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A CHARACTERIZATION OF CONE PRESERVING MAPPINGS OF QUASIORDERED SETS

IVAN CHAJDA AND ŠÁRKA HOŠKOVÁ

[Received: October 7, 2004]

Abstract. By a cone of a quasiordered set (A, Q), any $U_Q(a) = \{x \in A : \langle a, x\rangle \in Q\}$ is meant. A mapping $f : A \to B$ is a cone preserving mapping of (A, Q) into (B, Q') if $f(U_Q(a)) = U_{Q'}(f(a))$ for each $a \in A$. We characterize these mappings by using certain relational inclusions. The result can be applied for construction of a quotient quasiorder hypergroup.

Mathematics Subject Classification: 08A02, 08A30, 20N20

Keywords: quasiorder, quasiorder hypergroup, factorization of quasiordered set, quotient hypergroup

Let $H \neq 0$ be a set and "$*$" be a mapping of $H \times H$ into $\mathcal{P}^*(H)$ (the set of all non-void subsets of H). The pair $(H, *)$ is called *a hypergroupoid* (see, e. g., [2, 5]). For $A, B \in \mathcal{P}^*(H)$, we define $A * B = \bigcup\{a * b; a \in A, b \in B\}$.

A hypergroupoid $(H, *)$ is called *a hypergroup*, see [1,2,5], if $(a*b)*c = a*(b*c)$ for all $a, b, c \in H$, and the so-called *reproduction axiom*, i. e., $a * H = H = H * a$ for any $a \in H$, is satisfied.

For a binary relation R on A and $a \in A$ denote $U_R(a) = \{b \in A; \langle a, b\rangle \in R\}$. A binary relation Q on a set A is called *quasiorder* if it is reflexive and transitive. The set $U_Q(a)$ is called a *cone* of a. In the case when a quasiorder Q is an equivalence one can prove that $U_Q(A) = \{x \in H : \exists y \in A, \langle x, y\rangle \in \theta\}$ for any $A \subseteq H$. Analogously, for $M \subseteq A$ we set $U_Q(M) = \bigcap\{U_Q(a) : a \in M\}$. The concept of a quasiorder hypergroup can be found, e. g., in [1]:

Definition 1. Let (H, Q) be a quasiordered set. For $a, b \in H$ define

$$a \star b = U_Q(a) \cup U_Q(b). \tag{1}$$

Then $(H, \star)$ is a hypergroup which is called a *quasiorder hypergroup induced by* (H, Q).

To produce quotient hypergroups from a given general hypergroup, we need a concept of congruence. This was introduced independently by several authors but it was shown by J. Chvalina [4] that all of these definitions are equivalent to the following one:

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Definition 2. Let $\mathcal{H} = (H, *)$ be a hypergroupoid. An equivalence θ on H is called a *congruence on* $\mathcal{H}$ if $\langle a, b\rangle \in \theta$ and $\langle c, d\rangle \in \theta$ imply

$$U_\theta(a * c) = U_\theta(b * d).$$

Giving an equivalence θ on a set A, denote by h_θ the so-called *natural mapping* of A onto A/θ defined by $h_\theta(a) = [a]_\theta$.

It can be shown that if θ is a congruence on a hypergroupoid $(H, *)$, then $(H/\theta, *_\theta)$, where

$$[a]_\theta *_\theta [b]_\theta = h_\theta(a * b),$$

is also a hypegroupoid. If, moreover, $\mathcal{H} = (H, *)$ is a hypergroup, then $\mathcal{H}/\theta = (H/\theta, *_\theta)$ is a quotient hypergroup—see Chvalina [4, p. 155].

The following question was solved in [3]: Let (H, Q) be a quasiordered set and $\mathcal{H} = (H, \star)$ be the induced hypergroup. Let θ be an equivalence on $(H, \star)$.

(i) How to define a quotient relation Q/θ on the quotient set H/θ to be a quasiorder again?

(ii) Under what condition the quasiorder hypergroup $(H/\theta, \bullet)$ induced by the quasiordered set $(H/\theta, Q/\theta)$, i. e., $[a]_\theta \bullet [b]_\theta = U_{Q/\theta}([a]_\theta) \cup U_{Q/\theta}([b]_\theta)$, is the quotient hypergroup $\mathcal{H}/\theta$?

It can be visualized by the following diagram:

$$\begin{array}{ccc}
(H, Q) & \overset{\theta}{\dashrightarrow} & (H/\theta, Q/\theta) \\
\downarrow & & \vdots \\
\mathcal{H} = (H, \star) & \overset{\theta}{\longrightarrow} & \mathcal{H}/\theta = (H, \star)/\theta = (H/\theta, \bullet)
\end{array}$$

In Corollary 1 from [3] it was proved that if θ is a congruence, then the preceding diagram commutes.

We are going to use another approach to solve these problems in terms of relational products. Recall that for two binary relations R and S on a set A we have

$$R \circ S = \{\langle x, y\rangle \in A^2 : \exists z \in A \text{ with } \langle x, z\rangle \in R, \langle z, y\rangle \in S\}.$$

Further, let $f\colon A \to B$ be a mapping. Denote by θ_f the so-called *induced equivalence* (on A), i. e., $\langle x, y\rangle \in \theta_f$ if and only if $f(x) = f(y)$.

If $f\colon A \to B$ is a mapping and $M \subseteq A$, denote by $f(M) = \{f(m) : m \in M\}$. It is well-known that for any $X, Y \subseteq A$, $f(X \cup Y) = f(X) \cup f(Y)$.

We have to introduce a quotient quasiordered set. For this purpose, we introduce the following

Definition 3. Let R be a binary relation on a set A and θ be an equivalence on A. Define R/θ on A/θ as follows

$$\langle [a]_\theta, [b]_\theta \rangle \in R/\theta \text{ if and only if there exist } x \in [a]_\theta, y \in [a]_\theta \text{ with } \langle x, y \rangle \in R. \quad (2)$$

The couple $(A/\theta, R/\theta)$ is called a *quotient of* (A, R) *by* θ.

Lemma 1. *Let (H, Q) be a quasiordered set and $(H, \star)$, where $\star$ is given by* (1)*, be a corresponding quasiorder hypergroup. Let θ be an equivalence on H. Then θ is a congruence on $(H, \star)$ if and only if*

$$h_\theta(U_Q(a)) = U_{Q/\theta}(h_\theta(a)) \quad (= U_{Q/\theta}([a]_\theta)) \quad (3)$$

holds for any $a \in H$.

Proof. Assume first that θ is a congruence on $(H, \star)$. If $x \in U_Q(a)$, i. e., $\langle a, x \rangle \in Q$, then $\langle h_\theta(a), h_\theta(x) \rangle \in Q/\theta$, i. e., $h_\theta(x) \in U_{Q/\theta}(h_\theta(a))$, which means that

$$h_\theta(U_Q(a)) \subseteq U_{Q/\theta}(h_\theta(a)).$$

To prove the opposite inclusion consider $z \in U_{Q/\theta}(h_\theta(a))$.The mapping h_θ is a surjection, so there exists $x \in H$ such that $z = h_\theta(x)$. Therefore $h_\theta(x) \in U_{Q/\theta}(h_\theta(a))$, which implies $\langle h_\theta(a), h_\theta(x) \rangle \in Q/\theta$. From the definition of Q/θ there exist $b, y \in H$ such that $\langle b, y \rangle \in Q$, $h_\theta(b) = h_\theta(a)$, $h_\theta(x) = h_\theta(y) = z$. As θ is a congruence we get $U_\theta(a \star a) = U_\theta(b \star b)$, $U_\theta(x \star x) = U_\theta(y \star y)$ that is $h_\theta(U_Q(a)) = h_\theta(U_Q(b))$, $h_\theta(U_Q(x)) = h_\theta(U_Q(y))$. Further from $\langle b, y \rangle \in Q$ we have $U_Q(b) \supseteq U_Q(y)$. From this we obtain $z \in h_\theta(U_Q(y)) \subseteq h_\theta(U_Q(b)) = h_\theta(U_Q(a))$, which gives

$$U_{Q/\theta}(h_\theta(a)) \subseteq h_\theta(U(a)).$$

Assume now that (3) is true. We have to prove that θ is a congruence, i. e., $h_\theta(a \star c) = h_\theta(b \star d)$ for $\langle a, b \rangle \in \theta$, $\langle c, d \rangle \in \theta$. We have

$$\begin{aligned} h_\theta(a \star c) &= h_\theta(U_Q(a) \cup U_Q(c)) = h_\theta(U_Q(a)) \cup h_\theta(U_Q(c)) \\ &= U_{Q/\theta}(h_\theta(a)) \cup U_{Q/\theta}(h_\theta(c)) = U_{Q/\theta}(h_\theta(b)) \cup U_{Q/\theta}(h_\theta(d)) \\ &= h_\theta(U_Q(b)) \cup h_\theta(U_Q(d)) = h_\theta(U_Q(b) \cup U_Q(d)) = h_\theta(b \star d). \end{aligned}$$ □

Hence, we must solve the above mentioned problem (i) and classify the mappings having the property (ii). For this sake, let us introduce the following

Definition 4. Let $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ be quasiordered sets. A mapping $h\colon A \to B$ is called *cone preserving* if $h(U_R(a)) = U_S(h(a))$ for each $a \in A$.

Let us recall that for $\mathcal{A} = (A, R)$, $\mathcal{B} = (B, S)$ (where R or S is a binary relation on A or B, respectively) a mapping $h\colon A \to B$ is called a *homomorphism* if $\langle a, b \rangle \in R$ implies $\langle h(a), h(b) \rangle \in S$. It is called a *strong homomorphism* if, moreover, for every $\langle h(a), h(b) \rangle \in S$ there exist $c, d \in A$ such that $\langle c, d \rangle \in R$ and $h(c) = h(a)$, $h(d) = h(b)$.

The proof of the following lemma is almost evident (see, e. g., Lemma 2 in [3]):

Lemma 2. *Let $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ be quasiordered sets. If $h\colon A \to B$ is a cone preserving mapping, then h is a strong homomorphism of $\mathcal{A}$ into $\mathcal{B}$.*

It is an easy exercise to verify that the converse assertion is not valid in general.

We are ready to solve our first question.

Lemma 3. *Let $\mathcal{A} = (A, R)$ be a quasiordered set and θ an equivalence on A. If*

$$\theta \circ R \subseteq R \circ \theta,$$

then $\mathcal{A}/\theta = (A/\theta, R/\theta)$ is a quasiordered set.

Proof. It is clear from Definition 3 that if R is reflexive, then R/θ has this property. We have to show transitivity of R/θ. Suppose that $\langle [a]_\theta, [b]_\theta\rangle \in R/\theta$ and $\langle [b]_\theta, [c]_\theta\rangle \in R/\theta$. Then there exist $x \in [a]_\theta$, $y, y' \in [b]_\theta$ and $z \in [c]_\theta$ such that $\langle x, y\rangle \in R$, $\langle y', z\rangle \in R$. Moreover, $\langle y, y'\rangle \in \theta$ thus $\langle x, z\rangle \in R \circ \theta \circ R$. Due to the assumption, we have

$$\langle x, z\rangle \in R \circ \theta \circ R \subseteq R \circ R \circ \theta = R \circ \theta,$$

i. e., there exists $w \in A$ with $\langle x, w\rangle \in R$, $\langle w, z\rangle \in \theta$, thus $w \in [z]_\theta = [c]_\theta$. Altogether, $\langle [a]_\theta, [c]_\theta\rangle \in R/\theta$ proving transitivity of R/θ . □

In the remaining part of our paper, we will solve the problem whether a mapping f of a quasiordered set $\mathcal{A} = (A, R)$ into a quasiordered set $\mathcal{B} = (B, S)$ is a cone preserving mapping.

Theorem 1. *Let $\mathcal{A} = (A, R)$, $\mathcal{B} = (B, S)$ be quasiordered sets and $f\colon A \to B$ a surjective mapping. The following conditions are equivalent:*

(a) *f is a cone preserving mapping,*

(b) *f is a homomorphism of $\mathcal{A}$ onto $\mathcal{B}$ and the relation $\langle f(x), f(y)\rangle \in S$ implies that $\langle x, y\rangle \in R \circ \theta_f$.*

Proof. (a) $\Rightarrow$ (b): By Lemma 2, f is a homomorphism of $\mathcal{A}$ onto $\mathcal{B}$. Suppose $\langle f(x), f(y)\rangle \in S$. Then $f(y) \in U_S(f(x)) = f(U_R(x))$. Hence, there exists $z \in U_R(x)$ with $f(z) = f(y)$, i. e., $\langle x, z\rangle \in R$ and $\langle z, y\rangle \in \theta_f$ giving $\langle x, y\rangle \in R \circ \theta_f$.

(b) $\Rightarrow$ (a): Since f is a homomorphism of $\mathcal{A}$ onto $\mathcal{B}$, we have $f(U_R(x)) \subseteq U_S(f(x))$ for each $x \in A$. Suppose $z \in U_S(f(x))$. Since f is surjective, it follows that $z = f(w)$ and $\langle f(x), f(w)\rangle \in S$. By (b) we have $\langle x, w\rangle \in R \circ \theta_f$, i. e., there exists $c \in A$ with $\langle x, c\rangle \in R$ and $\langle c, w\rangle \in \theta_f$. Hence $f(c) = f(w) = z$ proving that $U_S(f(x)) \subseteq f(U_R(x))$. Thus f is cone preserving. □

As mentioned, an equivalence θ on a quasiordered set is a congruence of the induced quasiorder hypergroup if the quotient relation is a quasiorder again and the natural mapping h_θ is a cone preserving mapping. Hence, we can solve our second problem:

Theorem 2. *Let θ be an equivalence on a quasiordered set $\mathcal{A} = (A, R)$. If*

$$\theta \circ R \subseteq R \circ \theta \tag{4}$$

then $\mathcal{A}/\theta = (A/\theta, R/\theta)$ is a quasiordered set and the natural mapping h_θ is cone preserving.

Proof. By Lemma 3, (4) implies that $\mathcal{A}/\theta = (A/\theta, R/\theta)$ is a quasiordered set. Suppose now that

$$\langle [x]_\theta, [y]_\theta \rangle \in R/\theta.$$

Then there exists $a, b \in A$, such that $a \in [x]_\theta$, $b \in [y]_\theta$ and $\langle a, b \rangle \in R$, i. e., $\langle x, a \rangle \in \theta$ and $\langle x, b \rangle \in \theta \circ R$. By (4), $\langle x, b \rangle \in R \circ \theta$, i. e., there exists $w \in A$ such that $\langle x, w \rangle \in R$ and $\langle w, b \rangle \in \theta$. Hence and $[w]_\theta = [b]_\theta = [y]_\theta$, i. e., $\langle w, y \rangle \in \theta$. Altogether, we obtain $\langle x, y \rangle \in R \circ \theta$. Clearly $\theta_{h_\theta} = \theta$. Applying Theorem 1, we conclude that the natural mapping $h_\theta \colon a \mapsto [a]_\theta$ is cone preserving. □

By Lemma 2, every cone preserving mapping is a strong homomorphism. Hence, we will characterize which strong homomorphisms are cone preserving mappings.

Lemma 4. *Let $\mathcal{A} = (A, R)$, $\mathcal{B} = (B, S)$ be quasiordered sets and $f \colon A \to B$ be a surjective mapping. The following conditions are equivalent:*

(a) *f is a strong homomorphism of $\mathcal{A}$ onto $\mathcal{B}$;*

(b) *$\langle f(x), f(y) \rangle \in S$ if and only if $\langle x, y \rangle \in \theta_f \circ R \circ \theta_f$.*

Proof. (a) $\Rightarrow$ (b): Suppose $\langle f(x), f(y) \rangle \in S$. Since f is a strong homomorphism, there exist $a, b \in A$ with $\langle a, b \rangle \in R$, $f(a) = f(x)$, $f(b) = f(y)$, thus $\langle x, y \rangle \in \theta_f \circ R \circ \theta_f$. Conversely, if $\langle x, y \rangle \in \theta_f \circ R \circ \theta_f$ then there exist $c, d \in A$ with $f(x) = f(c)$, $\langle c, d \rangle \in R$, $f(d) = f(y)$. Since f is a homomorphism, it follows that $\langle f(c), f(d) \rangle \in S$, i. e., $\langle f(x), f(y) \rangle \in S$ as well.

(b) $\Rightarrow$ (a): Suppose $\langle a, b \rangle \in R$. Then also $\langle a, b \rangle \in \theta_f \circ R \circ \theta_f$ and, by virtue of (b), $\langle f(a), f(b) \rangle \in S$. Thus f is a homomorphism of $\mathcal{A}$ onto $\mathcal{B}$.

Suppose now $\langle c, d \rangle \in S$. Since f is surjective, there are $x, y \in A$ with $f(x) = c$, $f(y) = d$, i. e., $\langle f(x), f(y) \rangle \in S$. By (b) we have $\langle x, y \rangle \in \theta_f \circ R \circ \theta_f$, i. e., there exist $a, b \in A$ such that $f(x) = f(a)$, $f(y) = f(b)$ and $\langle a, b \rangle \in R$, i. e., f is a strong homomorphism. □

Theorem 3. *Let $\mathcal{A} = (A, R)$, $\mathcal{B} = (B, S)$ be quasiordered sets and f be a surjective strong homomorphism of $\mathcal{A}$ onto $\mathcal{B}$. The following assertions are equivalent:*

(a) *f is cone preserving,*

(b) *$\theta_f \circ R \subseteq R \circ \theta_f$.*

Proof. (a) $\Rightarrow$ (b): Suppose $\langle x, y \rangle \in \theta_f \circ R$. Then there exists $z \in A$ with $\langle x, z \rangle \in \theta_f$, $\langle z, y \rangle \in R$. Hence $f(x) = f(z)$ and $\langle f(z), f(y) \rangle \in S$; i. e., also $\langle f(x), f(y) \rangle \in S$. Thus

$f(y) \in U_S(f(x)) = f(U_R(x))$. Hence, there is $a \in A$ with $\langle x, a\rangle \in R$ and $f(a) = f(y)$ thus $\langle x, y\rangle \in R \circ \theta_f$ proving (b).

(b) $\Rightarrow$ (a): Suppose $\langle f(x), f(y)\rangle \in S$. By Lemma 4 we have $\langle x, y\rangle \in \theta_f \circ R \circ \theta_f$. Applying (b) we conclude $\langle x, y\rangle \in \theta_f \circ R \circ \theta_f \subseteq R \circ \theta_f \circ \theta_f = R \circ \theta_f$. By Theorem 1, f is a cone preserving mapping. □

Corollary 1. *Let $\mathcal{A} = (A, R)$, $\mathcal{B} = (B, S)$ be quasiordered sets and f be a surjective strong homomorphism of $\mathcal{A}$ onto $\mathcal{B}$. Suppose R is symmetric (i. e., an equivalence on A). Then f is cone preserving if and only if $\theta_f \circ R = R \circ \theta_f$.*

Proof. Due to Theorem 3, we need only to show the converse inclusion of that in (b). However, by symmetry of R and θ_f we have

$$R \circ \theta_f = R^{-1} \circ \theta_f^{-1} = (\theta_f \circ R)^{-1} \subseteq (R \circ \theta_f)^{-1} = \theta_f^{-1} \circ R^{-1} = \theta_f \circ R. \qquad \square$$

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ON A DECOMPOSITION FORMULA IN COMMUTATIVE GROUP RINGS

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Abstract. A necessary and sufficient condition is obtained when the p-component of the group of all normalized units can be decomposed into two special subgroups provided that the group basis has a standard proper decomposition. The established formula is useful for proving the niceness of certain subgroups in modular group rings and improves our identical claim in Hokkaido Math. J. (2000) as well.

We also point out a confusion due to Mollov in his reviewer's report (Zbl. Math. 2001) concerning our paper published in Rend. Sem. Mat. Univ. Padova (1999).

Mathematics Subject Classification: 16S34, 16U60, 20K10, 20K20, 20K21

Keywords: commutative group rings, decomposition, niceness

Suppose R is a commutative ring with 1 of prime characteristic p and suppose G is an abelian group with p-torsion part G_p. Throughout the text, $S(RG)$ shall designate the normed Sylow p-subgroup in the group ring RG. For H a subgroup of G, the symbol $I_p(RG; H)$ will denote the nil-radical of the relative augmentation ideal $I(RG; H)$ of the group ring RG with respect to H.

In [2], we have found a convenient direct decomposition of normalized p-units of the group $A \times B$ where A and B are arbitrary abelian groups (see, e. g., [3, 4]).

The aim of this paper is to inspect here the validity of the same formula when the product of the subgroups generating the basis is however not direct. The ratio deduced is next used for proving the niceness in commutative modular group rings.

We come now to the central statement motivating the present study.

Proposition 1 (Decomposition)**.** *Let $G = AB$ with $A \leq B$ and $B \leq G$. Then $S(RG) = S(RA)(1 + I_p(RG; B))$ if and only if $G_p = A_pB_p$.*

Proof. *Necessity.* For given $g_p \in G_p$ we write $g_p(r_1a_1 + r_2a_2 + \ldots + r_ta_t) = f_1b_1 + f_2g_2 + \ldots + f_tg_t$. The canonical forms yield $g_pa_1 = b_1$, $g_pa_2 = g_2$, …, $g_pa_t = g_t$. Further, we will distinguish two basic cases:

Case 1: $b_1 \in B_p$. Hence it is plain that $g_p = a_1^{-1}b_1 \in A_pB_p$ and everything is proved.

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Case 2: $b_1 \in B \setminus B_p$. Without loss of generality let $a_2 \in A_p$ and $g_2 \in G_p$; such elements exist because $r_1a_1 + r_2a_2 + \ldots + r_ta_t \in S(RA)$ and $f_1b_1 + f_2g_2 + \ldots + f_tg_t \in 1 + I_p(RG; B) \subseteq S(RG)$. Next, we distinguish four independent subcases (all other remaining possibilities have similar proofs):

2.1. $g_3 \in b_1G_p$, $g_3 = g_2b$ ($b \in B$), $g_2 \notin B$; $f_1 + f_3$ lies in the nil-ideal of R, $f_2 + f_3 = 0$.

In this situation we extract $b_1b^{-1} \in B_p$ and $g_pa_1a_2a_3^{-1} = b_1b^{-1} \in B$, i. e., in other words $g_p \in A_pB_p$.

2.2. $g_3 \in g_2B_p$, $g_2 \notin B$, $g_4 \in b_1G_p$, $g_5 = g_4b' \in G_p$ ($b' \in B$); $f_2 + f_3 = 0$, $f_1 + f_4$ belongs to the nil-ideal of R, $f_4 + f_5 = 0$.

Under these circumstances $b'b_1 \in B_p$, hence $g_pa_1a_5a_4^{-1} = b_1b' \in B_p$, i. e., equivalently $g_p \in A_pB_p$.

2.3. $g_3 = g_2b$ ($b \in B$), $g_5 = g_4b'$ ($b' \in B$), $g_4 \in b_1G_p$, $g_3 \in g_5G_p$; $f_2 + f_3 = 0$, $f_4 + f_5 = 0$, $f_3 + f_5$ lies in the nil-radical of R.

In that case $b_1^{-1}bb'^{-1} \in B_p$ and $g_p^{-1}a_1^{-1}a_2^{-1}a_3a_4a_5^{-1} = b_1^{-1}b'^{-1}b$ which substantiates our claim.

2.4. $g_3 = g_2b$ ($b \in B$), $g_5 = g_4b'$ ($b' \in B$), $g_5 \in b_1G_p$, $g_3 \in g_4G_p$; $f_2 + f_3 = 0$, $f_4 + f_5 = 0$, $f_3 + f_4$ belongs to the nil-radical of R.

Because this case is analogous to the last preceding one, by similar computations $b'bb_1^{-1} \in B_p$ and $g_p^{-1}a_1^{-1}a_2^{-1}a_3a_4^{-1}a_5 = b'bb_1^{-1}$.

Finally, in all cases, we deduce $g_p \in A_pB_p$ whence $G_p = A_pB_p$, as stated.

Sufficiency. Choosing $x \in S(RG)$, we write $x = f_1a_1b_1 + \ldots + f_ta_tb_t$. Furthermore in the canonical record there is a group member from G_p, say, e. g., a_tb_t; thus $a_tb_t = a_{tp}b_{tp}$ for some $a_{tp} \in A_p$ and $b_{tp} \in B_p$. Moreover, with no loss of generality, we may presume that the following relations are fulfilled (the remaining cases are analogous): for some $k \in \mathbf{N}$

$$a_1^{p^k}b_1^{p^k} = a_2^{p^k}b_2^{p^k} = \ldots = a_s^{p^k}b_s^{p^k} \neq 1 \qquad (s \in \mathbf{N}),$$

$(f_1 + f_2 + \ldots + f_s)^{p^k} = 0$; $a_{s+1}^{p^k}b_{s+1}^{p^k} = \ldots = a_t^{p^k}b_t^{p^k} = 1$, $(f_{s+1} + \ldots + f_t)^{p^k} = 1$. By the above listed assumptions $a_1a_2^{-1}a_{2p} = b_2b_1^{-1}b_{2p}$ for $a_{2p} \in A_p$, $b_{2p} \in B_p$, etc.; similarly $a_1a_s^{-1}a_{sp} = b_sb_1^{-1}b_{sp}$ for $a_{sp} \in A_p$ and $b_{sp} \in B_p$. Besides this, $a_{s+1}b_{s+1} = a_{s+1p}b_{s+1p}$ with $a_{s+1p} \in A_p$ and $b_{s+1p} \in B_p$, etc., $a_tb_t = a_{tp}b_{tp}$ with $a_{tp} \in A_p$ and $b_{tp} \in B_p$.

After this, we examine the sum $f_1a_1 + \ldots + f_sa_s = f_1a_1 + f_2a_1a_{2p} + \ldots + f_sa_1a_{sp} + f_2a_2 - f_2a_2b_2b_1^{-1}b_{2p} + \ldots + f_sa_s - f_sa_sb_sb_1^{-1}b_{sp} = f_1a_1 + f_2a_1a_{2p} + \ldots + f_sa_1a_{sp} + f_2a_2(1 - b_2b_1^{-1}b_{2p}) + \ldots + f_sa_s(1 - b_sb_1^{-1}b_{sp})$. It is a simple matter to see that $f_1a_1 + f_2a_1a_{2p} + \ldots + f_sa_1a_{sp}$ is a nil-element from $I(RA; A)$, i. e., it lies in $I_p(RA; A)$.

On the other hand, $f_{s+1}a_{s+1}b_{s+1} + \ldots + f_ta_tb_t = f_{s+1}a_{s+1p} + f_{s+1}a_{s+1p}(b_{s+p} - 1) + \ldots + f_ta_{tp} + f_ta_{tp}(b_{tp} - 1)$. Certainly, $f_{s+1}a_{s+1p} + \ldots + f_ta_{tp}$ is a p-element, i. e., it belongs to $S(RA)$. As a final step, we detect that $y = f_1a_1 + f_2a_1a_{2p} + \ldots + f_sa_1a_{sp} + f_{s+1}a_{s+1p} + \ldots + f_ta_{tp} \in S(RA)$. Thus, $x = y + f_1a_1(b_1 - 1) + \ldots + f_sa_s(b_s - 1) + f_2a_2(1 - b_2b_1^{-1}b_{2p}) + \ldots + f_sa_s(1 - b_sb_1^{-1}b_{sp}) + f_{s+1}a_{s+1p}(b_{s+1p} - 1) + \ldots + f_ta_{tp}(b_{tp} - 1) \in S(RA) + I(RG; B)$

whence we infer at once that $x \in S(RA)(1 + I_p(RG; B))$, as desired. The proof is now complete. □

Comment. At this point we shall give a detailed analysis of the arguments used in the proof of the necessity of Proposition 2.4 from [1]; the notions and notations are as in the considered assertion.

In fact, we have claimed that the non-existence of zero divisors in the field does imply that in the right hand-side of the canonical record of x (see, e. g., [1, p. 54, line 5] and [4, p. 8, lines 1, 2, 3 (-)]) there is an element from MB with nonzero coefficient in K, which is correct, because in the right hand-side all possible group relations if they eventually exist between the group elements of the two sums lead us to that; the existing of no zero divisors guarantees that all group members in these relations are of equal worth. If such relations do not exist, everything is done.

Such a claim is true even without this assumption on zero divisors by which we have just shown above or by exploiting the trick that if $0 \neq r \in R$, $\alpha_1, \ldots, \alpha_n \in R$, $r\alpha_2 = r\alpha_3 = \ldots = r\alpha_n = 0$ and $\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$ then $r = r\alpha_1 \neq 0$, i. e., the condition on zero divisors can be removed.

Moreover, the use of the necessity in Proposition 2.4 from [1] in the verification of Proposition 2.6 of [1] for infinitely many factors is correct, of course, since the infinite case follows immediately from the finite one by means of a standard transfinite induction. That is why this argumentation was omitted in the text of the manuscript of [1].

Remark 1. The technique shown above leads us to the following intersection identity:

$$[G_p S(RA)] \cap (1 + I_p(RG; B)) = B_p(1 + I_p(RA; A \cap B)).$$

Claim. *For any two abelian groups A and B, the following conditions are equivalent:*

(1) $A^{p^n} \cap B^{p^n} = (A \cap B)^{p^n}, \forall n \in \mathbf{N}$;
(2) $(AB)[p^n] = A[p^n]B[p^n], \forall n \in \mathbf{N}$;
(3) $(AB)_p = A_pB_p$ *and for each* $x = ab$ *when* $a \in A_p$ *and* $b \in B_p$ *the condition* $x^{p^m} = 1$ *holds* $\iff$ $a^{p^m} = 1$ *and* $b^{p^m} = 1$ *for any positive integer m.*

Proof. Evidently (2) $\Leftrightarrow$ (3). Now we consider the implication (1) $\Rightarrow$ (2). For this purpose, given $x \in (AB)[p^n]$, hence $x = ab$ with $a \in A$, $b \in B$ and $(ab)^{p^n} = 1$. Thus, $a^{p^n} = b^{-p^n} = c^{p^n}$ for $c \in A \cap B$ and we find $x = ac^{-1}cb \in A[p^n]B[p^n]$.

We treat now (2) $\Rightarrow$ (1). For this goal, choose $x \in A^{p^n} \cap B^{p^n}$. So, $x = a^{p^n} = b^{p^n}$ whenever $a \in A$ and $b \in B$. Henceforth $ab^{-1} \in (AB)[p^n] = A[p^n]B[p^n]$, i. e., $ab^{-1} = a_nb_n$ where $a_n \in A[p^n]$ and $b_n \in B[p^n]$. Observing that $aa_n^{-1} = b_nb \in A \cap B$, we see that $x = (aa_n^{-1})^{p^n} \in (A \cap B)^{p^n}$. The proof is complete . □

A direct consequence is the following.

Corollary. *Let* $G = AB$ *with* (1) *fulfilled. Then* $S(RG) = S(RA)(1 + I_p(RG; B))$.

Proof. Although we can copy our original proof from the foregoing main Proposition along with the Claim, we will give another confirmation of the attainment.

Indeed, given $x \in S(RG)$, hence $x = r_1a_1b_1+\ldots+r_ta_tb_t = r_1a_1(b_1-1)+r_1a_1+\ldots+r_ta_t(b_t-1)+r_ta_t = r_1a_1+\ldots+r_ta_t+r_1a_1(b_1-1)+\ldots+r_ta_t(b_t-1)$. Therefore there exists $k \in \mathbf{N}$ so that $1-(r_1a_1+\ldots+r_ta_t)^{p^k} = [r_1a_1(b_1-1)+\ldots+r_ta_t(b_t-1)]^{p^k} \in R^{p^k}A^{p^k} \cap I(R^{p^k}G^{p^k}; B^{p^k}) = I(R^{p^k}A^{p^k}; A^{p^k} \cap B^{p^k}) = I(R^{p^k}A^{p^k}; (A \cap B)^{p^k}) = I^{p^k}(RA; A \cap B)$. Consequently there is $v \in I(RA; A \cap B)$ such that $r_1a_1+\ldots+r_ta_t+v \in S(RA)$. Finally, $x = (r_1a_1+\ldots r_ta_t+v)(1+(r_1a_1+\ldots+r_ta_t+v)^{-1}(r_1a_1(b_1-1)+\ldots+r_ta_t(b_t-1)-v)) \in S(RA)(1+I_p(RG; B))$, as promised. The proof is complete. □

Critical Remark. In his reviewer's report [6], Mollov has claimed that our method of proof of [1, Proposition 2.4] contains an error. But his conclusion is obviously wrong by what we have already quoted above.

Now we are ready to study the niceness in group rings. Conforming with the classical definition for nice subgroups of p-primary groups (see, e. g., [5]), we shall say that the subgroup N of the arbitrary abelian group G is nice only when $\cap_{\alpha<\tau}(NG^{p^\alpha}) = NG^{p^\tau}$ for each prime p and for each limit ordinal number τ. If this equality is valid only for one single prime p, N is called p-nice in G.

