t,\varepsilon)=x(t,\varepsilon)-X_n(t,\varepsilon), \tag{2.34}$$

where $x(t,\varepsilon)$ is the exact solution of (1.1), (1.2), (1.3) and

$$X_n(t,\varepsilon)=\sum_{k=0}^n\varepsilon^k(x_k^i(t)+\Pi_k^ix(\nu_i)),\ \ i=\overline{1,p+1}.$$

It is easy to show that under some conditions $\|u(t,\varepsilon)\| \le c\varepsilon^{n+1}$, using the scheme of proof in [8], [1] with the necessary changes originating from the generalized initial and impulse conditions.

On this way the following theorem is proved.

Theorem 1: *Let the conditions H1-H4, H6 and H8 be fulfilled. The initial impulsive problem (1.1),(1.2), (1.3) has a unique asymptotic expansion of the solution in the form (2.1). The coefficients of the regular and singular series have the representation (2.26), under* $k = 0$ *and (2.32), (2.33) under* $k = 1, 2, 3, \cdots$ *if and only if* $\varphi(t)$ *satisfies the condition H5 and* $v,\ \ h_i,\ \ i = \overline{1,p}$ *satisfy H7 and H9.*

The next bound is true for the boundary functions

$$\|\Pi_k^i(\nu_i)\| \le \sigma exp(-\kappa\nu_i),\ \ i = \overline{1,p+1},\ \ k = 0,1,\cdots,$$

where σ *and* κ *are positive constants.*

Remark 1: *Let, instead of H6, the following condition be fulfilled*

H10: $\nu = (p+1)n$, $\mathrm{rank}M = m_1 < \min(\nu,(p+1)n)$.

Then $\mathrm{rank}P_M = \mathrm{rank}P_{M^*} = \nu - m_1 = k = d$. *The matrix* $\mathrm{rank}P_{M^*}$ *is* $k \times \nu$ *matrix and the matrix* R_0 *is a rectangular matrix. According to H8 (d = k), system (2.23) is always solvable as condition H9 is always real and* $R_0^+ = R_0^{-1}$. *The solution of problem (1.1),(1.2), (1.3) is presented in series (2.1) whose coefficients have the form (2.26), (2.32), (2.33).*

Remark 2: *Let instead of H6 the following condition be fulfilled*

H11: $\nu = (p+1)n$, $detM \neq 0$.

Then systems (2.13), (2.30) are always solvable, $P_{M^*} = 0$, $M^+ = M^{-1}$. *The coefficients of the formally asymptotic solution of the problem (1.1),(1.2), (1.3) have the representation*

$x_k^i(t) = \overline{x}_k^i(t)$, $\Pi_k^i(\nu_i) = \overline{\Pi}_k^i(\nu_i)$, $k = 0,1,2,\cdots$, $i = \overline{1,p+1}$.

3. Example

Let $t \in [0,2]$, $t \neq \tau_1$, $\tau_1 = 1$ and problem (1.1-1.3) have the following coefficients:

$$A = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix},\ \ A_1(t) = \begin{pmatrix} -t & 2t+1 \\ 2t & -4t+2 \end{pmatrix},\ \ \varphi(t) = \begin{cases} \begin{pmatrix} 1 & -1 \end{pmatrix}, & t \in [0,1] \\ 0, & t \in (1,2] \end{cases},$$

$$D = \begin{pmatrix} 2 & 6 \end{pmatrix},\ \ v = 1,\ \ N_1 = \begin{pmatrix} 0 & 0 \\ 1 & 3 \\ 2 & 1 \end{pmatrix},\ \ M_1 = \begin{pmatrix} -1 & 2 \\ 0 & 0 \\ 1 & -2 \end{pmatrix},\ \ h_1 = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}.$$

Then

$$A^+ = \frac{1}{10}\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix},\ \ P_A^1 = \frac{1}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix},\ \ P_{A^*}^1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \end{pmatrix},$$

Obviously the requirement H5 is fulfilled. Further $\overline{D} = \frac{3}{10}$, $\overline{D}^{-1} = \frac{10}{3}$, $B(t) = \frac{3}{10}$, $\Phi(t) = e^t$, $\Phi^{-1}(t) = e^{-t}$,

$$X(t) = \frac{1}{3}\begin{pmatrix} 2+e^{-3t} & 2-2e^{-3t} \\ 1-e^{-3t} & 1+2e^{-3t} \end{pmatrix},\ X^{-1}(t) = \frac{1}{3}\begin{pmatrix} 2+e^{3t} & 2-2e^{3t} \\ 1-e^{3t} & 1+2e^{3t} \end{pmatrix},$$

$$X_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t},\ P = \frac{1}{3}\begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix},\ \ I-P = \frac{1}{3}\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix},$$

$$K(\nu_i, s) = \begin{cases} \frac{1}{3}\begin{pmatrix} e^{-3(\nu_i-s)} & -2e^{-3(\nu_i-s)} \\ -e^{-3(\nu_i-s)} & 2e^{-3(\nu_i-s)}, \end{pmatrix} & 0 \le s \le \nu_i < \infty, \\ \frac{1}{3}\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}, & 0 \le \nu_i \le s < \infty. \end{cases}$$

$$Q_0 = \begin{pmatrix} 2 & 6 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},\ \ Q_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 3 \\ 2 & 1 \end{pmatrix},\ \ R_1 = \begin{pmatrix} 0 & 0 \\ -1 & 2 \\ 0 & 0 \\ 1 & -2 \end{pmatrix},\ \ \overline{h}_0 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 4 \end{pmatrix},$$

$$P = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix},\ \overline{D}(\varepsilon) = \begin{pmatrix} -4 & 0 \\ -3e^{-\frac{3}{\varepsilon}} & 0 \\ 0 & -2 \\ 3e^{-\frac{3}{\varepsilon}} & 1 \end{pmatrix},\ M = \begin{pmatrix} 2 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

then $M^+ = \dfrac{2}{30}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 10 & 20 \\ -6 & 0 & 0 & 0 \\ 0 & 0 & -10 & 10 \end{pmatrix}$, $P^1_M = \dfrac{1}{5}\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $P^1_{M^*} = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$,

$\overline{\overline{h}}_0 = (3\ 0\ 1\ 3)^T$, $P^1_{M^*}\overline{\overline{h}}_0 = 0$, i.e., the condition H7 is fulfilled.

Let $t \in [0,1]$, then $\nu_1 = \frac{t}{\varepsilon}$ and in accordance with (2.7) for $\alpha_0^1(t)$ and $\alpha_0^2(t)$ the following is obtained

$$\alpha_0^1(t) = e^t\eta_0^1 + \frac{1}{3}e^t + \frac{5}{3}t - \frac{1}{3},\ \alpha_0^2(t) = e^{t-1}\eta_0^2$$

$$x_0^1(t) = \frac{1}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t\eta_0^1 + \frac{1}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\left(\frac{1}{3}e^t + \frac{5}{3}t - \frac{1}{3}\right) - \frac{1}{5}\begin{pmatrix} -1 \\ 2 \end{pmatrix},\ \Pi_0^1(\nu_1) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3\nu_1}c_0^1,$$

and under $t \in (1,2]$, then $\nu_2 = \frac{t-1}{\varepsilon}$

$$x_0^2(t) = \frac{1}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t-1}\eta_0^2,\ \Pi_0^2(\nu_2) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3\nu_2}c_0^2.$$

From (2.14) the following is obtained

$$\eta_0^1 = \frac{2}{5}\xi_0 + \frac{3}{10},\ \eta_0^2 = \frac{7}{3},\ c_0^1 = \frac{1}{5}\xi_0 - \frac{3}{5},\ c_0^2 = \frac{2}{3}.$$

Further

$$R(\varepsilon) = -\frac{2}{25}e + \frac{3}{\varepsilon}e^{-\frac{3}{\varepsilon}},\ \overline{\overline{a}}_1(\varepsilon) = \frac{19}{150}e + \frac{24}{15} - \frac{9}{5}\frac{e^{\frac{-3}{\varepsilon}}}{\varepsilon}.$$

Then $R_0 = -\frac{2}{25}e$ and $\overline{\overline{a}}_{10} = \frac{19}{150}e + \frac{24}{15}$. In this case $R_0^+ = R_0^{-1} = -\frac{25}{2e}$, $\xi_0 = -\frac{19}{12} - \frac{20}{e}$ and $P_{R_0^*} = 0$, i.e., the condition H9 is fulfilled.

$$x^1(t,\varepsilon) = -\frac{8}{5}e\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^t + \begin{pmatrix} 10t+1 \\ 5t-7 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\left(-\frac{11}{12} - \frac{4}{e}\right)e^{-3\frac{t}{\varepsilon}} + O(\varepsilon),$$

$t \in [0,1]$.

$$x^2(t,\varepsilon) = \frac{7}{15}\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{t-1} + \frac{2}{3}\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-3\frac{t-1}{\varepsilon}} + O(\varepsilon),$$

$t \in (1,2]$.

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YELLOW-RED AND NONOBTUSE REFINEMENTS OF PLANAR TRIANGULATIONS

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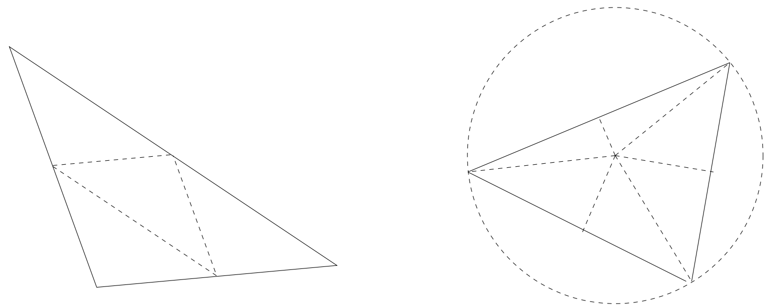
Abstract. A new yellow-red refinement of nonobtuse triangles, combining standard red and yellow refinements, is proposed. This approach is also adapted for the construction of nonobtuse global refinement of an initial triangulation in the case when some obtuse triangles in it are created by a grid generator.

Mathematical Subject Classification: 65N30, 65N50
Keywords: nonobtuse triangulation, polygonal domain, finite element method, discrete maximum principle, maximum angle condition, red and yellow refinements, grid generation

1. **Introduction**

In this paper, we consider conforming (planar) triangulations of a polygon into triangles, i.e., the union of all triangles of any particular triangulation is always equal to our polygon, and any two different triangles in any (particular) triangulation may only have a common edge, or a common vertex, or no common points (cf. [11]). In most cases namely such conforming triangulations are used in finite element modelling and finite element analysis.

A planar triangulation is called *nonobtuse* if all interior angles in all triangles of the triangulation are less than or equal to $\frac{\pi}{2}$. This condition was first introduced in [4] and [15], where it was particularly used to prove the discrete maximum principle for the piecewise linear finite element approximations to the Poisson equation with the Dirichlet boundary condition. Later this condition was imposed on the interior angles between faces of tetrahedra in tetrahedral partitions, and again used to prove the discrete maximum principle for the finite element solutions to a class of nonlinear elliptic problems for which the continuous maximum principle holds, see [12]. Obviously, a similar question, under which conditions on the triangulations used the



a) red refinement *b) yellow refinement*

Figure 1

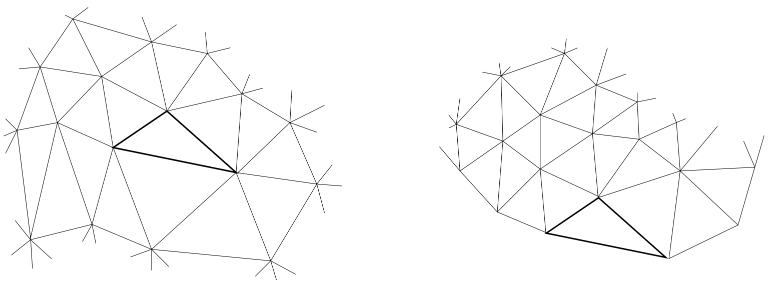
discrete maximum principle holds, naturally appears in the case of parabolic problems. For these problems there are also some other important qualitative properties closely related to the preservation of the maximum principle, we refer to [5], [6], and [14] in this respect.

Note that in nonobtuse triangulations, all triangles satisfy the maximum angle condition with the same constant, $\frac{\pi}{2}$. This condition is very important for the finite element analysis, since the optimum interpolation properties of the finite elements are preserved [11, Chapt. 4].

The construction of a conforming triangulation of a polygon into (arbitrary) triangles is, obviously, simple; moreover, there are proofs of the existence of nonobtuse triangulations of polygons [1], [7]. Thus, having a nonobtuse initial triangulation, and then applying the red refinement [2] and/or the yellow refinement [9] (see Figure 1), we can easily build a family of globally refined triangulations where no obtuse angles occur.

Nevertheless, in practice, the initial (coarse) triangulation of the polygonal domain is performed by a grid generator, and it is most probable that this initial triangulation is not nonobtuse. In this paper, we propose, first, a new way of refinement of a single nonobtuse triangle, combining the standard yellow and red refinement techniques (which we will call the yellow-red refinement). Thus, a nonobtuse triangulation can, in fact, be globally refined using red and/or yellow and/or yellow-red refinements for its triangles (in any desired combination). Further, evolving the approach used for yellow-red refinement, we also analyse a possibility of refining globally the given triangulation, initially containing several isolated obtuse triangles (cf. Figure 2), into the conforming nonobtuse triangulation. In particular, we present the algebraic conditions on the coordinates of vertices of a quadrilateral formed by obtuse and nonobtuse triangles in the triangulation (where those two triangles are adjacent along the longest side of the obtuse triangle), under which it is possible to refine the quadrilateral into nonobtuse subtriangles (cf. Figure 4a). In the partition proposed in this paper, the midpoints of non-common sides of the initial triangles and the center of the circumscribed circle around the obtuse triangle, which is inside of the quadrilateral, are used.

Thus, if the conditions presented are satisfied for all the obtuse triangles and nonobtuse triangles adjacent to them (in the above–described manner), occurring in the initial triangulation, we can conformly (globally) refine the given initial triangulation so that the next (refined) triangulation (after one refinement step) already becomes nonobtuse.



a) obtuse triangle is inside of solution domain *b) obtuse triangle is adjacent to the boundary*

Figure 2

It is also worth mentioning here, that the nonobtusity condition is only sufficient to prove the discrete maximum principle: it is possible to observe the validity of this principle also in situations when some of the interior angles in the triangulation are slightly obtuse. This issue, leading to so-called weakened acute type conditions, is addressed in [13] and [10] in the two- and three-dimensional cases, respectively.

