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THE METHOD OF LOWER AND UPPER SOLUTIONS TO THE DARBOUX PROBLEM FOR PARTIAL DIFFERENTIAL INCLUSIONS

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ABSTRACT. The method of lower and upper solutions combined with a fixed point theorem for condensing multivalued maps due to Martelli is used to investigate the existence of solutions to an hyperbolic differential inclusion.

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1. INTRODUCTION

This note deals with the existence of solutions to the following hyperbolic differential inclusion (Darboux problem):

$$\frac{\partial^2 u(x, y)}{\partial x \partial y} \in F(x, y, u(x, y)), \quad (x, y) \in J \times J, \quad (1.1)$$

$$u(x, 0) = f(x), \quad u(0, y) = g(y) \quad (1.2)$$

where $J = [0, T]$, $F : J \times J \times \mathbb{R} \rightarrow 2^{\mathbb{R}}$ is a compact and convex valued multivalued map and $f, g : J \rightarrow \mathbb{R}$.

The method of upper and lower solutions has been successfully applied to study the existence of multiple solutions for initial and boundary value problems of the first and second order partial differential equations. This method has been used only in the context of single-valued partial differential equations. We refer to the books by Carl and Heikkilae [6], Heikkilae and Lakshmikantham [10], Ladde et al. [15], to the papers by Agarwal [1], Agarwal and Sheng [2], Blakley and Pandit [5], Lakshmikantham and Pandit [14], Pandit [18] and the references cited therein.

In this note we shall establish an existence result for the problem (1.1), (1.2). Our approach is based on the existence of upper and lower solutions and on a fixed point theorem for condensing multivalued maps due to Martelli [17]. Other existence results for the Darboux problem for partial differential equations and inclusions by

the aid of other tools can be found in the recent monograph by Kamont [12] and in the papers [4, 7, 8, 13, 19].

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper. In the sequel, we set $\mathcal{J} = J \times J$.

In what follows, $C(\mathcal{J}, \mathbb{R})$ is the Banach space of continuous functions from $\mathcal{J}$ into $\mathbb{R}$ with the norm $\|z\|_\infty := \sup\{|z(x, y)| : (x, y) \in \mathcal{J}\}$ for each $z \in C(\mathcal{J}, \mathbb{R})$.

The condition

$$u \leq \bar{u} \quad \text{if and only if} \quad u(x, y) \leq \bar{u}(x, y) \quad \text{for all} \quad (x, y) \in \mathcal{J}$$

defines a partial ordering in $C(\mathcal{J}, \mathbb{R})$. If $\underline{u}, \bar{u} \in C(\mathcal{J}, \mathbb{R})$ and $\underline{u} \leq \bar{u}$, we put

$$[\underline{u}, \bar{u}] = \{u \in C(\mathcal{J}, \mathbb{R}) : \underline{u} \leq u \leq \bar{u}\}.$$

Let $(X, \|\cdot\|)$ be a Banach space. A multivalued map $G : X \rightarrow 2^X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \bigcup_{x \in B} G(x)$ is bounded in X for any bounded set B of X (i.e. $\sup_{x \in B} \{\sup\{\|y\| : y \in G(x)\}\} < \infty$). G is called upper semicontinuous (u. s. c.) on X if for each $x_* \in X$ the set $G(x_*)$ is a nonempty, closed subset of X , and if for each open set B of X containing $G(x_*)$, there exists an open neighbourhood V of x_* such that $G(V) \subseteq B$. G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$. If the multivalued map G is completely continuous with nonempty compact values, then G is u. s. c. if and only if G has a closed graph (i.e. $x_n \rightarrow x_*, y_n \rightarrow y_*, y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

In what follows, $CC(X)$ denotes the set of all nonempty compact and convex subsets of X .

A multivalued map $G : \mathcal{J} \rightarrow CC(X)$ is said to be measurable if for each $w \in X$ the function $Y : \mathcal{J} \rightarrow \mathbb{R}$ defined by

$$Y(x, y) = d(w, G(x, y)) = \inf\{\|w - v\| : v \in G(x, y)\}$$

is measurable.

Definition 2.1. A multivalued map $F : \mathcal{J} \times \mathbb{R} \rightarrow 2^{\mathbb{R}}$ is said to be an L^1 -Carathéodory map if

- (i) $(x, y) \mapsto F(x, y, u)$ is measurable for each $u \in \mathbb{R}$;
- (ii) $u \mapsto F(x, y, u)$ is upper semicontinuous for almost all $(x, y) \in \mathcal{J}$;
- (iii) For each $k > 0$, there exists $h_k \in L^1(\mathcal{J}, \mathbb{R}_+)$ such that

$$\|F(x, y, u)\| = \sup\{|v| : v \in F(x, y, u)\} \leq h_k(x, y)$$

for all $|u| \leq k$ and for almost all $(x, y) \in \mathcal{J}$.

An upper semi-continuous map $G : X \rightarrow 2^X$ is said to be condensing if for any subset $B \subseteq X$ with $\alpha(B) \neq 0$, we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompactness. For properties of the Kuratowski measure, we refer to Banas and Goebel [3].

We remark that a completely continuous multivalued map is the easiest example of a condensing map. For more details on multivalued maps see the books by Deimling [9] and Hu and Papageorgiou [11].

Definition 2.2. By a solution of (1.1), (1.2) we mean a function $u(\cdot, \cdot) \in C(\mathcal{J}, \mathbb{R})$ such that there exists $v \in L^1(\mathcal{J}, \mathbb{R})$ for which we have

$$u(x, y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v(t, s) dt ds \text{ for each } (x, y) \in \mathcal{J}$$

and with $v(t, s) \in F(t, s, u(t, s))$ a. e. on $\mathcal{J}$.

Now, we introduce the concept of lower and upper solutions to (1.1), (1.2). It will be the basic tool in the approach that follows.

Definition 2.3. A function $\underline{u}(\cdot, \cdot) \in C(\mathcal{J}, \mathbb{R})$ is said to be a lower solution of (1.1), (1.2) if there exists $v_1 \in L^1(\mathcal{J}, \mathbb{R})$ such that $v_1(t, s) \in F(t, s, \underline{u}(t, s))$ a. e. on $\mathcal{J}$, and for which we have

$$\underline{u}(x, y) \leq f(x) + g(y) - f(0) + \int_0^x \int_0^y v_1(t, s) dt ds \text{ for each } (x, y) \in \mathcal{J}.$$

A function $\bar{u}(\cdot, \cdot) \in C(\mathcal{J}, \mathbb{R})$ is said to be an upper solution of (1.1), (1.2) if there exists $v_2 \in L^1(\mathcal{J}, \mathbb{R})$ such that $v_2(t, s) \in F(t, s, \bar{u}(t, s))$ a. e. on $\mathcal{J}$, and for which we have

$$\bar{u}(x, y) \geq f(x) + g(y) - f(0) + \int_0^x \int_0^y v_2(t, s) dt ds \text{ for each } (x, y) \in \mathcal{J}.$$

For the multivalued map F and for each $u \in C(\mathcal{J}, \mathbb{R})$ we define $S_{F,u}^1$ by

$$S_{F,u}^1 = \{v \in L^1(\mathcal{J}, \mathbb{R}) : v(t, s) \in F(t, s, u(t, s)) \text{ for a. e. } (t, s) \in \mathcal{J}\}.$$

Our considerations are based on the following lemmas.

Lemma 2.4 ([16]). *Let F be a multivalued map satisfying (H1) and let Γ be a linear continuous mapping from $L^1(\mathcal{J}, \mathbb{R})$ to $C(\mathcal{J}, \mathbb{R})$, then the operator*

$$\Gamma \circ S_F : C(\mathcal{J}, \mathbb{R}) \rightarrow CC(C(\mathcal{J}, \mathbb{R})), \quad u \mapsto (\Gamma \circ S_F)(u) := \Gamma(S_{F,u})$$

is a closed graph operator in $C(\mathcal{J}, \mathbb{R}) \times C(\mathcal{J}, \mathbb{R})$.

Lemma 2.5 ([17]). *Let X be a Banach space and $N : X \rightarrow CC(X)$ be an u. s. c. and condensing map. If the set*

$$\Omega := \{u \in X : \lambda u \in N(u) \text{ for some } \lambda > 1\}$$

is bounded, then N has a fixed point.

We will need the following hypotheses:

- (H1) $F : \mathcal{J} \times \mathbb{R} \rightarrow CC(\mathbb{R})$ is an L^1 -Carathéodory multivalued map.
- (H2) there exist $\underline{u}$ and $\bar{u}$ in $C(\mathcal{J}, \mathbb{R})$ as lower and upper solutions, respectively, to the problem (1.1), (1.2) such that $\underline{u} \leq \bar{u}$.
- (H3) The functions $f, g : J \rightarrow \mathbb{R}$ are continuous with $f(0) = g(0)$.

3. MAIN RESULT

Now, we are able to state and prove our main theorem.

Theorem 3.1. *Assume that hypotheses (H1)–(H3) hold. Then problem (1.1), (1.2) has at least one solution on $\mathcal{J}$.*

Proof. Let us transform problem (1.1), (1.2) into a fixed point problem. Consider the following modified problem

$$\frac{\partial^2 u(x, y)}{\partial x \partial y} \in F(x, y, (\tau u)(x, y)), \quad (x, y) \in \mathcal{J} \quad (3.1)$$

$$u(x, 0) = f(x), \quad u(0, y) = g(y) \quad (3.2)$$

where $\tau : C(\mathcal{J}, \mathbb{R}) \rightarrow C(\mathcal{J}, \mathbb{R})$ is the truncation operator defined by

$$(\tau u)(x, y) = \begin{cases} \underline{u}(x, y) & \text{if } u(x, y) < \underline{u}(x, y); \\ u(x, y) & \text{if } \underline{u}(x, y) \leq u \leq \bar{u}(x, y); \\ \bar{u}(x, y) & \text{if } \bar{u}(x, y) < u(x, y). \end{cases}$$

Set

$$C_0(\mathcal{J}, \mathbb{R}) := \{u \in C(\mathcal{J}, \mathbb{R}) : u(x, 0) = f(x), u(0, y) = g(y)\}.$$

A solution to (3.1), (3.2) is a fixed point of the operator $N : C_0(\mathcal{J}, \mathbb{R}) \rightarrow 2^{C_0(\mathcal{J}, \mathbb{R})}$ defined by

$$N(u) := \left\{ h \in C_0(\mathcal{J}, \mathbb{R}) : h(x, y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v(t, s) dt ds, \right. \\ \left. v \in \tilde{S}_{F, \tau u}^1 \right\}$$

where

$$\tilde{S}_{F, \tau u}^1 = \{v \in S_{F, \tau u}^1 : v(t, s) \geq v_1(t, s) \text{ a. e. on } A_1 \text{ and } v(t, s) \leq v_2(t, s) \text{ a. e. on } A_2\},$$

$$S_{F, \tau u}^1 = \{v \in L^1(\mathcal{J}, \mathbb{R}) : v(t, s) \in F(t, s, (\tau u)(t, s)) \text{ for a. e. } (t, s) \in \mathcal{J}\},$$

$$A_1 = \{(t, s) \in \mathcal{J} : u(t, s) < \underline{u}(t, s) \leq \bar{u}(t, s)\},$$

$$A_2 = \{(t, s) \in \mathcal{J} : \underline{u}(t, s) \leq \bar{u}(t, s) < u(t, s)\}.$$

Remark 3.2. (i) For every $u \in C(\mathcal{J}, \mathbb{R})$, the set $S_{F, u}^1$ is nonempty (see Lasota and Opial [16]).

- (ii) For every $u \in C(\mathcal{J}, \mathbb{R})$, the set $\tilde{S}_{F,\tau u}^1$ is nonempty. Indeed, by (i) there exists $v \in S_{F,u}^1$. Set $w = v_1\chi_{A_1} + v_2\chi_{A_2} + v\chi_{A_3}$, where

$$A_3 = \{(t, s) \in \mathcal{J} : \underline{u}(t, s) \leq u(t, s) \leq \bar{u}(t, s)\}.$$

Then, by decomposability, $w \in \tilde{S}_{F,\tau u}^1$.

We shall show that N is a completely continuous multivalued map, u. s. c. with convex closed values. The proof will be divided into several steps.

Remark 3.3. It is clear that the fixed points of N are solutions to (3.1), (3.2).

STEP 1: $N(u)$ is convex for each $u \in C(\mathcal{J}, \mathbb{R})$.

Indeed, if h_1, h_2 belong to $N(u)$, then there exist $v^1, v^2 \in \tilde{S}_{F,\tau u}$ such that for each $(x, y) \in \mathcal{J}$ we have

$$h_i(x, y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v^i(t, s) dt ds, \quad i = 1, 2.$$

Let $0 \leq \alpha \leq 1$. Then for each $(x, y) \in \mathcal{J}$ we have

$$\begin{aligned} (\alpha h_1 + (1 - \alpha)h_2)(x, y) &= f(x) + g(y) - f(0) \\ &\quad + \int_0^x \int_0^y [\alpha v^1(t, s) + (1 - \alpha)v^2(t, s)] ds. \end{aligned}$$

Since $\tilde{S}_{F,\tau u}$ is convex (because F has convex values), we have $\alpha h_1 + (1 - \alpha)h_2 \in N(u)$.

STEP 2: N is bounded on bounded sets of $C(\mathcal{J}, \mathbb{R})$.

Indeed, it is enough to show that there exists a positive constant c such that for each $h \in N(u)$, $u \in B_r = \{u \in C(\mathcal{J}, \mathbb{R}) : \|u\|_\infty \leq r\}$ one has $\|h\|_\infty \leq c$.

If $h \in N(u)$, then there exists $v \in \tilde{S}_{F,\tau u}$ such that for each $(x, y) \in \mathcal{J}$ we have

$$h(x, y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v(t, s) dt ds.$$

By (H1) we have for each $(x, y) \in \mathcal{J}$ that

$$|h(x, y)| \leq |f(x)| + |g(y)| + |f(0)| + \int_0^x \int_0^y h_r(t, s) dt ds.$$

Then

$$\|h\|_\infty \leq \|f\|_\infty + \|g\|_\infty + |f(0)| + \int_0^T \int_0^T h_r(t, s) dt ds = c.$$

STEP 3: N sends bounded sets of $C(\mathcal{J}, \mathbb{R})$ into equicontinuous sets.

Let $(x_1, y_1), (x_2, y_2) \in \mathcal{J}$, $x_1 < x_2$, $y_1 < y_2$ and B_r be a bounded set of $C(\mathcal{J}, \mathbb{R})$. For each $u \in B_r$ and $h \in N(u)$, there exists $v \in \tilde{S}_{F,\tau u}$ such that

$$h(x, y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v(t, s) dt ds.$$

Thus we obtain

$$\begin{aligned} |h(x_2, y_2) - h(x_1, y_1)| &\leq |f(x_2) - f(x_1)| + |g(y_2) - g(y_1)| + \int_{x_1}^{x_2} \int_{y_1}^{y_2} |v(t, s)| dt ds \\ &\leq |f(x_2) - f(x_1)| + |g(y_2) - g(y_1)| + \int_{x_1}^{x_2} \int_{y_1}^{y_2} h_r(t, s) dt ds. \end{aligned}$$

As $(x_2, y_2) \rightarrow (x_1, y_1)$, the right-hand side of the above inequality tends to zero.

Taking into account Step 2, Step 3, and the Arzelà–Ascoli theorem, we conclude that N is completely continuous and therefore a condensing multivalued map.

STEP 4: N has closed graph.

Let $u_n \rightarrow u_*$, $h_n \in N(u_n)$, and $h_n \rightarrow h_*$. We shall prove that $h_* \in N(u_*)$. The inclusion $h_n \in N(u_n)$ means that there exists $v_n \in \tilde{S}_{F, \tau u_n}$ such that

$$h_n(x, y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v_n(t, s) dt ds, \quad (x, y) \in \mathcal{J}.$$

We must prove that there exists $g_* \in \tilde{S}_{F, \tau u_*}$ such that

$$h_*(x, y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v_*(t, s) dt ds, \quad (x, y) \in \mathcal{J}.$$

Now, we consider the linear continuous operator $\Gamma : L^1(\mathcal{J}, \mathbb{R}) \rightarrow C(\mathcal{J}, \mathbb{R})$,

$$v \mapsto \Gamma(v)(x, y) = \int_0^x \int_0^y v(t, s) dt ds, \quad (x, y) \in \mathcal{J}.$$

From Lemma 2.4, it follows that $\Gamma \circ \tilde{S}_F$ is a closed graph operator.

Clearly we have

$$\|(h_n(x, y) - f(x) - g(y) + f(0)) - (h_*(x, y) - f(x) - g(y) + f(0))\|_\infty \rightarrow 0$$

as $n \rightarrow \infty$. Moreover, from the definition of Γ , we have

$$(h_n(x, y) - f(x) - g(y) + f(0)) \in \Gamma(\tilde{S}_{F, \tau u_n}).$$

Since $u_n \rightarrow u_*$, it follows from Lemma 2.4 that, for some $v_* \in \tilde{S}_{F, \tau u_*}$,

$$h_*(x, y) - f(x) - g(y) + f(0) = \int_0^x \int_0^y v_*(t, s) dt ds, \quad (x, y) \in \mathcal{J}.$$

STEP 5: The set

$$\Omega := \{u \in C(\mathcal{J}, \mathbb{R}) : \lambda u \in N(u) \text{ for some } \lambda > 1\}$$

is bounded.

Let $u \in \Omega$. Then $\lambda u \in N(u)$ for some $\lambda > 1$. Thus there exists $v \in \tilde{S}_{F, \tau u}$ such that

$$u(x, y) = \lambda^{-1} f(x) + \lambda^{-1} g(y) - \lambda^{-1} f(0) + \lambda^{-1} \int_0^x \int_0^y v(t, s) dt ds, \quad (x, y) \in \mathcal{J}.$$

This implies that for each $(x, y) \in \mathcal{J}$ we have

$$|u(x, y)| \leq |f(x)| + |g(y)| + |f(0)| + \int_0^x \int_0^y |v(t, s)| dt ds.$$

The definition of τ implies that there exists $\phi \in L^1(\mathcal{J}, \mathbb{R}^+)$ such that $\|F(x, y, \tau u)\| = \sup\{|v| : v \in F(x, y, \tau u)\} \leq \phi(x, y)$ for each $u \in C(\mathcal{J}, \mathbb{R})$. Thus we obtain

$$\|u\|_\infty \leq \|f\|_\infty + \|g\|_\infty + |f(0)| + \int_0^T \int_0^T \phi(t, s) dt ds = K.$$

This shows that Ω is bounded. Set $X := C(\mathcal{J}, \mathbb{R})$. As a consequence of Lemma 2.5 we deduce that N has a fixed point which is a solution of (3.1), (3.2) on $\mathcal{J}$.

STEP 6: *The solution u of (3.1), (3.2) satisfies the relations*

$$\underline{u}(x, y) \leq u(x, y) \leq \bar{u}(x, y) \text{ for all } (x, y) \in \mathcal{J}.$$

Let u be a solution to (3.1), (3.2). We prove that $\underline{u}(x, y) \leq u(x, y)$ for all $(x, y) \in \mathcal{J}$.

Let us suppose the contrary. Then there exist $x_1, x_2, y_1, y_2 \in J$, $x_1 < x_2$, $y_1 < y_2$ such that $\underline{u}(x_1, y_1) = u(x_1, y_1)$ and $\underline{u}(x, y) > u(x, y)$ for all $(x, y) \in (x_1, x_2) \times (y_1, y_2)$. In view of the definition of τ there exists $v(\cdot, \cdot) \in F(\cdot, \cdot, \underline{u}(\cdot, \cdot))$ a. e. on $\mathcal{J}$ with $v(x, y) \geq v_1(x, y)$ a. e. on $(x_1, x_2) \times (y_1, y_2)$ such that

$$u(x, y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y v(t, s) dt ds, \quad (x, y) \in (x_1, x_2) \times (y_1, y_2).$$

Since $\underline{u}$ is a lower solution to (1.1), (1.2), then

$$\underline{u}(x, y) \leq f(x) + g(y) - f(0) + \int_0^x \int_0^y v_1(t, s) dt ds \quad (x, y) \in (x_1, x_2) \times (y_1, y_2).$$

It follows from the facts $u(x_1, y_1) = \underline{u}(x_1, y_1)$, $v(x, y) \geq v_1(x, y)$ that

$$\underline{u}(x, y) \leq u(x, y) \text{ for all } (x, y) \in (x_1, x_2) \times (y_1, y_2),$$

which is a contradiction since $u(x, y) < \underline{u}(x, y)$ for all $(x, y) \in (x_1, x_2) \times (y_1, y_2)$. Consequently, $\underline{u}(x, y) \leq u(x, y)$ for all $(x, y) \in \mathcal{J}$.

Analogously, we can prove that $u(x, y) \leq \bar{u}(x, y)$ for all $(x, y) \in \mathcal{J}$. This shows that problem (3.1), (3.2) has a solution in the interval $[\underline{u}, \bar{u}]$. Since $\tau(u) = u$ for all $u \in [\underline{u}, \bar{u}]$, we see that u is a solution to (1.1), (1.2). $\square$

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MULTISTAGE ITERATIONS AND SOLVABILITY OF LINEAR CAUCHY PROBLEMS

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ABSTRACT. New general unique solvability conditions of the Cauchy problem for many-dimensional systems of linear functional differential equations are established.

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1. INTRODUCTION

We are interested in conditions sufficient for the unique solvability of the n -dimensional initial value problem of the form

$$u'(t) = (\ell u)(t) + f(t), \quad t \in [a, b], \quad (1.1)$$

$$u(\tau) = c, \quad (1.2)$$

where $-\infty < a < b < +\infty$, $u : [a, b] \rightarrow \mathbb{R}^n$, $n \in \mathbb{N}$, for arbitrary $c \in \mathbb{R}^n$ and $f \in L([a, b], \mathbb{R}^n)$. In (1.1), $\ell : C([a, b], \mathbb{R}^n) \rightarrow L([a, b], \mathbb{R}^n)$ is a linear operator which is assumed to be $(\vec{\sigma}, \tau)$ -positive in the following sense.

Definition 1.1. [1, 3] An operator $\ell : C([a, b], \mathbb{R}^n) \rightarrow L([a, b], \mathbb{R}^n)$ is said to be $(\vec{\sigma}, \tau)$ -positive with some $\vec{\sigma} \in \{-1, 1\}^n$ and $\tau \in [a, b]$ if the relation

$$u(t) \geq_{\vec{\sigma}} 0 \quad \text{for all } t \in [a, b], \quad (1.3)$$

implies that

$$(\ell u)(t) \operatorname{sign}(t - \tau) \geq_{\vec{\sigma}} 0 \quad \text{for a. e. } t \in [a, b].$$

Here, the symbol $\geq_{\vec{\sigma}}$ stands for the partial ordering of the space $\mathbb{R}^n$ generated by the cone $\sigma_1 \mathbb{R}_+ \times \sigma_2 \mathbb{R}_+ \times \cdots \times \sigma_n \mathbb{R}_+$, i. e.,

$$\mathbb{R}^n \ni u = (u_k)_{k=1}^n \geq_{\vec{\sigma}} 0 \text{ if, and only if } \sigma_k u_k \geq 0 \text{ for all } k = 1, 2, \dots, n. \quad (1.4)$$

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In a similar way, we define the corresponding strict inequality:

$$\mathbb{R}^n \ni u = (u_k)_{k=1}^n >_{\vec{\sigma}} 0 \text{ if, and only if } \sigma_k u_k > 0 \text{ for all } k = 1, 2, \dots, n.$$

By a solution of problem (1.1), (1.2), as usual, an absolutely continuous function $u : [a, b] \rightarrow \mathbb{R}^n$ is meant which possesses property (1.2) at the point τ and satisfies equation (1.1) almost everywhere on $[a, b]$.

Remark 1.2. As noted in [3], a $(\vec{\sigma}, \tau)$ -positive linear operator from $C([a, b], \mathbb{R}^n)$ to $L([a, b], \mathbb{R}^n)$ is always bounded.

For the Cauchy problem (1.1), (1.2), we obtain new versions of the general unique solvability conditions established in [1, 3]. The main idea here, as the title of the paper suggests, is to use certain “multistage iterations,” in terms of which the main assumptions of the theorems are formulated.

2. UNIQUE SOLVABILITY OF PROBLEM (1.1), (1.2)

To formulate the theorems on the Cauchy problem (1.1), (1.2), we need to introduce some auxiliary functions related to the homogeneous problem

$$u'(t) = (\ell u)(t), \quad t \in [a, b], \quad (2.1)$$

$$u(\tau) = 0. \quad (2.2)$$

2.1. Multistage iterations for the homogeneous problem. With the homogeneous problem (2.1), (2.2), which corresponds, in a natural way, to the original Cauchy problem (1.1), (1.2), we associate the sequence of functions $\{y_k \mid k = 0, 1, 2, \dots\} \subset C([a, b], \mathbb{R}^n)$ determined by the recurrence relation

$$y_k(t) := \sum_{i=1}^r \alpha_i \int_{\tau}^t (\ell y_{k-i})(s) ds, \quad t \in [a, b], \quad k \geq r, \quad (2.3)$$

where $\alpha_1, \alpha_2, \dots, \alpha_r$ are certain fixed non-negative constants.

Here, $r \in \mathbb{N}$ is fixed, and $y_0, y_1, \dots, y_{r-1}$ are arbitrary absolutely continuous functions from $[a, b]$ to $\mathbb{R}^n$ chosen so that

$$y_k(t) \geq_{\vec{\sigma}} 0, \quad t \in [a, b], \quad k = 0, 1, \dots, r-1, \quad (2.4)$$

and

$$y_k(\tau) = 0, \quad k = 0, 1, \dots, r-1. \quad (2.5)$$

Remark 2.1. In the case where $r = 1$ and $\alpha_1 = 1$, equality (2.3) takes the form

$$y_k(t) = \int_{\tau}^t (\ell y_{k-1})(s) ds, \quad t \in [a, b], \quad k \in \mathbb{N}, \quad (2.6)$$

and thus coincides with the sequence used, e. g., in [1–3]. Formula (2.6) defines the standard iteration sequence used in studies of the uniqueness of the trivial solution of the integral functional equation

$$y(t) = \int_{\tau}^t (\ell y)(s) ds, \quad t \in [a, b],$$

which, obviously, is equivalent to the homogeneous problem (2.1), (2.2).

2.2. General theorems. The following general theorem on the solvability of problem (1.1), (1.2) is true.

Theorem 2.2. *Let the operator ℓ in equation (1.1) be $(\vec{\sigma}, \tau)$ -positive for some $\vec{\sigma} \in \{-1, 1\}^n$. Assume also that one can specify some integers r and m , $m \geq r \geq 1$, a real number $\varrho \in (1, +\infty)$, some constants $\{\beta_k\}_{k=0}^m \subset [0, +\infty)$ and $\{\alpha_i\}_{i=1}^r \subset [0, +\infty)$, and certain absolutely continuous vector-functions $y_0, y_1, \dots, y_{r-1}$ satisfying conditions (2.4), (2.5), and the relation*

$$\sum_{k=0}^{r-1} \beta_k y_k(t) >_{\vec{\sigma}} 0 \quad \text{for all } t \in [a, b] \setminus \{\tau\} \quad (2.7)$$

such that the differential functional inequality

$$\left[\sum_{k=0}^{r-1} \beta_k y'_k(t) + \sum_{k=0}^{m-1} \left(\sum_{v \in T_{r,m}(k)} \beta_{v+k} \alpha_v - \varrho \beta_k \right) (\ell y_k)(t) - \varrho \beta_m (\ell y_m)(t) \right] \text{sign}(t - \tau) \geq_{\vec{\sigma}} 0 \quad (2.8)$$

is satisfied for almost every t from $[a, b]$, where, by definition,

$$T_{r,m}(k) := \{v \in \mathbb{N} \mid v \leq r \leq v + k \leq m\} \quad (2.9)$$

for $r \in \mathbb{N}$, $m \geq r$, and $k = 0, 1, \dots, m-1$.

Then the Cauchy problem (1.1), (1.2) has a unique solution $u(\cdot)$ for arbitrary $f \in L([a, b], \mathbb{R}^n)$ and $c \in \mathbb{R}^n$, and this solution is representable as the uniformly convergent functional series

$$u(t) = \tilde{f}(t) + \int_{\tau}^t (\ell \tilde{f})(s) ds + \int_{\tau}^t \ell \left(\int_{\tau}^s (\ell \tilde{f})(\xi) d\xi \right) (s) ds + \dots, \quad t \in [a, b], \quad (2.10)$$

where, by definition,

$$\tilde{f}(t) := c + \int_{\tau}^t f(s) ds, \quad t \in [a, b]. \quad (2.11)$$

Moreover, if the function f and vector c in (1.1), (1.2) satisfy the additional condition

$$\int_{\tau}^t f(s) ds \geq_{\vec{\sigma}} -c \quad \text{for all } t \in [a, b], \quad (2.12)$$

then the unique solution $u(\cdot)$ of this problem satisfies relation (1.3).

Remark 2.3. It is not difficult to show that the set $T_{r,m}(k)$, where $k = 0, 1, \dots, m-1$, consists of those integers v which satisfy the inclusion $v \in \{r, r+1, \dots, m\} \cap \{k+1, k+2, \dots, k+r\}$, or, in other words,

$$T_{r,m}(k) = \{v \in \mathbb{N} \mid \max\{r, k+1\} \leq v \leq \min\{m, k+r\}\} \quad (2.13)$$

for $k = 0, 1, \dots, m-1$. Therefore, the differential inequality (2.8) can be rewritten in the form

$$\left[\sum_{k=0}^{r-1} \beta_k y'_k(t) + \sum_{k=0}^{m-1} \left(\sum_{v=\max\{r, k+1\}}^{\min\{m, k+r\}} \beta_{v+k} \alpha_v - \varrho \beta_k \right) (\ell y_k)(t) - \varrho \beta_m (\ell y_m)(t) \right] \text{sign}(t - \tau) \geq_{\vec{\sigma}} 0.$$

It seems, however, that form (2.8) is more convenient for applications.

Note that, due to the inequality $m \geq r$, for $k = 0, 1, 2, \dots, m-1$, we have

$$\max\{r, k+1\} \leq m, \quad \max\{r, k+1\} \leq k+r$$

and, hence, by (2.13), the set $T_{r,m}(k)$ is non-empty for any $k = 0, 1, \dots, m-1$.

Remark 2.4. Condition (2.7) implies, in particular, that

$$\sum_{k=0}^{r-1} \beta_k > 0, \quad (2.14)$$

and, therefore, we can always take 1 for one of the non-negative coefficients $\beta_0, \beta_1, \dots, \beta_{r-1}$ in Theorem 2.2. It is handier, however, to keep all these coefficients unfixed.

Remark 2.5. Theorem 2.2 implies, in particular, Theorem 1 from [1]. Both these statements, as we shall see in Section 3, are implied by Theorem 1 from [3]. This means that the theorems mentioned are, in fact, equivalent, although their corresponding assumptions are rather different from one another.

Remark 2.6. Similarly to [3], one can show that condition (2.8) is optimal in the sense that the assertion of Theorem 2.2, generally speaking, is not true if (2.8) is assumed with $\varrho = 1$.

The assumptions of the following version of Theorem 2.2 are somewhat simpler.

Theorem 2.7. *Let the operator ℓ in equation (1.1) be $(\vec{\sigma}, \tau)$ -positive for some $\vec{\sigma} \in \{-1, 1\}^n$. Assume also the existence of a natural number r , real number $\varrho \in (1, +\infty)$, some non-negative constants $\{\alpha_i\}_{i=1}^r$ and $\{\beta_k\}_{k=0}^r$, and certain absolutely continuous*

vector-functions $y_0, y_1, \dots, y_{r-1}$ satisfying conditions (2.4), (2.5), and (2.7) such that the differential functional inequality

$$\left[\sum_{k=0}^{r-1} \beta_k y'_k(t) + \sum_{k=0}^{r-1} (\beta_r \alpha_{r-k} - \varrho \beta_k) (\ell y_k)(t) - \varrho \beta_r (\ell y_r)(t) \right] \text{sign}(t - \tau) \geq_{\vec{\sigma}} 0 \quad (2.15)$$

is satisfied for almost every t from $[a, b]$.

Then the Cauchy problem (1.1), (1.2) has a unique solution $u(\cdot)$ for arbitrary $f \in L([a, b], \mathbb{R}^n)$ and $c \in \mathbb{R}^n$, and this solution is representable as the uniformly convergent functional series (2.10), where $\tilde{f}$ is the function defined by equality (2.11).

Moreover, if the function f and vector c in (1.1), (1.2) satisfy the additional condition (2.12), then the unique solution $u(\cdot)$ of this problem satisfies relation (1.3).

Note that the functions $y_0, y_1, \dots, y_{r-1}$ in Theorem 2.7 are given *a priori*, whereas the function y_r is constructed according to formula (2.3):

$$y_r(t) = \sum_{i=1}^r \alpha_i \int_{\tau}^t (\ell y_{r-i})(s) ds, \quad t \in [a, b].$$

2.3. Corollaries. Here, we present several corollaries derived from Theorems 2.2 and 2.7.

Corollary 2.8. Let the operator ℓ in equation (1.1) be $(\vec{\sigma}, \tau)$ -positive for some $\vec{\sigma} \in \{-1, 1\}^n$. Assume also the existence of an $r \in \mathbb{N}$, a real number $\gamma \in (1, +\infty)$, some non-negative constants $\{\beta_k\}_{k=0}^{r-1}$, and of certain absolutely continuous vector-functions $y_0, y_1, \dots, y_{r-1}$ satisfying conditions (2.4), (2.5), and (2.7) such that the differential functional inequality

$$\left[\sum_{k=0}^{r-1} \beta_k y'_k(t) - \gamma \sum_{i=1}^r \beta_{r-i} \left(\ell \left(\int_{\tau}^t (\ell y_{r-i})(s) ds \right) \right) (t) \right] \text{sign}(t - \tau) \geq_{\vec{\sigma}} 0 \quad (2.16)$$

holds for the functions $y_0, y_1, \dots, y_{r-1}$ at almost every point t from $[a, b]$.