Owing to the above decomposition formula, we are now in a position to obtain the following

Proposition 2 (Niceness)**.** *Suppose N is p-balanced, that is, p-nice and p-isotype subgroup of G. Then $1+I_p(RG; N)$ is nice in $S(RG)$ provided R is perfect.*

Proof. Consuming the definition for niceness of p-torsion groups, it is enough to show that for every limit ordinal τ one has $\bigcap_{\alpha<\tau}[S^{p^\alpha}(RG)(1+I_p(RG; N))] = S^{p^\tau}(RG)(1+I_p(RG; N))$. For this aim, take an arbitrary element x from the intersection. Thus we may write $x \in (r_{1\alpha}g_{1\alpha}+\ldots+r_{t\alpha}g_{t\alpha})(1+I_p(RG; N))$ and $x \in (f_1g_{1\beta}+\ldots+f_tg_{t\beta})(1+I_p(RG; N))$, where $r_{1\alpha}, \ldots, r_{t\alpha}; f_1, \ldots, f_t \in R$ and $g_{1\alpha}, \ldots, g_{t\alpha} \in G^{p^\alpha}$; $g_{1\beta}, \ldots, g_{t\beta} \in G^{p^\beta}$ for an arbitrary ordinal β with the property $\alpha < \beta < \tau$. That is why we can write $r_{1\alpha}g_{1\alpha}+\ldots+r_{t\alpha}g_{t\alpha} = (f_1g_{1\beta}+\ldots+f_tg_{t\beta})(e_1a_1+\ldots+e_ta_t)$, where $1 \neq e_1a_1+\ldots+e_ta_t \in 1+I_p(RG; N)$. We note that $r_{1\alpha}+\ldots+r_{t\alpha} = f_1+\ldots+f_t = e_1+\ldots+e_t = 1$ and that we may presume $a_1 \in N$. Moreover, let us assume that in R there exist zero divisors; otherwise if there are no zero divisors, the conclusions are simpler or similar to those presented above. Thus, suppose $f_1e_1 \neq 0$ and $f_1e_2 = \ldots = f_1e_t = 0$. Certainly, if $f_1e_1 = 0$, $f_1 = 0$, which is false. Besides, $f_2e_2 \neq 0$, $f_2e_3 \neq 0$ and $f_2e_1 = f_2e_4 = \ldots = f_2e_t = 0$; $f_3e_2 \neq 0$, $f_3e_3 \neq 0$ and $f_3e_1 = f_3e_4 = \ldots = f_3e_t = 0$, etc., due to the symmetry $f_{k-1}e_{k-1} \neq 0$, $f_{k-1}e_k \neq 0$ and $f_{k-1}e_1 = \ldots = f_{k-1}e_{k-2} = f_{k-1}e_{k+1} = \ldots = f_{k-1}e_t = 0$; $f_ke_{k-1} \neq 0$, $f_ke_k \neq 0$ and $f_ke_1 = \ldots = f_ke_{k-2} = f_{k-1}e_{k+1} = \ldots = f_{k-1}e_t = 0$ for some positive integer $k < t$. For the remaining cases, $f_{k+1}e_{k+1} \neq 0$ and $f_{k+1}e_1 = \ldots = f_{k+1}e_k = f_{k+1}e_{k+2} = \ldots = f_{k+1}e_t = 0$, …, $f_te_t \neq 0$ and $f_te_1 = \ldots = f_te_{t-1} = 0$. We note that $f_1e_1 = f_1, \ldots, f_te_t = f_t$.

The existence of the following relations is natural: $a_3 \in a_2N$ and $e_2 + e_3 = 0$, $a_5 \in a_4N$ and $e_4 + e_5 = 0$, etc., $a_k \in a_{k-1}N$, $e_{k-1} + e_k = 0$. Also, $e_1 + e_{k+1} + \ldots + e_t = 1$ and $a_{k+1}, a_{k+2}, \ldots, a_t \in N$ or $e_1 = 1$, $e_{k+1} + \ldots + e_t = 0$ and $a_{k+1} \in a_{k+2}N \in \ldots \in a_tN$.

So, we obtain $r_{1\alpha}g_{1\alpha} + \ldots + r_{t\alpha}g_{t\alpha} = f_1e_1g_{1\beta}a_1 + f_2e_2g_{2\beta}a_2 - f_2e_2g_{2\beta}a_3 + f_3e_2g_{3\beta}a_2 - f_3e_2g_{3\beta}a_3 + \ldots + f_{k-1}e_1g_{k-1\beta}a_{k-1} - f_{k-1}e_{k-1}g_{k-1\beta}a_k + f_ke_{k-1}g_{k\beta}a_{k-1} - f_ke_{k-1}g_{k\beta}a_k + f_{k+1}e_{k+1}g_{k+1\beta}a_{k+1} + \ldots + f_te_tg_{t\beta}a_t$.

For an extra difficulty, suppose also that $g_{1\beta}a_1$ does not lie in the support of x; e. g., $g_{1\beta}a_1 = g_{k+1\beta}a_{k+1}$ and $f_1e_1 + f_{k+1}e_{k+1} = 0$. But then $g_{k+2\beta}a_{k+2} \in g_{k+2\beta}a_{k+1}N = g_{k+2\beta}g_{k+1\beta}^{-1}g_{1\beta}a_1N \in G^{p^\beta}N$ and analogously $g_{t\beta}a_t \in G^{p^\beta}N$. If $g_{k+2\beta}a_{k+2} = \ldots = g_{t\beta}a_t \in G^{p^\beta}N$, then the coefficient in R of this element, equal to $f_{k+2}e_{k+2} + \ldots + f_te_t$, is nonzero. Otherwise if $f_{k+2}e_{k+2} + \ldots + f_te_t = (f_{k+2} + \ldots + f_t)(e_{k+2} + \ldots + e_t) = 0$, we derive $(f_1 + f_{k+1} + f_{k+2} + \ldots + f_t)(e_1 + e_{k+1} + e_{k+2} + \ldots + e_t) = 0$. Because $e_1 + e_{k+1} + e_{k+2} + \ldots + e_t = 1$, we have $f_1 + f_{k+1} + f_{k+2} + \ldots + f_t = 0$, hence $f_2 + f_3 + \ldots + f_k = 1$. Furthermore, since $e_2 + e_3 + \ldots + e_k = 0$, we get $1 = 1.1 = (f_1 + \ldots + f_t)(e_1 + \ldots + e_t) = (f_2 + f_3 + \ldots + f_k)(e_1 + e_{k+1} + e_{k+2} + \ldots + e_t) = 0$, which is a contradiction.

On the other hand, if $g_{k+2\,\beta}a_{k+2} = g_{2\beta}a_2$ and $f_{k+2}e_{k+2} + f_2e_2 = 0$, we deduce $g_{2\beta}a_3 \in g_{2\beta}a_2N \in G^{p^\beta}N$. In all that follows, in the right hand-side of the main equality does exist an element from $G^{p^\beta}N$. Since $\{\beta < \tau \geq \omega\}$ is an infinite set, and the support is finite, we can assume that all dependences are of the form presented.

And so, independently from the additional relations between the elements from the right hand-side if they eventually exist, we observe that $g_{1\alpha} \in \cap_{\beta<\tau}(G^{p^\beta}N) = G^{p^\tau}N$, $\ldots$, $g_{s\alpha} \in G^{p^\tau}N$ and $g_{s+1\,\alpha} \in g_{s+2\,\alpha}N$, $\ldots$, $g_{t-1\,\alpha} \in g_{t\alpha}N$ together with $r_{s+1\alpha} + r_{s+2\,\alpha} = 0, \ldots, r_{t-1\alpha} + r_{t\alpha} = 0$; $s \in N$. Because x is a p-torsion element, let $g_{s+1\alpha} \in G_p$, $g_{s+3\,\alpha} \in G_p$, $\ldots$, $g_{t-1\alpha} \in G_p$ with $r_{s+1\alpha} + r_{s+3\,\alpha} + \ldots + r_{t-1\alpha} - 1 \in \text{rad}\,(R)$, the nil-radical of R. Moreover, since $1 = r_{1\alpha} + r_{2\alpha} + \ldots + r_{s\alpha} \notin \text{rad}\,(R)$, the ratios $g_{1\alpha} \in g_{2\,\alpha}G_p \in \ldots \in g_{s\alpha}G_p$ are impossible. Thereby it is a real matter to presume that $g_{1\alpha} \in g_{s+2\,\alpha}G_p$. This, along with $g_{1\alpha} \in G^{p^\tau}N$ and $G_p \ni g_{s+1\,\alpha} \in g_{s+2\,\alpha}N$, leads us to $g_{1\alpha} \in (G^{p^\tau}N) \cap (G_pN) = N(G^{p^\tau}N)_p$. But the p-isotypity of N in G means $(G^{p^\tau}N)_p = G_p^{p^\tau}N_p$ whence $g_{1\alpha} \in G_p^{p^\tau}N$. By symmetry $g_{2\,\alpha}, \ldots, g_{s\alpha} \in G_p^{p^\tau}N$ eventually when $r_{2\alpha} \notin \text{rad}\,(R), \ldots, r_{s\alpha} \notin \text{rad}\,(R)$. Bearing in mind that $r_{s+1\,\alpha}g_{s+1\,\alpha} + \ldots + r_{t\alpha}g_{t\alpha} \in I(RG; N)$, that $r_{1\alpha}g_{1\alpha} + r_{2\alpha}g_{2\alpha} + \ldots + r_{s\alpha}g_{s\alpha} = r_{1\alpha}(g_{1\alpha} - 1) + r_{2\alpha}(g_{2\alpha} - 1) + \ldots + r_{s\alpha}(g_{s\alpha} - 1) + 1 = r_{1\alpha}(b_{p\tau}u_1 - 1) + r_{2\alpha}(b_{2\tau}u_2 - 1) + \ldots + r_{s\alpha}(b_{s\tau}u_s - 1) + 1 = 1 + r_{1\alpha}(b_{p\tau} - 1) + r_{1\alpha}b_{p\tau}(u_1 - 1) + \ldots + r_{s\alpha}(b_{s\tau} - 1) + r_{s\alpha}b_{s\tau}(u_s - 1)$ where $b_{p\tau} \in G_p^{p^\tau}$, $b_{2\tau} \in G^{p^\tau}$, $\ldots$, $b_{s\tau} \in G^{p^\tau}$; $u_1, u_2, \ldots, u_s \in N$, and that $1 + r_{1\alpha}(b_{p\tau} - 1) + \ldots + r_{s\alpha}(b_{s\tau} - 1) \in S^{p^\tau}(RG)$, we conclude that $r_{1\alpha}g_{1\alpha} + \ldots + r_{t\alpha}g_{t\alpha} \in S^{p^\tau}(RG) + I(RG; N) = S^{p^\tau}(RG)(1 + I(RG; N))$. So, $x \in S^{p^\tau}(RG)(1 + I_p(RG; N))$ and this ends the inclusion. The proof is over in all generality. □

Example. The following example demonstrably shows that in the evidence of Proposition on Niceness the condition for nonidentity of $1 \neq e_1a_1 + \ldots + e_ta_t \in 1 + I_p(RG;N)$ is essential and cannot be ignored. If yes, the proof fails. In fact, it starts with an element $r_{1\alpha}g_{1\alpha} + \ldots + r_{t\alpha}g_{t\alpha} \in S^{p^\tau}(RG) = \cap_{\alpha<\tau}S^{p^\alpha}(RG); \tau$ is a limit ordinal, and by the end of the proof has decomposed this element so that, assuming re-indexing of the subscripts, there is some s with $1 \leq s \leq t$ such that $g_{1\alpha}, \ldots, g_{s\alpha} \in G_p^{p^\tau}N$, and $r_{s+1\alpha}g_{s+1\alpha} + \ldots + r_{t\alpha}g_{t\alpha} \in I(RG;N)$. In order to substantiate our claim, suppose that $G = K \times N$, where K contains a non-trivial torsion-free element g and distinct p-torsion elements c_1 and c_2 such that $gc_i \in G^{p^\tau}; i = 1,2$. It is really not hard to find such a group K. We further see that the concrete element $1 + gc_1 - gc_2 \in S^{p^\tau}(RG) = S(R^{p^\tau}G^{p^\tau})$, thus the proof may have started with this element. But $gc_i \notin G_p^{p^\tau}N$ is clear since otherwise $gc_i \in (G_p^{p^\tau}N) \cap K \subseteq (K_p \times N) \cap K = K_p$ whence $g \in K_p$ and $g = 1$, contrary to the choice. So, only 1 lies in $G_p^{p^\tau}N$. This means that $gc_1 - gc_2$ must lie in $I(RG;N)$. But we observe that this is false because $gc_1 - gc_2 = gc_1(1 - c_1^{-1}c_2) \in I(RG;N)$ implies $c_1^{-1}c_2 \in K \cap N = 1$ hence $c_1 = c_2$, against our hypothesis.

Remark 2. W. May showed in [5] that if N is a nice p-subgroup of the abelian group G, then $1 + I_p(RG;N)$ is nice in $S(RG)$ provided R is a perfect field. The above affirmation extends May's result when N is not p-primary, however, and R is not necessarily a field. It may be successfully applied for arguing the simply presented structure of $S(RG)/G_p$, but this is a problem of some other investigation.

Problem. *Does it follow that* $1 + I_p(RG;N)$ *is nice in* $S(RG)$ *provided* N *is* p*-nice in* G *only?*

Corrigendum

The misspelled word "droped" from of [1, p. 51] should read as "dropped".

Also, in [2, p. 258, line 20 (+)], the expression "C_β" must be replaced by "G_β"; on p. 260, the words "ensure" and "choosen" should be replaced by "ensures" and "chosen", repsectively, and "expanson" on p. 261 should replaced by "expansion."

In [3, p. 223, line 13 (+)], the expression "B_α" must be replaced by "G_α" and on p. 224, the term "$\coprod_{\alpha<\lambda}\coprod_{\mu<\alpha}G_\mu$" should be replaced by "$\prod_{\alpha<\lambda}\coprod_{\mu<\alpha}G_\mu$."

Finally, in [4, p. 9, line 5 (-)], the sign "=" should be read as "$\subset$"; the choice realized there is possible because any subgroup of a σ-summable p-group with equal length is also σ-summable. Finally, the second expression "$g_{in}^{\varepsilon_n}$" on p. 12, line 5 (+), should be removed.

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PSEUDOCONJUGACY CRITERIA FOR HALF-LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

ONDŘEJ DOŠLÝ

Abstract. Using the variational principle, we present sufficient conditions for the existence of a pair of pseudoconjugate points in an interval $I \subseteq \mathbb{R}$ relative to the half-linear second order differential equation

$$(*) \qquad (r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-2}x, \ p > 1.$$

The endpoints of the interval I are allowed to be singular (in particular, the case $I = \mathbb{R}$ is possible) and an important role is played by the principal solution of (*) and of its reciprocal equation.

Mathematics Subject Classification: 34C10

Keywords: half-linear differential equation, pseudoconjugate points, pseudoconjugacy criterion, variational method, principal solution, reciprocal equation, coprincipal solution

1. Introduction

In this paper we deal with the half-linear second order differential equation

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x, \ p > 1, \tag{1}$$

where r, c are continuous functions and $r(t) > 0$ in an interval under consideration. Oscillation theory of (1) attracted considerable attention in recent years and many of the results of the classical linear oscillation theory (the linear Sturm-Liouville differential equation is a special case $p = 2$ in (1)) have been extended to (1), see, e. g., [1, Chap. 3] and the references given therein.

Let x be a nontrivial solution of (1) such that $x(t_1) = 0$ for some $t_1 \in \mathbb{R}$. A point $t_2 > t_1$ is said to be the *(right) conjugate point of* t_1 (relative to (1)) if $x(t_2) = 0$. A value $t_2 > t_1$ is said to be the *(right) pseudoconjugate point* of t_1 (relative to (1)) if $x'(t_2) = 0$. Note that these definitions of (pseudo)conjugate points of t_1 are correct since by the homogeneity property of the solution space of (1), any nontrivial solution z of (1) for which $z(t_1) = 0$ is a constant multiple of x. Equation (1) is said

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to be *(pseudo)conjugate* on an interval I if this interval contains at least one pair of (pseudo)conjugate points.

An integral comparison theorem for (pseudo)conjugate points relative to a pair of *linear* second order differential equations of the form

$$(r(t)x')' + c(t)x = 0$$

was established by Leighton in [8]. A part of this statement concerning conjugate points was extended to half-linear equation (1) by Jaroš and Kusano [7] and a "pseudoconjugate" part in the recent paper of Rostás [12]. These statements compare (1) with the equation of the same form

$$(R(t)\Phi(y'))' + C(t)\Phi(y) = 0, \tag{2}$$

and read as follows.

Proposition 1. *Consider a pair of half-linear differential equations* (1) *and* (2).

(i) *Let y be a nontrivial solution of* (2) *such that $y(a) = 0 = y(b)$, i. e., b is a right conjugate point of t_1. If*

$$\int_a^b \{[R(t) - r(t)]|y'(t)|^p - [C(t) - c(t)]|y(t)|^p\}\, dt \le 0, \tag{3}$$

then any nontrivial solution of (1) *with $x(a) = 0$ has a zero point in (a, b) or it is a constant multiple of y.*

(ii) *Let y be a nontrivial solution of* (2) *such that $y(a) = 0 = y'(b)$ and $y'(t) \neq 0$ for $t \in [a, b)$, i. e., b is the first right pseudoconjugate point of a relative to* (2). *If* (3) *holds, then any nontrivial solution x of* (1) *with $x(a) = 0$ has the property that $x'(\xi) = 0$ for some $\xi \in (a, b]$ with $\xi = b$ only if x is a multiple of y.*

Note that a closer examination of the proof of the previous statements reveals the fact that part (ii) can be reformulated as follows (part (i) can be reformulated in a similar way, but we will not need this modified statement in our paper).

Lemma 1. *Suppose that there exists a piecewise differentiable function z for which $z(a) = 0 = z'(b)$ for $t \in [a, b)$ and*

$$\mathcal{F}_{rc}(z; a, b) := \int_a^b [r(t)|z'(t)|^p - c(t)|z(t)|^p]\, dt < 0. \tag{4}$$

Then the nontrivial solution x of (1) *given by $x(a) = 0$ satisfies $x'(\xi) = 0$ for some $\xi \in (a, b]$.*

The aim of this paper is to investigate the situation when the endpoints a, b are allowed to be *singular* points of the investigated equations. A typical example is $a = -\infty$, $b = \infty$ (but, of course, a, b may also be finite singularities, i. e., points where the existence and uniqueness of solvability of the initial value problem for the investigated equations is violated).

A singular version of Leighton's comparison theorem for *conjugate points* was established in several papers. It was discovered that the crucial role is played by the so-called *principal solution* of (half-)linear equations in these extensions.

As a classical paper concerning the *linear* Sturm-Liouville equation (1) is usually regarded the paper of Müller-Pfeiffer [10]. In that paper (and then in its improved version [11]) it is shown that under the assumptions

$$\int^{\infty} \frac{dt}{r(t)} = \infty = \int_{-\infty} \frac{dt}{r(t)}, \tag{5}$$

equation (1) is conjugate on $\mathbb{R}$ provided

$$\liminf_{t_1\downarrow-\infty,\, t_2\uparrow\infty} \int_{t_1}^{t_2} c(t)\,dt \geq 0 \quad \text{and} \quad c(t) \not\equiv 0 \text{ on } \mathbb{R}. \tag{6}$$

Observe that under (5) the function $\tilde{x}(t) \equiv 1$ is the principal solution of the one-term equation $(r(t)x')' = 0$ both at $-\infty$ and ∞ (for explanation see the next section) and that the integrand $c(t)$ in (6) can be written in the form $c(t)\tilde{x}^2(t)$ in this special case. This observation is also a link between (6) and the linear version of the part (i) of Proposition 1, since y is the principal solution at (regular points) a and b of (2).

The half-linear extension of (6) is given in [4], where equation (1) is considered as a perturbation of the equation

$$(r(t)\Phi(x'))' + \tilde{c}(t)\Phi(x) = 0. \tag{7}$$

It is supposed that (7) is disconjugate and its principal solutions at a and b coincide, i. e., $x_a = x_b = \tilde{x}$. Then (1) is conjugate on (a, b) provided

$$\liminf_{t_1\downarrow a,\, t_2\uparrow b} \int_{t_1}^{t_2} [c(t) - \tilde{c}(t)]\tilde{x}^p(t)\,dt \geq 0, \quad c(t) \not\equiv \tilde{c}(t) \text{ on } (a, b).$$

When $p = 2$, $(a, b) = \mathbb{R}$ and $\tilde{c}(t) \equiv 0$, this conjugacy criterion reduces to (6). Concerning a singular version of part (ii) of Proposition 1, the main idea of our proof is based on the fact that the so-called *coprincipal solution* of (1) at b plays the role of the solution satisfying $y'(b) = 0$ if b is a singular point (and equals this solution if b is a regular point).

The paper is organized as follows. In the next section we recall basic methods of the half-linear oscillation theory, including the concept of the principal solution of (1) which plays the fundamental role in our investigation. Section 3 contains the main results of the paper—two "singular" pseudoconjugacy criteria for (1), and the last section is devoted to examples which illustrate these criteria.

2. Preliminaries

In this section we recall some basic facts concerning equation (1). First of all let us mention that the linear oscillation theory extends verbatim to (1). In particular, the zero points of two linearly independent solutions of this equation separate each

other and hence (1) can be classified as *oscillatory* or *nonoscillatory* at a singular point b according to whether or not any solution has/does not have infinitely many zeros in any neighborhood of b. Similarly to the linear case, equation (1) cannot be oscillatory at a regular point since singular points are the only possible cluster points of a sequence of zero points of a solution of (1).

If $c(t) \neq 0$ in some interval I and x is a solution of (1), then $u = r\Phi(x')$ is a solution of the *reciprocal equation*

$$\left(\frac{1}{\Phi^{-1}(c(t))}\Phi^{-1}(u')\right)' + \frac{1}{r^{q-1}(t)}\Phi^{-1}(u) = 0, \tag{8}$$

where $\Phi^{-1}(s) = |s|^{q-2}s$, is the inverse function of Φ, q being the conjugate number of p, i. e., $q = p/(p-1)$. The Rolle Mean Value theorem immediately implies that (1) is oscillatory if and only if the reciprocal equation (8) is oscillatory. If a solution x satisfies $x(t) \neq 0$ in some interval I, then $w := r\Phi(x'/x)$ solves in I the Riccati-type differential equation

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0. \tag{9}$$

Moreover, if $w(t) \neq 0$ for $t \in I$, then $v = -\frac{1}{\Phi^{-1}(w)}$ is a solution of the equation

$$v' + r^{1-q}(t) + (q-1)c(t)|v|^p = 0, \tag{10}$$

which is the Riccati-type equation associated with reciprocal equation (8). We will also need the half-linear version of Picone's identity, which was established in [7]. We present here this identity in a modified form which is more suitable for our purpose.

Lemma 2. *Let w be a solution of Riccati-type equation* (9)*, which exists for $t \in [a,b]$. Then for any piecewise C^1 function y we have*

$$\int_a^b [r(t)|y'|^p - c(t)|y|^p]\,dt = w(t)|y|^p\Big|_a^b + p\int_a^b r^{1-q}(t)P(r^{q-1}(t)y', w(t)\Phi(y))\,dt, \tag{11}$$

where

$$P(u,v) := \frac{|u|^p}{p} - uv + \frac{|v|^q}{q} \geq 0 \tag{12}$$

with equality only if $v = \Phi(u)$.

The crucial role in our investigation is played by the concept of the principal solution of nonoscillatory half-linear equation (1). This concept was introduced by Mirzov [9] as follows. We describe this construction near the singular point $b = \infty$ as it is done in Mirzov's original paper, but this construction extends also when b is a finite singularity (or even a regular point). Suppose that (1) is nonoscillatory at ∞, i. e., there exists a solution $\hat{w}$ of associated Riccati-type equation (9) which is defined

on an interval $[T,\infty)$. Let $d > T$ and x_d be the solution of (1) given by the initial condition $x_d(d) = 0$, $x_d'(d) = -1$ and let $w_d = r\Phi(x_d'/x_d)$. Then for $T < d_1 < d_2$ we have $\hat{w}(t) > w_{d_1}(t) > w_{d_2}(t)$ for $t \in [T, d_1)$. This means that on every compact subinterval of $[T,\infty)$ there exists the uniform limit

$$\tilde{w}(t) := \lim_{d\to\infty} w_d(t),$$

and hence this function is a solution of (9). This solution is called the *eventually minimal* solution of (9). The corresponding solution $\tilde{x}$ of (1) given by the formula

$$\tilde{x}(t) = K \exp\left\{\int_T^t \Phi(\tilde{w}(s)/r(s))\right\} ds,$$

where K is a nonzero real constant, is called the *principal solution* (at ∞) of (1). Note that the largest zero point $\tilde{t}$ of the principal solution $\tilde{x}$ (if exists) plays essentially the role of the first left conjugate point of $t = \infty$ since any solution x of (1) linearly independent of $\tilde{x}$ has exactly one zero in $(\tilde{t},\infty)$, see [7]. Note also that if $t = b$ is a *regular point* of (1), then the principal solution at b is simply defined as a nontrivial solution satisfying $x(b) = 0$. Finally, a solution $\tilde{x}$ of nonoscillatory equation (1) is said to be *coprincipal* at b, if its quasiderivative $\tilde{u} = r\Phi(\tilde{x}')$ is the principal solution at b of the reciprocal equation (8). In the last two statements of this section we recall some properties of the principal and coprincipal solutions of (1). The first one is formulated in [5] in case $b = \infty$, but a closer examination of its proof reveals that the statement remains to hold also when b is a finite singularity.

Lemma 3. *Suppose that* $\int^b r^{1-q}(t)\,dt = \infty$, $c(t) > 0$ *near* b *and* (1) *is nonoscillatory at* b.

(i) *If* $\tilde{x}$ *is the principal solution of equation* (1) *at* b, *then it is also a coprincipal solution at* b.

(ii) *A solution* $\tilde{x}$ *of* (1) *is principal at* b *if and only if*

$$\int^b \frac{dt}{r(t)\tilde{x}^2(t)|\tilde{x}'(t)|^{p-2}} = \infty. \tag{13}$$

The following alternative integral characterization of the principal solution of (1) is proved in the recent paper [3]. Note that this paper actually formulates the statement with the power x^2 instead of x^p in the formula (14) given below. However, a closer examination of the proof given in [3] reveals that the statement remains to hold in this modified setting which better fits our treatment.

Lemma 4. *Suppose that* $c(t) > 0$ *for large* t, *equation* (1) *is nonoscillatory and both integrals* $\int^\infty r^{1-q}(t)\,dt$ *and* $\int^\infty c(t)\,dt$ *are convergent. Then* x *is the principal solution of* (1) *if and only if*

$$\int^\infty \frac{dt}{r^{q-1}(t)x^p(t)} = \infty. \tag{14}$$

3. Existence of pseudoconjugate points

Our main result reads as follows.

Theorem 1. *Suppose that* (2) *is disconjugate on the interval* $I = (a, b)$, $-\infty \le a < b \le \infty$, $C(t) \ge c(t) > 0$ *and* $0 < R(t) \le r(t)$ *near* a *and* b. *Further, let* y *be the positive principal solution of* (2) *at* a *and suppose that this solution is coprincipal at* b *and* $y'(t) > 0$ *for* $t \in (a, b)$. *If*

$$\limsup_{t_1 \downarrow a,\, t_2 \uparrow b} \mathcal{J}(y; t_1, t_2) < 0, \tag{15}$$

where

$$\mathcal{J}(y; t_1, t_2) := \int_{t_1}^{t_2} \left[(R(t) - r(t))y'^p(t) - (C(t) - c(t))\right] y^p(t)\, dt$$

then (1) *is pseudoconjugate on* I, *i. e., there exists* $a < d_1 < d_2 < b$ *and a nontrivial solution* x *of* (1) *such that* $x(d_1) = 0 = x'(d_2)$.

Proof. Let $a < t_0 < t_1 < t_2 < t_3 < b$ (these values will be specified later), and define a piecewise differentiable function z for which $z(t_0) = 0 = z'(t_3)$ as follows:

$$z(t) = \begin{cases} 0 & \text{for } t \in (a, t_0], \\ f(t) & \text{for } t \in [t_0, t_1], \\ y(t) & \text{for } t \in [t_1, t_2], \\ g(t) & \text{for } t \in [t_2, t_3], \\ g(t_3) & \text{for } t \in [t_3, b). \end{cases}$$

Here, f is a solution of (2) satisfying the boundary conditions

$$f(t_0) = 0, \qquad f(t_1) = y(t_1), \tag{16}$$

this solution exists in view of disconjugacy of (1). The function g is also the solution of (2), given by the boundary condition

$$g(t_2) = y(t_2), \quad g'(t_3) = 0, \tag{17}$$

i. e., the quasiderivative $u_g = R\Phi(g')$ is a solution of the reciprocal equation to (2)

$$\left(C^{1-q}(t)\Phi^{-1}(u')\right)' + R^{1-q}(t)\Phi^{-1}(u) = 0 \tag{18}$$

satisfying the boundary conditions

$$u_g'(t_2) = -C(t_2)\Phi(y(t_2)), \quad u_g(t_3) = 0. \tag{19}$$

The existence of this solution we prove as follows. The solution $w = R\Phi(y'/y)$ of the Riccati-type differential equation associated with (2) is positive on (a, b) (since $y'(t) > 0$ on (a, b)) and hence the solution $v = -(1/\Phi^{-1}(w))$ of the Riccati-type equation corresponding to the reciprocal equation (18) is negative on (a, b). Now,

let u be a solution of reciprocal equation (18) for which $u(t_3) = 0$ and suppose that $u'(s) = 0$ for some $a < s < t_3 < b$. Then integration by parts yields

$$\int_s^{t_3} [C^{1-q}(t)|u'|^p - R^{1-q}(t)|u|^p]\,dt = C^{1-q}u\Phi^{-1}(u')\Big|_s^{t_3} - \int_s^{t_3} u[(C^{1-q}(t)\Phi^{-1}(u'))' + R^{1-q}(t)\Phi^{-1}(u)]\,dt = 0,$$

and, at the same time, applying the Picone identity to the above integral we get

$$\int_s^{t_3} [C^{1-q}(t)|u'|^p - R^{1-q}(t)|u|^p]\,dt = -v(s)|u(s)|^p + q\int_s^{t_3} C(t)P(v\Phi^{-1}(u), C(t)u')\,dt > 0,$$

a contradiction. Hence the solution u satisfies $u'(t) \neq 0$ for $t < t_3$. Consequently, multiplying this solution by a suitable constant, we get the solution satisfying (19).

In the remaining part of the proof we will show that $\mathcal{F}(z; t_0, t_3) < 0$ if t_0, t_3 are sufficiently close to a, b, respectively, which then means that (1) is pseudoconjugate on $[t_0, t_3] \subset (a, b)$ by Lemma 1. Using integration by parts, we get

$$\begin{aligned}\mathcal{F}_{RC}(f; t_0, t_1) &:= \int_{t_0}^{t_1} [R(t)|f'(t)|^p - C(t)|f(t)|^p]\,dt \\ &= R(t)f(t)\Phi(f'(t))\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} f(t)\,[(R(t)\Phi(f'(t))' + C(t)\Phi(f(t))]\,dt \\ &= f^p(t_1)w_f(t_1),\end{aligned}$$

where $w_f = R\Phi(f'/f)$. Similarly,

$$\mathcal{F}_{RC}(y, t_1, t_2) = y^p(t_2)w_y(t_2) - y^p(t_1)w_y(t_1), \quad w_y = \frac{R\Phi(y')}{y},$$

$$\mathcal{F}_{RC}(g; t_2, t_3) = -g^p(t_2)w_g(t_2), \quad w_g = \frac{R\Phi(g')}{\Phi(g)}.$$

Summarizing these computations and taking into account (16), (17), we obtain

$$\mathcal{F}_{RC}(z; a, b) = (w_f - w_y)(t_1)y^p(t_1) + (w_y - w_g)(t_2)y^p(t_2).$$

From the definition of the principal solution at a, we have

$$w_f(t_1) \to w_y(t_1) \quad \text{as} \quad t_0 \to a+. \tag{20}$$

Note that w_f depends on t_0. Concerning the difference $w_y(t_2) - w_g(t_2)$, the functions $v_y = -(1/\Phi^{-1}(w_y))$ and $v_g = -(1/\Phi^{-1}(w_g))$ are solutions of the Riccati equation (10) associated with reciprocal equation (8), and again $v_g(t_2) \to v_y(t_2) \neq 0$ as $t_3 \to b-$,

since we suppose that y is the coprincipal solution at b, i. e., v_y is the eventually minimal solution of (10) at b. Hence also

$$w_g(t_2) \to w_y(t_2) \quad \text{as} \quad t_2 \to b-. \tag{21}$$

Summarizing the previous computations and using inequalities between R, r and C, c near a and b, we have

$$\begin{aligned}\mathcal{F}_{rc}(z; t_0, t_3) &\le \mathcal{F}_{RC}(z; t_0, t_3) + \mathcal{J}(y; t_1, t_2)\\ &= (w_f - w_y)(t_1)y^p(t_1) + (w_y - w_g)(t_2)y^p(t_2) + \mathcal{J}(y; t_1, t_2).\end{aligned}$$

Now, let $\varepsilon > 0$ be such that limsup in (15) is less than -4ε, i. e., $\mathcal{J}(y; t_1, t_3) < -3\varepsilon$ if t_1, t_2 are sufficiently close to a and b, respectively. According to (20) and (21), if $t_0 < t_1$, $t_3 > t_2$ are sufficiently close to a and b, respectively, we have

$$|(w_f - w_y)(t_1)y^p(t_1)| < \varepsilon, \quad |(w_y - w_g)(t_2)y^p(t_2)| < \varepsilon.$$

Consequently, for $t_0 < t_1 < t_2 < t_3$ chosen in the above specified way, we obtain

$$\mathcal{F}_{rc}(z) \le -3\varepsilon + \varepsilon + \varepsilon < 0.$$

Now the statement of the theorem follows from Lemma 1. □

In the next statement we show that if the inequality $R(t) \le r(t)$ holds on *whole interval* (a, b) (and not only near a and b as supposed in Theorem 1), the statement of this theorem can be modified as follows.