Other known refinements of triangles are green [2], and blue [8], they are used for local refinement procedures. For a comprehensive survey of the results on the grid generation issues, we refer to monograph [3].

2. Nonobtuse triangle and yellow-red refinement

2.1. **Criterion for a nonobtuse triangle.** In this section, we give a simple criterion for a triangle to be nonobtuse, and introduce the idea of yellow-red refinement.

We present a simple criterion, which employs the coordinates of vertices of a triangle, to decide whether this triangle is nonobtuse. From now on, ABC always stands for a triangle with vertices $A = (x_A, y_A)$, $B = (x_B, y_B)$, and $C = (x_C, y_C)$ (see Figure 3).

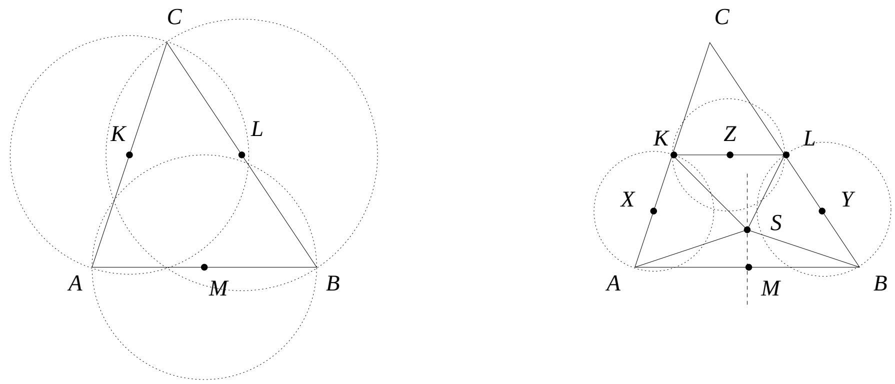
Lemma 1. *Triangle ABC is nonobtuse if and only if the following three inequalities hold*

$$(x_A - 2x_B + x_C)^2 + (y_A - 2y_B + y_C)^2 \geq (x_C - x_A)^2 + (y_C - y_A)^2, \tag{2.1a}$$

$$(x_C - 2x_A + x_B)^2 + (y_C - 2y_A + y_B)^2 \geq (x_B - x_C)^2 + (y_B - y_C)^2, \tag{2.1b}$$

$$(x_B - 2x_C + x_A)^2 + (y_B - 2y_C + y_A)^2 \geq (x_A - x_B)^2 + (y_A - y_B)^2. \tag{2.1c}$$

Proof. Conditions (2.1a)–(2.1c) imply that $|BK| \geq \frac{1}{2}|AC|$, $|AL| \geq \frac{1}{2}|CB|$, $|CM| \geq \frac{1}{2}|AB|$, where K, L, M are midpoints of AC, CB, AB, respectively. Geometrically,



a) acute triangle ABC   b) yellow-red refinement

Figure 3

it means that (2.1a) is equivalent to the case when vertex B does not lie inside of a circle having AC as its diameter (and K as its center), i.e., the angle at B is nonobtuse; the same holds for vertices A and C, from (2.1b) and (2.1c), respectively (cf. Figure 3a). $\square$

2.2. **Yellow-red refinement.** The idea of red refinement from [2] (cf. Figure 1a) consists of the usage of the midpoints of sides of the triangle: connecting them, we get four subtriangles similar to the original one; red refinement can be applied to any triangle.

The idea of yellow refinement was recently introduced in [9] for tetrahedra and their faces. It essentially uses the assumption that the center of the circumscribed circle of the given triangle lies in the closure of the triangle. Using this center and the midpoints of sides of the triangle, we get partition into four or six right subtriangles, depending on the value of the maximum angle in the given triangle (cf. Figure 1b). Thus, yellow refinement can only be applied to a nonobtuse triangle.

Now, we show that it is also possible to combine both techniques inside of a nonobtuse triangle. The idea of such a construction is explained (in terms of Figure 3b) as follows: let K, L, M be defined as before, the dashed line via point M be the mid-perpendicular to side AB, points X, Y, Z be the midpoints of AK, BL, KL, respectively (these three points are also centers of three dotted circles having AK, BL, and KL as their diameters). If we can take point S from trapezoid $AKLB$ ($\subset ABC$), lying on the mid-perpendicular via M, so that it does not lie inside of all three (dotted) circles, then triangle ABC is, obviously, decomposed into six nonobtuse subtriangles – KLC, KSA, KLS, LSB, SMA, and SMB (the last two are, moreover, right ones at the vertex M).

Remark 1. *In general, there exist many choices for the position of point S. Nevertheless, there may occur a situation when the only possible choice of S is to be at* *point M, and it only happens if ABC is the right triangle. In this case, the yellow-red refinement coincides with the standard red refinement of triangle ABC.*

2.3. **Nonobtuse refinements.** In this section, we discuss the issue of the construction of (refined) nonobtuse triangulations from the initially generated one, containing obtuse triangles; it is, particularly, shown how the yellow-red refinement technique can be adapted for this purpose.

2.4. **Nonobtuse refinement of two adjacent triangles.** The approach, used in Section 2.2, can be adapted to the problem of partitioning the triangulations with obtuse triangles. For this purpose, we make the following natural observation – we need not always make the refinement only in a single triangle, we can also perform the refinement, for example, for a pair of adjacent triangles.

To show how this observation and the yellow-red refinement can be employed, we assume (in denotations of Figure 4a) that triangle ADB is obtuse at $D = (x_D, y_D)$ and triangle ABC is nonobtuse as before. Also, let $X = (x_X, y_X)$, $Y = (x_Y, y_Y)$, and $Z = (x_Z, y_Z)$ be midpoints of AK, BL, and KL, respectively.

Lemma 2. *The coordinates of X, Y, and Z are given by the formulae*

$$x_X = \frac{3}{4}x_A + \frac{1}{4}x_C, \quad y_X = \frac{3}{4}y_A + \frac{1}{4}y_C; \tag{2.2a}$$

$$x_Y = \frac{3}{4}x_B + \frac{1}{4}x_C, \quad y_Y = \frac{3}{4}y_B + \frac{1}{4}y_C; \tag{2.2b}$$

$$x_Z = \frac{1}{2}x_C + \frac{1}{4}x_A + \frac{1}{4}x_B, \quad y_Z = \frac{1}{2}y_C + \frac{1}{4}y_A + \frac{1}{4}y_B\ . \tag{2.2c}$$

Further, let $S = (x_S, y_S)$ be the center of the circumscribed circle of ADB, it stands at the intersection of three mid-perpendiculars to sides AD, DB and AB, and lies outside of ADB, since it is obtuse.

Lemma 3. *The coordinates of S are given by the formulae*

$$x_S = \frac{(y_B - y_D)(x_A^2 + y_A^2 - x_D^2 - y_D^2) + (y_D - y_A)(x_B^2 + y_B^2 - x_D^2 - y_D^2)}{2 \cdot [(x_A - x_D)(y_B - y_D) - (x_B - x_D)(y_A - y_D)]}, \tag{2.3a}$$

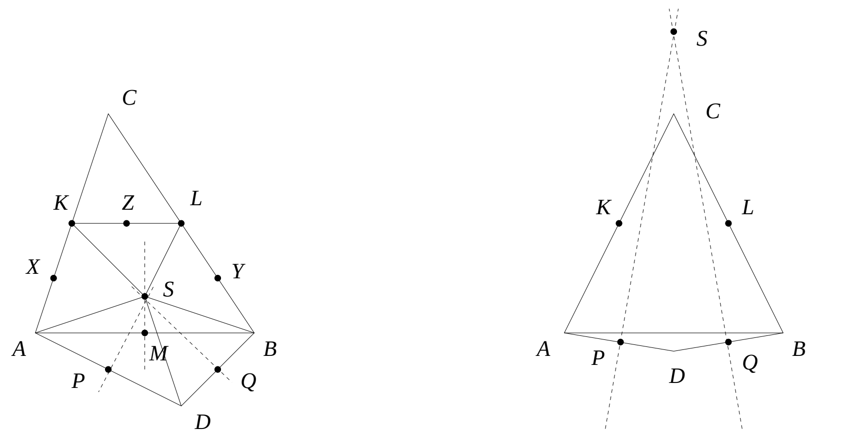
$$y_S = \frac{(x_D - x_B)(x_A^2 + y_A^2 - x_D^2 - y_D^2) + (x_A - x_D)(x_B^2 + y_B^2 - x_D^2 - y_D^2)}{2 \cdot [(x_A - x_D)(y_B - y_D) - (x_B - x_D)(y_A - y_D)]}. \tag{2.3b}$$

The calculation in both lemmas is straightforward, therefore omitted.

Further, the following theorem holds, where $|\cdot|$ denotes the standard distance between two points in plane.

Theorem 1. *Let a triangle ABC, adjacent to an obtuse triangle ADB, be nonobtuse. If*

$$\begin{aligned}&[(x_C - x_A)(y_A - y_B) + (y_C - y_A)(x_B - x_A)]\times \\ &\quad \times [(x_S - \frac{1}{2}(x_C + x_A))(y_A - y_B) + (y_S - \frac{1}{2}(y_C + y_A))(x_B - x_A)] \le 0\end{aligned} \tag{2.4}$$



a) refinement of quadrilateral is possible *b) refinement of quadrilateral is not possible*

Figure 4.

and

$$|SX| \geq \frac{1}{4}|AC| \ , \quad |SZ| \geq \frac{1}{4}|AB| \ , \quad |SY| \geq \frac{1}{4}|BC| \ , \tag{2.5}$$

where coordinates of points S, X, Y, and Z are given by (3.1)–(3.2)*, then subtriangles KLC, KSA, KLS, LSB, SAP, SPD, SDQ, and SQB are nonobtuse (the last four are, moreover, right triangles). Above, P and Q are the midpoints of sides AD and DB, respectively (see Figure 4a).*

Proof. Condition (3.3) means that points C and S are in different half-planes with respect to straight-line KL, i.e., S lies in the trapezoid $AKLB$. Further, condition (3.4) means that point S is outside of three circles defined as in Figure 3b. Thus, all the above mentioned subtriangles make a nonobtuse partition of a quadrilateral $ADBC$. $\square$

Remark 2. *Obviously, the conditions of Theorem 5 cannot be fulfilled in a general case. For example, in Figure 4b, two adjacent (obtuse and nonobtuse) triangles are such that the point S is completely outside of the quadrilateral made by the triangles. Therefore, a partition of the quadrilateral in the manner, proposed in Theorem 5, is not possible.*

Remark 3. *One of the other possibilities to get nonobtuse triangles, particularly, for the case of Figure 4b, would be to use a swapping of diagonals in the quadrilateral. Thus, in the decomposition of $ADBC$ we can employ two triangles ACD and BCD (which could be both nonobtuse) instead of ABC and ABD. However, the analysis of such an approach, is beyond the scope of the present paper.*

2.5. **On nonobtuse refinements of coarse triangulations.** In this subsection, we shortly discuss how the ideas from the previous subsection can be used to make the nonobtuse triangulation from the coarse triangulation, with obtuse triangles in it, built by the grid generator.

So far, various refinement techniques proposed – red, yellow, and yellow-red – are only applied to the single triangle. As we mentioned above, the red refinement, applied to the obtuse triangle, gives obtuse subtriangles; the other two techniques can only be applied to nonobtuse triangles by their definitions. However, the yellow-red technique can be used for the union of adjacent triangles, one of which is obtuse and the other is nonobtuse, as it was shown in Section 3.1.

Thus, we can make a refinement of the initial (generated) triangulation in the following manner. Using the criterion of Section 2.1, we check whether a triangle taken from the mesh is nonobtuse; and, thus, mark all obtuse triangles. If the (marked) obtuse triangle is adjacent to the boundary along its longest edge (see Figure 2b), we let the perpendicular from the interior vertex of this triangle onto the boundary, and add a new node and two right-angled triangles to the triangulation.

Further, if we have an obtuse triangle inside of the solution domain (see Figure 2a), we consider the quadrilateral formed by it and adjacent to it and try to apply the technique of Section 3.1.

Remark 4. *A similar question – how to build nonobtuse tetrahedral triangulations from triangulations containing some tetrahedra with obtuse interior angles between faces, naturally appears in the three-dimensional case. To the authors' knowledge this is an open problem. Another related open question is how to make local nonobtuse refinements of planar and tetrahedral triangulations. It is also interesting to make research on which geometrical properties of triangulations in the two- and three-dimensional cases provide the validity of the discrete maximum principle for parabolic problems.*

3. **Numerical experiments**

Numerical experiments on the effectivity of the proposed technique have been performed using Matlab in Intel 700 MgHz/128 RAM. We (pseudo)randomly generated 100 000 times the coordinates of vertices A, B, C, D (the time needed for this was 1131 seconds): the situation when one of triangles ABC or ABD had an obtuse angle at the vertex C or D, and the other triangle had a nonobtuse angle at D or C, respectively, occured 17 235 times. For those 17 235 "undesirable" cases, we could successfully apply 2 353 times the result of Theorem 5 (the time needed for that was 56 seconds). Also, the swapping procedure of Remark 7 was "successful" for 2 488 cases from those 17 235 "undesirable" cases (time costs for this checking was 24 seconds) , and there was no intersection between "successful" cases of Theorem 5 and Remark 7 (the time needed for this was 22 seconds).

Another interesting result was that the situation, when both angles C and D in the triangles ABC and ABD were obtuse, occurred 11 563 times, and we could "successfully" apply the swapping procedure from Remark 7 to 2 856 cases from those 11 563 (time costs in this experiment was 119 seconds).

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INVESTIGATION OF THE INTEGRAL MANIFOLDS OF SINGULARLY PERTURBED FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. A system of nonlinear singularly perturbed functional differential equations with time lag in fast and slow variables is considered. The conditions of existence of the integral manifolds are established, application of the method of integral manifolds in stability problem are analyzed.

