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SINGULAR BOUNDARY VALUE PROBLEMS WITH INTEGRABLE SINGULARITIES

R. P. AGARWAL AND DONAL O'REGAN

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Abstract. A new technique based on Schauder's fixed point theorem is given which establishes new existence results for singular two point boundary value problems where the singularity is integrable.

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Keywords: Singular problem, two-point problem, integrable singularity, Schauder theorem

1. Introduction

This paper presents a new approach to establishing the existence for the Dirichlet boundary value problem

$$\begin{aligned} y'' + f(t, y, y') = 0 \quad &\text{a. e. on } [0,1] \\ y(0) = y(1) = 0, \end{aligned} \tag{1.1}$$

where our nonlinearity f may be singular in the independent variable and may also be singular at $y = 0$. Problems of the form (1.1) have received a lot of attention in the literature; see [**1–4**] and the references therein. This paper presents a new approach based on Schauder's fixed point theorem and our results extend and complement those in the literature. Moreover it is easy to see that we could consider Sturm–Liouville boundary data in (1.1); however, since the arguments are essentially the same we will restrict our discussion to Dirichlet data. In Section 2, we also discuss a more general situation, namely

$$\begin{aligned} y'' + f(t, y, y') = 0 \quad &\text{a. e. on } [0,1] \\ y(0) = y'(1) = 0, \end{aligned}$$

where our nonlinearity f may be singular in the independent variable and may also be singular at $y = 0$ and $y' = 0$. Only a handful of papers (see [**1, 4**]) discuss the situation when f may be singular at $y = 0$ and $y' = 0$.

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2. Existence theory

Our first results concern

$$\begin{aligned} y'' + f(t, y, y') &= 0 \quad \text{a. e. on } [0,1] \\ y(0) = y(1) &= 0, \end{aligned} \tag{2.1}$$

where $f(t, y, z)$ may be singular at $y = 0$. We note that f may be singular also in the independent variable at some set $\Omega \subseteq [0,1]$ with measure zero.

Theorem 2.1. *Suppose the following conditions are satisfied:*

(2.2) $f : [0,1] \times (0,\infty) \times \mathbb{R} \to \mathbb{R}$ *with* $t \mapsto f(t, y, z)$ *measurable for every* $(y, z) \in (0,\infty) \times \mathbb{R}$ *and* $(y, z) \mapsto f(t, y, z)$ *continuous for a. e.* $t \in (0,1)$*;*

(2.3) *for any* $r > 0$, $\exists\, \psi_r : [0,1] \to \mathbb{R}$, $\psi_r > 0$ *a. e. on* $[0,1]$, $\psi_r \in L^1[0,1]$ *with* $f(t, y, z) \geq \psi_r(t)$ *a. e. on* $[0,1]$ *for every* $y \in (0, r]$ *and* $z \in [-r, r]$*;*

(2.4) *for any* $r > 0$ *with* $\int_0^1 G(t,s)\,\psi_r(s)\,ds \leq r$ *for* $t \in [0,1]$, $\exists\, h_r : [0,1] \to \mathbb{R}$, $h_r \geq 0$ *a. e. on* $[0,1]$, $h_r \in L^1[0,1]$ *with* $f(t, y, z) \leq h_r(t)$ *for a. e.* $t \in [0,1]$ *and* $y \in \left[\int_0^1 G(t,s)\,\psi_r(s)\,ds,\ r\right]$ *and* $z \in [-r, r]$*, where*

$$G(t,s) = \begin{cases} (1-t)\,s, & 0 \leq s \leq t \leq 1 \\ (1-s)\,t, & 0 \leq t \leq s \leq 1; \end{cases}$$

(2.5) $\exists\, M > 0$ *with* $M \geq \int_0^1 h_M(s)\,ds$ *and* $h_M(s) \geq \psi_M(s)$ *for a. e.* $s \in [0,1]$.

Then (2.1) *has a solution* $y \in W^{2,1}[0,1]$ *with* $y(t) > 0$ *for* $t \in (0,1)$.

Remark 2.1. In Theorem 2.1, it is possible to replace (2.4) by the following condition:

(2.6) *for any* $r > 0$ *with* $\int_0^1 G(t,s)\,\psi_r(s)\,ds \leq r$ *for* $t \in [0,1]$, *assume that* $h_r \in L^1[0,1]$ *where*

$$h_r(t) = \sup\left\{ f(t, y, z) :\ y \in \left[\int_0^1 G(t,s)\,\psi_r(s)\,ds, r\right] \ \text{ and } \ z \in [-r, r] \right\}.$$

Proof. Choose M so that (2.3), (2.4), and (2.5) hold. Let

$$Q = \Big\{ u \in C^1[0,1] : \int_0^1 G(t,s)\psi_M(s)ds \leq u(t) \leq \int_0^1 G(t,s)h_M(s)ds \\ \text{and } \ |u'(t)| \leq M \ \text{ for } \ t \in [0,1] \Big\}.$$

Clearly Q is a closed, convex subset of $C^1[0,1]$. To establish our result, we will apply Schauder's fixed point theorem to the operator $T : C^1[0,1] \to C^1[0,1]$; here, T is given by the equality

$$(Ty)(t) = \int_0^1 G(t,s)\, f(s, y(s), y'(s))\, ds.$$

We also note that (2.5) guarantees that

$$M \geq \int_0^1 G(s,s)\, h_M(s)\, ds.$$

First we show $T : Q \to Q$. To see this, let $u \in Q$, so, for $t \in [0,1]$, we have (by (2.5))

$$\int_0^1 G(t,s)\, \psi_M(s) ds \leq u(t) \leq \int_0^1 G(t,s)\, h_M(s) ds \leq \int_0^1 G(s,s)\, h_M(s) ds \leq M$$

and $|u'(t)| \leq M$. As a result, (2.3) yields

$$f(s, u(s), u'(s)) \geq \psi_M(s) \text{ a. e. on } [0,1],$$

so

$$(Tu)(t) = \int_0^1 G(t,s)\, f(s, u(s), u'(s))\, ds \geq \int_0^1 G(t,s)\, \psi_M(s) ds \text{ for } t \in [0,1].$$

Furthermore, (2.4) implies

$$f(s, u(s), u'(s)) \leq h_M(s) \text{ a. e. on } [0,1],$$

so

$$(Tu)(t) \leq \int_0^1 G(t,s)\, h_M(s) ds$$

and (by (2.5))

$$\begin{aligned} |(Tu)'(t)| &= \left| -\int_0^t s\, f(s, u(s), u'(s))\, ds + \int_t^1 (1-s)\, f(s, u(s), u'(s))\, ds \right| \\ &\leq \int_0^t s\, h_M(s)\, ds + \int_t^1 (1-s)\, h_M(s)\, ds \leq \int_0^1 h_M(s)\, ds \leq M \end{aligned}$$

for $t \in [0,1]$. Thus, $T : Q \to Q$. Next we show that T is continuous. Let $y_n \in Q$ and $y_n \to y$ in $C^1[0,1]$. Then, for $t \in [0,1]$, we have

$$\begin{aligned} |(Ty_n)(t) - (Ty)(t)| &\leq \int_0^1 G(s,s)\, |f(s, y_n(s), y_n'(s)) - f(s, y(s), y'(s))|\, ds \\ &\leq \int_0^1 |f(s, y_n(s), y_n'(s)) - f(s, y(s), y'(s))|\, ds \end{aligned}$$

and

$$\begin{aligned} |(Ty_n)'(t) - (Ty)'(t)| &\leq \int_0^t s |f(s, y_n(s), y_n'(s)) - f(s, y(s), y'(s))|\, ds \\ &\quad + \int_t^1 (1-s) |f(s, y_n(s), y_n'(s)) - f(s, y(s), y'(s))|\, ds \\ &\leq \int_0^1 |f(s, y_n(s), y_n'(s)) - f(s, y(s), y'(s))|\, ds. \end{aligned}$$

Thus, the Lebesgue dominated convergence theorem implies

$$\sup_{t\in[0,1]} |(Ty_n)(t) - (Ty)(t)| \le \int_0^1 |f(s, y_n(s), y_n'(s)) - f(s, y(s), y'(s))|\, ds \to 0$$

as $n \to \infty$ and

$$\sup_{t\in[0,1]} |(Ty_n)'(t) - (Ty)'(t)| \le \int_0^1 |f(s, y_n(s), y_n'(s)) - f(s, y(s), y'(s))|\, ds \to 0$$

as $n \to \infty$ because

$$\int_0^1 |f(s, y_n(s), y_n'(s)) - f(s, y(s), y'(s))|\, ds \le 2 \int_0^1 h_M(s)\, ds.$$

As a result, $T : Q \to Q$ is continuous. It remains to show that $T : Q \to Q$ is compact. This follows from the Arzela–Ascoli theorem and the following relations (here, $y \in Q$ and $t, t' \in [0, 1]$ with $t < t'$):

$$\sup_{t\in[0,1]} |(Ty)(t)| \le \sup_{t\in[0,1]} \int_0^1 G(s, s)\, h_M(s) \le M,$$

$$\sup_{t\in[0,1]} |(Ty)'(t)| \le \sup_{t\in[0,1]} \int_0^1 h_M(s) \le M,$$

$$|(Ty)(t) - (Ty)(t')| \le \int_0^1 |G(t, s) - G(t', s)|\, h_M(s)\, ds,$$

and

$$|(Ty)'(t) - (Ty)'(t')| \le \int_t^{t'} s\, h_M(s)\, ds + \int_t^{t'} (1 - s)\, h_M(s)\, ds = \int_t^{t'} h_M(s)\, ds.$$

Now Schauder's fixed point theorem guarantees that the mapping T has a fixed point in Q. □

Our next result is a more "applicable" version of Theorem 2.1.

Theorem 2.2. *Suppose that* (2.2) *and* (2.3) *hold and, in addition, the following conditions are satisfied:*

(2.7) $f(t, y, z) \le q(t)\,[g(y) + \tau(y)]\,\phi(z)$ *on* $[0, 1] \times (0, \infty) \times \mathbb{R}$ *with* $g > 0$ *continuous and nonincreasing on* $(0, \infty)$, $\tau \ge 0$ *continuous and nondecreasing on* $(0, \infty)$, $\phi \ge 0$ *continuous on* $(-\infty, \infty)$, *and* $q : [0, 1] \to \mathbb{R}$ *with* $q > 0$ *a. e. on* $[0, 1]$;

(2.8) *the relation*

$$\int_0^1 q(s)\, g(c_0 s(1 - s))\, ds < \infty$$

holds for any $c_0 > 0$;

(2.9) *there exists* $M > 0$ *such that*

$$M \geq \left[\sup_{w\in[-M,M]} \phi(w)\right] \int_0^1 q(s)\left[\tau(M) + g\left(\int_0^1 G(s,x)\,\psi_M(x)\,dx\right)\right] ds$$

and

$$q(t)\left[\tau(M) + g\left(\int_0^1 G(t,s)\,\psi_M(s)\,ds\right)\right] \sup_{w\in[-M,M]} \phi(w) \geq \psi_M(t)$$

for a. e. $t \in [0,1]$.

Then (2.1) *has a solution* $y \in W^{2,1}[0,1]$ *with* $y(t) > 0$ *for* $t \in (0,1)$.

Proof. The result follows from Theorem 2.1 once we show (2.4) and (2.5) hold. Notice for a. e. $t \in [0,1]$ and $y \in \left[\int_0^1 G(t,s)\,\psi_r(s)\,ds,\ r\right]$ and $z \in [-r,r]$, then (2.7) yields

$$f(t,y,z) \leq q(t)\left[\tau(r) + g\left(\int_0^1 G(t,s)\,\psi_r(s)\,ds\right)\right] \sup_{w\in[-r,r]} \phi(w).$$

If we take

$$h_r(t) = q(t)\left[\tau(r) + g\left(\int_0^1 G(t,s)\,\psi_r(s)\,ds\right)\right] \sup_{w\in[-r,r]} \phi(w),$$

then (2.4) is immediate if we show $h_r \in L^1[0,1]$. Also (2.9) guarantees (2.5). It remains to show that $h_r \in L^1[0,1]$. To see this, it is enough to establish the inclusion

$$q(t)\,g\left(\int_0^1 G(t,s)\,\psi_r(s)\,ds\right) \in L^1[0,1].$$

To show this, notice

$$\int_0^1 G(t,s)\,\psi_r(s)\,ds = t\,(1-t)\,\Theta_r(t),$$

where

$$\Theta_r(t) = \frac{1}{1-t}\int_t^1 (1-s)\,\psi_r(s)\,ds + \frac{1}{t}\int_0^t s\,\psi_r(s)\,ds.$$

Now since

$$\left|\frac{1}{t}\int_0^t s\,\psi_r(s)\,ds\right| \leq \int_0^t \psi_r(s)\,ds \to 0 \ \text{ as } \ t \to 0^+$$

and

$$\left|\frac{1}{1-t}\int_t^1 (1-s)\,\psi_r(s)\,ds\right| \leq \int_t^1 \psi_r(s)\,ds \to 0 \ \text{ as } \ t \to 1^-$$

we have that Θ_r extends to a continuous function on $[0,1]$. Thus, there exists $k_r > 0$ with $\Theta_r(t) \geq k_r > 0$ for $t \in [0,1]$. As a result,

$$\int_0^1 G(t,s)\,\psi_r(s)\,ds \geq k_r\,t\,(1-t) \ \text{ for } \ t \in [0,1],$$

so

$$h_r(t) \le q(t)\,[\tau(r) + g(k_r\,t\,(1-t))] \sup_{w\in[-M,M]} \phi(w),$$

and $h_r \in L^1[0,1]$ from (2.8). □

To show how Theorem 2.2 can be applied in practice, consider the problem

$$\begin{gathered} y'' + q(t)\,[g(y) + \tau(y)]\,\phi(y') = 0 \quad \text{a. e. on } [0,1], \\ y(0) = y(1) = 0. \end{gathered} \tag{2.10}$$

Theorem 2.3. *Assume that the following conditions are satisfied*:

(2.11) $g > 0$ *is continuous and nonincreasing on* $(0,\infty)$, $\tau \ge 0$ *is continuous and nondecreasing on* $(0,\infty)$, $\phi \ge 0$ *is continuous on* $(-\infty,\infty)$, $q : [0,1] \to \mathbb{R}$ *is measurable and* $q > 0$ *a. e. on* $[0,1]$;

(2.12) $\exists\, a_0 > 0$ *such that* $\phi(z) \ge a_0$ *for* $z \in (-\infty,\infty)$;

(2.13) $\int_0^1 q(s)\,g(c_0 s(1-s))\,ds < \infty$ *for any* $c_0 > 0$;

(2.14) *there exists* $M > 0$ *for which*

$$M \ge \left[\sup_{w\in[-M,M]} \phi(w)\right] \int_0^1 q(s)\left[\tau(M) + g\left(a_0\,g(M)\int_0^1 G(s,x)\,q(x)\,dx\right)\right] ds$$

and

$$\sup_{w\in[-M,M]} \phi(w)\left[\tau(M) + g\left(a_0\,g(M)\int_0^1 G(t,s)\,q(s)\,ds\right)\right] \ge a_0\,g(M)$$

for a. e. $t \in [0,1]$.

Then (2.10) *has a solution* $y \in W^{2,1}[0,1]$ *with* $y(t) > 0$ *for* $t \in (0,1)$.

Proof. The result follows from Theorem 2.2 once we notice that we can take $\psi_r(t) = q(t)\,g(r)\,a_0$. □

Remark 2.2. It is possible to replace (2.12) in Theorem 2.3 by the assumption

$$\text{for any } \ r > 0,\ \exists\, a_r > 0 \text{ with } \ \phi(z) \ge a_r \text{ for } \ z \in [-r,r]$$

provided a_0 in (2.14) is replaced by a_M.

Remark 2.3. If $g(y) = y^{-\alpha}$, $\alpha > 0$ and for $x \ge 0$ we have $\tau(x) = A\,x^p + B$, $A \ge 0$, $B \ge 0$, $p \ge 0$ and for $w \in \mathbb{R}$ we have $\phi(w) = C|w|^q + D$, $C \ge 0$, $D \ge 0$, $q \ge 0$ then (2.14) is satisfied if there exists a $M > 0$ with

$$M \ge (C\,M^q + D)\int_0^1 q(s)\left[A\,M^p + B + M^{\alpha^2}\left(a_0\int_0^1 G(s,x)q(x)\,dx\right)^{-\alpha}\right] ds$$

and

$$[C\,M^q + D]\left[A\,M^p + B + M^{\alpha^2}\left(a_0\int_0^1 G(t,x)q(x)\,dx\right)^{-\alpha}\right] ds \ge a_0\,M^{-\alpha}$$

for a. e. $t \in [0,1]$. Of course if $\alpha^2 + q < 1$ and $q + p < 1$, then this inequality is satisfied for M large.

As we remarked in the introduction, we could discuss Sturm Liouville data instead of Dirichlet data in (2.1). If the partial derivative of Green's function with respect to the first variable is of fixed sign, we can improve Theorem 2.1 considerably. To show what can be done, we consider the boundary value problem

$$\begin{aligned} y'' + f(t, y, y') &= 0 \quad \text{a. e. on } [0,1], \\ y(0) = y'(1) &= 0. \end{aligned} \tag{2.15}$$

It is easy to state and prove an analogue of Theorem 2.1 for (2.15). However, we will prove more since we will assume $f(t, y, z)$ may be singular at $y = 0$ and $z = 0$.

Theorem 2.4. *Suppose that the following conditions are satisfied:*

(2.16) $f : [0,1] \times (0,\infty) \times (0,\infty) \to \mathbb{R}$ *with* $t \mapsto f(t, y, z)$ *measurable for every* $(y, z) \in (0,\infty) \times (0,\infty)$ *and* $(y, z) \mapsto f(t, y, z)$ *continuous for a. e.* $t \in (0,1)$;

(2.17) *for any* $r > 0$, $\exists\, \psi_r : [0,1] \to \mathbb{R}$, $\psi_r > 0$ *a. e. on* $[0,1]$, $\psi_r \in L^1[0,1]$ *with* $f(t, y, z) \geq \psi_r(t)$ *a. e. on* $[0,1]$ *for every* $y \in (0, r]$ *and* $z \in (0, r]$;

(2.18) *for any* $r > 0$ *with* $\int_0^1 \psi_r(s)\, ds \leq r$, $\exists\, h_r : [0,1] \to \mathbb{R}$, $h_r \geq 0$ *a. e. on* $[0,1]$, $h_r \in L^1[0,1]$ *with* $f(t, y, z) \leq h_r(t)$ *for a. e.* $t \in [0,1]$, $y \in \left[\int_0^1 k(t,s)\, \psi_r(s)\, ds,\ r\right]$ *and* $z \in \left[\int_t^1 \psi_r(s)\, ds,\ r\right]$, *where*

$$k(t,s) = \begin{cases} s, & 0 \leq s \leq t \leq 1 \\ t, & 0 \leq t \leq s \leq 1. \end{cases}$$

(2.19) $\exists\, M > 0$ *with* $M \geq \int_0^1 h_M(s)\, ds$ *and* $h_M(s) \geq \psi_M(s)$ *for a. e.* $s \in [0,1]$;

Then (2.15) *has a solution* $y \in W^{2,1}[0,1]$ *with* $y(t) > 0$ *for* $t \in (0,1]$ (*and* $y'(t) > 0$ *for* $t \in [0,1)$).

Proof. Choose M so that (2.17), (2.18), and (2.19) hold. Let

$$\begin{aligned} (Ty)(t) &= \int_0^1 k(t,s)\, f(s, y(s), y'(s))\, ds \\ &= \int_0^t s\, f(s, y(s), y'(s))\, ds + t \int_t^1 f(s, y(s), y'(s))\, ds \end{aligned}$$

and

$$\begin{aligned} Q = \Big\{ u \in C^1[0,1] : \int_0^1 k(t,s)\psi_M(s)ds \leq u(t) \leq \int_0^1 k(t,s)h_M(s)ds \\ \text{and} \quad \int_t^1 \psi_M(s)\, ds \leq u'(t) \leq \int_t^1 h_M(s)\, ds \quad \text{for} \quad t \in [0,1] \Big\}. \end{aligned}$$

Notice that (2.19) yields $M \geq \int_0^1 s\, h_M(s)\, ds$. If $u \in Q$, then (2.17) (and (2.19)) implies

$$f(s, u(s), u'(s)) \geq \psi_M(s) \quad \text{a. e. on } [0,1],$$

and (2.18) implies

$$f(s,u(s),u'(s)) \le h_M(s) \quad \text{a. e. on } [0,1].$$

As a result, $Tu \in Q$ (note that $(Tu)'(t) = \int_t^1 f(s,y(s),y'(s))\,ds$), so $T : Q \to Q$. Essentially the same reasoning as in Theorem 2.1 guarantees that $T : Q \to Q$ is continuous and compact. □

Our next result is a more "applicable" version of Theorem 2.4.

Theorem 2.5. *Suppose that* (2.16) *and* (2.17) *hold and, in addition, the following conditions are satisfied:*

(2.20) *the estimate*

$$f(t,y,z) \le q(t)\,[g(y)+\tau(y)]\,[\phi(z)+\lambda(z)]$$

holds on $[0,1]\times(0,\infty)\times(0,\infty)$ *with* $g > 0$, $\phi > 0$ *continuous and nonincreasing on* $(0,\infty)$, $\tau \ge 0$, $\lambda \ge 0$ *continuous and nondecreasing on* $(0,\infty)$, $q : [0,1] \to \mathbb{R}$ *with* $q > 0$ *a. e. on* $[0,1]$;

(2.21) *the inequality*

$$\int_0^1 q(s)\,g(c_0\,s)\,\phi\left(\int_s^1 \psi_r(x)\,dx\right) ds < \infty$$

is true for any $r > 0$ *and* $c_0 > 0$;

(2.22) *there exists* $M > 0$ *such that*

$$M \ge \int_0^1 q(s)\left[\tau(M)+g\left(\int_0^1 k(s,x)\,\psi_M(x)\,dx\right)\right]\left[\lambda(M)+\phi\left(\int_s^1 \psi_M(x)\,dx\right)\right] ds$$

and

$$q(t)\left[\tau(M)+g\left(\int_0^1 k(t,s)\,\psi_M(s)\,ds\right)\right]\left[\lambda(M)+\phi\left(\int_t^1 \psi_M(s)\,ds\right)\right] \ge \psi_M(t)$$

for a. e. $t \in [0,1]$.

Then (2.15) *has a solution* $y \in W^{2,1}[0,1]$ *with* $y(t) > 0$ *for* $t \in (0,1]$ (*and, furthermore,* $y'(t) > 0$ *for* $t \in [0,1)$).

Proof. This follows from Theorem 2.4 once one notices that one can take

$$h_r(t) = q(t)\left[\tau(r)+g\left(\int_0^1 k(t,s)\,\psi_r(s)\,ds\right)\right]\left[\lambda(r)+\phi\left(\int_t^1 \psi_r(s)\,ds\right)\right]$$

for any $r > 0$. We need only to check that

$$q(t)\,g\left(\int_0^1 k(t,s)\,\psi_r(s)\,ds\right)\phi\left(\int_t^1 \psi_r(s)\,ds\right) \in L^1[0,1].$$

To see this, notice that

$$\int_0^1 k(t,s)\,\psi_r(s)\,ds = t\,\Psi_r(t),$$

where

$$\Psi_r(t) = \frac{1}{t}\int_0^t s\,\psi_r(s)\,ds + \int_t^1 \psi_r(s)\,ds.$$

It is easy to see (as in Theorem 2.2) that Ψ_r extends to a continuous function on $[0,1]$. Thus, there exists $k_r > 0$ with $\Psi_r(t) \geq k_r > 0$ for $t \in [0,1]$. As a result, we have

$$\int_0^1 k(t,s)\,\psi_r(s)\,ds \geq k_r\,t \quad \text{for} \quad t \in [0,1],$$

so

$$h_r(t) \leq q(t)\,[\tau(r) + g(k_r\,t)]\left[\lambda(r) + \phi\left(\int_t^1 \psi_r(s)\,ds\right)\right],$$

and $h_r \in L^1[0,1]$ from (2.21). □

Remark 2.4. There is also an analogue of Theorem 2.3 for the boundary value problem

$$y'' + q(t)\,[g(y) + \tau(y)]\,[\phi(y') + \lambda(y')] = 0 \quad \text{a. e. on} \quad [0,1],$$
$$y(0) = y'(1) = 0.$$

We leave the details to the reader.

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KANTOROVICH–SCHURER BIVARIATE OPERATORS

DAN BĂRBOSU

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Abstract. Let p, q be two non-negative given integers. The sequence $(\tilde{K}_{m,n,p,q})_{m,n\in\mathbb{N}}$, $\tilde{K}_{m,n,p,q} : L_1([0,1]\times[0,1]) \to C([0,1]\times[0,1])$,

$$\left(\tilde{K}_{m,n,p,q} f\right)(x,y) = (m+p+1)(n+p+1) \times \sum_{k=0}^{m+p} \sum_{j=0}^{n+q} \tilde{p}_{m,k}(x)\tilde{p}_{nj}(y) \int_{\frac{k}{m+p+1}}^{\frac{k+1}{m+p+1}} \int_{\frac{j}{n+q+1}}^{\frac{j+1}{n+q+1}} f(s,t)ds\,dt$$

of bivariate Kantorovich-Schurer operators is constructed and some approximation properties of the sequence $\left\{\tilde{K}_{m,n,p,q} f\right\}_{m,n\in\mathbb{N}}$ are studied.

Mathematics Subject Classification: 41A36, 41A63

Keywords: Linear positive operator, Bernstein operator, Kantorovich operator, Schurer operator, bivariate operator, Korovkin theorem, first order modulus of smoothness, Shisha-Mond theorem

1. Preliminaries

Starting with the well-known Bernstein operator B_m, L. V. Kantorovich [**6**] introduced and studied the operator $K_m : L_1([0,1]) \to C([0,1])$ defined for any $m \in \mathbb{N}$ and any $f \in L_1([0,1])$ by

$$(K_m f)(x) = (m+1)\sum_{k=0}^{m} p_{m,k}(x) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)dt, \tag{1.1}$$

where $p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}$ are the fundamental Bernstein polynomials. Operators (1.1) are known in mathematical literature as the Kantorovich operators.

In 1962, F. Schurer [**9**] considering a given non-negative integer p constructed and studied a generalization of classical Bernstein operator. This generalization is the operator $\tilde{B}_{m,p} : C([0,1+p]) \to C([0,1])$ defined for any $m \in \mathbb{N}$ and any $f \in C([0,1+p])$ by

$$\left(\tilde{B}_{m,p} f\right)(x) = \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1.2}$$

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where $\tilde{p}_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k}$ are the fundamental Schurer polynomials.

Starting with the operator (1.2), in [**4**], we constructed the Kantorovich-Schurer operator $\tilde{K}_{m,p} : L_1([0,1]) \to C([0,1])$, defined for any $m \in \mathbb{N}$ and any $f \in L_1([0,1])$ by

$$\left(\tilde{K}_{m,p} f\right)(x) = (m+p+1) \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) \int_{\frac{k}{m+p+1}}^{\frac{k+1}{m+p+1}} f(t)dt \tag{1.3}$$

There, we established a convergence theorem for the sequence $\{\tilde{K}_{m,p} f\}$ and we gave quantitative estimations of the approximation order in terms of first order modulus of smoothness.