Theorem 2. *Suppose that the assumptions of Theorem 1 are satisfied, except for the assumption $C(t) \ge c(t) > 0$ near a, and let $R(t) \le r(t)$ for $t \in (a, b)$. If*

$$\liminf_{s_1 \downarrow a,\ s_2 \uparrow b} \int_{s_1}^{s_2} (c(t) - C(t))\, y^p(t)\, dt \ge 0 \quad \textit{and} \quad C(t) \not\equiv c(t) \textit{ for } t \in (a, b), \tag{22}$$

then (1) *is pseudoconjugate on I, i. e., there exist $a < d_1 < d_2 < b$ and a nontrivial solution x of* (1) *such that $x(d_1) = 0 = x'(d_2)$.*

Proof. We use essentially the same function z as defined in the proof of Theorem 1, with the following modification. Continuity of the functions C, c and (22) imply the existence of $\bar{t} \in (t_1, t_2)$ and $d, \varrho > 0$ such that $(c(t) - C(t))y^p(t) > d$ for $(\bar{t} - \varrho, \bar{t} + \varrho) \subset (t_1, t_2)$. Let α be any positive differentiable function with the compact support in $(\bar{t} - \varrho, \bar{t} + \varrho)$. Now, we modify the function z for $t \in [t_1, t_2]$ as follows:

$$z(t) = \begin{cases} y(t) & \text{for } t \in [t_1, t_2] \setminus [\bar{t} - \varrho, \bar{t} + \varrho], \\ y(t)(1 + \delta\alpha(t)) & \text{for } t \in [\bar{t} - \varrho, \bar{t} + \varrho], \end{cases}$$

where $\delta > 0$ will be specified later. With this modification, using Picone's identity from Lemma 2,

$$\begin{aligned}
\mathcal{F}_{RC}(z;t_1,t_2) &= \int_{t_1}^{t_2} \left[R(t)|z'|^p - C(t)z^p\right]\,dt = w_y z^p\Big|_{t_1}^{t_2} \\
&\quad + \int_{t_1}^{t_2} \left[R(t)|z'|^p - pw_y(t)z'\Phi(z) + (p-1)R^{1-q}(t)|w_y(t)|^q z^p\right]\,dt \\
&= w_y y^p\Big|_{t_1}^{t_2} + \int_{\bar{t}-\varrho}^{\bar{t}+\varrho} \left\{R(t)|y'+\delta(\alpha y)'|^p - pR(t)\frac{\Phi(y')}{\Phi(y)}z'y^{p-1}(1+\delta\alpha)^{p-1}\right. \\
&\qquad \left. +(p-1)R^{1-q}(t)\left|\frac{R(t)\Phi(y')}{\Phi(y)}\right|^q y^p(1+\delta\alpha)^p\right\}dt \\
&= w_y y^p\Big|_{t_1}^{t_2} + \int_{\bar{t}-\varrho}^{\bar{t}+\varrho} R(t)\left\{|y'|^p + p\delta(\alpha y)'\Phi(y') + o(\delta)\right. \\
&\quad \left. - p(y'+\delta(\alpha y)')\Phi(y')(1+(p-1)\delta\alpha+o(\delta)) + (p-1)|y'|^p(1+p\delta\alpha+o(\delta))\right\}dt \\
&= w_y y^p\Big|_{t_1}^{t_2} + \int_{\bar{t}-\varrho}^{\bar{t}+\varrho} R(t)\left\{|y'|^p + p\delta(\alpha y)'\Phi(y') + p|y'|^p - p\delta\Phi(y')(\alpha y)'\right. \\
&\quad \left. - p(p-1)\delta\alpha|y'|^p + (p-1)|y'|^p + (p-1)p\delta\alpha|y'|^p + o(\delta)\right\}dt \\
&= w_y y^p\Big|_{t_1}^{t_2} + o(\delta).
\end{aligned}$$

Consequently, using the rechniques from the proof of Theorem 1, we get

$$\begin{aligned}
\mathcal{F}_{RC}(z;t_0,t_3) &= \int_{t_0}^{t_3} \left[R(t)|z'|^p - C(t)|z|^p\right]\,dt \\
&= w_f|f|^p\,\Big|_{t_0}^{t_1} + w_y y^p\,\Big|_{t_1}^{t_2} + w_g|g|^p\,\Big|_{t_2}^{t_3} + o(\delta) \\
&= y^p(t_1)(w_f(t_1) - w_y(t_1)) \\
&\quad + y^p(t_2)(w_y(t_2) - w_g(t_2)) + o(\delta)
\end{aligned}$$

as $\delta \to 0+$. Concerning the interval (t_1,t_2), we have

$$\begin{aligned}
\int_{t_1}^{t_2} (c(t)-C(t))\,z^p(t)\,dt &= \int_{t_1}^{\bar{t}-\varrho} (c(t)-C(t))\,y^p(t) \\
&\quad + \int_{\bar{t}-\varrho}^{\bar{t}+\varrho} (c(t)-C(t))\,y^p(t)(1+\delta\alpha(t))^p\,dt + \int_{\bar{t}+\varrho}^{t_2} (c(t)-C(t))\,y^p(t)\,dt \\
&= \int_{t_1}^{t_2} (c(t)-C(t))\,y^p(t)\,dt + p\delta\int_{\bar{t}-\varrho}^{\bar{t}+\varrho} (c(t)-C(t))\,y^p(t)\alpha(t)\,dt + o(\delta) \\
&\geq \int_{t_1}^{t_2} (c(t)-C(t))\,y^p(t)\,dt + \delta K + o(\delta),
\end{aligned}$$

where $K = pd \int_{\bar{t}-\varrho}^{\bar{t}+\varrho} \alpha(t)\,dt > 0$. Therefore

$$\int_{t_1}^{t_2} (c(t) - C(t))\, z^p(t)\, t \geq \int_{t_1}^{t_2} (c(t) - C(t))\, y^p(t)\, dt + K\delta + o(\delta).$$

Next we prove that the function f/y is increasing on (t_0, t_1). We have $(f/y)(t_0) = 0$, $(f/y)(t_1) = 1$ and $(f/y)'(t) \neq 0$ for $t \in (t_0, t_1)$. Indeed, if $(f/y)'(\tau) = 0$ for some $\tau \in (t_0, t_1)$, $w_f(\tau) = w_h(\tau)$ and this contradicts the unique solvability of the Riccati equation (9). Due to the monotonicity of f/y, the Second Mean Value Theorem of integral calculus implies that there exists $\xi \in (t_0, t_1)$ for which

$$\begin{aligned}\int_{t_0}^{t_1} [C(t) - c(t)]\, f^p(t)\, dt &= \int_{t_0}^{t_1} [C(t) - c(t)](f/y)^p(t) y^p(t)\, dt \\ &= \int_{\xi}^{t_1} [C(t) - c(t)] y^p(t)\, dt.\end{aligned}$$

Using the above computations and, again, the inequalities between the functions r, R and c, C, we get

$$\begin{aligned}\mathcal{F}_{rc}(z; t_0, t_3) &\leq \mathcal{F}_{RC}(z; t_0; t_3) - \int_{t_0}^{t_2} (c(t) - C(t)) z^p(t)\, dt \\ &\leq |w_f(t_1) - w_y(t_1)| y^p(t_1) + |w_y(t_2) - w_g(t_2)| y^p(t_2) \\ &\quad - \int_{\xi}^{t_2} (c(t) - C(t)) y^p(t)\, dt - K\delta + o(\delta).\end{aligned}$$

Now, let $\delta > 0$, a sufficiently small number, be such that $K\delta + o(\delta) =: \varepsilon > 0$. According to (22) the points t_1, t_2 can be chosen in such a way that

$$\int_{\xi}^{t_2} (c(t) - C(t))\, y^p(t)\, dt > -\frac{\varepsilon}{4}$$

whenever $\xi \in (a, t_1)$. Further, similarly as in the proof of Theorem 1

$$y^p(t_1) \left| w_f(t_1) - w_y(t_1) \right| < \frac{\varepsilon}{4}, \quad y^p(t_2) \left| w_y(t_2) - w_g(t_2) \right| < \frac{\varepsilon}{4}$$

if $t_0 < t_1$, $t_2 < t_3$ are sufficiently close to a and b, respectively. Consequently, for the above specified choice of $t_0 < t_1 < t_2 < t_3$ we have the inequality $\mathcal{F}_{rc}(z; t_0, t_3) < -(\varepsilon/4) < 0$ and Lemma 1 implies that (1) is pseudoconjugate on $[t_0, t_1] \subset (a, b)$. □

4. Examples

We finish the paper with two examples illustrating the results of the previous section.

Example 1. Consider the half-linear Euler equation with the so-called critical coefficient

$$(\Phi(x'))' + \frac{\gamma_p}{t^p} \Phi(x) = 0, \tag{23}$$

where $\gamma_p = (p-1)^p\, p^{-p}$. One solution of this equation can be computed explicitly if we look for this solution in the form $x(t) = t^\lambda$. Then substituting into (23) we find that this function is a solution of (23) if λ is a root of the equation

$$G(\lambda) := (p-1)(|\lambda|^p - \Phi(\lambda)) + \gamma_p = 0,$$

and by direct computation we have that $\lambda = (p-1)/p$ is a double root of G, hence $x(t) = t^{(p-1)/p}$ is a solution of (23). Using Lemma 3, it is not difficult to verify that x is the principal solution at $t = 0$, and principal as well as coprincipal solution at ∞. Consequently, equation (23) may play the role of equation (2) in Theorems 1, 2, and hence the equation

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + c(t)\right)\Phi(x) = 0$$

with $c(t) \geq 0$ for $t \in (0, \delta)$, $\delta > 0$, and for large t, is pseudoconjugate on $(0, \infty)$ provided

$$\liminf_{t_1 \downarrow 0,\ t_2 \uparrow \infty} \int_{t_1}^{t_2} c(t) t^{p-1}\, dt \geq 0 \quad \text{and} \quad c(t) \not\equiv 0,\ t \in (0, \infty).$$

Example 2. In this example we consider the half-linear differential equation

$$\left((1+t^p)^{p-1}\Phi(x')\right)' + \frac{p-1}{1+t^p}\Phi(x) = 0, \quad t \in [0, \infty). \tag{24}$$

By direct computation one can verify that

$$x(t) = \frac{t}{\sqrt[p]{1+t^p}}, \quad u(t) = (1+t^p)^{p-1}\Phi(x'(t)) = \frac{1}{\sqrt[q]{1+t^p}} \tag{25}$$

are solutions of (24) and of its reciprocal equation

$$\left((1+t^p)^{q-1}\Phi^{-1}(u')\right)' + \frac{(p-1)^{q-1}}{1+t^p}\Phi^{-1}(u) = 0, \tag{26}$$

respectively. Moreover, using Lemma 3 and Lemma 4 (applied to (26), i. e., with p replaced by q and vice versa) one can verify by a direct computation that the solution x is principal at $t = 0$ and coprincipal at ∞.

Since the point $t = 0$ is the regular point of (24), the equation

$$\left((1+t^p)^{p-1}\Phi(x')\right)' + \left(\frac{p-1}{1+t^p} + c(t)\right)\Phi(x) = 0$$

with $c(t) \geq 0$ for large t is pseudoconjugate on $[0, \infty)$ provided

$$\liminf_{t \to \infty} \int_0^t c(s)\frac{s^p}{1+s^p}\, ds \geq 0 \quad \text{and} \quad c(t) \not\equiv 0,\ t \in (0, \infty).$$

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ON THE DIRECTIONAL DERIVATIVE AND DIRECTIONAL CONTINUITY OF SET VALUED MAPS

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Abstract. In this study, the directional lower and upper derivative sets of the set valued map the values of which have piecewise smooth boundary are investigated and obtained. Moreover, the connection between the directional continuity and the directional derivative sets of set-valued maps is investigated.

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Keywords: set-valued map, derivative sets, directional continuity

1. Introduction

The concepts of the directional lower and upper derivative sets of set valued maps and continuity of set-valued maps are studied in many papers (see, e. g., [2, 3, 5–7, 9, 12–21]).

The concepts of the directional lower and upper derivative sets of the set valued maps are based on the concepts of the lower and upper Bouligand cones [5, 6]. In this paper, when the value of set-valued map has a piecewise smooth boundary, the directional lower and upper derivative sets are investigated. Furthermore, the connection between the directional continuity and the directional derivative sets of the set-valued maps is investigated.

In what follows, $\mathrm{cl}\,(\mathbb{R}^m)$ (resp., $\mathrm{comp}\,(\mathbb{R}^m)$) denotes the set of all nonempty closed (resp., compact) subsets in $\mathbb{R}^m$.

Let $a(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ be a set valued map and let $x_0 \in \mathbb{R}^n$. It is said that $a(\cdot)$ is upper semi-continuous at x_0 if for all open neighbourhoods $N(a(x_0))$ of the set $a(x_0)$, there exists a neighbourhood $N(x_0)$ of x_0 such that $a(x) \subset N(a(x_0))$ for all $x \in N(x_0)$.

It is said that $a(\cdot)$ is lower semi-continuous at x_0 if for all $y_0 \in a(x_0)$ and for all open neighbourhoods $N(y_0)$ of y_0, there exists a neighbourhood $N(x_0)$ of x_0 such that $a(x) \cap N(y_0) \neq \varnothing$ for all $x \in N(x_0)$ [3, 18].

It is well-known that a set-valued map $a(\cdot) : \mathbb{R}^n \to \mathrm{comp}\,(\mathbb{R}^m)$ is upper semi-continuous at x_0 if and only if for all $\varepsilon > 0$, there exists a $\delta(\varepsilon, x_0) > 0$ such that $\|x - x_0\| < \delta(\varepsilon, x_0) \Rightarrow a(x) \subset a(x_0) + \varepsilon B$, where $B = \{y \in \mathbb{R}^m : \|y\| < 1\}$.

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Furthermore, $a(\cdot) : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^m)$ is lower semi-continuous at x_0 if and only if for all $\varepsilon > 0$, there exists a $\delta(\varepsilon, x_0) > 0$ such that $\|x - x_0\| < \delta(\varepsilon, x_0) \Rightarrow a(x_0) \subset a(x) + \varepsilon B$, where $B = \{y \in \mathbb{R}^m : \|y\| < 1\}$ [1, 4].

It is said that the set-valued map $a(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is locally bounded at $x_0 \in \mathbb{R}^n$ if $a(\cdot)$ is bounded in a neighbourhood of x_0.

2. Directional derivative sets of set-valued maps

Let $a(\cdot) : \mathbb{R}^n \to \operatorname{cl}(\mathbb{R}^m)$ be an upper semi-continuous set-valued map. Let us consider the following sets. For $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ and vector $f \in \mathbb{R}^n$, we set

$$Da(x, y) \mid (f) = \left\{ d \in \mathbb{R}^m : \liminf_{\delta \to +0} \frac{1}{\delta} \operatorname{dist}(y + \delta d, a(x + \delta f)) = 0 \right\},$$

$$D^* a(x, y) \mid (f) = \left\{ v \in \mathbb{R}^m : \lim_{\delta \to +0} \frac{1}{\delta} \operatorname{dist}(y + \delta d, a(x + \delta f)) = 0 \right\}.$$

Here for $x \in \mathbb{R}^n$, $D \subset \mathbb{R}^n$, $\operatorname{dist}(x, D) = \inf_{d \in D} \|x - d\|$. $Da(x, y) \mid (f)$ ($D^* a(x, y) \mid (f)$) is called the upper (lower) derivative set of the set valued map $a(\cdot)$ at (x, y) in the direction f.

Note that the directional upper (lower) derivative set of the set-valued map $a(\cdot)$ is closed and there is a connection between the upper (lower) derivative set of a set-valued map and the upper (lower) contingent cone which is used to investigate several problems in nonsmooth analysis (see, e. g., [3, 8, 15]).

It is obvious that $D^* a(x, y) \mid (f) \subset Da(x, y) \mid (f)$.

$$A = \operatorname{graph} a(\cdot) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in a(x)\}$$

denotes the graph of the set-valued map $a(\cdot)$. Since $a(\cdot)$ is upper semicontinuous, $\operatorname{graph} a(\cdot)$ is a closed set. It is possible to show that $Da(x, y) \mid (f) = D^* a(x, y) \mid (f) = \emptyset$ if $(x, y) \notin \operatorname{graph} a(\cdot)$, $Da(x, y) \mid (f) = D^* a(x, y) \mid (f) = \mathbb{R}^m$ if $(x, y) \in \operatorname{int}(\operatorname{graph} a(\cdot))$ where $\operatorname{int}(\operatorname{graph} a(\cdot))$ denotes the interior of $\operatorname{graph} a(\cdot)$.

Suppose that the set-valued map $a(\cdot)$ is given as

$$a(x) = \{y \in \mathbb{R}^m : c(x, y) \geq 0\} \tag{2.1}$$

where $c(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^m \to R$ is a continuous function on $\mathbb{R}^n \times \mathbb{R}^m$ and locally Lipschitz on $\mathbb{R}^m$. The lower and upper derivative of $c(\cdot, \cdot)$ at the point (x, y) in the direction (f, d) is denoted by $\frac{\partial^- c(x,y)}{\partial(f,d)}$ and $\frac{\partial^+ c(x,y)}{\partial(f,d)}$, respectively, and defined by

$$\frac{\partial^- c(x, y)}{\partial(f, d)} = \liminf_{\delta \to +0} [c(x + \delta f, y + \delta d) - c(x, y)] \delta^{-1},$$

$$\frac{\partial^+ c(x, y)}{\partial(f, d)} = \limsup_{\delta \to +0} [c(x + \delta f, y + \delta d) - c(x, y)] \delta^{-1},$$

respectively. If

$$\frac{\partial c(x,y)}{\partial(f,d)} = \lim_{\delta\to+0}\left[c(x+\delta f, y+\delta d) - c(x,y)\right]\delta^{-1}$$

exists and is finite, then $c(\cdot,\cdot)$ is called differentiable at the point (x,y) in the direction (f,d) and $\frac{\partial c(x,y)}{\partial(f,d)}$ denotes the derivative of $c(\cdot,\cdot)$ at the point (x,y) in the direction (f,d).

We introduce the sets

$$H_*^-(x,y)\mid(f) = \left\{d\in\mathbb{R}^m : \frac{\partial^+ c(x,y)}{\partial(f,d)} > 0\right\},$$

$$H_*(x,y)\mid(f) = \left\{d\in\mathbb{R}^m : \frac{\partial^+ c(x,y)}{\partial(f,d)} \geq 0\right\},$$

$$E_*^-(x,y)\mid(f) = \left\{d\in\mathbb{R}^m : \frac{\partial^- c(x,y)}{\partial(f,d)} > 0\right\},$$

$$E_*(x,y)\mid(f) = \left\{d\in\mathbb{R}^m : \frac{\partial^- c(x,y)}{\partial(f,d)} \geq 0\right\}.$$

Proposition 1. *Let the set-valued map $a(\cdot)$ be in the form* (2.1). *Then for all $(x,y)\in\partial A$ and $f\in\mathbb{R}^n$,*

$$\operatorname{cl} H_*^-(x,y)\mid(f) \subset Da(x,y)\mid(f) \subset H_*(x,y)\mid(f),$$
$$\operatorname{cl} E_*^-(x,y)\mid(f) \subset D^*a(x,y)\mid(f) \subset E_*(x,y)\mid(f),$$

where ∂A denotes the boudary of A and cl A *denotes the closure of A.*

Proposition 2. *Let $(x,y)\in\partial A$, $c(\cdot,\cdot)$ be differentiable at (x,y) and $\frac{\partial c(x,y)}{\partial y}\neq 0$. Then it is possible to show that*

$$Da(x,y)\mid(f) = D^*a(x,y)\mid(f)$$
$$= \left\{d\in\mathbb{R}^m : \left\langle\frac{\partial c(x,y)}{\partial x}, f\right\rangle + \left\langle\frac{\partial c(x,y)}{\partial y}, d\right\rangle \geq 0\right\},$$

where the symbol $\langle\cdot,\cdot\rangle$ denotes the inner product.

Remark 1. Now suppose that the set-valued map $a(\cdot):\mathbb{R}^n\to\operatorname{cl}(\mathbb{R}^m)$ is given by the relation

$$a(x) = \{y\in\mathbb{R}^m : \max_{i\in I} c_i(x,y)\geq 0\} \tag{2.2}$$

where I is a finite set and $c_i(\cdot,\cdot)$ is a continuous differentiable function for all $i\in I$. Then (see [10, 11]) $b(x,y) = \max_{i\in I} c_i(x,y)$ is a directional derivable function and

$$\frac{\partial c(x,y)}{\partial(f,d)} = \max_{i\in I_*(x,y)}\left[\left\langle\frac{\partial c_i(x,y)}{\partial x}, f\right\rangle + \left\langle\frac{\partial c_i(x,y)}{\partial y}, d\right\rangle\right],$$

where

$$I_*(x,y) = \{i_*\in I : c_{i_*}(x,y) = \max_{i\in I} c_i(x,y)\}.$$

In that case, it follows herefrom that

$$\begin{aligned} E_*^-(x,y) \mid (f) &= H_*^-(x,y) \mid (f) \\ &= \left\{ d \in \mathbb{R}^m : \max_{i \in I_*(x,y)} \left[\left\langle \frac{\partial c_i(x,y)}{\partial x}, f \right\rangle + \left\langle \frac{\partial c_i(x,y)}{\partial y}, d \right\rangle \right] > 0 \right\}, \\ E_*(x,y) \mid (f) &= H_*(x,y) \mid (f) \\ &= \left\{ d \in \mathbb{R}^m : \max_{i \in I_*(x,y)} \left[\left\langle \frac{\partial c_i(x,y)}{\partial x}, f \right\rangle + \left\langle \frac{\partial c_i(x,y)}{\partial y}, d \right\rangle \right] \geq 0 \right\}. \end{aligned}$$

Theorem 1. *Let the set-valued map $a(\cdot) : \mathbb{R}^n \to \mathrm{cl}\,(\mathbb{R}^m)$ be of form* (2.2)*, $(x,y) \in \partial A$, $f \in \mathbb{R}^n$ and $H_*^-(x,y) \mid (f) \neq \emptyset$. Then*

$$Da(x,y) \mid (f) = D^* a(x,y) \mid (f) = H_*(x,y) \mid (f).$$

Proof. It is obtained by using the preceding propositions and the remark. □

Remark 2. The above theorem is not true when $H_*^-(x,y) \mid (f) = \emptyset$ for $(x,y) \in \partial A$ and for $f \in \mathbb{R}^n$.

Example 1. Let us take the set-valued map $a(\cdot) : [0,1] \to \mathrm{cl}\,(\mathbb{R}^2)$, $x \mapsto a(x) = \{(y_1, y_2) \in \mathbb{R}^2 : -y_1^2 - y_2^2 \geq 0\}$. We know that $a(x) = \{(0,0)\}$ for all $x \in [0,1]$ and $b(\cdot,\cdot,\cdot) : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$, $(x, y_1, y_2) \mapsto b(x, y_1, y_2) = y_1^2 + y_2^2$ is a differentiable function. Then we obtain $H_*(x,0,0) \mid (1) = \mathbb{R}^2$, $H_*^-(x,0,0) \mid (1) = \emptyset$ and $Da(x,0,0) \mid (1) = \{(0,0)\}$ for $(x,0,0) \in \partial A$.

3. Directional continuity of set-valued maps

Theorem 2. *Let $a(\cdot) : \mathbb{R}^n \to \mathrm{comp}\,(\mathbb{R}^m)$ be a set valued map. If* graph $a(\cdot)$ *is closed and $a(\cdot)$ is locally bounded at $x_0 \in \mathbb{R}^n$, then $a(\cdot)$ is upper semi continuous at x_0* [4, 18].

Theorem 3. *Let $a(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ be a set valued map and let $x_0 \in \mathbb{R}^n$. $a(\cdot)$ be lower semi continuous at x_0 if and only if for all $\{x_n\}_{n=1}^\infty$ where $x_n \to x_0$ as $n \to \infty$ and for all $y_0 \in a(x_0)$, there exists a $\{y_n\}_{n=1}^\infty$ such that $y_n \in a(x_n)$ and $y_n \to y_0$ as $n \to \infty$* [3].

Definition 1. Let $x_0 \in \mathbb{R}^n$, $f \in \mathbb{R}^n$ and let $a(\cdot) : \mathbb{R}^n \to \mathrm{comp}\,(\mathbb{R}^m)$ be a set-valued map. If

$$\lim_{\delta \to 0+} a(x_0 + \delta f) = a(x_0),$$

then it is said that $a(\cdot)$ is continuous at x_0 in the direction f. If

$$\lim_{\delta \to 0+} a(x + \delta f) = a(x)$$

for all $x \in \mathbb{R}^n$, then it is said that $a(\cdot)$ is continuous on $\mathbb{R}^n$ in the direction f.

Proposition 3. *Let $x_0 \in \mathbb{R}^n$, $f \in \mathbb{R}^n$ and let $a(\cdot) : \mathbb{R}^n \to \mathrm{comp}(\mathbb{R}^m)$ be a set-valued map. The set valued map $\varphi(\cdot) : \mathbb{R} \to \mathrm{comp}(\mathbb{R}^m)$, $\delta \mapsto \varphi(\delta) = a(x_0 + \delta f)$, is continuous on the right at $\delta = 0$ if and only if $a(\cdot)$ is continuous at x_0 in the direction f.*

Proposition 4. *Let $x_0 \in \mathbb{R}^n$, $f \in \mathbb{R}^n$ and $a(\cdot) : \mathbb{R}^n \to \mathrm{comp}(\mathbb{R}^m)$ be a set-valued map. If the set valued map $a(\cdot)$ is upper continuous (resp., lower continuous) at x_0, then the set-valued map $\varphi(\cdot) : \mathbb{R} \to \mathrm{comp}(\mathbb{R}^m)$,*

$$\delta \longmapsto \varphi(\delta) = a(x_0 + \delta f),$$

is upper continuous (resp., lower continuous) at $\delta = 0$.

Hence, if the set-valued map $a(\cdot)$ is continuous at x_0, then the set-valued map $\varphi(\cdot) : \mathbb{R} \to \mathrm{comp}(\mathbb{R}^m)$, $\delta \mapsto \varphi(\delta) = a(x_0 + \delta f)$, is continuous at $\delta = 0$.

Proof. Let $f \in \mathbb{R}^n$ and let $a(\cdot)$ be continuous at x_0. We will show that there exists a $\delta(\varepsilon) > 0$ such that

$$|\delta| < \delta(\varepsilon) \quad \Longrightarrow \quad \varrho_H(\varphi(\delta), \varphi(0)) = \varrho_H(a(x_0 + \delta f), a(x_0)) < \varepsilon$$

for all $\varepsilon > 0$, where ϱ_H denotes Hausdorff distance. Then the set-valued map $\varphi(\cdot)$ will be continuous at $\delta = 0$.

Let $\varepsilon > 0$. Since the set-valued map $a(\cdot)$ is continuous at x_0, then there exists a $\delta_1 > 0$ such that

$$\|x - x_0\| < \delta_1 \quad \Longrightarrow \quad \varrho_H(a(x), a(x_0)) < \varepsilon. \tag{3.1}$$

Put $\delta_* = \delta_1 \|f\|^{-1}$. Then for all $\delta \in (-\delta_*, \delta_*)$,

$$\|x_0 + \delta f - x_0\| = |\delta| \, \|f\| < \|f\| . \frac{\delta_1}{\|f\|} = \delta_1.$$

It follows that from (3.1), for all $\delta \in (-\delta_*, \delta_*)$,

$$\varrho_H(a(x_0 + \delta f), a(x_0)) < \varepsilon.$$

Hence, we obtain that the set-valued map $\varphi(\cdot)$ is continuous at $\delta = 0$. □

Theorem 4. *Let $x_0 \in \mathbb{R}^n$ and let $a(\cdot) : \mathbb{R}^n \to \mathrm{comp}(\mathbb{R}^m)$ be a set-valued map. If the set-valued map $a(\cdot)$ is continuous at x_0, then $a(\cdot)$ is continuous at x_0 in the direction f for any $f \in \mathbb{R}^n$.*

Proof. Since the set-valued map $a(\cdot)$ is continuous at x_0, then the set valued map $\varphi(\cdot)$ is continuous at $\delta = 0$ from Proposition 4. From Proposition 3, the set-valued map $a(\cdot)$ is continuous at x_0 in the direction f for any $f \in \mathbb{R}^n$. □

Remark 3. The converse of the above theorem is not true. The following example will show this.

Example 2. Let $f(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be the function defined by

$$(x,y) \longmapsto f(x,y) = \begin{cases} xy/(x^{4/3}+y^6) & \text{for } x \neq 0,\ y \neq 0, \\ 0 & \text{for } x = 0,\ y = 0. \end{cases}$$

We will show that $f(\cdot,\cdot)$ is continuous at $(0,0)$ in the direction (f_1, f_2) where $f_1 \neq 0$ or $f_2 \neq 0$, but $f(\cdot,\cdot)$ is not continuous at $(0,0)$.

Let $(x_0, y_0) = (0,0)$. Let $(f_1, f_2) \in \mathbb{R}\times\mathbb{R}$ such that $f_1 \neq 0$ or $f_2 \neq 0$. Then

$$\begin{aligned} f((x_0,y_0)+\delta(f_1,f_2)) &= f(x_0+\delta f_1, y_0+\delta f_2) \\ &= f(\delta f_1, \delta f_2) \\ &= \delta^2 f_1 f_2/(\delta^{4/3} f_1^{4/3} + \delta^6 f_2^6) \\ &= \delta^{2/3} f_1 f_2/(f_1^{4/3} + \delta^{4\frac{2}{3}} f_2^6) \end{aligned}$$

and hence, as $\delta \to +0$,

$$\frac{\delta^{2/3} f_1 f_2}{f_1^{4/3} + \delta^{4\frac{2}{3}} f_2^6} \to 0 = f(0,0).$$

Therefore, $f(\cdot,\cdot)$ is continuous at $(0,0)$ in the direction (f_1, f_2).

Take $\{(x_n,y_n)\}_{n=1}^{\infty} = \{(\delta_n, \delta_n^{1/3})\}_{n=1}^{\infty}$, where $\delta_n \to 0$ as $n \to \infty$. Then $(x_n,y_n) \to (0,0)$. Since

$$f(x_n,y_n) = \frac{\delta_n^{4/3}}{\delta_n^{4/3}+\delta_n^2} = \frac{1}{1+\delta_n^{2/3}}$$

and $\left(1+\delta_n^{2/3}\right)^{-1} \to 1$ as $n \to \infty$, then it follows that

$$\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0).$$

Hence, $f(\cdot,\cdot)$ is not continuous at $(0,0)$.

Theorem 5. *If the set-valued map $a(\cdot) : \mathbb{R}^n \to \mathrm{comp}(\mathbb{R}^m)$ is continuous at $x_0 = (x_0^1, x_0^2, \ldots, x_0^n)$ in all directions $f \in \mathbb{R}^n$, then the set-valued map*

$$x^i \longmapsto a(x_0^1, x_0^2, \ldots, x_0^{i-1}, x^i, x_0^{i+1}, \ldots, x_0^n)$$

is continuous at x_0^i.