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Keywords: integral manifold, singular perturbation, functional differential equation

1. Preliminaries

A method of integral manifolds is an effective algorithm to study singularly perturbed ordinary differential equations [1–3], systems with time lag [4,5] and functional differential equations [6–8]. The case of systems with time lag only in fast variables is discussed in [4,5,7,8] and only in slow variables in [6,9]. In this paper we consider a singularly perturbed system of functional differential equations with time lag both in fast and in slow variables. For such systems we establish the existence conditions of stable, and center-stable, center, center-unstable integral manifolds and consider their application for the investigation of solution stability. For ordinary differential equations analogous problems were studied in [3,10].

Let R^n be n-dimensional Euclidean space, $C_\Delta = C[-\Delta, 0]$, $C_{\varepsilon\Delta} = C[-\varepsilon\Delta, 0]$ are the spaces of $[-\Delta, 0]$, $[-\varepsilon\Delta, 0]$ continuous n-dimensioned functions.

Consider the system of functional differential equations

$$\begin{aligned} \frac{dx}{dt} &= Ax_t + f(t, x_t, y_t, \varepsilon), \\ \varepsilon \frac{dy}{dt} &= B(t)y_t + g(t, x_t, y_t, \varepsilon), \end{aligned} \tag{1.1}$$

where $x \in R^n, y \in R^m, x_t = x(t+\theta), y_t = y(t+\varepsilon\theta)$, $-\Delta \le \theta \le 0$, $\Delta > 0$, $\varepsilon > 0$ is a small parameter. Functions $f(t,x,y,\varepsilon)$, $g(t,x,y,\varepsilon)$ are defined in domain $\{t \in R, x \in$

$C_\Delta, y \in C_{\varepsilon\Delta}, \varepsilon \in [0, \varepsilon_0]\}$ with values in R^n and R^m. A, B are linear operators, given by

$$A\varphi = \int_{-\Delta}^{0} [d\eta_1(\theta)]\varphi(\theta), \quad B(t)\varphi = \int_{-\Delta}^{0} [d\eta_2(t,\theta)]\varphi(\varepsilon\theta),$$

where $\eta_1(\theta)$ is $n \times n$ matrix, whose elements are functions of limited variation; $\eta_2(t,\theta)$ is $m \times m$ matrix, whose elements are functions of limited variation on θ for all t and uniformly continuous on t with respect to θ and bounded for $t \in R$.

Suppose that for system (1.1) the following conditions are valid:

I) all roots of the characteristic equation

$$\det\left(\lambda E - \int_{-\Delta}^{0} [d\eta_2(t,\theta)]e^{\lambda\theta}\right) = 0$$

are located in the left half-plane $\operatorname{Re}\lambda < -2\mu < 0$;

II) functions f, g are continuous, bounded by constant K and satisfy the following inequalities:

$$|f(t,x,y,\varepsilon) - f(t,\bar{x},\bar{y},\varepsilon)| \le L(\varepsilon)(|x-\bar{x}| + |y-\bar{y}|),$$
$$|g(t,x,y,\varepsilon) - g(t,\bar{x},\bar{y},\varepsilon)| \le L(\varepsilon)(|x-\bar{x}| + |y-\bar{y}|),$$

where $L(\varepsilon) \to 0$ if $\varepsilon \to 0$.

2. **System transformation**

Consider the linear autonomous functional differential equation

$$\frac{dx}{dt} = Ax_t \tag{2.1}$$

and its charateristc quasipolynomial

$$H(\lambda) = \det\left(\lambda E - \int_{-\Delta}^{0} [d\eta_1(\theta)]e^{\lambda\theta}\right).$$

Define the shift operator, corresponding to equation (2.1), by the relation $T(t)\varphi = x_t(\varphi)$, where $x_t(\varphi)$ is the solution of (2.1) with initial value $\varphi \in C_\Delta$ at $t = 0$.

Let us introduce the following notation:

$$\Lambda_1 = \{\lambda :\, H(\lambda) = 0, \operatorname{Re}\lambda > 0\}, \quad \Lambda_2 = \{\lambda :\, H(\lambda) = 0, \operatorname{Re}\lambda = 0\},$$
$$\Lambda_3 = \{\lambda :\, H(\lambda) = 0, \operatorname{Re}\lambda < 0\}.$$

It is known [11] that there is only a finite number of roots of equation $H(\lambda) = 0$ in any half-plane $\operatorname{Re}\lambda \ge \gamma$, so the sets Λ_1, Λ_2 are finite dimensional.

The sets $\Lambda_i, i = \overline{1,3}$ generate on the space C_Δ invariant under $T(t)$ subspaces P_1, P_2, Q. Subspaces P_1, P_2 correspond to the initial values of all those solutions of

(2.1) which are in the form $q(t)e^{\lambda t}$, where $\lambda \in \Lambda_i$, $i = 1, 2$, $q(t)$ is a polynomial in t. They are finite dmensional. Let their dimensions be equal to n_1, n_2 respectively.

We denote by $\Phi_i(\theta)$, $\Delta \le \theta \le 0$ is a basis of P_i, and Ψ_i, $0 \le \theta \le \Delta$ is a basis of $P_i^* \subset C[0, \Delta]$ of the initial values of solutions of the adjoint to (2.1) system

$$\frac{dy}{dt} = A^* y_t, \quad A^*\psi = -\int_{-\Delta}^{0} [d\eta_1^T(\theta)]\psi(-\theta).$$

For elements $\varphi \in C[-\Delta, 0]$, $\psi \in C[0, \Delta]$ we define the scalar product by

$$(\psi, \varphi) = \psi^T(0)\varphi(0) - \int_{-\Delta}^{0}\int_{0}^{\theta} \psi(\xi - \theta)[d\eta_1(\theta)]\varphi(\xi)d\xi.$$

It is known [11] that $n_i \times n_i$ matrix (Ψ_i, Φ_i) is nonsingular and we can take that $(\Psi_i, \Phi_i) = E$. Let B_i denote $n_i \times n_i$ matrix such that

$$\frac{d}{d\theta}\Phi_i(\theta) = \Phi_i(\theta)B, \quad i = 1, 2, \quad \theta \in [-\Delta, 0].$$

The space C_Δ can be decomposed into direct sum $C_\Delta = P_1 + P_2 + Q$. Every element $x_t \in C_\Delta$ can be represented in the form [11, 12]

$$x_t = \Phi_1 u_1(t) + \Phi_2 u_2(t) + z_t, \tag{2.2}$$

$$u_1(t) = (\Psi_1, x_t), \quad u_2(t) = (\Psi_2, x_t), \quad z_t \in Q, \quad (\Psi_i, z_t) = 0, i = 1, 2.$$

Define matrices

$$X_0(\theta) = \begin{cases} 0, & -\Delta \le \theta < 0, \\ E, & \theta = 0, \end{cases} \qquad Y_0(\theta) = \begin{cases} 0, & -\varepsilon\Delta \le \theta < 0, \\ E, & \theta = 0 \end{cases}$$

and shift operator $V(t, \sigma)$ for equation

$$\varepsilon\frac{dy}{dt} = \int_{-\Delta}^{0} [d\eta_2(t, \theta)]y(t + \varepsilon\theta). \tag{2.3}$$

Changing variables (2.2) in system (1.1) and using the variation of constants formula [12] we get the equivalent system of differential and integral equations

$$\begin{aligned} \frac{du_i}{dt} &= B_i u_i + F_i(t, u_1, u_2, y_t, z_t, \varepsilon), \quad i = 1, 2, \\ z_t &= T(t - \sigma)z_\sigma + \int_{\sigma}^{t} T(t - s)X_0^Q F(s, u_1, u_2, y_t, z_t, \varepsilon)ds, \\ y_t &= V(t, \sigma)y_\sigma + \frac{1}{\varepsilon}\int_{\sigma}^{t} V(t, s)Y_0 G(s, u_1, u_2, y_t, z_t, \varepsilon)ds, \end{aligned} \tag{2.4}$$

where

$$F(t, u_1, u_2, y_t, z_t, \varepsilon) = f(t, \Phi_1 u_1 + \Phi_2 u_2 + z_t, y_t, \varepsilon),$$

$$F_i(t,u_1,u_2,y_t,z_t,\varepsilon)=\Psi_i^T(0)F(t,u_1,u_2,y_t,z_t,\varepsilon), i=1,2,$$
$$G(t,u_1,u_2,y_t,z_t,\varepsilon)=g(t,\Phi_1u_1+\Phi_2u_2+z_t,y_t,\varepsilon),$$
$$X_0^{P_i}=\Phi_i\Psi_i^T(0), i=1,2, \quad X_0^Q=X_0-X_0^{P_1}-X_0^{P_2}.$$

The integrals in (2.4) for each θ are understood as the integrals in Euclidean spaces R^n and R^m.

From the definition of the sets $\Lambda_i, i=\overline{1,3}$ and under assumption II there exist positive constants K_1,K_2,α such that following inequalities are valid [12, 13]:

$$\begin{aligned}
&|e^{B_1t}|\le K_1e^{\alpha t}, \quad t\le 0,\\
&|e^{B_2t}|\le K_1e^{\frac{\alpha}{4}|t|}, \quad t\in R,\\
&|T(t)\varphi^Q|\le K_1e^{-\alpha t}|\varphi^Q|, \quad t\ge 0,\\
&|V(t,\sigma)\xi|\le K_2e^{-\frac{\mu}{\varepsilon}(t-\sigma)}|\xi|, \quad t\ge\sigma, \xi\in C_{\varepsilon\Delta}.
\end{aligned} \tag{2.5}$$

3. **Existence of the center and center-unstable integral manifolds**

Definition 1. *A set of points $M\subset R\times R^{n_1}\times R^{n_2}\times Q\times C_{\varepsilon\Delta}$ is said to be an integral manifold of system* (2.4) *if for each $\varepsilon\in[0,\varepsilon_0]$ and any point $(t_0,u_{10},u_{20},z_{t_0},y_{t_0})\in M$ it follows that $(t,u_1(t),u_2(t),z_t,y_t)\in M$ for all $t\ge t_0$, where $(u_1(t),u_2(t),z_t,y_t)$ is the solution of system* (2.4) *with the initial values $(t_0,u_{10},u_{20},z_{t_0},y_{t_0})$.*

Theorem 1. *Let conditions I-II hold. Then there exist positive constants ρ_0,η_0 such that for all $0<\rho<\rho_0$, $0<\eta<\eta_0$ and sufficiently small ε the system* (2.4) *has the center and center-unstable integral manifolds*

$$\begin{aligned}
M^*=\{(t,u_1,u_2,z,y): t\in R, u_1=r^*(t,u_2,\varepsilon), u_2\in R^{n_2},\\
z=v^*(t,u_2,\varepsilon), y=w^*(t,u_2,\varepsilon)\},\\
M^{*-}=\{(t,u_1,u_2,z,y): t\in R, u_1\in R^{n_1}, u_2\in R^{n_2},\\
z=v^{*-}(t,u_1,u_2,\varepsilon), y=w^{*-}(t,u_1,u_2,\varepsilon)\},
\end{aligned}$$

where functions $r^*,v^*,w^*,v^{*-},w^{*-}$ are continuous with respect to all variables and satisfy the Lipschitz condition by u_1,u_2 with constant η and not exceeding ρ.

Proof. The existence of the center manifold of system (2.4) was studied in [14]. Now we prove that system (2.4) has center-unstable integral manifold M^{*-}.

Define the sets of functions

$$S_z=\{v: R\times R^{n_1}\times R^{n_2}\times[0,\varepsilon_0]\to Q\},$$
$$S_y=\{w: R\times R^{n_1}\times R^{n_2}\times[0,\varepsilon_0]\to C_{\varepsilon\Delta}\},$$

which are bounded by ρ and satisfy the Lipschitz condition by u_1,u_2 with constant η.

Consider the following system for any $v \in S_z, w \in S_y$

$$\frac{du_i}{dt} = B_i u_i + F_i(t, u_1, u_2, v(t, u_1, u_2, \varepsilon), w(t, u_1, u_2, \varepsilon), \varepsilon), i = 1, 2. \tag{3.1}$$

Due to the conditions of functions f, v, w the following inequalities hold

$$|F_i(t, u_1, u_2, v(t, u_1, u_2, \varepsilon), w(t, u_1, u_2, \varepsilon), \varepsilon) -$$
$$-F_i(t, \bar{u}_1, \bar{u}_2, v(t, \bar{u}_1, \bar{u}_2, \varepsilon), w(t, \bar{u}_1, \bar{u}_2, \varepsilon), \varepsilon)| \leq$$
$$\leq L(\varepsilon) m_i (\nu_1 + \nu_2 + 2\eta)(|u_1 - \bar{u}_1| + |u_2 - \bar{u}_2|),$$

where $m_i = |\Psi_i^T(0)|$, $\nu_i = |\Phi_i|$, $i = 1, 2$.

Thus, for every point $(u_{10}, u_{20}) \in R^{n_1} \times R^{n_2}$ system (3.1) has a unique solution $u_1(t) = U_1(t, t_0, u_{10}, \varepsilon, v, w)$, $u_2(t) = U_2(t, t_0, u_{20}, \varepsilon, v, w)$ such that $u_1(t_0) = u_{10}$, $u_2(t_0) = u_{20}$.

Lemma 1. *Let conditions I-II hold. Then for all sufficient small ε the following inequality is valid*

$$|U_1 - \bar{U}_1| + |U_2 - \bar{U}_2| \leq K_1 e^{-\frac{\alpha}{2}(t-s)}(|u_1 - \bar{u}_1| + |u_2 - \bar{u}_2| + ||v - \bar{v}|| + ||w - \bar{w}||), t \leq s.$$

The proof of Lemma 1 can be easily obtained using the Gronwal inequality and properties of functions f, v, w.

Consider now the set $S = S_z \times S_y$ of functions (v, w) with the norm $\|(v, w)\| = \max(\|v\|, \|w\|)$. Define in set S the operator

$$H(v, w) = (H^1_{t,u_1,u_2}(v, w), H^2_{t,u_1,u_2}(v, w)),$$

where

$$H^1_{t,u_1,u_2}(v, w) = \int_{-\infty}^{t} T(t - s) X_0^Q F(s, u_1(s), u_2(s),$$
$$v(s, u_1, u_2, \varepsilon), w(s, u_1, u_2, \varepsilon), \varepsilon) ds, \tag{3.2}$$
$$H^2_{t,u_1,u_2}(v, w) = \frac{1}{\varepsilon} \int_{-\infty}^{t} V(t, s) Y_0 G(s, u_1(s), u_2(s),$$
$$v(s, u_1, u_2, \varepsilon), w(s, u_1, u_2, \varepsilon), \varepsilon) ds\,. \tag{3.3}$$

It follows from inequalities (2.5) and condition II that the integrals on the right side of (3.2), (3.3) converge uniformly with respect to $\theta \in [-\Delta, 0]$.