Then the conclusion of Theorem 2.7 is true for problem (1.1), (1.2).

The conditions involving iteration sequences of type (2.3) admit different variations. For example, the following statement is true.

Corollary 2.9. Let the operator ℓ in equation (1.1) be $(\vec{\sigma}, \tau)$ -positive for some $\vec{\sigma} \in \{-1, 1\}^n$. Assume that there exist certain absolutely continuous functions $y_0, y_1 : [a, b] \rightarrow \mathbb{R}^n$ satisfying the conditions

$$y_k(t) >_{\vec{\sigma}} 0 \text{ and } y_k(\tau) = 0 \text{ for all } t \in [a, b] \setminus \{\tau\}, \quad k = 0, 1, \quad (2.17)$$

and some constants $\varrho \in (1, +\infty)$, $m \in \mathbb{N}$ ($m \geq 2$), $\{\alpha_1, \alpha_2\} \subset [0, +\infty)$, $\alpha_1 \alpha_2 \neq 0$, and $\{\beta_0, \beta_1, \beta_2, \dots, \beta_m\} \subset (0, +\infty)$ such that

$$\alpha_1 \beta_{k+1} + \alpha_2 \beta_{k+2} = \varrho \beta_k, \quad k = 1, 2, \dots, m-2, \quad (2.18)$$

for which the relation

$$\begin{aligned} & \left[\beta_0 y'_0(t) + \beta_1 y'_1(t) + (\alpha_2 \beta_2 - \varrho \beta_0) (\ell y_0)(t) \right. \\ & \quad \left. + (\alpha_1 \beta_m - \varrho \beta_{m-1}) (\ell y_{m-1})(t) - \varrho \beta_m (\ell y_m)(t) \right] \text{sign}(t - \tau) \geq_{\vec{\sigma}} 0 \end{aligned} \quad (2.19)$$

holds for almost all $t \in [a, b]$, where the functions y_{m-1} and y_m are defined by the recurrence relation

$$y_k(t) := \alpha_1 \int_{\tau}^t (\ell y_{k-1})(s) ds + \alpha_2 \int_{\tau}^t (\ell y_{k-2})(s) ds, \quad t \in [a, b],$$

for $k = 2, 3, \dots, m$.

Then the conclusion of Theorem 2.7 is true for problem (1.1), (1.2).

Remark 2.10. As will be seen from the proof of Corollary 2.9, instead of (2.17), the absolutely continuous functions y_0 and y_1 can be assumed to be just non-negative in the sense of relation $\geq_{\vec{\sigma}}$ (see formula (1.4)) and such that

$$y_0(\tau) = y_1(\tau) = 0; \quad y_0(t) + y_1(t) >_{\vec{\sigma}} 0 \quad \text{for } t \in [a, b] \setminus \{\tau\},$$

i. e., one of these functions may vanish at points different from τ whenever the other one remains non-zero.

Corollary 2.11. If ℓ in equation (1.1) is $(\vec{\sigma}, \tau)$ -positive for some $\vec{\sigma} \in \{-1, 1\}^n$ and, furthermore, one can specify some $\varrho \in (1, +\infty)$, $\alpha \in (0, \varrho)$, and certain absolutely continuous vector-functions $z_0, z_1 : [a, b] \rightarrow \mathbb{R}^n$ satisfying the conditions

$$z_k(t) >_{\vec{\sigma}} 0 \text{ and } z_k(\tau) = 0 \text{ for all } t \in [a, b] \setminus \{\tau\}, \quad k = 0, 1, \quad (2.20)$$

and, for almost every t from $[a, b]$, the inequality

$$\left[z'_0(t) + z'_1(t) - (\varrho - \alpha) (\ell z_0)(t) - \alpha (\ell z_{m-1})(t) - \varrho (\ell z_m)(t) \right] \text{sign}(t - \tau) \geq_{\vec{\sigma}} 0, \quad (2.21)$$

where the z_{m-1} and z_m are the functions defined recursively by the relation

$$z_k(t) := (\varrho - \alpha) \int_{\tau}^t (\ell z_{k-1})(s) ds + \alpha \int_{\tau}^t (\ell z_{k-2})(s) ds, \quad t \in [a, b], \quad (2.22)$$

for $k = 2, 3, \dots, m$, then the assertion of Theorem 2.7 is true for problem (1.1), (1.2).

The following version of Corollary 2.11 is more similar to Theorem 3.1.

Corollary 2.12. If ℓ in equation (1.1) is $(\vec{\sigma}, \tau)$ -positive for some $\vec{\sigma} \in \{-1, 1\}^n$ and, furthermore, there exist some $\varrho \in (1, +\infty)$, $\alpha \in (0, \varrho)$, and certain absolutely continuous vector-functions $z_0, z_1 : [a, b] \rightarrow \mathbb{R}^n$ satisfying conditions (2.20) and, for almost every t from $[a, b]$, the inequality

$$\left[z'_0(t) + z'_1(t) - (\varrho - \alpha) (\ell z_0)(t) - \alpha (\ell z_1)(t) - \varrho (\ell z_2)(t) \right] \text{sign}(t - \tau) \geq_{\vec{\sigma}} 0,$$

where

$$z_2(t) := (\varrho - \alpha) \int_{\tau}^t (\ell z_1)(s) ds + \alpha \int_{\tau}^t (\ell z_0)(s) ds, \quad t \in [a, b],$$

then the assertion of Theorem 2.7 is true for problem (1.1), (1.2).

For the equation with deviating argument

$$u'(t) = \sum_{j=1}^N P_j(t) u(\omega_j(t)) + f(t), \quad t \in [a, b], \quad (2.23)$$

where $f \in L([a, b], \mathbb{R}^n)$, $N \in \mathbb{N}$, $\{P_j \mid j = 1, 2, \dots, N\} \subset L([a, b], GL_n(\mathbb{R}))$, and $\omega_j : [a, b] \rightarrow [a, b]$, $j = 1, 2, \dots, N$, are measurable functions, one can establish, e. g., the following statement.

Corollary 2.13. *Assume that, for a certain $\vec{\sigma} \in \{-1, 1\}^n$, the functions $P_j : [a, b] \rightarrow GL_n(\mathbb{R})$ and $\omega_j : [a, b] \rightarrow [a, b]$, $j = 1, 2, \dots, N$, in equation (2.23) satisfy the inequalities*

$$\sum_{v=1}^N P_v(t) \vec{\sigma} \operatorname{sign}(t - \tau) \geq_{\vec{\sigma}} 0 \quad \text{for a. e. } t \in [a, b] \quad (2.24)$$

and, furthermore, the condition

$$\begin{aligned} \sum_{i=1}^r \beta_{r-i} \sum_{j=1}^N P_j(t) \int_{\tau}^{\omega_j(t)} \sum_{v=1}^N P_v(s) g |\omega_v(s) - \tau|^{q_{r-i}} ds \cdot \operatorname{sign}(t - \tau) \\ \leq_{\vec{\sigma}} \alpha \sum_{k=0}^{r-1} \beta_k q_k |t - \tau|^{q_k-1} g \quad \text{for a. e. } t \in [a, b] \end{aligned} \quad (2.25)$$

holds with some $\{\beta_k \mid k = 0, 1, \dots, r-1\} \subset [0, +\infty)$, $\{q_k \mid k = 0, 1, \dots, r-1\} \subset [1, +\infty)$, $\alpha \in (0, 1)$, and $g \in \mathbb{R}^n$, $g >_{\vec{\sigma}} 0$.

Then, for arbitrary f from $L([a, b], \mathbb{R}^n)$ and $c \in \mathbb{R}^n$, the initial value problem (2.23), (1.2) has a unique solution $u(\cdot)$, and this solution is representable as the uniformly convergent series

$$\begin{aligned} u(t) = \tilde{f}(t) + \sum_{k_1=1}^N \int_{\tau}^t P_{k_1} u(\xi_1) \tilde{f}(\omega_{k_1}(\xi_1)) d\xi_1 \\ + \sum_{k_1=1}^N \int_{\tau}^t P_{k_1}(\xi_1) \int_{\tau}^{\omega_{k_1}(\xi_1)} \sum_{k_2=1}^N P_{k_2}(\xi_2) \tilde{f}(\omega_{k_2}(\xi_2)) d\xi_2 d\xi_1 + \dots, \quad t \in [a, b], \end{aligned}$$

with the function $\tilde{f} : [a, b] \rightarrow \mathbb{R}^n$ given by formula (2.11). Moreover, under the additional condition (2.12), the solution of problem (2.23), (1.2) satisfies inequality (1.3).

3. PROOFS

We need the following theorem from [3].

Theorem 3.1. [3] *Let us suppose that the linear operator ℓ in equation (1.1) is $(\vec{\sigma}, \tau)$ -positive for some $\vec{\sigma} \in \{-1, 1\}^n$. Assume also that there exist a number $\varrho \in (1, +\infty)$ and an absolutely continuous function $y : [a, b] \rightarrow \mathbb{R}^n$ such that*

$$y(\tau) = 0, \quad (3.1)$$

$$y(t) >_{\vec{\sigma}} 0 \quad \text{for all } t \in [a, b] \setminus \{\tau\}, \quad (3.2)$$

and, furthermore, the functional differential inequality

$$[y'(t) - \varrho(\ell y)(t)] \operatorname{sign}(t - \tau) \geq_{\vec{\sigma}} 0 \quad (3.3)$$

is satisfied for almost every t from $[a, b]$.

Then the Cauchy problem (1.1), (1.2) is uniquely solvable for arbitrary $c \in \mathbb{R}^n$ and $f \in L([a, b], \mathbb{R}^n)$. The unique solution $u(\cdot)$ of problem (1.1), (1.2) is, moreover, representable as the uniformly convergent functional series (2.10).

If, in addition, c and f satisfy condition (2.12), then the solution $u(\cdot)$ mentioned possesses property (1.3).

The idea of proof of the results stated in Sections 2.2 and 2.3 is to construct a suitable solution of the differential inequality (3.3) by using some or other properties of the “multistage” iteration sequence (2.3).

3.1. Proof of Theorem 2.2. Let us fix certain absolutely continuous vector-functions $\{y_k\}_{k=0}^{r-1} : [a, b] \rightarrow \mathbb{R}^n$ and construct the corresponding functions $\{y_k\}_{k=r}^m : [a, b] \rightarrow \mathbb{R}^n$ according to formula (2.3). We consider the case where

$$m \geq r. \quad (3.4)$$

Introduce the function

$$y(t) := \sum_{k=0}^m \beta_k y_k(t), \quad t \in [a, b], \quad (3.5)$$

with the coefficients $\{\beta_k\}_{k=0}^m \subset [0, +\infty)$ determined by the assumptions of the theorem. Note that, in view of (2.4), assumption (2.7) implies that (2.14) holds.

Let us show that, under our assumptions, function (3.5) satisfies inequality (2.8). Indeed, by virtue of (3.5), the corresponding function

$$w_{\varrho, y} := y' - \varrho \ell y \quad (3.6)$$

has the form

$$w_{\varrho, y} = \sum_{k=0}^m \beta_k [y'_k - \varrho \ell y_k],$$

whence, by (3.4),

$$\begin{aligned} w_{\varrho,y} &= \sum_{k=0}^{r-1} \beta_k [y'_k - \varrho \ell y_k] + \sum_{k=r}^m \beta_k [y'_k - \varrho \ell y_k] \\ &= \sum_{k=0}^{r-1} \beta_k y'_k + \sum_{k=r}^m \beta_k y'_k - \varrho \sum_{k=0}^m \beta_k \ell y_k. \end{aligned} \quad (3.7)$$

Note that the first sum in (3.7) contains the given functions $y_0, y_1, \dots, y_{r-1}$ only.

In view of formula (2.3) for the functions $y_r, y_{r+1}, \dots$, we have

$$y'_k(t) := \sum_{i=1}^r \alpha_i (\ell y_{k-i})(t), \quad t \in [a, b], \quad k \geq r,$$

and, therefore, equality (3.7) can be brought to the form

$$w_{\varrho,y} = \sum_{k=0}^{r-1} \beta_k y'_k + \sum_{k=r}^m \beta_k \sum_{i=1}^r \alpha_i \ell y_{k-i} - \varrho \sum_{k=0}^m \beta_k \ell y_k. \quad (3.8)$$

Now we need the following technical lemma.

Lemma 3.2. *For arbitrary vectors $x_0, x_1, \dots, x_{m-1}$ from $\mathbb{R}^n$, the equality*

$$\sum_{k=r}^m \beta_k \sum_{i=1}^r \alpha_i x_{k-i} = \sum_{j=0}^{m-1} \gamma_j x_j \quad (3.9)$$

holds, where the coefficients $\gamma_0, \gamma_1, \dots, \gamma_{m-1}$ are given by the formulae

$$\gamma_k := \sum_{v \in T_{r,m}(k)} \beta_{v+k} \alpha_v, \quad k = 0, 1, \dots, m-1, \quad (3.10)$$

and the sets $T_{r,m}(k)$, $k = 0, 1, \dots, m-1$, are defined by equality (2.9).

Proof of Lemma 3.2. It is clear from the inequalities $1 \leq i \leq r \leq k \leq m$ that the expression

$$\sum_{k=r}^m \beta_k \sum_{i=1}^r \alpha_i x_{k-i}$$

is, in fact, a linear combination of the values $x_0, x_1, \dots, x_{m-1}$. It therefore remains to determine the corresponding coefficients.

Since equality (3.9) should be satisfied for arbitrary values of $x_0, x_1, \dots, x_{m-1}$, we can put, in particular,

$$x_j := \delta_{j,s}, \quad j = 0, 1, \dots, m-1,$$

where s is an arbitrary integer between 0 and $m-1$, and $\delta_{j,s}$ is the Kronecker symbol,

$$\delta_{j,s} = \begin{cases} 1 & \text{for } j = s \\ 0 & \text{otherwise.} \end{cases}$$

In this case, (3.9) has the form

$$\sum_{k=r}^m \beta_k \sum_{i=1}^r \alpha_i \delta_{k-i,s} = \gamma_s. \quad (3.11)$$

It is easy to see that

$$\sum_{i=1}^r \alpha_i \delta_{k-i,s} = \begin{cases} \alpha_{k-s} & \text{if } k-s \in \{1, 2, \dots, r\} \\ 0 & \text{if } k-s \notin \{1, 2, \dots, r\} \end{cases}$$

and, hence,

$$\sum_{k=r}^m \beta_k \sum_{i=1}^r \alpha_i \delta_{k-i,s} = \sum_{k: r \leq k \leq m \text{ and } 1 \leq k-s \leq r} \beta_k \alpha_{k-s}. \quad (3.12)$$

In view of (3.11), the change of variable $v = k - s$ brings (3.12) to the form

$$\sum_{v: 1 \leq v \leq r \leq v+s \leq m} \beta_{v+s} \alpha_v = \gamma_s.$$

The last relation, according to the definition of the sets $T_{r,m}(s)$, $s = 0, 1, \dots, m-1$, can be rewritten as $\sum_{v \in T_{r,m}(s)} \beta_{v+s} \alpha_v = \gamma_s$, which is nothing but (3.10). This leads us immediately to the required equality (3.9) for arbitrary $x_0, x_1, \dots, x_{m-1}$. $\square$

Returning now to expression (3.6), we conclude that, in view of Lemma 3.2, formula (3.8) can be rewritten as

$$\begin{aligned} w_{\varrho,y} &= \sum_{k=0}^{r-1} \beta_k y'_k + \sum_{k=0}^{m-1} \gamma_k \ell y_k - \varrho \sum_{k=0}^m \beta_k \ell y_k \\ &= \sum_{k=0}^{r-1} \beta_k y'_k + \sum_{k=0}^{m-1} (\gamma_k - \varrho \beta_k) \ell y_k - \varrho \beta_m \ell y_m, \end{aligned} \quad (3.13)$$

where $\gamma_0, \gamma_1, \dots, \gamma_{m-1}$ are given by relation (3.10). In view of (3.10), equality (3.13) is equivalent to the relation

$$w_{\varrho,y} = \sum_{k=0}^{r-1} \beta_k y'_k + \sum_{k=0}^{m-1} \left(\sum_{v \in T_{r,m}(k)} \beta_{v+k} \alpha_v - \varrho \beta_k \right) \ell y_k - \varrho \beta_m \ell y_m.$$

Therefore, assumption (2.8) guarantees that function (3.6) satisfies the condition

$$w_{\varrho,y}(t) \operatorname{sign}(t - \tau) \geq_{\sigma} 0 \quad \text{for a. e. } t \in [a, b],$$

i. e., the functional differential inequality (3.3) holds for the function y given by (3.5).

We have thus constructed a solution, y , of the differential inequality (3.3). It will be possible to apply Theorem 3.1 if we show that, under our assumptions, the solution mentioned possesses properties (3.1) and (3.2). We prove this by using the following lemma.

Lemma 3.3. Assume that the operator $\ell : C([a, b], \mathbb{R}^n) \rightarrow L([a, b], \mathbb{R}^n)$ is $(\vec{\sigma}, \tau)$ -positive with respect to certain $\vec{\sigma} \in \{-1, 1\}^n$ and $\tau \in [a, b]$. Then, for arbitrary continuous functions $\{y_k\}_{k=0}^{r-1} : [a, b] \rightarrow \mathbb{R}^n$ satisfying conditions (2.4) and (2.5), the corresponding functions $y_r, y_{r+1}, \dots$ defined by formulae (2.3) also satisfy condition (2.4):

$$y_k(t) \geq_{\vec{\sigma}} 0, \quad t \in [a, b], \quad k \geq r.$$

Moreover, each of these functions vanishes at the point τ .

Proof of Lemma 3.3. The first assertion is derived from the definition of functions (2.3) similarly to the proof of Lemma 1 from [1]. The important difference is that, in our case, we need to make use of the non-negativeness of the coefficients $\alpha_1, \alpha_2, \dots, \alpha_r$ in formula (2.3) for the functions $y_r, y_{r+1}, \dots$. We omit the details here.

The property

$$y_k(\tau) = 0 \quad \text{for all } k = 0, 1, 2, \dots, m$$

is obvious from condition (2.5) and formula (2.3). $\square$

Application of Lemma 3.3 allows us to claim that, in view of the $(\vec{\sigma}, \tau)$ -positiveness of the operator ℓ , the inequality

$$\beta_k y_k(t) \geq_{\vec{\sigma}} 0, \quad t \in [a, b], \quad k = 0, 1, \dots, m, \quad (3.14)$$

is satisfied because all $\beta_k, k = 0, 1, \dots, m$, are non-negative. It follows from (3.14) that

$$\sum_{k=0}^m \beta_k y_k(t) - \sum_{k=0}^{r-1} \beta_k y_k(t) = \sum_{k=r}^m \beta_k y_k(t) \geq_{\vec{\sigma}} 0$$

and, hence,

$$\sum_{k=0}^m \beta_k y_k(t) \geq_{\vec{\sigma}} \sum_{k=0}^{r-1} \beta_k y_k(t) \quad (3.15)$$

for every t from $[a, b]$.

Inequality (3.15) yields

$$y(t) = \sum_{k=0}^m \beta_k y_k(t) \geq_{\vec{\sigma}} \sum_{k=0}^{r-1} \beta_k y_k(t), \quad t \in [a, b],$$

whence, by virtue of assumption (2.7), we obtain

$$y(t) \geq_{\vec{\sigma}} \sum_{k=0}^{r-1} \beta_k y_k(t) >_{\vec{\sigma}} 0, \quad t \in [a, b] \setminus \{\tau\},$$

i. e., y satisfies condition (3.2).

Thus, we have shown that function (3.5) satisfies the differential functional inequality (3.3) and possesses properties (3.1) and (3.2), i. e., the assumptions of Theorem 3.1 are satisfied. Application of Theorem 3.1 leads us to the assertion required. $\square$

3.2. Proof of Theorem 2.7. To obtain the assertion of Theorem 2.7, it suffices to apply Theorem 2.2 with $m = r$ and take into account the equalities

$$T_{r,r}(k) = \{r - k\}, \quad k = 0, 1, \dots, r - 1. \quad (3.16)$$

3.3. Proof of Corollary 2.8. We shall apply Theorem 2.7. For this purpose, let us put

$$\varrho = \sqrt{\gamma}, \quad \beta_r = 1, \quad (3.17)$$

and

$$y_r(t) := \sqrt{\gamma} \sum_{i=1}^r \beta_{r-i} \int_{\tau}^t (ly_{r-i})(s) ds, \quad t \in [a, b].$$

This function, as is easy to see, coincides with the function y_r defined by formula (2.3) where

$$\alpha_i = \beta_{r-i} \sqrt{\gamma}, \quad i = 1, 2, \dots, r. \quad (3.18)$$

It remains to notice that, in view of (3.17) and (3.18), the relations

$$\beta_r \alpha_{r-k} - \varrho \beta_k = 0, \quad k = 0, 1, \dots, r - 1,$$

are true and, therefore, the corresponding differential inequality (2.15) takes the form

$$\left[\sum_{k=0}^{r-1} \beta_k y'_k(t) - \varrho \beta_r (ly_r)(t) \right] \text{sign}(t - \tau) \geq_{\mathcal{S}} 0.$$

Application of Theorem 2.7 completes the proof. $\square$

3.4. Proof of Corollary 2.9. We shall show that, under the conditions assumed, the differential inequality (2.8) is true and Theorem 2.2 is applicable.

For an arbitrary integer k such that $0 \leq k \leq m - 1$, a number v belongs to the set $T_{2,m}(k)$ if, and only if either $v = 1$ or $v = 2$, i. e.,

$$\bigcup_{k=0}^{m-1} T_{2,m}(k) \subset \{1, 2\}.$$

Recall that, by assumption, $m \geq 2$.

More precisely, $1 \in T_{2,m}(k) \Leftrightarrow 1 \leq k \leq m - 1$ and $2 \in T_{2,m}(k) \Leftrightarrow 0 \leq k \leq m - 2$. Thus, the sets $T_{2,m}(k)$, $k = 0, 1, \dots, m - 1$, have the following structure:

$$T_{2,m}(k) = \begin{cases} \{2\} & \text{for } k = 0, \\ \{1, 2\} & \text{for } k = 1, 2, \dots, m - 2, \\ \{1\} & \text{for } k = m - 1. \end{cases}$$

Therefore, with these values of the parameters, the differential inequality (2.8) of Theorem 2.2 has the form

$$\begin{aligned} & \left[\beta_0 y'_0(t) + \beta_1 y'_1(t) + (\alpha_2 \beta_2 - \varrho \beta_0)(\ell y_0)(t) \right. \\ & \quad + \sum_{k=1}^{m-2} (\alpha_1 \beta_{k+1} + \alpha_2 \beta_{k+2} - \varrho \beta_k)(\ell y_k)(t) \\ & \quad \left. + (\alpha_1 \beta_m - \varrho \beta_{m-1})(\ell y_{m-1})(t) - \varrho \beta_m(\ell y_m)(t) \right] \text{sign}(t - \tau) \geq_{\vec{\sigma}} 0. \end{aligned} \quad (3.19)$$

Recall that the numbers $\beta_1, \beta_2, \dots, \beta_m$, and α_1, α_2 , by assumption, satisfy relations (2.18).

It follows from (2.18) that the inner sum in (3.19) is equal to zero, i. e., in this case, (2.8) coincides with (2.19). It now remains to apply Theorem 2.2, which leads us immediately to the assertion desired. $\square$

3.5. Proof of Corollary 2.11. First of all note that sequence (2.22) is a particular case of (2.3) with a suitable choice of coefficients. More precisely,

$$z_k = y_k, \quad k = 2, 3, \dots$$

with $y_0 = z_0, y_1 = z_1, r = 2$, and

$$\alpha_2 = \alpha, \quad \alpha_1 = \alpha. \quad (3.20)$$

We can apply Corollary 2.9. Indeed, let us set

$$\beta_k = \beta, \quad k = 0, 1, \dots, m,$$

where β is a certain positive constant. Since, by (3.20), $\alpha_1 + \alpha_2 = \varrho$, it is obvious that relations (2.18) are satisfied. Inequality (2.19) in our case has form (2.21). $\square$

3.6. Proof of Corollary 2.12. It suffices to put $m = 2$ in Corollary 2.11. $\square$

3.7. Proof of Corollary 2.13. Let us put $\gamma = \frac{1}{\alpha}$ and, for $k = 0, 1, \dots, r-1$,

$$y_k(t) = |t - \tau|^{q_k}, \quad t \in [a, b]. \quad (3.21)$$

Functions (3.21), obviously, satisfy condition (2.7) with arbitrary non-negative constants $\beta_0, \beta_1, \dots, \beta_{r-1}$ such that $\sum_{k=0}^{r-1} \beta_k > 0$.

By using the formulae

$$y'_k(t) = q_k |t - \tau|^{q_k-1} \text{sign}(t - \tau), \quad t \in [a, b], \quad k = 0, 1, \dots, r-1,$$

it is easy to verify that, in this case, (2.25) coincides with inequality (2.16). The $(\vec{\sigma}, \tau)$ -positiveness of the linear operator

$$C([a, b], \mathbb{R}^n) \ni u \mapsto \ell u := \sum_{v=1}^N P_v(\cdot) u(\omega_v(\cdot)), \quad t \in [a, b],$$

under condition (2.24), is a consequence of Lemma 2 from [1]. Therefore, the required assertion follows from Corollary 2.8. $\square$

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ON FUZZY P -CONTINUOUS MULTIFUNCTIONS

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ABSTRACT. In this paper, by using the fuzzy property P , some characterisations and properties of certain types of upper (lower) fuzzy continuous multifunctions including upper (lower) fuzzy continuous, upper (lower) fuzzy almost continuous, upper (lower) fuzzy c -continuous, upper (lower) fuzzy almost c -continuous, upper (lower) fuzzy c^* -continuous, upper (lower) fuzzy s -continuous, upper (lower) fuzzy almost s -continuous, upper (lower) fuzzy ℓ -continuous, upper (lower) fuzzy almost ℓ -continuous functions are given.

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1. INTRODUCTION

It is well known that several types of fuzzy upper (lower) continuous multifunctions are given in literature. By using property P , our main goal here is to give some characterisations and properties of certain types of upper (lower) fuzzy continuous multifunctions including upper (lower) fuzzy continuous, upper (lower) fuzzy almost continuous, upper (lower) fuzzy c -continuous, upper (lower) fuzzy almost c -continuous, upper (lower) fuzzy c^* -continuous, upper (lower) fuzzy s -continuous, upper (lower) fuzzy almost s -continuous, upper (lower) fuzzy ℓ -continuous, upper (lower) fuzzy almost ℓ -continuous functions.

Fuzzy sets on a universe X will be denoted by Greek letters as μ, ρ, η , etc. Fuzzy points will be denoted by x_ε, y_ν , etc. For any fuzzy points x_ε and any fuzzy set μ , we write $x_\varepsilon \in \mu$ iff $\varepsilon \leq \mu(x)$. A fuzzy set x_ε is called quasi-coincident with a fuzzy set ρ , denoted by $x_\varepsilon q \rho$, iff $\varepsilon + \rho(x) > 1$. A fuzzy set μ is called quasi-coincident with a fuzzy set ρ , denoted by $\mu q \rho$, iff there exists a $x \in X$ such that $\mu(x) + \rho(x) > 1$. In this paper we use the concept of a fuzzy topological space as introduced by [1]. By $\text{int}(\mu)$, $\text{cl}(\mu)$ and $\text{co}(\mu)$, we mean the interior of μ , the closure of μ , and the complement of μ .

Let $F : X \rightarrow Y$ be a fuzzy multifunction from a fuzzy topological space X to a fuzzy topological space Y . For any fuzzy set $\mu \leq X$, $F^+(\mu)$ and $F^-(\mu)$ are defined by $F^+(\mu) = \{x \in X : F(x) \leq \mu\}$, $F^-(\mu) = \{x \in X : F(x) q \mu\}$. We know that $F^-(\text{co}(\beta)) = \text{co}(F^+(\beta))$ for any fuzzy set $\beta \leq Y$ [8].

2. FUZZY P -CONTINUOUS MULTIFUNCTIONS

Definition 1. Let (X, τ) be a fuzzy topological space and let $\mu \leq X$ be a fuzzy set. Then it is said that

- (i) μ is a fuzzy P -set if μ possesses fuzzy property P ,
- (ii) μ has fuzzy P -complement if $\text{co}(\mu)$ possesses fuzzy property P .

The following definition of fuzzy P -continuity is considered for fuzzy multisetting from [6] in the classical sense.

Definition 2. Let $F : X \rightarrow Y$ be a fuzzy multifunction from a fuzzy topological space (X, τ) to a fuzzy topological space (Y, ν) . Then it is said that F is fuzzy lower (upper) P -continuous if for each $x_\varepsilon \in X$ and for each fuzzy set μ having a P -complement such that $x_\varepsilon \in F^-(\mu)$ ($x_\varepsilon \in F^+(\mu)$), there exists an open fuzzy set ρ containing x_ε such that $\rho \leq F^-(\mu)$ ($\rho \leq F^+(\mu)$).

The following table gives us the list of some types of lower (upper) fuzzy P -continuous multifunctions with property P .

The definitions of lower (upper) fuzzy c -continuous, lower (upper) fuzzy almost c -continuous, lower (upper) fuzzy c^* -continuous, lower (upper) fuzzy s -continuous, lower (upper) fuzzy ℓ -continuous and lower (upper) fuzzy almost ℓ -continuous multifunctions are considered for fuzzy multisetting from [2, 3, 10, 4, 5] and [7], respectively.

The definitions of lower (upper) fuzzy continuous and lower (upper) fuzzy almost continuous multifunctions are considered from [8] and [9], respectively.

n°	Fuzzy P -set	Fuzzy lower P -con.	Fuzzy upper P -con.
1.	f. closed	f. l. con.	f. u. con.
2.	f. regular closed	f. l. almost con.	f. u. almost con.
3.	f. closed compact	f. l. c -con.	f. u. c -con.
4.	f. regular closed compact	f. l. almost c -con.	f. u. almost c -con.
5.	f. closed countable compact	f. l. c^* -con.	f. u. c^* -con.
6.	f. closed connected	f. l. s -con.	f. u. s -con.
7.	f. regular closed connected	f. l. almost s -con.	f. u. almost s -con.
8.	f. closed Lindelöf	f. l. ℓ -con.	f. u. ℓ -con.
9.	f. regular closed Lindelöf	f. l. almost ℓ -con.	f. u. almost ℓ -con.

We know that a net $(x_{\varepsilon_\alpha}^\alpha)$ in a fuzzy topological space (X, τ) is said to be eventually in the fuzzy set $\mu \leq X$ if there exists an index $\alpha_0 \in J$ such that $x_{\varepsilon_\alpha}^\alpha \in \mu$ for all $\alpha \geq \alpha_0$.

The following theorem gives us some characterisation of fuzzy lower (upper) P -continuous multifunctions.

Theorem 1. Let $F : X \rightarrow Y$ be a fuzzy multifunction from a fuzzy topological space (X, τ) to a fuzzy topological space (Y, ν) . Then the following statements are equivalent:

- (i) F is a lower (upper) fuzzy P -continuous multifunction.
- (ii) For each $x_\varepsilon \in X$ and for each fuzzy set μ having a P -complement such that $F(x_\varepsilon) \not\leq \mu$ ($F(x_\varepsilon) \leq \mu$), there exists an open fuzzy set ρ containing x_ε such that if $y_\beta \in \rho$, then $F(y_\beta) \not\leq \mu$ ($F(y_\beta) \leq \mu$).
- (iii) $F^-(\mu)$ ($F^+(\mu)$) is an open fuzzy set for any fuzzy set $\mu \leq Y$ having a P -complement.
- (iv) $F^+(\text{co}(\mu))$ ($F^-(\text{co}(\mu))$) is a closed fuzzy set for any fuzzy set $\mu \leq Y$ having a P -complement.
- (v) For each $x_\varepsilon \in X$ and for each net $(x_{\varepsilon_\alpha}^\alpha)$ which converges to x_ε in X and for each fuzzy set $\rho \leq Y$ having a P -complement such that $x_\varepsilon \in F^-(\rho)$ ($x_\varepsilon \in F^+(\rho)$), the net $(x_{\varepsilon_\alpha}^\alpha)$ is eventually in $F^-(\rho)$ ($F^+(\rho)$).

Proof. (i) $\Leftrightarrow$ (ii). This statement is obvious.

(i) $\Leftrightarrow$ (iii). Let $x_\varepsilon \in F^-(\mu)$ and let μ be a fuzzy set having a P -complement. From (i), there exists an open fuzzy set ρ containing x_ε such that $\rho \leq F^-(\mu)$. It follows that $x_\varepsilon \in \text{int}(F^-(\mu))$ and (iii) holds.

The converse can be shown easily.

(iii) $\Leftrightarrow$ (iv). Since $F^-(\text{co}(\mu)) = \text{co}(F^+(\mu))$ and $F^+(\text{co}(\mu)) = \text{co}(F^-(\mu))$, the proof is clear.

(i) $\Rightarrow$ (v). Let $x_{\varepsilon_\alpha}^\alpha$ be a net which converges to x_ε in X and let $\rho \leq Y$ be any fuzzy set having a P -complement such that $x_\varepsilon \in F^-(\rho)$. Since F is a lower fuzzy P -continuous multifunction, it follows that there exists an open fuzzy set $\mu \leq X$ containing x_ε such that $\mu \leq F^-(\rho)$. Since $x_{\varepsilon_\alpha}^\alpha$ converges to x_ε , it follows that there exists an index $\alpha_0 \in J$ such that $x_{\varepsilon_\alpha}^\alpha \in \mu$ for all $\alpha \geq \alpha_0$. From here, we obtain that $x_{\varepsilon_\alpha}^\alpha \in \mu \leq F^-(\rho)$ for all $\alpha \geq \alpha_0$. Thus, the net $(x_{\varepsilon_\alpha}^\alpha)$ is eventually in $F^-(\rho)$.

