

# MATHEMATICAL NOTES

A Publication of the University of Miskolc

VOLUME 4, NUMBER 1 (2003)



MISKOLC UNIVERSITY PRESS



## CONTENTS

**V. M. EVTUKHOV AND L. I. KUSICK**

- Asymptotic behaviour of solutions of some systems of quasilinear differential equations . . . . . 3

**DANIELA INOAN**

- Stability with respect to the domain of a nonlinear variational inequality . 25

**JÓZEF KALINOWSKI**

- On functions preserving rank of matrices . . . . . 35

**GEORGHE MICLĂUS**

- The Libera generalized integral operator and Hardy spaces . . . . . 39

**EDGAR PEREIRA**

- Block eigenvalues and solutions of differential matrix equations . . . . . 45

**N. A. PERESTYUK AND V. YU. SLUSARCHUK**

- Oscillation of nonlinear differential-integral equation in a Banach space with respect to its subspace . . . . . 53

**NUSRAT RAJABOV, MIKLÓS RONTÓ, AND LUTFIA RAJABOVA**

- On some two dimensional Volterra type linear integral equations with super-singularity . . . . . 65



## ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF SOME SYSTEMS OF QUASILINEAR DIFFERENTIAL EQUATIONS

V. M. EVTUKHOV AND L. I. KUSICK

[Received: January 15, 2003]

**ABSTRACT.** We investigate the asymptotic behaviour of solutions of some systems of differential equations appearing in studies of  $n$ th order quasilinear differential equations.

*Mathematics Subject Classification:* 34E05

**Keywords:** quasilinear differential equations, asymptotic representation of the solution

### 1. INTRODUCTION

Let us consider the system of differential equations

$$\frac{dy}{dx} = [W(x) + R(x)]y + F(x, y), \quad (1.1)$$

where  $W, R \in C([x_0, +\infty[; \mathbb{R}^{n \times n})$ ,  $F \in C([a, +\infty[ \times \mathbb{R}^n; \mathbb{R}^n)$ ,

$$W(x) = \text{diag}[W_1(x), \dots, W_s(x)]$$

with

$$W_i(x) = \begin{pmatrix} \omega_{i1}(x)I_2 & I_{i2}(x) & O_2 & \dots & O_2 & O_2 \\ O_2 & \omega_{i2}(x)I_2 & I_{i3}(x) & \dots & O_2 & O_2 \\ O_2 & O_2 & \omega_{i3}(x)I_2 & \dots & O_2 & O_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O_2 & O_2 & O_2 & \dots & \omega_{i, n_i-1}(x)I_2 & I_{i, n_i}(x) \\ O_2 & O_2 & O_2 & \dots & O_2 & \omega_{i, n_i}(x)I_2 \end{pmatrix} \quad (1.2)$$

for  $i = 1, \dots, r$ ,

$$W_i(x) = \begin{pmatrix} \omega_{i1}(x) & 1 & 0 & \dots & 0 & 0 \\ 0 & \omega_{i2}(x) & 1 & \dots & 0 & 0 \\ 0 & 0 & \omega_{i3}(x) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \omega_{i n_i - 1}(x) & 1 \\ 0 & 0 & 0 & \dots & 0 & \omega_{i n_i}(x) \end{pmatrix} \quad (1.3)$$

for  $i = r + 1, \dots, s$ .

Here,  $r \in \{0, \dots, s\}$  and  $\sum_{i=1}^r 2n_i + \sum_{i=r+1}^s n_i = n$ ,<sup>1</sup>

$$I_{ik}(x) = \begin{pmatrix} l_{ik}^{(2)}(x) & -l_{ik}^{(1)}(x) \\ l_{ik}^{(1)}(x) & l_{ik}^{(2)}(x) \end{pmatrix} \quad (k = 2, \dots, n_i, \quad i \in \{1, \dots, r\}),$$

$I_2$  is the identity matrix,  $O_2$  is the zero matrix of the second order;  $\omega_{ik}$ ,  $l_{ik}^{(1)}$ ,  $l_{ik}^{(2)} \in C([x_0, +\infty[; \mathbb{R})$  are so that

$$\lim_{x \rightarrow +\infty} l_{ik}^{(1)}(x) = 0, \quad \lim_{x \rightarrow +\infty} l_{ik}^{(2)}(x) = 1 \quad (k = 2, \dots, n_i, \quad i \in \{1, \dots, r\}),$$

$$\lim_{x \rightarrow +\infty} \omega_{ik}(x) = \omega_i^0 = \text{const} \quad (k = 1, \dots, n_i, \quad i \in \{1, \dots, r\}).$$

We can reduce the quasilinear equation

$$u^{(n)} = \sum_{k=0}^{n-1} [p_{0k}(t) + p_{1k}(t)]u^{(k)} + q(t, u, u', \dots, u^{(n-1)}) \quad (1.4)$$

to form (1.1) by means of some real transformation if  $q : [a, +\infty[ \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $p_{1k} : [a, +\infty[ \rightarrow \mathbb{R}$  ( $k = 0, \dots, n-1$ ) are “small” in some integral sense for continuous functions,  $p_{0k} : [a, +\infty[ \rightarrow \mathbb{R}$  ( $k = 0, \dots, n-1$ ) are continuously differentiable functions such that there are  $\varphi, \psi \in C^2([a, +\infty[; ]0, +\infty[)$  for which

$$\lim_{t \rightarrow +\infty} a_k(t) = a_{0k} = \text{const}, \quad \lim_{t \rightarrow +\infty} b_k(t) = b_{0k} = \text{const} \quad (k = 0, \dots, n-1),$$

where

$$a_k(t) = \varphi^{-1}(t)\psi^{-1-k}(t)(\varphi(t)\psi^k(t))', \quad b_k(t) = \psi^{k-n}(t)p_{0k}(t) \quad (k = 0, \dots, n-1).$$

Then, in system (1.1), each block (1.2) corresponds to complex conjugate roots of multiplicity  $n_i$  of the algebraic equation

$$\prod_{j=0}^{n-1} (\lambda + a_{0j}) = \sum_{k=1}^{n-1} b_{0k} \prod_{j=0}^{k-1} (\lambda + a_{0j}) + b_{00} \quad (1.5)$$

and each block (1.3) corresponds to a real root of (1.5) of multiplicity  $n_i$ .

<sup>1</sup>In case  $r = 0$ , the matrix  $W$  consists of the blocks  $W_i$  of type (1.3), whereas for  $r = s$  the matrix  $W$  consists of the blocks  $W_i$  of type (1.2). If  $k > m$ , we assume that  $\sum_{j=k}^m [\dots] = 0$ .

In this paper, we obtain asymptotic representations for solutions of the system of quasilinear differential equations of form (1.1). I. T. Kiguradze's results [1] on the asymptotic behaviour of solutions of (1.4), obtained for the case of simple roots of (1.5), are extended here to the case where multiple roots of (1.5) may exist. By virtue of the structure of equation (1.4) and that of the transformation reducing this equation to system (1.1), asymptotic formulas are obtained for solutions of system (1.1) with some consideration for small additive perturbations.

Many papers (see, e.g., [2–6]) are devoted to the study of the asymptotic equivalence between solutions of a quasilinear system and those of the corresponding linear system. Note that the results presented in this paper cannot be obtained by allying the results from [2–6].

## 2. BASIC ASSUMPTIONS ON THE PRINCIPAL PART OF THE ABRIDGED SYSTEM (1.1)

Let us consider the following auxiliary linear system of differential equations corresponding to system (1.1):

$$\frac{du}{dx} = W(x)u. \quad (2.1)$$

In order to obtain an asymptotic representation of the fundamental system of solutions of (2.1), we need some notation.

We write

$$N_1 = 0, \quad N_i = \sum_{k=1}^{i-1} m_k \quad (i = 2, \dots, s),$$

where

$$m_i = \begin{cases} 2n_i & \text{if } i \in \{1, \dots, r\}, \\ n_i & \text{if } i \in \{r+1, \dots, s\}. \end{cases}$$

We introduce two sets

$$J_0 = \{(i, j) : j = 1, \dots, n_i, 1 \leq i \leq s\}, \quad J = \{(i, k) : k = 1, \dots, m_i, 1 \leq i \leq s\}$$

and denote by  $e_{pv}$ , where  $(p, v) \in J$ , the  $n$ -dimensional vector whose  $N_p + v$ th component equals 1, whereas the others are equal to zero.

Setting, for every  $i \in \{1, \dots, s\}$ ,

$$d_{im}(x) = \exp \int_a^x (\omega_{im}(s) - \omega_{im-1}(s)) ds \quad (m = 2, \dots, n_i),$$

we define the functions  $B_{mk}^i$  ( $1 \leq m \leq k \leq n_i$ ,  $1 \leq i \leq s$ ) and  $B_{mk}^{i\mu}$  ( $1 \leq m \leq k \leq n_i$ ,  $1 \leq i \leq r$ ;  $\mu = 1, 2$ ) by the following recurrence relations:

$$B_{kk}^i(x) = 1, \quad B_{kk}^{i1}(x) = 0, \quad B_{kk}^{i2}(x) = 1 \quad (1 \leq k \leq n_i),$$

$$B_{mk}^i(x) = \int_{\beta_{mk}^i}^x B_{m+1k}^i(t) d_{im+1}(t) dt, \quad (1 \leq m < k \leq n_i),$$

$$B_{mk}^{i\mu}(x) = \int_{\alpha_{mk}^{i\mu}}^x \left[ B_{m+1k}^{i\mu}(t) l_{im+1}^{(2)}(t) + (-1)^{3-\mu} B_{m+1k}^{i3-\mu}(t) l_{im+1}^{(1)}(t) \right] d_{im+1}(t) dt \\ (1 \leq m < k \leq n_i),$$

where

$$\beta_{mk}^i = \begin{cases} a & \text{if } B_{mk0}^i = +\infty, \\ +\infty & \text{if } B_{mk0}^i < +\infty, \end{cases} \quad B_{mk0}^i = \int_a^{+\infty} |B_{m+1k}^i(t)| d_{im+1}(t) dt,$$

$$\alpha_{mk}^{i\mu} = \begin{cases} a & \text{if } B_{mk0}^{i\mu} = +\infty, \\ +\infty & \text{if } B_{mk0}^{i\mu} < +\infty, \end{cases}$$

$$B_{mk0}^{i\mu} = \int_a^{+\infty} \left| B_{m+1k}^{i\mu}(t) l_{im+1}^{(2)}(t) + (-1)^{3-\mu} B_{m+1k}^{i3-\mu}(t) l_{im+1}^{(1)}(t) \right| d_{im+1}(t) dt.$$

Using these functions, we also introduce the functions  $D_{kj}^i$  ( $1 \leq k \leq j \leq n_i$ ,  $i = r + 1, \dots, s$ ) and  $D_{kj}^{i\mu}$  ( $1 \leq k \leq j \leq n_i$ ,  $i = 1, \dots, r$ ;  $\mu = 1, 2$ ) by the recurrence relations

$$D_{kk}^i(x) = 1 \quad (k = 1, \dots, n_i),$$

$$D_{kj}^i(x) = - \sum_{m=k+1}^j B_{km}^i(x) D_{mj}^i(x) \quad (1 \leq k < j \leq n_i);$$

$$D_{kk}^{i1}(x) = 1, \quad D_{kk}^{i2}(x) = 0 \quad (k = 1, \dots, n_i),$$

$$D_{kj}^{i\mu}(x) = - \sum_{m=k+1}^j \left( B_{km}^{i2}(x) D_{mj}^{i\mu}(x) + (-1)^\mu B_{km}^{i1}(x) D_{mj}^{i3-\mu}(x) \right) \\ (1 \leq k < j \leq n_i; \mu = 1, 2).$$

Integrating the system “upside down,” we obtain

$$u_{N_i+2k-1}(x) = \exp \int_a^x \omega_{ik}(s) ds \sum_{j=k}^{n_i} \left( C_{N_i+2j-1} B_{kj}^{i2}(x) - C_{N_i+2j} B_{kj}^{i1}(x) \right),$$

$$u_{N_i+2k}(x) = \exp \int_a^x \omega_{ik}(s) ds \sum_{j=k}^{n_i} \left( C_{N_i+2j-1} B_{kj}^{i1}(x) + C_{N_i+2j} B_{kj}^{i2}(x) \right)$$

$$(k = 1, \dots, n_i, \quad i = 1, \dots, r);$$

$$u_{N_i+k}(x) = \exp \int_a^x \omega_{ik}(s) ds \sum_{j=k}^{n_i} C_{N_i+j} B_{kj}^i(x) \quad (k = 1, \dots, n_i, \quad i = r+1, \dots, s),$$

where  $(C_m)_{m=1}^n = C$  is a constant vector. Setting  $C = e_{pv}$  ( $v = 1, \dots, m_p$ ;  $p = 1, \dots, s$ ) we find for (2.1) a fundamental matrix of solutions of the form  $\Phi(x) = \text{diag} [\Phi_1(x), \dots, \Phi_s(x)]$ , where

$$\Phi_i(x) = \begin{pmatrix} \Phi_{11}^i(x) & \Phi_{12}^i(x) & \dots & \Phi_{1n_i}^i(x) \\ O^i & \Phi_{22}^i(x) & \dots & \Phi_{2n_i}^i(x) \\ \vdots & \vdots & \ddots & \vdots \\ O^i & O^i & \dots & \Phi_{n_i n_i}^i(x) \end{pmatrix}$$

with

$$O^i = \begin{cases} O_2 & \text{for } i \in \{1, \dots, r\}, \\ 0 & \text{for } i \in \{r+1, \dots, s\} \end{cases}$$

and

$$\Phi_{kj}^i(x) = \begin{cases} \begin{pmatrix} B_{kj}^{i2}(x) & -B_{kj}^{i1}(x) \\ B_{kj}^{i1}(x) & B_{kj}^{i2}(x) \end{pmatrix} \exp \int_a^x \omega_{ik}(s) ds & \text{if } 1 \leq k \leq j \leq n_i, \quad i = 1, \dots, r \\ B_{kj}^i(x) \exp \int_a^x \omega_{ik}(s) ds & \text{if } 1 \leq k \leq j \leq n_i, \quad i = r+1, \dots, s. \end{cases}$$

It is not difficult to see that the inverse matrix for  $\Phi(x)$  is given by the equality  $\Phi^{-1}(x) = \text{diag} [\Phi_1^{-1}(x), \dots, \Phi_s^{-1}(x)]$ , where

$$\Phi_i^{-1}(x) = \begin{pmatrix} (\Phi_i^{-1}(x))_{11} & (\Phi_i^{-1}(x))_{12} & \dots & (\Phi_i^{-1}(x))_{1n_i} \\ O^i & (\Phi_i^{-1}(x))_{22} & \dots & (\Phi_i^{-1}(x))_{2n_i} \\ \vdots & \vdots & \ddots & \vdots \\ O^i & O^i & \dots & (\Phi_i^{-1}(x))_{n_i n_i} \end{pmatrix} \quad (i = 1, \dots, s)$$

with

$$(\Phi_i^{-1}(x))_{kj} = \begin{cases} \begin{pmatrix} D_{kj}^{i1}(x) & -D_{kj}^{i2}(x) \\ D_{kj}^{i2}(x) & D_{kj}^{i1}(x) \end{pmatrix} \exp\left(-\int_a^x \omega_{ij}(s)ds\right), \\ \text{if } 1 \leq k \leq j \leq n_i, \ i = 1, \dots, r, \\ \\ D_{kj}^i(x) \exp\left(-\int_a^x \omega_{ij}(s)ds\right), \\ \text{if } 1 \leq k \leq j \leq n_i, \ i = r+1, \dots, s. \end{cases}$$

In what follows, we assume that the following condition  $(S_B)$  is satisfied for system (1.1):

$(S_B)$  For every  $i \in \{1, \dots, s\}$ ,

$$B_{1m}^i(x) \neq 0 \quad (m = 1, \dots, n_i)$$

in some neighbourhood of  $+\infty$  and, moreover, for arbitrary  $j$  and  $m$  such that  $1 \leq j \leq m \leq n_i$ , the relations

$$\limsup_{x \rightarrow +\infty} \left| \frac{B_{1j}^i(x) B_{jm}^{i\mu}(x)}{B_{1m}^i(x)} \right| < +\infty \quad \text{if } i \in \{1, \dots, r\}, \ \mu = 1, 2,$$

$$\limsup_{x \rightarrow +\infty} \left| \frac{B_{1j}^i(x) B_{jm}^i(x)}{B_{1m}^i(x)} \right| < +\infty \quad \text{if } i \in \{r+1, \dots, s\}$$

are true.

**Lemma 1.** If condition  $(S_B)$  is satisfied, then there are constants  $c_1 \geq 1$  and  $x_1 \geq x_0$  such that for all  $i \in \{1, \dots, s\}$  and all natural  $k$  and  $j$  with  $1 \leq k \leq j \leq n_i$ , the estimates

$$\begin{aligned} \sum_{m=k}^j \left| B_{1m}^i(x) D_{mj}^{i\mu}(x) \right| &\leq (2c_1)^{j-k} \left| B_{1j}^i(x) \right| \quad \text{if } i \in \{1, \dots, r\} \ (\mu = 1, 2), \\ \sum_{m=k}^j \left| B_{1m}^i(x) D_{mj}^i(x) \right| &\leq (2c_1)^{j-k} \left| B_{1j}^i(x) \right| \quad \text{if } i \in \{r+1, \dots, s\} \end{aligned} \quad (2.2_i)$$

are valid for  $x \geq x_1$ .

Lemma 1 is proved similarly to Lemma 1 from [7].

*Proof of Lemma 1.* In view of condition  $(S_B)$ , there are constants  $x_1 \geq x_0$  and  $c_1 \geq 1$  such that the relations

$$\left| \frac{B_{1j}^i(x) B_{jm}^{i\mu}(x)}{B_{1m}^i(x)} \right| \leq \frac{c_1}{2} \quad (i = 1, \dots, r; \ \mu = 1, 2),$$

$$\left| \frac{B_{1j}^i(x) B_{jm}^i(x)}{B_{1m}^i(x)} \right| \leq c_1 \quad (i = r+1, \dots, s)$$

are satisfied.

Let us show that (2.2<sub>i</sub>) is true for  $x_1 \geq x_0$ . We fix an arbitrary number  $i \in \{1, \dots, s\}$  and prove (2.2<sub>i</sub>) by induction in  $j - k$  ( $0 \leq j - k \leq n_i - k$ ).

Indeed, let  $i \in \{1, \dots, r\}$ . Since  $D_{kk}^{i1}(x) = 1$  and  $D_{kk}^{i2}(x) = 0$  for  $k = 1, \dots, n_i$ , we see that, for all  $j, k \in \{1, \dots, n_i\}$  such that  $j - k = 0$ , estimate (2.2<sub>i</sub>) is true.

Let us suppose that (2.2<sub>i</sub>) holds for all  $j, k \in \{1, \dots, n_i\}$  that satisfy the condition  $j - k = l < n_i - k$ . We shall show that (2.2<sub>i</sub>) is valid for arbitrary  $j, k \in \{1, \dots, n_i\}$  satisfying the condition  $j - k = l + 1 < n_i - k$ . By the definition of the functions  $D_{kj}^{i\mu}$ ,  $1 \leq k \leq j \leq n_i$ ,  $i = 1, \dots, r$ ,  $\mu = 1, 2$ , using the assumption of the induction and taking condition (S<sub>B</sub>) into account, for  $j - k = l + 1$ , we obtain

$$\begin{aligned} \sum_{m=k}^j \left| B_{1m}^i(x) D_{mj}^{i1}(x) \right| &= \left| B_{1k}^i(x) D_{kj}^{i1}(x) \right| + \sum_{m=k+1}^j \left| B_{1m}^i(x) D_{mj}^{i1}(x) \right| \leq \\ &\leq \left| B_{1k}^i(x) \sum_{m=k+1}^j B_{km}^{i1}(x) D_{mj}^{i2}(x) - B_{km}^{i2}(x) D_{mj}^{i1}(x) \right| + (2c_1)^l \left| B_{1j}^i(x) \right| \leq \\ &\leq \left| B_{1k}^i(x) \sum_{m=k+1}^j B_{km}^{i1}(x) D_{mj}^{i2}(x) \right| + \left| B_{1k}^i(x) \sum_{m=k+1}^j B_{km}^{i2}(x) D_{mj}^{i1}(x) \right| + \\ &+ (2c_1)^l \left| B_{1j}^i(x) \right| \leq \frac{c_1}{2} \sum_{m=k+1}^j \left| B_{1m}^i(x) D_{mj}^{i2}(x) \right| + \frac{c_1}{2} \sum_{m=k+1}^j \left| B_{1m}^i(x) D_{mj}^{i1}(x) \right| + \\ &+ (2c_1)^l \left| B_{1j}^i(x) \right| \leq 2 \frac{c_1}{2} (2c_1)^l \left| B_{1j}^i(x) \right| + (2c_1)^l \left| B_{1j}^i(x) \right| \leq (2c_1)^{l+1} \left| B_{1j}^i(x) \right| \end{aligned}$$

for  $x \geq x_1$ . Similarly,

$$\sum_{m=k}^j \left| B_{1m}^i(x) D_{mj}^{i2}(x) \right| \leq (2c_1)^{j-k} \left| B_{1j}^i(x) \right|$$

for  $x \geq x_1$ .

If  $i \in \{r + 1, \dots, s\}$ , then

$$\begin{aligned} \sum_{m=k}^j \left| B_{1m}^i(x) D_{mj}^i(x) \right| &= \left| B_{1k}^i(x) D_{kj}^i(x) \right| + \sum_{m=k+1}^j \left| B_{1m}^i(x) D_{mj}^i(x) \right| \leq \\ &\leq \left| B_{1k}^i(x) \sum_{m=k+1}^j B_{km}^{i1}(x) D_{mj}^i(x) \right| + (2c_1)^l \left| B_{1j}^i(x) \right| \leq \\ &\leq c_1 \sum_{m=k+1}^j \left| B_{1m}^i(x) D_{mj}^i(x) \right| + (2c_1)^l \left| B_{1j}^i(x) \right| \leq c_1 (2c_1)^l \left| B_{1j}^i(x) \right| + \\ &+ (2c_1)^l \left| B_{1j}^i(x) \right| \leq (2c_1)^{l+1} \left| B_{1j}^i(x) \right| \end{aligned}$$

for  $x \geq x_1$ .

The lemma is proved.  $\square$

### 3. MAIN RESULTS

In order to formulate the theorems, besides the notation introduced in Section 2, we also need some other definitions.

For an arbitrary pair  $(i, k) \in J_0$ , we write

$$H_{i2k-1}(x) = H_{i2k}(x) = B_{1k}^i(x) \exp \int_a^x (\omega_{i1}(t) - \omega_{ik}(t)) dt \quad \text{if } i \in \{1, \dots, r\},$$

and

$$H_{ik}(x) = B_{1k}^i(x) \exp \int_a^x (\omega_{i1}(t) - \omega_{ik}(t)) dt \quad \text{if } i \in \{r+1, \dots, s\}.$$

We also define the matrix

$$\Delta(x) = \text{diag} [\Delta_1(x), \dots, \Delta_s(x)],$$

where  $\Delta_i(x) = \text{diag} [H_{i1}(x), \dots, H_{im_i}(x)]$  ( $i = 1, \dots, s$ ), and denote by  $\mathbb{R}_b^n$  the following set:

$$\mathbb{R}_b^n = \left\{ (z_j)_{j=1}^n \in \mathbb{R}^n : |z_1| \leq b, \dots, |z_n| \leq b \right\}.$$

**Definition 1.** Let  $\gamma = (\gamma_i)_{i=1}^s$  be an  $n$ -dimensional vector function with the continuous components  $\gamma_i : [a, +\infty] \rightarrow ]0, +\infty[$  ( $i = 1, \dots, s$ ).

We shall say that a system of continuous functions  $\mu_{im} : ]a, +\infty[ \rightarrow \mathbb{R}$  ( $m = 1, \dots, n_i$ ,  $i = 1, \dots, s$ ) satisfies the condition  $(M - L)_{pq}$  (where  $(p, q) \in J_0$ ) on the interval  $[x_0, +\infty[ \subset ]a, +\infty[$  with the weight coefficient  $\gamma$  if, for arbitrary  $(i, m) \in J_0$ , either

$$\inf \left\{ \int_{\tau}^x [\mu_{pq}(s) - \mu_{im}(s)] ds - \ln \frac{\gamma_i(x)}{\gamma_i(\tau)} : x \geq \tau \geq x_0 \right\} > -\infty$$

and

$$\lim_{x \rightarrow +\infty} \left[ \int_{x_0}^x [\mu_{pq}(s) - \mu_{im}(s)] ds - \ln \frac{\gamma_i(x)}{\gamma_i(x_0)} \right] = +\infty$$

or

$$\sup \left\{ \int_{\tau}^x [\mu_{pq}(s) - \mu_{im}(s)] ds - \ln \frac{\gamma_i(x)}{\gamma_i(\tau)} : x \geq \tau \geq x_0 \right\} < +\infty.$$

If condition  $(M - L)_{pq}$  holds with the weight coefficient  $\gamma = (\gamma_i)_{i=1}^s$ , we use the diagonal matrix

$$\Gamma(x) = \text{diag} [\gamma_1(x)I_{m_1}, \dots, \gamma_s(x)I_{m_s}],$$

where  $I_{m_i}$  ( $i = 1, \dots, s$ ) is the identity matrix of order  $m_i$ .

**Theorem 1.** Let condition  $(S_B)$  be satisfied and, for a certain pair  $(p, v) \in J$  and some  $n$ -dimensional vector function  $\gamma = (\gamma_i)_{i=1}^s$  with continuous components  $\gamma_i : [a, +\infty[ \rightarrow ]0, +\infty[$  ( $i = 1, \dots, s$ ), the following conditions hold:

(i) *The system of functions*

$$\mu_{ik}(x) = \omega_{i1}(x) + \frac{(B_{1k}^i(x))'}{B_{1k}^i(x)} \quad (k = 1, \dots, n_i, \quad 1 \leq i \leq s)$$

satisfies the condition  $(M - L)_{pq}$ , where  $q$  is defined by the equality<sup>2</sup>

$$q = \begin{cases} \left[ \frac{v+1}{2} \right] & \text{if } p \in \{1, \dots, r\} \\ v & \text{if } p \in \{r+1, \dots, s\} \end{cases}$$

on the interval  $[x_0, +\infty[ \subset ]a, +\infty[$  with the weight coefficient  $\gamma$ ;

(ii) *For every  $(i, k) \in J$  and all  $m \in \{1, \dots, v\}$ , one has*

$$\int_{x_0}^{+\infty} \gamma_i(x) \left| \frac{H_{ik}(x)}{H_{pm}(x)} r_{N_i+kN_p+m}(x) \right| dx < +\infty;$$

(iii) *For arbitrary  $(i, k), (l, m) \in J$*

$$\int_{x_0}^{+\infty} \frac{\gamma_i(x)}{\gamma_l(x)} \left| \frac{H_{ik}(x)}{H_{lm}(x)} r_{N_i+kN_l+m}(x) \right| dx < +\infty;$$

(iv) *There is a number  $b \in ]0, +\infty[$  and there exists a continuous function*

$$\phi : [a, +\infty[ \rightarrow \mathbb{R}_+$$

such that, for all  $(i, k) \in J$  and all  $z \in \mathbb{R}_b^n$ , the relations

$$|F_{N_i+k}(x, \Delta^{-1}(x)\Gamma^{-1}(x)\xi_{pq}(x)z + \Phi(x)e_{pv})| \leq \phi(x)$$

and

$$\int_{x_0}^{+\infty} \gamma_i(x) \phi(x) \left| \frac{H_{ik}(x)}{\xi_{pq}(x)} \right| dx < +\infty$$

are true.

