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STUDIES ON CONCAVE YOUNG-FUNCTIONS

N. K. AGBEKO

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ABSTRACT. We succeeded in isolating a special class of concave Young-functions enjoying the so-called *density-level property*. In this class there is a proper subset whose members have each the so-called degree of contraction denoted by c^* , and map bijectively the interval $[c^*, \infty)$ onto itself. We constructed the fixed point of each of these functions. Later we proved that every positive number b is the fixed point of a concave Young-function having b as degree of contraction. We showed that every concave Young-function is square integrable with respect to a specific Lebesgue measure. We also proved that the distance generated by the L^2 -norm is a metric in the set of concave Young-functions and then derived that the concave Young-functions possessing the density-level property constitute a dense set in the space of concave Young-functions.

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1. INTRODUCTION

Let $\varphi : (0, \infty) \rightarrow (0, \infty)$ be a right-continuous and decreasing function such that it is integrable on every finite interval $(0, x)$. It is easily seen that the function $\Phi : [0, \infty) \rightarrow [0, \infty)$, defined by

$$\Phi(x) = \int_0^x \varphi(t) dt, \quad (1.1)$$

is a nonnegative, increasing and concave function with $\Phi(0) = 0$. We further assume that $\Phi(\infty) = \infty$ (Φ is referred to as *concave Young-function* in the literature [4].) We note that if Φ is a concave Young-function, then so is $b\Phi$ for all positive constants b . We shall recall the following definition and result in [1].

Definition A. We say that for the concave Young-function Φ the maximal inequality is valid with some positive constant K_Φ (depending only on Φ) if for an arbitrary nonnegative submartingale $(X_n, \mathcal{F}_n)$, $n \geq 1$, the inequality

$$E\Phi(X_n^*) \leq K_\Phi (1 + EX_n)$$

holds for all $n \geq 1$, where $X_n^* = \max_{1 \leq k \leq n} X_k$.

Theorem B. *Let Φ be any concave Young-function. In order that Φ satisfy the maximal inequality, it is necessary and sufficient that*

$$A_\Phi(\infty) := \int_1^\infty \frac{\varphi(t)}{t} dt < \infty.$$

Moreover, if $A_\Phi(\infty) < \infty$, then $K_\Phi = \max(\Phi(1), A_\Phi(\infty))$.

Theorem C ([3, p. 205]). *Each non-empty subset B of a metric space X is a metric space, the distance in B being the same as in X .*

We shall say that a concave Young-function Φ satisfies the *density-level property* if $A_\Phi(\infty) < \infty$. The quantity $A_\Phi(\infty)$ will be referred to as *density-level* and the function $A_\Phi : [1, \infty) \rightarrow [0, \infty)$, defined by

$$A_\Phi(x) = \int_1^x \frac{\varphi(t)}{t} dt,$$

will be called *density-level function*.

For instance the concave Young-functions $\Phi_1(x) = \sqrt{x}$ and $\Phi_2(x) = \ln(x+1)$, defined for $x \in [0, \infty)$, have finite density-levels. The concave Young-function $\Phi_3(x) = 2x + 1 - e^{-x}$ is of infinite density-level. In fact, if we let $\varphi_3(x)$ stand for the derivative of function $\Phi_3(x)$, then

$$A_{\Phi_3}(\infty) = \int_1^\infty \frac{\varphi_3(t)}{t} dt \geq \int_1^\infty \frac{2}{t} dt = \infty.$$

Theorem B suggests that the set of concave Young-functions that satisfy the density-level property is a rather broad class.

Define function $A_\Phi^* : (0, \infty) \rightarrow (0, \infty]$ by

$$A_\Phi^*(b) = \int_b^\infty \frac{\varphi(x)}{x} dx,$$

where $\Phi \in \mathcal{Y}_{\text{conc}}$.

It is not difficult to see that $A_{\Phi_1}^*(b) < \infty$ and $A_{\Phi_3}^*(b) = \infty$, for any number $b \in (0, \infty)$, where functions $\Phi_1(x) = \sqrt{x}$ and $\Phi_3(x) = 2x + 1 - e^{-x}$ are defined for $x \in [0, \infty)$.

Remark 1. The function $x^{-1}\Phi(x)$ is decreasing on the interval $(0, \infty)$ and

$$0 \leq \lim_{x \rightarrow \infty} \frac{\Phi(x)}{x} < \infty.$$

Notation. $\mathcal{Y}_{\text{conc}}$ will stand for the collection of all concave Young-functions. $\mathcal{A}$ will denote the set of all functions $\Phi \in \mathcal{Y}_{\text{conc}}$ that satisfy the density-level property.

We note that $\mathcal{A}$ is a proper subset of $\mathcal{Y}_{\text{conc}}$, since the concave Young-function $\Phi_3 : [0, \infty) \rightarrow [0, \infty)$, defined above by $\Phi_3(x) = 2x + 1 - e^{-x}$, was shown to be of infinite density-level.

We recall the following fact: A function T from a metric space (M, ϱ) to itself is called a contraction if there is an α which satisfies $0 \leq \alpha < 1$ so that

$$\varrho(T(x), T(y)) \leq \alpha \varrho(x, y)$$

for all $x, y \in M$.

We also recall the following well-known principle.

Contraction Mapping Principle ([5]). *Let T be a contraction on a complete metric space (M, ϱ) . Then there is a unique point $x \in M$ (called fixed point) such that $T(x) = x$. Furthermore, if x_0 is any point in M and we define $x_{n+1} = T(x_n)$, then $\lim_{n \rightarrow \infty} x_n = x$.*

In this communication we study, among others, the closure of $\mathcal{A}$ under the composition operation. In a sense, Theorems 1 and 2 show that the concave Young-functions with the density-level property behave like left and right ideals with respect to the composition operation. We also realize that not every function $\Phi \in \mathcal{A}$ admits a fixed point. The investigation in this direction leads us to isolate a proper subset $\mathcal{A}_1$ of $\mathcal{A}$ such that every function $\Phi \in \mathcal{A}_1$ possesses the so-called *degree of contraction*, which is closely related to the fixed point of Φ if it exists. We show that every concave Young-function is square integrable with respect to a specific given Lebesgue measure, and we prove that the natural distance defined by the L^2 -norm satisfies the metric axioms in $\mathcal{Y}_{\text{conc}}$. We then demonstrate that the subset $\mathcal{A}$ proves to be a dense set in $\mathcal{Y}_{\text{conc}}$.

2. THE CLOSURE OF $\mathcal{A}$ UNDER ADDITION AND COMPOSITION OPERATIONS

Remark 2. For every number $s \in (0, \infty)$ we have that $s\varphi(s) < \Phi(s)$.

PROOF. Fix arbitrarily two numbers $s \in (0, \infty)$ and $b \in (0, s)$. Then by applying twice the fact that φ decreases on $(0, \infty)$, we have that

$$\begin{aligned} \Phi(s) &= \int_0^s \varphi(t) dt = \int_0^b \varphi(t) dt + \int_b^s \varphi(t) dt \geq b\varphi(b) + (s-b)\varphi(s) \\ &> b\varphi(s) + (s-b)\varphi(s) = s\varphi(s), \end{aligned}$$

as required. $\square$

The following remark is an immediate consequence of Theorem B.

Remark 3. Let $\Phi \in \mathcal{Y}_{\text{conc}}$. If $\Phi \in \mathcal{A}$, then $\Phi(x) \leq K_\Phi(1+x)$ for all $x \in (0, \infty)$, where $K_\Phi = \max(\Phi(1), A_\Phi(\infty))$.

Remark 4. The composition of two concave Young-functions is also a concave Young-function.

The following two lemmas are trivial.

Lemma 1. *For any number $b \in (0, \infty)$ and function $\Phi \in \mathcal{Y}_{\text{conc}}$, we have that $b\Phi \in \mathcal{A}$ if and only if $\Phi \in \mathcal{A}$. Moreover, $A_{b\Phi}(x) = bA_\Phi(x)$, $x \in [1, \infty)$.*

Lemma 2. Let functions Φ_1 and $\Phi_2 \in \mathcal{Y}_{\text{conc}}$ be arbitrary. Then Φ_1 and $\Phi_2 \in \mathcal{A}$ if and only if $\Phi_1 + \Phi_2 \in \mathcal{A}$. Furthermore,

$$A_{\Phi_1 + \Phi_2}(x) = A_{\Phi_1}(x) + A_{\Phi_2}(x), \quad x \in [1, \infty).$$

Theorem 1. Let functions Φ_1 and $\Phi_2 \in \mathcal{Y}_{\text{conc}}$ be arbitrary. If $\Phi_2 \in \mathcal{A}$, then $\Phi_1 \circ \Phi_2 \in \mathcal{A}$.

PROOF. Write φ_i for the derivative of Φ_i ($i \in \{1, 2\}$). Compute the density-level of the composition $\Phi_1 \circ \Phi_2$.

$$\begin{aligned} A_{\Phi_1 \circ \Phi_2}(\infty) &= \int_1^\infty \frac{\varphi_2(x) \varphi_1(\Phi_2(x))}{x} dx \\ &\leq \varphi_1(\Phi_2(1)) \int_1^\infty \frac{\varphi_2(x)}{x} dx = \varphi_1(\Phi_2(1)) A_{\Phi_2}(\infty) < \infty, \end{aligned}$$

via the monotonicity of function φ_1 . $\square$

Remark 5. Let $\Phi \in \mathcal{Y}_{\text{conc}}$. Then for Φ to belong to $\mathcal{A}$ it is necessary that

$$\lim_{t \rightarrow \infty} \varphi(t) = 0.$$

PROOF. Assume that $\Phi \in \mathcal{A}$ but $\lim_{t \rightarrow \infty} \varphi(t) = l_0 > 0$. Pick an arbitrarily fixed number $t \in (1, \infty)$. Then

$$\infty > A_\Phi(\infty) \geq \int_1^t \frac{\varphi(x)}{x} dx \geq \varphi(t) \log(t) > l_0 \log(t).$$

Passing to the limit, it will follow that $\infty = A_\Phi(\infty) < \infty$, which is absurd. This completes the proof. $\square$

The following remark suggests that if $\Phi \in \mathcal{Y}_{\text{conc}}$, then either $A_\Phi^*(b) = \infty$ for all $b \in (0, \infty)$, or $A_\Phi^*(b) < \infty$ for all $b \in (0, \infty)$.

Remark 6. Let $\Phi \in \mathcal{Y}_{\text{conc}}$. Then $A_\Phi^*(b) < \infty$ for every constant $b \in (0, \infty) \setminus \{1\}$ if and only if $A_\Phi(\infty) < \infty$.

PROOF. A simple computation shows that

$$A_\Phi^*(b) = \int_b^\infty \frac{\varphi(x)}{x} dx = \begin{cases} A_\Phi(\infty) + \int_b^1 \frac{\varphi(x)}{x} dx & \text{if } b < 1 \\ A_\Phi(\infty) - \int_1^b \frac{\varphi(x)}{x} dx & \text{if } b > 1, \end{cases}$$

which yields the result. $\square$

Theorem 2. Let functions Φ_1 and $\Phi_2 \in \mathcal{Y}_{\text{conc}}$ be arbitrary. If $\Phi_1 \in \mathcal{A}$, then $\Phi_1 \circ \Phi_2 \in \mathcal{A}$.

PROOF. We first show that

$$A_{\Phi_1}(\infty) = \int_{\Phi_2^{-1}(1)}^\infty \frac{\varphi_2(t) \varphi_1(\Phi_2(t))}{\Phi_2(t)} dt,$$

where Φ_2^{-1} is the inverse function of Φ_2 (whose existence is guaranteed by the continuity of Φ_2).

In fact, by definition we have that

$$A_{\Phi_1}(\infty) = \int_1^\infty \frac{\varphi_1(x)}{x} dx.$$

Now, setting $x = \Phi_2(t)$ we observe that $dx = \varphi_2(t) dt$ and thus

$$A_{\Phi_1}(\infty) = \int_{\Phi_2^{-1}(1)}^\infty \frac{\varphi_2(t) \varphi_1(\Phi_2(t))}{\Phi_2(t)} dt.$$

Next, compute the density-level of the composition $\Phi_1 \circ \Phi_2$. Remark 1 implies that

$$\begin{aligned} A_{\Phi_1 \circ \Phi_2}(\infty) &= \int_1^\infty \frac{\varphi_2(t) \varphi_1(\Phi_2(t))}{t} dt \\ &= \int_1^\infty \frac{\Phi_2(t)}{t} \frac{\varphi_2(t) \varphi_1(\Phi_2(t))}{\Phi_2(t)} dt \\ &\leq c \int_{\Phi_2^{-1}(1)}^\infty \frac{\varphi_2(t) \varphi_1(\Phi_2(t))}{\Phi_2(t)} dt = c A_{\Phi_1}(\infty), \end{aligned}$$

where $c = 1/\Phi_2^{-1}(1)$ (the second equality holds because of the claim shown above), which was to be proven. $\square$

Corollary 1. *Let $\Phi \in \mathcal{Y}_{\text{conc}}$ and $\alpha \in (0, 1)$ be arbitrary. Then $\Phi_\alpha \in \mathcal{A}$, where the function $\Phi_\alpha : [0, \infty) \rightarrow [0, \infty)$ is defined by $\Phi_\alpha(x) = \Phi^\alpha(x) = (\Phi(x))^\alpha$.*

Proposition 1. *Let $\Phi \in \mathcal{Y}_{\text{conc}}$ be arbitrary and fix any number $s \in (0, \infty)$. Then*

$$|\Phi(x) - \Phi(y)| \leq \varphi(s) |x - y|$$

for all numbers $x, y \in [s, \infty)$.

PROOF. Pick numbers $x, y \in [s, \infty)$ arbitrarily. Via the monotonicity of Φ it follows that

$$\begin{aligned} |\Phi(x) - \Phi(y)| &= \max(\Phi(x), \Phi(y)) - \min(\Phi(x), \Phi(y)) \\ &= \Phi(\max(x, y)) - \Phi(\min(x, y)). \end{aligned}$$

Hence the monotonicity of φ yields that

$$|\Phi(x) - \Phi(y)| = \int_{\min(x, y)}^{\max(x, y)} \varphi(t) dt \leq \varphi(s) (\max(x, y) - \min(x, y)) = \varphi(s) |x - y|.$$

This was to be proved. $\square$

We shall similarly show the following proposition.

Proposition 2. *Let $\Phi \in \mathcal{A}$ be arbitrary and fix any number $s \in [1, \infty)$. Then*

$$|A_\Phi(x) - A_\Phi(y)| \leq \varphi(s) |x - y|$$

for all numbers $x, y \in [s, \infty)$.

PROOF. Pick numbers $x, y \in [s, \infty)$ arbitrarily. Via the monotonicity of A_Φ it follows that

$$\begin{aligned} |A_\Phi(x) - A_\Phi(y)| &= \max(A_\Phi(x), A_\Phi(y)) - \min(A_\Phi(x), A_\Phi(y)) \\ &= A_\Phi(\max(x, y)) - A_\Phi(\min(x, y)) \\ &= \int_{\min(x, y)}^{\max(x, y)} \frac{\varphi(t)}{t} dt \leq \frac{\varphi(s)}{s} |x - y| \leq \varphi(s) |x - y|, \end{aligned}$$

because $t^{-1}\varphi(t)$ is a decreasing function. $\square$

Proposition 3. Let $x, y \in (0, \infty)$ and $\Delta \subset \mathcal{Y}_{\text{conc}}$ (with $\Delta \neq \emptyset$) be arbitrary. Then

$$\left| \sup_{\Phi \in \Delta} \Phi(x) - \sup_{\Phi \in \Delta} \Phi(y) \right| \leq \sup_{\Phi \in \Delta} |\Phi(x) - \Phi(y)|,$$

provided that $\sup_{\Phi \in \Delta} \Phi(t) < \infty$ for all $t \in (0, \infty)$.

PROOF. We first note that

$$\Phi(x) \leq |\Phi(x) - \Phi(y)| + \Phi(y) \quad \text{and} \quad \Phi(y) \leq |\Phi(x) - \Phi(y)| + \Phi(x).$$

Taking the supremum we can easily observe that

$$\sup_{\Phi \in \Delta} \Phi(x) \leq \sup_{\Phi \in \Delta} |\Phi(x) - \Phi(y)| + \sup_{\Phi \in \Delta} \Phi(y)$$

and

$$\sup_{\Phi \in \Delta} \Phi(y) \leq \sup_{\Phi \in \Delta} |\Phi(x) - \Phi(y)| + \sup_{\Phi \in \Delta} \Phi(x).$$

Combining these inequalities we have that

$$-\sup_{\Phi \in \Delta} |\Phi(x) - \Phi(y)| \leq \sup_{\Phi \in \Delta} \Phi(x) - \sup_{\Phi \in \Delta} \Phi(y) \leq \sup_{\Phi \in \Delta} |\Phi(x) - \Phi(y)|,$$

which yields the result. $\square$

We know that $k\Phi \in \mathcal{Y}_{\text{conc}}$ for any fixed $\Phi \in \mathcal{Y}_{\text{conc}}$ and all $k \geq 1$. Then

$$\sup_{\Phi \in \mathcal{Y}_{\text{conc}}} \Phi(x) \geq \sup_{k \geq 1} k\Phi(x) = \begin{cases} 0 & \text{if } x = 0, \\ \infty & \text{if } x \in (0, \infty), \end{cases}$$

meaning that there is no real function $g(x)$ such that $\Phi(x) \leq g(x)$ for all $\Phi \in \mathcal{Y}_{\text{conc}}$ and $x \in [0, \infty)$. Nevertheless, this is possible for their suitably normalised forms, as shown in the following lemma.

Lemma 3. The function $S : [0, \infty) \rightarrow [0, \infty)$, defined by

$$S(x) = \sup_{\Phi \in \mathcal{Y}_{\text{conc}}} (\Phi(1))^{-1} \Phi(x),$$

has the following properties:

- (1) $S(0) = 0$ and $S(1) = 1$.

(2) S is a non-decreasing function such that

$$(\Phi(1))^{-1} \Phi(x) \leq S(x)$$

for all $\Phi \in \mathcal{Y}_{\text{conc}}$ and $x \in [0, \infty)$.

(3) The identity

$$\sup_{\Phi \in \mathcal{Y}_{\text{conc}}} (1 + \Phi(1))^{-1} = 1$$

holds.

(4) For every number $x \in [0, \infty)$, the chain of inequalities $x \leq S(x) \leq x+1$ holds true.

(5) We have that $\lim_{x \rightarrow \infty} x^{-1} S(x) = 1$ and $\lim_{x \rightarrow \infty} S(x) = \infty$.

PROOF. The first part is obvious. We show that $S(x)$ is a non-decreasing function. In fact, pick arbitrarily two numbers x_1 and $x_2 \in [0, \infty)$ with $x_1 < x_2$. By the monotonicity we have that $\Phi(x_1) < \Phi(x_2)$. If we normalize this inequality suitably and then take the supremum on both sides over all $\Phi \in \mathcal{Y}_{\text{conc}}$ we can then observe that $S(x_1) \leq S(x_2)$. Thus S is a non-decreasing function. To show the identity in the third part we begin by establishing the inequality $(1 + \Phi(1))^{-1} \leq 1$, which holds for every $\Phi \in \mathcal{Y}_{\text{conc}}$. Then $\sup_{\Phi \in \mathcal{Y}_{\text{conc}}} (1 + \Phi(1))^{-1} \leq 1$. We also know that $k^{-1}\Phi \in \mathcal{Y}_{\text{conc}}$ for any fixed integer $k \geq 1$. Hence

$$(1 + k^{-1}\Phi(1))^{-1} \leq \sup_{\Phi \in \mathcal{Y}_{\text{conc}}} (1 + \Phi(1))^{-1}.$$

Passing to the limit we observe that $\lim_{k \rightarrow \infty} (1 + k^{-1}\Phi(1))^{-1} = 1$. Consequently $\sup_{\Phi \in \mathcal{Y}_{\text{conc}}} (1 + \Phi(1))^{-1} = 1$. The fourth part will be proved if we show that $S(x) \leq x+1$ and $S(x) \geq x$ for all $x \in [0, \infty)$. In fact, take arbitrarily a function $\Phi \in \mathcal{Y}_{\text{conc}}$. Clearly the equation of the tangent line of Φ at the point $(1, \Phi(1))$ is given by $y = \varphi(1)(x-1) + \Phi(1)$, $x \in [0, \infty)$. Via the concavity of Φ , it is obvious that $\Phi(x) \leq \varphi(1)(x-1) + \Phi(1)$, $x \in [0, \infty)$. Hence by Remark 2 we have $\Phi(x) \leq \varphi(1)x + \Phi(1) < \Phi(1)(x+1)$, $x \in [0, \infty)$. This implies that $S(x) < x+1$, for all $x \in [0, \infty)$. Finally fix arbitrarily a function $\Phi \in \mathcal{Y}_{\text{conc}}$. Then the function, defined on $[0, \infty)$ by $x + \Phi(x)$ (for any fixed $\Phi \in \mathcal{Y}_{\text{conc}}$), also belongs to $\mathcal{Y}_{\text{conc}}$. Hence

$$S(x) \geq \frac{x + \Phi(x)}{1 + \Phi(1)} \geq \frac{x}{1 + \Phi(1)}, \quad x \in [0, \infty).$$

Now taking the supremum over $\Phi \in \mathcal{Y}_{\text{conc}}$, the third part leads to the desired inequality $S(x) \geq x$. To complete the proof we just point out that the fifth part becomes obvious because of the fourth part. $\square$

Lemma 4. The function $H : [1, \infty) \rightarrow [0, \infty)$ defined by

$$H(x) = \sup_{\Phi \in \mathcal{A}} (\varphi(1))^{-1} \Phi(x)$$

is increasing and has the property that

$$|H(x) - H(y)| \leq |x - y|$$

for all $x, y \in [1, \infty)$.

3. THE FIXED POINTS OF A CLASS OF CONCAVE YOUNG-FUNCTIONS

We shall introduce the following notion (probably new, at least in the author's opinion). Some series of examples will justify that it is well founded.

Definition 1. Let $\Phi \in \mathcal{A}$ be arbitrary. The number $c^* \in (0, \infty)$ will be called the degree of contraction of function Φ if there is some constant $\alpha > 1$ such that both the identities

$$\int_{c^*}^{\infty} \frac{\varphi(t)}{t} dt = 1 \quad \text{and} \quad \int_{c^*}^{\alpha c^*} \frac{\varphi(t)}{t} dt = \varphi(c^*)$$

hold simultaneously. (In this case we shall say that Φ admits the number c^* as its degree of contraction.)

Example 1. For $\Phi(x) = \sqrt{x+1} - 1$, $x \in [0, \infty)$, the degree of contraction of Φ is equal to $\frac{4e^2}{e^4 - 2e^2 + 1} \approx 0.7240616609$ with

$$\alpha = \frac{(e^2 - 1)^2 e^{2/(e^2+1)-1}}{e^{4/(e^2+1)+2} - 2e^{2/(e^2+1)+1} + 1} \approx 3.175019732.$$

Example 2. For any fixed number $p \in (0, 1)$, the degree of contraction of the function $\Phi_p(x) = x^p$, $x \in [0, \infty)$, is equal to $\left(\frac{p}{1-p}\right)^{\frac{1}{1-p}}$ with $\alpha = \frac{1}{p^{1/(1-p)}}$.

Example 3. The degree of contraction of the function $\log(x+1)$, $x \in [0, \infty)$, equals $(e-1)^{-1}$ with $\alpha = \frac{e-1}{e^{1/e}-1} \approx 3.864191634$.

Proposition 4. Let $\Phi \in \mathcal{A}$ admit the number c^* as its degree of contraction. Then $\varphi(c^*) < 1$.

PROOF. By Definition 1, there is a number $\alpha > 1$ such that both the identities

$$\int_{c^*}^{\infty} \frac{\varphi(t)}{t} dt = 1 \quad \text{and} \quad \int_{c^*}^{\alpha c^*} \frac{\varphi(t)}{t} dt = \varphi(c^*)$$

hold simultaneously. Consequently we have that

$$1 = \int_{c^*}^{\infty} \frac{\varphi(t)}{t} dt = \int_{c^*}^{\alpha c^*} \frac{\varphi(t)}{t} dt + \int_{\alpha c^*}^{\infty} \frac{\varphi(t)}{t} dt > \int_{c^*}^{\alpha c^*} \frac{\varphi(t)}{t} dt = \varphi(c^*)$$

because

$$0 < \int_{\alpha c^*}^{\infty} \frac{\varphi(t)}{t} dt < \infty$$

by the assumption and the monotonicity of the function $t^{-1}\varphi(t)$ on the interval $(0, \infty)$. This was to be proved. $\square$

On the one hand it is not difficult to verify that 1 is the degree of contraction of function $\Phi(x) = \sqrt{x}$, $x \in [0, \infty)$, with $\alpha = 4$, and 1 is the unique solution of equation $\Phi(x) = x$ on interval $[1, \infty)$. On the other hand, we know, for instance, in Example 3, that $(e - 1)^{-1}$ is the degree of contraction of function $\log(x + 1)$. Nevertheless,

$$\log\left(\frac{1}{e-1} + 1\right) = \log\left(\frac{e}{e-1}\right) \neq \frac{1}{e-1}.$$

The question thus arises which are those functions $\Phi \in \mathcal{A}$ that are contractions. We shall provide a proper subset of $\mathcal{A}$ enjoying this property.

Theorem 3. *Let $\Phi \in \mathcal{Y}_{\text{conc}}$ and c^* be any positive number. In order that the equality $\Phi(c^*) = c^*$ hold, it is necessary and sufficient that the range of the function $\Phi|_{[c^*, \infty)} : [c^*, \infty) \rightarrow [0, \infty)$, defined by the formula*

$$\Phi|_{[c^*, \infty)}(x) = \Phi(x),$$

should equal the interval $[c^, \infty)$.*

PROOF. Suppose that $\Phi|_{[c^*, \infty)}(c^*) = \Phi(c^*) = c^*$. Obviously Φ is a bijection on $[0, \infty)$. Hence it follows that $\Phi|_{[c^*, \infty)}$ is an injection on $[c^*, \infty)$. Since $\Phi|_{[c^*, \infty)}$ is continuous on $[c^*, \infty)$ and tends increasingly to ∞ , we have that the range of function $\Phi|_{[c^*, \infty)}$ equals $[\Phi(c^*), \infty) = [c^*, \infty)$, by assumption. Conversely, assume that the range of $\Phi|_{[c^*, \infty)}$ equals interval $[c^*, \infty)$, but, on the contrary, there is some number $y \in (c^*, \infty)$ such that $\Phi(y) = \Phi|_{[c^*, \infty)}(y) = c^*$. By the assumption it is obvious that function $\Phi|_{[c^*, \infty)}$ is surjective on $[c^*, \infty)$. Moreover, $\Phi|_{[c^*, \infty)}$ maps bijectively the interval $[c^*, \infty)$ onto itself because it is also an injection. The monotonicity of Φ yields that $\Phi(c^*) = \Phi|_{[c^*, \infty)}(c^*) < \Phi(y) = c^*$. However, by the bijective property of $\Phi|_{[c^*, \infty)}$, we have that $\Phi(c^*) \geq c^*$. Consequently the inequality $c^* < c^*$ follows. This, however, is absurd. Therefore, we can conclude on the validity of the argument. $\square$

Notation. $\mathcal{A}_1$ will stand for the collection of all functions $\Phi \in \mathcal{A}$ mapping bijectively the interval $[c^*, \infty)$ onto itself, where c^* is the degree of contraction of Φ .

Theorem 4. *Let $\Phi \in \mathcal{A}_1$ with c^* its degree of contraction. Endow the interval $[c^*, \infty)$ with the metric $\varrho(\cdot, \cdot) : [c^*, \infty) \times [c^*, \infty) \rightarrow [0, \infty)$ defined by $\varrho(x, y) = |x - y|$. Then Φ is a contraction over the metric space $([c^*, \infty), \varrho)$.*

Moreover, the number c^ is the unique solution of the equation $\Phi(x) = x$ on $[c^*, \infty)$.*

PROOF. We first note that the pair $([c^*, \infty), \varrho)$ is a complete metric space. Combining Propositions 1 and 4 we can easily derive that Φ is a contraction on the metric space $([c^*, \infty), \varrho)$. But since $\Phi(c^*) = c^*$ (via Theorem 3), we deduce, referring to the Contraction Mapping Principle, that the degree of contraction c^* is the unique solution to the equation $\Phi(x) = x$ on the interval $[c^*, \infty)$. This completes the proof. $\square$

Theorem 5. For every number $b \in (0, \infty)$ there can be found some function $\Phi_b \in \mathcal{A}_1$ with degree of contraction b . Furthermore, number b is the unique solution of equation $\Phi_b(x) = x$ on the interval $[b, \infty)$.

PROOF. Let $b \in (0, \infty)$ be any number and define the function $\Phi_b(t) = \sqrt{bt}$, $t \in [0, \infty)$. Clearly, the derivative of $\Phi_b(t)$ is expressed by $\varphi(t) = \frac{\sqrt{b}}{2\sqrt{t}}$, $t \in (0, \infty)$. On the one hand, an easy calculation shows that

$$\int_b^\infty \frac{\varphi(t)}{t} dt = \sqrt{b} \int_b^\infty \frac{1}{2t\sqrt{t}} dt = \sqrt{b} \lim_{x \rightarrow \infty} \left(\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{x}} \right) = 1,$$

and

$$\int_b^{4b} \frac{\varphi(t)}{t} dt = \sqrt{b} \int_b^{4b} \frac{1}{2t\sqrt{t}} dt = \frac{1}{2} = \varphi(b)$$

i. e., number b is the degree of contraction of Φ_b . On the other hand, an easy substitution leads to

$$\Phi_b(b) = \sqrt{b^2} = b.$$

Then Theorem 3 implies that $\Phi_b \in \mathcal{A}_1$. Consequently Theorem 4 entails that Φ_b is a contraction over the metric space $([b, \infty), \varrho)$ and moreover, number b is the unique solution of equation $\Phi_b(x) = x$ on $[b, \infty)$, with ϱ being the metric induced by the absolute value function. This concludes the proof. $\square$

To end this section we should like to point out that the set

$$\{\Phi \in \mathcal{Y}_{\text{conc}} \setminus \mathcal{A} : \Phi \text{ admits a positive fix point}\}$$

is a non-empty set. In fact, it is not hard to check that the function Φ , defined by $\Phi(x) = \frac{x}{2} + \sqrt{x}$ whenever $x \in [0, \infty)$, belongs to $\mathcal{Y}_{\text{conc}} \setminus \mathcal{A}$ and $\Phi(4) = 4$.

4. IS THE SET $\mathcal{A}$ DENSE IN $\mathcal{Y}_{\text{conc}}$?

We shall answer this question in the affirmative.

Theorem 6. For any concave Young-function Φ , there exists a sequence $(\Phi_n) \subset \mathcal{A}$ such that (Φ_n) converges pointwise to Φ , i. e., $\lim_{n \rightarrow \infty} \Phi_n(x) = \Phi(x)$ whenever $x \in [0, \infty)$.

PROOF. Fix arbitrarily an index $n \geq 1$ and define $\Phi_n(x) = \Phi^{n/(n+1)}(x)$, $x \in [0, \infty)$. Obviously, $(\Phi_n) \subset \mathcal{Y}_{\text{conc}}$ because of Remark 4. So, on the one hand, Corollary 1 yields that $(\Phi_n) \subset \mathcal{A}$. On the other hand, we can easily see in the limit that

$$\lim_{n \rightarrow \infty} \Phi_n(x) = \lim_{n \rightarrow \infty} \Phi^{n/(n+1)}(x) = \Phi(x)$$

for every $x \in [0, \infty)$. Therefore, we conclude on the validity of the theorem. $\square$

Lemma 5. Let $\Phi \in \mathcal{Y}_{\text{conc}}$. Then there are constants $C_\Phi > 0$ and $B_\Phi \geq 0$ such that

$$A_\Phi(\infty) - B_\Phi \leq \int_0^\infty \frac{\Phi(t)}{(t+1)^2} dt \leq C_\Phi + A_\Phi(\infty).$$

PROOF. An integration by parts leads to

$$\int_0^\infty \frac{\Phi(t)}{(t+1)^2} dt = \left[\frac{-\Phi(t)}{t+1} \right]_0^\infty + \int_0^\infty \frac{\varphi(t)}{t+1} dt = \int_0^\infty \frac{\varphi(t)}{t+1} dt - B_\Phi, \quad (4.1)$$

where $0 \leq B_\Phi := \lim_{t \rightarrow \infty} \frac{\Phi(t)}{t+1} < \infty$, as $\frac{\Phi(t)}{t+1} < \frac{\Phi(t)}{t}$ for all $t \in (0, \infty)$. On the one hand,

$$\int_0^\infty \frac{\varphi(t)}{t+1} dt = \int_0^1 \frac{\varphi(t)}{t+1} dt + \int_1^\infty \frac{\varphi(t)}{t+1} dt \leq \int_0^1 \frac{\varphi(t)}{t+1} dt + A_\Phi(\infty). \quad (4.2)$$

On the other hand, by the monotonicity of function $\varphi(t)$ and by the change of variables, we have that

$$\int_0^\infty \frac{\varphi(t)}{t+1} dt \geq \int_0^\infty \frac{\varphi(t+1)}{t+1} dt = \int_1^\infty \frac{\varphi(x)}{x} dx = A_\Phi(\infty). \quad (4.3)$$

Consequently if we combine (4.1)–(4.3), one can observe that

$$A_\Phi(\infty) - B_\Phi \leq \int_0^\infty \frac{\Phi(t)}{(t+1)^2} dt \leq \int_0^1 \frac{\varphi(t)}{t+1} dt + B_\Phi + A_\Phi(\infty).$$

This leads to the desired result. $\square$

Lemma 5 suggests that the quantity $\int_0^\infty \frac{\Phi(t)}{(t+1)^2} dt$ and the density-level $A_\Phi(\infty)$ are equivalent, in the sense that they are both either finite or infinite. This gives rise to the following essential result.

Lemma 6. *Let $\Phi \in \mathcal{Y}_{\text{conc}}$ be arbitrary. Then*

$$\int_0^\infty \frac{(\Phi(x))^2}{(x+1)^4} dx < \infty.$$

PROOF. Clearly,

$$\begin{aligned} \int_0^\infty \frac{(\Phi(x))^2}{(x+1)^4} dx &= \int_0^1 \frac{(\Phi(x))^2}{(x+1)^4} dx + \int_1^\infty \frac{(\Phi(x))^2}{(x+1)^4} dx \\ &\leq \int_0^1 \frac{(\Phi(x))^2}{(x+1)^4} dx + \int_1^\infty \frac{(\Phi(x))^2}{x^4} dx. \end{aligned}$$

Integration by parts yields

$$\int_1^\infty \frac{(\Phi(x))^2}{x^4} dx = \frac{\Phi(1)}{3} + \frac{2}{3} \int_1^\infty \frac{\varphi(x) \Phi(x)}{x^3} dx \leq \frac{\Phi(1)}{3} + \frac{2\varphi(1) \Phi(1)}{3},$$

because $\varphi(x)$ and $\frac{\Phi(x)}{x}$ are decreasing functions. $\square$

Now endow the half line $[0, \infty)$ with a σ -algebra $\mathcal{M}$ containing the Borel sets. Define a Lebesgue measure $\mu : \mathcal{M} \rightarrow [0, \infty)$ by setting

$$\mu([0, x)) = \frac{1}{3} \left(1 - \frac{1}{(x+1)^3} \right)$$

for all $x \in [0, \infty)$. Let $L^2 := L^2([0, \infty), \mathcal{M}, \mu)$ be the collection of all (measurable) square integrable functions. We know (see [6, p. 326], Remark 11.37) that the pair (L^2, d) is not a metric space unless we identify functions which differ only on a set of measure zero, where the mapping $d : L^2 \times L^2 \rightarrow [0, \infty)$ is defined by

$$d(f, g) = \sqrt{\int_{[0, \infty)} (f - g)^2 d\mu} = \sqrt{\int_0^\infty \frac{(f(x) - g(x))^2}{(x+1)^4} dx}.$$

By Lemma 6, we observe that $\mathcal{Y}_{\text{conc}} \subset L^2$. Unfortunately, we note that this does not guarantee that the pair $(\mathcal{Y}_{\text{conc}}, d)$ is a metric space, for the reason mentioned above. Nevertheless, we shall prepare the ground for showing that $(\mathcal{Y}_{\text{conc}}, d)$ is actually a metric space.

Whenever $\Phi \in \mathcal{Y}_{\text{conc}}$ write G_Φ for the graph of Φ on $[0, \infty)$, i. e.,

$$G_\Phi = \{(x, \Phi(x)) : x \in [0, \infty)\}$$

and write $G_\Phi^{a||b}$ for the graph of Φ on the interval $[a, b)$, i. e.,

$$G_\Phi^{a||b} = \{(x, \Phi(x)) : x \in (a, b)\},$$

where $a < b$ are any non-negative numbers.

Lemma 7. *Let Φ and $\Psi \in \mathcal{Y}_{\text{conc}}$ be arbitrary with distinct graphs. Then*

$$|\{x \in (0, \infty) : \Phi(x) = \Psi(x)\}| \leq 1,$$

where $|B|$ stands for the cardinality of B whenever B is a set.

PROOF. Suppose on the contrary that

$$|\{x \in (0, \infty) : \Phi(x) = \Psi(x)\}| \geq 2.$$

Write

$$x_1 = \inf \{x \in (0, \infty) : \Phi(x) = \Psi(x)\}$$

and

$$x_2 = \inf \{x \in (0, \infty) \setminus \{x_1\} : \Phi(x) = \Psi(x)\}.$$

It is clear that $0 < x_1 < x_2$ and $\Phi(x_i) = \Psi(x_i)$, $i \in \{1, 2\}$. We point out that the two graphs are continuous. We show that the graph of one of the functions Φ and Ψ lies above the graph of the other on the interval $(0, x_1)$. In fact, without loss of generality we may assume on the contrary that $G_\Phi^{0||x_1}$ lies both above and below $G_\Psi^{0||x_1}$. Then necessarily the two graphs must cross each other in the interior of interval $(0, x_1)$, i. e. there is some $x_0 \in (0, x_1)$ such that $\Phi(x_0) = \Psi(x_0)$. This, however, is in contradiction with the minimality of x_1 . Hence we can assume that $G_\Phi^{0||x_1}$ lies above $G_\Psi^{0||x_1}$. By the continuity and the fact that $\Phi(x_1) = \Psi(x_1)$ we note that G_Φ crosses G_Ψ at point $(x_1, \Phi(x_1))$. Nevertheless, since both Φ and Ψ are unbounded increasing functions and $\Phi(x_2) = \Psi(x_2)$, the graph G_Φ must cross the graph G_Ψ at

point $(x_2, \Phi(x_2))$. This means that Φ must be convex on the interval (x_1, x_2) , which is absurd since these functions are concave. $\square$

Corollary 2. *Let Φ and $\Psi \in \mathcal{Y}_{\text{conc}}$ be arbitrary. Then among the following three assertions exactly one fulfills*

- (1) $\{x \in [0, \infty) : \Phi(x) = \Psi(x)\} = [0, \infty)$.
- (2) $\{x \in (0, \infty) : \Phi(x) \neq \Psi(x)\} = (0, \infty)$.
- (3) *There is a unique number $x^* \in (0, \infty)$ with $\Phi(x^*) = \Psi(x^*)$ such that*

$$\{x \in (0, \infty) \setminus \{x^*\} : \Phi(x) \neq \Psi(x)\} = (0, \infty) \setminus \{x^*\}.$$

Lemma 8. *Let Φ and $\Psi \in \mathcal{Y}_{\text{conc}}$ be arbitrary. Then in order that $\Phi(x) = \Psi(x)$ for all $x \in [0, \infty)$ it is necessary and sufficient that*

$$\int_{[0, \infty)} (\Phi - \Psi)^2 d\mu = 0.$$

PROOF. We first note that the sufficiency is obvious. To show the necessity let us assume that

$$\int_{[0, \infty)} (\Phi - \Psi)^2 d\mu = 0.$$

Then, on the one hand,

$$\mu(\{x \in [0, \infty) : \Phi(x) = \Psi(x)\}) = \mu([0, \infty)) = \frac{1}{3}$$

so that necessarily $\{x \in [0, \infty) : \Phi(x) = \Psi(x)\} \neq \emptyset$. On the other hand

$$\mu(\{x \in (0, \infty) : \Phi(x) \neq \Psi(x)\}) = 0.$$

Note that both the sets $\{x \in [0, \infty) : \Phi(x) = \Psi(x)\}$ and $\{x \in (0, \infty) : \Phi(x) \neq \Psi(x)\}$ cannot be non-empty at the same time (because of Corollary 2). Consequently,

$$\{x \in (0, \infty) : \Phi(x) \neq \Psi(x)\} = \emptyset$$

and, therefore, $\{x \in [0, \infty) : \Phi(x) = \Psi(x)\} = [0, \infty)$. $\square$

We are now in the position to state the result hereby.

Proposition 5. *The mapping $d : \mathcal{Y}_{\text{conc}} \times \mathcal{Y}_{\text{conc}} \rightarrow [0, \infty)$, defined by*

$$d(\Phi, \Psi) = \sqrt{\int_{[0, \infty)} (\Phi - \Psi)^2 d\mu} = \sqrt{\int_0^\infty \frac{(\Phi(x) - \Psi(x))^2}{(x+1)^4} dx},$$

satisfies the metric axioms, i. e. for any three functions Φ_1, Φ_2 and $\Phi_3 \in \mathcal{Y}_{\text{conc}}$

- (1) $d(\Phi_1, \Phi_2) \geq 0$ and $d(\Phi_1, \Phi_2) = 0$ if and only if $\Phi_1 = \Phi_2$.
- (2) $d(\Phi_1, \Phi_2) = d(\Phi_2, \Phi_1)$.
- (3) $d(\Phi_1, \Phi_2) \leq d(\Phi_1, \Phi_3) + d(\Phi_3, \Phi_2)$.

The pair $(\mathcal{Y}_{\text{conc}}, d)$ is a metric space and then by referring to Theorem C the pair $(\mathcal{A}, d)$ is also a metric space.

Theorem 7. Let $\Phi \in \mathcal{Y}_{\text{conc}}$ and write $\Phi_n = \Phi^{n/(n+1)}$, $n \geq 1$. Then

$$\lim_{n \rightarrow \infty} \int_{[0, \infty)} \Phi_n^2 d\mu = \int_{[0, \infty)} \Phi^2 d\mu.$$

PROOF. For every index $n \geq 1$, define $\Phi_n^* := (\Phi_n(1))^{-1} \Phi_n$. Clearly $(\Phi_n) \subset \mathcal{Y}_{\text{conc}}$ (see Corollary 1) and hence $(\Phi_n^*) \subset \mathcal{A}$ because of Lemma 1. Via Theorem 6 it follows that sequence (Φ_n) converges to Φ pointwise, which in turn entails that sequence (Φ_n^*) converges to $(\Phi(1))^{-1} \Phi$ pointwise. Write the function $Z(x) := x + 1$, $x \in [0, \infty)$. We obtain (by Lemma 3) that

$$\sup_{n \geq 1} \Phi_n^*(x) \leq S(x) \leq Z(x), \quad x \in [0, \infty).$$

Now, on the one hand, a simple computation shows that $Z \in L^2$. On the other hand, we can deduce from Lemma 6 that $(\Phi_n) \subset L^2$ and thus $(\Phi_n^*) \subset L^2$. Therefore, the Dominated Convergence Theorem guarantees that

$$\lim_{n \rightarrow \infty} \int_{[0, \infty)} \Phi_n^{*2} d\mu = (\Phi(1))^{-2} \int_{[0, \infty)} \Phi^2 d\mu.$$

Now we remark that for every index $n \geq 1$,

$$\int_{[0, \infty)} \Phi_n^2 d\mu = (\Phi(1))^2 \int_{[0, \infty)} \Phi_n^{*2} d\mu.$$

Passing to the limit we can conclude that

$$\lim_{n \rightarrow \infty} \int_{[0, \infty)} \Phi_n^2 d\mu = \int_{[0, \infty)} \Phi^2 d\mu.$$

This was to be proven. □

Theorem 8. The subset $\mathcal{A}$ is a dense set in $\mathcal{Y}_{\text{conc}}$.

PROOF. Let $\Phi \in \mathcal{Y}_{\text{conc}}$ be arbitrary. For every index $n \geq 1$, set $\Phi_n^* := (\Phi_n(1))^{-1} \Phi_n$, where $\Phi_n = \Phi^{n/(n+1)}$. We need to prove that

$$\lim_{n \rightarrow \infty} d(\Phi, \Phi_n) = \lim_{n \rightarrow \infty} \int_{[0, \infty)} (\Phi - \Phi_n)^2 d\mu = 0.$$

In fact, fix arbitrarily an index $n \geq 1$. Then

$$\int_{[0, \infty)} (\Phi - \Phi_n)^2 d\mu = \int_{[0, \infty)} \Phi_n^2 d\mu + \int_{[0, \infty)} \Phi^2 d\mu - 2 \int_{[0, \infty)} \Phi \Phi_n d\mu. \quad (4.4)$$

Then Lemma 3 entails that

$$(\Phi(1))^{-(2n+1)/(n+1)} \Phi \Phi_n \leq Z^{(2n+1)/(n+1)} \leq Z^2,$$

since $Z(x) \geq 1$ for all $x \in [0, \infty)$ and the sequence $(\frac{2n+1}{n+1})$ tends increasingly to 2. On the other hand,

$$\lim_{n \rightarrow \infty} (\Phi(1))^{-(2n+1)/(n+1)} \Phi(x) \Phi_n(x) = (\Phi(1))^{-2} \Phi^2(x)$$

for all $x \in [0, \infty)$. Then by means of The Dominated Convergence Theorem it follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{[0, \infty)} \Phi \Phi_n d\mu &= \lim_{n \rightarrow \infty} (\Phi(1))^{(2n+1)/(n+1)} \int_{[0, \infty)} (\Phi(1))^{-\frac{2n+1}{n+1}} \Phi^{\frac{2n+1}{n+1}} d\mu \\ &= \int_{[0, \infty)} \Phi^2 d\mu. \end{aligned} \quad (4.5)$$

Finally we note that

$$\lim_{n \rightarrow \infty} \int_{[0, \infty)} \Phi_n^2 d\mu = \int_{[0, \infty)} \Phi^2 d\mu, \quad (4.6)$$

by Theorem 7. Therefore, combining the results established in (4.4)–(4.6), we get $\lim_{n \rightarrow \infty} d(\Phi, \Phi_n) = 0$. We can thus conclude on the validity of the theorem. $\square$

If the integral representation (1.1) is such that its derivative is a right-continuous function, tending increasingly to infinity and assumes the value zero at the origin, then we speak of *convex Young-functions* (see, e. g. [2]). Clearly the inverse of every convex Young-function is a concave Young-function (and *vice versa*).

A convex Young-function Ψ is said to satisfy the growth condition if

$$\sup_{x>0} \frac{\Psi(\beta x)}{\Psi(x)} < \infty$$

for some number $\beta > 1$ which is equivalent to

$$\sup_{x>0} \frac{x\psi(x)}{\Psi(x)} := p < \infty$$

with ψ being the derivative of Ψ . (The quantity p is referred to as the power of Ψ .)

Open problem 1. Let $\Phi \in \mathcal{Y}_{\text{conc}}$ be arbitrary. In order that $\Phi(x) = x^p$, $x \in [0, \infty)$ for some $p \in (0, 1)$ it is necessary and sufficient that both $\Phi \in \mathcal{A}_1$ and its inverse Φ^{-1} satisfy the growth condition together with the property $\Phi^{-1}(1) = 1$.

Open problem 2. The converse of Remark 5 holds true.

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Author's Address

N. K. Agbeko:

INSTITUTE OF MATHEMATICS, UNIVERSITY OF MISKOLC, H-3515 MISKOLC–EGYETEMVÁROS, HUNGARY

E-mail address: matagbek@uni.miskolc.hu



A NOTE ON A NONLINEAR m -POINT BOUNDARY VALUE PROBLEM FOR p -LAPLACIAN DIFFERENTIAL INCLUSIONS

M. BENCHOHRA, S. K. NTOUYAS, AND A. OUAHAB

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ABSTRACT. In this note a selection theorem due to Bressan and Colombo for lower semi-continuous multi-valued operators with nonempty closed decomposable values combined with Schaefer's fixed point theorem is used to investigate the existence of positive solutions for m -point boundary value problems for one dimensional p -Laplacian differential inclusions.

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1. INTRODUCTION

This note is concerned with the existence of positive solutions for the following class of boundary value problems for m -point one dimensional p -Laplacian differential inclusions

$$(\varphi(y'))' \in F(t, y), \quad \text{a. e. } t \in J := [0, 1]; \quad (1.1)$$

$$y'(0) = \sum_{i=1}^{m-2} b_i y'(\xi_i), \quad y(1) = \sum_{i=1}^{m-2} a_i y'(\xi_i), \quad (1.2)$$

where $\varphi : \mathbb{R}_+^* \rightarrow \mathbb{R}_+$ defined by $\varphi(v) := |v|^{p-2}v$, $p > 1$, $F : J \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}_+)$ is a multi-valued map, $\mathcal{P}(\mathbb{R}_+)$ is the family of all subsets of $\mathbb{R}_+$, $\xi_i \in (0, 1)$, $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, and a_i, b_i , $i = 1, \dots, m-2$, are positive and satisfy $0 < \sum_{i=1}^{m-2} a_i < 1$, $\sum_{i=1}^{m-2} b_i < 1$. The study of multi-point boundary value problems for linear second order ordinary differential equations was initiated by Il'in and Moiseev [15, 16]. Then Gupta [10, 11] studied three-point boundary value problems for nonlinear ordinary differential equations, the m -point boundary value problem was studied by Gupta *et al.*, [12, 14], Ma [18]. Very recently, in a series of papers by Benchohra and Ntouyas (see [2–5]) some extensions to multi-point differential inclusions have been proposed with the aid of fixed point arguments in the cases when the right-hand side is convex

as well as nonconvex valued. Some existence results for one dimensional p -Laplacian differential equations are given in the papers by Bai and Fang [1] and Jian and Guo [17]. Our goal here is to give the existence of at least one positive solution for m -point boundary value problems for one dimensional p -Laplacian differential inclusions. Our approach relies on Schaefer's fixed point theorem combined with a selection theorem due to Bressan and Colombo [6] for lower semi-continuous operators with nonempty closed and decomposable values.

2. PRELIMINARIES

In this Section, we introduce notations, definitions, and preliminary facts from multi-valued analysis which are used throughout this paper.