Proof. Let $f_1 = (0,0,\ldots,0,1,0,\ldots,0)$ and $f_2 = (0,0,\ldots,0,-1,0,\ldots,0)$ where 1 and -1 are the ith components. Since the set-valued map $a(\cdot)$ is continuous at x_0 in all directions $f \in \mathbb{R}^n$, we see that

$$\begin{aligned} \lim_{\delta\to 0+} a(x_0+\delta f_1) &= a(x_0), \\ \lim_{\delta\to 0+} a(x_0+\delta f_2) &= a(x_0). \end{aligned}$$

Then we obtain

$$\lim_{\delta\to 0+} a(x_0^1, x_0^2, \ldots, x_0^{i-1}, x_0^i + \delta, x_0^{i+1}, \ldots, x_0^n) = a(x_0), \tag{3.2}$$

$$\lim_{\delta\to 0+} a(x_0^1, x_0^2, \ldots, x_0^{i-1}, x_0^i - \delta, x_0^{i+1}, \ldots, x_0^n) = a(x_0). \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\lim_{\Delta\to 0} a(x_0^1, x_0^2, \ldots, x_0^{i-1}, x_0^i + \Delta, x_0^{i+1}, \ldots, x_0^n) = a(x_0).$$

This imples that the set-valued map $x^i \mapsto a(x_0^1, x_0^2, \ldots, x_0^{i-1}, x^i, x_0^{i+1}, \ldots, x_0^n)$ is continuous at x_0^i. □

Theorem 6. *Let $x_0 \in \mathbb{R}^n$, $f \in \mathbb{R}^n$ and let $a(\cdot) : \mathbb{R}^n \to \mathrm{comp}\,(\mathbb{R}^m)$ be a set-valued map. Suppose that the set $\mathrm{graph}_D\, a(\cdot) = \{(x, y) \in D \times \mathbb{R}^m : y \in a(x)\}$ is compact for any compact set $D \subset \mathbb{R}^n$. If $D^*a(x_0, y_0) \mid (f) \neq \emptyset$ for all $y_0 \in a(x_0)$, then the set-valued map $a(\cdot)$ is continuous at x_0 in the direction f.*

Proof. First, we will show that the set-valued map $a(\cdot)$ is upper semi continuous at x_0 in the direction f.

Since the set $\bar{B}(x_0, 1) = \{x \in \mathbb{R}^n : \|x - x_0\| \leq 1\}$ is compact, it follows that the set $\mathrm{graph}_{\bar{B}(x_0,1)}\, a(\cdot)$ is compact. Then, we obtain that the set-valued map $a(\cdot) : \bar{B}(x_0, 1) \to \mathrm{comp}\,(\mathbb{R}^m)$ is locally bounded at x_0 and its graph is closed. From Theorem 2, the set-valued map $a(\cdot) : \bar{B}(x_0, 1) \to \mathrm{comp}\,(\mathbb{R}^m)$ is upper semi-continuous at x_0. Hence, $a(\cdot)$ is upper semi-continuous at x_0 in the direction f and then the set valued map $\delta \to a(x_0 + \delta f)$ is upper semi-continuous on the right at $\delta = 0$.

Now we will show that the set-valued map $\delta \mapsto a(x_0 + \delta f)$ is lower semi-continuous on the right at $\delta = 0$, i. e., we will show that for all $y_0 \in a(x_0)$ and for all $\{\delta_k\}_{k=1}^{\infty}$ where $\delta_k \to +0$ as $k \to \infty$, there exists $y_k \in a(x_0 + \delta_k f)$ such that $y_k \to y_0$ as $k \to \infty$.

Take $y_0 \in a(x_0)$. From the hypothesis of the theorem, $D^*a(x_0, y_0) \mid (f) \neq \emptyset$. Take $d_* \in D^*a(x_0, y_0) \mid (f)$. From the definition of $D^*a(x_0, y_0) \mid (f)$, there exists a $\delta_* > 0$ such that for all $\delta \in [0, \delta_*]$,

$$y(\delta) = y_0 + \delta d_* + \delta s(\delta) \in a(x_0 + \delta f)$$

where $s(\delta) \to 0$ as $\delta \to +0$. For any sequence $\{\delta_k\}_{k=1}^{\infty}$ where $\delta_k \to +0$ as $k \to \infty$, there exists a $k_0 \in \mathbb{N}$ such that $\delta_k \in [0, \delta_*]$ for all $k \geq k_0$. Then for all $k \geq k_0$,

$$\begin{aligned} y_k &= y(\delta_k) \\ &= y_0 + \delta_k d_* + \delta_k s(\delta_k) \in a(x_0 + \delta_k f). \end{aligned}$$

Hence, we obtain that $y_k \in a(x_0 + \delta_k f)$ and $y_k \to y_0$ as $k \to \infty$ and hence the set-valued map $\delta \to a(x_0 + \delta f)$ is lower semi-continuous on the right at $\delta = 0$.

Consequently, we obtain that the set-valued map $\varphi(\cdot) : \mathbb{R} \to \mathrm{comp}\,(\mathbb{R}^m)$, $\delta \mapsto \varphi(\delta) = a(x_0 + \delta f)$, is upper and lower semi-continuous on the right at $\delta = 0$ and then it is continuous on the right. Therefore, $a(\cdot)$ is continuous at x_0 in the direction f. □

Corollary 1. *Let* $[a,b] \subset \mathbb{R}$, $x_0 \in (a,b)$ *and let* $a(\cdot) : [a,b] \to \operatorname{comp}(\mathbb{R}^m)$ *be a set-valued map and let* $\operatorname{graph} a(\cdot)$ *be a compact set. If* $D^*a(x_0,y) \mid (1) \neq \emptyset$ *and* $D^*a(x_0,y) \mid (-1) \neq \emptyset$ *for all* $y \in a(x_0)$*, then* $a(\cdot)$ *is continuous at* x_0.

Proof. Since $D^*a(x_0,y) \mid (1) \neq \emptyset$ for all $y \in a(x_0)$, by Theorem 6,

$$\lim_{\delta \to 0+} a(x_0+\delta) = a(x_0) \tag{3.4}$$

and since $D^*a(x_0,y) \mid (-1) \neq \emptyset$ for all $y \in a(x_0)$, from Theorem 6,

$$\lim_{\delta \to 0+} a(x_0-\delta) = a(x_0). \tag{3.5}$$

Then from (3.4) and (3.5),

$$\lim_{\delta \to 0} a(x_0+\delta) = a(x_0).$$

Hence, $a(\cdot)$ is continuous at x_0. □

Remark 4. The following example shows that Theorem 6 and Corollary 1 are not true when the condition $D^*a(x,y) \mid (f) \neq \emptyset$ is replaced by the condition $Da(x,y) \mid (f) \neq \emptyset$.

Example 3. Let us set, for $x \in [-1,1]$,

$$a(x) = \begin{cases} [-1,1] & \text{for } x \in [-1,0], \\ \sin \frac{1}{x} & \text{for } x \in (0,1]. \end{cases}$$

One can show that the set-valued map $x \mapsto a(x)$ is not continuous at $x_0 = 0$ in the direction $f = 1$. However,

$$Da(0,y) \mid (1) = \begin{cases} (-\infty, 0] & \text{for } y = 1, \\ (-\infty, +\infty) & \text{for } y \in (-1,1), \\ [0, +\infty) & \text{for } y = -1, \end{cases}$$

$$Da(0,y) \mid (-1) = \begin{cases} (-\infty, 0] & \text{for } y = 1, \\ (-\infty, +\infty) & \text{for } y \in (-1,1), \\ [0, +\infty) & \text{for } y = -1, \end{cases}$$

and then for all $y \in [-1,1] = a(0)$, $Da(0,y) \mid (1) \neq \emptyset$, $Da(0,y) \mid (-1) \neq \emptyset$. On the other hand,

$$D^*a(0,y) \mid (-1) = \begin{cases} l(-\infty, 0] & \text{for } y = 1, \\ (-\infty, +\infty) & \text{for } y \in (-1,1), \\ [0, +\infty) & \text{for } y = -1, \end{cases}$$

and $D^*a(0,y) \mid (1) = \emptyset$ for all $y \in [-1,1]$.

We obtain the following corollary from Theorem 6 and Theorem 5.

Let $f_i^+ = (0,0,\ldots,0,1,0,\ldots,0)$ and $f_i^- = (0,0,\ldots,0,-1,0,\ldots,0)$ where $i = 1,2,\ldots,n$ and 1 and -1 are ith coordinates of the vectors f_i^+ and f_i^-, respectively.

Corollary 2. *Let $a(\cdot) : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^m)$ be a set valued map. Suppose that $x_0 \in \mathbb{R}^n$ and the set*

$$\operatorname{graph}_D a(\cdot) = \{(x, y) \in D \times \mathbb{R}^m : y \in a(x)\}$$

is compact for any compact set $D \subset \mathbb{R}^n$. If $D^*a(x_0, y) \mid (f_i^+) \neq \varnothing$ and $D^*a(x_0, y) \mid (f_i^-) \neq \varnothing$ for all $y_0 \in a(x_0)$ and for all $i = 1, 2, \ldots, n$, then the set-valued map $x \mapsto a(x)$ is continuous according to all coordinates.

Remark 5. We will show that the converse of Theorem 6 is not true in general.

Example 4. Take the set-valued map $a(\cdot) : [-1, 1] \to \operatorname{comp}(\mathbb{R})$,

$$x \longmapsto a(x) = \{y \in \mathbb{R} : \ \sqrt[3]{x} \leq y \leq 2\}.$$

One can show that $a(\cdot)$ is continuous at $x_0 = 0$ and then $a(\cdot)$ is continuous at $x_0 = 0$ in any direction f.

We know that $(0, 0) \in \operatorname{graph} a(\cdot)$ and

$$Da(0, 0) \mid (1) = \Big\{v \in \mathbb{R} : \ \exists \delta_k > 0 \text{ such that } \delta_k \to 0+ \text{ as } k \to \infty, \\ \exists y_k \in a(\delta_k) \ni \lim_{k\to\infty} y_k/\delta_k = v\Big\}.$$

Take a sequence $\{y_k\}_{k=1}^{\infty}$ such that $y_k \in a(\delta_k)$ and $\delta_k \to 0+$ as $k \to \infty$. Since $y_k \in a(\delta_k)$, we have $y_k \geq \sqrt[3]{\delta_k}$ and then

$$\lim_{k\to\infty} \frac{y_k}{\delta_k} \geq \lim_{k\to\infty} \frac{\sqrt[3]{\delta_k}}{\delta_k} = \lim_{k\to\infty} \frac{1}{\sqrt[3]{\delta_k^2}} \to \infty.$$

Then $Da(0, 0) \mid (1) = \varnothing$ and since $D^*a(0, 0) \mid (1) \subset Da(0, 0) \mid (1)$, then it follows that

$$D^*a(0, 0) \mid (1) = \varnothing.$$

Remark 6. The hyphothesis of the above theorem does not imply the continuity of $a(\cdot)$.

Example 5. Let us put, for all $(x, y) \in \mathbb{R} \times \mathbb{R}$,

$$f(x, y) = \begin{cases} 1 & \text{if } y = x^3, \ x > 0, \\ 0 & \text{in the other cases,} \end{cases}$$

and take the set-valued map $a(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \operatorname{comp}(\mathbb{R})$, $(x, y) \mapsto a(x, y) = \{f(x, y)\}$. One can show that the function $f(\cdot, \cdot)$ is not continuous at $(0, 0)$.

We have

$$\frac{\partial f(0, 0)}{\partial(f_1, f_2)} = \lim_{\delta\to 0+} \frac{f(\delta f_1, \delta f_2) - f(0, 0)}{\delta} = 0$$

where $f_1 \neq 0$ or $f_2 \neq 0$. Then we get

$$D^*a(0, 0, 0) \mid (f_1, f_2) = \left\{\frac{\partial f(0, 0)}{\partial(f_1, f_2)}\right\} = \{0\} \neq \varnothing.$$

4. Conclusion

The concepts of directional lower and upper sets of set-valued maps and continuity of the set-valued maps are basic concepts in set-valued analysis. In this study, the directional lower and upper derivative sets of the set valued map the values of which have piecewise smooth boundary are suggested and investigated. Moreover, the connection between the directional continuity and the directional derivative sets of set-valued maps is investigated.

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VELCRO BIFURCATION IN COMPETITION MODELS WITH GENERALIZED HOLLING FUNCTIONAL RESPONSE

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Abstract. A four dimensional system is considered describing the competition of three predator species for a single prey species. The predators functional response is of a generalized Holling III type. Predators begin to grow at the same threshold quantity as prey does. The system has a two dimensional simplex of equilibria which is attracting at low values of the carrying capacity K but gets destabilized as K grows and the predators that are less r-strategists lose ground.

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Keywords: velcro bifurcation, competition models, predator-prey systems

1. Introduction

The phenomenon of *zip bifurcation* was introduced by Farkas [2] in 1984 in conjunction with a system describing the competition of two predator species for a prey. It was assumed that the threshold quantities of prey at which predator growth rate became positive were equal. As a consequence, the system had a one dimensional continuum of equilibria (a "zip"). One of the predators could be identified as a K-strategist, the other one as an r-strategist. A species is called a K-strategist, roughly, if it has a relatively low growth rate and may survive with low carrying capacity K. A species is an r-strategist if it has a high growth rate (for more see [6]). Clearly, the model was not a structurally stable one, however, it illustrated the intuitively evident fact that at low values of the carrying capacity K both predators might survive but as K grew the K-strategist lost ground and only the r-strategist survived with the prey. Later a whole class of models were characterized that showed the phenomenon [3], see also [4], and some applications were published, see e.g. [5]. Recently [1] the phenomenon was generalized to a four dimensional system that arose in economy-politology. In this case the system had a two dimensional surface of equilibria that got destabilized as the value of a parameter was increased. We called this a *velcro bifurcation*. In general, suppose that a system of autonomous differential equations

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has a two dimensional differentiable manifold of Liapunov stable equilibria and in an interval of a bifurcation parameter this manifold is an attractor of the system. Increasing, say, the bifurcation parameter a curve sweeps through the manifold and the equilibria that are left behind get destabilized, i. e., the part of the manifold that is left behind becomes a repeller while the part yet untouched remains an attractor. In this case we say that the system undergoes a velcro bifurcation.

In the present paper we consider three predator species that compete for a single prey. The functional response of the predators is assumed to be of a generalized Holling III type. The Holling III functional response that is of the form $S^2/(S^2 + a^2)$,where S is the prey quantity, has the property that up to a certain value the growth of S is increasing predator efficiency and, as a consequence, the predator has a stabilizing effect on the population, above this value predator efficiency is decreasing (see May [6]). Here we replace Holling III by $S^n/(S^n + a^n)$ where $n > 1$ is an arbitrary integer. This generalized Holling III functional response preserves the basic properties but is somewhat finer.

In the next Section we introduce the model and establish the conditions under which a velcro bifurcation occurs and in Section 3 we present some computer simulations.

2. Velcro Bifurcation

Let us denote the quantity of prey at time t by $S(t)$, the quantity of predator i by $x_i(t)$, $i = 1, 2, 3$ and consider the following system that describes the competition of the three predator species for the prey

$$\begin{aligned} \dot{S} &= rS(1 - S/K) - \sum_{i=1}^{3} m_i x_i \frac{S^n}{a_i^n + S^n}, \\ \dot{x}_i &= m_i x_i \frac{S^n}{a_i^n + S^n} - d_i x_i, \qquad i = 1, 2, 3 \end{aligned} \tag{2.1}$$

where $r, K, m_i, a_i, d_i > 0$ are the intrinsic growth rate of prey, the carrying capacity, the maximal birth rate, the half saturation constant and the death rate of predator i, respectively, and $n \geq 2$ is an integer. In absence of predation prey grows up to the value of K. If predators are present prey quantity is less. As S tends to infinity, *per capita* predator birth rate tends to m_i (the shape of the predator functional response is a sigmoid one). For the survival of predator i it is , clearly, necessary that the maximal birth rate be larger than the death rate:

$$m_i > d_i, \qquad i = 1, 2, 3. \tag{2.2}$$

This will be assumed in the sequel. When $S = a_i$, then the *per capita* growth rate is half of the maximal. The lower the half saturation constant a_i is, the less prey is needed for the maintenance of the predator. Therefore, we shall consider a predator

with low half saturation constant a *K-strategist.* We assume that

$$a_1 < a_2 < a_3. \tag{2.3}$$

Predator i begins to grow only if the right hand side of equation $(2.1)_i$ is positive. This is the case if $S > S_i := a_i \sqrt[n]{d_i/(m_i - d_i)} = a_i \sqrt[n]{1/(b_i - 1)}$, where $b_i = m_i/d_i > 1$. We shall assume that the *threshold values of prey are equal for the three predators:*

$$S_T := S_1 = S_2 = S_3. \tag{2.4}$$

Conditions (2.3) and (2.4) imply that

$$b_1 < b_2 < b_3. \tag{2.5}$$

We shall consider a species with a high ratio of maximal birth rate to death rate as an *r-strategist.* Thus, in our system, predator 1 may be considered a K-strategist and predator 3 an r-strategist. Predator 2 is in between the two.

Applying the conditions above, we can rewrite our system in the form

$$\begin{aligned} \dot{S} &= rS(1 - S/K) - \sum_{i=1}^{3} m_i x_i \frac{S^n}{a_i^n + S^n}, \\ \dot{x}_i &= \beta_i \frac{(S^n - S_T^n)}{a_i^n + S^n} x_i, \qquad i = 1, 2, 3, \end{aligned} \tag{2.6}$$

where $\beta_i = m_i - d_i > 0$. Clearly, the positive orthant of (S, x_1, x_2, x_3) space is invariant. We shall consider system (2.6) restricted to $\mathbb{R}^4_+$.

Lemma 1. *System* (2.1) *and, as a consequence,* (2.6) *is dissipative, i. e., all solutions are bounded.*

Proof. Let us add the four equations of (2.1):

$$(S + x_1 + x_2 + x_3)^{\cdot} = rS(1 - S/K) - \sum_{i=1}^{3} d_i x_i.$$

If $S > K$ or if $S \le K$ but $\sum_{i=1}^{3} d_i x_i > rK$ then this derivative is negative. This means that the trajectories of the system cross the hyperplanes

$$S + x_1 + x_2 + x_3 = C$$

from outside to inside if C is sufficiently large. □

The equilibria of system (2.6) are $(0,0,0,0)$, $(K,0,0,0)$ and the points of the two-dimensional plane

$$E := \left\{ (S, x_1, x_2, x_3) \in \mathbb{R}^4_+ : S = S_T,\ \sum_{i=1}^{3} \frac{m_i S_T^n x_i}{a_i^n + S_T^n} = r S_T (1 - S_T/K), \right.$$
$$\left. x_i \geq 0,\ i = 1,2,3 \right\}$$

the latter having points in $\mathbb{R}^4_+$ only if

$$S_T < K. \tag{2.7}$$

We are to study now the stability of these equilibria. The Jacobian of the system is (summation goes always from 1 to 3)

$$J = \begin{bmatrix} r(1 - 2S/K) - n \sum \frac{m_i a_i^n S^{n-1} x_i}{(a_i^n + S^n)^2} & -\frac{m_1 S^n}{a_1^n + S^n} & -\frac{m_2 S^n}{a_2^n + S^n} & -\frac{m_3 S^n}{a_3^n + S^n} \\ n\beta_1 \frac{S^{n-1}}{(a_1^n + S^n)} x_1 & \frac{\beta_1 (S^n - S_T^n)}{a_1^n + S^n} & 0 & 0 \\ n\beta_2 \frac{S^{n-1}}{(a_2^n + S^n)} x_2 & 0 & \frac{\beta_2 (S^n - S_T^n)}{a_2^n + S^n} & 0 \\ n\beta_3 \frac{S^{n-1}}{(a_3^n + S^n)} x_3 & 0 & 0 & \frac{\beta_3 (S^n - S_T^n)}{a_3^n + S^n} \end{bmatrix}.$$

Substituting $(0,0,0,0)$ we obtain easily that the origin is unstable (a hyperbolic saddle).

Substituting $(K,0,0,0)$ we obtain similarly that this equilibrium is asymptotically stable if $K < S_T$ and it is unstable, a hyperbolic saddle if $K > S_T$. The result is in accordance to what has been said before. If the carrying capacity, i. e., the lim sup of prey abundance is less than the threshold value above which predator's quantity may grow, then the predators die out. If (2.7) holds, then the predators may survive. This is precisely the case when the system has equilibria in the interior of the positive orthant. (2.7) will be assumed in the sequel.

We turn now to the stability problem of the points in E. Substituting the coordinates of an arbitrary point of E into the Jacobian, a straightforward calculation leads one to the following characteristic polynomial:

$$D(\lambda) = \lambda^2 \left(\lambda^2 + \lambda \left(n S_T^{n-1} \sum \frac{m_i a_i^n x_i}{(a_i^n + S_T^n)^2} - r(1 - 2S_T/K) \right) \right.$$
$$\left. + n S_T^{2n-1} \sum \frac{m_i \beta_i x_i}{(a_i^n + S_T^n)^2} \right).$$

This means that each equilibrium point in E has zero as a double eigenvalue and two eigenvalues with real part negative or positive depending on the fact whether the quadratic polynomial in brackets is a stable polynomial or inversely. This polynomial

is stable iff the coefficient of λ is positive, i. e., the equilibrium point in E has two eigenvalues with negative real part if and only if

$$\sum \frac{m_i a_i^n x_i}{(a_i^n + S_T^n)^2} > \frac{r}{nS_T^{n-1}} (1 - 2S_T/K). \tag{2.8}$$

Note that (2.8) holds obviously for each point in E if $S_T < K < 2S_T$. In what follows K will be considered the bifurcation parameter.

Theorem 1. *Under the conditions* (2.2) – (2.4), (2.7) *if at a point of* E (2.8) *holds, then this point is stable in the Liapunov sense and a neighbourhood of this point is attractive in the sense that it has a tubular neighbourhood from where it attracts the solutions; if the inequality* (2.8) *is inverted, then the point is unstable.*

Proof. If (2.8) holds, then the equilibrium has a two dimensional centre manifold (clearly the simplex E) and a two dimensional stable manifold, and by continuity the same holds for each equilibrium in a neighbourhood of the point. The stable manifolds of the equilibria in this neighbourhood fill in a tubular neighbourhood in $\mathbb{R}^4_+$ (this can be proven analogously to how this was done in [2]), thus, from this tubular neighbourhood every solution tends to some equilibrium in this neighbourhood. The unstability proposition is obvious. □

Consider (2.8) with the equality sign:

$$\sum \frac{m_i a_i^n x_i}{(a_i^n + S_T^n)^2} = \frac{r}{nS_T^{n-1}} (1 - 2S_T/K). \tag{2.9}$$

This is the equation of a two dimensional plane in the $S = S_T$ section of $\mathbb{R}^4_+$. If K is small ($\leq 2S_T$), then (2.8) holds, obviously, in each point of E, i. e., E is an attractor of the system. However, if K is increased, the plane (2.9) may intersect E and, as a consequence, the stability properties of the equilibria in E may change. The following theorem explains what happens.

Theorem 2. *Suppose that conditions* (2.2) – (2.4) *and* (2.7) *hold and assume that*

$$a_1 < a_2 < a_3 < S_T / \sqrt[n]{n-1}; \tag{2.10}$$

(1) *if* $K \in \left(S_T, \frac{((n-2)a_1^n - 2S_T^n)S_T}{(n-1)a_1^n - S_T^n}\right)$, *then each equilibrium in* E *is stable in the Liapunov sense and* E *is an attractor of system* (2.6)*;*

(2) *if* $K \in \left(\frac{((n-2)a_3^n - 2S_T^n)S_T}{(n-1)a_3^n - S_T^n}, \infty\right)$, *then all these equilibria are unstable and* E *is a repeller;*

(3) *if* K *is increased from one end to the other one of the interval*

$$\left(\frac{\left((n-2)\,a_1^n - 2S_T^n\right)S_T}{(n-1)\,a_1^n - S_T^n}, \frac{\left((n-2)\,a_3^n - 2S_T^n\right)S_T}{(n-1)\,a_3^n - S_T^n}\right),$$

then the line of intersection of E and the plane (2.9) *is traveling through E from the vertex on the axis x_1 to the vertex on the axis x_3 and the equilibria left behind get destabilized; for*

$$K \in \left(\frac{\left((n-2)\,a_1^n - 2S_T^{\,n}\right) S_T}{(n-1)\,a_1^n - S_T^{\,n}}, \frac{\left((n-2)\,a_3^n - 2S_T^{\,n}\right) S_T}{(n-1)\,a_3^n - S_T^{\,n}} \right)$$

this line of intersection divides E into two parts, "the lower one" is a repeller and "the upper one" is an attractor of the system (see Figure 1 in the next Section), i. e., the system undergoes a velcro bifurcation.

Proof. Let us denote the respective coordinate intercepts of E by $(x_{E1}, 0, , 0)$, $(0, x_{E2}, 0)$, $(0, 0, x_{E3})$ and those of the plane (2.9) by $(x_{s1}, 0, 0)$, $(0, x_{s2}, 0)$, $(0, 0, x_{s3})$. A simple calculation yields

$$x_{\mathrm{Ei}} = \frac{a_i^n + S_T^{\,n}}{m_i} \frac{r}{S_T^{\,n-1}} \left(1 - S_T/K\right);$$

$$x_{si} = \frac{(a_i^n + S_T^{\,n})^2}{m_i a_i^n} \frac{1}{n} \frac{r}{S_T^{\,n-1}} \left(1 - 2S_T/K\right), \qquad i = 1, 2, 3.$$

Introduce the notation

$$K_i = \frac{\left((n-2)\,a_i^n - 2S_T^{\,n}\right) S_T}{(n-1)\,a_i^n - S_T^{\,n}}, \qquad i = 1, 2, 3.$$

By condition (2.10) the denominator and *a fortiori* the numerator are negative, hence K_i is positive, further, as the function $f(a) = \frac{\left((n-2)a^n - 2S_T^{\,n}\right)S_T}{(n-1)a^n - S_T^{\,n}}$ is increasing in the interval $a \in [0, S_T/\sqrt[n]{n-1})$ we have

$$2S_T < K_1 < K_2 < K_3. \tag{2.11}$$

Now, a simple calculation shows that for $2S_T < K < K_i$ we have $x_{\mathrm{Ei}} < x_{si}$, and $x_{\mathrm{Ei}} = x_{si}$ when $K = K_i$. This implies that for $2S_T < K < K_1$ the plane (2.9) is "below" E and reaches E at $x_{E1} = x_{s1}$ when $K = K_1$. (As K is increased both planes are displaced parallelly). This implies that for $2S_T < K < K_1$ (2.8) holds for every equilibrium in E. As K is increased further the plane (2.9) cuts into E, reaches $x_{E2} = x_{s2}$ at $K = K_2$, reaches $x_{E3} = x_{s3}$ at $K = K_3$ and after that cuts the plane of E outside the positive orthant so that now E will be "below" the plane (2.9). In the process, condition (2.8) holds with an inverted inequality sign in the part of E which is already below the plane (2.9). This means that the equilibria on this part of the plane have a two dimensional unstable manifold which fill a neighbourhood of this part of E. This proves the Theorem. □

Note that K_i is positive also if $S_T\sqrt[n]{2/(n-2)} < a_1 < a_2 < a_3$, $n > 2$, however, in this case the plane (2.9) never reaches E.

Note also that in case of the classical Holling III functional response, i. e., $n = 2$ condition (2.10) is just $a_1 < a_2 < a_3 < S_T$, the half saturation constants must be less than the threshold value of prey abundance. This means that predator birth rate half the maximal is not sufficient for predator survival. In case of the generalized Holling III functional response the half saturation constants must be even less in order to experience velcro bifurcation.

3. Computer Images

In this Section a numerical example will be presented. The classical Holling III functional response will be considered, i. e., $n = 2$. The following parameter values are chosen:

$$r = 1,\ m_1 = 5,\ m_2 = 6,\ m_3 = 7,\ a_1 = 0.5 < a_2 = \sqrt{0.5} < a_3 = \sqrt{0.75}$$
$$d_1 = d_2 = d_3 = 4.$$

As a consequence

$$\beta_1 = 1,\ \beta_2 = 2,\ \beta_3 = 3,\ b_1 = 5/4 < b_2 = 6/4 < b_3 = 7/4, \qquad S_T = 1.$$

Conditions (2.2)–(2.4) and (2.10) hold and the system is

$$\begin{aligned} \dot{S} &= S(1 - S/K) - 5x_1S^2/(0.25 + S^2) - 6x_2S^2/(0.5 + S^2) \\ &\quad - 7x_3S^2/(0.75 + S^2), \\ \dot{x}_1 &= \frac{S^2 - 1}{0.25 + S^2}x_1, \quad \dot{x}_2 = 2\frac{S^2 - 1}{0.5 + S^2}x_2, \quad \dot{x}_3 = 3\frac{S^2 - 1}{0.75 + S^2}x_3. \end{aligned} \tag{3.1}$$

$$E = \{(S, x_1, x_2, x_3) \in \mathbb{R}^4_+ : S = 1,\ x_1 + x_2 + x_3 = 0.25(1 - 1/K),\ x_i \geq 0\}.$$

The equation of the plane (2.9) is now

$$x_1/0.375 + x_2/0.225 + x_3/0.175 = 1. \tag{3.2}$$

The critical values of K are: $K_1 = 2.667$, $K_2 = 4$, $K_3 = 8$. Condition (2.11) is satisfied. We choose $K = 5$. By this, clearly, the points of the plane of E on the axes x_1 and x_2 have already been destabilized, but the vertex on the axis x_3 is still stable.

Figure 1 shows the plane of E cut by the plane (3.2). (The figures were produced by Maple V. The coordinates S, x_1, x_2, x_3 are denoted on the figures by x, y, z, w, respectively).

We choose now two sets of initial values. The first (denoted by A) is near the destabilized part of E (with x_3 coordinate small). The second (denoted by B) is near the still attractive part of E (with x_3 coordinate relatively large). The coordinates of (A) are given by $(S(0), x_1(0), x_2(0), x_3(0)) = (1, 0.08, 0.08, 0.06)$, and the coordinates of (B) are given by $(S(0), x_1(0), x_2(0), x_3(0)) = (1, 0.01, 0.05, 0.16)$. Figures 2, 3, and 4 show, respectively, the x_1, x_2, and x_3 coordinates of the solution. As can be seen, the solution initiating at (A) tends to an equilibrium on the still attractive portion of E far

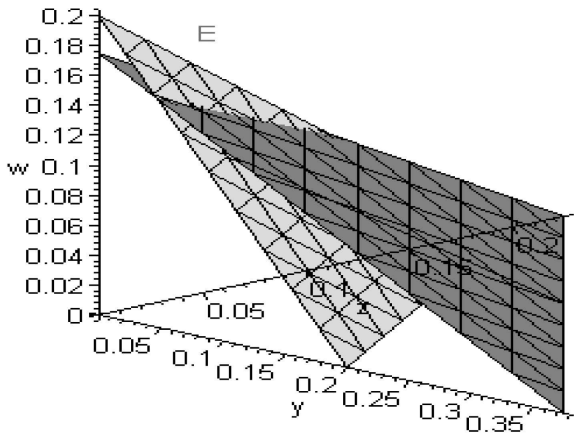

FIGURE 1. Plane of equilibria partly destabilized

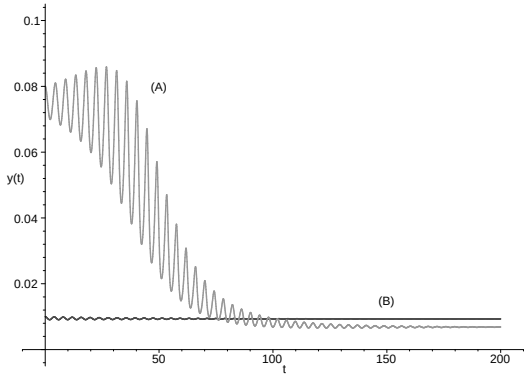

FIGURE 2. x_1 coordinates of solutions (A) and (B)

from (A) after exhibiting strong oscillations. In contrast, the solution corresponding to (B) tends to an equilibrium on the still attractive portion of E near (B).

Figure 3 shows the $x_2 = z$ coordinate of these solutions. The behaviour is similar to the previous one.

Figure 4 shows the $x_3 = w$ coordinate of these solutions. For solution (A) it is increasing considerably, for solution (B) it remains near the initial value.