(v) $\Rightarrow$ (i). Suppose that (i) is not true. There exists a point x_ε and a fuzzy set μ having a P -complement with $x_\varepsilon \in F^-(\mu)$ such that $\rho \not\leq F^-(\mu)$ for each open fuzzy set $\rho \leq X$ containing x_ε . Let $x_{\varepsilon_\rho} \in \rho$ and $x_{\varepsilon_\rho} \notin F^-(\mu)$ for each open fuzzy set $\rho \leq X$ containing x_ε . Then for the neighbourhood net (x_{ε_ρ}) , $x_{\varepsilon_\rho} \rightarrow x_\varepsilon$, but (x_{ε_ρ}) is not eventually in $F^-(\mu)$. This is a contradiction. Thus, F is a fuzzy lower (upper) P -continuous multifunction.

The proof of the fuzzy upper P -continuity of F is similar to that given above. $\square$

Example. Let $F : X \rightarrow Y$ be a fuzzy multifunction from a fuzzy topological space (X, τ) to a fuzzy topological space (Y, ν) . Then the following statements are equivalent to Theorem 1 when we take fuzzy P -set as fuzzy closed compact set:

- (i) F is a lower (upper) fuzzy c -continuous multifunction,

- (ii) For each $x_\varepsilon \in X$ and for each fuzzy open set μ having a compact complement such that $F(x_\varepsilon) \not\leq \mu$ ($F(x_\varepsilon) \leq \mu$), there exists an open fuzzy set ρ containing x_ε such that if $y_\beta \in \rho$, then $F(y_\beta) \not\leq \mu$ ($F(y_\beta) \leq \mu$),
- (iii) $F^-(\mu)$ ($F^+(\mu)$) is an open fuzzy set for any fuzzy open set $\mu \leq Y$ having a compact complement,
- (iv) $F^+(1 - \mu)$ ($F^-(1 - \mu)$) is a closed fuzzy set for any fuzzy open set $\mu \leq Y$ having a compact complement,
- (v) For each $x_\varepsilon \in X$ and for each net $(x_{\varepsilon_\alpha}^\alpha)$ which converges to x_ε in X and for each fuzzy open set $\rho \leq Y$ having a compact complement such that $x_\varepsilon \in F^-(\rho)$ ($x_\varepsilon \in F^+(\rho)$), the net $(x_{\varepsilon_\alpha}^\alpha)$ is eventually in $F^-(\rho)$ ($F^+(\rho)$).

Definition 3. Suppose that (X, τ) , (Y, ν) and (Z, ω) are fuzzy topological spaces. If $F_1 : X \rightarrow Y$ and $F_2 : Y \rightarrow Z$ are fuzzy multifunctions, then the fuzzy multifunction $F_2 \circ F_1 : X \rightarrow Z$ is defined by $(F_2 \circ F_1)(x_\varepsilon) = F_2(F_1(x_\varepsilon))$.

Theorem 2. Let (X, τ) , (Y, ν) , (Z, ω) be fuzzy topological spaces and let $F : X \rightarrow Y$ and $G : Y \rightarrow Z$ be fuzzy multifunctions. If $F : X \rightarrow Y$ is an upper fuzzy continuous multifunction and $G : Y \rightarrow Z$ is an upper fuzzy P -continuous multifunction, then $G \circ F : X \rightarrow Z$ is an upper fuzzy P -continuous multifunction.

Proof. Let $\mu \leq Z$ be any fuzzy set having a P -complement. From the definition of $G \circ F$, we have $(G \circ F)^+(\mu) = F^+(G^+(\mu))$. Since G is an upper fuzzy P -continuous multifunction, it follows that $G^+(\mu)$ is an open fuzzy set. Since F is an upper fuzzy continuous multifunction, it follows that $F^+(G^+(\mu))$ is an open fuzzy set. This shows that $G \circ F$ is an upper fuzzy P -continuous multifunction. $\square$

Theorem 3. Let $F : X \rightarrow Y$ be a fuzzy multifunction from a fuzzy topological space (X, τ) to a fuzzy topological space (Y, ν) . If F is a lower (upper) fuzzy P -continuous multifunction and $\mu \leq X$ is a fuzzy set, then the restriction multifunction $F|_\mu : \mu \rightarrow Y$ is a lower (upper) fuzzy P -continuous multifunction.

Proof. Suppose that $\beta \leq Y$ is a fuzzy set having a P -complement. Let $x_\varepsilon \in \mu$ and let $x_\varepsilon \in F|_\mu(\beta)$. Since F is a lower fuzzy P -continuous multifunction, it follows that there exists a fuzzy open set ρ such that $\rho \leq F^-(\beta)$. From here we obtain that $x_\varepsilon \in \rho \wedge \mu$ and $\rho \wedge \mu \leq F|_\mu(\beta)$. Thus, we show that the restriction multifunction $F|_\mu$ is lower fuzzy P -continuous.

The proof of the fuzzy upper P -continuity of $F|_\mu$ is similar to the above. $\square$

Theorem 4. Let $f : X \rightarrow Y$ be a fuzzy function from a fuzzy topological space (X, τ) to a fuzzy topological space (Y, ν) . Let $\{\gamma_\alpha : \alpha \in \Phi\}$ be an open cover of X . If the restriction multifunction $F_\alpha = F_{\gamma_\alpha}$ is a lower (upper) fuzzy P -continuous multifunction for each $\alpha \in \Phi$, then F is lower (upper) fuzzy P -continuous.

Proof. Let $\mu \leq Y$ be a fuzzy set having a P -complement. Since F_α is lower fuzzy P -continuous for each α , from Theorem 1, $F_\alpha^-(\mu) \leq \text{int}_{\gamma_\alpha}(F_\alpha^-(\mu))$ and from here

$F^-(\mu) \wedge \gamma_\alpha \leq \text{int}_{\gamma_\alpha}(F^{-1}(\mu) \wedge \gamma_\alpha)$ and $F^-(\mu) \wedge \gamma_\alpha \leq \text{int}(F^{-1}(\mu)) \wedge \gamma_\alpha$. Since $\{\gamma_\alpha : \alpha \in \Phi\}$ is an open cover of X , it follows that $F^-(\mu) \leq \text{int}(F^{-1}(\mu))$. Thus from Theorem 1, we obtain that F is a lower fuzzy P -continuous multifunction.

The proof of the fuzzy upper P -continuity of F is similar to the above. $\square$

Definition 4. Suppose that $F : X \rightarrow Y$ is a fuzzy multifunction from a fuzzy topological space X to a fuzzy topological space Y . The fuzzy graph multifunction $G_F : X \rightarrow X \times Y$ of F is defined as $G_F(x_\varepsilon) = \{x_\varepsilon\} \times F(x_\varepsilon)$ [8].

Theorem 5. Suppose that a finite product of fuzzy P -sets is a fuzzy P -set and X is a fuzzy P -set. Let $F : X \rightarrow Y$ be a fuzzy multifunction from a fuzzy topological space (X, τ) to a fuzzy topological space (Y, ν) . If the graph function of F is lower (upper) fuzzy P -continuous, then F is a lower (upper) fuzzy P -continuous multifunction.

Proof. For the fuzzy sets $\beta \leq X$, $\eta \leq Y$, we take

$$(\beta \times \eta)(z, y) = \begin{cases} 0 & \text{if } z \notin \beta \\ \eta(y) & \text{if } z \in \beta. \end{cases}$$

Let $x_\varepsilon \in X$ and let $\mu \leq Y$ be a fuzzy set having a P -complement such that $x_\varepsilon \in F^-(\mu)$. We obtain that $x_\varepsilon \in G_F^-(X \times \mu)$. Since fuzzy graph multifunction G_F is fuzzy lower P -continuous and X is a fuzzy P -set, it follows that there exists an open fuzzy set $\rho \leq X$ containing x_ε such that $\rho \leq G_F^-(X \times \mu)$. From here, we obtain that $\rho \leq F^-(\mu)$. Thus, F is a fuzzy lower P -continuous multifunction.

The proof of the fuzzy upper P -continuity of F is similar to the above. $\square$

Theorem 6. Suppose that (X, τ) and (X_α, τ_α) are fuzzy topological spaces and X_α is a fuzzy P -set where $\alpha \in J$. Let $F : X \rightarrow \prod_{\alpha \in J} X_\alpha$ be a multifunction from X to the product space $\prod_{\alpha \in J} X_\alpha$ and let $P_\alpha : \prod_{\alpha \in J} X_\alpha \rightarrow X_\alpha$ be the projection multifunction for each $\alpha \in J$ which is defined by $P_\alpha((x_\alpha)) = \{x_\alpha\}$ and let a product of fuzzy P -sets be a fuzzy P -set. If F is a fuzzy upper P -continuous multifunction, then $P_\alpha \circ F$ is a fuzzy upper P -continuous multifunction for each $\alpha \in J$.

Proof. Take any $\alpha_0 \in J$. Let $x_\varepsilon \in X$ and let $\mu_{\alpha_0} \leq X_{\alpha_0}$ be a fuzzy set having a P -complement such that $x_\varepsilon \in (P_{\alpha_0} \circ F)^+(\mu_{\alpha_0})$. We know that $x_\varepsilon \in (P_{\alpha_0} \circ F)^+(\mu_{\alpha_0}) = F^+(P_{\alpha_0}^+(\mu_{\alpha_0})) = F^+(\mu_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$. Since F is a fuzzy upper P -continuous multifunction and a product of fuzzy P -sets is a fuzzy P -set, it follows that there exists an open fuzzy set $\rho \leq X$ containing x_ε such that $\rho \leq F^+(\mu_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha) = F^+(P_{\alpha_0}^+(\mu_{\alpha_0})) = (P_{\alpha_0} \circ F)^+(\mu_{\alpha_0})$. This shows that $P_{\alpha_0} \circ F$ is a fuzzy upper P -continuous multifunction.

Thus we obtain that $P_\alpha \circ F$ is a fuzzy upper P -continuous multifunction for each $\alpha \in J$. $\square$

Theorem 7. Suppose that for each $\alpha \in J$, (X_α, τ_α) , (Y_α, ν_α) are fuzzy topological spaces and Y_α possesses property P for each $\alpha \in J$. Let $F_\alpha : X_\alpha \rightarrow Y_\alpha$ be a

multifunction for each $\alpha \in J$ and let $F : \prod_{\alpha \in J} X_\alpha \rightarrow \prod_{\alpha \in J} Y_\alpha$ be defined by $F((x_\alpha)) = \prod_{\alpha \in J} F_\alpha(x_\alpha)$ from the product space $\prod_{\alpha \in J} X_\alpha$ to the product space $\prod_{\alpha \in J} Y_\alpha$. If F is an upper fuzzy P -continuous multifunction and the product of fuzzy P -sets is a fuzzy P -set, then each F_α is an upper fuzzy P -continuous multifunction for each $\alpha \in J$.

Proof. Let $\mu_\alpha \leq Y_\alpha$ be a fuzzy set having a P -complement. Then $\mu_\alpha \times \prod_{\alpha \neq \beta} Y_\beta$ is a fuzzy set having a P -complement. Since F is an upper fuzzy P -continuous multifunction, it follows that $F^+(\mu_\alpha \times \prod_{\alpha \neq \beta} Y_\beta) = F^+(\mu_\alpha) \times \prod_{\alpha \neq \beta} X_\beta$ is an open fuzzy set. Consequently, we obtain that $F^+(\mu_\alpha)$ is an open fuzzy set. Thus, we show that F_α is an upper fuzzy P -continuous multifunction. $\square$

Theorem 8. Let $F : X \rightarrow Y$ be a fuzzy function from a fuzzy topological space (X, τ) to a fuzzy topological space (Y, ν) . Suppose that Y has a base of neighbourhood such that the complement of each fuzzy set of the base of neighbourhood is the finite union of P -sets. If F is lower fuzzy P -continuous multifunction, then F is lower fuzzy continuous.

Proof. Let $x_\varepsilon \in X$ and let μ be any fuzzy open set such that $x_\varepsilon \in F^-(\mu)$. Then there exists a fuzzy point $y_\nu \in \mu$ such that $x_\varepsilon \in F^-(y_\nu)$. Since Y has a base of neighbourhood such that the complement of each fuzzy set of the base of neighbourhood is finite unions of P -sets, it follows that there exists a neighbourhood β of y_ν such that $\beta \leq \mu$ and $\text{co}(\beta) = \bigvee_{i=1}^n \eta_i$ where each η_i is a P -set. From here we obtain that $\beta = \bigwedge_{i=1}^n \text{co}(\eta_i)$. Then $y_\nu \in \text{co}(\eta_i)$ and $x_\varepsilon \in F^-(\text{co}(\eta_i))$ for each $i = 1, 2, \dots, n$. Since F is a lower fuzzy P -continuous multifunction, it follows that there exists an open set ρ_i containing x_ε such that $\rho_i \leq F^-(\text{co}(\eta_i))$. We take $\rho = \bigwedge_{i=1}^n \rho_i$, then $\rho = \bigwedge_{i=1}^n \rho_i \leq \bigwedge_{i=1}^n F^-(\text{co}(\eta_i)) = F^-(\bigwedge_{i=1}^n \text{co}(\eta_i)) = F^-(\beta) \leq F^-(\mu)$. Thus, we obtain that F is a lower fuzzy P -continuous multifunction. $\square$

Theorem 9. Suppose that $(X_1, \tau_1), (X_2, \tau_2), (Y_1, \nu_1)$ and (Y_2, ν_2) are fuzzy topological spaces and $F_1 : X_1 \rightarrow Y_1, F_2 : X_2 \rightarrow Y_2$ are fuzzy multifunctions and suppose that $\eta \times \beta$ is a fuzzy P -set iff η and β are fuzzy P -sets for any fuzzy sets $\eta \leq Y_1, \beta \leq Y_2$. Let $F_1 \times F_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ be a fuzzy multifunction which is defined by $(F_1 \times F_2)(x_\varepsilon, y_\nu) = F_1(x_\varepsilon) \times F_2(y_\nu)$. Then $F_1 \times F_2$ is an upper fuzzy P -continuous multifunction iff F_1 and F_2 are upper fuzzy P -continuous multifunctions.

Proof. We know that $(\mu^* \times \beta^*)(x_\varepsilon, y_\nu) = \min\{\mu^*(x), \beta^*(y)\}$ for any fuzzy sets μ^*, β^* and for any fuzzy points x_ε, y_ν .

Let $\mu \times \beta \leq Y_1 \times Y_2$ be a fuzzy set having a P -complement. We know that $(F_1 \times F_2)^+(\mu \times \beta) = F_1^+(\mu) \times F_2^+(\beta)$ and $\rho \times \xi$ is a fuzzy P -set iff ρ and ξ are fuzzy P -sets for any fuzzy sets $\rho \leq Y_1, \xi \leq Y_2$. From Theorem 1, the proof is obtained. $\square$

Theorem 10. Suppose that $(X, \tau), (Y, \nu), (Z, \omega)$ are fuzzy topological spaces and $F_1 : X \rightarrow Y, F_2 : X \rightarrow Z$ are fuzzy multifunctions and suppose that $\eta \times \beta$ is a fuzzy P -set iff η and β are fuzzy P -sets for any fuzzy sets $\eta \leq Y, \beta \leq Z$. Let $F_1 \times F_2 : X \rightarrow Y \times Z$ be a fuzzy multifunction which is defined by $(F_1 \times F_2)(x_\varepsilon) = F_1(x_\varepsilon) \times F_2(x_\varepsilon)$. Then

$F_1 \times F_2$ is an upper fuzzy P -continuous multifunction iff F_1 and F_2 are upper fuzzy P -continuous multifunctions.

Proof. ($\Rightarrow$) Let $x_\varepsilon \in X$ and let $\mu \leq Y, \beta \leq Z$ be fuzzy sets having P -complements such that $x_\varepsilon \in F_1^+(\mu)$ and $x_\varepsilon \in F_2^+(\beta)$. Then we obtain that $F_1(x_\varepsilon) \leq \mu$ and $F_2(x_\varepsilon) \leq \beta$ and from here, $F_1(x_\varepsilon) \times F_2(x_\varepsilon) = (F_1 \times F_2)(x_\varepsilon) \leq \mu \times \beta$. We have $x_\varepsilon \in (F_1 \times F_2)^+(\mu \times \beta)$. Since $F_1 \times F_2$ is an upper fuzzy P -continuous multifunction, it follows that there exists an open fuzzy set ρ containing x_ε such that $\rho \leq (F_1 \times F_2)^+(\mu \times \beta)$. We obtain that $\rho \leq F_1^+(\mu)$ and $\rho \leq F_2^+(\beta)$. Thus we obtain that F_1 and F_2 are upper fuzzy P -continuous multifunctions.

($\Leftarrow$) Let $x_\varepsilon \in X$ and let $\mu \times \rho \leq Y \times Z$ be a fuzzy set having a P -complement such that $x_\varepsilon \in (F_1 \times F_2)^+(\mu \times \rho)$. We obtain that $(F_1 \times F_2)(x_\varepsilon) = F_1(x_\varepsilon) \times F_2(x_\varepsilon) \leq \mu \times \rho$. From here we have $F_1(x_\varepsilon) \leq \mu$ and $F_2(x_\varepsilon) \leq \rho$ and then $x_\varepsilon \in F_1^+(\mu)$ and $x_\varepsilon \in F_2^+(\rho)$. Since F_1 and F_2 are upper fuzzy P -continuous multifunctions, it follows that there exist fuzzy open sets η containing x_ε and β containing x_ε such that $\eta \leq F_1^+(\mu)$ and $\beta \leq F_2^+(\rho)$. We take $x_\varepsilon \in \eta \wedge \beta$. Then we obtain that $\eta \wedge \beta \leq F_1^+(\mu)$ and $\eta \wedge \beta \leq F_2^+(\rho)$. Hence we have $\eta \wedge \beta \leq (F_1 \times F_2)^+(\mu \times \rho)$. Thus, we show that $F_1 \times F_2$ is an upper fuzzy P -continuous multifunction. $\square$

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DYNAMICS OF NONLINEAR DIATOMIC LATTICES

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ABSTRACT. A model of nonlinear diatomic lattice is studied. We suppose small damping, small forcing and weak coupling between the lattices. We show the existence of breathers for undamped and unforced cases. The existence of chaos is shown for damped and forced cases. For lattices with nonsmall parameters, the existence of travelling waves is discussed.

Mathematics Subject Classification: Primary 34C25; Secondary 34C37, 37K60, 37L60

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1. INTRODUCTION

Let us consider a model of two one-dimensional interacting sublattices of harmonically coupled protons and heavy ions [11, 12, 15, 16]. It represents the Bernal-Flower filaments in ice or more complex biological macromolecules in membranes, in which only the degrees of freedom that contribute predominantly to proton mobility have been conserved. In these systems, each proton lies between a pair of ‘oxygens.’ The proton part of the Hamiltonian is

$$H_p = \sum_n \frac{1}{2} m \dot{u}_n^2 + U(u_n) + \frac{1}{2} k_1 (u_{n+1} - u_n)^2,$$

where u_n denotes the displacement of the n th proton with respect to the center of the oxygen pair and k_1 is the coupling between neighboring protons. Furthermore,

$$U(u) = \xi_0 \left(1 - u^2/d_0^2\right)^2$$

is the double-well potential with the potential barrier ξ_0 , and $2d_0$ is the distance between its two minima. Finally, m is the mass of protons.

Similarly, the oxygen part of the Hamiltonian is

$$H_o = \sum_n \frac{1}{2} M \dot{\varrho}_n^2 + \frac{1}{2} M \Omega_0^2 \varrho_n^2 + \frac{1}{2} K_1 (\varrho_{n+1} - \varrho_n)^2,$$

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where ϱ_n is the displacement between two oxygens, M is the mass of oxygens, Ω_0 is the frequency of the optical mode and K_1 is the harmonic coupling between neighboring oxygens.

The last part in the Hamiltonian of the model arises from the dynamic interaction between two sublattices and it is given by

$$H_{int} = \sum_n \chi \varrho_n (u_n^2 - d_0^2),$$

where χ measures the strength of the coupling. The Hamiltonian of the model is the sum of these three contributions

$$H = H_p + H_O + H_{int}.$$

We are also interested in the influence of external field and damping. For the model studied here, since a spatially homogeneous field is not coupled to the optical motion ϱ_n of the oxygens, a force term has to be considered only in the equation of motion of the protons.

Summarizing, we consider in this note the following coupled infinite chain of oscillators

$$\begin{aligned} \ddot{u}_n + \Gamma_1 \dot{u}_n &= \frac{k_1}{m} (u_{n+1} - 2u_n + u_{n-1}) + \frac{4\xi_0}{m d_0^2} u_n \left(1 - \frac{u_n^2}{d_0^2} \right) \\ &\quad - 2 \frac{\chi}{m} \varrho_n u_n + \frac{F}{m}, \\ \ddot{\varrho}_n + \Gamma_2 \dot{\varrho}_n &= \frac{K_1}{M} (\varrho_{n+1} - 2\varrho_n + \varrho_{n-1}) - \Omega_0^2 \varrho_n - \frac{\chi}{M} (u_n^2 - d_0^2), \end{aligned} \quad (1.1)$$

where F is the external force on the protons and Γ_1, Γ_2 are the damping coefficients for the proton and oxygen motions.

The contents of the paper is as follows. In Section 2, we study the weakly coupled undamped and unforced chain (1.1), i. e., with $\Gamma_1 = \Gamma_2 = 0$, $F = 0$ and the coupling parameters are small. We show under certain nonresonance conditions the existence of time periodic spatially localized solutions of (1.1), the so-called breathers [5, 9, 14]. Section 3 is devoted to the case when (1.1) is weakly forced and weakly damped with again weak coupling. By using a Melnikov method [10], we show under certain conditions the existence of spatially localized Smale horseshoes in (1.1), i. e., spatially localized chaos in (1.1). The localization of the Smale horseshoes is not exponential as for the breathers. Section 4 deals with (1.1) when the involved parameters are not small. The existence of travelling waves for (1.1) and also for its continuum limit is discussed.

2. WEAKLY COUPLED UNDAMPED AND UNFORCED LATTICES: THE EXISTENCE OF BREATHERS

We assume in this section that $\Gamma_1 = \Gamma_2 = F = 0$, $k_1/m = \varepsilon\mu_1$, $K_1/M = \varepsilon\mu_2$, $-2\chi/m = \varepsilon\mu_3$, $-\chi/M = \varepsilon\mu_4$ for a small parameter ε and constants μ_i , $i = 1, 2, 3, 4$.

Hence (1.1) has the form

$$\begin{aligned}\ddot{u}_n &= \varepsilon\mu_1(u_{n+1} - 2u_n + u_{n-1}) + a^2u_n(d_0^2 - u_n^2) + \varepsilon\mu_3\varrho_n u_n, \\ \ddot{\varrho}_n &= \varepsilon\mu_2(\varrho_{n+1} - 2\varrho_n + \varrho_{n-1}) - \Omega_0^2\varrho_n + \varepsilon\mu_4(u_n^2 - d_0^2),\end{aligned}\quad (2.1)$$

where $a^2 = \frac{4\xi_0}{md_0^4}$. Then for $\varepsilon = 0$, we get from (2.1) the uncoupled system

$$\ddot{u}_n = a^2(d_0^2 - u_n^2)u_n, \quad \ddot{\varrho}_n = -\Omega_0^2\varrho_n.$$

The equation

$$\dot{u} = v, \quad \dot{v} = a^2(d_0^2 - u^2)u \quad (2.2)$$

has a hyperbolic equilibrium $u = v = 0$ and centers $u = \pm d_0, v = 0$ [17]. Furthermore, equation (2.2) has two symmetric periodic solutions $(\gamma_\beta(t), \dot{\gamma}_\beta(t))$ and $(-\gamma_\beta(t), -\dot{\gamma}_\beta(t))$ around $(\pm d_0, 0)$ with periods β monotonically increasing from $\frac{\sqrt{2}\pi}{ad_0}$ to $+\infty$. They accumulate on two symmetric homoclinic solutions $(\gamma(t), \dot{\gamma}(t))$ and $(-\gamma(t), -\dot{\gamma}(t))$ for

$$\gamma(t) = \sqrt{2}d_0 \operatorname{sech} ad_0 t.$$

We assume that $\gamma_\beta(t)$ and $\gamma(t)$ are even functions. We are interested, in this section, in spatially localized time periodic solutions of (2.1) which are called breathers. We use the approach of [5, 9, 14]. For this reason, we take the exchange $u_n \leftrightarrow u_n + d_0$ in (2.1) to get

$$\begin{aligned}\ddot{u}_n + a^2u_n(u_n + d_0)(u_n + 2d_0) \\ - \varepsilon\mu_1(u_{n+1} - 2u_n + u_{n-1}) - \varepsilon\mu_3\varrho_n(u_n + d_0) &= 0, \\ \ddot{\varrho}_n + \Omega_0^2\varrho_n - \varepsilon\mu_2(\varrho_{n+1} - 2\varrho_n + \varrho_{n-1}) - \varepsilon\mu_4u_n(u_n + 2d_0) &= 0.\end{aligned}\quad (2.3)$$

Now we fix a constant $\tau > 1$ and consider the Banach spaces

$$\begin{aligned}X_i &= \left\{ \{(u_n(t), \varrho_n(t))\}_{n \in \mathbb{Z}} \in C^i(\mathbb{R}, \mathbb{R})^{2\mathbb{Z}} \mid u_n, \varrho_n \text{ are even,} \right. \\ &\quad \left. \beta\text{-periodic, and such that } \sup_n \tau^{|n|}(|u_n(\cdot)|_i + |\varrho_n(\cdot)|_i) < \infty \right\},\end{aligned}$$

where $i = 0, 2$ and $|\cdot|_i$ is the usual maximum norm on $C^i([0, \beta], \mathbb{R})$. The norm on X_i for $x = \{(u_n(t), \varrho_n(t))\}_{n \in \mathbb{Z}}$ is defined as $\|x\| = \sup_n \tau^{|n|}(|u_n(\cdot)|_i + |\varrho_n(\cdot)|_i)$. The left-hand side of (2.3) defines the mapping $F : X_2 \times \mathbb{R} \rightarrow X_0$ with $\varepsilon \in \mathbb{R}$. Hence (2.3) has the form

$$F(x, \varepsilon) = 0. \quad (2.4)$$

We take $x_0 \in X_2$ with $u_n = 0$ for $n \neq 0$, $u_0 = \gamma_\beta(t) - d_0$ and $\varrho_n = 0$ for any n . Then $F(x_0, 0) = 0$. Clearly $F \in C^\infty(X_2 \times \mathbb{R}, X_0)$. We solve (2.4) by using the implicit

function theorem. The linearization $D_x F(x_0, 0) : X_2 \rightarrow X_0$ has the form

$$\begin{aligned} &(\ddot{u}_n + 2a^2 d_0^2 u_n, \ddot{q}_n + \Omega_0^2 q_n), \quad n \neq 0, \\ &(\ddot{u}_0 + a^2(3\gamma_\beta(t)^2 - d_0^2)u_0, \ddot{q}_0 + \Omega_0^2 q_0). \end{aligned}$$

The equation

$$\ddot{v} + a^2(3\gamma_\beta(t)^2 - d_0^2)v = 0$$

has the solutions $v_1(t) = \dot{\gamma}_\beta(t)$ and $v_2(t) = D_\beta \gamma_\beta(t)$. Function $v_1(t)$ is odd and β -periodic, while function $v_2(t)$ is even and satisfies the relation $v_2(t + \beta) = v_2(t) - \dot{\gamma}_\beta(t + \beta)$. Since $\dot{\gamma}_\beta(t + \beta) \neq 0$, the function $v_2(t)$ is not β -periodic. We note that the existence of an even and β -periodic solution of the equation

$$\ddot{v} + a^2(3\gamma_\beta(t)^2 - d_0^2)v = h(t) \quad (2.5)$$

for an even and β -periodic continuous function $h(t)$, is equivalent to the existence of a solution of the following boundary value problem

$$\begin{aligned} \ddot{v} + a^2(3\gamma_\beta(t)^2 - d_0^2)v &= h(t) \\ \dot{v}(0) &= \dot{v}(\beta/2) = 0. \end{aligned} \quad (2.6)$$

Since the homogeneous problem of (2.6) with $h(t) = 0$ has the only zero solution, we get that (2.6) is uniquely solvable. Consequently, (2.5) has a unique even and β -periodic solution.

Furthermore, we can directly check that the equations

$$\begin{aligned} \ddot{u} + 2a^2 d_0^2 u &= h(t), \\ \ddot{q} + \Omega_0^2 q &= h(t) \end{aligned}$$

have unique even and β -periodic solutions for any even and β -periodic continuous function $h(t)$ if the following nonresonance conditions hold

$$2\sqrt{2} \frac{\sqrt{\xi_0}}{\sqrt{m}d_0} \beta = \sqrt{2}ad_0\beta \neq 2\pi k, \quad \Omega_0\beta \neq 2\pi k \quad (2.7)$$

for any $k \in \mathbb{N}$. Consequently, we see that $D_x F(x_0, 0) : X_2 \rightarrow X_0$ is continuously invertible if (2.7) hold. The implicit function theorem implies the following result.

Theorem 2.1. *Let $\tau > 1$ be given. If the conditions (2.7) hold, then the chain (2.1) has a β -periodic solution $\{(u_n(t), q_n(t))\}_{n \in \mathbb{Z}}$ for any ε small such that*

$$\sup_{t,n} (|u_n(t) - d_0| + |q_n(t)|) \tau^{|n|} < \infty.$$

Moreover, the relation $q_n(t) = O(\varepsilon)$ is true for any n . Furthermore, $u_n(t) = d_0 + O(\varepsilon)$ for $n \neq 0$ and $u_0(t) = \gamma_\beta(t) + O(\varepsilon)$.

Under assumptions of Theorem 2.1, we get more complicated dynamics of (2.1). Namely, if we start from $x_0 \in X_2$ such that $\varrho_n = 0$ for any n , $u_n = 0$ for large n while $u_n = \gamma_\beta(t) - d_0$ for several finite numbers of n . Then we get for any ε small, under conditions (2.7), the existence of multi-site breathers. Furthermore, if we take an infinite number of excitations, i. e., $u_n = \gamma_\beta(t) - d_0$ for infinite numbers of n in the above construction of x_0 , then we can repeat the proof of Theorem 2.1 for $\tau = 1$ to get β -periodic solutions of (2.1), which are not spatially localized. But they are near to $(\gamma_\beta(t), 0)$ in infinitely many modes n . Moreover, the same arguments hold when we consider $-d_0$ instead of d_0 in the above considerations, i. e., we take the exchange $u_n \leftrightarrow u_n - d_0$ in (2.1). Furthermore, if (2.7) hold for some β_0 , then (2.7) hold also for any β near to β_0 and Theorem 2.1 can be applied uniformly for such β and ε small. In particular, we get under assumptions of Theorem 2.1 in (2.1) for ε small infinitely many 1-parametric families of breathers.

Finally, if we start from x_0 such that either $u_n = -d_0$, $\varrho_n = 0$ or $u_n = \gamma_\beta(t) - d_0$ and $\varrho_n = 0$, then we consider $F(x, \varepsilon) : X_2 \times \mathbb{R} \rightarrow X_0$ for $\tau = 1$, and we can repeat the proof of Theorem 2.1 to get the next result.

Theorem 2.2. *If the conditions*

$$\Omega_0 \beta \neq 2\pi k \quad \text{for any } k \in \mathbb{N}$$

hold, then for any $E = \{e_n\}_{n \in \mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$ the chain (2.1) has a β -periodic, in general not localized spatially, solution $\{(u_{n,E}(t), \varrho_{n,E}(t))\}_{n \in \mathbb{Z}}$ for any ε small such that $\varrho_{n,E}(t) = O(\varepsilon)$ for any n , and $u_{n,E}(t) = O(\varepsilon)$ for n with $e_n = 1$ and $u_{n,E}(t) = \gamma_\beta(t) + O(\varepsilon)$ for n with $e_n = 0$.

We also note that the above results can be extended to the case when for (2.3) the element x_0 is such that $\varrho_n = 0$ for any n , and either $u_n(t) = \gamma_\beta(t) - d_0$ or $u_n = 0$ or $u_n = -d_0$ or $u_n = -2d_0$. Summarizing we see that the dynamics of (2.1) is rather complicated for $\varepsilon \neq 0$ small.