Then system (1) has at least one solution  $y_{pv}(x)$  that admits the following asymptotic representation for  $x \rightarrow +\infty$ :

$$y_{pv}(x) = \Phi(x)e_{pv} + \Delta^{-1}(x)\Gamma^{-1}(x)\bar{o}(\xi_{pq}(x)),$$

where

$$\xi_{pq}(x) = B_{1q}^p(x) \exp \int_a^x \omega_{p1}(s) ds.$$

*Proof.* Given the numbers  $(p, q) \in J_0$  and the  $n$ -dimensional vector function  $\gamma$ , we introduce the auxiliary functions

$$g_{im}(x, \tau) = \frac{\gamma_i(x)B_{1q}^i(x)B_{1q}^p(\tau)}{\gamma_i(\tau)B_{1q}^i(\tau)B_{1q}^p(x)} \exp \int_\tau^x (\omega_{i1}(s) - \omega_{p1}(s)) ds,$$

<sup>2</sup>Here,  $[\dots]$  denotes the integer part of the expression.

$m = 1, \dots, n_i, \quad i = 1, \dots, s$ , and

$$f_{pv}(x) = \frac{\Delta(x)\Phi(x)}{\xi_{pq}(x)} e_{pv}.$$

Due to the structure of the matrices  $\Delta$  and  $\Phi$ , the components of the vector  $f_{pv}(x) = ((f_{pv})_k)_{k=1}^n$  are determined as follows: for all  $\eta \in \{1, \dots, q\}$ ,

$$\begin{aligned} (f_{p2q-1})_{N_p+2\eta-1}(x) &= \frac{B_{1\eta}^p(x)B_{\eta q}^{p2}(x)}{B_{1q}^p(x)}, \\ (f_{p2q-1})_{N_p+2\eta}(x) &= \frac{B_{1\eta}^p(x)B_{\eta q}^{p1}(x)}{B_{1q}^p(x)}, \\ (f_{p2q})_{N_p+2\eta-1}(x) &= -\frac{B_{1\eta}^p(x)B_{\eta q}^{p1}(x)}{B_{1q}^p(x)}, \\ (f_{p2q})_{N_p+2\eta}(x) &= \frac{B_{1\eta}^p(x)B_{\eta q}^{p2}(x)}{B_{1q}^p(x)}, \\ (f_{p2q-1})_j(x) &= 0, \quad (f_{p2q})_j(x) = 0 \quad (j \neq N_p + 1, \dots, N_p + v) \end{aligned}$$

if  $p \in \{1, \dots, r\}$  and

$$\begin{aligned} (f_{pq})_{N_p+\eta}(x) &= \frac{B_{1\eta}^p(x)B_{\eta q}^p(x)}{B_{1q}^p(x)}, \\ (f_{pq})_j(x) &= 0 \quad (j \neq N_p + 1, \dots, N_p + q) \end{aligned}$$

if  $p \in \{r + 1, \dots, s\}$ .

In view of condition (i) of the theorem, the set  $J_0$  is decomposed as the union of two disjoint subsets  $J_{1pq}$  and  $J_{2pq}$ ,

$$J_0 = J_{1pq} \bigcup J_{2pq},$$

such that

$$\lim_{x \rightarrow +\infty} g_{im}(x, \tau) = 0 \quad \text{for all } \tau \geq x_0$$

and

$$\eta_{im} = \sup \{|g_{im}(x, \tau)| : x \geq \tau \geq x_0\} < +\infty$$

when  $(i, m) \in J_{1pq}$ , and

$$\eta_{im} = \sup \{|g_{im}(x, \tau)| : \tau \geq x \geq x_0\} < +\infty$$

when  $(i, m) \in J_{2pq}$ .

According to this decomposition of the set  $J_0$ , the matrix  $\Phi$  can be represented as the sum of two diagonal matrices

$$\Phi^{(\rho)}(x) = \text{diag} [\Phi_1^{(\rho)}(x), \dots, \Phi_s^{(\rho)}(x)] \quad (\rho = 1, 2),$$

where  $\Phi_i^{(\rho)}(x)$  ( $i \in \{1, \dots, s\}$ ,  $\rho = 1, 2$ ) are the  $m_i$ th order matrices with the elements

$$(\Phi_i^{(\rho)}(x))_{jm} = \begin{cases} (\Phi_i(x))_{jm} & \text{if } (i, m) \in J_{\rho pq}, \\ 0 & \text{if } (i, m) \notin J_{\rho pq} \end{cases}.$$

We consider

$$\alpha_\rho = \begin{cases} x_2 & \text{for } \rho = 1, \\ +\infty & \text{for } \rho = 2. \end{cases}$$

We shall show that, for a sufficiently large number  $x_2 \geq x_1$ , where  $x_1$  is as in Lemma 1, the system of integral equations

$$\begin{aligned} z(x) = & \sum_{\rho=1}^2 \int_{a_\rho}^x K^\rho(x, \tau) \left[ G(\tau) \{z(\tau) + \Gamma(\tau) f_{pv}(\tau)\} + \right. \\ & \left. + \frac{1}{\xi_{pq}(\tau)} \Gamma(\tau) \Delta(\tau) F(\tau, \Delta^{-1}(\tau) \Gamma^{-1}(\tau) \xi_{pq}(\tau) z(\tau) + \Phi(\tau) e_{pv}) \right] d\tau, \end{aligned} \quad (3.2)$$

where

$$K^\rho(x, \tau) = \frac{\xi_{pq}(\tau)}{\xi_{pq}(x)} \Gamma(x) \Delta(x) \Phi^{(\rho)}(x) \Phi^{-1}(\tau) \Delta^{-1}(\tau) \Gamma^{-1}(\tau),$$

$$G(\tau) = \Gamma(\tau) \Delta(\tau) R(\tau) \Delta^{-1}(\tau) \Gamma^{-1}(\tau),$$

has at least one solution  $z = (z_k)_{k=1}^n \in C([x_2, +\infty[; \mathbb{R}_b^n)$  vanishing when  $x \rightarrow +\infty$ .

In this system,

$$K^{(\rho)}(x, \tau) = \text{diag} [K_1^{(\rho)}(x, \tau), \dots, K_s^{(\rho)}(x, \tau)] \quad (\rho = 1, 2),$$

$$K_i^\rho(x, \tau) = \frac{\xi_{pq}(\tau) \gamma(x)}{\xi_{pq}(x) \gamma(\tau)} \Delta_i(x) \Phi_i^{(\rho)}(x) \Phi_i^{-1}(\tau) \Delta_i^{-1}(\tau).$$

Here,  $K_i^\rho$  consists of the blocks

$$(K_i^\rho(x, \tau))_{kj} = \begin{cases} \begin{pmatrix} (K_i^\rho(x, \tau))_{kj}^{(1)} & - (K_i^\rho(x, \tau))_{kj}^{(2)} \\ (K_i^\rho(x, \tau))_{kj}^{(2)} & (K_i^\rho(x, \tau))_{kj}^{(1)} \end{pmatrix} & \text{if } 1 \leq k \leq j \leq n_i, \\ O_2 & \text{if } 1 \leq j < k \leq n_i, \end{cases}$$

where

$$\begin{aligned} \left( K_i^{(\rho)}(x, \tau) \right)_{kj}^{(\mu)} &= \sum_{\substack{m=k \\ (i,m) \in J_{\rho pq}}}^j g_{im}(x, \tau) \left( \frac{B_{1k}^i(x) B_{km}^{i3-\mu}(x)}{B_{1m}^i(x)} \frac{B_{1m}^i(\tau) D_{mj}^{i1}(\tau)}{B_{1j}^i(\tau)} + \right. \\ &\quad \left. + (-1)^\mu \frac{B_{1k}^i(x) B_{km}^{i\mu}(x)}{B_{1m}^i(x)} \frac{B_{1m}^i(\tau) D_{mj}^{i2}(\tau)}{B_{1j}^i(\tau)} \right) \quad (\mu = 1, 2) \end{aligned}$$

for  $i = 1, \dots, r$ , and, for  $i = r + 1, \dots, s$ , it consists of the blocks

$$\left( K_i^{(\rho)}(x, \tau) \right)_{kj} = \begin{cases} \sum_{\substack{m=k \\ (i,m) \in J_{\rho pq}}}^j g_{im}(x, \tau) \frac{B_{1k}^i(x) B_{km}^i(x)}{B_{1m}^i(x)} \frac{B_{1m}^i(\tau) D_{mj}^i(\tau)}{B_{1j}^i(\tau)} & \text{if } 1 \leq k \leq j \leq n_i \\ 0 & \text{if } 1 \leq j < k \leq n_i. \end{cases}$$

The matrix  $G$  is given by

$$G(\tau) = \begin{pmatrix} G_{11}(\tau) & \dots & G_{1s} \\ \vdots & \ddots & \vdots \\ G_{s1}(\tau) & \dots & G_{ss} \end{pmatrix},$$

where  $G_{il}$  is the block of dimension  $m_i \times m_l$  having the elements

$$(G_{il}(\tau))_{kj} = \frac{\gamma_i(\tau)}{\gamma_l(\tau)} \frac{H_{ik}(\tau)}{H_{lj}(\tau)} r_{N_i+kN_l+j}(\tau), \quad (i, k), (l, j) \in J.$$

Considering the representations above, we rewrite (3.2) as

$$z_k(x) = \int_{x_2}^x \psi_{1k}(\tau, x, z(\tau)) d\tau + \int_x^{+\infty} \psi_{2k}(\tau, x, z(\tau)) d\tau \quad (k = 1, \dots, n), \quad (3.3)$$

where the functions  $\psi_{\rho k}$  ( $\rho = 1, 2$ ,  $k = 1, \dots, n$ ) are defined as follows: for all  $v \in \{1, \dots, n_\eta\}$ , we put

$$\begin{aligned} \psi_{\rho N_\eta+2v-1}(\tau, x, z) = & \sum_{i=1}^s \sum_{m=1}^{m_i} \sum_{l=1}^s \sum_{j=v}^{n_\eta} \left\{ \left( K_\eta^{(\rho)}(x, \tau) \right)_{vj}^{(1)} (G_{\eta l}(\tau))_{2j-1m} - \right. \\ & \left. - \left( K_\eta^{(\rho)}(x, \tau) \right)_{vj}^{(2)} (G_{\eta l}(\tau))_{2jm} \right\} z_{N_i+m} + \\ & + \sum_{m=1}^v \sum_{j=v}^{n_\eta} \left\{ \left( K_\eta^{(\rho)}(x, \tau) \right)_{vj}^{(1)} (G_{\eta p}(\tau))_{2j-1m} - \right. \\ & \left. - \left( K_\eta^{(\rho)}(x, \tau) \right)_{vj}^{(2)} (G_{\eta p}(\tau))_{2jm} \right\} \gamma_p(\tau) (f_{pv}(\tau))_{N_p+m} + \\ & + \sum_{j=v}^{n_\eta} \frac{\gamma_\eta(\tau)}{\xi_{pq}(\tau)} \left\{ \left( K_\eta^{(\rho)}(x, \tau) \right)_{vj}^{(1)} H_{\eta 2j-1}(\tau) \widetilde{F}_{N_\eta+2j-1}(\tau, z) - \right. \\ & \left. - \left( K_\eta^{(\rho)}(x, \tau) \right)_{vj}^{(2)} H_{\eta 2j}(\tau) \widetilde{F}_{N_\eta+2j}(\tau, z) \right\} \end{aligned}$$

and

$$\begin{aligned} \psi_{\rho N_\eta+2v}(\tau, x, z) = & \sum_{i=1}^s \sum_{m=1}^{m_i} \sum_{l=1}^s \sum_{j=v}^{n_\eta} \left\{ \left( K_\eta^{(\rho)}(x, \tau) \right)_{vj}^{(2)} (G_{\eta l}(\tau))_{2j-1m} + \right. \\ & \left. + \left( K_\eta^{(\rho)}(x, \tau) \right)_{vj}^{(1)} (G_{\eta l}(\tau))_{2jm} \right\} z_{N_i+m} + \\ & + \sum_{m=1}^v \sum_{j=v}^{n_\eta} \left\{ \left( K_\eta^{(\rho)}(x, \tau) \right)_{vj}^{(2)} (G_{\eta p}(\tau))_{2j-1m} + \right. \\ & \left. + \left( K_\eta^{(\rho)}(x, \tau) \right)_{vj}^{(1)} (G_{\eta p}(\tau))_{2jm} \right\} \gamma_p(\tau) (f_{pv}(\tau))_{N_p+m} + \\ & + \sum_{j=v}^{n_\eta} \frac{\gamma_\eta(\tau)}{\xi_{pq}(\tau)} \left\{ \left( K_\eta^{(\rho)}(x, \tau) \right)_{vj}^{(2)} H_{\eta 2j-1}(\tau) \widetilde{F}_{N_\eta+2j-1}(\tau, z) + \right. \\ & \left. + \left( K_\eta^{(\rho)}(x, \tau) \right)_{vj}^{(1)} H_{\eta 2j}(\tau) \widetilde{F}_{N_\eta+2j}(\tau, z) \right\} \end{aligned}$$

when  $\eta \in \{1, \dots, r\}$ , and we put

$$\begin{aligned} \psi_{\rho N_{\eta} + \nu}(\tau, x, z) = & \sum_{i=1}^s \sum_{m=1}^{m_i} \sum_{l=1}^s \sum_{j=\nu}^{n_{\eta}} \left( K_{\eta}^{(\rho)}(x, \tau) \right)_{\nu j} \left( G_{\eta l}(\tau) \right)_{j m} z_{N_i + m} + \\ & + \sum_{m=1}^{\nu} \sum_{j=\nu}^{n_{\eta}} \left( K_{\eta}^{(\rho)}(x, \tau) \right)_{\nu j} \left( G_{\eta p}(\tau) \right)_{j m} \gamma_p(\tau) \left( f_{p\nu}(\tau) \right)_{N_p + m} + \\ & + \sum_{j=\nu}^{n_{\eta}} \frac{\gamma_{\eta}(\tau)}{\xi_{pq}(\tau)} \left( K_{\eta}^{(\rho)}(x, \tau) \right)_{\nu j} H_{\eta j}(\tau) \widetilde{F}_{N_{\eta} + j}(\tau, z) \end{aligned}$$

if  $\eta \in \{r+1, \dots, s\}$ . Here,

$$\widetilde{F}_k(\tau, z) = F_k(\tau, \Delta^{-1}(\tau) \Gamma^{-1}(\tau) \xi_{pq}(\tau) z + \Phi(\tau) e_{p\nu}) \quad (k = 1, \dots, n).$$

Let us choose  $c_1$  as in Lemma 1 and introduce the constant

$$c = \sum_{i=1}^r (2c_1)^{n_i} n_i^2 \eta_i + \sum_{i=r+1}^s c_1 (2c_1)^{n_i-1} n_i^2 \eta_i,$$

where

$$\eta_i = \max\{\eta_{im} : 1 \leq m \leq n_i\} \quad (i = 1, \dots, s).$$

We set

$$\begin{aligned} \zeta(x, \tau) = & \frac{c_1}{c} \left\{ \sum_{i=1}^r \sum_{k=1}^{n_i} \sum_{j=k}^{n_i} \sum_{\substack{m=k \\ (i,m) \in J_{1pq}}}^j |g_{im}(x, \tau)| \left( \left| \frac{B_{1m}^i(\tau) D_{mj}^{i1}(\tau)}{B_{1j}^i(\tau)} \right| + \left| \frac{B_{1m}^i(\tau) D_{mj}^{i2}(\tau)}{B_{1j}^i(\tau)} \right| \right) + \right. \\ & \left. + \sum_{i=r+1}^s \sum_{k=1}^{n_i} \sum_{j=k}^{n_i} \sum_{\substack{m=k \\ (i,m) \in J_{1pq}}}^j |g_{im}(x, \tau)| \left| \frac{B_{1m}^i(\tau) D_{mj}^{i1}(\tau)}{B_{1j}^i(\tau)} \right| \right\} \end{aligned}$$

if  $J_{1pq} \neq \emptyset$ , and  $\zeta(x, \tau) = 0$  for  $J_{1pq} = \emptyset$ .

Furthermore, we set

$$\begin{aligned} g(\tau) = & c \left\{ \sum_{\eta=1}^s \sum_{\nu=1}^{n_{\eta}} \left( b \sum_{i=1}^s \sum_{m=1}^{m_i} \sum_{l=1}^s \sum_{j=\nu_{\eta}}^{m_{\eta}} |(G_{\eta l}(\tau))_{jm}| + \right. \right. \\ & \left. \left. + d_p \gamma_p(\tau) \sum_{m=1}^{\nu} \sum_{j=\nu_{\eta}}^{m_{\eta}} |(G_{\eta p}(\tau))_{jm}| + \sum_{j=\nu_{\eta}}^{m_{\eta}} \gamma_{\eta}(\tau) \phi(\tau) \left| \frac{H_{\eta j}(\tau)}{\xi_{pq}(\tau)} \right| \right) \right\}, \end{aligned}$$

where

$$\nu_{\eta} = \begin{cases} 2\nu - 1, & \text{if } \eta \in \{1, \dots, r\}, \\ \nu, & \text{if } \eta \in \{r+1, \dots, s\}, \end{cases} \quad d_p = \begin{cases} \frac{c_1}{2} & \text{for } p \in \{1, \dots, r\}, \\ c_1 & \text{for } p \in \{r+1, \dots, s\}. \end{cases}$$

Taking conditions (ii)–(iv) into account, we can choose the number  $x_2 \geq x_1$  so large that the inequality

$$\int_{x_2}^{+\infty} g(\tau) d\tau \leq b$$

be satisfied. Note that, by virtue of condition  $(S_B)$  and relation (3.1), for all  $z \in \mathbb{R}_b^n$ , the following estimates are true:

$$\begin{aligned} |\psi_{1k}(\tau, x, z)| &\leq \zeta(x, \tau) g(\tau) & \text{for } x_2 \leq \tau \leq x \ (k = 1, \dots, n), \\ |\psi_{2k}(\tau, x, z)| &\leq g(\tau) & \text{for } x_2 \leq x \leq \tau \ (k = 1, \dots, n). \end{aligned}$$

Therefore, by using (3.1), we get

$$\zeta(x, \tau) \leq 1 \quad \text{for } x \geq \tau \geq x_2$$

and

$$\lim_{x \rightarrow +\infty} \zeta(x, \tau) = 0 \quad \text{for } \tau \geq x_2.$$

Thus, for the chosen  $x_2 \geq x_1$ , system (3.3) satisfies all the conditions of Lemma 7.6 in the book [1] (Chapter II, §7, p. 201) and, therefore, it has at least one solution  $z_{pv}(x)$  vanishing as  $x \rightarrow +\infty$ .

Setting

$$z_{pv}(x) = \frac{\Gamma(x)\Delta(x)y_{pv}(x) - \Gamma(x)\Delta(x)\Phi(x)e_{pv}}{\xi_{pq}(x)}$$

in (3.2), we note that  $y_{pv}$  is a solution of system (1.1) for which the asymptotic representation

$$y_{pv}(x) = \Phi(x)e_{pv} + \Delta^{-1}(x)\Gamma^{-1}(x)\bar{o}(\xi_{pq}(x))$$

is valid as  $x \rightarrow +\infty$ .

The theorem is proved.  $\square$

For some fixed pair  $(p, l) \in J$ , let us now consider the system of  $n$ -dimensional vector functions

$$u_{N_p+j}(x) = \Phi(x)e_{pj} \quad (j = 1, \dots, l).$$

Using these vector functions, we consider the functions

$$Y_{pl}(x) = \sum_{i=1}^l C_i u_{N_p+i}(x),$$

where  $C_1, \dots, C_l$  are real constants.

Our question now is in which cases system (1.1) has a solution that is asymptotically close, as  $x \rightarrow +\infty$ , to  $Y_{pl}(x)$ .

In order to study this problem, we formulate the following.

**Definition 2.** We say that a system of continuous functions  $q_{im} : [a, +\infty[ \times [a, +\infty[ \rightarrow \mathbb{R}$  ( $m = 1, \dots, n_i$ ,  $i = 1, \dots, s$ ) satisfies condition  $(M - L)$  on  $[x_0, +\infty[ \times [x_0, +\infty[$  if, for arbitrary  $(i, m) \in J_0$ , either

$$\lim_{x \rightarrow +\infty} q_{im}(x, \tau) = 0 \quad (\tau \geq x_0)$$

and

$$\theta_{im} = \sup \{q_{im}(x, \tau) : x \geq \tau \geq x_0\} < +\infty, \quad (3.4)$$

or

$$\theta_{im} = \sup \{q_{im}(x, \tau) : \tau \geq x \geq x_0\} < +\infty. \quad (3.5)$$

**Theorem 2.** Let condition  $(S_B)$  be satisfied and, moreover,

$$\int_{x_0}^{+\infty} \left| \frac{H_{iv}(x)}{H_{jm}(x)} r_{N_i + \nu N_j + m}(x) \right| dx < +\infty$$

for all  $(i, \nu)$  and  $(j, m)$  in  $J$ . Furthermore, assume that, for some fixed pair  $(p, k) \in J_0$ , there exists a function  $U_{pl} \in C([a, +\infty[; ]0, +\infty[)$ , where

$$l = \begin{cases} 2k & \text{if } p \in \{1, \dots, r\} \\ k & \text{if } p \in \{r+1, \dots, s\} \end{cases}$$

such that, for each  $j \in \{1, \dots, l\}$ ,

$$U_{pl}(x) = \underline{O} \left( \sum_{i=N_p+1}^{N_p+j} |(u_{N_p+j}(x))_i| \right)$$

as  $x \rightarrow +\infty$ , and the following conditions hold:

(i) The system

$$q_{im}(x, \tau) = \frac{U_{pl}(\tau) B_{1m}^i(\tau)}{U_{pl}(x) B_{1m}^i(x)} \exp \int_{\tau}^x \omega_{i1}(s) ds \quad ((i, m) \in J_0)$$

satisfies condition  $(M - L)$  on  $[x_0, +\infty[ \times [x_0, +\infty[$ ;

(ii) For all  $(\eta, j) \in J$  and arbitrary  $m, \xi$  with  $1 \leq m \leq k$  and  $m \leq \xi \leq k$ ,

$$\int_{x_0}^{+\infty} \frac{1}{U_{pl}(x)} \left| \frac{H_{\eta j}(x)}{H_{p, 2m-\rho}(x)} r_{N_{\eta} + j N_p + 2m-\rho}(x) B_{m\xi}^{p\mu}(x) \right| \exp \int_a^x \omega_{pm}(s) ds \, dx < +\infty$$

if  $p \in \{1, \dots, r\}$  ( $\rho = 0, 1$ ,  $\mu = 1, 2$ ) and

$$\int_{x_0}^{+\infty} \frac{1}{U_{pl}(x)} \left| \frac{H_{\eta j}(x)}{H_{pm}(x)} r_{N_{\eta} + j N_p + m}(x) B_{m\xi}^p(x) \right| \exp \int_a^x \omega_{pm}(s) ds \, dx < +\infty$$

when  $p \in \{r+1, \dots, s\}$ ;

(iii) There is a number  $b \in ]0, +\infty[$  and there is a continuous function

$$\phi : [a, +\infty[ \rightarrow \mathbb{R}_+$$

such that, for every  $(i, \nu) \in J$  and arbitrary  $z \in \mathbb{R}_b^n$ ,

$$|F_{N_i+\nu}(x, \Delta^{-1}(x)U_{pl}(x)z + Y_{pl}(x))| \leq \phi(x)$$

and

$$\int_{x_0}^{+\infty} \frac{\phi(x)}{U_{pl}(x)} |H_{iv}(x)| dx < +\infty.$$

Then system (1.1) has at least one solution  $y_{pl}(x)$  that satisfies the asymptotic relation

$$y_{pl}(x) = Y_{pl}(x) + \Delta^{-1}(x)\bar{o}(U_{pl}(x))$$

when  $x \rightarrow +\infty$ .

*Proof.* By the definition of the vector function  $Y_{pl}(x) = (Y_{pl})_j)_{j=1}^n$ , its components are determined as follows: for every  $\xi \in \{1, \dots, k\}$ , we have

$$\begin{aligned} (Y_{pl})_{N_p+2\xi-1}(x) &= \sum_{\nu=\xi}^k [C_{2\nu-1}B_{\xi\nu}^{p2}(x) - C_{2\nu}B_{\xi\nu}^{p1}(x)] \exp \int_a^x \omega_{p\xi}(s)ds, \\ (Y_{pl})_{N_p+2\xi}(x) &= \sum_{\nu=\xi}^k [C_{2\nu-1}B_{\xi\nu}^{p1}(x) - C_{2\nu}B_{\xi\nu}^{p2}(x)] \exp \int_a^x \omega_{p\xi}(s)ds, \\ (Y_{pl})_j(x) &= 0 \quad (j \neq N_p+1, \dots, N_p+l) \end{aligned}$$

when  $p \in \{1, \dots, r\}$ , and

$$\begin{aligned} (Y_{pl})_{N_p+\xi}(x) &= \sum_{\nu=\xi}^k C_{\nu}B_{\xi\nu}^p(x) \exp \int_a^x \omega_{p\xi}(s)ds, \\ (Y_{pl})_j(x) &= 0 \quad (j \neq N_p+1, \dots, N_p+l) \end{aligned}$$

when  $p \in \{r+1, \dots, s\}$ .

Under condition (i) of the theorem, the set  $J_0$  decomposes as the union of disjoint subsets  $J_{1pl}$  and  $J_{2pl}$ ,

$$J_0 = J_{1pl} \bigcup J_{2pl},$$

such that if  $(i, m) \in J_{1pl}$ , then estimate (3.4) is true and if  $(i, m) \in J_{2pl}$ , then relation (3.5) is correct. Therefore,

$$\Phi(x) = \Phi^{(1)}(x) + \Phi^{(2)}(x)$$

with

$$\Phi^{(\rho)}(x) = \text{diag} [\Phi_1^{(\rho)}(x), \dots, \Phi_s^{(\rho)}(x)] \quad (\rho = 1, 2),$$

where  $\Phi_i^{(\rho)}(x)$  ( $i \in \{1, \dots, s\}$ ;  $\rho = 1, 2$ ) are  $m_i$ th order matrices with the elements

$$\left(\Phi_i^{(\rho)}(x)\right)_{jm} = \begin{cases} (\Phi_i(x))_{jm} & \text{for } (i, m) \in J_{\rho pl} \\ 0 & \text{for } (i, m) \notin J_{\rho pl}. \end{cases}$$

Let us show that, for a sufficiently large number  $x_3 \geq x_2$  (where  $x_2$  is as in Theorem 1), the system of integral equations

$$\begin{aligned} z(x) = & \sum_{\rho=1}^2 \int_{a_\rho}^x Q^\rho(x, \tau) \left[ G(\tau) \left\{ z(\tau) + \frac{1}{U_{pl}(\tau)} \right\} + \right. \\ & \left. + \frac{1}{U_{pl}(\tau)} \Delta(\tau) F(\tau, \Delta^{-1}(\tau) U_{pl}(\tau) z(\tau) + Y_{pl}(\tau)) \right] d\tau \end{aligned} \quad (3.5)$$

with

$$\begin{aligned} Q^\rho(x, \tau) &= \frac{U_{pl}(\tau)}{U_{pl}(x)} \Delta(x) \Phi^{(\rho)}(x) \Phi^{-1}(\tau) \Delta^{-1}(\tau), \\ G(\tau) &= \Delta(\tau) R(\tau) \Delta^{-1}(\tau), \end{aligned}$$

and

$$a_\rho = \begin{cases} x_3 & \text{if } \rho = 1 \\ +\infty & \text{if } \rho = 2, \end{cases}$$

has at least one solution  $z = (z_k)_{k=1}^n \in C([x_3, +\infty[; \mathbb{R}_b^n)$  vanishing when  $x \rightarrow +\infty$ .