It is an interesting question what the solutions do when $K > K_3$, i. e., when already all the equilibria in $\mathbb{R}^4_+$ have been destabilized. It is easy to see that the equilibrium point $(S_T, 0, 0, x_{\mathrm{Ei}})$ which is the "last stable one" undergoes a degenerate Andronov–Hopf bifurcation as K is increased through K_3. The fundamental Andronov–Hopf bifurcation theorem (see, e. g., [4]) does not apply since at the critical value K_3 of

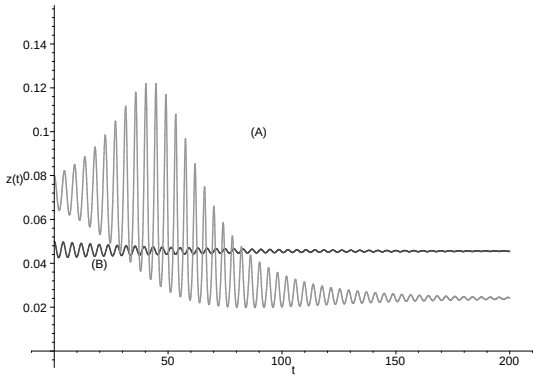

FIGURE 3. x_2 coordinates of solutions (A) and (B)

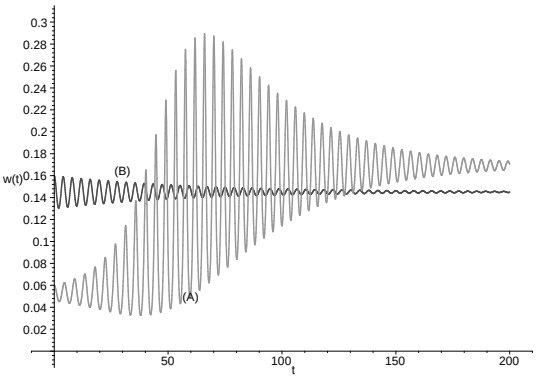

FIGURE 4. x_3 coordinates of solutions (A) and (B)

the bifurcation parameter this equilibrium has a double zero eigenvalue and a pair of imaginary eigenvalues. Computer simulations show that the closed orbit of the bifurcating periodic solution attracts solutions initiating in $\mathbb{R}^4_+ \setminus E$.

4. CONCLUSIONS

Three predator species compete for a single prey species. The prey species follows logistic dynamics with carrying capacity K in the absence of predation. The functional responses of the predators are of generalized Holling III type, i. e., increase of prey increases predator efficiency for relatively low prey abundance. All the three predators begin to grow at the same threshold quantity of prey denoted S_T. Predator 2 is more an r-strategist and less a K-strategist than predator 1, predator 3 is more an r-strategist and less a K-strategist than predator 2. We distinguish three values of K, denoted herein as $S_T < K_1 < K_2 < K_3$. For $K < S_T$, there are no equilibria corresponding to the survival of the predators. For $K > S_T$ the system has

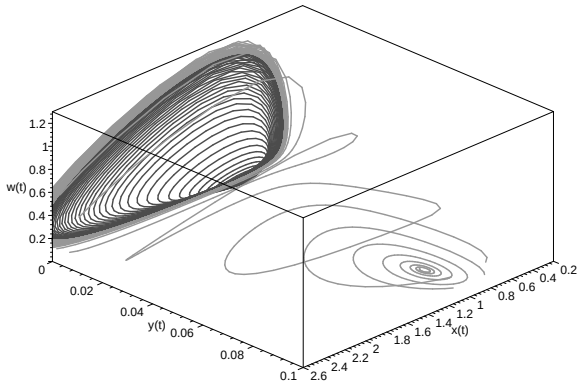

FIGURE 5. $K = 11$, trajectories tending to limit cycle in the (x, w) plane

an attracting simplex of equilibria in the positive orthant of the 4-dimensional space. This means that solutions tend to a point on this simplex determined by the initial conditions, i. e., the three predator species may coexist with the prey in the long run. Naturally, if neglected and stochastic effects are taken into consideration the limit point may drift on the simplex and this may lead to the extinction of one or two of the predators. As K is increased beyond K_1, the equilibria on the simplex begin to lose their stability starting with those that represent dominance of predator 1, the K-strategist, and then proceeding on with those that represent dominance of predator 2. This part of the simplex becomes a repeller of the system and all solutions tend to an equilibrium that represents dominance of predator 3, the r-strategist. For values of K larger than K_3 the first two predators disappear and only predator 3, the r-strategist survives with the prey. This is the phenomenon that we call a *velcro bifurcation.* The results support the intuitively expected fact that abundance of food is advantageous for r-strategists, and under such conditions a K-strategist loses because its positive properties are of no use. However, if the parameters of the system can be estimated with some exactness, the model tells us at what value of the carrying capacity the K-strategist and also the second predator begins to lose ground and at what value they disappear.

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FUNCTIONS PRESERVING RANK-k MATRICES OF ORDER n OVER FIELDS

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Abstract. Let $\mathbb{F}$ be an arbitrary field and n is an integer with $n \geq 2$. For a fixed positive integer k satisfying $k < n$, we determine the general form of all functions preserving rank-k matrices of order n. This article generalizes the recent results of J. Kalinowski [1, 2].

Mathematics Subject Classification: 15A03

Keywords: field, rank-k matrix, function

1. Introduction

Suppose $\mathbb{F}$ is an arbitrary field and $\mathbb{R}$ is the field of the real numbers. Let n be an integer with $n \geq 2$. For a function $f : \mathbb{F} \to \mathbb{F}$ and a matrix $A = [a_{ij}]$ over $\mathbb{F}$, denote the matrix $[f(a_{ij})]$ by A^f. We say that a function $f : \mathbb{F} \to \mathbb{F}$ preserves ranks of matrices if $\operatorname{rank} A^f = \operatorname{rank} A$ for all matrices (of any order) over $\mathbb{F}$, and preserves rank-k matrices of order n if $\operatorname{rank} A^f = \operatorname{rank} A$ for every rank-k matrix of order n.

Kalinowski [1] investigated that a monotonic and continuous function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ preserves ranks of matrices if and only if it is linear, i. e., $f(x) = cx$ for every $x \in \mathbb{R}$ and some non-zero $c \in \mathbb{R}$. Furthermore, in [2], Kalinowski generalized the result in [1] by removing any restrictions on the function f.

Inspired by [1, 2], in this article we prove the following two theorems which generalize the result in [2].

Theorem 1. *Let k be a fixed integer satisfying $2 \leq k < n$. Then $f : \mathbb{F} \to \mathbb{F}$ is a function preserving rank-k matrices of order n if and only if there exist a non-zero scalar c and an injective field endomorphism δ of $\mathbb{F}$ such that $f = c\delta$.*

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Theorem 2. *$f : \mathbb{F} \to \mathbb{F}$ is a function preserving rank-1 matrices of order n if and only if either f is a non-zero constant function or $f = c\kappa$, where c is a non-zero scalar and $\kappa : \mathbb{F} \to \mathbb{F}$ is a multiplicative function with $\kappa(0) = 0$ and $\kappa(1) = 1$.*

As pointed out by Marková [3], these results obtained in [1, 2] play a important role in the theory of g-calculus (see [4] for the concept of g-calculus and the relevant topics). Therefore, Theorems 1 and 2 will be helpful for studying extensively g-calculus.

We end this section by introducing the notation which will be used in the next section. Denote by $\oplus$ the usual direct sum of matrices. For a positive integer k, let I_k be the $k \times k$ identity matrix over $\mathbb{F}$

2. Proofs of Theorems 1 and 2

The Proof of Theorem 1. The "if" part is obvious. The proof of the "only if" part is divided into the following four steps.

Step 1: *$f(0) = 0$ and $f(d) \neq 0$ for every non-zero scalar d*. For any non-zero scalar d, it follows from $\operatorname{rank}(dI_k \oplus 0) = k$ and the definition of f that

$$\operatorname{rank}\begin{bmatrix} f(d) & f(0) & \cdots & \cdots & \cdots & f(0) \\ f(0) & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & f(d) & \ddots & & \vdots \\ \vdots & & \ddots & f(0) & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & f(0) \\ f(0) & \cdots & \cdots & \cdots & f(0) & f(0) \end{bmatrix} = k,$$

where the number of occurrences of $f(d)$ is equal to k. This, together with the inequality $2 \leq k < n$, completes the present step.

Step 2: *$f(1)f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{F}$*. For any $x, y \in \mathbb{F}$, since

$$\operatorname{rank}\left(\begin{bmatrix} 1 & x \\ y & xy \end{bmatrix} \oplus I_{k-1} \oplus 0\right) = k,$$

it follows from Step 1 (i. e., $f(0) = 0$) and the definition of f that

$$\operatorname{rank}\left(\begin{bmatrix} f(1) & f(x) \\ f(y) & f(xy) \end{bmatrix} \oplus f(1)I_{k-1} \oplus 0\right) = k,$$

and hence

$$\det\left(\begin{bmatrix} f(1) & f(x) \\ f(y) & f(xy) \end{bmatrix} \oplus f(1)I_{k-1}\right) = 0.$$

By direct computation, one shows that $f(1)^{k-1}(f(1)f(xy) - f(x)f(y)) = 0$. This, together with Step 1 (i. e., $f(1) \neq 0$), gives $f(1)f(xy) = f(x)f(y)$.

Step 3: $f(x+y) = f(x) + f(y)$ *for all* $x, y \in \mathbb{F}$. For any $x, y \in \mathbb{F}$, in view of the relation

$$\operatorname{rank}\left(\begin{bmatrix} 0 & 1 & x \\ 1 & 0 & y \\ 1 & 1 & x+y \end{bmatrix} \oplus I_{k-2} \oplus 0\right) = k,$$

it follows from Step 1 (i. e., $f(0) = 0$) and the definition of f that

$$\operatorname{rank}\left(\begin{bmatrix} 0 & f(1) & f(x) \\ f(1) & 0 & f(y) \\ f(1) & f(1) & f(x+y) \end{bmatrix} \oplus f(1)I_{k-2} \oplus 0\right) = k.$$

Furthermore,

$$\det\left(\begin{bmatrix} 0 & f(1) & f(x) \\ f(1) & 0 & f(y) \\ f(1) & f(1) & f(x+y) \end{bmatrix} \oplus f(1)I_{k-2}\right) = 0.$$

Thus, $f(1)^k\,(f(y) + f(x) - f(x+y)) = 0$. This, together with Step 1 (i. e., $f(1) \neq 0$), implies that $f(x+y) = f(x) + f(y)$.

Step 4: *there exist a non-zero scalar* c *and an injective field endomorphism* δ *of* $\mathbb{F}$ *such that* $f = c\delta$. If we denote $c = f(1)$ and $\delta = c^{-1}f$, then $f = c\delta$ and c is a non-zero scalar. Furthermore, it is easy to verify from Steps 1–3 that δ is an injective field endomorphism of $\mathbb{F}$.

The proof is complete. □

Proof of Theorem 2. *The "if" part*. If f is a non-zero constant function, then, clearly, f preserves rank-1 matrices of order n.

Now we prove the case $f = c\kappa$, where c is a non-zero scalar and $\kappa : \mathbb{F} \to \mathbb{F}$ is a multiplicative function with $\kappa(0) = 0$ and $\kappa(1) = 1$. For an arbitrary rank-1 matrix A, it can be written as $A = [a_i b_j]$, where $a_i, b_i \in \mathbb{F}$, $i = 1, \cdots, n$, and $a_p b_q \neq 0$ for some p, q. Hence $A^f = [c\kappa(a_i b_j)] = c[\kappa(a_i b_j)]$. Since κ is multiplicative, it can be concluded that $A^f = c[\kappa(a_i)\kappa(b_j)]$, which implies $\operatorname{rank} A^f \leq 1$. On the other hand, for any non-zero $d \in \mathbb{F}$, it follows from $dd^{-1} = 1$ and the multiplicativity of κ that $\kappa(d)\kappa(d^{-1}) = f(1)$. Using $\kappa(1) = 1$, we have $\kappa(d) \neq 0$. Therefore, $\kappa(a_p)\kappa(b_q) = \kappa(a_p b_q) \neq 0$ since $a_p b_q \neq 0$. In summary, $\operatorname{rank} A^f = 1$, i. e., f is a function preserving rank-1 matrices of order n.

The "only if" part. For any non-zero scalar d, it follows from $\operatorname{rank}(d \oplus 0) = 1$ and the definition of f that

$$\operatorname{rank}\begin{bmatrix} f(d) & f(0) & \cdots & f(0) \\ f(0) & f(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & f(0) \\ f(0) & \cdots & f(0) & f(0) \end{bmatrix} = 1. \tag{2.1}$$

Case 1. Suppose that $f(0) \neq 0$. Then, by (2.1), we have $f(d) = f(0)$. Since d is an arbitrary non-zero scalar, we can claim that f is a non-zero constant function.

Case 2. Suppose that $f(0) = 0$. Then, by (2.1), we have $f(d) \neq 0$ for any non-zero scalar d. For any $x, y \in \mathbb{F}$, since

$$\operatorname{rank}\left(\begin{bmatrix} 1 & x \\ y & xy \end{bmatrix} \oplus 0\right) = 1,$$

it follows from $f(0) = 0$ and the definition of f that

$$\operatorname{rank}\left(\begin{bmatrix} f(1) & f(x) \\ f(y) & f(xy) \end{bmatrix} \oplus 0\right) = 1.$$

Thus,

$$\det\begin{bmatrix} f(1) & f(x) \\ f(y) & f(xy) \end{bmatrix} = 0,$$

i. e., $f(1)f(xy) = f(x)f(y)$. If we put $c = f(1)$ and $\kappa = c^{-1}f$, then $f = c\kappa$ and c is a non-zero scalar. Furthermore, it is easily verified that κ is multiplicative function from $\mathbb{F}$ to itself such that with $\kappa(0) = 0$ and $\kappa(1) = 1$. This completes the proof. □

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PATTERN FORMATION IN A REACTION-DIFFUSION RATIO-DEPENDENT PREDATOR-PREY MODEL

MARCOS LIZANA AND JULIO J. MARÍN V.

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A . In this paper we study the effect of diffusion on the stability of the equilibria in a reaction-diffusion ratio-dependent predator-prey model and we explore under which parameter values Turing instability can occur giving rise to non-uniform stationary solutions. Moreover, their stability is studied.

Mathematics Subject Classification: 35K57, 92D25

Keywords: Turing instability, pattern formation, ratio-dependent predator-prey, reaction-diffusion system

1. I

In this paper we are going to study the following reaction diffusion ratio dependent predator prey model

$$\begin{aligned} \frac{\partial N}{\partial t} &= D_1 \Delta N + aN\left(1-\frac{N}{K}\right) - \frac{cNP}{mP+N}, \quad x \in \Omega, \ t>0, \\ \frac{\partial P}{\partial t} &= D_2 \Delta P + P\left(-d + \frac{fN}{mP+N}\right), \qquad x \in \Omega, \ t>0, \end{aligned} \tag{1.1}$$

subject to the Neumann boundary conditions

$$\frac{\partial N}{\partial \eta} = \frac{\partial P}{\partial \eta} = 0, \qquad x \in \partial\Omega, \quad t>0,$$

and initial conditions

$$N(x,0) = \varphi_1(x) \geq 0, \quad P(x,0) = \varphi_2(x) \geq 0, \quad x \in \Omega.$$

where a, K, c, m, f, d are positive constants and $N(x,t)$, $P(x,t)$ represent the population density of prey and predator at $x \in \Omega$ and at time t respectively. The prey grows with intrinsic growth rate a and carrying capacity K in the absence of predation. The predator consumes the prey with functional response of Michaelis-Menten type $cuy/(m+u)$, $u = x/y$ and contributes to its growth with rate $fuy/(m+u)$. The

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constant d is the death rate of predator, and $D_i > 0$ are constants, $i = 1, 2$; while Δ denotes the Laplace operator in $\Omega \subset \mathbb{R}^n$, Ω bounded and connected.

The motivation to consider the above described model comes from growing evidence [1, 2, 4, 7] that in some situations, specially when predators have to search for food and therefore have to share or compete for food, a more suitable general predator prey theory should be based on the so-called ratio-dependent theory, which can be roughly stated as that the *per capita* predator growth rate should be a function of the ratio of prey to predator abundance. This is supported by numerous field and laboratory experiments and observations [2–4].

Hsu et al. in [11] perform a global analysis of the Michaelis-Menten-type ratio-dependent predator-prey system without diffusion. Moreover, they discuss the main differences between the classical predator-prey models and the ratio dependent predator-prey system. In particular they brought into discussion the well-known "paradox of enrichment" or equivalently "the biological control paradox."

In this paper we will study the effect of diffusion on the stability of the equilibria in a reaction-diffusion ratio-dependent predator-prey model and we explore under which parameter values Turing instability can occur giving rise to non-uniform stationary solutions. Their stability is studied. Moreover, we give a comprehensive description, under which parameter values this pattern formation arises. In the concluding remark we will discuss the differences between the dynamics of this model and the classical one.

2. P

For simplicity, we undimensionalize the system (1.1) with the scaling $t \to at$, $N \to N/K$, $P \to mP/K$. Then system (1.1) takes the form

$$\begin{aligned} \frac{\partial N}{\partial t} &= d_1 \Delta N + N(1-N) - \frac{sNP}{P+N}, \quad x \in \Omega, \ t > 0, \\ \frac{\partial P}{\partial t} &= d_2 \Delta P + \delta P\left(-r + \frac{N}{P+N}\right), \qquad x \in \Omega, \ t > 0, \end{aligned} \tag{2.1}$$

where

$$s = \frac{c}{ma}, \quad \delta = \frac{f}{a}, \quad r = \frac{d}{f}, \quad d_1 = \frac{D_1}{a}, \quad d_2 = \frac{D_2}{a}.$$

We will show that the reaction-diffusion system (2.1) generates a dynamical system and it is biologically well-posed on a suitable Banach space.

Let us set $F = (F_1, F_2)$, $U = (N, P)$ and $D = \operatorname{diag}[d_1, d_2]$, where

$$F_1(N, P) = N(1-N) - \frac{sNP}{N+P}, \qquad F_2(N, P) = \delta P\left(-r + \frac{N}{P+N}\right).$$

Henceforth, considering also an initial condition, system (2.1) can be rewritten as

$$\begin{aligned} &\frac{\partial U(x,t)}{\partial t} = D\Delta U(x,t) + F(U), \quad x \in \Omega, \quad t > 0 \\ &\frac{\partial U}{\partial \eta}(x,t) = 0, \qquad x \in \partial\Omega, \quad t > 0 \\ &U(x,0) = \varphi(x), \qquad x \in \Omega. \end{aligned} \tag{2.2}$$

Let X be the Banach space $X_1 \times X_2$, where $X_i = C(\bar{\Omega}), i = 1, 2$. The norm on X is defined by $|\varphi| = |\varphi_1| + |\varphi_2|$. Let A_N^0 and A_P^0 be the differential operators $A_N^0 N = d_1 \Delta N$ and $A_P^0 P = d_2 \Delta P$, defined on the domains $D(A_N^0)$ and $D(A_P^0)$, respectively:

$$D(A_N^0) = \left\{ N \in C^2(\Omega) \cap C^1(\bar{\Omega}) : A_N^0 N \in C(\bar{\Omega}),\ \frac{\partial N}{\partial \eta}(x) = 0,\ x \in \partial\Omega \right\},$$

$$D(A_P^0) = \left\{ P \in C^2(\Omega) \cap C^1(\bar{\Omega}) : A_P^0 P \in C(\bar{\Omega}),\ \frac{\partial P}{\partial \eta}(x) = 0,\ x \in \partial\Omega \right\}.$$

The closures A_N of A_N^0, and A_P of A_P^0 in X_i generate analytic semigroups of bounded linear operators $T_N(t)$ and $T_P(t)$ for $t \geq 0$ such that $N(t) = T_N(t)\varphi_1$ and $P(t) = T_P(t)\varphi_2$ are solutions of the abstract linear differential equations in X_i given by

$$N'(t) = A_N N(t),\ P'(t) = A_P P(t).$$

An additional property of the semigroup is that for each $t > 0$, $T_N(t)$ and $T_P(t)$ are compact operators. In the language of partial differential equations $N(x,t) = [T_N(t)\varphi_1](x)$ and $P(x,t) = [T_P(t)\varphi_2](x)$ are classical solutions of the initial boundary value problem (2.2) with $F_1 = F_2 = 0$.

Let $\mathcal{T}(t) : X \to X$ be defined by $\mathcal{T}(t) = T_N(t) \times T_P(t)$. Then $\mathcal{T}(t)$ is a semigroup of operators on X generated by the operator $\mathcal{A} = A_N \times A_P$ defined on $D(\mathcal{A}) = D(A_N) \times D(A_P)$ and $U(x,t) = [\mathcal{T}(t)\varphi](x)$ is the solution of the linear system

$$\begin{aligned} &\frac{\partial U}{\partial t}(x,t) = D\Delta U(x,t), \quad x \in \Omega, \quad t > 0 \\ &\frac{\partial U}{\partial \eta}(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \quad U(x,0) = \varphi(x), \quad x \in \Omega. \end{aligned}$$

Observe that the nonlinear term F is twice continuously differentiable in U. Therefore, we can define the map $[F^*(\varphi)](x) = F(\varphi(x))$ which maps X into itself and equation (2.2) can be viewed as the abstract ordinary differential equation in X given by

$$u'(t) = \mathcal{A}u(t) + F^*(u(t)),\ u(0) = \varphi. \tag{2.3}$$

While a solution $u(t)$ de (2.3) can be obtained under the restriction that $\varphi \in D(\mathcal{A})$, a mild solution can be obtained for every $\varphi \in X$ by requiring only that $u(t)$ is a

continuous solution of the following integral equation

$$u(t) = \mathcal{T}(t)\varphi + \int_0^t \mathcal{T}(t-s)F^*(u(s))\,ds, \quad t \in [0,\beta), \tag{2.4}$$

where $\beta = \beta(\varphi) \leq \infty$. Restricting our attention to functions φ in the set

$$X_\Lambda = \left\{\varphi \in X : \varphi(x) \in \Lambda,\ x \in \overline{\Omega}\right\},$$

where $\Lambda = \left\{U = (N,P) \in R^2 : N \geq 0, P \geq 0\right\}$, and taking into account the definition of the functions F_i, we obtain that $F_1(0,P) = 0$ and $F_2(N,0) = 0$ for $U \in \Lambda$. Thus, Corollary 3.2 from [16, p. 129] implies that the Nagumo condition for the positive invariance of Λ is satisfied, i. e.,

$$\lim_{h\to 0^+} h^{-1}\,\mathrm{dist}\,(\Lambda, U + hF(U)) = 0, \quad U \in \Lambda. \tag{2.5}$$

On the other hand, the direct application of the strong parabolic maximum principle can be used to show that the linear semigroup $\mathcal{T}(t)$ leaves X_Λ positively invariant, i. e.,

$$\mathcal{T}(t)\,X_\Lambda \subset X_\Lambda, \quad t \geq 0. \tag{2.6}$$

Finally, conditions (2.5) and (2.6) together allow us to apply Theorem 3.1 from [16, p. 127], which gives us

Lemma 1. *For each $\varphi \in X_\Lambda$,* (2.1) *has a unique mild solution $u(t) = u(\varphi,t) \in X_\Lambda$ and a classical solution $U(x,t) = [u(t)](x)$. Moreover, the set X_Λ is positively invariant under the flow $\Psi_t(\varphi) = u(\varphi,t)$ induced by* (2.1).

So, the model (2.1) is biologically well-posed and its relevant dynamic is concentrated in X_Λ.

Finally, we are going to prove that all solutions of system (2.1) are bounded and therefore defined for all $t \geq 0$. Actually, from the following result by using the general theory of infinite dynamical system it follows that the relevant dynamic of the system (2.1) is concentrated in a compact set of the space X_Λ.

Theorem 1. *Let* (N,P) *be any solution of* (2.1). *Then*

$$\limsup_{t\to\infty} \max_{x\in\Omega} N(x,t) \leq 1, \qquad \limsup_{t\to\infty} \max_{x\in\Omega} P(x,t) \leq \frac{1}{r}\,.$$

P . From the first equation of the system (2.1), it follows that

$$\frac{\partial N}{\partial t} \leq d_1 \Delta N + N(1-N),$$

as long N is defined as a function of t.

Let z be the solution of the equation

$$z'(t) = z(t)(1-z(t)), \qquad z(0) = \max_{x\in\Omega} N(x,0).$$

From the comparison principle, we obtain $N(x,t) \le z(t)$. Now, taking into account that for any $\epsilon > 0$ there exists a $T_\epsilon > 0$ such that $z(t) < 1 + \epsilon$ for any $t \ge T_\epsilon$, which in turn implies that $N(x,t)$ is defined for all $t \ge 0$, and $\limsup_{t\to\infty} \max_{x\in\Omega} N(x,t) \le 1$.

Having in mind that for a given $\epsilon > 0$ there exists a $T_\epsilon > 0$ such that $N(x,t) \le 1+\epsilon$ for any $x \in \Omega$ and $t \ge T_\epsilon$, and by using the second equation of (2.1), we get

$$\frac{\partial P}{\partial t} - d_2 \Delta P \le \delta P\left(-r + \frac{1+\epsilon}{P}\right) = -\delta r P + \delta(1+\epsilon),$$

for any $x \in \Omega$ and $t \ge T_\epsilon$.

Let z be the solution of the following initial value problem

$$z'(t) = -\delta r z(t) + \delta(1+\epsilon), \qquad z(T_\epsilon) = \max_{x\in\Omega} P(x, T_\epsilon).$$

After a straightforward computation we get

$$z(t) \le \frac{1+\epsilon}{r} + z(T_\epsilon) e^{-\delta r (t - T_\epsilon)} \quad (\forall t \ge T_\epsilon).$$

Finally, by using the comparison principle we know that $P(x,t) \le z(t)$ as long as P is defined as a function of t. This, together with the previous inequality, implies that

$$\limsup_{t\to\infty} \max_{x\in\Omega} P(x,t) \le \frac{1}{r},$$

which completes the proof. □

3. A

In this section we will study the system (2.1) without diffusion, i. e.,

$$N'(t) = F_1(N,P), \qquad P'(t) = F_2(N,P). \tag{3.1}$$

In particular, we will focus our attention on the existence of equilibria and their local stability. This information will be crucial in the next section where we study the effect of the diffusion parameters on the stability of the steady states.

The equilibria of the system (3.1) are given by the solution of the following equations

$$N\left(1 - N - \frac{sP}{P+N}\right) = 0, \qquad \delta P\left(-r + \frac{N}{P+N}\right) = 0.$$

The system (3.1) has in the first quadrant the equilibrium points $(0,0)$ and $(1,0)$ for all values of the parameters. If $0 < r < 1$ and $0 < s < 1/(1-r)$, then (3.1) admits a nontrivial equilibrium, which is given by

$$(N^*, P^*) = \left(s(r-1) + 1, \ \frac{(1-r)[s(r-1)+1]}{r}\right).$$

We point out that for $r = 1$ we get that $(N^*, P^*) = (1,0)$.

Hereafter, we will assume that $(r, s) \in D$, where D is the region given by

$$D = \left\{(r, s) : 0 < r < 1,\ 0 < s < \frac{1}{1-r}\right\}.$$

In the system (3.1), the origin is a non-analytical complicated equilibrium point. The structure of a neighborhood of point $(0, 0)$ in the first quadrant of the plane (x, y) and the asymptotes of trajectories for $x, y \to 0$ depend on parameter values and change in an essential way with a change of parameter (see [6]).

A straightforward computation shows us that the equilibrium point $(1, 0)$ is locally asymptotically stable for $r > 1$, and unstable if $0 < r < 1$.

Linearizing the system (3.1) around the nontrivial equilibrium (N^*, P^*), we obtain that the characteristic equation is given by

$$\lambda^2 - \operatorname{trace} A\,\lambda + \det A = 0.$$

where

$$A = \begin{pmatrix} s(1-r^2) - 1 & -sr^2 \\ \delta(1-r)^2 & -\delta r(1-r) \end{pmatrix}.$$

Taking into account that $\operatorname{Re}\lambda < 0$ if and only if $\operatorname{trace} A < 0$ and $\det A > 0$, we get that (N^*, P^*) is locally asymptotically stable if and only if $r,\ s,\ \delta \in D_s$, where D_s is the set determined by the following inequalities:

$$0 < r < 1, \qquad 0 < s < \frac{1}{1-r^2} + \frac{\delta r}{1+r}, \qquad s < \frac{1}{1-r}, \qquad \delta > 0. \tag{3.2}$$

Let us set $f(r) = (1-r)^{-1}$ and

$$g_\delta(r) = \frac{r\delta}{1+r} + \frac{1}{1-r^2},$$

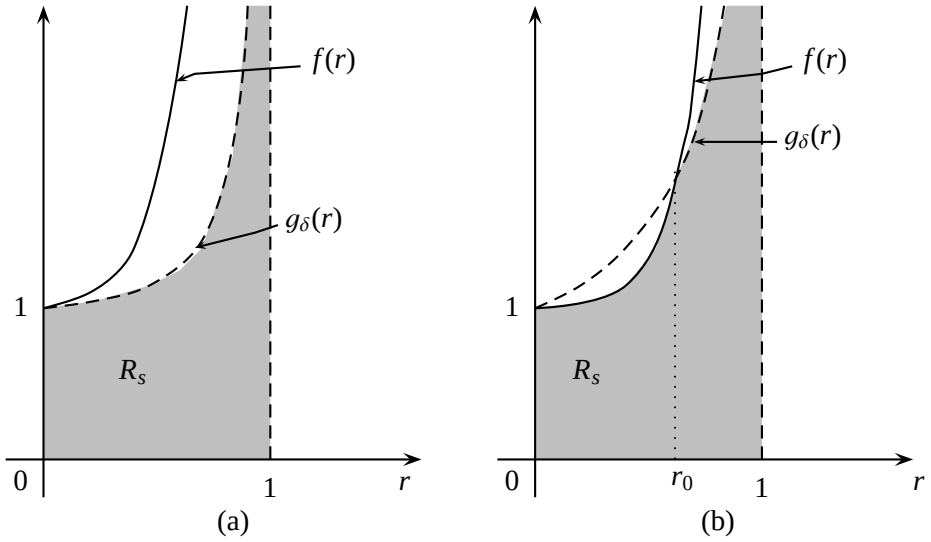
where δ is a positive parameter. We present on Fig. 1a and Fig. 1b the regions of the asymptotic stability of the nontrivial equilibrium.

4. T

It is obvious that the equilibria of the system (3.1) are solutions of (2.1). We shall focus our attention on the nontrivial equilibrium $U^* = (N^*, P^*)$ of the system (3.1). More concretely, in this section we will analyze the stability of nontrivial steady-state solutions of (2.1).

Definition 1 (see [14])**.** The equilibrium U^* of (2.1) is said to be diffusionally (Turing) unstable if it is an asymptotically stable equilibrium of (3.1) but it is unstable with respect to (2.1).

The stability of a homogeneous stationary solution U^* of (3.1) will be studied via the linearized stability analysis (see, e. g., [10, pp. 68–70]). Setting $W = U - U^*$ and



F 1. (a): R_s, the region of the local asymptotic stability for $0 < \delta \leq 1$; (b): R_s, the region of the local asymptotic stability for $\delta > 1$.

recalling that $A = F'(U^*)$, as given previously, the linearized system of the reaction-diffusion equation (2.1) around U^* is given by

$$\frac{\partial W}{\partial t} = D\Delta W + AW, \qquad \frac{\partial W}{\partial \eta}(x,t) = 0, \qquad x \in \partial\Omega,\ t > 0. \tag{4.1}$$

The trivial solution, $W = 0$, is asymptotically stable if and only if every solution of (4.1) decays to zero as $t \to \infty$.

Let $\phi_j(x)$ denote the jth eigenfunction of the Laplacian operator $-\Delta$ on Ω with no-flux boundary conditions. That is,

$$\Delta\phi_j + \lambda_j\phi_j = 0, \quad x \in \Omega, \qquad \boldsymbol{n}.\nabla\phi_j = 0,\ x \in \partial\Omega,$$

for scalars λ_j satisfying

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots.$$

The determination of the pairs $(\phi_j, \lambda j)$ is a standard problem (see, e. g., [9, pp. 205–208]). The differential operator $-\Delta$, with no-flux boundary conditions, is self-adjoint in $L_2(\Omega)$, i. e.,

$$\int_\Omega -\Delta\psi_1 \cdot \psi_2 dx = \int_\Omega -\Delta\psi_2 \cdot \psi_1 dx,$$

and it is easy to see that

$$\lambda_j = \frac{\int_\Omega |\nabla\phi_j|^2 dx}{\int_\Omega \phi_j^2 dx} > 0$$

for all $j \geq 1$. We can suppose without loss of generality that the $\phi_{j's}$ are normalized so that $\|\phi_j\|_{L_2(\Omega)} = 1$. Moreover, the set of ϕ_j forms an orthogonal basis for $L_2(\Omega)$ and any function may be expanded as a Fourier series or eigenfunction expansion

$$u(x) = \sum_{j=0}^{\infty} u_j \phi_j(x).$$

Using these preliminaries, we can solve (4.1) by expanding our solution W via

$$W(x,t) = \sum_{j=0}^{\infty} s_j(t)\phi_j(x) \tag{4.2}$$

where each $s_j(t) \in \mathbb{R}^2$. Substituting (4.2) into (4.1) and equating the coefficients at every ϕ_j, we have

$$\frac{ds_j}{dt} = B_j s_j,$$

where B_j is the matrix

$$B_j = A - \lambda_j D.$$

Now the trivial solution $W = 0$ of (4.1) is asymptotically stable if and only if each $s_j(t)$ decays to zero as $t \to \infty$. This is equivalent to the condition that each B_j has two eigenvalues with negative real parts for all j. The eigenvalues of the matrix B_j are given by

$$\det\left[B_j - \varrho I\right] = \varrho^2 - \operatorname{trace} B_j \varrho + \det B_j = 0.$$

Hereafter, we are going to assume that parameters $r, s, \delta \in D_s$; i. e. r, s, δ belong to the region where the nontrivial equilibrium U^* of the system (3.1) is asymptotically stable. Now we shall study the stability of U^* with respect to the system (2.1) in the (d_1, d_2) plane.