3. GENERAL WEAKLY COUPLED LATTICES: THE EXISTENCE OF CHAOS

We assume in this section that $\Gamma_1 = \varepsilon\delta_1$, $\Gamma_2 = \varepsilon\delta_2$, $F = \varepsilon f(t)$, $k_1/m = \varepsilon\mu_1$, $K_1/M = \varepsilon\mu_2$, $-2\chi/m = \varepsilon\mu_3$, $-\chi/M = \varepsilon\mu_4$ for a small parameter ε , constants $\delta_1 \geq 0$, $\delta_2 > 0$, μ_i , $i = 1, 2, 3, 4$, and a C^1 -smooth T -periodic function $f(t)$. Hence (1.1) has the form

$$\begin{aligned} \ddot{u}_n + \varepsilon\delta_1 \dot{u}_n + a^2 u_n (u_n^2 - d_0^2) &= \varepsilon\mu_1 (u_{n+1} - 2u_n + u_{n-1}) \\ &\quad + \varepsilon\mu_3 \varrho_n u_n + \varepsilon f(t), \\ \ddot{\varrho}_n + \varepsilon\delta_2 \dot{\varrho}_n + \Omega_0^2 \varrho_n &= \varepsilon\mu_2 (\varrho_{n+1} - 2\varrho_n + \varrho_{n-1}) \\ &\quad + \varepsilon\mu_4 (u_n^2 - d_0^2). \end{aligned} \tag{3.1}$$

We first consider the system

$$\begin{aligned} \ddot{u} + \varepsilon \delta_1 \dot{u} + a^2 u(u^2 - d_0^2) &= \varepsilon \mu_3 \varrho u + \varepsilon f(t), \\ \ddot{\varrho} + \varepsilon \delta_2 \dot{\varrho} + \Omega_0^2 \varrho &= \varepsilon \mu_4 (u^2 - d_0^2). \end{aligned} \quad (3.2)$$

We make the change of variable $\varrho \leftrightarrow \varrho - \frac{\varepsilon \mu_4 d_0^2}{\Omega_0^2}$ in (3.2) to get

$$\begin{aligned} \ddot{u} + \varepsilon \delta_1 \dot{u} + a^2 u(u^2 - d_0^2) &= \varepsilon \mu_3 \left(\varrho - \frac{\varepsilon \mu_4 d_0^2}{\Omega_0^2} \right) u + \varepsilon f(t), \\ \ddot{\varrho} + \varepsilon \delta_2 \dot{\varrho} + \Omega_0^2 \varrho &= \varepsilon \mu_4 u^2. \end{aligned}$$

To study a small T -periodic solution of the above system, we take its equivalent form

$$\begin{aligned} \ddot{u} + \varepsilon \delta_1 \dot{u} + a^2 u(u^2 - d_0^2) \\ = \varepsilon \mu_3 \left(\frac{\varepsilon \mu_4}{\Omega_\varepsilon} \int_{-\infty}^t e^{-\varepsilon \delta_2(t-s)/2} \sin \Omega_\varepsilon(t-s) u^2(s) ds - \frac{\varepsilon \mu_4 d_0^2}{\Omega_0^2} \right) u + \varepsilon f(t), \end{aligned} \quad (3.3)$$

where $\Omega_\varepsilon = \sqrt{\Omega_0^2 - \frac{\varepsilon^2 \delta_2^2}{4}}$ and $0 < \varepsilon < 2\Omega_0/\delta_2$. Now it is not difficult to prove in (3.3) by using the implicit function theorem the existence of a unique small T -periodic solution $u_\varepsilon(t) = O(\varepsilon)$, $\varrho_\varepsilon(t) = O(\varepsilon)$ of (3.2). Then we make in (3.1) the change of variables $u_n \leftrightarrow u_n + u_\varepsilon$, $\varrho_n \leftrightarrow \varrho_n + \varrho_\varepsilon$ to get the chain

$$\begin{aligned} \dot{u}_n &= v_n, \\ \dot{v}_n + \varepsilon \delta_1 v_n - a^2 u_n d_0^2 + a^2 u_n^3 + 3a^2 u_n^2 u_\varepsilon + 3a^2 u_n u_\varepsilon^2 \\ &= \varepsilon \mu_1 (u_{n+1} - 2u_n + u_{n-1}) + \varepsilon \mu_3 (\varrho_n u_n + \varrho_n u_\varepsilon + \varrho_\varepsilon u_n), \\ \dot{\varrho}_n &= \psi_n, \\ \dot{\psi}_n + \varepsilon \delta_2 \psi_n + \Omega_0^2 \varrho_n &= \varepsilon \mu_2 (\varrho_{n+1} - 2\varrho_n + \varrho_{n-1}) + \varepsilon \mu_4 (u_n^2 + 2u_\varepsilon u_n). \end{aligned} \quad (3.4)$$

We consider (3.4) as an ordinary differential equation in the Hilbert space

$$H := \left\{ z = \{(u_n, v_n, \varrho_n, \psi_n)\}_{n \in \mathbb{Z}} \mid \sum_{n \in \mathbb{Z}} (u_n^2 + v_n^2 + \varrho_n^2 + \psi_n^2) < \infty \right\}$$

with the norm $\|z\| = \sqrt{\sum_{n \in \mathbb{Z}} (u_n^2 + v_n^2 + \varrho_n^2 + \psi_n^2)}$. The inhomogeneous linearization of (3.4) at $z = 0$ has the form

$$\begin{aligned} \dot{u}_n &= v_n + h_{n1}(t), \\ \dot{v}_n + \varepsilon \delta_1 v_n + u_n(3a^2 u_\varepsilon^2 - a^2 d_0^2 - \varepsilon \mu_3 \varrho_\varepsilon) \\ &\quad - \varepsilon \mu_1 (u_{n+1} - 2u_n + u_{n-1}) - \varepsilon \mu_3 \varrho_n u_\varepsilon = h_{n2}(t), \\ \dot{\varrho}_n &= \psi_n + g_{n1}(t), \\ \dot{\psi}_n + \varepsilon \delta_2 \psi_n + \Omega_0^2 \varrho_n - \varepsilon \mu_2 (\varrho_{n+1} - 2\varrho_n + \varrho_{n-1}) \\ &\quad - 2\varepsilon \mu_4 u_\varepsilon u_n = g_{n2}(t) \end{aligned} \quad (3.5)$$

with $w(t) = \{(h_{n1}(t), h_{n2}(t), g_{n1}(t), g_{n2}(t))\}_{n \in \mathbb{Z}} \in C_b(\mathbb{R}, H)$, where $C_b(\mathbb{R}, H)$ is the Banach space of all bounded continuous functions from $\mathbb{R}$ to H with the norm $|w| = \sup_{\mathbb{R}} \|w(t)\|$. We look for a solution $z \in C_b(\mathbb{R}, H)$ of (3.4) for $\varepsilon > 0$ small. For this reason, we consider the Hilbert spaces $H_2 := H_1 \times H_1$ and

$$H_1 := \left\{ \{u_n\}_{n \in \mathbb{Z}} \mid \sum_{n \in \mathbb{Z}} u_n^2 < \infty \right\}$$

with the corresponding standard norms. We first study the equation

$$\begin{aligned} \dot{\varrho} &= \psi + g_1 \\ \dot{\psi} + \varepsilon \delta_2 \psi + A_\varepsilon \varrho &= g_2 \end{aligned} \quad (3.6)$$

on H_2 for $(g_1, g_2) \in C_b(\mathbb{R}, H_2)$. Here,

$$A_\varepsilon \varrho = \{\Omega_0^2 \varrho_n - \varepsilon \mu_2 (\varrho_{n+1} - 2\varrho_n + \varrho_{n-1})\}_{n \in \mathbb{Z}}.$$

Clearly $A_\varepsilon : H_1 \rightarrow H_1$ is symmetric and positive definite for ε small. Then for any small ε , there is a symmetric and positive definite $B_\varepsilon : H_1 \rightarrow H_1$ such that

$$B_\varepsilon^2 = A_\varepsilon - \frac{\varepsilon^2 \delta_2^2}{4} I.$$

We take the operators $\cos B_\varepsilon t$ and $\sin B_\varepsilon t$ from H_1 to H_1 . For any $\varrho \in H_1$, we consider the function

$$\phi(t) := |\cos B_\varepsilon t \varrho|^2 + |\sin B_\varepsilon t \varrho|^2.$$

Then we have

$$\dot{\phi}(t) = -2 \langle \cos B_\varepsilon t \varrho, B_\varepsilon \sin B_\varepsilon t \varrho \rangle + 2 \langle \sin B_\varepsilon t \varrho, B_\varepsilon \cos B_\varepsilon t \varrho \rangle = 0.$$

Hence

$$|\cos B_\varepsilon t \varrho|^2 + |\sin B_\varepsilon t \varrho|^2 = \varrho,$$

and then $\|\cos B_\varepsilon t\| \leq 1$ and $\|\sin B_\varepsilon t\| \leq 1$.

Now, the equation

$$\begin{aligned} \dot{\varrho} &= \psi \\ \dot{\psi} + \varepsilon \delta_2 \psi + A_\varepsilon \varrho &= 0 \end{aligned} \quad (3.7)$$

has the form $\ddot{\varrho} + \varepsilon \delta_2 \dot{\varrho} + A_\varepsilon \varrho = 0$ which has the general solution

$$e^{-\varepsilon \delta_2 t/2} \left[\cos B_\varepsilon t \varrho_1 + \sin B_\varepsilon t \varrho_2 \right]$$

for $\varrho_{1,2} \in H_1$. Consequently, the fundamental solution of (3.7) has the form

$$V_\varepsilon(t) = e^{-\varepsilon \delta_2 t/2} W_\varepsilon(t)$$

with uniformly bounded $W_\varepsilon(t)$ for $\varepsilon > 0$ small. Thus, the only bounded solution of (3.6) has the form

$$(\varrho(t), \psi(t)) = \int_{-\infty}^t e^{-\varepsilon \delta_2 (t-s)/2} W_\varepsilon(t-s) (g_1(s), g_2(s)) ds.$$

Hence

$$|(\varrho, \psi)| \leq K_1 |g_1, g_2| / \varepsilon$$

for a constant $K_1 > 0$ independent of $\varepsilon > 0$ small.

Furthermore, it is not difficult to see that the linear system

$$\begin{aligned} \dot{u}_n &= v_n + h_{n1}(t) \\ \dot{v}_n + \varepsilon \delta v_n - a^2 d_0^2 u_n &= h_{n2}(t) \end{aligned} \quad (3.8)$$

has a unique solution $\{(u_n(t), v_n(t))\}_{n \in \mathbb{Z}} \in C_b(\mathbb{R}, H_2)$ such that

$$| \{(u_n(t), v_n(t))\}_{n \in \mathbb{Z}} | \leq K_2 | \{(h_{n1}(t), h_{n2}(t))\}_{n \in \mathbb{Z}} |$$

for a constant $K_2 > 0$ independent of $\varepsilon > 0$ small.

Now we return to (3.5). Summarizing the above arguments, we see, by using the Banach contraction mapping principle for $\varepsilon > 0$ small, that (3.5) has for any $w(t) \in C_b(\mathbb{R}, H)$ a unique solution $z \in C_b(\mathbb{R}, H)$ such that $|z| \leq K_3 |w| / \varepsilon$ for a constant $K_3 > 0$ independent of $\varepsilon > 0$ small. Since the system (3.5) is T -periodic, then we get from the proof of Theorem 2.1 of [4, p. 288] that (3.5) has an exponential dichotomy on $\mathbb{R}$ in the space H for any $\varepsilon > 0$ sufficiently small. Consequently, we get the next result.

Theorem 3.1. *The T -periodic solution $u_n(t) = u_\varepsilon(t)$, $\varrho_n(t) = \varrho_\varepsilon(t) \forall n \in \mathbb{Z}$ of (3.1) is hyperbolic in H for any $\varepsilon > 0$ sufficiently small, i. e., the zero equilibrium of (3.4) in H is hyperbolic.*

Now we look for more complicated solutions of (3.1). For this reason, we shift in (3.4) the time $t \leftrightarrow t + \alpha$ to get the system

$$\begin{aligned} \dot{u}_n &= v_n, \\ \dot{v}_n + \varepsilon \delta_1 v_n - a^2 u_n d_0^2 + a^2 u_n^3 + 3a^2 u_n^2 u_\varepsilon(t + \alpha) \\ &\quad + 3a^2 u_n u_\varepsilon^2(t + \alpha) = \varepsilon \mu_1(u_{n+1} - 2u_n + u_{n-1}) \\ &\quad + \varepsilon \mu_3(\varrho_n u_n + \varrho_n u_\varepsilon(t + \alpha) + \varrho_\varepsilon(t + \alpha) u_n), \\ \dot{\varrho}_n &= \psi_n, \\ \dot{\psi}_n + \varepsilon \delta_2 \psi_n + \Omega_0^2 \varrho_n &= \varepsilon \mu_2(\varrho_{n+1} - 2\varrho_n + \varrho_{n-1}) + \varepsilon \mu_4(u_n^2 + 2u_\varepsilon(t + \alpha) u_n). \end{aligned} \quad (3.9)$$

We look for a solution of (3.9) for $\varepsilon > 0$ small such that $u_n \sim 0$, $v_n \sim 0$ for $n \neq 0$ and $u_0 \sim \gamma$, $v_0 \sim \dot{\gamma}$.

Let $(\varrho_0, \psi_0) = \{(\varrho_n^0, \psi_n^0)\}_{n \in \mathbb{Z}}$ be the unique bounded solution of (3.6) for $g_1 = 0$ and $g_2 = \{g_{n2}\}_{n \in \mathbb{Z}}$ with $g_{n2} = 0$ for $n \neq 0$ and

$$g_{02} = \varepsilon \mu_4(\gamma^2 + 2u_\varepsilon(t + \alpha)\gamma).$$

Let us put $u_n^0 = v_n^0 = 0$ for $n \neq 0$ and $u_0^0 = \gamma$, $v_0^0 = \dot{\gamma}$. Now we make in (3.9) the change of variables $u_n \leftrightarrow u_n + u_n^0$, $v_n \leftrightarrow v_n + v_n^0$, $\varrho_n \leftrightarrow \varrho_n + \varrho_n^0$, $\psi_n \leftrightarrow \psi_n + \psi_n^0$ to get,

for $n \neq 0$, the system

$$\begin{aligned}
\dot{u}_n &= v_n, \\
\dot{v}_n + \varepsilon \delta_1 v_n - a^2 u_n d_0^2 + a^2 u_n^3 + 3a^2 u_n^2 u_\varepsilon(t + \alpha) + 3a^2 u_n u_\varepsilon^2(t + \alpha) \\
&= \varepsilon \mu_1(u_{n+1} + u_{n+1}^0 - 2u_n + u_{n-1} + u_{n-1}^0) \\
&\quad + \varepsilon \mu_3((\varrho_n + \varrho_n^0)u_n + (\varrho_n + \varrho_n^0)u_\varepsilon(t + \alpha) \\
&\quad + \varrho_\varepsilon(t + \alpha)u_n), \\
\dot{\varrho}_n &= \psi_n, \\
\dot{\psi}_n + \varepsilon \delta_2 \psi_n + \Omega_0^2 \varrho_n &= \varepsilon \mu_2(\varrho_{n+1} - 2\varrho_n + \varrho_{n-1}) \\
&\quad + \varepsilon \mu_4(u_n^2 + 2u_\varepsilon(t + \alpha)u_n).
\end{aligned} \tag{3.10}$$

For the mode $n = 0$, we first note that the system

$$\begin{aligned}
\dot{u}_0 &= v_0 \\
\dot{v}_0 + a^2(3\gamma^2 - d_0^2)u_0 &= h(t)
\end{aligned}$$

for $h(t) \in C_b(\mathbb{R}, \mathbb{R})$ has a solution $(u_0, v_0) \in C_b(\mathbb{R}, \mathbb{R}^2)$ (see [10]) if and only if

$$\int_{-\infty}^{\infty} h(t) \dot{\gamma}(t) dt = 0,$$

and such a solution is unique if $\int_{-\infty}^{\infty} u_0(t) \dot{\gamma}(t) dt = 0$. Consequently, for the mode $n = 0$, we get from (3.9) the equations

$$\begin{aligned}
\dot{u}_0 &= v_0, \\
\dot{v}_0 + a^2(3\gamma^2 - d_0^2)u_0 &= h(t) - \dot{\gamma}(t) \int_{-\infty}^{\infty} h(t) \dot{\gamma}(t) dt / \int_{-\infty}^{\infty} \dot{\gamma}(t)^2 dt, \\
\int_{-\infty}^{\infty} u_0(t) \dot{\gamma}(t) dt &= 0, \\
\dot{\varrho}_0 &= \psi_0, \\
\dot{\psi}_0 + \varepsilon \delta_2 \psi_0 + \Omega_0^2 \varrho_0 &= \varepsilon \mu_2(\varrho_1 - 2\varrho_0 + \varrho_{-1}) \\
&\quad + \varepsilon \mu_4(u_0^2 + 2u_0 \gamma + 2u_\varepsilon(t + \alpha)u_0),
\end{aligned} \tag{3.11}$$

and

$$\int_{-\infty}^{\infty} h(t) \dot{\gamma}(t) dt = 0 \tag{3.12}$$

for

$$\begin{aligned} h(t) = & -a^2(u_0^3 + 3u_0^2\gamma) - \varepsilon\delta_1\dot{\gamma} - 3a^2(u_0 + \gamma)^2u_\varepsilon(t + \alpha) - \varepsilon\delta_1v_0 \\ & - 3a^2(u_0 + \gamma)u_\varepsilon^2(t + \alpha) + \varepsilon\mu_1(u_1 - 2(u_0 + \gamma) + u_{-1}) \\ & + \varepsilon\mu_3((\varrho_0 + \varrho_0^0)(u_0 + \gamma) + (\varrho_0 + \varrho_0^0)u_\varepsilon(t + \alpha) \\ & + \varrho_\varepsilon(t + \alpha)(u_0 + \gamma)). \end{aligned} \quad (3.13)$$

Now for $\varepsilon > 0$ small, we can solve (3.10) and (3.11) to get the solution

$$z = \left\{ (u_n(t), v_n(t), \varrho_n(t), \psi_n(t)) \right\}_{n \in \mathbb{Z}} \in C_b(\mathbb{R}, H)$$

such that $z = O(\varepsilon)$. Then we put this z into (3.13) to get the function $h_{\varepsilon, \alpha} \in C_b(\mathbb{R}, \mathbb{R})$. We note $h_{\varepsilon, \alpha}(t) = O(\varepsilon)$ uniformly for $\varepsilon > 0$ small and $\alpha, t \in \mathbb{R}$. Clearly $h_{\varepsilon, \alpha}(t)$ is T -periodic in α . Then from (3.12) we get the bifurcation equation

$$Q(\varepsilon, \alpha) := \frac{1}{\varepsilon} \int_{-\infty}^{\infty} h_{\varepsilon, \alpha}(t) \dot{\gamma}(t) dt = 0. \quad (3.14)$$

We need to study the limit of $h_{\varepsilon, \alpha}(t)$ as $t \rightarrow 0$. If we put

$$\lim_{\varepsilon \rightarrow 0} u_\varepsilon(t)/\varepsilon = w(t), \quad \lim_{\varepsilon \rightarrow 0} \varrho_\varepsilon(t)\varepsilon = \zeta(t),$$

then from (3.2) we get

$$\ddot{w} - a^2 d_0^2 w = f(t), \quad \ddot{\zeta} + \Omega_0^2 \zeta = -\mu_4 d_0^2.$$

Hence $\zeta = -\mu_4 d_0^2 / \Omega_0^2$ and

$$w(t) = -\frac{1}{2ad_0} \int_{-\infty}^t e^{-ad_0(t-s)} f(s) ds - \frac{1}{2ad_0} \int_t^{\infty} e^{ad_0(t-s)} f(s) ds. \quad (3.15)$$

It is clear that $w(t)$ is T -periodic. Furthermore, since $\gamma(t) \rightarrow 0$ as $t \rightarrow \pm\infty$ exponentially, from the formula over (3.8) of the bounded solution of (3.6) we see that $\lim_{\varepsilon \rightarrow 0} (\varrho_0, \psi_0)/\varepsilon = \{(\varrho_{0n}, \psi_{0n})\}_{n \in \mathbb{Z}}$ with $\varrho_{0n} = \psi_{0n} = 0$ for $n \neq 0$ and

$$\ddot{\varrho}_{00} + \Omega_0^2 \varrho_{00} = \mu_4 \gamma(t)^2,$$

i. e.,

$$\varrho_{00}(t) = \frac{\mu_4}{\Omega_0} \int_{-\infty}^t \sin \Omega_0(t-s) \gamma(s)^2 ds.$$

Summarizing, from (3.13) we get

$$\begin{aligned} M(\alpha) := Q(0, \alpha) = & \int_{-\infty}^{\infty} \left[-\delta_1 \dot{\gamma}(t) - 3a^2 \gamma(t)^2 w(t + \alpha) \right. \\ & \left. - 2\mu_1 \gamma(t) \right] \dot{\gamma}(t) dt = -\frac{4}{3} \delta_1 a d_0^3 + a^2 \int_{-\infty}^{\infty} \gamma(t)^3 \dot{w}(t + \alpha) dt. \end{aligned} \quad (3.16)$$

Clearly, $M(\alpha)$ is T -periodic. We note that similarly we can prove that

$$\lim_{\varepsilon \rightarrow 0} \frac{\partial}{\partial \alpha} Q(\varepsilon, \alpha) / \varepsilon = M'(\alpha)$$

uniformly for $\alpha \in \mathbb{R}$. Summarizing, we get the next result.

Theorem 3.2. *Let M be given by (3.16). If there is a simple zero α_0 of M , i. e., $M(\alpha_0) = 0$ and $M'(\alpha_0) \neq 0$, then (3.1) has for any $\varepsilon > 0$ small a bounded solution $z(t)$ with small u_n, ϱ_n for $n \neq 0$ and (u_0, ϱ_0) near $(\gamma(t - \alpha_0), 0)$.*

Now, it is not difficult to prove, as in the finite-dimensional case [10], that

$$\left(z(t) - \left\{ (u_\varepsilon(t), \dot{u}_\varepsilon(t), \varrho_\varepsilon(t), \dot{\varrho}_\varepsilon(t)) \right\}_{n \in \mathbb{Z}} \right) \rightarrow 0$$

exponentially fast as $t \rightarrow \pm\infty$ in H . Moreover, near $z(t)$, we can construct the Smale horseshoe. Consequently, we get in this case chaos in (3.1) with corresponding infinitely many periodic orbits with arbitrarily large periods. This Smale horseshoe of (3.1) is spatially localized but not exponentially as in breathers.

To be more concrete, we take

$$f(t) = \Upsilon \cos \omega t$$

for $\Upsilon > 0$. Then (3.15) gives

$$w(t) = -\frac{\Upsilon}{\omega^2 + a^2 d_0^2} \cos \omega t,$$

and the formula (3.16) has now the form

$$M(\alpha) = -\frac{4}{3} \delta_1 a d_0^3 + \frac{\omega \Upsilon \pi \sqrt{2}}{a} \operatorname{sech} \frac{\omega \pi}{2 a d_0} \sin \omega \alpha.$$

Consequently, if

$$4 \sqrt{2} \delta_1 \sqrt{\varepsilon_0} d_0 < 3 \sqrt{m} \omega \Upsilon \pi \operatorname{sech} \frac{\omega \pi}{2 a d_0}, \quad (3.17)$$

then $M(\alpha)$ has a simple zero, so then (3.1) is chaotic for any $\varepsilon > 0$ small. We note that the inequality (3.17) gives sufficient conditions between the magnitude of the forcing Υ and the damping δ_1 in order to get chaos in (3.1) for $\varepsilon > 0$ small. So chaos is generated by the proton part of (3.1). If $\delta_1 = 0$, then (3.1) is always chaotic for $f(t) = \Upsilon \cos \omega t$. Furthermore, if $f(t) = 0$, i. e., there is no forcing, then it is not difficult to prove that (3.1) has no nonconstant periodic solutions in space H .

Finally, we note that similarly we can study the case when more than one modes are excited. We do not carry out such computations here.

4. CONCLUDING REMARKS

In this note, we have studied weakly coupled diatomoc lattices presented by model (1.1). In the case where (1.1) is unforced and undamped, the existence of infinitely many time-periodic and spatially localized solutions (the so-called breathers) has been shown. For small damping and forcing, we have proved the existence of infinitely many spatially localized Smale horseshoes, i. e., the existence of chaos in (1.1). The localization of the Smale horseshoes is not exponential as is the case for breathers. The proofs of these results are based on the use of the implicit function theorem. This would allow us to establish profiles of these solutions by using analytic-numeric methods as in [7].

For general ‘non-small’ parameters involved in (1.1), the study of dynamics of (1.1) is rather difficult. The existence of travelling waves could give some answers. There are two possibilities to handle this problem.

The first one is to consider directly in (1.1) travelling wave solutions

$$u_n(t) = \phi(vt - n), \quad \varrho_n(t) = \psi(vt - n)$$

to get the system

$$\begin{aligned} v^2 \ddot{\phi}(t) + \Gamma_1 v \dot{\phi}(t) &= \frac{k_1}{m} (\phi(t+1) - 2\phi(t) + \phi(t-1)) + \frac{4\xi_0}{md_0^2} \phi(t) \left(1 - \frac{\phi^2(t)}{d_0^2} \right) \\ &\quad - 2\frac{\chi}{m} \psi(t)\phi(t) + \frac{F}{m}, \\ v^2 \ddot{\psi}(t) + \Gamma_2 v \dot{\psi}(t) &= \frac{K_1}{M} (\psi(t+1) - 2\psi(t) + \psi(t-1)) \\ &\quad - \Omega_0^2 \psi(t) - \frac{\chi}{M} (\phi(t) - d_0^2), \end{aligned} \quad (4.1)$$

where now F is a constant external force. Equations of types similar to (4.1) are studied in [6].

The second one is to take the continuum limit of (1.1) to obtain the partial differential equation

$$\begin{aligned} u_{tt} + \Gamma_1 u_t &= \frac{k_1}{m} b^2 u_{xx} + \frac{4\xi_0}{md_0^2} u \left(1 - \frac{u^2}{d_0^2} \right) - 2\frac{\chi}{m} \varrho u + \frac{F}{m}, \\ \varrho_{tt} + \Gamma_2 \varrho_t &= \frac{K_1}{M} b^2 \varrho_{xx} - \Omega_0^2 \varrho - \frac{\chi}{M} (u^2 - d_0^2), \end{aligned} \quad (4.2)$$

where again F is a constant external force, $x = bn$ is the continuum space variable and $b > 0$ is the lattice spacing. The travelling wave solutions

$$u(x, t) = \phi(vt - x), \quad v(x, t) = \psi(vt - x)$$

of (4.2) satisfy the system

$$\begin{aligned}\ddot{\phi}\left(v^2 - \frac{k_1}{m}b^2\right) + \Gamma_1 v \dot{\phi} &= \frac{4\xi_0}{md_0^2}\phi\left(1 - \frac{\phi^2}{d_0^2}\right) - 2\frac{\chi}{m}\psi\phi + \frac{F}{m}, \\ \ddot{\psi}\left(v^2 - \frac{K_1}{M}b^2\right) + \Gamma_2 v \dot{\psi} &= -\Omega_0^2\psi - \frac{\chi}{M}(\phi^2 - d_0^2).\end{aligned}\quad (4.3)$$

System (4.3) is numerically studied in [11, 16]. System (4.1) was not yet investigated. But for certain values of parameters in (4.1-4.3), the bifurcation methods as in this paper can be applied to find analytically either periodic, homoclinic or heteroclinic solutions of (4.1-4.3).

For instance, if

$$k_1/m = K_1/M \quad \text{and} \quad v^2 \sim K_1 b^2/M,$$

then by using the theory of singularly perturbed ordinary differential equations [3], the dynamics of (4.3) is reduced to the system

$$\begin{aligned}\Gamma_1 v \dot{\phi} &= \frac{4\xi_0}{md_0^2}\phi\left(1 - \frac{\phi^2}{d_0^2}\right) - 2\frac{\chi}{m}\psi\phi + \frac{F}{m}, \\ \Gamma_2 v \dot{\psi} &= -\Omega_0^2\psi - \frac{\chi}{M}(\phi^2 - d_0^2)\end{aligned}\quad (4.4)$$

for

$$v^2 = K_1 b^2/M.$$

System (4.4) admits the Lyapunov function

$$L(\phi, \psi) = \frac{\xi_0}{md_0^2}\left(1 - \frac{\phi^2}{d_0^2}\right)^2 - \frac{\chi}{M}\psi(\phi^2 - d_0^2) - \Omega_0^2\frac{\psi^2}{2} + \frac{F}{m}\phi.$$

Hence the limit sets of (4.4) are its equilibria. In general they are hyperbolic. So for

$$v^2 \sim K_1 b^2/M,$$

they persist in (4.3) and consequently, on bounded sets, these hyperbolic equilibria are limit sets of (4.3).

Similarly, we can study the case

$$v^2 \sim K_1 b^2/M \quad \text{when} \quad k_1/m \neq K_1/M.$$

This was numerically studied in [11]. The reduced system is now

$$\begin{aligned}\ddot{\phi}\left(v^2 - \frac{k_1}{m}b^2\right) + \Gamma_1 v \dot{\phi} &= \frac{4\xi_0}{md_0^2}\phi\left(1 - \frac{\phi^2}{d_0^2}\right) - 2\frac{\chi}{m}\psi\phi + \frac{F}{m}, \\ \Gamma_2 v \dot{\psi} &= -\Omega_0^2\psi - \frac{\chi}{M}(\phi^2 - d_0^2)\end{aligned}\quad (4.5)$$

for $v^2 = K_1 b^2 / M$. According to [3], the dynamics of bounded solutions of (4.3) is now approximated by (4.5). To study bounded solutions of (4.5), we consider its equivalent form given by the integro-differential equation

$$\ddot{\phi} \left(v^2 - \frac{k_1}{m} b^2 \right) + \Gamma_1 v \dot{\phi} = \frac{4\xi_0}{m d_0^2} \phi \left(1 - \frac{\phi^2}{d_0^2} \right) + \frac{2\chi^2}{v M m \Gamma_2} \phi \int_{-\infty}^t e^{-\frac{\Omega_0^2(t-s)}{\Gamma_2 v}} (\phi(s)^2 - d_0^2) ds + \frac{F}{m}. \quad (4.6)$$

Equation (4.6) can be studied similarly to (3.3) but we do not investigate it any further in this note. We postpone this to another paper.

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STABILITY OF A DELAYED RATIO-DEPENDENT PREDATOR-PREY SYSTEM

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ABSTRACT. A ratio-dependent predator-prey model is considered in which the predator growth rate depends on past quantities of the prey. Conditions for stability of an equilibrium and its bifurcation are established when special parameters are taken into account.

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1. INTRODUCTION

In [15, 16, 24], a possible generalisation of the traditional Volterra predator-prey system is considered,

$$\begin{aligned}\dot{S}_1 &= S_1 \cdot \alpha(S_1) - S_2 \cdot V(S_1), \\ \dot{S}_2 &= S_2 \cdot K(S_1)\end{aligned}\tag{1.1}$$

where the dot means differentiation with respect to time t , $S_1(t)$ and $S_2(t)$ are the quantities (or densities) of preys and predators, respectively. Here,

- α is smooth with $\alpha'(S_1) < 0$, $S_1 \geq 0$ and $\alpha(0) > 0 > \lim_{+\infty} \alpha$;
- K and V are nonnegative and increasing, and $K(0) = 0 = V(0)$.

If we replace the argument of K and V by the ratio of S_1 and S_2 , then we arrive at a ratio-dependent predator-prey system, which is capable of producing richer and more reasonable or acceptable dynamics. The substantial difference from the classical Kolmogorov model [15, 16] is due to the following two facts related to the system:

- Equilibrium abundances are positively correlated along a gradient of enrichment [1];
- These models do not produce the so-called paradox of enrichment (see [11, pp. 490, 502] and [17, p. 391]), more exactly, it either completely disappears or enrichment is related to stability in a more complicated way [21].

A simple ratio-dependent type mathematical model of two-species interaction was first presented in [10, 19, 21].

Let us consider a model of ratio-dependent type which has the form

$$\begin{aligned}\dot{S}_1 &= f_1(S_1, S_2), \\ \dot{S}_2 &= f_2(S_1, S_2)\end{aligned}\tag{1.2}$$

with the initial conditions $S_i(0) > 0$ ($i \in \{1, 2\}$), where

$$f_1(S_1, S_2) := \begin{cases} \alpha S_1 \left(1 - \frac{S_1}{K}\right) - \frac{\beta S_1 S_2}{\varepsilon S_2 + S_1} & \text{if } S_1^2 + S_2^2 > 0 \\ 0 & \text{if } S_1 = S_2 = 0 \end{cases}$$

and

$$f_2(S_1, S_2) := \begin{cases} -\gamma S_2 + \frac{\delta S_1 S_2}{\varepsilon S_2 + S_1} & \text{if } S_1^2 + S_2^2 > 0 \\ 0 & \text{if } S_1 = S_2 = 0. \end{cases}$$

Here, $\alpha > 0$ is the intrinsic growth rate of S_1 in the absence of the S_2 -population and without environmental limitation. In the absence of the S_2 -population, the S_1 -population grows logistically to the carrying capacity $K > 0$; the functional response of the S_2 -population is of Michaelis–Menten–Holling type with satiation coefficient or conversion rate $\delta > 0$. The specific mortality

$$E(S_2) = -\gamma S_2\tag{1.3}$$

of the S_2 -population in the absence of the S_1 -population depends on the quantity of S_2 . $\gamma > 0$ is the death rate of the S_2 -population.

If we take polar coordinates $x = r \cos \varphi$, $y = r \sin \varphi$, then a routine calculation shows that $f_i \in C^0(\mathbb{R}_+^2, \mathbb{R})$, however, it is easy to see that $f_i \in C^1(\mathbb{R}_+^2, \mathbb{R})$ ($i \in \{1, 2\}$), therefore solution of (1.2) with positive initial condition exists and is unique.

Let $S_1 \cdot M_1(S_1, S_2) := f_1(S_1, S_2)$ and $S_2 \cdot M_2(S_1, S_2) := f_2(S_1, S_2)$, then the system (1.2) is written in the Kolmogorov form of

$$\dot{\mathbf{S}} = \begin{pmatrix} S_1 \cdot M_1(\mathbf{S}) \\ S_2 \cdot M_2(\mathbf{S}) \end{pmatrix}\tag{1.4}$$

(see [19]), which is useful since one can check the following two properties of this system:

- M_1 and M_2 are smooth functions, therefore the positive quadrant of the phase space $[S_1, S_2]$ is an invariant region (see [22], pp. 198–203 and 230–231);
- $\frac{\partial M_1}{\partial S_2}(S_1, S_2) = -\frac{\beta}{\varepsilon + S_1} < 0$ and $\frac{\partial M_2}{\partial S_1}(S_1, S_2) = -\frac{\delta \varepsilon}{(\varepsilon + S_1)^2} > 0$ ($S_1, S_2 > 0$), i. e., in fact the system is a predator-prey system with prey S_1 and predator S_2 .