Considering two given non-negative integers p and q, in [**2**], we constructed the bivariate Schurer operator $\tilde{B}_{m,n,p,q} : C([0,1+p]\times[0,1+q]) \to C([0,1]\times[0,1])$

$$\left(\tilde{B}_{m,n,p,q} f\right)(x,y) = \sum_{k=0}^{m+p} \sum_{j=0}^{n+q} \tilde{p}_{m,k}(x) \tilde{p}_{n,j}(y) f\left(\frac{k}{m}, \frac{j}{n}\right). \tag{1.4}$$

The purposes of the present paper are the following:

(i) To construct the bivariate Kantorovich-Schurer operator

$$\tilde{K}_{m,n,p,q} : L_1([0,1]\times[0,1]) \to C([0,1]\times[0,1]);$$

(ii) To establish a convergence theorem for the sequence $\{\tilde{K}_{m,n,p,q} f\}_{m,n\in\mathbb{N}}$;

(iii) To give quantitative estimations of the approximation order in terms of first order modulus of smoothness for bivariate functions.

2. The construction of bivariate Kantorovich–Schurer operators

Let $p \geq 0, q \geq 0$ be two given integers and let $\tilde{K}_{m,p} : L_1([0,1]) \to C([0,1])$, $\tilde{K}_{n,q} : L_1([0,1]) \to C([0,1])$ defined for any $m, n \in \mathbb{N}$ and any $g \in L_1([0,1])$, $h \in L_1([0,1])$, respectively, by:

$$\left(\tilde{K}_{m,p} g\right)(x) = (m+p+1) \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) \int_{\frac{k}{m+p+1}}^{\frac{k+1}{m+p+1}} g(s)ds \tag{2.1}$$

and

$$\left(\tilde{K}_{n,q} h\right) = (n+q+1) \sum_{j=0}^{n+q} \tilde{p}_{n,j}(y) \int_{\frac{j}{n+q+1}}^{\frac{j+1}{n+q+1}} h(t)dt, \tag{2.2}$$

where $\tilde{p}_{m,k}(x)$, $\tilde{p}_{n,j}(y)$ are the fundamental Schurer polynomials.

The parametric extensions (for this term, see [**3**] or [**5**]) of (2.1) and (2.2) are the operators $\tilde{K}^x_{m,p}$, $\tilde{K}^y_{n,q} : L_1([0,1]\times[0,1]) \to C([0,1]\times[0,1])$, defined for any $m, n \in \mathbb{N}$

and any $f \in L_1([0,1]\times[0,1])$ as follows:

$$\left(\tilde{K}^{x}_{m,p} f\right)(x,y) = (m+p+1)\sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) \int_{\frac{k}{m+p+1}}^{\frac{k+1}{m+p+1}} f(s,y)ds, \tag{2.3}$$

$$\left(\tilde{K}^{y}_{n,j} f\right)(x,y) = (n+q+1)\sum_{j=0}^{n+q} \tilde{p}_{n,j}(y) \int_{\frac{j}{n+q+1}}^{\frac{j+1}{n+q+1}} f(x,t)dt. \tag{2.4}$$

Lemma 2.1. *The parametric extensions of the univariate Kantorovich-Schurer operator, defined by* (2.3) *and* (2.4) *are linear and positive operators.*

Proof. The assertion follows from the definitions of $\tilde{K}^{x}_{m,p}$ and $\tilde{K}^{y}_{n,q}$. □

Lemma 2.2. *The parametric extensions of the Kantorovich-Schurer operator commute on* $L_1([0,1]\times[0,1])$. *Their product is the bivariate Kantorovich-Schurer operator* $\tilde{K}_{m,n,p,q} : L_1([0,1]\times[0,1]) \to C([0,1]\times[0,1])$ *defined for any* $m,n \in \mathbb{N}$ *and any* $f \in L_1([0,1]\times[0,1])$ *by the relation*

$$\left(\tilde{K}_{m,n,p,q} f\right)(x,y) = (m+p+1)(n+q+1) \times \sum_{k=0}^{m+p}\sum_{j=0}^{n+q} \tilde{p}_{m,k}(x)\tilde{p}_{n,j}(y) \int_{\frac{k}{m+p+1}}^{\frac{k+1}{m+p+1}} \int_{\frac{j}{n+q+1}}^{\frac{j+1}{n+q+1}} f(s,t)ds\,dt. \tag{2.5}$$

Proof. We arrive to the desired result by direct computation, taking into account definitions (2.3), (2.4) and Lemma 2.1. □

Lemma 2.3. *The bivariate Kantorovich-Schurer operator* (2.5) *is linear and positive.*

Proof. The product of linear and positive linear operators is a linear and positive operator. We apply next Lemma 2.1. □

3. Convergence properties of the sequence $\{\tilde{K}_{m,n,p,q} f\}_{m,n\in\mathbb{N}}$

In what follows, $e_{ij}(x,y) = x^i y^j$ $(i,j \in \mathbb{N}, 0 \le i+j \le 2)$ denotes the test functions. We need the following auxiliary result (the first Korovkin Theorem for approximation of bivariate continuous functions [**12**]).

Theorem 3.1. *Let* a,b,c,d *be real numbers satisfying the inequalities* $a < b$, $c < d$ *and let* $(L_{m,n})_{m,n\in\mathbb{N}}$ *be a sequence of linear and positive operators* $L_{m,n} : C([a,b]\times[c,d]) \to C([a,b]\times[c,d])$ *having the properties*

$$(L_{m,n}e_{00})(x,y) = 1 + u_{m,n}(x,y), \tag{3.1}$$

$$\left(L_{m,n}(e_{10}-x)^2\right)(x,y) = v_{m,n}(x,y), \tag{3.2}$$

$$\left(L_{m,n}(e_{01}-y)^2\right)(x,y) = w_{m,n}(x,y). \tag{3.3}$$

If

$$\lim_{m,n\to\infty} u_{m,n}(x,y) = \lim_{m,n\to\infty} v_{m,n}(x,y) = \lim_{m,n\to\infty} w_{m,n}(x,y) = 0$$

uniformly on $[a,b]\times[c,d]$*, then the sequence* $\{L_{m,n}f\}_{m,n\in\mathbb{N}}$ *converges to* f *uniformly on* $[a,b]\times[c,d]$ *as* $m,n\to\infty$.

In our earlier paper [**4**], we proved

Lemma 3.1 (Lemma 3 in [**4**])**.** *For any* $x\in[0,1+p]$*, the operator* $\tilde{K}_{m,p}$ *satisfies the relations*

$$\left(\tilde{K}_{m,p}e_0\right)(x) = 1, \tag{3.4}$$

$$\left(\tilde{K}_{m,m}e_1\right)(x) = \frac{m+p}{m+p+1}x + \frac{1}{2(m+p+1)}, \tag{3.5}$$

$$\left(\tilde{K}_{m,p}e_2\right)(x) = \frac{m+p}{(m+p+1)^2}\left\{(m+p)x^2 + x(2-x)\right\} + \frac{1}{3(m+p+1)^2}. \tag{3.6}$$

Lemma 3.2. *The parametric extension* $\tilde{K}^x_{m,p}$ *satisfies the identities* (3.4), (3.5), *and* (3.6).

Proof. We make use of the definition (2.3) of $\tilde{K}^x_{m,p}$ and of Lemma 3.1. □

Remark 3.1. The parametric extension $\tilde{K}^y_{n,q}$ satisfies identities similar to the identities (3.4), (3.5), and (3.6).

Lemma 3.3. *The bivariate Kantorovich-Schurer operator* $\tilde{K}_{m,n,p,q}$ *defined by* (2.5) *satisfies the equalities*

$$\left(\tilde{K}_{m,n,p,q}e_{00}\right)(x,y) = 1, \tag{3.7}$$

$$\left(\tilde{K}_{mm,n,p,q}e_{10}\right)(x,y) = \frac{m+p}{m+p+1}\,x + \frac{1}{2(m+p+1)}, \tag{3.8}$$

$$\left(\tilde{K}_{m,n,p,q}e_{01}\right)(x,y) = \frac{n+q}{n+q+1}\,y + \frac{1}{2(n+q+1)}, \tag{3.9}$$

$$\begin{aligned}\left(\tilde{K}_{m,n,p,q}e_{20}\right)(x,y) &= \frac{m+p}{(m+p+1)^2}\left\{(m+p)x^2 + x(2-x)\right\}\\ &\quad + \frac{1}{3(m+p+1)^2},\end{aligned} \tag{3.10}$$

$$\begin{aligned}\left(\tilde{K}_{m,n,p,q}e_{02}\right)(x,y) &= \frac{n+q}{(n+q+1)^2}\left\{(n+q)y^2 + y(2-y)\right\}\\ &\quad + \frac{1}{3(n+q+1)^2}.\end{aligned} \tag{3.11}$$

Proof. Taking into account definition (2.5) and Lemma 3.2, we arrive at the desired equalities. □

Lemma 3.4. *The bivariate Kantorovich-Schurer operator* (2.5) *satisfies the relations*

$$\left(\tilde{K}_{m,n,p,q}(e_{10}-x)^2\right)(x,y) = \frac{m+p-1}{(m+p+1)^2}\, x(1-x) + \frac{1}{3(m+p+1)^2}, \tag{3.12}$$

$$\left(\tilde{K}_{m,n,p,q}(e_{01}-y)^2\right)(x,y) = \frac{n+q-1}{(n+q+1)^2}\, y(1-y) + \frac{1}{3(n+q+1)^2}. \tag{3.13}$$

Proof. Since $\tilde{K}_{m,n,p,q}$ is linear, we have

$$\begin{aligned}\left(\tilde{K}_{m,n,p,q}(e_{10}-x)^2\right)(x,y) &= \left(\tilde{K}_{m,n,p,q}e_{20}\right)(x,y)\\ &\quad - 2x\left(\tilde{K}_{m,n,p,q}e_{10}\right)(x,y) + x^2\left(\tilde{K}_{m,n,p,q}e_{10}\right)(x,y).\end{aligned}$$

Next, applying Lemma 3.3, we get relation (3.12). Equality (3.13) is proved in a similar way. □

Theorem 3.2. *The sequence* $\{\tilde{K}_{m,n,p,q}f\}_{m,n\in\mathbb{N}}$ *converges to* f *uniformly on* $[0,1]\times[0,1]$ *for any* $f \in L_1([0,1]\times[0,1])$.

Proof. We apply Theorem 3.1 with

$$\begin{aligned}u_{m,n}(x,y) &= 0,\\ v_{m,n}(x,y) &= \frac{m+p-1}{(m+p+1)^2}\, x(1-x) + \frac{1}{3(m+p+1)^2},\\ w_{m,n}(x,y) &= \frac{n+q-1}{(n+q+1)^2}\, y(1-y) + \frac{1}{3(n+q+1)^2}.\end{aligned}$$

□

4. Estimation of the rate of convergence

Here, we focus on estimating the rate of convergence for $\{\tilde{K}_{m,n,p,q}f\}$ in terms of the first order modulus of smoothness for bivariate functions. We first recall the following.

Definition 4.1. Let $a,b,c,d \in \mathbb{R}$ be given so that $a<b, c<d$ and let $f : [a,b]\times[c,d] \to \mathbb{R}$ be a bounded function. The function $\omega_f : [0,b-a]\times[0,d-c] \to \mathbb{R}$, defined for any $(\delta_1,\delta_2) \in [0,b-a]\times[0,d-c]$

$$\begin{aligned}\omega_f(\delta_1,\delta_2) = \sup\{|f(x,y)-f(x',y')| : (x,y),\\ (x',y') \in [0,b-a]\times[0,d-c],\ |x-x'|\le\delta_1,\ |y-y'|\le\delta_2\}\end{aligned} \tag{4.1}$$

is called first order modulus of smoothness of function f.

The first order modulus of smoothness for bivariate functions has properties similar to the properties of the first order modulus of smoothness for univariate functions. Some of them are contained in

Lemma 4.1. *The first order modulus of smoothness for bivariate functions* (4.1) *has the following properties:*

(i) $\omega_f(\delta_1,\delta_2)\le\omega_f(\delta_1',\delta_2')$ *for all* (δ_1,δ_2) *and* (δ_1',δ_2') *from* $[0,b-a]\times[0,d-c]$ *such that* $\delta_1<\delta_1'$ *and* $\delta_2<\delta_2'$;

(ii) $\omega_f(\lambda_1\delta_1,\lambda_2\delta_2)\le(1+\lambda_1)(1+\lambda_2)\omega_f(\delta_1,\delta_2)$ *for all* $(\delta_1,\delta_2)\in[0,b-a]\times[0,d-c]$ *and* $\lambda_1,\lambda_2\in\mathbb{R}_+^*$.

The following version of the Shisha–Mond theorem [**12**] for estimating the rate of convergence is known.

Theorem 4.1. *Let* $(L_{m,n})_{m,n\in\mathbb{N}}$ *be a sequence of bivariate linear positive operators mapping the space* $C([a,b]\times[c,d])$ *into itself and reproducing the constant functions. Then, for any* $f\in C([a,b]\times[c,d])$ *and any* $(\delta_1,\delta_2)\in[0,b-a]\times[0,d-c]$, *the estimate*

$$\begin{aligned}|L_{m,n}f(x,y)-f(x,y)|\le\Bigg\{1+\delta_1^{-1}\sqrt{(L_{m,n}(e_{10}-x)^2)(x,y)}+\\ +\delta_2^{-1}\sqrt{(L_{m,n}(e_{01}-y)^2)(x,y)}+\\ +\delta_1^{-1},\delta_2^{-1}\sqrt{(L_{m,n}(e_{10}-x)^2)(x,y)\cdot(L_{m,n}(e_{01}-y)^2)(x,y)}\Bigg\}\cdot\omega_f(\delta_1,\delta_2)\end{aligned}\tag{4.2}$$

is true.

Now, we are ready to prove the main result of this section.

Theorem 4.2. *For any* $f\in C([0,1]\times[0,1])$ *and any* $(x,y)\in[0,1]\times[0,1]$, *the operator* $\tilde{K}_{m,n,p,q}$ *satisfies the relation*

$$\begin{aligned}\left|\tilde{K}_{m,n,p,q}f(x,y)-f(x,y)\right|\le\\ \le4\omega_f\left(\frac{\sqrt{3(m+p-1)x(1-x)+1}}{\sqrt{3}(m+p+1)},\frac{\sqrt{3(n+q-1)y(1-y)+1}}{\sqrt{3}(n+q+1)}\right)\\ \le4\omega_f\left(\frac{\sqrt{3m+3p+1}}{2\sqrt{3}(m+p+1)},\frac{\sqrt{3n+3q+1}}{2\sqrt{3}(n+q+1)}\right).\end{aligned}\tag{4.3}$$

Proof. Applying Theorem 4.1 and Lemma 3.4, we get

$$\begin{aligned}\left|\tilde{K}_{m,n,p,q}f(x,y)-f(x,y)\right|\le\\ \le\Bigg\{1+\frac{\delta_1^{-1}}{\sqrt{3}(m+p+1)}\sqrt{3(m+p+1)x(1-x)+1}+\\ +\frac{\delta_2^{-1}}{\sqrt{3}(n+q+1)}\sqrt{3(n+q+1)y(1-y)+1}+\frac{\delta_1^{-1}\delta_2^{-1}}{3(m+p+1)(n+q+1)}\times\\ \times\sqrt{(3(m+p+1)x(1-x)+1)(3(n+q+1)y(1-y)+1)}\Bigg\}\omega_f(\delta_1,\delta_2)\end{aligned}\tag{4.4}$$

for any $(\delta_1, \delta_2) \in [0, 1+p] \times [0, 1+q]$. Choosing in (4.4)

$$\delta_1 \frac{\sqrt{3(m+p+1)x(1-x)+1}}{\sqrt{3}(m+p+1)}, \qquad \delta_2 = \frac{\sqrt{3(n+q+1)y(1-y)+1}}{\sqrt{3}(n+q+1)},$$

we get the first inequality (4.3). Taking into account that $x(1-x) \leq 1/4,\ y(1-y) \leq 1/4$ for any $(x, y) \in [0, 1] \times [0, 1]$, we obtain the second inequality (4.3). □

5. The bivariate Kantorovich operators

The Kantorovich bivariate operator $K_{m,n} : L_1([0, 1] \times [0, 1]) \to C([0, 1] \times [0, 1])$, defined for any $f \in L_1([0, 1] \times [0, 1])$ and any $m, n \in \mathbb{N}$ by the formula

$$(K_{m,n} f)(x, y) = (m+1)(n+1) \sum_{k=0}^{m} \sum_{j=0}^{n} p_{mk}(x) p_{nj}(y) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} \int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} f(s, t) ds\, dt \quad (5.1)$$

is the Kantorovich-Schurer bivariate operator $\tilde{K}_{m,n,0,0}$. All the proprieties of $K_{m,n}$ can be obtained from the properties of $\tilde{K}_{m,n,p,q}$ for $p = q = 0$. We formulate them as corollaries of the corresponding properties of $\tilde{K}_{m,n,p,q}$.

Corollary 5.1. *The sequence $\{K_{m,n} f\}_{m,n \in \mathbb{N}}$ converges to f, uniformly on $[0, 1] \times [0, 1]$, for any $f \in L_1([0, 1] \times [0, 1])$.*

Proof. The assertion follows from Theorem 3.2 for $p = q = 0$. □

Corollary 5.2. *For any $f \in C([0, 1] \times [0, 1])$ and any $(x, y) \in [0, 1] \times [0, 1]$, the Kantorovich bivariate operator* (5.1) *satisfies the relation*

$$\begin{aligned} \left|(K_{m,n} f)(x, y) - f(x, y)\right| \leq \\ \leq 4\omega_f \left(\frac{\sqrt{3(m-1)x(1-x)+1}}{\sqrt{3}(m+1)}, \frac{\sqrt{3(n-1)y(1-y)+1}}{\sqrt{3}(n+1)} \right) \leq \\ \leq 4\omega_f \left(\frac{\sqrt{3m+1}}{2\sqrt{3}(m+1)}, \frac{\sqrt{3n+1}}{2\sqrt{3}(n+1)} \right). \quad (5.2) \end{aligned}$$

Proof. Application of Theorem 4.2 with $p = q = 0$. □

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A COMMON FIXED POINT THEOREM FOR NONSELF MAPPINGS

VASILE BERINDE

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Abstract. A general common fixed point theorem for nonself discontinuous mappings is given. It contains, as particular cases, many classical and recent results in fixed point theory.

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Keywords: Banach space, generalized quasi contraction, common fixed point, nonself mappings

1. Introduction

Banach's fixed point theorem is one of the most useful results in fixed point theory. In a metric space setting it can be briefly stated as follows.

Theorem B. *Let (X,d) be a complete metric space and $T : X \to X$ a strict contraction, i. e., a map satisfying*

$$d(Tx,Ty) \leq \alpha\, d(x,y), \quad \forall\ x,y \in X, \tag{1.1}$$

where $0 < \alpha < 1$ is a constant. Then T has a unique fixed point in X.

Theorem B, together with its local variants, has many applications in solving nonlinear functional equations, but has one drawback the contraction condition (1.1) forces T to be continuous on the entire X.

In 1968, Kannan [**6**] obtained a fixed point theorem for mappings T that need not be continuous.

Theorem K. *Let (X,d) be a complete metric space and $T : X \to X$ a mapping for which there exists $a \in \left(0, \frac{1}{2}\right)$ such that*

$$d(Tx,Ty) \leq a\left[d(x,Tx) + d(y,Ty)\right], \quad \text{for all} \quad x,y \in X. \tag{1.2}$$

Then T has a unique fixed point in X.

Example 1. Let X be the set of reals with the usual norm and $T : X \to X$ given by $Tx = 0$ if $x \in (-\infty, 2]$, and $Tx = -\frac{1}{2}$ if $x \in (2, \infty)$. Then T satisfies (1.2) with $a = \frac{1}{5}$ and T is not continuous.

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Following Kannan's theorem, a great many papers were devoted to obtaining fixed point theorems for various contractive conditions that do not require the continuity of T, see, for example, Rus [**11**].

One of the most general contractive conditions obtained in this way, for which the Picard iteration still converges to the unique fixed point, was given by Ćirić [**4**].

Theorem C1. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping such that*

$$d(Tx, Ty) \leq h \cdot \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \tag{1.3}$$

for all $x, y \in X$ and for some constant $0 < h < 1$.

Then T has a unique fixed point in X.

Remarks. **1**° As shown by Rhoades [**10**, Theorem 2], a contractive mapping satisfying (1.3) is still continuous *at the fixed point*.

2° The fixed point theorems for contractive definitions of the form (1.1)–(1.3) were unified by many authors, see for example Berinde [**1**], Rus [**11**]. For a recent comparison of various contractive type conditions we refer to Mészáros [**7**].

3° The set $0_T(x) = \{x, Tx, T^2x, \dots\}$ is called *the orbit* of T relative to x. It is shown in [**12**] that condition (1.3), in fact, ensures that the orbits of T are bounded.

For any $T : X \to X$ and $x, y \in X$, where X is a metric space, let us put

$$B(x, y) = d(x, y);$$

$$K(x, y) = \frac{1}{2}\left[d(x, Tx) + d(y, Ty)\right];$$

$$C(x, y) = \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

The following theorem formally unifies Banach's, Kannan's and Ćirić's fixed point theorems.

Theorem G. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping satisfying*

$$d(Tx, Ty) \leq \lambda E(x, y) \quad \textit{for all} \quad x, y \in X, \tag{1.4}$$

where λ is a constant, $0 < \lambda < 1$, and $E(x, y)$ is any of the expressions $B(x, y)$, $K(x, y)$ and $C(x, y)$.

Then T has a unique fixed point.

Remarks. **1**° Theorem G above can be extended by considering a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ ($\mathbb{R}_+$ denotes the set of nonnegative numbers) which preserves some essential properties of the function

$$\varphi(t) = \lambda t, \quad t \in \mathbb{R}_+ \qquad (0 < \lambda < 1) \tag{1.5}$$

appearing in (1.4) and by replacing condition (1.4) by a more general one:

$$d(Tx, Ty) \leq \varphi(E(x, y)) \quad \text{for all} \quad x, y \in X. \tag{1.6}$$

2° One of the first results of this kind was obtained by Browder [**3**]. The function φ involved in such fixed point theorems is usually called *comparison function* and is supposed to satisfy at least the following two conditions:

(i_φ) φ is nondecreasing, i. e., $t_1 < t_2 \Rightarrow \varphi(t_1) \leq \varphi(t_2)$;
(ii_φ) The sequence $\{\varphi^n(t)\}$ converges to zero for every $t \in \mathbb{R}_+$, where φ^n stands for the nth iterate of φ.

Example 2. It is easy to check that a comparison function φ needs to be neither linear nor continuous, by considering $\varphi_1(t) = \frac{t}{1+t}$, $t \in \mathbb{R}_+$ and $\varphi_2(t) = \frac{t}{2}$ if $0 \leq t < 1$ and $\varphi_2(t) = t - \frac{1}{3}$ if $t \geq 1$.

To prove our main result we shall need the following Lemma.

Lemma 1. *If φ satisfies* (i_φ) *and* (ii_φ) *and is such that*

$$t \leq \varphi(t) \quad \text{for a certain } t \in \mathbb{R}_+, \tag{1.7}$$

then $t = 0$.

Proof. Suppose the contrary, i. e., there exists $t > 0$ such that (1.7) is satisfied. Then, by induction, in view of (ii_φ), we get

$$t \leq \varphi^n(t), \qquad n \geq 1.$$

By virtue of (ii_φ), this implies that $t \leq \varphi^n(t) \to 0$ as $n \to \infty$, a contradiction. □

2. Contractions that are not self-mappings

All fixed point theorems stated in the previous section deal with a self-mapping of a metric space. However, in many applications of fixed point theory, either a mapping of a closed subset K of X is not a self-mapping of K or it is very difficult to verify the invariance condition $T(K) \subset K$.

It was thus an open problem for more than 20 years to extend Theorem C1 from self-mappings $T : K \to K$ satisfying (1.3) to the corresponding nonself-mappings $T : K \to X$, where $K \neq X$. Recently, Ćirić [**5**] solved this problem by considering an additional boundary condition, also known as Rothe's boundary condition, which, however, restricts his results to a Banach space setting.

Theorem C2. *Let E be a Banach space, K a nonempty closed subset of E, and ∂K the boundary of K. Let $T : K \to E$ be a nonself-mapping satisfying* (1.3) *for all $x, y \in K$. If*

$$T(\partial K) \subset K, \tag{2.1}$$

then T has a unique fixed point in K.

Very recently, Theorem C2 was extended by Rakočević [**9**] to a common fixed point theorem. Radovanovic [**8**] also considered a similar but more particular contractive condition.

The main aim of this paper is to unify the results of Ćirić and Rakočević, as well as many other related results, in the framework of a very general common fixed point theorem.

3. Main result

Let E be a normed linear space. For $x, y \in E$ we shall denote by

$$\text{seg}[x, y] = \{z \in E : z = (1-t)x + ty\,, \quad 0 \le t \le 1\}$$

the segment of extremities x and y. The proof of the next lemma is straightforward, see Rakočević [**9**].

Lemma 2. *If $u \in E$ and $z \in \text{seg}[x, y]$, then*

$$\|u - z\| \le \max\{\|u - x\|\,,\ \|u - y\|\}\,.$$

Now we can state the main result of this paper.

Theorem 1. *Let E be a Banach space, K a nonempty closed subset of E and ∂K the boundary of K. Let $S : K \to E$ and $T : E \to E$, $T : K \to K$. Suppose that $\partial K \neq \emptyset$, T is continuous, and that S and T satisfy the following conditions:*

1° *There exists a continuous comparison function φ such that, for every $x, y \in K$,*

$$d(Sx, Sy) \le \varphi(M(x, y))\,, \tag{3.1}$$

where

$$M(x, y) = \max\{d(Tx, Ty), d(Tx, Sx), d(Ty, Sy), d(Tx, Sy), d(Ty, Sx)\}; \tag{3.2}$$

2° *T and S are weakly commutative, i. e.,*

$$d(TSx, STx) \le d(Tx, Sx), \quad \text{for every} \quad x \in K, \tag{3.3}$$

and, moreover,

$$S(K) \cap K \subset T(K)\,, \tag{3.4}$$

$$S(\partial K) \subset K \tag{3.5}$$

and

$$T(\partial K) \supset \partial K\,. \tag{3.6}$$

Then T and S have a unique common fixed point, provided that T and S have bounded orbits.

Proof. Let $x_0 \in \partial K$. Then $Sx_0 \in K$ by (3.5) and by (3.4) it results that there exists $x_1 \in K$ such that $Tx_1 = Sx_0$.

Consider Sx_1. If $Sx_1 \in K$, (3.4) again implies there exists $x_2 \in K$ such that $Tx_2 = Sx_1$. If $Sx_1 \notin K$, then by (3.6) there exists $x_2 \in \partial K$ such that $Tx_2 \in \partial K \cap \text{seg}\,[Tx_1, Sx_1]$. Hence, by induction, we construct a sequence $\{x_n\}$ of points in K as follows.

If $Sx_n \in K$, then $Tx_{n+1} = Sx_n$ for some $x_{n+1} \in K$, by (3.4). If $Sx_n \notin K$, then, by (3.6), we can pick $x_{n+1} \in \partial K$ such that

$$Tx_{n+1} \in \partial K \cap \operatorname{seg}[Tx_n, Sx_n].$$

We shall prove that both $\{Tx_n\}$ and $\{Sx_n\}$ are Cauchy sequences.