Consider any sequence of numbers $t_n < t$ such that $t_n \to -\infty$ when $n \to \infty$. It follows from (2.4) that

$$A_n = \int_{t_n}^{t} T(t - s) Y_0^Q F(s, u_1, u_2, v, w, \varepsilon) ds \in Q,$$

$$B_n = \frac{1}{\varepsilon} \int_{t_n}^{t} V(t, s) Y_0 G(s, u_1, u_2, v, w, \varepsilon) ds \in C_{\varepsilon\Delta}\,.$$

Then $H^1_{t,u_1,u_2}(v,w) \in Q$, $H^2_{t,u_1,u_2}(v,w) \in C_{\varepsilon\Delta}$ since $A_n \to H^1_{t,u_1,u_2}(v,w)$, $B_n \to H^2_{t,u_1,u_2}(v,w)$ when $n \to \infty$ uniformly with respect to $\theta \in [-\Delta, 0]$.

By means of similar arguments to those in [14] one can easily make sure that the following statement is valid.

Lemma 2. *Let conditions I-II hold. Then for all sufficiently small ε the operator H is a contraction mapping of S into itself.*

Denote by (v^{*-}, w^{*-}) the unique fixed point of the operator H. Let $u_1(t) = U_1(t, t_0, u_{10}, \varepsilon, v^{*-}, w^{*-})$, $u_2(t) = U_2(t, t_0, u_{20}, \varepsilon, v^{*-}, w^{*-})$ be a solution of system (3.1). Then functions v^{*-}, w^{*-} satisfy such system

$$
\begin{aligned}
v^{*-}(t, u_1(t), u_2(t), \varepsilon) &= \int\limits_{-\infty}^{t} T(t-s) X_0^Q F(s, u_1(s), u_2(s), \\
&\quad v^{*-}(s, u_1(s), u_2(s), \varepsilon), w^{*-}(s, u_1(s), u_2(s), \varepsilon), \varepsilon) ds, \\
w^{*-}(t, u_1(t), u_2(t), \varepsilon) &= \frac{1}{\varepsilon} \int\limits_{-\infty}^{t} V(t,s) Y_0 G(s, u_1(s), u_2(s), \\
&\quad v^{*-}(s, u_1(s), u_2(s), \varepsilon), w^{*-}(s, u_1(s), u_2(s), \varepsilon), \varepsilon) ds.
\end{aligned} \tag{3.4}
$$

Let $(t_0, u_{10}, u_{20}) \in R \times R^{n_1} \times R^{n_2}$ be an arbitrary point. Using the following identity for solutions of system (3.1)

$$U_i(s, t_0, u_{i0}, \varepsilon, v, w) = U_i(s, t, U_i(t, t_0, u_{i0}, \varepsilon, v, w), \varepsilon, v, w), i = 1, 2$$

and also the property of the operators T, V

$$T(t-s) = T(t-t_0) T(t_0 - s), \quad V(t,s) = V(t, t_0) V(t_0, s)$$

we rewrite system (3.4) in the form

$$
\begin{aligned}
v^{*-}(t, u_1, u_2, \varepsilon) &= T(t-t_0) v^{*-}(t_0, u_{10}, u_{20}, \varepsilon) + \int\limits_{t_0}^{t} T(t-s) X_0^Q F(s, u_1, u_2, \\
&\quad v^{*-}(s, u_1, u_2, \varepsilon), w^{*-}(s, u_1, u_2, \varepsilon), \varepsilon) ds, \\
w^{*-}(t, u_1, u_2, \varepsilon) &= V(t, t_0) w^{*-}(t_0, u_{10}, u_{20}, \varepsilon) + \frac{1}{\varepsilon} \int\limits_{t_0}^{t} V(t,s) Y_0 G(s, u_1, u_2, \\
&\quad v^{*-}(s, u_1, u_2, \varepsilon), w^{*-}(s, u_1, u_2, \varepsilon), \varepsilon) ds.
\end{aligned} \tag{3.5}
$$

It follows from (3.5) that functions $v^{*-}(s, u_1, u_2, \varepsilon), w^{*-}(s, u_1, u_2, \varepsilon)$ satisfy the third and fourth equations of system (2.4) for all $t \geq t_0$. Thus, the set M^{*-} is an integral manifold for (2.4). $\square$

Corollary 1. *The flow of the initial system* (1.1) *on the center-stable manifold* M^{*-} *is governed by the system of ordinary differential equations*

$$\frac{du_i}{dt} = B_i u_i + F_i(t, u_1, u_2, v^{*-}(t, u_1, u_2, \varepsilon), w^{*-}(t, u_1, u_2, \varepsilon), \varepsilon), i = 1, 2, \tag{3.6}$$

which is regular and has no time lag.

4. Existence of the stable and center-stable integral manifolds

Further we suppose that such equalities are valid:

$$f(t, 0, 0, \varepsilon) = 0, \quad g(t, 0, 0, \varepsilon) = 0.$$

Theorem 1. *Let conditions I-II be valid. Then there exist positive constants* l_0, N *such that for all* $0 < l < l_0$ *and sufficiently small* ε *system* (2.4) *has the stable and center-stable manifolds*

$$M^{+} = \{(t, u_1, u_2, z, y) : \, u_1 = r_1^{+}(t, z, y, \varepsilon),$$
$$u_2 = r_2^{+}(t, z, y, \varepsilon), z \in Q, y \in C_{\varepsilon\Delta}\},$$
$$M^{*+} = \{(t, u_1, u_2, z, y) : \, u_1 = r^{*+}(t, u_1, z, y, \varepsilon), u_2 \in R^{n_2}, z \in Q, y \in C_{\varepsilon\Delta}\},$$

where functions r_1^{+}, r_2^{+}, r^{*+} *are continuous with respect to all variables and satisfy the Lipschitz condition by* u_2, y, z *with constant* l.

The following estimates are valid for the arbitrary solution of the system (2.4), *whose initial values lie on a stable manifold* M^{+}, *when* $t \geq t_0$:

$$\begin{aligned} |u_i| &\leq N(|y_{t_0}| + |z_{t_0}|)e^{-\frac{\alpha}{2}(t-t_0)}, i = 1, 2, \\ |z_t| &\leq N(|y_{t_0}| + |z_{t_0}|)e^{-\frac{\alpha}{2}(t-t_0)}, \\ |y_t| &\leq N(|y_{t_0}| + |z_{t_0}|)e^{-\frac{\alpha}{2}(t-t_0)}. \end{aligned} \tag{4.1}$$

Proof. Denote by

$$\Omega := \{r_i : \, R \times Q \times C_{\varepsilon\Delta} \times [0, \varepsilon_1] \to R^{n_i}\}, i = 1, 2$$

the sets of all continuous functions with respect to all variables, which satisfy the Lipschitz condition on z, y with constant l and condition $r_i(t, 0, 0, \varepsilon) = 0$. For arbitrary $r_1 \in \Omega_1, r_2 \in \Omega_2$ we consider the following system

$$\begin{aligned} z_t &= T(t - t_0)z_{t_0} + \int_{t_0}^{t} T(t - s)X_0^Q F[s, z_s, y_s, \varepsilon]ds, \\ y_t &= V(t, t_0)y_{t_0} + \frac{1}{\varepsilon}\int_{t_0}^{t} V(t, s)Y_0 G[s, z_s, y_s, \varepsilon]ds, \end{aligned} \tag{4.2}$$

where

$$F[s, z_s, y_s, \varepsilon] = F(s, r_1(s, z_s, y_s, \varepsilon), r_2(s, z_s, y_s, \varepsilon), z_s, y_s, \varepsilon),$$
$$G[s, z_s, y_s, \varepsilon] = G(s, r_1(s, z_s, y_s, \varepsilon), r_2(s, z_s, y_s, \varepsilon), z_s, y_s, \varepsilon).$$

Let us prove the existence of the solution of system (4.2) with the help of the successive approximations method

$$z_t^0 = 0, \quad y_t^0 = 0,$$

$$z_t^{j+1} = T(t-t_0)z_{t_0} + \int\limits_{t_0}^{t} T(t-s)X_0^Q F[s, z_s^j, y_s^j, \varepsilon]ds,$$

$$y_t^{j+1} = V(t,t_0)y_{t_0} + \frac{1}{\varepsilon}\int\limits_{t_0}^{t} V(t,s)Y_0 G[s, z_s^j, y_s^j, \varepsilon]ds, \quad j = 0, 1, \ldots.$$

With the help of the mathematical induction method we ensure that the following inequalities are valid:

$$\begin{aligned} |z_t^{m+1} - z_t^m| &\le \frac{\bar{K}}{2^m} e^{-\frac{\alpha}{2}(t-t_0)}(|z_{t_0}| + |y_{t_0}|), \\ |y_t^{m+1} - y_t^m| &\le \frac{\bar{K}}{2^m} e^{-\frac{\alpha}{2}(t-t_0)}(|z_{t_0}| + |y_{t_0}|), \end{aligned} \tag{4.3}$$

where $\bar{K} = \max(K_1, K_2)$, $m = 0, 1, \ldots$.

From estimates (2.5) for $\varepsilon < \dfrac{2\mu}{\alpha}$ we obtain that the inequalities (4.3) are valid for $m = 0$. Suppose that (4.3) are valid for $m = j$. Then obtain

$$|z_t^{j+1} - z_t^j| \le \int\limits_{t_0}^{t} K_1 e^{-\alpha(t-s)} L(\varepsilon)(\nu_1 l + \nu_2 l + 1)(|z_t^j - z_t^{j-1}| + |y_t^j - y_t^{j-1}|)ds \le$$

$$\le \frac{\bar{K}}{2^{j-1}} \frac{4L(\varepsilon)(\nu_1 l + \nu_2 l + 1)}{\alpha}(|z_{t_0}| + |y_{t_0}|)e^{-\frac{\alpha}{2}(t-t_0)},$$

$$|y_t^{j+1} - y_t^j| \le \int\limits_{t_0}^{t} K_2 e^{-\frac{\mu}{\varepsilon}(t-s)} L(\varepsilon)(\nu_1 l + \nu_2 l + 1)(|z_t^j - z_t^{j-1}| + |y_t^j - y_t^{j-1}|)ds \le$$

$$\le \frac{\bar{K}}{2^{j-1}} \frac{4L(\varepsilon)(\nu_1 l + \nu_2 l + 1)}{2\mu - \varepsilon\alpha}(|z_{t_0}| + |y_{t_0}|)e^{-\frac{\alpha}{2}(t-t_0)}.$$

For $\varepsilon < \dfrac{\mu}{\alpha}$ and $L(\varepsilon) < \dfrac{\alpha}{8\bar{K}(\nu_1 l + \nu_2 l + 1)}$ we obtain that (4.3) are valid for $m = j+1$. Thus, inequalities (4.3) are valid for all natural m.

It follows from (4.3) that sequences z_t^m, y_t^m uniformly converge to the solutions $z_t(t_0, z_{t_0}, y_{t_0}, r_1, r_2)$, $y_t(t_0, z_{t_0}, y_{t_0}, r_1, r_2)$ of the system (4.2). After summation of inequaities (4.3) on m, it is obtained the following uniform estimates:

$$\begin{aligned} |z_t(t_0, z_{t_0}, y_{t_0}, r_1, r_2)| &\le 2\bar{K}(|z_{t_0}| + |y_{t_0}|)e^{-\frac{\alpha}{2}(t-t_0)}, \\ |y_t(t_0, z_{t_0}, y_{t_0}, r_1, r_2)| &\le 2\bar{K}(|z_{t_0}| + |y_{t_0}|)e^{-\frac{\alpha}{2}(t-t_0)}, \end{aligned} \tag{4.4}$$

for $t \ge t_0$.

Denote by $z_t(t_0,\bar z_{t_0},\bar y_{t_0},r_1,r_2)$, $y_t(t_0,\bar z_{t_0},\bar y_{t_0},r_1,r_2)$ the solution of system (4.2) with initial value $(\bar z_{t_0},\bar y_{t_0})$. Similarly to (4.3), (4.4) one can show that the following estimations are valid:

$$
\begin{gathered}
|z_t(t_0,z_{t_0},y_{t_0},r_1,r_2)-z_t(t_0,\bar z_{t_0},\bar y_{t_0},r_1,r_2)|\le\\
2\bar K(|z_{t_0}-\bar z_{t_0}|+|y_{t_0}-\bar y_{t_0}|)e^{-\frac{\alpha}{2}(t-t_0)},\\
\\
|y_t(t_0,z_{t_0},y_{t_0},r_1,r_2)-y_t(t_0,\bar z_{t_0},\bar y_{t_0},r_1,r_2)|\le\\
2\bar K(|z_{t_0}-\bar z_{t_0}|+|y_{t_0}-\bar y_{t_0}|)e^{-\frac{\alpha}{2}(t-t_0)},
\end{gathered}
\tag{4.5}
$$

for $t\ge t_0$.

Let us denote by $\Omega=\Omega_1\times\Omega_2$ the set of function (r_1,r_2) with the norm

$$
\|(r_1,r_2)\|=\max(\|r_1\|,\|r_2\|),\quad \|\cdot\|=\sup_{t,z,y,\varepsilon}|\cdot|.
$$

Let define the operator

$$
H(r_1,r_2)=(H^1_{r_1,r_2}(t_0,z_{t_0},y_{t_0},\varepsilon),H^2_{r_1,r_2}(t_0,z_{t_0},y_{t_0},\varepsilon)),
$$

where

$$
H^i_{r_1,r_2}(t_0,z_{t_0},y_{t_0},\varepsilon)=-\int_{t_0}^{\infty}e^{B_i(t-s)}F[s,z_s,y_s,\varepsilon]ds,\quad i=1,2. \tag{4.6}
$$

From estimates (2.5) and inequalities (4.4), (4.5) it follows that for sufficiently small ε the operator H maps Ω into itself.