The ratio-dependent predator-prey model (1.2) has been studied by several authors recently, and very rich dynamics has been observed. In [10] and later in [2], the authors restricted their analysis to parameter values that ensure that the origin as equilibrium behaves as a saddle point and they established conditions for persistence

of the model, and showed the existence of eight qualitatively different types of system behaviours realized for various parameter values. In [13] and later in [26] the authors studied the analytical behaviour at the origin and showed that this equilibrium can be either a saddle point or an attractor of certain trajectories, and shown that the origin is indeed a critical point of higher order. In [17] the global behaviour of solutions was investigated and it was showed that if the positive equilibrium is locally asymptotically stable, then the system does not have any nontrivial positive periodic solutions.

Now we are going to show that with respect to certain circumstances this model can have a periodic solution. This happens by incorporating a delay effect into the systems. For the justification and biological relevance of the delay in ratio-dependent systems see [6]. In [14] the most famous delay in the predator-equation was introduced, i. e., $K\left(\frac{S_1(t)}{S_2(t)}\right)$ was replaced by $K\left(\frac{S_1(t-\tau)}{S_2(t-\tau)}\right)$. Starting from the evidence that in such a system the present growth rate of a predator depends not only on the present quantity of food but also on past quantities, we will introduce an infinite distributed delay into the second equation of system (1.2) for prey density, i. e., we replace S_1 in the second equation by

$$R(t) := \int_{-\infty}^t S_1(\tau)\rho(t-\tau)d\tau, \quad t \in [0, +\infty), \quad (1.5)$$

where $\rho : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is the so-called *density function* satisfying the relation

$$\int_0^{+\infty} \rho(s)ds = 1. \quad (1.6)$$

Due to [5, 4, 7, 8, 18, 25], we assume that the influence of the past is fading away exponentially, i. e., $\rho(t) := a \cdot \exp(-at)$ ($a \in (0, +\infty)$) and $a \int_{-\infty}^t \exp(-a(t-\tau))d\tau = a \int_0^{+\infty} \exp(-as)ds = 1$ hold. The smaller a is, the longer the time interval is in the past in which the values of S_1 are taken into account, i. e., $1/a$ is the “measure of the influence of the past”.

Clearly, if $\gamma < \delta < \frac{\gamma\beta}{\beta-\varepsilon\alpha}$ for $\gamma > \varepsilon\alpha$ or $\delta > \gamma$ for $\gamma \leq \varepsilon\alpha$, then (1.2) has three equilibria, two on the boundary of the positive quadrant: $(0, 0)^T$, whose local stability cannot be directly studied (the system cannot be linearised), $(K, 0)^T$ that is unstable and a unique equilibrium with positive coordinates: $(\overline{S}_1, \overline{S}_2)^2 := \frac{K}{\alpha\varepsilon\delta}(\beta(\gamma - \delta) + \delta\varepsilon\alpha)\left(1, \frac{\delta-\gamma}{\gamma\varepsilon}\right)^T$ which may or may not be stable. To study the local stability of equilibrium $(0, 0)^T$ there are several ways, e. g. via introducing the new variable $S := S_1/S_2$ or making a time scale change $dt =: (\varepsilon S_2 + S_1)d\tau$ (cf. [17] and [26]). First we give conditions for the asymptotic stability of the equilibrium of the delayed system and show that under some conditions the increase of the delay $\frac{1}{a}$ destabilises the originally stable equilibrium by a Poincaré–Andronov–Hopf bifurcation (see [9]). While the system without delay has no periodic solutions apart from the trivial one, this one has.

2. THE SYSTEM WITH DELAY

We consider the model of predator-prey interaction with time delay which is given by

$$\begin{aligned} \dot{S}_1 &= \alpha S_1 \left(1 - \frac{S_1}{K}\right) - \frac{\beta S_1 S_2}{\varepsilon S_2 + S_1} \\ \dot{S}_2 &= -\gamma S_2 + \frac{\delta R S_2}{\varepsilon S_2 + R} \end{aligned} \quad (2.1)$$

where R is given by (1.5). Since

$$\dot{R}(t) = a(S_1(t) - R(t)) \quad (t \in [0, +\infty)),$$

we see that (2.1) is equivalent in its qualitative dynamical behaviour to the three dimensional system of ordinary differential equations

$$\begin{aligned} \dot{S}_1 &= \alpha S_1 \left(1 - \frac{S_1}{K}\right) - \frac{\beta S_1 S_2}{\varepsilon S_2 + S_1} \\ \dot{S}_2 &= -\gamma S_2 + \frac{\delta R S_2}{\varepsilon S_2 + R} \\ \dot{R} &= a(S_1 - R) \end{aligned} \quad (2.2)$$

on $[0, \infty)$ in the following sense (see also [7]). If $(S_1, S_2) : [0, +\infty) \rightarrow \mathbb{R}^2$ is the solution of (2.1) corresponding to the continuous and bounded initial function $\tilde{S}_1 : (-\infty, 0] \rightarrow \mathbb{R}$ and the initial value $S_2^0 := S_2(0)$ (i. e., $S_1(t) := \tilde{S}_1(t)$ ($t < 0$)), then $(S_1, S_2, R) : [0, +\infty) \rightarrow \mathbb{R}^3$ is the solution of (2.2) satisfying the initial conditions $S_1(0) = \tilde{S}_1(0)$, $S_2(0) = S_2^0$ and $R(0) = R^0 := a \int_{-\infty}^0 \tilde{S}_1(\tau) \exp(a\tau) d\tau$ and vice versa. (Clearly, if the initial values $S_1(0)$, S_2^0 and R^0 related to system (2.2) are prescribed then, the function $\tilde{S}_1$ is not uniquely determined.) There are seven parameters in (2.2), thus if we introduce new variables and time with substitutions such as

$$S_1 =: Ku, \quad S_2 =: \frac{K}{\varepsilon}v, \quad R =: Kw, \quad t =: \alpha\tau,$$

then (2.2) takes the following simpler, dimensionless form

$$\begin{aligned} u' &= u(u - 1) - \frac{buv}{v + u} \\ v' &= -cdv + \frac{cvw}{v + w} \\ w' &= \mu(u - w) \end{aligned} \quad (2.3)$$

where $b := \frac{\beta}{\alpha\varepsilon}$, $c := \frac{\delta}{\alpha}$, $d := \frac{\gamma}{\alpha}$ and $\mu := \frac{a}{\alpha}$, and where the prime, this time, denotes differentiation with respect to the variable τ . With these notations, if $0 < d < 1 < b < \frac{1}{1-d}$ for $b > 1$ or if $0 < d < 1$ for $b \leq 1$, then system (2.3) has the following equilibria: firstly, the origin, whose local stability cannot be directly investigated and is of no interest, secondly, the points $(1, 0, 1)^T$ and the unique positive $(\bar{u}, \bar{v}, \bar{w})^T :=$

$(b(d-1)+1, \frac{1-d}{d}(b(d-1)+1), b(d-1)+1)^T$, which represent the extinction state of predator and the coexistence state of predator and prey, respectively.

In order to check the stability of the last two equilibria we linearise system (2.3) at these points. The coefficient matrix is

$$A(u, v, w) := \begin{bmatrix} 1 - 2u - \frac{bv^2}{(v+u)^2} & -\frac{bu^2}{(v+u)^2} & 0 \\ 0 & \frac{cw^2}{(v+w)^2} - cd & \frac{cv^2}{(v+w)^2} \\ \mu & 0 & -\mu \end{bmatrix}.$$

In particular,

$$A(1, 0, 1) = \begin{bmatrix} -1 & -b & 0 \\ 0 & c(1-d) & 0 \\ \mu & 0 & -\mu \end{bmatrix}, \quad A(\bar{u}, \bar{v}, \bar{w}) = \begin{bmatrix} ek-1 & -bd^2 & 0 \\ 0 & cdk & ck^2 \\ \mu & 0 & -\mu \end{bmatrix},$$

where $e := b(1+d)$, $k := 1-d$, and the characteristic polynomials take the form

$$p_{101}(\lambda) := \lambda^3 + (1 + \mu - c(1-d))\lambda^2 + (\mu - \mu c(1-d) - c(1-d))\lambda + \mu c(1-d)$$

and

$$p_{\bar{u}\bar{v}\bar{w}}(\lambda) := \lambda^3 + (\mu + cdk - ek + 1)\lambda^2 + (cdk(\mu + 1) - ek(\mu - cdk) + \mu)\lambda + \mu cdk(ek + 1 + bdk).$$

Applying the Routh–Hurwitz criterion, we conclude that they are stable polynomials if and only if the following inequalities hold:

$$1 + \mu > c(1-d), \quad (2.4a)$$

$$\mu(1 - c(1-d)) > c(1-d), \quad (2.4b)$$

$$\mu c(1-d) > 0, \quad (2.4c)$$

$$P_{101}(\mu) := (1 - c(1-d))\mu^2 + (1 - 4c(1-d) + c^2(1-d))\mu + c(1-d)(c-1) > 0$$

and

$$\mu + cdk + 1 > ek \quad (2.5a)$$

$$\mu(cdk - ek + 1) > -cdk(1 - ek) \quad (2.5b)$$

$$\mu cdk(ek + 1 + bdk) > 0, \quad (2.5c)$$

and

$$P_{\bar{u}\bar{v}\bar{w}}(\mu) := (cdk - ek + 1)\mu^2 + (cdk(cdk + 2 - 2ek - bdk) + ek(ek - 2) + 1)\mu + cdk(1 + cdk(1 + ek) - ek) > 0. \quad (2.5d)$$

If we assume that

$$1 - ek > 0 \quad (2.6a)$$

i. e.,

$$\frac{1}{1-d} > b(1+d), \quad (2.6b)$$

then (2.5a) and (2.5b) hold. Clearly, (2.5c) holds automatically, thus (2.5d) and (2.6a) (resp. (2.6b)) altogether form a sufficient condition of asymptotic stability of the equilibrium $(\bar{u}, \bar{v}, \bar{w})^T$.

In view of (2.6a), the following three cases can be distinguished.

2.1. Case $ek < 1$. In this case the inequalities (2.5a-c) hold true. If

$$cdk(cdk + 2 - 2ek - bdk) + ek(ek - 2) + 1 \geq 0, \quad (2.7)$$

then (2.5d) holds for all μ and $(\bar{u}, \bar{v}, \bar{w})^T$ is asymptotically stable. If (2.7) does not hold, then since the constant term of the quadratic polynomial $P_{\bar{u}\bar{v}\bar{w}}$ is positive, this polynomial has either no real roots or has two roots of the same sign. If $P_{\bar{u}\bar{v}\bar{w}}$ has no real roots or has two negative roots, then (2.7) holds again for all μ and the equilibrium $(\bar{u}, \bar{v}, \bar{w})^T$ is asymptotically stable. If $P_{\bar{u}\bar{v}\bar{w}}$ has two positive roots, $0 < \mu_1 < \mu_0$, say, then the equilibrium $(\bar{u}, \bar{v}, \bar{w})^T$ is asymptotically stable for large values of μ , i. e., for small delays. At μ_0 , the characteristic polynomial $p_{\bar{u}\bar{v}\bar{w}}$ has the form

$$p_{\bar{u}\bar{v}\bar{w}}(\lambda) \equiv (\lambda^2 + cdk(\mu_0 + 1) - ek(\mu_0 - cdk) + \mu_0)(\lambda + \mu_0 + cdk - ek + 1),$$

and its roots are

$$\lambda_0(\mu_0) = -\mu_0 - cdk + ek - 1 < 0 \quad \text{and} \quad \lambda_{1,2}(\mu_0) = \pm i\omega,$$

where

$$\omega := \sqrt{cdk(\mu_0 + 1) - ek(\mu_0 - cdk) + \mu_0}.$$

A routine calculation shows that

$$\begin{aligned} & \frac{d\Re(\lambda_1(\mu_0))}{d\mu} \\ &= - \frac{(cdk(\mu_0 + 1) - ek(\mu_0 - cdk) + \mu_0 - 3\omega^2)(cdk(ek + 1) + bdk - \omega^2)}{(cdk(\mu_0 + 1) - ek(\mu_0 - cdk) + \mu_0 - 3\omega^2)^2 + 4\omega^2(\mu_0 + cdk - ek + 1)^2}. \end{aligned}$$

Example. Let $b = 2.0000$, $d = 0.1000$, $c = 0.1000$, then the expression on the left hand side in (2.7) is negative, $\mu_0 = 0.4319$ and $\omega = 0.5970$, furthermore,

$$\frac{d\Re(\lambda_1(\mu_0))}{d\mu} = -\frac{0.2187}{2.5511} < 0,$$

therefore using μ as bifurcation parameter, the equilibrium $(\bar{u}, \bar{v}, \bar{w})^T$ loses its stability by a Poincaré-Andronov-Hopf bifurcation when μ is decreased below μ_0 , i. e., the delay is increased, while the other equilibrium becomes asymptotically stable.

2.2. Case $ek = 1$. In this case the inequalities (2.5a-c) hold true, (2.5d) is equivalent to

$$P_{\overline{u}\overline{v}\overline{w}}(\mu) := \mu^2 + dk(c - b)\mu + 2cdk > 0. \quad (2.8)$$

If $c - d \geq 0$, then (2.8) holds for all μ and $(\overline{u}, \overline{v}, \overline{w})^T$ is asymptotically stable. If $c < d$, then since the constant term of the quadratic polynomial $P_{\overline{u}\overline{v}\overline{w}}$ is positive, this polynomial has either no real roots or has two roots of the same sign. If $P_{\overline{u}\overline{v}\overline{w}}$ has no real roots or has two negative roots, then (2.8) holds again for all μ and the equilibrium $(\overline{u}, \overline{v}, \overline{w})^T$ is asymptotically stable. If $P_{\overline{u}\overline{v}\overline{w}}$ has two positive roots, $0 < \mu_1 < \mu_0$, say, then the equilibrium $(\overline{u}, \overline{v}, \overline{w})^T$ is asymptotically stable for large values of μ , i. e., for small delays. At μ_0 the characteristic polynomial $p_{\overline{u}\overline{v}\overline{w}}$ has the form

$$p_{\overline{u}\overline{v}\overline{w}}(\lambda) \equiv (\lambda^2 + cdk(\mu_0 + 1) - (\mu_0 - cdk) + \mu_0) \times (\lambda + \mu_0 + cdk),$$

whose roots are

$$\lambda_0(\mu_0) = -\mu_0 - cdk < 0, \quad \text{and} \quad \lambda_{1,2}(\mu_0) = \pm i\omega,$$

where $\omega := \sqrt{2cdk(\mu_0 + 1)}$. A routine calculation shows that

$$\begin{aligned} & \frac{d\Re(\lambda_1(\mu_0))}{d\mu} \\ &= - \frac{\omega^2 \left(3\omega^2 + cdk(4\mu_0 - 2cdk(1 + \mu_0) - 2 - 3cd^2k^2) \right) + 2c^2d^2k^2(2 + bdk)}{(2cdk(\mu_0 + 1) - 3\omega^2)^2 + 4\omega^2(\mu_0 + cdk)^2}. \end{aligned}$$

Example. Let $b = 0.999900000$, $c = 0.000000001$, then $d = 0.000050003$ and (2.8) is negative, $\mu_0 = 0.00054791$ and $\omega = 0.999999998$, furthermore,

$$\frac{d\Re(\lambda_1(\mu_0))}{d\mu} = -0.031314697 < 0,$$

therefore in this case the equilibrium $(\overline{u}, \overline{v}, \overline{w})^T$ loses its stability if μ is decreased below μ_0 . This loss of stability occurs again by a Poincaré-Andronov-Hopf bifurcation, while the other equilibrium becomes asymptotically stable.

2.3. Case $ek > 1$. In this case (2.5a-b) are not satisfied automatically. If we assumed that $cd > e$, then (2.5a) would hold but we have no guarantee that (2.5b) will hold, so it is not sure even for the value μ ($\mu < \mu_0$) that the polynomial $p_{\overline{u}\overline{v}\overline{w}}(\lambda)$ is stable.

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ANALYTICAL SOLUTION OF NEUTRON DIFFUSION EQUATION FOR A PWR TYPICAL LATTICE

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ABSTRACT. This paper introduces an analytical technique to obtain the two-group neutron flux distribution on a typical square cell in the symmetrical core of a pressurised water reactor, PWR. Mathematically, the flux distribution is a solution to a system of fourth order homogeneous diffusion equations with irregular boundary conditions in Cartesian and polar coordinates. The formulae for the solution, obtained explicitly in terms of hypergeometric functions, are suitable for computer programming leading to temperature distribution and thermal stress analysis on fuel rod, etc. The proposed technique is capable of calculating the neutron flux at the corners of the typical cell, where neither approximated equivalent cell methods nor numerical methods succeeded. An actual condition of steady state critical reactors is studied, and numerical results by the proposed technique are compared to those of an approximated equivalent cylindrical cell technique.

Mathematics Subject Classification: 35k57

Keywords: Nuclear fuel cell, analytical solution, neutron diffusion equation, irregular boundary conditions

1. INTRODUCTION

Many engineering applications are simulated mathematically as partial differential equations with nonsmooth irregular boundary conditions, described geometrically by a composite of different shapes such as triangles, rectangles, and circles [1–3], as in the theory of elasticity [4–6] or in neutron diffusion [7, 8]. Most of the customary numerical methods such as finite difference, finite elements and nodal methods fail at the corners as being singular points [9–12]. Also equivalent boundaries methods such as cylindrical [13] or hexagonal [14, 15] are affected by corners ignorance. Approximate analytical methods obtain the solution as an infinite series [16], which is exposed to round off and truncation errors, and require stability and convergence investigation of the calculations.

One of the most powerful ways of treating multi-region reactors is by the group diffusion method. In this method the energies of all the neutrons are divided into

a number of energy groups. The neutrons within each group are then lumped together and their diffusion, scattering, absorption, and other interactions are described in terms of suitably averaged diffusion coefficients and cross-sections, which are collectively known as group constants.

The simplest group diffusion problem involves only one group of neutrons, which, for simplicity, are assumed to be all thermal neutrons. A more accurate procedure, particularly for thermal reactors, is to split the neutrons into two groups; in which case, thermal neutrons are included in one group called the thermal or slow group, and all the other neutrons are included in the fast group. This approach was utilised in [13] assuming only one-dimensional composite problem. In this work, the approach used in [13] is extended to two dimensions. Moreover, the fuel rod and the moderator are treated as separate regions as will be indicated in the following section.

It must be emphasised that the two-group method is not an exact technique for handling reactor criticality calculations [17]. Nevertheless, it does provide a relatively simple way of obtaining reasonable estimates of the critical mass, flux distributions, and so on.

2. PROBLEM STATEMENT

2.1. Governing equations. Consider a two-region thermal reactor consisting of homogeneous fuel rods submerged in moderator, which is divided into symmetrical cells. Every cell is a fuel lump consisting of a rod with circular cross-section of radius a in a square moderator with side length $2b$. Due to the symmetry of the system, the cell can be divided into triangular regions and confine the solution to one single triangle, say OAB , as shown in Figure 1.

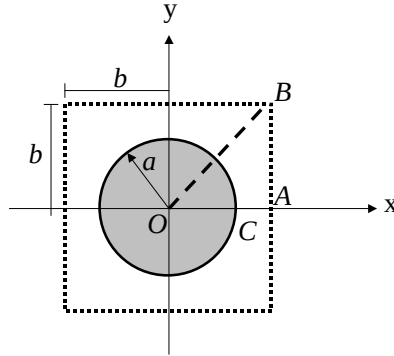


FIGURE 1. PWR typical lattice.

The steady state diffusion equation for the fast neutron flux in the fuel region ϕ_1^F is given by

$$-D_1^F \nabla^2 \phi_1^F + (\Sigma_{a1}^F + \Sigma_{12}^F) \phi_1^F - \nu \Sigma_{f1}^F \phi_1^F - \nu \Sigma_{f2}^F \phi_2^F = 0, \quad (2.1)$$

and that for the thermal flux ϕ_2^F in the fuel is described by

$$-D_2^F \nabla^2 \phi_2^F + \Sigma_{a2}^F \phi_2^F - \Sigma_{12}^F \phi_1^F = 0. \quad (2.2)$$

The diffusion equation for the fast neutron in the moderator region ϕ_1^M is

$$-D_1^M \nabla^2 \phi_1^M + (\Sigma_{a1}^M + \Sigma_{12}^M) \phi_1^M = 0, \quad (2.3)$$

and for the thermal flux in the moderator ϕ_2^M is

$$-D_2^M \nabla^2 \phi_2^M + \Sigma_{a2}^M \phi_2^M - \Sigma_{12}^M \phi_1^M = 0, \quad (2.4)$$

where

- ∇^2 is the Laplace operator,
- F and M stand for fuel and moderator, respectively,
- subscripts 1 and 2 stand for fast and thermal fluxes, respectively,
- $D_1^F, D_2^F, D_1^M, D_2^M$ are diffusion coefficients for fuel and moderator in group 1 and 2, respectively,
- $\Sigma_{a1}^F, \Sigma_{a2}^F, \Sigma_{a1}^M, \Sigma_{a2}^M$ are macroscopic absorption cross-sections,
- $\Sigma_{12}^F, \Sigma_{12}^M$ are macroscopic slowing down cross-sections,
- $\Sigma_{f1}^F, \Sigma_{f2}^F$ are macroscopic fission cross-sections,
- ν is the number of neutrons produced per fission.

The macroscopic cross-sections are assumed to be constants which characterise the material properties of the fuel and the moderator.

Since all cells are identical in an infinite uniform lattice, there can be no net flow of neutrons from one cell to another. It follows that the current density is zero along the boundary of each cell, also both fast and thermal fluxes and currents must be continuous at the fuel-moderator interface.

2.2. Boundary conditions. As mentioned in the previous section, both fast and thermal fluxes and currents must be continuous at the fuel-moderator interface, therefore the following conditions must be satisfied at $r = a$:

$$\phi_1^F = \phi_1^M, \quad (2.5)$$

$$\phi_2^F = \phi_2^M, \quad (2.6)$$

$$D_1^F \frac{\partial \phi_1^F}{\partial r} = D_1^M \frac{\partial \phi_1^M}{\partial r}, \quad (2.7)$$

and

$$D_2^F \frac{\partial \phi_2^F}{\partial r} = D_2^M \frac{\partial \phi_2^M}{\partial r}. \quad (2.8)$$

Likewise, the symmetrical conditions for the triangular region OAB in Figure 1 must be satisfied as follows:

(i) Along the line $y = 0$;

$$\frac{\partial \phi_j^i}{\partial y} = 0, \quad i = F, M, \text{ and } j = 1, 2 \quad (2.9)$$

(ii) Along the line $x = b$;

$$\frac{\partial \phi_j^M}{\partial x} = 0, \quad j = 1, 2 \quad (2.10)$$

(iii) Along the line $y = x$, where $0 < x < b$

$$\frac{\partial \phi_j^i}{\partial n} = 0, \quad (2.11)$$

where n is the outward normal to the line $y = x$.

The major difficulty of this problem is that criticality conditions yield some of mathematical relations expressed in Cartesian coordinates and others expressed in polar coordinates.

On the one hand, expressing the entire problem in polar coordinates introduces some inaccuracy at the outer boundary of the moderator (at $x = b$), on the other hand, a mathematical statement in the Cartesian coordinates introduces inaccuracy at the interface of both the fuel and the moderator.

In the following section, these difficulties will be discussed in some detail in the frame of the proposed solution approach.

3. AN APPROACH TO SOLUTION

To find the fast and thermal fluxes which satisfy the criticality conditions of the reactor, it is necessary to solve the two-group equations for fuel and moderator, in the confined triangular region.

Beginning with the fuel, the coupled second-order differential equations (2.1) and (2.2) are decoupled by solving equation (2.1) for ϕ_2^F to get

$$\phi_2^F = \frac{-1}{\nu \Sigma_{f2}^F} (D_1^F \nabla^2 \phi_1^F + \Sigma_1^F \phi_1^F). \quad (3.1)$$

Substituting this result into equation (2.2), one obtains a fourth-order equation for ϕ_1^F alone; namely,

$$\tau_f L_f^2 \nabla^2 \nabla^2 \phi_1^F - (\tau_f - L_f^2) \nabla^2 \phi_1^F + (k_\infty - 1) \phi_1^F = 0, \quad (3.2)$$

where

$$\Sigma_1^F = -(\Sigma_{a1}^F + \Sigma_{12}^F - \nu \Sigma_{f1}^F),$$

$$\tau_f = \frac{D_1^F}{\Sigma_1^F}, \quad L_f^2 = \frac{D_2^F}{\Sigma_{a2}^F}, \quad \text{and} \quad k_\infty = \frac{\Sigma_{12}^F \nu \Sigma_{f2}^F}{\Sigma_1^F \Sigma_{a2}^F},$$

Equation (3.2) is a fourth-order differential equation with constant coefficients and can be factored to get

$$(\nabla^2 - \mu^2)(\nabla^2 - \lambda^2)\phi_1^F = 0, \quad (3.3)$$

where

$$\mu^2 = \frac{(\tau_f - L_f^2) \pm \sqrt{(\tau_f - L_f^2)^2 - 4\tau_f L_f^2 (k_\infty - 1)}}{2\tau_f L_f^2}$$

and

$$-\lambda^2 = \mu^2 - \frac{\tau_f - L_f^2}{\tau_f L_f^2}.$$

Equation (3.3) can be split into the following two equations

$$(\nabla^2 - \mu^2)X = 0 \quad (3.4)$$

$$(\nabla^2 - \lambda^2)Y = 0. \quad (3.5)$$

The general solution for ϕ_1^F is then a linear combination of the independent functions X and Y ; namely

$$\phi_1^F = AX + CY, \quad (3.6)$$

where A and C are unknown constants. If the Laplace operator ∇^2 in Cartesian form is used, then

$$X(x, y) = (\cosh \alpha x)(\cosh \alpha y) \quad (3.7)$$

$$Y(x, y) = (\cosh \beta x)(\cosh \beta y), \quad (3.8)$$

where

$$\alpha = \frac{\mu}{\sqrt{2}}, \text{ and } \beta = \frac{\lambda}{\sqrt{2}}$$

Having the fast flux in the fuel been determined, the thermal flux ϕ_2^F can be found from equation (3.1), in the form

$$\phi_2^F = AS_1X + CS_2Y \quad (3.9)$$

where

$$S_1 = \frac{\Sigma_{12}^F / \Sigma_{a2}^F}{\mu^2 L_f^2 - 1} \quad \text{and} \quad S_2 = \frac{\Sigma_{12}^F / \Sigma_{a2}^F}{\lambda^2 L_f^2 - 1}.$$

We now focus on the equations for fast neutron and thermal flux in the moderator. Equation (2.3) may be written in the form

$$\nabla^2 \phi_1^M - k_{1M}^2 \phi_1^M = 0, \quad (3.10)$$

where

$$k_{1M}^2 = \frac{\Sigma_{a1}^M + \Sigma_{12}^M}{D_1^M}.$$

In general, the solution to equation (3.10) will be indicated by

$$\phi_1^M = FZ_1(x, y), \quad (3.11)$$

where F is an unknown constant, and

$$Z_1(x, y) = (\cosh \gamma(b - x)) (\cosh \gamma(b - y)), \quad (3.12)$$

where $\gamma = k_1^M / \sqrt{2}$. The thermal flux in the moderator ϕ_2^M can be found by assuming a solution to equation (2.4) in the form

$$\phi_2^M = S_3 \phi_1^M + GZ_2, \quad (3.13)$$

where G is an unknown constant and

$$S_3 = \frac{\Sigma_{12}^M / D_2^M}{k_{2M}^2 - k_{1M}^2}, \quad k_{2M}^2 = \frac{\Sigma_{a2}^M}{D_2^M}.$$

The function $Z_2(x, y)$ is then the solution of the equation

$$\nabla^2 Z_2 - k_{2M}^2 Z_2 = 0, \quad (3.14)$$

whose solution may be written in the form

$$Z_2(x, y) = (\cosh \omega(b - x)) (\cosh \omega(b - y)),$$

where $\omega = k_2^M / \sqrt{2}$.

The general solution which satisfies the symmetrical conditions (2.9) through (2.11), for the fast and thermal fluxes in both the fuel and moderator displayed in equations (3.6), (3.9), (3.11), and (3.13) can be conveniently written in the following matrix form:

$$\begin{bmatrix} \phi_1^F \\ \phi_2^F \\ \phi_1^M \\ \phi_2^M \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ S_1 & S_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & S_3 & 1 \end{bmatrix} \begin{bmatrix} AX \\ CY \\ FZ_1 \\ GZ_2 \end{bmatrix}. \quad (3.15)$$

To calculate the four unknowns A, C, F , and G , criticality of the reactor in equations (2.5) through (2.8) is utilised.

In the preceding equations, the functions X, Y, Z_1, Z_2 , are transformed into polar coordinates, then their magnitudes, and their partial derivatives are evaluated at the fuel-moderator interface where $r = a$.

Inserting the solutions of the fluxes, namely, equations (3.6), (3.9), (3.11), and (3.13), into equations (2.5) through (2.8) gives the following set of equations:

$$AX + CY - FZ_1 = 0, \quad (3.16)$$

$$AD_1^F \frac{\partial X}{\partial r} + CD_1^F \frac{\partial Y}{\partial r} - FD_1^M \frac{\partial Z_1}{\partial r} = 0, \quad (3.17)$$

$$AS_1 X + CS_2 Y - FS_3 Z_1 - GZ_2 = 0, \quad (3.18)$$

$$AD_2^F S_1 \frac{\partial X}{\partial r} + CD_2^F S_2 \frac{\partial Y}{\partial r} - FD_2^M S_3 \frac{\partial Z_1}{\partial r} - GD_2^M \frac{\partial Z_2}{\partial r} = 0. \quad (3.19)$$

In view of Cramer's rule regarding linear, homogeneous algebraic equations, the determinant of the coefficients multiplying the unknowns A, C, F , and G is zero, then

$$\begin{vmatrix} X & Y & -Z_1 & 0 \\ D_1^F \frac{\partial X}{\partial r} & D_1^F \frac{\partial Y}{\partial r} & -D_1^M \frac{\partial Z_1}{\partial r} & 0 \\ S_1 X & S_2 Y & -S_3 Z_1 & -Z_2 \\ D_2^F S_1 \frac{\partial X}{\partial r} & D_2^F S_2 \frac{\partial Y}{\partial r} & -D_2^M S_3 \frac{\partial Z_1}{\partial r} & -D_2^M \frac{\partial Z_2}{\partial r} \end{vmatrix} = 0. \quad (3.20)$$

Equation (3.20) provides a relation, admittedly a somewhat complicated one, which must be satisfied if the reactor is to be critical. Criticality can be obtained by adjusting either the physical properties or the size of the reactor in such a way that the critical determinant given by (3.20) is precisely zero.

To find the fast and thermal fluxes it is necessary to determine the constants A, C, F , and G . It is not possible, however, to find the absolute values of all of these constants, since it can be shown that if the determinant of the coefficients in equation (3.20) is zero, as it must be for criticality, then these four equations are no longer independent. Therefore it is possible to determine only three of the constants in terms of A . Thus the absolute value of the flux cannot be found from these equations. While it is possible to specify the shape of the flux in a critical reactor, its magnitude depends upon the operating power of the system and is not determined by the group equations. To find the constant A it is necessary to compute the reactor power P , where

$$P = \kappa \left(\Sigma_{f1}^F \int_v \phi_1^F dv + \Sigma_{f2}^F \int_v \phi_2^F dv \right). \quad (3.21)$$

It is found that

$$C = \frac{AX}{\beta Y} (D_1^F \kappa - D_1^M \xi), \quad (3.22)$$

$$F = \frac{AX}{\beta Z_1} (D_1^F \kappa - D_1^M \varepsilon), \quad (3.23)$$

and

$$G = \frac{AX}{\beta Z_2} (D_1^F (S_2 - S_3) \kappa + D_1^F (S_2 - S_3) \varepsilon + D_1^M (S_1 - S_2) \xi), \quad (3.24)$$

where

$$\beta = D_1^M \xi - D_1^F \varepsilon, \quad \kappa = \frac{\partial X}{\partial r}, \quad \xi = \frac{\partial Z_1}{\partial r}, \quad \text{and} \quad \varepsilon = \frac{\partial Y}{\partial r}.$$

All of the functions appearing in these equations are evaluated, at the fuel-moderator interface at $r = a$, (note that, numerically, it is preferable to consider $\theta = 45^\circ$) where

$$\begin{aligned} \frac{\partial X}{\partial r} &= \frac{\alpha}{2} [F_1 \sinh(\alpha r F_1) + F_2 \sinh(\alpha r F_2)] \\ \frac{\partial Y}{\partial r} &= \frac{\beta}{2} [F_1 \sinh(\beta r F_1) + F_2 \sinh(\beta r F_2)] \\ \frac{\partial Z_1}{\partial r} &= \frac{\gamma}{2} [-F_1 \sinh(\gamma(2b - r F_1)) + F_2 \sinh(\gamma r F_2)] \end{aligned}$$

$$\frac{\partial Z_2}{\partial r} = \frac{\omega}{2} [-F_1 \sinh(\omega(2b - rF_1)) + F_2 \sinh(\omega r F_2)]$$

and,

$$F_1 = \sin \theta + \cos \theta, \quad F_2 = \sin \theta - \cos \theta.$$

The solution written in the matrix form of equation (3.15) satisfies all conditions in (2.5) through (2.11) on the triangle region OAB , except only a part of condition (2.9) at the moderator region ($a \leq x \leq b$). Along the sediment line CA , the normal slopes cannot be exactly determined but approximately

$$\left[\frac{\partial \phi_1^M}{\partial y} \right]_{\substack{a \leq x \leq b, \\ y=0}} = -\gamma F (\cosh \gamma(b-x)) (\sinh \gamma b) \approx 0, \quad (3.25)$$

also,

$$\left[\frac{\partial \phi_2^M}{\partial y} \right]_{\substack{a \leq x \leq b, \\ y=0}} = -\gamma S_3 F (\cosh \gamma(b-x)) (\sinh \gamma b) - \omega G (\cosh \omega(b-x)) (\sinh \omega b) \approx 0. \quad (3.26)$$

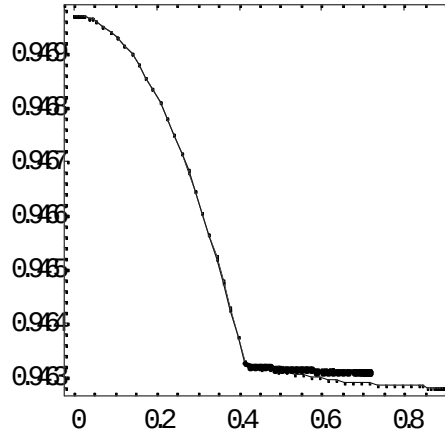


FIGURE 2. Fast flux distributions by the proposed and equivalent cylinder techniques.