Let us first prove that

$$Tx_{n+1} \neq Sx_n \Rightarrow Tx_n = Sx_{n-1}. \tag{3.7}$$

Suppose the contrary, that is, $Tx_n \neq Sx_{n-1}$. Then $x_n \in \partial K$ and (3.5) implies $Sx_n \in K$, i. e. $Tx_{n+1} = Sx_n$, a contradiction. This proves (3.7).

By setting

$$\begin{aligned} B(n,k) &= \{Tx_j, Sx_j : n \leq j \leq n+k\}, \\ b(n,k) &= \operatorname{diam}(B(n,k)), \\ B(n) &= \{Tx_j, Sx_j : n \leq j\}, \\ b(n) &= \operatorname{diam}(B(n)), \end{aligned}$$

we obtain that $b(n,k) \uparrow b(n)$ as $k \to \infty$ and $\{b(n)\}$ is a decreasing sequence with positive terms, hence $b = \lim_{n\to\infty} b(n)$ exists.

In order to prove that $\{Tx_n\}$ and $\{Sx_n\}$ are Cauchy sequences we must show that $b = 0$. We claim that

$$b(n,k) \leq \varphi(b(n-2, k+2)), \; n \geq 2, k \geq 0, \tag{3.8}$$

and consider the following three cases.

Case 1. $b(n,k) = d(Tx_i, Sx_j)$ *with* $n \leq i, j \leq n+k$.

If $Tx_i = Sx_{i-1}$, then, by (3.1), we get

$$b(n,k) = d(Sx_{i-1}, Sx_j) \leq \varphi(M(x_{i-1}, x_j)) \leq \varphi(b(n-2, k+2))$$

because φ is monotonically increasing.

If $Tx_i \neq Sx_{i-1}$, then $Tx_{i-1} = Sx_{i-2}$ and

$$Tx_i \in \operatorname{seg}[Tx_{i-1}, Sx_{i-1}] = \operatorname{seg}[Sx_{i-2}, Sx_{i-1}].$$

Thus,

$$\begin{aligned} b(n,k) = d(Tx_i, Sx_j) &\leq \max\{d(Sx_{i-2}, Sx_j), d(Sx_{i-1}, Sx_j)\} \\ &\leq \max\{\varphi(M(x_{i-2}, x_j)), \varphi(M(x_{i-1}, x_j))\} \\ &= \varphi\Big(\max\{M(x_{i-2}, x_j), M(x_{i-1}, x_j)\}\Big) \leq \varphi(b(n-2, k+2)). \end{aligned}$$

Case 2. $b(n,k) = d(Tx_i, Tx_j)$ *with* $n \leq i, j \leq n+k$.

If $Tx_j = Sx_{j-1}$, then Case 2 reduces to Case 1. If $Tx_j \neq Sx_{j-1}$, then as in Case 1 we have $Tx_{j-1} = Sx_{j-2}$ and

$$Tx_j \in \partial K \cap \operatorname{seg}[Sx_{j-2}, Sx_{j-1}].$$

Hence,

$$b(n,k) = d(Tx_i, Tx_j) \leq \max\{d(Tx_i, Sx_{j-2}), d(Tx_i, Sx_{j-1})\}$$

and so Case 2 also reduces to Case 1.

Case 3. $b(n,k) = d(Sx_i, Sx_j)$, *with* $n \leq i, j \leq n+k$.

Then

$$b(n,k) = d(Sx_i, Sx_j) \leq \varphi(M(x_i, x_j)) \leq \varphi(b(n,k)),$$

which by Lemma 1 implies $b(n,k) = 0$. Hence, due to $Sx_i = Tx_i$, $b(n,k) = d(Sx_j, Tx_i)$, which means Case 1. Therefore, (3.8) is proved.

Now, having in view the continuity of φ, we let $k \to \infty$ in (3.8), and obtain

$$b(n) \leq \varphi(b(n-2)).$$

Letting $n \to \infty$ in the previous inequality we obtain $b \leq \varphi(b)$ which, by Lemma 1, implies $b = 0$. This shows that both $\{Tx_n\}$ and $\{Sx_n\}$ are Cauchy sequences.

As $Tx_n \in K$ and K is a closed subset of the Banach space E, we conclude that

$$\lim_{n\to\infty} Tx_n = p \in K.$$

Since

$$d(Tx_n, Sx_n) \leq b(n) \to 0 \quad \text{as} \quad n \to \infty$$

we also have $\lim Sx_n = p$. As T is continuous, we obtain

$$\lim_{n\to\infty} T(Sx_n) = T(\lim_{n\to\infty} Sx_n) = Tp \in K,$$

and in view of the weak commutativity (3.3), we have

$$d(STx_n, Tp) \leq d(STx_n, TSx_n) + d(TSx_n, Tp) \leq$$

$$\leq d(Tx_n, Sx_n) + d(TSx_n, Tp) \to 0, \text{ as } n \to \infty. \tag{3.9}$$

This shows that

$$\lim_{n\to\infty} (ST)(x_n) = Tp, \tag{3.10}$$

and therefore, by (3.9) and (3.10), we have

$$M(Tx_n, p) \to d(Tp, Sp) \quad \text{as} \quad n \to \infty$$

and

$$d(Tp, Sp) \leq \varphi(d(Tp, Sp)),$$

which, again by Lemma 1, yields $d(Tp, Sp) = 0$, i. e.,

$$Tp = Sp. \tag{3.11}$$

We shall prove that Sp (and also Tp) is a common fixed point for S and T. Indeed, by (3.11) and (3.3) it results that

$$TSp = STp = SSp. \tag{3.12}$$

Now, by (3.1), (3.11), and (3.12), we have

$$d(S\,S\,p, S\,p) \le \varphi(M\,(S\,p, p)) = \varphi(d(S\,S\,p, S\,p)),$$

which yields $S\,S\,p = S\,p$. It follows from (3.12) that $S\,p$ is a fixed point of T as well. To prove the uniqueness, relation (3.1) is used. □

Remarks. **1**° For $T = 1_E$ (the identity map) and φ given by (1.5), from Theorem 1 we obtain Theorem C2 of Ćirić.

2° For $\varphi(t) = \lambda t$ with $0 < \lambda < 1$, Theorem 1 implies Theorem 2 of Rakočević [**9**]. It is known (see Lemma 4.3.1 in [**11**]) that if T is a generalized strict φ-contraction, i. e., T satisfies (1.6) with $E(x, y) \equiv C(x, y)$, and φ is a strict comparison function, then T has bounded orbits.

It is, however, an open question whether or not two mappings S and T satisfying (3.1) have bounded orbits.

3° By considering other comparison functions in Theorem 1, we obtain various related fixed point theorems as well as common fixed point theorems. Moreover, all the results in Rakočević [**9**] can be extended in a similar way. We restrict our study to Theorem 3.

4° The continuity assumption of T in Theorem 1 can be weakened to obtain a more general result similar to Theorem 3 of Rakočević [**9**].

Theorem 2. *Let E be a Banach space, K a nonempty closed subset of E, and $\partial K \neq \varnothing$ the boundary of K. Let $S : K \to E$, $T : E \to E$, and $T : K \to K$.*

Suppose that T^m is continuous for some fixed positive integer m, S and T satisfy (3.1), (3.4), (3.5), (3.6), *and, moreover, T and S commute, i. e.,*

$$T\,S\,x = S\,T\,x \quad \textit{for every} \quad x \in K\,.$$

Then S and T have a unique common fixed point in K, provided that T and S have bounded orbits.

Proof. Let $\{x_n\}$, $S\,x_n$ and $T\,x_n$ be constructed as in the proof of Theorem 1. Hence,

$$\lim_{n\to\infty} S\,x_n = \lim_{n\to\infty} T\,x_n = p \in K.$$

For every $n \ge 1$, we have

$$\begin{aligned} d(T^m S\,x_n, S\,T^{m-1}p) &= d(S\,T^m x_n, S\,T^{m-1}p) \le \\ &\le \varphi(M\,(T^m x_n, T^{m-1}p)) \\ &= \varphi\Big(\max\Big\{d(T^m T\,x_n, T^m p), d(T^m T\,x_n, T^m S\,x_n), \\ &\qquad d(T^m p, S\,T^{m-1}p), d(T^m T\,x_n, S\,T^{m-1}p), d(T^m p, T^m S\,x_n)\Big\}\Big)\,. \end{aligned}$$

Then, by the continuity of T^m,

$$d(T^m p, S\,T^{m-1}p) \le \varphi(d\,(T^m p, S\,T^{m-1}p))\,,$$

and hence $T^m p = ST^{m-1}p$, by Lemma 1. So, similarly to the method used in the proof of Theorem 1, it can be shown that $T^m p$ is a common fixed point of T and S. The uniqueness is proved by using the contraction condition (3.1). □

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ON A NONLINEAR PARABOLIC DIFFERENTIAL EQUATION

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Abstract. Our aim is to examine the nonlinear parabolic differential equation $u_{xx} - g(t,x)f(u_t,u_x) = 0$. We present three examples for the solution of the equation of some special forms. A maximum principle and some uniqueness results are given. Moreover, the approximate solution of the equation with $g(t,x) = 1$, obtained by the difference method is investigated.

Mathematics Subject Classification: 34A12, 34A45, 34K28

Keywords: Parabolic partial differential equation, maximum principle, uniqueness, difference method

1. Introduction

We consider the differential equation

$$u_{xx} - g(t,x)\ f(u_t^{(k)}, u_x) = 0, \tag{1.1}$$

where $k = 1, 2$,

$$u = u(t,x), \quad u_t^{(k)} = \frac{\partial^k u}{\partial t^k}, \quad (t,x) \in \mathbb{R} \times \mathbb{R}.$$

Throughout the paper we shall assume that the function $g(t,x) > 0$ and function f are homogeneous of the first degree, more precisely,

$$f(\lambda u, \lambda v) = \lambda\, f(u,v) \quad \text{for } \lambda > 0, \quad uv \neq 0,$$

and, moreover, the function f satisfies the condition

$$u\, f(u,v) > 0, \qquad uv \neq 0$$

and all the functions and derivatives involved here exist and are continuous in $\mathbb{R} \times \mathbb{R}$. When $k = 1$, equation (1.1) is a parabolic second order partial differential equation and if $k = 2$, it is a hyperbolic one.

For $g(t,x) = -q(x)$, equation (1.1) has solutions of the form $u(t,x) = e^t v(x)$ and $v = v(x)$ satisfies the second order differential equation

$$v'' + q(x) f(v, v') = 0. \tag{1.2}$$

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A special case of equation (1.1) is the nonlinear parabolic differential equation

$$(\Phi_p(u_x))_x - g(t,x)\,\Phi_p(u_t) = 0, \tag{1.3}$$

where $\Phi_p(w) = |w|^{p-1}w$, $p > 0$, and the function Φ_p is increasing. The function v in the solutions of the form $u(t,x) = e^t v(x)$ with $g(t,x) = -q(x)$ satisfies the relation

$$(\Phi_p(v'))' + q(x)\,\Phi_p(v) = 0. \tag{1.4}$$

We shall consider the solvability of equation (1.1) for $k = 1$ and that of equation (1.3) with the conditions

$$\begin{aligned} u(0,x) &= \gamma(x), \\ u(t,0) &= \alpha(t), \\ u(t,l) &= \beta(t), \quad l > 0 \end{aligned} \tag{1.5}$$

and $\gamma(0) = \alpha(0)$, $\gamma(l) = \beta(0)$, $\gamma \in \mathbf{C}([0,l])$, $\alpha, \beta \in \mathbf{C}([0,T])$, $T > 0$. We suppose that $u(t,x)$ has continuous derivatives in the domain $\mathscr{D} = \{(t,x) : t \in [0,T],\ x \in [0,l]\}$ and $u(t,x)$ is continuous on the boundary of $\mathscr{D}$.

First we give three examples in which the solutions of (1.3) of some special forms are presented. A maximum principle and some uniqueness results are given for the solution of (1.1) and (1.3). In the last section, the approximate solution of (1.3) with $g(t,x) = 1$ obtained by the difference method is examined.

2. Solutions of a special form

We give the solution of the parabolic partial differential equation (1.1) or (1.3) provided the solution is of a special form.

Example 1. Let us consider the solution of (1.3) of the form $u(t,x) = \exp(at + bx)$, where a and b are constants. In this case, equation (1.3) gives

$$p\,|b|^{p+1} - g(t,x)\,\Phi_p(a) = 0.$$

If $g(t,x) > 0$, then it is obvious that $a > 0$. If $g(t,x) = 1$, then

$$a = p^{\frac{1}{p}}\,|b|^{\frac{p+1}{p}} \quad \text{or} \quad b = \pm\left(\frac{a^p}{p}\right)^{\frac{1}{p+1}}$$

and the solution has the form

$$u(t,x) = \exp\left(at \pm \left(\frac{a^p}{p}\right)^{\frac{1}{p+1}} x\right)$$

or

$$u(t,x) = \exp\left(p^{\frac{1}{p}}\,|b|^{\frac{p+1}{p}}\,t + bx\right).$$

In the plane (t,x), the solution $u(t,x) = e^C$ is constant on the straight lines $at \pm \left(\frac{a^p}{p}\right)^{\frac{1}{p+1}} x = C$, $C = \text{const}$.

Example 2. Let us consider the solution of (1.3) of the form $u(t,x) = v(x)\tau(t)$ with $g(t,x) = r(x)s(t)$.

Substituting u into equation (1.3) we have

$$p\,|v'|^{p-1}v''\,\Phi_p(\tau) = r(x)s(t)\Phi_p(v)\Phi_p(\tau').$$

We suppose that $v(x) \neq 0$ and $\tau(t) \neq 0$. Separating the variables, one can get for v

$$p\,|v'|^{p-1}v'' - \mu\, r(x)\,\Phi_p(v) = 0, \qquad \mu = \text{const}, \tag{2.1}$$

and for τ

$$\frac{\tau'}{\tau} = \left|\frac{\mu}{s(t)}\right|^{\frac{1}{p}-1}\frac{\mu}{s(t)}. \tag{2.2}$$

For the solution of (2.1) we refer to [**1–3**]. The most important property of these solutions is that for any given initial condition at $t_0 \in I$,

$$x(t_0) = x_0,$$
$$x'(t_0) = x_0',$$

there exists a unique solution $x(t)$ defined for all $t \in I$. If $\mu\, r(x) < 0$ (this yields that $\mu\, s(t) < 0$), then v is oscillatory.

From this we can see that the solution of (2.2) has the form

$$\tau(t) = K\,\exp\left(\int_0^t \Phi_{\frac{1}{p}}\left(\frac{\mu}{s(w)}\right)\,dw\right), \quad K = \text{const}.$$

Let us consider the special case where $\mu > 0,\ r(x) > 0,\ s(t) > 0$, and

$$\alpha(t) = 1 + t, \qquad t \in [0,T],$$
$$v(0) = 1,$$
$$\tau(0) = 1,$$

then from this it follows that

$$s(t) = \mu\,(1+t)^p,$$
$$v(x) = \gamma(x),$$
$$\tau(t) = 1 + t,$$
$$\beta(t) = (1+t)\,\gamma(l)$$

and function γ ($x \in [0,l]$) satisfies equation (2.1). For the solution of differential equation (1.3) of the form $u(t,x) = v(x)\,\tau(t)$, the relation

$$u(t,x) = (1+t)\,\gamma(x)$$

holds.

Example 3. Let us consider the solution of (1.3) of the form $u(t,x) = v(x) + \tau(t)$ with $g(t,x) = r(x)\,s(t)$.

In this case, equation (1.3) gives

$$p\,|v'|^{p-1}v'' = r(x)s(t)\,\Phi_p(\tau).$$

Separating the variables, we obtain

$$p\,|v'|^{p-1}v'' = \kappa\, r(x), \qquad \kappa = \text{const}, \tag{2.3}$$

and

$$s(t)\,\Phi_p(\tau) = \kappa. \tag{2.4}$$

From equation (2.3), we get

$$v(x) = \Phi_p(\kappa)\int_0^x \Phi_{\frac{1}{p}}\left(\int_0^\eta r(\xi)\,d\xi\right)d\eta$$

and from (2.4),

$$\tau(t) = \Phi_p(\kappa)\int_0^t \Phi_p^{-1}(s(\tau))\,d\tau.$$

In the special case where $r(x) = s(t) = 1$ ($g(t,x) = 1$), we have

$$v(x) = \Phi_p(\kappa)\int_0^x \Phi_{\frac{1}{p}}(\eta)\,d\eta = \frac{p}{p+1}\Phi_p(\kappa)\,x^{\frac{p+1}{p}},$$
$$\tau(t) = \Phi_p(\kappa)\,t,$$

and, therefore the solution of (1.3) has the form

$$u(t,x) = \Phi_p(\kappa)\left[\frac{p}{p+1}x^{\frac{p+1}{p}} + t\right].$$

3. Results

Theorem 1. *Let us suppose that there exists a solution of* (1.1) *for* $k = 1$ *or* (1.3) *in the domain* $\mathscr{D}_0 = \{(t,x) : t \in [0,T)\,,\ x \in [0,l]\}$ *with the boundary conditions* (1.5). *Then the solution assumes its maximum on* $\partial\mathscr{D}_0$, *the boundary of* $\mathscr{D}_0$.

Proof. We suppose the opposite that solution u assumes its maximum at an inner point (t^*, x^*) of $\mathscr{D}_0$ or on the line $t = T$. By this assumption,

$$u(t^*,x^*) - \max_{(t,x)\in\partial\mathscr{D}_0} u(t,x) = \delta > 0.$$

For the auxiliary function

$$w(t,x) = u(t,x) + \frac{\delta}{2}\frac{T-t}{T},$$

we have

$$u(t,x) < w(t,x) < u(t,x) + \frac{\delta}{2}.$$

For any point $(\bar{t}, \bar{x})$ on $\mathscr{B} = \{(t, x) : t = 0,\ x = 0,\ x = l\}$, we see that

$$w(t^*, x^*) \geq u(t^*, x^*) = \max_{(t,x)\in\partial\mathscr{D}_0} u(t, x) + \delta \geq u(\bar{t}, \bar{x}) + \delta$$
$$\geq w(\bar{t}, \bar{x}) - \frac{\delta}{2}\frac{T - t}{T} + \delta \geq w(\bar{t}, \bar{x}) + \frac{\delta}{2} > w(\bar{t}, \bar{x}).$$

Therefore, $w(t, x)$ assumes its maximum at $(\tilde{t}, \tilde{x}) \in \mathscr{D}_0 \setminus \mathscr{B}$. At this point,

$$w_x = 0, \quad w_{xx} \leq 0, \quad w_t \geq 0,$$

which implies that

$$u_x = 0, \quad u_{xx} \leq 0, \quad u_t = w_t + \frac{\delta}{2T} > 0. \tag{3.1}$$

From this observation, it follows that

$$u_{xx} - g(t, x)\ f(u_t, u_x) < 0 \quad \text{or} \quad (\Phi_p(u_x))_x - g(t, x)\,\Phi_p(u_t) < 0 \ \text{ at } \ (\tilde{t}, \tilde{x}),$$

which is a contradiction.

We remark that $w_t > 0$ and also $u_t > 0$ at $\tilde{t} = T$. □

For the minimum of the solution of (1.1) or (1.3) in $\mathscr{D}_0$ we can formulate a similar statement, namely that $u(t, x)$ assumes its minimum on $\mathscr{B}$ in both cases.

Theorem 2. *There are no any two solutions u, v of* (1.3) *with $g(t, x) = 1$ such that $u = v$ on $\mathscr{B}$ and $u \neq v$, $u_x > v_x$ and $u_t > v_t$ in $\mathscr{D}_0$.*

Proof. We suppose that u and v are different solutions of the differential equation with $g(t, x) = 1$, then

$$(\Phi_p(u_x))_x - \Phi_p(u_t) = 0$$

and

$$(\Phi_p(v_x))_x - \Phi_p(v_t) = 0,$$

which gives that

$$(\Phi_p(u_x) - \Phi_p(v_x))_x - (\Phi_p(u_t) - \Phi_p(v_t)) = 0. \tag{3.2}$$

By the Lagrange Mean Value Theorem, there exists some $\xi \in (a, b)$ such that

$$\Phi_p(b) - \Phi_p(a) = (b - a)\ p\ |\xi|^{p-1}. \tag{3.3}$$

Let us introduce the notation $w = u - v$; for equation (3.2) we can write

$$(A(t, x)\ w_x)_x - B(t, x)\ w_t = 0, \tag{3.4}$$

where $A(t, x) = p\,|\xi_1|^{p-1}$ for $v_x < \xi_1 < u_x$ and $B(t, x) = p\,|\xi_2|^{p-1}$ for $v_t < \xi_2 < u_t$. Introducing the new variable ϑ instead of x, by

$$\vartheta = \int^x \frac{d\zeta}{A(t, \zeta)},$$

we get for $w(t, x) = \bar{w}(t, \vartheta)$ that $A(t, x)\ w_x = \bar{w}_\vartheta$ and equation (3.4) is transformed to

$$\bar{w}_{\vartheta\vartheta} - Q(t, \vartheta)\ \bar{w}_t = 0 \tag{3.5}$$

where $Q(t,\vartheta)=\bar{A}(t,\vartheta)\,\bar{B}(t,\vartheta)$, $A(t,x)=\bar{A}(t,\vartheta)$ and $B(t,x)=\bar{B}(t,\vartheta)$.

From the condition $u=v$ on $\mathscr{B}$, it follows that $\bar{w}(t,\vartheta)=0$ on $\mathscr{B}$, which, by Theorem 1, implies that $\bar{w}(t,\vartheta)=0$ and $u=v$ in $\mathscr{D}_0$. This contradicts the assumption on u and v. □

4. Application of the difference method

We shall use the difference method for the determination of the approximate solution of the parabolic differential equation

$$(\Phi_p(u_x))_x-\Phi_p(u_t)=0 \tag{4.1}$$

with conditions (1.5) .

Let m be a positive integer and

$$\frac{l}{m}=h,$$
$$x_i=ih, \qquad i=0,1,2,\ldots,m.$$

Obviously, $x_0=0$ and $x_m=l$.

Let us denote by $u_i(t)$ the solution of the first order system of ordinary difference-differential equations

$$\Phi_p\left(\frac{du_i}{dt}\right)=\frac{1}{h}\left[\Phi_p\left(\frac{u_{i+1}-u_i}{h}\right)-\Phi_p\left(\frac{u_i-u_{i-1}}{h}\right)\right]$$

or equivalently

$$\Phi_p\left(\frac{du_i}{dt}\right)=\frac{1}{h^{p+1}}\left[\Phi_p(u_{i+1}-u_i)-\Phi_p(u_i-u_{i-1})\right] \tag{4.2}$$

with the initial conditions

$$\begin{aligned} u_i(0)&=\gamma(x_i), \qquad i=1,2,\ldots,m-1,\\ u_0(t)&=\alpha(t),\\ u_m(t)&=\beta(t). \end{aligned} \tag{4.3}$$

Thus, system (4.2) involves $m-1$ equations with $m-1$ unknowns.

We intend to show that for an arbitrary T problem (4.2) (4.3) has a uniquely determined solution. We also show that $u(t,x_i)$ can be approximated by $u_i(t)$ with arbitrary accuracy, i. e., for every $\varepsilon>0$, there exists an $h(\varepsilon)>0$ such that $|u(t,x_i)-u_i(t)|<\varepsilon$ when $h<h(\varepsilon)$ in $0\le t\le T$, $i=1,2,\ldots,m-1$.

Lemma 1. *For $p>0$, the function $\bar{u}_i(t)=u(t,x_i)$ satisfies the system*

$$Q_i(t)=\Phi_p\left(\frac{d\bar{u}_i}{dt}\right)-\frac{1}{h^{p+1}}\left[\Phi_p(\bar{u}_{i+1}-\bar{u}_i)-\Phi_p(\bar{u}_i-\bar{u}_{i-1})\right], \quad i=1,2,\ldots,m-1$$

where $|Q_i(t)|\le Q_0(h)$, with $Q_0(h)\to 0$ as $h\to 0$.

Proof. Applying the Taylor formula for $\bar{u}_{i+1}$ and $\bar{u}_{i-1}$

$$\bar{u}_{i+1}(t) = u(t, x_i + h) = u(t, x_i) + hu_x(t, x_i) + \frac{h^2}{2}u_{xx}(t, x_i + \theta_1 h),$$
$$\bar{u}_{i-1}(t) = u(t, x_i - h) = u(t, x_i) - hu_x(t, x_i) + \frac{h^2}{2}u_{xx}(t, x_i - \theta_2 h),$$

where $0 < \theta_1, \theta_2 < 1$ we have

$$\bar{u}_{i+1}(t) - \bar{u}_i(t) = hu_x(t, x_i) + \frac{h^2}{2}u_{xx}(t, x_i + \theta_1 h), \tag{4.4}$$
$$\bar{u}_i(t) - \bar{u}_{i-1}(t) = hu_x(t, x_i) - \frac{h^2}{2}u_{xx}(t, x_i - \theta_2 h). \tag{4.5}$$

We rewrite problem (4.1) by

$$\Phi_p\left((\bar{u}_i)_t\right) = \left(\Phi_p\left(u_x(t, x_i)\right)\right)_x = p\left|u_x(t, x_i)\right|^{p-1} u_{xx}(t, x_i).$$

From this, together with (4.4) and (4.5), it follows that

$$Q_i(t) = p\left|u_x(t, x_i)\right|^{p-1} u_{xx}(t, x_i) - \frac{1}{h^{p+1}}\left[\Phi_p\left(hu_x(t, x_i) + \frac{h^2}{2}u_{xx}(t, x_i + \theta_1 h)\right) - \Phi_p\left(hu_x(t, x_i) - \frac{h^2}{2}u_{xx}(t, x_i - \theta_2 h)\right)\right].$$

By the Lagrange Mean Value Theorem (3.3), we can write

$$\Phi_p\left(hu_x(t, x_i) + \frac{h^2}{2}u_{xx}(t, x_i + \theta_1 h)\right) - \Phi_p\left(hu_x(t, x_i) - \frac{h^2}{2}u_{xx}(t, x_i - \theta_2 h)\right) = \frac{h^2}{2}\left[u_{xx}(t, x_i + \theta_1 h) + u_{xx}(t, x_i - \theta_2 h)\right] p\left|\xi\right|^{p-1},$$

where

$$\xi = hu_x(t, x_i) + \frac{h^2}{2}u_{xx}(t, x_i + \theta_3 h), \quad -\theta_2 < \theta_3 < \theta_1.$$

Then we have

$$Q_i(t) = p\left|u_x(t, x_i)\right|^{p-1} u_{xx}(t, x_i) - \frac{p}{2}\left[u_{xx}(t, x_i + \theta_1 h) + u_{xx}(t, x_i - \theta_2 h)\right]\left|\eta\right|^{p-1},$$

where

$$\eta = u_x(t, x_i) + \frac{h}{2}u_{xx}(t, x_i + \theta_3 h) = u_x(t, x_i) + O(h), \qquad h \to 0,$$

and

$$\begin{aligned} Q_i(t) = & \, p\,|u_x(t,x_i)|^{p-1}\,u_{xx}(t,x_i) \\ & - \frac{p}{2}\left[u_{xx}(t,x_i+\theta_1 h)+u_{xx}(t,x_i-\theta_2 h)\right]|u_x(t,x_i)+O(h)|^{p-1} \\ = & \, \frac{p}{2}\,|u_x(t,x_i)|^{p-1}\Big[u_{xx}(t,x_i)-u_{xx}(t,x_i+\theta_1 h) \\ & + u_{xx}(t,x_i)-u_{xx}(t,x_i-\theta_2 h)\Big]+O(h^{p-1}). \end{aligned}$$

We note that $u_x(t,x_i)$ is bounded, $u_{xx}(t,x_i)$ is uniformly continuous in $\mathscr{D}_0$ and $p>1$, then for any $\varepsilon>0$ there exists a function $h(\varepsilon)$ such that $|Q_i(t)|<Q_0(h)$ when $h<h(\varepsilon)$, $0\le t\le T$. □

Now our goal is to state a maximum principle for problem (4.2), (4.3) .