Let $(r_1,r_2),(\bar r_1,\bar r_2)\in\Omega$. Using the estimates (2.5), (4.5) and properties of functions f,r_1,r_2, we obtain

$$
\begin{gathered}
|H^i_{r_1,r_2}(t_0,z_{t_0},y_{t_0},\varepsilon)-H^i_{\bar r_1,\bar r_2}(t_0,\bar z_{t_0},\bar y_{t_0},\varepsilon)|\le\\
\le\frac{\bar K(\nu_1+\nu_2)}{\alpha}L(\varepsilon)(\|r_1-\bar r_1\|+\|r_2-\bar r_2\|).
\end{gathered}
$$

Thus, operator H is a contraction on Ω if

$$
L<\frac{\alpha}{\bar K(\nu_1+\nu_2)}.
$$

Let us denote by (r_1^+,r_2^+) the unique fixed point of H on Ω. We are going to prove that M^+ is an integral manifold. Let (z_t,y_t) be the solution of the system (4.2), where $r_i=r_i^+$, $i=1,2$. Then functions r_i^+, $i=1,2$ satisfy equations

$$
r_i^+=-\int_t^{\infty}e^{B_i(t-s)}F[s,z_s,y_s,\varepsilon]ds,\quad i=1,2. \tag{4.7}
$$

Differentiating these equations we obtain that functions r_i^+, $i=1,2$ satisfy the first and the second equations of system (2.4). Thus, the set M^+ is an integral manifold for system (2.4).

From (4.5) and properties of the functions r_1^+,r_2^+ it follows that estimates (4.1) are valid for $N=\max(2\bar K,4l\bar K)$. The existence of the center-stable manifold M^{*+} is proved similarly to the case of stable manifold. □

5. The stability problem

In this item we study the behaviour of solutions of system (1.1) when $t \to \infty$.

Theorem 1. *Let the conditions I-II are valid. Then for all sufficiently small ε and arbitrary solution $(u_1(t), u_2(t), z_t, y_t)$ of system (2.4) there exists a solution*

$$(h_1(t), h_2(t), v^{*-}(t, h_1, h_2, \varepsilon), w^{*-}(t, h_1, h_2, \varepsilon))$$

of system (2.4), starting from a center-unstable manifold that following inequalities hold

$$\begin{aligned} |u_i(t) - h_i(t)| &\le K_3 \varphi e^{-\frac{\alpha}{2}(t-t_0)}, \quad i = 1, 2, \\ |z_t - v^{*-}(t, h_1, h_2, \varepsilon)| &\le K_3 \varphi e^{-\frac{\alpha}{2}(t-t_0)}, \\ |y_t - w^{*-}(t, h_1, h_2, \varepsilon)| &\le K_3 \varphi e^{-\frac{\alpha}{2}(t-t_0)}, \end{aligned} \tag{5.1}$$

where $\varphi = |z_{t_0} - v^{*-}(t, h_1(t_0), h_2(t_0), \varepsilon)| + |y_{t_0} - w^{*-}(t, h_1(t_0), h_2(t_0), \varepsilon)|$, $K_3 > 0$.

Proof. Let consider arbitrary solution $(u_1(t), u_2(t), z_t, y_t)$ of system (2.4) with initial values $(t_0, u_{10}, u_{20}, z_{t_0}, y_{t_0})$ and a solution $(h_1(t), h_2(t), \xi_t, \psi_t)$ of system (2.4) with initial values $(c_1, c_2, v^{*-}(t_0, c_1, c_2, \varepsilon), w^{*-}(t_0, c_1, c_2, \varepsilon))$, where $c_1 = h_1(t_0)$, $c_2 = h_2(t_0)$. Changing the variables in system (2.4)

$$p_1(t) = u_1(t) - h_1(t), \quad p_2(t) = u_2(t) - h_2(t), \quad q_t = z_t - \xi_t, \quad \omega_t = y_t - \psi_t, \tag{5.2}$$

we obtain the following system

$$\begin{aligned} \frac{dp_i}{dt} &= B_i p_i + \bar{F}_i(t, p_1, p_2, q_t, \omega_t, \varepsilon), \quad i = 1, 2, \\ q_t &= T(t - t_0) q_{t_0} + \int\limits_{t_0}^{t} T(t - s) X_0^Q \bar{F}(s, p_1, p_2, q_s, \omega_s, \varepsilon) ds, \\ \omega_t &= V(t, t_0)\omega_{t_0} + \frac{1}{\varepsilon} \int\limits_{t_0}^{t} V(t, s) Y_0 \bar{G}(s, p_1, p_2, q_s, \omega_s, \varepsilon) ds, \end{aligned} \tag{5.3}$$

where

$$\bar{F}_i = F_i(t, p_1 + h_1, p_2 + h_2, q_t + \xi_t, \omega_t + \psi_t, \varepsilon) - F_i(t, h_1, h_2, \xi_t, \psi_t, \varepsilon), i = 1, 2$$

and functions $\bar{F}, \bar{G}$ have the same form.

According to theorem 1, the system (5.3) has a stable integral manifold. This manifold is described by functions $p_i = r_i^+(t, q_t, \omega_t, \varepsilon)$, $i = 1, 2$, which satisfy the following conditions

$$r_i^+(t, 0, 0, \varepsilon) = 0,$$
$$|r_i^+(t, q_t, \omega_t, \varepsilon) - r_i^+(t, \bar{q}_t, \bar{\omega}_t, \varepsilon)| \le l(|q_t - \bar{q}_t| + |\omega_t - \bar{\omega}_t|).$$

For arbitrary solution $(p_1, p_2, q_t, \omega_t)$ of system (5.3), which belongs to M^+, it follows that

$$\begin{aligned} |p_i(t)| &\le N e^{-\frac{\alpha}{2}(t-t_0)}(|q_{t_0}| + |\omega_{t_0}|), \quad i = 1, 2, \\ |q_t| &\le N e^{-\frac{\alpha}{2}(t-t_0)}(|q_{t_0}| + |\omega_{t_0}|), \end{aligned} \tag{5.4}$$

$$|\omega_t| \le N e^{-\frac{\alpha}{2}(t-t_0)}(|q_{t_0}| + |\omega_{t_0}|).$$

Let us now show that representations (5.2) are valid if solutions $(p_1(t), p_2(t), q_t, \omega_t)$ are on the stable manifold M^+. It is sufficient to show that (5.2) holds for $t = t_0$. In this case we obtain

$$\begin{array}{ll} r_1^+(t_0, q_{t_0}, \omega_{t_0}, \varepsilon) = u_{10} - c_1, & r_2^+(t_0, q_{t_0}, \omega_{t_0}, \varepsilon) = u_{20} - c_2, \\ q_{t_0} = z_{t_0} - v^{*-}(t_0, c_1, c_2, \varepsilon), & \omega_{t_0} = y_{t_0} - w^{*-}(t_0, c_1, c_2, \varepsilon). \end{array} \tag{5.5}$$

Next we show that system (5.5) has the solution with respect to $(c_1, c_2, q_{t_0}, w_{t_0})$ for any $(u_{10}, u_{20}, z_{t_0}, \omega_{t_0})$.

Let S denote the sphere in the space $R^{n_1} \times R^{n_2}$ which is defined by

$$|c_1 - u_{10}| + |c_2 - u_{20}| \le 4l(|z_{t_0} - v^{*-}(t_0, u_{10}, u_{20}, \varepsilon)| + |y_{t_0} - w^{*-}(t_0, u_{10}, u_{20}, \varepsilon)|).$$

Let consider the operator $J = (J_1, J_2)$ on S, where

$$J_i(c_1, c_2) = u_{i0} - r_i^*(t_0, z_{t_0} - v^{*-}(t_0, c_1, c_2, \varepsilon), y_{t_0} - w^{*-}(t_0, c_1, c_2, \varepsilon)), i = 1, 2.$$

It is easy to obtain that for each $\varepsilon \in [0, \varepsilon_1]$ and fixed u_{10}, u_{20} the operator J is equicontinuous on S and maps S into itself if constant η_0 in the theorem 1 satisfy the following condition $\eta_0 < \dfrac{1}{8l}$.

According to Schauder theorem it follows that the mapping J has at least one fixed point in S. Let (c_1^*, c_2^*) denote this fixed point. Then

$$c_1^*, c_2^*, q_{t_0}^* = z_{t_0} - v^{*-}(t_0, c_1^*, c_2^*, \varepsilon), \quad \omega_{t_0}^* = y_{t_0} - w^{*-}(t_0, c_1^*, c_2^*, \varepsilon)$$

satisfy the system (5.5). □

The next assertion follows from theorem 1.

Theorem 2. *Let the conditions I-II are valid. Then for all sufficiently small ε the trivial solution of system* (1.1) *is stable (asymtotic stable, unstable) if and only if stable (asymtotic stable, unstable) the trivial solution of system* (3.6) *on center-unstable manifold.*

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ON NONLINEAR BOUNDARY VALUE PROBLEMS WITH IMPULSES

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Abstract. This work deals with a nonlinear boundary value problem with impulses. Such a problem was investigated by many authors, see for example [1] or [2] and references therein. The main purpose of this paper is to prove the equivalence between impulsive problems and properly constructed boundary value problems without impulses. Using this equivalence we can transfer results known for boundary value problems of ordinary differential equations to boundary value problems of differential equations with impulses. Here, we use this approach and prove the lower and upper solutions method for nonlinear impulsive problems with impulsive functions having positive or negative derivatives.

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1. **Introduction**

Let us consider the interval $J = [a, b] \subset \mathrm{R}$, where $a = t_0 < t_1 < \ldots < t_p < t_{p+1} = b$. We will work with the Banach spaces $C(J)$ (the space of functions x continuous on J with the norm $\|x\|_C = \max_{t\in J} |x(t)|$), $C^1(J)$ (the space of functions x having continuous first derivatives on J with the norm $\|x\|_{C^1} = \|x\|_C + \|x'\|_C$), $L(J)$ (the space of functions y Lebesgue integrable on J with the norm $\|y\|_1 = \int_a^b |y(t)|dt$) and with the set $AC(J)$ (the set of functions absolutely continuous on J). We say that $f : J \times \mathrm{R} \to \mathrm{R}$ fulfils the Carathéodory conditions on $J \times \mathrm{R}$, if f has the following properties: (i) for each $x \in \mathrm{R}$ the function $f(\cdot, x)$ is measurable on J; (ii) for almost each $t \in J$ the function $f(t, \cdot)$ is continuous on R; (iii) for each compact set $K \subset \mathrm{R}$ the function $m_K(t) = \sup_{x\in K} |f(t, x)|$ is Lebesgue integrable on J. For the set of functions satisfying the Carathéodory conditions on $J \times \mathrm{R}$ we write $Car(J \times \mathrm{R})$. For a subset Ω of a Banach space, $\mathrm{cl}(\Omega)$ and $\partial\Omega$ stand for the closure and the boundary of Ω, respectively.

We will investigate the impulsive problem

$$u'(t) = f(t, u(t)) \quad \text{for a. e. } t \in (t_j, t_{j+1}), \quad j = 0, \ldots, p, \tag{1.1}$$

$$u(t_j+) = I_j(u(t_j)), \quad j = 1, \dots, p, \tag{1.2}$$

$$h(u(a), u(b)) = 0, \tag{1.3}$$

where $f \in Car(J \times \mathrm{R})$, $I_j \in C^1(\mathrm{R})$, $I_j' \neq 0$, $j = 1, \dots, p$, and $h \in C(\mathrm{R}^2)$.

Together with problem (1.1) - (1.3) we will study the problem without impulses

$$x'(t) = g(t, x(t)) \quad \text{for a. e. } t \in J, \tag{1.4}$$

$$h(x(a), w(x(b))) = 0, \tag{1.5}$$

where $g \in Car(J \times \mathrm{R})$, $w \in C^1(\mathrm{R})$, $w' \neq 0$, and $h \in C(\mathrm{R}^2)$.

We prove that problems (1.1) - (1.3) and (1.4), (1.5) are equivalent and by means of this we get the lower and upper functions method as well as the existence of solutions to problem (1.1) - (1.3). Let us note that the existence results for problem (1.1) - (1.3) were proven also by E. Liz in [2], but by another approach and for increasing impulsive functions only. Here, we extend the lower and upper functions method and the existence results to the case of decreasing impulsive functions. Our proofs need no techniques or results from the theory of impulsive differential equations.

Definition 1. By AC^* we mean a set of functions $u : J \to \mathrm{R}$, which are absolutely continuous on each (t_i, t_{i+1}), $i = 0, \dots, p$, $u(t_j) = u(t_j-)$, $j = 1, \dots, p+1$, $u(a) = u(a+)$. A function $u \in AC^*$ which satisfies conditions (1.1) - (1.3) is called a solution of problem (1.1) - (1.3). A function $x \in AC(J)$ which satisfies conditions (1.4), (1.5) is called a solution of problem (1.4), (1.5).

2. Nonlinear boundary value problems without impulses

To keep this paper self-contained, let us show the ideas of the lower and upper functions method for problem (1.4), (1.5).

Definition 2. A function $\alpha_1 \in AC^*$ ($\alpha_2 \in AC^*$) is called a lower (upper) function of problem (1.4), (1.5) provided the conditions

$$(\alpha_i'(t) - g(t, \alpha_i(t)))(-1)^i \geq 0 \quad \text{for a. e. } t \in (t_j, t_{j+1}), \quad j = 0, \dots, p, \tag{2.1}$$

$$(\alpha_i(t_j+) - \alpha_i(t_j))(-1)^i \geq 0, \quad j = 1, \dots, p, \tag{2.2}$$

$$h(\alpha_i(a), w(\alpha_i(b)))(-1)^i \geq 0, \quad i = 1, 2 \tag{2.3}$$

are satisfied.

We will assume a certain relation between lower and upper functions.
First, let

$$\alpha_1(t) \leq \alpha_2(t) \text{ for each } t \in J, \tag{2.4}$$

$$\alpha(t,x) = \begin{cases} \alpha_1(t) & \text{if} \quad x < \alpha_1(t) \\ x & \text{if} \quad \alpha_1(t) \le x \le \alpha_2(t) \\ \alpha_2(t) & \text{if} \quad \alpha_2(t) < x \end{cases} \quad \text{for each } t \in J \text{ and } x \in \mathrm{R}, \tag{2.5}$$

$$\tilde{g}(t,x) = g(t, \alpha(t,x)), \quad g \in Car(J \times \mathrm{R}). \tag{2.6}$$

Further, we assume that

$$w \in C^1(\mathrm{R}),\ w' > 0 \text{ on } \mathrm{R}, \tag{2.7}$$

$$h \in C(\mathrm{R}^2),\ h \text{ is nonincreasing in its second variable} \tag{2.8}$$

and consider an auxiliary problem

$$x'(t) = \tilde{g}(t, x(t)) \text{ for a. e. } t \in J, \tag{2.9}$$

$$x(a) = \alpha(a, x(a) - h(x(a), w(x(b)))). \tag{2.10}$$

Proposition 1. *Let us suppose (2.4) - (2.8) hold. Let x be a solution of problem (2.9), (2.10) and let α_1, α_2 be lower and upper functions of problem (1.4), (1.5). Then*

$$\alpha_1 \le x \le \alpha_2 \text{ on } J \tag{2.11}$$

and consequently x is a solution of (1.4), (1.5), as well.