Here,

$$Q^{(\rho)}(x, \tau) = \text{diag} [Q_1^{(\rho)}(x, \tau), \dots, Q_s^{(\rho)}(x, \tau)] \quad (\rho = 1, 2),$$

$$Q_i^\rho(x, \tau) = \frac{U_{pl}(\tau)}{U_{pl}(x)} \Delta_i(x) \Phi_i^{(\rho)}(x) \Phi_i^{-1}(\tau) \Delta_i^{-1}(\tau) \quad (i = 1, \dots, s; \rho = 1, 2),$$

For  $i = 1, \dots, r$ , the matrices  $Q_i^\rho$  consist of the blocks

$$\left(Q_i^\rho(x, \tau)\right)_{\nu j} = \begin{cases} \begin{pmatrix} \left(Q_i^\rho(x, \tau)\right)_{\nu j}^{(1)} & -\left(Q_i^\rho(x, \tau)\right)_{\nu j}^{(2)} \\ \left(Q_i^\rho(x, \tau)\right)_{\nu j}^{(2)} & \left(Q_i^\rho(x, \tau)\right)_{\nu j}^{(1)} \end{pmatrix} & \text{if } 1 \leq \nu \leq j \leq n_i \\ O_2 & \text{if } 1 \leq j < \nu \leq n_i, \end{cases}$$

where

$$\begin{aligned} \left( Q_i^{(\rho)}(x, \tau) \right)_{kj}^{(\mu)} &= \sum_{\substack{m=k \\ (i,m) \in J_{\rho pq}}}^j q_{im}(x, \tau) \left( \frac{B_{1k}^i(x) B_{km}^{i3-\mu}(x)}{B_{1m}^i(x)} \frac{B_{1m}^i(\tau) D_{mj}^{i1}(\tau)}{B_{1j}^i(\tau)} + \right. \\ &\quad \left. + (-1)^\mu \frac{B_{1k}^i(x) B_{km}^{i\mu}(x)}{B_{1m}^i(x)} \frac{B_{1m}^i(\tau) D_{mj}^{i2}(\tau)}{B_{1j}^i(\tau)} \right) \quad (\mu = 1, 2), \end{aligned}$$

and, for  $i = r + 1, \dots, s$ , they consist of the blocks

$$\left( Q_i^o(x, \tau) \right)_{vj} = \begin{cases} \sum_{\substack{m=v \\ (i,m) \in J_{\rho pl}}}^j q_{im}(x, \tau) \frac{B_{1v}^i(x) B_{vm}^i(x)}{B_{1m}^i(x)} \frac{B_{1m}^i(\tau) D_{mj}^i(\tau)}{B_{1j}^i(\tau)} & \text{if } 1 \leq v \leq j \leq n_i \\ 0 & \text{if } 1 \leq j < v \leq n_i. \end{cases}$$

Using the notation

$$z_m(x) = \int_{x_3}^x \lambda_{1m}(\tau, x, z(\tau)) d\tau + \int_x^{+\infty} \lambda_{2m}(\tau, x, z(\tau)) d\tau \quad (m = 1, \dots, n), \quad (3.6)$$

we rewrite (3.5) as (3.6), where the functions  $\lambda_{\rho m}$  ( $\rho = 1, 2$ ,  $m = 1, \dots, n$ ) are given in the following way: for all  $v \in \{1, \dots, n_\eta\}$ , we have

$$\begin{aligned} \lambda_{\rho N_\eta + 2v - 1}(\tau, x, z) &= \sum_{i=1}^s \sum_{m=1}^{m_i} \sum_{\xi=1}^s \sum_{j=v}^{n_\eta} \left\{ \left( Q_\eta^{(\rho)}(x, \tau) \right)_{vj}^{(1)} \left( G_{\eta\xi}(\tau) \right)_{2j-1m} - \right. \\ &\quad \left. - \left( Q_\eta^{(\rho)}(x, \tau) \right)_{vj}^{(2)} \left( G_{\eta\xi}(\tau) \right)_{2jm} \right\} z_{N_i+m} + \\ &+ \sum_{m=1}^l \sum_{j=v}^{n_\eta} \left\{ \left( Q_\eta^{(\rho)}(x, \tau) \right)_{vj}^{(1)} \left( G_{\eta p}(\tau) \right)_{2j-1m} - \right. \\ &\quad \left. - \left( Q_\eta^{(\rho)}(x, \tau) \right)_{vj}^{(2)} \left( G_{\eta p}(\tau) \right)_{2jm} \right\} \frac{1}{U_{pl}(\tau)} \left( Y_{pl}(\tau) \right)_{N_p+m} + \\ &+ \sum_{j=v}^{n_\eta} \frac{1}{U_{pl}(\tau)} \left\{ \left( Q_\eta^{(\rho)}(x, \tau) \right)_{vj}^{(1)} H_{\eta 2j-1}(\tau) \widetilde{F}_{N_\eta+2j-1}(\tau, z) - \right. \\ &\quad \left. - \left( Q_\eta^{(\rho)}(x, \tau) \right)_{vj}^{(2)} \sum H_{\eta 2j}(\tau) \widetilde{F}_{N_\eta+2j}(\tau, z) \right\} \end{aligned}$$

and

$$\begin{aligned}
\lambda_{\rho N_{\eta}+2\nu}(\tau, x, z) &= \sum_{i=1}^s \sum_{m=1}^{m_i} \sum_{\xi=1}^s \sum_{j=\nu}^{n_{\eta}} \left\{ \left( Q_{\eta}^{(\rho)}(x, \tau) \right)_{\nu j}^{(2)} \left( G_{\eta \xi}(\tau) \right)_{2j-1 m} + \right. \\
&\quad \left. + \left( Q_{\eta}^{(\rho)}(x, \tau) \right)_{\nu j}^{(1)} \left( G_{\eta \xi}(\tau) \right)_{2j m} \right\} z_{N_i+m} + \\
&\quad + \sum_{m=1}^l \sum_{j=\nu}^{n_{\eta}} \left\{ \left( Q_{\eta}^{(\rho)}(x, \tau) \right)_{\nu j}^{(2)} \left( G_{\eta p}(\tau) \right)_{2j-1 m} + \right. \\
&\quad \left. + \left( Q_{\eta}^{(\rho)}(x, \tau) \right)_{\nu j}^{(1)} \left( G_{\eta p}(\tau) \right)_{2j m} \right\} \frac{1}{U_{pl}(\tau)} \left( Y_{pl}(\tau) \right)_{N_p+m} + \\
&\quad + \sum_{j=\nu}^{n_{\eta}} \frac{1}{U_{pl}(\tau)} \left\{ \left( Q_{\eta}^{(\rho)}(x, \tau) \right)_{\nu j}^{(2)} H_{\eta 2j-1}(\tau) \widetilde{F}_{N_{\eta+2j-1}}(\tau, z) + \right. \\
&\quad \left. + \left( Q_{\eta}^{(\rho)}(x, \tau) \right)_{\nu j}^{(1)} H_{\eta 2j}(\tau) \widetilde{F}_{N_{\eta+2j}}(\tau, z) \right\},
\end{aligned}$$

when  $\eta \in \{1, \dots, r\}$ ; and we have

$$\begin{aligned}
\lambda_{\rho N_{\eta}+\nu}(\tau, x, z) &= \sum_{i=1}^s \sum_{m=1}^{m_i} \sum_{\xi=1}^s \sum_{j=\nu}^{n_{\eta}} \left( Q_{\eta}^{(\rho)}(x, \tau) \right)_{\nu j} \left( G_{\eta \xi}(\tau) \right)_{j m} z_{N_i+m} + \\
&\quad + \sum_{m=1}^l \sum_{j=\nu}^{n_{\eta}} \left( Q_{\eta}^{(\rho)}(x, \tau) \right)_{\nu j} \left( G_{\eta p}(\tau) \right)_{j m} \frac{1}{U_{pl}(\tau)} \left( Y_{pl}(\tau) \right)_{N_p+m} + \\
&\quad + \sum_{j=\nu}^{n_{\eta}} \frac{1}{U_{pl}(\tau)} \left( Q_{\eta}^{(\rho)}(x, \tau) \right)_{\nu j} H_{\eta j}(\tau) \widetilde{F}_{N_{\eta+j}}(\tau, z)
\end{aligned}$$

when  $\eta \in \{r+1, \dots, s\}$ . Here,

$$\widetilde{F}_j(\tau, z) = F_j(\tau, \Delta^{-1}(\tau) U_{pl}(\tau) z + Y_{pl}(\tau)) \quad (j = 1, \dots, n).$$

Let  $c_1$  be chosen as in Lemma 1 and

$$\begin{aligned} \zeta(x, \tau) = & \frac{c_1}{c_2} \left\{ \sum_{i=1}^r \sum_{v=1}^{n_i} \sum_{j=v}^{n_i} \sum_{\substack{m=v \\ (i,m) \in J_{1pl}}}^j |q_{im}(x, \tau)| \left( \left| \frac{B_{1m}^i(\tau) D_{mj}^{i1}(\tau)}{B_{1j}^i(\tau)} \right| + \right. \\ & \left. + \left| \frac{B_{1m}^i(\tau) D_{mj}^{i2}(\tau)}{B_{1j}^i(\tau)} \right| \right) + \sum_{i=r+1}^s \sum_{v=1}^{n_i} \sum_{j=v}^{n_i} \sum_{\substack{m=v \\ (i,m) \in J_{1pl}}}^j |q_{im}(x, \tau)| \left| \frac{B_{1m}^i(\tau) D_{mj}^i(\tau)}{B_{1j}^i(\tau)} \right| \right\} \end{aligned}$$

if  $J_{1pl} \neq \emptyset$ , and  $\zeta(x, \tau) = 0$  in the case where  $J_{1pl} = \emptyset$ . We have put

$$c_2 = \sum_{i=1}^r (2c_1)^{n_i} n_i^2 \theta_i + \sum_{i=r+1}^s c_1 (2c_1)^{n_i-1} n_i^2 \theta,$$

where

$$\theta_i = \max\{\theta_{im} : 1 \leq m \leq n_i\} \quad (i = 1, \dots, s).$$

It is easy to see that

$$\zeta(x, \tau) \leq 1$$

for  $x \geq \tau \geq x_3$ .

For  $p \in \{1, \dots, r\}$ , we set

$$\begin{aligned} q(\tau) = & c_3 \left\{ \sum_{\eta=1}^s \sum_{v=1}^{n_\eta} \left( b \sum_{i=1}^s \sum_{m=1}^{m_i} \sum_{\xi=1}^s \sum_{j=v_\eta}^{m_\eta} |(G_{\eta\xi}(\tau))_{jm}| + \right. \right. \\ & + \frac{1}{U_{pl}(\tau)} \sum_{\theta=1}^l C_\theta \sum_{m=1}^k \sum_{j=2v-1}^{m_\eta} \sum_{\xi=m}^k \exp \int_a^\tau \omega_{pm}(s) ds \left( |B_{m\xi}^{p1}(\tau)| + |B_{m\xi}^{p2}(\tau)| \right) \times \\ & \left. \left. \times \left( |(G_{\eta p}(\tau))_{j2m-1}| + |(G_{\eta p}(\tau))_{j2m}| \right) + \sum_{j=2v-1}^{m_\eta} \frac{\phi(\tau)}{U_{pl}(\tau)} |H_{\eta j}(\tau)| \right) \right\} \end{aligned}$$

and, for  $p \in \{r+1, \dots, s\}$ , we put

$$\begin{aligned} q(\tau) = & c_3 \left\{ \sum_{\eta=1}^s \sum_{v=1}^{n_\eta} \left[ b \sum_{i=1}^s \sum_{m=1}^{m_i} \sum_{\xi=1}^s \sum_{j=v_\eta}^{m_\eta} |(G_{\eta\xi}(\tau))_{jm}| + \right. \right. \\ & + \frac{1}{U_{pl}(\tau)} \sum_{\theta=1}^l C_\theta \sum_{m=1}^k \sum_{j=v}^{m_\eta} \sum_{\xi=m}^k \exp \int_{x_0}^\tau \omega_{pm}(s) ds |B_{m\xi}^p(\tau)| |(G_{\eta p}(\tau))_{jm}| + \\ & \left. \left. + \sum_{j=2v-1}^{m_\eta} \frac{\phi(\tau)}{U_{pl}(\tau)} |H_{\eta j}(\tau)| \right] \right\}. \end{aligned}$$

Using now the conditions assumed in Theorem 2, we choose the number  $x_3$  so large that the relation  $\int_{x_3}^{+\infty} q(\tau) d\tau \leq b$  for each  $z \in \mathbb{R}_b^n$  be satisfied. Then

$$|\lambda_{1m}(\tau, x, z)| \leq \zeta(x, \tau) q(\tau) \quad \text{for } x_3 \leq \tau \leq x \quad (m = 1, \dots, n)$$

and

$$|\lambda_{2m}(\tau, x, z)| \leq q(\tau) \quad \text{for } x_3 \leq x \leq \tau \quad (m = 1, \dots, n).$$

Hence, system (3.6) has at least one solution  $z_{pl}(x)$  such that  $\lim_{x \rightarrow +\infty} z_{pl}(x) = 0$ . To complete the proof, we define the function  $y_{pl}(x)$  as follows:

$$z_{pl}(x) = \frac{\Delta(x)y_{pl}(x) - \Delta(x)Y_{pl}(x)}{U_{pl}(x)}.$$

The theorem is proved. □

*Remark 1.* If the function  $U_{pl}(x)$  from Theorem 2 satisfies the condition

$$\sum_{i=N_p+1}^{N_p+j} |(u_{N_p+j}(x))_i| = O(U_{pl}(x)) \quad \text{for each } j \in \{1, \dots, l\}$$

when  $x \rightarrow +\infty$ , then condition (ii) of Theorem 2 can be dropped.

#### REFERENCES

- [1] KIGURADZE, I. T. and CHANTURIA T. A.: *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Nauka, Moscow, 1990.
- [2] CHANTURIA, T. A.: *On the asymptotic properties of solutions of perturbed linear systems of differential equations*, Ann. Matem. Pure Appl., **94**, (1972), 41–62.
- [3] KITAMURA YUICHI: *Remarks on the asymptotic relationships of two systems of ordinary differential equations*, Hiroshima Math. J., **6**, (1976), 403–420.
- [4] BRAUER, F.: *On the asymptotic behaviour of perturbed linear systems*, J. Diff. Equat., **6**, (1960), 142–153.
- [5] BRAUER, F. and WONG, J.: *On the asymptotic relationships between solutions of two systems of ordinary differential equations*, J. Diff. Equat., **6**, (1969), 527–534.
- [6] KOSTIN, A.V.: *On existence of finite partial solutions and vanishing solutions of systems of ordinary differential equations*, Differents. Uravnenija, **1**(5), (1965), 585–604.
- [7] EVTUKHOV, V.M.: *Asymptotic integrating of solutions of a certain class of systems of linear differential equations*, Nelinijni kolyvannja, **3**, No 3, (2000), 334–357.

#### Authors' Addresses

**V. M. Evtukhov:**

ODESSA I. I. MECHNIKOV NATIONAL UNIVERSITY, 65026 ODESSA, UKRAINE,  
E-mail address: emden@farlep.net

**L. I. Kusick:**

Current address: Odessa National Maritime University, 34, Mechnikova Str., 65029, Odessa, Ukraine,  
E-mail address: emden@farlep.net



## STABILITY WITH RESPECT TO THE DOMAIN OF A NONLINEAR VARIATIONAL INEQUALITY

DANIELA INOAN

[Received: August 22, 2002]

**ABSTRACT.** In this paper we investigate the stability with respect to the domain perturbations of a class of nonlinear variational inequality.

*Mathematics Subject Classification:* Primary 47J20, 90C31; Secondary 49Q12

*Keywords:* parametric variational inequalities, mapping method

### 1. INTRODUCTION

The mathematical theory of shape optimization benefits of a vast literature; the optimal shape design for systems governed by PDEs was studied in a great many papers, see for example [1], [2], [3]. Also, problems where variational or hemivariational inequalities appear were treated in [1], [4], [5], [6] and others.

A problem of shape optimization for a system described by a variational inequality can be looked at as a problem of optimal control in which the role of the control is played by sets from a class of admissible domains and the variational inequality appears as the state equation. In this context it is interesting to study the behavior of solutions to the variational inequality when the domain is perturbed.

The purpose of this paper is to study the stability with respect to the domain perturbations of a nonlinear variational inequality of the form

$$\begin{aligned} &\text{Find } u_\Omega \in K(\Omega) \text{ such that} \\ &\langle \mathcal{A}(\Omega, u_\Omega), v - u_\Omega \rangle \geq 0, \quad \forall v \in K(\Omega), \end{aligned}$$

where  $\Omega$  is a bounded open subset of  $\mathbb{R}^N$ ,  $K(\Omega)$  is a closed, convex set in the Sobolev space  $H^1(\Omega)$  and  $\mathcal{A}$  is a nonlinear operator of a special form,  $\mathcal{A}(\Omega, u) \in (H^1(\Omega))^*$ . If  $\Omega_0$  is fixed in the class of the admissible domains and  $u_0$  is the solution of  $(VI)_{\Omega_0}$ , the following problem arises: Is there a neighborhood  $V_0$  of  $\Omega_0$  (in a sense that will be specified) and a mapping  $\theta$  defined on this, continuous at  $\Omega_0$ , with  $\theta(\Omega_0) = u_0$  and such that  $\theta(\Omega)$  is a solution of  $(VI)_\Omega$  for each  $\Omega \in V_0$ ?

General stability results for parametric variational inequalities under small perturbations of the parameter have been given in [7], [8]. One of these results is presented in Section 3 and used in the paper.

In order to define the topological space of the parameters as a space of functions we use the mapping method (Section 2) for which the basic results were established by Murat and Simon in [3].

Finally, in Section 4 we present the main result of the paper and an example for a linear variational inequality.

## 2. THE MAPPING METHOD

The mapping method consists in defining the class of admissible domains as images of a fixed set. The main notions and properties were established in [3]; we present some of them following [4], [2].

Let  $C \subset \mathbb{R}^N$  be a bounded, open set, with  $\partial C$  of class  $W^{i,\infty}$ ,  $i \geq 1$  and such that  $\text{int}\bar{C} = C$ .

We consider the following spaces:

$$\begin{aligned} W^{k,\infty}(\mathbb{R}^N)^N &= \{\phi \mid D^\alpha \phi \in L^\infty(\mathbb{R}^N)^N \forall \alpha \text{ with } 0 \leq |\alpha| \leq k\} \\ \mathcal{F}^{k,\infty} &= \{S : \mathbb{R}^N \rightarrow \mathbb{R}^N \mid S \text{ bijective, } S - I, S^{-1} - I \in W^{k,\infty}(\mathbb{R}^N)^N\} \\ \mathcal{O}^{k,\infty} &= \{\Omega \mid \Omega = S(C), S \in \mathcal{F}^{k,\infty}\} \end{aligned}$$

and the norm in  $W^{k,\infty}(\mathbb{R}^N)^N$ :

$$\|S\|_{k,\infty} = \text{ess sup}_{x \in \mathbb{R}^N} \left( \sum_{0 \leq |\alpha| \leq k} |D^\alpha S|_N^2 \right)^{1/2}$$

$\|\cdot\|_N$  represents the norm on  $\mathbb{R}^N$ . The norm on  $L^\infty(\mathbb{R}^N)$  (or  $L^\infty(\mathbb{R}^N)^{N^2}$ ) will be denoted by  $\|\cdot\|_\infty$ .

$\mathcal{O}^{k,\infty}$  consists in a family of bounded and open sets. If  $\partial C$  is of class  $W^{k,\infty}$ ,  $k \geq 1$  then  $\partial\Omega$  is of class  $W^{k,\infty}$  also.

We define on  $\mathcal{O}^{k,\infty} \times \mathcal{O}^{k,\infty}$  a function

$$\delta_{k,\infty}(\Omega_1, \Omega_2) = \inf_{S \in \mathcal{F}^{k,\infty}, S(\Omega_1) = \Omega_2} (\|S - I\|_{k,\infty} + \|S^{-1} - I\|_{k,\infty}).$$

It can be proved (see [4], [2]) that there exists a positive constant  $\mu_k$  such that

$$d_{k,\infty}(\Omega_1, \Omega_2) = (\min\{\delta_{k,\infty}(\Omega_1, \Omega_2), \mu_k\})^{1/2}$$

is a complete metric on  $\mathcal{O}^{k,\infty}$ .

**Remark 1.** It is known (see [3]) that  $\Omega_n \rightarrow \Omega$  in  $\mathcal{O}^{k,\infty}$  if and only if there exist  $S_n, S \in \mathcal{F}^{k,\infty}$  such that  $\Omega_n = S_n(C)$ ,  $\Omega = S(C)$  and  $S_n \rightarrow S$ ,  $S_n^{-1} \rightarrow S^{-1}$  in  $W^{k,\infty}(\mathbb{R}^N)^N$ .

The next Lemma (see [4]) summarizes some properties used in the paper:

**Lemma 1.** (a) If  $S \in \mathcal{F}^{k,\infty}$ ,  $\Omega = S(C)$ , then  $u \in H^1(\Omega)$  if and only if  $u \circ S \in H^1(C)$ . If  $u_n \rightarrow u$  in  $H^1(\Omega)$  (or in  $H^1(C)$ ) then  $u_n \circ S \rightarrow u \circ S$  in  $H^1(C)$  (or  $u_n \circ S^{-1} \rightarrow u \circ S^{-1}$  in  $H^1(\Omega)$ ).

(b) Let  $k \geq 1$ ,  $u \in H^1(\mathbb{R}^N)$ . The mapping  $S \mapsto u \circ S$  is continuous from  $\mathcal{F}^{k,\infty}$  to  $H^1(\mathbb{R}^N)$ .

(c) Let  $k \geq 1$ . The mappings  $S \mapsto JS^{-1}$  and  $S \mapsto \det JS$  are continuous from  $\mathcal{F}^{k,\infty}$  to  $W^{k-1,\infty}(\mathbb{R}^N)$  (with  $JS$  we denoted the Jacobian matrix of  $S$ ).

(d) If  $u, v \in H^1(\Omega)$  and  $\Omega = S(C)$  we have:

$$\|u \circ S - v \circ S\|_{H^1(C)} \leq (\|\det JS^{-1}\|_\infty)^{1/2} (\|JS\|_\infty + 1) \|u - v\|_{H^1(\Omega)}.$$

### 3. PARAMETRIC VARIATIONAL INEQUALITIES

Let  $H$  be a real, reflexive Banach space and denote with  $H^*$  its dual. Let  $W$  be a topological space, let  $T : W \times H \rightarrow H^*$  and let  $K : W \rightarrow 2^H$  be a set-valued map. For a given parameter  $w \in W$  we consider the variational inequality:

(VIP) $_w$  Find an element  $x(w) \in K(w)$  such that

$$\langle T(w, x(w)), y - x(w) \rangle \geq 0, \quad \forall y \in K(w).$$

For a fixed  $w_0 \in W$  (the initial value of the parameter), suppose that  $x_0 \in K(w_0)$  is the solution of the corresponding problem (VIP) $_{w_0}$ . We say that the problem (VIP) $_{w_0}$  is *stable under perturbations* if there exists a neighborhood  $W_0$  of  $w_0$  and a mapping  $x : W_0 \rightarrow H$ , continuous at  $w_0$ , with  $x(w_0) = x_0$  and such that, for each  $w \in W_0$ ,  $x(w)$  is a solution of (VIP) $_w$ .

**Definition 1.** The map  $T : W \times H \rightarrow H^*$  is called consistent in  $w$  at  $(w_0, x_0)$  if, for each  $0 < r \leq 1$ , there exists a neighborhood  $W_r$  of  $w_0$  and a function  $\beta : W_r \rightarrow \mathbb{R}$ , continuous at  $w_0$ , with  $\beta(w_0) = x_0$  such that for each  $w \in W_r$ , there exists  $y_w \in K(w)$  such that

$$\|y_w - x_0\| \leq \beta(w)$$

and

$$\langle T(w, y_w), z - y_w \rangle + \beta(w) \|z - y_w\| \geq 0,$$

for each  $z \in K(w)$  such that  $r < \|z - y_w\| \leq 2$ .

**Definition 2.** The maps  $T(w, \cdot) : H \rightarrow H^*$  are called uniformly strongly monotone on  $W_0 \subset W$  if there exists a positive constant  $\alpha$  such that for all  $w \in W_0$  and  $x, y \in H$ ,  $x \neq y$  we have :

$$\langle T(w, x) - T(w, y), x - y \rangle \geq \alpha \|x - y\|^2.$$

The following Theorem is a particular case of a Theorem proved in [7].

**Theorem 1.** In the above notations, let the set  $K(w)$  be closed and convex for each  $w \in W$ . Consider  $w_0 \in W$  and  $x_0 \in K(w_0)$  fixed. Suppose that:

- (i)  $x_0$  is a solution of (VIP) $_{w_0}$ ;
- (ii)  $T$  is consistent in  $w$  at  $(w_0, x_0)$ ;

(iii) there exists a neighborhood  $V$  of  $w_0$  such that the maps  $T(w, \cdot)$  are uniformly strongly monotone, continuous from the line segments of  $H$  to the weak topology of  $H^*$ , for all  $w \in V$  and  $x \in K(w)$ .

Then the problem  $(VIP)_{w_0}$  is stable under perturbations.

#### 4. STABILITY WITH RESPECT TO THE DOMAIN OF PERTURBATIONS

Let  $\Omega \subset \mathbb{R}^N$  be a bounded and open set. We consider the problem:

$(VI)_\Omega$  Find  $u_\Omega \in K(\Omega)$  such that

$$\begin{aligned} & \int_{\Omega} A(x, \nabla u_\Omega(x)) \cdot (\nabla v(x) - \nabla u_\Omega(x)) dx \\ & + \int_{\Omega} a(x, u_\Omega(x))(v(x) - u_\Omega(x)) dx \geq 0, \quad \forall v \in K(\Omega) \end{aligned}$$

in the following notations and hypotheses:

(H<sub>1</sub>)  $K(\Omega) \subset H^1(\Omega)$  is a closed, convex, nonempty set,

(H<sub>2</sub>)  $A = (a_1, \dots, a_N)$  with  $a_j : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ ,  $a : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$  having the properties:

(P1)  $a_j(\cdot, \cdot)$ ,  $j = 1, \dots, N$  and  $a(\cdot, \cdot)$  are measurable with respect to the first variable and continuous with respect to the second one,

(P2)  $|a_j(x, \xi)| \leq c(k(x) + \|\xi\|_N)$  and  $|a(x, \eta)| \leq c_1(k_1(x) + |\eta|)$  a.e.  $x \in \mathbb{R}^N$ , for all  $\xi \in \mathbb{R}^N$ , for all  $\eta \in \mathbb{R}$ , with  $c, c_1$  positive constants and  $k, k_1$  functions in  $L^2(D)$  (for any bounded and open set  $D$ ).

$$(P3) \sum_{j=1}^N a_j(x, \xi) \xi_j \geq c_2 \|\xi\|_N^2 - c_3, \text{ a.e. } x \in \mathbb{R}^N, \text{ for all } \xi \in \mathbb{R}^N,$$

$$(P4) \sum_{j=1}^N (a_j(x, \xi) - a_j(x, \tilde{\xi}))(\xi_j - \tilde{\xi}_j) \geq \gamma_1 \|\xi - \tilde{\xi}\|_N^2, \text{ a.e. } x \in \mathbb{R}^N, \text{ for all } \xi, \tilde{\xi} \in \mathbb{R}^N$$

and  $(a(x, \eta) - a(x, \tilde{\eta}))(\eta - \tilde{\eta}) \geq \gamma_2 |\eta - \tilde{\eta}|^2$ , a.e.  $x \in \mathbb{R}^N$ , for all  $\eta, \tilde{\eta} \in \mathbb{R}$ .