$C([0, 1], \mathbb{R})$ is the Banach space of all continuous functions from $[0, 1]$ into $\mathbb{R}$ with the norm

$$\|y\|_{\infty} := \sup\{|y(t)| : 0 \leq t \leq 1\}.$$

$AC([0, 1], \mathbb{R})$ is the space of all absolutely continuous functions y from $[0, 1]$ into $\mathbb{R}$. $L^1(J, \mathbb{R})$ denotes the Banach space of functions $y : J \rightarrow \mathbb{R}$ which are Lebesgue integrable normed by

$$\|y\|_{L^1} = \int_0^1 |y(t)| dt.$$

Let $\mathcal{A}$ be a subset of $[0, 1] \times \mathbb{R}$. $\mathcal{A}$ is $\mathcal{L} \otimes \mathcal{B}$ measurable if $\mathcal{A}$ belongs to the σ -algebra generated by all sets of the form $\mathcal{N} \times D$ where $\mathcal{N}$ is Lebesgue measurable in $[0, 1]$ and D is Borel measurable in $\mathbb{R}$. A subset $\mathcal{I}$ of $L^1([0, 1], \mathbb{R})$ is decomposable if for all $u, v \in \mathcal{I}$ and $\mathcal{N} \subset [0, 1]$ measurable the function $u\chi_{\mathcal{N}} + v\chi_{[0, 1] - \mathcal{N}} \in \mathcal{I}$, where $\chi_{[0, 1]}$ stands for the characteristic function of $[0, 1]$.

Let E be a Banach space, X a nonempty closed subset of E and $G : X \rightarrow \mathcal{P}(E)$ a multi-valued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{x \in X : G(x) \cap B \neq \emptyset\}$ is open for any open set B in E . G has a fixed point if there is $x \in X$ such that $x \in G(x)$. For more details on multi-valued maps we refer to the books by Deimling [7], Górniewicz [9] and Hu and Papageorgiou [19].

Definition 1. Let Y be a separable metric space and let $N : Y \rightarrow \mathcal{P}(L^1([0, b], \mathbb{R}))$ be a multi-valued operator. We say N has the property (BC) if

- (1) N is lower semi-continuous (l.s.c.);
- (2) N has nonempty closed and decomposable values.

Let $F : J \times \mathbb{R}^+ \rightarrow \mathcal{P}(\mathbb{R}^+)$ be a multi-valued map with nonempty compact values. Assign to F the multi-valued operator

$$\mathcal{F} : C([0, 1], \mathbb{R}^+) \rightarrow \mathcal{P}(L^1([0, 1], \mathbb{R}^+))$$

by letting

$$\mathcal{F}(y) = \{w \in L^1([0, 1], \mathbb{R}) : w(t) \in F(t, y(t)) \text{ for a. e. } t \in [0, 1]\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated with F .

Definition 2. Let $F : J \times \mathbb{R}^+ \rightarrow \mathcal{P}(\mathbb{R}^+)$ be a multi-valued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Niemytzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Next we state a selection theorem due to Bressan and Colombo.

Theorem 1 ([6]). Assume that Y is a separable metric space and let $N : Y \rightarrow \mathcal{P}(L^1([0, 1], \mathbb{R}))$ be a multi-valued operator which has the property (BC). Then N has a continuous selection, i.e. there exists a continuous function (single-valued) $g : Y \rightarrow L^1(J, \mathbb{R})$ such that $g(y) \in N(y)$ for every $y \in Y$.

Let us introduce the following hypotheses which are assumed hereafter:

- (H1) $F : [0, 1] \times \mathbb{R}^+ \rightarrow \mathcal{P}(\mathbb{R}^+)$ is a nonempty compact valued multi-valued map such that:
- (a) $(t, y) \mapsto F(t, y)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;
 - (b) $y \mapsto F(t, y)$ is lower semi-continuous for a. e. $t \in [0, 1]$;
- (H2) There exists a function $h \in L^1([0, 1], \mathbb{R}_+)$ such that

$$\|F(t, y)\| := \sup\{|v| : v \in F(t, y)\} \leq h(t) \text{ for a. e. } t \in [0, 1] \text{ and for } y \in \mathbb{R}.$$

3. MAIN RESULT

Let us start by defining what we mean by a solution of problem (1.1)–(1.2).

Definition 3. A function $y \in C^1((0, 1), \mathbb{R})$ with $\varphi(y') \in AC((0, 1), \mathbb{R})$ is said to be a solution of (1.1), (1.2) if there exists $v(t) \in L^1(J, \mathbb{R})$ such that y satisfies the equation $(\varphi(y'))' = v(t)$ a. e. on J and the condition (1.2).

Theorem 2. Suppose that hypotheses (H1), (H2) are satisfied. Then the m -point BVP (1.1), (1.2) has at least one positive solution.

PROOF. (H1) and (H2) imply by Lemma 2.2 in Frigon [8] that F is of the lower semi-continuous type. Then from Theorem 1 there exists a continuous function $f : C([0, 1], \mathbb{R}) \rightarrow L^1([0, 1], \mathbb{R})$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in C([0, 1], \mathbb{R})$. Consider the following problem

$$(\varphi(y'))' = f(y(t)), \quad \text{a. e. } t \in J, \quad (3.1)$$

$$y'(0) = \sum_{i=1}^{m-2} b_i y'(\xi_i), \quad y(1) = \sum_{i=1}^{m-2} a_i y'(\xi_i). \quad (3.2)$$

Clearly, if y is a solution of problem (3.1), (3.2), then y is a solution to problem (1.1), (1.2).

Transform the problem (3.1), (3.2) into a fixed point problem. Consider the operator $N : C([0, 1], \mathbb{R}^+) \rightarrow C([0, 1], \mathbb{R})$ defined by:

$$\begin{aligned} N(y)(t) = & - \int_0^t \psi \left(\int_0^s f(y(\tau)) d\tau \right) ds - tB \sum_{i=1}^{m-2} b_i \psi \left(\int_0^{\xi_i} f(y(\tau)) d\tau \right) \\ & + A \left\{ \int_0^1 \psi \left(\int_0^s f(y(\tau)) d\tau \right) ds - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \psi \left(\int_0^s f(y(\tau)) d\tau \right) ds \right. \\ & \left. + B \sum_{i=1}^{m-2} b_i \psi \left(\int_0^{\xi_i} f(y(\tau)) d\tau \right) \left(1 - \sum_{i=1}^{m-2} a_i \xi_i \right) \right\} \end{aligned}$$

where ψ is the inverse of the function φ defined by $\psi(w) := |w|^{q-2}w$, with $q = \frac{p}{p-1} > 1$ and

$$A = \left(1 - \sum_{i=1}^{m-2} a_i \right)^{-1}, \quad B = \left(1 - \sum_{i=1}^{m-2} b_i \right)^{-1}.$$

The fixed points of the operator N are solutions to problem (3.1), (3.2) (see [1]). It is clear that $N(y)(t) \geq 0$ on J for any $y \in C([0, 1], \mathbb{R}^+)$. We shall first show that N is completely continuous. The proof will be given in three steps.

Step 1: N is continuous. Let $\{y_n\}$ be a sequence such that $y_n \rightarrow y$ in $C([0, 1], \mathbb{R})$. Set

$$L(y)(t) := \int_0^t |f(y(s))| ds.$$

Then

$$|L(y_n)(s) - L(y(s))| \leq \int_0^t |f(y_n(s)) - f(y(s))| ds \leq \int_0^1 |f(y_n(s)) - f(y(s))| ds.$$

Since f is a continuous function, it follows that

$$\|L(y_n) - L(y)\|_\infty \leq \|f(y_n(\cdot)) - f(y(\cdot))\|_{L^1} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Then

$$\begin{aligned}
|N(y_n(t)) - N(y(t))| &\leq \int_0^1 |\psi(L(y_n(t))) - \psi(L(y(t)))| ds \\
&\quad + tB \sum_{i=1}^{m-2} b_i |\psi(L(y_n(\xi_i))) - \psi(L(y(\xi_i)))| \\
&\quad + A \int_0^1 |\psi(L(y_n(s))) - \psi(L(y(s)))| ds \\
&\quad + A \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} |\psi(L(y_n(s))) - \psi(L(y(s)))| ds \\
&\quad + AB \sum_{i=1}^{m-2} b_i |\psi(L(y_n(\xi_i))) - \psi(L(y(\xi_i)))| \left(1 - \sum_{i=1}^{m-2} a_i \xi_i \right).
\end{aligned}$$

Since ψ is a continuous function, then

$$\begin{aligned}
\|N(y_n) - N(y)\|_\infty &\leq \|\psi(L(y_n)) - \psi(L(y))\|_\infty + B \sum_{i=1}^{m-2} b_i \|\psi(L(y_n)) - \psi(L(y))\|_\infty \\
&\quad + A \|\psi(L(y_n)) - \psi(L(y))\|_\infty + A \sum_{i=1}^{m-2} a_i \|\psi(L(y_n)) - \psi(L(y))\|_\infty \\
&\quad + AB \sum_{i=1}^{m-2} b_i \|\psi(L(y_n)) - \psi(L(y))\|_\infty \left(1 - \sum_{i=1}^{m-2} a_i \xi_i \right) \text{ as } n \rightarrow \infty.
\end{aligned}$$

Step 2: N maps bounded sets into bounded sets in $C([0, 1], \mathbb{R})$. Indeed, it is enough to show that, for any $q > 0$, there exists a positive constant ℓ such that, for each $y \in B_q = \{y \in C([0, 1], \mathbb{R}) : \|y\|_\infty \leq q\}$, we have $\|N(y)\|_\infty \leq \ell$. From (H2), we have

$$\left| \int_0^1 f(y(s)) ds \right| \leq \|h\|_{L^1} := q^*,$$

and

$$\begin{aligned}
|N(y)(t)| &\leq \int_0^1 |\psi(L(y(s)))| ds + tB \sum_{i=1}^{m-2} b_i |\psi(L(y(\xi_i)))| \\
&\quad + A \int_0^1 |\psi(L(y(s)))| ds + A \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} |\psi(L(y(s)))| ds \\
&\quad + AB \sum_{i=1}^{m-2} b_i |\psi(L(y(\xi_i)))| \left(1 - \sum_{i=1}^{m-2} a_i \xi_i \right).
\end{aligned}$$

Then

$$\begin{aligned}
\|N(y)\|_\infty &\leq \sup_{w \in [-q^*, q^*]} |\psi(w)| + B \sum_{i=1}^{m-2} b_i \sup_{w \in [-q^*, q^*]} |\psi(w)| \\
&\quad + A \sup_{w \in [-q^*, q^*]} |\psi(w)| + A \sum_{i=1}^{m-2} a_i \sup_{w \in [-q^*, q^*]} |\psi(w)| \\
&\quad + AB \sum_{i=1}^{m-2} b_i \sup_{w \in [-q^*, q^*]} |\psi(w)| \left(1 - \sum_{i=1}^{m-2} a_i \xi_i \right) := \ell.
\end{aligned}$$

Step 3: *N maps bounded sets into equicontinuous sets of $C([0, 1], \mathbb{R})$.* Let $l_1, l_2 \in [0, 1]$, $l_1 < l_2$ and B_q be a bounded set of $C([0, 1], \mathbb{R})$ as in Step 2. Let $y \in B_q$, then

$$\begin{aligned}
|N(y)(l_2) - N(y)(l_1)| &\leq (l_2 - l_1) \sup_{w \in [-q^*, q^*]} |\psi(w)| \\
&\quad + B(l_2 - l_1) \sum_{i=1}^{m-2} b_i \left| \psi \left(\int_0^{\xi_i} f(y(\tau)) d\tau \right) \right|.
\end{aligned}$$

As $l_2 \rightarrow l_1$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzelà-Ascoli theorem, we can conclude that $N : C([0, 1], \mathbb{R}) \rightarrow C([0, 1], \mathbb{R})$ is completely continuous.

Step 4: *The set*

$$\mathcal{E}(N) := \{y \in C([0, 1], \mathbb{R}) : y = \lambda N(y), \text{ for some } 0 < \lambda < 1\}$$

is bounded.

The reasoning used in the proof of Step 2 shows that the set $\mathcal{E}(N)$ is bounded.

Set $X := C([0, 1], \mathbb{R})$. As a consequence of Schaefer's fixed point theorem [20, p. 29] we deduce that N has a fixed point y which is a solution to problem (3.1), (3.2), and hence, a solution to problem (1.1), (1.2).

□

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*Authors' Addresses***M. Benchohra:**

LABORATOIRE DE MATHÉMATIQUES, UNIVERSITÉ DE SIDI BEL ABBÈS, BP 89, 22000 SIDI BEL ABBÈS,
ALGÉRIE

E-mail address: benchohra@univ-sba.dz

S. K. Ntouyas:

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOANNINA, 451 10 IOANNINA, GREECE

E-mail address: sntouyas@cc.uoi.gr

A. Ouahab:

LABORATOIRE DE MATHÉMATIQUES, UNIVERSITÉ DE SIDI BEL ABBÈS, BP 89, 22000 SIDI BEL ABBÈS,
ALGÉRIE



SECTIONALLY RESIDUATED LATTICES

I. CHAJDA

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ABSTRACT. The concept of residuum is relativized in the so-called sections of a given lattice. It is shown that such a concept still has a majority of good properties of residuum. The results correspond to previous ones involved in sectionally pseudocomplemented lattices.

Mathematics Subject Classification: 06D15

Keywords: Residuated lattice, sectionally pseudocomplemented lattice, sectional adjointness property

Residuated lattices were introduced by Ward and Dilworth [6] and studied by several authors. Two monographs contain a compendium on residuated lattices. They are that by Blyth and Janowitz [2] (where it is renamed as a residuated Abelian semi-group with a unit) and the book by R. Bělohlávek [1].

In this short note we will compare a certain modification of a residuated lattice with already introduced concepts (see [3, 4]).

At first, we recall the basic concept:

By a *residual lattice* is meant an algebra $\mathcal{L} = (L; \vee, \wedge, \otimes, \rightarrow, 0, 1)$ such that

- (i) $(L; \vee, \wedge, 0, 1)$ is a bounded lattice,
- (ii) $(L; \otimes, 1)$ is a commutative monoid,
- (iii) it satisfies the so-called *adjointness property*: $x \otimes z \leq y$ if and only if $z \leq x \rightarrow y$.

Let us note (see, e. g., [1]) that $x \rightarrow y$ is the greatest element of the set $\{z; x \otimes z \leq y\}$. Moreover, if we consider $x \otimes y = x \wedge y$, then $x \rightarrow y$ is the relative pseudo-complement of x with respect to y , i. e., for $\otimes = \wedge$ residuated lattices are just relatively pseudo-complemented lattices.

It is well known that every relatively pseudo-complemented lattice is distributive. An extension of relative pseudo-complementation for the non-distributive case was already involved in [3, 4]:

Definition 1. A lattice $\mathcal{L} = (L; \vee, \wedge, 1)$ with the greatest element 1 is *sectionally pseudo-complemented* if each interval $[y, 1]$ is a pseudo-complemented lattice.

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From now on, denote by $x \circ y$ the pseudo-complement of $x \vee y$ in the interval $[y, 1]$. Naturally, $x \vee y \in [y, 1]$ thus $\mathcal{L} = (L; \vee, \wedge, 1)$ is sectionally pseudo-complemented if and only if “ $\circ$ ” is an (everywhere defined) operation on L . The identities characterizing sectionally pseudo-complemented lattices are presented in [3], i. e., the class of these lattices is a variety in the signature $\{\vee, \wedge, \circ, 1\}$. We are going to apply a similar approach for the adjointness property:

Definition 2. An algebra $\mathcal{L} = (L; \vee, \wedge, \otimes, \rightarrow, 1)$ is called a *sectionally residuated lattice* if

- (i) $(L; \vee, \wedge, 1)$ is a lattice with the greatest element 1;
- (ii) $(L; \otimes, 1)$ is a commutative monoid;
- (iii) it satisfies the *sectional adjointness property*: $(x \vee y) \otimes z = y$ if and only if $y \leq z \leq x \rightarrow y$.

Example 1. Consider the lattice N_5 on Figure 1. Then it is not relatively pseudo-complemented since N_5 is not distributive. e. g. a relative pseudo-complement of c with respect to a does not exist. Thus, considering $\otimes = \wedge$, N_5 is not a residual lattice. On the contrary, for $\otimes = \wedge$ it is a sectionally residuated lattice where the operation $\rightarrow$

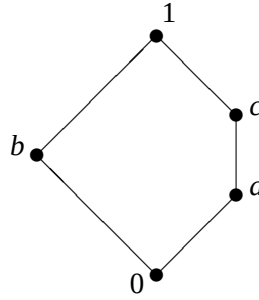


FIGURE 1

is determined by the table given below. The reader can verify the sectional adjointness property.

$\rightarrow$	0	a	c	b	1
0	1	1	1	1	1
a	b	1	1	b	1
c	b	a	1	b	1
b	c	a	c	1	1
1	0	a	c	b	1

The following result follows directly by Definition 2:

Lemma 1. Let $\mathcal{L} = (L; \vee, \wedge, \otimes, \rightarrow, 1)$ be a sectionally residuated lattice. Then $x \rightarrow y$ is the greatest element of the set $\{z; (x \vee y) \otimes z = y\}$.

This immediately yields the following facts:

$$(x \vee y) \otimes (x \rightarrow y) = y, \quad (1)$$

$$(x \vee y) \otimes y = y, \quad (2)$$

$$y \leq x \rightarrow y. \quad (3)$$

Lemma 2. Let $\mathcal{L} = (L; \vee, \wedge, \otimes, \rightarrow, 1)$ be a sectionally residuated lattice. Then $x \leq y$ if and only if $x \rightarrow y = 1$.

PROOF. Suppose $x \leq y$. Then $x \vee y = y$ and, by Lemma 1, $x \rightarrow y$ is the greatest element of the set $\{z; y \otimes z = y\}$. By Definition 2, $y \otimes 1 = 1$ thus $x \rightarrow y = 1$. Conversely, suppose $x \rightarrow y = 1$. Then, by (1), we have $y = (x \vee y) \otimes (x \rightarrow y) = (x \vee y) \otimes 1 = x \vee y$ whence $x \leq y$. $\square$

Lemma 3. In a sectionally residuated lattice, the following identities are satisfied: $x \rightarrow x = 1$, $x \rightarrow 1 = 1$, $0 \rightarrow x = 1$, and $1 \rightarrow x = x$.

PROOF. The first three identities follow directly by Lemma 2. Further, by Lemma 1, $1 \rightarrow x$ is the greatest element of the set $\{z; 1 \otimes z = x\} = \{x\}$ thus $1 \rightarrow x = x$. $\square$

Lemma 4. In a sectionally residuated lattice, $a \otimes b = a$ if and only if $a \leq b$.

PROOF. Putting $x = y = a$ and $z = b$ in the sectional adjointness property, the assumption $a \otimes b = a$ yields $(a \vee a) \otimes b = a$ iff $a \leq b \leq a \rightarrow a = 1$ thus $a \leq b$. Conversely, $a \leq b$ implies by Lemma 3 $a \leq b \leq 1 = a \rightarrow a$ and, by sectional adjointness, $a \otimes b = (a \vee a) \otimes b = a$. $\square$

Applying Lemma 2 and Lemma 4, we get

Corollary 1. In a sectionally residuated lattice,

- (a) $x \otimes y = x$ if and only if $x \rightarrow y = 1$;
- (b) $x \otimes x = x$.

Lemma 5. In a sectionally residuated lattice, $x \wedge y \leq x \otimes y$.

PROOF. By (3) we have $x \wedge y \leq x \rightarrow (x \wedge y)$. Applying sectional adjointness, we infer $x \otimes (x \wedge y) = (x \vee (x \wedge y)) \otimes (x \wedge y) = x \wedge y$ and, analogously, $y \otimes (x \wedge y) = x \wedge y$. Hence, by Corollary 1 (b),

$$x \otimes y \otimes (x \wedge y) = x \otimes (x \wedge y) \otimes y \otimes (x \wedge y) = (x \wedge y) \otimes (x \wedge y) = x \wedge y$$

and, by Lemma 4, $x \wedge y \leq x \otimes y$. $\square$

Theorem 1. Let $\mathcal{L} = (L; \vee, \wedge, \otimes, \rightarrow, 1)$ be a sectionally residuated lattice. Then $\otimes = \wedge$ and $\rightarrow = \circ$, i. e., it is a sectionally pseudo-complemented lattice.

PROOF. Replacing y by $x \wedge y$ in the sectional adjointness property, we obtain

$$x \otimes z = x \wedge y \quad \text{iff} \quad x \wedge y \leq z \leq x \rightarrow (x \wedge y).$$

However, $x \rightarrow (x \wedge y)$ is the greatest element of the set $\{t; (x \vee (x \wedge y)) \otimes t = x \wedge y\} = \{t; x \otimes t = x \wedge y\}$. By Lemma 5, $x \wedge t \leq x \otimes t = x \wedge y$, thus the greatest t of this property satisfies $t \geq y$.

Thus $y \leq x \rightarrow (x \wedge y)$, i. e.,

$$x \wedge y \leq y \leq x \rightarrow (x \wedge y)$$

and, by the sectional adjointness,

$$x \otimes y = (x \wedge (x \vee y)) \otimes y = x \wedge y.$$

Hence, $x \rightarrow y$ is the pseudocomplement of $x \vee y$ in the interval $[y, 1]$, i. e., $x \rightarrow y = x \circ y$. $\square$

Recall that a lattice $\mathcal{L} = (L; \vee, \wedge)$ is $\wedge$ -semidistributive if $a \wedge b_1 = a \wedge b_2$ implies $a \wedge b_1 = a \wedge (b_1 \vee b_2)$.

Corollary 2. *An algebraic lattice $\mathcal{L}$ is sectionally residuated if and only if $\mathcal{L}$ is $\wedge$ -semidistributive.*

PROOF. By Proposition 2.4 from [5], an algebraic lattice $\mathcal{L}$ is $\wedge$ -semidistributive if and only if $\mathcal{L}$ is sectionally pseudocomplemented. Hence, if $\mathcal{L}$ is sectionally residuated and algebraic, then, by this and Theorem 1, $\mathcal{L}$ is $\wedge$ -semidistributive. Conversely, if $\mathcal{L}$ is algebraic and $\wedge$ -semidistributive, then $\mathcal{L}$ is sectionally pseudocomplemented and hence sectionally residuated for $\otimes = \wedge$ and $\rightarrow = \circ$. $\square$

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Author's Address

I. Chajda:

PALACKÝ UNIVERSITY OF OLOMOUČ, DEPARTMENT OF ALGEBRA AND GEOMETRY, TOMKOVA 40, 779 00 OLOMOUČ, CZECH REPUBLIC

E-mail address: chajda@inf.upol.cz



UPPER AND LOWER SLIGHTLY α -CONTINUOUS MULTIFUNCTIONS

ERDAL EKICI

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ABSTRACT. The aim of this paper is to introduce and study upper and lower slightly α -continuous multifunctions as a generalization of upper and lower α -continuous multifunctions, respectively, due to Neubrunn (1988). Some characterizations and several properties concerning upper (lower) slightly α -continuous multifunctions are obtained. The relationships between upper (lower) slightly α -continuous multifunctions and upper (lower) α -continuous multifunctions are also discussed.

Mathematics Subject Classification: 54C05, 54C60, 54C08, 54C10

Keywords: α -open, slightly α -continuity, multifunction

1. INTRODUCTION

Various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good many of them have been extended to the setting of multifunctions. In 1988, Neubrunn introduced the concept of α -continuous multifunctions [7]. α -continuity which is a weaker form of continuity, in ordinary topology was extended to multifunctions.

The purpose of the present paper is to define upper (lower) slightly α -continuous multifunctions and to obtain several characterizations of upper (lower) slightly α -continuous multifunctions and basic properties of such multifunctions. Moreover, the relationships between upper (lower) slightly α -continuous multifunctions and upper (lower) α -continuous multifunctions are also discussed.

2. PRELIMINARIES

Throughout this paper, spaces (X, τ) and (Y, ν) (or simply X and Y) always mean topological spaces in which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X . For a subset A of (X, τ) , $\text{cl}(A)$ and $\text{int}(A)$ represent the closure of A with respect to τ and the interior of A with respect to τ , respectively.

A subset A is said to be α -open [8] (resp., preopen [4]) if $A \subset \text{int}(\text{cl}(\text{int}(A)))$, (resp., $A \subset \text{int}(\text{cl}(A))$). The complement of a α -open (resp., preopen) set is said to be

α -closed (preclosed). The family of all α -open (resp., preopen) sets of X is denoted by $\alpha O(X)$ (resp., $PO(X)$).

It is shown in (Njastad, 1965) that $\alpha O(X)$ is a topology for X .

By a multifunction $F : X \rightarrow Y$, we mean a point-to-set correspondence from X into Y , and always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \rightarrow Y$, following [1, 2] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$. For each $A \subset X$, $F(A) = \bigcup_{x \in A} F(x)$. Then F is said to be a surjection if $F(X) = Y$, or equivalently if for each $y \in Y$ there exists an $x \in X$ such that $y \in F(x)$.

Moreover, $F : X \rightarrow Y$ is called upper semi-continuous (resp., lower semi-continuous) if $F^+(V)$ (resp., $F^-(V)$) is open in X for every open set V of Y [10].

For a multifunction $F : X \rightarrow Y$, the graph multifunction $G_F : X \rightarrow X \times Y$ is defined as follows $G_F(x) = \{x\} \times F(x)$ for every $x \in X$ and the subset $\{\{x\} \times F(x) : x \in X\} \subset X \times Y$ is called the multigraph of F and is denoted by $G(F)$.

Definition 1. A multifunction $F : X \rightarrow Y$ is said to be:

- (1) Upper α -continuous at $x \in X$ [7] if for each open set V of Y containing $F(x)$, there exists $U \in \alpha O(X)$ containing x such that $F(U) \subset V$;
- (2) Lower α -continuous at $x \in X$ [7] if for each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in \alpha O(X)$ containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$;
- (3) Upper (lower) α -continuous if it has this property at each point of X .

3. SLIGHTLY α -CONTINUOUS MULTIFUNCTIONS

Definition 2. A multifunction $F : X \rightarrow Y$ is said to be:

- (1) Upper slightly α -continuous at $x \in X$ if for each clopen set V of Y containing $F(x)$, there exists $U \in \alpha O(X)$ containing x such that $F(U) \subset V$;
- (2) Lower slightly α -continuous at $x \in X$ if for each clopen set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in \alpha O(X)$ containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$;
- (3) Upper (lower) slightly α -continuous if it has this property at each point of X .

The following theorem states some characterizations of upper slightly α -continuous multifunctions.

We know that a net (x_α) in a topological space (X, τ) is called eventually in the set $U \subset X$ if there exists an index $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$.

Definition 3. A sequence (x_n) is said to α -converge to a point x if for every α -open set V containing x , there exists an index n_0 such that for $n \geq n_0$, $x_n \in V$. This is denoted by $x_n \rightarrow_\alpha x$.

Theorem 1. Let $F : X \rightarrow Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, ν) . Then the following statements are equivalent.

- (i) F is upper slightly α -continuous;
- (ii) For each $x \in X$ and for each clopen set V such that $x \in F^+(V)$, there exists an α -open set U containing x such that $U \subset F^+(V)$;
- (iii) For each $x \in X$ and for each clopen set V such that $x \in F^+(Y \setminus V)$, there exists an α -closed set H such that $x \in X \setminus H$ and $F^-(V) \subset H$;
- (iv) $F^+(V)$ is an α -open set for any clopen set $V \subset Y$;
- (v) $F^-(V)$ is an α -closed set for any clopen set $V \subset Y$;
- (vi) $F^-(Y \setminus V)$ is an α -closed set for any clopen set $V \subset Y$;
- (vii) $F^+(Y \setminus V)$ is an α -open set for any clopen set $V \subset Y$;
- (viii) For each $x \in X$ and for each net (x_α) which α -converges to x in X and for each clopen set $V \subset Y$ such that $x \in F^+(V)$, the net (x_α) is eventually in $F^+(V)$.

PROOF. (i) $\Leftrightarrow$ (ii). Clear.

(ii) $\Leftrightarrow$ (iii). Let $x \in X$ and let V be a clopen such that $x \in F^+(Y \setminus V)$. By (ii), there exists an α -open set U containing x such that $U \subset F^+(Y \setminus V)$. Then $F^-(V) \subset X \setminus U$. Take $H = X \setminus U$. We have $x \in X \setminus H$ and H is α -closed.

The converse is similar.

(i) $\Leftrightarrow$ (iv). Let $x \in F^+(V)$ and let V be a clopen set. From (i), there exists an α -open set U_x containing x such that $U_x \subset F^+(V)$. It follows that $F^+(V) = \bigcup_{x \in F^+(V)} U_x$ and hence $F^+(V)$ is α -open.

The converse can be shown easily.

(iv) $\Rightarrow$ (v). Let $V \subset Y$ be a clopen set. We have that $Y \setminus V$ is a clopen set. From (iv), $F^+(Y \setminus V) = X \setminus F^-(V)$ is an α -open set. Then it is obtained that $F^-(V)$ is an α -closed set.

(v) $\Rightarrow$ (iv). similar to the above.

(iv) $\Leftrightarrow$ (vi), (v) $\Leftrightarrow$ (vii). Since $F^-(Y \setminus V) = X \setminus F^+(V)$ and $F^+(Y \setminus V) = X \setminus F^-(V)$, the proof is clear.

(i) $\Rightarrow$ (viii). Let (x_α) be a net which α -converges to x in X and let $V \subset Y$ be any clopen set such that $x \in F^+(V)$. Since F is an upper slightly α -continuous multifunction, it follows that there exists an α -open set $U \subset X$ containing x such that $U \subset F^+(V)$. Since (x_α) α -converges to x , it follows that there exists an index $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. From here, we obtain that $x_\alpha \in U \subset F^+(V)$ for all $\alpha \geq \alpha_0$. Thus, the net (x_α) is eventually in $F^+(V)$.

(viii) $\Rightarrow$ (i). Suppose that (i) is not true. There exists a point x and a clopen set V with $x \in F^+(V)$ such that $U \not\subset F^+(V)$ for each α -open set $U \subset X$ containing x . Let $x_U \in U$ and $x_U \notin F^+(V)$ for each α -open set $U \subset X$ containing x . Then for the α -neighbourhood net (x_U) , $x_U \rightarrow_\alpha x$, but (x_U) is not eventually in $F^+(V)$. This is a contradiction. Thus, F is an upper slightly α -continuous multifunction. $\square$

Remark 1. For a multifunction $F : X \rightarrow Y$ from a topological space (X, τ) to a topological space (Y, ν) , the following implications hold:

upper semi-continuity $\implies$ upper α -continuity $\implies$ upper slightly α -continuity

However, the converse implications are not true in general.

Example 1. Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4, 5\}$. Let τ and ν , respectively, be the topologies on X and on Y given by $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\nu = \{\emptyset, Y, \{1, 2\}, \{3, 4\}, \{3, 4, 5\}, \{1, 2, 3, 4\}\}$. Define the multifunction $F : X \rightarrow Y$ by $F(a) = \{1, 3, 4, 5\}$, $F(b) = \{1, 5\}$ and $F(c) = \{1, 2, 3\}$. Then F is a lower slightly α -continuous multifunction but F is not lower α -continuous.

Example 2. Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4, 5\}$. Let τ and ν , respectively, be the topologies in X and Y given by the formulae $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\nu = \{\emptyset, Y, \{1, 2\}, \{3, 4\}, \{3, 4, 5\}, \{1, 2, 3, 4\}\}$. Define the multifunction $F : X \rightarrow Y$ by $F(a) = \{3, 4, 5\}$, $F(b) = \{1, 2\}$ and $F(c) = \{1, 2, 3\}$. Then F is an upper slightly α -continuous multifunction but F is not upper α -continuous.

The other implication is not reversible as shown in [7].

Recall that a space is 0-dimensional if its topology has a base consisting of clopen sets.

Theorem 2. Let $F : X \rightarrow Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, ν) . Suppose that Y is a 0-dimensional space. If F is an upper (lower) slightly α -continuous multifunction, then F is an upper (lower) α -continuous.

PROOF. Let $x \in X$ and let V be any open set such that $x \in F^+(V)$. Since Y is a 0-dimensional space, then there exists a clopen set G containing x such that $x \in F^+(G) \subset F^+(V)$. Since F is an upper slightly α -continuous multifunction, there exists an α -open set U containing x such that $U \subset F^+(G) \subset F^+(V)$. Thus, we obtain that F is upper α -continuous.

The proof of lower continuity is similar. □

The following theorem states some characterizations of a lower slightly α -continuous multifunction.

Theorem 3. Let $F : X \rightarrow Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, ν) . Then the following statements are equivalent.

- (i) F is lower slightly α -continuous;
- (ii) For each $x \in X$ and for each clopen set V such that $x \in F^-(V)$, there exists an α -open set U containing x such that $U \subset F^-(V)$;
- (iii) For each $x \in X$ and for each clopen set V such that $x \in F^-(Y \setminus V)$, there exists an α -closed set H such that $x \in X \setminus H$ and $F^+(V) \subset H$;
- (iv) $F^-(V)$ is an α -open set for any clopen set $V \subset Y$;
- (v) $F^+(V)$ is an α -closed set for any clopen set $V \subset Y$;
- (vi) $F^+(Y \setminus V)$ is an α -closed set for any clopen set $V \subset Y$;

- (vii) $F^-(Y \setminus V)$ is an α -open set for any clopen set $V \subset Y$;
- (viii) For each $x \in X$ and for each net (x_α) which α -converges to x in X and for each clopen set $V \subset Y$ such that $x \in F^-(V)$, the net (x_α) is eventually in $F^-(V)$.

PROOF. It can be obtained similarly as Theorem 4. $\square$

Theorem 4. Let $F : X \rightarrow Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, ν) and let $F(X)$ be endowed with subspace topology. If F is an upper slightly α -continuous multifunction, then $F : X \rightarrow F(X)$ is an upper slightly α -continuous multifunction.

PROOF. Since F is an upper slightly α -continuous, $F^+(V \cap F(X)) = F^+(V) \cap F^+(F(X)) = F^+(V)$ is α -open for each clopen subset V of Y . Hence $F : X \rightarrow F(X)$ is an upper slightly α -continuous multifunction. $\square$

Suppose that (X, τ) , (Y, ν) and (Z, ω) are topological spaces. It is known that if $F_1 : X \rightarrow Y$ and $F_2 : Y \rightarrow Z$ are multifunctions, then the multifunction $F_2 \circ F_1 : X \rightarrow Z$ is defined by $(F_2 \circ F_1)(x) = F_2(F_1(x))$ for each $x \in X$.

Theorem 5. Let (X, τ) , (Y, ν) , (Z, ω) be topological spaces and let $F : X \rightarrow Y$ and $G : Y \rightarrow Z$ be multifunctions. If $F : X \rightarrow Y$ is an upper (lower) α -continuous multifunction and $G : Y \rightarrow Z$ is an upper (lower) semi-continuous multifunction, then $G \circ F : X \rightarrow Z$ is an upper (lower) slightly α -continuous multifunction.

PROOF. Let $V \subset Z$ be any clopen set. From the definition of $G \circ F$, we have $(G \circ F)^+(V) = F^+(G^+(V))$ ($(G \circ F)^-(V) = F^-(G^-(V))$). Since G is an upper (lower) semi-continuous multifunction, it follows that $G^+(V)$ ($G^-(V)$) is an open set. Since F is an upper (lower) α -continuous multifunction, it follows that $F^+(G^+(V))$ ($F^-(G^-(V))$) is an α -open set. It shows that $G \circ F$ is an upper (lower) slightly α -continuous multifunction. $\square$

Lemma 1. If $A \in PO(X)$ and $B \in \alpha O(X)$, then $A \cap B \in \alpha O(A)$ [5].

Theorem 6. Let $F : X \rightarrow Y$ be a multifunction and let $U \in PO(X)$. If F is a lower (upper) slightly α -continuous multifunction, then the restriction multifunction $F|_U : U \rightarrow Y$ is a lower (upper) slightly α -continuous multifunction.

PROOF. Suppose that $V \subset Y$ is a clopen set. Let $x \in U$ and let $x \in F|_U^-(V)$. Since F is a lower slightly α -continuous multifunction, it follows that there exists $G \in \alpha O(X)$ containing x such that $G \subset F^-(V)$. From here we obtain that $x \in G \cap U \in \alpha O(U)$ and $G \cap U \subset F|_U^-(V)$. Thus, we show that the restriction multifunction $F|_U$ is a lower slightly α -continuous.

The proof of the upper slightly α -continuity of $F|_U$ is similar to the above. $\square$

Lemma 2. If $A \in \alpha O(Y)$ and $Y \in \alpha O(X)$, then $A \in \alpha O(X)$ [5].

Theorem 7. Let $\{U_\lambda : \lambda \in \Lambda\}$ be an α -open cover of a space X . Then a multifunction $F : X \rightarrow Y$ is upper slightly α -continuous (resp., lower slightly α -continuous) if and only if the restriction $F|_{U_\lambda} : U_\lambda \rightarrow Y$ is upper slightly α -continuous (resp., lower slightly α -continuous) for each $\lambda \in \Lambda$.

PROOF. We prove only the case when F is an upper slightly α -continuous.

($\Rightarrow$) Let $\lambda \in \Lambda$ and V be any clopen set of Y . Since F is upper slightly α -continuous, $F^+(V)$ is α -open in X . By Lemma 12, $(F|_{U_\lambda})^+(V) = F^+(V) \cap U_\lambda$ is α -open in U_λ and hence $F|_{U_\lambda}$ is slightly α -continuous.

($\Leftarrow$) Let V be any clopen set of Y . Since $F|_{U_\lambda}$ is slightly α -continuous for each $\lambda \in \Lambda$, $(F|_{U_\lambda})^+(V) = F^+(V) \cap U_\lambda$ is α -open in U_λ . By Lemma 14, $(F|_{U_\lambda})^+(V)$ is α -open in X for each $\lambda \in \Lambda$. We obtain that $F^+(V) = \bigcup_{\lambda \in \Lambda} (F|_{U_\lambda})^+(V)$ is α -open in X . Hence F is upper slightly α -continuous. $\square$

Lemma 3. For a multifunction $F : X \rightarrow Y$, the following holds:

$$(1) G_F^+(A \times B) = A \cap F^+(B);$$

$$(2) G_F^-(A \times B) = A \cap F^-(B)$$

for any subsets $A \subset X$ and $B \subset Y$ [9].

Theorem 8. Let $F : X \rightarrow Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, ν) . If the graph multifunction of F is an upper slightly α -continuous multifunction, then F is an upper slightly α -continuous multifunction.

PROOF. Let $x \in X$ and let $V \subset Y$ be a clopen set such that $x \in F^+(V)$. We obtain that $x \in G_F^+(X \times V)$ and that $X \times V$ is a clopen set. Since graph multifunction G_F is an upper slightly α -continuous, it follows that there exists an α -open set $U \subset X$ containing x such that $U \subset G_F^+(X \times V)$. Since $U \subset G_F^+(X \times V) = X \cap F^+(V) = F^+(V)$, we obtain that $U \subset F^+(V)$. Thus, F is upper slightly α -continuous multifunction. $\square$

Theorem 9. A multifunction $F : X \rightarrow Y$ is lower slightly α -continuous if $G_F : X \rightarrow X \times Y$ is lower slightly α -continuous.

PROOF. Suppose that G_F is lower slightly α -continuous. Let $x \in X$ and V be any clopen set of Y such that $x \in F^-(V)$. Then $X \times V$ is clopen in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$. Since G_F is lower slightly α -continuous, there exists an α -open set U containing x such that $U \subset G_F^-(X \times V)$. By the previous lemma, we have $U \subset F^-(V)$. This shows that F is lower slightly α -continuous. $\square$

Theorem 10. Suppose that (X, τ) and (X_α, τ_α) are topological spaces where $\alpha \in J$. Let $F : X \rightarrow \prod_{\alpha \in J} X_\alpha$ be a multifunction from X to the product space $\prod_{\alpha \in J} X_\alpha$ and let $P_\alpha : \prod_{\alpha \in J} X_\alpha \rightarrow X_\alpha$ be the projection multifunction for each $\alpha \in J$ which is defined by $P_\alpha((x_\alpha)) = \{x_\alpha\}$. If F is an upper (lower) slightly α -continuous multifunction, then $P_\alpha \circ F$ is an upper (lower) slightly α -continuous multifunction for each $\alpha \in J$.

PROOF. Take any $\alpha_0 \in J$. Let V_{α_0} be a clopen set in $(X_{\alpha_0}, \tau_{\alpha_0})$. Then $(P_{\alpha_0} \circ F)^+(V_{\alpha_0}) = F^+(P_{\alpha_0}^+(V_{\alpha_0})) = F^+(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$ (respectively, $(P_{\alpha_0} \circ F)^-(V_{\alpha_0}) = F^-(P_{\alpha_0}^-(V_{\alpha_0})) = F^-(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$). Since F is an upper (lower) slightly α -continuous multifunction and since $V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha$ is a clopen set, it follows that $F^+(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$ (respectively, $F^-(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$) is α -open in (X, τ) . It shows that $P_{\alpha_0} \circ F$ is an upper (lower) slightly α -continuous multifunction.

Hence, we obtain that $P_\alpha \circ F$ is an upper (lower) slightly α -continuous multifunction for each $\alpha \in J$. $\square$

Theorem 11. Suppose that for each $\alpha \in J$, (X_α, τ_α) , (Y_α, ν_α) are topological spaces. Let $F_\alpha : X_\alpha \rightarrow Y_\alpha$ be a multifunction for each $\alpha \in J$ and let $F : \prod_{\alpha \in J} X_\alpha \rightarrow \prod_{\alpha \in J} Y_\alpha$ be defined by $F((x_\alpha)) = \prod_{\alpha \in J} F_\alpha(x_\alpha)$ from the product space $\prod_{\alpha \in J} X_\alpha$ to the product space $\prod_{\alpha \in J} Y_\alpha$. If F is an upper (lower) slightly α -continuous multifunction, then each F_α is an upper (lower) slightly α -continuous multifunction for each $\alpha \in J$.

PROOF. Let $V_\alpha \subset Y_\alpha$ be a clopen set. Then $V_\alpha \times \prod_{\alpha \neq \beta} Y_\beta$ is a clopen set. Since F is an upper (lower) slightly α -continuous multifunction, it follows that $F^+(V_\alpha \times \prod_{\alpha \neq \beta} Y_\beta) = F_\alpha^+(V_\alpha) \times \prod_{\alpha \neq \beta} Y_\beta$ ($F^-(V_\alpha \times \prod_{\alpha \neq \beta} Y_\beta) = F_\alpha^-(V_\alpha) \times \prod_{\alpha \neq \beta} Y_\beta$) is an α -open set. Consequently, we obtain that $F_\alpha^+(V_\alpha)$ ($F_\alpha^-(V_\alpha)$) is an α -open set. Thus, we show that F_α is an upper (lower) slightly α -continuous multifunction. $\square$

For two multifunctions $F_1 : X_1 \rightarrow Y_1$ and $F_2 : X_2 \rightarrow Y_2$, the product multifunction $F_1 \times F_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is defined as follows: $(F_1 \times F_2)(x_1, x_2) = F_1(x_1) \times F_2(x_2)$ for every $x_1 \in X_1$ and $x_2 \in X_2$.

Theorem 12. Suppose that $F_1 : X_1 \rightarrow Y_1$, $F_2 : X_2 \rightarrow Y_2$ are multifunctions. If $F_1 \times F_2$ is an upper (lower) slightly α -continuous multifunction, then F_1 and F_2 are upper (lower) slightly α -continuous multifunctions.

PROOF. Let $K \subset Y_1$, $H \subset Y_2$ be clopen sets. It is known that $K \times H$ is a clopen set and $(F_1 \times F_2)^+(K \times H) = F_1^+(K) \times F_2^+(H)$. Since $F_1 \times F_2$ is an upper slightly α -continuous multifunction, it follows that $F_1^+(K) \times F_2^+(H)$ is an α -open set. From here, $F_1^+(K)$ and $F_2^+(H)$ are α -open sets. Hence, it is obtained that F_1 and F_2 are upper slightly α -continuous multifunctions.

The proof of the lower slightly α -continuity of F_1 and F_2 is similar to the above argument. $\square$

Theorem 13. Suppose that (X, τ) , (Y, ν) , (Z, ω) are topological spaces and $F_1 : X \rightarrow Y$, $F_2 : X \rightarrow Z$ are multifunctions. Let $F_1 \times F_2 : X \rightarrow Y \times Z$ be a multifunction which is defined by $(F_1 \times F_2)(x) = F_1(x) \times F_2(x)$ for each $x \in X$. If $F_1 \times F_2$ is an upper (lower) slightly α -continuous multifunction, then F_1 and F_2 are upper (lower) slightly α -continuous multifunctions.

PROOF. Let $x \in X$ and let $K \subset Y$, $H \subset Z$ be clopen sets such that $x \in F_1^+(K)$ and $x \in F_2^+(H)$. Then we obtain that $F_1(x) \subset K$ and $F_2(x) \subset H$ and from here, $F_1(x) \times F_2(x) \subset$

$(F_1 \times F_2)(x) \subset K \times H$. We have $x \in (F_1 \times F_2)^+(K \times H)$. Since $F_1 \times F_2$ is upper slightly α -continuous multifunction, it follows that there exists an α -open set U containing x such that $U \subset (F_1 \times F_2)^+(K \times H)$. We obtain that $U \subset F_1^+(K)$ and $U \subset F_2^+(H)$. Thus, we obtain that F_1 and F_2 are upper slightly α -continuous multifunctions.

The proof of the lower slightly α -continuity of F_1 and F_2 is similar to the above. $\square$

Definition 4. Let (X, τ) be a topological space. X is said to be a strongly normal space if for any disjoint closed subsets K and F of X , there exists two clopen sets U and V such that $K \subset U$, $F \subset V$ and $U \cap V = \emptyset$.

Example 3. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then (X, τ) is a strongly normal space.

Recall that a multifunction $F : X \rightarrow Y$ is said to be punctually closed if, for each $x \in X$, $F(x)$ is closed.

Theorem 14. If Y is strongly normal space and $F_i : X_i \rightarrow Y$ is an upper slightly α -continuous multifunction such that F_i is punctually closed for $i = 1, 2$, then a set $\{(x_1, x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) \neq \emptyset\}$ is an α -closed set in $X_1 \times X_2$.

PROOF. Let $A = \{(x_1, x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) \neq \emptyset\}$ and $(x_1, x_2) \in (X_1 \times X_2) \setminus A$. Then $F_1(x_1) \cap F_2(x_2) = \emptyset$. Since Y is strongly normal and F_i is punctually closed for $i = 1, 2$, there exist disjoint clopen sets V_1, V_2 such that $F_i(x_i) \subset V_i$ for $i = 1, 2$. Since F_i is upper slightly α -continuous, $F_i^+(V_i)$ is α -open for $i = 1, 2$. Put $U = F_1^+(V_1) \times F_2^+(V_2)$, then U is α -open and $(x_1, x_2) \in U \subset (X_1 \times X_2) \setminus A$. This shows that $(X_1 \times X_2) \setminus A$ is α -open and hence A is α -closed in $(X_1 \times X_2)$. $\square$

Theorem 15. Let F and G be upper slightly α -continuous and punctually closed multifunctions from a topological space X to a strongly normal topological space Y . Then the set $K = \{x : F(x) \cap G(x) \neq \emptyset\}$ is α -closed in X .

PROOF. Let $x \in X \setminus K$. Then $F(x) \cap G(x) = \emptyset$. Since F and G are punctually closed multifunctions and Y is a strongly normal space, it follows that there exist disjoint clopen sets U and V containing $F(x)$ and $G(x)$, respectively. Since F and G are upper slightly α -continuous, then the sets $F^+(U)$ and $G^+(V)$ are α -open and contain x . Let $H = F^+(U) \cup G^+(V)$. Then H is an α -open set containing x and $H \cap K = \emptyset$. Hence, K is α -closed in X . $\square$

Definition 5. A space X is said to be α -Hausdorff if for each pair of distinct points x and y in X , there exist disjoint α -open sets U and V in X such that $x \in U$ and $y \in V$ [6].

Theorem 16. Let $F : X \rightarrow Y$ be an upper slightly α -continuous multifunction and punctually closed from a topological space X to a strongly normal topological space Y and let $F(x) \cap F(y) = \emptyset$ for each distinct pair $x, y \in X$. Then X is a α -Hausdorff space.

PROOF. Let x and y be any two distinct points in X . Then we have $F(x) \cap F(y) = \emptyset$. Since Y is a strongly normal space, it follows that there exist disjoint clopen sets U and V containing $F(x)$ and $F(y)$ respectively. Thus $F^+(U)$ and $F^+(V)$ are disjoint α -open sets containing x and y , respectively. Thus, it is obtained that X is α -Hausdorff. $\square$

4. SOME PROPERTIES

Definition 6. A space X is said to be mildly compact if every clopen cover of X has a finite subcover [11].

Definition 7. A space X is said to be α -compact if every α -open cover of X has a finite subcover [3].

Theorem 17. Let $F : X \rightarrow Y$ be an upper slightly α -continuous surjective multifunction such that $F(x)$ is mildly compact for each $x \in X$. If X is an α -compact space, then Y is mildly compact.

PROOF. Let $\{V_\lambda : \lambda \in \Lambda\}$ be a clopen cover of Y . Since $F(x)$ is mildly compact for each $x \in X$, there exists a finite subset $\Lambda(x)$ of Λ such that $F(x) \subset \bigcup\{V_\lambda : \lambda \in \Lambda(x)\}$. Put

$$V(x) = \bigcup\{V_\lambda : \lambda \in \Lambda(x)\}.$$

Since F is an upper slightly α -continuous, there exists an α -open set $U(x)$ of X containing x such that $F(U(x)) \subset V(x)$. Then the family $\{U(x) : x \in X\}$ is a α -open cover of X and since X is α -compact, there exists a finite number of points, say, $x_1, x_2, x_3, \dots, x_n$ in X such that $X = \bigcup\{U(x_i) : i = 1, 2, 3, \dots, n\}$. Hence we have

$$Y = F(X) = F\left(\bigcup_{i=1}^n U(x_i)\right) = \bigcup_{i=1}^n F(U(x_i)) \subset \bigcup_{i=1}^n V(x_i) = \bigcup_{i=1}^n \bigcup_{\lambda \in \Lambda(x_i)} V_\lambda.$$

This shows that Y is mildly compact. $\square$

Definition 8. Let $F : X \rightarrow Y$ be a multifunction. The multigraph $G(F)$ is said to be α -co-closed if for each $(x, y) \notin G(F)$, there exist α -open set U and clopen set V containing x and y , respectively, such that $(U \times V) \cap G(F) = \emptyset$.

Definition 9. A space X is said to be co-Hausdorff if for each pair of distinct points x and y in X , there exist disjoint clopen sets U and V in X such that $x \in U$ and $y \in V$.

Theorem 18. If a multifunction $F : X \rightarrow Y$ is an upper slightly α -continuous multifunction such that $F(x)$ is mildly compact relative to Y for each $x \in X$ and Y is co-Hausdorff space, then the multigraph $G(F)$ of F is α -co-closed in $X \times Y$.

PROOF. $(x, y) \notin G(F)$. That is $y \notin F(x)$. Since Y is co-Hausdorff, for each $z \in F(x)$, there exist disjoint clopen sets $V(z)$ and $U(z)$ of Y such that $z \in U(z)$ and $y \in V(y)$.