Proof. Let us put

$$z(t) = \alpha_1(t) - x(t) \text{ for each } t \in J.$$

According to (2.10) we have $x(a) \in [\alpha_1(a), \alpha_2(a)]$, which means that $z(a) \le 0$. Suppose that there exists $q_1 \in (a, t_1)$ such that

$$z(q_1) > 0. \tag{2.12}$$

Since z is continuous on $[a, t_1)$, we can find $q_0 \in [a, q_1)$ such that

$$z(q_0) = 0 \text{ and } z > 0 \text{ on } (q_0, q_1]. \tag{2.13}$$

In view of (2.1) we have

$$z'(t) = \alpha_1'(t) - x'(t) \le g(t, \alpha_1(t)) - \tilde{g}(t, x(t)) = 0$$

for a. e. $t \in (q_0, q_1]$. Therefore

$$0 \ge \int_{q_0}^{q_1} z'(t)dt = z(q_1) - z(q_0) = z(q_1),$$

which contradicts (2.12). Thus we get

$$z \le 0 \text{ on } [a, t_1]. \tag{2.14}$$

By (2.2) and (2.14), the inequalities $\alpha_1(t_1+) \le \alpha_1(t_1) \le x(t_1) = x(t_1+)$ are true, and so $z(t_1+) \le 0$.

Suppose that there exists $q_1 \in (t_1, t_2)$ such that (2.12) is true. Then we can find $q_0 \in (t_1, q_1)$ such that (2.13) is valid and we get a contradiction to (2.12) as before. In such a way we can argue at each interval $(t_j, t_{j+1}]$, $j = 1, \ldots, p$, and get $z \le 0$ on J which means that

$$\alpha_1 \le x \text{ on } J.$$

The second inequality in (2.11) can be proved similarly putting $z = x - \alpha_2$ on J. Due to (2.11), we have

$$x'(t) = \tilde{g}(t, x(t)) = g(t, x(t)) \text{ for a. e. } t \in J.$$

It remains to prove that x fulfils (1.5). It is sufficient to show that

$$\alpha_1(a) \le x(a) - h(x(a), w(x(b))) \le \alpha_2(a), \tag{2.15}$$

because (2.10) implies (1.5) then.
Suppose on the contrary, that

$$\alpha_1(a) > x(a) - h(x(a), w(x(b))).$$

Since (2.10) is valid, we get $x(a) = \alpha_1(a)$. Then, by (2.7) and (2.8), we have $0 < h(x(a), w(x(b))) \le h(\alpha_1(a), w(\alpha_1(b)))$, which contradicts (2.3). The second inequality in (2.15) can be proved similarly. This completes the proof. □

We can easily see that the following modification of Proposition 1 is true.

Proposition 2. *The assertion of Proposition 1 remains valid if we replace the conditions (2.7) and (2.8) with*

$$w \in C^1(\mathrm{R}),\ w' < 0 \textit{ on } \mathrm{R} \tag{2.16}$$

and

$$h \in C(\mathrm{R}^2),\ h \textit{ is nondecreasing in its second variable.} \tag{2.17}$$

Theorem 1. *Let α_1 and α_2 be lower and upper functions of problem (1.4), (1.5). Further, suppose that $\alpha_1 \le \alpha_2$ on J and that either $w' > 0$ and h is nonincreasing in its second variable or $w' < 0$ and h is nondecreasing in its second variable. Then there exists a solution x of problem (1.4), (1.5) such that*

$$\alpha_1 \le x \le \alpha_2 \textit{ on } J. \tag{2.18}$$

Proof. Consider the integral equation

$$x(t) = \alpha(a, x(a) - h(x(a), w(x(b)))) + \int_a^t \tilde{g}(s, x(s))ds, \tag{2.19}$$

which is equivalent to problem (2.9), (2.10). Further, define the set

$$\Omega = \{x \in C(J) : \|x\|_C \le M\},$$

where $M = \sup_{t\in J} |\alpha_1(t)| + \sup_{t\in J} |\alpha_2(t)| + \int_a^b \lambda(s)ds$ and $\lambda(t) = \sup\{|g(t, x)| : x \in [\alpha_1(t), \alpha_2(t)]\}$. Clearly Ω is a nonempty, convex, closed and bounded set in $C(J)$. We can check that the operator $T : \Omega \to C(J)$ given by

$$(Tx)(t) = \alpha(a, x(a) - h(x(a), w(x(b)))) + \int_a^t \tilde{g}(s, x(s))ds$$

is continuous and that $\mathrm{cl}(T(\Omega))$ is a compact set in $C(J)$ and T maps Ω to itself. Thus, according to the Schauder fixed point theorem, there is a point $x \in \Omega$ such that

$$Tx = x,$$

which means that the function x is a solution of (2.19) and consequently a solution of (2.9), (2.10). Propositions 1 and 2 imply that x is a solution of (1.4), (1.5) and satisfies (2.18). □

Now, suppose

$$\alpha_2(t) \le \alpha_1(t) \text{ for each } t \in J, \tag{2.20}$$

$$\alpha(t,x) = \begin{cases} \alpha_2(t) & \text{if} \quad x < \alpha_2(t) \\ x & \text{if} \quad \alpha_2(t) \le x \le \alpha_1(t) \\ \alpha_1(t) & \text{if} \quad \alpha_1(t) < x \end{cases} \quad \text{for each } t \in J \text{ and } x \in \mathrm{R} \tag{2.21}$$

and consider an auxiliary problem (2.9),

$$x(b) = \alpha(b, x(b) + h(x(a), w(x(b)))). \tag{2.22}$$

Proposition 3. *Let conditions (2.20), (2.21), (2.6) and*

$$h \in C(\mathrm{R}^2),\ h \text{ is nondecreasing in its first variable, } w \in C^1(\mathrm{R}) \tag{2.23}$$

be satisfied. Let x be a solution of problem (2.9), (2.22) and let α_1, α_2 be lower and upper functions of problem (1.4), (1.5). Then

$$\alpha_2 \le x \le \alpha_1 \text{ on } J \tag{2.24}$$

and consequently x is a solution of (1.4), (1.5), as well.

Proof. This proof is similar to that of Proposition 1. We put

$$z(t) = x(t) - \alpha_1(t) \text{ for all } t \in J$$

and prove $z(t) \le 0$ for all $t \in J$. In view of (2.22) we have $x(b) \in [\alpha_2(b), \alpha_1(b)]$, which means that $z(b) \le 0$. Suppose that there is $q_0 \in (t_p, b)$ such that

$$z(q_0) > 0. \tag{2.25}$$

Since z is continuous on $(t_p, b]$, we can find $q_1 \in (q_0, b]$ such that

$$z(q_1) = 0 \text{ and } z > 0 \text{ on } [q_0, q_1). \tag{2.26}$$

In view of (2.1) we have

$$z'(t) = x'(t) - \alpha_1'(t) \ge \tilde{g}(t, x(t)) - g(t, \alpha_1(t)) = 0 \text{ for a. e. } t \in [q_0, q_1).$$

Therefore

$$0 \le \int_{q_0}^{q_1} z'(t)dt = z(q_1) - z(q_0) = -z(q_0),$$

which contradicts (2.25). Similarly, we can proceed to the initial point $a = t_0$. To prove the first inequality in (2.24), we work with

$$z(t) = \alpha_2(t) - x(t) \text{ for each } t \in J.$$

Due to (2.24), we have

$$x'(t) = \tilde{g}(t, x(t)) = g(t, x(t)) \text{ for a. e. } t \in J.$$

Finally, we need to prove

$$\alpha_2(b) \le x(b) + h(x(a), w(x(b))) \le \alpha_1(b). \tag{2.27}$$

If $\alpha_1(b) < x(b) + h(x(a), w(x(b)))$, then $x(b) = \alpha_1(b)$ and

$$0 < h(x(a), w(x(b))) \le h(\alpha_1(a), w(\alpha_1(b))),$$

which contradicts (2.3). For the first inequality in (2.27) we argue similarly. □

Theorem 2. *Let α_1 and α_2 be lower and upper functions of problem (1.4), (1.5). Further, suppose that $\alpha_1 \ge \alpha_2$ on J and that h is nondecreasing in its first variable. Then there exists a solution x of problem (1.4), (1.5) such that*

$$\alpha_2 \le x \le \alpha_1 \ \text{ on } J. \tag{2.28}$$

Proof. Starting with the integral equation

$$x(t) = \alpha(b, x(b) + h(x(a), w(x(b)))) + \int_b^t \tilde{g}(s, x(s))ds$$

and using Proposition 3 instead of Propositions 1, 2, we can argue as in the proof of Theorem 1. □

3. Relation between impulsive problem (1.1)-(1.3) and problem (1.4), (1.5)

Consider the impulsive functions $I_j \in C^1(\mathrm{R})$, $I'_j \ne 0$, $j = 1, \dots, p$, and define functions $w_i : \mathrm{R} \to \mathrm{R}$, $i = 0, \dots, p$:

$$\begin{aligned} &w_0 = \mathrm{id}_{\mathrm{R}},\\ &w_1 = I_1,\\ &w_2 = I_2(I_1) = I_2(w_1),\\ &\dots\\ &w_{p-1} = I_{p-1}(w_{p-2}),\\ &w = w_p = I_p(w_{p-1}). \end{aligned} \tag{3.1}$$

Further, having $f \in Car(J \times \mathrm{R})$, define

$$g(t,x) = \begin{cases} \frac{f(t,w_0(x))}{w'_0(x)} & \text{for} \quad t \in [a, t_1] \\ \frac{f(t,w_1(x))}{w'_1(x)} & \text{for} \quad t \in (t_1, t_2] \\ & \dots \\ \frac{f(t,w_p(x))}{w'_p(x)} & \text{for} \quad t \in (t_p, b] \end{cases} \quad \text{for } x \in \mathrm{R}. \tag{3.2}$$

Conversely, having $g \in Car(J \times \mathrm{R})$, define

$$f(t,u)=\begin{cases} g(t,w_0^{-1}(u))w_0'(w_0^{-1}(u)) & \text{for} \quad t\in[a,t_1] \\ g(t,w_1^{-1}(u))w_1'(w_1^{-1}(u)) & \text{for} \quad t\in(t_1,t_2] \\ \quad\cdots & \\ g(t,w_p^{-1}(u))w_p'(w_p^{-1}(u)) & \text{for} \quad t\in(t_p,b] \end{cases} \quad \text{for } u\in \mathrm{R}. \tag{3.3}$$

Theorem 3. *a) Let u be a solution of problem (1.1) - (1.3) and let g be defined by (3.1), (3.2). Then a function x given by*

$$x(t)=\begin{cases} w_0^{-1}(u(t)) & \textit{for} \quad t\in[a,t_1] \\ w_1^{-1}(u(t)) & \textit{for} \quad t\in(t_1,t_2] \\ \quad\cdots & \\ w_p^{-1}(u(t)) & \textit{for} \quad t\in(t_p,b], \end{cases} \tag{3.4}$$

is a solution of problem (1.4), (1.5).
b) Let x be a solution of problem (1.4), (1.5) and let f be defined by (3.1), (3.3). Then a function u given by

$$u(t)=\begin{cases} w_0(x(t)) & \textit{for} \quad t\in[a,t_1] \\ w_1(x(t)) & \textit{for} \quad t\in(t_1,t_2] \\ \quad\cdots & \\ w_p(x(t)) & \textit{for} \quad t\in(t_p,b], \end{cases} \tag{3.5}$$

is a solution of problem (1.1) - (1.3).
Proof. a) Let u be a solution of problem (1.1) - (1.3). Then, in view of (3.4) and (3.2)

$$x'(t)=\frac{u'(t)}{w_i'(x(t))}=\frac{f(t,w_i(x(t)))}{w_i'(x(t))}=g(t,x(t)) \text{ for a. e. } t\in(t_i,t_{i+1}),$$

$i=0,\dots,p$. Since $u(a)=x(a)$ and $u(b)=w_p(x(b))=w(x(b))$, we have by (1.3), $h(x(a),w(x(b)))=0$.
Next, $x(t_j+)=w_j^{-1}(u(t_j+))=w_j^{-1}(I_j(u(t_j)))=w_{j-1}^{-1}(I_j^{-1}(I_j(u(t_j))))=w_{j-1}^{-1}(u(t_j))=x(t_j)$, $j=1,\dots,p$. This together with $u\in AC^*$ implies $x\in AC(J)$ and therefore x is a solution of (1.4), (1.5).
b) Let x be a solution of (1.4), (1.5). Then, by (3.5) and (3.3), $u'(t)=w_i'(x(t))x'(t)=w_i'(x(t))g(t,x(t))=w_i'(w_i^{-1}(u(t)))g(t,w_i^{-1}(u(t)))=f(t,u(t))$ for a. e. $t\in(t_i,t_{i+1})$, $i=0,\dots,p$.
We have $x(a)=u(a)$, $w(x(b))=w_p(x(b))=u(b)$ and then $h(u(a),u(b))=0$. Finally, $u(t_j+)=w_j(x(t_j+))=w_j(x(t_j))=I_j(w_{j-1}(x(t_j)))=I_j(u(t_j))$, $j=1,\dots,p$.
Since $x\in AC(J)$, then, in view of (3.5), $w_i(x(t))$ are absolutely continuous on (t_i,t_{i+1}), $i=0,\dots,p$ and $u(t_j-)=u(t_j)$, $j=1,\dots,p+1$. Therefore u is a solution of (1.1) - (1.3). □

4. Lower and upper functions method to problem (1.1)-(1.3)

4.1 Increasing impulsive functions

Let us suppose

$$I_j \in C^1(\mathrm{R}),\, I_j' > 0, \quad j = 1, \dots, p. \tag{4.1}$$

Definition 3. A function $\sigma_1 \in AC^*$ ($\sigma_2 \in AC^*$) is called a lower (upper) function of problem (1.1) - (1.3), provided the conditions

$$(\sigma_i'(t) - f(t, \sigma_i(t)))(-1)^i \geq 0 \text{ for a. e. } t \in (t_j, t_{j+1}), \quad j = 0, \dots, p, \tag{4.2}$$

$$(\sigma_i(t_j+) - I_j(\sigma_i(t_j)))(-1)^i \geq 0, \quad j = 1, \dots, p, \tag{4.3}$$

$$h(\sigma_i(a), \sigma_i(b))(-1)^i \geq 0, \quad i = 1, 2 \tag{4.4}$$

are satisfied.