**Theorem 2.** *In the conditions stated above, the variational inequality  $(VI)_\Omega$  has at least one solution (see [9], p. 74).*

Let  $S \in \mathcal{F}^{k, \infty}$  such that  $\Omega = S(C)$ . Making the transform  $x = S(X)$  in  $(VI)_\Omega$  we get an equivalent problem on the fixed set  $C$ :

Find  $u_S \in K_S \subset H^1(C)$  such that

$$\begin{aligned} & \int_C A(S(X), JS^{-t}(X) \nabla u_S(X)) \cdot JS^{-t}(X) (\nabla v(X) - \nabla u_S(X)) \det JS(X) dX \\ & + \int_C a(S(X), u_S(X))(v(X) - u_S(X)) \det JS(X) dX \geq 0, \quad \forall v \in K_S, \end{aligned}$$

where  $K_S = SK(\Omega) = \{u \circ S \mid u \in K(\Omega)\}$  is closed and convex.

We define the operator  $\mathcal{A} : \mathcal{F}^{k,\infty} \times H^1(C) \rightarrow (H^1(C))^*$  as

$$\begin{aligned} \langle \mathcal{A}(S, u), v \rangle &= \int_C A(S(X), JS^{-t}(X) \nabla u(X)) \cdot JS^{-t}(X) \nabla v(X) \det JS(X) dX \\ &\quad + \int_C a(S(X), u(X)) v(X) \det JS(X) dX, \quad \forall v \in H^1(C). \end{aligned}$$

Then the variational inequality can be written:

$$(VI)_S \quad \text{Find } u_S \in K_S \text{ such that } \langle \mathcal{A}(S, u_S), v - u_S \rangle \geq 0, \quad \forall v \in K_S.$$

We will apply Theorem 1 to this family of variational inequalities, considering  $S \in \mathcal{F}^{k,\infty}$  as the parameter.

Other hypotheses that we impose are:

(H<sub>3</sub>)  $|a_j(x, \xi) - a_j(\tilde{x}, \tilde{\xi})| \leq \psi(x, \tilde{x})(\|\xi\|_N + \|\tilde{\xi}\|_N) + \phi(x, \tilde{x})\|\xi - \tilde{\xi}\|_N + \sigma(x, \tilde{x})$ , for all  $j = 1, \dots, N$ ,  $x, \tilde{x}, \xi, \tilde{\xi} \in \mathbb{R}^N$ ; where  $\psi(\cdot, \cdot)$ ,  $\phi(\cdot, \cdot)$ ,  $\sigma(\cdot, \cdot)$  are nonnegative functions belonging to  $C(\mathbb{R}^N \times \mathbb{R}^N) \cap L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$  and  $\psi(x, \tilde{x}) = \psi(\tilde{x}, x)$ ,  $\psi(x, x) = \sigma(x, x) = 0$ .

(H<sub>4</sub>)  $|a(x, \eta) - a(\tilde{x}, \tilde{\eta})| \leq \chi(x, \tilde{x})(|\eta| + |\tilde{\eta}|) + \mu(x, \tilde{x})|\eta - \tilde{\eta}|$ , where  $\chi$  has the same properties as  $\psi$  and  $\mu$  has the same properties as  $\phi$ .

**Lemma 2.** Suppose that the hypotheses (H<sub>1</sub>)-(H<sub>4</sub>) take place. If  $\psi \in C(\mathbb{R}^N \times \mathbb{R}^N) \cap L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$  and  $S_n, S_0 \in \mathcal{F}^{k,\infty}$  with  $S_n \rightarrow S_0$ ,  $S_n^{-1} \rightarrow S_0^{-1}$  in  $W^{k,\infty}(\mathbb{R}^N)^N$ , then

$$\|\psi(S_n(\cdot), S_0(\cdot)) - \psi(S_0(\cdot), S_0(\cdot))\|_{L^\infty(C)} \rightarrow 0$$

when  $n \rightarrow \infty$ .

*Proof.* We have that  $\psi$  is uniformly continuous on every bounded set of  $\mathbb{R}^N \times \mathbb{R}^N$ , which implies:

For each  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for each  $x_1, \tilde{x}_1, x_2, \tilde{x}_2 \in D$  (a bounded closed subset of  $\mathbb{R}^N$ ), with  $\|x_1 - x_2\|_N < \delta$  and  $\|\tilde{x}_1 - \tilde{x}_2\|_N < \delta$  we have

$$|\psi(x_1, \tilde{x}_1) - \psi(x_2, \tilde{x}_2)| < \varepsilon.$$

$S_n \rightarrow S_0$  in  $L^\infty(C)$ , that is  $\inf_{E \subset C, |E|=0} \sup_{X \in C \setminus E} \|S_n(X) - S_0(X)\|_N \rightarrow 0$ .

For  $\delta > 0$  there exists  $n_0 \in \mathbb{N}$  such that for  $n > n_0$

$$\inf_{E \subset C, |E|=0} \sup_{X \in C \setminus E} \|S_n(X) - S_0(X)\|_N < \delta.$$

This implies that, for  $n > n_0$ , there exists a set  $E_n \subset C$ ,  $|E_n| = 0$  such that

$$\|S_n(X) - S_0(X)\|_N < \delta \text{ for each } X \in C \setminus E_n.$$

Using the uniform continuity and the fact that  $S_n(C)$  and  $S_0(C)$  are in a bounded set of  $\mathbb{R}^N$ , we get: There exists  $n_0 \in \mathbb{N}$  such that for all  $n > n_0$ , exists  $E_n \subset C$ ,  $|E_n| = 0$  with

$$|\psi(S_n(X), S_0(X)) - \psi(S_0(X), S_0(X))| < \varepsilon, \quad \forall X \in C \setminus E_n$$

Denote  $E = \cup_{n>n_0} E_n$ ,  $|E| = 0$ ,  $C \setminus E \subset C \setminus E_n$ , so

$$\begin{aligned} & \sup_{X \in C \setminus E} |\psi(S_n(X), S_0(X)) - \psi(S_0(X), S_0(X))| \\ & \leq \sup_{X \in C \setminus E_n} |\psi(S_n(X), S_0(X)) - \psi(S_0(X), S_0(X))| < \varepsilon \end{aligned}$$

for all  $n > n_0$ . Finally, for  $\varepsilon > 0$  there exists  $n_0 \in \mathbb{N}$  such that for all  $n > n_0$  we have

$$\inf_{E \subset C, |E|=0} \sup_{X \in C \setminus E} |\psi(S_n(X), S_0(X)) - \psi(S_0(X), S_0(X))| \leq \varepsilon,$$

which completes the proof.  $\square$

Suppose that the next hypothesis is satisfied in all the paper:

**(H<sub>5</sub>)** There exists a neighborhood  $V_0$  of  $S_0$  and a positive constant  $\delta$  such that for all  $S_1, S_2 \in V_0$  and  $u_1 \in K_{S_1}$ , there exists  $u_2 \in K_{S_2}$  such that

$$\|u_1 - u_2\|_{H^1(C)} \leq \delta(\|S_1 - S_2\|_{k,\infty} + \|S_1^{-1} - S_2^{-1}\|_{k,\infty}).$$

**Example 1.** Let  $K(\Omega) = \{v \in H_0^1(\Omega) \mid \nabla v = 0, v(x) \geq \varphi(x) \text{ a.e. } x \in \Omega\}$ , with  $\varphi \in H_0^1(\Omega) \cap C(\Omega)$  a Lipschitz function with  $\nabla \varphi = 0$  on  $\Omega$ .

We have:

$$K_S = \{\check{v} \circ S \mid \check{v} \in K(\Omega)\} = \{v \in H_0^1(C) \mid \nabla v = 0, v(X) \geq \varphi(S(X)) \text{ a.e. } X \in C\}$$

Let  $S_1, S_2 \in \mathcal{F}^{k,\infty}$ ,  $u_1 \in K_{S_1}$ . This means  $\nabla u_1 = 0$  and  $u_1 \geq \varphi(S_1(X))$  a.e. on  $C$ . We define  $u_2$  as:

$$u_2(X) = \begin{cases} u_1(X), & \text{if } u_1(X) \geq \varphi(S_2(X)) \\ \varphi(S_2(X)), & \text{if } u_1(X) < \varphi(S_2(X)). \end{cases}$$

We have  $u_2 \in K_{S_2}$ . Obviously  $|u_1(X) - u_2(X)| \leq |\varphi(S_1(X)) - \varphi(S_2(X))|$ . Moreover,

$$\begin{aligned} \|u_1 - u_2\|_{H^1(C)}^2 &= \int_C |u_1(X) - u_2(X)|^2 dX \leq \int_C |\varphi(S_1(X)) - \varphi(S_2(X))|^2 dX \\ &\leq L^2 \int_C |S_1(X) - S_2(X)|^2 dX \leq L^2 \|S_1 - S_2\|_{k,\infty}^2 |C|^{1/2} \end{aligned}$$

which shows that **(H<sub>5</sub>)** is satisfied.

**Example 2.** A special case when **(H<sub>5</sub>)** is trivially satisfied is the case when  $K = K_S$  is independent of  $S \in \mathcal{F}^{k,\infty}$ , for all  $k \geq 1$ . For example (see [4]), this happens for

$$K(\Omega)_1 = \{v \in H^1(\Omega) \mid v|_{\partial\Omega} = c, v \geq d \text{ a.e. in } \Omega\}$$

$$K(\Omega)_2 = \{v \in H^1(\Omega) \mid v \geq f \text{ a.e. in } \Omega\},$$

where  $c, d, f$  are constants. Then:

$$SK(\Omega)_1 = \{v \in H^1(C) \mid v|_{\partial C} = c, v \geq d \text{ a.e. in } C\}$$

$$SK(\Omega)_2 = \{v \in H^1(C) \mid v \geq f \text{ a.e. in } C\},$$

which do not depend on  $S$ .

We fix now an initial value  $S_0 \in \mathcal{F}^{k,\infty}$ ,  $u_0 \in K$  a solution of  $(VI)_{S_0}$  and we prove the main theorem of the paper.

(In order to prove the main theorem, we suppose that:

**(H<sub>6</sub>)**  $|a_j(x, \xi) - a_j(x, \tilde{\xi})| \leq L_1 \|\xi - \tilde{\xi}\|_N$  and  $|a(x, \eta) - a(x, \tilde{\eta})| \leq L_2 |\eta - \tilde{\eta}|$ , a.e.  $x \in \mathbb{R}^N$ , for all  $\xi, \tilde{\xi} \in \mathbb{R}^N$  and  $\eta, \tilde{\eta} \in \mathbb{R}$ , with  $L_1, L_2$  positive constants.)

**Theorem 3.** *Suppose that (H<sub>1</sub>)-(H<sub>5</sub>) are satisfied. Let  $S_0 \in \mathcal{F}^{k,\infty}$  and  $u_0 \in K_{S_0}$  fixed. If  $u_0$  is a solution of the variational inequality  $(VI)_{S_0}$ , then  $(VI)_{S_0}$  is stable under perturbations, that is: there exists a neighborhood  $W_0$  of  $S_0$  and a mapping  $\theta : W_0 \rightarrow H^1(C)$  such that for each  $S \in W_0$ ,  $\theta(S) = u_S$  is a solution of  $(VI)_S$ ,  $\theta(S_0) = u_0$  and  $\theta$  is continuous in  $S_0$ .*

*Proof.* We use Theorem 1 with  $W := \mathcal{F}^{k,\infty}$ ,  $H = H^1(C)$ ,  $K : W \rightarrow 2^H$ ,  $K(S) = K_S$  a set-valued map and  $T : W \times H \rightarrow H^*$ ,  $T(S, u) = \mathcal{A}(S, u)$ . We check the hypotheses of Theorem 1.

(i) is obvious.

We prove now (iii), that is we show that there exists a neighborhood  $V$  of  $S_0$  such that the mappings  $\mathcal{A}(S, \cdot)$  are uniformly strongly monotone for each  $S \in V$ .

We have:

$$\begin{aligned} & \langle \mathcal{A}(S, v) - \mathcal{A}(S, u), v - u \rangle \\ &= \int_{\Omega} [A(x, \nabla(v \circ S^{-1})(x)) - A(x, \nabla(u \circ S^{-1})(x))] \cdot \nabla(v \circ S^{-1} - u \circ S^{-1})(x) dx \\ &+ \int_{\Omega} [a(x, (v \circ S^{-1})(x)) - a(x, (u \circ S^{-1})(x))](v \circ S^{-1} - u \circ S^{-1})(x) dx \\ &\geq \int_{\Omega} \{ \gamma_1 \|\nabla(v \circ S^{-1})(x) - \nabla(u \circ S^{-1})(x)\|_N^2 + \gamma_2 |(v \circ S^{-1})(x) - (u \circ S^{-1})(x)|^2 \} dx \\ &\geq \gamma \|v \circ S^{-1} - u \circ S^{-1}\|_{H^1(\Omega)}^2 \geq \frac{\gamma}{\|\det JS^{-1}\|_{\infty} (1 + \|JS\|_{\infty})^2} \|v - u\|_{H^1(C)}^2 \\ &\geq \tilde{\gamma} \|v - u\|_{H^1(C)}^2. \end{aligned}$$

In this evaluation we made use of the transform  $X = S^{-1}(x)$ , the hypothesis (P4) and of the continuity of the mappings  $S \mapsto JS$  and  $S \mapsto \det JS^{-1}$  (see Lemma 1, c).

Next the continuity of  $\mathcal{A}(S, \cdot)$  from  $H^1(C)$  with the strong topology to  $(H^1(C))^*$  with the weak topology will be proved.

Let  $u_n \rightarrow u$  in  $H^1(C)$  and let  $v \in H^1(C)$ . We get:

$$\begin{aligned} & |\langle \mathcal{A}(S, u_n), v \rangle - \langle \mathcal{A}(S, u), v \rangle| \\ &\leq \|\det JS\|_{\infty} \{ \|\phi(S(\cdot), S(\cdot))\|_{\infty} \|JS^{-1}\|_{\infty}^2 \\ &+ \|\mu(S(\cdot), S(\cdot))\|_{\infty} \} \|u_n - u\|_{H^1(C)} \|v\|_{H^1(C)} \rightarrow 0. \end{aligned}$$

There is still to be proved (ii) from Theorem 1, that is the consistency of  $T$ .

Let  $0 < r \leq 1$ . We consider  $S \in V_0$  and in  $(\mathbf{H}_5)$  we put  $S_1 := S_0$ ,  $S_2 := S$ ,  $u_1 := u_0$ . Then there exists  $u_S \in K_S$  such that

$$\|u_S - u_0\|_{H^1(C)} \leq \delta(\|S - S_0\|_{k,\infty} + \|S^{-1} - S_0^{-1}\|_{k,\infty}).$$

We define

$$\beta(S) = \max \{ \sqrt{\delta(\|S - S_0\| + \|S^{-1} - S_0^{-1}\|)}, 2\|\mathcal{A}(S, u_S) - \mathcal{A}(S_0, u_0)\|_{(H^1(C))^*} \}.$$

Obviously  $\beta(S_0) = 0$  (we take  $u_{S_0} = u_0$ ). We prove next that  $\beta$  is continuous at  $S_0$ .

For  $\sqrt{\delta(\|S - S_0\| + \|S^{-1} - S_0^{-1}\|)}$  this is obvious.

For the second term, we consider  $S_n \rightarrow S_0$  in  $\mathcal{F}^{k,\infty}$ . Then, according to  $(\mathbf{H}_5)$ ,  $u_{S_n} \rightarrow u_0$  in  $H^1(C)$ .

We evaluate  $|\langle \mathcal{A}(S_n, u_{S_n}) - \mathcal{A}(S_0, u_0), v \rangle|$  for  $v \in H^1(C)$ .

We have

$$|\langle \mathcal{A}(S_n, u_{S_n}) - \mathcal{A}(S_0, u_{S_n}), v \rangle| \leq (\alpha_{1n} + \alpha_{2n} + \alpha_{3n} + \alpha_{4n})\|v\|_{H^1(C)}$$

where

$$\begin{aligned} \alpha_{1n} &= \|\det JS_n\|_\infty N \{ \|\psi(S_n(\cdot), S_0(\cdot))\|_\infty \|JS_n^{-t}\|_\infty (\|JS_n^{-t}\|_\infty + \|JS_0^{-t}\|_\infty) \\ &\quad + \|\phi(S_n(\cdot), S_0(\cdot))\|_\infty \|JS_n^{-t}\|_\infty \|JS_n^{-t} - JS_0^{-t}\|_\infty \} \|u_{S_n}\|_{H^1(C)} \\ &\quad + \|\det JS_n\|_\infty N \|\sigma(S_n(\cdot), S_0(\cdot))\|_\infty |C|^{1/2} \|JS_n^{-t}\|_\infty \\ &\quad + \|\det JS_n\|_\infty \|JS_n^{-t} - JS_0^{-t}\|_\infty cN \{ (\int_C |k(S_0(X))|^2 dX)^{1/2} + \|JS_0^{-t}\|_\infty \} \|u_{S_n}\|_{H^1(C)}, \\ \alpha_{2n} &= \|\det JS_n - \det JS_0\|_\infty \|JS_0^{-t}\|_\infty cN \{ (\int_C |k(S_0(X))|^2 dX)^{1/2} \\ &\quad + \|JS_0^{-t}\|_\infty \|u_{S_n}\|_{H^1(C)} \}, \\ \alpha_{3n} &= \|\det JS_n\|_\infty \|\chi(S_n(\cdot), S_0(\cdot))\|_\infty 2 \|u_{S_n}\|_{H^1(C)}, \\ \alpha_{4n} &= \|\det JS_n - \det JS_0\|_\infty c_1 \{ (\int_C |k_1(S_0(X))|^2 dX)^{1/2} + \|u_{S_n}\|_{H^1(C)} \} \end{aligned}$$

Next, as in the proof of (iii), one can obtain

$$|\langle \mathcal{A}(S_0, u_{S_n}) - \mathcal{A}(S_0, u_0), v \rangle| \leq \alpha_{5n} \|v\|_{H^1(C)}$$

where

$$\alpha_{5n} = \|\det JS_0\|_\infty \{ \|\phi(S_0(\cdot), S_0(\cdot))\|_\infty \|JS_0^{-t}\|_\infty^2 + \|\mu(S_0(\cdot), S_0(\cdot))\|_\infty \} \|u_{S_n} - u_0\|_{H^1}$$

From the hypotheses imposed it is clear that  $\alpha_n = \sum_{i=1}^5 \alpha_{in} \rightarrow 0$  when  $n \rightarrow 0$ . So we get:

$$\begin{aligned} & \|\mathcal{A}(S_n, u_{S_n}) - \mathcal{A}(S_0, u_0)\|_{(H^1(C))^*} \\ &= \sup \{ |\langle \mathcal{A}(S_n, u_{S_n}) - \mathcal{A}(S_0, u_0), v \rangle| : v \in H^1(C), \|v\|_{H^1(C)} \leq 1 \} \\ &\leq \sup \{ \alpha_n \|v\|_{H^1(C)} : v \in H^1(C), \|v\|_{H^1(C)} \leq 1 \} \leq \alpha_n \rightarrow 0. \end{aligned}$$

From this the continuity of  $\beta$  at  $S_0$  is clear.

Let  $W_r \subset V_0$  be a neighbourhood of  $S_0$  such that for each  $S \in W_r$

$$\beta(S) \leq 1 \quad \text{and} \quad r - 4\beta(S)\|\mathcal{A}(S_0, u_0)\|_{(H^1(C))^*} \geq 0.$$

Let  $v \in K_S$  with  $r \leq \|v - u_S\|$ . We have:

$$\begin{aligned} & \langle \mathcal{A}(S, u_S), v - u_S \rangle + \beta(S)\|v - u_S\| \\ &= \langle \mathcal{A}(S, u_S) - \mathcal{A}(S_0, u_0), v - u_S \rangle + \langle \mathcal{A}(S_0, u_0), v - u_S \rangle + \beta(S)\|v - u_S\| \\ &\geq -\|\mathcal{A}(S, u_S) - \mathcal{A}(S_0, u_0)\|\|v - u_S\| + \langle \mathcal{A}(S_0, u_0), v - u_S \rangle + \beta(S)\|v - u_S\| \\ &\geq -\frac{1}{2}\beta(S)\|v - u_S\| + \beta(S)\|v - u_S\| + \langle \mathcal{A}(S_0, u_0), v - u_0 \rangle + \langle \mathcal{A}(S_0, u_0), u_0 - u_S \rangle \\ &\geq \frac{1}{2}\beta(S)\|v - u_S\| + \langle \mathcal{A}(S_0, u_0), v - v_0 \rangle + \langle \mathcal{A}(S_0, u_0), v_0 - u_0 \rangle \\ &\quad - \|\mathcal{A}(S_0, u_0)\|\|u_0 - u_S\| \\ &\geq \frac{1}{2}\beta(S)\|v - u_S\| - \|v - v_0\|\|\mathcal{A}(S_0, u_0)\| - \|u_0 - u_S\|\|\mathcal{A}(S_0, u_0)\| \\ &\geq \frac{1}{2}\beta(S)[r - 4\beta(S)\|\mathcal{A}(S_0, u_0)\|] \geq 0. \end{aligned}$$

(Here we considered  $v_0 \in K_{S_0}$  such that  $\|v - v_0\| \leq \delta(\|S - S_0\| + \|S^{-1} - S_0^{-1}\|)$ ).

On the other hand we have  $\|u_S - u_0\| \leq \beta^2(S) \leq \beta(S)$ , which concludes the proof.  $\square$

**Example 3.** We present a linear variational inequality for which all the previous hypotheses are satisfied. Consider the problem:

$$\text{Find } u_\Omega \in K(\Omega) \text{ such that } a_\Omega(u_\Omega, v - u_\Omega) \geq 0, \quad \forall v \in K(\Omega),$$

where

$$a_\Omega(u, v) = \int_{\Omega} [B(x)\nabla u(x) \cdot \nabla v(x) + b(x)u(x)v(x)]dx$$

with  $B \in C(\mathbb{R}^N)^{N^2} \cap L^\infty(\mathbb{R}^N)^{N^2}$ ,  $b \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ ,  $b(x) \geq \tilde{b} > 0$  and

$$\sum_{j=1}^N \sum_{k=1}^N b_{jk}(x) \xi_k \xi_j \geq \alpha \|\xi\|_N^2$$

a.e.  $x \in \mathbb{R}^N$ , for all  $\xi \in \mathbb{R}^N$ . It can be easily shown that, with

$$a_j(x, \xi) = \sum_{k=1}^N b_{jk}(x) \xi_k = B_j(x) \cdot \xi \text{ and } a(x, \eta) = b(x)\eta$$

the hypotheses  $(H_1)$ – $(H_4)$  are satisfied.

#### REFERENCES

- [1] SOKOŁOWSKI, J. and ZOLESIO, J.P.: *Introduction to Shape Optimization. Shape Sensitivity Analysis*, Springer-Verlag, 1992.
- [2] PIRONNEAU, O.: *Optimal Shape Design for Elliptic Systems*, Springer-Verlag, New-York, 1984.
- [3] MURAT, F. and SIMON, J.: *Sur le Contrôle par un Domaine Géométrique*, Preprint No. 76015, University of Paris, 1976.
- [4] LIU W. B. and RUBIO J. E.: *Optimal shape design for systems governed by variational inequalities, Part 1: Existence theory for elliptic case, Part 2: Existence theory for evolution case*, J. of Optimization Theory and Applications, **69**, (1991), 351–371, 373–396.
- [5] DENKOWSKI, Z. and MIGÓRSKI S.: *Optimal shape design problems for a class of systems described by hemivariational inequalities*, J. of Global Optimization, **12**, (1998), 37–59.
- [6] GASINSKI, L.: *Mapping Method in Optimal Shape Design Problems Governed by Hemivariational Inequalities*, Shape Optimization and Optimal Design (edited by J. Cagnol, M.P. Polis, J.P. Zolesio), Dekker, New York, (2001), 277–288.
- [7] KASSAY, G. and KOLUMBÁN, J.: *Multivalued Parametric Variational Inequalities with  $\alpha$ -Pseudomonotone Maps*, J. of Optimization Theory and Applications, **107**, No.1 (2000), 35–50.
- [8] ALT, W. and KOLUMBÁN, I.: *An implicit function theorem for a class of monotone generalized equations*, Kybernetika, **29**, (1993), 210–221.
- [9] SHOWALTER, R. E.: *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, American Mathematical Society, Mathematical Surveys and Monographs, **49**, 1997.

#### Author's Address

##### Daniela Inoan:

DEPARTMENT OF MATHEMATICS, TECHNICAL UNIVERSITY OF CLUJ-NAPOCA, 15, C. DAICOVICIU STR., 3400 CLUJ, ROMANIA

E-mail address: Daniela.Inoan@math.utcluj.ro



*Mathematical Notes, Miskolc*, Vol. 4, No. 1 (2003), pp. 35–37

HU ISSN 1586-8850

## ON FUNCTIONS PRESERVING RANK OF MATRICES

JÓZEF KALINOWSKI

[Received: June 18, 2002]

**ABSTRACT.** The paper is motivated by papers of E. Pap [4] and A. Marková [3]. We give a complete characterization of functions preserving ranks of all matrices, without assuming any regularity conditions.

*Mathematics Subject Classification:* 15A03

*Keywords:* rank of matrices

### 1. INTRODUCTION

The notion of  $g$ -calculus has been introduced by E. Pap in [4]. A. Marková in [3] observed that a fundamental role in this calculus is played by functions preserving ranks of matrices.

First of all let us introduce

**Definition 1.** *Function  $f : \mathbb{R} \rightarrow \mathbb{R}$  preserves rank of matrices if and only if the rank of a matrix  $A = [a_{ij}]$  is equal to rank of the matrix  $f(A) := [f(a_{ij})]$ , for every matrix  $A$ , where  $i = 1, 2, \dots, n$ ,  $j = 1, 2, \dots, m$ .*

We proved in [1] that a monotonic and continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  with  $f(0) = 0$  preserves ranks of matrices if and only if it is linear, i.e.  $f(x) = c \cdot x$ ,  $x \in \mathbb{R}$ , for some constant  $c \neq 0$ .

In the present paper we show that the characterization holds true even without admitting additional assumptions on function  $f$ .