Then $\{U(z) : z \in F(x)\}$ is clopen cover of $F(x)$ and since $F(x)$ is mildly compact, there exists a finite number of points, say, $z_1, z_2, z_3, \dots, z_n$ in $F(x)$ such that

$$F(x) \subset \bigcup \{U(z_i) : i = 1, 2, 3, \dots, n\}.$$

Put

$$U = \bigcup \{U(z_i) : i = 1, 2, 3, \dots, n\} \text{ and } V = \bigcap \{V(y_i) : i = 1, 2, 3, \dots, n\}.$$

Then U and V are clopen in Y such that $F(x) \subset U$, $y \in V$ and $U \cap V = \emptyset$. Since F is an upper slightly α -continuous multifunction, there exists α -open W containing x such that $F(W) \subset U$. We have $(x, y) \in W \times V \subset (X \times Y) \setminus G(F)$. We obtain that $(W \times V) \cap G(F) = \emptyset$ and hence $G(F)$ is α -co-closed in $X \times Y$. $\square$

Theorem 19. *Let $F : X \rightarrow Y$ be a multifunction having α -co-closed multigraph $G(F)$. If B is a mildly compact subset relative to Y , then $F^-(B)$ is α -closed in $X \times Y$.*

PROOF. Let $x \in X \setminus F^-(B)$. For each $y \in B$, $(x, y) \notin G(F)$ and there exist an α -open set $U(y) \subset X$ and a clopen set $V(y) \subset Y$ containing x and y , respectively, such that $F(U(y)) \cap V(y) = \emptyset$. That is $U(y) \cap F^-(V(y)) = \emptyset$. Then $\{V(y) : y \in B\}$ is clopen cover of B and since B is mildly compact relative to Y , there exists a finite subset B_0 of B such that $B \subset \bigcup \{V(y) : y \in B_0\}$. Put

$$U = \bigcap \{U(y) : y \in B_0\}.$$

Then U is α -open in X , $x \in U$ and $U \cap F^-(B) = \emptyset$; i. e., $x \in U \subset X \setminus F^-(B)$. This shows that $F^-(B)$ is α -closed in X . $\square$

Recall that a multifunction $F : X \rightarrow Y$ is said to be punctually connected if, for each $x \in X$, $F(x)$ is connected.

Definition 10. A space X is called α -connected provided that X is not the union of two disjoint nonempty α -open sets.

Theorem 20. *Let F be a multifunction from an α -connected topological space X onto a topological space Y such that F is punctually connected. If F is an upper slightly α -continuous multifunction, then Y is a connected space.*

PROOF. Let $F : X \rightarrow Y$ be a an upper slightly α -continuous multifunction from a α -connected topological space X onto a topological space Y . Suppose that Y is not connected and let $Y = H \cup K$ be a partition of Y . Then both H and K are open and closed subsets of Y . Since F is an upper slightly α -continuous multifunction, $F^+(H)$ and $F^+(K)$ are α -open subsets of X . In view of the fact that $F^+(H)$, $F^+(K)$ are disjoint and F is punctually connected, $X = F^+(H) \cup F^+(K)$ is a partition of X . This is contrary to the α -connectedness of X . Hence, it is obtained that Y is a connected space. $\square$

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*Author's Address***Erdal Ekici:**

DEPARTMENT OF MATHEMATICS, CANAKKALE ONSEKIZ MART UNIVERSITY, TERZIOGLU CAMPUS, 17100 CANAKKALE, TURKEY

E-mail address: eekici@comu.edu.tr



BANACH–STEINHAUS TYPE THEOREM IN LOCALLY CONVEX SPACES FOR LINEAR Σ -LOCALLY LIPSCHITZIAN OPERATORS

SAMIR LAHRECH

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ABSTRACT. In the past, all Banach–Steinhaus type results have been established only for some special classes of locally convex spaces, e.g., barrelled spaces ([2], [3], [4]), s-barrelled spaces ([5]), strictly s-barrelled spaces ([6]), etc. Recently, Cui Chengri and Songho Han ([1]) have obtained a Banach–Steinhaus type result for linear bounded operators which is valid in every locally convex space.

In this paper we would like to prove the same result, but for linear locally Lipschitzian operators.

Mathematics Subject Classification: 47B37, 46A45

Keywords: Banach–Steinhaus theorem, locally convex spaces, linear locally Lipschitzian operators

Let (X, λ) and (Y, μ) be locally convex spaces. Assume that the locally convex topology μ is generated by the family $(q_\beta)_{\beta \in I}$ of semi-norms on Y . Let $B(X_\lambda)$ denote the family of bounded set in (X, λ) and let $\sigma \subset B(X_\lambda)$.

For a linear mapping $T : X \rightarrow Y$, a semi-norm p on Y and $C \in \sigma$, set

$$L(p, C)(T) = \sup_{h \in C} p(Th).$$

Let $T : X \rightarrow Y$ be a linear operator. T is said to be σ -locally Lipschitzian if

$$\forall C \in \sigma \quad \forall \beta \in I \quad L(\beta, C)(T) \equiv L(q_\beta, C)(T) < \infty.$$

By $\text{Lip}(X_\lambda, Y_\mu, \sigma)$ we denote the vector space of σ -locally Lipschitzian operators. Note that $\text{Lip}(X_\lambda, Y_\mu, \sigma)$ is a locally convex space under the locally convex topology $\tau(\mu, \sigma)$ generated by the family of semi-norms $L(\beta, C)$, $\beta \in I$, $C \in \sigma$.

An operator $T : X \rightarrow Y$ is said to be sequentially continuous if $\{x_n\}$ is a sequence in X such that $x_n \rightarrow x$ then $Tx_n \rightarrow Tx$; T is said to be bounded if T sends bounded sets into bounded sets. Clearly, continuous operators are sequentially continuous; sequentially continuous operators are bounded, and linear bounded operators are σ -locally Lipschitzian but in general, converse implications fail. Let X' , X^s , X^b and X_σ^L denote the families of continuous linear functionals, sequentially continuous linear

functionals, bounded linear functionals and σ -locally Lipschitzian functionals on X , respectively. In general, the inclusions $X' \subset X^s \subset X^b \subset X_\sigma^L$ are strict.

Let $\theta(X, X_\sigma^L)$ denote the topology of uniform convergence on $\sigma(X_\sigma^L, X)$ -Cauchy sequences in X_σ^L . Note that if $\sigma = B(X_\lambda)$, then $X^b = X_\sigma^L$ and consequently, $\theta(X, X_\sigma^L) = \theta(X, X^b)$.

Theorem 1. *Let (X, λ) , (Y, μ) be locally convex spaces and $T_n : X \rightarrow Y$ σ -locally Lipschitzian operators, $n \in N$. If $\text{weak-lim}_n T_n x = Tx$ exists at each $x \in X$, then the limit operator T maps $\theta(X, X_\sigma^L)$ -bounded sets into bounded sets.*

PROOF. Let $y' \in Y'$. Then $\lim_n y'(T_n x) = y'(Tx)$ for each $x \in X$. So $(y' \circ T_n)_{n \in N}$ is a $\sigma(X_\sigma^L, X)$ -Cauchy sequence in X_σ^L . Suppose that B is $\theta(X, X_\sigma^L)$ -bounded subset of X and $\{x_k\} \subset B$. Then $\frac{1}{k}x_k \rightarrow 0$ in $(X, \theta(X, X_\sigma^L))$, so $\lim_k \frac{1}{k}y'(T_n x_k) = 0$ uniformly in $n \in N$.

Now fix $\varepsilon > 0$. There is a $k_0 \in N$ such that $|\frac{1}{k}y'(T_n x_k)| < \frac{\varepsilon}{2}$ for all $n \in N$ and all $k \geq k_0$. Fix a $k \geq k_0$. Since $\lim_n y'(T_n x_k) = y'(Tx_k)$ there is an $n_0 \in N$ such that $|y'(T_{n_0} x_k) - y'(Tx_k)| < \frac{\varepsilon}{2}$. Therefore,

$$|y'(Tx_k)| \leq |y'(Tx_k) - y'(T_{n_0} x_k)| + |y'(T_{n_0} x_k)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2k} < \varepsilon.$$

This shows that $\{y'(Tx) : x \in B\}$ is bounded. Since $y' \in Y'$ is arbitrary, $T(B)$ is μ -bounded by the classical Mackey theorem. Thus, we obtain the proof. $\square$

Theorem 1 shows that the limit operator T can be bounded even if the sequence $(T_n)_n$ is not bounded.

Now we have the following useful result.

Theorem 2. *For a locally convex space X the following conditions are equivalent.*

- (1) *For every locally convex space Y and for every sequence T_n of σ -locally Lipschitzian linear operators from X into Y such that $\text{weak-lim}_n T_n x = Tx$ exists at each $x \in X$, the limit operator T is σ -locally Lipschitzian.*
- (2) *$(X_\sigma^L, \sigma(X_\sigma^L, X))$ is sequentially complete.*

PROOF. (1) $\Rightarrow$ (2). Let $\{f_n\}$ be a $\sigma(X_\sigma^L, X)$ -Cauchy sequence in X_σ^L . Then, there exists a linear functional f such that for every $x \in X$ $\lim_n f_n(x) = f(x)$. Consequently, $f \in X_\sigma^L$ by (1).

(2) $\Rightarrow$ (1). Let Y be a locally convex space and $\{T_n\}$ a sequence of σ -locally Lipschitzian linear operators from X into Y such that $\text{weak-lim}_n T_n x = Tx$ exists at each $x \in X$. Let $y' \in Y'$, $C \in \sigma$. Then $\lim_n y'(T_n x) = y'(Tx)$ at each $x \in X$. Since $y' \circ T_n \in X_\sigma^L$ for all $n \in N$, $y' \circ T \in X_\sigma^L$ by (2). Therefore $\{y'(Tx) : x \in C\}$ is bounded and hence $T(C)$ is μ -bounded by the classical Mackey theorem. Thus, T is σ -locally Lipschitzian. $\square$

Remark 1. The proof of [1, Theorem 4] seems to be incorrect and, hence, the result is not correct because even if $\lim_n y'(T_n x) = y'(Tx)$ for each $x \in X$, $\{y' \circ T_n : n \in N\}$

is not necessarily conditionally $\sigma(X^b, X)$ -sequentially compact. Indeed, $y' \circ T$ is not necessarily in X^b . Consequently, $y' \circ T_n$ do not converge to $y' \circ T$ in $(X^b, \sigma(X^b, X))$.

Let X^θ denote the space of linear $\theta(X, X_\sigma^L)$ -bounded functionals on X . By $\eta(X, X^\theta)$ we denote the topology of uniform convergence on conditionally $\sigma(X^\theta, X)$ -sequentially compact sets of X^θ .

Now we have a useful proposition as follows.

Theorem 3. *Let (X, λ) , (Y, μ) be locally convex spaces and $T_n : X \rightarrow Y$ σ -locally Lipschitzian operators, $n \in N$. If $\text{weak-lim}_n T_n x = Tx$ exists at each $x \in X$, then the limit operator T maps $\eta(X, X^\theta)$ -bounded sets into bounded sets.*

PROOF. Let $y' \in Y'$. Then $\lim_n y'(T_n x) = y'(Tx)$ for each $x \in X$. It follows from Theorem 2 that $T \in X^\theta$. Consequently, $y' \circ T \in X^\theta$. On the other hand, since $(y' \circ T_n)_{n \in N}$ is $\sigma(X_\sigma^L, X)$ -Cauchy sequence in X_σ^L , then $y' \circ T_n \in X^\theta$. Therefore, $\{y' \circ T_n : n \in N\}$ is conditionally $\sigma(X^\theta, X)$ -sequentially compact.

Suppose that B is a $\eta(X, X^\theta)$ -bounded subset of X and $\{x_k\} \subset B$. Then $\frac{1}{k}x_k \rightarrow 0$ in $(X, \eta(X, X^\theta))$, so $\lim_k \frac{1}{k}y'(T_n x_k) = 0$ uniformly in $n \in N$.

Now fix $\varepsilon > 0$. There is a $k_0 \in N$ such that $|\frac{1}{k}y'(T_n x_k)| < \frac{\varepsilon}{2}$ for all $n \in N$ and all $k \geq k_0$. Fix a $k \geq k_0$. Since $\lim_n y'(T_n x_k) = y'(Tx_k)$ there is an $n_0 \in N$ such that $|y'(T_{n_0} x_k) - y'(Tx_k)| < \frac{\varepsilon}{2}$. Therefore,

$$|y'(Tx_k)| \leq |y'(Tx_k) - y'(T_{n_0} x_k)| + |y'(T_{n_0} x_k)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2k} < \varepsilon.$$

This shows that $\{y'(Tx) : x \in B\}$ is bounded. Since $y' \in Y'$ is arbitrary, $T(B)$ is μ -bounded by the classical Mackey theorem. Thus, we achieve the proof. $\square$

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Author's Address

Samir Lahrech:

UNIVERSITÉ MOHAMED 1ER, FACULTÉ DES SCIENCES, DÉPARTEMENT DE MATHÉMATIQUE, OUJDA, MOROCCO



SOLUTIONS OF SINGULAR ANTIPERIODIC BOUNDARY VALUE PROBLEMS

OTO PŘIBYL

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ABSTRACT. Sufficient conditions for the existence of a solution of the equation

$$(g(x'(t)))' = f(t, x(t), x'(t))$$

with the antiperiodic conditions $x(0) + x(T) = 0$, $x'(0) + x'(T) = 0$ are established. Our nonlinearity f may be singular at its phase variables. The proofs are based on a combination of regularity and sequential techniques and use the topological transversality principle.

Mathematics Subject Classification: 34B15, 34B16

Keywords: singular second-order differential equation, g-Laplacian, antiperiodic boundary conditions, topological transversality principle, Vitali's convergence theorem

1. INTRODUCTION

Let T be a positive number, $J = [0, T]$, $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$. We will consider the differential equation

$$(g(x'(t)))' = f(t, x(t), x'(t)), \quad (1.1)$$

together with the boundary conditions of the antiperiodic type

$$x(0) + x(T) = 0, \quad x'(0) + x'(T) = 0. \quad (1.2)$$

Here f satisfies the local Carathéodory conditions on $J \times \mathbb{R}_0^2$ ($f \in \text{Car}(J \times \mathbb{R}_0^2)$) and f may be singular at the zero value of its phase variables.

A function $x \in C^1(J)$ is said to be a *solution of the boundary value problem* (BVP for short) (1.1), (1.2) if $g(x') \in AC(J)$ (that is $g(x')$ is absolutely continuous on J), x satisfies the boundary conditions (1.2) and (1.1) is satisfied a. e. on J .

The object of the work is to give assumptions on the functions g and f in (1.1) which guarantee the solvability of BVP (1.1), (1.2). We recall that by our definition, solutions of BVP (1.1), (1.2) have the “maximal” smoothness (that is, $x \in C^1(J)$ and $g(x') \in AC(J)$). Also any solution and its derivative ‘pass through’ the singular points of f at inner points of the interval J . Since our existence results are proved

by regularization and sequential methods, we will use Vitali's convergence theorem (see [3]) in limiting processes since it is impossible to find an integrable majorant for applying the Lebesgue Dominated Convergence Theorem. Boundary conditions of the antiperiodic type appear in the literature usually as the special case of nonlocal boundary conditions. They were considered for regular (in phase variables) n -order differential equations (see, e. g., [4, 5]), systems of differential equations (see, e. g., [8]) and second-order functional differential equations (see, e. g., [11]). In the literature there are only a few papers dealing with BVPs whose nonlinearities are singular at the phase variables and their solutions and their derivatives 'pass through' singular points at inner points of the interval J . Such BVPs were discussed for higher order differential equations (see, e. g., [1, 9]) and second-order differential equations (see, e. g., [2, 10, 12–14]). From now on, $\|x\| = \max\{|x(t)|; t \in J\}$, $\|x\|_L = \int_0^T |x(t)| dt$ and $\|x\|_\infty = \text{ess sup}\{|x(t)|; t \in J\}$ stands for the norm in $C^0(J)$, $L_1(J)$ and $L_\infty(J)$, respectively. For a measurable set $M \subset \mathbb{R}$, $\mu(M)$ denotes the Lebesgue measure of M . Throughout the paper, we assume that the following assumptions are satisfied:

(H₁) $g \in C^0(\mathbb{R})$ is an increasing and odd function on $\mathbb{R}$ and $\lim_{u \rightarrow \infty} g(u) = \infty$;

(H₂) $f \in \text{Car}(J \times \mathbb{R}_0^2)$ and there exists $a \in \mathbb{R}_+$ such that

$$a \leq f(t, u, v) \text{ for a. e. } t \in J \text{ and all } (u, v) \in \mathbb{R}_0^2;$$

(H₃) For a. e. $t \in J$ and all $(u, v) \in \mathbb{R}_0^2$,

$$f(t, u, v) \leq h(|u|) \left[\omega_1(g(|v|)) + \omega_2(g(|v|)) \right]$$

where $\omega_1 : [0, \infty) \rightarrow \mathbb{R}_+$ is continuous and increasing and

$$\int_0^\infty \frac{1}{\omega_1(s)} ds = \infty, \quad (1.3)$$

$h, \omega_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are continuous and decreasing and

$$\int_0^1 h(s) ds < \infty, \quad \int_0^1 \omega_2(s) ds < \infty. \quad (1.4)$$

The paper is organized as follows. Section 2 deals with a sequence of auxiliary regular BVPs to problem (1.1), (1.2). We give a priori bounds for their solutions x_n and prove their existence by the topological transversality principle (see, e. g., [6] or [7]). In addition, we prove that the sequence $\{f_n(t, x_n(t), x'_n(t))\}$ is uniformly integrable on J . In Section 3, applying Vitali's convergence theorem, existence results to BVP (1.1), (1.2) are proved. We also include an example to illustrate our theory.

*Since $h, \omega_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are decreasing, the condition (1.4) implies that $\int_0^V h(s) ds < \infty$ and $\int_0^V \omega_2(s) ds < \infty$ for all $V > 0$.

2. AUXILIARY REGULAR BVPs AND THEIR PROPERTIES

For each $n \in \mathbb{N}$ and $\lambda \in [0, 1]$, define f_n and $h_n(\lambda; \cdot) \in \text{Car}(J \times \mathbb{R}^2)$ by

$$f_n(t, u, v) = \begin{cases} f(t, u, v) \text{ for } (t, u, v) \in J \times \left(\mathbb{R} \setminus \left[-\frac{1}{n}, \frac{1}{n}\right]\right)^2; \\ \frac{n}{2} \left[\left(u + \frac{1}{n}\right) \cdot f(t, 1/n, v) - \left(u - \frac{1}{n}\right) \cdot f(t, -1/n, v) \right] \\ \text{for } (t, u, v) \in J \times \left[-\frac{1}{n}, \frac{1}{n}\right] \times \left(\mathbb{R} \setminus \left[-\frac{1}{n}, \frac{1}{n}\right]\right); \\ \frac{n}{2} \left[\left(v + \frac{1}{n}\right) \cdot f_n(t, u, 1/n) - \left(v - \frac{1}{n}\right) \cdot f_n(t, u, -1/n) \right] \\ \text{for } (t, u, v) \in J \times \mathbb{R} \times \left[-\frac{1}{n}, \frac{1}{n}\right] \end{cases}$$

and

$$h_n(\lambda; t, u, v) = \lambda f_n(t, u, v) + (1 - \lambda)a \quad \text{for } (t, u, v) \in J \times \mathbb{R}^2.$$

Then

$$a \leq h_n(\lambda; t, u, v) \leq f_n(t, u, v) \text{ for a. e. } t \in J \text{ and each } (u, v) \in \mathbb{R}^2 \quad (2.1)$$

and

$$f_n(t, u, v) \leq h(|u|) \left[\omega_1(g(|v|) + g(1)) + \omega_2(g(|v|)) \right] \quad (2.2)$$

for a. e. $t \in J$ and each $(u, v) \in \mathbb{R}_0^2$.

Consider the family of regular differential equations

$$(g(x'(t)))' = h_n(\lambda; t, x(t), x'(t)) \quad (2.3)$$

depending on the parameters $n \in \mathbb{N}$ and $\lambda \in [0, 1]$.

Lemma 1. *Let x be a solution of BVP (2.3), (1.2) with some $\lambda \in [0, 1]$ and $n \in \mathbb{N}$ in (2.3). Then*

(a) x' is increasing on J and

$$g(|x'(t)|) \geq a|\xi - t| \quad \text{for } t \in [0, T] \quad (2.4)$$

where $\xi \in (0, T)$ is the unique zero of x' ;

(b) x has a unique zero η on $[0, T)$, $\eta \neq \xi$ and if $\xi < \eta$ then

$$|x(t)| \geq \begin{cases} \frac{S}{\xi} t & \text{for } t \in [0, \xi] \\ \frac{S}{\eta - \xi} |t - \eta| & \text{for } t \in (\xi, T] \end{cases} \quad (2.5)$$

and if $\eta < \xi$ then

$$|x(t)| \geq \begin{cases} \frac{S}{\xi - \eta} |t - \eta| & \text{for } t \in [0, \xi] \\ \frac{S}{T - \xi} (T - t) & \text{for } t \in (\xi, T] \end{cases} \quad (2.6)$$

where

$$S = \frac{1}{2a} \int_0^{\frac{aT}{2}} g^{-1}(s) ds.^\dagger \quad (2.7)$$

PROOF. Since $(g(x'(t)))' = h_n(\lambda; t, x(t), x'(t)) \geq a$ for a. e. $t \in J$ by (2.1) and g is increasing on $\mathbb{R}$, we see that x' is increasing on J and from $x'(0) = -x'(T)$ it follows that x' has a unique zero $\xi \in (0, T)$ and then $x' < 0$ on $[0, \xi)$ and $x' > 0$ on $(\xi, T]$. Next, we have

$$\begin{aligned} -g(x'(t)) &= \int_t^\xi (g(x'(s)))' ds \geq a(\xi - t), \quad t \in [0, \xi], \\ g(x'(t)) &= \int_\xi^t (g(x'(s)))' ds \geq a(t - \xi), \quad t \in (\xi, T] \end{aligned}$$

which gives (2.4). Since x is decreasing on $[0, \xi]$ and increasing on $[\xi, T]$, we see that if $x(0) < 0$ (resp. $x(0) > 0$), then $x(T) = -x(0) > 0$ (resp. $x(T) < 0$), and so x vanishes at a unique $\eta \in (\xi, T)$ (resp. $\eta \in (0, \xi)$). In the case of $x(0) = 0$, we have $x(T) = 0$ and $x < 0$ on $(0, T)$. We have proved that x has a unique zero η in the interval $[0, T]$. Now from (2.4) we get

$$\begin{aligned} x(T) - x(\xi) &= \int_\xi^T x'(t) dt \geq \int_\xi^T g^{-1}(a(t - \xi)) dt = \frac{1}{a} \int_0^{a(T-\xi)} g^{-1}(s) ds, \\ x(\xi) - x(0) &= \int_0^\xi x'(t) dt \leq - \int_0^\xi g^{-1}(a(\xi - t)) dt = -\frac{1}{a} \int_0^{a\xi} g^{-1}(s) ds, \end{aligned}$$

and so

$$-x(\xi) - \frac{1}{a} \int_0^{a\xi} g^{-1}(s) ds \geq -x(0) = x(T) \geq x(\xi) + \frac{1}{a} \int_0^{a(T-\xi)} g^{-1}(s) ds.$$

We get

$$\begin{aligned} x(\xi) &\leq -\frac{1}{2a} \left[\int_0^{a\xi} g^{-1}(s) ds + \int_0^{a(T-\xi)} g^{-1}(s) ds \right] \\ &\leq -\frac{1}{2a} \int_0^{a \max\{\xi, T-\xi\}} g^{-1}(s) ds. \end{aligned}$$

Since $\max\{\xi, T - \xi\} \geq T/2$, we have $|x(\xi)| \geq S$ where S is defined by (2.7). (2.5) for $\xi < \eta$ or (2.6) for $\eta < \xi$ now follows from the convexity of x on J . $\square$

Lemma 2. *Let x be a solution of BVP (2.3), (1.2) with some $\lambda \in [0, 1]$ and $n \in \mathbb{N}$ in (2.3). Then there exists a positive constant P independent of λ and n such that*

$$\|x'\| < P. \quad (2.8)$$

[†]Here and subsequently, $g^{-1} : \mathbb{R} \rightarrow \mathbb{R}$ denotes the inverse function to g .

PROOF. By Lemma 1, x' is increasing on J and has a unique zero $\xi \in (0, T)$. Hence

$$\|x'\| = \max\{-x'(0), x'(T)\}. \quad (2.9)$$

We are going to give upper bounds for $-x'(0)$ and $x'(T)$. Since

$$(g(x'(t)))' = h_n(\lambda; t, x(t), x'(t)) \leq f_n(t, x(t), x'(t))$$

for a. e. $t \in J$ by (2.1), (2.2) shows that

$$\frac{(g(x'(t)))'}{\omega_1(-g(x'(t)) + g(1)) + \omega_2(-g(x'(t)))} \leq h(|x(t)|) \quad \text{for a. e. } t \in [0, \xi] \quad (2.10)$$

and

$$\frac{(g(x'(t)))'}{\omega_1(g(x'(t)) + g(1)) + \omega_2(g(x'(t)))} \leq h(|x(t)|) \quad \text{for a. e. } t \in [\xi, T] \quad (2.11)$$

because of $x' < 0$ on $[0, \xi]$, $x'(t) > 0$ on $(\xi, T]$ and g is an odd function by (H_1) . To find upper estimates for $\int_0^\xi h(|x(t)|) dt$ and $\int_\xi^T h(|x(t)|) dt$ we must distinguish two cases according to the values of ξ and η :

Case 1. If $\xi < \eta$, then by (2.5)

$$\int_0^\xi h(|x(t)|) dt \leq \int_0^\xi h\left(\frac{S}{\xi} t\right) dt \leq \frac{T}{S} \int_0^S h(s) ds \quad (2.12)$$

and

$$\begin{aligned} \int_\xi^T h(|x(t)|) dt &\leq \int_\xi^T h\left(\frac{S}{\eta - \xi} |t - \eta|\right) dt \\ &\leq \int_\xi^\eta h\left(\frac{S}{\eta - \xi} (\eta - t)\right) dt + \int_\eta^T h\left(\frac{S}{T} (t - \eta)\right) dt \\ &\leq \frac{\eta - \xi}{S} \int_0^S h(s) ds + \frac{T}{S} \int_0^{\frac{T-\eta}{T}} h(s) ds \\ &\leq \frac{2T}{S} \int_0^S h(s) ds. \end{aligned} \quad (2.13)$$

Case 2. If $\eta < \xi$, then by (2.6)

$$\begin{aligned} \int_0^\xi h(|x(t)|) dt &\leq \int_0^\xi h\left(\frac{S}{\xi - \eta} |t - \eta|\right) dt \\ &\leq \int_0^\eta h\left(\frac{S}{T} (\eta - t)\right) dt + \int_\eta^\xi h\left(\frac{S}{\xi - \eta} (t - \eta)\right) dt \\ &\leq \frac{T}{S} \int_0^{\frac{\eta}{T}} h(s) ds + \frac{\xi - \eta}{S} \int_0^S h(s) ds \\ &\leq \frac{2T}{S} \int_0^S h(s) ds, \end{aligned} \quad (2.14)$$

and

$$\int_{\xi}^T h(|x(t)|) dt \leq \int_{\xi}^T h\left(\frac{S}{T-\xi}(T-t)\right) dt \leq \frac{T}{S} \int_0^S h(s) ds. \quad (2.15)$$

Let

$$Q = 1 + \frac{2T}{S} \int_0^S h(s) ds.$$

Since

$$\begin{aligned} \int_0^{\xi} \frac{(g(x'(t)))'}{\omega_1(-g(x'(t)) + g(1)) + \omega_2(-g(x'(t)))} dt \\ = \int_0^{-g(x'(0))} \frac{1}{\omega_1(s + g(1)) + \omega_2(s)} ds \end{aligned} \quad (2.16)$$

and

$$\begin{aligned} \int_{\xi}^T \frac{(g(x'(t)))'}{\omega_1(g(x'(t)) + g(1)) + \omega_2(g(x'(t)))} dt \\ = \int_0^{g(x'(T))} \frac{1}{\omega_1(s + g(1)) + \omega_2(s)} ds, \end{aligned} \quad (2.17)$$

integrating (2.10) over $[0, \xi]$ and using (2.16) we get

$$\int_0^{-g(x'(0))} \frac{1}{\omega_1(s + g(1)) + \omega_2(s)} ds < Q \quad (2.18)$$

and integrating (2.11) over $[\xi, T]$ and using (2.17) we get

$$\int_0^{g(x'(T))} \frac{1}{\omega_1(s + g(1)) + \omega_2(s)} ds < Q. \quad (2.19)$$

Since $\int_0^{\infty} \frac{1}{\omega_1(s)} ds = \infty$ by (1.3), we see that $\int_0^{\infty} \frac{1}{\omega_1(s+g(1))} ds = \infty$ and then

$$\int_0^{\infty} \frac{1}{\omega_1(s + g(1)) + \omega_2(s)} ds = \infty$$

as well. Therefore there exists $P_1 \in \mathbb{R}_+$ such that

$$\int_0^u \frac{1}{\omega_1(s + g(1)) + \omega_2(s)} ds > Q \quad \text{for } u \geq P_1.$$

Now from the last inequality, (2.18) and (2.19) we deduce that

$$\max\{-g(x'(0)), g(x'(T))\} < P_1.$$

Hence $\max\{-x'(0), x'(T)\} < g^{-1}(P_1)$ and then (2.9) shows that is true (2.8) with $P = g^{-1}(P_1)$.

The lemma is proved. $\square$

Lemma 3. For each $n \in \mathbb{N}$ there exists a solution of BVP (2.3), (1.2) with $\lambda = 1$ in (2.3).

PROOF. Fix $n \in \mathbb{N}$ and let P be a positive constant given in Lemma 2. Set

$$\Omega = \{x \in C^1(J); \|x\| < RT, \|x'\| < R\}$$

where $R = \max\{P, g^{-1}(aT)\}$. Then Ω is an open, bounded and convex subset of the Banach space $C^1(J)$. Define the operators

$$\Pi : [0, 1] \times \bar{\Omega} \rightarrow C^0(J) \quad \text{and} \quad \mathcal{K} : [0, 1] \times \bar{\Omega} \rightarrow C^1(J)$$

by the formulas

$$\Pi(\lambda, x)(t) = g^{-1} \left(\frac{1}{2} \int_0^t h_n(\lambda; s, x(s), x'(s)) ds - \frac{1}{2} \int_t^T h_n(\lambda; s, x(s), x'(s)) ds \right)$$

and

$$\mathcal{K}(\lambda, x)(t) = \frac{1}{2} \int_0^t \Pi(\lambda, x)(s) ds - \frac{1}{2} \int_t^T \Pi(\lambda, x)(s) ds.$$

It is easy to check that x is a solution of BVP (2.3), (1.2) with $\lambda = 1$ in (2.3) if and only if x is a fixed point of the operator $\mathcal{K}(1, \cdot)$. We now show that $\mathcal{K}(1, \cdot)$ has a fixed point. By the topological transversality principle, it is sufficient to verify that:

- (i) $\mathcal{K}$ is a compact operator,
- (ii) $\mathcal{K}(\lambda, x) \neq x$ for each $(\lambda, x) \in [0, 1] \times \partial\Omega$ and
- (iii) $\mathcal{K}(0, \cdot)$ is a constant operator and $\mathcal{K}(0, x) = \hat{x}$ for $x \in \bar{\Omega}$ with an $\hat{x} \in \Omega$.

Since $h_n(\lambda; \cdot) \in \text{Car}(J \times \mathbb{R}^2)$, there exists $q \in L_1(J)$ such that

$$|h_n(\lambda; t, u, v)| \leq q(t) \quad (2.20)$$

for a. e. $t \in J$ and each $\lambda \in [0, 1]$, $u \in [-RT, RT]$, $v \in [-R, R]$.

Let $\{\lambda_k\} \subset [0, 1]$, $\{x_k\} \subset \Omega$ be convergent sequences such that $\lim_{k \rightarrow \infty} \lambda_k = \lambda_*$ and $\lim_{k \rightarrow \infty} x_k = x_*$. Since

$$\begin{aligned} & |g(\Pi(\lambda_k, x_k)(t)) - g(\Pi(\lambda_*, x_*)(t))| \\ & \leq \int_0^T |h_n(\lambda_k; s, x_k(s), x'_k(s)) - h_n(\lambda_*; s, x_*(s), x'_*(s))| ds, \end{aligned}$$

using the Lebesgue dominated convergence theorem we see that

$$\lim_{k \rightarrow \infty} \|g(\Pi(\lambda_k, x_k)) - g(\Pi(\lambda_*, x_*))\| = 0.$$

Then from the continuity of g^{-1} on $\mathbb{R}$ it follows that

$$\lim_{k \rightarrow \infty} \|\Pi(\lambda_k, x_k) - \Pi(\lambda_*, x_*)\| = \lim_{k \rightarrow \infty} \|g^{-1}(g(\Pi(\lambda_k, x_k))) - g^{-1}(g(\Pi(\lambda_*, x_*)))\| = 0.$$

Now from the inequalities

$$|\mathcal{K}(\lambda_k, x_k)(t) - \mathcal{K}(\lambda_*, x_*)(t)| \leq \int_0^T |\Pi(\lambda_k, x_k)(s) - \Pi(\lambda_*, x_*)(s)| ds$$

and

$$\begin{aligned} |(\mathcal{K}(\lambda_k, x_k))'(t) - (\mathcal{K}(\lambda_*, x_*))'(t)| &= |\Pi(\lambda_k, x_k)(t) - \Pi(\lambda_*, x_*)(t)| \\ &\leq \|\Pi(\lambda_k, x_k) - \Pi(\lambda_*, x_*)(s)\|, \end{aligned}$$

we get $\lim_{k \rightarrow \infty} \mathcal{K}(\lambda_k, x_k) = \mathcal{K}(\lambda_*, x_*)$ in $C^1(J)$, which proves that $\mathcal{K}$ is a continuous operator.

Let now $\{(\bar{\lambda}_k, \bar{x}_k)\} \subset [0, 1] \times \bar{\Omega}$ and consider the sequence $\{\mathcal{K}(\bar{\lambda}_k, \bar{x}_k)\}$. Then (see (2.20))

$$|\mathcal{K}(\bar{\lambda}_k, \bar{x}_k)(t)| \leq T g^{-1}(\|q\|_L), \quad |(\mathcal{K}(\bar{\lambda}_k, \bar{x}_k))'(t)| \leq g^{-1}(\|q\|_L),$$

and

$$\left| g \left[(\mathcal{K}(\bar{\lambda}_k, \bar{x}_k))'(t_1) \right] - g \left[(\mathcal{K}(\bar{\lambda}_k, \bar{x}_k))'(t_2) \right] \right| \leq \int_{t_1}^{t_2} q(t) dt \quad (2.21)$$

for $k \in \mathbb{N}$, $t \in J$ and $0 \leq t_2 \leq t_1 \leq T$. Therefore $\{\mathcal{K}(\bar{\lambda}_k, \bar{x}_k)\}$ is bounded in $C^1(J)$ and from (2.21) we deduce that $\{(\mathcal{K}(\bar{\lambda}_k, \bar{x}_k))'(t)\}$ is equicontinuous on J . By the Arzelà-Ascoli theorem, there is a convergent subsequence of $\{\mathcal{K}(\bar{\lambda}_k, \bar{x}_k)\}$ in $C^1(J)$. Hence $\mathcal{K}([0, 1] \times \bar{\Omega})$ is relatively compact and since $\mathcal{K}$ is continuous, that is a compact operator.

Assume that $\mathcal{K}(\lambda_0, x_0) = x_0$ for some $(\lambda_0, x_0) \in [0, 1] \times \partial\Omega$. Then x_0 is a solution of BVP (2.3), (1.2) with $\lambda = \lambda_0$ in (2.3), and so $\|x'_0\| < P$ by Lemma 2. From $x_0(0) = -x_0(T)$ it follows that x_0 has a zero $\eta \in J$. Now

$$|x_0(t)| = |x_0(t) - x_0(\eta)| \leq \|x'_0\| |t - \eta|$$

yields $\|x_0\| < PT$, contrary to $x_0 \in \partial\Omega$. Hence $\mathcal{K}(\lambda, x) \neq x$ for all $(\lambda, x) \in [0, 1] \times \partial\Omega$.

Finally, for $x \in \Omega$ and $t \in J$

$$\mathcal{K}(0, x)(t) = \frac{1}{2} \int_0^t g^{-1}(a(s - T/2)) ds - \frac{1}{2} \int_t^T g^{-1}(a(s - T/2)) ds. \quad (2.22)$$

Since g^{-1} is increasing and odd on $\mathbb{R}$, we get

$$|(\mathcal{K}(0, x))'(t)| = |g^{-1}(a(t - T/2))| = g^{-1}(a|t - T/2|) \leq g^{-1}(aT/2)$$

for $t \in J$. Hence (iii) holds with $\hat{x}(t) \in \Omega$ defined by the righthand side of equality (2.22). $\square$

Lemma 4. For each $n \in \mathbb{N}$, let x_n be a solution of BVP (2.3), (1.2) with $\lambda = 1$ in (2.3) and let η_n be the unique zero of x_n lying in $[0, T)$ and ξ_n be the unique zero of x'_n . Then there exists a positive constant Δ independent of n such that

$$\Delta \leq \frac{|\xi_n - \eta_n|}{2} \quad \text{for } n \in \mathbb{N}. \quad (2.23)$$

PROOF. Applying Lemmas 1 and 2, there exist positive constants S and P such that

$$|x_n(\xi_n)| \geq S, \quad \|x'_n\| < P, \quad n \in \mathbb{N}.$$

Next (for $n \in \mathbb{N}$)

$$S \leq |x_n(\xi_n)| = |x_n(\xi_n) - x_n(\eta_n)| \leq \|x'_n\| |\xi_n - \eta_n| < P |\xi_n - \eta_n|.$$

Consequently (2.23) holds with $\Delta = \frac{S}{2P}$. $\square$

Lemma 5. *For each $n \in \mathbb{N}$, let x_n be a solution of BVP (2.3), (1.2) with $\lambda = 1$ in (2.3) and let ξ_n be the unique zero of x'_n . Then for any subsequence $\{x_{k_n}\}$ of $\{x_n\}$ satisfying*

$$|x_{k_n}(T)| \leq \frac{S}{T+1} \quad \text{for } n \in \mathbb{N} \quad (2.24)$$

with S given by (2.7), there exists its subsequence $\{x_{l_n}\}$ and a positive constant c independent of n such that

$$c \leq \xi_{l_n} \leq T - c \quad \text{for } n \in \mathbb{N}. \quad (2.25)$$

PROOF. By Lemma 1, $|x_n(\xi_n)| \geq S$ for $n \in \mathbb{N}$ and Lemma 2 guarantees the existence of a positive constant P such that $\|x'_n\| < P$ for $n \in \mathbb{N}$. Let $\{x_{k_n}\} \subset \{x_n\}$ be an arbitrary subsequence satisfying (2.24). Consider the sequences

$$\{x_{k_n}(\xi_{k_n})\}, \{x_{k_n}(T)\} \subset [-PT, PT]. \quad (2.26)$$

On account of the Bolzano–Weierstrass theorem, we see that there exists a subsequence $\{l_n\}$ of $\{k_n\}$ such that $\{x_{l_n}(\xi_{l_n})\}$ and $\{x_{l_n}(T)\}$ are convergent in $\mathbb{R}$. Let

$$\lim_{n \rightarrow \infty} x_{l_n}(\xi_{l_n}) = \alpha, \quad \lim_{n \rightarrow \infty} x_{l_n}(T) = \beta.$$

Naturally,

$$|\beta| \leq \frac{S}{T+1} < |\alpha|. \quad (2.27)$$

Also

$$|x_{l_n}(T) - x_{l_n}(\xi_{l_n})| < P(T - \xi_{l_n}) \quad (2.28)$$

and

$$|x_{l_n}(T) + x_{l_n}(\xi_{l_n})| = |x_{l_n}(\xi_{l_n}) - x_{l_n}(0)| < P \xi_{l_n}. \quad (2.29)$$

Assume that $\sup\{\xi_{l_n}; n \in \mathbb{N}\} = T$ (resp. $\inf\{\xi_{l_n}; n \in \mathbb{N}\} = 0$). Then (2.28) (resp. (2.29)) gives $\beta - \alpha = 0$ (resp., $\beta + \alpha = 0$). Therefore

$$|\alpha| = |\beta|, \quad (2.30)$$

contrary to (2.27). Hence there exists a positive constant c such that relation (2.25) is satisfied. $\square$

Lemma 6. *For each $n \in \mathbb{N}$, let x_n be a solution of BVP (2.3), (1.2) with $\lambda = 1$ in (2.3). Then there exists a subsequence $\{x_{k_n}\} \subset \{x_n\}$ such that the sequence*

$$\{f_{k_n}(t, x_{k_n}(t), x'_{k_n}(t))\} \subset L_1(J)$$

is uniformly integrable (UI) on J , that is for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_{\mathcal{M}} |f_{k_n}(t, x_{k_n}(t), x'_{k_n}(t))| dt < \varepsilon, \quad n \in \mathbb{N}$$

whenever $\mathcal{M} \subset J$ is a measurable set and $\mu(\mathcal{M}) < \delta$.

PROOF. For $n \in \mathbb{N}$, let $\eta_n \in [0, T)$ and $\xi_n \in (0, T)$ be the unique zero of x_n on $[0, T)$ and x'_n , respectively. Let

$$\Lambda = \min \left\{ a, \frac{S}{T+1} \right\} \quad (2.31)$$

where a appears in (H_2) and S is given by (2.7). By Lemmas 2 and 4, there exist positive constants P and Δ such that

$$\|x'_n\| < P, \quad 2\Delta \leq |\xi_n - \eta_n| \quad \text{for } n \in \mathbb{N}. \quad (2.32)$$

In addition (2.1), (2.2) and (2.32) give

$$a \leq f_n(t, x_n(t), x'_n(t)) \leq h(|x_n(t)|) \left[\omega_1(g(P) + g(1)) + \omega_2(g(|x'_n(t)|)) \right]$$

for a. e. $t \in J$ and each $n \in \mathbb{N}$. Since

$$h(|x_n(t)|) \leq \frac{1}{\omega_2(g(P))} h(|x_n(t)|) \omega_2(g(|x'_n(t)|))$$

for a. e. $t \in J$ and any $n \in \mathbb{N}$, we see that to prove our lemma it suffices to show that there exists a subsequence $\{x_{k_n}\} \subset \{x_n\}$ such that the sequence

$$\left\{ h(|x_{k_n}(t)|) \omega_2(g(|x'_{k_n}(t)|)) \right\} \quad (2.33)$$

is UI on J .

From the structure of the measurable sets in $\mathbb{R}$ we deduce that the sequence (2.33) is UI on J if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any at most countable sequence $\{(a_j, b_j)\}_{j \in \mathbb{J}}$ of mutually disjoint intervals $(a_j, b_j) \subset J$ such that $\sum_{j \in \mathbb{J}} (b_j - a_j) < \delta$, we have

$$\sum_{j \in \mathbb{J}} \int_{a_j}^{b_j} h(|x_{k_n}(t)|) \omega_2(g(|x'_{k_n}(t)|)) dt < \varepsilon, \quad n \in \mathbb{N}.$$

For this, let $\{(a_j, b_j)\}_{j \in \mathbb{J}}$ be an at most countable sequence of mutually disjoint intervals $(a_j, b_j) \subset J$ and assume that

$$b_j - a_j \leq \Delta/2 \quad \text{for each } j \in \mathbb{J}. \quad (2.34)$$

In addition, by (2.32), we have

$$\frac{|\xi_n - \eta_n|}{2} \geq \Delta > \frac{\Delta}{2} \geq b_j - a_j \quad \text{for } n \in \mathbb{N}, \quad j \in \mathbb{J}. \quad (2.35)$$

It is easy to see that there exists a subsequence $\{x_{k_n}\} \subset \{x_n\}$ such that one of the following two cases may occur, namely, Case 1 where $\xi_{k_n} > \eta_{k_n}$ for $n \in \mathbb{N}$, and Case 2 where $\xi_{k_n} < \eta_{k_n}$ for $n \in \mathbb{N}$.

Case 1. Let $\xi_{k_n} > \eta_{k_n}$ for $n \in \mathbb{N}$. Since (see (2.32))

$$\xi_{k_n} - t \geq \xi_{k_n} - \frac{\xi_{k_n} + \eta_{k_n}}{2} \geq \frac{\xi_{k_n} - \eta_{k_n}}{2} \geq \Delta, \quad t \in [0, (\xi_{k_n} + \eta_{k_n})/2]$$

and

$$t - \eta_{k_n} \geq \frac{\xi_{k_n} + \eta_{k_n}}{2} - \eta_{k_n} \geq \frac{\xi_{k_n} - \eta_{k_n}}{2} \geq \Delta, \quad t \in [(\xi_{k_n} + \eta_{k_n})/2, \xi_{k_n}],$$

we have (by Lemma 1)

$$g(|x'_{k_n}(t)|) \geq \begin{cases} \Lambda \Delta & \text{for } t \in [0, (\eta_{k_n} + \xi_{k_n})/2], \\ \Lambda |t - \xi_{k_n}| & \text{for } t \in [(\eta_{k_n} + \xi_{k_n})/2, T] \end{cases} \quad (2.36)$$

and

$$|x_{k_n}(t)| \geq \begin{cases} \Lambda |t - \eta_{k_n}| & \text{for } t \in [0, (\eta_{k_n} + \xi_{k_n})/2], \\ \Lambda \Delta & \text{for } t \in [(\eta_{k_n} + \xi_{k_n})/2, \xi_{k_n}], \\ \Lambda (T - t) & \text{for } t \in [\xi_{k_n}, T] \end{cases} \quad (2.37)$$

where Λ is given by (2.31).

For $n \in \mathbb{N}$, set

$$\mathbb{K}_{k_n} = \{j \in \mathbb{J}; b_j \leq \xi_{k_n}\}, \quad \mathbb{L}_{k_n} = \mathbb{J} \setminus \mathbb{K}_{k_n} \quad (2.38)$$

If $j \in \mathbb{K}_{k_n}$ and (see (2.35))

(a) $b_j \leq \eta_{k_n}$, then (see (2.36) and (2.37))

$$\begin{aligned} \int_{a_j}^{b_j} h(|x_{k_n}(t)|) \omega_2(g(|x'_{k_n}(t)|)) dt &\leq \omega_2(\Lambda \Delta) \int_{a_j}^{b_j} h(\Lambda(\eta_{k_n} - t)) dt \\ &= \frac{\omega_2(\Lambda \Delta)}{\Lambda} \int_{\Lambda(\eta_{k_n} - b_j)}^{\Lambda(\eta_{k_n} - a_j)} h(s) ds; \end{aligned}$$

(b) $a_j \leq \eta_{k_n} < b_j < (\eta_{k_n} + \xi_{k_n})/2$, then

$$\begin{aligned} \int_{a_j}^{b_j} h(|x_{k_n}(t)|) \omega_2(g(|x'_{k_n}(t)|)) dt \\ \leq \omega_2(\Lambda \Delta) \left(\int_{a_j}^{\eta_{k_n}} h(\Lambda(\eta_{k_n} - t)) dt + \int_{\eta_{k_n}}^{b_j} h(\Lambda(t - \eta_{k_n})) dt \right) \\ = \frac{\omega_2(\Lambda \Delta)}{\Lambda} \left(\int_0^{\Lambda(\eta_{k_n} - a_j)} h(s) ds + \int_0^{\Lambda(b_j - \eta_{k_n})} h(s) ds \right); \end{aligned}$$

(c) $\eta_{k_n} < a_j < b_j \leq (\eta_{k_n} + \xi_{k_n})/2$, then

$$\begin{aligned} \int_{a_j}^{b_j} h(|x_{k_n}(t)|) \omega_2(g(|x'_{k_n}(t)|)) dt &\leq \omega_2(\Lambda \Delta) \int_{a_j}^{b_j} h(\Lambda(t - \eta_{k_n})) dt \\ &= \frac{\omega_2(\Lambda \Delta)}{\Lambda} \int_{\Lambda(a_j - \eta_{k_n})}^{\Lambda(b_j - \eta_{k_n})} h(s) ds; \end{aligned}$$

(d) $\eta_{k_n} < a_j < (\eta_{k_n} + \xi_{k_n})/2 < b_j$, then

$$\begin{aligned} \int_{a_j}^{b_j} h(|x_{k_n}(t)|) \omega_2(g(|x'_{k_n}(t)|)) dt &\leq \omega_2(\Lambda \Delta) \int_{a_j}^{(\eta_{k_n} + \xi_{k_n})/2} h(\Lambda(t - \eta_{k_n})) dt \\ &\quad + h(\Lambda \Delta) \int_{(\eta_{k_n} + \xi_{k_n})/2}^{b_j} \omega_2(\Lambda(\xi_{k_n} - t)) dt \\ &= \frac{\omega_2(\Lambda \Delta)}{\Lambda} \int_{\Lambda(a_j - \eta_{k_n})}^{\Lambda(\xi_{k_n} - \eta_{k_n})/2} h(s) ds + \frac{h(\Lambda \Delta)}{\Lambda} \int_{\Lambda(\xi_{k_n} - b_j)}^{\Lambda(\xi_{k_n} - \eta_{k_n})/2} \omega_2(s) ds; \end{aligned}$$

(e) $(\eta_{k_n} + \xi_{k_n})/2 \leq a_j$, then (see (2.38))

$$\begin{aligned} \int_{a_j}^{b_j} h(|x_{k_n}(t)|) \omega_2(g(|x'_{k_n}(t)|)) dt &\leq h(\Lambda \Delta) \int_{a_j}^{b_j} \omega_2(\Lambda(\xi_{k_n} - t)) dt \\ &= \frac{h(\Lambda \Delta)}{\Lambda} \int_{\Lambda(\xi_{k_n} - b_j)}^{\Lambda(\xi_{k_n} - a_j)} \omega_2(s) ds. \end{aligned}$$

Summarizing, we have proved that

$$\begin{aligned} \sum_{j \in \mathbb{K}_{k_n}} \int_{a_j}^{b_j} h(|x_{k_n}(t)|) \omega_2(g(|x'_{k_n}(t)|)) dt &\leq \frac{2 h(\Lambda \Delta)}{\Lambda} \int_{\mathcal{M}_{k_n}^1} \omega_2(s) ds + \frac{2 \omega_2(\Lambda \Delta)}{\Lambda} \int_{\mathcal{M}_{k_n}^2} h(s) ds, \quad (2.39) \end{aligned}$$

where $\mathcal{M}_{k_n}^i \subset [0, \Lambda T]$, $\mu(\mathcal{M}_{k_n}^i) \leq \Lambda \sum_{j \in \mathbb{J}} (b_j - a_j)$ for $n \in \mathbb{N}$ and $i = 1, 2$.

The situation is more complicated for $j \in \mathbb{L}_{k_n}$. Passing to a subsequence if necessary, one of the following subcases of Case 1 occurs (see Lemma 5), namely, Case 1.1 where $|x_{k_n}(T)| > \Lambda$ for $n \in \mathbb{N}$ and Case 1.2 where $|x_{k_n}(T)| \leq \Lambda$ and $c \leq \xi_{k_n} \leq T - c$ for $n \in \mathbb{N}$ with a positive constant c .