Theorem 4. *Suppose that (4.1) holds.*
a) Let σ_1 and σ_2 be lower and upper functions of problem (1.1) - (1.3) and let g be defined by (3.1), (3.2). Then functions α_1 and α_2 given by

$$\alpha_i(t) = \begin{cases} w_0^{-1}(\sigma_i(t)) & \textit{for} \quad t \in [a, t_1] \\ w_1^{-1}(\sigma_i(t)) & \textit{for} \quad t \in (t_1, t_2] \\ & \cdots \\ w_p^{-1}(\sigma_i(t)) & \textit{for} \quad t \in (t_p, b]. \end{cases} \tag{4.5}$$

are lower and upper functions of problem (1.4), (1.5).
b) Let α_1 and α_2 be lower and upper functions of problem (1.4), (1.5) and let f be defined by (3.1), (3.3).
Then functions σ_1 and σ_2 given by

$$\sigma_i(t) = \begin{cases} w_0(\alpha_i(t)) & \textit{for} \quad t \in [a, t_1] \\ w_1(\alpha_i(t)) & \textit{for} \quad t \in (t_1, t_2] \\ & \cdots \\ w_p(\alpha_i(t)) & \textit{for} \quad t \in (t_p, b], \end{cases} \tag{4.6}$$

are lower and upper functions of problem (1.1) - (1.3).

Proof. We will prove the assertion for lower functions only. Choose arbitrary $j \in \{1, \dots, p\}$ and $i \in \{0, \dots, p\}$. Let σ_1 be a lower function of (1.1) - (1.3).
Then

$$\alpha_1'(t) = \frac{\sigma_1'(t)}{w_i'(\alpha_1(t))} \leq \frac{f(t, \sigma_1(t))}{w_i'(\alpha_1(t))} = g(t, \alpha_1(t)) \text{ for a. e. } t \in (t_i, t_{i+1}).$$

Since $\sigma_1(a) = \alpha_1(a)$ and $\sigma_1(b) = w_p(\alpha_1(b)) = w(\alpha_1(b))$, we have

$$h(\alpha_1(a), w(\alpha_1(b))) = h(\sigma_1(a), \sigma_1(b)) \leq 0.$$

Further, $\alpha_1(t_j+) = w_j^{-1}(\sigma_1(t_j+)) \leq w_j^{-1}(I_j(\sigma_1(t_j))) = w_{j-1}^{-1}(\sigma_1(t_j)) = \alpha_1(t_j)$.
Thus α_1 is a lower function of (1.4), (1.5).
Now, let α_1 be a lower function of (1.4), (1.5). Then $\sigma_1'(t) = w_i'(\alpha_1(t))\alpha_1'(t) \leq w_i'(\alpha_1(t))g(t, \alpha_1(t)) = w_i'(w_i^{-1}(\sigma_1(t)))g(t, w_i^{-1}(\sigma_1(t))) = f(t, \sigma_1(t))$, for a. e. $t \in$

(t_i, t_{i+1}).
As before, we have

$$h(\sigma_1(a), \sigma_1(b)) = h(\alpha_1(a), w(\alpha_1(b))) \leq 0.$$

Finally, $\sigma_1(t_j+) = w_j(\alpha_1(t_j+)) \leq w_j(\alpha_1(t_j)) = I_j(w_{j-1}(\alpha_1(t_j))) = I_j(\sigma_1(t_j))$.
So, σ_1 is a lower function of (1.1) - (1.3). □

Theorem 5. *Let σ_1 and σ_2 be lower and upper functions of problem (1.1) - (1.3) and let (4.1) be fulfilled. Further, let one of the following conditions be satisfied:*

(1) *$\sigma_1 \leq \sigma_2$ on J and h is nonincreasing in its second variable;*
(2) *$\sigma_2 \leq \sigma_1$ on J and h is nondecreasing in its first variable.*

Then there exists a solution of problem (1.1) - (1.3) lying between σ_1 and σ_2 on J.

Proof. We will investigate only the case $\sigma_1 \leq \sigma_2$ on J. The second case can be proved similarly. Consider problem (1.4), (1.5), where g is defined by (3.1), (3.2). We can see that $f \in Car(J \times \mathrm{R})$ implies $g \in Car(J \times \mathrm{R})$, and that, according to (4.1), we get $w \in C^1(\mathrm{R})$, $w' > 0$. Theorem 4 implies that the functions α_1, α_2 given by (4.5) are lower and upper functions of (1.4), (1.5).
Since the functions w_i, $i = 0, \ldots, p$, are increasing, it follows that $\alpha_1 \leq \alpha_2$ on J if and only if $\sigma_1 \leq \sigma_2$ on J. Therefore, by Theorem 1, problem (1.4), (1.5) has a solution x which satisfies (2.18). Thus, in view of Theorem 3, the function u given by (3.5) is a solution of (1.1) - (1.3) and $\sigma_1 \leq u \leq \sigma_2$ on J. □

4.2 Decreasing impulsive function

Let us suppose

$$I_j \in C^1(\mathrm{R}),\, I_j' < 0, \quad j = 1, \ldots, p. \tag{4.7}$$

Definition 4. A function $\sigma_1 \in AC^*$ ($\sigma_2 \in AC^*$) is called a generalized lower (upper) function of problem (1.1) - (1.3), provided the conditions

$$(\sigma_i'(t) - f(t, \sigma_i(t)))(-1)^{i+j} \geq 0 \quad \text{for a. e. } t \in (t_j, t_{j+1}), \quad j = 0, \ldots, p, \tag{4.8}$$

$$(\sigma_i(t_j+) - I_j(\sigma_i(t_j)))(-1)^{i+j} \geq 0, \quad j = 1, \ldots, p, \tag{4.9}$$

$$h(\sigma_i(a), \sigma_i(b))(-1)^i \geq 0, \quad i = 1, 2. \tag{4.10}$$

Theorem 6. *Suppose that (4.7) is satisfied.*
a) Let σ_1 and σ_2 be generalized lower and upper functions of problem (1.1) - (1.3) and let g be defined by (3.1), (3.2). Then α_1, α_2 given by (4.5) are lower and upper functions of (1.4), (1.5).
b) Let α_1, α_2 be lower and upper functions of (1.4), (1.5) and let f be defined by (3.1), (3.3). Then σ_1, σ_2 given by (4.6) are generalized lower and upper functions of problem (1.1) - (1.3).

Proof. We can use the arguments from the proof of Theorem 4 having in mind that conditions (3.1) and (4.7) imply that

$$(-1)^j w_j' > 0,\, j = 0, \ldots, p. \tag{4.11}$$

We will prove the assertion a) for σ_2, only. So, choose arbitrary $j \in \{0, \dots, p\}$ and consider a generalized upper function σ_2 of (1.1) - (1.3). Then

$$\alpha_2'(t) - g(t, \alpha_2(t)) = \frac{\sigma_2'(t)}{w_j'(\alpha_2(t))} - \frac{f(t, \sigma_2(t))}{w_j'(\alpha_2(t))} = (\sigma_2'(t) - f(t, \sigma_2(t)))\frac{(-1)^j}{|w_j'(\alpha_2(t))|} \geq 0$$

for a. e. $t \in (t_j, t_{j+1})$.
Further, choose $j \in \{1, \dots, p\}$ and suppose that j is odd. Then, by (4.9), $\sigma_2(t_j+) \leq I_j(\sigma_2(t_j))$, and in view of (4.11) we have

$$w_j^{-1}(\sigma_2(t_j+)) \geq w_j^{-1}(I_j(\sigma_2(t_j))).$$

Therefore, $\alpha_2(t_j+) = w_j^{-1}(\sigma_2(t_j+)) \geq w_j^{-1}(I_j(\sigma_2(t_j))) = w_{j-1}^{-1}(\sigma_2(t_j)) = \alpha_2(t_j)$. Similarly for j if it is even. As in the proof of Theorem 4, we get $h(\alpha_2(a), w(\alpha_2(b))) \geq 0$. □

Theorem 7. *Let σ_1 and σ_2 be generalized lower and upper functions of problem (1.1) - (1.3) and let (4.7) be fulfilled. Further, let one of the following conditions be satisfied:*

(1) *$(\sigma_1 - \sigma_2)(-1)^i \leq 0$ on $(t_i, t_{i+1}]$ for $i = 0, \dots, p$ and h is nonincreasing (nondecreasing) in its second variable if p is even (odd);*
(2) *$(\sigma_1 - \sigma_2)(-1)^i \geq 0$ on $(t_i, t_{i+1}]$ for $i = 0, \dots, p$ and h is nondecreasing in its first variable.*

Then there exists a solution of problem (1.1) - (1.3) lying between σ_1 and σ_2 on J.

Proof. We follow the proof of Theorem 5 working with $(-1)^i w_i$, $i = 0, \dots, p$ and using Theorem 6 instead of Theorem 4. □

We can also study, instead of problem (1.4), (1.5), problem (1.4),

$$h(w(x(a)), x(b)) = 0, \tag{4.12}$$

where $g \in Car(J \times \mathrm{R})$ is given by

$$g(t,x) = \begin{cases} \frac{f(t,w_0(x))}{w_0'(x)} & \text{for} \quad t \in (t_p, b] \\ \frac{f(t,w_1(x))}{w_1'(x)} & \text{for} \quad t \in (t_{p-1}, t_p] \\ \dots & \\ \frac{f(t,w_p(x))}{w_p'(x)} & \text{for} \quad t \in [a, t_1] \end{cases} \quad \text{for } x \in \mathrm{R}, \tag{4.13}$$

where

$$\begin{aligned} &w_0 = \mathrm{id}_{\mathrm{R}}, \\ &w_1 = I_1^{-1}, \\ &w_2 = I_2^{-1}(I_1^{-1}) = I_2^{-1}(w_1), \\ &\dots \\ &w_{p-1} = I_{p-1}^{-1}(w_{p-2}), \\ &w = w_p = I_p^{-1}(w_{p-1}). \end{aligned} \tag{4.14}$$

Working with problem (1.4), (4.12) we can prove the following modification of Theorem 7.

Theorem 8. *Let σ_1 and σ_2 be generalized lower and upper function of problem (1.1) - (1.3) and let (4.7) be fulfilled. Further, let one of the following conditions be satisfied:*

(1) $(\sigma_1 - \sigma_2)(-1)^{p-i} \le 0$ *on* $(t_i, t_{i+1}]$ *for* $i = 0, \dots, p$ *and* h *is nonincreasing in its second variable;*
(2) $(\sigma_1 - \sigma_2)(-1)^{p-i} \ge 0$ *on* $(t_i, t_{i+1}]$ *for* $i = 0, \dots, p$ *and* h *is nondecreasing (nonincreasing) in its first variable if* p *is even (odd).*

Then there exists a solution of problem (1.1) - (1.3) lying between σ_1 *and* σ_2 *on* J*.*

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ON THE MEAN VALUE PROBLEM FOR LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. We establish some unique solvability results concerning the mean value problem for a linear functional differential equation in a Banach space.

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1. Problem description

We consider the boundary value problem

$$u'(t) = (lu)(t) + q(t), \qquad t \in [a, b]\,, \tag{1.1}$$

$$\frac{1}{b-a}\int_a^b u(\sigma)\,\mathrm{d}\sigma = u(\tau) + \eta, \tag{1.2}$$

where $q \in L\big([a,b], X\big)$ and $\eta \in X$. Here and below, τ is a certain (fixed) point from $[a,b]$, $\langle X, \|\cdot\|\rangle$ is a Banach space, and $l : C\big([a,b], X\big) \to L\big([a,b], X\big)$ is a linear operator satisfying the following

Assumption 1. *There exists a function $\beta \in L([a,b], \mathbb{R})$ such that*

$$\big\|(lu)(t)\big\| \le \beta(t) \max_{\sigma\in[a,b]} \|u(\sigma)\| \tag{1.3}$$

for all $u \in C([a,b], X)$ and a.e. $t \in [a,b]$.

Remark 2. It follows immediately from (1.3) that, under Assumption 1, the operator l is continuous.

By a solution of (1.1), (1.2), we mean an absolutely continuous function $x : [a,b] \to X$ possessing property (1.2) and satisfying (1.1) almost everywhere on $[a,b]$. It is natural to refer to system (1.1), (1.2) as to the *linear inhomogeneous mean value problem.*

Along with (1.1), (1.2), we shall also consider the corresponding *homogeneous mean value problem*

$$u'(t) = (lu)(t), \qquad t \in [a, b]\,, \tag{1.4}$$

$$\frac{1}{b-a}\int_a^b u(\sigma)\,\mathrm{d}\sigma = u(\tau). \tag{1.5}$$

2. **Notation**

Let $\langle X, \|\cdot\|\rangle$ be a Banach space. The following basic notation is used throughout the paper.

(1) $\mathcal{B}(X)$ is the algebra of all linear bounded operators on X.
(2) 1_X is the unity in $\mathcal{B}(X)$.
(3) $\ker A$ (resp., $\operatorname{im} A$) is the kernel (resp., image) of $A \in \mathcal{B}(X)$.
(4) $r(A)$ is the spectral radius of $A \in \mathcal{B}(X)$.
(5) $C\big([a, b], X\big)$ is the Banach space of all the continuous mappings from $[a, b]$ to X with the norm

$$C\big([a, b], X\big) \ni u \longmapsto \max_{t\in[a,b]} \|u(t)\|;$$

(6) $L\big([a, b], X\big)$ is the Banach space of all the Bochner integrable mappings from $[a, b]$ to X with the norm

$$L\big([a, b], X\big) \ni u \longmapsto \int_a^b \|u(t)\|\,\mathrm{d}t.$$

3. **Definitions, assumptions, and subsidiary results**

Let us introduce a definition.