Theorem 3. *Consider a solution $u_i(t)$, $i=1,2,\dots,m-1$, of* (4.2), (4.3), *where the function γ is positive, increasing, and convex. Then the maxima of $u_i(t)$, $i=1,2,\dots,m-1$, cannot be greater than the maxima of $\alpha(t)$, $\beta(t)$ and $h(x)$; moreover, the minima of $u_i(t)$, $i=1,2,\dots,m-1$, cannot be less than the minima of $\alpha(t)$, $\beta(t)$, and $\gamma(x)$.*

Proof. Let us suppose that there exists a $u_k(t^*)$ where $t^*>0$ and $k\neq 0$, $k\neq m$ such that $\max_{i,t}u_i(t)=u_k(t^*)$. Then the following two cases are possible:

(i) At least one of the inequalities $u_{k+1}-u_k<0$ and $u_k-u_{k-1}>0$ holds, moreover $u_k'(t_0)\ge 0$ if $t^*\neq T$ and $u_k'(t^*)>0$ if $t^*=T$. In (4.2) we have different signs on the left side and on the right side, which is a contradiction. By the convexity of function γ we see that

$$\Phi_p\left(u_i'(O)\right)=\frac{1}{h^{p+1}}\left[\Phi_p(\gamma_{i+1}-\gamma_i)-\Phi_p(\gamma_i-\gamma_{i-1})\right]>0,$$

then the maxima of $u_i(t)$ cannot be taken at $t^*=0$. This means that $k=0$ or $k=m$ can be taken for any $t^*\in(0,T]$.

(ii) At least one of equalities $u_{k+1}-u_k=0$ and $u_k-u_{k-1}=0$ holds. We assume that $u_{k+1}-u_k=0$. Then stepping from index k to $k+1$, either we obtain a contradiction or we get

$$u_k=u_{k+1}=\cdots=u_{m-1}=u_m=\beta(t^*),$$

which we had to prove.

The proof concerning the minima is similar except that $t^*=0$ is also allowed as $u_i'(0)>0$. □

Now we consider the existence and uniqueness of a solution of problem (4.2), (4.3).

Theorem 4. *Let us suppose that function γ is continuous, increasing, convex and*

$$\min_{x,h} \Phi_{\frac{1}{p}} \left\{ \frac{1}{h^{p+1}} \left[\Phi_p \left(\gamma(x+h) - \gamma(x) \right) - \Phi_p \left(\gamma(x) - \gamma(x-h) \right) \right] \right\} > L,$$

where L is independent of h, moreover, $u_i(0) = \gamma_i$ with $\gamma_i = \gamma(ih)$ for $i = 1, 2, \ldots, m-1$, and $u_0(t) \equiv \alpha(t)$, $u_m(t) \equiv \beta(t)$, where $\alpha < \beta$, $\alpha' > 0$, $\beta' > 0$ for all $t \geq 0$. Then problem (4.2), (4.3) *has a uniquely determined solution for $0 \leq t \leq T$, where T is positive.*

Proof. By the conditions on γ, we get that

$$\frac{du_i}{dt} = \Phi_{\frac{1}{p}} \left\{ \frac{1}{h^{p+1}} \left[\Phi_p \left(u_{i+1} - u_i \right) - \Phi_p \left(u_i - u_{i-1} \right) \right] \right\} \tag{4.6}$$

are continuous and satisfy the Lipschitz condition for any $t < \tau$, with a small $\tau > 0$. This implies that the solution exists and is unique for $t < \tau$. The conditions on γ also gives that

$$\frac{du_i}{dt}(0) = \Phi_{\frac{1}{p}} \left\{ \frac{1}{h^{p+1}} \left[\Phi_p \left(\gamma_{i+1} - \gamma_i \right) - \Phi_p \left(\gamma_i - \gamma_{i-1} \right) \right] \right\} > L,$$

then

$$u_i'(t) > 0 \quad \text{for} \quad t < \tau.$$

Since $u_i(0) = \gamma(x_i) > 0$, $i = 1, 2, \ldots, m-1$, then either $u_i'(t)$ remains positive for $t > 0$ or there is a smallest value τ_h for which $u_k'(\tau_h) = 0$ for some $k = i$.

Taking the derivative of (4.6) we obtain

$$|u_k'|^{p-1} u_k'' = \frac{1}{h^{p+1}} \left\{ |\Delta u_{k+1}|^{p-1} \left(u_{k+1}' - u_k' \right) - |\Delta u_k|^{p-1} \left(u_k' - u_{k-1}' \right) \right\}, \tag{4.7}$$

where $\Delta u_k = u_k - u_{k-1}$. At $t = \tau_h$ we have that $\Delta u_k = \Delta u_{k+1}$. For small $\varepsilon > 0$, in interval $(\tau_h - \varepsilon, \tau_h)$ we obtain that $|\Delta u_k| > 0$, $|\Delta u_{k+1}| > 0$, $u_k' = o(1)$, $u_{k+1}' - u_k' > 0$, $u_k' - u_{k-1}' < 0$. From these it follows that in (4.7) the right side is positive while the left side is negative as $u_k'' < 0$ for $t \in (\tau_h - \varepsilon, \tau_h)$.

In the case $u_{k+1}'(\tau_h) = 0$, passing from k to $k+1$ and carrying on, we obtain

$$u_{m-1}' = u_m' = \beta'(\tau_h) = 0,$$

which is a contradiction. From the argument above, it follows that such a finite τ_h does not exist.

Consequently, we have that

$$\Delta u_1 < \Delta u_2 < \cdots < \Delta u_{m-1} < \Delta u_m.$$

□

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BOUNDS FOR THE SINGULAR VALUES OF THE SCHWARZ MATRIX

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Abstract. Upper bounds for the squares of singular values of the Schwarz matrix are obtained by using matricial (and matrix) norms and inequalities concerning the spectral radii of matrices.

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Keywords: Singular values, matricial norms, matrix norms, Schwarz matrix

1. Introduction

Recently, singular values and singular value decomposition have attracted much attention because of their applications in various areas, namely, in control theory and systems theory [**5**], in biomedical engineering [**3**], and in signal and image processing. The application of singular values is illustrated by examples in controllability and observability problems [**5**], in obtaining the fetal ECG from that of the mother [**3**], and in processing, storage and retrieval of information [**2**].

Here, we are interested in establishing bounds for the singular values of the Schwarz matrix, S.

A special choice of the elements of S (involving Hurwitz determinants) guarantees that the characteristic polynomial of S is a given polynomial $a(x)$ [**1**, p. 211]. In this case, it can be shown that S is similar to the usual companion matrix associated with $a(x)$. Thus, the Schwarz matrix is a real canonical form to which any non-degenerate matrix is similar [**1**, p. 212]. However, such form of S is not relevant for the purpose of this paper, so hereafter we consider the Schwarz matrix, S (of order m) given as

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follows:

$$S = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ -s_m & 0 & \ddots & \ddots & \vdots \\ 0 & -s_{m-1} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & -s_2 & -s_1 \end{bmatrix},$$

with $s_i \in \mathbb{R}$ $(i = 1, \ldots, m)$. The Schwarz matrix is of interest in its own right [**1**, p. 212]. It is known that this matrix satisfies the Lyapunov equation (see [**1**, p. 208] and [**9**, p. 450]). It is also proved that (depending on the normalizing factor) the continuant matrix associated with a certain polynomial recurrence involving Hurwitz polynomials, has the Schwarz form [**8**].

Such a matrix, nowadays referred to as the Schwarz matrix, was first presented in [**12**] and is used extensively in control theory [**8**]. It also appears in circuit theory under the name "Bückner matrix" [**8**].

The tools used in this paper are the matricial and matrix norms and inequalities concerning the spectral radius of a matrix and the spectral radius of a convenient matricial norm [**6, 7, 11**].

2. Results

Singular values, v_j $(j = 1, \ldots, m)$, of S, are the positive square roots of the eigenvalues of $S^T S$. Thus it is useful to make known the structure of such a matrix, which we present in the following Lemma.

Lemma 1. *The matrix $S^T S$ associated with Schwarz matrix S, of order $m \geq 5$, is of the form*

$$S^T S = \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix},$$

where P is the following symmetric matrix:

$$P = \begin{bmatrix} s_m^2 & 0 & -s_m & 0 & \cdots & 0 \\ 0 & 1 + s_{m-1}^2 & 0 & -s_{m-1} & \ddots & \vdots \\ -s_m & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & -s_{m-1} & \ddots & \ddots & 0 & -s_5 \\ \vdots & \ddots & \ddots & 0 & 1 + s_4^2 & 0 \\ 0 & \cdots & 0 & -s_5 & 0 & 1 + s_3^2 \end{bmatrix} \in Q_{m-2}(\mathbb{R}),$$

$Q = \begin{bmatrix} O \\ Q_F \end{bmatrix} \in Q_{(m-2)\times 2}(\mathbb{R})$, *with* $Q_F = \begin{bmatrix} -s_4 & 0 \\ 0 & -s_3 \end{bmatrix}$, *$O$ the null matrix of order* $(m-4)\times 2$, *and* $R = \begin{bmatrix} 1+s_2^2 & s_1 s_2 \\ s_1 s_2 & 1+s_1^2 \end{bmatrix} \in Q_2(\mathbb{R})$.

PROOF. The proof goes by induction with respect to the order of the matrix S. □

Remark 1. If the order of S is $l = 2, 3, 4$, elementary calculations show that $S_l^T S_l$ is given by the formulae

$$S_2^T S_2 = \begin{bmatrix} s_2^2 & s_1 s_2 \\ s_1 s_2 & 1 + s_1^2 \end{bmatrix}, \qquad S_3^T S_3 = \begin{bmatrix} s_3^2 & 0 & -s_3 \\ 0 & 1 + s_2^2 & s_1 s_2 \\ -s_3 & s_1 s_2 & 1 + s_1^2 \end{bmatrix},$$

and

$$S_4^T S_4 = \begin{bmatrix} s_4^2 & 0 & -s_4 & 0 \\ 0 & 1 + s_3^2 & 0 & -s_3 \\ -s_4 & 0 & 1 + s_2^2 & s_1 s_2 \\ 0 & -s_3 & s_1 s_2 & 1 + s_1^2 \end{bmatrix}.$$

In the following proposition we establish two upper bounds for the squares of singular values of the Schwarz matrix S.

Matrix norms will be needed in what follows. For $A = [a_{ij}] \in \mathscr{M}_{s,s}(\mathbb{C})$ we define the well-known matrix norms $\|A\|_\infty := \max_{i=1,2,\ldots,s} \sum_{j=1}^{s} |a_{ij}|$ and $\|A\|_1 := \max_{j=1,2,\ldots,s} \sum_{i=1}^{s} |a_{ij}|$. Since the matrices R and P are symmetric, by taking into account the form of Q, we conclude that the $\|\cdot\|_1$ and $\|\cdot\|_\infty$ norms of these matrices coincide; therefore, we just write $\|\cdot\|$.

Proposition 1. *The singular values, v_j $(j = 1, \ldots, m)$, of the Schwarz matrix S satisfy the inequalities*

$$v_j^2 \leq \frac{\|P\| + \|R\| + \sqrt{(\|P\| + \|R\|)^2 - 4\left(\|P\|\,\|R\| - \|Q\|^2\right)}}{2} \tag{1}$$

and

$$v_j^2 \leq \|Q\| + \max\{\|P\|, \|R\|\}. \tag{2}$$

PROOF. Consider the matrix $S^T S = \left[\begin{smallmatrix} P & Q \\ Q^T & R \end{smallmatrix}\right]$ from Lemma 1, partitioned into the four block matrices P, Q, Q^T, and R. Taking the matrix norm $\|\cdot\|$ of each block of $S^T S$, we obtain the matricial norm of $S^T S$, denoted by $M(S^T S)$. More precisely,

$$M(S^T S) = \begin{bmatrix} \|P\| & \|Q\| \\ \|Q^T\| & \|R\| \end{bmatrix},$$

where

$$\|P\| = \max_{k=5,\ldots,m-2} \Big\{ |s_m^2| + |s_m|,\ |1 + s_{m-1}^2| + |s_{m-1}|,\ |s_{k+2}| + |1 + s_k^2| + |s_k|, \\ |s_6| + |1 + s_4^2|,\ |s_5| + |1 + s_3^2| \Big\};$$

$\|Q\| = \max\{|s_3|, |s_4|\}$, $\|Q^T\| = \|Q\|$, and

$$\|R\| = \max\Big\{|1 + s_2^2| + |s_1 s_2|,\ |1 + s_1^2| + |s_1 s_2|\Big\} = 1 + |s_1 s_2| + \max\{s_1^2,\ s_2^2\}.$$

Bearing in mind the relations [**6, 7, 11**]

$$v_j^2 \leq \varrho(S^T S) \leq \varrho\,[M(S^T S)] \leq \|M(S^T S)\| \qquad (j = 1, \ldots, m) \tag{3}$$

concerning the spectral radius ϱ, and the matricial (and matrix) norms, we must compute $\varrho\,[M(S^T S)]$ and $\|M(S^T S)\|$. Hence, from $\det[M(S^T S) - \lambda I] = 0$, we obtain

$$\varrho[M(S^T S)] = \frac{\|P\| + \|R\| + \sqrt{(\|P\| + \|R\|)^2 - 4\left(\|P\|\,\|R\| - \|Q\|^2\right)}}{2},$$

and, applying relation (3), we get inequality (1). On the other hand, since

$$\|M(S^T S)\| = \|Q\| + \max\{\|P\|, \|R\|\},$$

applying relation (3) once again, we obtain inequality (2). □

Example 1. Consider the Schwarz matrix

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 27 & 0 & 1 \\ 0 & 0 & -16 & -1 \end{bmatrix}.$$

Since

$$S^T S = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 730 & 0 & 27 \\ 1 & 0 & 257 & 16 \\ 0 & 27 & 16 & 2 \end{bmatrix},$$

we have $P = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 730 \end{smallmatrix}\right]$, $Q = Q^T = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 27 \end{smallmatrix}\right]$, and $R = \left[\begin{smallmatrix} 257 & 16 \\ 16 & 2 \end{smallmatrix}\right]$, with the ∞-norm equal to 730, 27, and 273, respectively. In this way, applying inequality (1), for $\gamma = 1, \ldots, 4$, we obtain

$$v_\gamma^2 \leq \frac{730 + 273 + \sqrt{(730 + 273)^2 - 4\,(730 \times 273 - 27^2)}}{2}$$

or, alternatively, $v_\gamma^2 \lesssim 731.58966$, and so $v_\gamma \lesssim 27.047914$. Applying inequality (2), we obtain $v_\gamma^2 \lesssim 27 + \max\{730, 273\}$ and, therefore, $v_\gamma \lesssim 27.514$.

Note that the singular values of S calculated directly are 27.037, 16.0623, 1, and 0.0368428.

3. Final Remarks

As has been said above, there are reasons justifying our interest in establishing upper bounds for the singular values of the Schwarz matrix, namely, these topics have applications in control theory. Note that inequalities (1) and (2) presented in Proposition 1 are of the Parodi type [**10**] and of the Carmichael and Mason type [**4**], respectively.

Inequality (1) gives a bound more accurate than inequality (2), but the values involved therein are more difficult to compute.

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ON THE SOLVABILITY OF THE CAUCHY PROBLEM FOR LINEAR INTEGRAL-DIFFERENTIAL EQUATIONS

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Abstract. New general unique solvability conditions of the Cauchy problem for many-dimensional systems of linear integral differential equations are established. The conditions obtained are optimal in some sense.

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Keywords: Integral-differential equation, differential inequality, Cauchy problem

1. Introduction

The purpose of this paper is to establish new efficient conditions sufficient for the unique solvability of the Cauchy problem for some classes of many-dimensional systems of linear integral-differential equations.

The proof of the main Theorem 3.1 of Section 3.1 is based on the application of Theorem 2 from [**4**] (see Theorem 4.2 in this paper) established by using a result of [**5, 7**]. Note that similar statements were obtained in [**1, 6**] by different methods.

2. Notation

The following notation is used.

(i) $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{N} = \{1, 2, 3, \dots\}$.

(ii) $\|x\| := \max_{1 \le k \le n} |x_k|$ for $x = (x_k)_{k=1}^n \in \mathbb{R}^n$.

(iii) $C([a, b], \mathbb{R}^n)$ is the Banach space of the continuous functions $[a, b] \to \mathbb{R}^n$ equipped with the standard norm

$$C([a, b], \mathbb{R}^n) \ni u \longmapsto \max_{s \in [a,b]} \|u(s)\|.$$

(iv) $L([a, b], \mathbb{R}^n)$ is the Banach space of Lebesgue integrable functions $u : [a, b] \to \mathbb{R}^n$ with the norm

$$L([a, b], \mathbb{R}^n) \ni u \longmapsto \int_a^b \|u(s)\| \, ds.$$

(v) $\mathscr{L}(\mathbb{R}^n)$ is the algebra of square real matrices of dimension n.

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(vi) The relation $x \leq_{\vec{\sigma}} y$ for

$$\vec{\sigma} := \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \end{pmatrix}$$

and $\{x, y\} \subset \mathbb{R}^n$ means that $\sigma_k(y_k - x_k) \geq 0$ for all $k = 1, 2, \ldots, n$. Similarly, the relation $x <_{\vec{\sigma}} y$ means that $\sigma_k(y_k - x_k) > 0$ for every $k = 1, 2, \ldots, n$. The symbols "$\geq_{\vec{\sigma}}$" and "$>_{\vec{\sigma}}$" are defined by analogy.

3. Initial value problem for the integral-differential equation

We consider the system of n linear inhomogeneous integral-differential equations of the form

$$u'(t) = \sum_{j=1}^{N} \int_a^b H_j(t, s)u(\omega_j(t, s))ds + f(t), \qquad t \in [a, b], \tag{3.1}$$

with the initial condition

$$u(\tau) = c, \tag{3.2}$$

and the corresponding homogeneous Cauchy problem

$$u'(t) = \sum_{j=1}^{N} \int_a^b H_j(t, s)u(\omega_j(t, s))ds, \qquad t \in [a, b], \tag{3.3}$$

$$u(\tau) = 0, \tag{3.4}$$

where $H_j : [a, b]\times[a, b] \to \mathscr{L}(\mathbb{R}^n)$ and $f : [a, b] \to \mathbb{R}^n$ are integrable functions, and $\omega_j : [a, b]\times[a, b] \to [a, b]$ are measurable functions, $j = 1, 2, \ldots, N$. By a solution of problem (3.1), (3.2) (respectively, (3.3), (3.4)), according to the definition adopted in the modern theory of functional-differential equations [**2**], an absolutely continuous function $u : [a, b] \to \mathbb{R}^n$ is meant which possesses property (3.2) (respectively, (3.4)) at the point τ and satisfies relation (3.1) (respectively, (3.3)) almost everywhere on $[a, b]$. Various equations, e. g.,

$$u'(t) = \lambda(t) \int_a^t r(s)u(h(s))ds + f(t), \qquad t \in (-\infty, \infty),$$

$$u'(t) = \int_{\alpha(t)}^{\beta(t)} r(s)u(\phi(|t - s|))ds + f(t), \qquad t \in (-\infty, \infty),$$

can be rewritten in form (3.1).

3.1. Main theorem

Here, we state the main theorem on the unique solvability of the initial value problem (3.1), (3.2), which can be used to obtain efficient conditions for concrete equations (see, e. g., Corollary 3.6).

The conditions of the Theorem 3.1 below are formulated in terms of the functions obtained by the consecutive application of the linear operation defined by the right-hand side of equation (3.3) to a given function with suitable properties. More precisely, let us fix some absolutely continuous function $y_0 : [a, b] \to \mathbb{R}^n$ which satisfies the conditions

$$y_0(\tau) = 0 \tag{3.5}$$

and

$$y_0(t) >_{\vec{\sigma}} 0, \qquad t \in [a, b] \setminus \{\tau\}, \tag{3.6}$$

and consider the sequence of functions given by the recurrence relation

$$y_k(t) = \sum_{j=0}^{N} \int_{\tau}^{t} \left(\int_{a}^{b} H_j(\xi, s) y_{k-1}(\omega_j(\xi, s)) ds \right) d\xi, \qquad t \in [a, b], \quad k = 1, 2, \dots. \tag{3.7}$$

The following general theorem is true.

Theorem 3.1. *Let the functions $\omega_j : [a, b] \times [a, b] \to [a, b]$ be measurable and $H_j : [a, b] \times [a, b] \to \mathscr{L}(\mathbb{R}^n)$, $j = 1, 2, \dots, N$, be integrable and such that*

$$\sum_{j=1}^{N} \int_{a}^{b} H_j(t, s)\vec{\sigma}\, ds \operatorname{sign}(t - \tau) \geq_{\vec{\sigma}} 0 \quad \text{for a. e.} \quad t \in [a, b] \tag{3.8}$$

with some constant vector $\vec{\sigma} = \operatorname{col}(\sigma_1, \sigma_2, \dots, \sigma_n)$, $\{\sigma_k \mid k = 1, 2, \dots, n\} \subset \{-1, 1\}$. Assume also that there exist some real constants $\alpha \in (0, 1)$ and $\varrho \in (1, +\infty)$, integers $k \geq 0$ and $r \geq 1$, and a certain absolutely continuous function $y_0 : [a, b] \to \mathbb{R}^n$ satisfying relations (3.5) *and* (3.6) *such that the integral-differential inequality*

$$\left[y_0'(t) - \frac{\varrho^{k+1}}{1 - \alpha} \sum_{j=1}^{N} \left(\int_{a}^{b} H_j(t, s) [\varrho^r y_{k+r}(\omega_j(t, s)) - \alpha y_k(\omega_j(t, s))] ds \right) \right] \operatorname{sign}(t - \tau) \geq_{\vec{\sigma}} 0 \tag{3.9}$$

holds for almost every t from $[a, b]$.

Then the Cauchy problem (3.3), (3.4) *has only the trivial solution, the inhomogeneous Cauchy problem* (3.1), (3.2), *has a unique solution $u(\cdot)$ for arbitrary $c \in \mathbb{R}^n$ and $f \in L([a, b], \mathbb{R}^n)$, and this solution is representable as the uniformly convergent functional series*

$$u(t) = \sum_{k=0}^{\infty} f^{[k]}(t), \qquad t \in [a, b], \tag{3.10}$$

where, by definition,

$$f^{[k]}(t) := \sum_{j=1}^{N} \int_{\tau}^{t} \left(\int_{a}^{b} H_j(\xi, s) f^{[k-1]}(\omega_j(\xi, s)) ds \right) d\xi, \qquad t \in [a, b],\ k = 1, 2, \ldots,$$

and

$$f^{[0]}(t) := c + \int_{\tau}^{t} f(s) ds, \qquad t \in [a, b].$$

Moreover, if the vector-function f and vector c satisfy the additional condition

$$\int_{\tau}^{t} f(s) ds \geq_{\vec{\sigma}} -c \tag{3.11}$$

for all t from $[a, b]$, then the unique solution $u(\cdot)$ of the inhomogeneous problem (3.1), (3.2) *satisfies the relation*

$$u(t) \geq_{\vec{\sigma}} 0, \qquad t \in [a, b]. \tag{3.12}$$

We recall that, everywhere in this paper, notation (vi) of Section 2 is used. For example, condition (3.11) means that the inequality

$$\sigma_k \left[c_k + \int_{\tau}^{t} f_k(s) ds \right] \geq 0$$

is true for all t from $[a, b]$ and $k = 1, 2, \ldots, n$.

Remark 3.2. Under the condition of Theorem 3.1, the uniform convergence of the sequence of functions (3.7) to the function equal identically to zero, i. e., to the unique solution of the homogeneous Cauchy problem (3.3), (3.4), follows from the proof of [**7**, Theorem 2], which theorem implies Theorems 3.1 and 4.2 of [**4**] used in this paper.

Remark 3.3. Condition (3.9) is optimal in the sense that the assertion of Theorem 3.1, generally speaking, is not true if (3.9) is assumed with $\varrho = 1$, i. e., has the form

$$\left[(1 - \alpha) y_0'(t) - \sum_{j=1}^{N} \left(\int_{a}^{b} H_j(t, s) [y_{k+r}(\omega_j(t, s)) - \alpha y_k(\omega_j(t, s))] ds \right) \right] \operatorname{sign}(t - \tau) \geq_{\vec{\sigma}} 0. \tag{3.13}$$

This statement is justified by the example below.

Example 3.4. Consider the homogeneous integral-differential equation

$$u'(t) = \frac{2 \operatorname{sign}(t - \tau)}{(a - \tau)^2 + (b - \tau)^2} \int_{a}^{b} u(s) ds, \quad t \in [a, b], \tag{3.14}$$

where τ is a given point from (a, b). Clearly, equation (3.14) can be rewritten as (3.3) if we put

$$n = 1, \quad N = 1,$$
$$H_1(t, s) = \frac{2\,\mathrm{sign}(t-\tau)}{(a-\tau)^2 + (b-\tau)^2}, \quad (t, s) \in [a, b] \times [a, b]. \tag{3.15}$$

Let us set

$$y_0(t) = |t - \tau|, \qquad t \in [a, b]. \tag{3.16}$$

Obviously, this function is absolutely continuous and satisfies conditions (3.5) and (3.6) with $n = 1$ and $\vec{\sigma} = 1$. We construct the sequence of functions $y_1, y_2, \ldots$ defined by the formula (3.7). In this case, it has the form

$$\begin{aligned} y_k(t) &= \int_\tau^t \frac{2\,\mathrm{sign}(\xi-\tau)}{(a-\tau)^2 + (b-\tau)^2} \left(\int_a^b y_{k-1}(s)ds \right) d\xi \\ &= \frac{2|t-\tau|}{(a-\tau)^2 + (b-\tau)^2} \int_a^b y_{k-1}(s)ds, \qquad t \in [a, b]. \end{aligned} \tag{3.17}$$

It is clear that, by virtue of (3.16),

$$y_1(t) = \frac{2|t-\tau|}{(a-\tau)^2 + (b-\tau)^2} \int_a^b |s-\tau|ds = |t-\tau|, \qquad t \in [a, b].$$

Arguing by induction, we obtain that

$$y_k(t) = |t - \tau|, \qquad t \in [a, b], \tag{3.18}$$

for every $k \geq 0$ It is obvious that, for $\vec{\sigma} = 1$ and N, n, and H_1 given by formulae (3.15), the equality

$$\sum_{j=1}^N \int_a^b H_j(t, s)\vec{\sigma}\, ds\ \mathrm{sign}(t-\tau) = \frac{2}{(a-\tau)^2 + (b-\tau)^2} \tag{3.19}$$

is true and, hence, condition (3.8) is fulfilled.