Case 1.1. Let $|x_{k_n}(T)| > \Lambda$ for $n \in \mathbb{N}$. Since $\xi_{k_n} > \eta_{k_n}$, we see that $|x_{k_n}|$ is decreasing on $[\xi_{k_n}, T]$ and then $|x_{k_n}(t)| > \Lambda$ for $t \in [\xi_{k_n}, T]$. Let $j \in \mathbb{L}_{k_n}$. If

a) $a_j < \xi_{k_n} < b_j$, then (see (2.37))

$$\begin{aligned} & \int_{a_j}^{b_j} h(|x_{k_n}(t)|) \omega_2(g(|x'_{k_n}(t)|)) dt \\ & \leq h(\Lambda \Delta) \int_{a_j}^{\xi_{k_n}} \omega_2(\Lambda(\xi_{k_n} - t)) dt + h(\Lambda) \int_{\xi_{k_n}}^{b_j} \omega_2(\Lambda(t - \xi_{k_n})) dt \\ & = \frac{h(\Lambda \Delta)}{\Lambda} \int_0^{\Lambda(\xi_{k_n} - a_j)} \omega_2(s) ds + \frac{h(\Lambda)}{\Lambda} \int_0^{\Lambda(b_j - \xi_{k_n})} \omega_2(s) ds; \end{aligned}$$

b) $\xi_{k_n} \leq a_j$, then

$$\begin{aligned} & \int_{a_j}^{b_j} h(|x_{k_n}(t)|) \omega_2(g(|x'_{k_n}(t)|)) dt \leq h(\Lambda) \int_{a_j}^{b_j} \omega_2(\Lambda(t - \xi_{k_n})) dt \\ & = \frac{h(\Lambda)}{\Lambda} \int_{\Lambda(a_j - \xi_{k_n})}^{\Lambda(b_j - \xi_{k_n})} \omega_2(s) ds. \end{aligned}$$

It follows that

$$\sum_{j \in \mathbb{L}_{k_n}} \int_{a_j}^{b_j} h(|x_{k_n}(t)|) \omega_2(g(|x'_{k_n}(t)|)) dt \leq \frac{\max\{h(\Lambda), h(\Lambda \Delta)\}}{\Lambda} \int_{\mathcal{M}_{k_n}^3} \omega_2(s) ds \quad (2.40)$$

where $\mathcal{M}_{k_n}^3 \subset [0, \Lambda T]$ and $\mu(\mathcal{M}_{k_n}^3) \leq \Lambda \sum_{j \in \mathbb{J}} (b_j - a_j)$ for $n \in \mathbb{N}$.

Case 1.2. Let $|x_{k_n}(T)| \leq \Lambda$ for $n \in \mathbb{N}$ and

$$c \leq \xi_{k_n} \leq T - c \quad \text{for } n \in \mathbb{N}. \quad (2.41)$$

Since

$$T - t \geq T - \frac{T + \xi_{k_n}}{2} \geq \frac{T - \xi_{k_n}}{2} \geq \frac{c}{2}, \quad t \in [\xi_{k_n}, (\xi_{k_n} + T)/2],$$

and

$$t - \xi_{k_n} \geq \frac{T + \xi_{k_n}}{2} - \xi_{k_n} \geq \frac{T - \xi_{k_n}}{2} \geq \frac{c}{2}, \quad t \in [(\xi_{k_n} + T)/2, T],$$

then from (2.36) and (2.37) we have

$$g(|x'_{k_n}(t)|) \geq \begin{cases} \Lambda |t - \xi_{k_n}| & \text{for } t \in [(\eta_{k_n} + \xi_{k_n})/2, (\xi_{k_n} + T)/2], \\ \Lambda c/2 & \text{for } t \in [(\xi_{k_n} + T)/2, T] \end{cases} \quad (2.42)$$

and

$$|x_{k_n}(t)| \geq \begin{cases} \Lambda \Delta & \text{for } t \in [(\eta_{k_n} + \xi_{k_n})/2, \xi_{k_n}], \\ \Lambda c/2 & \text{for } t \in [\xi_{k_n}, (\xi_{k_n} + T)/2], \\ \Lambda(T - t) & \text{for } t \in [(\xi_{k_n} + T)/2, T]. \end{cases} \quad (2.43)$$

Let $j \in \mathbb{L}_{k_n}$ and assume that $b_j - a_j \leq c/4$. Then $(T - \xi_{k_n})/2 \geq c/2 > c/4 \geq b_j - a_j$ for $n \in \mathbb{N}$. If

(a) $a_j \leq \xi_{k_n} < b_j < (\xi_{k_n} + T)/2$,[‡] then (see (2.42))

$$\begin{aligned} & \int_{a_j}^{b_j} h(|x_{k_n}(t)|) \omega_2(g(|x'_{k_n}(t)|)) dt \\ & \leq h(\Lambda\Delta) \int_{a_j}^{\xi_{k_n}} \omega_2(\Lambda(\xi_{k_n} - t)) dt + h(\Lambda c/2) \int_{\xi_{k_n}}^{b_j} \omega_2(\Lambda(t - \xi_{k_n})) dt \\ & = \frac{h(\Lambda\Delta)}{\Lambda} \int_0^{\Lambda(\xi_{k_n} - a_j)} \omega_2(s) ds + \frac{h(\Lambda c/2)}{\Lambda} \int_0^{\Lambda(b_j - \xi_{k_n})} \omega_2(s) ds; \end{aligned}$$

(b) $\xi_{k_n} < a_j < b_j \leq (\xi_{k_n} + T)/2$, then

$$\begin{aligned} \int_{a_j}^{b_j} h(|x_{k_n}(t)|) \omega_2(g(|x'_{k_n}(t)|)) dt & \leq h(\Lambda c/2) \int_{a_j}^{b_j} \omega_2(\Lambda(t - \xi_{k_n})) dt \\ & = \frac{h(\Lambda c/2)}{\Lambda} \int_{\Lambda(a_j - \xi_{k_n})}^{\Lambda(b_j - \xi_{k_n})} \omega_2(s) ds; \end{aligned}$$

(c) $\xi_{k_n} < a_j < (\xi_{k_n} + T)/2 < b_j$, then (see (2.42) and (2.43))

$$\begin{aligned} & \int_{a_j}^{b_j} h(|x_{k_n}(t)|) \omega_2(g(|x'_{k_n}(t)|)) dt \\ & \leq h(\Lambda c/2) \int_{a_j}^{(\xi_{k_n} + T)/2} \omega_2(\Lambda(t - \xi_{k_n})) dt \\ & \quad + \omega_2(\Lambda c/2) \int_{(\xi_{k_n} + T)/2}^{b_j} h(\Lambda(T - t)) dt \\ & = \frac{h(\Lambda c/2)}{\Lambda} \int_{\Lambda(a_j - \xi_{k_n})}^{\Lambda(T - \xi_{k_n})/2} \omega_2(s) ds + \frac{\omega_2(\Lambda c/2)}{\Lambda} \int_{\Lambda(b_j - \xi_{k_n})}^{\Lambda(T - \xi_{k_n})/2} h(s) ds; \end{aligned}$$

(d) $(\xi_{k_n} + T)/2 \leq a_j$, then

$$\begin{aligned} \int_{a_j}^{b_j} h(|x_{k_n}(t)|) \omega_2(g(|x'_{k_n}(t)|)) dt & \leq \omega_2(\Lambda c/2) \int_{a_j}^{b_j} h(\Lambda(T - t)) dt \\ & = \frac{\omega_2(\Lambda c/2)}{\Lambda} \int_{\Lambda(T - b_j)}^{\Lambda(T - a_j)} h(s) ds. \end{aligned}$$

[‡]We see that then $a_j \geq b_j - \Delta > \xi_{k_n} - \Delta \geq \xi_{k_n} - (\xi_{k_n} - \eta_{k_n})/2 = (\xi_{k_n} + \eta_{k_n})/2$.

It follows that

$$\begin{aligned} \sum_{j \in \mathbb{L}_{k_n}} \int_{a_j}^{b_j} h(|x_{k_n}(t)|) \omega_2(g(|x'_{k_n}(t)|)) dt \\ \leq \frac{2 h(\Lambda c/2)}{\Lambda} \int_{\mathcal{N}_{k_n}^1} \omega_2(s) ds + \frac{2 \omega_2(\Lambda c/2)}{\Lambda} \int_{\mathcal{N}_{k_n}^2} h(s) ds \end{aligned} \quad (2.44)$$

where $\mathcal{N}_{k_n}^i \subset [0, \Lambda T]$, $\mu(\mathcal{N}_{k_n}^i) \leq \Lambda \sum_{j \in \mathbb{J}} (b_j - a_j)$ for $n \in \mathbb{N}$ and $i = 1, 2$.

Summarizing, since $h, \omega_2 \in L_1([0, \Lambda T])$, from (2.39), (2.40) and (2.44) we deduce that $\{h(|x_{k_n}(t)|) \omega_2(g(|x'_{k_n}(t)|))\}$ is UI on J .

Case 2. Let $\eta_{k_n} > \xi_{k_n}$ for $n \in \mathbb{N}$. Define (for $n \in \mathbb{N}$ and $t \in J$)

$$\eta_{k_n}^* = T - \eta_{k_n}, \quad \xi_{k_n}^* = T - \xi_{k_n}$$

and

$$y_{k_n}(t) = x_{k_n}(T - t).$$

Then $\eta_{k_n}^* < \xi_{k_n}^*$, $\eta_{k_n}^*$ and $\xi_{k_n}^*$ is a zero of y_{k_n} and y'_{k_n} , respectively. Next

$$|\eta_{k_n}^* - \xi_{k_n}^*| = |(T - \eta_{k_n}) - (T - \xi_{k_n})| = |\xi_{k_n} - \eta_{k_n}| \geq \Delta/2, \quad n \in \mathbb{N}$$

where Δ is given by Lemma 4. In addition, by Lemma 1 (see (2.4) and (2.5)) we have (for $n \in \mathbb{N}$)

$$g(|y'_{k_n}(t)|) = g(|x'_{k_n}(T - t)|) \geq \Lambda|(T - t) - \xi_{k_n}| = \Lambda|t - \xi_{k_n}^*|, \quad t \in [0, T]$$

and

$$\begin{aligned} |y_{k_n}(t)| &= |x_{k_n}(T - t)| \\ &\geq \begin{cases} \Lambda|(T - t) - \eta_{k_n}| = \Lambda|\eta_{k_n}^* - t|, & \xi_{k_n} \leq T - t \leq T \quad (\iff t \in [0, \xi_{k_n}^*]), \\ \Lambda(T - t), & 0 \leq T - t < \xi_{k_n} \quad (\iff t \in (\xi_{k_n}^*, T)). \end{cases} \end{aligned}$$

By Case 1, the sequence

$$\{h(|y_{k_n}(t)|) \omega_2(g(|y'_{k_n}(t)|))\}$$

is UI on J . Since

$$\sum_{j \in \mathbb{J}} \int_{a_j}^{b_j} h(|x_{k_n}(t)|) \omega_2(g(|x'_{k_n}(t)|)) dt = \sum_{j \in \mathbb{J}} \int_{T-b_j}^{T-a_j} h(|y_{k_n}(s)|) \omega_2(g(|y'_{k_n}(s)|)) ds$$

and

$$\sum_{j \in \mathbb{J}} (b_j - a_j) = \sum_{j \in \mathbb{J}} [(T - a_j) - (T - b_j)],$$

we see that the sequence $\{h(|x_{k_n}(t)|) \omega_2(g(|x'_{k_n}(t)|))\}$ is UI on J .

□

3. EXISTENCE RESULTS AND AN EXAMPLE

Theorem. *Let assumptions $(H_1) - (H_3)$ be satisfied. Then BVP (1.1), (1.2) has a solution.*

PROOF. By Lemma 3, for each $n \in \mathbb{N}$, there exists a solution x_n of BVP (2.3), (1.2) with $\lambda = 1$ in (2.3). Let $\eta_n \in [0, T)$ and $\xi_n \in (0, T)$ be the unique zero of x_n on $[0, T)$ and x'_n , respectively. Without loss of generality, we can assume that $\xi_n < \eta_n$ (then $\eta_n > 0$) for $n \in \mathbb{N}$. Now Lemma 1 shows that, for each $n \in \mathbb{N}$, we have

$$g(|x'_n(t)|) \geq \Lambda |t - \xi_n| \quad \text{for } t \in J \quad (3.1)$$

and

$$|x_n(t)| \geq \begin{cases} \Lambda t & \text{for } t \in [0, \xi_n], \\ \Lambda |t - \eta_n| & \text{for } t \in (\xi_n, T], \end{cases} \quad (3.2)$$

where Λ is given by (2.31). Applying Lemmas 2 and 4, there exist positive constants P and Δ such that the inequalities (2.32) are satisfied. In addition, without loss of generality we can also assume that the sequence $\{f_n(t, x_n(t), x'_n(t))\}$ is UI on J by Lemma 6, which guarantees that $\{g(x'_n(t))\}$ is equicontinuous on J , and then from

$$|x'_n(t_2) - x'_n(t_1)| = |g^{-1}(g(x'_n(t_2))) - g^{-1}(g(x'_n(t_1)))|, \quad t_1, t_2 \in J,$$

the uniform continuity of g^{-1} on $[-g(P), g(P)]$ and the fact, that g^{-1} is increasing, we deduce that $\{x'_n(t)\}$ is equicontinuous on J as well. On account of the Arzelà-Ascoli theorem and the Bolzano-Weierstrass theorem, passing to subsequences if necessary, we can assume that $\{x_n\}$ is convergent in $C^1(J)$ and $\{\eta_n\}, \{\xi_n\}$ are convergent in $\mathbb{R}$. Let $\lim_{n \rightarrow \infty} x_n = x$, $\lim_{n \rightarrow \infty} \eta_n = \eta$ and $\lim_{n \rightarrow \infty} \xi_n = \xi$. Then $x \in C^1(J)$, x satisfied (1.2) and $\Delta \leq |\xi - \eta|$. Letting $n \rightarrow \infty$ in (3.1) and (3.2), we get

$$|x'(t)| \geq g^{-1}(\Lambda |t - \xi|) \quad \text{for } t \in J \quad (3.3)$$

and

$$|x(t)| \geq \begin{cases} \Lambda t & \text{for } t \in [0, \xi], \\ \Lambda |t - \eta| & \text{for } t \in (\xi, T]. \end{cases} \quad (3.4)$$

From the construction of the functions $f_n \in \text{Car}(J \times \mathbb{R}^2)$ it follows the existence of a $\mathcal{U} \subset J$, $\mu(\mathcal{U}) = 0$, such that $f_n(t, \cdot, \cdot)$ is continuous on $\mathbb{R}^2$ for each $t \in J \setminus \mathcal{U}$ and $n \in \mathbb{N}$. Then (see (3.3) and (3.4))

$$\lim_{n \rightarrow \infty} f_n(t, x_n(t), x'_n(t)) = f(t, x(t), x'(t)), \quad t \in J \setminus (\mathcal{U} \cup \{0, \eta, \xi\}).$$

Therefore $f(t, x(t), x'(t)) \in L_1(J)$ and

$$\lim_{n \rightarrow \infty} \int_0^t f_n(s, x_n(s), x'_n(s)) ds = \int_0^t f(s, x(s), x'(s)) ds$$

for $t \in J$ by Vitali's convergence theorem. Now, taking the limit as $n \rightarrow \infty$ in the equalities

$$g(x'_n(t)) = g(x'_n(0)) + \int_0^t f_n(s, x_n(s), x'_n(s)) ds, \quad t \in J, \quad n \in \mathbb{N},$$

we obtain

$$g(x'(t)) = g(x'(0)) + \int_0^t f(s, x(s), x'(s)) ds, \quad t \in J.$$

Hence $g(x') \in AC(J)$ and x is a solution of (1.1). We have proved that x is a solution of BVP (1.1), (1.2). $\square$

If $f \in C^0(J \times \mathbb{R}_0^2)$, then the theorem given above immediately yields

Corollary. *Let $f \in C^0(J \times \mathbb{R}_0^2)$ and assumptions $(H_1) - (H_3)$ be satisfied. Then there exists a solution x of BVP (1.1), (1.2). If $\eta \in [0, T)$ and $\xi \in (0, T)$ denotes the (unique) zero of x on $[0, T)$ and x' , respectively, then $g(x') \in C^1(J \setminus \mathcal{U})$ and (1.1) is satisfied for $t \in J \setminus \mathcal{U}$ where $\mathcal{U} = \{0, \xi, T\}$ if $\eta = 0$ and $\mathcal{U} = \{\eta, \xi\}$ if $\eta > 0$.*

Example. Let $p \in (0, \infty)$, $q_i \in L_\infty(J)$ be nonnegative, $\alpha_j \in (0, 1)$ and $\gamma_j \in (0, p)$ ($i = 1, 2, 3$ and $j = 1, 2$). Consider the differential equation

$$(|x'(t)|^p \operatorname{sign} x'(t))' = \arctan(1 + |x(t)|) + \frac{q_1(t)}{|x(t)|^{\alpha_1}} + q_2(t) |x'(t)|^{\gamma_1} + \frac{q_3(t)}{|x(t)|^{\alpha_2} |x'(t)|^{\gamma_2}}. \quad (3.5)$$

Problem (3.5), (1.2) is the special case of BVP (1.1), (1.2) with $g(u) = |u|^p \operatorname{sign} u$ and

$$f(t, u, v) = \arctan(1 + |u|) + \frac{q_1(t)}{|u|^{\alpha_1}} + q_2(t) |v|^{\gamma_1} + \frac{q_3(t)}{|u|^{\alpha_2} |v|^{\gamma_2}}.$$

Assumptions $(H_1) - (H_3)$ are satisfied with $a = \arctan 1$, $\omega_1(u) = 2 + u^{\gamma_1/p}$, $\omega_2(u) = 1/u^{\gamma_2/p}$, and

$$h(u) = K(1 + 1/u^{\alpha_1} + 1/u^{\alpha_2})$$

where $K = \max\{\frac{\pi}{2}, \|q_j\|_\infty; j = 1, 2, 3\}$. Applying the theorem, we show that there exists a solution of BVP (3.5), (1.2).

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Author's Address

Oto Přibyl:

BRNO UNIVERSITY OF TECHNOLOGY, FACULTY OF CIVIL ENGINEERING, INSTITUTE OF MATHEMATICS, VEVEŘÍ
331/95, 602 00 BRNO, CZECH REPUBLIC

E-mail address: pribyl.o@fce.vutbr.cz



INTEGRABILITY BY QUADRATURES OF HAMILTONIAN SYSTEMS AND PICARD–FUCHS TYPE EQUATIONS: THE MODERN DIFFERENTIAL-GEOMETRIC ASPECTS

Y. A. PRYKARPATSKY, A. M. SAMOILENKO, D. L. BLACKMORE,
AND A. K. PRYKARPATSKY

This paper is dedicated to the 95-th birthday anniversary (1909-1993) of the greatest mathematician and physicist of the late 20th century, our teacher and academician Nikolay N. Bogoliubov

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ABSTRACT. There is developed a symplectic theory approach for partially solving the problem of algebraic-analytical construction of integral submanifold imbeddings for integrable via the abelian and nonabelian Liouville–Arnold theorems for Hamiltonian systems on canonically symplectic phase spaces. The fundamental role of so-called Picard–Fuchs type equations is revealed and their differential-geometric and algebraic properties are studied in detail. Some interesting examples of integrable Hamiltonian systems demonstrating the algorithm of investigating the integral submanifolds imbedding mapping are studied in detail.

Mathematics Subject Classification: Primary 34A30, 34B01 Secondary 34B15

Keywords: Liouville–Arnold integrability by quadratures, Mishchenko–Fomenko noncommutative integrability, Hamiltonian dynamical systems, Picard–Fuchs equations, integral submanifold imbedding problem, hyperelliptic algebraic curves, symplectic structures, Koelerian structures

1. INTRODUCTION

As it is well-known [1, 4], the integrability by quadratures of a differential equation in space $\mathbb{R}^n$ is a method of seeking its solutions by means of a finite number of algebraic operations (together with the inversion of functions) and “quadratures”, i. e., calculation of integrals of known functions.

Assume that our differential equation is given as a Hamiltonian dynamical system on some appropriate symplectic manifold $(M^{2n}, \omega^{(2)})$, $n \in \mathbb{Z}_+$, in the form

$$du/dt = \{H, u\}, \quad (1.1)$$

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where $u \in M^{2n}$, $H : M^{2n} \rightarrow \mathbb{R}$ is a sufficiently smooth Hamiltonian function [1, 4] with respect to the Poisson bracket $\{\cdot, \cdot\}$ on $\mathcal{D}(M^{2n})$, dual to the symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$, and $t \in \mathbb{R}$ is the evolution parameter. More than one hundred and fifty years ago French mathematicians and physicists, first E. Bour and next J. Liouville, proved the first “integrability by quadratures” theorem which in modern terms [33] can be formulated as follows.

Theorem 1. *Let $M^{2n} \simeq T^*(\mathbb{R}^n)$ be a canonically symplectic phase space and there be given a dynamical system (1.1) with a Hamiltonian function $H : M^{2n} \times \mathbb{R}_t \rightarrow \mathbb{R}$, possessing a Poissonian Lie algebra $\mathcal{G}$ of $n \in \mathbb{Z}_+$ invariants $H_j : M^{2n} \times \mathbb{R}_t \rightarrow \mathbb{R}$, $j = \overline{1, n}$, such that*

$$\{H_i, H_j\} = \sum_{s=1}^n c_{ij}^s H_s, \quad (1.2)$$

and for all $i, j, k = \overline{1, n}$ the $c_{ij}^s \in \mathbb{R}$ are constants on $M^{2n} \times \mathbb{R}_t$. Suppose further that

$$M_h^{n+1} := \{(u, t) \in M \times \mathbb{R}_t : h(H_j) = h_j, j = \overline{1, n}, h \in \mathcal{G}^*\}, \quad (1.3)$$

the integral submanifold of the set $\mathcal{G}$ of invariants at a regular element $h \in \mathcal{G}^*$ is a well defined connected submanifold of $M \times \mathbb{R}_t$. Then, if:

- (i) All functions of $\mathcal{G}$ are functionally independent on M_h^{n+1} ;
 - (ii) $\sum_{s=1}^n c_{ij}^s h_s = 0$ for all $i, j = \overline{1, n}$;
 - (iii) The Lie algebra $\mathcal{G} = \text{span}_{\mathbb{R}}\{H_j : M^{2n} \times \mathbb{R}_t \rightarrow \mathbb{R} : j = \overline{1, n}\}$ is solvable,
- the Hamiltonian system (1.1) on M^{2n} is integrable by quadratures.

As a simple corollary of the Bour-Liouville theorem one gets the following

Corollary 1. *If a Hamiltonian system on $M^{2n} = T^*(\mathbb{R}^n)$ possesses just $n \in \mathbb{Z}_+$ functionally independent invariants in involution, that is a Lie algebra $\mathcal{G}$ is abelian, then it is integrable by quadratures.*

In the autonomous case when a Hamiltonian $H = H_1$, and invariants $H_j : M^{2n} \rightarrow \mathbb{R}$, $j = \overline{1, n}$, are independent of the evolution parameter $t \in \mathbb{R}$, the involutivity condition $\{H_i, H_j\} = 0$, $i, j = \overline{1, n}$, can be replaced by the weaker one $\{H, H_j\} = c_j H$ for some constants $c_j \in \mathbb{R}$, $j = \overline{1, n}$.

The first proof of Theorem 1 was based on a result of S. Lie, which can be formulated as follows.

Theorem 2 (S. Lie). *Let vector fields $K_j \in \Gamma(M^{2n})$, $j = \overline{1, n}$, be independent in some open neighborhood $U_h \in M^{2n}$, generate a solvable Lie algebra $\mathcal{G}$ with respect to the usual commutator $[\cdot, \cdot]$ on $\Gamma(M^{2n})$ and $[K_j, K] = c_j K$ for all $j = \overline{1, n}$, where $c_j \in \mathbb{R}$, $j = \overline{1, n}$, are constants. Then the dynamical system*

$$du/dt = K(u), \quad (1.4)$$

where $u \in U_h \subset M^{2n}$, is integrable by quadratures.

Example 1 (Motion of three particles on $\mathbb{R}$ under a uniform potential field). The motion of three particles on the axis $\mathbb{R}$ pairwise interacting via a uniform potential field $Q(\|\cdot\|)$ is described as a Hamiltonian system on the canonically symplectic phase space $M = T^*(\mathbb{R}^3)$ with the following Lie algebra $\mathcal{G}$ of invariants on M^{2n} :

$$\begin{aligned} H = H_1 &= \sum_{j=1}^3 p_j^2/2m_j + \sum_{i<j=1}^3 Q(\|q_i - q_j\|), \\ H_2 &= \sum_{j=1}^3 q_j p_j, \quad H_3 = \sum_{j=1}^3 p_j, \end{aligned} \quad (1.5)$$

where $(q_j, p_j) \in T^*(\mathbb{R})$, $j = \overline{1, 3}$, are coordinates and momenta of particles on the axis $\mathbb{R}$. The commutation relations for the Lie algebra $\mathcal{G}$ are

$$\{H_1, H_3\} = 0, \quad \{H_2, H_3\} = H_3, \quad \{H_1, H_2\} = 2H_1, \quad (1.6)$$

hence it is clearly solvable. Taking a regular element $h \in \mathcal{G}^*$, such that $h(H_j) = h_j = 0$, for $j = 1$ and 3 , and $h(H_2) = h_2 \in \mathbb{R}$ being arbitrary, one obtains the integrability of the problem above in quadratures.

In 1974 V. Arnold proved [4] the following important result known as the commutative (abelian) Liouville–Arnold theorem.

Theorem 3 (J. Liouville – V. Arnold). *Suppose a set $\mathcal{G}$ of functions $H_j : M^{2n} \rightarrow \mathbb{R}$, $j = \overline{1, n}$, on a symplectic manifold M^{2n} is abelian, that is*

$$\{H_i, H_j\} = 0 \quad (1.7)$$

for all $i, j = \overline{1, n}$. If on the compact and connected integral submanifold

$$M_h^n = \{u \in M^{2n} : h(H_j) = h_j \in \mathbb{R}, \quad j = \overline{1, n}, \quad h \in \mathcal{G}^*\}$$

with $h \in \mathcal{G}$ being regular, all functions $H : M^{2n} \rightarrow \mathbb{R}$, $j = \overline{1, n}$, are functionally independent, then M_h^n is diffeomorphic to the n -dimensional torus $T^n \simeq M^{2n}$, and the motion on it with respect to the Hamiltonian $H = H_1 \in \mathcal{G}$ is a quasi-periodic function of the evolution parameter $t \in \mathbb{R}$.

A dynamical system satisfying the hypotheses of Theorem 3 is called completely integrable.

In 1978 Mishchenko and Fomenko [2] proved the following generalization of the Liouville–Arnold Theorem 3:

Theorem 4 (A. Mishchenko – A. Fomenko). *Assume that on a symplectic manifold $(M^{2n}, \omega^{(2)})$ there is a nonabelian Lie algebra $\mathcal{G}$ of invariants $H_j : M \rightarrow \mathbb{R}$, $j = \overline{1, k}$, with respect to the dual Poisson bracket on M^{2n} , that is*

$$\{H_i, H_j\} = \sum_{s=1}^k c_{ij}^s H_s, \quad (1.8)$$

where all values $c_{ij}^s \in \mathbb{R}$, $i, j, s = \overline{1, k}$, are constants, and the following conditions are satisfied:

- (i) The integral submanifold $M_h^r := \{u \in M^{2n} : h(H_j) = h \in \mathcal{G}^*\}$ is compact and connected at a regular element $h \in \mathcal{G}^*$;
- (ii) All the functions $H_j : M^{2n} \rightarrow \mathbb{R}$ $j = \overline{1, k}$, are functionally independent on M^{2n} ;
- (iii) The Lie algebra $\mathcal{G}$ of invariants satisfies the following relationship:

$$\dim \mathcal{G} + \text{rank } \mathcal{G} = \dim M^{2n}, \quad (1.9)$$

where $\text{rank } \mathcal{G} = \text{rank } \mathcal{G}_h$ is the dimension of a Cartan subalgebra $\mathcal{G}_h \subset \mathcal{G}$.

Then the submanifold $M_h^r \subset M^{2n}$ is $r = \text{rank } \mathcal{G}$ -dimensional, invariant with respect to each vector field $K \in \Gamma(M^{2n})$, generated by an element $H \in \mathcal{G}_h$, and diffeomorphic to the r -dimensional torus $\mathbb{T}^r \simeq M_h^r$, on which the motion is a quasiperiodic function of the evolution parameter $t \in \mathbb{R}$.

The simplest proof of the Mishchenko–Fomenko Theorem 4 can be obtained from the well-known [3, 16] classical Lie–Cartan theorem.

Theorem 5 (S. Lie – E. Cartan). Suppose that a point $h \in \mathcal{G}^*$ for a given Lie algebra $\mathcal{G}$ of invariants $H_j : M^{2n} \rightarrow \mathbb{R}$, $j = \overline{1, k}$, is not critical, and the rank $\|\{H_i, H_j\} : i, j = \overline{1, k}\| = 2(n - r)$ is constant in an open neighborhood $U_h \in \mathbb{R}^n$ of the point $\{h(H_j) = h_j \in \mathbb{R} : j = \overline{1, k}\} \subset \mathbb{R}^k$. Then in the neighborhood $(h \circ H)^{-1} : U_h \subset M^{2n}$ there exist $k \in \mathbb{Z}_+$ independent functions $f_s : \mathcal{G} \rightarrow \mathbb{R}$, $s = \overline{1, k}$, such that the functions $F_s := (f_s \circ H) : M^{2n} \rightarrow \mathbb{R}$, $s = \overline{1, k}$, satisfy the following relationships:

$$\{F_1, F_2\} = \{F_3, F_4\} = \dots = \{F_{2(n-r)-1}, F_{2(n-r)}\} = 1, \quad (1.10)$$

with all other brackets $\{F_i, F_j\} = 0$, where $(i, j) \neq (2s - 1, 2s)$, $s = \overline{1, n - r}$. In particular, $(k + r - n) \in \mathbb{Z}_+$ functions $F_j : M^{2n} \rightarrow \mathbb{R}$, $j = \overline{1, n - r}$, and $F_s : M^{2n} \rightarrow \mathbb{R}$, $s = \overline{1, k - 2(n - r)}$, compose an abelian algebra $\mathcal{G}_\tau$ of new invariants on M^{2n} , independent on $(h \circ H)^{-1}(U_h) \subset M^{2n}$.

As a simple corollary of the Lie–Cartan Theorem 5 one obtains the following: in the case of the Mishchenko–Fomenko Theorem where $\text{rank } \mathcal{G} + \dim \mathcal{G} = \dim M^{2n}$, that is $r + k = 2n$, the abelian algebra $\mathcal{G}_\tau$ (it is not a subalgebra of $\mathcal{G}$!) of invariants on M^{2n} is just $n = 1/2 \dim M^{2n}$ -dimensional, giving rise to its local complete integrability in $(h \circ H)^{-1}(U_h) \subset M^{2n}$ via the abelian Liouville–Arnold Theorem 3. It is also evident that the Mishchenko–Fomenko nonabelian integrability Theorem 4 reduces to the commutative (abelian) Liouville–Arnold case when a Lie algebra $\mathcal{G}$ of invariants is just abelian, since then $\text{rank } \mathcal{G} = \dim \mathcal{G} = 1/2 \dim M^{2n} = n \in \mathbb{Z}_+$, the standard complete integrability condition.

All the cases of integrability by quadratures described above pose the following fundamental question: How can one effectively construct by means of algebraic-analytical methods the corresponding integral submanifold imbedding

$$\pi_h : M_h^r \rightarrow M^{2n}, \quad (1.11)$$

where $r = \text{rank } \mathcal{G}$, thereby making it possible to express the solutions of an integrable flow on M_h^r as some exact quasi-periodic functions on the torus $\mathbb{T}^r \simeq M_h^r$.

Below we shall describe an algebraic-analytical algorithm for resolving this question for the case when a symplectic manifold M^{2n} is diffeomorphic to the canonically symplectic cotangent phase space $T^*(\mathbb{R}) \simeq M^{2n}$.

2. GENERAL SETTING

Our main object of study will be differential systems of vector fields on the cotangent phase space $M^{2n} = T^*(\mathbb{R}^n)$, $n \in \mathbb{Z}_+$, endowed with the canonical symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$, where by $\omega^{(2)} = d(\text{pr}^* \alpha^{(1)})$, and

$$\alpha^{(1)} := \langle p, dq \rangle = \sum_{j=1}^n p_j dq_j, \quad (2.1)$$

is the canonical 1-form on the base space $\mathbb{R}^n$, lifted naturally to the space $\Lambda^1(M^{2n})$, $(q, p) \in M^{2n}$ are canonical coordinates on $T^*(\mathbb{R}^n)$, $\text{pr} : T^*(\mathbb{R}^n) \rightarrow \mathbb{R}^n$ is the canonical projection, and $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^n$.

Assume further that there is also given a Lie subgroup G (not necessarily compact), acting symplectically via the mapping $\varphi : G \times M^{2n} \rightarrow M^{2n}$ on M^{2n} , generating a Lie algebra homomorphism $\varphi_* : T(\mathcal{G}) \rightarrow \Gamma(M^{2n})$ via the diagram

$$\begin{array}{ccc} \mathcal{G} \times \mathcal{G} \simeq T(\mathcal{G}) & \xrightarrow{\varphi_*(u)} & T(M^{2n}) \\ \downarrow & & \downarrow \\ \mathcal{G} & \xrightarrow{\varphi(u)} & M^{2n} \end{array} \quad (2.2)$$

where $u \in M^{2n}$. Thus, for any $a \in \mathcal{G}$ one can define a vector field $K_a \in \Gamma(M^{2n})$ as follows:

$$K_a = \varphi_* \cdot a. \quad (2.3)$$

Since the manifold M^{2n} is symplectic, one can naturally define for any $a \in \mathcal{G}$ a function $H_a \in \mathcal{D}(M^{2n})$ as follows:

$$-i_{K_a} \omega^{(2)} = dH_a, \quad (2.4)$$

whose existence follows from the invariance property

$$L_{K_a} \omega^{(2)} = 0 \quad (2.5)$$

for all $a \in \mathcal{G}$. The following lemma [1] is useful in applications.

Lemma 1. *If the first homology group $H_1(\mathcal{G}; \mathbb{R})$ of the Lie algebra $\mathcal{G}$ vanishes, then the mapping $\Phi : \mathcal{G} \rightarrow \mathcal{D}(M^{2n})$ defined as*

$$\Phi(a) := H_a \quad (2.6)$$

for any $a \in \mathcal{G}$ is a Lie algebra homomorphism of $\mathcal{G}$ and $\mathcal{D}(M^{2n})$ (endowed with the Lie structure induced by the symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$). In this case $\mathcal{G}$ is said to be Poissonian.

As the mapping $\Phi : \mathcal{G} \rightarrow \mathcal{D}(M^{2n})$ is evidently linear in $\mathcal{G}$, expression (2.6) naturally defines a momentum mapping $l : M^{2n} \rightarrow \mathcal{G}^*$ as follows: for any $u \in M^{2n}$ and all $a \in \mathcal{G}$

$$(l(u), a)_{\mathcal{G}} := H_a(u), \quad (2.7)$$

where $(\cdot, \cdot)_{\mathcal{G}}$ is the standard scalar product on the dual pair $\mathcal{G}^* \times \mathcal{G}$. The following characteristic equivariance [1] lemma holds.

Lemma 2. *The diagram*

$$\begin{array}{ccc} M^{2n} & \xrightarrow{l} & \mathcal{G}^* \\ \varphi_g \downarrow & & \downarrow \text{Ad}_{g^{-1}}^* \\ M^{2n} & \xrightarrow{l} & \mathcal{G}^* \end{array} \quad (2.8)$$

commutes for all $g \in G$, where $\text{Ad}_{g^{-1}}^ : \mathcal{G}^* \rightarrow \mathcal{G}^*$ is the corresponding co-adjoint action of the Lie group G on the dual space $\mathcal{G}^*$.*

Take now any vector $h \in \mathcal{G}^*$ and consider a subspace $\mathcal{G}_h \subset \mathcal{G}$, consisting of elements $a \in \mathcal{G}$, such that $\text{ad}_a^* h = 0$, where $\text{ad}_a^* : \mathcal{G}^* \rightarrow \mathcal{G}^*$ is the corresponding Lie algebra $\mathcal{G}$ representation in the dual space $\mathcal{G}^*$.

The following lemmas hold.

Lemma 3. *The subspace $\mathcal{G}_h \subset \mathcal{G}$ is a Lie subalgebra of $\mathcal{G}$, called here a Cartan subalgebra.*

Lemma 4. *Assume a vector $h \in \mathcal{G}^*$ is chosen in such a way that $r = \dim \mathcal{G}_h$ is minimal. Then the Cartan Lie subalgebra $\mathcal{G}_h \subset \mathcal{G}$ is abelian.*

In Lemma 4 the corresponding element $h \in \mathcal{G}^*$ is called regular and the number $r = \dim \mathcal{G}_h$ is called the rank $\mathcal{G}$ of the Lie algebra $\mathcal{G}$.

About twenty years ago Mishchenko and Fomenko [2] proved the following important noncommutative (nonabelian) Liouville–Arnold theorem.

Theorem 6. *On a symplectic space $(M^{2n}, \omega^{(2)})$ let there be given a set of smooth functions $H_j \in \mathcal{D}(M^{2n})$, $j = \overline{1, k}$, whose linear span over $\mathbb{R}$ comprises a Lie algebra $\mathcal{G}$ with respect to the corresponding Poisson bracket on M^{2n} . Suppose also that the set*

$$M_h^{2n-k} := \{u \in M^{2n} : h(H_j) = h_j \in \mathbb{R}, j = \overline{1, k}, h \in \mathcal{G}^*\}$$

with $h \in \mathcal{G}^*$ regular, is a submanifold of M^{2n} , and on M_h^{2n-k} all the functions $H_j \in \mathcal{D}(M^{2n})$, $j = \overline{1, k}$, are functionally independent. Assume also that the Lie algebra $\mathcal{G}$ satisfies the following condition:

$$\dim \mathcal{G} + \text{rank } \mathcal{G} = \dim M^{2n}. \quad (2.9)$$

Then the submanifold $M_h^r := M_h^{2n-k}$ is rank $\mathcal{G} = r$ -dimensional and invariant with respect to each vector field $K_{\bar{a}} \in \Gamma(M^{2n})$ with $\bar{a} \in \mathcal{G}_h \subset \mathcal{G}$. Given a vector field $K = K_{\bar{a}} \in \Gamma(M^{2n})$ with $\bar{a} \in \mathcal{G}_h$ or $K \in \Gamma(M^{2n})$ such that $[K, K_a] = 0$ for all $a \in \mathcal{G}$, then, if the submanifold M_h^r is connected and compact, it is diffeomorphic to the r -dimensional torus $T^r \simeq M_h^r$ and the motion of the vector field $K \in \Gamma(M^{2n})$ on it is a quasiperiodic function of the evolution parameter $t \in \mathbb{R}$.

The easiest proof of this result can be obtained from the well-known [3] classical Lie–Cartan theorem, mentioned in the Introduction. Below we shall only sketch the original Mishchenko–Fomenko proof which is heavily based on symplectic theory techniques, some of which have been discussed above.

SKETCH OF PROOF. Define a Lie group G naturally as $G = \exp \mathcal{G}$, where $\mathcal{G}$ is the Lie algebra of functions $H_j \in \mathcal{D}(M^{2n})$, $j = \overline{1, k}$, in the theorem, with respect to the Poisson bracket $\{\cdot, \cdot\}$ on M^{2n} . Then for an element $h \in \mathcal{G}^*$ and any $a = \sum_{j=1}^k c_j H_j \in \mathcal{G}$, where $c_j \in \mathbb{R}$, $j = \overline{1, k}$, the following equality

$$(h, a)_{\mathcal{G}} := \sum_{j=1}^k c_j h(H_j) = \sum_{j=1}^k c_j h_j \quad (2.10)$$

holds. Since all functions $H_j \in \mathcal{D}(M^{2n})$, $j = \overline{1, k}$, are independent on the level submanifold $M_h^r \subset M^{2n}$, this evidently means that the element $h \in \mathcal{G}^*$ is regular for the Lie algebra $\mathcal{G}$. Consequently, the Cartan Lie subalgebra $\mathcal{G}_h \subset \mathcal{G}$ is abelian. The latter is proved by means of simple straightforward calculations. Moreover, the corresponding momentum mapping $l : M^{2n} \rightarrow \mathcal{G}^*$ is constant on M_h^r and satisfies the following relation:

$$l(M_h^r) = h \in \mathcal{G}^*. \quad (2.11)$$

From this it can be shown that all vector fields $K_{\bar{a}} \in \Gamma(M^{2n})$, $\bar{a} \in \mathcal{G}_h$, are tangent to the submanifold $M_h^r \subset M^{2n}$. Thus the corresponding Lie subgroup $G_h := \exp \mathcal{G}_h$ acts naturally and invariantly on M_h^r . If the submanifold $M_h^r \subset M^{2n}$ is connected and compact, it follows from (2.9) that $\dim M_h^r = \dim M^{2n} - \dim \mathcal{G} = \text{rank } \mathcal{G} = r$, and one obtains via the Arnold theorem [4] that $M_h^r \simeq \mathbb{T}^r$ and the motion of the vector field $K \in \Gamma(M^{2n})$ is a quasiperiodic function of the evolution parameter $t \in \mathbb{R}$, thus proving the theorem. $\square$

As a nontrivial consequence of the Lie–Cartan theorem mentioned before and of Theorem 6, one can prove the following dual theorem about abelian Liouville–Arnold integrability.

Theorem 7. *Let a vector field $K \in \Gamma(M^{2n})$ be completely integrable via the non-abelian scheme of Theorem 6. Then it is also Liouville–Arnold integrable on M^{2n} and possesses, under some additional conditions, yet another abelian Lie algebra $\mathcal{G}_h$ of functionally independent invariants on M^{2n} , for which $\dim \mathcal{G}_h = n = (1/2) \dim M^{2n}$.*

The available proof of the theorem above is quite complicated, and we shall comment on it in detail later on. We mention here only that some analogs of the reduction Theorem 6 for the case where $M^{2n} \simeq \mathcal{G}^*$, so that an arbitrary Lie group G acts symplectically on the manifold, were proved also in [6–10, 34]. Notice here, that in the case when equality (2.10) is not satisfied one can then construct in the usual way the reduced manifold $\bar{M}_h^{2n-k-r} := M_h^{2n-k}/G_h$ on which there exists a symplectic structure $\bar{\omega}_h^{(2)} \in \Lambda^2(\bar{M}_h^{2n-k-r})$, defined as

$$r_h^* \bar{\omega}_h^{(2)} = \pi_h^* \omega^{(2)} \quad (2.12)$$

with respect to the following compatible reduction-imbedding diagram:

$$\bar{M}_h^{2n-k-r} \xleftarrow{r_h} M_h^{2n-k} \xrightarrow{\pi_h} M^{2n} \quad (2.13)$$

where $r_h : M_h^{2n-k} \rightarrow \bar{M}_h^{2n-k-r}$ and $\pi_h : M_h^{2n-k} \rightarrow M^{2n}$ are, respectively, the corresponding reductions and imbedding mappings. The nondegeneracy of the 2-form $\bar{\omega}_h^{(2)} \in \Lambda^2(\bar{M}_h)$ defined by (2.13), follows simply from the expression

$$\begin{aligned} \ker(\pi_h^* \omega^{(2)}(u)) &= T_u(M_h^{2n-k}) \cap T_u^\perp(M_h^{2n-k}) \\ &= \text{span}_{\mathbb{R}} \left\{ K_{\bar{a}}(u) \in T_u(\bar{M}_h^{2n-k-r} := M_h^{2n-k}/G_h) : \bar{a} \in \mathcal{G}_h \right\} \end{aligned} \quad (2.14)$$

for any $u \in M_h^{2n-k}$, since all vector fields $K_{\bar{a}} \in \Gamma(M^{2n})$, $\bar{a} \in \mathcal{G}_h$, are tangent to $\bar{M}_h^{2n-k-r} := M_h^{2n-k}/G_h$. Thus, the reduced space $\bar{M}_h^{2n-k-r} := M_h^{2n-k}/G_h$ with respect to the orbits of the Lie subgroup G_h action on M_h^{2n-k} will be a $(2n - k - r)$ -dimensional symplectic manifold. The latter evidently means that the number $2n - k - r = 2s \in \mathbb{Z}_+$ is even as there is no symplectic structure on odd-dimensional manifolds. This obviously is closely connected with the problem of existence of a symplectic group action of a Lie group G on a given symplectic manifold $(M^{2n}, \omega^{(2)})$ with a symplectic structure $\omega^{(2)} \in \Lambda^{(2)}(M^{2n})$ being *a priori* fixed. From this point of view one can consider the inverse problem of constructing symplectic structures on a manifold M^{2n} , admitting a Lie group G action. Namely, owing to the equivariance property (2.8) of the momentum mapping $l : M^{2n} \rightarrow \mathcal{G}^*$, one can obtain the induced symplectic structure $l^* \Omega_h^{(2)} \in \Lambda^2(\bar{M}_h^{2n-k-r})$ on $\bar{M}_h^{2n-k-r}$ from the canonical symplectic structure $\Omega_h^{(2)} \in \Lambda^{(2)}(\text{Or}(h; G))$ on the orbit $\text{Or}(h; G) \subset \mathcal{G}^*$ of a regular element $h \in \mathcal{G}^*$.

Since the symplectic structure $l^*\Omega_h^{(2)} \in \Lambda^2(\bar{M}_h)$ can be naturally lifted to the 2-form $\tilde{\omega}^{(2)} = (r_h^* \circ l^*)\Omega_h^{(2)} \in \Lambda^2(M_h^{2n-k})$, the latter being degenerate on M_h^{2n-k} can apparently be nonuniquely extended on the whole manifold M^{2n} to a symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$, for which the action of the Lie group G is *a priori* symplectic. Thus, many properties of a given dynamical system with a Lie algebra $\mathcal{G}$ of invariants on M^{2n} are deeply connected with the symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$ the manifold M^{2n} is endowed with, and in particular, with the corresponding integral submanifold imbedding mapping $\pi_h : M_h^{2n-k} \rightarrow M^{2n}$ at a regular element $h \in \mathcal{G}^*$. The problem of direct algebraic-analytical construction of this mapping was in part solved in [11] in the case where $n = 2$ for an abelian algebra $\mathcal{G}$ on the manifold $M^4 = T^*(\mathbb{R}^2)$. The treatment of this problem in [11] was extensively based both on the classical Cartan studies of integral submanifolds of ideals in Grassmann algebras and on the modern Galisot–Reeb–François results for a symplectic manifold $(M^{2n}, \omega^{(2)})$ structure, on which there exists an involutive set $\mathcal{G}$ of functionally independent invariants $H_j \in \mathcal{D}(M^{2n})$, $j = \overline{1, n}$. In what follows below we generalize the Galisot–Reeb–François results to the case of a nonabelian set of functionally independent functions $H_j \in \mathcal{D}(M^{2n})$, $j = \overline{1, k}$, comprising a Lie algebra $\mathcal{G}$ and satisfying the Mishchenko–Fomenko condition (2.9): $\dim \mathcal{G} + \text{rank } \mathcal{G} = \dim M^{2n}$. This makes it possible to devise an effective algebraic-analytical method of constructing the corresponding integral submanifold imbedding and reduction mappings, giving rise to a wide class of exact, integrable by quadratures, solutions of a given integrable vector field on M^{2n} .

3. INTEGRAL SUBMANIFOLD IMBEDDING PROBLEM FOR AN ABELIAN LIE ALGEBRA OF INVARIANTS

We shall consider here only a set $\mathcal{G}$ of commuting polynomial functions $H_j \in \mathcal{D}(M^{2n})$, $j = \overline{1, n}$, on the canonically symplectic phase space $M^{2n} = T^*(\mathbb{R}^n)$. Due to the Liouville–Arnold theorem [4], any dynamical system $K \in \Gamma(M^{2n})$ commuting with corresponding Hamiltonian vector fields K_a for all $a \in \mathcal{G}$, will be integrable by quadratures in case of a regular element $h \in \mathcal{G}^*$, which defines the corresponding integral submanifold $M_h^n := \{u \in M^{2n} : h(H_j) = h_j \in \mathbb{R}, j = \overline{1, n}\}$ which is diffeomorphic (when compact and connected) to the n -dimensional torus $\mathbb{T}^n \simeq M_h^n$. This in particular means that there exists some algebraic-analytical expression for the integral submanifold imbedding mapping $\pi_h : M_h^n \rightarrow M^{2n}$ into the ambient phase space M^{2n} , which one should find in order to properly demonstrate integrability by quadratures.

The problem formulated above was posed and in part solved (as was mentioned above) for $n = 2$ in [11] and in [13] for a Henon–Heiles dynamical system which had previously been integrated [14, 15] using other tools. Here we generalize the approach of [11] for the general case $n \in \mathbb{Z}_+$ and proceed further in Section 3 to solve

this problem in the case of a nonabelian Lie algebra $\mathcal{G}$ of polynomial invariants on $M^{2n} = T^*(\mathbb{R}^n)$, satisfying all the conditions of the Mishchenko–Fomenko Theorem 6.

Define now the basic vector fields $K_j \in \Gamma(M^{2n})$, $j = \overline{1, n}$, generated by basic elements $H_j \in \mathcal{G}$ of an abelian Lie algebra $\mathcal{G}$ of invariants on M^{2n} , as follows:

$$-i_{K_j}\omega^{(2)} = dH_j \quad (3.1)$$

for all $j = \overline{1, n}$. It is easy to see that the condition $\{H_j, H_i\} = 0$ for all $i, j = \overline{1, n}$, yields also $[K_i, K_j] = 0$ for all $i, j = \overline{1, n}$. Taking into account that $\dim M^{2n} = 2n$, one obtains the equality $(\omega^{(2)})^n = 0$ identically on M^{2n} . This makes it possible to formulate the following Galisau–Reeb result.