### 2. MAIN RESULTS

It follows immediately from properties of determinants:

**Example 1.** *The linear function  $f(x) = c \cdot x$  on  $\mathbb{R}$  with a constant  $c \neq 0$  is a function preserving rank of matrices.*

Without additional assumptions on  $f$  (i.e. continuity or others) the following holds true

**Lemma 1.** *If rank  $r(A) = r(f(A))$  for all square matrices of degree less or equal to 3, then there exists a constant  $c \neq 0$  such that  $f(x) = c \cdot x$ .*

*Proof.* Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function preserving rank of square matrices of degree less or equal to 3. For matrix  $A = [a]$  of degree 1 it follows that  $f(a) = 0$  if and only if  $a = 0$ . Let  $f(1) = c$ . From the above it follows that  $c \neq 0$ . From properties of determinants it follows that  $r(A) = r(\frac{1}{c}f(A))$ . Consider the function

$$g := \frac{1}{c} \cdot f \quad \text{on } \mathbb{R}, \quad (2.1)$$

so that  $g(1) = 1$ . For any square matrix

$$B = \begin{bmatrix} 1 & b \\ a & a \cdot b \end{bmatrix},$$

where  $a, b \in \mathbb{R}$ , we obtain  $r(B) = 1$ . Then also  $r(g(B)) = 1$ . Thus

$$\det \begin{bmatrix} 1 & g(b) \\ g(a) & g(a \cdot b) \end{bmatrix} = 0$$

or  $g(a \cdot b) - g(a) \cdot g(b) = 0$ , i.e.  $g$  solves the multiplicative Cauchy equation

$$g(x \cdot y) = g(x) \cdot g(y) \quad \text{for all } x, y \in \mathbb{R}. \quad (M)$$

Similarly for any square matrix of the form

$$C = \begin{bmatrix} 0 & 1 & a \\ 1 & 0 & b \\ 1 & 1 & a + b \end{bmatrix}$$

with arbitrary  $a, b \in \mathbb{R}$  the rank  $r(C) = 2$ . Hence also  $r(g(C)) = 2$  and thus

$$\det \begin{bmatrix} 0 & 1 & g(a) \\ 1 & 0 & g(b) \\ 1 & 1 & g(a + b) \end{bmatrix} = 0,$$

i.e.  $\det C = g(a) + g(b) - g(a + b) = 0$ . In other words the additive Cauchy functional equation

$$g(x + y) = g(x) + g(y) \quad \text{for all } x, y \in \mathbb{R} \quad (A)$$

is fulfilled by  $g$ . Now, from [2] (Theorem 1, page 356), it follows that the only functions satisfying simultaneously (A) and (M) are  $g = 0$  or  $g = id$ . Since  $g(1) = 1$ , we see that in our case  $g(x) = x$ ,  $x \in \mathbb{R}$ . From Definition 1 we obtain that  $f(x) = c \cdot x$  for all  $x \in \mathbb{R}$ .  $\square$

The following theorem completely describes all functions preserving rank of matrices.

**Theorem 1.** *Function  $f : \mathbb{R} \rightarrow \mathbb{R}$  preserves rank of matrices if and only if  $f(x) = c \cdot x$ , where  $c \neq 0$  is a constant.*

*Proof.* It follows immediately from Lemma 1 and Example 1.  $\square$

**REFERENCES**

- [1] KALINOWSKI, J.: *On rank equivalence and preserving rank operators*, Novi Sad J. Math., **32**(1), (2002) (in press).
- [2] KUCZMA, M.: *Functional Equations in a Single Variable*, Monografie Mat. 46, Polish Sci. Publ., (PWN), Warszawa, 1968.
- [3] MARKOVÁ, A.: *Some remarks on the pseudo-linear algebra*, Tatra Mountains Math. Publ., **6**, (1995), 123–130.
- [4] PAP, E.: *g-calculus*, Zb. Prir.-Mat. Fak., Univ. Novom Sadu, Ser. Mat. **23**, (1993), 145–156.

*Author's Address***Józef Kalinowski:**

DEPARTMENT OF MATHEMATICS, SILESIAN UNIVERSITY, 40-007 KATOWICE, UL. BAN-KOWA 14, POLAND  
E-mail address: kalinows@ux2.math.us.edu.pl



## THE LIBERA GENERALIZED INTEGRAL OPERATOR AND HARDY SPACES

GHEORGHE MICLĂUS

[Received: October 14, 2002]

**ABSTRACT.** In this paper, we determine the Hardy spaces to which  $L_\gamma^n(f)$  belongs,  $n \in \mathbb{N}^*$ , if  $f \in H^p$ , where  $L_\gamma^n = \underbrace{L_\gamma \circ L_\gamma \circ \cdots \circ L_\gamma}_n$  is the Libera generalized integral operator (2). As a corollary we obtain the Hardy spaces for some classes of analytic functions.

*Mathematics Subject Classification:* 30C35, 30C45

*Keywords:* Hardy spaces, Libera generalized operator

### 1. INTRODUCTION

Let  $f$  be an analytic function in the unit disc  $U = \{z \mid |z| < 1\}$  and let  $H(U)$  denote the set of all analytic functions in  $U$ . In 1965, R. J. Libera [5] had studied the operator  $L : H(U) \rightarrow H(U)$  defined by

$$L(f)(z) = \frac{2}{z} \int_0^z f(t) dt \quad (1)$$

showing that  $L(S^*) \subset S^*$ , where  $S^*$  is the class of starlike functions,  $L(K) \subset K$ , where  $K$  is the class of convex functions and  $L(C) \subset C$ , if  $C$  is the class of close-to-convex functions. In 1969, S. D. Benardi generalized these results investigating in [1] a more general integral operator defined by  $L_\gamma : H(U) \rightarrow H(U)$

$$L_\gamma(f)(z) = \frac{\gamma + 1}{z^\gamma} \int_0^z f(t) t^{\gamma-1} dt. \quad (2)$$

He showed that if  $\gamma \in \mathbb{N}^*$ , then  $L_\gamma$  preserves these properties. In this paper we determine the Hardy spaces, to which  $L_\gamma^n(f)$  belongs,  $n \in \mathbb{N}^*$ , if  $f \in H^p$ , where  $L_\gamma^n = \underbrace{L_\gamma \circ L_\gamma \circ \cdots \circ L_\gamma}_n$ .

## 2. PRELIMINARIES

For  $f \in H(U)$  and  $z = re^{i\theta}$ , we set

$$M_p(r, f) = \begin{cases} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}, & 0 < p < \infty \\ \sup_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})|, & \text{for } p = \infty. \end{cases}$$

A function  $f \in H(U)$  is said to be of Hardy spaces  $H^p$  ( $0 < p < \infty$ ) if  $M_p(r, f)$  remains bounded as  $r \rightarrow 1^-$ .  $H^\infty$  is the class of bounded analytic functions in the unit disc.

We shall need the following well-known lemmas [2].

**Lemma 1.** *If  $f' \in H^p$ ,  $0 < p < 1$ , then  $f \in H^{\frac{p}{1-p}}$ . If  $f' \in H^p$ ,  $1 \leq p$ , then  $f \in H^\infty$ .*

**Lemma 2.** *If  $f \in H^p$  and  $g \in H^q$ , then  $fg \in H^\lambda$ ,  $\lambda = \frac{pq}{p+q}$ .*

**Lemma 3** (Integral theorem of Hardy–Littlewood). *If  $f \in H^p$  and  $F = \int_0^z f(t)dt$ , then  $f \in H^{\frac{p}{1-p}}$  for  $0 < p < 1$ , and  $f \in H^\infty$  for  $p \geq 1$ .*

## 3. MAIN RESULTS

Let  $\gamma \in \mathbb{C}$ ,  $\operatorname{Re} \gamma > 0$ , let  $f$  be an analytic function in  $U$  and let  $L_\gamma$  be the Libera generalized integral operator (Bernardi's operator) defined by (2).

**Theorem 1.** *If  $f \in H^p$ , then:*

- (i) *if  $n < \frac{1}{p}$ ,  $n \in \mathbb{N}^*$ , then  $L_\gamma^n(f) \in H^\lambda$ ,  $\lambda = \frac{p}{1-np}$ ;*
- (ii) *if  $n \geq \frac{1}{p}$ ,  $n \in \mathbb{N}^*$ , then  $L_\gamma^n(f) \in H^\infty$ , where  $L_\gamma^n = \underbrace{L_\gamma \circ L_\gamma \circ \dots \circ L_\gamma}_n$ ;*
- (iii)  $[L_\gamma(f)]' \in H^p$ .

*Proof.* Assertions (i) and (ii). If  $f$  is analytic,  $f(z) = z + a_2 z^2 + \dots$ , then there exists a unique function

$$L_\gamma(f)(z) = \frac{\gamma+1}{z^\gamma} \int_0^z f(t) t^{\gamma-1} dt,$$

which is analytic in  $U$ . Indeed,

$$g(z) = \frac{f(z)}{z} = 1 + a_2 z + \dots$$

is analytic in  $U$ . Let  $h(z) = z[g(z)]^{\frac{1}{\gamma+1}}$  be the branch that is 1 for  $z = 0$ , and we have that  $h(z) = z + b_2 z^2 + \dots$  is analytic in  $U$ ,  $\frac{h(z)}{z} \neq 0$ ,  $[h(z)]^{\gamma+1} = (a_2 z + \dots)$ . Hence

$$\begin{aligned} L_\gamma(f)(z) &= \frac{\gamma+1}{z^\gamma} \int_0^z f(t) t^{\gamma-1} dt = \frac{\gamma+1}{z^\gamma} \int_0^z \frac{h^{\gamma+1}(t)}{t} dt = \\ &= \frac{\gamma+1}{z^\gamma} \int_0^z t^\gamma (1 + a_2 t + \dots) dt = z + \frac{\gamma+1}{\gamma+2} a_2 z^2. \end{aligned}$$

For the integral operator  $L_\gamma$  we have  $L_\gamma = A \circ B$ , where

$$A(f)(z) = \frac{\gamma+1}{z^\gamma} f(z), \quad \gamma \in \mathbb{C}, \quad \operatorname{Re} \gamma > 0$$

and

$$B(f)(z) = \int_0^z f(t) t^{\gamma-1} dt.$$

We determine the Hardy class for  $A$  and  $B$ . If  $f \in H^p$ ,  $\gamma = a + ib$ ,  $a > 0$ , then

$$M_p(r, A) \leq \frac{\gamma+1}{r^a} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}},$$

and hence  $A(f) \in H^p$ .

From integral theorem of Hardy–Littlewood (Lemma 3), we have  $B(f) \in H^{\frac{p}{1-p}}$  for  $f \in H^p$ ,  $p < 1$ , and  $B(f) \in H^\infty$ , for  $f \in H^p$ ,  $p \geq 1$ . Because  $L_\gamma(f) = A(B(f))$  we have for  $f \in H^p$ ,  $p < 1$  that  $B(f) \in H^{\frac{p}{1-p}}$  and  $A(B(f)) \in H^{\frac{p}{1-p}}$ . Hence  $L_\gamma(f) \in H^{\frac{p}{1-p}}$  for  $p < 1$  and  $L_\gamma(f) \in H^\infty$ , for  $p \geq 1$ . Suppose that  $n < \frac{1}{p}$ ,  $n \in \mathbb{N}^*$  and  $L_\gamma^k(f) \in H^{\frac{p}{1-kp}}$  for  $1 \leq k \leq n-1$ . Then

$$L_\gamma^n(f) \in H^\lambda, \quad \lambda = \frac{\frac{p}{1-(n-1)p}}{1 - \frac{p}{1-(n-1)p}} = \frac{p}{1-np}.$$

If  $n \geq \frac{1}{p}$ , then there is a  $k \in \mathbb{N}^*$  such that  $k < \frac{1}{p} \leq k+1 \leq n$  and  $L_\gamma^k(f) \in H^{\frac{p}{1-kp}}$  and  $L_\gamma^{k+1}(f) \in H^\infty$ .

*Assertion (iii).* We have

$$\left[ L_\gamma(f)(z) \right]' = \frac{1}{z} \left[ (\gamma+1)f(z) - \gamma L_\gamma(f)(z) \right]. \quad (3)$$

Applying the Minkowski inequality, we obtain

$$\begin{aligned}
 M_p^p(r, [L_\gamma(f)]) &= \frac{1}{2\pi} \int_0^{2\pi} \left| [L_\gamma(f)(re^{i\theta})] \right|^p d\theta = \\
 &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|re^{i\theta}|^p} |(\gamma + 1)f(re^{i\theta}) - \gamma L_\gamma(f)(re^{i\theta})|^p d\theta \leq \\
 &\leq \frac{1}{2\pi} \frac{|\gamma + 1|^p}{r^p} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta + \frac{1}{2\pi} \frac{|\gamma|^p}{r^p} \int_0^{2\pi} |L_\gamma(f)(re^{i\theta})|^p d\theta.
 \end{aligned}$$

The value  $M_p^p(r, [L_\gamma(f)]')$  is bounded for

$$\min \left\{ p, \frac{p}{1-p} \right\} = p.$$

Hence,  $[L_\gamma(f)]' \in H^p$ . That results are best possible because from  $[L_\gamma(f)]' \in H^q$ ,  $q > p$  applying (3) we should conclude that  $f \in H^q$ .  $\square$

For  $n = 1$ , these results were obtained in [4].

**Remark 1.** Since the results for Hardy classes do not depend on  $\gamma$ , the results remain the same for  $L = L_{\gamma_1} \circ L_{\gamma_2} \circ \dots \circ L_{\gamma_n}$ , where  $\operatorname{Re} \gamma_i > 0$ ,  $i = 1, n$ .

For  $\gamma = 1$ ,  $L_\gamma = L$  is the Libera operator (1), and we have the following

**Corollary 1.** If  $f$  is an analytic function in  $U$ ,  $\gamma \in \mathbb{C}$ ,  $\operatorname{Re} \gamma_i > 0$  and  $f \in H^p$  then:

- (i) if  $n < \frac{1}{p}$ ,  $n \in \mathbb{N}^*$ , then  $L^n(f) \in H^\lambda$ ,  $\lambda = \frac{p}{1-np}$ ;
- (ii) if  $n \geq \frac{1}{p}$ ,  $n \in \mathbb{N}^*$ , then  $L^n(f) \in H^\infty$ ;
- (iii)  $[L^n(f)]' \in H^p$ .

**Corollary 2.** If  $f \in K$ ,  $f \neq \frac{z}{1-ze^{i\tau}}$ ,  $\tau \in \mathbb{R}$  then  $L_\gamma(f)$  is a bounded function.

It is well-known that if  $f \in K$  (a convex function) then  $f \in H^{1+\varepsilon}$ ,  $\varepsilon = \varepsilon(f) > 0$ . Hence, we obtain from Theorem 1 that  $L_\gamma(f) \in H^\infty$ .

Therefore,  $L_\gamma$  transforms  $K$  into its subclass of bounded functions, excepting extremal functions.

**Corollary 3.** If  $f \in S^*$ ,  $f(z) \neq \frac{z}{(1+ze^{i\tau})^2}$ , then

- (i)  $L_\gamma(f) \in H^1$ ;

$$(ii) \quad L_\gamma^2(f) \in H^\infty \text{ if } L_\gamma(f) \neq \frac{z}{(1 + ze^{i\tau})^2}, \quad \tau \in \mathbb{R}.$$

It is known that if  $f \in S^*$ , then  $f \in H^{\frac{1}{2}+\varepsilon}$ ,  $\varepsilon = \varepsilon(f) > 0$ . Hence  $L_\gamma(f) \in H^1$ . If

$$L_\gamma(f) = \frac{z}{(1 + ze^{i\tau})^2},$$

then  $L_\gamma^2(f) \in H^\infty$ .

In other words,  $L_\gamma$  transforms  $S^*$  into its subclass and  $L_\gamma^2$  transforms  $S^*$  into a subclass of bounded functions, with the exception of extremal functions.

**Corollary 4.** *If  $f \in \mathbb{C}$ ,  $f(z) \neq \frac{p(z)}{(1 + ze^{i\tau})^2}$ ,  $\text{Rep}(z) > 0$ , then*

- (i)  $L_\gamma(f) \in H^1$ ;
- (ii)  $L_\gamma^2(f) \in H^\infty$  if  $L_\gamma(f) \neq \frac{p(z)}{(1 + ze^{i\tau})^2}$ ,  $\text{Rep}(z) > 0$ .

It is known that if  $f \in \mathbb{C}$  and  $f(z) \neq \frac{p(z)}{(1 + ze^{i\tau})^2}$ ,  $\text{Rep}(z) > 0$  then  $f \in H^{\frac{1}{2}+\varepsilon}$ ,  $\varepsilon = \varepsilon(f) > 0$ . Hence,  $L_\gamma(f) \in H^1$ . If  $L_\gamma(f) \neq \frac{p(z)}{(1 + ze^{i\tau})^2}$ ,  $\text{Rep}(z) > 0$  from Theorem 1 we obtain  $L_\gamma^2(f) \in H^\infty$ .

## REFERENCES

- [1] BERNARDI, S. D.: *Convex and starlike univalent functions*, Trans. Amer. Math. Soc., **135**, (1969), 429–446.
- [2] DUREN, P. L.: *Theory of  $H^p$  spaces*, Academic Press, New York and London, 1970.
- [3] EENIGENBURG, P. J. and KEOGH, F.R.: *The Hardy classes of some univalent functions and their derivatives*, Mich. Math. J., **17**, (1970), 335–346.
- [4] FEKETE, O.: *Some integral operators and Hardy spaces*, Mathematica (Cluj), **29**(52), (1987), 29–31.
- [5] LIBERA R.J.: *Some classes of regular univalent functions*, Proc. Amer. Math. Soc., **16**, (1965), 755–58.
- [6] MILLER, S. S. and MOCANU, P. T. and READE, M. O.: *The Hardy classes for functions in the class  $MV[a, k]$* , J. Math. Anal. Appl., **51**, (1975), 33–42.

## Author's Address

**Gheorghe Miclăuș:**

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, NORTH UNIVERSITY OF BAIA MARE, ROMANIA

E-mail address: miclaus5@yahoo.com



## BLOCK EIGENVALUES AND SOLUTIONS OF DIFFERENTIAL MATRIX EQUATIONS

EDGAR PEREIRA

[Received: January 08, 2002]

**ABSTRACT.** We present an application of block eigenvalues of the block companion matrix of a matrix polynomial  $P(X)$  to obtain a general solution of the differential matrix equation associated with  $P(X)$ .

*Mathematics Subject Classification:* 15A24, 34A99, 65L05

*Keywords:* matrix polynomials, block eigenvalues, differential matrix equations

### 1. INTRODUCTION

The preliminary theory on matrix polynomials revisited next can be found in [4], [1], [6], [3] and [5].

Let

$$P(X) = X^m + A_1X^{m-1} + \dots + A_m \quad (1.1)$$

be a monic (right) matrix polynomial of degree  $m$  in the indeterminate  $X$  with the coefficients  $A_1, \dots, A_m$  being  $n \times n$  complex matrices. An  $n \times n$  matrix  $X_1$ , such that  $P(X_1) = 0$ , is a (right) solvent of  $P(X)$ . Furthermore for each nonsingular  $n \times n$  matrix formed of leading vectors of Jordan chains of  $P(\lambda)$  it is possible to construct one solvent of  $P(X)$  and the total number of solvents will be the number of such nonsingular matrices.

The matrix

$$C = \begin{bmatrix} 0_n & I_n & \dots & 0_n \\ \vdots & & \ddots & \\ -A_m & -A_{m-1} & \dots & -A_1 \end{bmatrix}, \quad (1.2)$$

(where  $0_n$  and  $I_n$  are the null matrix and the identity matrix of order  $n$ , respectively) associated with the coefficients of  $P(X)$ , is said to be the block companion matrix of  $P(X)$ . Moreover, if  $X_1, \dots, X_m$  are  $m$  solvents of  $P(X)$ , then the respective block

Vandermonde matrix is

$$V(X_1, \dots, X_m) = \begin{bmatrix} I_n & \dots & I_n \\ X_1 & \dots & X_m \\ \vdots & & \vdots \\ X_1^{m-1} & \dots & X_m^{m-1} \end{bmatrix}$$

and if the matrix  $V(X_1, \dots, X_m)$  is nonsingular, then we say that these  $m$  solvents form a complete set of solvents of  $P(X)$ .

We consider now a differential matrix equation, i. e., homogeneous ordinary differential equation of order  $m$  having  $n \times n$  matrix coefficients, written by

$$P\left(\frac{d}{dt}\right)x = x^{(m)}(t) + A_1x^{(m-1)}(t) + \dots + A_mx(t) = 0. \quad (1.3)$$

An important result on this is that if  $X_1, \dots, X_m$  are a complete set of solvents of the matrix polynomial  $P(X)$  associated with  $P\left(\frac{d}{dt}\right)x$  (that is, having the same coefficients), then every solution of

$$P\left(\frac{d}{dt}\right)x = 0$$

is of the form

$$x(t) = e^{X_1 t} z_1 + e^{X_2 t} z_2 + \dots + e^{X_m t} z_m, \quad (1.4)$$

where  $z_1, z_2, \dots, z_m \in \mathbb{C}^n$  ([5], p. 525).

## 2. BLOCK EIGENVALUES

We recall the classical equation of matrix theory,  $AX = XB$ , where  $A, B$  are given complex square matrices (see [2], p. 215). We work here with a variation of this equation that happens when  $X$  spans an invariant subspace of  $A$ , and only  $A$  is given. It appears in the computation of eigenvalues (see [8], p. 587).

**Definition 1.** Given a matrix  $A$  of order  $p$ , if a matrix  $Y$  of order  $q < p$  is such that

$$AW = WY \quad (2.1)$$

for a rectangular matrix  $W$  of full rank. We say that  $Y$  is a (right) block eigenvalue of  $A$  and  $W$  is a corresponding (right) block eigenvector of dimension  $q \times p$ .

This definition can be restricted to block matrices, of order  $mn$ , partitioned in blocks of order  $n$ . In this case the block eigenvalues are of the same order as the blocks of the block matrix that is  $n$  and the corresponding block eigenvector is a vector of blocks of dimension  $mn \times n$  (see [7]).

A block eigenvalue has the property that any similar block is also a block eigenvalue, and it is clear that a block eigenvector  $W$  spans an invariant subspace of  $A$ , since being of full rank is equivalent to having linearly independent columns.

Next we have the strong relationship between a matrix and a block eigenvalue (see also [9], Corollary II).

**Theorem 1.** Let  $A$  be a matrix, then a matrix  $Y$  is a block eigenvalue of  $A$ , if and only if the eigenvalues of  $Y$  are also eigenvalues of  $A$ , and for each common eigenvalue  $\alpha$ , the corresponding partial multiplicities  $k_1(Y), \dots, k_n(Y)$  in  $Y$ , and  $k_1(A), \dots, k_m(A)$  in  $A$ , where the integers  $k_i$  are in decreasing order of magnitude, satisfy

- (i)  $n \leq m$ ;
- (ii)  $k_i(Y) \leq k_i(A)$ ,  $i = 1, \dots, n$ .

*Proof.* First note that  $n$  and  $m$  are the geometric multiplicities of  $\alpha$  in  $Y$ , and in  $A$ , or the number of Jordan blocks, of  $\alpha$  in  $J_Y$ , and in  $J_A$ , the Jordan normal forms of  $Y$  and  $A$ , respectively. And that the  $k_i$  are the orders of these Jordan blocks. Let now  $Y = TJ_YT^{-1}$ , where  $T$  is a nonsingular matrix. Then suppose that  $AW = WY$ , with  $W$  of full rank, thus  $AWT = WYT = WTJ_YT^{-1}T = WTJ_Y$ . Since  $WT$  is still of full rank, it follows that the linearly independent columns of  $WT$  are eigenvectors or generalized eigenvectors of  $A$ , with respect to the eigenvalues of  $J_Y$ , thus the eigenvalues of  $J_Y$  (and of  $Y$ ) are also of  $A$ . Furthermore, from  $AWT = WTJ_Y$ , it follows that  $J_Y$  is a submatrix of  $J_A$ . Therefore, for each common eigenvalue  $\alpha$ , the corresponding geometric multiplicities  $m$  in  $A$  and  $n$  in  $J_Y$ , and hence in  $Y$ , satisfy  $n \leq m$ . Also the orders of the Jordan blocks of  $J_A$  and of  $J_Y$  corresponding to  $\alpha$ , satisfy  $k_i(Y) \leq k_i(A)$ ,  $i = 1, \dots, n$ . Conversely, suppose that the eigenvalues of  $Y$  (and hence of  $J_Y$ ), are common to  $A$ . And supposing (i) and (ii) we can write  $AZ = ZJ_Y$ , where the columns of  $Z$ , eigenvectors or generalized eigenvectors of  $A$ , corresponding to the eigenvalues of  $J_Y$ , are linearly independent. Hence  $Z$  is of full rank, thus  $AZT^{-1} = ZJ_YT^{-1} = ZT^{-1}Y$ , with  $ZT^{-1}$  of full rank, and the conclusion is that  $Y$  is a block eigenvalue of  $A$ .  $\square$

We observe that, for each eigenvalue of a complex matrix, the respective number of partial multiplicities gives the geometric multiplicity, and therefore the number of Jordan blocks of the Jordan normal form of the matrix, for this eigenvalue. These partial multiplicities are the sizes of these Jordan blocks, hence we can conclude that, if we have the partial multiplicities of all the eigenvalues of a complex matrix, we can write its Jordan normal form. Considering that, we define a set of block eigenvalues in which this information can be obtained.

**Definition 2.** Let  $A$  be a matrix, and let  $Y_1, \dots, Y_k$  be a set of block eigenvalues of  $A$ . We say that this set is a complete set of block eigenvalues, if the eigenvalues, and respective partial multiplicities, of these block eigenvalues are the eigenvalues, with the same partial multiplicities, of the matrix  $A$ .

**Theorem 2.** A set of block eigenvalues  $Y_1, \dots, Y_k$  of a matrix  $A$ , is a complete set, if and only if there is a set of corresponding block eigenvectors  $W_1, \dots, W_k$ , such that the matrix  $\begin{bmatrix} W_1 & \cdots & W_k \end{bmatrix}$  is of full rank, and

$$A \begin{bmatrix} W_1 & \cdots & W_k \end{bmatrix} = \begin{bmatrix} W_1 & \cdots & W_k \end{bmatrix} \text{diag}(Y_1, \dots, Y_k), \quad (2.2)$$

where  $\text{diag}(Y_1, \dots, Y_k)$  is a block diagonal matrix of the same order of  $A$ .

*Proof.* Let  $Y_1, \dots, Y_k$  be a complete set of block eigenvalues, and let

$$D = \text{diag}(Y_1, \dots, Y_k)$$

be a block diagonal matrix. Since the eigenvalues of the  $Y_1, \dots, Y_k$ , and their partial multiplicities, are the same as those of  $A$ . The same happens to  $D$ , the direct sum of the  $Y_1, \dots, Y_k$ . Consequently  $A$  and  $D$  have the same Jordan normal form, and therefore they are similar, so that there is a nonsingular matrix  $R$ , such that  $AR = RD$ . Writing  $R = \begin{bmatrix} R_1 & \cdots & R_k \end{bmatrix}$ , with the number of columns of each  $R_i$ ,  $i = 1, \dots, k$  being equal to the order of  $Y_i$ , it follows that,  $AR_i = R_i Y_i$ , for  $i = 1, \dots, k$ , and it is obvious that each  $R_i$  is of full rank, and thus, it is a block eigenvector corresponding to  $Y_i$ . Conversely, let  $W_1, \dots, W_k$  be a set of right block eigenvectors corresponding to  $Y_1, \dots, Y_k$ , and let

$$A \begin{bmatrix} W_1 & \cdots & W_k \end{bmatrix} = \begin{bmatrix} W_1 & \cdots & W_k \end{bmatrix} \text{diag}(Y_1, \dots, Y_k)$$

with the matrix  $\begin{bmatrix} W_1 & \cdots & W_k \end{bmatrix}$  being of full rank, then  $A$  and  $\text{diag}(Y_1, \dots, Y_k)$  are similar, and hence their Jordan normal form, and the partial multiplicities of their eigenvalues, are common. Thus the  $Y_1, \dots, Y_k$  are a complete set.  $\square$

### 3. SOLUTIONS OF MATRIX DIFFERENTIAL EQUATIONS

Now we consider block eigenvalues of the block companion matrix, in order to obtain a general solution to the previously mentioned differential matrix equation.