Let us show that condition (3.9) with $\varrho = 1$ (i. e., inequality (3.13)) holds for $\vec{\sigma} = 1$ and N, n, and H_1 given by formulae (3.15). Indeed, in this case, according to (3.18), the expression in the left-hand side of inequality (3.13) takes the form

$$\begin{aligned} &\left[(1-\alpha)y_0'(t) - \sum_{j=1}^N \left(\int_a^b H_j(t, s)[y_{k+r}(\omega_j(t, s)) - \alpha y_k(\omega_j(t, s))]ds \right) \right] \mathrm{sign}(t-\tau) \\ &\quad = (1-\alpha) \left(\mathrm{sign}(t-\tau) - \frac{2\,\mathrm{sign}(t-\tau)}{(a-\tau)^2 + (b-\tau)^2} \int_a^b |s-\tau|ds \right) \mathrm{sign}(t-\tau) \\ &\quad = (1-\alpha)\,(1-1) = 0 \end{aligned} \tag{3.20}$$

and, hence, (3.9) is satisfied in the form of an equality. However, the function

$$u(t) = \lambda|t - \tau|, \qquad t \in [a, b],$$

where $\lambda \in \mathbb{R}$ is arbitrary, is a non-trivial solution of the homogeneous Cauchy problem (3.14), (3.4).

Thus, in Theorem 3.1, one cannot replace condition (3.9) with $\varrho > 1$ by condition (3.13) because the latter does not guarantee the unique solvability of the Cauchy problem under consideration.

3.2. Corollaries

Conditions (3.5) and (3.6) are satisfied, e. g., for the function

$$y_0(t) = |t - \tau|^q \vec{\sigma}, \qquad t \in [a, b], \tag{3.21}$$

where q is some natural number and $\vec{\sigma} = (\sigma_k)_{k=1}^n$ is an n-dimensional vector such that $\sigma_k \in \{-1, 1\}$ for all $k = 1, 2, \dots, n$. The following results then follow from Theorem 3.1.

Corollary 3.5. *Let the functions $\omega_j : [a, b] \times [a, b] \to [a, b]$ be measurable, $H_j : [a, b] \times [a, b] \to \mathscr{L}(\mathbb{R}^n)$ be integrable, and inequality* (3.8) *be satisfied for some $\vec{\sigma} \in \{-1, 1\}^n$. Moreover, assume that there exist a real number $\gamma \in (0, 1)$ and some $q \in \mathbb{N}$ such that the inequality*

$$\sum_{j_1=1}^{N} \sum_{j_2=1}^{N} \int_\tau^t \left(\int_a^b H_{j_1}(\xi, p) \left(\int_\tau^{\omega_{j_1}(\xi,p)} \left(\int_a^b H_{j_2}(\eta, s)\vec{\sigma} |\omega_{j_2}(\eta, s) - \tau|^q ds \right) d\eta \right) dp \right) d\xi$$
$$\leq_{\vec{\sigma}} \gamma \sum_{j_1=1}^{N} \int_\tau^t \left(\int_a^b H_{j_1}(\eta, s)\vec{\sigma} |\omega_{j_1}(\eta, s) - \tau|^q ds \right) d\eta \tag{3.22}$$

is true for almost every $t \in [a, b]$

Then for arbitrary $c \in \mathbb{R}^n$ and $f \in L([a, b], \mathbb{R}^n)$, the inhomogeneous Cauchy problem (3.1), (3.2) *is uniquely solvable, and homogeneous Cauchy problem* (3.3), (3.4) *has only trivial solution. Moreover, if the function f and vector c satisfy inequality* (3.11) *for all $t \in [a, b]$, then the unique solution of problem* (3.1), (3.2) *satisfies condition* (3.12)*.*

Corollary 3.6. *Assume that the functions $\omega_j : [a, b] \times [a, b] \to [a, b]$ are measurable and the integrable $H_j : [a, b] \times [a, b] \to \mathscr{L}(\mathbb{R}^n)$ satisfy* (3.8) *with some vector $\vec{\sigma} \in \{-1, 1\}^n$. Let, moreover, there exist some constants $\gamma \in (0, 1)$ and $q \in \mathbb{N}$ such that the inequality*

$$\sum_{j=1}^{N} \int_\tau^t \left[\int_a^b |\omega_j(\eta, s) - \tau|^q H_j(\eta, s) ds \right] d\eta \, \vec{\sigma} \leq_{\vec{\sigma}} \gamma |t - \tau|^q \vec{\sigma} \tag{3.23}$$

is fulfilled for a. e. $t \in [a, b]$.

Then the conclusion of Corollary 3.5 is true for the inhomogeneous problem (3.1), (3.2) *and homogeneous problem* (3.3), (3.4)*.*

3.3. The case of a two-dimensional system

Let us obtain conditions sufficient for the unique solvability of the initial value problem

$$u_1'(t) = \int_a^b h_{11}(t,s)u_1(\omega_{11}(t,s))ds + \int_a^b h_{12}(t,s)u_2(\omega_{12}(t,s))ds + f_1(t), \tag{3.24}$$

$$u_2'(t) = \int_a^b h_{21}(t,s)u_1(\omega_{21}(t,s))ds + \int_a^b h_{22}(t,s)u_2(\omega_{22}(t,s))ds + f_2(t), \tag{3.25}$$

$$u_1(\tau) = c_1, \tag{3.26}$$

$$u_2(\tau) = c_2, \tag{3.27}$$

where $t \in [a,b]$, the functions f_i and h_{ij}, $i,j = 1,2$, are integrable, ω_{ij}, $i,j = 1,2$, are measurable, and c_1 and c_2 are arbitrary real constants.

The following assertion is true.

Corollary 3.7. *Assume that the functions $\omega_{ij} : [a,b] \times [a,b] \to [a,b]$ are measurable, $h_{ij} : [a,b] \times [a,b] \to \mathbb{R}$ are integrable for all $i,j = 1,2$, and, moreover, the following conditions are satisfied with certain $\{\sigma_1,\sigma_2\} \subset \{-1,1\}$:*

$$\left(\int_a^b h_{11}(t,s)ds + \sigma_1\sigma_2 \int_a^b h_{12}(t,s)ds\right)\operatorname{sign}(t-\tau) \geq 0 \quad \text{for a. e. } t \in [a,b]$$

and

$$\left(\sigma_1\sigma_2 \int_a^b h_{21}(t,s)ds + \int_a^b h_{22}(t,s)ds\right)\operatorname{sign}(t-\tau) \geq 0 \quad \text{for a. e. } t \in [a,b].$$

Let there exist some constants $\gamma \in (0,1)$ and $q \in \mathbb{N}$ such that the inequalities

$$\int_\tau^t \left[\int_a^b |\omega_1(\eta,s) - \tau|^q h_{11}(\eta,s)ds\right]d\eta + \sigma_1\sigma_2 \int_\tau^t \left[\int_a^b |\omega_2(\eta,s) - \tau|^q h_{12}(\eta,s)ds\right]d\eta \leq \gamma|t-\tau|^q \tag{3.28}$$

and

$$\sigma_1\sigma_2 \int_\tau^t \left[\int_a^b |\omega_1(\eta,s) - \tau|^q h_{21}(\eta,s)ds\right]d\eta + \int_\tau^t \left[\int_a^b |\omega_2(\eta,s) - \tau|^q h_{22}(\eta,s)ds\right]d\eta \leq \gamma|t-\tau|^q \tag{3.29}$$

hold for almost every $t \in [a,b]$.

Then the Cauchy problem (3.1), (3.2), (3.3), (3.4) *has a unique solution for arbitrary* $\{f_1, f_2\} \subset L([a,b],\mathbb{R})$ *and real* c_1, c_2. *Furthermore, if the inequalities*

$$\sigma_1\left(\int_\tau^t f_1(s)ds + c_1\right) \geq 0,$$
$$\sigma_2\left(\int_\tau^t f_2(s)ds + c_2\right) \geq 0$$

are true for all $t \in [a,b]$, *then the unique solution* (u_1, u_2) *of problem* (3.1), (3.2), (3.3), (3.4) *satisfies the inequalities*

$$\sigma_1 u_1(t) \geq 0,$$
$$\sigma_2 u_2(t) \geq 0$$

for all $t \in [a,b]$.

4. Proofs of the statements of Sections 3.1 and 3.2

We recall the following definition [**7**].

Definition 4.1. An operator $l : C([a,b],\mathbb{R}^n) \to L([a,b],\mathbb{R}^n)$ is said to be $(\vec{\sigma},\tau)$-*positive* with some $\vec{\sigma} \in \{-1,1\}^n$ and $\tau \in [a,b]$ if relation (3.12) implies that

$$(lu)(t)\operatorname{sign}(t-\tau) \geq_{\vec{\sigma}} 0 \quad \text{for a. e.} \quad t \in [a,b].$$

In other words, this means that the inequalities

$$(l_k u)(t)\operatorname{sign}(t-\tau) \geq 0, \quad k = 1,2,\dots,n,$$

are true for a. e. $t \in [a,b]$ if

$$\sigma_k u_k(t) \geq 0, \quad k = 1,2,\dots,n,$$

for all $t \in [a,b]$. Here, $l_k : C([a,b],\mathbb{R}^n) \to L([a,b],\mathbb{R})$, $k = 1,2,\dots,n$, are the components of the operator $l : C([a,b],\mathbb{R}^n) \to L([a,b],\mathbb{R}^n)$.

We need the following theorem on the unique solvability of the Cauchy problem established in [**3, 4**].

Theorem 4.2. *Let us suppose that the linear operator* l *in the equation*

$$u'(t) = (lu)(t) + f(t), \qquad t \in [a,b], \tag{4.1}$$

is $(\vec{\sigma},\tau)$*-positive for some* $\vec{\sigma} \in \{-1,1\}^n$. *Assume also that there exist real constants* $\alpha \in (0,1)$ *and* $\varrho \in (1,+\infty)$, *integers* $k \geq 0$ *and* $r \geq 1$, *and an absolutely continuous function* $y_0 : [a,b] \to \mathbb{R}^n$ *with properties* (3.5) *and* (3.6) *such that the integral-differential inequality*

$$\left[y_0'(t) - \frac{\varrho^{k+1}}{1-\alpha} l\left(\varrho^r y_{k+r} - \alpha y_k\right)(t)\right]\operatorname{sign}(t-\tau) \geq_{\vec{\sigma}} 0 \tag{4.2}$$

is satisfied for almost every t *from* $[a,b]$.

Then the homogeneous Cauchy problem

$$u'(t) = (lu)(t), \qquad t \in [a,b], \tag{4.3}$$
$$u(\tau) = 0$$

has only the trivial solution, and the corresponding inhomogeneous problem (4.1), (3.2) *is uniquely solvable for arbitrary* $c \in \mathbb{R}^n$ *and* $f \in L([a,b],\mathbb{R}^n)$. *The unique solution* $u(\cdot)$ *of problem* (4.1), (3.2) *is, moreover, representable as the uniformly convergent functional series* (3.10).

If, in addition, c *and* f *satisfy condition* (4.1), *then the above-mentioned solution* $u(\cdot)$ *possesses property* (3.12).

4.1. Proof of Theorem 3.1

The assertion of Theorem 3.1 follows from Theorem 2 of [**4**], which is formulated here as Theorem 4.2, because assumption (3.8) guarantees that the linear operator

$$C([a,b],\mathbb{R}^n) \ni u \longmapsto lu := \sum_{j=1}^{N} \int_a^b H_j(\cdot,s)u(\omega_j(\cdot,s))ds \tag{4.4}$$

is $(\vec{\sigma},\tau)$-positive in the sense of Definition 4.1. Indeed, the following lemma is true.

Lemma 4.3. *For arbitrary measurable functions* $\omega_j : [a,b]\times[a,b] \to [a,b]$ *and integrable functions* $H_j : [a,b]\times[a,b] \to \mathscr{L}(\mathbb{R}^n)$, $j = 1,2,\dots,N$, *satisfying condition* (3.8) *with some column* $\vec{\sigma} \in \{-1,1\}^n$, *the linear operator* (4.4) *is* $(\vec{\sigma},\tau)$*-positive.*

Proof. Let us assume that a function $u \in C([a,b],\mathbb{R}^n)$ satisfies condition (3.12). Consider the expression

$$\sigma_k \sum_{j=1}^{N} \int_a^b H_j(t,s)u(\omega_j(t,s))ds\,\operatorname{sign}(t-\tau)$$
$$= \sigma_k \sum_{j=1}^{N}\sum_{\nu=1}^{n} \int_a^b \sigma_\nu h_{k\nu}^j(t,s)\operatorname{sign}(t-\tau)\sigma_\nu u_\nu(\omega_j(t,s))ds, \quad t \in [a,b], \tag{4.5}$$

where $h_{k\nu}^j$, $k,\nu = 1,2,\dots,n$, are the corresponding coordinates of the matrix H_j, $j = 1,2,\dots,N$, i. e.,

$$H_j(t,s) = \begin{pmatrix} h_{11}^j(t,s) & \dots & h_{1n}^j(t,s) \\ \dots & \dots & \dots \\ h_{n1}^j(t,s) & \dots & h_{nn}^j(t,s) \end{pmatrix}, \qquad (t,s) \in [a,b]^2.$$

Every kth component of the vector

$$\sum_{j=1}^{N} \int_a^b H_j(t,s)\vec{\sigma}\,ds\ \operatorname{sign}(t-\tau)$$

has the form

$$\sum_{j=1}^{N}\sum_{\nu=1}^{n}\int_a^b \sigma_\nu h_{k\nu}^j(t,s)ds\,\mathrm{sign}(t-\tau)$$

and, therefore, by (3.8),

$$\sigma_k\sum_{j=1}^{N}\sum_{\nu=1}^{n}\int_a^b \sigma_\nu h_{k\nu}^j(t,s)ds\,\mathrm{sign}(t-\tau)\geq 0$$

for all k. Thus, in view of (3.12) and (4.5),

$$\mathrm{diag}(\sigma_1,\sigma_2,\dots,\sigma_n)\sum_{j=1}^{N}\int_a^b H_j(t,s)u(\omega_j(t,s))ds\ \mathrm{sign}(t-\tau)\geq 0$$

componentwise, i. e.,

$$\sum_{j=1}^{N}\int_a^b H_j(t,s)u(\omega_j(t,s))ds\ \mathrm{sign}(t-\tau)\geq_{\vec{\sigma}} 0$$

for a. e. $t\in[a,b]$.

Since u is an arbitrary function satisfying condition (3.12), we have shown that operator (4.4) is $(\vec{\sigma},\tau)$-positive in the sense of Definition 4.1. □

4.2. Proof of Corollary 3.5

To prove Corollary 3.5, it is sufficient to apply Corollary 5 from [**4**] putting $k=1$ and $r=1$ and defining the function y_0 by formula (3.21).

4.3. Proof of Corollary 3.6

Corollary 3.6 follows from Corollary 6 of [**4**]. Inequality (3.23) is fulfilled for operator (4.4) and the function y_0 given by formula (3.21). Indeed, it is easy to verify that the derivative y_0' of function (3.21) is given by the formula

$$y_0'(t)=q|t-\tau|^{q-1}\,\mathrm{sign}\,(t-\tau)\,,\quad t\in[a,b].$$

Therefore,

$$y_0'(t)\,\mathrm{sign}(t-\tau)\geq_{\vec{\sigma}} 0,\quad t\in[a,b],$$

because, obviously,

$$q|t-\tau|^{q-1}\vec{\sigma}\geq_{\vec{\sigma}} 0,\quad t\in[a,b].$$

4.4. Proof of Corollary 3.7

Let us put $n = 2$ and $N = 4$ in equation (3.1) and define the matrix-valued functions H_j, $1 \le j \le 4$, by the formulae

$$H_1(t,s) = \begin{pmatrix} h_{11}(t,s) & 0 \\ 0 & 0 \end{pmatrix}, \qquad H_2(t,s) = \begin{pmatrix} 0 & h_{12}(t,s) \\ 0 & 0 \end{pmatrix},$$

$$H_3(t,s) = \begin{pmatrix} 0 & 0 \\ h_{21}(t,s) & 0 \end{pmatrix}, \qquad H_4(t,s) = \begin{pmatrix} 0 & 0 \\ 0 & h_{22}(t,s) \end{pmatrix}$$

for a. e. t and s from $[a,b]$. Let $\omega_1 = \omega_{11}$, $\omega_2 = \omega_{12}$, $\omega_3 = \omega_{21}$, and $\omega_4 = \omega_{22}$. Then, as is easy to see, the inhomogeneous Cauchy problem (3.24), (3.25), (3.26), (3.27) takes form (3.1), (3.2). Applying Corollary 3.6, we obtain the required assertion.

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RECEDING HORIZON CONTROL METHOD APPLIED TO ANTIVIRAL TREATMENT OF AIDS

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Abstract. This paper investigates the stabilization of a desired equilibrium point of a model describing the interaction of the immune system (CD4 cells and cytotoxic T lymphocytes CTL) with human immunodeficiecy virus HIV with sampled-data controllers. The control variable is the drug dose. Two cases of anti-HIV drugs are considered: **1**° reverse transcriptase inhibitors (RTI), and **2**° protease inhibitors (PI). The controller is constructed by a variant of the receding horizon control (RHC) method. We use the results obtained by [**3**] giving sufficient conditions which guarantee that the receding horizon controller that stabilizes the approximate discrete-time model also stabilizes the exact discrete-time model for sufficiently small discretization parameters. Results of simulations are discussed.

Mathematics Subject Classification: 93D15, 93C57, 92D30

Keywords: Receding horizon control, AIDS, feedback stabilization, sampled data systems

1. Introduction

Mathematical models which describe the interaction of the immune system with human immunodeficiency virus HIV have recently become the object of scientific research (see, e. g., [**1, 2, 4, 15, 16**]). Anti-retroviral drug therapy has successfully been used to significantly suppress viral replication and to delay disease progression in many patients. Currently, these drugs act by two mechanisms: reverse transcriptase inhibitors (RTI) interfere with the process of reverse transcriptase and prevent the virus from infecting a cell; protease inhibitors (PI) prevent the assembly of new infectious virus by an infected cell. Optimal control theory is used to optimize the administration of drugs by considering the drug as a controller $u(t)$. One way to design an open-loop optimal controller $u^*(t)$ in connection with the HIV/AIDS model is to use Pontryagin's Maximum Principle (see, e. g., [**4, 5, 11**]). However, some drawbacks may arise during the application of this method: **1**° the optimization is performed over a finite time horizon, and no care is taken of the evolution of the

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process behind this time horizon; **2**° the optimal controller is obtained as continuous-time controller in spite of the patient taking the drug doses in a discrete manner; **3**° the optimal controller is given in open-loop form and it does not deal with the changes that may happen in the system during the treatment. Therefore, it seems to be better to design a *feedback controller* based on the *discrete-time* model.

To overcome the first mentioned problem, the application of the receding horizon method seems to be obvious. Nonlinear receding horizon control has received much interest in the academic community in the past years [**7,9,10,12**] due to the capacity of handling constraints and obtaining a stabilizing state feedback controller. Receding horizon control RHC is a form of the control in which the current control action is obtained by solving on-line, at each sampling instant, a finite horizon open-loop optimal control problem, using the current state of the plant as the initial state; the optimization yields an optimal control sequence and the first control in this sequence is applied to the plant.

As far as the second problem is concerned, one has to investigate sampled data systems. The stabilization of sampled data nonlinear systems is studied by several papers (see, e. g., [**6, 13, 14**]). For nonlinear systems it is unusual to assume the knowledge of the exact discrete-time model, hence we actually use the approximate discrete-time model most of the time. If we use the approximate discrete-time model, then the sampling period T and/or integration parameter h should be sufficiently small in order to have a small mismatch between the exact and approximate models.

Stabilization of sampled data nonlinear systems by receding horizon control via approximate discrete-time models is studied in [**3, 8**]. In these papers sufficient conditions are established which guarantee that the receding horizon controller that renders the origin to be asymptotically stable for the approximate model also stabilizes the exact discrete-time model for sufficiently small discretization parameters for the cases of fixed sampling period and varying integration parameter [**3**] and when these two parameters coincide [**8**]. The aim of this paper is to investigate the stabilization of the HIV/AIDS model by receding horizon controller in both (RTI) and (PI) cases using the results obtained in [**3, 8**].

2. Receding horizon control for sampled data systems

In this section, we shall give a summary of the idea of the receding horizon control method and the main results of [**3**].

Consider a nonlinear control system described by

$$\dot{x}(t) = f\left(x(t), u(t)\right), \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^m$, $f : \mathbb{R}^n \times U \to \mathbb{R}^n$, with $f(0,0) = 0$, U is closed, and $0 \in U$. The system is to be controlled digitally using piecewise constant control functions $u(t) = u(kT) =: u_k$, if $t \in [kT, (k+1)T)$, $k \in \mathbb{N}$, where $T > 0$ is the sampling period. The function f is assumed to have enough regularity to guarantee

the existence and uniqueness of solutions of (1) with $u(t) \equiv \bar{u}$, $x(0) = x_0$, defined on $[0, T]$. Let $t \in [0, T] \mapsto \phi(t, x_0, \bar{u})$ denote this solution. Then the exact discrete-time model of the system can be defined as

$$x_{k+1} = F_T^e(x_k, u_k), \tag{2}$$

where $F_T^e(x, u) := \phi(T, x, u)$.

We emphasize that F_T^e in (2) is not known in most cases and, therefore, the controller design can be carried out by means of an approximate discrete-time model

$$x_{k+1} = F_{T,h}^a(x_k, u_k). \tag{3}$$

The results of [**3**] and [**8**] cover two cases: (i) $T \neq h$, and, in this case T is fixed and the family of approximate discrete-time models is generated by a numerical integration method with adjustable integration step h where, $\sum_{i=0}^{m-1} h_i = T$ and $h_i \leq h$ [**3**]; (ii) $T = h$, and T can be adjusted [**8**].

It is unreasonable to consider the case when $T = h$ to obtain the approximate discrete-time model for the HIV/AIDS model since, the sampling period T cannot be adjusted arbitrarily. Therefore, we shall study the case when $T \neq h$ using the results obtained by [**3**].

We assume that both $F_{T,h}^a$ and F_T^e are continuous in x and u and satisfy a local Lipschitz condition.

The problem is to define a state-feedback controller at least in a neighborhood $\bar{\mathscr{X}} \subset \mathbb{R}^n$ of the origin

$$v_{T,h}^a : \bar{\mathscr{X}} \longrightarrow U$$

using the approximate model (3) which practically asymptotically stabilizes the origin for the exact model (2).

Theorem 1 ([**6,13**])**.** *Suppose that, in a given compact neighborhood of the origin, the following conditions are satisfied:*

(i) *the origin is an asymptotically stable equilibrium of the family of the closed-loop systems* $(F_{T,h}^a, v_{T,h}^a)$ *with a uniformly continuous Lyapunov function* $V_{T,h}^a(\cdot)$*;*

(ii) *the family of controllers* $v_{T,h}^a$ *is uniformly bounded;*

(iii) *the family* $F_{T,h}^a$ *is consistent with* F_T^e.

Then the family of the closed-loop systems $(F_T^e, v_{T,h}^a)$ *is practically asymptotically stable near the origin.*

Thus, a state-feedback controller $v_{T,h}^a$ has to be defined using the approximate model (3) for which all conditions of Theorem 1 are satisfied. In order to define such a state feedback, let equation (3) be subject to the cost function

$$J_{T,h}^a(N, x, \mathbf{u}) = \sum_{k=0}^{N-1} T l_h(\phi_{T,h}^a(k, x, \mathbf{u}), u(k)) + g(\phi_{T,h}^a(N, x, \mathbf{u})), \tag{4}$$

under the constraints

$$u(k) \in U,$$
$$\phi^a_{T,h}(N; x, \mathbf{u}) \in \mathscr{G}_\eta = \{x : g(x) \le \eta\}, \tag{5}$$

where l_h and g are the running and terminal costs, $\mathbf{u}$= $\{u_0,u_1,...,u_{N-1}\}$ is the control sequence and $\phi^a_{T,h}(k, x_0, \mathbf{u})$ is the trajectory of the approximate model.

Consider the optimization problem

$$P^a_{T,h}(N, x) : \ \min\left\{J^a_{T,h}(N, x, \mathbf{u}),\ u(k) \in U,\ \phi^a_{T,h}(N; x, \mathbf{u}) \in \mathscr{G}_\eta\right\}. \tag{6}$$

If this optimization problem has a solution denoted by $\mathbf{u}^*$= $\{u^*_0,u^*_1,\dots,u^*_{N-1}\}$, then the first element of $\mathbf{u}^*$, i. e., u^*_0 is applied at the state x. Since the optimal solution of $P^a_{T,h}(N, x)$ naturally depends on x, in this way a feedback has been defined on the basis of the approximate discrete-time model, i. e.,

$$v^a_{T,h}(x) := u^*_0. \tag{7}$$

We shall introduce the following notation

$$V^a_{T,h}(x) = J^a_{T,h}(N, x, \mathbf{u}^*). \tag{8}$$

Assumption A1. *The running and the terminal cost functions satisfy the following conditions:*

(i) *l_h is continuous with respect to x and u, uniformly in small h;*

(ii) *there exist $h^* > 0$ and two class-$\mathscr{K}_\infty$ functions φ_1 and φ_2 such that the inequality*

$$\varphi_1(\|x\|) + \varphi_1(\|u\|) \le l_h(x, u) \le \varphi_2(\|x\|) + \varphi_2(\|u\|)$$

holds for all x, u and $h \in (0, h^*]$;

(iii) *g is a continuous, strictly positive definite radially unbounded function.*

Assumption A2. *There exist an $\eta > 0$ and $h^* > 0$ such that for all $x \in \mathscr{G}_\eta$ there exists a $k(x) \in U$ such that inequality*

$$T l_h(x, k(x)) + g\left(F^a_{T,h}(x, k(x))\right) \le g(x) \tag{9}$$

holds true for all $h \in (0, h^*]$.

Assumption A3. *Let $\Delta > 0$, $0 < T \le T^*$ be fixed. The exact discrete-time model is practically asymptotically controllable from $\mathscr{B}_\Delta = \{x \in \mathbb{R}^n : \|x\| \le \Delta\}$ to the origin.*

Using Theorem 1, the following theorem is proven in [**3**].

Theorem 2 ([**3**])**.** *Let us suppose that $F^a_{T,h}$ is consistent with F^e_T and, in addition, assumptions* A1–A3 *hold. Then, in a given compact set, the family of the closed-loop systems F^e_T brought about the receding horizon control* (7) *is practically asymptotically stable near the origin.*

3. The HIV/AIDS model

The mathematical model proposed by [**1**] and [**4**] is the nonlinear control system described by

$$\begin{aligned}
\dot{\overline{x}}(t) &= \lambda - d\overline{x}(t) - \beta(t)\overline{x}(t)\overline{v}(t),\\
\dot{\overline{y}}(t) &= \beta(t)\overline{x}(t)\overline{v}(t) - a\overline{y}(t) - p\overline{y}(t)\overline{z}(t),\\
\dot{\overline{v}}(t) &= q(t)\overline{y}(t) - r\overline{v}(t),\\
\dot{\overline{z}}(t) &= c\overline{y}(t)\overline{z}(t) - b\overline{z}(t).
\end{aligned} \tag{10}$$

Here, $X(t) = (\overline{x}(t), \overline{y}(t), \overline{v}(t), \overline{z}(t))' \in \mathbb{R}^4$, $t \in \mathbb{R}_{\geq 0}$, and the variables $\overline{x}(t)$, $\overline{y}(t)$, $\overline{v}(t)$ and $\overline{z}(t)$ represent the uninfected CD4 cells, infected cells, free viruses, and the HIV-antigen specific cytoxic T lymphocytes (CTL). Moreover, it is assumed that the parameters β and q present an exponential characteristic with respect to controller u as follows:

1° *The first case* (RTI)

$$\beta(t) = \beta_0 e^{-u(t)}, \quad q = q_0;$$

2° *The second case* (PI)

$$q(t) = q_0 e^{-u(t)}, \quad \beta = \beta_0,$$

where u is the drug dose, $u \in U \subset \mathbb{R}_{\geq 0}$ with a given closed set U. In the case of absence of treatments (i. e., $\beta = \beta_0$ and $q = q_0$), (10) leads to the model proposed by [**15**]. In order to avoid numerical difficulties during the integration, a normalization of the variables is introduced.