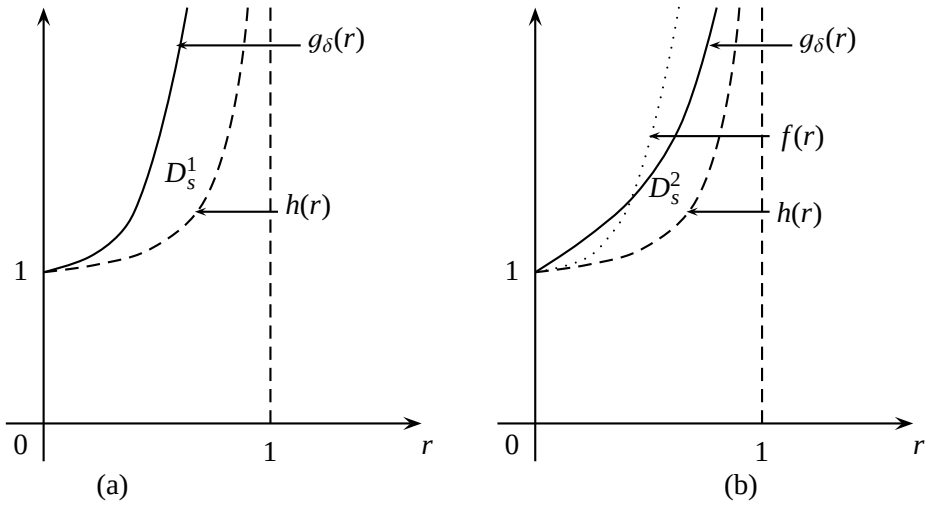
Taking into account that r, s, $\delta \in D_s$, it follows that $\operatorname{trace} A < 0$ and $\det A > 0$. Therefore, $\operatorname{trace} B_j = \operatorname{trace} A - \lambda_j(d_1 + d_2) < 0$, due to $\lambda_j \geq 0$, $j = 0, 1, 2, \ldots$, and $d_1, d_2 > 0$. Henceforth, for the Turing instability to occur, it should be satisfied that $\det B_j \leq 0$, for some $j \geq 1$, where $\det B_j = (A_{11} - \lambda_j d_1)(A_{22} - \lambda_j d_2) - A_{12}A_{21}$.

For fixed λ let us denote the hyperbola in the (d_1, d_2) plane by

$$H_\lambda : (\lambda d_1 - A_{11})(\lambda d_2 - A_{22}) - A_{12}A_{21} = 0.$$

We know that $A_{22} = -\delta r(1 - r) < 0$ on the admissible region. Hence, the location of the graph of the hyperbola H_λ on the (d_1, d_2)-plane is dictated by the sign of $A_{11} = s(1 - r^2) - 1$. A straightforward computation gives us that the graph of the function $h(r) = 1/(1 - r^2)$ lies strictly below the boundary of the region of asymptotic stability for any $\delta > 0$, see Fig. 2a and Fig. 2b.

Let us suppose that $A_{11} < 0$. In this case $\det B_j > 0$ for any $j \geq 0$ and $d_1 > 0$, $d_2 > 0$. We disregard this situation because we are looking for conditions of the Turing instability, see Fig. 3a. Assuming that $A_{11} > 0$, we obtain that there exist



F 2. (a): $0 < \delta \leq 1$; (b): $\delta > 1$.

positive parameters d_1 and d_2 where U^* is diffusionally unstable. That region is depicted on Fig. 3b.

From Fig. 3b, it follows that the set of $(d_1, d_2) \in \mathbb{R}^2_+$ satisfying that $\det B_j \leq 0$ for some $j \in \mathbb{N}$ consists of all points which are above the graph of the hyperbola H_{λ_j}. Clearly, for each $j \in \mathbb{N}$ this set is nonempty and therefore we can always choose $(d_1, d_2) \in \mathbb{R}^2_+$ in such a way that U^* is diffusionally unstable. Let us fix $d_2 > 0$. Since $\lambda_j \to \infty$ as $j \to \infty$, then there exists a $k \in \mathbb{N}$ such that $d_k^* = \frac{\det A}{\lambda_k A_{11}} < d_2$. Therefore, the point (d_A, d_2) belongs to the hyperbola H_{λ_k}, where

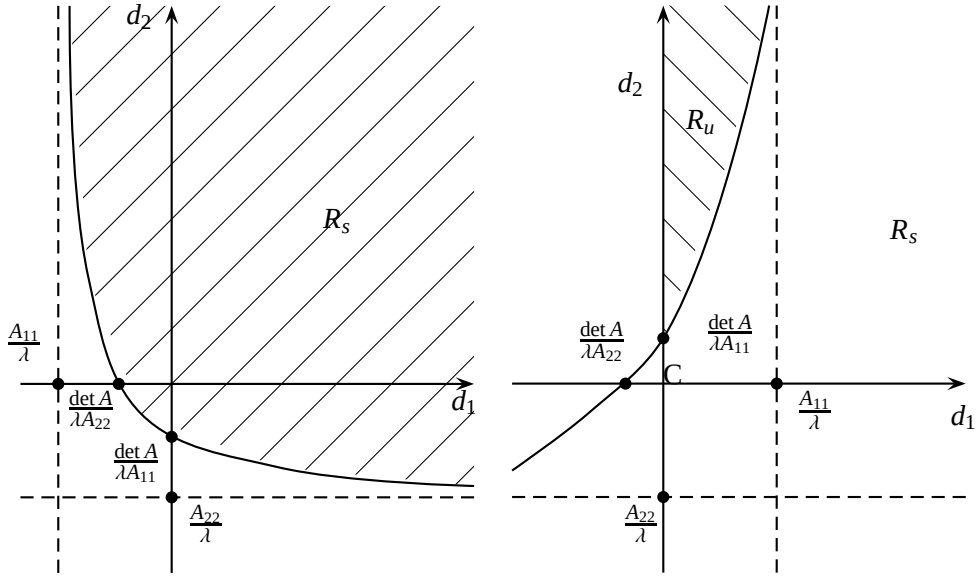
$$d_A = \frac{A_{11}\lambda_k d_2 - \det A}{\lambda_k(\lambda_k d_2 - A_{22})}.$$

Moreover, if $0 < d_1 < d_A$, then (d_1, d_2) will lie above the graph of H_{λ_k} and the homogeneous steady-state solution $U^* = (N^*, P^*)$ will be diffusionally unstable. We can also remark that if $d_2 \to \infty$, we have

$$\frac{A_{11}\lambda_k d_2 - \det A}{\lambda_k(\lambda_k d_2 - A_{22})} \to \frac{A_{11}}{\lambda_k}.$$

5. P

In this section we shall show how the diffusion-driven instability phenomenon gives rise to nonhomogeneous steady-state solutions of (2.1) that bifurcate from the uniform stationary solution. For this purpose, we start by introducing a definition.



F 3. (a): R_s, the stability region for $A_{11} < 0$, $r, s, \delta \in D_s$, and $\lambda > 0$; (b): R_s, the stability region, and R_u, the instability region for $A_{11} > 0$, $r, s, \delta \in D_s$, and $\lambda > 0$.

Consider the following reaction-diffusion system

$$\frac{\partial U}{\partial t} = D\Delta U + F(U), \qquad \frac{\partial U}{\partial \eta}(x,t) = 0, \qquad x \in \partial\Omega,\ t > 0 \tag{5.1}$$

where $U \in \mathbb{R}^2$, D is a 2×2 nonnegative diagonal matrix and $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth function, where $\partial/\partial\eta$ denotes the normal derivate. Assume that U^* is an uniform stationary solution of (5.1), i. e., $F(U^*) = 0$.

Definition 2. We say that U^* undergoes a Turing bifurcation at $\mu_0 \in (0, \infty)$ if the solution U^* changes its stability at μ_0 and in some neighborhood of μ_0 there exists a one-parameter family of nonconstant stationary solution of systems (5.1).

Now we use Theorem 13.5 from [17] for to determine the nonhomogeneous stationary solutions of (5.1), in this case take d_2 as bifurcation parameter.

Theorem 2. *Let υ_{1k} and υ_{2k} be the eigenvectors of B_k corresponding to the eigenvalues λ_{1k} and λ_{2k}, respectively. Assume that*

(1) $r, s, \delta \in D_s^i$, $i = 1, 2$,
(2) $\upsilon_{1k} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ *and* υ_{2k} *is not parallel to* $\begin{pmatrix} \xi_1 \\ 0 \end{pmatrix}$,
(3) $0 < d_1 < D^*$, *where* $D^* = A_{11}{\lambda_k}^{-1}$.

Then there exists a $k \in \mathbb{N}$ *such that at*

$$d_2^* = \frac{A_{22}\lambda_k d_1 - \det A}{\lambda_k(\lambda_k d_1 - A_{11})}$$

the uniform steady-state solution U^* *of* (5.1) *undergoes the Turing bifurcation.*

P . Hereafter, the role of the space X will be played by

$$X = \left\{ W \in C(\Omega, \mathbb{R}^2) \times C(\Omega, \mathbb{R}^2) : \frac{\partial W}{\partial \eta}(x,t) = 0,\ t > 0\ x \in \partial\Omega \right\}$$

with the supremum norm involving the first and second derivatives, and $Y = C(\Omega, \mathbb{R}^2)$ with the usual supremum norm. However, when choosing the subspace Z, we shall use the orthogonality induced by the scalar product

$$\langle V, W \rangle = \int_\Omega (V_1(x)W_1(x) + V_2(x)W_2(x))dx,$$

where $V = (V_1, V_2)$ y $W = (W_1, W_2)$.

Setting $W = U - U^*$, where U^* is a nontrivial homogeneous steady-state solution of (5.1), we get

$$W_t = D\Delta W + AW + G(W), \quad \frac{\partial W}{\partial \eta}(x,t) = 0, \quad t > 0, \quad x \in \partial\Omega, \tag{5.2}$$

where A is the Jacobian matrix of F in U^* and $G(W) = F(U^* + W) - AW$.

For any nonhomogeneous stationary solution U of (5.1), $W = U - U^*$ satisfies the elliptic equation

$$D\Delta W + AW + G(W) = 0, \quad \frac{\partial W}{\partial \eta}(x,t) = 0,\ t > 0\ x \in \partial\Omega. \tag{5.3}$$

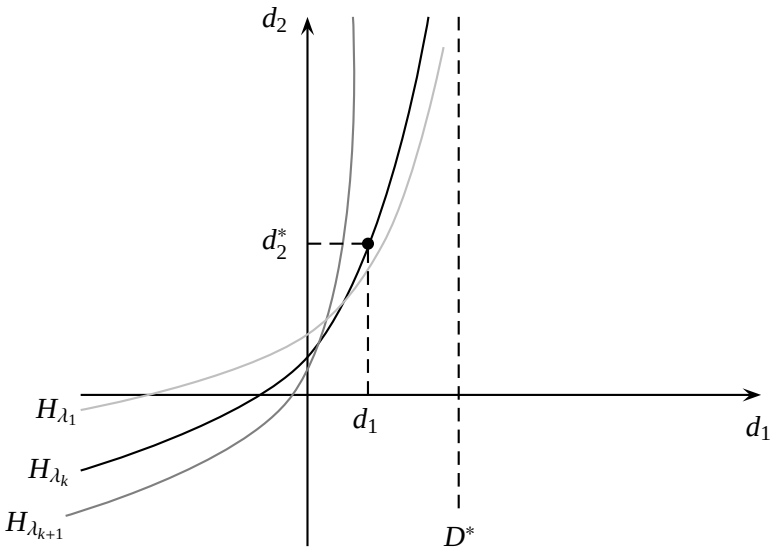
Taking into account this observation, define the function $f : \mathbb{R} \times X \to Y$ and linear operator L_0 considered in Theorem 13.5 of [17] as follows: $f(d_2, W) = D\Delta W + AW + G(W)$ and

$$L_0 = D_2 f(d_2^*, 0) = \frac{\partial f(d_2^*, 0)}{\partial W},$$

where d_2 is the diffusion coefficient of the susceptible class. The spectrum of the linear operator L_0 is given by the eigenvalues λ_{ij} of the matrices

$$B_j = A - \lambda_j D$$

evaluated at $d_2 = d_2^*$, where $i = 1, 2$, and $j = 0, 1, 2, \ldots$. Since $0 < d_1 < D^*$, there exists a unique $k \in \mathbb{N}$ such that (d_1, d_2^*) belongs to the hyperbola H_{λ_k}.



F 4. Turing Bifurcation. When $d_1 < D^*$, the uniform steady-state solution U^* of (5.1) undergoes a Turing bifurcation at $d_2 = d_2^*$

In other words, $\det B_j > 0$ for $j \neq k$ and $\det B_j = 0$ just for $j = k$. Therefore, for $i = 1, 2$ and $j = 0, 1, 2, \ldots, k-1, k+1, \ldots$ all eigenvalues λ_{ij} have negative real parts. For $j = k$, one eigenvalue, say λ_{1k}, is zero and the other one is negative, i. e., $\lambda_{2k} < 0$.

Since υ_{1k} is the eigenvector of B_k corresponding to the zero eigenvalue λ_{1k}, the eigenfunction of the linear operator L_0 corresponding to $\lambda_{1k} = 0$ is given by $\psi_k = \upsilon_{1k}\Phi_k(x)$ which is a non-uniform stationary solution of the linearized system (4.1), i. e.,

$$D\Delta\psi_k(x) + A\psi_k(x) = 0, \quad \frac{\partial\psi_k}{\partial\eta}(x) = 0, \ x \in \partial\Omega.$$

Therefore, the null subspace $N(L_0)$ of the operator $D_2 f(d_2^*, 0)$ is one-dimensional, spanned by ψ_k. Because of the orthogonality of the system, $\Phi_n(x)$, $n = 0, 1, 2, \ldots$ obtained by solving the eigenvalue problem

$$\Delta\Phi_n(x) + \lambda_n\Phi_n(x) = 0, \quad x \in \Omega,$$
$$\boldsymbol{n}.\nabla\Phi_n(x) = 0, \quad x \in \partial\Omega.$$

The range $R(L_0)$ of this operator is given by the relation

$$R(L_0) = \{U \in [C(\Omega, \mathbb{R})]^2 : \text{the Fourier expansion of } U \text{ does not contain the term } \Phi_n(x)\} \cup \{\upsilon_{2k}\Phi_n(x)\},$$

and has codimension one. So conditions (i) and (ii) of Theorem 13.5 from [17] are satisfied. It still remains to verify condition (iii). Let

$$L_1 = D_1 D_2 f(d_2^*, 0) = \frac{\partial}{\partial d}\left(\frac{\partial f}{\partial W}\right)(d_2^*, 0).$$

Then $L_1 = \left(\begin{smallmatrix} \Delta & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and

$$L_1\psi_k = \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} \upsilon_{1k}\Phi_n(x) = -\lambda_n \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix} \Phi_n(x),$$

with $\xi_1 \neq 0$, and $\left(\begin{smallmatrix} \xi_1 \\ 0 \end{smallmatrix}\right)$ not being parallel to υ_{2k}. Then, $L_1\psi_{1k} \notin R(L_0)$ and condition (iii) of Theorem 13.5 from [17] is satisfied. So, by choosing $Z = R(L_0)$ we conclude that there exists a $\gamma > 0$ and a C^1 curve $(d, \phi) : (-\gamma, \gamma) \to \mathbb{R} \times Z$ with $d(0) = d_2^*$ and $\phi(0) = 0$ such that

$$W(x, s) = s\upsilon_{1k}\Phi_n(x) + s\phi(x, s)$$

is a solution of the elliptic equation (5.3) with $d_2 = d(s)$, $s \in (-\gamma, \gamma)$. Finally, taking into account that $W = U - U^*$, we obtain

$$U(x, s) = U^* + s\upsilon_{1k}\Phi_n(x) + O(s^2)$$

are non-uniform stationary solutions of (5.1) with $d_2 = d(s)$, and $s \in (-\gamma, \gamma)$.

Therefore, at $d_2 = d_2^*$, the uniform steady-state solution U^* undergoes a Turing bifurcation.

□

6. S

In this section we will study the stability of the one parameter family of non-uniform stationary solution $U(x, s)$ of the system (2.1) that arise from the bifurcation of the homogeneous steady state U^*.

We showed that λ_{1k} is a L_1-simple eigenvalue of L_0, where $L_1 = D_1 D_2 f(d_2^*, 0)$ and $L_0 = D_2 f(d_2^*, 0)$. On the other hand, for $|\varepsilon|$ and $|s|$ small enough, the operators $D_2 f(d_2^* + \varepsilon, 0)$ and $D_2 f(d(s), s\psi_k + s\phi(x, s))$ are close to L_0. Applying Lemma 13.7 of [17], we obtain that there exist functions

$$d \longmapsto (\varrho(d), \psi_c(d)), \qquad s \longmapsto (\eta(s), \psi_b(s))$$

defined on neighborhoods of d_2^* and 0, respectively, such that

$$D_2 f(d, 0)\psi_c(d) = \varrho(d)\psi_c(d),$$
$$D_2 f(d(s), s\psi_k + s\phi(x, s))\psi_b(s) = \eta(s)\psi_b(s),$$

and

$$(\varrho(d_2^*), \psi_c(d_2^*)) = (0, \psi_k) = (\eta(0), \psi_b(0)).$$

Note that the functions

$$\eta(s) = \eta(D_2 f(d(s), s\psi_k + s\phi(x, s))),$$
$$\psi_b(s) = \psi_b(D_2 f(d(s), s\psi_k + s\phi(x, s))),$$
$$\varrho(d) = \eta(D_2 f(d, 0)), \;\; \psi_c(d) = \psi_b(D_2 f(d, 0))$$

given by Lemma 13.7 of [17] are smooth functions.

The following result is the Crandall-Rabinowitz Theorem 1.16 from [8, p. 165].

Theorem 3. *Let the assumptions of Theorem 13.5 of* [17] *hold, and let the functions $\varrho(d)$ and $\eta(s)$ be defined as above. Then $\varrho'(d_2^*) \neq 0$, and if $\eta(s) \neq 0$ for s close to 0, then*

$$\lim_{s\to 0} \frac{sd'(s)\varrho'(d_2^*)}{\eta(s)} = -1. \tag{6.1}$$

First we determine $\varrho'(d_2^*)$. It is known that $\varrho(d_2)$ satisfies the equation

$$\varrho^2(d_2) - \text{trace } B_k\varrho(d_2) + \det B_k = 0.$$

Differentiating implicitly the former equation with respect to d_2, we have

$$\varrho'(d_2) = \frac{\lambda_k A_{11} - \lambda_k^2 d_1 - \lambda_k\varrho(d_2)}{2\varrho(d_2) - \text{trace } B_k}.$$

Evaluating at d_2^*, we obtain

$$\varrho'(d_2^*) = \frac{\lambda_k^2 d_1 - \lambda_k A_{11}}{\text{trace}A - \lambda_k(d_1 + d_2^*)} = \frac{\lambda_k(\lambda_k d_1 - A_{11})}{\text{trace } A - \lambda(d_1 + d_2^*)}.$$

Since $A_{11} > 0$ and $0 < d_1 < A_{11}\lambda_k{}^{-1}$, we see that $\lambda_k d_1 - A_{11} < 0$ and trace $A - \lambda_k(d_1 + d_2^*) < 0$. Therefore,

$$\varrho'(d_2^*) > 0.$$

Proposition. *Let $(d(s), U(x, s))$ be the one parameter family of bifurcating solutions given by the formula*

$$U(x, s) = W^* + s\upsilon_{1k}\Phi_n(x) + O(s^2).$$

Assume that the conditions of Theorem 2 are satisfied, $d'(0) \neq 0$, and that the eigenvalues $\eta(s)$ of the nonhomogeneous steady state bifurcating from the critical value $\lambda_{1k} = 0$ are non-zero for small $|s| \neq 0$. Then if $d(s) < d_2^*$ the corresponding solution $U(x, s)$ is stable and if $d(s) > d_2^*$, the corresponding solution $U(x, s)$ is unstable.

P . We know that $\varrho'(d_2^*) > 0$. Let us determine the sign of $\eta(s)$. Since $d'(0) \neq 0$, we may assume that $d'(0) > 0$. Then by continuity we have that $d'(s) > 0$ for $|s|$ small enough. Therefore, using (6.1), it follows that $\eta(s) < 0$ for $s > 0$ small enough, which in turn implies that the bifurcating solution is asymptotically stable. For small $s < 0$, $\eta(s) > 0$. Hence, the bifurcating nonhomogeneous stationary solution is unstable.

The case $d(s) < 0$ can be analyzed similarly . This completes the proof of our claim. □

7. D

In this paper, we discussed the main mathematical features exhibited by the reaction-diffusion system (1.1). More concretely, we showed that when $A_{11} = s(1 - r^2) - 1$ is positive, a nontrivial geotemporal dynamics of the reaction-diffusion ratio-dependent predator-prey model (1.1) can be obtained. In the case where the $0 < d_1 < A_{11}/\lambda_j$, we showed that for a wide range of parameter values and diffusion coefficients d_1 and d_2, see Fig. 3b, the nonlinear system (1.1) can exhibit stable spatially heterogeneous solutions which arise from Turing bifurcations. It is worth pointing out that a Turing bifurcation can not occur for a large diffusive coefficient of the prey, nevertheless the diffusive coefficient of the predator can be large enough.

The existence of this pattern formation for system (1.1) shows that the reaction-diffusion ratio-dependent predator-prey model exhibits features which were not possible for the classical model. More specifically, one can show that for a classical Lotka–Volterra prey-predator system with diffusion on a finite domain and zero flux, boundary condition cannot give rise to temporally or spatially inhomogeneous solutions asymptotically as $t \to \infty$.

In conclusion, we note that the mathematical analysis of model (1.1) shows how a reaction-diffusion ratio-dependent predator-prey model can stably regulate its growth around either spatially homogeneous or heterogeneous solutions through a Turing instability mechanism.

A

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SUBDIRECT IRREDUCIBILITY OF ALGEBRAS AND ACTS WITH AN ADDITIONAL UNARY OPERATION

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Abstract. As a generalization of algebras with involution [1], we consider algebras $(A; \mathcal{F} \cup \{\varphi\})$ with an additional unary operation φ of finite degree k. A subdirectly irreducible algebra $(A; \mathcal{F} \cup \{\varphi\})$ is a subdirect product of subdirectly irreducible algebras $(A/\varphi(\Theta); \mathcal{F}), \ldots, (A/\varphi^d(\Theta); \mathcal{F})$ where Θ is a certain congruence, d is a divisor of k and φ^d is the identical mapping. For a congruence modular algebra $(A; \mathcal{F})$ also the unique smallest congruence of $(A; \mathcal{F} \cup \{\varphi\})$ described. Examples are S-acts with an additional unary operation. In this case more information is given on the subdirectly irreducible components.

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Keywords: subdirectly irreducible algebra, congruence, unary operation, S-act

1. Preliminaries

The purpose of this note is to generalize the notion of involution of universal algebras and to study the structure of subdirectly irreducible universal algebras with generalized involution, that is, with an additional unary operation of finite degree.

Let $\mathbb{V}_\varphi$ be a variety of universal algebras $(A; \mathcal{F} \cup \{\varphi\})$ with finitary operations $f \in \mathcal{F}$ and a unary operation $\varphi : a \to \varphi(a)$ subject to the identities

(i) $E_\lambda = E_\lambda(f_{\lambda_1}, \ldots, f_{\lambda n_\lambda}) \mid f_{\lambda_1}, \ldots, f_{\lambda n_\lambda} \in \mathcal{F}),\ \lambda \in \Lambda$,

involving only operations from $\mathcal{F}$,

(ii) $E_\varphi : \varphi^k(a) = a$,

where $k \geq 2$ is a fixed integer, the *degree of* φ,

(iii) $E_f : \varphi(f(a_1, \ldots, a_n)) = f(\varphi(a_{\pi_f(1)}, \ldots, \varphi(a_{\pi_f(n)})))$ for all $f \in \mathcal{F}$,

where π_f is a permutation of n elements (n the arity of the operation f) assigned to each operation $f \in \mathcal{F}$.

Each permutation π_f can be written into a product of disjoint cyclic permutations $c(\pi_f)_1, \ldots, c(\pi_f)_r$ admitting also cyclic permutations of length 1.

These identities may induce some further ones. The identity $f = \varphi^k(f)$ is a consequence of (ii), and *this is not trivial if and only if the degree k of* φ *is not a multiple*

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of the lengths $c(\pi_f)_1, \dots, c(\pi_f)_r$. The identity $f = \varphi^k(f)$ may be viewed as a *generalized commutativity*; in the case of a binary operation $f(x_1, x_2)$, cyclic permutation $c(\pi_f) = (12) = \pi_f$ and odd degree $k = 2\ell + 1$ we have

$$f(x_1, x_2) = \varphi^k(f(x_1, x_2)) = \varphi^{2\ell+1}(f(x_1, x_2)) = \varphi(f(x_1, x_2)) = f(x_2, x_1),$$

that is, the commutativity of the operation f.

Moreover, if $f :\to a$ and $g : a \to g(a)$ are 0-ary and unary operations, then by (iii) it follows that

$$\varphi(f) :\to a \qquad \text{and} \qquad \varphi(g(a)) = g(\varphi(a)).$$

In particular, for the zero element 0 and the unity element 1 (if they exist) we have

$$\varphi(0) = 0 \qquad \text{and} \qquad \varphi(1) = 1,$$

and also

$$\varphi(-a) = -\varphi(a)$$

for the additive inverse of an element a.

We are going to scrutinize the interrelations between the variety $\mathbb{V}_\varphi$ and the variety $\mathbb{V}$ of algebras $(A; \mathcal{F})$ subject to the identities (i).

Let us observe that for any algebra $(A; \mathcal{F} \cup \{\varphi\})$ and congruence $\Theta \in \mathrm{Con}(A; \mathcal{F})$ of the algebra $(A; \mathcal{F}) \in \mathbb{V}$ also

$$\varphi(\Theta) = \{(\varphi(a), \varphi(b)) \mid (a, b) \in \Theta\}$$

is a congruence of $(A; \mathcal{F})$, as seen readily from (iii).

Proposition 1. (a) *Let* $(A; \mathcal{F} \cup \{\varphi\}) \in \mathbb{V}_\varphi$ *and* $\Theta \in \mathrm{Con}(A; \mathcal{F})$. *Then* $\Theta = \varphi(\Theta)$ *if and only if* $\Theta \in \mathrm{Con}(A; \mathcal{F} \cup \{\varphi\})$.

(b) *The lattice* $\mathrm{Con}(A; \mathcal{F} \cup \{\varphi\})$ *is a sublattice of the lattice* $\mathrm{Con}(A; \mathcal{F})$.

(c) *If* $\Theta \in \mathrm{Con}(A; \mathcal{F})$, *then* $\varphi(\Theta) \vee \cdots \vee \varphi^k(\Theta)$ *and* $\varphi(\Theta) \wedge \cdots \wedge \varphi^k(\Theta)$ *belong to* $\mathrm{Con}(A; \mathcal{F} \cup \{\varphi\})$.

Proof. The statements correspond to those of [1, Proposition 3.1] and their proofs are straightforward. □

As a motivation for the subsequent investigations we give an example.

Example. Let G be an abelian group, and consider the direct sum $A = G \oplus G$. On A we define a unary operation φ by

$$\varphi(a, b) = (b, a + b) \qquad (\forall a, b \in G).$$

Since

$$\varphi((a, b) + (c, d)) = (b + d, a + b + c + d) = \varphi(a, b) + \varphi(c, d),$$

the operation fulfils the requirement of condition (iii). One readily sees that if A is an elementary 2-group, then

$$\varphi^3(a, b) = (a + 2b, 2a + 3b) = (a, b),$$

so the degree of φ is 3. Clearly, $(0, G)$ defines a congruence of $(A; +)$,

$$\varphi(0, G) = \{(a, a) \mid a \in G\}$$

gives the diagonal congruence of A and $\varphi^2(0, G)$ is the congruence determined by $(G, 0)$. If $|G| = 2$, then all the three factors of $(A; +)$ by these congruences are subdirectly irreducible, and $(A; +)$ is a subdirect product of these factors. Nevertheless, $(A; \{+, \varphi\})$ itself is subdirectly irreducible.

For elementary $3-, 5-, 7-, 11-$ groups, the degree of φ is 8, 20, 16, 10, respectively. If G is a cyclic group of 4 elements or a direct sum of such groups, then the degree of φ is 12.

2. Subdirect decompositions

As seen in [1], a subdirectly irreducible algebra $(A; \mathcal{F} \cup \{\varphi\})$ with involution φ (that is the degree of φ is $k = 2$) is either subdirectly irreducible as an algebra $(A; \mathcal{F}) \in \mathbb{V}$ or a certain subdirect product of two subdirectly irreducible algebras of $\mathbb{V}$. We shall generalize this result, and describe the subdirectly irreducible algebra $(A; \mathcal{F} \cup \{\varphi\}) \in \mathbb{V}_\varphi$ as a subdirect product of subdirectly irreducible algebras of $\mathbb{V}$ and see that the number d of the subdirect components is a divisor of the degree k of φ.

The smallest congruence of an act A will be denoted by ω_A or briefly by ω. We shall call the unique atom of the lattice of congruences of a subdirectly irreducible algebra the *heart* of that algebra. This terminology corresponds to that of ring theory: the unique atom in the lattice of ideals of a subdirectly irreducible ring used to be called the heart of that ring.

Theorem 2. *If $(A; \mathcal{F} \cup \{\varphi\})$ is a subdirectly irreducible algebra with heart χ_φ, then there exists a congruence $\Theta \in \mathrm{Con}(A; \mathcal{F})$ such that $\chi_\varphi \not\leq \Theta$ and $(A; \mathcal{F})$ is isomorphic to a subdirect product of subdirectly irreducible algebras*

$$(A/\varphi(\Theta); \mathcal{F}),\ (A/\varphi^2(\Theta); \mathcal{F}),\ \ldots,\ (A/\varphi^d(\Theta); \mathcal{F}), \tag{1}$$

where d is the least positive integer such that $\varphi^d(\Theta) = \Theta$. The heart of $(A/\varphi^i(\Theta); \mathcal{F})$ is $\chi_i = (\chi_\varphi \vee \varphi^i(\Theta))/\varphi^i(\Theta)$ for each $i = 1, \ldots, d$. The isomorphism between $(A; \mathcal{F})$ and the subdirect product $\bigotimes_{i=1}^{d}(A/\varphi^i(\Theta); \mathcal{F})$ is given by the correspondence

$$a \longmapsto ([a]_{\varphi(\Theta)}, \ldots, [a]_{\varphi^d(\Theta)}) \qquad (\forall a \in A),$$

where $[a]_{\varphi^i(\Theta)}$ stands for the $\varphi^i(\Theta)$-coset represented by $a \in A$.

Proof. If $(A; \mathcal{F})$ is subdirectly irreducible, then $d = 1$ and $\Theta = \omega = \varphi(\omega) = \varphi(\Theta)$ does the job.

Suppose, next, that $(A; \mathcal{F})$ is not subdirectly irreducible. Since $\chi_\varphi \in \mathrm{Con}(A; \mathcal{F})$, there exists a congruence $\Theta \in \mathrm{Con}(A; \mathcal{F})$ such that $\omega \neq \Theta \neq \varphi(\Theta)$, $\chi_\varphi \not\leq \Theta$ and by an appropriate choice of Θ it can be achieved (Zorn's Lemma) that $(A/\Theta; \mathcal{F})$ is subdirectly irreducible and in this case the heart of $(A/\Theta; \mathcal{F})$ is obviously $\chi_1 = (\chi_\varphi \vee \Theta)/\Theta$. Let d be the smallest integer such that $\varphi^d(\Theta) = \Theta$. Such an integer d

does exist, and d must divide k, the degree of φ. Moreover, by (ii) also $(A/\varphi^i(\Theta);\mathcal{F})$ is subdirectly irreducible with heart χ_i for every $i = 1,\dots,d$. Since $\chi_\varphi \not\subseteq \Theta$ and $\varphi(\Theta) \wedge \varphi^2(\Theta) \wedge \cdots \wedge \varphi^d(\Theta)$ is in $\mathrm{Con}(A; \mathcal{F} \cup \{\varphi\})$, we conclude that

$$\varphi(\Theta) \wedge \varphi^2(\Theta) \wedge \cdots \wedge \varphi^d(\Theta) = \omega.$$

Thus, $(A; \mathcal{F})$ has the desired subdirect product representation.