Theorem 8. *Assume that an element $h \in \mathcal{G}^*$ is chosen to be regular and a Lie algebra $\mathcal{G}$ of invariants on M^{2n} is abelian. Then there exist differential 1-forms $h_j^{(1)} \in \Lambda^1(U(M_h^n))$, $j = \overline{1, n}$, where $U(M_h^n)$ is some open neighborhood of the integral submanifold $M_h^n \subset M^{2n}$, satisfying the following properties:*

- (i) $\omega^{(2)}|_{U(M_h^n)} = \sum_{j=1}^n dH_j \wedge h_j^{(1)}$;
- (ii) *The exterior differentials $dh_j^{(1)} \in \Lambda^2(U(M_h^n))$ belong to the ideal $\mathcal{I}(\mathcal{G})$ in the Grassmann algebra $\Lambda(U(M_h^n))$ generated by the 1-forms $dH_j \in \Lambda^1(U(M_h^n))$, $j = \overline{1, n}$.*

PROOF. Consider the following identity on M^{2n} :

$$\left(\otimes_{j=1}^n i_{K_j}\right)(\omega^{(2)})^{n+1} = 0 = \pm(n+1)! \left(\wedge_{j=1}^n dH_j\right) \wedge \omega^{(2)}, \quad (3.2)$$

which implies that the 2-form $\omega^{(2)} \in \mathcal{I}(\mathcal{G})$. Therefore, one can find 1-forms $h_j^{(1)} \in \Lambda^1(U(M_h^n))$, $j = \overline{1, n}$, satisfying the condition

$$\omega^{(2)}|_{U(M_h^n)} = \sum_{j=1}^n dH_j \wedge h_j^{(1)}. \quad (3.3)$$

Since $\omega^{(2)} \in \Lambda^2(U(M_h^n))$ is nondegenerate on M^{2n} , it follows that all 1-forms $h_j^{(1)}$, $j = \overline{1, n}$, in (3.3) are independent on $U(M_h^n)$, proving part (i) of the Theorem. As $d\omega^{(2)} = 0$ on M^{2n} , from (2.3) one gets that

$$\sum_{j=1}^n dH_j \wedge dh_j^{(1)} = 0 \quad (3.4)$$

on $U(M_h^n)$, hence it is obvious that $dh_j^{(1)} \in \mathcal{I}(\mathcal{G}) \subset \Lambda(U(M_h^n))$ for all $j = \overline{1, n}$, proving part (ii) of the Theorem. $\square$

Now we proceed to study properties of the integral submanifold $M_h^n \subset M^{2n}$ of the ideal $\mathcal{I}(\mathcal{G})$ in the Grassmann algebra $\Lambda(U(M_h^n))$. In general, the integral submanifold M_h^n is completely described [16] by means of the imbedding

$$\pi_h : M_h^n \rightarrow M^{2n} \quad (3.5)$$

and using this, one can reduce all vector fields $K_j \in \Gamma(M^{2n})$, $j = \overline{1, n}$, on the submanifold $M_h^n \subset M^{2n}$, since they are all evidently in its tangent space. If $\bar{K}_j \in \Gamma(M_h^n)$, $j = \overline{1, n}$, are the corresponding pulled-back vector fields $K_j \in \Gamma(M^{2n})$, $j = \overline{1, n}$, then by definition, the equality

$$\pi_{h*} \circ \bar{K}_j = K_j \circ \pi_h \quad (3.6)$$

holds for all $j = \overline{1, n}$. Similarly one can construct 1-forms $\bar{h}_j^{(1)} := \pi_h^* \circ h_j^{(1)} \in \Lambda^1(M_h^n)$, $j = \overline{1, n}$, which are characterized by the following Cartan–Jost [16] Theorem.

Theorem 9. *The following assertions are true:*

- (i) *The 1-forms $\bar{h}_j^{(1)} \in \Lambda^1(M_h^n)$, $j = \overline{1, n}$, are independent on M_h^n ;*
- (ii) *The 1-forms $\bar{h}_j^{(1)} \in \Lambda^1(M_h^n)$, $j = \overline{1, n}$, are exact on M_h^n and satisfy $\bar{h}_j^{(1)}(\bar{K}_j) = \delta_{ij}$, $i, j = \overline{1, n}$.*

PROOF. As the ideal $\mathcal{I}(\mathcal{G})$ is by definition vanishing on $M_h^n \subset M^{2n}$ and closed on $U(M_h^n)$, the integral submanifold M_h^n is well defined in the case of a regular element $h \in \mathcal{G}^*$. This implies that the imbedding (3.5) is nondegenerate on $M_h^n \subset M^{2n}$, or the 1-forms $\bar{h}_j^{(1)} := \pi_h^* \circ h_j^{(1)}$, $j = \overline{1, n}$, will persist in being independent if they are 1-forms $h_j^{(1)} \in \Lambda^1(U(M_h^n))$, $j = \overline{1, n}$, proving part (i) of the theorem. Using property (ii) of Theorem 8, one sees that on the integral submanifold $M_h^n \subset M^{2n}$ all 2-forms $d\bar{h}_j^{(1)} = 0$, $j = \overline{1, n}$. Consequently, owing to the Poincaré lemma [1, 16], the 1-forms $\bar{h}_j^{(1)} = d\bar{t}_j \in \Lambda^1(M_h^n)$, $j = \overline{1, n}$, for some mappings $\bar{t}_j : M_h^n \rightarrow \mathbb{R}$, $j = \overline{1, n}$, defining global coordinates on an appropriate universal covering of M_h^n . Consider now the following identity based on the representation (3.3):

$$i_{K_j} \omega^{(2)}|_{U(M_h^n)} = - \sum_{i=1}^n h_i^{(1)}(K_j) dH_i := -dH_j, \quad (3.7)$$

which holds for any $j = \overline{1, n}$. As all $dH_j \in \Lambda^1(U(M_h^n))$, $j = \overline{1, n}$, are independent, from (3.7) one infers that $h_i^{(1)}(K_j) = \delta_{ij}$ for all $i, j = \overline{1, n}$. Recalling now that for any $i = \overline{1, n}$, $K_i \circ \pi_h = \pi_{h*} \circ K_i$, one readily computes that

$$\bar{h}_i^{(1)}(\bar{K}_j) = \pi_h^* h_i^{(1)}(\bar{K}_j) := h_i^{(1)}(\pi_{h*} \circ K_j) := h_i^{(1)}(K_j \circ \pi_h) = \delta_{ij}$$

for all $i, j = \overline{1, n}$, proving part (ii) of the Theorem. $\square$

The following is a simple consequence of Theorem 9:

Corollary 2. *Suppose that the vector fields $K_j \in \Gamma(M^{2n})$, $j = \overline{1, n}$, are parametrized globally along their trajectories by means of the corresponding parameters $t_j : M^{2n} \rightarrow \mathbb{R}$, $j = \overline{1, n}$, that is on the phase space M^{2n}*

$$d/dt_j := K_j \quad (3.8)$$

for all $j = \overline{1, n}$. Then the following important equalities hold (up to constant normalizations) on the integral submanifold $M_h^n \subset M^{2n}$:

$$t_j|_{M_h^n} = \bar{t}_j, \quad (3.9)$$

where $1 \leq j \leq n$.

We consider a completely integrable via Liouville–Arnold Hamiltonian system on the cotangent canonically symplectic manifold $(T^*(\mathbb{R}^n), \omega^{(2)})$, $n \in \mathbb{Z}_+$, possessing exactly $n \in \mathbb{Z}_+$ functionally independent and Poisson commuting algebraic polynomial invariants $H_j : T^*(\mathbb{R}^n) \rightarrow \mathbb{R}$, $j = \overline{1, n}$. Due to the Liouville–Arnold theorem this Hamiltonian system can be completely integrated by quadratures in quasi-periodic functions on its integral submanifold when taken compact. It is equivalent to the statement that this compact integral submanifold is diffeomorphic to a torus $\mathbb{T}^n$, which makes it possible to formulate the problem of integrating the system by means of searching the corresponding integral submanifold imbedding mapping $\pi_h : M_h^n \rightarrow T^*(\mathbb{R}^n)$, where by definition

$$M_h^n := \{(q, p) \in T^*(\mathbb{R}^n) : H_j(q, p) = h_j \in \mathbb{R}, j = \overline{1, n}\}. \quad (3.10)$$

Since $M_h^n \simeq \mathbb{T}^n$, and the integral submanifold (1.1) is invariant subject to all Hamiltonian flows $K_j : T^*(\mathbb{R}^n) \rightarrow T(T^*(\mathbb{R}^n))$, $j = \overline{1, n}$, where

$$i_{K_j} \omega^{(2)} = -dH_j, \quad (3.11)$$

there exist corresponding “action–angle” coordinates $(\varphi, \gamma) \in (\mathbb{T}_\gamma^n, \mathbb{R}^n)$ on the torus $\mathbb{T}_\gamma^n \simeq M_h^n$, specifying its imbedding $\pi_\gamma : \mathbb{T}_\gamma^n \rightarrow T^*(\mathbb{R}^n)$ by means of a set of smooth functions $\gamma \in \mathcal{D}(\mathbb{R}^n)$, where

$$\mathbb{T}_\gamma^n := \{(q, p) \in T^*(\mathbb{R}^n) : \gamma_j(H) = \gamma_j \in \mathbb{R}, j = \overline{1, n}\}. \quad (3.12)$$

The induced by (3.12) mapping $\gamma : \mathbb{R}^n \ni h \rightarrow \mathbb{R}^n$ is of great interest for many applications and was studied still earlier by Picard and Fuchs subject to the corresponding differential equations it satisfies:

$$\partial \gamma_j(h) / \partial h_i = F_{ij}(\gamma; h), \quad (3.13)$$

where $h \in \mathbb{R}^n$ and $F_{ij} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, $i, j = \overline{1, n}$, are some almost everywhere smooth functions. In the case when the right hand side of (1.5) is a set of algebraic functions on $\mathbb{C}^n \times \mathbb{C}^n \ni (\gamma; h)$, all Hamiltonian flows $K_j : T^*(\mathbb{R}^n) \rightarrow T(T^*(\mathbb{R}^n))$, $j = \overline{1, n}$, are said to be algebraically completely integrable in quadratures. In general equations like (3.13) were studied in [19,31], a recent example can be found in [18].

It is clear enough that Picard–Fuchs equations (3.13) are related to the associated canonical transformation of the symplectic 2-form $\omega^{(2)} \in \Lambda^2(T^*(\mathbb{R}^n))$ in a neighborhood $U(M_h^n)$ of the integral submanifold $M_h^n \subset T^*(\mathbb{R}^n)$. To make it more precise, denote $\omega^{(2)}(q, p) = d \operatorname{pr}^* \alpha^{(1)}(q; p)$, where for $(q, p) \in T^*(\mathbb{R}^n)$

$$\alpha^{(1)}(q; p) := \sum_{j=1}^n p_j dq_j = \langle p, dq \rangle \in \Lambda^1(\mathbb{R}^n) \quad (3.14)$$

is the canonical Liouville 1-form on $\mathbb{R}^n$, $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^n$, $\operatorname{pr} : T^*(\mathbb{R}^n) \rightarrow \mathbb{R}^n$ is the bundle projection. One can now define a mapping

$$dS_q : \mathbb{R}^n \rightarrow T_q^*(\mathbb{R}^n), \quad (3.15)$$

such that $dS_q(h) \in T_q^*(\mathbb{R}^n)$ is an exact 1-form for all $q \in M_h^n$ and $h \in \mathbb{R}^n$, yielding

$$(dS_q)^*(d \operatorname{pr}^* \alpha^{(1)}) = (dS_q)^* \omega^{(2)} := d^2 S_q \equiv 0. \quad (3.16)$$

Thereby the mapping (3.15) defines a so-called generating function [1, 2] $S_q : \mathbb{R}^n \rightarrow \mathbb{R}$, satisfying on M_h^n the relationship

$$\operatorname{pr}^* \alpha^{(1)}(q; p) + \langle t, dh \rangle = dS_q(h), \quad (3.17)$$

where $t \in \mathbb{R}^n$ is the set of evolution parameters. From (3.17) one gets right away that equality

$$S_q(h) = \int_{q^{(0)}}^q \langle p, dq \rangle \Big|_{M_h^n} \quad (3.18)$$

holds for any $q, q^{(0)} \in M_h^n$. On the other hand, one can define one more generating function $S_\mu : \mathbb{R}^n \rightarrow \mathbb{R}$, such that

$$dS_\mu : \mathbb{R}^n \rightarrow T_\mu^*(M_h^n), \quad (3.19)$$

where $\mu \in M_h^n \simeq \otimes_{j=1}^n \mathbb{S}_j^1$ are global separable coordinates existing on M_h^n owing to the Liouville–Arnold theorem. Thus one can write the following canonical relationship:

$$\langle w, d\mu \rangle + \langle t, dh \rangle = dS_\mu(h), \quad (3.20)$$

where $w_j := w_j(\mu_j; h) \in T_{\mu_j}^*(\mathbb{S}_j^1)$ for every $j = \overline{1, n}$. Whence it follows readily that

$$S_\mu(h) = \sum_{j=1}^n \int_{\mu_j^{(0)}}^{\mu_j} w(\lambda; h) d\lambda, \quad (3.21)$$

satisfying on $M_h^n \subset T^*(\mathbb{R}^n)$ the following relationship

$$dS_\mu + d\mathcal{L}_\mu = dS_q \Big|_{q=q(\mu; h)} \quad (3.22)$$

for some mapping $\mathcal{L}_\mu : \mathbb{R}^n \rightarrow \mathbb{R}$. As a result of (3.21) and (3.22) one gets that the following important expressions

$$t_i = \partial S_\mu(h) / \partial h_i, \quad \langle p, \partial q / \partial \mu_i \rangle = w_i + \partial \mathcal{L}_\mu / \partial \mu_i \quad (3.23)$$

hold for all $i = \overline{1, n}$. A construction similar to the above can be done subject to the imbedded torus $\mathbb{T}_\gamma^n \subset T^*(\mathbb{R}^n)$:

$$d\tilde{S}_q(\gamma) := \sum_{j=1}^n p_j dq_j + \sum_{i=1}^n \varphi_i d\gamma_i, \quad (3.24)$$

where owing to (3.15) $\tilde{S}_q(\gamma) := S_q(\xi \cdot \gamma)$, $\xi \cdot \gamma(h) = h$, for all $(q; \gamma) \in U(M_h^n)$. For angle coordinates $\varphi \in \mathbb{T}_\gamma^n$ one obtains from (3.24) that

$$\varphi_i = \partial \tilde{S}_q(\gamma) / \partial \gamma_i \quad (3.25)$$

for all $i = \overline{1, n}$. As $\varphi_i \in \mathbb{R}/2\pi\mathbb{Z}$, $i = \overline{1, n}$, from (3.26) one derives that

$$\frac{1}{2\pi} \oint_{\sigma_j^{(h)}} d\varphi_i = \delta_{ij} = \frac{1}{2\pi} \frac{\partial}{\partial \gamma_i} \oint_{\sigma_j^{(h)}} d\tilde{S}_q(\gamma) = \frac{1}{2\pi} \frac{\partial}{\partial \gamma_i} \oint_{\sigma_j^{(h)}} \langle p, dq \rangle \quad (3.26)$$

for all canonical cycles $\sigma_j^{(h)} \subset M_h^n$, $j = \overline{1, n}$, constituting a basis of the one dimensional homology group $H^1(M_h^n; \mathbb{Z})$. Thereby, owing to (3.26), it follows that for all $i = \overline{1, n}$ “action” variables can be found as

$$\gamma_i = \frac{1}{2\pi} \oint_{\sigma_i^{(h)}} \langle p, dq \rangle \quad (3.27)$$

Recall now that $M_h^n \simeq \mathbb{T}_\gamma^n$ are diffeomorphic also to $\otimes_{j=1}^n \mathbb{S}_j^1$, where $\mathbb{S}_j^1$, $j = \overline{1, n}$, are some one-dimensional real circles. The evolution along any of the vector fields $K_j : T^*(\mathbb{R}^n) \rightarrow T(T^*(\mathbb{R}^n))$, $j = \overline{1, n}$, on $M_h^n \subset T^*(\mathbb{R}^n)$ is known [1, 2] to be a linear winding around the torus $\mathbb{T}_\gamma^n$, that can be interpreted also this way: the above introduced independent of every other global coordinate on circles $\mathbb{S}_j^1$, $j = \overline{1, n}$, are such that the resulting evolution undergoes a quasiperiodic motion. These coordinates being still called Hamilton–Jacobi ones prove to be very important for accomplishing the complete integrability by quadratures via solving the corresponding Picard–Fuchs type equations.

Let us denote these separable coordinates on the integral submanifold $M_h^n \simeq \otimes_{j=1}^n \mathbb{S}_j^1$ by $\mu_j \in \mathbb{S}_j^1$, $j = \overline{1, n}$, and define the corresponding imbedding mapping $\pi_h : M_h^n \rightarrow T^*(\mathbb{R}^n)$ as

$$q = q(\mu; h), \quad p = p(\mu; h). \quad (3.28)$$

There exist two important cases subject to the imbedding (3.28).

The first case is related to the integral submanifold $M_h^n \subset T^*(\mathbb{R}^n)$ which can be parametrized as a manifold by means of the base coordinates $q \in \mathbb{R}^n$ of the cotangent bundle $T^*(\mathbb{R}^n)$. This can be explained as follows: the canonical Liouville 1-form $\alpha^{(1)} \in \Lambda^1(\mathbb{R}^n)$, in accordance with the diagram

$$\begin{array}{ccccc}
 T^*(M_h^n) & \simeq & T^*\left(\otimes_{j=1}^n \mathbf{S}_j^1\right) & \xleftarrow{\pi^*} & T^*(\mathbb{R}^n) \\
 \text{pr} \downarrow & & \text{pr} \downarrow & & \text{pr} \downarrow \\
 M_h^n & \simeq & \otimes_{j=1}^n \mathbf{S}_j^1 & \xrightarrow{\pi} & \mathbb{R}^n
 \end{array} \quad (3.29)$$

is mapped by the imbedding mapping $\pi = \text{pr} \cdot \pi_h : M_h^n \rightarrow \mathbb{R}^n$ not depending on a set of parameters $h \in \mathbb{R}^n$, into the 1-form

$$\alpha_h^{(1)} = \pi^* \alpha^{(1)} = \sum_{j=1}^n w_j(\mu_j; h) d\mu_j, \quad (3.30)$$

where $(\mu, w) \in T^*\left(\otimes_{j=1}^n \mathbf{S}_j^1\right) \simeq \otimes_{j=1}^n T^*(\mathbf{S}_j^1)$. The imbedding mapping $\pi : M_h^n \rightarrow \mathbb{R}^n$ due to equality (3.30) makes the function $\mathcal{L}_\mu : \mathbb{R}^n \rightarrow \mathbb{R}$ to be zero giving rise to the generating function $S_\mu : \mathbb{R}^n \rightarrow \mathbb{R}$, enjoying the condition

$$dS_\mu = dS_q|_{q=q(\mu; h)}, \quad (3.31)$$

where as before

$$S_\mu(h) = \sum_{j=1}^n p_j dq_j + \sum_{j=1}^n t_j dh_j \quad (3.32)$$

and $\det \|\partial q(\mu; h) / \partial \mu\| \neq 0$ almost everywhere on M_h^n for all $h \in \mathbb{R}^n$. Similarly to (3.23), one gets from (3.32) that

$$t_j = \partial S_\mu(h) / \partial h_j \quad (3.33)$$

for $j = \overline{1, n}$. Concerning the second part of the imbedding mapping (3.28) we arrive due to equality (3.30) at the following simple result:

$$p_i = \sum_{j=1}^n w_j(\mu_j; h) \partial \mu_j / \partial q_i, \quad (3.34)$$

where $i = \overline{1, n}$ and $\det \|\partial \mu / \partial q\| \neq 0$ almost everywhere on $\pi(M_h^n)$ due to the local invertibility of the imbedding mapping $\pi : M_h^n \rightarrow \mathbb{R}^n$. Thus, we can claim that the problem of complete integrability in the first case is solved iff the only imbedding mapping $\pi : M_h^n \rightarrow \mathbb{R}^n \subset T^*(\mathbb{R}^n)$ is constructed. This case was in detail considered in [11], where the corresponding Picard–Fuchs type equations were built based on an extension of Galisot–Reeb and Françoise results [18]. Namely, similarly to (3.13), these equations are defined as follows:

$$\partial w_j(\mu_j; h) / \partial h_k = P_{kj}(\mu_j, w_j; h), \quad (3.35)$$

where $P_{kj} : T^*\left(\otimes_{j=1}^n \mathbf{S}_j^1\right) \times \mathbb{C}^n \rightarrow \mathbb{C}$, $k, j = \overline{1, n}$, are some algebraic functions of their arguments.

Concerning *the second case* when the integral submanifold $M_h^n \subset T^*(\mathbb{R}^n)$ cannot be imbedded almost everywhere into the base space $\mathbb{R}^n \subset T^*(\mathbb{R}^n)$, the relationship like (3.32) does not take place, and we are forced to consider the usual canonical transformation from $T^*(\mathbb{R}^n)$ to $T^*(\mathbb{R}^n)$ based on a mapping $d\mathcal{L}_q : \otimes_{j=1}^n \mathbb{S}_j^1 \rightarrow T^*(\mathbb{R}^n)$, where $\mathcal{L}_q : \otimes_{j=1}^n \mathbb{S}_j^1 \rightarrow \mathbb{R}$ enjoys for all $\mu \in \otimes_{j=1}^n \mathbb{S}_j^1 \simeq M_h^n \ni q$ the following relationship

$$\text{pr}^* \alpha^{(1)}(q; p) = \sum_{j=1}^n w_j d\mu_j + dL_q(\mu). \quad (3.36)$$

In this case we can derive for any $\mu \in \otimes_{j=1}^n \mathbb{S}_j^1$ the previously introduced hereditary generating function $\mathcal{L}_\mu : \mathbb{R}^n \rightarrow T^*(\otimes_{j=1}^n \mathbb{S}_j^1)$ as

$$d\mathcal{L}_\mu = d\mathcal{L}_q|_{q=q(\mu; h)}, \quad (3.37)$$

satisfying evidently the following canonical transformation condition:

$$dS_q(h) = \sum_{j=1}^n w_j(\mu_j; h) d\mu_j + \sum_{j=1}^n t_j dh_j + dL_\mu(h), \quad (3.38)$$

for almost all $\mu \in \otimes_{j=1}^n \mathbb{S}_j^1$ and $h \in \mathbb{R}^n$. Based on (3.38) one can derive the following relationships:

$$\partial L_\mu(h) / \partial h_j = \langle p, \partial q / \partial h_j \rangle|_{M_h^n} \quad (3.39)$$

for all $j = \overline{1, 2}, \mu \in \otimes_{j=1}^n \mathbb{S}_j^1$ and $h \in \mathbb{R}^n$. Whence the following important analytical result,

$$t_s = \sum_{j=1}^n \int_{\mu_j^{(0)}}^{\mu_j} (\partial w_j(\lambda; h) / \partial h_s) d\lambda, \quad (3.40)$$

$$\sum_{j=1}^n p_j(\mu; h) (\partial q_j / \partial \mu_s) = w_s + \partial \mathcal{L}_\mu(h) / \partial \mu_s,$$

holds for all $s = \overline{1, 2}$ and $\mu, \mu^{(0)} \in \otimes_{j=1}^n \mathbb{S}_j^1$ with parameters $h \in \mathbb{R}^n$ being fixed. Thereby we have found a natural generalization of relationships (3.34) subject to the extended integral submanifold imbedding mapping $\pi_h : M_h^n \rightarrow T^*(\mathbb{R}^n)$ in the form (3.28).

Assume now that functions $w_j : \mathbb{C} \times \mathbb{C}^n \rightarrow \mathbb{C}$, $j = \overline{1, n}$, satisfy in general Picard–Fuchs equations like (3.35), having the following [3] algebraic solutions:

$$w_j^{n_j} + \sum_{k=0}^{n_j-1} c_{j,k}(\lambda; h) w_j^k = 0, \quad (3.41)$$

where $c_{j,k} : \mathbb{C} \times \mathbb{C}^n \rightarrow \mathbb{C}$, $k = \overline{0, n_j-1}$, $j = \overline{1, n}$, are some polynomials in $\lambda \in \mathbb{C}$. Each algebraic curve of (3.41) is known to be in general topologically equivalent

due to the Riemann theorem [20] to some Riemannian surface $\Gamma_h^{(j)}$, of genus $g_j \in \mathbb{Z}_+$, $j = \overline{1, n}$. Thereby, one can realize the local diffeomorphism $\varrho : M_h^n \rightarrow \otimes_{j=1}^n \Gamma_h^{(j)}$, mapping homology group basis cycles $\sigma_j^{(h)} \subset M_h^n$, $j = \overline{1, n}$, into homology subgroup $H_1(\otimes_{j=1}^n \Gamma_h^{(j)}; \mathbb{Z})$ basis cycles $\sigma_j(\Gamma_h) \subset \Gamma_h^{(j)}$, $j = \overline{1, n}$, satisfying the following relationships:

$$\varrho(\sigma_j^{(h)}) = \sum_{k=1}^n n_{jk} \sigma_k(\Gamma_h), \quad (3.42)$$

where $n_{jk} \in \mathbb{Z}$, $k = \overline{1, j}$ and $j = \overline{1, n}$, are some fixed integers. Based on (3.42) and (3.37) one can write down, for instance, expressions (3.27) as follows:

$$\gamma_i = \frac{1}{2\pi} \sum_{j=1}^n n_{ij} \oint_{\sigma_j(\Gamma_h)} w_j(\lambda; h) d\lambda, \quad (3.43)$$

where $i = \overline{1, n}$. Subject to the evolution on $M_h^n \subset T^*(\mathbb{R}^n)$ one can easily obtain from (3.39) that

$$dt_i = \sum_{j=1}^n (\partial w_j(\mu_j; h) / \partial h_i) d\mu_j \quad (3.44)$$

at $dh_i = 0$ for all $i = \overline{1, n}$, giving rise evidently to a global τ -parametrization of the set of circles $\otimes_{j=1}^n \mathbf{S}_j^1 \subset \otimes_{j=1}^n \Gamma_h^{(j)}$, that is one can define some inverse algebraic functions to Abelian type integrals (3.37) as

$$\mu = \mu(\tau; h), \quad (3.45)$$

where as before, $\tau = (t_1, t_2, \dots, t_n) \in \mathbb{R}^n$ is an a vector of evolution parameters. Recalling now expressions (3.28) for integral submanifold mapping $\pi_h : M_h^n \rightarrow T^*(\mathbb{R}^n)$, one can at last write down final expressed by “quadratures” mappings for evolutions on $M_h^n \subset T^*(\mathbb{R}^n)$ as follows:

$$q = q(\mu(\tau; h)) = \tilde{q}(\tau; h), \quad p = p(\mu(\tau; h)) = \tilde{p}(\tau; h), \quad (3.46)$$

where obviously, a vector $(\tilde{q}, \tilde{p}) \in T^*(\mathbb{R}^n)$ is quasiperiodic in each variable $t_i \in \tau$, $i = \overline{1, n}$.

Theorem 10. *Every completely integrable Hamiltonian system admitting an algebraic submanifold $M_h^n \subset T^*(\mathbb{R}^n)$ possesses a separable canonical transformation (3.38) which is described by differential algebraic Picard–Fuchs type equations whose solution is a set of algebraic curves (3.41).*

Therefore, the main ingredient of this scheme of integrability by quadratures is finding the Picard–Fuchs type equations (3.35) corresponding to the integral submanifold imbedding mapping (3.28) depending in general on $\mathbb{R}^n \ni h$ -parameters for the case when the integral submanifold $M_h^n \subset T^*(\mathbb{R}^n)$ cannot be imbedded into the base space $\mathbb{R}^n \subset T^*(\mathbb{R}^n)$ of the phase space $T^*(\mathbb{R}^n)$. Based now on Theorem 8 one

can find 1-forms $h_j^{(1)} \in \Lambda^1(T^*(\mathbb{R}^n))$, $j = \overline{1, n}$, enjoying the following identity on $T^*(\mathbb{R}^n)$:

$$\omega^{(2)}(q, p) := \sum_{j=1}^n dp_j \wedge dq_j = \sum_{j=1}^n dH_j \wedge h_j^{(1)}. \quad (3.47)$$

The 1-forms $h_j^{(1)} \in \Lambda^1(T^*(\mathbb{R}^n))$, $j = \overline{1, n}$, possess the stated important property: pullbacked to the integral submanifold (3.10) gives rise to the global linearization

$$\pi_h^* h_j^{(1)} := \bar{h}_j^{(1)} = dt_j \quad (3.48)$$

where $\bar{h}_j^{(1)} \in \Lambda^1(M_h^n)$, and $\pi_{h*} d/dt_j = K_j \cdot \pi_h$ for all $j = \overline{1, n}$. Expressions (2.2) combined with those in (3.44) give rise easily to the following set of relationships

$$\bar{h}_j^{(1)} = \sum_{i=1}^n (\partial w_j(\mu_j; h) / \partial h_i) d\mu_i \quad (3.49)$$

at $dh_j = 0$ for all $j = \overline{1, n}$ on $M_h^n \simeq \otimes_{j=1}^n \mathbb{S}_j^1 \subset \otimes_{j=1}^n \Gamma_h^{(j)}$ for all $j = \overline{1, n}$. Since we are interested in the integral submanifold imbedding mapping (3.25) being locally diffeomorphic in a neighborhood $U(M_h^n) \subset T^*(\mathbb{R}^n)$, the Jacobian $\det \|\partial q(\mu; h) / \partial \mu\| \neq 0$ almost everywhere in $U(M_h^n)$. On the other hand, as it was proved in [4], the set of 1-forms $\bar{h}_j^{(1)} \in \Lambda^1(M_h^n)$, $j = \overline{1, n}$, can be in general, represented in $U(M_h^n)$ as

$$\bar{h}_j^{(1)} = \sum_{k=1}^n \bar{h}_{jk}^{(1)}(q, p) dq_k \Big|_{M_h^n}, \quad (3.50)$$

where $\bar{h}_{jk}^{(1)} : T^*(\mathbb{R}^n) \rightarrow \mathbb{R}$, $k, j = \overline{1, n}$, are some algebraic expressions of their arguments. Thereby, one easily finds from (3.50) and (3.49) that

$$\partial w_i(\mu_i; h) / \partial h_j = \sum_{k=1}^n \bar{h}_{jk}^{(1)}(q(\mu; h), p(\mu; h)) (\partial q_k(\mu; h) / \partial \mu_i) \quad (3.51)$$

for all $i, j = \overline{1, n}$. Subject to p -variables in (3.51) we must, owing to (3.40), use the expressions

$$\sum_{j=1}^n p_j(\mu; h) (\partial q_j / \partial \mu_s) = w_s + \partial \mathcal{L}_\mu(h) / \partial \mu_s, \quad (3.52)$$

$$\partial \mathcal{L}_\mu(h) / \partial h_j = \langle p, \partial q / \partial h_j \rangle \Big|_{M_h^n},$$

being true for $s = \overline{1, n}$ and all $\mu \in \otimes_{j=1}^n \mathbb{S}_j$, $h \in \mathbb{R}^n$ in the neighborhood $U(M_h^n) \subset T^*(\mathbb{R}^n)$ chosen before. Thereby, we arrived at the following form of equations (3.51):

$$\partial w_i(\mu_i; h) / \partial h_j = \bar{P}_{ji}(\mu, w; h), \quad (3.53)$$

where for all $i, j = \overline{1, n}$ expressions

$$\bar{P}_{ji}(\mu, w; h) := \sum_{k=1}^n \bar{h}_{jk}^{(1)}(q(\mu; h), p(\mu; h)) (\partial q_k / \partial \mu_i) \quad (3.54)$$

depend correspondingly only on $\Gamma_h^{(i)} \ni (\mu_i, w_i)$ -variables for each $i \in \overline{1, n}$ and all $j = \overline{1, n}$. This condition can be evidently written as follows:

$$\partial \bar{P}_{ji}(\mu, w; h) / \partial \mu_k = 0, \quad \partial \bar{P}_{ji}(\mu, w; h) / \partial w_k = 0 \quad (3.55)$$

for $j, i \neq k \in \overline{1, n}$ at almost all $\mu \in \otimes_{j=1}^n \mathbb{S}_j^1$ and $h \in \mathbb{R}^n$. The set of conditions (3.51) gives rise in general to a system of algebraic-differential equations subject to the imbedding mapping $\pi_h : M_h^n \rightarrow \mathbb{R}^n$ defined analytically by (3.28) and the generating function (3.37). As a result of solving these equations we obtain evidently, owing to (3.53) and (3.55), the following system of Picard–Fuchs type equations:

$$\partial w_i(\mu_i; h) / \partial h_j = P_{ji}(\mu_i, w_i; h) \quad (3.56)$$

where, in general, mappings

$$P_{ji} : \Gamma_h^{(i)} \times \mathbb{R}^n \rightarrow \mathbb{C} \quad (3.57)$$

are some algebraic expressions. Since the set of algebraic curves (3.41) must enjoy the system (3.56), we can retrieve this set solving the Picard–Fuchs type equations (3.56). The latter gives rise due to (3.39) and (3.28) to the integrability of all flows on $M_h^n \subset T^*(\mathbb{R}^n)$ by quadratures as was mentioned in Section 1.

Theorem 11. *Let there be given a completely integrable Hamiltonian system on the coadjoint manifold $T^*(\mathbb{R}^n)$ whose integral submanifold $M_h^n \subset T^*(\mathbb{R}^n)$ is described by Picard–Fuchs type algebraic equations (3.56). The corresponding imbedding mapping $\pi_h : M_h^n \rightarrow T^*(\mathbb{R}^n)$ (3.28) is a solution of a compatibility condition subject to the differential-algebraic relationships (3.55) on the canonical transformations generating function (3.37).*

To show that the scheme described above really leads to an algorithmic procedure of constructing the Picard–Fuchs type equations (3.56) and the corresponding integral submanifold imbedding mapping $\pi_h : M_h^n \rightarrow T^*(\mathbb{R}^n)$ in the form (3.28), we apply it below in Section 6 to some Hamiltonian systems including a so-called truncated Fock–Plank Hamiltonian system on the canonically symplectic cotangent space $T^*(\mathbb{R}^n)$. Making use of representations (3.21) and (3.28) and equation (3.31), we have shown above that the set of 1-forms (3.30) is reduced to the following purely differential-algebraic relations on $M_{h,\tau}^{2n}$:

$$\partial w_i(\mu_i; h) / \partial h_j = \mathbf{P}_{ji}(\mu_i, w_i; h), \quad (3.58)$$

which generalize similar relations from [18, 31], where the characteristic functions $\mathbf{P}_{ji} : T^*(M_h^n) \rightarrow \mathbb{R}$, $i, j = \overline{1, n}$, are defined as follows:

$$\mathbf{P}_{ji}(\mu_i, w_i; h) := \bar{\mathbf{P}}_{ji}(\mu, w; h)|_{M_h^n}. \quad (3.59)$$

It is clear that the above set of purely differential-algebraic relationships (3.33) and (3.34) makes it possible to write explicitly some first order compatible differential-algebraic equations, whose solution yields the first half of the desired imbedding (3.5) for the integral submanifold $M_h^n \subset M^{2n}$ in an open neighborhood $M_{h,\tau}^{2n} \subset M^{2n}$. As a result of the above computations one can formulate the following main theorem.

Theorem 12. *The imbedding (3.5) for the integral submanifold $M_h^n \subset M^{2n}$ (compact and connected), parametrized by a regular parameter $h \in \mathcal{G}^*$, is an algebraic solution (up to diffeomorphism) to the set of characteristic Picard–Fuchs type equations (3.35) on $T^*(M_h^n)$, and can be represented in the general case [19] in the following algebraic-geometric form:*

$$w_j^{n_j} + \sum_{s=1}^n c_{js}(\lambda; h) w_j^{n_j-s} = 0, \quad (3.60)$$

where $c_{js} : \mathbb{R} \times \mathcal{G}^* \rightarrow \mathbb{R}$, $s, j = \overline{1, n}$ are algebraic expressions, depending only on the functional structure of the original abelian Lie algebra $\mathcal{G}$ of invariants on M^{2n} . In particular, if the right-hand side of the characteristic equations (3.35) is independent of $h \in \mathcal{G}^*$, then this dependence will be linear in $h \in \mathcal{G}^*$.

It should be noted here that some ten years ago an attempt was made in [18, 19] to describe the explicit algebraic form of the Picard–Fuchs type equations (3.35) by means of straightforward calculations for the well-known completely integrable Kowalewskaya top Hamiltonian system. The idea suggested in [18, 19] was in some aspects very close to that devised independently and thoroughly analyzed in [11] which did not consider the explicit form of the algebraic curves (3.37) starting from an abelian Lie algebra $\mathcal{G}$ of invariants on a canonically symplectic phase space M^{2n} .

As is well known, a set of algebraic curves (3.37), prescribed via the above algorithm, to a given *a priori* abelian Lie algebra $\mathcal{G}$ of invariants on the canonically symplectic phase space $M^{2n} = T^*(\mathbb{R}^n)$ can be realized by means of a set of n_j -sheeted Riemannian surfaces $\Gamma_h^{n_j}$, $j = \overline{1, n}$, covering the corresponding real-valued cycles $\mathbb{S}_j^1$, $j = \overline{1, n}$, which generate the corresponding homology group $H_1(\mathbb{T}^n; \mathbb{Z})$ of the Arnold torus $\mathbb{T}^n \simeq \otimes_{j=1}^n \mathbb{S}_j^1$ diffeomorphic to the integral submanifold $M_h^n \subset M^{2n}$.

Thus, upon solving the set of algebraic equations (3.37) with respect to the functions $w_j : \mathbb{S}_j^1 \times \mathcal{G}^* \rightarrow \mathbb{R}$, $j = \overline{1, n}$, from (3.29) one obtains a vector parameter $\tau = (t_1, \dots, t_n) \in \mathbb{R}^n$ on M_h^n explicitly described by means of the following abelian

type equations:

$$t_j = \sum_{s=1}^n \int_{\mu_s^0}^{\mu_s} d\lambda \partial w_s(\lambda; h) / \partial h_j = \sum_{s=1}^n \int_{\mu_s^0}^{\mu_s} d\lambda \bar{\mathbf{P}}_{js}(\lambda, w_s; h), \quad (3.61)$$

where $j = \overline{1, n}$, $(\mu^0; h) \in (\otimes_{j=1}^n \Gamma_h^{n_j}) \times \mathcal{G}^*$. Using expression (3.28) and recalling that the generating function $S : M_h^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a one-valued mapping on an appropriate covering space $(M_h^n; H_1(M_h^n; \mathbb{Z}))$, one can construct via the method of Arnold [4] the so called action-angle coordinates on M_h^n . Denote the basic oriented cycles on M_h^n by $\sigma_j \subset M_h^n$, $j = \overline{1, n}$. These cycles together with their duals generate homology group $H_1(M_h^n; \mathbb{Z}) \simeq H_1(\mathbb{T}^n; \mathbb{Z}) = \oplus_{j=1}^n \mathbb{Z} j$. By virtue of the diffeomorphism $M_h^n \simeq \otimes_{j=1}^n \mathbb{S}_j^1$ described above, there is a one-to-one correspondence between the basic cycles of $H_1(M_h^n; \mathbb{Z})$ and those on the algebraic curves $\Gamma_h^{n_j}$, $j = \overline{1, n}$, given by (3.37):

$$\varrho : H_1(M_h^n; \mathbb{Z}) \rightarrow \oplus_{j=1}^n \mathbb{Z} j \sigma_{h,j}, \quad (3.62)$$

where $\sigma_{h,j} \subset \Gamma_h^{n_j}$, $j = \overline{1, n}$ are the corresponding real-valued cycles on the Riemann surfaces $\Gamma_h^{n_j}$, $j = \overline{1, n}$.

Assume that the following meanings of the mapping (3.37) are prescribed:

$$\varrho(\sigma_i) := \oplus_{j=1}^n n_{ij} \sigma_{h,j} \quad (3.63)$$

for each $i = \overline{1, n}$, where $n_{ij} \in \mathbb{Z}$, $i, j = \overline{1, n}$ - some fixed integers. Then following the Arnold construction [4, 18], one obtains the set of so-called action-variables on $M_h^n \subset M^{2n}$:

$$\gamma_j := \frac{1}{2\pi} \oint_{\sigma_j} dS = \sum_{s=1}^n n_{js} \oint_{\sigma_{h,s}} d\lambda w_s(\lambda; h), \quad (3.64)$$

where $j = \overline{1, n}$. It is easy to show [4, 16] that expressions (3.41) naturally define an a. e. differentiable invertible mapping

$$\xi : \mathcal{G}^* \rightarrow \mathbb{R}^n, \quad (3.65)$$

which enables one to treat the integral submanifold M_h^n as a submanifold $M_\gamma^n \subset M^{2n}$, where

$$M_\gamma^n := \{u \in M^{2n} : \xi(h) = \gamma \in \mathbb{R}^n\}. \quad (3.66)$$

But, as was demonstrated in [18, 32], functions (3.43) do not in general generate a global foliation of the phase space M^{2n} , as they are connected with both topological and analytical constraints. Since functions (3.41) are evidently also commuting invariants on M^{2n} , one can define a further canonical transformation of the phase space M^{2n} , generated by the following relationship on $M_{h,\tau}^{2n}$:

$$\sum_{j=1}^n w_j d\mu_j + \sum_{j=1}^n \varphi_j d\gamma_j = dS(\mu; \gamma), \quad (3.67)$$

where $\varphi = (\varphi_1, \dots, \varphi_n) \in \mathbb{T}^n$ are the so-called angle-variables on the torus $\mathbb{T}^n \simeq M_h^n$ and $S : M_\gamma^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is the corresponding generating function. Whence it follows easily from (3.28) and (3.38) that

$$\begin{aligned} \varphi_j &:= \partial S(\mu; \gamma) / \partial \gamma_j = \sum_{s=1}^n \partial S(\mu; \gamma(h)) / \partial h_s \partial h_s / \partial \gamma_j = \sum_{s=1}^n t_s \omega_{sj}(\gamma), \\ \frac{1}{2\pi} \oint_{\sigma_j} d\varphi_k &= \delta_{jk}, \end{aligned} \quad (3.68)$$

where $\Omega := \{\omega_{sj} : \mathbb{R}^n \rightarrow \mathbb{R}, s, j = \overline{1, n}\}$ is the so-called [4] frequency matrix, which is a. e. invertible on the integral submanifold $M_\gamma^n \subset M^{2n}$. As an evident result of (3.45), we claim that the evolution of any vector field $K_a \in \Gamma(M^{2n})$ for $a \in \mathcal{G}$ on the integral submanifold $M_\gamma^n \subset M^{2n}$ is quasiperiodic with a set of frequencies generated by the matrix $\Omega \stackrel{\text{a. e.}}{\in} \text{Aut}(\mathbb{R}^n)$ defined above. As examples showing the effectiveness of the above method of construction of integral submanifold imbeddings for abelian integrable Hamiltonian systems, one can verify the Liouville–Arnold integrability of all Henon–Heiles and Neumann type systems described in detail in [21, 22]; however, we shall not dwell on this here.

4. INTEGRAL SUBMANIFOLD IMBEDDING PROBLEM FOR A NONABELIAN LIE ALGEBRA OF INVARIANTS

We shall assume below that there is given a Hamiltonian vector field $K \in \Gamma(M^{2n})$ on the canonically symplectic phase space $M^{2n} = T^*(\mathbb{R}^n)$, $n \in \mathbb{Z}_+$, which is endowed with a nonabelian Lie algebra $\mathcal{G}$ of invariants, satisfying all the conditions of the Mishchenko–Fomenko Theorem 6, that is

$$\dim \mathcal{G} + \text{rank } \mathcal{G} = \dim M^{2n}. \quad (4.1)$$

Then, as was proved above, an integral submanifold $M_h^r \subset M^{2n}$ at a regular element $h \in \mathcal{G}^*$ is $\text{rank } \mathcal{G} = r$ -dimensional and diffeomorphic (when compact and connected) to the standard r -dimensional torus $\mathbb{T}^r \simeq \otimes_{j=1}^r \mathbb{S}_j^1$. It is natural to ask the following question: How does one construct the corresponding integral submanifold imbedding

$$\pi_h : M_h^r \rightarrow M^{2n}, \quad (4.2)$$

which characterizes all possible orbits of the dynamical system $K \in \Gamma(M^{2n})$?

Having gained some experience in constructing the imbedding (4.2) in the case of the abelian Liouville–Arnold theorem on integrability by quadratures, we proceed below to study the integral submanifold $M_h^r \subset M^{2n}$ by means of Cartan’s theory [3, 12, 16, 22] of the integrable ideals in the Grassmann algebra $\Lambda(M^{2n})$. Let $\mathcal{I}(\mathcal{G}^*)$ be an ideal in $\Lambda(M^{2n})$, generated by independent differentials $dH_j \in \Lambda^1(M^{2n})$, $j = \overline{1, k}$, in an open neighborhood $U(M_h^r)$, where by definition, $r = \dim \mathcal{G}$. The ideal $\mathcal{I}(\mathcal{G}^*)$ is obviously Cartan integrable [16, 23] with the integral submanifold $M_h^r \subset M^{2n}$ (at a

regular element $h \in \mathcal{G}^*$, on which it vanishes, that is $\pi_h^* \mathcal{I}(\mathcal{G}^*) = 0$. The dimension $\dim M_h^r = \dim M^{2n} - \dim \mathcal{G} = r = \text{rank } \mathcal{G}$ due to condition (4.1) imposed on the Lie algebra $\mathcal{G}$. It is useful to note here that, owing to the inequality $r \leq k$ for the rank $\mathcal{G}$, one readily obtains from (3.1) that $\dim \mathcal{G} = k \geq n$. Since each base element $H_j \in \mathcal{G}$, $j = \overline{1, k}$, generates a symplectically dual vector field $K_j \in \Gamma(M^{2n})$, $j = \overline{1, k}$, one can try to study the corresponding differential system $K(\mathcal{G})$ which is also Cartan integrable in the entire open neighborhood $U(M_h^r) \subset M^{2n}$. Denote the corresponding dimension of the integral submanifold by $\dim M_h^r = \dim K(\mathcal{G}) = k$. Consider now an abelian differential system $K(\mathcal{G}_h) \subset K(\mathcal{G})$, generated by the Cartan subalgebra $\mathcal{G}_h \subset \mathcal{G}$ and its integral submanifold $M_h^r \subset U(M_h^r)$. Since the Lie subgroup $G_h = \exp \mathcal{G}_h$ acts on the integral submanifold M_h^r invariantly (see Section 2) and $\dim M_h^r = \text{rank } \mathcal{G} = r$, it follows that $\bar{M}_h^r = M_h^r$. On the other hand, the system $K(\mathcal{G}_h) \subset K(\mathcal{G})$ by definition means that the integral submanifold M_h^r is an invariant part of the integral submanifold $M_h^k \subset U(M_h^r)$ with respect to the Lie group $G = \exp \mathcal{G}$ -action on M_h^k . In this case one has the following result.

Lemma 5. *There exist just $(n - r) \in \mathbb{Z}_+$ vector fields $\tilde{F}_j \in K(\mathcal{G})/K(\mathcal{G}_h)$, $j = \overline{1, n - r}$, for which*

$$\omega^{(2)}(\tilde{F}_i, \tilde{F}_j) = 0 \quad (4.3)$$

on $U(M_h^r)$ for all $i, j = \overline{1, n - r}$.

PROOF. It is obvious that the matrix $\omega(\tilde{K}) := \{\omega^{(2)}(\tilde{K}_i, \tilde{K}_j) : i, j = \overline{1, k}\}$ has on $U(M_h^r)$ the rank $\omega(\tilde{K}) = k - r$, since $\dim_{\mathbb{R}} \ker(\pi_h^* \omega^{(2)}) = \dim_{\mathbb{R}}(\pi_{h*} K(\mathcal{G}_h)) = r$ on M_h^r at the regular element $h \in \mathcal{G}^*$. Let us now complexify the tangent space $T(U(M_h^r))$ using its even dimensionality. Whence one can easily deduce that on $U(M_h^r)$ there exist just $(n - r) \in \mathbb{Z}_+$ vectors (not vector fields!) $\tilde{K}_j^{\mathbb{C}} \in K^{\mathbb{C}}(\mathcal{G})/K^{\mathbb{C}}(\mathcal{G}_h)$, $j = \overline{1, n - r}$, from the complexified [24] factor space $K^{\mathbb{C}}(\mathcal{G})/K^{\mathbb{C}}(\mathcal{G}_h)$. To show this, let us reduce the skew-symmetric matrix $\omega(\tilde{K}) \in \text{Hom}(\mathbb{R}^{k-r})$ to its self-adjoint equivalent $\omega(\tilde{K}^{\mathbb{C}}) \in \text{Hom}(\mathbb{C}^{n-r})$, having taken into account that $\dim_{\mathbb{R}} \mathbb{R}^{k-r} = \dim_{\mathbb{R}} \mathbb{R}^{k+r-2r} = \dim_{\mathbb{R}} \mathbb{R}^{2(n-r)} = \dim_{\mathbb{C}} \mathbb{C}^{n-r}$. Let now $f_j^{\mathbb{C}} \in \mathbb{C}^{n-r}$, $j = \overline{1, n - r}$, be eigenvectors of the nondegenerate self-adjoint matrix $\omega(\tilde{K}^{\mathbb{C}}) \in \text{Hom}(\mathbb{C}^{n-r})$, that is

$$\omega(\tilde{K}^{\mathbb{C}}) f_j^{\mathbb{C}} = \tilde{\lambda}_j f_j^{\mathbb{C}}, \quad (4.4)$$

where $\tilde{\lambda}_j \in \mathbb{R}$, $j = \overline{1, n - r}$, and for all $i, j = \overline{1, n - r}$, $\langle f_i^{\mathbb{C}}, f_j^{\mathbb{C}} \rangle = \delta_{i,j}$. The above obviously means that in the basis $\{f_j^{\mathbb{C}} \in K^{\mathbb{C}}(\mathcal{G})/K^{\mathbb{C}}(\mathcal{G}_h) : j = \overline{1, n - r}\}$ the matrix $\omega(\tilde{K}^{\mathbb{C}}) \in \text{Hom}(\mathbb{C}^{n-r})$ is strictly diagonal and representable as

$$\omega(\tilde{K}^{\mathbb{C}}) = \sum_{j=1}^{n-r} \tilde{\lambda}_j f_j^{\mathbb{C}} \otimes_{\mathbb{C}} f_j^{\mathbb{C}}, \quad (4.5)$$

where $\otimes_{\mathbb{C}}$ is the usual Kronecker tensor product of vectors from $\mathbb{C}^{n-r}$. Owing to the construction of the complexified matrix $\omega(\tilde{K}^{\mathbb{C}}) \in \text{Hom}(\mathbb{C}^{n-r})$, one sees that the space $K^{\mathbb{C}}(\mathcal{G})/K^{\mathbb{C}}(\mathcal{G}_h) \simeq \mathbb{C}^{n-r}$ carries a Kähler structure [24] with respect to which the following expressions

$$\omega(\tilde{K}) = \text{Im } \omega(\tilde{K}), \quad \langle \cdot, \cdot \rangle_{\mathbb{R}} = \text{Re } \langle \cdot, \cdot \rangle \quad (4.6)$$

hold. Making use now of the representation (4.5) and expressions (4.6), one can find vector fields $\tilde{F}_j \in K(\mathcal{G})/K(\mathcal{G}_h)$, $j = \overline{1, n-r}$, such that

$$\omega(\tilde{F}) = \text{Im } \omega(\tilde{F}^{\mathbb{C}}) = J, \quad (4.7)$$

holds on $U(M_h^r)$, where $J \in \text{Sp}(\mathbb{C}^{n-r})$ is the standard symplectic matrix, satisfying the complex structure [24] identity $J^2 = -I$. By virtue of the normalization conditions $\langle f_j^{\mathbb{C}}, f_j^{\mathbb{C}} \rangle = \delta_{i,j}$, for all $i, j = \overline{1, n-r}$, one easily infers from (3.7) that $\omega^{(2)}(\tilde{F}_i, \tilde{F}_j) = 0$ for all $i, j = \overline{1, n-r}$, where by definition

$$\tilde{F}_j := \text{Re } \tilde{F}_j^{\mathbb{C}} \quad (4.8)$$

for all $j = \overline{1, n-r}$, and this proves the lemma. $\square$

Assume now that the Lie algebra $\mathcal{G}$ of invariants on M^{2n} has been split into a direct sum of subspaces as

$$\mathcal{G} = \mathcal{G}_h \oplus \tilde{\mathcal{G}}_h, \quad (4.9)$$

where $\mathcal{G}_h$ is the Cartan subalgebra at a regular element $h \in \mathcal{G}^*$ (being abelian) and $\tilde{\mathcal{G}}_h \simeq \mathcal{G}/\mathcal{G}_h$ is the corresponding complement to $\mathcal{G}_h$. Denote a basis of $\mathcal{G}_h$ as $\{\bar{H}_i \in \mathcal{G}_h : i = \overline{1, r}\}$, where $\dim \mathcal{G}_h = \text{rank } \mathcal{G} = k \in \mathbb{Z}_+$, and correspondingly, a basis of $\tilde{\mathcal{G}}_h$ as $\{\tilde{H}_j \in \tilde{\mathcal{G}}_h \simeq \mathcal{G}/\mathcal{G}_h : j = \overline{1, k-r}\}$. Then, owing to the results of Section 2, the following relationships hold:

$$\{\bar{H}_i, \bar{H}_j\} = 0, \quad h(\{\bar{H}_i, \tilde{H}_s\}) = 0 \quad (4.10)$$

in the open neighborhood $U(M_h^r) \subset M^{2n}$ for all $i, j = \overline{1, r}$ and $s = \overline{1, k-r}$. We have as yet had nothing to say of expressions $h(\{\tilde{H}_s, \tilde{H}_m\})$ for $s, m = \overline{1, k-r}$. Making use of the representation (4.8) for our vector fields (if they exist) $\tilde{F}_j \in K(\mathcal{G})/K(\mathcal{G}_h)$, $j = \overline{1, n-r}$, one can write the following expansion:

$$\tilde{F}_i = \sum_{j=1}^{k-r} c_{ji}(h) \tilde{K}_j, \quad (4.11)$$

where $i_{\tilde{K}_j} \omega^{(2)} := -d\tilde{H}_j$, $c_{ji} : \mathcal{G}^* \rightarrow \mathbb{R}$, $i = \overline{1, n-r}$, $j = \overline{1, k-r}$, are real-valued functions on $\mathcal{G}^*$, being defined uniquely as a result of (4.11). Whence it clearly follows that there exist invariants $\tilde{f}_s : U(M_h^r) \rightarrow \mathbb{R}$, $s = \overline{1, n-r}$, such that

$$-i_{\tilde{F}_s} \omega^{(2)} = \sum_{j=1}^{k-r} c_{js}(h) d\tilde{H}_j := d\tilde{f}_s, \quad (4.12)$$

where $\tilde{f}_s = \sum_{j=1}^{k-r} c_{js}(h) \tilde{H}_j$, $s = \overline{1, n-r}$, holds on $U(M_h^r)$.