In this model the infected cells are produced by rate $\beta\overline{x}\,\overline{v}$ and die at rate $a\overline{y}$. Free viruses are produced from infected cells $q\overline{y}$ and decline at rate $r\overline{v}$, therefore the average life time of an infected cell is $1/a$ and the average life time of free virus particle is $1/r$. The total number of virus particles produced from one cell is q/a. Uninfected cells are produced at a constant rate λ, from a pool of precursor cells and die at rate $d\overline{x}$.

The model takes into account the production of CTL by the organism, the rate of CTL proliferation in response to antigen is given by $c\overline{y}\,\overline{z}$ and decay at rate $b\overline{z}$, and the infected cells are killed by CTLs at rate $p\overline{y}\,\overline{z}$. The parameter c denotes the CTL responsiveness, defined earlier as the growth rate of specific CTLs after encountering infected cells. The parameter p specifies the rate at which CTLs cells kill infected cells. If $\overline{y} > b/c$, the CTL response will increase.

The stationary points of model (10) in the absence of the drug doses, i. e., at $u = 0$, are:

$$X_0^{(1)} = \left[\frac{\lambda}{d}, 0, 0, 0\right]$$
$$X_0^{(2)} = \left[\frac{ar}{q_0\beta_0}, \frac{\lambda}{a} - \frac{dr}{q_0\beta_0}, \frac{q_0\lambda}{ar} - \frac{d}{\beta_0}, 0\right]$$
$$X_0^{(3)} = \left[\frac{c\lambda r}{bq_0\beta_0 + cdr}, \frac{b}{c}, \frac{bq_0}{cr}, \frac{-a(bq_0\beta_0 + cdr) + cq_0\lambda\beta_0}{p(bq_0\beta_0 + cdr)}\right].$$

The matrix of the linearized system is

$$\begin{pmatrix} -d - \beta_0 v_0 & 0 & -\beta_0 x_0 & 0 \\ \beta_0 v_0 & -a - pz_0 & \beta_0 x_0 & -py_0 \\ 0 & q_0 & -r & 0 \\ 0 & cz_0 & 0 & cy_0 - b \end{pmatrix}, \tag{11}$$

where (x_0, y_0, v_0, z_0) is any stationary point. Before infection the system is in the invariant hyperplane $\overline{y} = \overline{v} = \overline{z} = 0$ and it has one stationary point $X_0^{(1)}$. If the basic reproductive ratio of model (10), $R_0 = \frac{\lambda q_0\beta_0}{adr}$, is smaller than 1, then at the beginning of the infection, each virus-infected cell produces on average less than one newly infected cell. Thus, the infection cannot spread and the system returns to the uninfected state $X_0^{(1)}$ (stable case). If R_0 is larger than 1, then initially each virus-infected cell produces on the average more than one newly infected cell (unstable case).

The values of parameters of the model obtained from [**4**] are presented in Table 1.

Parameters	d	λ	a	p	b	c	r	q_0	β_0
Values	1	1	0.8	0.05	0.01	0.1	0.01107	1	1

TABLE 1

In this case, R_0 is greater than one and the stationary points become:

$$X_0^{(1)} = [1,\ 0\ ,0,\ 0]$$
$$X_0^{(2)} = [0.008856,\ 1.23893,\ 111.918,\ 0] \tag{12}$$
$$X_0^{(3)} = [0.0996669,\ 0.1,\ 9.03342,\ 164.067]\ .$$

Since the first stationary point $X_0^{(1)}$ is the healthy one, we shall consider the stabilization of the system around this point. The eigenvalues of the matrix of the linearized system at $X_0^{(1)}$ are $\{0.669455, -1.48052, -1, -0.01\}$.

Now we shall consider the first case (RTI) and the second case (PI) can be shown in a similar way. For convenience we transform the point $X_0^{(1)}$ into the origin with $x = \bar{x} - \frac{\lambda}{d},\ y = \bar{y},\ v = \bar{v},\ z = \bar{z}$, then (10) takes the form

$$\begin{aligned}
\dot{x}(t) &= -dx(t) - \beta_0 e^{-u(t)}\left(x(t) + \frac{\lambda}{d}\right)v(t) \\
\dot{y}(t) &= \beta_0 e^{-u(t)}\left(x(t) + \frac{\lambda}{d}\right)v(t) - ay(t) - py(t)z(t) \\
\dot{v}(t) &= q_0 y(t) - rv(t) \\
\dot{z}(t) &= cy(t)z(t) - bz(t).
\end{aligned} \tag{13}$$

Let the approximate discrete-time model be defined by the second order Taylor series method in the following way. Let $\bar{X} = (x^a(k), y^a(k), v^a(k), z^a(k))'$, $\bar{u} = u(k)$. With $\bar{X} = X_0$, let

$$\begin{aligned}
x_{k+1} = x_k + h\left[-dx_k - \beta_0 e^{-\bar{u}}\left(x_k + \frac{\lambda}{d}\right)v_k\right] + \frac{h^2}{2}\left[\left(-d - \beta_0 e^{-\bar{u}}v_k\right)\right. \\
\left.\left(-dx_k - \beta_0 e^{-\bar{u}}\left(x_k + \frac{\lambda}{d}\right)v_k\right) - \beta_0 e^{-\bar{u}}\left(x_k + \frac{\lambda}{d}\right)(q_0 y_k - rv_k)\right],
\end{aligned}$$

$$\begin{aligned}
y_{k+1} = y_k + h\left[\beta_0 e^{-\bar{u}}\left(x_k + \frac{\lambda}{d}\right)v_k - ay_k - py_k z_k\right] \\
+ \frac{h^2}{2}\left[\beta_0 e^{-\bar{u}}v_k\left(-dx_k - \beta_0 e^{-\bar{u}}\left(x_k + \frac{\lambda}{d}\right)v_k\right)\right. \\
- (a + pz_k)\left(\beta_0 e^{-\bar{u}}\left(x_k + \frac{\lambda}{d}\right)v_k - ay_k - py_k z_k\right) \\
\left.+\beta_0 e^{-\bar{u}}\left(x_k + \frac{\lambda}{d}\right)(q_0 y_k - rv_k) - py_k\left(cy_k z_k - bz_k\right)\right],
\end{aligned}$$

$$\begin{aligned}
v_{k+1} = v_k + h\left[q_0 y_k - rv_k\right] + \frac{h^2}{2}\left[-r\left(q_0 y_k - rv_k\right)\right. \\
\left.+ q_0\left(e^{-\bar{u}}\left(x_k + \frac{\lambda}{d}\right)v_k - ay_k - py_k z_k\right)\right],
\end{aligned}$$

and

$$\begin{aligned}
z_{k+1} = z_k + h\left[cy_k z_k - bz_k\right] + \frac{h^2}{2}\left[\left(cy_k - b\right)\left(cy_k z_k - bz_k\right)\right. \\
\left.+ cz_k\left(\beta_0 e^{-\bar{u}}\left(x_k + \frac{\lambda}{d}\right)v_k - ay_k - py_k z_k\right)\right],
\end{aligned}$$

$k = 0, 1, ..., m-1$, and let $X^a(k+1) = (x_m, y_m, v_m, z_m)'$ where $h = T/m$. The running and the terminal costs can be chosen such that the final state constraints (5) and Assumptions A1-A2 are satisfied:

1° *Case* (RTI)

$$l_h(X,u) = 0.01u^2 + x^2 + y^2 + \left(\sqrt{10}v\right)^6 + z^2$$
$$g(X) = 100(x^2 + y^2 + v^4 + z^2)$$

2° *Case* (PI)

$$l_h(X,u) = 0.001u^2 + x^2 + y^2 + 1000v^2 + z^2$$
$$g(X) = 100(x^2 + y^2 + v^4 + z^2).$$

Simulations for the continuous-time system in both cases were carried out using ODE45 program in MATLAB when $h = \frac{T}{m}$, $T = 0.5$, $m = 50$ for (RTI) and $T = 0.5$, $m = 30$ for (PI).

4. Numerical results

We assume that the infection started near the equilibrium point $X_0^{(1)}$ with the values $\overline{x}_0 = 1$, $\overline{y}_0 = 0$, $\overline{v}_0 = 0.001$ and $\overline{z}_0 = 0.03$. For comparison purposes, we present numerical simulations of two mechanisms of treatment (RTI) and (PI) when RHC strategy is used. We vary initiation of the treatment beginning with T_1 and T_2 time units after the onset of infection (dashed and solid lines types on the Figures, respectively). Also, the non-treated case ($u = 0$) is shown (dotted lines on the Figures).

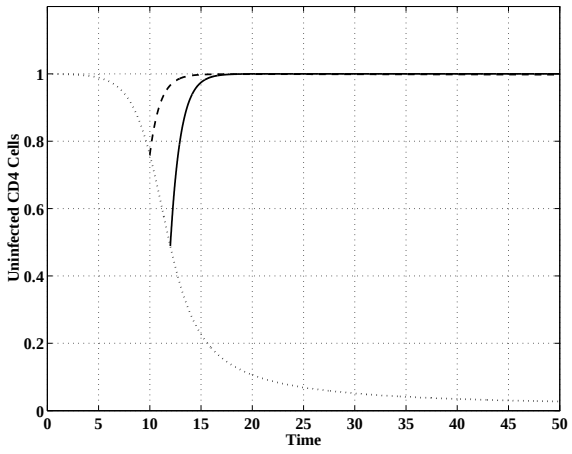

Figure 1a

On Figures 1a and 1b, the number of uninfected CD4 cells $\overline{x}$ is seen to decay rapidly for non-treated case ($u = 0$) and tends to the value λ/d when RHC is used. Moreover, comparing the two kinds of treatment one can observe slower convergence to the equilibrium in the (PI) case than in the (RTI) one.

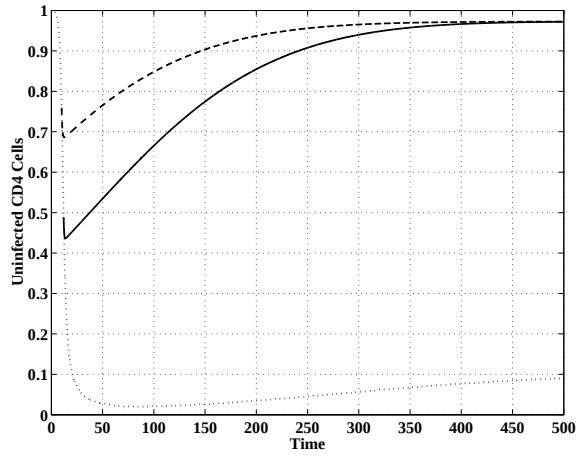

FIGURE 1B

The variation of the number of infected cells $\bar{y}$ with time is shown in Figures 2a and 2b. It is found that $\bar{y}$ increases when $u = 0$, and tends to zero when RHC is applied. Further, it is seen that the convergence of $\bar{y}$ is faster in the (RTI) case than in the (PI) case.

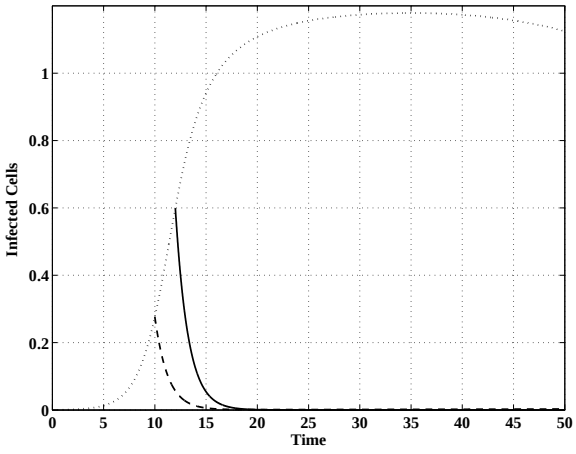

FIGURE 2A

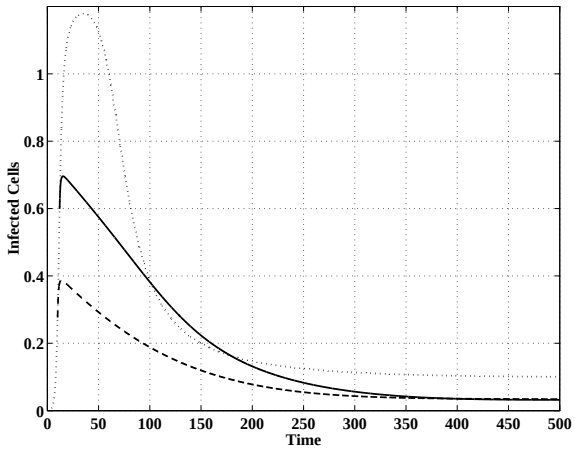

FIGURE 2B

Figures 3a and 3b show that the amount of free viruses $\bar{v}$ decreases under the application of RHC in both the (RTI) and the (PI) cases and it rapidly increases if there is no control.

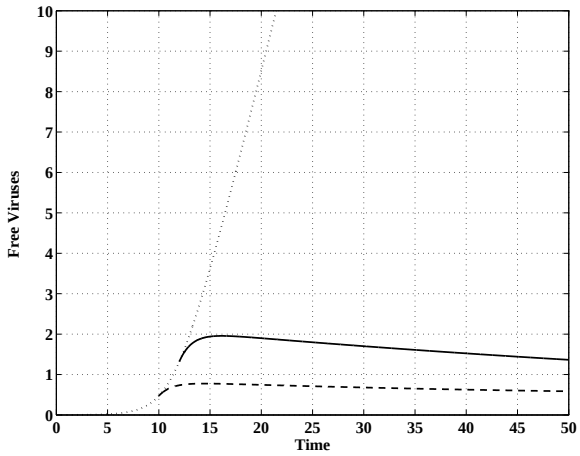

FIGURE 3A

The CTL response $\bar{z}$ can be seen in Figures 4a and 4b. It is found that the amount of CTL rapidly increases for the non-treated case. Also, when the RHC is applied,

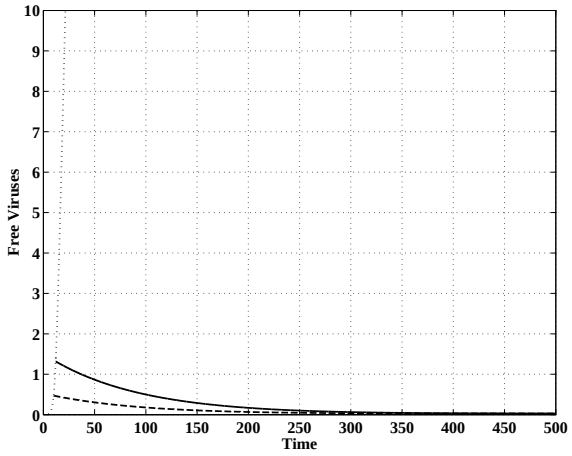

FIGURE 3B

the amount of CTL increases if $\overline{y} > b/c$ and decreases if $\overline{y} < b/c$ for both the (RTI) and the (PI) cases.

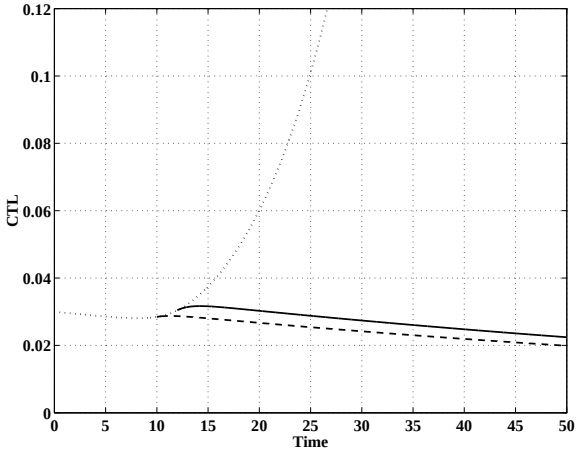

FIGURE 4A

The control as a function of the time produced by RHC method for both the (RTI) and the (PI) cases is shown in Figures 5a and 5b. The total amount of drugs during finite intervals is given in Table 2 when $T_1 = 10$ and $T_2 = 12$.

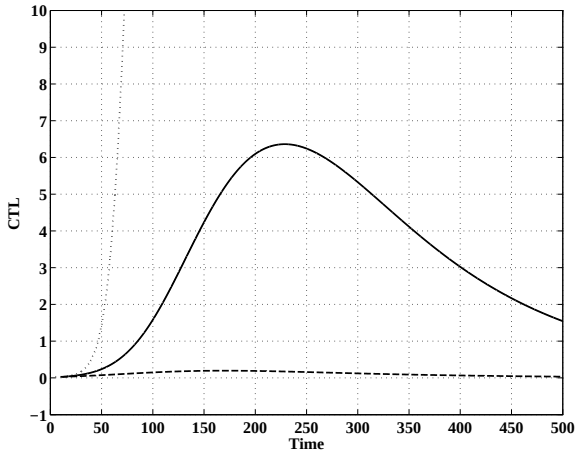

FIGURE 4B

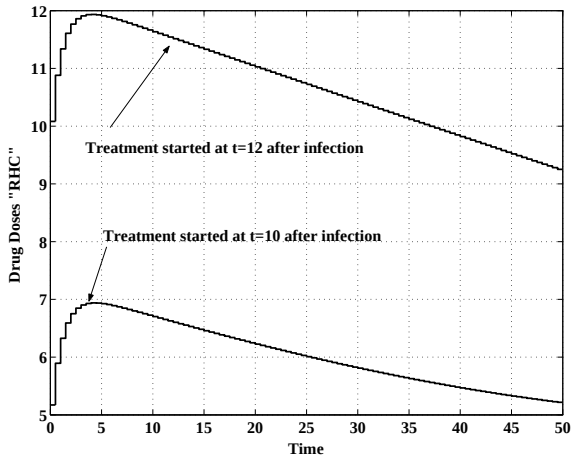

FIGURE 5A

We can say that the system can be stabilized by a smaller amount of drug doses if we start the treatment earlier.

5. CONCLUSIONS

The stabilizing property of the RHC method for sampled data nonlinear systems via their approximate models was proven. The proposed method was applied for the

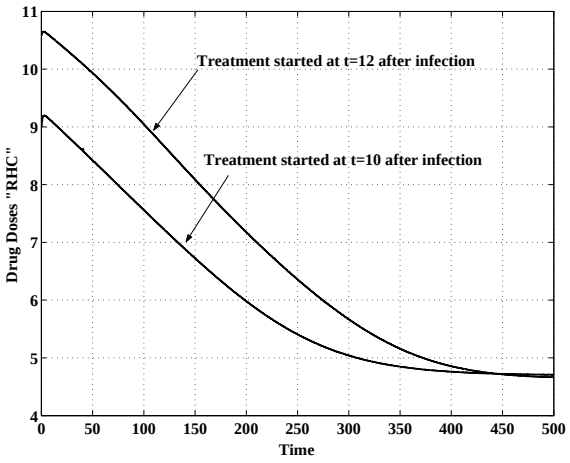

FIGURE 5B

stabilization of the uninfected equilibrium point of the HIV/AIDS model. Two types of treatment were considered. The results of simulations show that the proposed method can effectively be applied to eliminate some drawbacks of the approaches previously published in the literature.

Case	Interval	Total Amount of Drugs (T_1)	Total Amount of Drugs (T_2)
RTI	$[0, 50]$	603	1068.25
PI	$[0, 500]$	6037.5	6861.37

TABLE 2

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BIFURCATIONS OF EQUILIBRIA OF A NON-LINEAR AGE STRUCTURED MODEL

J. Z. FARKAS

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Abstract. In [**6**], M. E. Gurtin and R. C. MacCamy investigated a non-linear age-structured population dynamical model, which served as one of the basic non-linear population dynamical models in the last three decades. They described a characteristic equation but they did not use it to discuss stability of equilibria of the system in certain special cases. In a recent paper [**4**] M. Farkas deduced a characteristic equation in another form. This characteristic equation enabled us to prove results about the stability of stationary age distributions of the system, [**3**]. In the present paper we are going to investigate how equilibria arise and change their stability as a basic parameter of the system varies.

Mathematics Subject Classification: 70K20, 92B05

Keywords: stability of equilibria, bifurcation

1. Introduction

We consider the following model equation:

$$p'_a(a,t) + p'_t(a,t) = -\mu(a,P(t))p(a,t), \tag{1.1}$$

where $p(a,t)$ denotes the density of population of age a, $(0 \le a \le m < \infty)$ at time t, $\mu(a,P(t))$ denotes the mortality of individuals of age a at the population quantity $P(t)$, which quantity at time t is given by the relation

$$P(t) = \int_0^m p(a,t)da. \tag{1.2}$$

The number of newborns at time t is given by the equation

$$p(0,t) = \int_0^m \beta(a,P(t))p(a,t)da, \tag{1.3}$$

with the fertility function $\beta(a,P(t))$. Finally, we assume a given age distribution at time $t = 0$,

$$p(a,0) := p_0(a). \tag{1.4}$$

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The functions $\beta(\cdot,\cdot)$, $\mu(\cdot,\cdot) \in C^1$ and, naturally, they are non-negative. We assume a finite maximal age denoted by m. This assumption does not make any restriction in a biological sense setting m as a "very large number", but lets us avoid some mathematical problems.

The dynamics of the system (1.1)–(1.4) is determined by the vital rates $\beta(a, P(t))$ and $\mu(a, P(t))$ depending on the age a and on the total population quantity $P(t)$. Many papers deal with models where $P(t)$ is replaced by a weighted size function $S(t) = \int \gamma(a)p(a,t)da$, or more generally a finite number of such functions $S_i(t) = \int \gamma_i p(a,t)da$, $i = 1..n$, [**2, 7**].

Theorem 1 (M. E. Gurtin & R. C. MacCamy [**6**])**.** *Any stationary solution $p_1(a)$ of the system is determined uniquely by the root P_1 of the following equation*

$$R(P) = \int_0^m \beta(a,P)\pi(a,P)da = 1.$$

Here $\pi(a,P) = e^{-\int_0^a \mu(s,P)ds}$ is the probability for any individual to reach the age a, and $R(P)$ is the net reproduction number. Then

$$p_1(a) = \frac{P_1\pi(a,P_1)}{\int_0^m \pi(a,P_1)da}.$$

Definition 1. The stationary solution $p_1(a)$ of the system is said to be asymptotically stable if there exists a $\delta > 0$ such that whenever $|p(a,0) - p_1(a)|_{L^1} < \delta$, one has $\lim_{t\to\infty} |p(a,t) - p_1(a)|_{L^1} = 0$.

The stability of the trivial equilibrium is mainly characterised by the following theorem.

Theorem 2 (M. Iannelli [**7**])**.** *If $R(0) < 1$, then the trivial equilibrium $p_1(a) \equiv 0$ is asymptotically stable, and if $R(0) > 1$, then it is unstable.*

As we mentioned earlier in [**3**] we proved some results about the stability of stationary solutions. One of them characterises the stability in a case when the vital rates assume a special form, namely,

$$\mu(a,P(t)) = m(a), \qquad \beta(a,P(t)) = b(a)f(P(t)), \tag{1.5}$$

a factorisation property for $\beta(\cdot,\cdot)$.

Theorem 3 (J. Z. Farkas [**3**])**.** *If $R'(P_1) < 0$, then the stationary solution corresponding to the population quantity $P_1 = \int_0^m p_1(a)da$ is asymptotically stable, and if $R'(P_1) > 0$, then it is unstable.*

In [**1**], J. M. Cushing investigated a general non-linear McKendrick model. He proved that at the critical value of the bifurcating parameter $n = R(0) = 1$ two positive equilibria arise from the trivial equilibrium and he studied stability both for the trivial and the positive equilibria in a small neighbourhood of the critical parameter value

by means of the principle of linearised stability. He used implicit function techniques under smoothness conditions on the vital rates as functionals of age and population density.

In the recent paper we are going to investigate a family of systems for which a "global" theorem proves the linear stability or instability of equilibria. Saddle-node bifurcation "far" from the trivial equilibrium and transcritical bifurcation near the trivial equilibrium are shown.

2. Bifurcations of equilibria

Now consider a family of systems with the following vital rates

$$\mu(a,P) = m(a), \quad \beta_c(a,P) = b(a)f_c(P), \quad f_c(P) = \frac{1}{P+1} + cP \tag{2.1}$$

with the bifurcation parameter $c > 0$, $m, \beta_c \in C^1$. Setting $K = \int_0^m b(a)\pi(a)da$, we get the equation

$$Kf_c(P_1) - 1 = \frac{cKP_1^2 + cKP_1 + 1}{P_1 + 1} - 1 = 0 \tag{2.2}$$

for the population quantity P_1. Observe that now $R(0) = K$ holds.

From the equation

$$cKP_1^2 + (cK - 1)P_1 + K - 1 = 0, \tag{2.3}$$

we get

$$P_1^{1,2} = \frac{(1 - cK) \pm \sqrt{c^2K^2 - 4cK^2 + 2cK + 1}}{2cK}, \tag{2.4}$$

and we seek for a positive equilibrium.

2.1. The case $K > 1$

In (2.4), the relation

$$c^2K^2 - 4cK^2 + 2cK + 1 > 0, \tag{2.5}$$

holds if

$$0 < c < 2 - \frac{1}{K} - 2\sqrt{1 - \frac{1}{K}}$$

or

$$c > 2 - \frac{1}{K} + 2\sqrt{1 - \frac{1}{K}}. \tag{2.6}$$

We first consider the interval $0 < c < 2 - \frac{1}{K} - 2\sqrt{1 - \frac{1}{K}}$, and, in view of the inequality $\frac{1}{K} > 2 - \frac{1}{K} - 2\sqrt{1 - \frac{1}{K}}$, we now have $1 - cK > 0$. The inequality $1 - cK > \sqrt{c^2K^2 - 4cK^2 + 2cK + 1}$ shows that, in this interval of values of the parameter c, there exist exactly two non-trivial equilibria.

At the parameter value $c = 2 - \frac{1}{K} - 2\sqrt{1-\frac{1}{K}}$ we have $c^2K^2 - 4cK^2 + 2cK + 1 = 0$ and $1 - cK > 0$, so, at this value of c, there exists only one positive equilibrium with total population quantity $P_1 = \frac{1-K+K\sqrt{1-\frac{1}{K}}}{2K-1-2K\sqrt{1-\frac{1}{K}}}$.

For $2 - \frac{1}{K} - 2\sqrt{1-\frac{1}{K}} < c < 2 - \frac{1}{K} + 2\sqrt{1-\frac{1}{K}}$ there are not any positive equilibria, and this holds for $c > 2 - \frac{1}{K} + 2\sqrt{1-\frac{1}{K}}$ because $1 - cK < 0$ and $cK - 1 > \sqrt{c^2K^2 - 4cK^2 + 2cK + 1}$.