**Theorem 3.** *Let  $P(X)$  be a matrix polynomial and let  $C$  be the associated block companion matrix, if the matrices  $Y_1, \dots, Y_k$  are a complete set of block eigenvalues of  $C$ , and  $W_1, \dots, W_k$  are the corresponding block eigenvectors. Then every solution of  $P\left(\frac{d}{dt}\right)x = 0$  is of the form*

$$x(t) = (W_1)_1 e^{Y_1 t} z_1 + \dots + (W_k)_1 e^{Y_k t} z_k, \quad (3.1)$$

where  $(W_i)_1$  is the top submatrix of  $n$  rows of  $W_i$ , for  $i = 1, 2, \dots, k$  and  $z_1, \dots, z_k \in \mathbb{C}^n$ .

*Proof.* From [5], p. 512, we have

$$x(t) = P e^{Ct} z,$$

with  $P = \begin{bmatrix} I_n & 0_n & \cdots & 0_n \end{bmatrix}$  and  $z \in \mathbb{C}^{mn}$  is arbitrary. Now let  $W_1, \dots, W_k$  be block eigenvectors of  $C$  corresponding to the block eigenvalues  $Y_1, \dots, Y_k$ , thus we have from Theorem 2

$$C \begin{bmatrix} W_1 & \cdots & W_k \end{bmatrix} = \begin{bmatrix} W_1 & \cdots & W_k \end{bmatrix} \text{diag}(Y_1, \dots, Y_k),$$

now we write  $W = \begin{bmatrix} W_1 & \cdots & W_k \end{bmatrix}$  and it follows that

$$\begin{aligned} x(t) &= P e^{Ct} z = P e^{W \text{diag}(Y_1, \dots, Y_k) W^{-1} t} z = \\ &= P W e^{\text{diag}(Y_1, \dots, Y_k) t} W^{-1} z = P W \text{diag}(e^{Y_1 t}, \dots, e^{Y_k t}) W^{-1} z. \end{aligned}$$

Considering that  $PW = \begin{bmatrix} (W_1)_1 & \cdots & (W_k)_1 \end{bmatrix}$  and by writing

$$W^{-1}z = \begin{bmatrix} z_1 \\ \vdots \\ z_k \end{bmatrix},$$

we get

$$\begin{aligned} x(t) &= \begin{bmatrix} (W_1)_1 e^{Y_1 t} & \cdots & (W_k)_1 e^{Y_k t} \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_k \end{bmatrix} \\ &= (W_1)_1 e^{Y_1 t} z_1 + \cdots + (W_k)_1 e^{Y_k t} z_k. \end{aligned}$$

□

The goal here is to get a general solution when it is not possible to achieve it with solvents. We see this in the following example.

*Example 1.* Consider the differential equation

$$P\left(\frac{d}{dt}\right)x = x^{(2)}(t) + A_1 x^{(1)}(t) + A_2 x(t),$$

with coefficients given by

$$A_1 = \begin{bmatrix} -98/25 & 108/25 & -112/25 \\ 4/5 & -24/5 & -4/5 \\ 22/25 & 38/25 & -182/25 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 89/25 & -294/25 & 316/25 \\ -7/5 & 42/5 & -8/5 \\ -46/25 & -59/25 & 251/25 \end{bmatrix},$$

the associated matrix polynomial is

$$P(X) = X^2 + A_1 X + A_2,$$

where  $m = 2$  and  $n = 3$ . We have that

$$V_1 = \begin{bmatrix} 2 & 2 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} -2 & -2 & 5 & 3 \\ 2 & 2 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

are the Jordan chains of  $P(\lambda)$  and the respective Jordan blocks are

$$J_1 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

It can be verified that there are no nonsingular matrices of order 3 with the leading vectors of  $V_1$  and  $V_2$  (Jordan chains of  $P(\lambda)$ ), hence  $P(X)$  has no solvents at all,

therefore a general solution of  $P\left(\frac{d}{dt}\right)x = 0$  in terms of solvents does not exist. On the other hand, we have  $C = S \operatorname{diag}(J_1, J_2)S^{-1}$ , where

$$S = \begin{bmatrix} V_1 & V_2 \\ V_1 J_1 & V_2 J_2 \end{bmatrix}$$

is nonsingular. Thus we have from Theorem 2 that  $J_1$  and  $J_2$  form a complete set of block eigenvalues of  $C$  and

$$\begin{bmatrix} V_1 \\ V_1 J_1 \end{bmatrix} \text{ and } \begin{bmatrix} V_2 \\ V_2 J_2 \end{bmatrix}$$

are the corresponding block eigenvectors. Hence from Theorem 3 it follows that every solution of  $P\left(\frac{d}{dt}\right)x = 0$  is of the form

$$x(t) = V_1 e^{J_1 t} z_1 + V_2 e^{J_2 t} z_2.$$

The considered block eigenvalues are in a canonical form, but in general this is not necessary. In fact, if  $Y_1$  and  $Y_2$  are any matrices similar to  $J_1$  and  $J_2$ , respectively, then  $Y_1$  and  $Y_2$  are also block eigenvalues, as pointed out before. Thus if we write  $Y_1 = T_1^{-1} J_1 T_1$  and  $Y_2 = T_2^{-1} J_2 T_2$ , with  $T_1$  and  $T_2$  nonsingular. It follows that

$$\begin{aligned} C &= S \operatorname{diag}(T_1 Y_1 T_1^{-1}, T_2 Y_2 T_2^{-1}) S^{-1} = \\ &= S \operatorname{diag}(T_1, T_2) \operatorname{diag}(Y_1, Y_2) \operatorname{diag}(T_1, T_2)^{-1} S^{-1} = \\ &= U \operatorname{diag}(Y_1, Y_2) U^{-1}, \end{aligned}$$

where  $U = S \operatorname{diag}(T_1, T_2)$  is nonsingular and so  $Y_1$  and  $Y_2$  are a complete set of block eigenvalues of  $C$  (from Theorem 2) with

$$\begin{bmatrix} V_1 T_1 \\ V_1 J_1 T_1 \end{bmatrix} \text{ and } \begin{bmatrix} V_2 T_2 \\ V_2 J_2 T_2 \end{bmatrix}$$

being the corresponding block eigenvectors. Hence from Theorem 3 every solution of  $P\left(\frac{d}{dt}\right)x = 0$  can be written in the form

$$x(t) = V_1 T_1 e^{Y_1 t} z_1 + V_2 T_2 e^{Y_2 t} z_2.$$

Numerical procedures to compute a complete set of block eigenvalues can be found in [7].

**Acknowledgement.** *The author is grateful to J. Vitória and T. P. Lima for helpful discussions. This work was partially supported by ISR Polo de Coimbra.*

#### REFERENCES

- [1] DENNIS, J. E., TRAUB, J. F. and WEBER, R. P.: *The algebraic theory of matrix polynomials*, SIAM J. Numer. Anal., **13**, (1976), 831–845.
- [2] GANTMACHER, F.: *The Theory of Matrices, I, II*, Chelsea Publishing Co., New York, 1960.
- [3] GOHBERG, I., LANCASTER, P. and RODMAN, L.: *Matrix Polynomials*, Academic Press, New York, 1982.

- [4] LANCASTER, P.: *Lambda-Matrices and Vibrating Systems*, Pergamon Press, Oxford, 1966.
- [5] LANCASTER, P. AND TISMENETSKY, M.: *The Theory of Matrices*, 2nd edition, Academic Press, New York, 1985.
- [6] MARCUS, A. S. and MEREUTSA, I. V.: *On the complete  $n$ -tuple of roots of the operator equation corresponding to a polynomial operator bundle*, Math. USSR Izv., **7**, (1973), 1105–1128.
- [7] PEREIRA, E. and VITÓRIA, J.: *Deflation for block eigenvalues of block partitioned matrices with an application to matrix polynomials of commuting matrices*, Comput. Math. Appl., **42**, (2001), 1177–1188.
- [8] WILKINSON, J. H.: *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford, London, 1965.
- [9] ZABALLA, I.: *Matrices with prescribed rows and invariant factors*, Linear Algebra Appl., **87**, (1987), 113–146.

*Author's Address*

**Edgar Pereira:**

DEPARTAMENTO DE INFORMÁTICA, UNIVERSIDADE DA BEIRA INTEIROR, 6200 - COVILHÁ, PORTUGAL  
E-mail address: edgar@noe.ubi.pt



## OSCILLATION OF NONLINEAR DIFFERENTIAL-INTEGRAL EQUATION IN A BANACH SPACE WITH RESPECT TO ITS SUBSPACE

N.A. PERESTYUK AND V.YU. SLUSARCHUK

[Received: January 17, 2003]

**ABSTRACT.** The conditions for oscillation of the solution of a nonlinear differential-integral equation in a Banach space with respect to its subspace are established.

*Mathematics Subject Classification:* 34C15, 34G20

*Keywords:* differential-integral equation, Banach space, oscillation with respect to a subspace, impulse system

### 1. INTRODUCTION

Let  $E$  be a real Banach space and let  $E_1$  be an arbitrary subspace of  $E$  with  $\text{codim} E_1 = 1$ . Let also  $\varphi$  be a linear continuous functional on  $E$  with  $\ker \varphi = E_1$ . Finally, let  $P$  be a class of functions  $p : [0, +\infty) \rightarrow R$ , which are non-decreasing on  $[0, +\infty)$  and left continuous on  $(0, +\infty)$  and such that

$$\lim_{t \rightarrow 0^+} p(t) = p(0) \quad \text{and} \quad \sup_{t \geq 0} (p(t+1) - p(t)) < +\infty.$$

Consider now the following differential-integral equation

$$\frac{d_- x(t)}{dt} - \frac{d_+ x(t)}{dt} + \sum_{k=1}^n \int_s^t [dp_k(u)] F_k(x(u)) = 0, \quad t \geq s \geq 0, \quad (1.1)$$

where:  $n \in N$ ;  $p_k \in P$ ,  $k = 1, 2, \dots, n$ ;  $F_k : E \rightarrow E$ ,  $k = 1, 2, \dots, n$

are continuous mappings; the integrals in (1.1) are Lebesgue-Stieltjes ones;  $\frac{d_- x(t)}{dt}$  and  $\frac{d_+ x(t)}{dt}$  are the left and right derivatives of the solution  $x(t)$  of (1.1) at the point  $t$ , respectively. Observe that at the point  $t = 0$  the left derivative of  $x(t)$  is  $\frac{d_- x(0)}{dt}$ , which is compatible with the requirement  $\lim_{t \rightarrow 0^+} p_k(t) = p_k(0)$ ,  $k = 1, 2, \dots, n$ , and the continuity of  $F_k$ ,  $k = 1, 2, \dots, n$ , on  $E$ .

As it is known (see, for example, [6]), each function  $p_k(t)$ ,  $t \geq 0$ ,  $k = 1, 2, \dots, n$ , can be represented in the form

$$p_k(t) = p_{c,k}(t) + p_{s,k}(t) + p_{d,k}(t), \quad k = 1, 2, \dots, n, \quad (1.2)$$

where  $p_{c,k}(t)$  is an absolutely continuous function on each segment of the form  $[0, a]$  with  $a \in (0, +\infty)$ , while  $p_{s,k}(t)$  and  $p_{d,k}(t)$  are the singular and discrete components of  $p_k(t)$ , respectively.

If in (1.3) for every  $t \geq 0$

$$p_{s,k}(t) \equiv \text{const}, \quad k = 1, 2, \dots, n, \quad (1.3)$$

then equation (1.1) is equivalent to a differential equation with impulse disturbance, while if in (1.3) for every  $t \geq 0$

$$p_{s,k}(t) \equiv \text{const} \quad \text{and} \quad p_{d,k}(t) \equiv \text{const}, \quad k = 1, 2, \dots, n,$$

then equation (1.1) is equivalent to a differential equation without impulse disturbance. Such differential equations have been studied in [3] and [8], when  $\dim E < +\infty$ , and in [2]. Finally, note that if, in (1.3), for every  $t \geq 0$ ,

$$p_{c,k}(t) \equiv \text{const} \quad \text{and} \quad p_{s,k}(t) = \text{const}, \quad k = 1, 2, \dots, n,$$

then equation (1.1) is equivalent to a difference equation, the theory of which has been developed in [5].

The purpose of this article is to give conditions for oscillation of the solution of equation (1.1) with respect to the subspace  $E_1$  of the Banach space  $E$ . The notion of such oscillation is stated and used successfully in [10] – [14] and [16] and is defined as follows.

**Definition 1.** A solution  $x(t)$  of equation (1.1) is said to be oscillatory with respect to the subspace  $E_1$  of the Banach space  $E$ , if for every number  $a > 0$  there are points  $\tau_1, \tau_2 \in (a, +\infty)$  such that

$$\varphi(\tau_1)\varphi(\tau_2) < 0.$$

As it follows from [10] – [14] and [16], this definition is a convenient tool for the investigation of oscillation of trajectories of dynamical systems with infinite dimensional phase space. Note that, in the case when  $\dim E_1 = 1$ , the above definition of the oscillation of a solution of (1.1) with respect to  $E_1$  is equivalent to the corresponding one given the real solutions of the differential equations considered, for example, in [9].

An interesting class of equations is the class of differential equations with impulse disturbance, which so far has not been investigated sufficiently. Therefore, it is very important for the theory of systems with impulse, as well as for the theory of differential-integral equations to investigate the oscillatory nature of solutions of equations of form (1.1).

## 2. PRELIMINARIES

In order to state our main results, we need the following notions and assumptions.

(I) Assume that for any number  $t_0 \geq 0$  and for any set  $\{x_1, x_2\} \subset E$  equation (1.1) admits a unique solution  $x(t)$ , which is defined and continuous on  $[0, +\infty)$  and satisfies the following two conditions:

$$x(t_0) = x_1 \quad (2.1)$$

and

$$\frac{d_+ x(t_0)}{dt} = x_2, \quad (2.2)$$

where, if  $(x_1, x_2) \in E_1^2$ , then  $x(t) \in E_1$  for all  $t \geq t_0$ .

A solution of (1.1), satisfying (2.1) and (2.2), will be denoted by

$$x(t, t_0, x_1, x_2).$$

For some sufficient conditions, concerning the requirement (I), see [17].

(II) Consider the sets

$$E_2 = \{x \in E : \varphi(x) > 0\} \text{ and } E_3 = \{x \in E : \varphi(x) < 0\},$$

and the continuous mappings

$$F_k : E \rightarrow E, \quad k = 1, 2, \dots, n. \quad (2.3)$$

Assume that the sets  $E_1, E_2$  and  $E_3$  are invariant ones with respect to each of the mappings (2.3), i.e.,

$F_k E_i \subset E_i$  for all  $k = 1, 2, \dots, n$ , and all  $i = 1, 2, 3$ .

(III) Let  $D_k$  be the set of all points of  $[0, +\infty)$  in which the function  $p_k \equiv p_k(t)$  is differentiable, and let  $D$  be the intersection of the sets  $D_k$ ,  $k = 1, 2, \dots, n$ . Denote by  $\Phi_k$  the set of functions  $z \equiv z_k(t)$  which are continuous on  $[0, +\infty)$  and differentiable on  $D$  with values in  $E_i$ ,  $i = 1, 2, 3$ , for each of which

$$|\varphi(z_k(t))|, \quad k = 1, 2, \dots, n,$$

is a monotone non-decreasing on  $[0, +\infty)$  function.

Note that, because of the inclusion  $p \in P$ , the set  $[0, +\infty)/D_k$  has zero Lebesgue measure.

## 3. MAIN RESULTS

Here we formulate and prove a sufficient result (Theorem 1) and a necessary result (Theorem 2) as follows.

**Theorem 1.** Consider equation (1.1), subject to the conditions (I) – (III), and assume that for  $k = 1, 2, \dots, n$ :

$$(i) \quad \inf_{s \geq t \geq 0} \frac{\varphi(F_k(z(s)))}{\varphi(F_k(z(t)))} > 0 \quad \text{for all } z \in \Phi_2 \cup \Phi_3,$$

(ii) *The improper integrals*

$$\int_0^{+\infty} \frac{d\varphi(z(t))}{\varphi(F_k(z(t)))}$$

converge for all  $z \in \Phi_2 \cup \Phi_3$  and

$$(iii) \sum_{k=1}^n \int_0^{+\infty} t dp_k(t) = +\infty.$$

Then all solutions  $x(t, 0, x_1, x_2)$ , where  $(x_1, x_2) \in E^2 \setminus E_1^2$ , of equation (1.1) are oscillatory with respect to the subspace  $E_1$ .

*Proof.* Let  $z(t)$  be non-oscillatory with respect to  $E_1$  solution of (1.1), for which

$$\left( z(0), \frac{d_+ z(0)}{dt} \right) \notin E_1^2. \quad (3.1)$$

Then, without loss of generality, we can (and do) suppose that for some  $a \in (0, +\infty)$

$$\varphi(z(a)) > 0 \quad \text{and} \quad \varphi(z(t)) \geq 0, \quad t \geq a. \quad (3.2)$$

Since, in view of (1.1), for arbitrary  $t \geq a$  and  $s > t$

$$\frac{d_- z(s)}{ds} - \frac{d_+ z(t)}{dt} + \sum_{k=1}^n \int_t^s [dp_k(u)] = 0,$$

and consequently for  $s \geq t \geq 0$

$$\varphi\left(\frac{d_- z(s)}{ds}\right) - \varphi\left(\frac{d_+ z(t)}{dt}\right) + \sum_{k=1}^n \int_t^s [dp_k(u)] \varphi(F_k(z(u))) = 0, \quad (3.3)$$

and, by (I) – (III) and (3.2), we have

$$\sum_{k=1}^n \int_t^s [dp_k(u)] \varphi(F_k(z(u))) \geq 0$$

for all  $t$  and  $s$  with  $s > t \geq a$ , we conclude that the function  $\varphi\left(\frac{d_+ z(t)}{dt}\right)$  is non-increasing on  $[a, +\infty)$ . Thus, the function  $\varphi(z(t))$  is concave on  $[a, +\infty)$  (see, [18], page 17) and, in view of (3.2) and the concavity of the function  $\varphi(z(t))$  on  $[a, +\infty)$ , we derive that

$$\lim_{t \rightarrow +\infty} \varphi\left(\frac{d_+ z(t)}{dt}\right) = c, \quad c \in [0, +\infty). \quad (3.4)$$

Now, (3.4) and (3.3) imply that function  $\varphi(z(t))$  is monotone non-decreasing on  $[a, +\infty)$  and for all  $t \geq a$

$$\varphi\left(\frac{d_+z(t)}{dt}\right) = c + \sum_{k=1}^n \int_t^{+\infty} [dp_k(u)] \varphi(F_k(z(u))). \quad (3.5)$$

Denote by  $\delta$  the least of the numbers

$$\inf_{s \geq t \geq 0} \frac{\varphi(F_k(z(s)))}{\varphi(F_k(z(t)))} > 0, \quad k = 1, 2, \dots, n \text{ and } z \in \Phi_2 \cup \Phi_3$$

which, by (i), is positive. Taking into account the inclusion  $z \in \Phi_2$ , (ii) and (3.5), we see that

$$\begin{aligned} +\infty &> \int_a^{+\infty} \frac{d\varphi(z(t))}{\varphi(F_k(z(t)))} > \int_{a+1}^{+\infty} \frac{d\varphi\left(\frac{d_+z(t)}{dt}\right)}{\varphi(F_k(z(t)))} = \\ &= \int_{a+1}^{+\infty} \frac{1}{\varphi(F_k(z(t)))} \left( c + \sum_{k=1}^n \int_t^{+\infty} [dp_k(u)] \varphi(F_k(z(u))) \right) dt \geq \\ &\geq \int_{a+1}^{+\infty} \left( \int_t^{+\infty} [dp_k(u)] \frac{\varphi(F_k(z(u)))}{\varphi(F_k(z(t)))} \right) dt \geq \delta \int_{a+1}^{+\infty} (t - a - 1) dp_k(t), \end{aligned}$$

which implies that the improper integrals

$$\int_{a+1}^{+\infty} t dp_k(t), \quad k = 1, 2, \dots, n,$$

are convergent. But this contradicts (iii). The contradiction obtained shows that the assumption (2.3) is false and the proof of the theorem is complete.  $\square$

In order to state our next result, we need the following

**Definition 2.** A mapping  $h : E \rightarrow E$  is called Lipschitzian, if for arbitrary  $b \in E$  and  $r \in (0, +\infty)$  there exists a constant  $M > 0$  such that

$$\|h(x) - h(y)\|_E \leq M \|x - y\|_E$$

for all  $x, y \in B(b, r) = \{x \in E : \|x - b\|_E \leq r\}$ .

**Theorem 2.** Consider equation (1.1), subject to the conditions (I) – (III), and assume that:

- (iv) the mappings  $F_k$ ,  $k = 1, 2, \dots, n$ , are:
  - (a) locally Lipschitzian, or
  - (b) completely continuous;
- (v) the functions  $p_k \in P$ ,  $k = 1, 2, \dots, n$ , are continuous on  $[0, +\infty)$ ;
- (vi) all solutions  $x(t, 0, x_1, x_2)$ , where  $(x_1, x_2) \in E^2 \setminus E_1^2$ , of equation (1.1) are oscillatory with respect to  $E$ .

Then condition (iii) of Theorem 1 holds, i. e.,

$$(iii) \sum_{k=1}^n \int_0^{+\infty} t dp_k(t) = +\infty.$$

*Proof. Case 1.* The mappings  $F_k$ ,  $k = 1, 2, \dots, n$ , are locally Lipschitzian, i.e. (iv,a) holds. In this case, assume that (iii) is false, i.e.

$$0 \leq \sum_{k=1}^n \int_0^{+\infty} t dp_k(t) < +\infty. \quad (3.6)$$

For an arbitrary vector  $y \in E_2 \cup E_3$ , and a closed ball  $B(y, r)$ ,  $r > 0$ , for which  $E_1 \cap B(y, r) = \emptyset$ , consider the equation

$$z(t) = y - \sum_{k=1}^n \int_t^{+\infty} [dp_k(u)] (u-t) F_k(z(u)), \quad t \geq a, \quad (3.7)$$

where  $a$  is chosen in such a way that

$$\sum_{k=1}^n \int_a^{+\infty} [dp_k(u)] (u-a) \sup_{x \in B(y, r)} \|F_k(x)\|_E \leq r \quad (3.8)$$

and

$$\begin{aligned} & \sup_{t \geq a} \left\| \sum_{k=1}^n \int_t^{+\infty} [dp_k(u)] (u-t) F_k(z_1(u)) - \sum_{k=1}^n \int_t^{+\infty} [dp_k(u)] (u-t) F_k(z_2(u)) \right\|_E \leq \\ & \leq \frac{1}{2} \sup_{t \geq a} \|z_1(t) - z_2(t)\|_E \end{aligned} \quad (3.9)$$

for all  $E$ -valued functions  $z_i(t)$ ,  $i = 1, 2$  which are continuous and bounded on  $[a, +\infty)$  for which

$$\sup_{t \geq a} \|z_i(t) - y\|_E \leq r, \quad i = 1, 2.$$

Note that, because of (iv,a) and (3.3), the relations (3.8) and (3.9) are valid.

Next consider a Banach space  $X$  of  $E$ -valued functions  $x = x(t)$  which are continuous and bounded on  $[a, +\infty)$  with the norm

$$\|x\|_E = \sup_{t \geq a} \|x(t)\|_E.$$

Furthermore, consider a bounded, closed and convex set  $Y$  of all functions  $x \in X$ , such that  $x \in B(y, r)$  for all  $t \geq a$ , and an operator  $U : X \rightarrow X$  defined by the formula

$$(Ux)(t) = y - \sum_{k=1}^n \int_t^{+\infty} [dp_k(u)] (u-t) F_k(x(u)), \quad t \geq a \quad (3.10)$$

Using (3.8) and (3.9), from (3.10) it follows that  $UY \subset X$  and  $\|Uv - Uw\|_X \leq \frac{1}{2} \|v - w\|_X$  for all  $v, w \in Y$ . Hence, applying the contraction mapping principle (see,

[15], page 72), we conclude that there is a function  $z \in Y$  which is a solution of equation (3.7). The same function  $z$  is also a solution to equation (1.1). Indeed, by (3.6) and (v), we see that

$$\begin{aligned} \frac{d_- z(t)}{dt} &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (z(t) - z(t - \varepsilon)) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left( - \sum_{k=1}^n \int_t^{+\infty} [dp_k(u)] (u - t) F_k(z(u)) + \right. \\ &\quad \left. + \sum_{k=1}^n \int_{t-\varepsilon}^{+\infty} [dp_k(u)] (u - t + \varepsilon) F_k(z(u)) \right) = \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left( \varepsilon \sum_{k=1}^n \int_t^{+\infty} [dp_k(u)] F_k(z(u)) + \sum_{k=1}^n \int_{t-\varepsilon}^t [dp_k(u)] (u - t + \varepsilon) F_k(z(u)) \right) = \\ &= \sum_{k=1}^n \int_t^{+\infty} [dp_k(u)] F_k(z(u)) + \lim_{\varepsilon \rightarrow 0} \sum_{k=1}^n \int_{t-\varepsilon}^t [dp_k(u)] \frac{u-t}{\varepsilon} F_k(z(u)) + \\ &\quad + \lim_{\varepsilon \rightarrow 0} \sum_{k=1}^n \int_{t-\varepsilon}^t [dp_k(u)] F_k(z(u)), \end{aligned}$$

where the two last terms tend to zero, because the integrand functions are bounded on  $[t - \varepsilon, t]$ ,  $\bigvee_{t-\varepsilon}^t [p_k] = p_i$ ,  $\bigvee_{t-\varepsilon}^t [p_k] = p_k(t) - p_k(t - \varepsilon)$ ,  $k = 1, 2, \dots, n$ , and the functions  $p_k(t)$ ,  $k = 1, 2, \dots, n$  are continuous on  $[0, +\infty)$ . Therefore, it follows that

$$\frac{d_- z(t)}{dt} = \sum_{k=1}^n \int_t^{+\infty} [dp_k(u)] F_k(z(u)). \quad (3.11)$$

In the same manner we see that

$$\begin{aligned} \frac{d_+ z(s)}{ds} &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (z(s + \varepsilon) - z(s)) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left( - \sum_{k=1}^n \int_{s+\varepsilon}^{+\infty} [dp_k(u)] (u - s - \varepsilon) F_k(z(u)) + \right. \\ &\quad \left. + \sum_{k=1}^n \int_s^{+\infty} [dp_k(u)] (u - s) F_k(z(u)) \right) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left( \varepsilon \sum_{k=1}^n \int_s^{+\infty} [dp_k(u)] F_k(z(u)) + \right. \\ &\quad \left. + \sum_{k=1}^n \int_s^{s+\varepsilon} [dp_k(u)] F_k(z(u)) \right) = \\ &= \sum_{k=1}^n \int_s^{+\infty} [dp_k(u)] F_k(z(u)) + \lim_{\varepsilon \rightarrow 0} \sum_{k=1}^n \int_s^{s+\varepsilon} [dp_k(u)] \frac{u-s}{\varepsilon} F_k(z(u)) - \\ &\quad - \lim_{\varepsilon \rightarrow 0} \sum_{k=1}^n \int_s^{s+\varepsilon} [dp_k(u)] F_k(z(u)) = \sum_{k=1}^n \int_s^{+\infty} [dp_k(u)] F_k(z(u)). \end{aligned} \quad (3.12)$$

Subtracting (3.12) from (3.11), we obtain

$$\begin{aligned} \frac{d_- z(t)}{dt} - \frac{d_+ z(s)}{ds} &= \sum_{k=1}^n \int_t^{+\infty} [dp_k(u)] F_k(z(u)) - \\ &\quad - \sum_{k=1}^n \int_s^{+\infty} [dp_k(u)] F_k(z(u)) = - \sum_{k=1}^n \int_s^t [dp_k(u)] F_k(z(u)), \end{aligned}$$

which was to be shown. As an element of  $Y$ , the function  $z(t)$  is not oscillatory with respect to  $E_1$ . By (I) – (III), there is a solution  $y(t)$  of equation (1.1), which coincides with  $z(t)$  on  $[a, +\infty)$ . For this solution it holds

$$\left(y(0), \frac{d_+ y(0)}{dt}\right) \in E^2 \setminus E_1^2$$

and, due to the uniqueness of a solution  $x(t)$  of (1.1), we see that

$$x(0) = y(0) \quad \text{and} \quad \frac{d_+ x(0)}{dt} = \frac{d_+ y(0)}{dt}.$$

Note that, if

$$\left(x(0), \frac{d_+ x(0)}{dt}\right) \in E_1^2,$$

then equation (1.1) has a solution  $x(t)$  for which

$$\left(x(0), \frac{d_- x(0)}{dt}\right) \in E_1^2 \quad \text{for all } t \geq 0.$$

From the above observation it follows that, if (iii) does not hold, then equation (1.1) has a non-oscillatory solution  $x = x(t)$  with respect to  $E_1$  for which

$$\left(x(0), \frac{d_+ x(0)}{dt}\right) \in E^2 \setminus E_1^2.$$

But this is a contradiction, which, in the case when the mapping  $F_k$ ,  $k = 1, 2, \dots, n$  are locally Lipschitzian, proves the necessity of the condition (iii) for the oscillation of solution of (1.1) with respect to  $E_1$ .