To proceed further, let us look at the following identity which is similar to (3.2):

$$\left(\otimes_{j=1}^r i_{\bar{K}_j}\right) \left(\otimes_{s=1}^{n-r} i_{\tilde{F}_s}\right) (\omega^{(2)})^{n+1} = 0 = \pm(n+1)! \left(\wedge_{j=1}^r d\bar{H}_j\right) \left(\wedge_{s=1}^{n-r} d\tilde{f}_s\right) \wedge \omega^{(2)}, \quad (4.13)$$

on $U(M_h^r)$. Whence, the following result is easily obtained using Cartan theory [3, 16]:

Lemma 6. *The symplectic structure $\omega^{(2)} \in \Lambda^2(U(M_h^r))$ has the following canonical representation:*

$$\omega^{(2)}|_{U(M_h^r)} = \sum_{j=1}^r d\bar{H}_j \wedge \bar{h}_j^{(1)} + \sum_{s=1}^{n-r} d\tilde{f}_s \wedge \tilde{h}_s^{(1)}, \quad (4.14)$$

where $\bar{h}_j^{(1)}, \tilde{h}_s^{(1)} \in \Lambda^1(U(M_h^r))$, $j = \overline{1, r}$, $s = \overline{1, n-r}$.

Expression (4.14) obviously means, that on $U(M_h^r) \subset M^{2n}$ the differential 1-forms $\bar{h}_j^{(1)}, \tilde{h}_s^{(1)} \in \Lambda^1(U(M_h^r))$, $j = \overline{1, r}$, $s = \overline{1, n-r}$, are independent together with exact 1-forms $d\bar{H}_j$, $j = \overline{1, r}$, and $d\tilde{f}_s$, $s = \overline{1, n-r}$. Since $d\omega^{(2)} = 0$ on M^{2n} identically, from (4.14) one obtains that the differentials $d\bar{h}_j^{(1)}, d\tilde{h}_s^{(1)} \in \Lambda^2(U(M_h^r))$, $j = \overline{1, r}$, $s = \overline{1, n-r}$, belong to the ideal $\mathcal{I}(\tilde{\mathcal{G}}_h) \subset \mathcal{I}(\mathcal{G}^*)$, generated by exact forms $d\tilde{f}_s$, $s = \overline{1, n-r}$, and $d\bar{H}_j$, $j = \overline{1, r}$, for all regular $h \in \mathcal{G}^*$. Consequently, one obtains the following analog of the Galisau-Reeb Theorem 8.

Theorem 13. *Let a Lie algebra $\mathcal{G}$ of invariants on the symplectic space M^{2n} be nonabelian and satisfy the Mishchenko–Fomenko condition (4.1). At a regular element $h \in \mathcal{G}^*$ in some open neighborhood $U(M_h^r)$ of the integral submanifold $M_h^r \subset M^{2n}$ there exist differential 1-forms $\bar{h}_j^{(1)}$, $j = \overline{1, n}$, and $\tilde{h}_s^{(1)}$, $s = \overline{1, n-r}$, satisfying the following properties:*

- (i) $\omega^{(2)}|_{U(M_h^r)} = \sum_{j=1}^r d\bar{H}_j \wedge \bar{h}_j^{(1)} + \sum_{s=1}^{n-r} d\tilde{f}_s \wedge \tilde{h}_s^{(1)}$ where $\bar{H}_j \in \mathcal{G}$, $j = \overline{1, r}$, is a basis of the Cartan subalgebra $\mathcal{G}_h \subset \mathcal{G}$ (being abelian), and $\tilde{f}_s \in \mathcal{G}$, $s = \overline{1, n-r}$, are invariants from the complementary space $\tilde{\mathcal{G}}_h \simeq \mathcal{G}/\mathcal{G}_h$;
- (ii) The 1-forms $\bar{h}_j^{(1)} \in \Lambda^1(U(M_h^r))$, $j = \overline{1, r}$, and $\tilde{h}_s^{(1)} \in \Lambda^1(U(M_h^r))$, $s = \overline{1, n-r}$, are exact on M_h^r and satisfy the equations: $\bar{h}_j^{(1)}(\bar{K}_i) = \delta_{i,j}$ for all $i, j = \overline{1, r}$, $\bar{h}_j^{(1)}(\tilde{F}_s) = 0$ and $\tilde{h}_s^{(1)}(\bar{K}_j) = 0$ for all $j = \overline{1, r}$, $s = \overline{1, n-r}$, and $\tilde{h}_s^{(1)}(\tilde{F}_m) = \delta_{s,m}$ for all $s, m = \overline{1, n-r}$.

PROOF. Obviously we need to prove only the last statement (ii). Making use of Theorem 13, one can find on the integral submanifold $M_h^r \subset M^{2n}$ the differential 2-forms $d\bar{h}_j^{(1)} \in \Lambda^2(U(M_h^r))$, $j = \overline{1, r}$, and $d\tilde{h}_s^{(1)} \in \Lambda^2(U(M_h^r))$, $s = \overline{1, n-r}$, are identically vanishing. This means in particular, owing to the classical Poincaré lemma [1, 4, 16], that there exist some exact 1-forms $d\bar{t}_{h,j} \in \Lambda^1(U(M_h^r))$, $j = \overline{1, r}$, and $d\tilde{t}_{h,s} \in \Lambda^1(U(M_h^r))$, $s = \overline{1, n-r}$, where $\bar{t}_{h,j} : M_h^r \rightarrow \mathbb{R}$, $j = \overline{1, r}$, and $\tilde{t}_{h,s} : M_h^r \rightarrow \mathbb{R}$,

$s = \overline{1, n-r}$, are smooth independent a. e. functions on M_h^r ; they are one-valued on an appropriate covering of the manifold $M_h^r \subset M^{2n}$ and supply global coordinates on the integral submanifold M_h^r . Using representation (4.14), one can easily obtain that

$$-i_{\bar{K}_i} \omega^{(2)}|_{U(M_h^r)} = \sum_{j=1}^r d\bar{H}_j \bar{h}_j^{(1)}(\bar{K}_i) + \sum_{s=1}^{n-r} d\tilde{f}_s \tilde{h}_s^{(1)}(\bar{K}_i) = d\bar{H}_i \quad (4.15)$$

for all $i = \overline{1, r}$ and

$$-i_{\tilde{F}_m} \omega^{(2)}|_{U(M_h^r)} = \sum_{j=1}^r d\bar{H}_j \bar{h}_j^{(1)}(\tilde{F}_m) + \sum_{s=1}^{n-r} d\tilde{f}_s \tilde{h}_s^{(1)}(\tilde{F}_m) = d\tilde{f}_m \quad (4.16)$$

for all $m = \overline{1, n-r}$. Whence, from (4.15) it follows on that on $U(M_h^r)$,

$$\bar{h}_j^{(1)}(\bar{K}_i) = \delta_{i,j}, \quad \tilde{h}_s^{(1)}(\bar{K}_i) = 0 \quad (4.17)$$

for all $i, j = \overline{1, r}$ and $s = \overline{1, n-r}$, and similarly, from (4.16) it follows that on $U(M_h^r)$,

$$\bar{h}_j^{(1)}(\tilde{F}_m) = 0, \quad \tilde{h}_s^{(1)}(\tilde{F}_m) = 0 \quad (4.18)$$

for all $j = \overline{1, r}$ and $s, m = \overline{1, n-r}$. Thus the theorem is proved. $\square$

Having now defined global evolution parameters $t_j : M^{2n} \rightarrow \mathbb{R}$, $j = \overline{1, r}$, of the corresponding vector fields $\bar{K}_j = d/dt_j$, $j = \overline{1, r}$, and local evolution parameters $\tilde{t}_s : M^{2n} \cap U(M_h^r) \rightarrow \mathbb{R}$, $s = \overline{1, n-r}$, of the corresponding vector fields $\tilde{F}_s|_{U(M_h^r)} := d/d\tilde{t}_s$, $s = \overline{1, n-r}$, one can easily see from (4.18) that the equalities

$$t_j|_{U(M_h^r)} = \bar{t}_j, \quad \tilde{t}_s|_{U(M_h^r)} = \tilde{t}_{h,s} \quad (4.19)$$

hold for all $j = \overline{1, r}$, $s = \overline{1, n-r}$, up to constant normalizations. Thereby, one can develop a new method, similar to that of Section 3, for studying the integral submanifold imbedding problem in the case of the nonabelian Liouville–Arnold integrability theorem.

Before starting, it is interesting to note that the system of invariants

$$\mathcal{G}_\tau := \mathcal{G}_h \oplus \underset{\mathbb{R}}{\text{span}} \left\{ \tilde{f}_s \in \mathcal{G}/\mathcal{G}_h : s = \overline{1, n-r} \right\}$$

constructed above, comprises a new involutive (abelian) complete algebra $\mathcal{G}_\tau$, to which one can apply the abelian Liouville–Arnold theorem on integrability by quadratures and the integral submanifold imbedding theory devised in Section 3, in order to obtain exact solutions by means of algebraic-analytical expressions. Namely, the following corollary holds.

Corollary 3. *Assume that a nonabelian Lie algebra $\mathcal{G}$ satisfies the Mishchenko–Fomenko condition (4.1) and $M_h^r \subset M^{2n}$ is its integral submanifold (compact and connected) at a regular element $h \in \mathcal{G}^*$, is diffeomorphic to the standard torus $\mathbb{T}^r \simeq M_{h,\tau}^n$.*

Assume also that the dual complete abelian algebra $\mathcal{G}_\tau$ ($\dim \mathcal{G}_\tau = n = 1/2 \dim M^{2n}$) of independent invariants constructed above is globally defined. Then its integral submanifold $M_{h,\tau}^n \subset M^{2n}$ is diffeomorphic to the standard torus $\mathbb{T}^n \simeq M_{h,\tau}^n$, and contains the torus $\mathbb{T}^r \simeq M_h^r$ as a direct product with some completely degenerate torus $\mathbb{T}^{n-r}$, that is $M_{h,\tau}^n \simeq M_h^r \times \mathbb{T}^{n-r}$.

Thus, having successfully applied the algorithm of Section 3 to the algebraic-analytical characterization of integral submanifolds of a nonabelian Liouville–Arnold integrable Lie algebra $\mathcal{G}$ of invariants on the canonically symplectic manifold $M^{2n} \simeq T^*(\mathbb{R}^n)$, one can produce a wide class of exact solutions represented by quadratures – which is just what we set out to find. At this point it is necessary to note that up to now the (dual to $\mathcal{G}$) abelian complete algebra $\mathcal{G}_\tau$ of invariants at a regular $h \in \mathcal{G}^*$ was constructed only in some open neighborhood $U(M_h^r)$ of the integral submanifold $M_h^r \subset M^{2n}$. As mentioned before, the global existence of the algebra $\mathcal{G}_\tau$ strongly depends on the possibility of extending these invariants to the entire manifold M^{2n} . This possibility is in 1-1 correspondence with the existence of a global complex structure [24] on the reduced integral submanifold $\tilde{M}_{h,\tau}^{2(n-r)} := M_h^k/G_h$, induced by the reduced symplectic structure $\pi_\tau^* \omega^{(2)} \in \Lambda^2(M_h^k/G_h)$, where $\pi_\tau : M_h^k \rightarrow M^{2n}$ is the imbedding for the integrable differential system $K(\mathcal{G}) \subset \Gamma(M^{2n})$, introduced above. If this is the case, the resulting complexified manifold ${}^{\mathbb{C}}\tilde{M}_{h,\tau}^{n-r} \simeq \tilde{M}_{h,\tau}^{2(n-r)}$ will be endowed with a Kählerian structure, which makes it possible to produce the dual abelian algebra $\mathcal{G}_\tau$ as a globally defined set of invariants on M^{2n} . This problem will be analyzed in more detail in Section 6.

5. EXAMPLES

Below we consider some examples of nonabelian Liouville–Arnold integrability by quadratures covered by Theorem 6.

Example 2 (Point vortices in the plane). Consider $n \in \mathbb{Z}_+$ point vortices on the plane $\mathbb{R}^2$, described by the Hamiltonian function

$$H = -\frac{1}{2\pi} \sum_{i \neq j=1}^n \xi_i \xi_j \ln \|q_i - p_j\| \quad (5.1)$$

with respect to the following partially canonical symplectic structure on $M^{2n} \simeq T^*(\mathbb{R}^n)$:

$$\omega^{(2)} = \sum_{j=1}^n \xi_j dp_j \wedge dq_j, \quad (5.2)$$

where $(p_j, q_j) \in \mathbb{R}^2$, $j = \overline{1, n}$, are coordinates of the vortices in $\mathbb{R}^2$. There exist three additional invariants

$$\begin{aligned} P_1 &= \sum_{j=1}^n \xi_j q_j, & P_2 &= \sum_{j=1}^n \xi_j p_j, \\ P &= \frac{1}{2} \sum_{j=1}^n \xi_j (q_j^2 + p_j^2), \end{aligned} \quad (5.3)$$

satisfying the following Poisson bracket conditions:

$$\begin{aligned} \{P_1, P_2\} &= -\sum_{j=1}^n \xi_j, & \{P_1, P\} &= -P_2, & \{P_2, P\} &= P_1, \\ \{P, H\} &= 0 = \{P_j, H\}. \end{aligned} \quad (5.4)$$

It is evident, that invariants (5.1) and (5.3) comprise on $\sum_{j=1}^n \xi_j = 0$ a four-dimensional Lie algebra $\mathcal{G}$, whose rank $\mathcal{G} = 2$. Indeed, assume that a regular vector $h \in \mathcal{G}^*$ is chosen, and parametrized by real values $h_j \in \mathbb{R}$, $j = \overline{1, 4}$, where

$$h(P_i) = h_i, \quad h(P) = h_3, \quad h(H) = h_4, \quad (5.5)$$

and $i = \overline{1, 2}$. Then, one can easily verify that the element

$$Q_h = \left(\sum_{j=1}^n \xi_j \right) P - \sum_{i=1}^n h_i P_i \quad (5.6)$$

belongs to the Cartan Lie subalgebra $\mathcal{G}_h \subset \mathcal{G}$, that is

$$h(\{Q_h, P_i\}) = 0, \quad h(\{Q_h, P\}) = 0. \quad (5.7)$$

Since $\{Q_h, H\} = 0$ for all values $h \in \mathcal{G}^*$, we claim that $\mathcal{G}_h = \text{span}_{\mathbb{R}} \{H, Q_h\}$ — the Cartan subalgebra of $\mathcal{G}$. Thus, $\text{rank } \mathcal{G} = \dim \mathcal{G}_h = 2$, and it comes right away that the condition (4.1)

$$\dim M^{2n} = 2n = \text{rank } \mathcal{G} + \dim \mathcal{G} = 6 \quad (5.8)$$

holds only if $n = 3$. Thereby, the following theorem is proved.

Theorem 14. *The three-vortex problem (5.1) on the plane $\mathbb{R}^2$ is nonabelian Liouville–Arnold integrable by quadratures on the phase space $M^6 \simeq T^*(\mathbb{R}^3)$ with the symplectic structure (5.2).*

As a result, the corresponding integral submanifold $M_h^2 \subset M^6$ is two-dimensional and diffeomorphic (when compact and connected) to the torus $\mathbb{T}^2 \simeq M_h^2$, on which the motions are quasiperiodic functions of the evolution parameter.

Concerning Corollary 3, the dynamical system (5.1) is also abelian Liouville–Arnold integrable with an extended integral submanifold $M_{h,\tau}^3 \subset M^6$, which can be found via the scheme suggested in Section 4. Using simple calculations, one obtains an additional invariant $Q = (\sum_{j=1}^3 \xi_j)P - \sum_{i=1}^3 P_i^2 \notin \mathcal{G}$, which commutes with

H and P of $\mathcal{G}_h$. Therefore, there exists a new complete dual abelian algebra $\mathcal{G}_\tau = \text{span}_{\mathbb{R}} \{Q, P, H\}$ of independent invariants on M^6 with $\dim \mathcal{G}_\tau = 3 = 1/2 \dim M^6$, whose integral submanifold $M_{h,\tau}^3 \subset M^6$ (when compact and connected) is diffeomorphic to the torus $\mathbb{T}^3 \simeq M_h^2 \times \mathbb{S}^1$.

Note also here that the above additional invariant $Q \in \mathcal{G}_\tau$ can be naturally extended to the case of an arbitrary number $n \in \mathbb{Z}_+$ of vortices as follows: $Q = (\sum_{j=1}^n \xi_j)P - \sum_{i=1}^n P_i^2 \in \mathcal{G}_\tau$, which obviously also commutes with invariants (5.1) and (5.3) on the entire phase space M^{2n} .

Example 3 (A material point motion in a central field). Consider the motion of a material point in the space $\mathbb{R}^3$ under a central potential field whose Hamiltonian

$$H = \frac{1}{2} \sum_{j=1}^3 p_j^2 + Q(\|q\|), \quad (5.9)$$

contains a central field $Q : \mathbb{R}_+ \rightarrow \mathbb{R}$. The motion takes place in the canonical phase space $M^6 = T^*(\mathbb{R}^3)$, and possesses three additional invariants:

$$P_1 = p_2 q_3 - p q, \quad P_2 = p_3 q_1 - p_1 q_3, \quad P_3 = p_1 q_2 - p_2 q_1, \quad (5.10)$$

satisfying the following Poisson bracket relations:

$$\{P_1, P_2\} = P_3, \quad \{P_3, P_1\} = P_2, \quad \{P_2, P_3\} = P_1. \quad (5.11)$$

Since $\{H, P_j\} = 0$ for all $j = \overline{1, 3}$, one sees that the problem under consideration has a four-dimensional Lie algebra $\mathcal{G}$ of invariants, isomorphic to the classical rotation Lie algebra $\mathfrak{so}(3) \times \mathbb{R} \simeq \mathcal{G}$. Let us show that at a regular element $h \in \mathcal{G}^*$ the Cartan subalgebra $\mathcal{G}_h \subset \mathcal{G}$ has the dimension $\dim \mathcal{G}_h = 2 = \text{rank } \mathcal{G}$. Indeed, one easily verifies that the invariant

$$P_h = \sum_{j=1}^3 h_j P_j \quad (5.12)$$

belongs to the Cartan subalgebra $\mathcal{G}$, that is

$$\{H, P_h\} = 0, \quad h(\{P_h, P_j\}) = 0 \quad (5.13)$$

for all $j = \overline{1, 3}$. Thus, as the Cartan subalgebra $\mathcal{G}_h = \text{span}_{\mathbb{R}} \{H \text{ and } P_h \subset \mathcal{G}\}$, one gets $\dim \mathcal{G}_h = 2 = \text{rank } \mathcal{G}_h$, and the Mishchenko–Fomenko condition (4.1)

$$\dim M^6 = 6 = \text{rank } \mathcal{G} + \dim \mathcal{G} = 4 + 2 \quad (5.14)$$

holds. Hence one can prove its integrability by quadratures via the nonabelian Liouville–Liouville–Arnold Theorem 6 and obtain the following theorem:

Theorem 15. *It follows from Theorem 6 that the free material point motion in $\mathbb{R}^3$ is a completely integrable by quadratures dynamical system on the canonical symplectic phase space $M^6 = T^*(\mathbb{R}^3)$. The corresponding integral submanifold $M_h^2 \subset$*

M^6 at a regular element $h \in \mathcal{G}^*$ (if compact and connected) is two-dimensional and diffeomorphic to the standard torus $\mathbb{T}^2 \simeq M_h^2$.

Making use of the integration algorithm devised in Section 2 and 3, one can readily obtain the corresponding integral submanifold imbedding mapping $\pi_h : M_h^2 \rightarrow M^6$ by means of algebraic-analytical expressions and transformations.

There are clearly many other interesting nonabelian Liouville–Arnold integrable Hamiltonian systems on canonically symplectic phase spaces that arise in applications, which can similarly be integrated using algebraic-analytical means. We hope to study several of these systems in detail elsewhere.

6. EXISTENCE PROBLEM FOR A GLOBAL SET OF INVARIANTS

It was proved in Section 4, that locally, in some open neighborhood $U(M_h^r) \subset M^{2n}$ of the integral submanifold $M_h^r \subset M^{2n}$ one can find by algebraic-analytical means just $n - r \in \mathbb{Z}_+$ independent vector fields $\tilde{F}_j \in K(\mathcal{G})/K(\mathcal{G}_h) \cap \Gamma(U(M_h^r))$, $j = \overline{1, n-r}$, satisfying condition (4.3). Since each vector field $\tilde{F}_j \in K(\mathcal{G})/K(\mathcal{G}_h)$, $j = \overline{1, n-r}$, is generated by an invariant $\tilde{H}_j \in \mathcal{D}(U(M_h^r))$, $j = \overline{1, n-r}$, it follows readily from (4.3) that

$$\{\tilde{H}_i, \tilde{H}_j\} = 0 \quad (6.1)$$

for all $i, j = \overline{1, n-r}$. Thus, in an open neighborhood $U(M_h^r)$ there exist just $n - r$ invariants in addition to $\tilde{H}_j \in \mathcal{D}(U(M_h^r))$, $j = \overline{1, n-r}$, all of which are in involution. Denote as before this new set of invariants as $\mathcal{G}_\tau$, keeping in mind that $\dim \mathcal{G}_\tau = r + (n - r) = n \in \mathbb{Z}_+$. Whence, in an open neighborhood $U(M_h^r) \subset M^{2n}$ we have constructed the set $\mathcal{G}_\tau$ of just $n = 1/2 \dim M^{2n}$ invariants commuting with each other, thereby guaranteeing via the abelian Liouville–Arnold theorem its local complete integrability by quadratures. Consequently, there exists locally a mapping $\pi_\tau : M_{h,\tau}^k \rightarrow M^{2n}$, where $M_{h,\tau}^k := U(M_h^r) \cap M_\tau^k$ is the integral submanifold of the differential system $K(\mathcal{G})$, and one can therefore describe the behavior of integrable vector fields on the reduced manifold $\bar{M}_{h,\tau}^{2(n-r)} := M_{h,\tau}^{k-r}/G_h$. For global integrability properties of a given set $\mathcal{G}$ of invariants on $(M^{2n}, \omega^{(2)})$, satisfying the Mishchenko–Fomenko condition (4.1), it is necessary to have the additional set of invariants $\tilde{H}_j \in \mathcal{D}(U(M_h^r))$, $j = \overline{1, n-r}$, extended from $U(M_h^r)$ to the entire phase space M^{2n} . This problem evidently depends on the existence of extensions of vector fields $\tilde{F}_j \in \Gamma(U(M_h^r))$, $j = \overline{1, n-r}$, from the neighborhood $U(M_h^r) \subset M^{2n}$ to the whole phase space M^{2n} . On the other hand, as stated before, the existence of such a continuation depends intimately on the properties of the complexified differential system $K^\mathbb{C}(\mathcal{G})/K^\mathbb{C}(\mathcal{G}_h)$, which has a nondegenerate complex metric $\omega(\tilde{K}^\mathbb{C}) : T(\bar{M}_{h,\tau}^{2(n-r)})^\mathbb{C} \times T(\bar{M}_{h,\tau}^{2(n-r)})^\mathbb{C} \rightarrow \mathbb{C}$, induced by the symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$. This point can be clarified further by using the notion [24–27] of a Kähler manifold and some of the associated constructions presented above. Namely,

consider the local isomorphism $T(\bar{M}_{h,\tau}^{2(n-r)})^{\mathbb{C}} \simeq T({}^{\mathbb{C}}\bar{M}_{h,\tau}^{n-r})$, where ${}^{\mathbb{C}}\bar{M}_{h,\tau}^{n-r}$ is the complex $(n-r)$ -dimensional local integral submanifold of the complexified differential system $K^{\mathbb{C}}(\mathcal{G})/K^{\mathbb{C}}(\mathcal{G}_h)$. This means that the space $T(\bar{M}_{h,\tau}^{2(n-r)})$ is endowed with the standard almost complex structure

$$J : T(\bar{M}_{h,\tau}^{2(n-r)}) \rightarrow T(\bar{M}_{h,\tau}^{2(n-r)}), \quad J^2 = -1, \quad (6.2)$$

such that the 2-form $\omega(\tilde{K}) := \operatorname{Im} \omega(\tilde{K}^{\mathbb{C}}) \in \Lambda^2(\bar{M}_{h,\tau}^{2(n-r)})$ induced from the above metric on $T({}^{\mathbb{C}}\bar{M}_{h,\tau}^{n-r})$ is closed, that is $d\omega(\tilde{K}) = 0$. If this is the case, the almost complex structure on the manifold $T(\bar{M}_{h,\tau}^{2(n-r)})$ is said to be integrable. Define the proper complex manifold ${}^{\mathbb{C}}\bar{M}_{h,\tau}^{n-r}$, on which one can then define global vector fields $\tilde{F}_j \in K(\mathcal{G})/K(\mathcal{G}_h)$, $j = \overline{1, n-r}$, which are being sought for the involutive algebra $\mathcal{G}_\tau$ of invariants on M^{2n} to be integrable by quadratures via the abelian Liouville–Arnold theorem. Thus the following theorem can be obtained.

Theorem 16. *A nonabelian set $\mathcal{G}$ of invariants on the symplectic space $M^{2n} \simeq T^*(\mathbb{R}^n)$, satisfying the Mishchenko–Fomenko condition (2.9), admits algebraic-analytical integration by quadratures for the integral submanifold imbedding $\pi_h : M_h^r \rightarrow M^{2n}$ if the corresponding complexified reduced manifold ${}^{\mathbb{C}}\bar{M}_{h,\tau}^{n-r} \simeq \bar{M}_{h,\tau}^{2(n-r)} = M_{h,\tau}^{k-r}/G_h$ of the differential system $K^{\mathbb{C}}(\mathcal{G})/K^{\mathbb{C}}(\mathcal{G}_h)$ is Kählerian with respect to the standard almost complex structure (5.1) and the nondegenerate complex metric $\omega(\tilde{K}^{\mathbb{C}}) : T(\bar{M}_{h,\tau}^{2(n-r)})^{\mathbb{C}} \times T(\bar{M}_{h,\tau}^{2(n-r)})^{\mathbb{C}} \rightarrow \mathbb{C}$ induced by the symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$ is integrable, that is $d \operatorname{Im} \omega(\tilde{K}^{\mathbb{C}}) = 0$.*

Theorem 16 shows, in particular, that nonabelian Liouville–Arnold integrability by quadratures does not in general imply integrability via the abelian Liouville–Arnold theorem; it actually depends on certain topological obstructions associated with the Lie algebra structure of invariants $\mathcal{G}$ on the phase space M^{2n} . We hope to explore this intriguing problem in another place.

7. SUPPLEMENT

In this section we consider some examples of investigation of integral submanifold imbedding mappings for abelian Liouville–Arnold integrable Hamiltonian systems on $T^*(\mathbb{R}^2)$.

7.1. The Henon–Heiles system. This flow is governed by the Hamiltonian

$$H_1 = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + q_1q_2^2 + \frac{1}{3}q_1^3 \quad (7.1)$$

on the canonically symplectic phase space $M^4 = T^*(\mathbb{R}^2)$ with the symplectic structure

$$\omega^{(2)} = \sum_{j=1}^2 dp_j \wedge dq_j. \quad (7.2)$$

As is well known, there exists the following additional invariant that commutes with (7.1):

$$H_2 = p_1 p_2 + 1/3 q_2^3 + q_1^2 q_2, \quad (7.3)$$

that is $\{H_1, H_2\} = 0$ on the entire space M^4 .

Take a regular element $h \in \mathcal{G} := \{H_j : M^4 \rightarrow \mathbb{R}, j = \overline{1, 2}\}$, with fixed values $h(H_j) = h_j \in \mathbb{R}, j = \overline{1, 2}$. Then the integral submanifold

$$M_h^2 := \{(q, p) \in M^4 : h(H_j) = h_j \in \mathbb{R}, j = \overline{1, 2}\}, \quad (7.4)$$

if compact and connected, is diffeomorphic to the standard torus $\mathbb{T}^2 \simeq \mathbb{S}^1 \times \mathbb{S}^1$ owing to the Liouville–Arnold theorem, and one can find cyclic (separable) coordinates $\mu_j \in \mathbb{S}^1, j = \overline{1, 2}$, on the torus such that the symplectic structure (7.2) will take the form:

$$\omega^{(2)} = \sum_{j=1}^2 dw_j \wedge d\mu_j, \quad (7.5)$$

where the conjugate variables $w_j \in T^*(\mathbb{S}^1), j = \overline{1, 2}$, on M_h^2 depend only on the corresponding variables $\mu_j \in \mathbb{S}^1, j = \overline{1, 2}$. In this case it is evident that the evolution along M_h^2 will be separable and representable by means of quasi-periodic functions of the evolution parameters.

To show this, recall that the fundamental determining equations (3.34) based on the 1-forms $\bar{h}_j^{(1)} \in \Lambda(M_h^2), j = \overline{1, 2}$, satisfy the identity

$$\sum_{j=1}^2 dH_j \wedge_j \bar{h}_j^{(1)} = \sum_{j=1}^2 dp_j \wedge dq_j. \quad (7.6)$$

Here

$$\bar{h}_j^{(1)} = \sum_{k=1}^2 \bar{h}_{jk}(q, p) dq_k, \quad (7.7)$$

where $j = \overline{1, 2}$. Substituting (7.7) into (7.6), one obtains

$$\bar{h}_1^{(1)} = \frac{p_1 dq_1}{p_1^2 - p_2^2} + \frac{p_2 dq_2}{p_1^2 - p_2^2}, \quad \bar{h}_2^{(1)} = \frac{p_2 dq_1}{p_2^2 - p_1^2} + \frac{p_1 dq_2}{p_2^2 - p_1^2}. \quad (7.8)$$

On the other hand, the following implication holds on $M_h^2 \subset M^4$:

$$\alpha_h^{(1)} = \sum_{j=1}^2 w_j(\mu_j; h) d\mu_j \implies \sum_{j=1}^2 p_j dq_j := \alpha^{(1)}, \quad (7.9)$$

where we have assumed that the integral submanifold M_h^2 admits the local coordinates in the base manifold $\mathbb{R}^2$ endowed with the canonical 1-form $\alpha_h^{(1)} \in \Lambda(M_h^2)$ as given in (7.9). Thus, making use of the imbedding $\pi_h : M_h^2 \rightarrow T^*(\mathbb{R}^2)$ in the form

$$q_j = q_j(\mu; h), \quad p_j = p_j(\mu; h), \quad (7.10)$$

$j = \overline{1, 2}$, one readily finds that the equalities

$$p_j = \sum_{k=1}^2 w_k(\mu_k; h) \partial \mu_k / \partial q_j \quad (7.11)$$

hold for $j = \overline{1, 2}$ on the entire integral submanifold M_h^2 .

Substituting (7.11) into (7.8) and using the characteristic relationships (3.34), one obtains after simple but cumbersome calculations the following differential-algebraic expressions:

$$\partial q_1 / \partial \mu_1 - \partial q_2 / \partial \mu_1 = 0, \quad \partial q_1 / \partial \mu_2 + \partial q_2 / \partial \mu_2 = 0, \quad (7.12)$$

whose simplest solutions are

$$q_1 = (\mu_1 + \mu_2) / 2, \quad q_2 = (\mu_1 - \mu_2) / 2. \quad (7.13)$$

Using expressions (7.11) one finds that

$$p_1 = w_1 + w_2, \quad p_2 = w_1 - w_2, \quad (7.14)$$

where

$$w_1 = \sqrt{h_1 + h_2 - 4/3\mu_1^3}, \quad w_2 = \sqrt{h_1 - h_2 - 4/3\mu_2^3}. \quad (7.15)$$

Consequently, one obtains the separable [15] Hamiltonian functions (7.1) and (7.3) in the vicinity of the cotangent space $T^*(M_h^2)$:

$$h_1 = \frac{1}{2}w_1^2 + \frac{1}{2}w_2^2 + \frac{2}{3}(\mu_1^3 + \mu_2^3), \quad h_2 = \frac{1}{2}w_1^2 - \frac{1}{2}w_2^2 + \frac{2}{3}(\mu_1^3 - \mu_2^3), \quad (7.16)$$

which generate the following separable motions on $M_h^2 \subset T^*(\mathbb{R}^2)$:

$$\begin{aligned} d\mu_1/dt &:= \partial h_1 / \partial w_1 = \sqrt{h_1 + h_2 - 4/3\mu_1^3}, \\ d\mu_2/dt &:= \partial h_1 / \partial w_2 = \sqrt{h_1 - h_2 - 4/3\mu_2^3} \end{aligned} \quad (7.17)$$

for the Hamiltonian (7.1), and

$$\begin{aligned} d\mu_1/dx &:= \partial h_2 / \partial w_1 = \sqrt{h_1 + h_2 - 4/3\mu_1^3}, \\ d\mu_2/dt &:= \partial h_1 / \partial w_2 = -\sqrt{h_1 - h_2 - 4/3\mu_2^3} \end{aligned} \quad (7.18)$$

for the Hamiltonian (7.3), where $x, t \in \mathbb{R}$ are the corresponding evolution parameters.

Analogously, one can show that there exists [28, 29] a similar to (7.13) and (7.14) integral submanifold imbedding for the following integrable modified Henon–Heiles involutive system:

$$\begin{aligned} H_1 &= \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + q_1q_2^2 + \frac{16}{3}q_1^3, \\ H_2 &= 9p_2^4 + 36q_1p_2^2q_2^2 - 12p_1p_2q_2^3 - 2q_2^4(q_2^2 + 6q_1^2), \end{aligned} \quad (7.19)$$

where $\{H_1, H_2\} = 0$ on the entire phase space $M^4 = T^*(\mathbb{R}^2)$.

Based on considerations similar to the above, one can deduce the following [29] expressions:

$$\begin{aligned} q_1 &= -\frac{1}{4}(\mu_1 + \mu_2) - \frac{3}{8}\left(\frac{w_1 + w_2}{\mu_1 - \mu_2}\right)^2, \\ q_2^2 &= -2\sqrt{h_2}/(\mu_1 - \mu_2), \quad w_1 = \sqrt{2/3\mu_1^3 - 4/3\sqrt{h_2} - 8h_1}, \\ p_1 &= \frac{1}{2\sqrt{-6(\mu_1 + \mu_2 + 4q)}} \left[\frac{-2\sqrt{h_2}}{\mu_1 - \mu_2} - \mu_1\mu_2 + 4(\mu_1 + \mu_2)q_1 + 32q_1^2 \right], \\ p_2 &= \sqrt{h_2}(\mu_1 + \mu_2 + 4q_1)/(3(\mu_1 - \mu_2)), \quad w_2 = \sqrt{2/3\mu_2^3 + 4/3\sqrt{h_2} - 8h_1}, \end{aligned} \quad (7.20)$$

thereby solving explicitly the problem of finding the corresponding integral submanifold imbedding $\pi_h : M_h^2 \rightarrow T^*(\mathbb{R}^2)$ that generates separable flows in the variables $(\mu, w) \in T^*(M_h^2)$.

7.2. A truncated 4-dimensional Focker–Plank Hamiltonian system on $T^*(\mathbb{R}^2)$ and its integrability by quadratures. Consider the following dynamical system on the canonically symplectic phase space $T^*(\mathbb{R}^2)$:

$$\left. \begin{aligned} dq_1/dt &= p_1 + \alpha(q_1 + p_2)(q_2 + p_1), & dq_2/dt &= p_2, \\ dp_1/dt &= -(q_1 + p_2) - \alpha[q_2p_1 + 1/2(p_1^2 + p_2^2 + q_2^2)], \\ dp_2/dt &= -(q_2 + p_1), \end{aligned} \right\} = K_1(q, p), \quad (7.21)$$

where $K_1 : T^*(\mathbb{R}^2) \rightarrow T(T^*(\mathbb{R}^2))$ is the corresponding vector field on $T^*(\mathbb{R}^2) \ni (q, p)$, $t \in \mathbb{R}$ is an evolution parameter, called a truncated four-dimensional Focker–Plank flow. It is easy to verify that functions $H_j : T^*(\mathbb{R}^2) \rightarrow \mathbb{R}$, $j = 1, 2$, where

$$H_1 = 1/2(p_1^2 + p_2^2 + q_1^2) + q_1p_2 + \alpha(q_1 + p_2)[q_2p_1 + 1/2(p_1^2 + p_2^2 + q_2^2)] \quad (7.22)$$

and

$$H_2 = 1/2(p_1^2 + p_2^2 + q_2^2) + q_2p_1 \quad (7.23)$$

are functionally independent invariants with respect to the flow (7.21). Moreover, the invariant (7.22) is the Hamiltonian function for (7.21), that is the relationship

$$i_{K_1}\omega^{(2)} = -dH_1 \quad (7.24)$$

holds on $T^*(\mathbb{R}^2)$, where the symplectic structure $\omega^{(2)} \in \Lambda^2(T^*(\mathbb{R}^2))$ is given as follows:

$$\omega^{(2)} := d(\text{pr}^* \alpha^{(1)}) = \sum_{j=1}^2 dp_j \wedge dq_j, \quad (7.25)$$

with $\alpha^{(1)} \in \Lambda^1(\mathbb{R}^2)$ to be the canonical Liouville form on $\mathbb{R}^2$:

$$\alpha^{(1)}(q; p) = \sum_{j=1}^2 p_j dq_j \quad (7.26)$$

for any $(q, p) \in T^*(\mathbb{R}^2) \simeq \Lambda^1(\mathbb{R}^2)$.

The invariants (7.22) and (7.23) commute evidently with each other subject to the associated Poisson bracket on $T^*(\mathbb{R}^2)$, i. e., $\{H_1, H_2\} = 0$. Thereby, owing to the abelian Liouville–Arnold theorem [1, 2], the dynamical system (7.21) is completely integrable by quadratures on $T^*(\mathbb{R}^2)$, and we can apply the scheme devised in Section 2 to the commuting invariants (7.22) and (7.23) subject to the symplectic structure (7.25). One easily calculates that

$$\omega^{(2)} = \sum_{i=1}^2 dH_i \wedge h_i^{(1)}, \quad (7.27)$$

where the corresponding 1-forms $\pi_h^* h_i^{(1)} := \bar{h}_i^{(1)} \in \Lambda^1(M_h^2)$, $i = \overline{1, 2}$, are given as

$$\begin{aligned} \bar{h}_1^{(1)} &= \frac{p_2 dq_1 - (p_1 + q_2) dq_2}{p_1 p_2 - (p_1 + q_2)(q_1 + p_2) - \alpha h_2 (p_1 + q_2)}, \\ \bar{h}_2^{(1)} &= \frac{-[(q_1 + p_2)(1 + \alpha p_2) + \alpha h_2] dq_1 + (p_1 + \alpha[h_2 + (q_2 + p_1)(q_1 + p_2)]) dq_2}{p_1 p_2 - (q_2 + p_1)(\alpha h_2 + q_1 + p_2)}, \end{aligned}$$

and an invariant submanifold $M_h^2 \subset T^*(\mathbb{R}^2)$ is defined as

$$M_h^2 := \{(q, p) \in T^*(\mathbb{R}^2) : H_i(q, p) = h_i \in \mathbb{R}, i = \overline{1, 2}\} \quad (7.28)$$

for some parameters $h \in \mathbb{R}^2$ and based now on expressions (7.28), and (3.38) one can easily construct functions $\bar{P}_{ij}(w; h)$, $i, j = \overline{1, 2}$, in (3.53), defined on $T^*(M_h^2) \simeq T^*(\otimes_{j=1}^2 \mathbb{S}_j^1)$ subject to the integral submanifold imbedding mapping $\pi_h : M_h^2 \rightarrow T^*(\mathbb{R}^2)$ in coordinates $\mu \in \otimes_{j=1}^2 \mathbb{S}_j^1 \subset \otimes_{j=1}^2 \Gamma_h^{(j)}$, which we do not write in detail due to their long and cumbersome form. Having applied then the criterion (3.55), we arrive at the following compatibility relations subject to the mappings $q : (\otimes_{j=1}^2 \mathbb{S}_j^1) \times \mathbb{R}^2 \rightarrow$

$\mathbb{R}^2$ and $p : (\otimes_{j=1}^2 \mathbb{S}_j^1) \times \mathbb{R}^2 \rightarrow T_q^*(\mathbb{R}^2) :$

$$\begin{aligned} \partial q_1 / \partial \mu_1 - \partial q_2 / \partial \mu_2 &= 0, & w_1 \partial \mathcal{L}_\mu / \partial w_1 - w_2 \partial \mathcal{L}_\mu / \partial w_2 &= 0, \\ \partial^2 q_1 / \partial \mu_2 \partial h_2 + \partial^2 w_2 / \partial \mu_2 \partial h_2 &= 0, \\ \partial w_1 / \partial h_1 (\partial q_1 / \partial h_1) &= \partial w_2 / \partial h_1 (\partial q_2 / \partial h_1), \\ w_1 \partial w_1 / \partial h_1 - w_2 \partial w_2 / \partial h_2 &= 0, \\ \partial (w_1 \partial w_1 / \partial h_2) / \partial h_2 - \alpha^2 \partial q_1 / \partial \mu_1 &= 0, \dots \end{aligned} \quad (7.29)$$

and so on, subject to variables $\mu \in \otimes_{j=1}^2 \mathbb{S}_j^1$ and $h \in \mathbb{R}^2$. Solving equations like (7.29), one can find right away that the expressions

$$\begin{aligned} p_1 &= w_1, & p_2 &= w_2, \\ q_1 &= c_1 + \mu_1 - w_2(\mu_2; h), & q_2 &= c_2 + \mu_2 - w_1(\mu_1; h), \\ \mathcal{L}_\mu(h) &= -w_1 w_2, \end{aligned} \quad (7.30)$$

where $c_j(h_1, h_2) \in \mathbb{R}^1$, $j = \overline{1, 2}$, are constant, hold on $T^*(M_h^2)$, giving rise to the following Picard–Fuchs type equations in the form (3.56):

$$\begin{aligned} \partial w_1(\mu_1; h) / \partial h_1 &= 1/w_1, & \partial w_1(\mu_1; h) / \partial h_2 &= \alpha^2 h_2 / w_1, \\ \partial w_2(\mu_2; h) / \partial h_1 &= 0, & \partial w_2(\mu_2; h) / \partial h_2 &= 1/w_2. \end{aligned} \quad (7.31)$$

The Picard–Fuchs equations (7.31) can be easily integrated by quadratures as follows:

$$w_1^2 + k_1(\mu_1) - \alpha^2 h_2 - 2h_1 = 0, \quad w_2^2 + k_2(\mu_2) - 2h_2 = 0, \quad (7.32)$$

where $k_j : \mathbb{S}_j^1 \rightarrow \mathbb{C}$, $j = \overline{1, 2}$, are still unknown functions. For them to be determined explicitly, it is necessarily to substitute (7.30) into expressions (7.22) and (7.23), making use of (7.32), which amounts to the following results:

$$k_1 = \mu_1^2, \quad k_2 = \mu_2^2 \quad (7.33)$$

under the condition that $c_1 = -\alpha h_2$, $c_2 = 0$. Thereby, we have constructed, owing to (7.32), the corresponding algebraic curves $\Gamma_h^{(j)}$, $j = \overline{1, 2}$, (3.41) in the explicit form:

$$\begin{aligned} \Gamma_h^{(1)} &:= \{(\lambda, w_1) : w_1^2 + \lambda^2 - \alpha^2 h_2^2 - 2h_1 = 0\}, \\ \Gamma_h^{(2)} &:= \{(\lambda, w_2) : w_2^2 + \lambda^2 - 2h_2 = 0\}, \end{aligned} \quad (7.34)$$

where $(\lambda, w_j) \in \mathbb{C}$, $j = \overline{1, 2}$, and $h \in \mathbb{R}^2$ are arbitrary parameters. Making use now of expressions (7.35) and (7.30), one can construct in explicit form the integral submanifold imbedding mapping $\pi_h : M_h^2 \rightarrow T^*(\mathbb{R}^2)$ for the flow (7.21):

$$\begin{aligned} q_1 &= \mu_1 - \sqrt{2h_2 - \mu_2^2 - \alpha h_2^2}, & p_1 &= w_1(\mu_1; h), \\ q_2 &= \mu_2 - \sqrt{2h_1 - \alpha^2 h_2^2 - \mu_1^2}, & p_2 &= w_2(\mu_2; h), \end{aligned} \quad (7.35)$$

where $(\mu, w) \in \otimes_{j=1}^2 \Gamma_h^{(j)}$. As mentioned before in Section 2, formulas (7.35) together with explicit expressions (3.40) make it possible right away to find solutions to the truncated Fock–Plank flow (7.21) by quadratures, thereby completing its integrability.

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Authors’ Addresses

Y. A. Prykarpatsky:

INSTITUTE OF MATHEMATICS, NAS OF UKRAINE, 01601 KYIV, UKRAINE

Current address: Department of Applied Mathematics, AGH University of Science and Technology, 30-059 Kraków, Poland

E-mail address: yarchyk@imath.kiev.ua

A. M. Samoilenko:

INSTITUTE OF MATHEMATICS, NAS OF UKRAINE, 01601 KYIV, UKRAINE

E-mail address: sam@imath.kiev.ua

D. L. Blackmore:

DEPARTMENT OF MATHEMATICAL SCIENCES, NJIT, NEWARK, NJ 07102, USA

E-mail address: deblac@m.njit.edu, deblac@chaos.njit.edu

A. K. Prykarpatsky:

INSTITUTE FOR APPLIED PROBLEMS OF MECHANICS AND MATHEMATICS, NAS OF UKRAINE, 79601 LVIV, UKRAINE

Current address: Department of Applied Mathematics, AGH University of Science and Technology, 30059 Kraków, Poland

E-mail address: pryk.anat@ua.fm, prykanat@cybergal.com



ON THE INITIAL VALUE PROBLEM FOR SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS WITH ARGUMENT DEVIATIONS

ANDREI RONTÓ

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ABSTRACT. We establish new efficient conditions sufficient for the unique solvability of the Cauchy problem for many-dimensional systems of linear differential equations with arbitrary argument deviations.

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Keywords: Functional differential equation, Cauchy problem, initial value problem, equation with deviated argument

1. INTRODUCTION

In the class of functional differential equations, the question on the solvability of the initial value problem, as is well-known, is non-trivial even in the linear case. For example, the simplest scalar functional differential equation

$$u'(t) = \frac{u(b)}{b} + q(t), \quad t \in [0, b], \quad (1.1)$$

has no solutions satisfying the initial condition

$$u(0) = c,$$

provided that the function $q \in L_1([0, b], \mathbb{R})$ and the real constant c are related by the inequality

$$\int_0^b q(s)ds \neq -c.$$

From the standpoint of the theory of ordinary differential equations, the fact indicated may seem somewhat strange if we note that (1.1) is a scalar linear equation with the constant coefficient b^{-1} , and this coefficient, moreover, becomes arbitrarily small when the length of the interval, b , increases to $+\infty$. Such phenomena are, however, not uncommon, if not typical, for functional differential equations.

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At present, there are not but a few sharp and efficient conditions guaranteeing the unique solvability of the initial value problem for differential equations with arbitrary argument deviations, and most of them are available for the one-dimensional case only (see, e. g., [4] and references therein).

In this paper, we continue the study initiated in [2, 3, 6] and establish some new conditions guaranteeing the unique solvability of the problem

$$u'_k(t) = \sum_{j=1}^n r_{kj}(t)u_j(\omega_{kj}(t)) + q_k(t), \quad t \in [a, b], \quad k = 1, 2, \dots, n, \quad (1.2)$$

$$u_k(\tau) = c_k, \quad k = 1, 2, \dots, n. \quad (1.3)$$

Here, $n \in \mathbb{N}$, τ is a given point from $[a, b]$, $q_k : [a, b] \rightarrow \mathbb{R}$ and $r_{kj} : [a, b] \rightarrow \mathbb{R}$, $k, j = 1, 2, \dots, n$, are Lebesgue integrable functions, $\{c_k \mid k = 1, 2, \dots, n\} \subset \mathbb{R}$ are given constants, and ω_{kj} , $k, j = 1, 2, \dots, n$, are arbitrary measurable transformations of the interval $[a, b]$ to itself.

By a solution of problem (1.2), (1.3), as usual [1], we mean a vector-function $u = (u_k)_{k=1}^n : [a, b] \rightarrow \mathbb{R}^n$ with absolutely continuous components u_k , $k = 1, 2, \dots, n$, satisfying relations (1.2) almost everywhere on $[a, b]$ and possessing properties (1.3).

2. NOTATION

The following notation is used throughout the paper.