Now we are going to examine the stability of the equilibria. We first consider the positive equilibrium at the parameter value $c = 2-\frac{1}{K}-2\sqrt{1-\frac{1}{K}}$ with total population quantity $P_1 = \frac{1-cK}{2cK} = \frac{1-K+K\sqrt{1-\frac{1}{K}}}{2K-1+2K\sqrt{1-\frac{1}{K}}}$. We have

$$f_c'(P) = c - \frac{1}{P^2 + 2P + 1}, \tag{2.7}$$

and an elementary calculation gives

$$f_c'(P_1) = 0. \tag{2.8}$$

It is easy to show that for $P > P_1$ we have $f_c'(P) > 0$ and for $P < P_1$ we get $f_c'(P) < 0$. Applying Theorem 3 we get that one of the positive equilibria is asymptotically stable and the other one is unstable at the parameter interval $0 < c < 2 - \frac{1}{K} - 2\sqrt{1-\frac{1}{K}}$. The trivial equilibrium is unstable by Theorem 2 because $R(0) = K > 1$. Summarising, we have the following

Theorem 4 (Saddle-node bifurcation)**.** *For $c > 2 - \frac{1}{K} - 2\sqrt{1-\frac{1}{K}}$, there is no positive equilibrium, for $c_0 = 2 - \frac{1}{K} - 2\sqrt{1-\frac{1}{K}}$ we have one positive equilibrium. For $c < c_0$, there are two curves of positive equilibria in a neighbourhood of*

$$(P, c) = \left(\frac{1 - K + K\sqrt{1-\frac{1}{K}}}{2K - 1 - 2K\sqrt{1-\frac{1}{K}}},\ c_0 \right).$$

The upper equilibria are unstable and the lower equilibria are asymptotically stable.

2.2. The case $K < 1$

Now $c^2K^2 - 4cK^2 + 2cK + 1 > 0$ for every $c > 0$. If $0 < c < \frac{1}{K}$, then $1 - cK > 0$ and in view of the relation $1 - cK < \sqrt{c^2K^2 - 4cK^2 + 2cK + 1}$, we get one positive non-trivial equilibrium with total population quantity

$$P_1 = \frac{1 - cK + \sqrt{c^2K^2 - 4cK^2 + 2cK + 1}}{2cK}.$$

For $c \geq \frac{1}{K}$ we have $1 - cK \leq 0$ and $1 - cK + \sqrt{c^2K^2 - 4cK^2 + 2cK + 1} \leq 0$, so there are no positive equilibria.

The positive equilibrium above tends to zero if c tends to $\frac{1}{K}$ and tends to infinity if c tends to 0. For this positive equilibrium,

$$f_c'(P_1) = c - \frac{1}{P_1^2 + 2P_1 + 1} > 0 \tag{2.9}$$

and thus, it is unstable. The trivial equilibrium is asymptotically stable because $R(0) = K < 1$.

Theorem 5 (Transcritical bifurcation)**.** *There are no positive equilibria for $c \geq \frac{1}{K}$ and, for $c < \frac{1}{K}$, there is a curve of equilibria which is unstable, and the trivial equilibrium is stable.*

2.3. The case $K = 1$

Now we have

$$P^{1,2} = \frac{(1 - c) \pm \sqrt{c^2 - 2c + 1}}{2c}, \tag{2.10}$$

whence $P^1 = 0$ and $P^2 = \frac{1}{c} - 1$ so for $c < 1$ there exists a non-trivial equilibrium. We have

$$f_c'(P^2) = c - \frac{1}{\left(\frac{1-c}{c}\right)^2 + 2\frac{1-c}{c} + 1} = c - c^2 > 0; \tag{2.11}$$

the positive equilibrium is unstable.

For the trivial equilibrium $R(0) = K = 1$; but $f_c'(0) = c - 1$ means that for small P we have $R(P) > 1$ if $c > 1$ and $R(P) < 1$ for $c < 1$. This means that the trivial equilibrium is stable for $c < 1$ and it is unstable for $c > 1$.

Theorem 6 (Transcritical bifurcation)**.** *For $c \geq 1$ we have only the trivial equilibrium, for $c < 1$ a curve of positive unstable equilibria emerges.*

3. Discussion

In this paper, we have considered the well-known age-structured model of Gurtin an MacCamy in a special case where the mortality of the individuals is age-dependent only and the fertility function is separable in the variables a of age and P of total population quantity. In that case, Theorem 3 establishes the stability of the positive equilibrium, while the stability of the trivial equilibrium depends on the constant

$$K = \int_0^m b(a)e^{-\int_0^a m(s)ds}da = R(0)$$

in Theorem 2. We showed that a degenerate transcritical bifurcation occurs, that is, for $K \leq 1$, a curve of positive unstable equilibria bifurcates from the trivial one at the critical parameter value $c_0 = \frac{1}{K}$ and as c tends to 0 the total population quantity at the

equilibrium tends to infinity. For $K > 1$, we have showed that a saddle-node bifurcation occurs at the critical parameter value $c_0 = 2 - \frac{1}{K} - 2\sqrt{1 - \frac{1}{K}}$. For $c < c_0$, two curves of positive equilibria appear, from which one contains unstable equilibrium points with total population quantity tending to infinity, and the other curve consists of positive stable equilibrium points.

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STABILITY OF SETS FOR IMPULSIVE DIFFERENTIAL SYSTEMS

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Abstract. The problems of sets' stability for impulsive differential systems are considered. A short survey of the recent results connected with the stability properties of sets for impulsive differential systems is given.

Mathematics Subject Classification: 34D20

Keywords: Impulsive differential system, invariant set, integral set, stability of invariant set, stability of integral sets

1. Stability of invariant sets for discontinuous dynamical systems

In the present paper, we give a survey of the recent results connected with the stability properties of sets for impulsive differential systems and characterise the methods of investigation of the sets' stability.

Let us first consider the system of differential equations (discontinuous dynamical system [**1, 2**])

$$\begin{aligned} \frac{dx}{dt} &= f(x), \quad x \notin \Gamma, \\ \Delta x_{|x\in\Gamma} &= Tx - x = I(x), \end{aligned} \tag{1.1}$$

where $x \in \bar{D} \subset \mathbb{R}^n$, $f(x) \in C[\bar{D}, \mathbb{R}^n]$, D is a bounded domain in $\mathbb{R}^n$, $t \in [0, \infty)$, T is an operator, $T : \Gamma \to Q$, Γ is a surface without contact for the system $\frac{dx}{dt} = f(x)$, $\Gamma \subset \bar{D}$ (possibly, $\Gamma = \partial D$), and Q is a closed subset of $\bar{D}$.

We shall assume that the conditions for existence and uniqueness of solutions of (1.1) are fulfilled and also suppose that each solution of (1.1) exists on $[0, +\infty)$. Denote by $x(t) = x(t, x_0)$ the solution of (1.1) satisfying initial condition $x(0) = x_0$.

We indicate the following kinds of motions of the system (1.1): **1**° motions without impulse effect; **2**° motions whose trajectories hit the set Γ no more than finitely many times; and **3**° motions whose trajectories hit the set Γ infinitely many times. Let Ω be the set of such points x_0 that trajectories $\{x(t, x_0) : t \in [0, +\infty)\}$ meet Γ infinitely many times. Let us put $S_\delta(M) = \{x \in \bar{D} : \varrho(x, M) < \delta\}$ and $U_\delta(M) = \{x \in \bar{D} : 0 < \varrho(x, M) < \delta\}$ for any set $M \subseteq \bar{D}$.

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We shall say that a set $M \subseteq \bar{D}$ is an invariant set of system (1.1) if $x_0 \in M$ implies that $x(t,\ x_0) \in M$ for $t \geq 0$.

A closed invariant set M of the system (1.1) is said to be stable if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $x_0 \in S_\delta(M)$ implies $x(t,\ x_0) \in S_\varepsilon(M)$, $t \geq 0$, for any solution $x(t) = x(t,\ x_0)$ of (1.1).

A closed invariant set M of the system (1.1) is said to be asymptotically stable if M is stable and there exists $\delta_0 > 0$ such that the relation $x_0 \in S_{\delta_0}(M)$ implies that $\lim_{t\to\infty} \varrho(x(t,\ x_0),\ M) = 0$ for any solution $x(t) = x(t,\ x_0)$ of (1.1).

A closed invariant set M of the system (1.1) is said to be stable (asymptotically stable) with respect to the set $B \subset \bar{D}$ if the solutions beginning in B satisfy the conditions of the definition of stability (asymptotic stability) for the set M.

The closed invariant set M is said to be unstable if conditions of the definition of stability fail to hold.

To establish stability results for invariant sets of the system (1.1) we use an auxiliary function $V(x)$ possessing certain properties.

Let $V(x)$ be a function continuously differentiable in $\bar{D}$ and such that

$$V(x) = 0,\ \ x \in M_0,\ M_0 \subset \bar{D},\ M_0 \neq \varnothing; \qquad V(x) > 0,\ x \in \bar{D} \setminus M_0. \tag{1.2}$$

Denote by K_0 the class of continuous functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(s) = 0$ if and only if $s = 0$.

We formulate some conditions sufficient for the stability of invariant sets of system (1.1).

Theorem 1 (see [**3**]). (A) *Assume that there exists a function $V(x)$ satisfying condition* (1.2) *and the conditions*

$$\langle \operatorname{grad} V(x),\ f(x)\rangle \leq 0 \quad (x \notin \Gamma), \tag{1.3}$$

$$V(x + I(x)) \leq V(x) \quad (x \in \Gamma). \tag{1.4}$$

Then M_0 is a stable invariant set for system (1.1).

(B) *Assume that there exists a function $V(x)$ satisfying conditions* (1.2), (1.4), *and*

$$\langle \operatorname{grad} V(x),\ f(x)\rangle \leq -\varphi(V(x)) \quad (x \notin \Gamma), \tag{1.5}$$

where $\varphi(s) \in K_0$. Then M_0 is an asymptotically stable invariant set of the system (1.1).

Let M_0 be a closed invariant set such that $M_0 \cap \Omega \neq \varnothing$, $M_0 \neq \Omega$.

Theorem 2 (see [**3**]). *If there exists a function $V(x)$ satisfying conditions* (1.2), (1.3), *and*

$$V(x + I(x)) - V(x) \leq -\psi(V(x)),\ \ x \in \Gamma,$$

where $\psi(s) \in K_0$, then M_0 is an asymptotically stable invariant set of system (1.1) *with respect to set Ω.*

The proofs of the statements of this theorem are based on reasoning similar to [**1, 2, 4**].

The following theorem gives more interesting conditions for stability properties of the invariant set in suggestion that

$$t_{k+1}^{x_0} - t_k^{x_0} \geq \theta > 0, \quad k \geq 1. \tag{1.6}$$

We denote by $t_k^{x_0}$, $k = 1, 2, \ldots$, the moments at which the discontinuous trajectory $x(t, x_0)$ meets the set Γ.

Theorem 3. *Assume that for differential system* (1.1), *there exists a function* $V(x)$ *satisfying conditions* (1.2), (1.5), *and the condition*

$$V(x + I(x)) \leq \psi(V(x)), \quad x \in \Gamma,$$

where $\psi(s) \in K_0$.

(A) *If there exists* $a_0 > 0$ *such that, for all* $a \in (0, a_0]$,

$$\int_a^{\psi(a)} \frac{ds}{\varphi(s)} \leq \theta,$$

where θ *is the number from* (1.6), *then* M_0 *is a stable invariant set of the system* (1.1).

(B) *If there exists* $a_0 > 0$ *such that, for all* $a \in (0, a_0]$,

$$\int_a^{\psi(a)} \frac{ds}{\varphi(s)} \leq \theta - \gamma$$

for some $\gamma > 0$, *then* M_0 *is an asymptotically stable invariant set of system* (1.1).

Consider some conditions for instability of the set $M_0 \subset \bar{D}$ [**3**].

Theorem 4. *Assume that there exists a function* $V(x)$ *continuously differentiable on the set* $S_h(M_0)$ *and such that* $\underline{V}(x) = 0$, $x \in M_0$ *and, for any* δ $(0 < \delta < h)$, $U_\delta(M_0) \cap \Pi \neq \varnothing$, *where* $\Pi = \{x \in \bar{D} : V(x) > 0\}$. *Assume also that, on the set* Π, *the function* $V(x)$ *satisfies the conditions*

$$\langle \operatorname{grad} V(x), \ f(x) \rangle \geq \varphi(V(x)) \quad (x \notin \Gamma)$$

and

$$V(x + I(x)) \geq V(x) \quad (x \in \Gamma),$$

where $\varphi(s) \in K_0$. *Then* M_0 *is an unstable set for system* (1.1).

Theorem 5. *Assume that there is a continuously differentiable function* $V(x)$ *defined on the set* $S_h(M_0)$ *such that* $\underline{V}(x) = 0$, $x \in M_0$, *and for any* δ $(0 < \delta < h)$, $U_\delta(M_0) \cap \Pi \neq \varnothing$, *where* $\Pi = \{x \in \bar{D} : V(x) > 0\}$. *Assume also that, on the set* Π, *the function* $V(x)$ *satisfies the conditions*

$$\langle \operatorname{grad} V(x), \ f(x) \rangle \geq \varphi(V(x)), \quad (x \notin \Gamma)$$

and

$$V(x + I(x)) \geq \psi(V(x)) \quad (x \in \Gamma),$$

where $\varphi(s) \in K_0$, $\psi(s) \in K_0$.

If for any $a \in (0, a_0]$, *where* $a_0 = \sup_{x \in \Pi} V(x)$, *and some* $\gamma > 0$

$$\int_{\psi(a)}^{a} \frac{ds}{\varphi(s)} \leq \theta - \gamma,$$

then M_0 *is an unstable set for system* (1.1).

Analogous results may be established in the case when the sequence $\{t_k^{x_0}\}$, $k = 1, 2, \ldots$, satisfies the condition $t_{k+1}^{x_0} - t_k^{x_0} \leq \theta_1$.

The results above are natural generalisations of the stability results for impulsive systems with unique equilibrium positions. If the invariant set of system (1.1) is a point $x = 0$, then the results mentioned above give the well-known sufficient conditions for stability, asymptotic stability, and instability of the trivial solution (see the survey of the corresponding results in [**5**]).

2. Stability of integral sets of the differential systems with impulsive perturbations

Consider now the impulsive differential system

$$\begin{aligned} \frac{dx}{dt} &= f(t, x), \quad t \neq t_i(x), \\ \Delta x|_{t=t_i(x)} &= I_i(x), \end{aligned} \tag{2.1}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$, $f(t, x)$, $I_i(x)$, $i = 1, 2, \ldots$, are continuous vector-functions, $t_i(x)$ are continuous functions such that $0 < t_1(x) < t_2(x) < \ldots$, $x \in \mathbb{R}^n$, $t_i(x) \to \infty$ as $i \to \infty$ uniformly in $x \in \mathbb{R}^n$. We assume that, for any point $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, there exists a unique solution $x(t) = x(t, t_0, x_0)$ of (2.1) on $[t_0, \infty)$ satisfying $x(t_0) = x_0$, and that the integral curve of each solution meets each of the hypersurfaces $t = t_i(x)$ only once, i. e., the phenomenon of "beating" is not observed. Sufficient conditions for the absence of the phenomenon of "beating" are well-known (see, e. g., [**1, 2, 6**]).

We recall [**1, 2**] that any solution $x(t, t_0, x_0)$ of (2.1) is a piecewise continuous function with points of discontinuity θ_i^x, $\theta_i^x = t_i(x(\theta_i^x))$, $i = 1, 2, \ldots$;

$$\begin{aligned} x\left(\theta_i^x, t_0, x_0\right) &= x\left(\theta_i^x - 0, t_0, x_0\right), \\ x\left(\theta_i^x + 0, t_0, x_0\right) &= x\left(\theta_i^x, t_0, x_0\right) + I_i\left(x\left(\theta_i^x, t_0, x_0\right)\right). \end{aligned}$$

We shall say that set $M \subset \mathbb{R}_+ \times \mathbb{R}^n$ is an integral set of system (1.1) if, for any point $(t_0, x_0) \in M$, it follows that $(t, x(t, t_0, x_0)) \in M$ for $t \geq t_0$.

Note that, together with a point $(t^\bullet, x^\bullet) \in M$ belonging to the surface of discontinuity ($t^\bullet = t_i(x^\bullet)$), the set M contains also the point $(t^\bullet, x^\bullet + I_i(x^\bullet))$.

For each $i = 1, 2, \ldots$, we denote by Λ_i the set of time moments when the integral curves belonging to M meet the hypersurface $t = t_i(x)$, i. e, the set of solutions of the equation $\tau = t_i(x(\tau))$, $(\tau, x(\tau)) \in M$. If, for instance, the integral set is the integral curve $\{(t, x) : t \in \mathbb{R}^+, x = x(t, 0, x_0)\}$ then Λ_i consists of a single point: $\Lambda_i = \{\theta_i^x\}$. In the case where $t_i(x) \equiv \tau_i$, $i = 1, 2, \ldots$, each set Λ_i also consists of a single element: $\Lambda_i = \{\tau_i\}$.

Let $M(t) = \{x \in \mathbb{R}^n : (t, x) \in M\}$. If $t = \theta_i \in \Lambda_i$, $(\theta_i, x) \in M$ and $\theta_i = t_i(x)$, then, as it was noted above, the set M contains both points: (θ_i, x) and $(\theta_i, x + I_i(x))$ and, therefore, the section $M(\theta_i)$ contains both points x and $x + I_i(x)$.

For $\theta_i \in \Lambda_i$, we distinguish two subsets of the section: $M(\theta_i)$ and $M_{i+1}(\theta_i)$. The properties that a point $x \in \mathbb{R}^n$ belongs to $M(\theta_i)$ or $M_{i+1}(\theta_i)$ are defined in the following way. Let $\Omega_i = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n : t_{i-1}(x) < t \leq t_i(x)\}$, $i = 1, 2, \ldots$, $t_0(x) \equiv 0$. If $(\theta_i, x) \in M \cap \Omega_i$, then $x \in M_i(\theta_i)$; if $(\theta_i, x) \in M \cap \Omega_{i+1}$ or (θ_i, x) is a boundary point for $M \cap \Omega_{i+1}$ then $x \in M_{i+1}(\theta_i)$; $M(\theta_i) = M_i(\theta_i) \cup M_{i+1}(\theta_i)$, $(\theta_i \in \Lambda_i)$. It is possible that $M_i(\theta_i) \cap M_{i+1}(\theta_i) \neq \varnothing$.

Assume that $M(t) \neq \varnothing$, $M(t) \subseteq Q \subset \mathbb{R}^n$, $t \in \mathbb{R}_+$, where Q is a compact set. Furthermore we shall suppose that $Q_{\varepsilon_0} \subset B_h$, where $Q_{\varepsilon_0} = \{x \in \mathbb{R}^n : \varrho(x, Q) < \varepsilon_0\}$, $B_h = \{x \in \mathbb{R}^n : \|x\| \leq h\}$, for some $\varepsilon_0 > 0$, $h > 0$.

In the case when impulsive system (1.1) is a system with impulses at fixed times, i. e., $t_i(x) \equiv \tau_i$, $i = 1, 2, \ldots$, we differentiate two subsets of the section of integral set M only for points τ_i: if $(\tau_i, x) \in M \cap \Omega_i$,then $x \in M_i(\tau_i)$; if $(\tau_i, x) \in M$ and (τ_i, x) is a boundary point of $M \cap \Omega_{i+1}$, then $x \in M_{i+1}(\tau_i)$.

We shall say that the solution $y(t)$ of (2.1) belongs to ε-neighbourhood of the integral set M for $t \geq t_0$ if there exists a sequence $\{\theta_i\}$, $i = 1, 2, \ldots$, $\theta_i \in \Lambda_i$, such that $|\theta_i^y - \theta_i| < \varepsilon$ and $\varrho(y(t, t_0, y_0), M(t)) < \varepsilon$ for $t \geq t_0$, $t \notin \bigcup_{i=1}^{\infty}(\theta_i - \varepsilon, \theta_i + \varepsilon)$ (here, θ_i^y, $i = 1, 2, \ldots$, are points of discontinuity of the solution $y(t)$ and $\varrho(y, M(t))$ is a distance between y and the set $M(t)$).

With every point $(t_0, y_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, we associate the set $M_{y_0}(t_0)$ as follows:

$$M_{y_0}(t_0) = \begin{cases} M(t_0) & \text{if } t_0 \notin \bigcup_{i=1}^{\infty} \Lambda_i, \\ M_i(t_0) & \text{if } t_0 \in \Lambda_i,\ (t_0, y_0) \in \Omega_{,i} \\ M_{i+1}(t_0) & \text{if } t_0 \in \Lambda_i,\ (t_0, y_0) \in \Omega_{i+1}. \end{cases}$$

We give the following definitions of stability and asymptotic stability of an integral set [7].

The integral set M of system (2.1) is stable if for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$ there exists such a number $\delta = \delta(\varepsilon, t_0) > 0$ that, for any solution $y(t) = y(t, t_0, y_0)$ of system (2.1), $\varrho(y_0, M_{y_0}(t_0)) < \delta$ implies that $y(t)$ belongs to the ε-neighbourhood of the integral set M for $t \geq t_0$ (it is assumed that $t_0 \neq \theta_i^y$).

The stable integral set M of system (2.1) is asymptotically stable if there exists such a number $\delta_0 > 0$ that for any solution $y(t) = y(t, t_0, y_0)$ of (2.1) that satisfies the inequality $\varrho(y_0, M_{y_0}(t_0)) < \delta_0$, the following holds: $\lim_{t\to\infty} \varrho(y(t), M(t)) = 0$.

These definitions of the stability properties of the integral set of the impulsive system (2.1) agree with the corresponding definitions of stability properties of a single discontinuous solution of (2.1) [**1, 2**].

There are various ways for the investigation of stability of integral sets for impulsive systems. One of them consists of the reduction of the problem to the consideration of a simpler case impulsive system - a system with impulses at fixed times.

Suppose that, in (2.1), the functions $f(t,\ x)$ and $I_i(x)$ satisfy the conditions

$$\|f(t,\ x) - f(t, y)\| + \|I_i(x) - I_i(y)\| \le K\|x - y\|$$

uniformly with respect to $t \in \mathbb{R}_+$, $i = 1, 2, \ldots$, for all $x, y \in \mathbb{R}^n$, and

$$\sup_{t\ge 0}\|f(t,\ 0)\| + \sup_{i\ge 1}\|I_i(0)\| < +\infty$$

Suppose that, for $x \in B_h$, the equations of surfaces of discontinuity of solutions can be written in the form

$$t = t_i(x) = \tau_i + \tau_i(x),\ i = 1, 2, \ldots \quad ,$$

and the following conditions are satisfied:

$$\tau_{i+1} - \tau_i \ge d > 0, \quad 0 \le \tau_i(x) \le l < d, \quad |\tau_i(x) - \tau_i(y)| \le l\,\|x - y\|, \quad i = 1, 2, \ldots,$$

where l is a sufficiently small number.

It is proved in [**7**] that, under such assumptions, the investigation of stability of integral sets of impulsive differential system (2.1) can be reduced to the investigation of stability of integral set of some differential system with impulses at fixed times. One can point out the system with impulses at fixed times and the integral set $\hat{M}$ (constructed by system (2.1) and the set M) such that the stability (resp., asymptotic stability) of $\hat{M}$ for constructed system implies the stability (resp., asymptotic stability) of the set M for system (2.1). Note that definitions of stability properties of sets in this particular case of impulsive system coincide with well-known definitions for ordinary differential equations [**8**]. As it was mentioned, investigation of a set's stability for the systems with fixed times of impulses is simpler than in the general case of impulsive system.

To illustrate the possible character of such conditions we adduce one variant of sufficient conditions for stability (asymptotic stability) of an integral set of an impulsive differential system with fixed time moments sequence of impulse action [**7**].

Consider the system

$$\begin{aligned} &\frac{dx}{dt} = F(t,\ x),\ t \ne \tau_i, \\ &\Delta x_{|t=\tau_i} = J_i(x), \end{aligned} \tag{2.2}$$

where $x \in \mathbb{R}^n,\ t \in \mathbb{R}_+$, $i = 1, 2, \ldots$, $0 = \tau_0 < \tau_1 < \ldots,\ \tau_i \to \infty$ as $i \to \infty$.

As above, assume that system (2.2) satisfies the conditions of existence and uniqueness of solutions, and that any solution of system (2.2) is defined on $[t_0, +\infty)$.

Let M be an integral set of system (2.2). As above, suppose that, for each $t \in \mathbb{R}_+$, the section $M(t)$ is not empty and is contained in a compact set $Q \subset \mathbb{R}^n$; $Q_{\varepsilon_0} \subseteq B_h$.

Consider a continuously differentiable function $V(t, x)$ defined in the domain $Z = \{(t, x) : t \in \mathbb{R}_+, x \in Q_{\varepsilon_0}\}$ and possessing the properties

$$V(t, x) = 0, \ (t, x) \in M; \qquad V(t, x) > 0, \ (t, x) \notin M;$$
$$V(t, x) \geq a(\varrho(x, M(t))), \tag{2.3}$$

where $a(s)$ $(s \geq 0)$ is a continuous increasing function, $a(0) = 0$.

Assume that there exists $\mu > 0$ $(0 < \mu < \varepsilon_0)$ such that, for $x \in Q_\mu$, one has $x + J_i(x) \in Q_{\varepsilon_0}$, $i = 1, 2, \ldots$.

Consider, for example, the case where the sequence $\{\tau_i\}$, $i = 1, 2, \ldots$, satisfies the condition $\tau_{i+1} - \tau_i \geq \theta$, $\theta > 0$.

Theorem 6. (A) *Let there exist a continuously differentiable function $V(t, x)$ satisfying conditions* (2.3) *in the domain Z and such that*

$$\frac{\partial V(t, x)}{\partial t} + \langle \operatorname{grad}_x V(t, x), F(t, x) \rangle \leq -\varphi(V(t, x)), \tag{2.4}$$
$$V(\tau_i, x + J_i(x)) \leq \psi(V(\tau_i, x)), \ i = 1, 2, \ldots \tag{2.5}$$

where $\varphi(s)$, $\psi(s) \in K_0$. If there exists a positive number a_0 such that, for an arbitrary $a \in (0, a_0]$,

$$\int_a^{\psi(a)} \frac{ds}{\varphi(s)} \leq \theta,$$

then the set M is stable with respect to system (2.2).

(B) *Let there exist a continuously differentiable function $V(t, x)$ satisfying in the domain Z conditions* (2.3), (2.4), (2.5).

If there exist positive numbers a_0 and γ such that

$$\int_a^{\psi(a)} \frac{ds}{\varphi(s)} \leq \theta - \gamma$$

for all $a \in (0, a_0]$, then the set M is asymptotically stable with respect to system (2.2).

The form of these conditions is similar to conditions quoted above for the stability of an invariant set of system (1.1) (Theorem 3).

It must be mentioned that problems of sets' stability for impulsive differential equations were studied by many authors, see, e. g., [**9–13**].

The work [**9**] studies sufficient conditions of stability, asymptotic stability, instability, and global stability of the trivial invariant torus of the system

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = f(\varphi, x), \quad \Delta x|_{t=\tau_i} = I(\varphi, x), \tag{2.6}$$

where $a(\varphi) \in C_{\mathrm{Lip}}(\mathscr{T}_m)$, the functions $f(\varphi, x)$ and $I_i(\varphi, x)$ are defined in the domain $Z = \{\varphi \in \mathscr{T}_m, x \in \bar{J}_h\}$ (here, $\bar{J}_h = \{x \in \mathbb{R}^n : \|x\| \leq h, h > 0\}$), are 2π-periodic with

respect to each of their variables φ_ν, $\nu = 1, \ldots, m$, and, moreover, $f(\varphi, 0) \equiv 0$, $I(\varphi, 0) \equiv 0$ for all $\varphi \in \mathscr{T}_m$. The sequence $\{\tau_i\}$ is assumed to be such that $\tau_i > \tau_{i-1}$, $i = 1, 2, \ldots$, and $\lim_{i\to\infty} \tau_i = \infty$.