**Case 2.** The mappings  $F_k$ ,  $k = 1, 2, \dots, n$ , are completely continuous, i.e., (iv, b) holds.

Assume that (iii) does not hold. Then (3.6) is satisfied. Let  $y \in E_2 \cup E_3$  be an arbitrary vector and let  $r > 0$  be a number such that  $E_1 \cap B(y, r) = \emptyset$ . Next, consider the equation (3.7), where  $a \in [0, +\infty)$  is such that (3.8) holds. Such a choice of  $a$  is possible because of (3.6) and the compactness of the mappings  $F_k$ ,  $k = 1, 2, \dots, n$ . Further, as in Case 1, consider the Banach space  $X$ , the bounded, closed and convex set  $Y$  of all functions  $x \in X$  for which  $x(t) \in B(y, r)$  for all  $t \geq a$ , and the operator  $U : X \rightarrow X$  defined by the formula (3.10). By (3.10) and (3.8), it follows that  $UY \subset X$ .

Consider now the function

$$\delta(t) = L \sum_{k=1}^n \int_t^{+\infty} (u-t) dp_k(u), \quad t \geq a,$$

where

$$L = \max_{\substack{1 \leq k \leq n \\ x \in B(y, r)}} \|F_k(x)\|_E.$$

For arbitrary  $y \in Y$  and  $t, s \in [a, +\infty)$ ,  $t \geq s$ , we see that

$$\begin{aligned} & \| (Uy)(t) - (Uy)(s) \|_E = \\ & = \left\| \sum_{k=1}^n \int_t^{+\infty} [dp_k(u)] (u-t) F_k(y(u)) - \sum_{k=1}^n \int_s^{+\infty} [dp_k(u)] (u-s) F_k(y(u)) \right\|_E = \\ & = \left\| \sum_{k=1}^n \int_s^t [dp_k(u)] (u-t) F_k(y(u)) - (t-s) \sum_{k=1}^n \int_t^{+\infty} [dp_k(u)] F_k(y(u)) \right\|_E \leq \\ & \leq L(t-s) \sum_{k=1}^n \int_s^t V[p_k] + L(t-s) \delta(t). \end{aligned}$$

Hence, in view of the fact that

$$\sup_{\substack{1 \leq k \leq n \\ t \geq 0}} \int_t^{t+1} V[p_k] < +\infty,$$

and because of the boundedness of the function  $\delta(t)$  on  $[a, +\infty)$ , we conclude that the set of all  $z \in UY$  is equicontinuous on  $[a, +\infty)$ . Since for every  $z \in UY$

$$\|z(t) - y\|_E \leq \delta(t), \quad t \geq a$$

and

$$\lim_{t \rightarrow +\infty} \delta(t) = 0,$$

in view of (i) and the generalized Arzelà theorem (see, [4], page 110), we conclude that the set  $UY$  is relatively compact. Thus,  $U : Y \rightarrow Y$  is a completely continuous mapping. Consequently, according to Schauder's Fixed Point Theorem (see, [7], p. 37), the mapping  $U$  has a fixed point  $z \in Y$ , which is a solution of the equation (1.1) on  $[a, +\infty)$ .

Finally, using arguments similar to those in Case 1, we come again to a contradiction. Therefore, the necessity of the condition (iii) for the oscillation of the solution of (1.1) with respect to  $E_1$  in the case of compact mapping  $F_k$ ,  $k = 1, 2, \dots, n$ , is also established.

The proof of the theorem is complete.  $\square$

#### 4. APPLICATIONS

Let  $T$  be an arbitrary countable set of real numbers  $t_n$ ,  $n \in N$  with

$$0 < t_1 < t_2 < \dots < t_n < \dots \quad \text{and} \quad \lim_{n \rightarrow +\infty} t_n = +\infty.$$

Moreover, let for  $m \in N$

$$c_k : [0, +\infty) \setminus T \rightarrow [0, +\infty), \quad k = 1, 2, \dots, m$$

be bounded and continuous functions,

$$q_k : T \rightarrow [0, +\infty), \quad k = m+1, m+2, \dots, 2m,$$

be arbitrary mappings with a bounded set of values, and

$$G_k : E \rightarrow E, \quad k = 1, 2, \dots, 2m,$$

be continuous mappings, for which

$$G_k E_i \subset E_i \quad \text{for all } k = 1, 2, \dots, 2m \quad \text{and } i = 1, 2, 3.$$

Then, consider an impulse system, described by the equations

$$\begin{cases} \frac{d^2 x(t)}{dt^2} + \sum_{k=1}^m c_k(t) G_k(x(t)) = 0, & t \in [0, +\infty) \setminus T, \\ \frac{d_+ x(t)}{dt} - \frac{d_- x(t)}{dt} + \sum_{k=m+1}^{2m} q_k(t) G_k(x(t-0)) = 0, & t \in T, \\ x(t+0) = x(t-0) = x(t), & t \in T. \end{cases} \quad (4.1)$$

Furthermore, let

$$p_k = p_k(t) \in P, \quad k = 1, 2, \dots, 2m,$$

be functions defined as follows

$$p_k(t) = \begin{cases} \int_0^t c_k(s) ds & \text{for } k = 1, 2, \dots, m, \\ \sum_{s \in [0, t) \cap T} q_k(s) & \text{for } k = m+1, m+2, \dots, 2m, \end{cases}$$

where  $\int_0^t C_k(s) ds$  is a Lebesgue integral, while

$$p_k(t) = 0, \quad k = m+1, m+2, \dots, 2m, \quad \text{if } [0, t) \cap T = \emptyset,$$

and consider the equation

$$\frac{d_- x(t)}{dt} - \frac{d_+ x(t)}{dt} + \sum_{k=1}^n \int_s^1 [dp_k(u)] G_k(x(u)), \quad t \geq s \geq 0. \quad (4.2)$$

It is not difficult to see that every solution of system (4.1) is a solution of equation (4.2) and vice versa. Therefore, Theorem 1 can be used to investigate the oscillation of solutions of system (4.1). As an application of Theorem 1 to system (4.1), we state the following result.

**Theorem 3.** Consider system (4.1) and assume that for arbitrary elements  $x_1, x_2 \in E$  system (4.1) has a unique solution  $x(t, 0, x_1, x_2)$  defined on  $[0, +\infty)$ ;

$$\inf_{s \geq t \geq 0} \frac{\varphi(G_k(z(s)))}{\varphi(G_k(z(t)))} > 0, \quad k = 1, 2, \dots, 2m,$$

for all  $z \in \Phi_2 \cup \Phi_3$ ;  
the improper integrals

$$\int_0^{+\infty} \frac{d\varphi(z(t))}{\varphi(G_k(z(t)))}, \quad k = 1, 2, \dots, 2m,$$

converge for all  $z \in \Phi_2 \cup \Phi_3$ , and

$$\int_0^{+\infty} t \sum_{k=1}^m c_k(t) dt + \sum_{t \in T} t \sum_{k=m+1}^{2m} q_k(t) = +\infty.$$

Then for arbitrary  $(x_1, x_2) \in E^2 \setminus E_1^2$  solution  $x(t, 0, x_1, x_2)$  of system (4.1) oscillates with respect to  $E_1$ .

**Remark 1.** 1. For sufficient conditions, ensuring the fulfillment of the conditions (B) and (C) of Theorem 3, see [16].

2. As it is shown in [16], in the case of locally Lipschitzian or completely continuous mappings  $G_k$ ,  $k = 1, 2, \dots, 2m$ , the fulfillment of the condition (D) is also a sufficient condition for oscillation of solutions  $x(t, 0, x_1, x_2)$ , with  $(x_1, x_2) \in E^2 \setminus E_1^2$ , of system (4.1) with respect to  $E_1$ .

3. Necessary and sufficient conditions for oscillation of system (4.1), when  $m = 1$ ,  $\dim E = 1$  and condition (C) is not satisfied, can be found in [15].

A special case of Theorems 1 and 2 is the following result, due to F. Atkinson [1]:

**Theorem 4.** Let  $p(t)$  be a continuous and positive function for  $t \geq 0$ , and let  $m$  be an integer greater than 1. Then the condition

$$\int_0^{+\infty} tp(t) dt = +\infty$$

is a necessary and sufficient one for oscillation of all (except trivial) solutions of the equation

$$y''(t) + p(t)y^{2m-1}(t) = 0.$$

#### REFERENCES

- [1] ATKINSON, F.: On second order non-linear oscillations, Pacific J. Math., 5, (1995), 643-647.
- [2] DALETSKII, YU. L. and KREYN M. G.: Stability of Solutions of Differential Equations in a Banach Space, Moscow, 1970. (in Russian)
- [3] HALANAY, A. and WEXLER, D.: Qualitative Theory of Sampled-data Systems, Editura Acad. RSR, Bucharest, 1968 (in Rumanian). Russian edition in 1971 by "Mir", Moscow.
- [4] KOLMOGOROV, A. N. and FOMIN, S. V.: Elements of Theory of Functions and Functional Analysis, Moscow, 1968. (in Russian)
- [5] MARTYNYUK, D. I.: Lectures on Qualitative Theory of Difference Equations, Kiev, 1972. (in Russian)

- [6] NATANSON, I. P.: *Theory of Real-variable Functions*, Moscow, 1974. (in Russian)
- [7] NIRENBERG, L.: *Lectures on Nonlinear Functional Analysis*, Moscow, 1977. (in Russian)
- [8] SAMOYLENKO, A. M. and PERESTYUK, N. A.: *Differential Equations with Impulse Disturbance*, Kiev, 1987. (in Russian)
- [9] SHEVELO, V. N.: *Oscillation of Solutions of Differential Equations with a Deviating Argument*, Kiev, 1978. (in Russian)
- [10] SLUSARCHUK, V. YU.: *Oscillation of Solutions of Differential-difference Equations in a Banach Space* (Ukrainian), "Nonlinear Differential Equations and their Applications", Kiev, 1993, pp. 66-70. (in Ukrainian)
- [11] SLUSARCHUK, V. YU.: *Oscillation of solutions of differential equations with impulse disturbance in a Banach space*, "Constructive methods of investigation of differential equations", Kiev, 1993, pp. 174-178. (in Ukrainian)
- [12] SLUSARCHUK, V.YU.: *Sufficient conditions for oscillation of trajectories of impulse systems with unfixed moments of impulse disturbance*, "Integral transformations and their applications to boundary-value problems", Kiev, 1994, pp. 192-197. (in Ukrainian)
- [13] SLUSARCHUK, V.YU.: *Oscillation of solutions of the differential equation  $\Delta^2 x(n) + \sum_{k=1}^m p_k(n) q_k(x(n)) = 0$  in a Banach space*, In: *Systems of Evolutionary Equations with Aftereffect*, Institute of Mathematics, NAS of Ukraine, Kiev, 1995, pp. 98–102. (in Ukrainian)
- [14] SLUSARCHUK, V. YU.: *Oscillation of solutions of difference equations in a Banach space*, Proc. of Int. Math. Conference dedicated to the memory of Hans Han, Chernivtsi, 1995, pp. 169–175. (in Ukrainian)
- [15] SLUSARCHUK, V. YU.: *Necessary and sufficient conditions for oscillation of solutions of nonlinear impulse system with multiplicatively divided right-hand part*, Ukrain. Math. J., **47**, (1995), 381-398. (in Ukrainian)
- [16] SLUSARCHUK, V. YU.: *Necessary and sufficient conditions for oscillation of solutions of nonlinear differential equations with impulse disturbance in a Banach space*, Ukrain. Math. J., **51**, (1999), 98-109. (in Russian)
- [17] SLUSARCHUK, V. YU.: *General theorems for existence and uniqueness of solutions of differential equations with impulse disturbance* (Russian), Ukrain. Math. J., **52** (2000), 954-964. (in Russian)
- [18] USHAKOV, R. R. and HACET, B. I.: *Convex Functions and Inequalities*, Kiev, 1986. (in Ukrainian)

### Authors' Addresses

#### N.A. Perestyuk:

MECHANICAL AND MATHEMATICAL FACULTY, KYIV NATIONAL TARAS SHEVCHENKO UNIVERSITY, VOLODYMYRSKA, 64, 01033, KYIV, UKRAINE,  
*E-mail address:* pmo@mechmat.univ.kiev.ua

#### V.Yu. Slusarchuk:

*Current address:* Mechanical and Mathematical Faculty, Kyiv National Taras Shevchenko University, Volodymyrska, 64, 01033, Kyiv, Ukraine,  
*E-mail address:* pmo@mechmat.univ.kiev.ua



## ON SOME TWO DIMENSIONAL VOLTERRA TYPE LINEAR INTEGRAL EQUATIONS WITH SUPER-SINGULARITY

NUSRAT RAJABOV, MIKLÓS RONTÓ, AND LUTFIA RAJABOVA

[Received: December 22, 2002]

**ABSTRACT.** In this paper the two dimensional linear Volterra type of integral equations containing singularity or super-singularity in the kernel function is considered. For different values of parameters in the integral equations existence theorems are proved for homogeneous and inhomogeneous equations.

*Mathematics Subject Classification:* 45E99

*Keywords:* two dimensional Volterra type integral equations, singularity, super-singularity, linear independent solutions

### 1. PROBLEM DESCRIPTION

Let  $D$  denote the rectangle  $D = \{(x, y) : a < x < x_0, b_0 < y < b\}$ ,  $\Gamma_1 = \{a < x < x_0, y = b\}$ ,  $\Gamma_2 = \{x = a, b_0 < y < b\}$ . In the domain  $D$  we consider the following integral equation

$$U(x, y) + \lambda \int_a^x \frac{U(t, y)}{(t-a)^\alpha} dt - \mu \int_y^b \frac{U(x, s)}{(b-s)^\beta} ds + \delta \int_a^x \frac{dt}{(t-a)^\alpha} \int_y^b \frac{U(t, s)}{(b-s)^\beta} ds(x, y) = f(x, y) \quad (1.1)$$

where  $\alpha > 0$ ,  $\beta > 0$  and  $\lambda, \mu, \delta$  are constants, the right-hand side function  $f(x, y) \in C(\bar{D})$  is a given function in  $\bar{D}$ , such that it vanishes on the lines  $\Gamma_1, \Gamma_2$ .

If the integral equation (1.1) has a solution and the corresponding integrals are convergent, then passing to the limit in (1.1) as  $y \rightarrow b$  and  $x \rightarrow a$ , we obtain on  $\Gamma_2$  and  $\Gamma_1$  the following two one-dimensional Volterra type integral equations:

$$U(a, y) - \mu \int_y^b \frac{U(a, s)}{(b-s)^\beta} ds = f(a, y), \quad (1.2)$$

$$U(x, b) + \lambda \int_a^x \frac{U(t, b)}{(t-a)^\alpha} dt = f(x, b). \quad (1.3)$$

The equations (1.2) and (1.3) are studied in detail in [1], [2].

It should be noted that the integral equation (1.1) under the following restrictions for the pairs of parameters  $\alpha < 1, \beta < 1$ ;  $\alpha < 1, \beta = 1$ ;  $\alpha = 1, \beta < 1$  and  $\alpha > 1, \beta > 1$  was investigated in [3], [4].

In this paper the general solution of the integral equation (1.1) is constructed for the pairs  $\alpha = 1, \beta > 1$  and  $\alpha > 1, \beta = 1$ . It will be established that for certain values of parameters  $\lambda$  and  $\mu$  the corresponding homogeneous integral equation (1.1) has an infinite number of linear independent solutions and for other values of  $\lambda, \mu$  the homogeneous integral equation (1.1) has no other solutions than the trivial solution. Under some additional conditions it will be proved that the inhomogeneous integral equation (1.1) for certain values of  $\lambda, \mu$  is solvable, and for some other values of  $\lambda, \mu$  has a unique solution. In the case when  $\delta = -\lambda\mu$  in (1.1), the solution can be found using a well known approach.

It seems to the authors that the integral equation (1.1) with super-singularity has not been studied yet.

## 2. SINGULARITY ON THE FIRST AND SUPER-SINGULARITY ON THE SECOND VARIABLE

**Theorem 2.1** *Let us suppose that  $\alpha = 1, \beta > 1, \mu > 0, \lambda < 0$ , and  $\delta = -\lambda\mu$  in integral equation (1.1). Moreover, assume that the behaviour of the right-hand side function  $f(x, y) \in C(\bar{D})$  is determined by the following asymptotic formulae*

$$f(x, y) = o[\exp(-\mu\omega_b^\beta(y))(b-y)^{\gamma_1}], \gamma_1 > \beta \quad (2.1)$$

$$\omega_b^\beta(y) = [(\beta-1)(b-y)^{\beta-1}]^{-1},$$

as  $y \rightarrow b$  and

$$f(x, y) = o[(x-a)^{\gamma_2}], \gamma_2 > |\lambda|, \quad (2.2)$$

as  $x \rightarrow a$ .

*Then the corresponding homogeneous integral equation (1.1) in the class of functions  $C(\bar{D})$  has an infinite number of linear independent solutions. The non-homogeneous integral equation (1.1) is always solvable in  $C(\bar{D})$  and its general solution containing four arbitrary functions is given by the following formula*

$$\begin{aligned} U(x, y) = & \exp[-\mu\omega_b^\beta(y)]\varphi_1(x) + (x-a)^{-\lambda}\psi_1(y) + \\ & + (x-a)^{-\lambda} \exp[-\mu\omega_b^\beta(y)] \int_y^b \exp[\mu\omega_b^\beta(s)]\psi_2(s)ds + \\ & + (x-a)^{-\lambda} \exp[-\mu\omega_b^\beta(y)] \int_a^x (t-a)^\lambda \varphi_2(t)dt + f(x, y) - \end{aligned}$$

$$\begin{aligned}
& -\lambda \int_a^x \left(\frac{t-a}{x-a}\right)^\lambda \frac{f(t,y)}{t-a} dt + \mu \int_y^b \frac{\exp[\mu(\omega_b^\beta(s) - \omega_b^\beta(y))]}{(b-s)^\beta} f(x,s) ds - \\
& -\lambda\mu \int_a^x \left(\frac{t-a}{x-a}\right)^\lambda \frac{dt}{t-a} \int_y^b \frac{\exp[\mu(\omega_b^\beta(s) - \omega_b^\beta(y))]}{(b-s)^\beta} f(t,s) ds \equiv \\
& \equiv T_{1,\beta}[\varphi_1(x), \varphi_2(x), \psi_1(y), \psi_2(y), f(x,y)], \tag{2.3}
\end{aligned}$$

In (2.3)  $\varphi_j(x)$ ,  $\psi_j(y)$ ,  $j = 1, 2$  are arbitrary functions defined on  $\Gamma_1$ ,  $\Gamma_2$ , such that  $\varphi_j(x) \in C(\Gamma_1)$ ,  $\psi_j(y) \in C(\Gamma_2)$  ( $j = 1, 2$ ) satisfying the following conditions as  $x \rightarrow a$ ,  $y \rightarrow b$ :

$$\varphi_1(x) = o[(x-a)^{\gamma_3}], \gamma_3 > 0 \quad \text{as } x \rightarrow a \tag{2.4}$$

$$\psi_1(y) = o[(b-y)^{\gamma_4}], \gamma_4 > \beta - 1 \quad \text{as } y \rightarrow b \tag{2.5}$$

$$\varphi_2(x) = o[(x-a)^{\gamma_5} \exp(-\mu\omega_b^\beta(y))], \gamma_5 > |\lambda| \quad \text{as } x \rightarrow a \tag{2.5}$$

$$\psi_2(y) = o[\exp(-\mu\omega_b^\beta(y))(b-y)^{\gamma_6}], \gamma_6 > \beta - 1 \quad \text{as } y \rightarrow b. \tag{2.7}$$

*Proof.* The proof of Theorem 2.1 essentially uses the theory of partial differential equations of hyperbolic type with singular and super-singular lines [5], [6]. For the values of parameters  $\alpha = 1$ ,  $\beta > 1$ ,  $\lambda < 0$ ,  $\mu > 0$  let the integral equation (1.1) have a differentiable solution in the domain  $D$ . Then by differentiating (1.1) with respect to  $x$  and  $y$ , we get the following equality

$$\frac{\partial^2 U}{\partial x \partial y} + \frac{\lambda}{x-a} \frac{\partial U}{\partial x} + \frac{\mu}{(b-y)^\beta} \frac{\partial U}{\partial x} - \frac{\delta}{(x-a)(b-y)^\beta} U = \frac{\partial^2 f}{\partial x \partial y}.$$

Thus we reduce the given problem to finding a solution of the so-called model type second order linear hyperbolic equation with one singular line on  $\Gamma_1$  and one super-singular line on  $\Gamma_2$ . The theory of the model equation obtained above is constructed in [5-6]. According to [5-6], the solution of the model equation for  $\lambda < 0$ ,  $\mu > 0$ ,  $\delta = -\lambda\mu$  has the form:

$$\begin{aligned}
U(x,y) &= \exp[-\mu\omega_b^\beta(y)]\varphi_1(x) + (x-a)^{-\lambda}\psi_1(y) + \\
&+ (x-a)^{-\lambda} \exp[-\mu\omega_b^\beta(y)] \int_y^b \exp[\mu\omega_b^\beta(s)]\psi_2(s)ds + \\
&+ (x-a)^{-\lambda} \exp[-\mu\omega_b^\beta(y)] \int_a^x (t-a)^\lambda \varphi_2(t)dt -
\end{aligned}$$

$$- \int_a^x \left( \frac{t-a}{x-a} \right)^\lambda dt \int_y^b \exp[\mu(\omega_b^\beta(s) - \omega_b^\beta(y))] \frac{\partial^2 f}{\partial t \partial s} ds.$$

Performing the necessary transformations in this expression and using the conditions of Theorem 2.1, we get the integral representation (2.3), which is obtained by supposing that  $\frac{\partial^2 f}{\partial x \partial y} \in C(\bar{D})$ . By direct computation one can check that under the conditions of the theorem the solution having the form (2.3) satisfies the integral equation (1.1) even in the case when  $f(x, y) \in C(\bar{D})$ .  $\square$

**Theorem 2.2** *Let us suppose that  $\alpha = 1$ ,  $\beta > 1$ ,  $\mu < 0$ ,  $\lambda > 0$ ,  $\delta = -\lambda\mu$  in the integral equation (1.1) and  $f(x, y) \in C(D)$ . In addition, assume that on the lines  $\Gamma_1$ ,  $\Gamma_2$  the function  $f(x, y) \in C(D)$  possesses the following asymptotic properties*

$$f(x, y) = o[(x-a)^{\gamma_7}], \quad \gamma_7 > 0, \quad (2.8)$$

as  $x \rightarrow a$  and

$$f(x, y) = o[(b-y)^{\gamma_8}], \quad \gamma_8 > \beta - 1 \quad (2.9)$$

as  $y \rightarrow b$ .

Then on the one hand, the corresponding homogeneous integral equation (2.1) has only the trivial solution in the class of functions  $C(\bar{D})$ . On the other hand, the non-homogeneous integral equation (1.1) is always uniquely solvable in  $C(\bar{D})$  and its solution is given by formula

$$U(x, y) = T_{1,\beta}[0, 0, 0, 0, f(x, y)]. \quad (2.10)$$

**Theorem 2.3.** *Let us assume that in the integral equation (1.1)  $\alpha = 1$ ,  $\beta > 1$ ,  $\mu < 0$ ,  $\lambda < 0$ ,  $\delta = -\lambda\mu$  and the asymptotic behaviour of the function  $f(x, y) \in C(\bar{D})$  is determined according to (2.2) and (2.9) as  $x \rightarrow a$  and as  $y \rightarrow b$ , respectively. Then the corresponding homogeneous integral equation (1.1) in  $C(\bar{D})$  has an infinite number of linear independent solutions. Moreover, the non-homogeneous equation (1.1) in  $C(\bar{D})$  is also always solvable and its general solution contains two arbitrary functions defined on  $\Gamma_2$  and given by the following formula*

$$U(x, y) = T_{1,\beta}[0, 0, \psi_1(y), \psi_2(y), f(x, y)], \quad (2.11)$$

where  $\psi_1(y)$ ,  $\psi_2(y)$  arbitrary functions defined on  $\Gamma_2$  such that  $\psi_1(y)$  has the property (2.5) and  $\psi_2(y)$  satisfy the condition

$$\psi_2(y) = o[(b-y)^{\gamma_9}], \quad \gamma_9 > \beta - 1. \quad (2.12)$$

as  $y \rightarrow b$

**Theorem 2.4.** *If in the integral equation (1.1)  $\alpha = 1$ ,  $\beta > 1$ ,  $\mu > 0$ ,  $\lambda > 0$ ,  $\delta = -\lambda\mu$  and the right hand side function  $f(x, y) \in C(\bar{D})$  vanishing on  $\Gamma_2$  has the asymptotic property (2.1) as  $y \rightarrow b$ , then the corresponding homogeneous integral*

equation (1.1) in  $C(\bar{D})$  is always solvable. In this case the general solution contains two arbitrary functions defined on  $\Gamma_1$  and can be written in the form

$$U(x, y) = T_{1,\beta}[\varphi_1(x), \varphi_2(x), 0, 0, f(x, y)], \quad (2.13)$$

where  $\varphi_1(x)$ ,  $\varphi_2(x)$  are arbitrary functions defined on  $\Gamma_1$  such that  $\varphi_1(x)$  satisfy condition (2.4) as  $x \rightarrow a$  and for  $\varphi_2(x) \in C(\bar{\Gamma}_1)$  holds the equality

$$\varphi_2(x) = o[(x - a)^{\gamma_{10}}], \quad \gamma_{10} > 0.$$

as  $x \rightarrow a$ .