Definition 3. Given a linear operator $l : C\big([a, b]\,, X\big) \to L\big([a, b]\,, X\big)$, we define the function $l^{\square} : [a, b] \to \mathcal{B}(X)$ by putting

$$l^{\square} = l1_X. \tag{3.1}$$

Remark 4. In other words, the action of the restriction of l to the subspace of constant functions is nothing but the multiplication by $l^{\square}$.

From now on, we make the following

Assumption 5. *The linear operator $\Lambda_{l,\tau} : X \to X$ defined by the formula*

$$\Lambda_{l,\tau} := \int_a^b \left(\int_\tau^t l^{\square}(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}t \tag{3.2}$$

is invertible.

The invertibility of operator (3.2) allows us to put

$$[\Gamma_{l,\tau} y](t) := \int_\tau^t y(\sigma)\,\mathrm{d}\sigma - \int_\tau^t l^{\square}(\sigma)\,\mathrm{d}\sigma\, \Lambda_{l,\tau}^{-1} \int_a^b \left(\int_\tau^s y(\sigma)\,\mathrm{d}\sigma \right) \mathrm{d}s \tag{3.3}$$

for $y \in L([a,b],X)$ and $t \in [a,b]$. [The integral in (3.2) is understood in the Bochner sense, whereas $l^{\square}$ is related to l according to formula (3.1).]

Consequently, it is easy to see that formula (3.3) determines a linear, continuous operator $\Gamma_{l,\tau}$ from $L([a,b],X)$ to $C([a,b],X)$.

Lemma 6. *The inclusion*

$$X \subset \ker \Gamma_{l,\tau} l \tag{3.4}$$

holds.

Remark 7. Relation (3.4) from Lemma 6 should be understood in the sense that the constant mappings $[a,b] \to X$ are identified with the corresponding vectors from X.

Proof of Lemma 6. According to Definition 3, for an arbitrary $x \in X$, we have

$$(lx)(t) = l^{\square}(t)x, \qquad t \in [a,b].$$

Therefore, for all $x \in X$, formula (3.3) yields

$$\begin{aligned}
\Gamma_{l,\tau} lx &= \int_\tau^t (lx)(\sigma)\,\mathrm{d}\sigma - \int_\tau^t l^{\square}(\sigma)\,\mathrm{d}\sigma\, \Lambda_{l,\tau}^{-1} \int_a^b \left(\int_\tau^s (lx)(\sigma)\,\mathrm{d}\sigma \right) \mathrm{d}s \\
&= \int_\tau^t l^{\square}(\sigma)x\,\mathrm{d}\sigma - \int_\tau^t l^{\square}(\sigma)\,\mathrm{d}\sigma\, \Lambda_{l,\tau}^{-1} \int_a^b \left(\int_\tau^s l^{\square}(\sigma)x\,\mathrm{d}\sigma \right) \mathrm{d}s \\
&= \left[\int_\tau^t l^{\square}(\sigma)\,\mathrm{d}\sigma - \int_\tau^t l^{\square}(\sigma)\,\mathrm{d}\sigma\, \Lambda_{l,\tau}^{-1} \int_a^b \left(\int_\tau^s l^{\square}(\sigma)\,\mathrm{d}\sigma \right) \mathrm{d}s \right] x,
\end{aligned}$$

which, combined with (3.2), implies that $\Gamma_{l,\tau} lx = 0$. ☐

Remark 8. Lemma 6 claims that $\operatorname{im} l|_X \subset \ker \Gamma_{l,\tau}$.

Lemma 9. *For an arbitrary y from $L([a,b],X)$, the equality*

$$\big(\Gamma_{l,\tau} u\big)(\tau) = 0 \tag{3.5}$$

holds.

Proof. Relation (3.5) is an immediate consequence of formula (3.3). ☐

Let T_f denote the "shift" mapping

$$u \longmapsto T_f u := u + f, \tag{3.6}$$

where f is a fixed element of $L([a,b]\,,X)$.

Lemma 10. *Let us assume that* $l : C([a,b]\,,X) \to L([a,b]\,,X)$ *and* $A : L([a,b]\,,X) \to C([a,b]\,,X)$ *are certain linear operators satisying the condition*[1]

$$X \subset \ker Al. \tag{3.7}$$

Then, for every $N \geq 1$,

$$(T_\xi\, Al)^N = T_\xi (Al)^N, \tag{3.8}$$

where $\xi \in X$ *is arbitrary and the mapping* $T_\xi : C([a,b]\,,X) \to C([a,b]\,,X)$ *is defined according to relation* (3.6).

Proof. Let us fix arbitrary $\xi \in X$ and $u_0 \in C([a,b]\,,X)$, and set

$$u_N := (T_\xi Al)^N u_0, \qquad N = 0, 1, 2, \ldots.$$

In view of definition (3.6) and condition (3.7), we have

$$u_1 = \xi + Alu_0$$

and

$$u_2 = \xi + Al[\xi + Alu_0] = \xi + A[l\xi + lAlu_0] = \xi + (Al)^2 u_0.$$

Arguing similarly, we can show that

$$u_N = \xi + (Al)^N u_0$$

for all $u_0 \in C\big([a,b], X\big)$, $\xi \in X$, and $N \in \mathbb{N}$. This means that equality (3.8) is true. □

4. Reduction of the homogeneous problem (1.4), (1.5) to a family of fixed-point type equations

Lemma 11. *Let* l *in* (1.1) *be such that the corresponding operator* (3.2) *is invertible. If* $u \in C\big([a,b]\,,X\big)$ *is a solution of the mean value problem* (1.4), (1.5), *then the equalities*

$$u = \xi + \Gamma_{l,\tau} lu, \tag{4.1}$$

$$\int_a^b \left[\int_\tau^t (lu)(\sigma)\,\mathrm{d}\sigma\right] \mathrm{d}t = 0 \tag{4.2}$$

hold with

$$\xi = u(\tau). \tag{4.3}$$

Conversely, if equations (4.1) *and* (4.2) *hold with some* $\xi \in X$, *then* u *is a solution of problem* (1.4), (1.5) *and, furthermore, equality* (4.3) *holds.*

Prior to the proof of Lemma 11, we establish the following

Lemma 12. *For an arbitrary* ξ *from* X, *every solution of equation* (4.1), *if there are any, satisfies condition* (1.5).

[1] Relation (3.7) should be understood similarly to (3.4) in Lemma 6; see Remark 7.

Proof of Lemma 12. Let u be a solution of (4.1) for some $\xi \in X$. Then, by virtue of (4.1) and Lemma 9, relation (4.3) holds. In view of (4.1) and (4.3), we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b u(\sigma)\,\mathrm{d}\sigma - u(\tau) &\\ &= \frac{1}{b-a}\int_a^b \Big[\xi + \big(\Gamma_{l,\tau} lu\big)(\sigma)\Big]\,\mathrm{d}\sigma - \xi - \big(\Gamma_{l,\tau} lu\big)(\tau)\\ &= \frac{1}{b-a}\int_a^b \big(\Gamma_{l,\tau} lu\big)(\sigma)\,\mathrm{d}\sigma - \big(\Gamma_{l,\tau} lu\big)(\tau),\end{aligned}$$

whence, according to Lemma 9, it follows that

$$\frac{1}{b-a}\int_a^b u(\sigma)\,\mathrm{d}\sigma - u(\tau) = \frac{1}{b-a}\int_a^b \big(\Gamma_{l,\tau} lu\big)(\sigma)\,\mathrm{d}\sigma. \tag{4.4}$$

On the other hand, considering (3.3) and (3.2), we obtain the equality

$$\int_a^b \big(\Gamma_{l,\tau} lu\big)(\sigma)\,\mathrm{d}\sigma = 0,$$

which, combined with (4.4), proves our lemma. □

Proof of Lemma 11. Let u be a solution of system (4.1), (4.2). Then, according to Lemma 12, u satisfies (1.5). Differentiation of equality (4.1), in view of (3.3), yields

$$u'(t) = \big(lu\big)(t) - l^{\square}(t)\,\Lambda_{l,\tau}^{-1}\int_a^b \left(\int_\tau^s (lu)(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}s, \qquad t \in [a,b],$$

whence we see that condition (4.2) guarantees the fulfillment of (1.4), i.e., u satisfies both (1.4) and (1.5).

Conversely, if u is a solution of problem (1.4), (1.5), then

$$u(t) = \xi + \int_\tau^t \big(lu\big)(s)\,\mathrm{d}s, \qquad t \in [a,b] \tag{4.5}$$

with

$$\xi := u(\tau).$$

Therefore, by virtue of (4.5) and (1.5),

$$\begin{aligned}\int_a^b \left(\int_\tau^s (lu)(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}s &= \int_a^b \big[u(s) - \xi\big]\,\mathrm{d}s\\ &= \int_a^b u(s)\,\mathrm{d}s - (b-a)\xi\\ &= 0.\end{aligned} \tag{4.6}$$

Taking into account (3.3) and (4.6), we obtain

$$\begin{aligned}\big(\Gamma_{l,\tau} l u\big)(t) &= \int_\tau^t \big(lu\big)(s)\,\mathrm{d}s \\ &\qquad - \int_\tau^t l^{\square}(\sigma)\,\mathrm{d}\sigma\,\Lambda_{l,\tau}^{-1}\int_a^b \left(\int_\tau^s \big(lu\big)(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}s \\ &= \int_\tau^t \big(lu\big)(s)\,\mathrm{d}s, \qquad t \in [a,b],\end{aligned}$$

which means that (4.5) coincides with (4.1). Thus, we have shown that u satisfies (4.1), (4.2). □

5. **Iteration method for equation** (4.1)

The purpose of this section is to prove a statement (Lemma 13 below) is related to the iteration sequence[2]

$$u_{k+1} := \xi + \Gamma_{l,\tau} l u_k, \quad k = 0, 1, \ldots \tag{5.1}$$

associated with equation (4.1). We suppose that Assumption 5 holds and, therefore, the expression in the right-hand side of (5.1) is well-defined.

Lemma 13. *For an arbitrary u_0 from $C([a,b],X)$, sequence* (5.1) *can be represented alternatively as*

$$u_k = \xi + \big(\Gamma_{l,\tau} l\big)^k u_0, \quad k = 1, 2, \ldots. \tag{5.2}$$

Proof. It suffices to take into account Lemma 6 and apply Lemma 10 with $A := \Gamma_{l,\tau}$. □

It follows from Lemma 13 that the structure of operator (3.3) guarantees the simplest possible dependence on the parameter ξ in the iteration method corresponding to the parametrised equation (1.3).

Let us put

$$\rho\,(l,\tau) := r\,(\Gamma_{l,\tau} l) \tag{5.3}$$

and introduce the following

Assumption 14. $\rho\,(l,\tau) < 1$.

Relation (5.3) makes sense because, as is easy to see from (3.3), the composition $\Gamma_{l,\tau} l$ is a continuous linear mapping from $C([a,b],X)$ to $C([a,b],X)$.

Lemma 15. *Under Assumptions 1, 5, and 14, equation* (4.1) *possesses the unique solution*

$$u(t) = \xi, \qquad t \in [a,b] \tag{5.4}$$

for every ξ from X.

[2] u_0 in (5.1) is chosen in an arbitrary way

Proof. By virtue of Lemma 13, the iteration sequence (5.1) corresponding to equation (4.1) admits representation (5.2). Assumption 14 guarantees that

$$\lim_{k\to+\infty} \left(\Gamma_{l,\tau} l\right)^k u_0 = 0$$

uniformly on $[a,b]$ for every u_0 from $C([a,b],X)$. In view of Lemma 13, this means that, uniformly on $[a,b]$,

$$\lim_{k\to+\infty} u_k = 0.$$

The uniqueness of solution (5.4) is obvious from Assumption 14. Indeed, if w is another solution of equation (4.1) (with the same value of ξ), then the difference

$$z(t) := \xi - w(t), \qquad t \in [a,b]$$

satisfies the equation

$$z = \Gamma_{l,\tau} l z, \tag{5.5}$$

which, by virtue of Assumption 14, has only the trivial solution. □

6. **Unique solvability of problem** (1.1)**,** (1.2)

We are now in a position to establish a theorem on the unique solvability of the mean value problems (1.4), (1.5) and (1.1), (1.2).

Proposition 16. *Under 5, and 14, problem* (1.4)*,* (1.5) *has only the trivial solution.*

Proof. According to Lemma 11, every solution u of problem (1.4), (1.5), if there are any, satisfies equation (4.1) for some $\xi \in X$. Lemma 15 guarantees the unique solvability of equation (4.1) for an arbitrary $\xi \in X$. Furthermore, the solution u of (4.1) is represented by formula (5.4),

$$u(t) = \xi, \qquad t \in [a,b].$$

According to Lemma 11, function (5.4) is a solution of the mean value problem (1.4), (1.5) if, and only if

$$\int_a^b \left[\int_\tau^s (l\xi)(\sigma)\,\mathrm{d}\sigma\right] \mathrm{d}s = 0$$

or, which is the same,

$$\Lambda_{l,\tau}\xi = 0. \tag{6.1}$$

By virtue of Assumption 5, relation (6.1) yields immediately $\xi = 0$, whence, according to (4.1), it follows that $u = 0$. □

Proposition 17. *Under Assumptions 1, 5, and 14, problem* (1.1)*,* (1.2) *has a unique solution for arbitrary* $q \in L\big([a,b]\,,X\big)$ *and* $d \in X$*.*

Proof. One can show that Assumption 1 guarantees the complete continuity of the mapping

$$C([a,b],X) \ni u \longmapsto \int_{\tau}^{\cdot} \big(lu\big)(s)\,\mathrm{d}s.$$

It is, therefore, easy to apply the Riesz–Schauder theory to conclude that (1.1), (1.2) is uniquely solvable for all q and η provided that the homogeneous mean value problem (1.4), (1.5) has only the trivial solution (see also [1]). Application of Proposition 16 completes the proof. □

Remark 18. It is not difficult to show that Assumption 1 can be dropped in Proposition 17, for which purpose one should establish slightly more general versions of Lemmata 11 and 13.

Remark 19. Proposition 17 is similar, in a sense, to Corollary 6.2 from [2, p. 162] (for the periodic problem) and Theorem 4.3.1 from [3, p. 146] (for a more general two-point problem), which concern linear systems of ordinary differential equations.

Remark 20. It can be shown that Assumption 14 is strict in the sense that it cannot be replaced by the inequality $\rho(l,\tau) \leq 1$.

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