The last assertion is obvious. □

In the next two theorems we shall use the assumption that the congruence lattice of the algebra $(A; \mathcal{F})$ is modular. This is always so for groups, rings, and in many other cases, but in general the congruence lattice of an algebra, in particular of an S-act, need not be modular.

Remark. In [1, Theorem 3.2] *it has to be assumed that the congruence lattice of* $(A; \mathcal{F})$ *is modular* (this was used on line 8 of its proof). This inaccuracy, however, does not affect the further results of [1].

Theorem 3. *Let* $(A; \mathcal{F} \cup \{\varphi\})$ *be an algebra as in Theorem 2. If* $(A; \mathcal{F})$ *is congruence modular and* $\chi_\varphi \wedge \Theta = \omega$, *then* χ_φ *is an atom in* $\mathrm{Con}(A; \mathcal{F})$ *and there is a one-to-one correspondence between the* χ_φ*-cosets of* $(A; \mathcal{F})$ *and the* χ_i*-cosets of* $(A/\varphi^i(\Theta);\mathcal{F})$ *given by the correspondence*

$$[a]_{\chi_\varphi} \longmapsto [[a]_{\varphi^i(\Theta)}]_{\chi_i},$$

which can be expressed also in terms of relations as

$$a\chi_\varphi b \longmapsto [a]_{\varphi^i(\Theta)}\chi_i[b]_{\varphi^i(\Theta)}.$$

Proof. If $d = 1$, then there is nothing to prove.

Suppose that $d > 1$, and let $\kappa \in \mathrm{Con}(A; \mathcal{F})$ be a congruence such that $\kappa < \chi_\varphi$. Then either $\kappa \le \Theta$ and $\kappa \le \chi_\varphi \wedge \Theta = \omega$, or $\Theta < \kappa \vee \Theta$. In the latter case, taking into account that $(\chi_\varphi \vee \Theta)/\Theta$ is the heart of $(A/\Theta;\mathcal{F})$, we have $\chi_\varphi \vee \Theta \le \kappa \vee \Theta$ whence $\kappa \vee \Theta = \chi_\varphi \vee \Theta$. Now, the sublattice $\{\omega, \kappa, \chi_\varphi, \Theta, \chi_\varphi \vee \Theta\}$ is not modular unless $\kappa = \omega$. This proves that χ_φ is an atom in $\mathrm{Con}(A; \mathcal{F})$.

In view of the relation $\chi_\varphi \wedge \Theta = \omega$, a moment's reflection shows the validity of the further statements. □

Theorem 4. *Let* $(A; \mathcal{F} \cup \{\varphi\})$ *be an algebra as in Theorem 2. If* $(A; \mathcal{F})$ *is congruence modular and* $\chi_\varphi \wedge \Theta \ne \omega$, *then*

(1) $\bigwedge_{i\ne j=1}^{d}(\chi_\varphi \wedge \varphi^j(\Theta))$ *is an atom in* $\mathrm{Con}(A; \mathcal{F})$,

(2) $\bigvee_{i=1}^{d}(\bigwedge_{i\ne j=1}^{d}(\chi_\varphi \wedge \varphi^j(\Theta))) = \chi_\varphi$,

(3) $\bigwedge_{i\ne j=1}^{d}\left(\chi_\varphi \wedge \varphi^j(\Theta)\right) \wedge \bigvee_{i\ne j=1}^{d}(\bigwedge_{j\ne k=1}^{d}(\chi_\varphi \wedge \varphi^k(\Theta))) = \omega$.

Proof. (1) It suffices to prove the statement for $i = d$. Since A/Θ is subdirectly irreducible with heart $(\chi_\varphi \vee \Theta)/\Theta$, the congruence $\chi_\varphi \wedge \Theta$ covers Θ in $\mathrm{Con}(A; \mathcal{F})$.

Let us consider a congruence $\kappa \in \mathrm{Con}(A; \mathcal{F})$ such that $\chi_\varphi \wedge \Theta < \kappa \leq \chi_\varphi$. Then $\kappa \not\leq \Theta$ and by the subdirectly irreducibility of A/Θ we have

$$\chi_\varphi \vee \Theta \leq \kappa \vee \Theta \leq \chi_\varphi \vee \Theta,$$

that is, $\kappa \vee \Theta = \chi_\varphi \vee \Theta$. Thus by $\kappa \leq \chi_\varphi$ the modularity of $\mathrm{Con}(A; \varphi)$ yields

$$\chi_\varphi = (\chi_\varphi \vee \Theta) \wedge \chi_\varphi = (\kappa \vee \Theta) \wedge \chi_\varphi = \kappa \vee (\Theta \wedge \chi_\varphi) = \kappa,$$

showing that χ_φ covers $\chi_\varphi \wedge \Theta$.

Next, we consider a congruence $\tau \in \mathrm{Con}(A; \mathcal{F})$ such that

$$\omega \neq \tau \leq \bigwedge_{i=1}^{d-1} (\chi_\varphi \wedge \varphi^i(\Theta)).$$

Since χ_φ covers $\chi_\varphi \wedge \Theta$, by $\tau \neq \omega = \bigwedge_{i=1}^{d} (\chi_\varphi \wedge \varphi^i(\Theta))$ we have $\tau \vee (\chi_\varphi \wedge \Theta) = \chi_\varphi$. If $\tau \neq \bigwedge_{i\neq j=1}^{d} (\chi_\varphi \wedge \varphi^j(\Theta))$ then $\{\omega, \tau, \bigwedge_{i\neq j=1}^{d} (\chi_\varphi \wedge \varphi^j(\Theta)), \chi_\varphi \wedge \Theta, \chi_\varphi\}$ is a non-modular sublattice in $\mathrm{Con}(A; \mathcal{F})$. This contradiction proves the assertion in (1).

(2) Since $\chi_\varphi \wedge \Theta \neq \omega$ and $\varphi(\bigvee_{i=1}^{d} (\chi_\varphi \wedge \varphi^i(\Theta))) = \bigvee_{i=1}^{d} (\chi_\varphi \wedge \varphi^i(\Theta))$, the assertion holds.

(3) Assume that the expression in (3) is not equal ω. Then, by (1), it equals $\bigwedge_{i\neq j=1}^{d} \left(\chi_\varphi \wedge \varphi^j(\Theta)\right)$, Hence by ii) we have

$$\bigwedge_{i\neq j=1}^{d} (\chi_\varphi \wedge \varphi^j(\Theta)) \leq \bigvee_{i\neq j=1}^{d} \left(\bigwedge_{j\neq k=1}^{d} (\chi_\varphi \wedge \varphi^k(\Theta)) \right) = \chi_\varphi.$$

This, however, contradicts $\varphi(\chi_\varphi) = \chi_\varphi$. □

Restricting the considerations to an algebra $(A; \mathcal{F})$ the congruences of which are determined by certain subalgebras (ideals, normal subgroups, etc.), then $(A; \mathcal{F})$ is congruence modular. We shall use the language and notions of ring theory, and denote the unique minimal ideal (i. e., the *heart*) of $(A; \mathcal{F} \cup \{\varphi\})$ by H_φ, and write I for the ideal determined by the congrunce Θ. Then Theorems 3 and 4 have the following meaning.

Corollary 5. *If $H_\varphi \wedge I = 0$, then H_φ is an atom in the lattice of ideals of $(A; \mathcal{F})$ and H_φ is isomorphic to the heart $(H_\varphi + \varphi^i(I))/\varphi^i(I)$ of $(A/\varphi^i(I); \mathcal{F})$ for each $i = 1, \ldots d$.*

If $H_\varphi \wedge I \neq 0$, then $\bigcap_{i\neq j=1}^{d} (H_\varphi \cap \varphi^i(I))$ is an atom in the lattice of ideals of $(A; \mathcal{F})$ for each $i = 1, \ldots, d$, and H_φ is isomorphic to the direct sum of these ideals.

3. *S*-acts

An S-act A over a semigroup S is a set A with scalar multiplication sa ($s \in S$, $a \in A$) subject to the rule $s(ta) = (st)a$. A one-element subact is called a *sink*. We call an S-act A *simple*, if it has only two congruences: the all congruence ι_A and the

equality congruence ω_A. Recall that an S-act is *strongly connected* if $A = Sa$ for every element $a \in A$. For details the reader may consult [2].

An S-act A is, in fact, an algebra $(A; S)$ where each element s of the semigroup S is a unary operation $s : a \mapsto sa,\ a \in A$, subject to the identity

$$(st)a = s(ta) \qquad (\forall s, t \in S,\ \forall a \in A).$$

Having one more operation φ on A subject to conditions (ii) and (iii) of Sec. 1, the (S, φ)-act $(A; S \cup \{\varphi\})$ satisfies the identity

$$\varphi(sa) = s(\varphi(a) \qquad (\forall s \in S,\ \forall a \in A),$$

in other words, φ is an automorphism of the S-act A of order k.

Let $(A : S \cup \{\varphi\})$ be a subdirectly irreducible (S, φ)-act with heart χ_φ, and $|A| > 1$ Beside the decomposition Theorems 2, 3 and 4 we can say more on the structure of the (S, φ)-act A in terms of the S-act A.

First, we make some obserations.

Let us consider the intersection $B = \cap(C_\lambda \mid \lambda \in \Lambda)$ of all (S, φ)-subacts C_λ of A with $|C| > 1$. If the corresponding Rees congruence ϱ_B to B is ω, then A is not subdirectly irreducible, a contradiction. Hence $\chi_\varphi \leq \varrho_B$, and so χ_ϱ *is a partition of* B *and each element* $a \in A \setminus B$ *forms a* χ_φ*-coset.* The set T of all (S, φ)-sinks of A is an (S, φ)-subact, and so either $|T| \leq 1$ or $B \subseteq T$. Since A is a subdirectly irreducible (S, φ)-act, the set T cannot have more than 2 elements. By the minimality of B *in the latter case only* $|B| = |T| = 2$ *is possible.*

Thus, *there exists a unique smallest* (S, φ)*-subact* B *of* A *with* $|B| > 1$, and one of the following cases may occur.

Case 1. $\chi_\varphi = \iota_A$.

$\chi_\varphi \neq \iota_A$ and

Case 2. There is no (S, φ)-sink in B.

Case 3. There is exactly one (S, φ)-sink in B.

Case 4. B consists of two (S, φ)-sinks.

Next, we discuss these cases.

Case 1. The subdirectly irreducible (S, φ)-act A is simple. ¿From Theorem 2 we infer that $\Theta = \omega$. Since $\varphi(\omega) = \omega$, it follows that $d = 1$ and that the S-act A is subdirectly irrecducible by Theorem 3. Thus, A *is a simple* S*-act which may possess one* S*-sink.*

Case 2. Let C be a non-empty S-subact of B. Since B does not contain (S, φ)-sinks, necessarily

$$C \cup \varphi(C) \cup \cdots \cup \varphi^{d-1}(C) = B$$

and

$$C \cap \varphi(C) \cap \cdots \cap \varphi^{d-1}(C) = \varnothing.$$

Let j_1 be the least integer such that $C \cap \varphi^{j_1}(C) \neq \varnothing$. If such an integer j_1 does not exist, then C is disjoint to each $\varphi^i(C)$ for every $i = 1, \ldots, d-1$, and so is $\varphi^i(C)$ to each $\varphi^j(C)$ where $i \neq j \in \{1, \ldots, d\}$. Further, let j_2 be the next integer such that

$C \cap \varphi^{j_1}(C) \cap \varphi^{j_2}(C) \neq \emptyset$, and so on. In this way we get integers $0 < j_1 < j_2 < \cdots < d-1$ such that

$$D = C \cap \varphi^{j_1}(C) \cap \cdots \cap \varphi^{j_r}(C) \neq \emptyset,$$

and the number r is maximal. One readily sees that

$$D \cap \varphi^i(D) \neq \emptyset \text{ if } i = j_1 \text{ and } j_2 = 2j_1, \ldots, j_r = rj_1,\ d = (r+1)j_1.$$

By the same token we see that $D \cap \varphi^i(D) \neq \emptyset$ if and only if $i = j_s$ for some divisor s of d. Now $D \cap \varphi^i(D) = \emptyset$ for every $i \notin \{j_s, 2j_2, \ldots, d\}$. Hence $D, \varphi(D), \ldots, \varphi^{j_1-1}(D)$ are disjoint S-subacts. Since

$$D = \varphi^{j_1}(D) = \varphi^{j_2}(D) = \cdots = \varphi^{j_r}(D),$$

we conclude that B is the disjoint union, that is, *the coproduct*

$$B = D \coprod \varphi(D) \coprod \ldots \coprod \varphi^{j_1-1}(D).$$

By the minimality of B the disjoint components $D, \varphi(D), \ldots, \varphi^{j_1-1}(D)$ are strongly connected S-acts. Observe that $\varphi^{j_s}(a) \in D$ for every $a \in D$, but φ^i need not be the identical mapping. Thus, in Case 2, *the (S, φ)-subact B is a coproduct of j_1 S-subacts $D, \varphi(D), \cdots \cup \varphi^{j_1-1}(D)$, and each component is a strongly connected S-subact*.

Case 3. We can proceed as in Case 2, but now all the intersections contain the sink $\{0\}$ and so for any S-subact C of at least two elements and for

$$D = C \cap \varphi^{j_1}(C) \cap \cdots \cap \varphi^{j_r}(C)$$

the (S, φ)-subact B is the 0-disjoint union of the S-subacts $D, \varphi(D), \ldots, \varphi^{j_1-1}(D)$ and $D = Sa$ for every element $a \in D \setminus \{0\}$.

Case 4. Now χ_φ *is the Rees congruence determined by B.*

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HERMITE INTERPOLATING POLYNOMIALS

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Abstract. A method is described to construct Hermite polynomials fitting on to 2, 4, 8 points of a 1D, 2D, 3D functions, respectively, up to their rth derivatives.

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1. Introduction

The Hermite interpolation is a well-known theoretical tool for approximating functions. Although the theory and practice of Hermite interpolation is well developed for univariate functions (see, e. g., [1, 4, 6]), fewer results is known for the multivariate case (see, e. g., [3, 5]). We emphasize the importance of the work of Lorentz [3], which surveys the well-known theory on the multivariate case. Here we describe a computational method to construct Hermite polynomials fitting on to 2, 4, 8 points of a 1D, 2D, 3D functions, respectively, up to their rth derivatives.

2. One-dimensional polynomial

The aim of this section is to construct a polynomial $H_r(x)$ on to the points x_0 and $x_1$of the function $y(x)$ up to its rth derivatives.

$$\frac{d^m H_r(x_l)}{dx^m} = \frac{d^m y(x_l)}{dx^m}, \qquad l = 0, 1, \quad 0 \le m \le r.$$

Let us seek for it in the form

$$H_r(x) = \boldsymbol{A}_{r,1}\boldsymbol{X}_{r,1}(x) + \boldsymbol{A}_{r,2}C_r(x)\boldsymbol{X}_{r,2}(x). \tag{1}$$

The number of the prescribed conditions is $2r + 2$. For the sake of a convenient form of writing the vectors, we define the $\boldsymbol{e}_{r,j+1}$ unit vector $(0 \le j \le r)$ the $(j + 1)$th entry of which is equal to 1 and the rest are equal zero. Then the vectors in (1) can

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be written in the form

$$\boldsymbol{A}_{r,1} = \sum_{j=0}^{r} a_{j+1}\boldsymbol{e}_{r,j+1}, \qquad \boldsymbol{A}_{r,2} = \sum_{j=0}^{r} a_{r+2+j}\boldsymbol{e}_{r,j+1},$$

$$\boldsymbol{X}_{r,1}(x) = \sum_{j=0}^{r} (x-x_0)^j\,\boldsymbol{e}_{r,j+1}, \qquad \boldsymbol{X}_{r,2}(x) = \sum_{j=0}^{r} (x-x_1)^j\boldsymbol{e}_{r,j+1},$$

where $C_r(x) = (x-x_0)^{r+1}$. To determine the constants a, the derivatives of the function $H_r(x)$ must be calculated at the values x_0 and x_1,

$$\boldsymbol{A}_{r,1}\boldsymbol{X}_{r,1}(x) = \sum_{j=0}^{r} a_{j+1}(x-x_0)^j,$$

$$\boldsymbol{A}_{r,1}\frac{d^m\boldsymbol{X}_{r,1}(x)}{dx^m} = \sum_{j=0}^{r-m}\frac{(m+j)!}{j!}\,(x-x_0)^j\,a_{m+1+j},$$

Let us set $x_1 - x_0 = h$. Then

$$\boldsymbol{A}_{r,1}\frac{d^m\boldsymbol{X}_{r,1}(x_0)}{dx^m} = m!a_{m+1},$$

$$\boldsymbol{A}_{r,1}\frac{d^m\boldsymbol{X}_{r,1}(x_1)}{dx^m} = \sum_{j=0}^{r-m}\frac{(m+j)!}{j!}h^j a_{m+1+j}, \qquad 0 \le m \le r.$$

To compute the derivatives of the second term in $H_r(x)$, the following notation will be used: $f_r(x) = (x-x_0)^{r+1}$, $g_j(x) = (x-x_1)^j$,

$$\frac{d^t f_r(x)}{dx^t} = f_r^{(t)}(x) = \frac{(r+1)!}{(r+1-t)!}(x-x_0)^{r+1-t},$$

$f_r^{(r+1)}(x) = (r+1)!$, $f_r^{(t)}(x_0) = 0$ if $t < r+1$. Let $t = m-l$. Then $f_r^{(m-l)}(x_0) = 0$ if $r+1 > m-l$, and

$$f_r^{(m-l)}(x_1) = \frac{(r+1)!}{(r+1-m+l)!}h^{r+1-m+l}, \qquad m-l \le r+1;$$

moreover,

$$g_j^{(l)}(x) = \frac{j!}{(j-l)!}(x-x_1)^{j-l}, \quad l \le j, \quad g_j^{(j)}(x) = j!,$$

$$g_j^{(l)}(x_1) = \begin{cases} 0 & \text{if } l \ne j, \\ j! & \text{if } l = j. \end{cases}$$

The derivative of their product is

$$\frac{d^m}{dx^m}\left[f_r(x)g_j(x)\right] = \sum_{l=0}^{m}\binom{m}{l}f_r^{(m-l)}(x)g_j^{(l)}(x),$$

and $(d^m/dx^m)[f_r(x_0)g_j(x_0)] = 0$ for $0 \le m \le r$.

One has $g_j^{(l)}(x_1) = 0$ if $l \ne j$ and $g_j^{(j)} = j!$; therefore $m - l = m - j$, and then

$$f_r^{(m-j)}(x_1) = \frac{(r+1)!}{(r+1-m+j)!} h^{r+1-m+j}, \qquad 0 \le m \le r.$$

The derivative at the point x_1 will be

$$\frac{d^m}{dx^m}\left[f_r(x_1)g_j(x_1)\right] = \sum_{j=0}^{m} \frac{m!}{(m-j)!} \frac{(r+1)!}{(r+1-m+j)!} h^{r+1-m+j}, \qquad 0 \le m \le r,$$

and at x_0

$$\frac{d^m}{dx^m}\left[f_r(x_0)g_j(x_0)\right] = \sum_{l=l_{\min}}^{l_{\max}} \binom{m}{l} f_r^{(m-l)}(x_0) g_j^{(l)}(x_0).$$

If $m - l \ne r + 1$, then $f_r^{(m-l)}(x_0) = 0$ and $f_r^{(r+1)}(x_0) = (r+1)!$, so $m - l = r + 1$, $l = m - (r+1)$, and

$$g_j^{(m-(r+1))}(x_0) = \frac{j!}{(r+1-m+j)!} (-h)^{r+1-m+j},$$

whence

$$\frac{d^m}{dx^m}\left[f_r(x_0)g_j(x_0)\right] = \\ = \binom{m}{m-(r+1)} (r+1)! \sum_{j=m-(r+1)}^{r} \frac{j!}{(r+1-m+j)!} (-h)^{r+1-m+j}$$

for $r + 1 \le m \le 2r + 1$. The derivatives at the point x_1 are $g_j^{(l)}(x_1) = 0$ if $l \ne j$, therefore, with $m - l = m - j$, there should be

$$f_r^{(m-j)}(x_1) = \frac{(r+1)!}{(r+1-m+j)!} h^{r+1-m+j},$$

whence

$$\frac{d^m}{dx^m}\left[f_r(x_1)g_j(x_1)\right] = \ = m!(r+1)! \sum_{j=m-(r+1)}^{r} \frac{h^{r+1-m+j}}{(m-j)!(r+1-m+j)!}$$

for $r + 1 \le m \le 2r + 1$. Returning to the original notation, we get

$$C_r(x)\boldsymbol{A}_{r,2}\boldsymbol{X}_{r,2}(x) = \sum_{j=0}^{r} a_{r+2+j} f_r(x) g_j(x).$$

Summing up the results above, we get

$$\begin{aligned}
\boldsymbol{A}_{r,1}\frac{d^m\boldsymbol{X}_{r,1}(x_0)}{dx^m} &= m!a_{m+1}, && 0\le m\le r,\\
\boldsymbol{A}_{r,1}\frac{d^m\boldsymbol{X}_{r,1}(x_1)}{dx^m} &= \sum_{j=0}^{r-m}\frac{(m+j)!}{j!}\,h^j\,a_{m+1+j}, && 0\le m\le r,\\
\boldsymbol{A}_{r,1}\frac{d^m\boldsymbol{X}_{r,1}(x_0)}{dx^m} &= 0, && m>r,\\
\boldsymbol{A}_{r,2}\frac{d^m}{dx^m}\left[C_r(x_0)\boldsymbol{X}_{r,2}(x_0)\right] &= 0, && 0\le m\le r,\\
\boldsymbol{A}_{r,2}\frac{d^m}{dx^m}\left[C_r(x_1)\boldsymbol{X}_{r,2}(x_1)\right] &= m!(r+1)!\sum_{j=0}^{m}\frac{h^{r+1-m+j}}{(m-j)!(r+1-m+j)!}\,a_{r+2+j}
\end{aligned}$$

for $0\le m\le r$,

$$\begin{aligned}
\boldsymbol{A}_{r,2}\frac{d^m}{dx^m}\left[C_r(x_0)\boldsymbol{X}_{r,2}(x_0)\right] = \binom{m}{m-(r+1)}&(r+1)!\times\\
&\times\sum_{j=m-(r+1)}^{r}\frac{j!\,(-h)^{r+1-m+j}}{(r+1-m+j)!}a_{r+2+j}
\end{aligned}$$

for $r+1\le m\le 2r+1$, and

$$\boldsymbol{A}_{r,2}\frac{d^m}{dx^m}\left[C_r(x_1)\boldsymbol{X}_{r,2}(x_1)\right] = m!(r+1)!\sum_{j=m-(r+1)}^{r}\frac{h^{r+1-m+j}}{(m-j)!(r+1-m+j)!}a_{r+2+j}$$

for $r+1\le m\le 2r+1$. Finally, the derivatives of the Hermite polynomial are

$$\frac{d^mH_r(x)}{dx^m} = \boldsymbol{A}_{r,1}\frac{d^m\boldsymbol{X}_{r,1}}{dx^m} + \boldsymbol{A}_{r,2}\frac{d^m}{dx^m}\left[C_r(x)\boldsymbol{X}_{r,2}(x)\right].$$

Example 1. For $r=2$,

$$\begin{aligned}
H_2(x_0) &= a_1,\\
\frac{dH_2(x_0)}{dx} &= a_2,\\
\frac{d^2H_2(x_0)}{dx^2} &= a_3 2,\\
H_2(x_1) &= a_1 + a_2h + a_3h^2 + a_4h^3,\\
\frac{dH_2(x_1)}{dx} &= a_2 + a_3 2h + a_4 3h^2 + a_5h^3,\\
\frac{d^2H_2(x_1)}{dx^2} &= a_3 2 + a_4 6h + a_5 6h^2 + a_6 2h^3.
\end{aligned}$$

3. Two-dimensional polynomial

The aim of this section is to construct a polynomial $H_r(x, y)$ on to the points (x_0, y_0), (x_0, y_1), (x_1, y_0), (x_1, y_1), which fit the given values up to its rth derivatives. The conditions can be written as follows:

$$\frac{\partial^{m+n} H_r(x_s, y_t)}{\partial x^m \partial y^n} = \frac{\partial^{m+n} u(x_s, y_t)}{\partial x^m \partial y^n}$$

for $0 \le m, n \le r$, $0 \le m + n \le r$, $s, t = 0, 1$. The number of the prescribed conditions at one point is $\frac{1}{2}r(r + 3) + 1$, and in the four points is $2r^2 + 6r + 4$.

Let us seek for it in the form

$$H_r(x, y) = \boldsymbol{A}(x)\boldsymbol{Y}_{r,1}(y) + \boldsymbol{B}(x)D_r(y)\boldsymbol{Y}_{r,2}(y).$$

For the sake of a convenient form of writing the vectors, let us define the unit vector of $j + 1$ entries, $\boldsymbol{e}_{j,q+1}$ $(0 \le q \le j)$, the $(q + 1)$th entry of which is equal to 1 and the rest are equal to zero. Then the vectors in $H_r(x, y)$ can be written as follows:

$$\boldsymbol{A}(x) = \sum_{j=0}^{r} A_j(x)\boldsymbol{e}_{r,r+1-j}, \qquad \boldsymbol{B}(x) = \sum_{j=0}^{r} B_j(x)\boldsymbol{e}_{r,r+1-j},$$

$$\boldsymbol{Y}_{j,1}(y) = \sum_{i=0}^{j} (y - y_0)^i \boldsymbol{e}_{j,i+1}, \qquad \boldsymbol{Y}_{j,2}(y) = \sum_{i=0}^{j} (y - y_1)^i \boldsymbol{e}_{j,i+1},$$

where $D_j(y) = (y - y_0)^{j+1}$, $0 \le j \le r$. The functions $\boldsymbol{A}(x)$ and $\boldsymbol{B}(x)$ are composed as

$$A_j(x) = \boldsymbol{A}_{j,1}\boldsymbol{X}_{j,1}(x) + \boldsymbol{A}_{j,2}C_j(x)\boldsymbol{X}_{j,2}(x),$$
$$B_j(x) = \boldsymbol{B}_{j,1}\boldsymbol{X}_{j,1}(x) + \boldsymbol{B}_{j,2}C_j(x)\boldsymbol{X}_{j,2}(x),$$

where

$$\boldsymbol{X}_{j,1}(x) = \sum_{i=0}^{j} (x - x_0)^i \boldsymbol{e}_{j,i+1}, \qquad \boldsymbol{X}_{j,2}(x) = \sum_{i=0}^{j} (x - x_1)^i \boldsymbol{e}_{j,i+1},$$

and $C_j(x) = (x - x_0)^{j+1}$, $0 \le j \le r$. The distances between the coordinates are $x_1 - x_0 = h$, $y_1 - y_0 = g$.

For writing the vectors $\boldsymbol{A}_{j,1}$, $\boldsymbol{A}_{j,2}$, $\boldsymbol{B}_{j,1}$, $\boldsymbol{B}_{j,2}$ conveniently, we introduce the notation

$$l_j = (r - j)\frac{r + 3 + j}{2}, \qquad n_r = 1/2(r + 1)(r + 2), \qquad L_j = l_j + n_r.$$

With this notation, we get

$$\boldsymbol{A}_{j,1} = \sum_{i=0}^{j} a_{l_j+i+1}\boldsymbol{e}_{j,i+1}, \qquad \boldsymbol{A}_{j,2} = \sum_{i=0}^{j} a_{L_j+i+1}\boldsymbol{e}_{j,i+1},$$

$$\boldsymbol{B}_{j,1} = \sum_{i=0}^{j} b_{l_j+i+1}\boldsymbol{e}_{j,i+1}, \qquad \boldsymbol{B}_{j,2} = \sum_{i=0}^{j} b_{L_j+i+1}\boldsymbol{e}_{j,i+1}.$$

The number of the constants a, b in $\boldsymbol{A}(x)$, $\boldsymbol{B}(x)$ is $4n_r = 2r^2+6r+4$, which agrees with the number of conditions; consequently, $H_r(x,y)$ is the minimal degree polynomial satisfying the requirements.

The derivatives of the above vectors should be computed,

$$\frac{d^m \boldsymbol{X}_{j,1}(x)}{dx^m} = \sum_{q=0}^{j-m} \frac{(m+q)!}{q!}(x-x_0)^q \boldsymbol{e}_{j,m+1+q}, \qquad 0 \le m \le j,$$

$$\frac{d^t \boldsymbol{X}_{j,2}(x)}{dx^t} = \sum_{q=0}^{j-t} \frac{(t+q)!}{q!}(x-x_1)^q \boldsymbol{e}_{j,t+1+q}, \qquad 0 \le t \le j,$$

$$\frac{d^l C_j(x)}{dx^l} = C_j^{(l)}(x) = \frac{(j+1)!}{(j+1-l)!}(x-x_0)^{j+1-l}, \qquad 0 \le l \le j+1,$$

$$\frac{d^m}{dx^m}\left[C_j(x)\boldsymbol{X}_{j,2}(x)\right] = \sum_{l=l_{\min}}^{l_{\max}} \binom{m}{l} C_j^{(l)}(x) \boldsymbol{X}_{j,2}^{(m-l)}(x).$$

The limits of the sums are

$$l_{\min} = 0,\ l_{\max} = m \qquad \text{if } 0 \le m \le j,$$
$$l_{\min} = m-j,\ l_{\max} = j+1 \qquad \text{if } j+1 \le m \le 2j+1.$$

The derivatives of $A_j(x)$ and $B_j(x)$ are

$$\frac{d^m A_j(x)}{dx^m} = \boldsymbol{A}_{j,1}\frac{d^m \boldsymbol{X}_{j,1}(x)}{dx^m} + \boldsymbol{A}_{j,2}\frac{d^m}{dx^m}\left[C_j(x)\boldsymbol{X}_{j,2}(x)\right],$$
$$\frac{d^m B_j(x)}{dx^m} = \boldsymbol{B}_{j,1}\frac{d^m \boldsymbol{X}_{j,1}(x)}{dx^m} + \boldsymbol{B}_{j,2}\frac{d^m}{dx^m}\left[C_j(x)\boldsymbol{X}_{j,2}(x)\right].$$

To compute the coefficients a and b at the points x_0 and x_1, the following formulas are used:

$$\frac{d^m A_j(x_0)}{dx^m} = m! a_{l_j+m+1}, \qquad 0 \le m \le j,$$

$$\frac{d^m A_j(x_1)}{dx^m} = \sum_{q=0}^{j-m} \frac{(m+q)!}{q!} h^q a_{l_j+m+1+q} + \\ + m! \sum_{l=0}^{m} \frac{(j+1)!}{l!\,(j+1-l)!} h^{j+1-l} a_{L_j+m+1-l}, \qquad 0 \le m \le j,$$

$$\frac{d^m A_j(x_0)}{dx^m} = \frac{m!}{[m-(j+1)]!} \times \\ \times \sum_{q=0}^{2j+1-m} \frac{[m-(j+1)+q]!}{q!}(-h)^q a_{L_j+m-j+q},$$

$$j+1 \le m \le 2j+1,$$

$$\frac{d^m A_j(x_1)}{dx^m} = m! \sum_{l=m-j}^{j+1} \frac{(j+1)!}{(j+1-l)!} h^{j+1-l} a_{L_j+m+1-l}$$

$$j+1 \le m \le 2j+1.$$

Analogous formulas are valid for $B_j(x)$. The derivatives of the Hermite polynomial are

$$\frac{\partial^{m+n} H_r(x,y)}{\partial x^m \partial y^n} = \frac{d^m \boldsymbol{A}(x)}{dx^m} \frac{d^n \boldsymbol{Y}_{r,1}(y)}{dy^n} + \frac{d^m \boldsymbol{B}(x)}{dx^m} \frac{d^n}{dy^n} \left[D_r(y) \boldsymbol{Y}_{r,2}(y) \right].$$

The following formulas are also needed:

$$\frac{d^n \boldsymbol{Y}_{j,1}(y)}{dy^n} = \sum_{s=0}^{j-n} \frac{(n+s)!}{s!} (y-y_0)^s \boldsymbol{e}_{j,n+1+s}, \qquad 0 \le n \le j,$$

$$\frac{d^t \boldsymbol{Y}_{j,2}(y)}{dy^t} = \sum_{s=0}^{j-t} \frac{(t+s)!}{s!} (y-y_1)^s \boldsymbol{e}_{j,t+1+s}, \qquad 0 \le t \le j,$$

$$\frac{d^l D_j(y)}{dy^l} = D_j^{(l)}(y) = \frac{(j+1)!}{(j+1-l)!} (y-y_0)^{j+1-l}, \qquad 0 \le l \le j+1.$$

Thus, we can write

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} \left[\boldsymbol{A}(x) \boldsymbol{Y}_{r,1}(y) \right] = \sum_{j=0}^{r-n} A_j(x) \frac{(r-j)!}{(r-n-j)!} (y-y_0)^{r-n-j}, \tag{2}$$

and the values of these are

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} \left[\boldsymbol{A}(x_0) \boldsymbol{Y}_{r,1}(y_0) \right] = n! \frac{d^m A_{r-n}(x_0)}{dx^m}, \tag{3}$$

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} \left[\boldsymbol{A}(x_0) \boldsymbol{Y}_{r,1}(y_1) \right] = \sum_{j=0}^{r-n} \frac{d^m A_j(x_0)}{dx^m} \frac{(r-j)!}{(r-n-j)!} g^{r-n-j}, \tag{4}$$

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} \left[\boldsymbol{A}(x_1) \boldsymbol{Y}_{r,1}(y_0) \right] = n! \frac{d^m A_{r-n}(x_1)}{dx^m}, \tag{5}$$

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} \left[\boldsymbol{A}(x_1) \boldsymbol{Y}_{r,1}(y_1) \right] = \sum_{j=0}^{r-n} \frac{d^m A_j(x_1)}{dx^m} \frac{(r-j)!}{(r-n-j)!} g^{r-n-j} \tag{6}$$

for $0 \le n \le r$, $0 \le m \le 2r+1$.