4. VALIDATION

The problem considered in this paper was treated in [16], where an approximate analytical solution was obtained. Another mathematical treatment based on an equivalent cylindrical cell was introduced in [13]. The solution obtained using the approach proposed in this work was validated by comparing some numerical results with solutions obtained by the above two methods.

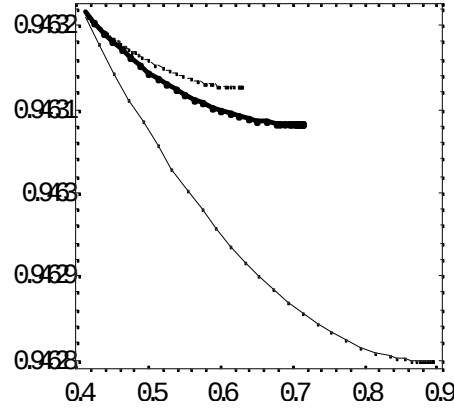


FIGURE 3. Fast flux distributions by the proposed and equivalent cylinder techniques, at moderator region.

Firstly, the approximated equivalent cylindrical cell technique, which is used now in reactors, is utilised to compare the numerical results of the obtained solution, as well as to compare the stability and convergence of the solution.

Secondly, the approximate analytical solution is introduced to show what the proposed solution is like in an explicit and simple form.

The approximate analytical and equivalent cylindrical cell solutions are highlighted in the following subsections.

4.1. Equivalent cylindrical cell. In the equivalent cylindrical cell approach, the actual cell which consists of a cylindrical fuel rod surrounded by a square moderator is replaced by a cylindrical cell of the same volume [13], see Figure 4.

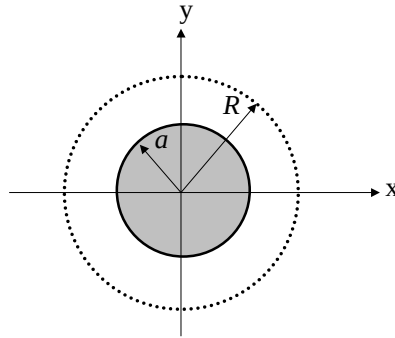


FIGURE 4. Equivalent cylindrical cell.

The solution of the system of equations (2.1) to (2.4) with the critical conditions (2.5) to (2.8), in piecewise cylindrical equivalent cell. The solution obtained may be

recast into matrix form (3.15) as follows:

$$X = I_0(\mu r) \quad (4.1)$$

$$Y = I_0(\lambda r) \quad (4.2)$$

$$Z_1 = K_1(k_{1m}R)I_0(k_{1m}r) + I_1(k_{1m}R)K_0(k_{1m}r) \quad (4.3)$$

$$Z_2 = K_1(k_{2m}R)I_0(k_{2m}r) + I_1(k_{2m}R)K_0(k_{2m}r), \quad (4.4)$$

where $R = \frac{2b}{\sqrt{\pi}}$, and I_0 , I_1 , K_0 , and K_1 are the Bessel functions.

4.2. Approximate analytical method. An approximate analytical method, which uses polar coordinates centred at the centre of the cell, was used in [16] for solving the system of equations given by (2.1) to (2.4). The solution is obtained as an infinite series of products of pairs of functions such that, for every pair, one of the functions depends solely on the angular variable while the other function depends on the radial variable and on the two energy groups. The solution is summarised in the matrix form (3.15) as

$$AX = \sum_{k=0}^{\infty} \cos 4k\theta (a_k Z_{4k}(B_0 r)) \quad (4.5)$$

$$CY = \sum_{k=0}^{\infty} \cos 4k\theta (b_k Z_{4k}(B_1 r)) \quad (4.6)$$

$$FZ_1 = \sum_{k=0}^{\infty} \cos 4k\theta [c_k W_{4k}(B_2 r) + d_k Z_{4k}(B_2 r)] \quad (4.7)$$

$$GZ_2 = \sum_{k=0}^{\infty} \cos 4k\theta [e_k W_{4k}(B_3 r) + f_k Z_{4k}(B_3 r)], \quad (4.8)$$

where W_{4k} , and Z_{4k} are Bessel functions.

The coefficients $S_1, S_2, S_3, a_k, b_k, c_k, d_k, e_k$, and f_k are determined from the boundary and critical conditions.

The results obtained from this solution are in an excellent agreement with our results. However, the solution form obtained using the proposed technique is expressed in a simpler and explicit form.

Each of the three techniques introduced above has its own arbitrary constants. To satisfy equivalence between two of them, arbitrary constants must be determined. One of the assumptions is that, at the cell centre, the equivalent techniques should have the same flux values.

Both the proposed technique and the equivalent cylindrical cell approach have the same flux values at the cell centre as:

$$[\phi_1^F]_{r=0} = A + C, \quad (4.9)$$

and

$$\left[\phi_2^F\right]_{r=0} = AS_1 + CS_2. \quad (4.10)$$

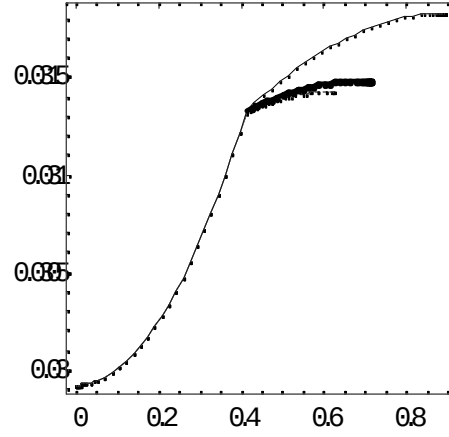


FIGURE 5. Thermal flux distributions by the proposed and equivalent cylinder techniques.

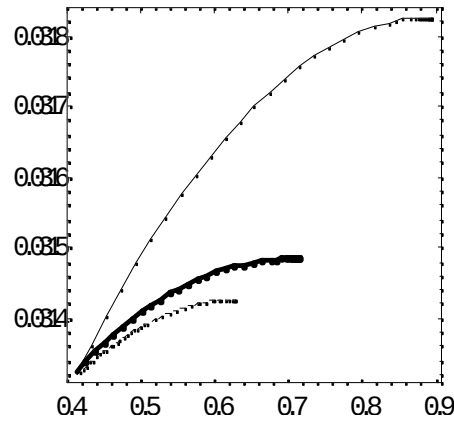


FIGURE 6. Thermal flux distributions by the proposed and equivalent cylinder techniques, at moderator region.

4.3. Study of a typical case. In a steady state symmetrical PWR typical lattice, two-group constants are the following ones:

$$\begin{aligned}
D_1^F &= 1.13, & D_2^F &= 0.519, & D_1^M &= 21.2, & D_2^M &= 1.71, \\
\Sigma_{a1}^F &= 0.0309, & \Sigma_{a2}^F &= 0.283, & \Sigma_{a1}^M &= 0.000127, & \Sigma_{a2}^M &= 0.00432, \\
\Sigma_{12}^F &= -0.00902, & \Sigma_{12}^M &= 0.0157, & \Sigma_{f1}^F &= 0.0094, & \Sigma_{f2}^F &= 0.211, \\
\nu &= 2.42, & a &= 0.4119, & b &= 0.6331.
\end{aligned}$$

As a consequence, the equations of ϕ_1^F , ϕ_1^M , ϕ_2^F , and ϕ_2^M for the proposed technique become (roughly):

$$\phi_1^F = \cosh(0.08x) \cosh(0.08y) - 0.053 \cosh(0.52x) \cosh(0.52y), \quad (4.11)$$

$$\phi_1^M = 0.95 \cosh(0.019(0.6331 - x)) \cosh(0.019(0.6331 - y)), \quad (4.12)$$

$$\phi_2^F = -0.033 \cosh(0.08x) \cosh(0.08y) + 0.063 \cosh(0.52x) \cosh(0.52y), \quad (4.13)$$

and

$$\begin{aligned}
\phi_2^M &= 4.87 \cosh(0.019(0.6331 - x)) \cosh(0.019(0.6331 - y)) \\
&\quad - 4.84 \cosh(0.036(0.6331 - x)) \cosh(0.036(0.6331 - y)). \quad (4.14)
\end{aligned}$$

The normal derivatives in equations (3.25) and (3.26) are of order less than 10^{-10} .

The corresponding solution sets for the equivalent cylindrical cell approach are

$$\phi_1^F = I_0(0.11r) - 0.053I_0(0.73r), \quad (4.15)$$

$$\phi_1^M = 0.01(K_1(0.011)I_0(0.016r) + I_1(0.011)K_0(0.016r)), \quad (4.16)$$

$$\phi_2^F = -0.03I_0(0.11r) + 0.06I_0(0.73r), \quad (4.17)$$

and

$$\begin{aligned}
\phi_2^M &= 0.056(K_1(0.011)I_0(0.016r) + I_1(0.011)K_0(0.016r)) \\
&\quad - 0.1(K_1(0.021)I_0(0.029r) + I_1(0.021)K_0(0.029r)). \quad (4.18)
\end{aligned}$$

The numerical results are showed on the figures enclosed.

Figures 2 and 5 show that both fast and thermal fluxes (ϕ_1^F and ϕ_2^F) by the proposed or equivalent cylindrical cell techniques have the same distribution in fuel region.

Figures 3 and 6 show that, in the moderator region, both fast and thermal fluxes (ϕ_1^M and ϕ_2^M) by equivalent cylindrical cell technique take middle positions between $\theta = 0^\circ$ and $\theta = 45^\circ$.

Note 1. The dashed line stands for the flux at $\theta = 0$, and the continuous one stands for the flux at $\theta = 45^\circ$ by the proposed technique, while the thick line stands for the flux by the equivalent cylindrical cell technique.

Figure 7 specifies the fast flux (ϕ_1^F) distribution over the typical cell by the proposed technique, and Figure 8 specifies the fast flux distribution over the equivalent cylindrical cell.

Figures 9 and 10, respectively, describe the thermal flux distribution of the typical cell found by the proposed technique and the equivalent cylindrical cell.

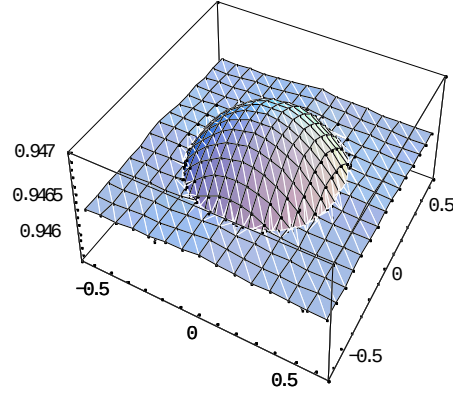


FIGURE 7. Fast flux distribution of the proposed technique.

Figures 11 and 12 show the variations of fast and thermal fluxes (ϕ_1^M, ϕ_2^M), respectively, against the angle θ , at different radii.

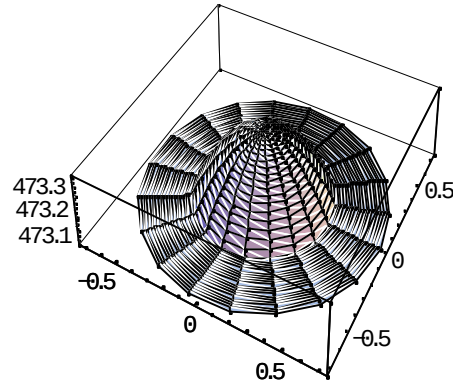


FIGURE 8. Fast flux distribution of equivalent cylinder technique.

5. CONCLUSIONS

An analytical technique was introduced for solving the two-group neutron diffusion equations in the critical reactor's typical lattice. Mathematically, it is a solution of the fourth order homogeneous diffusion equations with non-smooth (or irregular) boundary conditions in both the Cartesian and polar coordinates. The solution was obtained in the explicit form suitable for computer programming and other purposes such as analysis of temperature distribution, thermal stress on fuel rod, etc. By using this technique, one can compute the neutron flux at the corners of the typical cell, in

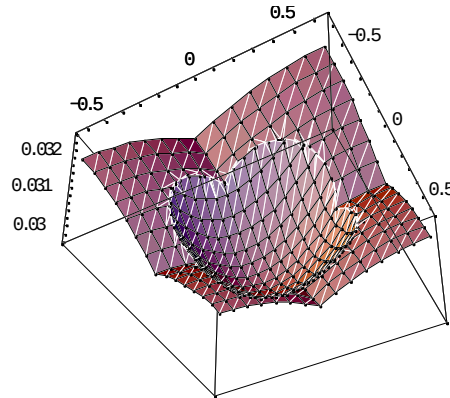


FIGURE 9. Thermal flux distribution of the proposed technique.

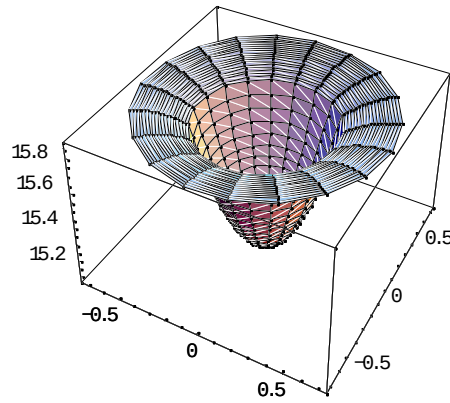


FIGURE 10. Thermal flux distribution of equivalent cylinder technique.

which case neither approximate equivalent cell methods nor numerical methods succeed. The proposed technique also avoids round off and truncation errors analysis and stability and convergence investigation in calculations, in which other approximate analytical methods may fail.

Future work:

- (i) Studying temperature distribution of PWR typical lattice and stress analysis in fuel rod using the proposed technique.
- (ii) Extending the proposed technique in three dimensions.
- (iii) Studying flux distributions of non-symmetrical PWR typical lattice.
- (iv) Treating the problem as a stochastic process, since the macroscopic cross-sections are not constant but random variables.

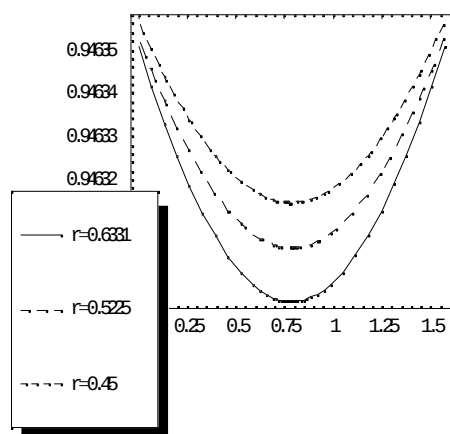


FIGURE 11. Variations of fast flux in moderator region against angle θ , at different values of r .

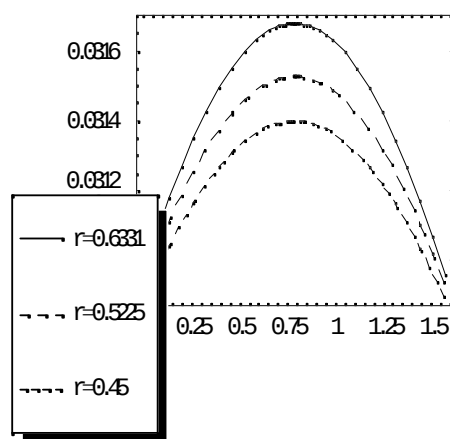


FIGURE 12. Variations of thermal flux in moderator region against angle θ , at different values of r .

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A GENERALIZATION OF THE WALLIS FORMULA

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ABSTRACT. We prove a generalization of the Wallis formula which is connected with the generalized π (denoted by $\hat{\pi}(n)$, $n > 0$ is real) introduced by Elbert in [1].

Mathematics Subject Classification: Primary 34C10; Secondary 40A20

Keywords: Wallis formula, generalized π

1. INTRODUCTION

The infinite product of the form

$$\frac{\pi}{2} = \begin{cases} \lim_{k \rightarrow \infty} \frac{2^2}{1 \cdot 3} \cdot \frac{4^2}{3 \cdot 5} \cdot \dots \cdot \frac{(2k)^2}{(2k-1)(2k+1)} \\ \lim_{k \rightarrow \infty} 2 \cdot \frac{2 \cdot 4}{3^2} \cdot \frac{4 \cdot 6}{5^2} \cdot \dots \cdot \frac{(2k-2) \cdot 2k}{(2k-1)^2} \end{cases} \quad (1)$$

is known as the Wallis formula (or product) [5]. This formula is the most remarkable expression for the number π . The formula is proved in an elementary way on the basis of recurrence formula of $\int \sin^\alpha(t) dt$.

In this paper we give a generalization and prove it similarly as in the original case.

2. THE MAIN RESULT

Theorem (Generalized Wallis formula). *Let $n > 0$ be an arbitrary real number. Then*

$$\frac{\hat{\pi}(n)}{2} = \begin{cases} \lim_{k \rightarrow \infty} \frac{1}{n} \prod_{i=1}^k \frac{i^2(n+1)^2}{i^2(n+1)^2 - n^2} \\ \lim_{k \rightarrow \infty} \frac{n+1}{n} \prod_{i=1}^{k-1} \frac{i(i+1)(n+1)^2}{[(i+1)(n+1)-1][(i+1)(n+1)-n]} \end{cases} \quad (2)$$

where

$$\hat{\pi}(n) = 2 \cdot \frac{\frac{\pi}{n+1}}{\sin \frac{\pi}{n+1}}. \quad (2a)$$

In the case where $n = 1$, (2) is the original Wallis formula (1).

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Proof. Consider the half-linear differential equation

$$y''|y'|^{n-1} + y^{n*} = 0, \quad y = y(x), \quad (-\infty < x < \infty), \quad (3)$$

where $y^{n*} = |y|^n \cdot \text{sign } y$, $n > 0$.

In [1] Elbert introduced the generalized sine function $S_n(x)$ as the solution of the differential equation (3) satisfying the initial conditions $y(0) = 0$, $y'(0) = 1$. The function $S_n(x)$ is periodic with the period $2\hat{\pi}$, and

$$0 < S_n(x) < 1 \quad \text{for } 0 < x < \frac{\hat{\pi}}{2}$$

and

$$S_n\left(\frac{\hat{\pi}}{2}\right) = 1, \quad S'_n\left(\frac{\hat{\pi}}{2}\right) = 0.$$

It is proved that the generalized Pythagorean formula holds

$$|S_n(x)|^{n+1} + |S'_n(x)|^{n+1} = 1. \quad (4)$$

First we show the following simple integral formula

$$\int_0^{\frac{\hat{\pi}(n)}{2}} S_n^\alpha(t) dt = \frac{\alpha - n}{\alpha} \int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{\alpha-(n+1)}(t) dt \quad (\alpha \geq n+1). \quad (5)$$

From the differential equation (3), we have

$$\int S_n^{n*}(t) dt = -\frac{1}{n} S_n^{n*} + \text{const.} \quad (6)$$

By partial integration we obtain from (6) and (4) that

$$\begin{aligned} \int_0^{\frac{\hat{\pi}(n)}{2}} S_n^\alpha(t) dt &= \int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{\alpha*}(t) S_n^{\alpha-n}(t) dt = \\ &= \int_0^{\frac{\hat{\pi}(n)}{2}} S_n^\alpha(t) dt = \int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{\alpha*}(t) S_n^{\alpha-n}(t) dt = \\ &= \frac{\alpha - n}{n} \int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{\alpha-(n+1)}(t) dt + \frac{\alpha - n}{n} \int_0^{\frac{\hat{\pi}(n)}{2}} S_n^\alpha(t) dt. \end{aligned} \quad (7)$$

Rearranging (7) we obtain (5).

From (6), we get

$$\int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{n*}(t) dt = \frac{1}{n}. \quad (8)$$

In the cases where $\alpha = k(n+1)$ and $\alpha = k(n+1) + n$, ($k = 1, 2, \dots$), we obtain from (5) the relations

$$\int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{k(n+1)}(t) dt = \frac{k - \nu}{k} \int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{(k-1)(n+1)}(t) dt \quad (9)$$

and

$$\int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{k(n+1)+n}(t) dt = \frac{k}{k+\nu} \int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{(k-1)(n+1)+n}(t) dt, \quad (10)$$

where $\nu = \frac{n}{n+1}$, $\nu \in (0, 1)$.

Repeating (9) and (10) and using (8) we get

$$\int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{k(n+1)}(t) dt = \frac{\hat{\pi}(n)}{2} \prod_{i=1}^k \frac{i-\nu}{i}, \quad (k = 1, 2, \dots) \quad (11)$$

and

$$\int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{k(n+1)+n}(t) dt = \frac{1}{n} \prod_{i=1}^k \frac{i}{i+\nu}. \quad (12)$$

In the case where $n = 1$, formulas (11) and (12) are the well-known recurrence formulas ($k(n+1)$ is the even, $k(n+1)+n$ is the odd equivalent of the index).

Since $0 < S_n(x) < 1$ on $(0, \frac{\hat{\pi}(n)}{2})$, we have

$$\begin{aligned} \int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{k(n+1)+n}(t) dt &< \int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{k(n+1)}(t) dt \\ &< \int_0^{\frac{\hat{\pi}(n)}{2}} S_n^{(k-1)(n+1)+n}(t) dt \quad (k = 1, 2, \dots). \end{aligned} \quad (13)$$

By (11) and (12), formula (13) can be written in the form

$$A_k < \frac{\hat{\pi}(n)}{2} < B_k \quad (k = 1, 2, \dots), \quad (14)$$

where

$$A_k = \frac{1}{n} \prod_{i=1}^k \frac{i^2}{i^2 - \nu^2} \quad (15)$$

and

$$B_k = \frac{1}{\nu} \prod_{i=1}^{k-1} \frac{i(i+1)}{\left[i + 1 - \frac{\nu}{n}\right][i + 1 - \nu]}. \quad (16)$$

Clearly,

$$B_k = \frac{k+\nu}{k} A_k. \quad (17)$$

The sequence A_k is strictly increasing because

$$\frac{i^2(n+1)^2}{i^2(n+1)^2 - n^2} > 1 \quad (i = 1, 2, \dots, k).$$

Hence, by (14), the limit $\lim_{k \rightarrow \infty} A_k$ exists.

The sequence B_k is strictly decreasing because

$$\frac{i(i+1)}{\left[i+1-\frac{v}{n}\right][i+1-v]} = \frac{i(i+1)}{i(i+1)+\frac{v^2}{n}} < 1 \quad (i = 1, 2, \dots, k-1).$$

By (14), the limit $\lim_{k \rightarrow \infty} B_k$ exists.

We estimate the difference of B_k and A_k , (14) and (17)

$$0 < B_k - A_k = \frac{v}{k} A_k < \frac{v}{k} \frac{\hat{\pi}(n)}{2}. \quad (18)$$

With increasing n , $B_k - A_k$ can be diminished, so $\frac{\hat{\pi}(n)}{2}$ can be approximated with optional accuracy with the A_k, B_k sequence, that is

$$\lim_{k \rightarrow \infty} A_k = \frac{\hat{\pi}(n)}{2} = \lim_{k \rightarrow \infty} B_k,$$

which proves our theorem. $\square$

Remark. The integrals (11) and (12) may be computed by using the gamma-functions.

Second Proof. We take

$$I(n, \alpha) = \int_0^{\frac{\hat{\pi}(n)}{2}} S_n^\alpha(t) dt. \quad (19)$$

Consider $\varphi_n(t)$, the inverse function of $S_n(t)$. From (4),

$$\frac{d\varphi_n(t)}{dt} = \frac{1}{\sqrt[n+1]{1-t^{n+1}}} \quad (0 \leq t < 1). \quad (20)$$

From (19), using the substitution $\tau = S_n(t)^{n+1}$, we get

$$I(n, \alpha) = \frac{1}{n+1} \int_0^1 \tau^{\frac{\alpha-n}{n+1}} (1-\tau)^{-\frac{1}{n+1}} d\tau. \quad (21)$$

Consider the Euler first integral [2]

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt \quad (22)$$

where x, y are positive. This integral can be evaluated using gamma-function:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \quad (23)$$

In the right side of (20), there is the Euler first integral. Using (23), we get

$$I(n, \alpha) = \frac{1}{n+1} \frac{\Gamma\left(\frac{\alpha+1}{n+1}\right)\Gamma\left(\frac{n}{n+1}\right)}{\Gamma\left(\frac{\alpha}{n+1}\right)} \quad (\alpha > 0). \quad (24)$$

In the case $\alpha = k(n+1) + n$, ($k = 1, 2, \dots$), we have

$$I(n, k(n+1) + n) = \frac{1}{n+1} \frac{\Gamma(k+1)\Gamma\left(\frac{n}{n+1}\right)}{\Gamma\left(k + \frac{n}{n+1}\right)}.$$

Using the elementary formula for the gamma-function

$$x\Gamma(x) = \Gamma(x+1), \quad (25)$$

we get

$$I(n, k(n+1) + n) = \frac{1}{n} \prod_{i=1}^k \frac{i}{i+v},$$

which is equivalent to (12). If $\alpha = k(n+1)$, applying (22), we obtain

$$I(n, k(n+1)) = \frac{\Gamma\left(k + \frac{1}{n+1}\right)\Gamma\left(\frac{n}{n+1}\right)}{\Gamma(n+1)}.$$

In view of (25), we have

$$I(n, k(n+1)) = \frac{1}{n+1} \Gamma\left(\frac{1}{n+1}\right) \Gamma\left(\frac{n}{n+1}\right) \prod_{i=1}^k \frac{i-v}{i}. \quad (26)$$

With the help of the Euler functional equation

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x},$$

and (2a), we get

$$\frac{1}{n+1} \Gamma\left(\frac{n}{n+1}\right) \Gamma\left(\frac{n}{n+1}\right) = \frac{\hat{\pi}(n)}{2}.$$

Substituting this into (26), we obtain (11). $\square$

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ON THE INF-TYPE EXTREMALITY SOLUTIONS TO HAMILTON–JACOBI EQUATIONS, THEIR REGULARITY PROPERTIES, AND SOME GENERALIZATIONS

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AND M. PYTEL-KUDELA

This paper is dedicated to the memory of our friend Prof. Jerzy Zagrodzinski (†2001)

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ABSTRACT. The extremality problem representation of the Lax type [1] is studied in detail for some class of Hamilton–Jacobi equations in the many-dimensional case. The regularity properties of solutions of the Cauchy problem in the class of convex lower semicontinuous functions are established. A generalisation to a wider class of functions is obtained.

The Hamilton–Jacobi equation on the sphere is considered, and its exact solutions are found in terms of a Lax type extremality problem. Some generalisation of the results for the general case of many-dimensional Hamilton–Jacobi equations is obtained by using the Fan–Brouwer fixed point techniques in a Banach space.

Mathematics Subject Classification: Primary 34A30, 34B05; Secondary 34B15

Keywords: extremality problem, Lax type representation, Hamiltonian dynamical systems, fixed point problem, exact solutions

1. THE EXTREMALITY PROBLEM FUNCTIONAL ANALYSIS: CONVEX BSC-CLASS SOLUTIONS

1.1. Introduction. It is well-known that equations like

$$u_t + f(t, x; u, \nabla u) = 0 \quad (0.1)$$

for $u : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$ with $f : \mathbb{R}_+ \times J^{(1)}(\mathbb{R}^n) \rightarrow \mathbb{R}$ being some fixed mapping on the jet-manifold $J^{(1)}(\mathbb{R}^n; \mathbb{R})$ are called *Hamilton–Jacobi equations*, and are related to the motion of certain mechanical systems. As was shown earlier in [1–3], these equations possess the stabilisation property as $t \rightarrow \infty$, namely, a solution to (0.1) for any Cauchy data $u|_{t=0^+} = v$ from some appropriate class of functions tends to these Cauchy data.

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In case the mapping $f : J^{(1)}(\mathbb{R}^n; \mathbb{R}) \rightarrow \mathbb{R}$ is smooth and does not depend on variables $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ and function $u : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$, S. Kruzhkov [8–9] obtained the following analytical representation for the solution to (0.1) with the smooth Cauchy data $v : \mathbb{R}^n \rightarrow \mathbb{R}$:

$$u(x, t) = v(x - t\nabla f(p_0)) + t[\langle p_0, \nabla f(p_0) \rangle - f(p_0)], \quad (0.2)$$

where $u|_{t=0^+} = v$ and the function $p_0 = \nabla v(x - t\nabla f(p_0))$.

In this work, we develop the theory of equations like (0.1) for some special kinds of mappings $f : J^{(1)}(\mathbb{R}^n; \mathbb{R}) \rightarrow \mathbb{R}$, namely,

$$u_t + f(x; u, \nabla u) = 0 \quad (0.3)$$

and demonstrate the inf-type extremality structure of their solutions which was first observed by P. Lax (see [1]). In particular, we prove that this inf-type extremality structure really gives rise to solutions of (0.3) for the Cauchy data from the class $\text{BSC}(\mathbb{R}^n)$ of convex lower semicontinuous functions on $\mathbb{R}^n$. In particular, for the case where

$$f(x; u, \nabla u) := \frac{1}{2} \langle \nabla u, \nabla u \rangle, \quad u \in \text{BSC}(\mathbb{R}^n),$$

we prove in Section 1 the aforementioned Lax result and give its generalisation to a wider class of Cauchy data.

In Section 2, we study the extremality structure of solutions to equations (0.3) which reduce to a fixed point problem and show its well-posedness. Another generalisation considered in this work is related to a Hamilton-Jacobi equation (0.3) on the n -dimensional sphere $\mathbb{S}^n$. The corresponding inf-type extremality solution to this Hamilton-Jacobi equation is proved to exist also for the BSC-class of functions on $\mathbb{S}^n$.

1.2. Problem setting. The review article [1] devoted to viscosity solutions of first and second order partial differential equations contains the following exact formula, suggested by P. Lax,

$$u(x, t) = \inf_{y \in \mathbb{R}^n} \left\{ v(y) + \frac{1}{2t} \|x - y\|^2 \right\}, \quad (1.1)$$

for the solutions to the Hamilton–Jacobi nonlinear partial differential equation:

$$\frac{\partial u}{\partial t} + \frac{1}{2} \|\nabla u\|^2 = 0, \quad u|_{t=0^+} = v, \quad (1.2)$$

with Cauchy data $v \in \text{BSC}(\mathbb{R}^n)$ being chosen as a properly convex and lower semicontinuous function. Here, $\|\cdot\| = \langle \cdot, \cdot \rangle$ is the usual norm in $\mathbb{R}^n$, $n \in \mathbb{Z}$, and $t \in \mathbb{R}_+$ is a positive evolution parameter. It is also stated in [1] that there is no exact proof of the Lax formula (1.1) based on general properties of the Hamilton-Jacobi equation (1.2).

The present section is devoted to such an exact proof of the Lax formula (1.1) and to the study of some of its properties.

1.3. Analysis of the Hamilton-Jacobi dynamics. Consider the following canonical Hamiltonian system associated naturally [2] with (1.2):

$$\frac{dx}{dt} = \frac{\partial H_0}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H_0}{\partial x}, \quad (2.1)$$

where the Hamiltonian function $H_0 \in C^2(T(\mathbb{R}^n); \mathbb{R})$ is

$$H_0(x, p) = \frac{1}{2} \|p\|^2 \quad (2.2)$$

for $(x, p) \in T^*(\mathbb{R}^n)$, $T^*(\mathbb{R}^n)$ being the canonical phase space of coordinates. The solution to (2.1) with Cauchy data at $(x_0, p_0) \in T^*(\mathbb{R}^n)$ is given for all $t \in \mathbb{R}_+$ as follows:

$$x = x_0 + p_0 t, \quad p = p_0. \quad (2.3)$$

Introduce now the so-called “action function” $u : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$ which can be defined [2] locally as

$$du = -H_0(x, p)dt + \langle p, dx \rangle, \quad (2.4)$$

where, by (2.3), $p = (x - x_0)/t$, and let $u|_{t=0^+} = v \in \text{BSC}(\mathbb{R}^n)$. From (2.4) one obtains immediately that

$$\frac{\partial u}{\partial t} = -H_0(x, p), \quad \frac{\partial u}{\partial x} = p \quad (2.5)$$

for all points $(x, p) \in T^*(\mathbb{R}^n)$. Substituting (2.2) into (2.5), one gets the following.

Lemma 1.1. *The action function $u : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfies exactly the Hamilton-Jacobi equation (1.2), that is*

$$\frac{\partial u}{\partial t} + \frac{1}{2} \|\nabla u\|^2 = 0, \quad u|_{t=0^+} = v. \quad (2.6)$$

Now we shall proceed to computing an expression for the action function $u : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$, defined by (2.4). From (2.4) one finds that

$$\begin{aligned} u(x, t) &= \int_0^t d\tau \left(\frac{du}{d\tau} \right) \Big|_{\substack{x=x_0+p_0\tau \\ p=p_0}} + v(x_0)|_{x=x_0+p_0t} = \\ &= \int_0^t d\tau \left(\left\langle p, \frac{dx}{d\tau} \right\rangle - H_0(x, p) \right) \Big|_{\substack{x=x_0+p_0\tau \\ p=p_0}} + v(x_0)|_{x=x_0+p_0t} = \\ &= \left(\frac{1}{2} \|p_0\|^2 t + v(x_0) \right) \Big|_{x=x_0+p_0t}. \end{aligned} \quad (2.7)$$

Since, due to (2.5), at a fixed $x \in \mathbb{R}^n$, the function

$$\frac{\partial u(x, t)}{\partial x} \Big|_{t=0^+} = p_0(x), \quad (2.8)$$

is defined, where $p_0 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is some mapping naturally defined from (2.3), (2.8) and (2.7), one arrives at the formula

$$u(x, t) = v(x - p_0(x_0)t) + \frac{t}{2} \|p_0(x_0)\|^2. \quad (2.9)$$

Here, for some $x_0 \in C(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}^n)$, the equation

$$x_0 = x - tp(x_0)$$

holds, giving an unwieldy solution to the Hamilton-Jacobi equation (2.6). The expression (2.9) can be easily transformed into the following useful form:

$$u(x, t) = v(\xi) + \frac{1}{2t} \|x - \xi\|^2, \quad (2.10)$$

where the mapping $\xi : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$ is defined as a solution to the functional equations

$$\xi(x, t) := x - tp_0(\xi(x, t)), \quad p_0(x) := \left. \frac{\partial u(x, t)}{\partial x} \right|_{t=0^+} \quad (2.11)$$

for any $x \in \mathbb{R}^n$, and $t \in \mathbb{R}_+$.