**Remark 2.1.** Under the conditions of Theorem 2.1 the solution of the form (2.3) vanishes on  $\Gamma_1$ ,  $\Gamma_2$  according to the following asymptotic formula

$$U(x, y) = o[(x - a)^{\gamma_{11}}], \quad \gamma_{11} > 0, \quad \text{as } x \rightarrow a,$$

$$U(x, y) = o[(b - y)^{\gamma_{12}}], \quad \gamma_{12} > \beta - 1, \quad \text{as } y \rightarrow b.$$

**Remark 2.2.** If the conditions of Theorem 2.2 hold, then the solution of the form (2.10) vanishes on  $\Gamma_1$ ,  $\Gamma_2$  satisfying the following relations

$$U(x, y) = o[(b - y)^{\gamma_{13}}], \quad \gamma_{13} > \beta - 1, \quad (2.14)$$

$$\begin{aligned} U(a, y) &= f(o, y) - f(o, y) + \mu \int_y^b \frac{\exp[\mu(\omega_b^\beta(s) - \omega_b^\beta(y))]}{(b - s)^\beta} f(a, s) ds - \\ &- \mu \int_y^b \frac{\exp[\mu(\omega_b^\beta(s) - \omega_b^\beta(y))]}{(b - s)^\beta} f(a, s) ds = 0 \end{aligned}$$

as  $x \rightarrow a$ ,  $y \rightarrow b$ .

**Remark 2.3.** Under the conditions of Theorem 2.3 the solution of the form (2.11) vanishes on  $\Gamma_1$ ,  $\Gamma_2$  satisfying the following relations as  $x \rightarrow a$ ,  $y \rightarrow b$

$$U(x, y) = o[(x - a)^{|\lambda|}], \quad \text{as } x \rightarrow a,$$

$$U(x, y) = o[(b - y)^\beta], \quad \text{as } y \rightarrow b.$$

**Remark 2.4.** If the conditions of Theorem 2.4 are satisfied, then the solution (2.13) vanishes on  $\Gamma_1$ ,  $\Gamma_2$  such that the following asymptotic formulae hold as  $x \rightarrow a$ ,  $y \rightarrow b$

$$U(x, y) = o[(x - a)^{\gamma_{14}}], \quad \gamma_{14} > 0, \quad \text{as } x \rightarrow a,$$

$$U(x, y) = o[\exp(-\mu\omega_b^\beta(y))], \quad \text{as } y \rightarrow b.$$

### 3. SUPERSINGULARITY ON THE FIRST AND SINGULARITY ON THE SECOND VARIABLE

**Theorem 3.1.** *Let us assume that in the integral equation (1.1)  $\alpha > 1$ ,  $\beta = 1$ ,  $\mu > 0$ ,  $\lambda < 0$ ,  $\delta = -\lambda\mu$ , and the right hand side function  $f(x, y) \in C(\bar{D})$  satisfies the following asymptotic relations*

$$f(x, y) = o[\exp(\lambda\omega_a^\alpha(x))(x - a)^{\gamma_{15}}], \quad \gamma_{15} > \alpha - 1 \quad (3.1)$$

as  $x \rightarrow a$  and

$$f(x, y) = o[(b - y)^{\gamma_{16}}], \quad \gamma_{16} > \mu, \quad (3.2)$$

as  $y \rightarrow b$ .

Then the corresponding homogeneous integral equation (1.1) in  $C(\bar{D})$  has an infinite number of linear independent solutions. Besides, the non-homogeneous integral equation (1.1) in  $C(\bar{D})$  is always solvable and its general solution containing four arbitrary functions defined on  $\Gamma_1$ ,  $\Gamma_2$  has the form

$$\begin{aligned} U(x, y) = & (b - y)^\mu \varphi_1(x) + \exp[\lambda\omega_a^\alpha(x)]\psi_1(y) + \\ & + (b - y)^\mu \exp[\lambda\omega_a^\alpha(x)] \int_y^b (b - s)^{-\mu} \psi_2(s) ds + (b - y)^\mu \exp[\lambda\omega_a^\alpha(x)] \times \\ & \times \int_x^a \exp[-\lambda\omega_a^\alpha(t)] \varphi_2(t) dt + f(x, y) - \lambda \int_a^x \frac{\exp[\lambda(\omega_a^\alpha(x) - \omega_a^\alpha(t))]}{(t - a)^\alpha} f(t, y) dt + \\ & + \mu \int_y^b \left( \frac{b - y}{b - s} \right)^\mu \frac{f(x, s)}{b - s} ds - \mu \lambda \int_y^b \left( \frac{b - y}{b - s} \right)^\mu \frac{ds}{b - s} \times \\ & \times \int_a^x \frac{\exp[\lambda(\omega_a^\alpha(x) - \omega_a^\alpha(t))]}{(t - a)^\alpha} f(t, s) ds \equiv \\ & \equiv T_{\alpha, 1}[\varphi_1(x), \varphi_2(x), \psi_1(y), \psi_2(y), f(x, y)], \end{aligned} \quad (3.3)$$

where  $\omega_a^\alpha(x) = [(\alpha - 1)(x - a)^{\alpha-1}]^{-1}$ , and the arbitrary functions  $\varphi_j(x) \in C(\bar{\Gamma}_1)$  and  $\psi_j(y) \in C(\Gamma_2)$ ,  $j = 1, 2$  satisfy the following asymptotic relations as  $x \rightarrow a$ ,  $y \rightarrow b$

$$\varphi_1(x) = o[(x - a)^{\gamma_{17}} \exp(\lambda\omega_a^\alpha(x))], \quad \gamma_{17} > \alpha - 1, \quad \text{as } x \rightarrow a, \quad (3.4)$$

$$\psi_1(y) = o[(b - y)^{\gamma_{18}}], \quad \gamma_{18} > 0, \quad \text{as } y \rightarrow b, \quad (3.5)$$

$$\varphi_2(x) = o[\exp(\lambda\omega_a^\alpha(x))(x - a)^{\gamma_{19}}], \quad \gamma_{19} > \alpha - 1, \quad \text{as } x \rightarrow a, \quad (3.6)$$

$$\psi_2(y) = o[(b - y)^{\gamma_{20}}], \quad \gamma_{20} > \mu, \quad \text{as } y \rightarrow b. \quad (3.7)$$

From the integral representation (3.3) it follows:

**Remark 3.1.** If the conditions of Theorem 3.1 are valid, then any solution of the integral equation (1.1) vanishes on  $\Gamma_1$  and  $\Gamma_2$  as  $x \rightarrow a$ ,  $y \rightarrow b$  in the following manner

$$U(x, y) = o[(x - a)^{\gamma_{21}}], \gamma_{21} > \alpha - 1, \text{ as } x \rightarrow a,$$

$$U(x, y) = o[(b - y)^{\gamma_{22}}], \gamma_{22} > 0, \text{ as } y \rightarrow b.$$

**Theorem 3.2.** Let us assume that  $\alpha > 1$ ,  $\beta = 1$ ,  $\mu < 0$ ,  $\lambda > 0$ ,  $\delta = -\lambda\mu$  hold in the integral equation (1.1) and the function  $f(x, y) \in C(\bar{D})$  has the asymptotic behaviour

$$f(x, y) = o[(x - a)^{\gamma_{23}}], \gamma_{23} > \alpha - 1,$$

as  $x \rightarrow a$  and

$$f(x, y) = o[(b - y)^{\gamma_{24}}], \gamma_{24} > 0$$

as  $y \rightarrow b$ .

Then the corresponding homogeneous integral equation (1.1) in  $C(\bar{D})$  has only the trivial solution, while the inhomogeneous equation (1.1) is uniquely solvable in  $C(\bar{D})$ . The solution is given by the formula

$$U(x, y) = T_{\alpha, 1}[0, 0, 0, 0, f(x, y)]. \quad (3.8)$$

From Theorem 3.2 it follows:

**Remark 3.2.** Under the conditions of Theorem 3.2 any solution of the integral equation (1.1) vanishes on  $\Gamma_1$  and  $\Gamma_2$  having the asymptotic behaviour

$$U(x, y) = o[(x - a)^{\gamma_{25}}], \gamma_{25} > \alpha - 1,$$

as  $x \rightarrow a$  and

$$\begin{aligned} U(x, b) &= f(x, b) - \lambda \int_a^x \frac{\exp[\lambda(\omega_a^\alpha(x) - \omega_a^\alpha(t))]}{(t - a)^\alpha} f(t, b) dt - \\ &- f(x, b) + \lambda \int_a^x \frac{\exp[\lambda(\omega_a^\alpha(x) - \omega_a^\alpha(t))]}{(t - a)^\alpha} f(t, b) dt = 0 \end{aligned}$$

as  $y \rightarrow b$ .

**Theorem 3.3.** Let us suppose that in the integral equation (1.1) the relations  $\alpha > 1$ ,  $\beta = 1$ ,  $\lambda < 0$ ,  $\mu < 0$ ,  $\delta = -\lambda\mu$  hold and the function  $f(x, y) \in C(\bar{D})$  fulfils the following asymptotic relations

$$f(x, y) = o[(x - a)^{\gamma_{26}}], \gamma_{26} > \alpha - 1, \text{ as } x \rightarrow a$$

and

$$f(x, y) = o[(b - y)^{\gamma_{27}}], \gamma_{27} > |\mu| \text{ as } y \rightarrow b$$

Then the homogeneous integral equation (1.1) in  $C(\bar{D})$  has an infinite number of linear independent solutions. Moreover, the non-homogeneous equation (1.1) in  $C(\bar{D})$

is always solvable and its general solution contains two arbitrary functions and has the form

$$U(x, y) = T_{\alpha, 1}[0, 0, \psi_1(y), \psi_2(y), f(x, y)], \quad (3.9)$$

where  $\psi_1(y)$ ,  $\psi_2(y)$  are arbitrary functions such that  $\psi_1(y)$  satisfy the condition (3.5) while for the function  $\psi_2(y)$  holds the following relation

$$\psi_2(y) = o[(b - y)^{\gamma_{28}}], \gamma_{28} > 0$$

as  $y \rightarrow b$ .

From the integral representation (3.9) it follows:

**Remark 3.3.** If the conditions of Theorem 3.3 hold then any solution of the integral equation (1.1) vanishes on  $\Gamma_1$ ,  $\Gamma_2$  with the following asymptotic behaviour

$$U(x, y) = o[\exp(\lambda\omega_a^\alpha(x))], \text{ as } x \rightarrow a,$$

$$U(x, y) = o[(b - y)^{\gamma_{29}}], \gamma_{29} > 0.$$

In the case when the parameters  $\lambda$  and  $\mu$  are positive, we have the assertion:

**Theorem 3.4.** Let us assume that in the integral equation (1.1) the relations  $\alpha > 1$ ,  $\beta = 1$ ,  $\lambda > 0$ ,  $\mu > 0$ ,  $\delta = -\lambda\mu$ , hold and for the function  $f(x, y) \in C(\bar{D})$  the following asymptotic relations

$$f(x, y) = o[(x - a)^{\gamma_{30}} \exp[-\lambda\omega_a^\alpha(x)]], \gamma_{30} > \alpha - 1,$$

as  $x \rightarrow a$  and

$$f(x, y) = o[(b - y)^{\gamma_{31}}], \gamma_{31} > \mu, \text{ as } y \rightarrow b$$

hold. Then the homogeneous integral equation (1.1) has an infinite number of linear independent solutions in the class of functions  $C(\bar{D})$ . The non-homogeneous equation (1.1) is also solvable in  $C(\bar{D})$ . The general solution contains two arbitrary functions

$$U(x, y) = T_{\alpha, 1}[\varphi_1(x), \varphi_2(x), 0, 0, f(x, y)], \quad (3.10)$$

where  $\varphi_1(x)$ ,  $\varphi_2(x)$  are arbitrary functions satisfying the following conditions

$$\varphi_1(x) = o[(x - a)^{\gamma_{32}}], \gamma_{32} > \alpha - 1,$$

$$\varphi_2(x) = o[(x - a)^{\gamma_{33}}], \gamma_{33} > \alpha - 1$$

as  $x \rightarrow a$ .

From Theorem 3.4 it follows:

**Remark 3.4.** Under the conditions of Theorem 3.4 any solution of the integral equation (1.1) vanishes on  $\Gamma_1$ ,  $\Gamma_2$  with the following asymptotic properties

$$U(x, y) = o[(x - a)^{\gamma_{34}}], \gamma_{34} > \alpha - 1, \text{ as } x \rightarrow a,$$

$$U(x, y) = o[(b - y)^{\gamma_\mu}], \text{ as } y \rightarrow b.$$

#### 4. SERIES REPRESENTATION OF THE SOLUTION WITH RESPECT TO ONE OF THE VARIABLES

It is easy to see that in the case when  $\delta \neq -\lambda\mu$  the problem of finding the solution of the integral equation (1.1) is reduced to the solution of this equation with  $\lambda = 0$ ,  $\mu = 0$ .

For example in the case when  $\alpha > 1$ ,  $\beta = 1$ ,  $\mu > 0$ ,  $\lambda < 0$ ,  $\delta \neq -\lambda\mu$ , finding the solution of equation (1.1) is reduced to studying the solution of the following integral equation

$$V(x, y) + \delta_1 \int_a^x \frac{dt}{(t-a)^\alpha} \int_y^b \frac{V(t, s)}{b-s} ds = G(x, y), \quad (4.1)$$

where  $\delta_1 = \delta + \lambda\mu$ ,

$$\begin{aligned} V(x, y) &= (b-y)^{-\mu} \exp[-\lambda\omega_a^\alpha(x)] U(x, y), \\ G(x, y) &= (b-y)^{-\mu} \psi_1(y) + \exp[-\lambda\omega_a^\alpha(x)] \varphi_1(x) + \\ &+ \int_a^x \exp(-\lambda\omega_a^\alpha(t)) \varphi_2(t) dt + \int_y^b (b-s)^{-\mu} \psi_2(s) ds + \\ &+ (b-y)^{-\mu} \exp[-\lambda\omega_a^\alpha(x)] f(x, y) - \lambda(b-y)^{-\mu} \times \\ &\times \int_a^x \frac{\exp[-\lambda\omega_a^\alpha(t)] f(t, y)}{(t-a)^\alpha} dt + \mu \exp[-\lambda\omega_a^\alpha(x)] \times \\ &\times \int_y^b \frac{f(x, s) ds}{(b-s)^{\mu+1}} - \lambda\mu \int_y^b \frac{ds}{(b-s)^{\mu+1}} \int_a^x \frac{\exp[-\lambda\omega_a^\alpha(t)]}{(t-a)^\alpha} f(t, s) ds, \end{aligned}$$

$\varphi_j(x)$ ,  $\psi_j(y)$ ,  $j = 1, 2$  are arbitrary functions defined on  $\Gamma_1$ ,  $\Gamma_2$  such that  $\varphi_j(x) \in C(\Gamma_1)$ ,  $\psi_j(y) \in C(\Gamma_2)$ ,  $j = 1, 2$  and these functions satisfy the conditions (3.4)-(3.6) as  $x \rightarrow a$ ,  $y \rightarrow b$ .

Let us search the solution of the integral equation (4.1) in the following form

$$V(x, y) = \sum_{k=0}^{\infty} (b-y)^{k+\delta} V_k(x), \quad \delta > 0, \quad (4.2)$$

where  $V_k(x)$  are unknown functions.

Suppose that the function  $G(x, y)$  has the following series representation

$$G(x, y) = \sum_{k=0}^{\infty} (b-y)^{k+\delta} G_k(x), \quad (4.3)$$

where  $G_k(x)$  are given functions. Moreover, assume that the series (4.3) converge absolutely and uniformly.

In this case in finding the solutions of the two-dimensional integral equation (4.1) one should determine the solutions of the following one-dimensional Volterra type integral equations

$$V_k(x) + \frac{\delta_1}{k + \delta} \int_a^x \frac{V_k(t)dt}{(t-a)^\alpha} = G_k(x), \quad (4.4)$$

$k = 0, 1, 2, \dots$ , which were investigated in [1-2] in detail.

Using the technique from [1-2] for solving (4.4), we find that

$$V_k(x) = \exp\left[\frac{\delta_1}{k + \delta} \omega_a^\alpha(x)\right] c_k + G_k(x) - \frac{\delta_1}{k + \delta} \int_a^x \frac{\exp\left[\frac{\delta_1}{k + \delta} (\omega_a^\alpha(x) - \omega_a^\alpha(t))\right]}{(t-a)^\alpha} G_k(t) dt, \quad (4.5)$$

where  $c_0, c_1, c_2, \dots$  are arbitrary constants.

The solutions of the form (4.5) were obtained, supposing that  $\delta_1 < 0$  and the functions  $G_k(x) \in C(\bar{\Gamma}_0)$  vanish in the point  $x = c$  having the following asymptotic behaviour

$$G_k(x) = o\left[\exp\left(\frac{\delta_1}{k + \delta} \omega_a^\alpha(x)\right) (x-a)^{\gamma_{35}}\right], \quad (4.6)$$

$\gamma_{35} > \alpha - 1$ ,  $k = 0, 1, 2, \dots$

In this case, we have the following assertion.

**Theorem 4.1.** *Let us assume that in the integral equation (4.1) relations  $\alpha > 1$ ,  $\mu > 0$ ,  $\lambda < 0$ ,  $\delta_1 = \delta + \lambda\mu \neq 0$ ,  $\delta_1 < 0$  ( $\delta < |\lambda|\mu$ ) hold and moreover for the function  $G(x, y)$  the series representation (4.3) holds, where  $G_k(x)$  are given functions vanishing on  $\Gamma_1$  with the asymptotic behaviour (4.6), where  $\delta > 0$ . Then the corresponding homogeneous integral equation (4.1) in the class of functions having the series representation of the form (4.2) has an infinite number of linear independent solutions of the form*

$$V_k(x, y) = \exp\left[\frac{\delta_1}{k + \delta} \omega_a^\alpha(x)\right] (b-y)^{k+\delta}, \quad k = 0, 1, \dots, \delta > 0.$$

The non-homogeneous integral equation (4.1) in the class of functions having the series form (4.2) is always solvable and this solution is given by the following formula

$$V(x, y) = \sum_{k=0}^{\infty} \exp\left[\frac{\delta_1}{k + \delta} \omega_a^\alpha(x)\right] (b-y)^{k+\delta} c_k + G(x, y) - \sum_{k=0}^{\infty} \frac{\delta_1}{k + \delta} \left( \int_a^x \frac{\exp\left[\frac{\delta_1}{k + \delta} (\omega_a^\alpha(x) - \omega_a^\alpha(t))\right]}{(t-a)^\alpha} G_k(t) dt \right) (b-y)^{k+\delta},$$

where  $c_k$  ( $k = 0, 1, 2, \dots$ ) are arbitrary constants.

**Theorem 4.2.** *Let us suppose that in the equation (1.1)  $\alpha > 1$ ,  $\beta = 1$ ,  $\mu > 0$ ,  $\lambda < 0$ ,  $\delta_1 = \delta + \lambda\mu \neq 0$ ,  $\delta_1 > 0$  ( $\delta > |\lambda|\mu$ ) and the function  $G(x, y)$  is presented in the form (4.3), where  $G_k(x)$  are given functions such that  $G_k(x) \in C(\Gamma_1)$ . Then the corresponding homogeneous integral equation (4.1) is not solvable in the class of functions having the series representation of the form (4.2). In this case the non-homogeneous integral equation (4.1) has a unique solution of the series form (4.2), which is given by the following formula*

$$V(x, y) = G(x, y) - \int_a^x \frac{G_1(x, y, t) dt}{(t - a)^\alpha}, \quad (4.7)$$

where

$$G_1(x, y, t) = \sum_{k=0}^{\infty} (b - y)^{k+\delta} \frac{\delta_1}{k + \delta} \exp\left[\frac{\delta_1}{k + \delta} (\omega_a^\alpha(x) - \omega_a^\alpha(t))\right] G_k(t).$$

**Remark 3.5.** If the conditions of Theorem 4.1 are met, then the solution having the series form (4.2) vanishes on  $\Gamma_1$ ,  $\Gamma_2$  and its behaviour is determined by the following asymptotic formula

$$V(x, y) = o[\exp(-\varepsilon\omega_a^\alpha(x))],$$

as  $x \rightarrow a$ , where  $\varepsilon$  is a sufficiently small positive constant satisfying the conditions  $\varepsilon < \frac{|\delta_1|}{\delta+k}$  and

$$V(x, y) = o[(b - y)^{\gamma_{36}}], \gamma_{36} > 0, \text{ as } y \rightarrow b.$$

**Remark 3.6.** Under the conditions of Theorem 4.2, the solution of the series form (4.2) vanishes also and its asymptotic behaviour is determined from the following equality

$$V(0, y) = G(0, y) - G(0, y) = 0, \text{ as } x \rightarrow a,$$

i.e.,

$$V(x, y) = o((b - y)^{\gamma_{37}}), \gamma_{37} > 0, \text{ as } y \rightarrow b.$$

The proof of the statements above can be carried out by using the technique proposed in [5, 6].

**Acknowledgement.** *The research of the first and second authors was supported in part by the Hungarian NATO Science and Technology Fellowship Programme through grants 2041/NATO/2001 and 2042/NATO/2001, respectively.*

## REFERENCES

- [1] RAJABOV, N.: *General Volterra type integral equations with left and right fixed singular and super-singular point in kernel*, News Academy of Science the Republic of Tajikistan, Department Physics, Mathematics, Chemistry and Geology Sciences, **1**, (2001), 30–46. (in Russian)
- [2] RAJABOV, N.: *On Volterra type integral equations*, Russian Doklady Acad. Sciences, **383**(3), (2002), 314–317.

- [3] RAJABOV, N. and RAJABOVA, L.: *On a class of two dimensional linear Volterra type integral equations with boundary fixed singular kernels*, ISAAC Conference on Complex Analysis, Differential Equations and Related Topics, September 17-21, 2002, Yerevan, Armenia, Book of Abstracts, 52–53.
- [4] RAJABOV, N. and RAJABOVA, L.: *An explicit solution to a class of two dimensional Volterra type linear integral equations with fixed boundary singular kernels*, ICM 2002, Beijing 2002, August 20-28, Abstract of Short Communications and Poster Session, 229–229.
- [5] RAJABOV, N.: *Integral representation and boundary value problems for differential equations with singular lines and singular surfaces*, Vol. 4, Dushanbe, 1986. (in Russian)
- [6] RAJABOV, N.: *An Introduction to the Theory of Partial Differential Equations with Super-Singular Coefficients*, Tehran University Publications, Tehran, 1997.

### *Authors' Addresses*

**Nusrat Rajabov:**

PRESIDIUM OF THE ACADEMY OF SCIENCES OF TAJIKISTAN, 734025 DUSHANBE, RUDAKI AVE. 33, TAJIKISTAN  
*E-mail address:* nusrat@ac.tajik.net

**Miklós Rontó:**

INSTITUTE OF MATHEMATICS, UNIVERSITY OF MISKOLC, 3515 MISKOLC-EGYETEMVÁROS, HUNGARY  
*E-mail address:* matronto@gold.uni-miskolc.hu

**Lutfia Rajabova:**

TAJK TECHNICAL UNIVERSITY, 734025 DUSHANBE, ACAD. RAJABOV AVE. 10, TAJIKISTAN  
*E-mail address:* n.rajabov@tajik.net

## EDITORIAL BOARD

Miklós RONTÓ, Editor in Chief, Institute of Mathematics, University of Miskolc; H-3515 MISKOLC, Hungary; [matronto@gold.uni-miskolc.hu](mailto:matronto@gold.uni-miskolc.hu)

Gabriella BOGNÁR, Institute of Mathematics, University of Miskolc; H-3515 MISKOLC, Hungary; [matvbg@gold.uni-miskolc.hu](mailto:matvbg@gold.uni-miskolc.hu)

Mátyás DOMOKOS, A. Rényi Institute of Mathematics, Hungarian Academy of Sciences; P. O. Box 127, H-1053 BUDAPEST, Hungary; [domokos@renyi.hu](mailto:domokos@renyi.hu)

Miklós FARKAS, School of Mathematics, Budapest University of Technology and Economics; 3 Műegyetem rkp. H-1111 BUDAPEST, Hungary; [fm@math.bme.hu](mailto:fm@math.bme.hu)

Michal FEČKAN, Department of Mathematical Analysis, Comenius University; Mlynská dolina, 842 15 BRATISLAVA, Slovakia; [Michal.Feckan@fmph.uniba.sk](mailto:Michal.Feckan@fmph.uniba.sk)

Aurél GALÁNTAI, Institute of Mathematics, University of Miskolc; H-3515 MISKOLC, Hungary; [matgal@gold.uni-miskolc.hu](mailto:matgal@gold.uni-miskolc.hu)

Chaitan P. GUPTA, Department of Mathematics, University of Nevada; RENO, NV 89557, USA; [chaitang@hotmail.com](mailto:chaitang@hotmail.com)

István GYŐRI, Department of Mathematics and Computing, University of Veszprém; P. O. Box 158, H-8201 VESZPRÉM, Hungary; [gyori@almos.vein.hu](mailto:gyori@almos.vein.hu)

László HATVANI, Bolyai Institute, University of Szeged; Aradi vértanúk tere 1, H-6720 SZEGED, Hungary; [hatvani@math.u-szeged.hu](mailto:hatvani@math.u-szeged.hu)

József KOLUMBÁN, Department of Mathematics and Computer Science, Babeş-Bolyai University; 1 Kogalniceanu str., 3400 CLUJ-NAPOCA, Romania; [kolumban@math.ubbcluj.ro](mailto:kolumban@math.ubbcluj.ro)

Nikolai PERESTYUK, Kiev National University Mechanical-Mathematical Department; 64 Vladimirska Str., 01006 KIEV, Ukraine; [pmo@mechmat.mechmat.kiev.ua](mailto:pmo@mechmat.mechmat.kiev.ua)

Nikolai ROZOV, Moscow State University, Mechanical and Mathematical Department; 119899 MOSCOW, Russia; [rozov@rozov.mccme.ru](mailto:rozov@rozov.mccme.ru)

Anatolij SAMOILENKO, Institute of Mathematics, Ukrainian Academy of Sciences; 3 Tereshenkovskaya Str., 01601 KIEV, Ukraine; [sam@imath.kiev.ua](mailto:sam@imath.kiev.ua)

Emilio SPEDICATO, Department of Mathematics, University of Bergamo; Piazza Rosate 2, 24127 BERGAMO, Italy; [emilio@unibg.it](mailto:emilio@unibg.it)

Gisbert STOYAN, Department of Numerical Analysis, Eötvös Lóránd University of Sciences; 1D Pázmány P., H-1117 BUDAPEST, Hungary; [stoyan@cs.elte.hu](mailto:stoyan@cs.elte.hu)

György SZEIDL, Department of Mechanics, University of Miskolc; H-3515 MISKOLC, Hungary; [Gyorgy.SZEIDL@uni-miskolc.hu](mailto:Gyorgy.SZEIDL@uni-miskolc.hu)

Jenő SZIGETI, Institute of Mathematics, University of Miskolc; H-3515 MISKOLC, Hungary; [matszj@gold.uni-miskolc.hu](mailto:matszj@gold.uni-miskolc.hu)

Zsolt TUZA, Computer and Automation Institute, Hungarian Academy of Sciences; 13-17 Kende Str., H-1111 BUDAPEST, Hungary; [tuza@sztaki.hu](mailto:tuza@sztaki.hu)

## LOCAL EDITORIAL BOARD

M. RONTÓ, G. BOGNÁR, A. GALÁNTAI,  
J. SZIGETI, G. SZEIDL

The Mathematical Notes of the University of Miskolc is abstracted in Zentralblatt für Mathematik, Mathematical Reviews, and Referativnyj Zhurnal

Responsible for publication: Rector of the University of Miskolc  
Published by the Miskolc University Press under the leadership of Dr. József PÉTER  
Responsible for duplication: Works manager Mária KOVÁCS  
Number of copies printed: 200  
Put to the Press on May 20, 2003  
Number of permission: TU 2003-560-ME

**HU e-ISSN 1787-2413**