- (1) $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{R} = (-\infty, \infty)$.
- (2) $L_1([a, b], \mathbb{R}^n)$ is the Banach space of Bochner integrable functions $u : [a, b] \rightarrow \mathbb{R}^n$ endowed by the norm

$$L_1([a, b], \mathbb{R}^n) \ni u = (u_k)_{k=1}^n \mapsto \max_{k=1,2,\dots,n} \int_a^b |u_k(t)| dt.$$

- (3) $C([a, b], \mathbb{R}^n)$ is the Banach space of continuous functions $u : [a, b] \rightarrow \mathbb{R}^n$ endowed by the norm

$$C([a, b], \mathbb{R}^n) \ni u = (u_k)_{k=1}^n \mapsto \max_{k=1,2,\dots,n} \max_{t \in [a,b]} |u_k(t)|. \quad (2.1)$$

- (4) $C_\tau([a, b], \mathbb{R}^n)$, where $\tau \in [a, b]$, is the subspace of $C([a, b], \mathbb{R}^n)$ constituted by those functions u for which $u(\tau) = 0$.
- (5) $r(A)$ is the spectral radius of a bounded linear operator A .
- (6) $\text{GL}_n(\mathbb{R})$ is the algebra of the n -dimensional square matrices with real elements.

3. CONDITIONS FOR THE UNIQUE SOLVABILITY OF PROBLEM (1.2), (1.3)

In this section, we present our main results concerning the initial value problem (1.2), (1.3). The proofs are given later, in Section 5.

3.1. Existence of Green's operator. The following statement provides conditions sufficient for the unique solvability of the initial value problem (1.2), (1.3).

Theorem 3.1. *Let there exist some $\alpha \in [1, +\infty)$ and $\{\gamma_k \mid k = 1, 2, \dots, n\} \subset (0, +\infty)$ such that one of the following conditions is satisfied:*

$$\max_{k=1,2,\dots,n} \text{vrai max}_{t \in [a,b] \setminus \{\tau\}} \frac{1}{\gamma_k |t - \tau|^{\alpha-1}} \sum_{j=1}^n \gamma_j |r_{kj}(t)| |\omega_{kj}(t) - \tau|^\alpha < \alpha, \quad (3.1a)$$

$$\max_{k=1,2,\dots,n} \sup_{t \in [a,b] \setminus \{\tau\}} \frac{\text{sign}(t - \tau)}{\gamma_k |t - \tau|^\alpha} \sum_{j=1}^n \gamma_j \int_\tau^t |r_{kj}(s)| |\omega_{kj}(s) - \tau|^\alpha ds < 1, \quad (3.1b)$$

$$\max_{k=1,2,\dots,n} \text{vrai max}_{t \in [a,b] \setminus \{\tau\}} \sum_{l=1}^n \frac{|r_{kl}(t)|}{\gamma_k |t - \tau|^{\alpha-1}} \sum_{j=1}^n \gamma_j \left| \int_\tau^{\omega_{kl}(t)} |r_{lj}(s)| |\omega_{lj}(s) - \tau|^\alpha ds \right| < \alpha. \quad (3.1c)$$

Then the initial value problem (1.2), (1.3) is uniquely solvable for arbitrary constants $\{c_k \mid k = 1, 2, \dots, n\} \subset \mathbb{R}$ and integrable functions $q_k : [a, b] \rightarrow \mathbb{R}$, $k = 1, 2, \dots, n$.

None of the strict inequalities (3.1a), (3.1b), and (3.1c) assumed in Theorem 3.1 can be replaced by the corresponding non-strict inequality. For instance, the condition

$$\max_{k=1,2,\dots,n} \text{vrai max}_{t \in [a,b] \setminus \{\tau\}} \frac{1}{\gamma_k |t - \tau|^{\alpha-1}} \sum_{j=1}^n \gamma_j |r_{kj}(t)| |\omega_{kj}(t) - \tau|^\alpha \leq \alpha \quad (3.2)$$

does not guarantee the unique solvability of problem (1.2), (1.3) because, as follows from the example given below, for any $\alpha \in [1, +\infty)$ and $\tau \in [a, b]$, one can specify a functional differential system (1.2) for which relation (3.2) is satisfied but the Cauchy problem (1.3) is not uniquely solvable.

Example. Given $\tau \in [a, b]$ and $\alpha \in [1, +\infty)$, consider the scalar functional differential equation

$$u'(t) = \frac{\alpha |t - \tau|^{\alpha-1}}{|\theta - \tau|^\alpha} \text{sign}(t - \tau) u(\theta), \quad t \in [a, b], \quad (3.3)$$

where $\theta \in [a, b]$ is an arbitrary point different from τ . This equation, clearly, has form (1.2) with $n = 1$, $q_1(t) = 0$, $\omega_{11}(t) = \theta$, and

$$r_{11}(t) = \frac{\alpha |t - \tau|^{\alpha-1}}{|\theta - \tau|^\alpha} \text{sign}(t - \tau)$$

for almost every $t \in [a, b]$. It is easy to verify that, for an arbitrary λ , the function

$$u(t) = \lambda |t - \tau|^\alpha, \quad t \in [a, b],$$

is a solution of the homogeneous Cauchy problem

$$u(\tau) = 0$$

for equation (3.3). However, condition (3.2) is satisfied for this equation because

$$\text{vrai max}_{t \in [a,b] \setminus \{\tau\}} \frac{1}{\gamma_1 |t - \tau|^{\alpha-1}} \gamma_1 |r_{11}(t)| |\omega_{11}(t) - \tau|^\alpha = \alpha.$$

One can verify that the non-strict versions of inequalities (3.1b) and (3.1c) are also satisfied in this case.

3.2. Positivity of Green's operator. The conditions given below guarantee not only existence but also the positivity of Green's operator associated with the homogeneous Cauchy problem (1.2), (1.3).

Theorem 3.2. *Let there exist some $\{\sigma_1, \sigma_2, \dots, \sigma_n\} \subset \{-1, 1\}$ such that*

$$\sigma_k \sigma_j r_{kj}(t) \text{sign}(t - \tau) \geq 0 \quad (3.4)$$

*for almost every $t \in [a, b]$ and all $k, j = 1, 2, \dots, n$. Moreover, assume the existence of some constants $\alpha \in [1, +\infty)$ and $\{\gamma_k \mid k = 1, 2, \dots, n\} \subset (0, +\infty)$ such that one of the following conditions is satisfied for all $k = 1, 2, \dots, n$:**

$$\text{vrai max}_{t \in [a,b] \setminus \{\tau\}} \frac{\sigma_k \text{sign}(t - \tau)}{\gamma_k |t - \tau|^{\alpha-1}} \sum_{j=1}^n \sigma_j \gamma_j r_{kj}(t) |\omega_{kj}(t) - \tau|^\alpha < \alpha, \quad (3.5a)$$

$$\sup_{t \in [a,b] \setminus \{\tau\}} \frac{\sigma_k}{\gamma_k |t - \tau|^\alpha} \sum_{j=1}^n \sigma_j \gamma_j \int_\tau^t r_{kj}(s) |\omega_{kj}(s) - \tau|^\alpha ds < 1, \quad (3.5b)$$

$$\text{vrai max}_{t \in [a,b] \setminus \{\tau\}} \frac{\sigma_k \text{sign}(t - \tau)}{\gamma_k |t - \tau|^{\alpha-1}} \sum_{j=1}^n r_{kj}(t) \sum_{l=1}^n \sigma_l \gamma_l \int_\tau^{\omega_{kj}(t)} r_{jl}(s) |\omega_{jl}(s) - \tau|^\alpha ds < \alpha. \quad (3.5c)$$

Then the initial value problem (1.2), (1.3) is uniquely solvable for arbitrary constants $\{c_k \mid k = 1, 2, \dots, n\} \subset \mathbb{R}$ and integrable functions $q_k : [a, b] \rightarrow \mathbb{R}$, $k = 1, 2, \dots, n$. If, in addition, q_k and c_k , $k = 1, 2, \dots, n$, possess the property

$$\sigma_k \left(\int_\tau^t q_k(s) ds + c_k \right) \geq 0, \quad t \in [a, b], \quad k = 1, 2, \dots, n,$$

then the unique solution $u = (u_k)_{k=1}^n$ of problem (1.2), (1.3) satisfies the condition

$$\sigma_k u_k(t) \geq 0, \quad t \in [a, b], \quad k = 1, 2, \dots, n.$$

The example presented in Section 3.1 shows that conditions (3.5a), (3.5b), and (3.5c) are optimal in the sense that none of them can be weakened to the corresponding non-strict inequality.

4. GENERAL STATEMENTS

The theorems presented above rely upon some general results on linear integral functional operators. The results mentioned are presented in this section.

*We emphasise that one and the same condition should be satisfied for all k from 1 to n .

4.1. Spectra of linear integral operators with inner superpositions. Let $R_i : [a, b] \rightarrow \text{GL}_n(\mathbb{R})$, $i = 1, 2, \dots, N$, where $N \in \mathbb{N}$, be matrix-valued functions with Lebesgue integrable components, and $\omega_i : [a, b] \rightarrow [a, b]$, $i = 1, 2, \dots, N$, be measurable functions.

For every u from $C_\tau([a, b], \mathbb{R}^n)$,[†] we put

$$(Bu)(t) := \sum_{i=1}^N \int_{\tau}^t R_i(s)u(\omega_i(s))ds, \quad t \in [a, b]. \quad (4.1)$$

Obviously, B is a bounded linear mapping of the space $C_\tau([a, b], \mathbb{R}^n)$ to itself. One can show that operator (4.1) is completely continuous in the space mentioned.

Until the end of this section, we fix a diagonal matrix

$$\Sigma = \text{diag} \{\sigma_1, \sigma_2, \dots, \sigma_n\} \quad (4.2)$$

with components $\{\sigma_1, \sigma_2, \dots, \sigma_n\} \subset \{-1, 1\}$. For the sake of brevity we introduce a definition.

Definition 4.1. For $\{x, y\} \subset \mathbb{R}^n$ we write $x \geq_\Sigma y$ (resp., $x >_\Sigma y$) if, and only if $\sigma_k x_k \geq \sigma_k y_k$ (resp., $\sigma_k x_k > \sigma_k y_k$) for all $k = 1, 2, \dots, n$.

The following statement provides a convenient way to estimate the spectral radius of operator (4.1).

Proposition 4.1. *Let each of the functions $R_v : [a, b] \rightarrow \text{GL}_n(\mathbb{R})$, $v = 0, 1, \dots, N$, satisfy the condition*

$$\Sigma R_v(t) \Sigma \text{sign}(t - \tau) \geq 0 \quad \text{for a. e. } t \in [a, b] \quad (4.3)$$

with respect to the given matrix (4.2). Furthermore, assume that there exists an absolutely continuous function $y : [a, b] \rightarrow \mathbb{R}^n$ possessing the properties

$$y(\tau) = 0; \quad y(t) >_\Sigma 0 \quad \text{for all } t \in [a, b] \setminus \{\tau\} \quad (4.4)$$

and

$$y'(t) \text{sign}(t - \tau) \geq_\Sigma 0 \quad \text{for a. e. } t \in [a, b], \quad (4.5)$$

and such that, with a certain constant $\varrho \in (1, +\infty)$, the integral relation

$$y(t) \geq_\Sigma \varrho \sum_{i=1}^N \int_{\tau}^t R_i(s)y(\omega_i(s))ds \quad (4.6)$$

is true for every $t \in [a, b]$.

Then the spectral radius $r(B)$ of the operator $B : C_\tau([a, b], \mathbb{R}^n) \rightarrow C_\tau([a, b], \mathbb{R}^n)$ given by formula (4.1) is less than 1.

Remark 4.1. The inequality in (4.3) and similar relations below is understood componentwise.

[†]See notation (4), Section 2.

Proposition 4.1 is obtained from [3, Corollary 6] and the argument given in the proof of [6, Theorem 1]. The details are omitted here.

Proposition 4.2. *Let the functions $R_v : [a, b] \rightarrow GL_n(\mathbb{R})$, $v = 0, 1, \dots, N$, satisfy condition (4.3). Moreover, there exist some constants $\gamma \in [0, 1)$ and $\alpha \in [1, +\infty)$ and an n -dimensional vector g with the property*

$$g >_{\Sigma} 0 \quad (4.7)$$

such that

$$\sum_{v=1}^N \int_{\tau}^t R_v(s) g |\omega_v(s) - \tau|^{\alpha} ds \leq_{\Sigma} \gamma |t - \tau|^{\alpha} g, \quad t \in [a, b]. \quad (4.8)$$

Then the spectral radius $r(B)$ of operator (4.1) in the space $C_{\tau}([a, b], \mathbb{R}^n)$ is less than 1.

PROOF. Let $y : [a, b] \rightarrow \mathbb{R}^n$ be the function given by the formula

$$y(t) := |t - \tau|^{\alpha} g, \quad t \in [a, b], \quad (4.9)$$

where α and g are the values appearing in the formulation of Theorem 4.2. Function (4.9) obviously possesses properties (4.4). One can verify that, for almost every $t \in [a, b]$,

$$y'(t) = \alpha |t - \tau|^{\alpha-1} \text{sign}(t - \tau) g,$$

and, hence, function (4.9) also satisfies condition (4.5). Furthermore, it follows from assumption (4.8) that function (4.9) satisfies condition (4.6) with $\varrho := \gamma^{-1}$.[‡] Therefore, it remains to apply Proposition 4.1. $\square$

Proposition 4.3. *Let the functions $R_v : [a, b] \rightarrow GL_n(\mathbb{R})$, $v = 0, 1, \dots, N$, satisfy condition (4.3). Let, moreover, there exist some constants $\gamma \in [0, 1)$ and $\alpha \in [1, +\infty)$ and n -dimensional vector g with property (4.7) such that the following relation is satisfied for almost every t from $[a, b]$:*

$$\sum_{v=1}^N R_v(t) \sum_{i=1}^N \int_{\tau}^{\omega_v(t)} R_i(s) g |\omega_i(s) - \tau|^{\alpha} ds \cdot \text{sign}(t - \tau) \leq_{\Sigma} \gamma \alpha |t - \tau|^{\alpha-1} g. \quad (4.10)$$

Then the spectral radius $r(B)$ of operator (4.1) in the space $C_{\tau}([a, b], \mathbb{R}^n)$ is less than 1.

To establish the theorem formulated above, one should combine Theorem 4.1, [3, Corollary 7], and [3, Remark 6].

[‡]It is sufficient to consider the case where $\gamma > 0$ because if $\gamma = 0$, then the corresponding relation (4.6) is satisfied with an arbitrary $\varrho \in (1, +\infty)$.

Proposition 4.4. *Assume that there exists a certain diagonal matrix (4.2) with $\{\sigma_1, \sigma_2, \dots, \sigma_n\} \subset \{-1, 1\}$ for which the functions $R_v : [a, b] \rightarrow \text{GL}_n(\mathbb{R})$, $v = 0, 1, \dots, N$, satisfy condition (4.3) and, moreover, let there exist some constants $\gamma \in [0, 1)$ and $\alpha \in [1, +\infty)$ and n -dimensional vector g with property (4.7) such that*

$$\sum_{v=1}^N R_v(t) g |\omega_v(t) - \tau|^\alpha \text{sign}(t - \tau) \leq_\Sigma \gamma \alpha |t - \tau|^{\alpha-1} g \quad \text{for a. e. } t \in [a, b]. \quad (4.11)$$

Then the spectral radius $r(B)$ of operator (4.1) in the space $C_\tau([a, b], \mathbb{R}^n)$ is less than 1.

Similarly to Proposition 4.3, the assertion of Proposition 4.4 is deduced from [3, Corollary 8] and [3, Remark 6].

4.2. Solvability of linear initial value problems. The combination of the statements presented in Section 4.1 with Lemma 4.1 given below leads one to efficient conditions of solvability of the Cauchy problem (1.3),

$$u(\tau) = c, \quad (4.12)$$

for the functional differential equation of the form

$$u'(t) = \sum_{i=1}^N R_i(t) u(\omega_i(t)) + q(t), \quad t \in [a, b], \quad (4.13)$$

where $N \in \mathbb{N}$, the functions $R_i : [a, b] \rightarrow \text{GL}_n(\mathbb{R})$, $i = 1, 2, \dots, N$, and $q : [a, b] \rightarrow \mathbb{R}^n$ are integrable, and $\omega_i : [a, b] \rightarrow [a, b]$, $i = 1, 2, \dots, N$, are measurable.

Lemma 4.1. *If an absolutely continuous function $u : [a, b] \rightarrow \mathbb{R}^n$ is a solution of the Cauchy problem (4.13), (4.12), then it satisfies the equation*

$$u(t) = (Bu)(t) + c + \int_\tau^t q(s) ds, \quad t \in [a, b]. \quad (4.14)$$

Conversely, every continuous solution $u(\cdot)$ of equation (4.14) is also a solution of problem (4.12), (4.13)

The assertion of Lemma 4.1, which is easily verified by direct computation, allows one to interpret the space of solutions of the homogeneous problem

$$\begin{aligned} u'(t) &= \sum_{i=1}^N R_i(t) u(\omega_i(t)), \quad t \in [a, b], \\ u(\tau) &= 0, \end{aligned}$$

as the set of fixed points of operator (4.1) considered in the space $C_\tau([a, b], \mathbb{R}^n)$.

Lemma 4.2. *Let the functions $R_i : [a, b] \rightarrow \text{GL}_n(\mathbb{R})$, $i = 1, 2, \dots, N$, satisfy condition (4.3) for some matrix (4.2) with elements $\{\sigma_1, \sigma_2, \dots, \sigma_n\} \subset \{-1, 1\}$. Then*

$$\sum_{i=1}^N R_i(t) u(\omega_i(t)) \text{sign}(t - \tau) \geq_{\Sigma} 0 \quad \text{for a. e. } t \in [a, b], \quad (4.15)$$

provided that the continuous function $u : [a, b] \rightarrow \mathbb{R}^n$ satisfies the condition

$$u(t) \geq_{\Sigma} 0, \quad t \in [a, b]. \quad (4.16)$$

PROOF. Indeed, let $u : [a, b] \rightarrow \mathbb{R}^n$ be an arbitrary continuous function possessing property (4.16), or, which is the same (see Definition 4.1), such that the component-wise inequality

$$\Sigma u(t) \geq 0, \quad t \in [a, b], \quad (4.17)$$

is true. Since $\{\sigma_k \mid k = 1, 2, \dots, n\} \subset \{-1, 1\}$, it is obvious that Σ^2 is the identity matrix and, hence, according to Definition 4.1, relation (4.15) can be rewritten in the form

$$\Sigma \sum_{i=1}^N R_i(t) \Sigma u(\omega_i(t)) \text{sign}(t - \tau) \geq 0 \quad \text{for a. e. } t \in [a, b]. \quad (4.18)$$

In view of (4.17), we see that $\Sigma u(\omega_i(t)) \geq 0$ for every $i = 1, 2, \dots, N$ and almost every $t \in [a, b]$. Therefore, (4.18) is true provided that each of the functions $R_i : [a, b] \rightarrow \text{GL}_n(\mathbb{R})$, $i = 1, 2, \dots, N$, satisfies condition (4.3). $\square$

Lemma 4.3. *Condition (4.3) is satisfied if, and only if*

$$\sigma_k \sigma_j r_{kj}^{(\nu)}(t) \text{sign}(t - \tau) \geq 0 \quad (4.19)$$

for almost every $t \in [a, b]$ and all $k, j = 1, 2, \dots, n$ and $\nu = 1, 2, \dots, N$, where $r_{kj}^{(\nu)}$ stand for the corresponding elements of the matrix $R_{\nu} = (r_{kj}^{(\nu)})_{k,j=1}^n$.

PROOF. It is not difficult to verify that, for all $\nu = 1, 2, \dots, N$ and almost every $t \in [a, b]$, the matrix $\Sigma R_{\nu}(t) \Sigma$ in the left-hand side of (4.3) has the form

$$\Sigma R_{\nu}(t) \Sigma = \begin{pmatrix} r_{11}^{(\nu)}(t) & \sigma_1 \sigma_2 r_{11}^{(\nu)}(t) & \dots & \sigma_1 \sigma_n r_{1n}^{(\nu)}(t) \\ \sigma_2 \sigma_1 r_{21}^{(\nu)}(t) & r_{11}^{(\nu)}(t) & \dots & \sigma_2 \sigma_n r_{2n}^{(\nu)}(t) \\ \dots & \dots & \dots & \dots \\ \sigma_n \sigma_1 r_{n1}^{(\nu)}(t) & \sigma_n \sigma_2 r_{n2}^{(\nu)}(t) & \dots & r_{nn}^{(\nu)}(t) \end{pmatrix},$$

and, therefore, (4.3) means nothing but the fulfilment of (4.19) for almost every $t \in [a, b]$ and all k and j from 1 to n . $\square$

The following statement summarises the conditions established by the propositions above.

Proposition 4.5. Assume that the matrix-valued functions $R_v : [a, b] \rightarrow \text{GL}_n(\mathbb{R})$, $v = 0, 1, \dots, N$, satisfy condition (4.3) with respect to some matrix (4.2) with components $\{\sigma_1, \sigma_2, \dots, \sigma_n\} \subset \{-1, 1\}$. Moreover, let there exist some constants $\gamma \in [0, 1)$ and $\alpha \in [1, +\infty)$ and vector $g \in \mathbb{R}^n$ with property (4.7) such that one of conditions (4.8), (4.10), and (4.11) is true.

Then the spectral radius of operator (4.1) in $C_\tau([a, b], \mathbb{R}^n)$ admits the estimate

$$r(B) < 1.$$

PROOF. One should apply Proposition 4.2, 4.3, or 4.4, depending on the conditions assumed. $\square$

Combining Proposition 4.5 with with Lemma 4.1, we arrive at

Theorem 4.1. Under conditions of Proposition 4.5, the initial value problem (4.12), (4.13) has a unique solution $u(\cdot)$ for any $q \in L_1([a, b], \mathbb{R}^n)$ and $c \in \mathbb{R}^n$. Moreover, if q and c possess the property

$$\int_{\tau}^t q(s) ds \geq_{\Sigma} -c, \quad t \in [a, b], \quad (4.20)$$

then the solution $u(\cdot)$ of problem (4.12), (4.13) satisfies condition (4.16).

PROOF. The validity of the first assertion of the theorem is a consequence of Proposition 4.5 and Lemma 4.1. The second assertion follows from Lemma 4.2. $\square$

For systems (4.13) with coefficients not satisfying conditions of type (4.3), the following somewhat weaker statement is true.

Theorem 4.2. Assume that, for some constants $\gamma \in [0, 1)$ and $\alpha \in [1, +\infty)$ and vector $g = (g_k)_{k=1}^n \in \mathbb{R}^n$ with strictly positive components, one of the following conditions is satisfied:[§]

$$\sum_{v=1}^N \int_{\tau}^t |R_v(s)| g |\omega_v(s) - \tau|^{\alpha} ds \cdot \text{sign}(t - \tau) \leq \gamma |t - \tau|^{\alpha} g, \quad t \in [a, b], \quad (4.21)$$

$$\begin{aligned} \sum_{v=1}^N \text{sign}(\omega_v(t) - \tau) |R_v(t)| \sum_{i=1}^N \int_{\tau}^{\omega_v(t)} |R_i(s)| g |\omega_i(s) - \tau|^{\alpha} ds \\ \leq \gamma \alpha |t - \tau|^{\alpha-1} g \quad \text{for a. e. } t \in [a, b], \end{aligned} \quad (4.22)$$

and

$$\sum_{v=1}^N |R_v(t)| g |\omega_v(t) - \tau|^{\alpha} \leq \gamma \alpha |t - \tau|^{\alpha-1} g \quad \text{for a. e. } t \in [a, b]. \quad (4.23)$$

[§]In (4.21) and similar relations below, we use the notation $|M| = (|m_{kj}|)_{k,j=1}^n$ for a matrix $M = (m_{kj})_{k,j=1}^n$.

Then the initial value problem (4.12), (4.13) has a unique solution $u(\cdot)$ for any $q \in L_1([a, b], \mathbb{R}^n)$ and $c \in \mathbb{R}^n$.

It should be noted that, in contrast to Theorem 4.1, the solution u of problem (4.12), (4.13) may not possess property (4.16) under assumptions of Theorem 4.2 even if q and c satisfy condition (4.20).

4.3. Auxiliary statements and the proof of Theorem 4.2. We need the following lemma.

Lemma 4.4. Let $A : C_\tau([a, b], \mathbb{R}^n) \rightarrow C_\tau([a, b], \mathbb{R}^n)$ be the operator defined by the formula

$$(Au)(t) := \sum_{i=1}^N \int_{\tau}^t |R_i(s)| \operatorname{sign}(s - \tau) u(\omega_i(s)) ds, \quad t \in [a, b]. \quad (4.24)$$

Then the componentwise estimate

$$-(Au)(t) \leq (Bu)(t) \leq (Au)(t), \quad t \in [a, b], \quad (4.25)$$

is true for an arbitrary continuous function $u : [a, b] \rightarrow \mathbb{R}^n$ possessing the properties

$$u(\tau) = 0; \quad u(t) \geq 0, \quad t \in [a, b]. \quad (4.26)$$

Recall that the operator $B : C_\tau([a, b], \mathbb{R}^n) \rightarrow C_\tau([a, b], \mathbb{R}^n)$ is determined by the matrix-valued functions $R_i : [a, b] \rightarrow \operatorname{GL}_n(\mathbb{R})$, $i = 1, 2, \dots, N$, according to formula (4.1).

PROOF OF LEMMA 4.4. Indeed, let u satisfy (4.26). Then, according to (4.1), we have

$$-\sum_{i=1}^N \int_{\tau}^t |R_i(s)| u(\omega_i(s)) ds \leq (Bu)(t) \leq \sum_{i=1}^N \int_{\tau}^t |R_i(s)| u(\omega_i(s)) ds$$

if $t \geq \tau$, and, similarly,

$$\sum_{i=1}^N \int_{\tau}^t |R_i(s)| u(\omega_i(s)) ds \leq (Bu)(t) \leq -\sum_{i=1}^N \int_{\tau}^t |R_i(s)| u(\omega_i(s)) ds$$

if $t \leq \tau$. This, in view of (4.24), leads us to (4.25). $\square$

The following statement is, in fact, Theorem 5.3 from [5].

Theorem 4.3. Let K be a normal and reproducing cone in a real Banach space X and A and B be linear operators in X such that

$$\{Au + Bu, Au - Bu\} \subseteq K \quad \text{for an arbitrary } u \in K. \quad (4.27)$$

Then the relation

$$r(B) \leq r(A) \quad (4.28)$$

is true for the spectral radii of A and B .

Remark 4.2. Theorem 5.3 from [5] contains the assumption that the mapping A should leave the cone K invariant, i. e.,

$$A(K) \subseteq K. \quad (4.29)$$

It is easy to see that inclusion (4.29) follows immediately from condition (4.27) and, thus, can be omitted in the formulation.

Now we are in position to prove Theorem 4.2.

PROOF OF THEOREM 4.2. Let us set

$$X := C_\tau([a, b], \mathbb{R}^n) \quad (4.30)$$

and

$$K := \{u = (u_k)_{k=1}^n \in C_\tau([a, b], \mathbb{R}^n) \mid u_k(t) \geq 0 \text{ for all } k = 1, 2, \dots, n \text{ and } t \in [a, b]\}. \quad (4.31)$$

Obviously, set (4.30) (see notation (4), Section 2) is a Banach space with respect to the uniform norm (2.1). It is also easy to verify that set (4.31) is a cone in X and, moreover, this cone is normal and reproducing (see, e. g., [5], §§ 3.2 and 3.3, for the terminology).

It follows from Lemma 4.4 that relation (4.25) holds for the operators A and B given by formulae (4.1) and (4.24), respectively. Taking (4.31) into account, we conclude that the operators mentioned have property (4.27), and therefore Theorem 4.3 ensures the validity of estimate (4.28).

Relations (4.21) and (4.22), as is easy to see, can be rewritten as

$$\sum_{v=1}^N \int_{\tau}^t |R_v(s)| \operatorname{sign}(s - \tau) g |\omega_v(s) - \tau|^\alpha ds \leq \gamma |t - \tau|^\alpha g, \quad t \in [a, b], \quad (4.32)$$

and

$$\begin{aligned} \sum_{v=1}^N |R_v(t)| \sum_{i=1}^N \int_{\tau}^{\omega_v(t)} |R_i(s)| \operatorname{sign}(s - \tau) g |\omega_i(s) - \tau|^\alpha ds \leq \\ \leq \gamma \alpha |t - \tau|^{\alpha-1} g, \quad t \in [a, b], \end{aligned} \quad (4.33)$$

respectively. Therefore, assumptions (4.21), (4.22), and (4.23) guarantee the fulfilment of the corresponding relations (4.8), (4.10), and (4.11) with Σ equal to the identity matrix and R_i , $i = 1, 2, \dots, N$, replaced by the functions

$$[a, b] \ni t \mapsto |R_i(t)| \operatorname{sign}(t - \tau), \quad i = 1, 2, \dots, N.$$

Applying Proposition 4.4 to the operator $A : C_\tau([a, b], \mathbb{R}^n) \rightarrow C_\tau([a, b], \mathbb{R}^n)$ given by the formula (4.24), we conclude that its spectral radius admits the estimate

$$r(A) < 1. \quad (4.34)$$

In view of (4.28), inequality (4.34) implies that $r(B) < 1$. Now, similarly to Theorem 4.1, it remains to apply Lemma 4.1. $\square$

5. PROOFS OF THE RESULTS OF SECTION 3

Let us put $N := n^2$ and, for any $v = 1, 2, \dots, N$,

$$k_v := \left\lfloor \frac{v-1}{n} \right\rfloor + 1 \quad (5.1)$$

and

$$j_v := v - n \left\lfloor \frac{v-1}{n} \right\rfloor, \quad (5.2)$$

where the symbol $\lfloor \cdot \rfloor$ stands for the integer part of a number. Furthermore, define the functions $\omega_v : [a, b] \rightarrow [a, b]$ and $R_v : [a, b] \rightarrow \text{GL}_n(\mathbb{R})$, $v = 1, 2, \dots, n^2$, by setting

$$\omega_v(t) := \omega_{k_v j_v}(t) \quad (5.3)$$

and

$$R_v(t) := r_{k_v j_v}(t) \Pi_v, \quad (5.4)$$

where $v = 1, 2, \dots, n^2$, $t \in [a, b]$, and Π_v is the n -dimensional matrix the only non-zero element of which is equal to 1 and located at row k_v and column j_v , $v = 1, 2, \dots, n^2$; in other words,

$$\Pi_v = \begin{pmatrix} \delta_{1,k_v} \delta_{1,j_v} & \delta_{1,k_v} \delta_{2,j_v} & \dots & \delta_{1,k_v} \delta_{n,j_v} \\ \delta_{2,k_v} \delta_{1,j_v} & \delta_{2,k_v} \delta_{2,j_v} & \dots & \delta_{2,k_v} \delta_{n,j_v} \\ \dots & \dots & \dots & \dots \\ \delta_{n,k_v} \delta_{1,j_v} & \delta_{n,k_v} \delta_{2,j_v} & \dots & \delta_{n,k_v} \delta_{n,j_v} \end{pmatrix}, \quad (5.5)$$

where $\delta_{k,j}$, $k, j = 1, 2, \dots, n$, is the Kronecker symbol.

5.1. Lemmata. Let us establish several auxiliary statements.

Lemma 5.1. *For any v from 1 to n^2 , the equality*

$$(k_v - 1)n + j_v = v \quad (5.6)$$

is true. Furthermore,

$$\{k_v \mid v = 1, 2, \dots, n^2\} = \{1, 2, \dots, n\} \quad (5.7)$$

and

$$\{j_v \mid v = 1, 2, \dots, n^2\} = \{1, 2, \dots, n\}. \quad (5.8)$$

PROOF. Equality (5.6) is obvious from formulae (5.1) and (5.2). To establish (5.7) and (5.8), one should note that (5.1) can be rewritten as the system of equalities

$$\begin{aligned} k_1 &= k_2 = \dots = k_n = 1, \\ k_{n+1} &= k_{n+2} = \dots = k_{2n} = 2, \\ k_{2n+1} &= k_{2n+2} = \dots = k_{3n} = 3, \\ &\dots\dots\dots \\ k_{(n-1)n+1} &= k_{(n-1)n+2} = \dots = k_{n^2} = n, \end{aligned} \quad (5.9)$$

and, similarly, relations (5.2) mean that

$$\begin{aligned} j_1 &= j_{n+1} = j_{2n+1} = \dots = j_{(n-1)n+1} = 1, \\ j_2 &= j_{n+2} = j_{2n+2} = \dots = j_{(n-1)n+2} = 2, \\ j_3 &= j_{n+3} = j_{2n+3} = \dots = j_{(n-1)n+3} = 3, \\ &\dots\dots\dots \\ j_n &= j_{2n} = j_{3n} = \dots = j_{n^2} = n. \end{aligned} \quad (5.10)$$

Relations (5.9) and (5.10) lead us to the required equalities (5.7) and (5.8). $\square$

Lemma 5.2. *A number $v \in \{1, 2, \dots, n^2\}$ satisfies the condition $k_v = k$, where $k \in \{1, 2, \dots, n\}$, if, and only if it satisfies the inequality*

$$1 \leq v - (k-1)n \leq n, \quad (5.11)$$

in which case

$$j_v = v - (k-1)n + 1. \quad (5.12)$$

PROOF. Taking (5.9) and (5.10) into account, we conclude that condition (5.14) is necessary and sufficient for the fulfilment of the equality $k = k_v$. Equality (5.12) is an immediate consequence of relation (5.6). $\square$

For all k from 1 to n , put

$$e_k := \text{col}(\underbrace{0, 0, \dots, 0}_{k-1}, 1, 0, \dots, 0).$$

Lemma 5.3. *Let $\{m_{kj} \mid k, j = 1, 2, \dots, n\} \subset \mathbb{R}$ be arbitrary constants and $v = (v_j)_{j=1}^n$ be an arbitrary vector. Then the equality*

$$\sum_{v=1}^{n^2} m_{k_v j_v} e_k^* \Pi_v v = \sum_{j=1}^n m_{kj} v_j \quad (5.13)$$

is true for any k from 1 to n .

Recall that the $n \times n$ matrices Π_v , $v = 1, 2, \dots, n^2$, appearing in equality (5.13) are defined by formula (5.5). The asterisk in (5.13) and similar formulae below marks the transposed vector.

PROOF OF LEMMA 5.3. In view of (5.5), for any v from 1 to n^2 , we have

$$e_k^* \Pi_v v = \sum_{j=1}^n \delta_{k,k_v} \delta_{j,j_v} = \delta_{k,k_v} v_{j_v} = \begin{cases} v_{j_v} & \text{if } k = k_v, \\ 0 & \text{in the contrary case.} \end{cases}$$

By virtue of Lemma 5.2, an integer v satisfies the equality $k_v = k$ if, and only if it is represented in the form

$$v = (k-1)n + p, \quad (5.14)$$

where $1 \leq p \leq n$. Therefore, for any v and k , we have

$$m_{k_v j_v} e_k^* \Pi_v v = \begin{cases} m_{k,p+1} v_{p+1} & \text{if } v \text{ satisfies (5.14) and } 1 \leq p \leq n, \\ 0 & \text{in the contrary case.} \end{cases} \quad (5.15)$$

The last relation yields the equality

$$\sum_{v=1}^{n^2} m_{k_v j_v} e_k^* \Pi_v v = \sum_{p=1}^n m_{k,p} v_p,$$

which coincides with (5.13). $\square$

The following lemma establishes some relations for the functions $R_v : [a, b] \rightarrow \text{GL}_n(\mathbb{R})$, $v = 1, 2, \dots, n^2$, defined by formula (5.4).

Lemma 5.4. *Let $v = (v_j)_{j=1}^n \in \mathbb{R}^n$ be an arbitrary vector and ϕ_v , $v = 1, 2, \dots, n^2$, be arbitrary constants. Then the equalities*

$$e_k^* \sum_{v=1}^{n^2} \phi_v R_v(t) v = \sum_{j=1}^n \phi_{(k-1)n+j} r_{kj}(t) v_j \quad (5.16)$$

and

$$e_k^* \sum_{v=1}^{n^2} \phi_v |R_v(t)| v = \sum_{j=1}^n \phi_{(k-1)n+j} |r_{kj}(t)| v_j \quad (5.17)$$

are true for any k from 1 to n and almost every $t \in [a, b]$.

PROOF. Application of Lemma 5.3 with

$$m_{kj} := r_{kj}(t) \phi_{(k-1)n+j},$$

where $t \in [a, b]$ and $k, j = 1, 2, \dots, n$, yields

$$\sum_{v=1}^{n^2} \phi_{(k_v-1)n+j_v} r_{k_v j_v}(t) e_k^* \Pi_v v = \sum_{j=1}^n \phi_{(k-1)n+j} r_{kj}(t) v_j. \quad (5.18)$$

Taking equality (5.6) into account, one can rewrite (5.18) in the form

$$\sum_{v=1}^{n^2} \phi_v r_{k_v j_v}(t) e_k^* \Pi_v v = \sum_{j=1}^n \phi_{(k-1)n+j} r_{kj}(t) v_j. \quad (5.19)$$

It follows from (5.4) that $r_{k_{\nu}j_{\nu}}(t)\Pi_{\nu} = R_{\nu}(t)$ for any ν and t , and, hence (5.19) means that (5.16) is satisfied. Relation (5.17) is proved similarly. $\square$

Lemma 5.5. *Let $v^{(\nu)} = (v_j^{(\nu)})_{j=1}^n$, $\nu = 1, 2, \dots, n^2$, be arbitrary n -dimensional vectors. Then the equalities*

$$e_k^* \sum_{\nu=1}^{n^2} R_{\nu}(t) v^{(\nu)} = \sum_{j=1}^n v_j^{((k-1)n+j)} r_{kj}(t) \quad (5.20)$$

and

$$e_k^* \sum_{\nu=1}^{n^2} |R_{\nu}(t)| v^{(\nu)} = \sum_{j=1}^n v_j^{((k-1)n+j)} |r_{kj}(t)| \quad (5.21)$$

are true for any k from 1 to n and almost every $t \in [a, b]$.

PROOF. In view of the equality

$$v^{(\nu)} = \sum_{i=1}^n v_i^{(\nu)} e_i,$$

for almost every $t \in [a, b]$ and all $k = 1, 2, \dots, n$, we have

$$\begin{aligned} e_k^* \sum_{\nu=1}^{n^2} R_{\nu}(t) v^{(\nu)} &= e_k^* \sum_{\nu=1}^{n^2} R_{\nu}(t) \sum_{i=1}^n v_i^{(\nu)} e_i \\ &= \sum_{i=1}^n \left(e_k^* \sum_{\nu=1}^{n^2} v_i^{(\nu)} R_{\nu}(t) e_i \right). \end{aligned} \quad (5.22)$$

In view of Lemma 5.4 applied with $v := e_i$ and $\phi_{\nu} := v_i^{(\nu)}$, $\nu = 1, 2, \dots, n^2$, equality (5.22) yields

$$\begin{aligned} e_k^* \sum_{\nu=1}^{n^2} R_{\nu}(t) v^{(\nu)} &= \sum_{i=1}^n \left(\sum_{j=1}^n v_i^{((k-1)n+j)} r_{kj}(t) \delta_{i,j} \right) \\ &= \sum_{j=1}^n \left(\sum_{i=1}^n v_i^{((k-1)n+j)} \delta_{i,j} \right) r_{kj}(t) \\ &= \sum_{j=1}^n v_j^{((k-1)n+j)} r_{kj}(t) \end{aligned}$$

for all k from 1 to n and almost every $t \in [a, b]$, which proves (5.20). Relation (5.21) is established by analogy. $\square$

Lemma 5.6. *The equality*

$$\omega_{(k-1)n+j}(t) = \omega_{kj}(t) \quad (5.23)$$

is satisfied for almost every $t \in [a, b]$ and all k and j from 1 to n .

PROOF. Let $\{k, j\} \subset \{1, 2, \dots, n\}$ be arbitrary. Then, obviously, $k = k_v$ and $j = j_v$ for $v := (k-1)n + j$. Applying the equality (5.6) of Lemma 5.1 and using the definition (5.3) of the functions ω_v , $v = 1, 2, \dots, n^2$, we obtain that

$$\omega_{(k_v-1)n+j_v}(t) = \omega_v(t) = \omega_{k_v j_v}(t)$$

for almost every $t \in [a, b]$. This yields the required property (5.23). $\square$

Lemma 5.7. *Problem (1.2), (1.3) is equivalent to problem (4.12), (4.13), where $N = n^2$ and $\omega_v : [a, b] \rightarrow [a, b]$ and $R_v : [a, b] \rightarrow \text{GL}_n(\mathbb{R})$, $v = 1, 2, \dots, N$, are defined by formulae (5.3) and (5.4), respectively.*

PROOF. Let $u = (u_k)_{k=1}^n : [a, b] \rightarrow \mathbb{R}^n$ be an arbitrary continuous vector-function. Applying formula (5.20) of Lemma 5.5 with $v^{(v)} := u(\omega_v(t))$, $v = 1, 2, \dots, n^2$, for any k from 1 to n , we obtain

$$e_k^* \sum_{v=1}^{n^2} R_v(t) u(\omega_v(t)) = \sum_{j=1}^n u_j(\omega_{(k-1)n+j}(t)) r_{kj}(t)$$

and therefore, by Lemma 5.6,

$$e_k^* \sum_{v=1}^{n^2} R_v(t) u(\omega_v(t)) = \sum_{j=1}^n r_{kj}(t) u_j(\omega_{kj}(t)) \quad (5.24)$$

for almost every t from $[a, b]$. Comparing the right-hand side member of (5.24) with that of (1.2) and taking the arbitrariness of u and k into account, we arrive at the desired conclusion. $\square$

Lemma 5.8. *Assume that the functions $r_{kj} : [a, b] \rightarrow \mathbb{R}$, $k, j = 1, 2, \dots, n$, satisfy relations (3.4) for almost every $t \in [a, b]$ and all $k, j = 1, 2, \dots, n$. Then the corresponding functions $R_v : [a, b] \rightarrow \text{GL}_n(\mathbb{R})$, $v = 1, 2, \dots, n^2$, given by formula (5.4) satisfy condition (4.3) with respect to matrix (4.2).*

PROOF. Indeed, according to Lemma 4.3, condition (4.3) is satisfied if, and only if the components $r_{kj}^{(v)} : [a, b] \rightarrow \mathbb{R}$, $k, j = 1, 2, \dots, n$, of the matrices $R_v : [a, b] \rightarrow \text{GL}_n(\mathbb{R})$, $v = 1, 2, \dots, n^2$, satisfy inequalities (4.19) almost everywhere on $[a, b]$. By virtue of (5.4), for any $v = 1, 2, \dots, n^2$ and almost every $t \in [a, b]$, we have

$$r_{kj}^{(v)}(t) = \begin{cases} r_{k_v j_v}(t) & \text{if } k = k_v \text{ and } j = j_v, \\ 0 & \text{in the contrary case.} \end{cases} \quad (5.25)$$

In view of (5.25), the fulfilment of all the inequalities (4.19) means that

$$\sigma_{k_v} \sigma_{j_v} r_{k_v j_v}(t) \text{sign}(t - \tau) \geq 0 \quad (5.26)$$

for all $v = 1, 2, \dots, n^2$ and almost every $t \in [a, b]$. However, it follows from Lemma 5.1 that equalities (5.7) and (5.8) are true and, therefore, the fulfilment of

conditions (3.4) for all $k, j = 1, 2, \dots, n$ and almost every $t \in [a, b]$ ensures that relation (5.26) is satisfied for all $v = 1, 2, \dots, n^2$ and almost every $t \in [a, b]$. We have thus shown that functions (5.4) satisfy condition (4.3). $\square$

5.2. Proof of Theorem 3.1. Let us show that, under the conditions assumed, the required assertion for the Cauchy problem (1.2), (1.3) follows from Theorem 4.2 applied to the corresponding problem (4.12), (4.13).

Case 1: *Condition (3.1a) holds.* Assumption (3.1a) ensures that the functions $\omega_v : [a, b] \rightarrow [a, b]$ and $R_v : [a, b] \rightarrow \text{GL}_n(\mathbb{R})$, $v = 1, 2, \dots, n^2$, given by formulae (5.3) and (5.4) satisfy condition (4.23) of Theorem 4.2 with

$$g := \text{col}(\gamma_1, \gamma_2, \dots, \gamma_n) \quad (5.27)$$

and

$$\gamma := \frac{1}{\alpha} \max_{k=1,2,\dots,n} \text{vrai max}_{t \in [a,b] \setminus \{\tau\}} \frac{1}{\gamma_k |t - \tau|^{\alpha-1}} \sum_{j=1}^n \gamma_j |r_{kj}(t)| |\omega_j(t) - \tau|^\alpha. \quad (5.28)$$

Indeed, let us put $v_j = \gamma_j$, $j = 1, 2, \dots, n$. Applying equality (5.17) of Lemma 5.4 and taking formulae (5.3) and (5.12) into account, we obtain that

$$e_k^* \sum_{v=1}^{n^2} \omega_v(t) |R_v(t)| g = \sum_{j=1}^n \omega_{(k-1)n+j}(t) |r_{kj}(t)| \gamma_j \quad (5.29)$$

for almost every $t \in [a, b]$ and all k from 1 to n . By virtue of Lemma 5.6, relation (5.29) can be brought to the form

$$e_k^* \sum_{v=1}^{n^2} \omega_v(t) |R_v(t)| g = \sum_{j=1}^n \omega_{kj}(t) |r_{kj}(t)| \gamma_j.$$

Therefore, property (4.23) in our case is equivalent to the fulfilment of the inequality

$$\sum_{j=1}^n |r_{kj}(t)| \gamma_j |\omega_{kj}(t) - \tau|^\alpha \leq \gamma \alpha \gamma_k |t - \tau|^{\alpha-1}$$

for almost every $t \in [a, b]$ and all k from 1 to n . However, this condition is satisfied in view of (3.1a) and (5.28).

Case 2: *Condition (3.1b) holds.* In this case, condition (4.21) of Theorem 4.2 is satisfied. Indeed, let us define the vector g by formula (5.27), where $\gamma_1, \gamma_2, \dots, \gamma_n$ are the positive constants appearing in (3.1b). Then condition (4.21) is equivalent to the system of relations

$$\sum_{v=1}^N \int_{\tau}^t e_k^* |R_v(s)| g |\omega_v(s) - \tau|^\alpha ds \cdot \text{sign}(t - \tau) \leq \gamma |t - \tau|^\alpha \gamma_k \quad (5.30)$$

valid for all $k = 1, 2, \dots, n$ and $t \in [a, b]$.

Applying equality (5.17) of Lemma 5.4 with $v_j := \gamma_j$, $j = 1, 2, \dots, n$, and $\phi_v := |\omega_v(s) - \tau|^\alpha$, $v = 1, 2, \dots, n^2$, and using Lemma 5.6, we show that the equalities

$$\begin{aligned} \sum_{v=1}^N e_k^* |\omega_v(s) - \tau|^\alpha |R_v(s)| g &= \sum_{j=1}^n |\omega_{(k-1)n+j}(s) - \tau|^\alpha |r_{kj}(s)| \gamma_j \\ &= \sum_{j=1}^n |\omega_{kj}(s) - \tau|^\alpha |r_{kj}(s)| \gamma_j \end{aligned}$$

are satisfied for any $k = 1, 2, \dots, n$ and almost every $s \in [a, b]$. Therefore, relation (5.30) in our case means that, for all $k = 1, 2, \dots, n$ and $t \in [a, b]$, the inequality

$$\sum_{j=1}^n \gamma_j \int_{\tau}^t |r_{kj}(s)| |\omega_{kj}(s) - \tau|^\alpha ds \cdot \text{sign}(t - \tau) \leq \gamma \gamma_k |t - \tau|^\alpha. \quad (5.31)$$

is true. Setting now

$$\gamma := \max_{k=1,2,\dots,n} \sup_{t \in [a,b] \setminus \{\tau\}} \frac{\text{sign}(t - \tau)}{\gamma_k |t - \tau|^\alpha} \sum_{j=1}^n \gamma_j \int_{\tau}^t |r_{kj}(s)| |\omega_{kj}(s) - \tau|^\alpha ds$$

and taking assumption (3.1b) into account, we conclude that (5.31) is satisfied for all $t \in [a, b]$ and k from 1 to n . As we have already seen above, this means that relation (4.21) is true.

Case 3: *Condition (3.1c) holds.* Let us show that, in this case, condition (4.22) is satisfied with the vector g defined by formula (5.27) and

$$\begin{aligned} \gamma &:= \frac{1}{\alpha} \max_{k=1,2,\dots,n} \text{vrai max}_{t \in [a,b] \setminus \{\tau\}} \sum_{l=1}^n \frac{\text{sign}(\omega_{kl}(t) - \tau)}{\gamma_k |t - \tau|^{\alpha-1}} |r_{kl}(t)| \\ &\quad \times \sum_{j=1}^n \gamma_j \int_{\tau}^{\omega_{kl}(t)} |r_{lj}(s)| |\omega_{lj}(s) - \tau|^\alpha ds. \end{aligned} \quad (5.32)$$

Indeed, in the case indicated, condition (4.22) for functions (5.3) and (5.4) means that, for all $k = 1, 2, \dots, n$ and almost every $t \in [a, b]$,

$$\begin{aligned} \sum_{v=1}^N \text{sign}(\omega_v(t) - \tau) e_k^* |R_v(t)| \sum_{i=1}^N \int_{\tau}^{\omega_v(t)} |R_i(s)| g |\omega_i(s) - \tau|^\alpha ds \\ \leq \gamma \alpha |t - \tau|^{\alpha-1} \gamma_k \end{aligned} \quad (5.33)$$

or, which is the same,

$$\sum_{v=1}^{n^2} e_k^* |R_v(t)| v^{(v)}(t) \leq \gamma \alpha |t - \tau|^{\alpha-1} \gamma_k, \quad (5.34)$$

where

$$v^{(v)}(t) := \sum_{i=1}^{n^2} \int_{\tau}^{\omega_v(t)} |R_i(s)| g |\omega_i(s) - \tau|^\alpha ds \operatorname{sign}(\omega_v(t) - \tau)$$

for any $v = 1, 2, \dots, n^2$. Applying formula (5.21) of Lemma 5.5, we obtain

$$\begin{aligned} e_k^* \sum_{v=1}^{n^2} |R_v(t)| v^{(v)}(t) &= \sum_{j=1}^n v_j^{((k-1)n+j)}(t) |r_{kj}(t)| \\ &= \sum_{j=1}^n |r_{kj}(t)| e_j^* \left(\sum_{i=1}^{n^2} \int_{\tau}^{\omega_{(k-1)n+j}(t)} |R_i(s)| g |\omega_i(s) - \tau|^\alpha ds \right) \operatorname{sign}(\omega_{(k-1)n+j}(t) - \tau), \end{aligned}$$

whence, by Lemma 5.6, it follows that

$$\begin{aligned} e_k^* \sum_{v=1}^{n^2} |R_v(t)| v^{(v)}(t) \\ = \sum_{j=1}^n |r_{kj}(t)| \int_{\tau}^{\omega_{kj}(t)} \left(e_j^* \sum_{i=1}^{n^2} |R_i(s)| g |\omega_i(s) - \tau|^\alpha \right) ds \operatorname{sign}(\omega_{kj}(t) - \tau) \quad (5.35) \end{aligned}$$

for almost every t from $[a, b]$. On the other hand, by virtue of formula (5.27) and Lemmata 5.4 and 5.6, the equality

$$\begin{aligned} e_j^* \sum_{i=1}^{n^2} |R_i(s)| g |\omega_i(s) - \tau|^\alpha &= \sum_{l=1}^n \gamma_l |r_{jl}(s)| |\omega_{(j-1)n+l}(s) - \tau|^\alpha \\ &= \sum_{l=1}^n \gamma_l |r_{jl}(s)| |\omega_{jl}(s) - \tau|^\alpha \quad (5.36) \end{aligned}$$

is true for almost every $s \in [a, b]$ and all $j = 1, 2, \dots, n$. Using (5.36) and (5.35), we obtain that

$$\begin{aligned} e_k^* \sum_{v=1}^{n^2} |R_v(t)| v^{(v)}(t) \\ = \sum_{j=1}^n |r_{kj}(t)| \left(\sum_{l=1}^n \gamma_l \int_{\tau}^{\omega_{kj}(t)} |r_{jl}(s)| |\omega_{jl}(s) - \tau|^\alpha ds \right) \operatorname{sign}(\omega_{kj}(t) - \tau) \quad (5.37) \end{aligned}$$

for all $k = 1, 2, \dots, n$ and almost every $t \in [a, b]$. Substituting (5.37) into (5.34), we bring the latter relation to the form

$$\sum_{j=1}^n |r_{kj}(t)| \left(\sum_{l=1}^n \gamma_l \int_{\tau}^{\omega_{kj}(t)} |r_{jl}(s)| |\omega_{jl}(s) - \tau|^{\alpha} ds \right) \text{sign}(\omega_{kj}(t) - \tau) \leq \gamma \alpha |t - \tau|^{\alpha-1} \gamma_k. \quad (5.38)$$

Taking (5.32) into account, we conclude that (5.38) is satisfied for all $k = 1, 2, \dots, n$ and almost every $t \in [a, b]$ in view of condition (3.1c).