For the expression (2.10) to be interpreted more exactly, it is useful to recall that Hamiltonian equations (2.1) are completely equivalent to the following shortened extremal Lagrange action principle:

$$\delta \tilde{u}[x_0; \tilde{x}, t] \Big|_{\substack{\tilde{x}|_{\tau=0^+} = x_0 \in \mathbb{R}^n \\ \tilde{x}|_{\tau=t} = x \in \mathbb{R}^n \\ \tilde{x} \in C^1(\mathbb{R}_+; \mathbb{R}^n)}} = 0, \quad \tilde{u}[x_0; \tilde{x}, t] := \int_0^t d\tau L_0(\tilde{x}, \dot{\tilde{x}}) + v(x_0), \quad (2.12)$$

where, by definition, the Lagrangian function is

$$L_0(\tilde{x}, \dot{\tilde{x}}) := \langle p, \dot{\tilde{x}} \rangle - H_0(\tilde{x}, p) \Big|_{\tilde{x} = \partial H_0(\tilde{x}, p) / \partial p}. \quad (2.13)$$

Based on (2.4) and (2.13), one infers easily that the extremum expression

$$\begin{aligned} \tilde{u}(x, t) &:= \inf_{x_0 \in \mathbb{R}^n} \left\{ v(x_0) + \frac{1}{2t} \|x - x_0\|^2 \right\} \\ &= v(\tilde{\xi}) + \frac{1}{2t} \|x - \tilde{\xi}\|^2 \end{aligned} \quad (2.14)$$

holds if it is assumed that the infimum in the parenthesis exists and is attained at a unique point $x_0 = \tilde{\xi}(x, t)$ for fixed $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$. For the above motivation to be validated, we shall study in detail properties of the solution $\tilde{\xi} = \tilde{\xi}(x, t)$ to the extremal problem (2.14) aiming to prove that $\tilde{\xi}(x, t) = \xi(x, t)$ for any $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$, where

$$\xi(x, t) := x - tp_0(\xi(x, t)), \quad p_0(x) := \left. \frac{\partial u(x, t)}{\partial x} \right|_{t=0^+},$$

as it was found in (2.11).

1.4. Analysis of the extremality problem. Let us consider the problem (2.14) in the case when a function $v : \mathbb{R}^n \rightarrow \mathbb{R}$ is properly convex and semicontinuous from below, that is $v \in \text{BSC}(\mathbb{R}^n)$. Then the following lemma (similar to lemma A5 in [1]) is true.

Lemma 1.2. *There exists a unique solution $x_0 = \tilde{\xi}(x, t) \in \mathbb{R}^n$ to the extremum problem (2.14) characterised by the inequality*

$$\frac{1}{t} \langle \tilde{\xi} - x, \tilde{\xi} - y \rangle \leq v(y) - v(\tilde{\xi}) \quad (3.1)$$

for all $y \in \mathbb{R}^n$, $t \in \mathbb{R}_+$.

Proof. We first prove the inequality (3.1) assuming the existence of a solution to (2.14). Let us take any

$$z = \tilde{\xi} + \vartheta(y - \tilde{\xi}) = \vartheta y + (1 - \vartheta)\tilde{\xi},$$

where $\vartheta \in (0, 1)$. Then one easily obtains the inequality

$$\begin{aligned} v(\tilde{\xi}) + \frac{1}{2t} \|x - \tilde{\xi}\|^2 &\leq v(\tilde{\xi} + \vartheta(y - \tilde{\xi})) \\ &+ \frac{1}{2t} \langle \tilde{\xi} + \vartheta(y - \tilde{\xi}) - x, \tilde{\xi} + \vartheta(y - \tilde{\xi}) - x \rangle \\ &\leq (1 - \vartheta)v(\tilde{\xi}) + \vartheta v(y) + \frac{1}{2t} \langle \tilde{\xi} - x, \tilde{\xi} - x \rangle \\ &\quad + \frac{1}{t} \vartheta \langle \tilde{\xi} - x, y - \tilde{\xi} \rangle + \frac{\vartheta^2}{2t} \langle y - \tilde{\xi}, y - \tilde{\xi} \rangle, \end{aligned} \quad (3.2)$$

whence we find that

$$v(\tilde{\xi}) - v(y) + \frac{1}{t} \langle \tilde{\xi} - x, \tilde{\xi} - y \rangle \leq \frac{\vartheta}{2t} \|y - \tilde{\xi}\|^2, \quad (3.3)$$

for any $\vartheta \in (0, 1)$. Passing to the limit in (3.3) as $\vartheta \rightarrow 0$, one gets exactly the inequality (3.1).

Now we shall proceed to the proof of the existence of a solution to (2.4). We use a standard minimizing sequence $\tilde{\xi}_j \in \mathbb{R}^n$, $j \in \mathbb{Z}_+$, satisfying the inequality

$$v(\tilde{\xi}_j) + \frac{1}{2t} \langle \tilde{\xi}_j - x, \tilde{\xi}_j - x \rangle \leq \tilde{u}(x, t) + \frac{1}{j} \quad (3.4)$$

for any $j \in \mathbb{Z}_+$ and prove first that it is a Cauchy sequence.

One deduces now that

$$\begin{aligned} \|\tilde{\xi}_j - \tilde{\xi}_i\|^2 &= 2\|\tilde{\xi}_j - x\|^2 + 2\|\tilde{\xi}_i - x\|^2 - 4\left\|\frac{\tilde{\xi}_j + \tilde{\xi}_i}{2} - x\right\|^2 \leq \\ &\leq 4t\left[\frac{1}{i} + \frac{1}{j} + 2\tilde{u}(x, t) - v(\tilde{\xi}_j) - v(\tilde{\xi}_i)\right] + 8t\left(v\left(\frac{\tilde{\xi}_j + \tilde{\xi}_i}{2}\right) - \tilde{u}(x, t)\right) \leq \\ &\leq 4t\left(\frac{1}{i} + \frac{1}{j}\right), \end{aligned} \quad (3.5)$$

using inequalities (3.4) and the convexity of $v \in \text{BSC}(\mathbb{R}^n)$. Inequality (3.5) means that for any fixed $t \in \mathbb{R}_+$, there exists the limit element $\lim_{j \rightarrow \infty} \tilde{\xi}_j = \tilde{\xi} \in \mathbb{R}^n$, by virtue of the completeness of the Euclidean space $\mathbb{R}^n$. On the other hand, the lower semicontinuity of $v : \mathbb{R}^n \rightarrow \mathbb{R}$ yields

$$\begin{aligned} \tilde{u}(x, t) &\leq v(\tilde{\xi}) + \frac{1}{2t}\|\tilde{\xi} - x\|^2 \\ &\leq \liminf_{j \rightarrow \infty} \left(v(\tilde{\xi}_j) + \frac{1}{2}\|\tilde{\xi}_j - x\|^2\right) \leq \tilde{u}(x, t), \end{aligned} \quad (3.6)$$

which implies the following identity related to (2.14):

$$\tilde{u}(x, t) \equiv v(\tilde{\xi}) + \frac{1}{2t}\|\tilde{\xi} - x\|^2. \quad (3.7)$$

The uniqueness of the solution $\tilde{\xi} \in \mathbb{R}^n$ of problem (2.14) for fixed $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$ is proved as follows. If $\tilde{\xi}_1$ and $\tilde{\xi}_2 \in \mathbb{R}^n$ are two different solutions of (2.14), then it follows from (3.1) that

$$\begin{aligned} v(\tilde{\xi}_1) - v(\tilde{\xi}_2) + \frac{1}{t}\langle \tilde{\xi}_1 - x, \tilde{\xi}_1 - \tilde{\xi}_2 \rangle &\leq 0, \\ v(\tilde{\xi}_2) - v(\tilde{\xi}_1) + \frac{1}{t}\langle \tilde{\xi}_2 - x, \tilde{\xi}_2 - \tilde{\xi}_1 \rangle &\leq 0, \end{aligned} \quad (3.8)$$

for any $t \in \mathbb{R}_+$. Summing inequalities (3.8), one gets readily that

$$\frac{1}{t}\|\tilde{\xi}_1 - \tilde{\xi}_2\| \leq 0 \quad (3.9)$$

for any $t \in \mathbb{R}_+$ and, hence, $\tilde{\xi}_1 \equiv \tilde{\xi}_2$. Thus, the solution $\tilde{\xi} : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$ of problem (2.14) is unique. $\square$

Now we shall investigate differential properties of the solution $\tilde{\xi} : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$, which will be used essentially in the main required equality $\tilde{\xi} = \xi$, thus ensuring that the extremum function $\tilde{u} : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$ belongs to $\text{BSC}(\mathbb{R}^n)$ and coincides with the solution $\tilde{u} : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$ of problem (1.2).

The following lemma is almost obvious from inequality (3.1).

Lemma 1.3. *The mappings $\tilde{P}_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $(1 - \tilde{P}_t) : \mathbb{R}^n \rightarrow \mathbb{R}^n$, where, by definition, $\tilde{P}_t x := \tilde{\xi}(x, t)$ for any $t \in \mathbb{R}_+$, are Lipschitzian, that is, for any $x, y \in \mathbb{R}^n$*

$$\|\tilde{P}_t x - \tilde{P}_t y\| \leq \|x - y\|, \quad \|(1 - \tilde{P}_t)x - (1 - \tilde{P}_t)y\| \leq \|x - y\|. \quad (3.10)$$

Proof. From (3.1) one easily obtains

$$\|\tilde{P}_t x - \tilde{P}_t y\|^2 + \|(1 - \tilde{P}_t)x - (1 - \tilde{P}_t)y\|^2 \leq \|x - y\|^2, \quad (3.11)$$

which yields inequalities (3.10). $\square$

Consider now the minimum function $\tilde{u} : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$ for problem (2.14), which is realized at a unique element $\tilde{\xi} = \tilde{P}_t \in \mathbb{R}^n$, $t \in \mathbb{R}_+$. We can formulate the following useful lemma.

Lemma 1.4. *The mapping $\tilde{u}_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined as $\tilde{u}_t(x) := \tilde{u}(x, t)$ for any $x \in \mathbb{R}^n$, with $t \in \mathbb{R}_+$ fixed, is convex and differentiable with respect to $x \in \mathbb{R}^n$, and satisfies the equality*

$$\nabla \tilde{u}_t(x) = \frac{1}{t} (x - \tilde{P}_t(x)). \quad (3.12)$$

Moreover, as $t \rightarrow 0^+$, for any $x \in \mathbb{R}^n$ the following limits exist:

$$\lim_{t \rightarrow 0^+} \tilde{u}_t(x) = v(x), \quad \lim_{t \rightarrow 0^+} \tilde{P}_t(x) = x. \quad (3.13)$$

Proof. We first verify equalities (3.13) making use of the following Fenchel characterisation [3] of convex lower semicontinuous functions: A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex and lower semicontinuous if it coincides with its second conjugate via a Fenchel function, that is $f^{**}(x) = f(x)$, $x \in \mathbb{R}^n$, where, by definition,

$$f^*(p) := \sup_{x \in \mathbb{R}^n} \{\langle x, p \rangle - f(x)\}, \quad (3.14)$$

for any $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $p \in \mathbb{R}^n$. Since (2.14) yields

$$\tilde{u}(x, t) \leq \left\{ v(y) + \frac{1}{2t} \|y - x\|^2 \right\} \Big|_{y=x} = v(x)$$

for any $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, we get the following chain of inequalities using the semicontinuity of the function $v : \mathbb{R}^n \rightarrow \mathbb{R}$:

$$\begin{aligned} \frac{1}{2t} \|\tilde{\xi}(x, t) - x\|^2 &\leq v(x) + v^*(p) - \langle p, x \rangle + \langle p, x - \tilde{\xi}(x, t) \rangle \leq \\ &\leq \frac{1}{4t} \|\tilde{\xi}(x, t) - x\|^2 + v(x) + v^*(p) - \langle p, x \rangle + t\|p\|^2. \end{aligned} \quad (3.15)$$

Whence one arrives at the inequality

$$\|\tilde{\xi}(x, t) - x\|^2 \leq 4t \left(v(x) + v^*(p) - \langle p, x \rangle + t\|p\|^2 \right), \quad (3.16)$$

which means evidently that $\lim_{t \rightarrow 0^+} \tilde{\xi}(x, t) = x$ for all $x \in \mathbb{R}^n$.

Note that this result can also be obtained by making use of inequality (3.1) and the convexity of the function $v : \mathbb{R}^n \rightarrow \mathbb{R}$.

Since the function $v : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex and lower semi-continuous, one can write the following chain of inequalities:

$$\begin{aligned} \tilde{u}(x, t) &\leq \left\{ v(y) + \|x - y\|^2 \frac{1}{2t} \right\} \Big|_{y=x} = v(x) \leq \liminf_{t \rightarrow 0^+} v(\tilde{\xi}(x, t)) = \\ &= \liminf_{t \rightarrow 0^+} \left\{ \tilde{u}(x, t) - \frac{1}{2t} \|x - \tilde{\xi}\|^2 \right\} \leq \liminf_{t \rightarrow 0^+} \tilde{u}(x, t). \end{aligned} \quad (3.17)$$

Thus, applying the $\limsup_{t \rightarrow 0^+}$ operation to the left hand side of (3.17), one gets easily that

$$\limsup_{t \rightarrow 0^+} \tilde{u}(x, t) \leq v(x) \leq \liminf_{t \rightarrow 0^+} \tilde{u}(x, t),$$

hence the limit

$$\lim_{t \rightarrow 0^+} \tilde{u}(x, t) = v(x)$$

exists. To prove now equality (3.12), we note that the inequality

$$\tilde{u}(x, t) - \tilde{u}(y, t) \leq \left\langle \frac{1}{t}(x - \tilde{\xi}(x, t)), x - y \right\rangle \quad (3.18)$$

holds for any $x, y \in \mathbb{R}^n$ and fixed $t \in \mathbb{R}_+$. From (3.18) and (3.10), applying the change of variables $x \rightleftharpoons y$, one arrives at the following inequality:

$$\begin{aligned} \tilde{u}(x, t) - \tilde{u}(y, t) &\geq \left\langle \frac{1}{t}(y - \tilde{\xi}(y, t)), x - y \right\rangle = \left\langle \frac{1}{t}(x - \tilde{\xi}(x, t)), x - y \right\rangle + \\ &+ \left\langle \frac{1}{t}(y - x + \tilde{\xi}(x, t) - \tilde{\xi}(y, t)), x - y \right\rangle = \left\langle \frac{1}{t}(x - \tilde{\xi}(x, t)), x - y \right\rangle + \\ &+ \left\langle \frac{1}{t}(1 - \tilde{P}_t)y - \frac{1}{t}(1 - \tilde{P}_t)x, x - y \right\rangle \geq \left\langle \frac{1}{t}(x - \tilde{P}_t x), x - y \right\rangle - \frac{1}{t} \|x - y\|^2. \end{aligned} \quad (3.19)$$

It follows from (3.19) that, for all $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$,

$$\left| \tilde{u}(x, t) - \tilde{u}(y, t) - \left\langle \frac{1}{t}(1 - \tilde{P}_t)x, x - y \right\rangle \right| \leq \frac{1}{t} \|x - y\|^2, \quad (3.20)$$

that is,

$$\frac{1}{t}(1 - \tilde{P}_t)x = \nabla \tilde{u}(x, t),$$

which is the result desired. As a consequence of expressions (3.12) and (2.14), one obtains the equality

$$\lim_{t \rightarrow 0^+} \nabla \tilde{u}(x, t) = \tilde{p}_0(x) \equiv \frac{x - \tilde{\xi}(x, t)}{t} \quad (3.21)$$

for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$, defining a certain function $\tilde{p}_0 : \mathbb{R}^n \rightarrow \mathbb{R}^n$. The latter proves exactly that $\xi(x, t) \equiv \tilde{\xi}(x, t)$ and $u(x, t) \equiv \tilde{u}(x, t)$ for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$ if only one identifies $p_0(x) = \tilde{p}_0(x)$, $x \in \mathbb{R}^n$. $\square$

Summing up the results above, we obtain the following characterisation theorem.

Theorem 1.5. *The solution of the extremum problem (2.14) realized at the point $\tilde{\xi} = \tilde{\xi}(x, t) \in \mathbb{R}^n$ is a convex, lower semicontinuous function $\tilde{u} : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$ coinciding exactly with expression (2.10) containing the vector $\xi = \xi(x, t) \in \mathbb{R}^n$ equal to the vector $\tilde{\xi} = \tilde{\xi}(x, t) \in \mathbb{R}^n$ constructed above.*

Having stated in Theorem 1.5 the identity between two expressions (2.10) and (2.14), we obtain our ultimate theorem.

Theorem 1.6. *The extremum Lax expression (1.1) solves the Hamilton–Jacobi equation (1.2) for the Cauchy data from the class of convex lower semicontinuous functions $\text{BSC}(\mathbb{R}^n)$.*

2. THE EXTREMALITY PROBLEM FUNCTIONAL ANALYSIS REVISITED:

THE BSC-CLASS SOLUTIONS

2.1. A general description of results. This subsection deals with the study of the validity of the Lax formula (1.1) of Section 1 for the solution to Hamilton–Jacobi nonlinear partial differential equation (1.2), Section 1, with the Cauchy data $v : \mathbb{R}^n \rightarrow \mathbb{R}$ being a lower semicontinuous function which is not necessarily convex. We shall prove that the Lax formula solves the Cauchy problem (1.2), Section 1, at any point $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$ fixed, save for an exceptional set of points Q of the F_δ type having Lebesgue measure zero.

2.2. Problem setting and formulation of results. Suppose that $n \in \mathbb{Z}_+$ and define $\text{BSC}(\mathbb{R}^n)$ as the set of all lower semicontinuous functions $v : \mathbb{R}^n \rightarrow \mathbb{R}$ for which

$$v^-(y) = o(\|y\|^2) \quad (1.1)$$

as $\|y\| \rightarrow \infty$, where $v^- := \frac{1}{2}(|v| - v)$ and $y \in \mathbb{R}^n$. Let us put $\Lambda := \mathbb{R}^n \times \mathbb{R}_+$ and let $\mathcal{B}_{loc}(\Lambda)$ be the set of all locally bounded functions $u : \Lambda \rightarrow \mathbb{R}$. Consider now the following essentially nonlinear operator

$$u(x) := L(v)(x, t) = \inf_{y \in \mathbb{R}^n} \left\{ v(y) + \frac{1}{2t} \|x - y\|^2 \right\} \quad (1.2)$$

from $\text{BSC}(\mathbb{R}^n)$ to $\mathcal{B}_{loc}(\Lambda)$, where $z := (x, t) \in \Lambda$ is an arbitrary point.

Remark 2.1. It follows from (1.1) that

$$L(v)(z) \leq v(0) + \frac{\|x\|^2}{2t}, \quad (1.3)$$

and therefore, $L : \text{BSC}(\mathbb{R}^n) \rightarrow \mathcal{B}_{loc}(\Lambda)$, that is operator (1.2) is well-defined.

Below we prove the Lax formula (1.1), Section 1, for the solution to the Hamilton–Jacobi equation (1.2), Section 1, that is, the formula $u(z) = L(v)(z)$ for almost all $z \in \Lambda$ does solve the Cauchy problem (1.2), Section 1. Namely, we develop a background

for stating the Lax formula (1.1), Section 1, and prove the validity of Theorem 2.2 below.

Theorem 2.2. *Let $v \in \text{BSC}(\mathbb{R}^n)$ and $u = L(v)$. Then:*

- (i) *The function $u \in \mathcal{B}_{\text{loc}}(\Lambda) \cap \text{Lip}_{1,\text{loc}}(\Lambda)$ and, for almost all $(x, t) \in \Lambda$, is differentiable in the usual sense;*
- (ii) *The set of points where the function $u : \Lambda \rightarrow \mathbb{R}$ is not differentiable is contained in some exceptional set Q of type F_δ having Lebesgue measure zero;*
- (iii) *The derivatives $\partial u / \partial t$ and ∇u are continuous on $\Lambda \setminus Q$;*
- (iv) *For all $(x, t) \in \Lambda \setminus Q$, the function $u : \Lambda \rightarrow \mathbb{R}$ solves the Cauchy problem (1.2), Section 1;*
- (v) *For all $x \in \mathbb{R}^n$, we have $\lim_{t \rightarrow 0^+} u(x, t) = v(x)$.*

2.3. Regularity properties of the operator L . We divide our proof of Theorem 2.2 into several steps and formulate some regularity lemmas.

Definition 2.3. For any $z \in \Lambda$, we put

$$A_z := \left\{ y_0 \in \mathbb{R}^n : \psi(z; y_0) = \inf_{y \in \mathbb{R}^n} \psi(z; y) \right\}, \quad (2.1)$$

where

$$\psi(z; y) := v(y) + \frac{\|x - y\|^2}{2t}, \quad (2.2)$$

for $z = (x, t) \in \Lambda$, and, for arbitrary $K \subset \Lambda$, we set

$$A(K) := \bigcup_{z \in K} A_z. \quad (2.3)$$

Remark 2.4. Since $v \in \text{BSC}(\mathbb{R}^n)$, we see that the mapping $\psi_z : \mathbb{R}^n \ni y \mapsto \psi(z; y) \in \mathbb{R}$ is also lower semicontinuous and, hence, the sets A_z , $z \in \Lambda$, are nonempty. In the sequel, for the sake of convenience, we put

$$r(z; y) = \frac{\|x - y\|^2}{2t}, \quad (2.4)$$

where $y \in \mathbb{R}^n$ and $z = (x, t) \in \Lambda$.

Lemma 2.5. *Suppose K is a nonempty compact subset of Λ . Then $A(K)$ is nonempty and compact in $\mathbb{R}^n$. In particular, for arbitrary $z \in \Lambda$, the set A_z is compact.*

Proof. Let us first prove that the set $A(K)$ is bounded. In accordance with the notation above, we have that, for all $z \in \Lambda$,

$$u(z) = \inf_{y \in \mathbb{R}^n} \psi(z; y). \quad (2.5)$$

Thus, having taken into account (2.1) and (2.2), one gets that

$$A(K) = \{y \in \mathbb{R}^n : \exists z \in K \quad \psi(z; y) = u(z)\}. \quad (2.6)$$

From (2.6) one easily deduces that for any $y \in A(K)$

$$\inf_{z \in K} \psi(z; y) \leq \sup_{z \in K} u(z),$$

whence, by virtue of the local boundedness of $u : \Lambda \rightarrow \mathbb{R}$ (see Remark 1.1), it follows that there exists a constant $c_0 \in \mathbb{R}_+$ such that, for any $y \in A(K)$,

$$\inf_{z \in K} \psi(z; y) \leq c_0. \quad (2.7)$$

It is also evident that there exist constants $c_1, c_2 \in \mathbb{R}_+$ such that

$$\inf_{z \in K} r(z; y) \geq c_1 \|y\|^2 - c_2. \quad (2.8)$$

From (2.7) and (2.8) it follows that for any $y \in A(K)$

$$v(y) + c_1 \|y\|^2 - c_2 \leq \inf_{z \in K} \psi(z; y) \leq c_0,$$

and therefore, for any $y \in A(K)$

$$c_1 \|y\|^2 - v^-(y) \leq c_0 + c_2. \quad (2.9)$$

The last inequality, in view of condition (1.1), implies the boundedness of the set $A(K)$.

We shall now prove that the set $A(K)$ is closed. Let a sequence $\{y_m \in \mathbb{R}^n : m \in \mathbb{Z}_+\} \subset A(K)$ be convergent and $\tilde{y} := \lim_{m \rightarrow \infty} y_m$. From (2.5) and (2.6) we get that there exists a sequence $\{z_m \in K : m \in \mathbb{Z}_+\}$ such that, for any $y \in \mathbb{R}^n$ and $m \in \mathbb{Z}_+$,

$$\psi(z_m; y_m) \leq \psi(z_m; y). \quad (2.10)$$

Taking, if necessary, a suitable subsequence, we may assume that the sequence $\{z_m \in K : m \in \mathbb{Z}_+\}$ is also convergent. Putting $z := \lim_{m \rightarrow \infty} z_m$ and taking into account that $v \in \text{BSC}(\mathbb{R}^n)$, one concludes immediately that, for any $y \in \mathbb{R}^n$,

$$\begin{aligned} \psi(z; \bar{y}) &= v(\bar{y}) + r(z; \bar{y}) \leq \lim_{k \rightarrow \infty} \psi(z_k, y_k) \leq \lim_{k \rightarrow \infty} \psi(z_k, y) = \\ &= v(y) + \lim_{k \rightarrow \infty} r(z_k; y) = v(y) + r(z; y) + \psi(z; y). \end{aligned}$$

Thus, for any $y \in \mathbb{R}^n$, we have $\psi(z; \bar{y}) \leq \psi(z; y)$, that is, $\bar{y} \in A_z \subset A(K)$. Since $z \in K$, this proves the lemma. $\square$

Lemma 2.6. *The function $u : \Lambda \rightarrow \mathbb{R}$ (2.5) belongs to the space $\text{Lip}_{1, \text{loc}}(\Lambda)$ of locally Lipschitzian functions.*

Proof. Let us fix an arbitrary closed ball $K \subset \Lambda$ and take $z_1, z_2 \in \bar{K}$, $y_1 \in A_{z_1}$, $y_2 \in A_{z_2}$. Then

$$\begin{aligned} u(z_1) &= v(y_1) + r(z_1; y_1), \\ u(z_2) &= v(y_2) + r(z_2; y_2), \end{aligned} \quad (2.11)$$

and, by the definition of infimum, we get

$$\begin{aligned} u(z_1) &\leq v(y_2) + r(z_1; y_2), \\ u(z_2) &\leq v(y_1) + r(z_2; y_1). \end{aligned} \quad (2.12)$$

From (2.11) and (2.12) it follows immediately that

$$\begin{aligned} u(z_1) - u(z_2) &\leq r(z_1; y_2) - r(z_2; y_2), \\ u(z_2) - u(z_1) &\leq r(z_2; y_1) - r(z_1; y_1), \end{aligned}$$

whence the inequality

$$\begin{aligned} |u(z_1) - u(z_2)| &\leq \max_{j=1,2} |r(z_1; y_j) - r(z_2; y_j)| \\ &\leq \sup_{y \in A(K)} |r(z_1; y) - r(z_2; y)|. \end{aligned} \quad (2.13)$$

follows.

Since $r \in C^\infty(\Lambda \times \mathbb{R}^n; \mathbb{R})$ and $A(\bar{K})$ is compact by virtue of Lemma 1.1, we deduce that there exists a constant $\bar{c} \in \mathbb{R}_+$, depending only on the ball $\bar{K}$, such that for any $z_1, z_2 \in \bar{K}$, $y \in A(\bar{K})$ one has

$$|r(z_1; y) - r(z_2; y)| \leq \bar{c} \|z_1 - z_2\|.$$

Thus, our lemma is proved. $\square$

Lemma 2.7. *For all $x \in \mathbb{R}^n$,*

$$\lim_{t \rightarrow 0^+} u(x, t) = v(x).$$

Proof. Take any $x \in \mathbb{R}^n$, $t_1 \leq t_2 \in \mathbb{R}_+$ and $z_j := (x, t_j) \in \Lambda$, $j = \overline{1, 2}$. Then, for any $y \in \mathbb{R}^n$, we have $\psi(z_1; y) \geq \psi(z_2; y)$ and, hence,

$$u(x, t_1) \geq u(x, t_2) \quad (2.14)$$

for any $x \in \mathbb{R}^n$ and $t_1 \leq t_2 \in \mathbb{R}_+$. It is also evident that, for any $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$,

$$u(x, t) \leq v(x). \quad (2.15)$$

Now, from (2.14) and (2.11), we easily get that for all $x \in \mathbb{R}^n$ the limit $\bar{v}(x) = \lim_{t \rightarrow 0^+} u(x, t)$ exists and satisfies inequality

$$\bar{v}(x) \leq v(x). \quad (2.16)$$

Let now $x \in \mathbb{R}^n$ and a sequence $\{t_j \in \mathbb{R}_+ : j \in \mathbb{Z}_+\}$ be convergent to zero. Put $z_k := (x, t_k) \in \Lambda$, $k \in \mathbb{Z}_+$ and choose a sequence $\{y_k \in \mathbb{R}^n : k \in \mathbb{Z}_+\}$ with $y_k \in A_{z_k}$, $k \in \mathbb{Z}_+$. Then evidently for all $k \in \mathbb{Z}_+$, $x \in \mathbb{R}^n$

$$u(x, t_k) = v(y_k) + r(z_k; y_k). \quad (2.17)$$

Using the same considerations as in Lemma 1.1 when proving the boundedness of the set $A(K)$, it is easy to see that $\{y_k \in \mathbb{R}^n : k \in \mathbb{Z}_+\}$ is also bounded. Passing to a

subsequence if necessary, we may assume that the sequence $\{y_k \in \mathbb{R}^n : k \in \mathbb{Z}_+\}$ is convergent.

Put $\bar{y} := \lim_{k \rightarrow \infty} y_k$. Then, by virtue of the lower semicontinuity of the function $v \in \text{BSC}(\mathbb{R}^n)$, we obtain

$$\lim_{k \rightarrow \infty} v(y_k) \geq v(\bar{y}). \quad (2.18)$$

Hence the sequence $\{v(y_k) : k \in \mathbb{Z}_+\}$ is bounded from below. As a result, using (2.17) and (2.18), one arrives at the boundedness of the sequence $\{r(z_k; y_k) : k \in \mathbb{Z}_+\}$, which leads to the following limit relations:

$$\|x - \bar{y}\|^2 = \lim_{k \rightarrow \infty} \|x - y_k\|^2 = \lim_{k \rightarrow \infty} 2t_k r(z_k; y_k) = 0.$$

Thus it is evident that $\bar{y} \equiv x \in \mathbb{R}^n$. Due to (2.17), we see that $u(x, t_k) \geq v(y_k)$ for any $x \in \mathbb{R}^n$, $t_k \in \mathbb{R}_+$ and $y_k \in A_{z_k}$, $k \in \mathbb{Z}_+$, and consequently, taking into account (2.14) and the equality $\bar{y} \equiv x \in \mathbb{R}^n$, one gets the following limit relations:

$$\bar{v}(x) = \lim_{k \rightarrow \infty} u(x, t_k) \geq \lim_{k \rightarrow \infty} v(y_k) \geq v(\bar{y}) = v(x).$$

This obviously implies that $\bar{v}(x) \geq v(x)$ for any $x \in \mathbb{R}^n$. Thus, on account of (2.16), the required assertion is proved. $\square$

Lemma 2.8. *Take $\{K_m \subset \Lambda : m \in \mathbb{Z}_+\}$ as a sequence of embedded compact subsets of Λ , where, for any $m \in \mathbb{Z}_+$, $K_m \subset K_{m+1}$. Then*

$$\bigcap_{j=0}^{\infty} A(K_j) = A\left(\bigcap_{j=0}^{\infty} K_j\right).$$

Proof. Let us set $K := \bigcap_{j=0}^{\infty} K_j$ and $B := \bigcap_{j=0}^{\infty} A(K_j)$. It is evident that it is sufficient to prove the inclusion $B \subset A(K)$. By virtue of Lemma 1.1, the set B is compact. Suppose also here that $B \neq \emptyset$, and choose $y \in B$. Evidently there exists a sequence $\{y_k \in \mathbb{R}^n : k \in \mathbb{Z}_+\}$ such that $y := \lim_{k \rightarrow \infty} y_k$ and $y_m \in A(K_m)$, $m \in \mathbb{Z}_+$. It follows from the consideration above that there exists a sequence $\{z_m \in K_m : m \in \mathbb{Z}_+\}$ such that $y_m \in A_{z_m}$ for any $m \in \mathbb{Z}_+$. Clearly, the sequence may be assumed to be convergent.

Let $\lim_{m \rightarrow \infty} z_m = z$; it is evident that $z \in K$. Using Definition (2.2) and taking into account the continuity of the mapping $u : \Lambda \rightarrow \mathbb{R}$, one obtains

$$\begin{aligned} \psi(z; y) = v(y) + r(z; y) &\leq \lim_{m \rightarrow \infty} v(y_m) + \lim_{m \rightarrow \infty} r(z_m; y_m) = \\ &= \lim_{m \rightarrow \infty} \psi(z_m; y_m) = \lim_{m \rightarrow \infty} u(z_m) = u(z), \end{aligned}$$

which yields the equality $\psi(z; y) = u(z)$ for $z \in K$, or $y \in A_z \subset A(K)$, thus proving the lemma. $\square$

2.4. Analysis of the exceptional set Q . Denote now by $\text{diam } M$ the diameter of a set M in a metrics space. Let us set $a(z) := \text{diam } A_z$ for any $z \in \Lambda$ and put

$$Q := \{z \in \Lambda : a(z) > 0\}. \quad (3.1)$$

One of the crucial points in the proof of Theorem 2.2 is based on the following statement.

Proposition 2.9. *The set Q is a set of type F_δ , and it has Lebesgue measure zero.*

The proof of Proposition 2.9 follows from the lemmas formulated and proved below.