To compute the derivatives of the second term in $H_r(x,z)$, the next *interim* notation will be used: $f_r(y) = D_r(y)$, $g_j(y) = (y-y_1)^{r-j}$. The derivatives of these are

$$f_r^{(l)}(y) = \frac{(r+1)!}{(r+1-l)!} (y-y_0)^{r+1-l}, \qquad 0 \le l \le r+1,$$

$f_r^{(r+1)}(y) = (r+1)!$, $f_r^{(l)}(y_0) = 0$ if $0 \le l < r+1$, $f_r^{(l)}(y) = 0$ if $l > r+1$, $g_j^{(t)}(y) = (r-j)!((r-j-t)!)^{-1}(y-y_1)^{r-j-t}$,

$$g_j^{(t)}(y) = \begin{cases} (r-j)! & \text{if } t = r-j, \\ 0 & \text{if } t \ne r-j, \end{cases}$$

$$\frac{d^n f_r(y)}{dy^n} = \sum_{l=l_{\min}}^{l_{\max}} \binom{n}{l} f_r^{(l)}(y) g_j^{(n-l)}(y),$$

$l_{\min} = \max\{n-(r-j),\ 0\}$, $l_{\max} = n$ if $r-n \le j \le r$ and $0 \le n \le r$; $l_{\min} = n+j-r$, $l_{\max} = r+1$ if $0 \le j \le 2r+1-n$, $r+1 \le n \le 2r+1$. It is easy to see that

$$\frac{d^n}{dy^n}\left[f_r(y_0)g_j(y_0)\right] = 0, \qquad 0 \le n \le r,$$

$$\frac{d^n}{dy^n}\left[f_r(y_1)g_j(y_1)\right] = \binom{n}{n+j-r}\frac{(r+1)!(r-j)!}{(2r+1-n-j)!}g^{2r+1-n-j}, \quad 0 \le n \le r,\ r-n \le j \le r,$$

$$\frac{d^n}{dy^n}\left[f_r(y_0)g_j(y_0)\right] = \binom{n}{r+1}\frac{(r+1)!(r-j)!}{(2r+1-n-j)!}(-g)^{2r+1-n-j}$$

for $r+1 \le n \le 2r+1$, $0 \le j \le 2r+1-n$,

$$\frac{d^n}{dy^n}\left[f_r(y_1)g_j(y_1)\right] = \binom{n}{n+j-r}\frac{(r+1)!(r-j)!}{(2r+1-n-j)!}g^{2r+1-n-j}$$

for $r+1 \le n \le 2r+1$, $0 \le j \le 2r+1-n$. The derivatives of the second term in $H_r(x,y)$ are

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n}\left[\boldsymbol{B}(x)\, D_r(y)\boldsymbol{Y}_{r,2}(y)\right] = \sum_{j=0}^{r} \frac{d^m B_j(x)}{dx^m}\,\frac{d^n}{dy^n}\left[f_r(y)g_j(y)\right],$$

and their values of at x, y_0 and y_1 are

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n}\left[\boldsymbol{B}(x)\, D_r(y_0)\boldsymbol{Y}_{r,2}(y_0)\right] = 0 \tag{7}$$

for $0 \le m \le 2r+1$, $0 \le n \le r$;

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n}\left[\boldsymbol{B}(x)\, D_r(y_1)\boldsymbol{Y}_{r,2}(y_1)\right] = (r+1)! \sum_{j=r-n}^{r} \frac{d^m B_j(x)}{dx^m} \times \\ \times \binom{n}{n+j-r}\frac{(r-j)!}{(2r+1-n-j)!}g^{2r+1-n-j} \tag{8}$$

for $0 \le m \le 2r+1$, $0 \le n \le r$;

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n}\left[\boldsymbol{B}(x)\, D_r(y_0)\boldsymbol{Y}_{r,2}(y_0)\right] = (r+1)!\binom{n}{r+1}\sum_{j=0}^{2r+1-n}\frac{d^m B_j(x)}{dx^m}\times \\ \times\frac{(r-j)!}{(2r+1-n-j)!}(-g)^{2r+1-n-j} \tag{9}$$

for $0 \le m \le 2r+1$, $r+1 \le n \le 2r+1$;

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n}\left[\boldsymbol{B}(x)\, D_r(y_1)\boldsymbol{Y}_{r,2}(y_1)\right] = (r+1)!\sum_{j=0}^{2r+1-n}\frac{d^m B_j(x)}{dx^m}\binom{n}{n+j-r}\times \\ \times\frac{(r+1)!}{(2r+1-n-j)}g^{2r+1-n-j} \tag{10}$$

for $0 \le m \le 2r+1$, $r+1 \le n \le 2r+1$. Applying the formulas (2), (3), (4), (5), (6) and (7), (8), (9), (10), the derivatives of $H_r(x,y)$ can be calculated and upon these the coefficients a, b can be determined.

Example 2 ($r = 2$).

$$\begin{aligned}
H_2(x_0,y_0) &= \underline{a}_1,\\
H_2^{(x)}(x_0,y_0) &= \underline{a}_2,\\
H_2^{(x,x)}(x_0,y_0) &= \underline{a}_3 2,\\
H_2^{(y)}(x_0,y_0) &= \underline{a}_4,\\
H_2^{(x,y)}(x_0,y_0) &= \underline{a}_5,\\
H_2^{(y,y)}(x_0,y_0) &= \underline{a}_6 2,\\
H_2(x_1,y_0) &= a_1 + a_2 h + a_3 h^2 + \underline{a}_7 h^3,\\
H_2^{(x)}(x_1,y_0) &= a_2 + a_3 2h + a_7 3h^2 + \underline{a}_8 h^3,\\
H_2^{(x,x)}(x_1,y_0) &= a_3 2 + a_7 6h + a_8 6h^2 + \underline{a}_9 2h^3,\\
H_2^{(y)}(x_1,y_0) &= a_4 + a_5 h + \underline{a}_{10} h^2,\\
H_2^{(x,y)}(x_1,y_0) &= a_5 + a_{10} 2h + \underline{a}_{11} h^2,\\
H_2^{(y,y)}(x_1,y_0) &= a_6 2 + \underline{a}_{12} 2h,\\
H_2(x_0,y_1) &= a_1 + a_4 g + a_6 g^2 + \underline{b}_1 g^3,\\
H_2^{(x)}(x_0,y_1) &= a_2 + a_5 g + a_{12} g^2 + \underline{b}_2 g^3,\\
H_2^{(x,x)}(x_0,y_1) &= a_3 2 + a_{10} 2g - a_{11} 2hg + \underline{b}_3 2g^3,\\
H_2^{(y)}(x_0,y_1) &= a_4 + a_6 2g + b_1 3g^2 + \underline{b}_4 g^3,
\end{aligned}$$

$$
\begin{aligned}
H_2^{(x,y)}(x_0,y_1) &= a_5 + a_{12}2g + b_2 3g^2 + \underline{b}_5 2g^3,\\
H_2^{(y,y)}(x_0,y_1) &= a_6 2 + b_1 6g + b_4 6g^2 + \underline{b}_6 2g^3,\\
H_2(x_1,y_1) &= a_1 + a_2 h + a_3 h^2 + a_4 g + a_5 hg + a_6 g^2 + a_7 h^3 + a_{10} h^2 g + a_{12} hg^2 +,\\
&\quad + b_1 g^3 + b_2 hg^3 + b_3 h^2 g^3 + \underline{b}_7 h^3 g^3,\\
H_2^{(x)}(x_1,y_1) &= a_2 + a_3 2h + a_5 g + a_7 3h^2 + a_8 h^3 + a_{10} 2hg + a_{11} h^2 g + a_{12} g^2 +,\\
&\quad + b_2 g^3 + b_3 2hg^3 + b_7 3h^2 g^3 + \underline{b}_8 h^3 g^3,\\
H_2^{(x,x)}(x_1,y_1) &= a_3 2 + a_7 6h + a_8 6h^2 + a_9 2h^3 + a_{10} 2g + a_{11} 4hg + b_3 2g^3 + b_7 6hg^3 +,\\
&\quad + b_8 6h^2 g^3 + \underline{b}_9 2h^3 g^3,\\
H_2^{(y)}(x_1,y_1) &= a_4 + a_5 h + a_6 2g + a_{10} h^2 + a_{12} 2hg + b_1 3g^2 + b_2 3hg^2 + b_3 3h^2 g^2 +,\\
&\quad + b_4 g^3 + b_5 hg^3 + b_7 3h^3 g^2 + \underline{b}_{10} h^2 g^3,\\
H_2^{(x,y)}(x_1,y_1) &= a_5 + a_{10} 2h + a_{11} h^2 + a_{12} 2g + b_2 3g^2 + b_3 6hg^2 + b_5 g^3 + b_7 9h^2 g^2 +,\\
&\quad + b_8 3h^3 g^2 + b_{10} 2hg^3 + \underline{b}_{11} h^2 g^3,\\
H_2^{(y,y)}(x_1,y_1) &= a_6 2 + a_{12} 2h + b_1 6g + b_2 6hg + b_3 6h^2 g + b_4 6g^2 + b_5 6hg^2 + b_6 2g^3 +,\\
&\quad + b_7 6h^3 g + b_{10} 6h^2 g^2 + \underline{b}_{12} 2hg^3.
\end{aligned}
$$

4. Three-dimensional polynomial

The aim of this section is to construct a polynomial $H_r(x,y,z)$ which fits the given values on to the coordinate points $x_0, x_1, y_0, y_1, z_0, z_1$ up to its rth derivatives. The conditions can be written as follows:

$$\frac{\partial^{m+n+p} H_r(x_l, y_s, z_t)}{\partial x^m \partial y^n \partial z^p} = \frac{\partial^{m+n+p} u(x_l, y_s, z_t)}{\partial x^m \partial y^n \partial z^p}$$

for $l, s, t = 0, 1$; $0 \le m, n, p \le r$; $0 \le m+n+p \le r$. The number of the jth derivatives of a function having three independent variables is the number of conditions at one point, $s_j = \frac{1}{2}(j^2 + 3j) + 1$; up to its rth derivatives, the number is equal to $S_r = \sum_{j=0}^{r} s_j = \frac{1}{6}(r^3 + 6r^2 + 11r + 6)$, and for the eight points we have $8S_r$.

Let us seek for it in the form

$$H_r(x,y,z) = \boldsymbol{A}(x,z)\boldsymbol{Y}_{r,1}(y) + \boldsymbol{B}(x,z)D_r(y)\boldsymbol{Y}_{r,2}(y).$$

For the sake of a convenient form of writing the vectors, let us define the unit vector of $j+1$ entries $\boldsymbol{e}_{j,q+1}$ $(0 \le q \le j)$ the $q+1$-*th* entry of which is 1, and all the

rest are zero. Then the vectors in $H_r(x,y,z)$ can be written by using the formulas

$$\boldsymbol{Y}_{j,1}(y)=\sum_{i=0}^{j}(y-y_0)^i\,\boldsymbol{e}_{j,i+1},\qquad \boldsymbol{Y}_{j,2}(y)=\sum_{i=0}^{j}(y-y_1)^i\boldsymbol{e}_{j,i+1},$$

$$D_j(y)=(y-y_0)^{j+1},\quad 0\le j\le r.$$

$$\boldsymbol{A}(x,z)=\sum_{j=0}^{r}A_j(x,z)\boldsymbol{e}_{r,r+1-j},\qquad \boldsymbol{B}(x,z)=\sum_{j=0}^{r}B_j(x,z)\boldsymbol{e}_{r,r+1-j},$$

$$A_j(x,z)=\boldsymbol{A}_{j,1}(z)\boldsymbol{X}_{j,1}(x)+\boldsymbol{A}_{j,2}(z)C_j(x)\,\boldsymbol{X}_{j,2}(x),\quad 0\le j\le r,$$

$$B_j(x,z)=\boldsymbol{B}_{j,1}(z)\boldsymbol{X}_{j,1}(x)+\boldsymbol{B}_{j,2}(z)C_j(x)\,\boldsymbol{X}_{j,2}(x),\quad 0\le j\le r,$$

$$\boldsymbol{X}_{j,1}(x)=\sum_{i=0}^{j}(x-x_0)^i\,\boldsymbol{e}_{j,i+1},\qquad \boldsymbol{X}_{j,2}(x)=\sum_{i=0}^{j}(x-x_1)^i\boldsymbol{e}_{j,i+1},$$

$$C_j(x)=(x-x_0)^{j+1},\quad 0\le j\le r,$$

$$\boldsymbol{A}_{j,1}(z)=\sum_{q=0}^{j}A_{j,1,q}(z)\boldsymbol{e}_{j,j+1-q},\qquad \boldsymbol{A}_{j,2}(z)=\sum_{q=0}^{j}A_{j,2,q}(z)\boldsymbol{e}_{j,j+1-q},$$

$$\boldsymbol{B}_{j,1}(z)=\sum_{q=0}^{j}B_{j,1,q}(z)\,\boldsymbol{e}_{j,j+1-q},\qquad \boldsymbol{B}_{j,2}(z)=\sum_{q=0}^{j}B_{j,2,q}(z)\boldsymbol{e}_{j,j+1-q},$$

$$A_{j,1,q}(z)=\boldsymbol{\alpha}_{j,1,2k}\boldsymbol{Z}_{q,1}(z)+\boldsymbol{\alpha}_{j,1,2k+1}E_j(z)\boldsymbol{Z}_{q,2}(z),\quad 0\le q\le j,$$

$$A_{j,2,q}(z)=\boldsymbol{\alpha}_{j,2,2k}\boldsymbol{Z}_{q,1}(z)+\boldsymbol{\alpha}_{j,2,2k+1}E_j(z)\boldsymbol{Z}_{q,2}(z),\quad 0\le q\le j,$$

$$B_{j,1,q}(z)=\boldsymbol{\beta}_{j,1,2k}\boldsymbol{Z}_{q,1}(z)+\boldsymbol{\beta}_{j,1,2k+1}E_j(z)\boldsymbol{Z}_{q,2}(z),\quad 0\le q\le j,$$

$$B_{j,2,q}(z)=\boldsymbol{\beta}_{j,2,2k}\boldsymbol{Z}_{q,1}(z)+\boldsymbol{\beta}_{j,2,2k+1}E_j(z)\boldsymbol{Z}_{q,2}(z),\quad 0\le q\le j,$$

and $\boldsymbol{Z}_{q,1}(z)=\sum_{i=0}^{q}(z-z_0)^i\,\boldsymbol{e}_{q,i+1}$, $\boldsymbol{Z}_{q,2}(z)=\sum_{i=0}^{q}(z-z_1)^i\,\boldsymbol{e}_{q,i+1}$, $E_q(z)=(z-z_0)^{q+1}$, $0\le q\le j$. The vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are composed by constant components a and b. To give the connection, the following notation will be applied:

$$l_{r,q}=(r-q)\frac{r+3+q}{2},\qquad l_{j,q}=(j-q)\frac{r+3+q}{2}+\sum_{i=j+1}^{r}n_i,$$

$$n_i=\frac{1}{2}(i+1)(i+2),\qquad N_r=\sum_{i=0}^{r}n_i=\frac{1}{6}\left(r^3+6r^2+11r+6\right)$$

$$L_{r,q}=2N_r+l_{r,q},\qquad L_{j,q}=2N_r+l_{j,q},\quad k=j-q;$$

$$\boldsymbol{\alpha}_{j,1,2k}=\sum_{i=0}^{q}a_{l_{j,q}+i+1}\boldsymbol{e}_{q,i+1},\qquad \boldsymbol{\alpha}_{j,1,2k+1}=\sum_{i=0}^{q}a_{N_r+l_{j,q}+i+1}\boldsymbol{e}_{q,i+1},$$

$$\boldsymbol{\alpha}_{j,2,2k} = \sum_{i=0}^{q} a_{L_{j,q}+i+1}\boldsymbol{e}_{q,i+1}, \qquad \boldsymbol{\alpha}_{j,2,2k+1} = \sum_{i=0}^{q} a_{N_r+L_{j,q}+i+1}\boldsymbol{e}_{q,i+1}.$$

The relations between $\boldsymbol{\beta}$ and b are analogous to the previously given formulas where a is replaced by b.

The number of the constants a, b in $\boldsymbol{A}_{j,1}$, $\boldsymbol{A}_{j,2}$, $\boldsymbol{B}_{j,1}$, $\boldsymbol{B}_{j,2}$ is $8N_r$, which agrees with the number of conditions $8N_r = 8S_r$. Consequently, $H_r(x, y, z)$ is the minimal degree polynomial satisfying the requirements.

The distances between the coordinates are$x_1 - x_0 = h$, $y_1 - y_0 = g$, $z_1 - z_0 = l$.The derivatives of the $H_r(x, y, z)$ function are needed to compute the coefficients a and b (rather tedious calculations are omitted here to save space):

$$\frac{d^p A_{j,1,q}(z_0)}{dz^p} = p! a_{l_{j,q}+p+1}, \quad 0 \le p \le q, \tag{11}$$

$$\frac{d^p A_{j,1,q}(z_1)}{dz^p} = \sum_{i=0}^{q-p} \frac{(p+i)!}{i!} l^i a_{l_{j,q}+p+i+1} + p!\,(q+1)! \sum_{i=0}^{p} \frac{l^{q+1-p+i} a_{N_r+l_{j,q}+i+1}}{(p-i)!(q+1-p+i)!}, \quad 0 \le p \le q,$$

$$\frac{d^p A_{j,1,q}(z_0)}{dz^p} = \binom{p}{p-(q+1)}(q+1)! \sum_{i=p-(q+1)}^{q} \frac{i!\,(-l)^{q+1-p+i}}{(q+1-p+i)!} a_{N_r+l_{j,q}+i+1}, \tag{12}$$

$$\frac{d^p A_{j,1,q}(z_1)}{dz^p} = p!(q+1)! \sum_{i=p-(q+1)}^{q} \frac{l^{q+1-p+i}}{(p-i)!(q+1-p+i)!} a_{N_r+l_{j,q}+i+1}, \quad q+1 \le p \le 2q+1.$$

The derivatives of (11) are similar to the previous ones,

$$\frac{d^p A_{j,2,q}(z_0)}{dz^p} = p! a_{L_{j,q}+p+1}, \tag{13}$$

$$\frac{d^p A_{j,2,q}(z_1)}{dz^p} = \sum_{i=0}^{q-p} \frac{(p+i)!}{i!} l^i a_{L_{j,q}+p+i+1} + p!\,(q+1)! \sum_{i=0}^{p} \frac{l^{q+1-p+i} a_{N_r+L_{j,q}+i+1}}{(p-i)!\,(q+1-p+i)!}, \quad 0 \le p \le q,$$

$$\frac{d^p A_{j,2,q}(z_0)}{dz^p} = \binom{p}{p-(q+1)}(q+1)! \sum_{i=p-(q+1)}^{q} \frac{i!\,(-l)^{q+1-p+i}\, a_{N_r+L_{j,q}+i+1}}{(q+1-p+i)!}, \tag{14}$$

$$\frac{d^p A_{j,2,q}(z_1)}{dz^p} = p!\,(q+1)! \sum_{i=p-(q+1)}^{q} \frac{l^{q+1-p+i} a_{N_r+L_{j,q}+i+1}}{(p-i)!\,(q+1-p+i)!}, \quad q+1 \le p \le 2q+1.$$

To obtain similar formulas for $B_{j,1,q}$ and $B_{j,2,q}$, a should be replaced by b.

For the derivatives of $A_j(x,z)$, expressions (11), (12), (13), (14) are to be used,

$$\frac{\partial^{m+p}}{\partial x^m \partial z^p}\left[\boldsymbol{A}_{j,1}(z)\boldsymbol{X}_{j,1}(x_0)\right] = 0 \quad \text{for } m \neq j-q, \tag{15}$$

$$\frac{\partial^{m+p}}{\partial x^m \partial z^p}\left[\boldsymbol{A}_{j,1}(z)\boldsymbol{X}_{j,1}(x_0)\right] = \frac{d^p A_{j,1,j-m}(z)}{dz^p} m!,$$

$$\frac{\partial^{m+p}}{\partial x^m \partial z^p}\left[\boldsymbol{A}_{j,1}(z)\boldsymbol{X}_{j,1}(x_1)\right] = \sum_{q=0}^{j-m} \frac{d^p A_{j,1,q}(z)}{dz^p} \frac{(j-q)!\, h^{j-q-m}}{(j-q-m)!}, \quad 0 \le m \le j;$$

$$\frac{\partial^{m+p}}{\partial x^m \partial z^p}\left[\boldsymbol{A}_{j,2}(z)C_j(x_0)\boldsymbol{X}_{j,2}(x_0)\right] = 0, \quad 0 \le m \le j, \tag{16}$$

$$\frac{\partial^{m+p}}{\partial x^m \partial z^p}\left[\boldsymbol{A}_{j,2}(z)C_j(x_1)\boldsymbol{X}_{j,2}(x_1)\right] = \sum_{q=j-m}^{j} \frac{d^p A_{j,2,q}(z)}{dz^p} \times$$

$$\times \frac{m!\,(j+1)!\, h^{2j+1-m-q}}{(m-(j-q))!(2j+1-m-q)!}, \quad 0 \le m \le j,\ j-q < m.$$

$$\frac{\partial^{m+p}}{\partial x^m \partial z^p}\left[\boldsymbol{A}_{j,2}(z)C_j(x_0)\boldsymbol{X}_{j,2}(x_0)\right] = \frac{(m)!}{(m-(j+1))!} \times \tag{17}$$

$$\times \sum_{q=0}^{2j+1-m} \frac{d^p A_{j,2,q}(z)}{dz^p} \frac{(j-q)!\,(-h)^{2j+1-m-q}}{(2j+1-m-q)!},$$

$$\frac{\partial^{m+p}}{\partial x^m \partial z^p}\left[\boldsymbol{A}_{j,2}(z)C_j(x_1)\boldsymbol{X}_{j,2}(x_1)\right] = m!(j+1)! \times$$

$$\times \sum_{q=0}^{2j+1-m} \frac{d^p A_{j,2,q}(z)}{dz^p} \frac{h^{2j+1-m-q}}{(m-(j-q))!(2j+1-m-q)},$$

where $j+1 \le m \le 2j+1$. The derivatives of $B_j(x,z)$ will be obtained by replacing A by B in formulas (15), (16), and (17).

Finally, the derivatives of the first and second terms in the Hermite polynomials are to be given, and these are as follows:

$$\frac{\partial^{m+n+p}}{\partial x^m \partial y^n \partial z^p}\left[\boldsymbol{A}(x,z)\boldsymbol{Y}_{r,1}(y_0)\right] = n! \frac{\partial^{m+p} A_{r-n}(x,z)}{\partial x^m \partial z^p},$$

$$\frac{\partial^{m+n+p}}{\partial x^m \partial y^n \partial z^p}\left[\boldsymbol{A}(x,z)\boldsymbol{Y}_{r,1}(y_1)\right] = \sum_{j=0}^{r-n} \frac{\partial^{m+p} A_j(x,z)}{\partial x^m \partial z^p} \frac{(r-j)! g^{r-j-n}}{(r-j-n)!}, \quad 0 \le n \le r;$$

$$\frac{\partial^{m+n+p}}{\partial x^m \partial y^n \partial z^p}\left[\boldsymbol{B}(x,z)D_r(y_0)\boldsymbol{Y}_{r,2}(y_0)\right] = 0, \quad 0 \le n \le r,$$

$$\frac{\partial^{m+n+p}}{\partial x^m \partial y^n \partial z^p}\left[\boldsymbol{B}(x,z)D_r(y_1)\boldsymbol{Y}_{r,2}(y_1)\right] = (r+1)!\sum_{j=r-n}^{r}\frac{\partial^{m+p}B_j(x,z)}{\partial x^m \partial z^p}\times$$
$$\times\binom{n}{n+j-r}\frac{(r-j)!\,g^{2r+1-n-j}}{(2r+1-n-j)!},\quad 0\le n\le r;$$
$$\frac{\partial^{m+n+p}}{\partial x^m \partial y^n \partial z^p}\left[\boldsymbol{B}(x,z)D_r(y_0)\boldsymbol{Y}_{r,2}(y_0)\right] = \binom{n}{r+1}(r+1)!\sum_{j=0}^{2r+1-n}\frac{\partial^{m+p}B_j(x,z)}{\partial x^m \partial z^p}\times$$
$$\times\frac{(r-j)!(-g)^{2r+1-n-j}}{(2r+1-n-j)!},$$
$$\frac{\partial^{m+n+p}}{\partial x^m \partial y^n \partial z^p}\left[\boldsymbol{B}(x,z)D_r(y_1)\boldsymbol{Y}_{r,2}(y_1)\right] = (r+1)!\sum_{j=0}^{2r+1-n}\frac{\partial^{m+p}B_j(x,z)}{\partial x^m \partial z^p}\binom{n}{n+j-r}\times$$
$$\times\frac{(r-j)!\,g^{2r+1-n-j}}{(2r+1-n-j)!},\quad r+1\le n\le 2r+1.$$

Example 3 ($r=2$). In order to save space, we show below only the first 40 coefficients:

$$H_2(x_0,y_0,z_0) = \underline{a}_1,$$
$$H_2^{(z)}(x_0,y_0,z_0) = \underline{a}_2,$$
$$H_2^{(z,z)}(x_0,y_0,z_0) = \underline{a}_3 2,$$
$$H_2^{(x)}(x_0,y_0,z_0) = \underline{a}_4,$$
$$H_2^{(x,z)}(x_0,y_0,z_0) = \underline{a}_5,$$
$$H_2^{(x,x)}(x_0,y_0,z_0) = \underline{a}_6 2,$$
$$H_2^{(y)}(x_0,y_0,z_0) = \underline{a}_7,$$
$$H_2^{(y,z)}(x_0,y_0,z_0) = \underline{a}_8,$$
$$H_2^{(x,y)}(x_0,y_0,z_0) = \underline{a}_9,$$
$$H_2^{(y,y)}(x_0,y_0,z_0) = \underline{a}_{10} 2,$$
$$H_2(x_0,y_0,z_1) = a_1 + a_2 l + a_3 l^2 + \underline{a}_{11} l^3,$$
$$H_2^{(z)}(x_0,y_0,z_1) = a_2 + a_3 2l + a_{11} 3l^2 + \underline{a}_{12} l^3,$$
$$H_2^{(z,z)}(x_0,y_0,z_1) = a_3 2 + a_{11} 6l + a_{12} 6l^2 + \underline{a}_{13} 2l^3,$$
$$H_2^{(x)}(x_0,y_0,z_1) = a_4 + a_5 l + \underline{a}_{14} l^2,$$
$$H_2^{(x,z)}(x_0,y_0,z_1) = a_5 + a_{14} 2l + \underline{a}_{15} l^2,$$
$$H_2^{(x,x)}(x_0,y_0,z_1) = a_6 2 + \underline{a}_{16} 2l,$$

$$
\begin{aligned}
H_2^{(y)}(x_0,y_0,z_1) &= a_7 + a_8 l + \underline{a}_{17} l^2,\\
H_2^{(y,z)}(x_0,y_0,z_1) &= a_8 + a_{17} 2l + \underline{a}_{18} l^2,\\
H_2^{(x,y)}(x_0,y_0,z_1) &= a_9 + \underline{a}_{19} l,\\
H_2^{(y,y)}(x_0,y_0,z_1) &= a_{10} 2 + \underline{a}_{20} 2l,\\
H_2(x_1,y_0,z_0) &= a_6 h^2 + a_4 h + a_1 + \underline{a}_{21} h^3,\\
H_2^{(z)}(x_1,y_0,z_0) &= a_{16} h^2 + a_5 h + a_2 + \underline{a}_{22} h^3,\\
H_2^{(z,z)}(x_1,y_0,z_0) &= (a_{14} 2 - a_{15} 2l)\, h + a_3 2 + \underline{a}_{23} 2h^3,\\
H_2^{(x)}(x_1,y_0,z_0) &= a_6 2h + a_4 + a_{21} 3h^2 + \underline{a}_{24} h^3,\\
H_2^{(x,z)}(x_1,y_0,z_0) &= a_{16} 2h + a_5 + a_{22} 3h^2 + \underline{a}_{25} h^3,\\
H_2^{(x,x)}(x_1,y_0,z_0) &= a_6 2 + a_{21} 6h + a_{24} 6h^2 + \underline{a}_{26} h^3,\\
H_2^{(y)}(x_1,y_0,z_0) &= a_9 h + a_7 + \underline{a}_{27} h^2,\\
H_2^{(y,z)}(x_1,y_0,z_0) &= a_{19} h + a_8 + \underline{a}_{28} h^2,\\
H_2^{(x,y)}(x_1,y_0,z_0) &= a_9 + a_{27} 2h + \underline{a}_{29} h^2,\\
H_2^{(y,y)}(x_1,y_0,z_{01}) &= \left(a_{10} + \underline{a}_{30} h\right) 2,\\
H_2(x_1,y_0,z_1) &= a_1 + a_2 l + a_3 l^2 + a_{11} l^3 + \left(a_4 + a_5 l + a_{14} l^2\right) h + (a_6 + a_{16} l)\, h^2 +,\\
&\quad + \left(a_{21} + a_{22} l + a_{23} l^2 + \underline{a}_{31} l^3\right) h^3,\\
H_2^{(z)}(x_1,y_0,z_1) &= a_2 + a_3 2l + a_{11} 3l^2 + a_{12} l^3 + a_{16} h^2 + \left(a_5 + a_{14} 2l + a_{15} l^2\right) h +,\\
&\quad + \left(a_{22} + a_{23} 2l + a_{31} 3l^2 + \underline{a}_{32} l^3\right) h^3,\\
H_2^{(z,z)}(x_1,y_0,z_1) &= a_3 2 + a_{11} 6l + a_{12} 6l^2 + a_{13} 2l^3 + (a_{14} 2 + a_{15} 4l)\, h +,\\
&\quad + \left(a_{23} 2 + a_{31} 6l + a_{32} 6l^2 + \underline{a}_{33} 2l^3\right) h^3,\\
H_2^{(x)}(x_1,y_0,z_1) &= a_4 + a_5 l + a_{14} l^2 + (a_6 + a_{16} l)\, 2h +,\\
&\quad \left(a_{21} + a_{22} l + a_{23} l^2 + a_{31} l^3\right) 3h^2 + \left(a_{24} + a_{25} l + \underline{a}_{34} l^2\right) h^3,\\
H_2^{(x,z)}(x_1,y_0,z_1) &= a_5 + a_{14} 2l + a_{15} l^2 + a_{16} 2h +,\\
&\quad + \left(a_{22} + a_{23} 2l + a_{31} 3l^2 + a_{32} l^3\right) 3h^2 + \left(a_{25} + a_{34} 2l + \underline{a}_{35} l^2\right) h^3,\\
H_2^{(x,x)}(x_1,y_0,z_1) &= a_6 2 + a_{16} 2l + \left(a_{24} + a_{25} l + a_{34} l^2\right) 6h^2 +,\\
&\quad + \left(a_{21} + a_{22} l + a_{23} l^2 + a_{31} l^3\right) 6h + \left(a_{26} + \underline{a}_{36} l\right) h^3,\\
H_2^{(y)}(x_1,y_0,z_1) &= a_7 + a_8 l + a_{17} l^2 + (a_9 + a_{19} l)\, h +,
\end{aligned}
$$

$$+\left(a_{27}+a_{28}l+\underline{a}_{37}l^2\right)h^2,$$

$$H_2^{(y,z)}(x_1,y_0,z_1)=a_8+a_{17}2l+a_{18}l^2+a_{19}h+,$$

$$+\left(a_{28}+a_{37}2l+\underline{a}_{38}l^2\right)h^2,$$

$$H_2^{(x,y)}(x_1,y_0,z_1)=a_9+a_{19}l+\left(a_{27}+a_{28}l+a_{37}l^2\right)2h+\left(a_{29}+\underline{a}_{39}l\right)h^2,$$

$$H_2^{(y,y)}(x_1,y_0,z_1)=a_{10}2+a_{20}2l+\left(a_{30}+\underline{a}_{40}l\right)2h.$$

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