We have thus shown that, under the conditions assumed, Theorem 4.2 can be applied to problem (4.12), (4.13) with $N = n^2$ and the functions ω_v and R_v , $v = 1, 2, \dots, n^2$, given by formulae (5.3) and (5.4). Using the theorem mentioned and taking Lemma 5.7 into account, we arrive at the desired assertion on the unique solvability of the original Cauchy problem (1.2), (1.3).

5.3. Proof of Theorem 3.2. We shall show that the assertion of Theorem 3.2 for the Cauchy problem (1.2), (1.3) can be obtained from Theorem 4.1 applied to the corresponding problem (4.12), (4.13).

Indeed, by virtue of assumption (3.4) and Lemma 5.8, the functions $\omega_v : [a, b] \rightarrow [a, b]$ and $R_v : [a, b] \rightarrow \text{GL}_n(\mathbb{R})$, $v = 1, 2, \dots, n^2$, defined by formulae (5.3) and (5.4), satisfy condition (4.3) with Σ given by (4.2). Therefore, it is sufficient to show that, under our assumptions on r_{kj} and ω_{kj} , $k, j = 1, 2, \dots, n$, some of conditions (4.8), (4.10), and (4.11) are satisfied.

Let us put

$$g := \text{col}(\sigma_1 \gamma_1, \sigma_2 \gamma_2, \dots, \sigma_n \gamma_n), \quad (5.39)$$

where $\gamma_1, \gamma_2, \dots, \gamma_n$ are the positive constants appearing in conditions (3.5a), (3.5b), and (3.5c). Obviously, vector (5.39) possesses property (4.7).

Consider the following three cases.

Case 1: *Condition (3.5a) holds.* Let us prove that, under assumption (3.5a), the functions ω_v and R_v , $v = 1, 2, \dots, n^2$, in the corresponding system (4.13) satisfy condition (4.11) of Theorem 4.1.

Indeed, applying formula (5.16) of Lemma 5.4, we show that condition (4.11) is equivalent to the fulfilment of the relation

$$\begin{aligned} \sigma_k \sum_{j=1}^n \sigma_j \gamma_j r_{kj}(t) |\omega_{(k-1)n+j}(t) - \tau|^{\alpha} \text{sign}(t - \tau) &\leq \sigma_k \gamma \sigma_k \gamma_k \alpha |t - \tau|^{\alpha} \\ &= \gamma \gamma_k \alpha |t - \tau|^{\alpha} \end{aligned} \quad (5.40)$$

for all $k = 1, 2, \dots, n$ and almost every $t \in [a, b]$. By Lemma 5.6, relation (5.40) means that

$$\sigma_k \sum_{j=1}^n \sigma_j \gamma_j r_{kj}(t) |\omega_{kj}(t) - \tau|^\alpha \operatorname{sign}(t - \tau) \leq \gamma_k \gamma \alpha |t - \tau|^\alpha \quad (5.41)$$

for all $k = 1, 2, \dots, n$ and almost every $t \in [a, b]$. However, the existence of a constant $\gamma \in [0, 1)$ such that (5.41) holds for all $k = 1, 2, \dots, n$ and almost every $t \in [a, b]$ is a consequence of assumption (3.5a) because one can put

$$\gamma := \max_{k=1,2,\dots,n} \operatorname{vrai} \max_{t \in [a,b] \setminus \{\tau\}} \frac{\sigma_k \operatorname{sign}(t - \tau)}{\alpha \gamma_k |t - \tau|^{\alpha-1}} \sum_{j=1}^n \sigma_j \gamma_j r_{kj}(t) |\omega_{kj}(t) - \tau|^\alpha.$$

Case 2: *Condition (3.5b) holds.* Let us show that functions (5.3) and (5.4) in this case satisfy condition (4.8).

It follows from Lemma 5.4 that condition (4.8) with g given by formula (5.39) means that

$$\sigma_k \sum_{j=1}^n \int_{\tau}^t \sigma_j \gamma_j |\omega_{(k-1)n+j}(s) - \tau|^\alpha r_{kj}(s) ds \leq \gamma |t - \tau|^\alpha \gamma_k,$$

or, equivalently (see Lemma 5.6)

$$\sigma_k \sum_{j=1}^n \sigma_j \gamma_j \int_{\tau}^t |\omega_{kj}(s) - \tau|^\alpha r_{kj}(s) ds \leq \gamma_k \gamma |t - \tau|^\alpha \quad (5.42)$$

for all $t \in [a, b]$ and k from 1 to n . A constant $\gamma \in [0, 1)$ such that (5.42) holds for all k and t exists due to assumption (3.5b).

Case 3: *Condition (3.5c) holds.* Assumption (3.5c) ensures that the functions ω_v and R_v , $v = 1, 2, \dots, n^2$, in the corresponding system (4.13) satisfy condition (4.10) of Theorem 4.1.

Indeed, let us define the vector g by formula (5.39). Condition (4.10) for functions (5.3) and (5.4) in our case means that, for all $k = 1, 2, \dots, n$ and almost every $t \in [a, b]$,

$$\begin{aligned} \sigma_k \operatorname{sign}(t - \tau) \sum_{v=1}^N e_k^* R_v(t) \sum_{i=1}^N \int_{\tau}^{\omega_v(t)} R_i(s) g |\omega_i(s) - \tau|^\alpha ds \\ \leq \gamma \gamma_k \alpha |t - \tau|^{\alpha-1} \end{aligned} \quad (5.43)$$

or, which is the same,

$$\sigma_k \operatorname{sign}(t - \tau) \sum_{v=1}^{n^2} e_k^* R_v(t) w^{(v)}(t) \leq \gamma \gamma_k \alpha |t - \tau|^{\alpha-1}, \quad (5.44)$$

where

$$w^{(v)}(t) := \sum_{i=1}^{n^2} \int_{\tau}^{\omega_v(t)} R_i(s) g |\omega_i(s) - \tau|^\alpha ds$$

for any $v = 1, 2, \dots, n^2$. Formula (5.20) of Lemma 5.5 implies that, for almost every $t \in [a, b]$ and all $k = 1, 2, \dots, n$,

$$\begin{aligned} e_k^* \sum_{v=1}^{n^2} R_v(t) w^{(v)}(t) &= \sum_{j=1}^n w_j^{((k-1)n+j)}(t) r_{kj}(t) \\ &= \sum_{j=1}^n r_{kj}(t) e_j^* \left(\sum_{i=1}^{n^2} \int_{\tau}^{\omega_{(k-1)n+j}(t)} R_i(s) g |\omega_i(s) - \tau|^\alpha ds \right), \end{aligned}$$

whence, by Lemma 5.6,

$$e_k^* \sum_{v=1}^{n^2} R_v(t) w^{(v)}(t) = \sum_{j=1}^n r_{kj}(t) \int_{\tau}^{\omega_{kj}(t)} \left(e_j^* \sum_{i=1}^{n^2} R_i(s) g |\omega_i(s) - \tau|^\alpha \right) ds. \quad (5.45)$$

However, it follows from Lemmata 5.4 and 5.6 and formula (5.39) that the equality

$$\begin{aligned} e_j^* \sum_{i=1}^{n^2} R_i(s) g |\omega_i(s) - \tau|^\alpha &= \sum_{l=1}^n \sigma_l \gamma_l r_{jl}(s) |\omega_{(j-1)n+l}(s) - \tau|^\alpha \\ &= \sum_{l=1}^n \sigma_l \gamma_l r_{jl}(s) |\omega_{jl}(s) - \tau|^\alpha \end{aligned}$$

is true for almost every $s \in [a, b]$ and all $j = 1, 2, \dots, n$, and, therefore, equality (5.45) yields

$$e_k^* \sum_{v=1}^{n^2} R_v(t) w^{(v)}(t) = \sum_{j=1}^n r_{kj}(t) \int_{\tau}^{\omega_{kj}(t)} \left(\sum_{l=1}^n \sigma_l \gamma_l r_{jl}(s) |\omega_{jl}(s) - \tau|^\alpha \right) ds. \quad (5.46)$$

In view of (5.46), relation (5.44) can be rewritten as

$$\begin{aligned} \sigma_k \operatorname{sign}(t - \tau) \sum_{j=1}^n r_{kj}(t) \int_{\tau}^{\omega_{kj}(t)} \left(\sum_{l=1}^n \sigma_l \gamma_l r_{jl}(s) |\omega_{jl}(s) - \tau|^\alpha \right) ds \\ \leq \gamma \gamma_k \alpha |t - \tau|^{\alpha-1}. \end{aligned} \quad (5.47)$$

It remains to note that the existence of a constant $\gamma \in [0, 1)$ such that (5.47) is true for almost every $t \in [a, b]$ and all $k = 1, 2, \dots, n$ is guaranteed by condition (3.5c).

Thus, in each of the three cases considered above, the conditions of imposed on problem (1.2), (1.3) in Theorem 3.2 guarantee that the corresponding problem (4.12),

(4.13) satisfies all the assumptions of Theorem 4.1. It therefore remains to use Theorem 4.1 and Lemma 5.7.

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Author's Address

Andrei Rontó:

INSTITUTE OF MATHEMATICS, NATIONAL ACADEMY OF SCIENCES OF UKRAINE, 3 TERESCHENKOVSKAYA ST., 01601 KIEV, UKRAINE

Current address: Mathematical Institute, Academy of Sciences of the Czech Republic, Žižkova 22, CZ-61662 Brno, Czech Republic

E-mail address: ronto@ipm.cz



DISCRETE CROUZEIX-VELTE DECOMPOSITION FOR THE DISK DOMAIN

GY. STRAUBER

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ABSTRACT. In this paper, a staggered grid approximation of the Stokes problem on the unit disk is investigated. A full description of the approximation and the discrete Crouzeix-Velte decomposition is given in the case of homogeneous Dirichlet boundary conditions. Further we consider the Uzawa algorithm combined with the conjugate gradient method and FFT algorithm for solving the discrete Poisson equations. Computational results show the effectiveness of this discretization.

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Keywords: Stokes problem, Crouzeix-Velte decomposition, unit disk, staggered grid approximation

1. INTRODUCTION

The Crouzeix-Velte decomposition is a decomposition into three orthogonal subspaces of the customary Sobolev space $(H_0^1(\Omega))^d$ of vector functions defined over a Lipschitzian domain $\Omega \subset \mathbb{R}^d$ which was first described in [4] and later, independently, in [18]. This decomposition is similar to the well-known Helmholtz-Weyl decomposition but different from the latter; not only by the prescribed boundary conditions (originally homogeneous Dirichlet conditions) but also by the properties of the functions in the three subspaces. E.g., along with the subspaces of divergence-free and rotation-free functions, there is a third subspace the functions of which have both nonzero divergence and rotation or are the zero function. In [18] and [5], the decomposition was used to get more information about the inf-sup constant of the Stokes problem. The optimal inf-sup constant can be characterized by the third subspace alone [12].

For the numerical solution of the Stokes problem and its connection to the inf-sup problem, see [1], [9], [2]. In [3], additional results were gained about the inf-sup constant for ring and disk domains.

If using finite difference methods for approximating the Stokes equations, and if the discrete scheme admits a discrete Crouzeix-Velte decomposition, then the same conclusions as in the continuous case can be drawn. The subspaces of the discrete

rotation-free and discrete divergence-free functions are in a similar relation as the corresponding subspaces of $(H_0^1)^n$, and there is a third orthogonal subspace consisting of discrete harmonic functions. The optimal constant of the discrete inf-sup condition can therefore be computed in this much smaller third space. These results are of interest for the iterative solution of stationary problems for the Stokes and linear elasticity equations, and for long-term calculations in the time-dependent case for the following reasons: if there exists a Crouzeix-Velte decomposition, then independently of the domain and its triangulation, there is an eigenvalue 1 of high multiplicity (of the order of inner grid points). Then, the error components lying in the eigensubspace corresponding to this eigenvalue can be removed by one step of a simple damped Jacobi iteration. The remaining error is connected only with boundary effects. Since in the two-dimensional case the spectrum is symmetric with respect to $1/2$, the optimal iteration parameters can be calculated using only a lower estimate of the inf-sup constant. Moreover, the conjugate gradient iteration automatically takes advantage of such a spectrum and converges faster than for discrete spaces without a decomposition.

In [13], the Uzawa and Arrow-Hurwitz iterations are investigated and are shown to reach the third Crouzeix-Velte subspace after at most 2 steps, and an improved convergence estimate is derived for the conjugate gradient method using the estimates of the inf-sup constant. In [16], the well-known staggered grid approximation of the Stokes problem is investigated. The existence of the discrete Crouzeix-Velte decomposition is proved in the case of a non-equidistant grid for the Shortley-Weller approximation, for a finite volume scheme and in the case of second order approximation and homogeneous Dirichlet boundary conditions. The first order staggered grid approximation in a special case is investigated in [12]. This result was generalized in [6] to general two and three-dimensional domains. In [14], the Crouzeix-Velte decomposition is generalized with respect to boundary conditions and boundary value problems. Moreover, in the discrete case, the gap between the lower-order staggered grid and finite volume approximations and the high-order Scott-Vogelius elements is filled by proving that Fortin-Soulie elements also admit such a decomposition.

The aim of the present paper is to investigate the staggered grid approximation of the Stokes problem in a polar coordinate system, and to prove the existence of the discrete Crouzeix-Velte decomposition for the unit disk, in the case of second order approximation and homogeneous Dirichlet boundary conditions. Further we present computational results using the Uzawa algorithm (as outer iteration) and solving the discrete Poisson equations by the conjugate gradient method (as inner iteration) with an effective preconditioning matrix and FFT algorithm.

The outline of the paper is as follows. In Section 2, the necessary notations and the Crouzeix-Velte decomposition in the continuous case as well as the Stokes problem in the polar coordinate system are introduced. In Section 3, the discrete Stokes

problem is described. In Section 4, the discrete Crouzeix-Velte decomposition is investigated, and it is shown that this decomposition exists for the disk. Finally, in Section 5 we show the computational results. The results show that the discretization obeying a discrete Crouzeix-Velte decomposition leads to effectively solvable systems of algebraic equations. The average number of the outer iterations using the Uzawa algorithm was 3-5, and the number of inner iterations needed to reach the stopping criterion of the conjugate gradient method, using the best preconditioning matrix, was only 2.

2. NOTATIONS

2.1. The Stokes problem on the unit disk. Let Ω be the unit disk

$$\Omega = \{(r, \varphi) | 0 \leq r < 1, 0 \leq \varphi < 2\pi\},$$

and consider the following Stokes problem:

$$\Delta_{r\varphi} u - \frac{u}{r^2} - \frac{2}{r^2} \frac{\partial v}{\partial \varphi} - \frac{\partial p}{\partial r} = f_1, \quad (2.1)$$

$$\Delta_{r\varphi} v - \frac{v}{r^2} + \frac{2}{r^2} \frac{\partial u}{\partial \varphi} - \frac{1}{r} \frac{\partial p}{\partial \varphi} = f_2, \quad (2.2)$$

$$\operatorname{div} \vec{u} = \frac{1}{r} \left(\frac{\partial}{\partial r}(ru) + \frac{\partial v}{\partial \varphi} \right) = 0, \quad (2.3)$$

where $(u, v) = \vec{u}$ and $(f_1, f_2) = \vec{f}$ and

$$\Delta_{r\varphi} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}.$$

On the boundary, homogeneous Dirichlet boundary conditions are imposed:

$$\vec{u} = 0, \text{ on } \partial\Omega, \quad (2.4)$$

The problem consists in finding a vector-function $\vec{u}(x)$ and a scalar function $p(x)$ that satisfy the system of partial differential equations above. The function $\vec{u}$ is the velocity of fluid and p is the pressure. We mention that the appearance of v in equation (2.1) and the appearance of u in equation (2.2) cause difficulty during the numerical solution of (2.1)-(2.4).

A unique weak solution $\vec{u} \in V$ and $p \in P$ exists when, for example, $\vec{f} \in (L_2(\Omega))^n$, (see, e.g., [11],[17]), where $P := L_{2,0}(\Omega)$ the subspace of $L_2(\Omega)$ of square integrable functions with zero integral over Ω and $V := (H_0^1(\Omega))^n$ with the Hilbert space

$$L_2(\Omega) = \{\phi | (\phi, \phi) < \infty\}, (\phi, \psi) = \int_{\Omega} \phi \psi d\Omega$$

and the Sobolev space

$$H_0^1(\Omega) = \{\phi | \phi \in L_2(\Omega), D^1 \phi \in L_2(\Omega), \phi = 0 \text{ on } \partial\Omega\},$$

where D^1 denotes any derivative of the first order.

For existence and uniqueness of the classical solution see [10]. For the numerical solution by finite element methods see [9].

2.2. The Crouzeix-Velte decomposition. For an $n > 2$, let $(\cdot, \cdot)$ denote the Euclidean scalar product in $\mathbb{R}^n$, moreover, let $A, B, C \in \mathbb{R}^{n \times n}$ be matrices satisfying

$$A = B + C, \quad (2.5)$$

$$A = A^T > 0, \quad (2.6)$$

$$B = B^T \geq 0, \quad C = C^T \geq 0, \quad (2.7)$$

$$\delta := \dim \ker B \geq 1, \quad \varrho := \dim \ker C \geq 1. \quad (2.8)$$

Then, with a suitable subspace $W \subset \mathbb{R}^n$ (which may turn out to be empty) and with orthogonality to be understood in the sense of the scalar product $(A \cdot, \cdot)$, the following orthogonal decomposition of $\mathbb{R}^n$ can be derived (see [15]):

$$\mathbb{R}^n = \ker B \oplus \ker C \oplus W. \quad (2.9)$$

We call this decomposition the algebraic Crouzeix-Velte decomposition of $\mathbb{R}^n$. It is of interest due to the following correspondences connected with the matrices in (2.5):

$$A \sim -\Delta, \quad B \sim -\text{grad div}, \quad C \sim \text{curl rot},$$

where Δ is the (vector) Laplace operator and where the sign $\sim$ expresses only an analogy between a differential operator and a matrix, and is not necessarily a (good) approximation. In this sense (2.5) corresponds to the well-known identity

$$-\Delta = -\text{grad div} + \text{curl rot} \quad (2.10)$$

of vector analysis.

In the discrete case, the velocity space (which approximates $(H^1(\Omega))^d$ or a subspace of the latter) will be denoted by V_h , the pressure space will be denoted by P_h , and div_h and rot_h will be written for the discrete equivalents of the divergence and rotation operator, Δ_h will denote the discrete Laplace operator. The matrix corresponding to the mapping $-\text{div}_h$ from the velocity space into the pressure space is denoted by $\tilde{B}_h$ and we introduce the following notations: $\tilde{C}_h$ for the matrix of the operator rot_h and A_h for the matrix of the operator $-\Delta_h$. If $A_h, \tilde{B}_h, \tilde{C}_h$ matrices satisfy the following:

$$\begin{aligned} A_h &= B_h + C_h, \\ A_h &= A_h^T > 0, \end{aligned} \quad (2.11)$$

where $B_h = \tilde{B}_h^T \tilde{B}_h$ and $C_h = \tilde{C}_h^T \tilde{C}_h$ and $\ker \tilde{B}_h \neq \emptyset$ and $\ker \tilde{C}_h \neq \emptyset$, then a discrete Crouzeix-Velte decomposition exists and (2.9) takes the form

$$V_h = \ker \text{div}_h \oplus \ker \text{rot}_h \oplus W = V_{h,0} \oplus V_{h,1} \oplus V_{h,\beta}.$$

P_h is decomposed similarly into three orthogonal subspaces:

$$P_h = \ker \operatorname{grad}_h \oplus \operatorname{div}_h \ker \operatorname{rot}_h \oplus \operatorname{div}_h V_{h,\beta} = P_{h,0} \oplus P_{h,1} \oplus P_{h,\beta}.$$

After discretization by the finite element or finite difference methods, the Stokes problem takes the following form ([9],[16]):

$$\begin{pmatrix} A_h & \tilde{B}_h^T \\ \tilde{B}_h & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \quad (2.12)$$

For the specific formulae of these matrices in a polar coordinate system see below Section 3. With these matrices the Schur complement is $S_h := \tilde{B}_h A_h^{-1} \tilde{B}_h^T$. For the numerical solution of the discretized Stokes problem, the eigenvalue problem for S_h is important:

$$S_h p_h = \lambda_h p_h$$

If the discrete Crouzeix-Velte decomposition exists, then $\lambda_h \in [0, 1]$ ([12]).

The analytical case of the Crouzeix-Velte decomposition for two-dimensional domains was investigated, and for the circle domain, the spectrum of the Schur complement was proved to be $0, 1/2, 1$, where $1/2$ and 1 have infinite multiplicity, in [3], [8] and [4]. For the analytical case a three-dimensional ball is described in [18]. The results of the spectrum for tube domains are contained in [7].

3. THE STAGGERED GRID APPROXIMATION ON A UNIT DISK

The difference approximation on a staggered grid will be considered here. In this case, where Ω is the unit circle subdivided by an equidistant grid according to r and φ into $n \times m$ cells. The cell midpoints are pressure nodes, the pressure vector is denoted by p_h and its components by p_{ij} , with $i = 1, \dots, n$; $j = 1, \dots, m$. The sides of the cells contain as their midpoints the velocity nodes: nodes of the u -components of the velocity are on the sides according to r , nodes of the v -components are on the sides according to φ . The velocity vector is $\vec{u}_h$, its components are denoted by u_h and v_h and their components by u_{ij} and v_{ij} , $i = 1, \dots, n$, $j = 1, \dots, m$. The boundary values are $u_{n,j} := 0$ where $j = 1, \dots, m$, further u_h and v_h are periodical with respect to φ , that is

$$u_{i,0} = u_{i,m}, \quad v_{i,0} = v_{i,m}, \quad i = 1, \dots, n. \quad (3.1)$$

For the approximation of the Stokes problem we need the discrete gradient, the discrete divergence operator and the vector Laplace operator. Moreover, we will also define the discrete rotation and curl operators. For the gradient:

$$(\operatorname{grad} p) = \begin{pmatrix} \frac{\partial p}{\partial r} \\ \frac{1}{r} \frac{\partial p}{\partial \varphi} \end{pmatrix} \quad (3.2)$$

we shall use the approximation:

$$(\text{grad}_h p_h)_{ij} := \begin{pmatrix} (p_{i+1,j} - p_{i,j})/h_r \\ \frac{1}{\bar{r}_i}(p_{i,j+1} - p_{i,j})/h_\varphi \end{pmatrix} \quad (3.3)$$

where $h_r := 1/n$, $h_\varphi := 1/m$ and $\bar{r}_i := (i - 1/2)h_r$, $i = 1, \dots, n$, $j = 1, \dots, m$. For the approximation of the divergence operator, we shall use:

$$(\text{div}_h \vec{u}_h)_{ij} := \frac{r_i u_{i,j} - r_{i-1} u_{i-1,j}}{\bar{r}_i h_r} + \frac{v_{i,j} - v_{i,j-1}}{\bar{r}_i h_\varphi}, \quad (3.4)$$

where $r_i := ih_r$, $i = 1, \dots, n$, $j = 1, \dots, m$ and $r_0 u_{0,j} = 0$.

For the rotation:

$$(\text{rot } \vec{u}) = \frac{1}{r} \left(\frac{\partial}{\partial r}(rv) - \frac{\partial u}{\partial \varphi} \right) \quad (3.5)$$

we shall use the approximation:

$$(\text{rot}_h \vec{u}_h)_{ij} := \frac{\bar{r}_{i+1} v_{i+1,j} - \bar{r}_i v_{i,j}}{r_i h_r} - \frac{u_{i,j+1} - u_{i,j}}{r_i h_\varphi}, \quad (3.6)$$

where $i = 1, \dots, n-1$, $j = 1, \dots, m$, and

$$(\text{rot}_h \vec{u}_h)_{nj} := \frac{\bar{r}_n v_{n,j}}{r_n h_r / 2}$$

where $j = 1, \dots, m$. For scalar functions, we define the operator curl as follows:

$$(\text{curl } \psi) := \begin{pmatrix} \frac{1}{r} \frac{\partial \psi}{\partial \varphi} \\ -\frac{\partial \psi}{\partial r} \end{pmatrix},$$

and we shall approximate this operator in the following way:

$$(\text{curl } \psi)_{ij} := \begin{pmatrix} \frac{1}{r_i} \frac{\psi_{ij} - \psi_{i,j-1}}{h_\varphi} \\ -\frac{\psi_{ij} - \psi_{i-1,j}}{h_r} \end{pmatrix}.$$

Since

$$(\text{grad div } \vec{u} - \text{curl rot } \vec{u}) = \begin{pmatrix} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r}(ru) + \frac{1}{r} \frac{\partial v}{\partial \varphi} \right) - \frac{1}{r} \frac{\partial}{\partial \varphi} \left(\frac{1}{r} \frac{\partial}{\partial r}(rv) - \frac{1}{r} \frac{\partial u}{\partial \varphi} \right) \\ \frac{1}{r} \frac{\partial}{\partial \varphi} \left(\frac{1}{r} \frac{\partial}{\partial r}(ru) + \frac{1}{r} \frac{\partial v}{\partial \varphi} \right) + \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r}(rv) - \frac{1}{r} \frac{\partial u}{\partial \varphi} \right) \end{pmatrix},$$

that is

$$(\text{grad div } \vec{u} - \text{curl rot } \vec{u}) = \begin{pmatrix} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) - \frac{u}{r^2} - \frac{2}{r^2} \frac{\partial v}{\partial \varphi} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} \\ \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v}{\partial r} \right) - \frac{v}{r^2} + \frac{2}{r^2} \frac{\partial u}{\partial \varphi} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \varphi^2} \end{pmatrix},$$

identity (2.10) holds in polar coordinate system in the following form:

$$\begin{pmatrix} \Delta_{r\varphi} u - \frac{u}{r^2} - \frac{2}{r^2} \frac{\partial v}{\partial \varphi} \\ \Delta_{r\varphi} v - \frac{v}{r^2} + \frac{2}{r^2} \frac{\partial u}{\partial \varphi} \end{pmatrix} = (\text{grad div } \vec{u} - \text{curl rot } \vec{u}). \quad (3.7)$$

Using identity (3.7) and the approximation of the divergence (3.4), rotation (3.6) and curl (3.7), we can approximate the Stokes problem (2.1), (2.2), (2.3), (2.4) in the following form:

$$\begin{pmatrix} \tilde{A}_h & \tilde{B}_h^T \\ \tilde{B}_h & 0 \end{pmatrix} \begin{pmatrix} \vec{u} \\ p \end{pmatrix} = \begin{pmatrix} \vec{f} \\ 0 \end{pmatrix}. \quad (3.8)$$

where $\tilde{B}_h$ corresponds to the negative divergence operator and $-\tilde{A}_h$ corresponds to the left side of identity (3.7) and is the following:

$$\begin{aligned} -(\tilde{A}_h \vec{u}_h)_{ij} &: = \frac{1}{h_r} \left(\frac{1}{\bar{r}_{i+1}} ((ru)_r)_{i+1,j} - \frac{1}{\bar{r}_i} ((ru)_r)_{i,j} \right) + \\ &+ \frac{1}{h_r} \left(\frac{1}{\bar{r}_{i+1}} (v_\varphi)_{i+1,j} - \frac{1}{\bar{r}_i} (v_\varphi)_{i,j} \right) + \\ &+ \frac{1}{h_\varphi r_i^2} \left((u_\varphi)_{i,j+1} - (u_\varphi)_{i,j} \right) - \\ &- \frac{1}{h_\varphi r_i^2} \left(((\bar{r}v)_{\bar{r}})_{i+1,j} - ((\bar{r}v)_{\bar{r}})_{i+1,j-1} \right) \end{aligned} \quad (3.9)$$

$$\begin{aligned} -(\tilde{A}_h \vec{v}_h)_{ij} &: = \frac{1}{h_r} \left(\frac{1}{r_i} ((\bar{r}v)_{\bar{r}})_{i+1,j} - \frac{1}{r_{i-1}} ((\bar{r}v)_{\bar{r}})_{i,j} \right) + \\ &+ \frac{1}{h_r} \left(\frac{1}{r_{i-1}} (u_\varphi)_{i-1,j+1} - \frac{1}{r_i} (u_\varphi)_{i,j+1} \right) + \\ &+ \frac{1}{h_\varphi \bar{r}_i^2} \left((v_\varphi)_{i,j+1} - (v_\varphi)_{i,j} \right) - \\ &- \frac{1}{h_\varphi \bar{r}_i^2} \left(((ru)_r)_{i,j+1} - ((ru)_r)_{i,j} \right), \end{aligned} \quad (3.10)$$

where $i = 1, \dots, n$, $j = 1, \dots, m$ and taking into account (3.1), and we used the following notations:

$$\begin{aligned} (u_\varphi)_{i,j} &: = \frac{u_{i,j} - u_{i,j-1}}{h_\varphi}, \\ (v_\varphi)_{i,j} &: = \frac{v_{i,j} - v_{i,j-1}}{h_\varphi}, \\ ((ru)_r)_{i,j} &: = \frac{r_i u_{i,j} - r_{i-1} u_{i-1,j}}{h_r}, \\ ((\bar{r}v)_{\bar{r}})_{i,j} &: = \frac{\bar{r}_i v_{i,j} - \bar{r}_{i-1} v_{i-1,j}}{h_r}. \end{aligned} \quad (3.11)$$

We also introduce the following notation: $\tilde{C}_h$ for the matrix of the operator rot_h .

For pressure vectors p_h, q_h and velocity vectors $\vec{u}_h = (u_h, v_h)^T$, $\vec{w}_h = (t_h, s_h)^T$ the following discrete scalar products and the corresponding norms are introduced: and

$$\begin{aligned} (p_h, q_h)_{0,r,h} &:= \sum_{i=1}^n \sum_{j=1}^m p_{ij} q_{ij} r_i h_r h_\varphi, \\ \|p_h\|_{0,r,h}^2 &:= (p_h, p_h)_{0,r,h}, \end{aligned} \quad (3.12)$$

and

$$\begin{aligned} (\vec{u}_h, \vec{w}_h)_{0,h} &:= \sum_{i=1}^n \sum_{j=1}^m (u_{ij} t_{ij} r_i + v_{ij} s_{ij} \bar{r}_i) h_r h_\varphi \\ \|\vec{u}_h\|_{0,h}^2 &:= (\vec{u}_h, \vec{u}_h)_{0,h}. \end{aligned} \quad (3.13)$$

Theorem 1. $(\tilde{A}_h \vec{u}_h, \vec{u}_h)_{0,h} = \|\tilde{B}_h \vec{u}_h\|_{0,\bar{r},h}^2 + \|\tilde{C}_h \vec{u}_h\|_{0,r,h}^2$ holds for all vectors $\vec{u}_h := (u_h, v_h)^T \in \vec{V}_h$

PROOF. We apply partial summation to $(\tilde{A}_h \vec{u}_h, \vec{u}_h)_{0,h}$:

$$\begin{aligned}
(\tilde{A}_h \vec{u}_h, \vec{u}_h)_{0,h} &= \sum_{i=1}^{n-1} \sum_{j=1}^m \frac{h_\varphi}{\bar{r}_{i+1} h_r} (r_{i+1} u_{i+1,j} - r_i u_{i,j})^2 + \\
&+ \sum_{j=1}^m \frac{h_\varphi}{\bar{r}_1 h_r} (r_1 u_{1,j})^2 + \sum_{i=1}^n \sum_{j=1}^m \frac{h_r}{r_i h_\varphi} (u_{i,j+1} - u_{i,j})^2 - \\
&- \sum_{i=1}^n \sum_{j=1}^m \left(\frac{r_i}{\bar{r}_{i+1}} (v_{i+1,j} u_{i,j} - v_{i+1,j-1} u_{i,j}) - \frac{r_i}{\bar{r}_i} (v_{i,j} u_{i,j} - v_{i,j-1} u_{i,j}) \right) - \\
&- \sum_{i=1}^n \sum_{j=1}^m \left(\frac{\bar{r}_{i+1}}{r_i} (v_{i+1,j-1} u_{i,j} - v_{i+1,j} u_{i,j}) - \frac{\bar{r}_i}{r_i} (v_{i,j-1} u_{i,j} - v_{i,j} u_{i,j}) \right) + \\
&+ \sum_{i=1}^{n-1} \sum_{j=1}^m \frac{h_\varphi}{r_i h_r} (\bar{r}_{i+1} v_{i+1,j} - \bar{r}_i v_{i,j})^2 + \\
&+ \sum_{j=1}^m \frac{h_\varphi}{r_n h_r} (\bar{r}_n v_{n,j})^2 + \sum_{i=1}^n \sum_{j=1}^m \frac{h_r}{\bar{r}_i h_\varphi} (v_{i,j+1} - v_{i,j})^2 - \\
&- \sum_{i=1}^n \sum_{j=1}^m \left(\frac{\bar{r}_i}{r_{i-1}} (u_{i-1,j+1} v_{i,j} - u_{i-1,j} v_{i,j}) - \frac{\bar{r}_i}{r_i} (u_{i,j+1} v_{i,j} - u_{i,j} v_{i,j}) \right) - \\
&- \sum_{i=1}^n \sum_{j=1}^m \left(\frac{r_{i-1}}{\bar{r}_i} (u_{i-1,j} v_{i,j} - u_{i-1,j+1} v_{i,j}) - \frac{r_i}{\bar{r}_i} (u_{i,j} v_{i,j} - u_{i,j+1} v_{i,j}) \right).
\end{aligned} \tag{3.14}$$

$\|\tilde{B}_h \vec{u}_h\|_{0,\bar{r},h}^2$ may be written in the following form:

$$\begin{aligned}
\|\tilde{B}_h \vec{u}_h\|_{0,\bar{r},h}^2 &= \sum_{i=1}^n \sum_{j=1}^m \left(\frac{h_\varphi}{\bar{r}_i h_r} (r_i u_{i,j} - r_{i-1} u_{i-1,j})^2 + \right. \\
&+ \left. \frac{h_r}{\bar{r}_i h_\varphi} (v_{i,j} - v_{i,j-1})^2 + \frac{2}{\bar{r}_i} (r_i u_{i,j} - r_{i-1} u_{i-1,j}) (v_{i,j} - v_{i,j-1}) \right).
\end{aligned} \tag{3.15}$$

$\|\tilde{C}_h \vec{u}_h\|_{0,r,h}^2$ can be written as follows:

$$\begin{aligned}
\|\tilde{C}_h \vec{u}_h\|_{0,r,h}^2 &= \sum_{i=1}^n \sum_{j=1}^m \left(\frac{h_r}{r_i h_\varphi} (u_{i,j+1} - u_{i,j})^2 + \frac{h_\varphi}{r_i h_r} (\bar{r}_{i+1} v_{i+1,j} - \bar{r}_i v_{i,j})^2 - \right. \\
&- \left. \frac{2}{r_i} (u_{i,j+1} - u_{i,j}) (\bar{r}_{i+1} v_{i+1,j} - \bar{r}_i v_{i,j}) \right).
\end{aligned} \tag{3.16}$$

Performing some index shifts and taking into account the periodicity in the direction φ and the homogeneous boundary conditions we get:

$$(\tilde{A}_h \vec{u}_h, \vec{u}_h)_{0,h} - \|\tilde{B}_h \vec{u}_h\|_{0,\bar{r},h}^2 - \|\tilde{C}_h \vec{u}_h\|_{0,r,h}^2 = 0.$$

□

Remark 1. $\tilde{A}_h$ a symmetric matrix in the sense of the scalar product (3.13). This means that Theorem can be described in matrix terms as follows:

$$\begin{aligned} (D_{\tilde{A}} \tilde{A}_h \vec{u}_h, \vec{u}_h) &= (D_{\tilde{B}} \tilde{B}_h \vec{u}_h, \tilde{B}_h \vec{u}_h) + (D_{\tilde{C}} \tilde{C}_h \vec{u}_h, \tilde{C}_h \vec{u}_h) = \\ &= (\tilde{B}_h^T D_{\tilde{B}} \tilde{B}_h \vec{u}_h, \vec{u}_h) + (\tilde{C}_h^T D_{\tilde{C}} \tilde{C}_h \vec{u}_h, \vec{u}_h), \end{aligned} \quad (3.17)$$

where $(.,.)$ is the Euclidean scalar product and $D_{\tilde{A}}, D_{\tilde{B}}, D_{\tilde{C}}$ are diagonal matrices corresponding to (3.13) and (3.12):

$$\begin{aligned} D_{\tilde{A}} &= \text{diag}(r_1 I_{m \times m}, \dots, r_{n-1} I_{m \times m}, \bar{r}_1 I_{m \times m}, \dots, \bar{r}_n I_{m \times m}), \\ D_{\tilde{B}} &= \text{diag}(\bar{r}_1 I_{m \times m}, \dots, \bar{r}_n I_{m \times m}), \\ D_{\tilde{C}} &= \text{diag}(r_1 I_{m \times m}, \dots, r_n I_{m \times m}). \end{aligned} \quad (3.18)$$

From (3.17) we get:

$$D_{\tilde{A}} \tilde{A}_h = \tilde{B}_h^T D_{\tilde{B}} \tilde{B}_h + \tilde{C}_h^T D_{\tilde{C}} \tilde{C}_h. \quad (3.19)$$

That is $D_{\tilde{A}} \tilde{A}_h$ is a symmetric matrix and can be written in the following form:

$$D_{\tilde{A}} \tilde{A}_h =: A_h = B_h + C_h, \quad (3.20)$$

where $B_h = \hat{B}_h^T \hat{B}_h$ and $C_h = \hat{C}_h^T \hat{C}_h$ with the notation $\hat{B}_h = D_{\tilde{B}}^{1/2} \tilde{B}_h$, $\hat{C}_h = D_{\tilde{C}}^{1/2} \tilde{C}_h$.

4. NUMERICAL RESULTS

Using the notations (3.20), our problem consists in finding the solution of the following algebraic system:

$$A_h \vec{u}_h + \hat{B}_h^T p_h = \vec{f}_h, \quad (4.1)$$

$$\hat{B}_h \vec{u}_h = g_h. \quad (4.2)$$

We use the Uzawa-algorithm ([9]) to solve (4.1), (4.2):

$$\begin{aligned} p_h^{(0)} &:= 0, \\ p_h^{(i+1)} &:= p_h^{(i)} + \omega(\hat{B}_h \vec{u}_h^{(i)} - g_h) \\ \vec{u}_h^{(i)} &:= A_h^{-1}(\vec{f}_h - \hat{B}_h^T p_h^{(i)}) \\ i &= 0, 1, 2, \dots \end{aligned} \quad (4.3)$$

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Author's Address

Gy. Strauber:

COLLEGE OF DUNAÚJVÁROS, DEPT. OF INFORMATICS, TÁNCICS M. U. 1/A, H-2400 DUNAÚJVÁROS, HUNGARY
E-mail address: strauber@mail.poliud.hu

(4.3) can be written in the following form:

$$\begin{aligned} p_h^{(0)} &= 0, \\ p_h^{(i+1)} &= p_h^{(i)} + \omega(\psi_h - S_h p_h^{(i)}) \\ i &= 0, 1, 2, \dots \end{aligned} \quad (4.4)$$

where $\psi_h = \hat{B}_h A_h^{-1} \vec{f}_h - g_h$ and S_h is the discrete Schur complement operator, that is $S_h = \hat{B}_h A_h^{-1} \hat{B}_h^T$. Since the discrete Crouzeix-Velte decomposition exists, using the Uzawa-algorithm we can reach the third Crouzeix-Velte subspace after at most 2 steps ([13]). In this subspace the spectrum of the Schur complement is closer, and the algorithm performs effective convergence. We introduce the following notations: the optimal iteration parameter is $\omega_{opt}^h := 2/(\underline{\lambda}_h + \bar{\lambda}_h)$, where $\underline{\lambda}_h$ and $\bar{\lambda}_h$ are the smallest and the largest of the eigenvalues different from 0 and 1 of the discrete Schur complement; and $\omega = 2$ is the optimal iteration parameter for the undiscretized Uzawa-algorithm, see ([13]). Table 1 shows the values of the optimal iteration parameter (ω_{opt}^h) and the smallest and largest eigenvalues ($\underline{\lambda}_h, \bar{\lambda}_h$).

Table 1

$n = m$	5	10	20	40	60
$\underline{\lambda}_h$	0.5133234	0.5036472	0.5009311	0.5002340	0.5001041
$\bar{\lambda}_h$	0.5248821	0.5188427	0.5154488	0.5138629	0.5133509
ω_{opt}^h	1.926401	1.95601	1.967768	1.972198	1.973447

In a first numerical experiment in (4.1), (4.2) we took random values for the exact solution of p_h which was projected to the orthogonal complement of the kernel of $\hat{B}_h^T$. Here ψ_h was calculated as $\psi_h := S_h p_h$. After $i = 1, 2$ initial steps with $\omega = 1$, the optimal parameters have been taken. In Table 2 $\langle i; \omega_{opt}^h \rangle$ means the number of iterations (included the initial steps) in the case of i initial steps and after that iterations with ω_{opt}^h . $\langle i; \omega = 2 \rangle$ means the number of iterations in the case of i initial steps followed by iterations with $\omega = 2$. For comparison we show the number of iterations in the case of $\omega = 1$ iteration parameter in all steps, which is in widespread use, denoted by $\langle 1; \omega = 1 \rangle$. The numerical convergence rate $q := (\|e^{(it)}\|/\|e^{(0)}\|)^{1/it}$ is also shown, where $e^{(0)}$ is the initial pressure error and $e^{(it)}$ is the final error after it iterations. The stopping criterion is that the initial pressure error in the Euclidean norm has been reduced by a factor of 10^{-5} . In the inner iteration (in the conjugate gradient method for details see below), it is necessary to use a stronger stopping criterion: that the initial error in the Euclidean norm has been decreased at least by a factor of 10^{-6} . (If in the outer iteration the initial pressure error is reduced by a factor of 10^{-10} , then in the inner iteration it is necessary to decrease the initial error at least by a factor of 10^{-11} .)

Table 2

$n = m$	5	10	20	40	60
$\langle 1; \omega = 1 \rangle$	14	15	16	17	17
q	0.4179	0.4172	0.4131	0.4166	0.4129
$\langle 1; \omega_{opt}^h \rangle$	4	4	4	4	4
q	0.0056	0.0057	0.0046	0.0041	0.0041
$\langle 1; \omega = 2 \rangle$	5	4	4	4	4
q	0.0291	0.0138	0.0105	0.0090	0.0083
$\langle 2; \omega_{opt}^h \rangle$	4	5	5	5	5
q	0.0029	0.0044	0.0041	0.0034	0.0032
$\langle 2; \omega = 2 \rangle$	5	5	5	5	5
q	0.0144	0.0106	0.0090	0.0074	0.0067

In the second numerical experiment, an algebraic Stokes problem with known solution was generated. (The exact solution is: $u_{exact} = r \cos \varphi + 2r \sin \varphi + 3$, $v_{exact} = 4r \cos \varphi - r \sin \varphi + 2$ and $p_{exact} = 2r \cos \varphi$. In this case $\text{div}(u, v) = 0$ and the solvability condition of (4.1), (4.2) is satisfied.) The number of outer iterations and the numerical convergence rate are shown in Table 3 depending on the iteration parameter. Here, iteration was stopped if the initial pressure error had been reduced by a factor of 10^{-6} . In the inner iteration the initial error in the Euclidean norm was decreased by a factor of 10^{-7} .

Table 3

$n = m$	5	10	20	40	60	80
$\langle 1; \omega = 1 \rangle$	21	22	23	24	25	25
q	0.4693	0.5022	0.5606	0.5784	0.6354	0.6143
$\langle 1; \omega_{opt}^h \rangle$	5	5	5	5	5	-
q	0.0341	0.0690	0.0742	0.0932	0.0110	-
$\langle 1; \omega = 2 \rangle$	5	4	4	4	3	3
q	0.0342	0.0283	0.0312	0.0422	0.0043	0.00731
$\langle 2; \omega_{opt}^h \rangle$	5	6	6	6	6	-
q	0.0112	0.0690	0.0742	0.0932	0.0092	-
$\langle 2; \omega = 2 \rangle$	6	5	5	4	4	4
q	0.0341	0.0283	0.0312	0.0087	0.0043	0.0073

Because of the large amount of memory needed, ω_{opt}^h was not calculated in the case of $n = m = 80$.

Instead of the calculation of A_h^{-1} in (4.3) we used the fast Fourier transformation in combination with the preconditioned conjugate gradient method. For this, let us introduce the following notations:

$$\begin{aligned}
 \vec{u}_{h,F} &:= Q_{2n-1}^* \vec{u}_h, & p_{h,F} &:= Q_n^* p_h, \\
 A_{h,F} &:= Q_{2n-1}^* A_h Q_{2n-1}, & \hat{B}_{h,F} &:= Q_n^* \hat{B}_h Q_{2n-1},
 \end{aligned} \tag{4.5}$$

where Q_k , the matrix of the Fourier transformation, is a block-diagonal matrix with k blocks $Q = (q_{j,l})_{j,l=1}^m$ and $q_{j,l} = \sqrt{1/m} e^{ih_\varphi lj}$. Now, instead of (4.3) the following iteration for the transformed variables can be used:

$$\begin{aligned}\vec{f}_{h,F} &:= Q_{2n-1}^* \vec{f}_h, & g_{h,F} &:= Q_n^* g_h, \\ p_{h,F}^{(0)} &:= 0,\end{aligned}\quad (4.6)$$

$i = 0, 1, 2, \dots, \max_{it} :$

$$\begin{aligned}\left[p_{h,F}^{(i+1)} \right] &:= p_{h,F}^{(i)} + \omega(\hat{B}_{h,F} \vec{u}_{h,F}^{(i)} - g_{h,F}) \\ \vec{u}_{h,F}^{(i)} &:= A_{h,F}^{-1}(\vec{f}_{h,F} - \hat{B}_{h,F}^* p_{h,F}^{(i)}) \\ \vec{u}_h &:= Q_{2n-1} \vec{u}_{h,F}, & p_h &:= Q_n p_{h,F},\end{aligned}\quad (4.7)$$

Observe that in a cartesian coordinate system, $A_{h,F}$ would be block-diagonal with tridiagonal blocks. In our case of polar coordinates, however, $A_{h,F}$ has 7 nonzero diagonals, the main diagonal, the $\pm m$ -th, the $\pm(m-1) * n$ -th and the $\pm(m-1) * n + m$ -th diagonal (using the Matlab notation). To solve the equation $A_{h,F} x = b$ the preconditioned conjugate gradient method was used combined with incomplete Gauss-elimination. (Additional diagonals were not used in the incomplete Gauss-elimination. This version is sometimes called ILU(0).) In the numerical experiments several ILU(0)-type preconditioning matrices fitting the structure of $A_{h,F}$ were investigated and T_{opt} was found to be the optimal preconditioner from the 5 preconditioners considered, resulting in the fastest convergence with a few number of iterations. T_{opt} is a tridiagonal matrix consisting of the main diagonal and the m -th and $-m$ -th diagonal of $A_{h,F}$. In the numerical experiments above, this preconditioning matrix was used. Table 4 displays the number of inner iterations needed to reach the stopping criterion of the conjugate gradient method, in that the initial error in the Euclidean norm has been decreased by a factor of 10^{-4} . In the Table T_{opt} shows that the matrix T_{opt} was used as the preconditioning matrix, and the unpreconditioned conjugate gradient method is denoted by $T = I$. For comparison we show the number of iterations with the preconditioning matrix, denoted by T_D , which contains only the main diagonal of $A_{h,F}$ and two other preconditioning matrices denoted by T_* and T_{**} . Here T_* is a tridiagonal matrix consisting of the main diagonal and the $(m-1) * n + m$ -th and $-((m-1) * n + m)$ -th diagonal of $A_{h,F}$ and T_{**} is a pentadiagonal matrix with the main diagonal and the $\pm m$ -th and $\pm(m-1) * n + m$ -th diagonal of $A_{h,F}$.

Table 4

$n = m$	5	10	20	50	100	200	500	640
T_{opt}	9	12	12	12	8	5	5	4
$T = I$	25	59	76	197	362	-	-	-
T_D	19	39	72	106	125	-	-	-
T_*	19	39	72	107	125	-	-	-
T_{**}	7	8	8	8	5	3	2	2

Because of the large amount of computational time needed, the number of the iteration was not calculated in the case of $n = m \geq 200$, and the preconditioning matrices are $T = I, T_D$ or T_* .

In Table 5 we show the computational time of the conjugate gradient method (inner iteration) to reach our stopping criterion 10^{-4} , and the necessary memory for the incomplete Gauss-elimination using the different preconditioning matrices and $n = m = 50$.

Table 5

$n = m = 50$	computational time (s)	memory needed (byte)
T_{opt}	32.28	2400
$T = I$	202.70	-
T_D	163.95	800
T_*	326.47	2400
T_{**}	63.92	4000

The full computational time of the second numerical experiment in the case of $\langle 1; \omega = 2 \rangle$ and $n = m = 50$ is the following: the time of the FFT and IFFT and the calculation of ψ_h are altogether 25, 155 s and the full time of the outer iteration (see (4.7), $\max_{it} = 3$) is 121.125 s. Here the stopping criterion of the inner iteration is 10^{-4} and the stopping criterion of the outer iteration is 10^{-3} and we used T_{opt} as the preconditioning matrix.

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Egyetemváros, Hungary;
mechszgy@gold.uni-miskolc.hu

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University of Miskolc; H-3515 Miskolc–
Egyetemváros, Hungary;
matszj@gold.uni-miskolc.hu

ZSOLT TUZA, Computer and Automation
Institute, Hungarian Academy of Sciences;
13-17 Kende St., H-1111 Budapest, Hungary;
tuza@sztaki.hu

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