The paper [**9**] also considers stability conditions for the trivial invariant torus of the burst dynamical system

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = f(\varphi, x), \quad \Delta x|_{\varphi\in\Gamma} = I(\varphi, x), \tag{2.7}$$

where Γ is a subset of the torus $\mathscr{T}_m$ (a manifold of dimension $m - 1$).

It is important to note that, for solving the problem on stability of an invariant torus, it is sufficient to consider the functions $V(\varphi, x)$, $V(\varphi, I(\varphi, x))$, and $\frac{dV}{dt}$ in the region $Z_\Omega = \{\varphi \in \Omega, x \in \bar{J}_h\}$ only, and not everywhere in the region Z (here, $\Omega = \bigcup_{\varphi\in\mathscr{T}_m} \Omega_\varphi \subset \mathscr{T}_m$, Ω_φ is the ω-limit set of the positive half-trajectory of the solution $\varphi_t(\varphi)$, $\varphi \in \mathscr{T}_m$, $t \in [0, +\infty)$ of the first equation of system (2.6) or (2.7)).

In [**10**], the stability of a set $M \subset \mathbb{R}_+ \times \mathbb{R}^n$ is investigated for an impulsive system of the general form (2.1) and, in [**13**], for the system

$$\frac{dx}{dt} = f(t, x(t), x(t-h)), \ t > t_0, \ t \neq \tau_k(x(t)),$$
$$x(t) = \varphi_0(t), \ t \in [t_0 - h, t_0],$$
$$\Delta x(t)|_{t=\tau(x(t))} = I_k(x(t)), \ t > t_0, \quad k = 1, 2, \ldots$$

Note that the definitions of stability of a set $M \subset \mathbb{R}_+ \times \mathbb{R}^n$ for the system (2.1) used in [**10, 13**] correspond to the definitions of stability properties of sets for ordinary differential equations (without impulses) [**8**] and agree with the definitions of stability properties for the trivial (continuous) solution of an impulsive equation [**1, 2**].

The methods of investigation of the stability of sets in the papers cited above are based on Lyapunov's direct method and its generalisations. As one can see [**11–13**], it is also efficient to apply the comparison principle together with the Lyapunov second method.

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AN OSCILLATION CRITERION FOR HALF-LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. We investigate oscillatory properties of the half-linear second order differential equation

$$(\Phi(x'))' + c(t)\Phi(x) = 0,$$

where $\Phi(x) := |x|^{p-2}x$. This equation is regarded as a perturbation of the generalized Euler equation

$$(\Phi(x'))' + \frac{\gamma_p}{t^p}\Phi(x) = 0,$$

where $\gamma_p = \left(\frac{p-1}{p}\right)^p$.

Mathematics Subject Classification: 34C10

Keywords: Half-linear equation, variational principle, oscillation

1. Introduction

In this paper, we investigate oscillatory properties of the half-linear second order differential equation

$$(\Phi(x'))' + c(t)\Phi(x) = 0, \tag{1}$$

where $\Phi(x) := |x|^{p-2}x$, $p > 1$, $t \in I := [T, \infty)$, and c is a real-valued continuous function.

Oscillation theory of (1) attracted considerable attention in the past years and it was shown that many of the oscillation criteria for the linear Sturm–Liouville second order differential equation

$$x'' + c(t)x = 0 \tag{2}$$

(which is the special case $p = 2$ of (1)) can be extended to (1).

In these criteria, equation (1) is essentially viewed as a perturbation of the one-term (nonoscillatory) equation

$$(\Phi(x'))' = 0 \tag{3}$$

and the nonoscillation or oscillation of (1) is "measured" by the "smallness" of the function c.

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In this paper we investigate equation (1) not as a perturbation of (3), but as a perturbation of the generalized Euler equation

$$(\Phi(x'))' + \frac{\gamma_p}{t^p}\Phi(x) = 0, \tag{4}$$

where $\gamma_p = \left(\frac{p-1}{p}\right)^p$ is the so-called *critical constant* in this equation. In particular, this equation is nonoscillatory for $\gamma \le \gamma_p$ and oscillatory for $\gamma > \gamma_p$, see [**3**]. A similar idea was used in [**2, 4, 5**]. In these papers, nonoscillation criteria were mostly based on the Riccati technique consisting in the fact that if x is a nonzero solution of (1) then

$$w(t) = \frac{\Phi(x'(t))}{\Phi(x(t))}$$

solves the generalized Riccati equation

$$w' + c(t) + (p-1)|w|^q = 0,$$

where q is the conjugate number to p, i. e., $\frac{1}{p} + \frac{1}{q} = 1$.

Here we use the variational principle which is based on the relationship between disconjugacy of (1) in $[a, b]$ and the positivity of the "p-degree" functional

$$\mathscr{F}(y; a, b) = \int_a^b [|y'|^p - c(t)|y|^p]\, dt \tag{5}$$

in the class of (sufficiently smooth) functions satisfying $y(a) = 0 = y(b)$. In particular, equation (1) is *disconjugate* in $[a, b]$, i. e., any nontrivial solution has at most one zero in $[a, b]$) if and only if $\mathscr{F}(y; a, b) > 0$ for every nontrivial function $y \in W^{1,p}(a, b)$ with $y(a) = 0 = y(b)$. More details will be given in the next section. The variational approach was used, e. g., in [**6**].

The main result of this paper is the oscillation criterion which answers in the affirmative way the conjecture posed in [**2**].

This paper is organized as follows. In the Section 2, we recall the properties of equation (4) needed for our investigation. We also formulate some auxiliary results and give known results. In Section 3, we present the main result of the paper, an oscillation criterion for (1). Section 4 contains some remarks about possible extensions of the results of this paper.

2. Preliminaries

First we return to Euler equation (4). In the critical case $\gamma = \gamma_p$, this equation possesses the solution $\tilde{x}(t) = t^{\frac{p-1}{p}}$ and any linearly independent solution satisfies the relation

$$x(t) \sim t^{\frac{p-1}{p}} \lg^{\frac{2}{p}} t,$$

where $f_1 \sim f_2$ for a pair of functions f_1, f_2 means that $L := \lim_{t\to\infty} \frac{f_1(t)}{f_2(t)}$ exists and $0 < |L| < \infty$.

Now we formulate exactly the relationship between the positivity of the functional $\mathscr{F}$ given in (5) and diconjugacy of (1). This statement is proved, e. g., in [**6**] and we formulate it here in a slightly modified form as needed in the next section.

Lemma 1. *Equation* (1) *is oscillatory if and only if for every* $T \in \mathbb{R}$ *there exists* $0 \not\equiv y \in W_0^{1,p}(T, \infty)$ *such that*

$$\mathscr{F}(y; T, \infty) = \int_T^\infty [|y'(t)|^p - c(t)|y(t)|^p]\, dt \le 0.$$

The following lemma sumarizes the results of Došlý and Lomtatidze [**2**] where (1) is viewed as a perturbation of the Euler equation (4).

Lemma 2. *The following statements hold.*

(i) *Suppose that* $c(t) \ge 0$ *and*

$$\liminf_{t\to\infty} \lg t \int_t^\infty \left(c(s) - \frac{\gamma_p}{s^p}\right) s^{p-1}\, ds > \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}. \tag{6}$$

Then (1) *is oscillatory.*

(ii) *If*

$$\limsup_{t\to\infty} \lg t \int_t^\infty \left(c(s) - \frac{\gamma_p}{s^p}\right) s^{p-1}\, ds < \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} \tag{7}$$

and

$$\liminf_{t\to\infty} \lg t \int_t^\infty \left(c(s) - \frac{\gamma_p}{s^p}\right) s^{p-1}\, ds > -\frac{3}{2}\left(\frac{p-1}{p}\right)^{p-1}, \tag{8}$$

then (1) *is nonoscillatory.*

Recall also the results of the paper by Elbert and Schneider based on the connection of oscillatory behaviour between the half-linear differential equation

$$(\Phi(x'))' + \left(\frac{p-1}{p}\right)^p \frac{1}{t^p}\left[1 + \frac{p}{2(p-1)}\delta(t)\right]\Phi(x) = 0 \tag{9}$$

(where $\delta(t)$ is a piecewise continuous function on (t_0, ∞), for some $t_0 \ge 0$) and the linear differential equation

$$(tz')' + \frac{\delta(t)}{t} z = 0. \tag{10}$$

Lemma 3. *Suppose that* $\int^\infty \frac{\delta(t)}{t}\, dt$ *is convergent and* $\int_t^\infty \frac{\delta(s)}{s}\, ds \ge 0$ *for large* t.

(i) *If* $p \ge 2$ *and the linear differential equation* (10) *is nonoscillatory, then the half-linear differential equation* (9) *is also nonoscillatory.*

(ii) *If* $p \in (1, 2]$ *and equation* (9) *is nonoscillatory, then equation* (10) *is also nonoscillatory.*

Note that the equivalence of nonoscillation of (9) and (10) is an open problem. However, under some restrictions on the function $\delta(t)$ or on the corresponding linear differential equation (10), one can establish a closer connection between (9) and (10). For more details, see [**5**].

3. Oscillation criterion

In this section, we present the main result of the paper - an oscillation criterion for (1). This equation is written in the form

$$(\Phi(x'))' + \frac{\gamma_p}{t^p}\Phi(x) + \left(c(t) - \frac{\gamma_p}{t^p}\right)\Phi(x) = 0 \tag{11}$$

and it is viewed as a perturbation of the generalized Euler equation (4).

Theorem 1. *Suppose that*

$$\liminf_{t\to\infty} \frac{1}{\lg t}\int_1^t \left(c(s) - \frac{\gamma_p}{s^p}\right) s^{p-1}\lg^2 s\, ds > 2\left(\frac{p-1}{p}\right)^{p-1}. \tag{12}$$

Then (1) *is oscillatory.*

Proof. According to Lemma 1, it suffices to find, for any $T \in \mathbb{R}$, a function $y \in W^{1,p}(t,\infty)$ with a compact support in (T,∞), say $[t_0,t_3]$, such that the functional $\mathscr{F}$ given by (5) is negative. To this end, let $T \in \mathbb{R}$ be arbitrary and $T < t_0 < t_1 < t_2 < t_3$. Further, let f be arbitrary sufficiently smooth function satisfying the boundary conditions

$$f(t_0) = 0, \quad f(t_1) = h(t_1)$$

and let

$$g(t) = h(t_2)\frac{t^{\frac{p-1}{p}}\lg^{\frac{2}{p}}\frac{t_3}{t}}{t_2^{\frac{p-1}{p}}\lg^{\frac{2}{p}}\frac{t_3}{t_2}},$$

where $h(t) = t^{\frac{p-1}{p}}\lg^{\frac{2}{p}} t$. Define a test function y as follows:

$$y(t) = \begin{cases} 0 & \text{if} \quad T \le t \le t_0, \\ f(t) & \text{if} \quad t_0 \le t \le t_1, \\ h(t) & \text{if} \quad t_1 \le t \le t_2, \\ g(t) & \text{if} \quad t_2 \le t \le t_3, \\ 0 & \text{if} \quad t \ge t_3. \end{cases}$$

For the computation to follow, we put

$$\hat{g}(t) = t^{\frac{p-1}{p}}\lg^{\frac{2}{p}}\frac{t_3}{t}$$

and

$$K(t_2,t_3) = t_2^{\frac{p-1}{p}}\lg^{\frac{2}{p}}\frac{t_3}{t_2}.$$

Then, the function g can be written in the form

$$g(t) = \frac{h(t_2)}{K(t_2,t_3)}\hat{g}(t).$$

Further, we set

$$\begin{aligned}
L = \mathscr{F}(f;t_0,t_1) &= \int_{t_0}^{t_1}\left[|f'(t)|^p - \frac{\gamma_p}{t^p}|f(t)|^p\right]dt - \int_{t_0}^{t_1}\left[c(t) - \frac{\gamma_p}{t^p}\right]|f(t)|^p\,dt,\\
G(h;t_1,t_2) &= \int_{t_1}^{t_2}\left[|h'(t)|^p - \frac{\gamma_p}{t^p}|h(t)|^p\right]dt,\\
G(g;t_2,t_3) &= \int_{t_2}^{t_3}\left[|g'(t)|^p - \frac{\gamma_p}{t^p}|g(t)|^p\right]dt,\\
H(h;t_1,t_2) &= \int_{t_1}^{t_2}\left[c(t) - \frac{\gamma_p}{t^p}\right]|h(t)|^p\,dt,\\
H(g;t_2,t_3) &= \int_{t_2}^{t_3}\left[c(t) - \frac{\gamma_p}{t^p}\right]|g(t)|^p\,dt\\
2 &= \int_{t_2}^{t_3}\left[c(t) - \frac{\gamma_p}{t^p}\right]|h(t)|^p\left|\frac{g(t)}{h(t)}\right|^p\,dt.
\end{aligned}$$

Then we have

$$\begin{aligned}
\mathscr{F}(y;t_0,t_3) &= \int_{t_0}^{t_3}\left[|y'(t)|^p - c(t)|y(t)|^p\right]dt\\
&= \int_{t_0}^{t_3}\left[|y'(t)|^p - \frac{\gamma_p}{t^p}|y(t)|^p\right]dt - \int_{t_0}^{t_3}\left[c(t) - \frac{\gamma_p}{t^p}\right]y(t)|^p\,dt\\
&= \int_{t_0}^{t_1}\left[|f'(t)|^p - \frac{\gamma_p}{t^p}t)|^p\right]dt - \int_{t_0}^{t_1}\left[c(t) - \frac{\gamma_p}{t^p}\right]|f(t)|^p\,dt\\
&+ \int_{t_1}^{t_2}\left[|h'(t)|^p - \frac{\gamma_p}{t^p}|h(t)|^p\right]dt + \int_{t_2}^{t_3}\left[|g'(t)|^p - \frac{\gamma_p}{t^p}|g(t)|^p\right]dt\\
&- \int_{t_1}^{t_2}\left[c(t) - \frac{\gamma_p}{t^p}\right]|h(t)|^p\,dt - \int_{t_2}^{t_3}\left[c(t) - \frac{\gamma_p}{t^p}\right]|g(t)|^p\,dt\\
&= \mathscr{F}(f;t_0,t_1) + G(h;t_1,t_2) + G(g;t_2,t_3) - H(h;t_1,t_2) - H(g;t_2,t_3).
\end{aligned}$$

Let us start with computing $G(h;t_1,t_2)$. Since

$$h'(t) = \frac{p-1}{p}t^{-\frac{1}{p}}\lg^{\frac{2}{p}}t\left(1 + \frac{2}{(p-1)\lg t}\right),$$

then, using the asymptotic formula

$$(1+x)^p = 1 + px + \frac{1}{2}p(p-1)x^2 + o(x^2) \quad \text{as } x \to 0, \tag{13}$$

we get

$$|h'(t)|^p = \left(\frac{p-1}{p}\right)^p t^{-1}\lg^2 t\left[1+\frac{2p}{(p-1)\lg t}+\frac{2p}{(p-1)\lg^2 t}+o\left(\frac{1}{\lg^2 t}\right)\right].$$

Hence,

$$\begin{aligned}
G(h;t_1,t_2) &= \int_{t_1}^{t_2}\left[|h'(t)|^p-\frac{\gamma_p}{t^p}|h(t)|^p\right]dt\\
&= \int_{t_1}^{t_2}\left[\left(\frac{p-1}{p}\right)^p t^{-1}\lg^2 t\left\{1+\frac{2p}{(p-1)\lg t}+\frac{2p}{(p-1)\lg^2 t}+o\left(\frac{1}{\lg^2 t}\right)\right\}\right.\\
&\quad\left.-\frac{1}{t^p}\left(\frac{p-1}{p}\right)^p t^{p-1}\lg^2 t\right]dt\\
&= 2\left(\frac{p-1}{p}\right)^{p-1}\int_{t_1}^{t_2}\left[\frac{1}{t}\lg t+\frac{1}{t}+o\left(\frac{1}{t}\right)\right]dt\\
&= 2\left(\frac{p-1}{p}\right)^{p-1}\left[\lg t|_{t_1}^{t_2}+\frac{1}{2}\lg^2 t|_{t_1}^{t_2}+o(\lg t_2)\right]\\
&= 2\left(\frac{p-1}{p}\right)^{p-1}\lg t|_{t_1}^{t_2}+\left(\frac{p-1}{p}\right)^{p-1}\lg^2 t|_{t_1}^{t_2}+o(\lg t_2)
\end{aligned}$$

as $t_2\to\infty$. Concerning $G(h;t_1,t_2)$, we obtain

$$\begin{aligned}
G(g;t_2,t_3) &= \frac{h^p(t_2)}{K^p(t_2,t_3)}\int_{t_2}^{t_3}\left[|\hat g'(t)|^p-\frac{\gamma_p}{t^p}|\hat g(t)|^p\right]dt\\
&= \frac{\lg^2 t_2}{\lg^2\frac{t_3}{t_2}}\left(\frac{p-1}{p}\right)^p\int_{t_2}^{t_3}\left\{\frac{1}{t}\lg^2\frac{t_3}{t}\left(1-\frac{2}{(p-1)\lg t}\right)^p-\frac{1}{t}\lg^2\frac{t_3}{t}\right\}dt \qquad (14)\\
&= 2\left(\frac{p-1}{p}\right)^{p-1}\frac{\lg^2 t_2}{\lg^2 t|_{t_2}^{t_3}}\left[\lg t|_{t_2}^{t_3}-\lg t_3\lg t|_{t_2}^{t_3}+\frac{1}{2}\lg^2 t|_{t_2}^{t_3}+o(\lg t_3)\right].
\end{aligned}$$

In the last line of the previous computation we used again the formula (13). Now, let t_2 be fixed for a moment. We will show that

$$\lim_{t_3\to\infty} G(g;t_2,t_3) = -\left(\frac{p-1}{p}\right)^{p-1}\lg^2 t_2. \qquad (15)$$

From (14), we have

$$\begin{aligned}
&\frac{\lg t\big|_{t_2}^{t_3} - \lg t_3 \lg t\big|_{t_2}^{t_3} + \frac{1}{2}\lg^2 t\big|_{t_2}^{t_3} + o(\lg t_3)}{\lg^2 t\big|_{t_2}^{t_3}} \\
&\quad = \frac{\lg t_3 - \lg t_2 - \lg t_3(\lg t_3 - \lg t_2) + \frac{1}{2}\lg^2 t_3 - \frac{1}{2}\lg^2 t_2 + o(\lg t_3)}{\lg^2 t_3 - \lg^2 t_2} \\
&\quad = \frac{-\frac{1}{2}\lg^2 t_3 + \lg t_3 - \lg t_2 + \lg t_3 \lg t_2 - \frac{1}{2}\lg^2 t_2 + o(\lg t_3)}{\lg^2 t_3 - \lg^2 t_2} \\
&\quad = \frac{-\frac{1}{2} + \frac{1}{\lg t_3} - \frac{\lg t_2}{\lg^2 t_3} + \frac{\lg t_2}{\lg t_3} - \frac{1}{2}\frac{\lg^2 t_2}{\lg^2 t_3} + o\left(\frac{1}{\lg t_3}\right)}{1 - \frac{\lg^2 t_2}{\lg^2 t_3}} \longrightarrow -\frac{1}{2}
\end{aligned}$$

as $t_3 \to \infty$ and, thus, (15) holds indeed. In the next part of the proof, we show that the function $\frac{g}{h}$ is strictly monotonic on (t_2, t_3). Since

$$\frac{g(t)}{h(t)} = \frac{h(t_2)}{K(t_2, t_3)} \frac{\hat{g}(t)}{h(t)}$$

and

$$\left(\frac{\hat{g}(t)}{h(t)}\right)' = \left(\frac{\lg^{\frac{2}{p}} \frac{t_3}{t}}{\lg^{\frac{2}{p}} t}\right)' = \left[\left(\frac{\lg \frac{t_3}{t}}{\lg t}\right)^{\frac{2}{p}}\right]' = -\frac{2}{p}\left(\frac{\lg \frac{t_3}{t}}{\lg t}\right)^{\frac{2-p}{p}} \frac{\lg t_3}{t \lg^2 t} < 0,$$

we see that $\left(\frac{g}{h}\right)' < 0$. Hence, $\frac{g}{h}$ is monotonic on (t_2, t_3). Now, by the second mean value theorem of the integral calculus, there exists $\xi \in (t_2, t_3)$ such that

$$\begin{aligned}
\int_{t_2}^{t_3} \left[c(t) - \frac{\gamma_p}{t^p}\right] |g(t)|^p \, dt &= \int_{t_2}^{t_3} \left[c(t) - \frac{\gamma_p}{t^p}\right] |h(t)|^p \left|\frac{g(t)}{h(t)}\right|^p \, dt \\
&= \int_{t_2}^{\xi} \left[c(t) - \frac{\gamma_p}{t^p}\right] |h(t)|^p.
\end{aligned}$$

Consequently,

$$\begin{aligned}
H(h; t_1, t_2) + H(g; t_2, t_3) &= \int_{t_1}^{t_3} \left(c(t) - \frac{\gamma(t)}{t^p}\right) |h(t)|^p \, dt \\
&= \int_{t_1}^{\xi} \left(c(t) - \frac{\gamma(t)}{t^p}\right) |h(t)|^p \, dt \\
&= \int_{t_1}^{\xi} \left(c(t) - \frac{\gamma(t)}{t^p}\right) t^{p-1} \lg^2 t \, dt.
\end{aligned}$$

Let us now return to the computation of the functional $\mathscr{F}$.

$$\begin{aligned}\mathscr{F}(y;t_0,t_3) &= L + G(h;t_1,t_2) + G(g;t_2,t_3) - \int_{t_1}^{\xi}\left(c(t)-\frac{\gamma_p}{t^p}\right)t^{p-1}\lg^2 t\,dt\\ &= \lg t_2\left[\frac{L}{\lg t_2} + 2\left(\frac{p-1}{p}\right)^{p-1} - 2\left(\frac{p-1}{p}\right)^{p-1}\frac{\lg t_1}{\lg t_2} + \left(\frac{p-1}{p}\right)^{p-1}\lg t_2\right.\\ &\quad -\left(\frac{p-1}{p}\right)^{p-1}\frac{\lg^2 t_1}{\lg t_2} + o(1) + \frac{G(g;t_2,t_3)}{\lg t_2}\\ &\quad \left.-\frac{1}{\lg t_2}\int_{t_1}^{\xi}\left(c(t)-\frac{\gamma_p}{t^p}\right)t^{p-1}\lg^2 t\,dt\right]\\ &\le \lg t_2\left[\frac{L}{\lg t_2} + 2\left(\frac{p-1}{p}\right)^{p-1} + \left(\frac{p-1}{p}\right)^{p-1}\lg t_2 + o(1) + \frac{G(g;t_2,t_3)}{\lg t_2}\right.\\ &\quad \left.-\frac{1}{\lg\xi}\int_1^{\xi}\left(c(t)-\frac{\gamma_p}{t^p}\right)t^{p-1}\lg^2 t\,dt + \frac{1}{\lg t_2}\int_1^{t_1}\left(c(t)-\frac{\gamma_p}{t^p}\right)t^{p-1}\lg^2 t\,dt\right].\end{aligned}$$

Now, let $\varepsilon > 0$ be such that the lower limit in (12) is greater than $2\left(\frac{p-1}{p}\right)^{p-1} + 6\varepsilon$ and let $t_1 > t_0$ be arbitrary. We can choose $t_2 > t_1$ in such a way that $\frac{L}{\lg t_2} < \varepsilon$, the term $o(1)$ is $< \varepsilon$ and

$$\frac{1}{\lg\xi}\int_1^{\xi}\left(c(s)-\frac{\gamma_p}{s^p}\right)s^{p-1}\lg^2 s\,ds > 2\left(\frac{p-1}{p}\right)^{p-1} + 5\varepsilon \quad \text{whenever } \xi > t_2.$$

Finally, according to (15), we take $t_3 > t_2$ such that

$$\frac{G(g;t_2,t_3)}{\lg t_2} < -\left(\frac{p-1}{p}\right)^{p-1}\lg t_2 + \varepsilon.$$

Summarizing all the estimates above, we have

$$\begin{aligned}\mathscr{F}(y;t_0,t_3) \;\le\; &\lg t_2\left[\varepsilon + 2\left(\frac{p-1}{p}\right)^{p-1} + \left(\frac{p-1}{p}\right)^{p-1}\lg t_2 + \varepsilon\right.\\ &\left.-\left(\frac{p-1}{p}\right)^{p-1}\lg t_2 + \varepsilon - 2\left(\frac{p-1}{p}\right)^{p-1} - 5\varepsilon + \varepsilon\right] = -\varepsilon < 0\end{aligned}$$

and, by Lemma 1, equation (1) is oscillatory. □

4. Remarks and comments

In this concluding section, we formulate some open problems and suggestions for the further investigation of problems similar to the one solved in Theorem 1.

4.1

Corollary 1 in [**2**] states that (1) is oscillatory if (6) holds, i. e., the constant $\frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$ is 4 times less than the constant $2\left(\frac{p-1}{p}\right)^{p-1}$ in (12). This is a typical phenomenon of oscillation theory. Oscillation criteria proved by using the variational principle contain the oscillation constant which is 4–times larger than the constant in the same criterion but proved by the Riccati technique. Based on this observation, we conjecture that the constant $2\left(\frac{p-1}{p}\right)^{p-1}$ in (12) can be replaced by the constant $\frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$ and the statement of Theorem 1 remains valid. The proof of this modified statement of Theorem 1 via the Riccati technique is the subject of the present investigation.

4.2

Corollary 2 in [**2**] gives the nonoscillation counterpart of the above mentioned Corollary 1 of that paper: Equation (1) is nonoscillatory provided (7) and (8) hold. We conjecture that a similar nonoscillation counterpart can be formulated also for Theorem 1 as follows.

Conjecture 1. *Suppose that*

$$\limsup_{t\to\infty} \frac{1}{\lg t} \int_t^\infty \left(c(s) - \frac{\gamma_p}{s^p}\right) s^{p-1} \lg^2 s \, ds < \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$$

and

$$\liminf_{t\to\infty} \frac{1}{\lg t} \int_t^\infty \left(c(s) - \frac{\gamma_p}{s^p}\right) s^{p-1} \lg^2 s \, ds > -\frac{3}{2}\left(\frac{p-1}{p}\right)^{p-1}.$$

Then (1) *is nonoscillatory.*

4.3

In oscillation and nonoscillation criteria of the above-mentioned papers [**2, 5**], where the half-linear equation

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0 \tag{16}$$

is viewed as a perturbation of a nonoscillatory equation

$$(r(t)\Phi(x'))' + \tilde{c}(t)\Phi(x) = 0, \tag{17}$$

a function h appeared which *is a solution* of (16) (in case $r(t) \equiv 1$ and $\tilde{c}(t) = \frac{\gamma_p}{t^p}$ it is $h(t) = t^{\frac{p-1}{p}}$). In our main result, Theorem 1, the function

$$h(t) = t^{\frac{p-1}{p}} \lg^{\frac{2}{p}} t$$

is *not* a solution of (4) (equation (1) is viewed as a perturbation of this Euler equation), but it is only close to a solution of this equation, in a certain sense. This

observation suggests another direction for the next investigation of (16) as a perturbation of (17)—to look for (non)oscillation criteria where $\int^t (c(s) - \tilde{c}(s))\,|h(s)|^p\,ds$ or $\int_t^\infty (c(s) - \tilde{c}(s))\,|h(s)|^p\,ds$ appear and h generally is *not* a solution of (17) but it is only close to a solution of (17), in a certain sense.

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