Denote by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$ and for $z \in \Lambda$, $e \in \mathbb{S}^{n-1}$, put

$$A_z^e := \{\lambda = \langle w, e \rangle : w \in A_z\}, \quad G_e := \{x, t \in \Lambda : \langle x, e \rangle = 0\}, \quad (3.2)$$

and define, for $e \in \mathbb{S}^{n-1}$, the mapping

$$\Lambda \ni z \mapsto a_e(z) := \text{diam } A_z^e \in \overline{\mathbb{R}}_+. \quad (3.3)$$

In addition, for any fixed $e \in \mathbb{S}^{n-1}$ and $z \in G_e$, we define the mappings

$$\begin{aligned} \mathbb{R} \ni \xi &\mapsto c_{e,z}(\xi) = \text{diam } A_{z+(\xi e, 0)}^e \in \overline{\mathbb{R}}_+, \\ \mathbb{R} \ni \xi &\mapsto c_{e,z}^+(\xi) = \sup A_{z+(\xi e, 0)}^e \in \overline{\mathbb{R}}, \\ \mathbb{R} \ni \xi &\mapsto c_{e,z}^-(\xi) = \inf A_{z+(\xi e, 0)}^e \in \overline{\mathbb{R}}. \end{aligned} \quad (3.4)$$

Lemma 2.10. *The mappings a , a_e , $c_{e,z}$, $c_{e,z}^+$, and $-c_{e,z}^-$ are upper semicontinuous for any $e \in \mathbb{S}^{n-1}$ and $z \in G_e$.*

Proof. Take, for instance, the mapping $a: \Lambda \rightarrow \overline{\mathbb{R}}_+$. For the other mappings, the proof is completely similar.

Let $\{z_m \in \Lambda : m \in \mathbb{Z}_+\}$ and $\lim_{m \rightarrow \infty} z_m = z \in \Lambda$. Let us put $K := \{z\} \cup \{z_m : m \in \mathbb{Z}_+\}$. Since K is compact, we see that, due to Lemma 1.1, $A(K)$ is also compact. Thus, for any $m \in \mathbb{Z}_+$, $a(z_m) = \text{diam } A_{z_m} \leq \text{diam } A(K) < \infty$, whence it follows that the sequence $\{a(z_m) \in \mathbb{R} : m \in \mathbb{Z}_+\}$ is bounded.

Put

$$\alpha := \overline{\lim_{m \rightarrow \infty}} a(z_m); \quad (3.5)$$

then there exists a subsequence $\{z_{m_p} \in \Lambda : p \in \mathbb{Z}_+\}$ for which

$$\alpha := \lim_{p \rightarrow \infty} a(z_{m_p}) \quad (3.6)$$

together with sequences $\{y_p \in A(K_p) : p \in \mathbb{Z}_+\}$ and $\{\tilde{y}_p \in A(K_p) : p \in \mathbb{Z}_+, K_p := \{z\} \cup \{z_{m_j} : j = \overline{p, \infty}\}\}$ satisfying for all $p \in \mathbb{Z}_+$ the condition

$$a(z_{m_p}) = \|y_p - \tilde{y}_p\|. \quad (3.7)$$

It is evident that, without loss of generality, we may assume that the sequences $\{y_p \in A(K_p) : p \in \mathbb{Z}_+\}$ and $\{\tilde{y}_p \in A(K_p) : p \in \mathbb{Z}_+\}$ are convergent.

Put $y = \lim_{p \rightarrow \infty} y_p$ and $\tilde{y} = \lim_{p \rightarrow \infty} \tilde{y}_p$. Then, taking into account Lemma 2.8, one obtains that $y, \tilde{y} \in \bigcap_{p=0}^{\infty} A(K_p) = A(\bigcap_{p=0}^{\infty} K_p) = A_z$. Having used relations (3.5)–(3.7), one arrives at the relation

$$\overline{\lim}_{m \rightarrow \infty} a(z_m) = \lim_{p \rightarrow \infty} \|y_p - \tilde{y}_p\| = \|y - \tilde{y}\| \leq \text{diam } A_z = a(z),$$

which evidently proves the lemma. $\square$

Let us now define, for an arbitrary $e \in \mathbb{S}^{n-1}$ (see (3.3)), the set

$$Q^e := \{z \in \Lambda : a_e(z) > 0\}. \quad (3.8)$$

Lemma 2.11. *The sets Q and Q^e are sets of F_δ type, that is, they are at most countable unions of closed sets in $\mathbb{R}^{n+1}$. In particular, the sets Q and Q^e are Lebesgue measurable.*

Proof. We confine ourselves to the case of the set Q . The proof for the set Q^e is completely similar.

The mapping $a : \Lambda \rightarrow \overline{\mathbb{R}}_+$ is upper semicontinuous, hence, the following sets are closed for all $m \in \mathbb{Z}_+$:

$$Q_m = \left\{ (x, t) \in \Lambda : a(z) \geq \frac{1}{m+1}, t \geq \frac{1}{m+1} \right\}.$$

Now the obvious equality $Q = \bigcup_{m=0}^{\infty} Q_m$ proves the lemma. $\square$

Lemma 2.12. *For arbitrary $e \in \mathbb{S}^{n-1}$, $z \in G_e$, the set*

$$P_{e,z} := \{\xi \in \mathbb{R} : c_{e,z}(\xi) \neq 0\} \quad (3.9)$$

is countable.

Proof. Fix $e \in \mathbb{S}^{n-1}$ and $z = (x, t) \in G_e$ and put, for the sake of convenience, $c(\xi) := c_{e,z}(\xi)$ and $c^\pm(\xi) := c_{e,z}^\pm(\xi)$ for $\xi \in \mathbb{R}$, and $P := P_{e,z}$. We also set, for any $\xi \in \mathbb{R}$, $\Delta_\xi := (c^-(\xi), c^+(\xi))$.

It is evident that $|\Delta_\xi| = c^+(\xi) - c^-(\xi) = c(\xi)$ for any $\xi \in \mathbb{R}$, where $|\Delta_\xi| = \text{diam } \Delta_\xi$, and $P = \{\xi \in \mathbb{R} : \Delta_\xi \neq \emptyset\}$. This means that the lemma follows if one proves that, for any $\xi \neq \eta \in \mathbb{R}$, we have $\Delta_\xi \cap \Delta_\eta = \emptyset$.

Put $z_\xi := z + (\xi e, 0)$, $z_\eta := z + (\eta e, 0)$ and $\xi < \eta$. Then for any $y \in \mathbb{R}^n$

$$\begin{aligned} \psi(z_\eta; y) - \psi(z_\xi; y) &= \frac{1}{2t} \left(\|x - y + \eta e\|^2 - \|x - y + \xi e\|^2 \right) \\ &= \frac{1}{2t} (\eta - \xi)(\eta + \xi - 2\langle y, e \rangle). \end{aligned} \quad (3.10)$$

Since the set A_{z_ξ} is compact and

$$c^+(\xi) = \sup\{\langle y, e \rangle : y \in A_{z_\xi}\},$$

it follows that there exists an element $\bar{y} \in A_{z_\xi}$ such that $\langle \bar{y}, e \rangle = c^+(\xi)$.

Put now

$$E_\xi := \{y \in \mathbb{R}^n : \langle y, e \rangle < c^+(\xi)\}.$$

Since $\xi \in \mathbb{R}^n$ is fixed, it is easy to see that for all $y \in E_\xi$

$$\frac{1}{2t}(\eta - \xi)(\eta + \xi - 2\langle \bar{y}, e \rangle) < \frac{1}{2t}(\eta - \xi)(\eta + \xi - 2\langle y, e \rangle), \quad (3.11)$$

whence, by virtue of (3.10), (3.11) and the inequality $\psi(z_\xi; y) \geq \psi(z_\xi; \bar{y})$ valid for any $y \in \mathbb{R}^n$, one obtains

$$\psi(z_\eta; y) > \psi(z_\xi; \bar{y}) + \frac{1}{2t}(\eta - \xi)(\eta + \xi - 2\langle \bar{y}, e \rangle) = \psi(z_\eta; \bar{y}).$$

Thus, for any $\xi < \eta$, we have $A_{z_\eta} \subset \mathbb{R}^n \setminus E_\xi$, which results in the inequality $c^-(\eta) \geq c^+(\xi)$. The latter is equivalent to the equality

$$\Delta_\xi \cap \Delta_\eta = \emptyset$$

for any $\xi \neq \eta \in \mathbb{R}$, so the lemma is proved. $\square$

Lemma 2.13. *The set Q has zero Lebesgue measure.*

Proof. Let $e \in \mathbb{S}^{n-1}$ and χ_{Q^e} be the characteristic function of the corresponding set Q^e . It follows from Lemma 2.11 that the function χ_{Q^e} is locally integrable in Λ . Let K_e^0 and K be compact sets in G_e and $\mathbb{R}$, respectively, and

$$K_e := \{z + (\xi e, 0) \in \mathbb{R}^{n+1} : z \in K_e^0, \xi \in K\}.$$

The Fubini theorem [5, 6] implies that

$$\int_{K_e} \chi_{Q^e} d\mu_{n+1} = \int_{K_e^0} d\mu_n(x) \int_K \chi_{Q^e}(z + (\xi e, 0)) d\mu_1(\xi) \quad (3.12)$$

for any $e \in \mathbb{S}^{n-1}$, where μ_k is the Lebesgue measure in $\mathbb{R}^k$, $k \in \mathbb{Z}_+$. It is now easy to see that for arbitrary $e \in \mathbb{S}^{n-1}$, $z \in G_e$,

$$\{\xi \in \mathbb{R} : \chi_{Q^e}(z + (\xi e, 0)) \neq 0\} = \{\xi \in \mathbb{R} : c_{e,z}(\xi) \neq 0\} = P_{e,z}.$$

By virtue of Lemma 2.12, the set $P_{e,z}$, at fixed $e \in \mathbb{S}^{n-1}$, $z \in G_e$, is countable, and therefore, for $z \in G_e$, we have

$$\int_K \chi_{Q^e}(z + (\xi e, 0)) d\mu_1(\xi) = 0.$$

Hence, using (3.12) and the condition $\chi_{Q^e} \geq 0$, one concludes that the function χ_{Q^e} is equal to zero almost everywhere on K . Taking into account the arbitrariness of the compact sets K_e^0 and K , one easily gets that χ_{Q^e} is equal to zero almost everywhere on Λ . Fix now an arbitrary orthonormal basis $\{e_j \in \mathbb{S}^{n-1} : j = \overline{1, n}\}$ in the space $\mathbb{R}^n$. It is easy to see, by (3.1) and (3.8), that $Q = \bigcup_{j=\overline{1, n}} Q^{e_j}$. By virtue of the results proved above, each set Q^{e_j} , $j = \overline{1, n}$, is of zero Lebesgue measure and, hence, the set Q has zero Lebesgue measure as well. This proves the lemma. $\square$

By using Lemmas 2.11 and 2.13, one readily obtains the proof of Proposition 2.9.

2.5. Analysis of the differential operator L . For Theorem 1.3 to be proved completely, we introduce a function $\rho : \Lambda \rightarrow \mathbb{R}^n$ such that for any $z \in \Lambda$

$$\Lambda \ni z \mapsto \rho(z) \in A_z. \quad (4.1)$$

The existence of such a function (4.1) follows from the axiom of choice [3].

Define the differential operator

$$Hu = \frac{\partial u}{\partial t} + \frac{1}{2} \|\nabla u\|^2, \quad (4.2)$$

where mapping $u : \Lambda \rightarrow \mathbb{R}$ is differentiable almost everywhere on Λ .

Lemma 2.14. *A function $\rho : \Lambda \rightarrow \mathbb{R}^n$ satisfying condition (4.1) is continuous at all points $z \in \Lambda \setminus Q$.*

Proof. Take $z \in \Lambda \setminus Q$ and choose a convergent sequence $\{z_m \in \Lambda : m \in \mathbb{Z}_+\}$ such that $\lim_{m \rightarrow \infty} z_m = z$. Owing to Lemma 2.7, the sequence

$$\{\rho(z_m) \in \mathbb{R}^n : m \in \mathbb{Z}_+\}$$

is bounded.

Fix now an arbitrary convergent subsequence $\{\rho(z_{m_k}) \in \mathbb{R}^n : k \in \mathbb{Z}_+\}$ of the sequence $\{\rho(z_m) \in \mathbb{R}^n : m \in \mathbb{Z}_+\}$. The lemma will be proved if we show that

$$\lim_{k \rightarrow \infty} \rho(z_{m_k}) = \rho(z). \quad (4.3)$$

For any $y \in \mathbb{R}^n$ and $k \in \mathbb{Z}_+$,

$$v(\rho(z_{m_k})) + r(z_{m_k}; \rho(z_{m_k})) \leq v(y) + r(z_{m_k}; y), \quad (4.4)$$

whence, passing to the limit as $k \rightarrow \infty$, one gets

$$v(\bar{y}) + r(z; \bar{y}) \leq v(y) + r(z; y), \quad (4.5)$$

where, by definition, $\lim_{k \rightarrow \infty} \rho(z_{m_k}) = \bar{y}$. From (4.5) it follows that $\bar{y} \in A_z$. Since $z \in \Lambda \setminus Q$ and the set A_z is a singleton, one concludes immediately that $\bar{y} = \rho(z)$. Thus the lemma is proved. $\square$

Lemma 2.15. *The function $u := L(v)$ is differentiable at all points $z \in \Lambda \setminus Q$ and*

$$(Hu)(z) = (Hr)(z; y)|_{y=\rho(z)} = 0. \quad (4.6)$$

Proof. Choose $z \in \Lambda \setminus Q$, $e \in \mathbb{S}^n$, and a sequence $\{\tau_k \in \mathbb{R}_+ : k \in \mathbb{Z}_+\}$ convergent to zero. Put also $z_k := (z + \tau_k e) \in \Lambda$, $k \in \mathbb{Z}_+$. From the definition of the function $u = L(v)$ one obtains the inequalities

$$\begin{aligned} u(z) &\leq v(\rho(z_k)) + r(z; \rho(z_k)), \\ u(z_k) &\leq v(\rho(z)) + r(z_k; \rho(z)) \end{aligned} \quad (4.7)$$

for all $k \in \mathbb{Z}_+$. As a result of (4.7), having taken into account that, for all $k \in \mathbb{Z}_+$,

$$\begin{aligned} u(z) &= v(\rho(z) + r(z; \rho(z))), \\ u(z_k) &= v(\rho(z_k) + r(z_k; \rho(z_k))), \end{aligned}$$

one gets, also for all $k \in \mathbb{Z}_+$, the following inequalities:

$$\begin{aligned} \frac{u(z + \tau_k e) - u(z)}{\tau_k} &\leq \frac{r(z + \tau_k e; \rho(z)) - r(z; \rho(z))}{\tau_k}, \\ \frac{u(z + \tau_k e) - u(z)}{\tau_k} &\geq \frac{r(z + \tau_k e; \rho(z_k)) - r(z; \rho(z_k))}{\tau_k}. \end{aligned} \quad (4.8)$$

Lemma 2.14 now implies that $\lim_{k \rightarrow \infty} \rho(z_k) = \rho(z)$. Taking this into account and passing in (4.8) to the limit when $k \rightarrow \infty$, one arrives at the relation

$$\lim_{k \rightarrow \infty} \frac{u(z + \tau_k e) - u(z)}{\tau_k} = \frac{\partial}{\partial \tau} r(z + \tau e; \rho(z))|_{\tau=0}. \quad (4.9)$$

Since the right hand side of equality (4.9) is independent of the choice of the sequence $\{\tau_k \in \mathbb{R}_+ : k \in \mathbb{Z}_+\}$, the mapping $\tau \mapsto u(z + \tau e)$ is differentiable at zero, and the equality

$$\frac{\partial}{\partial \tau} u(z + \tau e)|_{\tau=0} = \frac{\partial}{\partial \tau} r(z + \tau e; \rho(z))|_{\tau=0}$$

is true for all $z \in \Lambda \setminus Q$. Thus, we obtain the proof of the lemma. $\square$

The final steps of proof of Theorem 2.2. It is now easy to complete the proof of Theorem 1.2. The validity of (i) follows from Lemmas 2.6 and 2.15; (ii) follows from Proposition 2.9 and Lemma 2.15; (iv) follows from Lemma 2.15; and (v) follows from Lemma 2.15.

To prove statement (iii) of Theorem 2.2, we note that Lemma 2.15 implies that the relation

$$\nabla u(z) = \nabla r(z; y)|_{y=\rho(z)}, \quad (4.10)$$

is true for all $z \in \Lambda \setminus Q$. From Lemma 2.14 (by the continuity of the mapping $\rho : \Lambda \setminus Q \rightarrow \mathbb{R}^n$) and (4.10) we now immediately obtain assertion (iii). Thus, the theorem is proved completely. $\square$

2.6. Generalizations of the inf-type Lax formula. There are many applications in modern mathematical physics of the following nonlinear Hamilton-Jacobi type first order partial differential equation:

$$\frac{\partial u}{\partial t} + \frac{1}{2} \|\nabla u\|^2 + \frac{\beta u}{2} \|x\|^2 + \frac{1}{2} \langle Jx, x \rangle = 0, \quad (5.1)$$

where $t \in \mathbb{R}_+$, $\beta \in \mathbb{R}_+$, and $J : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a diagonal positive definite matrix, with the Cauchy data

$$u|_{t=0^+} = v \in \text{BSC}(\mathbb{R}^n). \quad (5.2)$$

It is natural to seek for a generalisation of the Lax formula (0.1) to represent the solution of the Cauchy problem (5.1), (5.2) by using the dynamical systems approach

developed in Section 1 (see also [4, 7]). As a result of such an analysis, we obtain an inf-type representation for problem (5.1), (5.2), which is described below. Some related results concerning generalised solutions to Hamilton-Jacobi equations can be also found in [8, 9].

Proposition 2.16. *The Cauchy problem (5.1), (5.2) admits the following inf-type fixed point problem representation:*

$$u(x, t) = \inf_{y \in \mathbb{R}^n} \{v(y) + P_u(x, t; y)\}, \quad (5.3)$$

where, by definition, for any $(z; y) \in \Lambda \times \mathbb{R}^n$ the nonlocal functional

$$P(z; y) : \text{BSC}(\mathbb{R}^n; \mathbb{R}) \ni u \rightarrow P_u(z; y) \in \mathbb{R}$$

is given by

$$P_u(x, t; y) := \frac{1}{4} \frac{d}{d\tau} \|\alpha(\tau|x, t; y)\|^2 \Big|_{0^+}^t - \frac{\beta}{16} (\|x\|^4 - \|y\|^4), \quad (5.4)$$

and the mapping $\alpha : \mathbb{R}_+ \times (\Lambda \times \mathbb{R}^n) \rightarrow \mathbb{R}^n$ is the functional solution to the following nonlinear second order ordinary differential equation (with respect to the first variable $\tau \in \mathbb{R}_+$):

$$\ddot{\alpha} + \beta u(\alpha, \tau) \alpha + \beta \|\alpha\|^2 \dot{\alpha} = -J\alpha, \quad \alpha|_{\tau=0^+} = y, \quad \alpha|_{\tau=t} = x, \quad (5.5)$$

for any $y, x \in \mathbb{R}^n$ and fixed $u : \Lambda \rightarrow \mathbb{R}$.

Letting $\beta \rightarrow 0^+$ in the exact expressions (5.3)–(5.5), we get that the fixed point problem (5.3) becomes that of the Lax type exact form. Consider the fixed point problem (5.3) and discuss conditions under which it possesses a unique solution $u \in \text{BSC}(\mathbb{R}^n; \mathbb{R})$ satisfying the Cauchy data

$$\lim_{t \rightarrow 0^+} u(x, t) = v(x)$$

for all $x \in \mathbb{R}^n$.

For the infimum expression (5.3) to exist at almost all points $z := (x; t) \in \Lambda$, one needs to find conditions at which the function $P_u(z; \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$ will be convex and lower semicontinuous at each $u \in \text{BSC}(\mathbb{R}^n; \mathbb{R})$. Since

$$P_u(z; y) = \frac{1}{2} \langle \dot{\alpha}(t|z; y), \alpha(t|z; y) \rangle - \frac{1}{2} \langle \dot{\alpha}(0|z; y), \alpha(0|z; y) \rangle - \frac{\beta}{16} (\|x\|^4 - \|y\|^4),$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)^T \in \mathbb{R}^n$ and for every $j = \overline{1, n}$ the set of equations

$$\ddot{\alpha}_j + \beta u(\alpha, \tau) \alpha_j + \beta \|\alpha\|^2 \dot{\alpha}_j = -\omega_j^2 \alpha_j, \quad \alpha_j|_{\tau=0^+} = y_j, \quad \alpha_j|_{\tau=t} = x_j, \quad (5.6)$$

holds, where $\omega_j^2 \in \mathbb{R}_+$, $j = \overline{1, n}$, are some fixed eigenvalues of the matrix $J : \mathbb{R}^n \rightarrow \mathbb{R}^n$ in (5.1), one gets easily from (5.6) that

$$P_u(z; y) = \frac{1}{2} \int_0^t \|\dot{\alpha}\|^2 ds - \frac{\beta}{2} \int_0^t \|\alpha\|^2 (u(\alpha; s) + \langle \dot{\alpha}, \alpha \rangle) ds - \frac{1}{2} \int_0^t \langle J\alpha, \alpha \rangle ds - \frac{\beta}{16} (\|x\|^4 - \|y\|^4) \quad (5.7)$$

for all $(z; y) \in \Lambda \times \mathbb{R}^n$. Thereby, making use of expression (5.7) (for $\beta \in \mathbb{R}_+$), we can claim that $P_u(z; \cdot) \in \text{BSC}(\mathbb{R}^n)$, and that the mapping $\mathcal{P}_v : \text{BSC}(\mathbb{R}^n) \rightarrow \text{BSC}(\mathbb{R}^n)$ is well-defined, where, for any $u \in \text{BSC}(\mathbb{R}^n)$,

$$\mathcal{P}_v(u) := \inf_{y \in \mathbb{R}^n} \{v(y) + P_u(z; y)\}. \quad (5.8)$$

In view of (5.8) and (5.3), we arrive at the following fixed point problem:

$$\mathcal{P}_v(u) = u \quad (5.9)$$

for $u \in \text{BSC}(\mathbb{R}^n)$ with an arbitrary, but fixed, function $v \in \text{BSC}(\mathbb{R}^n)$.

In relation to the nonlinear mapping (5.8) one can define the topological space $\mathcal{B}_{loc}(\mathbb{R}^n)$ of all locally bounded functions $u : \Lambda \times \mathbb{R}^n \rightarrow \mathbb{R}$ with the topology of uniform consequence over all compact subsets of $\Lambda \times \mathbb{R}^n$. For the further analysis of problem (5.9), we need the Fan-Browder fixed point theorem [11], which reads as follows.

Theorem 2.17. *In a topological vector space $\mathcal{B}$, let $\mathcal{P}_v$ be a continuous mapping of $\mathcal{B}$ into $\mathcal{B}$ such that there exists a non-empty subset $\bar{A} \subset \mathcal{B}$ satisfying the following condition:*

The set $\mathcal{B} \setminus \left(\bigcup_{u \in \bar{A}} \mathcal{P}_v u\right)$ is either compact or empty, and the subset $\text{conv } \bar{A}$ is compact.

Then there exists a point $\bar{u} \in \mathcal{B}$ such that $\bar{u} = \mathcal{P}_v \bar{u}$.

Keeping in mind the conditions above, we define, for any $v \in \text{BSC}(\mathbb{R}^n)$, the subset

$$A_v := \bigcup_{u \in \mathcal{B}_{loc}(\mathbb{R}^n)} \mathcal{P}_v(u) \subset \mathcal{B}_{loc}(\mathbb{R}^n) \quad (5.10)$$

and formulate the following two lemmas.

Lemma 2.18. *The set $\text{conv } A_v \subset \mathcal{B}_{loc}(\mathbb{R}^n)$ is compact and convex in $\mathcal{B}_{loc}(\mathbb{R}^n)$ for any $v \in \text{BSC}(\mathbb{R}^n)$.*

Lemma 2.19. *The set $\mathcal{B} \setminus \left(\bigcup_{u \in A_v} \mathcal{P}_v u\right)$ is compact in $\mathcal{B}_{loc}(\mathbb{R}^n)$ for any $v \in \text{BSC}(\mathbb{R}^n)$.*

In connection with these lemmas, we introduce the following sets, $\bar{A} = A_v$ and $\mathcal{B} = \mathcal{B}_{loc}(\mathbb{R}^n)$, and formulate the following statement.

Proposition 2.20. *The fixed point problem (5.9) with an arbitrary, but fixed, function $v \in \text{BSC}(\mathbb{R}^n)$ restricted to the topological space $\mathcal{B}_{\text{loc}}(\mathbb{R}^n)$, is uniquely solvable, with the solution belonging to the space $\text{BSC}(\mathbb{R}^n)$. Moreover, this solution satisfies the standard viscosity condition*

$$v(x) = \lim_{u \rightarrow 0^+} \bar{u}(x, t)$$

for all $(x, y) \in \Lambda$.

Thereby, the Hamilton–Jacobi equation (5.1) with Cauchy data from the functional space $\text{BSC}(\mathbb{R}^n)$ possesses a unique solution representable as the fixed point problem (5.3), (5.4), which is solvable for any $v \in \text{BSC}(\mathbb{R}^n)$.

In the case where the Cauchy data belong to the space $\text{BSC}(\mathbb{S}^n)$, where $\mathbb{S}^n$ is the N -dimensional sphere imbedded smoothly into the space $\mathbb{R}^{N+1}$, and $\nabla u \in T^*(\mathbb{S}^N)$, the Hamilton–Jacobi equation (1.2) on the sphere $\mathbb{S}^n$ defines [12] the evolution of a function $u \in \text{BSC}(\mathbb{S}^N)$, which can be also represented in the Lax inf-type form. Below we will analyse this problem making use of some results from Section 1.

2.7. Solution of extremality problem on $\mathbb{S}^N$. As is well-known, the following extended finite-dimensional Hamiltonian system is closely related to the Hamilton–Jacobi equation (I.1.2) on the sphere $\mathbb{S}^N$:

$$dq/d\tau = \partial H(q, p)/\partial p, \quad dp/d\tau = -\partial H(q, p)/\partial q, \quad (6.1)$$

where $H(q, p) = 1/2 \|p\|^2 \|q\|^2$, $(q, p) \in T^*(\mathbb{R}^{N+1})$, $\tau \in \mathbb{R}_+$. This system is considered together with the following constraints:

$$(q, p) \in T^*(\mathbb{S}^N) := \{q \in \mathbb{R}^{N+1} : \|q\|^2 - 1 = 0, \langle q, p \rangle = 0\}.$$

From (6.1) we find that, for $\tau \in (0, t]$, on $T^*(\mathbb{R}^{N+1})$,

$$dq/d\tau = p \|q\|^2, \quad dp/d\tau = -\|p\|^2 q. \quad (6.2)$$

Having taken the Cauchy data $q|_{\tau=0^+} = y \in \mathbb{S}^N$, $q|_{\tau=t} = x \in \mathbb{S}^N$, from (6.2), one easily obtains that, for all $t \in \mathbb{R}_+$, the expression

$$\|p\| = t^{-1} \arccos \langle y, x \rangle \quad (6.3)$$

is independent of $\tau \in (0, t]$. Consider now the following extremality expression for the Hamilton–Jacobi equation (1.2), Section 1, extended in a natural way on the entire space $\mathbb{R}^{N+1}$:

$$\begin{aligned} \tilde{u}(x, t) &= \inf_{\substack{\{q(\tau) \in \mathbb{S}^N : q|_{\tau=0^+} = y \in \mathbb{S}^N, \\ q|_{\tau=t} = x \in \mathbb{S}^N\}}} \left\{ v(y) + \int_0^t d\tau (\langle p, dq/dt \rangle - H(q, p)) \right\} \\ &= \inf_{y \in \mathbb{S}^N} \left\{ v(y) + \frac{t}{2} \|p(t; x, t|y)\|^2 \right\}, \quad (6.4) \end{aligned}$$

where we have put $q = q(\tau; x, t|y)$, $p = p(\tau; x, t|y)$ for $\tau \in (0, t]$, $x, y \in \mathbb{S}^N$, and made use of the equality $d\|p\|/d\tau = 0$ implied by equations (6.2).

On the other hand, for the quantity $\|p\| \in \mathbb{R}^{N+1}$, we have formula (6.3), which, together with (6.4), leads us to the inf-type Lax expression

$$\tilde{u}(x, t) = \inf_{y \in \mathbb{S}^N} \left\{ v(y) + \frac{1}{2t} \arccos^2 \langle y, x \rangle \right\}, \quad (6.5)$$

which we suggest as a “candidate” for a solution of the Hamilton-Jacobi equation (1.2), Section 1, on the sphere $\mathbb{S}^N$ with the Cauchy data $v \in \text{BSC}(\mathbb{S}^N)$. The required equality

$$\tilde{u}(x, t) = u(x, t),$$

for almost all $x \in \mathbb{S}^N$ and $t \in \mathbb{R}_+$, can be obtained by an argument similar to that in Section 1. Here, we shall consider the case where $v \in \text{BSC}(\mathbb{S}^N)$.

2.8. Analysis of the extremality problem on $\mathbb{S}^N$. Here we shall prove the equality

$$\tilde{u}(x, t) = u(x, t)$$

for all $x \in \mathbb{S}^N$ and $t \in \mathbb{R}_+$, based on the properties of the Cauchy data for (1.2), Section 1, and the exact inf-type expression (6.5). It is easy to show [3, 4] that there exists a point $\tilde{\xi}(x, t) \in \mathbb{S}^N$, such that

$$\tilde{u}(x, t) = v(\tilde{\xi}(x, t)) + \frac{1}{2t} \arccos^2 \langle \tilde{\xi}(x, t), x \rangle, \quad (7.1)$$

for any $x \in \mathbb{S}^N$ and a fixed $t \in \mathbb{R}_+$. At the same time, one can show that the exact solution $u : \mathbb{S}^N \times \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfies the following expression involving differential forms:

$$du(x, t) = \langle p(x, t), dx \rangle - 1/2 \|p(x, t)\|^2 dt, \quad (7.2)$$

which is completely equivalent to the Hamilton-Jacobi equation (1.2), Section 1, where $p : \mathbb{S}^N \times \mathbb{R}_+ \rightarrow \mathbb{R}^{N+1}$ fulfils (6.2). Thus, based on (6.2) and (7.2), one gets easily that

$$\begin{aligned} u(x, t) &= v(\xi(x, t)) + \int_0^t (du(q(\tau; x, t|y)), t)/d\tau d\tau = \\ &= v(\xi(x, t)) + 1/2 \int_0^t d\tau \|p(\tau; x, t|y)\|^2, \end{aligned} \quad (7.3)$$

that is, for all $x \in \mathbb{S}^N$ and fixed $t \in \mathbb{R}_+$, we have

$$u(x, t) = v(\xi(x, t)) + \frac{1}{2t} \arccos^2 \langle \xi(x, t), x \rangle. \quad (7.4)$$

Here, $\xi : \mathbb{S}^N \times \mathbb{R}_+ \rightarrow \mathbb{S}^N$ is a mapping defined as follows:

$$(x - \xi \langle x, \xi \rangle) \langle \xi, x \rangle = tp_0(\xi)(1 - \langle x, \xi \rangle)^{1/2}, \quad (7.5)$$

where the function $p_0 : \mathbb{S}^N \rightarrow \mathbb{R}^{N+1}$ satisfies the following relation implied by (7.3):

$$\nabla u(\xi, t)|_{t=0^+} := p_0(\xi), \quad (7.6)$$

depending only on the Cauchy data $v \in \text{BSC}(\mathbb{S}^N)$. Therefore, it is sufficient to prove the equality for all $x \in \mathbb{S}^N$ and $t \in \mathbb{R}_+$, and that will yield the required equality

$$\tilde{u}(x, t) = u(x, t).$$

It is easy to establish the following lemma.

Lemma 2.21. *Expression (7.1) is $\mathbb{S}^N \ni x$ -differentiable for each $t \in \mathbb{R}_+$, and the following equality holds:*

$$\nabla \tilde{u}(\tilde{\xi}, t)|_{t=0^+} = \tilde{p}_0(\tilde{\xi}), \quad (7.7)$$

where the relation

$$(x - \tilde{\xi} \langle x, \tilde{\xi} \rangle) \langle \tilde{\xi}(x, t), x \rangle = t \tilde{p}_0(\tilde{\xi}) (1 - \langle x, \tilde{\xi} \rangle)^{1/2}, \quad (7.8)$$

is fulfilled for any $x \in \mathbb{S}^N$ and $t \in \mathbb{R}_+$.

In view of (7.8) and (7.6), we conclude that, with

$$p_0(x) = \tilde{p}_0(x),$$

for any $x \in \mathbb{S}^N$, the equality $\tilde{\xi}(x, t) = \xi(x, t)$ holds for all $t \in \mathbb{R}_+$. This means, by (7.1) and (7.2), that the inf-type Lax expression (6.4) solves the Hamilton-Jacobi equation (1.2), Section 1, on the sphere $\mathbb{S}^N$.

The argument above proves the following statement.

Proposition 2.22. *The inf-type expression*

$$u(x, t) = \inf_{y \in \mathbb{S}^N} \left\{ v(y) + \frac{1}{2t} \arccos^2 \langle x, y \rangle \right\},$$

solves the Hamilton-Jacobi equation (1.2), Section 1, on the sphere $\mathbb{S}^N$ with the Cauchy data $v \in \text{BSC}(\mathbb{S}^N)$ having the standard limit solution property.

It is also easy to see that the method suggested above can be applied to the problem of finding an inf-type solution of an oscillator Hamilton-Jacobi equation on the sphere $\mathbb{S}^N$. This problem will produce an inf-type solution based on properties of the well-known K. Neumann N -oscillator dynamical system, whose complete Liouville-Arnold integrability was proved by J. Moser [10, 12].

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